HARMONIC ANALYSIS, ERGODIC THEORY AND COUNTING FOR THIN GROUPS

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Abstract. For a geometrically finite group $\Gamma$ of $G = \text{SO}(n, 1)$, we survey recent developments on counting and equidistribution problems for orbits of $\Gamma$ in a homogeneous space $H\backslash G$ where $H$ is trivial, symmetric or horospherical. Main applications are found in an affine sieve on orbits of thin groups as well as in sphere counting problems for sphere packings invariant under a geometrically finite group. In our sphere counting problems, spheres can be ordered with respect to a general conformal metric.

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1. Introduction

In these notes, we will discuss counting and equidistribution problems for orbits of thin groups in homogeneous spaces.

Let $G$ be a connected semisimple Lie group and $H$ a closed subgroup. We consider the homogeneous space $V = H\backslash G$ and fix the identity coset $x_0 = [e]$. Let $\Gamma$ be a discrete subgroup of $G$ such that the orbit $x_0\Gamma$ is discrete, and let $\{B_T : T > 1\}$ be a family of compact subsets of $V$ whose volume tends to infinity as $T \to \infty$. Understanding the asymptotic of $\#(x_0\Gamma \cap B_T)$ is a fundamental problem which bears many applications in number theory and geometry.
When $\Gamma$ is a lattice in $G$, i.e., when $\Gamma \backslash G$ admits a finite invariant measure, this problem is well understood: assuming that the boundaries of $B_T$’s are sufficiently regular and that $H \cap \Gamma$ is a lattice in $H$, $\#(x_0 \Gamma \cap B_T)$ is asymptotically proportional to the volume of $B_T$, computed with respect to a suitably normalized $G$-invariant measure in $V$.

When $\Gamma$ is a thin group, i.e., a Zariski dense subgroup which is not a lattice in $G$, this type of counting problem is little understood in a general setting. In the present notes, we focus on the case when $G$ is the special orthogonal group $\text{SO}(n,1)$ and $H$ is either a symmetric subgroup or a horospherical subgroup (the case of $H$ being the trivial subgroup will be treated as well). In this case, we have a more or less good understanding for the counting problem for groups $\Gamma$ satisfying a certain finiteness property.

By the fundamental observation of Duke, Rudnick and Sarnak [14], the counting problem for $x_0 \Gamma \cap B_T$ can be well approached via the following equidistribution problem:

Describe the asymptotic distribution of $\Gamma \backslash \Gamma Hg$ in $\Gamma \backslash G$ as $g \rightarrow \infty$.

The assumption that $H$ is either symmetric or horospherical is made so that we can approximate the translate $\Gamma \backslash \Gamma Hg$ locally by the matrix coefficient function in the quasi-regular representation space $L^2(\Gamma \backslash G)$. This idea goes back to Margulis’s thesis in 1970 [28] (also see [23]), but a more systematic formulation in our setting is due to Eskin and McMullen [16].

The fact that the trivial representation is contained in $L^2(\Gamma \backslash G)$ for $\Gamma$ lattice is directly related to the phenomenon that, when $\Gamma \backslash \Gamma H$ is of finite volume, the translate $\Gamma \backslash \Gamma Hg$ becomes equidistributed in $\Gamma \backslash G$ with respect to a $G$-invariant measure as $g \rightarrow \infty$ in $H \backslash G$. When $\Gamma$ is not a lattice, the minimal subrepresentation, that is, the subrepresentation with the slowest decay of matrix coefficients of $L^2(\Gamma \backslash G)$, is infinite dimensional and this makes the distribution of $\Gamma \backslash \Gamma Hg$ much more intricate, and understanding it requires introducing several singular measures in $\Gamma \backslash G$. A key input is the work of Roblin [40] on the asymptotic of matrix coefficients for $L^2(\Gamma \backslash G)$, which he proves using ergodic theoretic methods.

There is a finer distinction among thin groups depending on the size of their critical exponents. When the critical exponent $\delta$ of $\Gamma$ is bigger than $(n - 1)/2$, the work of Lax and Phillips [26], together with the classification of the unitary dual of $G$, implies that the minimal subrepresentation of $L^2(\Gamma \backslash G)$ is isolated; in other words, $L^2(\Gamma \backslash G)$ possesses a
spectral gap property. Developing Harish-Chandra’s work on harmonic analysis on $G$ in combination with Roblin’s work on ergodic theory, we obtain an effective version of the asymptotic of matrix coefficients for $L^2(\Gamma\backslash G)$ in joint work with Mohammadi [31]. This enables us to state an effective equidistribution of $\Gamma\backslash \Gamma H g$ in $\Gamma\backslash G$. Similar to the condition that $\Gamma\backslash \Gamma H$ is of finite volume in the case of $\Gamma$ lattice, there is also certain restriction on the size of the orbit $\Gamma\backslash \Gamma H$ in order to deduce such an equidistribution. A precise condition is that the skinning measure of $\Gamma\backslash \Gamma H$, introduced in [35], is finite. When the skinning measure, say $\mu_{\text{PS}}^H$, is compactly supported, the passage from the asymptotic of the matrix coefficient to the equidistribution of $\Gamma\backslash \Gamma H g$ can be done by the so-called usual thickening methods. However when $\mu_{\text{PS}}^H$ is not compactly supported, this step requires a genuinely different strategy from the lattice case via the study of the transversal intersections, carefully done in [35].

The error term in our effective equidistribution result of $\Gamma\backslash \Gamma H g$ depends only on the spectral gap of $\Gamma$. This enables us to state the asymptotic of $\#(x_0 \Gamma_d \gamma \cap B_T)$ effectively in a uniform manner for all $\gamma \in \Gamma$ and for any family $\{\Gamma_d \vartriangleleft \Gamma\}$ of subgroups of finite index which has a uniform spectral gap. When $\Gamma$ is a subgroup of an arithmetic subgroup of $G$, the work of Salehi-Golsefidy and Varju [41], extending an earlier work of Bourgain, Gamburd and Sarnak [6], implies that the family $\{\Gamma_d\}$ of congruence subgroups of $\Gamma$ is such a family.

A recent development on an affine sieve [7] then tells us that such a uniform effective counting statement can be used to describe the distribution of almost prime vectors.

One of the most beautiful applications of the study of thin orbital counting problem can be found in Apollonian circle packings. We will describe this application as well as its higher dimensional analogues. The ordering in counting circles can be done not only in the Euclidean metric, but also in general conformal metrics. It is due to this flexibility that we can also describe the asymptotic number of circles in the ideal triangle of the hyperbolic plane ordered by the hyperbolic area in the last section.

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2. \( \Gamma \)-invariant conformal densities and measures on \( \Gamma \setminus G \)

In the whole article, let \((\mathbb{H}^n, d)\) be the \(n\)-dimensional real hyperbolic space with constant curvature \(-1\) and \(\partial(\mathbb{H}^n)\) its geometric boundary. Set \(G := \text{Isom}^+(\mathbb{H}^n) \cong \text{SO}(n, 1)^0\). Let \(\Gamma\) be a discrete torsion-free subgroup of \(G\). We assume that \(\Gamma\) is non-elementary, or equivalently, \(\Gamma\) has no abelian subgroup of finite index.

We review some basic geometric and measure theoretic concepts for \(\Gamma\) and define several locally finite Borel measures (i.e., Radon measures) on \(\Gamma \setminus G\) associated to \(\Gamma\)-invariant conformal densities on \(\partial(\mathbb{H}^n)\). When \(\Gamma\) is a lattice, these measures all coincide with each other, being simply a \(G\)-invariant measure. But for a thin group \(\Gamma\), they are all different and singular, and appear in our equidistribution and counting statements. General references for this section are [38], [10], [37], [45], [46], [40].

2.1. Limit set and geometric finiteness. We denote by \(\delta_\Gamma = \delta\) the critical exponent of \(\Gamma\), i.e., the abscissa of convergence of the Poincare series \(\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))}\) for \(o \in \mathbb{H}^n\). We have \(0 < \delta \leq n - 1\). The limit set \(\Lambda(\Gamma)\) is defined to be the set of all accumulation points of \(\Gamma(z)\) in the compactification \(\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial(\mathbb{H}^n), z \in \mathbb{H}^n\). As \(\Gamma\) is discrete, \(\Lambda(\Gamma)\) is contained in the boundary \(\partial(\mathbb{H}^n)\).

**Definition 2.1.** (1) The convex core \(C(\Gamma)\) of \(\Gamma\) is the quotient by \(\Gamma\) of the smallest convex subset of \(\mathbb{H}^n\) containing all geodesics connecting points in \(\Lambda(\Gamma)\).

(2) \(\Gamma\) is called geometrically finite (resp. convex cocompact) if the unit neighborhood of the convex core of \(\Gamma\) has finite volume (resp. compact).

A lattice is clearly a geometrically finite group and so is a discrete group admitting a finite sided convex fundamental domain in \(\mathbb{H}^n\). An important characterization of a geometrically finite group is given in terms of its limit set. For this, we need to define: a point \(\xi \in \Lambda(\Gamma)\) is called a parabolic limit point for \(\Gamma\) if \(\xi\) is a unique fixed point of an element of \(\Gamma\) in \(\mathbb{H}^n\) and a radial limit point if the projection of a geodesic ray \(\xi_t\) toward \(\xi\) in \(\Gamma \setminus \mathbb{H}^n\) meets a compact subset for an unbounded sequence of time \(t\).

Now \(\Gamma\) is geometrically finite if and only if \(\Lambda(\Gamma)\) consists only of parabolic and radial limit points [10]. For \(\Gamma\) geometrically finite, its critical exponent \(\delta\) is equal to the Hausdorff dimension of \(\Lambda(\Gamma)\), and is \(n - 1\) only when \(\Gamma\) is a lattice in \(G\) [45].
2.2. Conformal densities. To define a conformal density, we first recall the Busemann function $\beta_\xi(x, y)$ for $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$:

$$
\beta_\xi(x, y) = \lim_{t \to \infty} d(x, \xi_t) - d(y, \xi_t)
$$

where $\xi_t$ is a geodesic toward $\xi$. Hence $\beta_\xi(x, y)$ measures a signed distance between horospheres based at $\xi$ passing through $x$ and $y$ (a horosphere based at $\xi$ is a Euclidean sphere in $\mathbb{H}^n$ tangent at $\xi$).

**Definition 2.2.** A $\Gamma$-invariant conformal density of dimension $\delta_\mu > 0$ is a family $\{\mu_x : x \in \mathbb{H}^n\}$ of finite positive measures on $\partial(\mathbb{H}^n)$ satisfying:

1. For any $\gamma \in \Gamma$, $\gamma_\ast \mu_x = \mu_{\gamma(x)}$;
2. $\frac{d\mu_x}{d\mu_y}(\xi) = e^{\delta_\mu \beta_\xi(y, x)}$ for all $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$.

It is easy to construct such a density of dimension $n - 1$, as we simply need to set $m_x$ to be the $\text{Stab}_G(x)$-invariant probability measure on $\mathbb{H}^n$. This is a unique up to scaling, and does not depend on $\Gamma$. We call it the Lebesgue density.

How about in other dimensions? A fundamental work of Patterson [37], generalized by Sullivan [45], shows the following by an explicit construction:

**Theorem 2.3.** There exists a $\Gamma$-invariant conformal density of dimension $\delta_\Gamma = \delta$.

Assuming that $\Gamma$ is of divergence type, i.e., its Poincare series diverges at $s = \delta$, Patterson’s construction can be summarized as follows: Fixing $o \in \mathbb{H}^n$, for each $x \in \mathbb{H}^n$, consider the finite measure on $\mathbb{H}^n$ given by

$$
\nu_{x, s} = \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))}} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(o))} \delta_\gamma(o).
$$

Then $\nu_x$ is the (unique) weak-limit of $\nu_{x, s}$ as $s \to \delta^+$, and $\{\nu_x : x \in \mathbb{H}^n\}$ is the desired density of dimension $\delta$.

In the following, we fix a $\Gamma$-invariant conformal density $\{\nu_x\}$ of dimension $\delta$, and call it the Patterson-Sullivan density (or simply the PS density). It is known to be unique up to scaling, when $\Gamma$ is of divergence type, e.g., geometrically finite groups.

Denoting by $\Delta$ the hyperbolic Laplacian on $\mathbb{H}^n$, the PS density is closely related to the bottom of the spectrum of $\Delta$ for its action on smooth functions on $\Gamma \setminus \mathbb{H}^n$. If we set $\phi_0(x) := |\nu_x|$ for each $x \in \mathbb{H}^n$, then

$$
\phi_0(x) = \int_{\xi \in \Lambda(\Gamma)} d\nu_x(\xi) = \int_{\xi \in \Lambda(\Gamma)} \frac{d\nu_x}{d\nu_o}(\xi) d\nu_o(\xi) = \int_{\xi \in \Lambda(\Gamma)} e^{\delta \beta_{\xi}(o, x)} d\nu_o(\xi).
$$
Since \( \gamma_x \nu_x = \nu_{\gamma(x)} \), we note that \( \phi_0(\gamma(x)) = \phi_0(x) \), i.e., \( \phi_0 \) is a function on \( \Gamma \backslash \mathbb{H}^n \), which is positive everywhere! Furthermore we have

1. \( \Delta(\phi_0) = \delta(n - 1 - \delta) \phi_0 \);
2. if \( \Gamma \) is geometrically finite, then \( \phi_0 \in L^2(\Gamma \backslash \mathbb{H}^n) \) if and only if \( \delta > (n - 1)/2 \).

2.3. Measures on \( \Gamma \backslash G \) associated to a pair of conformal densities. For \( v \in T^1(\mathbb{H}^n) \), we denote by \( v^+ \) and \( v^- \) the forward and backward endpoints of the geodesic determined by \( v \). Fixing \( o \in \mathbb{H}^n \), the map \( v \mapsto (v^+, v^-, s = \beta_{v^-}(o, v)) \) gives a homeomorphism between \( T^1(\mathbb{H}^n) \) and \( (\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n) - \text{diagonal}) \times \mathbb{R} \). Therefore we may use the coordinates \( (v^+, v^-, s = \beta_{v^-}(o, v)) \) of \( v \) in order to define measures on \( T^1(\mathbb{H}^n) \).

Let \( \{\mu_x\} \) and \( \{\mu'_x\} \) be \( \Gamma \)-invariant conformal densities on \( \partial(\mathbb{H}^n) \) of dimensions \( \delta_\mu \) and \( \delta_{\mu'} \) respectively. After Roblin [40], we define a measure \( m^{\mu,\mu'} \) on \( T^1(X) \) associated to \( \{\mu_x\} \) and \( \{\mu'_x\} \) by

\[
d\tilde{m}^{\mu,\mu'}(v) = e^{\delta_\mu \beta_{v^+}(o,x)} e^{\delta_{\mu'} \beta_{v^-}(o,x)} d\mu_o(v^+)d\mu'_o(v^-)ds.
\]

(2.4)

It follows from the \( \Gamma \)-invariant conformal properties of \( \{\mu_x\} \) and \( \{\mu'_x\} \) that the definition of \( \tilde{m}^{\mu,\mu'} \) is independent of the choice of \( o \in \mathbb{H}^n \) and that \( \tilde{m}^{\mu,\mu'} \) is left \( \Gamma \)-invariant. Hence it induces a Radon measure \( m^{\mu,\mu'} \) on the quotient space \( T^1(\Gamma \backslash \mathbb{H}^n) = \Gamma \backslash T^1(\mathbb{H}^n) \).

We will lift the measure \( m^{\mu,\mu'} \) to \( \Gamma \backslash G \). This lift depends on the choice of subgroups \( K, M \) and \( A = \{a_t\} \) of \( G \). Here \( K \) is a maximal compact subgroup of \( G \) and \( M \) is the stabilizer of a vector \( X_0 \in T^1(\mathbb{H}^n) \) based at \( o \in \mathbb{H}^n \) with \( K = G_o \). Via the isometric action of \( G \), we may identify the quotient spaces \( G/K \) and \( G/M \) with \( \mathbb{H}^n \) and \( T^1(\mathbb{H}^n) \) respectively. Let \( A = \{a_t\} \) be the one parameter subgroup of diagonalizable elements of \( G \) such that the right multiplication by \( a_t \) on \( G/M \) corresponds to the geodesic flow on \( T^1(\mathbb{H}^n) \) for time \( t \).

By abuse of notation, we use the same notation \( m^{\mu,\mu'} \) for the \( M \)-invariant extension of \( m^{\mu,\mu'} \) on \( \Gamma \backslash G/M = \Gamma \backslash T^1(\mathbb{H}^n) \) to \( \Gamma \backslash G \), that is, for \( \Psi \in C_c(\Gamma \backslash G) \),

\[
m^{\mu,\mu'}(\Psi) = \int_{x \in \Gamma \backslash G/M} \Psi^M(x)dm^{\mu,\mu'}(x)
\]

where \( \Psi^M(x) = \int_M \Psi(xm)dm \) for the probability \( M \)-invariant measure \( dm \) on \( M \).

The measures \( m^{\mu,\mu'} \) on \( \Gamma \backslash G \) where \( \mu \) and \( \mu' \) are the PS-density \( \{\nu_x\} \) or the Lebesgue density \( \{m_x\} \) are of special importance. We name them as follows:
• Bowen-Margulis-Sullivan measure: \( m_{\text{BMS}} := m_{\nu,\nu} \);
• Burger-Roblin measure: \( m_{\text{BR}} := m_{m,\nu} \);
• Burger-Roblin *-measure: \( m_{\text{BR}*} := m_{\nu,m} \);
• Haar measure: \( m_{\text{Haar}} := m_{m,m} \).

For brevity, we refer to these as BMS, BR, BR*, Haar measures respectively. As the naming indicates, \( m_{\text{Haar}} \) turns out to be a \( G \)-invariant measure. For \( g \in G \), we use the notation \( g^\pm \) for \((gM)^\pm\) with \( gM \) considered as a vector in \( T^1(\mathbb{H}^n) \). It is clear from the definition that the supports of BMS, BR, BR* measures are respectively given by \( \{ g \in \Gamma \backslash G : g^\pm \in \Lambda(\Gamma) \} \), \( \{ g \in \Gamma \backslash G : g^- \in \Lambda(\Gamma) \} \), and \( \{ g \in \Gamma \backslash G : g^+ \in \Lambda(\Gamma) \} \). In particular, the support of BMS measure is contained in the convex core of \( \Gamma \). Sullivan showed that \( |m_{\text{BMS}}| < \infty \) if \( \Gamma \) is geometrically finite.

The BMS, BR and BR* measures are respectively invariant under \( A, N^+ \) and \( N^- \) where \( N^+ \) (resp. \( N^- \)) denotes the expanding (resp. contracting) horospherical subgroup of \( G \) for \( a_t \):

\[ N^\pm = \{ g \in G : a_t g a_{-t} \to e \text{ as } t \to \pm \infty \} . \]

The finiteness of \( m_{\text{BMS}} \) turns out to be a critical condition for the ergodic theory on \( \Gamma \backslash G \).

**Theorem 2.5.** Suppose that \( |m_{\text{BMS}}| < \infty \) and that \( \Gamma \) is Zariski dense.

1. \( m_{\text{BMS}} \) is \( A \)-mixing: for any \( \Psi_1, \Psi_2 \in L^2(\Gamma \backslash G) \),

\[ \lim_{t \to \infty} \int_{\Gamma \backslash G} \Psi_1(ga_t)\Psi_2(g) \, dm_{\text{BMS}}(g) = \frac{1}{|m_{\text{BMS}}|} m_{\text{BMS}}(\Psi_1) \cdot m_{\text{BMS}}(\Psi_2) . \]

2. \( m_{\text{BR}} \) is the unique \( N^+ \)-invariant ergodic measure on \( \Gamma \backslash G \) which is not supported on a closed \( N^+ \)-orbit.

3. \( m_{\text{BR}} \) is a finite measure if and only if \( \Gamma \) is a lattice in \( G \).

The claim (1) is due to Flaminio and Spatzier [17]. For \( M \)-invariant functions, this was earlier proved by Babillot [1] and in this case the Zariski density assumption is not needed. (3) is proved in [35], using Ratner’s measure classification of finite measures invariant under unipotent flows [39]. Namely, we show that \( m_{\text{BR}} \) is not one of those homogeneous measures that her classification theorem lists for finite invariant measures. The claim (2) is a far reaching generalization by Roblin [40] of an earlier theorem of Burger [11] in the case of a convex cocompact surface.

In the spirit of Ratner’s measure classification theorem [39], we pose the following question:
Problem 2.6. Under the assumption of Theorem 2.5, let U be a connected unipotent subgroup of G or more generally a connected subgroup generated by unipotent one-parameter subgroups.

1. Classify all locally finite U-invariant ergodic measures in $\Gamma \backslash G$.
2. Describe the closures of U-orbits in $\Gamma \backslash G$.

The emphasis here is that we want to understand not only finite measures but all Radon measures. This seems to be a very challenging question. We mention a recent related result of [30]: if $\Gamma$ is a convex cocompact subgroup of $G$ and $U$ is a connected unipotent subgroup of $N^+$ of dimension $k$, $m^{BR}$ is $U$-ergodic if $\delta > (n-1) - k$.

3. Matrix coefficients for $L^2(\Gamma \backslash G)$

Let $K$ be a maximal compact subgroup of $G$ and fix a Cartan decomposition $G = KA^+K$. We parametrize $A^+ = \{a_t : t \geq 0\}$ so that $a_t$ corresponds to the geodesic flow on $T^1(\mathbb{H}^n) = G/M$ where $M := C_K(A^+)$.

Denote by $\hat{G}$ the unitary dual, i.e., the set of equivalence classes of all irreducible unitary representations of $G$. A representation $\pi \in \hat{G}$ is said to be tempered if for any $K$-finite vectors $v_1, v_2$ of $\pi$, the matrix coefficient function $g \mapsto \langle \pi(g)v_1, v_2 \rangle$ belongs to $L^{2+\epsilon}(G)$ for any $\epsilon > 0$. We write $\hat{G} = \hat{G}_{\text{temp}} \cup \hat{G}_{\text{non-temp}}$ as the disjoint union of tempered representations and non-tempered representations.

The well-known classification of the unitary dual $\hat{G}$ says that $\hat{G}_{\text{non-temp}}$ consists of the trivial representation and the complementary series representations [25]. By choosing a Casimir operator $C$ of the Lie algebra of $G$ normalized so that it acts on $C^\infty(G)^K = C^\infty(\mathbb{H}^n)$ by the negative Laplacian, we can parametrize the complementary series representations as $\{\mathcal{H}_s : s \in ((n-1)/2, (n-1))\}$ where $C$ acts on $\mathcal{H}_s^\infty$ by the scalar $s(s-n+1)$.

Let $\Gamma$ be a discrete, torsion-free, non-elementary subgroup of $G$. The right translation action of $G$ on $L^2(\Gamma \backslash G, m^{\text{Haar}})$ gives rise to a unitary representation, as $m^{\text{Haar}}$ is $G$-invariant. For $\Psi_1, \Psi_2 \in L^2(\Gamma \backslash G)$, the matrix coefficient function is a smooth function on $G$ defined by

$$g \mapsto \langle g, \Psi_1, \Psi_2 \rangle := \int_{\Gamma \backslash G} \Psi_1(xg)\overline{\Psi_2(x)}dm^{\text{Haar}}(x).$$

Understanding the asymptotic expansion of $\langle a_t, \Psi_1, \Psi_2 \rangle$ (as $t \to \infty$) is a basic problem in harmonic analysis as well as a main tool in our approach to the counting problem.
The quality of the error term in this type of the asymptotic expansion usually depends on Sobolev norms of $\Psi_i$'s. For $\Psi \in C^\infty(\Gamma \backslash G)$ and $d \in \mathbb{N}$, the $d$-th Sobolev norm of $\Psi$ is given by $S_d(\Psi) = \sum \|X(\Psi)\|_2$ where the sum is taken over all monomials $X$ in some fixed basis of the Lie algebra of $G$ of order at most $d$ and $\|X(\Psi)\|_2$ denotes the $L^2$-norm of $\Psi$.

3.1. Lattice case: $|m_{\text{Haar}}| < \infty$. Set

$$L_0^2(\Gamma \backslash G) := \{ \Psi \in L^2(\Gamma \backslash G) : \int \Psi dm_{\text{Haar}} = 0 \}.$$ 

When $\Gamma$ is a lattice, we have $L^2(\Gamma \backslash G) = \mathbb{C} \oplus L_0^2(\Gamma \backslash G)$. It is well-known that there exists $(n - 1)/2 < s_0 < (n - 1)$ such that $L_0^2(\Gamma \backslash G)$ does not contain any complementary series representation $\mathcal{H}_s$ of parameter $s \geq s_0$ [3]. This implies the following (cf. [24, Prop 5.3]):

**Theorem 3.1.** Suppose $\Gamma$ is a lattice in $G$, that is, $|m_{\text{Haar}}| < \infty$. There exists $\ell \in \mathbb{N}$ such that for any $\Psi_1, \Psi_2 \in L^2(\Gamma \backslash G) \cap C(\Gamma \backslash G)^\infty$, we have, as $t \to \infty$,

$$\langle a_t, \Psi_1, \Psi_2 \rangle = \frac{1}{m_{\text{Haar}}} m_{\text{Haar}}(\Psi_1) m_{\text{Haar}}(\Psi_2) + O(S_t(\Psi_1) S_t(\Psi_2) e^{(s_0 - n + 1)t})$$

where $m_{\text{Haar}}(\Psi_i) = \int_{\Gamma \backslash G} \Psi_i(x) dm_{\text{Haar}}(x)$ for $i = 1, 2$.

We note that the constant function $1/\sqrt{|m_{\text{Haar}}|}$ is a unit vector in $L^2(\Gamma \backslash G)$ and the main term $\frac{1}{m_{\text{Haar}}} m_{\text{Haar}}(\Psi_1) m_{\text{Haar}}(\Psi_2)$ is simply the product of the projections of $\Psi_1$ and $\Psi_2$ to the minimal subrepresentation space, which is $\mathbb{C}$, of $L^2(\Gamma \backslash G)$.

3.2. Discrete groups with $|m_{\text{BMS}}| < \infty$. When $\Gamma$ is not a lattice, we have $L^2(\Gamma \backslash G) = L_0^2(\Gamma \backslash G)$. By the well-known decay of the matrix coefficients of unitary representations with no $G$-invariant vectors due to Howe and Moore [21], we have, for any $\Psi_1, \Psi_2 \in L^2(\Gamma \backslash G)$,

$$\lim_{t \to \infty} \langle a_t, \Psi_1, \Psi_2 \rangle = 0.$$

However we have a much more precise description on the decay of $\langle a_t, \Psi_1, \Psi_2 \rangle$ due to Roblin:

**Theorem 3.2.** [10] Let $\Gamma$ be Zariski dense with $|m_{\text{BMS}}| < \infty$. For any $\Psi_1, \Psi_2 \in C_c(\Gamma \backslash G)$,

$$\lim_{t \to \infty} e^{(n - 1 - \delta)t} \langle a_t, \Psi_1, \Psi_2 \rangle = \frac{m_{\text{BR}}^\ast(\Psi_1) \cdot m_{\text{BR}}(\Psi_2)}{|m_{\text{BMS}}|}.$$
Roblin proved this theorem for $M$-invariant functions using the mixing of the geodesic flow due to Babillot [1]. His proof extends without difficulty to general functions, based on the $A$-mixing in $\Gamma \backslash G$ stated as in Theorem 2.5.

3.3. **Geometrically finite groups with** $\delta > (n - 1)/2$. In this subsection, we assume that $\Gamma$ is a geometrically finite group with $\delta > (n - 1)/2$. This assumption is needed for the existence of a spectral gap for $L^2(\Gamma \backslash G)$. Together with the aforementioned fact about $\hat{\mathcal{G}}_{\text{non-tem}}$, the works of Lax and Phillips [26] and of Sullivan [37] imply:

**Theorem 3.3.** There exist 

$$\frac{(n - 1)}{2} < s_k \leq \cdots \leq s_1 < s_0 \leq (n - 1)$$

such that

$$L^2(\Gamma \backslash G) = \mathcal{H}_{s_0} \oplus \cdots \oplus \mathcal{H}_{s_k} \oplus \mathcal{W}$$

where $\mathcal{W}$ consists of tempered representations. Moreover $\delta = s_0(n - 1 - s_0)$.

Note that the minimal representation $\mathcal{H}_{s_0}$ is of multiplicity one in $L^2(\Gamma \backslash G)$. The difference $s_1 - s_0 > 0$ is called the spectral gap for $\Gamma$. Theorem 3.3 implies that for $\Psi_1, \Psi_2 \in L^2(\Gamma \backslash G)$, the leading term of the asymptotic expansion of the matrix coefficient $\langle a_t \Psi_1, \Psi_2 \rangle$ is determined by $\langle a_t P_{s_0}(\Psi_1), P_{s_0}(\Psi_2) \rangle$ where $P_{s_0}$ is the projection operator from $L^2(\Gamma \backslash G)$ to $\mathcal{H}_{s_0}$.

Building up on the work of Harish-Chandra on the asymptotic behavior of the Eisenstein integrals (cf. [49], [50]), we obtain an asymptotic formula for $\langle a_t v, w \rangle$ for all $K$-isotypic vectors $v, w \in \mathcal{H}_{s_0}$. This extension alone does not quite explain the leading term of $\langle a_t P_{s_0}(\Psi_1), P_{s_0}(\Psi_2) \rangle$ in terms of functions $\Psi_1$ and $\Psi_2$; however, with the help of Theorem 3.2, we are able to prove the following:

**Theorem 3.4.** [31] There exist $\eta_0 > 0$ (depending only on the spectral gap for $\Gamma$) and $\ell \in \mathbb{N}$ such that for any $\Psi_1, \Psi_2 \in C^\infty_c(\Gamma \backslash G)$, as $t \to \infty$,

$$e^{(n-1-\delta)t} \langle a_t \Psi_1, \Psi_2 \rangle = \frac{m^{\text{BR}}(\Psi_1) \cdot m^{\text{BR}}_*(\Psi_2)}{|m^{\text{BMS}}|} + O(S_t(\Psi_1)S_t(\Psi_2)e^{-\eta_0 t}).$$

We reiterate that the leading term in Theorem 3.4 is from the ergodic theory and the error term is from the harmonic analysis and the spectral gap.

3.4. **Effective mixing for** $m^{\text{BMS}}$. Theorem 3.4 can be used to obtain an effective mixing for the BMS measure (i.e., an effective version of Theorem 2.5(1)) for geometrically finite groups with $\delta > (n - 1)/2$ [31].

For geometrically finite groups with $\delta \leq (n - 1)/2$, Lax-Phillips showed that the $L^2$-spectrum of the Laplacian is continuous [26] and
hence we cannot expect Theorem 3.4 to hold for such groups. However, a recent work of Guillarmou and Mazzeo [20] establishes meromorphic extensions of the resolvents of the Laplacian and of the Poincare series.

For $\Gamma$ convex cocompact, Stoyanov obtained, via the spectral properties of Ruelle transfer operators, an effective mixing of the geodesic flow for the BMS measure, regardless of the size of the critical exponent (see [47]). We remark that the idea of using spectral estimates of Ruelle transfer operators in obtaining an effective mixing was originated in Dolgopyat’s work [13].

It will be interesting to see if the results in [20] can be used to answer the question:

**Problem 3.5.** Prove an effective mixing of the geodesic flow for the BMS measure for all geometrically finite groups.

We close this section by posing the following:

**Problem 3.6.** Find a suitable analogue of Theorem 3.4 for a higher rank simple Lie group $G$ such as $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ or $\text{SL}_3(\mathbb{R})$.

4. **Distribution of $\Gamma$ in $G$**

For a family of compact subsets $B_T$ in $G$ which is getting larger and larger as $T \to \infty$, the study of the asymptotic of $\#(\Gamma \cap B_T)$ can be approached directly by Theorems 3.2 and 3.4. The link is given by the following function on $\Gamma \backslash G \times \Gamma \backslash G$:

$$F_T(g, h) := \sum_{\gamma \in \Gamma} \chi_{B_T}(g^{-1} \gamma h).$$

Observing that $F_T(e, e) = \#(\Gamma \cap B_T)$, we will explain how the asymptotic behavior of $F_T(e, e)$ is related to the matrix coefficient function for $L^2(\Gamma \backslash G)$. For $g = k_1 a_t k_2 \in KA^+ K$, we have $dm^{\text{Haar}}(g) = \xi(t) dt dk_1 dk_2$ where $\xi(t) = e^{(n-1)t} (1 + O(e^{-\beta t}))$ for some $\beta > 0$.

Now assume that $\Gamma$ is geometrically finite with $\delta > (n-1)/2$, so that Theorem 3.4 holds. For any real-valued $\Psi_1, \Psi_2 \in C_c^\infty(\Gamma \backslash G)$, if we set
\( \Psi_i^k(g) := \Psi_i(gk) \), then we can deduce from Theorem 3.4 that

\[
\langle F_T, \Psi_1 \otimes \Psi_2 \rangle_{\Gamma \times \Gamma} = \int_{g \in B_T} \langle \Psi_1, g \Psi_2 \rangle_{L^2(\Gamma)} \, dm^{\text{Haar}}(g)
\]

\[
= \int_{k_1a_1k_2 \in B_T} \langle \Psi_1^{k_1^{-1}}, a_t, \Psi_2^{k_2} \rangle \xi(t) \, dk_1 \, dk_2
\]

\[
= \frac{1}{|m^{\text{BMS}|}} \int_{k_1a_1k_2 \in B_T} \left( m_*^{\text{BR}}(\Psi_1^{k_1^{-1}}) m_*^{\text{BR}}(\Psi_2^{k_2}) e^{\delta t} + O(e^{(\delta-\eta)t}) \right) \, dk_1 \, dk_2
\]

for some \( \eta > 0 \), with the implied constant depending only on Sobolev norms of \( \Psi_i \)'s.

Therefore, if \( F_T(e,e) \) can be effectively approximated by \( \langle F_T, \Psi_\epsilon \otimes \Psi_\epsilon \rangle \) for an approximation of identity \( \Psi_\epsilon \) in \( \Gamma \), a condition which depends on the regularity of the boundary of \( B_T \) relative to the Patterson-Sullivan density, then \( F_T(e,e) \) can be computed by evaluating the integral (4.1) for \( \Psi_1 = \Psi_2 = \Psi_\epsilon \) and by taking \( \epsilon \) to be a suitable power of \( e^{-t} \).

To state our counting theorem more precisely:

**Definition 4.2.** Define a Borel measure \( \mathcal{M}_G \) on \( G \) as follows: for \( \psi \in C_c(G) \),

\[
\mathcal{M}_G(\psi) = \frac{1}{|m^{\text{BMS}|}} \int_{k_1a_1k_2 \in K A^+ K} \psi(k_1a_1k_2) e^{\delta t} \, d\nu_o(k_1) \, dt \, d\nu_o(k_2^{-1})
\]

here \( \nu_o \) is the \( M \)-invariant lift to \( K \) of the PS measure on \( \partial(\mathbb{H}^n) = K/M \) viewed from \( o \in \mathbb{H}^n \) with \( K = \text{Stab}_G(o) \).

**Definition 4.3.** For a family \( \{ B_T \subset G \} \) of compact subsets with \( \mathcal{M}_G(B_T) \) tending to infinity as \( T \to \infty \), we say that \( \{ B_T \} \) is effectively well-rounded with respect to \( \Gamma \) if there exists \( p > 0 \) such that for all small \( \epsilon > 0 \) and \( T \gg 1 \):

\[
\mathcal{M}_G(B_{T,\epsilon}^+ - B_{T,\epsilon}^-) = O(e^p \cdot \mathcal{M}_G(B_T))
\]

where \( B_{T,\epsilon}^+ = G_* B_T G_\epsilon \) and \( B_{T,\epsilon}^- = \cap_{g_1,g_2 \in G_*} g_1 B_T g_2 \). Here \( G_\epsilon \) denotes a symmetric \( \epsilon \)-neighborhood of \( e \) in \( G \).

**Theorem 4.4.** [31] If \( \{ B_T \} \) is effectively well-rounded with respect to \( \Gamma \), then there exists \( \eta_0 > 0 \) such that for any subgroup \( \Gamma' \subset \Gamma \) of finite index and for any \( \gamma \in \Gamma \),

\[
\#(\Gamma' \gamma \cap B_T) = \frac{1}{|\Gamma' : \Gamma|} \mathcal{M}_G(B_T) + O(\mathcal{M}_G(B_T)^{1-\eta_0})
\]
where $\eta_0 > 0$ depends only on a uniform spectral gap for $\Gamma$ and $\Gamma'$, and the implied constant is independent of $\Gamma'$ and $\gamma$.

For instance, consider the set $B_T = \Omega_1 A_T \Omega_2$ where $\Omega_i \subset K$ and $A_T = \{a_t : 0 \leq t \leq T\}$. If there exists $\beta' > 0$ such that the PS-measures of the $\epsilon$-neighborhoods of $\Omega_1$ and $\Omega_2^{-1}$ are at most of order $\epsilon^{\beta'}$ for all small $\epsilon > 0$, then $B_T$ is effectively well-rounded and hence one can deduce the following:

**Theorem 4.5.** [31] As $T \to \infty$,

$$
\#(\Gamma' \gamma \cap \Omega_1 A_T \Omega_2) = \Xi(\Gamma, \Omega_1, \Omega_2) \frac{\nu_0(\Omega_1)\nu_0(\Omega_2^{-1})}{[\Gamma : \Gamma']} e^{\delta T} + O(e^{(\delta-\eta_0)T})
$$

where $\Xi(\Gamma, \Omega_1, \Omega_2) := \frac{\nu_0(\Omega_1)\nu_0(\Omega_2^{-1})}{[\Gamma : \Gamma']} m_{BMS}$ and the implied constant is independent of $\Gamma'$ and $\gamma \in \Gamma$.

The fact that the only dependence of $\Gamma'$ on the right hand side of the above formula is on the index $[\Gamma : \Gamma']$ is of crucial importance for our intended applications to an affine sieve (cf. subsection 6.6). Bourgain, Kontorovich and Sarnak [8] showed Theorem 4.5 for the case $G = SO(2,1)$ via an explicit computation of matrix coefficients using the Gauss hypergeometric functions, and Vinogradov [48] generalized their method to $G = SO(3,1)$. We note that in view of Theorem 3.2, the non-effective version of Theorem 4.5 holds for any Zariski dense $\Gamma$ with $|m_{BMS}| < \infty$; this was obtained in [40] (see also [35]) with a different proof.

5. Asymptotic distribution of $\Gamma \backslash \Gamma H a_t$

The counting problem in $H \backslash G$ for $H$ a non-trivial group can be approached via studying the asymptotic distribution of translates $\Gamma \backslash H a_t$ as mentioned in the introduction. We will assume in this section that $H$ is either a symmetric subgroup or a horospherical subgroup of $G$. In this case, we can relate the distribution of $\Gamma \backslash \Gamma H a_t$ to the matrix coefficient function of $L^2(\Gamma \backslash G)$.

We fix a generalized Cartan decomposition $G = HAK$ (for $H$ horospherical, it is just an Iwasawa decomposition) where $K$ is a maximal compact subgroup and $A$ is a one-dimensional subgroup of diagonalizable elements. As before, we parametrize $A = \{a_t : t \in \mathbb{R}\}$ so that the right multiplication by $a_t$ on $G/M = T^1(\mathbb{H}^n)$ corresponds to the geodesic flow for time $t$.

When $H$ is horospherical, we will assume that $H = N^+$, i.e., the expanding horospherical subgroup with respect to $a_t$. 

For a symmetric subgroup $H$, isomorphic to $\text{SO}(k, 1) \times \text{SO}(n-k)$ for $0 \leq k \leq n - 1$, an $H$-orbit in $G/K$ (resp. $G/M$) is (resp. the normal bundle to) a complete totally geodesic subspace of $\mathbb{H}^n$ of dimension $k$. The right multiplication by $a_t$ on $G/M$ corresponds to the geodesic flow and the image of $Ha_t$ in $G/M$ represents the expansion of the totally geodesic subspace of dimension $k$ by distance $t$.

5.1. **Measures on $\Gamma \backslash \Gamma H$ associated to a conformal density.** The leading term in the description of the asymptotic distribution of $\Gamma \backslash \Gamma Ha_t$ turns out to be a new measure on $\Gamma \backslash \Gamma H$ associated to the PS density (see [33]). We assume that $\Gamma \backslash \Gamma H$ is closed in $\Gamma \backslash G$ in the rest of this section.

For a $\Gamma$-invariant conformal density $\{\mu_x\}$ on $\partial(\mathbb{H}^n)$ of dimension $\delta_{\mu}$, define a measure $\tilde{\mu}_H$ on $H/(H \cap M)$ by

$$d\tilde{\mu}_H([h]) = e^{\delta_{\mu} \beta([h])} d\mu_o([h]^+)$$

where $o \in \mathbb{H}^n$ and $[h] \in H/H \cap M$ is considered as an element of $G/M = T^1(\mathbb{H}^n)$ under the injective map $H/(H \cap M) \to G/M$.

By abuse of notation, we use the same notation $\tilde{\mu}_H$ for the $H \cap \Gamma$-invariant lift of $\tilde{\mu}_H$ to $H$: for $\psi \in C_c(\Gamma \backslash G)$,

$$\tilde{\mu}_H(\psi) = \int_{H/(H \cap M)} \psi^{H \cap M}(x) d\tilde{\mu}_H(x)$$

where $\psi^{H \cap M}(x) = \int_{H \cap M} \psi(xm) d_{H \cap M}(m)$ for the $H \cap M$-invariant probability measure $d_{H \cap M}$. This definition is independent of the choice of $o \in \mathbb{H}^n$ and the measure $\tilde{\mu}_H$ is $H \cap \Gamma$-invariant from the left, and hence induces a measure $\mu_H$ on $(H \cap \Gamma) \backslash H$ or equivalently on $\Gamma \backslash \Gamma H$.

For the PS density $\{\nu_x\}$ and the Lebesgue density $\{m_x\}$, the following two locally finite measures on $\Gamma \backslash \Gamma H$ are of special importance:

- **Skinning measure:** $\mu_{\text{PS}}^H = \nu_H$;
- **$H$-invariant measure:** $\mu_{\text{Leb}}^H = m_H$.

We remark that $\mu_{\text{PS}}^H$ is different from $\mu_{\text{Leb}}^H$ in general even when $H \cap \Gamma$ is a lattice in $H$.

5.2. **Finiteness of $\mu_{\text{PS}}^H$.** The finiteness of the skinning measure $\mu_{\text{PS}}^H$ turns out to be the precise replacement for the finiteness of the volume measure $\mu_{\text{Leb}}^H$, in extending the equidistribution statement from the cases of $\Gamma$ lattices to those of thin groups.

When is the skinning measure $\mu_{\text{PS}}^H$ finite? This question is completely answered in [35] for $\Gamma$ geometrically finite. First, when $H$ is a horospherical subgroup, the support of $\mu_{\text{PS}}^H$ is compact. When $H$ is a symmetric subgroup, i.e., isomorphic to $\text{SO}(k, 1) \times \text{SO}(n-k)$ locally,
the answer to this question depends on the notion of the parabolic corank of $\Gamma \cap H$: let $\Lambda_p(\Gamma)$ denote the set of all parabolic limit points of $\Gamma$. For $\xi \in \Lambda_p(\Gamma)$, the stabilizer $\Gamma_\xi$ has a free abelian subgroup of finite index, whose rank is defined to be the rank of $\Gamma_\xi$.

**Definition 5.1.** The parabolic corank of $\Gamma \cap H$ in $\Gamma$ is defined to be the maximum of the difference $\text{rank}(\Gamma_\xi) - \text{rank}(\Gamma \cap H)_\xi$ over all $\xi \in \Lambda_p(\Gamma) \cap \partial(\mathbb{H}^k)$.

**Theorem 5.2.** [35] Let $\Gamma$ be geometrically finite.

1. $\mu^\text{PS}_H$ is compactly supported if and only if the parabolic corank of $\Gamma \cap H$ is zero.
2. $|\mu^\text{PS}_H| < \infty$ if and only if $\delta$ is bigger than the parabolic corank of $\Gamma \cap H$.

As we show $\text{rank}(\Gamma_\xi) - \text{rank}(\Gamma \cap H)_\xi \leq n - k$ for all $\xi \in \Lambda_p(\Gamma) \cap \partial(\mathbb{H}^k)$, we have:

**Corollary 5.3.** [35] If $\delta > (n - k)$, then $|\mu^\text{PS}_H| < \infty$.

### 5.3. Distribution of $\Gamma \backslash \Gamma H a_t$.

We first recall:

**Theorem 5.4.** Suppose that $\Gamma$ is a lattice in $G$ and that $H \cap \Gamma$ is a lattice in $H$. In other words, $|m^\text{Haar}_H| < \infty$ and $|m^\text{Leb}_H| < \infty$. Then there exist $\eta_0 > 0$ (depending only on the spectral gap for $\Gamma$) and $\ell \in \mathbb{N}$ such that for any $\Psi \in C_\infty^c(\Gamma \backslash G)$, as $t \to \infty$,

$$
\int_{\Gamma \backslash H} \Psi(ha_t) d\mu^\text{Leb}_H(h) = \frac{|\mu^\text{Leb}_H|}{|m^\text{Haar}_H|} \cdot m^\text{Haar}(\Psi) + O(S_\ell(\Psi) \cdot e^{-\eta_0 t}).
$$

The non-effective statement is due to [14] (also see [16]). For the effective statement, see [14] for the case when $H \cap \Gamma \backslash H$ is compact and [2] in general.

An analogue of Theorem 5.4 for discrete groups which are not necessarily lattices is given as follows:

**Theorem 5.5.** [35] Let $\Gamma$ be Zariski dense with $|m^\text{BMS}| < \infty$. Suppose $|\mu^\text{PS}_H| < \infty$. Then for any $\Psi \in C_\infty^c(\Gamma \backslash G)$,

$$
\lim_{t \to \infty} e^{(n-1-\delta)t} \int_{h \in \Gamma \backslash H} \Psi(ha_t) d\mu^\text{Leb}_H(h) = \frac{|\mu^\text{PS}_H|}{|m^\text{BMS}|} m^\text{BR}(\Psi).
$$

**Theorem 5.6.** [31] Let $\Gamma$ be geometrically finite with $\delta > (n - 1)/2$. Suppose $|\mu^\text{PS}_H| < \infty$. Then there exist $\eta_0 > 0$ and $\ell \in \mathbb{N}$ such that for any $\Psi \in C_\infty^c(\Gamma \backslash G)$, as $t \to \infty$,

$$
e^{(n-1-\delta)t} \int_{h \in \Gamma \backslash H} \Psi(ha_t) d\mu^\text{Leb}_H(h) = \frac{|\mu^\text{PS}_H|}{|m^\text{BMS}|} m^\text{BR}(\Psi) + O(S_\ell(\Psi) \cdot e^{-\eta_0 t}).$$
In the case when $\mu_{PS}^H$ is compactly supported, we can show that there exists a compact subset $O_H$ (depending on $\Psi$) of $\Gamma \backslash \Gamma H$ such that for any $t \in \mathbb{R}$, $\Psi(ha_t) = 0$ for all $h \notin O_H$. Therefore $\int O_H \Psi(ha_t)d\mu_{Leb}^H = \int_{\mathcal{O}_H} \Psi(ha_t)d\mu_{Leb}^H$. Now the assumption on $H$ being either symmetric or horospherical ensures the wave front property of [16] which can be used to establish, as $t \to \infty$,

$$\int_{\mathcal{O}_H} \Psi(ha_t)d\mu_{Leb}^H \approx \langle a_t\Psi, \rho_{\mathcal{O}_H, \varepsilon} \rangle_{L^2(\Gamma \backslash G)}$$

where $\rho_{\mathcal{O}_H, \varepsilon} \in C_c^\infty(\Gamma \backslash G)$ is an $\varepsilon$-approximation of $\mathcal{O}_H$. Therefore the estimates on the matrix coefficients in Theorems 3.2 and 3.4 can be used to establish Theorems 5.5 and 5.6.

The case when $\mu_{PS}^H$ is not compactly supported turns out to be much more intricate, the main reason being that we are taking the integral with respect to $\mu_{Leb}^H$ as well as multiplying the weight factor $e^{-(n-1-\delta)\beta}$ in the left hand side of Theorem 5.5 whereas the finiteness assumption is made on the skinning measure $\mu_{PS}^H$. In this case, the proofs are done by comparing the two measures $(a_t)_*\mu_{PS}^H$ and $(a_t)_*\mu_{Leb}^H$ via the the transversal intersections of the orbits $\Gamma \backslash \Gamma Ha_t$ with the weak-stable horospherical foliations. A key reason that this approach works is that these transversal intersections are governed by the topological property of the orbit $\Gamma \backslash \Gamma Ha_t$, independent of the measures put on $\Gamma \backslash \Gamma H$.

In the special case of $n = 2, 3$ and $H$ horospherical, Theorem 5.6 was proved in [27] by a different method.

6. DISTRIBUTION OF $\Gamma$ ORBITS IN $H \backslash G$ AND AFFINE SIEVE

6.1. Distribution of $\Gamma$ orbits in $H \backslash G$. Let $H$ be as in the previous section (i.e., symmetric or horospherical subgroup) and let $\{B_T\}$ be a family of compact subsets in $H \backslash G$ which is getting larger and larger as $T \to \infty$. We assume that $[e]\Gamma$ is discrete in $H \backslash G$. The study of the asymptotic of $\#([e]\Gamma \cap B_T)$ can be now approached by Theorems 5.5 and 5.6 via the following counting function on $\Gamma \backslash G$:

$$F_T(g) := \sum_{\gamma \in H \backslash G \cap \Gamma \backslash \Gamma} \chi_{B_T}(\gamma g)$$

as $F_T(e) = \#([e]\Gamma \cap B_T)$. Depending on the subgroup $H$, we have $G = HA^+K$ or $G = HA^+K \cup HA^-K$. For the sake of simplicity, we will consider the sets $B_T$ contained in $HA^+K$. We have for $g = ha_tk$, $dm_{Haar}(g) = \rho(t)d\mu_{Leb}^H(h)dtdk$ where $\rho(t) = e^{(n-1)\beta}(1 + O(e^{-\beta t}))$ for some $\beta > 0$. Let $\mu$ be a $G$-invariant measure on $H \backslash G$ normalized so that $dm_{Haar} = d\mu_{Leb}^H \otimes d\mu$ locally. Then, under the assumption that $\Gamma$
is geometrically finite with \( \delta > (n - 1)/2 \) and \( |\mu_{H}^{PS}| < \infty \), we deduce from Theorem \ref{thm:1}:

\[
\langle F_T, \Psi \rangle = \int_{g \in B_T} \int_{h \in \Gamma \setminus \Gamma H} \Psi(hg) d\mu_{H}^{Leb}(h) d\mu(g)
\]

\[
= \int_{[e]a \in B_T} \int_{h \in \Gamma \setminus \Gamma H} \Psi^{\epsilon}(ha_{\epsilon}) d\mu_{H}^{Leb}(h) \rho(t) dk
\]

\[
= \int_{[e]a \in B_T} \left( \frac{|\mu_{H}^{PS}|}{|m^{BR}|} m^{BR}(\Psi^{\epsilon}) e^{\delta t} + O(e^{(\delta - \eta_{0})t}) \right) dt dk
\]

for some \( \eta_{0} > 0 \).

Therefore, similarly to the discussion in section \ref{sec:4}, if \( F_T(\epsilon) \) can be effectively approximated by \( \langle F_T, \Psi_{\epsilon} \rangle \) for an approximation of identity \( \Psi_{\epsilon} \) in \( \Gamma \setminus G \), a condition which depends on the regularity of the boundary of \( B_T \) relative to the PS density, then \( F_T(\epsilon) \) can be computed by evaluating the integral (6.1) for \( \Psi = \Psi_{\epsilon} \) and by taking \( \epsilon \) a suitable power of \( e^{-t} \).

For \( H \) horospherical or symmetric, we have \( G = HA^{+}K \) or \( G = HA^{+}K \cup HA^{-}K \) (as a disjoint union except for the identity element) where \( A^{\pm} = \{ a_{x,t} : t \geq 0 \} \).

**Definition 6.2.** Define a Borel measure \( \mathcal{M}_{H \setminus G} \) on \( H \setminus G \) as follows: for \( \psi \in C_{c}(H \setminus G) \),

\[
\mathcal{M}_{H \setminus G}(\psi) = \begin{cases} 
\frac{|\mu_{H}^{PS}|}{|m^{BR}|} \int_{a_{x,k} \in A^{+}K} \psi([e]a_{x,k}) e^{\delta t} d\nu_{\omega}(k^{-1}) & \text{if } G = HA^{+}K \\
\sum_{|m^{BR}|} \frac{|\mu_{H}^{PS}|}{|m^{BR}|} \int_{a_{x,k} \in A^{-}K} \psi([e]a_{x,k}) e^{\delta t} d\nu_{\omega}(k^{-1}) & \text{otherwise,}
\end{cases}
\]

where \( o \in \mathbb{H}^{n} \) is the point fixed by \( K \), \( \nu_{\omega} \) is the right \( M \)-invariant measure on \( K \), which projects to the PS-measure \( \nu_{o} \) on \( K/M = \partial(\mathbb{H}^{n}) \) and \( \mu_{H \setminus G}^{PS} \) is the skinning measure on \( \Gamma \setminus \Gamma H \) in the negative direction.

**Definition 6.3.** For a family \( \{ B_T \subset H \setminus G \} \) of compact subsets with \( \mathcal{M}_{H \setminus G}(B_T) \) tending to infinity as \( T \to \infty \), we say that \( \{ B_T \} \) is effectively well-rounded with respect to \( \Gamma \) if there exists \( p > 0 \) such that for all small \( \epsilon > 0 \) and \( T \gg 1 \):

\[
\mathcal{M}_{H \setminus G}(B_{T,\epsilon}^{+} - B_{T,\epsilon}^{-}) = O(e^{p} \cdot \mathcal{M}_{H \setminus G}(B_T))
\]

where \( B_{T,\epsilon}^{+} = B_{T}G_{\epsilon} \) and \( B_{T,\epsilon}^{-} = \cap_{g \in G, B_{T}g} \).

**Theorem 6.4.** \[31\] When \( H \) is symmetric, we assume that \( |\mu_{H}^{PS}| < \infty \). Suppose that \( \{ B_T \} \) is effectively well-rounded with respect to \( \Gamma \). Then for any subgroup \( \Gamma' < \Gamma \) of finite index such that \( H \cap \Gamma' = H \cap \Gamma \) and for any \( \gamma \in \Gamma', \) there exists \( \eta_{0} > 0 \) (depending only on a uniform spectral
gap of $\Gamma$ and $\Gamma'$) such that
\[
\#([e]^{\Gamma'} \cap B_T) = \frac{1}{|\Gamma : \Gamma'|} \mathcal{M}_{H \setminus G}(B_T) + O(\mathcal{M}_{H \setminus G}(B_T)^{1-\eta_0})
\]
with the implied constant independent of $\Gamma'$ and $\gamma \in \Gamma$.

For a given family $\{B_T\}$, understanding its effective well-roundedness may be the hardest part of the work in general. We consider an example of the family $\{B_T = [e]A_T \Omega\}$ where $\Omega \subset K$ and $A_T = \{a_t : 0 \leq t \leq T\}$. In this case, it is rather simple to formulate the effective well-rounded condition: there exists $\beta' > 0$ such that the PS-measure of the $\epsilon$-neighborhood of $\partial(\Omega^{-1})$ is at most of order $\epsilon^{\beta'}$ for all small $\epsilon > 0$. Of course, this holds when $\Omega = K$ or when $\partial(\Omega^{-1})$ does not intersect the limit set $\Lambda(\Gamma)$.

Then, setting $\Xi(\Gamma, \Omega) := \frac{|\mu^\text{PS}_H \nu_0(\Omega^{-1})|}{\delta^{|m_{\text{BMS}}|}}$, one can deduce from (6.1), Theorem 6.5. [31] Under the same assumption on $\mu^\text{PS}_H$ and $\Gamma'$, there exists $\eta_0 > 0$ (depending only on a uniform spectral gap of $\Gamma$ and $\Gamma'$) such that for any $\gamma \in \Gamma$,
\[
\#([e]^{\Gamma'} \cap [e]A_T \Omega) = \Xi(\Gamma, \Omega) \frac{e^\delta T}{|\Gamma : \Gamma'|} + O(e^{(\delta - \eta)T})
\]
with implied constant independent of $\Gamma'$ and $\gamma$.

We note that in view of Theorem 4.1, the non-effective version of this theorem holds for any Zariski dense $\Gamma$ with $|m_{\text{BMS}}| < \infty$ and $|\mu^\text{PS}_H| < \infty$, as proved in [35].

6.2. Affine sieve. For applications to an affine sieve, we consider the case when the homogeneous space $H \setminus G$ is defined over $\mathbb{Z}$. More precisely, we assume that $G$ is defined over $\mathbb{Z}$ and acts linearly and irreducibly on a finite dimensional vector space $W$ defined over $\mathbb{Z}$ in such a way that $G(\mathbb{Z})$ preserves $V(\mathbb{Z})$. Let $w_0 \in W(\mathbb{Z})$ be a non-zero vector such the stabilizer of $w_0$ is a symmetric subgroup or the stabilizer of the line $\mathbb{R}w_0$ is a parabolic subgroup. We set $V := w_0G$. Let $\Gamma$ be a geometrically finite subgroup with $\delta > (n - 1)/2$, which is contained in $G(\mathbb{Z})$.

Let $F$ be an integer-valued polynomial on the orbit $w_0 \Gamma$. Salehi-Golsefidy and Sarnak [42], generalizing [6], showed that for some $R > 1$, the set of $x \in w_0 \Gamma$ with $F(x)$ has at most $R$ prime factors is Zariski dense in $w_0G$.

For a square-free integer $d$, let $\Gamma_d < \Gamma$ be a subgroup which contains $\{\gamma \in \Gamma : \gamma \equiv e \mod d\}$ and satisfies $\text{Stab}_{\Gamma_d}(w_0) = \text{Stab}_{\Gamma}(w_0)$. Salehi-Golsefidy and Varju [41], generalizing [6], showed that the family of Cayley graphs of $\Gamma/\Gamma_d$’s (with respect to the projections of a fixed
generating set for $\Gamma$) forms expanders as $d$ runs through square-free integers with large prime factors. As $\delta > (n - 1)/2$, this implies that the family $\{\Gamma_d : d \text{ square-free}\}$ has a uniform spectral gap, as shown in [7] (see also [22]). Hence we can apply Theorem 6.5 to $\Gamma_d$’s. By a recent development on an affine linear sieve on homogeneous spaces ([6], [32]), we are able to deduce: let $F$ be an integer-valued polynomial on $w_0\Gamma$ which is weakly primitive, i.e., $\gcd\{F(x) : x \in w_0\Gamma\} = 1$. Letting $F = F_1F_2\cdots F_r$ be a factorization into irreducible polynomials, assume that all $F_j$’s are integral on $w_0\Gamma$. Let $\lambda$ be the log of the largest eigenvalue of $a_1$ on the $\mathbb{R}$-span of $w_0G$.

**Theorem 6.6.** [31] For any norm $\|\cdot\|$ on $V$,

1. $\{x \in w_0\Gamma : \|x\| < T, F_j(x) \text{ is prime for } j = 1, \cdots, r\} \ll \frac{T^{\delta/\lambda}}{(\log T)^r}$;

2. there exists $R = R(F, w_0\Gamma) \geq 1$ such that

$$\{x \in w_0\Gamma : \|x\| < T, F(x) \text{ has at most } R \text{ prime factors}\} \gg \frac{T^{\delta/\lambda}}{(\log T)^r}.$$

In view of the above discussion, it is also possible to state Theorem 6.6 for more general sets $B_T$, instead of the norm balls.

### 7. Application to sphere packings

In this section we will discuss counting problems for sphere packings in $\mathbb{R}^n$ as an application of an orbital counting problem for thin groups of $\text{SO}(n, 1)$. By a sphere packing in the Euclidean space $\mathbb{R}^n$ for $n \geq 1$, we simply mean a union of (possibly intersecting) $n - 1$-dimensional spheres; here an $n - 1$ dimensional plane is regarded as a sphere of infinite radius.

Fixing a sphere packing $\mathcal{P}$ in $\mathbb{R}^n$, a basic problem is to understand the asymptotic of the number $\#\{S \in \mathcal{P} : \text{Radius}(S) > t\}$ or equivalently $\#\{S \in \mathcal{P} : \text{Vol}(S) > t\}$ where $\text{Vol}(S)$ means the volume of the ball enclosed by $S$. For the sake of brevity, we will simply refer $\text{Vol}(S)$ to the volume of $S$. We will consider this problem in a more general setting, that is, allowing the volume of the sphere $S$ to be computed in various conformal metrics.

Let $U$ be an open subset of $\mathbb{R}^n$ and $f$ a positive continuous function on $U$. A conformal metric associated to the pair $(U, f)$ is a metric on $U$ of the form $f(x)dx$ where $dx$ is the Euclidean metric on $\mathbb{R}^n$. Given a conformal metric $(U, f)$, we set $\text{Vol}_f(S) := \int_B f(x)^n dx$ where $B$ is the ball enclosed by $S$. When $S$ is a plane, we put $\text{Vol}_f(S) = \infty$. 


For any compact subset $E$ of $U$ and $t > 0$, set

$$N_t(P, f, E) := \#\{S \in P : \text{Vol}_f(S) > t, \ E \cap S \neq \emptyset\}.$$ 

For simplicity, we will oftentimes omit $P$ in the notation $N_t(P, f, E)$. When $(U, f) = (\mathbb{R}^n, 1)$, i.e., the standard Euclidean metric, we simply write $N_t(E)$ instead of $N_t(1, E)$. We call $P$ locally finite if $N_t(E) < \infty$ for all $E$ bounded.

In order to approach the problem on the computation of the asymptotic of $N_t(f, E)$, a crucial condition is that $P$ admits enough symmetries of Mobius transformations of $\mathbb{R}^n$. By the Poincare extension, the group $\text{MG}(\mathbb{R}^n)$ of Mobius transformations of $\mathbb{R}^n$ can be identified with the isometry group of the upper half space $\mathbb{H}^{n+1} = \{(z, r) : z \in \mathbb{R}^n, r > 0\}$. We assume that $P$ is invariant under a non-elementary discrete subgroup $\Gamma$ of $G := \text{Isom}^+(\mathbb{H}^{n+1})$. As before, let $\delta$ denote the critical exponent of $\Gamma$ and $\{\nu_x : x \in \mathbb{H}^{n+1}\}$ a Patterson-Sullivan density for $\Gamma$.

We recall from [36]:

**Definition 7.1** (The $\Gamma$-skinning size of $P$). For a sphere packing $P$ invariant under $\Gamma$, define $0 \leq \text{sk}_\Gamma(P) \leq \infty$ as follows:

$$\text{sk}_\Gamma(P) := \sum_{i \in I} \int_{s \in \text{Stab}_\Gamma(S_i^\dagger)} e^{\delta_+ (\alpha, s)} d\nu_o(s^+)$$

where $o \in \mathbb{H}^{n+1}$, $\{S_i : i \in I\}$ is a set of representatives of $\Gamma$-orbits in $P$, and $S_i^\dagger \subset T^1(\mathbb{H}^{n+1})$ is the set of unit normal vectors to the convex hull of $S_i$.

**Definition 7.2.** For the pair $(U, f)$, we define a Borel measure $\omega_{\Gamma, f}$ on $U$: for $\psi \in C_c(U)$ and for $o \in \mathbb{H}^{n+1}$,

$$\omega_{\Gamma, f}(\psi) = \int_{z \in U} \psi(z) f(z)^\delta e^{\delta_+ (\alpha, z, 1)} d\nu_o(z).$$

Alternatively, we have a simple formula:

$$d\omega_{\Gamma, f} = f(z)^\delta (|z|^2 + 1)^\delta d\nu_{e_{n+1}}$$

where $e_{n+1} = (0, \cdots, 0, 1) \in \mathbb{H}^{n+1}$.

**Example 7.3.**

1. For the spherical metric $(\mathbb{R}^n, \frac{2}{1+|z|^2})$ (also called the chordal metric) on $\mathbb{R}^n$, $\text{Vol}_f(S)$ is the spherical volume of the ball enclosed by $S$ and $d\omega_{\Gamma, f} = 2^\delta d\nu_{e_{n+1}}$.

2. For the hyperbolic metric $(\mathbb{H}^n = \{z \in \mathbb{R}^n : z_n > 0\}, \frac{1}{z_n})$, $\text{Vol}_f(S)$ is the hyperbolic volume of the ball enclosed by $S$ and $d\omega_{\Gamma, f} = \frac{(1+|z|^2)^\delta}{z_n^\delta} d\nu_{e_{n+1}}$. 
Definition 7.4. By an infinite bouquet of tangent spheres glued at a point \( \xi \in \mathbb{R}^n \cup \{\infty\} \), we mean a union of two collections, each consisting of infinitely many pairwise internally tangent spheres with the common tangent point \( \xi \) and their radii tending to 0, such that the spheres in each collection are externally tangent to the spheres in the other at \( \xi \).

We set \( v_n := \text{Vol}(B(0,1)) \) to be the Euclidean volume of the unit ball \( B(0,1) := \{x \in \mathbb{R}^n : \|x\| < 1\} \); \( v_n \) is equal to \( \frac{(2\pi)^{n/2}}{2 \cdot 4 \cdot \ldots \cdot n} \) if \( n \) is even and \( \frac{2(2\pi)^{(n-1)/2}}{1 \cdot 3 \cdot \ldots \cdot n} \) if \( n \) is odd.

Theorem 7.5. Let \( \mathcal{P} \) be a locally finite sphere packing invariant under a geometrically finite group \( \Gamma \) with finitely many \( \Gamma \)-orbits. In the case of \( \delta \leq 1 \), we also assume that \( \mathcal{P} \) has no infinite bouquet of spheres glued at a parabolic fixed point of \( \Gamma \).

Then for any conformal metric \((U,f)\) and for any compact subset \( E \) of \( U \) whose boundary has zero Patterson-Sullivan density, as \( t \to 0 \),

\[
\lim_{t \to 0} N_t(f,E)^{\delta/n} = \frac{\text{sk}_\Gamma(\mathcal{P}) \cdot v_n^{\delta/n} \cdot \omega_{\Gamma,f}(E)}{\delta \cdot |m_{\text{BMS}}|}.
\]

The assumption on the non-existence of an infinite bouquet in \( \mathcal{P} \) is to ensure that the \( \Gamma \)-skinning size for \( \mathcal{P} \) is finite.

In the case when \((U,f) = (\mathbb{R}^n,1)\), Theorem 7.5 was proved in [36] for the case of \( n = 2 \) and the proof given there extends easily for any \( n \geq 2 \) using the general equidistribution result in [35].

To give an interpretation of Theorem 7.5 as a special case of the orbital counting problem discussed in the subsection 6.1, fixing \( H \) to be the stabilizer of a sphere \( S_0 \) in \( \mathcal{P} \), we may think of \( H \backslash G \) as the space of all totally geodesic planes in \( \mathbb{H}^{n+1} \). Then a key point is to describe a particular subset \( B_t(E) \) in \( H \backslash G \) such that \( N_t(E) \) is same as \( \#[e] \Gamma \cap B_t(E) \).

The extension to a general conformal metric \((U,f)\) is possible basically due to the uniform continuity of \( f \) on a compact subset \( E \) and a covering argument. We give a brief sketch as follows (the argument below is established in a discussion with Shah): denote by \( Q_z(\eta) \) the cube \( \{z' \in \mathbb{R}^n : \max_{1 \leq i \leq n} |z_i - z'_i| \leq \eta\} \) centered at \( z \in \mathbb{R}^n \) with radius \( \eta \).

First, Theorem 7.5 for \( f = 1 \), together with the uniform continuity of \( f \) on \( E \), implies that for any \( \epsilon > 0 \), there exists \( \eta = \eta(\epsilon) > 0 \) (depending only on \( E \) and \( f \) ) such that for any cube \( Q_z(\eta) \) centered at \( z \in E \)

\[
\frac{N_t(f,Q_z(\eta))^{\delta/n}}{v_n^{\delta/n}} = (1 + O(\epsilon)) \frac{\text{sk}_\Gamma(\mathcal{P})}{\delta \cdot |m_{\text{BMS}}|} f(z)^{\delta} \omega_{\Gamma}(Q_z(\eta)). \tag{7.6}
\]

Let \( k \) be the minimal integer such that a \( k \)-dimensional sphere, say, \( P \) charges a positive PS density. As the PS density is atom-free, we
have \( k > 0 \) and the limit set \( \Lambda(\Gamma) \) is contained in \( P \) (see [40, Prop. 3.1]). We cover \( E \cap P \) with cubes \( \{Q_z(\eta) : z \in I_\eta\} \) with disjoint interiors for a finite subset \( I_\eta \) of \( E \cap P \). As each cube is centered at a point of \( P \) which is a sphere, the intersection of its boundary with \( P \) is contained in a \((k-1)\)-dimensional sphere for all small \( \eta > 0 \). It follows that the boundary of each cube has zero PS density by the minimality assumption on \( k \). Let \( C(\eta) := \{Q_z(\eta) : z \in \tilde{I}_\eta\} \) be a covering of \( E \) with \( \tilde{I}_\eta \supset I_\eta \). Note that the boundary of each cube in \( C(\eta) \) has zero PS density. We can find compact subsets \( E_{\pm} \) of \( U \) with \( E_{-} \subset E \subset E_{+} \) and a positive integer \( m_\epsilon \) so that \( \omega_{\Gamma,f}(E_{+} - E_{-}) < \epsilon \) and \( E_{+} \) (resp. \( E_{-} \)) contains all cubes centered at \( E \) (resp. \( E_{-} \)) and of size less than \( \eta \) possibly except at most \( m_\epsilon \) number of such cubes.

We may also assume that \( f(z) = (1 + O(\epsilon))f(z') \) for all \( z' \in Q_z(\eta) \in C(\eta) \) where the implied constant is uniform for all \( z \in E \). We may now apply Theorem 7.6 for \( f = 1 \) to each cube in the covering of \( C(\eta) \) to obtain that

\[
\frac{N_t(f,E)}{\nu_{\delta/n}^\delta} = (1 + O(\epsilon)) \frac{skr(P)}{\delta \cdot |m_{\text{BMS}}|} \sum_{z \in I_\eta} f(z)^\delta \omega_{\Gamma}(Q_z(\eta))
\]

\[
= (1 + O(\epsilon)) \frac{skr(P)}{\delta \cdot |m_{\text{BMS}}|} \omega_{\Gamma,f}(E_{+}) + O(\epsilon)
\]

\[
= (1 + O(\epsilon)) \frac{skr(P)}{\delta \cdot |m_{\text{BMS}}|} \omega_{\Gamma,f}(E) + O(\epsilon).
\]

As \( \epsilon > 0 \) is arbitrary, this proves Theorem 7.5.

8. On Apollonian circle packings

8.1. Construction. In the case of Apollonian circle packings in the plane \( \mathbb{R}^2 = \mathbb{C} \), Theorem 7.5 can be made more explicit as the measure \( \omega_{\Gamma,f} \) turns out to be the \( \delta \)-dimensional Hausdorff measure with respect to \( (U,f) \), restricted to \( \Lambda(\Gamma) \).

We begin by recalling Apollonian circle packings, whose construction is based on the following theorem of Apollonius of Perga:

Theorem 8.1 (Apollonius, 200 BC). Given 3 mutually tangent circles in the plain (with distinct tangent points), there exist precisely two circles tangent to all three circles.

Consider four mutually tangent circles in the plane with distinct points of tangency. By the above Apollonius’ theorem, one can add four new circles each of which is tangent to three of the given ones.
Continuing to repeatedly add new circles tangent to three of the previous circles, we obtain an infinite circle packing, called an Apollonian circle packing. Figure 8.1 shows the first three generations of this procedure where each circle is labeled with its curvature (the reciprocal of its radius).

8.2. Apollonian packing and Hausdorff measure. We start by recalling:

**Definition 8.2.** For $s > 0$, the $s$-dimensional Hausdorff (also known as covering) measure $\mathcal{H}^s$ of a closed subset $E$ of $\mathbb{R}^2$ is defined as follows:

$$\mathcal{H}^s(E) := \lim_{\epsilon \to 0} \inf \left\{ \sum_{i \in I} \text{diam}(D_i)^s : E \subset \bigcup_{i \in I} D_i, \text{diam}(D_i) \leq \epsilon \right\}.$$ 

The Hausdorff dimension of $E$ is then given as

$$\text{dim}_H(E) := \sup \{ s : \mathcal{H}^s(E) = \infty \} = \inf \{ s \geq 0 : \mathcal{H}^s(E) = 0 \}.$$ 

For $s$ a positive integer, the $s$-dimensional Hausdorff measure is proportional to the usual Lebesgue measure on $\mathbb{R}^s$.

The residual set $\text{Res}(\mathcal{P})$ is defined to be the closure of the union of all circles in $\mathcal{P}$. Its Hausdorff dimension, say $\alpha$, is independent of $\mathcal{P}$ and known to be approximately 1.30568(8) [29].

**Theorem 8.3.** Let $\mathcal{P}$ be any Apollonian circle packing. For any conformal metric $(U, f)$ and for any compact subset $E \subset U$ with smooth boundary, we have

$$\lim_{t \to 0} t^{\alpha/2} \cdot \# \{ C \in \mathcal{P} : \text{area}_f(C) > t, C \cap E \neq \emptyset \} = c_A \cdot \mathcal{H}_f^\alpha(\text{Res}(\mathcal{P}) \cap E)$$

where $c_A > 0$ is independent of $\mathcal{P}$ and $d\mathcal{H}_f^\alpha(z) = f(z)^\alpha \cdot d\mathcal{H}^\alpha(z)$.

The symmetry group $\Gamma_{\mathcal{P}} := \{ g \in \text{PSL}_2(\mathbb{C}) : g(\mathcal{P}) = \mathcal{P} \}$ satisfies the following:

1. $\Gamma_{\mathcal{P}}$ is geometrically finite.
2. The limit set of $\Gamma_{\mathcal{P}}$ coincides with $\text{Res}(\mathcal{P})$; in particular, its critical exponent is $\alpha$.
3. There are only finitely many $\Gamma_{\mathcal{P}}$-orbits of circles in $\mathcal{P}$. 
Let $\nu_{P,j}$ denote the PS measure viewed from $j = (0, 0, 1) \in \mathbb{H}^3$ for the group $\Gamma_P$. As $\Gamma_P$ has no rank 2 parabolic limit points and $\alpha > 1$, Sullivan’s work \cite{Sullivan} implies that the $\alpha$-dimensional Hausdorff measure $\mathcal{H}^\alpha$ is a locally finite measure on $\text{Res}(P)$ and that

$$\frac{1}{|\nu_{P,j}|} (|z|^2 + 1)^\alpha \, d\nu_{P,j} = d\mathcal{H}^\alpha.$$ 

Therefore Theorem 8.3 is a special case of Theorem 7.5. Moreover the constant $c_A$ is given by

$$c_A = \frac{\pi^{\alpha/2} \cdot \text{sk}_{\Gamma_P}(P) \cdot |\nu_{P,j}|}{\alpha \cdot |m_{\text{BMS}}|}$$

for any Apollonian circle packing $P$. We propose to call $c_A$ the Apollonian constant.

**Problem 8.4.** Compute (or estimate) $c_A$!

When $P$ is a bounded Apollonian circle packing, the existence of the asymptotic formula $\#\{C \in P : \text{area}(C) > t\} \sim c_P \cdot t^{\alpha/2}$ was first shown in \cite{Vas06} without an error term, and later in \cite{Vas08} with an error term (see also \cite{Sullivan}). In view of \cite{BoMo06}, Theorem 5.6 can be used to prove an effective circle count in a compact region $E$ for general Apollonian packings, provided the boundary of $E$ satisfies a regularity property.

There is also a beautiful arithmetic aspect of Apollonian circle packings which is entirely omitted in this article. We refer to \cite{Wolff}, \cite{Vav03}, \cite{BoMo03}, \cite{BoMo04}, etc.

### 8.3. Apollonian sphere packing for $n = 3$.

Given $n + 1$ mutually tangent spheres in $\mathbb{R}^n$ with disjoint interiors, it is known that there is a unique sphere, called a dual sphere, passing through their points of tangency and orthogonal to all $(n+1)$ spheres \cite[Thm 7.1]{Bar}. Hence for $(n+2)$ mutually tangent spheres with disjoint interiors $S_1, \ldots, S_{n+2}$ in $\mathbb{R}^n$, there are $(n+2)$ dual spheres, say, $\tilde{S}_1, \ldots, \tilde{S}_{n+2}$. The Apollonian group $\mathcal{A} = \mathcal{A}(S_1, \ldots, S_{n+2})$ is generated by the inversions with respect to $\tilde{S}_i$, $1 \leq i \leq n+2$.

Only for $n = 2$ or 3, the Apollonian group $\mathcal{A}$ is a discrete subgroup of $\text{MG}(\mathbb{R}^n)$ \cite[Thm 4.1]{Bar} and in this case its orbit $P := \cup_{i=1}^{n+2} \mathcal{A}(S_i)$ consists of spheres with disjoint interiors. For $n = 2$, $P$ is an Apollonian circle packing. For $n = 3$, $P$ is called an Apollonian sphere packing. Note that $\mathcal{A}$ is geometrically finite and that $P$ is locally finite, as the spheres in $P$ have disjoint interiors, hence Theorem 7.5 applies to $P$. The critical exponent of $\mathcal{A}$ for $n = 3$ has been estimated to be $2.473946(5)$ in \cite{BoMo02}. 

8.4. **Dual Apollonian Cluster Ensemble for any** $n \geq 2$. Given $(n + 2)$ mutually tangent spheres with disjoint interiors $S_1, \cdots, S_{n+2}$ in $\mathbb{R}^n$, let $\mathcal{A}^*$ denote the group generated by the inversions with respect to $S_i$, $1 \leq i \leq n + 2$. The dual Apollonian group $\mathcal{A}^*$ is a discrete geometrically finite subgroup of $\text{MG}(\mathbb{R}^n)$ for all $n \geq 2$ and the orbit $\mathcal{P} := \bigcup_{i=1}^{n+2} \mathcal{A}^*(S_i)$ is a sphere packing, in our sense, consisting of spheres nested in $S_i$'s. We note that $\mathcal{P}$ is locally finite, as nested spheres are getting smaller and smaller and hence Theorem 7.5 applies to $\mathcal{P}$.

9. **Packing circles of the ideal triangle in $\mathbb{H}^2$**

Consider an ideal triangle $\mathcal{T}$ in $\mathbb{H}^2$ i.e., a triangle whose sides are hyperbolic lines connecting vertices on the boundary of $\mathbb{H}^2$. An ideal triangle exists uniquely up to hyperbolic congruences. Consider $\mathcal{P}(\mathcal{T})$ the circle packing of an ideal triangle by filling in largest inner circles. The notation $\overline{\mathcal{P}(\mathcal{T})}$ denotes the closure of $\mathcal{P}(\mathcal{T})$ and $\text{area}_{\text{Hyp}}(C)$ is the hyperbolic area of the disk enclosed by $C$.

**Theorem 9.1** (Packing circles of the ideal triangle). Let $\mathcal{T}$ be the ideal triangle of $\mathbb{H}^2$. Then

$$\lim_{t \to 0} t^{\alpha/2} \cdot \# \{ C \in \mathcal{P}(\mathcal{T}) : \text{area}_{\text{Hyp}}(C) > t \} = c_A \cdot \int_{\overline{\mathcal{P}(\mathcal{T})}} y^{-\alpha} \, d\mathcal{H}^\alpha(z)$$

where $c_A$ denotes the Apollonian constant.

Fix the Apollonian circle packing $\mathcal{P}_0$ generated by two vertical lines $x = \pm 1$ and the unit circle $\{ |z| = 1 \}$. The corresponding Apollonian group $\Gamma_0 = \Gamma(\mathcal{P}_0)$ is generated by the inversions with respect to horizontal lines $y = 0$ and $y = -2i$ and the circles $\{ |z - (\pm 1 - i)| = 1 \}$. We set $\mathbb{R}_+^2 := \{ x + iy : y > 0 \}$. Now for the conformal metric $(U, f) = (\mathbb{R}_+^2, 1/y) = \mathbb{H}^2$, we note that $\{ C \in \mathcal{P}(\mathcal{T}) : \text{area}_{\text{Hyp}}(C) > t \} = \{ C \in \mathcal{P}_0 : \text{area}_{f}(C) > t, C \cap \mathcal{T} \neq \emptyset \}$.

However Theorem 9.1 does not immediately follow from Theorem 8.3 since the ideal triangle $\mathcal{T}$ is not a compact subset of $\mathbb{H}^2$.

We need to understand the $\mathcal{H}^\alpha_f$-measure of neighborhoods of cusps in the triangle for $f = 1/y$. For the next two theorems, consider a conformal metric $(\mathbb{R}_+^2, f)$.

**Theorem 9.2.** If $f(x + iy) \ll y^{-k}$ for some $k > \alpha^{-1}$ with implied constant independent of $|x| \leq 1$, then for any $\eta > 0$,

$$\mathcal{H}^\alpha_f \{|x| \leq 1, y > \eta \} < \infty.$$
Moreover for any Borel $E \subset \{ |x| \leq 1, y > \eta \}$ (not necessarily compact) with smooth boundary,
\[
\lim_{t \to \infty} t^{\alpha/2} \cdot \# \{ C \in \mathcal{P}_0 : \text{area}_f(C) > t, C \cap E \neq \emptyset \} \sim c_A \cdot \mathcal{H}_f^\alpha(E).
\]

Proof. It suffices to show the claim for any $\eta > 0$ fixed. We put $\eta = 1$ and set $U_R := \{ |x| \leq 1, y > R \}$. Define $F_t(E) := \{ C \in \mathcal{P}_0 : \text{area}_f(C) > t, C \cap E \neq \emptyset \}$ and $E_n := \{ |x| \leq 1, n \leq y < n + 1 \}$. Then $F_t(U_1) = \cup_{n \geq 1} F_t(E_n)$.

For $C \in E_n$, $C - (n-1)i \in E_1$ and
\[
\text{area}(C - (n-1)i) = \int_{C-(n-1)i} f(z)^2 dz
\]
\[
= \int_C f(z + (n-1)i)^2 dz \ll n^{-2k} \text{area}(C).
\]

Hence we get an injective map $F_t(E_n)$ to $F_{tn^{-2k}}(E_1)$ and hence for $R_0 \geq 1$,
\[
\#F_t(U_{R_0}) = \sum_{n \geq R_0} \#F_{tn^{-2k}}(E_1).
\]

By Theorem 8.3, for $\epsilon > 0$, there exists $t_\epsilon$ such that for all $t < t_\epsilon$,
\[
\#F_t(E_1) \leq t^{-\alpha/2} c_A(\mathcal{H}_f^\alpha(E_1) + \epsilon)
\]
and hence there exists $N_\epsilon > 1$ such that for all $t \leq 1$ and $n > N_\epsilon$,
\[
\#F_{tn^{-2k}}(E_1) \leq t^{-\alpha/2} n^{-ka} c_A(\mathcal{H}_f^\alpha(E_1) + \epsilon)
\]
and hence
\[
\#F_t(U_{N_\epsilon} t^{\alpha/2}) \leq \left( \sum_{n \geq N_\epsilon} n^{-ka} \right) c_A(\mathcal{H}_f^\alpha(E_1) + \epsilon).
\]

Since $\left( \sum_{n \geq N_\epsilon} n^{-ka} \right) < \infty$ as $ka > 1$, if $N_\epsilon \gg 1$ is sufficiently large, we can make $\#F_t(U_{N_\epsilon} t^{\alpha/2}) \leq \epsilon$ and $\mathcal{H}_f^\alpha(U_{N_\epsilon}) < \epsilon$. Hence
\[
\limsup \#F_t(U_1) t^{\alpha/2} = \limsup \# \sum_{1 \leq n < N_\epsilon} F_t(E_n) t^{\alpha/2} + O(\epsilon) = c_A \cdot \mathcal{H}_f^\alpha(U_1) + O(\epsilon).
\]

As $\epsilon > 0$ is arbitrary, we have
\[
\limsup \#F_t(U_1) t^{\alpha/2} = c_A \cdot \mathcal{H}_f^\alpha(U_1).
\]

We note that the boundary of $\mathbb{R}^2_+$ meets with $\text{Res}(\mathcal{P}_0)$ at three points $1, -1, \infty$ and these three points are in one $\Gamma_0$-orbit. Note that $\Gamma_0$ contains an element $\gamma_0$: $\gamma_0(x + yi) = x + (y + 2)i$. \qed
Theorem 9.3. Let \((\mathbb{R}^2_+, f)\) be a conformal metric such that \(f(x+iy) \asymp y^{-k}\) for \(k \in \mathbb{R}\).

1. If \(\alpha - 1 < k < 2 - \alpha - 1\), we have \(\mathcal{H}_f^\alpha(\mathbb{R}^2_+) < \infty\).
2. If either \(k < \alpha - 1\) or \(k > 2 - \alpha - 1\), we have \(\mathcal{H}_f^\alpha(\mathbb{R}^2_+) = \infty\).

Proof. Let \(\gamma(z) = \frac{z-i}{z-i-1}\). Then \(\gamma \in \Gamma_0\) and \(\gamma(\infty) = 1\). Hence if we set \(U_\eta := \{ |x| \leq 1, y > \eta \}\), \(\gamma(U_\eta)\) is a neighborhood of 1 for all large \(\eta > 1\). We will show that \(\mathcal{H}_f^\alpha(\gamma(U_\eta)) < \infty\) if \(k < 2 - \alpha - 1\).

We have

\[
\mathcal{H}_f^\alpha(\gamma(U_\eta)) = \int_{z \in \gamma(U_\eta)} f(z)^\alpha d\mathcal{H}(z)
= \int_{w \in U_\eta} f(\gamma(w))|\gamma'(w)|^\alpha d\mathcal{H}(w)
\]

Since \(|\gamma'(w)| = |w - 1 + i|^{-2}\) and \(\mathfrak{Re}(\gamma(w)) \asymp y^{-1}\), we have

\[f(\gamma(w))|\gamma'(w)| \asymp y^{k-2} \text{.}\]

Hence by Lemma 9.2 if \(2 - k > \alpha - 1\),

\[\mathcal{H}_f^\alpha(\gamma(U_\eta)) < \infty \text{.}\]

\(\square\)

Note that \(\mathcal{T} := \mathcal{P}_0 \cap \mathbb{R}^2_+\) is a circle packing of the curvilinear triangle made by largest inner circles.

As \(\alpha > 1\), Theorem 9.1 is a special case of the following:

Theorem 9.4. Let \((\mathbb{R}^2_+, f)\) be a conformal metric such that \(f(x+iy) \asymp y^{-k}\) where \(\alpha - 1 < k < 2 - \alpha - 1\). Then for any Borel subset \(E \subset \mathbb{R}^2_+\) (not necessarily compact) with smooth boundary, we have

\[
\lim_{t \to \infty} t^{\alpha/2} \cdot \# \{ C \in \mathcal{T} : \text{area}_f(C) > t, C \cap E \neq \emptyset \} \sim c_A \cdot \mathcal{H}_f^\alpha(E) \text{.}
\]

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