ATTRACTORS AND THEIR STABILITY WITH RESPECT TO
ROTATIONAL INERTIA FOR NONLOCAL EXTENSIBLE BEAM
EQUATIONS

TAKAYUKI NIIMURA
Department of Mathematics
Hokkaido University
Sapporo, 060-0810, Japan

(Communicated by Irena Lasiecka)

ABSTRACT. In this paper we consider the nonlinear beam equations accounting
for rotational inertial forces. Under suitable hypotheses we prove the existence,
regularity and finite dimensionality of a compact global attractor and an ex-
ponential attractor. The main purpose is to trace the behavior of solutions of
the nonlinear beam equations when the effect of the rotational inertia fades
away gradually. A natural question is whether there are qualitative differences
would appear or not. To answer the question, we deal with the rotational in-
ertia with a parameter \( \alpha \) and consider the difference of behavior between the
case \( 0 < \alpha \leq 1 \) and the case \( \alpha = 0 \). The main novel contribution of this paper
is to show the continuity of global attractors and exponential attractors with
respect to \( \alpha \) in some sense.

1. Introduction. In this paper we consider the following models of extensible
beams

\[
(1 - \alpha \Delta)u_{tt} + \Delta^2 u - M \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = F(x, u) + G(u_t, \Delta u_t),
\]

where \( M \) is a scalar function, \( F \) represents forcing terms and \( G \) represents damping
terms. The equation (1) has been extensively studied in different contexts. First
we note that, for the linear case (that is, \( M = F = G = 0 \)), when \( \alpha = 0 \), the
form of (1) is known as the Euler-Bernoulli beam equation, while for \( \alpha > 0 \), one
has the so-called Rayleigh beam. The latter model was introduced for improving
the former model by taking into account the effect of rotational inertia. Another
context from which the equation (1) has arisen is the simplified “large deflection"
model for a beam, by taking \( F = G = 0 \), \( \alpha > 0 \) and \( M(s) = a + bs \), where \( a, b \) are
constants related to the forces applied on the system. The large deflection model
takes into account both in-plane and out-of-plane dynamics, and reflects the effect
of stretching on bending. The equation (1) can be viewed as a simplification of
the large deflection model when we suppose in-plane accelerations to be negligible.
With respect to more detailed arguments, we refer to [31], [26]. Also, as another
context, we can see the equation (1) as a simplification of the so-called scalar von

2010 Mathematics Subject Classification. Primary: 35B35, 35B40, 35B41, 37L30; Secondary:
74H40.

Key words and phrases. Long-time behavior of solutions, attractors, stability, extensible beam.

2561
Karman equation [13]. If one assumes that the second strain invariant is small, the scalar von Karman equation simplifies to what is known as a Berger plate equation, with same nonlocal nonlinear structure $M(s) = a + bs$.

In the case $\alpha = 0$, the model has been historically known as a Krieger or Krieger-Woinowsky beam,

$$u_{tt} + \Delta^2 u - M \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = F(x, u) + G(u_t, \Delta u_t).$$

(2)

For example, see [34], [45]. The equation (2) has been extensively studied by many researchers since the model was proposed. In particular, the pioneering works related to extensible beams and plates ($\alpha \equiv F \equiv G \equiv 0$ and $M = a + bs$) were done by Ball [4], [5] and Dickey [16], [17] in which existence, uniqueness, regularity and stability of the solution was studied. Also, the equation (2) in more general setting in mathematical point of views has been treated by many authors, for instance, in [35], [15], [7], [28], [43], [8], [6], [33] and [48]. Our purpose in this paper is to draw a comparison between the long-time behavior for dynamical systems generated by the specific type of equation (1) with rotational inertia ($\alpha > 0$) and the one without rotational inertia ($\alpha = 0$).

More recently, the equation (2) also has been studied in the context of the piston-theory;

$$(1 - \alpha \Delta) u_{tt} + \Delta^2 u + \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u + \beta u_t = p + \nabla \cdot u.$$

This model is a non-gradient problem, owing to the non-dissipative lower order term $\nabla \cdot u$. Howell-Lasiecka-Webster [25] studied this model under the abstract setting ($\alpha = 0$) and showed the existence of global attractor and generalized exponential attractors, with clamped boundary condition, by taking $F = p(x) - Lu$, $G = -\beta u_t$ ($\beta > 0$) and $M(s) = as + b$ ($L$ is an operator related to the lower order term and $a$, $b$ is constants). They also established a proper exponential attractor by assuming the coefficient of damping $\beta$ is large enough. Howell-Toundykov-Webster [26] considered the model with clamped-free boundary condition. They addressed both the case with rotational inertia ($\alpha > 0$) and the case without one ($\alpha = 0$), showed the existence of the solutions, by utilizing the semigroup-method in the former case and Galerkin-method in the later case. For the former, they prove the existence of global attractor. However, the latter one, even the uniqueness is still an open problem.

Our main goal in this paper is to examine the continuity of attractors when $\alpha \to 0$. That is, we show that the family of global attractors $\mathcal{A}_{\alpha}$ and exponential attractors $\mathcal{A}_{\text{exp}, \alpha}$ are continuous with respect to the parameter $\alpha$ in some sense. Specifically, we consider the model (1) which exhibits a nonlinear damping of the form

$$N \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u_t,$$

where $N > 0$ is a nonlinear function. To our best knowledge, this kind of nonlocal damping was firstly used in Lange and Perla Menzala [32] as a type of nonlocal beam equations in $\Omega = \mathbb{R}^n$. Later Cavalcanti et al. [9] also dealt with the same type of damping for viscoelastic problem defined on a bounded or unbounded domain $\Omega \subset \mathbb{R}^n$ and showed exponential decay for the problem. More recently, Silva-Narciso [39] treated nonlinear fractional damping $N(\int_{\Omega} |\nabla u|^2 \, dx)(-\Delta)^{\theta} u_t$ ($0 \leq \theta \leq 1$) in the context of the Krieger-Woinowsky beam and showed the existence of a
global attractor and a generalized exponential attractor to (2) with the supported boundary condition and the initial condition. Our work here is motivated by their result.

On the other hand, there exists another type of nonlocal nonlinear damping (so-called energy damping)

$$\gamma \left[ \int_{\Omega} (|\Delta u|^2 + |u_t|^2) \, dx \right]^q \Delta u_t,$$

where, $\gamma > 0$ and $q \geq 1$. Damping of the above type was introduced by Balakrishnan and Taylor [2] as a model for vibrating extensible beams with nonlocal nonlinear dissipative structure, in the process of studying damping phenomena in flight structures. Recently, Silva-Narciso-Vicente [40] treated the above damping in higher-dimensional system and studied the global existence, uniqueness and stabilization with clamped boundary condition, by taking $F = f(u)$ and $M \equiv \text{const}$. For related models and works, we also refer to [36], [46], [3] and [47].

The question for the stability of attractors naturally arises in the context of the relation between Euler-Bernoulli beam model and Rayleigh beam model. Also, this kind of problem has been considered from various viewpoints. Actually, besides rotational inertia, such a problem also appears in quasi-static problems in wave or plate vibrations and has been studied by many researchers. About the results for the von Karman equations, we refer to [13]. We also refer to the problem in the Krieger-Woinowski beam by Zelati [14]. In [13], the stability problems were well treated under the effect of internal dissipation and they showed the upper semicontinuity of global attractors. More recently, this problem was also studied under the effects of dissipation, one is geometrically constrained, the other is spatially localized. Moreover, upper semicontinuity was also obtained in [23]. In other words, we believe that it is interesting to consider the continuity problem with nonlinear damping. In particular, we concentrate upon here the construction of the exponential attractors for the equations (1) that are in the state space and continuous with respect to the rotational inertia parameter.

Now we are in a position to state our problem explicitly. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following equation

$$(1 - \alpha \Delta)u_{tt} + \Delta^2 u - M \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u - N \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u_t + f(u) = h \tag{3}$$

with the simply supported boundary condition

$$u = \Delta u = 0 \quad \text{on} \ [0, \infty) \times \partial \Omega, \tag{4}$$

and initial conditions

$$u(\cdot, 0) = u_0(\cdot), \ u_t(\cdot, 0) = u_1(\cdot) \quad \text{in} \ \Omega, \tag{5}$$

where $0 \leq \alpha \leq 1$, $M$ and $N$ are scalar functions specified later, $f(u)$ is a nonlinear source term and $h$ is an external forcing term. For this problem, we shall show the well-posedness and the existence of the global attractor and a proper exponential attractor. Moreover, we wish to reveal the continuous dependance of the attractors on the parameter $\alpha$ in some sense. Basically, we follow the arguments in [39] for each $\alpha$. Concerning the global attractor, we can apply to the abstract theory given in [12] to our problem. For the construction of exponential attractors, we invoke the arguments treated in [20]. However, for the construction of the family with
continuous dependence on the parameter, we can not invoke their arguments as done in [20], so we construct the exponential attractors by modifying the discussion.

The remainder of this paper is organized as follows. In Section 2 we introduce some notations on the function spaces and operators, and state our results. Section 3 is devoted to show that the problem (3)-(5) is well-posed. In Section 4 we review the basic terminologies and definitions of infinite-dimensional dynamical system and examine the existence of the global attractors and construct the exponential attractors. Finally, Section 5 is dedicated to the proof of the stability with respect to rotational inertia.

2. Preliminaries.

2.1. Notations and definitions. We first introduce some notation concerning the function spaces and operators that will be used throughout the remainder of this paper.

We denote by $L^2(\Omega)$ the set of square integrable functions with the usual $L^2$-inner product $(\cdot, \cdot)$ and by $L^p(\Omega)$ the set of p-th power integrable functions with the usual $L^p(\Omega)$-norm $\| \cdot \|_p$. We set $V = H^2(\Omega) \cap H_0^1(\Omega)$ with the inner product $(\Delta \cdot, \Delta \cdot)$ and the norm $\| \Delta \cdot \|_2$.

We define the operator $A : D(A) \to L^2(\Omega)$:

$$ Au := \Delta^2 u \quad \text{with the domain} \quad D(A) := \{ u \in H^4(\Omega) \mid u, \Delta u \in H_0^1(\Omega) \}, $$

where $\Delta$ is the Laplace operator with the Dirichlet boundary condition. Obviously, $A$ is self-adjoint in $D(A)$ and strictly positive on $D(A)$. Hence as is well-known $A$ has the inverse operator with the domain $L^2(\Omega)$ and it is compact. Thus from the spectral theory there exists an orthonormal basis $(\omega_j)_{j \in \mathbb{N}}$ in $L^2(\Omega)$ composed by eigenfunctions of $A$ such that $A\omega_j = \lambda_j \omega_j$ with $0 < \lambda_1 \leq \lambda_2 \ldots$ and $\lambda_j \to \infty$ as $j \to \infty$.

Moreover, we can define the fractional powers $A^s$, $s \in \mathbb{R}$, of $A$ with domains $D(A^s)$ being Hilbert spaces with the inner products and the norms defined by

$$(u,v)_{D(A^s)} = (A^s u, A^s v) \quad \text{and} \quad \|u\|_{D(A^s)} = \|A^s u\|_2, \quad u,v \in D(A^s).$$

The embedding $D(A^{s_1}) \hookrightarrow D(A^{s_2})$ is continuous if $s_1 \geq s_2$, is compact if $s_1 > s_2$, and it holds that

$$\|A^{s_2} u\|_2 \leq \lambda_1^{s_2-s_1}\|A^{s_1} u\|_2, \quad u \in D(A^{s_1}).$$

In particular, one has $D(A^0) = L^2(\Omega)$, $D(A^{1/4}) = H_0^1(\Omega)$, and

$$ D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega) \quad \text{with} \quad A^{1/2} u = -\Delta u, \quad u \in D(A^{1/2}). $$

Then we can convert the concrete system (3) to an abstract evolitional problem\footnote{One should note that to change boundary conditions can lead to dramatic change in the dynamics and may raise the necessity of an entirely different model.} given by

$$(1 + \alpha A^{1/2}) u_{tt} + Au + M(\|A^{1/4} u\|_2^2) A^{1/2} u + N(\|A^{1/4} u\|_2^2) A^{1/2} u_t + f(u) = h, \quad (u(0), u_t(0)) = (u_0, u_1).$$

(6)

The long-time dynamics of (6) is considered on a Hilbert space $\mathcal{H}_\alpha := D(A^{1/2}) \times H_\alpha$ equipped with the norm $\|(u,v)\|_{\mathcal{H}_\alpha}^2 := \|A^{1/2} u\|_2^2 + \|v\|_{H_\alpha}^2$. 

where if $\alpha > 0$, $H_\alpha$ is the Hilbert space $H^1_0(\Omega)$ with the norm
\[ \|v\|_{H_\alpha}^2 := \|v\|_2^2 + \alpha \|A^{1/4}v\|_2^2, \]
and if $\alpha = 0$, $H_0$ is the Hilbert space $L^2(\Omega)$. For the convenience of descriptions, we also introduce the notation $H_{\alpha,\nu} := D(A^{1/2} + \nu^4) \times H_{\alpha,\nu}$ ($\nu \in \mathbb{R}$) equipped with the norm
\[ \|(u,v)\|_{H_{\alpha,\nu}}^2 := \|A^{1/4}u\|_2^2 + \|v\|_{H_{\alpha,\nu}}^2, \]
where if $\alpha > 0$, $H_{\alpha,\nu}$ is the Hilbert space $D(A^{1/2} + \nu^4)$ with the norm
\[ \|v\|_{H_{\alpha,\nu}}^2 := \|A^{1/4}v\|_2^2 + \alpha \|A^{(1+\nu)/4}v\|_2^2, \]
and if $\alpha = 0$, $H_{0,\nu}$ is the Hilbert space $D(A^{\nu/4})$.

Now we give the definitions of global attractor, minimal attractor, fractal dimension, exponential attractor and unstable manifold.

**Definition 2.1.** Let $X$ be a complete linear metric space. A bounded set $A \subset X$ is said to be a global attractor of the dynamical system $(X,S(t))$ if and only if the following properties hold:

i. $A$ is an invariant set; that is, $S(t)A = A$, $\forall t \geq 0$.

ii. $A$ is uniformly attracting; that is, for all bounded set $D \subset X$
\[ \lim_{t \to \infty} h(S(t)D,A) = 0, \]
where $h(A,B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X$ is the Hausdorff semidistance.

**Definition 2.2.** Let $X$ be a complete linear metric space. A bounded set $A_{\text{min}} \subset X$ is said to be a minimal attractor of the dynamical system $(X,S(t))$ if and only if the following properties hold:

i. $A_{\text{min}}$ is a positively invariant set; that is $S(t)A_{\text{min}} \subseteq A_{\text{min}}$, $\forall t \geq 0$.

ii. $A_{\text{min}}$ attracts every point $x$ in $X$; that is,
\[ \lim_{t \to \infty} \text{dist}_X(S(t)x, A_{\text{min}}) = 0 \quad \text{for any } x \in X; \]

iii. $A_{\text{min}}$ is minimal; that is, $A_{\text{min}}$ has no proper subsets possessing the above properties.

**Definition 2.3.** Let $X$ be a complete linear metric space and $K$ be a compact set in $X$. The fractal (box-counting) dimension $\dim_f K$ of $K$ is defined by the formula
\[ \dim_f K := \lim_{\epsilon \to 0} \sup \log n(K,\epsilon) \log(1/\epsilon), \]
where, $n(K,\epsilon)$ is the minimal number of closed sets of the radius $\epsilon$ that cover $K$.

**Definition 2.4.** Let $X$ be a complete linear metric space. A compact set $A_{\text{exp}} \subset X$ is said to be a exponential attractor of the dynamical system $(X,S(t))$ if and only if $A_{\text{exp}}$ is a positively invariant set of finite fractal dimension and for every bounded set $D \subset X$ there exists positive constants $t_D, C_D$ and $\gamma_D$ such that
\[ h(S(t)D, A_{\text{exp}}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad \forall t \geq t_D. \]
Definition 2.5. Let $Y$ be a subset of the phase space $X$ of the dynamical system $(X, S(t))$. Then the **unstable manifold** $M^u(Y)$ emanating from $Y$ is defined as the set of points $x \in X$ such that there exists a trajectory $\gamma = \{S(t)x = u(t) : t \in \mathbb{R}\}$ with the properties

$$
\lim_{t \to -\infty} \text{dist}(u(t), Y) = 0.
$$

2.2. **Assumptions and results.**

2.2.1. **Well-posedness.** First of all, we list assumptions that we shall use for proving the well-posedness:

- **(H1)** $M$ and $N$ are $C^1$-functions on $[0, \infty)$ with
  
  $$
  M(\tau) \ge 0 \quad \text{and} \quad N(\tau) > 0, \quad \forall \tau \ge 0.
  $$

- **(H2)** $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function such that $f(0) = 0$, and
  
  $$
  |f'(u)| \le \sigma_1(1 + |u|^{p/2}), \quad \forall u \in \mathbb{R},
  $$
  
  for some constant $\sigma_1 > 0$, and the power $p$ satisfying

  $$
  p > 0 \quad \text{if} \quad 1 \le n \le 4 \quad \text{and} \quad 0 < p \le \frac{8}{n-4} \quad \text{if} \quad n \ge 5.
  $$

Besides, let us suppose that there exists a constant $l_0 \ge 0$ such that

$$
\hat{f}(u) := \int_0^u f(s)ds \ge -\frac{\lambda_1}{8}|u|^2 - l_0, \quad \forall u \in \mathbb{R}.
$$

Then the well-posedness of (6) is given by the following:

**Theorem 2.6.** Let $T > 0$ be an arbitrary number, $h \in L^2(\Omega)$ and $0 \le \alpha \le 1$. Also, we assume (H1) and (H2). Then, we have:

i. If the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$, then the problem (6) has a strong solution in the class

$$
u \in L^\infty(0, T; D(A)), \quad u_t \in L^\infty(0, T; D(A^{1/2})) \quad u_{tt} \in L^\infty(0, T; D(A^{1/4})).$$

ii. If the initial data $(u_0, u_1) \in \mathcal{H}_\alpha$, then the problem (6) has a weak solution in the class

$$
u \in L^\infty(0, T; D(A^{1/2})), \quad u_t \in L^\infty(0, T; H_\alpha) \cap L^2(0, T; D(A^{1/4}))$$

satisfying

$$(u, u_t) \in C([0, T]; \mathcal{H}_\alpha).$$

Moreover, the above weak solution has a more regularity

$$
u \in L^\infty(0, T; D(A^{3/4})), \quad u_t \in L^\infty(0, T; H_{\alpha,1}) \cap L^2(0, T; D(A^{1/2}))$$

for initial data $(u_0, u_1) \in \mathcal{H}_{\alpha,1}$.

iii. Both weak and strong solutions depend continuously on the initial data in $\mathcal{H}_\alpha$. More precisely, if $z = (u, u_t)$, $\tilde{z} = (\tilde{u}, \tilde{u}_t)$ are two solutions corresponding to the initial data $z_0 = (u_0, u_1)$, $\tilde{z}_0 = (\tilde{u}_0, \tilde{u}_1)$ lying in $\mathcal{H}_\alpha$, then

$$
\|z(t) - \tilde{z}(t)\|_{\mathcal{H}_\alpha} \le c^CT\|z_0 - \tilde{z}_0\|_{\mathcal{H}_\alpha}, \quad \forall t \in [0, T],
$$

for some positive constant $C = C(\|z_0\|, \|\tilde{z}_0\|)$. In particular, the problem (6) has uniqueness.
2.2.2. Stability properties of solutions. The well-posedness of the problem (6) provide
the family of evolution operators \( S_\alpha(t) : H_\alpha \to H_\alpha \) defined by
\[
S_\alpha(t)(u_0, u_1) = (u(t), u_t(t)), \quad t \geq 0,
\]
where \((u, u_t)\) is the unique weak solution of (6). \( S_\alpha \) are nonlinear \( C_0 \)-semi-groups,
and is locally Lipschitz continuous on the phase space \( H_\alpha \). Hence, the problem (6)
generates a dynamical system \((H_\alpha, S_\alpha(t))\) and we study the asymptotic behavior of
solutions for the problem (6) through the dynamical system \((H_\alpha, S_\alpha(t))\).

Now, we add assumptions for establishing stability properties of solutions:

\( \text{(H3)} \) There exists a constant \( l_1 \geq 0 \) such that
\[
\tilde{f}(u) \leq f(u)u + \frac{\lambda_1}{8}|u|^2 + l_1, \quad \forall u \in \mathbb{R}.
\]  \hfill (16)

\( \text{(H4)} \) There exists a constant \( l_2 \geq 0 \) such that
\[
\tilde{M}(\tau) := \int_0^\tau M(s)ds \leq 2M(\tau)\tau + \frac{\lambda_1/2}{4}\tau + 2l_2, \quad \forall \tau \geq 0.
\]  \hfill (17)

Then we can state the stability properties of solutions as follows:

**Theorem 2.7.** Let us assume that the hypothesis of Theorem 2.6 holds. Besides,
we suppose that \( (\text{H3}) \) and \( (\text{H4}) \) hold. Then we have

i. The dynamical system \( H_\alpha \) generated from the problem (6) possesses the global
attractor \( A_\alpha \subset H_\alpha \) and it is compact and connected.

ii. The global attractor \( A_\alpha \) is an unstable manifold \( A_\alpha = M^u(N) \), emanating
from the set \( N \) consisting of stationary points of \( S_\alpha \), namely,
\[
N = \{(u, 0) \in H_\alpha \mid Au + M(\|u\|_2^2)A^{1/2}u + f(u) = h\}.
\]

iii. There exists the minimal attractor \( A_{\alpha_{\min}} \) to the dynamical system \((H_\alpha, S_\alpha(t))\),
which is the set of the stationary points, that is, \( A_{\alpha_{\min}} = N \).

In addition, we consider the following slightly stronger requirement on \( p \) than (9):

\( \text{(H5)} \) If in (9) from \( (\text{H2}) \) we consider either of the conditions below
\[
p > 0 \text{ if } 1 \leq n \leq 4 \quad \text{or else} \quad 0 < p < \frac{8}{n - 4} \text{ if } n \geq 5.
\]  \hfill (18)

Then we also have the following:

**Theorem 2.8.** Let us assume that the hypothesis of Theorem 2.6 and Theorem 2.7
hold. Besides, we suppose that \( (\text{H5}) \) holds. Then we have

i. The compact global attractor \( A_\alpha \) has a finite fractal dimension.

ii. Any full trajectory \((u(t), u_t(t))\) that belongs to the global attractor \( A_\alpha \) enjoys
the following regularity properties,
\[
(u_t, u_{tt}) \in L^\infty(\mathbb{R}; H_\alpha).
\]

Specifically, there exists \( R_1 > 0 \) such that
\[
\|A^{1/2}u_{tt}(t)\|_2^2 + \|u_{tt}(t)\|_2^2 + \alpha\|A^{1/2}u_{tt}\|_2^2 \leq R_1^2, \quad t \in \mathbb{R},
\]  \hfill (19)

where \( R_1 \) does not depend on \( \alpha \). In addition, the global attractor \( A_\alpha \) lies in
\( D(A) \times D(A^{1/2}) \).
where we set \( E_{2568} \).

2.2.3. **Main Theorem.**

**Theorem 2.9.** Let the assumptions of Theorem 2.8 be in force. Then we have

i. The family of attractors \( A_\alpha \) is upper semi-continuous at the point 0, that is,

\[
h_{\mathcal{H}_\alpha}(A_\alpha, A_0) \equiv \sup_{y \in A_\alpha} \inf_{z \in A_0} \|y - z\|_{\mathcal{H}_\alpha} \to 0
\]

as \( \alpha \to 0^+ \).

ii. There exist exponential attractors \( A_{\exp, \alpha} \) for \( (\mathcal{H}_\alpha, S_\alpha(t)) \), for which the estimate

\[
H_{\mathcal{H}_\alpha}(A_{\exp, \alpha}, A_{\exp, 0}) \equiv \max\{h_{\mathcal{H}_\alpha}(A_{\exp, \alpha}, A_{\exp, 0}), h_{\mathcal{H}_\alpha}(A_{\exp, 0}, A_{\exp, \alpha})\} \leq C\alpha^n
\]

holds with some exponent \( 0 < \kappa < 1 \) and constant \( C > 0 \).

In the next sections we begin with the proofs of these statements.

3. **Proof of Theorem 2.6.** Let \((\omega_m)\) be the basis in \( L^2(\Omega) \), \( W_m \) be the space generated by \( \omega_1, \ldots, \omega_m \), and for given \( C^2 \)-functions \( y_{jm}(t) \), set

\[
u_m(t) = \sum_{j=1}^m y_{jm}(t)\omega_j.
\]

For \((u_0, u_1) \in D(A) \times D(A^{1/2})\), we consider the following problem:

\[
(u_{it}^m, \omega) + \alpha(u_{tt}^m, A^{1/2} \omega) + (A^{1/2} u_m^m, A^{1/2} \omega) + M(\|A^{1/4} u_m^m\|_2^2)(A^{1/2} u_m^m, \omega) + N(\|A^{1/4} u_m^m\|_2^2)(u_{tt}^m, A^{1/2} \omega) + (f(u_m^m), \omega) = (h, \omega), \quad \forall \omega \in W_m,
\]

\[
u_m(0) = u_{0m} \to u_0 \text{ in } D(A) \text{ and } u_{tm}(0) = u_{1m} = u_1 \text{ in } D(A^{1/2}).
\]

By standard methods for ordinal differential equations, we can prove the existence of \( C^2 \)-class solutions to the approximate problem on some interval \([0, T_m]\) and this solution can be extended to the closed interval \([0, T]\) by using the first energy estimate (30) below.

3.1. **Strong solutions. The First Energy:** Taking \( \omega = u_{tm}\) in (21), we infer

\[
d\frac{d}{dt} E_{\alpha}(t) + N(\|A^{1/4} u_{m}(t)\|_2^2)\|A^{1/4} u_{tm}(t)\|_2^2 = 0,
\]

where we set \( E_{\alpha}(t) = E_{\alpha}(u_m(t), u_{tm}(t)) \). Here the energy \( E_{\alpha}(t) = E_{\alpha}(u(t), u_t(t)) \) \((u, u_t) \in \mathcal{H}_\alpha\) is defined by

\[
E_{\alpha}(t) := \frac{1}{2}(\|A^{1/2} u(t)\|_2^2 + \|u_t(t)\|_2^2 + \alpha\|A^{1/4} u_t(t)\|_2^2)
\]

\[
+ \tilde{M}(\|A^{1/4} u(t)\|_2^2) + \int_{\Omega} (f(u(t)) - hu(t)) dx,
\]

where the definition of \( \tilde{M} \) is given by (17). Integrating (22) from 0 to \( t \) \((\leq T)\) we get

\[
E_{\alpha}(t) + \int_0^t N(\|A^{1/4} u_m(s)\|_2^2)\|A^{1/4} u_{tm}(s)\|_2^2 ds = E_{\alpha}(0).
\]


Now, using Young’s inequality with \( \varepsilon = \frac{\lambda_1}{4} \) and the condition (10) we have

\[
\int_\Omega \left( \bar{f}(u^m(t)) - hu^m(t) \right) dx \\
\geq -\frac{\lambda_1}{8} \|u^m(t)\|_2^2 - l_0|\Omega| - \frac{1}{2} \left( \frac{\lambda_1}{4} \|u^m(t)\|_2^2 + \frac{4}{\lambda_1} \|h\|_2^2 \right) \\
\geq -\frac{1}{8} \|A^{1/2}u^m(t)\|_2^2 - l_0|\Omega| - \frac{1}{8} \|A^{1/2}u^m(t)\|_2^2 - \frac{2}{\lambda_1} \|h\|_2^2 \\
= -\frac{1}{4} \|A^{1/2}u^m(t)\|_2^2 - \frac{2}{\lambda_1} \|h\|_2^2 - l_0|\Omega|
\]  

(25)

hence

\[
\frac{1}{4} \left( \|A^{1/2}u^m(t)\|_2^2 + \|u^m(t)\|_2^2 + \alpha \|A^{1/4}u^m(t)\|_2^2 \right) \leq E^m_\alpha(t) + \frac{2}{\lambda_1} \|h\|_2^2 + l_0|\Omega|.
\]  

(26)

Compounding (24) with (27), we have

\[
\|A^{1/2}u^m(t)\|_2^2 + \|u^m(t)\|_2^2 + \alpha \|A^{1/4}u^m(t)\|_2^2 \leq C_1 \quad \forall t \in [0, T], \ m \in \mathbb{N},
\]  

(28)

where \( C_1 = C_1(\|A^{1/2}u_0\|_2, \|A^{1/4}u_1\|_2, \|h\|_2, |\Omega|, l_0 > 0 \). Since \( N(\tau) > 0 \) and above the estimate, there exists a positive constant \( N_0 = N_0(\|u_0, u_1\|_{\mathcal{H}_a}) > 0 \) such that \( N(\|A^{1/4}u(t)\|_2^2) \geq N_0 \) for any \( t \in [0, T] \). Going back to (24) and combining this uniform boundedness with the easy relation \( E_\alpha^m \leq E_1^m(t) \ (t \geq 0) \), we derive

\[
E^m_\alpha(t) + N_0 \int_0^t \|A^{1/4}u^m(s)\|_2^2 \, ds \leq E^m_\alpha(0) \leq E^m_1(0).
\]  

(29)

Combining (27) with (29) we conclude

\[
\|A^{1/2}u^m(t)\|_2^2 + \|u^m(t)\|_2^2 + \alpha \|A^{1/4}u^m(t)\|_2^2 + \int_0^t \|A^{1/4}u^m(s)\|_2^2 \, ds \leq C_1
\]  

(30)

for any \( t \in [0, T] \) and \( m \in \mathbb{N} \), and some constant \( C_1 > 0 \) depending on the norm of the initial data in \( \mathcal{H}_a \).

**The Second Energy:** Differentiating (21) with respect to \( t \) and substituting \( \omega = u^m_{tt} \), it holds that

\[
\frac{1}{2} \frac{d}{dt} \left( \|A^{1/2}u^m_t(t)\|_2^2 + \|u^m(t)\|_2^2 + \alpha \|A^{1/4}u^m_t(t)\|_2^2 \right)
\]

\[
+ M \left( \|A^{1/4}u^m(t)\|_2^2 \right) \|A^{1/4}u^m_t(t)\|_2^2 + N \left( \|A^{1/4}u^m(t)\|_2^2 \right) \|A^{1/4}u^m_{tt}(t)\|_2^2 = \sum_{j=1}^4 I_j
\]  

(31)

where

\[
I_1 = -\left( \frac{d}{dt} N \left( \|A^{1/4}u^m(t)\|_2^2 \right) \right) \left( A^{1/2}u^m_t(t), u^m_{tt}(t) \right),
\]

\[
I_2 = -\left( \frac{d}{dt} M \left( \|A^{1/4}u^m(t)\|_2^2 \right) \right) \left( A^{1/2}u^m_t(t), u^m_{tt}(t) \right),
\]

\[
I_3 = -\int_\Omega f^\prime(u^m(t))u^m_t(t)u^m_{tt}(t) \, dx,
\]

\[
I_4 = \left( \frac{d}{dt} M \left( \|A^{1/4}u^m(t)\|_2^2 \right) \right) \|A^{1/4}u^m(t)\|_2^2.
\]
Further, utilizing the condition (8), generalized Hölder inequality with the condition (21) to obtain a strong solution satisfying (21) and

\[ \max_{t \in [0, T]} \left\{ \| M'(\tau) \|, \| N(\tau) \|, \| N'(\tau) \| \right\} = C(\| u_0, u_1 \|_{H_\alpha}) < \infty, \]

where \( C_1 \) is the constant appeared in (30). In the following \( C > 0 \) denotes various constants which depends on the initial data in \( H_\alpha \), but not on \( T > 0 \). Using Young’s inequality and the self-adjointness of operator \( A^{1/4} \), we have

\[
|I_1| = \left| \{ M'(\| A^{1/4}u^m(t) \|_2^2) \} \left( A^{1/4}u^m(t), u^m_t(t) \right) \right| \leq C \left( \| A^{1/4}u^m(t) \|_2^2 + \| u^m_t(t) \|_2^2 \right)
\]

and

\[
|I_2| = \left| \{ M'(\| A^{1/4}u^m(t) \|_2^2) \} \left( A^{1/4}u^m(t), u^m_t(t) \right) \right| \leq C \| A^{1/4}u^m(t) \|_2 \| A^{1/4}u^m_t(t) \|_2 \| A^{1/2}u^m(t) \|_2 \| u^m_t(t) \|_2 \leq C \left( \| A^{1/2}u^m_t(t) \|_2^2 + \| u^m_t(t) \|_2^2 \right).
\]

Further, utilizing the condition (8), generalized Hölder inequality with \( \frac{p}{2(p+2)} + \frac{1}{p+2} = 1 \), Young’s inequality, the estimate (30) and the embedding \( V = H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^{p+2}(\Omega) \), we infer

\[
|I_3| \leq \sigma_1 \left( \| (1 + \| u^m(t) \|^{p/2}) u^m_t(t) u^m(t) \|_1 \right) \leq \sigma_1 \left( \| A^{1/2}u^m(t) \|_2 \| u^m_t(t) \|_2 \right) \leq C \left( \| A^{1/2}u^m_t(t) \|_2^2 + \| u^m_t(t) \|_2^2 \right).
\]

It is easy to see that

\[
|I_4| \leq C \| A^{1/2}u^m(t) \|_2^2
\]

Since \( N(\tau) > 0 \), the estimate (30) implies that \( N(\| A^{1/4}u^m(t) \|_2^2) \geq N_0 > 0 \) for all \( t \in [0, T] \), where \( N_0 = N_0(\| u_0, u_1 \|_{H_\alpha}) \) is a constant, so that we can derive

\[ I_1 \geq N_0 \| A^{1/4}u^m(t) \|_2^2. \]

Using these five estimates in (31), there exists a constant \( C > 0 \) such that

\[
\frac{1}{2} \frac{d}{dt} \left( \| A^{1/2}u^m(t) \|_2^2 + \| u^m_t(t) \|_2^2 + \alpha \| A^{1/4}u^m_t(t) \|_2^2 \right) + M \left( \| A^{1/4}u^m(t) \|_2^2 \right) \left( \| A^{1/4}u^m(t) \|_2^2 \right) \leq C \left( \| A^{1/2}u^m(t) \|_2^2 + \| u^m_t(t) \|_2^2 + \alpha \| A^{1/4}u^m_t(t) \|_2^2 \right.
\]

\[
+ M \left( \| A^{1/4}u^m(t) \|_2^2 \right) \left( \| A^{1/4}u^m(t) \|_2^2 \right).
\]

The estimates (30), (32) are sufficient to pass the limit in the approximate equation (21) to obtain a strong solution satisfying (21) and

\[
(1 + \alpha A^{1/2})u_t + Au + M \left( \| A^{1/4}u \|_2^2 \right) A^{1/2}u + N \left( \| A^{1/4}u \|_2^2 \right) A^{1/2}u = f(u) = h
\]

in \( L^\infty(0, T; L^2(\Omega)) \).
3.2. **Weak solutions.** Let \( z_0 = (u_0, u_1) \in \mathcal{H}_\alpha \). Then, since \( D(A) \times D(A^{1/2}) \) is dense in \( \mathcal{H}_\alpha \), there exists \((u^k_0, u^k_1) \in D(A) \times D(A^{1/2})\) such that
\[
(u^k_0, u^k_1) \to (u_0, u_1) \quad \text{in} \quad \mathcal{H}_\alpha.
\]
For each regular initial data \((u^k_0, u^k_1)\) there exists a strong solution \( u^k(t) \) satisfying the estimate (30). Also, taking the multiplier \( A^{1/2}u^k \) in (21) it easy to derive the following estimate
\[
\|A^{3/4}u^k(t)\|^2 + \|A^{1/4}u^k(t)\|^2 + \alpha\|A^{1/2}u^k(t)\|^2 + \int_0^t \|A^{1/2}u_t(s)\|^2 ds \leq C,
\]
for any \( t \in [0,T] \) and \( k \in \mathbb{N} \), where \( C = C(\|z_0\|_{\mathcal{H}_\alpha}, h, |\Omega|, T) \).

Furthermore, the difference of strong solutions \( w(t) := u^k(t) - u^j(t) \) satisfies the estimate
\[
\|A^{1/2}w(t)\|^2 + \|w_t(t)\|^2 + \alpha\|A^{1/4}w_t(t)\|^2 \\
\leq C\left( \|A^{1/2}w(0)\|^2 + \|w_t(0)\|^2 + \alpha\|A^{1/4}w_t(0)\|^2 \right)
\]
for all \( t \in [0,T] \) and some positive constant \( C = C(\|z_0\|_{\mathcal{H}_\alpha}, T) \). This implies that
\[
(u^k, u^k) \to (u(t), u_t(t)) \quad \text{in} \quad C([0,T], \mathcal{H}_\alpha).
\]
We omit the details of estimate (33) here, because they are identical to that concerning the continuous dependence presented in the below. These estimates are enough to conclude that the limit of approximate solutions satisfy (12) and the following weak formulation:
\[
(u_{tt}, \omega) + \alpha(u_{tt}, A^{1/2}\omega) + (A^{1/2}u_t, A^{1/2}\omega) + M\left( \|A^{1/4}u\|^2_2 \right) (A^{1/2}u, \omega) \\
+ N\left( \|A^{1/4}u\|^2_2 \right) (u_t, A^{1/2}\omega) + (f(u), \omega) = (h, \omega), \quad \forall \omega \in V = H^2(\Omega) \cap H^1_0(\Omega).
\]

3.3. **Uniqueness of strong and weak solutions.** Let \( z = (u, u_t) \) and \( \tilde{z} = (\tilde{u}, \tilde{u}_t) \) be two (strong or weak) solutions corresponding to the initial data \( z_0 = (u_0, u_1) \) and \( \tilde{z}_0 = (\tilde{u}_0, \tilde{u}_1) \), respectively. Putting \( w = u - \tilde{u} \), we see that the function \((w, w_t) = z - \tilde{z} \) verifies
\[
(w_{tt}(t), \omega) + (A^{1/2}w(t), A^{1/2}\omega) + \alpha(A^{1/4}w_{tt}(t), A^{1/4}\omega) \\
+ N(\|A^{1/4}u(t)\|^2_2)(A^{1/4}w_t(t), A^{1/4}\omega) = -M(\|A^{1/4}u(t)\|^2_2)(A^{1/2}w(t), \omega) \\
- \{N(\|A^{1/4}u(t)\|^2_2) - N(\|A^{1/4}\tilde{u}(t)\|^2_2)\}(A^{1/2}\tilde{u}_t(t), \omega) \\
- \{M(\|A^{1/4}u(t)\|^2_2) - M(\|A^{1/4}\tilde{u}(t)\|^2_2)\}(A^{1/2}\tilde{u}_t(t), \omega) - (f(u(t) - f(\tilde{u}(t)), \omega),
\]
with the initial data \((w(0), w_t(0)) = z_0 - \tilde{z}_0 \), in the strong or weak sense.

We first deal with strong solutions. Substituting \( \omega = w_t(t) \) in (36), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|A^{1/2}w(t)\|^2_2 + \|w_t(t)\|^2_2 + \alpha\|A^{1/4}w_t(t)\|^2_2 \right) \\
+ N(\|A^{1/4}u(t)\|^2_2)(A^{1/4}w_t(t), A^{1/4}\omega) = \sum_{j=1}^4 J_j,
\]
where
\begin{align*}
J_1 &= -M \langle \|A^{1/4}u(t)\|_2^2 \rangle \langle A^{1/2}w(t), w_t(t) \rangle, \\
J_2 &= -\left\{ N \langle \|A^{1/4}u(t)\|_2^2 \rangle - N \langle \|A^{1/4}\tilde{u}(t)\|_2^2 \rangle \right\} \langle A^{1/4}\tilde{u}_t(t), A^{1/4}w_t(t) \rangle, \\
J_3 &= -\left\{ M \langle \|A^{1/4}u(t)\|_2^2 \rangle - M \langle \|A^{1/4}\tilde{u}(t)\|_2^2 \rangle \right\} \langle A^{1/2}\tilde{u}_t(t), w_t(t) \rangle, \\
J_4 &= -(f(u(t)) - f(\tilde{u}(t)), w_t(t)).
\end{align*}

Since \(N(\tau) > 0\), the estimate (30) implies that \(N(\|A^{1/4}u(t)\|_2^2) \geq N_0 > 0\) for all \(t \in [0, T]\), where \(N_0 = N_0(\|\alpha\|_{\tilde{u}_t})\). Then (37) leads to
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + N_0 \|A^{1/4}w_t(t)\|_2^2 \leq \sum_{j=1}^4 J_j. \tag{42}
\]

Next we estimate \(J_1\), \(J_2\), \(J_3\), and \(J_4\). Analogously to the estimate of \(I_2\) we have
\[
|J_1| \leq C \left( \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 \right)
\]
In the next two estimates we shall use the mean value theorem, Young’s inequality and the estimate (30). Then we get
\[
|J_2| \leq C \left[ \|A^{1/4}u(t)\|_2^2 - \|A^{1/4}\tilde{u}(t)\|_2^2 \right] \left[ \|A^{1/4}\tilde{u}_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \right]
\]
\[
\leq C \left[ \|A^{1/4}u(t)\|_2^2 + \|A^{1/4}\tilde{u}(t)\|_2^2 \right] \left[ \|A^{1/4}w(t)\|_2 \|A^{1/4}\tilde{u}_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \right]
\]
\[
\leq \epsilon \|A^{1/4}w_t(t)\|_2^2 + \frac{C^2}{4\epsilon} \|A^{1/4}\tilde{u}_t(t)\|_2^2 \|A^{1/2}w(t)\|_2^2
\]
for any \(\epsilon > 0\) and
\[
|J_3| \leq C \left[ \|A^{1/4}u(t)\|_2^2 - \|A^{1/4}\tilde{u}(t)\|_2^2 \right] \left[ \|A^{1/4}\tilde{u}_t(t)\|_2 \|w_t(t)\|_2 \right]
\]
\[
\leq C \left[ \|A^{1/4}u(t)\|_2^2 + \|A^{1/4}\tilde{u}(t)\|_2^2 \right] \left[ \|A^{1/4}w(t)\|_2 \|A^{1/2}\tilde{u}_t(t)\|_2 \|w_t(t)\|_2 \right]
\]
\[
\leq C \langle \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 \rangle.
\]

From the condition (8), we can immediately see that there exists a constant \(\sigma_0 > 0\) such that
\[
|f(u) - f(v)| \leq \sigma_0 (1 + |u|^{p/2} + |v|^{p/2}) |u - v|, \quad \forall u, v \in \mathbb{R}, \tag{43}
\]
and we can estimate \(J_4\) likewise \(I_5\)
\[
|J_4| \leq \sigma_0 \langle \|\Omega\| \frac{p}{p-2} + \|u(t)\|_{p+2}^{p/2} + \|\tilde{u}(t)\|_{p+2}^{p/2} \rangle \|w(t)\|_{p+2} \|w_t(t)\|_2
\]
\[
\leq C \langle \|A^{1/2}w(t)\|_2 \|w_t(t)\|_2 \rangle
\]
\[
\leq C \langle \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 \rangle.
\]

Using these four estimates in (42) and taking \(\epsilon > 0\) small enough, there exists a constant \(C > 0\) such that
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + N_0 \|A^{1/4}w_t(t)\|_2^2
\]
\[
\leq C(1 + \|A^{1/4}\tilde{u}_t(t)\|_2^2) \langle \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \rangle \tag{44}
\]
for any \(t \in [0, T]\). From the estimate (30) the function \(1 + \|A^{1/4}u(t)\|_2^2\) is integrable on \([0, T]\). Then integrating (44) on \([0, t]\) and using Gronwall’s inequality, we arrive
Let \( z \) be a semiflow on a Banach space \( X \).

\[ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 + N_0 \int_0^t \|A^{1/4}w_t(s)\|_2^2 \, ds \leq e^{C_T} \left( \|A^{1/2}w(0)\|_2^2 + \|w_t(0)\|_2^2 + \alpha \|A^{1/4}w_t(0)\|_2^2 \right) \]

(45)

for some positive constant \( C_T = C_T(\|(u_0, u_1)\|_{\mathcal{H}_a}) \). (45) shows the continuous dependence of strong solutions on the initial data in \( \mathcal{H}_a \).

The same conclusion holds for weak solutions by using the density argument. In fact, if we consider the initial data \( z_0 = (u_0, u_1), z_0 = (\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}_a \), then similarly to (34) there exist sequences of strong solutions \( z^k = (w^k, u^k) \) and \( \tilde{z}^k = (\tilde{w}^k, \tilde{u}^k) \) such that

\[ (z^k, \tilde{z}^k) \to (z, \tilde{z}) \] in \( C([0, T], \mathcal{H}_a \times \mathcal{H}_a) \).

The difference \( z^k - \tilde{z}^k := (w^k, u^k) \) satisfies (45) for each \( k \in \mathbb{N} \), hence the estimate (14) holds for the difference of weak solutions \( z - \tilde{z} \) after passing the limit as \( k \to \infty \).

Particularly, we have uniqueness of both strong and weak solutions. This completes the proof of Theorem 2.6.

\[ \square \]

4. Stability and long-time dynamics. In this section we show the existence of attractors and clarifying their properties. First of all, we introduce a couple of notions of infinite-dimensional dynamical systems.

4.1. Review on dynamical systems. Here, we introduce some important concepts we shall need for proving the statement on stability properties:

**Definition 4.1.** Let \( S \) be a semiflow on a Banach space \( X \).

i. A closed set \( B \subset X \) is said to be absorbing for \( (X, S(t)) \) if and only if for any bounded set \( D \subset X \) there exists \( t_0(D) \) such that \( S(t)D \subset B \) for all \( t \geq t_0(D) \).

ii. \( (X, S(t)) \) is said to be dissipative if and only if it possesses a bounded absorbing set \( B \).

iii. \( (X, S(t)) \) is said to be asymptotically compact if and only if there exists an attracting compact set \( K \); that is for any bounded set \( D \) we have

\[ \lim_{t \to \infty} h_X(S(t)D, K) = 0. \]

**Definition 4.2.** Let \( B \subseteq X \) be a positively invariant set of a dynamical system \( (X, S(t)) \).

i. The continuous functional \( \Phi \) defined on \( B \) is said to be the Lyapunov function for the dynamical system \( (X, S(t)) \) on \( B \) if and only if the function \( t \mapsto \Phi(S(t)z) \) is a non-increasing function for any \( z \in B \).

ii. The Lyapunov function \( \Phi \) is said to be strict on \( B \) if and only if for \( z \in B \), the equation \( \Phi(S(t)z) = \Phi(z) \) for all \( t > 0 \) implies that \( S(t)z = z \) for all \( t > 0 \); that is, \( z \) is a stationary point of \( (X, S(t)) \).

iii. The dynamical system \( (X, S(t)) \) is said to be gradient if and only if there exists a strict Lyapunov function for \( (X, S(t)) \) on the whole phase space \( X \).

The concept introduced below is one of the form so-called quasi-stable or compact perturbation of contraction operator.

**Definition 4.3.** (See [13]) A dynamical system \( (X, S(t)) \) is said to be quasi-stable on a set \( B \subset X \) if and only if there exist a Banach space \( X_T \) (the space depends on \( T > 0 \)), a map \( \phi_T : B \to X_T \) and nonnegative functions \( a(\cdot), b(\cdot), c(\cdot) \) on \([0, \infty)\) such that (i) the image \( \phi_T(B) \) is compact in \( X_T \), (ii) \( a(\cdot) \) and \( c(\cdot) \) are locally bounded on
where the energy $E$ and $T > 0$ the following relation

$$\|S(T)x_1 - S(T)x_2\|_X \leq a(T)\|x_1 - x_2\|_X$$

and

$$\|S(T)x_1 - S(T)x_2\| \leq b(T)\|x_1 - x_2\|_X + c(T)\|\phi_Tx_1 - \phi_Tx_2\|_{X_T}$$

holds.

We show the existence of attractors and their structure by using the following criteria:

**Proposition 1.** (See [13]) Suppose that the dynamical system $(X, S(t))$ possesses the following properties:

i. The dynamical system $(X, S(t))$ is dissipative.

ii. The dynamical system $(X, S(t))$ is asymptotically compact.

Then the system $(X, S(t))$ possesses the global attractor.

**Proposition 2.** (See [13]) Let a dynamical system $(X, S(t))$ possesses a compact global attractor $A$. Assume also that the Lyapunov function $\Phi$ exists on $A$. Then

i. $A = \mathcal{M}(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary points of the dynamical system.

ii. $A_{\min} = \mathcal{N}$, where $A_{\min}$ is the minimal attractor of the dynamical system $(X, S(t))$.

4.2. **Global attractor and minimal attractor.** Let us show the existence of the global attractor of $(\mathcal{H}_\alpha, S_\alpha(t))$ according to the Proposition 1. First we show the dissipativity of dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.

4.2.1. **Dissipativity of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.**

**Proposition 3.** Let us assume that $(u_0, u_1) \in \mathcal{H}_\alpha$ and the hypotheses of Theorem 2.6 and 2.7 hold. Then, if $z = (u, u_t)$ is a weak solution corresponding to the initial data $z_0 = (u_0, u_1)$, we have

$$\|z(t)\|_{\mathcal{H}_\alpha}^2 \leq K_1e^{-\delta t} + K_2, \quad \forall t > 0,$$

for some positive constants $K_1 = K_1(\|z_0\|_{\mathcal{H}_\alpha})$ and $K_2 = K_2(h, l_0, l_1, |\Omega|)$ and a small constant $\delta > 0$.

**Proof.** We deal with only strong solutions, because the same conclusion follows easily for weak solutions by using the density argument.

The strong solution $z = (u, u_t)$ satisfies the following estimates

$$\frac{1}{4}\|z(t)\|_{\mathcal{H}_\alpha}^2 = \frac{1}{4}\left(\|A^{1/2}u(t)\|_2^2 + \|u_t(t)\|_2^2 + \alpha\|A^{1/4}u(t)\|_2^2\right) \leq E_\alpha(t) + \frac{2}{\lambda_1}\|h\|_2^2 + l_0|\Omega|,$$

(47)

and

$$\frac{d}{dt}E_\alpha(t) + N(\|A^{1/4}u(t)\|_2^2)\|A^{1/4}u_t(t)\|_2^2 = 0,$$

where the energy $E_\alpha$ is given in (23). From $N(\tau) > 0$ and the estimate (30), we get

$$\frac{d}{dt}E_\alpha(t) \leq -N_0\|A^{1/4}u_t(t)\|_2^2,$$

(48)
for some positive constant $N_0 > 0$ depending on the initial data in $\mathcal{H}_\alpha$. Now we define for any $\epsilon > 0$ a perturbed energy

$$E_{\alpha, \epsilon}(t) := E_\alpha(t) + \epsilon \Psi_\alpha(t) \quad \text{with} \quad \Psi_\alpha(t) := (u_t(t), u(t)) + \alpha (A^{1/4} u_t(t), A^{1/4} u(t)).$$

In the following we use $C_0, C_1, C_2$ to denote several positive constants appearing in the estimates. Firstly, we claim that there exists a constant $C > 0$ such that

$$\frac{d}{dt} \Psi_\alpha(t) \leq E_\alpha(t) + C_0 \|A^{1/4} u_t(t)\|_2^2 + l_1 |\Omega| + l_2.$$  \hspace{1cm} (49)

In fact, taking derivative of the function $\Psi_\alpha(t)$, using the weak formulation (21), adding and subtracting $E_\alpha(t)$ into the resulting expression, we obtain

$$\frac{d}{dt} \Psi_\alpha(t) = -E_\alpha(t) + \frac{3}{2} \left( \|u_t(t)\|_2^2 + \alpha \|A^{1/4} u_t(t)\|_2^2 \right) - \frac{1}{2} \|A^{1/2} u(t)\|_2^2 + \sum_{j=1}^{3} L_j,$$  \hspace{1cm} (50)

where

$$L_1 = \frac{1}{2} \bar{M} \left( \|A^{1/4} u(t)\|_2^2 \right) - M \left( \|A^{1/4} u(t)\|_2 \right) \|A^{1/4} u(t)\|_2^2,$$

$$L_2 = -N \left( \|A^{1/4} u(t)\|_2 \right) (A^{1/4} u_t(t), A^{1/4} u(t)),$$

$$L_3 = \int_{\Omega} \tilde{f}(u(t)) dx - (f(u(t)), u(t)).$$

Now we estimate $L_1, L_2$ and $L_3$. From the condition (17) and embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4})$, we get

$$|L_1| \leq \frac{1}{8} \|A^{1/2} u(t)\|_2^2 + l_2.$$

Using Young’s inequality and the embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4})$, we obtain

$$|L_2| \leq N \left( \|A^{1/4} u(t)\|_2 \right) \|A^{1/4} u_t(t)\|_2 \|A^{1/2} u(t)\|_2 \leq \frac{1}{8} \|A^{1/2} u(t)\|_2^2 + C_0 \|A^{1/4}_t(t)\|_2^2.$$

From the condition (16) we see that

$$|L_3| \leq \frac{1}{8} \|A^{1/2} u(t)\|_2^2 + C_0 |\Omega|.$$

Inserting these last three estimates in (50) and using the embedding $D(A^{1/4}) \hookrightarrow L^2(\Omega)$, then we get (49).

Choosing $\epsilon > 0$ small enough such that $\epsilon \leq \frac{N_0}{C_0}$, then we have

$$\frac{d}{dt} E_{\alpha, \epsilon}(t) \leq -\epsilon E_\alpha(t) + \epsilon (l_1 |\Omega| + l_2), \quad \forall t > 0.$$  \hspace{1cm} (51)

On the other hand, using Young’s inequality and the estimate (30), there exists a constant $C_1 > 0$ such that

$$|E_{\alpha, \epsilon}(t) - E_\alpha(t)| \leq \epsilon C_1 (E_\alpha(t) + \|h\|_2^2 + |\Omega|), \quad \forall t > 0, \quad \forall \epsilon > 0.$$  \hspace{1cm} (52)

Let us take and fix $\epsilon > 0$ small enough such that $\epsilon \leq \min \{ \frac{N_0}{C_0}, \frac{1}{2C_1} \}$. Then the estimate (52) implies

$$-\frac{1}{2} (\|h\|_2^2 + |\Omega|) + \frac{1}{2} E_\alpha(t) \leq E_{\alpha, \epsilon}(t) \leq \frac{3}{2} E_\alpha(t) + \frac{1}{2} (\|h\|_2^2 + |\Omega|).$$  \hspace{1cm} (53)
and combining (51) with (53), we have
\[ E_\alpha(t) \leq 3E_\alpha(0)e^{-\frac{2}{3}t} + C, \quad \forall t > 0, \tag{54} \]
where \( C = C(\|h\|_2, |\Omega|) > 0 \). Therefore, again using (27) we conclude that (46) holds true.

**Remark 1.** From Proposition 3, we immediately see that

\[ B_\alpha := \{ z \in \mathcal{H}_\alpha \mid \|z\|_{\mathcal{H}_\alpha} \leq K_2 + \delta' \} \]

is absorbing set of the system \((\mathcal{H}_\alpha, S_\alpha(t))\) and combining (51) with (53), we have
\[ \|S_\alpha(t)z\|_{\mathcal{H}_\alpha} \leq C, \tag{55} \]
where \( \delta' \) is an arbitrary positive constant, \( K_2 = K_2(h, l_0, l_1, l_2, |\Omega|) \) and \( C \) is a positive constant which does not depend on \( t \geq 0 \) and \( \alpha \in [0, 1] \).

4.2.2. **Existence of the attractors and its structures.** There remains to prove the asymptotically compactness. For showing the property, we use the other equivalent description below (for the proof and discussion we refer to [11]):

**Proposition 4.** Assume that \( X \) is a Banach space and \((X, S(t))\) is a dissipative dynamical system. Then the following assertions are equivalent.

- \((X, S(t))\) is asymptotically compact.
- There exists a decomposition \( S(t) = S^1(t) + S^2(t) \), where \( S^1(t) \) is continuous mapping in \( X \) such that
\[ r_D(t) = \sup \{ \|S^1(t)x\|_X \mid x \in D \} \to 0 \quad (t \to \infty) \]

for every bounded set \( D \) and \( S^2(t) \) is uniformly compact for large \( t \); that is, for any bounded set \( D \) there exists \( t_0 = t_0(D) \) such that the set \( \bigcup_{\tau \geq t_0} S^2(\tau)D \) is relatively compact in \( X \).

**The semigroup decomposition.** Now we decompose the solution \( S_\alpha(t)z \) with initial data \( z \in B_\alpha \) into the sum
\[ S_\alpha(t)z = S^1_\alpha(t)z + S^2_\alpha(t)z \tag{56} \]
where
\[ S^1_\alpha(t)z = (\xi(t), \xi_\alpha(t)) \quad \text{and} \quad S^2_\alpha(t)z = (\zeta(t), \zeta_\alpha(t)) \]
respectively solve the problems
\[ \begin{align*}
\xi_{tt} + \alpha A^{1/2}\xi_{tt} + A\xi + M(\|A^{1/4}u\|_2^2)A^{1/2}\xi + N(\|A^{1/4}u\|_2^2)A^{1/2}\xi_t &= 0, \\
(\xi_0(t), \xi_\alpha(0)) &= z, 
\end{align*} \tag{57} \]
and
\[ \begin{align*}
\zeta_{tt} + \alpha A^{1/2}\zeta_{tt} + A\zeta + M(\|A^{1/4}u\|_2^2)A^{1/2}\zeta + N(\|A^{1/4}u\|_2^2)A^{1/2}\zeta_t + f(u) &= h, \\
(\zeta_0(t), \zeta_\alpha(0)) &= 0. 
\end{align*} \tag{58} \]

This decomposition gives the asymptotic regularity of \( S_\alpha \), for initial data \( z \in B_\alpha \).

First, we begin from the exponential decay of \( S^1_\alpha \).

**Proposition 5.** For every \( \alpha \in [0, 1] \), there exists \( \delta > 0 \) (independent of \( \alpha \)) such that
\[ \|S^1_\alpha(t)z\|_{\mathcal{H}_\alpha} \leq Ce^{-\delta t}\|z\|_{\mathcal{H}_\alpha}. \]
Proof. Here, the generic positive constant $C$ may depend on the radius $B_\alpha$. The proof is similar to the one of Lemma , so we will give a quick sketch. One defines the functionals
\begin{equation}
E^1_\alpha(t) := \frac{1}{2} \left\{ \| A^{1/2} \xi(t) \|_2^2 + \| \xi_t(t) \|_2^2 + \alpha \| A^{1/4} \xi_t(t) \|_2^2 + M \left( \| A^{1/4} u(t) \|_2 \right) \| A^{1/4} \xi(t) \|_2^2 \right\}
\end{equation}
and
\begin{equation}
E^1_{\alpha, \epsilon}(t) := E^1_\alpha(t) + \epsilon \Psi^1_\alpha(t) \quad \text{with} \quad \Psi^1_\alpha(t) := (\xi_t(t), \xi(t)) + \alpha (A^{1/4} \xi_t(t), A^{1/4} \xi(t))
\end{equation}
for any $\epsilon > 0$. The former functional satisfies the relation below
\begin{equation}
\frac{d}{dt} E^1_\alpha(t) + N \left( \| A^{1/4} u(t) \|_2^2 \right) \| A^{1/4} \xi_t(t) \|_2^2 = L^1_1,
\end{equation}
where
\begin{equation}
L^1_1 = \left( \frac{1}{2} \frac{d}{dt} M \left( \| A^{1/4} u(t) \|_2^2 \right) \right) (A^{1/2} \xi(t), \xi_t(t)).
\end{equation}
Since $S_\alpha(t)z = (u(t), u_t(t))$ is uniformly bounded by the inequality (30), we can estimate $L^1_1$
\begin{align}
|L^1_1| & \leq |M'(\|A^{1/4} u(t)\|_2^2)|(|A^{1/4} u(t), A^{1/4} u_t(t))(A^{1/2} \xi(t), \xi_t(t))| \\
& \leq C \| u_t(t) \|_2 \| A^{1/2} \xi(t) \|_2 \| \xi_t(t) \|_2 \\
& \leq \epsilon' \| A^{1/4} \xi_t(t) \|_2^2 + \frac{C}{4 \epsilon'} \| u_t(t) \|_2^2 \| A^{1/2} \xi(t) \|_2^2.
\end{align}
Inserting the estimate in (59), we have
\begin{equation}
\frac{d}{dt} E^1_\alpha(t) \leq (\epsilon' - N_0) \| A^{1/4} \xi_t(t) \|_2^2 + C \epsilon' \| u_t(t) \|_2^2 E^1_\alpha(t).
\end{equation}
Now we consider the derivative of the latter functional. Likewise the estimation of the derivative of function $\Psi_\alpha(t)$, we obtain the following
\begin{equation}
\frac{d}{dt} \Psi^1_\alpha(t) \leq E^1_\alpha(t) - \frac{1}{2} \| A^{1/2} \xi(t) \|_2^2 + \frac{3}{2} \left( \| \xi_t(t) \|_2^2 + \alpha \| A^{1/4} \xi_t(t) \|_2^2 \right) + L^1_2
\end{equation}
where
\begin{equation}
L^1_2 = N \left( \| A^{1/4} u(t) \|_2^2 \right) (A^{1/4} \xi_t(t), A^{1/4} \xi(t)).
\end{equation}
Using the same argument in $L_2$, we get
\begin{align}
|L^1_2| & \leq N \left( \| A^{1/4} u(t) \|_2^2 \right) \| A^{1/4} \xi_t(t) \|_2^2 \| A^{1/4} \xi(t) \|_2^2 \\
& \leq \frac{1}{2} \| A^{1/2} \xi(t) \|_2^2 + C \| A^{1/4} \xi_t(t) \|_2^2.
\end{align}
Combining the estimates (60) and (62), and choosing $\epsilon, \epsilon' > 0$ small enough, then the derivative of the function $E^1_{\alpha, \epsilon}$ is controlled by
\begin{equation}
\frac{d}{dt} E^1_{\alpha, \epsilon}(t) \leq -\delta E^1_\alpha(t), \quad \forall t > 0.
\end{equation}
Similarly to the proof of the Proposition , we arrive at the desire estimate. \qed

Next we show the boundedness of $S^2_\alpha(t)z$ in more regular space.

**Proposition 6.** For every $\alpha \in [0, 1]$, we have the estimate:
\begin{equation}
\sup_{t \geq 0} \| S^2_\alpha(t)z \|_{\mathcal{H}_{\alpha, \lambda}} \leq C.
\end{equation}
Proof. Set
\[ E_2^\alpha(t) := \|A^{3/4}\zeta(t)\|_2^2 + \|A^{1/4}\zeta(t)\|_2^2 + \alpha \|A^{1/2}\zeta(t)\|_2^2 \]
and
\[
E_{\alpha,\epsilon}^2(t) := E_2^\alpha(t) + \epsilon \{ (A^{1/4}\zeta(t), A^{1/4}\zeta(t)) + \alpha(A^{1/2}\zeta(t), A^{1/2}\zeta(t)) \}
+ \frac{\epsilon}{2} N(\|A^{1/4}u(t)\|_2^2) \|A^{1/2}\zeta(t)\|_2^2 + M(\|A^{1/4}u(t)\|_2^2) \|A^{1/2}\zeta(t)\|_2^2,
\]
where \( \epsilon > 0 \) to be determined. By easy calculation, we have
\[
E_{\alpha,\epsilon}^2(t) \geq \left( 1 - \frac{cm}{2} \right) E_2^\alpha(t) \quad \text{where} \quad m := \max \left\{ \frac{1}{\lambda_1}(1 + \alpha \lambda_1^{1/2}), 1 \right\}
\]
and we can immediately see that \( E_{\alpha,\epsilon}^2(t) \leq 2E_{\alpha,\epsilon}^2(t) \) for \( \epsilon \leq m^{-1} \).

Next, we multiply (58) by \( \epsilon A^{1/2}\zeta + 2A^{1/2}\zeta \) (\( \epsilon \leq m^{-1} \)), we get
\[
\frac{d}{dt} E_{\alpha,\epsilon}^2(t) + \epsilon E_{\alpha,\epsilon}^2(t) - 2\epsilon(\|A^{1/4}\zeta(t)\|_2^2 + \|A^{1/2}\zeta(t)\|_2^2)
+ 2N(\|A^{1/4}u(t)\|_2^2) \|A^{1/2}\zeta(t)\|_2^2 = \sum_{k=1}^6 L_k^2,
\]
where
\[
L_1^2 = \left( \frac{d}{dt} M(\|A^{1/4}u(t)\|_2^2) \right) \|A^{1/2}\zeta(t)\|_2^2,
L_2^2 = \left( \frac{\epsilon}{2} \frac{d}{dt} N(\|A^{1/4}u(t)\|_2^2) \right) \|A^{1/2}\zeta(t)\|_2^2,
L_3^2 = \epsilon^2 \left( \frac{1}{2} N(\|A^{1/4}u(t)\|_2^2) \|A^{1/2}\zeta(t)\|_2^2 + (A^{1/2}\zeta(t), \zeta(t)) \right),
L_4^2 = \epsilon(h - f(u(t)), A^{1/2}\zeta(t)),
L_5^2 = 2(h - f(u(t)), A^{1/2}\zeta(t)) + \epsilon^2(\|A^{1/2}\zeta(t)\|_2^2),
\]
It is easy to estimate these terms. Indeed, from the estimates (30) and (55), we learn that
\[
|L_j^2| \leq C_{B_\alpha,\epsilon,M,N,f,h} \quad \text{for each} \quad j = 1, 2, 3, 4
\]
and
\[
|L_5^2| \leq \epsilon \|A^{1/2}\zeta(t)\|_2^2 + C_{B_\alpha,\epsilon,f,h}
\]
for any \( \epsilon > 0 \). Substituting the above estimates in (63), we arrive at
\[
\frac{d}{dt} E_{\alpha,\epsilon}^2(t) + \epsilon E_{\alpha,\epsilon}^2(t) + (N_0 - C_{\lambda_1}) \|A^{1/2}\zeta(t)\|_2^2 \leq C_{B_\alpha,\epsilon,M,N,f,h}.
\]
Since \( E_{\alpha,\epsilon}^2(t) \leq 2E_{\alpha,\epsilon}^2(t) \) for \( \epsilon \leq m^{-1} \) and \( E_{\alpha,\epsilon}^2(0) = 0 \), choosing \( \epsilon > 0 \) small enough so that \( \epsilon := \min \left\{ \frac{1}{m}, \frac{N_0}{C_{\lambda_1}} \right\} \), we can obtain the desired estimate by applying the Gronwall’s inequality.

Thanks to the general theory of attractors, we are now ready to prove the existence of the global attractor \( A_\alpha \). Though the proof is not new by itself, we give a quick review for the reader’s convenience. For the details, we refer to [11]. Till the end of the paper, we will set
\[
\mathbb{B}_{\alpha,\nu}(r) := \{ z \in \mathcal{H}_{\alpha,\nu} : \|z\|_{\mathcal{H}_{\alpha,\nu}} \leq r \}.
\]
Proposition 5 and Proposition 6 show that \( S_\alpha \) admits a decomposition
\[
S_\alpha(t)z = S^1_\alpha(t)z + S^2_\alpha(t)z,
\]
for every $z \in B_\alpha$. $S^1_\alpha$ is exponentially stable, while

$$S^2_\alpha(t)z \subset B_{\alpha,1}(r_0)$$

for some $r_0 > 0$ independent of $\alpha$. The embedding of $\mathcal{H}_{\alpha,1}$ into $\mathcal{H}_\alpha$ is compact, so $B_{\alpha,1}(r_0)$ is a compact subset of $\mathcal{H}_\alpha$. We then obtain the existence of connected global attractor $\mathcal{A}_\alpha \subset \mathcal{H}_\alpha$.

**Remark 2.** The decomposition indicates that the closed ball $B_{\alpha,1}(r_0)$ is uniformly exponential attracting set for $(\mathcal{H}_\alpha, S_\alpha(t))$, namely, for every bounded set $B \subset \mathcal{H}_\alpha$,

$$h_{\mathcal{H}_\alpha}(S_\alpha(t)B, B_{\alpha,1}(r_0)) \leq C_{r_0}e^{-\delta(t-t_0)}.$$  \hspace{1cm} (63)

holds, where $t_{B_\alpha} > 0$ is a time $B_\alpha$ such that $S_\alpha(t)B \subset B_\alpha$ for every $t \geq t_{B_\alpha}$.

**Remark 3.** It is worth noting that, from the well-posedness result (13), any semi-trajectory $(u(t), u_t(t))$ emanating from $B_{\alpha,1}(r_0)$ has more regularity as in (13). Hence, in the same way as in the proof of Proposition 5 and Remark 1, There exists a ball $B_{\alpha,1}(R_0)$ ($R_0 > r_0$) such that $B_{\alpha,1}(R_0)$ is an absorbing set for the system $(\mathcal{H}_{\alpha,1}, S_\alpha(t))$.

We conclude this section with comment on the structure of the global attractor and minimal attractor. It is easy to check that the energy $E_\alpha$ is a Lyapunov function for the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$ and we can immediately see that $(\mathcal{H}_\alpha, S_\alpha(t))$ is gradient. Thus we are able to get the conclusions on the structures of the global attractor and minimal attractor from the Proposition 2.

4.3. **Proof of Theorem 2.8.** Our aim in this section is to prove the existence of an exponential attractor, for all $\alpha \in [0, 1]$. For establishing the existence, we derive the following important inequality, which leads to the quasi-stability of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.

4.3.1. **Quasi-stability of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.**

**Proposition 7.** Let the assumptions of Theorem 2.6 and 2.8 be in force. Given a bounded set $B \subset (\mathcal{H}_\alpha, S_\alpha(t))$ we consider two weak solutions $z^1 = (u, u_t), z^2 = (v, v_t)$ corresponding to initial data $z^1_0 = (u_0, u_1), z^2_0 = (v_0, v_1)$ lying in $B$. Then the following inequality holds:

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}_\alpha} \leq Ce^{-\delta t}\|z^1_0 - z^2_0\|_{\mathcal{H}_\alpha} + C\int_0^t e^{-\delta(t-s)}\|w(s)\|_2^2 ds,$$  \hspace{1cm} (64)

for all $t > 0$, where $C = C_B$ and $\delta = \delta_B$ are positive constants, and $w = u - v$.

**Proof.** First of all we fix a bounded set $B \subset \mathcal{H}_\alpha$ and consider two weak solutions $z^1 = (u, u_t), z^2 = (v, v_t)$ with the initial data $z^1_0, z^2_0 \in B$, that is $\|z^1_0\|_{\mathcal{H}_\alpha}, \|z^2_0\|_{\mathcal{H}_\alpha} \leq R$, where $R > 0$ depends on the size of $B$. Putting the difference $z^1 - z^2 = (w, w_t)$ and proceeding exactly as in the proof of the a priori estimates we get the following inequality

$$\frac{1}{2} \frac{d}{dt}\left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + NR\|A^{1/4}w_t(t)\|_2^2 \leq \sum_{j=1}^4 J_j,$$  \hspace{1cm} (65)

for some constant $N_R > 0$, where we use the global estimate (30) and $N(\tau) > 0$. Here, the expressions for $J_j, j = 1, 2, 3, 4$, are the same ones given in (38)-(41).
Using the relation
\[ \frac{1}{2} \frac{d}{dt} \left[ M \left( \| A^{1/4} u(t) \|_2^2 \right) \| A^{1/4} w(t) \|_2^2 \right] = M' \left( \| A^{1/4} u(t) \|_2^2 \right) (A^{1/2} u(t), u_t(t)) \| A^{1/4} w(t) \|_2^2 - J_1, \]

we can rewrite (65) as follows:
\[ \frac{d}{dt} F_\alpha(t) + N_R \| A^{1/4} w_t(t) \|_2^2 \leq J_2 + J_3 + J_4 + J_5, \] (66)

where we set
\[ F_\alpha(t) := \frac{1}{2} \left[ \| A^{1/2} w(t) \|_2^2 + \| w_t(t) \|_2^2 + \alpha \| A^{1/4} w_t(t) \|_2^2 \right. \]
\[ + M \left( \| A^{1/4} u(t) \|_2^2 \| A^{1/4} w(t) \|_2^2 \right) \]

and
\[ J_5 := M' \left( \| A^{1/4} u(t) \|_2^2 \right) (A^{1/2} u(t), u_t(t)) \| A^{1/4} w(t) \|_2^2. \] (68)

Next we shall estimate the right-hand side of (66). To simplify the notation we use \( C_R \) to denote various positive constants depending on \( R > 0 \), but not on time. Firstly, since \( M, N \in C^1([0, \infty)) \), then the estimate (30) implies
\[ \max_{\tau \in [0, C_R]} \{ |M(\tau)|, |M'(\tau)|, |N(\tau)|, |N'(\tau)| \} < \infty. \]

Likewise the proof of the a priori estimate, applying the mean value theorem, Young’s inequality with \( \epsilon > 0 \), the estimate (30) and the embedding \( D(A^{1/2}) \hookrightarrow D(A^{1/4}) \hookrightarrow \mathcal{L}_2(\Omega) \), we obtain
\[ |J_2| \leq C_R \| A^{1/4} u(t) \|_2 + \| A^{1/4} v(t) \|_2 \| A^{1/4} w(t) \|_2 \| A^{1/4} v_t(t) \|_2 \| A^{1/4} w_t(t) \|_2 \]
\[ \leq C_R \| A^{1/2} w(t) \|_2 \| A^{1/4} v_t(t) \|_2 \| A^{1/4} w_t(t) \|_2 \]
\[ \leq \epsilon' \| A^{1/4} w_t(t) \|_2^2 + C_{R, \epsilon'} \| A^{1/4} v_t(t) \|_2^2 \| A^{1/2} w(t) \|_2^2 \]

for any \( \epsilon' > 0 \). Adding the interpolation inequality, we have
\[ |J_3| \leq C_R \| A^{1/4} u(t) \|_2 + \| A^{1/4} v(t) \|_2 \| A^{1/4} w(t) \|_2 \| A^{1/2} v(t) \|_2 \| w_t(t) \|_2 \]
\[ \leq C_R \| w(t) \|_2^{1/2} \| A^{1/2} w(t) \|_2^{1/2} \| A^{1/4} w_t(t) \|_2 \]
\[ \leq (C_{R, \epsilon'} \| w(t) \|_2 + \epsilon' \| A^{1/2} w(t) \|_2) \| A^{1/4} w_t(t) \|_2 \]
\[ \leq \epsilon' \| A^{1/4} w_t(t) \|_2^2 \| A^{1/2} w(t) \|_2^2 + C_{R, \epsilon'} \| w(t) \|_2^2. \]

Using the same conditions in the proof of the uniqueness, Young’s inequality and Nirenberg-Gagliard’s inequality, we have
\[ |J_4| \leq \sigma_0 \left( \| w(t) \|_2^{p/2} + \| v(t) \|_2^{p/2} \right) \| w(t) \|_2 \| w_t(t) \|_2 \]
\[ \leq C_R \| w(t) \|_2 \| A^{1/4} w_t(t) \|_2 \]
\[ \leq C_{R, \sigma} \| A^{1/2} w(t) \|_2^{1/2} \| w(t) \|_2^{1/2} \| A^{1/4} w_t(t) \|_2 \]
\[ \leq (\epsilon' \| A^{1/2} w(t) \|_2 + C_{R, \sigma, \epsilon'} \| w(t) \|_2) \| A^{1/4} w_t(t) \|_2 \]
\[ \leq \epsilon' \| A^{1/4} w_t(t) \|_2^2 + \epsilon' \| A^{1/2} w(t) \|_2^2 + C_{R, \epsilon'} \| w(t) \|_2^2, \]
Substituting these four estimates in (66) and choosing \( \theta \) where
\[
\epsilon > 0 \quad \text{small enough, we see that there exists constants } N_R, C_R > 0 \text{ such that}
\]
\[
\frac{d}{dt} F_\alpha(t) \leq -N_R \| A^{1/4} w(t) \|^2_2 + C_R \| A^{1/4} v(t) \|^2_2 \| A^{1/2} w(t) \|^2_2
\]
\[
+ \epsilon \| A^{1/2} w(t) \|^2_2 + \| w(t) \|^2_2, \quad t > 0.
\]

Now we define the functional
\[
F_{\alpha, \eta}(t) := F_\alpha(t) + \eta \Phi_\alpha(t) \quad \text{with } \Phi_\alpha(t) := (w_\alpha(t), w(t)) + \alpha (A^{1/4} w_\alpha(t), A^{1/4} w(t))
\]
where \( \eta > 0 \) will be fixed later. We first show that there exists a constant \( C_R, D_R > 0 \) such that
\[
\frac{d}{dt} \Phi_\alpha(t) \leq -F_\alpha(t) + C_R \| A^{1/4} w(t) \|^2_2 - D_R \| A^{1/2} w(t) \|^2_2 + \| w(t) \|^2_2, \quad t > 0. \quad (70)
\]
Indeed, taking derivative of \( \Phi_\alpha(t) \), using the weak formulation for \( w \), adding and subtracting \( F_\alpha(t) \) in the resulting expression and neglecting unnecessary terms, we arrive at
\[
\frac{d}{dt} \Phi_\alpha(t) \leq -F_\alpha(t) + \frac{3}{2} \left( |w_\alpha(t)|^2 + \alpha \| A^{1/4} w_\alpha(t) \|^2_2 \right) - \frac{1}{2} \| A^{1/2} w(t) \|^2_2 + \sum_{j=1}^4 K_j, \quad (71)
\]
where
\[
K_1 = -N \left( \| A^{1/4} u(t) \|^2_2 \right) (A^{1/4} w(t), A^{1/4} w_\alpha(t)),
\]
\[
K_2 = -\left\{ M \left( \| A^{1/4} u(t) \|^2_2 \right) - M \left( \| A^{1/4} v(t) \|^2_2 \right) \right\} (A^{1/4} v(t), A^{1/4} w(t)),
\]
\[
K_3 = -\left\{ N \left( \| A^{1/4} u(t) \|^2_2 \right) - N \left( \| A^{1/4} v(t) \|^2_2 \right) \right\} (v(t), A^{1/2} w(t)),
\]
\[
K_4 = -(f(u(t)) - f(v(t)), w(t)).
\]

First we estimate \( K_1 \). From the estimate (30) and the uniform boundedness of \( N \), Young’s inequality with \( \epsilon > 0 \) and the embedding \( D(A^{1/2}) \hookrightarrow D(A^{1/4}) \) we get
\[
|K_1| \leq C_R \| A^{1/4} w_\alpha(t) \|_2 \| A^{1/2} w(t) \|_2
\]
\[
\leq \epsilon \| A^{1/2} w(t) \|^2_2 + C_R \epsilon \| A^{1/4} w_\alpha(t) \|^2_2.
\]

Proceeding almost the same way as \( J_1, J_2, \) and \( J_4 \), but replacing the function \( w_\alpha \) by \( w \), we derive
\[
|K_2| \leq \epsilon \| A^{1/2} w(t) \|^2_2 + C_R \epsilon \| w(t) \|^2_2,
\]
\[
|K_3| \leq \epsilon \| A^{1/2} w(t) \|^2_2 + C_R \epsilon \| w(t) \|^2_2,
\]
\[
|K_4| \leq C_R \| w(t) \|_p \| w(t) \|_2
\]
\[
\leq \epsilon \| A^{1/2} w(t) \|^2_2 + C_R \epsilon \| w(t) \|^2_2.
\]

for any \( \epsilon > 0 \) and some positive constant \( C_R \). Going back to (71) and inserting these four estimates, we result that (70) holds, after choosing \( \epsilon > 0 \) small enough and using the embedding \( D(A^{1/4}) \hookrightarrow L^2(\Omega) \).
Combining (69) with (70), noting that \( \|A^{1/2}w(t)\|_2^2 \leq F_\alpha(t) \), and taking \( \eta, \epsilon' > 0 \) small enough such that \( \eta < \frac{N_0}{C_R} \), and \( \epsilon' < D_R \), there exists a constant \( C_R > 0 \) such that
\[
\frac{d}{dt}F_{\alpha,\eta}(t) \leq -\eta F_\alpha(t) + C_R\|A^{1/4}v(t)\|_2^2 F_\alpha(t) + C_R\|w(t)\|_2^2,
\]
for all \( t > 0 \) \( \epsilon' < D_R \) and \( \eta < \frac{N_0}{C_R} \).

On the other hand, by taking \( C_1 := \max\{1, 1/\lambda_1\} > 0 \), it is readily to see that
\[
|F_{\alpha,\eta}(t) - F_\alpha(t)| \leq \eta C_1 F_\alpha(t), \quad \forall t \geq 0, \quad \forall \eta > 0.
\]
Then taking and fixing \( \eta > 0 \) such that \( \eta \leq \min\{\frac{1}{2C_1}, \frac{N_0}{C_R}\} \), (73) implies that
\[
\frac{1}{2} F_\alpha(t) \leq F_{\alpha,\eta}(t) \leq \frac{3}{2} F_\alpha(t), \quad \forall t \geq 0.
\]
Compounding (72) with (74) we get
\[
\frac{d}{dt}F_{\alpha,\eta}(t) \leq \phi_\eta(t)F_{\alpha,\eta}(t) + C_R\|w(t)\|_2^2, \quad t > 0,
\]
where we define
\[
\phi_\eta(t) := -\frac{\eta}{3} + C_R\|A^{1/4}v(t)\|_2^2.
\]
Applying Gronwall’s inequality, we deduce that
\[
F_{\alpha,\eta}(t) \leq e^{\int_0^t \phi_\eta(s)ds} \left( F_{\alpha,\eta}(0) + C_R \int_0^t e^{\int_0^s \phi_\eta(\xi)ds} \|w(t)\|_2^2 ds \right).
\]
Moreover, from the estimate (30) we also have
\[
\int_0^t \phi_\eta(s)ds = -\frac{\eta t}{3} + C_R \int_0^t \|A^{1/4}v(t)\|_2^2 ds \leq -\frac{\eta t}{3} + \tilde{C}_R,
\]
for some positive constant \( \tilde{C}_R \). Thus (76) leads to
\[
F_{\alpha,\eta}(t) \leq C_R F_{\alpha,\eta}(0)e^{-\frac{\eta t}{3}} + C_R \int_0^t e^{-\frac{\eta}{3}(t-s)}\|w(s)\|_2^2 ds,
\]
for all \( t > 0 \), and some constant \( C_R > 0 \). Lastly, from (67) we note that there exists a constant \( C_R > 0 \) such that
\[
\|z_1(t) - z_2(t)\|_{\mathcal{H}_\alpha} \leq F_\alpha(t) \leq (1 + C_R)\|z_1(t) - z_2(t)\|_{\mathcal{H}_\alpha}, \quad \forall t \geq 0.
\]
Therefore, combining (74) with (77)-(78), we get the conclusion that the stability inequality (64) holds true for some constants \( C_R > 0 \) and \( \delta_R > 0 \). The proof of Proposition 7 is complete.

From now on, we check that the inequality (64) leads to the quasi-stability of the dynamical system \((\mathcal{H}_\alpha, S_\alpha(t))\). Let us define the map \( \phi_{T,\alpha} : \mathcal{Y} \to X_T \) as
\[
(\phi_{T,\alpha}z)(\cdot) := u(\cdot) \in X_T := C([0, T], L(\Omega)), \quad z \in \mathcal{Y}.
\]
Then we can rewrite (64) as follows:
\[
\|S_\alpha(T)z^1 - S_\alpha(T)z^2\|_{\mathcal{H}_\alpha} \leq C_\beta e^{-\delta_T\|z^1 - z^2\|_{\mathcal{H}_\alpha}} + C_B \|\phi_{T,\alpha}z^1 - \phi_{T,\alpha}z^2\|_{X_T}.
\]
Then we can find that \( \phi_{T,\alpha} \) satisfies the desire property. Now, given a sequence of initial data \( z^i \in \mathcal{Y} \), we write \( S_\alpha(t)z^i = (u^i(t), u^i_1(t)) \). Since \( \mathcal{Y} \) is bounded and the estimate (30), the sequence \((u^i)_{i \in \mathbb{N}} = \{u^i(t) \mid t \in [0, T] \} \) is uniformly bounded on \( L^2(\Omega) \) and equicontinuous. Compound these properties with the compactness of
the embedding $D(A^{1/2}) \hookrightarrow L^2(\Omega)$, there exists a subsequence still denoted by $(u')$ such that

$$
(u') \text{ converges strongly in } X_T, \ T > 0.
$$

Therefore $(\mathcal{H}_\alpha, S_\alpha(t))$ is quasi-stable.

We conclude this subsection with comment on the properties of the global attractor stated in Theorem 2.8. The quasi-stability of the system $(\mathcal{H}_\alpha, S_\alpha(t))$ immediately leads the finite dimensionality and more regularity of the global attractor $\mathcal{A}_\alpha$ as in Theorem 2.8. Also, we are instantly able to see that (19) holds.

4.3.2. Construction of exponential attractors. In this subsection, we consider the existence of exponential attractors. Mentioning beforehand, the way of the construction of exponential attractors used in the following is essentially based on the method given in [20]. However, for the constructing exponential attractors with continuous dependence on the parameter $\alpha$, we need to grasp the construction (the detail will be treated in the next section). So, along with the reader's convenience, we will introduce their method in brief.

For the beginning, we recall that each system $(\mathcal{H}_\alpha, S_\alpha(t))$ possesses an exponentially attracting set $\mathbb{B}_{\alpha,1}(R)$ ($R = R_0$) which is compact in $\mathcal{H}_\alpha$ and hereinafter, we denote the ball by $\mathbb{B}_\alpha$. Hence, it is sufficient to consider the dynamical system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha)$ and we show that each system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha)$ possesses an exponential attractor.

From now on, we consider a discrete dynamical system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S^m_\alpha)$, where $S^m_\alpha := S_\alpha(mT)$ and $T > 0$ is a constant determined in the following. From the inequality (79) and (45), the flow $S_\alpha$ satisfies

$$
\|S_\alpha z^1 - S_\alpha z^2\|_{\mathcal{H}_\alpha} \leq C_Re^{-\delta_k T}\|z^1 - z^2\|_{\mathcal{H}_\alpha} + CR\|\phi_{T,\alpha}z^1 - \phi_{T,\alpha}z^2\|_{X_T} \tag{80}
$$

and

$$
\|\phi_{T,\alpha}z^1 - \phi_{T,\alpha}z^2\|_{X_T} \leq L\|z^1 - z^2\|_{\mathcal{H}_\alpha} \quad (L = L(R, T)). \tag{81}
$$

Then we choose $T = T_0 > 0$ so that $C_RC^{-\delta T_0} =: q < 1/2$ holds and define constant $0 < \nu < 1$ as $\nu := 2C_R(q + \rho L)$, where $\rho > 0$ satisfies $0 < \rho < (1 - 2C_Rq)/2LC_R$. Under the above condition, we can construct an exponential attractor for the system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S^m_\alpha)$ as a collection of the central points of a finite covering balls for each $S^m_\alpha \mathbb{B}_\alpha$. That is, for each $k = 0, 1, 2, \ldots$, there exists a finite covering of $S^m_\alpha \mathbb{B}_\alpha$ such that

$$
S^k_\alpha \mathbb{B}_\alpha \subset \bigcup_{i=1}^{N^k} \mathcal{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; Rv^k), \tag{82}
$$

where $\tilde{z}_{k,i}$ are the central points $\tilde{z}_{k,i} := S^k_\alpha z_{k,i} \in S^m_\alpha \mathbb{B}_\alpha, 1 \leq i \leq N^k_\rho$. Then we can see that a set defined below is an exponential attractor for the system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S^m_\alpha)$:

$$
\mathcal{A}_{\exp,\alpha}^* := \bigcup_{k,i=0}^{\infty} S^k_{\alpha} E^k_{\alpha},
$$

where $E^k_{\alpha} := \{S^k_{\alpha} z_{k,i} \mid z_{k,i} \in \mathbb{B}_\alpha, i = 1, 2, \ldots, N^k_\rho\}$. The proof of $\mathcal{A}_{\exp,\alpha}^*$ satisfying the desire conditions, we refer to [20]. Here, we only sketch the procedure for constructing the above covering.
Proof. We prove the relation (82) utilizing the inductive argument. The case $k = 0$ holds clearly. Now we assume that the relation (82) holds for $k$. Then, it is easy to see that

$$S_{\alpha}^{k+1} \mathbb{B}_{\alpha} = S_{\alpha}(S_{\alpha}^{k} \mathbb{B}_{\alpha}) \subset \bigcup_{i=1}^{N_{\rho}} S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k})$$

and it is sufficient to prove that, for each set $S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k})$, there exists a covering such that

$$S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k}) \subset \bigcup_{j=1}^{N_{\rho}} \overline{B}^{H_{\alpha}}(\tilde{z}_{k+1,j}; R^{k+1}).$$

For constructing such a covering, we consider the image of $S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k})$ through $\phi_{T,\alpha}$. From the continuity (81), it is clear that

$$\phi_{T,\alpha}(S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k})) \subset V^{T}(\phi_{T,\alpha} \tilde{z}_{k,i}; RL^{k}),$$

where

$$V^{T}(\phi_{T,\alpha} \tilde{z}_{k,i}; RL^{k}) := \{ x \in \mathbb{T} \mid \| x - \phi_{T,\alpha} \tilde{z}_{k,i} \|_{\mathbb{T}} \leq RL^{k}, x \text{ is equicontinuous, } \| A^{1/2}x \|_{\mathbb{T}} \leq C \}.$$ 

Since $V^{T}(\phi_{T,\alpha} \tilde{z}_{k,i}; RL^{k})$ and $\phi_{T,\alpha}(S_{\alpha} \overline{B}(\tilde{z}_{k,i}; R^{k}))$ are compact in $\mathbb{T}$, there exists a covering balls such that

$$V^{T}(\phi_{T,\alpha} \tilde{z}_{k,i}; RL^{k}) \cap \phi_{T,\alpha}(S_{\alpha} \overline{B}(\tilde{z}_{k,i}; R^{k})) \subset \bigcup_{j=1}^{N_{\rho}} U^{T}(RL^{k}; \phi_{T,\alpha} w_{k+1,j}),$$

where $U^{T}(RL^{k}; \phi_{T,\alpha} w_{k+1,j}) := \{ w \in S_{\alpha} \overline{B}(\tilde{z}_{k,i}; R^{k}) \mid \| \phi_{T,\alpha} w - \phi_{T,\alpha} w_{k+1,j} \|_{\mathbb{T}} \leq RL^{k} \}$ and $w_{k+1,j} = S_{\alpha}^{k+1} \tilde{z}_{k+1,j}$. Then, from the inequality (79), we see that, for any $z \in S_{\alpha} \overline{B}(\tilde{z}_{k,i}; R^{k})$, there exists $j$ ($1 \leq j \leq N_{\rho}$) such that

$$\| S_{\alpha} z - S_{\alpha} w_{k+1,j} \|_{H_{\alpha}} \leq 2C_{R} R L^{k} + 2C_{R} L R L^{k} \leq R^{k+1}$$

and this implies that $S_{\alpha} \overline{B}^{H_{\alpha}}(\tilde{z}_{k,i}; R^{k})$ is covered with $\bigcup_{j=1}^{N_{\rho}} \overline{B}^{H_{\alpha}}(\tilde{z}_{k+1,j}; R^{k})$. This holds for every $i = 1, 2, ..., N^{k}_{\alpha}$, hence the statement for $k + 1$ is also true.

Thanks to the way of the construction of the discrete of exponential attractor, one can see that

$$A_{\exp,\alpha} := \bigcup_{0 \leq t \leq T} S_{\alpha}(t) A_{\exp,\alpha}^{r}$$

is a compact positively invariant set with respect to $S_{\alpha}(\cdot)$. Adding the Hölder continuity on time (we show it in the following), we have $\dim_{f}^{H_{\alpha}} A_{\exp,\alpha} \leq c(\dim_{f}^{H} [0, T] + \dim_{f}^{H_{\alpha}} A_{\exp,\alpha}) < \infty$. We also have

$$h_{H_{\alpha}}(S_{\alpha}(t) \mathbb{B}_{\alpha}, A_{\exp,\alpha}) \leq C e^{-\gamma t}, \quad t \geq 0$$

for some $\gamma = \gamma_{\mathbb{B}_{\alpha}} > 0$. Thus $A_{\exp,\alpha}$ is an exponential attractor.

We would like to finish this section by checking the mapping $t \mapsto S_{\alpha}(t)z$ is Hölder continuous in $H_{\alpha}$ for every $z \in A_{\exp,\alpha}^{r}$ in order to do it, we estimate $u_{tt}$ for given $z \in \mathbb{B}_{\alpha}$.
Lemma 4.4. Let us assume that \( z_0 \in \mathbb{B}_\alpha \) and the hypotheses Theorems 2.6 and 2.7 hold. Then, for weak solution \((u, u_t)\) corresponding to the initial data \( z_0 \in \mathbb{B}_\alpha \), we have

\[
\int_t^{t+T} \| A^{-1/4}u_{tt}(s) \|_2^2 ds + \alpha \int_t^{t+T} \| u_{tt}(s) \|_2^2 ds \leq C(R, T) \quad (\forall t \geq 0).
\]

Proof. Multiply (6) by \( A^{-1/2}u_{tt} \), we get

\[
(h, A^{-1/2}u_{tt}(t)) - (A^{3/4}u(t), A^{-1/4}u_{tt}(t)) - (f(u(t), A^{-1/2}u_{tt}(t)) - M(\|A^{1/4}u(t)\|_2^2)(A^{1/2}u(t), A^{-1/2}u_{tt}(t)) - N(\|A^{1/4}u(t)\|)(A^{1/2}u(t), A^{-1/2}u_{tt}(t)).
\]

In view of the Remark 3, we end up with the inequality

\[
\frac{1}{2} \left( \| A^{-1/4}u_{tt}(t) \|_2^2 + \alpha \| u_{tt}(t) \|_2^2 \right) \leq C_R + N_0 \| A^{1/2}u_{t}(t) \|_2^2.
\]

Integrating over \((t, t+T)\), we obtain

\[
\int_t^{t+T} \| A^{-1/4}u_{tt}(s) \|_2^2 ds + \alpha \int_t^{t+T} \| u_{tt}(s) \|_2^2 ds \leq C(R, T).
\]

Now we show the Hölder continuity of the mapping \( t \mapsto S_\alpha(t)z \). For given \( z = (u_0, u_1) \in A^*_{\exp, \alpha} \subset \mathbb{B}_\alpha \), we can see from Lemma 4.4 that

\( (u_t, u_{tt}) \in L^2(0, T; D(A^{1/2})) \times L^2(0, T; H_{\alpha, -1}), \forall T > 0. \)

Hence we have

\[
\| S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0 \|_{H_{\alpha, -1}} \leq \left( \int_0^T \| (u_t(s), u_{tt}(s)) \|_{H_{\alpha, -1}}^2 ds \right)^{1/2} |t_2 - t_1|^{1/2} \leq C(R, T)|t_2 - t_1|^{1/2},
\]

for every \( t_1, t_2 \in [0, T] \), so \( t \mapsto S_\alpha(t)z_0 \) is Hölder continuous in \( H_{\alpha, -1} \). Utilizing the boundedness of \( A^*_{\exp, \alpha} \subset \mathbb{B}_\alpha \) in \( H_{\alpha, 1} \) and interpolation inequality, we find

\[
\| S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0 \|_{H_{\alpha, -1}} \leq C_N \| S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0 \|_{H_{\alpha, -1}}^{1/2} \| S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0 \|_{H_{\alpha, 1}}^{1/2} \leq C_N R \| S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0 \|_{H_{\alpha, -1}}^{1/2} \leq C_{N, R} |t_2 - t_1|^{1/4}, \forall t_1, t_2 \in [0, T].
\]

Thus \( t \mapsto S_\alpha(t)z_0 \) is Hölder continuous in \( H_{\alpha} \).

5. Stability of attractors with respect to the rotational inertia.

5.1. Upper semicontinuity of the global attractors \( A_\alpha \) when \( \alpha \to 0 \). In order to show the upper semicontinuity of attractors, we use the following criterion:

Proposition 8. (See [13]) Assume that a dynamical system \((X_\alpha, S_\alpha(t))\) possesses a compact global attractor \( A_\alpha \) for every \( \alpha \in [0, 1] \). Assume that the following conditions hold:
• There exists a compact set $K \subset X$ such that $A_\alpha \subset K$ for all $\alpha \in [0, 1]$. 

• If $\alpha_k \to 0$, $x_k \in A_{\alpha_k}$ and $x_k \to x_0$, then $S_{\alpha_k}(t_0)x_k \to S(t_0)x_0$ for some $t_0 > 0$.

Then the family of attractors $A_\alpha$ is upper semicontinuous at the point $\alpha = 0$, that is,

$$h(A_\alpha, A) = \sup_{y \in A_\alpha} \inf_{z \in A} \|y - z\| \to 0$$

as $\alpha \to 0^+$. 

In the remainder of this section we check that the system $(H_\alpha, S_\alpha(t))$ satisfies the condition of this proposition.

**Lemma 5.1.** Let $T > 0$ be an arbitrary number. Assume that the hypotheses of Theorems 2.6 and 2.7 hold. Let $z^\alpha(t)$, $\alpha \in (0, 1]$ be a solution to (6) ($\alpha > 0$) with initial data $z^\alpha_0 = (u^\alpha_0, w^\alpha_0) \in B_\alpha = B_\alpha(R_0)$ and $z^0(t)$ solve (6) ($\alpha = 0$) with $z_0^0 = (u^0_0, w^0_0) \in B_0 = B_0(R_0)$. Then

$$\|z^\alpha(t) - z^0(t)\|_{H_\alpha, t} \leq C_{r_\alpha, r} \|z^\alpha_0 - z^0_0\|_{H_{\alpha, t}}^{1/2} + \alpha^{1/4} C_{r_\alpha, t}, \quad \forall t \in [0, T], \quad \forall \alpha \in (0, 1].$$

(83)

In particular, if $z^\alpha_0 \to z^0_0$ in $H_0$ as $\alpha \to 0^+$, then

$$\lim_{\alpha \to 0^+} \sup_{t \in [0, T]} \|z^\alpha(t) - z^0(t)\|_{H_\alpha} = 0 \quad \text{for any } T > 0.$$  

(84)

**Proof.** Set $w(t) := u^\alpha(t) - u^0(t)$. Then we have

$$\alpha A^{1/2} w^\alpha_{tt} + w_{tt} + A w + N \left( \|A^{1/4} u^\alpha\|^2 \right) A^{1/2} w_t$$

$$= -M \left( \|A^{1/4} u^\alpha\|^2 \right) A^{1/2} w - \left( N \left( \|A^{1/4} u^\alpha\|^2 \right) - N \left( \|A^{1/4} u^0\|^2 \right) \right) A^{1/2} w^0$$

$$- \left( M \left( \|A^{1/4} u^\alpha\|^2 \right) - M \left( \|A^{1/4} u^0\|^2 \right) \right) A^{1/2} w^0 - (f(u^\alpha) - f(u^0)),$$

Multiplying this by $A^{-1/2} w_t$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|A^{1/4} w(t)\|^2 + \|A^{-1/4} w_t(t)\|^2 \right) + \alpha \left( A^{-1/4} w^\alpha_{tt}(t), A^{1/4} w_t(t) \right)$$

$$+ N \left( \|A^{1/4} u(t)\|^2 \right) \|w_t(t)\|^2 = \sum_{j=1}^4 \tilde{J}_j,$$

where the expressions for $\tilde{J}_j$ ($j = 1, 2, 3, 4$) are almost the same ones given in (38)-(41), but replacing the function $w_t$ by $A^{-1/2} w_t$. As in the proof of the uniqueness, we have

$$\frac{1}{2} \frac{d}{dt} \left( \|A^{1/4} w(t)\|^2 + \|A^{-1/4} w_t(t)\|^2 \right) + \alpha \left( A^{-1/4} w^\alpha_{tt}(t), A^{1/4} w_t(t) \right)$$

$$+ N \left( \|A^{1/4} u(t)\|^2 \right) \|w_t(t)\|^2 \leq C(1 + \|A^{1/4} u^\alpha(t)\|^2) \left( \|A^{1/4} w(t)\|^2 + \|A^{-1/4} w_t(t)\|^2 \right),$$

(85)
where the $C$ and $N_0$ are constants. Integrating (85) on $[0, t]$ and using Gronwall’s inequality we arrive at
\[
\|A^{1/4}w(t)\|_2^2 + \|A^{-1/4}w_t(t)\|_2^2 + N_0 \int_0^t \|w_t(s)\|_2^2 ds \\
\leq e^{C_T} \left( \|A^{1/4}w(0)\|_2^2 + \|A^{-1/4}w_t(0)\|_2^2 \right) + \alpha \int_0^t \|A^{-1/4}w_t(s)\|_2^2 ds + \alpha \int_0^t \|A^{1/4}w_t(s)\|_2^2 ds .
\]
Since we take the each initial datum $z_0^\alpha$ from the bounded set $\mathbb{B}_\alpha \subset \mathcal{H}_{\alpha,1}$ and from Lemma 4.4, we see that $\int_0^t \|A^{-1/4}w_t(s)\|_2^2 ds$, $\int_0^t \|A^{1/4}w_t(s)\|_2^2 ds < \infty$. Hence, utilizing the boundedness of $\mathbb{B}_\alpha$ in $\mathcal{H}_{\alpha,1}$ and interpolation inequality, we find
\[
\|z_\alpha(t) - z_0(t)\|_{\mathcal{H}_\alpha} \leq \|z_\alpha(t) - z_0(t)\|_{\mathcal{H}_\alpha} + \sqrt{\alpha} R_0 \\
\leq \|z_\alpha(t) - z_0(t)\|_{\mathcal{H}_{\alpha,1}^1}^{1/2} \|z_\alpha(t) - z_0(t)\|_{\mathcal{H}_{\alpha,1}^-1}^{1/2} + \sqrt{\alpha} R_0 \\
\leq Cr_0, T \|z_\alpha - z_0\|_{\mathcal{H}_{\alpha,1}^-1}^{1/2} + \alpha^{1/4} Cr_0, T .
\]

From the continuity of the semiflows $S_\alpha$ on the parameter $\alpha$, we can conclude the upper semicontinuity of $\mathcal{A}_\alpha$. Indeed, from Theorem 2.8 every attractor $\mathcal{A}_\alpha$ is included in a bounded set $K \subset D(A) \times D(A^{1/2})$, that is,
\[
\|Au_\alpha\|_2^2 + \|A^{1/2}v_\alpha\|_2^2 \leq R_1 ,
\]
where $R_1$ does not depend on $\alpha$ and $t$. Since $D(A) \times D(A^{1/2})$ is compactly embedded into $D(A^{1/2}) \times L^2(\Omega)$, the set $K$ is compact in $D(A^{1/2}) \times L^2(\Omega)$. Thus we are able to obtain the conclusion of Proposition 8.

5.2. Continuity of exponential attractors when $\alpha \to 0$. At the end of this section, we verify the existence of the family of exponential attractors $\{\mathcal{A}_{\exp, \alpha}\}_{\alpha \in [0, 1]}$ possessing the Hölder continuity with respect to the parameter $\alpha$. In order to do it, we define the scaling operators $j_\alpha : \mathcal{H}_{0,1} \to \mathcal{H}_{\alpha,1}$ as
\[
j_\alpha z_0 = j_\alpha(u_0, u_1) := (u_0, (1 + \sqrt{\alpha} A^{1/4})^{-1} u_1).
\]
Then the scaling operators satisfy the following properties:
• $j_\alpha \mathbb{B}_0 \subset \mathbb{B}_\alpha$.
• $\|j_\alpha z - z\|_{\mathcal{H}_0} \leq \sqrt{\alpha} R$ (where $R > 0$ is the radius of the ball $\mathbb{B}_0$).

We can show the above properties by using the relation below
\[
\|v\|_2^2 \leq \|v\|_2^2 + \alpha \|A^{1/4}v\|_2^2 \leq (1 + \sqrt{\alpha} A^{1/4}) \|v\|_2^2 \quad (v \in D(A^{1/4})).
\]
Moreover, the distance between $\mathbb{B}_0$ and $\mathbb{B}_\alpha$ is estimated as
\[
H_{\mathcal{H}_0}(\mathbb{B}_0, \mathbb{B}_\alpha) \leq \sqrt{\alpha} R .
\]
Indeed, it is sufficient to check that $h_{\mathcal{H}_0}(\mathbb{B}_0, \mathbb{B}_\alpha) \leq \sqrt{\alpha} R$ since $\mathbb{B}_\alpha$ is included in $\mathbb{B}_0$. Taking $z$ from $\mathbb{B}_0$ arbitrary and fix it, then we see that
\[
d_{\mathcal{H}_0}(z, \mathbb{B}_\alpha) \leq \|z - j_\alpha z\|_{\mathcal{H}_0} \leq \sqrt{\alpha} R .
\]
The element $z$ is arbitrary, so the estimate surely holds.

From now on, we get into the proof of Main Theorem. The construction below is based on the arguments in [20], but we need to modify the process of the estimating
of the difference between $A_{\exp,\alpha}$ and $A_{\exp,0}$. So, along with the reader’s convenience, we show the process of the construction in the following.

**Proof.** We recall that the exponential attractor for the system $(\mathcal{H}_0, \mathcal{B}_0, S_0^m)$ is constructed as follows:

$$A_{\exp,0}^* = \bigcup_{k,l=0}^{\infty} S_0^k E_0^l,$$  

(87)

where $E_0^k = \{S_0^k z_{k,i} | z_{k,i} \in \mathcal{B}_0, \ i = 1, 2, \ldots, N_0^k\}$, $\rho > 0$ and $N_0^k$ are constants satisfying the conditions appeared in the subsection 4.3.2. More precisely, the above set is constructed as a collection of the central points of finite covering balls for each $S_0^k \mathcal{B}_0$; that is, under the conditions (80) and (81), we are able to construct an $R_0^k$-radius covering of $S_0^k \mathcal{B}_0$ by a finite number of closed balls of $\mathcal{H}_{0,1}$ with center points in $S_0^k \mathcal{B}_0$ as follows

$$S_0^k \mathcal{B}_0 \subset \bigcup_{i=1}^{N_0^k} \overline{B}^{\mathcal{H}_0}(\tilde{z}_{k,i}; R_0^k),$$  

(88)

where $R > 0$ is the radius of the ball $\mathcal{B}_0$ and $\tilde{z}_{k,i}$ are the central points $\tilde{z}_{k,i} := S_0^k z_{k,i} \in S_0^k \mathcal{B}_0$, $1 \leq i \leq N_0^k$. Then the set defined in (87) gives an exponential attractor for $(\mathcal{H}_0, \mathcal{B}_0, S_0^m)$.

Now, we are going to construct an exponential attractor for $(\mathcal{H}_\alpha, \mathcal{B}_\alpha, S_\alpha^m)$, $0 < \alpha \leq 1$, on the basis of the set $\cup_{k=0}^{\infty} E_0^k$. From the property of the scaling operators and Proposition 5.1, $S_\alpha^k \mathcal{B}_\alpha$ ($k \in \mathbb{N}$) has a covering as follows:

$$S_\alpha^k \mathcal{B}_\alpha \subset \bigcup_{i=1}^{N_\alpha^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; R_\alpha^k + \tilde{L}^k \alpha^{1/4}),$$  

(89)

where $\tilde{z}_{k,i}^\alpha = S_\alpha^k z_{k,i}$, $\tilde{L} = \tilde{L}(R, T)$ is a constant depending on $R$ and $T > 0$. Indeed, for any $w = S_\alpha^k z \in S_\alpha^k \mathcal{B}_\alpha$, there exists $z_{k,i} \in \mathcal{B}_\alpha$ such that the difference $S_\alpha^k z - S_\alpha^k z_{k,i}$ on the topology $\mathcal{H}_\alpha$ is estimated as below

$$\|S_\alpha^k z - S_\alpha^k z_{k,i}\|_{\mathcal{H}_\alpha} \leq \|S_\alpha^k z - S_\alpha^k z_{k,i}\|_{\mathcal{H}_\alpha} + \|S_0^k z - S_0^k z_{k,i}\|_{\mathcal{H}_\alpha} + \|S_0^k z_{k,i} - S_\alpha^k z_{k,i}\|_{\mathcal{H}_\alpha} \leq R_\alpha^k + \tilde{L}^k \alpha^{1/4}.$$

So, right-hand-side of (89) is a covering balls of $S_\alpha^k \mathcal{B}_\alpha$ and let us construct an exponential attractor for the system $(\mathcal{H}_\alpha, \mathcal{B}_\alpha, S_\alpha^m)$ by modifying the above covering (89). Now, we define the integer $m_\alpha$ as the largest one satisfying $\tilde{L}^k \alpha^{1/4} \leq R_\alpha^k$. Then, it is clear that

$$S_\alpha^k \mathcal{B}_\alpha \subset \bigcup_{i=1}^{N_\alpha^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; 2R_\alpha^k) \quad (0 \leq k \leq m_\alpha).$$

For $k > m_\alpha$, we redo the procedure of constructing appropriate covering balls for each $S_\alpha^k \mathcal{B}_\alpha$ by restaring the same argument handled in the subsection 4.3.2 from the balls $\overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; 2R_\alpha^m) (1 \leq i \leq N_\alpha^m)$. Then we are able to construct inductively for every $m_\alpha < k < \infty$, a $2R_\alpha^k$-radius covering of $S_\alpha^k \mathcal{B}_\alpha$ such that

$$S_\alpha^k \mathcal{B}_\alpha \subset \bigcup_{i=1}^{N_\alpha^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; 2R_\alpha^k), \quad (\tilde{z}_{k,i}^\alpha = S_\alpha^k z_{k,i} \in S_\alpha^k \mathcal{B}_\alpha).$$
We redefine $E^k_\alpha$ as

$$E^k_\alpha := \begin{cases} 
    \{ S^k_\alpha z_{k,i} | z_{k,i} \in B_0, \ i = 1, 2, \ldots, N^k_\rho \} & (0 \leq k \leq m_\alpha) \\
    \{ S^k_\alpha z'_{k,i} | z'_{k,i} \in B_\alpha, \ i = 1, 2, \ldots, N^k_\rho \} & (m_\alpha < k < \infty)
\end{cases}$$

and we obtain a new exponential attractor $A^*_{\exp,\alpha}$ as follows:

$$A^*_{\exp,\alpha} := \bigcup_{k,l=0}^{\infty} S^k_\alpha E^l_\alpha.$$ 

We can verify the above set $A^*_{\exp,\alpha}$ satisfies the definition of exponential attractor by the same way (for the details, we refer to [20]), so we only check (20).

Now, we recall that, from the remark 3, there exists the time $t_{R_0} > 0$ such that the ball $B_0$ is absorbed by itself and we put $l_0 := \lceil t_R \rceil + 1$, where $t_R$ is an integer part of the number $t_R$. Then, it is easy to see

$$A^*_{\exp,\alpha} \subset \bigcup_{k+l \leq m_\alpha} S^k_\alpha E^{l+1}_{\alpha} \cup \bigcup_{k,l=0}^{l_0-1} S^k_\alpha E^{l}_{\alpha},$$

and the semi-distance $h_{R_0}(A^*_{\exp,\alpha}, A^*_{\exp,0})$ is estimated from above by

$$h_{R_0}\{(\bigcup_{k+l \leq m_\alpha} S^k_\alpha E^{l}_{\alpha}) \cup (\bigcup_{i=1}^{l_0-1} S^{m_\alpha+i}_{\alpha}) \} = \max_{k,l,i} \{ h_{R_0}(S^k_\alpha E^{l}_{\alpha}, A^*_{\exp,\alpha}), h_{R_0}(S^{m_\alpha+i}_{\alpha}, A^*_{\exp,0}) \}.$$ (90)

So, in order to prove the statement, it is sufficient to estimate the right-hand-side of (90), for each $k,l,i$. From the definition of the set $E^l_{\alpha}$, it is easy to see that

$$h_{R_0}(S^k_\alpha E^l_{\alpha}, A_{\exp,0}) \leq h_{R_0}(S^k_\alpha E^0_{\alpha}, S^k_0 E^0_{0}) \leq \tilde{L}^{k+l+1/4} \leq m_\alpha.$$ (91)

By using triangle inequality, (83) and (86), we have

$$h_{R_0}(S^k_\alpha E_{\alpha}, A^*_{\exp,\alpha}) \leq h_{R_0}(S^k_\alpha E_{\alpha}, S^l_\alpha E_{\alpha}) + h_{R_0}(S^l_\alpha E_{\alpha}, A^*_{\exp,0}) \leq R \tilde{L}^{1/4} + R \tilde{L} \leq 2R \tilde{L}^{1/4} (m_\alpha + 1 \leq l \leq m_\alpha + l_0 - 1).$$ (92)

Combining (91) with (92), we get

$$h_{R_0}\{(\bigcup_{k+l \leq m_\alpha} S^k_\alpha E^{l}_{\alpha}) \cup (\bigcup_{i=1}^{l_0-1} S^{m_\alpha+i}_{\alpha}) \} \leq C_{R,T} \tilde{L}^{m_\alpha+1/4}.$$ (93)

Since $m_\alpha = \lceil -\log_\alpha L \rceil$, we obtain that $\tilde{L}^{m_\alpha + 1/4} \leq C_{R,T} \alpha^\kappa$ with the exponent $\kappa = \frac{-\log \alpha L}{-\log \alpha - \log \nu}$.

The converge estimate $h_{R_0}(A^*_{\exp,0}, A^*_{\exp,\alpha}) \leq C_{R,T} \alpha^\kappa$ is also verified in a similar way. Thus, we conclude that (20) holds true.

Acknowledgments. I’m grateful to Professor Hideo Kubo for a lot of instructions and encouragements. I also would like to thank Professor Syuichi Jimbo and Professor Nao Hamamuki for giving me many comments. Finally I would like to thank the referees for their remarkable comments and suggestions that inspired and encouraged myself to reach the correct version of the current paper.
REFERENCES

[1] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Application 25, North-Holland Publishing Co., Amsterdam, 1992.

[2] A. V. Balakrishnan and L. W. Taylor, Distributed parameter nonlinear damping models for flight structures, *Proceedings Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB*, (1989).

[3] A. V. Balakrishnan, A theory of nonlinear damping in flexible structures, *Stabilization of Flexible Structures*, (1988), 1–12.

[4] J. M. Ball, Initial-boundary value problems for an extensible beam, *J. Math. Anal. Appl.*, 42 (1973), 61–90.

[5] J. M. Ball, Stability theory for an extensible beam, *J. Differential Equations*, 14 (1973), 399–418.

[6] A. C. Biazutti and H. R. Crippa, Global attractor and inertial set for the beam equation, *Appl. Anal.*, 55 (1994), 61–78.

[7] P. Biler, Remark on the decay for damped string and beam equations, *Nonlinear Anal.*, 10 (1986), 839–842.

[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, *Commun. Contemp. Math.*, 6 (2004), 705–731.

[9] M. M. Cavalcanti, V. N. Domingos Cavalcanti and T. F. Ma, Exponential decay of the viscoelastic Euler-Bernoulli equation with a nonlocal dissipation in general domains, *Differential Integral Equations*, 17 (2004), 495–510.

[10] I. Chueshov and S. Kolbasin, Plate models with state-dependent damping coefficient and their quasi-static limits, *Nonlinear Anal.*, 73 (2010), 1626–1644.

[11] I. D. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, ACTA, Kharkov, 1999, 436 pp.

[12] I. Chueshov and I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, *Mem. Amer. Math. Soc.*, 195 (2008).

[13] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations. Well-Posedness and Long-Time Dynamics*, Springer Monographs in Mathematics, Springer, New York, 2010.

[14] M. Coti Zelati, Global and exponential attractors for the singularly perturbed extensible beam, *Discrete Contin. Dyn. Syst.*, 25 (2009), 1041–1060.

[15] E. H. de Brito, The damped elastic stretched string equation generalized: Existence, uniqueness, regularity, and stability, *Applicable Anal.*, 13 (1982), 219–233.

[16] R. W. Dickey, Free vibrations and dynamic buckling of the extensible beam, *J. Math. Anal. Appl.*, 29 (1970), 443–454.

[17] R. W. Dickey, Dynamic stability of equilibrium states of the extensible beam, *Proc. Amer. Math. Soc.*, 41 (1973), 94–102.

[18] A. Eden and A. J. Milani, Exponential attractor for extensible beam equations, *Nonlinearity*, 6 (1993), 457–479.

[19] A. Eden, V. Kalantarov and A. Miranville, Finite-dimensional attractors for a general class of nonautonomous wave equations, *Appl. Math. Lett.*, 13 (2000), 17–22.

[20] M. Efendiev and A. Yagi, Continuous dependence on a parameter of exponential attractors for chemotaxis-growth syste, *J. Math. Soc. Japan*, 57 (2006), 167–181.

[21] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, 25. American Mathematical Society, Providence, RI, 1988.

[22] J. S. Howell, I. Lasiecka and J. T. Webster, Quasi-Stability and exponential attractors for a non-gradient system-applications to piston-theoretic plates with internal damping, *Evolution Equations and Control Theory*, 5 (2016), 567–603.
STABILITY WITH RESPECT TO ROTATIONAL INERTIA

[26] J. S. Howell, D. Toundykov and J. T. Webster, A cantilevered extensible beam in axial flow: Semigroup well-posedness and postflutter regimes, SIAM Journal on Mathematical Analysis, 50 (2018), 2048–2085.

[27] A. Kh. Khanmamedov, A global attractor for the plate equation with displacement-dependent damping, Nonlinear Anal., 74 (2011), 1607–1615.

[28] S. Kouemou Patcheu, On a global solution and asymptotic behaviour for the generalized damped extensible beam equation, J. Differential Equations, 135 (1997), 299–314.

[29] S. Kolbasin, Attractors for Kirchhoff’s equation with a nonlinear damping coefficient, Nonlinear Anal., 71 (2009), 2361–2371.

[30] H. Lange and G. Perla Menzala, Rates of decay of a nonlocal beam equation, Differential Integral Equations, 10 (1997), 1075–1092.

[31] J. E. Lagnese and G. Leugering, Uniform stabilizability of a full von Karman system with nonlinear boundary feedback, J. Differential Equations, 91 (1991), 355–388.

[32] T. F. Ma and V. Narciso, Global attractor for a model of extensible beam with nonlinear damping and source terms, Nonlinear Anal., 73 (2010), 3402–3412.

[33] T. F. Ma, V. Narciso and M. L. Pelicer, Long-time behavior of a model of extensible beams with nonlinear boundary dissipations, J. Math. Anal. Appl., 396 (2012), 694–703.

[34] L. A. Medeiros, On a new class of nonlinear wave equations, J. Math. Anal. Appl., 69 (1979), 252–262.

[35] C. L. Mu and J. Ma, On a system of nonlinear wave equations with Balakrishnan-Taylor damping, Z. Angew. Math. Phys., 65 (2014), 91–113.

[36] J. E. Muñoz Rivera, Global solution and regularizing properties on a class of nonlinear evolution equation, J. Differential Equations, 128 (1996), 103–124.

[37] G. R. Sell and Y. C. You, Dynamics of Evolutionary Equations, Applied Mathematical Sciences, 143. Springer-Verlag, New York, 2002.

[38] M. A. J. da Silva and V. Narciso, Attractors and their properties for a class of nonlocal extensible beams, Discrete Contin. Dyn. Syst., 35 (2015), 985–1008.

[39] M. A. Jorge Silva, V. Narciso and A. Vicente, On a beam model related to flight structures with nonlocal energy damping, Discrete Contin. Dyn. Syst. B, 24 (2019), 3281–3298.

[40] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl.(4), 146 (1987), 65–96.

[41] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1988.

[42] C. F. Vasconcellos and L. M. Teixeira, Existence uniqueness and stabilization for a nonlinear plate system with nonlinear damping, Ann. Fac. Sci. Toulouse Math.(6), 8 (1999), 173–193.

[43] D. X. Wang and J. W. Zhang, Global attractor for a nonlinear plate equation with supported boundary conditions, J. Math. Anal. Appl., 363 (2010), 468–480.

[44] Y. C. You, Inertial manifolds and stabilization of nonlinear beam equations with Balakrishnan-Taylor damping, Abstr. Appl. Anal., 1 (1996), 83–102.

[45] W. Zhang, Nonlinear damping model: Response to random excitation, 5th Annual NASA Spacecraft Control Laboratory Experiment (SCOLE) Workshop, (1988), 27–38.

[46] Z. J. Yang, On an extensible beam equation with nonlinear damping and source terms, J. Differential Equations, 254 (2013), 3903–3927.

Received November 2018; revised December 2019.

E-mail address: ni-mura@math.sci.hokudai.ac.jp