On Inflation Rules for Mosseri–Sadoc Tilings

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Abstract

We give the inflation rules for the decorated Mosseri–Sadoc tiles in the projection class of tilings \( T^{(MS)} \). Dehn invariants related to the stone inflation of the Mosseri–Sadoc tiles provide eigenvectors of the inflation matrix with eigenvalues equal to \( \tau = \frac{1 + \sqrt{5}}{2} \) and \( (-\tau^{-1}) \).

Keywords: Quasiperiodic tiling; Icosahedral tiling; Inflation rules; Dehn invariant

1. Introduction

Kramer has introduced the icosahedrally symmetric tiling of the (3-dimensional) space by seven (proto)tiles [1]. Sadoc and Mosseri have rebuilt these prototiles and reduced their number to four: \( z, h, s \) and \( a \). But the inflation class (inflation specie [2]) of the tilings of the space by the four Mosseri–Sadoc prototiles has lost the icosahedral symmetry [3]. In Ref. [4] a projection class (projection specie [2]) of the tilings \( T^{(MS)} \) by the Mosseri–Sadoc prototiles has been locally derived from the canonical icosahedrally projected (from the lattice \( D_6 \)) local isomorphism class of the tilings \( T^*(2F) \). In Ref. [4] it has been shown that the projection class of the locally isomorphic tilings \( T^*(2F) \) (the tilings of the 3dimensional space by the six tetrahedra with all edges parallel to the 2fold symmetry axis of an icosahedron, of two lengths, the standard one denoted by \( \frac{1}{2} \), the “short” edge, and \( \tau \), the “long” edge, [3]) by the “golden” tetrahedra [8] can be locally transformed into the tilings \( T^{(MS)} \), \( T^*(2F) \rightarrow T^{(MS)} \). The class \( T^{(MS)} \) of locally isomorphic tilings by Mosseri–Sadoc tiles has been defined by the projection [4]. The important property is that the mini-
Fig. 1. Decorated tiles $a$, $m$, $r$, $z$ and $s$ of the tilings $T^{(MS)}$ obtained from the decorated eight tiles $A^*$, $B^*$, $C^*$, $D^*$, $F^*b$, $F^*r$, $G^*b$ and $G^*r$ of the tilings $T^{(2F)}$. The arrows which decorate the eight $T^{(2F)}$-tiles are along their edges, the arrows which decorate the (composite) tiles of the projection species $T^{(MS)}$ are drawn on certain distance in order to be distinguishable from the previous. The “white” arrow (by the tile $a$) is marking the edge $\tau a$, the “long” edge in the $\tau T^{(2F)}$-class of tilings.

Fig. 2. The tiles $r$ and $m$ appear in $T^{(MS)}$ always as a union, $h = r \cup m$. The decoration of the tile $h$ is determined by the decoration of the tiles $r$ and $m$.

Fig. 3. Inflation rule of the decorated tile $a$: $\tau a = a \cup s \cup a$. The “white” arrow marks the edge $\tau a^2$, the “long” edge in the $\tau T^{(2F)}$-class of tilings.

Fig. 4. Inflation rule of the decorated tile $r$: $\tau r = z \cup s \cup m \cup r$.

Fig. 5. Inflation rule of the decorated tile $z$: $\tau z = \tau r \cup a$. The white arrows are marking the “short” and “long” edges in the $\tau T^{(2F)}$-class of tilings.

Table of Invariants

| Invariant | Value |
|-----------|-------|
| $\alpha$  | $\frac{\tau}{\tau + 2} = \frac{1}{\sqrt{5}}$ |

Thus, the space of Dehn invariants of the Mosseri–Sadoc tiles is one-dimensional, there is only one independent lateral angle. For the Dehn invariants applied to the inflation, see Ref. [8].

For the vector of volumes of the Mosseri–Sadoc tiles one obtains

$$v_{MS} = \text{Vol} \begin{pmatrix} z \\ h \\ s \\ a \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4\tau + 2 \\ 6\tau + 4 \\ 4\tau + 3 \\ 2\tau + 1 \end{pmatrix} .$$

We show that the inflation matrix for Mosseri–Sadoc tiles $z$, $h$, $s$ and $a$ can be uniquely reconstructed from the Dehn invariants (and the volumes).

Denote the inflation matrix by $\mathcal{M}_{MS}$.

The vectors $d_{MS}$ and $v_{MS}$ are eigenvectors of the inflation matrix, with the eigenvalues $\tau$ and $\tau^2$ correspondingly (the eigenvalue is equal to the inflation...
Fig. 6. Inflation rule of the decorated tile $s$: $\tau s = \tau z \cup a$. The white arrow is marking the “long” edge in the $\tau T^{(2F)}$–class of tilings.

Fig. 7. Inflation rule of the decorated tile $m$: $\tau m = a \cup s \cup z \cup a$. The white arrow is marking the “long” edge in the $\tau T^{(2F)}$–class of tilings.

factor to the power which is the dimension of the corresponding invariant).

Explicitly,

$$M_{MS} \begin{pmatrix} 4\tau + 2 \\ 6\tau + 4 \\ 4\tau + 3 \\ 2\tau + 1 \end{pmatrix} = \begin{pmatrix} 16\tau + 10 \\ 26\tau + 16 \\ 18\tau + 11 \\ 8\tau + 5 \end{pmatrix}$$

(6)

for the vector of volumes and

$$M_{MS} \begin{pmatrix} \tau \\ 2 \\ \tau - 1 \\ -\tau \end{pmatrix} = \begin{pmatrix} \tau + 1 \\ 2\tau \\ 1 \\ -\tau - 1 \end{pmatrix}$$

(7)

for the the vector of Dehn invariants.

Assume that the inflation matrix is rational. Then, decomposing eqs. (6) and (7) in powers of $\tau$, we obtain four vector equations for $M_{MS}$ or a matrix equation

$$M_{MS} \begin{pmatrix} 4 & 2 & 1 & 0 \\ 6 & 4 & 0 & 2 \\ 4 & 3 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 16 & 10 & 1 & 1 \\ 26 & 16 & 2 & 0 \\ 18 & 11 & 0 & 1 \\ 8 & 5 & -1 & -1 \end{pmatrix}$$

(8)

The solution of this equation is unique and we find the matrix [8].

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