This paper considers how the finite dimensions of a photonic crystal placed inside a resonator or waveguide affect the law of electron beam instability. The dispersion equations describing e-beam instability in the finite photonic crystal placed inside the resonator or waveguide (a bounded photonic crystal) are obtained. Two cases are considered: the conventionally considered case, when diffraction is suppressed, and the case of direct and diffracted waves having almost equal amplitudes. The instability law is shown to be responsible for increase of increment of instability and decrease of length, at which instability develops, for the case when amplitude of diffracted wave is comparable with that of direct one, that happens in the vicinity of $\pi$-point of dispersion curve. Application of photonic crystals for development of THz sources at electron beam current densities available at modern accelerators is discussed.
I. INTRODUCTION

THz sources are in demand for a variety of applications: information and communications technology, biology and medicine, non-destructive investigations and homeland security, food and agricultural products quality control, global environmental monitoring, space research and ultra-fast computing. High-power tunable THz sources are very important devices to bring promising prospects to a wide use; high efficiency and compactness are also highly desired (see \[1\], [2] and references therein).

Sources of THz radiation are studied by numerous authors through expansion to THz range of approaches and principles, which generally serve for microwave sources, namely: travelling wave tubes, backward wave oscillators, generators utilizing diffraction or Smith-Purcell radiation, etc \[3–6\]. A slow-wave structure (SWS) is generally used in such devices to make the electromagnetic wave phase velocity less than the speed of light so that beam electrons can interact with the wave and convert their energy to radiation. A diffraction grating, a helical line, a corrugated waveguide, a photonic crystal, a multipin structure or a spatially periodic structure of any type, they all could work as SWS and enable frequency tuning by change of their geometry \[7–11\].

The general feature for all the above listed radiation sources is the instability of an electron beam in a spatially periodic media, which results in beam self-modulation and radiation of electromagnetic waves. The increment \(\delta_0 \sim Im k_z\) (\(k_z\) is the longitudinal wave number), which describes the electron beam instability responsible for radiation process in such devices, conventionally is determined by the unperturbed density \(\rho_{00}\) of electrons in the beam as follows: \(\delta_0 \sim \sqrt[3]{\rho_{00}}\) \[12–14\]. The threshold current density \(j_{thr}\), which is required for coherent electromagnetic oscillations to grow, in this case depends on the beam-wave interaction length \(L\) as \(j_{thr} \sim L^{-3}\).

Shift to THz range for conventional microwave devices faces several difficulties limiting the output power due to drastic decrease in dimensions of the interaction area. SWS period tends to submillimeter range and requirements to electron beam quality and guiding precision become more strict. The amplitude of the harmonic, which is in synchronism with the beam, decreases with the distance from the SWS surface on the scale

\[
\Delta = \frac{\lambda \beta \gamma}{2 \pi},
\]

where \(\lambda\) is the wavelength, \(\beta = \frac{v}{c}\) is the Lorentz factor, \(v\) and \(c\) are the speed of electron beam and light, respectively. Thus, only electrons moving close to the structure can efficiently interact with the wave; for example, \(\delta \approx 0.1\) mm for 100 keV electron beam and \(\lambda = 1\) mm. Small dimensions constrain applicable electron beam currents and, therefore, available output power. Therefore, approaches enabling increase of efficiency and transverse dimensions of interaction area (electron-beam cross section) are of high priority for THz source development. Such approaches were for the first time ever proposed for X-ray range by the concept of volume free electron laser (VFEL) \[9, 15–17\], which enables increase of both the efficiency and the transverse coherence area, and simultaneous reduction of threshold current density and operation length. Successive expansion of VFEL concept to microwave \[8, 18\], terahertz \[19, 20\] and visible light \[21\] ranges includes both theoretical and experimental studies; diversely designed SWS are used, namely: diffraction gratings, photonic crystals, etc.

Another law of electron beam instability was discovered in \[9, 15–17\]. It was found there that for an electron beam moving in a crystal for the case, when Bragg diffraction could occur, the electron beam instability increment \(\delta_0\) could turn out to be determined by \(\delta_0 \sim \sqrt[3]{\rho_{00}}\) rather then conventional \(\delta_0 \sim \sqrt[3]{\rho_{00}}\) \[12–14\]. The law of electron beam instability inherent for volume free electron laser \[8, 15–17\] is revealed when distributed feedback is formed by Bragg diffraction of beam-induced electromagnetic wave by a spatially periodic SWS. This law gives for the threshold current density the very different dependence on interaction length \(L\) as follows: \(j_{thr} \sim L^{-3-2s}\), where \(s\) is the number of additional waves arisen due to diffraction (when one additional wave arises then \(j_{thr} \sim L^{-5}\)).

For a certain slow-wave structure this law reveals only for narrow range of parameters (electron beam energy and radiation frequency), for which the group velocity of the excited wave is close to zero i.e. synchronism of beam and wave is achieved in the vicinity of \(\pi\)-point on the SWS dispersion curve \(k(\omega) = 0\) (\(k\) is the wavenumber, \(\omega\) is the radiation frequency). For example, when in addition to the electromagnetic wave, which is excited by electron beam via its interaction with SWS, one more wave, which propagates in the direction determined by the Bragg’s law, is present, one could expect that the threshold current density \(j_{thr}\) is defined by law \(j_{thr} \sim L^{-5}\) rather than \(j_{thr} \sim L^{-3}\) for proper combination of electron beam energy and radiation frequency. In particular, the law \(j_{thr} \sim L^{-5}\) can be observed in conventional backward wave oscillator with corrugated waveguide in case when the excited wave and the diffracted one propagate in opposite directions along the system axis.

First lasing of VFEL, which uses the above described instability law, was presented in \[7\]. Frequency tuning in this generator is analyzed in \[8\]. Theoretical study of the instability of electron beams moving in natural and artificial (photonic) crystals was carried out for the ideal case of an infinite medium (see the review \[9\] and \[3, 7, 15–18, 22\]). It is known that the discrete structure of the modes in waveguides and resonators is crucial for effective generation
in the microwave range \[23\text{–}25\]. This paper considers how the finite dimensions of a photonic crystal placed inside a resonator or waveguide affect the law of electron beam instability.

The dispersion equation describing e-beam instability in the finite photonic crystal placed inside the resonator or waveguide (a bounded photonic crystal) is obtained. The instability law is shown to be valid and caused by mixing of the electromagnetic field modes in the finite volume due to the periodic disturbance produced by the photonic crystal.

II. EQUATIONS DESCRIBING MOTION OF A RELATIVISTIC ELECTRON BEAM IN A BOUNDED PHOTONIC CRYSTAL

To describe generation of induced radiation (i.e. electron beam instability) in either a photonic or a natural crystal one should start from Maxwell equations:

\[
\text{rot} \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{j}, \quad \text{rot} \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t},
\]

where \(\vec{E}\) and \(\vec{H}\) are the strength of the electric and the magnetic field, respectively; \(\vec{j}\) and \(\rho\) are the current and charge densities; \(D_i(\vec{r}, t') = \int \varepsilon_{il}(\vec{r}, t-t') E_i(\vec{r}, t') \, dt'\) or \(D_i(\vec{r}, \omega) = \varepsilon_{il}(\vec{r}, \omega) E_i(\vec{r}, \omega)\), where indices \(i, l = 1, 2, 3\) correspond to \(x, y, z\); \(\varepsilon_{il}(\vec{r}, \omega)\) is the dielectric permittivity tensor of the photonic crystal.

The current and charge densities are defined as:

\[
\vec{j}(\vec{r}, t) = e \sum_\alpha \vec{v}_\alpha \delta(\vec{r} - \vec{r}_\alpha(t)), \quad \rho(\vec{r}, t) = e \sum_\alpha \delta(\vec{r} - \vec{r}_\alpha(t)) = e \rho_b(\vec{r}, t),
\]

where \(e\) is the electron charge, \(\rho_b\) is the beam density (the number of electrons per 1 cm\(^3\)). The velocity \(\vec{v}_\alpha\) of electron with number \(\alpha\) reads as

\[
\frac{d\vec{v}_\alpha}{dt} = \frac{e}{m \gamma_\alpha} \left\{ \vec{E}(\vec{r}_\alpha(t), t) + \frac{1}{c} \left[ \vec{v}_\alpha(t) \times \vec{H}(\vec{r}_\alpha(t), t) \right] - \frac{\vec{v}_\alpha}{c^2} \left[ \vec{v}_\alpha(t) \vec{E}(\vec{r}_\alpha(t), t) \right] \right\},
\]

where \(\gamma_\alpha = \left(1 - \frac{\vec{v}_\alpha^2}{c^2}\right)^{-\frac{1}{2}}\) is the Lorentz factor, \(\vec{E}(\vec{r}_\alpha(t), t)\) (\(\vec{H}(\vec{r}_\alpha(t), t)\)) is the electric (magnetic) field strength at point \(\vec{r}_\alpha\), where the electron with number \(\alpha\) is located. Note that equation (4) can be written as follows (26):

\[
\frac{d\rho_\alpha}{dt} = m \frac{d\gamma_\alpha \vec{v}_\alpha}{dt} = e \left\{ \vec{E}(\vec{r}_\alpha(t), t) + \frac{1}{c} \left[ \vec{v}_\alpha(t) \times \vec{H}(\vec{r}_\alpha(t), t) \right] \right\},
\]

where \(\rho_\alpha\) is the particle momentum.

From equations (4) one can obtain

\[
-\Delta \vec{E} + \nabla \left( \nabla \cdot \vec{E} \right) + \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial \vec{j}}{\partial t}.
\]

The dielectric permittivity tensor can be presented in the form \(\varepsilon(\vec{r}) = 1 + \chi(\vec{r})\), where \(\chi(\vec{r})\) is the susceptibility. For \(\chi << 1\), equation (6) can be rewritten as follows

\[
\Delta \vec{E}(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} \int \varepsilon(\vec{r}, t - t') \vec{E}(\vec{r}, t') \, dt' = 4\pi \left( \frac{1}{c^2} \frac{\partial \vec{j}(\vec{r}, t)}{\partial t} + \nabla \rho(\vec{r}, t) \right).
\]

In the general case, the susceptibility of the photonic crystal reads

\[
\chi(\vec{r}) = \sum_i \chi_{cell}(\vec{r} - \vec{r}_i),
\]
where $\chi_{\text{cell}}(\vec{r} - \vec{r}_0)$ is the susceptibility of the crystal unit cell. The susceptibility of an infinite perfect crystal $\chi(\vec{r})$ can be expanded into a Fourier series as follows: $\chi(\vec{r}) = \sum_{\vec{r}} \chi_{\text{cell}}(\vec{r})$, where $\vec{r}$ is the reciprocal lattice vector of the crystal.

Let us consider in details a practically important case of a bounded photonic crystal; to be more specific, let us study a photonic crystal placed inside a waveguide of rectangular cross-section with smooth walls. The eigenfunctions and eigenvalues of a rectangular waveguide are well-studied [27, 28]. Suppose we study a photonic crystal placed inside a waveguide of rectangular cross-section with smooth walls. The eigenfunctions $\chi(\vec{r})$ and $\tilde{\vec{r}}$ can be expanded into a Fourier series as follows: $\chi(\vec{r}) = \sum_{\vec{r}} \chi_{\text{cell}}(\vec{r})$, where $\vec{r}$ is the reciprocal lattice vector of the crystal.

As a result, the following system of equations can be written

$$
\vec{E}(\vec{r}_\perp, k_z, \omega) = \sum_{m\lambda} C_{mn}^{\lambda}(k_z, \omega) Y_{mn}(\vec{r}_\perp, k_z).
$$

As a result, the following system of equations can be written

$$
\frac{k_0^2}{c^2} \sum_{m'n'} \int Y_{mn}^{*}(\vec{r}_\perp, k_z) \hat{\chi}(\vec{r}) Y_{m'n'}(\vec{r}_\perp, k_z') e^{-i(k_z-k_z')z} \frac{d^2 r_\perp}{k_0^2} C_{mn}^{\lambda}(k_z, \omega) dk_z' dz =
$$

where $\frac{k_0^2}{c^2} = k_z^2 + k_{n\lambda}^2$. The beam current and density, those appear on the right-hand side of (10), are the complicated functions of $\vec{E}$. To study the system instability, it is sufficient to consider it in the approximation linear over perturbation, i.e., one can expand the expressions for $\hat{j}$ and $\rho$ over $\vec{E}$ amplitude and confine oneself with the linear approximation.

As a result, a closed system of equations comes out. For further consideration, one should find the expressions for corrections to beam current density $\delta \hat{j}$ and beam charge density $\delta \rho$, which arise due to beam perturbation by the field. Considering the Fourier transforms of current and charge densities $\hat{j}(\vec{k}, \omega)$ and $\rho(\vec{k}, \omega)$, one can obtain from (10) that

$$
\delta \hat{j}(\vec{k}, \omega) = e^{-i\vec{k}\vec{r}_0 \rho} \left\{ \delta \tilde{\vec{u}}_{\alpha} \left( \omega - \vec{k} \tilde{\vec{u}}_{\alpha} \right) + \vec{u}_{\alpha} \frac{\delta \hat{\vec{u}}_{\alpha} \left( \omega - \vec{k} \tilde{\vec{u}}_{\alpha} \right)}{\omega - \vec{k} \tilde{\vec{u}}_{\alpha}} \right\},
$$

where $\vec{r}_0$ is the initial coordinate of the electron, $\tilde{\vec{u}}_{\alpha}$ is the unperturbed velocity of the electron.

For simplicity, let us consider a cold beam, for which $\tilde{\vec{u}}_{\alpha} \approx \vec{u}$, where $\vec{u}$ is the mean velocity of the beam. The general case of a hot beam can be obtained by averaging $\delta \hat{j}(\vec{k}, \omega)$ over distribution of particle the velocities $\tilde{\vec{u}}_{\alpha}$ in the beam.

According to (1), velocity $\delta \tilde{\vec{u}}_{\alpha}$ is determined by field $\vec{E}(\vec{r}_\alpha, \omega)$, where $\vec{r}_\alpha$ is the electron location point. The Fourier transform of $\vec{E}(\vec{r}_\alpha, \omega)$ has a form

$$
\vec{E}(\vec{r}_\alpha, \omega) = \frac{1}{(2\pi)^2} \int \vec{E}(\vec{k}', \omega) e^{i\vec{k}' \vec{r}_\alpha} d^2 k'.
$$

As a result, $\delta \hat{j}(\vec{k}, \omega)$ includes the following sum over the beam particles $\sum_{\alpha} e^{-i(\vec{k} - \vec{k}')} \delta(\vec{k} - \vec{k}')$, Suppose that the electrons in an unperturbed beam are uniformly distributed over the area occupied by the beam. Therefore

$$
\sum_{\alpha} e^{-i(\vec{k} - \vec{k}')} \delta(\vec{k} - \vec{k}') = (2\pi)^2 \rho_{00} \delta(\vec{k} - \vec{k}'),
$$

where $\rho_{00}$ is the unperturbed beam density (the number of electrons per 1 cm$^3$). As a result, the following expression for $\delta \hat{j}(\vec{k}, \omega)$ can be obtained [30, 31]:

$$
\delta \hat{j}(\vec{k}, \omega) = e^{-i\vec{k}\vec{r}_0 \rho} \left\{ \delta \tilde{\vec{u}}_{\alpha} \left( \omega - \vec{k} \tilde{\vec{u}}_{\alpha} \right) + \vec{u}_{\alpha} \frac{\delta \hat{\vec{u}}_{\alpha} \left( \omega - \vec{k} \tilde{\vec{u}}_{\alpha} \right)}{\omega - \vec{k} \tilde{\vec{u}}_{\alpha}} \right\},
$$
The expression for $\rho_k(\vec{k}, \omega)$ one can obtain using the continuity equation. Expression \ref{eq:rho_k}, the inverse Fourier transform of $\vec{E}(\vec{k}, \omega)$, and the expansion enable writing the system of equations as follows:

\begin{equation}
\left[ (\vec{k}_{\perp}^2 + \vec{r}_{a}^2) - \frac{\omega^2}{c^2} \right] C_{mn}^\lambda (k_{z}, \omega) - \frac{\omega^2}{c^2} \frac{1}{2\pi} \sum_{\lambda'} \int \vec{Y}_{mn}^\lambda (\vec{r}_{\perp}, k_{z}) \hat{\chi} (\vec{r}) \vec{Y}_{mn'}^{\lambda'} (\vec{r}_{\perp}, k'_{z}) e^{-i(k_{z} - k'_{z})z} d^{2}r_{\perp} C_{mn'}^{\lambda'} (k'_{z}, \omega) dk'_{z} dz =
\end{equation}

where $\vec{Y}_{mn}^\lambda (\vec{r}_{\perp}, k_{z}) = \int e^{-i\vec{k}_{\perp}\vec{r}_{\perp}} \vec{Y}_{mn}^\lambda (\vec{r}_{\perp}, k_{z}) d^{2}r_{\perp}$.

The system of equations in the approximation linear over perturbation describes the electromagnetic field modes, which are induced by an electron beam in the finite volume of a rectangular waveguide due to the periodic disturbance produced by a photonic crystal.

\section{Radiative Instability of a Relativistic Electron Beam Moving in a Bounded Photonic Crystal}

The above obtained system of equations enables to derive the dispersion equation for a bounded photonic crystal and to analyze conditions, when electron beam instability presents. Let us consider the sums in the left-hand side of equation:

\begin{equation}
\sum_{\lambda'} \int \vec{Y}_{mn}^{\lambda'} (\vec{r}_{\perp}, k_{z}) \hat{\chi} (\vec{r}) \vec{Y}_{mn}^{\lambda'} (\vec{r}_{\perp}, k'_{z}) e^{-i(k_{z} - k'_{z})z} d^{2}r_{\perp} C_{mn'}^{\lambda'} (k'_{z}, \omega) dk'_{z} dz =
\end{equation}

and analyze integrals

\begin{equation}
\int \vec{Y}_{mn}^{\lambda} (\vec{r}_{\perp}, k_{z}) \hat{\chi} (\vec{r}) \vec{Y}_{mn'}^{\lambda'} (\vec{r}_{\perp}, k'_{z}) e^{-i(k_{z} - k'_{z})z} d^{2}r_{\perp} dz.
\end{equation}

to evaluate what terms mostly contribute to the considered sums. Using \ref{eq:eigenfunctions} and representing eigenfunctions $\vec{Y}_{mn}^{\lambda} (\vec{r}_{\perp}, k_{z})$ of a rectangular waveguide by combinations of sines and cosines of the form $\sin \frac{\pi n}{a} x$, $\cos \frac{\pi n}{a} x$, $\sin \frac{\pi n}{b} y$, $\cos \frac{\pi n}{b} y$ (i.e., in fact, the combinations $e^{i\frac{\pi n}{a} x}$, $e^{i\frac{\pi n}{b} y}$) integrals can be transformed to the expressions of the form as follows:

\begin{equation}
I_x = \int e^{-i\frac{\pi n}{a} x} \sum_{i} \hat{\chi}_{cell} (x - x_i, y - y_i, z - z_i) e^{i\frac{\pi n'}{a} x} dx,
\end{equation}

\begin{equation}
I_y = \int e^{-i\frac{\pi n}{b} y} \sum_{i} \hat{\chi}_{cell} (x - x_i, y - y_i, z - z_i) e^{i\frac{\pi n'}{b} y} dy
\end{equation}

Substitution of variables $x - x_i = \eta_1$ in \ref{eq:ix} and $y - y_i = \eta_2$ in \ref{eq:iy} produces the sums of the form

\begin{equation}
S_x = \sum_{f_1=1}^{N_x} e^{-i\frac{\pi}{a} (m - m')} d_{x} f_{1}, S_y = \sum_{f_2=1}^{N_y} e^{-i\frac{\pi}{b} (n - n')} d_{y} f_{2},
\end{equation}

where $d_{x}$ and $d_{y}$ are the periods of the photonic crystal along $x$ and $y$ axes, $N_x = \frac{a}{d_{x}}$ and $N_y = \frac{b}{d_{y}}$ are the number of cells along $x$ and $y$ axes, respectively; coordinates of different cells $x_i = d_{x} f_{1}$, $y_i = d_{y} f_{2}$ are determined by integers.
The major contribution to the sums in the left-hand side of equation (13) comes from the amplitudes $\pm d_m$. This condition determines the dispersion equation.

$\sin \frac{\pi (m-m')}{2a} \approx 0$ one can obtain $S_x = N_x$. When $m - m' = 1$, a simple reasoning enables to get what $S_x$ is equal to: factor $d_x N_x = a$ and hence, the numerator is equal to $\sin \frac{\pi}{2} = 1$, while in the denominator $\sin \frac{\pi d_x}{2a} \approx \frac{\pi}{2N_x}$.

Therefore, the ratio $\frac{S_x(m-m'=1)}{S_x(m-m'=0)} = \frac{2\pi}{\pi} \approx 0.6$. With growing difference $(m - m')$, the contribution to the sum of the next terms diminishes provided the following equality is fulfilled

$$\frac{\pi (m-m') d_x}{2a} = \pi P,$$

where $P = \pm 1, \pm 2\ldots$; in these cases the sum $S_x = N_x$.

Fulfillment of conditions declared by equalities like (19) is equivalent to fulfillment of conditions $k_{xm} - k'_{xm'} = \tau_x$ (i.e., $k'_{xm'} = k_{xm} - \tau_x$) and $k_{yn} - k'_{yn'} = \tau_y$ (i.e., $k'_{yn'} = k_{yn} - \tau_y$), where $\tau_x = \frac{2\pi}{\lambda} F$ and $\tau_y = \frac{2\pi}{\lambda} F'$ are $x$- and $y$-components of the reciprocal lattice vector of the photonic crystal, respectively. $F, F' = 0, \pm 1, \pm 2\ldots$. Therefore, the major contribution to the sums in the left-hand side of equation (13) comes from the amplitudes $C_{m'n'}(k', \omega) \equiv C\chi \left( \kappa_{mn} - k, \alpha \right)$.

Hereinafter, when describing electron beam instability, we consider only modes those satisfy the equalities similar to (19). The contribution of other modes is supposed to be suppressed. Thus, system of equations (13) reads as follows:

$$\left( \kappa_{mn}^2 - \frac{\omega^2}{c^2} \right) C\chi \left( \kappa_{mn}, \omega \right) = \frac{\omega^2}{c^2} \sum_{\lambda, \tau} \chi_{mn}^{(\lambda)} (\tau) C^{\lambda} \left( \kappa_{mn} - \tau, \omega \right)$$

$$- \frac{\omega^2}{c^2} \left( \kappa_{mn}^2 - \frac{\omega^2}{c^2} \right) \chi_{mn}^{(\lambda)} (\tau) C^{\lambda} \left( \kappa_{mn}, \omega \right) = 0$$

where $\chi_{mn}^{(\lambda)} (\tau) = \frac{1}{2\pi} \int \bar{\chi}_{mn}^{(\lambda)} (\bar{\kappa}_{\perp}, k_z) (\bar{\kappa}_{\perp}, k_z - \tau \gamma) L_{2\kappa}^{\lambda} (\bar{\kappa}_{\perp}, k_z - \tau \gamma) d^2k_{\perp} d\sigma \chi (\kappa_{\perp}, \tau) = \sum_{x_i y_i} \int \chi_{\text{cell}} (x - x_i, y - y_i, z) e^{-i\gamma x} d\zeta,$

$m'$ and $n'$ are determined by conditions like (19), $\omega_L$ is the Langmuir frequency, $\omega_L^2 = \frac{4e^2 \mu_0}{m}.$

Since this system of equations is homogeneous, for the existence of a nontrivial solution the system determinant must vanish. This condition determines the dispersion equation.

Note that expression $\kappa_{mn} - \kappa$ is very much similar to Bragg condition and even converts to it, when the transversal dimensions of a waveguide $a$ and $b$ tend to infinity. Actually, when $a, b \to \infty$ the spectrum of eigenvalues becomes continuous and the set of wave vectors $\kappa_{mn}$ converts to wave vector $\kappa'$ of the wave propagating in the direction determined by Bragg condition $\kappa' - \kappa = \tau$, which in more familiar notation reads as

$$2d_g \sin \theta_B = m\lambda_B,$$

where $d_g$ is the diffraction grating period, $\theta_B$ is the Bragg angle, which determines the direction of diffracted wave propagation, $\lambda_B$ is the wavelength of the diffracted wave, $m$ is an integer number. Smooth rotation of diffraction grating with respect to electron velocity at angle $\varphi$ converts $d_g$ in (22) to $d_g \cos \varphi$, thus making $\lambda_B$ and $\theta_B$ in (22) slowly changing in a wide range.

System of equations (21) is similar to that describing instability of a beam passing through an infinite crystal [30, 31]. However, the coefficients appearing in these two systems are differently defined: for an infinite crystal, the wave vectors have continuous spectrum of eigenvalues rather than the discrete spectrum relevant for a bounded photonic crystal. These equations enable one to define dependence $k(\omega)$ for the waves propagating in the crystal.
Matching the incident wave packet with the set of waves propagating inside the photonic crystal by the boundary conditions, one can obtain the explicit solutions of the considered equations: the result obtained is formally analogous to that given in \[32\].

According to \[21\], the expression within the square brackets acts as dielectric permittivity $\varepsilon$ of the crystal in case when diffraction can be neglected:

$$
\varepsilon = n^2 = 1 + \chi_{mn}^{\lambda\lambda} (0) - \frac{\omega_L^2 (k_{mn}^2 - \omega^2)}{\omega^2 \gamma c^2 (\omega - k_{mn} u)} \left\{ \frac{1}{2\pi} \left| \int \bar{u} \bar{Y}_{\lambda}^{\lambda} (\bar{k}_{\perp}, k_z) d^2 k_\perp \right| ^2 \right\}, \quad (23)
$$

$n$ is the refractive index i.e. photonic crystal acts as a medium that can be described by a ceratin refractive index $n$ or the dielectric permittivity $\varepsilon$. The refractive index of the photonic crystal in the absence of the beam $n_0$ is determined by $n_0^2 = \varepsilon_0 = 1 + \chi_{mn}^{\lambda\lambda} (0)$. Two terms contribute to dielectric permittivity \[23\]: scattering of waves by the unit cell of the crystal and scattering of waves by beam electrons. The latter is given by the term proportional to $\omega_L^2$ and increases when $\omega \rightarrow k_\perp u$.

### A. Electron beam instability in a bounded photonic crystal beyond conditions enabling Bragg’s diffraction

Let us first assume that there are no waves for which Bragg’s diffraction conditions are fulfilled. Then the amplitudes of diffracted waves are assumed to be small. In this case the sum over $\tau$ in \[21\] can be dropped, and the conditions for the wave existence are given by putting equal to zero the expression left to $C^\lambda (\bar{k}_{mn}, \omega)$. This requirement can be written in the form

$$(\omega - k_z u)^2 \left( k_{mn}^2 - \frac{\omega^2}{c^2} n_0^2 \right) = - \frac{\omega_L^2 k_{mn}^2 c^2 - \omega^2}{\gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \bar{u} \bar{Y}_{\lambda}^{\lambda} (\bar{k}_{\perp}, k_z) d^2 k_\perp \right| ^2 \right\},$$

where velocity $\bar{u}|_o z$, therefore,

$$
\left( k_z^2 - \left( \frac{\omega^2}{c^2} n_0^2 - \chi_{mn}^2 \right) \right) (\omega - k_z u)^2 = - \frac{\omega_L^2 \left( k_{mn}^2 c^2 - \omega^2 \right)}{\gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \bar{u} \bar{Y}_{\lambda}^{\lambda} (\bar{k}_{\perp}, k_z) d^2 k_\perp \right| ^2 \right\} \quad (24)
$$

Since the nonlinearity is insignificant, let us consider the spectrum of the waves of equation \[24\] with zero right-hand side as zero-order approximation.

Let us concern with the case when $\omega - k_z u \rightarrow 0$ (i.e., the Cherenkov radiation condition is fulfilled) and

$$
\left( k_z^2 - \left( \frac{\omega^2}{c^2} n_0^2 - \chi_{mn}^2 \right) \right) \rightarrow 0, \quad (\omega - k_z u) \rightarrow 0.
$$

The frequency of radiation produced by the electron beam is determined by interception of dispersion curve $\omega(k_z)$ with the beam line $(\omega - k_z u) = 0$ (see Fig.1) that means synchronism of wave and beam electrons. Interceptions can occur at different values of $v_{gr} = \frac{d\omega}{dk_z}$ at $v_{gr} > 0$ wave travels forward along beam velocity (such situation is typical for travelling wave tubes); wave with $v_{gr} < 0$ moves backward enabling generation of backward wave oscillators \[33\].

The roots of equation \[25\] are

$$k_{1z} = \pm \frac{\omega}{c} \sqrt{\frac{n_0^2 - \chi_{mn}^2 c^2}{\omega^2}}, \quad k_{2z} = -k_{1z}, \quad k_{2z} = \frac{\omega}{u}. \quad (26)
$$

Since $k_{2z} = \frac{\omega}{u} > 0$ in view of the Cherenkov condition, we are concerned with propagation in the photonic crystal of the wave with $k_{1z} > 0$ (i.e. only sign ”+” rests in \[23\]). In this case in the equation for $k_{z}$, one can take $(k_z - k_{1z}) (k_z + k_{1z}) \approx 2k_{1z} (k_z - k_{1z})$ and rewrite equation \[24\] as follows:

$$
(k_z - k_{1z}) (k_z - k_{2z})^2 = - \frac{\omega_L^2 \omega^2 n_0^2 - 1}{2k_{1z} u^2 c^4} \left\{ \frac{1}{2\pi} \left| \int \bar{u} \bar{Y}_{\lambda}^{\lambda} (\bar{k}_{\perp}, k_z) d^2 k_\perp \right| ^2 \right\} \quad (27)
$$
Figure 1. Schematic drawing of dispersion curve $\omega(k_z)$. Point marked 1 corresponds to $v_{gr} = 0$ (\(\pi\)-point), it matches $k_zD = \pi$, where $D$ is the period of periodic structure. Points marked 2 correspond to beam-wave synchronism points. Left plot is for TWT, right plot corresponds to BWO case.

i.e.,

$$ (k_z - k_{1z})(k_z - k_{2z})^2 = -A, \quad (28) $$

$$ A = \frac{\omega^2 \omega^2 (n_0^2 - 1)}{2k_{1z}u^2 \gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \tilde{u} Y_{mn} \left( \bar{k}_{\perp}, k_z \right) d^2 k_{\perp} \right|^2 \right\}, \quad (29) $$

where $A$ is real and $A > 0$ (as to enable Cherenkov effect, it is necessary to have $n_0^2 > 1$).

Thus, $k_z$ satisfies cubic equation (28). Roots $k_{1z}$ and $k_{2z}$ coincide ($k_{1z} = k_{2z}$), when the particle velocity satisfies condition as follows:

$$ u = \frac{c}{\sqrt{n_0^2 - \frac{\pi n_0^2 c^2}{\omega^2}}}. \quad (30) $$

For $k_{1z} = k_{2z}$ substitution $\xi = k_z - k_{1z}$ in (28) gives equation

$$ \xi^3 = -A, \quad (31) $$

which has three solutions

$$ \xi^{(1)} = -\sqrt[3]{A}, \quad \xi^{(2,3)} = \frac{1}{2} \left( 1 \pm i\sqrt{3} \right) \sqrt[3]{A}. \quad (32) $$

The state corresponding to solution $\xi^{(3)} = \frac{1}{2} \left( 1 - i\sqrt{3} \right) \sqrt[3]{A}$ grows with $z$ growing that indicates the presence of instability in a beam [12, 34]. In this case the beam instability increment

$$ \delta_0^{(3)} \sim Im k_z = Im \xi^{(3)} \sim \sqrt[3]{\omega_L^2} \sim \sqrt[3]{\rho b_0} \quad (33) $$

and the threshold current density (a minimum beam current density required for oscillations to start spontaneously) is determined by the well-known law $j_{thr} \sim \frac{1}{L}$ (see equation (11.7) in [13] or equation (8.63) in [14]).

**B. Electron beam radiative instability in a bounded photonic crystal in the vicinity of \(\pi\)-point**

Let us now suppose that in a bounded photonic crystal there is a wave, for which Bragg’s diffraction conditions are fulfilled. In this case the roots of dispersion equation are close to each other that happens, when $k_z$ is in the vicinity of $\pi$-point [33, 37]. Such assumptions mean that in (21) the wave amplitude $C_{mn} \left( \bar{k}_{mn} + \tau \right)$ is comparable with the amplitude $C_{mn} \left( \bar{k}_{mn} \right)$. This is in contrast to assumptions used to derive (27) and, for this case conclusions obtained
in subsection III A are not valid. We also assume that \( \chi \ll 1 \), thus only the equations for these two amplitudes remain in (24) similar to the standard diffraction theory for an infinite crystal [35, 36].

When \( \chi \sim 1 \) one should consider more waves (see for example [39]) and the number of equations in (24) would be greater.

Analysis of diffraction of wave \( \lambda \), which electric vector is parallel to the plane \((y, z)\) (a TM-wave) gives:

\[
\left[ k_{mn}^2 - \frac{\omega^2}{c^2} \varepsilon_0 \right] C^\lambda \left( \vec{k}_{mn} + \vec{\tau}, \omega \right) - \frac{\omega^2}{c^2} \lambda_{mn}^\lambda \left( -\vec{\tau} \right) C^\lambda \left( \vec{k}_{mn} + \vec{\tau}, \omega \right) = 0,
\]

(34)

\[
\left[ \left( \vec{k}_{mn} + \vec{\tau} \right) - \frac{\omega^2}{c^2} \varepsilon_0 \right] C^\lambda \left( \vec{k}_{mn} + \vec{\tau}, \omega \right) - \frac{\omega^2}{c^2} \lambda_{mn}^\lambda \left( \vec{\tau} \right) C^\lambda \left( \vec{k}_{mn}, \omega \right) = 0.
\]

Since the term containing \((\omega - (\vec{k} + \vec{\tau}) \vec{u})^{-1}\) is small when \((\omega - \vec{k} \vec{u})\) vanishes, in the second equation it is dropped.

The dispersion equation defining the relation between \( k_z \) and \( \omega \) is obtained by equating to zero the determinant of the system (34) and has a form:

\[
\left[ \left( k_{mn}^2 - \frac{\omega^2}{c^2} \varepsilon_0 \right) \left( \vec{k}_{mn} + \vec{\tau} \right)^2 - \frac{\omega^4}{c^4} \varepsilon_0 \right] \left( \omega - k_z \vec{u} \right)^2 =
\]

\[
- \frac{\omega^2}{c^2} \left\{ \frac{1}{2\pi} \int \vec{u} \vec{Y}_{mn}^\lambda \left( \vec{k}_{\perp}, k_z \right) d^2 k_{\perp} \right\} \left( k_{mn}^2 c^2 - \omega^2 \right) \left( \vec{k}_{mn} + \vec{\tau} \right)^2 - \frac{\omega^2}{c^2} \varepsilon_0 \right).
\]

(35)

The right-hand side of equation (35) is small, therefore, one can again seek the solution near the points, where the right-hand side is zero, that means fulfillment of Cherenkov radiation condition and excitation of the wave, which can propagate in the waveguide:

\[
\left( k_z^2 - \frac{\omega^2}{c^2} \varepsilon_0 - \varepsilon_{mn}^2 \right) \left( (k_z + \tau)^2 - \frac{\omega^2}{c^2} \varepsilon_0 - (\vec{\tau}_{mn} + \vec{\tau}_{\perp})^2 \right) - \frac{\omega^4}{c^4} \varepsilon_0 \chi \chi_{\tau-\tau} = 0
\]

(36)

The roots for system of equations (36) are sought in the vicinity of condition \( k_{mn}^2 \approx \left( \vec{k}_{mn} + \vec{\tau} \right)^2 \), by substitution \( \xi = k_z - k_{z0} \), which gives the expressions as follows:

\[
k_z = k_{z0} + \xi, \quad k_z^2 = k_{z0}^2 + 2k_{z0} \xi + \xi^2, \quad k_z^2 = \frac{\omega^2}{c^2} \varepsilon_0 - \varepsilon_{mn}^2; \quad k_{z0} = \frac{\omega}{c} \sqrt{\varepsilon_0 - \varepsilon_{mn}^2 \frac{c^2}{\omega^2}}
\]

(37)

\[
(k_z + \tau_z)^2 = [(k_{z0} + \tau_z) + \xi]^2 = (k_{z0} + \tau_z)^2 + 2(k_{z0} + \tau_z) \xi + \xi^2.
\]

Hence, transformation of expressions as follows

\[
(k_{z0} + \tau_z)^2 + (\vec{\tau}_{mn} + \vec{\tau}_{\perp})^2 + 2(k_{z0} + \tau_z) + 2(k_{z0} + \tau_z) \xi + \xi^2 =
\]

\[
\left( \vec{k}_{mn} + \vec{\tau} \right)^2 + 2(k_{z0} + \tau_z) \xi + \xi^2 = k_{0mn}^2 + 2k_{0mn}^2 \tau + \tau^2 + 2(k_{z0} + \tau_z) \xi + \xi^2.
\]

(38)

enables to render the first equation in (36) as follows

\[
2k_{z0} \xi \left( 2(k_{z0} + \tau_z) \xi + \left( 2k_{0mn}^2 \tau + \tau^2 \right) \right) - \frac{\omega^2}{c^4} \chi \chi_{\tau-\tau} = 0,
\]

which is equivalent to

\[
4k_{z0} (k_{z0} + \tau_z) \xi^2 + 2k_{z0} \left( 2k_{0mn}^2 \tau + \tau^2 \right) \xi - \frac{\omega^4}{c^4} \chi \chi_{\tau-\tau} = 0.
\]

(39)
Thus, the second-order equation

\[ \xi^2 + \left( \frac{2k_{0mn}^2 + \tau^2}{k_{z0} + \tau_z} \right) \xi - \frac{\omega^4}{c^4} \frac{\chi_{\tau} \chi_{-\tau}}{4k_{z0} (k_{z0} + \tau_z)} = 0. \]  

(40)

enables to get the following solutions for \( \xi \):

\[ \xi_{1,2} = -\frac{\left( 2k_{0}^2 + \tau^2 \right)}{4 (k_{z0} + \tau_z)} \pm \sqrt{\frac{(2k_{0}^2 + \tau^2)}{4 (k_{z0} + \tau_z)}}^2 + \frac{\omega^4}{c^4} \frac{\chi_{\tau} \chi_{-\tau}}{4k_{z0} (k_{z0} + \tau_z)}. \]  

(41)

When \( (k_{z0} + \tau_z) = -|k_{z0} + \tau_z| \), the second term in (41) could become equal to zero. At the same time, the second equation in (40) must hold:

\[ \omega - k_z u = \omega - k_{z0} u - \xi u = 0. \]

Consequently,

\[ \xi_3 = \frac{\omega - k_{z0} u}{u} = \frac{\omega}{u} - k_{z0} = \frac{\omega}{u} \sqrt{\varepsilon_0 - \frac{\varepsilon_{mn}^2 c^2}{\omega^2}} = \frac{\omega}{u} \left( 1 - \beta \sqrt{\varepsilon_0 - \frac{\varepsilon_{mn}^2 c^2}{\omega^2}} \right). \]  

(42)

If \( \varepsilon_0 < 1 \), then \( \xi_3 = \frac{\omega}{u} - k_{z0} > 0 \). Let solutions \( \xi_1 \) and \( \xi_2 \) coincide \( (\xi_1 = \xi_2) \). This is possible at point

\[ \frac{2k_{0}^2 + \tau^2}{4 (k_{z0} + \tau_z)} = \frac{\omega^2}{c^2} \frac{\sqrt{\chi_{\tau} \chi_{-\tau}}}{4k_{z0} |k_{z0} + \tau_z|}, \]

here \( k_{z0} + \tau_z < 0 \), and the following equality is fulfilled

\[ \frac{\omega}{u} - k_{z0} = \frac{\omega^2}{c^2} \frac{\sqrt{\chi_{\tau} \chi_{-\tau}}}{4k_{z0} |k_{z0} + \tau_z|}, \]

i.e.,

\[ \frac{\omega}{u} = k_{z0} + \frac{\omega^2}{c^2} \frac{\sqrt{\chi_{\tau} \chi_{-\tau}}}{4k_{z0} |k_{z0} + \tau_z|}, \]  

(43)

where \( k_{z0} = \frac{\omega}{c} \sqrt{\varepsilon_0 - \frac{\varepsilon_{mn}^2 c^2}{\omega^2}} \). When \( \varepsilon_0 < 1 \) ratio \( \frac{\omega}{u} > k_{z0} \) (since \( u < c \)), and for solution \( \frac{\omega}{u} = k_{z0} - \frac{\omega^2}{c^2} \frac{\sqrt{\chi_{\tau} \chi_{-\tau}}}{4k_{z0} |k_{z0} + \tau_z|} \) the Cherenkov condition is not fulfilled.

Now let us consider the solution \( \frac{\omega}{u} = k_{z0} + \frac{\omega^2}{c^2} \frac{\sqrt{\chi_{\tau} \chi_{-\tau}}}{4k_{z0} |k_{z0} + \tau_z|} \). At \( \tau_z < 0 \) the difference \( k_{z0} + \tau_z \) can be reduced to make the sum on the right-side equal to \( \frac{\omega}{u} \), thus providing equality of all four dispersion equation roots.

Note that for backward diffraction, which is conventional for frequently used one-dimensional generators with a corrugated metal waveguide (traveling-wave tube, backward-wave oscillator), such coincidence of roots is impossible. Indeed, suppose the solutions \( \xi_1 \) and \( \xi_2 \) coincide in case of backward Bragg diffraction \( (|\tau_z| \approx 2k_{z0}, \tau_z < 0) \). Then by substituting the expressions for

\[ k_{z0} = \frac{\omega}{c} \sqrt{\varepsilon_0 - \frac{\varepsilon_{mn}^2 c^2}{\omega^2}} \text{ and } \varepsilon_0 = n_0^2 = 1 + \frac{\chi_{\tau}}{\chi_{-\tau}} (0) \]

and retaining the first-order infinitesimal terms, the relation

\[ \frac{\omega}{u} \approx k_{z0} + \frac{\omega^2}{c^2} \frac{\chi_{-\tau}}{2k_{z0}} \]

can be reduced to the form

\[ \frac{\omega}{u} \approx \frac{\omega}{c} \left( 1 - \frac{\frac{\chi_{\tau}}{\chi_{-\tau}} (0)}{2} - \frac{\varepsilon_{mn}^2 c^2}{2\omega^2} + \frac{\omega |\chi_{-\tau}|}{c \omega^2} \right) < \frac{\omega}{u}. \]
i.e., the equality does not hold and the four-fold degeneracy is impossible. Only the case of three-fold degeneration considered in subsection III A is possible. However, if \( \varepsilon_0 > 1 \) and is appreciably large, then in a one-dimensional case, the four-fold degeneracy of roots is also possible in a finite photonic crystal. Thus, the left-side of equation (35) has four solutions (\( \xi_1, \xi_2, \) and a double degenerated \( \xi_3 \)). Hence, equation (35) can be written as follows:

\[
(\xi - \xi_1) (\xi - \xi_2) (\xi - \xi_3)^2 = -B,
\]

where \( \xi_1, \xi_2 \) and \( \xi_3 \) are defined by (41) and (42), respectively, \( B \) is real, \( B > 0 \), \( B = \omega^2 L_4 k_0 \),

\[
B = \frac{\omega^2 L}{4k z_0 (k z_0 + \tau_2 ) u^2 \gamma c^4} \left\{ \frac{1}{2\pi} \int \bar{\underline{u}} Y_{mn} \left( \bar{k}_\perp, k_z \right) d^2 k_\perp \right\} \left( k_{mn}^2 c^2 - \omega^2 \right) \left( \left( \bar{k}_{mn} + \tau \right)^2 - \frac{\omega^2}{c^2} \varepsilon_0 \right).
\]

If suppose \( \xi_1 = \xi_2 = \xi_3 \), then equation (44) converts to

\[
(\xi - \xi_1)^4 = -B, \text{ i.e. } \xi - \xi_1 = \sqrt[4]{-B},
\]

thus defining four solutions as follows:

\[
\xi^{(1)}_1 = \frac{1}{\sqrt[4]{2}} (1 + i) \sqrt[4]{B}, \quad \xi^{(2)}_1 = \frac{1}{\sqrt[4]{2}} (-1 + i) \sqrt[4]{B}, \quad \xi^{(3)}_1 = -\frac{1}{\sqrt[4]{2}} (1 + i) \sqrt[4]{B}, \quad \xi^{(4)}_1 = -\frac{1}{\sqrt[4]{2}} (-1 + i) \sqrt[4]{B}.
\]

Degeneration of roots of dispersion equation puts the synchronism point to \( \pi \)-point of dispersion curve (see Fig. 2). In this point amplitudes of direct and diffracted waves are comparable to each other.

![Figure 2. Schematic drawing of dispersion curve \( \omega(k_z) \) in case, when roots of dispersion equation are equal.](image)

Solutions \( \xi^{(3)}_1 \) and \( \xi^{(4)}_1 \) have negative imaginary parts and cause exponential growth of field amplitude, thus they are responsible for beam instability. Increment of beam instability in case of four roots degeneracy:

\[
\delta_0^{(4)} \sim Im k_z = Im \xi^{(3,4)}_1 \sim \sqrt[4]{\omega_L} \sim \sqrt[4]{\rho_0}.
\]

To compare increments \( \delta_0^{(3)} \) and \( \delta_0^{(4)} \) let us consider case when solutions \( \xi_1 \neq \xi_2 \) and \( \xi_1 - \xi_2 \gg \chi \tau \) and study (41) for \( \xi \to \xi_1 \), thus reducing the order of equation:

\[
(\xi - \xi_1) (\xi - \xi_3)^2 = -\frac{B}{\xi_1 - \xi_2}.
\]

The obtained equation is conformable to (28). Hence, the ratio \( \frac{\delta_0^{(4)}}{\delta_0^{(3)}} \) can be expressed as:

\[
\frac{\delta_0^{(4)}}{\delta_0^{(3)}} = \sqrt[4]{\frac{2}{3}} \sqrt[4]{\xi_1 - \xi_2} \frac{1}{\sqrt[4]{B}}.
\]
For the sake of evaluations (45) can be replaced by the approximate expression for \( B \) as follows:

\[
B \approx \frac{\omega_\parallel^2}{\omega^2 \gamma} k_0 \chi_0^4,
\]

(51)

hence, evaluation for \( \delta_0^{(4)} \) and \( \delta_0^{(3)} \) read:

\[
\delta_0^{(4)} \approx \frac{1}{\sqrt{2}} \sqrt{k_0 \chi_0^4 \frac{\omega^2}{\omega^2 \gamma}} = \frac{1}{\sqrt{2}} \frac{k_0 \chi_0}{\sqrt{\gamma}} \sqrt{\frac{\omega_L}{\omega}}
\]

\[
\delta_0^{(3)} \approx \frac{\sqrt{3}}{2} \sqrt{k_0 \chi_0^3 \frac{\omega^2}{\omega^2 \gamma}} = \frac{\sqrt{3}}{2} \delta_0^{(4)} \gamma^{-\frac{1}{2}} \sqrt{\frac{\omega_L}{\omega}}
\]

and ratio \( \frac{\delta_0^{(4)}}{\delta_0^{(3)}} \) can be evaluated by

\[
\frac{\delta_0^{(4)}}{\delta_0^{(3)}} = \sqrt{\frac{2}{3}} \frac{\gamma^{\frac{1}{2}}}{\sqrt{\omega_L}}.
\]

(52)

For terahertz range even for a beam with current density \( j \sim 10^8 \, \text{A/cm}^2 \) (\( \omega_L \sim 10^8 \, \text{Hz} \)) expression (52) gives \( \frac{\delta_0^{(4)}}{\delta_0^{(3)}} \gg 1 \).

The typical length, at which instability develops, is inverse to the increment value, therefore, along with (52) the following relations are also valid:

\[
\frac{L^{(3)}}{L^{(4)}} = \frac{\delta_0^{(4)}}{\delta_0^{(3)}} = \sqrt{\frac{2}{3}} \frac{\gamma^{\frac{1}{2}}}{\sqrt{\omega_L}}, \quad \frac{L^{(3)}}{L^{(4)}} = \frac{1}{\delta_0^{(3)}} = \sqrt{\frac{2}{3}} \frac{\gamma^{\frac{1}{2}}}{k_0 \chi_0} \sqrt{\frac{\omega^2}{\omega_L^2}},
\]

(53)

(54)

providing evaluation for lengths ratio, at which instability develops, \( \frac{L^{(3)}}{L^{(4)}} \gg 1 \). Thus, generation in the vicinity of \( \pi \)-point (or, equally, in case, when amplitudes of direct and diffracted waves approximate to each other) gives a advantage of shorter length, at which instability develops.

Let us analyze what values \( L^{(3)} \) and \( L^{(4)} \) can possess at typical parameters of modern accelerators and radiation frequency 1 terahertz (\( \lambda = 3 \cdot 10^{-2} \, \text{cm} \)). Suppose electron beam energy 8 Mev, bunch transversal size 250 \( \mu \text{m} \) \( \times \) 250 \( \mu \text{m} \), bunch charge 25 pC [38]. Let us also consider a photonic crystal for modulated by parallel metallic threads, which are parallel to the waveguide boundary \((y, z)\). According to [3] for such a photonic crystal with threads of 10 \( -3 \) cm diameter spaced 6 \( \cdot \) 10 \( -2 \) cm the susceptibility value is \( \chi_0 \approx 3 \cdot 10^{-1} \). For selected parameters, according to (54), length \( L^{(4)} \approx 70 \, \text{cm} \), while \( L^{(3)} \) is more than 10 times larger (\( L^{(3)} \approx 900 \, \text{cm} \)). Increase of electron beam current density could make length \( L^{(4)} \) even smaller. The same could be provided by increase susceptibility value, but for \( \chi_0 > 1 \) detailed numerical analysis is necessary [33].

The threshold generation conditions, i.e., the values of the electron current and other parameters of the beam, at which radiation begins to exceed the losses, can be obtained by solving the boundary-value problem similar to how it was made in [10]. For instance the expression for the generation threshold under the conditions of two–wave diffraction in the case of cold beam reads:

\[
\frac{1}{4 \gamma_0 \omega^2 m} \frac{4 \pi e^2}{u} \left[ \frac{j_{thr}}{4} \int \frac{d^2 k_{\perp}}{2} f(y) = 16 \left( \frac{\gamma_0 c}{u B_\parallel} \right)^3 \frac{\beta_1}{k^3 \chi^2} L \right],
\]

(55)

where \( \gamma \) is the beam Lorentz factor; \( \vec{n} \) is the unperturbed velocity vector of the beam particles; \( \vec{n} \) is the unit normal vector to the crystal surface (directed toward the interior); \( L \) is the crystal thickness; \( \chi_0 \) and \( \chi_r \) are the Fourier expansion coefficients of the crystal dielectric susceptibility; \( \beta_1 = \gamma_1 / \gamma_0 \) is the diffraction asymmetry factor; \( \beta_1 = \frac{\gamma_1}{\gamma_0} = \frac{\vec{n} \cdot (\vec{k} + \vec{\ell})}{nk} \), \( \gamma_0 \) and \( \gamma_1 \) are the cosines of the angles between the normal vector \( \vec{n} \) and the wave vectors of the
transmitted $\vec{k}$ and diffracted $\vec{k} + \vec{\tau}$ waves, respectively; the subscript $\perp$ denotes the projection of the vector on the plane perpendicular to $\vec{u}$; $f(y)$ is the spectral function depending on detuning from the synchronism conditions defined in \[16\]:

$$f(y) = \sin y \frac{(2y + \pi n) \sin y - y(y + \pi n) \cos y}{y^2(y + \pi n)^3},$$

where $y = \frac{kRe(\xi_2)L}{2}$ and $\xi_2$ is the root of the dispersion equation in the absence of the electron beam. Expression \[15\] provides for $j_{thr}$ the dependance on the crystal length $L$ as follows: $j_{thr} \sim \frac{1}{L^3}$, which is different from that cited in subsection \[11A\] and \[13, 14\].

The analysis \[9, 16, 17\] shows that with increasing the number of diffracted waves, the law established in \[15, 40, 41\] is still valid: the instability increment appears to be proportional to $\rho^{-1}_{s_{\omega_0}}$, where $s$ is the number of waves emerging through diffraction. As a result, the abrupt decrease in the threshold generation current also remains in this case (the threshold generation current $j_{thr} \sim \frac{1}{(kL)^4(s_{\omega_0}-L)^2}$, where $L$ is the length of the interaction area).

IV. CONCLUSION

Combining the photonic crystal-based structures with vacuum electronic devices opens the way for creation of a family of radiation sources: volume FELs, photonic BWOs, etc.

The dispersion equations describing electron beam instability in a bounded photonic crystal are obtained for two cases: the conventionally considered case, when diffraction is suppressed, and the case when amplitude of diffracted wave is comparable with that of direct one that happens in the vicinity of $\pi$-point of dispersion curve. Such instability law enables application of photonic crystals for development of THz sources at electron beam current densities available at modern accelerators.

Some beneficial options are additionally available, namely: use for radiation generation of multiple either pencil-like or sheet electron beams instead of single annular or sheet one and, thus, establishing the beam-wave interaction within the whole crystal cross-section and increasing the efficiency of radiation source.

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