Trace anomaly and Hawking effect in 2D dilaton gravity theories

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We investigate the classical and semiclassical features of generic 2D, matter-coupled, dilaton gravity theories. In particular, we show that the mass, the temperature and the flux of Hawking radiation associated with 2D black holes are invariant under dilaton-dependent Weyl rescalings of the metric. The relationship between quantum anomalies and Hawking radiation is discussed.

1. INTRODUCTION

These notes summarize the content of two of our papers dealing with two-dimensional (2D) dilaton gravity theories [1,2]. Our presentation in Santa Margherita was essentially based on the first paper [1], whereas the content of the second one [2] was briefly sketched as work in progress. For sake of completeness it seems to us advisable to give a full account of the recent results of Ref. [2] in these notes.

2D dilaton gravity theories have become very popular in recent years. They represent simple toy models for studying 4D black hole physics and its related challenging issues such as the ultimate fate of black holes or the loss of quantum coherence in the evaporation process. Moreover, the relation with non-critical string theory and the renormalizability of gravity in two space-time dimensions give to these theories an intrinsic interest, which goes beyond black hole physics. Even though 2D dilaton gravity contains, as particular cases, models with different features (e.g. the Callan-Giddings-Harvey-Strominger (CGHS) model [3], the Jackiw-Teitelboim (JT) model [4]), a simple, unified description of the general theory still exists. In this paper we will discuss three crucial issues of 2D dilaton gravity, which, as we shall see, are deeply interconnected. These issues are the equivalence of 2D dilaton gravity models under Weyl rescalings of the metric, the existence of black hole solutions and the relationship between quantum anomalies and Hawking radiation.

This paper is structured as follows: In Sect. 2 we consider the generic, matter-coupled, classical 2D dilaton gravity theory. In particular, we study the behaviour of physical observables under Weyl transformations and the black hole solutions of the theory. In Sect. 3 we investigate the semiclassical theory. In Sect. 4 we discuss some particular cases of the generic model. In Sect. 5 we present our conclusions.

2. CLASSICAL 2D DILATON GRAVITY

The most general action of 2D dilaton gravity, conformally coupled to a set on N matter scalar fields has the form [3]

\[ S[g, \phi, f] = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ D(\phi) R(g) \right. \]
\[ + H(\phi)(\nabla \phi)^2 + \lambda^2 V(\phi) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \]  

(1)

where \( D, H, V \) are arbitrary functions of the dilaton \( \phi \) and \( \lambda \) is a constant. An important issue in the context of 2D dilaton gravity is the equivalence of models connected by conformal transformations of the metric. From a purely field theoretical point of view, performing dilaton-dependent Weyl rescalings of the metric in the 2D dilaton gravity action we should get equivalent models, since these transformations are nothing but reparametrizations of...
the field space. The space-time interpretation of this equivalence presents, however, some problems because geometrical objects such as the scalar curvature of the space-time or the equation for the geodesics are not invariant under Weyl transformations. Let us consider the following Weyl transformation of the metric

$$g_{\mu \nu} = e^{P(\phi)} \hat{g}_{\mu \nu},$$

(2)

where the function $P$ is constrained by requiring the transformation (2) to be non-singular and invertible. Whereas the matter part of the action (1) is invariant under the transformation (2), the gravitational part is not, but it maintains its form, with the functions $D, H, V$ transforming as ($' = d/d\phi$)

$$\hat{D} = D, \quad \hat{H} = H + D'P', \quad \hat{V} = e^P V.$$  

(3)

The transformation laws (2) and (3) enable us to find out how the physical parameters characterizing the solutions of the theory transform under the Weyl transformation (2). Following Mann (4), one can define the conserved quantity

$$M = \frac{1}{2\lambda} \int dV \exp \left( - \int d\tau \frac{H(\tau)}{D'(\tau)} \right) - \frac{1}{2\lambda} (\nabla D)^2 \exp \left( - \int d\tau \frac{H(\tau)}{D'(\tau)} \right).$$

(4)

$M$ is constant whenever the equations of motion are satisfied and, in this case, it can be interpreted as the mass of the solution. Using Eqs. (2), (3), one can easily demonstrate that the mass $M$ given by the expression (4) is invariant under Weyl transformations of the metric.

Choosing $P = -\int d\phi [H(\tau)/D'(\tau)],$ we can always achieve $\dot{H} = 0$. The generic static solutions in this conformal frame have already been found in Ref. (5).

$$ds^2 = \left( \ddot{J} - \frac{2M}{\lambda} \right) dt^2 + \left( \ddot{J} - \frac{2M}{\lambda} \right)^{-1} dr^2,$$

$$\dot{D}(\phi) = \lambda r,$$

(5)

where $d\dot{J}/d\dot{D} = \dot{V}$ and $M$ is the mass of the solution given by Eq. (1). If the equation $\dot{J} = 2M/\lambda$ has at least one solution $\phi = \phi_0$, with $\dot{J}$ monotonic, one is lead to interpret the solution as a black hole. However, a rigorous proof of the existence of black holes involves a detailed analysis of the global structure of the space-time. The dilaton $\phi$ gives a coordinate-independent notion of location and it can therefore be used to define the asymptotic region, the singularities and the event horizon of our 2D space-time. Moreover, the natural coupling constant of the theory is $D^{-1/2}$ so that we have a natural division of our space-time in a strong-coupling region ($D = 0$) and a weak-coupling region ($D = \infty$). These considerations enable us to identify the weak-coupling region $D = \infty$ as the asymptotic region of our space-time. This notion of location is Weyl-invariant because $D$ behaves as a scalar under the Weyl transformation (2). One can now write down a set of conditions that, if satisfied, makes the interpretation of the solution (3) as a black hole meaningful. One weakness of this kind of approach is that we need to use the scalar curvature $R$ to define the singularities and the asymptotic behaviour of the space-time. $R$ is not Weyl rescaling invariant and cannot be taken as a good quantity for a conformal invariant characterization of black holes. One has to perform the analysis in a particular conformal frame. In the conformal frame defined by Eq. (3) black holes exist provided the function $\dot{V}$ behaves asymptotically as

$$\dot{V} \sim D^\alpha, \quad -1 < \alpha \leq 1.$$  

(6)

A broader class of models whose static solutions can be interpreted as black holes can be obtained choosing a conformal frame in which the metric is asymptotically Minkowskian (4).

The Hawking temperature of the generic black hole solution of the action (1) is given by

$$T = \frac{\lambda}{4\pi} K(\phi_0),$$

(7)

where

$$K(\phi) = V(\phi) \exp \left( - \int d\tau \frac{H(\tau)}{D'(\tau)} \right).$$

(8)

The temperature is invariant under Weyl transformations. This can be easily checked using Eqs. (3) and taking into account that the transformations (3) do not change the position $\phi_0$ of the
event horizon. It is interesting to note that the mass \(\Theta\) and the temperature \(\Theta\) are invariant not only under Weyl transformations but also under reparametrizations of the dilaton field.

3. QUANTUM ANOMALIES AND HAWKING RADIATION

It is well-known that in quantizing the scalar matter fields \(f\) in a fixed background geometry the Weyl rescaling and/or part of the diffeomorphism invariance of the classical action for the matter fields has to be explicitly broken. If one decides to preserve diffeomorphism invariance the semiclassical action is given by the usual Liouville-Polyakov action, covariant, dilaton-dependent counterterms to the action but one has still the freedom of adding local action is given by the usual Liouville-Polyakov action, covariance, dilaton-dependent counterterms to the action but one has still the freedom of adding local counterterms. In the conformal frame where the metric is asymptotically Minkowskian, it is interesting to note that the asymptotic radiation rate is invariant under Weyl transformations (see Eq. (8)). Though the trace dependence of the particular conformal frame chosen.

The black hole radiation can now be studied working in the conformal gauge \(ds^2 = -e^{2\rho}dx^+dx^-\) and considering a black hole formed by collapse of a \(f\)-shock-wave, travelling in the \(x^+\) direction and described by a classical stress-energy tensor \(T_{++} = M\delta(x^+-x_0^+)\). The classical solution describing the collapse of the shock-wave, for \(x^+ \leq x_0^+\), is given by

\[
e^{2\rho} = K \exp \left( -\int_0^\tau d\tau \frac{H(\tau)}{D'(\tau)} \right),
\]

\[
\int_0^\tau \frac{d\tau}{K(\tau)} \frac{\lambda}{2} \left( x^+ - x^- \right),
\]

and, for \(x^+ \geq x_0^+\), it is given by

\[
e^{2\rho} = \exp \left( -\int_0^\tau d\tau \frac{H(\tau)}{D'(\tau)} \right) \left( K - \frac{2M}{\lambda} \right) F'(x^-),
\]

\[
F'(x^-) = \frac{dF}{dx^-} = \left( \frac{K}{K - \frac{2M}{\lambda}} \right) x^+=x_0^+,
\]

where \(K\) is given as in Eq. (9). The next step in our semiclassical calculation is to use the effective action \(\Theta\) to derive the expression for the quantum contributions of the matter to the stress-energy tensor. The flux of Hawking radiation across spatial infinity is given by \(< T_{--} >\) evaluated on the asymptotic \(D = \infty\) region. For the class of models in Eq. (9) and for the shock-wave solution described previously a straightforward calculation leads to

\[
<T_{--}>_{as} = \frac{N}{24(F')^2} \{F,x^-\},
\]

where \(\{F,x^-\}\) denotes the Schwarzian derivative of the function \(F(x^-)\). This is a Weyl rescaling and dilaton reparametrization invariant result for the Hawking flux. In fact the function \(F(x^-)\) is defined entirely in terms of the function \(K(\phi)\) (see Eq. (13)), which in turn is invariant under both transformations (see Eq. (9)). Though the trace anomaly is Weyl rescaling dependent, the Hawking radiation seen by an asymptotic observer is independent of the particular conformal frame chosen. When the horizon \(\phi_0\) is approached, the Hawking flux reaches the thermal value

\[
<T_{--}>_{as} = \frac{N}{12} \frac{\lambda^2}{16} |K(\phi_0)|^2,
\]

which is the result found in Ref. (10), written in a manifest Weyl rescaling and dilaton reparametrization invariant form.
4. PARTICULAR CASES

The general model (1) contains, as particular cases, models that have already been investigated in the literature. In this section we will show how previous results for the CGHS model and the JT theory can be obtained as particular cases of our formulation.

4.1. String inspired dilaton gravity

This is the most popular 2D dilaton gravity model. In its original derivation [3] the action has the form (1) with

\[ D = V = e^{-2\phi}, \quad H = 4e^{-2\phi}. \]

(14)

The model admits asymptotically flat black hole solutions. Using Eq. (7), (13) we find for the temperature and magnitude of the Hawking effect

\[ T = \frac{\lambda}{4\pi}, \quad <T_{--}>_\text{H} = \frac{N\Lambda^2}{12} \frac{1}{16}. \]

(15)

This result coincides, after the redefinition \( \lambda \to 2\lambda \) with the CGHS result [3]. All the 2D dilaton gravity models obtained from the CGHS model using Weyl transformation are characterized by the same values of mass, temperature and Hawking radiation rate. In particular, this is true for the model investigated in Ref. [9]. This model is characterized by \( H = 0 \) and its black hole solutions are described by a Rindler space-time. Also the models of Ref. [10] can be obtained from the CGHS model through a Weyl transformation of the metric [8]. The black hole solutions of these models have, therefore, the same values of mass, temperature and Hawking radiation rate as the CGHS black holes.

4.2. The Jackiw-Teitelboim theory

The JT theory is obtained from the action (1) by taking \( H = 0 \) and \( 2D = V = 2\exp(-2\phi) \). The model admits black hole solutions with asymptotic anti-de Sitter behaviour. More precisely, as shown in Ref. [11], the black hole space-time is obtained from a particular parametrization of 2D anti-de Sitter space-time endowed with a boundary. Eqs. (6) and (13) give now

\[ T = \frac{1}{2\pi \sqrt{2M\Lambda}}, \quad <T_{--}^\text{H}>_\text{as} = \frac{N}{24}M\Lambda. \]

The same result for the Hawking radiation rate has been obtained in Ref. [11] performing the canonical quantization of the scalar fields \( f \) in the anti-de Sitter background geometry.

5. CONCLUSIONS

In this paper we have discussed classical and semiclassical features of generic 2D dilaton gravity theories. We have shown that the usual relationship between conformal anomalies and Hawking radiation can be extended to a broad class of 2D dilaton gravity models. In particular, we have found a simple, general formula for the magnitude of the Hawking effect. We have also shown that physical observables associated with 2D black holes such as the mass, the temperature and the Hawking radiation rate are invariant both under Weyl transformations and dilaton reparametrizations. This implies a conformal equivalence between 2D dilaton gravity models. An important issue we have not discussed in this paper is the physical meaning of this equivalence. For a detailed discussion of this issue see Ref. [3].

REFERENCES

1. M. Cadoni, Phys. Rev. D 53 (1996) 4413.
2. M. Cadoni, hep-th/9610201.
3. C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. D 45 (1992) 1005.
4. R. Jackiw, in Quantum Theory of gravity, S.M. Christensen, ed. (Adam Hilger, Bristol, 1984); R. Jackiw, ibidem.
5. R. B. Mann, Phys. Rev. D 47 (1993) 4438.
6. D. Louis-Martinez and G. Kunstatter, Phys. Rev. D 49 (1994) 5227.
7. A. Strominger, Phys. Rev. D 46, 4396 (1992).
8. J.C. Russo, L. Susskind, L. Thorlacius, Phys. Rev. D 46 (1992) 3444.
9. M. Cadoni, S. Mignemi Phys. Lett. B 358 (1995) 217.
10. A. Fabbri, J.G. Russo, hep-th/9510109.
11. M. Cadoni and S. Mignemi, Phys. Rev. D 51 (1995) 4319.