On the weakness of short-range interactions in Fermi gases

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Received: 24 May 2022 / Revised: 14 October 2022 / Accepted: 7 December 2022 / Published online: 20 December 2022 © The Author(s) 2022

Abstract
Ultracold quantum gases of equal-spin fermions with short-range interactions are often considered free even in the presence of strongly binding spin-up–spin-down pairs. We describe a large class of many-particle Schrödinger operators with short-range pair interactions, where this approximation can be justified rigorously.

Keywords Fermi gas · Short-range interactions · Zero-range limit of many-particle Schrödinger operators · Contact interactions for fermions · Ultracold quantum gases

1 Introduction
Short-range interactions among equal-spin fermions in ultracold quantum gases are often neglected, while at the same time the interaction between particles of opposite spin is modeled by zero-range (i.e., contact) interactions [6, 10, 20]. This can be justified by the fact that zero-range interactions among spinless (or equal spin) fermions are prohibited by the Pauli principle (see Theorem 5.1), and by the recent approximation results for zero-range interactions in terms of short-range potentials [2, 11, 12]. In the present paper, we give a more direct analysis of the weakness of short-range interactions among spinless fermions in terms of estimates for the resolvent difference of free and interacting Hamiltonians. Our main results hold for all space dimensions $d \leq 3$.

We consider fermionic $N$-particle systems in the Hilbert space $\mathcal{H}_f = \wedge^N L^2(\mathbb{R}^d)$.
described by Schrödinger operators

\[ H_\varepsilon = -\Delta - \lambda_\varepsilon \sum_{i<j} V_\varepsilon(x_i - x_j), \]

(1)

where \( V_\varepsilon(r) = \varepsilon^{-2} V(r/\varepsilon), \) \( V(-r) = V(r), \) \( \lambda_\varepsilon > 0 \) and \( d \in \{1, 2, 3\}. \) We are primarily interested in the case, where the interaction strength of two distinct particles, in their center-of-mass frame described by

\[ -2\Delta - \lambda_\varepsilon V_\varepsilon, \quad \text{in } L^2(\mathbb{R}^d), \]

(2)

is independent of \( \varepsilon \) in the sense that (2) has a ground state energy \( E_\varepsilon \) that is fixed or convergent \( E_\varepsilon \to E \) with limit \( E < 0. \) It is well known in the spectral theory of Schrödinger operators what this means for \( \lambda_\varepsilon \) [17, 22]. In fact, \( \varepsilon \mapsto \lambda_\varepsilon \) and \( V \) can be chosen, depending on \( d, \) in such a way that the resolvent of (2)—and hence the spectrum of (2)—has a limit as \( \varepsilon \to 0. \) Since \( E < 0, \) the limit of (2) describes a non-trivial point interaction at the origin [1].

Our main result, Theorem 3.1, can be described in simplified form as follows. Assuming \( V \in L^1 \cap L^2(\mathbb{R}^d) \) with \( V \geq 0, \) \( C_V = \sup_{r \in \mathbb{R}^d} V(r)|r|^2 < \infty, \) some further decay of \( V \) in the case \( d = 1, \) and

\[ \limsup_{\varepsilon \to 0} \lambda_\varepsilon < \frac{d^2}{C_V N}, \]

(3)

we show that

\[ H_\varepsilon \to -\Delta \quad (\varepsilon \to 0) \]

(4)

in norm resolvent sense. The rate of convergence depends on the size of \( \lambda_\varepsilon \) and—to some extent—on the decay of \( V(x) \) as \( |x| \to \infty. \) Given sufficient decay of \( V, \) the regularity of \( H^2(\mathbb{R}^d)-\text{functions}, \) and hence the dimension \( d, \) begins to play a role. If we choose \( \lambda_\varepsilon \) as described above, where \( E_\varepsilon \to E < 0, \) then condition (3) is satisfied for all \( N \) if \( d \leq 2, \) and for some \( N \geq 3 \) if \( d = 3. \) Surprisingly, this particular choice of \( \lambda_\varepsilon \) conspires with the regularity of Sobolev functions in such a way that

\[ \|(H_\varepsilon + z)^{-1} - (-\Delta + z)^{-1}\| = O(\varepsilon^2) \quad (\varepsilon \to 0) \]

(5)

with the bound \( O(\varepsilon^2) \) independent of the space dimension \( d. \) Another choice for the coupling constant, consistent with (3) as well, is the one where \( \lambda_\varepsilon \) is a positive constant smaller than \( d^2/C_V N. \) Then, operator (2), upon a rescaling, is proportional to \( \varepsilon^{-2} \) and hence the (negative) binding energy \( E_\varepsilon \) diverges. In this case, the limit \( \varepsilon \to 0 \) amounts to a combined short-range and strong interaction limit, which is interesting and relevant physically [20].

In summary, we can say that fully spin-polarized Fermi gases in \( d \leq 2 \) with short-range interactions—the spin-up–spin-down interaction strength being fixed—are asymptotically free in the limit of zero-range interaction. This is true in dimension...
\( d = 3 \) as well for suitable \( V \) and small \( N \geq 3 \), depending on \( V \). The result remains correct even in a suitable combined limit of short-range and strong interaction.

We conclude with some remarks on the literature: For single particles, the approximation of point-like disturbances by short-range potentials is discussed at length and in rich detail in [1], see also [5]. For systems of \( N \geq 3 \) particles in \( d \leq 2 \) dimensions, it was recently shown in a series of papers that contact interactions (of TMS-type) can be approximated by rescaled two-body potentials in the norm resolvent sense of \( N \)-particle Hamiltonians \([2, 11–14]\). From these results, the mere convergence \((4)\) can be derived by a reduction in the Hilbert space to antisymmetric wave functions. This works for the very special choice of \( \lambda_\varepsilon \) needed for the approximation of contact interactions, and for \( d \leq 2 \), only. For \( d = 3 \), systems of \( N \geq 3 \) distinct (or bosonic) particles with two-body short-range interactions are prone to suffer collapse, a phenomenon known as Thomas effect. See Proposition 4.1. In order to avoid this effect, suitable many-body forces are required \([3, 8, 9]\).

This work is organized as follows. In Sect. 2, we present an explicit estimate of the norm of \((H_{\varepsilon=1} + z)^{-1} - (-\Delta + z)^{-1}\) in terms of the pair potential \( V \). This estimate is then used in Sect. 3 to prove \((4)\) and \((5)\). The proofs benefit from the methods and tools developed in \([11, 12]\). Section 4 gives examples illustrating our results, in particular for \( d = 3 \) and \( N = 3 \). Finally, in Sect. 5, we prove the impossibility of fermionic contact interaction in \( d \geq 2 \). This improves, for fermions, a well-known result about the impossibility of contact interactions in dimensions \( d \geq 4 \) \([23]\).

2 The resolvent difference

Let \( H_0 = -\Delta \) in \( \mathcal{H}_f \) and let \( R_0(z) = (H_0 + z)^{-1} \). We consider \( N \)-body Hamiltonians \( H = H_0 - W \) in \( \mathcal{H}_f \) with

\[
W = \sum_{i < j} V_{ij},
\]

where \( V_{ij} \) denotes multiplication with \( V(x_i - x_j) \) and \( V \in L^1 \cap L^2(\mathbb{R}^d) \) is real-valued and even. There is no scaling parameter and no coupling constant. We assume \( d \leq 3 \) and hence \( H \) is self-adjoint on \( D(H) = D(H_0) \) \([21]\). The result of this section, Proposition 2.1, is based on a suitable factorization \( W = A^*B \) and an iterated (second) resolvent identity related to the Konno–Kuroda formula \([18]\). We start by constructing the factorization.

Let

\[
\mathcal{X}_t := L^2_{\text{odd}}(\mathbb{R}^d, \, dr) \otimes L^2(\mathbb{R}^d, \, dR) \otimes \bigwedge_{i=3}^N L^2(\mathbb{R}^d, \, dx_i),
\]

where \( L^2_{\text{odd}} \) denotes the subspace of odd functions from \( L^2 \). The integration variables \( r \) and \( R \) in \((6)\) correspond to the relative and center-of-mass coordinates of the fermion positions \( x_1 \) and \( x_2 \). This change of coordinates is implemented isometrically by the
operator $\mathcal{H} : \mathcal{H}^\ell \to \mathcal{X}_f$ given by

$$\left( \mathcal{H} \psi \right) (r, R, x_3, ..., x_N) := \psi \left( R - \frac{r}{2}, R + \frac{r}{2}, x_3, ..., x_N \right). \quad (7)$$

From $\langle \varphi, V_{ij} \psi \rangle = \langle \varphi, V_{12} \psi \rangle$ and from $\langle \varphi, V_{12} \psi \rangle = \langle \mathcal{H} \varphi, \mathcal{H} V_{12} \psi \rangle = \langle \mathcal{H} \varphi, (V \otimes 1) \mathcal{H} \psi \rangle$, it follows that

$$W = \left( \frac{N}{2} \right)^{-1} \mathcal{H}^* (V \otimes 1) \mathcal{H}. \quad (8)$$

We therefore write $V = vu$ with

$$v(r) := |V(r)|^{1/2}, \quad u(r) := J |V(r)|^{1/2}, \quad J := \text{sgn}(V),$$

and we set

$$A := \left( \frac{N}{2} \right)^{1/2} (v \otimes 1) \mathcal{H}, \quad \quad (9)$$

$$B := \left( \frac{N}{2} \right)^{1/2} (u \otimes 1) \mathcal{H} = JA. \quad \quad (10)$$

Recall that $V \in L^1(\mathbb{R}^d)$ and hence $u, v \in L^2(\mathbb{R}^d)$. The domain $D(A)$ of $A : D(A) \subset \mathcal{H}^\ell \to \mathcal{X}_f$ is determined by the domain of the multiplication operator $v \otimes 1$, so it follows that $A$ and $B$ are densely defined and closed on $D(A) \supset D(H_0)$. Hence, $H$ can be rewritten in the form

$$H = H_0 - A^* B. \quad \quad (11)$$

By an iteration of the resolvent identity, we find

$$(H + z)^{-1} = R_0(z) + R_0(z)WR_0(z) + R_0(z)W(H + z)^{-1}WR_0(z). \quad \quad (12)$$

Upon setting $W = A^* B$, Identity (12) can be written in the form

$$(H + z)^{-1} = R_0(z) + (AR_0(z))^* S(z) BR_0(z), \quad \quad (13)$$

$$S(z) = 1 + B(H + z)^{-1} A^*. \quad \quad (14)$$

The following proposition is our tool for proving norm resolvent convergence in the next section.

**Proposition 2.1** Suppose there exists $\delta > 0$ such that $H_0 - (1 + \delta) W \geq 0$ and suppose that $V \geq 0$, $V \leq 0$, or that $\sum_{i<j} |V_{ij}| \leq CH_0$ with some $C > 0$. Then, $H \geq 0$ and,
for all \( z > 0 \),

\[
\|(H + z)^{-1} - (H_0 + z)^{-1}\| \leq C_\delta \left( \begin{array}{c} N \\ 2 \end{array} \right) \|v(-\Delta + z)^{-1}\|^2_{\text{odd}},
\]

where \( v = |V|^{1/2} \), and \( C_\delta \) is a function of \( C \) and \( \delta \). Here, \( \|\cdot\|_{\text{odd}} \) denotes the operator norm in \( L^2_{\text{odd}}(\mathbb{R}^d, dr) \).

**Proof** Suppose, temporarily, that \( H \geq 0 \) and let \( z > 0 \). Then, from (13), the definition of \( A \), and from \( \|(\Delta_r \otimes 1 + z)\| \leq 1 \), it follows that

\[
\|(H + z)^{-1} - R_0(z)\| \leq \left( \begin{array}{c} N \\ 2 \end{array} \right) \|v(-\Delta + z)^{-1}\|^2_{\text{odd}}\||S(z)||.
\]

It remains to prove \( H \geq 0 \) and \( \|S(z)\| \leq C_\delta \) under the various assumptions on \( V \).

In the case \( V \leq 0 \), we have \( J = -1 \), \( B = -A \), and hence, \( H = H_0 + A^*A \geq A^*A \geq 0 \). By Lemma 2.2, it follows that \( \|S(z)\| \leq 2 \).

In the case \( V \geq 0 \), we have \( J = 1 \), \( B = A \), and hence the assumption \( H_0 - (1 + \delta)W \geq 0 \) implies that

\[
H \geq \delta W = \delta A^*A.
\]

By Lemma 2.2, it follows that \( A(H + \mu)^{-1}A^* \leq 1/\delta \), and hence \( \|S(z)\| \leq 1 + \delta^{-1} \), by (14).

It remains to consider the case where \( V \) changes sign. The assumption \( H_0 - (1 + \delta)W \geq 0 \) implies that

\[
H = \frac{\delta}{1 + \delta} H_0 + \frac{1}{1 + \delta} (H_0 - (1 + \delta)W) \geq \frac{\delta}{1 + \delta} H_0.
\]

Combining this with the assumption on \( |V| \), that is, with \( CH_0 \geq \sum_{i < j} |V_{ij}| = A^*A \), we find that

\[
H \geq \frac{\delta}{1 + \delta} \frac{1}{C} A^*A.
\]

This, by Lemma 2.2, implies \( A(H + \mu)^{-1}A^* \leq C(1 + \delta)/\delta \) and the desired bound on \( \|S(z)\| \) follows from (14). \( \square \)

**Lemma 2.2** Let \( H \), \( A \) be any two closed operators in a Hilbert space, with \( H^* = H \geq 0 \) and \( D(A) \supseteq D(H) \). If \( H \geq \lambda A^*A \) for some \( \lambda > 0 \), then \( D(A) \supseteq D(H^{1/2}) \) and for all \( \mu > 0 \),

\[
A(H + \mu)^{-1}A^* \leq \frac{1}{\lambda}.
\]
Proof The assumption means that \( \| (H + \mu)^{1/2} \psi \|^2 \geq \lambda \| A \psi \|^2 \) for \( \mu > 0 \) and all \( \psi \in D(H) \). By an approximation argument, this inequality extends to all \( \psi \in D(H^{1/2}) \), and \( D(A) \supset D(H^{1/2}) \) follows from the closedness of \( A \). Upon rewriting the inequality in terms of \( \varphi = (H + \mu)^{1/2} \psi \in \mathcal{H}_f \), the assertion follows. \(\square\)

3 The resolvent convergence

We now apply the results from the previous section to Schrödinger operators with rescaled two-body potentials, that is

\[
H_\varepsilon = -\Delta - \lambda_\varepsilon \sum_{i < j} V_{\varepsilon,ij},
\]

where \( \lambda_\varepsilon > 0 \) and \( V_\varepsilon(r) = \varepsilon^{-2} V(r/\varepsilon) \). The Schrödinger operator \( \varepsilon^2 H_\varepsilon \) is unitarily equivalent to \( S_\lambda := -\Delta - \lambda \sum_{i < j} V_{ij} \) with \( \lambda = \lambda_\varepsilon \). We therefore define

\[
\lambda_{\text{max}} := \sup \{ \lambda \geq 0 \mid S_\lambda \geq 0 \}.
\]

Note that \( S_\lambda \geq 0 \) for all \( \lambda \leq \lambda_{\text{max}} \). This follows from the fact that \( E_\lambda := \inf \sigma(S_\lambda) \) as a function of \( \lambda \) is concave (hence continuous) and \( E_0 = 0 \).

For Theorem 3.1 to be non-void, we need that \( \lambda_{\text{max}} > 0 \). This can be achieved, e.g., by assuming that

\[
C_V := \sup_{x \in \mathbb{R}^d} V(x)|x|^2 < \infty.
\]

Then, \( \lambda_{\text{max}} \geq d^2/(C_V N) > 0 \), by the Hardy inequality for fermionic wave functions \( \psi \in \mathcal{H}_f \) [15],

\[
\sum_{i < j} \int \frac{|\psi(x_1, \ldots, x_N)|^2}{|x_i - x_j|^2} \, dx \leq \frac{N}{d^2} \|
\nabla \psi \|^2.
\]

Statement as well as proof of Theorem 3.1 depends on the regularity of \( H^2 \)-Sobolev functions. Explicitly, we use the embedding \( H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d) \), valid for \( s \in I_d \), where \( I_1 = [0, 1] \), \( I_2 = [0, 1) \) and \( I_3 = [0, 1/2) \), and we use Lemma 3.2, which improves the embedding in the case \( d = 2 \). Here, \( C^{0,s}(\mathbb{R}^d) \) denotes the space of continuous functions that are uniformly Hölder continuous of exponent \( s \).

**Theorem 3.1** Suppose that \( V \geq 0, V \leq 0 \), or that \( \sup_{x \in \mathbb{R}^d} |V(x)||x|^2 < \infty \). If \( \lambda_0 := \limsup_{\varepsilon \to 0} \lambda_\varepsilon < \lambda_{\text{max}} \), then \( H_\varepsilon \geq 0 \) for \( \varepsilon \) small enough, and for all \( z > 0 \),

\[
\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = o(\lambda_\varepsilon \varepsilon^{d-2}) \quad (\varepsilon \to 0).
\]

Moreover, the following is true:
(a) If \( \int |V(r)||r|^{2s} \, dr < \infty \) for some \( s \in I_d \), then
\[
\| (H_\epsilon + z)^{-1} - (H_0 + z)^{-1} \| = O(\lambda_\epsilon \varepsilon^{d-2+2s}) \quad (\varepsilon \to 0).
\]
(b) If \( d = 2 \) and \( \int |V(r)||r|^2 \log |r| \, dr < \infty \), then
\[
\| (H_\epsilon + z)^{-1} - (H_0 + z)^{-1} \| = O(\lambda_\epsilon \varepsilon^d \log \varepsilon) \quad (\varepsilon \to 0).
\]

In the situation described in the Introduction, where \( \inf \sigma(-2\Delta - \lambda_\epsilon V_\epsilon) \) has a limit \( E < 0 \), the bounds (a) and (b) reveal a surprising interplay between \( \lambda_\epsilon \) and the (optimal) regularity of \( H^2(\mathbb{R}^d) \)-functions: if, depending on \( d \), we choose \( \lambda_\epsilon = O(\varepsilon) \), \( \lambda_\epsilon = O(1/|\log \varepsilon|) \), and \( \lambda_\epsilon = O(1) \) for \( d = 1, d = 2, \) and \( d = 3 \), respectively, then, for all \( d \in \{1, 2, 3\} \),
\[
\| (H_\epsilon + z)^{-1} - (H_0 + z)^{-1} \| = O(\varepsilon^2), \quad (\varepsilon \to 0)
\]
provided \( V \) decays fast enough, e.g., as in the hypothesis of (b), and \( N \) is small if \( d = 3 \).

**Remarks**

1. Part (a) of the theorem shows that, for a large class of potentials,
\[
H_\epsilon \to H_0 \quad (\varepsilon \to 0)
\]
in the norm resolvent sense, provided that \( \lambda_\epsilon \varepsilon^{d-2+2s} \to 0 \) as \( \varepsilon \to 0 \). This is not true for the Hamiltonians \( \tilde{H}_\epsilon \) defined by (15) on the enlarged Hilbert space \( L^2(\mathbb{R}^{Nd}) \), as shown in the next section.

2. For the convergence (19) in norm resolvent sense to hold, it is necessary that \( \inf \sigma(H_\epsilon) \to \inf \sigma(H_0) = 0 \). Therefore, the assumption \( \limsup \lambda_\epsilon \to 0 \) in Theorem 3.1 cannot be relaxed significantly. Strong resolvent convergence, by contrast, has much weaker spectral implications and hence—given some decay of \( V \)—much less is needed of \( \lambda_\epsilon \), see Proposition 3.3.

3. A weaker result, similar to Theorem 3.1, could be derived from [11, 12]. Indeed, for suitable \( \lambda_\epsilon \) we know from [11, 12] that \( H_\epsilon \to H \) in norm resolvent sense, where \( H = -\Delta \) on \( \mathcal{H}_f \). Information on the rate of convergence can also be found in these papers.

**Proof** We are going to apply Proposition 2.1 to the Hamiltonian (15), and we assume that \( V \) changes sign, the other cases being easier. Due to the unitary equivalence of \( \varepsilon^2 H_\epsilon \) and \( S_{\lambda_\epsilon} \), the hypotheses of Proposition 2.1 are equivalent to
\[
H_0 - (1 + \delta)\lambda_\epsilon \sum_{i<j} V_{ij} \geq 0 \quad (20)
\]
\[
C H_0 - \lambda_\epsilon \sum_{i<j} |V_{ij}| \geq 0 \quad (21)
\]
for some \( \delta, C > 0 \). Both (20) and (21) are true for \( \varepsilon \) small enough. This follows from \( \lambda_0 = \limsup_{\varepsilon \to 0} \lambda_\varepsilon < \lambda_{\max} \), from \( \sup_{x \in \mathbb{R}^d} |V(x)||x|^2 < \infty \), and from the Hardy inequality for fermions (17). Hence, by Proposition 2.1, for \( \varepsilon > 0 \) small enough,

\[
\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| \leq C_\delta \left( \frac{N}{2} \right) \lambda_\varepsilon \varepsilon^{d-2} \|v_\varepsilon(-\Delta + z)^{-1}\|_{\text{odd}},
\]

where \( v_\varepsilon(x) := \varepsilon^{-d/2}|V(x/\varepsilon)|^{1/2} \).

By the Sobolev embedding \( H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d) \), valid for \( s \in I_d \), the elements \( \psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d) \) are Hölder continuous (of exponent \( s \)) odd functions. It follows that \( \psi(0) = 0 \) and that

\[
|\psi(x)| = |\psi(x) - \psi(0)| \leq C_s \|\psi\|_{H^2}|x|^s.
\]

Therefore, for all \( \psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d) \),

\[
\|v_\varepsilon \psi\|^2 \leq C_s^2 \|\psi\|^2_{H^2} \int |v_\varepsilon(x)|^2 |x|^{2s} \, dx
= \varepsilon^{2s} C_s^2 \|\psi\|^2_{H^2} \int |V(x)||x|^{2s} \, dx.
\]

This is true for all \( s \in I_d \) and, combined with (22), it proves statement (a) of the theorem. To prove (b), we use Lemma 3.2 in (23) (rather than \( H^2 \hookrightarrow C^{0,s} \)) and then (24) becomes \( C \varepsilon^2 \log \varepsilon \int |V(x)||x|^2(1 + |\log |x||) \, dx \), where the integral is finite by the assumptions on \( V \).

It remains to prove (18). Equation (24) with \( s = 0 \) implies that \( \|v_\varepsilon(-\Delta + z)^{-1}\|_{\text{odd}} = O(1) \), which can be improved as follows: let \( \chi_k \) denote the characteristic function of the ball \( |x| \leq k \) in \( \mathbb{R}^d \) and let \( (v_\varepsilon \chi_k)_\varepsilon = v_\varepsilon \chi_{k_\varepsilon} \). Then, by (24),

\[
\|(v\chi_k)_\varepsilon(-\Delta + z)^{-1}\|_{\text{odd}}^2 = O(\varepsilon^{2s}) = o(1) \quad (\varepsilon \to 0)
\]

for any \( s > 0 \) in \( I_d \). On the other hand,

\[
\|(v - v\chi_k)_\varepsilon(-\Delta + z)^{-1}\|_{HS} \leq C \|v - v\chi_k\| = o(1) \quad (k \to \infty)
\]

uniformly in \( \varepsilon > 0 \), where \( HS \) refers to Hilbert–Schmidt norm. The combination of (22), (25) and (26) proves (18) and concludes the proof of the theorem. \( \square \)

In the proof of Theorem 3.1, we have used the following lemma, which can probably be found in the literature, but we are not aware of suitable reference.

**Lemma 3.2** For all \( u \in H^2(\mathbb{R}^2) \) and all \( x, y \in \mathbb{R}^2 \), \( y \neq 0 \), we have

\[
|u(x + y) - u(x)| \leq \frac{1}{2\sqrt{\pi}} |y|(2 + |\log |y||)^{1/2} \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right)^{1/2}.
\]
Remark  By our method of proof, this inequality can be generalized to derivatives $\partial^\alpha u$, $|\alpha| \leq k$, of functions $u \in H^s(\mathbb{R}^n)$ with $s - (n/2) = k + 1$, $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$.

Proof  We first note that $u \in H^2(\mathbb{R}^2)$ implies that $\hat{u} \in L^1(\mathbb{R}^2)$, and hence, for all $x \in \mathbb{R}^2$,

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{u}(p) \exp(ipx) \, dp.$$ 

Therefore, by Cauchy–Schwarz,

$$2\pi \left| u(x+y) - u(x) \right| \frac{1}{|y|} \leq \int_{\mathbb{R}^2} \left| \frac{\exp(ipy) - 1}{|y|} \right| |\hat{u}(p)| \, dp \leq I(y)^{1/2} \left( \| \Delta u \|^2 + \| \nabla u \|^2 \right)^{1/2}, \quad (27)$$

where

$$I(y) := \int_{\mathbb{R}^2} \left| \frac{\exp(ipy) - 1}{|y|^2} \right| \frac{1}{|p|^4 + |p|^2} \, dp.$$ 

To estimate the integral $I(y)$, we may assume that $y = (|y|, 0)$. Then, upon the substitution $q = p|y|$, we find that, for any $Q > 0$,

$$I(y) = \int_{\mathbb{R}^2} \left| \frac{\exp(iq_1) - 1}{|q|^2 + |y|^2} \right|^2 \, dq \leq \int_{|q| \leq Q} \frac{q_1^2}{|q|^4 + |y|^2 |q|^2} \, dq + \int_{|q| > Q} \frac{4}{|q|^4} \, dq, \quad (28)$$

where $|\exp(iq_1) - 1| \leq |q_1|$ and $|\exp(iq_1) - 1| \leq 2$ was used, respectively. Both integrals on the right of (28) can be computed explicitly. For the first one, we obtain

$$\int_{|q| \leq Q} \frac{q_1^2}{|q|^4 + |y|^2 |q|^2} \, dq = \frac{1}{2} \int_{|q| \leq Q} \frac{1}{|q|^2 + |y|^2} \, dq = \frac{\pi}{2} \log \left( 1 + \frac{Q^2}{|y|^2} \right) \leq \pi \left( |\log |y|| + |\log Q| + \frac{1}{2} \log(2) \right), \quad (29)$$

where the inequality follows from $\log(1 + t) \leq |\log t| + \log 2$, valid for all $t > 0$. The second integral on the right side of (28) equals $4\pi/Q^2$. Choosing $Q = 2\sqrt{2}$, we find from (28) and (29) that

$$I(y) \leq \pi ( |\log |y|| + c)$$

with $c = \frac{1}{2} + \log 4 < 2$. This concludes the proof. \qed
We conclude this section with the proposition announced in Remark 2. It is a consequence of Theorem 5.1 concerning the essential self-adjointness of the Laplacian. In all the following $\Omega = \mathbb{R}^{Nd} \setminus \Gamma$, where

$$
\Gamma := \bigcup_{i < j} \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \mid x_i = x_j \},
$$

Furthermore, $C^\infty_0(\Omega) := \{ \psi \in C^\infty(\mathbb{R}^{Nd}) \mid \text{supp} \psi \subset \Omega \}$.

**Proposition 3.3** Let $d \geq 2$ and suppose that $V \in L^2(\mathbb{R}^d)$. Suppose there exists $s \geq 0$ such that $\int V(r)^2 |r|^{2s} \, dr < \infty$ and $\lim_{\varepsilon \to 0} \lambda_\varepsilon \varepsilon^{s+d/2-2} < \infty$. Then, $H_\varepsilon \to H_0$ in the strong resolvent sense as $\varepsilon \to 0$.

**Proof** In view of $\|(H_\varepsilon + i)^{-1}\| = 1 = \|R_0(i)\|$ it suffices to prove that $(H_\varepsilon + i)^{-1}\psi \to R_0(i)\psi$ for $\psi$ from a dense subset of $\mathcal{H}_f$. By Theorem 5.1, the set of all $\psi = (H_0 + i)\varphi$ with $\varphi \in C^\infty_0(\Omega) \cap \mathcal{H}_f$ is dense in $\mathcal{H}_f$, and for such $\psi$,

$$
\|(H_\varepsilon + i)^{-1}\psi - R_0(i)\psi\| \leq \|\lambda_\varepsilon \sum_{i < j} V_{\varepsilon,ij}\varphi\| \leq \lambda_\varepsilon \left(\frac{N}{2}\right) \|V_{\varepsilon,12}\|,
$$

where the (anti-)symmetry of $\varphi$ was used in the second inequality. Let $\tilde{\varphi} := \mathcal{K}\varphi$, that is

$$
\tilde{\varphi}(r, R, x') = \varphi(R - r/2, R + r/2, x'),
$$

where $x' = (x_3, \ldots, x_N)$. Like $\varphi$, $\tilde{\varphi}$ is compactly supported and hence $\text{supp} \tilde{\varphi} \subset \mathbb{R}^{d} \times B_{N-1}$ for some ball $B_{N-1} \subset \mathbb{R}^{d(N-1)}$. It follows that, for any $c > 0$,

$$
e^{4-d} \|V_{\varepsilon,12}\|^2 = \int_{B_{N-1}} dR \, dx' \int_{|r| \leq c} |V(r)|^2 \left|\tilde{\varphi}(\varepsilon r, R, x')\right|^2 \, dr
$$

$$
+ \int_{B_{N-1}} dR \, dx' \int_{|r| > c} |V(r)|^2 \left|\tilde{\varphi}(\varepsilon r, R, x')\right|^2 \, dr. \tag{31}
$$

By assumption on $\varphi$, $\tilde{\varphi}(r, R, x') = 0$ for $r < \text{dist}(\text{supp} \varphi, \Gamma)$. This means that the first summand vanishes for $\varepsilon c < \text{dist}(\text{supp} \varphi, \Gamma)$ and that

$$
\left|\tilde{\varphi}(r, R, x')\right| \leq C(\tilde{\varphi}, s)|r|^s
$$

for each $s \geq 0$. It follows that

$$
\lambda_\varepsilon \|V_{\varepsilon,12}\| \leq \lambda_\varepsilon \varepsilon^{s+d/2-2} C(\tilde{\varphi}, s) \left|B_{N-1}\right|^{1/2} \left(\int_{|r| > c} |V(r)|^2 \, dr\right)^{1/2},
$$
where the integral can be made arbitrarily small by choosing $c$ large. By assumption, \( \limsup_{\varepsilon \to 0} \lambda_\varepsilon \varepsilon^{d/2 - 2} < \infty \); hence, it follows that \( \limsup_{\varepsilon \to 0} \lambda_\varepsilon \| V_\varepsilon 12 \varphi \| = 0 \), and the proof is complete.

\[ \square \]

4 Examples and discussion

To put Theorem 3.1 into a broader perspective and to demonstrate its dependence on the Pauli principle, we now view $H_\varepsilon$ as the restriction $H_\varepsilon = \tilde{H}_\varepsilon |_{\mathcal{H}_f}$, where $\tilde{H}_\varepsilon$ denotes the Schrödinger operator defined by expression (15) on the enlarged Hilbert space $L^2(\mathbb{R}^{Nd})$. We shall give choices for $\lambda_\varepsilon$ and $V$, where $\tilde{H}_\varepsilon$, in contrast to $H_\varepsilon$, has a limit $\tilde{H}$ describing non-trivial contact interactions or no limit at all.

In the cases $d = 1$ and $d = 2$ we choose, for simplicity, a two-body potential $V \in L^\infty(\mathbb{R}^d)$ with compact support and $\int V(r) \, dr = 1$. Suppose further that

\[
\lambda_\varepsilon = g \varepsilon > 0 \quad \text{if } d = 1,
\]

\[
\lambda_\varepsilon^{-1} = \frac{|\log(\varepsilon)|}{4\pi} + a \quad \text{if } d = 2.
\]

Then, $\lambda_{\text{max}} > 0$ and $\lambda_0 = \limsup_{\varepsilon \to 0} \lambda_\varepsilon = 0$. So, the hypotheses of Theorem 3.1 are satisfied and hence $H_\varepsilon \to H_0$ in norm resolvent sense. On the other hand, by [11, 12], $\tilde{H}_\varepsilon \to \tilde{H}$, where $\tilde{H}$ describes non-trivial contact interactions. That is, $\tilde{H}$ is a self-adjoint extension of $-\Delta |C_0^\infty(\mathbb{R}^{Nd} \setminus \Gamma)$ distinct from $-\Delta$. See (30) for the definition of $\Gamma$.

We now turn to the more interesting case of $N$ particles in $d = 3$ dimensions. In the following, $N$ is exhibited in the notation: we write $H_{N,\varepsilon}$ for $H_\varepsilon$ and $\tilde{H}_{N,\varepsilon}$ for $\tilde{H}_\varepsilon$. For the coupling constant and the two-body potential, we choose $\lambda_\varepsilon = 2$ and

\[
V(r) := \begin{cases} 
2/|r| - 1 & \text{if } |r| \leq 1 \\
0 & \text{if } |r| > 1.
\end{cases} \tag{32}
\]

Then, $0 \leq V(r) \leq |r|^{-2}$ and hence $C_V = \sup V(r)|r|^2 \leq 1$. It follows that

\[
\lambda_0 = \limsup_{\varepsilon \to 0} \lambda_\varepsilon = 2
\]

\[
\lambda_{\text{max}} \geq \frac{d^2}{N C_V} \geq \frac{9}{N}.
\]

Thus, for $N \leq 4$ we have $\lambda_0 < \lambda_{\text{max}}$ and hence, by Theorem 3.1, $H_{N,\varepsilon} \to H_0$ in norm resolvent sense. On the other hand, concerning $\tilde{H}_{N,\varepsilon}$ the following can be said:

**Proposition 4.1** With the above notations, in the case $d = 3$ we have
(a) For $N = 2$, $\tilde{H}_{2,\varepsilon} \rightarrow \tilde{H}_2$ in norm resolvent sense, where $\tilde{H}_2$ is a non-trivial self-adjoint extension of $-\Delta |C_0^\infty(\mathbb{R}^6 \setminus \Gamma)$.

(b) For each $N \geq 3$, there exists a constant $C_N < 0$ such that

$$\sigma(\tilde{H}_{N,\varepsilon}) = [C_N \varepsilon^{-2}, \infty).$$

**Remark** The divergence of the ground state energy established in Part (b) is known as Thomas effect [24].

**Proof** With respect to center-of-mass and relative coordinates $R = (x_1 + x_2)/2$ and $r = x_2 - x_1$, the Schrödinger operator for $N = 2$ takes the form

$$\tilde{H}_{2,\varepsilon} = -\Delta_R/2 \otimes 1 + 1 \otimes h_\varepsilon,$$

$$h_\varepsilon = -2\Delta_r - \lambda_\varepsilon V_\varepsilon.$$ 

By construction of $V$, $h = -\Delta - V \geq 0$ and $z = 0$ is not an eigenvalue but a resonance energy. This means that the Birman–Schwinger operator $V^{1/2}(-\Delta)^{-1}1^{1/2}$ has the (simple) eigenvalue 1, but the corresponding solution $\psi$ of $(-\Delta - V)\psi = 0$ fails to be square-integrable. Explicitly, in the present case, $\psi(x) = e^{-|x|}$ for $|x| \leq 1$ and $\psi(x) = e^{-1/|x|}$ for $|x| > 1$. These properties of $V$ imply that $h_\varepsilon \rightarrow -2\Delta_0$ in norm resolvent sense, as $\varepsilon \rightarrow 0$, where $\Delta_0$ denotes a self-adjoint extension of $\Delta |C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ that is distinct from the free Laplacian [1]. Since $-\Delta_R/2 \geq 0$, it follows that $\tilde{H}_{2,\varepsilon} \rightarrow -\Delta_R/2 \otimes 1 + 1 \otimes (-2\Delta_0)$ in norm resolvent sense, which proves assertion (a) (for details see [14]).

In the case (b), we use that the Schrödinger operator $\tilde{H}_{N,\varepsilon}$ is unitarily equivalent to $\varepsilon^{-2}\tilde{H}_{N,\varepsilon=1}$. For $N = 3$, the presence of a zero-energy resonance in the two-body Hamiltonian leads to non-empty (in fact, infinite) discrete spectrum in the three-particle Hamiltonian with center-of-mass motion removed [7, 16, 19]. This is the Efimov effect. It means, in particular, that $C_3 := \inf \sigma(\tilde{H}_{3,\varepsilon=1}) < 0$. By the HVZ theorem, $C_N := \inf \sigma(\tilde{H}_{N,\varepsilon=1}) \leq C_3$ for all $N \geq 3$.

\[\square\]

### 5 Absence of contact interactions for $d \geq 2$

In space dimensions $d \geq 2$, zero-range interactions among equal-spin fermions are prohibited by the Pauli principle. This is true in the very strong form of Theorem 5.1. For a related result in the physics literature concerning two fermions in $d = 2$, see [4].

Let $\Gamma_{ij} := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} | x_i = x_j\}$ and $\Omega_{ij} = \mathbb{R}^{Nd} \setminus \Gamma_{ij}$. Recall from Sect. 3 that $\Gamma = \bigcup_{i < j} \Gamma_{ij}$ and

$$\Omega = \mathbb{R}^{Nd} \setminus \Gamma = \bigcap_{i < j} \Omega_{ij},$$

\[\square\]
Theorem 5.1 If $d \geq 2$, then $C_0^\infty(\Omega) \cap \mathcal{H}_f$ is dense in $H^2(\mathbb{R}^N)$ with respect to the norm of $H^2$. This means that

$$H^2_0(\Omega) \cap \mathcal{H}_f = H^2(\mathbb{R}^N) \cap \mathcal{H}_f,$$

and it implies that the Laplacian $\Delta$ in $\mathcal{H}_f$ is essentially self-adjoint on $C_0^\infty(\Omega) \cap \mathcal{H}_f$.

Remarks 1. The main point of Theorem 5.1 is that elements of $C_0^\infty(\Omega)$ vanish in an entire neighborhood of the collision set $\Gamma$. The elements of $C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}_f$ vanish on $\Gamma$ too. But the weaker statement, that $C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}_f$ is dense in $H^2(\mathbb{R}^N) \cap \mathcal{H}_f$, is true for all $d \geq 1$ and it easily follows from the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$.

2. For $d = 1$, the assertion of the theorem is false. To see this, consider a sequence $(\psi_n)$ in $C_0^\infty(\Omega)$ with $\psi_n \to \psi$ in the norm of $H^2$. Then, $\nabla \psi_n \to \nabla \psi$ in the norm of $H^1$. Since the trace operators $T_{ij} : H^1(\mathbb{R}^N) \to L^2(\Gamma_{ij})$ are continuous, and since, clearly, $\nabla \psi_n = 0$ on all hyperplanes $\Gamma_{ij}$, it follows that

$$\nabla \psi = 0 \quad \text{on all } \Gamma_{ij}, \quad (33)$$

or, more precisely, $T_{ij} \nabla \psi = 0$ in $L^2(\Gamma_{ij})$. We now give an example of an antisymmetric wave function $\psi \in H^2(\mathbb{R}^N)$ without property (33), which proves that $C_0^\infty(\Omega) \cap \mathcal{H}_f$ is not dense in $H^2(\mathbb{R}^N) \cap \mathcal{H}_f$.

Let $|x|^2 := \sum_{i=1}^N x_i^2$ and

$$\psi(x_1, \ldots, x_N) := e^{-|x|^2} \prod_{i<j} (x_j - x_i).$$

Apart from the Gaussian, this is a Vandermonde determinant. This shows that $\psi$ is antisymmetric. On the hyperplane $\Gamma_{12}$, we have

$$\left. \frac{\partial \psi}{\partial x_1} \right|_{x_1=x_2} = -e^{-|x|^2} \prod_{j=3}^N (x_j - x_1) \prod_{2 \leq i < j} (x_j - x_i),$$

which shows that $\nabla \psi$ does not vanish on $\Gamma_{12}$.

The proof of Theorem 5.1 is based on the following lemmas, Lemma 5.2 being the heart of it.

Lemma 5.2 If $d \geq 2$, then there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^d, [0, 1])$ with $u_n(x) = 1$ if $|x| \leq 1/n$, and, in the limit $n \to \infty$, $\text{diam}(\text{supp} \ u_n) \to 0$ as well as

$$\int |\nabla u_n(x)|^2 \ dx \to 0 \quad (34)$$

$$\int |x|^2 |\Delta u_n(x)|^2 \ dx \to 0. \quad (35)$$
Proof In the case \( d \geq 3 \), we may choose any function \( u \in C^\infty(\mathbb{R}^d, [0, 1]) \) with \( u(x) = 1 \) for \( |x| \leq 1 \) and define \( u_n(x) = u(nx) \). Then, with the substitution \( y = nx \), in the limit \( n \to \infty \),

\[
\int |\nabla u_n(x)|^2 \, dx = \int |\nabla u(y)|^2 \, dy \cdot n^{2-d} \to 0,
\]

\[
\int |x|^2 |\Delta u_n(x)|^2 \, dx = \int |y|^2 |\Delta u(y)|^2 \, dy \cdot n^{2-d} \to 0.
\]

In the case \( d = 2 \), we define \( u_n(0) := 1 \) and for \( |x| > 0 \) we set

\[
u_n(x) := g \left( \frac{\log(n|x|)}{\log \log n} \right),\]

where \( g \in C^\infty(\mathbb{R}, [0, 1]) \) with

\[
g(s) := \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1. \end{cases}
\]

It follows that \( u_n(x) = 1 \) for \( |x| \leq 1/n \), \( u_n(x) = 0 \) for \( |x| \geq (\log n)/n \) and hence that \( u_n \in C^\infty(\mathbb{R}^d, [0, 1]) \). Moreover,

\[
\frac{1}{2\pi} \int |\nabla u_n(x)|^2 \, dx = \int_{1/n}^{(\log n)/n} g' \left( \frac{\log(nr)}{\log \log n} \right)^2 \frac{dr}{r} \cdot \frac{1}{(\log \log n)^2} = \int_0^1 g'(s)^2 \, ds \cdot \frac{1}{\log \log n}.
\]

On the other hand, using that on radially symmetric functions

\[
r^2 \Delta = \left( r \frac{\partial}{\partial r} \right)^2,
\]

we find

\[
\frac{1}{2\pi} \int |x|^2 |\Delta u_n(x)|^2 \, dx = \int_{1/n}^{(\log n)/n} \left( r \frac{\partial}{\partial r} g \left( \frac{\log(nr)}{\log \log n} \right) \right)^2 \frac{dr}{r} = \int_{1/n}^{(\log n)/n} g'' \left( \frac{\log(nr)}{\log \log n} \right)^2 \frac{dr}{r} \cdot \frac{1}{(\log \log n)^4} = \int_0^1 g''(s)^2 \, ds \cdot \frac{1}{(\log \log n)^3}.
\]

This concludes the proof. \( \Box \)
Lemma 5.3 Suppose that \( d \geq 2 \) and let \( \psi \in C_0^\infty(\mathbb{R}^{Nd}) \) with \( \psi = 0 \) on \( \Gamma \). Then, for each pair \( i, j \in \{1, \ldots, N\}, i \neq j \) and for each \( \varepsilon > 0 \), there exists \( \psi_\varepsilon \in C_0^\infty(\Omega_{ij}) \) with \( \text{supp} \psi_\varepsilon \subset \text{supp} \psi, \psi_\varepsilon = 0 \) on \( \Gamma \), and

\[
\|(-\Delta + 1)(\psi - \psi_\varepsilon)\| < \varepsilon.
\]

Proof We may assume that \((i, j) = (1, 2)\) and we introduce the relative and center-of-mass coordinates

\[
r := x_2 - x_1, \quad R := \frac{1}{2}(x_1 + x_2).
\]

Then,

\[
\Delta = 2\Delta_r + \frac{1}{2}\Delta_R + \Delta_{x'},
\]

where \( x' := (x_3, \ldots, x_N) \). Let \( \psi \in C_0^\infty(\mathbb{R}^{Nd}) \) with \( \psi = 0 \) on \( \Gamma \) and let

\[
\psi_n(x_1, \ldots, x_N) = \psi(x_1, \ldots, x_N) \cdot (1 - u_n(x_2 - x_1))
\]

where \( u_n \) is given by Lemma 5.2. In the following \( u_n \) also denotes the function \((x_1, \ldots, x_N) \mapsto u_n(x_2 - x_1)\). Then, \( \psi - \psi_n = \psi u_n \) and hence

\[
\|(-\Delta + 1)(\psi - \psi_n)\| \leq \|\psi u_n\| + \|\Delta(\psi u_n)\|.
\]

Clearly \( \|\psi u_n\| \to 0 \) because \( |\psi u_n| \leq |\psi| \) and because \( \psi u_n \to 0 \) pointwise as \( n \to \infty \). On the other hand, using (36) and the fact that \( u_n \) depends on \( r \) only,

\[
\Delta(\psi u_n) = 2\Delta_r(\psi u_n) + \frac{1}{2}(\Delta_R \psi) u_n + (\Delta_{x'} \psi) u_n
\]

where the first term equals

\[
2\Delta_r(\psi u_n) = 2(\Delta_r \psi) u_n + 4(\nabla_r \psi)(\nabla_r u_n) + 2\Delta_r u_n.
\]

By the pointwise convergence \( u_n \to 0 \), as explained above, \((\Delta_r \psi) u_n, (\Delta_R \psi) u_n\) and \((\Delta_{x'} \psi) u_n\) have vanishing \( L^2 \)-norm in the limit \( n \to \infty \). It remains to show that

\[
\|\nabla_r \psi \cdot \nabla_r u_n\| \to 0 \quad (n \to \infty)
\]

\[
\|\psi \cdot \Delta_r u_n\| \to 0 \quad (n \to \infty).
\]

From Lemma 5.2, we know that

\[
\|\nabla_r \psi \cdot \nabla_r u_n\|^2 \leq \int |\nabla_r \psi(R, x')|^2 |\nabla u_n(r)|^2 \, dr \, dR \, dx'
\]
\[
\leq \sup_{\mathbf{r} \in \mathbb{R}^d} \int |\nabla_r \psi(\mathbf{r}, R, \mathbf{x}')|^2 \, d\mathbf{R} \, d\mathbf{x}' \cdot \|\nabla u_n\|^2 \to 0 \quad (n \to \infty).
\]

For \( \psi \Delta u_n \), we use that \( \psi(0, R, x') = 0 \) and hence that
\[
\psi(\mathbf{r}, R, \mathbf{x}') = \int_0^1 (\nabla_r \psi)(tr, R, x') \cdot r \, dt.
\]

It follows that
\[
\int |\psi \Delta u_n|^2 \, dr \, d\mathbf{R} \, d\mathbf{x}' \\
\leq \int dr \, d\mathbf{R} \, d\mathbf{x}' \left( \int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, dt \right) |r|^2 |\Delta u_n(r)|^2 \\
\leq C \int |r|^2 |\Delta u_n(r)|^2 \, dr \to 0, \quad (n \to \infty)
\]
by Lemma 5.2, because
\[
C := \sup_{\mathbf{r} \in \mathbb{R}^d} \int d\mathbf{R} \, d\mathbf{x}' \int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, dt < \infty.
\]

\( \square \)

**Proof of Theorem 5.1** For given \( \psi \in H^2(\mathbb{R}^{Nd}) \cap \mathcal{H}_f \) and \( \varepsilon > 0 \) it suffices to find \( \phi_\varepsilon \in C_0^\infty(\Omega) \) with \( \|\psi - \phi_\varepsilon\|_{H^2} < \varepsilon \). Let \( P_f : L^2(\mathbb{R}^{Nd}) \to L^2(\mathbb{R}^{Nd}) \) denote the orthogonal projection onto \( \mathcal{H}_f \). Then, \( \psi_\varepsilon := P_f \phi_\varepsilon \) belongs to \( C_0^\infty(\Omega) \cap \mathcal{H}_f \) and
\[
\|\psi - \psi_\varepsilon\|_{H^2} = \|P_f(\psi - \phi_\varepsilon)\|_{H^2} \leq \|\psi - \phi_\varepsilon\|_{H^2} < \varepsilon
\]
because \( P_f \) is an orthogonal projection in \( H^2 \) (if suitably normed). To find \( \phi_\varepsilon \), we may assume that \( \psi \in C_0^\infty(\mathbb{R}^{Nd}) \cap \mathcal{H}_f \), which is dense in \( H^2(\mathbb{R}^{Nd}) \cap \mathcal{H}_f \), and we use Lemma 5.3 repeatedly. That is, we use \( \{\sigma_k \mid k = 0, \ldots, n\} \) to denote the set of \( n := \binom{N}{2} \) pairs \((i, j)\), we define \( \Omega_0 := \mathbb{R}^{Nd} \) and
\[
\Omega_k := \cap_{j=1}^k \Omega_{\sigma_j}, \quad k = 1 \ldots n.
\]
Then, we construct smooth functions \((\gamma_k)_{k=0}^n\) recursively with \( \gamma_0 := \psi \), \( \text{supp}(\gamma_k) \subset \text{supp}(\gamma_{k-1}) \cap \Omega_k \), \( \gamma_k = 0 \) on \( \Gamma' \), and \( \|(-\Delta + 1)(\gamma_k - \gamma_{k-1})\| < \varepsilon/n \). This is achieved with the help of Lemma 5.3. The function \( \phi_\varepsilon := \gamma_n \) has the desired properties. \( \square \)

**Acknowledgements** We thank Semjon Wugalter for pointing out reference [17].

**Funding** Open Access funding enabled and organized by Projekt DEAL.

**Declarations**
Conflict of interest  On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data sharing  Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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