Theory of Slow Light Enhanced Four-Wave Mixing in Photonic Crystal Waveguides

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Abstract: The equations for Four-Wave-Mixing in a Photonic Crystal waveguide are derived accurately. The dispersive nature of slow-light enhancement, the impact of Bloch mode reshaping in the nonlinear overlap integrals and the tensor nature of the third order polarization are therefore taken into account. Numerical calculations reveal substantial differences with simpler models, which increase with decreasing group velocity. We predict that the gain for a 1.3 mm long, unoptimized GaInP waveguide will exceed 10 dB if the pump power exceeds 1 W.

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References and links
1. Slow Light: Science and Applications, J. B. Khurgin and R.S. Tucker, Eds., (CRC Press, Boca Raton, 2009).
2. M. Santagiustina, "Governing the speed of light: recent advances and future perspectives of slow and fast light in microwave-Photonics", in Proc. 2009 IEEE Intern. Top. Meet. on Microwave Photonics, (Valencia, Spain, 2009) Th3.1.
3. T. Baba, "Slow light in photonic crystals", Nat. Photon. 2, 465–473 (2008).
4. N. A. R. Bhat, J. E. Sipe, "Optical pulse propagation in nonlinear photonic crystals”, Phys. Rev. E 64, 056604 (2001).
5. M. Soljacic, M. Ibanescu, S. G. Johnson, Y. Fink, J. D. Joannopoulos, "Optimal bistable switching in nonlinear photonic crystals”, Phys. Rev. E 65, 055601(R) (2002).
6. T. Kamalakis, T. Sphicopoulos, "A New Formulation of Coupled Propagation Equations in Periodic Nanophotonic Waveguides for the Treatment of Kerr-Induced Nonlinearities", IEEE J. Quantum Electron. 43, 923–933 (2007).
7. T.F. Krauss, "Slow light in photonic crystal waveguides”, J. Phys. D: Appl. Phys. 40, 2666–2670 (2007).
8. B. Corcoran, C. Monat, C. Grillet, D. Moss, B. J. Eggleton, T. White, L. O’Faolain, T. Krauss T, "Green light emission in silicon through slow-light enhanced third-harmonic generation in photonic-crystal waveguides”, Nat. Phot. 3, 206-210 (2009).
9. S. Combric, Q. Vy Tran, C. Husko, P. Colman, and A. De Rossi, "High quality GaInP nonlinear photonic crystals with minimized nonlinear absorption”, Appl. Phys. Lett. 95, 221108 (2009).
10. C. Husko, S. Combric, Q. Tran, F. Raineri, C. Wong, A. De Rossi, "Non-trivial scaling of self-phase modulation and three-photon absorption in III-V photonic crystal waveguides”, Opt. Express 17, 22442-22451 (2009).
11. V. Eckhouse, I. Cestier, G. Eisenstein, S. Combric, P. Colman, A. De Rossi, M. Santagiustina, C. G. Someda, G. Vadalà, "Highly efficient four wave mixing in GaInP photonic crystal waveguides”, Opt. Lett. 35, 1440-1442 (2010).
12. T. Hasegawa, T. Nagashima, N. Sugimoto, “Determination of nonlinear coefficient and group-velocity dispersion of bismuth-based high nonlinear optical fiber by four-wave mixing”, Opt. Commun. 281, 782–787 (2008).
13. M. D. Pelusi, F. Luan, E. Magi, M. R. E. Lamont, D. J. Moss, B. J. Eggleton, J. S. Sanghera, L. B. Shaw, I. D. Aggarwal, "High bit rate all-optical signal processing in a fiber photonic wire", Opt. Express 16, 11506–11512 (2008).
14. M. Ebnali-Heidari, C. Monat, C. Grillet, M. K Moravvej-Farshi, "A proposal for enhancing four-wave mixing in slow light engineered photonic crystal waveguides and its application to optical regeneration", Opt. Express 17, 18340–18353 (2009).
15. http://ab-initio.mit.edu/photons/
16. N. C. Panoiu, J. F. McMillan, C. W. Wong, "Theoretical Analysis of Pulse Dynamics in Silicon Photonic Crystal Wire Waveguides", IEEE J. Quantum Electron. 16, 257–266 (2010).
17. D. Michaelis, U. Peschel, C. Wächter, A. Brauer, "Reciprocity theorem and perturbation theory for photonic crystal waveguides", Phys. Rev. E 68, 065601(R) (2003).
18. R. Boyd, Nonlinear Optics, Chapt. 4 (Academic Press, San Diego, 2003).
19. P. Yeh, "Electromagnetic propagation in birefringent layered media" J. Opt. Soc. Am. 69, 742 (1979).
20. K. Sakoda, Optical Properties of Photonic Crystals, Chapt. 2 (Springer, Berlin, 2005).
21. B. Lombardet, L.A. Dunbar, R. Ferrini, R. Houdré, "Bloch wave propagation in two-dimensional photonic crystals: Influence of the polarization", Opt. Q. Electr. 37, 293–307 (2005).
22. G.P. Agrawal, Nonlinear fiber optics, Chapt. 10 (Academic Press, San Diego, 2001).
23. D. C. Hutchings, B. S. Wherrett, "Polarisation dichroism of nonlinear refraction in zinc-blende semiconductors", Opt. Commun. 111, 507–512 (1994).

1. Introduction

Slow light (SL) can enable interesting applications in photonics [1] and microwave-photonics [2] and it is also expected to enhance nonlinear phenomena. In particular, photonic crystal waveguides (PhCWs) present SL propagation [3] which is predicted to enhance self-phase modulation (SPM) [4, 5, 6]. One intuitive but powerful picture of this effect represents pulses subject to a spatial compression which locally increases the power density [7], pretty much as cars in a highway get closer to each other as their speed is decreased. The dependence of nonlinearity on group velocity has been recently observed in PhCW for third harmonic generation [8] and for SPM and three photon absorption [9][10].

Efficient four-wave mixing (FWM) was reported recently in 1.3 mm long, III-V semiconductor PhCW [11], with a conversion efficiency comparable to that of about 1 m-long highly nonlinear fiber [12] and about 2 cm-long chalcogenide fiber photonic wire [13]. That result confirms the theoretical prediction [14] of enhanced (FWM) on the basis of the square group index scaling factor. However, it must be pointed out that a simple square group index scaling, to model the FWM enhancement, does not actually take into account several fundamental features of the phenomenon.

The first feature is that the group index in the waveguide is a function of frequency and, therefore, it is not the same for the various waves involved in the FWM. Observe in fact in Fig. 1a the numerically calculated [15] group index (red dots) of a GaInP membrane PhCW. The calculation is carried out with pumps placed at $N = 6$ different wavelengths approaching the band edge: $\lambda_{1M} = [1570 + 10 (M - 1)]\text{nm}$, $M = 1, ..., 6$. The second pump wavelength is $\lambda_{2M} = \lambda_{1M} + 2\text{nm}$; the signal and idler frequency are calculated according to $\omega_{3M} = \omega_{1M} - \Delta\omega_M$, $\omega_{4M} = \omega_{2M} + \Delta\omega_M$, where $\Delta\omega_M = \omega_{1M} - \omega_{2M}$, thus satisfying the FWM frequency condition $\omega_{1M} + \omega_{2M} = \omega_{3M} + \omega_{4M}$. The waveguide parameters are: $a = 480\text{nm}$ (crystal period), $d = 0.38a$ (hole diameter) and $h = 170\text{nm}$ (PhC slab height).

Moreover, the modal superposition of the interacting fields must be also considered. In PhCWs this overalap is a function of frequency and, in particular, as the frequency approaches the bandgap, the mode spreads into the hole region as shown in Fig. 1a where the intensity distribution of the electric field within a cell of the PhCW is shown at three different wavelengths. Finally, the PhCWs mode is not constant but periodic along the propagation direction and so, differently from other photonic waveguides (slabs, fibers etc.), the careful determination of the nonlinear effective coefficients is more complicated [4][6][16].
In this paper, an accurate calculation of all nonlinear effective coefficients necessary to evaluate the FWM interaction in the SL regime of a PhCW, is carried out. The derivation is performed through a perturbation approach, similarly to [17], directly from Maxwell’s equations. The resulting SPM coefficient corresponds to that obtained by previous derivations [4, 6]. The XPM coefficient is consistent with the one obtained for multimode propagation [16]. The FWM effective coefficients for a PhCW are determined for the first time to the best of our knowledge.

2. Derivation of the nonlinear propagation equations

The starting point of our analysis are Maxwell’s equations in the frequency domain where the linear permittivity \( \varepsilon(\mathbf{r}) \) is a spatial function describing the PhCWs structure and \( \mathbf{P}_{NL}(\mathbf{r}, \omega) \) accounts for the nonlinear response:

\[
\nabla \times \mathbf{E}(\mathbf{r}, \omega) = j \omega \mu \mathbf{H}(\mathbf{r}, \omega), \quad \nabla \times \mathbf{H}(\mathbf{r}, \omega) = -j \omega \varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}, \omega) - j \omega \mathbf{P}_{NL}(\mathbf{r}, \omega).
\]

(1)

It is assumed that four signals are propagating in the fundamental TE mode of the PhCW, at frequencies that satisfy the FWM condition

\[
\omega_1 + \omega_2 = \omega_3 + \omega_4.
\]

Here, we also used the hypothesis that \( \varepsilon(\mathbf{r}) \) is a slowly varying function of \( \mathbf{r} \) and therefore has the explicit dependence on space and replaced the frequency \( \omega_0 \) with the subscript \( i \).

We assume that the nonlinearity is small enough so that complex amplitude, \( A_i \), in the direction \( \hat{z} \) is slowly varying in comparison to \( \exp(i \beta \hat{z}) \) and to the Bloch mode within the cell \( \{ e_i, h_i \} \). Furthermore, the Bloch modes are normalized: \( \int_V (e_i \times h_i^* + e_i^* \times h_i) \cdot \hat{z} dV = 4a \). Let us stress that, with this choice, \( |A_i|^2 = P_i \) is the active power propagating in the \( z \) direction at frequency \( \omega_i \) [6]. In the following, we will show that this is the natural choice for normalizing the Bloch modes when group velocity is substantially different from the phase velocity. Similarly as in ref. [17] we consider Eqs. (1) calculated at frequency \( \omega_0 \); the second equation is scalarly multiplied by \( e_i^* \exp(-i \beta \hat{z}) \) and then subtracted from the first, multiplied by \( h_i^* \exp(i \beta \hat{z}) \). The result is integrated over the volume of the PhCW unit cell to obtain:

\[
\frac{\partial A_i}{\partial z} \int_V \hat{z} \cdot [e_i \times h_i^* + e_i^* \times h_i] dV = j \omega_i \int_V e_i^* \exp(-i \beta \hat{z}) \cdot \mathbf{P}_{NL}(\mathbf{r}, \omega_i) dV
\]

(4)

Here, we also used the hypothesis that \( A_i, \partial A_i/\partial z \) are slowly varying functions of \( z \) and therefore can be taken constant over one unit cell.

We now introduce the explicit form of the third-order nonlinear polarization \( \mathbf{P}_{NL}(\mathbf{r}, \omega_i) \). Using the notation of [18] it reads:

\[
\mathbf{P}_{NL}(\mathbf{r}, \omega_i) = \varepsilon_0 \chi^{(3)}(\mathbf{r}; \omega_i, \omega_j, -\omega_j, \omega_i) \cdot \mathbf{E}_j \mathbf{E}_j^* \mathbf{E}_i + \varepsilon_0 \sum_{j=1, \neq i}^4 \left[ \chi^{(3)}(\mathbf{r}; \omega_i, -\omega_i, \omega_j, \omega_j) \cdot \mathbf{E}_j \mathbf{E}_j^* \mathbf{E}_i \right] + \varepsilon_0 \chi^{(3)}(\mathbf{r}; \omega_i, -\omega_i, \omega_j, \omega_j) \cdot \mathbf{E}_j \mathbf{E}_j^* \mathbf{E}_i.
\]

(5)
where \(i,j,k,l = \{1,2,3,4\}\), with the constraint that \(i,j,k,l\) are all different in the last term. The susceptibility tensor is real because multi-photon absorption can be neglected in GaInP at \(\lambda \approx 1.6\mu m\). On the right hand side (RHS) of Eq. (6), the first term represents SPM, the summand term XPM, and the last the non-degenerate FWM. Here, third and other harmonic generations are neglected by assuming that they will not be phase matched. For the sake of brevity the tensor explicit dependence on position and frequencies will also be omitted till the end of the derivation. Inserting Eq. (5) in Eq. (4) we obtain:

\[
4d \frac{\partial A_i}{\partial z} = \frac{i\omega_d \epsilon_0}{4} \left[ |A_i|^2 A_i \int \chi^{(3)}(\gamma_i) e_i e_i e_i dV + \sum_{j=1,j\neq i}^4 |A_j|^2 A_j \int \chi^{(3)}(\gamma_j) e_i e_j e_i dV + A_i A_i^* A_i \exp(-i\sigma_i \Delta \beta z) \int \chi^{(3)}(\gamma_i) e_i e_i^* e_i dV \right].
\]  

(6)

Here \(\Delta \beta = \beta_3 + \beta_4 - \beta_1 - \beta_2\) is the linear phase mismatch and \(\sigma_i = \pm 1\), where the plus (minus) sign applies for \(i = 3,4\) (\(i = 1,2\)). In Eq. (6) the SL enhancement of the nonlinear response is hidden in the integrals. In order to make this dependence explicit, we use the identity between the electromagnetic energy velocity \(v_e\) and the group velocity that holds in lossless homogeneous media, in periodic ones \[19\] and in PhCWs \[20\]. By projecting the energy velocity along the axis unit vector \(\hat{z}\) and using the property that the space-time average magnetic and electric energies are equal for Bloch modes \[21\], \(\mu_0/4 \int f dV \hat{h}_i \cdot \hat{h}_i = 1/4 \int f dV \hat{e}_i \cdot \hat{d}_i\), the following is obtained:

\[
v_{ei} \cdot \hat{z} = \frac{1}{4} \int f dV (\epsilon_i \hat{h}_i + e_i \hat{d}_i) \cdot \hat{z} dV = \frac{4a}{f \int \epsilon_0 \epsilon_i(r) |e_i|^2 dV} = \frac{2a}{\epsilon_0 W_i} = v_{gi}. \]

(7)

Note that energies appearing above are normalized consistently with the choice \(|A_i|^2 = \rho_i\). We can now normalize the terms on the RHS of Eq. (6) multiplying them by the factors \(\eta_i^2\) (SPM), \(\eta_i^2 \eta_j^2\) (XPM) and \(\eta_i \eta_j \eta_k \eta_l (FWM)\), with \(\eta_i = \sqrt{2a/(\epsilon_0 W_i v_{gi})} = 1, \forall i\). The nonlinear coefficients of Eq. (6) are now cast in their canonical form so that the equations governing the FWM in PhCW are obtained in a form similar to nonlinear fiber optics \[22\]:

\[
\frac{dA_i}{dz} = i\gamma |A_i|^2 A_i + 2 \sum_{j=1,j\neq i}^4 \gamma_{ij} |A_j|^2 A_i + 2\gamma_{ij} A_j^* A_i \exp(-i\sigma_i \Delta \beta z), \quad i = 1,2,3,4.
\]

(8)

with the effective nonlinear coefficients and the relative effective volumes taking the form:

\[
\gamma = \frac{n_2 O_0 a}{c V_i^2}; \quad \frac{1}{V_i} = \frac{n_2 O_0 a}{c V_i^2} \int \chi^{(3)}(\gamma_i) e_i e_i e_i dV; \quad \gamma_{ij} = \frac{n_2 O_0 a}{c V_{ij}}; \quad \frac{1}{V_{ij}} = \frac{n_2 O_0 a}{c V_{ij}} \int \chi^{(3)}(\gamma_{ij}) e_i e_j e_i dV; \quad \gamma_{Fi} = \frac{n_2 O_0 a}{c V_{Fi}}; \quad \frac{1}{V_{Fi}} = \frac{4}{\eta_0 W_0} \left( \int \chi^{(3)}(\gamma_{Fi}) e_i e_i e_i dV \right)^{1/2};
\]

(9)-(11)

and where \(n_2 = 3 \chi^{(3)}_{xxx} / (4 \epsilon_0 c)\) is the bulk, nonlinear refractive index coefficient for a linear state of polarization \[18\].

So, for each nonlinear effect (SPM, XPM and FWM) we determined: 1) the correct enhancement factor due to SL; 2) the correct overlap integral. The obtained SPM and XPM coefficients are consistent to those previously found \[8,16\]; the FWM coefficient is derived for the first time to the best of our knowledge. By observing Eqs. (9)-(11) a general rule can be remarked: the enhancement factor due to the SL is always given by the geometric mean of the group indexes of the waves interacting through the tensor \(\chi^{(3)}\).
3. Numerical Results

Fig. 1. Comparison with model in ref. [14]. (a) Square of the group index $n_g^2$ and SL enhancement factor $S$, as a function of the wavelength. In the insets the intensity of electric field $|e|^2$ of the Bloch mode within an elementary cell is shown at three different wavelengths. (b) Wavelength dependence of the SL scaling factor and of the effective FWM nonlinear coefficient, both normalized to their respective value at $\lambda_{11} = 1570\text{nm}$: $\gamma_{\text{norm}} = |\gamma_{44}^{\lambda_{11}}|/\gamma_{44}(\lambda_{11})$, $S_{\text{norm}} = S(\lambda_{11})/S(\lambda_{11})$.

The theoretical findings of the previous section are applied to the PhCW of Fig. 1(a). To the aim of explicitly calculating the tensor products in Eqs. (9,10,11) the theory of ref. [23], that can be generally applied to zinc-blend semiconductors (group symmetry $4\overline{3}m$), is exploited and then $\chi^{(3)}_{\text{xxxx}} = 2\chi^{(3)}_{\text{xyxy}} = 2\chi^{(3)}_{\text{xyyx}} = \chi^{(3)}_{\text{xxxx}}$. For signal wavelengths $1.52\mu m < \lambda < 1.62\mu m$ and given that $E_g \simeq 1.9eV$ for GaInP, the frequencies $\omega_{i,j,k,l}$ at which the tensor elements are to be calculated satisfy the condition $0.4 < h\omega_{i,j,k,l}/E_g < 0.43$. Then, although Kleinmann symmetry [18] is not satisfied, it is found that the dichroism parameter, defined by $\chi^{(3)}_{\text{xyxy}}/\chi^{(3)}_{\text{xxxx}}$, can be approximated to 0.28 [23]. Thus all tensors can be determined from the above relations, from the knowledge of the nonlinear refractive index in GaInP, $n_2 = 10^{-17}m^2/W$ and through [18]:

$$e_i \cdot \chi^{(3)}(r):e_j e_k = \sum_m \left[ e_{im}^* D \sum_{nop} \chi^{(3)}_{mnop}(r; \omega_i, -\omega_j, -\omega_k, \omega_l) e_{jn} e_{lo} e_{kp} \right]$$

(12)

where the summations over the indexes $m,n,o,p$ are made on all possible values of the coordinate axes $\{x,y,z\}$ and $D$ is the frequency degeneracy factor [18] which represents the number of distinct permutations of the three frequencies $\{\omega_i, -\omega_j, -\omega_k\}$ ($D = 3$ for SPM, $D = 6$ for XPM and FWM).

We numerically determined, for all previously defined pump and signal wavelengths the Bloch mode electrical field and the dispersion relation [15], then calculating the FWM SL scaling factor $S = \prod_{k=1}^{4} n_p^{1/2}$ and the effective nonlinear coefficient $\gamma_{44}$ according to Eq. (11). In Fig. 1(b) $S$ can be compared to the square of the group index at the mean frequency, which is the approximation used in refs. [11] [14]; a slight discrepancy appears at the band edge (where the SL effect is strong). Note that no particular dispersion engineering of the PhCW has been realized on purpose; $\Delta \beta$ could still be reduced by design, to increase FWM efficiency. To evaluate the effective nonlinear enhancement of PhCW, in Fig. 1(b) the wavelength dependence of $S$ is compared to that of $\gamma$, revealing that the enhancement of FWM coefficient does not follow the pure SL scaling $S$, the large difference deriving from the decrease in the modal overlap integrals.
By following [22] we can finally determine the nonlinear phase matching and gain coefficient for optical parametric amplification of the wave at frequency $\omega_3$:

$$\Delta \kappa = \Delta \beta - \sum_{i,j=1}^{2} [\gamma_i + 2i - j|\gamma_i - 2\gamma_j - 2\gamma_i|] P_i = \Delta \beta + \gamma_{pm} P_0, \quad g = \left[ \frac{\gamma_{gm}^2 P_0^2 - \Delta \kappa^2}{4} \right]^{1/2}, \quad (13)$$

where the last terms of each of Eqs. (13) are obtained for $P_1 = P_2 = P_0/2$ and $\gamma_{gm}^2 = \gamma_{F3} \gamma_{F4}$. The effective coefficients $\gamma_{pm}$ and $\gamma_{gm}$ describe the strength of the nonlinearity contribution, respectively, to the phase matching and to the maximum gain. It is remarkable that, differently from fiber optics [22], $\gamma_{pm} \neq \gamma_{gm}$. Figure (2a) compares the maximum achievable gain coefficient $\gamma_{gm} P_0$ to the actual one $g$, which is limited by the phase mismatch. As the pump power increases from 0.9 W to 1.4 W the phase mismatch is almost completely canceled by the nonlinear phase terms and the maximum gain is approached. This fact leads to a dramatic increase of the FWM gain $G_3 = P_3(L)/P_3(0) = 1 + \gamma_{gm}^2 P_0^2 / g^2 \sinh^2(gL)$ which is shown in fig. (2b) for $L = 1.3 \text{ mm}$.

4. Conclusions

We have derived the nonlinear equations which describe four-wave mixing in photonic crystal waveguides directly from Maxwell’s equations. These equations are exact in the limit in which we can neglect the changes that the nonlinearity induces in the Bloch modes describing the field in the photonic crystal waveguide, a situation by far verified in practice. We demonstrate rigorously the explicit dependence of the nonlinear enhancement on the group index and that in four-wave mixing (where the fields involved have different group indexes) the dispersive nature is rigorously accounted for by the geometric mean of the group indexes of the modes involved. Moreover, we demonstrate a substantial correction arising from Bloch mode reshaping in the nonlinear field overlap. Finally we account for the tensor nature of the nonlinear polarization. As an example, we calculated the gain for a 1.3 mm long waveguide operated at moderately small group velocity ($v_g > c/20$). Particularly, when $n_g \simeq 12$ and the coupled pump power $> 1 \text{ W}$, the expected gain exceeds 10 dB even if the waveguide is unoptimized.

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