M-theory realization of a N=1 supersymmetric chiral gauge theory in four dimensions

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Abstract

We study M5-brane configuration of the chiral gauge theory whose Type IIA brane configuration with orientifold 6-plane(O6) is studied by various authors. Much of the paper is devoted to studying M-theory picture of SO/Sp gauge theory with fundamental flavors realized in Type IIA setup with O6-plane. The Higgs branch of N=2 SO/Sp gauge theory is studied and the curve corresponding to rotated brane configuration is presented. In the chiral gauge theory, the middle NS'5-brane on top of the O6 plane is realized as a detached rational curve. Depending on a location of the rational curve in $x^7$ direction, the same curve plus the rational curve can be interpreted as describing the Coulomb branch of $SU(2N_c)$ chiral gauge theory, $SO(2N_c)/Sp(N_c)$ gauge theory with $N_f/(N_f+4)$ fundamental hypermultiplets. Various consistency checks for this proposal are made. By introducing two more rational curves corresponding to NS'5-branes, one can produce a non-trivial fixed point which mediates chiral non-chiral transition.

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1 Introduction

Recently there has been much progress in understanding non-perturbative aspects of supersymmetric gauge theories using brane configuration. The configuration where D-branes are suspended between NS-branes was firstly studied in [1], and the subsequent study of similar configurations in various dimensions deepen our understanding on supersymmetric gauge theories [4].

Especially interesting examples are the brane realizations of supersymmetric gauge theories in four dimensions with four or eight supercharges. One particular success is that the N=1 Seiberg's duality can be reproduced in this approach. Much of the gauge theories studied via brane setup are non-chiral, but there have been fruitful attempts to understand chiral gauge theories using brane dynamics[6, 8, 9]. For four dimensional gauge theories, one can lift the brane setup in Type IIA to the eleven dimensions where D4-branes and NS-5 branes altogether turn into M-5 branes[2]. Several aspects of supersymmetric gauge theories are clear in the eleven dimensions such as the Riemann surface associated with the Coulomb branch of a specific gauge theory.

Even though there are several constructions of chiral gauge theories in ten-dimensional brane configurations, so far the corresponding eleven-dimensional construction is lacking. The purpose of this paper is to propose the M-theory realization of a particular chiral gauge theory constructed in [10, 11, 12]. Orientifold 6-plane is introduced to construct the chiral gauge theory. The brane configuration with orientifold plane leads to the understanding of the SO/Sp gauge theories as well as the construction of SU gauge theories with symmetric/antisymmetric matter[4, 7]. One can introduce either orientifold 4-plane or orientifold 6-plane (O6) in the brane setup without breaking further supersymmetry, but the relevant one for our study is orientifold 6-plane. Much of the paper will be devoted to understanding the SO/Sp gauge theories with orientifold 6-plane, since this is closely related to the study of the chiral gauge theory, as we will see.

The content of this paper is as follows. In section 2, we introduce the basic brane setup with orientifold 6-plane and review how to construct the chiral gauge theory. In section 3, we study M-theory interpretation of SO/Sp gauge theories with fundamental flavors and see how the Higgs branch of the N=2 gauge theories are realized in M-theory setup. This constitutes a part of the moduli space of the chiral gauge theory. In section 4, we propose the rotated N=1 brane configuration of SO/Sp gauge theories starting from N=2 brane setup.
In section 5, we propose the 11-d interpretation of the chiral gauge theory. Closely related chiral-nonchiral transitions are discussed.

After we obtained the results in section 4, we receive the preprint where the same problem is solved by a slightly different approach [13]. We understand that the M5-brane construction of the chiral gauge theory treated in this paper is known to [14].

2 Relevant Brane Configuration with Orientfold Plane

In order to have N=2 supersymmetry in 4-d, we need three ingredients in Type IIA string theory; NS5-brane whose world volume spans $x^0, x^1, x^2, x^3, x^4, x^5$, D4-branes whose world volume spans $x^0, x^1, x^2, x^3, x^6$, and D6-branes whose world volume spans $x^0, x^1, x^2, x^3, x^7, x^8, x^9$. The $SO(1,9)$ Lorentz symmetry is broken to $SO(1,3) \times SO(2)_{4,5} \times SO(3)_{7,8,9}$ where $SO(n)_{i_1, \ldots, i_n}$ is a rotational group in $x^{i_1}, \ldots, x^{i_n}$-direction. $SO(3)_{7,8,9}$ can be identified with $SU(2)_R$ symmetry of the N=2 gauge theory. We can introduce O6-plane parallel to D6-brane without breaking supersymmetry further. We have two types of O6-plane, one carries +4 Ramond-Ramond (RR) charge and the other carries −4 RR charge if we normalize the RR charge of a D4-brane to be +1. The former gives SO gauge group on the world volume of D4-brane and the latter gives Sp gauge group. In the presence of O6-plane, the configuration should be symmetric under the $Z_2$ action $(x^4, x^5, x^6) \to (-x^4, -x^5, -x^6)$. Throughout the paper, we use the covering space of the $Z_2$ action when we count the number of branes. Thus if we have $2N_c$ D4-branes and $2N_f$ D6-branes, the resulting gauge group is $SO(2N_c)/Sp(N_c)$ with $N_f$ N=2 hypermultiplets where $Sp(N_c)$ has rank $N_c$.

One can rotate a NS5-brane by an arbitrary angle $\theta$, which characterizes the rotation between $(x^4, x^5)$-plane and $(x^8, x^9)$-plane. The supersymmetry is reduced to N=1. The rotated 5-brane with $\theta = \pi/2$ is denoted by NS'5-brane whose world volume is in the $x^0, x^1, x^2, x^3, x^8, x^9$-direction. $SO(1,9)$ spacetime Lorentz symmetry is broken to $SO(1,3) \times SO(2)_{4,5} \times SO(2)_{8,9}$. The first factor is the Lorentz symmetry of the 4-d theory. The two $SO(2)$ factors can be identified as $R$ symmetries of the N=1 theory.

The chiral gauge theory we are interested in is constructed by placing NS'5-brane on top of the O6-plane. O6-plane changes its RR charge as it traverses the NS'5-brane. This is the T-dual version of [8] where it is shown that O4-plane changes its RR charge as it traverses a NS5-brane. Anomaly cancellation of the corresponding field theory or the charge conservation requires that 4 physical half D6-branes and the mirror are stuck on the negatively charged
Figure 1: The left figure illustrates the basic brane configuration with N=2 supersymmetry. \( v = x^4 + ix^5 \) and \( w = x^8 + ix^9 \). In the right figure the right NS5-brane is rotated by \( \theta \).

Figure 2: The configuration for the chiral gauge theory. Left figure describes the brane configuration in \((v, w, x^6)\)-plane while right figure describes same configuration in \((x^6, x^7)\)-plane. Note that 8 half D6-branes are stuck on the half of O6-plane.
half of the O6-plane. If we have $2N_c$ D4-branes, the gauge group is $SU(2N_c)$ since the orientifold identifies the left $SU(2N_c)_L$ and the right $SU(2N_c)_R$. Among the degrees of freedom that lead to the bifundamentals in the product gauge group $SU(2N_c)_L \times SU(2N_c)_R$, only one antisymmetric and one conjugate symmetric tensor survive under the orientifold projection. Thus we have $SU(2N_c)$ gauge group with one antisymmetric tensor $A$, one conjugate symmetric tensor $\tilde{S}$ and 8 fundamentals $T$ which come from 8 half D6-branes stuck on the half of O6-plane. This is indeed an anomaly free chiral theory. In the presence of D6-branes we have vectorlike fundamental flavors $Q, \tilde{Q}$.

We can rotate left NS5-brane by $\theta$ and right NS5-brane by $-\theta$. The superpotential is given by

$$W = \tilde{S}TT + mX^2 + XA\tilde{S},$$

where $m = tan\theta$ and $X$ is the adjoint of $SU(2N_c)$. The first term can be deduced by considering the flat directions as we will see shortly. The remaining terms are the superpotential of the product gauge group which are invariant under the orientifold projection. The resulting superpotential after integrating out the massive adjoint in case $m \neq 0$ is

$$W = \tilde{S}TT - \frac{1}{2m}(A\tilde{S})^2.$$ 

One can easily incorporate the superpotential due to the presence of D6-branes.

In the limit $m \to \infty$, we have Coulomb branch and $SU(2N_c)$ is generically broken to $U(1)^{N_c}$. In the brane picture, this corresponds to pairwise movement of D4-branes in $x^4, x^5$-direction compatible with the orientifold projection. In the Coulomb branch, the D4-branes are reconnected and stretched between the NS5-brane and its mirror. If $m$ is not zero, the mass of the adjoint does not allow any Coulomb branch. But we have several Higgs branches. Meson branches arise in the usual way by splitting D4-branes between D6-branes. In addition to that, there are interesting baryonic branches. In the flat directions where the baryon operator $A^n\tilde{Q}^{2N_c-n}$ gets an expectation value, the gauge group is broken to $Sp(n)$ with a symmetric tensor(adjoint) with a superpotential $W = mX^2$. In the brane picture, this corresponds to moving NS'5-brane in the positive $x^7$-direction. On the other hand, in the flat direction where the baryon operator $\tilde{S}^nQ^{2N_c-n}$ gets an expectation value, the gauge group is reduced to $SO(2n)$ with an adjoint with $W = mX^2$. This corresponds to moving NS'5-brane in the negative $x^7$-direction. The fundamentals arising from half D6-branes become

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3 We enumerate bosonic degrees of freedom in each N=1 supermultiplet.

4 Alternatively one can consider the global symmetry. The flavor symmetry coming from 8 half D6-branes is \(SO(8)\) and the first term of (3) should be introduced to have the right flavor symmetry.
massive, as is obvious from the brane picture. In case $m = 0$, these baryonic branches have an accidental N=2 supersymmetry.

3 Higgs branch of N=2 SO/Sp gauge theories

The M-theory lifting of the brane configuration with O6-plane was discussed in [7]. O6-plane with $-4 RR$ charge ($O6^-$) is lifted to Atiyah-Hitchin space in M-theory[17, 18]. For O6-plane with $+4 RR$ charge ($O6^+$), the corresponding M-theory interpretation is less clear, but it appears to be a $D_4$ singularity which cannot be blown up [3]. In order to describe the curve for the Coulomb branch of a gauge theory we just need to concentrate on the complex structure of these spaces. For $O6^-$ plane, far away from the origin, it can be described by

$$xy = \Lambda_{N=2}^{4N_c+4}v^{-4}. \quad (3)$$

In the usual convention of M-theory setting $v = x^4 + ix^5, t = exp(-\frac{x^6+ix^{10}}{R})$ where $R$ is the radius of the 11-th circle, we have $x \sim t^{-1}$ for a finite value of $y$ and $y \sim t$ for a finite value of $x$.

The curve for pure $Sp(N_c)$ gauge theory can be written as

$$x + y = B_{N_c}(v^2) + \Lambda_{N=2}^{2N_c+2} \frac{2}{v^2}, \quad (4)$$

or

$$y^2 - y(B_{N_c}(v^2) + \Lambda_{N=2}^{2N_c+2} \frac{2}{v^2}) + \Lambda_{N=2}^{4N_c+4}v^{-4} = 0 \quad (5)$$

using (3). Here $B_{N_c}(x)$ is a polynomial of degree $N_c$ in $x$ and $\frac{1}{v^4}$ term is allowed due to $v^{-4}$ contribution of the orientifold. We can interpret $\frac{1}{v^4}$ term as a manifestation that close to $v = 0$, (3) does not provide a good description of the orientifold background[4]. We can incorporate the $2N_f$ D6-branes by considering

$$xy = \Lambda_{N=2}^{4N_c+4-2N_f}v^{-4} \prod_{i=1}^{f}(v^2 - m_i^2). \quad (6)$$

Then the curve for the $Sp(N_c)$ gauge theory with $N=2 N_f$ hypermultiplet is given by

$$x + y = B_{N_c}(v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \frac{2}{v^2} \prod_{j=1}^{f} \frac{im_j}{v^2} \quad (7)$$

or

$$y^2 - y(B_{N_c}(v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \frac{2}{v^2} \prod_{j=1}^{f} \frac{im_j}{v^2}) + \Lambda_{N=2}^{4N_c+4-2N_f}v^{-4} \prod_{j=1}^{f}(v^2 - m_j^2) = 0. \quad (8)$$
Note that (7) is invariant under the $Z_2$ projection $v \rightarrow -v, x \leftrightarrow y$. Again nonzero $\frac{1}{v^2}$ term tell us that (3) is not a good description near $v = 0$.

The Type IIA brane configuration is invariant under the rotations $SO(2)_{4,5}$ and $SO(3)_{7,8,9}$ if the order parameters are also rotated accordingly. These can be interpreted as classical $U(1)$ and $SU(2)R$-symmetry of the 4-d world volume theory of the brane. In the M-theory construction, $SU(2)_{7,8,9}$ is preserved but $U(1)_{4,5}$ is broken. We can preserve the discrete subgroup $Z_{4N_c+4-2N_f}$ of $U(1)_{4,5}$ if we modify the $U(1)_{4,5}$ action so that the variables $x$ and $y$ have charges $4N_c$. The full $U(1)$ symmetry is restored if we assign the instanton charge $(4N_c + 4 - 2N_f)$ to the factor $\Lambda_{N=2}^{2N_c+2-N_f}$, reflecting the $U(1)_R$ anomaly. The modified $U(1)_{4,5}$ charges are

$$\begin{align*}
x & \quad y & \quad v & & \Lambda_{N=2}^{2N_c+2-N_f} \\
4N_c & \quad 4N_c & \quad 2 & & 4N_c + 4 - 2N_f
\end{align*}$$

(9)

For $N=2$ $Sp(N_c)$ gauge theory, the flavor symmetry is $SO(2N_f)$. But from (8), $SO(2N_f)$ symmetry is not clear. This can be seen by some variable changes. In case all $m_i$ vanish in (8), we have

$$xy = v^{-4+2N_f}$$

if we set $\Lambda_{N=2} = 1$ for simplicity. If we perform the variable change

$$\bar{x}' = \frac{-x + y}{2} v, \bar{y}' = -(x + y),$$

(11) becomes

$$(\bar{x}')^2 = \frac{v^2}{4}(\bar{y}')^2 - v^{-2+2N_f}.$$  

(12)

This can be recast to the standard $D_{N_f}$ singularity

$$x^2 + y^2 z = z^{n-1}$$

(13) by setting $z = v^2, x = i\bar{x}'$ and $y = \frac{\bar{y}'}{2}$. In case $N_f = 0$, (12) becomes $(\bar{x}')^2 = \frac{z}{4}(\bar{y}')^2 - \frac{1}{z}$. With the variable change $\bar{y}' = \bar{y} + \frac{z}{2}$, we have

$$(\bar{x}')^2 = \frac{z}{4}(\bar{y}')^2 + y.$$  

(14)

(14) defines the Atiyah-Hitchin space as a complex manifold in one of its complex structure[19]. With the new variables (14), the curve for the $Sp(N_c)$ gauge theory can be written as

$$\bar{y}' = B_{N_c}(z)$$

(15)
If all $m_i$ in (6) vanish, corresponding D6-branes are located at the same position in $x^4, x^5$-direction, but they can be separated in the $x^6$-direction. When all $m_i$ vanish, we saw that the surface develops $D_{N_f}$-singularity. The separation of the D6-brane in the $x^6$-direction corresponds to the resolution of singularity. The resolution of $D_{N_f}$ singularity makes it possible to identify the Higgs branch of N=2 $Sp(N_c)$ gauge theory with $N_f$ flavors. This will be the topic of the next subsection.

Note that the variable change (11) is singular at $x = y = v = 0$. Actually the description $xy = v^{-4+2N_f}$ is not valid near the origin and we have some difficulty in understanding the blowup just from this approximate description. As above, if we go to the correct description, we can straightforwardly obtain the resolution of $D_{N_f}$. One interesting work is done in [20] in light of this subtlety. The Atiyah-Hitchin space looks like a Taub-Nut space with a negative mass parameter asymptotically. The M-theory configuration corresponding $O6^-$-plane with D6-branes can be described approximately by a Taub-Nut space with a negative mass parameter superposed by multi Taub-Nut space corresponding to D6-branes. There are several two cycles in this space and the intersection matrix of the independent cycles is shown to be that of $D$-singularities. In one of the complex structures, this space can be described by the equation $xy = v^{-4+2N_f}$ if the number of D6-branes is $2N_f$.

For $SO(2N_c)$ gauge theory, the corresponding $O6^+$ plane is described by $xy = \Lambda_{N=2}^{4N-c-4}v^4$ and in the presence of $2N_f$ D6-branes the configuration is described by

$$xy = \Lambda_{N=2}^{4N-c-4}v^4 \prod_{i=1}^{N_f}(v^2 - m_i^2)$$

in M-theory. The Coulomb branch of $SO(2N_c)$ gauge theory with $N_f$ N=2 hypermultiplets is described by the curve

$$x + y = B_{N_c}(v^2),$$

or

$$y^2 - B_{N_c}(v^2)y + \Lambda_{N=2}^{4N-c-4+2N_f}v^4 \prod_{i=1}^{f}(v^2 - m_i^2) = 0.$$ (18)

### 3.1 Resolution of $D_n$ singularity

The (mixed Coulomb-) Higgs branches of the $Sp(N_c)$ gauge theory with $N_f$ fundamentals were analyzed in [13]. They are classified by an integer $r = 1, 2 \ldots \left\lfloor \frac{N_f}{2} \right\rfloor$. The $r$-th Higgs branch has quaternionic dimension $2rN_f - (2r^2 + r)$ and emanates from a $(N_c-r)$-dimensional
except at the inverse image of the singular point $x$. This is mapped onto the singular complex subspace of the Coulomb branch where there is an unbroken gauge group $Sp(r)$. Higgs branches are not corrected by quantum effects, but there are interesting subtleties in the way the Higgs branches emanates from the Coulomb branch. In the classical picture, Higgs branch emanates from the locus where $r$ of the $\phi_a$'s vanish where $\phi_a$'s appearing in $B_{Ne}(v^2) = \prod_{a=1}^{Ne}(v^2 - \phi_a^2)$. This will be true quantum mechanically for values $r$ where $Sp(r)$ gauge theory with $N_f$ flavors is not asymptotically free, i.e., for $r \leq \frac{N_f - 2}{2}$. For $r = \lfloor \frac{N_f}{2} \rfloor$, the low energy theory is asymptotically free and can be affected by strong coupling effects. We will see that the Higgs branch emanates from the locus where only the low energy theory is asymptotically free and can be affected by strong coupling effects.

The blowup of $D_{N_f}$ surface is explained in detail at \[19\], and we just quote their results. Starting from the singular surface of type $D_{N_f}$, $x^2 + y^2 z = z^{N_f - 1}$ in $x$-$y$-$z$ space, we obtain a 3-fold covered by $N_f$ open subsets $U_1, U_2, \cdots, U_{N_f}$ with coordinates $(s_1, t_1, z_1) = (x, y, z)$, $(s_2 = y, t_2 = \bar{x}, z_2), \ldots, (s_{N_f}, t_{N_f}, z_{N_f})$. These open sets are glued together by transition relations; $(s_j, t_j, z_j) = (s_{j+1}t_{j+1}z_{j+1}, s_{j+1}, t_{j+1}^{-1})$ for $j = 1, \ldots, N_f - 4$, $(s_{N_f - 3}, t_{N_f - 3}, z_{N_f - 3}) = (s_{N_f - 2}t_{N_f - 2}z_{N_f - 2}, s_{N_f - 2}, t_{N_f - 2}^{-1})$, and $(s_{N_f - 2}, t_{N_f - 2}, z_{N_f - 2}) = (z_{N_f - 1}t_{N_f - 1}, s_{N_f - 1}, t_{N_f - 1}^{-1})$. The projection to the $x$-$y$-$z$ space is given by

$$
\begin{align*}
x &= s_{j+1}^t \bar{z}_{j+1}^{-1} = s_{j+1}t_{j+1}z_{j+1}^{-1} \\
y &= s_{j+1}^{-1}t_{j+1}\bar{z}_{j+1}^{-1} = s_{j+1}^{-1}t_{j+1}z_{j+1}^{-1} \\
z &= s_{j+1}^{-1}\bar{z}_{j+1} = s_{j+1}^{-1}\bar{z}_{j+1}
\end{align*}
$$

on $U_1, \ldots, U_{N_f - 3}, U_{N_f - 1}$. We need the expression of $y, z$ on $U_{N_f - 2}$ and $U_{N_f}$:

$$
\begin{align*}
y &= \begin{cases}
z^{N_f - 2}s_{N_f - 2}t_{N_f - 2} = s_{N_f}z^{N_f - 2} & N_f : \text{even} \\
t_{N_f - 2}z^{N_f - 1} = s_{N_f}t_{N_f}z^{N_f - 1} & N_f : \text{odd}
\end{cases}
\end{align*}
$$

$$
\begin{align*}
z &= s_{N_f - 2}t_{N_f - 2}z_{N_f - 2} = s_{N_f}z_{N_f}.
\end{align*}
$$

The resolved $D_{N_f}$ surface is given by

$$
\begin{align*}
s_i + t_i^2z_i &= s_i^{N_f - 1 - i}z_i^{N_f - i} & \text{in } U_i \quad (i \neq N_f - 2, N_f) \\
s_{N_f - 2} + t_{N_f - 2}z_{N_f - 2} &= s_{N_f - 2}z_{N_f - 2}^2 & \text{in } U_{N_f - 2} \\
1 + s_{N_f}^2t_{N_f}z_{N_f} &= z_{N_f}^2 & \text{in } U_{N_f}
\end{align*}
$$

This is mapped onto the singular $D_{N_f}$ surface by \[19\], and the map is an isomorphism except at the inverse image of the singular point $x = y = z = 0$. The inverse image
consists of $N_f$ rational curves $C_1, \ldots, C_{N_f}$ where $C_i$ (i.e., $s_i = t_i = 0$), and also the $t_{i+1}$-axis in $U_{i+1}$ (i.e. $s_{i+1} = z_{i+1} = 0$). $C_{N_f-1}$ and $C_{N_f}$ are the loci $t_{N_f-2} = z_{N_f-2} ± 1 = 0$ parallel to the $s_{N_f-2}$-axis in $U_{N_f-2}$. $C_i$ and $C_i$ (i.e., $s_i = t_i = 0$), while $C_{N_f-2}$ intersects also with $C_{N_f-1}$ and $C_{N_f}$ at $s_{N_f-2} = t_{N_f-2} = z_{N_f-2} ± 1 = 0$. There is no other intersection of distinct $C_i$’s. The self-intersection of $C_i$ in the resolved surface can be shown to be $-2$.

### 3.2 N=2 Higgs branch of $Sp(N_c)$ gauge theory

In the Type IIA brane setup, Higgs branch is described by D4-branes suspended between D6-branes where they can move in $x^7, x^8, x^9$ directions. Likewise, in M-theory the transition to the Higgs branch occurs where the 5-brane intersects with the D6-branes. This is possible only when $y = {1 \over 2} \bar{y}' = {1 \over 2} B_{N_c}(z)$ of \[15\] passes through the singular point $x = y = z = 0$. Thus we need $B_{N_c}(z = 0) = 0$ or

$$B_{N_c}(z) = z^r (z^{N_c-r} + u_1 z^{N_c-r-1} + \cdots + u_{N_c-r}),$$ \[(25)\]

and we can assume that $u_{N_c-r}$ is non-zero.\(^5\) The terms in the bracket describes the Coulomb branch which is broken to $U(1)^{N_c-r}$ and this branch does not meet the D6-brane. The term $B_{N_c}(z) = z^r$ near $z = 0$ describes Higgs branch and we should use the resolved $D_{N_f}$ surface in order to describe this. Since higher terms of \[(25)\] are negligible, we can just consider the equation $B_{N_c} = z^r$.\(^6\) Let us look at the $F \equiv y - z^r$ in the $2j$-th patch $U_{2j}, 2j \leq N_f - 3$.

$$F = y - z^r \quad (26)$$  
$$= s_{2j}^j z_{2j}^{j-1} (1 - s_{2j}^{r-j} z_{2j}^{r-j+1}) \quad j \leq r \quad (27)$$  
$$= s_{2j}^r z_{2j}^{r-1} (1 - z_{2j}) \quad j = r \quad (28)$$  
$$= s_{2j}^r z_{2j}^{r-1} (-1 + s_{2j}^{r-r} z_{2j}^{r-1-r}) \quad j \geq r + 1 \quad (29)$$

The curve consists of several components. One component, to be denoted by $C$, is the zero of the last factor of the above equations \[(27),(28)\] and \[(29)\]. This extends to the region away from $x = y = z = 0$ reaching infinity. $F$ also has zero at $s_{2j} = 0$ and $z_{2j} = 0$. The defining equation of the surface is

$$s_{2j}^2 + t_{2j}^2 z_{2j} = s_{2j}^{N_f-1-2j} z_{2j}^{N_f-2j}. \quad (31)$$

\(^5\)For simplicity, we set $\Lambda_{N=2}$ to $1$ in this subsection otherwise stated.

\(^6\)We will follow the presentation of \[16\] closely.
There are two branches of zero of $F$ for each $j$; $s_{2j} = z_{2j} = 0$ and $s_{2j} = t_{2j} = 0$ which corresponds to the rational curves $C_{2j-1}$ and $C_{2j}$ respectively. Near the first branch $s_{2j} = z_{2j} = 0$, $(t_{2j}, z_{2j})$ is a good coordinate, i.e. $s_{2j}$ can be uniquely expressed in terms of $t_{2j}$ and $z_{2j}$ by the defining equation. Since $t_{2j} \neq 0$ generically, $s_{2j} \sim z_{2j}$ near $C_{2j-1}$. Hence $F \sim z_{2j}^{2j-1}$ for $j \leq r$ while $F \sim z_{2j}^{2j}$ for $j > r$. Thus, the zero of $F$ at $C_{2j-1}$ is of order $2j - 1$ for $j \leq r$ and order $2r$ for $j > r$. Near the second branch $s_{2j} = t_{2j} = 0$, $(t_{2j}, z_{2j})$ is again a good coordinate, and $s_{2j} \sim t_{2j}^2$ for $z_{2j} \neq 0$. Thus, $F \sim t_{2j}^{2j}$ for $j \leq r$ and $F \sim t_{2j}^{2j}$ for $j > r$ near $C_{2j}$. Namely, $F$ has a zero at $C_{2j}$ of order $2j$ for $j \leq r$ and $2r$ for $j > r$. By looking at the equation for $j = r$, we see that the infinite curve $C$ and the rational curve $C_{2r}$ meet at the point $s_{2r} = t_{2r} = 0, z_{2r} = 1$. For $U_{2j-1}$ the analysis is similar and we obtain the same result. In summary, in the patches $U_1, \ldots, U_{N_f-3}$, $F$ has zeros at $C_1, C_2, \ldots, C_{2r-1}, C_{2r}, C_{2r+1}, \ldots, C_{N_f-3}$ and $C$ of order $1, 2, \ldots, 2r-1, 2r, 2r, \ldots, 2r$ and 1 respectively.

Let us now look at the function $F$ in $U_{N_f-2}$. If $2r \leq N_f - 3$, from (20)-(21) we can see that $y$ is divisible by $z^r$, and $y/z^r - 1$ of $F = z^r(y/z^r - 1)$ has a single zero at $C$. Now, we consider the zero of $z^r = (s_{N_f-2}t_{N_f-2}z_{N_f-2})^r$. By looking at the defining equation of the surface

$$s_{N_f-2}(z_{N_f-2}^2 - 1) = t_{N_f-2}z_{N_f-2}^2$$

we see that there are four branches of zero: $s_{N_f-2} = z_{N_f-2} = 0$, $s_{N_f-2} = t_{N_f-2} = 0$, $t_{N_f-2} = z_{N_f-2} - 1 = 0$ and $t_{N_f-2} = z_{N_f-2} + 1 = 0$, which corresponds to $C_{N_f-3}, C_{N_f-2}, C_{N_f-1}$ and $C_{N_f}$ respectively. Near $C_{N_f-3}$ where $s_{N_f-2} = z_{N_f-2} = 0$ and $t_{N_f-2} \neq 0$, the surface is coordinatized by $(t_{N_f-2}, z_{N_f-2})$ and $F \sim s_{N_f-2}^r z_{N_f-2}^r \sim z_{N_f-2}^{2r}$ has zero at $C_{N_f-3}$ of order $2r$, as we have seen. Near $C_{N_f-2}$ where $s_{N_f-2} = t_{N_f-2} = 0$ and $z_{N_f-2} \neq 0$, the surface is again coordinatized by $(t_{N_f-2}, z_{N_f-2})$ and $F \sim s_{N_f-2}^r t_{N_f-2}^r \sim t_{N_f-2}^{2r}$ has zero at $C_{N_f-2}$ of order $2r$.
Near $C_{N_f-1}$ or $C_{N_f}$ where $t_{N_f-2} = z_{N_f-2} \mp 1 = 0$ and $s_{N_f-2} \neq 0$, the surface is coordinatized by $(s_{N_f-2}, z_{N_f-2})$, and $F \sim t_{N_f-2} \sim (z_{N_f-2} \mp 1)$" has zero at $C_{N_f-1}$ and $C_{N_f}$ of order $r$. The zero of $F$ in the part in $U_{N_f-1}$ and $U_{N_f}$ can be seen in the same way, and the analysis agrees with that in $U_{N_f-2}$. Once the curve degenerates and rational curves are generated, they can move in $(x^{7}, x^{8}, x^{9})$-directions. This motion together with the integration of the chiral two-form field over such rational curves parametrize the Higgs branch of the 4-d theory [2]. Since the total rational components are $C_i$ times the order of $F$ in $C_i$, the quaternionic dimension of the $r$-th Higgs branch is $\sum_{i=1}^{2r} i + 2r \times (N_f - 2r - 2) + 2r = 2rN_f - r(2r + 1)$ as expected.

For $N_f = 2r, 2r + 1, 2r + 2$, the analysis is a bit different since the counting in $U_{N_f-2}$ is modified. For $N_f = 2r + 2$, with $y = \frac{a}{2} z^r$ the counting in $U_1, U_2 . . . U_{N_f-3}$ is the same as before and we have zeroes of order $1, 2, . . . , k, . . . , N_f - 3$ in $C_1, C_2, . . . , C_k, . . . , C_{N_f-3}$ respectively. In $U_{N_f-2}$, we have

$$y - \frac{a}{2} z^r = (s_{N_f-2}t_{N_f-2}z_{N_f-2})^{r-1}(s_{N_f-2}t_{N_f-2} - \frac{a}{2} s_{N_f-2}t_{N_f-2}z_{N_f-2}).$$ (32)

At $C_{N_f-3}$, $y - \frac{a}{2} z^r \sim (s_{N_f-2}t_{N_f-2})^{r-1}s_{N_f-2} \sim z_{N_f-2}^{2r-1}$ which gives zeroes of order $N_f - 3$ as obtained above. At $C_{N_f-2}$, $y - \frac{a}{2} z^r \sim (s_{N_f-2}t_{N_f-2})^{r}(1 - \frac{a}{2} z_{N_f-2}) \sim (t_{N_f-2})^{2r}$ and we see that the infinite curve $C$ meet with $C_{N_f-2}$ at $z_{N_f-2} = \frac{2}{a}$. At $C_{N_f-1}$ where $t_{N_f-2} = z_{N_f-2} - 1 = 0$, $y - \frac{a}{2} z^r \sim (t_{N_f-2})^{r}(1 - \frac{a}{2} z_{N_f-2})$ and the order of zero is $r$ if $a \neq 2$ and $r + 1$ if $a = 2$. At $C_{N_f}$ where $t_{N_f-2} = z_{N_f-2} + 1 = 0$, $y - \frac{a}{2} z^r \sim (t_{N_f-2})^{r}(1 - \frac{a}{2} z_{N_f-2})$ and the order of zero is $r$ if $a \neq -2$ and $r + 1$ if $a = -2$. Thus the quaternionic dimension we get is $\frac{N_f(N_f-1)}{2} - 1$ if $a \neq \pm 2\Lambda_{N_f=2}^{2N_f+2-N_f}$ and $\frac{N_f(N_f-1)}{2} - 1$ if $a = \pm 2\Lambda_{N_f=2}^{2N_f+2-N_f}$ if we recover the scale factor. The former coincides with the $(r = \frac{N_f}{2})$-th Higgs branch and the latter coincides with the dimension of the $(r = \frac{N_f}{2})$-th Higgs branch. Also this result agrees with the findings of [21]; the $(r = \frac{N_f}{2})$-th Higgs branch emanates from the locus where only $\frac{N_f-2}{2}$ of $\phi_a$’s vanish and the product of nonvanishing $\phi_a^2$’s is $\pm 2\Lambda_{N_f=2}^{2N_f+2-N_f}$. In [21], this result is obtained using O4-plane while we use O6-plane. For $r = \frac{N_f-1}{2}$, similar calculation shows that the Higgs-branch emanates from the locus where all $\frac{N_f-1}{2}$ of $\phi_a$’s vanish and gives the correct dimension $\frac{N_f(N_f-1)}{2}$.

\footnote{Previously the coefficient of $z^r$ is not important but for $N_f = 2r + 2$ this is going to be important. The $\frac{1}{2}$ factor comes from (3) the relation between $y$ and $y', y = \frac{y'}{2}$.}
3.3 The N=2 Higgs branch of $SO(2N_c)$ gauge theory

Our understanding of the Higgs branch of the N=2 $SO(2N_c)$ gauge theory is incomplete, since we do not have a good understanding of the M-theory geometry corresponding to $O6^+$-orientifold. According to [15], the (mixed Coulomb)-Higgs branches are again classified by an integer $r = 1, 2, \ldots, \left\lfloor \frac{N_f}{2} \right\rfloor$ where $N_f$ is the number of hypermultiplets. The $r$-th Higgs branch has quaternionic dimension $2rN_f - (2r^2 - r)$ and emanates from a $(N_c - r)$-dimensional complex subspaces of the Coulomb branch, where there is an unbroken gauge group $SO(2r)$.

Since $SO(2N_c)$ gauge theory are asymptotically free when $N_f \leq 2N_c - 2$, the $SO(2r)$ factors at the root of the $r$-branches are all infrared-free, and will remain unbroken quantum mechanically. The lifting of D6-brane configuration is described by $xy = \Lambda^{4N_c - 4 - 2N_f} y^{4 + 2N_f}$ in M-theory and the space has $D_{4+N_f}$ singularity where $D_4$ singularity cannot be blownup. This means that we can blowup the singularity until we get at $D_4$ singularity. Again the curve for the Coulomb branch is given by $y = \frac{1}{2}B_{N_c}(z)$. If we have ordinary $D_{4+N_f}$ singularity, then we can have $N_f + 4$ blowup rational curves $C_1, C_2, \ldots, C_{N_f}, C_{N_f+1}, C_{N_f+2}, C_{N_f+3}, C_{N_f+4}$. If we have $F = y - z^r$ near singularity, we obtain the zeroes of $1, 2, \ldots, 2r, 2r, 2r, 2r, r, r$ from $C_1, C_2, \ldots, C_{2r}, \ldots, C_{N_f}, C_{N_f+1}, C_{N_f+2}, C_{N_f+3}, C_{N_f+4}$ respectively. If the $D_4$ singularity cannot be blowup, we should exclude the zeroes obtained from $C_{N_f+1}$ to $C_{N_f+4}$. This is especially obvious when we just consider the pure $SO(2N_c)$ gauge theory with $N_f = 0$. Since this theory does not have a Higgs branch, we should not have any contribution from the frozen $D_4$ singularity. Thus the quaternionic dimension of the $r$-th Higgs branch is given by $\Sigma_{i=1}^{2r} i + 2r(N_f - 2r) = 2rN_f - 2r^2 + r$ as desired.

Unlike the case of $Sp(N_c)$ case where we can understand $SO(2N_f)$ flavor in various ways in M-theory, we do not understand how the $Sp(N_f)$ flavor symmetry is realized for $SO(2N_c)$ gauge group in M-theory picture. This requires better understanding of $D_{4+N_f}$ singularity with frozen $D_4$ singularity. One intriguing fact is that the geometry of usual
$D_{4+N_f}$ singularity and that of $D_{4+N_f}$ singularity with frozen $D_4$ singularity are the same. The geometry is determined uniquely by the hyperkähler property of space metric in M-theory. There should be some additional ingredient in M-theory which freezes $D_4$ singularity thereby producing Sp-symmetry, but currently that ingredient is not well understood.

4 N=1 rotated curve

4.1 $Sp(N_c)$ gauge group

We can break N=2 to N=1 by giving a bare mass $\mu$ to the adjoint chiral multiplet. For small $\mu$, we can use the Seiberg-Witten curve to analyze the structure of vacua. Such analysis is done in [13]. Most of the Coulomb branch is lifted except for discrete points where the underlying curve is degenerated into a curve of genus zero. The remaining vacua are in the locus where $r = \lfloor \frac{N_f}{2} \rfloor$ of the parameters $\phi_a$ in $B_{N_c} = \prod_{a=1}^{N_f} (v^2 - \phi_a^2)$ vanish as well as in the locus where $r = N_f - N_c - 2$ of them vanishing. The former is called the A-branch root and the latter is called the B-branch root in [21]. The B-branch is a singlet which is invariant under discrete $Z_{2N_c+2-N_f}R$-symmetry, whereas the A-branch root consists of $2N_c + 2 - N_f$ points. For $N_f$ odd these are related by $Z_{2N_c+2-N_f}$, but for $N_f$ even they fall into two separate orbits, each having $N_c + 1 - \frac{N_f}{2}$ points. On the other hand, pure Yang-Mills theory has $N_c + 1$ supersymmetric vacua related by the discrete $R$-symmetry.

In the Type IIA setup with O6-plane describing SO/Sp gauge theories, we can rotate the left NS5-brane by $\theta$ and the right NS5-brane by $-\theta$. This corresponds to giving a mass for the adjoint chiral multiplet in N=2 theory, thereby reducing N=2 to N=1 [22]. Both of the 5-branes are extended in the $x^8, x^9$-directions, and we need additional complex coordinates

$$w = x_8 + ix_9$$

in order to describe the corresponding configuration of the M5-brane. Since the two NS5-branes correspond to the two asymptotic regions with $v \to \infty$, where $y \sim v^{2N_c}$ and $x \sim v^{2N_c}$ respectively. Thus the rotation requires the boundary conditions

$$w \to \mu v, \quad y \sim v^{2N_c} \sim \lambda^{2N_c}, \quad v \to \infty$$

$$w \to -\mu v, \quad x \sim v^{2N_c}, \quad y \sim v^{-4+2N_f-2N_c} \sim \lambda^{4-2N_f+2N_c}, \quad v \to \infty.$$

We can identify $\mu$ as the mass of the adjoint chiral multiplet by using $R$ symmetry. The N=2 configuration is invariant under the rotation $U(1)_{4,5}$ and $SU(2)_{7,8,9}$. After the rotation,
$SU(2)_{7,8,9}$ is broken to $U(1)_{8,9}$ if $\mu$ is assigned $(-2,2)$ charge of $U(1)_{4,5} \times U(1)_{8,9}$. Since this is the same as the $R$-charge of the mass of the adjoint field and since there is no other parameters charged under $U(1)_{8,9}$, the two quantities should be identified.

Consider $Sp(N_c)$ gauge theory with $N_f$ hypermultiplets. The charges of the coordinates and parameters are the following.

\[
\begin{array}{ccc}
 & U(1)_{4,5} & U(1)_{8,9} \\
v & 2 & 0 \\
w & 0 & 2 \\
y & 4N_c & 0 \\
x & 4N_c & 0 \\
\mu & -2 & 2 \\
\Lambda_{N=2}^{2N_c+2-N_f} & 4N_c + 4 - 2N_f & 0 \\
\end{array}
\]

(36)

The combination of $\frac{1}{2}U(1)_{4,5} + \frac{1}{2}U(1)_{8,9}$ makes $\mu$ invariant, and its $Z_{2N_c+2-N_f}$ subgroup makes $\Lambda_{N=2}^{2N_c+2-N_f}$ invariant. In field theory, the mass of the adjoint chiral multiplet lifts the Coulomb branch. Thus rotating the branes can only be done at points where the curve degenerates to a curve of genus zero. Then we can introduce an auxiliary complex parameter $\lambda$ and identify the genus zero curve as the complex $\lambda$ plane with some points deleted or points at infinity added. $v, w$ and $t$ can be expressed as a rational function of $\lambda$[24]. Here we assume that the rotated curve consists of a single component. The case where the curve consists of more than one components will be treated separately. For rotated curve $C$, we can consider $\hat{C}$, the projection of $C$ into $x, y, v$ space. Since $\mu$ is the only parameter, we cannot deform the $N=2$ curve equation without breaking $U(1)_{8,9}$ symmetry. Thus the projection of the rotated curve $\hat{C}$ on $x, y, v$ remains the same $N=2$ curve [23]. $v$ has two poles corresponding to the two NS5-branes and we take two poles to occur at $\lambda = 0$ and $\lambda = \infty$. Thus we have

\[v = b(\lambda + \frac{1}{\lambda}).\]

(37)

There might be a constant term in the right hand side, but this should vanish if we take $Z_2$ acts as $\lambda \to -\frac{1}{\lambda}$. We take the asymptotic conditions as

\[w \sim \mu v, \quad \lambda \to \infty\]

\[w \sim -\mu v, \quad \lambda \to 0.\]

(38)

This determines

\[w = \mu b(\lambda - \frac{1}{\lambda}).\]

(39)
Thus \( w \) is invariant under the \( Z_2 \) action \( \lambda \to -\frac{1}{\lambda} \) while \( v \) is odd. Asymptotic conditions for \( x, y \) are given by

\[
\begin{align*}
y & \sim v^{2N_c} \sim \lambda^{2N_c}, \quad \lambda \to \infty \quad \text{(40)} \\
x & \sim v^{2N_c}, \quad y \sim v^{-4+2N_f-2N_c} \sim \lambda^{4-2N_f+2N_c}, \quad \lambda \to 0 \quad \text{(41)}
\end{align*}
\]

where \( x, y \) satisfy (3) and (7). Thus \( y = \lambda^{2N_c+4-2N_f} \frac{P_{2N_f}(\lambda)}{P_2(\lambda)} \) where \( P_i(\lambda) \) is the \( i \)-th polynomial of \( \lambda \). Now under the \( Z_2 \) action \( \lambda \to -\frac{1}{\lambda} \), \( y \) is maped to \( x \) so \( x = (\frac{1}{\lambda})^{2N_c+4-2N_f} \frac{P_{2N_f}(\lambda)}{P_2(-\frac{1}{\lambda})} \). The expression should satisfy (3).

\[
\begin{align*}
xy &= \Lambda_{N=2}^{4N_c+4-2N_f} v^{-4} \prod_{i=1}^{N_f} (v^2 - m_i^2) \quad \text{(42)} \\
&= \Lambda_{N=2}^{4N_c+4-2N_f} b^{-4+2N_f} (\lambda + \frac{1}{\lambda})^{-4} \prod_{i=1}^{N_f} ((\lambda + \frac{1}{\lambda})^{2} - \frac{m_i^2}{b^2}). \quad \text{(43)}
\end{align*}
\]

From this, we can set \( P_4(\lambda) = \lambda^2(\lambda + \frac{1}{\lambda})^2 \) and \( P_{2N_f}(\lambda) = \Lambda_{N=2}^{4N_c+4-2N_f} b^{-4+2N_f} \prod_{i=1}^{N_f} ((\lambda + \frac{1}{\lambda})^2 - \frac{m_i^2}{b^2}) \). This can be satisfied if we choose \( P_{2N_f}(\lambda) = C \prod_{i=1}^{N_f} (\lambda^2 - a_i^2) \) with \( a_i + \frac{1}{a_i} = \pm \frac{m_i}{b} \) and \( C^2 (-1)^{N_f} \prod_{i=1}^{N_f} a_i^2 = \Lambda_{N=2}^{4N_c-4+2N_f} b^{4-2N_f} \). We choose

\[
a_i = \frac{2}{\frac{m_i}{b} + \sqrt{(\frac{m_i}{b})^2 - 2}}. \quad \text{(44)}
\]

Therefore we have

\[
\begin{align*}
y &= \frac{\lambda^{2N_c+2-2N_f} C \prod_{i=1}^{N_f} (\lambda^2 - a_i^2)}{(\lambda + \frac{1}{\lambda})^2} \quad \text{(45)} \\
x &= \frac{(\frac{1}{\lambda})^{2N_c+2-2N_f} C \prod_{i=1}^{N_f} ((\frac{1}{\lambda})^2 - a_i^2)}{(\lambda + \frac{1}{\lambda})^2}.
\end{align*}
\]

From \( x + y = B_{N_c}(v^2) = v^{2N_c} + \cdots \), we have \( C = b^{2N_c} \) comparing the highest order term in \( \lambda \). Therefore we obtain

\[
b^{4N_c-2N_f+4} = \frac{\Lambda_{N=2}^{4N_c+4-2N_f}}{(-1)^{N_f} \prod_{i=1}^{N_f} a_i^2}. \quad \text{(46)}
\]

Note that with \( a_i \) in (44), in the limit of \( m_{N_f} \to \infty \), the expression (46) is correctly reduced to \( N_f - 1 \) case with the replacement

\[
\Lambda_{N=2}^{4N_c+4-2N_f} (-m_i^2) \to \Lambda_{N=2}^{4N_c+4-(N_f-1)}. \quad \text{(47)}
\]

This is consistent with the one-loop matching of the gauge coupling.
In case $m_i = 0$, $a_i = \pm i$ and $b_{N_{2}}^{4N_{c}+4-2N_{f}} = \Lambda_{N_{2}}^{4N_{c}+4-2N_{f}}$. One can check that with the expression (43), there’s a special $u_i$’s for $B_{N_c}(v^2) = v^{2N_c} + u_1 v^{2N_c-2} + \ldots u_{N_c}$ where $x + y = B_{N_c}(v^2) + \prod_{i=1}^{N_f} \frac{\sin^{2N_c+2}}{\sqrt{2}}$ holds. $b^2$ has $2N_c + 2 - N_f$ solutions. The $Z_{2N_c+2-N_f}$ rotational symmetry is completely broken by $b^2$ for odd $N_f$ and is broken to $Z_2$ for even $N_f$, since its generator acts $b^2$ as $b^2 \rightarrow e^{\frac{4\pi i}{2N_c+2-N_f}} b^2$. For $N_f = 0$, apparently we obtain $2N_c + 2$ solutions from (46). But we know that there are only $N_c + 1$ vacua for pure Yang-Mills theory. The apparent discrepancy is resolved if we carefully look at the equation $x + y = B_{N_c}(v^2) + \frac{2\Lambda_{N_{2}}^{2N_{c}+2}}{v^2}$. $b$ should satisfy $b^{2N_c+2} = (-1)^{N_c} \Lambda_{N_{2}}^{2N_{c}+2}$ in order to cancel the $\frac{1}{v^2}$ term.

The above result is consistent with the $N=1$ vacua of the $Sp(N_c)$ gauge theory coming from the $A$-branch. The structure of the above solution is similar to [21], and we can follow their argument. One can show that $B_{N_c}(v^2)$ defining the projected curve in $x$-$y$-$v$ plane is of the form $B_{N_c}(v^2) = cv^{2r} + \ldots, c \neq 0$, where $r \geq \frac{N_f-1}{2}$ for $N_f$ odd and $r = \frac{N_f-2}{2}$ for $N_f$ even which is the property of the $A$-branch. This can be seen as follows. As $\lambda \rightarrow \pm i$, $v \rightarrow 0$ and $y \rightarrow v^{N_f-2}$ as can be seen from (37) and (43). On the other hand, equation (7) implies

$$y \sim -\frac{c v^{2r}}{2} \pm \sqrt{\frac{c^2 v^{4r}}{4} - \Lambda_{N_{2}}^{4N_c+4-2N_f} v^{2N_f-4}}. \quad (48)$$

The two statements are consistent only when $r \geq \frac{N_f-2}{2}$. In particular, if $N_f$ is even, equation (37) and (43) shows that $y$ is a single valued function of $v$ near $v \sim 0$. This is possible only if the two terms in the square root of (48) cancel. Thus $r = \frac{N_f-2}{2}$ and $c = \pm 2\Lambda_{N_{2}}^{2N_c+2-N_f}$.

So far we assume that the infinite curve consists of a single component. But suppose the curve is factorized so that the infinities are separated in covering space. When such factorization occurs, we just have to rotate each component. One can see that the factorization is unique and is given by

$$(y - v^{2N_c})(y - \Lambda_{N_{2}}^{4N_c+4-2N_f} v^{2N_f-2N_c-4}) = 0. \quad (49)$$

This gives $B_{N_c}(v^2) = v^{2N_c} + \Lambda_{N_{2}}^{4N_c+4-2N_f} v^{2N_f-2N_c-4}$. From this, we see that the factorization is possible only when $N_f \geq N_c + 2$. Also this shows that the curve belongs to $r = N_f - N_c - 2$ Higgs branch. The rotated curve is just the union of

$$y = v^{2N_c} \quad (50)$$

$$x = \Lambda_{N_{2}}^{4N_c+4-2N_f} v^{2N_f-2N_c-4}$$

$$w = \mu v$$
\[ x = v^{2N_c} \]  
\[ y = \Lambda^{4N_c+4-2N_f} v^{2N_f-2N_c-4} \]  
\[ w = -\mu v \]  

4.2 \( SO(2N_c) \) gauge theory

Similar analysis can be done for \( SO(2N_c) \) case. First consider the rotated curve consisting of one component in covering space. For such a rotated curve \( C \), the projection of the rotated curve \( \tilde{C} \) on \( x, y, v \) remains the same N=2 curve. \( v \) and \( w \) are given by the same expressions as before

\[ v = b(\lambda + \frac{1}{\lambda}). \]  
\[ w = \mu b(\lambda - \frac{1}{\lambda}). \]  

Asymptotic conditions for \( x, y \) are given by

\[ y \sim v^{2N_c} \sim \lambda^{2N_c}, \quad \lambda \to \infty \]  
\[ x \sim v^{2N_c}, \quad y \sim v^{4+2N_f-2N_c} \sim \lambda^{4-2N_f+2N_c}, \lambda \to 0 \]  

where \( x, y \) satisfy (16) and (17). By the similar calculation to \( Sp(N_c) \) case, we obtain

\[ y = \lambda^{2N_c-2-2N_f}(\lambda + \frac{1}{\lambda})^2 C \prod_{i=1}^{N_f}(\lambda^2 - a_i^2) \]  
\[ x = (\frac{1}{\lambda})^{2N_c-2-2N_f}(\lambda + \frac{1}{\lambda})^2 C \prod_{i=1}^{N_f}(\frac{1}{\lambda^2} - a_i^2) \]  

with \( C^2(-1)^{N_f} \prod_{i=1}^{N_f} a_i^2 = \Lambda^{4N_c-4-2N_f} b^{4+2N_f} \) and

\[ b^{4N_c-2N_f-4} = \frac{\Lambda^{4N_c-4-2N_f}}{(-1)^{N_f} \prod_{i=1}^{N_f} a_i^2}. \]  

In the limit of \( m_{N_f} \to \infty \), we correctly reproduce the formula for \( N_f - 1 \) case. If all \( m_i \) vanish in (16), we have \( b^{4N_c-2N_f-4} = \Lambda^{4N_c-4-2N_f} \) and \( b^2 \) has \( 2N_c - 2 - N_f \) solutions. The \( Z_{2N_c-2-N_f} \) rotational symmetry is completely broken by \( b^2 \) for odd \( N_f \) and is broken to \( Z_2 \) for even \( N_f \). For \( N_f = 0 \), we obtain \( 2N_c - 2 \) solutions. This agrees with the Witten index calculation for \( N_c \geq 3 \), For \( N_c = 2 \), we have 4 vacua since \( SO(4) \sim SU(2) \times SU(2) \) and each
SU(2) gives two vacua. Thus we should have two more vacua, which arise in the reducible component as we will see shortly.

By the similar argument to \( Sp(N_c) \) gauge group, we can see that \( B_{N_c}(v^2) \) defining the projected curve in \( (x, y, v) \) plane is of the form \( B_{N_c}(v^2) = cv^{2r} + \cdots, c \neq 0 \) where \( r \geq \frac{N_f + 2}{2} \). For \( N_f \) even, \( r = \frac{N_f + 2}{2} \) and \( c = \pm 2 \Lambda_{N_{N_{c}}}^{2N_{c} - 2 - N_f} \). For \( N_f \) even, the maximal value \( r \) allowable for the Higgs branch is \( \frac{N_f}{2} \). Thus in order to have \( B_{N_c}(v^2) = v^{N_f}v^2(v^{2N_c - N_f - 2} + \cdots) \) the \( v^2 \) factor should be attributed to the Coulomb branch. This implies that the \( N=1 \) vacua are emanated from the \( \frac{N_f}{2} \)-th Higgs branch where the residual Coulomb branch is constrained to the \( N_c - 1 - \frac{N_f}{2} \)-dimensional subspace.

If the infinite curve is reducible in covering space, it is uniquely given by

\[
(y - v^{2N_c})(y - \Lambda_{N=2}^{4N_c - 4 - 2N_f}v^{2N_f - 2N_c + 4}) = 0. \tag{56}
\]

and the rotated curve is given by the union of

\[
y = v^{2N_c} \tag{57}
\]
\[
x = \Lambda_{N=2}^{4N_c - 4 - 2N_f}v^{2N_f - 2N_c + 4}
\]
\[
w = \mu v
\]

and

\[
x = v^{2N_c} \tag{58}
\]
\[
y = \Lambda_{N=2}^{4N_c - 4 - 2N_f}v^{2N_f - 2N_c + 4}
\]
\[
w = -\mu v.
\]

Alternative we can have \( w = -\mu v \) in (57) and \( w = \mu v \) in (58). These two possibilities are related by discrete \( R \) symmetry. This reducible curve exists for \( N_c = 2 \) even with \( N_f = 0 \) and two possibilities mentioned above provide two additional vacua for pure \( SO(4) \) gauge theory with \( N=1 \) supersymmetry.

In [25], there appear the expressions for rotated curves of \( SO(2N_c)/Sp(N_c) \) gauge theories. Since their Type IIA configuration is the same as our configuration, their results should be consistent with ours. Indeed, from (52), we have \( (w + \mu v)(w - \mu v) = -4\mu^2 b^2 \) if we eliminate \( \lambda \). This agrees with (5.3) of [26]. Also if we change the variables,

\[
\tilde{y} = \frac{y}{\prod_{i=1}^{N_f}(v + m_i)} = \lambda^{2N_c - 2}v^2 \frac{C}{b^{N_f + 2}} \prod_{i=1}^{N_f}(\lambda + \frac{1}{a_i}) \tag{59}
\]
\[ = v^2 \lambda^{2N_c-2} \frac{\Lambda^{2N_c-2-N_f} \prod_{i=1}^{N_f} (1 - \frac{a_i}{\lambda})}{(-1)^{N_f} \prod_{i=1}^{N_f} a_i \prod_{i=1}^{N_f} (\lambda + \frac{1}{a_i})} \]  

(60)

using (55) (52), the expression coincides with (5.4) of [25] up to constant which can be absorbed into \( \Lambda_{N=2} \) in our expression. And \( \tilde{x} = \prod_{i=1}^{N_f} \frac{x}{v-m_i} \) gives the corresponding expression in [25]. Note that \( \tilde{x} \tilde{y} \propto v^4 \). This change of variable corresponds to moving D6-branes outside of NS5-branes, thereby creating semi-infinite D4-branes. Indeed from (58), we have after the change of variables,

\[ \tilde{y}^2 \prod_{i=1}^{N_f} (v + m_i) - B_{N_c}(v^2) \tilde{y} + v^4 \prod_{i=1}^{N_f} (v - m_i) = 0. \]  

(61)

which describes \( SO(2N_c) \) gauge theory where hypermultiplets are realized by semi-infinite D4-branes in Type IIA setting. Thus our expression represents M-theory lifting of the Type IIA configuration with D6-branes while the expressions in [25] represents M-theory picture where D6-branes are moved to infinity, thereby producing semi-infinite D4-branes. In [25], there are several consistent checks for their expressions. They checked that in a suitable limit, the expressions recover the Type IIA brane configuration they started with. They checked that decoupling of a flavor works correctly and the expressions encode N=1 duality properly. All those checks in [25] can be consistent checks for our expressions as well. The same relation holds for the rotated curve for \( Sp(N_c) \) gauge group.

5 Chiral gauge theory

In order to get the M5-brane picture of the chiral gauge theory, it’s better to start with the N=2 \( SU(2N_c) \) gauge theory with symmetric matter and consider the situation where the middle NS5-brane decouples. The corresponding brane configuration is depicted in Fig 3. This theory is explored in detail in [4]. The bifundamentals of the product gauge group gives the symmetric matter under the orientifold projection. The curve describing Coulomb branch is given by

\[ y^3 + y^2 \prod_{i=1}^{2N_c} (v - a_i) + y^2 \prod_{i=1}^{2N_c} (v + a_i) + v^6 = 0 \]  

(62)

with \( xy = v^4 \). The coefficient in front of \( y^2 \) describes the position of D4-branes on the left of the middle NS5-brane and the coefficient in front of \( y \) describes the D4-branes on the right.

\footnote{Again, we set \( \Lambda_{N=2} = 1 \).}
Figure 5: The left figure is the brane configuration of $SU(2N_c)$ gauge theory with symmetric matter. The right figure is the chiral gauge theory configuration.

of the middle NS5-brane in the classical limit. Generically the positions of left D4-branes in $v$ are different from the positions of right D4-branes and we cannot pull away the middle NS5-brane in $x^7$-direction without breaking gauge symmetry.

However, if the curve has the special form

$$y^3 + y^2 \prod_{i=1}^{N_c} (v^2 - a_i^2) + yv^2 \prod_{i=1}^{N_c} (v^2 - a_i^2) + v^6 = 0,$$

(63)

the positions of left D4-branes match the positions of right D4-branes. In this case, a left D4-brane and the corresponding right D4-brane can be recombined into a single 4-brane and the middle 5-brane decouples. This means that the curve is factorized as

$$(y + v^2)(y^2 + \prod_{i=1}^{N_c} (v^2 - a_i^2) - v^2)y + v^4) = 0.$$  

(64)

Without the orientifold projection, this describes the special case of $SU(2N_c) \times SU(2N_c)$ gauge theory. The first factor corresponds to the middle NS5-brane and we can rotated it as we wish. But due to the orientifold projection, we have only two possibilities for the middle 5-brane. One is the NS5-brane and the other is NS$'5$-brane where the rotated angle is $\frac{\pi}{2}$. And in both cases, central NS5-brane and NS$'5$-brane are described by a decoupled rational curve. If the middle 5-brane is a NS$'5$-brane, the brane configuration describes the chiral gauge theory.

If we look at the brane configuration of the chiral gauge theory, one can see that each left D4-branes match a right D4-branes and recombine into a single D4-brane. Again NS$'5$-brane is decoupled and is described by a rational curve in M-theory picture. Thus the suggested curve for the chiral gauge theory is the union of $v = 0, x = 0, y = 0$ (isomorphic to $w$-plane) and

$$y^2 + \prod_{i=1}^{N_c} (v^2 - a_i^2) - v^2 + v^4 = 0.$$  

(65)
with \( w = 0 \). An interesting thing is that the second factor can be interpreted as two ways; N=2 \( SO(2N_c) \) pure gauge theory or N=2 \( Sp(N_c) \) gauge theory with 4 fundamental N=2 massless hypermultiplets which are decomposed as 8 N=1 chiral multiplets. Recall that the general form of \( Sp(N_c) \) curve is

\[
y^2 + (B_{N_c}(v^2) + 2A)y + v^{-4} \prod_{i=1}^{N_f} (v^2 - a_i^2) = 0 \tag{66}
\]

with \( A = \frac{2}{v^4} \prod_{i=1}^{N_f} b_i \). If we have a massless hypermultiplets then some \( b_i \) vanishes to give \( A = 0 \).

One can easily incorporate the D6-branes. The curve is a union of a rational curve isomorphic to \( w \)-plane and

\[
y^2 + B_{N_c}(v^2)y + v^4 \prod_{i=1}^{N_f} (v^2 - a_i^2) = 0 \tag{67}
\]

with \( w = 0 \). Again (67) can be interpreted as N=2 \( SO(2N_c) \) gauge theory with \( N_f \) flavors or \( Sp(N_c) \) gauge theory with \( N_f + 4 \) hypermultiplets where 4 of them are massless. This is consistent with the Type IIA brane configuration. If the middle NS'5-brane is located at \( x^7 = 0 \), we have the chiral gauge theory. But if the NS'5-brane is moved to negative direction, we have the N=2 \( Sp(N_c) \) gauge theory with \( N_f + 4 \) hypermultiplets. If the NS'5-brane is moved to positive direction, we have the N=2 \( SO(2N_c) \) gauge theory with \( N_f \) hypermultiplets. Also we have seen how the N=2 Higgs branch of the SO/Sp gauge theory could be understood in M-theory picture. These Higgs branches are a part of the moduli space of the chiral gauge theory.

Since the NS'5-brane is decoupled, the rotated curve for the chiral gauge theory can be obtained by attaching a rational curve to the rotated curve of (67). Again the rotated curve for \( SO(2N_c) \) gauge theory with \( N_f \) flavors is the same as that for \( Sp(N_c) \) gauge theory with \( N_f + 4 \) flavors where four of them are massless hypermultiplets.

In [11], using the brane construction of the chiral gauge theory the chiral non-chiral transition is discussed. We can put two additional NS'5-branes on the O6-plane. Those NS'5-branes are constrained to move in \( x^7 \)-direction. Only when a pair of NS'5-branes coincide, they can move in \( x^4, x^5, x^6 \)-directions pairwise. Once such movement occurs, we have non-chiral gauge theory with gauge group either \( SU(2N_c) \times Sp(N_c) \) or \( SU(2N_c) \times SO(2N_c) \) depending on the sign of the orientifold D4-branes intersect. The matter contents of the

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\[ ^9 \text{Similar observation is made in [12].} \]
resulting theory are a symmetric/antisymmetric tensor for the $SO/Sp$ group and a pair (chiral and its conjugate) of bifundamental fields. From the discussions above, we can describe the chiral gauge theory configuration where two additional NS$'$-branes are located on orientifold. Since NS$'$-branes are factorized in the chiral gauge theory setup, we can just attach two rational curves in $x^7$ direction to the curve (67). Again as we traverse a NS$'$-brane, the interpretation of the ambient space $xy = v^4$ is changed. In one case, we have Atiyah-Hitchin space with the lifting of 8 half D6-branes. In the other case, we have frozen $D_4$ singularity. We can make two NS$'$-branes coincident, thereby producing a nontrivial fixed point which mediates chiral non-chiral transition. But since we only know the low energy behavior of the nontrivial fixed point, not much can be said about the fixed point just using M-theory description obtained in this paper. It is also quite interesting to understand the other non-chiral branch in M5-brane picture. This involves the understanding of the rotation of NS5-branes in N=2 configuration by $\frac{\pi}{2}$ in M-theory picture.

6 Discussion

We have explored the M5-brane realization of one specific chiral gauge theory. Since the Type IIA brane configuration of the chiral gauge theory is constructed using orientifold 6-plane, a part of the paper is devoted to studying the corresponding M-theory configuration of the orientifold 6-planes. The geometric interpretation in M-theory of the orientifold with $-4$ RR charge ($O6^-$) is the Atiyah-Hitchin space. This clear geometric interpretation makes
easy the analysis of the gauge theory constructed using $O6^-$-plane. On the other hand, the corresponding geometry in M-theory of the orientifold of $+4$ RR charge ($O6^+$) is less clear and we do not have the complete understanding of the gauge theory arising from it. The $O6^+$-plane is lifted to the $D_4$ singularity which cannot be blownup, but we do not understand why it cannot be blownup. Related problem is that we cannot distinguish geometrically usual $D_4$-singularity which corresponds to $4$ D6-branes on top of the $O6^-$-plane and the frozen $D_4$ singularity corresponding to the $O6^+$-plane. Similar problem arise if we consider the $SO(2N_c)/Sp(N_c)$ gauge theory with $N_f/(N_f + 4)$ with $4$ massless fundamental flavors. The same curve describing the both theories and by moving the detached rational curve from the chiral gauge theory configuration, we can move to either theories. But we cannot tell in which case we have the degrees of freedom corresponding to the physical D6-branes just by looking the curve. To resolve this issue would be in important step to understand M-theory beyond its supergravity approximation.

One special feature of the chiral gauge theory is the presence of half D6-branes. It is argued in \cite{26} that the half D-branes play an important role in the realization of the chiral symmetry. But in usual brane configurations, half D-branes have the other half D6-branes to form whole D-branes. The configuration considered here is a rare case where half D6-branes do not have their pairs. And indeed these half D6-branes give the chiral spectrum in the gauge theory. It would be interesting to consider more examples where half D-branes which do not have their pairs give rise to a chiral spectrum.

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