Partial Sublinear Time Approximation and Inapproximation for Maximum Coverage *

Bin Fu

Department of Computer Science
University of Texas - Rio Grande Valley, Edinburg, TX 78539, USA
bin.fu@utrgv.edu

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Abstract

We develop a randomized approximation algorithm for the classical maximum coverage problem, which given a list of sets \( A_1, A_2, \cdots, A_n \) and integer parameter \( k \), select \( k \) sets \( A_{i_1}, A_{i_2}, \cdots, A_{i_k} \) for maximum union \( A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \). In our algorithm, each input set \( A_i \) is a black box that can provide its size \( |A_i| \), generate a random element of \( A_i \), and answer the membership query \( (x \in A_i) \) in \( O(1) \) time. Our algorithm gives \((1 - \frac{1}{e})\)-approximation for maximum coverage problem in \( O(p(n)) \) time, which is independent of the sizes of the input sets. No existing \( O(q(m)) \) time \((1 - \frac{1}{e})\)-approximation algorithm for the maximum coverage has been found for any function \( p(n) \) that only depends on the number of sets, where \( m = \max(|A_1|, \cdots, |A_n|) \) (the largest size of input sets). The notion of partial sublinear time algorithm is introduced. For a computational problem with input size controlled by two parameters \( n \) and \( m \), a partial sublinear time algorithm for it runs in a \( O(p(n)m^{1-\epsilon}) \) time or \( O(q(m)n^{1-\epsilon}) \) time. The maximum coverage has a partial sublinear time \( O(p(n)) \) constant factor approximation. On the other hand, we show that the maximum coverage problem has no partial sublinear \( O(q(m)n^{1-\epsilon}) \) time constant factor approximation algorithm. It separates the partial sublinear time computation from the conventional sublinear time computation by disproving the existence of sublinear time approximation algorithm for the maximum coverage problem.

Key words: Maximum Coverage, Greedy Method, Approximation, Partial Sublinear Time.

1. Introduction

The maximum coverage problem is a classical NP-hard problem with many applications [8, 16], and is directly related to set cover problem, one of Karp’s twenty-one NP-complete problems [18]. The input has several sets and a number \( k \). The sets may have some elements in common. You must select at most \( k \) of these sets such that the maximum number of elements are covered, i.e. the union of the selected sets has a maximum size. The greedy algorithm for maximum coverage chooses sets according to one rule: at each stage, choose a set which contains the largest number of uncovered elements. It can be shown that this algorithm achieves an approximation ratio of \((1 - \frac{1}{e})\) [8, 17]. Inapproximability results show that the greedy algorithm is essentially the best-possible polynomial time approximation algorithm for maximum coverage [11]. The existing implementation for the greedy \((1 - \frac{1}{e})\)-approximation algorithm for the maximum coverage problem needs \( \Omega(n^m) \) time for a

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list of \( n \) sets \( A_1, \ldots, A_n \) with \( m = |A_1| = |A_2| = \cdots = |A_n| \) \cite{17, 25}. We have not found any existing \( O(p(n)m^{1-\epsilon}) \) time algorithm for the same ratio \( (1 - \frac{1}{2}) \) of approximation for any function \( p(n) \) that only depends on the number of sets. The variant versions and methods for this problem have been studied in a series of papers \cite{1, 5, 6, 19, 23}.

This paper sticks to the original definition of the maximum coverage problem, and studies its complexity under several concrete models. In the first model, each set is accessed as a black box that only provides random elements and answers membership queries. When \( n \) input sets \( A_1, A_2, \cdots, A_n \) are given, our model allows random sampling from each of them, and the cardinality \( |A_i| \) (or approximation for \( |A_i| \)) of each \( A_i \) is also part of the input. The results of the first model can be transformed into other conventional models. A set could be a set of points in a geometric shape. For example, a set may be all lattice points in a \( d \)-dimensional rectangular shape. If the center position, and dimension parameters of the rectangle are given, we can count the number of lattice points and provide a random sample for them.

A more generalized maximum coverage problem was studied under the model of submodular set function subject to a matroid constraint \cite{2, 12, 22}, and has same approximation ratio \( 1 - \frac{1}{2} \). The maximum coverage problem in the matroid model has time complexity \( O(r^3n^2m) \) \cite{12}, and \( O(r^2n^3m + n^7) \) \cite{2}, respectively, according to the analysis in \cite{12}, where \( r \) is the rank of matroid, \( n \) is the number of sets, and \( m \) is the size of the largest set. The maximum coverage problem in the matroid model has the oracle query to the submodular function \cite{2} and is counted \( O(1) \) time per query. Computing the size of union of input sets is \#P-hard if each input set as a black box is a set of high dimensional rectangular lattice points since \#DNF is \#P-hard \cite{24}. Thus, the generalization of submodular function in the matroid model does not characterize the computational complexity for these types of problems. Our model can be applied to this high dimensional space maximum coverage problem.

When each set \( A_i \) is already saved in an efficient data structure such as B-tree, we can also provide an efficient random sample, and make a membership query to each \( A_i \) in an \( O(\log |A_i|) \) time. This model also has practical importance because B-tree is often used to collect a large set of data.

In this paper, we develop a randomized algorithm to approximate the maximum coverage problem. We show an approximation algorithm for maximum coverage problem with \( (1 - \frac{1}{2}) \)-ratio. Our computational time is independent of the size of each set if the membership checking for each input set takes one step. Our algorithms are suitable to estimate the maximum coverage when there are multiple big data sets, and each data set is stored in a efficient data structure that can support efficient random sampling and membership query. The widely used B-tree in modern data base clearly fits our algorithm. Our algorithms are considered to be sublinear time algorithm when the number of sets is much less than the maximal size of the sets. Our model and algorithm are suitable to support online computation.

Sublinear time algorithms have been found for many computational problems, such as checking polygon intersections \cite{3}, estimating the cost of a minimum spanning tree \cite{4, 9, 10}, finding geometric separators \cite{13}, property testing \cite{14, 15}, etc.

The notion of partial sublinear time computation is introduced in this paper. For a function \( f(.) \) that maps a list of sets to nonnegative integers, a \( O(p(n)m^{1-\epsilon}) \) time or \( O(q(m)n^{1-\epsilon}) \) time approximation to \( f(.) \) is a partial sublinear time computation. The maximum coverage has a partial sublinear time constant factor approximation scheme. On the other hand, we show that the maximum coverage problem has no partial sublinear \( O(q(m)n^{1-\epsilon}) \) approximation algorithm. Thus, the partial sublinear time computation is separated from the conventional sublinear time computation.

The paper is organized as follows: In Section 2, we define our model of computation and complexity. In Section 3, we give an overview of our method for approximating maximum coverage problem. In Section 4, we give randomized greedy approximation for the maximum coverage problem. In Section 5, a faster algorithm is presented with one round random sampling, which is different from the multiple rounds random sampling used in Section 4. In Section 6 , we introduce the notion of partial sublinear time computation, and prove inapproximability for maximum coverage if the time is \( O(q(m)n^{1-\epsilon}) \). In Section 7, we show a special case of maximum coverage problem that all input sets have the same size, and prove that it is as hard as the general case. In Section 8, the algorithm
is implemented in more concrete data model for the maximum coverage problem. An input set can be stored in a sorted array, unsorted array, B-tree, or hashing function. A set may be represented by a small set of parameters if it is a set of high dimensional points such as a set of lattice points in a rectangle shape.

2. Computational Model and Complexity

In this section, we show our model of computation, and the definition of complexity. Assume that \(A_1\) and \(A_2\) are two sets. Their union \(A_1 \cup A_2\) contains the elements in either \(A_1\) or \(A_2\). Define \(A_2 - A_1\) to be the set of elements in \(A_2\), but not in \(A_1\). Define their intersection \(A_1 \cap A_2\) to be the set of elements in both \(A_1\) and \(A_2\). For example, \(A_1 = \{3, 5\}\) and \(A_2 = \{1, 3, 7\}\), then \(A_1 \cup A_2 = \{1, 3, 5, 7\}\), \(A_2 - A_1 = \{1, 7\}\), and \(A_1 \cap A_2 = \{3\}\). For a finite set \(A\), we use \(|A|\), cardinality of \(A\), to be the number of distinct elements in \(A\). For a real number \(x\), let \([x]\) be the least integer \(y \geq x\), and \([x]\) be the largest integer \(z \leq x\). For example, \([3.2]\) = 4, and \([3.2]\) = 3. Let \(N = \{0, 1, 2, \cdots\}\) be the set of nonnegative integers, \(R = (-\infty, \infty)\) be the set of all real numbers, and \(R^+ = [0, +\infty)\) be the set of all nonnegative real numbers. An integer \(s\) is a \((1 + \epsilon)\)-approximation for \(|A|\) if \((1 - \epsilon)|A| \leq s \leq (1 + \epsilon)|A|\).

**Definition 1.** The type 0 model of randomized computation for our algorithm is defined below: An input \(L\) is a list of sets \(A_1, A_2, \cdots, A_n\) that support the following operations:

1. The cardinality of \(A_i\) is \(m_i = |A_i|\) for \(i = 1, 2, \cdots, n\).
2. The largest cardinality of input set is \(m = \max\{m_i : 1 \leq i \leq n\}\) and also part of the input.
3. Function RandomElement\((A_i)\) returns a random element \(x\) from \(A_i\) for \(i = 1, 2, \cdots, n\).
4. Function Query\((x, A_i)\) function returns 1 if \(x \in A_i\), and 0 otherwise.

**Definition 2.** Let parameters \(\alpha_L\) and \(\alpha_R\) be in \([0, 1)\). Let \(x\) be a random variable in set \(A\) such that for each \(y \in A\), \((1 - \alpha_L) \cdot \frac{1}{|A|} \leq \text{Prob}\(x = y\) \leq (1 + \alpha_R) \cdot \frac{1}{|A|}\). The random variable \(x\) is called \((\alpha_L, \alpha_R)\)-biased random variable for \(A\). An \((\alpha_L, \alpha_R)\)-biased generator RandomElement\((A)\) is considered a \((\alpha_L, \alpha_R)\)-biased random variable for \(A\).

Definition 3 gives the type 1 model, which is a generalization of type 0 model. It is suitable to apply our algorithm for high dimensional problems that may not give uniform random sampling or exact set size. For example, it is not trivial to count the number of lattice points or generate a random lattice point in a \(d\)-dimensional ball with center not at a lattice point.

**Definition 3.** The type 1 model of randomized computation for our algorithm is defined below: Let real parameters \(\alpha_L, \alpha_R, \delta_L, \delta_R\) be in \([0, 1)\). An input \(L\) is a list of sets \(A_1, A_2, \cdots, A_n\) that support the following operations:

1. An approximate cardinality of \(A_i\) is \(s_i\) with \((1 - \delta_L)|A_i| \leq s_i \leq (1 + \delta_R)|A_i|\) for \(i = 1, 2, \cdots, n\).
2. The largest approximate cardinality of input sets is \(s = \max\{s_i : 1 \leq i \leq n\}\).
3. Function RandomElement\((A_i)\) is a \((\alpha_L, \alpha_R)\)-biased random generator for \(A_i\) for \(i = 1, 2, \cdots, n\).
4. Function Query\((x, A_i)\) function returns 1 if \(x \in A_i\), and 0 otherwise.

The main problem, which is called maximum coverage, is that given a list of sets \(A_1, \cdots, A_n\) and an integer \(k\), find \(k\) sets from \(A_1, A_2, \cdots, A_n\) to maximize the size of the union of the selected sets in the computational model defined in Definition 1 or Definition 3. For real number \(a \in [0, 1]\), an approximation algorithm is a \((1 - a)\)-approximation for the maximum coverage problem that has
input of integer parameter \( k \) and a list of sets \( A_1, \ldots, A_n \) if it outputs a sublist of sets \( A_{i_1}, A_{i_2}, \ldots, A_{i_k} \) such that |\( A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \)| \( \geq (1-\alpha)|A_j| \) \( \cup A_{j_2} \cup \cdots \cup A_{j_k} | \), where \( A_{j_1}, A_{j_2}, \ldots, A_{j_k} \) is an optimal solution with maximum size of union.

We use the triple \((T(\cdot), R(\cdot), Q(\cdot))\) to characterize the computational complexity, where

- \( T(\cdot) \) is a function for the number of steps that each access to RandomElement(\( \cdot \)) or Query(\( \cdot \)) is counted one step,
- \( R(\cdot) \) is a function to count the number of random samples from \( A_i \) for \( i = 1, 2, \ldots, n \). It is measured by the total number of times to access those functions RandomElement(\( A_i \)) for all input sets \( A_i \), and
- \( Q(\cdot) \) is a function to count the number of queries to \( A_i \) for \( i = 1, \ldots, A_n \). It is measured by the total number of times to access those functions Query(\( x, A_i \)) for all input sets \( A_i \).

The parameters \( \epsilon, \gamma, k, m, n \) can be used to determine the three complexity functions, where \( m = \max(|A_1|, \ldots, |A_n|) \) (the largest cardinality of input sets), \( \epsilon \) controls the accuracy of approximation, and \( \gamma \) controls the failure probability of a randomized algorithm. Their types could be written as \( T(\epsilon, \gamma, k, n), R(\epsilon, \gamma, k, n), \) and \( Q(\epsilon, \gamma, k, n) \). All of the complexity results of this paper at both model 0 and model 1 are independent of parameter \( m \).

**Definition 4.** For a list \( L \) of sets \( A_1, A_2, \ldots, A_n \) and real \( \alpha_L, \alpha_R, \delta_L, \delta_R \in [0, 1) \), it is called \(((\alpha_L, \alpha_R), (\delta_L, \delta_R))\)-list if each set \( A_i \) is associated with a number \( s_i \) with \( (1-\delta_L)|A_i| \leq s_i \leq (1+\delta_R)|A_i| \) for \( i = 1, 2, \ldots, n \), and the set \( A_i \) has a \((\alpha_L, \alpha_R)\)-biased random generator RandomElement(\( A_i \)).

## 3. Outline of Our Methods

For two sets \( A \) and \( B \), we develop a randomized method to approximate the cardinality of the difference \( B - A \). We approximate the size of \( B - A \) by sampling a small number of elements from \( B \) and calculating the ratio of the elements in \( B - A \) by querying the set \( A \). The approximate \( |A \cup B| \) is the sum of an approximate \( |A| \) and an approximate of \( |B - A| \).

A greedy approach will be based on the approximate difference between a new set and the union of sets already selected. Assume that \( A_1, A_2, \ldots, A_n \) is the list of sets for the maximum coverage problem. After \( A_1, \ldots, A_t \) have been selected, the greedy approach needs to check the size \( |A_j - (A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_t})| \) before selecting the next set. Our method to estimate \( |A_j - (A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_t})| \) is based on randomization in order to make the time independent of the sizes of input sets. Some random samples are selected from set \( A_j \). We control the accuracy of the approximate cardinality of the set difference so that it is enough to achieve the approximation ratio \( 1 - \frac{1}{\epsilon} \) for the maximum coverage problem.

During the accuracy analysis, Chernoff Bound (see [21]) plays an important role. It shows how the number of samples determines the accuracy of approximation.

**Theorem 5.** Let \( X_1, \ldots, X_n \) be \( n \) independent random 0-1 variables, where \( X_i \) takes 1 with probability at least \( p \) for \( i = 1, \ldots, n \). Let \( X = \sum_{i=1}^{n} X_i \), and \( \mu = E[X] \). Then for any \( \delta > 0 \),
\[
\Pr(X < (1-\delta)pn) < e^{-\frac{\delta^2}{2}}pn.
\]

**Theorem 6.** Let \( X_1, \ldots, X_n \) be \( n \) independent random 0-1 variables, where \( X_i \) takes 1 with probability at most \( p \) for \( i = 1, \ldots, n \). Let \( X = \sum_{i=1}^{n} X_i \). Then for any \( \delta > 0 \),
\[
\Pr(X > (1+\delta)pn) < \left[ \frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \right]^{pn}.
\]
Corollary 7 ([20]). Let $X_1, \ldots, X_n$ be $n$ independent random 0-1 variables and $X = \sum_{i=1}^{n} X_i$.

1. If $X_i$ takes 1 with probability at most $p$ for $i = 1, \ldots, n$, then for any $\frac{1}{3} > \epsilon > 0$, $\Pr(X > pn + cn) < e^{-\frac{1}{4}n\epsilon^2}$.
2. If $X_i$ takes 1 with probability at least $p$ for $i = 1, \ldots, n$, then for any $\epsilon > 0$, $\Pr(X < pn - cn) < e^{-\frac{1}{4}n\epsilon^2}$.

We define the function $\mu(x)$ in order to simply the probability mentioned in Corollary 7.

$$\mu(x) = e^{-\frac{1}{4}x^2}$$

A well known fact in probability theory is the inequality

$$\Pr(E_1 \cup E_2 \cup \cdots \cup E_m) \leq \Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_m),$$

where $E_1, E_2, \ldots, E_m$ are $m$ events that may not be independent. In the analysis of our randomized algorithm, there are multiple events such that the failure from any of them may fail the entire algorithm. We often characterize the failure probability of each of those events, and use the above inequality to show that the whole algorithm has a small chance to fail, after showing that each of them has a small chance to fail.

Our algorithm performance will depend on the initial accuracy of approximation to each set size, and how biased the random sample from each input set. This consideration is based on the applications to high dimensional geometry problems which may be hard to count the exact number of elements in a set, and is also hard to provide perfect uniform random source. We plan to release more applications to high dimensional geometry problems that need approximate counting and biased random sampling.

Overall, our method is an approximate randomized greedy approach for the maximum coverage problem. The numbers of random samples is controlled so that it has enough accuracy to derive the classical approximation ratio $1 - \frac{1}{e}$. The main results are stated at Theorem 9 (type 1 model) and Corollary 10 (type 0 model).

**Definition 8.** Let the maximum coverage problem have integer parameter $k$, and a list $L$ of sets $A_1, A_2, \cdots, A_n$ as input. We always assume $k \leq n$. Let $C^*(L,k) = |A_{t_1} \cup A_{t_2} \cup \cdots \cup A_{t_k}|$ be the union size of an optimal solution $A_{t_1}, \cdots, A_{t_k}$.

**Theorem 9.** Let $\rho$ be a constant in $(0,1)$. For parameters $\epsilon \in (0,1)$ and $\alpha_L, \alpha_R, \delta_L, \delta_R \in [0,1-\rho]$, there is an algorithm to give a $(1 - \frac{1}{e})$ approximation for the maximum cover problem, such that given a $((\alpha_L, \alpha_R), (\delta_L, \delta_R))$-list $L$ of finite sets $A_1, \cdots, A_n$ and an integer $k$, it returns $z$ and $H \subseteq \{1,2,\cdots,n\}$ that satisfy

1. $|\bigcup_{j \in H} A_j| \geq (1 - \frac{1}{e})C^*(L,k)$ and $|H| = k$,
2. $((1 - \alpha_L)(1 - \delta_L) - \epsilon)|\bigcup_{j \in H} A_j| \leq z \leq ((1 + \alpha_R)(1 + \delta_R) + \epsilon)|\bigcup_{j \in H} A_j|$, and
3. Its complexity is $(T(\epsilon, \gamma, k, n), R(\epsilon, \gamma, k, n), Q(\epsilon, \gamma, k, n))$ with

$$T(\epsilon, \gamma, k, n) = O\left(\frac{k^5}{\epsilon^2} (k \log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),$$

$$R(\epsilon, \gamma, k, n) = O\left(\frac{k^4}{\epsilon^2} (k \log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),$$

$$Q(\epsilon, \gamma, k, n) = O\left(\frac{k^5}{\epsilon^2} (k \log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),$$

where $\beta = \frac{(1-\alpha_L)(1-\delta_L)}{(1+\alpha_R)(1+\delta_R)}$. 

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Corollary 10 gives the importance case that we have exact sizes for all input sets, and uniform random sampling for each of them. Such an input is called \(((0,0),(0,0))\)-list according to Definition 4.

**Corollary 10.** For parameter \(\epsilon \in (0,1)\), there is an algorithm to give a \((1 - \frac{1}{k})\) approximation for the maximum cover problem, such that given a \(((0,0),(0,0))\)-list \(L\) of finite sets \(A_1, \ldots, A_n\) and an integer \(k\), it returns \(z\) and \(H \subseteq \{1,2,\ldots,n\}\) that satisfy

1. \(|\bigcup_{j \in H} A_j| \geq (1 - \frac{1}{k})C^*(L,k)\) and \(|H| = k\),
2. \((1 - \epsilon)|\bigcup_{j \in H} A_j| \leq z \leq (1 + \epsilon)|\bigcup_{j \in H} A_j|\), and
3. Its complexity is \((T(\epsilon,\gamma,k,n),R(\epsilon,\gamma,k,n),Q(\epsilon,\gamma,k,n))\) with

\[
T(\epsilon,\gamma,k,n) = O\left(\frac{k^5}{\epsilon^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),
\]
\[
R(\epsilon,\gamma,k,n) = O\left(\frac{k^4}{\epsilon^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),\text{ and}
\]
\[
Q(\epsilon,\gamma,k,n) = O\left(\frac{k^5}{\epsilon^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right).
\]

**Proof:** Since \(\alpha_L = \alpha_R = \delta_L = \delta_R = 0\) implies \(\beta = 1\), it follows from Theorem 9.

### 4. Randomized Algorithm for Maximum Coverage

We give a randomized algorithm for approximating the maximum coverage. It is based on an approximation to the cardinality of set difference. The algorithms are described at type 1 model, and has corollaries for type 0 model.

#### 4.1. Randomized Algorithm for Set Difference Cardinality

In this section, we develop a method to approximate the cardinality of \(B - A\) based on random sampling. It will be used as a submodule to approximate the maximum coverage.

**Definition 11.** Let \(R = x_1, x_2, \ldots, x_w\) be a list of elements from set \(B\), and let \(L\) be a list of sets \(A_1, A_2, \ldots, A_u\). Define \(\text{test}(L,R) = |\{j : 1 \leq j \leq w, \text{ and } x_j \notin (A_1 \cup A_2 \cup \cdots \cup A_u)\}|\).

The Algorithm ApproximateDifference(.) gives an approximation \(s\) for the size of \(B - A\). It is very time consuming to approximate \(|B - A|\) when \(|B - A|\) is much less than \(|B|\). The algorithm ApproximateDifference(.) returns an approximate value \(s\) for \(|B - A|\) with a range in \([(1 - \delta)|B - A| - \epsilon|B|, (1 + \delta)|B - A| + \epsilon|B|]\), and will not lose much accuracy when it is applied to approximate the maximum coverage by controlling the two parameters \(\delta\) and \(\epsilon\).

**Algorithm 1 : RandomTest(L,B,w)**

Input: \(L\) is a list of sets \(A_1, A_2, \ldots, A_u\), \(B\) is another set with a random generator RandomElement\((B)\), and \(w\) is an integer to control the number of random samples from \(B\).

1. For \(i = 1\) to \(w\) let \(x_i = \text{RandomElement}(B)\);
2. For \(i = 1\) to \(w\)
   3. Let \(y_i = 0\) if \((x_i \in A_1 \cup A_2 \cup \cdots \cup A_u)\), and 1 otherwise;
4. Return \(t = y_1 + \cdots + y_w\);
Corollary 7, with probability at most $P_3$. Return $t = \frac{1}{w} \cdot s_2$. Let $R = \text{RandomTest}(L, B, w)$;

End of Algorithm

Algorithm 2: ApproximateDifference($L, B, s_2, \epsilon, \gamma$)

Input: $L$ is a list of sets $A_1, A_2, \cdots, A_u$, $B$ is another set with a random generator RandomElement($B$), integer $s_2$ is an approximation for $|B|$ with $(1 - \delta_L)|B| \leq s_2 \leq (1 + \delta_R)|B|$, and $\epsilon$ and $\gamma$ are real parameters in $(0, 1)$, where $\delta \in [0, 1]$.

Steps:
1. Let $w$ be an integer with $\mu(\frac{\epsilon}{w}) \leq \frac{w}{\gamma}$, where $\mu(x)$ is defined in equation (1).
2. Let $t = \text{RandomTest}(L, B, w)$;
3. Return $s = \frac{1}{w} \cdot s_2$

End of Algorithm

Lemma 12 shows how Algorithm ApproximateDifference(.) returns an approximation $s$ for $|B - A|$ with a small failure probability $\gamma$, and its complexity depends on the accuracy $\epsilon$ of approximation and probability $\gamma$. Its accuracy is controlled for the application to the approximation algorithms for maximum coverage problem.

Lemma 12. Assume that real number $\epsilon \in (0, 1]$, $B$ is a set with $(\alpha_L, \alpha_R)$-biased random generator RandomElement($B$) and an approximation $s_2$ for $|B|$ with $(1 - \delta_L)|B| \leq s_2 \leq (1 + \delta_R)|B|$, and $L$ is a list of sets $A_1, A_2, \cdots, A_u$. Then

1. If $R = x_1, x_2, \cdots, x_w$ be a list of elements generated by RandomElement($B$), and $\mu(\frac{\epsilon}{w}) \leq \frac{w}{\gamma}$, then with probability at most $\gamma$, the value $s = \frac{1}{w} \cdot s_2$ fails to satisfy inequality (3)

   $$(1 - \alpha_L)(1 - \delta_L)|B - A| - \epsilon|B| \leq s \leq (1 + \alpha_R)(1 + \delta_R)|B - A| + \epsilon|B|,$$

   where $A = A_1 \cup A_2 \cup \cdots \cup A_u$ is the union of sets in the input list $L$.

2. With probability at most $\gamma$, the algorithm ApproximateDifference(.) returns $s$ such that the returned value $s$ fails to satisfy inequality (3), and

3. If the implementation of RandomTest(.) in Algorithm 1 is used, then the complexity of ApproximateDifference(.) is $T_D(\epsilon, \gamma, u), R_D(\epsilon, \gamma, u), Q_D(\epsilon, \gamma, u)$ with $T_D(\epsilon, \gamma, u) = O(\frac{\mu}{\epsilon} \log \frac{1}{\gamma})$, $R_D(\epsilon, \gamma, u) = O(\frac{1}{\epsilon} \log \frac{1}{\gamma})$, and $Q_D(\epsilon, \gamma, u) = O(\frac{1}{\epsilon^2} \log \frac{1}{\gamma})$.

Proof: Let $A = A_1 \cup A_2 \cdots \cup A_u$. The $w$ random elements from $B$ are via the $(\alpha_L, \alpha_R)$-biased random generator RandomElement($B$). We get $t$ to be the number of the $w$ items in $B - A$. Value $s = \frac{1}{w} \cdot s_2$ is an approximation for $|B - A|$. Let $p = \frac{|B - A|}{|B|}, p_L = (1 - \alpha_L)p$, and $p_R = (1 + \alpha_R)p$. By Corollary 7, with probability at most $P_1 = \mu(\frac{\epsilon}{w})$, we have $t > p_Rw + \frac{\epsilon}{3} \cdot w = (1 + \alpha_R)pw + \frac{\epsilon}{3} \cdot w$.

If $t \leq (1 + \alpha_R)pw + \frac{\epsilon}{3} \cdot w$, then the value

$$s = \frac{t}{w} \cdot s_2 \leq \frac{(1 + \alpha_R)pw + \frac{\epsilon}{3} \cdot w}{w} \cdot s_2 \leq ((1 + \alpha_R)p + \frac{\epsilon}{3})s_2$$

$$\leq ((1 + \alpha_R)p + \frac{\epsilon}{3})(1 + \delta_R)|B| \leq (1 + \alpha_R)(1 + \delta_R)|B - A| + \frac{\epsilon}{3} \cdot (1 + \delta_R)|B|$$

$$\leq ((1 + \alpha_R)p + \frac{\epsilon}{3})(1 + \delta_R)|B| \leq (1 + \alpha_R)(1 + \delta_R)|B - A| + \epsilon|B|.$$

By Corollary 7, with probability at most $P_2 = \mu(\frac{\epsilon}{w})$, we have $t < p_Lw - \frac{\epsilon}{3} \cdot w = (1 - \alpha)pw - \frac{\epsilon}{3} \cdot w$.

If $t \geq (1 - \alpha_L)pw - \frac{\epsilon}{3} \cdot w$, then the value

$$s = \frac{t}{w} \cdot s_2 \geq \frac{(1 - \alpha_L)pw - \frac{\epsilon}{3} \cdot w}{w} \cdot s_2 \geq ((1 - \alpha_L)p - \frac{\epsilon}{3})s_2$$

$$\geq ((1 - \alpha_L)p - \frac{\epsilon}{3})(1 - \delta_L)|B| \geq (1 - \alpha_L)(1 - \delta_L)|B - A| - \frac{\epsilon}{3} \cdot |B|$$

$$\geq (1 - \alpha_L)(1 - \delta_L)|B - A| - \epsilon|B|.$$

7
By line 1 of ApproximateDifference(.), we need \( w = O(\frac{1}{\epsilon^2} \log \frac{1}{\gamma}) \) random samples in \( B \) so that the total failure probability is at most \( P_1 + P_2 \leq 2 \cdot \frac{\gamma}{\epsilon^2} \) (by inequality (2)). The number of queries to \( A \) is \( w \). Thus, the number of total queries to \( A_1, A_2, \ldots, A_u \) is \( uw \).

Therefore, we have its complexity \( (T_D(\epsilon, \gamma), R_D(\epsilon, \gamma), Q_D(\epsilon, \gamma)) \) with
\[
T_D(\epsilon, \gamma) = O(uw) = O\left(\frac{u}{\epsilon^2} \log \frac{1}{\gamma}\right),
\]
\[
R_D(\epsilon, \gamma) = w = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\gamma}\right), \quad \text{and}
\]
\[
Q_D(\epsilon, \gamma) = O(uw) = O\left(\frac{u}{\epsilon^2} \log \frac{1}{\gamma}\right).
\]

This completes the proof of the Lemma 12.

### 4.2. A Randomized Algorithm for Set Union Cardinality

We describe a randomized algorithm for estimating the cardinality for set union. It will use the algorithm for set difference developed in Section 4.1. The following lemma gives an approximation for the union size of sets. Its accuracy is enough when it is applied in the approximation algorithms for maximum coverage problem.

**Lemma 13.** Assume \( \epsilon, \delta_L, \delta_R, \delta_{2,L}, \delta_{2,R}, \alpha_L, \alpha_R \in [0,1] \), \( (1 - \delta_L) \leq (1 - \alpha_L)(1 - \delta_{2,L}) \) and \( (1 + \delta_R) \geq (1 + \alpha_R)(1 + \delta_{2,R}) \). Assume that \( L \) is a list of sets \( A_1, A_2, \ldots, A_n \), and \( X_2 \) is set with an \((\alpha_L, \alpha_R)\)-biased random generator RandomElement(X2). Let integers \( s_1 \) and \( s_2 \) satisfy \( (1 - \delta_L)|X_1| \leq s_1 \leq (1 + \delta_R)|X_1| \), and \( (1 - \delta_{2,L})|X_2| \leq s_2 \leq (1 + \delta_{2,R})|X_2| \), then

i. If \( t \) satisfies \( (1 - \alpha_L)(1 - \delta_{2,L})|X_2 - X_1| - \epsilon|X_2| \leq t \leq (1 + \alpha_R)(1 + \delta_{2,R})|X_2 - X_1| + \epsilon|X_2| \), then \( s_1 + t \) satisfies
\[
(1 - \delta_L - \epsilon)|X_1 \cup X_2| \leq s_1 + t \leq (1 + \delta_R + \epsilon)|X_1 \cup X_2|.
\] (4)

ii. If \( t = \text{ApproximateDifference}(L, X_2, s_2, \epsilon, \gamma) \), with probability at most \( \gamma \), \( s_1 + t \) does not have inequality (4),

where \( X_1 = A_1 \cup A_2 \cup \cdots \cup A_u \).

**Proof:** Assume that \( s_1 \) and \( s_2 \) satisfy
\[
(1 - \delta_L)|X_1| \leq s_1 \leq (1 + \delta_R)|X_1|, \quad \text{and}
\]
\[
(1 - \delta_{2,L})|X_2| \leq s_2 \leq (1 + \delta_{2,R})|X_2|.
\] (5)
(6)

Since \( (1 + \delta_R) \geq (1 + \alpha_R)(1 + \delta_{2,R}) \), we have
\[
s_1 + t \leq (1 + \delta_R)|X_1| + (1 + \alpha_R)(1 + \delta_{2,R})|X_2 - X_1| + \epsilon|X_2|
\]
\[
\leq (1 + \delta_R)(|X_1| + |X_2 - X_1|) + \epsilon|X_2|
\]
\[
= (1 + \delta_R)|X_1 \cup X_2| + \epsilon|X_2|
\]
\[
\leq (1 + \delta_R + \epsilon)|X_1 \cup X_2|.
\]

Since \( (1 - \delta_L) \leq (1 - \alpha_L)(1 - \delta_{2,L}) \), we have
\[
s_1 + t \geq (1 - \delta_L)|X_1| + (1 - \alpha_L)(1 - \delta_{2,L})|X_2 - X_1| - \epsilon|X_2|
\]
\[
\geq (1 - \delta_L)(|X_1| + |X_2 - X_1|) - \epsilon|X_2|
\]
\[
= (1 - \delta_L)|X_1 \cup X_2| - \epsilon|X_2|
\]
\[
\geq (1 - \delta_L - \epsilon)|X_1 \cup X_2|.
\]

Case ii follows from Case i, and Lemma 12. By executing \( t = \text{ApproximateDifference}(X_1, X_2, \epsilon, \gamma) \), we have \( (1 - \alpha_L)(1 - \delta_{2,L})|X_2 - X_1| - \epsilon|X_2| \leq t \leq (1 + \alpha_R)(1 + \delta_{2,R})|X_2 - X_1| + \epsilon|X_2| \). The probability to fail inequality (4) is at most \( \gamma \) by Lemma 12.
4.3. Approximation to the Maximum Coverage Problem

In this section, we show that our randomized approach to the cardinality of set union can be applied to the maximum coverage problem. Lemma 15 gives the approximation performance of greedy method for the maximum coverage problem. It is adapted to a similar result[17] with our approximation accuracy to the size of set difference.

Definition 14. For a list \( L \) of sets \( A_1, A_2, \ldots, A_n \), define its initial \( h \) sets by \( L(h) = A_1, A_2, \ldots, A_h \), and the union of sets in \( L \) by \( U(L) = A_1 \cup A_2 \cup \cdots \cup A_n \).

Lemma 15. Let \( L' \) be a sublist of sets \( A_1, A_2, \ldots, A_{t_k} \) selected from the list \( L \) of sets \( A_1, A_2, \ldots, A_n \). If each subset \( A_{t_j} (j = 0, 2, \ldots, k - 1) \) in \( L' \) satisfies \( |A_{t_{j+1}} - U(L'(j))| \geq \theta \cdot \delta \cdot (L, k) - \delta C^*(L, k) \), then \( |U(L')| \geq (1 - (1 - \frac{\theta}{k})^l)C^*(L, k) - \delta C^*(L, k) \) for \( l = 1, 2, \ldots, k \).

Proof: It is proven by induction. It is trivial at \( l = 1 \) as \( L'(0) = \emptyset \). Assume \( |U(L'(l))| \geq (1 - (1 - \frac{\theta}{k})^l)C^*(L, k) - \delta C^*(L, k) \).

Let \( A_{t_{j+1}} \) satisfy \( |A_{t_{j+1}} - U(L'(l))| \geq \theta \cdot \delta \cdot (L, k) - \delta C^*(L, k) \). Therefore,

\[
|U(L'(l+1))| = |U(L'(l))| + |A_{t_{j+1}} - U(L'(l))| \geq |U(L'(l))| + \theta \cdot \delta C^*(L, k) - \delta C^*(L, k) \geq (1 - \frac{\theta}{k})|U(L'(l))| + \frac{\theta C^*(L, k)}{k} - \delta C^*(L, k) \geq (1 - \frac{\theta}{k})((1 - (1 - \frac{\theta}{k})^l)C^*(L, k) - l \cdot \delta C^*(L, k)) + \frac{\theta C^*(L, k)}{k} - \delta C^*(L, k) \geq (1 - \frac{\theta}{k})((1 - (1 - \frac{\theta}{k})^l)C^*(L, k) + \frac{\theta C^*(L, k)}{k}) - l \cdot \delta C^*(L, k) - \delta C^*(L, k) \geq (1 - (1 - \frac{\theta}{k})^l)C^*(L, k) - (l + 1) \cdot \delta C^*(L, k). \]

Definition 16. If \( L' \) is a list of sets \( B_1, B_2, \ldots, B_u \), and \( B_{u+1} \) is another set, define \( \text{Append}(L', B_{u+1}) \) to be the list \( B_1, B_2, \ldots, B_u, B_{u+1} \), which is to append \( B_{u+1} \) to the end of \( L' \).

In Algorithm ApproximateMaximumCover(), there are several virtual functions including RandomSample(), ApproximateSetDifferenceSize(), and ProcessSet(), which have variant implementations and will be given in Virtual Function Implementations 1,2 and 3. We use a virtual function ApproximateSetDifferenceSize\((L', A_i, s_i, \epsilon, \gamma, k, n)\) to approximate \(|A_i - \cup A_j\) is in \( L' A_j\)|. We will have variant implementations for this function, and get different time complexity. One implementation will be given at Lemma 19, and the other one will be given at Lemma 22. Another function ProcessSet\((A_j)\) also has variant implementations. Its purpose is to process a newly selected set \( A_j \) to list \( L' \) of existing selected sets, and may sort it in one of the implementations. The function RandomSample() is also virtual and will have two different implementation.

Algorithm 3 : ApproximateMaximumCover\((L, k, \xi, \gamma)\)

Input: a list \( ((\alpha_L, \alpha_R), (\delta_L, \delta_R))\)-list \( L \) of \( n \) sets \( A_1, A_2, \ldots, A_n \), an integer parameter \( k \), and two real parameters \( \xi, \gamma \in (0, 1) \). Each \( A_i \) has a \((\alpha_L, \alpha_R)\)-biased random generator RandomElement\((A_i)\), and an approximation \( s_i \) for \(|A_i|\).

Steps:
1. Let \( H = \emptyset \), and list \( L' \) be empty;

2. Let \( z = 0 \);

3. Let \( \epsilon' = \frac{\xi}{4k} \);

4. For \( i = 1 \) to \( n \) let \( R_i = \text{RandomSamples}(A_i, \xi, \gamma, k, n) \);

5. For \( j = 1 \) to \( k \) do

   6. {

      7. Let \( s_j^* = -1 \);

      8. For each \( A_i \) in \( L \),

         9. {

            10. Let \( s_{i,j} = \text{ApproximateSetDifferenceSize}(L', A_i, s_{i,j}, R_i, \epsilon', \gamma, k, n) \);

            11. If \( s_{i,j} > s_j^* \) then let \( s_j^* = s_{i,j} \) and \( t_j = i \);

         }

      12. }

      13. Let \( H = H \cup \{ t_j \} \);

      14. Let \( z = z + s_{t_j,j} \);

      15. ProcessSet(\( A_{t_j} \));

      16. Let \( L' = \text{Append}(L', A_{t_j}) \);

      17. Remove \( A_{t_j} \) from list \( L \);

   }

18. }

19. Return \( z \) and \( H \);

**End of Algorithm**

The algorithm \( \text{ApproximateMaximumCover(.)} \) is a randomized greedy approach for the maximum coverage problem. It adds the set \( A_{t_j} \) that has an approximate largest \( |A_{t_j} - (\cup_{A_k \in L'} A_k)| \) to the existing partial solution saved in the list \( L' \). The accuracy control for the estimated size of the set \( A_{t_j} - (\cup_{A_k \in L'} A_k) \) will be enough to achieve the same approximation ratio as the classical deterministic algorithm. Since \( s_j^* \) starts from \(-1\) at line 7 in the algorithm \( \text{ApproximateMaximumCover(.)} \), each iteration picks up one set from the input list \( L \), and remove it from \( L \). By the end of the algorithm, \( L' \) contains \( k \) sets if \( k \leq n \).

Lemma 17 shows the approximation accuracy for the maximum coverage problem if \( s_{i,j} \) is accurate enough to approximate \( |A_i - U(L')| \). It may be returned by \( \text{ApproximateDifference(.)} \) with a small failure probability and complexity shown at Lemma 19.

**Lemma 17.** Let \( \xi \in (0, 1) \), and \( \alpha_L, \alpha_R, \delta_L, \delta_R \in [0, 1) \), \( L \) be a \( ((\alpha_L, \alpha_R), (\delta_L, \delta_R)) \)-list of sets \( A_1, \cdots, A_n \), and \( L' \) be the sublist \( L' \) after algorithm \( \text{ApproximateMaximumCover(.)} \) is completed. If every time \( s_{i,j} \) in the line 10 of the algorithm \( \text{ApproximateMaximumCover(.)} \) satisfies

\[
(1 - \alpha_L)(1 - \delta_L)|A_i - U(L'(j))| - \epsilon'|A_i| \leq s_{i,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_i - U(L'(j))| + \epsilon'|A_i|,
\]

then it returns an integer \( z \) and a size \( k \) subset \( H \subseteq \{1, 2, \cdots, n\} \) that satisfy

1. \( |\cup_{j \in H} A_j| \geq (1 - (1 - \frac{\beta}{k})^k - \xi)C^*(L, k) \), and
2. \((1 - \alpha_L)(1 - \delta_L) - \xi)\|\bigcup_{j \in H} A_j\| \leq z \leq (1 + \alpha_R)(1 + \delta_R) + \xi)\|\bigcup_{j \in H} A_j\|,

where \(\beta = \frac{(1 - \alpha_L)(1 - \delta_L)}{(1 + \alpha_R)(1 + \delta_R)}\).

**Proof:** The randomized greedy algorithm selects a subset that is close to have the best improvement for coverage. Let \(L'_*\) be the list \(L'\) after the algorithm \(\text{ApproximateMaximumCover(.)}\) is completed \((L'\) is dynamic list during the execution of the algorithm, and \(L'_*\) is static after the algorithm ends). The list \(L'_*\) contains \(k\) subsets \(A_1, \ldots, A_k\). According to the algorithm, \(L'_*(j)\) is the list of \(j\) subsets \(A_1, \ldots, A_j\) that have been appended to \(L'\) after the first \(j\) iterations for the loop from line 5 to line 18.

Assume that for each \(s_{i,j}\) we have

\[
(1 - \alpha_L)(1 - \delta_L)|A_i - U(L'_*(j))| - \epsilon'|A_i| \leq s_{i,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_i - U(L'_*(j))| + \epsilon'|A_i|.
\]

If set \(A_{v_j}\) makes \(|A_{v_j} - U(L'_*(j))|\) be the largest, we have

\[
|A_{v_j} - U(L'_*(j))| \geq \frac{|C^*(L,k) - U(L'_*(j))|}{k}.
\]

A special case of inequality (14) is inequality (16)

\[
(1 - \alpha_L)(1 - \delta_L)|A_{v_j} - U(L'_*(j))| - \epsilon'|A_{v_j}| \leq s_{v_j,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_{v_j} - U(L'_*(j))| + \epsilon'|A_{v_j}|.
\]

Since \(s_{v_j,j} \leq s_{t,j}\), we have \((1 - \alpha_L)(1 - \delta_L)|A_{v_j} - U(L'_*(j))| - \epsilon'|A_{v_j}| \leq s_{v_j,j} \leq s_{t,j} \leq s_{t,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_{v_j} - U(L'_*(j))| + \epsilon'|A_{v_j}|\) by inequalities (14) and (16).

Therefore, \((1 - \alpha_L)(1 - \delta_L)|A_{v_j} - U(L'_*(j))| - \epsilon'|A_{v_j}| \leq (1 + \alpha_R)(1 + \delta_R)|A_{v_j} - U(L'_*(j))| + \epsilon'|A_{v_j}|\). Since \(|A_{v_j}| \leq C^*(L,k)\) and \(|A_{v_j}| \leq C^*(L,k)|\), we have \(\frac{(1 - \alpha_L)(1 - \delta_L)}{(1 + \alpha_R)(1 + \delta_R)} \cdot |A_{v_j} - U(L'_*(j))| - \frac{2\epsilon'}{k} C^*(L,k) \leq |A_{v_j} - U(L'_*(j))|\). By inequality (15), we have

\[
\beta \cdot \frac{|C^*(L,k) - U(L'_*(j))|}{k} - 2\epsilon'C^*(L,k) \leq |A_{v_j} - U(L'_*(j))|,
\]

where \(\beta = \frac{(1 - \alpha_L)(1 - \delta_L)}{(1 + \alpha_R)(1 + \delta_R)}\).

By Lemma 15 and line 3 in \(\text{ApproximateMaximumCover(.)}\), we have \(L'_*\) with

\[
|U(L'_*)| \geq (1 - (1 - \beta)C^*(L,k) - k \cdot 2\epsilon'C^*(L,k)
\]

\[
\geq (1 - (1 - \frac{\beta}{k})^k - \xi)C^*(L,k).
\]

Case 2 of this lemma can be proven by a simple induction. We just need to prove after the \(i\)-th iteration of the for loop from line 5 to line 18 of this algorithm,

\[
((1 - \alpha_L)(1 - \delta_L) - i\epsilon')\|\bigcup_{j \in H} A_j\| \leq z \leq (1 + \alpha_R)(1 + \delta_R) + i\epsilon')\|\bigcup_{j \in H} A_j\|.
\]

It is trivial right after the initialization (line 1 to line 3 of the algorithm) since \(H = \emptyset\) and \(z = 0\). Assume inequality (17) holds after the \(i\)-th iteration of the loop from line 5 to line 18. By inequality (14) and Lemma 13 we have following inequality (18) after the \((i + 1)\)-th iteration of the loop from line 5 to line 18.

\[
((1 - \alpha_L)(1 - \delta_L) - (i + 1)\epsilon')\|\bigcup_{j \in H} A_j\| \leq z \leq (1 + \alpha_R)(1 + \delta_R) + (i + 1)\epsilon')\|\bigcup_{j \in H} A_j\|.
\]

Thus, at the end of the algorithm, we have

\[
((1 - \alpha_L)(1 - \delta_L) - k\epsilon')\|\bigcup_{j \in H} A_j\| \leq z \leq (1 + \alpha_L)(1 + \delta_R) + k\epsilon')\|\bigcup_{j \in H} A_j\|.
\]
Thus, by the end of the algorithm, we have the following inequality (20):

\[ ((1 - \alpha L)(1 - \delta L) - \xi) \cup_{j \in H} A_j \leq z \leq ((1 + \alpha R)(1 + \delta R) + \xi) \cup_{j \in H} A_j. \] (20)

We need Lemma 18 to transform the approximation ratio given by Lemma 17 to constant \((1 - \frac{1}{e})\) to match the classical ratio for the maximum coverage problem.

Lemma 18. For all integer \(k \geq 2\), and \(b \in [0, 1]\), we have

1. \((1 - \frac{b}{k})^k \leq \frac{1}{e^b} - \frac{np}{2e^b k}\), and

2. If \(\xi \leq \frac{np}{2e^b k}\), then \(1 - (1 - \frac{b}{k})^k - \xi > 1 - \frac{1}{e^b}\), where \(\eta = e^{-\frac{1}{k}}\).

Proof: Let function \(f(x) = 1 - \frac{b}{k} - e^{-x}\). We have \(f(0) = 0\). Taking differentiation, we get \(\frac{df(x)}{dx} = -\eta + e^{-x} > 0\) for all \(x \in (0, \frac{1}{4})\).

Therefore, for all \(x \in (0, \frac{1}{4})\),

\[ e^{-x} \leq 1 - \eta x. \] (21)

The following Taylor expansion can be found in standard calculus textbooks. For all \(x \in (0, 1)\),

\[ \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots. \] (22)

Therefore, we have

\[ (1 - \frac{b}{k})^k = e^{k \ln(1 - \frac{b}{k})} = e^{k(-\frac{b}{k} - \frac{b^2}{2k} - \frac{b^3}{3k} - \cdots)} = e^{-b - \frac{b^2}{2k} - \frac{b^3}{3k} - \cdots} \] (23)

\[ \leq e^{-b - \frac{b}{2k}} = e^{-b} \cdot e^{-\frac{b}{2k}} \] (24)

\[ \leq e^{-b} \cdot (1 - \eta \cdot \frac{b}{2k}) \leq \frac{1}{e^b} - \frac{np}{2e^b k}. \] (25)

Note that the transition from (24) to (24) is based on inequality (21).

The part 2 follows from part 1. This is because \(1 - (1 - \frac{b}{k})^k - \xi \geq 1 - \frac{1}{e^b} + \frac{np}{2e^b k} - \xi \geq 1 - \frac{1}{e^b} + \frac{np}{2e^b k}\).

4.4. Multiple Rounds Random Sampling for Maximum Coverage

Theorem 20 gives the performance of our randomized greedy approximation algorithm for the maximum coverage problem. It uses multiple rounds of random samplings since there is a series of executions for calling \texttt{ApproximateDifference(.)} via \texttt{ApproximateSetDifferenceSize(.)}, which is given at Virtual Function Implementation 1. This shows maximum coverage problem has a \(p(n)\) time \((1 - \frac{1}{e})\)-approximation for \((\emptyset, (0, 0))\)-list as input (see Definition 4) under the model that each input set \(A_i\) provides \(O(1)\)-time to generate a random sample and answer one membership query.

Algorithm 4 : Virtual Function Implementation 1

The parameters \(L', A_i, s_i, R_i, e', \gamma, k, n\) follow from those in Algorithm 3.

RandomSamples\((A_i, R_i, \xi, \gamma, k, n)\):

\{ Let \(R_i = \emptyset\); \}

RandomTest\(.)\{ the same as that defined at Algorithm 1 \}
ApproximateSetDifferenceSize($L', A_i, s_i, R_i, \epsilon', \gamma, k, n$);
{
    Return ApproximateDifference($L', A_i, s_i, R_i, \epsilon', \gamma$);
}

ProcessSet($A_i$) { } (Do nothing to set $A_i$)

End of Algorithm

Lemma 19 gives the complexity of the ApproximateMaximumCover(.) using multiple rounds of random samplings from the input sets. It also gives a small failure probability of the algorithm. Its complexity will be improved and shown at Lemma 23 in Section 5.

**Lemma 19.** Let $\xi \in (0, 1)$, and $\alpha_L, \alpha_R, \delta_L, \delta_R \in [0, 1)$. Assume that the algorithm ApproximateMaximumCover(.) is executed with Virtual Function Implementation 1. Let $L'_s$ be the list $L'$ after the completion of ApproximateMaximumCover(.) Then

1. With probability at most $\gamma$, there is a value $s_{i,j}$ in the line 10 of the algorithm ApproximateMaximumCover(.) does not satisfies

   $$(1 - \alpha_L)(1 - \delta_L)|A_i - U(L'_s(j))| - \epsilon'|A_i| \leq s_{i,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_i - U(L'_s(j))| + \epsilon'|A_i|, \text{ and}$$

2. The algorithm has complexity $(T_1(\xi, \gamma, k, n), R_1(\xi, \gamma, k, n), Q_1(\xi, \gamma, k, n)$ with

   $$T_1(\xi, \gamma, k, n) = \text{kn}T_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k),$$

   $$R_1(\xi, \gamma, k, n) = \text{kn}R_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k), \text{ and}$$

   $$Q_1(\xi, \gamma, k, n) = \text{kn}Q_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k),$$

   where $(T_D(.), R_D(.), Q_D(.))$ are the complexity functions defined in Lemma 12 and $\beta$ is the same as that in Lemma 17, $\gamma' = \frac{\gamma}{\text{kn}}$ and $\epsilon'$ is the same as that in ApproximateMaximumCover(.)

**Proof:** By Lemma 12, for each $s_{i,j}$ we have

$$(1 - \alpha_L)(1 - \delta_L)|A_i - U(L'_s(j))| - \epsilon'|A_i| \leq s_{i,j} \leq (1 + \alpha_R)(1 + \delta_R)|A_i - U(L'_s(j))| + \epsilon'|A_i| \quad (26)$$

with probability at most $\gamma'$ to fail. The total probability that one of the $\text{kn}$ cases fails is at most

$$P_1 = \text{kn}\gamma' = \text{kn} \cdot \frac{\text{kn}}{\text{kn}} \leq \frac{\beta}{\text{kn}}.$$

By Lemma 12, the function ApproximateDifference(.) at line 8 has complexity

$$(T_D(\epsilon', \gamma', k), R_D(\epsilon', \gamma', k), Q_D(\epsilon', \gamma', k)).$$

The algorithm’s complexity is $(T_1(\xi, \epsilon, \gamma, k, n), R_1(\xi, \epsilon, \gamma, k, n), Q_1(\xi, \epsilon, \gamma, k, n))$ with

$$T_1(\xi, \epsilon, \gamma, k, n) = \text{kn}T_D(\epsilon', \gamma', k) = \text{kn}T_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k),$$

$$R_1(\xi, \epsilon, \gamma, k, n) = \text{kn}R_D(\epsilon', \gamma', k) = \text{kn}R_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k), \text{ and}$$

$$Q_1(\xi, \epsilon, \gamma, k, n) = \text{kn}Q_D(\epsilon', \gamma', k) = \text{kn}Q_D(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k).$$
Theorem 20. Let $\rho$ be a constant in $(0,1)$. For parameters $\epsilon \in (0,1)$ and $\alpha_L, \alpha_R, \delta_L, \delta_R \in [0,1-\rho]$, there is an algorithm to give a $(1 - \frac{1}{e^\beta})$-approximation for the maximum cover problem ($\beta$ is defined in Lemma 17), such that given a $((\alpha_L, \alpha_R), (\delta_L, \delta_R))$-list $L$ of finite sets $A_1, \ldots, A_n$ and an integer $k$, it returns $z$ and $H \subseteq \{1, 2, \ldots, n\}$ that satisfy

1. $|\cup_{j \in H} A_j| \geq (1 - \frac{1}{e^\beta})C^*(L,k)$ and $|H| = k$,
2. $((1 - \alpha_L)(1 - \delta_L) - \epsilon)|\cup_{j \in H} A_j| \leq z \leq ((1 + \alpha_R)(1 + \delta_R) + \epsilon)|\cup_{j \in H} A_j|$, and
3. Its complexity is $(T_C(\epsilon, \gamma, k, n), R_C(\epsilon, \gamma, k, n), Q_C(\epsilon, \gamma, k, n))$ where
   
   $T_C(\epsilon, \gamma, k, n) = O\left(\frac{k^6 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right)$,
   
   $R_C(\epsilon, \gamma, k, n) = O\left(\frac{k^5 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right)$, and
   
   $Q_C(\epsilon, \gamma, k, n) = O\left(\frac{k^6 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right)$.

Proof: Select $\xi = \min(\frac{\epsilon \eta \beta}{4e^\beta}, \epsilon)$, where $\eta$ is defined in Lemma 18. The accuracy of approximation follows from Lemma 18, Lemma 17 and Lemma 19. The complexity follows from the complexity functions $T_D(\cdot), R_D(\cdot),$ and $Q_D(\cdot)$ in Lemma 12. Since $T_D(\epsilon, \gamma, k) = O\left(\frac{k^6 n}{\epsilon^2} \log \frac{1}{\gamma}\right)$, we have

$$T_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^6 n}{\epsilon^2} \log \frac{4kn}{\gamma}\right)$$

(27)

$$= O\left(\frac{k^5 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right)$$

(28)

$$= O\left(\frac{k^5 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right).$$

(29)

Similarly,

$$R_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^4 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right)$$

(30)

$$Q_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^3 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right).$$

(31)

Thus,

$$T_C(\epsilon, \gamma, k, n) = knT_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^6 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right),$$

$$R_C(\epsilon, \gamma, k, n) = knR_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^5 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right),$$

and

$$Q_C(\epsilon, \gamma, k, n) = knQ_D\left(\frac{\xi}{4k}, \frac{\gamma}{4kn}, k\right) = O\left(\frac{k^6 n}{\epsilon^2} (\log n + \log \frac{1}{\gamma})\right).$$

5. Faster Algorithm for Maximum Coverage

In this section, we describe an improved approximation algorithm for the maximum coverage problem. It has slightly less time and keeps the same approximation ratio. We showed the multi-round
random sampling approach in Section 4.4 with Theorem 20. A single round random sampling approach is given in this section with an improved time complexity. It may help us under how two different approaches affect the algorithm complexity.

In this section, we control the total number of random samples from each set. The random samples from each set $S_i$ will be generated in the beginning of algorithm. We show how to remove the samples that are already in the selected sets saved in the list $L'$ of the algorithm for ApproximateMaximumCover().

**Definition 21.** Assume that $\varepsilon, \gamma \in (0, 1)$ and $k, n \in N$. Let $L$ be a list of sets $A_1, A_2, \cdots, A_n$.

- Define $h^*(k, n)$ to be the number of subsets of size at most $k$ in $\{1, 2, \cdots, n\}$.
- Define $\gamma_{k, n} = \frac{\gamma}{nh(k, n)}$.
- Define $g(\varepsilon, \gamma, k, n) = R_D(\varepsilon, \gamma_{k, n}, k)$, where $R_D(.)$ is defined in Lemma 12.

**Lemma 22.** Assume parameters $\varepsilon, \gamma, \alpha_L, \alpha_R, \delta_L, \delta_R \in (0, 1)$ and $k, n \in N$. Let function $g(\varepsilon, \gamma, k, n)$ be defined as in Definition 21. Let $L$ be a list of sets $A_1, A_2, \cdots, A_n$ such that each $A_i$ has a $(\alpha_L, \alpha_R)$-biased random generator RandomElement($A_i$), and an approximation $s_j$ for $|A_i|$ with $(1 - \delta_L)|A_i| \leq s_j \leq (1 + \delta_R)|A_i|$.

1. The function $g(.)$ has $g(\varepsilon, \gamma, k, n) = O(\frac{1}{\gamma}(k \log(\frac{3n}{k}) + \log \frac{1}{\gamma}))$.

2. Let $R_i$ be a list of $w = g(\varepsilon, \gamma, k, n)$ random samples from each set $A_i$ in the input list via the $(\alpha_L, \alpha_R)$-biased generator RandomElement($A_i$), then with failure probability at most $\gamma$, the value $s = \frac{\sum_i r_{i,j}}{w} \cdot s_i$ with $r_{i,j} = \text{test}(L^*, R_i)$ (see Definition 11) satisfies the inequality (32)

\[
(1 - \alpha_L)(1 - \delta_L)|A_i| - A| - \varepsilon|A_i| \leq s \leq (1 + \alpha_R)(1 + \delta_R)|A_i| - A| + \varepsilon|A_i|, \tag{32}
\]

for every sublist $L^* = A_{i_1}, A_{i_2}, \cdots, A_{i_j}$ with $j \leq k$ of $L$, where $A = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_j}$.

**Proof:** Let $A_1, \cdots, A_n$ be the input list of sets, and $k$ be the integer parameter in the input. Let $V_k$ be the class of subsets from $\{1, 2, \cdots, n\}$ with size at most $k$. In other words, $V_k = \{H : H \subseteq \{1, 2, \cdots, n\} \text{ and } |H| \leq k\}$. Thus, we have $|V_k| = h^*(k, n) = \sum_{i=0}^{k} \binom{n}{i}$. Let $h(k, n) = k\binom{n}{k}$ if $k \leq \frac{n}{2}$, and $2^n$ otherwise. Clearly, $h^*(k, n) \leq h(k, n)$. By the classical Stirling formula $k! \sim \sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k$, we have

\[
\binom{n}{k} \leq \frac{n^k}{k!} = O((\frac{en}{k})^k). \tag{33}
\]

Let $\gamma_{k, n}$ be given as Definition 21. There are two cases to be discussed.

Case 1: $1 \leq k \leq \frac{n}{2}$. By inequality (33), we have

\[
\log \frac{1}{\gamma_{k, n}} = \log \frac{nh^*(k, n)}{\gamma} \leq \log \frac{nh(k, n)}{\gamma} \leq \log k \log(\frac{3n}{k}) + \log \frac{1}{\gamma} \tag{34}
\]

\[
= O(k \log(\frac{3n}{k}) + \log kn + \frac{1}{\gamma}) \tag{35}
\]

\[
= O(k \log(\frac{3n}{k}) + \log(k^2 \frac{3n}{k}) + \log \frac{1}{\gamma}) \tag{36}
\]

\[
= O(k \log\frac{3n}{k} + 2 \log k + \log \frac{3n}{k} + \log \frac{1}{\gamma}) \tag{37}
\]

\[
= O(k \log\frac{3n}{k} + \log \frac{1}{\gamma}). \tag{38}
\]
Case 2: \( n \geq k > \frac{n}{2} \). It is trivial that
\[
\log \frac{1}{\gamma_{k,n}} = \log \frac{n h^*(k, n)}{\gamma} \leq \log \frac{n h(k, n)}{\gamma} = \log \frac{n 2^n}{\gamma} \tag{39}
\]
\[
= O(k \log (\frac{3n}{k}) + \log \frac{1}{\gamma}). \tag{40}
\]
Thus, \( \frac{1}{\gamma_{k,n}} = O(k \log (\frac{3n}{k}) + \log \frac{1}{\gamma}) \) for all \( 1 \leq k \leq n \). Thus,
\[
g(\epsilon, \gamma, k, n) = R_D(\epsilon, \gamma_{k,n}, k) = O(\frac{1}{\epsilon^2} \log \frac{1}{\gamma_{k,n}}) = O(\frac{1}{\epsilon^2} (k \log (\frac{3n}{k}) + \log \frac{1}{\gamma})). \tag{41}
\]
By Lemma 12, with \( g(\epsilon, \gamma, k, n) = R_D(\epsilon, \gamma_{k,n}, k) \) random samples from each set \( A_i \), the probability that one of at most \( n h^*(k, n) \) cases fails to satisfy inequality (32) is at most \( n h^*(k, n) \cdot \gamma_{k,n} \leq \gamma \) by inequality (2).

The random samples from each input set \( A_i \) are collected in the beginning of the algorithm of ApproximateMaximumCoverage(\( \cdot \)), and are stored in the list \( R_i \). Virtual Function Implementation 2 gives such an consideration.

**Algorithm 5 : Virtual Function Implementation 2**

The parameters \( L', A_i, s_i, R_i, \xi, \gamma, k, n \) follow from those in Algorithm 3.

**RandomSamples** \( (A_i, R_i, \xi, \gamma, k, n) \)

\{
    Generate a list \( R_i \) of \( g(\epsilon, \gamma, k, n) \) random samples of \( A_i \);
    Mark all elements of \( R_i \) as white.
\}

**RandomTest** \( (L', A_i, w) \)

\{
    Let \( A_{t_j} \) be the newly picked set saved in \( L' \) (\( L' = A_{t_1}, A_{t_2}, \cdots, A_{t_j} \));
    For each \( a \) in the list \( R_i \) of random samples from \( A_i \)
    \{
        If \( a \in A_{t_j} \) then mark \( a \) as black in \( R_i \);
    \}
    Let \( r_{i,j} \) be the number of white items in \( R_i \) (it may have multiplicity);
    return \( r_{i,j} \);
\}

**ApproximateSetDifferenceSize** \( (L', A_i, R_i, s_i, \epsilon', \gamma, k, n) \)

\{
    Let \( w = g(\epsilon, \gamma, k, n) \);
    Let \( r_{i,j} = \text{RandomTest}(L', A_i, w) \);
    Return \( s = \frac{r_{i,j}}{w} \cdot s_i \);
\}

**ProcessSet** \( (A_i) \) \{ \} (Do nothing to set \( A_i \))

**End of Algorithm**

Lemma 22 shows approximation for maximum coverage is possible via one round random samplings from input sets. It shows how to control the number of random samples from each input set to guarantee small failure probability of the approximation algorithm. It slightly reduces the complexity by multiple rounds of random samplings described in Theorem 20.

Lemma 23 gives the complexity for the algorithm with Virtual Function Implementation 2.
Lemma 23. Assume that $\epsilon, \gamma \in (0, 1)$ and $k, n \in \mathbb{N}$. Let $k$ be an integer parameter and $L$ be a list of sets $A_1, A_2, \ldots, A_n$ for a maximum coverage problem. Let $L'_s$ be the sublist $L'$ after algorithm ApproximateMaximumCover(.) is completed. Assume that the algorithm uses Virtual Function Implementation 2

1. After the $j$-th iteration of the loop from line 5 to line 18 of the algorithm ApproximateMaximumCover(.), the returned value $r_{i,j}$ from RandomTest($L'_s(j)$, $w$) is equal to test($L'_s(j)$, $R$) (see Definition 11).

2. The algorithm with Virtual Function Implementation 2 has complexity $(T_2(\xi, \gamma, k, n), R_2(\xi, \gamma, k, n), Q_2(\xi, \gamma, k, n))$ such that for each set $A_i$, it maintains $t_{i,j} = |R_i - U(L'_s(j))|$, where

\[
T_2(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right),
\]

(42)

\[
R_2(\xi, \gamma, k, n) = O\left(\frac{k^2}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right),
\]

and

(43)

\[
Q_2(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right),
\]

(44)

where $R_i$ is the list of random samples from $A_i$ and defined in Lemma 22.

Proof: Let $A_1, \ldots, A_n$ be the input list of sets, and $k$ be the integer parameter in the input. Consider $g(\epsilon', \gamma, k, n)$ random (white) samples for each set as in Lemma 22, where $\epsilon'$ is defined in algorithm ApproximateMaximumCover(.). After a set $A_{t_j}$ is added to $L'$, all the random samples in $R_i$ will be checked if they are from $A_{t_j}$. For each white random sample $x$ in $R_i$ for all $t_j \neq i$, change $x$ to black if $x \in A_{t_j}$. Thus, it takes $kg(\epsilon', \gamma, k, n)$ time. Count the white samples left in $R_i$ and save it in the variable $r_{i,j}$. A simple induction can prove Part 1. In the beginning all elements in $R_i$ are white. Assume that after $j$-iterations of the loop from line 5 to line 18 of the algorithm ApproximateMaximumCover(.), the number of white elements of $R_i$ is test($L'_s(j)$, $R_i$). After $(j + 1)$ iterations, list $L'$ has $j + 1$ sets and $A_{t_{j+1}}$ as the last added. Since each white random sample of $R_i$ in $A_{t_{j+1}}$ is changed to black for all $t_{j+1} \neq i$, we have $r_{i,j+1} = \text{test}(L'_s(j + 1), R_i)$. Thus, the returned value $r_{i,j}$ from RandomTest($L'$, $A_i$, $w$) is equal to test($L'_s(j)$, $R_i$).

By Lemma 22, the algorithm has complexity $(T_2(\xi, \gamma, k, n), Q_2(\xi, \gamma, k, n), R_2(\xi, \gamma, k, n))$ with

\[
T_2(\xi, \gamma, k, n) = kg(\epsilon', \gamma, k, n)n = O\left(\frac{k^3}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right),
\]

(45)

\[
R_2(\xi, \gamma, k, n) = g(\epsilon', \gamma, k, n)n = O\left(\frac{k^2}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right),
\]

and

(46)

\[
Q_2(\xi, \gamma, k, n) = kg(\epsilon', \gamma, k, n)n = O\left(\frac{k^3}{\xi^2}(k \log \frac{3n}{k} + \log \frac{1}{\gamma})n\right).
\]

(47)

(48)

Theorem 9 states that the improved approximation algorithm for the maximum coverage problem has a reduced complexity while keeping the same approximation ratio $\frac{1 - \eta}{L}$ for $((0, 0), (0, 0))$-list as input (see Definition 4). The algorithm is based on one round samplings from the input sets. Now we give the proof of Theorem 9.

Proof: [Theorem 9]. Use $g(\epsilon', \gamma, k, n)$ random samples for each set $S_i$. It follows from Lemma 22, and Lemma 17. Select $\xi = \min\left(\frac{\epsilon n^2}{2^{2i}T}, \epsilon\right)$, where $\gamma$ is defined in Lemma 18 and $\beta$ is the same as that in Lemma 17. With the condition $\epsilon \in (0, 1)$, the accuracy of approximation follows from Lemma 18.
By Lemma 23, its complexity is \((T(\epsilon, \gamma, k, n), R(\epsilon, \gamma, k, n), Q(\epsilon, \gamma, k, n))\)

\[
T(\epsilon, \gamma, k, n) = T_2(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),
\]

\[
R(\epsilon, \gamma, k, n) = R_2(\xi, \gamma, k, n) = O\left(\frac{k^2}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),
\]

\[
Q(\epsilon, \gamma, k, n) = Q_2(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right).
\]

We have Theorem 24 that gives a slightly less approximation ratio and has a less time complexity.

**Theorem 24.** Let \(\rho\) be a constant in \((0,1)\) For parameters \(\xi \in (0, 1)\) and \(\alpha_L, \alpha_R, \delta_L, \delta_R \in [0, 1 - \rho]\), there is an algorithm to give a \((1 - (1 - \frac{\beta}{k})^k - \xi)\)-approximation for the maximum cover problem, such that given a \((\alpha_L, \alpha_R), (\delta_L, \delta_R)\)-list \(L\) of finite sets \(A_1, \cdots, A_n\) and an integer \(k\), it returns \(z\) and \(H \subseteq \{1, 2, \cdots, n\}\) that satisfy

1. \(|\cup_{j \in H} A_j| \geq (1 - (1 - \frac{\beta}{k})^k - \xi)C^*(L, k)\) and \(|H| = k\),
2. \((1 - \alpha_L)(1 - \delta_L) - \xi|)\cup_{j \in H} A_j| \leq z \leq ((1 + \alpha_R)(1 + \delta_R) + \xi)|\cup_{j \in H} A_j|,\) and
3. Its complexity is \((T(\xi, \gamma, k, n), R(\xi, \gamma, k, n), Q(\xi, \gamma, k, n))\) with

\[
T(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),
\]

\[
R(\xi, \gamma, k, n) = O\left(\frac{k^2}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),\] and

\[
Q(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),
\]

where \(\beta = \frac{(1 - \alpha_L)(1 - \delta_L)}{(1 + \alpha_R)(1 + \delta_R)}\).

**Proof:** The accuracy of approximation follows from Lemma 22, and Lemma 17. By Lemma 23, its complexity is \((T(\xi, \gamma, k, n), R(\xi, \gamma, k, n), Q(\xi, \gamma, k, n))\) with

\[
T(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),
\]

\[
R(\xi, \gamma, k, n) = O\left(\frac{k^2}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right),\] and

\[
Q(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k \log\frac{3n}{k} + \log\frac{1}{\gamma})n\right).
\]

**Corollary 25.** For parameter \(\xi \in (0, 1)\), there is an algorithm to give a \((1 - (1 - \frac{1}{k})^k - \xi)\)-approximation for the maximum cover problem, such that given a \((0, 0), (0, 0)\)-list \(L\) of finite sets \(A_1, \cdots, A_n\) and an integer \(k\), it returns \(z\) and \(H \subseteq \{1, 2, \cdots, n\}\) that satisfy

1. \(|\cup_{j \in H} A_j| \geq (1 - (1 - \frac{1}{k})^k - \xi)C^*(L, k)\) and \(|H| = k\),
2. \((1 - \xi)|\cup_{j \in H} A_j| \leq z \leq (1 + \xi)|\cup_{j \in H} A_j|, \) and

3. Its complexity is \((T(\xi, \gamma, k, n), R(\xi, \gamma, k, n), Q(\xi, \gamma, k, n))\) with

\[
T(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right),
\]

\[
R(\xi, \gamma, k, n) = O\left(\frac{k^2}{\xi^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right), \quad \text{and}
\]

\[
Q(\xi, \gamma, k, n) = O\left(\frac{k^3}{\xi^2}(k\log(\frac{3n}{k}) + \log \frac{1}{\gamma})n\right).
\]

6. Inapproximability of Partial Sublinear Time Computation

In this section, we introduce the concept of partial sublinear time computation. The maximal coverage has a partial sublinear constant factor approximation algorithm. On the other hand, we show that an inapproximability result for maximum coverage if the time is \(q(m)n^{1-\epsilon}\), where \(n\) is the number of sets. This makes the notion of partial sublinear computation different from conventional sublinear computation.

6.1. Model for Inapproximation

We define a more generalized randomized computation model than that given by Definition 1. In the model given by definition 26, it allows to fetch the \(j\)-th element from an input set \(A_i\). As we study a sublinear time computation, its model is defined by Definition 26. It is more powerful than that given in Definition 1. The inapproximation result derived in this model also implies a similar result in the model of Definition 1.

Definition 26. A randomized computation \(T(., .)\) for the maximum coverage problem is a tree \(T\) that takes an input \(k\) of integer and a list of finite sets defined in Definition 1.

1. Each node of \(T((L, k), .)\) (with input list \(L\) of sets and integer \(k\) for the maximum coverage) allows any operation in Definition 1.

2. A fetching statement \(x = A_i[j]\) \((1 \leq j \leq s_i = |A_i|)\) lets get the \(a_{i,j} \in A_i\), where set \(A_i\) contains the elements \(a_{i,1}, \ldots, a_{i,s_i}\) (which may be unsorted).

3. A branching point of \(p\) that has \(s\) children \(p_1, p_2, \ldots, p_s\) and is caused by the following two cases

   - RandomElement\((A_i)\) returns a random element in \(A_i = \{a_{i,1}, \ldots, a_{i,s_i}\}\) such that \(p_j\) is the case that \(a_{i,j}\) is selected in \(A_i\), and \(s = s_i\).
   - RandomNumber\((s)\) returns a random element in \(\{0, 1, \ldots, s - 1\}\) for an integer \(s > 0\) such that \(p_j\) is the case that \(j + 1\) is returned.

4. A computation path is determined by a series of numbers \(r_0, r_1, r_2, \ldots, r_m\) such that the \(r_j\) corresponds to the \(j\)-th branching point for \(j = 1, 2, \ldots, m - 1\), and \(r_0\) is the root, and \(r_m\) is a leaf.

5. A partial path \(p\) is an initial part of a path that starts from the root \(r_0\) of computation to a node \(q\).

6. The root node \(r_0\) has weight \(w(r_0) = 1\).

7. If a partial path \(p\) from root \(r_0\) to a node \(q\) that has children \(p_1, \ldots, p_s\), and weight \(w(q)\). Then \(w(p_1) = w(p_2) = \cdots = w(p_s) = \frac{w(q)}{s}\), where \(w(p_i)\) is the weight for \(p_i\).

8. The weight of a path from the root \(r_0\) to a leaf \(q\) has the weight \(w(q)\), which is the weight of \(q\).
9. The output of the randomized computation \( T((L, k), .) \) (with input \((L, k)\)) on a path \( p \) is defined to be \( T((L, k), p) \).

In Definition 27, we give the concept of a shared path for randomized computation under two different inputs of lists of sets. Intuitively, the computation of the two shared paths with different inputs does not have any difference, gives both the same output, and has the same weight.

**Definition 27.**

- Let \( L \) be a list of sets \( A_1, \cdots, A_n \), and \( L' \) be another input list of \( n \) sets \( A_1', \cdots, A_n' \). If \( |A_i| = |A_i'| \) for \( i = 1, 2, \cdots, n \), then \( L \) and \( L' \) are called equal size list of sets.
- Let \( L \) be a list of sets \( A_1, \cdots, A_n \) and \( L' \) an another input of \( n \) sets \( A_1', \cdots, A_n' \) such that they are of equal size. A partial path \( p \) is shared by \( T((L, k), .) \) and \( T((L', k), .) \) if
  - path \( p \) gets the same result for \( Query(x, A_i) \) and \( Query(x, A_i') \) for all queries along \( p \),
  - path \( p \) gets the same result for fetching between \( x = A_i[j] \) and \( x = A_i'[j] \),
  - path \( p \) gets the same result for each random access to \( \text{RandomElement}(A_i) \) and \( \text{RandomElement}(A_i') \), and
  - path \( p \) gets the same result for each random access to \( \text{RandomNumber}(s) \).

**Definition 28.** Let \( T((L, k), .) \) be a randomized computation for the maximum coverage with a input list \( L \) and an integer \( k \).

- Define \( P(1) \) to be the set that contains the partial path with the root.
- If \( p \in P(a) \) and \( p \) is from root \( r_0 \) to a branching point \( q \) with children \( q_1, \cdots, q_t \), then each partial path from \( r_0 \) to \( q_i \) belongs to \( P(a + 1) \) for \( i = 1, 2, \cdots, t \).
- \( P(a + 1) \) contains all paths (from the root to leave) of length at most \( a + 1 \) nodes.

**Lemma 29.** Let \( T((L, k), .) \) be a randomized computation for the maximum coverage with a input \((L, k)\).

1. \( \sum_{p \in P(a)} w(p) = 1 \) for all \( a \geq 1 \).
2. \( \sum_{path \ p} w(p) = 1 \).

**Proof:** It can be proven via an induction. It is true for \( a = 1 \) by definition. Assume \( \sum_{p \in P(a)} w(p) = 1 \). We have \( \sum_{p \in P(a+1)} w(p) = 1 \) by item 7 of Definition 26. Statement ii follows from Statement i.

**Definition 30.** A partial sublinear \( O(t_1(n)t_2(m)) \) time \((u(n, m), v(n, m))\)-approximation algorithm \( T(., .) \) that satisfies the following conditions:

1. It runs \( O(t_1(n)t_2(m)) \) steps along every path.
2. The two functions have \( t_1(n) = o(n) \) or \( t_2(m) = o(m) \), and
3. The sum of weights \( w(p) \) of paths \( p \) that satisfy \( \frac{C^*(L, k)}{u(n, m)} - v(n, m) \leq T((L, k), p) \leq u(n, m) \cdot C^*(L, k) + v(n, m) \) is at least \( \frac{3}{4} \), where \( m = \max\{|A_i| : i = 1, \cdots, n\} \).
6.2. Inapproximation for Maximum Coverage

We derive an inapproximability result for the maximum coverage problem in partial sublinear \( p(m)n^{1-\epsilon} \) time model. It contrasts the partial sublinear time \( O(p(n)) \) constant factor approximation for it.

**Theorem 31.** For nondecreasing functions \( t_1(n), t_2(m), v(m) : N \to R^+ \) with \( v(m) = o(m) \) and \( t_1(n) = o(n) \), the function the maximum coverage problem has no partial sublinear \( t_1(n)t_2(m) \) time \( (u,v(m)) \)-approximation for any fixed \( u > 0 \).

**Proof:** It is proven by contradiction. Let \( u \) be a fixed positive integer. Assume that the maximum coverage problem \( C^*(L,k) \) has a partial sublinear \( t_1(n)t_2(m) \) time \( (u,v(m)) \)-approximation by a randomized computation \( T(.,.) \).

Let

\[
c_1 = 200, \quad k = d = c_1u^2. \tag{49}
\]

Since \( v(m) = o(m) \), select \( m \) to be an integer such that

\[
c_1 u \cdot v(m) < m, \quad \text{and} \quad m \equiv 0 (\mod d). \tag{50}
\]

Select an integer \( n \) to be large enough such that

\[
\max(2d, c_1 \cdot d \cdot t_1(n)t_2(m)) < n. \tag{53}
\]

Let \( N_m \) be the set of of integers \( \{1, 2, \ldots, m\} \). Let sets \( A_1 = A_2 = \cdots = A_n = \{1, 2, \ldots, \frac{m}{d}\} \). Let \( L \) be the list of sets \( A_1, A_2, \ldots, A_n \).

For each \( A_i \), define \( Q(A_i) = \sum_{path \ p \ in \ T((L,k),)} \text{queries } A_i \cdot w(p) \). If there are more than \( \frac{n}{2} \) sets \( A_i \) with \( Q(A_i) > \frac{0.01n}{2d} \), then

\[
\sum_{i=1}^{n} Q(A_i) > \frac{0.01n}{2d}. \tag{54}
\]

For a path \( p \), define \( H(p) \) to be the number of sets \( A_i \) queried by \( p \). Clearly, \( H(p) \leq t_1(n)t_2(m) \) since each path runs in at most \( t_1(n)t_2(m) \) steps. We have

\[
\sum_{i=1}^{n} Q(A_i) \leq \sum_{p} w(p)H(p) \leq t_1(n)t_2(m) \sum_{p} w(p) = t_1(n)t_2(m). \tag{55}
\]

By inequalities (54) and (55), we have \( \frac{0.01n}{2d} < t_1(n)t_2(m) \), which implies \( n < 200 \cdot d \cdot t_1(n)t_2(m) \). This contradicts inequality (53). Therefore, there are at least \( \frac{n}{2} \) sets \( A_j \) with \( Q(A_j) < \frac{0.01n}{d} \). Let \( J \) be the set \( \{j : Q(A_j) < \frac{0.01n}{d}\} \). We have \( |J| \geq \frac{n}{2} \) \( \geq d \) (by inequality (53)).

Let \( i_1 < i_2 < \cdots < i_d \) be in \( J \). Define the list \( L' \) of sets \( A'_1, A'_2, \ldots, A'_n \) with

\[
A'_{i_1} = \{1, 2, \ldots, \frac{m}{d}\},
\]

\[
A'_{i_2} = \{\frac{m}{d} + 1, \frac{m}{d} + 2, \ldots, \frac{2m}{d}\},
\]

\[
\ldots
\]

\[
A'_{i_d} = \{\frac{(d-1)m}{d} + 1, \frac{(d-1)m}{d} + 2, \ldots, m\}, \quad \text{and}
\]

\[
A'_j = A_j \quad \text{for every } j \in \{1,2,\ldots,n\} - \{i_1,i_2,\ldots,i_d\}.
\]

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In this section, we show that the special case of maximum coverage problem with equal size sets input is as hard as the general maximum coverage problem. This gives a hard core for the maximum coverage problem. When \( A_1, \ldots, A_n \) are of the same size \( m \), the input size is measured as \( nm \).

The classical set cover problem is that given a set \( U \) of elements (called the universe) and a collection \( S \) of sets whose union equals the universe \( U \), identify the smallest sub-collection of \( S \) whose union equals the universe. Clearly, the two lists \( L \) and \( L' \) are of the same size, and we have

\[
C^*(L', k) = \frac{m}{d}, \quad \text{and} \quad C^*(L, k) = \frac{m}{d}.
\]

A shared path has the same weight in both \( T((L, k), .) \) and \( T(L', .) \). We have \( \{ p : p \text{ is shared by } T((L, k), .) \text{ and } T(L', .) \} \) has weight of at least 0.99 in total.

There exists a \( z \), which is equal to \( T((L, k), p) = T((L', k), p) \) for some shared path \( p \), such that \( z \) is a \((u, v(m))\)-approximation for both \( C^*(L', k) \) and \( C^*(L, k) \). Therefore,

\[
\frac{C^*(L', k)}{u} - v(m) \leq z \leq u \cdot C^*(L, k) + v(m), \quad \text{and} \quad \frac{C^*(L, k)}{u} - v(m) \leq z \leq u \cdot C^*(L', k) + v(m).
\]

Therefore,

\[
\frac{C^*(L', k)}{u} - v(m) \leq z \leq u \cdot C^*(L, k) + v(m).
\]

By equations (56) and (57),

\[
\frac{m}{u} - v(m) \leq \frac{um}{d} + v(m).
\]

Therefore,

\[
\frac{m(d - u^2)}{du} \leq 2v(m).
\]

By inequality (49), and equation (50), and inequality (62), we have

\[
\frac{m}{2u} \leq \frac{m(d - \frac{d^2}{2})}{du} \leq \frac{m(d - u^2)}{du} \leq 2v(m).
\]

Therefore,

\[
m \leq 4u \cdot v(m).
\]

This brings a contradiction by inequality (51).

Theorem 31 implies there is no \( O((nm)^{1-\epsilon}) \) time approximation for the maximum coverage problem. Thus, Theorem 31 gives a natural example that has partial sublinear time constant factor approximation, but has no sublinear time approximation.

A lot of computational problems can be represented by a function on a list of sets. For example, any bipartite graph \( G(V_1, V_2, E) \) can be represented by a list of sets \( A_1, A_2, \ldots, A_n \), where \( n = |V_1| \) and \( V_1 = \{ v_1, v_2, \ldots, v_n \} \), and each \( A_i \) is a subset of \( V_2 \) with \( m = |V_2| \) such that each one of \( A_i \) has an edge adjacent to \( v_i \in V_1 \). If we define \( f(A_1, A_2, \ldots, A_n) \) to be length of longest path in \( G(V_1, V_2, E) \), the function \( f(.) \) is a NP-hard. If \( g(A_1, A_2, \ldots, A_n) \) is defined to be number of paths of longest paths in \( G(V_1, V_2, E) \), the function \( g(.) \) is a \#P-hard.

7. Hardness of Maximum Coverage with Equal Size Sets

In this section, we show that the special case of maximum coverage problem with equal size sets input is as hard as the general maximum coverage problem. This gives a hard core for the maximum coverage problem. When \( A_1, \ldots, A_n \) are of the same size \( m \), the input size is measured as \( nm \).

The classical set cover problem is that given a set \( U \) of elements (called the universe) and a collection \( S \) of sets whose union equals the universe \( U \), identify the smallest sub-collection of \( S \) whose union equals the universe.
**Definition 32.** The \( s \)-equal size maximum coverage problem is the case of maximum coverage problem when the input list of sets are all of the same size \( s \). The equal size maximum coverage problem is the case of maximum coverage problem when the input list of sets are all of the same size.

**Theorem 33.** Let \( c \) be an positive real number and \( s \) be an integer parameter.

i. There is a polynomial time reduction from a set cover problem with set sizes bounded by \( s \) to \( s \)-equal size maximum coverage problem.

ii. Assume there is a polynomial time \( c \)-approximation algorithm for \( s \)-equal size maximum coverage problem, then there is a polynomial time \((c - o(1))\)-approximation algorithm for the maximum coverage problem with input sets \( A_1, \ldots, A_n \) of size \( |A_i| \leq s \) for \( i = 1, 2, \ldots, n \).

**Proof:** We use the following two cases to prove the two statements in the theorem, respectively.

i. Let \( A_1, A_2, \ldots, A_n \) be the input for a set cover problem, and none of \( A_1, A_2, \ldots, A_n \) is empty set. Without loss of generality, assume \( t = |A_1| = \max(|A_1|, |A_2|, \ldots, |A_n|) \). Let \( A_0 \) be a new set with \( |A_0| = t \) and \( A_0 \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = \emptyset \). Construct a new list of sets \( A_0, A_1', A_2', \ldots, A_n' \) such that each \( A_i' = A_i \cup A_0[t - |A_i|] \) for \( i = 1, 2, \ldots, n \), where \( A_0[u] \) is the first \( u \) elements of \( A_0 \) (under an arbitrary order for the elements in \( A_0 \)). It is easy to see that \( A_1, A_2, \ldots, A_n \) has a \( k \) sets solution if and only if \( A_0, A_1', A_2', \ldots, A_n' \) has a \( k \) sets solution for the set cover problem.

ii. Since maximum coverage problem has a polynomial time \((1 - \frac{1}{k})\)-approximation algorithm, we assume that \( c \) is a fixed positive real number. When \( k \) is fixed, a brute force polynomial time solution is possible to find the optimal solution for the maximum coverage problem. Therefore, we assume that \( \frac{1}{k} = o(1) \). Let \( A_1, A_2, \ldots, A_n \) be an input for a maximum coverage problem with a integer parameter \( k \). Without loss of generality, assume \( |A_1| \) is the largest as Case i. Let \( A_1^* = A_1 \) and \( A_j^* = (A_1 - A_j)||A_0||A_j||A_j \cup A_j \) for \( j = 2, 3, \ldots, n \). Consider the maximum coverage problem \( A_1^*, A_2^*, A_3^*, \ldots, A_n^* \). Assume that the maximum coverage problem \( A_1^*, A_2^*, A_3^*, \ldots, A_n^* \) has a \( c \)-approximation \( A_1'^*, A_2'^*, \ldots, A_n'^* \).

1) \( i \in \{i_1, i_2, \ldots, i_k\} \) \( (A_i^* \) is one of the sets in the solution). We have that \( A_{i_1}, \ldots, A_{i_k} \) is a \( c \)-approximation for the maximum coverage problem for the input \( A_1, A_2, \ldots, A_n \).

2) \( i \notin \{i_1, i_2, \ldots, i_k\} \). Let \( A_j^* \) be that set in the solution such that \( |A_j^* - \cup_{j \neq i} A_i^*| \) is the least. Clearly, \( |A_j^* - \cup_{j \neq i} A_i^*| \leq \frac{|A_j^* - \cup_{j \neq i} A_i^*|}{k} \). Thus, \( |A_1 \cup (\cup_{j \neq i} A_i^*)| = |A_1^* \cup (\cup_{j \neq i} A_i^*)| \geq (1 - \frac{1}{k})|A_1^* \cup (\cup_{j \neq i} A_i^*)| \cup A_i^* \). Therefore, we have a \((c - o(1))\)-approximation \( |A_1 \cup (\cup_{j \neq i} A_i^*)| \) for the maximum coverage problem with the input \( A_1, A_2, \ldots, A_n \).

Our partial sublinear time algorithm can be also applied to the equal size maximum coverage problem, which has size \( nm \) controlled by two parameters \( n \) and \( m \). Our algorithm has a time complexity independent of \( m \) in the first model that gives \( O(1) \) time random element, and \( O(1) \) answer for any membership query. Our partial sublinear time approximation algorithm for the maximum coverage problem becomes sublinear time algorithm when \( m \geq n^c \) for a fixed \( c > 0 \).

### 8. Maximum Coverage on Concrete Models

In this section, we show some data structures that can support the efficient implementation of the algorithm. We define the time complexity of a randomized algorithm in our computation model.

**Definition 34.** Assume that the complexity for getting one random sample from set \( A_i \) is \( r(|A_i|) \) and the complexity for making one membership query for set \( A_i \) is \( q(|A_i|) \). If an algorithm has a complexity \( T(\epsilon, \gamma, k, n), R(\epsilon, \gamma, k, n), Q(\epsilon, \gamma, k, n) \), define its time complexity by \( T(\epsilon, \gamma, k, n) + R(\epsilon, \gamma, k, n) r(m) + Q(\epsilon, \gamma, k, n) q(m) \).
Theorem 20 can be restated in the following format.

**Theorem 35.** Assume that each set of size $m$ can generate a random element in $r(m)$ time and answer a membership query in $q(m)$ time. Then there is a randomized algorithm such that with probability at most $\gamma$, $\text{ApproximateMaximumCover}(A_1, A_2, \ldots, A_n, \epsilon, \gamma)$ does not produce an output of a $(1 - \frac{1}{\epsilon})$-approximation. Furthermore, its complexity is $T(\epsilon, \gamma, k, n) + R(\epsilon, \gamma, k, n)r(m) + Q(\epsilon, \gamma, k, n)q(m)$, where $T(\cdot), R(\cdot),$ and $Q(\cdot)$ are the same as those in Theorem 9.

**Proof:** It follows from Definition 34, Lemma 22 and Theorem 9.

8.1. Maximum Coverage on High Dimension Space

In this section, we apply the randomized algorithm for high dimensional maximum coverage problem. It gives an application to a #P-hard problem.

**Definition 36.** An axis aligned rectangular shape $R$ is called integer rectangle if all of its corners are lattice points. A special integer rectangle is called 0-1-rectangle if each corner $(x_1, x_2, \ldots, x_n)$ has $x_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$.

**Geometric Integer Rectangular Maximum Coverage Problem:** Given a list of integer rectangles $R_1, R_2, \ldots, R_n$ and integer parameter $k$, find $k$ of them $R_{i_1}, \ldots, R_{i_k}$ that has the largest number of lattice points. The 0-1 Rectangle Maximum Coverage problem is the Geometric Integer Rectangular Maximum Coverage Problem with each rectangle to be 0-1 rectangle.

This problem is #P-hard even at the special case $k = n$ for counting the total number of lattice points in the union of the $n$ rectangles. A logical formula is considered to be in DNF if and only if it is a disjunction of one or more conjunctions of one or more literals. Counting the number of assignments to make a DNF true is #P-hard [24].

**Proposition 37.** The 0-1 Rectangle Maximum Coverage problem is #P-hard.

**Proof:** The reduction is from #DNF, which is #P-hard [24], to the 0-1 Rectangle Maximum Coverage problem. For each conjunction of literals $x_1^* x_2^* \cdots x_k^*$ (each $x_i^* \in \{x_i, \overline{x_i}\}$ is a literal), all of the satisfiable assignments to this term form the corners of a 0-1 rectangle.

**Theorem 38.** There is a polynomial time $(1 - \frac{1}{d})$-approximation algorithm for the $d$-dimensional Geometric Integer Rectangular Maximum Coverage Problem, and has time complexity $O(T(\epsilon, \gamma, k, n) + R(\epsilon, \gamma, k, n)d + Q(\epsilon, \gamma, k, n)d)$, where $T(\cdot), R(\cdot),$ and $Q(\cdot)$ are the same as those in Theorem 9.

**Proof:** For each integer rectangular shape $R_i$, we can find the number of lattice points in $R_i$. It takes $O(d)$ time to generate a random lattice at a $d$-dimensional rectangle, and answer a membership query to an input rectangle. There is uniform random generator for the set of lattice points in $R_i$. The input list of sets is a $((0, 0), (0, 0))$-list as there is a perfect uniform random sampling, and has the exact number of lattice points for each set. It follows from Theorem 9.

8.2. Maximum Coverage with Sorted List

In this section, we discuss that for each input set for the maximum coverage problem is in a sorted array. We have the Theorem 39.

**Theorem 39.** Assume each input set is in a sorted list. Then with probability at least $\frac{1}{4}$, $\text{ApproximateMaximumCover}(A_1, A_2, \ldots, A_n, \epsilon, \gamma)$ outputs a $(1 - \frac{1}{\epsilon})$-approximation in time $O(T(\epsilon, \gamma, k, n) + R(\epsilon, \gamma, k, n) + Q(\epsilon, \gamma, k, n) \log m)$, where $T(\cdot), R(\cdot),$ and $Q(\cdot)$ are the same as those in Theorem 9, and $n = \max\{|A_1|, |A_2|, \ldots, |A_n|\}$.

**Proof:** The input list of sets is a $((0, 0), (0, 0))$-list as a sorted list provides perfect uniform random sampling, and has the exact number of items for each set. It takes $O(1)$ steps to get a random sample from each input set $A_i$, and $O(\log m)$ steps to check membership. It follows from Lemma 22 and Theorem 9.
8.3. Maximum Coverage with Input as Unsorted Arrays

In this section, we show our approximation when each input set is an unsorted array of elements. A straightforward method is to sort each set or generate a B-tree for each set. This would take $O(m \log m)$ time, where $m$ is the size of the largest set.

The following implementations will be used to support the case when each input set is unsorted. Whenever a set is selected to the solution, it will be sorted so that it will be efficient to test if other random samples from the other sets belong to it.

**Algorithm 6 : Virtual Function Implementation 3**

The parameters $L', A_i, s_i, R_i, \epsilon', \gamma, k, n$ follow from those in Algorithm 3.

```
RandomSamples(A_i, R_i, \xi, \gamma, k, n)
{ The same as that in Virtual Function Implementation 2; }
RandomTest(.) { the same as that in Virtual Function Implementation 2; }
ApproximateSetDifferenceSize(L', A_i, R_i, s_i, R_i, \epsilon', \gamma, k, n)
{ The same as that in Virtual Function Implementation 2; }
ProcessSet(A_i)
{ Sort A_i; }
```

End of Algorithm

**Lemma 40.** Assume that $\epsilon, \gamma \in (0, 1)$ and $k, n \in \mathbb{N}$. The algorithm can be implemented with sort merge for the function Merge(.) in complexity $(T_3(\xi, \gamma, k, n), Q_3(\xi, \gamma, k, n), R_3(\xi, \gamma, k, n))$ such that for each set $A_i$, it maintains $t_{i,j} = |R_i - U(L'(j))|$, where

$$
T_3(\xi, \gamma, k, n) = O(kg(\epsilon', \gamma, k, n)n + km(\log k + \log m)),
$$

(65)

$$
Q_3(\xi, \gamma, k, n) = kg(\epsilon', \gamma, k, n)n,
$$

(66)

$$
R_3(\xi, \gamma, k, n) = g(\epsilon', \gamma, k, n)n,
$$

(67)

$\epsilon'$ is defined in algorithm ApproximateMaximumCover(.), and $R_i$ is the set of random samples from $A_i$ and defined in Lemma 22.

**Proof:** Let $A_1, \ldots, A_n$ be the input list of sets, and $k$ be the integer parameter in the input.

Consider $g(\epsilon', \gamma, k, n)$ random samples for each set as in Lemma 22, where $\epsilon'$ is defined in algorithm ApproximateMaximumCover(.). After a set $A_i$ is selected, it takes $O(m \log m)$ to sort the elements in newly selected set, and adjust the random samples according to ApproximateSetDifferenceSize(.) in Virtual Function Implementation 3.

It let $R_j$ become $R_j - A_i$ for all $j \neq i$. Thus, it takes $kg(\epsilon', \gamma, k, n)n$ time. Mark those samples that are in the selected sets, and count the samples left (unmarked). It is similar to update $R_j$ and $t_{i,j}$ as in the proof of Lemma 23. The approximation $t_{i,j}$ to $|A_i - U(L'(j))|$ can be computed as $\frac{1}{w} \cdot s_i$ ($w = g(\epsilon, \gamma, k, n)$) as in line 3 in ApproximateDifference(.)

**Theorem 41.** Assume each input set is an unsorted list. Then with probability at least $\frac{3}{4}$, ApproximateMaximumCover($A_1, A_2, \ldots, A_n, \epsilon, \gamma$) outputs a $(1 - \frac{1}{\gamma})$-approximation. Its time complexity is $O(\frac{\epsilon^3}{\gamma^2}(k \log (\frac{2n}{k}) + \log \frac{1}{\gamma})n + km(\log k + \log m)))$, where $m = \max\{|A_1|, |A_2|, \ldots, |A_n|\}$. 

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Proof: At line 16 in ApproximateMaximumCover(), we build up a B-tree to save all the elements in the sets that have been collected to $L'$. The total amount time to build up $L'$ in the entire algorithm is $O(km(\log k + \log m))$.

Use $g(\epsilon', \gamma, k, n)$ random samples for each set $S_i$. It follows from Lemma 22, Lemma 17, and Lemma 40. Select $\xi = \min\left(\frac{m\eta}{\epsilon'}, \epsilon\right)$, where $\eta$ is defined in Lemma 18 and $\beta$ is the same as that in Lemma 17. With the condition $\epsilon \in (0, 1)$, the accuracy of approximation follows from Lemma 18.

### 8.4. Maximum Coverage with B-Tree

In this section, we discuss an online model. The sorted array approach is not suitable for the online model as insertion or deletion may take $\Omega(m)$ steps in the worst case. Therefore, we discuss the following model.

The B-tree implementation is suitable for dynamic sets that can support insertion and deletion for their members. It is widely used in database as a fundamental data structure. It takes $O(\log m)$ time for query, insertion, or deletion. B-tree can be found in a standard textbook of algorithm (for example, [7]).

We slightly revise the B-tree structure for its application to the maximum coverage problem. Each set for the maximum coverage problem is represented by a B-tree that saves all data in the leaves. Let each internal node $t$ contain a number $C(t)$ for the number of leaves under it. Each element has a single leaf in the B-tree (we do not allow multiple copies to be inserted in a B-tree for one element). We also let each node contain the largest values from the subtrees with roots at its children. It takes $O(\log m)$ time to generate a random element, and $O(\log m)$ time to check if an element belongs to a set.

**Definition 42.** For two real numbers $a$ and $b$, define $N[a, b]$ to be the set of integers $x$ in $[a, b]$.

**Algorithm 7 :** Rand($T, t$)

Input a B-tree $T$ with a node $t$ in $T$.

Steps:

1. If $t$ is a leaf, return $t$;
2. Let $t_1, \cdots, t_k$ be the children of $t$;
3. Select a random integer $i \in N[1, C(t_1) + \cdots + C(t_k)]$;
4. Partition $N[1, C(t_1) + \cdots + C(t_k)]$ into $I_1 = N[1, C(t_1)], I_2 = N[C(t_1) + 1, C(t_1) + C(t_2)], \cdots, I_k = N[C(t_1) + \cdots + C(t_k - 1) + 1, C(t_1) + \cdots + C(t_k)]$;
5. Find $I_j$ with $i \in I_j$ and return Rand($T, t_j$);

End of Algorithm

**Lemma 43.** There is a B-tree implementation such that it can generate a random element in $O(\log m)$ time, where $m$ is the number of elements saved in the B-tree.

Proof: All data are in the leaves. Let each internal node $t$ contain a number $C(t)$ for the number of leaves below it. Start from the root. For each internal node $t$ with children $t_1, \cdots, t_k$. With probability $\frac{C(t_k)}{C(t_1) + \cdots + C(t_k)}$, go the next node $N_i$. Clearly, a trivial induction can show that each leaf has an equal chance to be returned.
Each set is represented by a B-tree that saves all data in the leaves. Let each internal node contain a number for the number of leaves below it. It takes $O(\log m)$ time to generate a random element, and $O(\log m)$ time to check if an element belongs to a set. We have $r(m) = O(\log m)$ and $q(m) = O(\log m)$.

**Theorem 44.** Assume each input set is in a B-tree. Then we have

- it takes $O(\log m)$ time for insertion and deletion, and
- with probability at least $\frac{3}{4}$, $\text{ApproximateMaximumCover}(A_1, A_2, \ldots, A_n, \epsilon, \gamma)$ outputs a $(1 - \frac{1}{e})$-approximation in time $O(T(\epsilon, \gamma, n) + R(\epsilon, \gamma, n) \log m + Q(\epsilon, \gamma, n) \log m)$, where $T(\epsilon, R(\epsilon, \gamma, n)$, and $Q(\epsilon, \gamma, n)$ are the same as those in Theorem 9, and $m = \max \{|A_1|, |A_2|, \ldots, |A_n|\}$.

**Proof:** The input list of sets is a $((0,0), (0,0))$-list as B-tree provide perfect uniform random sampling, and has the exact number of items for each set. It follows from Lemma 22 and Theorem 9.

### 8.5. Maximum Coverage with Hashing Table

Each set $S_j$ is saved in an unsorted array $A_i[\cdot]$. A hashing table $H_i[\cdot]$ is used to indicate if an element belongs to a set. We can let each cell $j$ of hashing table to contain a linked list that holds all the elements $x$ in $S_i$ with $H_i(x) = j$.

**Definition 45.** Let $\alpha(m)$ and $\beta(m)$ be two functions from $N$ to $N$. A set $S$ of $m$ elements is saved in a $(\alpha(\cdot), \beta(\cdot))$-hashing table $H[1..M]$ if the following conditions are satisfied:

- There is an integer $M \leq \alpha(m)m$.
- There is a hashing function $h(\cdot)$ with range $\{1, 2, \ldots, M\}$ such that there are most $\beta(m)$ elements in $S$ to be mapped to the same value by the function $h(\cdot)$. In other words, $|x : x \in S$ and $h(x) = j| \leq \beta(m)$ for every $j \in \{1, 2, \ldots, M\}$.
- The table $H[1..M]$ is of size $M \leq \alpha(m)m$ such that entry $H[j]$ points to a B-tree that stores all the elements $x \in S$ with $h(x) = j$.

Assume each input set is saved in a $(\alpha(\cdot), \beta(\cdot))$-hashing table. Each set $S_i$ is saved in an unsorted array $A_i[\cdot]$. A hashing table $H_i[\cdot]$ is used to indicate if an element belong to a set. It takes $O(1)$ time to generate a random element of $S_i$ by accessing $A_i[\cdot]$. It takes $O(\log \beta(m))$ time to check the membership problem by accessing the hashing table $H_i(\cdot)$. This method makes it easy to add and delete elements from the set. It takes $O(\log \beta(m))$ time for insertion and deletion when the $A_i[\cdot]$ and $H[\cdot]$ are not full. It needs to increase the hashing table size when it is full, and take $O(m(\alpha(m) + \log \beta(m)))$ time to build a new table. Thus, it takes $O(\alpha(m) + \log \beta(m))$ amortized time for insertion and deletion.

The array size $A_i[\cdot]$ and hashing table size $H_i[\cdot]$ are larger than the size of the set $S_i$. When one of them is full, it will be doubled by applying for a double size memory. If its usage is less than half, it can be shrunk by half. We show the existence of a $(O(1), O(\log n))$-Hashing Table for a set of size $m$ under some assumption.

**Definition 46.** For a hashing table $H[1..M]$ of size $M$ and a hashing function $h(\cdot)$ with range $\{1, 2, \ldots, M\}$, function $h(\cdot)$ is $d$-uniform if $\text{Prob}(h(x) = j) \leq \frac{d}{M}$ for every $j \in \{1, 2, \ldots, M\}$, where $d$ is a real number in $[1, +\infty)$.

**Theorem 47.** Let $\alpha(m)$ and $\beta(m)$ be two functions from $N$ to $N$. Assume each input set is saved in a $(\alpha(\cdot), \beta(\cdot))$-hashing table. Then there is a $O(T(\epsilon, \gamma, k, n) + R(\epsilon, \gamma, k, n) + Q(\epsilon, \gamma, k, n) \log \beta(m))$ time randomized algorithm such that with probability at most $\frac{1}{4}$, $\text{ApproximateMaximumCover}(A_1, A_2, \ldots, A_n, \epsilon, \gamma)$ does not output a $(1 + \epsilon)$-approximate size of $A_1 \cup A_2 \cup \cdots \cup A_n$, where $T(\cdot), R(\cdot)$, and $Q(\cdot)$ are the same as those in Theorem 9.
Proof: It takes O(1) time to generate a random element, and O(β(m)) time to make a membership query when a set is saved in a (α(·), β(·))-hashing table. It follows from Theorem 20.

We tend to believe that a set can be saved in a (O(1), O(1))-hashing table. Assume that each set can be saved in a (O(1), O(1))-hashing table. It takes O(1) time to generate a random element of \( S \) by accessing \( A_i[\cdot] \). It takes O(1) time to check the membership problem by accessing the hashing table \( H_i(\cdot) \). We have \( r(m) = O(1) \) and \( q(m) = O(1) \).

**Proposition 48.** Let \( d \) be a fixed real in \([1, +∞)\). Let \( a_1 ≤ a_2 \) be fixed real numbers in \((0, +∞)\), and \( c \) be fixed real numbers in \((0, 1)\). Let \( h(\cdot) \) be a \( d \)-uniform hashing function for some \( M \) in the range \([a_1m, a_2m]\) for \( a_1 \) and \( a_2 \). Then every set of size \( m \) has a \((O(1), O(\frac{b(m)\log m}{\log \log m}))\)-Hashing Table via \( h(\cdot) \) with probability at least \( 1 - o(1) \) for any nondecreasing unbounded function \( b(m) : N → N \).

Proof: Consider an arbitrary integer \( j \) with \( 1 ≤ j ≤ M \) in the hashing table. Let \( S \) have the elements \( x_1, x_2, \cdots, x_m \). Let \( M \) be an arbitrary \( a_1m ≤ M ≤ a_2m \) for two fixed \( a_1, a_2 \in (0, +∞) \). The probability that \( h(x) = j \) is at most \( \frac{d}{M} \leq \frac{a_1m}{a_1m} = p \). Let \( y(m) = \frac{b(m)\log m}{\log \log m} \). By Theorem 6, with probability at most \( P_1 = \left(\frac{e^{y(m)}{(1+cg(m))^{1+cg(m)}}}{(1+cg(m))^{1+cg(m)}} \right)^{pm} \leq \left(\frac{e^{y(m)}{(1+cg(m))^{1+cg(m)}}}{(1+cg(m))^{1+cg(m)}} \right)^{\frac{d}{M}} = o(\frac{1}{m}) \), we have \( \{|a_i : h(a_i) = j| \geq (1 + y(m))^{pm} = (1 + y(m))^{\frac{d}{M}} = O(y(m))\} \).

With probability at most \( M \cdot P_1 = o(1), \) one of the \( M \) positions in the hashing table has more than \( cy(m) \) elements of \( S \) to be mapped by \( h(\cdot) \) for some fixed \( c > 0 \). Thus, set \( S \) has a \((O(1), O(y(m)))\)-Hashing Table via \( h[\cdot] \) at probability at least \( 1 - o(1) \).

**Proposition 49.** Let \( d \) be a fixed real in \([1, +∞)\). Let \( 1 ≤ b \) be fixed real numbers in \((0, +∞)\), and \( c \) be fixed real numbers in \((0, 1)\). Let \( h(\cdot) \) be a \( d \)-uniform hashing function for some \( M \) in the range \([m^{1+}, am^{1+}+] \). Then every set of size \( m \) has a \((m^c, O(1))\)-Hashing Table via \( h(\cdot) \) with probability at least \( 1 - o(1) \).

Proof: Consider a position \( j \) with \( 1_j ≤ M \) in the hashing table. Let \( S \) have the elements \( x_1, x_2, \cdots, x_m \). Select a constant \( c = ec_1 \) with \( c_1 = \frac{1}{mg(m)} \). Let \( g(m) = m^c \). Let \( M ≥ mg(m) \). Let \( p = \frac{d}{M} \). By Theorem 6, with probability at most \( P_1 = \left(\frac{e^{g(m)}{(1+c(m))^{1+c(m)}}}{(1+c(m))^{1+c(m)}} \right)^{pm} \), \( \{|a_i : h(a_i) = j| \geq (1 + c(m))^{pm} \} \).

On the other hand, \( (1 + c(m))^{pm} ≤ 2cg(m)pm = 2cg(m) \cdot \frac{d}{M} \cdot m ≤ 2cg(m) \cdot \frac{d}{mg(m)} \cdot m = 2cd \).

Let \( k = 2cd = 2ec_1d = 2e \cdot \frac{100}{e} \cdot d = 200d \).

We have
\[
P_1 = \left(\frac{e^{g(m)}{(1+c(m))^{1+c(m)}}}{(1+c(m))^{1+c(m)}} \right)^{pm} \leq \left(\frac{e^{g(m)}{(1+c(m))^{1+c(m)}}}{(1+c(m))^{1+c(m)}} \right)^{\frac{d}{M}} \leq \left(\frac{e^{g(m)}{(1+c(m))^{1+c(m)}}}{(1+c(m))^{1+c(m)}} \right)^{\frac{d}{M}} \leq \left(\frac{e^{g(m)}{(1+c(m))^{1+c(m)}}}{(1+c(m))^{1+c(m)}} \right)^{\frac{d}{M}} \leq \left(\frac{1}{c_1g(m)} \right)^{cd} \leq \left(\frac{1}{c_1m^c} \right)^{cd} \leq \left(\frac{1}{c_1m^c} \right)^{cd}.
\]

With probability at most \( MP_1 ≤ 2g(m)m(\frac{1}{c_1m^c}e) = o(1) \), one of the \( M \) positions in the hashing table has more than \( k \) elements of \( S \) to be mapped by \( h(\cdot) \).

9. Conclusions

We developed a randomized greedy approach for the maximum coverage problem. It obtains the same approximation ratio \((1 - \frac{1}{k})\) as the classical approximation for the maximum coverage problem, while its computational time is independent of the cardinalities of input sets under the model that
each set answers query and generates one random sample in $O(1)$ time. It can be applied to find approximate maximum volume by selecting $k$ objects among a list of objects such as rectangles in high dimensional space. It can provide an efficient online implementation if each set is saved in a B-tree. Our approximation ratio depends on the how much the random sampling is biased, and the initial approximation accuracy for the input set sizes. The two accuracies are determined by the parameters $\alpha_L, \alpha_R$, $\delta_L$ and $\delta_R$ in a $((\alpha_L, \alpha_R), (\delta_L, \delta_R))$-list. It seems that our method can be generalized to deal with more general version of the maximum coverage problems, and it is expected to obtain more results in this direction. The notion of partial sublinear time algorithm will be used to characterize more computational problems than the sublinear time algorithm.

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