The Peters polynomials are a generalization of Boole polynomials. In this paper, we consider Peters and poly-Cauchy mixed type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities of those polynomials associated with special polynomials.

1. Introduction

The Peters polynomials are defined by the generating function to be

$\sum_{n=0}^{\infty} S_n(x; \lambda, \mu) \frac{t^n}{n!} = \left(1 + (1 + t)^\lambda\right)^{-\mu} (1 + t)^x$, \hspace{1cm} (see [14]).

The first few of them are given by

$S_0(x; \lambda, \mu) = 2^{-\mu}$, $S_1(x; \lambda, \mu) = 2^{-\mu+1} (2x - \lambda \mu)$, \hspace{1cm} (4).

If $\mu = 1$, then $S_n(x; \lambda) = S_n(x; \lambda, 1)$ are called Boole polynomials.

In particular, for $\mu = 1, \lambda = 1$, $S_n(x; 1, 1) = Ch_n(x)$ are Changhee polynomials which are defined by

$\sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} = \frac{1}{t+2} (1 + t)^x$, \hspace{1cm} (see [8]).

The generating functions for the poly-Cauchy polynomials of the first kind $C_n^{(k)}(x)$ and of the second kind $\hat{C}_n^{(k)}(x)$ are, respectively, given by

$Lif_k(\log (1+t))(1+t)^{-x} = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!}$, \hspace{1cm} (2)

and

$Lif_k(-\log (1+t))(1+t)^x = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x) \frac{t^n}{n!}$, \hspace{1cm} (3)

where $Lif_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$, $(k \in \mathbb{Z})$, (see [10, 11]).

In this paper, we consider the poly-Cauchy of the first kind and Peters (respectively the poly-Cauchy of the second kind and Peters) mixed-type polynomials as follows:

$\left(1 + (1 + t)^\lambda\right)^{-\mu} Lif_k(\log (1+t))(1+t)^{-x} = \sum_{n=0}^{\infty} CP_n^{(k)}(x; \lambda, \mu) \frac{t^n}{n!}$, \hspace{1cm} (4)

and

$\left(1 + (1 + t)^\lambda\right)^{-\mu} Lif_k(-\log (1+t))(1+t)^x = \sum_{n=0}^{\infty} \hat{C}P_n^{(k)}(x; \lambda, \mu) \frac{t^n}{n!}$, \hspace{1cm} (5)
For \( \alpha \in \mathbb{Z}_{\geq 0} \), the Bernoulli polynomials of order \( \alpha \) are defined by the generating function to be
\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)} (x) \frac{t^n}{n!}, \quad \text{see [1-8]}.\]

As is well known, the Frobenius-Euler polynomials of order \( \alpha \) are given by
\[
\left( \frac{1 - \lambda}{e^t - \lambda} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)} (x|\lambda) \frac{t^n}{n!}, \quad \text{see [1-13]},
\]
where \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \) and \( \alpha \in \mathbb{Z}_{\geq 0} \).

When \( x = 0 \), \( CP_n^{(k)} (0; \lambda, \mu) \) (or \( \hat{CP}_n^{(k)} (0; \lambda, \mu) \)) are called the poly-Cauchy of the first kind and Peters (or the poly-Cauchy of the second kind and Peters) mixed-type numbers.

The higher-order Cauchy polynomials of the first kind are defined by the generating function to be
\[
\left( \frac{t}{\log(1+t)} \right)^\alpha (1+t)^{-x} = \sum_{n=0}^{\infty} C_n^{(\alpha)} (x) \frac{t^n}{n!}, \quad (\alpha \in \mathbb{Z}_{\geq 0}),
\]
and the higher-order Cauchy polynomials of the second kind are given by
\[
\left( \frac{t}{(1+t) \log(1+t)} \right)^\alpha (1+t)^x = \sum_{n=0}^{\infty} \hat{C}_n^{(\alpha)} (x) \frac{t^n}{n!}, \quad (\alpha \in \mathbb{Z}_{\geq 0}).
\]

The Stirling number of the first kind is given by
\[
(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l.
\]

Thus, by (10), we get
\[
(\log(1+t))^m = m! \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!}, \quad m \in \mathbb{Z}_{\geq 0}, \quad \text{see [14]}.\]

It is easy to show that
\[
x^{(n)} = x(x+1) \cdots (x+n-1) = (-1)^n (-x)_n = \sum_{l=0}^{n} S_1(n,l) (-1)^{n-l} x^l.
\]

Let \( \mathbb{C} \) be the complex number field and let \( \mathcal{F} \) be the algebra of all formal power series in the variable \( t \) over \( \mathbb{C} \) as follows:
\[
\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}.
\]

Let \( \mathbb{P} = \mathbb{C}[x] \) and let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L, p(x) \rangle \) denotes the action of linear functional \( L \) on the polynomial \( p(x) \), and we recall that the vector space operations on \( \mathbb{P}^* \) are defined by \( \langle L + M, p(x) \rangle = \langle L, p(x) \rangle + \langle M, p(x) \rangle \), \( \langle cL, p(x) \rangle = c \langle L, p(x) \rangle \), where \( c \) is complex constant in \( \mathbb{C} \).

For \( f(t) \in \mathcal{F} \), let us define the linear functional on \( \mathbb{P} \) by setting
\[
\langle f(t), x^n \rangle = a_n, \quad (n \geq 0).
\]

Then, by (13) and (14), we get
\[
\langle t^k, x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad \text{see [14, 15]},
\]
where \( \delta_{n,k} \) is the Kronecker’s symbol.
Let \( f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^n \rangle}{k!} t^k \). Then, by (13), we see that \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \).

The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of umbral algebra. The order \( O(f) \) of the power series \( f(t) \) (\( \neq 0 \)) is the smallest integer for which the coefficient of \( t^k \) does not vanish. If \( O(f(t)) = 1 \), then \( f(t) \) is called a delta series; if \( O(f(t)) = 0 \), then \( f(t) \) is called an invertible series. For \( f(t), g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \), there exists a unique sequence \( s_n(x) \) of polynomials such that

\[
\langle g(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, \quad (n, k \geq 0).
\]

The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \).

For \( f(t), g(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), we have

\[
\langle f(t) g(t) | p(x) \rangle = \langle f(t) | g(t) p(x) \rangle = \langle f(t) | f(t) p(x) \rangle,
\]

and

\[
f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^n \rangle t^k k!, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!},
\]

By (10), we get

\[
p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle, \quad (k \geq 0).
\]

where \( p^{(k)}(0) = \frac{d^k p(x)}{dx^k} \bigg|_{x=0} \).

Thus, by (17), we have

\[
t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (\text{see } [8, 10, 14]).
\]

Let \( s_n(x) \sim (g(t), f(t)) \). Then the following equations are known in [14]:

\[
\frac{1}{g(\overline{f(t)})} x^\overline{f(t)} = \sum_{n=0}^\infty s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbb{C},
\]

where \( \overline{f(t)} \) is the compositional inverse for \( f(t) \) with \( f(\overline{f(t)}) = t \),

\[
s_n(x) = \sum_{j=0}^n \frac{1}{j!} \left( \frac{\overline{f(t)}^j}{g(\overline{f(t))}} \right) x^n x^j,
\]

\[
s_n(x+y) = \sum_{j=0}^n \binom{n}{j} s_j(x) P_{n-j}(y), \quad \text{where } P_n(x) = g(t) s_n(x),
\]

and

\[
s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x), \quad f(t) s_n(x) = ns_{n-1}(x), \quad (n \geq 0),
\]

and

\[
\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \langle \overline{f(t)} | x^{n-l} \rangle s_l(x).
\]
As is well known, the transfer formula for \( p_n(x) \sim (1, f(t)), q_n(x) \sim (1, g(t)), (n \geq 1) \), is given by

\[
q_n(x) = x^\left( f(t) \right) x^{-1} p_n(x).
\]

For \( s_n(x) \sim (g(t), f(t)), r_n(x) \sim (h(t), l(t)), \) let

\[
s_n(x) = \sum_{m=0}^{\infty} C_{n,m} r_n(x),
\]

where

\[
C_{n,m} = \frac{1}{m!} \left( \frac{h(f(t))}{g(f(t))} \right)^m | x^n |, \quad (\text{see } [14]).
\]

It is known that

\[
\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \quad ex^\mu p(x) = p(x + y)
\]

where \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathcal{P} \) (see \([8,10,14]\)).

In this paper, we consider Peters and poly-Cauchy mixed type polynomials with umbral calculus viewpoint and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give some interesting identities of those polynomials associated with special polynomials.

2. poly-Cauchy and Peters mixed-type polynomials

From \((2), (3), \) and \((19), \) we note that

\[
CP_n^{(k)}(x; \lambda, \mu) \sim \left( 1 + e^{-\lambda t} \right)^\mu \frac{1}{\text{Li}_{f_k}(-t)} e^{-t} - 1
\]

and

\[
\hat{CP}_n^{(k)}(x; \lambda, \mu) \sim \left( 1 + e^{\lambda t} \right)^\mu \frac{1}{\text{Li}_{f_k}(-t)} e^t - 1.
\]

It is not difficult to show that

\[
\left( 1 + e^{-\lambda t} \right)^\mu
\]

\[
= 2^\mu \left( 1 + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-\lambda t)^j}{j!} \right)^\mu
\]

\[
= \sum_{i=0}^{\infty} \sum_{j_1+\cdots+j_i=j} 2^{\mu-i} \binom{\mu}{i} (j_1+1, \cdots, j_i+1) \frac{(-\lambda t)^{j+i}}{(j+i)!}
\]

and

\[
\left( 1 + (1 + t)^\lambda \right)^{-\mu}
\]

\[
= 2^{-\mu} \left( 1 + \frac{1}{2} \sum_{j=0}^{\infty} \left( \lambda \frac{j+1}{j+1} \right)^{t+1} \right)^{-\mu}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j_1+\cdots+j_i=j} 2^{-(\mu+i)} \binom{\mu}{i} \frac{\lambda}{(j_1+1)} \cdots \frac{\lambda}{(j_i+1)} t^{j+i}.
\]
From (14), we have

\[
CP_n^{(k)}(y; \lambda, \mu) = \langle \left(1 + (1 + t)^k\right)^{-\mu} \text{Lift}_k(\log(1 + t))(1 + t)^{-y} \mid x^n \rangle
\]

\[
= \left\langle \left(1 + (1 + t)^k\right)^{-\mu} \sum_{l=0}^{n} \binom{n}{l} C_l^{(k)}(y) x^{n-l} \right\rangle
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} C_l^{(k)}(y) \left( \sum_{m=0}^{-\mu} S_m(0; \lambda, \mu) \frac{t^m}{m!} x^{n-l} \right)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} S_{n-l}(0; \lambda, \mu) C_l^{(k)}(y).
\]

Therefore, by (32), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), we have

\[
CP_n^{(k)}(x; \lambda, \mu) = \sum_{l=0}^{n} \binom{n}{l} S_{n-l}(0; \lambda, \mu) C_l^{(k)}(x).
\]

Alternatively,

\[
(33) \quad CP_n^{(k)}(y; \lambda, \mu) = \left\langle \sum_{l=0}^{\infty} CP_l^{(k)}(y; \lambda, \mu) \frac{t^l}{l!} x^n \right\rangle
\]

\[
= \left\langle \text{Lift}_k(\log(1 + t))(1 + (1 + t)^k)^{-\mu} (1 + t)^{-y} x^n \right\rangle
\]

\[
= \left\langle \text{Lift}_k(\log(1 + t)) \sum_{l=0}^{n} \binom{n}{l} S_l(-y; \lambda, \mu) x^{n-l} \right\rangle
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} S_l(-y; \lambda, \mu) \text{Lift}_k(\log(1 + t)) x^{n-l}
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} S_l(-y; \lambda, \mu) C_l^{(k)}(0).
\]

Therefore, by (33), we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), let \( C_{n-l}(0) = C_{n-l}^{(k)} \). Then we have

\[
CP_n^{(k)}(x; \lambda, \mu) = \sum_{l=0}^{n} \binom{n}{l} C_{n-l}(0; \lambda, \mu) \tilde{C}_l^{(k)}(x).
\]

**Remark.** By the same method, we get

\[
(34) \quad \tilde{CP}_n^{(k)}(x; \lambda, \mu) = \sum_{l=0}^{n} \binom{n}{l} S_{n-l}(0; \lambda, \mu) \tilde{C}_l^{(k)}(x),
\]

and

\[
(35) \quad \check{CP}_n^{(k)}(x; \lambda, \mu) = \sum_{l=0}^{n} \binom{n}{l} S_{n-l} \tilde{C}_l^{(k)}(x; \lambda, \mu).
\]
From (20) and (28), we have

\[ CP_n^{(k)}(x; \lambda, \mu) = \frac{1}{n!} \langle \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \text{Lif}_k (\log (1 + t)) (-\log (1 + t))^j \rangle \big| x^n \big| x^j \]

From (31), we note that

\[ \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \text{Lif}_k (\log (1 + t)) (-\log (1 + t))^j \big| x^n \big| x^j \]

\[ = \sum_{m=0}^{n-j} \frac{(-1)^j}{m!(m+1)^k} \sum_{l=0}^{n-j-m} \frac{(m+j)!}{(l+m+j)!} S_1 (l + m + j, m + j) \times (n)_{l+m+j} \left( \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \big| x^{n-l-m-j} \right) \]

\[ = \sum_{m=0}^{n-j} \frac{(-1)^j}{m!(m+1)^k} \sum_{l=0}^{n-j-m} \frac{(m+j)!}{(l+m+j)!} \sum_{r=0}^{\infty} \sum_{r_1+\cdots+r_j=n-j-m-l-1}^{\infty} 2^{-i} (-1)^j \frac{(m+j)!}{l!(m+1)^k (l+m+j)!} \times \left( \left( \frac{-\mu}{i} \right) \left( \frac{\lambda}{r_1+1} \right) \cdots \left( \frac{\lambda}{r_j+1} \right) S_1 (l + m + j, m + j) \right) \big| x^j \big| \]

Therefore, by (30) and (37), we obtain the following theorem.

**Theorem 3.** For \( n \geq 0 \), we have

\[ CP_n^{(k)}(x; \lambda, \mu) = \frac{(-1)^j}{n!} \sum_{m=0}^{n-j} \sum_{l=0}^{n-j-m} \sum_{r=0}^{\infty} \sum_{r_1+\cdots+r_j=n-j-m-l-1}^{\infty} \frac{2^{-i}}{m!(m+1)^k (l+m+j)!} \times \left( \left( \frac{-\mu}{i} \right) \left( \frac{\lambda}{r_1+1} \right) \cdots \left( \frac{\lambda}{r_j+1} \right) S_1 (l + m + j, m + j) \right) \big| x^j \big| \]

**Remark.** By the same method as Theorem 3, we get

\[ \tilde{CP}_n^{(k)}(x; \lambda, \mu) = \frac{(-1)^j}{n!} \sum_{m=0}^{n-j} \sum_{l=0}^{n-j-m} \sum_{r=0}^{\infty} \sum_{r_1+\cdots+r_j=n-j-m-l-1}^{\infty} \frac{2^{-i}}{m!(m+1)^k (l+m+j)!} \times \left( \left( \frac{-\mu}{i} \right) \left( \frac{\lambda}{r_1+1} \right) \cdots \left( \frac{\lambda}{r_j+1} \right) S_1 (l + m + j, m + j) \right) \big| x^j \big| \]
From (23), we note that
\begin{equation}
(1 + e^{-\lambda t})^\mu \frac{1}{\text{Li}_k(-t)} CP_n^{(k)}(x; \lambda, \mu) \sim (1, e^{-t} - 1)
\end{equation}
and
\begin{equation}
x^n \sim (1, t).
\end{equation}

By (21), (39) and (40), we get
\begin{equation}
(1 + e^{-\lambda t})^\mu \frac{1}{\text{Li}_k(-t)} CP_n^{(k)}(x; \lambda, \mu)
\end{equation}

\begin{align*}
&= t \left( \frac{t}{e^t - 1} \right)^n n^{n-1} \\
&= (-1)^n x \left( \frac{-t}{e^t - 1} \right)^n n^{n-1} \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l B_{l}^{(n)} \left( \frac{n-1}{l} \right) x^{n-l}.
\end{align*}

Thus, by (41), we see that
\begin{equation}
CP_n^{(k)}(x; \lambda, \mu)
\end{equation}

\begin{align*}
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \left( 1 + e^{-\lambda t} \right)^{-\mu} \text{Li}_k(-t) x^{n-l} \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \sum_{m=0}^{n-l} \frac{(-1)^m}{(m+1)^k} \left( 1 + e^{-\lambda t} \right)^{-\mu} x^{n-l-m} \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} \sum_{i=0}^{\infty} \sum_{j_1 + \ldots + j_n = j} 2^{-\mu-i} \binom{-\mu}{i} \\
&\quad \times \binom{j+i}{j_1+1, \ldots, j_i+1} \left( \frac{-\lambda t}{j+i} \right)^{j+i} x^{n-l-m} \\
&= (-1)^n \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \sum_{i=0}^{\infty} \sum_{j_1 + \ldots + j_i = n-l-m-i} (-1)^{n-r} \frac{2^{-\mu-i} \lambda^{n-l-m-r}}{(m+1)^k} \binom{n-1}{l} \\
&\quad \times \binom{n-l}{m} (-\mu) \binom{n-l-m-r}{j_1+1, \ldots, j_i+1} \binom{n-l-m}{r} B_l^{(n)} t^r.
\end{align*}

Therefore, by (42), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0 \), we have
\begin{equation}
CP_n^{(k)}(x; \lambda, \mu)
\end{equation}

\begin{align*}
&= \frac{\lambda^n}{2^\mu} \sum_{r=0}^{n-1} (-\lambda^{-1})^r \left( \sum_{l=0}^{n-r-l} \sum_{m=0}^{n-r-l-m} \sum_{i=0}^{\infty} \sum_{j_1 + \ldots + j_i = n-r-l-m-i} (-1)^{n-r} \frac{2^{-i} \lambda^{l-m}}{(m+1)^k} \binom{n-1}{l} \\
&\quad \times \binom{n-l}{m} (-\mu) \binom{n-r-l-m}{j_1+1, \ldots, j_i+1} \binom{n-l-m-r}{r} B_l^{(n)} t^r.
\end{align*}
Remark. By the same method as Theorem 4, we get

\[
\hat{C}_n^{(k)}(x; \lambda, \mu) = \frac{\lambda^n}{2^n} \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-l} \sum_{i=0}^{\ell} \sum_{j_1, \ldots, j_i = 0}^{n-l-m-i} \frac{(-1)^m 2^{-i} \lambda^{-m}}{(m+1)^2} \left( \begin{array}{c} n-1 \n-\ell \n-m \end{array} \right) \times \left( \begin{array}{c} n-l \n-m \end{array} \right) \left( \begin{array}{c} n-l-m-r \end{array} \right) B_{\ell}^{(m)} \right) x^r.
\]

From (12), we note that

\[
x^{(n)} = x(x+1) \ldots (x+n-1) \sim (1, 1-e^{-t}).
\]

Thus, by (44), we see that

\[
(-1)^n x^{(n)} = (-x)_n = \sum_{m=0}^{n} S_1(n, m) (-x)^m \sim (1, e^{-t} - 1),
\]

and

\[
(1 + e^{-\lambda})^\mu \frac{1}{\text{Li}_k(-t)} C_n^{(k)}(x; \lambda, \mu) \sim (1, e^{-t} - 1).
\]

From (45), and (46), we have

\[
(1 + e^{-\lambda})^\mu \frac{1}{\text{Li}_k(-t)} C_n^{(k)}(x; \lambda, \mu) = \sum_{l=0}^{n} S_1(n, l) (-x)^l
\]
Thus, by (47), we get

\begin{equation}
CP_n^{(k)}(x; \lambda, \mu) \\
= \sum_{l=0}^{n} S_1(n, l) (-1)^l \left( 1 + e^{-\lambda t} \right)^{-\mu} \text{Li}_{k-1}(-t) x^l
\end{equation}

By the same method as Theorem 5, we get

\begin{equation}
\text{It is easy to see that}
\end{equation}

\begin{equation}
(1 + e^{\lambda t})^\mu \frac{1}{\text{Li}_{k-1}(-t)} CP_n^{(k)}(x; \lambda, \mu) \sim (1, e^t - 1)
\end{equation}

and

\begin{equation}
(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n, l) x^l \sim (1, e^t - 1).
\end{equation}

By the same method as Theorem 5 we get

\begin{equation}
\hat{CP}_n^{(k)}(x; \lambda, \mu) \\
= 2^{-\mu} \sum_{r=0}^{n} \lambda^{-r} \left\{ \sum_{l=r}^{n} \sum_{m=0}^{l-r-m} \sum_{j_1 + \cdots + j_i = l-r-m-i}^{\infty} \frac{(-1)^m 2^{-i} \lambda^{l-m}}{(m+1)^k} \right\} \left( \begin{array}{c} l \\ m \end{array} \right) \left( \begin{array}{c} l - m - r \\ j_1 + \cdots + j_i + 1 \end{array} \right) x^r S_1(n, l) \bigg) x^r.
\end{equation}
Theorem 6. For

\[ CP_n^{(k)} (x; \lambda, \mu) = \sum_{j=0}^{n} \frac{1}{j!} \left( 1 + (1 + t)^k \right)^{-\mu} \left. \text{Lif}_k \left( \log (1 + t) \right) \left( -\log (1 + t) \right)^j \right| x^n x^j. \]

Now, we observe that

\[ CP_n^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{\infty} \frac{\left. CP_m^{(k)} (0; \lambda, \mu) \right| x^n}{m!} e^n \]

By (20) and (28), we have

\[ \left( 1 + (1 + t)^k \right)^{-\mu} \left. \text{Lif}_k \left( \log (1 + t) \right) \left( -\log (1 + t) \right)^j \right| x^n \]

Therefore, by (52) and (53), we obtain the following theorem.

Theorem 6. For \( n \geq 0 \), we have

\[ CP_n^{(k)} (x; \lambda, \mu) = \sum_{j=0}^{n} (-1)^j \left\{ \sum_{m=0}^{n} \binom{n}{m} S_1 (n - m, j) CP_m^{(k)} (0; \lambda, \mu) \right\} x^j. \]

Remark. By the same method as Theorem 6, we get

\[ \hat{CP}_n^{(k)} (x; \lambda, \mu) = \sum_{j=0}^{n} \left( \sum_{m=0}^{n} \binom{n}{m} S_1 (n - m, j) \hat{CP}_m^{(k)} (0; \lambda, \mu) \right) x^j. \]

From (21), we have

\[ CP_n^{(k)} (x + y; \lambda, \mu) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} CP_{n-j}^{(k)} (x; \lambda, \mu) y^j \]

and

\[ \hat{CP}_n^{(k)} (x + y; \lambda, \mu) = \sum_{j=0}^{n} \binom{n}{j} \hat{CP}_{n-j}^{(k)} (x; \lambda, \mu) (y)_j. \]

By (22) and (28), we get

\[ (e^{-t} - 1) CP_n^{(k)} (x; \lambda, \mu) = nCP_{n-1}^{(k)} (x; \lambda, \mu) \]

and

\[ (e^{-t} - 1) CP_n^{(k)} (x; \lambda, \mu) = CP_n^{(k)} (x - 1; \lambda, \mu) - CP_n^{(k)} (x; \lambda, \mu). \]

Therefore, by (57) and (58), we obtain the following theorem.

Theorem 7. For \( n \geq 0 \), we have

\[ CP_n^{(k)} (x - 1; \lambda, \mu) - CP_n^{(k)} (x; \lambda, \mu) = nCP_{n-1}^{(k)} (x; \lambda, \mu). \]

Remark. By the same method as Theorem 7, we get

\[ \hat{CP}_n^{(k)} (x + 1; \lambda, \mu) - \hat{CP}_n^{(k)} (x; \lambda, \mu) = n\hat{CP}_{n-1}^{(k)} (x; \lambda, \mu). \]
From (22), (28), and (29), we have

\begin{align}
CP_{n+1}^{(k)}(x; 1, \mu) &= -xCP_{n}^{(k)}(x + 1; 1, \mu) + \mu \sum_{m=0}^{n} \left( \frac{-1}{2} \right)^{m+1} (n)_m CP_{n-m}^{(k)}(x; 1, \mu) \\
&\quad + 2^{-\mu} \sum_{r=0}^{n} (-1)^r \left\{ \sum_{m-r}^{n} \sum_{l=0}^{m-r} \frac{2^{-i}}{(m-l+2)^{k}} \left( \frac{m}{l} \right) \right\} \times \left( \frac{-\mu}{i} \right) \left( \binom{l-r}{j_1+1, \ldots, j_r+1} \right) S_1(n, m) (x+1)^r,
\end{align}

and

\begin{align}
\hat{CP}_{n+1}^{(k)}(x; 1, \mu) &= xCP_{n}^{(k)}(x - 1; 1, \mu) + \mu \sum_{m=0}^{n} \left( \frac{-1}{2} \right)^{m+1} (n)_m \hat{CP}_{n-m}^{(k)}(x; 1, \mu) \\
&\quad - 2^{-\mu} \sum_{r=0}^{n} \left\{ \sum_{m-r}^{n} \sum_{l=0}^{m-r} \frac{(-1)^{m-l-2-i}}{(m-l+2)^k} \left( \frac{m}{l} \right) \left( \frac{-\mu}{i} \right) \right\} \times \left( \binom{l-r}{j_1+1, \ldots, j_r+1} \right) S_1(n, m) (x-1)^r.
\end{align}

By (14) and (27), we get

\begin{align}
CP_{n}^{(k)}(y; \lambda, \mu) &= \left\{ \sum_{l=0}^{\infty} CP_{l}^{(k)}(y; \lambda, \mu) \frac{t^l}{l!} \right\} x^n \\
&= \left\{ (1 + (1+t)^\lambda)^{-\mu} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \mid x \cdot x^{n-1} \right\} \\
&= \frac{\partial}{\partial \lambda} \left\{ (1 + (1+t)^\lambda)^{-\mu} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \right\} x^{n-1} \\
&= \left\{ (1 + (1+t)^\lambda)^{-\mu} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \mid x^{n-1} \right\} \\
&\quad + \left\{ (1 + (1+t)^\lambda)^{-\mu} \frac{\partial}{\partial \lambda} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \mid x^{n-1} \right\} \\
&\quad + \left\{ (1 + (1+t)^\lambda)^{-\mu} \text{Li}_{k} (\log (1+t)) \left( \frac{\partial}{\partial \lambda} (1+t)^{-y} \right) \mid x^{n-1} \right\} \\
&= -\mu \lambda \left\{ (1 + (1+t)^\lambda)^{-\mu} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \mid x^{n-1} \right\} \\
&\quad - y \left\{ (1 + (1+t)^\lambda)^{-\mu} (\log (1+t)) (1+t)^{-y} \mid x^{n-1} \right\} \\
&\quad + \left\{ (1 + (1+t)^\lambda)^{-\mu} \frac{\partial}{\partial \lambda} \text{Li}_{k} (\log (1+t)) (1+t)^{-y} \mid x^{n-1} \right\}
\end{align}

\begin{align}
&= -\mu \lambda CP_{n-1}^{(k)}(y - \lambda + 1; \lambda, \mu + 1) - yCP_{n-1}^{(k)}(y + 1; \lambda, \mu) \\
&\quad + \left\{ (1 + (1+t)^\lambda)^{-\mu} \frac{\text{Li}_{k-1} (\log (1+t)) - \text{Li}_{k} (\log (1+t))}{(1+t) \log (1+t)} (1+t)^{-y} \mid x^{n-1} \right\}.
\end{align}
Now, we observe that

\[
\left(1 + \left(1 + t^\lambda \right) - \mu \frac{\text{Lif}_k (\log (1 + t)) - \text{Lif}_k (\log (1 + t))}{(1 + t) \log (1 + t)} (1 + t)^{-y} \right)x^{n-1} \right)\right] = \binom{n}{l} \left(\hat{C}^{(1)}_{n-1-l} (0) \right) \times \left(1 + (1 + t)^\lambda \right) - \mu \frac{\text{Lif}_k (\log (1 + t)) - \text{Lif}_k (\log (1 + t))}{t} (1 + t)^{-y} \right)x^l \right) \right] = \binom{n}{l} \left(\hat{C}^{(1)}_{n-1-l} (0) \right) \times \left(1 + (1 + t)^\lambda \right) - \mu \frac{\text{Lif}_k (\log (1 + t)) - \text{Lif}_k (\log (1 + t))}{t} (1 + t)^{-y} \right)x^{l+1} \right) \right] = \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{C}^{(1)}_{n-1-l} (0) \left\{ \text{CP}^{(k-1)}_{l+1} (y; \lambda, \mu) - \text{CP}^{(k)}_{l+1} (y; \lambda, \mu) \right\}.

Therefore, by (62) and (63), we obtain the following theorem.

**Theorem 8.** For \( n \geq 0 \), we have

\[
\text{CP}^{(k)}_n (x; \lambda, \mu) = -\mu \lambda \text{CP}^{(k)}_{n-1} (x - \lambda + 1; \lambda, \mu + 1) - x \text{CP}^{(k)}_{n-1} (x + 1; \lambda, \mu) + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{C}^{(1)}_{n-1-l} \left\{ \text{CP}^{(k-1)}_{l+1} (x; \lambda, \mu) - \text{CP}^{(k)}_{l+1} (x; \lambda, \mu) \right\}.
\]

where \( \hat{C}^{(1)}_{n-1-l} = \hat{C}^{(1)}_{n-1-l} (0) \).

**Remark.** By the same method as Theorem 8, we get

\[
\hat{\text{CP}}^{(k)}_n (x; \lambda, \mu) = -\mu \lambda \hat{\text{CP}}^{(k)}_{n-1} (x + \lambda - 1; \lambda, \mu + 1) + x \hat{\text{CP}}^{(k)}_{n-1} (x - 1; \lambda, \mu) + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{C}^{(1)}_{n-1-l} \left\{ \hat{\text{CP}}^{(k-1)}_{l+1} (x; \lambda, \mu) - \hat{\text{CP}}^{(k)}_{l+1} (x; \lambda, \mu) \right\}.
\]
By (23), we get

\[
\frac{d}{dx} CP_n^{(k)} (x; \lambda, \mu) = \sum_{l=0}^{n-1} \binom{n}{l} \langle \sum_{m=1}^{\infty} \left( \frac{-1}{m} \right)^m t^m \left| x^{n-l} \right\rangle CP_l^{(k)} (x; \lambda, \mu) \]

\[
= \sum_{l=0}^{n-1} \binom{n}{l} \left( \frac{-1}{m} \right)^m \langle t^m \left| x^{n-l} \right\rangle CP_l^{(k)} (x; \lambda, \mu) \]

\[
= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l} CP_l^{(k)} (x; \lambda, \mu) (n - l - 1)!
\]

\[
= n! \sum_{l=0}^{n-1} \left( \frac{-1}{m} \right)^{n-l} \binom{n}{l} \frac{1}{(n-l)!} CP_l^{(k)} (x; \lambda, \mu).
\]

By the same method as (65), we get

\[
\frac{d}{dx} \hat{CP}_n^{(k)} (x; \lambda, \mu) = n! \sum_{l=0}^{n-1} \left( \frac{-1}{m} \right)^{n-l-1} \frac{1}{(n-l)!} \hat{CP}_l^{(k)} (x; \lambda, \mu).
\]

Now, we compute the following equation in two different ways:

\[
\left\langle \left( 1 + (1 + t)^{\lambda} \right)^{-m} \text{Lif}_k (- \log (1 + t)) (\log (1 + t))^m \right| x^n \right\rangle.
\]

On the one hand,

\[
\left\langle \left( 1 + (1 + t)^{\lambda} \right)^{-m} \text{Lif}_k (- \log (1 + t)) (\log (1 + t))^m \right| x^n \right\rangle
\]

\[
= \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1 (l+m, m) \left\langle \left( 1 + (1 + t)^{\lambda} \right)^{-m} \text{Lif}_k (- \log (1 + t)) \right| x^{n-l-m} \right\rangle
\]

\[
= \sum_{l=0}^{n-m} m! \binom{n}{l} S_1 (n-l, m) \hat{CP}_l^{(k)} (0; \lambda, \mu).
\]
On the other hand,

\[
(68) \quad \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \mathcal{L}_k \left( - \log (1 + t) \right) \left( \log (1 + t) \right)^m x^n
\]

\[
= \left\{ \partial_t \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \mathcal{L}_k \left( - \log (1 + t) \right) \left( \log (1 + t) \right)^m \right\} x^{n-1}
\]

\[
= \left\{ \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \mathcal{L}_k \left( - \log (1 + t) \right) \left( \log (1 + t) \right)^m \right\} x^{n-1}
\]

\[
+ \left\{ (1 + (1 + t)^{\lambda})^{-\mu} \left( \partial_t \mathcal{L}_k \left( - \log (1 + t) \right) \right) \left( \log (1 + t) \right)^m \right\} x^{n-1}
\]

\[
+ \left\{ \left( 1 + (1 + t)^{\lambda} \right)^{-\mu} \mathcal{L}_k \left( - \log (1 + t) \right) \left( \partial_t \left( \log (1 + t) \right)^m \right) \right\} x^{n-1}
\].

Therefore, by (67) and (68), we obtain the following theorem.

**Theorem 9.** For \( n \in \mathbb{N} \) with \( n \geq 2 \), let \( n - 1 \geq m \geq 1 \). Then we have

\[
m \sum_{l=0}^{n-m} \binom{n}{l} S_1 \left( n - l, m \right) CP_t^{(k)} \left( 0; \lambda, \mu \right)
\]

\[
= \left\{ \frac{n-1-m}{l} \right\} S_1 \left( n - l, m - 1 \right) \hat{CP}_t^{(k)} \left( -1; \lambda, \mu \right)
\]

Remark. By the same method as Theorem 9, we get

\[
m \sum_{l=0}^{n-m} \binom{n}{l} S_1 \left( n - l, m \right) CP_t^{(k)} \left( 0; \lambda, \mu \right)
\]

\[
= \left\{ \frac{n-1-m}{l} \right\} S_1 \left( n - l, m - 1 \right) \hat{CP}_t^{(k)} \left( -1; \lambda, \mu \right)
\]

where \( n - 1 \geq m \geq 1 \).

Let us consider the following two Sheffer sequences:

\[
(69) \quad CP_t^{(k)} \left( x; \lambda, \mu \right) \sim \left( 1 + e^{-\lambda t} \right)^{\mu} \frac{1}{\mathcal{L}_k (-t)} e^{-t} - 1
\]

and

\[
(70) \quad B_t^{(s)} \left( x \right) \sim \left( \frac{e^t - 1}{t} \right)^s , \quad \left( s \in \mathbb{Z}_{\geq 0} \right)
\].
Let

\[(71)\quad CP_n^{(k)}(x; \lambda, \mu) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x).\]

Then, by (26), we get

\[(72)\quad C_{n,m} = \frac{1}{m!} \langle \left( e^{-\log(1 + t)} - 1 - \log(1 + t) \right) \left( e^{\lambda \log(1 + t)} \right)^m x^n \rangle
= (-1)^m \frac{m!}{m!} \langle \left( 1 + (1 + t)^\lambda \right)^{-\mu} \text{Lif}_k(\log(1 + t)) \rangle
\times \langle \left( 1 + (1 + t)^\lambda \right)^{-s} \left( \frac{t}{\log(1 + t)} \right)^s (\log(1 + t))^m x^n \rangle
= (-1)^m \sum_{l=0}^{n-m} \frac{m!}{l+m} \binom{n}{l+m} S_1(l+m,m) \sum_{i=0}^{n-l-m} \binom{l}{i} C_{i}^{(s)} C_{n-l,m}^{(k)}(s; \lambda, \mu) x^{n-l-m-i}
= (-1)^m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \sum_{i=0}^{l} \binom{l}{i} C_{i}^{(s)} C_{l-i}^{(k)}(s; \lambda, \mu).
\]

Therefore, by (71) and (72), we obtain the following theorem.

**Theorem 10.** For \(n \geq 0\), we have

\[CP_n^{(k)}(x; \lambda, \mu) = \sum_{m=0}^{n} (-1)^m \binom{n-m}{l+i} \binom{n}{l} S_1(n-l,m) C_{i}^{(s)} C_{l-i}^{(k)}(s; \lambda, \mu) B_m^{(s)}(x).\]

**Remark.** By the same method as Theorem 10, we have

\[(73)\quad \hat{CP}_n^{(k)}(x; \lambda, \mu) = \sum_{m=0}^{n} \binom{n-m}{l+i} \binom{n}{l} S_1(n-l,m) \hat{C}_{i}^{(s)} \hat{C}_{l-i}^{(k)}(s; \lambda, \mu) B_m^{(s)}(x).\]

For \(CP_n^{(k)}(x; \lambda, \mu) \sim \left(1 + e^{-\lambda t}\right)^\mu e^{-t} - 1, H_n^{(s)}(x|\lambda) \sim \left(\frac{e^{-\lambda}}{1-\lambda}\right)^s, t\),
\(s \in \mathbb{Z}_{\geq 0}, \lambda \in \mathbb{C}\) with \(\lambda \neq 1\), let us assume that

\[(74)\quad CP_n^{(k)}(x; \lambda, \mu) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x; \lambda).\]
From (26), we have

\[ C_{n,m} = \frac{(-1)^m}{m!} \sum_{l=0}^{n-m} m! \binom{n}{l+m} \min\{s,n-l,m\} \left( \sum_{i=0}^{s-l-m} \frac{(\lambda - 1)^i}{i!} \right) \times \left(1 + (1 + t)^\lambda \right)^{-\mu} \text{L}_{\mu k}(\log (1 + t)) \]

\[ \times (1 + t)^{-s} \left(1 + \frac{\lambda}{\lambda - 1} \right)^{s-l-m} \frac{1}{(1 + t)^s} x^{n-l-m} \]

\[ = \frac{(-1)^m}{m!} \sum_{l=0}^{n-m} m! \binom{n}{l+m} \min\{s,n-l,m\} \sum_{i=0}^{s-l-m} \left( \frac{\lambda}{\lambda - 1} \right)^i \times \left(1 + (1 + t)^\lambda \right)^{-\mu} \text{L}_{\mu k}(\log (1 + t)) \left(1 + t\right)^{-s} x^{n-l-m} \]

Therefore, by (75) and (76), we obtain the following theorem.

**Theorem 11.** For \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), \( n \geq 0 \), we have

\[ CP_n^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{n} (-1)^m \binom{n-m}{l+m} \left( \sum_{i=0}^{s-l-m} \left( \frac{\lambda}{\lambda - 1} \right)^i \right) S_l (n-l, m) CP_{l-i}^{(k)} (s; \lambda, \mu) \]

\[ = \sum_{m=0}^{n} \left( \frac{1}{1-\lambda} \right)^i S_l (n-l, m) CP_{l-i}^{(k)} (0; \lambda, \mu) \right) H_m^{(s)} (x; \lambda). \]

**Remark.** By the same method as Theorem 11 we get

\[ \hat{CP}_n^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{\infty} C_{n,m} x^{(m)} \]

For \( CP_n^{(k)} (x; \lambda, \mu) \sim (1 + e^{-\lambda t})^\mu \frac{1}{\text{L}_{\mu k}(\log (1 + t))} e^{-t} - 1 \) and \( x^{(n)} \sim (1, 1 - e^{-t}) \), let us assume that

\[ CP_n^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{\infty} C_{n,m} x^{(m)}. \]
By (26), we get
\begin{align*}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{(1 + e^{\lambda \log(1+t)})^m} \right| \frac{1 - e^{\lambda \log(1+t)}}{t^{m+1}} \right\rangle x^n \\
&= \frac{1}{m!} \left\langle \left(1 + (1 + t)^{\lambda} \right)^{-\mu} \right| \frac{\text{Li}_k(\log(1+t))}{t^m} \right\rangle x^n \\
&= (-1)^m \frac{1}{m!} \left\langle \left(1 + (1 + t)^{\lambda} \right)^{-\mu} \right| \frac{\text{Li}_k(\log(1+t))}{t^m} \right\rangle x^{n-m} \\
&= (-1)^m \binom{n}{m} CP_{n-m}^{(k)} (0; \lambda, \mu).
\end{align*}
Therefore, by (77) and (78), we obtain the following theorem.

**Theorem 12.** For $n \geq 0$, we have
\[ CP_{n}^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} CP_{n-m}^{(k)} (0; \lambda, \mu) x^m. \]

**Remark.** By the same method as Theorem 12, we get
\[ \check{C}P_{n}^{(k)} (x; \lambda, \mu) = \sum_{m=0}^{n} \binom{n}{m} \check{C}P_{n-m}^{(k)} (0; \lambda, \mu) (x)_m. \]

**References**

[1] M. Can, M. Cenkci, V. Kurt, Y. Simsek, *Twisted Dedekind type sums associated with Barnes’ type multiple Frobenius-Euler L-functions*, Adv. Stud. Contemp. Math. 18 (2009), no. 2, 135-160.

[2] L. Carlitz, *A note on Bernoulli and Euler polynomials of the second kind*, Scripta Math. 25 (1961), 323-330.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[4] R. Dere, Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math 22 (2012), no. 3, 433-438.

[5] H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly 79 (1972), 44-51.

[6] D. S. Kim, T. Kim, *Some identities of Bernoulli and Euler polynomials arising from umbral calculus*, Adv. Stud. Contemp. Math. 23 (2013), no. 1, 159-171.

[7] D. S. Kim, T. Kim, Y. H. Kim, D. V. Dolgy, *A note on Eulerian polynomials associated with Bernoulli and Euler numbers and polynomials*, Adv. Stud. Contemp. Math. 22 (2012), no. 3, 379-389.

[8] T. Kim, D. S. Kim, T. Mansour, S.-H. Rim, M. Schork, *Umbral calculus and Sheffer sequences of polynomials*, J. Math. Phys. 54 (2013), 083504.

[9] D. S. Kim, T. Kim, D. V. Dolgy, S. H. Rim, *Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus*, Adv. Difference Eq. 2013 (2013), 2013:73.

[10] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on \( \mathbb{Z}_p \)*, Russ. J. Math. Phys. 16 (2009), no. 4, 484-491.

[11] D. S. Kim, T. Kim, S.-H. Lee, *Poly-Cauchy Numbers and polynomials with umbral calculus viewpoint*, Int. Journal of Math. Analysis, Vol. 7, 2013, no. 45, 2235-2253.

[12] T. Komatsu, *On poly-Cauchy numbers and polynomials*, available at [http://carma.newcastle.edu.au/alicon/pdfs/Takao_Komatsu-alicon.pdf](http://carma.newcastle.edu.au/alicon/pdfs/Takao_Komatsu-alicon.pdf).

[13] D. Merlini, R. Sprugnoli and M. C. Verri, *The Cauchy numbers*, Discrete Math. 306(2006), 1906-1920.

[14] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q- Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. 18 (2009), no. 1, 41-48.

[15] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984, x+193 pp. ISBN: 0-12-594380-6.
[16] S. Roman, G.-C. Rota, *The umbral calculus*, Advances in Math. 27 (1978), no. 2, 95-188.

**Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea**
*E-mail address: dskim@sogang.ac.kr*

**Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea**
*E-mail address: tkkim@kw.ac.kr*