Observational signatures of spherically-symmetric black hole spacetimes

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Abstract. A binary system composed of a supermassive black hole and a pulsar orbiting around it is studied. The motivation for this study arises from the fact that pulsar timing observations have proven to be a powerful tool in identifying physical features of the orbiting companion. In this study, taking into account a general spherically-symmetric metric, we present analytic calculations of the geodesic motion, and the possible deviations with respect to the standard Schwarzschild case of General Relativity. In particular, the advance at periastron is studied with the aim of identifying corrections to General Relativity. A discussion of the motion of a pulsar very close the supermassive central black hole in our Galaxy (Sgr A*) is reported.

1. Introduction
It is now widely believed that supermassive black holes (SMBHs) reside at the centres of all galaxies and that their estimated masses are in the range of a few million to up to tens of billions of solar masses. Earth’s closest SMBH candidate is found at the Galactic Centre, Sagittarius A* (Sgr A*), which astronomers have been observing for several decades [1]. It is expected that the mathematical description of astrophysical black holes (BHs) is based on solutions to the Einstein field equations, and therefore founded on general relativity (GR). However, there also exist many BH solutions in extended and alternative theories of gravity, and to-date observational constraints, most notably in the strong-field regime, are lacking [2]. One promising probe of strong-field gravity is the direct imaging of the shadow cast by a SMBH[3]. Direct observation of the event horizon of our own SMBH, Sgr A*, will soon be obtained by the Event Horizon Telescope Collaboration (EHTC)[4, 5]. Pulsars provide an additional independent observational tool to help improve the understanding of the properties of Sgr A* (e.g., its mass, spin and even geometry), providing considerably stronger constraints than is possible with event horizon-scale imaging alone. The case of a pulsar orbiting around a SMBH is particularly interesting since one can in principle combine precision timing measurements with measurements of geodesic motion around the BH, i.e., in the strong-field regime. Such timing measurements can contribute to fixing strict ranges on the parameters of a given class of gravity theories and therefore facilitate the selection of viable theories without imposing any a priori assumptions. However, in this study theories of gravity entirely distinct from GR are investigated, and consequently the parametrisation of Rezzolla and Zhidenko (RZ) [6] is employed. The RZ parametrization is a general representation of BH spacetimes in arbitrary metric theories of gravity. In the case of spherically symmetric spacetimes, the parametrization makes use of a coordinate compactification in terms of a rapidly-convergent continued-fraction expansion defined in the radial direction between the event horizon and spatial infinity. The focus of the present
study is restricted to the spherically symmetric case and general expressions for the dynamics of a test-particle in general BH spacetimes, such as the motion of a pulsar orbiting around the SMBH candidate Sgr A*, are derived. In particular, explicit general expressions for the advance of the periastron at zeroth order of the parametrization is provided. Furthermore, periastron-advance formulae is also given for Einstein-Maxwell-Axion-Dilaton theory \cite{7}.

2. Parametrization Framework
In what follows the RZ parametrization \cite{6} for a generic spherically symmetric BH spacetime is briefly reviewed and subsequently used to determine the dynamics of a test particles. The Lagrangian may be written as \cite{7}

\[ 2\mathcal{L} = \frac{N(r)^2}{N(r)} - \frac{B(r)}{N(r)} r^2 - r^2 \dot{\vartheta}^2 - r^2 \sin^2 \vartheta \dot{\varphi}^2, \tag{1} \]

In the RZ parametrization, the function \( N(r) \) is then expressed as \( N(x) = x A(x) \) where \( A(x) > 0 \) for \( 0 \leq x \leq 1 \), with \( x := 1 - \frac{r_0}{r} \), so that \( x = 0 \) is the position of the event horizon and \( x = 1 \) corresponds to spatial infinity. Furthermore, \( A \) and \( B \) may be expressed in terms of the parameters \( \epsilon, a_i, \) and \( b_i (i \in [0, n], \) where \( n \) is the expansion order), such that

\[ A(x) = 1 - \epsilon(1 - x) + (a_0 - \epsilon)(1 - x)^2 + \tilde{A}(x)(1 - x)^3, \tag{2} \]
\[ B(x) = 1 + b_0(1 - x) + \tilde{B}(x)(1 - x)^2, \tag{3} \]

where the functions \( \tilde{A} \) and \( \tilde{B} \) describe the metric near the horizon (i.e., \( x \approx 0 \)) and at spatial infinity (i.e., \( x \approx 1 \)). It is evident that the metric is finite in both limits \cite{6}. The functions (2)–(3) can then be expanded via a Padé approximation of continuous as

\[ \tilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{1 + \ldots}}}, \quad \tilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \ldots}}}, \tag{4} \]

where \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) are dimensionless constants that can be fixed once the generalised metric is matched to a specific metric. Finally, the parameter \( \epsilon \) in equations (2)–(3) measures the deviations of the position of the event horizon in the general metric from the corresponding location in a Schwarzschild spacetime, i.e., \( \epsilon = \frac{2M - r_0}{r_0} = -\left(1 - \frac{2M}{r_0}\right) \).

3. Applications of the RZ parametrization: Einstein-Maxwell-Axion-Dilaton metric
The Einstein-Maxwell-Axion-Dilaton (EMAD) metric considered in this study is spherically symmetric and is constructed from a simplification of the axisymmetric EMAD solution \cite{8} in the case of a vanishing axion field. Solutions of this type arise from string theory. When the axion field vanishes and the BH is spherically symmetric, the EMAD BH ine element takes the following form

\[ ds^2 = 1 - \frac{2M}{r_0} \rho^2 dt^2 - \frac{r^2}{b^2 + r^2} dr^2 - d\Omega^2, \tag{5} \]

where \( \mu := M - \hat{b} \). Here \( \hat{b} \) is the dilaton parameter and \( M \) the BH mass \cite{9}, and \( \rho^2 = r^2 + 2\hat{b}r \), where \( r \equiv r(\rho) \). Recalling that the location of the event horizon for this BH is given by \( \rho_{0,\text{EMAD}} = 2 \left(M - \hat{b}\right) \),

The expression for \( \epsilon \) in terms of the axion-dilaton parameters may be written as \( \epsilon_{\text{EMAD}} = \sqrt{1 + \frac{\hat{b}}{\mu} - 1} \).

In a similar manner, expanding the metric coefficients at spatial infinity gives the values for \( a_0 \) and \( b_0 \) as

\[ a_{0,\text{dil}} = \frac{\hat{b}}{2\mu}, \quad \text{and} \quad b_{0,\text{dil}} = 0. \tag{6} \]
zeroth order in the RZ parametrization is obtained as the precession per orbit is the only non-zero expansion terms are for the periastron advance within the RZ parametrization, first consider the expansion of the metric when calculated. The orbits in this case remain closed if the magnitude of the angle swept out by the orbit is 2π. If this is not the case, then the inner turning points are precessing and the amount of this precession per orbit is  δϕ_{prec} = Δϕ−2π . To illustrate how to derive a theory-independent expression for the periastron advance within the RZ parametrization, first consider the expansion of the metric when the only non-zero expansion terms are a0 and b0 . The expression for the periastron advance, Δϕ, at zeroth order in the RZ parametrization is obtained as

\[
(0) \Delta \tilde{\phi} = 3\sigma + \sigma \left( 9 - 26\sigma + 20\sigma^2 \right) a_0 + 20a^2 (1 - \sigma) \epsilon ,
\]

where Δ\tilde{\phi} := Δϕ/2π and σ = r_g/ℓ with ℓ is the semi-latus rectum of the orbits. The first term in the above equation is simply the GR contribution, while the remaining two terms represent the deviations from GR in a general BH spacetime at zeroth order in the RZ parametrization.

4. Application to astrophysical test cases

In this sections both the RZ parameterization and the expression derived for the periastron advance are tested. To do this, a set of four test objects (four pulsar toy models) are considered. Their properties have been chosen to have a range of semi-major axes, reasonably high eccentricities and moderate-to-short orbital periods. In this respect, these toy models are idealised, but not altogether unrealistic: future advances in instrumental sensitivity could, in principle, enable the detection of Galactic Centre pulsars with such properties [12]. In modelling the toy pulsar-SMBH system it is hereafter assumed that the mass of the central SMBH is known to some degree of precision.

The properties of these pulsars are collected in Table 1, which also reports, besides the eccentricities e and semi-major axes a, the values of the periastron advances Δϕ_{GR} and (0) Δϕ at the zeroth order, in the RZ expansion. The models I, II, III are in principle already measurable with present astronomical observations [13], while Toy IV is, at the present time, an optimistic model. Since the RZ parametrisation is constructed to be most accurate at the event horizon and at spatial infinity, the results presented in Table 1 are, albeit weakly, dependent on the specific form of the parametrization. Note that in evaluating the periastron advance, specific values for the coefficients ϵ, ai and bi must be specified, since these coefficients cannot yet be constrained by astrophysical observations and hereafter chosen to be a0 = ϵ = 10^{-3}. In particular, looking at Table 1 one can establish how well the RZ parametrization works in the vicinity of the event horizon (this is especially true for Toy IV). This is illustrated in Fig. 1 for the EMAD BH (assuming b = 5 × 10^{-8}) using models Toy II and Toy IV, where the periastron advance (light-blue shaded areas around the solid blue line) and of the orbital period (light-red shaded areas around the red line) assuming an accuracy of 10^{-4}. In summary, the results reported in Figs. 1 demonstrate that by using a general description of test-particle motion in arbitrary BH spacetimes,
Figure 1. Constraints set by the pulsar orbits for toy models II and IV in the EMAD metric (5) at zeroth order, with dilaton parameter fixed at $\hat{b} = 5 \times 10^{-8}$. The values of the coefficients $a_0$ and $\epsilon$ are constrained by the observations of the periastron advance (light-blue shaded areas around the solid blue line) and of the orbital period (light-red shaded areas around the solid red line) assuming an accuracy of $10^{-4}$.

such as the RZ parameterization, future observations of pulsars near a SMBH can help impose tighter constraints on different theories of gravity, and even potentially facilitate ruling out certain theories (and, by extension, related classes and extensions thereof) entirely.

Furthermore, an illustrative example demonstrating the facility of the parametric framework to constrain the parameters of a theory of gravity, is presented. Figures 2 present the deviations from GR of the periastron advance as a function of the semi-major axis length, for models Toy I, Toy II (left panels) and model Toy IV (right panels) (Fig. 2). Multiple coloured curves, coloured from violet through to red, denote, for a fixed value of the theory parameter, how the relative difference in the periastron advance varies as a function of the semi-major axis length. These 51 coloured lines are uniformly logarithmically spaced between the stated parameter value in the upper right of each panel (uppermost red line) and 0.01 times that value (bottommost violet line), i.e., 25 lines per decade in the theory parameter. The horizontal black dashed line at $10^{-7}$ represents a potential astrophysical measurement precision [13]. It is immediately clear that for models Toy I and Toy II the relative differences in the periastron advance are insensitive to the semi-major axis length and practically indistinguishable, as is evident from the horizontal, parallel theory parameter lines. This can be interpreted as near-Keplerian pulsar motion, i.e., the weak-field limit of the RZ parametrization. The vertical blue line (right panel) denotes the semi-major axis position of model Toy IV. Since this model places the pulsar much closer to the event horizon of the BH (1 AU), and as is clear from the theory parameter lines steeper gradients, the pulsar motion can be considered as occurring in the transition region between the weak-field and strong-field regimes. The intersection of the uppermost plotted theory parameter line with the vertical blue and magenta lines provides an upper limit for the theory parameters $\hat{b} < 6.68 \times 10^{-8}$ for Toy I and Toy II, whereas for Toy IV case this yields $\hat{b} < 7.762 \times 10^{-8}$. In particular, one can see from Fig. 2 that it is possible to provide constraints on the two theories. More specifically, one can place an upper limit on the value of the parameters $\alpha$ and $\hat{b}$. It is assumed that the error on the measurements is of the order of $10^{-7}$ [13]. Figure 2 present the relative differences from $\left(0\right)\Delta\tilde{\varphi}$ at first order of the expansion in GR with respect to the semi-major axes for the four Toy models. What emerges from the figures is that the deviations from GR are much more remarkable if one considers, as seen before, small semi-major axes and high eccentricities. In addition, one may also estimate the various values of the parameters that characterise
the specific theory through the location of the different level curves. It can be seen from Figs. 2 that for a given measurement precision, as the pulsar semi-major axis location is shifted towards the BH event horizon, the range over which the theory parameter can be probed (and effectively constrained) is increased. Therefore, a pulsar orbiting in the immediate vicinity of a BH event horizon (i.e., in the truly strong-field regime) can provide much tighter constraints on the theory parameters. It is clear that in order to be proven useful, such a pulsar must be close to the SMBH, with an orbital period of only a few hours or less. Observational detection of such pulsars is in principle achievable with present day radio telescopes.

5. Conclusions

Although GR has proven to be a reliable theory of gravity in several different fields of application, and across several different scales, it is by no means a complete one. Furthermore, there exist other theories of gravity which reproduce not only the results of classical GR, but also the results of modern astrophysical observations where pure GR fails to be as predictive. There is therefore a pressing need, both theoretically and observationally, to begin to, at the very least, impose constraints (and even potentially exclude) particular theories of gravity. Owing to their very narrow mass range, extreme compactness and rapid, stable rotation periods, pulsars are one of the best candidates to probe strong-field gravity in the truly nonlinear regime. Measurement of a pulsar in the strong-field regime, i.e., near a BH event horizon, would enable not just a highly accurate determination of the BH properties, but also provide an accurate probe of the spacetime structure and geometry. Given the rapid increase in recent efforts to perform astronomical observations of the Galactic Center, the prospect of detecting a pulsar orbiting in close proximity to Sgr A* is promising. Such a detection would provide the most accurate measurements of the physical parameters (e.g. mass, spin, and even quadrupole moment) of Sgr A* [14]. Theoretical studies of pulsar motion and timing in both GR and alternative theories of gravity are therefore of great importance. However, given the breadth of available theories of gravity

Figure 2. Relative difference of the zeroth order periastron advance with respect to the GR value plotted as a function of semi-major axis distance for Toy models I,II (left panel) and IV (right panel). Different coloured lines for a fixed value of the theory parameter the variation in this relative difference as a function of semi-major axis. The 51 lines are equally spaced (logarithmically) between the stated inset upper limits of $\hat{b} \leq 6.68 \times 10^{-8}$ (left panel) and $\hat{b} \leq 7.762 \times 10^{-8}$ (right panel). In the left panel solid lines at 175.4 AU (blue) and 43.85 AU (magenta), and in the right panel (blue) at 1 AU denotes the semi-major axis position of Toy model I, II and IV respectively.
in the present literature, it is most expedient to perform any such studies in a manner which assumes neither a specific theory of gravity nor any particular solution to any particular theory. It is also most desirable to have a representation in which the classical GR limit is recovered. Consequently, this study has presented an analysis of test-particle (i.e., pulsar) dynamics in several different BH spacetimes, using a theory-independent approach. This approach makes use of a general mathematical representation of BH spacetimes. In the case of spherically symmetric spacetimes, i.e., those considered in the present study, a rapidly-convergent continued fraction expansion in terms of a compactified radial coordinate has been employed. Using this parametrization, general expressions for the dynamics of a test-particle in general BH spacetimes were derived. This formalism was applied to a set of astrophysical test case: (i) four hypothetical pulsar toy models. Next, the periastron advance properties of four pulsar toy models were investigated. It was shown that, pulsars with smaller semi-major axis lengths (i.e. orbiting closer to Sgr A*) indeed exhibit quantifiable deviations from GR. Finally, the particular cases of EMAD theory was employed to show that measurements of the relative difference of the periastron advance (from GR) can provide another avenue through which to constrain the parameters of different theories of gravity. Therefore, using pulsar observations presents the possibility to strongly constrain the parameters of all theories which are purely geometrical (i.e., not containing, e.g., exotic particles and scalar fields).

In conclusion, pulsar observations can in principle accurately constrain the properties of their central SMBH companion. In this study, using a theory-independent parametric framework, it has been shown that pulsar timing and dynamics also presents the possibility to constrain (and even potentially exclude) theories of gravity [7].

Acknowledgements

Support comes from the ERC Synergy Grant “BlackHoleCam - Imaging the Event Horizon of Black Holes” (Grant 610058)

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