Legal Assignments and fast EADAM with consent via classical theory of stable matchings

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Abstract

Gale and Shapley’s college admission problem and concept of stability [13] have been extensively studied, applied, and extended. In school choice problems, mechanisms often aim to obtain an assignment that is more favorable to students. We investigate two extensions introduced in this context – legal assignments [24] and the EADAM algorithm [18] – through the lens of classical theory of stable matchings. In any instance, the set \( L \) of legal assignments is known to contain all stable assignments. We prove that \( L \) is exactly the set of stable assignments in another instance, and that essentially any optimization problem over \( L \) can be solved within the same time bound needed for solving them over the set of stable assignments. A key tool for these results is an algorithm that finds the student-optimal legal assignment. We then generalize our algorithm to obtain the output of EADAM with any given set of consenting students without sacrificing the running time, hence improving over known algorithms in both theory and practice. Lastly, we investigate how larger the set \( L \) can be compared to the set of stable matchings in the one-to-one case, and connect legal matchings with certain concepts and open problems in the literature.

Keywords: stable matchings, distributive lattice, rotations, school choice problem, legal assignments, EADAM algorithm, Latin marriages.

1 Introduction

Stable matchings and stable assignments are fundamental paradigms in operations research and the design of matching markets. Since the seminal work of Gale and Shapley [13], these concepts have received widespread attention for their mathematical elegance and broad applicability (see e.g. [15, 28, 30]). Those two facets are tightly connected. A detailed understanding of the lattice structure of stable matchings led to many fast algorithms for e.g. enumerating all stable matchings [14] and finding a stable matching that maximizes some linear profit function [16]. In turns, these algorithmic results propel the application of stable matchings to many matching markets, such as college admission, assignment of residents to hospitals [27], and kidney transplant [29].

One of the most important applications, the school choice problem, considers the assignment of high school students to public schools. After the pioneering work of Abdulkadiroğlu and Sönmez [1], many school districts, such as New York City and Boston, subsequently adopted the student-optimal stable mechanism for its fairness (no priority violation or stability) and strategy-proofness (for students). The mechanism asks students to report their (strict) preferences of the schools and schools to report their priorities\(^1\) (preferences with ties) over the students. It then randomly breaks ties in the latter to obtain an instance of the

\(^1\)Priorities are preferences with ties, as schools usually rank students based on categorical information such as demographics, test scores, etc.
stable assignment problem and performs the Gale-Shapley algorithm\(^2\) to obtain the student-optimal stable assignment.

However, in this setting, schools are often perceived as commodities, and only students’ welfare matters. Hence, enforcing stability implies a loss of efficiency. Abdulkadiroğlu et al. [2] demonstrate the magnitude of such efficiency loss with empirical data from the New York City school system, where over 4,000 eighth graders in their sample could improve their assignments if stability constraints were relaxed. Striving to regain this loss in welfare for the students, many alternative concepts and mechanisms have therefore been introduced and extensively studied (see e.g. [3, 11, 18, 19, 24]).

Those mechanisms lead therefore to solutions outside the well-structured set of stable matchings. As a consequence, ad-hoc structural studies and algorithms must be presented. Unfortunately, properties of the former and performance of the latter rarely match theory of and algorithms for stable matchings [18, 19, 31]. This harms the applicability of such mechanisms, especially if school districts were to address two of the major concerns economists have regarding the current school choice design. One concern is that the way ties are broken in priorities affects the quality of the outcome [11]. The other concern is that the constraint on the number of schools a student can rank [26] puts mechanisms at risk of manipulation [7]. Hence, if school districts were to remove the capping on the number of choices and/or were to test different tiebreakers, fast algorithms for those alternative mechanisms and an improved understanding of the structure of feasible solutions would be useful in applications.

This paper focuses on two concepts introduced to regain the loss of welfare in the school choice problem: legal assignments [24] and EADAM with consent [18]. Legal assignments form a superclass of stable assignments. They share many interesting properties with stable assignments, e.g. lattice structure and consequently the existence of a student-optimal legal assignment. EADAM operates by iteratively asking for students’ consent to “waive” their priority at certain schools and re-run the Gale-Shapley algorithm. The output of EADAM is constrained efficient [31]. That is, the assignment does not violate any nonconsenting students’ priorities, but any other assignment that is weakly preferred by all students does. Hence, when all students consent, the output of EADAM is Pareto efficient for students, and it is known to coincide with the student-optimal legal assignment [24].

The goal of our paper is to use classical theory of stable matchings to achieve a better structural and algorithmic understanding of these two extensions. In fact, although both EADAM and legal assignments have been further analyzed and extended by several authors (see e.g. [8, 10, 19, 4, 32]), our knowledge of those two concepts is far from complete. In particular, the knowledge of a lattice structure alone gives little information on how to exploit it for algorithmic purposes, e.g. how to find the legal assignment that maximizes some linear profit function\(^3\). Moreover, little is known on how to exploit the structure of legal assignments to obtain the output of EADAM when not all students consent, since the output assignment may not be legal.

1.1 Our Contribution

Our first contribution deals with the structure of legal assignments. We prove in Section 3 that the set of legal assignments coincide with that of stable assignments in a subinstance of the original one. Hence, legal assignments inherit all structural properties of stable assignments. This result, in particular, greatly simplifies the treatment from [24], where the lattice structure is proved from scratch.

As our second contribution, in Section 6 we show how to obtain the aforementioned subinstance in time linear in the number of edges of the input. Hence, in order to solve an optimization problem over the set of legal assignments (e.g. to find the already mentioned school-optimal, or other assignments of interest as the egalitarian, profit-optimal, minimum regret), one can resort to the broad literature on algorithms developed for the same problem on the set of stable assignments (see e.g. [23] for a collection of those results). Since the worst-case running time of those algorithm is at least linear in the number of the edges, the complexity of those problems over the set of legal assignments does not exceed their complexity over the set of stable assignments. To achieve this second contribution, we first extend classical concepts of rotations and rotation

\(^2\)In some literature, Gale-Shapley algorithm is referred to as Deferred acceptance. In this paper, we stick to Gale-Shapley.

\(^3\)A typical example are strongly stable matchings, which have been known for a long time to form a distributive lattice [22], but only recently was this structure exploited for algorithmic purposes [21].
digraphs in Section 4; we then developed a pair of algorithms in Section 5, which we name \textit{rotate-remove} (Algorithm 1) and \textit{reverse rotate-remove} (Algorithm 2), that respectively find the school-optimal and student-optimal legal assignments.

Our third contribution is a fast algorithm for EADAM with consent. Algorithmic results above imply that, when all students consent, EADAM can be implemented as to run with the same time complexity as that of Gale-Shapley’s. However, when only some students consent, the output of EADAM may no longer be legal (see Example 7.3). We show in Section 7 how to modify \textit{reverse rotate-remove} to produce the output of EADAM again within the same time bound as Gale-Shapley’s. Computational tests performed in Section 7.5 confirm that our algorithms run significantly faster in practice.

As last contribution, we show that relaxing the stability condition to legality can greatly increase the size of the set of feasible matchings. We provide instances with one stable matching and exponentially many (in the number of men and women) legal matchings. This is achieved by an exploration of the connection between legal matchings and \textit{Latin marriages}, introduced by [6] in relationship with a classical, long-standing open question of Knuth [20] on the maximum number of stable matchings an instance can have. We defer details to Section 8.

Our algorithm implementation for (1) finding student-optimal and school-optimal legal assignments and for obtaining the legal subinstance; and for (2) EADAM with consent can be found online\textsuperscript{4}.

We remark that our work is completely self-contained, except for classical results on stable matchings that can be found e.g. in [15] and, for results in Section 7 and Section 8, facts on EADAM from Tang and Yu [31] and on Latin instances from [6], respectively.

1.2 Literature Review

There is a vast amount of literature on mechanism design for the school choice problem, balancing their focus among strategy-proofness, efficiency and stability. From a theoretical prospective, Ergin [12] shows that under certain acyclicity conditions on the priority structure, the student-optimal stable assignment is also Pareto efficient for the students. Kesten [18] interprets these cycles as the existence of interrupting pairs and proposes the EADAM mechanism, which improves efficiency while simultaneously maintaining stability by obtaining students’ consent to waive their priorities.

Extending upon Kesten’s framework, many researchers offer new perspectives. Tang and Yu [31] propose a simplified algorithm for EADAM, which repeatedly runs Gale-Shapley algorithm after fixing the assignments of underdemanded schools. Bando [5] shows an algorithm which iteratively runs Gale-Shapley algorithm after fixing the assignments of the set of last proposers. Bando [5] also shows that when restricting to one-to-one setting, his algorithm finds the student-optimal \textit{von Neumann-Morgenstern (vNM) stable matching}. vNM stable set is a concept proposed by Von Neumann and Morgenstern [33] for cooperative games. In the college admission problem, the definition of legal assignments by Morrill [24] corresponds to vNM stable set. In one-to-one setting, results from [9] and [34] show existence and uniqueness of the vNM stable set. Morrill [24] further proves the existence and uniqueness result in the one-to-many setting, as well as the fact that they have a lattice structure. Wako [34] presents an algorithm that finds the man- and woman-optimal vNM stable matchings, and show that vNM stable matchings coincide with stable matchings in another instance. However, Wako [34] points out that his algorithm does not directly apply to the one-to-many setting, and he poses as an open question to find one such algorithm.

Our results answer this open question. We remark that, although there is a standard reduction of one-to-many instances to one-to-one instances [15, 28] such that the set of stable assignments of the former and the set of stable matchings of the latter correspond, such one-to-one mapping fails for the set of legal assignments (see Example 3.2). So we need to directly tackle the one-to-many setting.

2 Basic Mathematical Models

We introduce here basic notions and facts. We point readers to the book by Gusfield and Irving [15] for a more comprehenive introduction on stable marriage and stable assignment problems.

\textsuperscript{4}(1). \url{https://github.com/xz2569/LegalAssignments}. (2).\url{https://github.com/xz2569/FastEADAM}. 
2.1 Stable Marriage Problem

An instance of the stable marriage problem is a pair \((G, <)\) where \(G\) is a bipartite graph with bipartition \((A, B)\) and \(<\) denotes \(\{<_v\}_{v \in V}\), with \(<_v\) being a strict ordering of the neighbors of \(v\) in \(G\). Elements of \(A\) are referred to as men, and elements of \(B\) as women. For \(a \in A, b, b' \in B\), we say \(a\) strictly prefers \(b\) to \(b'\) if \(b >_a b'\), and we say that \(a\) prefers \(b\) to \(b'\) and write \(b \geq_a b'\) if \(b >_a b'\) or \(b = b'\). Similar definitions are employed for \(a >_b a'\) and \(a \geq_b a'\). If \(a \in A\) and \(b \in B\) are matched in matching \(M\), we say that \(a\) and \(b\) are partners in \(M\) and we write \(b = M(a)\) and \(a = M(b)\). If \(x \in A \cup B\) is not matched in \(M\), we write \(M(x) = \emptyset\). If \(M(a) >_a M'(a)\) (resp. \(M(a) \geq_a M'(a)\)), we say that \(a\) strictly prefers (resp. prefers) \(M\) to \(M'\) and similarly for \(b\). A pair \((a, b) \in A \times B\) is said to block a matching \(M\), or to be a blocking pair for \(M\), if \(a >_a M(b)\) and \(b >_a M(a)\). In particular, a blocking pair is an edge of \(G\). Hence, we often drop the brackets and simply write \(ab\), also omitting to specify each time that \(a \in A\) and \(b \in B\). Similarly, we say that any matching containing \(ab\) blocks \(M\). A matching \(M\) is stable for \((G, <)\) if there is no edge of \(G\) that blocks \(M\). We denote by \(\mathcal{M}(G)\) the set of matchings of \(G\), and \(S(G, <)\) as the set of stable matchings of \((G, <)\).

2.2 Stable Assignment Problem

An instance of the stable assignment problem, also known as the college assignment problem, is a triple \((G, <, q)\) where \(G\) and \(<\) are defined as in the stable marriage problem, and \(q = \{q_b\}_{b \in B} \in \mathbb{N}^B\) denotes the maximum number of vertices in \(A\) that can be assigned to each \(b \in B\). \(q_b\) is called the quota of \(b\). Elements of \(A\) are referred to as students and elements of \(B\) as schools. The definition of \(a\) prefers \(b\) to \(b'\), or \(b \geq_a b'\) and of the symmetric concepts for \(b \in B\) are immediate extension of the equivalent concepts in the marriage case.

An assignment \(M\) for an instance \((G, <, q)\) is a collection of edges of \(G\) such that: at most one edge of \(M\) is incident to \(a\) for each \(a \in A\); at most \(q_b\) edges of \(M\) are incident to \(b\) for each \(b \in B\). We write \(M(b) = \{a : ab \in M\}\). We call \(ab\) a blocking pair for \(M\) if \(b >_a M(a)\), and either \(a >_b a'\) for some \(a' \in M(b)\) or \(|M(b)| < q_b\). In this case, we say that \(ab\) blocks \(M\), and similarly, we say that \(M'\) blocks \(M\) for every assignment \(M'\) containing edge \(ab\). An assignment is stable if it is not blocked by any edge of \(G\). Extending the notations from the stable marriage problem, let \(\mathcal{M}(G, q)\) be the set of assignments of \((G, q)\), and let \(S(G, <, q)\) be the set of stable assignments of \((G, <, q)\). For a subgraph \(G'\) of \(G\), we denote by \((G', <, q)\) the stable assignment instance whose preference lists are those induced by \(<\) on \(G'\) and quotas are those obtained by restricting \(q\) to nodes in \(G'\).

Assignments output by Gale-Shapley’s algorithm(s) play a special role. We refer e.g. to [15] for details on those algorithms.

**Theorem 2.1.** The student-proposing (resp. school-proposing) Gale-Shapley algorithm outputs a stable assignment \(M_0\) (resp. \(M_2\)) such that \(M_0(a) \geq_a M'(a)\) (resp. \(M'(a) \geq_a M_2(a)\)) for any \(a \in A\) and stable assignment \(M'\).

2.3 Reduction: Stable Marriages to Stable Assignments

A stable assignment instance \((G, <, q)\) can be transformed into a stable marriage problem \((H_G, <_G)\) via the following well-known reduction [15, 28]. For each school \(b \in B\), create \(q_b\) copies of \(b\), say \(b^1, \ldots, b^{q_b}\), and replace \(b\) in the preference list of each adjacent \(a \in A\) by the \(q_b\) copies in exactly this order. The preference list of each \(b^j\) is identical to the preference list of \(b\). We call these copies seats of the schools and denote their collection by \(B_H\). With this reduction, we can construct a map \(\pi : \mathcal{M}(G, q) \rightarrow \mathcal{M}(H_G)\) that induces a bijection between \(S(G, <, q)\) and \(S(H_G, <_G)\). Given \(M \in \mathcal{M}(G, q)\), assume for some \(b \in B\), \(M(b) = \{a^1, \ldots, a^j\}\) and \(a^1 \geq_b a^2 \geq_b \cdots \geq_b a^j\). Define \(\pi(M)(b^i) = a^i\) for \(i \in [j]\) and \(\pi(M)(b^i) = \emptyset\) for \(i \in [j, q_b]\). For the sake of shortness, we often abbreviate \(M_H = \pi(M)\).
3 Legal Assignments are Stable Assignments in Disguise

Throughout the section, we fix an instance \((G, <, \pi)\) of the stable assignment problem and let \(M := M(G, \pi)\), \(S := S(G, <, \pi)\). For a set \(\mathcal{M}' \subseteq \mathcal{M}\), define \(I(\mathcal{M}')\) as the set of assignments that are blocked by some assignment \(M' \in \mathcal{M}'\) and define \(L(\mathcal{M}')\) as the set of assignments that are not blocked by any assignment in \(\mathcal{M} \setminus I(\mathcal{M}')\). That is, \(L(\mathcal{M}') := \mathcal{M} \setminus I(\mathcal{M} \setminus I(\mathcal{M}'))\). We say a set \(\mathcal{M}'\) has the legal property if \(I(\mathcal{M}') = \mathcal{M} \setminus \mathcal{M}'\).

We devote this section to the proof of the following theorem.

**Theorem 3.1.** Let \((G, <, \pi)\) be an instance of the stable assignment problem. There exists a unique set \(\mathcal{L} \subseteq M(G, \pi)\) that has the legal property. We call \(\mathcal{L}\) the set of legal assignments. This set coincides with the set of stable assignments in \((G_L, <, \pi)\), where \(G_L\) is a subgraph of \(G\). Moreover, \(E(G_L) = \cup\{M : M \in S(G_L, <, \pi)\} = \cup\{M : M \in \mathcal{L}\}\).

As introduced in Section 2, there is a one-to-one correspondence between stable assignments in \((G, <, \pi)\) and stable matchings in the reduced instance \((H_G, <, G)\). One could think of proving Theorem 3.1 by showing the (simpler) results for the stable marriage instance \((H_G, <, G)\), and then deducing the set of legal assignments of \((G, <, \pi)\) from the set of legal matchings of \((H_G, <, G)\). Unfortunately, the bijection between stable assignments and stable matchings does not extend to the legal setting, as next example shows.

**Example 3.2.** Consider an instance with 4 students and 2 schools, each with 2 seats. Let \(a_1, b_1, b_1'\) represent students, schools, and seats respectively. The preference lists are given as follows. In this and all following examples, when it is clear whose preference list we are referring to, the subscript in \(>\) is dropped.

\[
\begin{align*}
   a_1 & : b_1 > b_2 \\
   a_2 & : b_2 > b_1 \\
   a_3 & : b_2 > b_1 \\
   a_4 & : b_1 > b_2
\end{align*}
\]

Since all preference lists are complete, we can restrict our attention to the 6 assignments where all students are matched. One can easily verify that \(M = \{a_1b_1, a_2b_2, a_3b_2, a_4b_1\}\) is the only stable assignment, and all other assignments are blocked by some pair in \(M\), hence they are all illegal. Thus, \(\mathcal{L} = \{M\}\). Now, consider the reduction to the stable marriage problem. The preference lists can be expanded:

\[
\begin{align*}
   a_1 & : b_1 > b_2 > b_1' > b_2' \quad b_1' & : a_3 > a_4 > a_2 > a_1 \\
   a_2 & : b_2 > b_2' > b_1 > b_1' \quad b_2' & : a_3 > a_4 > a_2 > a_1 \\
   a_3 & : b_1 > b_2 > b_1' > b_2' \quad b_3 & : a_2 > a_4 > a_3 > a_1 \\
   a_4 & : b_1 > b_2 > b_2' > b_1' \quad b_3' & : a_2 > a_4 > a_3 > a_1
\end{align*}
\]

The corresponding matchings and their blocking pairs are:

| matchings | blocking pairs | classification |
|-----------|----------------|----------------|
| \(M_{1H}\) | \(a_1b_1', a_2b_1', a_3b_2', a_4b_2'\) | \(a_2b_2, a_2b_2', a_4b_1, a_4b_1'\) | illegal |
| \(M_{2H}\) | \(a_1b_2', a_2b_1, a_3b_2', a_4b_2'\) | \(a_3b_1'\) | legal |
| \(M_{3H}\) | \(a_1b_1', a_2b_1', a_3b_2', a_4b_1'\) | none | stable |
| \(M_{4H}\) | \(a_1b_2', a_2b_1, a_3b_1', a_4b_2'\) | \(a_2b_2, a_2b_2', a_3b_2', a_4b_1\) | illegal |
| \(M_{5H}\) | \(a_1b_2', a_2b_1', a_3b_1', a_4b_2'\) | \(a_2b_1', a_2b_2\) | illegal |
| \(M_{6H}\) | \(a_1b_2', a_2b_1, a_3b_1', a_4b_1'\) | \(a_3b_1'\) | illegal |

\(M_{3H}\) is the only stable matching. All other matchings except for \(M_{2H}\) are blocked by some edge in \(M_{3H}\) (underlined). Hence, one easily verifies that \(\mathcal{L} = \{M_{2H}, M_{3H}\}\), but \(\pi^{-1}(M_{2H}) \neq M\).

Before proving Theorem 3.1, we show some preliminary results. The proof of Lemma 3.5 is immediate from Lemma 4.3. However, the proof of the latter requires concepts and machinery developed later in Section 4 – hence, we postpone the proof of Lemma 3.5 to that section.

**Lemma 3.3.** Consider an instance of stable marriage problem \((G, <)\) with \(G = G(A \cup B, E)\). Let \(M, M' \in M(G)\). Call an edge \(ab \in M \cup M'\) irregular if both \(a\) and \(b\) strictly prefer \(M\) to \(M'\) or both strictly prefer \(M'\) to \(M\). Suppose \(M\) does not block \(M'\) and \(M'\) does not block \(M\). Then:
1) there are no irregular edges;
2) \(G[M\triangle M']\) is a disjoint union of singletons and cycles;
3) a node is matched in \(M\) if and only if it is matched in \(M'\).

Proof. 1) Assume \(a_1b_1\) is an irregular edge and assume wlog both endpoints strictly prefer \(M\) to \(M'\). Then \(a_1b_1 \notin M\), because otherwise \(M\) blocks \(M'\). Starting from \(i = 2\), iteratively define \(a_i = M(b_{i-1})\) and \(b_i = M'(a_i)\). Repeatedly using the assumption that \(M\) and \(M'\) do not block each other, we deduce that, for all \(i \geq 2\), \(a_i\) strictly prefers \(M'\) to \(M\), and vice versa \(b_i\) strictly prefers \(M\) to \(M'\). Moreover, \(a_i \neq \emptyset\) and \(b_i \neq \emptyset\). Since \(M, M'\) are matchings, there exists \(\ell \geq 2\) such that \(a_\ell = a_1\). Hence, \(a_1 = a_\ell\) strictly prefers \(M'\) to \(M\), a contradiction.

2) Note that the degree of each node in \(G[M\triangle M']\) is at most 2. Suppose the thesis does not hold, then \(G[M\triangle M']\) contains a path, say wlog \(a_1, b_1, a_2, b_2, \ldots\), whose endpoints have degree 1 in \(G[M\triangle M']\). Assume wlog that \(a_1b_1 \in M'\). Since \(a_1\) is unmatched in \(M, a_1\) strictly prefers \(M'\) to \(M\). In addition, since \(a_1b_1 \in M'\) does not block \(M\), \(b_1\) strictly prefers \(M\) to \(M'\). We can iterate and conclude, similarly to part 1), that all nodes \(a_i\) strictly prefer \(M'\) to \(M\), and vice versa all nodes \(b_i\) strictly prefer \(M\) to \(M'\). Suppose first that \(a_2b_2\) is the last edge of the path. Then \(b_2\) strictly prefers \(M'\) to \(M\) as \(a_2b_2 \in M'\) and \(M(b_2) = \emptyset\), a contradiction. Similarly, if the last edge is \(b_2a_{k+1}\), \(a_{k+1}\) strictly prefers \(M\) as \(a_{k+1}\) is unmatched in \(M'\), again a contradiction.

3) Immediately from 2).

Lemma 3.4. Let \((G(A \cup B, E), <, q)\) be an instance of the stable assignment problem. Let \(M, M' \in \mathcal{M}(G, q)\) be such that \(M\) does not block \(M'\), and \(M'\) does not block \(M\). Fix \(a \in A\) that is matched in \(M\). Then \(a\) is matched in \(M'\). Let therefore \(b = M(a), \overline{b} = M'(a)\). If \(b >_a \overline{b}\), then there exists \(\pi \in M(\overline{b})\) such that \(a >_b \pi\) and \(\overline{b} >_a M'(\pi)\).

Proof. We first claim that \(M_H'\) does not block \(M_H\), and \(M_H\) does not block \(M_H'\), where \(M_H = \pi(M)\) and \(\pi\) is the mapping defined in Section 2. It then follows from Lemma 3.3, part 3) and the definition of mapping \(\pi\) that \(a\) is matched in \(M'\). It is enough to show that \(M_H'\) does not block \(M_H\). Assume by contradiction that there exists \(w^i \in M_H'\) that blocks \(M_H\). That means \(b^i \succ \pi M_H(\pi)\) and \(\pi \succ b^i M_H(\pi)\). If \(M_H(\pi) = b^i\) where \(b^i \neq \emptyset\), then \(b >_\pi M(\overline{b})\) and \(w^i \succ \pi M(\overline{b})\). If \(w^i = \emptyset\) blocks \(M\), a contradiction. So assume \(M_H(\pi) = b^j\) for some \(j \in \{q\}\). Since \(b^j \succ \pi b^j\), we have \(j > i\) by construction of \((H_G, <_G)\). Then by the definition of \(\pi\), \(M_H(b^j) >_b M_H(b^i) = \pi\), a contradiction.

Let \(\overline{b}^i := M_H'(a)\) and \(b^i := M_H(a)\). By Lemma 3.3, part 2), there exists a cycle \(C = a, b^i, \ldots, \overline{b}^i \in H_G[M_H'\triangle M_H]\), and this cycle has no irregular edges. Since \(b^i >_a \overline{b}^i\), (i) all nodes from \(A \cap C\) strictly prefer \(M_H\) to \(M_H'\), and vice-versa (ii) all nodes from \(B_H \cap C\) strictly prefer \(M_H'\) to \(M_H\). Recall that \(B_H\) is the collection of seats in the reduced instance \((H_G, <_G)\). Let \(\overline{b}^i \in C\) be such that all nodes of \(C \cap B_H\) that precede \(\overline{b}^i\) in \(C\) are not seats of \(\overline{b}\), while all nodes that follow \(\overline{b}^i\) in \(C \cap B_H\) are seats of \(\overline{b}\). Note that \(\overline{b}^i\) is well-defined, since \(C\) terminates with \(\overline{b}\) (hence possibly \(j = i\)). Let \(\overline{\pi} := M_H(\overline{b}^i)\), i.e. \(C = a, b^i, \ldots, \pi, \overline{b}^i, \ldots, \overline{b}\). By (i) above, \(\overline{b}^i = M_H(\overline{\pi}) >_\pi M_H(\overline{\pi})\), hence \(\overline{b}^i >_\pi M'(\overline{\pi})\), as required. Moreover, \(\pi = M_H(\overline{b}^i) \leq \pi, M_H'(\overline{b}^i) \leq \pi, M_H'(\overline{b}^i) = \pi\) (where the strict preference follows from (ii) and the non-strict one from the definition of mapping \(\pi\)), hence \(a >_\pi \overline{a}\), as required.

Lemma 3.5. Let \((G, <, q)\) be an instance of the stable assignment problem. Let \(ab\) be an edge of \(G\) that is not in any stable assignment. Assume that there exist stable assignments \(M_1\) and \(M_2\) such that \(M_1(a) >_a b\) and \(b >_a M_2(a)\). Then, there exists a stable assignment \(M'\) such that \(M'(a) >_a b\) and \(a' >_a a\) for all \(a' \in M'(b)\).

The previous facts on stable assignments will prove useful for the proof of Theorem 3.1. Lemmas 3.6 and 3.10 mirror similar ones that already appeared in [24], but we prove them here within our framework to make our treatment self-contained. Define \(L^0 = \mathcal{S}\) and iteratively \(L^i = L(L^{i-1})\) for \(i \in \mathbb{N}\).

Lemma 3.6. There exists \(k \in \mathbb{N}\) such that \(L^k = L^{k+1}\).

Proof. For \(i \in \mathbb{N} \cup \{0\}\), let \(I^i = I(L^i)\). We show by induction on \(i\) that \(L^i \subseteq L^{i+1}\), which concludes the proof. Clearly \(L^0 = \mathcal{S} \subseteq L^1\). Now fix \(i \in \mathbb{N}\). Since \(L^{i-1} \subseteq L^i\), we deduce \(I^i \supseteq I(L^{i-1})\). Hence, \(L^{i+1} = L(I^i) = M \setminus I(M \setminus I^i) \supseteq M \setminus I(M \setminus I^{i-1}) = L^i\),
where in the containment relation we use $\mathcal{I}^i \supseteq \mathcal{I}^{i-1}$ and therefore $\mathcal{I}(\mathcal{M} \setminus \mathcal{I}^i) \subseteq \mathcal{I}(\mathcal{M} \setminus \mathcal{I}^{i-1})$. 

For $k \in \mathbb{N}$ that satisfies Lemma 3.6, we let $\mathcal{L} := \mathcal{L}^k$. For $\mathcal{M}' \subseteq \mathcal{M}$, let $E(\mathcal{M}') := \cup \{M : M \in \mathcal{M}'\}$ and $\overline{E}(\mathcal{M}') := \cup \{M : M \in \mathcal{M} \setminus \mathcal{M}'\}$.

Lemma 3.7. Assume $\mathcal{M}_0 \subseteq \mathcal{M}$ satisfies $\mathcal{L}(\mathcal{M}_0) = \mathcal{M}_0$. Then $\mathcal{L}_0 = S(G', <, \mathbf{q})$, where $G' := G[\overline{E}(\mathcal{I}(\mathcal{M}_0))]$.

Proof. Suppose $M \in \mathcal{M} \setminus \mathcal{M}_0 = \mathcal{M} \setminus \mathcal{L}(\mathcal{M}_0)$. Then there is a matching $M' \in \mathcal{M} \setminus \mathcal{I}(\mathcal{M}_0)$ and an edge $e \in M'$ that blocks $M$. But then $e \in \cup \{M : M \in \mathcal{M} \setminus \mathcal{I}(\mathcal{M}_0)\} = \mathcal{E}(\mathcal{M}_0)$. Hence, $M \notin S(G', <, \mathbf{q})$. Conversely, suppose $M \in \mathcal{M} \setminus S(G', <, \mathbf{q})$. If $M \in \mathcal{M}(G', \mathbf{q})$, then $M$ is blocked by some $e \in \overline{E}(\mathcal{I}(\mathcal{M}_0))$. This means a matching in $\mathcal{M} \setminus \mathcal{I}(\mathcal{M}_0)$ blocks $M$, implying $M \notin \mathcal{L}(\mathcal{M}_0) = \mathcal{M}_0$. If $M \notin \mathcal{M}(G', \mathbf{q})$, then $M$ contains an edge that is not in $E(G')$. This implies $M \in \mathcal{I}(\mathcal{M}_0)$ and thus $M \notin \mathcal{L}(\mathcal{M}_0) = \mathcal{M}_0$. □

Lemma 3.8. $E(G_L) = \cup \{M : M \in \mathcal{L}\}$, where $G_L = G[\overline{E}(\mathcal{I}(\mathcal{L}))]$.

Proof. The containment relationship $E(G_L) \supseteq \cup \{M : M \in \mathcal{L}\}$ is clear from definition. So it suffices to show $E(G_L) \subseteq \cup \{M : M \in \mathcal{L}\}$. Assume by contradiction that there exists an edge $ab \in E(G_L) \setminus \cup \{M : M \in \mathcal{L}\}$. Let $M \in \mathcal{M} \setminus \mathcal{I}(\mathcal{L})$ be an assignment such that $ab \in M$. Let $M_0$ (resp. $M_L$) be the stable assignment output by the students (resp. schools) proposing Gale-Shapley algorithm in $G_L$. Since $\mathcal{L} = \mathcal{L}(\mathcal{L})$, we have $M_0, M_L \in \mathcal{L}$ by Lemma 3.7. By construction, $ab \notin M_0 \cup M_L$. In the following, when talking about a specific execution of the Gale-Shapley algorithm, we say that $a$ rejects $b$ if during the execution, $a$ rejects the proposal by $b$, possibly after having temporarily accepted it. We distinguish three cases.

**Case a:** $b >_a M_0(a) := \overline{a}$. By the choice of $M$, we know that $M_0$ and $M$ do not block each other. Note that this case contains all and only the edges that have been rejected by some (equivalently, any) execution of the student-proposing Gale-Shapley algorithm on $G_L$. Among all those edges, pick the one $ab$ that is last rejected by some execution of the algorithm. Apply Lemma 3.4 (with $M = M$ and $M' = M_0$) and conclude that there exists $\pi \in M(\overline{a})$ such that $a >_\pi \overline{a}$ and $b >_\pi M_0(\pi)$. $b >_\pi M_0(\pi)$ implies that $b$ rejected $\pi$ during the execution of Gale-Shapley in consideration. Hence, when $a$ proposes to $\overline{a}$, either $\overline{a}$ still has to be rejected by $b$, or it has been rejected before. In the latter case, $\overline{a}$ has her quota filled and rejects some other student when $a$ proposes to $\overline{a}$. Hence the following events happen in this order during the execution of Gale-Shapley: $a$ is rejected by $b$; $a$ proposes to $\overline{a}$; $\overline{a}$ rejects a student. This contradicts our assumption that $ab$ is the last rejected edge.

**Case b:** $M_0(a) >_a b >_a M_L(a)$. By Lemma 3.5, there exists a stable assignment $M'$ such that $M'(a) >_a b$ and $a' >_a a$ for all $a' \in M'(b)$. Again by choice of $M$, $M$ and $M'$ do not block each other. We can therefore apply Lemma 3.4 (with the roles of $M$ and $M'$ inverted) and conclude that there exists $\overline{a} \in M'(b)$ with $a >_\pi \overline{a}$, a contradiction.

**Case c:** $M_L(a) >_a b$. Using Lemma 3.4 (with $M = M_L$ and $M' = M$) we deduce that there exists $\overline{a} \in M_L(b)$ such that $a >_\pi \overline{a}$ and $b >_\pi M(\overline{a}) = \overline{b}$. Hence, in some (equivalently, any) iteration of the school-proposing Gale-Shapley algorithm, $a$ rejects $b$. Since this is the last case that still needs to be considered, we may assume edges $E(G_L) \setminus \cup \{M : M \in \mathcal{L}\}$ are exactly those rejected by some execution of the school-proposing Gale-Shapley algorithm. Among all such edges, take $ab$ that is the last rejected by some execution. Applying Lemma 3.4 again (with $a = \overline{a}$, $M = M_L$, $M' = M$), we know $\pi >_\pi a'$ for some $a' \in M_L(\overline{a})$. This implies that $\pi$ rejected $\overline{a}$ during the execution of Gale-Shapley in consideration. Hence, when $b$ proposes to $\overline{a}$, either $\overline{a}$ still has to be rejected by $\overline{a}$, or it has been rejected before. In the latter case, when $b$ proposes to $\overline{a}$, $\overline{a}$ rejects the school it temporarily accepted. Hence, the following events happen during the considered execution in this order: $a$ rejects $b$; $b$ proposes to $\overline{a}$; $\overline{a}$ rejects a school, contradicting the choice of $ab$. □

Lemma 3.9. $\mathcal{L}$ has the legal property. That is, $\mathcal{I}(\mathcal{L}) = \mathcal{M} \setminus \mathcal{L}$.

Proof. Clearly $\mathcal{I}(\mathcal{L}) \subseteq \mathcal{M} \setminus \mathcal{L}$. Now take $M \in \mathcal{M} \setminus \mathcal{I}(\mathcal{L})$. Then $M \subseteq \cup \{M' : M' \in \mathcal{M} \setminus \mathcal{I}(\mathcal{L})\} = E(G_L)$. Hence, $M$ is an assignment of $G_L$ not blocked by any assignment from $\mathcal{L} = S(G_L, <, \mathbf{q})$, where the last equality holds by Lemma 3.7. By Lemma 3.8, $M$ is not blocked by any edge in $E(G_L)$, and we conclude that $M \in S(G_L, <, \mathbf{q}) = \mathcal{L}$. □

Because of Lemma 3.9, we say that $(\mathcal{L}, \mathcal{I} := \mathcal{M} \setminus \mathcal{L})$ is a legal partition of $\mathcal{M}$.

Lemma 3.10. $\mathcal{L}$ is the unique subset of $\mathcal{M}$ with the legal property.
Proof. Assume by contradiction that there exists a set $L' \subseteq M$, $L' \neq L$ with the legal property. Let $I' := M \setminus L'$. If $L \subseteq L'$, we must have $I' \subseteq I$. Take any $M \in I \setminus I'$, it must be blocked by some assignment in $L \subseteq L'$. But we also have $M \in L'$, which contradicts the assumption that $L'$ has the legal property. Similarly, we cannot have $L' \subseteq L$. Thus, sets $A := \{M : M \in I \cap I'\}$ and $B := \{M : M \in L \cap I'\}$ are both non-empty. In addition, let $C := L \cap L'$. It is also non-empty because all stable assignments are contained in any set with the legal property. In particular, $L^0 \subseteq C$. Note that every assignment in $B$ is blocked by some assignment from $A$. Moreover, $(\dagger)$ no assignments from $A \cup B$ can be blocked by any assignments from $C$. Now take the first $i \in \mathbb{N}$ such that $L' \cap B \neq \emptyset$, and note that $i \geq 1$. Let $M \in L' \cap B$. All assignments blocking $M$ must be contained in $I(L'^{-i})$. Thus, we can pick $M' \in I(L'^{-i}) \cap A$. Hence, $M'$ is blocked by some assignment in $L'^{-i-1} \subseteq C$ (containment relation due to the choice of $i$), contradicting $(\dagger)$. \qed

Proof of Theorem 3.1. Immediately from Lemmas 3.7, 3.8, 3.9, and 3.10. \qed

Since a graph may have exponentially many assignments, we cannot efficiently deduce graph $G_L$ by explicitly keeping track of $L^0, L^1, \cdots$, etc. In Section 6, we will show an $O(|E|)$ algorithm for computing $G_L$. Some machinery for this algorithm is developed in Section 4 and Section 5.

We then introduce the classical concept of rotations in stable marriages and investigate its extension to stable assignments.

\section{Lattice and Rotations}

In this section, we first present the known lattice structure associated to stable assignments. We then introduce and investigate an extension of the classical concept of rotations from stable marriage (see e.g. [15]) to stable assignment. Due to the natural asymmetry of the problem, we distinguish between school- and student-rotations, treated in Section 4.1 and 4.2, respectively. For the sake of readability, we keep technical details to the minimum, postponing most of them to the appendix, which also includes a treatment of rotations in the marriage case. Even though definitions we introduce do not explicitly rely on the latter, proofs do so extensively, and are therefore also deferred to the appendix. The main take-home message of this section is the following. Informally speaking, a rotation exposed in a stable matching $M$ is a certain $M$-alternating cycle $C$ such that $M \triangle C$ is a stable matching. We show in this section that rotations in the stable marriage instance $(H_G, <_G)$ associated to a stable assignment instance $(G, <, \mathbf{q})$ (see Section 2.3) behave in a very structured manner. Indeed, constructing $M \triangle C$ from $M \in (H_G, <_G)$ will have the following effect on $M_H' \in S(G, <, \mathbf{q})$: each school $b$ will either not change its assigned students, or replace its least preferred student only. This allows us to define an extension of rotations directly on $(G, <, \mathbf{q})$, and show that it inherits many properties of rotations in the marriage setting.

Throughout the section, fix a stable assignment instance $(G, <, \mathbf{q})$, with $G = (A \cup B, E)$. We say a pair $ab \in A \times B$ is stable if there exists a stable assignment where $a$ is assigned to $b$. Given $M, M' \in M(G, <, \mathbf{q})$, we say $M$ dominates $M'$ (and write $M \succeq M'$) if for every student $a \in A$, $M(a) \succeq M'(a)$. If moreover $M \neq M'$, we say that $M$ strictly dominates $M'$ and write $M \succ M'$. The following fact is well-known (see e.g. [15]).

\begin{theorem}
$S(G, <, \mathbf{q})$ endowed with the dominance relation $\succeq$ forms a distributive lattice. The stable assignment $M_0$ (resp. $M_z$) such that $M_0 \succeq M$ (resp. $M \succeq M_z$) for all $M \in S(G, <, \mathbf{q})$ is called the student-optimal (resp. school-optimal) stable assignment. Moreover, if $M, M' \in S(G, <, \mathbf{q})$ and $M \succeq M'$, then for every school $b \in B$, $a' \succ_b a$ for all $a \in M(b) \setminus M'(b)$ and $a' \in M'(b) \setminus M(b)$.
\end{theorem}

Note that the student-optimal (resp. school-optimal) stable assignment coincides with the one output by the Gale-Shapley algorithm with students (resp. schools) proposing, as described in Theorem 2.1 (hence, the notation describing those assignments coincide).

\subsection{Student-rotations}

Let $M \in \mathcal{M}(G, \mathbf{q})$. For a student $a$, define $s_M(a)$ to be the first school $b \neq M(a)$ on $a$’s preference list such that $a \succ_b a'$ for some $a' \in M(b)$. Note that if $M \in S(G, <, \mathbf{q})$, we must have $M(a) >_a b$. Also note that such $b$ might not exist. If $s_M(a)$ exists, define $next_M(a)$ as the least preferred student $s_M(a)$ is assigned to.
in $M$, i.e. $\text{next}_M(a) \in M(s_M(a))$ and for all $a'' \in M(s_M(a))$, $a'' \geq_b \text{next}_M(a)$. Note that if $s_M(a)$ has no assigned students then $\text{next}_M(a) = \emptyset$.

Given distinct $a_0, \ldots, a_{r-1} \in A$ and $b_0, \ldots, b_{r-1} \in B$, a cycle $b_0, a_0, b_1, a_1, \ldots, b_{r-1}, a_{r-1}$ of $G$ is a student-rotation exposed in $M$ if $a_i b_i \in M$, and $s_M(a_i) = b_{i+1}$ for all $i = 0, \ldots, r-1$ (hence $a_{i+1} = \text{next}_M(a_i)$ – indices are taken modulo $r$). Note that student-rotations exposed in $M$ are in one-to-one correspondence with directed cycles in the student-rotation digraph $D_A$, with vertex set $A \cup \{\emptyset\}$ and arcs $\{(a, \text{next}_M(a)) : a \in A\}$. Given $M \in S(G, <, q)$, and a rotation $\rho = b_0, a_0, \ldots, b_{r-1}, a_{r-1}$ exposed in $M$, define $M' = M/\rho$ to be the assignment where $M'(a) = M(a)$ for all $a \in M \setminus \rho$, and $M'(a) = \{b_{i+1}\}$ if $a = a_i \in \rho$ (indices again are taken modulo $r$). This mapping of $M$ to $M/\rho$ is called the elimination of $\rho$ from $M$. The following lemmas extend classical results on rotations in the marriage setting to the stable assignment setting.

**Lemma 4.2.** Let $M \in S(G, <, q)$, $\rho$ be a rotation exposed in $M$, and $M' = M/\rho$. Then $M' \in S(G, <, q)$ and $M \succ M'$. Moreover, every stable assignment can be generated by a sequence of student-rotation eliminations, starting from the student-optimal stable assignment, and every such sequence contains the same student-rotations.

**Lemma 4.3.** Let $a \in A, b \in B$. A pair $ab$ is stable if and only if $a$ is assigned to $b$ in the school-optimal stable assignment or, for some student-rotation $b_0, a_0, b_1, a_1, \ldots, b_{r-1}, a_{r-1}$ and some $i \in \{0, \ldots, r-1\}$, we have $a = a_i$ and $b = b_i$. Equivalently, a pair $ab$ is stable if and only if it is a pair in the student-optimal stable assignment or for some student-rotation $b_0, a_0, b_1, a_1, \ldots, b_{r-1}, a_{r-1}$ and some $i$, we have $a = a_i$ and $b = b_{i+1}$.

We can now give the proof of Lemma 3.5.

**Proof of Lemma 3.5.** By Lemma 4.3, there exists a stable assignment $M'$ and a student-rotation $\rho = b_1, a, b_2, b_3, \ldots$ exposed in $M'$ such that $b_1 >_a b >_a b_2$. By definition of student-rotation, we have $ab_1 \in M'$ and $a' >_b a$ for all $a' \in M'(b)$. □

### 4.2 School-rotations

For $M \in M(G, q)$ and $b \in B$, let $s_M(b)$ be the first student $a$ on $b$’s preference list satisfying $b >_a M(a)$. Note that if $M \in S(G, <, q)$, we must have $a' >_b a$ for all $a' \in M(b)$. If $s_M(b)$ exists, let also $\text{next}_M(b) := M(s_M(b))$. Note that it is possible to have $\text{next}_M(b) = \emptyset$. Also note that, unlike in student-rotations, $s$ and $\text{next}$ are defined over nodes of $B$. Hence, no confusion can arise with the notation.

Given distinct $a_0, \ldots, a_{r-1} \in A$ and $b_0, \ldots, b_{r-1} \in B$, a cycle $a_0, b_0, a_1, b_1, \ldots, a_{r-1}, b_{r-1}$ of $G$ is a school-rotation exposed in $M$ if $a_i b_i \in M$ and $s_M(b_i) = a_{i+1}$ for all $i = 0, \ldots, r-1$ (indices are again taken modulo $r$). Define the school-rotation digraph $D_B$ with vertices $A \cup B \cup \{\emptyset\}$ and arcs $(a, b)$ and $(a, b')$ if $s_M(b) = a$ and $M(a) = b'$ (hence $\text{next}_M(b) = b'$). School-rotations are in one-to-one correspondence with directed cycles in $D_B$. Note the asymmetry between the definition of $D_B$ and that of $D_A$, whose vertices are all and only the nodes in $A$. We include nodes of $A$ in $D_B$ to keep track of $s_M(b_i)$, which is otherwise not immediately deducible from arcs in $D_B$. The elimination of rotation $\rho$ maps $M$ to the assignment $M' = M/\rho$ with $M'(a) = M(a)$ for $a \in A \setminus \rho$ and $M'(a_i) = b_{i-1}$ for $a = a_i \in \rho$. Moreover, note that if $b, a = s_M(b)$ belong to a rotation $\rho$ exposed at $M$, then $a$ is the least preferred student of $b$ in $M/\rho$.

When it is clear whether we are referring to the rotation digraph associated with students or schools, we will simply say “rotation digraph”, dropping the prefix.

**Lemma 4.4.** Let $M \in S(G, <, q)$. If there is a school-rotation $\rho$ exposed in $M$, then $M' = M/\rho \in S(G, <, q)$ and $M \prec M'$. Conversely, if there is no school-rotation exposed in $M$, $M$ is the student-optimal stable assignment.

For an instance $(G, <, q)$, we denote by $R(G, <, q)$ the set of its student-rotations, and by $\mathcal{SR}(G, <, q)$ the set of its school-rotations.

**Lemma 4.5.** $|R(G, <, q)| = |\mathcal{SR}(G, <, q)|$. There is a bijection $\sigma : R(G, <, q) \to \mathcal{SR}(G, <, q)$ such that for each $M \in S(G, <, q)$, $\rho \in R(G, <, q)$, we have $M = (M/\rho)/\sigma(\rho)$. 

9
5 Algorithms for Student- and School-Optimal Legal Assignments

This section is devoted to present fast algorithms for finding the student-optimal and the school-optimal legal assignments, whose existence is implied by Theorem 3.1 and Theorem 4.1.

Throughout the section, we again fix a stable assignment instance \((G, \prec, q)\), with \(G = (A \cup B, E)\). Let \(G_L\) be as defined in Theorem 3.1. We say an edge \(e \in E\) is legal if \(e \in E(G_L)\), or illegal otherwise. We denote by \(L(G, \prec, q)\) to be the set of legal assignments of instance \((G, \prec, q)\) and by \(M^L_0\) (resp. \(M^L_{\pi}\)) the student-optimal (resp. school-optimal) legal assignment.

We first introduce two algorithms, \textit{rotate-remove} and \textit{reverse rotate-remove}. We then show in Section 5.2 that they output the school- and student-optimal legal assignments, respectively. Lastly, in Section 5.3, we give an efficient implementation of the algorithms that shows they run in time complexity \(O(|E|)\).

5.1 The Rotate-Remove and Reverse Rotate-Remove Algorithms

The key idea of \textit{rotate-remove} relies on the following two lemmas, which enables us to identify illegal edges and to eliminate them without changing the set of legal assignments. Note that \(\emptyset\) is always a sink in the rotation digraph, although it may be an isolated vertex.

Lemma 5.1. Let \(M \in S(G, \prec, q)\). If \(a\) is a sink in the student-rotation digraph \(D_A\) of \(M\) and \((a', a) \in A(D_A)\), then \(ab\) is an illegal edge where \(b = s_M(a')\).

\textit{Proof.} By Theorem 3.1, \(M \in S(G, \prec, q) \subseteq L(G, \prec, q) = S(G_L, \prec, q)\). In \((G_L, \prec, q)\), consider any sequence of student-rotations, \(\rho_1, \rho_2, \cdots, \rho_k\), whose elimination from \(M\) gives the school-optimal legal assignment \(M^L_\pi\). The existence of such sequence follows from Lemma 4.2. Let \(M^t = M/\rho_1/\cdots/\rho_k\) with \(M^0 = M\). If \(a = \emptyset\), by definition of student-rotations, we have \(b \notin \rho_i\) for all \(i \in [k]\). Now consider the case where \(a \neq \emptyset\). Since \(M^t \supseteq M^i\) for all \(i \leq j\), by Theorem 4.1, we have \(s_{M^i}(a) = \emptyset\) in \((G_L, \prec, q)\) for all \(i = 0, 1, \cdots, k\). We again have \(b \notin \rho_i\) for all \(i \in [k]\). Thus, we deduce \(M(b) = M^L_\pi(b)\). Now assume by contradiction that \(a'b \in M'\) for some legal assignment \(M'\). Note that \(M\) and \(M'\) do not block each other given that both are legal assignments. Moreover, since \(M(a') >_w b = M'(a')\) by construction of \(D_A\), we can apply Lemma 3.4 (with \(a = a', b = b\)) and conclude that there exist \(\pi \in M(b)\) such that \(b >_\pi M'(\pi)\). However, \(M' \geq M^L_\pi\) implies \(M'(\pi) \geq M^L_\pi(\pi)\) and \(\pi \in M(b)\) implies \(M^L_\pi(\pi) = M(\pi) = b\). Hence, \(M'(\pi) \geq b\), a contradiction. \(\square\)

Lemma 5.2. Let \(e\) be an illegal edge of \((G, \prec, q)\), and \(\tilde{G} = G[E(G) \setminus \{e\}]\). Then \(L = L(\tilde{G}, \prec, q)\).

\textit{Proof.} Let \(M' = \{M \in \mathcal{M} : e \in M\}\), and we know \(M' \subseteq \mathcal{M} \setminus L\) by Theorem 4.1. Also let \(\tilde{M} := \mathcal{M}(\tilde{G}, \prec, q)\), and note that \(\tilde{M} = \mathcal{M} \setminus M'\). It is clear that \((L, \tilde{M} \setminus L)\) is a legal partition of \(\tilde{M}\). By Lemma 3.10 on the uniqueness of the legal partition, we have \(L(\tilde{G}, \prec, q) = L\). \(\square\)

Iteration \(i\) of the \textit{rotate-remove} algorithm starts with a subgraph \(G^i\) of \(G\) and \(M_i \in S(G^i, \prec, q)\), with \(G^0 = G\) and \(M_0 = M^L_\pi\). Let \(D_{A^i}\) be the student-rotation digraph associated with \(M_i\) in \((G^i, \prec, q)\). During each iteration, the algorithm either identifies an exposed student-rotation from \(D_{A^i}\) and eliminates it from \(M_i\), or finds an illegal edge, by locating a sink in \(D_{A^i}\), and removes it as to produce a new instance \((G^{i+1}, \prec, q)\). If both cases are present, we are free to choose between the two. When neither of these can be found, the algorithm outputs the current assignment. We give a formal description in Algorithm 1.

\textit{Reverse rotate-remove} is very similar to \textit{rotate-remove} but deals with school-rotations and the school-rotation digraph instead. Recall that the school-rotation digraph has nodes corresponding to both students and schools, unlike the student-rotation digraph whose nodes are students only. We give a “school version” of Lemma 5.1 below. Its proof, very similar to that of Lemma 5.1, is omitted. Algorithm 2 presents a formal description of \textit{reverse rotate-remove}. In Example 5.4, we show an application of both algorithms.

Lemma 5.3. Let \(M \in S(G, \prec, q)\). If \(b \in B\) is a sink in the school-rotation digraph \(D_B\) of \(M\) and \((b', a), (a, b) \in A(D_B)\). Then \(ab'\) is an illegal edge.
Algorithm 1 \textit{rotate-remove} for school-optimal legal assignment

\textbf{Require:} \((G(A \cup B, E), <, q)\)
1: Let \(G^0 = G\).
2: Find the school-optimal stable assignment \(M_0\) of \((G^0, <, q)\).
3: Set \(i = 0\) and set \(D_{A^i}\) to be the rotation digraph of \(M_0\) in \((G^0, <, q)\).
4: \textbf{while} \(D_{A^i}\) still has an arc \textbf{do}
5: Find (i) a cycle \(C^i\) of \(D_{A^i}\) or (ii) an arc \((a', a) \in A(D_{A^i})\) where \(a\) is a sink in \(D_{A^i}\).
6: \textbf{if} (i) is found \textbf{then}
7: Let \(\rho^i\) be the corresponding student-rotation. Set \(M_{i+1} = M_i/\rho^i\), and \(G^{i+1} = G^i\).
8: \textbf{else if} (ii) is found \textbf{then}
9: Define \(G_{i+1}\) from \(G_i\) by removing \(a'_{M}(a')\), and set \(M_{i+1} = M_i\).
10: \textbf{end if}
11: Set \(i = i + 1\) and set \(D_{A^i}\) to be the rotation digraph of \(M_i\) in \((G^i, <, q)\).
12: \textbf{end while}
13: Output \(M_i\).

\textbf{Example 5.4.} We apply \textit{rotate-remove} and \textit{reverse rotate-remove} to the following instance with 6 students and 3 schools, where each school has a quota of 2.

\begin{align*}
    a_1 &: \quad b_2 \succ b_3 \succ b_1 \\
    a_2 &: \quad b_1 \succ b_2 \succ b_3 \\
    a_3 &: \quad b_3 \succ b_1 \succ b_2 \\
    a_4 &: \quad b_1 \succ b_2 \succ b_3 \\
    a_5 &: \quad b_3 \succ b_2 \succ b_1 \\
    a_6 &: \quad b_1 \succ b_3 \succ b_2
\end{align*}

\begin{align*}
    b_1 &: \quad a_4 \succ a_3 \succ a_5 \succ a_2 \succ a_6 \\
    b_2 &: \quad a_3 \succ a_2 \succ a_6 \succ a_1 \succ a_4 \\
    b_3 &: \quad a_6 \succ a_1 \succ a_5 \succ a_2 \succ a_4 \succ a_3
\end{align*}

The stable student- and school-optimal stable assignments coincide, and are given by \{\(a_1b_2, a_2b_2, a_3b_1, a_4b_1, a_5b_3, a_6b_3\)\} (squared entries above).

\textbf{Rotate-remove}. On \(a_1\)'s preference list, \(b_3\) is the first school after \(M_0(a_1)\). In addition, \(b_3\) prefers \(a_1\) to \(a_5\), who is \(b_3\)'s least preferred student among \(M_0(b_3)\). Thus, \(s_{M_0}(a_1) = b_3\) and \(next_{M_0}(a_1) = a_5\). After working out \(next_{M_0}(\cdot)\) of all the students, we have the rotation digraph \(D_{A^0}\) for the first iteration of \textit{rotate-remove}:

\begin{center}
\begin{tikzpicture}
  \node (a3) at (0,0) {$a_3$};
  \node (a1) at (1,1) {$a_1$};
  \node (a5) at (2,0) {$a_5$};
  \node (a2) at (3,1) {$a_2$};
  \node (a4) at (4,0) {$a_4$};
  \node (a6) at (0,-1) {$a_6$};
  \draw[->] (a3) -- (a1);
  \draw[->] (a1) -- (a5);
  \draw[->] (a5) -- (a2);
  \draw[->] (a2) -- (a4);
  \draw[->] (a4) -- (a1);
  \draw[->] (a1) -- (a6);
\end{tikzpicture}
\end{center}

Here, we find a case (ii) with \(a' = a_1\) and \(a = a_5\). So we set \(M_1 = M_0\), remove \(a'M_0(a) = a_1b_3\) from the instance, and update the rotation digraph \(D_{A^1}\) for the next iteration:

\footnote{Note that here we are overloading notation, since \(M_0\) in the algorithm represents the assignment at the 0\textsuperscript{th} iteration, and should not be confused with the student-optimal stable assignment. Since most of the time we will deal with \textit{reverse-rotate-remove} this overload of notation will not appear again.}
Algorithm 2 reverse rotate-remove for student-optimal legal assignment

Require: \((G(A \cup B, E), <, q)\)

1: Let \(G^0 = G\).
2: Find the student-optimal stable assignment \(M_0\) of \((G^0, <, q)\).
3: Set \(i = 0\) and set \(D_{B^0}\) to be the rotation digraph of \(M_0\) in \((G^0, <, q)\).

4: while \(D_{B^i}\) still has an arc do
5: Find (i) a cycle \(C^i\) of \(D_{B^i}\) or (ii) arcs \((b', a)\) and \((a, b) \in A(D_{B^i})\) where \(b\) is a sink in \(D_{B^i}\).
6: if (i) is found then
7: Let \(\rho^i\) be the corresponding school-rotation. Set \(M_{i+1} = M_i/\rho^i\), and \(G^{i+1} = G^i\).
8: else if (ii) is found then
9: Define \(G^{i+1}\) from \(G^i\) by removing \(ab'\) and set \(M_{i+1} = M_i\).
10: end if
11: Set \(i = i + 1\), and set \(D_{B^i}\) to be the rotation digraph of \(M_i\) in \((G^i, <, q)\).
12: end while
13: Output \(M_i\).

Now, we have a case (i), with the corresponding student-rotation \(\rho^1 = b_2, a_1, b_1, a_3\). Eliminating \(\rho^1\) from \(M_1\), we have \(M_2 = M_1/\rho^1 = \{a_1b_1, a_2b_2, a_3b_2, a_4b_1, a_5b_3, a_6b_3\}\). In the next iteration, the rotation digraph \(D_{A^2}\) only contains sinks. Thus, the algorithm terminates and output \(M_2\) as the school-optimal legal assignment.

Reverse rotate-remove. The first student on \(b_1\)’s preference list that prefers \(b_1\) to his assigned school under \(M_0\) is \(a_2\). Thus, \(s_{M_0}(b_1) = a_2\) and \(next_{M_0}(b_1) = b_2\). After working out \(s_{M_0}(\cdot)\) and \(next_{M_0}(\cdot)\) of all the schools, we have the rotation digraph \(D_{B^0}\) for the first iteration:

Here, we find a case (ii) with \(b' = b_1, a = a_2\) and \(b = b_2\). So we set \(M_1 = M_0\), remove \(ab' = a_2b_1\) from the instance, and update the rotation digraph \(D_{B^1}\) for the next iteration:

Now, we have a case (i), with the corresponding school-rotation \(\rho^1 = a_6, b_3, a_4, b_1\). Eliminating \(\rho^1\) from \(M_1\), we have \(M_2 = M_1/\rho^1 = \{a_1b_2, a_2b_2, a_3b_3, a_4b_1, a_5b_3, a_6b_1\}\). In the next iteration, the rotation digraph \(D_{B^2}\) only contains sinks. Thus, the algorithm terminates and output \(M_2\) as the student-optimal legal assignment. 

\(\diamond\)
5.2 Correctness of the Algorithms

Theorem 5.5. Algorithm 1 (resp. Algorithm 2) finds the school-optimal (resp. student-optimal) legal assignment.

Proof. We focus on the statement for Algorithm 2, the other follows analogously. We first show, by induction on the iteration \( i \) of the algorithm, that \( M_i \in \mathcal{S}(G^i, <, q) \) and \( L(G^i, <, q) = L \). This is obvious for \( i = 0 \). Assume the claim is true for \( i - 1 \geq 0 \) and now consider iteration \( i \). If the condition at Step 6 is satisfied, then \( \rho^{i-1} \) is a school-rotation exposed in \( M_{i-1} \), and \( M_i = M_{i-1}/\rho^{i-1} \in \mathcal{S}(G^{i-1}, <, q) \) by induction and Lemma 4.4. Moreover, since \( G^i = G^{i-1}, \) we have \( S(G^i, <, q) = S(G^{i-1}, <, q) \) and \( L(G^i, <, q) = L(G^{i-1}, <, q) = L \). If conversely the condition at Step 8 is satisfied, \( M_i = M_{i-1} \) is unchanged and the edge removed from \( G^{i-1} \) is illegal by Lemma 5.3. Hence, \( M_i = M_{i-1} \in \mathcal{S}(G^{i-1}, <, q) \subseteq \mathcal{S}(G^i, <, q) \) and \( L(G^i, <, q) = L(G^{i-1}, <, q) = L \) by induction and Lemma 5.2.

In order to conclude the proof, observe that, at the end of the algorithm, the rotation digraph \( D_B^- \) only has sinks. We claim that the assignment output -- call it \( M^* \) -- is Pareto efficient for the students in the final instance \( (G^*, <, q) \), that is, there is no other assignment in \( (G^*, <, q) \), legal or not, that dominates it. Assume by contradiction that there is \( M \in \mathcal{M}(G^*, q) \) such that \( M > M^* \). Since all students (weakly) prefer \( M \) to \( M^* \), there is a student \( a \) such that \( b := M(a) >_a M^*(a) \). Then \( s_{M^*}(b) \) exists, contradicting the fact that \( b \) is a sink in \( D_B^- \) (it is possible that \( s_{M^*}(b) \neq a \), as there may be other nodes that precede \( a \) in \( b \)'s list and have the required property, but it is a contradiction regardless). Since we know that legal assignments form a lattice with respect to the partial order given by \( \geq \), \( M^* \) is the student-optimal legal assignment. \( \square \)

Note that the previous theorem in particular implies that the output of Algorithm 2 (resp. Algorithm 1) is unique, regardless of how we choose between Step 6 and Step 8 at each iteration, when multiple possibilities are present.

5.3 Time Complexity

In this section, we show how to implement Algorithm 1 and Algorithm 2 so as to run in time \( O(|E|) \). We start by observing that the Gale-Shapley algorithm for stable assignment problems can be implemented as to run with the same asymptotic time complexity as the one for stable marriage problems. This does not follow from the mapping \( \pi \) defined in Section 2, which may increase the number of vertices and edges by a factor of \(|V|\). The proof can be found in the appendix.

Lemma 5.6. The Gale-Shapley algorithm with students or schools proposing can be implemented to run in time \( O(|E|) \).

Note that reverse-remove and reverse rotate-remove algorithms seem to require the complete rotation digraphs at each step. However, this is too expensive to obtain and forbids us from achieving the same time complexity bound as in Lemma 5.6. Instead, in our implementation, we will only locally build and update the rotation digraph until a cycle or an illegal edge is found.

In Example 5.8, we show what to maintain and update throughout the iterations of Algorithm 2 for fast implementation. Those for Algorithm 1 follow analogously. Full details, including correctness and time complexity analysis, can be found in the proof of Theorem 5.7 included in the appendix. We remark here that our implementation only requires simple data structures, such as arrays and linked lists.

Theorem 5.7. Algorithm 1 and 2 can be implemented as to run in time \( O(|E|) \).

Example 5.8. Consider the following instance with 5 students and 5 schools, where each school has quota 1. The student-optimal stable assignment is \( M_0 = \{a_1b_4, a_2b_3, a_3b_2, a_4b_1, a_5b_5\} \), denoted succinctly by \( (4, 3, 2, 1, 5) \) (ordered list of school to which each student is matched).

\[
\begin{align*}
a_1 : & \ b_1 > b_3 > b_2 > b_5 > b_4 \\
a_2 : & \ b_2 > b_1 > b_3 > b_4 > b_5 \\
a_3 : & \ b_3 > b_4 > b_1 > b_2 > b_5 \\
a_4 : & \ b_4 > b_3 > b_2 > b_1 > b_5 \\
a_5 : & \ b_4 > b_3 > b_2 > b_5 > b_1 \\
b_1 : & \ a_4 > a_5 > a_3 > a_2 > a_1 \\
b_2 : & \ a_3 > a_5 > a_4 > a_1 > a_2 \\
b_3 : & \ a_2 > a_5 > a_1 > a_4 > a_3 \\
b_4 : & \ a_1 > a_5 > a_2 > a_3 > a_4 \\
b_5 : & \ a_5 > a_1 > a_2 > a_3 > a_4
\end{align*}
\]
For the fast implementation of reverse rotate-remove, at each iteration $i$, together with $M_i$, we will additionally keep:

(i) a partial list $T^i$ of sinks of $D_{Bi}$, stored as a 0/1 Boolean array of dimension $|B|$, together with a position $f$ such that $b_f$ is the first school that is not in $T^i$;
(ii) a chain $P^i$ of $D_{Bi}$, stored as a doubly-linked list, together with a Boolean array $W^i$ recording whether a school $b$ is in $P^i$;
(iii) for each $b \in B$, a position $p_b$ with the following property: in determining $s_{M_i}(b)$, one does not need to check if $b \succ_a M_i(a)$ for all $a \succeq b(p_b)$, where $b(p_b)$ is the student at position $p_b$ on $b$’s preference list.

In Table 1, we outline the updates occurred at all steps (denoted by $j$) of all iterations (denoted by $i$) during the fast execution of reverse rotate-remove. A cell is left blank if no update happens. The steps of iteration $i$ illustrate the steps in extending the chain $P^i$. $W^i$ can be easily deduced from $P^i$ and is therefore not included in the table.

| $(i,j)$ | $P^i$                  | $\{p_b\}_{b \in B}$ | $M_i$       | $T^i$ | $f$ |
|---------|------------------------|----------------------|-------------|-------|-----|
| (0.0)   | $\emptyset \rightarrow b_1$ | [1, 1, 1, 1] | (4, 3, 2, 1, 5) | $\emptyset$ | 1   |
| (0.1)   | $b_1, a_5, b_5$         | 2, 1, 1, 1   |            |       |     |
| (1.0)   | $b_1$                  | [2, 1, 1, 1, 6] | $b_5$      |       |     |
| (1.1)   | $b_1, a_3, b_2$         | [3, 1, 1, 1, 6] |            |       |     |
| (1.2)   | $b_1, a_3, b_2, a_5, b_5$ | [3, 2, 1, 1, 6] |            |       |     |
| (2.0)   | $b_1, a_3, b_2$         | [3, 3, 1, 1, 6] |            |       |     |
| (2.1)   | $b_1, a_3, b_2, a_4, (b_1)$ | [3, 3, 1, 1, 6] |            |       |     |
| (3.0)   | $\emptyset \rightarrow b_1$ | [4, 3, 1, 2, 5] |            |       |     |
| (3.1-2) | $b_1, a_2, b_3, a_5, b_5$ | [4, 3, 2, 1, 6] |            |       |     |
| (4.0)   | $b_1, a_2, b_3$         | [4, 3, 3, 2, 6] |            |       |     |
| (4.1-2) | $b_1, a_2, b_3, a_1, b_4, a_5, b_5$ | [4, 3, 3, 2, 6] |            |       |     |
| (5.0)   | $b_1, a_2, b_3, a_1, b_4$ | [4, 3, 3, 3, 6] |            |       |     |
| (5.1)   | $b_1, a_2, b_3, a_1, b_4, (a_2)$ | [4, 3, 3, 3, 6] |            |       |     |
| (6.0)   | $b_1$                  | [3, 3, 3, 3, 6] | (3, 4, 1, 2, 5) |       |     |
| (6.1-2) | $b_1, a_2, b_4, a_3, (b_1)$ | [4, 3, 3, 4, 6] |            |       |     |
| (7.0)   | $\emptyset \rightarrow b_1$ | (3, 1, 4, 2, 5) |            |       |     |
| (7.1-3) | $b_1, a_1, b_3, a_4, b_2, (a_1)$ | [5, 4, 4, 4, 6] |            |       |     |
| (8.0)   | $b_1$                  | [4, 4, 4, 4, 6] | (2, 1, 4, 3, 5) |       |     |
| (8.1-2) | $b_1, a_1, b_2, a_2, (b_1)$ | [5, 5, 4, 4, 6] |            |       |     |
| (9.0)   | $\emptyset \rightarrow b_1$ | (1, 2, 4, 3, 5) |            |       |     |
| (10.0)  | $\emptyset \rightarrow b_2$ | [6, 5, 4, 4, 6] | $b_1, b_2, b_5$ | 2     |     |
| (11.0)  | $\emptyset \rightarrow b_3$ | [6, 6, 4, 4, 6] | $b_1, b_2, b_5$ | 3     |     |
| (11.1-2)| $b_3, a_3, b_4, a_4, (b_3)$ | [6, 6, 5, 5, 6] |            |       |     |
| (12.0)  | $\emptyset \rightarrow b_3$ | (1, 2, 3, 4, 5) |            |       |     |
| (13.0)  | $\emptyset \rightarrow b_4$ | [6, 6, 6, 5, 6] | $b_1, b_2, b_3, b_5$ | 4     |     |
| (14.0)  | $\emptyset$            | [6, 6, 6, 6, 6] | $b_1, b_2, b_3, b_4, b_5$ | $\infty$ |     |

Table 1: Iterations of reverse rotate-remove of Example 5.8.
When extending the chain $P^i$, if $P^i = \emptyset$, as in (0.0) and (10.0), we add the first school not in $T^i$ to the chain, which is achieved by repeatedly checking if $b_f \in T^i$ and while so, update $f := f + 1$. If $P^i$ is non-empty with $b$ at the tail, we rely on $p_b$ to find $s_{M^i}(b)$. That is, we repeatedly update $p_b := p_b + 1$ until either $p_b > 5$ or $a_b := b(p_b)$ satisfies $b > a M_i(a)$. So $p_b$ strictly increases every time an extension happens with $b$ at the tail. The only time that $p_b$ will decrease is when $b$ points to a directed cycle, as the school $b_1$ in (5.1) and (7.3). In such case, $p_b$ is decremented by 1 after the rotation elimination, as seen in (6.0) and (8.0). This is because it is possible to have $s_{M^i+1}(b) = s_{M^i}(b)$. There are two scenarios, corresponding to Step 8 and Step 6 in Algorithm 2, where we stop extending the chain $P^i$: one is when the tail $b$ is a sink, implied by having $p_b > 5$; the other is when the additional node is already in the chain, which can be checked against $W_i$. In the latter case, such nodes are written as (node) in Table 1.

6 An $O(|E|)$ Algorithm for Computing $G_L$

We miss one more ingredient before showing how to build graph $G_L$ efficiently. Throughout the section, we fix an instance $(G, <, q)$ and abbreviate $M := M(G, q)$, $S := S(G, <, q)$, and $L := L(G, <, q)$. We start with a preliminary fact.

**Lemma 6.1.** Let $e$ be an illegal edge of $(G, <, q)$, and $\bar{G} = G[E(G) \setminus \{e\}]$. Then $\mathcal{R}(G, <, q) \subseteq \mathcal{R}(\bar{G}, <, q)$ and $\mathcal{SR}(G, <, q) \subseteq \mathcal{SR}(\bar{G}, <, q)$.

**Proof.** Fix $M \in S$. Since $S \subseteq S(\bar{G}, <, q)$, $M$ is also a stable assignment of $(\bar{G}, <, q)$. First consider any student-rotation $\rho \in \mathcal{R}(G, <, q)$ exposed in $M$. We want to show that $\rho$ is also exposed at $M$ in $(G, <, q)$. Assume $\rho = b_0, a_0, b_1, a_1, \ldots, b_r, a_r$. By Lemma 4.3, we deduce $a_i b_{i+1}, a_{i+1} b_{i+1} \in E(\bar{G})$ for all $i = 0, 1, \ldots, r - 1$. Hence, $b_{i+1} = s_M(a_i)$ and next$_M(a_i) = a_{i+1}$ hold true in $\bar{G}$ as well. We conclude that $\rho \in \mathcal{R}(\bar{G}, <, q)$. A similar argument shows $\mathcal{SR}(G, <, q) \subseteq \mathcal{SR}(\bar{G}, <, q)$. □

**Theorem 6.2.** Given an instance of the stable assignment problem $(G(V, E), <, q)$, $G_L$ can be found in time $O(|E|)$, where $G_L$ is the subgraph as defined in Theorem 3.1.

**Proof.** By Theorem 3.1 and Lemma 4.3, $E(G_L)$ is given by edges in $M_L^\sigma$, plus all pairs $a_i b_{i+1}$ for some student-rotation $\rho = b_0, a_0, \ldots, a_k \in \mathcal{R}(G_L, <, q)$. By Lemma 4.2, there exists exactly one set $\mathcal{R}_1$ of student-rotations whose elimination leads from $M^\sigma_0$ to $M_0$; one set $\mathcal{R}_2$ leading from $M_0$ to $M_z$; and one set $\mathcal{R}_3$ leading from $M_z$ to $M^\sigma_z$; and those set are disjoint and their union gives $\mathcal{R}(G_L, <, q)$. $\mathcal{R}_3$ is computed during the execution of Algorithm 1, hence in time $O(|E|)$ by Theorem 5.7. By Lemma 6.1 and Lemma 4.2, $\mathcal{R}_2$ coincide with the set $\mathcal{R}(G, <, q)$, which can be computed in time $O(|E|)$ by classical algorithms, see e.g. [15]. Algorithm 2 computes in time $O(|E|)$ (again by Theorem 5.7) the set of school-rotations $\mathcal{SR}_1$ whose sequential elimination starting from $M_0$ leads to $M^\sigma_0$. By Lemma 4.5, we can compute $\mathcal{R}_1$ from $\mathcal{SR}_1$ in time $O(|E|)$, concluding the proof. □

7 An $O(|E|)$ Algorithm for EADAM with Consent

In this section, we will formally introduce EADAM algorithm. Instead of the original version by Kesten [18], we focus on a simplified and outcome-equivalent version introduced by Tang and Yu [31]. Then in Section 7.2 we show how to modify our reverse rotate-remove algorithm to accommodate for nonconsenting students. In Section 7.3 we give a proof that our modification outputs the same assignment as Tang and Yu [31]’s simplified EADAM algorithm with any given set of consenting students. Together with Theorem 5.7, this implies the following.

**Theorem 7.1.** EADAM with consent on a stable assignment instance $(G(A \cup B, E), <, q)$ can be implemented as to run in time $O(|E|)$.

We also compare our algorithm with previous versions of EADAM algorithm through computational experiments. The theoretical advantage of reverse rotate-remove is reflected by computational results on random instances presented in Section 7.5.
7.1 Simplified EADAM

Kesten [18]'s original EADAM algorithm iteratively re-runs Gale-Shapley’s procedure after identifying and removing the last interruption caused by consenting interrupters. Informally speaking, an interrupter is a student who, by applying to school \( b \), interrupts a desirable assignment between school \( b \) and another student at no gain to himself. Removing such interruptions is crucial in neutralizing their adverse effects on the outcome. In this algorithm, Gale-Shapley algorithm is run as a subroutine in rounds: at each round, every student that is currently unassigned proposes to the next school on his preference list. We refer the reader to Kesten [18] for details.

Tang and Yu [31] offer a new perspective on Kesten [18]'s algorithm. The key concept is that of underdemanded schools for an assignment \( M \), i.e. \( b \in B \) is underdemanded if there is no student \( a \) that strictly prefers \( b \) to \( M(a) \). Tang and Yu [31] observe that, at the student-optimal stable matching \( M \), student \( a \) that is matched to underdemanded schools is not Pareto improvable. That is, if a matching \( M' \) Pareto dominates \( M \), it must be that \( M(a) = M'(a) \).

With this observation, they develop the simplified EADAM algorithm and show that it is output-equivalent to Kesten [18]'s original mechanism. The algorithm takes as input an instance \((G,\prec,q)\) and a list of consenting students. It iteratively re-runs the Gale-Shapley procedure, identifies underdemanded schools, and fixes their assignments via deletion of edges. If a non-consenting student is matched to an underdemanded school, more edges are removed from the instance in order to respect students’ priorities. A precise description is presented in Algorithm 3 and an example is given in Example 7.3. Note that, at each iteration \( i \), \( M^i \) is a stable assignment in \( G^i \). The following theorem collects some results from Tang and Yu [31], demonstrating the transparency of the consenting incentives and properties of the output. Recall that an assignment is constrained efficient if it does not violate any nonconsenting students’ priorities, but any other assignment that is weakly preferred by all students does.

**Theorem 7.2.** Under the simplified EADAM,

1. the assignment of a student does not change whether she consents or not;
2. the assignment output is Pareto efficient when all students consent and is constrained efficient otherwise;

**Example 7.3.** Each school in this example has a quota of 1. Their preference lists are given below. All students are consenting except for \( a_3 \).

\[
\begin{align*}
\text{a}_1 & : \ b_1 > b_2 > b_3 > b_4 \\
\text{a}_2 & : \ b_1 > b_2 > b_3 > b_4 \\
\text{a}_3 & : \ b_3 > b_2 > b_4 > b_1 \\
\text{a}_4 & : \ b_3 > b_1 > b_2 > b_4 \\
\text{b}_1 & : \ a_4 > a_2 > a_1 > a_3 \\
\text{b}_2 & : \ a_2 > a_3 > a_1 > a_4 \\
\text{b}_3 & : \ a_1 > a_4 > a_3 > a_2 \\
\text{b}_4 & : \ a_3 > a_1 > a_2 > a_4
\end{align*}
\]

In the first round, the student-proposing Gale-Shapley algorithm outputs the student-optimal stable assignment \( M^0 \) = \( \{a_1b_3, a_2b_2, a_3b_4, a_4b_1\} \). \( b_4 \) is the underdemanded school, so simplified EADAM settles its assignment to student \( a_3 \) by removing edges \( a_3b_3 \) and \( a_3b_2 \) from the instance, as in Step 8. However, since \( a_3 \) is not consenting, edges \( b_2a_1, b_2a_4 \), and \( b_3a_2 \) are also to be removed as in Step 10.

The second round of Gale-Shapley outputs the assignment \( M^1 \) = \( \{a_1b_3, a_2b_2, a_3b_4, a_4b_1\} \). \( b_2 \) is the underdemanded school and its assigned student \( a_2 \) is consenting. So simplified EADAM simply fixes their assignment by removing edge \( a_2b_1 \) from the instance, as in Step 8.

Running Gale-Shapley again on the updated instance, we obtain the assignment \( M^2 \) = \( \{a_1b_1, a_2b_2, a_3b_4, a_4b_3\} \). All schools are underdemanded and the algorithm terminates.

Note that using tools developed in previous sections, one can show that \( a_2b_1 \) is actually a legal edge and the resulting assignment is illegal.

7.2 Reverse Rotate-Remove with Consent

In reverse rotate-remove, the key idea is to reroute arcs that point to students who are assigned to sinks in the rotation digraph. This allows us to identify school-rotations in the underlying legalized instance.
Algorithm 3 simplified EADAM

Require: \((G(A \cup B, E), <, q)\), consenting students \(\overline{A} \subseteq A\)

1: Let \(G^0 = G\) and \(i = 0\).

2: repeat

3: Run student-proposing Gale-Shapley on \((G^i, <, q)\) to obtain assignment \(M^i\).

4: Identify underdemanded schools \(B^i\) and their students \(A^i := \cup_{b \in B} M^i(b)\).

5: Set \(G^{i+1} = G^i\).

6: for \(a \in A^i\) do

7: for \(b \in B\) such that \(ab \in E(G^{i+1})\) and \(b >_a M^i(a)\) do

8: remove edge \(ab\) from \(G^{i+1}\).

9: if \(a \notin \overline{A}\) then

10: remove edge \(a'b\) from \(G^{i+1}\) for all \(a' \in A\) such that \(a >_b a'\) and \(a'b \in E(G^i)\).

11: end if

12: end for

13: end for

14: Set \(i = i + 1\).

15: until \(B^{i-1} = B\)

16: Output \(M^{i-1}\).

Assume for instance that \((b', a), (a, b) \in A(D_B)\), and \(b\) is a sink. Upon such rerouting, \(a\)'s priority might be violated. In particular, if \(b'\) successfully participates in a school-rotation, then \(ab'\) will be a blocking pair for the new assignment. Hence, if \(a\) is not consenting, we can no longer freely reroute the arc. In fact, in order to respect \(a\)'s priority (i.e., to avoid \(ab'\) becoming a blocking pair), \(b'\) cannot be assigned to any student \(a'\) such that \(a >_{b'} a'\). This means that the arc coming out of \(b'\) cannot be rerouted to any other student, essentially marking \(b'\) a sink.

A detailed description of our algorithm is presented in Algorithm 4. As in Algorithm 2, when both case (i) and (ii) are present at Step 5 of some iteration, we are free to choose between Step 6 and Step 8. These choices do not affect the final assignment output, as shown in Lemma 7.5. A fast implementation is provided later in Section 7.4. A step-by-step application of our algorithm on the instance given in Example 7.3 is outlined in Example 7.4.

Example 7.4. Consider the instance given in Example 7.3, from the student-optimal stable assignment \(\{a_1b_3, a_2b_2, a_3b_4, a_4b_1\}\), we can construct the rotation digraph:

Since \(b_4\) is a sink, we will remove edge \(a_3b_2\) as in Step 9, in the hope of rerouting the arc coming out of \(b_2\). However, because \(a_3\) is not consenting, we have to additionally remove edges \(a_1b_2\) and \(a_4b_2\) as in Step 11. This completely removes the possibilities of rerouting, essentially marking \(b_2\) a sink, as seen in the rotation digraph of the updated instance:

Now, \(b_2\) is a sink. Since its assigned student \(a_2\) is consenting, the algorithm simply removes edge \(a_2b_1\) in Step 9, resulting in the following updated rotation digraph:
Algorithm 4 reverse rotate-remove with consent

Require: \((G(A \cup B, E), <, q)\), consenting students \(\overline{A} \subseteq A\)

1: Let \(G_0 = G\).
2: Find the student-optimal stable assignment \(M_0\) of \((G_0, <, q)\).
3: Set \(i = 0\) and set \(D_{B^i}\) to be the rotation digraph of \(M_0\) in \((G_0, <, q)\).
4: while \(D_{B^i}\) still has an arc do
5: Find (i) a cycle \(C_i\) of \(D_{B^i}\) or (ii) arcs \((b', a)\) and \((a, b) \in A(D_{B^i})\) where \(b\) is a sink in \(D_{B^i}\).
6: if (i) is found then
7: Let \(\rho^i\) be the corresponding school-rotation. Set \(M_{i+1} = M_i/\rho^i\), and \(G^{i+1} = G^i\).
8: else if (ii) is found then
9: Define \(G^{i+1}\) from \(G^i\) by removing \(ab'\), and set \(M_{i+1} = M_i\).
10: if \(a \notin \overline{A}\) then
11: Remove from \(G^{i+1}\) edges \(a'b'\) for all \(a'\) such that \(a >_{B^i} a'\).
12: end if
13: end if
14: Set \(i = i + 1\), and set \(D_{B^i}\) to be the rotation digraph of \(M_i\) in \((G^i, <, q)\).
15: end while
16: Output \(M_i\).

Now, we can eliminate the school-rotation (i.e. trading schools between \(a_1\) and \(a_4\)), and update the assignment to be \(\{a_1, b_1, a_2b_2, a_3b_4, a_4b_1\}\). After the assignment update, the new rotation digraph only contains sinks, and thus the algorithm terminates. This final assignment coincides with the assignment output from the simplified EADAM algorithm.

It is straightforward to see, from our rotation-based algorithm, that there is a clear separation between the consenting student and the students participating in a Pareto-improvement cycle. This confirms the result in Theorem 7.2, part 1 that students have no incentive not to consent.

Lemma 7.5. The output of Algorithm 4 for a given instance of the stable assignment problem is unique.

Proof. Note that the execution of Algorithm 4 is not univocally determined since, when both case (i) and case (ii) apply in Step 5 of some iteration, we are free to choose whether to enter the if clause at Step 6 or the else clause at Step 8, to enter. Let \(\mathcal{E}\) be all possible executions of the algorithm. We want to show the outputs of any two executions coincide. We call iteration \(i\) the \(i\)-th repetition of the while loop from Step 4. Hence, in iteration \(i\), the algorithm takes from the previous iteration graph \(G^{i-1}\), assignment \(M_{i-1}\) with rotation digraph \(D_{B^{i-1}}\), and creates \(G^i\), \(M_i\) and \(D_{B^i}\).

Assume by contradiction that there are two executions \(\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}\) that output two distinct assignments \(M_1\) and \(M_2\). Also assume that among all executions that output \(M_1\) and \(M_2\), \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are the ones that have the most number of initial iterations in common, where we identify an iteration by the edges removed or the rotation eliminated. That is, \(\mathcal{E}_1\) and \(\mathcal{E}_2\) coincide in the first \(i-1\) iterations, and start to diverge at iteration \(i\); and \(i\) is the latest possible iteration to diverge so that \(\mathcal{E}_1\) and \(\mathcal{E}_2\) output two distinct assignments.

For \(j \in \mathbb{N}\), let \(M^j_1\) (resp. \(M^j_2\)) and \(D^j_{B_1}\) (resp. \(D^j_{B_2}\)) be the assignment and rotation digraph created at iteration \(j\) of \(\mathcal{E}_1\) (resp. \(\mathcal{E}_2\)). Now consider the following cases.
Assume at iteration $i$ of execution $E_1$, a rotation $\rho$, corresponding to a directed cycle $C \subseteq D_{B_{i-1}}^1$, is eliminated from $M_{i-1}^1$. Since $E_1$ and $E_2$ are identical up to iteration $i - 1$, we also have $C \subseteq D_{B_{i-1}}^2$. Now in execution $E_2$, cycle $C$ remains in $D_{B_{i}}^2$, for all $j \geq i$ until $\rho$ is eliminated. This is because for all $j \geq i$, we have $s_{M_{i-1}^2}(b) = s_{M_{i}^2}(b)$ for all $b \in C$. Moreover, the termination criterion of the algorithm implies that there must be an iteration $k$ of $E_2$ where $\rho$ is eliminated. Now consider any iteration $j \in [i, k)$ of $E_2$. If an edge $a'b'$ is removed, we claim that $b' \notin C$; and if a rotation corresponding to cycle $C^j$ is eliminated, we claim that $V(C^j) \cap V(C) = \emptyset$. Both claims follow from the fact that every node $b \in B$ has outdegree at most 1 in $D_{B_{i}}^2$. Let $I^j$ denote the $j^{th}$ iteration of $E_2$. Then there is an execution $E'_2 \subseteq E$ which has iterations in order $(I^1, \ldots, I^j, I^{k-1}, I^{k}, \ldots, I^{k+1}, \ldots)$ that outputs $M_2$ but shares more initial iterations with $E_1$, a contradiction.

Assume $(b', a), (a, b) \in D_{B_{i-1}}^1$, where $b$ is a sink, and $E_1$ enters Step 8 at iteration $i$. With the same reasoning, in execution $E_2$, we have $(b', a), (a, b) \in D_{B_{i-1}}^2$ with $b$ being a sink. Moreover, $(b', a), (a, b) \in D_{B_{i}^j}$ for all iteration $j \in [i, k]$, where $k$ is the iteration of $E_2$ during which $ab'$ is removed from the instance. Again, because every node $b \in B$ has outdegree at most 1 in $D_{B_{i}^j}$, at any iteration $j \in [i, k]$, if an edge $ab$ is removed, we must have $b \neq b'$; and if a rotation corresponding to cycle $C^j$ is eliminated, we must have $b' \notin V(C^j)$. Thus, as in the previous case, we can construct another execution $E_2' \subseteq E$ from $E_2$ by bringing iteration $k$ before iteration $i$ such that $E_2'$ outputs $M_2$ but shares more common iterations with $E_1$, a contradiction.

Since there are only two possibilities for iteration $i$ of $E_1$ and we have ruled out both cases, it is impossible to have $E_1$ and $E_2$ that output different assignments. □

7.3 Equivalence Between Reverse Rotate-Remove with Consent and Simplified EADAM

The following lemmas show an interesting connection between underdemanded schools and sinks in rotation digraphs.

Lemma 7.6. Consider the school-rotation digraph $D_B$ associated with instance $(G, <, q)$ at a stable assignment $M$. $b$ is a sink in $D_B$ if and only if it is an underdemanded school at $M$.

Proof. Let $a$ be $b$’s least preferred student among $M(b)$. $b$ is a sink implies that for all students $a'$ such that $a >_b a'$, we have $M(a') >_a b$. On the other hand, stability of $M$ implies that for all $a'$ such that $a' >_b a$ and $a' \notin M(b)$, we have $M(a') >_a b$. These two cases conclude the proof for the “only if” direction. The other is clear from the construction of the rotation digraph. □

Lemma 7.7. In Algorithm 4, if $b$ is a sink in $D_{B_{j+1}}$, it remains a sink in $D_{B_{j}}$ for all $j \geq i$. Moreover, if $a \in M_i(b)$, $M_j(a) = M_{j+1}(a)$ for all $j \geq i$.

Proof. The first part follows from the observation that $M_j \succeq M_i$ for all iterations $j \geq i$. For any $a \in M_i(b)$, since $b$ is a sink in $D_{B_{j+1}}$, for all $j \geq i$, $(a, b)$ is not part of a directed cycle of $D_{B_{j+1}}$ for any $j \geq i$. Thus, the assignment of $a$ remains for all iterations $j \geq i$. □

The following theorem, together with Lemma 7.5, can be used to prove that Algorithm 4 is outcome-equivalent to the simplified EADAM, and hence outcome-equivalent to Kesten’s original EADAM itself. As shown in Example 7.3, the output may not be legal, hence we cannot rely on the structure or legal matchings here. The full proof can be found in the appendix. Here we sketch the main arguments.

Because of Lemma 7.5, it is sufficient to show that a certain sequence of iterations of reverse rotate-remove with consent leads to an output that is the same as the one from simplified EADAM. The particular sequence is as follows. In the initial iterations, we may eliminate any rotation that is found in the rotation digraph, but we can only enter the else clause in Step 8 if the sink $b$, described in case (ii) of Step 5, is a sink in $D^0$. Assume iteration $j_1$ ends with $D^{j_1}$ that has no cycles, and none of its sinks with positive indegree is a sink in $D^0$. Similarly, in the next set of iterations, we can only enter the else clause if the sink $b$ in case (ii) is a sink in $D^{j_1}$. Define $j_2, j_3, \ldots$ analogously. The proof is then concluded by showing that for all $i \geq 1$, $M_{j_i} = M^i$.

Theorem 7.8. The outputs of Algorithm 4 and Algorithm 3 coincide.
7.4 Fast Implementation of Reverse Rotate-Remove with Consent

The fast implementation is heavily based on the implementation presented in Section 5.3. Therefore, we deferred its proof of Lemma 7.10 to the appendix, but demonstrate it in Example 7.9.

**Example 7.9.** Consider the instance in Example 5.8. Assume $a_5$ is not consenting. In Table 2, we outline the updates, similar to those in Example 5.8. When school $b$ points to the nonconsenting student $a_5$ (whose partner $b_5$ is a sink) in the rotation digraph, in addition to remove $a_5$ and $b_5$ from the chain $P^i$, we also remove $b$ from $P^i$, set $T^i := T^i \cup \{b\}$, and update $p_b := 6$ in lieu of the edge removals in Step 11. Such updates can be seen in $(1.0), (2.0), (3.0)$, and $(4.0)$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
(i, j) & P^i & \{p_b\}_{b \in B} & M_i & T^i & f \\
\hline
(0.0) & \emptyset \rightarrow b_1 & [1, 1, 1, 1] & (4, 3, 2, 1, 5) & \emptyset & 1 \\
(0.1) & b_1, a_5, b_5 & [2, 1, 1, 1, 1] & & & \\
\hline
(1.0) & \emptyset \rightarrow b_2 & [6, 1, 1, 1, 6] & b_1, b_5 & 2 \\
(1.1) & b_2, a_5, b_5 & [2, 1, 1, 1, 6] & & \\
\hline
(2.0) & \emptyset \rightarrow b_3 & [6, 6, 1, 1, 6] & b_1, b_2, b_5 & 3 \\
(2.1) & b_3, a_5, b_5 & [6, 6, 2, 1, 6] & & \\
\hline
(3.0) & \emptyset \rightarrow b_4 & [6, 6, 6, 1, 6] & b_1, b_2, b_3, b_5 & 4 \\
(3.1) & b_4, a_5, b_5 & [6, 6, 6, 2, 6] & & \\
\hline
(4.0) & \emptyset & [6, 6, 6, 6, 6] & b_1, b_2, b_3, b_4, b_5 & \infty \\
\hline
\end{array}
\]

Table 2: Iterations of reverse rotate-remove of Example 7.9

**Lemma 7.10.** Algorithm 4 can be implemented as to run in time $O(|E|)$.

*Proof of Theorem 7.1.* It follows immediately from Theorem 7.8 and Lemma 7.10.

7.5 Computational Experiments

Since Gale-Shapley’s algorithm on stable assignment instances can be implemented to run in time $O(|E|)$ by Lemma 5.6, the original EADAM runs in time $O(|E|^2)$, and the simplified EADAM runs in time $O(|E||V|)$. This is because both algorithm run Gale-Shapley’s routine iteratively, and the original EADAM repeats the routine for at most $|E|$ times, whereas the simplified EADAM repeats the routine for at most $|V|$ times. We remark that although mechanism design, rather than computational complexity, is the primary interest of Kesten’s paper, computation efficiency is nevertheless crucial in putting the mechanism into practice, especially for large markets such as the New York school system. In fact, Tang and Yu [31] mention computational tractability as one of the contributions of their simplified version.

One major advantage of our reverse rotate-remove with consent is that instead of following the iterative structure of Kesten’s algorithm, we update assignment locally using the structural results (lattice structure and rotations) of stable assignments. Our algorithm runs in time $O(|E|)$ as shown in Lemma 7.10.

To further demonstrate the computational advantage of our algorithm, we randomly generated instances of varying sizes, and recorded the running time of all three algorithms. The running time of Gale-Shapley algorithm is also recorded as a benchmark. The number of students in our instances ranges from 500 to 30,000, and the corresponding number of schools ranges from 5 to 300. For each instance size, 100 instances $(G, <, q)$ are obtained by randomly generating $G$, $<$, and $q$. The quota of each schools is randomly selected between 50 and 150 uniformly. We tested scenarios where each student’s chance of consenting is 10%, 30%, 50%, 80%, and 100%. The experiments were carried out on a computing node with 1 core and 4GB RAM.

A visual representation of the running times of different algorithms can be found in Figure 1. The shaded areas are 95% confidence intervals of each algorithm for given instance sizes. Our algorithms perform
Theorem 8.1. Let \((G, <)\) be a Latin instance. Then \(G_L = G\). Moreover, let \((G, <)\) have \(n\) men and \(n\) women. Then there exists an instance \((G', <')\) with \(n + 1\) men and \(n + 1\) women such that \(|S(G', <')| = 1\) and \(L(G', <') = \{M \cup \bar{a}b : M \in S(G, <)\}\), where \(\bar{a}\) and \(b\) are the additional \((n + 1)\)th man and woman.

Benjamin et al. [6] provide, for each \(n\) that is a power of 2, a Latin instance \((G, <)\) with \(|S(G, <)| = \omega(2^n)\). Hence, there are instances \((G', <')\) with \(n\) men and \(n\) women such that \(|S(G', <')| = 1\) and \(|L(G', <')| > 2^n\). Up to a different constant in the basis, the asymptotic ratio between the quantities \(|L(G, <)|\) and \(|S(G, <)|\) cannot be increased, as it has been recently shown that there exists a constant \(c > 1\) such that each instance of the stable marriage problem with \(n\) men and women has \(O(e^n)\) stable matchings [17]. We believe that future investigations of the relationship between Latin instances and legal matchings may provide further advancement on the question of Knuth [20].

The theorem below [6] gives a necessary and sufficient condition for a matching to be stable in a Latin instance.

Theorem 8.2. Let \(M\) be a matching on Latin ranking matrix \(Q\). \(M\) is stable if and only if there do not exist row \(a\) and column \(b\) such that \(Q(M(b), b) > Q(a, b) > Q(a, M(a))\) or \(Q(M(b), b) < Q(a, b) < Q(a, M(a))\).

The following lemma shows that in a Latin instance, the set of legal matchings is exactly the set of stable matchings.

Lemma 8.3. Let \((G, <)\) be a Latin instance. Then \(G_L = G\).

Proof. Let \(M^i = \{ab : Q(a, b) = i\}\), where \(Q\) is the Latin ranking matrix of the instance. \(M^i\) is a matching by properties of Latin squares. By construction, for any row \(a\) and column \(b\), \(Q(M^i(b), b) = i = Q(a, M^i(a))\) and thus \(M^i\) must be stable by Theorem 8.2. Since \(\cup_{i \in [n]} M^i = E(G)\), by Theorem 3.1, \(G_L = G\).}

As we show next, the set of stable matchings of a Latin instance can be “masked” into the set of legal matchings of an auxiliary instance with one more man and woman, such that the auxiliary instance has only one stable matching. The construction is as follows: given a Latin instance \((G(A \cup B, E), <)\), construct an auxiliary instance \((G'(A' \cup B', E'), <')\), where \(A' = A \cup \{\bar{a}\}, B' = B \cup \{\bar{b}\}, E' = A' \times B',\) and \(<'\) is defined as follows:

(i) every \(a \in A\) ranks \(\bar{b}\) in the last position, and \(<'_a\) restricted to \(B\) is exactly \(<_a\).

(ii) \(\bar{a}\) can have arbitrary rankings of \(B'\) as long as \(\bar{b}\) is the least preferred.
(iii) every $b \in B$ ranks $\tilde{a}$ in the second place, and $<_{b}'$ restricted to $A$ is exactly $<_b$.
(iv) $\tilde{b}$ can also have arbitrary rankings of $A'$ as long as $\tilde{a}$ is ranked the first.

We first show the following facts before concluding the proof of Theorem 8.1.

**Lemma 8.4.** Given a Latin instance $(G, <)$, define $(G', <')' as above. $|S(G', <')| = 1$.

**Proof.** Let $M \in S(G', <')$. We will first show $M(\tilde{b}) = \tilde{a}$. Assume by contradiction that $M(\tilde{b}) = a$ for some $a \in A$. Let $b$ be $a$’s least preferred partner in $B$. Then $b >_{a}' \tilde{b} = M(a)$ by construction. By the symmetric nature of Latin instances, $a$ must be $b$’s most preferred partner in $A$, which means $a >_{b}' M(b)$. But then $ab$ is a blocking pair of $M$, contradicting stability. Next, we want to show every woman in $A$ is matched to her most preferred man. Assume by contradiction that the claim is not true for some $b \in B$. Then $\tilde{a} >_{b}' M(b)$. Since $b >_{a}' \tilde{b}$ by construction, $\tilde{a}b$ blocks $M$, which again contradicts stability. Hence, $S(G', <')$ contains exactly one stable matching, namely the one where every woman is matched to her most preferred man according to $<'. \square$

**Lemma 8.5.** Let $(G, <)$ and $(G', <')$ be as before. $L(G', <') = \{M \cup \tilde{a}b : M \in S(G, <)\}$

**Proof.** Let $M_0$ be the only stable matching of $(G', <')$. Since every woman in $A'$ is matched to her most preferred man in $B'$ as shown in the proof of Lemma 8.4, $M_0$ is also the woman-optimal legal matching of $L(G', <')$. In addition, since $\tilde{b}$ is the least preferred woman of every man by construction of $G'$, $\tilde{b}$ is a sink in the rotation digraph of $M_0$ and remains a sink during the execution of Algorithm 2. Thus, $\tilde{a}$ is matched $\tilde{b}$ in the man-optimal legal matching of $L(G', <')$. Thus, $\tilde{a}b \in M$ for all $M \in L(G', <')$, and according to Theorem 3.1, all edges in $E' := \{\tilde{ab} : a \in A\} \cup \{\tilde{a}b : b \in B\}$ are illegal edges. By Lemma 5.2, we have $L(G', <') = L(G[E \setminus E'], <') = \{M \cup \tilde{a}b : M \in L(G, <)\}$, where the last equality is because $E \setminus E' = E(G) \cup \{\tilde{a}b\}$. Finally, by Lemma 8.3, we have $L(G, <) = S(G, <)$ and thus, $L(G', <') = \{M \cup \tilde{a}b : M \in S(G, <)\}. \square$

**Example 8.6.** Consider the following Latin ranking matrix $Q$ and the associated instance.

|     | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|-----|-------|-------|-------|-------|
| $a_1$ | [1]   | 2     | 3     | 4     |
| $a_2$ | 2     | 1     | [4]   | 3     |
| $a_3$ | 3     | [4]   | 1     | 2     |
| $a_4$ | 4     | 3     | 2     | [1]   |

Let $M = \{a_1b_1, a_2b_3, a_3b_2, a_4b_4\}$ be the matching corresponding to the cells boxed in the Latin ranking matrix. $M$ is unstable, since $a_3b_1$ is a blocking pair. Equivalently, we can apply Theorem 8.2 on the Latin ranking matrix. Consider $a = a_3, b = b_1$, we have $Q(M(b), b) = 1 < Q(a, b) = 3 < Q(a, M(a)) = 4$, also implying that $M$ is unstable.

$(G, <)$ has 10 stable matchings. Now consider the auxiliary instance $(G', <')$, whose preference lists are given in Example 5.8. $(G', <')$ has only one stable matching, which is $\{a_1b_4, a_2b_3, a_3b_2, a_4b_1, \tilde{a}b\}$, but its legalized instance $(G_L', <_L)$ has 10 stable matchings. $\diamond$

Lemmas 8.3, 8.4 and 8.5 immediately imply Theorem 8.1.

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Figures.

(a) all students consent

(b) 80% students consent

(c) 50% students consent

(d) 30% students consent

(e) 10% students consent

Figure 1: Comparing EADAM, simplified EADAM, and reverse rotate-remove with consent in one-to-many setting, where schools have an average of 100 seats. Average run time of simplified EADAM and reverse rotate-remove with consent included for largest instance in our experiment.
| Consent Level | Run Time (min) |
|--------------|---------------|
| (a) all students consent | 2.86, 3.15, 231.61 |
| (b) 80% students consent | 2.87, 100.20 |
| (c) 50% students consent | 2.60, 2.61, 57.70 |
| (d) 30% students consent | 2.69, 2.70, 19.25 |
| (e) 10% students consent | 2.69, 2.70, 19.25 |

Figure 2: Comparing simplified EADAM (sEADAM) and reverse rotate-remove with consent (RRR) in random instances whose sizes are similar to those of the New York City school system. Run time of Gale-Shapley (GS) included as a benchmark. Each line represents one instance.
References

[1] Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

[2] Atila Abdulkadiroğlu, Parag A Pathak, and Alvin E Roth. Strategy-proofness versus efficiency in matching with indifferences: Redesigning the nyc high school match. *American Economic Review*, 99(5):1954–78, 2009.

[3] Atila Abdulkadiroğlu, Yeon-Koo Che, and Yosuke Yasuda. Expanding” choice” in school choice. *American Economic Journal: Microeconomics*, 7(1):1–42, 2015.

[4] Mustafa Oğuz Afacan, Zeynel Harun Aliogulları, and Mehmet Barlo. Sticky matching in school choice. *American Economic Review*, 100(4):1860–74, 2010.

[5] Keisuke Bando. On the existence of a strictly strong nash equilibrium under the student-optimal deferred acceptance algorithm. *Games and Economic Behavior*, 87:269–287, 2014.

[6] Arthur T Benjamin, Cherlyn Converse, and Henry A Krieger. How do i marry thee? let me count the ways. *Discrete Applied Mathematics*, 59(3):285–292, 1995.

[7] Caterina Calsamiglia, Guillaume Haeringer, and Flip Klijn. Constrained school choice: An experimental study. *American Economic Review*, 100(4):1860–74, 2010.

[8] Umut Dur, A Arda Gitmez, and O Yılmaz. School choice under partial fairness. Technical report, Tech. rep., Working paper, North Carolina State University, 2015,[19], 2015.

[9] Lars Ehlers. von neumann–morgenstern stable sets in matching problems. *Journal of Economic Theory*, 134(1):537–547, 2007.

[10] Lars Ehlers and Thayer Morrill. (il)legal assignments in school choice. Technical report, 2017.

[11] Aytek Erdil and Haluk Ergin. What’s the matter with tie-breaking? improving efficiency in school choice. *American Economic Review*, 98(3):669–89, 2008.

[12] Haluk I Ergin. Efficient resource allocation on the basis of priorities. *Econometrica*, 70(6):2489–2497, 2002.

[13] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[14] Dan Gusfield. Three fast algorithms for four problems in stable marriage. *SIAM Journal on Computing*, 16(1):111–128, 1987.

[15] Dan Gusfield and Robert W Irving. *The stable marriage problem: structure and algorithms*. MIT press, 1989.

[16] Robert W Irving, Paul Leather, and Dan Gusfield. An efficient algorithm for the optimal stable marriage. *Journal of the ACM (JACM)*, 34(3):532–543, 1987.

[17] Anna R Karlin, Shayan Oveis Gharan, and Robbie Weber. A simply exponential upper bound on the maximum number of stable matchings. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 920–925. ACM, 2018.

[18] Onur Kesten. School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348, 2010.

[19] Andrew Kloosterman and Peter Troyan. Efficient and essentially stable assignments. 2016.
[20] Donald Ervin Knuth. *Mariages stables et leurs relations avec d'autres problèmes combinatoires: introduction à l’analyse mathématique des algorithmes*. Montréal: Presses de l’Université de Montréal, 1976.

[21] Adam Kunysz, Katarzyna Paluch, and Pratik Ghosal. Characterisation of strongly stable matchings. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 107–119. Society for Industrial and Applied Mathematics, 2016.

[22] David F Manlove. The structure of stable marriage with indifference. *Discrete Applied Mathematics*, 122(1-3):167–181, 2002.

[23] David F Manlove. *Algorithmics of matching under preferences*, volume 2. World Scientific, 2013.

[24] Thayer Morrill. Which school assignments are legal? Technical report, 2016.

[25] Yusuke Narita. Match or mismatch: learning and inertia in school choice. Technical report, Working paper, 2016.

[26] Parag A Pathak and Tayfun Sönmez. School admissions reform in chicago and england: Comparing mechanisms by their vulnerability to manipulation. *American Economic Review*, 103(1):80–106, 2013.

[27] Alvin E Roth. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of political Economy*, 92(6):991–1016, 1984.

[28] Alvin E Roth and Marilda Sotomayor. Two-sided matching. *Handbook of game theory with economic applications*, 1:485–541, 1992.

[29] Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. Pairwise kidney exchange. *Journal of Economic theory*, 125(2):151–188, 2005.

[30] Tayfun Sönmez and M Utku Ünver. Matching, allocation, and exchange of discrete resources. In *Handbook of social Economics*, volume 1, pages 781–852. Elsevier, 2011.

[31] Qianfeng Tang and Jingsheng Yu. A new perspective on kesten’s school choice with consent idea. *Journal of Economic Theory*, 154:543–561, 2014.

[32] Qianfeng Tang and Yongchao Zhang. Weak stability and pareto efficiency in school choice. 2017.

[33] John Von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior: 3d Ed*. Princeton University Press, 1953.

[34] Jun Wako. A polynomial-time algorithm to find von neumann-morgenstern stable matchings in marriage games. *Algorithmica*, 58(1):188–220, 2010.

A Rotations

A.1 Basic Results for Stable Marriages

Throughout this section, fix a stable marriage instance \((G, <)\) with \(G = G(A \cup B, E)\) and write \(M := M(G)\), \(S := S(G, <)\). If \(ab \in M\) for some \(M \in S\), we say that \(a\) is a stable partner of \(b\), and that \(ab\) is a stable edge or stable pair.

Both the definition of dominance relation between matchings and Theorem 4.1 immediately specialize to the marriage case, as well as student-rotations and school-rotations. Moreover, those concepts coincide with classical ones, see e.g. [15]. In particular, we call the school-rotations woman-rotations and student-rotations just rotations. Because of the symmetric nature of men and women in stable marriage problems, a woman-rotation is a rotation in the instance obtained from \((G, <)\) by switching the sets of men and women.
Part 1,2,3 of next theorem collect results that can be found in e.g. Gusfield and Irving [15]. They are stated in terms of rotations, but immediately extend to woman-rotations. Part 4 is immediate by the symmetric role of rotations and woman-rotations.

**Theorem A.1.** Consider an instance \((G(A \cup B, E), <)\) of the stable marriage problem.

1. Let \(M \in S\). If \(M \neq M_z\), there is at least one rotation \(\rho\) exposed in \(M\), and so let \(M' = M/\rho\). Then, \(M' \in S\) and \(M \succ M'\).
2. Every stable matching can be generated by a sequence of rotation eliminations, starting from \(M_0\), and every such sequence contains the same rotations.
3. For any \(a \in A\) and \(b \in B\), there is at most one rotation \(\rho = b_0, a_0, b_1, \ldots, b_{r-1}, a_{r-1}\) such that \(a = a_i\) and \(b = b_i\), for some \(i \in \mathbb{N}\), and at most one rotation such that \(a = a_i\) and \(b = b_{i+1}\) (indices are taken modulo \(r\)).
4. There exists a bijection \(f\) between rotations and woman-rotations with the following properties: if \(\rho\) is a rotation exposed in \(M \in S\), then \(f(\rho)\) is a woman-rotation exposed in \(M/\rho \in S\), and \((M/\rho)/f(\rho) = M\); if \(\rho'\) is a woman-rotation exposed in \(M \in S\), then \(f^{-1}(\rho')\) is a rotation exposed in \(M/\rho' \in S\), and \((M/\rho')/f^{-1}(\rho') = M\).

**A.2 Extension to Stable Assignments and Proofs of Lemmas 4.2, 4.3, 4.4, and 4.5**

Recall that map \(\pi\) induces a bijection between \(S(G, <, q)\) and \(S(H_G, <, G)\) and that we use \(M_H\) to denote \(\pi(M)\) for \(M \in M(G, q)\). Moreover, this bijection is known to be dominance-preserving, i.e. for \(M, M' \in S(G, <, q)\), we have \(M_H \succeq M_H'\) if and only if \(M \succeq M'\). Next lemma shows that rotations associated to stable matchings in \((H_G, <, G)\) have a very structured manner of moving students from/to seats of schools. See Figure 3 for a visual representation.

**Lemma A.2.** Let \(M_H \in S(H_G, <, G)\).

1) Let \(a \in A\) such that \(M_H(a) = b^i\) for some \(b \in B\) and \(i \in [q_b - 1]\). Then \(s_{M_H}(a) = b^{i+1}\).

2) Let \(\rho\) be a rotation exposed in \(M_H\), and \(M_H' = M_H/\rho\). Then, for each \(b \in B\), there exists an index \(i \in [q_b + 1]\) such that \(M_H(b^i) = M_H'(b^i)\) for all \(j \in [i - 1]\), and \(M_H(b^j) = M_H'(b^{j+1})\) for all \(j = i, \ldots, q_b - 1\).

**Proof.** By construction, \(b^{i+1}\) is the seat immediately after \(b^i\) on \(a\)'s preference list and by definition of mapping \(\pi\), we know \(a \succ b^i\), \(M_H(b^{i+1})\). This proves part 1). For part 2), if no seats of \(b\) are in rotation \(\rho\), then \(i = q_b + 1\) is the required index. Hence, assume \(b^i \in \rho\) for some \(j \in [q_b]\), and let \(i\) be minimum index with this property. Then \(M_H(b^i) = M_H'(b^j)\) for all \(j \in [i - 1]\) by the choice of \(i\). Moreover, by part 1), we have that \(a_i := M_H(b^i) \neq \emptyset\) is matched to \(b^{i+1}\) in \(M_H'\). Hence, \(b^{i+1}\) \(\in \rho\). Similarly, \(a_{i+1} := M_H(b^{i+1}) \neq \emptyset\) is matched to \(b^{i+2}\) in \(M_H'\). Hence we can iterate and conclude that \(a_j := M_H(b^j) \neq \emptyset\) is matched to \(b^{j+1}\) in \(M_H\) for all \(j = i, \ldots, q_b - 1\). \(\square\)

**Lemma A.3.** There exists a bijection \(\mu\) between student-rotations in \((G, <, q)\) and rotations in \((H_G, <, G)\) such that \(\rho\) is a rotation exposed in \(M_H \in S(H_G, <, G)\) if and only if \(\mu(\rho)\) is exposed in \(M \in S(G, <, q)\). Moreover, in such case we have \((M/\mu(\rho))_H = M_H/\rho\).

**Proof.** We first define \(\mu\) and show that, for a rotation \(\rho\) in \((H_G, <, G)\) and \(M_H \in S(H_G, <, G)\) at which \(\rho\) is exposed, we have that \(\mu(\rho)\) is a rotation exposed at \(M\) and \((M/\mu(\rho))_H = M_H/\rho\). Fix \(\rho, M_H\) as above. For every school \(b\) such that \(b'_\ell \in \rho\) for some \(j \in [q_b]\), let \(a_1, \ldots, a_{q_b}\) be the students assigned to \(b_1, \ldots, b_{q_b}\) in \(M_H\). From Lemma A.2, we know \(a_j \neq \emptyset\) for all \(j \in [q_b]\) and \(s_{M_H}(a_j) = b^{j+1}\) for all \(j \in [q_b - 1]\). Thus, \(\rho\) can be represented as

\[
\rho = b_0^{i_0} a_0^{i_0}, \ldots, b_0^{q_b} a_0^{q_b}, b_1^{i_1} a_1^{i_1}, \ldots, b_1^{q_b} a_1^{q_b}, \ldots, b_{r-1}^{i_{r-1}} a_{r-1}^{i_{r-1}}, \ldots, b_{r-1}^{q_{r-1}} a_{r-1}^{q_{r-1}},
\]

where for each \(\ell, i_\ell\) is the index as in Lemma A.2. Observe that all students participating in the rotation \(\rho\), except for the ones that are least preferred by their assigned schools, are assigned to different seats of
the same school in $M_H$ and $M'_H := M_H/\rho$. Thus, $\mu(\rho) := b_0, a_0^{q_0}, b_1, a_1^{q_1}, \ldots, b_{r-1}, a_{r-1}^{q_{r-1}}$ is a student-rotation exposed in $M$ and $M/\mu(\rho) = M'$. Note moreover that $\mu(\rho)$ is a function of $\rho$ only, and not of $M_H$. We now prove that $\mu$ is injective, by showing that there do not exist distinct rotations $\rho_1, \rho_2$ of $(H_G, \prec_G)$ such that $\mu(\rho_1) = \mu(\rho_2)$. Assume such rotations exist and write $\mu(\rho_1) = \mu(\rho_2) = b, a, \ldots$. Then both $\rho_1$ and $\rho_2$ rotate $a$ from $b$, which contradicts Theorem A.1, part 3. Last, assume student-rotation $\rho'$ is exposed in $M$ for some $M \in \mathcal{S}(G,\prec, q)$. Let $\rho' = b_0, a_0^{q_0}, b_1, a_1^{q_1}, \ldots, b_{r-1}, a_{r-1}^{q_{r-1}}$. Then, one can check that

$$\rho_0 := b_0, a_0^{q_0}, \ldots, b_0^{q_0}, a_0^{q_1}, b_1, a_1^{q_1}, \ldots, b_1^{q_1}, a_1^{q_1}, \ldots, b_{r-1}^{q_{r-1}}, a_{r-1}^{q_{r-1}}$$

is a rotation exposed in $M_H$ and $\mu(\rho_0) = \rho'$, where for all $j \in [r]$, $i_j$ is the rank of $a_{j-1}^{q_{j-1}}$ in $M''(b_1) \cup \{a_{j-1}^{q_{j-1}}\}$ by the preference ordering of $b_j$. This shows that $\mu$ is surjective – hence a bijection – and the “if” part of the statement, concluding the proof.

**Proof of Lemma 4.2.** The first statement follows from Lemma A.3 and Theorem A.1, part 1 immediately. Now let $M_0$ be the student-optimal stable assignment of $(G, \prec, q)$. For any stable assignment $M \neq M_0$, we want to show that $M$ can be generated from $M_0$ through a sequence of student-rotations. Since $\pi$ is stability- and dominance-preserving, $M_{0H}$ is the man-optimal stable matching in $(H_G, \prec_G)$, and $M_H$ is a stable matching in $(H_G, \prec_G)$. By Theorem A.1, part 2, $M_H$ can be generated by a sequence of rotations from $M_{0H}$. Assume $\rho_0, \rho_1, \ldots, \rho_k$ is one of such sequences. That is, $M_H = M_{0H}/\rho_0/\rho_1/\ldots/\rho_k$. Then iteratively applying Lemma A.3, we have $M = M_0/\mu(\rho_1)/\mu(\rho_1)/\ldots/\mu(\rho_k)$. Hence, $\mu(\rho_1), \mu(\rho_1), \ldots, \mu(\rho_k)$ is the sequence of student-rotations desired. Finally, to prove all such sequences contain the same set of student-rotations, assume by contradiction that there is one sequence $\rho_0, \rho_1, \ldots, \rho_k$ whose sequential elimination from $M_0$ leads to $M$ but $\{\rho_0, \rho_1, \ldots, \rho_k\} \neq \{\mu(\rho_0), \mu(\rho_1), \ldots, \mu(\rho_k)\}$. Again we have $M_H = M_{0H}/\mu^{-1}(\rho_0)/\mu^{-1}(\rho_1)/\ldots/\mu^{-1}(\rho_k)$ by iteratively applying Lemma A.3. Since $\mu$ is a bijection, we must have $\{\mu^{-1}(\rho_0), \mu^{-1}(\rho_1), \ldots, \mu^{-1}(\rho_k)\} \neq \{\rho_0, \rho_1, \ldots, \rho_k\}$. However, this contradicts Theorem A.1, part 2, and thus all such sequences must contain the same set of student-rotations. This concludes the proof.

**Proof of Lemma 4.3.** We first prove the second half of the lemma. For the “if” part, if $a$ is assigned to $b$ in $M_0$, where $M_0$ is the student-optimal stable assignment, then $ab$ is a stable pair by definition. Assume $M$ is a stable assignment and $\rho = b_0, a_0, b_1, a_1, \ldots, b_{r-1}, a_{r-1}$ is a student-rotation exposed in $M$ with $a = a_i, b = b_{i+1}$ for some $i$. Then, from Lemma A.3, we know $M' := M/\rho$ is a stable assignment. Since $M'(a) = b$,
ab is again a stable pair. For the “only if” part, assume ab is a stable pair. That is, ab ∈ M for some stable assignment M. Pick M to be maximal w.r.t. to the dominance relation ≥, i.e., there is no other stable assignment M′ such that M′ ⊇ M and M′(a) = b. If M = M₀, we are done. So assume M ≠ M₀, then by Lemma 4.2, M can be generated by a sequence of student-rotation eliminations starting from M₀. At least one of those rotations must be ρ = a₀, a₁, b₁, a₂, b₂, ···, aᵣ₋₁, bᵣ₋₁ with a = aᵣ, b = bᵣ₊₁ for some i, otherwise M(a) ≠ b.

For the first half of the lemma, the “if” part follows from the definition of student-rotations. For the “only if” portion, assume ab is a stable pair and in particular, assume ab ∈ M for some stable assignment M. Pick M to be minimal one w.r.t. to dominance relation ≥. If M is the school-optimal stable assignment, we are done. So assume it is not. We will show that there must be a student-rotation exposed in M. Consider the reduced marriage instance (H₅, <₅). Assume by contradiction that there exists a school-rotation ρ exposed in M₅. By Lemma A.3, μ(ρ) is a student-rotation exposed in M. Now we want to show that ab is a pair in one of these student-rotations. Assume by contradiction that ab is not a pair in any student-rotations exposed in M. Let ρ′ be any one of these student-rotations, then M′ := M/ρ′ is dominated by M and ab ∈ M′, contradicting the choice of M.

We now show the school-rotation counterpart of Lemma A.3.

**Lemma A.4.** There exists a bijection μ_w between school-rotations in (G, <, q) and woman-rotations in (H₅, <₅) such that ρ is a woman-rotation exposed in M₅ ∈ S(H₅, <₅) if and only if μ_w(ρ) is exposed in M. Moreover, in such case we have (M/μ_w(ρ))H = M₅/ρ.

**Proof.** We start by defining μ_w and show that, for a woman-rotation ρ in (H₅, <₅) and M₅ ∈ S(H₅, <₅) at which ρ is exposed, we have that μ_w(ρ) is a school-rotation exposed at M and (M/μ(ρ))H = M₅/ρ. Fix ρ, M₅ as above. We will follow the same notations as in the proof of Lemma A.3. First, we show that for every seat b_j ∈ ρ with j < qρ, we have s_{M_j}(b_j) = a_j₊, where a_j₊ := H(b_j₊). Assume by contradiction that s_{M_j}(b_j) = a′ for some other a′. Then we must have a′ > b_j₊, a_j₊ > a′ M(a′), and a′ ̸∈ M(b). This implies, by construction of (H₅, <₅), that a′ > b_j₊ a_j₊₊ and b_j₊₊ > a′ M(a′), making a′ b_j₊₊ a blocking pair of M₅, contradicting stability. Thus, we can write

\[ ρ = a_0^{i_0}, b_0^{q_0}, a_1^{i_1}, b_1^{q_1}, \ldots, a_r^{i_r}, b_r^{q_r} \]

Observe that for a given school, among its assigned students that are in ρ, only the highest ranked one is assigned to a seat in a different school in M₅ := M₅/ρ. Moreover, by the definition of woman-rotations and school-rotations, we have s_{M₅}(bₗ) = s(bₗ) for all ℓ = 0, 1, ···, r − 1. Thus,

\[ μ_w(ρ) := a_0^{i_0}, b_0, a_1^{i_1}, b_1, \ldots, a_r^{i_r}, b_r \]

is exposed in M and M/μ_w(ρ) = M′. Again, note that μ_w(ρ) is a function of ρ only and not of M₅. To see that μ_w is surjective, assume ρ′ is a school-rotation exposed in an assignment M′. Then the inverse mapping μ_w⁻¹(ρ′) follows simply from the construction of μ_w defined above. Next, we will show μ_w is injective. Assume by contradiction that there exists a school-rotation ρ′ = a₀, b₀, a₁, b₁, ··· and woman-rotations ρ₁ ≠ ρ₂ such that ρ′ = μ_w(ρ₁) = μ_w(ρ₂). Then both ρ₁ and ρ₂ moves a₁ to q₀, contradicting Theorem A.1, part 3. Thus, μ_w is a bijection, concluding the proof.

**Proof of Lemma 4.4** The first part follows from Lemma A.4 and Theorem A.1, part 1. For the second part, assume that there are no school-rotations exposed in M. That means M₅ also has no exposed woman-rotations. By Theorem A.1, part 1, M₅ must be the man-optimal stable matching in (H₅, <₅). Moreover, since π is dominance preserving, M must be the student-optimal stable assignment.

**Proof of Lemma 4.5.** Let M₀ and Mₕ be the student- and school-optimal stable assignments. Consider a sequence of rotations in (H₅, <₅) to generate Mₕ from M₀. By Theorem A.1, part 2, all such sequences contain the same set of rotations, call it q. By Lemma A.3, ∪_{ρ ∈ q} μ(ρ) = R(G, <, q). Similarly, there is a sequence of woman-rotations to generate M₀ from Mₕ, and denote this set of woman-rotations by q′. Then by Lemma A.4, ∪_{ρ ∈ q′} μ(ρ) = S(R(G, <, q). Clearly |q| = |q′|, as the role of men and women is symmetric in marriage problems. By Theorem A.1, part 4, we know there is a bijection between q and q′, which together with bijection μ in Lemma A.3 and bijection μ_w in Lemma A.4 concludes the proof.
B Details of implementations

B.1 Proof of Lemma 5.6

Proof. We consider the case where students propose, the other one following in a similar fashion. Since each student proposes to each school at most once, there are $O(|E|)$ proposals. In each iteration, we keep a list of students that have schools remaining on their preference lists, and pick the first of this list at each iteration. In addition, for each student, we keep a pointer to the last school they proposed to, which gives us faster access to the school he will propose to next. Since there are $O(|E|)$ rejections, these updates take $O(|E|)$ time.

Each school $b \in B$ maintains three pieces of information throughout the iterations. They are: a Boolean array $f_b$ of size $|A|$ recording its currently assigned students; an integer $c_b$ recording the number of students it is assigned to; and the least preferred student $w_b$ it is assigned to. Now consider the iteration where $a$ proposes to $b$. If $c_b < q_b$, information update is simple and requires constant time. Assume $c_b = q_b$ where rejection happens. If $w_b > b$ a, there is nothing to update. We are left with the case where $a > b$ $w_b$. In such case, $b$ rejects $w_b$ and accepts $a$, so the update on $f_b$ is straightforward. Since the least preferred student of each school improves over the course of the algorithm, we may update $w_b$ by looking forwards on $b$'s preference list, starting from $w_b$, until reaching a student $a$ such that $t_b(a) = 1$. Thus, throughout the algorithm, all such updates require $O(\deg(b))$ time for each school $b$. All together, the algorithm runs in time $O(|E|)$. \hfill \Box

B.2 Proof of Theorem 5.7

Proof. We show details for Algorithm 2, as those for Algorithm 1 follow in a similar fashion. For simplicity, we call “school-rotations” simply “rotations” throughout the proof. We can preprocess the input in time $O(M)$, and an assignment $M$ and an assignment $M'$ to compute $B$. We propose to $w$ to $M$ and $w$ to $M'$, and if $w$ is assigned to $b$ in the former but to $a$ in the latter, we replace $w$ by $a$ and $b$ by $a$. Since each $a$ proposes to each school at most once, this replacement takes constant time. We denote by $s_M(b)$ the student at the $b$ position recording the number of neighbors $b$ has in $G$. Assume schools are sorted as $B = \{b_1, b_2, \ldots, b_M\}$.

At each iteration $i$, we keep the assignment $M_i$ as an $|A|$-dimensional array with the $k$th student position recording the school the $k$th student is assigned to; a partial list $T^i$ of sinks of $D_B$, stored as a 0/1 Boolean array of dimension $|B|$; a position $f$ such that $b_f$ is the first school that is not in $T^i$; a chain $P_i$ of $D_B$, stored as a doubly-linked list; a Boolean array $W^i$ recording whether a school $b$ is in $P^i$; and for each $b \in B$, a position $p_b$ such that, in determining $s_M(b)$, we do not need to scan $ab$ for all $a$ such that $a \geq b(p_b)$. We initialize $M_0 = M_0$, $T^0 := \emptyset$, $f := 1$, $P^0 := \emptyset$, $W^0 := \emptyset$, and $p_b$ to be the position of the least preferred student in $M_0(b)$ on $b$'s preference list for every $b \in B$. Clearly the initialization takes $O(|E|)$ time.

We start by showing, for each iteration $i$, how to update the aforementioned pieces of information through two series of operations: those underlined in the text, which require constant time, and those wave underlined. Second, we show the correctness of these updates. Lastly, we bound the running time of the algorithm by investigating the number of times we repeat each of underlined operations and the total time needed to perform wave underlined operations.

For each iteration of the while loop, we perform the following updates.

- If $P^i$ is empty, we select the first school that is not in $T^i$ and add it to $P^i$. This school can be obtained by checking if $b_f \in T^i$ and, while $b_f \in T^i$, updating $f := f + 1$. So we may assume $P^i$ is non-empty, and represented as $P^i = b_0, a_1, b_1, \ldots, a_k, b_k$.
- Within the iteration, we extend $P^i$ and simultaneously maintain $W^i$, by finding $a_{k+1} = s_M(b_k)$, $b_{k+1} = M_i(a_{k+1})$, \ldots until we reach a node $b_j$ such that either (1) $b_j$ is a sink (step 8); or (2)
next_M(b_j) = b_ℓ for some ℓ < j (step 6). In particular,

a) In finding s_M(b_k), we repeatedly update p_b := p_{b_k} + 1 until p_b > deg(b_k) (i.e., b_k is a sink and we are in case (1) above) or by scanning of b_k(p_b)b_k we deduce s_M(b_k) = b_k(p_b).

b) If s_M(b_k) is found, we check if b_{k+1} := next_M(b_k) ∈ W^i. If this happens, we are in case (2) above, otherwise we set k := k + 1, and go to a).

• In case (1), a_jb_{j-1} is removed from G^i as an illegal edge. We achieve this by setting P^{i+1} := P^i \{ a_j, b_j \}, W^{i+1} := W^i \{ b_j \}, and we update T^{i+1} := T^i \cup \{ b_j \}.

• In case (2), a school-rotation – corresponding to the directed cycle C^i = b_ℓ, · · · , b_j, a_{j+1} – is found and eliminated, as to construct M_{i+1} from M_i. We update p_{b_{j+1}} := p_{b_{j+1}} - 1 if ℓ > 0, and set P^{i+1} := P^i \{ C^i \}, W^{i+1} := W^i \{ C^i \}.

This shows that storing and updating T^i, f, P^i, W^i, \{ p_b \}_{b \in B}, together with M_i, are sufficient for the execution of the algorithm.

We will now argue about the correctness of these updates. In both cases (1) and (2), P^{i+1} is a chain of D_{B^{i+1}}, and W^{i+1}, M_{i+1} are correctly computed. Moreover, because of (\dagger), sinks of D_{B^i} are also sinks in D_{B^{i+1}}, justifying the update on T^i and f. Lastly, consider any node b whose associated position p_b is updated in this iteration. There are two scenarios. The first scenario is when looking for s_M(b), where p_b is repeatedly updated until p_b > deg(b) or until b(p_b) is added to the chain P^i. In either case, because of (\dagger) and the fact that every time p_b is updated, it is incremented only by 1, the updated p_b remains a good choice for our purpose. The second scenario is when b = b_{ℓ-1}, where p_b is updated to be p_b - 1. In this case, we found a rotation ρ with b \notin ρ and next_M(b) ∈ ρ. We carry out the decrement because it is possible to have s_M(b) = s_M(b) and thus re-scanning of s_M(b)b is required. No further decrements on p_b is needed again because of (\dagger).

Finally, we will argue about the time complexity. First note that the number of iterations is clearly bounded by the number of edges plus the number of rotations, hence by O(|E|). The number of updates on P^i, W^i and T^i in case (1) is then also O(|E|). Since f only increases, we update f := f + 1 at most O(|V|) times. The number of times we check if f_j ∈ T^i is given by the number of positive answers (proportional to the number of updates of f) plus the number of negative answers (proportional to the number of iterations), hence O(|E|). The number of times \{ p_b \}_{b \in B}, is updated is given by the number of times we update p_b := p_b + 1 (proportional to the number of edges) plus twice the number of times we update p_b := p_b - 1 (proportional to the number of rotations), hence O(|E|). We claim that we scan each edge at most once, with O(|E|) exceptions. From the update on p_b, we see that the only time an edge ab is scanned more than once is when a rotation is eliminated and b = b_{ℓ-1}, a = a_ℓ. We call this an exception. Since every rotation corresponds to at most one exception, the number of exceptions does not exceed the number of rotations, which is O(|E|).

Note that each time we check if p_b > deg(b), we either find a sink (which happens at most once per iteration), or we scan an edge (which has been shown to happen O(|E|) times). Hence, the number of times we compare p_b and deg(b) is O(|E|). In addition, the number of times we check if next_M(b) ∈ W^i is upper bounded by the number of edge scans, hence O(|E|). The number of individual entry updates when constructing M_{i+1} from M_i, P^{i+1} from P^i and W^{i+1} from W^i in case (2) is upper bounded by the number of edges in all rotations from \mathcal{SR}(G_L, <, q), which is O(|E|) from Theorem A.1, part 3 and Lemma A.4, concluding the proof.

B.3 Proof of Lemma 7.10

Proof. The implementation follows as in the proof of Theorem 5.7. The only modification regards the update of T^{i+1} in case (1) considered in the proof, which is when extending the chain P^i, we encounter a node b_j that is a sink. If a_j consents, then the update on T^{i+1} remains unchanged, which is to set T^{i+1} := T^i \cup \{ b_j \}; however, if a_j is nonconsenting, we set T^{i+1} := T^i \cup \{ b_j, b_{j-1} \}. Correctness analysis and the counting arguments used for time complexity analysis in the proof of Theorem 5.7 remain valid and can be extended to conclude the proof here.
C Proof of Outcome-Equivalence

C.1 Proof of Theorem 7.8

Proof. To distinguish between notations in Algorithm 3 and Algorithm 4, we use $M_i$ (subscripts) to represent the assignment at iteration $i$ in Algorithm 4, and $M^i$ (superscripts) for the assignment in Algorithm 3. We use $G^i$ to represent instances generated during Algorithm 3, and $H^i$ to represent instances generated during Algorithm 4. Underdemanded schools $B^i$’s are for Algorithm 3 only; and rotation digraph $D^i_{B}$ for Algorithm 4 is abbreviated by $D^i$. We call each while loop of Algorithm 3 and Algorithm 4 an iteration and a loop, respectively. Note that loop $i$ takes assignment $M_{i-1}$ from the previous loop and produce $M_i$. Similarly, iteration $i$ starts with $M^{i-1}$ and outputs $M^i$.

Because of Lemma 7.5, it suffices to show that there is a specific execution of Algorithm 4 whose output coincides with that of Algorithm 3. In particular, we consider an execution of Algorithm 4 in which loops are partitioned in consecutive batches, as follows. In the first batch, we eliminate any rotation that is found in the rotation digraph, but we can only enter the else clause in Step 8 if the sink $b$, described in case (ii) of Step 5, is a sink in $D^0$. After repeating this, we arrive at a loop $j_1$ where $D^{j_1}$ has no cycles, and none of its sinks with positive indegree is a sink in $D^0$. This is when the second batch starts. We call $j_1$ the last iteration of the first batch. Similarly, in the next batch, we can only enter the else clause if the sink $b$ in case (ii) is a sink in $D^{j_1}$, and define $j_2,j_3,...$ analogously. Let $k$ be the last iteration of Algorithm 3. In the following paragraphs, we will show by induction that for all $i = 0,1,\ldots,k$, we have $M_{j_i} = M^i$, where $j_0 = 0$, and $D^{j_i}$ is the rotation digraph of $M^i$ in $(G^i, <, \mathbf{q})$ (Note that by construction, $M^i$ is stable in $G^i$).

The base case is when $i = j_1 = 0$. The claim holds because $M^0 = M_0$ is the student-optimal stable assignment in $(G^0, <, \mathbf{q})$, and $D^0$ is the rotation digraph of $M^0$ in $(G^0, <, \mathbf{q})$ since $G^0 = H^0$. Next, assume the claim is true for all indices $t \leq i$, and we show it for $i+1$. That is, we consider the loops in the $(i+1)^{th}$ batch of Algorithm 4. As described above, in this batch, we can only enter the else clause if the sink $b$ in case (ii) is a sink in $D^{j_i}$, and those are exactly $B^i$ (i.e., underdemanded schools in $M^i$) by inductive hypothesis and by Lemma 7.6.

Fix $\tilde{b} \in B^i$. Define the following sets: $S$ is the set of $a \in A$ such that $(\tilde{b}, a) \in A(D^b)$ for some loop $h \in [j_i, j_{i+1}]$; $E^i$ (resp. $E^i_{\mathbf{q}}$) is the set of edges removed in some iteration $1,\ldots,i$ of Algorithm 3 at Step 8 (resp. Step 10). We will show the following:

(i). If $\tilde{b} \in B^i$, then $S = \emptyset$.

If $\tilde{b} \in B^i$, it is a sink in $D^{j_i}$ by Lemma 7.6 and the inductive hypothesis. Moreover, it remains a sink in $D^h$ for all $h \geq j_i$ by Lemma 7.7. Thus, $S = \emptyset$.

(ii). If $(\tilde{b}, a) \in D^\tilde{b}$ for some $\tilde{b} \in [j_i, j_{i+1}]$, then $a\tilde{b} \notin E^i_{\mathbf{q}}$.

Assume by contradiction that for some $a \in S$, $a\tilde{b} \in E^i_{\mathbf{q}}$. Then, there must be an iteration $t \leq i$ and a nonconsenting student $a'$ such that $a' >_T a$, $M^t(a') \in B^t \setminus B^{t-1}$, and $a >_{a'} M^t(a')$ in $G^i$. Hence, in Algorithm 3, both $a\tilde{b}$ and $a'\tilde{b}$ are removed when graph $G^{t+1}$ is created. Since $M^i$ is stable in $(G^i, <, \mathbf{q})$ and $a\tilde{b} \in E(G^i)$, we must have $M^i(\tilde{b}) >_T a'$. By inductive hypothesis, we know $M^t = M_{j_t}$; and by Lemmas 7.6 and 7.7, we know that school $M_{j_t}(a')$ is a sink in $D^t$, and $M_t(a') = M_{j_t}(a')$ remains a sink for all $t' \geq j_t$. Thus, there must be $j \in [j_t, j_i]$ such that $(\tilde{b}, a') \in A(D^j)$. In particular, $M_j(a')$ is a sink in $D^j$. Hence, we can wlog assume $j$ is the loop during which Algorithm 4 removes the edge $a\tilde{b}$. Since $a' >_T a$, we must have $(\tilde{b}, a') \notin A(D^t)$ for all $r \leq j$. In addition, since $a'$ is nonconsenting, Step 11 of Algorithm 4 removes edge $a\tilde{b}$ from the instance at the same loop $j$. Hence, $(\tilde{b}, a) \notin A(D^h)$ for any loop $h$, which implies $a \notin S$, a contradiction.

(iii). Let $a \in S$, and $(\tilde{b}, a) \in D^h$ for some $h \in [j_i, j_{i+1}]$. If $M_h(a) \in B^i$, then $(\tilde{b}, a)$ is not part of a directed cycle in $D^h$ for all $h' \geq j_i + 1$. If $(\tilde{b}, a) \in D^{j_i+1}$, then $M_{j_i}(a) \notin B^i$.

The first part follows from Lemma 7.6 and Lemma 7.7. The second part follows from the definition of $j_{i+1}$.

(iv). Let $a \in S$ and $(\tilde{b}, a) \in D^h$ for some $h \in [j_i, j_{i+1}]$. If $M_h(a) \notin B^i$ and $M_h \subseteq E(G^{i+1})$, then $a = s_{M_h}(\tilde{b})$ in $(G^{i+1}, <, \mathbf{q})$.

We will first show that $a\tilde{b} \in E(G^{i+1})$. If not, then we must have $a\tilde{b} \notin E^{i+1}$, as we already showed in (ii) that $a\tilde{b} \notin E^{i+1}$. However, $a\tilde{b} \notin E^{i+1}$ implies $M_{j_i}(a) = M^i(a) \in B^i$ and thus $M_h(a) \in B^i$ by Lemma 7.6 and Lemma 7.7, a contradiction. Since $a = s_{M_h}(\tilde{b})$ in $(H^h, <, \mathbf{q})$, we only need to show that $a$ is the first student, in $G^{i+1}$, on $\tilde{b}$’s preference list that prefers $\tilde{b}$ to his assigned school. Assume by contradiction that there exists

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another student $a'$ such that $a' >_b a, \overline{a'} >_{a'} M_h(a')$, and $a'\overline{b} \in E(G_i^{i+1})$ but $a'\overline{b} \notin E(H^h)$. If $a'\overline{b}$ is removed at Step 11 of Algorithm 4, then $a\overline{b}$ must also be removed at the same loop, contradicting $a = s_{M_h}(\overline{b})$. Thus, $a'\overline{b}$ must be removed at Step 9 of some loop. That is, there must exist a loop $\overline{h} < h$, where $\overline{h} \in [j, j+1)$ for some $j \leq i$, such that $(\overline{b}, a') \in A(D^\overline{h})$ and $M_{\overline{h}}(a') \in B^\overline{h}$. However, this means $a'\overline{b} \in E^{i+1} \subseteq E^{i+1}$, which contradicts the assumption that $a'\overline{b} \in E(G^{i+1})$.

We now combine (i), (ii), (iii), and (iv) for all schools in $B$. (iv) and the first part of (iii) imply that if $D^h$ has a directed cycle for some $h \in [j, j+1]$, then the directed cycle corresponds to a rotation exposed at $M_h$ in $(G_i^{i+1}, <, q)$. That is, if $M_{j_i} = M^i \in S(G_i^{i+1}, <, q)$, all assignments obtained in this batch of Algorithm 4 are stable assignments in $(G_i^{i+1}, <, q)$. To see $M^i \in S(G_i^{i+1}, <, q)$, consider the edges removed from $G_i$ to obtain $G_i^{i+1}$ in Algorithm 3. It is easy to see that none of the edges removed are in $M^i$. Thus, $M^i$ is an assignment in $G_i^{i+1}$. Moreover, since the order in preference lists remains from iteration to iteration, $M^i$ is stable in $(G_i^{i+1}, <, q)$. Hence, the last assignment obtained at this batch, $M_{j_{i+1}}$ in particular, is a stable assignment in $(G_i^{i+1}, <, q)$. Furthermore, by the second part of (iii) and (iv), we know $D^{j_{i+1}}$ is the rotation digraph of $M_{j_{i+1}}$ in $(G_i^{i+1}, <, q)$. By choice of $j_{i+1}$, $D^{j_{i+1}}$ contains no directed cycles, which means there are no exposed rotations at $M_{j_{i+1}}$ in $(G_i^{i+1}, <, q)$. Thus, $M_{j_{i+1}}$ must be the student-optimal stable assignment in $(G_i^{i+1}, <, q)$, which coincides with $M^{i+1}$. This concludes the proof of the induction.

Finally, consider the last batch of Algorithm 4. By Lemma 7.6, sinks in $D^{j_k}$ are exactly underdemanded schools in $M^k$. Moreover, by the termination criterion of Algorithm 3, all schools are underdemanded in $M^k$. Thus, $D^{j_k}$ only has sinks and Algorithm 4 may terminates with $M_{j_k}$, which is the same assignment as $M^k$. □