BPS Conditions of Supermembrane on the PP-wave

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Abstract
We study the BPS conditions in the closed supermembranes on the maximally supersymmetric pp-wave background. In particular, the 1/2 and 1/4 BPS states are discussed in detail. Moreover, we comment on the zero-modes in the invariant mass formulae of the theory.

Keywords: supermembranes, matrix theory, M-theory, pp-waves
1 Introduction

The matrix model approach \[1\] to M-theory seems a great success. In a recent progress, the pp-wave backgrounds \[2,3,4\] and Penrose limits \[5,6\] are attractive subjects. It has been shown that the Green-Schwarz (GS) string theory on the pp-wave is exactly-solvable \[7, 8, 9\]. Also, the matrix model on the eleven dimensional maximally supersymmetric pp-wave solution \[2\] has been proposed in Ref. \[10\] and further consideration has been done \[11\].

In our previous paper \[12\], we have studied the supermembrane \[13, 14,15\] on the maximally supersymmetric pp-wave background, and derived the supercharges and associated algebra with the central charges by carefully treating the surface terms by the use of the Dirac bracket procedure.

In this paper, we consider the BPS states from the superalgebra derived in our previous work. We use triangular decomposition for the supercharge matrix, and derive the BPS conditions by analyzing the rank of the supercharge matrix. In particular, the 1/2 BPS and 1/4 BPS states are investigated in detail. We also find that the zero-modes of the membrane variables do not appear in the BPS conditions.

In section 2, the BPS states are studied by investigating the rank of the supercharge matrix. In particular, we consider the cases of 1/2 and 1/4 BPS states, concretely. Section 3 is devoted to conclusions and discussions.

2 Supercharge Matrix and BPS States

In our previous work \[12\], we have studied supermembranes on the maximally supersymmetric pp-wave and derived the expressions of the supercharges \(Q^+\) and \(Q^-\). We calculated the associated superalgebra. The results are written as follows:

\[
\begin{align*}
\left\{ \frac{1}{\sqrt{2}} Q^\alpha_\gamma, \frac{1}{\sqrt{2}} (Q^-)^\alpha_\gamma \right\}_{DB}^T &= -\delta_{\alpha\beta}, \\
i \left\{ \frac{1}{\sqrt{2}} Q^+\alpha_\gamma, \frac{1}{\sqrt{2}} (Q^-)^+\alpha_\gamma \right\}_{DB}^T &= i \sum_{I=1}^{3} \left[ \left( P_0^I + \frac{\mu}{3} X_0^I \gamma_{123} \right) \gamma_I e^{-\frac{4}{5} \gamma_{123}^T} \right]_{\alpha\beta}, \\
+&i \sum_{I'=4}^{9} \left[ \left( P_0^{I'} - \frac{\mu}{6} X_0^{I'} \gamma_{123} \right) \gamma_{I'} e^{-\frac{4}{5} \gamma_{123}^T} \right]_{\alpha\beta} - i \sum_{I,J=1}^{3} \int d^2 \sigma \partial_a S^a_{IJ} \left( \gamma_{I'} e^{-\frac{4}{5} \gamma_{123}^T} \right)_{\alpha\beta} \\
-2i &\sum_{I=1}^{3} \sum_{I'=4}^{9} \int d^2 \sigma \partial_a S^a_{I'I} \left( \gamma_{I'} e^{-\frac{4}{5} \gamma_{123}^T} \right)_{\alpha\beta},
\end{align*}
\]

1
\[ i \left\{ \frac{1}{\sqrt{2}} Q_{\alpha}^+ \frac{1}{\sqrt{2}} (Q^+)_{\beta} \right\}_{DB} = 2H\delta_{\alpha\beta} \quad (2.3) \]

\[ + \frac{\mu}{3} \sum_{i,j=1}^{3} M^{ij}_0 (\gamma_{i,j}\gamma_{123})_{\alpha\beta} - \frac{\mu}{6} \sum_{i',j'=4}^{9} M^{i'j'}_0 (\gamma_{i',j'}\gamma_{123})_{\alpha\beta} \]

\[-2 \sum_{i=1}^{3} \int d^2 \sigma \varphi X_i (\gamma')_{\alpha\beta} - 2 \sum_{i'=4}^{9} \int d^2 \sigma \varphi X_{i'} (\gamma' e^{\frac{i'}{6}\gamma_{123}})_{\alpha\beta} \]

\[+2 \sum_{i=1}^{3} \int d^2 \sigma \partial_a S_i^\alpha (\gamma')_{\alpha\beta} + 2 \sum_{i'=4}^{9} \int d^2 \sigma \partial_a S_i^a (\gamma' e^{\frac{i'}{6}\gamma_{123}})_{\alpha\beta} \]

\[+2 \sum_{i,j=1}^{3} \sum_{i',j'=4}^{9} \int d^2 \sigma \partial_a S_{ij,i'}^{a'j'} (\gamma^{ij,i'}^{j'} e^{\frac{i'}{6}\gamma_{123}})_{\alpha\beta} \]

\[+2 \sum_{i,j,k=1}^{3} \sum_{i',j',k'=4}^{9} \int d^2 \sigma \partial_a S_{ijk}^{a,i,j'} (\gamma^{ij,j'}^{j'} e^{\frac{i'}{6}\gamma_{123}})_{\alpha\beta} \]

\[+2 \mu \sum_{i,j,k=1}^{3} \sum_{i',j',k'=4}^{9} \int d^2 \sigma \partial_a U_{ij,k'}^{a,i,j'} (\gamma^{ij,j'}^{j'} e^{\frac{i'}{6}\gamma_{123}})_{\alpha\beta} \]

\[+2 \mu \sum_{i'=4}^{9} \int d^2 \sigma \partial_a U_{i'}^{a,i} (\gamma' e^{\frac{i}{6}\gamma_{123}})_{\alpha\beta} \].

Here \( M^{ij}_0 \) and \( M^{i'j'}_0 \) are defined by

\[ M^{ij}_0 \equiv \int d^2 \sigma \left( X^i P^j - P^i X^j - \frac{1}{2} S^{ij} \psi \right), \quad (2.4) \]

\[ M^{i'j'}_0 \equiv \int d^2 \sigma \left( X^{i'} P^{j'} - P^{i'} X^{j'} - \frac{1}{2} S^{i'j'} \psi \right), \quad (2.5) \]

where \( P^r (\equiv wD_r X^r) \) and \( S_{\alpha} (\equiv iw\psi_{\alpha}^q) \) are the canonical momenta of \( X^r \) and \( \psi \), respectively.

The zero-modes of \( P^r \) and \( X^r \) are defined by

\[ P^r_0 \equiv \int d^2 \sigma w D_r X^r, \quad X^r_0 \equiv \int d^2 \sigma w X^r, \quad (2.6) \]

and describe the motion of the membrane’s center of mass. Also, the Hamiltonian \( H \) is expressed by

\[ H = \int d^2 \sigma w \left[ \frac{1}{2} \left( \frac{P^r}{w} \right)^2 + \frac{1}{4} \{ X^r, X^s \}^2 + \frac{1}{2} \left( \frac{\mu}{3} \right)^2 \sum_{i=1}^{3} (X^i)^2 + \frac{1}{2} \left( \frac{\mu}{6} \right)^2 \sum_{i'=4}^{9} (X^{i'})^2 \right. \]

\[ + \frac{\mu}{6} \sum_{i,j,k=1}^{3} \epsilon_{ijk} X^K \{ X^i, X^j \} - w^{-1} \frac{\mu}{4} S_{123} \psi - w^{-1} S_{r} \{ X^r, \psi \}. \quad (2.7) \]
Other quantities in the above algebra are defined by

\[ S_{rs}^a \equiv -\frac{1}{2} \epsilon^{ab} X^r \partial_b X^s, \quad (2.8) \]

\[ \varphi \equiv w\{w^{-1}P^r, X^r\} + iw\{\psi^T, \psi\}, \quad (2.9) \]

\[ S^a_r \equiv \epsilon^{ab} \left( w^{-1} X_r P_s \partial_b X^s + X_r i \psi^T \partial_b \psi + \frac{3}{8} i X^s \partial_b \left( \psi^T \gamma_{rs} \psi \right) \right), \quad (2.10) \]

\[ S_{rstu}^a \equiv i \frac{48}{3} \epsilon^{ab} X^r \partial_b \left( \psi^T \gamma_{stu} \psi \right), \quad (2.11) \]

\[ U_{JKI}^a J^I' \equiv -\frac{1}{12} \sum_{I=1}^3 \epsilon_{JKI} \epsilon^{ab} X^I \partial_b \left( X^J X^I' \right), \quad (2.12) \]

\[ U_{j'}^a \equiv -\frac{1}{2} \epsilon^{ab} X^I \partial_b \left[ \frac{1}{3} \sum_{I=1}^3 (X^I)^2 - \frac{1}{6} \sum_{j'=4}^9 (X^j)^2 \right]. \quad (2.13) \]

In order to study the BPS states in the supermembrane theory on the pp-wave background, let us construct the supercharge matrix with \(32 \times 32\) components

\[ i\{Q_\alpha, Q^T_\beta\}_{DB} \equiv \begin{pmatrix} i\{Q^-_\alpha, (Q^-)^T_\beta\}_{DB} & i\{Q^-_\alpha, (Q^+_\beta)_T\}_{DB} \\ i\{Q^+_\alpha, (Q^-)^T_\beta\}_{DB} & i\{Q^+_\alpha, (Q^+_\beta)_T\}_{DB} \end{pmatrix}. \quad (2.14) \]

By the use of the formula of the triangular decomposition, which can be applied for an arbitrary matrix,

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}, \quad (2.15) \]

we can decompose the supercharge matrix as follows:

\[ \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp \left( -\frac{m}{12} \gamma_{123} \right) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp \left( \frac{m}{12} \gamma_{123} \right) \end{pmatrix} \begin{pmatrix} 1 & N_1 \\ 0 & 1 \end{pmatrix}. \quad (2.16) \]

Each block part is a matrix with \(16 \times 16\) components. The component matrix \(m\) in the

*Hereafter, we use the expressions of the supercharges redefined by rearranging the factor \(1/\sqrt{2}\) into the overall factor of the fermion \(\psi\).
triangular decomposed expression is given by
\[
m_{\gamma\delta} = \left[ 2H - \sum_{r=1}^{9} (P_r)^2 - \sum_{r,s=1}^{9} z_{rs} z_{rs}^* + \frac{2}{3} \mu \sum_{I,J,K=1}^{3} \epsilon_{IJK} X_0^I z^J z^K \\
- \left( \frac{\mu}{3} \right)^2 \sum_{i=1}^{3} (X_0^i)^2 - \frac{\mu}{6} \sum_{i'=4}^{9} (X_0^{i'})^2 \right] \delta_{\gamma\delta} + 2 \sum_{r=1}^{9} \left[ z_r + \sum_{s=1}^{9} P_s^* z_{rs} \right] (\gamma^r)_{\gamma\delta}
\]
\[-2 \sum_{r=1}^{9} d^2 \sigma \varphi X^r (\gamma^r)_{\gamma\delta} + \sum_{r,s,t,u=1}^{9} z_{rstu} (\gamma^{rstu})_{\gamma\delta} + 2 \sum_{r,s,t,u=1}^{9} z_{rstu} (\gamma^{rstu})_{\gamma\delta}
\]
\[+2 \mu \sum_{I,K=1}^{3} \sum_{i'=4}^{9} \left[ U_{IJK},j' - \frac{1}{6} \sum_{I=1}^{9} \epsilon_{IJK} (X_0^I z_{i'j'} + X_0^{i'} z_{i'j'}) \right] (\gamma_{IJK})_{\gamma\delta}
\]
\[+4 \mu \sum_{i'=4}^{9} \left[ \frac{1}{2} U_{i'} + \frac{3}{3} \sum_{I=1}^{9} X_0^I z_{i'I'} - \frac{1}{6} \sum_{i'=4}^{9} X_0^{i'} z_{i'j'} \right] (\gamma_{i'})_{\gamma\delta}
\]
\[+\frac{\mu}{3} \sum_{i',j'=4}^{9} \left[ M_{i'}^{j'} - (X_0^i P_0^j - X_0^j P_0^i) \right] (\gamma_{ij})_{\gamma\delta}
\]
\[-\frac{\mu}{6} \sum_{i',j'=4}^{9} \left[ M_{i'}^{j'} - (X_0^{i'} P_0^{j'} - X_0^{j'} P_0^{i'}) \right] (\gamma_{ij})_{\gamma\delta},
\]
where the central charges \( z_{rs}, z_r, z_{rstu}, U_{IJK},j' \) and \( U_{i'} \) are written as

\[
z_{rs} = \int d^2 \sigma \partial_\alpha s_{\alpha}^{rs} = -\frac{1}{2} \int d^2 \sigma w \{ X_r, X_s \},
\]
\[
z_r = \int d^2 \sigma \partial_\alpha s_{\alpha}^{rs} = \sum_{s=1}^{9} \int d^2 \sigma w \{ w^{-1} X_r P_s, X^s \},
\]
\[+i \int d^2 \sigma w \{ X_r \psi^T, \psi \} + \frac{3}{8} i \sum_{s=1}^{9} \int d^2 \sigma w \{ X^s, \psi^T \gamma_{rs} \psi \},
\]
\[
z_{rstu} = \int d^2 \sigma \partial_\alpha s_{\alpha}^{rstu} = \frac{i}{48} \int d^2 \sigma w \{ X_r, \psi^T \gamma_{rstu} \psi \},
\]
\[
U_{IJK},j' = -\frac{1}{12} \sum_{i=1}^{3} \epsilon_{IJK} \int d^2 \sigma w \{ X_i^j, X^i X^{j'} \},
\]
\[
U_{i'} = -\frac{1}{2} \int d^2 \sigma w \left\{ X_i^j, \frac{1}{3} \sum_{i=1}^{3} (X_i^j)^2 - \frac{1}{6} \sum_{j'=4}^{9} (X_i^{j'})^2 \right\},
\]
and constraint \( \varphi \) is given by
\[
\varphi = w \{ w^{-1} P_r, X^r \} + i w \{ \psi^T, \psi \}.
\]

Also, the component matrices \( N_1 \) and \( N_2 \) in the right and left triangle matrices are respectively
Here, we should note on the invariant mass $M$, which plays an important role in studying the BPS states in the theory. Let us recall that the invariant mass $M$ of a supermembrane is defined by

$$ M^2 \equiv - P^\mu_0 P^\mu_0 = 2H - \sum_{r=1}^{9} P^r_0 P^r_0, \quad (P^+_0 = 1). $$

Thus the first two terms of Eq. (2.17) can be replaced with the invariant mass $M^2$ of a supermembrane. In particular, if we consider 1/2 BPS states of the closed supermembrane in the flat space (i.e., the $\mu \to 0$ limit), then BPS condition

$$ M^2 = 2 \sum_{r,s=1}^{9} z_{rs} z^{rs}, $$

arises from the requirement that the coefficient of $\delta_{\gamma^k}$ equals zero. This condition implies that a mass should be proportional to its charge in the BPS state.

Hereafter, we shall restrict ourselves to the case of the closed membrane.

In flat case the Hamiltonian contains only the momentum zero-modes $P^r_0$’s and not $X^r_0$’s zero-modes $X^r_0$’s and fermion zero-modes $\psi_0$’s. However, the invariant mass $M^2$ includes no zero-modes since the momentum zero-modes are subtracted by definition.

In the pp-wave case, due to the presence of the additional terms in the action (3-point coupling, boson mass terms and fermion mass term), the Hamiltonian includes the zero-modes $X^r_0$ and $\psi_0$ in addition to the momentum zero-mode $P^r_0$ even if we consider closed membranes. In fact, we will see that these zero-modes are subtracted and do not appear in our considerations.
for the BPS states. This result might be understood if one notes that the $U(1)$ parts decouple from $SU(N)$ parts even in the pp-wave background as noted in Refs. [10, 11].

From now on, we will show that the matrix “m” is independent of the zero-modes. Let us notice the part in “m” proportional to $\delta_\gamma\delta$. First we will decompose the coordinate $X^r$ and momentum $P^r$ around their averaged values

$$X^r = X_0^r + \tilde{X}^r, \quad X_0^r = \int d^2\sigma w X^r, \quad (2.28)$$

$$P^r = wP_0^r + \tilde{P}^r, \quad P_0^r = \int d^2\sigma P^r. \quad (2.29)$$

Then the focusing part can be rewritten as

$$\tilde{H} \equiv H - \frac{1}{2} \sum_{r=1}^9 (P_0^r)^2 + \frac{\mu}{3} \sum_{I,J,K=1}^3 \epsilon_{IJK} X_0^I z_J^K$$

$$- \frac{1}{2} \left( \frac{\mu}{3} \right)^2 \sum_{i=1}^3 (X_0^i)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 \sum_{i'=4}^9 (X_0^i')^2$$

$$= \int d^2\sigma w \left[ \frac{1}{2} \left( \frac{\tilde{P}^r}{w} \right)^2 + \frac{1}{2} \left( \tilde{X}^r, \tilde{X}^s \right)^2 + \frac{\mu}{6} \sum_{i,j,k=1}^3 \epsilon_{ijk} \tilde{X}^k \{ \tilde{X}^i, \tilde{X}^j \} \right.$$

$$+ \frac{1}{2} \left( \frac{\mu}{3} \right)^2 \sum_{i=1}^3 (\tilde{X}^i)^2 + \frac{1}{2} \left( \frac{\mu}{6} \right)^2 \sum_{i'=4}^9 (\tilde{X}^i')^2$$

$$\left. - w^{-1} \frac{\mu}{4} S\gamma_{123} w S\gamma_{r} \{ \tilde{X}^r, \psi \} \right]. \quad (2.30)$$

The $\tilde{H}$ is interpreted as a Hamiltonian where bosonic zero modes $X_0$ and $P_0$ are subtracted. As a result, the BPS condition on the pp-wave corresponding to one (2.27) in the flat case, is given by

$$2\tilde{H} = 2 \sum_{r,s=1}^9 z_{rs} z_{rs}^r. \quad (2.31)$$

Now, we may understand the subtracted Hamiltonian $\tilde{H}$ as the improved invariant mass $\tilde{M}$ defined by

$$\tilde{M}^2 \equiv 2\tilde{H}$$

$$= M^2 + \frac{2}{3} \mu \sum_{I,J,K=1}^3 \epsilon_{IJK} X_0^I z_J^K$$

$$- \left( \frac{\mu}{3} \right)^2 \sum_{i=1}^3 (X_0^i)^2 - \left( \frac{\mu}{6} \right)^2 \sum_{i'=4}^9 (X_0^i')^2, \quad (2.32)$$
where the zero-modes $X_0^r$’s are completely subtracted. Therefore, we find that the zero-modes in the first square bracket of (2.17) are spurious.

Next, in the similar way we can subtract the zero modes from the $U_{JKI'}, J'$ and $U_{I'}, I'$, and obtain the following expressions

$$
\tilde{U}_{JKI', J'} \equiv -\frac{1}{12} \sum_{i=1}^{3} \epsilon_{iJK} \int d^2 \sigma \ w \{ \tilde{X}^J' , \tilde{X}^I' \tilde{X}^J' \}
= U_{JKI', J'} - \frac{1}{6} \sum_{i=1}^{3} \epsilon_{iJK} (X_0^I' z_{I' J'} + X_0^J' z_{I J'}) ; \quad (2.33)
$$

$$
\frac{1}{2} \tilde{U}_{I'} \equiv -\frac{1}{4} \int d^2 \sigma \ w \left\{ \tilde{X}^I' , \frac{1}{3} \sum_{i=1}^{3} (\tilde{X}^I')^2 - \frac{1}{6} \sum_{j'=4}^{9} (\tilde{X}^{J'})^2 \right\}
= \frac{1}{2} U_{I'} + \frac{1}{3} \sum_{i=1}^{3} X_0^I' z_{I' I'} - \frac{1}{6} \sum_{j'=4}^{9} X_0^J' z_{J' J'} . \quad (2.34)
$$

Also, the charges $z_r, z_{rs}$ and $z_{rstu}$ are rewritten as

$$
\tilde{z}_r \equiv \sum_{s=1}^{9} \int d^2 \sigma \ w \{ w^{-1} \tilde{X}^r \tilde{P}_s , \tilde{X}^s \}
+i \int d^2 \sigma \ w \{ \tilde{X}_r \psi^T , \psi \} + 3 \frac{9}{8} i \sum_{s=1}^{9} \int d^2 \sigma \ w \{ \tilde{X}^s , \psi^T \gamma_{rs} \psi \}
= z_r + 2 \sum_{s=1}^{9} P_0^s \tilde{z}_{rs} - X_0^r \int d^2 \sigma \ \tilde{\varphi} , \quad (2.35)
$$

$$
\tilde{\varphi} \equiv \sum_{r=1}^{9} w \{ w^{-1} \tilde{P}_r , \tilde{X}^r \} + i w \{ \psi^T , \psi \} , \quad (2.36)
$$

$$
\tilde{z}_{rs} \equiv -\frac{1}{2} \int d^2 \sigma \ w \{ \tilde{X}^r , \tilde{X}^s \} = z_{rs} , \quad (2.37)
$$

$$
\tilde{z}_{rstu} \equiv -\frac{i}{48} \int d^2 \sigma \ w \{ \tilde{X}_r [ , \psi^T \gamma_{rstu} ]\psi \} = z_{rstu} . \quad (2.38)
$$

Thus, it is shown that the zero-modes in square brackets containing $z_r, U_{JKI', J'}, U_{I'}$ of (2.17) are also spurious. Moreover, we can easily show that the other terms in (2.17) do not contain any bosonic zero-modes, and so we obtain the drastically simplified result for the matrix “m”
\[ m_{\gamma\delta} = 2 \left[ \tilde{H} - \sum_{r,s=1}^{9} z_{rs} z^{rs} \right] \delta_{\gamma\delta} + 2 \sum_{r=1}^{9} \left[ \tilde{z}_r - \int d^2 \sigma \tilde{\phi} \tilde{X}^r \right] (\gamma^r)_{\gamma\delta} \]
\[ + \sum_{r,s,t,u=1}^{9} \left[ z_{rs} z_{tu} + 2 z_{rstu} \right] (\gamma^{rstu})_{\gamma\delta} \]
\[ + 2\mu \sum_{j,k=1}^{3} \sum_{l', j'=4}^{9} \tilde{U}_{jKt', j'} (\gamma_{jKt', j'})_{\gamma\delta} + 2\mu \sum_{l'=4}^{9} \tilde{U}_{l'} (\gamma_{l'23})_{\gamma\delta} \]
\[ + \frac{\mu}{3} \sum_{l,j=1}^{3} \tilde{M}_0^{ij} (\gamma_{ij23})_{\gamma\delta} - \frac{\mu}{6} \sum_{l', j'=4}^{9} \tilde{M}_0^{ij'} (\gamma_{ij23})_{\gamma\delta}, \] (2.39)

where \( \tilde{M}_0^{ij} \) and \( \tilde{M}_0^{ij'} \) are defined by
\[ \tilde{M}_0^{ij} \equiv \int d^2 \sigma \left( \tilde{X}^i \tilde{P}^j - \tilde{P}^i \tilde{X}^j - \frac{1}{2} S_{\gamma^{ij}} \psi \right), \]
\[ \tilde{M}_0^{ij'} \equiv \int d^2 \sigma \left( \tilde{X}^i \tilde{P}^{j'} - \tilde{P}^{i} \tilde{X}^{j'} - \frac{1}{2} S_{\gamma^{ij'}} \psi \right). \]

Thus, we have proven that the zero-modes of the variables \( X \) and \( P \) in \( m \) (2.17) have been subtracted and \( m \) can be described in terms of oscillation-modes in \( X \) and \( P \) only. That is, the matrix \( m \) represents the excited modes of variables. The first three terms in Eq. (2.39) have the same forms as in the flat case, if \( 2\tilde{H} \) is replaced with the invariant mass \( \mathcal{M}^2 \). In the case of the pp-wave, additional four terms proportional to \( \mu \) appear. These terms impose the BPS conditions for the charges on the additional extended objects only living on the pp-wave.

We can find the BPS states by analyzing the rank of the supercharge matrix \( m \). The 1/2 and 1/4 BPS cases will be considered in the following subsections. When we consider the BPS states, the fermion parts are omitted in the discussion because we do not take the winding of fermion into account.

### 2.1 1/2 BPS States

The 1/2 BPS states can be described by the condition \( m=0 \), which means some constraints
\[ 2\tilde{H} - 2 \sum_{r,s=1}^{9} z_{rs} z^{rs} = 0, \] (2.40)
\[ \tilde{z}_r = 0, \] (2.41)
\[ \tilde{U}_{jKt', j'} = \tilde{U}_{l'} = 0, \] (2.42)
\[ \tilde{M}_0^{ij} = \tilde{M}_0^{ij'} = 0. \] (2.43)
We neglected the term $z_{rs}z_{tu} \gamma^{rstu}$ by imposing proper conditions (for example, $z_{1r} \neq 0$ and otherwise is zero). The first equation (2.40) is the BPS condition indicating that the mass is proportional to its charge. The BPS condition also indicates that the oscillation-modes are absent. But there are modes linear to the worldvolume coordinates $\sigma^a$ ($a = 1, 2$) and $z_{rs} \neq 0$. Then the second condition (2.41) is automatically satisfied. The third condition should mean that the charge of the extended objects only living on the pp-wave. That is, this case corresponds to the (transverse) M2-brane in the flat space. It has been also shown in the flat space [16] that the longitudinal M2-brane is contained in 1/2 BPS states in the $SO(10,1)$ covariant way. Thus, it would be possible to say that the longitudinal M2-brane might be contained in this case though it is not seen manifestly in our $SO(9)$ formalism. In conclusion, 1/2 BPS states are transverse and longitudinal M2-branes which are the same objects as the flat case.

It is also possible to obtain the remaining unbroken supercharges under the 1/2 BPS conditions. Note the supercharge matrix can be rewritten as

$$i \{ \left( \begin{array}{c} Q^+_{\alpha} \\ Q^-_{\beta} \end{array} \right), \left( \begin{array}{c} Q^+_{\beta} \\ Q^-_{\alpha} \end{array} \right) \} = \left( \begin{array}{cc} -1 & 0 \\ 0 & m \end{array} \right),$$

(2.44)

$$\left( \begin{array}{c} Q^+_{\alpha} \\ Q^-_{\alpha} \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \exp \left( \frac{\mu}{16} \gamma_{123} \tau \right) \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ -N_2 & 1 \end{array} \right) \left( \begin{array}{c} Q^+_{\alpha} \\ Q^-_{\alpha} \end{array} \right).$$

(2.45)

We can easily find the unbroken supersymmetry and its charge is given by

$$Q^+ = -e^{\frac{\mu}{16} \gamma_{123} \tau} \left( N_2 Q^- - Q^+ \right).$$

(2.46)

2.2 1/4 BPS States

The 1/4 BPS conditions are that the rank of the matrix “m” is eight. In order to study the rank of “m”, we decompose the $16 \times 16$ $SO(9)$ gamma matrices $\gamma^r$ ($r = 1, \ldots, 9$) into $8 \times 8$ gamma matrices as

$$\gamma^\tilde{r} = \left( \begin{array}{cc} 0 & \tilde{\gamma}^\tilde{r} \\ (\tilde{\gamma}^\tilde{r})^T & 0 \end{array} \right) \quad (\tilde{r} = 1, \ldots, 8), \quad \gamma^9 = \left( \begin{array}{cc} 1 \times & 0 \\ 0 & -1 \times \end{array} \right),$$

(2.47)
where $\tilde{\gamma}^\ell$'s and $\gamma^9$ are real and symmetric matrices with $8 \times 8$ components, and satisfy the commutation relations

$$
\tilde{\gamma}^\ell (\tilde{\gamma}^\ell)^T + \tilde{\gamma}^{\ell'} (\tilde{\gamma}^{\ell'})^T = 2\delta^{\ell\ell'},
$$
$$
(\tilde{\gamma}^\ell)^T \tilde{\gamma}^\ell + (\tilde{\gamma}^{\ell'})^T \tilde{\gamma}^{\ell'} = 2\delta^{\ell\ell'}.
$$

By the use of the above matrices, $\tilde{U}_{J'}(\gamma_{J'123})$ and $\tilde{U}_{JKI',J'}(\gamma^{JKI',J'})$ can be rewritten as

$$
\sum_{J' = 4}^{9} \tilde{U}_{J'}(\gamma_{J'123}) = \begin{pmatrix}
\sum_{J' = 4}^{8} \tilde{U}_{J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^{2}(\tilde{\gamma}^3)^T & \tilde{U}_9(\tilde{\gamma}^2)^T \tilde{\gamma}^3 \\
-\tilde{U}_9(\tilde{\gamma}^3)^T \tilde{\gamma}^2 & \sum_{J' = 4}^{8} \tilde{U}_{J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^1(\tilde{\gamma}^2)^T \tilde{\gamma}^3
\end{pmatrix},
$$

$$
\sum_{J,K=1}^{3} \sum_{J',J'=4}^{9} \tilde{U}_{JKI',J'}(\gamma^{JKI',J'}) = \begin{pmatrix}
\tilde{U}_{JKI',J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^2(\tilde{\gamma}^{J'})^T & 0 \\
0 & \tilde{U}_{JKI',J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^1(\tilde{\gamma}^2)^T \tilde{\gamma}^{J'}
\end{pmatrix}
+ \sum_{J,K=1}^{3} \sum_{J' = 4}^{8} \begin{pmatrix}
0 & -2\tilde{U}_{JKI',J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^2(\tilde{\gamma}^{J'})^T \\
-2\tilde{U}_{JKI',J'}(\tilde{\gamma}^{J'})^T \tilde{\gamma}^1(\tilde{\gamma}^2)^T \tilde{\gamma}^{J'} & 0
\end{pmatrix}.
$$

We would like to consider the rank of the matrix “$m$”, but the general analysis is more complicated and difficult. Here we shall present some special solutions concretely.

To begin, we consider only $\mu$-dependent parts by letting $\mu$-independent parts vanish. For simplicity, we impose further conditions $\tilde{M}^{\ell}_{0J} = \tilde{M}^{\ell'}_{0J'} = \tilde{U}_{JKI',J'} = \tilde{U}_9 = 0$. Then the matrix “$m$” can be written as

$$
m = 2\mu \sum_{J' = 4}^{9} \tilde{U}_{J'}(\gamma_{J'123}) - 2 \cdot \frac{\mu}{6} \sum_{J' = 4}^{8} \tilde{M}^{\ell'}_{0J'}(\gamma_{J'123})
$$

$$
= 2\mu \begin{pmatrix}
\sum_{J' = 4}^{8} (\tilde{U}_{J'} - \frac{1}{6}\tilde{M}^{\ell'}_{0J'}) \tilde{\gamma}^{J'}(\tilde{\gamma}^{J'})^T \\
0 & \sum_{J' = 4}^{8} (\tilde{U}_{J'} + \frac{1}{6}\tilde{M}^{\ell'}_{0J'}) \tilde{\gamma}^{J'}(\tilde{\gamma}^{J'})^T
\end{pmatrix}.
$$

We can easily find the conditions that the rank of the matrix “$m$” is eight, those are

$$
\tilde{U}_{J'} = +\frac{1}{6}\tilde{M}^{\ell'}_{0J'} \quad (J' = 4, 5, \cdots, 8),
$$

$$
or \quad \tilde{U}_{J'} = -\frac{1}{6}\tilde{M}^{\ell'}_{0J'} \quad (J' = 4, 5, \cdots, 8).
$$
These conditions (2.52) and (2.53) show that the brane charge \( \tilde{U}_I \) equals to the angular momentum \( \tilde{M}_9^{ij} \). These are 1/4 BPS solutions as special solutions of the general 1/4 BPS states. We may consider such solutions as rotating membranes which are 1/4 BPS. In fact, the superalgebra includes the angular momentum operators, which might be considered as the remnants of the \( AdS_7 \times S^4 \) or \( AdS_4 \times S^7 \) backgrounds, and such rotating solutions can exist in the theory. Furthermore, (2.52) and (2.53) mean the “stringy exclusion principle” which is also a remnant in \( AdS \) space physics. That is, configurations with larger angular momenta than the given brane charges cannot exist. The central charges in the superalgebra may also indicate other extended objects only living on the pp-wave, which might be expected as a fuzzy membrane, a giant graviton \([10]\) or other 1/4 BPS states \([17]\) coming from Myers effects \([18]\).

In the above case, only the \( \mu \)-dependent parts have been studied, but we should note that such a situation can be also realized by taking the large \( \mu \) limit without imposing certain conditions on \( \mu \)-independent parts. The large \( \mu \) limit has been discussed in \([11]\) where the existence of such 1/4 BPS states is stated at least in this limit. Conversely speaking, in our considerations we could avoid the large \( \mu \) limit by requiring \( \mu \)-independent parts to satisfy vanishing conditions.

Moreover, we comment that it is possible to discuss the 1/4 BPS states in the same way as in the flat space \([10]\) if we consider \( \tilde{M}_0^{ij} = \tilde{M}_0^{ij} = \tilde{U}_I = \tilde{U}_I = U_J K_J , J' = 0 \) (i.e., \( \mu = 0 \)). Also, it would be possible to apply the similar considerations for \( \tilde{U}_J K_J , J' \).

Finally, the unbroken supercharges of the 1/4 BPS states are given by

\[
Q^{1(\mp)} = \left( \frac{1 \mp \gamma^9}{2} \right) Q^1, \tag{2.54}
\]

where the minus (−) sign corresponds to the case in (2.52) and the plus (+) one to (2.53), respectively.

### 3 Conclusions and Discussions

In this paper, we have obtained the BPS conditions of the supermembrane on the pp-wave background from the viewpoint of the rank of the supercharge matrix by the use of triangular decomposition. In the pp-wave case the Hamiltonian and invariant mass contain more zero-modes than in the flat case, but we have seen that these zero-modes are spurious and play no roles in our considerations of the BPS states. Moreover, we have in detail studied the BPS conditions for 1/2 and 1/4 BPS states. The conditions for 1/2 BPS states are the same ones as
in the flat space. For the 1/4 BPS states, we could not present general solutions but a special solution as an example of the 1/4 BPS states, which is peculiar in the pp-wave case.

It is an interesting future work to investigate more general BPS mass formulae systematically.

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