TRANSLATORS OF THE GAUSS CURVATURE FLOW

MUHITTIN EVREN AYDIN AND RAFAEL LÓPEZ

ABSTRACT. A $K^\alpha$-translator is a surface in Euclidean space $\mathbb{R}^3$ that moves by translations in a spatial direction and under the $K^\alpha$-flow, where $K$ is the Gauss curvature and $\alpha$ is a constant. We classify all $K^\alpha$-translators that are rotationally symmetric. In particular, we prove that for each $\alpha$ there is a $K^\alpha$-translator intersecting orthogonally the rotation axis. We also describe all $K^\alpha$-translators invariant by a uniparametric group of helicoidal motions and the translators obtained by separation of variables.

1. Introduction and results

The flow by powers of the Gauss curvature $K$ was initiated by Chow [8] after the articles of Firey and Tso ([11], [15]). These works were the starting point of the theory of the flow by the Gaussian curvature ([1], [16]), a topic of high activity in geometric analysis that continues to the present. Given a smooth immersion $X : \Sigma \to \mathbb{R}^3$ of a strictly convex surface $\Sigma$ in Euclidean space $\mathbb{R}^3$, we consider the $K^\alpha$-flow as a one-parameter family of smooth immersions $X_t = X(\cdot, t) : \Sigma \to \mathbb{R}^3$, $t \in [0, T)$ such that $X_0 = X$ and satisfying the flow

$$\frac{\partial}{\partial t} X(p, t) = -K(p, t)^\alpha N(p, t), \quad (p, t) \in \Sigma \times [0, T),$$

where $\alpha \in \mathbb{R}$ is a constant, $N(p, t)$ is the unit normal of $X(p, t)$ and $K(p, t)$ is the Gauss curvature at $X(p, t)$. An interesting problem is the evolution of a surface through the flow. As an example for the reader, if $\alpha = 1$ the surface becomes spherical ([3]).

If the surface moves under the $K^\alpha$-flow along a spatial direction $\vec{v} \in \mathbb{R}^3$, then $X_0$ satisfies $K^\alpha = \lambda \langle N, \vec{v} \rangle$ for some positive constant $\lambda$. The vector $\vec{v}$ is assumed to be unitary and it is called the speed of the flow. After a dilation, we can assume that $\lambda = 1$. A surface $\Sigma$ is called a translator by the $K^\alpha$-flow with speed $\vec{v}$, or simply, a $K^\alpha$-translator in case that $\vec{v}$ is understood, if

$$K^\alpha = \langle N, \vec{v} \rangle. \quad (1)$$

The notion of translator by positive powers of the Gauss curvature appeared in [17]. See also [9, 14].

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In this paper we want to find examples of $K^\alpha$-translators under the geometric condition that the surface is defined kinematically as the movement of a curve by a uniparametric family of rigid motions of $\mathbb{R}^3$. Following Darboux [10, Livre I], we consider surfaces parametrized by $\Psi(s,t) = A(t) \cdot \gamma(s) + \beta(t)$, where $\gamma$ and $\beta$ are two spatial curves and $A(t)$ is an orthogonal matrix. The cases that we are interesting are: rotational surfaces ($A(t)$ is a uniparametric group of rotations and $\beta$ is constant), helicoidal surfaces ($A(t)$ is a uniparametric group of helicoidal motions), translation surfaces ($A(t)$ is the identity) and ruled surfaces ($\gamma$ is a straight-line). Since we have left to one side the study of the evolution of the surface, we do not require that $K$ to be positive but only that $K^\alpha$ has sense. For example, $K$ may be negative if $\alpha \in \mathbb{Z}$. This will be implicitly assumed in all the results.

In the flow by powers of the Gaussian curvature, it is known that the $K^{1/4}$-flow is special because it has an interpretation in affine differential geometry as the affine normal flow of a convex surface ([2, 6]). However, in base of our proofs, this case will appear as natural because when we express the equation (1) for each one of the above three types of surfaces and when $\alpha$ is exactly 1/4, this equation is greatly simplified due to a cancellation of terms. This makes the arguments easier than in the general case of $\alpha$.

Planes are examples of $K^\alpha$-translators for any $\alpha > 0$ provided the plane is parallel to the speed of the flow. Throughout this paper, we will discard planes as examples of $K^\alpha$-translators. Other examples that we will not consider in the class of $K^\alpha$-translators are the surfaces whose Gauss curvature is constant, where now the equation (1) says that the unit normal vector $N$ makes a constant angle with a fixed direction. Similarly, we discard the case $\alpha = 0$.

It is important to point out that there is not a priori relation between the speed vector $\vec{v}$ and the special parametrization of each of the above Darboux surfaces. So, if one considers the study of translators one can prescribed the speed $\vec{v}$, usually, the vector $(0,0,1)$ in the literature. However, if one considers one of the above types of Darboux surfaces, the parametrization has no relation with $\vec{v}$. This can be clearly seen for rotational surfaces (also helicoidal surfaces). The rotation axis of the surface is, initially, independent of the vector $\vec{v}$. However, as we will see, for rotational and helicoidal surfaces, the speed must be parallel to the axis (Propositions 2.1 and 3.1). Something similar occurs for translations surfaces where, if one prescribes that the surface is $z = f(x) + g(y)$, the speed may be arbitrary.

The organization of this paper is as follows. In Section 2, we obtain all rotational $K^\alpha$-translators. Recall that in [17], Urbas obtains these surfaces for $\alpha \in (0, 1/2]$ using the Legendre transform (see also [13] for a similar calculation). Our approach uses simple geometric arguments and it holds for any $\alpha$ (Theorem 2.2). In particular, for each value of $\alpha$, ...
we prove the existence of rotational examples intersecting orthogonally
the rotation axis (Corollary 2.4). Section 3 is devoted to the study of
$K^\alpha$-translators of helicoidal type. Although we are not able to obtain explicit parametrizations of these surfaces, we do a first integration of
the generating curve (Theorem 3.2), also in terms of the Bour function
(Theorem 3.3). In Section 4, we obtain the classification of all solutions
of (1) obtained by separation of variables $z = f(x) + g(y)$, where
$(x, y, z)$ are the canonical coordinate system of $\mathbb{R}^3$. Although these
solutions depend on a particular choice of coordinates of $\mathbb{R}^3$, our result
holds for any speed vector $\vec{v}$. In particular, we prove that there are
$K^\alpha$-translators only if $\alpha = 1/4$ (Theorems 4.1 and 4.3). Besides we
provide new examples of $K^{1/4}$-translators of type $z = f(x) + g(y)$, we
also give new examples of $K^{1/4}$-translators obtained by separation of
variables of type $z = f(x)g(y)$ (Example 4.4). Finally, in Section 5, we
investigate the existence of ruled $K^\alpha$-translators, proving that there
are not ruled $K^\alpha$-translators, except trivial cases (Theorem 5.1).

The authors have extended the results of this paper for (spacelike
and timelike) $K^\alpha$-translators in Lorentz-Minkowski space ($\mathbb{H}$).

2. ROTATIONAL $K^\alpha$-TRANSLATORS

In this section, we classify the surfaces of revolution satisfying (1).
A first question is if there is a relation between the rotation axis of the
surface and the speed vector $\vec{v}$. As it is expectable, we prove that the
vector $\vec{v}$ must be parallel to the rotation axis.

Proposition 2.1. Let $\Sigma$ be a $K^\alpha$-translator. If $\Sigma$ is a surface of rev-
olution, then the rotation axis is parallel to the speed vector $\vec{v}$.

Proof. After a rigid motion, we can assume that the rotation axis $L$
is the $z$-axis. The generating curve of a rotational surface is a curve
included in the coordinate $xz$-plane which can be assumed that it is a
graph on the $x$-axis, except in the case that this curve is a straight-
line parallel to the $z$-axis. In this particular situation, the surface is a
circular cylinder where we know that $K = 0$. Thus a circular cylinder
satisfies (1) if $\langle N, \vec{v} \rangle = 0$, that is, the vector $v$ is parallel to the
$z$-axis, proving the result for this particular case.

Suppose now the general case that the generating curve writes as
$z = f(x)$ where $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ is a smooth function. Then a
parametrization of $\Sigma$ is

$$X(r, \theta) = (r \cos \theta, r \sin \theta, f(r)), \quad (2)$$

where $\theta \in \mathbb{R}$. The unit normal vector $N$ is

$$N(r, \theta) = \frac{1}{\sqrt{1 + f'^2}}(-f' \cos \theta, -f' \sin \theta, 1)$$
and the principal curvatures of $\Sigma$ are
\[
\kappa_1 = \frac{f''}{(1 + f'^2)^{3/2}}, \quad \kappa_2 = \frac{f'}{r(1 + f'^2)^{1/2}}.
\]
If $\vec{v} = (v_1, v_2, v_3)$, then (1) writes as
\[
K^\alpha = (\kappa_1 \kappa_2)^\alpha = \left(\frac{f' f''}{r(1 + f'^2)^2}\right)^\alpha = \frac{1}{\sqrt{1 + f'^2}}(-v_1 f' \cos \theta - v_2 f' \sin \theta + v_3),
\]
or equivalently
\[
A_0 + A_1 \cos \theta + A_2 \sin \theta = 0,
\]
where
\[
A_0 = \left(\frac{f' f''}{r(1 + f'^2)^2}\right)^\alpha - v_3 (1 + f'^2)^{-1/2}, \quad A_1 = v_1 f'(1 + f'^2)^{-1/2}, \quad A_2 = v_2 f'(1 + f'^2)^{-1/2}.
\]
Since the functions $\{1, \cos \theta, \sin \theta\}$ are linearly independent, we conclude $A_0 = A_1 = A_2 = 0$. If $f'$ is constantly 0, then $f$ is constant and the surface is a plane. This case was initially discarded. Therefore from $A_1 = A_2 = 0$, we find that $v_1 = v_2 = 0$, concluding that $\vec{v}$ is parallel to $L$. □

After a rigid motion, we will assume that the rotation axis is the $z$-axis and consequently from this proposition, that $\vec{v} = (0, 0, 1)$ after a symmetry about the $xy$-plane if necessary. Then (1) is
\[
\left(\frac{f' f''}{r(1 + f'^2)^2}\right)^\alpha = \frac{1}{(1 + f'^2)^{1/2}}.
\]
Set $g = f'(1 + f'^2)^{1/2}$. Notice that $\kappa_1 = g'$ and $r \kappa_2 = g$. In terms of $g$, the above equation becomes
\[
\frac{g}{(1 - g^2)^{1/2}} g' = r.
\]
A first integration gives
\[
g^2 = \begin{cases} 
1 - m e^{-r^2}, & m > 0, \\
1 - \left(m - \frac{2 \alpha - 1}{2 \alpha} r^2\right)^{\frac{2\alpha}{2\alpha - 1}}, & m \in \mathbb{R}, \quad \alpha \neq \frac{1}{2}, \end{cases}
\]
In the case $\alpha = 1/2$, the condition $g \ast 2 \geq 0$ says that the domain of $r$ is when $r^2 \geq \log(m)$. Otherwise, we need to distinguish if $2\alpha/(2\alpha - 1)$ is negative or positive, that is, $\alpha$ belongs to $(0, 1/2)$ or not. If $\alpha \in (0, 1/2)$, the parenthesis in (3) must be positive, yielding $r^2 > 2\alpha/(2\alpha - 1)m$. On the other hand, using now that $g^2 \geq 0$, we have $r^2 \geq 2\alpha/(2\alpha - 1)(m - 1) \geq 0$, so this is the restriction on $r$ because $2\alpha/(2\alpha - 1) < 0$. If $\alpha \notin [0, 1/2]$, then $2\alpha/(2\alpha - 1) > 0$. Since the parenthesis in (3) must be positive because $g^2 < 1$, then we obtain $r^2 < 2\alpha/(2\alpha - 1)m$ and
from the fact that $g^2 \geq 0$, the restriction is $r^2 \geq 2\alpha/(2\alpha - 1)(m - 1)$. To summarize, we have

$$\begin{cases} 
\frac{2\alpha}{2\alpha - 1}(m - 1) \leq r^2, & \alpha \in (0, \frac{1}{2}) \\
\frac{2\alpha}{2\alpha - 1}(m - 1) \leq r^2 < \frac{2\alpha}{2\alpha - 1}m, & \alpha \not\in [0, \frac{1}{2}].
\end{cases}$$

(4)

Hence we deduce

$$f'(r) = \begin{cases} 
\pm \left(\frac{1}{m}e^{r^2} - 1\right)^{1/2}, & \alpha = \frac{1}{2} \\
\pm \left((m - \frac{2\alpha - 1}{2\alpha}r^2)^\frac{2\alpha}{2\alpha - 1} - 1\right)^{1/2}, & \alpha \neq \frac{1}{2}.
\end{cases}$$

(5)

As conclusion, we have the classification of all $K^\alpha$-translators that also are surfaces of revolution.

**Theorem 2.2.** Let $\Sigma$ be a $K^\alpha$-translator. If $\Sigma$ is a surface of revolution about the $z$-axis, then $\Sigma$ is a circular cylinder of arbitrary radius or $\Sigma$ parametrizes as (2) where

$$f(r) = \begin{cases} 
\pm \int_r^\infty \left(\frac{1}{m}e^{t^2} - 1\right)^{1/2} dt, & m > 0, \alpha = \frac{1}{2} \\
\pm \int_r^\infty \left((m - \frac{2\alpha - 1}{2\alpha}t^2)^\frac{2\alpha}{2\alpha - 1} - 1\right)^{1/2} dt, & m \in \mathbb{R}, \alpha \neq \frac{1}{2}.
\end{cases}$$

(6)

Furthermore, the maximal domain of the function $f(r)$ is

(1) $[\sqrt{\log m}, \infty)$, if $\alpha = 1/2$.

(2) $[\sqrt{\frac{2\alpha}{2\alpha - 1}m}, \infty)$, if $\alpha \in (0, 1/2)$.

(3) $[\sqrt{\frac{2\alpha}{2\alpha - 1}(m - 1)}, \sqrt{\frac{2\alpha}{2\alpha - 1}m})$, if $\alpha \not\in [0, 1/2]$. In this case, we have

$$\lim_{r \to \sqrt{\frac{2\alpha}{2\alpha - 1}(m - 1)}} f'(r) = 0, \quad \lim_{r \to \sqrt{\frac{2\alpha}{2\alpha - 1}m}} f(r) = \infty.$$ 

In all these cases, we understand that if in the radicand in the left-end of the interval is negative, then the value of this end is 0.

**Proof.** It remains to prove the limits in the case (3). The first limit is consequence of (4) and (5). For the second limit, note that the maximal domain of $f$ is a bounded interval and that $\lim_{r \to \sqrt{\frac{2\alpha}{2\alpha - 1}m}} f(r) = \infty$ by (5).

We point that it is expectable that in the case $\alpha \not\in [0, 1/2]$ the domain cannot be $[0, \infty)$ because there are no entire graphs that are $K^\alpha$-translators if $\alpha > 1/2$ ([17 Sect. 4]) and if $\alpha < 0$ ([18 Th. 6.1]). It deserves to note the case $\alpha = 1/4$ because it is possible to integrate explicitly (see also [14]). See Figure 1.

**Corollary 2.3.** Rotational $K^{1/4}$-translators form a uniparametric family of surfaces parametrized by (2), where

$$f(r) = \frac{1}{2} \left(r\sqrt{m + r^2 - 1} + (m - 1) \log \left(\sqrt{m + r^2 - 1} + r\right)\right) + c, \quad m, c \in \mathbb{R}.$$
The maximal domain of $f$ is $[\sqrt{1-m}, \infty)$ if $m < 1$ and $[0, \infty)$ if $m \geq 1$. For the value $m = 1$, $f$ is the parabola $f(r) = r^2/2$, the graphic of $f(r)$ intersects orthogonally the rotation axis and the surface is a paraboloid.

By (4), we point out that the maximal domain of $f$ is not $[0, \infty)$ in general. However, an interesting case to investigate is if there are generating curves that meet orthogonally the rotation axis. We prove that this occurs for all cases of $\alpha$. See Figure 2.

**Corollary 2.4.** For each $\alpha$, there are rotational $K^\alpha$-translators whose generating curves intersect orthogonally the rotation axis. These surfaces are unique up to vertical translations. Furthermore,

1. If $\alpha \in (0, \frac{1}{2}]$, the maximal domain is $[0, \infty)$, \(\lim_{r \to \infty} f(r) = \infty\) and
   \[
   f(r) = (1 - 2\alpha) \left( \frac{1 - 2\alpha}{2\alpha} \right)^{1/2\alpha} r^{1/2\alpha} + o(r^{1/2\alpha}).
   \]

2. If $\alpha \not\in [0, \frac{1}{2}]$, the maximal domain is $[0, \sqrt{\frac{2\alpha}{2\alpha - 1}})$, with
   \[
   \lim_{r \to \sqrt{\frac{2\alpha}{2\alpha - 1}}} f(r) = \infty.
   \]

**Proof.** The condition on the orthogonality with the rotation axis requires that $f$ is defined at $r = 0$ and $f'(0) = 0$. From (5), it is immediate that $m$ must be 1 and the same occurs in the particular case $\alpha = 1/2$. This solution is $C^2$ at $r = 0$ because from (5), we have $\lim_{r \to 0} f''(r) = 1$. Hence $f$ is $C^\infty$ in its domain by regularity (5, 12). The uniqueness is consequence of the solvability of (6).

The behaviour of $f$ at infinity is consequence of (5) and the L’Hôpital rule. Indeed, if $\delta = 1/(1 - 2\alpha)$, then

\[
\lim_{r \to \infty} \frac{f(r)}{r^\delta} = \lim_{r \to \infty} \frac{f'(r)}{\delta r^{\delta - 1}} = \frac{1}{\delta} \left( \frac{1 - 2\alpha}{2\alpha} \right)^{1/2\alpha}.
\]
Figure 2. Generating curves of rotational $K^\alpha$-translators intersecting orthogonally the rotation axis. Left: $\alpha = 1/2$ and the maximal domain is $[0, \infty)$. Right: $\alpha = 1$ and the maximal domain is $[0, \sqrt{2})$.

We point out that in some particular cases, the integrals in (6) can be explicitly solved. Here, we denote by $f_\alpha$ to emphasize the parameter $\alpha$ where we also assume $m = 1$.

(1) Case $\alpha = 1$. Then

$$f'(r) = \frac{r\sqrt{4 - r^2}}{2 - r^2}$$

and

$$f_1(r) = \mp \sqrt{4 - r^2} \mp \sqrt{2} \tanh^{-1} \left( \frac{1}{\sqrt{2}} \sqrt{4 - r^2} \right),$$

defined on $[0, 2)$.

(2) Case $\alpha = 1/3$. Now

$$f'(r) = \pm \frac{1}{2} r \sqrt{4 + r^2}$$

and the solution is

$$f_{1/3}(r) = \pm \frac{1}{6} (4 + r^2)^{3/2},$$

defined on $[0, \infty)$.

(3) For $\alpha = 1/6$, we have

$$f'(r) = \pm (\sqrt{2r^2 + 1} - 1)^{1/2},$$

and

$$f_{1/6}(r) = \pm \frac{\sqrt{2r^2 + 1} - 1 (2r^2 - \sqrt{2r^2 + 1} - 1)}{3r},$$

and defined on $[0, \infty)$.

3. Helicoidal $K^\alpha$-translators

Consider a helicoidal surface $\Sigma$ in $\mathbb{R}^3$ with axis $z$ whose generating curve $\gamma$ is included in the $xz$-plane and pitch $h$. Without loss of generality, we can assume that $\gamma(r) = (r, 0, f(r))$ where $f: I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$
is a smooth function. Then \( \Sigma \) parametrizes by
\[
X(r, \theta) = (r \cos \theta, r \sin \theta, f(r) + h \theta), \quad r \in I, \theta \in \mathbb{R}.
\] (7)
If \( D = r^2(1 + f'^2) + h^2 \), then the unit normal vector \( N \) is
\[
N = \frac{1}{\sqrt{D}} (h \sin \theta - rf' \cos \theta, -h \cos \theta - rf' \sin \theta, r)
\]
and the Gauss curvature is
\[
K = \frac{r^3 f' f'' - h^2}{D^2}.
\]
Then \( \Sigma \) is a \( K^\alpha \)-translator if
\[
\left( \frac{r^3 f' f'' - h^2}{D^2} \right)^\alpha = D^{-1/2} (v_1 (h \sin \theta - rf' \cos \theta) - v_2 (h \cos \theta + rf' \sin \theta) + v_3 r).
\] (8)
As a first conclusion, we prove that \( \bar{v} \) must be parallel to the \( z \)-axis.
The proof is similar of Proposition 2.1.

**Proposition 3.1.** Let \( \Sigma \) be a \( K^\alpha \)-translator. If \( \Sigma \) is a helicoidal surface, then the axis is parallel to the speed vector \( \bar{v} \).

**Proof.** Equation (8) can be written as \( A_0 + A_1 \cos \theta + A_2 \sin \theta = 0 \).
From \( A_1 = A_2 = 0 \), we obtain
\[
v_1 h - v_2 rf' = 0, \quad v_1 rf' + v_2 h = 0.
\]
Combining both equations, we have \( v_2(h^2 + r^2 f'^2) = 0 \). Thus \( v_2 = 0 \) and hence, \( v_1 = 0 \).

From this proposition, we can assume that \( \bar{v} = (0, 0, 1) \). Then (8) is
\[
\left( \frac{r^3 f' f'' - h^2}{D^2} \right)^\alpha = \frac{r}{\sqrt{D}}.
\] (9)
We will obtain a first integration of this equation. For this, let
\[
g(r) = r^2(1 + f'(r)^2) + h^2.
\]
Then \( 2g' = g + r^3 f' f'' - h^2 \) and thus (9) is equivalent to
\[
r g' = 2g + 2r^{\frac{1}{2}} g^{\frac{4\alpha - 1}{2\alpha}}.
\]
If \( \alpha = 1/2 \), the solution is \( g(r) = mr^2 e^{r^2} \), with \( m \in \mathbb{R} \). If \( \alpha \neq 1/2 \), the solution of this equation is
\[
g(r) = r^2 \left( \frac{1 - 2\alpha}{2\alpha} r^2 + m \right)^{1 - 2\alpha}, \quad m \in \mathbb{R}.
\]
In terms of the function \( f \), we have
\[
1 + f'^2 = \left( \frac{1 - 2\alpha}{2\alpha} r^2 + m \right)^{1 - 2\alpha} - \frac{h^2}{r^2}.
\]
In particular, this gives a restriction on the domain of \( f(r) \).
Theorem 3.2. Let $\Sigma$ be a $K^\alpha$-translator. If $\Sigma$ is a helicoidal surface about the $z$-axis and pitch $h$, then $\Sigma$ parametrizes as \( f \), where

$$ f(r) = \begin{cases} \pm \int \left( me^{r^2 - \frac{h^2}{r^2}} - 1 \right)^{1/2} dt, m > 0, & \alpha = \frac{1}{2} \\ \pm \int \left( \frac{1-2\alpha}{2\alpha} f^2 + m \right) \frac{2\alpha}{r^2} - \frac{h^2}{r^2} - 1 \right)^{1/2} dt, m \in \mathbb{R}, & \alpha \neq \frac{1}{2}. \end{cases} \quad (7) $$

We now show examples of $f'$ for some choices of $\alpha$. In all cases, the pitch is $h = 1$. See Figures 3 and 4.

1. Case $\alpha = 1/2$. We take $m = 1$ in (10). The function $f$ is defined provided $r^2 \geq \log((1 + r^2)/r^2)$, that is, $[r_0, \infty)$, where $r_0 \approx 0.898$.

2. Case $\alpha = 1/4$. Let $m = 0$. Then $1 + f'^2 = -1/r^2 + r^2$. Now the restriction on $r$ is $r^4 - r^2 - 1 \geq 0$, that is, the domain is $[r_0, \infty)$ with $r_0 \approx 1.272$. This case appeared in [14].

3. Case $\alpha = 1$. Let $m = 0$ in (10). Then $1 + f'^2 = 4/r^4 - 1/r^2$ and the domain of $f$ is $[0, r_0]$ with $r_0 \approx 1.249$. Here $\lim_{r \to 0} f(r) = \lim_{r \to 0} f'(r) = \infty$.

Following Lee [14] we may also use the approach of Bour [7]. More explicitly, the Bour coordinates are defined by $r \mapsto s$ and $\theta \mapsto t$, $t = \theta + \Theta$, where

$$ ds = \left( 1 + \frac{r^2}{r^2 + h^2} f'^2 \right)^{1/2} dr, \quad d\Theta = \frac{h}{r^2 + h^2} f'dr. $$

The first fundamental form is now $I = ds^2 + U^2 dt^2$, where the so-called Bour function is introduced by the relation $U^2 = r^2 + h^2$. Using the Bour function $U$, the terms $r$, $f$ and $\Theta$ can be determined by

$$ \begin{cases} r = \sqrt{U^2 - h^2}, \\ df^2 = \frac{U^2}{(U^2 - h^2)} \left( U^2 \left( 1 - \left( \frac{dU}{ds} \right)^2 \right) - h^2 \right) ds^2, \\ d\Theta = \frac{h}{U^2} df. \end{cases} \quad (11) $$

With the above discussion, the Gauss curvature is now $K = -(d^2U/ds^2)/U$ and $\langle N, (0, 0, 1) \rangle = dU/ds$. Then (11) is now
Figure 4. Helicoidal $K^\alpha$-translators: $\alpha = 1/2$ (left), $\alpha = 1/4$ (middle) and $\alpha = 1$ (right).

\[ -\frac{1}{U} \frac{d^2U}{ds^2} = \left( \frac{dU}{ds} \right)^\frac{1}{\alpha} \]

or equivalently

\[ \frac{1}{dU/ds} \frac{d^2U}{ds^2} = -U \left( \frac{dU}{ds} \right)^\frac{1-\alpha}{\alpha}. \]

Setting $P = dU/ds$ and $\frac{dP}{dU} = (ds/dU)(d^2U/ds^2)$, we have

\[ \frac{dP}{dU} = -UP^{1-\alpha}. \]

A first integration is

\[ ds = \begin{cases} m^{-1/e^2} U^{1/2} dU, & \alpha = 1/2, \\ (m - \frac{2\alpha-1}{2s} U^2)^{\frac{\alpha}{1-\alpha}} dU, & \alpha \neq 1/2, \end{cases} \]

where $m \in \mathbb{R}$ is a integration constant with $m > 0$ if $\alpha = 1/2$. Because $s$ can be viewed a function of $U$, we may interchange their roles. Therefore, we obtain again a classification of helicoidal $K^\alpha$-translators in terms of the Bour function $U$.

**Theorem 3.3.** Let $\Sigma$ be a $K^\alpha$-translator. If $\Sigma$ is a helicoidal surface about the z-axis and pitch $h$, then $\Sigma$ parametrizes as

\[ X(U,t) = (\sqrt{U^2 - h^2} \cos(t - \Theta(U)), \sqrt{U^2 - h^2} \sin(t - \Theta(U)), f(U) + h(t - \Theta(U))). \]
where $U$ is the Bour function, $d\Theta = hU^{-2}df$ and $df$ is
\[
\begin{cases}
\frac{+m^{-1}U}{U^2 - h^2} \left( U^2 \left( 1 - m^2 e^{-U^2} \right) - h^2 \right)^{\frac{1}{2}} e^{\frac{U^2}{2h^2}} dU, \quad \alpha = 1/2 \\
\frac{+U}{U^2 - h^2} \left( U^2 \left( 1 - \left( m - \frac{2\alpha - 1}{2\alpha - 1} U^2 \right) \frac{2\alpha - 1}{2\alpha - 1} \right) - h^2 \right)^{\frac{1}{2}} \left( m - \frac{2\alpha - 1}{2\alpha - 1} U^2 \right)^{\frac{2\alpha - 1}{2\alpha - 1}} dU, \quad \alpha \neq 1/2,
\end{cases}
\]
with $m \in \mathbb{R}$ ($m > 0$ if $\alpha = 1/2$).

Proof. Because we see $s$ as a function of $U$ in (12), $U$ can be considered as a new variable. Then we have the first equality in (11), where $r$ depends on this new variable $U$ and as do $f$ and $\Theta$. Considering this, together $\theta = t - \Theta$ in (7), we have the parametrization of the helicoidal $K^\alpha$-translator. Up to $\alpha = 1/2$ or not, from (12) we complete the proof. \[\square\]

Again, for some particular values of $\alpha$, (12) can be explicitly integrated. We present the cases $\alpha = 1/4$ (14), $\alpha = 1/3$ and $\alpha = 1$.

1. Case $\alpha = 1/4$. The solution is
\[
s = \begin{cases}
\frac{1}{8} \left( U\sqrt{m + U^2 + m \cosh^{-1}(U)} \right), \quad U \geq \sqrt{-m}, \quad m < 0, \\
\frac{1}{8} U^2, \quad U > 0, \quad m = 0, \\
\frac{1}{8} \left( U\sqrt{m + U^2 + m \sinh^{-1}(U)} \right), \quad U > 0, \quad m > 0.
\end{cases}
\]

2. Case $\alpha = 1/3$. The solution is
\[
s = mU + \frac{U^3}{6},
\]where $U \in (0, \infty)$.

3. Case $\alpha = 1$. Then,
\[
s = \begin{cases}
\sqrt{\frac{2}{m}} \tanh^{-1} \left( \frac{U}{\sqrt{2m}} \right), \quad U \leq \sqrt{2m}, \quad m > 0 \\
-\sqrt{\frac{2}{m}} \tan^{-1} \left( \frac{U}{\sqrt{2m}} \right), \quad U > 0, \quad m = 0 \\
-\sqrt{\frac{2}{m}} \tanh \left( \sqrt{2ms} \right), \quad |s| \leq (2m)^{-1/2}, \quad m < 0.
\end{cases}
\]

In particular, we can express $U$ in terms of $s$,
\[
U = \begin{cases}
\sqrt{\frac{m}{2}} \tanh \left( \sqrt{2ms} \right), \quad |s| \leq (2m)^{-1/2}, \quad m > 0 \\
\sqrt{\frac{m}{2}}, \quad s > 0, \quad m = 0 \\
-\sqrt{\frac{m}{2}} \tan \left( \sqrt{-2ms} \right), \quad |s| \leq (-2m)^{-1/2}, \quad m < 0.
\end{cases}
\]

4. $K^\alpha$-translators of translation type

By a translation surface of $\mathbb{R}^3$ we mean a surface given by the sum of two curves contained in two coordinate planes. This is a particular case of a Darboux surface where $A(t)$ is the identity. After a rigid motion, the surface is the sum of the curves $\gamma(x) = (x, 0, f(x))$ and
\[ \beta(y) = (0, y, g(y)), \] where \( f: I \subset \mathbb{R} \to \mathbb{R} \) and \( g: J \subset \mathbb{R} \to \mathbb{R} \) are smooth functions in one variable. Thus the surface parametrizes by

\[ X(x, y) = (x, y, f(x) + g(y)), \quad x \in I, y \in J. \tag{14} \]

If we see the surface \( \Sigma \) as the graph of \( z = f(x) + g(y) \), then the problem of finding all translation surfaces that are \( K^\alpha \)-translator is equivalent to ask which are the solutions of (1) obtained by separation of variables \( z = f(x) + g(y) \). In this section we classify all \( K^\alpha \)-translators of translation type.

We calculate all terms of equation (1). Let us observe that once we have the parametrization (14), we cannot prescribed the speed \( \vec{v} \) because the parametrization (14) was previously fixed after a rigid translation type. If the speed is \( \vec{v} \), we have the parametrization (14), we cannot prescribed the speed \( \vec{v} \) because the parametrization (14) was previously fixed after a rigid translation type.

\[ \frac{(f''g'')^\alpha}{(1 + f'^2 + g'^2)^2} = \frac{v_3 - v_1f' - v_2g'}{(1 + f'^2 + g'^2)^{1/2}}, \tag{15} \]

where the prime denotes the derivatives with respect to the corresponding variables \( x \) or \( y \) in each case.

In order to clarify the arguments, we separate the case \( \alpha = 1/4 \). This case appears because the denominators in (15) are cancelled.

**Theorem 4.1.** Let \( \Sigma \) be a \( K^{1/4} \)-translator with speed \( \vec{v} \) of translation type parametrized by (14). Up to a change of the roles of \( f \) and \( g \), the function \( f \) is \( f(x) = x^2/m + ax + b \), \( a, b, m \in \mathbb{R} \), \( m \neq 0 \) and \( g \) is one of the following functions depending on the speed \( \vec{v} \):

1. If the speed is \( \vec{v} = (0, 0, 1) \), then \( g(y) = my^2/4 + cy + d \), \( c, d \in \mathbb{R} \).
2. If the speed is \( \vec{v} = (0, 1, v_3) \), then

\[ g(y) = \frac{v_3(2c + my)}{m} - \frac{(\frac{3}{2})^{2/3}(2c + my)^{2/3}}{m} + d, \tag{16} \]

where \( c, d \in \mathbb{R} \).

**Proof.** Now (15) is

\[ (f''g'')^{1/4} = v_3 - v_1f' - v_2g'. \tag{17} \]

Let us observe the symmetry of the roles of \( f \) and \( g \), so it is enough to distinguish cases according to the function \( f \). Notice that \( f'' \) is not constantly 0 because the case \( K = 0 \) was initially discarded.

1. Case \( f'' = 2/m \neq 0 \) is a non-zero constant, \( m \neq 0 \). In particular, \( f(x) = x^2/m + ax + b \), \( a, b \in \mathbb{R} \). From (17), \( v_1 = 0 \) and \( (2g''/m)^{1/4} = v_3 - v_2g' \). Since \( g'' \neq 0 \) (otherwise, \( K = 0 \)), then \( g'' = \frac{m}{2}(v_3 - v_2g')^2 \). If \( v_2 = 0 \), we assume that \( v_3 = 1 \) and the solution is \( g(y) = my^2/4 + cy + d \), \( c, d \in \mathbb{R} \). If \( v_2 \neq 0 \), then the solution of this equation is (16).
(2) Suppose that \( f'' \) is not constant. Differentiating successively \( f'' \) with respect to \( x \) and next with respect to \( y \), we obtain 
\[
f'''(x)g'''(y) = 0 \text{ for all } x, y.
\]
If at some \( x \), \( f''(x) \neq 0 \), then \( g'' \) is constant in some interval and we are in the previous case (1) interchanging the roles of \( f \) and \( g \). Thus \( f'' = 0 \) in its domain, which it is a contradiction because \( f'' \) is not a constant function.

Let us observe that the surfaces of the case (1) of Theorem 4.1 are affinities of the paraboloid \( z = x^2 + y^2 \) and that the speed \( \vec{v} = (0, 0, 1) \). Other consequence of this result is that we find \( K^{1/4} \)-translators where the speed \( \vec{v} \) is parallel to the \( xy \)-plane by choosing \( \vec{v} = (0, 1, 0) \). If we assume that the speed \( \vec{v} \) is \( (0, 0, 1) \), as usually is taken as a convention in the literature (e.g. [9, 17]), then changing the roles of \( y \) and \( z \), we can provide examples of translation surfaces \( X(x, z) = (x, f(x) + g(z), z) \) whose speed is \( (0, 0, 1) \).

**Example 4.2.** Let \( \alpha = 1/4 \) and \( \vec{v} = (0, 0, 1) \). Suppose that a \( K^{1/4} \)-translator parametrizes as \( y = f(x) + g(z) \). In this case, \( f'' \) is
\[
(f''g'')^{1/4} = g'.
\]
Hence \( g' > 0 \) and
\[
g'' = \frac{1}{f''} = m
\]
for some constant \( m \neq 0 \). Integrating,
\[
f(x) = \frac{1}{2m} x^2 + ax + b, \quad g(z) = -\frac{1}{2m} (-3mz + c)^{2/3} + d,
\]
with \( -3mz + c > 0, c, d \in \mathbb{R} \). For example, choose \( m = 1 \) and \( a = b = c = d = 0 \). Then the surface is
\[
X(x, z) = (x, \frac{1}{2} x^2 - \frac{1}{2} (-3z)^{2/3}, z).
\]
If we see this surface as the graph on the \( xy \)-plane, and after a symmetry about the \( z \)-plane, we have
\[
z = \frac{1}{3} (x^2 - 2y)^{3/2}.
\]

We now consider the general case \( \alpha \neq 1/4 \).

**Theorem 4.3.** If \( \alpha \neq 1/4 \), there are not \( K^\alpha \)-translators that are surfaces of translation.

**Proof.** Again by the symmetry of the roles of \( f \) and \( g \), we discuss according to the function \( f \). Recall that \( f'' \) (and consequently, \( g'' \)) cannot be constantly 0 because then \( K \) would be 0. Then \( (15) \) is
\[
(g'')^\alpha = (1 + f'^2 + g'^2)^{\frac{m-1}{2}} (v_3 - v_1 f' - v_2 g') (f'')^{-\alpha}.
\]
(18)
We differentiate \((18)\) with respect to \(z\), and after some manipulations, we arrive to
\[
(4\alpha - 1)f'f''P - (1 + f'^2 + g'^2) \left( v_1 (f'')^{1-\alpha} + \alpha (f'')^{-\alpha - 1} f''' P \right) = 0,
\]
where \(P = v_3 - v_1 f' - v_2 g'\). The above equation is a polynomial equation on \(g' = g'(y)\) of degree 3, which we write as
\[
A_0 + A_1 g' + A_2 g'^2 + A_3 g'^3 = 0,
\]
where all coefficients \(A_i\) are functions on the variable \(x\). Therefore they must vanish because \(g' \neq 0\). We have
\[
A_1 = -v_2 (4\alpha - 1) f' f'' + \alpha v_2 (1 + f'^2) (f'')^{-\alpha - 1} f''',
A_3 = \alpha (f'')^{-\alpha - 1} f''' v_2.
\]
Since \(4\alpha - 1 \neq 0\), we obtain \(v_2 = 0\). Now
\[
A_0 = (4\alpha - 1)(v_3 - v_1 f')f'f'' - (1 + f'^2)(v_1 (f'')^{1-\alpha} + \alpha (f'')^{-\alpha - 1} f''' (v_3 - v_1 f')),
A_2 = v_1 (f'')^{-\alpha} + \alpha (f'')^{-\alpha - 1} f''' (v_3 - v_1 f').
\]
Using \(A_2 = 0\) into \(A_1\), we arrive to \((4\alpha - 1)(v_3 - v_1 f')f'f'' = 0\), obtaining a contradiction. This completes the proof.

Translations surfaces appear as surfaces obtained by the translation of a curve along another curve. As we have seen, in case that the two curves are included in coordinate planes, then the surface can be written as \(z = f(x) + g(y)\). As we said, the problem to classify all translation surfaces that are \(K^p\)-translators is equivalent to solve equation \((17)\) by separation of variables. Other way of separation of variables is assuming that \(z = f(x)g(y)\), for two smooth functions \(f\) and \(g\). However, the equation \((17)\) is difficult to solve in all its generality. We only show an example where we can obtain non-trivial examples if \(\alpha = 1/4\).

**Example 4.4.** Assume \(\alpha = 1/4\) and the speed is \(\vec{v} = (0, 0, 1)\). Instead to assume \(z = f(x)g(y)\), we suppose \(x = f(z)g(y)\). Then the parametrization of the surface is \(X(z, y) = (f(z)g(y), y, z)\), \(z \in I\), \(y \in J\) and \((1)\) is
\[
fgf''g'' - f'^2 g'^2 = f'^4 g^4. \tag{19}
\]
Because \(\alpha = 1/4\), then \(K > 0\) and so, \(f''(x)g''(y) \neq 0\). We divide \((19)\) with \(f'^2 g''\), obtaining
\[
\frac{ff''}{f'^2} - \frac{g'^2}{gg''} = f'^2 g^3 g'''. \tag{20}
\]
Differentiating successively \((20)\) with respect to \(z\) and \(y\), we have
\[
2 f' f'' \left( \frac{g^3}{g''} \right)' = 0.
\]
Because \( f'f'' \neq 0 \), then \( g'' = ag^3 \), where \( a \in \mathbb{R}, a \neq 0 \). Now (20) is
\[
\frac{ff''}{f'^2} - \frac{f'^2}{a} = \frac{g'^2}{ag^4} = b,
\]
for some nonzero constant \( b \). Then \( g'^2 = abg^4 \) and differentiating and using that \( g'' = ag^3 \), we deduce that \( b = 1/2 \). For the function \( g \), we have
\[
g'^2 = \frac{a}{2}g^4,
\]
in particular, \( a > 0 \). The solution of this equation is \( g(y) = \pm \frac{2}{ay+c}, c \in \mathbb{R} \). We come back to (19), obtaining
\[
ff'' - \frac{1}{2}f'^2 = \frac{f'^4}{a},
\]
We see \( f' \) as a function of \( f \), \( f' = p(f) \). Then \( p' = \frac{dp}{df} = \frac{f''}{f'}, \) so
\[
p' - \frac{p}{2f} = \frac{p^3}{af},
\]
which is an ODE of Bernoulli type. The solution is
\[
f'(z) = \pm \frac{\sqrt{am}\sqrt{f}}{\sqrt{1-2m^2f}}.
\]
The solution \( f(z) \) of this equation together the above function \( g(y) \) provide an example of a \( K^{1/4} \)-translator given by \( x = f(z)g(y) \).

5. \( \text{Ruled } K^\alpha \)-translators

In this section we study the solutions of (1) when the surface is ruled. A parametrization of a ruled surface \( \Sigma \) is
\[
X(s,t) = \gamma(s) + tw(s), \quad s \in I \subset \mathbb{R}, t \in \mathbb{R},
\]
where \( w = w(s) \) is a smooth function with \( |w(s)| = 1 \) for all \( s \in I \) and \( \gamma \) is a curve parametrized by arc-length.

A case to discard is that \( \Sigma \) is a cylindrical surface because \( K = 0 \). Thus, \( w = w(s) \) is a non-constant function. In such a case, we can choose \( \gamma \) to be the striction curve, that is, \( \langle \gamma'(s), w'(s) \rangle = 0 \) for all \( s \in I \). Then
\[
K = -\frac{\lambda^2}{(\lambda^2 + t^2)^2}, \quad \lambda = \frac{\langle \gamma', w, w' \rangle}{|w'|^2},
\]
where the parenthesis \( \langle \cdot, \cdot, \cdot \rangle \) denotes the determinant of the three vectors. In particular, we are assuming that \( \lambda \) is not constantly 0. On the other hand, the unit normal vector \( N \) is
\[
N = \frac{\lambda w' + tw' \times w}{|w'|\sqrt{\lambda^2 + t^2}}.
\]
In particular, $K$ is negative, so we are assuming that $\alpha$ is an integer. Then (1) writes as

$$(-1)^\alpha \frac{\lambda^{2\alpha}}{(\lambda^2 + t^2)^{2\alpha}} = \frac{1}{|w'|\sqrt{\lambda^2 + t^2}} (\lambda \langle w', \vec{v} \rangle + t (w', w, \vec{v})).$$

or equivalently,

$$-(-1)^\alpha \lambda^{2\alpha} |w'| + \lambda \langle w', \vec{v} \rangle (\lambda^2 + t^2)^{\frac{4\alpha - 1}{2}} + (w', w, \vec{v}) t (\lambda^2 + t^2)^{\frac{4\alpha - 1}{2}} = 0. \quad (21)$$

The Wronskian of the functions $\{1, (\lambda^2 + t^2)^{\frac{4\alpha - 1}{2}}, t (\lambda^2 + t^2)^{\frac{4\alpha - 1}{2}}\}$ is

$$(4\alpha - 1)(\lambda^2 + t^2)^{4\alpha-3} \left(4\alpha t^2 - \lambda^2\right).$$

Since $\alpha \neq 1/4$, the Wronskian if not 0, proving these three functions are linearly independent. This yields a contradiction with (21). As a conclusion, we have the following result.

**Theorem 5.1.** There are not $K^\alpha$-translators that are ruled surfaces.

Let us notice that this result of non-existence has discarded the case that the surface is cylindrical (the rulings are parallel to $\vec{v}$) or that $\lambda = 0$, that is, the surface is developable. The condition $\lambda = 0$ together $(w, w', \vec{v}) = 0$ is equivalent to the case that the base curve is a planar curve parallel to $\vec{v}$, obtaining that the surface is part of a plane.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE. FIRAT UNIVERSITY. ELAZIG 23200 TURKEY

Email address: meaydin@firat.edu.tr

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA. UNIVERSIDAD DE GRANADA. AVENIDA FUENTENUEVA, S/N. GRANADA 18071 SPAIN

Email address: rcamino@ugr.es