Weak diameter coloring of graphs on surfaces

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Abstract

Consider a graph $G$ drawn on a fixed surface, and assign to each vertex a list of colors of size at least two if $G$ is triangle-free and at least three otherwise. We prove that we can give each vertex a color from its list so that each monochromatic connected subgraph has bounded weak diameter (i.e., diameter measured in the metric of the whole graph $G$, not just the subgraph). In case that $G$ has bounded maximum degree, this implies that each connected monochromatic subgraph has bounded size. This solves a problem of Esperet and Joret for planar triangle-free graphs, and extends known results in the general case to the list setting, answering a question of Wood.

The colorings in this paper are not necessarily proper. The weak diameter of a subgraph $H$ of a graph $G$ is the maximum distance in $G$ between vertices of $V(H)$; note that the distances are measured in the whole graph $G$, not in the subgraph $H$. For a non-negative integer $\ell$, a weak diameter-$\ell$ coloring of a graph $G$ is an assignment of colors to its vertices such that each monochromatic connected subgraph has weak diameter at most $\ell$ (in particular, a weak diameter-0 coloring is just a proper coloring). We say a class of graphs $G$ has weak diameter chromatic number at most $k$ if for some $\ell$, every graph in $G$ has a weak diameter-$\ell$ coloring using at most $k$ colors.

Weak diameter coloring arises in the context of asymptotic dimension of graph classes: Denoting by $G^r$ the graph obtained from $G$ by joining by an edge each pair of distinct vertices at distance at most $r$, the class $\mathcal{G}$ has asymptotic dimension at most $d$ if for every $r \geq 1$, the class $\{G^r : G \in \mathcal{G}\}$ has weak diameter chromatic number at most $d + 1$. In a recent breakthrough, Bonamy et al. [1] proved that graphs of bounded treewidth have asymptotic dimension 1 and all proper minor-closed classes have asymptotic dimension at most 2. As a special case:

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**Theorem 1** (Bonamy et al. [1]). *For any surface $\Sigma$, the class of graphs drawn on $\Sigma$ has weak diameter chromatic number at most 3.*

In graphs with bounded maximum degree, the notion of weak diameter coloring coincides with the well-studied notion of *clustered coloring*. For a positive integer $s$, a *coloring with clustering $s$* is an assignment of colors to vertices such that each monochromatic component has size at most $s$ (a coloring with clustering 1 is just a proper coloring). A class of graphs $\mathcal{G}$ has *clustered chromatic number* at most $k$ if for some $s$, every graph in $\mathcal{G}$ has a coloring with clustering at most $s$ using at most $k$ colors. We refer the reader to an extensive survey by Wood [5] for further background on clustered coloring.

**Observation 2.** Let $k \geq 1$ be an integer and let $\mathcal{G}$ be a class of graphs.

- If $\mathcal{G}$ has clustered chromatic number at most $k$, then $\mathcal{G}$ also has weak diameter chromatic number at most $k$.
- If $\mathcal{G}$ has weak diameter chromatic number at most $k$ and bounded maximum degree, then $\mathcal{G}$ also has clustered chromatic number at most $k$.

Hence, by Theorem 1, the class of graphs of bounded maximum degree drawn on any fixed surface $\Sigma$ has clustered chromatic number at most three. This was proved earlier by Esperet and Joret [3], who also asked the following question.

**Question 3.** Consider a surface $\Sigma$ and a positive integer $\Delta$. Is it true that the class of triangle-free graphs of maximum degree $\Delta$ than can be drawn on $\Sigma$ has clustered chromatic number at most two?

It is also natural to consider the list versions of these notions. Given an assignment $L$ of lists of colors to vertices of a graph $G$, a coloring of $G$ is an $L$-coloring if the color of each vertex $v \in V(G)$ belongs to $L(v)$. We say that a class of graphs $\mathcal{G}$ has *weak diameter choosability* at most $k$ if for some $\ell$, every graph in $\mathcal{G}$ has a weak diameter-$\ell$ $L$-coloring from any assignment $L$ of lists of size at least $k$; and *clustered choosability* at most $k$ if for some $s$, every graph in $\mathcal{G}$ has an $L$-coloring with clustering at most $s$ from any assignment $L$ of lists of size at least $k$.

For any surface $\Sigma$, the class of graphs drawn on $\Sigma$ has clustered choosability at most four [2]. This bound cannot be improved (even for non-list coloring), as for every $s$, there exists a planar graph that has no coloring with clustering at most $s$ using at most three colors. However, it has been asked by Wood whether this can be improved for graphs of bounded maximum degree (matching the result of Esperet and Joret [3] in the non-list setting).
**Question 4** (Wood [5, Open Problem 18]). Consider a surface $\Sigma$ and a positive integer $\Delta$. Is it true that the class of graphs of maximum degree $\Delta$ than can be drawn on $\Sigma$ has clustered choosability at most three?

We answer Questions 3 and 4 in positive, in the more general setting of weak diameter choosability.

**Theorem 5.** For every surface $\Sigma$,

(i) the class of graphs drawn on $\Sigma$ has weak diameter choosability at most three, and

(ii) the class of triangle-free graphs drawn on $\Sigma$ has weak diameter choosability at most two.

It is tempting to ask whether Theorem 5 could be strengthened by replacing weak diameter with diameter, where we measure the diameter of each monochromatic connected subgraph inside the subgraph (not in the ambient graph). However, such a strengthening of both (i) and (ii) is false even for the class of planar graphs and non-list coloring, as we show in Section 1.

We suspect the following substantial relaxation of the “triangle-free” assumption could be sufficient to make coloring by two colors possible.

**Question 6.** For a non-negative integer $r$, let $G_r$ be the class of plane graphs with no separating triangles such that each vertex is at distance at most $r$ from a face of length at least four. Does $G_r$ have weak diameter chromatic number at most two?

The importance of the presence of non-triangular faces can be seen from the following standard example: Let $G_n$ be the $n \times n$ grid with a diagonal added to each face (so that all faces except for the outer one are triangles). By the HEX lemma, any coloring of $G_n$ by colors red and blue contains a red path from the left side to the right side of the grid, or a blue path from the top side to the bottom side of the grid. Hence, any coloring of $G_n$ by two colors contains a monochromatic component of weak diameter at least $n - 1$.

Moreover, let us remark that Question 6 has positive answer in the case that the graph has exactly one non-triangular face, since plane graphs where all vertices are at a bounded distance from a fixed face have bounded treewidth and consequently asymptotic dimension one.

The rest of the paper is organized as follows. In Section 1 we show that weak diameter cannot be replaced by diameter in Theorem 5. In Section 2 we give the proof of Theorem 5, deferring the parts specific to the triangle-free case and to the non-triangle-free case to sections Section 3 and 4.
1 Counterexamples for diameter coloring

In this section, for every positive integer \( \ell \) we construct

- a planar triangle-free graph \( G_\ell \) such that any coloring of \( G_\ell \) by two colors contains a monochromatic component of diameter at least \( \ell \), and

- a planar graph \( G'_\ell \) such that any coloring of \( G'_\ell \) by three colors contains a monochromatic component of diameter at least \( \ell \).

This shows that it is necessary to consider the weak diameter in both cases of Theorem 5.

1.1 The triangle-free case

For a positive integer \( k \), consider the following graphs \( H_{0,k}, H_{1,k}, \ldots, \) each with two distinct interface vertices \( u \) and \( v \): The graph \( H_{0,k} \) consists just of the vertices \( u \) and \( v \). For \( i \geq 1 \), the graph \( H_{i,k} \) consists of a path \( P = v_1 \ldots v_k \) such that for \( j = 1, \ldots, k \), \( v_j \) is adjacent to \( u \) if \( j \) is odd and to \( v \) if \( j \) is even, and \( k \) copies of \( H_{i−1,k} \) such that the \( j \)-th one has interface vertices \( v_j \) and \( v \) if \( j \) is odd and \( v_j \) and \( u \) if \( j \) is even; see Figure 1. Note that \( H_{i,k} \) is planar, triangle-free, and can be drawn so that its interface vertices are incident with the outer face. For a coloring \( \varphi \) and a vertex \( x \), let \( r_\varphi(x) \) denote the maximum distance between \( x \) and another vertex in the same monochromatic component of \( \varphi \).

**Lemma 7.** For all integers \( i \geq 0 \) and \( k \geq 1 \), if a coloring \( \varphi \) assigns colors 1 and 2 to vertices of the graph \( H_{i,k} \) and both interface vertices \( u \) and \( v \) receive color 1, then
• \( \varphi \) contains a component of color 2 of diameter at least \( k-1 \), or
• \( u \) and \( v \) are in the same monochromatic component of \( \varphi \), or
• \( r_\varphi(u) + r_\varphi(v) \geq i \).

Proof. We prove the claim by induction on \( i \). The case \( i = 0 \) is trivial, and thus we can assume \( i \geq 1 \). Let \( P = v_1 v_2 \ldots v_k \) be the path in \( H_{i,k} \) from the definition. Suppose that \( \varphi \) does not contain a component of color 2 of diameter at least \( k-1 \); then the whole path \( P \) cannot be colored by color 2, and thus \( \varphi(v_j) = 1 \) for some \( j \in \{1, \ldots, k\} \). By symmetry, we can assume that \( j \) is odd. Consider the copy of \( H_{i-1,k} \) with interface vertices \( v \) and \( v_j \), and let \( \varphi' \) be the restriction of \( \varphi \) to this copy. Then \( \varphi' \) also cannot contain a component of color 2 of diameter at least \( k-1 \).

Moreover, suppose that \( u \) and \( v \) are in different monochromatic components of \( \varphi \); since \( uv \) is an edge and \( \varphi(u) = \varphi(v_j) = 1 \), this implies that \( v \) and \( v_j \) are not in the same monochromatic component of \( \varphi' \).

By the induction hypothesis, it follows that \( r_{\varphi'}(v_j) + r_{\varphi'}(v) \geq i-1 \). Since \( u \) and \( v \) are in different monochromatic components, a monochromatic path from \( u \) cannot pass through \( v \), and thus \( r_{\varphi}(u) \geq r_{\varphi'}(v_j) + 1 \). Similarly, \( r_{\varphi}(v) \geq r_{\varphi'}(v) \). Therefore, \( r_{\varphi}(u) + r_{\varphi}(v) \geq (r_{\varphi'}(v_j) + 1) + r_{\varphi'}(v) \geq i \). \( \square \)

Let \( W_\ell \) be the \((\ell + 1) \times (\ell + 1)\) grid with diagonals added to the 4-faces. Let \( G_\ell \) be the graph obtained from \( W_\ell \) by replacing each diagonal \( uv \) by a copy of \( H_{\ell+1,2\ell} \) with interface vertices \( u \) and \( v \), see the right part of Figure 1. Consider any 2-coloring of \( G_\ell \). By the HEX lemma, the corresponding 2-coloring of \( W_\ell \) contains a monochromatic path \( Q \) (say in color 1) joining the opposite sides of the grid. If both ends of \( Q \) belong to the same monochromatic component of \( \varphi \) on \( G_\ell \), then this component has diameter greater than \( \ell \). Otherwise, there exists an edge \( uv \in E(Q) \) such that \( u \) and \( v \) belong to different monochromatic components of \( \varphi \), and thus also to different monochromatic components of \( \varphi \) restricted to the copy of \( H_{\ell+1,2\ell} \) with interface vertices \( u \) and \( v \). By Lemma 7 this implies that either one of the monochromatic components of \( u \) and \( v \) has diameter at least \( \ell \), or \( \varphi \) contains a component of color 2 of diameter at least \( \ell \).

1.2 The non-triangle-free case

For the case of general planar graphs colored by three colors, we use a similar construction. The graph \( H'_{i,k} \) whose recursive construction is depicted in Figure 2 has the following property: Suppose that the interface vertices \( u \) and \( v \) receive colors 1 and 2, respectively. Then

• the \( k \)-vertex path is colored by 3 (resulting in a monochromatic component of diameter at least \( k-1 \), or
• a vertex \( x \) of the path has color 1, there exists a copy of \( H'_{i-1,k} \) with interface vertices \( x \) and \( v \) of colors 1 and 2, and the monochromatic component of \( x \) additionally contains the edge \( xu \), or

• a symmetric situation with a vertex \( x \) of the path receiving color 2.

As in the previous case, this implies that any coloring of \( H'_{\ell+1,2\ell} \) by three colors where the interface vertices receive a different color contains a monochromatic component of diameter at least \( \ell \). The graph \( G'_{\ell} \) is then obtained by concatenating \( \ell \) copies of this graph, as depicted on the right side of Figure 2.

2 The proof of Theorem 5

In this section we present the common parts of the proofs of Theorem 5 (i) and (ii). We use \( c \) to denote the number of colors in each list and \( t \) the lower bound on the girth of the considered graph, where \( c = 3 \) and \( t = 3 \) for the proof of Theorem 5 (i), while \( c = 2 \) and \( t = 4 \) for Theorem 5 (ii). Observe that \( t = \frac{2c}{c-1} \) in both cases.

The starting point of our proof is a standard island argument. A non-empty set \( I \subseteq V(G) \) is a \( c \)-island if every vertex in \( I \) has less than \( c \) neighbors outside of \( I \). For real numbers \( a \) and \( b \), we say that a graph \( G \) is \((a,b)\)-sparse if \( |E(G)| \leq a|V(G)| + b \), and hereditarily \((a,b)\)-sparse if every induced subgraph of \( G \) is \((a,b)\)-sparse. In [2] we proved the following claim.

Lemma 8. For all positive integers \( c \) and \( b \), any real number \( \varepsilon > 0 \), and every surface \( \Sigma \), there exists a positive integer \( s \) such that the following claim holds: Every \((c-\varepsilon,b)\)-sparse graph \( G \) with \( V(G) \neq \emptyset \) drawn on \( \Sigma \) contains a \( c \)-island of size at most \( s \).

The presence of \( c \)-islands can be used to obtain clustered colorings.
Corollary 9. For all positive integers $c$ and $b$, any real number $\varepsilon > 0$, and every surface $\Sigma$, the class of hereditarily $(c-\varepsilon, b)$-sparse graphs drawn on $\Sigma$ has clustered choosability at most $c$.

Proof. Let $s$ be the constant from Lemma 8. We show that every hereditarily $(c-\varepsilon, b)$-sparse graph $G$ drawn on $\Sigma$ has an $L$-coloring with clustering at most $s$ for any assignment $L$ of lists of size $c$. We prove the claim by induction on the number of vertices of $G$. The claim is trivial if $G$ has no vertices. Otherwise, by Lemma 8, $G$ contains a $c$-island $I$ of size at most $s$. By the induction hypothesis, $G-I$ has an $L$-coloring with clustering at most $s$. We color $I$ so that each vertex $v \in I$ chooses a color from $L(v)$ different from the colors of its neighbor outside of $I$. This ensures that any newly arising monochromatic components are contained in $I$, and thus they have size at most $s$.

To this end, let us introduce the notion of sparsifiers.

A multiassignment for a graph $S$ is an assignment of multisets to vertices of $S$. For a multiassignment $B$ to vertices of a graph $S$, we say that a coloring of $S$ is $B$-opaque if for each color $a$, each connected subgraph of $S$ of color $a$ contains at most one vertex $v$ such that $a \in B(v)$ and if there is such a vertex $v$, then $a$ appears in $B(v)$ with multiplicity one. The motivation for this definition is as follows: If $S$ is an induced subgraph of a colored graph $G$ and $B(u)$ consists of colors that appear on the neighbors of $u$ in $V(G) \setminus V(S)$, then $B$-opacity implies that no two monochromatic components of $G-V(S)$ are contained in the same monochromatic component of $G$.

A $c$-sparsifier is a pair $(S, \gamma)$, where $S$ is a connected graph and $\gamma : V(S) \to \mathbb{Z}_+^*$ assigns an integer $\gamma(v) \geq \deg v$ to each vertex $v \in V(S)$, with the following property: For any assignment $L$ of lists of size $c$ to vertices of $S$ and a multiassignment $B$ of lists to vertices of $S$ such that $|B(v)| \leq \gamma(v) - \deg v$ for each $v \in V(S)$, there exists a $B$-opaque $L$-coloring of $S$. The size of the sparsifier is $|V(S)|$. An appearance of a $c$-sparsifier $(S, \gamma)$ in a graph $G$ drawn on a surface is an injective function $h : V(S) \to V(G)$ such that

- for $u, v \in V(S)$, we have $uv \in E(S)$ if and only if $h(u)h(v) \in E(G)$ (i.e., $h$ shows $S$ is an induced subgraph of $G$),
- for $u \in V(S)$, we have $\deg_G h(u) \leq \gamma(u)$, and
- for $u \in V(S)$, every face of $G$ incident with $h(u)$ is bounded by a cycle of length $t = \frac{2c}{c-\varepsilon}$.
We write $G - h$ for the graph obtained from $G$ by deleting all vertices in the image of $h$. Appearances $h_1$ and $h_2$ of $c$-sparsifiers $(S_1, \gamma_1)$ and $(S_2, \gamma_2)$ are independent if $h_1(u) \neq h_2(v)$ and $h_1(u)h_2(v) \notin E(G)$ for every $u \in V(S_1)$ and $v \in V(S_2)$.

The definition of a sparsifier and its appearance is motivated by the following properties.

**Lemma 10.** Let $G$ be a graph drawn on a surface, let $c \geq 2$, $p \geq 1$ and $\ell \geq 0$ be integers, let $L$ be an assignment of lists of size $c$ to vertices of $G$, and let $h_1, \ldots, h_m$ be pairwise-independent appearances of $c$-sparsifiers of size at most $p$ in $G$. If $G - \{h_1, \ldots, h_m\}$ has a weak diameter-$\ell$ $L$-coloring $\varphi$, then $G$ has a weak diameter-$(\ell + 2p)$ $L$-coloring.

**Proof.** For $i \in \{1, \ldots, m\}$, we extend $\varphi$ to the image of $h_i$ as follows. Let $(S_i, \gamma_i)$ be the $c$-sparsifier with appearance $h_i$. For $u \in V(S_i)$, let $L_i(u) = L(h_i(u))$ and $B_i(u) = \{\varphi(v) : vh_i(u) \in E(G), v \in V(G - \{h_1, \ldots, h_m\})\}$. Since $\deg_G h_i(u) \leq \gamma_i(u)$, we have $|B_i(u)| \leq \gamma_i(u) - \deg_S u$. By the definition of a $c$-sparsifier, there exists a $B_i$-opaque $L_i$-coloring $\psi_i$ of $S_i$, and for each $u \in V(S_i)$, we define $\varphi(h_i(u)) = \psi_i(u)$.

Since $\psi_i$ is $B_i$-opaque for each $i$, each monochromatic component of $G$ in the coloring $\varphi$ is either contained in the image of $h_i$ for some $i$, or it is obtained from a monochromatic component of $G - \{h_1, \ldots, h_m\}$ by adding disjoint non-adjacent connected subgraphs with at most $p$ vertices. We conclude that each monochromatic component of $G$ has weak diameter at most $\ell + 2p$. \hfill \Box

A system $h_1, \ldots, h_m$ of pairwise-independent appearances of $c$-sparsifiers of size at most $p$ in a graph $G$ drawn on a surface is maximal if there does not exist an appearance of a $c$-sparsifier of size at most $p$ in $G$ independent of $h_1, \ldots, h_m$. A graph $G$ is $(c, p)$-sparsifier-free if no $c$-sparsifier of size at most $p$ has an appearance in $G$.

**Lemma 11.** Let $G$ be a graph drawn on a surface, let $c \geq 2$ and $p \geq 1$ be integers, and let $h_1, \ldots, h_m$ be pairwise-independent appearances of $c$-sparsifiers of size at most $p$ in $G$. Let $t = \frac{2p}{c-1}$. If the system $h_1, \ldots, h_m$ is maximal, $|V(G)| > t$ and $G$ does not contain any separating cycle of length $t$, then every induced subgraph $G'$ of $G - \{h_1, \ldots, h_m\}$ is $(c, p)$-sparsifier-free.

**Proof.** Suppose for a contradiction that $h$ is an appearance of a $c$-sparsifier $(S, \gamma)$ of size at most $p$ in $G'$. By the last condition in the definition of an appearance, all faces incident with the $h$-images of vertices of $S$ are bounded by $t$-cycles. Since $G$ does not contain separating $t$-cycles, $|V(G)| > t$, and $G'$ is an induced subgraph of $G$, these faces are also faces of $G$. In particular, all the vertices in the image of $h$ have the same degree in $G$ as in $G'$. Hence, $h$ is also an appearance of $(S, \gamma)$ in $G$ independent from $h_1, \ldots, h_m$, contradicting the maximality of the system. \hfill \Box
Finally, we will need the following lemma, whose proof is specific to the cases \( c \in \{2, 3\} \) and is given in Sections \([3, 4]\)

**Lemma 12.** For \( c \in \{2, 3\} \), there exists a constant \( \varepsilon_c > 0 \) such that the following claim holds. Let \( G \) be a graph of minimum degree at least \( c \) and girth at least \( t = \frac{2c}{c-1} \) drawn on a surface of Euler genus \( g \) with no non-contractible cycles of length at most four. Suppose that \( G \) is \((c, 4)\)-sparsifier-free and does not contain separating cycles of length \( t \). Then \( G \) is \((c - \varepsilon_c, 10(g + 3))\)-sparse.

Let us now combine these claims.

**Corollary 13.** For \( c \in \{2, 3\} \) and every surface \( \Sigma \), the class \( \mathcal{G}_{c, \Sigma} \) of graphs of girth at least \( t = \frac{2c}{c-1} \) drawn on \( \Sigma \) with no non-contractible cycles of length at most four and no separating cycles of length \( t \) has weak diameter choosability at most \( c \).

**Proof.** Let \( \varepsilon_c > 0 \) be the constant from Lemma 12 and let \( g \) be the Euler genus of \( \Sigma \). By Corollary 9 there exists \( s \) such that every hereditarily \((c - \varepsilon_c, 10(g + 3))\)-sparse graph drawn on \( \Sigma \) has a coloring with clustering at most \( s \) from any assignment of lists of size \( c \).

Consider a graph \( G \in \mathcal{G}_{c, \Sigma} \) and an assignment \( L \) of lists of size \( c \) to vertices of \( G \). Let \( h_1, \ldots, h_m \) be a maximal system of pairwise-independent \( c \)-sparsifiers of size at most \( 4 \) in \( G \). Let \( G_0 = G - \{h_1, \ldots, h_m\} \). We claim that \( G_0 \) is hereditarily \((c - \varepsilon_c, 10(g + 3))\)-sparse. Hence, we need to prove that every induced subgraph \( G' \) of \( G_0 \) is \((c - \varepsilon_c, 10(g + 3))\)-sparse. We prove the claim by induction on \( |V(G')| \). If \( |V(G')| \leq t \), then the claim is trivial since \( 10(g + 3) \geq 10 \geq |E(G')| \). In particular, we can assume that \( |V(G)| > t \), and Lemma 11 implies \( G' \) is \((c, 4)\)-sparsifier-free. If a vertex \( v \in V(G') \) has degree at most \( c - 1 \), then \( |E(G')| \leq (c - 1) + |E(G' - v)| \leq (c - 1) + (c - \varepsilon_c) |V(G' - v)| + 10(g + 3) \leq (c - \varepsilon_c) |V(G')| + 10(g + 3) \) by the induction hypothesis. On the other hand, if \( G' \) has minimum degree at least \( c \), then \( G' \) is \((c - \varepsilon_c, 10(g + 3))\)-sparse by Lemma 12.

By Corollary 9, \( G_0 \) has an \( L \)-coloring \( \varphi \) with clustering at most \( s \). Then \( \varphi \) is also a weak diameter-\((s - 1)\) coloring. By Lemma 10, \( G \) has a weak diameter-\((s + 7)\) \( L \)-coloring.

Next, we need to take care of separating \( t \)-cycles. These are generally dealt with using standard precoloring arguments, but the cases where a vertex has \( c \) precolored neighbors turn out to be somewhat problematic and require us to handle the following special case separately. For a cycle \( K \) in a plane graph \( G \), let \( G_K \) denote the subgraph of \( G \) drawn in the closed disk bounded by \( K \). The 3-base is the plane drawing of \( K_4 \) and the 4-base is the plane drawing of \( K_{2,3} \). For \( t \in \{3, 4\} \), a finite plane graph \( G \) is a \( t \)-stack if it is either a cycle of length \( t \), or if there exists a \( t \)-base \( H \subseteq G \) such that the outer face of \( G \) is equal to the outer face of \( H \) and for each internal face of
Figure 3: A 3-stack and a 4-stack.

$H$ bounded by a $t$-cycle $K$, the graph $G_K$ is a $t$-stack. See Figure 3 for an example of a 3-stack and a 4-stack. Let $C$ be the cycle bounding the outer face of a $t$-stack $G$, let $\psi$ be a coloring of $C$ and let $\varphi$ be a coloring of $G$ that extends $\psi$. We say that $\varphi$ is $\psi$-opaque if no monochromatic component of $\varphi$ on $G$ contains vertices belonging to two distinct monochromatic components of $\psi$ on $C$. The proof of the following lemma is specific to the cases $c \in \{2, 3\}$ and is given in Sections 3 and 4.

**Lemma 14.** For $c \in \{2, 3\}$, let $t = \frac{2c}{c-1}$. Let $G$ be a $t$-stack and let $L$ be an assignment of lists of size $c$ to vertices of $G$. Then every $L$-coloring $\psi$ of the cycle $C$ bounding the outer face of $G$ extends to a weak diameter-$4$ $\psi$-opaque $L$-coloring $\varphi$ of $G$.

A cycle $C$ in a graph $G$ is $c$-solitary if every vertex $v \in V(G) \setminus V(C)$ has fewer than $c$ neighbors in $C$. Given a coloring $\psi$ of $C$, we say that a coloring $\varphi$ of $G$ properly extends $\psi$ if the restriction of $\varphi$ to $C$ is equal to $\psi$ and $\varphi(u) \neq \varphi(v)$ for every $uv \in E(G)$ such that $u \in V(C)$ and $v \notin V(C)$. For a graph $G$ drawn on a surface of non-zero Euler genus and a contractible cycle $K$ in $G$, let $G_K$ denote the subgraph of $G$ drawn in the unique closed disk in the surface bounded by $K$.

**Lemma 15.** For $c \in \{2, 3\}$ and every surface $\Sigma$, there exists a positive integer $\ell$ such that the following claim holds. Let $G$ be a graph of girth at least $t = \frac{2c}{c-1}$ drawn on $\Sigma$ without non-contractible cycles of length at most four and let $L$ be an assignment of lists of size $c$ to vertices of $G$. Suppose that either $C$ is an empty graph, or $\Sigma$ is the plane and $C$ is a cycle of length $t$ bounding the outer face of $G$, and let $\psi$ be an $L$-coloring of $C$. If $C$ is $c$-solitary, then $\psi$ properly extends to a weak diameter-$\ell$ $L$-coloring of $G$. 

10
Proof. Let $\Sigma_0$ be the sphere. By Corollary 13 there exists $\ell'$ such that every graph from $G_{c,\Sigma} \cup G_{c,\Sigma_0}$ has a weak diameter-$\ell'$ coloring from any assignment of lists of size $c$. Let $\ell = 2\ell' + 22$.

We prove the claim by induction on the number of vertices of $G$. Suppose first that there exists a separating $t$-cycle $K$ in $G$ (necessarily contractible, since $t \leq 4$) such that $K$ is $c$-solitary in $G_K$. Let $G_1$ be the graph obtained from $G$ by deleting the vertices and edges drawn in the open disk bounded by $K$. By the induction hypothesis, $\psi$ properly extends to a weak diameter-$\ell$ $L$-coloring $\varphi_1$ of $G_1$. Using the induction hypothesis again, the restriction of $\varphi_1$ to $K$ properly extends to a weak diameter-$\ell$ $L$-coloring $\varphi_2$ of $G_K$ ($G_K$ is drawn in the plane rather than in $\Sigma$ when $\Sigma$ has positive genus, but this is not a problem, as we included the genus-0 case in the choice of $\ell'$). Since $\varphi_2$ properly extends the restriction of $\varphi_1$, each monochromatic component of $G$ in the $L$-coloring $\varphi_1 \cup \varphi_2$ is contained in $G_1$ or $G_K$, and thus has weak diameter at most $\ell$.

Hence, we can assume there is no such separating $t$-cycle. Observe that this implies that for each separating $t$-cycle, the graph $G_K$ is a $t$-stack. Let $K_1, \ldots, K_m$ be separating $t$-cycles in $G$ such that the open disks bounded by them are inclusionwise-maximal, and observe that these open disks are disjoint. Let $G'$ be the graph obtained from $G$ by deleting vertices and edges drawn in these disks. Then $G'$ has no separating $t$-cycles, and thus $G' \in G_{c,\Sigma}$. By Corollary 13 $G' - V(C)$ has a weak diameter-$\ell'$ $L$-coloring $\varphi'$. For $i \in \{1, \ldots, m\}$, Lemma 14 implies the restriction $\psi_i$ of $\varphi' \cup \psi$ to $K_i$ extends to a weak diameter-$4$ $\psi_i$-opaque $L$-coloring $\varphi_i$ of $G'_K$. Then $\varphi'' = \varphi' \cup \varphi_1 \cup \ldots \cup \varphi_m$ is a weak diameter-$(\ell' + 8)$ $L$-coloring of $G - V(C)$.

Let $\varphi$ be the $L$-coloring that matches $\psi$ on $C$, $\varphi(v) \in L(v)$ is chosen as an arbitrary color different from the colors of the neighbors of $v$ in $C$ for every vertex $v \in V(G) \setminus V(C)$ with at least one neighbor in $C$ (this is possible, since $C$ is $c$-solitary), and $\varphi(v) = \varphi''(v)$ for each vertex $v$ at distance at least two from $C$. Every monochromatic component in $\varphi$ not contained in $C$ is obtained from a disjoint union of connected monochromatic subgraphs in $\varphi''$ by adding neighbors of vertices of $C$, and thus the distance between any two vertices of the resulting monochromatic component is at most $2(\ell' + 8) + 6 = \ell$.

To finish the proof, we need to deal with non-contractible cycles of length at most 4.

Proof of Theorem 5. For $c \in \{2, 3\}$, let $t = \frac{2c}{c-1}$. For a non-negative integer $g$, let $\ell_g'$ be the maximum of the constants $\ell$ from Lemma 15 over all surfaces of Euler genus at most $g$, and let us define $\ell_0 = \ell_0'$ and $\ell_g = \max(\ell_g', 2\ell_{g-1} + 4)$. We prove by induction on $g$ that any graph $G$ of girth at least $t$ drawn on a surface of Euler genus at most $g$ has a weak diameter-$\ell_g$ $L$-coloring from any assignment $L$ of lists of size $c$. 

11
If \( g = 0 \), then \( G \) does not contain any non-contractible cycles, and thus the claim follows from Lemma \([15]\) (with \( C \) being an empty graph, and considering the drawing of \( G \) in the plane instead of on the sphere). Hence, suppose that \( g > 0 \). If \( G \) does not contain any non-contractible cycle of length at most 4, then the claim again follows from Lemma \([15]\). Hence, suppose \( K \) is a non-contractible cycle of length at most 4 in \( G \). Then each component of the graph \( G - V(K) \) can be drawn on a surface of Euler genus at most \( g - 1 \), and by the induction hypothesis, \( G - V(K) \) has a weak diameter-\( \ell_{g-1} \)-\( L \)-coloring. We extend this \( L \)-coloring to \( G \) by choosing the colors of vertices of \( K \) from their lists arbitrarily; each monochromatic component of the resulting \( L \)-coloring has weak diameter at most \( 2\ell_{g-1} + 4 \leq \ell_g \), as required.

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\Box
\]

3 The triangle-free case

Let us now provide the proofs of Lemmas \([12]\) and \([14]\) in the case \( c = 2 \). Let \( S_1 \) be a single vertex and \( \gamma_1 \) the function assigning to this vertex the value 3, and let \( S_2 \) be the 4-cycle and \( \gamma_2 \) the function assigning to all its vertices the value 4.

**Lemma 16.** Both \((S_1, \gamma_1)\) and \((S_2, \gamma_2)\) are 2-sparsifiers of size at most 4.

**Proof.** Consider \( i \in \{1, 2\} \), let \( L \) be an assignment of lists of size 2 to vertices of \( S_i \), and let \( B \) be a multiassignment of a list of size 3 in case \( i = 1 \) and of lists of size 2 in case \( i = 2 \). In the case \( i = 1 \), choose \( \varphi(v) \in L(v) \) to be different from the color that appears in \( B(v) \) twice (if any). Clearly, \( \varphi \) is \( B \)-opaque.

In the case \( i = 2 \), we choose the \( L \)-coloring of the 4-cycle \( S_2 \) as follows. Let \( R \) be the set of vertices \( v \in V(S_2) \) such that \( B(v) \) contains some color \( a \) with multiplicity two; for such a vertex, set \( L'(v) = L(v) \setminus \{a\} \). For any vertex \( v \in V(S_2) \setminus R \), set \( L'(v) = L(v) \). Orient the cycle \( S_2 \) arbitrarily, and let \( S' \) be the graph obtained from \( S_2 \) by, for each vertex \( v \in R \), deleting the edge that follows it in \( S_2 \) in this orientation. Note that \( |L'(v)| \geq \deg_{S'} v \) for each \( v \in V(S_2) \), and that either \( S' \) is a 4-cycle, or the first vertex \( u \) of each component of \( S' \) according to the orientation of the 4-cycle satisfies \( |L'(u)| > \deg_{S'} u \). Consequently, \( S' \) has a proper \( L' \)-coloring \( \varphi \). We claim that \( \varphi \) is \( B \)-opaque. Indeed, consider distinct vertices \( v_1, v_2 \in V(S_2) \) such that \( \varphi(v_1) = \varphi(v_2) = a \in B(v_1) \cap B(v_2) \). Clearly, \( v_1, v_2 \in V(S_2) \setminus R \), and thus the vertices following \( v_1 \) and \( v_2 \) in \( S_2 \) have colors different from \( a \). Hence, \( v_1 \) and \( v_2 \) are not in the same monochromatic component.

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\Box
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**Proof of Lemma \([12]\) in the case \( c = 2 \).** Let \( \varepsilon_2 = 1/3000 \). We can assume \( |V(G)| > 5 \), as otherwise the claim holds trivially. Since \( G \) is triangle-free and has minimum degree at least two, every face of \( G \) has length at least four. Let \( \beta = \sum_{f \in F} (|f| - 4) \), where the sum is over all faces. By the
generalized Euler’s formula, we have $|E(G)| \leq |V(G)| + |F| + g - 2$, and since $2|E(G)| = \sum_{f \in F} |f| = 4|F| + \beta$, we conclude that

$$|E(G)| < 2|V(G)| - \frac{\beta}{2} + 2g.$$ 

Since $G$ is simple, does not have separating or non-contractible 4-cycles, and $|V(G)| > 5$, every vertex of degree two is incident with a face of length at least five. Since $G$ is $(2, 4)$-sparsifier-free, $(S_1, \gamma_1)$ has no appearance in $G$, and thus each vertex of degree three is also incident with a face of length at least five. Hence, the number $n_3$ of vertices of degree at most three is at most $5\beta$.

Let us give each vertex of degree at most three the charge 1, any vertex of degree $d \geq 4$ charge $d - 4$, and any face $f$ the charge $|f| - 4$. By the generalized Euler’s formula, the sum of charges is at most

$$\left( \sum_{v \in V(G)} (\deg v - 4) \right) + 3n_3 + \sum_{f \in F} (|f| - 4) = 4(|E(G)| - |V(G)| - |F|) + 3n_3$$

$$\leq 4g - 8 + 3n_3 < 15\beta + 4g.$$ 

Each vertex $v$ of degree $d \neq 4$ now sends $1/11$ to each adjacent vertex and each vertex opposite to $v$ over a 4-face; this still leaves $v$ with at least

$$\max(d - 4, 1) - 2d/11 \geq \max(9d/11 - 4, 1 - 2d/11) \geq 1/11$$

units of charge. Each face $|f|$ of length at least five sends $1/11$ to each incident vertex, still keeping $|f| - 4 - |f|/11 > 0$ units of charge. Afterwards, each vertex $v$ of degree four which received charge sends $1/99$ to each adjacent vertex and each vertex opposite to $v$ over a 4-face; note that $v$ keeps at least $1/11 - 8/99 = 1/99$ units of charge. Since each face has non-negative final charge and the total amount of charge did not change, we conclude that the sum of the final charges of vertices is less than $15\beta + 4g$. Note that each vertex has non-negative final charge, and vertices of degree other than four have final charge at least $1/11$.

We claim that vertices of degree four have charge at least $1/99$. Consider for a contradiction a vertex $v$ of degree four with smaller final charge. All incident faces must be 4-faces, only incident with vertices of degree four, and the faces incident those must also have length four. This is not possible, since $(S_2, \gamma_2)$ does not have an appearance in $G$.

Since every vertex has final charge at least $1/99$, we have $|V(G)|/99 \leq 15\beta + 4g$, and thus $\beta \geq |V(G)|/1500 - g$. Consequently, $|E(G)| < 2|V(G)| - \beta/2 + 2g < (2 - 1/3000)|V(G)| + 3g$.

We finish this section by proving a strengthening Lemma 14 for $c = 2$. Introducing this strengthening requires the following additional definitions. Let $C$ be the 4-cycle bounding the outer face of a 4-stack $G$, and let $\varphi$
be a coloring of \( G \). We say that a monochromatic component \( Q \) of \( \varphi \) is \( C \)-transversal if \( V(Q) \cap V(C) \neq \emptyset \) and \( V(Q) \setminus V(C) \neq \emptyset \). We say that \( \varphi \) is \( v \)-compliant for some \( v \in V(C) \) if either no monochromatic component of \( \varphi \) is \( C \)-transversal, or there exists a unique such component \( Q \) and the following conditions hold

(C1) \( \forall v \in V(Q) \),

(C2) \( \varphi(v) \neq \varphi(v') \), where \( v' \) is the unique non-neighbor of \( v \) on \( C \),

(C3) every vertex in \( V(Q) \setminus V(C) \) has a neighbor in \( V(Q) \setminus V(Q) \).

We say that a vertex \( v \in V(C) \) is \( G \)-active if every vertex in \( V(G) \setminus V(C) \) with two neighbors on \( C \) is adjacent to \( v \).

**Lemma 17.** Let \( G \) be a 4-stack with the outer face bounded by a 4-cycle \( C \), let \( L \) be an assignment of lists of size two to vertices of \( G \), let \( \psi \) be an \( L \)-coloring of \( C \), and let \( v \in V(C) \) be \( G \)-active. Then \( \psi \) extends to a weak diameter-4 \( \psi \)-opaque \( v \)-compliant \( L \)-coloring \( \varphi \) of \( G \).

**Proof.** We prove the lemma by induction on \( |V(G)| \). The basic case \( G = C \) is trivial. Hence, we can assume \( G \neq C \). Let \( v' \) be the unique vertex of \( C \) non-adjacent to \( v \), and let \( X = \{x_1, \ldots, x_{m+1}\} \) be the set of all common neighbors of \( v \) and \( v' \) in \( G \), numbered so that for every \( i \in \{1, \ldots, m\} \) the cycle \( C_i = x_i x_{i+1} v' \) does not contain any vertices of \( X \) in its interior. In particular, we have \( x_1, x_{m+1} \in V(C) \). Let \( G_i = G_{C_i} \), and note that \( x_i \) and \( x_{i+1} \) are \( G_i \)-active.

For \( i = 2, \ldots, m \), choose a color \( \psi(x_i) \in L(x_i) \setminus \{\psi(v')\} \). Let \( v_1 = x_2 \) and \( v_m = x_m \). For \( i = 2, \ldots, m-1 \), let \( v_i = x_i \) if \( \psi(x_{i+1}) = \psi(v) \) and \( v_i = x_{i+1} \) otherwise; note that \( \psi(v_i) \neq \psi(v) \) unless \( \psi(x_i) = \psi(v) = \psi(x_{i+1}) \). By the induction hypothesis, the restriction of \( \psi \) to \( V(G_i) \) extends to a weak diameter-4 \( \psi \)-opaque \( v_i \)-compliant \( L \)-coloring \( \varphi_i \) of \( G_i \) for every \( i \in \{1, \ldots, m\} \). Let \( \varphi \) be the \( L \)-coloring of \( G \) such that \( \varphi_i \) is the restriction of \( \varphi \) to \( G_i \) for every \( i \). We show that \( \varphi \) satisfies the lemma.

Note that every \( v \)-compliant coloring of \( G \) that extends \( \psi \) is necessarily \( \psi \)-opaque. Thus it suffices to show that every monochromatic component \( Q \) of \( \varphi \) has weak diameter at most four, and that if \( Q \) is \( C \)-transversal then \( Q \) satisfies the conditions (C1)-(C3) above.

Suppose first that \( V(Q) \cap V(C) = \emptyset \). If \( Q \) is a monochromatic component of \( G_i \) for some \( i \) then the weak diameter of \( Q \) is at most four by the choice of \( \varphi_i \). Thus we may assume that \( Q \) contains vertices in both \( V(G_i) \setminus V(G_{i-1}) \) and \( V(G_{i-1}) \setminus V(G_i) \) for some \( i \in \{2, \ldots, m\} \). Since \( V(Q) \cap V(C) = \emptyset \), it follows that \( x_i \in V(C) \) and for \( j \in \{i-1, i\} \) the restriction of \( Q \) to \( G_j \) is \( C_j \)-transversal. Since \( \varphi_{i-1} \) and \( \varphi_i \) are \( \psi \)-opaque, \( V(Q) \cap V(C_{i-1} \cup C_i) \) is \( \{x_i\} \) and \( V(Q) \subseteq V(G_{i-1} \cup G_i) \). By (C3), every vertex of \( Q \) is a neighbor of some
vertex in \( \{v, v', x_{i-1}, x_{i+1}\} \), and thus the weak diameter of \( Q \) is at most four as desired.

It remains to consider the case when \( Q \) is \( C \)-transversal. Consider any edge \( uw \in E(G) \) such that \( u \in V(C) \), \( w \notin V(C) \) and \( \varphi(u) = \varphi(w) \):

- If \( w = x_i \) for some \( i \in \{2, \ldots, m\} \), then since \( \psi(x_i) \neq \psi(v') \), we have \( u = v \) and \( \psi(v) \neq \psi(v') \).

- Otherwise, \( w \in V(G_i) \setminus V(C_i) \) for some \( i \in \{1, \ldots, m\} \). By (C1) for \( \varphi_i \), we have \( \psi(v_i) = \psi(u) \), and the choice of \( v_1 \) and \( v_m \) and the property (C2) of \( \varphi_i \) imply \( u \notin \{x_1, x_{m+1}\} \). Since \( \psi(v_i) \in L(v_i) \setminus \{\psi(v')\} \), it follows that \( u = v \) and \( \psi(v) \neq \psi(v') \).

In either case, we conclude that \( \varphi \) satisfies (C1) and (C2). It remains to check that (C3) holds, i.e. every vertex \( w \in V(Q) \setminus V(C) \) has a neighbor in \( V(C) \setminus V(Q) \). If \( w \in X \) then \( v' \) is such a neighbor. Hence, assume that \( w \in V(G_i) \setminus V(C_i) \) for some \( i \), and thus the restriction of \( Q \) to \( G_i \) is \( C_i \)-transversal. Recall that \( v \in V(Q) \); by (C1) for \( \varphi_i \), we have \( v_i \in V(Q) \), and thus \( \psi(v_i) = \psi(v) \). Moreover, (C2) for \( \varphi_i \) implies \( \psi(x_i) \neq \psi(x_{i+1}) \). It follows from the choice of \( v_i \) that \( i \in \{1, m\} \), and thus \( V(C_i) \setminus V(Q) \subseteq V(C_i) \setminus \{v_i\} \subseteq V(C) \). By (C3) for \( \varphi_i \), the vertex \( w \) has a neighbor in \( V(C_i) \setminus V(Q) \subseteq V(C) \setminus V(Q) \), as desired. \( \square \)

### 4 The non-triangle-free case

Next, let us consider the case \( c = 3 \). Let \( S_1 \) be a single vertex and \( \gamma_1 \) the function assigning to this vertex the value 5, and let \( S_2 \) be the 4-cycle with one chord and \( \gamma_2 \) the function assigning to all its vertices the value 6.

**Lemma 18.** Both \((S_1, \gamma_1)\) and \((S_2, \gamma_2)\) are 3-sparsifiers of size at most 4.

**Proof.** Consider \( i \in \{1, 2\} \), let \( L \) be an assignment of lists of size 3 to vertices of \( S_i \), and let \( B \) be a multiassignment of a list of size 5 in case \( i = 1 \) and of list of size \( 6 - \deg_{S_2} v \) to each vertex \( v \in V(S_2) \) in case \( i = 2 \). In the case \( i = 1 \), choose \( \varphi(v) \in L(v) \) to be different from the (at most two) colors that appear in \( B(v) \) more than once. Clearly, \( \varphi \) is \( B \)-opaque.

In the case \( i = 2 \), we choose the \( L \)-coloring of \( S_2 \) as follows. Let \( R \) consist of the vertices \( u \in V(S_2) \) such that \( B(u) \) contains at most two distinct colors. For \( u \in R \), let \( \varphi(u) \in L(u) \) be chosen different from the colors in \( B(u) \). For \( v \in V(S_2) \setminus R \), let \( L'(v) \) consist of the colors in \( L(v) \) that appear in \( B(v) \) at most once. Note that if \( \deg_{S_2} v = 2 \), then \( |B(v)| \leq 4 \) and since \( B(v) \) contains at least three distinct colors, we have \( |L'(v)| \geq 2 \); and if \( \deg_{S_2} v = 3 \), then \( |B(v)| \leq 3 \) and \( L'(v) = L(v) \) has size three. In particular, \( |L'(v)| \geq \deg_{S_2} v \) for each \( v \in V(S_2) \setminus R \). Hence, we can choose \( \varphi \) on \( V(S_2) \setminus R \) to be a proper \( L' \)-coloring of \( S_2 - R \). Additionally, in case neither of the vertices
$x, y \in V(S_2)$ of degree two belongs to $R$ and at least one vertex $z$ of degree three belongs to $R$ (so $S_2 - R$ is either a path or consists of two isolated vertices), we can choose $\varphi$ so that $\varphi(x) \neq \varphi(y)$.

If distinct vertices $v_1, v_2 \in V(S_2)$ both receive the same color $a \in B(v_1) \cap B(v_2)$, then $v_1, v_2 \notin R$ and $v_1v_2 \notin E(S_2)$, and thus $v_1$ and $v_2$ are the vertices of $S_2$ of degree two. Moreover, by the last condition in the choice of $\varphi$, since $\varphi(v_1) = \varphi(v_2)$, we have $R = \emptyset$, and thus no other vertex of $S_2$ has color $a$; hence, $v_1$ and $v_2$ do not belong to the same monochromatic component. It follows that $\varphi$ is $B$-opaque.

**Proof of Lemma 12 in the case $c = 3$.** Let $\epsilon_3 = 1/1000$. We can assume $|V(G)| > 4$, as otherwise the claim holds trivially. Since $G$ has minimum degree at least three, every face of $G$ has length at least four. Let $\beta = \sum_{f \in F}(|f| - 3)$, where the sum is over all faces. By the generalized Euler’s formula, we have $|E(G)| \leq |V(G)| + |F| + g - 2$, and since $2|E(G)| = \sum_{f \in F} |f| = 3|F| + \beta$, we conclude that

$$|E(G)| < 3|V(G)| - \beta + 3g.$$  

Since $G$ is simple, does not have separating or non-contractible triangles, and $|V(G)| > 4$, every vertex of degree three is incident with a face of length at least four. Since $G$ is $(3,4)$-sparsifier-free, $(S_1, \gamma_1)$ has no appearance in $G$, and thus each vertex of degree at most five is also incident with a face of length at least four. Hence, the number $n_5$ of vertices of degree at most five is at most $4\beta$.

Let us give each vertex of degree at most five the charge 1, any vertex of degree $d \geq 6$ charge $d - 6$, and any face $f$ the charge $2|f| - 6$. By the generalized Euler’s formula, the sum of charges is at most $6g - 12 + 4n_3 < 16\beta + 6g$. Each vertex $v$ of degree $d \neq 6$ now sends $1/8$ to each adjacent vertex; this still leaves $v$ with at least $1/8$ units of charge. Each face of length at least four sends $1/8$ to each incident vertex, still keeping its charge nonnegative. After this, each vertex of degree six which received charge sends $1/56$ to each adjacent vertex. Since each face has non-negative final charge and the total amount of charge did not change, we conclude that the sum of the final charges of vertices is less than $16\beta + 6g$. Note that each vertex has non-negative final charge, and vertices of degree other than six have final charge at least $1/8$.

We claim that vertices of degree six have final charge at least $1/56$. Consider for a contradiction a vertex $v$ of degree six with smaller final charge. All incident faces must be triangles, only incident with vertices of degree six, and the faces incident those must also be triangles. Moreover, since $G$ does not contain separating or non-contractible triangles, the neighbors of $v$ form an induced 6-cycle. This is not possible, since $(S_2, \gamma_2)$ does not have an appearance in $G$. 

16
Since every vertex has final charge at least $1/56$, we have $|V(G)|/56 \leq 16\beta + 6g$, and thus $\beta \geq |V(G)|/1000 - g$. Consequently, $|E(G)| < 3|V(G)| - \beta + 3g < (3 - 1/1000)|V(G)| + 4g$.

We now show that Lemma 14 holds for $c = 3$. Note that in this case, the condition that the resulting coloring is $\psi$-opaque is trivially satisfied, since $\psi$ cannot have distinct components of the same color on the triangle $C$. We prove the following stronger statement. In a coloring $\varphi$ of a $3$-stack $G$ with the outer face bounded by a triangle $C$, a vertex $v \in V(C)$ is a singleton if no adjacent vertex in $V(G) \setminus V(C)$ has the color $\varphi(v)$.

**Lemma 19.** Let $G$ be a $3$-stack with the outer face bounded by the triangle $C$, and let $u$ be a vertex of $C$. Let $L$ be an assignment of lists of size three to vertices of $G$, and let $\psi$ be an $L$-coloring of $C$. Then $\psi$ extends to a weak diameter-$2$ $L$-coloring of $G$ in which vertices in $V(C) \setminus \{u\}$ are singletons and the monochromatic component containing $u$ is contained in the neighborhood of each of the vertices in $V(C) \setminus \{u\}$; and moreover, if $\psi$ only uses at most two distinct colors on $C$, then $u$ is also a singleton.

**Proof.** We prove the claim by induction on the number of vertices of $G$. The claim is clear if $G = C$, and thus we can assume that there exists a vertex $v \in V(G)$ adjacent to all vertices of $C$. Let $C_1$, $C_2$, and $C_3$ be the three triangles in $G[V(C) \cup \{v\}]$ distinct from $C$, where $u \not\in V(C_3)$. Choose a color $\psi(v) \in L(v)$ distinct from the colors of the two vertices in $V(C) \setminus \{u\}$, and distinct from $\psi(u)$ if $\psi$ only uses at most two distinct colors on $C$.

For $i \in \{1, 2, 3\}$, extend $\psi$ to a weak diameter-$2$ $L$-coloring of $G_{C_i}$ by the induction hypothesis, with the vertex $v$ playing the role of $u$. This ensures that the vertices in $V(C) \setminus \{u\}$ are singletons in the resulting $L$-coloring of $G$, and if $\psi(v) \neq \psi(u)$ (which is always the case if $\psi$ only uses at most two colors on $C$), then also $u$ is a singleton. Consider the monochromatic component $Q$ of the vertex $v$ in the resulting coloring:

- If $\psi(v) \neq \psi(u)$, then by the induction hypothesis, for $1 \leq i < j \leq 3$, $Q \cap (V(G_{C_i}) \cup V(G_{C_j}))$ is contained in the neighborhood of a vertex of $C$, and thus $Q$ has weak diameter at most two.

- If $\psi(v) = \psi(u)$, then $u$ and $v$ are singletons in the colorings of $G_{C_1}$ and $G_{C_2}$, and thus $Q = (Q \cap V(G_{C_3})) \cup \{u\}$. By the induction hypothesis, we conclude that $Q$ is contained in the neighborhood of each of the vertices in $V(C) \setminus \{u\}$.

\[\square\]

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