Grothendieck meeting [Wess & Bagger]:

Wess & Bagger [Supersymmetry and supergravity: Chs. IV, V, VI, VII, XXII] (2nd ed.)

reconstructed in complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry

I. The construction under a trivialization of the Weyl spinor bundles

by complex conjugate pairs of covariantly constant sections

Chien-Hao Liu and Shing-Tung Yau

Abstract

Forty-six years after the birth of supersymmetry in 1973 from works of Julius Wess and Bruno Zumino, the standard quantum-field-theorists and particle physicists’ language of ‘superspaces’, ‘supersymmetry’, and ‘supersymmetric action functionals in superspace formulation’ as given in Chapters IV, V, VI, VII, XXII of the classic on supersymmetry and supergravity: Julius Wess & Jonathan Bagger: Supersymmetry and Supergravity (2nd ed.), is finally polished, with only minimal mathematical patches added for consistency and accuracy in dealing with nilpotent objects from the Grassmann algebra involved, to a precise setting in the language of complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry. This is completed after the lesson learned from D(14.1) (arXiv:1808.05011 [math.DG]) and the notion of ‘\( d = 3+1, N = 1 \) towered superspaces’ as complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-schemes, their distinguished sectors, and purge-evaluation maps first developed in SUSY(1) (= D(14.1.Supp.1)) (arXiv:1902.06240 [hep-th]) and further polished in the current work. While the construction depends on a choice of a trivialization of the spinor bundle by covariantly constant sections, as long as the transformation law and the induced isomorphism under a change of trivialization of the spinor bundle by covariantly constant sections are understood, any object or structure thus defined or constructed is mathematically well-defined. The construction can be generalized to all other space-time dimensions with simple or extended supersymmetries. This is part of the mathematical foundation required to study fermionic D-branes in the Ramond-Neveu-Schwarz formulation.

Key words: superspace, tower, supersymmetry; complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \) function-ring, complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-scheme; tame superfield, small chiral superfield, Wess-Zumino model; vector superfield, supersymmetric gauge theory; chiral map, nonlinear sigma model

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Chien-Hao Liu dedicates this work to the memory of two founding fathers of Supersymmetry:
Prof. Bruno Zumino (1923 - 2014), whom he came across in his topic course on Quantum Groups during U.C. Berkeley years, and Prof. Julius Wess (1934 - 2007), whose classic book with Jonathan Bagger on Supersymmetry and Supergravity motivates the current work.

[Supersymmetry: $Q$ and $\bar{Q}$]

$^\sharp$Works of Julius Wess and Bruno Zumino (1973):

[W-Z1] J. Wess and B. Zumino, A Lagrangian model invariant under supergauge transformations, Phys. Lett. 49B (1974), 52–54.

[W-Z2] ———, Supergauge transformations in four-dimensions, Nucl. Phys. 70 (1974), 39–50.

[W-Z3] ———, Supergauge invariant extension of quantum electrodynamics, Nucl. Phys. B78 (1974), 1–13.

$^\natural$The classic book by Julius Wess and Jonathan Bagger:

[Wess & Bagger] J. Wess and J. Bagger, Supersymmetry and supergravity, 2nd ed., revised and expanded, Princeton Univ. Press, 1992.

$^\flat$Illustration inspired by particle physics and art works of Maurits Cornelis Escher (1898-1972): Day and Night, (1938), woodcut, and Swans, (1956), wood engraving on thin Japanese paper.
0. Introduction and outline

Following [L-Y1] (D(1)) and [L-Y2] (D(11.1)), one naturally comes to the realization that

- From the aspect of modern Algebraic Geometry in the spirit of Alexander Grothendieck, a fermionic D-brane in the Ramond-Neveu-Schwarz (RNS) formulation is described by a differentiable map

\[ \hat{f} : (\hat{X}^{Az}, \hat{E}, \hat{\nabla}) \rightarrow Y \]

from an Azumaya/matrix supermanifold/superscheme with a fundamental module \( \hat{X}^{Az} := (\hat{X}, \hat{O}_{\hat{X}}^{Az} := \text{End}_{\hat{E}}(\hat{E})) \) with a connection \( \hat{\nabla} \) on \( \hat{E} \) (cf. the world-sheet of a fermionic D-brane with the Chan-Paton bundle with a connection) to a smooth manifold \( Y \) (cf. the target space-time).

Cf. [L-Y3] (D(11.2)) and [L-Y4] (D(14.1)). The structure sheaf \( \hat{O}^{Az}_{\hat{X}} \) of the Azumaya superscheme \( \hat{X}^{Az} \) is a sheaf of Azumaya algebras over a complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-scheme \( \hat{X} \). Thus, any construct of \( \hat{X} \) would give rise to a notion/definition of fermionic D-branes in RNS formulation in the end. The question is

Q. Which construct of \( \hat{X} \), or its enhancement if necessary, truly reflects particle physicists’ perception and practical language (albeit possibly requiring some mathematical patches to make everything precise) of superspaces and supersymmetries, as in, e.g.

- [G-G-R-S] S.J. Gates, Jr., M.T. Grisaru, M. Ročček, and W. Siegel, Superspace – one thousand and one lessons in supersymmetry, Frontiers Phys. Lect. Notes Ser. 58, Benjamin/Cummings Publ. Co., Inc., 1983.
- [We] P. West, Introduction to supersymmetry and supergravity, extended 2nd ed., World Scientific, 1990.
- [Wess & Bagger] J. Wess and J. Bagger, Supersymmetry and Supergravity, 2nd ed., Princeton Univ. Press, 1992.

so that the corresponding notion of ‘fermionic D-branes’ rings with the fermionic D-branes from superstring theory. (Cf. See footnote 1 for more words on what we are looking for here.) In this way, our pursuit in the study of fermionic D-branes brings us back to a more fundamental question we have to resolve first before going on.

The goal of the current work is to provide a construct of superspaces that answers the above question. (The construct is in the \( d = 3 + 1, N = 1 \) case but similar arguments extend it to other space-time dimensions with simple (i.e. \( N = 1 \)) or extended (i.e. \( N \geq 2 \)) supersymmetries.) With lessons learned from [L-Y4] (D(14.1), arXiv:1808.05011 [math.DG]) and [L-Y5] (SUSY(1) (= D(14.1.Suppl.1), arXiv:1902.06246 [hep-th]) as the foundation, we re-do one of the classics on supersymmetry and supergravity for quantum-field-theorists and particle physicists:

- [Wess & Bagger] Julius Wess & Jonathan Bagger: Supersymmetry and Supergravity
  2nd ed., revised and expanded, Princeton Univ. Press, 1992,

  Chapter IV Superfields
  Chapter V Chiral superfields
  Chapter VI Vector superfields
  Chapter VII Gauge invariant interactions, \( (U(1) \) part)
  Chapter XXII Chiral models and ähler geometry
from the aspect of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry. In this first part of the work, we present the construction as close as can be to the original presentation except mathematically more favored notations and necessary mathematical patches for accuracy and consistency. In particular, the construction of the full function-ring of a superspace, i.e. the function ring of a towered superspace, will depend on a choice of a trivialization of the spinor bundle by covariantly constant sections. As long as the transformation law and/or the induced isomorphism under a change of trivialization of the spinor bundle by covariantly constant sections.

Almost twenty-five years later, in fall 2017 when the D-project stepped into her second decade and the focus was turned to fermionic D-branes, one would expect that after all these years’ joint effort from both physics-friendly mathematicians and mathematics-friendly physicists, there should be some mathematical work on supersymmetry and superspaces that one could take as the clean, clear, and solid starting point to address fermionic D-branes in parallel to Ramond-Neveu-Schwarz fermionic strings. To make sure such a study of fermionic D-branes is linked and superspaces that one could take as the clean, clear, and solid starting point to address fermionic D-branes in parallel to Ramond-Neveu-Schwarz fermionic strings. To make sure such a study of fermionic D-branes is linked to superstring theory, two minimal requirements on such a sought-for mathematical work on supersymmetry are

1. Since

   - in addition to the ordinary commuting coordinate functions, a superspace has also fermionic/anticommuting coordinate functions, which are nilpotent,
   - the underlying topology of a superspace remains a smooth manifold,
   - either complex spinors or complexified spinors were involved,

   this sought-for work should be in the realm of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry, in the same spirit as how Alexander Grothendieck built up the modern language of Algebraic Geometry.

2. Since [Wess & Bagger: *Supersymmetry and supergravity*] (2nd ed.) by Julius Wess and Jonathan Bagger is one of the classical textbooks that were taken by string theorists as the standard language for supersymmetry, the sought-for work should reproduce chapters in [Wess & Bagger] of classical (as opposed to quantum) level in a direct, fluent, and effortless way. In particular, all the expressions in these chapters should have a well-defined mathematical meaning and all the formulas and computations are re-derivable in this sought-for mathematical work.

Painfully, up to summer 2018, (i.e. 47 years since the appearance of supersymmetry in string theory in 1971 and 45 years since a 4-dimensional supersymmetric quantum field theory was constructed in 1973), a mathematical work that met the above minimal requirements had not yet been in existence. It was when I was in such an embarrassing situation that a train of lucks were given to us.

(a) Through the *Topic course in Supersymmetry* given by Girma Hailu at the Department of Physics, Harvard University, and related discussions with him, fall 2018, it was realized that to construct the physically correct function-ring of a superspace, a Grassmann-algebra tower must be built over the ordinary superspace. It is because of this tower that physicists can take only even objects but still with anticommuting fields included. With this as the starting point and taking into account all its mathematical consequences, we were led to the “(tower construction) + (purge-evaluations in the end)” way to produce results in [Wess & Bagger] in the realm of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry; cf. [L-Y5] (SUSY(1)). Through SUSY(1), one also learns why and how physicists could so cleverly bypass all the sign-factors in $\mathbb{Z}/2$-graded geometry, as compared to [L-Y4] (D(14.1)).

(b) In March 2019, after a highlight of the construction to Albrecht Klemm at the Center, he pointed out to me that many of the multiple-spinors type expressions that appear in supersymmetric quantum field theories are already contracted in some way in physicists’ interpretation of these expressions. This led us to a guiding question:

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1 *Special acknowledgements from C.-H.L.* The time-and-place was some year in 1990s at the Department of Mathematics in Evens Hall, University of California, Berkeley. Motivated by the emerging dominating mathematical topic at that time: *Quantum invariants of low-dimensional manifolds*, which had arisen from supersymmetry and topological field theories on the physics side — particularly works of Edward Witten —, Prof. Nicolai Reshetikhin, a then young star on quantum invariants in low dimensional topology, organized a seminar on Quantum invariants. In one of the beginning meetings of the seminar, Prof. Reshetikhin started to explain Supersymmetry to a group of serious mathematicians, including Alexander Givental and Alan Weinstein, and enthusiastic graduate students. Alas! Not far into the intended lecture, tons of questions and puzzles already arose from the audience that in the end Prof. Reshetikhin had to quit the lecture he had prepared and rather announced that he would try to invite someone from the Department of Physics to come to the seminar to explain supersymmetry to mathematicians. Unfortunately, no physicists showed up to give a lecture on supersymmetry for the mathematical mind and that intended introduction of supersymmetry to mathematicians was thus ended in the middle and never finished.

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is understood, any object or structure thus defined or constructed is mathematically well-defined. In this way — and 46 years after the birth of supersymmetry in 1973 from works of Julius Wess and Bruno Zumino — the standard physics language of ‘superspaces’, ‘supersymmetry’, and ‘supersymmetric action functionals in superspace formulation’ is finally polished, with only minimal mathematical patches added for consistency and accuracy in dealing with nilpotent objects from the Grassmann algebra involved, to a precise setting in the language of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry.

Once a mathematical presentation of superspace and supersymmetry that matches [Wess & Bagger] is completed, the immediate next question is: What is the intrinsic description of the same, without resorting to a choice of a trivialization of the spinor bundles by covariantly constant sections? This and the similar construction for supersymmetric non-Abelian gauge theory ([Wess & Bagger: Chapter VII, non-Abelian part]) are the themes for the sequels.

**Convention.** References for standard notations, terminology, operations and facts are:

1. algebraic geometry: [Hart]; $C^\infty$-algebraic geometry: [Jo]; supergeometry: [Ke];
2. spinors and supersymmetry (mathematical aspect): [Ch], [De], [D-F1], [D-F2], [Fr], [Harv], [S-W];
3. supersymmetry (physical aspect, especially $d = 4, N = 1$ case): [Wess & Bagger; W-B], [G-G-R-S], [We]; also [Argu], [Argy], [Bi], [B-T-T], [RR-vN], [St], [S-S].

Q. *Should one enact a purge-evaluation map not at the end of making sense of the supersymmetric action functional, as did in SUSY (1), but rather much earlier before that? If so, then how?*

(c) In May 2019, the two days’ intensive discussions with Pei-Ming Ho at the Department of Physics, National Taiwan University, and the presentation of SUSY (1) in the String Theory Seminar there left me with other input, including operational interpretation of these bi-spinor or triple-spinor type expressions. After all, for physicists, it is the quantum theory that matters most. Expressions that appear in a supersymmetric action functional may or may not have easy explanation classically but as long as their meaning is clear at the quantum level, it remains a very good expression. Indeed, in one aspect of superspace, there is hiddenly/implicitly already an infinite tower over the ordinary superspace. Such quantum-level picture of a superspace from physicists’ view could be difficult to realize through $C^\infty$-Algebraic Geometry alone but may serve as a final goal.

(d) Finally, in fall 2019, Jesse Thaler gave a topic course on Supersymmetric quantum field theories at the Department of Physics, Massachusetts Institute of Technology. This is a SQFT course given by a superspace advocate. From the very first lecture and through the seven lectures in September 2019, he explained carefully what a $d = 3 + 1, N = 1$ superspace is from a physicist’s eye and how it can be used in a very elegant manner to construct supersymmetric action functionals. After the exercises done in D(14.1) and SUSY (1) and inputs from Girma, Albrecht, and Pei-Ming, this gave me a rare chance to re-examine and compare step by step physicists’ way of working with supersymmetry and superspace and what’s enforced by complexified $\mathbb{Z}/2$-graded Algebraic Geometry. His very careful counting of independent degrees of freedom (either on-shell or off-shell) whenever a new type of superfield is introduced indicates very clearly that something crucial is still missing in the setting of SUSY (1). There, as a mathematical must, additional independent degrees of freedom have to be thrown in to keep the collection of superfields (resp. chiral superfields) to honestly form a ring. It is through such re-checking, cross examinations, and his generous answer to my questions that it becomes clear that there is still room for improvement beyond SUSY (1) to realize [Wess & Bagger] in complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry.

It is through the above sequence of unexpected occurrences — with right people, at right time, in right order, and at right place — and these physicists’ input along the way that the current work appears. As if these lucks were not enough, while the current work is in the intensive editing stage, Enno Keesler suggested weekly Friday meetings on Supersymmetry at the Center of Mathematical Sciences and Applications, spring semester 2020, in which we teach each other our different, “orthogonal-to-each-other”, tower-vs-base aspects of supergeometry and supersymmetry. The various questions he raised about [Wess & Bagger] in the meetings were turned to part of the motivations to improve the writing of the current notes. His insight on supergeometry as explained in his book [Ke], *Supergeometry, super Riemann surfaces and the superconformal action functional*, particularly the emphasis on the construction over bases (as opposed to the towered construction we are advocating in [L-Y5] (SUSY (1)) and the current work, which looks most natural from the particle physics aspect), the notion of (relative) ‘underlying even manifolds’, and the various setup/design in [Ke: Part II] are sure to have a significant influence to one’s understanding of supergeometry.
For clarity, the real line as a real 1-dimensional manifold is denoted by $\mathbb{R}^1$, while the field of real numbers is denoted by $\mathbb{R}$. Similarly, the complex line as a complex 1-dimensional manifold is denoted by $\mathbb{C}^1$, while the field of complex numbers is denoted by $\mathbb{C}$.

The inclusion ‘$\mathbb{R} \subset \mathbb{C}$’ is referred to the field extension of $\mathbb{R}$ to $\mathbb{C}$ by adding $\sqrt{-1}$, unless otherwise noted.

All manifolds are paracompact, Hausdorff, and admitting a (locally finite) partition of unity. We adopt the index convention for tensors from differential geometry. In particular, the tuple coordinate functions on an $n$-manifold is denoted by, for example, $(y^1, \cdots, y^n)$. However, all summations are expressed explicitly; no up-low index summation convention is used.

‘differentiable’, ‘smooth’, and $C^\infty$ are taken as synonyms.

Section $s$ of a sheaf or vector bundle vs. dummy labelling index $s$

group action vs. action functional in a quantum field theory

sheaves $\mathcal{F}, \mathcal{G}$ vs. curvature tensor $F_\nabla$, gauge-symmetry group $\mathcal{G}_{\text{gauge}}$

coordinate-function index, e.g. $(y^1, \cdots, y^n)$ for a real manifold vs. the exponent of a power, e.g. $a_0 y^r + a_1 y^{r-1} + \cdots + a_{r-1} y + a_r \in \mathbb{R}[y]$.

dimension $n$ or $2n$ vs. nilpotent component $n$ of an element of a ring.

Various brackets: $[A, B] := AB - BA$, $\{A, B\} := AB + BA$, $[A, B] := AB - (-1)^{p(A)p(B)} BA$, where $p(\bullet)$ is the parity of $\bullet$.

We adopt the following convention as in the work of Deligne and Freed [D-F2: §6]:

Convention [cohomological degree vs. parity] We treat elements $f$ of $\mathbb{Z}/2$-graded ring as of cohomological degree 0 and the exterior differential operator $d$ as of cohomological degree 1 and even. In notation, $c.h.d(f) = 0$ and $c.h.d(d) = 1$, $p(d) = 0$. Under such $(\mathbb{Z} \times (\mathbb{Z}/2))$-bi-grading,

$$ab = (-1)^{c.h.d(a)c.h.d(b)}(-1)^{p(a)p(b)} ba$$

for objects $a, b$ homogeneous with respect to the bi-grading. Here, $a$ and $b$ are not necessarily of the same type.

The current SUSY(2.1) continues the study in

[L-Y4] $N = 1$ fermionic D3-branes in RNS formulation I. $C^\infty$-Algebrogometric foundations of $d = 4$, $N = 1$ supersymmetry, SUSY-rep compatible hybrid connections, and $\hat{D}$-chiral maps from a $d = 4$ $N = 1$ Azumaya/matrix superspace, arXiv:1808.05011 [math.DG]. (D(14.1))

[L-Y5] Physicists’ $d = 3 + 1$, $N = 1$ superspace-time and supersymmetric QFTs from a tower construction in complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry and a purge-evaluation/index-contracting map, arXiv:1902.06246 [hep-th]. (D(14.1.Supp.1)=SUSY(1))

Notations and conventions follow ibidem whenever applicable.
Outline

0 Introduction

1 The function-ring of a $d = 3 + 1$, $N = 1$ towered superspace and its distinguished even subrings (under a trivialization of the spinor bundle by covariantly constant sections)

   1.1 Basics of the Clifford algebra, the spin group, Clifford modules, and Weyl spinors behind $d = 3 + 1$, $N = 1$ supersymmetry (cf. [Wess & Bagger: Appendix A])

   1.2 The function-ring of a $d = 3 + 1$, $N = 1$ towered superspace and its distinguished even subrings (under a trivialization of the spinor bundle by covariantly constant sections) (cf. [Wess & Bagger: Chapter IV and Appendices A & B])

   1.3 The chiral and the antichiral condition on $C^\infty(\hat{X})$

2 Purge-evaluation maps and the Fundamental Theorem on supersymmetric action functionals via a superspace formulation

3 The small function-ring of $\hat{X}$ and the Wess-Zumino model (cf. [Wess & Bagger: Chapter V])

   3.1 The small function-ring $C^\infty(\hat{X})^{small}$ and its chiral and antichiral sectors

   3.2 The Wess-Zumino model

4 Supersymmetric $U(1)$ gauge theory with matter on $X$ in terms of $\hat{X}$ (cf. [Wess & Bagger: Chapter VI and Chapter VII, $U(1)$ part])

   4.1 Vector superfields and their associated (even left) connection

   4.2 Vector superfields in Wess-Zumino gauge

   4.3 Supersymmetry transformations of a vector superfield in Wess-Zumino gauge

   4.4 Supersymmetric $U(1)$ gauge theory with matter on $X$ in terms of $\hat{X}$

5 $d = 3 + 1$, $N = 1$ nonlinear sigma models (cf. [Wess & Bagger: Chapter XXII])

   5.1 Smooth maps from $\hat{X}^{small}$ to a smooth manifold $Y$

   5.2 Chiral maps from $\hat{X}^{small}$ to a complex manifold $Y$

   5.3 The action functional for chiral maps — $d = 3 + 1$, $N = 1$ nonlinear sigma models with a superpotential
1 The function-ring of a $d = 3 + 1$, $N = 1$ towered superspace and its distinguished even subrings (under a trivialization of the spinor bundle by covariantly constant sections)

In this section we review and upgrade the notion of ‘towered superspace’ from [L-Y5: Sec. 1] (SUSY(1)). The representation theory of the Lorentz group, particularly the vector representation and the spinor representations, is among the key constituents in how particle physicists think about a superspace and superfields, thus in Sec. 1.1 we recall some basics of the Clifford algebra, the spin group, Clifford modules, and Weyl spinors behind $d = 3 + 1$, $N = 1$ supersymmetry and set up the notations. In Sec. 1.2 we recall why one needs the notion of a towered superspace and how they are constructed. Three distinguished sectors thereof that are related to physicists’ notion of scalar superfields are identified. In Sec. 1.3 we discuss the chiral conditions and the antichiral conditions on these sectors and compare the corresponding chiral superfields to chiral multiplets from representations of supersymmetry algebra.

1.1 Basics of the Clifford algebra, the spin group, Clifford modules, and Weyl spinors behind $d = 3 + 1$, $N = 1$ supersymmetry

(cf. [Wess & Bagger: Appendix A])

In this subsection we highlight basics of the Clifford algebra, the spin group, Clifford modules, and Weyl spinors behind the $d = 3 + 1$, $N = 1$ supersymmetry and introduce some notations used in [Wess & Bagger: Appendix A] and the current notes. Readers are referred to, e.g., [Harv], [L-M], [Mo] for details.

The Clifford algebra and the spin group

Let $V \cong \mathbb{R}^4$ be a 4-dimensional vector space over $\mathbb{R}$ with an inner product $\langle , \rangle$ of signature type $(- + + +)$, and

\[ T^*V := \oplus_{k\geq 0} V^\otimes k \] is the tensor algebra associated to $V$,

\[ \wedge^* V := \oplus_{k\geq 0} \wedge^k V = T^*V/(v \otimes v : v \in V) \] is the Grassmann algebra (synonymously, exterior algebra) associated to $V$, where $(v \otimes v : v \in V)$ is the bi-ideal in $T^*V$ generated by elements as indicated, and

\[ Cl(V, \langle , \rangle) := T^*V/(v \otimes v + \langle v, v \rangle \cdot 1 : v \in V) \] is the Clifford algebra associated to $(V, \langle , \rangle)$, where $(v \otimes v + \langle v, v \rangle \cdot 1 : v \in V)$ is the bi-ideal in $T^*V$ generated by elements as indicated. Denote the multiplicative group of multiplicatively invertible elements of $Cl(V, \langle , \rangle)$ by $Cl^*(V, \langle , \rangle)$.

The tensor algebra $T^*V$ associated to $V$ is naturally $\mathbb{Z}/2$-graded: $T^*V = T^{\text{even}}V \oplus T^{\text{odd}}V := (\oplus_{k\geq 0, \text{even}} V^\otimes k) \oplus (\oplus_{k\geq 1, \text{odd}} V^\otimes k)$. After passing to quotients, all $T^*V$, $\wedge^*V$, and $Cl(V, \langle , \rangle)$ are (unital, associative) $\mathbb{Z}/2$-graded algebras over $\mathbb{R}$. The quotient maps $T^*V \to \wedge^*V$ and $T^*V \to Cl(V, \langle , \rangle)$ together induce a canonical isomorphism $\wedge^*V \cong Cl(V, \langle , \rangle)$ via the above vector-space isomorphism. This renders both $\wedge^*V$ and $Cl(V, \langle , \rangle)$ algebras over $\mathbb{R}$ with norm squared\footnote{Here, for any vector space $W$ with an inner product, (i.e. nondegenerate pairing) $\langle , \rangle$, we call $\langle w, w \rangle$ the norm squared of $w \in W$.} denoted $\| \cdot \|^2$. By definition, the spin group $SO(1,3)$ is the subgroup of the multiplicative group $Cl^*(V, \langle , \rangle)$ even of


\[ CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \] generated by elements in \( V \) of norm squared \( \pm 1 \) (i.e. the unit "sphere" in \( (V, \langle \cdot, \cdot \rangle) \)). Its connected component that contains the identity element will be denoted \( \text{Spin}^0(1,3) \) and called the \textit{identity-component} of the spin group. The isometry on \( (V, \langle \cdot, \cdot \rangle) \), \( v \mapsto -v \), induces an algebra-automorphism \( \sim : CL(V, \langle \cdot, \cdot \rangle) \to CL(V, \langle \cdot, \cdot \rangle) \), which in turn defines a \textit{twisted Adjoint representation} \( \widetilde{Ad} \) of \( CL^*(V, \langle \cdot, \cdot \rangle) \) on \( CL(V, \langle \cdot, \cdot \rangle) \) (as a representation of a Lie group on a vector space)

\[ \widetilde{Ad}_a b := ab \alpha^{-1} \quad \text{for} \quad a \in CL^*(V, \langle \cdot, \cdot \rangle) \quad \text{and} \quad b \in CL(V, \langle \cdot, \cdot \rangle). \]

In terms of \( \widetilde{Ad} \), the spin group is characterized by

\[ \text{Spin}(1,3) = \{ a \in CL^*(V, \langle \cdot, \cdot \rangle)^{\text{even}} \mid \widetilde{Ad}_a(V) \subset V, \|a\|^2 = \pm 1 \} \]

\[ \text{Spin}^0(1,3) = \{ a \in CL^*(V, \langle \cdot, \cdot \rangle)^{\text{even}} \mid \widetilde{Ad}_a(V) \subset V, \|a\|^2 = 1 \}. \]

Furthermore, the \( \text{Spin}(1,3) \)-action on \( V \) via the twisted Adjoint representation preserves the inner product \( \langle \cdot, \cdot \rangle \). This gives rise to the double covers

\[ \text{Spin}(1,3) \longrightarrow SO(1,3) \quad \text{and} \quad \text{Spin}^0(1,3) \longrightarrow SO^+(1,3) \]

with kernel \( \{ 1, -1 \} \cong \mathbb{Z}/2 \). Here, \( SO(1,3) \) is the isometry group of \( (V, \langle \cdot, \cdot \rangle) \) that preserves a fixed orientation of \( V \); \( SO^+(1,3) \subset SO(1,3) \) is its subgroup that preserves in addition a specified time direction on \( V \), which is identical to the connected component of \( SO(1,3) \) that contains the identity element.

**Weyl spinors with a symplectic pairing \( \varepsilon \)**

Up to equivalences, \( CL(V, \langle \cdot, \cdot \rangle) \) has a unique irreducible complex representation \( S \), of complex dimension 4 (i.e. the Dirac spinors), while \( CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \) has two inequivalent irreducible complex representations \( S' \) and \( S'' \), both of complex dimension 2, that are complex conjugate to each other. \( S \cong S' \oplus S'' \) as \( CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \)-modules under the inclusion \( CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \subset CL(V, \langle \cdot, \cdot \rangle) \). The application of \( CL(V, \langle \cdot, \cdot \rangle)^{\text{odd}} \) on \( S \) exchanges \( S' \) and \( S'' \). Elements in \( S' \) and \( S'' \) are called Weyl spinors in physics literature. Let \( \sim : CL(V, \langle \cdot, \cdot \rangle) \to CL(V, \langle \cdot, \cdot \rangle) \) be an involution on \( CL(V, \langle \cdot, \cdot \rangle) \) generated by the correspondence \( v_1 \otimes \cdots \otimes v_p \mapsto (-v_p) \otimes \cdots \otimes (-v_1) \). Then, up to a complex constant, there is a unique complex symplectic bilinear form \( \varepsilon \) on \( S' \) (resp. \( S'' \)) such that \( \varepsilon (s_1, s_2) = \varepsilon (s_1, \bar{s}_2) \) for \( a \in CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \) and \( s_1, s_2 \in S' \) (resp. \( S'' \)). The restriction of \( CL(V, \langle \cdot, \cdot \rangle)^{\text{even}} \) to \( \text{Spin}^0(1,3) \) gives a group-isomorphism \( \text{Spin}^0(1,3) \cong SL(2, \mathbb{C}) \).

**Remark 1.1.1. [choice of \( \varepsilon \)]** Through the above highlights, note that, up to an equivalence of representations, the above construction is canonically and uniquely associated to \( (V, \langle \cdot, \cdot \rangle) \), except for the choice of the complex symplectic bilinear form \( \varepsilon \) on \( S' \) and \( S'' \). The symplectic complex bilinear form \( \varepsilon \) on \( S \) can be chosen to be real and hence pass to \( \varepsilon \) on \( S'' \). Under the reality constraint, the ambiguity remains a positive constant in \( \mathbb{R}_{>0} \). The isomorphisms \( S' \cong S'^{\vee} \) and \( S'' \cong S''^{\vee} \) determined by \( \varepsilon \) via the \( \mathbb{C} \)-linear map \( s \mapsto s^\vee = \varepsilon (\cdot, s) \) can thus be canonically specified only up to a complex constant. The induced symplectic form, still denoted by \( \varepsilon \), on \( S'^{\vee} \) is defined by requiring \( \varepsilon (s'^\vee, t'^\vee) = -\varepsilon (s, t) \) for \( s, t \in S'^{\vee} \). Similarly for the induced symplectic form, also denoted by \( \varepsilon \), on \( S''^{\vee} \).

---

3Note for mathematicians There can be three other conventions in physics literature to set the isomorphisms \( S' \cong S'^{\vee} \) and \( S'' \cong S''^{\vee} \) via \( \varepsilon \): (2) \( s \mapsto \varepsilon (s, \cdot) \) for \( s \in S' \) or \( S'' \); (3) \( s' \mapsto \varepsilon (\cdot, s') \) and \( s'' \mapsto \varepsilon (\cdot, s'') \) for \( s' \in S' \) and \( s'' \in S'' \); and (4) \( s' \mapsto \varepsilon (s', \cdot) \) and \( s'' \mapsto \varepsilon (s'', \cdot) \) for \( s' \in S' \) and \( s'' \in S'' \). Since \( \varepsilon \) is symplectic, different conventions lead to a discrepancy by factors of \( -1 \). Here, we follow the convention of [Wess & Bagger: Appendix Eq. (A.9)].

4One may define the induced symplectic form \( \varepsilon \) on \( S'^{\vee} \) (resp. \( S''^{\vee} \)) by requiring \( \varepsilon (s'^\vee, t'^\vee) = \varepsilon (s, t) \) for \( s, t \in S' \) (resp. \( S'' \)). However, since we choose to define \( s'^\vee \) as \( \varepsilon (\cdot, s) \), rather than \( \varepsilon (s, \cdot) \), for \( s \in S' \), it is more natural to require \( \varepsilon (s'^\vee, t'^\vee) = -\varepsilon (s, t) \) (\( = \varepsilon (t, s) \)). This is also the convention used in [Wess & Bagger].
The Clifford multiplication and the $Spin^0(1,3)$-module-isomorphism $V_C^\vee \simeq S' \otimes \mathbb{C} S''$

The representation of $Cl(V,\langle , \rangle)$ on $S \simeq S' \oplus S''$ (i.e. the Clifford multiplication) and the inclusion $V \subset Cl(V,\langle , \rangle)^{\text{odd}}$ induces a homomorphism

$$V_C := V \otimes \mathbb{R} \mathbb{C} \longrightarrow Hom_{\mathbb{C}}(S', S'') \simeq S'^\vee \otimes \mathbb{C} S''$$

as $Spin^0(1,3)$-modules. This homomorphism turns out to be an isomorphism. Together with the $Spin^0(1,3)$-module isomorphisms $S' \simeq S'^\vee$ and $S'' \simeq S''\vee$, one has the $Spin^0(1,3)$-module isomorphism $V_C^\vee \simeq S' \otimes \mathbb{C} S''$.

The explicit presentation in [Wess & Bagger: Appendix A]

An explicit presentation of the above is given by [Wess & Bagger: Appendix A], which we recall here and will be used in the current notes.

Definition 1.1.2. [Weyl spinors with a symplectic pairing $\varepsilon$]

Under the isomorphism $Spin^0(1,3) \simeq SL(2, \mathbb{C})$, the two inequivalent Weyl spinor representations $S'$ and $S''$ of $Spin^0(1,3)$ are given respectively by the fundamental representation $\mathbb{C}^2$ of $SL(2, \mathbb{C})$ and its complex conjugate. The representation of $SL(2, \mathbb{C})$ on $S'^\vee$ (resp. $S''\vee$) is equivalent to that on $S'$ (resp. $S''$) and is given by $\eta \mapsto (m^{-1})^\dagger \eta$ for $m \in SL(2, \mathbb{C})$ (resp. $\bar{\eta} \mapsto (\bar{m})^{-1} \bar{\eta}$) for $m \in SL(2, \mathbb{C})$. Here $(\cdot)^\dagger$ is the transpose of a matrix $(\cdot)$. In terms of components of Weyl spinors, these representations are given explicitly by

on $S'$: $$(\psi_\alpha)_\alpha \mapsto (\sum_\beta m^\alpha_\beta \psi_\beta)_\alpha,$$
on $S''$: $$(\bar{\psi}_\dot{\alpha})_\dot{\alpha} \mapsto (\sum_\beta \bar{m}^\dot{\alpha}_\beta \bar{\psi}_\dot{\beta})_\dot{\alpha};$$

on $S'^\vee$: $$(\psi_\alpha)^\alpha \mapsto (\sum_\beta m^{-1}_\alpha^\beta \psi_\beta)_\alpha,$$
on $S''\vee$: $$(\bar{\psi}_\dot{\alpha})^{\dot{\alpha}} \mapsto (\sum_\beta \bar{m}^{-1}_\dot{\alpha}^\dot{\beta} \bar{\psi}_\dot{\beta})^{\dot{\alpha}},$$

for $m = (m^\alpha_\beta, \bar{m}^\dot{\alpha}_\dot{\beta}) \in SL(2, \mathbb{C})$; cf. [W-B: Appendix A, Eq. (A.1)].

We fix the symplectic pairing $\varepsilon$ on $S'$ and $S''$ to be the standard/defining symplectic pairing and on $S'^\vee$ and $S''\vee$ the negative standard symplectic pairing: (as anticommuting 2-tensors)

$$\varepsilon^{12} = -\varepsilon^{21} = -\varepsilon^{\dot{1}\dot{2}} = 1, \quad \varepsilon_{12} = -\varepsilon_{21} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon_{\dot{2}\dot{1}} = -1.$$ 

By construction, $\varepsilon$ on $S'$, $S''$, $S'^\vee$, and $S''\vee$ are invariant under the $SL(2, \mathbb{C})$-action. Also note that

$$\sum_\beta \varepsilon^\alpha_\beta \varepsilon^\beta_\gamma = \delta^\alpha_\gamma \quad \text{and} \quad \sum_\beta \varepsilon^\dot{\alpha}_{\dot{\beta}} \varepsilon^\dot{\beta}_{\dot{\gamma}} = \delta^\dot{\alpha}_{\dot{\gamma}},$$

for our choice of $\varepsilon$. Here, $\delta^\alpha_{\gamma}$ is the Kronecker delta: $\delta^\alpha_{\gamma} = 1$ for $\alpha = \gamma$, else $0$; similarly for $\delta^\dot{\alpha}_{\dot{\gamma}}$.

Definition 1.1.3. [raising/lowering Weyl spinorial indices via $\varepsilon$]

Continuing Definition 1.1.2, we adopt the following rule to raise or lower a spinorial index:

$$\psi^\alpha = \sum_{\beta=1,2} \varepsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}^\dot{\alpha} = \sum_{\dot{\beta}=1,2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \quad \psi_\alpha = \sum_{\beta=1,2} \varepsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}_\dot{\alpha} = \sum_{\dot{\beta}=1,2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}},$$

and similarly for any object with upper or lower, undotted or dotted spinorial indices.

Note for mathematicians Here we respect physicists' standard notation that the conjugate Weyl spinor in $S''$ is denoted with a bar “” together with a dotted spinor index $\dot{\beta}$. Though this may look redundant for mathematicians, there is a good reason for this: In situations physicists want to take a 2-component Weyl spinor as a whole, unbar versus bar distinguishes the two Weyl spinors in inequivalent representations of the Lorentz group, for example, $\theta \theta := \sum_\alpha \theta^\alpha \theta_\alpha$ versus $\bar{\theta} \bar{\theta} := \sum_\dot{\alpha} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}$, while in situations that involve only spinorial indices, undotted versus dotted distinguishes the two inequivalent spinor representations the expression must transform accordingly under the covering Spin group of the Lorentz group, for example, (1) $\varepsilon^{\alpha\gamma}$ versus $\varepsilon^\beta_{\dot{\delta}}$ and (2) $\sigma^\alpha_{\dot{\beta}}$. Employing both notational conventions on a spinor component reinforces the distinction of the two inequivalent Weyl spinors.
Note that, by convention from differential geometry, tensorial indices from $V$ or $V_\mathbb{C}$ can be raised or lowered via the inner product $\langle \cdot, \cdot \rangle$. (Note that the inner product on $V$ extends to a complex inner product, still denoted $\langle \cdot, \cdot \rangle$, on $V_\mathbb{C}$ via $\mathbb{C}$-bi-linearity.)

**Remark 1.1.4.** [$S^\vee \simeq S'$ and $S''^\vee \simeq S''$] Let

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Then

$$(m^{-1})^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1};$$

$$(\bar{m}^{-1})^t = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}. $$

This realizes the equivalences of $Spin^0(1,3)$-modules $S^\vee \simeq S'$ and $S''^\vee \simeq S''$ that respect the rule of raising and lowering spinorial indices via $\varepsilon$.

Let $M_2(\mathbb{C})$ be the algebra of $2 \times 2$ matrices over $\mathbb{C}$. The dual vector space with inner product $(V^\vee, \langle \cdot, \cdot \rangle)$ is realized as the $\mathbb{R}$-subspace of Hermitian matrices in $M_2(\mathbb{C})$

$$p = (p_0, p_1, p_2, p_3)^t \in V^\vee \mapsto \tilde{p} := \begin{pmatrix} -p_0 + p_3 \\ p_1 + \sqrt{-1} p_2 \\ p_1 - \sqrt{-1} p_2 \\ -p_0 - p_3 \end{pmatrix},$$

with the inner product realized by the quadratic form $\langle \tilde{p}, \tilde{\rho} \rangle := -\det(\tilde{p})$. Under this isomorphism, $SL(2, \mathbb{C})$ acts on $V^\vee$ by

$$\tilde{p} \mapsto m \tilde{p} \bar{m}^t =: m \tilde{p} m^\dagger$$

for $m \in SL(2, \mathbb{C})$. It preserves $\langle \tilde{p}, \tilde{p} \rangle$ and realizes the double covering map $SL(2, \mathbb{C}) \to SO^\dagger(1,3)$. The fact that $SL(2, \mathbb{C})$ acts on $\tilde{p}$ from the left by multiplication by $m$ and from the right by multiplication by $\bar{m}^t$ implies that the correspondence $p \mapsto \tilde{p}$ induces an isomorphism

$$V^\vee_\mathbb{C} \xrightarrow{\sim} S' \otimes S''$$

as (left) $SL(2, \mathbb{C})$-modules. This realizes the Clifford multiplication. Denote

$$\sigma^0 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma^4 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(the *Pauli matrices*). Then, one may write

$$\tilde{p} = p_0 \sigma^0 + p_1 \sigma^1 + p_2 \sigma^2 + p_3 \sigma^3 =: (p_{\dot{\alpha} \dot{\beta}})_{\dot{\alpha} \dot{\beta}}.$$

The double indices $\alpha \dot{\beta}$ for entries of $\tilde{p}$ is justified by the isomorphism $V_\mathbb{C}^\vee \simeq S' \times S''$. The conversion rules are

$$p_{\alpha \dot{\beta}} = \sum_{\mu = 0}^3 \overline{\sigma^\mu}_{\alpha \dot{\beta}} p_\mu \quad \text{and} \quad p^\mu = -\frac{1}{2} \sum_{\alpha = 1,2; \beta = 1,2} \bar{\sigma}^{\mu \dot{\beta} \alpha} p_{\alpha \dot{\beta}}, \quad \text{cf. \cite[Eq. (A.13)]{Wess & Bagger}}$$

where $\bar{\sigma}^{\mu \dot{\beta} \alpha} := \sum_{\gamma \delta} \varepsilon^{\alpha \gamma} \varepsilon^{\dot{\beta} \delta} \sigma^\mu_{\gamma \delta}$.
1.2 The function-ring of a $d = 3 + 1$, $N = 1$ towered superspace and its distinguished subrings (under a trivialization of the spinor bundle by covariantly constant sections) (cf. [Wess & Bagger: Chapter IV and Appendices A & B])

We recall in this subsection how we come to the notion of ‘towered superspace’ from [L-Y5: Sec. 1] (SUSY(1)) for terminology and notations and explain some upgrade on the notion of ‘physics sector’ $^6$ Readers are referred to ibidem for more details.

$d = 3 + 1$, $N = 1$ towered superspaces and their function ring

Let

- $X = \mathbb{R}^{3+1}$ be the 3+1-dimensional Minkowski space-time; as a $C^\infty$-scheme $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of smooth functions on $X$; with coordinate functions $(x^\mu)_\mu = (x^0, x^1, x^2, x^3)$ and the metric $ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$; tangent bundle $T_X$, cotangent bundle $T^*_X$, their corresponding sheaf of sections $\mathcal{T}_X$, $\mathcal{T}^*_X$ and complexification $\mathcal{T}^{\mathbb{C}}_X := \mathcal{T}_X \otimes \mathcal{O}_X \mathbb{C}$ and $\mathcal{T}^{*\mathbb{C}}_X := \mathcal{T}^*_X \otimes \mathcal{O}_X \mathbb{C}$, all equipped with the Levi-Civita connection; here $\mathcal{O}^{\mathbb{C}}_X$ is the sheaf of complex-valued smooth functions on $X$ and $(X, \mathcal{O}^{\mathbb{C}}_X)$ as a complexified $C^\infty$-scheme with $C^\infty$ hull $\mathcal{O}_X \subset \mathcal{O}^{\mathbb{C}}_X$;

- $P_X$ the principal Lorentz-frame bundle over $X$ with the Levi-Civita connection, $\mathcal{P}_X$ the corresponding principal sheaf of Lorentz frames with a connection;

- $S'$, $S''$ be the Weyl spinor bundles and $S'$, $S''$ the corresponding sheaf of sections, all equipped with the induced connection and covariantly constant sympletic pairing $\varepsilon$ that respect the complex conjugation $\bar{\varepsilon}$: $S' \leftrightarrow S''$, $S' \leftrightarrow S''$; $S'^\vee$, $S''^\vee$ the dual spinor bundle of $S'$ and $S''$ respectively; $S'^\vee$ and $S''^\vee$ the corresponding sheaves.

- Recall Definition 1.1.3 and Remark 1.1.4. Let $(\theta_1, \theta_2)$ be a pair of independent covariantly constant sections of $S'$. This then gives rise to pairs of covariantly constant sections $(\theta^1, \theta^2)$ in $S'^\vee$, $(\bar{\theta}_1, \bar{\theta}_2)$ in $S''$, and $(\bar{\theta}^1, \bar{\theta}^2)$ in $S''^\vee$.

When in need of taking a copy of spinor bundle in a construction, i.e. complex rank-2 bundle on $X$ with a fixed isomorphism to $S'$, $(\theta_1, \theta_2)$ then passes to $(\bar{\theta}_1, \bar{\theta}_2)$ in the copy, which in turn gives rise to $(\bar{\theta}^1, \bar{\theta}^2)$, $(\bar{\theta}_1, \bar{\theta}_2)$, and $(\bar{\theta}^1, \bar{\theta}^2)$ respectively in the various corresponding copies of spinor bundles via taking dual or complex conjugation.

From the very start, physicists’ design of superfields (Abdus Salam and John Strathdee: [SS: Eqs. (I.15) & (I.16)] (1978)) wants them to form a ring, i.e. addition and multiplication of superfields make sense and are still superfields. In essence, a ‘towered superspace’ is the ‘space’ that takes this ring as its function ring. From the fact that a single ‘superfield’ on $X$ combines both bosonic field(s) and fermionic field(s) on $X$ into its components (and hence “super”) and the number and details of bosonic fields and fermionic fields depend on the supersymmetric

$^6$In [L-Y5: Sec. 1] (SUSY(1)) we define the function ring of the physics sector as subring generated by the chiral superfield and the antichiral superfield. Then in [L-Y5: Sec. 3] (SUSY(1)) we define ‘vector superfield’ from this physics sector to construct a supersymmetric $U(1)$ gauge theory coupled with matters. It will turn out that to fit [Wess & Bagger] exactly, one has to look for the notion of ‘vector superfields’ outside this subring! It is for this reason that in the current new work we abandon the name “physics sector” assigned to this subring since that would give a wrong impression or implication that elements outside this subring is not physics-relevant. In the current work, this subring is given a new name the ‘small function-ring’ of the towered superspace in question. It is the subring that is involved in Wess-Zumino models (cf. Sec. 3) and $d = 3 + 1$, $N = 1$ nonlinear sigma models (cf. Sec. 5).
quantum-field-theory model one wants to construct, this towered superspace depends on the supersymmetric quantum-field-theory model one wants to construct as well. As a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme, this towered superspace can be described level-by-level as follows: (The presentation here is for the case of $d = 3 + 1$, $N = 1$ supersymmetric quantum field theories and assuming that all fermionic fields are described by sections of Weyl spinor bundles $S'$, $S''$. Other situations are similar.)

(1) [fundamental/ground level] This is the super homogeneous space $\hat{X}$ from the quotient of the super Poincaré group by the Lorentz subgroup. As a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme, it has the underlying topology $X$ and the structure sheaf

$$\mathcal{O}_{\hat{X}} := \hat{\mathcal{O}}_X := \Lambda^*_{\hat{\mathcal{O}}_X}(S'^\vee \oplus S''^\vee),$$

with the $\mathbb{Z}/2$-grading given by

$$\Lambda^*_{\hat{\mathcal{O}}_X}(S'^\vee \oplus S''^\vee) = \Lambda^\text{even}_{\hat{\mathcal{O}}_X}(S'^\vee \oplus S''^\vee) \oplus \Lambda^\text{odd}_{\hat{\mathcal{O}}_X}(S'^\vee \oplus S''^\vee) =: \hat{\mathcal{O}}_X^\text{even} \oplus \hat{\mathcal{O}}_X^\text{odd}$$

and the $C^\infty$-hull given by

$$\mathcal{O}_X \oplus \Lambda^\text{even,}\geq 2(S'^\vee \oplus S''^\vee).$$

This is the sheaf of Grassmann algebras generated by the $\mathcal{O}_{\hat{X}}^\mathbb{C}$-module $S'^\vee \oplus S''^\vee$. Covariantly constant sections of $S'^\vee \oplus S''^\vee$ provide fermionic/anticommuting coordinate functions on $\hat{X}$ and the action of the super Poincaré group on $\hat{X}$ is realized as automorphisms on the function ring $C^\infty(\hat{X}) = \hat{\mathcal{O}}_X(X)$ of $X$. This gives a representation of the super Poincaré algebra in the $\mathbb{Z}/2$-graded Lie algebra of derivations on $C^\infty(\hat{X})$. In particular, the infinitesimal supersymmetry generators $Q_\alpha, Q_\beta$ and the supersymmetrically-invariant-flow generators $D_\alpha, D_\beta$ of [Wess & Bagger: Eqs. (4.4) and (4.5)] are specific derivations on $C^\infty(\hat{X})$. (Cf. Example 1.2.5.)

(2) [field/upper level(s)] The Spin-Statistics Theorem from Quantum Statistical Mechanics and Quantum Field Theory says that fermionic fields must be anticommuting by nature. Thus, every time a fermionic field appears in the problem, we have to ask:

**Q. Does it already lie in the existing sheaf of Grassmann algebras of the problem?**

If not, then we have to enlarge the generating $\mathcal{O}_{\hat{X}}^\mathbb{C}$-module of the existing sheaf of Grassmann algebras to include the new anticommuting field(s). This gives rise to an inclusion system of function rings or, contravariantly equivalently a projection system of complexified $\mathbb{Z}/2$-graded $C^\infty$-schemes over $\hat{X}$. This is why and how a towered superspace appears. The new generators to the enlarged function ring are themselves fermionic fields over $X$. Specifically, when these fermionic fields are sections of different copies of spinor sheaves, we distinguish them by a subscript $S'_{\text{field}, i} \oplus S''_{\text{field}, i}$, $i = 1, \ldots, l$, and construct the towered superspace as a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme

$$\hat{X}^\# := (X, \hat{\mathcal{O}}_{\hat{X}}^\#) := (X, \Lambda^*_{\hat{\mathcal{O}}_X^\#} \mathcal{F})$$

over $\hat{X}$ with

$$\mathcal{F} := (S'^\text{coordinates} \oplus S''\text{coordinates}) \oplus \bigoplus_{i=1}^l (S'_{\text{field}, i} \oplus S''_{\text{field}, i}).$$

We say that $S'_{\text{field}, i} \oplus S''_{\text{field}, i}$ contributes to the $i$-th field level of $\hat{X}^\#$. The total level number $l$ is the number of distinct types/species/generations of fermionic fields in a $d = 3+1$, $N = 1$ supersymmetric field theory one wants to construct. It can be different theory-by-theory.
Finally, when physicists working on supersymmetry introduce ‘Grassmann number’ parameter \((\eta, \bar{\eta}) := (\eta^1, \eta^2, \bar{\eta}^1, \bar{\eta}^2)\) in their computation, these ‘Grassmann number’ parameter are meant to be independent of anything else. Thus, they should be thought of as constant sections of another copy of \(S'^\vee \oplus S''^\vee\). Since they are used in the computation, it is appealing in practice (if not in concept) to incorporate them into the function ring of the towered superspace and think of \(\hat{X}^{\hat{\mathbb{H}}}\) as over this basic complexified \(\mathbb{Z}/2\)-graded \(C^\infty\)-scheme. Thus, we make a final adjustment to \(\hat{X}^{\hat{\mathbb{H}}}\) by redefining

\[
\hat{X}^{\hat{\mathbb{H}}} := (X, \hat{\mathcal{O}}^{\hat{\mathbb{H}}}_X) := (X, \bigwedge^\bullet \mathcal{O}^\mathbb{C}_X \mathcal{F})
\]

with

\[
\mathcal{F} := (S'^\vee_{\text{parameter}} \oplus S''^\vee_{\text{parameter}}) \oplus (S'^\vee_{\text{coordinates}} \oplus S''^\vee_{\text{coordinates}}) \oplus \bigoplus_{i=1}^l (S'_\text{field},i \oplus S''_\text{field},i).
\]

We say that \(S'^\vee_{\text{parameter}} \oplus S''^\vee_{\text{parameter}}\) contributes to the Grassmann parameter level of \(\hat{X}^{\hat{\mathbb{H}}}\). Similar to the fundamental level \(\hat{X}\) of the tower, this level depends only on the supersymmetry in question (here, \(d = 3 + 1, N = 1\)). Together they form the universal base for all \(d = 3 + 1, N = 1\) towered superspaces.

These motivate the following definitions in [L-Y5: Sec. 1.3] (SUSY(1)):

**Definition 1.2.1.** [\(d = 4, N = 1\) towered superspace \(\hat{X}^{\hat{\mathbb{H}}}\) with \(l\) field-theory levels] The complexified \(\mathbb{Z}/2\)-graded \(C^\infty\)-scheme

\[
\hat{X}^{\hat{\mathbb{H}}} := (X, \hat{\mathcal{O}}^{\hat{\mathbb{H}}}_X) := (X, \bigwedge^\bullet \mathcal{O}^\mathbb{C}_X \mathcal{F})
\]

with

\[
\mathcal{F} := (S'^\vee_{\text{parameter}} \oplus S''^\vee_{\text{parameter}}) \oplus (S'^\vee_{\text{coordinates}} \oplus S''^\vee_{\text{coordinates}}) \oplus \bigoplus_{i=1}^l (S'_\text{field},i \oplus S''_\text{field},i)
\]

is called the \(d = 4, N = 1\) towered superspace with \(l\) field-theory levels. Here, all \(S'_i\) (resp. \(S''_i, S'^\vee_i, S''^\vee_i\)) are copies of \(S'\) (resp. \(S''_i, S'^\vee_i, S''^\vee_i\)). When \(l\) is implicit in the problem, we will denote \(\hat{X}^{\hat{\mathbb{H}}}\) simply by \(\hat{X}^{\hat{\mathbb{H}}}\). By convention, we will keep the parameter level suppressed when not activated for use in a discussion.

Figure 1-2-1. (Cf. [L-Y5: Definition/Explanation 1.3.2] (SUSY(1)).)

Note that, as an \(\mathcal{O}^\mathbb{C}_X\)-modules,

\[
\hat{\mathcal{O}}^{\hat{\mathbb{H}}}_X = \bigwedge^\bullet \mathcal{O}^\mathbb{C}_X (S'^\vee_{\text{coordinates}} \oplus S''^\vee_{\text{coordinates}}) \otimes \mathcal{O}^\mathbb{C}_X (S'^\vee_{\text{parameter}} \oplus S''^\vee_{\text{parameter}}) \otimes \bigotimes_{i=1}^l \otimes \mathcal{O}^\mathbb{C}_X (S'_\text{field},i \oplus S''_\text{field},i).
\]

Each factor \(\bigwedge^\bullet \mathcal{O}^\mathbb{C}_X (\cdots)\) in the \(\otimes \mathcal{O}^\mathbb{C}_X\)-decomposition contributes to a level/layer/floor of the towered superspace \(\hat{X}^{\hat{\mathbb{H}}}\).

\(^7\)Mathematically this means that \(S'^\vee_{\text{coordinates}}\) is isomorphic to \(S'^\vee\) with a fixed isomorphism; and similarly for all other spinor sheaves that appear as direct summands of \(\mathcal{F}\).
Figure 1-2-1. (Cf. [L-Y5: Figure 1-4-1].) The space-time coordinate functions $x^\mu$, $\mu = 0, 1, 2, 3$, and the fermionic coordinate functions $\theta^\alpha$, $\bar{\theta}^{\dot{\beta}}$, $\alpha = 1, 2$, $\dot{\beta} = \dot{1}, \dot{2}$, generate the function ring of the fundamental superspace $\hat{X}$ as a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme. Over it sits a supertower with Grassmann-number level and other field-theory levels that are needed for the construction of supersymmetric quantum field theories. From the direct-sum expression of the generating sheaf $F := (S'^\vee\text{coordinates} \oplus S'^{''\vee}\text{coordinates}) \oplus (S'^\vee\text{parameter} \oplus S'^{''\vee}\text{parameter}) \oplus \bigoplus_{i=1}^l (S'_\text{field,i} \oplus S''_\text{field,i})$ of the structure sheaf $\hat{\mathcal{O}}_{\hat{X}}$ of $\hat{X}$, one may think of each field-theory level as contributing a floor-[$i$]

$$\hat{X}_{[i]}^{\text{Double}} := \left( X, \bigwedge_{\mathcal{O}_{\hat{X}}} \left( S'^\vee\text{coordinates} \oplus S'^{''\vee}\text{coordinates} \oplus S'_\text{field,i} \oplus S''_\text{field,i} \right) \right)$$

over $\hat{X}$ and these field-theory floors are glued by the $\mathbb{Z}/2$-graded version of fibered product over $\hat{X}$ to give $\hat{X}^{\hat{\mathbb{H}}}$. Each field-theory floor $\hat{X}_{[i]}^{\text{Double}}$ has some distinguished sectors $X_{[i]}^\bullet$ that are purely even. They generate some distinguished sectors $\hat{X}_{[i]}^{\hat{\mathbb{H}}, \bullet}$ of $\hat{X}^{\hat{\mathbb{H}}}$ that are also purely even. These distinguished sectors are where physics-relevant superfields on $X$ lie.
Definition 1.2.2. [derivation on $\hat{X}$ applied to $\mathcal{C}^\infty(\hat{X})$] Let $\xi \in \text{Der}_\mathcal{C}(\hat{X})$ be a derivation on $\hat{X}$ over $\mathbb{C}$ and $\check{f} \in \mathcal{C}^\infty(\hat{X})$. Then we define $\xi \check{f} \in \mathcal{C}^\infty(\hat{X})$ via the built-in inclusion $\text{Der}_\mathcal{C}(\hat{X}) \hookrightarrow \text{Der}_\mathcal{C}(\hat{X})$.

Definition 1.2.3. [complex conjugation vs. twisted complex conjugation] The complex conjugation $- : \mathcal{O}_X^\mathbb{C} \rightarrow \mathcal{O}_X^\mathbb{C}$ and $S' \rightarrow S''$, $S'' \rightarrow S'$, of Weyl spinors extends canonically to a complex conjugation

$$\hat{-} : \mathcal{O}_X^\mathbb{C} \rightarrow \mathcal{O}_X^\mathbb{C},$$

by setting

$$\hat{f} + \hat{g} = \check{f} + \check{g};$$

$$\hat{f} \hat{g} = \check{g} \check{f}.$$

and a twisted complex conjugation

$$\hat{\dag} : \mathcal{O}_X^\mathbb{C} \rightarrow \mathcal{O}_X^\mathbb{C},$$

by setting

$$\hat{f} + \hat{g} = \check{f} + \check{g};$$

$$\hat{f} \hat{g} = \check{g} \check{f}.$$

Caution that the order of multiplication is preserved under the complex conjugate $-$ but is reversed under the twisted complex conjugate $\hat{\dag}$.

Definition 1.2.4. [standard coordinate functions on $\hat{X}$] Let $\hat{X} = \mathbb{R}^4$. Then, the standard coordinate functions $(x, \theta, \bar{\theta})$ on $\hat{X}$ extends uniquely to a tuple of coordinate functions

$$(x^\mu, \theta^\alpha, \bar{\theta}^\beta, \eta^\gamma, \bar{\eta}^\bar{\gamma}, \phi^1_\gamma, \bar{\phi}^1_{\bar{\gamma}}, \ldots ; \phi^l_\gamma, \bar{\phi}^l_{\bar{\gamma}}) =: (x, \theta, \bar{\theta}, \eta, \bar{\eta}, \phi, \bar{\phi})$$

on $\hat{X}$ via the $\varepsilon$-tensor $\varepsilon : S' \otimes \mathcal{O}_X^\mathbb{C} S' \rightarrow O_{\hat{X}}^\mathbb{C}$, $S'' \otimes \mathcal{O}_X^\mathbb{C} S'' \rightarrow O_{\hat{X}}^\mathbb{C}$, and the fixed isomorphisms $S'_\varepsilon \simeq S'$, $S''_\varepsilon \simeq S''$.

Explicitly, regard $S'_{\text{parameter}}$ as a copy of $S'_{\text{coordinates}}$, $S''_{\text{parameter}}$ as a copy of $S''_{\text{coordinates}}$, $S'_{\text{field},i}$ as a copy of $(S'_{\text{coordinates}})^\vee = S'_{\text{coordinates}}^\ast$, and $S''_{\text{field},i}$ as a copy of $(S''_{\text{coordinates}})^\vee = S''_{\text{coordinates}}$ under the fixed isomorphisms. Then, $(\eta^{\gamma'}, \bar{\eta}^{\bar{\gamma}'}) = (\theta^{\alpha'}, \bar{\theta}^{\beta'})$ and $(\phi^i_\gamma, \bar{\phi}^i_{\bar{\gamma}}) = (\theta^i_\alpha, \bar{\theta}^i_{\bar{\beta}})$ for all $i$, where $\theta_\alpha = \sum_\gamma \varepsilon_{\alpha \gamma} \theta^\gamma$, $\theta_\beta = \sum_\delta \varepsilon_{\beta \delta} \theta^\delta$.

We shall call $(x, \theta, \bar{\theta}, \eta, \bar{\eta}, \phi, \bar{\phi})$ the standard coordinate functions on $\hat{X}$.

In terms of this, $C^\infty(\hat{X}) = C^\infty(X)^\mathbb{C}[\theta, \bar{\theta}, \eta, \bar{\eta}, \phi, \bar{\phi}]_{\text{anti-c}}$ and an $\check{f} \in C^\infty(\hat{X})$ has a $(\theta, \bar{\theta})$-expansion

$$\check{f} = \check{f}_0 + \sum_\alpha \theta^\alpha \check{f}_\alpha + \sum_\beta \bar{\theta}^\beta \check{f}_\beta + \theta^1 \theta^2 \check{f}_{(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \check{f}_{(\alpha \beta)}$$

$$+ \bar{\theta}^1 \bar{\theta}^2 \check{f}_{(12)} + \sum_\beta \theta^1 \theta^2 \bar{\theta}^\beta \check{f}_{(12 \beta)} + \sum_\alpha \theta^\alpha \theta^1 \theta^2 \check{f}_{(\alpha 12)} + \theta^1 \theta^2 \theta^1 \theta^2 \check{f}_{(1212)}$$

with coefficients $\check{f}_\bullet \in C^\infty(X)^\mathbb{C}[\eta, \bar{\eta}, \phi, \bar{\phi}]_{\text{anti-c}}$.  

14
Example 1.2.5. [special derivations on \( C^\infty(\hat{X}) \)] Recall from [L-Y4: Sec. 1.4] (D(14.1)) the standard infinitesimal supersymmetry generators on \( \hat{X} \)

\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - \sqrt{-1} \sum_{\mu=0}^{3} 3 \sum_{\beta=1}^{2} \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \bar{Q}_\beta = - \frac{\partial}{\partial \bar{\theta}^\beta} + \sqrt{-1} \sum_{\mu=0}^{3} 3 \sum_{\alpha=1}^{2} \theta^\alpha \sigma^\mu_{\alpha \beta} \frac{\partial}{\partial \bar{x}^\mu}
\]

and derivations that are invariant under the flow that generate supersymmetries on \( \hat{X} \)

\[
e^\alpha = \frac{\partial}{\partial \theta^{\alpha'}} + \sqrt{-1} \sum_{\mu=0}^{3} 3 \sum_{\beta=1}^{2} \sigma^\mu_{\alpha' \beta} \bar{\theta}^\beta \frac{\partial}{\partial x^\mu} \quad \text{and} \quad e^{\beta'} = - \frac{\partial}{\partial \bar{\theta}^{\beta'}} - \sqrt{-1} \sum_{\mu=0}^{3} 3 \sum_{\alpha=1}^{2} \theta^\alpha \sigma^\mu_{\alpha' \beta'} \frac{\partial}{\partial \bar{x}^\mu}.
\]

Then one can check directly that they satisfy the following anticommutator relations

\[
\{Q_\alpha, \bar{Q}_\beta\} = 2\sqrt{-1} \sum_\mu \sigma^\mu_{\alpha \beta} \frac{\partial}{\partial x^\mu}, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0;
\]

\[
\{e^\alpha, e^{\beta'}\} = -2\sqrt{-1} \sum_\mu \sigma^\mu_{\alpha' \beta'} \frac{\partial}{\partial x^\mu}, \quad \{e^\alpha, e^{\beta'}\} = \{e^{\alpha'}, e^{\beta'}\} = 0,
\]

\[
\{e^\alpha, Q_\beta\} = \{e^\alpha, \bar{Q}_\beta\} = \{e^{\alpha'}, Q_\beta\} = \{e^{\alpha'}, \bar{Q}_\beta\} = 0,
\]

for \( \alpha, \beta = 1, 2; \alpha', \beta' = 1, 2'; \) and \( \alpha'', \beta'' = 1'', 2''. \) See ibidem for more details. (Cf. [Wess & Bagger: Eqs. (4.4), (4.5), (4.6), (4.7), (4.8)] and related discussion therein.)

A subtlety: The \( \mathcal{P}_X \)-module structure on \( \hat{\mathcal{O}}_X \) from a partial twisting by \( \mathcal{T}_{X_S}^C \)

While under the fixed trivialization of \( S' \) by the covariantly constant sections \( \theta^\alpha \) and of \( S'' \) by their complex conjugate \( \bar{\theta}^\beta \) the structure sheaf \( \hat{\mathcal{O}}_X \) of \( \hat{X} \), as a sheaf of complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-rings, is isomorphic to the sheaf of Grassmann algebras \( \bigwedge_{\mathcal{O}_X}^\bullet \mathcal{F} \), where

\[
\mathcal{F} = (S'_{\text{coordinates}} \oplus S''_{\text{coordinates}}) \oplus (S'_{\text{parameter}} \oplus S''_{\text{parameter}}) \oplus \bigoplus_{l=1}^I (S'_{\text{field}, i} \oplus S''_{\text{field}, i}) \quad \text{for some } l,
\]

this isomorphism does not respect the \( \mathcal{P}_X \)-module structure physicists intended for \( \hat{\mathcal{O}}_X \)! This is what makes an intrinsic construction of the function ring \( C^\infty(\hat{X}) = \Gamma(\hat{\mathcal{O}}_X) \) that matches [Wess & Bagger] subtle. While the built-in/natural \( \mathcal{P}_X \)-module structure on \( \bigwedge_{\mathcal{O}_X}^\bullet \mathcal{F} \) is induced by the spinor representations of the Lorentz group on the spinor bundles \( S' \) and \( S'' \),

\[
\text{\( \cdot \) the \( \mathcal{P}_X \)-module structure physicists use for } \hat{\mathcal{O}}_X \text{ is somewhat the "sheaf of Grassmann algebras } \bigwedge_{\mathcal{O}_X}^\bullet \mathcal{F} \text{ partially twisted by the complexified cotangent sheaf } \mathcal{T}_{X_S}^C \text{ of } X \".}
\]

Since \( \mathcal{T}_{X_S}^C \) is itself a nontrivial \( \mathcal{P}_X \)-module, this creates a difference between the \( \mathcal{P}_X \)-module structure on \( \hat{\mathcal{O}}_X \) and that on \( \bigwedge_{\mathcal{O}_X}^\bullet \mathcal{F} \). For the current work, we will explicitly spell out such a partial-twisting-by-\( \mathcal{T}_{X_S}^C \) in the next theme for all the superfields that will be used to reconstruct [Wess & Bagger].

Before proceeding to the next theme, it is important to note that it is with respect to this partially twisted \( \mathcal{P}_X \)-module structure on \( \hat{\mathcal{O}}_X \), not the natural \( \mathcal{P}_X \)-module structure on \( \bigwedge_{\mathcal{O}_X}^\bullet \mathcal{F} \) that physicists defines the following notions:

Lemma/Definition 1.2.6. [(Lorentz-)scalar superfield, spinor superfield, ... on \( X \)]

The partially twisted \( \mathcal{P}_X \)-structure on \( \hat{\mathcal{O}}_X \) induces a direct-sum decomposition as a \( \mathcal{P}_X \)-module

\[
\hat{\mathcal{O}}_X = \mathcal{M}_{\text{trivial}} + \mathcal{M}_{\text{spinor}} + \mathcal{M}_{\text{vector}} + \cdots,
\]
where $\mathcal{M}_{\text{trivial}}$ is a trivial $\mathcal{P}_X$-module, $\mathcal{M}_{\text{spinor}}$ is a direct sum of $\mathcal{P}_X$-modules associated to spinor representations of the Lorentz group, $\mathcal{M}_{\text{vector}}$ is a direct sum of $\mathcal{P}_X$-modules associated to the vector representation of the Lorentz group, etc. A section of $\mathcal{M}_{\text{trivial}}$ is called a Lorentz-scalar superfield or simply scalar superfield on $X$; a section of $\mathcal{M}_{\text{spinor}}$ is called a spinor superfield on $X$; a section of $\mathcal{M}_{\text{vector}}$ is called a vector superfield on $X$; etc.

The details of the decomposition require a further study of the partial twist of $\bigwedge^\bullet_{\hat{X}} \mathcal{F}$ by $T^*_X \mathbb{C}$, which we won’t pursue in the current work.

**Terminology 1.2.7.** [scalar superfield, vector superfield vs. Lemma/Definition 1.2.6] In addition to Lemma/Definition 1.2.6, there is a second naming system assigned to the term ‘scalar superfield’, ‘vector superfield’, etc. that physicists also use: naming by the lowest dynamical component fields of a superfield. E.g. a Lorentz-scalar superfield in the sense of Lemma/Definition 1.2.6 that has its lowest dynamical component(s) in the trivial representation of the Lorentz group is called a scalar superfield; a Lorentz-scalar superfield in the sense of Lemma/Definition 1.2.6 that has its lowest dynamical components in the vector representation of the Lorentz group is called a vector superfield; etc.. It is this second sense that is used in [Wess & Bagger: Chapter VI] when defining ‘vector superfield’, which we will follow in Sec. 4.

**Convention 1.2.8.** [towered superspace] To keep the notation simple and being enough to demonstrate the reconstruction of [Wess & Bagger], for the rest of the work we will assume $\hat{X} = \hat{X}$ unless otherwise noted. Also, the parameter level of $\hat{X}$ will be suppressed when not in use for a discussion, computation, or expression.

**Tame, medium, and small scalar superfields on $X$**

As a polynomial in $\mathbb{C}[\theta, \bar{\theta}, \vartheta, \bar{\vartheta}, \theta, \bar{\theta}, \vartheta, \bar{\vartheta}]_{\text{anti-c}}$, a general element in $\mathbb{C}[\hat{X}]$ has $2^8 = 256$ monomial summands. Some special classes of elements, with much fewer monomial summands, play more prominent role in the construction of supersymmetric quantum field theories. In this theme we spell out a few such classes. Each class forms an even subring of $\mathbb{C}[\hat{X}]$.

**Definition 1.2.9.** [tame (Lorentz-)scalar superfield] An $\tilde{f} \in \mathbb{C}[\hat{X}]$ is called a tame scalar superfield on $X$ if it is a Lorentz-scalar superfield on $X$ in the sense of Lemma/Definition 1.2.6 and, as a polynomial in $\mathbb{C}[\theta, \bar{\theta}, \vartheta, \bar{\vartheta}]_{\text{anti-c}}$, it satisfies the following property

\[
\cdot (\vartheta, \bar{\vartheta})\text{-degree} \leq (\theta, \bar{\theta})\text{-degree} \quad \text{for every summand of } \tilde{f}.
\]

*Note for mathematicians* The naming system by the $\mathcal{P}_X$-module structure is favored from the perspective of complexified $\mathbb{Z}/2$-graded $\mathbb{C}$-Algebraic Geometry while the naming by the representation of the lowest dynamical component(s) is favored from the perspective of representation theory of supersymmetry algebra. One should tell exactly which sense the term is in from the context.

Since $\tilde{f}$ is a combination of both bosonic and fermionic fields on $X$, it is conceptually more accurate to call $\tilde{f}$ a superfield on $X$, rather than on $\hat{X}$ or $\hat{X}$.
Explicitly, a tame scalar superfield $\tilde{f}$ on $X$ is of the following form

$$
\tilde{f} = f(0) + \sum_{\alpha} \theta^\alpha f(\alpha) + \sum_{\beta} \bar{\theta}^\beta \tilde{f}(\beta) + \theta^1 \theta^2 \tilde{f}_{(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \tilde{f}_{(\alpha \beta)} + \bar{\theta}^1 \bar{\theta}^2 \tilde{f}_{(\bar{1} \bar{2})}
$$

where $\alpha = 1, 2, 3, \beta = 1, 2, 3$;

$$
\sigma^{\mu \alpha} = \sum_{\gamma} e^\gamma e^\gamma \sigma^{\mu \gamma}, \quad \bar{\sigma}^{\mu \beta} = \sum_{\gamma} e^\gamma e^\gamma \bar{\sigma}^{\mu \gamma}, \quad \bar{\sigma}^{\nu \beta} = \sum_{\gamma} e^\gamma e^\gamma \sigma^{\nu \gamma};
$$

and, for the forty-one coefficients $f^*_{(\theta, \bar{\theta}, \alpha, \beta)}$ of the $(\theta, \bar{\theta}, \alpha, \beta)$-monomial summands of $\tilde{f}$,

$$
(f(0)), f(\alpha), f(\beta), f(\gamma), f(12), f(\alpha \beta), f''(0), f''(\alpha), f''(\beta), f''(\gamma), f''(12), f''(\alpha \beta) \in C^\infty(X)^C
$$

is a trivial $\mathcal{P}_X$-module, while

$$(f''(\mu))_\mu, (f''(\mu))_\mu, (f''(\nu))_\mu \in (C^\infty(X)^C)_\mu \cong T_X^C$$

Here the trivialization $(C^\infty(X)^C)_\mu \cong T_X^C$ as $\mathcal{O}_X^C$-modules is induced by the fixed isomorphism $T_X^C \cong S' \otimes_{\mathcal{O}_X} S''$ as $\mathcal{P}_X$-modules and the trivialization of $S''$ by $\theta^\gamma$'s and $S''$ by $\bar{\theta}^\gamma$'s. This is the partial twist of $\bigwedge_{\mathcal{O}_X^C} F$ by $T_X^C$ we refer to in the previous theme. This partial twist justifies the $\tilde{f}$ as expressed above to be a Lorentz-scalar superfield in the sense of Lemma/Definition 1.2.6.

Note that among the forty-one coefficients, twenty-nine

$$
f(0), f'(0), f(12), f(\alpha \beta), f''(0), f''(0), f''(\alpha), f''(\beta), f''(\gamma), f''(12), f''(\alpha \beta) \in C^\infty(X)^C
$$

are related to bosonic fields on $X$ and twelve

$$
f(\alpha), f(\beta), f''(\alpha), f''(\beta) \in C^\infty(X)^C
$$

to fermionic fields on $X$.

**Lemma/Definition 1.2.10. [tame sector $\hat{X}_{\mathbb{Z},tone}$ of $\hat{X}_{\mathbb{Z}}$]** The collection of tame scalar superfields on $X$ as defined in Definition 1.2.9 is an even subring of the complexified $\mathbb{Z}/2$-graded $C^\infty$-ring $C^\infty(\hat{X}_{\mathbb{Z}})$. Denote this subring (also a $C^\infty(X)^C$-subalgebra of $C^\infty(\hat{X}_{\mathbb{Z}})$) by $C^\infty(\hat{X}_{\mathbb{Z}})^{tame}$. Then, the $C^\infty$-hull of $C^\infty(\hat{X}_{\mathbb{Z}})_{tame}$ restricts to the $C^\infty$-hull of $C^\infty(\hat{X}_{\mathbb{Z}})^{tame}$, which is given by

$$
C^\infty(\hat{X}_{\mathbb{Z}})^{tame} = \{ \tilde{f} \in C^\infty(\hat{X}_{\mathbb{Z}})^{tame} \mid \tilde{f}(0) \in C^\infty(X) \}.
$$

Denote by $\hat{X}_{\mathbb{Z}}^{tame}$ the complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme with the underlying topology $X$ and function ring $C^\infty(\hat{X}_{\mathbb{Z}})^{tame}$. Then there is a built-in dominant morphism $\hat{X}_{\mathbb{Z}}^{tame} \to \hat{X}_{\mathbb{Z}}^{tame}$. We will call $\hat{X}_{\mathbb{Z}}^{tame}$ the tame sector of $\hat{X}_{\mathbb{Z}}$.  
Proof. This follows from the fact that both the conditions on tame scalar superfields (1) Lorentz scalar and (2) \((\vartheta, \bar{\vartheta})\)-degree \(\leq (\theta, \bar{\theta})\)-degree for every summand are closed under the multiplication in \(C^\infty(\hat{X}^\mathbb{B})\).

Explicitly, let
\[
\bar{A} = A_0 + \sum_{\alpha} \vartheta^\alpha \partial_{\alpha} A_{(\alpha)} + \sum_{\beta} \vartheta^\beta \bar{\partial}_\beta A_{(\bar{\beta})} \\
+ \vartheta^1 \vartheta^2 (A'_0 + \vartheta_1 \vartheta_2 A_{(12)}) + \sum_{\alpha, \beta} \vartheta^\alpha \bar{\partial}_\beta \left( \sum_{\mu} \sigma_{\alpha \beta}^\mu A_{(\mu)} + \vartheta_\alpha \bar{\partial}_\beta A_{(\alpha \beta)} \right) + \vartheta_1 \vartheta_2 (A''_0 + \bar{\vartheta}_1 \bar{\vartheta}_2 A_{(12)}) \\
+ \sum_{\beta} \vartheta^1 \vartheta^2 \bar{\partial}_\beta \left( \sum_{\alpha, \mu} \vartheta^\alpha \sigma^{\mu \beta}_{\alpha} A'_{(\mu \beta)} + \bar{\partial}_\beta A'_{(\beta)} + \vartheta_1 \vartheta_2 \bar{\partial}_\beta A_{(12 \beta)} \right) \\
+ \sum_{\alpha} \vartheta^\alpha \bar{\partial}_\beta \left( \vartheta_{(\alpha)} f_\alpha'' + \sum_{\beta, \mu} \vartheta_\beta \sigma^{\mu \beta}_{\alpha} A''_{(\mu \beta)} + \vartheta_\alpha \bar{\partial}_\beta A_{(\alpha \beta)} \right) \\
+ \vartheta^1 \vartheta^2 \bar{\partial}_\beta \left( A''_0 + \vartheta_1 \vartheta_2 A''_{(12)} + \sum_{\alpha, \beta, \mu} \vartheta_\alpha \bar{\partial}_\beta \sigma^{\mu \beta \alpha}_{\alpha} A''_{(\mu \beta \alpha)} + \bar{\vartheta}_1 \bar{\vartheta}_2 A''_{(12 \beta \alpha)} + \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_1 \bar{\vartheta}_2 A''_{(12 \beta \alpha \beta)} \right)
\]

and
\[
\bar{B} = B_0 + \sum_{\alpha} \vartheta^\alpha \partial_{\alpha} B_{(\alpha)} + \sum_{\beta} \vartheta^\beta \bar{\partial}_\beta B_{(\bar{\beta})} \\
+ \vartheta^1 \vartheta^2 (B'_0 + \vartheta_1 \vartheta_2 B_{(12)}) + \sum_{\alpha, \beta} \vartheta^\alpha \bar{\partial}_\beta \left( \sum_{\mu} \sigma_{\alpha \beta}^\mu B_{(\mu)} + \vartheta_\alpha \bar{\partial}_\beta B_{(\alpha \beta)} \right) + \vartheta_1 \vartheta_2 (B''_0 + \bar{\vartheta}_1 \bar{\vartheta}_2 B_{(12)}) \\
+ \sum_{\beta} \vartheta^1 \vartheta^2 \bar{\partial}_\beta \left( \sum_{\alpha, \mu} \vartheta^\alpha \sigma^{\mu \beta}_{\alpha} B'_{(\mu \beta)} + \bar{\partial}_\beta B'_{(\beta \beta)} + \vartheta_1 \vartheta_2 \bar{\partial}_\beta B_{(12 \beta)} \right) \\
+ \sum_{\alpha} \vartheta^\alpha \bar{\partial}_\beta \left( \vartheta_{(\alpha)} f''_\alpha + \sum_{\beta, \mu} \vartheta_\beta \sigma^{\mu \beta}_{\alpha} B''_{(\mu \beta)} + \vartheta_\alpha \bar{\partial}_\beta B_{(\alpha \beta)} \right) \\
+ \vartheta^1 \vartheta^2 \bar{\partial}_\beta \left( B''_0 + \vartheta_1 \vartheta_2 B''_{(12)} + \sum_{\alpha, \beta, \mu} \vartheta_\alpha \bar{\partial}_\beta \sigma^{\mu \beta \alpha}_{\alpha} B''_{(\mu \beta \alpha)} + \bar{\vartheta}_1 \bar{\vartheta}_2 B''_{(12 \beta \alpha)} + \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_1 \bar{\vartheta}_2 B''_{(12 \beta \alpha \beta)} \right)
\]

\(\in C^\infty(\hat{X}^\mathbb{B})_{tame}\)

be tame scalar superfields on \(X\). Here, \(A_\bullet = A_\bullet(x)\) and \(B_\bullet = B_\bullet(x) \in C^\infty(X)^\mathbb{C}\). Then, their product is given by
\[ AB = BA \]
\[ = A_{(0)}B_{(0)} + \sum_\alpha \theta^\alpha \partial_\alpha (A_{(\alpha)}B_{(0)} + A_{(0)}B_{(\alpha)}) + \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta (A_{(\beta)}B_{(0)} + A_{(0)}B_{(\beta)}) \]
\[ + \theta^1 \bar{\theta}^2 \left\{ (A'_{(0)}B_{(0)} + A_{(0)}B'_{(0)}) + \partial_1 \bar{\theta}_2 \left( A_{(12)}B_{(0)} - A_{(1)}B_{(2)} - A_{(2)}B_{(1)} + A_{(0)}B_{(12)} \right) \right\} \]
\[ + \sum_{\alpha, \beta} \left\{ \sum_{\mu} \sigma^\mu_{\alpha \beta} (A_{[\mu]}B_{(0)} + A_{(0)}B_{[\mu]}) + \partial_\alpha \bar{\theta}_\beta \left( A_{(\alpha \beta)}B_{(0)} - A_{(\alpha \beta)}B_{(0)} - A_{(\beta \alpha)}B_{(\alpha \beta)} + A_{(0)}B_{(\alpha \beta)} \right) \right\} \]
\[ + \bar{\theta}^1 \bar{\theta}^2 \left\{ (A''_{(0)}B_{(0)} + A_{(0)}B''_{(0)}) + \bar{\theta}_1 \partial_2 \left( A_{(12)}B_{(0)} - A_{(1)}B_{(2)} - A_{(2)}B_{(1)} + A_{(0)}B_{(12)} \right) \right\} \]
\[ + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left\{ \partial_\alpha \left( A''_{(\alpha)}B_{(0)} + A''_{(0)}B_{(\alpha)} + A_{(\alpha)}B''_{(0)} + A_{(0)}B''_{(\alpha)} \right) \right\} \]
\[ + \sum_{\beta, \mu} \bar{\theta}^1 \bar{\theta}^2 (A''_{(\beta)}B_{(0)} + A_{(0)}B''_{(\beta)} + A_{(\beta)}B''_{(0)} + A_{(0)}B''_{(\beta)}) \]
\[ + \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left\{ \partial_\alpha \left( A''_{(\alpha \beta)}B_{(0)} + A''_{(\alpha \beta)}B_{(0)} + A_{(\alpha \beta)}B''_{(0)} + A_{(0)}B''_{(\alpha \beta)} \right) \right\} \]
\[ + \theta^1 \bar{\theta}^1 \bar{\theta}^2 \left\{ A''_{(0)}B_{(0)} + A''_{(0)}B''_{(0)} + 2 \sum_{\mu, \nu} \eta^\mu_{\nu} A_{[\mu]}B_{[\nu]} + A''_{(0)}B''_{(0)} + A_{(0)}B''_{(0)} \right\} \]
\[ + \bar{\theta}_1 \partial_2 \left( A''_{(12)}B_{(0)} - A''_{(1)}B_{(2)} - A''_{(2)}B_{(1)} + A''_{(0)}B_{(12)} \right) \]
\[ + \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left( A''_{(\alpha \beta)}B_{(0)} + A''_{(\alpha \beta)}B_{(0)} - A''_{(\beta \alpha)}B_{(\alpha \beta)} + A_{(\alpha \beta)}B''_{(0)} + A_{(0 \beta)}B''_{(0)} + A_{(0)}B''_{(0)} \right) \]
\[ + \bar{\theta}_1 \partial_2 \left( A''_{(12)}B_{(0)} - A''_{(12)}B_{(0)} - A''_{(212)}B_{(12)} + A''_{(0)}B''_{(12)} \right) \]
\[ + \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left( A''_{(\alpha \beta)}B_{(0)} - A''_{(\alpha \beta)}B_{(0)} - A''_{(\beta \alpha)}B_{(\alpha \beta)} + A_{(\alpha \beta)}B''_{(0)} + A_{(0 \beta)}B''_{(0)} + A_{(0)}B''_{(0)} \right) \]
\[ + \theta_1 \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4 \left( A_{(12)}B_{(212)} - A_{(12)}B_{(212)} - A_{(12)}B_{(212)} - A_{(212)}B_{(12)} + A_{(0)}B_{(12)} \right) \]
\[ + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left( A_{(\alpha \beta)}B_{(0)} - A_{(\alpha \beta)}B_{(0)} - A_{(\beta \alpha)}B_{(\alpha \beta)} + A_{(\alpha \beta)}B''_{(0)} + A_{(0 \beta)}B''_{(0)} + A_{(0)}B''_{(0)} \right) \]
\[ \in C^\infty (\tilde{X})^{\text{tame}}. \]

**Definition 1.2.11. [medium (Lorentz-)scalar superfield]** A tame scalar superfield \( \tilde{f} \in C^\infty (\tilde{X}) \) is called a medium scalar superfield on \( X \) if in addition

- \( f''_{(0)} = 0 \)

as a polynomial in \( C^\infty (X)^{C[\theta, \bar{\theta}, \varphi, \partial \varphi]}^{\text{anti-c}} \).
Explicitly, a medium scalar superfield \( \tilde{f} \) on \( X \) is of the following form

\[
\tilde{f} = f_0 + \sum_{\alpha} \theta^\alpha \partial_\alpha f_0 + \sum_{\beta} \bar{\theta}^\beta \bar{\partial}_\beta f_0 + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta f_{0(\alpha \beta)} + \theta^1 \theta^2 \bar{\partial}_1 \bar{\partial}_2 f_{12(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \left( \sum_{\mu} \bar{\sigma}_{\alpha \beta}^\mu A_{\mu[\alpha]} + \bar{\partial}_\alpha \bar{\partial}_\beta A_{\alpha \beta(12)} \right) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 A_{12(12)}
\]

\[
\sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \left( \sum_{\mu} \bar{\sigma}_{\alpha \beta}^\mu A_{\mu[\alpha]} + \bar{\partial}_\alpha \bar{\partial}_\beta A_{\alpha \beta(12)} \right) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 A_{12(12)}
\]

\[
\in C^\infty(X)^C[\theta, \bar{\theta}, \vartheta, \bar{\vartheta}]_{\text{anti-c}}.
\]

**Lemma/Definition 1.2.12. [medium sector \( \hat{X}^{\hat{\bullet}, \text{medium}} \) of \( \hat{X}^{\hat{\bullet}} \)]** The collection of medium scalar superfields on \( X \) as defined in Definition 1.2.11 is an even subring of \( C^\infty(\hat{X}^{\hat{\bullet}}) \). Denote this subring (also a \( C^\infty(X)^C{-}\)subalgebra of \( C^\infty(\hat{X}^{\hat{\bullet}}) \)) by \( C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}} \). Then, the \( C^\infty \)-hull of \( C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}} \) restricts to the \( C^\infty \)-hull of \( C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}} \), which is given by

\[
C^\infty \text{-hull}(C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}}) = \{ \tilde{f} \in C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}} \mid f_0 \in C^\infty(X) \}.
\]

Denote by \( \hat{X}^{\hat{\bullet}, \text{medium}} \) the complexified \( Z/2 \)-graded \( C^\infty \)-scheme with the underlying topology \( X \) and function ring \( C^\infty(\hat{X}^{\hat{\bullet}})^{\text{medium}} \). Then there is a built-in dominant morphism \( \hat{X}^{\hat{\bullet}, \text{tame}} \rightarrow \hat{X}^{\hat{\bullet}, \text{medium}} \). We will call \( \hat{X}^{\hat{\bullet}, \text{medium}} \) the medium sector of \( \hat{X}^{\hat{\bullet}} \).

**Proof.** This follows from the fact that the conditions on medium scalar superfields (1) Lorentz scalar, (2) \( (\theta, \bar{\theta}) \)-degree \( \leq (\vartheta, \bar{\vartheta}) \)-degree for every summand, and (3) \( f_0' = f_0'' = 0 \) as a polynomial in \( C^\infty(X)^C[\theta, \bar{\theta}, \vartheta, \bar{\vartheta}]_{\text{anti-c}} \) are closed under the multiplication in \( C^\infty(\hat{X}^{\hat{\bullet}}) \).

\( \square \)

Explicitly, let

\[
\tilde{A} = A_0 + \sum_{\alpha} \theta^\alpha \partial_\alpha A_0 + \sum_{\beta} \bar{\theta}^\beta \bar{\partial}_\beta A_0 + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta A_{0(\alpha \beta)} + \sum_{\mu} \bar{\sigma}_{\alpha \beta}^\mu A_{\mu(\alpha \beta)} + \bar{\partial}_\alpha \bar{\partial}_\beta A_{\alpha \beta(12)}.
\]
be medium scalar superfields on $X$. Then, their product is given by

$$
\bar{A} \tilde{B} = \bar{B} \tilde{A} = A_{(0)} B_{(0)} + \sum_{\alpha} \theta^\alpha \partial_\alpha \left( A_{(\alpha)} B_{(0)} + A_{(0)} B_{(\alpha)} \right) + \sum_{\bar{\beta}} \bar{\theta}^\beta \bar{\partial}_{\bar{\beta}} \left( A_{(1)\bar{\beta}} B_{(0)} + A_{(0)} B_{(1)\bar{\beta}} \right) \\
+ \theta^1 \theta^2 \partial_1 \partial_2 \left( A_{(12)} B_{(0)} - A_{(1)} B_{(2)} - A_{(2)} B_{(1)} + A_{(0)} B_{(12)} \right) \\
+ \sum_{\alpha, \bar{\beta}} \left\{ \sum_{\mu} \sigma^{\mu \alpha}_{\beta \bar{\beta}} \left( A_{[\mu]} B_{(0)} + A_{(0)} B_{[\mu]} \right) + \partial_\alpha \bar{\partial}_{\bar{\beta}} \left( A_{(\alpha)\bar{\beta}} B_{(0)} - A_{(\alpha)} B_{(\bar{\beta})} - A_{(\bar{\beta})} B_{(\alpha)} + A_{(0)} B_{(\alpha)\bar{\beta}} \right) \right\} \\
+ \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 \left( A_{(12)} B_{(0)} - A_{(1)} B_{(2)} - A_{(2)} B_{(1)} + A_{(0)} B_{(12)} \right) \\
+ \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \left\{ \sum_{\alpha, \mu} \sigma^{\mu \alpha}_{\beta \bar{\beta}} \left( A'_{[\mu]} B_{(0)} + A_{[\mu]} B_{(\alpha)} + A_{(\alpha)} B_{[\mu]} + A_{(0)} B'_{[\mu]} \right) + \bar{\theta}^\beta \left( A'_{(\beta)} B_{(0)} + A_{(0)} B'_{(\beta)} \right) \\
+ \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 \left( A'_{(12)} B_{(0)} + A_{(1)\bar{\beta}} B_{(\beta)} + A_{(1)\bar{\beta}} B_{(12)} + A_{(0)} B_{(12)\bar{\beta}} \right) \right\} \\
+ \sum_{\bar{\alpha}} \theta^\alpha \bar{\theta}^\beta \{ \partial_\alpha \left( A''_{(\alpha)} B_{(0)} + A_{(0)} B''_{(\alpha)} \right) + \sum_{\bar{\beta}, \mu} \sigma^{\mu \alpha}_{\beta \bar{\beta}} \left( A''_{[\mu]} B_{(0)} + A_{[\mu]} B_{(\bar{\beta})} + A_{(\bar{\beta})} B_{[\mu]} + A_{(0)} B''_{[\mu]} \right) \\
+ \bar{\partial}_1 \bar{\theta}_1 \bar{\partial}_2 \left( A_{(1)\bar{\alpha}} B_{(0)} + A_{(\alpha)\bar{1}} B_{(\bar{1})} + A_{(\alpha)\bar{1}} B_{(12)} + A_{(0)} B_{(\alpha)\bar{1}\bar{2}} \right) \right\} \\
+ \theta^1 \theta^2 \bar{\theta}^\beta \{ \left( A''_{(0)} B_{(0)} + 2 \sum_{\mu, \nu} \eta^{\mu \nu} A_{[\mu]} B_{[\nu]} + A_{(0)} B''_{(0)} \right) \\
+ \bar{\theta}^1 \bar{\theta}_2 \left( A''_{(12)} B_{(0)} - A''_{(1)} B_{(2)} - A''_{(2)} B_{(1)} + A_{(1)} B''_{(1)} + A_{(2)} B''_{(1)} \right) \\
+ \sum_{\alpha, \bar{\beta}} \partial_\alpha \bar{\partial}_{\bar{\beta}} \left( A''_{[\mu]} B_{(0)} + A_{[\mu]} B_{(\beta)} - A''_{[\mu]} B_{(\alpha)} + A_{(\beta)} B_{[\mu]} + A_{(\alpha)} B'_{[\mu]} \right) \\
+ \bar{\theta}^1 \bar{\theta}_2 \left( A''_{(12)} B_{(0)} - A''_{(1)} B_{(2)} - A''_{(2)} B_{(1)} + A_{(1)} B''_{(1)} + A_{(2)} B''_{(1)} \right) \\
+ \bar{\theta}^1 \bar{\theta}_2 \left( A''_{(12)} B_{(0)} - A''_{(1)} B_{(2)} - A''_{(2)} B_{(1)} + A_{(1)} B''_{(1)} + A_{(2)} B''_{(1)} \right) \\
+ \bar{\theta}^1 \bar{\theta}_2 \left( A''_{(12)} B_{(0)} - A''_{(1)} B_{(2)} - A''_{(2)} B_{(1)} + A_{(1)} B''_{(1)} + A_{(2)} B''_{(1)} \right) \right\} \\
\in C^\infty(\hat{X}^\hat{M})_{\text{medium}}.
Lemma/Definition 1.2.14. [small sector $\hat{\mathbb{X}}^{\text{anti-c}}$ and small function-ring of $\hat{\mathbb{X}}^{\text{anti-c}}$] The collection of all scalar superfields on $X$ as defined in Definition 1.2.13 is an even subring of the complexified $\mathbb{Z}/2$-graded $C^\infty$-ring $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$. Denote this subring (also a $C^\infty(X)^C$-subalgebra of $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$) by $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}}$. Then, the $C^\infty$-hull of $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$ restricts to the $C^\infty$-hull of $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}}$, which is given by
\[C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}} = \left\{ \hat{f} \in C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}} \mid \hat{f}(0) \in C^\infty(X) \right\}.
\]
Denote by $\hat{\mathbb{X}}^{\text{anti-c}}^{\text{small}}$ the complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme with the underlying topology $X$ and function ring $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}}$. Then there is a built-in dominant morphism $\hat{\mathbb{X}}^{\text{anti-c}}^{\text{medium}} \to \hat{\mathbb{X}}^{\text{anti-c}}^{\text{small}}$. We will call $\hat{\mathbb{X}}^{\text{anti-c}}^{\text{small}}$ the small sector of $\hat{\mathbb{X}}^{\text{anti-c}}$ and $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})^{\text{small}}$ the small function-ring of the towered superspace $\hat{\mathbb{X}}^{\text{anti-c}}$.

Proof. This follows from the fact that the conditions on small scalar superfields (1) Lorentz scalar, (2) $(\theta, \bar{\theta})$-degree $\leq (\theta, \bar{\theta})$-degree for every summand, (3) $f''(0) = f''(\beta) = f''(\alpha) = f''(\tilde{\alpha}) = f''(\tilde{\beta}) = f''(\tilde{\tilde{\alpha}}) = f''(\tilde{\tilde{\beta}}) = 0$ as a polynomial in $C^\infty(X)^C[\theta, \bar{\theta}, \theta, \bar{\theta}]^{\text{anti-c}}$ are closed under the multiplication in $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$. Cf. [L-Y5: Lemma 1.4.2] (SUSY(1)).

The term ‘short’ is better reserved for the future when we address BPS short multiplet in the situation with extended (i.e. $N \geq 2$) supersymmetries. Conceptually one may better call a small scalar superfield on $X$ a ‘curtained’ or ‘trimmed’ or ‘diminished’ scalar superfield on $X$ since this is exactly what it is. However, when naming the subring they form in $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$, it seems more appealing to call it the small function-ring of the towered superspace $\hat{\mathbb{X}}^{\text{anti-c}}$ than any other name. It is for this reason, we choose the term ‘small scalar superfield’ here. Such superfields are called superfields in the physical sector of $\hat{\mathbb{X}}^{\text{anti-c}}$ or synonymously physical superfields in [L-Y5: Definition 1.4.1] (SUSY(1)) as they form the subring in $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$ generated by chiral superfields and antichiral superfields; cf. [L-Y5: Sec. 1.4] (SUSY(1)). However, during the topic course Supersymmetric quantum field theories given by Jesse Thaler at MIT, fall 2019, we gradually realized that the best candidate in the setting of [L-Y5] (SUSY(1)) for the notion of ‘vector superfield’ in [Wess & Bagger: Chapter VI] is not in this subring; cf. Sec. 4.1 of the current notes versus [L-Y5: Sec. 3] (SUSY(1)). With this upgraded understanding and since the name ‘physical’ will almost for sure mislead one to think that elements in $C^\infty(\hat{\mathbb{X}}^{\text{anti-c}})$ that do not lie in this subring are “not physical” or “not relevant to physics”, we now fix ourselves to call this subring the “small function-ring” or the “small sector” of the towered superspace $\hat{\mathbb{X}}^{\text{anti-c}}$; cf. Lemma/Definition 1.2.14.
Explicitly, let
\[
\begin{align*}
\dot{A} &= A(0) + \sum_\alpha \theta^\alpha \partial_\alpha A(\alpha) + \sum_\beta \theta^\beta \partial_\beta A(\beta)
\end{align*}
\]
\[
\begin{align*}
&+ \theta^1 \theta^2 \partial_1 \partial_2 A_{(12)} + \sum \left( \sum_{\alpha, \beta} \sigma_{\alpha \beta} A_{[\mu]} + \partial_\alpha \partial_\beta A_{(\alpha \beta)} \right) + \theta^1 \theta^2 \partial_1 \partial_2 A_{(12)} \\
&+ \sum_{\beta} \theta^1 \theta^2 \partial_\beta \left( \sum_{\alpha, \mu} \partial_\alpha \sigma_{\alpha \beta} A_{[\mu]} + \partial_1 \partial_2 A_{(\beta)} \right) + \sum_\alpha \theta^\alpha \partial^1 \partial^2 \left( \sum_{\beta, \mu} \partial_{\beta} \sigma_{\alpha \beta} A_{[\mu]} + \partial_\alpha \partial_1 \partial_2 A_{(\alpha \beta)} \right) \\
&+ \theta^1 \theta^2 \partial^1 \partial^2 \left( \dot{A}_{(0)} + \sum \partial_\alpha \partial_\beta \sigma_{\alpha \beta} A_{[\mu]} + \partial_1 \partial_2 \partial_1 \partial_2 A_{(12)} \right)
\end{align*}
\]

and
\[
\begin{align*}
\dot{B} &= B(0) + \sum \theta^\gamma \partial_\gamma B(\gamma) + \sum_\delta \theta^\delta \partial_\delta B(\delta)
\end{align*}
\]
\[
\begin{align*}
&+ \theta^1 \theta^2 \partial_1 \partial_2 B_{(12)} + \sum \left( \sum_{\gamma, \delta} \sigma_{\gamma \delta} B_{[\nu]} + \partial_\gamma \partial_\delta B_{(\gamma \delta)} \right) + \theta^1 \theta^2 \partial_1 \partial_2 B_{(12)} \\
&+ \sum \theta^1 \theta^2 \partial_\delta \left( \sum_{\gamma, \nu} \partial_\gamma \sigma_{\gamma \delta} B_{[\nu]} + \partial_1 \partial_2 \partial_1 \partial_2 B_{(12)} \right) + \sum_\gamma \theta^\gamma \partial^1 \partial^2 \left( \sum_{\delta, \nu} \partial_{\delta} \sigma_{\gamma \delta} B_{[\nu]} + \partial_\gamma \partial_1 \partial_2 B_{(12)} \right) \\
&+ \theta^1 \theta^2 \partial^1 \partial^2 \left( \dot{B}_{(0)} + \sum \partial_\gamma \partial_\delta \sigma_{\gamma \delta} B_{[\nu]} + \partial_1 \partial_2 \partial_1 \partial_2 B_{(12)} \right)
\end{align*}
\]

be elements in $C^\infty(\hat{X}^\mathbb{N})^{small}$. Here, $A^\bullet = A^\bullet(x)$ and $B^\bullet = A^\bullet(x) \in C^\infty(X)^C$. Then,
\[ A\bar{B} = \bar{B}A \]
\[ \begin{align*}
&= A_{(0)}B_{(0)} + \sum_{\alpha} \theta^\alpha \partial_\alpha \left( A_{(\alpha)}B_{(0)} + A_{(0)}B_{(\alpha)} \right) + \sum_{\beta} \bar{\theta}^\beta \partial_\beta \left( A_{(\beta)}B_{(0)} + A_{(0)}B_{(\beta)} \right) \\
&\quad + \theta^1 \theta^2 \bar{\partial}_1 \bar{\partial}_2 \left( A_{(12)}B_{(0)} - A_{(1)}B_{(2)} - A_{(2)}B_{(1)} + A_{(0)}B_{(12)} \right) \\
&\quad + \sum_{\alpha,\beta} \left\{ \sum_{\mu} \sigma^\mu_{\alpha\beta} \left( A_{[\mu]}B_{(0)} + A_{(0)}B_{[\mu]} \right) + \partial_\alpha \bar{\theta}^\beta \left( A_{(\alpha)}B_{(0)} - A_{(\alpha)}B_{\beta} - A_{(\beta)}B_{(\alpha)} + A_{(0)}B_{(0\beta)} \right) \right\} \\
&\quad + \theta^1 \theta^2 \bar{\partial}_1 \bar{\partial}_2 \left( A_{(12)}B_{(0)} - A_{(1)}B_{(2)} - A_{(2)}B_{(1)} + A_{(0)}B_{(12)} \right) \\
&\quad + \sum_{\alpha} \theta^\alpha \bar{\partial}^2 \left\{ \sum_{\beta,\mu} \bar{\theta}^\beta \sigma^\mu_{\beta\alpha} \left( A''_{[\mu]}B_{(0)} + A_{[\mu]}B_{(\alpha)} + A_{(\alpha)}B_{[\mu]} + A_{(0)}B''_{[\mu]} \right) \\
&\quad + \partial_\alpha \bar{\partial}_1 \bar{\partial}_2 \left( A_{(1\alpha)}B_{(0)} + A_{(\alpha)}B_{(1)} + A_{(1\alpha)}B_{(2\beta)} + A_{(2\beta)}B_{(1)} \\
&\quad + A_{(1)}B_{(2\beta)} + A_{(2\beta)}B_{(1\beta)} + A_{(\beta)}B_{(12\beta)} + A_{(0)}B_{(12\beta)} \right) \right\} \\
&\quad + \theta^1 \theta^2 \bar{\partial}^2 \left\{ A''_{(0)}B_{(0)} + 2 \sum_{\mu,\nu} \eta^{\mu\nu} A_{[\mu]}B_{[\nu]} + A_{(0)}B''_{(0)} \right\} \\
&\quad + \sum_{\alpha,\beta} \partial_\alpha \bar{\partial}_1 \bar{\partial}_2 \left( A_{(1\alpha)}B_{(0)} + A_{(\alpha)}B_{(1)} + A_{(1\alpha)}B_{(2\beta)} + A_{(2\beta)}B_{(1)} \\
&\quad + A_{(1)}B_{(2\beta)} + A_{(2\beta)}B_{(1\beta)} + A_{(\beta)}B_{(12\beta)} + A_{(0)}B_{(12\beta)} \right) \right\} \\
&\in C^{\infty}(\hat{X}^{\hat{m}})_{\text{small}}.
\end{align*} \]

For a brief comparison:

| class of superfield \( \in C^{\infty}(\hat{X}^{\hat{m}}) \) | general | tame scalar | medium scalar | small scalar |
|-------------------------------------------------|-------|------------|---------------|-------------|
| number of components in \( C^{\infty}(X)^C \) | \( 2^8 = 256 \) | 41 | 39 | 33 |

1.3 The chiral and the antichiral condition on \( C^{\infty}(\hat{X}^{\hat{m}}) \) and its subrings

**Definition 1.3.1. [chiral and antichiral function on \( \hat{X}^{\hat{m}} \) and its sectors]** Recall the supersymmetrically invariant derivations on \( \hat{X} \)

\[
e_{\alpha'} := \frac{\partial}{\partial \theta^\alpha} + \sqrt{-1} \sum_{\beta,\mu} \sigma^\mu_{\alpha\beta} \bar{\theta}^\beta \partial_\mu, \quad e_{\beta'} := -\frac{\partial}{\partial \theta^\beta} - \sqrt{-1} \sum_{\alpha,\mu} \theta^\alpha \sigma^\mu_{\alpha\beta} \bar{\theta}_\mu.
\]

An \( \hat{f} \in C^{\infty}(\hat{X}^{\hat{m}}) \) is called **chiral** (resp. **antichiral**) if

\[
e_{\alpha'} \hat{f} = e_{\alpha'} \hat{f} = 0 \quad \text{(resp. } e_{\beta'} \hat{f} = e_{\beta'} \hat{f} = 0 \text{)}.
\]

\( \hat{f} \) is called a **tame chiral superfield** (resp. medium chiral superfield, small chiral superfield) on \( X \) if in addition \( \hat{f} \in C^{\infty}(\hat{X}^{\hat{m}})^{\text{tame}} \) (resp. \( C^{\infty}(\hat{X}^{\hat{m}})^{\text{medium}}, C^{\infty}(\hat{X}^{\hat{m}})^{\text{small}} \)).
Example 1.3.2. [basic chiral and antichiral function on $\hat{X}$]  
(1) Let

$$x^\mu := x^\mu + \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \hat{\theta}^\beta \quad \text{and} \quad x'^\mu := x^\mu - \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \hat{\theta}^\beta \in C^\infty(\hat{X}),$$

for $\mu = 0, 1, 2, 3$. (In collective short-hand, $x' = x + \sqrt{-1} \theta \sigma \hat{\theta}$ and $x'' = x - \sqrt{-1} \theta \sigma \hat{\theta}$.) Then $x'^\mu$'s are chiral and $x''^\mu$ are antichiral.

(2) $\theta^\alpha, \alpha = 1, 2$, are chiral and $\hat{\theta}^\beta, \beta = 1, 2$, are antichiral.

Note that since $x^\mu$, $x'^\mu$, and $x''^\mu$ differ only by an even nilpotent element in $C^\infty(\hat{X})$, any of the collection $(x, \theta, \hat{\theta})$, or $(x', \theta, \hat{\theta})$, $(x'', \theta, \hat{\theta})$ generate $C^\infty(\hat{X})$ as a complexified $\mathbb{Z}/2$-graded $C^\infty$-ring and, hence, can serve as coordinate functions on $\hat{X}$.

Definition 1.3.3. [standard chiral coordinate functions and standard antichiral coordinate functions on $\hat{X}$] - with abuse]  
For convenience but with slight abuse of the terminology, we shall call $(x', \theta, \hat{\theta})$ the standard chiral coordinate functions on $\hat{X}$ (despite that $\hat{\theta}^\beta$ are not chiral) and $(x'', \theta, \hat{\theta})$ the standard antichiral coordinate functions on $\hat{X}$ (despite that $\theta_\alpha$ are not antichiral).

The following lemma gives a characterization of chiral functions and antichiral functions on $\hat{X}$:

Lemma 1.3.4. [chiral function and antichiral function on $\hat{X}$]  
(1) $\tilde{f} \in C^\infty(\hat{X})$ is chiral if and only if, as an element of $C^\infty(X)^C[\theta, \tilde{\theta}, \bar{\theta}]^{anti-c}$, $\tilde{f}$ is of the following form

$$\tilde{f} = \tilde{f}(0)(x, \tilde{\theta}, \bar{\theta}) + \sum_\gamma \theta^\gamma \tilde{f}(\gamma)(x, \tilde{\theta}, \bar{\theta}) + \theta^1 \theta^2 \tilde{f}(12)(x, \tilde{\theta}, \bar{\theta}) + \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma \delta} \tilde{f}(0)(x, \tilde{\theta}, \bar{\theta})$$

$$+ \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1 \bar{\theta}^\delta \sigma^\gamma_{\delta \nu} \tilde{f}(\gamma)(x, \tilde{\theta}, \bar{\theta}) - \theta^1 \theta^2 \tilde{f}(\cdot)(x, \tilde{\theta}, \bar{\theta}).$$

In particular, a chiral $\tilde{f} \in C^\infty(\hat{X})$ has four independent components in $C^\infty(X)^C[\theta, \tilde{\theta}, \bar{\theta}]^{anti-c}$:

$$\tilde{f}(0), \tilde{f}(\gamma), \gamma = 1, 2, \tilde{f}(12).$$

In terms of the standard chiral coordinate functions $(x', \theta, \tilde{\theta}, \bar{\theta})$ on $\hat{X}$,

$$\tilde{f} = \tilde{f}(0)(x', \tilde{\theta}, \bar{\theta}) + \sum_\gamma \theta^\gamma \tilde{f}(\gamma)(x', \tilde{\theta}, \bar{\theta}) + \theta^1 \theta^2 \tilde{f}(12)(x', \tilde{\theta}, \bar{\theta}),$$

which is independent of $\tilde{\theta}$.

(2) $\bar{f} \in C^\infty(\hat{X})$ is antichiral if and only if, as an element of $C^\infty(X)^C[\theta, \tilde{\theta}, \bar{\theta}]^{anti-c}$, $\bar{f}$ is of the following form

$$\bar{f} = \bar{f}(0)(x, \tilde{\theta}, \bar{\theta}) + \sum_\delta \bar{\theta}^\delta \bar{f}(\delta)(x, \tilde{\theta}, \bar{\theta}) + \theta^1 \theta^2 \tilde{f}(12)(x, \tilde{\theta}, \bar{\theta}) - \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma \delta} \bar{f}(0)(x, \tilde{\theta}, \bar{\theta})$$

$$+ \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1 \bar{\theta}^\delta \sigma^\gamma_{\delta \nu} \bar{f}(\delta)(x, \tilde{\theta}, \bar{\theta}) - \theta^1 \theta^2 \bar{f}(\cdot)(x, \tilde{\theta}, \bar{\theta}).$$

In particular, an antichiral $\bar{f} \in C^\infty(\hat{X})$ has four independent components in $C^\infty(X)^C[\theta, \tilde{\theta}, \bar{\theta}]^{anti-c}$:

$$\bar{f}(0), \bar{f}(\delta), \delta = 1, 2, \bar{f}(12).$$

In terms of the standard antichiral coordinate functions $(x'', \theta, \tilde{\theta}, \bar{\theta})$ on $\hat{X}$,

$$\bar{f} = \bar{f}(0)(x'', \tilde{\theta}, \bar{\theta}) + \sum_\delta \bar{\theta}^\delta \bar{f}(\delta)(x'', \tilde{\theta}, \bar{\theta}) + \theta^1 \theta^2 \tilde{f}(12)(x'', \tilde{\theta}, \bar{\theta}),$$

which is independent of $\tilde{\theta}$.  

25
Proof. Given

\[
\bar{f} = \bar{f}(0) + \sum_{\alpha} \theta^\alpha f(\gamma) + \sum_{\delta} \bar{\theta}^\delta \bar{f}(\delta) + \theta^1 \theta^2 \bar{f}(12) + \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \bar{f}(\gamma \delta) + \bar{\theta}^1 \bar{\theta}^2 \bar{f}(12)
\]

+ \sum_{\delta} \theta^1 \theta^2 \bar{\theta}^\delta \bar{f}(12\delta) + \sum_{\gamma} \theta^\gamma \bar{\theta}^1 \bar{\theta}^2 \bar{f}(\gamma12) + \theta^1 \theta^2 \bar{\theta} \bar{\theta} \bar{f}(1212) \in C^{\infty}(\bar{\mathcal{X}}),

where \(\bar{f}(\bullet) \in C^{\infty}(X)^{\mathbb{C}[\bar{\theta}, \bar{\theta}]}\) anti-c, one has

\[-e_{\beta^\nu} \bar{f} = \bar{f}(\beta) + \sum_{\gamma} \theta^\gamma \left( -\bar{f}(\gamma) + \sqrt{-1} \sum_{\nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(0) \right) - \sum_{\delta} \bar{\theta}^\delta \bar{e}_{\beta^\delta} \bar{f}(12) + \theta^1 \theta^2 \left( \bar{f}(12\beta) - \sqrt{-1} \sum_{\gamma, \nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(\gamma) \right) + \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \left( \bar{e}_{\beta^\delta} \bar{f}(\gamma12) + \sqrt{-1} \sum_{\nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(\delta) \right) - \sum_{\gamma, \nu} \theta^1 \theta^2 \theta^2 \sum_{\gamma, \nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(\gamma12) - \sqrt{-1} \theta^1 \theta^2 \bar{\theta} \bar{\theta} \theta^2 \sum_{\gamma, \nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(\gamma12).
\]

Thus, \(e_1^\nu \bar{f} = e_2^\nu \bar{f} = 0\) if and only if

\[
\bar{f}(\beta) = \bar{f}(12) = \bar{f}(\gamma12) = 0,
\]

\[
\bar{f}(\gamma\delta) = \sqrt{-1} \sum_{\nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(0), \quad \bar{f}(12\beta) = \sqrt{-1} \sum_{\gamma, \nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(\gamma),
\]

\[
\bar{f}(1212) = -\frac{\sqrt{-1}}{2} \sum_{\gamma, \delta, \mu} \sigma^{\mu\gamma}_{\beta} \partial_{\mu} \bar{f}(\gamma\delta),
\]

for \(\gamma = 1, 2, \beta = 1, 2\). The last equation simplifies to \(\bar{f}(1212) = -\Box \bar{f}(0)\) after plugging in the equation \(\bar{f}(\gamma\delta) = \sqrt{-1} \sum_{\nu} \sigma^{\nu\gamma}_{\beta} \partial_{\nu} \bar{f}(0)\) in the system. This proves the first part of Statement (1). The second part of Statement (1) follows from the observation that

\[
(e_{\alpha^\mu} = \frac{\partial}{\partial \theta^\mu} + 2\sqrt{-1} \sum_{\beta, \mu} \sigma^{\mu\beta}_{\alpha \beta} \partial_{\mu} \quad \text{and} \quad e_{\beta^\nu} = -\frac{\partial}{\partial \theta^\beta})
\]

in the coordinate system \((x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) for \(\bar{\mathcal{X}}\).

Similar argument proves Statement (2). In which case,

\[
\tilde{e}_{\alpha^\mu} = \frac{\partial}{\partial \theta^\mu} \quad \text{(and} \quad \tilde{e}_{\beta^\nu} := -\frac{\partial}{\partial \theta^\beta} - 2\sqrt{-1} \sum_{\alpha, \mu} \bar{\theta}^\alpha \sigma^{\mu\alpha}_{\beta \alpha} \partial_{\mu}\)),
\]

in the coordinate system \((x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) for \(\bar{\mathcal{X}}\).

\[\square\]

Corollary 1.3.5. [tame chiral superfield, tame antichiral superfield] (1) A tame superfield \(\tilde{f}\) on \(X\) is chiral if and only if, as an element of \(C^{\infty}(X)^{\mathbb{C}[\theta, \bar{\theta}, \bar{\theta}, \bar{\theta}]}\) anti-c, \(\tilde{f}\) is of the following form

\[
\tilde{f} = f(0)(x) + \sum_{\gamma} \theta^\gamma \theta_\gamma f(\gamma)(x) + \theta^1 \theta^2 (f'(0)(x) + \theta_1 \theta_2 f(12)(x))
\]

+ \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^{\nu\gamma}_{\beta} \partial_\nu \bar{f}(0)(x) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^\gamma \bar{\theta}^\delta \theta_\gamma \sigma^{\nu\gamma}_{\beta} \partial_\nu f(\gamma)(x) - \theta^1 \theta^2 \bar{\theta} \bar{\theta} \Box f(0)(x).
\]

In particular, a chiral \(\tilde{f} \in C^{\infty}(\bar{\mathcal{X}})^{\text{tame}}\) has five independent components in \(C^{\infty}(X)^{\mathbb{C}}:\)

\[
f(0), \ f(\gamma), \ \gamma = 1, 2, \ f'(0), \ f(12).
\]

\[\square\]
In terms of the standard antichiral coordinate functions \( (x', \bar{\theta}, \bar{\theta}, \bar{\bar{\theta}}) \) on \( \hat{X}^\text{\textregistered} \),

\[
\hat{f} = f_{(0)}(x') + \sum_{\gamma} \theta^\gamma \bar{\theta}_\gamma f_{(\gamma)}(x') + \theta^1 \bar{\theta}^2 (f''_{(0)}(x') + \bar{\bar{\theta}}_1 \bar{\bar{\theta}}_2 f_{(12)}(x')) ,
\]

which is independent of \( \bar{\theta} \) and \( \bar{\bar{\theta}} \).

(2) A tame superfield \( \hat{f} \) on \( X \) is antichiral if and only if, as an element of \( C^\infty(X)^\text{\textregistered}[\theta, \bar{\theta}, \bar{\bar{\theta}}]^\text{anti-c} \), \( \hat{f} \) is of the following form

\[
\hat{f} = f_{(0)}(x) + \sum_{\delta} \bar{\theta}^\delta \bar{\bar{\theta}}_\delta f_{(\delta)}(x) + \bar{\bar{\theta}}^1 \bar{\theta}^2 (f''_{(0)}(x) + \bar{\bar{\theta}}_1 \bar{\bar{\theta}}_2 f_{(12)}(x))
- \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(0)}(x) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^\gamma \bar{\theta}^\delta \bar{\bar{\theta}}_\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(\delta)}(x) - \theta^1 \bar{\theta}^2 \bar{\bar{\theta}}^1 \bar{\bar{\theta}}^2 \Box f_{(0)}(x) .
\]

In particular, an antichiral \( \hat{f} \in C^\infty(\hat{X}^\text{\textregistered})^\text{tame} \) has five independent components in \( C^\infty(X)^\text{\textregistered} \):

\[
f_{(0)} , \ f_{(\delta)} , \ \delta = 1,2 , \ f'_{(0)} , \ f_{(12)} .
\]

In terms of the standard antichiral coordinate functions \( (x'', \theta, \bar{\theta}, \bar{\bar{\theta}}) \) on \( \hat{X}^\text{\textregistered} \),

\[
\hat{f} = f_{(0)}(x'') + \sum_{\delta} \bar{\theta}^\delta \bar{\bar{\theta}}_\delta f_{(\delta)}(x'') + \bar{\bar{\theta}}^1 \bar{\theta}^2 (f''_{(0)}(x'') + \bar{\bar{\theta}}_1 \bar{\bar{\theta}}_2 f_{(12)}(x'')) ,
\]

which is independent of \( \theta \) and \( \bar{\theta} \).

**Corollary 1.3.6.** [medium chiral = small chiral, medium antichiral = small antichiral] (1) A medium chiral superfield coincides with a small chiral superfield. A medium or small superfield \( \hat{f} \) on \( X \) is chiral if and only if, as an element of \( C^\infty(X)^\text{\textregistered}[\theta, \bar{\theta}, \bar{\bar{\theta}}]^\text{anti-c} \), \( \hat{f} \) is of the following form

\[
\hat{f} = f_{(0)}(x) + \sum_{\gamma} \theta^\gamma \partial_{\gamma} f_{(\gamma)}(x) + \theta^1 \bar{\theta}^2 \partial_1 \bar{\theta}_2 f_{(12)}(x)
+ \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(0)}(x) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^\gamma \bar{\theta}^\delta \bar{\bar{\theta}}_\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(\gamma)}(x) - \theta^1 \bar{\theta}^2 \bar{\bar{\theta}}^1 \bar{\bar{\theta}}^2 \Box f_{(0)}(x) .
\]

In particular, a chiral \( \hat{f} \in C^\infty(\hat{X}^\text{\textregistered})^\text{medium} \) or \( C^\infty(\hat{X}^\text{\textregistered})^\text{small} \) has four independent components in \( C^\infty(X)^\text{\textregistered} \):

\[
f_{(0)} , \ f_{(\gamma)} , \ \gamma = 1,2 , \ f_{(12)} .
\]

In terms of the standard chiral coordinate functions \( (x', \theta, \bar{\theta}, \bar{\bar{\theta}}) \) on \( \hat{X}^\text{\textregistered} \),

\[
\hat{f} = f_{(0)}(x') + \sum_{\gamma} \theta^\gamma \partial_{\gamma} f_{(\gamma)}(x') + \theta^1 \bar{\theta}^2 \partial_1 \bar{\theta}_2 f_{(12)}(x') ,
\]

which is independent of \( \bar{\theta} \) and \( \bar{\bar{\theta}} \).

(2) A medium antichiral superfield coincides with a small antichiral superfield. A medium or small superfield \( \hat{f} \) on \( X \) is antichiral if and only if, as an element of \( C^\infty(X)^\text{\textregistered}[\theta, \bar{\theta}, \bar{\bar{\theta}}]^\text{anti-c} \), \( \hat{f} \) is of the following form

\[
\hat{f} = f_{(0)}(x) + \sum_{\delta} \bar{\theta}^\delta \bar{\bar{\theta}}_\delta f_{(\delta)}(x) + \bar{\bar{\theta}}^1 \bar{\theta}^2 \partial_1 \bar{\bar{\theta}}_2 f_{(12)}(x)
- \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(0)}(x) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^\gamma \bar{\theta}^\delta \bar{\bar{\theta}}_\delta \sigma^\nu_{\gamma\delta} \partial_{\nu} f_{(\delta)}(x) - \theta^1 \bar{\theta}^2 \bar{\bar{\theta}}^1 \bar{\bar{\theta}}^2 \Box f_{(0)}(x) .
\]

In particular, an antichiral \( \hat{f} \in C^\infty(\hat{X}^\text{\textregistered})^\text{medium} \) or \( C^\infty(\hat{X}^\text{\textregistered})^\text{small} \) has four independent components in \( C^\infty(X)^\text{\textregistered} \):

\[
f_{(0)} , \ f_{(\delta)} , \ \delta = 1,2 , \ f_{(12)} .
\]
In terms of the standard antichiral coordinate functions \((x'', \bar{\theta}, \bar{\vartheta}, \bar{\bar{\vartheta}})\) on \(\widehat{\mathbb{X}}^\oplus\),

\[
\bar{f} = f_{(0)}(x'') + \sum_{\delta} \bar{\theta}^\delta \bar{\bar{\theta}}^\delta f_{(\delta)}(x'') + \bar{\theta}^1 \bar{\bar{\theta}}^{\bar{\bar{1}}} \bar{\bar{\bar{\theta}}}_{(12)} f_{(12)}(x'') ,
\]

which is independent of \(\theta\) and \(\vartheta\).

**Corollary 1.3.7.** [medium or small chiral = chiral multiplet] A medium or small chiral superfield matches with the chiral multiplet from representations of \(d = 3 + 1, N = 1\) supersymmetry algebra. Medium or small chiral superfields on \(X\) form a complexified \(C^\infty\)-ring. Similar statements hold for medium or small antichiral superfields.

**Corollary 1.3.8.** [small ring generated by small chiral and small antichiral] The small function ring \(C^\infty(\widehat{\mathbb{X}})_{\text{small}}^\oplus\) of \(\widehat{\mathbb{X}}^\oplus\) is the subring of \(C^\infty(\widehat{\mathbb{X}})_{\text{generated by the small chiral superfields and the small antichiral superfields.}}\)

**Remark 1.3.9.** [mathematical patch to [Wess & Bagger]: nilpotency of ‘F-component’] Caution that chiral functions on \(\widehat{\mathbb{X}}^\oplus\) of the following form

\[
\bar{f} = f_{(0)}(x') + \sum_{\gamma} \theta^\gamma \vartheta_{\gamma} f_{(\gamma)}(x') + \theta^1 \theta^2 f_{(0)}'(x')
\]

in coordinate functions \((x', \theta, \bar{\theta}, \vartheta, \bar{\bar{\vartheta}})\) do not form a ring as they are not closed under multiplication: The product \(\bar{A} B\) of general \(\bar{A}\) and \(B\) of the above form will have an additional nilpotent summand

\[
- \theta^1 \theta^2 \vartheta_1 \vartheta_2 (A_{(1)} B_{(2)} + A_{(2)} B_{(1)})
\]

do that not fit in. It follows that if one demands that chiral superfields that match with the chiral multiplets (which have only four independent components) to form a ring, then their ‘F-component’ (cf. [Wess & Bagger: Eq. (5.3)]) must be nilpotent Lorentz-scalar of the form \(\vartheta_1 \vartheta_2 f_{(12)}\), where \(f_{(12)} \in C^\infty(X)\). This is the first mathematical patch that one is forced to make if one wants to bring [Wess & Bagger] in complexified \(\mathbb{Z}/2\)-graded Algebraic Geometry; ([L-Y5: Sec. 1.2] (SUSY(1))))

Similarly for antichiral functions on \(\widehat{\mathbb{X}}^\oplus\) of the following form

\[
\bar{f} = f_{(0)}(x'') + \sum_{\delta} \bar{\delta} \bar{\delta} f_{(\delta)}(x'') + \bar{\theta}^1 \bar{\bar{\theta}}^{\bar{\bar{1}}} f_{(0)}''(x'')
\]

in coordinate functions \((x'', \theta, \bar{\theta}, \vartheta, \bar{\bar{\vartheta}})\).

**Chirality on \(C^\infty(\widehat{\mathbb{X}})\)** and its sectors under the twisted complex conjugation

**Lemma 1.3.10.** \([Q_\alpha, \bar{Q}_\beta, e_\alpha, e_\beta\] under twisted complex conjugation] Under the twisted complex conjugation \(^{\dagger}\) on \(C^\infty(\widehat{\mathbb{X}}) = C^\infty(\widehat{\mathbb{X}})_{\text{even}} + C^\infty(\widehat{\mathbb{X}})_{\text{odd}}\),

\[
Q^\dagger_\alpha = \bar{Q}_\bar{\bar{\alpha}}, \quad \bar{Q}^\dagger_{\bar{\bar{\alpha}}} = Q_{\alpha}, \quad e^\dagger_{\alpha'} = e_{\alpha'}, \quad e^\dagger_{\beta'} = e_{\beta'}
\]

\(^{11}\)In [L-Y5] (SUSY(1)), we take this as a starting point to understand how physicists think of the function ring of a superspace and its “more physically relevant sector”.

\(^{12}\)Note for physicists It turns that, as demonstrated in the current notes, except the additional introduction of purge-evaluation maps (which physicists already took implicitly whenever in need), this is the only mathematical patch to make precise (of the non-quantum part of) [Wess & Bagger].
on $C^\infty(\hat{X}^\oplus)$ in the sense that

$$(Q_\alpha \tilde{f})^\dagger = Q_\alpha \tilde{f}^\dagger, \quad (\bar{Q}_\beta \tilde{f})^\dagger = Q_\beta \tilde{f}^\dagger, \quad (e_{\alpha'} \tilde{f})^\dagger = e_{\alpha'} \tilde{f}^\dagger, \quad (e_{\beta''} \tilde{f})^\dagger = e_{\beta''} \tilde{f}^\dagger$$

for $\tilde{f} \in C^\infty(\hat{X}^\oplus)$. Similarly,

$$Q_\alpha^\dagger = -Q_\alpha, \quad \bar{Q}_\beta^\dagger = -Q_\beta, \quad e_{\alpha'}^\dagger = -e_{\alpha'}, \quad e_{\beta''}^\dagger = -e_{\beta''}$$

on $C^\infty(\hat{X}^\oplus)$ odd in the sense that

$$(Q_\alpha \tilde{f})^\dagger = -Q_\alpha \tilde{f}^\dagger, \quad (\bar{Q}_\beta \tilde{f})^\dagger = -Q_\beta \tilde{f}^\dagger, \quad (e_{\alpha'} \tilde{f})^\dagger = -e_{\alpha'} \tilde{f}^\dagger, \quad (e_{\beta''} \tilde{f})^\dagger = -e_{\beta''} \tilde{f}^\dagger$$

for $\tilde{f} \in C^\infty(\hat{X}^\oplus)$ odd.

Proof. This follows from the following basic identities

$$\left(\frac{\partial}{\partial \bar{\theta}^\alpha}(\theta^\nu \tilde{A})\right)^\dagger = (-1)^{p(\tilde{A})} \frac{\partial}{\partial \bar{\theta}^\alpha}(\bar{\theta}^\nu \tilde{A})^\dagger, \quad \left(\frac{\partial}{\partial \theta^\beta}(\theta^\nu \tilde{A})\right)^\dagger = (-1)^{p(\tilde{A})} \frac{\partial}{\partial \theta^\beta}(\bar{\theta}^\nu \tilde{A})^\dagger,$$

$$\left(\partial_\mu \tilde{B}\right)^\dagger = \partial_\mu \tilde{B}^\dagger, \quad \sigma_{\alpha\beta}^\mu = \sigma_{\beta\alpha}^\mu,$$

$$\left(\theta^\nu \tilde{B}\right)^\dagger = (-1)^{p(\tilde{B})} \bar{\theta}^\nu \tilde{B}^\dagger, \quad \left(\bar{\theta}^\nu \tilde{B}\right)^\dagger = (-1)^{p(\tilde{B})} \theta^\nu \tilde{B}^\dagger,$$

for $\tilde{A}, \tilde{B}$ parity-homogeneous elements of $C^\infty(\hat{X}^\oplus)$. \hfill \Box

Since the twisted complex conjugation leaves each of the tame, the medium, and the small sectors of $\hat{X}^\oplus$ invariant, one has the following consequence:

**Corollary 1.3.11.** [swapping of ‘chiral’ and ‘antichiral’ under twisted complex conjugation] (Cf. [L-Y5: Lemma 2.1.7] (SUSY(1)).) The twisted complex conjugation $^\dagger$ on $C^\infty(\hat{X}^\oplus)$ takes chiral elements to antichiral elements, and vice versa. The same holds for the restriction of $^\dagger$ on the tame subring $C^\infty(\hat{X}^\oplus)^{\text{tame}}$, the medium subring $C^\infty(\hat{X}^\oplus)^{\text{medium}}$, and the small subring $C^\infty(\hat{X}^\oplus)^{\text{small}}$ of $C^\infty(\hat{X}^\oplus)$.

2. Purge-evaluation maps and the Fundamental Theorem on supersymmetric action functionals via a superspace formulation

We highlight [L-Y5: Sec. 1.5] (SUSY(1)) on the purge-evaluation maps and the Fundamental Theorem on supersymmetric action functionals via a superspace formulation, with two additional remarks.

**Purge-evaluation maps**

The action functional of a supersymmetric quantum field theory model via a superspace formulation is itself a (Lorentz-)scalar superfield $\in C^\infty(\hat{X}^\oplus) = C^\infty(X)^C[\tilde{\theta}, \tilde{\bar{\theta}}, \tilde{\vartheta}, \tilde{\bar{\vartheta}}]^{\text{anti-c}}$. After taking an appropriate fermionic integration:

$$\int d\bar{\theta}^\alpha d\bar{\vartheta}^\dagger d\theta^\beta d\vartheta^\dagger, \quad \text{or} \quad \int d\bar{\theta}^\alpha d\vartheta^\dagger, \quad \text{or} \quad \int d\bar{\vartheta}^\dagger d\vartheta^\dagger,$$

it becomes an element in $C^\infty(X)^C[\theta, \bar{\theta}, \tilde{\vartheta}, \tilde{\bar{\vartheta}}]^{\text{anti-c}}$. In general, a summand of which may still contain an even nilpotent factor of the form $\hat{v}_1 \hat{\vartheta}_2, \hat{\vartheta}_1 \hat{\varphi}_2$, $\alpha=1,2, \beta=1,2, \varphi=1,2$, or $\hat{y}_1 \hat{\varphi}_2 \hat{\vartheta}_1 \hat{\varphi}_2$. Such a nilpotent factor has to be “removed” properly to obtain a final supersymmetric action functional of fields on $X$. This removal is realized by a purge-evaluation:
Definition 2.1. [purge-evaluation map $P^{ev}$] A purge-evaluation map
\[ P^{ev} : C^\infty(X)^{\mathbb{C}}[\theta, \bar{\theta}]^{\text{anti-c}} \longrightarrow C^\infty(X)^{\mathbb{C}} \]
is a $C^\infty(X)^{\mathbb{C}}$-module homomorphism that sends monomials $\theta^d \bar{\theta}_1 \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4$ to a constant $c_{d_1d_2d_3d_4} \in \mathbb{C}$. This induces a map, also denoted by $P^{ev}$ and called purge-evaluation map
\[ P^{ev} : C^\infty(\hat{X}) \longrightarrow C^\infty(\hat{X}) \]
(Cf. [L-Y5: Definition 1.5.1] (SUSY(1)) for a slightly different formulation.) Any $P^{ev}$ thus defined has the following property: (Cf. [L-Y5: Lemma 1.5.2] (SUSY(1)).)

Lemma 2.2. [property of $P^{ev}$] (1) Let $\xi \in Der_{\mathbb{C}}(\hat{X})$ be a derivation on $\hat{X}$. Then for $\check{f} \in C^\infty(\hat{X})$,
\[ P(\check{f}) = \xi P(\check{f}). \]
(2) For $\check{f} \in C^\infty(\hat{X})$,
\[ \int d\theta^1 \bar{\theta}^1 d\theta^2 \bar{\theta}^2 P^{ev}(\check{f}) = P^{ev}(\int d\theta^2 \bar{\theta}^1 \check{f}). \]
(3) For $\check{f} \in C^\infty(\hat{X})$ chiral (resp. antichiral),
\[ \int d\theta^2 \bar{\theta}^1 P^{ev}(\check{f}) = P^{ev}(\int d\theta^2 \bar{\theta}^1 \check{f}) \quad (\text{resp.} \quad \int d\theta^2 \bar{\theta}^1 P^{ev}(\check{f}) = P^{ev}(\int d\theta^1 \bar{\theta}^2 \check{f})). \]

Proof. Statement (1) follows from the fact that $\xi \in Der_{\mathbb{C}}(\hat{X})$ has no $(\theta, \bar{\theta})$-dependence while $P^{ev}$ applies to $\check{f} (\theta, \bar{\theta})$-degree-by-(\theta, \bar{\theta})-degree with $P^{ev}(\theta^d \bar{\theta}_1 \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4)$ constant, and hence $\xi(P^{ev}(\check{f}(\bullet))) = P^{ev}(\check{f}(\bullet))$, where $\check{f}(\bullet) \in C^\infty(X)^{\mathbb{C}}[\theta, \bar{\theta}]^{\text{anti-c}}$ is a component of $\check{f}$ as a polynomial in $(\theta, \bar{\theta})$. Statement (2) and Statement (3) follow from the definition of fermionic integration on $\hat{X}$. \hfill \Box

The Fundamental Theorem on supersymmetric action functionals

The following Fundamental Theorem is truly amazing and yet so elegantly simple! It is the tool physicists use for the construction of supersymmetric quantum field theory models via superspace. Readers are referred to, e.g. [Bi: Sec. 4.3] and [B-T-T: Sec. 2.9] for the original account by physicists.

Theorem 2.3. [fundamental: supersymmetric functional] Up to a boundary term\footnote{Though ignored in the current work, it should be noted that such a boundary term becomes an important part of understanding when one studies supersymmetric quantum field theory with boundary.} on $X$,
\[ (1) \quad S_1(\check{f}) := \int_{\hat{X}} d^4x d\theta^2 d\bar{\theta}_1 d\theta^2 d\bar{\theta}_1 \check{f} \]
is a functional on $C^\infty(\hat{X})$ that is invariant under supersymmetries;
\[ (2) \quad S_2(\check{f}) := \int_{\hat{X}} d^4x d\theta^2 d\bar{\theta}_1 \check{f} \quad \text{(resp.} \quad S_3(\check{f}) := \int_{\hat{X}} d^4x d\theta^2 d\bar{\theta}_1 \check{f} \quad \text{)} \]
is a functional on $C^\infty(\hat{X})^{\text{ch}}$ (resp. $C^\infty(\hat{X})^{\text{ach}}$) that is invariant under supersymmetries.

13Though ignored in the current work, it should be noted that such a boundary term becomes an important part of understanding when one studies supersymmetric quantum field theory with boundary.
Let $\mathcal{P}^{cv}$ be a purge-evaluation map. Then, up to a boundary term on $X$,

\[
S'_1(\tilde{f}) := \int_X d^4x \bar{\theta}^3 \bar{\theta}^1 d\theta^2 d\theta^1 \mathcal{P}^{cv}(\tilde{f})
\]
is a functional on $C^\infty(\tilde{X})$ that is invariant under supersymmetries;

\[
S'_2(\tilde{f}) := \int_X d^4x \bar{\theta}^2 \bar{\theta}^1 d\theta^2 d\theta^1 \mathcal{P}^{cv}(\tilde{f}) \quad \text{(resp. } S'_3(\tilde{f}) := \int_X d^4x \bar{\theta}^2 \bar{\theta}^1 \mathcal{P}^{cv}(\tilde{f}) \text{)}
\]
is a functional on $C^\infty(\tilde{X})$ isochronous (resp. $C^\infty(\tilde{X})$ isoachronous) that is invariant under supersymmetries.

**Proof.** We give the proof of Statement (1') and Statement (2'). (The proof is identical to that of [L-Y5: Theorem 1.5.3] (SUSY(1)) and is repeated here due to that the Theorem is so fundamental.) A similar, simpler argument proves Statement (1) and Statement (2).

For Statement (1'), since $Q_\alpha, Q_{\bar{\beta}} \in \text{Der}_C(\tilde{X})$, it follows from the invariance of $d^4x \bar{\theta}^3 \bar{\theta}^1 d\theta^2 d\theta^1$, $d^4x \bar{\theta}^2 \bar{\theta}^1 d\theta^1$, and $d^4x \bar{\theta}^2 \bar{\theta}^1$ under the flow that generates supersymmetries, Lemma 2.2, and basic calculus that

\[
\delta_{Q_\alpha} S'_1(\tilde{f}) := \int_X d^4x \bar{\theta}^3 \bar{\theta}^1 d\theta^2 d\theta^1 \mathcal{P}^{cv}(Q_\alpha \tilde{f}) = \int_X d^4x \bar{\theta}^3 \bar{\theta}^1 d\theta^2 d\theta^1 Q_\alpha \mathcal{P}^{cv}(\tilde{f})
\]

\[
= -\sqrt{-1} \int_X d^4x \bar{\theta}^3 \bar{\theta}^1 d\theta^2 d\theta^1 \sum_{\beta, \mu} \sigma^\mu_{\alpha \beta} \bar{\theta}^3 \partial_\mu (\mathcal{P}^{cv}(\tilde{f}))
\]

\[
= -\sqrt{-1} \int_X d^4x d\theta^2 d\theta^1 \sum_\alpha \sigma^\alpha_{\beta} \bar{\theta}^3 \mathcal{P}^{cv}(\tilde{f}) = -\sqrt{-1} \int_X dB_\alpha,
\]

where $B_\alpha = B_\alpha^0 dx^1 \wedge dx^2 \wedge dx^3 - B_\alpha^1 dx^0 \wedge dx^2 \wedge dx^3 + B_\alpha^2 dx^0 \wedge dx^1 \wedge dx^3 - B_\alpha^3 dx^0 \wedge dx^1 \wedge dx^2$ is a 3-form on $X$ with

\[
B_\alpha^0 = \int d^2\theta \bar{\theta}^1 d\theta^1 \sum_{\beta} \sigma^\alpha_{\beta} \bar{\theta}^3 \mathcal{P}^{cv}(\tilde{f}).
\]

The proof that $\delta_{Q_\alpha} S'_1(\tilde{f})$ is also a boundary term is similar.

For Statement (2'), note that for $\tilde{f}$ chiral, $\delta_{Q_\beta} S'_2(\tilde{f}) = 0$ always, for $\alpha = 1, 2$, and, thus one only needs to check the variation $\delta_{Q_{\bar{\beta}}} S'_2(\tilde{f})$:

\[
\delta_{Q_{\bar{\beta}}} S'_2(\tilde{f}) := \int_X d^4x \bar{\theta}^2 \bar{\theta}^1 d\theta^2 d\theta^1 \mathcal{P}(Q_{\bar{\beta}} \tilde{f}) = \int_X d^4x \bar{\theta}^2 \bar{\theta}^1 d\theta^2 d\theta^1 \mathcal{P}^{cv}(\tilde{f})(\alpha^\beta + 2\sqrt{-1} \sum_\alpha \theta^\alpha \sigma^\alpha_{\beta} \partial_\beta(\mathcal{P}^{cv}(\tilde{f})))
\]

\[
= 2\sqrt{-1} \int_X d^4x \bar{\theta}^2 \bar{\theta}^1 d\theta^2 d\theta^1 \sum_\alpha \theta^\alpha \sigma^\alpha_{\beta} \bar{\theta}^2 \mathcal{P}(\tilde{f})
\]

\[
= 2\sqrt{-1} \int_X d^4x d\theta^2 d\theta^1 \sum_\alpha \theta^\alpha \sigma^\alpha_{\beta} \mathcal{P}^{cv}(\tilde{f}) = 2\sqrt{-1} \int_X dC_{\beta},
\]

where $C_{\beta} = C_{\beta}^0 dx^1 \wedge dx^2 \wedge dx^3 - C_{\beta}^1 dx^0 \wedge dx^2 \wedge dx^3 + C_{\beta}^2 dx^0 \wedge dx^1 \wedge dx^3 - C_{\beta}^3 dx^0 \wedge dx^1 \wedge dx^2$ is a 3-form on $X$ with

\[
C_{\beta}^0 = \int d^2\theta \bar{\theta}^1 d\theta^1 \sum_\alpha \theta^\alpha \sigma^\alpha_{\beta} \mathcal{P}^{cv}(\tilde{f}).
\]

For $\tilde{f}$ antichiral, $\delta_{Q_{\bar{\beta}}} S'_3(\tilde{f}) = 0$ always, for $\beta = 1, 2$, and the variation $\delta_{Q_\alpha} S'_3(\tilde{f})$, $\alpha = 1, 2$, can be computed similarly to show that it is a boundary term on $X$.

This completes the proof.

\[\square\]

**Remark 2.4.** [application of Theorem 2.3] In an application of Theorem 2.3, the $\tilde{f}$ in Statements (1), (2), (1'), and (2') usually comes from a functional of elements in $C^\infty(\tilde{X})$. 

31
Having reviewed 'purge-evaluation maps' and the 'Fundamental Theorem', we give two remarks/impose two questions below on purge-evaluation maps that deserve to be understood better.

**Remark: A purge-evaluation map merged into an exotic ring?**

To begin, we impose a guiding question:

Q. Is it possible to construct a new function ring on a towered superspace $\mathcal{X}$ that accommodates both the anticommuting Grassmann coordinate functions $\theta, \bar{\theta}, \vartheta, \bar{\vartheta}$ on $\mathcal{X}$ and a purge-evaluation map $C^\infty(X)=[\theta, \bar{\theta}, \vartheta, \bar{\vartheta}] \rightarrow C^\infty(X)$ defined via $P^{cv}: C^\infty(X)[\theta, \bar{\theta}] \rightarrow C^\infty(X)$?

The design that $P^{cv}(\vartheta, \bar{\vartheta}, \vartheta, \bar{\vartheta})$ are constants in $\mathbb{C}$ motivates one to consider first a relevant $\mathbb{C}$-algebra generated by the variables $\vartheta, \bar{\vartheta}$. Together with

1. the fact the variables $\vartheta, \bar{\vartheta}$ are in spinor representations $S', S''$ of the Lorentz group and there are pairings $\varepsilon$ chosen on the spinor bundles,
2. the isomorphism of representations of the Lorentz group: $V^{\vee}_C \simeq S' \oplus S''$,
3. compatibility with the twisted complex conjugation, and
4. a Fierz-type identity: $\sum_{\alpha, \beta} \theta^\alpha \sigma_{\alpha \beta} \bar{\vartheta}^\beta = 2 \theta^{\dot{1}} \bar{\theta}^{\dot{1}} \bar{\vartheta}^{\dot{2}} \bar{\vartheta}^{\dot{2}}$, 

(and some sense of naturality), one is led almost uniquely to the following $\mathbb{C}$-algebra:

**Definition 2.5. [basic exotic $\mathbb{Z}/2$-graded $\mathbb{C}$-algebra $\mathcal{R}$]** Let $\mathcal{R}$ be the $\mathbb{C}$-algebra with the underline $\mathbb{C}$-vector space $(\mathbb{C} \oplus (S' \oplus S'') \oplus V^{\vee}_C, +)$ and multiplication $\cdot$ defined as follows.

(Recall that $\varepsilon_{12} = \varepsilon_{12} = -1$ and $\varepsilon_{21} = \varepsilon_{21} = 1$.)

(a) In terms of the basis $(1, \vartheta_1, \bar{\vartheta}_1, \vartheta_2, \bar{\vartheta}_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$:

\[
\begin{align*}
\vartheta_\alpha \cdot \vartheta_\beta &= -\varepsilon_{\alpha \beta}, & \bar{\vartheta}_\alpha \cdot \bar{\vartheta}_\beta &= -\bar{\varepsilon}_{\alpha \beta}, & \vartheta_\alpha \cdot \bar{\vartheta}_\beta &= \sigma_{\alpha \beta}, & \bar{\vartheta}_\alpha \cdot \vartheta_\beta &= \varepsilon_{\alpha \beta}, \\
\vartheta_\alpha \cdot \sigma_{\gamma \delta} &= \sigma_{\gamma \delta} \cdot \vartheta_\alpha = \varepsilon_{\gamma \alpha} \bar{\vartheta}_\delta, & \bar{\vartheta}_\alpha \cdot \sigma_{\gamma \delta} &= -\varepsilon_{\delta \beta} \vartheta_\gamma, & \sigma_{\alpha \beta} \cdot \sigma_{\gamma \delta} &= |\varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta}|.
\end{align*}
\]

Explicitly,

| $(\cdot)$ | $(\cdot)$ |
|---------|---------|
| 1       | 1       |
| $\vartheta_1$ | $\vartheta_1$ | 0 | 1 | $\sigma_{11}$ | $\sigma_{12}$ |
| $\bar{\vartheta}_1$ | $\bar{\vartheta}_1$ | $\bar{\vartheta}_1$ | $\sigma_{11}$ | $\sigma_{12}$ |
| $\sigma_{11}$ | $\sigma_{11}$ | 0 | $-\bar{\vartheta}_1$ | 0 | $-\sigma_{11}$ | $\bar{\vartheta}_1$ | 0 | 1 | 0
| $\sigma_{12}$ | $\sigma_{12}$ | 0 | $-\bar{\vartheta}_1$ | $\vartheta_1$ | 0 | 0 | 1 | 0
| $\sigma_{21}$ | $\sigma_{21}$ | $\bar{\vartheta}_1$ | 0 | 0 | $-\vartheta_2$ | 0 | 1 | 0 | 0
| $\sigma_{22}$ | $\sigma_{22}$ | $\bar{\vartheta}_2$ | 0 | $\vartheta_2$ | 0 | 1 | 0 | 0 | 0

32
(b) Or, equivalently, in terms of the basis \((1, \vartheta_1, \vartheta_2, \bar{\vartheta}_1, \bar{\vartheta}_2, \sigma^0, \sigma^1, \sigma^2, \sigma^3)\):

\[
\begin{align*}
\vartheta_\alpha \cdot \vartheta_\beta &= -\varepsilon_{\alpha \beta}, & \vartheta_\alpha \cdot \bar{\vartheta}_\beta &= -\bar{\vartheta}_\beta \cdot \vartheta_\alpha &= -\frac{1}{2} \sum_\mu \sigma^\mu \bar{\vartheta}_\mu \bar{\vartheta}_\beta, & \vartheta_\alpha \cdot \bar{\vartheta}_\beta &= \varepsilon_{\alpha \beta}; \\
\vartheta_\alpha \cdot \sigma^\mu &= \sigma^\mu \cdot \vartheta_\alpha = \sum_\gamma \delta \bar{\vartheta}_\delta \sigma_\gamma \varepsilon_\alpha, & \bar{\vartheta}_\beta \cdot \sigma^\mu &= \sigma^\mu \cdot \bar{\vartheta}_\beta = \sum_\gamma \delta \bar{\vartheta}_\gamma \sigma_\delta \varepsilon_\delta,
\end{align*}
\]

\[
\sigma^\mu \cdot \sigma^\nu = \sigma^\nu \cdot \sigma^\mu = \sum_{\alpha, \beta, \gamma, \delta} |\varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta}| \sigma_\alpha^{\mu} \sigma_\beta^{\nu}.
\]

Explicitly,

| (i) \(\cdot\) (\(\dagger\)) | 1 | \(\vartheta_1\) | \(\vartheta_2\) | \(\bar{\vartheta}_1\) | \(\bar{\vartheta}_2\) | \(\sigma^0\) | \(\sigma^1\) | \(\sigma^2\) | \(\sigma^3\) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | \(\vartheta_1\) | 0 | 1 | -\frac{1}{2} \sigma^0 + \frac{1}{2} \sigma^3 | \frac{1}{2} \sigma^1 + \frac{\sqrt{2}}{2} \sigma^2 | -\vartheta_2 | -\vartheta_2 | \sqrt{-1} \vartheta_1 | -\vartheta_2 |
| \(\vartheta_2\) | -1 | 0 | \frac{1}{2} \sigma^1 - \frac{\sqrt{2}}{2} \sigma^2 | -\frac{1}{2} \sigma^0 - \frac{1}{2} \sigma^3 | \vartheta_1 | -\vartheta_1 | \sqrt{-1} \vartheta_2 | -\vartheta_1 |
| \(\bar{\vartheta}_1\) | \(\bar{\vartheta}_1\) | \frac{1}{2} \sigma^0 - \frac{1}{2} \sigma^3 | -\frac{1}{2} \sigma^1 + \frac{\sqrt{2}}{2} \sigma^2 | 0 | 1 | -\vartheta_2 | \vartheta_1 | -\sqrt{-1} \vartheta_1 | -\vartheta_2 |
| \(\bar{\vartheta}_2\) | \(\bar{\vartheta}_2\) | -\frac{1}{2} \sigma^1 - \frac{\sqrt{2}}{2} \sigma^2 | \frac{1}{2} \sigma^0 + \frac{1}{2} \sigma^3 | 1 | 0 | \vartheta_1 | -\vartheta_2 | -\sqrt{-1} \vartheta_2 | -\vartheta_1 |
| \(\sigma^0\) | \(\sigma^0\) | -\vartheta_2 | \vartheta_1 | -\vartheta_2 | \vartheta_1 | 2 | 0 | 0 | 0 |
| \(\sigma^1\) | \(\sigma^1\) | -\vartheta_2 | \vartheta_1 | \vartheta_1 | -\vartheta_2 | 0 | 2 | 0 | 0 |
| \(\sigma^2\) | \(\sigma^2\) | \sqrt{-1} \vartheta_1 | \sqrt{-1} \vartheta_2 | -\sqrt{-1} \vartheta_1 | -\sqrt{-1} \vartheta_2 | 0 | 2 | 0 | 0 |
| \(\sigma^3\) | \(\sigma^3\) | -\vartheta_2 | -\vartheta_1 | \vartheta_1 | -\vartheta_2 | 0 | 0 | 0 | -2 |

This gives a \(\mathbb{Z}/2\)-graded, \(\mathbb{Z}/2\)-commutative, unital, non-associative algebra over \(\mathbb{C}\). We will call it the basic exotic \(\mathbb{Z}/2\)-graded \(\mathbb{C}\)-algebra and denote it by \(\hat{R}^\beta\).

One can check directly that

Lemma 2.6. [\(\hat{R}^\beta\) under twisted complex conjugation \(^1\)] For \(r_1, r_2 \in \hat{R}^\beta\), \((r_1 \cdot r_2)^\dagger = r_2^\dagger \cdot r_1^\dagger\).

Remark 2.7. [from Grassmann number to exotic number] Conceptually, one should think of the \(\mathbb{Z}/2\)-graded \(\mathbb{C}\)-algebra \(\hat{R}^\beta\) as diminished/flattened from the Grassmann algebra \(\bigwedge^\bullet \mathbb{C}(S' \oplus S'')\) via the purge-evaluation defined by the \(\mathbb{C}\)-vector-space-homomorphism associated to the assignment

\[
\widehat{P}^{ev} : \bigwedge^\bullet \mathbb{C}(S' \oplus S'') \longrightarrow \hat{R}^\beta
\]

\[
\begin{align*}
1 &\mapsto 1, \\
\vartheta_\alpha &\mapsto \vartheta_\alpha, \\
\bar{\vartheta}_\beta &\mapsto \bar{\vartheta}_\beta, \\
\sigma_{\alpha \beta} &\mapsto \sigma_{\alpha \beta}, \\
\vartheta_1 \vartheta_2 &\mapsto 1, \\
\bar{\vartheta}_1 \bar{\vartheta}_2 &\mapsto -1, \\
\vartheta_1 \vartheta_2 \bar{\vartheta}_\beta &\mapsto \bar{\vartheta}_\beta, \\
\vartheta_\alpha \vartheta_1 \bar{\vartheta}_2 &\mapsto -\vartheta_\alpha, \\
\vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 &\mapsto -1,
\end{align*}
\]

for \(\alpha = 1, 2\) and \(\beta = 1, 2\). \(\widehat{P}^{ev}\) is a \(\mathbb{C}\)-vector-space projection map from \(\bigwedge^\bullet \mathbb{C}(S' \oplus S'')\) onto its sub-\(\mathbb{C}\)-vector-space \(\hat{R}^\beta\). While \(\widehat{P}^{ev}\) is not a \(\mathbb{C}\)-algebra-homomorphism, it is an \(SO^{(1,3)}\)-module-homomorphism.
Definition 2.8. [exotic function-ring of towered superspace] Let \( \mathcal{X}^\oplus \) be a ringed-space with the underlying topology \( X \) and the function ring

\[
C^\infty(\mathcal{X}^\oplus) := C^\infty(X)^\mathbb{C}[\theta, \bar{\theta}; \vartheta, \bar{\vartheta}]^{exotic}
\]

be the extension-over-\( \mathbb{C} \) of \( C^\infty(X)^\mathbb{C}[\theta, \bar{\theta}]^{anti-c} \) by \( \bar{R}^\flat \) with \( \theta, \bar{\vartheta} \) anticommuting with \( \theta, \bar{\theta} \).

This is a \( \mathbb{Z}/2 \)-graded, \( \mathbb{Z}/2 \)-commutative nonassociative ring. We shall call \( \mathcal{X}^\oplus \) an exotic towered superspace and its function ring the exotic function-ring of a towered superspace.

The twisted complex conjugation \((\cdot)\dagger\) is naturally defined on \( C^\infty(\mathcal{X}^\oplus) \), with the property that \((\hat{f} \bullet \hat{g})\dagger = \hat{g}\dagger \bullet \hat{f}\dagger\), for \( \hat{f}, \hat{g} \in C^\infty(\mathcal{X}^\oplus) \), where \( \bullet \) is the multiplication on \( C^\infty(\mathcal{X}^\oplus) \).

One can formulate (1) the notion of chiral and antichiral superfields on \( X \), (2) the notion of the small exotic function-ring of \( \mathcal{X}^\oplus \), which is a complexified \( C^\infty \)-ring — in particular, commutative and associative — contained in \( C^\infty(\mathcal{X}^\oplus) \) as a subring, and (3) Theorem 2.3 all in terms of elements in \( C^\infty(\mathcal{X}^\oplus) \). In particular, for Item (3), there is no need to introduce the additional purge-evaluation map now since that data is already merged into the construction of the exotic function-ring of the towered superspace. This can be used to directly reproduce the Wess-Zumino model in [Wess & Bagger: Chapter V]. In this simple case, the nonassociativity of the \((\vartheta, \bar{\vartheta})\)-part is completely overridden by the nilpotency of the \((\theta, \bar{\theta})\)-part.

Unfortunately, when attempting to extend its application to the construction of supersymmetric gauge theories, one no longer has such a luck. The nonassociativity of \( C^\infty(\mathcal{X}^\oplus) \) brings in new technical issues that do not look to have any simple cure for the time being.

Remark: A canonical/standard purge-evaluation map with respect to \((\vartheta, \bar{\vartheta})\)?

So far in the consideration of purge-evaluation maps, we only take into account the requirement from physics that the density over \( X \) of an action functional has to be real-valued (or complex-valued plus its complex conjugation) without any nilpotent factors.

Q. How additional physical considerations restrict the choice of the purge-evaluation map?

For example, assume that \( \mathcal{P}^{ev}(\psi_1^\dagger, \psi_2^\dagger, \psi_3^\dagger, \psi_4^\dagger) \in \mathbb{R} \). Then while the absolute-value of the value \( \mathcal{P}^{ev} \) takes may be absorbed into field-redefinitions, the sign of the value may not be. In that case, what physics consideration select the sign of the values of the purge-evaluation map?

To get a sense of this, let us consider the Wess-Zumino model quoted from [Wess & Bagger: Eq. (5.13)] (in the case of one single chiral superfield):

\[
\mathcal{L} = \sqrt{-1} \sum_{\mu} \partial_{\mu} \bar{\psi} \partial^{\mu} \psi + A^\dagger \Box A - \frac{1}{2} m \psi \psi - \frac{1}{2} m^* \bar{\psi} \bar{\psi} - g \psi \psi A - m^* \bar{\psi} \bar{\psi} A^* - F^* F.
\]

Here, \((A, \psi, F)\) is a chiral multiplet and \( m, g \in \mathbb{C} \). To make the nilpotent feature of the independent components \( A, \psi, F \) of the chiral superfield manifest, set

\[
A \sim f_0(0) , \quad A^\dagger \sim \bar{f}_0(0) , \quad \psi_\alpha \sim \vartheta_\alpha f_0(0) , \quad \bar{\psi}_\alpha \sim \bar{\vartheta}_\alpha \bar{f}_0(0) ,
\]

\[
F \sim -\vartheta_1 \vartheta_2 f_{(12)} , \quad F^* \sim (-\vartheta_1 \vartheta_2 f_{(12)})\dagger = \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}_{(12)},
\]

and apply the rule of short-hand-to-long-hand notation change ([Wess & Bagger: Eq. (A.21)]) and the rule of raising-or-lowering spinorial indices ([Wess & Bagger: Eq. (A.9)]):

\[
\psi := \sum_\alpha \psi_\alpha \psi_\alpha = -2 \psi_1 \psi_2 = -2 \vartheta_1 \vartheta_2 f_{(1)} f_{(2)} ,
\]

\[
\bar{\psi} := \sum_\alpha \bar{\psi}_\alpha \bar{\psi}_\alpha = 2 \bar{\psi}_1 \bar{\psi}_2 = 2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}_{(1)} \bar{f}_{(2)} ,
\]

the density \( \mathcal{L} \) of the action functional of the Wess-Zumino model is then converted to

\[
\mathcal{L} = \sqrt{-1} \sum_{\alpha, \beta, \mu} \bar{\vartheta}_\beta \vartheta_\alpha \partial_{\mu} \bar{f}_{(3)}(\psi_\alpha \vartheta_\beta f_0(0) + \bar{f}_0(0) \Box f_0) + m \vartheta_1 \vartheta_2 f_{(1)} f_{(2)} - m \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}_{(1)} \bar{f}_{(2)}
\]

\[+ 2g \vartheta_1 \vartheta_2 f_{(1)} f_{(2)} f_0(0) - 2g \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}_{(1)} \bar{f}_{(2)} \bar{f}_0(0) + \vartheta_1 \vartheta_2 \vartheta_1 \vartheta_2 f_{(12)} f_{(12)} \).

34
Which can be converted further to a manifestly real expression

\[ \mathcal{L} = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} \bar{\vartheta}_\beta \vartheta_\alpha \left( \partial_\mu \bar{f}(\beta) \sigma^{\mu \beta \alpha} f(\alpha) - \bar{f}(\beta) \sigma^{\mu \beta \alpha} \partial_\mu f(\alpha) \right) - \sum_\mu \partial_\mu \bar{f}(0) \varphi^\mu f(0) \\
+ m \vartheta_1 \vartheta_2 f(1) f(2) - m \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) + 2g \vartheta_1 \vartheta_2 f(1) f(2) f(0) - 2\bar{g} \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) \bar{f}(0) \\
+ \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) \bar{f}(12) \]

up to boundary terms on \( X \). The Hamiltonian density \( \mathcal{H} \) over the spatial 3-space, identified as, say, \( \{0\} \times \mathbb{R}^3 \subset X \), on the configuration space of the system is given by

\[
\mathcal{H} = \frac{\delta \mathcal{L}}{\delta (\partial_0 f(0))} \cdot \partial_0 f(0) + \frac{\delta \mathcal{L}}{\delta (\partial_0 \bar{f}(0))} \cdot \partial_0 \bar{f}(0) + \sum_\alpha \frac{\delta \mathcal{L}}{\delta (\partial_0 f(\alpha))} \cdot \partial_0 f(\alpha) + \sum_\beta \frac{\delta \mathcal{L}}{\delta (\partial_0 \bar{f}(\beta))} \cdot \partial_0 \bar{f}(\beta) - \mathcal{L}
\]

\[
= -\frac{\sqrt{-1}}{2} \sum_{\mu=1}^3 \bar{\vartheta}_\beta \vartheta_\alpha \left( \partial_\mu \bar{f}(\beta) \sigma^{\mu \beta \alpha} f(\alpha) - \bar{f}(\beta) \sigma^{\mu \beta \alpha} \partial_\mu f(\alpha) \right) + \sum_{\mu=1}^3 \partial_\mu \bar{f}(0) \varphi^\mu f(0) \\
m \vartheta_1 \vartheta_2 f(1) f(2) - m \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) - 2g \vartheta_1 \vartheta_2 f(1) f(2) f(0) + 2\bar{g} \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) \bar{f}(0) \\
- \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) \bar{f}(12) \]

with the final expression restricted to the spatial slice \( \{0\} \times \mathbb{R}^3 \subset X \). The Hamiltonian density over the spatial 3-space is meant to be the energy density along a trajectory in the phase space following the equation of motion and hence better be real positive-or-bounded-below. Thus,

1. If setting \( \vartheta_1 \vartheta_2 \sim \pm 1 \) by convention, then it is required that \( \vartheta_1 \vartheta_2 \sim \mp 1 \) in order to be compatible with the twisted complex conjugation. This also helps keep \( \mathcal{H} \) real.

2. The summand \( -\vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{f}(1) \bar{f}(2) \) in \( \mathcal{H} \) is a potential energy term. Since \( \bar{f}(12) \bar{f}(12) \geq 0 \), it is required that \( -\vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \sim \text{positive real number} \). One may thus set \( \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \sim -1 \) to meet this requirement.

3. For the nilpotent factor \( \bar{\vartheta}_\beta \vartheta_\alpha \) in the kinetic terms of the spinor component fields \( f(\alpha), \alpha = 1, 2 \), exchanging the chirality/handedness of spinor fields \( \psi \leftrightarrow \tilde{\psi} \) changes the sign in front of the kinetic term. Thus, the sign of \( \mathcal{P}^{CD}(\vartheta_\beta \vartheta_\alpha) \) is only a matter of convention and we may set \( \vartheta_\beta \vartheta_\alpha = -\vartheta_\alpha \vartheta_\beta \sim \pm 1 \) either choice by convention.

Thus overall up to positive constant factors, which can be absorbed into a redefinition of fields or coupling constants, the following four choices are physically acceptable:

- \( \vartheta_1 \vartheta_2 \sim 1, \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \sim -1, \bar{\vartheta}_\beta \vartheta_\alpha \sim \pm 1 \);
- \( \vartheta_1 \vartheta_2 \sim -1, \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \sim 1, \vartheta_\beta \vartheta_\alpha \sim \pm 1 \).

3 The small function-ring of \( \hat{X} \) and the Wess-Zumino model
(cf. [Wess & Bagger: Chapter V])

We reconstruct in this section

- [Wess & Bagger: Chapter V. Chiral superfields]

in the complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry setting of Sec. 1. The same construction was made in [L-Y5: Sec. 2] (SUSY(1)) with slightly different terminology; cf. footnote 6 and footnote 10.

The small function-ring \( \mathcal{C}^\infty(\hat{X} \small) \) and its chiral and antichiral sectors

Recall from Sec. 1.2 the following complexified \( C^\infty \)-subrings of \( \mathcal{C}^\infty(\hat{X}) \). Their elements give the superfields involved in the superspace formulation of the Wess-Zumino model.
The small function-ring \( C^\infty(\hat{X}^{\oplus})_{\text{small}} \) of \( \hat{X}^{\oplus} \), which consists of elements of \( C^\infty(\hat{X}^{\oplus}) \) of the following form

\[
\hat{f}(x) = f(0) + \sum_\alpha \theta^\alpha \partial_\alpha f_0(\alpha) + \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta f_0(\beta) + \theta^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) + \sum_{\alpha, \beta} \theta^1 \bar{\theta}^2 \sum_{\mu} \sigma^\mu_{\alpha \beta} f_{(\mu)} + \partial_\alpha \bar{\theta} \bar{\partial}_\beta f_0(\alpha) + \bar{\theta}^1 \bar{\partial}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) + \sum_{\alpha, \beta, \mu} \theta^1 \bar{\theta}^2 \sum_{\nu} \sigma^\nu_{\alpha \beta} f_{(\nu)} + \partial_{\nu} \bar{\theta} \bar{\partial}_\beta f_0(\alpha) + \bar{\theta}^1 \bar{\partial}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x)
\]

\[\in C^\infty(X)_{[\theta, \bar{\theta}, \partial, \bar{\partial}]_{\text{anti-c}}} \]

The small chiral function-ring \( C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ch}} \) of \( \hat{X}^{\oplus} \), which consists of elements of \( C^\infty(\hat{X}^{\oplus})_{\text{small}} \) of the following form

\[
\hat{f}(x) = f(0) + \sum_\gamma \theta^\gamma \partial_\gamma f_0(\gamma) + \theta^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) + \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma \delta} \partial_\nu f_0(\gamma) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1 \bar{\theta}^2 \bar{\partial}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) - \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma \delta} \partial_\nu f_0(\gamma) + \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1 \bar{\theta}^2 \bar{\partial}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x)
\]

\[\in C^\infty(X)_{[\theta, \bar{\theta}, \partial, \bar{\partial}]_{\text{ch}}} \]

The small antichiral function-ring \( C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ach}} \) of \( \hat{X}^{\oplus} \), which consists of elements of \( C^\infty(\hat{X}^{\oplus})_{\text{small}} \) of the following form

\[
\hat{f}(x) = f(0) + \sum_\delta \bar{\theta}^\delta \bar{\partial}_\delta f_0(\delta) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) - \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_{\gamma \delta} \partial_\nu f_0(\gamma) - \sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1 \bar{\theta}^2 \bar{\partial}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x)
\]

Recall also that, as a ring, \( C^\infty(\hat{X}^{\oplus})_{\text{small}} \) is generated by \( C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ch}} \cup C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ach}} \) and that the twisted complex conjugation \( \hat{} \) takes \( C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ch}} \) and \( C^\infty(\hat{X}^{\oplus})_{\text{small}, \text{ach}} \) to each other.

We now proceed to construct the Wess-Zumino model on \( X \) in terms of \( C^\infty(\hat{X}^{\oplus})_{\text{small}} \).

Relevant basic computations/formulae

Let

\[
f(x) = f(0) + \sum_\alpha \theta^\alpha \partial_\alpha f_0(\alpha) + \theta^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) + \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} \partial_\mu f_0(\alpha)
\]

be a chiral function on \( \hat{X}^{\oplus}_{\text{small}} \), determined by \( f(0), f_{(12)} \). It follows from Corollary 1.3.11 that its twisted complex conjugate \( \hat{f} \) is the antichiral function on \( X_{\text{phys}} \) determined by \( f(0), f_{(12)} \) by

\[
\hat{f}(x) = f(0) - \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta f_{(\beta)}(x) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} \partial_\mu f_0(\alpha)
\]

be a chiral function on \( \hat{X}^{\oplus}_{\text{small}} \), determined by \( f(0), f_{(12)} \). It follows from Corollary 1.3.11 that its twisted complex conjugate \( \hat{f} \) is the antichiral function on \( X_{\text{phys}} \) determined by \( f(0), f_{(12)} \) by

\[
\hat{f}(x) = f(0) - \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta f_{(\beta)}(x) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 f_{(12)}(x) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} \partial_\mu f_0(\alpha)
\]
Consequently, (recall that $\Box := -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$)

\[
\tilde{f}^i \tilde{f}^j = \tilde{f}^i \tilde{f}^j = f_{(0)}(x) f_{(0)}(x) + \sum_{\alpha} \theta^\alpha \partial_0 f_{(0)}(x) f_{(\alpha)}(x) - \sum_{\beta} \bar{\theta}^\beta \partial_\beta \overline{f_{(\beta)}}(x) f_{(12)}(x) + \theta^1 \theta^2 \theta_1 \theta_2 f_{(0)}(x) f_{(12)}(x)
\]

\[
+ \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \left\{ \sqrt{-1} \sum_{\mu} \sigma^\mu_{\alpha \beta} \left( \overline{f_{(0)}}(x) \partial_\mu f_{(\alpha)}(x) - \partial_\mu \overline{f_{(0)}}(x) f_{(\alpha)}(x) \right) \right. + \partial_\alpha \bar{\theta}^\beta \overline{f_{(\beta)}}(x) f_{(\alpha)}(x) \left. \right\}
\]

\[
+ \bar{\theta}^1 \theta^2 \bar{\theta}^1 \overline{f_{(12)}}(x) f_{(0)}(x)
\]

\[
+ \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^1 \overline{f_{(12)}}(x) f_{(0)}(x)
\]

\[
+ \theta^1 \theta^2 \overline{f_{(12)}}(x) f_{(0)}(x)
\]

\[
+ \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} \left( - \partial_\mu \overline{f_{(\beta)}}(x) f_{(\alpha)}(x) + \overline{f_{(\beta)}}(x) \partial_\mu \overline{f_{(\beta)}}(x) \right)
\]

\[
+ \theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2 \overline{f_{(12)}}(x) f_{(12)}(x) \right) ;
\]

\[
\tilde{f}^2 = f_{(0)}(x)^2 + 2 \sum_{\alpha} \theta^\alpha \partial_0 f_{(0)}(x) f_{(\alpha)}(x) + 2 \theta^1 \theta^2 \theta_1 \theta_2 \left( f_{(0)}(x) f_{(12)}(x) - f_{(1)}(x) f_{(2)}(x) \right)
\]

\[
+ \text{(terms of } \theta\text{-degree } \geq 1) ;
\]

\[
\tilde{f}^3 = f_{(0)}(x)^3 + 3 \sum_{\alpha} \theta^\alpha \partial_0 f_{(0)}(x)^2 f_{(\alpha)}(x) + 3 \theta^1 \theta^2 \theta_1 \theta_2 \left( f_{(0)}(x)^2 f_{(12)}(x) - 2 f_{(0)}(x) f_{(1)}(x) f_{(2)}(x) \right)
\]

\[
+ \text{(terms of } \theta\text{-degree } \geq 1) ;
\]

\[
(\tilde{f}^1)^2 = \overline{f_{(0)}}(x)^2 - 2 \sum_{\beta} \bar{\theta}^\beta \partial_\beta \overline{f_{(\beta)}}(x) + 2 \bar{\theta}^1 \theta^2 \theta_1 \theta_2 \left( \overline{f_{(0)}}(x) \overline{f_{(12)}}(x) - \overline{f_{(1)}}(x) \overline{f_{(2)}}(x) \right)
\]

\[
+ \text{(terms of } \theta\text{-degree } \geq 1) ;
\]

\[
(\tilde{f}^1)^3 = \overline{f_{(0)}}(x)^3 - 3 \sum_{\beta} \bar{\theta}^\beta \partial_\beta \overline{f_{(\beta)}}(x)^2 + 3 \bar{\theta}^1 \theta^2 \theta_1 \theta_2 \left( \overline{f_{(0)}}(x)^2 \overline{f_{(12)}}(x) - 2 \overline{f_{(0)}}(x) \overline{f_{(1)}}(x) \overline{f_{(2)}}(x) \right)
\]

\[
+ \text{(terms of } \theta\text{-degree } \geq 1) .
\]

(Cf. [Wess & Bagger: Eqs. (5.9), (5.7), (5.8)].)

The action functional of the Wess-Zumino model

The action functional of the Wess-Zumino model is given by:

\[
(\text{caution that } (d\theta^2 d\theta^1)^1 = d\theta^1 d\theta^2 = -d\theta^2 d\theta^1)
\]
\[ S(\tilde{f}) := \int_X d^4x \left\{ -\frac{1}{4} \int d\bar{\theta}^2 d\theta^1 d\bar{\theta}^2 d\theta^1 \left( \tilde{f}^\dagger \tilde{f} \right) + \int d\bar{\theta}^2 d\theta^1 \left( \lambda \tilde{f} + \frac{1}{2} m \tilde{f}^2 + \frac{1}{3} \bar{g} \tilde{f}^3 \right) - \int d\bar{\theta}^2 d\theta^1 \left( \bar{\lambda} \tilde{f}^\dagger + \frac{1}{2} \bar{m} (\tilde{f}^\dagger)^2 + \frac{1}{3} \bar{g} (\tilde{f}^\dagger)^3 \right) \right\} \]

\[ = \int_X d^4x \left\{ \frac{1}{3} \nabla_0 f_0(x) \cdot f_0(x) + \frac{1}{4} \nabla_0 f_0(x) \cdot \nabla f_0(x) - \frac{1}{2} \sum \partial_\mu f_0(x) \partial^\mu f_0(x) + \frac{\sqrt{-1}}{4} \sum_{\alpha, \beta, \mu} \bar{\theta}_\alpha \bar{\theta}_\beta \cdot \bar{\sigma}^{\mu \beta \alpha} \left( \bar{f}_0(x) \partial_\mu f_0(x) - f_0(x) \partial_\mu \bar{f}_0(x) \right) + \sqrt{-1} \partial_1 \partial_2 \bar{\phi}_1 \cdot \bar{f}_0(x) f_0(x) \right\}. \]

It follows from Theorem 2.3 that up to a space-time boundary term, this functional is supersymmetric. After imposing the purge-evaluation map

\[ \mathcal{P}^{ev} : \partial_1 \partial_2 \rightarrow 1, \quad \partial_\alpha \bar{\partial}_\beta \rightarrow -1, \quad \bar{\partial}_1 \bar{\partial}_2 \rightarrow -1, \quad \partial_1 \partial_2 \bar{\partial}_1 \bar{\partial}_2 \rightarrow -1. \]

to remove the even nilpotent factors \( \partial_1 \partial_2, \partial_\alpha \bar{\partial}_\beta, \bar{\partial}_1 \bar{\partial}_2, \partial_1 \partial_2 \bar{\partial}_1 \bar{\partial}_2 \) in the expression and integration by parts to fix the kinetic terms, \( S(\tilde{f}) \) becomes

\[ S(\tilde{f}) := \int_X d^4x \mathcal{P}^{ev} \left\{ -\frac{1}{4} \int d\bar{\theta}^2 d\theta^1 d\bar{\theta}^2 d\theta^1 \left( \tilde{f}^\dagger \tilde{f} \right) + \int d\bar{\theta}^2 d\theta^1 \left( \lambda \tilde{f} + \frac{1}{2} m \tilde{f}^2 + \frac{1}{3} \bar{g} \tilde{f}^3 \right) - \int d\bar{\theta}^2 d\theta^1 \left( \bar{\lambda} \tilde{f}^\dagger + \frac{1}{2} \bar{m} (\tilde{f}^\dagger)^2 + \frac{1}{3} \bar{g} (\tilde{f}^\dagger)^3 \right) \right\} \]

\[ = \int_X d^4x \left\{ \bar{f}_0(x) \cdot f_0(x) + \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} \bar{\sigma}^{\mu \beta \alpha} f_0(x) \partial_\mu \bar{f}_0(x) + \frac{1}{4} f_0(x) f_0(x) \right\}. \]

This is [Wess & Bagger: Eq. (5.11)] in the setting of Sec. 1 and Sec. 2.

The component field \( f_{1(2)} \) (and hence \( \bar{f}_{1(2)} \)) has no kinetic term and thus is non-dynamical. It can be removed from the action functional by solving its equations of motion from \( S(\tilde{f}) \)

\[ \frac{1}{4} f_{1(2)}(x) + \bar{m} f_0(x) + \bar{g} \bar{f}_0(x)^2 + \bar{\lambda} = 0, \]

\[ \frac{1}{4} \bar{f}_{1(2)}(x) + m f_0(x) + g f_0(x)^2 + \lambda = 0 \]

and plugging back into \( S(\tilde{f}) \). This gives another form of the action functional that involves only dynamical component fields:
\[ S(f_0, (f_a)_\alpha) := \int_X d^4x \left\{ \sum_{\alpha, \beta, \mu} \partial^\mu \beta \alpha f_{(\alpha)}(x) \partial_\mu \overline{f}_{(\beta)}(x) \right. \\
\left. - \left( m_{(f_1)}(x) f_{(2)}(x) + 2g f_{(0)}(x) f_{(1)}(x) f_{(2)}(x) \\
+ \overline{m} f_{(1)}(x) \overline{f}_{(2)}(x) + 2 \overline{g} f_{(0)}(x) \overline{f}_{(1)}(x) \overline{f}_{(2)}(x) \right) \\
- 4 \left( m f_{(0)}(x) + g f_{(0)}(x)^2 + \lambda \right) \cdot \left( \overline{m} f_{(0)}(x) + \overline{g} f_{(0)}(x)^2 + \overline{\lambda} \right) \\
+ \text{(space-time boundary terms)} \right\}. \]

(Cf. [Wess & Bagger: Eq. (5.13)].)

4 Supersymmetric U(1) gauge theory with matter on X in terms of \( \overline{\mathbb{X}} \) (cf. [Wess & Bagger: Chapter VI and Chapter VII, U(1) part])

We reconstruct in this section

- [Wess & Bagger: Chapter VI. Vector superfields and
  Chapter VII. Gauge invariant interactions, U(1) part]

in the complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry setting of Sec. 1.

4.1 Vector superfields and their associated (even left) connection

Since all the bundles and sheaves involved in the construction of supersymmetric gauge theories in the current notes are trivialized, we will directly take a connection as a differential operator acting on \( C^\infty(\overline{\mathbb{X}}) \) to keep our focus on [Wess & Bagger]. Readers are referred to [L-Y5: Sec. 3.1] (SUSY(1)=D(14.1.Sup.1)) and references ibidem for more words on connections and gauge theories in the superworld.

Definition 4.1.1. [even left connection on \( C^\infty(\overline{\mathbb{X}}) \)-module] (Cf. [L-Y4:Definition 2.1.2] (D(14.1)).) Let \( \hat{M} \) be an \( C^\infty(\overline{\mathbb{X}}) \)-module. An even left connection \( \hat{\nabla} \) on \( \hat{M} \) is a \( \mathbb{C} \)-bilinear pairing

\[ \hat{\nabla} : \text{Der}_C(\overline{\mathbb{X}}) \times \hat{M} \rightarrow \hat{M} \]

such that

1. \( C^\infty(\overline{\mathbb{X}}) \)-linearity in the \( \text{Der}_C(\overline{\mathbb{X}}) \)-argument

\[ \hat{\nabla}_{f_1 \xi_1 + f_2 \xi_2} s = f_1 \hat{\nabla}_{\xi_1} s + f_2 \hat{\nabla}_{\xi_2} s, \text{ for } f_1, f_2 \in C^\infty(\overline{\mathbb{X}}), \xi_1, \xi_2 \in \text{Der}_C(\overline{\mathbb{X}}), \text{ and } s \in \hat{M}; \]

2. \( \mathbb{C} \)-linearity in the \( \hat{M} \)-argument

\[ \hat{\nabla}_{c \xi} (c_1 s_1 + c_2 s_2) = c_1 \hat{\nabla}_{\xi} s_1 + c_2 \hat{\nabla}_{\xi} s_2, \text{ for } c, c_1, c_2 \in \mathbb{C}, \xi \in \text{Der}_C(\overline{\mathbb{X}}), \text{ and } s_1, s_2 \in \hat{M}; \]

3. \( \mathbb{Z}/2 \)-graded Leibniz rule in the \( \hat{M} \)-argument

\[ \hat{\nabla}_{\xi} (f s) = (\xi f) s + (-1)^{p(\xi)p(f)} f \cdot \hat{\nabla}_{\xi} s, \]

for \( f \in C^\infty(\overline{\mathbb{X}}), \xi \in \text{Der}_C(\overline{\mathbb{X}}) \) parity homogeneous and \( s \in \hat{M} \).

\[ 14 \text{In [L-Y4: Definition 2.1.2] (D(14.1)), a left connection on } \hat{E} \text{ is required to satisfy the generalized } \mathbb{Z}/2 \text{-graded Leibniz rule in the } \hat{E} \text{-argument: } \hat{\nabla}_{\xi} (f s) = (\xi f) s + (-1)^{p(\xi)p(f)} f \cdot \hat{\nabla}_{\xi} s, \text{ for } f \in \hat{\mathcal{O}}_{\hat{E}}, \xi \in T_{\hat{\mathcal{O}}_{\hat{E}}} \text{ parity homogeneous and } s \in \hat{E}, \text{ where } \hat{\omega}(\hat{\nabla}) \text{ is the parity-conjugation of } \hat{\nabla} \text{ induced by } f; \text{i.e., } \hat{\omega}(\hat{\nabla}) = \hat{\nabla}, \text{ if } f \text{ is even, or } \hat{\nabla} := (\text{odd part of } \hat{\nabla}) - (\text{odd part of } \hat{\nabla}) \text{ if } f \text{ is odd; (cf. [L-Y4: Definition 1.3.1] (D(14.1))). When } \hat{\nabla} \text{ is even, } \hat{\omega}(\hat{\nabla}) = \hat{\nabla} \text{ always and the general } \mathbb{Z}/2 \text{-graded Leibniz rule reduces to the } \mathbb{Z} \text{-graded Leibniz rule.} \]
As an operation on the pairs \((\xi, s)\), a connection \(\nabla\) on \(\tilde{M}\) is applied to \(\xi\) from the right while applied to \(s\) from the left\(^{15}\) cf. [L-Y4: Lemma 1.3.7 & Remark 1.3.8] (D\((14.1)\)).

Note that since \(\tilde{\nabla}\) is even, the parity of \(\tilde{\nabla}\xi\) is the same as that of \(\xi\).

**Lemma/Definition 4.1.2. [curvature tensor of (even left) connection]**

(Cf. [L-Y4: Lemma/Definition 2.1.9] (D\((14.1)\)).) Continuing Definition 4.1.1. Let \(\tilde{\nabla}\) be an even left connection on \(\tilde{M}\). Then the correspondence

\[
F^\Theta : (\xi_1, \xi_2; s) \mapsto (\nabla_{\xi_1}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) s,
\]

for \(\xi_1, \xi_2 \in \text{Der}_{\mathcal{C}}(\tilde{\mathcal{X}})\) parity-homogeneous and \(s \in \tilde{M}\), satisfies the following tensorial property on \(\tilde{\mathcal{X}}\):

\[
\begin{align*}
(\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) s &= f \cdot (\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) s,
(\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) (fs) &= (-1)^{p(f)p(\xi_1)} f \cdot (\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) s,
(\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) (s) &= (-1)^{p(f)(p(\xi_1)+p(\xi_2))} f \cdot (\nabla_{\xi_{1}}, \nabla_{\xi_2} - \nabla_{[\xi_1, \xi_2]}) s,
\end{align*}
\]

for \(f \in C^\infty(\tilde{\mathcal{X}})\), \(\xi_1, \xi_2 \in \text{Der}_{\mathcal{C}}(\tilde{\mathcal{X}})\), \(s \in \tilde{M}\) all parity homogeneous. \(F^\Theta\) is called the curvature tensor on \(\tilde{\mathcal{X}}\) associated to the even left connection \(\tilde{\nabla}\) on \(\tilde{M}\).

**Proof.** This is a special case of [L-Y4: Lemma/Definition 2.1.9] (D\((14.1)\)) with the odd part of \(\tilde{\nabla}\) vanishes.

---

**Definition 4.1.3. [vector superfield]** A vector superfield \(\tilde{V}\) is an element in \(C^\infty(\tilde{\mathcal{X}})\) of the following form

\[
\tilde{V} = V_0 + \sum_{\alpha} \theta^\alpha \theta_\alpha V_\alpha + \sum_{\alpha, \beta} \overline{\theta}^\alpha \overline{\theta}_\beta V_{(\beta)} + \theta^1 \theta^2 \theta_1 \theta_2 V_{(12)} + \sum_{\alpha, \beta, \mu} \theta^\alpha \theta^\beta \sigma^\mu V_\mu + \overline{\theta}^1 \overline{\theta}^2 \overline{\theta}_1 \overline{\theta}_2 V'_{(12)}
\]

\[
+ \sum_{\beta} \theta^1 \theta^2 \overline{\theta}_\beta \left( \overline{\theta}^{(-1)} \sum_{\alpha, \mu} \theta_\alpha \sigma^{\alpha\mu} \partial_\mu V_\alpha \right.
\]

\[
+ \sum_{\alpha} \theta^\alpha \theta^1 \theta^2 \left( \partial_\alpha V''_0 + \overline{\theta}^{(-1)} \sum_{\beta, \mu} \partial_\mu \sigma^{\alpha\beta} \overline{\theta}_\beta V_\beta \right)
\]

that satisfies the realness condition

\[
\tilde{V}^\dagger = \tilde{V}.
\]

Here, \(\tilde{V}\) is expressed in coordinate functions \((x, \theta, \overline{\theta}, \vartheta)\) of \(\tilde{\mathcal{X}}\) with the components

\[
V'_\alpha = V'_{\alpha}(x).
\]

The set of vector superfields in \(C^\infty(\tilde{\mathcal{X}})\) form a \(C^\infty(\mathcal{X})\)-module\(^{16}\).

---

\(^{15}\)In the \(\mathbb{Z}/2\)-graded world, it is instructive to denote \(\nabla_{\xi}s\) as \(\xi \nabla s\) or \(\xi \nabla s\) (though we do not adopt it as a regularly used notation in this work). In particular, from \(\nabla_{\xi}s\) to \(f_{\nabla}s\), \(f\) and \(\nabla\) do not pass each other.

\(^{16}\)In [L-Y5: Sec. 3] (SUSY (1)), we took the attitude that all physics-related superfields should be connected to elements in the subring of \(C^\infty(\tilde{\mathcal{X}})\) generated by the small chiral elements and the small antichiral elements in \(C^\infty(\tilde{\mathcal{X}})\). Under such a restriction, the most natural definition for the candidate for physicists’ vector superfields has more degrees of freedom than that defined by physicists, particularly in [Wess & Bagger]. To distinguish them, we call it pre-vector superfield in [L-Y5] (SUSY (1)) and had to introduce linear constraints to reduce a pre-vector superfield to a vector superfield. Surprisingly, that still defines a supersymmetric gauge theory mimicking [Wess & Bagger]. For the current work (SUSY (2.1)), we find a better picture. In [L-Y5] (SUSY(1)), we look for vector
Explicitly, a vector superfield can be expressed as

\[
\hat{V} = V_0 + \sum_{\alpha} \theta^\alpha \partial_\alpha V_\alpha - \sum_{\beta} \bar{\theta}^\beta \partial_\beta \hat{V}_{(\beta)} + \theta^1 \theta^2 \partial_1 \partial_2 V_{(12)} + \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma_{\alpha\beta}^\mu V_\mu + \bar{\theta}^1 \bar{\theta}^2 \partial_1 \partial_2 \hat{V}_{(12)}
\]

\[+ \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \left( \sum_{\alpha,\mu} \partial_\alpha \sigma_{\mu}^{\alpha\beta} \partial_\mu V_\alpha + \bar{\sigma}_{\beta} \hat{V}_{(\beta)} \right) + \sum_{\beta,\mu} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \left( \partial_\alpha V_\mu - \sum_{\beta,\mu} \bar{\sigma}_{\alpha\beta} \partial_\mu V_\beta \right) + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 V_{(0)}^\sim
\]

with

\[V_0, \ V_\mu, \ V_{(0)}^\sim \in C^\infty(X), \text{ i.e. real-valued.}\]

Thus, while a small (resp. chiral/antichiral) superfield has 33 (resp. 4) independent complex components in \(C^\infty(X)^C\), a vector superfield \(\hat{V}\) has 16 independent real-valued components from \(V_0, V_\mu, V_{(0)}^\sim \in C^\infty(X), \mu = 0, 1, 2, 3\), and \(V_\alpha, V_{(12)}, V_{(\gamma)}^\prime \in C^\infty(X)^C, \alpha, \gamma = 1, 2\). Note that by definition a vector superfield is contained in \(C^\infty(\hat{X}^\mathbb{M})^\text{medium}\), but in general not in \(C^\infty(\hat{X}^\mathbb{M})^\text{small}\).

For physicists working on supersymmetric gauge theories, the following class of even left connections (adapted to the current \(U(1)\) case) is the major concern.

**Definition 4.1.4.** \((\text{left, even) connection associated to vector superfield})\] With the above setting, let \(\hat{V} \in C^\infty(\hat{X}^\mathbb{M})\) be a vector superfield on \(X\). Then, one can define an (even left) connection \(\nabla_{\hat{V}}\) on \(C^\infty(\hat{X}^\mathbb{M})\) (as a left \(C^\infty(\hat{X}^\mathbb{M})\)-module) associated to \(\hat{V}\) as follows.

1. Firstly, we acquire the compatibility with the chiral structure on \(C^\infty(\hat{X}^\mathbb{M})\) by setting

   \[\nabla_{\hat{V}} = e_{\beta'}\]

2. Secondly, we set

   \[\nabla_{e_{\alpha'}} = e^{-\hat{V}} \circ e_{\alpha'} \circ e_\hat{V} = e_{\alpha'} + e^{-\hat{V}} (e_{\alpha'} e_\hat{V})\]

   Thus, in a way \(\hat{V}\) is an indication of the twisting of the original antichiral structure of \(C^\infty(\hat{X}^\mathbb{M})\) to the one selected by \(\nabla_{e_{\alpha'}}\).

3. Finally, we set

   \[\nabla_{\sigma_\mu} = -\frac{\sqrt{-1}}{4} \sum_{\beta,\alpha} \bar{\sigma}_{\mu}^{\beta\alpha} \left\{ \nabla_{e_{\alpha'}} \nabla_{e_{\beta'}} \right\},\]

where \(\bar{\sigma}_{\mu} = (\bar{\sigma}_{\mu}^{\beta\alpha})_{\beta\alpha} = \sum_{\nu} \eta_{\mu\nu} \bar{\sigma}_{\nu}^{\beta\alpha}\) satisfies

\[\sum_{\beta,\alpha} \bar{\sigma}_{\mu}^{\beta\alpha} \sigma_{\nu}^{\alpha\beta} = -2 \delta_{\nu}^{\mu}, \text{ for } \mu, \nu = 0, 1, 2, 3.\]

superfields in the ring generated by chiral superfields and antichiral superfields. Under these constraints there is no other choice to make than what is done in [L-Y\textsuperscript{5}](SUSY (1)). Why do we need to require vector superfields to be in this ring? After all, unlike chiral superfields or antichiral superfields that form a ring, vector superfields do not form a ring: the multiplication of two vector superfields in general is no longer a vector superfield. If we choose them still in the grand function-ring of the towered superspace but not require them to lie in the above subring, then what shall we get? It is to answer this question that leads us in the end to the setting presented here. It matches now with [Wess & Bagger].

\[e^{\hat{V}} = e^{-\hat{V}} \circ e_{\alpha'} \circ e_\hat{V} \text{ or } e_\hat{V} \circ e_{\alpha'} \circ e^{-\hat{V}}\]

as the definition of \(\nabla_{e_{\alpha'}}\) is dictated by how one would construct the action functional for the gauge-invariant kinetic term for the chiral superfield in the supersymmetric \(U(1)\)-gauge theory with matter. The former is consistent with the setting in Sec. 4.4 while the latter isn’t. Cf. Lemma 4.2.6 vs. Sec. 4.4, theme ‘Explicit computations/formulae for \(\hat{V}\) in Wess-Zumino gauge’.
Explicitly,
\[
\bar{\sigma}_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{\sigma}_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \bar{\sigma}_2 := \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}, \quad \bar{\sigma}_3 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

This is indeed a flatness condition on the curvature of \(\hat{\nabla}\bar{V}\) in the fermionic directions \((e_{\alpha'}, e_{\beta''})\).

(Cf. Lemma 4.1.5 for the precise statement.)

Since \(\bar{V}\) is even, \(\hat{\nabla}\bar{V}\) as defined is even as well. In this way a vector superfield \(\bar{V} \in \mathcal{C}^\infty(\bar{X}^\mathbb{R})\) determines an even left connection \(\hat{\nabla}\bar{V}\) on \(\mathcal{C}^\infty(\bar{X}^\mathbb{R})\). \(\hat{\nabla}\bar{V}\) is called the connection on \(\mathcal{C}^\infty(\bar{X}^\mathbb{R})\) associated to \(\bar{V}\); For simplicity of notations, we often denote \(\hat{\nabla}\bar{V}\) by \(\hat{\nabla}\), keeping \(\bar{V}\) implicit.

**Lemma 4.1.5.** [flatness of \(\hat{\nabla}\bar{V}\) along fermionic directions] Let \(\hat{\nabla} = \hat{\nabla}\bar{V}\) be the connection on \(\mathcal{C}^\infty(\bar{X}^\mathbb{R})\) associated to a vector superfield \(\bar{V}\). Let \(F_{\bar{V}}\) be the curvature 2-tensor of \(\hat{\nabla}\) and denote \(F_{\bar{V}}(e_{\alpha'}, e_{\beta'})\) by \(F^{\alpha'}_{\beta'}\) (resp. \(F^{\alpha'}_{\beta''}\), \(F^{\alpha''}_{\beta''}\)). Then with respect to the supersymmetrically invariant coframe \((e^i)_1\) on \(\bar{X}\), the components of the curvature tensor \(F_{\bar{V}}\) of \(\hat{\nabla}\bar{V}\) in purely fermionic directions all vanish: For \(\alpha', \beta' = 1', 2'\) and \(\alpha'', \beta'' = 1'', 2''\),
\[
F^{\alpha'}_{\beta'} = F^{\alpha''}_{\beta''} = F^{\alpha'}_{\beta''} = 0.
\]

**Proof.** See [L-Y5: proof of Lemma 3.1.9] (SUSY(1)).

From the perspective of a physicist, one indeed defines the connection \(\hat{\nabla}\bar{V}\) as in Definition 4.1.4 so that Lemma 4.1.5 holds.

### 4.2 Vector superfields in Wess-Zumino gauge

We prove the existence of Wess-Zumino gauge for a vector superfield \(\bar{V}\) in the sense of Definition 4.1.3 and work out the connection \(\hat{\nabla}\bar{V}\) on \(\mathcal{C}^\infty(\bar{X}^\mathbb{R})\), as a (left) \(\mathcal{C}^\infty(\bar{X}^\mathbb{R})\)-module, for \(\bar{V}\) in Wess-Zumino gauge.

**Gauge transformations of a vector superfield**

Under a gauge transformation specified by a small chiral superfield \(\bar{\Lambda} \in \mathcal{C}^\infty(\bar{X}^\mathbb{R})^{small}\), a vector superfield \(\bar{V}\) transforms as
\[
\bar{V} \rightarrow \bar{V} + \delta \bar{\Lambda} \bar{V} := \bar{V} - \sqrt{-1}(\bar{\Lambda} - \bar{\Lambda}')\).
\]

Explicitly in terms of the standard coordinate functions \((x, \theta, \bar{\theta}, \bar{\phi})\), let
\[
\bar{V} = V(0) + \sum_\alpha \theta^\alpha \partial_\alpha V(\alpha) - \sum_\beta \bar{\theta}^\beta \partial_\beta V(\beta) + \theta^1 \theta^2 \partial_1 \partial_2 V(12) + \sum_\alpha \theta^\alpha \bar{\theta}^\beta \sigma^\mu_\alpha \partial_\mu V(\mu) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 V(12)
\]
\[
+ \sum_\beta \theta^\beta \bar{\theta}^\beta \left( \sqrt{-1} \sum_\alpha \partial_\alpha \sigma^\mu_\beta \partial_\mu V(\alpha) + \bar{\partial}_\beta V(\beta) \right)
\]
\[
+ \sum_\alpha \theta^\alpha \bar{\theta}^\beta \left( \bar{\partial}_\beta V(\alpha) - \sqrt{-1} \sum_\mu \bar{\partial}_\mu \sigma^\beta_\alpha \partial_\mu V(\beta) \right) + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 V(0) \in \mathcal{C}^\infty(\bar{X}^\mathbb{R})
\]
be a vector superfield and
\[
\bar{\Lambda} = \Lambda(0) + \sum_\alpha \theta^\alpha \partial_\alpha \Lambda(\alpha) + \theta^1 \theta^2 \partial_1 \partial_2 \Lambda(12) + \sqrt{-1} \sum_\alpha \theta^\alpha \bar{\theta}^\beta \sigma^\mu_\alpha \partial_\mu \Lambda(0)
\]
\[
+ \sqrt{-1} \sum_\beta \theta^\beta \bar{\theta}^\beta \partial_\alpha \sigma^\mu_\alpha \partial_\mu \Lambda(\alpha) - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \Lambda(0) \in \mathcal{C}^\infty(\bar{X}^\mathbb{R})^{small,ch},
\]

* Cf. [Wess & Bagger: Eq. (6.4)].
where $\Box := -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$, be a small chiral superfield. The twisted complex conjugate $\tilde{\Lambda}$ of $\Lambda$ is given by

$$
\tilde{\Lambda} = \Lambda(0) - \sum_{\beta} \bar{\theta}^\beta \bar{\delta}_\beta \Lambda(\beta) + \bar{\theta}^1 \bar{\theta}^2 \bar{\delta}_1 \bar{\delta}_2 \Lambda(12) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} \partial_\mu \Lambda(0) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \bar{\theta}^\alpha \theta^\beta \tilde{\sigma}^{\mu \beta}_{\alpha \beta} \partial_\mu \Lambda(\beta) - \theta^1 \bar{\theta}^2 \bar{\bar{\delta}} \Box \Lambda(0) \in C^\infty(\tilde{\mathcal{X}}^{\tilde{\beta}})_{\text{small, ach}}.
$$

Then,

$$
\tilde{V} + \delta_\Lambda \tilde{V} := \tilde{V} - \sqrt{-1}(\tilde{\Lambda} - \tilde{\Lambda}^t) = (V(0) - \sqrt{-1}(\Lambda(0) - \Lambda(0))) + \sum_{\alpha} \theta^\alpha \partial_\alpha (V_\alpha(0) - \sqrt{-1}\Lambda_\alpha(0)) - \sum_{\beta} \bar{\theta}^\beta \bar{\delta}_\beta (V_\beta(0) + \sqrt{-1}\Lambda_\beta(0))
$$

$$
+ \theta^1 \bar{\theta}^2 \theta_1 \theta_2 (V(12) - \sqrt{-1}\Lambda(12)) + \sum_{\alpha, \beta, \mu} \theta^\alpha \theta^\beta \sigma^\mu_{\alpha \beta} (V_\mu + \partial_\mu (\Lambda(0) + \Lambda(0)))
$$

$$
+ \bar{\theta}^1 \bar{\theta}^2 \bar{\delta}_1 \bar{\delta}_2 (V^t(12) + \sqrt{-1}\Lambda^t(12))
$$

$$
+ \sum_{\beta} \theta^1 \bar{\theta}^2 \theta_1 \theta_2 \left( V(0) + \delta_\Lambda V(0) \right) + \sum_{\alpha} \theta^\alpha \partial_\alpha (V_\alpha(0) + \delta_\Lambda V_\alpha(0)) - \sum_{\beta} \bar{\theta}^\beta \bar{\delta}_\beta (V_\beta(0) + \delta_\Lambda V_\beta(0))
$$

$$
+ \theta^1 \bar{\theta}^2 \theta_1 \theta_2 (V(12) + \delta_\Lambda V(12)) + \sum_{\alpha, \beta, \mu} \theta^\alpha \theta^\beta \sigma^\mu_{\alpha \beta} (V_\mu + \delta_\Lambda V_\mu) + \theta^1 \bar{\theta}^2 \theta_1 \theta_2 (V(12) + \delta_\Lambda V(12))
$$

is another vector superfield in the sense of Definition 4.1.3.

A comparison of $\delta_\Lambda V(0)$ against $\delta_\Lambda V(0)$ implies that if one expresses a vector superfield $\tilde{V} \in C^\infty(\tilde{\mathcal{X}}^{\tilde{\beta}})$ in the following shifted form, which can always be made:

**Definition 4.2.1. [vector superfield in shifted expression]** By re-defining the $\theta^1 \theta^2 \theta_1 \theta_2$-component $V^\sim$, any vector superfield in $C^\infty(\tilde{\mathcal{X}}^{\tilde{\beta}})$ can be expressed in the following form

$$
\tilde{V} = V(0) + \sum_{\alpha} \theta^\alpha \partial_\alpha V_\alpha(0) - \sum_{\beta} \bar{\theta}^\beta \bar{\delta}_\beta (V_\beta(0)) + \theta^1 \bar{\theta}^2 \theta_1 \theta_2 V(12) + \sum_{\alpha, \beta, \mu} \theta^\alpha \theta^\beta \sigma^\mu_{\alpha \beta} V_\mu + \bar{\theta}^1 \bar{\theta}^2 \bar{\delta}_1 \bar{\delta}_2 V^t(12)
$$

$$
+ \sum_{\beta} \theta^1 \bar{\theta}^2 \theta_1 \theta_2 (V_\beta(0)) + \sum_{\alpha} \theta^\alpha \theta^\beta \sigma^{\mu \beta}_{\alpha \beta} (V_\mu + \bar{\delta}_\beta (V_\beta(0)))
$$

$$
+ \theta^1 \bar{\theta}^2 \theta_1 \theta_2 (V^\sim(0) + \delta_\Lambda V^\sim(0)) \in C^\infty(\tilde{\mathcal{X}}^{\tilde{\beta}})
$$

We call a vector superfield of such form a vector superfield in the shifted expression.

---

* Cf. [Wess & Bagger: Eq. (6.2)].

* Cf. [Wess & Bagger: Eq. (6.3)].
Then the components $V''_\alpha, \alpha=1,2; V_\sim(0) \in C^\infty(X)^C$ are invariant under gauge transformations:

\[
\begin{align*}
\delta_\Lambda : & \\
V(0) & \rightarrow V(0) - \sqrt{-1}(\Lambda(0) - \overline{\Lambda}(0)), \\
V(\alpha) & \rightarrow V(\alpha) - \sqrt{-1}\Lambda(\alpha), \\
V(12) & \rightarrow V(12) - \sqrt{-1}\Lambda(12), \\
V[\mu] & \rightarrow V[\mu] + \partial_\mu(\Lambda(0) + \overline{\Lambda}(0)), \\
V'' & \rightarrow V'', \quad V_\sim \rightarrow V_\sim.
\end{align*}
\]

Remark 4.2.2. [R-linearity] An $\mathbb{R}$-linear combination\(^{18}\) of vector superfields in the shifted expression is also a vector superfield in the shifted expression.

It follows that, for a given vector superfield $\tilde{V}$ in the shifted expression, if one chooses a small chiral superfield $\tilde{\Lambda} \in C^\infty(\tilde{X})^{\text{small,ch}}$ with

\[
Im\Lambda(0) = -\frac{1}{2}V(0), \quad \Lambda(\alpha) = -\sqrt{-1}V(\alpha), \quad \Lambda(12) = -\sqrt{-1}V(12),
\]

which always exists, then after the gauge transformation specified by $\tilde{\Lambda}$, $\tilde{V}$ becomes

\[
\tilde{V}' = \sum_{\alpha,\beta,\mu} \theta^\alpha \overline{\theta}^\beta \sigma_{\alpha \beta}^\mu (V[\mu] + 2\partial_\mu Re\Lambda(0)) + \sum_{\beta} \theta^1 \theta^2 \overline{\theta}^\beta \mu(\beta) V'' + \sum_{\alpha} \theta^\alpha \overline{\theta}^1 \overline{\theta}^2 \overline{\partial}_\alpha V'' + \theta^1 \theta^2 \overline{\theta}^2 V_\sim.
\]

Here, $Re\Lambda(0)$ and $Im\Lambda(0)$ are the real part and the imaginary part of $\Lambda(0) \in C^\infty(X)^C$ respectively.

We summarize the above discussion into the following definition and lemma:

**Definition 4.2.3. [vector superfield in Wess-Zumino gauge]** A vector superfield $\tilde{V} \in C^\infty(\tilde{X})$ that is of the following form in the standard coordinate functions $(x, \theta, \bar{\theta}, \varphi, \bar{\varphi})$ on $\tilde{X}$

\[
\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \overline{\theta}^\beta \sigma_{\alpha \beta}^\mu V[\mu] + \sum_{\beta} \theta^1 \theta^2 \overline{\theta}^\beta \overline{\partial}_\beta V'' + \sum_{\alpha} \theta^\alpha \overline{\theta}^1 \overline{\theta}^2 \overline{\partial}_\alpha V'' + \theta^1 \theta^2 \overline{\theta}^2 V_\sim
\]

is called a vector superfield in Wess-Zumino gauge.

**Lemma 4.2.4. [vector superfield representative in Wess-Zumino gauge]** Given any vector superfield $\tilde{V} \in C^\infty(\tilde{X})$, there exists a unique small chiral superfield $\tilde{\Lambda} \in C^\infty(\tilde{X})^{\text{small,ch}}$ depending on $\tilde{V}$ with $Re\Lambda(0) = 0$ such that the gauge transformation specified by $\tilde{\Lambda}$ takes $\tilde{V}$ to a vector superfield in Wess-Zumino gauge. In particular, any vector superfield can be transformed to a vector superfield in Wess-Zumino gauge by a gauge transformation.

**Lemma 4.2.5. [naturality]** (1) The set of vector superfields in Wess-Zumino gauge is a $C^\infty(X)$-submodule of $C^\infty(\tilde{X})$. (2) If a vector superfield $\tilde{V}$ expressed in terms of the standard coordinate functions $(x, \theta, \bar{\theta}, \varphi, \bar{\varphi})$ on $\tilde{X}$ is in Wess-Zumino gauge, then it remains in Wess-Zumino gauge when re-expressed in terms of the chiral coordinate functions $(x', \theta, \bar{\theta}, \varphi)$ or the antichiral coordinate functions $(x'', \theta, \bar{\theta}, \varphi)$ on $\tilde{X}$.

\(^{18}\)However, caution that a $C^\infty(X)$-linear combination of vector superfields in the shifted expression in general is not directly a vector superfield in the shifted expression. One has to convert it accordingly.
Proof. Statement (1) is clear. We focus on Statement (2).

Recall that, in shorthand, \( x' = x + \sqrt{-1} \theta \sigma \theta \). When in Wess-Zumino gauge, a vector superfield \( \tilde{V} \) in terms of the standard coordinate functions \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) is written as

\[
\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha\beta} V_{[\mu]}(x) + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\vartheta} (\bar{V}^\mu_{(\beta)}(x) + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\vartheta} \partial_{\alpha} \tilde{V}''(x) + \theta^1 \theta^2 \bar{\theta}^2 \bar{\vartheta} \tilde{V}_0''(x)
\]

To re-express \( \tilde{V} \) in terms of the chiral coordinate functions \((x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \( \tilde{X} \), one substitutes \( x \) in

\[
V_{[\mu]}(x') \quad \text{by} \quad x' = \sqrt{-1} \theta \sigma \theta
\]

and use the \( C^\infty \)-hull structure of \( C^\infty(\tilde{X}) \) to expand it in \( x' \). Due to the product structure of \( \theta^\alpha, \bar{\theta}^\beta \), this will only influence the coefficient of \( \theta^1 \theta^2 \bar{\theta}^1 \bar{\vartheta} \) and, hence, keep the vector superfield in Wess-Zumino gauge. Explicitly, the result is

\[
\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha\beta} V_{[\mu]}(x') + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\vartheta} (\bar{V}^\mu_{(\beta)}(x') + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\vartheta} \partial_{\alpha} \tilde{V}''(x') + \theta^1 \theta^2 \bar{\theta}^2 \bar{\vartheta} \tilde{V}_0''(x') \biggr) - 2\sqrt{-1} \sum_{\mu} (\partial^\mu V_{[\mu]}(x'))
\]

Similar argument goes when re-expressing \( \tilde{V} \) in terms of the antichiral coordinate functions \((x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \( \tilde{X} \), with \( x'' = x - \sqrt{-1} \theta \sigma \theta \). The explicit expression is given by

\[
\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha\beta} V_{[\mu]}(x'') + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\vartheta} (\bar{V}^\mu_{(\beta)}(x'') + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\vartheta} \partial_{\alpha} \tilde{V}''(x'') + \theta^1 \theta^2 \bar{\theta}^2 \bar{\vartheta} \tilde{V}_0''(x'') \biggr) + 2\sqrt{-1} \sum_{\mu} (\partial^\mu V_{[\mu]}(x''))
\]

This completes the proof.

\[\square\]

In the shifted expression, once a vector superfield is rendered a vector superfield \( \tilde{V} \) in Wess-Zumino gauge, a gauge transformation specified by \( \tilde{\Lambda} \) with

\[
Im \Lambda_{(0)} = \Lambda_{(\alpha)} = \Lambda_{(12)} = 0,
\]

(i.e.,

\[
\tilde{\Lambda} = \Lambda_{(0)} + \sqrt{-1} \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha\beta} \partial_{\mu} \Lambda_{(0)} - \theta^1 \theta^2 \bar{\theta}^1 \bar{\vartheta} \partial_{\vartheta} \Lambda_{(0)}
\]

specified by a real-valued component \( \Lambda_{(0)} \) will send \( \tilde{V} \) to another vector superfield \( \tilde{V}' \) still in Wess-Zumino gauge with only the components \( V_{[\mu]} \) of \( \tilde{V} \) transformed, by

\[
V_{[\mu]} \rightarrow V_{[\mu]} + 2 \partial_{\mu} \Lambda_{(0)}.
\]

Thus, the residual gauge symmetries on vector superfields in Wess-Zumino gauge are the usual \( U(1) \) gauge symmetries.

**Lemma 4.2.6. [restriction \( \nabla^V \) of \( \nabla \) to \( X^C \)]** Let \( \tilde{V} \) be a vector superfield in Wess-Zumino gauge. Then the restriction \( \nabla^V \) of \( \nabla \) to \( X^C \) is given by \( \partial_{\mu} - \frac{\sqrt{-1}}{2} V_{[\mu]} \), \( \mu = 0, 1, 2, 3 \).

**Proof.** This follows from the same argument of [L-Y5: proof of Lemma 3.2.6] (SUSY(1)). See also the full expression of \( \nabla^V \) in the next theme.

\[\square\]
Explicit formula of $\hat{\nabla}_V$ for $V$ in Wess-Zumino gauge

Note that the powers of a vector superfield $V \in C^\infty(\mathbb{X})$ in Wess-Zumino gauge can be computed easily:

$$\hat{V} = \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \delta_{\alpha \beta} V_{[\mu]} + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\theta}^\beta V_{(\beta)} + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \partial_{\alpha} V''_{(\alpha)} + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 V_{(0)},$$

$$\hat{V}^2 = 2 \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \sum_{\mu, \nu} \eta^{\mu \nu} V_{[\mu]} V_{[\nu]},$$

$$\hat{V}^3 = 0.$$

It follows that

$$e^V = 1 + \hat{V} + \frac{1}{2} \hat{V}^2$$

$$= 1 + \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\theta}^\beta \delta_{\alpha \beta} V_{[\mu]} + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\theta}^\beta V_{(\beta)} + \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \partial_{\alpha} V''_{(\alpha)} + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 (V_{(0)} + \sum_{\mu, \nu} \eta^{\mu \nu} V_{[\mu]} V_{[\nu]}$$

and $\hat{\nabla}_e V$, $I = \mu, \alpha', \beta''$, can be computed less tediously for $V$ in Wess-Zumino gauge. The result is given below in the standard coordinate functions $(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\mathbb{X}$.

First,

$$\hat{\nabla}_e V_{\mu} := e_{\beta \nu}$$

by definition.

Next, $\hat{\nabla}_e V_{\alpha}$, can be computed slightly easier in the antichiral coordinate functions $(x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ first and then converted back to $(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$:

$$\hat{\nabla}_e V_{\alpha} := e^{-V} \circ e_{\alpha'} \circ e^V = e_{\alpha'} + e^{-V}(e_{\alpha'} e^V)$$

$$= e_{\alpha'} + \sum_{\beta, \mu} \bar{\theta}^\beta \delta_{\alpha \beta} V_{[\mu]} - \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \bar{\theta}^\beta \varepsilon_{\alpha \beta} V_{(\beta)} + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 V''_{(\beta)}$$

$$+ \sum_{\gamma} \theta^\gamma \bar{\theta}^1 \bar{\theta}^2 \left( - \varepsilon_{\alpha \gamma} V_{(0)} + \sqrt{-1} \sum_{\beta, \mu, \nu} \sigma_{\alpha}^\beta \sigma_{\gamma \beta} \partial_{\mu} V_{[\nu]} \right) - \sqrt{-1} \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \sum_{\beta} \bar{\partial}_\beta \sigma_{\beta}^{\alpha \beta} \partial_{\mu} \bar{\partial}_{(\beta)} V_{(\beta)}.$$

The identity

$$(\sigma^\alpha \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_{\alpha \gamma} := \sum_{\beta, \delta} \varepsilon^{\beta \delta} (\sigma_{\alpha}^\beta \sigma_{\gamma \beta} + \sigma_{\alpha}^\nu \sigma_{\gamma \beta}) = 2 \eta^{\mu \nu} \varepsilon_{\alpha \gamma}$$

is used in the computation to simplify the coefficient function of $\theta^\gamma \bar{\theta}^1 \bar{\theta}^2$. Note that terms quadratic in components of $V$ cancel each other and only terms linear in components of $V$ remain in the end.

Finally,

$$\hat{\nabla}_e V_{\mu} = - \sqrt{-1} \sum_{\beta, \alpha} \bar{\theta}^\beta \delta_{\alpha \beta} \left( \hat{\nabla}_{e_{\alpha'}} V_{\beta \mu} \right) = \partial_{\mu} - \sqrt{-1} \sum_{\beta, \alpha} \bar{\theta}^\beta \Theta_{\beta \alpha}$$

where

$$\Theta_{\alpha \beta} := \{ e^{-V}(e_{\alpha'} e^V), e_{\beta \nu} \} = (e_{\beta \nu} e^{-V})(e_{\alpha'} e^V) + e^{-V}(e_{\beta \nu} e_{\alpha'} e^V).$$

After some (pages of) straightforward computations, one obtains the explicit expression for $\Theta_{\alpha \beta}$ in the standard coordinate functions $(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$: 

46
Lemma 4.3.1. [chiral function and antichiral function on \(\hat{X}\)]

In this subsection we repeat the discussion of \([L-Y5: \text{Sec. 3.3}]

(\text{SUSY(1)}) for the upgraded notion of

4.3 Supersymmetry transformations of a vector superfield in Wess-Zumino gauge

In this subsection we repeat the discussion of \([L-Y5: \text{Sec. 3.3}]

(\text{SUSY(1)}) for the upgraded notion of vector superfield. Readers are recommended to read this subsection along with \([A-R: \text{Sec. 4.3}], \text{pp. 76, 77}\) of Argurio and \([G-G-R-S: \text{Sec. 4.2.1, from before Eq. (4.2.7) to after Eq. (4.2.13)}\) of Gates, Grosaru, Roček, and Siegel for comparison.

Recall the Grassmann parameter level of \(\hat{X}\). The setup, discussions, and results in Sec. 1.3, Sec. 4.1 and Sec. 4.2 can be generalized straightforwardly to the case where the Grassmann parameter level is turned on. In particular, recall the additional coordinate functions \((\eta, \bar{\eta}) := (\eta^1, \eta^2; \bar{\eta}^1, \bar{\eta}^2)\) on \(\hat{X}\) from the parameter level, then: (cf. Lemma 1.3.4)

**Lemma 4.3.1.** [chiral function and antichiral function on \(\hat{X}\) with parameter level activated]

(1) \(\bar{f} \in C^\infty(\hat{X})\) is chiral if and only if, as an element of \(C^\infty(X)^C[\theta, \bar{\theta}, \eta, \bar{\eta}, \varphi, \bar{\varphi}]\) anti-c., \(\bar{f}\) is of the following form

\[
\bar{f} = \bar{f}_0(x, \eta, \bar{\eta}, \varphi, \bar{\varphi}) + \sum_{\gamma} \theta^\gamma \bar{f}_{(\gamma)}(x, \eta, \bar{\eta}, \varphi, \bar{\varphi})
\]

\[+
\theta^1\bar{\theta}^2 \bar{f}_{(12)}(x, \eta, \bar{\eta}, \varphi, \bar{\varphi}) + \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\nu_\gamma \bar{\sigma}^\delta_\nu \bar{f}_0(x, \eta, \bar{\eta}, \varphi, \bar{\varphi})
\]

\[+
\sqrt{-1} \sum_{\delta, \gamma, \nu} \theta^1\bar{\theta}^2 \bar{\sigma}^\nu_\gamma \partial_{\bar{\nu}} \bar{f}_{(\gamma)}(x, \eta, \bar{\eta}, \varphi, \bar{\varphi}) - \theta^1\bar{\theta}^2 \bar{\theta}^2 \Box \bar{f}_0(x, \eta, \bar{\eta}, \varphi, \bar{\varphi}).
\]
In particular, a chiral \( f \in \mathcal{C}^\infty(\hat{X}^\oplus) \) has four independent components in \( \mathcal{C}^\infty(X)^C[\eta, \bar{\eta}, \theta, \bar{\theta}]^{\text{anti-c}} \):

\[
\tilde{f}(0), \quad \tilde{f}(\gamma), \quad \gamma = 1, 2, \quad \tilde{f}(12).
\]

In terms of the standard chiral coordinate functions \((x', \theta, \tilde{\theta}, \eta, \bar{\eta}, \bar{\theta})\) on \(\hat{X}^\oplus\),

\[
\tilde{f} = \tilde{f}(0)(x', \eta, \bar{\eta}, \bar{\theta}) + \sum_{\gamma} \theta^\gamma \tilde{f}(\gamma)(x', \eta, \bar{\eta}, \bar{\theta}) + \theta^1 \theta^2 \tilde{f}(12)(x', \eta, \bar{\eta}, \bar{\theta}),
\]

which is independent of \(\bar{\theta}\).

(2) \( \tilde{f} \in \mathcal{C}^\infty(\hat{X}^\oplus) \) is antichiral if and only if, as an element of \( \mathcal{C}^\infty(X)^C[\theta, \tilde{\theta}, \eta, \bar{\eta}, \bar{\theta}]^{\text{anti-c}} \), \( \tilde{f} \) is of the following form

\[
\tilde{f} = \tilde{f}(0)(x, \eta, \bar{\eta}, \bar{\theta}) + \sum_{\delta} \bar{\theta}^\delta \tilde{f}(\delta)(x, \eta, \bar{\eta}, \bar{\theta})
\]

\[
+ \bar{\theta}^1 \bar{\theta}^2 \tilde{f}(12)(x, \eta, \bar{\eta}, \bar{\theta}) - \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \sigma^\gamma_{\delta \nu} \partial_\nu \tilde{f}(0)(x, \eta, \bar{\eta}, \bar{\theta})
\]

\[
+ \sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \bar{\sigma}^\gamma_{\delta \nu} \partial_\nu \tilde{f}(\delta)(x, \eta, \bar{\eta}, \bar{\theta}) - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \Box \tilde{f}(0)(x, \eta, \bar{\eta}, \bar{\theta}).
\]

In particular, an antichiral \( \tilde{f} \in \mathcal{C}^\infty(\hat{X}^\oplus) \) has four independent components in \( \mathcal{C}^\infty(X)^C[\eta, \bar{\eta}, \theta, \bar{\theta}]^{\text{anti-c}} \):

\[
\tilde{f}(0), \quad \tilde{f}(\delta), \quad \delta = 1, 2, \quad \tilde{f}(12).
\]

In terms of the standard antichiral coordinate functions \((x'', \theta, \tilde{\theta}, \eta, \bar{\eta}, \bar{\theta})\) on \(\hat{X}^\oplus\),

\[
\tilde{f} = \tilde{f}(0)(x'', \eta, \bar{\eta}, \bar{\theta}) + \sum_{\delta} \bar{\theta}^\delta \tilde{f}(\delta)(x'', \eta, \bar{\eta}, \bar{\theta}) + \bar{\theta}^1 \bar{\theta}^2 \tilde{f}(12)(x'', \eta, \bar{\eta}, \bar{\theta}),
\]

which is independent of \(\theta\).

We shall use this to understand how supersymmetries act on vector superfields in Wess-Zumino gauge.

Consider the infinitesimal supersymmetry transformation

\[
\delta_{(\eta, \bar{\eta})} \tilde{V} := (\eta Q + \bar{\eta} \bar{Q}) \tilde{V} := \left( \sum_{\alpha} \eta^\alpha Q_\alpha - \sum_{\beta} \bar{\eta}^\beta \bar{Q}_{\beta} \right) \tilde{V}
\]

of \( \tilde{V} \). Then \( (\delta_{(\eta, \bar{\eta})} \tilde{V})^\dagger = \delta_{(\eta, \bar{\eta})} \tilde{V} \). However, for \( \tilde{V} \) in Wess-Zumino gauge, \( (\eta Q + \bar{\eta} \bar{Q}) \tilde{V} \) remains a vector superfield (in the sense of reality condition and matching of the independent components with the vector multiplet from representations of the \(d = 3 + 1, N = 1\) supersymmetry algebra) but in general no longer in Wess-Zumino gauge (in the sense of the pattern as a \((\theta, \tilde{\theta})\)-polynomial). This can be remedied by a gauge transformation: (e.g., [Argu: Sec. 4.3.1], [G-G-R-S: Sec. 4.2a.1], [W-B: Chap. VII, Exercise (8)], and [We: Sec. 15.3, Eq. (15.78)])

**Lemma 4.3.2.** [existence and uniqueness of correcting gauge transformation] Let \( \tilde{V} \) be a vector superfield in Wess-Zumino gauge. Then there exists a chiral function \( \tilde{\Lambda}_{(\eta, \bar{\eta})} \in \mathcal{C}^\infty(\hat{X}^\oplus)^{\text{ch}} \) (now with the parameter level activated) depending \(\mathbb{C}\)-multilinearly on \((\eta, \bar{\eta})\) and \( \tilde{V} \) such that the gauge transformation

\[
(\eta Q + \bar{\eta} \bar{Q}) \tilde{V} - \sqrt{-1}(\tilde{\Lambda}_{(\eta, \bar{\eta})} \tilde{V} - \tilde{\Lambda}_{(\eta, \bar{\eta})}^I \tilde{V})
\]

of \( (\eta Q + \bar{\eta} \bar{Q}) \tilde{V} \) is in Wess-Zumino gauge. Furthermore, one may require that the \((\theta, \tilde{\theta})\)-degree-0 component \( \tilde{\Lambda}_{(\eta, \bar{\eta}) \tilde{V}, (0) \} \) of \( \tilde{\Lambda}_{(\eta, \bar{\eta}) \tilde{V}} \) vanish, in which case \( \tilde{\Lambda}_{(\eta, \bar{\eta}) \tilde{V}} \) is unique.
Proof. When $\bar{\mathcal{V}}$ is in Wess-Zumino gauge, $(\theta, \bar{\theta})$-degree-0 component $((\eta Q + \bar{\eta}Q)\bar{\mathcal{V}})_{(0)}$ of $(\eta Q + \bar{\eta}Q)\bar{\mathcal{V}}$ is always zero. It follows from Lemma 4.3.1 and the same reasoning as the explicit computation in Sec. 4.2 that leads to Lemma 4.2.4 that there is a unique chiral function $\bar{\Lambda} \in C^{\infty}\left(\hat{X}^\text{fr} \right)^{ch}$ associated to $(\eta Q + \bar{\eta}Q)\bar{\mathcal{V}}$ with the $(\theta, \bar{\theta})$-degree-0 component $\bar{\Lambda}_{(0)} = 0$ such that $(\eta Q + \bar{\eta}Q)\bar{\mathcal{V}} = \sqrt{-1}(\bar{\Lambda} - \bar{\Lambda}^\dagger)$ is in Wess-Zumino gauge. The same explicit computation implies also that this unique $\Lambda$ depends $\mathbb{C}$-multilinearly on $(\eta, \bar{\eta})$ and $\bar{\mathcal{V}}$. This proves the lemma.

\[\square\]

Definition 4.3.3. [supersymmetry in Wess-Zumino gauge] Continuing Lemma 4.3.2. Set

\[
(\eta Q + \bar{\eta}Q)\bar{\mathcal{V}} - \sqrt{-1}(\bar{\Lambda}_{(\eta, \bar{\eta}; \bar{\mathcal{V}})} - \bar{\Lambda}^\dagger_{(\eta, \bar{\eta}; \bar{\mathcal{V}})}) = \sum_{\alpha} \eta^\alpha Q^\alpha W \bar{\mathcal{V}} - \sum_{\beta} \bar{\eta}^\beta \bar{Q}^\beta W \bar{\mathcal{V}}.
\]

This defines (infinitesimal) supersymmetry transformations in Wess-Zumino gauge $Q^\alpha W$, $\bar{Q}^\beta W$ that take a vector superfield in Wess-Zumino gauge to another in Wess-Zumino gauge.

Explicitly, let

\[
\bar{\mathcal{V}} = \sum_{\gamma, \delta, \nu} \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} + \sum_{\delta} \delta^{\theta^2 \bar{\theta}^3 \bar{\gamma} \delta} \bar{\mathcal{V}}_{(\delta)} + \sum_{\gamma} \delta^{\theta \bar{\theta} \bar{\gamma} \gamma} V''_{(\gamma)} + \theta^1 \bar{\theta} \bar{\gamma} \gamma V_{(0)}
\]

in $C^{\infty}(\hat{X}^\text{fr})$ be a vector superfield in Wess-Zumino gauge. Then,

\[
\delta_{\eta Q + \bar{\eta}Q} \bar{\mathcal{V}} := (\eta Q + \bar{\eta}Q)\bar{\mathcal{V}}
\]

\[
= \left( \sum_{\alpha} \eta^\alpha \frac{\partial}{\partial \eta^\alpha} - \sqrt{-1} \sum_{\alpha, \beta, \mu} \eta^\alpha \sigma^{\mu}_a \bar{\sigma}^\delta_{a\beta} \partial_{\mu} \right) \bar{\mathcal{V}} + \left( \sum_{\beta} \bar{\eta}^\beta \frac{\partial}{\partial \bar{\theta}^\beta} - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \sigma^{\mu}_{\bar{a}} \bar{\sigma}^\delta_{\bar{a}\beta} \partial_{\mu} \right) \bar{\mathcal{V}}
\]

\[
= \sum_{\gamma, \delta, \nu} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} - \sum_{\delta} \delta^{\theta^2 \bar{\theta}^3 \bar{\gamma} \delta} \bar{\mathcal{V}}_{(\delta)} + \theta^1 \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} \right) + \theta^1 \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} \right)
\]

Recall Lemma 4.3.1 and let $\bar{\Lambda} = \bar{\Lambda}_{(\eta, \bar{\eta}; \bar{\mathcal{V}})}$ be the unique chiral function in $C^{\infty}(\hat{X}^\text{fr})$ with

\[
\bar{\Lambda}_{(0)} = 0, \quad \bar{\Lambda}_{(\gamma)} = -\sqrt{-1} \sum_{\beta, \nu} \bar{\eta}^\beta \delta^{\gamma\nu} \bar{\sigma}^\nu_{\gamma\nu} V_{[\nu]}, \quad \bar{\Lambda}_{(12)} = -\sqrt{-1} \sum_{\beta} \bar{\eta}^\beta \bar{\sigma}^\nu_{\gamma\nu} \bar{\mathcal{V}}_{(\beta)}.
\]

I.e.

\[
\bar{\Lambda}_{(\eta, \bar{\eta}; \bar{\mathcal{V}})} = -\sqrt{-1} \sum_{\gamma, \delta, \nu} \theta^\gamma \bar{\theta}^\delta \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} - \sqrt{-1} \theta^1 \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} \right) + \theta^1 \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\delta} V_{[\nu]} \right)
\]

Then,

\[
\bar{\Lambda}^\dagger_{(\eta, \bar{\eta}; \bar{\mathcal{V}})} = -\sqrt{-1} \sum_{\delta, \alpha, \nu} \bar{\theta}^\delta \eta^\alpha \delta^{\gamma\nu} \bar{\sigma}^\nu_{\gamma\nu} V_{[\nu]} + \sqrt{-1} \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\nu} V_{[\nu]} \right) + \theta^1 \bar{\theta} \bar{\gamma} \gamma \left( \sum_{\alpha} \eta^\alpha \delta^{\gamma\delta} \bar{\sigma}^\nu_{\gamma\nu} V_{[\nu]} \right)
\]
\[
\delta_{\eta Q + \eta \tilde{Q}} \tilde{V} + \delta_{\tilde{A}_{(\eta, \bar{\eta}; \bar{V})}} \tilde{V} = (\eta Q + \eta \tilde{Q}) \tilde{V} - \sqrt{-1}(\tilde{A}_{(\eta, \bar{\eta}; \bar{V})} - \tilde{A}^\dagger_{(\eta, \bar{\eta}; \bar{V})}) \\
= \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \left(- \sum_{\alpha} \eta^\alpha \bar{\theta}^\delta \varepsilon_{\alpha \gamma} V^\dagger_{(\delta)} + \sum_{\beta} \bar{\eta}^\beta \partial_\gamma \varepsilon_{\beta \delta} V''_{(\gamma)}\right) \\
+ \sum_{\delta, \beta} \theta^\delta \bar{\theta}^\gamma \left(\varepsilon_{\bar{\beta} \delta} V''_{(0)} - \sqrt{-1} \sum_{\gamma, \mu, \nu} \sigma^\gamma_{\beta} \gamma^\nu_{\delta \mu} F_{\mu \nu}\right) \\
+ \sum_{\gamma, \alpha} \theta^\gamma \bar{\theta}^\beta \left(\varepsilon_{\alpha \gamma} V''_{(0)} + \sqrt{-1} \sum_{\delta, \mu, \nu} \sigma^\mu_{\alpha} \gamma^\nu_{\delta \gamma} F_{\mu \nu}\right) \\
+ \sqrt{-1} \theta^\gamma \bar{\theta}^\beta \left(\sum_{\alpha, \delta, \mu} \eta^\alpha \bar{\theta}^\delta \sigma^\mu_{\alpha} \partial_\mu V''_{(\delta)} + \sum_{\beta, \gamma, \mu} \bar{\eta}^\beta \partial_\gamma \sigma^\mu_{\beta} \partial_\mu V''_{(\gamma)}\right),
\]

where \(F_{\mu \nu} := \partial_\mu V^{(0)}_{[\nu]} - \partial_\nu V^{(0)}_{[\mu]}\), now resumes in Wess-Zumino gauge.

From the last expression and Definition 4.3.3, one reads off

\[
Q^{\alpha}_{WZ} \tilde{V} = - \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \varepsilon_{\alpha \gamma} V''_{(\delta)} - \sum_{\gamma} \theta^\gamma \bar{\theta}^\beta \left(\varepsilon_{\alpha \gamma} V''_{(0)} + \sqrt{-1} \sum_{\delta, \mu, \nu} \sigma^\mu_{\alpha} \gamma^\nu_{\delta \gamma} F_{\mu \nu}\right) \\
+ \sqrt{-1} \theta^\gamma \bar{\theta}^\beta \left(\sum_{\delta, \mu} \eta^\alpha \bar{\theta}^\delta \sigma^\mu_{\alpha} \partial_\mu V''_{(\delta)}\right),
\]

\[
\bar{Q}^{\beta}_{WZ} \tilde{V} = - \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \varepsilon_{\bar{\beta} \delta} V''_{(\gamma)} + \sum_{\beta} \theta^\beta \bar{\theta}^\gamma \left(\varepsilon_{\bar{\beta} \delta} V''_{(0)} - \sqrt{-1} \sum_{\gamma, \mu, \nu} \sigma^\gamma_{\beta} \gamma^\nu_{\beta \delta} F_{\mu \nu}\right) \\
- \sqrt{-1} \theta^\beta \bar{\theta}^\gamma \left(\sum_{\gamma, \mu} \bar{\eta}^\gamma \partial_\gamma \sigma^\mu_{\bar{\beta}} \partial_\mu V''_{(\gamma)}\right).
\]

The supersymmetry algebra generated by \(Q^{\alpha}_{WZ}\)’s, \(Q_{WZ}^{\bar{\beta}}\)’s, and \(\partial_{\mu}\)’s is now closed only up to a gauge transformation.

### 4.4 Supersymmetric \(U(1)\) gauge theory with matter on \(X\) in terms of \(\hat{X}\)

With the preparations in Sec. 4.1 and Sec. 4.2, we are now ready to construct\(^{19}\) a supersymmetric \(U(1)\) gauge theory with matter on \(X\) in terms of functions on \(\hat{X}\).

#### Two basic derived\(^{20}\) superfields: gaugino superfield and kinetic-term superfield

Unlike chiral or antichiral superfields, a vector superfield \(\tilde{V}\) contains no components that involve spacetime derivatives. For that reason, to construct a supersymmetric action functional for components of \(\tilde{V}\), one needs to work out appropriate derived superfields from \(\tilde{V}\) first.

**Lemma/Definition 4.4.1. [gaugino superfield]** (Cf. [Wess & Bagger: Eq. (6.7)].)

*Let \(\tilde{V} \in C^\infty(\hat{X})\) be a vector superfield. Define\(^{21}\)\]*

\(^{19}\)Note for mathematicians. See [L-Y5: Sec. 3.5, footnote 28] (SUSY(1)).

\(^{20}\)Here, we are not using the term \textit{‘derived’} in any deeper sense. We only mean that such superfields arise from the combination of more basic superfields such as small chiral superfields and vector superfields. For example, the superpotential is a polynomial (or more generally holomorphic function) of small chiral superfields and thus can be regarded as a “derived” superfield. Caution that these derived superfields may go beyond the small function-ring \(C^\infty(\hat{X})\) small and lie only in \(C^\infty(\hat{X})\).

\(^{21}\)The design here is made so that \(W_{\alpha} = \partial_\alpha V''_{(\alpha)} + \text{terms of } (\theta, \bar{\theta})\)-degree \(\geq 1\) and \(W_{\beta} = \bar{\partial}_\beta V''_{(\beta)} + \text{terms of } (\theta, \bar{\theta})\)-degree \(\geq 1\). Caution that, while \(\epsilon_{\alpha} = \partial/\partial \theta^\alpha + \cdots\), \(\epsilon_{\beta} = -\partial/\partial \bar{\theta}^\beta + \cdots\).
\[ W_\alpha := e_{2\nu}e_{1\nu}e_{\alpha}\tilde{V} \quad (\text{resp. } W_\beta := e_{1\nu}e_{2\nu}e_{\beta}\tilde{V}) \]

\( \alpha = 1,2, \beta = \dot{1},\dot{2}. \) Then (1) \( W_\alpha \) (resp. \( W_\beta \)) is chiral (resp. antichiral). (2) \( W_\alpha \) and \( \tilde{W}_\beta \) are invariant under gauge transformations on \( \tilde{V} \).

\( W_\alpha, \tilde{W}_\beta \) are called the gaugino superfields associated to the vector superfield \( \tilde{V} \).

**Proof.** The same as [L-Y5: Proof of Lemma/Definition 3.5.1] (SUSY(1)) but now for vector superfield in the sense of Definition 4.1.3. Details are repeated below due to the importance of these quantities.

For Statement (1),
\[
\begin{align*}
e_{1\nu}W_\alpha &= -e_{2\nu}(e_{1\nu})^2e_{\alpha}\tilde{V} = 0, \\
e_{2\nu}W_\alpha &= (e_{2\nu})^2e_{1\nu}e_{\alpha}\tilde{V} = 0
\end{align*}
\]

since \((e_{1\nu})^2 = (e_{2\nu})^2 = 0\). Similarly for the antichirality of \( \tilde{W}_\beta \).

For Statement (2), under a gauge transformation \( \tilde{V} \rightarrow \tilde{V} - \sqrt{-1}(\tilde{\Lambda} - \tilde{\Lambda}^\dagger) \) on \( \tilde{V} \) specified by a small chiral superfield \( \tilde{\Lambda} \in C^\infty(\tilde{X}\tilde{\bar{M}})^{small,ch} \),
\[
W_\alpha \rightarrow e_{2\nu}e_{1\nu}e_{\alpha}\left(\tilde{V} - \sqrt{-1}(\tilde{\Lambda} - \tilde{\Lambda}^\dagger)\right) = W_\alpha - \sqrt{-1}(e_{2\nu}e_{1\nu}e_{\alpha}\tilde{\Lambda} - e_{1\nu}e_{2\nu}e_{\alpha}\tilde{\Lambda}^\dagger) = W_\alpha
\]

since \( \tilde{\Lambda}^\dagger \) is antichiral (thus, \( e_{\alpha}\tilde{\Lambda}^\dagger = 0 \)) and \( \tilde{\Lambda} \) is chiral (thus, \( e_{1\nu}\tilde{\Lambda} = e_{2\nu}\tilde{\Lambda} = 0 \)). Similarly for \( \tilde{W}_\beta \).

It follows that in the construction of a supersymmetric \( U(1) \)-gauge theory with matter, one may assume that the vector superfield \( \tilde{V} \) is in Wess-Zumino gauge, which encodes the component fields
\[
V_{(\mu)}, \quad V''_{(\alpha)}, \quad \tilde{V}_{(0)}
\]
on \( X \). Here, \( \mu = 0,1,2,3 \), and \( \alpha = 1,2 \). For \( \tilde{V} \) in Wess-Zumino gauge, \( \tilde{V}^3 = 0 \) and its exponential \( e^{\tilde{V}} \) is simply the polynomial \( 1 + \tilde{V} + \frac{1}{2!}\tilde{V}^2 \) in \( \tilde{V} \).

Under a gauge transformation specified by a small chiral superfield \( \tilde{\Lambda} \), a small chiral superfield \( \tilde{\Phi} \in C^\infty(\tilde{X}\tilde{\bar{M}})^{small,ch} \) transforms as
\[
\tilde{\Phi} \rightarrow e^{\sqrt{-1}\tilde{\Lambda}}\tilde{\Phi}
\]
while \( \tilde{\Phi}^\dagger \in C^\infty(\tilde{X}\tilde{\bar{M}})^{small,ach} \) transforms as
\[
\tilde{\Phi}^\dagger \rightarrow \tilde{\Phi}^\dagger e^{-\sqrt{-1}\tilde{\Lambda}^\dagger}.
\]

It follows that

**Lemma/Definition 4.4.2. [gauge-invariant kinetic term for small chiral superfield]** Let
\( \tilde{V} \in C^\infty(\tilde{X}\tilde{\bar{M}}) \) be a vector superfield and \( \tilde{\Phi} \in C^\infty(\tilde{X}\tilde{\bar{M}})^{small,ch} \) be a small chiral superfield on \( X \). Then the product
\[
\tilde{\Phi}^\dagger e^{\tilde{V}} \tilde{\Phi}
\]
is gauge-invariant. Since the expression of the product in \( (x,\theta,\bar{\theta},\vartheta,\bar{\vartheta}) \) involves space-time derivatives \( (\partial_{\mu}, \mu = 0,1,2,3) \) of components of \( \tilde{\Phi} \), this product is called the gauge-invariant kinetic term for the small chiral superfield \( \tilde{\Phi} \).

**Proof.** By construction, under the gauge transformation specified by a small chiral superfield \( \tilde{\Lambda} \),
\[
\tilde{\Phi}^\dagger e^{\tilde{V}} \tilde{\Phi} \rightarrow \left(\tilde{\Phi}^\dagger e^{-\sqrt{-1}\tilde{\Lambda}^\dagger}\right) e^{\tilde{V} - \sqrt{-1}(\tilde{\Lambda} - \tilde{\Lambda}^\dagger)} e^{\sqrt{-1}\tilde{\Lambda}} \tilde{\Phi}
\]
\[
= \tilde{\Phi}^\dagger e^{-\sqrt{-1}\tilde{\Lambda}^\dagger + \tilde{V} - \sqrt{-1}(\tilde{\Lambda} - \tilde{\Lambda}^\dagger) + \sqrt{-1}\tilde{\Lambda}} \tilde{\Phi} = \tilde{\Phi}^\dagger e^{\tilde{V}} \tilde{\Phi}.
\]

\( \square \)
Note that, for $\Phi$ a small chiral superfield and $V$ a vector superfield on $X$, $W_\alpha$ and $\bar{W}_\beta$ in general are no longer tame while $\tilde{\Phi}^1 e^V \tilde{\Phi}$ always lies in $C^\infty(\tilde{X})$ medium.

Explicit computations/formulae for $\tilde{V}$ in Wess-Zumino gauge

Let

$$\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} V_{[\nu]}(x) + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \tilde{V}_{(\beta)}'(x) + \sum_{\alpha} \theta^0 \bar{\theta}^1 \bar{\theta}^2 \theta_{\alpha} V''_{(\alpha)}(x) + \theta^1 \bar{\theta}^2 \bar{\theta}^2 V_{(0)}(x),$$

be a vector superfield in Wess-Zumino gauge, in the standard coordinate functions $(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\tilde{X}$. Recall the chiral coordinate functions $(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ and the antichiral coordinate functions $(x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\tilde{X}$, where

$$x'^\mu := x^\mu + \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta \quad \text{and} \quad x''^\mu := x^\mu - \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta.$$

It is convenient to compute $W_\alpha$ and $W_1 W_2$ in the chiral coordinates $(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ and $\bar{W}_\beta$ and $\bar{W}_2 \bar{W}_1$ in the antichiral coordinates $(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$.

- $W_\alpha$: (in chiral coordinate functions $(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\tilde{X}$)

Through the $C^\infty$-hull structure of $C^\infty(\tilde{X})$, one can express $\tilde{V}$ in terms of the chiral coordinate functions $(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\tilde{X}$ as

$$\tilde{V} = \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha \beta} V_{[\nu]}(x') + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \tilde{V}_{(\beta)}'(x') + \sum_{\alpha} \theta^0 \bar{\theta}^1 \bar{\theta}^2 \theta_{\alpha} V''_{(\alpha)}(x') + \theta^1 \bar{\theta}^2 \bar{\theta}^2 \left( V_{(0)}'(x') - 2 \sqrt{-1} \sum_{\mu,\nu} \eta^{\mu \nu} \partial_{\mu} V_{[\nu]}(x') \right).$$

Recall that $e_{\alpha} x'^\mu = 2 \sqrt{-1} \sum_{\beta} \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta$. Then, a straightforward computation gives

$$e_{\alpha'} x''^\mu = e_{\alpha'} x' = 0,$$

$$W_\alpha := e_{2\nu} e_{\alpha'} (e_{\alpha'} \tilde{V}) = (-1)^2 \left( \frac{\partial}{\partial \theta^2} \frac{\partial}{\partial \bar{\theta}^1} \left( (e_{\alpha'} \tilde{V})(x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta}) \right) \right) = \theta^\alpha V''_{(\alpha)}(x') + \sum_{\gamma} \theta^\gamma \left( 2 \sqrt{-1} \sum_{\delta,\beta,\mu,\nu} \epsilon^{\delta \beta \gamma} \sigma^\nu_{\delta \beta} \partial_{\mu} V_{[\nu]}(x') - 2 \sqrt{-1} \sum_{\mu,\nu} \eta^{\mu \nu} \partial_{\mu} V_{[\nu]}(x') \right).$$

Since $e_{1\nu} x'^\mu = e_{2\nu} x' = 0$, $W_\alpha = 0$.

Applying the family of identities from raising or lowering the spinor index in the defining identity of the Dirac/Pauli matrices

$$\left( \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu \right)_{\alpha}^\gamma = -2 \eta^{\mu \nu} \delta_{\alpha}^\gamma$$

52
and a relabelling of the $\mu$, $\nu$ indices to the summation of terms of involving $\partial_{\mu}V_{(\nu)}(x')$, one has the simplification in the end

$$W_\alpha = \partial_\alpha V''_{(\alpha)}(x') - \sum_\gamma \theta^\gamma \left\{ \sqrt{-1} \sum_{\mu,\nu,\beta,\delta} \varepsilon^{\beta\delta} \sigma^\mu_{\alpha\beta} \sigma^\nu_{\gamma\delta} F_{\mu\nu}(x') + \varepsilon_{\alpha\gamma} V_{(0)}(x') \right\}$$

$$- 2 \sqrt{-1} \theta^1 \theta^2 \sum_{\delta,\beta,\mu} \partial_{\delta} \varepsilon^{\beta\delta} \sigma^\mu_{\alpha\beta} \partial_\mu V''_{(\delta)}(x'),$$

where $F_{\mu\nu} := \partial_\mu V_{(\nu)} - \partial_\nu V_{(\mu)}$.

- $W_1 W_2$: (in chiral coordinate functions $(x', \theta, \tilde{\theta}, \vartheta, \tilde{\vartheta})$ on $\hat{X}$)

From the expression of $W_\alpha$ above,

$$W_1 W_2 = \partial_1 \partial_2 V'_{(1)}(x')V''_{(2)}(x')$$

$$+ \sum_\gamma \varepsilon^{\gamma} \left\{ \partial_1 V''_{(1)}(x') \left( \sqrt{-1} \sum_{\mu,\nu,\beta,\delta} \varepsilon^{\beta\delta} \sigma^\mu_{\gamma\beta} \sigma^\nu_{\gamma\delta} F_{\mu\nu}(x') + \varepsilon_{\gamma}(x') \right)$$

$$- \partial_2 V''_{(2)}(x') \left( \sqrt{-1} \sum_{\mu,\nu,\beta,\delta} \varepsilon^{\beta\delta} \sigma^\mu_{\gamma\beta} \sigma^\nu_{\gamma\delta} F_{\mu\nu}(x') + \varepsilon_{1\gamma} V_{(0)}(x') \right) \right\}$$

$$+ \theta^1 \theta^2 \left\{ 2 \sqrt{-1} \left( \partial_1 \sum_{\beta,\delta,\mu} \partial_{\beta} \varepsilon^{\beta\delta} \sigma^\mu_{2\beta,\delta} \partial_\mu V''_{(\delta)}(x') V'_{(1)}(x') \right.$$

$$- \partial_1 \sum_{\beta,\delta,\mu} \partial_{\beta} \varepsilon^{\beta\delta} \sigma^\mu_{1\beta,\delta} \partial_\mu V''_{(\delta)}(x') V'_{(2)}(x') \right\}$$

$$+ \sum_\gamma \varepsilon^{\gamma} \left( \sqrt{-1} \sum_{\mu,\nu,\gamma,\delta} \varepsilon^{\gamma\delta} \sigma^\mu_{\gamma\delta} \sigma^\nu_{\gamma\delta} F_{\mu\nu}(x') + \varepsilon_{1\gamma} V_{(0)}(x') \right)$$

$$\left\{ \sqrt{-1} \sum_{\mu',\nu',\gamma',\delta'} \varepsilon^{\gamma'\delta'} \sigma^\mu_{\gamma'\delta'} \sigma^\nu_{\gamma'\delta'} F_{\mu'\nu'}(x') + \varepsilon_{2\gamma'} V_{(0)}(x') \right\} \right\}. \right.$$
In summary,
\[
W_1W_2 = \partial_1 \partial_2 V_{(1)}(x') V_{(2)}''(x') + \sum_\gamma \theta^\gamma \left\{ \partial_1 V_{(1)}''(x') \left( \sqrt{-1} \sum_{\mu, \nu, \beta, \delta} \varepsilon^{\beta \delta} \sigma_{2 \beta}^\mu \sigma_{\gamma \delta}^\nu F_{\mu \nu}(x') + \varepsilon_{2 \gamma}(x') \right) \\
- \partial_2 V_{(2)}''(x') \left( \sqrt{-1} \sum_{\mu, \nu, \beta, \delta} \varepsilon^{\beta \delta} \sigma_{1 \beta}^\mu \sigma_{\gamma \delta}^\nu F_{\mu \nu}(x') + \varepsilon_{1 \gamma}(x') \right) \right\}
\]
\[
+ \theta^1 \theta^2 \left\{ V_{(0)}'(x')^2 - 2 \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta_{\alpha} \tilde{\theta}_{\beta} \sigma_{\alpha \beta}^\mu V''_{(\alpha)}(x') \partial_\mu V''_{(\beta)}(x')
- 2 \sum_{\mu, \nu, \mu', \nu'} \eta_{\mu \nu} \eta_{\mu' \nu'} F_{\mu \nu}(x') F_{\mu' \nu'}(x') + \sqrt{-1} \sum_{\mu, \nu, \mu', \nu'} \varepsilon_{\mu \nu \mu' \nu'} F_{\mu \nu}(x') F_{\mu' \nu'} \right\}.
\]

The substitutions
\[
\theta \theta \rightsquigarrow -2\theta^1 \theta^2, \quad m, n, k, l \rightsquigarrow \mu, \nu, \mu', \nu', \quad v_n \rightsquigarrow -V_{[\mu]}, \quad \lambda_n \rightsquigarrow \sqrt{-1} \partial_\alpha V''_{(\alpha)},
\]
plus an appropriate raising-and-lowering of the spinor and the space-time indices turn [Wess & Bagger: Eq. (6.13)] into the expression for the \(\theta^1 \theta^2\)-component above up to a factor of \(-\frac{1}{2}\):
\[
W^\alpha W_\alpha \quad \text{in [Wess & Bagger: Eq. (6.13) via Eq. (6.7)]}
\]
\[
= (\cdots) + \theta \theta \left( -2 \sqrt{-1} \partial_{\alpha \beta} \partial_{\mu \nu} x - \frac{1}{2} v_{mn} + D^2 + \frac{\sqrt{-1}}{2} v^{mn} v^{kl} \varepsilon_{mnkl} \right)
\]
\[
\rightsquigarrow (\cdots) - \frac{1}{2} \cdot \theta^1 \theta^2 \text{(the \(\theta^1 \theta^2\)-component computed above)}.
\]

\[
\tilde{W}_\beta \quad \text{and} \quad \tilde{W}_2 \tilde{W}_1: \quad \text{(in antichiral coordinate functions \((x'', \theta, \tilde{\theta}, \tilde{\vartheta}, \tilde{\vartheta})\) on \(\tilde{X}^{\tilde{\alpha}}\))}
\]
The formulae for \(\tilde{W}_\beta\) and \(\tilde{W}_2 \tilde{W}_1\) follow either by similar computations or by taking the twisted complex conjugate on \(W_\alpha\) and \(W_1 W_2\) respectively. The results are listed below.

In terms of the antichiral coordinate functions \((x'', \theta, \tilde{\theta}, \tilde{\vartheta}, \tilde{\vartheta})\) on \(\tilde{X}^{\tilde{\alpha}}\),
\[
\tilde{V} = \sum_{\alpha, \beta} \theta^\alpha \tilde{\theta}^\beta \sum_\mu \sigma_{\alpha \beta}^\mu V_{[\mu]}(x'') + \sum_\beta \theta^1 \theta^2 \tilde{\theta}^\beta \tilde{\theta}^\gamma \tilde{V}_{(\beta)}'(x'') + \sum_\alpha \theta^\alpha \tilde{\theta}^1 \tilde{\theta}^\gamma \partial_\alpha V''_{(\alpha)}(x'')
+ \theta^1 \theta^2 \tilde{\theta}^1 \tilde{\theta}^2 \left( V_{(0)}'(x'') + 2 \sqrt{-1} \sum_{\mu, \nu} \eta_{\mu \nu} \partial_\nu V_{[\mu]}(x'') \right);
\]
\[
\epsilon_{\gamma \nu} \tilde{V} = \sum_{\gamma, \nu} \theta^\gamma \sigma_{\gamma \beta}^\nu V_{[\nu]}(x'') - \theta^1 \theta^2 \tilde{\theta}^\beta \tilde{\theta}^\gamma \tilde{V}_{(\beta)}'(x'') + \sum_\gamma \theta^\alpha \tilde{\theta}^1 \tilde{\theta}^\delta \varepsilon_{\delta \beta} V''_{(\gamma)}(x'')
- \sum_\delta \theta^1 \theta^2 \tilde{\theta}^1 \tilde{\theta}^2 \left\{ 2 \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \varepsilon^{\alpha \gamma} \sigma_{\alpha \beta}^\mu \varepsilon_{\gamma \delta} \partial_\mu V_{[\nu]}(x'') + \varepsilon_{\delta \beta} \left( V_{(0)}''(x'') + 2 \sqrt{-1} \sum_{\mu, \nu} \eta_{\mu \nu} \partial_\mu V_{[\nu]}(x'') \right) \right\}
+ 2 \sqrt{-1} \theta^1 \theta^2 \tilde{\theta}^1 \tilde{\theta}^2 \sum_{\gamma, \nu} \theta^\gamma \sigma_{\gamma \beta}^\nu \partial_\nu V''_{(\gamma)}(x'');
\]
\[
\tilde{W}_\beta := \epsilon_{\nu} e_2 e_{\beta \nu} \tilde{V}
= \tilde{\theta}^\beta \tilde{V}_{(\beta)}'(x'') + \sum_\beta \tilde{\theta}^\beta \left\{ \varepsilon_{\delta \beta} V_{(0)}''(x'') + \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \varepsilon^{\alpha \gamma} \sigma_{\alpha \beta}^\mu \varepsilon_{\gamma \delta} F_{\mu \nu}(x'') \right\}
- \sqrt{-1} \theta^1 \theta^2 \tilde{\theta}^1 \tilde{\theta}^2 \sum_{\gamma, \nu} \theta^\gamma \sigma_{\gamma \beta}^\nu \partial_\nu V''_{(\gamma)}(x''),
\]

54
where $F_{\mu\nu} := \partial_\mu V_{[\nu]} - \partial_\nu V_{[\mu]}$; and

$$\bar{W}_2 \bar{W}_1 = - \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_3 \bar{\vartheta}_4 \{ \bar{\vartheta}_1 \bar{V}_2 (x'') - \bar{\vartheta}_1 \bar{V}_1 (x') + \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \epsilon^{\alpha \gamma} \sigma_{\alpha \gamma} \bar{\vartheta}_2 \partial_\alpha F_{\mu \nu} (x'') \}
- \bar{\vartheta}_1 \bar{V}_0 (x'') + \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \epsilon^{\alpha \gamma} \sigma_{\alpha \gamma} \bar{\vartheta}_1 \partial_\alpha F_{\mu \nu} (x'') \}
+ \bar{\vartheta}_1 \bar{\vartheta}_2 \{ \bar{V}_0 (x'') - \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \epsilon^{\alpha \gamma} \sigma_{\alpha \gamma} \bar{\vartheta}_2 \partial_\alpha F_{\mu \nu} (x'') \}
+ 2 \sum_{\mu, \nu, \mu', \nu'} \eta^{\mu \nu} \eta^{\mu' \nu'} \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_3 \bar{\vartheta}_4 \eta^{\mu \nu} \bar{V}_{\mu' \nu'}. \quad (3)$$

where $\epsilon^{\mu \nu \nu'}$ is the space-time volume-element tensor with respect to the frame $(\partial_\mu)_\alpha$ with $\epsilon^{0123} = 1$.

- $\bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_3 \bar{\vartheta}_4$: (in standard coordinate functions $(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$ on $\bar{X}^{\mathbb{R}}$)

Let

$$\bar{\Phi} = \Phi (0) + \sum_{\alpha} \theta^\alpha \partial_\alpha \Phi (0) + \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \Phi (12) + \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\vartheta}_2 \partial_\alpha \sigma_{\alpha \beta} \partial_{\mu} \Phi (0)$$

be a chiral superfield on $X^{\text{physics}}$, determined by $(\Phi (0), (\Phi (0))_{\alpha}, (\Phi (12))_{\alpha})$ and

$$\bar{\Phi}^\dagger = \bar{\Phi} (0) - \sum_{\beta} \bar{\vartheta}_1 \bar{\vartheta}_2 \Phi (\beta) + \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{\vartheta}_1 \bar{\vartheta}_2 \Phi (12) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \bar{\vartheta}_2 \partial_\alpha \sigma_{\alpha \beta} \partial_{\mu} \Phi (0)$$

be its twisted complex conjugate, which is antichiral. Note also that, for the vector superfield $\bar{V}$ in the Wess-Zumino gauge,

$$\bar{V}^2 = 2 \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \sum_{\mu, \nu} \eta^{\mu \nu} \bar{V}_{\mu} \bar{V}_{\nu}, \quad \bar{V}^3 = 0,$$

and hence the exponential $e^{\bar{V}}$ is given by

$$e^{\bar{V}} = 1 + \bar{V} + \frac{1}{2} \bar{V}^2$$

A straightforward computation gives

$$e^{\bar{V}} \bar{\Phi} = \Phi (0) + \sum_{\gamma} \theta^\gamma \partial_\gamma \Phi (0) + \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \Phi (12) + \sqrt{-1} \sum_{\gamma, \beta, \mu} \theta^\gamma \bar{\vartheta}_2 \partial_\gamma \sigma_{\gamma \beta} \partial_{\mu} \Phi (0)$$

$$+ \sum_{\beta} \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \{ \sum_{\gamma, \mu} \theta^\gamma \sigma_{\gamma \beta} \partial_{\mu} \Phi (0) + \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{V}_0 (0) \} + \sum_{\gamma} \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{V}_0 (\gamma) \Phi (0)$$

$$+ \sum_{\beta} \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \{ \sum_{\mu, \nu} \eta^{\mu \nu} \bar{V}_{\mu} \bar{V}_{\nu} + \sqrt{-1} \sum_{\mu, \nu} \eta^{\mu \nu} \bar{V}_{\mu} \partial_{\nu} \Phi (0) \}$$

$$- \theta^1 \theta^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \bar{V}_0 (\gamma) \Phi (0) - \frac{1}{2} \bar{V}^2 \Phi (12) \Phi (12) \Phi (12) \}.$$
For comparison with [Wess & Bagger: Eq. (7.7)], replace $\hat{\beta} = \hat{\beta}$ into the expression for the $\theta^2\hat{\beta}^2\beta^2$-component, and extract some boundary terms:

\[
\Phi^i e^\bar{V} = (\text{terms of total } (\theta, \hat{\beta})\text{-degree}\leq 3) + \theta^1 \theta^2 \hat{\beta}^2 \sum_{\alpha, \beta} \partial_\alpha \hat{\beta} \left( \sum_{\mu} \theta^\mu \partial_\mu \bar{\Phi}(0) - \bar{\Phi}(0) \right) - \sqrt{-1} \sum_{\alpha, \beta} \hat{\beta} \left( \sum_{\mu} \theta^\mu \partial_\mu \bar{\Phi}(0) - \bar{\Phi}(0) \right) - \sqrt{-1} \sum_{\alpha, \beta} \hat{\beta} \left( \sum_{\mu} \theta^\mu \partial_\mu \bar{\Phi}(0) - \bar{\Phi}(0) \right)
\]

\[
-4 \Box \bar{\Phi}(0) - 2 \sqrt{-1} t \sum_{\mu, \nu} \eta^{\mu\nu} \partial_\mu \bar{\Phi}(0) V_{[\nu]} \bar{\Phi}(0)
\]

\[
+ 2 \sqrt{-1} t \bar{\Phi}(0) \sum_{\mu, \nu} \eta^{\mu\nu} V_{[\nu]} \partial_\nu \bar{\Phi}(0) + \bar{\Phi}(0) (t \bar{V} + t^2 \sum_{\mu, \nu} \eta^{\mu\nu} V_{[\nu]} V_{[\nu]}) \bar{\Phi}(0)
\]

\[
- \partial_1 \partial_2 t \bar{\Phi}(0) (V''_{(1)} \bar{\Phi}(2) + V''_{(2)} \bar{\Phi}(1))
\]

\[
+ 2 \sqrt{-1} \sum_{\alpha, \beta} \partial_\alpha \hat{\beta} \left( \sum_{\mu} \theta^\mu \partial_\mu \bar{\Phi}(0) - \bar{\Phi}(0) \right) - t \sum_{\mu} \hat{\beta} \left( \sum_{\mu} \theta^\mu \partial_\mu \bar{\Phi}(0) - \bar{\Phi}(0) \right)
\]

\[
+ \partial_1 \partial_2 t \left( \bar{\Phi}(1) \bar{V}'_{(2)} + \bar{\Phi}(2) \bar{V}'_{(1)} \right) \bar{\Phi}(0) + \partial_1 \partial_2 \bar{\Phi}(12) \bar{\Phi}(12)
\]

Then the substitutions

\[
\theta \hat{\beta} \hat{\beta} \rightarrow -4 \theta^1 \theta^2 \hat{\beta}^2, \quad n \rightarrow \mu, \quad A \rightarrow \Phi(0), \quad \psi_\alpha \rightarrow \frac{1}{\sqrt{2}} \partial_\alpha \Phi(0), \quad F \rightarrow -\frac{1}{\sqrt{2}} \partial_1 \partial_2 \Phi(12),
\]

\[
v_n \rightarrow -V_{[\nu]}, \quad \lambda_\alpha \rightarrow \frac{\sqrt{-1}}{2} \partial_\alpha V''(\alpha), \quad D \rightarrow -\frac{1}{2} \bar{V}(\bar{V}) \quad F^* \rightarrow \frac{1}{2} \partial_1 \partial_2 \bar{V}(\bar{V}), \quad \bar{\lambda}_\beta \rightarrow -\frac{\sqrt{-1}}{2} \partial_\beta \bar{V}'(\beta)
\]

turn [Wess & Bagger: Eq. (22.7)] into the expression for the $\theta^1 \theta^2 \hat{\beta}^2$-component above:
explicit formula for \( \Phi^A \) in \( \theta \bar{\partial} \bar{\partial} \)

\[
\begin{align*}
FF^* + A & \Box A^* \\
+ & \sqrt{-1} \partial_\alpha \bar{\psi} \sigma^\alpha \psi \\
+ & \frac{1}{2} v_\alpha \bar{\psi} \sigma^\alpha \psi \\
+ & \sqrt{-1} 2 t v^\alpha (A^* \partial_\alpha A - \partial_\alpha A^*) \\
- & \sqrt{-1} 2 t (A \bar{\psi} \psi - A^* \lambda \psi) \\
+ & \frac{1}{2} (tD - \frac{1}{2} t^2 v_\alpha v^\alpha) A^* A \\
\end{align*}
\]

Finally, up to boundary terms, there is another expression for the \( \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \)-component of \( \tilde{\Phi}^1 e^{tV} \tilde{\Phi} \) that is mathematically more appealing:

\[
\tilde{\Phi}^1 e^{tV} \tilde{\Phi} = (\text{terms of total } (\theta, \bar{\theta})\text{-degree } \leq 3) + \text{(space-time boundary terms)}
\]

\[
+ \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \left\{ - \sum_{\mu} \partial_\mu (\partial^\mu \Phi(0) + V_{[\mu]} \Phi(0)) + \sqrt{-1} \sum_{\alpha, \beta} \partial_\mu (\partial_\alpha \bar{\partial}_{\beta} \bar{\partial}^\mu \Phi(\alpha) V_{(\beta)}) \right. \\
+ 4 \sum_{\mu, \nu} \eta^{\mu \nu} (\partial_\mu + \frac{\sqrt{-1}}{2} tV_{[\mu]} \Phi(0)) \cdot (\partial_\nu - \frac{\sqrt{-1}}{2} tV_{[\nu]} \Phi(0)) \\
- 2 \sqrt{-1} \sum_{\alpha, \beta, \mu} \partial_\alpha \bar{\partial}_{\beta} \bar{\partial}^\mu \Phi(\alpha) \cdot (\partial_\mu - \frac{\sqrt{-1}}{2} tV_{[\mu]} \Phi(0)) \\
- \partial_1 \bar{\partial}_2 t \bar{\Phi}(0) (V_{(1)} \Phi(2) + V_{(2)} \Phi(1)) + \bar{\partial}_1 \bar{\partial}_2 t (\bar{\Phi}(1) V_{(2)} + \bar{\Phi}(2) V_{(1)}) \Phi(0) \\
\left. + \bar{\partial}_1 \bar{\partial}_2 \bar{\Phi}(12) \Phi(12) + t \Phi(0) V_{(0)} \Phi(0) \right\},
\]

where

\[
\nabla^tV \Phi(0) := (\partial_\mu - \frac{\sqrt{-1}}{2} tV_{[\mu]} \Phi(0), \quad \nabla^tV \Phi(\alpha) := (\partial_\mu - \frac{\sqrt{-1}}{2} tV_{[\mu]} \Phi(\alpha),
\]

\[
\nabla^tV \Phi(\beta) := (\partial_\mu + \frac{\sqrt{-1}}{2} tV_{[\mu]} \Phi(\beta),
\]

are the covariant derivative of the component fields along \( \partial/\partial x^\mu \) associated to the connection \( \nabla^tV \) associated to \( tV \). Note that this is consistent with Lemma 4.2.6; cf. footnote 17.

A supersymmetric action functional for \( U(1) \) gauge theory with matter on \( X \)

Now restore the electric charge \( e_m \) in the discussion. Then the gauge-invariant kinetic term for the matter chiral superfield \( \tilde{\Phi} \) becomes

\[
\tilde{\Phi}^1 e^{-mV} \tilde{\Phi}.
\]
Thus, replacing $\Lambda$ with $e_m \Lambda$ and $\bar{V}$ with $e_m \bar{V}$ in the above discussion and computations, we recover the charge $e_m$ case we well. It follows now from Theorem 2.3 that\textsuperscript{22}

\[
S(\bar{V}, \Phi) := \frac{\tau}{8} \int_{\tilde{X}} d^4x \, d\theta^2 d\bar{\theta}^2 \bar{W}_1 W_2 - \frac{\bar{\tau}}{8} \int_{\tilde{X}} d^4x \, d\bar{\theta}^2 \, d\theta^2 \, \bar{W}_2 W_1
\]

\[
- \frac{1}{4} \int_{\tilde{X}} d^4x \, d\theta^2 d\bar{\theta}^2 d\theta^2 d\bar{\theta}^2 \Phi_1 e^{e\alpha \bar{V} \Phi}
\]

\[
+ \int_{\tilde{X}} d^4x \, d\theta^2 d\bar{\theta}^2 \left( \lambda \Phi + \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3 \right) + \int d^4x \, d\bar{\theta}^2 d\theta^2 \left( \lambda \Phi + \frac{1}{2} m \Phi^2 + \frac{1}{3} \bar{g} \Phi^3 \right)
\]

gives a gauge-invariant action functional for the component fields $(\Phi_{(0)}; \Phi_{(\alpha)}; \Phi_{(12)}) \alpha = 1, 2$ of $\Phi$ (cf. small chiral matter) and $(V_{(\mu)}; V_{(\alpha)}; V_{(0)}) \mu = 0, 1, 2, 3, \alpha = 1, 2$ of $\bar{V}$ (cf. gauge field and gaugino field) on $X$ that is invariant under supersymmetries, up to boundary terms on $X$. Here,

\[
\tau := \frac{1}{g_{\text{gauge}}} - \sqrt{-1} \frac{\theta_{\text{gauge}}}{8\pi^2} \in \mathbb{C}
\]

is the complexified gauge coupling constant, $e_m \in \mathbb{R}$ is the matter charge, and $\lambda, m, g \in \mathbb{C}$ are the complex coupling constant as in the Wess-Zumino model for a small chiral superfield $\Phi$, cf. Sec. 3.

Explicitly, up to boundary terms on $X$,

\[
S(\bar{V}, \Phi) = S(V_{[\mu]}, \alpha = 1, 2, 3, V_{(\alpha)}, \alpha = 1, 2, V_{(0)}; \Phi_{(0)}, \Phi_{(\alpha)}, \alpha = 1, 2, \Phi_{(12)})
\]

\[
= \int_{X} d^4x \left\{ \frac{\tau + \bar{\tau}}{8} V_{(0)}^{\alpha 2} + \frac{\sqrt{-1}}{4} \sum_{\alpha, \beta, \mu} \partial_{\alpha} \bar{\partial}_{\beta} \bar{\sigma}^{\mu \beta \alpha} \left( - \tau V_{(0)}^{\alpha} \partial_{\mu} \bar{V}_{(\beta)} + \bar{\tau} V_{(\beta)}^{\mu} \partial_{\mu} V_{(\alpha)} \right) - \frac{\tau + \bar{\tau}}{4} \sum_{\mu, \nu} F_{\mu \nu} F_{\mu \nu} + \frac{\sqrt{-1}}{8} (\tau - \bar{\tau}) \sum_{\mu, \nu, \mu', \nu'} \epsilon^{\mu \nu \mu' \nu'} F_{\mu \nu} F_{\mu' \nu'}
\]

\[- \sum_{\mu, \nu} \eta^{\mu \nu} \left( \partial_{\mu} + \frac{\sqrt{-1}}{2} e_m V_{[\mu]} \right) \Phi_{(0)} \left( \partial_{\mu} - \frac{\sqrt{-1}}{2} e_m V_{[\mu]} \right) \Phi_{(0)}
\]

\[+ \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} \partial_{\alpha} \bar{\partial}_{\beta} \Phi_{(\beta)} \left( \partial_{\mu} - \frac{\sqrt{-1}}{2} e_m V_{[\mu]} \right) \Phi_{(\alpha)}
\]

\[+ \frac{1}{4} \partial_1 \partial_2 e_m \Phi_{(0)} \left( V_{(1)}^{\mu} \Phi_{(2)} + V_{(2)}^{\mu} \Phi_{(1)} \right) - \frac{1}{4} \partial_1 \partial_2 e_m \left( \Phi_{(1)} V_{(2)}^{\mu} + \Phi_{(2)} V_{(1)}^{\mu} \right) \Phi_{(0)}
\]

\[+ \frac{1}{4} \partial_1 \partial_2 e_m \Phi_{(0)} \left( \Phi_{(12)} - \Phi_{(1)} \Phi_{(2)} \right)
\]

\[+ \bar{\partial}_1 \bar{\partial}_2 \left( \bar{m} \left( \Phi_{(0)} \Phi_{(12)} - \Phi_{(1)} \Phi_{(2)} \right)
\]

\[+ \bar{g} \left( \Phi_{(0)}^{2} \Phi_{(12)} - 2 \Phi_{(0)} \Phi_{(1)} \Phi_{(2)} \right) + \lambda \Phi_{(12)} \right)
\]

\[+ \bar{\partial}_1 \bar{\partial}_2 \left( \bar{m} \left( \Phi_{(0)} \Phi_{(12)} - \Phi_{(1)} \Phi_{(2)} \right)
\]

\[+ \bar{g} \left( \Phi_{(0)}^{2} \Phi_{(12)} - 2 \Phi_{(0)} \Phi_{(1)} \Phi_{(2)} \right) + \lambda \Phi_{(12)} \right)
\}

After imposing the purge-evolution map

\[
P^{\mu \nu} : \bar{\partial}_1 \partial_2 \rightarrow 1, \quad \partial_{\alpha} \bar{\partial}_{\beta} \rightarrow -1, \quad \bar{\partial}_1 \bar{\partial}_2 \rightarrow -1, \quad \partial_1 \partial_2 \bar{\partial}_1 \bar{\partial}_2 \rightarrow 1.
\]

to remove the even nilpotent factors $\bar{\partial}_1 \partial_2, \partial_\alpha \bar{\partial}_{\beta}, \bar{\partial}_1 \bar{\partial}_2, \partial_1 \partial_2 \bar{\partial}_1 \bar{\partial}_2$ in the expression, the action functional

\textsuperscript{22}Note for mathematicians The coefficients are chosen to make the kinetic term of the complex scalar field $\Phi_{(0)}$ in the standard/normalized form: $\sum_{\mu} \partial_{\mu} \Phi_{(0)}^{\mu} \Phi_{(0)}$ and the kinetic term of the gauge field $V_{[\mu]}$ in the standard form $- \frac{1}{2 g_{\text{gauge}}} \text{Tr} \sum_{\mu, \nu} F_{\mu \nu} F_{\mu \nu}$ of the Yang-Mills theory with gauge coupling $g_{\text{gauge}}$. \]
becomes
\[ S(\tilde{V}, \Phi) = S(\{V_\mu, \mu = 0,1,2,3, \}; V_{(\alpha, \alpha = 1,2,3)}; \Phi(0), \Phi(\alpha), \alpha = 1,2,3 \} ) \]
\[ = \int \mathcal{A} \left\{ \frac{\tau + \bar{\tau}}{8} V_{(0)}^2 + \frac{\gamma}{4} \sum_{\alpha, \beta, \mu} g^{\mu \beta \alpha} \left( \tau V_{(0)}^{\mu \beta} \partial_\mu V_{(0)}^{\gamma \beta} - \bar{\tau} V_{(0)}^{\gamma \beta} \partial_\mu V_{(0)}^{\beta \mu} \right) \right. \]
\[ - \frac{\tau + \bar{\tau}}{8} \sum_{\mu, \nu} F^{\mu \nu} F_{\mu \nu} + \frac{\sqrt{-1}}{8} \sum_{\mu, \nu, \mu', \nu'} e^{\mu \nu \mu' \nu'} F_{\mu \nu} F_{\mu' \nu'} \]
\[ - \sum_{\mu, \nu} \eta^{\mu \nu} \left( \partial_\mu + \frac{\sqrt{-1}}{2} e_m V_{(\mu)} \right) \bar{\Phi}(0) \left( \partial_\mu - \frac{\sqrt{-1}}{2} e_m V_{(\mu)} \right) \Phi(0) \]
\[ - \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} \bar{\Phi}(\beta) \left( \partial_\mu - \frac{\sqrt{-1}}{2} e_m V_{(\mu)} \right) \Phi(\alpha) \]
\[ + \frac{1}{4} e_m \bar{\Phi}(0) \left( V_{(1)}^{\mu} \Phi(2) + V_{(2)}^{\mu} \Phi(1) \right) + \frac{1}{4} e_m \bar{\Phi}(1) \left( V_{(2)}^{\mu} + \Phi(0) V_{(1)}^{\mu} \right) \Phi(0) \]
\[ + \frac{1}{4} \Phi(12) \Phi(12) - \frac{1}{4} e_m \bar{\Phi}(0) V_{(0)}^2 \Phi(0) \]
\[ + m \left( \Phi(0) \Phi(12) - \Phi(1) \Phi(2) \right) \]
\[ + g \left( \Phi(0)^2 \Phi(12) - 2 \Phi(0) \Phi(1) \Phi(2) \right) + \lambda \Phi(12) \]
\[ + \bar{m} \left( \Phi(0) \Phi(12) - \Phi(1) \Phi(2) \right) \]
\[ + \bar{g} \left( \Phi(0)^2 \Phi(12) - 2 \Phi(0) \Phi(1) \Phi(2) \right) + \bar{\lambda} \Phi(12) \} . \]

Remark 4.4.3. [supersymmetric non-Abelian gauge theory] (Cf. [Wess & Bagger: Ch. VII, pp. 45, 46, 47].) The above construction can be generalized to the non-Abelian case. In particular, [Wess & Bagger: Eqs. (7.22), (7.24)] can be computed explicitly under the setting of Sec. 1. However, the proof of the existence of Wess-Zumino gauge is more technical. Thus, the discussion of supersymmetric non-Abelian gauge theories in the current setting deserves a separate work in its own right.

5 \quad d = 3 + 1, \quad N = 1 \quad \text{nonlinear sigma models}

(cf. [Wess & Bagger: Chapter XXII])

We reconstruct in this section

- [Wess & Bagger: Chapter XXII. Chiral models and Kähler geometry]

in the complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry setting of Sec. 1.

5.1 Smooth maps from \( \hat{X}_{\text{small}} \) to a smooth manifold \( Y \)

Basics of smooth maps from the small towered superspace \( \hat{X}_{\text{small}} \) to a smooth manifold \( Y \) from the aspect of complexified \( C^\infty \)-Algebraic Geometry are given in this subsection.

Smooth maps from the small towered superspace \( \hat{X}_{\text{small}} \) to a smooth manifold \( Y \)

Let \( Y \) be a smooth manifold, with its function ring \( C^\infty(Y) \) of smooth functions, its structure sheaf \( \mathcal{O}_Y \) of smooth functions, and their complexification \( C^\infty(Y)^{\mathbb{C}} \) and \( \mathcal{O}_Y^{\mathbb{C}} \) respectively.
Definition 5.1.1. [smooth map from small towered superspace] A smooth map $\tilde{f}$ from the small towered superspace $\hat{X}_\small$ to $Y$, in notation

$$\tilde{f} : \hat{X}_\small \to Y,$$

is a pair $(f, \hat{f})$, where $f : X \to Y$ is a smooth map of smooth manifolds and

$$\hat{f} : C^\infty(Y) \to C^\infty(\hat{X}_\small)$$

is a ring-homomorphism with its $(\theta, \bar{\theta})$-degree-zero component $\hat{f}_{(0)}$ identical to the $C^\infty$-ring-homomorphism $f^\#: C^\infty(Y) \to C^\infty(X)$ associated to $f$. As locally-ringed spaces, we may regard $\hat{X}_\small$ and $Y$ as equivalence classes of gluing systems of ring-homomorphisms and write $\hat{f} : \hat{O}_Y \to \hat{O}_X\small$; (cf. the similar setting in [L-Y1: Sec. 1] (D(1))).

The following lemma can be checked directly:

Lemma 5.1.2. [\hat{f} as C^\infty-ring-homomorphism and natural extension to C^\infty(Y)^\hat{\bigcirc}] The ring-homomorphism $\hat{f} : C^\infty(Y) \to C^\infty(\hat{X}_\small)$ is a $C^\infty$-ring-homomorphism from $C^\infty(Y)$ to the $C^\infty$-hull of $C^\infty(\hat{X}_\small)$. $\hat{f}$ extends naturally to a ring-homomorphism $C^\infty(Y)^\hat{\bigcirc} \to C^\infty(\hat{X}_\small)$ by the correspondence

$$h_1 + \sqrt{-1} h_2 \to \hat{f}(h_1) + \sqrt{-1} \hat{f}(h_2),$$

for $h_1, h_2 \in C^\infty(Y)$. We will denote this extension of $\hat{f}$ still by $\hat{f}$.

Due to the above lemma and with slight abuse of terminology, we will call $\hat{f} : C^\infty(Y) \to C^\infty(\hat{X}_\small)$ directly a $C^\infty$-ring-homomorphism. Also, depending on context, we will write $\hat{f} : C^\infty(Y)^\hat{\bigcirc} \to C^\infty(\hat{X}_\small)$ as a ring-homomorphism or $\hat{f} : \hat{O}_Y \to \hat{O}_X\small$ as an equivalence of gluing systems of ring-homomorphisms.

The components of $\hat{f}$ in the $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-expansion are $C^\infty(X)^\hat{\bigcirc}$-valued operation on $C^\infty(X)$. Their basic properties are summarized in the following proposition.

Proposition 5.1.3. [components of $\hat{f}$] Given a smooth map $\tilde{f} : \hat{X}_\small \to Y$, let

$$\hat{f} = f^\#_{(0)} + \sum_\alpha \theta^\alpha \partial_\alpha \hat{f}_{(\alpha)} + \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta \hat{f}_{(\beta)}$$

$$+ \theta^1 \theta^2 \vartheta_1 \vartheta_2 \hat{f}_{(12)}^{(12)} + \sum_{\alpha, \bar{\beta}} \theta^\alpha \bar{\theta}^{\bar{\beta}} \left( \sum_{\mu} \sigma^{\alpha}_{\mu} \hat{f}_{(\mu)}^{\alpha} + \vartheta_1 \bar{\vartheta}_2 \hat{f}_{(12\alpha)} \right)$$

$$+ \sum_\beta \bar{\theta}^1 \bar{\theta}^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \hat{f}_{(12\beta)}^{(12)} + \sum_\alpha \theta^\alpha \bar{\theta}^1 \bar{\vartheta}_1 \bar{\vartheta}_2 \hat{f}_{(12\alpha)}$$

$$+ \theta^1 \theta^2 \vartheta_1 \vartheta_2 \hat{f}_{(12)}^{(12)}$$

be the presentation of the associated $\hat{f}$ in terms of components in the $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-expansion and $D_Y^{[1,i]}$, $i \in \mathbb{Z}_{\geq 1}$ be the sheaf of differential operators on $Y$ of order $i$ with smooth coefficients and without the zero-th order term. (Note that $D_Y^{[1,1]}$ is the tangent sheaf $T_Y$ of $Y$.) Then $f^\#_{(0)} = f^\#: \hat{O}_Y \to \hat{O}_X$ is the equivalence class of gluing systems of $C^\infty$-ring-homomorphisms associated to $f : X \to Y$ underlying $\tilde{f}$;
and \( \tilde{f}^i - f^i_{(0)} \) is a smooth section of
\[
\sum_{\alpha} O_X^C \cdot \theta^\alpha \partial_\alpha \otimes f_{(0)}^* T_Y + \sum_{\beta} O_X^C \cdot \tilde{\theta}^\beta \tilde{\partial}_\beta \otimes f_{(0)}^* T_Y + O_X^C \cdot \theta^1 \theta^2 \partial_1 \partial_2 \otimes f_{(0)}^* D_Y^{[1,2]} \\
+ \sum_{\alpha, \beta} O_X^C \cdot \theta^\alpha \tilde{\theta}^\beta \left( \sum_{\mu} \sigma_{\alpha \beta}^\mu \otimes f_{(0)}^* T_Y + \partial_\alpha \tilde{\theta}_{\beta} \otimes f_{(0)}^* D_Y^{[1,2]} \right) + O_X^C \cdot \tilde{\theta}^\beta \tilde{\theta}^2 \tilde{\partial}_1 \tilde{\partial}_2 \otimes f_{(0)}^* D_Y^{[1,2]} \\
+ \sum_{\beta} O_X^C \cdot \theta^1 \theta^2 \tilde{\theta}^\beta \left( \sum_{\alpha, \mu} \sigma_{\alpha \beta}^{\mu \alpha} \otimes f_{(0)}^* D_Y^{[1,3]} + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 \otimes f_{(0)}^* D_Y^{[1,2]} \right) \\
+ \sum_{\alpha} O_X^C \cdot \theta^1 \theta^2 \tilde{\theta}^\beta \left( \sum_{\alpha, \mu} \sigma_{\alpha \beta}^{\mu \alpha} \otimes f_{(0)}^* D_Y^{[1,3]} + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 \otimes f_{(0)}^* D_Y^{[1,4]} \right)
\]
in \( \mathcal{O}_{X}^\text{small} \otimes O_X f_{(0)}^* D_Y = \mathcal{O}_{X}^\text{small} \otimes f_{(0)}^* \mathcal{O}_Y D_Y \), where \( D_Y \) is the sheaf of differential operators on \( Y \) of smooth coefficients.

In terms of \( \tilde{f}^i : O_Y \to \mathcal{O}_{X}^\text{small} \), a local chart \( V \) of \( Y \) with coordinates functions \((y^1, \cdots, y^n)\), and an open set \( U \subset X \) such that \( f_{(0)}(U) \subset V \), let
\[
\tilde{f}^i \equiv \tilde{f}^i(y^i) \\
= f^i_{(0)}(y^i) + \sum_{\alpha} \theta^\alpha \partial_\alpha f^i_{(0)}(y^i) + \tilde{\theta}^\beta \tilde{\partial}_\beta f^i_{(0)}(y^i) \\
+ \theta^1 \theta^2 \partial_1 \partial_2 f^i_{(12)}(y^i) + \sum_{\alpha, \beta} \sigma_{\alpha \beta}^\mu \otimes f^i_{(0)}(y^i) + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 f^i_{(12\beta)}(y^i)
\]
\[
+ \sum_{\alpha} O_X^C \cdot \theta^1 \theta^2 \tilde{\theta}^\beta \left( \sum_{\alpha, \mu} \sigma_{\alpha \beta}^{\mu \alpha} \otimes f^i_{(0)}(y^i) + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 f^i_{(12\beta)}(y^i) \right)
\]
\[
+ \sum_{\alpha} O_X^C \cdot \theta^1 \theta^2 \tilde{\theta}^\beta \left( \sum_{\alpha, \mu} \sigma_{\alpha \beta}^{\mu \alpha} \otimes f^i_{(0)}(y^i) + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 f^i_{(12\beta)}(y^i) \right)
\]
\[
+ \theta^1 \theta^2 \tilde{\theta}^\beta \left( f^i_{(0)}(y^i) + \sum_{\alpha, \beta, \mu} \sigma_{\alpha \beta}^{\mu \alpha} f^i_{(0)}(y^i) + \partial_\alpha \tilde{\theta}_{\beta} \tilde{\partial}_2 f^i_{(12\beta)}(y^i) \right)
\]
\[
=: f^i_{(0)}(y^i) + \tilde{n}^i \in C^\infty(\mathcal{O}_{X}^\text{small})
\]
where \( \tilde{n}_f^j \) is the nilpotent part of \( \tilde{f}^j(y') \), and denote \( \frac{\partial}{\partial y^i} \) by \( \partial_i \), for \( i = 1, \cdots, n \). Then, for \( h \in C^\infty(V) \),

\[
\tilde{f}^j(h) = f^j_{(0)}(h) + \sum_\alpha \theta^\alpha \varphi_\alpha \sum^n_{i=1} f^j_{(\alpha)}(y') \otimes \partial_i h + \sum_\beta \tilde{\theta}^\beta \tilde{\varphi}_\beta \sum^n_{j=1} f^j_{(\beta)}(y') \otimes \partial_j h
+ \theta^1 \theta^2 \cdot \tilde{\varphi}_1 \tilde{\varphi}_2 \left\{ \sum_i f^j_{(12)}(y') \otimes \partial_i + \frac{1}{2} \sum_{i,j} J^{ij}_{f,(12)} \otimes \partial_i \partial_j \right\} h
+ \sum_{\alpha, \beta} \theta^\alpha \tilde{\theta}^\beta \left\{ \sum_i f^j_{(\mu)}(y') \otimes \partial_i + \var_\alpha \tilde{\varphi}_\beta \left( \sum_i f^j_{(\alpha \beta)}(y') \otimes \partial_i + \frac{1}{2} \sum_{i,j} J^{ij}_{f,(\alpha \beta)} \otimes \partial_i \partial_j \right) \right\} h
+ \theta^1 \theta^2 \cdot \tilde{\varphi}_1 \tilde{\varphi}_2 \left\{ \sum_i f^j_{(12)}(y') \otimes \partial_i + \frac{1}{2} \sum_{i,j} J^{ij}_{f,(12)} \otimes \partial_i \partial_j \right\} h
+ \sum_{\alpha, \beta} \theta^1 \theta^2 \tilde{\theta}^\beta \left\{ \sum_{i,j} J^{ij}_{f,(\alpha \beta)} \otimes \partial_i \partial_j \right\} h
+ \theta^1 \theta^2 \tilde{\varphi}_1 \tilde{\varphi}_2 \left\{ \left( \sum_i f^j_{(0)}(y') \otimes \partial_i + \frac{1}{2} \sum_{i,j} J^{ij}_{f,(0)} \otimes \partial_i \partial_j \right) + \sum_\beta \tilde{\varphi}_\beta \left( \sum_i J^{ij}_{f,(\beta)} \otimes \partial_i \partial_j \right) \right\} h
+ \sum_{\alpha, \beta} \theta^1 \theta^2 \tilde{\varphi}_\beta \left( \sum_i J^{ij}_{(\alpha \beta)} \otimes \partial_i \partial_j \right) + \theta^1 \theta^2 \tilde{\varphi}_\beta \left( \sum_i J^{ij}_{(\alpha \beta)} \otimes \partial_i \partial_j \right) h
+ \theta^1 \theta^2 \tilde{\varphi}_\beta \left( \sum_i J^{ij}_{(\alpha \beta)} \otimes \partial_i \partial_j \right)
\]

\[\in \widehat{\mathcal{O}}_{U,small} \otimes f^j_{(0)} \mathcal{D}_V = \widehat{\mathcal{O}}_{U,small} \otimes \mathcal{O}_U f^j_{(0)} \mathcal{D}_V.\]

Here, \( J^j_{f,(\bullet)} \in C^\infty(U)^C \) are the components in the following expansions:

\[
\tilde{n}_f^j \tilde{n}_f^i = \theta^1 \theta^2 \theta^1 \theta^2 \sum_\alpha \theta^\alpha \tilde{\varphi}_\alpha J^{ij}_{f,(\alpha \beta)} + \sum_\alpha \theta^\alpha \tilde{\varphi}_\alpha J^{ij}_{f,(\alpha \beta)} + \theta^1 \theta^2 \tilde{\varphi}_1 \tilde{\varphi}_2 J^{ij}_{f,(12)}
\]

\[
\tilde{n}_f^j \tilde{n}_f^i = \sum_\alpha \theta^\alpha \tilde{\varphi}_\alpha J^{ij}_{f,(\alpha \beta)} + \theta^1 \theta^2 \tilde{\varphi}_1 \tilde{\varphi}_2 J^{ij}_{f,(12)}
\]

\[
\tilde{n}_f^j \tilde{n}_f^k \tilde{n}_f^i = \sum_\alpha \theta^\alpha J^{ij}_{f,(\alpha \beta)} + \theta^1 \theta^2 \tilde{\varphi}_1 \tilde{\varphi}_2 J^{ij}_{f,(12)}
\]

\[
\tilde{n}_f^i \tilde{n}_f^j \tilde{n}_f^k = \theta^1 \theta^2 \tilde{\varphi}_1 \tilde{\varphi}_2 J^{ij}_{f,(12)}
\]

\[62\]
which can be read off explicitly from the expansion

\[
\begin{align*}
\tilde{n}_i^\alpha \tilde{n}_i^\beta &= -\theta^i \theta^j \partial_1 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdot \left( f_{(1)}^i(y') f_{(2)}^j(y') + f_{(2)}^i(y') f_{(1)}^j(y') \right) \\
&\quad - \sum_{\alpha, \beta, \gamma} \theta^\alpha \theta^\beta \theta^\gamma \partial_1 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdot \left( f_{(1)}^\alpha(y') f_{(2)}^\beta(y') + f_{(2)}^\alpha(y') f_{(1)}^\beta(y') \right) - \partial^i \theta^j \partial_1 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdot \left( f_{(1)}^i(y') f_{(2)}^j(y') + f_{(2)}^i(y') f_{(1)}^j(y') \right) \\
&\quad + \sum_{\alpha, \beta} \theta^\alpha \theta^\beta \left\{ \sum_{\alpha, \mu} \theta_\alpha \sigma^{\mu \beta} \cdot \left( f_{(\alpha)}^\mu(y') f_{(\mu)}^i(y') + f_{(\mu)}^\mu(y') f_{(\alpha)}^i(y') \right) \\
&\quad + \partial_1 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdot \left( f_{(12)}^i(y') f_{(1)}^\alpha(y') + f_{(1)}^i(y') f_{(2)}^\alpha(y') \right) + f_{(2)}^i(y') f_{(2)}^\alpha(y') \\
&\quad + f_{(1)}^i(y') f_{(1)}^\alpha(y') + f_{(2)}^i(y') f_{(2)}^\alpha(y') + f_{(1)}^i(y') f_{(1)}^\alpha(y') \right\} \\
&\quad + \sum_{\alpha} \theta^\alpha \tilde{\alpha}_1 \tilde{\alpha}_2 \left\{ 2 \sum_{\mu, \nu} \eta^{\mu \nu} f_{(\mu)}^i(y') f_{(\nu)}^i(y') \\
&\quad + \sum_{\alpha, \beta, \gamma} \theta_\alpha \theta_\beta \tilde{\alpha}_1 \tilde{\alpha}_2 \cdot \left( - f_{(\alpha)}^\beta(y') f_{(\beta)}^\gamma(y') + f_{(\beta)}^\beta(y') f_{(\alpha)}^\gamma(y') \\
&\quad - f_{(\alpha)}^\gamma(y') f_{(\beta)}^\beta(y') + f_{(\beta)}^\gamma(y') f_{(\alpha)}^\beta(y') \\
&\quad + f_{(12)}^\beta(y') f_{(1)}^\alpha(y') + f_{(12)}^\beta(y') f_{(2)}^\alpha(y') + f_{(12)}^\beta(y') f_{(2)}^\alpha(y') \right) \right\},
\end{align*}
\]
\[ \bar{n}^i \bar{n}^j \bar{n}^k = -\sum_{\beta} \theta^i \theta^j \theta^k \partial_{\alpha} \partial_{\beta} \partial_{\alpha} \partial_{\beta} \cdot \left\{ \left( f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right) \right. \\
+ \left. f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right) + f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right\} \\
- \sum_{\alpha} \theta^i \theta^j \theta^k \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right. \\
+ \left. f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right) + f_{(3)}(y') \cdot \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right\} \\
+ \theta^i \theta^j \theta^k \cdot \left\{ \sum_{\alpha, \beta, \mu} \theta_{\alpha, \beta, \mu} \left( f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right) \\
+ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} \\
- \bar{\rho}_1 \partial_{\alpha} \bar{\rho}_2 \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right. \\
+ \left. f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} \\
+ \bar{\rho}_1 \partial_{\alpha} \bar{\rho}_2 \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right. \\
+ \left. f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} \\
+ \bar{\rho}_1 \partial_{\alpha} \bar{\rho}_2 \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right. \\
+ \left. f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} \\
+ \bar{\rho}_1 \partial_{\alpha} \bar{\rho}_2 \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right. \\
+ \left. f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} \\
+ \bar{\rho}_1 \partial_{\alpha} \bar{\rho}_2 \partial_{\alpha} \partial_{\alpha} \partial_{\alpha} \cdot \left\{ f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right. \\
+ \left. f_{(1)}(y') \cdot \left( f_{(2)}(y') f_{(3)}(y') + f_{(3)}(y') f_{(2)}(y') \right) \right\} . \\
\right\} \\
\]

\[ \bar{n}^i \bar{n}^j \bar{n}^k \bar{n}^l = \theta^i \theta^j \theta^k \theta^l \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \cdot \left\{ \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right. \\
+ \left. \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right\} + \sum_{\alpha, \gamma, \beta, \delta} \left\{ \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \left( f_{(1)}(y') f_{(2)}(y') + f_{(2)}(y') f_{(1)}(y') \right) \right\} . \\
\right\} \\
\]

\textbf{Proof.} By construction, \( f_{(0)} = f^2 \). Now let \( h_1, h_2 \in C^\infty(Y) \). Since \( \bar{f}^2 \) is a ring-homomorphism, one has

\[ \bar{f}^2(h_1 h_2) = \bar{f}^2(h_1) \bar{f}^2(h_2) . \]
The expansion of the above identity in terms of \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-components gives

\[
\begin{align*}
&f^\alpha_{(0)}(h_1 h_2) = f^\alpha_{(0)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(0)}(h_2), \\
f^\alpha_{(\beta)}(h_1 h_2) = f^\alpha_{(\beta)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(\beta)}(h_2), \\
f^\alpha_{(12)}(h_1 h_2) = f^\alpha_{(12)}(h_1) f^\alpha_{(0)}(h_2) - f^\alpha_{(1)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(2)}(h_1) f^\alpha_{(1)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(12)}(h_2), \\
f^\alpha_{(12\beta)}(h_1 h_2) = f^\alpha_{(12\beta)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(12)}(h_1) f^\alpha_{(\beta)}(h_2) + f^\alpha_{(1\beta)}(h_1) f^\alpha_{(2)}(h_2) + f^\alpha_{(2\beta)}(h_1) f^\alpha_{(1)}(h_2) \\
&\quad + f^\alpha_{(1)}(h_1) f^\alpha_{(2\beta)}(h_2) + f^\alpha_{(2)}(h_1) f^\alpha_{(1\beta)}(h_2) + f^\alpha_{(\beta)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(12\beta)}(h_2), \\
f^\alpha_{(0\beta)}(h_1 h_2) = f^\alpha_{(0\beta)}(h_1) f^\alpha_{(0)}(h_2) - f^\alpha_{(\beta)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(0\beta)}(h_2), \\
f^\alpha_{(012)}(h_1 h_2) = f^\alpha_{(012)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(012)}(h_1) f^\alpha_{(0)}(h_2) + f^\alpha_{(012)}(h_1) f^\alpha_{(0)}(h_2) \\
&\quad + f^\alpha_{(01)}(h_1) f^\alpha_{(02)}(h_2) + f^\alpha_{(02)}(h_1) f^\alpha_{(01)}(h_2) + f^\alpha_{(0\beta)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(0)}(h_1) f^\alpha_{(012\beta)}(h_2), \\
f^\alpha_{(1212)}(h_1 h_2) = f^\alpha_{(1212)}(h_1) f^\alpha_{(0)}(h_2) - f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) \\
&\quad - f^\alpha_{(212)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(212)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) + f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) \\
&\quad + f^\alpha_{(21)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(21)}(h_1) f^\alpha_{(12)}(h_2) + f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2), \\
&\quad - f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) - f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) - f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) + f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) \\
&\quad + f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) + f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2) + f^\alpha_{(12)}(h_1) f^\alpha_{(2)}(h_2) - f^\alpha_{(21)}(h_1) f^\alpha_{(1212)}(h_2). 
\end{align*}
\]

This shows that

- \(f^\alpha_{(\alpha)}, f^\alpha_{(\beta)}, \text{and } f^\alpha_{([\alpha])}\) satisfy the Leibniz rule and, hence, are \(C^\infty(X)^C\)-valued derivations on \(C^\infty(Y)\);

other \(f^\alpha_{(\alpha)}\) satisfy higher-order Leibniz rules and, hence, are \(C^\infty(X)^C\)-valued differential operators on \(C^\infty(Y)\): Inductively and recursively,

- \(f^\alpha_{(12)}, f^\alpha_{(\alpha\beta)}, f^\alpha_{([\alpha])}, f^\alpha_{([\alpha])}, f^\alpha_{(0)}\) are \(C^\infty(X)^C\)-valued second-order differential operators on \(C^\infty(Y)\);

- \(f^\alpha_{(12\beta)}, f^\alpha_{(012)}, \text{and } f^\alpha_{(0)}\) are \(C^\infty(X)^C\)-valued third-order differential operators on \(C^\infty(Y)\);

- \(f^\alpha_{(1212)}\) is a \(C^\infty(X)^C\)-valued fourth-order differential operator on \(C^\infty(Y)\).

Locally and explicitly, under the setting of the statement of the proposition, it follows from the \(C^\infty\)-hull structure of \(\tilde{\mathcal{U}}^\infty_{\text{small}}\) that

\[
\begin{align*}
f^\alpha(h) &= h(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) = h(f^\alpha_{(0)}(y_1) + \tilde{n}_f^1, \ldots, f^\alpha_{(0)}(y^n) + \tilde{n}_f^n) \\
&= h(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) + \sum_{i}(\partial_i h)(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) \cdot \tilde{n}_f^i \\
&\quad + \frac{1}{2} \sum_{i,j}(\partial_i \partial_j h)(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) \cdot \tilde{n}_f^i \tilde{n}_f^j \\
&\quad + \frac{1}{3} \sum_{i,j,k}(\partial_i \partial_j \partial_k h)(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) \cdot \tilde{n}_f^i \tilde{n}_f^j \tilde{n}_f^k \\
&\quad + \frac{1}{4} \sum_{i,j,k,l}(\partial_i \partial_j \partial_k \partial_l h)(f^\alpha_{(0)}(y_1), \ldots, f^\alpha_{(0)}(y^n)) \cdot \tilde{n}_f^i \tilde{n}_f^j \tilde{n}_f^k \tilde{n}_f^l.
\end{align*}
\]
After plugging in the \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-expansion of \(n_i^j\)'s and then simplifying the resulting expression, one obtains the explicit expression of \(f_j^i\) as differential operators of order as stated in the proposition and without the degree-zero term. This proves the proposition.

\[\square\]

### 5.2 Chiral maps from \(\hat{X}_{\hat{\mathbb{H}}, \text{small}}\) to a complex manifold \(Y\)

In this subsection we study smooth maps from the small towered superspace \(\hat{X}_{\hat{\mathbb{H}}, \text{small}}\) to a complex manifold \(Y\) and introduce the notion of `chiral maps'.

**Smooth functions on a complex manifold**

Let \(Y\) be a complex manifold of complex dimension \(n\). As a smooth real \(2n\)-manifold, its function-ring \(C^\infty(Y)\) is a \(C^\infty\)-ring. In this digression, we review how this structure is rephrased in terms of complex coordinates on \(Y\); cf. [L-Y4: Sec. 4] (D(14.1)).

**Definition 5.2.1. [smooth function in complex coordinates]** For a local coordinate chart \(V \subset Y\), the \(C^\infty\)-ring structure of \(C^\infty(V)\) in terms of the complex coordinate functions

\[(z^1, \ldots, z^n) = (y^1 + \sqrt{-1}y^2, \ldots, y^{2n-1} + \sqrt{-1}y^{2n})\]

We will call \((y^1, y^2, \ldots, y^{2n-1}, y^{2n})\) the real coordinates functions on \(V\) associated to the complex coordinate functions \((z^1, \ldots, z^n)\). Let \(h = h(y^1, \ldots, y^{2n}) \in C^\infty(V)\). Then, define

\[h_C := h_C(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n) := h(y^1, \ldots, y^{2n})\]

and call it the presentation of \(h \in C^\infty(V)\) in complex coordinate functions \((z^1, \ldots, z^n)\) on \(V\). Define

\[
\frac{\partial}{\partial z^i} := \frac{1}{2} \left( \frac{\partial}{\partial y^{2i-1}} - \sqrt{-1} \frac{\partial}{\partial z^{2i}} \right), \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2} \left( \frac{\partial}{\partial y^{2i-1}} + \sqrt{-1} \frac{\partial}{\partial z^{2i}} \right)
\]

and

\[
\frac{\partial}{\partial \bar{z}^i} h_C := \frac{1}{2} \left( \frac{\partial}{\partial y^{2i-1}} - \sqrt{-1} \frac{\partial}{\partial z^{2i}} \right) h, \quad \frac{\partial}{\partial \bar{z}^i} h_C := \frac{1}{2} \left( \frac{\partial}{\partial y^{2i-1}} + \sqrt{-1} \frac{\partial}{\partial z^{2i}} \right) h \in C^\infty(V)^C.
\]

**Lemma 5.2.2. [Taylor’s formula in complex coordinates]** Denote coordinate functions on \(V\) collectively by \(y := (y^1, \ldots, y^{2n})\), \(z := (z^1, \ldots, z^n)\), and \(\bar{z} := (\bar{z}^1, \ldots, \bar{z}^n)\). Then, for \(h \in C^\infty(V)\) and \(q \in V\) of coordinates \(y, z\), and \(a := (a^1, \ldots, a^{2n}) \in \mathbb{R}^{2n}\) such that points \(q_t\) of real coordinates \(y_t := y + t \cdot a\) are contained in \(V\) for all \(t \in [0, 1]\), the Taylor’s formula

\[h(y + a) = \frac{t}{d!} \sum_{l=0}^{d} \frac{1}{d!} \frac{\partial h}{\partial y^d}(y) a^d + \sum_{l=1}^{d} \frac{1}{d!} \frac{\partial^{l+1} h}{\partial y^d y^d}(y_{t_0}) a^d\]

for some \(t_0 \in [0, 1]\) depending on \(a\) has the following form in complex coordinates

\[h_C(z + u, \bar{z} + \bar{u}) = \frac{1}{d!} \sum_{l=0}^{d} \frac{1}{d!} \frac{\partial h_C}{\partial z^d z^d}(z, \bar{z}) u^d \bar{u}^d + \sum_{l=1}^{d} \frac{1}{d!} \frac{\partial^{l+1} h_C}{\partial z^d z^d}(z_{t_0}, \bar{z}_{t_0}) u^d \bar{u}^d\]

for some \(t_0 \in [0, 1]\) depending on \(u\). Here, \(d := (d_1, \ldots, d_{2n}) \) with \(d_i \in \mathbb{Z}_{\geq 0}\), \(|d| := d_1 + \cdots + d_{2n}\), \(d! := d_1! \cdots d_{2n}! \) with \(0! := 1\), \(\partial^{d} := (\partial/\partial y^1)^{d_1} \cdots (\partial/\partial y^{2n})^{d_{2n}}\) for \(|d| = d\), \(a^d := (a_1^1 \cdots a_n^{2n})\) and similarly \(u := (u_1, \ldots, u^n) \in \mathbb{C}^n\) such that points \(q_t\) of complex coordinates \(z_t := z + t \cdot u\) are contained in \(V\) for all \(t \in [0, 1]\), \(d_i = (d_{i,1}, \ldots, d_{i,n})\), \(i = 1, 2\), with \(d_{i,j} \in \mathbb{Z}_{\geq 0}\), \(|d_i| := d_{i,1} + \cdots + d_{i,n}\), \(d_i! := d_{i,1}! \cdots d_{i,n}!\), \(\partial^d/(\partial z_1 \cdots \partial z_{2n}) := (\partial/\partial z_1)^{d_{1,1}} \cdots (\partial/\partial z_n)^{d_{1,n}} \cdots (\partial/\partial \bar{z}_1)^{d_{2,1}} \cdots (\partial/\partial \bar{z}_{2n})^{d_{2,n}}\) for \(|d_1| + |d_2| = d\), \(u^d := (u_1)_{d_{1,1}} \cdots (u_{1,n})_{d_{1,n}}\), \(\bar{u}^d := (\bar{u})_{d_{2,1}} \cdots (\bar{u}_{2,n})_{d_{2,n}}\).
Chiral and antichiral maps from $\hat{X}^{\hat{\mathbb{R}},\text{small}}$ to a complex manifold $Y$

**Definition 5.2.3. [chiral/antichiral map]** Let $\hat{f} : \hat{X}^{\hat{\mathbb{R}},\text{small}} \to Y$ be a smooth map and

$$\bar{f}^\sharp : O^C_Y \to \hat{O}^{\hat{\mathbb{R}},\text{small}}_X$$

be the associated equivalence class of gluing systems of ring-homomorphisms. Then $\bar{f}$ is called *chiral* if it satisfies (1) $\bar{f}^\sharp(\hat{h}) = \bar{f}^\sharp(h)$ and (2) $\bar{f}^\sharp(O^C_Y,\text{hol}) \subset \hat{O}^{\hat{\mathbb{R}},\text{small},\text{ch}}_X$.

Similarly, $\bar{f}$ is called *antichiral* if it satisfies (1) $\bar{f}^\sharp(\hat{h}) = \bar{f}^\sharp(h)$ and (2) $\bar{f}^\sharp(O^C_Y,\text{hol}) \subset \hat{O}^{\hat{\mathbb{R}},\text{small},\text{ach}}_X$.

Note that if $\bar{f} : \hat{X}^{\hat{\mathbb{R}},\text{small}} \to Y$ is chiral (resp. antichiral) then $\bar{f}^\sharp(O^C_Y,\text{hol}) \subset \hat{O}^{\hat{\mathbb{R}},\text{small},\text{ach}}_X$. (resp. $\bar{f}^\sharp(O^C_Y,\text{hol}) \subset \hat{O}^{\hat{\mathbb{R}},\text{small},\text{ch}}_X$.)

**Remark 5.2.4. [naturality of chiral/antichiral map]** Recall that the most natural class of maps from a complex manifold to another complex manifold is the class of holomorphic maps or antiholomorphic maps. One should think the same for chiral or antichiral maps from $\hat{X}^{\hat{\mathbb{R}},\text{small}}$ to a complex manifold.

**Local presentation of the components of $\bar{f}^\sharp$ that defines a chiral map $\bar{f}$**

Proposition 5.1.3 can be adapted for a chiral map from $\hat{X}^{\hat{\mathbb{R}},\text{small}}$ to a complex manifold $Y$. For the simplicity of presentation, we let $Y = \mathbb{C}^n$ in a single complex coordinate chart and work out the explicit form of the action functional for chiral maps to a complex manifold $\bar{f} : \hat{X}^{\hat{\mathbb{R}},\text{small}} \to Y$ specified by the independent components $(f^i_{(0)}, f^i_{(\alpha)}, f^i_{(12)})$ of the associated $\bar{f}^\sharp : C^\infty(Y)^\mathbb{C} \to C^\infty(\hat{X}^{\hat{\mathbb{R}}})$.

(a) $h \in C^\infty(Y)^\mathbb{C}$

Continuing the discussion and the notations in the previous theme. Let

$$h_\mathbb{C}(z^1, \cdots, z^n, z^1, \cdots, z^n) := h(y^1, y^2, \cdots, y^{2n-1}, y^{2n}) \in C^\infty(Y)$$

be a Kähler potential of the Kähler metric on $Y = \mathbb{C}^n$, expressed in terms of the complex coordinates on $\mathbb{C}$, — here, for simplicity, we assume that the Kähler metric on $\mathbb{C}^n$ admits a Kähler potential that is defined on all $\mathbb{C}^n$ — and

$$\bar{f}^i := \bar{f}^\sharp(z^i)$$

$$= f^i_{(0)}(x) + \sum_\alpha \theta^\alpha \partial_\alpha f^i_{(\alpha)}(x) + \sqrt{-1} \sum_{\alpha,\beta,\mu} \theta^\alpha \bar{\theta}^\beta \sigma^\mu_{\alpha\beta} \partial_\mu f^i_{(0)}(x) + \theta^1 \theta^2 \partial_1 \partial_2 f^i_{(12)}(x)$$

$$+ \sqrt{-1} \sum_{\beta,\alpha,\mu} \theta^1 \theta^2 \bar{\theta}^\beta \partial_\alpha \sigma^\mu_{\beta\alpha} \partial_\mu f^i_{(0)}(x) - \theta^1 \theta^2 \bar{\theta}^\beta \partial_\beta \square f^i_{(0)}(x)$$

$$=: f^i_{(0)}(x) + \bar{h}^i_\bar{f}$$

$$\bar{f}^{\bar{i}} := \bar{f}^\sharp(\bar{z}^i)$$

$$= \bar{f}^{\bar{i}}_{(0)}(x) - \sum_\beta \bar{\theta}^\beta \bar{\theta}^\beta \bar{f}^{\bar{i}}_{(0)}(x) - \sqrt{-1} \sum_{\alpha,\beta,\mu} \bar{\theta}^\beta \bar{\theta}^\beta \partial_\mu \bar{f}^{\bar{i}}_{(0)}(x) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 \bar{f}^{\bar{i}}_{(12)}(x)$$

$$- \sqrt{-1} \sum_{\alpha,\beta,\mu} \bar{\theta}^1 \bar{\theta}^2 \bar{\theta}^\beta \partial_\alpha \sigma^\mu_{\beta\alpha} \partial_\mu \bar{f}^{\bar{i}}_{(0)}(x) - \theta^1 \theta^2 \bar{\theta}^\beta \partial_\beta \square \bar{f}^{\bar{i}}_{(0)}(x)$$

$$=: \bar{f}^{\bar{i}}_{(0)}(x) + \bar{h}^{\bar{i}}_\bar{f}.$$
Here, $\tilde{n}_j^{i\dagger}$ (resp. $\tilde{n}_j^{i\dagger}$) is the nilpotent component of $\tilde{f}^i$ (resp. $\tilde{f}^{i\dagger}$). They commute among themselves and satisfy

$$\tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger} = \tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger} = 0,$$

for $1 \leq i_1, i_2, i_3 \leq n$. Denote $(f_1^{(0)}(x), \ldots, f_n^{(0)}(x))$ collectively by $f^{(0)}(x)$ and $(\tilde{f}_1^{(0)}(x), \ldots, \tilde{f}_n^{(0)}(x))$ collectively by $\tilde{f}^{(0)}(x)$. Then it follows from Lemma 5.2.2 and the $C^\infty$-hull structure of $C^\infty(\tilde{\mathcal{M}})$ small that

$$\tilde{f}^2(h) = \tilde{f}^2(h_{C}) = h_{C}(f^{(0)}(x), \tilde{f}^{(0)}(x)) + \sum_{i=1}^{n} (\partial_x h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} + \sum_{j=1}^{n} (\partial_x h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} + \frac{1}{2} \sum_{i,j} (\partial_{x_i} h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger} + \frac{1}{2} \sum_{i,j} (\partial_{x_i} h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger}$$

$$= h_{C}(f^{(0)}(x), \tilde{f}^{(0)}(x)) + \sum_{i=1}^{n} (\partial_x h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} + \sum_{j=1}^{n} (\partial_x h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} + \frac{1}{2} \sum_{i,j} (\partial_{x_i} h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger} + \frac{1}{2} \sum_{i,j} (\partial_{x_i} h_{C})(f^{(0)}(x), \tilde{f}^{(0)}(x)) \cdot \tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger}$$

By definition,

$$\tilde{n}_j^{i\dagger} = \sum_{\alpha} \theta^\alpha \partial_\alpha f^{(1)}(x) + \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \tilde{\partial}_\beta \sigma^{\mu} \partial_\mu f^{(1)}(x) + \theta^1 \theta^2 \partial_1 \bar{\partial}_2 f^{(1)}(x)$$

$$+ \sqrt{-1} \sum_{\beta, \alpha, \mu} \theta^1 \theta^2 \tilde{\partial}_\alpha \tilde{\partial}_\mu f^{(1)}(x) - \theta^1 \theta^2 \partial_1 \bar{\partial}_2 \Box f^{(1)}(x).$$

$$\tilde{n}_j^{\mu\dagger} = (\tilde{n}_j^{i\dagger})^\dagger = \left(\sum_{\beta} \tilde{\theta}^\beta \bar{\partial}_\beta f^{(1)}(x) - \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \tilde{\partial}_\beta \sigma^{\mu} \partial_\mu f^{(1)}(x) + \bar{\theta}^1 \bar{\theta}^2 \partial_1 \bar{\partial}_2 f^{(1)}(x)$$

$$- \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^1 \theta^2 \tilde{\partial}_\alpha \tilde{\partial}_\mu f^{(1)}(x) - \theta^1 \theta^2 \tilde{\partial}_1 \tilde{\partial}_2 \Box f^{(1)}(x).$$

A straightforward computation gives the following explicit formulae:

$$\tilde{n}_j^{i\dagger} \tilde{n}_j^{i\dagger}$$

$$= - \theta^1 \theta^2 \partial_1 \bar{\partial}_2 (f^{(1)}(x)f^{(1)}(x) + f^{(1)}(x)f^{(1)}(x))$$

$$+ \sqrt{-1} \sum_{\beta, \alpha, \mu} \theta^1 \theta^2 \tilde{\partial}_\alpha \tilde{\partial}_\mu f^{(1)}(x) + \theta^1 \theta^2 \tilde{\partial}_1 \tilde{\partial}_2 f^{(1)}(x)$$

$$- 2 \theta^1 \theta^2 \tilde{\partial}_1 \tilde{\partial}_2 \sum_{\mu, \nu} \eta_{\mu\nu} \partial_\mu f^{(1)}(x) \partial_\nu f^{(1)}(x).$$
\[ \dot n_{ij}^{\prime\prime} \dot n_{ij}^{\prime\prime} \]
\[ = - \dot \theta^1 \dot \theta^2 \partial_{\dot \alpha} f^i_{(1)}(x) \left( f^j_{(2)}(x) + f^j_{(2)}(x) f^i_{(1)}(x) \right) \]
\[ - \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \beta} \left( f^i_{(1)}(x) \partial_{\mu} f^j_{(0)}(x) + \partial_{\mu} f^i_{(0)}(x) f^j_{(1)}(x) \right) \]
\[ - 2 \theta^1 \theta^2 \partial_{\mu} \left( \eta_{\mu \nu} \partial_{\mu} f^i_{(0)}(x) \partial_{\nu} f^j_{(0)}(x) \right) ; \]

\[ \ddot n_{ij}^{\prime\prime} \ddot n_{ij}^{\prime\prime} \]
\[ = \sum_{\alpha,\beta} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \alpha} f^i_{(1)}(x) f^j_{(1)}(x) \]
\[ - \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \beta} \left( f^i_{(1)}(x) f^j_{(2)}(x) + f^j_{(2)}(x) f^i_{(1)}(x) \right) \]
\[ + \sqrt{-1} \sum_{\alpha,\beta,\mu} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \gamma} \left( f^i_{(1)}(x) \partial_{\mu} f^j_{(0)}(x) + \partial_{\mu} f^i_{(0)}(x) f^j_{(1)}(x) \right) \]
\[ + \theta^1 \theta^2 \partial_{\mu} \left( 2 \eta_{\mu \nu} \partial_{\mu} f^i_{(0)}(x) \partial_{\nu} f^j_{(0)}(x) \right) ; \]

\[ \dddot n_{ij}^{\prime\prime} \dddot n_{ij}^{\prime\prime} \]
\[ = \sum_{\alpha} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \alpha} f^i_{(1)}(x) \cdot f^j_{(1)}(x) \cdot f^k_{(1)}(x) \]
\[ - \sqrt{-1} \sum_{\alpha,\beta,\mu} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \gamma} \left( f^i_{(1)}(x) f^j_{(2)}(x) + f^j_{(2)}(x) f^i_{(1)}(x) \right) \cdot f^k_{(1)}(x) \]
\[ + \theta^1 \theta^2 \partial_{\mu} \left( \eta_{\mu \nu} \partial_{\mu} f^i_{(0)}(x) \cdot f^j_{(1)}(x) \right) \cdot f^k_{(1)}(x) ; \]

\[ \ddot n_{ij}^{\prime\prime} \ddot n_{ij}^{\prime\prime} \ddot n_{ij}^{\prime\prime} \]
\[ = - \sum_{\alpha} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \alpha} \dot \partial_{\dot \alpha} f^i_{(0)}(x) \cdot f^j_{(1)}(x) \cdot f^k_{(1)}(x) \]
\[ + \sqrt{-1} \sum_{\alpha,\beta,\mu} \theta^\alpha \dot \theta^\beta \dot \partial_{\dot \gamma} \left( f^i_{(1)}(x) f^j_{(2)}(x) + f^j_{(2)}(x) f^i_{(1)}(x) \right) \cdot f^k_{(1)}(x) \]
\[ - \theta^1 \theta^2 \partial_{\mu} \left( \eta_{\mu \nu} \partial_{\mu} f^i_{(0)}(x) \cdot f^j_{(2)}(x) \right) \cdot f^k_{(1)}(x) ; \]

\[ \dddot n_{ij}^{\prime\prime} \dddot n_{ij}^{\prime\prime} \dddot n_{ij}^{\prime\prime} \]
\[ = \theta^1 \theta^2 \partial_{\mu} \left( \eta_{\mu \nu} \partial_{\mu} f^i_{(0)}(x) \cdot f^j_{(2)}(x) + f^k_{(1)}(x) \cdot f^j_{(2)}(x) \right) \cdot f^k_{(1)}(x) . \]
Substituting these expressions into the expansion of \( h_C(\vec{f}^1, \ldots, \vec{f}^n, \vec{f}^1, \ldots, \vec{f}^m) \), one obtains

\[
\begin{align*}
\vec{f}^2(h) &= \vec{f}^2(h_C) = h_C(\vec{f}^1, \ldots, \vec{f}^n, \vec{f}^1, \ldots, \vec{f}^m) \\
&= h_C(f_{00}(x), \bar{f}_{00}(x)) + \sum_{n,i} \theta^n \partial_i f_{(i)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) - \sum_{\beta,i} \theta^\beta \bar{\partial}_\beta \bar{f}_{(\beta)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad + \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \left\{ \sum_{i,j} f_{(12)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad - \frac{1}{2} \sum_{i,j} \left( f_{(1)}(x) f_{(2)}(x) + f_{(2)}(x) f_{(1)}(x) \right) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right\} \\
&\quad + \sum_{\alpha, \beta} \theta^\alpha \theta^\beta \sum_{i,j} f_{(i)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad + \frac{1}{2} \sum_{i,j} \left( f_{(1)}(x) f_{(2)}(x) + f_{(2)}(x) f_{(1)}(x) \right) f_{(\beta)}(x) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right\} \\
&\quad + \sum_{\beta} \theta^\alpha \theta^\beta \sum_{i,j} \theta^\mu \sigma^\mu_{\alpha \beta} \left( \sum_{i} \partial_{\alpha} f_{(i)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad + \frac{1}{2} \sum_{i,j} \left( f_{(1)}(x) f_{(2)}(x) + f_{(2)}(x) f_{(1)}(x) \right) f_{(\beta)}(x) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right) \\
&\quad + \theta_1 \theta_2 \left( \sum_{i,j} f_{(1)}(x) f_{(2)}(x) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad + \frac{1}{2} \sum_{i,j,k} \left( f_{(1)}(x) f_{(2)}(x) + f_{(2)}(x) f_{(1)}(x) \right) f_{(\beta)}(x) \cdot (\partial_x, \partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right) \right\} \\
&\quad + \sum_{\alpha} \theta^\alpha \sum_{\beta, \mu} \theta^\beta \sigma^\beta_{\alpha \mu} \left( - \sum_{i} \partial_{\mu} f_{(i)}(x) \cdot (\partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad - \frac{1}{2} \sum_{i,j} \left( f_{(i)}(x) f_{(j)}(x) + f_{(j)}(x) f_{(i)}(x) \right) f_{(\beta)}(x) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right) \\
&\quad + \theta_1 \theta_2 \left( \sum_{i,j} f_{(i)}(x) f_{(j)}(x) \cdot (\partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \\
&\quad \quad - \frac{1}{2} \sum_{i,j,k} \left( f_{(i)}(x) f_{(j)}(x) + f_{(j)}(x) f_{(i)}(x) \right) f_{(\beta)}(x) \cdot (\partial_x, \partial_x, \partial_x, h_C)(f_{00}(x), \bar{f}_{00}(x)) \right) \right\} \\
&\quad \text{(formula continued to the next page)}
\end{align*}
\]
\[ + \theta^i \theta^a \bar{\theta}^a \bar{\theta}^a \left\{ - \sum_{i=1}^{n} \square f_{(0)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) - \sum_{i} \square f_{(0)}^{(i)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \right. \\
- \sum_{\alpha, \beta, \mu, \nu} \eta^{\mu \nu} \partial_{\mu} f_{(0)}(x) \partial_{\nu} f_{(0)}^{(i)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
+ 2 \sum_{\alpha, \beta, \mu, \nu} \eta^{\mu \nu} \partial_{\mu} f_{(0)}^{(i)}(x) \partial_{\nu} f_{(0)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
- \sum_{\alpha, \beta, \mu, \nu} \eta^{\mu \nu} \partial_{\mu} f_{(0)}^{(i)}(x) \partial_{\nu} f_{(0)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
+ \sqrt{-1} \sum_{\alpha, \beta, \mu} \partial_{\alpha} \partial_{\beta} \bar{\partial}_{\alpha} \bar{\partial}_{\beta} \left( \sum_{i,j} \left( f_{(0)}^{(i)}(x) \partial_{\mu} f_{(0)}^{(j)}(x) - \partial_{\mu} f_{(0)}(x) f_{(0)}^{(i)}(x) \right) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \right) \\
- \frac{1}{2} \sum_{i,j,k} \left( f_{(0)}^{(i)}(x) \partial_{\mu} f_{(0)}^{(j)}(x) + \partial_{\mu} f_{(0)}^{(i)}(x) f_{(0)}^{(j)}(x) \right) f_{(0)}^{(k)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
+ \frac{1}{2} \sum_{i,j,k} \left( f_{(0)}^{(i)}(x) f_{(0)}^{(j)}(x) \partial_{\mu} f_{(0)}^{(k)}(x) + \partial_{\mu} f_{(0)}^{(i)}(x) f_{(0)}^{(k)}(x) \right) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
+ \partial_{1} \partial_{2} \bar{\partial}_{1} \bar{\partial}_{2} \left( \sum_{i,j} \left( f_{(12)}^{(i)}(x) f_{(12)}^{(j)}(x) \right) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \right) \\
- \frac{1}{2} \sum_{i,j,k} \left( f_{(12)}^{(i)}(x) f_{(12)}^{(j)}(x) + f_{(12)}^{(i)}(x) f_{(12)}^{(j)}(x) \right) f_{(12)}^{(k)}(x) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
- \frac{1}{2} \sum_{i,j,k} f_{(12)}^{(i)}(x) \left( f_{(12)}^{(j)}(x) f_{(12)}^{(k)}(x) + f_{(12)}^{(j)}(x) f_{(12)}^{(k)}(x) \right) \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \\
+ \frac{1}{2!} \sum_{i,j,k,l} \left( f_{(12)}^{(i)}(x) f_{(12)}^{(j)}(x) + f_{(12)}^{(i)}(x) f_{(12)}^{(j)}(x) \right) \left( f_{(12)}^{(k)}(x) f_{(12)}^{(l)}(x) + f_{(12)}^{(k)}(x) f_{(12)}^{(l)}(x) \right) \\
\left. \cdot (\partial_{x_i} h_c)(f_{(0)}(x), f_{(0)}(x)) \right\} \right\} \\
\text{(formula continued to the next page)}
\[
\begin{align*}
&= h_c(f_0(x), \overline{f_0(x)}) + \sum_{\alpha} \theta^\alpha \partial_{\alpha} f_{(\alpha)}(x) \cdot (\partial_x h_c)(f_0(x), \overline{f_0(x)}) - \sum_{\beta,i} \theta^\beta \partial_{\beta} f_{(\beta)}(x) \cdot (\partial_x h_c)(f_0(x), \overline{f_0(x)}) \\
&+ \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \left\{ \sum_i f_{(12)}(x) \cdot (\partial_x h_c)(f_0(x), \overline{f_0(x)}) - \sum_{i,j} f_{(1)}(x) f_{(2)}(x) \cdot (\partial_x \partial_x h_c)(f_0(x), \overline{f_0(x)}) \right\} \\
&+ \sum_{\alpha,\beta} \theta^\alpha \theta^\beta \left\{ \sqrt{-1} \sum_{\mu,\nu} \sigma^\mu_{\alpha,\beta} \left( \partial_{\mu} f_{(\alpha)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) - \partial_{\mu} f_{(\beta)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \right) \\
&+ \partial_1 \partial_2 \left( - \sum_{i,j} f_{(12)}(x) f_{(1)}(x) \cdot (\partial_x \partial_x h_c)(f_0(x), \overline{f_0(x)}) \right) \\
&+ \sum_{\beta,\mu} \theta^\beta \sigma^\mu_{\beta,\alpha} \left( \sum_i \partial_{\mu} f_{(\alpha)}(x) \cdot (\partial_x h_c)(f_0(x), \overline{f_0(x)}) - \sum_{i,j} \partial_{\mu} f_{(\beta)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \right) \\
&+ \partial_1 \partial_2 \left( \sum_{i,j} f_{(12)}(x) f_{(1)}(x) \cdot (\partial_x \partial_x h_c)(f_0(x), \overline{f_0(x)}) - \sum_{i,j} f_{(1)}(x) f_{(2)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \right) \\
&\quad - \sum_{i,j} \partial_{\mu} f_{(\alpha)}(x) f_{(1)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \\
&\quad + \sum_{i,j} \partial_{\mu} f_{(\beta)}(x) f_{(2)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \\
&+ \partial_1 \partial_2 \left( \sum_{i,j} f_{(12)}(x) f_{(1)}(x) \cdot (\partial_x \partial_x h_c)(f_0(x), \overline{f_0(x)}) - \sum_{i,j} f_{(1)}(x) f_{(2)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \right) \\
&\quad - \sum_{i,j} \partial_{\mu} f_{(\alpha)}(x) f_{(1)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \\
&\quad + \sum_{i,j} \partial_{\mu} f_{(\beta)}(x) f_{(2)}(x) \cdot (\partial_x \partial_{\nu} h_c)(f_0(x), \overline{f_0(x)}) \right\}
\]

(formula continued to the next page)
\[ + \theta^1 \theta^2 \partial^1 \partial^2 \left\{ - \sum_{i=1}^{n} \Box f^{0}_{(i)}(x) \cdot (\partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) - \sum_{i} \Box f^{0}_{(i)}(x) \cdot (\partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \right. \\
- \sum_{i,j,\mu,\nu} \eta^{\mu\nu} \partial_{\mu} f^{0}_{(i)}(x) \partial_{\nu} f^{0}_{(i)}(x) \cdot (\partial_{j}, \partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
+ 2 \sum_{i,j,\mu,\nu} \eta^{\mu\nu} \partial_{\mu} f_{(i)}(x) \partial_{\nu} f_{(i)}(x) \cdot (\partial_{j}, \partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
- \sum_{i,j,\mu,\nu} \eta^{\mu\nu} \partial_{\mu} \bar{f}_{(i)}(x) \partial_{\nu} f_{(i)}(x) \cdot (\partial_{j}, \partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
\left. + \sqrt{-1} \sum_{\alpha,\beta,\rho} \partial_{\alpha} \bar{\theta}^\beta \bar{\theta}^\rho \left( \sum_{i,j} \left( f_{(i)}(x) \partial_{\mu} f_{(j)}(x) \cdot (\partial_{i}, \partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \right) \right. \right. \\
- \sum_{i,j,k} \partial_{\mu} f^{0}_{(i)}(x) f_{(i)}(x) f_{(i)}(x) \cdot (\partial_{j}, \partial_{j}, \partial_{k}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
+ \sum_{i,j,k} f_{(i)}(x) f_{(j)}(x) \partial_{\mu} f_{(k)}(x) \cdot (\partial_{i}, \partial_{j}, \partial_{k}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
\left. \left. + \vartheta_{1} \vartheta_{2} \bar{\vartheta}_{1} \vartheta_{2} \left( \sum_{i,j} f_{(i)}(x) \bar{f}_{(j)}(x) \cdot (\partial_{i}, \partial_{j}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \right) \right. \right. \\
- \sum_{i,j,k} \partial_{\mu} f^{0}_{(i)}(x) f_{(i)}(x) f_{(j)}(x) \cdot (\partial_{j}, \partial_{j}, \partial_{k}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \\
+ \sum_{i,j,k} f_{(i)}(x) f_{(j)}(x) \partial_{\mu} f_{(k)}(x) \cdot (\partial_{i}, \partial_{j}, \partial_{k}, h_{C})(f^{0}_{(i)}(x), f^{0}_{(i)}(x)) \right) \right) \right) . \]

after relabeling the \( i, j, k, l \) indices of some of the terms and collecting the like terms. In particular, as in Proposition 5.1.3, the components of \( \mathcal{f} - f^{0} \) in the \( (\theta, \bar{\theta}, \vartheta, \vartheta) \)-expansion are \( C^\infty(X)^C \)-valued differential operators on \( C^\infty(Y)^C \) and the equality \( \left( \bar{f}^2(h) \right)^1 = \bar{f}^2(h) \) holds explicitly for \( \mathcal{f} \) chiral and \( h \) (real-valued) smooth function on \( Y \).

(b) \( h \) holomorphic on \( Y = C^n \)

When \( h \in C^\infty(Y)^C \) is holomorphic on \( Y \),
\[ \partial_{z_i} h_C = \partial_{z_i} \partial_{z_i} h_C = \partial_{z_i} \partial_{z_i} \partial_{z_k} h_C = \partial_{z_i} \partial_{z_i} \partial_{z_k} \partial_{z_l} h_C = 0 \]
and \( \bar{f}^2(h) \) reduces to
\[ \bar{f}^2(h) = \bar{f}^2(h_C) = h_C(\bar{f}^1, \ldots, \bar{f}^n) = h_C(f^{0}_{(1)}(x)) + \sum_{\alpha,\beta} \theta^\alpha \partial_{\alpha} f^{0}_{(1)}(x) \cdot (\partial_{\beta}, h_{C})(f^{0}_{(1)}(x)) \]
\[ + \theta^1 \theta^2 \theta^1 \theta^2 \left\{ \sum_{i} f_{(1)}^{(i)}(x) \cdot (\partial_{i}, h_{C})(f^{0}_{(1)}(x)) \right. \right. \\
- \sum_{i,j} f_{(1)}^{(i)}(x) f_{(2)}^{(j)}(x) \cdot (\partial_{i}, \partial_{j}, h_{C})(f^{0}_{(1)}(x), f^{0}_{(2)}(x)) \right. \right. \\
+ \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \theta^\beta \sigma^{\alpha \beta} \partial_{\alpha} f^{0}_{(i)}(x) \cdot (\partial_{\beta}, h_{C})(f^{0}_{(i)}(x)) \\
+ \sqrt{-1} \sum_{\beta,\alpha} \theta^\beta \theta^\alpha \sigma^{\alpha \beta} \left\{ \sum_{i,j} \partial_{\mu} f_{(i)}^{(j)}(x) \cdot (\partial_{i}, h_{C})(f^{0}_{(i)}(x)) \right. \right. \\
\left. \left. + \sum_{i,j} f_{(i)}^{(j)}(x) \partial_{\mu} f^{0}_{(i)}(x) \cdot (\partial_{i}, \partial_{j}, h_{C})(f^{0}_{(i)}(x)) \right) \right. \right. \\
- \theta^1 \theta^2 \theta^1 \theta^2 \left\{ \sum_{i=1}^{n} \Box f^{0}_{(i)}(x) \cdot (\partial_{z_i}, h_{C})(f^{0}_{(i)}(x)) \right. \right. \\
+ \sum_{i,j,\mu,\nu} \eta^{\mu\nu} \partial_{\mu} f^{0}_{(i)}(x) \partial_{\nu} f^{0}_{(i)}(x) \cdot (\partial_{z_i}, \partial_{z_j}, h_{C})(f^{0}_{(i)}(x)) \right) \right) \right) \right) \right) . \]

This is the chiral superfield in \( C^\infty(\mathcal{X}^C)^{\text{small}} \) determined by the four components
\[ \left( h_C(f^{0}_{(1)}(x)), f_{(i)}^{(j)}(x) \cdot (\partial_{z_i}, h_{C})(f^{0}_{(j)}(x)), f_{(1)}^{(2)}(x) \cdot (\partial_{z_1}, h_{C})(f^{0}_{(2)}(x)) - \sum_{i,j} f_{(1)}^{(i)}(x) f_{(2)}^{(j)}(x) \cdot (\partial_{z_i}, \partial_{z_j}, h_{C})(f^{0}_{(i)}(x)) \right)_{\alpha=1,2} . \]
Its twisted complex conjugate \( \tilde{f}^* (h)^\dagger \) is antichiral and is given by

\[
(\tilde{f}^* (h))^\dagger = \tilde{h}_C((\tilde{f}^* (z^1))^\dagger, \ldots, (\tilde{f}^* (z^n))^\dagger) = \tilde{h}_C(\tilde{f}^\dagger, \ldots, \tilde{f}^n)^\dagger
\]

\[
= \tilde{h}_C(\tilde{f}(0)) - \sum_{\beta} \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
+ \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x)) - \sum_{i,j} \tilde{f}^i(1)(x) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[
= \tilde{h}_C(\tilde{f}(0)) - \sum_{\beta} \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
+ \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x)) - \sum_{i,j} \tilde{f}^i(1)(x) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[
- \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[\vdots\]

\[\vdots\]

\[
= \tilde{h}_C(\tilde{f}(0)) - \sum_{\beta} \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
+ \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x)) - \sum_{i,j} \tilde{f}^i(1)(x) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[\vdots\]

\[\vdots\]

\[
- \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[\vdots\]

\[\vdots\]

\[
- \bar{\theta}^\beta \bar{\theta}^\dagger (\sum_i \tilde{f}^i(1)(x)) \cdot (\partial_{\beta} \tilde{h}_C)(\tilde{f}(0)(x))
\]

\[\vdots\]

\[\vdots\]

5.3 The action functional for chiral maps

— \( d = 3 + 1, \ N = 1 \) nonlinear sigma models with a superpotential

With the mathematical background/preparation in Sec. 5.1 and Sec. 5.2, we can now reconstruct

\[\textit{Definition 5.3.1. [action functional for chiral maps]} \]

Let

• \( Y = (\mathbb{C}^n, h) \) be the Kähler manifold \( \mathbb{C}^n \) with complex coordinates

\((z^1, \ldots, z^n) = (y^1 + \sqrt{-1}y^2, \ldots, y^{2n-1} + \sqrt{-1}y^n) \)

and a Kähler metric determined by a Kähler potential: a pluri-subharmonic function \( h \in C^\infty(\mathbb{C}^n) \),

• \( W \) be a holomorphic function on \( Y \), and

• \( \tilde{f} : \tilde{X}^{\tilde{\mathbb{C}}},_{\text{small}} \rightarrow Y \) be a chiral map, with the underlying \( C^\infty\)-ring-homomorphism

\[ \tilde{f}^\dagger : C^\infty(Y) \rightarrow C^\infty(\tilde{X}^{\tilde{\mathbb{C}}})_{\text{small}}. \]

Then, as a 4-dimensional sigma model on \( Y \), the \( N = 1 \) supersymmetric action functional for \( \tilde{f}^\dagger \)'s is given by the action functional

\[ S^h(W)(\tilde{f}) = \int_X d^4x \left( -\frac{1}{4} \int d\theta^2 d\bar{\theta}^2 d\theta^1 d\bar{\theta}^1 \tilde{f}^\dagger (h) + \int d\theta^2 d\theta^1 \tilde{f}^\dagger (W) - \int d\bar{\theta}^2 d\bar{\theta}^1 (\tilde{f}^\dagger (W))^\dagger \right). \]

Here the factor \( -\frac{1}{4} \) is added so that in the end the kinetic term for \( \tilde{f}(0) : X \rightarrow Y \) would have the correct/conventional coefficient. (Cf. [Wess & Bagger: Eq. (22.1)].)
The Kähler potential part

\[ f^2(h) = \begin{cases} \text{terms of total } (\theta, \bar{\theta})\text{-degree } 0, 1, 2, 3 \\ + \theta^1 \bar{\theta}^1 \bar{\theta}^1 \left\{ - \sum_i (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \square f_{(0)}^i(x) - \sum_i (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \square \bar{f}_{(0)}^i(x) \\ - \sum_{i,j} (\partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu f_{(0)}^i(x) \partial_\nu \bar{f}_{(0)}^j(x) \\ + \sum_{i,j} (\partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \left( 2 \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu f_{(0)}^i(x) \partial_\nu \bar{f}_{(0)}^j(x) \right) \\ - \sum_{i,j,k} (\partial_2, \partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \left( \sqrt{-1} \sum_{\alpha, \beta, \lambda} \partial_\alpha \bar{\partial}_\beta \bar{\partial}_\gamma \partial_\mu f_{(0)}^i(x) f_{(0)}^j(x) \partial_\mu \bar{f}_{(0)}^k(x) + \bar{\partial}_1 \partial_2 \bar{\partial}_1 \partial_2 f_{(1)}^i(x) f_{(2)}^j(x) \bar{f}_{(1)}(x) \bar{f}_{(2)}^k(x) \right) \\ + \sum_{i,j,k} (\partial_2, \partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \left( \sqrt{-1} \sum_{\alpha, \beta, \lambda} \partial_\alpha \bar{\partial}_\beta \bar{\partial}_\gamma \partial_\mu f_{(0)}^i(x) f_{(0)}^j(x) \partial_\mu \bar{f}_{(0)}^k(x) - \bar{\partial}_1 \partial_2 \bar{\partial}_1 \partial_2 f_{(1)}^i(x) f_{(2)}^j(x) \bar{f}_{(1)}(x) \bar{f}_{(2)}^k(x) \right) \right\} \right. \]

First observe that the sum of the first four summations has a conversion

\[- \sum_{i=1}^n (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \square f_{(0)}^i(x) - \sum_{j=1}^n (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \square \bar{f}_{(0)}^j(x) \]

\[- \sum_{i,j} (\partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu f_{(0)}^i(x) \partial_\nu \bar{f}_{(0)}^j(x) \]

\[- \sum_{i,j} (\partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \bar{f}_{(0)}^i(x) \partial_\nu f_{(0)}^j(x) \]

\[= - \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \left( \sum_i (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \partial_\nu f_{(0)}^i(x) + \sum_j (\partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \partial_\nu \bar{f}_{(0)}^j(x) \right) \]

\[+ 2 \sum_{i,j} (\partial_2, \partial_2, h_C)(f_{(0)}(x), \bar{f}_{(0)}(x)) \cdot \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu f_{(0)}^i(x) \partial_\nu \bar{f}_{(0)}^j(x). \]

The first summation \(- \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \bar{f}_{(0)}^i(x)\) gives rise to a boundary-term of the action functional \(S^b(\hat{f})\) on the space-time \(X\). Second and optionally, one may choose to present the kinetic term for spinor fields.
with only $\partial_{\mu} f^i_{(\alpha)}$'s or only $\partial_{\mu} f^i_{(\beta)}$'s. Suppose the former, then

$$\sqrt{-1} \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x)$$

$$= \sqrt{-1} \sum_{\mu} \eta^{\mu\nu} \partial_{\mu} \left( \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x) \right)$$

$$- \sqrt{-1} \sum_{i,j,k} (\partial_x, \partial_y, z_h)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x) f_{(0)}^j(x)$$

$$+ \sqrt{-1} \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x) f_{(0)}^j(x)$$

which contribute another boundary-term to the action functional $S(h \tilde{f})$ on the space-time $X$.

Thus, setting

$$\text{(space-time boundary terms)}$$

$$= - \sum_{\mu,\nu} \eta^{\mu\nu} \partial_{\mu} \left( \sum_{i,j} (\partial_x, h_C)(f_{(0)}(x), f_{(0)}(x)) \cdot \partial_{\nu} f_{(0)}^i(x) + \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \cdot \partial_{\nu} f_{(0)}^i(x) \right)$$

$$+ \sqrt{-1} \sum_{\mu} \partial_{\mu} \left( \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x) \right)$$

then, in a final form

$$\tilde{f}^i(h) = \left( \text{terms of total } (\theta, \bar{\theta})\text{-degree } = 0, 1, 2, 3 \right) + \left( \text{space-time boundary terms} \right)$$

$$+ \theta^i \theta^j \bar{\theta}^k \bar{\theta}^l \left\{ \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \cdot \left[ \sum_{\mu,\nu} \eta^{\mu\nu} \partial_{\mu} f_{(0)}^i(x) \right] \right. \right.$$

$$+ \left. \left. \sqrt{-1} \sum_{\mu,\nu} \eta^{\mu\nu} \partial_{\mu} \left( \sum_{i,j} (\partial_x, \partial_y, h_C)(f_{(0)}(x), f_{(0)}(x)) \sum_{\alpha,\beta,\mu} \partial_{\alpha} \bar{\theta}_{\beta} \sigma_{\mu} \int_{(\alpha)}(x) \partial_{\mu} f^i_{(\beta)}(x) \right) \right\} \right.$$
Let \( W = W(z^1, \ldots, z^n) \) be a holomorphic function on \( Y \). It follows from Sec. 5.2 that

\[
\tilde{f}^i(W) = W(\tilde{f}^1(z^1), \ldots, \tilde{f}^n(z^n)) = W(\tilde{f}^1, \ldots, \tilde{f}^n)
\]

\[
= W(\tilde{f}(0)(x)) + \sum_{\alpha} \theta^\alpha \partial_\alpha \left( (\partial_{\tilde{x}i}W)(f_{(0)}(x)) \cdot f_{(1)}(x) \right)
\]

\[
+ \theta^3 \sigma^i_3 \tilde{\partial}_3 \left( \sum_{i} (\partial_{\tilde{x}i}W)(f_{(0)}(x)) \cdot f_{(2)}(x) \right)
\]

\[
+ \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \tilde{\sigma}^\beta_\mu \cdot (\partial_{\tilde{x}i}W)(f_{(0)}(x)) \cdot \partial_\mu f_{(0)}(x)
\]

\[
+ \sqrt{-1} \sum_{\beta, \alpha, \mu} \theta^3 \sigma^i_3 \tilde{\sigma}^\beta_\mu \cdot \left( (\partial_{\tilde{x}i}W)(f_{(0)}(x)) \cdot \partial_\beta f_{(1)}(x) \right)
\]

\[
- \theta^4 \tilde{\tilde{\partial}} \tilde{\tilde{\partial}} \left( \sum_{i} (\partial_{\tilde{x}i}W)(f_{(0)}(x)) \cdot \Box f_{(0)}(x) \right)
\]

and

\[
(\tilde{f}^i(W))^* = W((\tilde{f}^i(z^1))^*, \ldots, (\tilde{f}^i(z^n))^*) = W(\tilde{f}^{i*}, \ldots, \tilde{f}^{n*})
\]

\[
= \tilde{f}(0)(x) - \sum_{\beta} \tilde{\partial}_3 \tilde{\partial}_3 \left( \sum_{i} (\partial_{\tilde{x}i}W)(\tilde{f}_{(0)}(x)) \cdot \tilde{f}_{(2)}(x) \right)
\]

\[
- \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \tilde{\sigma}^\beta_\mu \cdot (\partial_{\tilde{x}i}W)(\tilde{f}_{(0)}(x)) \cdot \partial_\mu \tilde{f}_{(0)}(x)
\]

\[
- \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^3 \sigma^i_3 \tilde{\sigma}^\beta_\mu \cdot \left( (\partial_{\tilde{x}i}W)(\tilde{f}_{(0)}(x)) \cdot \partial_\mu \tilde{f}_{(1)}(x) \right)
\]

\[
- \theta^4 \tilde{\tilde{\partial}} \tilde{\tilde{\partial}} \left( \sum_{i} (\partial_{\tilde{x}i}W)(\tilde{f}_{(0)}(x)) \cdot \Box \tilde{f}_{(0)}(x) \right)
\]
The action functional of \( d = 3 + 1, N = 1 \) sigma models with a superpotential

Combining the previous two themes, one has the action functional of \( d = 3 + 1, N = 1 \) sigma models with a superpotential given explicitly by

\[
S^{(h,W)}(\tilde{f}) = \int_X d^4x \left( -\frac{1}{4} \int d\bar{\theta}^2 d\theta^1 d\bar{\theta}^1 d\theta^1 \tilde{f}^2(h) + \left[ \int d\bar{\theta}^2 d\theta^1 \tilde{f}^2(W) - \int d\bar{\theta}^2 d\theta^1 (\tilde{f}^2(W)^{\dagger}) \right] \right)
\]

\[
= \int_X d^4x \left\{ - \sum_{i,j,m,v} (\partial_i \partial_j h_C) (f(0)(x), \tilde{f}(0)(x)) \eta^{\mu\nu} \partial_\mu f^i_{(0)}(x) \partial_\nu f^j_{(0)}(x) \\
+ \frac{\sqrt{-1}}{2} \sum_{i,j,a,b,m} (\partial_i \partial_j h_C) (f(0)(x), \tilde{f}(0)(x)) \cdot \partial_a \partial_\beta \cdot \tilde{\sigma}^{\mu\nu\alpha} \partial_\mu f^i_{(a)}(x) f^j_{(b)}(x) \\
- \frac{1}{4} \sum_{i,j} (\partial_i \partial_j h_C) (f(0)(x), \tilde{f}(0)(x)) \cdot \partial_1 \partial_2 \tilde{\partial}_1 \tilde{\partial}_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(12)}(x) \\
+ \sum_{i,j,k} (\partial_i \partial_j h_C) (f(0)(x), \tilde{f}(0)(x)) \cdot \left( \frac{1}{4} \partial_1 \partial_2 \tilde{\partial}_1 \tilde{\partial}_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(12)}(x) f^{(k)}_{(2)}(x) + \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta,m} \partial_\alpha \partial_\beta \cdot \tilde{\sigma}^{\mu\nu\alpha} \cdot \partial_\mu f^i_{(a)}(x) f^j_{(b)}(x) f^{k}_{(2)}(x) \right) \\
+ \frac{1}{4} \sum_{i,j,k} (\partial_i \partial_j \partial_k h_C) (f(0)(x), \tilde{f}(0)(x)) \cdot \partial_1 \partial_2 \tilde{\partial}_1 \tilde{\partial}_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(12)}(x) f^{(k)}_{(2)}(x) \\
- \frac{1}{4} \sum_{i,j,k,l} (\partial_i \partial_j \partial_k \partial_l h_C) (f(0)(x), \tilde{f}(0)(x)) \cdot \partial_1 \partial_2 \tilde{\partial}_1 \tilde{\partial}_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(12)}(x) f^{(1)}_{(2)}(x) \\
+ \sum_{i} (\partial_i W)(f(0)) \cdot \partial_1 \partial_2 \cdot f^{(1)}_{(12)}(x) - \sum_{i,j} (\partial_i \partial_j W)(f(0))(x) \cdot \partial_1 \partial_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(2)}(x) \\
- \sum_{i} (\partial_i W)(f(0))(x) \cdot \partial_1 \partial_2 \cdot f^{(1)}_{(12)}(x) + \sum_{i,j} (\partial_i \partial_j W)(f(0))(x) \cdot \partial_1 \partial_2 \cdot f^{(1)}_{(12)}(x) f^{(1)}_{(2)}(x) \right\},
\]

up to boundary terms. (Caution that \((d\bar{\theta}^2 d\theta^1)^{\dagger} = d\bar{\theta}^{1} d\theta^{2} = -d\bar{\theta}^{2} d\theta^{1}.)

Recall that in terms of the Kähler potential \( h_C \), the Kähler metric \( g \) on \( Y = \mathbb{C}^n \), the Christoffel symbols for the Levi-Civita connection associated to the metric, and the curvature tensor are given by the Basic Identities:

\[
g_{ij} = g_{ji} = \partial_i \partial_j h_C, \quad \sum_s g_{sk} \Gamma^s_{ji} = \partial_i g_{jk}, \quad \sum_s g_{st} \Gamma^s_{ji} = \partial_i g_{jk},
\]

\[
R_{kji} = \partial_j \partial_i g_{kl} - \sum_{s,t} g^{st} \partial_j g_{si} \partial_i g_{kt} = \partial_j \partial_i g_{kl} - \sum_{s,t} g_{st} \Gamma^s_{ij} \Gamma^t_{kl},
\]

(e.g. [K-N: vol. II: Sec. IX.5]). It follows that, up to boundary terms and with some re-arrangement of terms and relabelling,
\[
S^{(h,W)}(\bar{f}) = \int_X d^4x \left\{ -\sum_{i,j,\mu,\nu} g_{ij}(f(0)(x), \bar{f}(0)(x)) \eta^{\mu\nu} \partial_\mu f^{(i)}(x) \partial_\nu f^{(j)}(x) \\
+ \frac{\sqrt{-1}}{2} \sum_{i,j,\alpha,\beta,\mu} \partial_\alpha \bar{\partial}_\beta \cdot g_{ij}(f(0)(x), \bar{f}(0)(x)) \tilde{\sigma}^{\mu\beta\alpha} \partial_\mu f^{(i)}(x) f^{(j)}(x) \right\} \\
+ \frac{1}{4} \partial_1 \bar{\partial}_2 \bar{\partial}_1 \bar{\partial}_2 \sum_{i,j} g_{ij}(f(0)(x), \bar{f}(0)(x)) f^{i}(x) f^{j}(x) \\
+ \frac{1}{4} \partial_1 \bar{\partial}_2 \bar{\partial}_1 \bar{\partial}_2 \sum_{i,j} g_{ij}(f(0)(x), \bar{f}(0)(x)) \Gamma^i_{jk}(f(0)(x), \bar{f}(0)(x)) f^{j}(x) f^{k}(x) \\
+ \frac{1}{4} \partial_1 \bar{\partial}_2 \bar{\partial}_1 \bar{\partial}_2 \sum_{i,j} g_{ij}(f(0)(x), \bar{f}(0)(x)) \Gamma^j_{ik}(f(0)(x), \bar{f}(0)(x)) f^{i}(x) f^{k}(x) \right\},
\]

where

\[
D_\mu f^{(i)}(x) := \partial_\mu f^{(i)}(x) + \sum_{j,k} \Gamma^i_{jk}(f(0)(x), \bar{f}(0)(x)) \partial_\mu f^{j}(x) f^{k}(x)
\]

is the covariant derivative along \(\partial_\mu\) from the induced connection on \((S^' \oplus S^\nu) \otimes \phi f^*(\bar{f})\). This reproduces [Wess & Bagger: Eq. (22.10)] completely via the function ring of towered superspace as defined in the current notes without imposing the purge-evaluation map to remove the even nilpotent factors

\[
\partial_1 \bar{\partial}_2, \quad \partial_\alpha \bar{\partial}_\beta, \quad \bar{\partial}_1 \bar{\partial}_2, \quad \partial_1 \bar{\partial}_1 \bar{\partial}_1 \bar{\partial}_1
\]
in the expression.

In the form above the action functional \(S^{(h,W)}\) remains to take values in a Grassmann algebra, albeit even. To render it real-valued, one needs to impose a purge-evaluation map. Here we will take\(^{23}\)

\[
\mathcal{P}^{ev} : \partial_1 \bar{\partial}_2 \rightarrow 1, \quad \partial_\alpha \bar{\partial}_\beta \rightarrow -1, \quad \bar{\partial}_1 \bar{\partial}_2 \rightarrow -1, \quad \partial_1 \bar{\partial}_1 \bar{\partial}_1 \bar{\partial}_1 \rightarrow -1.
\]

\(^{23}\)Note that \(\partial_\alpha \bar{\partial}_\beta\)'s appear only in the kinetic term \(\sum_{\alpha,\beta,\mu,\nu} \partial_\alpha \bar{\partial}_\beta \phi^{\mu\beta\alpha} \tilde{\sigma} \partial_\mu f^{(i)}(x) f^{(j)}(x)\) for the mappino fields \((f^i_{(o)})_{\alpha,\beta}\). One is free to set them to be either all 1 or all -1. The choice here is to match [Wess & Bagger] even with the sign. For the other three, once setting \(\partial_1 \bar{\partial}_2 \rightarrow 1\), then the purge-evaluation rule, \(\bar{\partial}_1 \bar{\partial}_2 \rightarrow -1\) and \(\partial_1 \bar{\partial}_1 \bar{\partial}_1 \bar{\partial}_1 \rightarrow -1\), is governed by the invariance under the twisted complex conjugation and the wish to make the rule as close to product-preserving as possible. It turns out that this is also the choice that makes the final form of \(S^{(h,W)}(\bar{f})\) term-by-term sign-identical with [Wess & Bagger].
Then, in the next-to-final form and up to boundary terms,
\[
S^{(h,W)}(\tilde{f}) = \int_X d^4x \mathcal{P} e^\gamma \left( -\frac{1}{2} \int d^3\theta d^3\bar{\theta} d^4\phi d^4\bar{\phi} \tilde{f}^2(\phi) + \left[ \int d^3\theta d^3\bar{\theta} (\bar{\phi}^2 - \int d^3\theta d^3\bar{\theta} \tilde{f}^2(\phi)) \right] \right) 
\]
\[
= \int_X d^4x \left\{ -\sum_{i,j,\mu,\nu} g_{ij}(f_0(x), \bar{f}_0(x)) \eta^{\mu\nu} \partial_\mu \bar{f}^i(x) \partial_\nu f^j_0(x) 
\right.
\[
+ \frac{1}{2} \sum_{i,j,k,l} g_{k,j}(f_0(x), \bar{f}_0(x)) f^i_1(x) f^k_1(x) f^j_1(x) f^l_1(x) + \frac{1}{4} \sum_{i,j} g_{ij}(f_0(x), \bar{f}_0(x)) f^i_2(x) f^j_2(x) 
\right.
\[
- \sum_{i,j} (\partial_\mu \bar{f}(x)) f^i_1(x) f^j_1(x) - \sum_{i,j} \left( \partial_\mu \bar{f}(x) W_i \right) f^i_1(x) f^j_1(x) 
\right.
\[
+ \sum_i f^{i}_1(x) \left( \partial_\mu \bar{f}_0(x) \right) - \frac{1}{4} \sum_{i,j,k,l} g_{ij}(f_0(x), \bar{f}_0(x)) \Gamma_{ij}^{kl}(f_0(x), \bar{f}_0(x)) f^{i}_1(x) f^{j}_1(x) f^{k}_1(x) f^{l}_1(x) \right\} . 
\]

(Cf. [Wess & Bagger: Eq. (22.10)].)

Observe next that \(S^{(h,W)}(\tilde{f})\) contains no kinetic terms for \(f^{i}_1\)'s and \(f^{i}_2\)'s and, hence, these components are nondynamical and the equations of motion for them from the first variation of \(S^{(h,W)}(\tilde{f})\) with respect to \(f^{i}_{1(2)}\) or \(f^{i}_{1(2)}\) are purely algebraic in \(f^{i}_{1(2)}, f^{k}_{1(2)}\):
\[
\frac{1}{4} \sum_i \left( \partial_\mu \bar{f}(x) \right) f^{i}_{1(2)}(x) + (\partial_\mu \bar{f}(x)) f^{i}_{1(2)}(x) = 0, \quad \text{for all } i; 
\]
\[
\frac{1}{4} \sum_j \left( \partial_\mu \bar{f}(x) \right) f^{j}_{1(2)}(x) + (\partial_\mu \bar{f}(x)) f^{j}_{1(2)}(x) = 0, \quad \text{for all } j. 
\]

Which gives
\[
f^{i}_{1(2)}(x) = -4 \sum_j g^{ij}(f_0(x), \bar{f}_0(x)) \partial_{\mu\bar{f}} \bar{f}^{j}_{1(2)}(x) + \sum_{k,l} \Gamma_{ij}^{kl}(f_0(x), \bar{f}_0(x)) f^{k}_{1(2)}(x) f^{l}_{1(2)}(x), 
\]
\[
\bar{f}^{i}_{1(2)}(x) = -4 \sum_i g^{ij}(f_0(x), \bar{f}_0(x)) \partial_{\mu\bar{f}} \bar{f}^{j}_{1(2)}(x) + \sum_{k,l} \Gamma_{ij}^{kl}(f_0(x), \bar{f}_0(x)) f^{k}_{1(2)}(x) f^{l}_{1(2)}(x). 
\]

Plugging this into \(S^{(h,W)}(\tilde{f})\) to remove the nondynamical component fields \(f^{i}_{1(2)}\) and \(\bar{f}^{i}_{1(2)}\) and employing the Basic Identities, one obtains the final form of the action functional after two sets of cancellations — one leading to the curvature term and the other involving superpotential terms —

\[24\] Note for mathematicians. This is called by physicists “integrating out the non-dynamical \(f^{i}_{1(2)}\) and \(\bar{f}^{i}_{1(2)}\) from the perspective of path-integrals in Quantum Field Theory.
\[ S^{(h,W)}(f(0), (f(\alpha))_{\alpha=1,2}) = S^{(g,W)}(f(0), (f(\alpha))_{\alpha=1,2}) \]

\[
= \int_X d^4x \left\{ - \sum_{i,j,\mu} g_{ij}(f(0)(x), \overline{f(0)(x)}) \eta^{\mu\nu} \partial_\mu f_{(0)}^i(x) \partial_\nu \overline{f_{(0)}^j(x)} \\
- \frac{\sqrt{-1}}{4} \sum_{i,j,\alpha,\beta,\mu} g_{ij}(f(0)(x), \overline{f(0)(x)}) \sigma^{\mu\beta\alpha} \partial_\mu f_{(0)}^i(x) \overline{f_{(0)}^j(x)} \\
+ \frac{1}{2} \sum_{i,j,k,l} R_{ijkl}(f(0)(x), \overline{f(0)(x)}) f_{(1)}^i(x) f_{(2)}^j(x) \overline{f_{(1)}^k(x)} \overline{f_{(2)}^l(x)} \\
- 4 \sum_{i,j} g^{ij}(f(0)(x), \overline{f(0)(x)}) (\partial_{\bar{z}} W)(f(0)(x)) (\partial_{z} \overline{W})(f(0)(x)) \\
- \sum_{i,j} (D_{z_i} \partial_{\bar{z}_j} W)(f(0)(x), \overline{f(0)(x)}) f_{(1)}^i(x) f_{(2)}^j(x) - \sum_{i,j} (D_{z_i} \partial_{\bar{z}_j} \overline{W})(f(0)(x), \overline{f(0)(x)}) \overline{f_{(1)}^i(x)} \overline{f_{(2)}^j(x)} \right\} ,
\]

where

\[ D_{z_i} \partial_{\bar{z}_j} W = \partial_{\bar{z}_j} \partial_{z_i} W - \sum_k \Gamma_{i,j}^k \partial_{z_k} W \quad \text{and} \quad D_{z_i} \partial_{\bar{z}_j} \overline{W} = \partial_{\bar{z}_j} \partial_{z_i} \overline{W} - \sum_k \Gamma_{i,j}^k \partial_{\bar{z}_k} \overline{W} \]

are the induced covariant derivative on the complexified cotangent bundle \( T^* \mathbb{C}^Y \) of \( Y \) from the Levi-Civita connection on \( T^Y \). Up to some conventional coefficients, this is precisely the manifestly real expression for [Wess & Bagger: Eq. (22.12)] in the setting of the current notes.

**Remark 5.3.2. [for \( Y \) a general Kähler manifold]** The final expression of \( S^{(h,W)}(f(0), (f(\alpha))_{\alpha=1,2}) \) implies that \( S^{(h,W)}(f(0), (f(\alpha))_{\alpha=1,2}) = S^{(g,W)}(f(0), (f(\alpha))_{\alpha=1,2}) \) depends only on the Kähler metric \( g \), not the choice of the Kähler potential \( h \) that gives the metric. It follows that the \( d = 3 + 1, N = 1 \) supersymmetric sigma model \( S^{(g,W)}(f(0), (f(\alpha))_{\alpha=1,2}) \) is defined for all Kähler manifold \( Y \). (When \( Y \) is compact, the superpotential \( W = 0 \).)

Once a mathematical presentation of superspace and supersymmetry that matches the particle physicists' standard language of the topic as presented in [Wess & Bagger] is completed, the immediate next question is:

**Q. What is the intrinsic description of the same, without resorting to a choice of a trivialization of the spinor bundles by covariantly constant sections?**

That is the theme for the sequel.
References

[Argu] R. Argurio, Introduction to supersymmetry, lecture notes for PHYS-F-417, October, 2017.

[Argy] P. Argyres, Introduction to supersymmetry, Physics 661 lecture notes, Cornell University, fall, 2000.

[Bag1] J.A. Bagger, Supersymmetric sigma model, in Supersymmetry, K. Dietz, R. Flume, G.v. Gehlen and V. Rittenberg eds., NATO ASI Ser. B: Physics, vol. 125, 45-87, Plenum Press, 1985.

[Bag2] ———, Supersymmetry and superspace, three lectures given in the program Aspects of supersymmetry, Institute for Advanced Study, Princeton, July 2010; available through YouTube website https://www.youtube.com/

[Bal] W. Ballmann, Lectures on Kähler manifolds, European Math. Soc., 2006.

[Bi] A. Bilal, Introduction to supersymmetry, [arXiv:hep-th/0101055].

[B-T-T] D. Hertolini, J. Thaler, and Z. Thomas, TASI 2012: Super-tricks for superspace, [arXiv:1302.6229 [hep-ph]].

[Ca] É. Cartan, The theory of spinors, Hermann, 1966.

[Ch] C. Chevalley, The algebraic theory of spinors and Clifford algebras, Springer, 1997.

[Co] E.M. Corson, Introduction to tensors, spinors, and relativistic wave-equations (Relation structure), Hafner Publ., 1953.

[CB] Y. Choquet-Bruhat, Graded bundles and supermanifolds, Mono. Text. Phys. Sci. Lect. Notes 12, Bibliopolis, 1989.

[De] P. Deligne, Notes on spinors, in Quantum fields and strings: a course for mathematicians, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 99-135, American Math. Soc., 1999.

[D-F1] P. Deligne and D.S. Freed, Supersolutions, in Quantum fields and strings: a course for mathematicians, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 227-355, American Math. Soc., 1999.

[D-F2] ———, Sign manifesto, in Quantum fields and strings: a course for mathematicians, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 357-363, American Math. Soc., 1999.

[D-K] S.K. Donaldson and P.B. Kronheimer, The geometry of four manifolds, Oxford Univ. Press, 1990.

[Ei] D. Eisenbud, Commutative algebra – with a view toward algebraic geometry, GTM 150, Springer, 1994.

[E-H] D. Eisenbud and J. Harris, The geometry of schemes, GTM 197, Springer, 2000.

[Fr] D.S. Freed, Five lectures on supersymmetry, Amer. Math. Soc., 1999.

[F-P] D.Z. Freedman and A. Proeyen, Supergravity, Cambridge Univ. Press, 2012.

[F-T] D.Z. Freedman and P.K. Townsend, Antisymmetric tensor gauge theories and non-linear σ-models, Nucl. Phys. B177 (1981), 282–296.

[F-Z] S. Ferrara and B. Zumino, Supergauge invariant Yang-Mills theories, Nucl. Phys. B79 (1974), 413–421.

[G-G-R-S] S.J. Gates, Jr., M.T. Grisaru, M. Ročňek, and W. Siegel, Superspace – one thousand and one lessons in supersymmetry, Frontiers Phys. Lect. Notes Ser. 58, Benjamin/Cummings Publ. Co., Inc., 1983.

[G-H] P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley & Sons, 1978.

[Hai] G. Hailu, Quantum field theory III: Supersymmetry, course Physics 253cr given at Department of Physics, Harvard University, fall 2018.

[Hart] R. Hartshorne, Algebraic geometry, GTM 52, Springer, 1977.

[Harv] F.R. Harvey, Spinors and calibrations, Perspectives Math., vol. 9, Academic Press, 1990.

[Ho] P.-M. Ho, private discussion, Department of Physics, National Taiwan University, May 2019.

[H-K-P-T-V-V-Z] K. Hori, A. Klemm, P. Pandharipanda, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, Mirror symmetry, Clay Math. Mono. vol. 1, Amer. Math. Soc., 2003.

[Jo] D. Joyce, Algebraic geometry over C∞-rings, Mem. Amer. Math. Soc. no. 1256, Amer. Math. Soc., 2019. (arXiv:1001.0023 [math.AG]).

[Ke] E. Keßler, Supergeometry, super Riemann surfaces and the superconformal action functional, LNM 2230, Springer, 2019.

[KI] A. Klemm, private discussion, Center of Mathematical Sciences and Applications, Harvard University, March 2019.
S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan 15, Princeton Univ. Press, 1987.

S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I & vol. II, Interscience Publ., John Wiley & Sons, 1963 and 1969.

D.A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys 35 (1980), 1–64.

H.B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Math. Ser., 38, Princeton Univ. Press, 1989.

C.-H. Liu and S.-T. Yau, *Azumaya-type noncommutative spaces and morphisms therefrom: Polchinski’s D-branes in string theory from Grothendieck’s viewpoint*, [arXiv:0709.1515 [math.AG]].

C.-H. Liu and S.-T. Yau, *D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds*, [arXiv:1406.0929 [math.DG]].

C.-H. Liu and S.-T. Yau, *D-branes and Azumaya/matrix noncommutative differential geometry, II: Azumaya/matrix supermanifolds and differentiable maps therefrom - with a view toward dynamical fermionic D-branes in string theory*, [arXiv:1412.0771 [hep-th]].

C.-H. Liu and S.-T. Yau, *N = 1 fermionic D3-branes in RNS formulation I. C∞-Algebrogemetric foundations of d = 4, N = 1 supersymmetry, SUSY-rep compatible hybrid connections, and D-chiral maps from a d = 4 N = 1 Azumaya/matrix superspace*, [arXiv:1808.05011 [math.DG]].

C.-H. Liu and S.-T. Yau, *Physicists’ d = 3 + 1, N = 1 superspace-time and supersymmetric QFTs from a tower construction in complexified Z/2-graded C∞-Algebraic Geometry and a purge-evaluation/index-contracting map*, [arXiv:1902.06246 [hep-th]].

Y.I. Manin, *Gauge field theory and complex geometry*, Springer, 1988.

J.W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Math. Notes 44, Princeton Univ. Press, 1996.

R. Penrose and W. Rindler, *Spinors and space-time*, vol. 1, *Two-spinor calculus and relativistic fields*, Cambridge Univ. Press, 1984.

M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publ. Co., 1995.

F. Ruiz Ruiz and P. van Nieuwenhuizen, *Lectures on supersymmetry and supergravity in 2 + 1 dimensions and regularization of supersymmetric gauge theories*, in Recent developments in gravitation and mathematical physics, A. Garcia C. Lämmerzahl, A. Macias, T. Matos, D. Nuñez, eds., Science Network Publ., 1998.

R.D. Schafer, *An introduction to nonassociative algebras*, Academic Press, 1966; Dover edition, 1995.

M.J. Strassler, *An unorthodox introduction to supersymmetric gauge theory*, lectures given at TASI 2001, [arXiv:hep-th/0309149]

A. Salam and J.A. Strathdee, *Supersymmetry and superfields*, Fort. Phys. 26 (1978), 57–142.

S. Shnider and R.O. Wells, Jr., *Supermanifolds, super twistor spaces and super Yang-Mills fields*, Séminaire Math. Supér. 106. Press. Univ. Montréal, 1989.

J. Thaler, *Supersymmetric quantum field theories*, course Physics 8.831 given at Department of Physcis, Massachusetts Institute of Technology, fall 2019.

P. West, *Introduction to supersymmetry and supergravity*, extended 2nd ed., World Scientific, 1990.

J. Wess and J. Bagger, *Supersymmetry and supergravity*, 2nd ed., revised and expanded, Princeton Univ. Press, 1992.

J. Wess and B. Zumino, *A Lagrangian model invariant under supergauge transformations*, Phys. Lett. 40B (1974), 52–54.

J. Wess and B. Zumino, *Supergauge transformations in four-dimensions*, Nucl. Phys. 70 (1974), 39–50.

J. Wess and B. Zumino, *Supergauge invariant extension of quantum electrodynamics*, Nucl. Phys. B78 (1974), 1–13.

B. Zumino, *Supersymmetry and Kähler manifolds*, Phys. Lett. 87B (1979), 203–206.