NILPOTENT OPERATORS AND
WEIGHTED PROJECTIVE LINES

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Abstract. We show a surprising link between singularity theory and the invariant subspace problem of nilpotent operators as recently studied by C. M. Ringel and M. Schmidmeier, a problem with a longstanding history going back to G. Birkhoff. The link is established via weighted projective lines and (stable) categories of vector bundles on those.

The setup yields a new approach to attack the subspace problem. In particular, we deduce the main results of Ringel and Schmidmeier for nilpotency degree $p$ from properties of the category of vector bundles on the weighted projective line of weight type $(2, 3, p)$, obtained by Serre construction from the triangle singularity $x^2 + y^3 + z^p$. For $p = 6$ the Ringel-Schmidmeier classification is thus covered by the classification of vector bundles for tubular type $(2, 3, 6)$, and then is closely related to Atiyah’s classification of vector bundles on a smooth elliptic curve.

Returning to the general case, we establish that the stable categories associated to vector bundles or invariant subspaces of nilpotent operators may be naturally identified as triangulated categories. They satisfy Serre duality and also have tilting objects whose endomorphism rings play a role in singularity theory. In fact, we thus obtain a whole sequence of triangulated (fractional) Calabi-Yau categories, indexed by $p$, which naturally form an ADE-chain.

1. Introduction and main results

In recent work Ringel and Schmidmeier thoroughly studied the classification problem for invariant subspaces of nilpotent linear operators (in a graded and ungraded version) [19–21]. This problem has a long history and can actually be traced back to Birkhoff’s problem [2], dealing with the classification of subgroups of finite abelian $p$-groups. We note that Simson [23] determined the complexity for the classification of indecomposable objects, depending on the nilpotency degree. Even more generally, Simson considered the classification problem for chains of invariant subspaces, without however attempting an explicit classification. For additional information on the history of the problem we refer to [20, 23]. The main achievement of [20] is such an explicit classification for $p \leq 6$ where the case $p = 6$, yielding tubular type, is the most difficult one and very much related to the representation theory of so-called tubular algebras, a problem initiated and accomplished by Ringel in [18]. In the present paper we describe an unexpected access to the invariant subspace problem for graded nilpotent operators through the theory of weighted projective lines. Our approach covers all major results from [20]; it links the problem with other mathematical subjects (singularities, vector bundles, Cohen-Macaulay modules, Calabi-Yau categories) and largely enhances our knowledge about the original problem.

Let $X = X(2, 3, p)$ denote the weighted projective line of weight type $(2, 3, p)$, where the integer $p$ is at least 2. Following [7], the category coh-$X$ of coherent sheaves on $X$ is obtained by applying Serre’s construction [22] to the (suitably
graded) triangle singularity \( x_1^2 + x_2^3 + x_3^p \). We recall that the Picard group of \( \mathbb{Z} \) is naturally isomorphic to the rank one abelian group \( \mathbb{L} = \mathbb{L}(2, 3, p) \) on three generators \( x_1, x_2, x_3 \) subject to the relations \( 2x_1 = 3x_2 = px_3 \). Up to isomorphism the line bundles are therefore given by the system \( \mathcal{L} \) of twisted structure sheaves \( \mathcal{O}(\bar{x}) \) with \( \bar{x} \in \mathbb{L} \). A key aspect of our paper is a properly chosen subdivision of the system \( \mathcal{L} \) of all line bundles into two disjoint classes \( \mathcal{P} \) and \( \mathcal{F} \) of line bundles, called persistent and fading, respectively. This subdivision arises from the partition of \( \mathbb{L} \) into the subsets \( \mathbb{P} = \mathbb{Z} x_3 \sqcup (x_2 + \mathbb{Z} x_3) \) and \( \mathbb{F} = \mathbb{L} \setminus \mathbb{P} \), each consisting of cosets modulo \( \mathbb{Z} x_3 \).

By \( \text{vect-} \mathbb{X} \) we denote the full subcategory of the category \( \text{coh-} \mathbb{X} \) of coherent sheaves formed by all vector bundles, that is, all locally free sheaves. Let further \( \mathcal{F} \) denote the ideal of all morphisms in \( \text{vect-} \mathbb{X} \) factoring through a finite direct sum of fading line bundles. We recall that a Frobenius category is an exact category (Quillen’s sense) which has enough projectives and injectives, and where the projective and the injective objects coincide.

**Theorem A.** Assume \( \mathbb{X} \) has weight type \((2, 3, p)\) with \( p \geq 2 \). Then the following holds.

1. The category \( \text{vect-} \mathbb{X} \) is a Frobenius category with the system \( \mathcal{L} \) of all line bundles as the indecomposable projective-injective objects.
2. The factor category \( \text{vect-} \mathbb{X}/[\mathcal{F}] \) is a Frobenius category with the system \( \mathcal{P} \) of persistent line bundles as the indecomposable projective-injective objects.
3. The stable categories \( \text{vect-} \mathbb{X}/[\mathcal{L}] \) and \( (\text{vect-} \mathbb{X}/[\mathcal{F}])/(\mathcal{P}/[\mathcal{F}]) \) are naturally equivalent as triangulated categories, notation \( \text{vect-} \mathbb{X}/[\mathcal{F}] \).

We write \( \mathcal{P} \) to denote \( \mathcal{P} \) as a full subcategory of \( \text{vect-} \mathbb{X}/[\mathcal{F}] \).

**Lemma B.** The category \( \mathcal{P} \) is equivalent to the path category of the quiver

\[
\begin{align*}
\cdots & \o x \o x \o x \cdots \o x \o x \o x \cdots \\
\cdots & \o y \o y \o y \cdots \\
\end{align*}
\]

modulo the ideal given by all commutativities \( xy = yx \) and all nilpotency relations \( x^p = 0 \).

Hence the category of finite dimensional contravariant \( k \)-linear representations of the above quiver with relations is naturally isomorphic to the category \( \text{mod-} \mathcal{P} \) of finitely presented right \( k \)-modules over \( \mathcal{P} \). We call \( \mathcal{P} \) (and by abuse of language sometimes also the quiver \((1.1)\)) the (infinite) \( p \)-ladder. Clearly a right \( \mathcal{P} \)-module is exactly a morphism \( U \to M \) between two \( \mathbb{Z} \)-graded \( k[x]/(x^p) \)-modules, where \( x \) gets degree 1. The category \( \tilde{\mathcal{S}}(p) \) consists of all those morphisms which are monomorphisms. As a full subcategory \( \tilde{\mathcal{S}}(p) \) is extension-closed in \( \text{mod-} \mathcal{P} \), hence \( \tilde{\mathcal{S}}(p) \) inherits an exact structure which is actually Frobenius with the projectives from \( \text{mod-} \mathcal{P} \) as the projective-injective objects. Note further that \( \mathbb{Z} \) acts on \( \tilde{\mathcal{S}}(p) \) by grading shift, denoted by \( s \).

**Theorem C.** Assume \( \mathbb{X} \) has weight type \((2, 3, p)\). Then the functor

\[ \Phi: \text{vect-} \mathbb{X} \to \text{mod-} \mathcal{P}, \quad E \mapsto \mathcal{P}(-, E) \]

induces equivalences \( \Phi: \text{vect-} \mathbb{X}/[\mathcal{F}] \to \tilde{\mathcal{S}}(p) \) of Frobenius categories and \( \Phi: \text{vect-} \mathbb{X} \to \tilde{\mathcal{S}}(p) \) of triangulated categories, respectively. Moreover, under the functor \( \Phi \), the shift by \( x_3 \in \mathbb{L} \) on \( \text{vect-} \mathbb{X} \) corresponds to the shift \( s \) by \( 1 \in \mathbb{Z} \) on \( \text{mod-} \mathcal{P} \), \( s = \tilde{\mathcal{S}}(p) \).
This theorem implies (most of) the results from [20] from results on the categories vect-\(X\); it further has a significant number of additional consequences. For instance, we show that the triangulated category vect-\(X = \overline{S}(p)\), has Serre duality and admits a tilting object. Indeed, we give explicit constructions for two tilting objects \(T\) and \(T'\) in vect-\(X\) with non-isomorphic endomorphism rings, yielding by Theorem C explicit tilting objects for \(\overline{S}(p)\). The two tilting objects have \(2(p-1)\) pairwise indecomposable summands. The endomorphism algebra of \(T\) is the representation-finite Nakayama algebra \(A(2(p-1), 3)\) given by the quiver
\[
1 \overset{x}{\rightarrow} 2 \overset{x}{\rightarrow} 3 \overset{x}{\rightarrow} 4 \overset{x}{\rightarrow} \cdots \overset{x}{\rightarrow} 2p-3 \overset{x}{\rightarrow} 2(p-1)
\]
with all nilpotency relations \(x^3 = 0\). Let \([1, n]\) denote the linearly ordered set \(\{1, 2, \ldots, n\}\). Then the endomorphism ring of \(T'\) is given as the incidence algebra \(B(2, p-1)\) of the poset \([1, 2] \times [1, p-1]\), that is the algebra given by the fully commutative quiver
\[
\begin{array}{cccccc}
1 & 2 & 3 & \cdots & p-2 & p-1 \\
1' & 2' & 3' & \cdots & (p-2)' & (p-1)'.
\end{array}
\]
It is well-known that rectangular diagrams of this shape appear in singularity theory, see for instance [5, 6].

Algebraically, an established method to investigate the complexity of a singularity is due to R. Buchweitz [3], later revived by D. Orlov [17] who primarily deals with the graded situation. Given the \(L\)-graded triangle singularity \(S = k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^4)\) this amounts to consider the Frobenius category CM\(^L\)-\(S\) of \(L\)-graded maximal Cohen-Macaulay modules, its associated stable category \(\text{CM}^L\)-\(S\) and the triangulated category of graded singularities of \(S\) defined as the quotient \(D^b_{\text{sg}}(S) = D^b(\text{mod}^L\)-\(S)/D^b(\text{proj}^L\)-\(S)\). It is shown in [3, 17] that the two constructions yield naturally equivalent triangulated categories \(\text{CM}^L\)-\(S = D^b_{\text{sg}}(S)\). It further follows from [7] that sheafification yields natural equivalences \(\text{CM}^L\)-\(S \sim \text{vect-}X\) with the indecomposable projective \(L\)-graded \(S\)-modules corresponding to the line bundles on \(X\), and then inducing natural identifications
\[
D^b_{\text{sg}}(S) = \text{CM}^L\)-\(S = \text{vect-}X\), where \(X = \chi(2, 3, p)\).
\]
In particular, comparing the sizes of the triangulated categories \(D^b(\text{coh-}X)\) and \(\text{vect-}X\) by the ranks of their Grothendieck groups yields
\[
\text{rk}(\text{Ko}(\text{vect-}X)) - \text{rk}(\text{Ko}(\text{coh-}X)) = p - 6,
\]
a formula nicely illustrating the effects of (an \(L\)-graded version of) Orlov’s theorem [17]. Moreover, for each \(p\) the triangulated categories from (1.4) are fractional Calabi-Yau where, up to cancelation, the Calabi-Yau dimension equals \(1 - 2\chi_X\). Here \(\chi_X = 1/p - 1/6\) is the orbifold Euler characteristic of \(X\). By Theorem C all these assertions transfer to properties of \(\overline{S}(p)\). For details and further applications we refer to Section 4.

The structure of the paper is as follows. In Section 2 we recall some basic properties of weighted projective lines. The proofs of the main results will be given in Section 3. The final Section 4 is devoted to applications concerning the categories \(\overline{S}(p) = \text{vect-}X/\langle F \rangle\) and \(\overline{S}(p) = \text{vect-}X\).

2. Definitions and basic properties

We recall some basic notions and facts about weighted projective lines. We restrict our treatment to the case of three weights. So let \(p_1, p_2, p_3 \geq 2\) integers,
called weights. Denote by $S$ the commutative algebra

$$S = \frac{k[x_1, x_2, x_3]}{(x_1^p_1 + x_2^p_2 + x_3^p_3)} = k[x_1, x_2, x_3].$$

Let $L = \mathbb{L}(p_1, p_2, p_3)$ be the abelian group given by generators $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and defining relations $p_1 \vec{x}_1 = p_2 \vec{x}_2 = p_3 \vec{x}_3 = \text{c}$. The $L$-graded algebra $S$ is the appropriate object to study the triangle singularity $x_1^p_1 + x_2^p_2 + x_3^p_3$. The element $\text{c}$ is called the canonical element. Each element $\vec{x} \in L$ can be written in canonical form

$$(2.1) \quad \vec{x} = n_1 \vec{x}_1 + n_2 \vec{x}_2 + n_3 \vec{x}_3 + m \text{c}$$

with unique $n_i, m \in \mathbb{Z}, 0 \leq n_i < p_i$.

The algebra $S$ is $\mathbb{L}$-graded by setting $\text{deg } x_i = \vec{x}_i (i = 1, 2, 3)$, hence $S = \bigoplus_{\vec{x} \in L} S_{\vec{x}}$. By an $\mathbb{L}$-graded version of the Serre construction [22], the weighted projective line $\mathcal{X} = \mathcal{X}(p_1, p_2, p_3)$ of weight type $(p_1, p_2, p_3)$ is given by its category of coherent sheaves $\text{coh-}\mathcal{X} = \text{mod}^L(S)/\text{mod}^L_0(S)$, the quotient category of finitely generated $\mathbb{L}$-graded modules modulo the Serre subcategory of graded modules of finite length. The abelian group $\mathbb{L}$ is ordered by defining the positive cone $\{ \vec{x} \in \mathbb{L} \mid \vec{x} \geq 0 \}$ to consist of the elements of the form $n_1 \vec{x}_1 + n_2 \vec{x}_2 + n_3 \vec{x}_3$, where $n_1, n_2, n_3 \geq 0$. Then $\vec{x} \geq 0$ if and only if the homogeneous component $S_{\vec{x}}$ is non-zero, and equivalently, if in the normal form $(2.1)$ of $\vec{x}$ we have $m \geq 0$.

The image $\mathcal{O}$ of $S$ in $\text{mod}^L(S)/\text{mod}^L_0(S)$ serves as the structure sheaf of $\text{coh-}\mathcal{X}$, and $L$ acts on the above data, in particular on $\text{coh-}\mathcal{X}$, by grading shift. Each line bundle has the form $\mathcal{O}(\vec{x})$ for a uniquely determined $\vec{x}$ in $\mathbb{L}$, and we have natural isomorphisms

$$\text{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{x} - \vec{y}}.$$ 

Defining the dualizing element from $\mathbb{L}$ as $\vec{\omega} = \text{c} - (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)$, the category $\text{coh-}\mathcal{X}$ satisfies Serre duality in the form

$$\text{D Ext}^1(X, Y) = \text{Hom}(Y, X(\vec{\omega}))$$

functorially in $X$ and $Y$. Moreover, Serre duality implies the existence of almost split sequences for $\text{coh-}\mathcal{X}$ with the Auslander-Reiten translation $\tau$ given by the shift with $\vec{\omega}$.

For each line bundle $L$, the extension term of the almost split sequence

$$0 \longrightarrow L(\vec{\omega}) \longrightarrow E(L) \longrightarrow L \longrightarrow 0$$

is called the Auslander bundle corresponding to $L$. Dealing with three weights $p_i \geq 2 (i = 1, 2, 3)$ each Auslander bundle $E = E(L)$ is exceptional [11], that is, satisfies $\text{End}(E) = k$ and $\text{Ext}^1(E, E) = 0$.

The category vect-$\mathcal{X}$ carries the structure of a Frobenius category such that the system $\mathcal{L}$ of all line bundles is the system of all indecomposable projective-injectives, see [11]: A sequence $\eta: 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ in vect-$\mathcal{X}$ is distinguished exact if all the sequences $\text{Hom}(L, \eta)$ with $L$ a line bundle are exact (equivalently all the sequences $\text{Hom}(\eta, L)$ are exact).

Accordingly, see [8], the stable category

$$\text{vect-}\mathcal{X} = \text{vect-}\mathcal{X}/[\mathcal{L}]$$

is triangulated. It is shown in [11] that the triangulated category vect-$\mathcal{X}$ is Krull-Schmidt with Serre duality induced from the Serre duality of coh-$\mathcal{X}$. The triangulated category vect-$\mathcal{X}$ is homologically finite. Moreover, we will see later that vect-$\mathcal{X}$ has a tilting object.

It is shown in [7] that the quotient functor $q: \text{mod}^L(S) \rightarrow \text{coh-}\mathcal{X}$ induces an equivalence $\text{CM}^L(S) \rightarrow \text{vect-}\mathcal{X}$, where $\text{CM}^L(S)$ denotes the category of $L$-graded
(maximal) Cohen-Macaulay modules over $S$. Under this equivalence indecomposable graded projective modules over $S$ correspond to line bundles in $\text{vect-} \mathcal{X}$, resulting in a natural equivalence

$$\text{CM}^1(S) \simeq \text{vect-} \mathcal{X}$$

used from now on as an identification. Stable categories of (graded) Cohen-Macaulay modules play an important role in the analysis of singularities, see [3,9,10,17].

3. Proofs

From now on the weight type is always the triple $(p_1, p_2, p_3) = (2, 3, p)$ with $p \geq 2$. In this Section we provide the proofs for Theorem A, Theorem C and Lemma B. Note that only the proof for Lemma B is straightforward. By contrast the proofs for Theorems A and C are far from obvious. Additionally they behave quite sensitive with respect to a (re)arrangement of the steps involved.

Proof of Lemma B. To prepare the proof of Lemma B we observe that the category $\mathcal{P}$ has the shape of an infinite ladder:

$$
\cdots \to \mathcal{O}(-\vec{x}_3) \xrightarrow{x_3} \mathcal{O} \xrightarrow{x_3} \mathcal{O}(\vec{x}_3) \xrightarrow{x_3} \mathcal{O}(2\vec{x}_3) \xrightarrow{x_3} \cdots
$$

$$
\cdots \to \mathcal{O}(\vec{x}_2 - \vec{x}_3) \xrightarrow{x_3} \mathcal{O}(\vec{x}_2) \xrightarrow{x_3} \mathcal{O}(\vec{x}_2 + \vec{x}_3) \xrightarrow{x_3} \mathcal{O}(\vec{x}_2 + 2\vec{x}_3) \xrightarrow{x_3} \cdots
$$

where the upper bar (resp. lower bar) is formed by all line bundles $\mathcal{O}(n\vec{x}_3)$, (resp. $\mathcal{O}(\vec{x}_2 + n\vec{x}_3)$) for an arbitrary integer $n$.

Commutativity of the diagram (1.1) follows from the commutativity of $S$. Applying $\text{Hom}_S(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y} - \vec{x}}$ it follows that each morphism in $\mathcal{P}$, viewed as a full subcategory of $\text{vect-} \mathcal{X}$, is a linear combination of powers of $x_2$ and $x_3$. Next we observe that $\text{Hom}_S(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x} + \vec{c})) = 0$ holds for each $\vec{c} \in \mathcal{P}$. Indeed $\text{Hom}_S(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x} + \vec{c}))$ is generated by $x_2^3$ and $x_3^4$, moreover each of the two morphisms factors through a fading line bundle $(\mathcal{O}(\vec{x} + 2\vec{x}_2)$ and $\mathcal{O}(\vec{x} + \vec{x}_1)$, respectively. Finally, we have $x_2 x_3^{p-1} \neq 0$ (and hence $x_3^{p-1} \neq 0$) in $\text{vect-} \mathcal{X}/\{\mathcal{F}\}$ since there are no morphisms from $\mathcal{O}(\vec{x})$ to $\mathcal{O}(\vec{x} + \vec{x}_2 + (p-1)\vec{x}_3)$ factoring through a fading line bundle. Indeed, every $\vec{y} \in \mathbb{L}$ with $0 \leq \vec{y} \leq \vec{x}_2 + (p-1)\vec{x}_3$ is of the form $\vec{y} = a\vec{x}_2 + b\vec{x}_3$ with $a = 0, 1$ and $b = 0, \ldots, p - 1$, implying that $\vec{y}$ belongs to $\mathcal{P}$.

□

Lemma 3.1. Let $L$ be a line bundle. Then for each integer $n \geq 1$ the following sequence is exact in $\text{coh-} \mathcal{X}$.

$$0 \to L \xrightarrow{(x_1, x_2^p)} L(\vec{x}_1) \oplus L(n\vec{x}_2) \xrightarrow{(-x_2^p, x_1)} L(\vec{x}_1 + n\vec{x}_2) \to 0.$$

Proof. The exact sequence is obtained from the following pushout diagram in $\text{coh-} \mathcal{X}$ (3.1)

$$
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
0 & L \xrightarrow{x_2} L(n\vec{x}_2) \xrightarrow{x_2} S_2^{(n)} & 0 \\
0 & L(\vec{x}_1) \xrightarrow{x_2} L(\vec{x}_1 + n\vec{x}_2) \xrightarrow{x_2} S_2^{(n)} & 0 \\
& 0 & 0 & S_1 & S_1 & 0
\end{array}
$$
with $S_1$ a simple sheaf concentrated in $x_1$ and $S_2^{(n)}$ a sheaf of length $n$ concentrated in $x_2$. □

For the further discussion our next result is of central importance. It expresses a fundamental property of the partition $L = \mathcal{P} \sqcup \mathcal{F}$.

**Proposition 3.2.** Let $L$ be a persistent line bundle. Then the following holds:

1. The functor $F(L, -) = \text{Hom}(L, -)_{\mathcal{F}}$ is generated by $x_1, x_2^2$ if $L$ belongs to the upper bar of $\mathcal{P}$ and by $x_1, x_2$ if $L$ belongs to the lower bar of $\mathcal{P}$. Moreover, in the first case we have a short exact sequence
   $$
   \eta_1 : 0 \longrightarrow L \xrightarrow{(x_1, x_2^2)} L(\vec{x}_1) \oplus L(2\vec{x}_2) \xrightarrow{(-x_2, x_1)} L(\vec{x}_1 + 2\vec{x}_2) \longrightarrow 0,
   $$
   while in the second case the sequence
   $$
   \eta_2 : 0 \longrightarrow L \xrightarrow{(x_1, x_2)} L(\vec{x}_1) \oplus L(\vec{x}_2) \xrightarrow{(-x_2, x_1)} L(\vec{x}_1 + \vec{x}_2) \longrightarrow 0
   $$
   is exact. Moreover, with the exception of $L$, all terms of $\eta_1$ and $\eta_2$ belong to $\text{add}(\mathcal{F})$, and for each $F \in \text{add}(\mathcal{F})$ the sequences $\text{Hom}(\eta_2, F)$ and $\text{Hom}(\eta_1, F)$ are exact.

2. The functor $F(-, L)$ is generated by $x_1, x_2$ if $L$ belongs to the upper bar of $\mathcal{P}$ and is generated by $x_1, x_2^2$ if $L$ belongs to the lower bar of $\mathcal{P}$.

**Proof.** Let $\bar{E} = \text{Hom}(E, \mathcal{O})$ denote the dual bundle of $E$. Then $d : \text{vect-}\mathcal{X} \to \text{vect-}\mathcal{X}$, $E \mapsto \bar{E}(\vec{x}_2)$ defines a self-duality preserving the partition of $L$ into persistent and fading line bundles. It hence suffices to show assertion (1). Applying a suitable shift with $\vec{x} \in \mathbb{Z} \vec{x}_3$ we can assume that $L = \mathcal{O}$ or $L = \mathcal{O}(\vec{x}_2)$. In the first case each morphism $\mathcal{O} \to \mathcal{O}(\vec{y})$ with $\vec{y} \in \mathcal{F}$ factors through $\mathcal{O} \xrightarrow{(x_1, x_2^2)} \mathcal{O}(\vec{x}_1) \oplus \mathcal{O}(2\vec{x}_2)$. In case $L = \mathcal{O}(\vec{x}_2)$ each such morphism $\mathcal{O}(\vec{x}_2) \to \mathcal{O}(\vec{y})$ factors through $\mathcal{O}(\vec{x}_2) \xrightarrow{(x_1, x_2)} \mathcal{O}(\vec{x}_1 + \vec{x}_2) \oplus \mathcal{O}(2\vec{x}_2)$. The preceding lemma now yields the short exact sequences $\eta_2$ and $\eta_1$, respectively, whose middle and end terms are clearly fading. The exactness of the sequences $\text{Hom}(\eta_i, F)$, $i = 1, 2$, then immediately follows. □

**Proposition 3.3.** Let $\eta : 0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$ be a distinguished exact sequence in vect-$\mathcal{X}$. Then the sequence
   $$
   \Phi(\eta) : 0 \longrightarrow \Phi(X') \xrightarrow{\alpha^*} \Phi(X) \xrightarrow{\beta^*} \Phi(X'') \longrightarrow 0
   $$
   is an exact sequence in mod-$\mathcal{P}$.

**Proof.** $\alpha^*$ is injective: Let $L$ be a persistent line bundle and $f' : L \to X'$ a morphism with $\alpha f' \in [\mathcal{F}](L, X)$. Using Proposition 3.2 we obtain a commutative diagram with exact rows

(3.2) $$
\begin{array}{cccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & L_1 + L_2 & \xrightarrow{f''} & L_3 & \longrightarrow & 0 \\
\text{η} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & X'' & \longrightarrow & 0
\end{array}
$$

where $L_1, L_2$ and $L_3$ belong to $\mathcal{F}$. Since the sequence $\eta$ is distinguished exact in vect-$\mathcal{X}$, the morphism $f''$ lifts via $\beta$, so equivalently $f'$ extends to $L_1 + L_2$. Hence $f' \in [\mathcal{F}](L, X')$, as claimed.
we get a commutative diagram

$$\begin{array}{cccc}
0 & \rightarrow & X' & \overset{\alpha}{\rightarrow} & X & \overset{\beta}{\rightarrow} & X'' & \rightarrow & 0 \\
\downarrow{f} & & \downarrow{b} & & \downarrow{a} & & \downarrow{c} & & \downarrow{b} \\
L & \rightarrow & L_1 \oplus L_2 & & & & & & & \\
\end{array}$$

with $L_1, L_2 \in \mathcal{F}$. Now $b$ lifts via $\beta$ to a morphism $b: L_1 \oplus L_2 \rightarrow X$. It follows $\beta(f - ba) = 0$ and hence there exists $f': L \rightarrow X'$ with $\alpha f' = f - ba$ implying $\alpha_s(f') = f$ in $\mathcal{P}(L, X)$.

$\beta_s$ is surjective: This is obvious since $\eta$ is distinguished exact, and then already the mapping $\text{Hom}(L, \beta): \text{Hom}_\mathcal{X}(L, X) \rightarrow \text{Hom}_\mathcal{X}(L, X'')$ is surjective. \hfill \Box

**Proposition 3.4.** For each $E$ from $\text{vect-} \mathcal{X}$ the right $\mathcal{P}$-module $\mathcal{P}(E)$ is finitely presented, indeed finite dimensional. Moreover, for each persistent line bundle $L$ from the upper bar the morphism $x_2^*:\mathcal{P}(L(\bar{x}_2), E) \rightarrow \mathcal{P}(L, E)$, induced by $x_2: L \rightarrow L(\bar{x}_2)$, is a monomorphism.

**Proof.** In vect-$\mathcal{X}$ we choose a distinguished exact sequence $0 \rightarrow E' \rightarrow P \rightarrow E \rightarrow 0$ with $P$ from $\text{add}(\mathcal{L})$. By Proposition 3.3 the induced mapping $\Phi(P) \rightarrow \Phi(E)$ is surjective. Moreover, $\Phi(P)$ is a finitely generated projective module over $\mathcal{P}$, hence finite dimensional. This implies that $\Phi(E)$ is finite dimensional and finitely presented.

Next we show that all maps $x_2^*:\mathcal{P}(L(\bar{x}_2), E) \rightarrow \mathcal{P}(L, E)$ induced by $L \xrightarrow{x_2} L(\bar{x}_2)$, where $L$ is persistent from the upper bar, are injective. Let $f: L(\bar{x}_2) \rightarrow E$ be a morphism such that $fx_2 \in \mathcal{F}$. By Proposition 3.2 we get a commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{x_2} & L(\bar{x}_2) \\
\downarrow{(x_1, x_2)^*} & & \downarrow{f} \\
L(\bar{x}_1) \oplus L(2\bar{x}_2) & \xrightarrow{(g, h)} & E,
\end{array}
$$

and hence $fx_2 = g x_1 + h x_2$, that is, $(f - h x_2)x_2 = g x_1$. Using the pushout property of diagram (3.1) (with $n = 1$) we obtain a morphism $\ell: L(\bar{x}_1 + \bar{x}_2) \rightarrow E$ such that $\ell x_1 = f - h x_2$, and $f \in \mathcal{F}$ follows. \hfill \Box

Together with Proposition 3.3 we get

**Corollary 3.5.** Viewing $\Phi$ as a functor from the Frobenius category vect-$\mathcal{X}$ to the Frobenius category $\mathcal{S}(p)$ the functor is exact, that is, $\Phi$ sends distinguished exact sequences to distinguished exact sequences. \hfill \Box

**The kernel of $\Phi$.** Next, we are going to show that the kernel of $\Phi$ agrees with the ideal $[\mathcal{F}]$ of morphisms factoring through finite direct sums of fading line bundles.

**Lemma 3.6.** Let $\mathcal{X} = \mathcal{X}(2, 3, p)$ with $p \geq 2$. Then the factor group $\mathbb{L}/\mathbb{Z}\bar{x}_3$ is cyclic of order 6 and generated by the class of $\bar{y}$. Moreover both in $\tau$- and $\tau^-$-direction, the $\tau$-orbit of any line bundle in vect-$\mathcal{X}$ consists of persistent and fading bundles according to the 6-periodic pattern $+ - + - - -$, where $+$ and $-$ stand for persisting and fading, respectively.

**Proof.** By construction $\mathbb{L}/\mathbb{Z}\bar{x}_3$ is the abelian group on generators $\bar{x}_1, \bar{x}_2$ with relations $2\bar{x}_1 = 3\bar{x}_2 = 0$, hence $\mathbb{L}/\mathbb{Z}\bar{x}_3$ is cyclic of order 6. Further we have the following congruences modulo $\mathbb{Z}\bar{x}_3$:

$$0 \bar{y} \equiv 0, \quad 1\bar{y} \equiv \bar{x}_1 + 2\bar{x}_2, \quad 2\bar{y} \equiv \bar{x}_2, \quad 3\bar{y} \equiv \bar{x}_1, \quad 4\bar{y} \equiv 2\bar{x}_2, \quad 5\bar{y} \equiv \bar{x}_1 + \bar{x}_2$$
which immediately implies the last claim. □

For every line bundle $L$ denote by $E(L)$ the Auslander bundle associated with $L$, that is, the extension term of the almost split sequence

\[(3.5)\]

\[0 \rightarrow L(\bar{\omega}) \xrightarrow{\alpha} E(L) \xrightarrow{\beta} L \rightarrow 0.\]

**Lemma 3.7.** Let $E$ be an Auslander bundle. Then there exists a persistent line bundle which is a direct summand of the projective cover $P(E)$ of $E$, equivalently we have $\Phi E \neq 0$.

**Proof.** For $L = \mathcal{O}$ the projective cover of the Auslander bundle $E(\mathcal{O})$ in vect-$\mathcal{X}$ is given by the expression

\[(3.6)\]

\[P(E(\mathcal{O})) = \mathcal{O}(\bar{\omega}) \oplus \bigoplus_{i=1}^{3} \mathcal{O}(-\bar{x}_i),\]

and $\mathcal{O}(-\bar{x}_3)$ is persistent. The assertion clearly also holds for $L = \mathcal{O}(n\bar{x}_3)$ ($n \in \mathbb{Z}$).

By the preceding lemma it then suffices to show that after twisting (3.6) with $i\bar{\omega}$ (for $i = 1, \ldots, 5$) there will always exist a persistent line bundle on the right hand side. This follows from Table 1 with entries from $\mathbb{L}$ modulo $\mathbb{Z}\bar{x}_3$, where elements from $\mathbb{P}$ are boxed. Since each row in the table contains an element from $\mathbb{P}$ the claim follows. □

**Lemma 3.8.** Let $X$ be an indecomposable bundle of rank $\geq 2$. Then there exists an Auslander bundle $E$ and a morphism $u: E \rightarrow X$ such that $u \notin [\mathcal{L}]$.

**Proof.** Choose a line bundle $L'$ of maximal degree (=slope) such that there is a morphism $0 \neq h': L' \rightarrow X$. The almost split sequence $\eta: 0 \rightarrow L' \xrightarrow{\alpha} E \xrightarrow{\beta} L'(-\bar{\omega}) \rightarrow 0$ yields a morphism $h: E \rightarrow X$ with $h\alpha = h' \neq 0$. We show $h \notin [\mathcal{L}]$: Otherwise there would be a factorization $h = \sum_{i=1}^{n} b_i a_i$ with morphisms $E \xrightarrow{a_i} L_i \xrightarrow{b_i} X$ and line bundles $L_i$. Then we have

\[0 \neq h\alpha = \sum_{i=1}^{n} b_i a_i \alpha,\]

yielding an index $i$ with non-zero composition $\alpha a_i b_i$. In particular $\mu L' \leq \mu L_i$ and $\text{Hom}(L_i, X) \neq 0$. By the choice of $L'$ we get $\mu L' = \mu L$, thus $a_i \alpha$ is an isomorphism. This implies that $\eta$ splits, a contradiction. □

**Lemma 3.9.** Let $E$ be an Auslander bundle and $u: E \rightarrow X$ a morphism in vect-$\mathcal{X}$ with $\Phi u = 0$. Then $u \notin [\mathcal{F}]$. 

![Table 1. Persistent direct summands of $P(E)$](image)
Proof. We divide the proof into several steps.

(1) Let $E = E(L)$ (as in (3.5)). By (3.6) the projective cover of $E$ is given by

$$P(E) = L(\bar{\omega}) \oplus \bigoplus_{i=1}^{3} L(-\bar{x}_i).$$

By Lemma 3.7 at least one of the line bundles $L(\bar{\omega}), L(-\bar{x}_1), L(-\bar{x}_2), L(-\bar{x}_3)$ is persistent.

(2) Claim. Let $\vec{y} \in \{-\bar{\omega}, \vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Then there is an exact sequence

$$0 \to L(-\vec{y}) \xrightarrow{\alpha'} E \xrightarrow{\beta'} L(\vec{y} + \bar{\omega}) \to 0.$$

Indeed, if $\vec{y} = -\bar{\omega}$, we take the almost split sequence (3.5). If $\vec{y} = \vec{x}_i$, then there is an exact sequence

$$0 \to L(-\vec{x}_i) \xrightarrow{\pi_i} E \xrightarrow{\kappa_i} L(\vec{x}_i + \bar{\omega}) \to 0,$$

where $\pi_i$ is induced by $x_i: L(-\vec{x}_i) \to L$ such that $\beta x_i = x_1$, and similarly $\kappa_i$ is such that $\kappa_i x_i = x_1: L(\bar{\omega}) \to L(\bar{\omega} + \vec{x}_i)$. We have $\kappa_i \pi_i = 0$, since $\text{Hom}(L(-\vec{x}_i), L(\vec{x}_i + \bar{\omega})) = 0$. With $L' = \ker(\alpha)$ we obtain a commutative diagram

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & & & & & \gamma & \gamma \\
0 & L(\bar{\omega}) & \xrightarrow{x_i} E & \xrightarrow{\beta} L & 0 & 0 & 0 & 0 & 0 \\
0 & L(\vec{x}_i + \bar{\omega}) & \xrightarrow{\kappa_i} L(\vec{x}_i + \bar{\omega}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S_i & \xrightarrow{\delta} C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

with exact rows and columns. Note that $\gamma$ is not an isomorphism because otherwise $\beta$ would split. Since $S_i$ is simple then $\delta = 0$, hence $C = 0$, by the snake lemma, and consequently $\kappa_i$ is an epimorphism. As a non-zero map from a line bundle the map $\pi_i$ is a monomorphism, since further the degree is additive on the sequence (3.9), this sequence is exact.

(3) Let $\vec{y} \in \{-\bar{\omega}, \vec{x}_1, \vec{x}_2, \vec{x}_3\}$ so that $L(-\vec{y})$ is persistent, by step (1). It follows that there is a short exact sequence

$$0 \to L(-\vec{y}) \xrightarrow{a} L_1 \oplus L_2 \xrightarrow{b} L_3 \to 0$$

with fading line bundles $L_1, L_2, L_3$, and satisfying the properties of Proposition 3.2. Since $\Phi u = 0$ and thus $u \alpha' \in [F]$ we obtain a commutative square

\[
\begin{array}{ccc}
E & \xrightarrow{u} & X \\
\alpha' \downarrow & & \downarrow \alpha' \\
L(-\vec{y}) & \xrightarrow{a} & L_1 \oplus L_2.
\end{array}
\]
We next form the pushout diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
L(\bar{y} + \bar{\omega}) \quad \quad \quad L(\bar{y} + \bar{\omega}) \\
\downarrow \quad \beta' \\
E \quad \quad \quad \quad \quad E \\
\downarrow \quad \alpha' \\
0 \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\downarrow \\
0 \\
L(-\bar{y}) \quad \quad \quad L_1 \oplus L_2 \\
\downarrow \\
L_3 \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\downarrow \\
0 \\
0 \\
\end{array}
\]

Since \( u \) factors through \( \overline{E} \), it is sufficient to show that \( \overline{E} \in \text{add}(\mathcal{F}) \).

One checks easily that \( L(\bar{y} + \bar{\omega}) \) is fading. (For \( \bar{y} = \bar{\omega} \) this follows from the 6-periodic pattern in Lemma 3.6.) Therefore it is sufficient to show that \( \text{Ext}^1(L(\bar{y} + \bar{\omega}), L_i) = 0 \), by Serre duality equivalently, that \( \text{Hom}(L_i, L(\bar{y} + 2\bar{\omega})) = 0 \) (for \( i = 1, 2 \)).

By Proposition 3.2 we can assume that \( L_1 = L(-\bar{y} + \bar{x}_1) \), and \( L_2 = L(-\bar{y} + 2\bar{x}_2) \) if \( L(-\bar{y}) \) is from the upper bar, and \( L_2 = L(-\bar{y} + \bar{x}_2) \) if \( L(-\bar{y}) \) is from the lower bar. Therefore, one has to check whether \( \text{Hom}(\mathcal{O}, \mathcal{O}(2\bar{y} + 2\bar{\omega} - \bar{x})) \) is zero, that is, whether \( 2\bar{y} + 2\bar{\omega} - \bar{x} \not\leq 0 \) for \( \bar{x} \in \{\bar{x}_1, \bar{x}_2, 2\bar{x}_2\} \). There are two cases:

1. case. Assume that \( P(E) \) admits a direct summand \( L(-\bar{y}) \) which is a persistent line bundle from the upper bar. In this case \( \bar{x} \in \{\bar{x}_1, 2\bar{x}_2\} \). Table 1 shows that we can assume \( L = \mathcal{O}(i\bar{\omega}) \) for \( i = 0, 2, 3, 5 \), and the value of \( \bar{y} \) can also extracted from that table. In all these cases it is easy to see that the condition \( 2\bar{y} + 2\bar{\omega} - \bar{x} \not\leq 0 \) is satisfied.

2. case. Assume that each persistent line bundle summand of \( P(E) \) is from the lower bar. In this case \( \bar{x} \in \{\bar{x}_1, \bar{x}_2\} \). Table 1 shows that we can assume \( L = \mathcal{O}(\bar{\omega}) \) and \( \bar{y} = -\bar{\omega} \), or \( L = \mathcal{O}(4\bar{\omega}) \) and \( \bar{y} = \bar{x}_2 \). In these cases again the condition \( 2\bar{y} + 2\bar{\omega} - \bar{x} \not\leq 0 \) holds. \( \square \)

**Proposition 3.10.** Let \( X \in \text{vect-}\mathcal{X} \) be indecomposable such that \( \Phi X = 0 \). Then \( X \in \mathcal{F} \).

**Proof.** If \( X \) is a line bundle this is clear. Assume \( \text{rk} X \geq 2 \). By Lemma 3.8 we obtain a morphism \( u: E \to X \) where \( E \) is an Auslander bundle and \( u \not\in [\mathcal{L}] \). By Lemma 3.9 we get \( 0 \neq \Phi u: \Phi E \to \Phi X \), in particular \( \Phi X \neq 0 \). \( \square \)

Let \( \sigma_i, i = 1, 2, 3 \), denote the shift (line bundle twist) \( E \mapsto E(\bar{x}_i) \). Then the isomorphism classes of line bundles decompose into 6 orbits under the action of the group \( \langle \sigma_3 \rangle \).

**Corollary 3.11.** Let \( X \) be an indecomposable bundle of \( \text{rk} X \geq 2 \). Then the projective cover \( P(X) \) of \( X \) admits a persistent line bundle as a direct summand. Moreover \( P(X) \) admits line bundle summands from at least four pairwise distinct \( \langle \sigma_3 \rangle \)-orbits.

**Proof.** Since \( \Phi(X) = 0 \) if and only if \( P(X) \) contains a persistent line bundle, the first claim immediately follows. For each integer \( n \), then also \( P(X)(n\bar{\omega}) = P(X(n\bar{\omega})) \) contains a persistent line bundle. Recall that the class of \( \bar{\omega} \) generates \( \mathbb{L}/\mathbb{Z}\mathbb{X}_3 \) which is cyclic of order 6 and that the classes of 0 and \( 2\bar{\omega} \) represent the persistent members of \( \mathbb{L} \). Let \( U \) be the subset of \( \mathbb{L}/\mathbb{Z}\mathbb{X}_3 \) corresponding to the \( \langle \sigma_3 \rangle \)-orbits of line bundles.
in \( P(X) \). By the above, for each integer \( n \) the set \( U \) must contain \( n \) or \( n + 2 \). As is easily checked this implies \( |U| \geq 4 \), proving the claim.

Actually there are, up to cyclic permutation, just two possibilities for a four-element subset \( U \) as above, given by the two following patterns, where a black dot indicates membership in \( U \).

\[
\begin{array}{ccc}
\bullet & \bullet & \circ \\
\circ & \bullet & \bullet \\
\bullet & \circ & \bullet
\end{array}
\]

\[
\begin{array}{ccc}
\circ & \bullet & \bullet \\
\bullet & \bullet & \circ \\
\bullet & \circ & \bullet
\end{array}
\]

Lemma 3.12. Assume \( P \in \text{add}(\mathcal{L}) \). There exists an exact sequence \( \eta: 0 \to P \xrightarrow{\alpha} P_0 \xrightarrow{\beta} P_1 \to 0 \) in \( \text{coh-}X \) with \( P_0, P_1 \) from \( \text{add}(\mathcal{F}) \) such that:

1. for each persistent line bundle \( L' \) the sequence \( \text{Hom}(L', \eta) \) is exact;
2. for each fading line bundle \( L' \) the sequence \( \text{Hom}(\eta, L') \) is exact.

Proof. It suffices to show the statement if \( P = L \) is indecomposable. If \( L \in \mathcal{F} \), then one can take \( \eta: 0 \to L \to L \to 0 \to 0 \). Let now \( L \in \mathcal{P} \). Then let

\[
\eta = \eta_n: 0 \to L \xrightarrow{(x_1, x_2)} L(\vec{x}_1) \oplus L(n\vec{x}_2) \xrightarrow{(-x_2^2, x_1)} L(\vec{x}_1 + n\vec{x}_2) \to 0
\]

be one of the sequences from Proposition 3.2, where \( n = 2 \) if \( L \) is from the upper bar and \( n = 1 \) if \( L \) is from the lower bar. Condition (2) follows from 3.2. Let \( L' \) be a persistent line bundle and \( h: L' \to L(\vec{x}_1 + n\vec{x}_2) \) a morphism. Since \( L \) and \( L' \) are persistent, one shows as in the proof of Lemma B that \( h \) factors through \( x_1: L(n\vec{x}_2) \to L(\vec{x}_1 + n\vec{x}_2) \), in particular through the middle term of \( \eta \). This shows condition (1).

The next result constitutes a key step in our proof of Theorems A and C.

Proposition 3.13. Each morphism \( h: E \to F \) in \( \text{vect-}X \) with \( \Phi(h) = 0 \) belongs to the ideal \( \mathcal{D} \), that is, \( h \) factors through a member of \( \text{add}(\mathcal{F}) \).

Proof. Let \( P \xrightarrow{\pi} E \to 0 \) be a distinguished epimorphism with \( P \in \text{add}(\mathcal{L}) \). Since \( \Phi h = 0 \), the composition \( h \pi \) factors through an object \( \text{add}(\mathcal{F}) \), and by (2) of the preceding lemma we obtain a diagram

\[
\begin{array}{ccc}
\eta: 0 & \xrightarrow{\pi} & P & \xrightarrow{\alpha} & P_0 & \xrightarrow{\beta} & P_1 & \to 0 \\
\downarrow h \pi & & \downarrow \circ \gamma & & \downarrow \circ \delta & & \downarrow \circ \delta & & \downarrow \circ \delta \\
& F & & & & & & & \end{array}
\]

where \( \eta \) is an exact sequence in \( \text{coh-}X \) with \( P_0, P_1 \in \text{add}(\mathcal{F}) \). From this we get a commutative diagram

\[
\begin{array}{ccc}
\mu: 0 & \xrightarrow{(\pi, \alpha)} & P \oplus P_0 & \xrightarrow{(\sigma_1, \sigma_2)} & E \oplus P_0 & \xrightarrow{\gamma} & C & \to 0 \\
\downarrow (h, -\gamma) & & \downarrow \circ \beta & & \downarrow \circ \beta & & \downarrow \circ \beta & & \downarrow \circ \beta \\
& F & & & & & & & \end{array}
\]
in coh-\(\mathcal{X}\) whose row is exact. It suffices to show that the cokernel \(C\) lies in \(\text{add}(\mathcal{F})\).

To prove this consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & P & \xrightarrow{\alpha} & P_0 & \xrightarrow{\beta} & P_1 & \to & 0 \\
\eta: 0 & \to & P & \xrightarrow{\kappa=(\pi,\alpha)} & E & \oplus & P_0^\sigma=\langle\sigma_1,\sigma_2\rangle & \to & 0 \\
\mu: 0 & \to & E & \xrightarrow{(0,1)} & E & \oplus & P_0^\sigma=\langle\sigma_1,\sigma_2\rangle & \to & 0 \\
0 & \to & 0 & \to & E & \xrightarrow{(1,0)} & E & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

with exact rows and columns. For each persistent line bundle \(L\), applying the functor \(\text{Hom}(L, -)\) the first (compare part (1) of the preceding lemma) and the third row, and the first and the second column stay exact. It follows that also the third column stays exact implying that \(\text{Hom}(L, \mu)\) is exact for each \(L \in \mathcal{P}\), in particular \(\text{Hom}(L, \sigma)\) is an epimorphism for \(L \in \mathcal{P}\). We conclude that \(\Phi\sigma: \mathcal{I}(\sigma)\) is an epimorphism. Since \(\Phi\varepsilon = \Phi\pi\) is also an epimorphism, the composition \(0 = \Phi(\sigma\varepsilon) = \Phi(\sigma\Phi(\varepsilon))\) is an epimorphism as well, yielding \(\Phi C = 0\), equivalently \(C \in \text{add}(\mathcal{F})\).

\(\Phi\) is full. Our next lemma plays a key role in order to show that the functor \(\Phi: \text{vect-}\mathcal{X} \to \text{mod-}\mathcal{P}\) is full.

**Lemma 3.14.** We assume that \(\eta: 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E \to 0\) is a distinguished exact sequence in \(\text{vect-}\mathcal{X}\). Then \(\beta = \text{coker}(\alpha)\) holds in \(\text{vect-}\mathcal{X}/[F]\).

*Proof.* (1) \(\beta\) is an epimorphism in \(\text{vect-}\mathcal{X}/[F]\): To this end let \(f: E \to X\) be a morphism in \(\text{vect-}\mathcal{X}\) such that \(f\beta \in \ker \Phi\). Then \((\Phi f)(\Phi \beta) = 0\). By Proposition 3.3, \(\Phi \beta\) is an epimorphism, thus \(\Phi f = 0\), that is, \(f \in \ker \Phi\).

(2) Let \(h: E \to X\) be a morphism in \(\text{vect-}\mathcal{X}\) such that \(ha \in \ker \Phi\). We make use of the fact \(\ker \Phi = [F]\). Hence there is \(P \in \text{add}(\mathcal{F})\) such that \(h\alpha = \frac{E'}{\alpha} \to P \xrightarrow{\beta} X\). Since \(\eta\) is distinguished exact there is a morphism \(a': E \to P\) such that \(a'\alpha = a\). We obtain \((h - ba')\alpha = 0\). Since \(\eta\) is exact, there is a morphism \(h': E' \to X\) with \(h - ba' = h'\beta\), which leads to \(h = h'\beta\) modulo \([F]\) = \(\ker \Phi\). \(\square\)

**Proposition 3.15.** The functor \(\Phi: \text{vect-}\mathcal{X} \to \text{mod-}\mathcal{P}\) is full, and induces a full embedding \(\text{vect-}\mathcal{X}/[F] \to \mathcal{S}(p)\).

*Proof.* Let \(h: \Phi E \to \Phi F\) be a morphism in \(\text{mod-}\mathcal{P}\). Consider projective covers in \(\text{vect-}\mathcal{X}\):

\[
\begin{align*}
0 & \to E' \xrightarrow{\alpha} P \xrightarrow{\beta} E \to 0, \\
0 & \to F' \xrightarrow{\gamma} Q \xrightarrow{\delta} F \to 0.
\end{align*}
\]
By Proposition 3.3 we get a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & E' & \overset{\Phi_0}{\longrightarrow} & P & \overset{\Phi_\beta}{\longrightarrow} & E & \longrightarrow & 0 \\
0 & \longmapsto & F' & \overset{\Phi_\gamma}{\longrightarrow} & Q & \overset{\Phi_\delta}{\longrightarrow} & F & \longrightarrow & 0.
\end{array}
\]

We have \(\Phi(\delta u \alpha) = \Phi(\delta) \Phi(u) \Phi(\alpha) = 0\), hence \((\delta u) \alpha = 0\) in \(\text{vect-} \mathcal{X}/\ker \Phi\). By the preceding lemma there is a morphism \(v: E \to F\) with \(v \beta = \delta u\). Applying \(\Phi\) we get \((h - \Phi v) \Phi \beta = 0\). Since \(\Phi \beta\) is an epimorphism we get \(h = \Phi(v)\).

**Reflecting exactness.** The next proposition turns out to be crucial in comparing the exact structures of \(\text{vect-} \mathcal{X}, \text{vect-} \mathcal{X}/[\mathcal{F}], \mod \mathcal{P}\) and \(\tilde{S}(p)\).

**Proposition 3.16.** Let \(\eta: 0 \to E' \overset{\alpha}{\longrightarrow} E \overset{\beta}{\longrightarrow} E'' \to 0\) be a sequence in \(\text{vect-} \mathcal{X}\) such that \(\Phi(\eta)\) is exact in \(\mod \mathcal{P}\). Modifying terms by adding suitable summands from \(\text{add}(\mathcal{F})\), we can change \(\eta\) to a distinguished exact sequence \(\hat{\eta}\) in \(\text{vect-} \mathcal{X}\) such that \(\Phi(\eta)\) and \(\Phi(\hat{\eta})\) are isomorphic in \(\mod \mathcal{P}\) and, accordingly, \(\eta\) and \(\hat{\eta}\) are isomorphic in \(\text{vect-} \mathcal{X}/[\mathcal{F}]\).

**Proof.** Let \(P \overset{\pi}{\longrightarrow} E'' \to 0\) in \(\text{vect-} \mathcal{X}\) be a projective cover. Then \(\Phi \beta\) is an epimorphism, that is, in \(\text{vect-} \mathcal{X}/[\mathcal{F}]\) the morphism \(\beta\) can be lifted to \(E\). That is, there is a morphism \(\hat{\pi}\) such that \(\hat{\beta} \pi - \pi\) factors through a bundle \(P_f\) which is a direct sum of fading line bundles, say \(\beta \pi - \pi = [P \overset{h}{\longrightarrow} P_f \overset{\bar{h}}{\longrightarrow} E''\). It then follows that \(P_f \oplus E \overset{(\beta, -h)}{\longrightarrow} E''\) is a distinguished epimorphism. If \(K\) denotes its kernel we obtain a commutative diagram of distinguished exact sequences in \(\text{vect-} \mathcal{X}\):

\[
\begin{array}{ccccccccc}
\hat{\eta}: 0 & \longrightarrow & K & \overset{\hat{\alpha}}{\longrightarrow} & P_f \oplus E & \overset{(\beta, -h)}{\longrightarrow} & E'' & \longrightarrow & 0 \\
\eta: 0 & \longmapsto & E' & \overset{\alpha}{\longrightarrow} & E & \overset{\beta}{\longrightarrow} & E'' & \longrightarrow & 0.
\end{array}
\]

Applying \(\Phi\) to this diagram we obtain that \(\Phi((1,0)^t)\) and hence \(\Phi(\gamma)\) are isomorphisms. Since \(\Phi: \text{vect-} \mathcal{X} \to \mod \mathcal{P}\) is a full embedding, \(\gamma\) becomes an isomorphism in the factor category \(\text{vect-} \mathcal{X}/[\mathcal{F}]\). \(\square\)

**\(\Phi\) is dense.** The next lemma will serve as an induction step to prove that the functor \(\Phi\): \(\text{vect-} \mathcal{X} \to \tilde{S}(p)\) is dense.

**Lemma 3.17.** Let \(L\) be a persistent line bundle and \(\eta: 0 \to L(\mathcal{X}) \overset{\alpha}{\longrightarrow} E \overset{\beta}{\longrightarrow} L \to 0\) the corresponding almost split sequence in \(\text{vect-} \mathcal{X}\). Application of \(\Phi\) yields an exact sequence

\[
0 \longrightarrow \Phi(E) \overset{\Phi(\beta)}{\longrightarrow} \Phi(L) \overset{\pi}{\longrightarrow} S \longrightarrow 0
\]

in \(\mod \mathcal{P}\), where \(S\) is a simple module (not necessarily lying in \(\tilde{S}(p)\)).

**Proof.** By assumption \(\mathcal{X}\) has exactly three weights, therefore the Auslander bundle \(E\) is indecomposable and hence \(\Phi(\beta)\): \(\Phi(E) \to \Phi(L)\) is not an isomorphism since \(\Phi\) induces a full embedding \(\text{vect-} \mathcal{X}/[\mathcal{F}] \hookrightarrow \mod \mathcal{P}\). The modules \(\Phi(E)\) and \(\Phi(L)\) have local endomorphism rings; moreover, \(\Phi(L)\) is indecomposable projective. Denote by \(\pi: \Phi(L) \to S\) the natural projection on the simple top. Since the mapping \(\Phi(\beta)\) belongs to the radical of \(\mod \mathcal{P}\) we obtain \(\pi \beta = 0\).

We claim that the map \(\Phi(\beta): \Phi(E) \to \Phi(L)\) is injective. Indeed let \(L_1\) be a persistent line bundle and \(f: L_1 \to E\) such that \(\Phi(\beta f) = 0\). This yields a factorization \(\beta f = [L_1 \overset{\alpha}{\longrightarrow} P \overset{b}{\longrightarrow} L]\) with \(P\) from \(\text{add}(\mathcal{F})\). As a radical morphism \(b\)
then lifts via $\beta$, thus $b = \bar{b}$ for some morphism $\bar{b}: P \to E$. We obtain $\beta(f - ba) = 0$ such that $f - ba$ factors (via $\alpha$) over $L(\omega)$. Since $L \in P$, equation (3.4) from Lemma 3.6 shows that $L(\bar{\omega})$ is fading. It follows that $f$ belongs to $[F]$, proving the claim.

By the preceding argument we obtain an exact sequence $0 \to \Phi(E) \to \Phi(L) \to C \to 0$. We claim that the cokernel term $C$ is a simple $P$-module. We first show that $C$ — viewed as a representation of $P$ — has support $\{L\}$, and hence is semisimple. For each persisting line bundle $L_1$, not isomorphic to $L$, each morphism $\gamma: \Phi(L_1) \to C$ lifts by projectivity of $\Phi(L_1)$ to a morphism $\Phi(U): \Phi(L_1) \to \Phi(L)$. Since $\eta$ is almost split the non-isomorphism $U: L_1 \to L$ lifts via $\beta$, then implying that $\gamma = 0$. We have shown that $C \cong S^n$ where $S = S_L$ denotes the simple module concentrated in $L$. Moreover, $n \geq 1$ since $C \neq 0$. As an indecomposable projective module $\Phi(L)$ is local, and we conclude that $n = 1$, implying that $C$ is simple. □

**Proposition 3.18.** For each module $M$ in $\tilde{S}(p)$ there exists a bundle $E$ such that $\Phi(E)$ is isomorphic to $M$.

**Proof.** We argue by induction on the (finite) dimension $n$ of $M$. If $n = 0$, the assertion is evident. So assume that $n > 0$. Then we obtain an exact sequence $0 \to M' \to M \to S \to 0$ in $\text{mod-}P$, where $S$ is simple and $M' \in \tilde{S}(p)$.

Invoking Lemma 3.17 we obtain a commutative diagram in $\text{mod-}P$ with exact rows and columns

\[
\begin{array}{c}
0 & 0 \\
\Phi E & \Phi E \\
\Phi \beta & \\
\mu: 0 & M' & M & \Phi(L) & 0 \\
0 & M' & M & S & 0 \\
0 & 0 & \end{array}
\]

Since $\Phi(L)$ is projective the sequence $\mu$ splits yielding $M = M' \oplus \Phi(L)$. By induction $M'$ belongs to the image of $\Phi$, say $M' = \Phi(F')$. Summarizing we obtain an exact sequence

\[
0 \to \Phi E \xrightarrow{(\Phi \beta, \Phi U')} \Phi L \oplus \Phi F' \to M \to 0
\]

in $\text{mod-}P$. We put $\bar{x}_i = \bar{x}_i + \bar{\omega}$ and form the injective hull

\[
0 \to E \xrightarrow{a} L \oplus \bigoplus_{i=1}^3 L(\bar{x}_i) \to E(\bar{x}_1) \to 0
\]

of $E$ (compare (3.7)), where $a = (\beta, \kappa_1, \kappa_2, \kappa_3)'$, with $\beta$ from the almost split sequence (3.5) and the $\kappa_i$ like in (3.9). We obtain in $\text{coh-}X$ the exact sequence

\[
\gamma: 0 \to E \xrightarrow{(\beta, (\bar{x}_1), u')} L \oplus \bigoplus_{i=1}^3 L(\bar{x}_i) \oplus F' \to E(\bar{x}_1) \to 0,
\]

which is distinguished exact in $\text{vect-}X$. There are two cases:
1. case. \( L \) belongs to the upper bar. Then all line bundles \( L(\bar{x}_i) \) are fading \((i=1, 2, 3)\), and

\[
\Phi \gamma : 0 \to \Phi E \xrightarrow{(\Phi \beta, \Phi u')} \Phi L \oplus \Phi F' \to \Phi \left( E(\bar{x}_1) \right) \to 0
\]

is exact. Comparing this with (3.11) we obtain \( \Phi \left( E(\bar{x}_1) \right) \cong M \).

2. case. \( L \) belongs to the lower bar. Then \( L(\bar{x}_1) \) is persistent, and \( L(\bar{x}_2), L(\bar{x}_3) \) are fading. We obtain the diagram

\[
\begin{array}{ccc}
0 & \to & E \\
\downarrow & & \downarrow \\
0 & \to & \Phi(L(\bar{x}_1)) \oplus \Phi(F') \\
\downarrow & & \downarrow \\
0 & \to & \Phi(C) \\
\downarrow & & \\
0 & \to & \Phi(L(\bar{x}_1)) \\
\end{array}
\]

The sequence

\[
0 \to L(\bar{x}_1) \to \Phi C \to M \to 0
\]

is exact with all terms lying in \( \tilde{S}(p) \). This sequence splits since \( L(\bar{x}_1) \) is injective in \( \tilde{S}(p) \). We get \( \Phi C = M \oplus L(\bar{x}_1) \). Write \( C = \bigoplus_{i=1}^{n} C_i \) with all \( C_i \in \text{vect-}\mathcal{X} \) indecomposable. Since \( \Phi \) is full the \( \Phi C_i \neq 0 \) have local endomorphism rings. It follows, that \( M \) is the direct sum of some of the \( \Phi C_i \), hence \( M \) lies in the image of \( \Phi \).

For later applications we need a related result:

**Proposition 3.19.** Let \( L \) be a persisting line bundle from the upper bar and let \( S_L \) be the simple right \( \mathcal{X} \)-module concentrated in \( L \). Then \( S_L \) belongs to \( \tilde{S}(p) \) and has the form \( \Phi \left( E(L)(\bar{x}_1) \right) \), where \( E(L) \) denotes the Auslander bundle attached to \( L \).

Moreover, each simple \( \mathcal{X} \)-module belonging to \( \tilde{S}(p) \) has the above form.

**Proof.** This follows from the proof (1. case) of Proposition 3.18 (with \( M = S \) and hence \( F' = 0 \)).

**Frobenius structure and proof of Theorems A and C.** Define a sequence

\[
0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0
\]

in \( \text{vect-}\mathcal{X}/[\mathcal{F}] \) to be distinguished exact if it is isomorphic to a sequence which is induced by a distinguished exact sequence in \( \text{vect-}\mathcal{X} \).

We will prove now Theorems A and C. Part (1) from Theorem A was already shown before, part (3) is trivial.

By Propositions 3.4, 3.13, 3.15 and 3.18 the assignment \( E \mapsto \mathcal{X}(-, E) \) induces an equivalence of categories \( \Phi : \text{vect-}\mathcal{X}/[\mathcal{F}] \to \tilde{S}(p) \). It follows from Propositions 3.3 and 3.16 that a sequence \( \eta : 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0 \) in \( \text{vect-}\mathcal{X}/[\mathcal{F}] \) is distinguished exact if and only if \( \Phi(\eta) \) is exact in \( \tilde{S}(p) \). It follows (via \( \Phi \)) that the distinguished exact sequences give \( \text{vect-}\mathcal{X}/[\mathcal{F}] \) the structure of a Frobenius category, and moreover, such that the indecomposable projective-injective objects are given by the objects of \( \mathcal{X} \). This proves part (2) from Theorem A. Hence \( \Phi : \text{vect-}\mathcal{X}/[\mathcal{F}] \to \tilde{S}(p) \) is even an equivalence of Frobenius categories, which shows the first statement of Theorem C. The second statement of Theorem C is an immediate consequence of
the first together with Theorem A (3). The last assertion of Theorem C on the shift-commutation of $\Phi$ follows by construction.

4. Applications

Theorem C allows to obtain the main results from [20] and further properties as direct consequences of properties from the theory of weighted projective lines. Indeed, as a general rule, we will prove results first for the category vect-$\mathbb{X}$ or the stable category vect-$\mathbb{X}$ of vector bundles, and then export such results to $\tilde{S}(p)$ or $\tilde{S}(p)$.

Action of the Picard group. Obviously, the $L$-action on vect-$\mathbb{X}$ by line bundle twist (= grading shift) induces an $L$-action on vect-$\mathbb{X}$. By transport of structure, Theorem C then induces an $L$-action on $\tilde{S}(p)$. This action of the Picard group of $\mathbb{X}$ on vect-$\mathbb{X} = \tilde{S}(p)$ reveals a certain amount of symmetry of vect-$\mathbb{X}$ which is instrumental in proving most of the properties to follow. (An important example is the Calabi-Yau property to be discussed later. By contrast the treatment of the Fuchsian singularities in [10, 13] lacks this amount of symmetry and only yields a finite number of categories which are fractionally Calabi-Yau.)

**Proposition 4.1.** The Picard group $\mathbb{L} = L(2, 3, p)$ acts on $\tilde{S}(p)$. Let $s$ denote the automorphism induced by the grading shift of mod-$\mathcal{P}$. Then the generators $\tilde{x}_i$ of $L$ act as follows on $\tilde{S}(p)$

(i) $\tilde{x}_1$ acts as $\tau^3 s^3$,
(ii) $\tilde{x}_2$ acts as $\tau^2 s^2$,
(iii) $\tilde{x}_3$ acts as $s$.

The proof immediately follows from the next lemma.

**Lemma 4.2.** Let $\mathbb{L} = L(2, 3, p)$. Then $\mathbb{L}$ is generated by $\tilde{x}_3$ and $\tilde{\omega}$. Moreover, we have with $\tilde{x}_1 = \tilde{x}_1 + \tilde{\omega}$

(i) $\tilde{x}_1 = \tilde{x}_2 + \tilde{x}_3$,
(ii) $\tilde{x}_2 = 2\tilde{x}_3$,
(iii) $\tilde{x}_3 = 3\tilde{x}_3$.

**Proof.** (i) We have $\tilde{x}_2 + \tilde{x}_3 = 2\tilde{\omega} + \tilde{x}_2 + \tilde{x}_3 = 2\tilde{\omega} - 2\tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3 = \tilde{x}_1 + \tilde{\omega} + \tilde{x}_3 = \tilde{x}_3$.
(ii) $2\tilde{x}_3 = 2\tilde{\omega} - 2\tilde{x}_1 - 2\tilde{x}_2 = \tilde{\omega} - 2\tilde{x}_2 = \tilde{x}_2$.
(iii) $3\tilde{x}_3 = 3\tilde{\omega} - 3\tilde{x}_1 - 3\tilde{x}_2 = \tilde{x}_1$. □

**Corollary 4.3.** The suspension functor of $\tilde{S}(p)$ acts as $\tau^3 s^3$.

**Proof.** For weight type $(2, p, q)$ it is shown in [11] that the shift with $\tilde{x}_1$ serves as the suspension functor for vect-$\mathbb{X}$. □

Tilting objects and Orlov’s trichotomy. First we establish two tilting objects in vect-$\mathbb{X} = \tilde{S}(p)$ with non-isomorphic endomorphism rings.

**Proposition 4.4.** Assume $U$ is a simple right $\mathcal{P}$-module lying in $\tilde{S}(p)$. Then

$$\bigoplus_{a=0, \ldots, p-2, b=0,1} \tau^{4a+b}s^{3a+b}(U)$$

is a tilting object in $\tilde{S}(p)$ with endomorphism ring $A(2(p-1), 3)$; see (1.2).
Proof. Let $E$ be an Auslander bundle. With $\bar{x}_i = \bar{x}_i + \bar{\omega}$ we put
\[ M = \{ a\bar{x}_1 + b\bar{x}_3 \mid a = 0, \ldots, p - 2, \ b = 0, 1 \} \]
and define $T$ as the direct sum of all $E(\bar{x})$, with $\bar{x}$ in $M$. It is shown in [11] that
$T$ is a tilting object of vect-$\mathcal{X}$ with endomorphism ring $\text{End}(T) = A(2(p - 1), 3)$. Transferred to $\tilde{\mathcal{X}}(p)$ this yields the claim by Proposition 3.19.

For $\bar{x} \in M := \{ b\bar{x}_2 + c\bar{x}_3 \mid b = 0, 1; \ c = 0, \ldots, p - 2 \}$ we define $E(\bar{x})$ as the extension term of the unique non-split exact sequence $0 \to O(\bar{x}) \to E(\bar{x}) \to O(\bar{x}) \to 0$. By [11] the system $T = \bigoplus_{\bar{x} \in M} E(\bar{x})$ is a tilting object in vect-$\mathcal{X}$ with $\text{End}(T) = B(2, p - 1)$, the incidence algebra of the poset $\mathcal{P}$ that is the $2 \times (p - 1)$-rectangle with all commutativities. Note that such diagrams appear in singularity theory. By applying Theorem C we thus obtain the following result

**Proposition 4.5.** The category $\tilde{\mathcal{X}}(p) = \text{vect-$\mathcal{X}$}$ with $\mathcal{X} = \mathcal{X}(2, 3, p)$ has a tilting object $T$ whose endomorphism ring is the algebra $A(2(p - 1), 3)$ and $B(2, p - 1)$ are derived equivalent.

By $B'(2, p - 1)$ we denote the incidence algebra of the poset (fully commutative quiver)

\[
\begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \cdots & p - 3 & \rightarrow & p - 2 & \rightarrow & p - 1 \\
1' & \rightarrow & 2' & \rightarrow & 3' & \cdots & (p - 3)' & \rightarrow & (p - 2)' \\
\end{array}
\]

**Corollary 4.6.** Let $S$ be a simple $\mathcal{P}$-module belonging to $\tilde{\mathcal{X}}(p)$. Then the right perpendicular category $S^\perp$, consisting of all objects $X$ from $\tilde{\mathcal{X}}(p)$ satisfying $\text{Hom}(S,X[n]) = 0$ for each integer $n$, is triangulated with Serre duality. Moreover, the category $S^\perp$ has tilting objects $U$ and $U'$ such that $\text{End}(U) \cong A(2p - 3, 3)$ and $\text{End}(U') \cong B'(2, p - 1)$. In particular, the algebras $A(2p - 3, 3)$ and $B'(2, p - 1)$ are derived equivalent.

Proof. We switch to the category vect-$\mathcal{X}$, where we have to calculate the category $E^\perp$ for an Auslander bundle $E$. The first claim follows from Proposition 4.4 and its proof. For the second claim we use Proposition 4.5 and the fact that $E(\bar{x}_2 + (p - 2)\bar{x}_3)$ is an Auslander bundle.

By different methods the derived equivalence of the algebras $A(2p - 1, 3)$ and $B(2, p - 1)$ was shown independently by S. Ladjkani [12].

Returning to the context of Proposition 4.5, we want to give the tilting object $\Phi(T)$ of $\tilde{\mathcal{X}}(p)$ a more concrete shape. We briefly point out what the $\Phi(E(\bar{x}))$ are in the language of the category $\tilde{\mathcal{X}}(p)$. For this let $\mathcal{P}^\text{up}$, resp. $\mathcal{P}^\text{low}$, denote the full subcategory of $\mathcal{P}$ formed by the objects of the upper (resp. lower) bar. Moreover, we identify mod-$\mathcal{P}^\text{up}$ with the full subcategory of mod-$\mathcal{P}$ of all modules whose support is contained in $\mathcal{P}^\text{up}$. Further we identify mod-$\mathcal{P}$ with the category mod-$\mathcal{X}$ of finitely generated $\mathbb{Z}$-graded modules over the algebra $A = k[x]/(x^p)$ with $x$ having degree one.

**Lemma 4.7.** (a) The restriction functor $\rho : \text{mod-$\mathcal{P}$} \to \text{mod-$\mathcal{P}^\text{up}$}$ has an exact left adjoint $\lambda$ sending the indecomposable $\mathcal{P}^\text{up}$-projective $P(\bar{x} + \bar{\omega})$ to the indecomposable $\mathcal{P}$-projective $P(\bar{x})$.

(b) Putting $T^\text{up} = \bigoplus_{j=0}^{p-2} E(j\bar{x})$ and $T^\text{low} = \bigoplus_{j=0}^{p-2} E(j\bar{x}_1 + \bar{x}_2)$, the tilting object $T$ from the above proposition has the form $T = T^\text{up} \oplus T^\text{low}$. Moreover, with the above identifications this yields:

\[ \Phi(T^\text{up}) = \bigoplus_{j=0}^{p-2} x^{j+1} A(j) \quad \text{and} \quad \Phi(T^\text{low}) = \bigoplus_{j=0}^{p-2} \lambda(x^{j+1} A(j)). \quad \Box \]
Independently this tilting object in \( \tilde{S}(p) \) was constructed by Xiao-Wu Chen [4] with a direct argument not relying on Theorem C.

Since the Grothendieck group \( K_0(\text{coh-}\mathcal{X}) \) is free abelian of rank \( p + 4 \), see [7], we obtain from Proposition 4.4 or Proposition 4.5 the next result.

**Corollary 4.8.** The Grothendieck group of \( \text{vect-}\mathcal{X}(2, 3, p) = \tilde{S}(p) \) is free abelian of rank \( 2(p - 1) \). Moreover, we have \( \text{rk} \, K_0(\tilde{S}(p)) - \text{rk} \, K_0(\text{coh-}\mathcal{X}) = p - 6 \). \( \square \)

This result serves as a nice illustration of an \( \mathbb{L} \)-graded version of Orlov’s theorem [17]. For this we recall from the Introduction, (1.4) that there are natural equivalences

\[
(4.1) \quad \mathcal{T} := D_{Sg}^{b, \omega}(S) = \text{CM}^{\mathbb{L}}-S = \text{vect-}\mathcal{X} = \tilde{S}(p), \quad \text{where} \quad \mathcal{X} = \mathcal{X}(2, 3, p)
\]

where we have used Theorem C for the last identification. It follows from an \( \mathbb{L} \)-graded version of Orlov’s theorem [17] that the comparison between \( D^0(\text{coh-}\mathcal{X}) \) and any of the four triangulated categories above follows a trichotomy determined by the Gorenstein parameter of the singularity. In the present \( \mathbb{L} \)-graded setting this index equals \( 6 - p \), compare Corollary 4.8. (We have normalized the sign in order to make it equal to the sign of the Euler characteristic.)

**Proposition 4.9** (Orlov’s trichotomy). Let \( \mathcal{X} = \mathcal{X}(2, 3, p) \). Then the categories \( D^0(\text{coh-}\mathcal{X}) \) and \( \mathcal{T} \) are related as follows.

1. For \( \chi_X > 0 \) the category \( \mathcal{T} \) is triangle-equivalent to the right perpendicular category in \( D^0(\text{coh-}\mathcal{X}) \) with respect to an exceptional sequence of \( 6 - p \) members;
2. For \( \chi_X = 0 \) the category \( \mathcal{T} \) is triangle-equivalent to \( D^0(\text{coh-}\mathcal{X}) \);
3. For \( \chi_X < 0 \) the category \( D^0(\text{coh-}\mathcal{X}) \) is triangle equivalent to the right perpendicular category in \( \mathcal{T} \) with respect to an exceptional sequence of \( p - 6 \) members.

**Calabi-Yau dimension and Euler characteristic.** Let \( \mathcal{T} \) be a triangulated category with Serre duality. Let \( S \) denote the Serre functor of \( \mathcal{T} \). Assume the existence of a smallest integer \( n \geq 1 \) such that we have an isomorphism \( S^n \cong [m] \) of functors for some integer \( m \). (Here, \([m]\) denotes the \( m \)-fold suspension of \( \mathcal{T} \).) Then \( \mathcal{T} \) is called Calabi-Yau of fractional CY-dimension \( \frac{m}{n} \). Note that the “fraction” \( \frac{m}{n} \) is kept in uncanceled format. The bounded derived category \( D^b(\text{coh-}\mathcal{X}) \) of coherent sheaves on \( \mathcal{X}(2, 3, p) \) is almost never Calabi-Yau, the only exception being the tubular case \( p = 6 \), where we have fractional CY-dimension \( 6/6 \). It is therefore remarkable that the category \( \tilde{S}(p) = \text{vect-}\mathcal{X}(2, 3, p) \) is always fractional Calabi-Yau. Moreover, the CY-dimension only depends on the Euler characteristic of \( \mathcal{X} \). To show this the next lemma is useful.

**Lemma 4.10.** Let \( \mathbb{L} = \mathbb{L}(2, 3, p) \). The class of \( \bar{\omega} \) in \( \mathbb{L}/\mathbb{Z}\bar{x}_1 \) is of order \( \text{lcm}(3, p) \). Moreover, the equality

\[
\text{lcm}(3, p) \cdot \bar{\omega} = \left( \text{lcm}(3, p) \cdot \frac{p - 6}{3p} \right) \cdot \bar{x}_1
\]

holds in \( \mathbb{L} \).

**Proof.** Write \( n \geq 1 \) as \( n = a \cdot p + b \) with \( a, b \in \mathbb{Z} \) and \( 0 \leq b < p \). Then we have \( n\bar{\omega} = n(\bar{x}_1 - \bar{x}_2 - \bar{x}_4) = n\bar{x}_1 - n\bar{x}_2 - n\bar{x}_4 \); hence \( n\bar{\omega} \in \mathbb{Z}\bar{x}_1 \) if and only if \( 3 \mid n \) and \( p \mid n \), which is equivalent to \( \text{lcm}(3, p) \mid n \). This shows the first claim. The second follows from \( \text{lcm}(3, p) \cdot \bar{\omega} = \text{lcm}(3, p) \cdot \left( 1 - 2/3 - 2/p \right) \cdot \bar{x}_1 \). \( \square \)
Proposition 4.11. The category $\mathcal{S}(p)$ is Calabi-Yau of fractional Calabi-Yau dimension $d_p$ given as follows:

\[
\begin{align*}
    d_2 &= \frac{1}{3} (1 - 2 \cdot \chi_X), \\
    d_p &= \frac{\text{lcm}(3, p) \cdot (1 - 2 \cdot \chi_X)}{\text{lcm}(3, p)}, \text{ for } p \geq 3.
\end{align*}
\]

Here, $\chi_X = 1/p - 1/6 = (6 - p)/6p$ is the Euler characteristic of $X(2, 3, p)$, and $1 - 2 \cdot \chi_X = (4p - 6)/3p$.

Note that the nominator of $d_p$ is always an integer.

Proof. Assume first that $p \geq 3$. Then the Picard group $\mathbb{L} = L(2, 3, p)$ acts faithfully on $\text{vect-}X$. Indeed, if $E$ is an Auslander bundle with $E(\bar{x}) \simeq E$ in $\text{vect-}X$, then $p \geq 3$ implies $\bar{x} = 0$. (In case $p \geq 3$ inspection of the AR components shows the for two line bundles $L, L'$ the corresponding Auslander bundles $E(L), E(L')$ are isomorphic if and only if $L$ and $L'$ are.) Since shift by $\bar{x}$ serves as suspension $[1]$ and the Serre functor on $X$ is given by $S = \tau[-1]$, it follows from the preceding lemma that the fractional Calabi-Yau dimension of $\text{vect-}X$ is given by $\frac{n - m}{n}$ with $n = \text{lcm}(3, p)$ and $m = \text{lcm}(3, p) \cdot \frac{\bar{x}}{3p}$.

For $p = 2$ we have a similar formula, but in the resulting fraction $2/6$ the factor $2$ can be canceled, since in this case for two line bundles $L, L'$ the corresponding Auslander bundles $E(L), E(L')$ are isomorphic if and only if $L' \simeq L$ or $L' \simeq L(\bar{x}_1 - \bar{x}_3)$. \hfill \Box

Corollary 4.12. The category $\mathcal{S}(p)$ determines $\text{coh-}X$.

Recall that the Coxeter transformation of a triangulated category $T$ with Serre duality is the automorphism of the Grothendieck group of $T$ induced by the Auslander-Reiten translation $\tau = S[1]$, where $S$ denotes the Serre functor for $T$. From Lemma 4.10 we then deduce the following, see [11].

Proposition 4.13. The Coxeter transformation $\phi$ of $\mathcal{S}(p) = \text{vect-}X(2, 3, p)$ has order $h = 3$ for $p = 2$ and order $h = \text{lcm}(6, p)$ otherwise. Moreover, assuming $p \geq 3$ we have $\phi^{h/2} = -1$ if and only if $p$ is odd. \hfill \Box

Note that, in classical situations, $h$ is called the Coxeter number, a nomination which we extend to the present context.

Shape of the categories $\mathcal{S}(p)$ and $\mathcal{S}(p)$. In this subsection we show how the structural results of [20] for $\mathcal{S}(p)$ and $\mathcal{S}(p)$, in particular the assertions of the shape of Auslander-Reiten components, follow from Theorem C. The shape of the results depends sensibly on the (orbifold) Euler characteristic $\chi_X = 1/p - 1/6$ of $X(2, 3, p)$.

Note that $\chi_X > 0$ if and only if $p < 6, p = 6$ or $p > 6$, respectively.

The Auslander-Reiten components of $\text{vect-}X/[\mathcal{F}]$ or of $\text{vect-}X = \text{vect-}X/[\mathcal{L}]$ are those from $\text{vect-}X$ with all line bundles from $\mathcal{F}$ (resp. $\mathcal{L}$) removed. By transport of structure this allows to determine the Auslander-Reiten structure of $\mathcal{S}(p)$ and $\mathcal{S}(p)$, thus obtaining the corresponding results of [20].

Fundamental domain under shift. It is shown in [20] that identification $E = E(\bar{x}_3)$ yields for $p \leq 5$ the ungraded invariant subspace problem $S(p)$. More explicitly, with $X = X(2, 3, p)$ we have a covering functor $\text{vect-}X/[\mathcal{F}] = \mathcal{S}(p) \rightarrow S(p)$ with infinite cyclic covering group $G$ generated by the grading shift $\sigma_3 = \sigma(\bar{x}_3)$ with $\bar{x}_3$. In the next proposition we describe explicitly a fundamental domain in $\text{vect-}X/[\mathcal{F}]$ with respect to this $G$-action. From [20] one obtains a full embedding of the orbit category $(\text{vect-}X/[\mathcal{F}])/G \hookrightarrow \mathcal{S}(p)$. It is shown in [20] that for $p \leq 6$ this embedding
is actually an equivalence. It is conjectured that for $p \geq 7$ the above embedding is not dense.

To describe a fundamental domain with respect to the $G$-action, we recall from [7] that the slope of a vector bundle $X$ is defined by $\mu X = \deg X / \text{rk} X$, where the degree $\text{deg}$ is the linear form on $K_0(\mathcal{X})$ which is uniquely determined by $\text{deg} \mathcal{O}(\mathcal{F}) = \delta(\mathcal{F})$ and where $\delta : \mathbb{L} \to \mathbb{Z}$ is the homomorphism sending $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ to lcm$(6, p)/2$, lcm$(6, p)/3$, lcm$(6, p)/p$, respectively.

**Proposition 4.14.** Let $\mathcal{X}$ be of weight type $(2, 3, p)$, $p \geq 3$. Then the following holds:

(i) The indecomposable bundles $X$ not in $\mathcal{F}$ having slope in the range $0 \leq \mu X < \delta(\mathcal{F}_3)$ form a fundamental domain $\mathcal{D}$ of vect-$\mathcal{X}/[\mathcal{F}]$ with respect to the $\langle \sigma_3 \rangle$-action.

(ii) There are exactly 6 line bundles $L$ with slope in the range $0 \leq \mu L < \delta(\mathcal{F}_3)$.

(iii) $\mathcal{D}$ contains exactly two (persistent) line bundles, one of them from the upper bar the other one from the lower bar.

(iv) $\mathcal{D}$ contains exactly 6 Auslander bundles.

**Proof.** Assertion (i) follows from the formula $\mu(E(\mathcal{F}_3)) = \mu E + \delta(\mathcal{F}_3)$. For (ii) we recall that $\mathbb{Z}\mathcal{F}_3$ has index 6 in $\mathbb{L}$. Moreover, each $\langle \sigma_3 \rangle$-orbit $\{ L(n\mathcal{F}_3) | n \in \mathbb{Z} \}$ contains exactly one line bundle in the given slope range. Assertion (iii) amounts to determine all $\mathcal{F}$ of shape $n\mathcal{F}_3$, $n\mathcal{F}_3$ satisfying $0 \leq \delta(\mathcal{F}) < \delta(\mathcal{F}_3)$. Claim (iv) is a direct consequence of (ii). □

**Positive Euler characteristic.** This deals with the cases $p = 2, 3, 4$ and 5. Note that the treatment is related to [9], but except for $p = 5$ deals with a different situation.

**Proposition 4.15.** For $2 \leq p \leq 5$ let $\Delta = [2, 3, p]$ (resp. $\hat{\Delta}$) be the attached Dynkin (resp. extended Dynkin) diagram.

1. The Auslander-Reiten quiver of vect-$\mathcal{X}$ consists of a single standard component. The category of indecomposable vector bundles on $\mathcal{X}$ is equivalent to the mesh category of the Auslander-Reiten component $\mathcal{F}(\mathcal{X})$. (The vertices corresponding to persistent (fading) vector bundles will be called persistent (fading).)

2. The Auslander-Reiten quiver $\Gamma$ of vect-$\mathcal{X}/[\mathcal{F}] = \hat{\mathcal{S}}(p)$ consists of a single component. It is obtained from the translation quiver $\mathbb{Z}\hat{\Delta}$ by deleting the fading vertices and adjacent arrows. The category of indecomposable objects of $\hat{\mathcal{S}}(p)$ is equivalent to the mesh-category of $\Gamma$.

3. The category vect-$\mathcal{X} = \hat{\mathcal{S}}(p)$ is equivalent to D$^b$(mod-$\mathcal{F}$) for some quiver $\Delta_p$ with underlying Dynkin graph $\Delta_2 = A_2$, $\Delta_3 = D_4$, $\Delta_4 = E_6$ and $\Delta_5 = E_8$.

**Proof.** We only sketch the argument, for further details we refer to [11]. One first shows that the direct sum of all indecomposable bundles $E$ with slope in the range $0 \leq \mu E < -\delta(\mathcal{F})$ yields a tilting object $T$ for coh-$\mathcal{X}$. This allows to prove assertion (1). Assertion (2) then follows from (1) using Theorem C. For (3) we use that the indecomposable summands of $T$ which are not line bundles yield a tilting object $T'$ for vect-$\mathcal{X}$ whose endomorphism ring is as described in (3). □

By way of example we treat the cases $\hat{\mathcal{S}}(4)$ and $\hat{\mathcal{S}}(5)$. In Figure 1 we illustrate a fundamental domain in the Auslander-Reiten quiver of $\hat{\mathcal{S}}(4)$ modulo the shift action by $\mathbb{Z}\mathcal{F}_3$. The line bundles are the objects at the upper and lower boundary of the graph. In the following figures the fading line bundles are indicated by circles and the adjacent (fading) arrows are dotted. All other objects (in particular the persistent line bundles) are marked by fat points.
We have marked the indecomposable summands of a tilting object for $\tilde{S}(4)$ with endomorphism ring of Dynkin type $E_6$.

In Figure 2 we illustrate a fundamental domain in the Auslander-Reiten quiver of $\tilde{S}(5)$ modulo the shift action by $\mathbb{Z}x_3$. Here the line bundles are the objects at the lower boundary of the quiver.

**Figure 2. Fundamental domain for $\tilde{S}(5)$**

We have marked a tilting object for the stable category $\text{vect-5} = \tilde{S}(5)$ with endomorphism ring of Dynkin type $E_8$.

**Euler characteristic zero, the case $p = 6$.** For $\chi_2 = 0$, that is $p = 6$, the category $\text{coh-}X$ is tubular of type $(2, 3, 6)$. Hence the line bundles are exactly the objects in the tubes of integral slope and of $\tau$-period 6, see [16]. Passing to the factor category $\text{vect-}X/\mathcal{F} = \tilde{S}(p)$ all other Auslander-Reiten components remain unchanged, while the “line bundle components” get the shape from Figure 3.
the investigation of Fuchsian singularities in \[\text{line bundles is factored out in the Fuchsian case).}\]

Concerning the stable category \(\text{vect-}X = \mathcal{S}(6)\), we have the following result.

**Proposition 4.16.** Assume \(X\) has weight type \((2, 3, 6)\). Then there exists a tilting object in the stable category \(\text{vect-}X = \mathcal{S}(6)\) whose endomorphism ring is the canonical algebra \(\Lambda = \Lambda(2, 3, 6)\). In particular, we have triangle equivalences \(\text{vect-}X = \mathcal{S}(6) \cong \text{D}^b(\text{coh-}X)\).

**Proof.** We sketch the argument, leaving details to [11]. As shown in [7], the direct sum \(T\) of all line bundles \(\mathcal{O}(\tilde{x}_3 + \tilde{x})\) with \(\tilde{x}\) in the range \(0 \leq \tilde{x} \leq \tilde{c}\) is a tilting object for \(\text{coh-}X\) and \(\text{D}^b(\text{coh-}X)\). By [16] there is an auto-equivalence \(\rho\) of \(\text{D}^b(\text{coh-}X)\) acting on slopes \(q\) by \(q \mapsto 1/(1 + q)\). It follows that \(\rho T\) is a bundle whose indecomposable summands have slopes \(q\) in the range \(1/2 < q < 1\). It follows from this property that \(\rho T\) is a tilting object for \(\text{vect-}X\) having all the claimed properties. \(\square\)

Recall in this context that the category \(\mathcal{H} = \text{coh-}X\) is hereditary, yielding the very concrete description of \(\text{D}^b(\text{coh-}X)\) as the repetitive category \(\bigvee_{n \in \mathbb{Z}} \mathcal{H}[n]\), where each \(\mathcal{H}[n]\) is a copy of \(\mathcal{H}\) (objects written \(X[n]\) with \(X \in \mathcal{H}\)) and where morphisms are given by \(\text{Hom}(X[n], Y[m]) = \text{Ext}_{\mathcal{H}}^{m-n}(X, Y)\) and composition is given by the Yoneda product.

**Remark 4.17.** The classification of indecomposable bundles over the weighted projective line \(X = X(2, 3, 6)\) is very similar to Atiyah’s classification of vector bundles on a smooth elliptic curve, compare [11] and [16]. Indeed the relationship is very close: Assume the base field is algebraically closed of characteristic different from \(2\) and \(3\). If \(E\) is a smooth elliptic curve of \(j\)-invariant \(0\), it admits an action of the cyclic group \(G\) of order \(6\) such that the category \(\text{coh}_G(E)\) of \(G\)-equivariant coherent sheaves on \(E\) is equivalent to \(\text{coh-}X\). Thus \(\mathcal{S}(6)\) has the additional description as stable category \(\text{vect-}G\)-\(E\) of \(G\)-equivariant vector bundles on \(E\).

**Negative Euler characteristic.** Let \(\chi_X < 0\), accordingly \(p \geq 7\). Here, the classification problem for \(\text{vect-}X = \mathcal{S}(p)\) is wild. The study of these categories is related to the investigation of Fuchsian singularities in [10, 13] but, with the single exception \(p = 7\) yields a different stable category of vector bundles (since only one \(\tau\)-orbit of line bundles is factored out in the Fuchsian case).
It is shown in [14] that, for \( \mathbb{X} = \mathbb{X}(2,3,p) \) and \( p \geq 7 \), all Auslander-Reiten components for vect-\( \mathbb{X} \) have the shape \( \mathbb{Z}A_{\infty} \), and we have exactly \( \lfloor L/\mathbb{Z}\bar{\omega}\rfloor = p - 6 \) components containing a line bundle. Only the shape of these components is affected when passing to the factor category vect-\( \mathbb{X}/[\mathcal{F}] = \mathcal{S}(p) \).

**Proposition 4.18.** For \( p \geq 7 \) each Auslander-Reiten component of \( \text{vect-} \mathbb{X}(2,3,p) = \mathcal{S}(p) \) is of shape \( \mathbb{Z}A_{\infty} \). Moreover there is a natural bijection between the set of all Auslander-Reiten components to the set of all regular Auslander-Reiten components over the wild path algebra \( \Lambda_0 \) over the star \( [2,3,p] \).

**Proof.** Invoking stability arguments, all line bundles lie at the border of their Auslander-Reiten component in vect-\( \mathbb{X} \) [14]. Passage to the stable category then shows that all components in vect-\( \mathbb{X} \) have shape \( \mathbb{Z}A_{\infty} \). The argument implies, moreover, that there is a natural bijection between the set of Auslander-Reiten components in vect-\( \mathbb{X} \) and in vect-\( \mathcal{X} \), respectively. The claim then follows from [14]. \( \square \)

![Figure 4. Case \( p \geq 7 \). Fundamental domain for the “distinguished” components](image)

The picture is a nice illustration for Proposition 4.14. For \( p \geq 7 \) the class of \( \bar{x}_3 \) is a generator of \( L/\mathbb{Z}\bar{\omega} \) having order \( p - 6 \). Accordingly shift with \( \bar{x}_3 \) acts on the \( (p - 6) \)-element set of “distinguished” components by cyclic permutation. Figure 4 therefore shows a fundamental domain \( \mathcal{D} \) for the \( p - 6 \) “distinguished” components.

**ADE-chain.** The table below summarizes previous results and displays for vect-\( \mathbb{X} = \mathcal{S}(p) \) with \( \mathbb{X} \) of type \( (2,3,p) \) the fractional Calabi-Yau dimension, the Euler characteristic \( \chi \), the Coxeter number \( h \), the representation type, and the derived type of \( \mathcal{S}(p) \) for small values of \( p \).
The table expresses an interesting property of the sequence of triangulated categories \( \text{vect-X}(2,3,p) = \tilde{S}(p) \). For small values of \( p \), the category \( \text{vect-X} = \tilde{S}(p) \) yields Dynkin type. For \( p = 6 \) the sequence passes the ‘borderline’ of tubular type and then continues with wild type. While such situations occur frequently, it is quite rare that one gets an infinite sequence of categories \( T_n \) which all are fractional Calabi-Yau and where the size of \( T_n \), measured in terms of the Grothendieck group, is increasing with \( n \).

There is no formal definition of an ADE-chain. Below we therefore collect a number of desirable properties, fulfilled by the above sequence if we ‘interpolate’ with the categories \( \text{D}^b(\text{mod } A(2p - 3,3)) \):

1. The \( (T_n) \) form an infinite chain \( T_1 \subset T_2 \subset T_3 \subset \cdots \) of triangulated categories with Serre duality which are fractionally Calabi-Yau;
2. Each category \( T_n \) has a tilting object \( T_n \), hence a Grothendieck group which is finitely generated free of rank \( n \);
3. The endomorphism rings \( A_n = \text{End}(T_n) \) form an accessible chain of finite dimensional algebras in the sense of [15], that is, \( A_1 = k \) and for each integer \( n \) the algebra \( A_{n+1} \) is a one-point extension or coextension of \( A_n \) with an exceptional \( A_n \)-module.

**Table 2. An ADE-chain**

| \( p \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( p \) |
|---|---|---|---|---|---|---|---|---|---|
| CY-dim | \( \frac{1}{3} \) | \( \frac{2}{3} \) | \( \frac{10}{12} \) | \( \frac{14}{18} \) | \( \frac{6}{15} \) | \( \frac{22}{31} \) | \( \frac{26}{34} \) | \( \frac{10}{17} \) | \( \frac{\text{lcm}(3,p), (1-2 \chi_X)}{\text{lcm}(3,p)} \) |
| \( \chi_X \) | \( \frac{1}{3} \) | \( \frac{1}{12} \) | \( \frac{1}{12} \) | \( \frac{1}{12} \) | \( 0 \) | \( -\frac{1}{12} \) | \( -\frac{1}{12} \) | \( -\frac{1}{12} \) | \( \frac{1}{2} - \frac{1}{6} \) |
| \( h \) | 3 | 6 | 24 | 30 | 6 | 42 | 24 | 18 | \( \text{lcm}(6,p) \) |
| type | \( A_2 \) | \( D_4 \) | \( E_6 \) | \( E_8 \) | \( (2,3,6) \) | \( (2,3,7) \) | \( (2,3,8) \) | \( (2,3,9) \) | \( (2,3,p) \) |
| repr. type | repr.-finite | tubular | wild, new type |

The special role of the number 6. The numbers 6 and \( p - 6 \) play a special role in dealing with \( S(p) \) and \( \tilde{S}(p) \). We advise the reader in this context to check the paper [21] for the ubiquitous appearance of the number 6. Of course this ubiquity of the number 6 has its roots in the correspondence between \( \tilde{S}(p) \) and vect-X, where X has weight type \( (2,3,p) \). The following list displays a number of appearances of the two numbers:

1. The group \( \mathbb{L}/\mathbb{Z} \bar{\omega} \) is cyclic of order 6 generated by the class of \( \bar{\omega} \).
2. We have \( 6\bar{\omega} = (p - 6)\bar{x}_3 \), thus \( \tau^6 = \sigma_3^{p-6} \) holds in vect-X, \( \tilde{S}(p) \) and \( \tilde{S}(p) \).
3. The partition of line bundles into persistent and fading ones obeys the 6-periodic pattern \( + - + - - - \) in each \( \tau \)-orbit, where \( + \) and \( - \) stand for persisting and fading, respectively.
4. Euler characteristic \( \chi_X \) of X and fractional Calabi-Yau dimension \( d_p \) of \( \text{vect-X} = \tilde{S}(6) \) are given by \( \chi_X = 1/p - 1/6 \) and \( d_p = (4p - 6)/3p \) (up to cancelation), respectively.
5. The borderline between (derived) representation-finiteness and wildness for \( \tilde{S}(p) \) and \( \tilde{S}(p) \) is marked by \( p = 6 \).
6. If \( p \) and 6 are coprime, then the Auslander-Reiten translation \( \tau \) on \( \tilde{S}(p) \) has a unique \( (p-6) \)th root in the Picard group.

Additionally we refer to Proposition 4.14 for further occurrences of the number 6.
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