Beyond cusp anomalous dimension from integrability

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Abstract

We study the first sub-leading correction $O((\ln s)^0)$ to the cusp anomalous dimension in the high spin expansion of finite twist operators in $\mathcal{N} = 4$ SYM theory. Since this approximation is still governed by a linear integral equation (derived already from the Bethe Ansatz equations in a previous paper), we finalise it better in order to study the weak and strong coupling regimes. In fact, we emphasise how easily the weak coupling expansion can be obtained, confirms the known four loop result and predicts the higher orders. Eventually, we pay particular attention to the strong coupling regime showing agreement and predictions in comparison with string expansion; speculations on the 'universal' part (upon subtracting the collinear anomalous dimension) are brought forward.
1 Plan of the work

Let us consider a (massless) gauge theory with a certain field $\phi$ in the adjoint representation (or $q$ in the fundamental one). To fix ideas, we may focus our attention on the planar $sl(2)$ sector of $\mathcal{N} = 4$ SYM with local operators

$$\text{Tr}(\mathcal{D}^s \phi^L) + \ldots, \quad (1.1)$$

where $\mathcal{D}$ is the (symmetrised, traceless) covariant derivative acting in all possible ways on the $L$ bosonic fields $\phi$. At finite twist $L$ the high spin behaviour of the anomalous dimension

$$\gamma(g, s, L) = \Delta(g; s, L) - L - s, \quad (1.2)$$

will in general be determined by the universal (in the sense here of being twist and flavour independent [1, 2], though theory dependent) scaling function $f(g)$ (cf. for instance [3, 4, 5, 6, 2] and references therein):

$$\gamma(g, s, L) = f(g) \ln s + f_{sl}(g, L) + \ldots, \quad (1.3)$$

(with a factor $1/2$ for the field $q$ in the fundamental representation [5]) with this parametrisation of the 't Hooft coupling $\lambda = 8\pi^2 g^2$. The universal scaling $f(g)$ equals in the $L = 2$ sector twice the \textit{cusp anomalous dimension} of the light-like Wilson loops [4] and therefore inherits this name.

We denote above by dots terms which are going to zero, whilst we wish to investigate the sub-leading (constant) scaling $f_{sl}(g, L)$ (for QCD see also [7]). In fact, the leading scaling $f(g)$ has been constrained (for all the values of $g$) in [2] (cf. also [8] for the relation with the Bethe root density) by a linear integral equation derived from the \textit{asymptotic Bethe Ansatz} [9] and from the same Bethe Ansatz the sub-leading one $f_{sl}(g, L)$ has been shown in [10], after careful consideration [11], to enjoy a linear integral equation with the same kernel (but different inhomogeneous term; for an alternative approach using the non-linear integral equation [12], see [13]). It is, of course, non-universal in the sense again that it depends explicitly on the twist $L$ (and, if there are other representations than the adjoint one, on the representation carried by the field, as well as on the theory). Yet, it will be conjectured in the following with some evidence to be wrapping-free (at least for the smallest twist $L = 2$), thus sharing the destiny of $f(g)$ in this respect [1]. The main evidence is coming from the agreement with the recent string computations [14], which should seriously allow for gauge loop wrapping effects, as they live at very large coupling $g$. In addition, a very recent analysis [15] \textit{à la} Lüscher has proved for the scalar twist two operator that the wrapping correction at four loops goes to zero (like $\ln^2 s/s^2$).

In any case, thanks to Kotikov, Lipatov, Onishchenko and Velizhanin [10] it is by now well established that (in $\mathcal{N} = 4$ SYM)

$$f_{sl}(g) \equiv f_{sl}(g, 2) \quad (1.4)$$

is the same for the operators with the smallest twist $L = 2$ (and determines also that of some twist three operators, see [17]). In fact, all twist two operators belong to the same supermultiplet, and their anomalous dimension is expressed in terms of a 'universal' function defined by the scalars

\footnote{As already clear, but also stressed in the following it is not universal as $f(g)$.}
\(\gamma_{univ.}(s) \equiv \gamma^\phi(s)\) and with shifted Lorentz spin for gauginos \(\gamma^\psi(s) = \gamma_{univ}(s + 1)\) and gauge fields \(\gamma^A(s) = \gamma_{univ}(s + 2)\).

Still at the leading twist \(L = 2\) Dixon, Magnea and Sterman \[18\] have pointed out that, although not universal (as \(f(g)\)), its first logarithmic integral, \(f_{sl}^{(-1)}(g)\), equals the sub-leading correction to the logarithm of the (diverging) scattering amplitude (the following is just a sketchy formula whose precise definitions and regularisation rely on \[18\] or references therein, in particular for the connexion between \(f^{(-1)}_{sl}(g)\) and \(f_{sl}(g)\)^3).

\[
\ln \mathcal{A} = \frac{f^{(-2)}(g)}{\epsilon^2} - \frac{f^{(-1)}_{sl'}(g)}{\epsilon} + \ldots, \tag{1.5}
\]

in dimensional regularisation to \(D = 4 - 2\epsilon\), plus the first logarithmic integral of a universal function \(h(g)\) (times a representation depending factor we omit here because not extant in \(SYM_4\)). In terms of the by-logarithm-derivative function of \(f^{(-1)}_{sl'}(g)\), i.e. the so-called collinear anomalous dimension \(f_{sl'}(g)\), this means the important relation between sub-leading term

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\[
f_{sl}(g) = f_{sl'}(g) + h(g). \tag{1.6}
\]

In other words, this \((L = 2)\) sub-leading scaling in the anomalous dimension \((1.3)\) contains a universal part, \(h(g)\), which, once subtracted from it, yields (upon integration) the sub-leading correction in the amplitude \((1.5)\); both the sub-leading terms are not universal, albeit their difference ought to be. As we will see later (cf. \((3.12)\)), the linear integral equation leads us to a split form of \(f_{sl}(g, L)\) into a twist-dependent part and another independent of it, \(f_{sl}(g)\) (cf. \((3.12)\) below). The latter contains somehow the universal \(h(g)\) and thus might share with it, at very strong coupling, its logarithmic behaviour (cf. below \((5.2)\) and \((5.4)\) which are in agreement with the string theory calculation by \[19\]). This may lead us to the possible proposal

\[
h(g) = f(g) \left[ \ln \frac{1}{\sqrt{2g}} - \left( \frac{1}{2} - \frac{3}{2} \ln 2 \right) + \ldots \right], \tag{1.7}
\]

with an intriguing object between square parenthesis (and the presence of \(f(g)\) may be perhaps motivated by the relation of \(h(g)\) to a Wilson loop \[18\]). In fact, at the first order, recalling that \(f(g) = 2\sqrt{2g} + \ldots\), we would obtain

\[
h(g) = 2\sqrt{2g} \left[ \ln \frac{1}{\sqrt{2g}} - \left( \frac{1}{2} - \frac{3}{2} \ln 2 \right) + \ldots \right], \tag{1.8}
\]

which coincides with the tested function in string theory for gluons and quarks \[20\].

**Note added:** An interesting paper \[21\] appears today in the web archives. It seems to contain an analysis of the strong coupling expansions of \(f_{sl}(g)\) in overlap with some of the results of this work.

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[^3]: The same connexion holds between \(f^{(-1)}_{sl'}(g)\) and \(f_{sl'}(g)\). A similar one between \(f^{(-2)}(g)\) and \(f(g)\) (cf. \[18\] or references therein).
2 One loop results

The present analysis fully relies upon the main achievements of [10], where it was shown that the relevant integral equations which enter the computation of the anomalous dimension at large spin are linear. Let us briefly remind some results which will be useful later.

For what concerns the one loop results, we consider equation (3.52) of that paper. The meaning of such equation is that, in the high spin limit \( s \to \infty \), the one loop density of roots is approximated, up to orders \( o(s^0) \), by the function whose Fourier transform \( \tilde{\sigma}_0(k) \) reads as

\[
\tilde{\sigma}_0(k) = -2\pi \frac{\frac{L}{2} \left( 1 - e^{-\frac{|k|}{2}} \right) + e^{-\frac{|k|}{2}} \left( 1 - \cos \frac{k s}{\sqrt{2}} \right)}{\sinh \frac{|k|}{2}} - 4\pi \delta(k) \ln 2 + o(s^0).
\] (2.1)

We also showed that the one loop energy,

\[
E_0(s,L) = -\int_{-\infty}^{\infty} \frac{dk}{4\pi^2} \hat{e}(k) \tilde{\sigma}_0(k) - (L - 2)e(0),
\]

with the function

\[
e(u) = \frac{1}{u^2 + \frac{1}{4}} \Rightarrow \hat{e}(k) = 2\pi e^{-\frac{|k|}{2}},
\] (2.2)

in the high spin limit behaves as

\[
E_0(s,L) = 4 \ln s - 4(L - 2) \ln 2 + 4\gamma_E + o(s^0),
\] (2.3)

and the one loop anomalous dimension reads

\[
\gamma_0(s,L) = g^2 E_0(s,L) = 4g^2 \ln s - [4(L - 2) \ln 2 - 4\gamma_E]g^2 + g^2 o(s^0).
\] (2.4)

3 All loops results

In order to study the higher than one loop density of roots \( \tilde{\sigma}_H(k) \) it is convenient to define the quantity

\[
S(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi |k|} \tilde{\sigma}_H(k).
\] (3.1)
We then consider equation (4.11) of \[10\]. Passing to Fourier transforms, we obtain that the function $S(k)$ satisfies the linear equation

$$S(k) = \frac{L}{|k|} [1 - J_0(\sqrt{2}g k)] + \frac{1}{\pi |k|} \int_{-\infty}^{+\infty} \frac{dh}{|h|} \left[ \sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}g k) J_r(\sqrt{2}g h) \frac{1 - \text{sgn}(kh)}{2} e^{-\frac{|h|}{2}} + \right.$$  
$$+ \text{sgn}(h) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,\nu+1+2\nu}(g) (-1)^{r+\nu} e^{-\frac{|h|}{2}} \left( J_{r-1}(\sqrt{2}g k) J_{r+2\nu}(\sqrt{2}g h) - J_{r-1}(\sqrt{2}g h) J_{r+2\nu}(\sqrt{2}g k) \right) \right] \cdot$$  
$$\cdot \left[ \frac{\pi|h|}{\sinh \frac{|h|}{2}} S(h) - 4\pi \ln 2 \delta(h) - \pi(L-2) \frac{1 - e^{|h|}}{\sinh \frac{|h|}{2}} - 2\pi \frac{1 - e^{-\frac{|h|}{2}} \cos \frac{hs}{\sqrt{2}}}{\sinh \frac{|h|}{2}} \right] + o(s^0) ,$$

(3.3)

where the functions $c_{r,s}(g)$ enter the definition of the ”dressing factor” (for their definition, see e.g. \[2\]). As follows from the generalisation of the Kotikov-Lipatov identity \[8\], the relation of the series of Bessel functions the following system of equations

$$\gamma(g,s,L) = 2S(0).$$

(3.4)

We now want to put equation (3.3) in a form which is suitable for analysis of both the weak and the strong coupling limit. It is convenient to restrict to the domain $k \geq 0$ and expand $S(k)$ in series of Bessel functions

$$S(k) = \sum_{p=1}^{\infty} S_p(g) \frac{J_p(\sqrt{2}g k)}{k}$$

(3.5)

with the energy (3.4) given by

$$\gamma(g,s,L) = \sqrt{2}g S_1(g).$$

(3.6)

On the other hand, for what concerns the coefficients $S_p(g)$, after some simple calculations we find the following system of equations

$$S_{2p-1}(g) = 2\sqrt{2}g (\ln s + \gamma_E) \delta_{p,1} + 4(2p - 1) \int_{0}^{\infty} \frac{dh}{h} \frac{\tilde{J}_{2p-1}(\sqrt{2}g h)}{e^h - 1} -$$  
$$- 2(2p - 1)(L - 2) \int_{0}^{\infty} \frac{dh}{h} \frac{J_{2p-1}(\sqrt{2}g h)}{e^h + 1} - 2(2p - 1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m(g),$$

(3.7)

$$S_{2p}(g) = 4 + 8p \int_{0}^{\infty} \frac{dh}{h} \frac{J_{2p}(\sqrt{2}g h)}{e^h - 1} + 2(L - 2) - 4p(L - 2) \int_{0}^{\infty} \frac{dh}{h} \frac{J_{2p}(\sqrt{2}g h)}{e^h + 1} +$$  
$$+ 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}(g).$$

\[3\]The functions $F_0(u), F_H(u)$, appearing in (4.11) of \[10\], are related to $\sigma_0(u)$ and $\sigma_H(u)$ by the relations

$$\sigma_0(u) = \frac{d}{du} F_0(u), \quad \sigma_H(u) = \frac{d}{du} F_H(u).$$

(3.2)
In writing (3.7) we used the notations
\[ \tilde{J}_{2p-1}(x) = J_{2p-1}(x) - \delta_{p,1} \frac{x}{2}, \quad Z_{n,m}(g) = \int_0^\infty \frac{dh}{h} \frac{J_n(\sqrt{2}gh)J_m(\sqrt{2}gh)}{e^h - 1}. \] (3.8)

The structure of the forcing terms appearing in the linear system (3.7) suggests that its solution and - consequently - the (all-loops) anomalous dimension can be split as
\[ S_r(g) = S_r^{BES}(g) \ln s + (L - 2) S_r^{(1)}(g) + S_r^{extra}(g), \] (3.9)
where \( S_r^{BES}(g) \) is the (coefficient of the) solution of the BES equation [2], \( S_r^{(1)}(g) \) is the (coefficient of the) first generalised scaling density [22, 23, 24, 25, 26] and the extra coefficients \( S_r^{extra}(g) \) are solutions of the following system
\[
S_{2p-1}^{extra}(g) = 2\sqrt{2}g\gamma E_\delta \delta_{p,1} + 4(2p - 1) \int_0^\infty \frac{dh}{h} \frac{\tilde{J}_{2p-1}(\sqrt{2}gh)}{e^h - 1} - 2(2p - 1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{extra}(g),
\]
(3.10)

\[
S_{2p}^{extra}(g) = 4 + 8p \int_0^\infty \frac{dh}{h} \frac{J_{2p}(\sqrt{2}gh)}{e^h - 1} + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{extra}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{extra}(g).
\]

From the splitting (3.9) of their solution, we obtain that the all loops energy at high spin is
\[ \gamma(g, s, L) = f(g) \ln s + (L - 2)f^{(1)}(g) + f_{sl}(g) + o(s^0), \] (3.11)
where \( f(g) \) is the cusp anomalous dimension, \( f^{(1)}(g) \) is the first generalised scaling function and \( f_{sl}(g) = \sqrt{2}gS_1^{extra}(g) \) comes from the solution of (3.10). In other terms, the sub-leading correction is worth
\[ f_{sl}(g, L) = f_{sl}(g) + (L - 2)f^{(1)}(g), \] (3.12)
with explicit mention to its twist dependence (since \( f_{sl}(g) \) does not depend on \( L \)).

In the next sections we will analyse the weak and the strong coupling expansion of \( f_{sl}(g, L) \).

4 Weak coupling

Since the system (3.10) is linear (as the original linear integral equation [10]), it can be systematically and very easily expanded at small \( g \), giving the weak coupling (convergent) series of \( f_{sl}(g, L) \) (via (3.12)). This kind of computations can be automatised by using a program for the symbolic manipulation in order to go on to an arbitrary order [4].

\[ \text{In what follows we report the results up to six loops, but we easily reached loop ten. We invite the interested reader to contact the authors for having these expressions, since their writing would fill some pages in.} \]
In view of (3.12), it is natural to start by the twist two formula up to the order $g^{12}$:

$$f_{sl}(g) = \frac{\gamma_E}{(g)} - 24\zeta(3) \left( \frac{g}{\sqrt{2}} \right)^4 + \frac{16}{3} \left( \pi^2\zeta(3) + 30\zeta(5) \right) \left( \frac{g}{\sqrt{2}} \right)^6 + \frac{8}{15} \left( 7\pi^4\zeta(3) + 50\pi^2\zeta(5) + 2625\zeta(7) \right) \left( \frac{g}{\sqrt{2}} \right)^8 + \left( \frac{128}{35}\pi^6\zeta(3) + 192\pi^3\zeta(5) + \frac{832}{45}\pi^4\zeta(5) + \frac{560}{3}\pi^2\zeta(7) + 14112\zeta(9) \right) \left( \frac{g}{\sqrt{2}} \right)^{10} + \left( \frac{58552\pi^8\zeta(3)}{14175} + \frac{1184}{63}\pi^6\zeta(5) + \frac{5824}{45}\pi^4\zeta(7) + \frac{32}{3}\pi^2\left( 10\zeta(3)^3 + 147\zeta(9) \right) \right) \left( \frac{g}{\sqrt{2}} \right)^{12} + \ldots.$$  

(4.1)

And we end up with the general twist expansion (we also expanded $f^{(1)}(g)$ according to [13, 10])

$$f_{sl}(g, L) = f_{sl}(g) + (L - 2)f^{(1)}(g) = \left( \gamma_E - (L - 2)\ln 2 \right)f(g) + 8(2L - 7)\zeta(3) \left( \frac{g}{\sqrt{2}} \right)^4 + \frac{8}{3} \left( \pi^2\zeta(3)(L - 4) + 3(21L - 62)\zeta(5) \right) \left( \frac{g}{\sqrt{2}} \right)^6 + \frac{8}{15} \left( \pi^4\zeta(3)(3L - 13) + 75(46L - 127)\zeta(7) + 5(11L - 32)\pi^2\zeta(5) \right) \left( \frac{g}{\sqrt{2}} \right)^8 + \left( \frac{128}{945}\pi^6\zeta(3)(11L - 49) + 8 \left( 2695\zeta(9)L + 16\zeta(3)^3L - 7154\zeta(9) - 56\zeta(3)^3 \right) + \frac{40}{3}(25L - 64)\pi^2\zeta(7) + \frac{8}{45}(103L - 310)\pi^4\zeta(5) \right) \left( \frac{g}{\sqrt{2}} \right)^{10} + \left( \frac{32}{45}\pi^4\zeta(7)(295L - 772) + \frac{8}{3}\pi^2 \left( 1519\zeta(9)L + 24\zeta(3)^3L - 3626\zeta(9) - 88\zeta(3)^3 \right) + 8 \left( 33285\zeta(11)L + 536\zeta(3)^2\zeta(5)L - 85974\zeta(11) - 1728\zeta(3)^2\zeta(5) \right) + \frac{8}{945}(2023L - 6266)\pi^6\zeta(5) + \frac{8(2956L - 13231)\pi^8\zeta(3)}{14175} \right) \left( \frac{g}{\sqrt{2}} \right)^{12} + \ldots.$$  

(4.2)

Importantly, this expansion for twist $L = 2, 3$ agrees with those up to four loops present in the literature [16, 27, 13] and checked by different means (cf., for instance, [14] for a summary). Thanks to [15, 28] we can also argue that for twist two (at least up to four loops) $f_{sl}(g)$ enjoys the property of being wrapping free, and we would like to conjecture the same in general about $f_{sl}(g, L)$.

In fact, last but not least, after the completion of this work we became aware that a five

\footnote{We ought to thank Matteo Beccaria for this private communication.}
loop calculation (wrapping inclusive) on twist $L = 3$ would exactly confirm the expansion of this section.

5 Strong coupling

Let us pass to consider the strong coupling limit of (3.10). From the results in Appendix A, we get that the leading contribution at large $g$ in the forcing terms of (3.10) is

$$-2\sqrt{2}g \ln g \delta_{p,1},$$

which allows to write down the following analytic leading behaviour

$$f_{sl}(g) = -f(g) \ln g + \ldots$$

where $f(g)$ is the cusp anomalous dimension at strong coupling [29, 30] and the dots denotes contributions of order $g$ or smaller. This important point will be clarified in the remaining part of the section.

In order to analyse the next to leading terms of $f_{sl}(g)$ at large $g$, we solve numerically the linear system (3.10). Indeed, this system is amenable for the usual numerical treatment as developed in [31]. The kernel is the same as in the study of the universal scaling function, settling any issue about convergence. What changes now is the nature of the forcing terms, which is expected to produce some subtle effects at strong coupling. In particular, as explained in appendix A, the forcing term which contains $\tilde{J}_1(x)$ shows a logarithmic leading asymptotic behaviour for $g \to \infty$.

A (numerically) clean way to study the system at large $g$ is to explicitly subtract such a logarithmic dependence in equations (3.10). We are then led to consider the objects $\tilde{S}_{extra}^r(g)$, satisfying the linear system

\[
\tilde{S}_{2p-1}^{extra}(g) = 2\sqrt{2}g \left( \gamma_E - \ln \frac{2\sqrt{2}}{g} \right) \delta_{p,1} + 4(2p - 1) \int_0^\infty \frac{dh}{h} J_{2p-1}(\sqrt{2}gh) e^{-h} - \\
- 2(2p - 1) \sum_{m=1}^\infty Z_{2p-1,m}(g) \tilde{S}_m^{extra}(g),
\]

\[
\tilde{S}_{2p}^{extra}(g) = 4 + 8p \int_0^\infty \frac{dh}{h} J_{2p}(\sqrt{2}gh) e^{-h} + \\
+ 4p \sum_{m=1}^\infty Z_{2p,2m-1}(g) \tilde{S}_{2m-1}^{extra}(g) - 4p \sum_{m=1}^\infty Z_{2p,2m}(g) \tilde{S}_{2m}^{extra}(g).
\]

With such a subtraction we have a simple expression for $f_{sl}(g)$,

$$f_{sl}(g) = f(g) \ln \frac{2\sqrt{2}}{g} + \sqrt{2}g \tilde{S}_1^{extra}(g),$$
where $f(g)$ is the usual cusp anomalous dimension and $\tilde{S}_{1}^{\text{extra}}(g)$, supposed to be free of logarithms, can be easily studied numerically (a plot of the complete function $f_{\text{sl}}(g)$ in the range $g \in [0, 5]$ can be found in fig. 1 together with a magnification of the weak-coupling behaviour). From a general point of view, it is likely to expect that $\sqrt{2} g \tilde{S}_{1}^{\text{extra}}(g)$ can be expanded in powers of $g$ as follows,

$$\sqrt{2} g \tilde{S}_{1}^{\text{extra}}(g) = k_{1} g + k_{0} + \frac{k_{-1}}{g} + O(1/g^2), \quad g \to \infty. \quad (5.5)$$

The usual numerical analysis \cite{31}, including up to $L = 30$ Bessel functions in the Neumann expansion, gives the following best fit coefficients:

$$k_{1}^{\text{num}} = -2.828426 \pm 0.000001; \quad k_{0}^{\text{num}} = 0.3238 \pm 0.0001; \quad k_{-1}^{\text{num}} = -0.01194 \pm 0.00015. \quad (5.6)$$

Inserting this expansion into (5.4) we obtain

$$f_{\text{sl}}(g) = 2\sqrt{2} g \left[ \ln \frac{2\sqrt{2}}{g} - c_{1} - \frac{3}{2\sqrt{2}\pi g} \ln \frac{2\sqrt{2}}{g} - \frac{c_{0}}{2\sqrt{2}\pi g} - \frac{K}{8\pi^{2} g^{2}} \ln \frac{2\sqrt{2}}{g} + \frac{k_{-1}}{2\sqrt{2} g^{2}} + O(\frac{\ln g}{g^{3}}) \right] \quad (5.7)$$

where $K = \beta(2)$ is the Catalan’s constant and $c_{1} = -k_{1}/(2\sqrt{2})$, $c_{0} = k_{0}\pi$. Now, we may also infer the plausible analytic values\footnote{They seem also to be in agreement with the expansion by Freyhult and Zieme \cite{21}}

$$c_{1} = 1, \quad c_{0} = 6\ln 2 - \pi, \quad k_{-1} = \frac{4K - 9(\ln 2)^{2}}{4\sqrt{2}\pi^{2}} = -0.0118253 \ldots . \quad (5.8)$$

In particular, for twist two this value of $c_{0} = 6\ln 2 - \pi$ entails the non-published evaluation $c = 6\ln 2 + \pi$ by N. Gromov, reported in the added note of \cite{14}.

Eventually, for generic twist $L$ plugging into (3.12) the asymptotic value of the first generalised scaling function $f^{(1)}(g) = -1 + O(e^{-\frac{\pi}{\sqrt{2}}})$ \cite{22}, we end up with

$$f_{\text{sl}}(g, L) = 2\sqrt{2} g \left[ \ln \frac{2\sqrt{2}}{g} - c_{1} - \frac{3}{2\sqrt{2}\pi g} \ln \frac{2\sqrt{2}}{g} + \frac{c_{0} + (2 - L)\pi}{2\sqrt{2}\pi g} - \frac{K}{8\pi^{2} g^{2}} \ln \frac{2\sqrt{2}}{g} + \frac{k_{-1}}{2\sqrt{2} g^{2}} + O(\frac{\ln g}{g^{3}}) \right]. \quad (5.9)$$

This shows clearly how, in this asymptotic expansion, the only trace of the twist comes up in the piece $c_{0} + (2 - L)\pi$ and thus cancels out completely, at order $O(s^{0})$, in the asymptotic (large $g$) expansion of $\Delta - s = \gamma + L$. This cancellation is due to the simple asymptotic expansion $f^{(1)}(g) = -1 + O(e^{-\frac{\pi}{\sqrt{2}}})$ \cite{22} and holds for any $L$. Furthermore, it makes possible that the constant term in $\Delta - s = \gamma + L$ at order $O(s^{0})$ be $\frac{\pi}{\pi} = \frac{6\ln 2 + \pi}{\pi}$ for any twist and thus future comparison with string theory (which does not distinguish between null and small values of $L$ \cite{14}).

## 6 Summarising

In the present work we have performed a detailed study of the sub-leading (constant) contribution to the anomalous dimension of the twist operator in the large spin expansion.
To this aim, we fully exploited the results obtained in [10] in order to write down a linear integral equation for $f_{sl}(g, L)$. This approach allowed us to study its weak-coupling expansion up to the order $g^{20}$ (and even further), and to access the strong coupling behaviour by means of a modification of well-established numerical techniques. An entire range plot is depicted by Fig. 1.

The strong coupling regime is of particular interest, because it shows a leading logarithmic behaviour (with the ’t Hooft coupling), that we are able to single out analytically by inspecting the structure of the inhomogeneous term in the linear integral equation (see appendix A). Moreover, relying on the numerical analysis we are in the position to disentangle the terms of the asymptotic expansion; in particular, we confirm the string result in [14], additionally fix the constant $c$ for the twist 2 case and understand how this constant is the same for any other twist.

It is also important to stress that our computations, based on the asymptotic Bethe Ansatz, are in full agreement with previous findings [11, 27, 13, 14, 15, 28].

Finally, our formalism also allows the treatment and can be seen as a particular case of the high spin and large twist case, when the ratio between the logarithm of the spin and the twist is fixed: this is the subject of the contemporaneous paper [32].

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\section*{A Strong coupling of the extra terms}

We briefly analyse the strong coupling (asymptotic) behaviour of the forcing terms of the system (3.10) for the extra coefficients $S_r^\text{extra}(g)$. From the asymptotic formula

$$\int_0^\infty dh \ h^s \frac{J_r(\sqrt{2}gh)}{e^h - 1} = \sum_{m=0}^\infty \frac{2^{m+s-1}B_m}{m!(\sqrt{2}g)^{m-1}} \frac{\Gamma\left(\frac{r+m+s}{2}\right)}{\Gamma\left(1 + \frac{r-s-m}{2}\right)}, \quad r + s \geq 1,$$

(A.1)

where $B_n$ are the Bernoulli numbers, we deduce the large $g$ behaviour of all the forcing terms in (3.10), but the one involving $\tilde{J}_1$. The strong coupling limit of such term can be evaluated after writing it as

$$\int_0^\infty dx \ J_2(x) + J_0(x) - 1.$$

(A.2)

Then we compute separately the strong coupling limits of the addends containing $J_2$ and $J_0 - 1$, respectively. For the former, we use (A.1). For the latter, one uses the integral representation

$$J_0(x) - 1 = \frac{1}{2\pi} \int_{-\pi}^\pi d\theta \ (e^{ix\sin\theta} - 1)$$

(A.3)

and then integrates first on $x$, getting an expression in terms of $\psi$ functions, and finally on $\theta$, after developing the $\psi$ functions for large argument. With this procedure we get the asymptotic formula

$$g \to \infty \quad \Rightarrow \quad \int_0^\infty dh \ \frac{\tilde{J}_1(\sqrt{2}gh)}{e^h - 1} = -\frac{g}{\sqrt{2}} \ln \frac{g}{\sqrt{2}} + \frac{g}{\sqrt{2}} \left(\frac{1}{2} - \gamma_E\right) - \frac{1}{2} + O\left(\frac{1}{g}\right).$$

We see that such term at large $g$ behaves as $-\frac{g}{\sqrt{2}} \ln g$. Therefore, it dominates the strong coupling behaviour of the forcing terms of (3.10).
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