CROSSED PRODUCTS BY ENDMORPHISMS AND REDUCTION OF RELATIONS IN RELATIVE CUNTZ-PIMSNER ALGEBRAS

B. K. Kwaśniewski, A. V. Lebedev

Abstract

Starting from an arbitrary endomorphism $\alpha$ of a unital $C^*$-algebra $A$ we construct a crossed product. It is shown that the natural construction depends not only on the $C^*$-dynamical system $(A, \alpha)$ but also on the choice of an ideal $J$ orthogonal to $\ker \alpha$. The article gives an explicit description of the internal structure of this crossed product and, in particular, discusses the interrelation between relative Cuntz-Pimsner algebras and partial isometric crossed products. We present a canonical procedure that reduces any given $C^*$-correspondence to the 'smallest' $C^*$-correspondence yielding the same relative Cuntz-Pimsner algebra as the initial one. In the context of crossed products this reduction procedure corresponds to the reduction of $C^*$-dynamical systems and allow us to establish a coincidence between relative Cuntz-Pimsner algebras and crossed products introduced.

Keywords: $C^*$-algebra, endomorphism, partial isometry, orthogonal ideal, crossed product, covariant representation, $C^*$-correspondence, relative Cuntz-Pimsner algebra, reduction

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Introduction

The crossed product of a $C^*$-algebra $A$ by an automorphism $\alpha : A \to A$ is defined as a universal $C^*$-algebra generated by a copy of $A$ and a unitary element $U$ satisfying the relations

$$\alpha(a) = UaU^*, \quad \alpha^{-1}(a) = U^*aU, \quad a \in A.$$ 

On one hand, algebras arising in this way (or their versions adapted to actions of groups of automorphisms) are very well understood and became a part of a $C^*$-folklore [Ped79, KR86]. On the other hand, it is very symptomatic that, even though the first attempts on generalizing this kind of constructions to endomorphisms go back to 1970s, articles introducing different definitions of the related object appear almost continuously until the present-day, see, for example, [CK80], [Pas80], [Sta93], [Mur96], [Exel94], [Exel03], [Kwa05], [ABL11]. This phenomenon is caused by very fundamental problems one has to face when dealing with crossed products by endomorphisms. Namely, one has to answer the following questions:

(i) What relations should the element $U$ satisfy?

(ii) What should be used in place of $\alpha^{-1}$?

It is important that in spite of substantial freedom of choice (in answering the foregoing questions), all the above listed papers do however have a certain nontrivial intersection. They mostly agree, and simultaneously boast their greatest successes, in the case when dynamics is implemented by monomorphisms with a hereditary range. In view of the articles [BL05], [ABL11], [Kwa12], this coincidence is completely understood. It is shown in [BL05] that in the case of monomorphism with hereditary range there exists a unique non-degenerate transfer operator $\alpha_*$ for $(A, \alpha)$, called by authors of [BL05] a complete transfer operator, and the theory goes smooth with $\alpha_*$ as it takes over the role classically played by $\alpha^{-1}$. The $C^*$-dynamical systems of this sort will be called partially reversible.
If a pair \((A, \alpha)\) is of the above described type, then \(A\) is called a coefficient algebra. This notion was introduced in \([LO04]\) where its investigation and relation to the extensions of \(C^*\)-algebras by partial isometries was clarified. Further in \([BL05]\) a certain criterion for a \(C^*\)-algebra to be a coefficient algebra associated with a given endomorphism was obtained (see also \([Kwa12]\)). On the base of the results of these papers one naturally arrives at the construction of a certain crossed product which was implemented in \([ABL11]\). It was also observed in \([ABL11]\) that in the most natural situations the coefficient algebras arise as a result of a certain extension procedure on the initial \(C^*\)-algebra. Since the crossed product is (should be) an extension of the initial \(C^*\)-algebra one can consider the construction of an appropriate coefficient algebra as one of the most important intermediate steps in the procedure of construction of the crossed product itself (a detailed discussion of the philosophy of the arising construction is given in \([ABL11\) Section 5]).

It was also shown in \([ABL11\) Section 4] how different extension procedures lead to the most popular constructions of crossed products such as Cuntz-Krieger algebras \([CK80]\), Paschke’s crossed product \([Pas80]\), partial crossed product \([Exel94]\), Exel’s crossed product \([Exel03]\) and others.

The analysis of these extension procedures naturally leads to the next problem: can we extend a \(C^*\)-dynamical system associated with an arbitrary endomorphism to a partially reversible \(C^*\)-dynamical system? In the commutative \(C^*\)-algebra situation the corresponding procedure and the explicit description of maximal ideals of the arising \(C^*\)-algebra is given in \([KL08]\). On the base of this construction the general construction of the crossed product associated to an arbitrary endomorphism of a commutative \(C^*\)-algebra is presented in \([Kwa05]\). Further the general construction of an extension of a \(C^*\)-dynamical system associated with an arbitrary endomorphism to a partially reversible \(C^*\)-dynamical system is worked out in \([Kwa07]\). Therefore the mentioned results of \([BL05], [ABL11], [Kwa07]\), give us the key to construct a general crossed product starting from a \(C^*\)-dynamical system associated with an arbitrary endomorphism, and this is one of the main themes of the present article.

The most important novelties we incorporate to the theory of crossed products are

1) an explicit description of the crossed product based on the worked out matrix calculus presented in Section 2

and an observation of (in a way unexpected) phenomena that

2) in general the universal construction of the crossed product depends not only on the algebra \(A\) and an endomorphism \(\alpha\) one starts with but also on the choice of an (arbitrary) singled out ideal \(J\) orthogonal to the kernel of \(\alpha\) (see Section 1).

So in fact we have a variety of crossed products depending on \(J\).

On the appearance of \([KL07a]\) B. Solel noted to the authors that the crossed product constructed in \([KL07a]\) can also be modeled as a certain relative Cuntz-Pimsner algebra (Proposition 4.13 of the present article describes in essence the main idea of B. Solel’s remark). Thus we have naturally arrived at the discussion of interrelations between relative Cuntz-Pimsner algebras and crossed products, and this was the
theme of [KL07b]. Since relative Cuntz-Pimsner algebras are defined by means of $C^*$-correspondences the role of the latter objects in the whole picture should be clarified and in this way we necessarily come to the analysis of the interplay: crossed products – relative Cuntz-Pimsner algebras – $C^*$-correspondences. Corollary 4.14 of the present article states that if $X$ is a $C^*$-correspondence of a $C^*$-dynamical system $(A, \alpha)$ and $J$ is an ideal orthogonal to the kernel of $\alpha$ then the relative Cuntz-Pimsner algebra $O(J, X)$ and the crossed product $C^*(A, \alpha, J)$ of the present article are canonically isomorphic. This observation in its turn causes a problem. Namely, $O(J, X)$ is defined for ideals $J$ that are not necessarily orthogonal to the kernel of $\alpha$. Moreover, by means of $O(J, X)$ one can construct crossed products seemingly different from those introduced in the present article (see, for example, Stacey’s crossed product identified in Corollary 4.15). Therefore, one may guess that $O(J, X)$ is a more general object than $C^*(A, \alpha, J)$. At the same time it is known (see [MS98, Proposition 2.21], i.e. Proposition 4.12 of the present article) that when $J$ is not orthogonal to the kernel of $\alpha$ the algebra $O(J, X)$ possesses certain ‘degeneracy’. All this stimulates us to take a closer look and provide a more thorough analysis of the structure of $O(J, X)$ and its relation to $C^*(A, \alpha, J)$, and this is one more main goal of the article.

As we show the necessary apparatus of investigation of the noted vagueness in the relation between $O(J, X)$ and $C^*(A, \alpha, J)$ is reduction. The general scheme of reduction procedure (taking quotients) associated with ideals in $C^*$-correspondences and the corresponding reduction in relative Cuntz-Pimsner algebras as well as the analysis of this scheme was provided in the structure theorem of [FMR03] (see Theorem 5.1 of the present article). We add canonicity to this scheme by applying the reduction procedure to a sequence of ideals $J_n$, $n = 1, 2, \ldots, J_\infty$ (Definition 5.2) that are naturally generated by $J$ and $O(J, X)$. Being defined in this way the canonical procedure reduces any given $C^*$-correspondence to the ‘smallest’ $C^*$-correspondence yielding the same relative Cuntz-Pimsner algebra as the initial one. In the context of the crossed products this reduction procedure corresponds to the reduction of $C^*$-dynamical systems. Using this, on the one hand, we obtain the canonical $C^*$-dynamical system (in Section 6) and, on the other hand, eliminate the mentioned ‘degeneracy’ in $O(J, X)$ and simultaneously establish an isomorphism between $O(J, X)$ and appropriate crossed product introduced in the present article (Theorem 5.4 and Proposition 5.12) obviating in this way the mentioned vagueness in their interrelations. As a byproduct we also get a refinement of Stacey’s results (Example 5.14) and add clarity to Katsura’s canonical relations (Subsection 5.2).

We would also like to make a certain additional remark on the objects of the paper. It is generally agreed that a crossed product of a unital $C^*$-algebra $A$ by a $C^*$-mapping should be a $C^*$-algebra $B$ generated by a copy (or at least a homomorphic image) of the $C^*$-algebra $A$ and an operator $U$ implementing the dynamics. On the other hand when $A$ is non-unital, then it seems that there are various essentially different ideas of what the non-unital crossed product $B$ should be. In particular, one can add to questions (i), (ii) (related to irreversibility of dynamics) two more still open principle problems (related to the lack of unity):

(iii) What should $B$ be generated by? Should it be the set $A \cup AU$, cf. e.g. [Ped79], [KR86], [Sta93], [BRV10], or maybe $A \cup A_0U$ for a certain subspace $A_0$ of $A$, cf.
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[Exel94], [FMR03], if so what should $A_0$ be?

(iv) How should $U$ be related to $B$? Should it belong to the multiplier algebra $M(B)$ of $B$, cf. [Ped79], [KR86], [Sta93], an enveloping $W^*$-algebra $B^{**}$ of $B$, see e.g. [Exel94], [BRV10], or maybe something else?

A way to bypass this trouble, usually adopted by most of the authors, cf. e.g. [LiR04], [FMR03], [BRV10], is to consider the non-unital crossed products only for the so-called extendible systems, that is for systems which naturally extend from $A$ to the multiplier algebra $M(A)$. For such systems a non-unital crossed product is actually a subalgebra of a unital crossed product. In the present paper we drop the technicalities arising from consideration of extendible systems and non-trivial issues concerning non-extendible systems. We consider only unital crossed products. Nevertheless the general $C^*$-correspondences will be considered over arbitrary (not necessarily unital) $C^*$-algebras.

The present paper is based on [KL07a] and [KL07b], and in essence forms their unification, refinement and development.

The paper is organized as follows.

In the first section we discuss and clarify the relations that should be used in a definition of a covariant representation. In particular, we split the class of covariant representations into subclasses according to a certain ideal they determine arriving at the notion of a $J$-covariant representation. In Subsection 1.2 we establish existence of such representations and introduce the corresponding crossed products as universal algebras. Section 2 presents a matrix calculus which describes the internal algebraic structure of the crossed product serving simultaneously as its certain regular representation. This leads us in Subsection 2.2 to an explicit formula for the norm of elements of the crossed product introduced, thus providing us with one more its alternative definition (Definition 2.13). In Section 3 we give a series of isomorphism theorems. They provide an apparatus for verifying faithfulness of a given representation of crossed product and, in particular, establishing equivalence of different approaches to construction of crossed products (cf. Proposition 3.7). The isomorphism theorems are discussed as on the operator algebraic level so also on the dynamical topological level exploiting topological freeness of the arising $C^*$-dynamical systems (Theorems 3.1 and 3.11). In addition we present here an overview of existing crossed product constructions and their comparison with the crossed product of the present paper. This stimulates us to undertake deeper analysis of interrelation between different approaches and in this way we pass to the next main theme of the article. In Section 4 Subsections 4.1, 4.2, 4.3 we recall the indispensable notions and objects concerning $C^*$-correspondences and Cuntz-Pimsner algebras, while presentation of the crossed products as relative Cuntz-Pimsner algebras is given in Subsection 4.4. Analysis of interrelations: $C^*$-correspondences – relative Cuntz-Pimsner algebras – crossed products is implemented on the base of reduction. The principal results in this direction are given in Section 5. Applying the canonical reduction to $C^*$-correspondences we eliminate the ’degeneracy’ in relative Cuntz-Pimsner algebras (Theorem 5.4). Applying this procedure to $C^*$-dynamical systems we prove coincidence between the corresponding relative Cuntz-Pimsner algebras and the crossed products of the present article (Proposition 5.12). Finally in Section 6
starting from a triple \((A, \alpha, J)\) we construct a \(C^*\)-dynamical system \((A_J, \alpha_J)\) which is canonical in the sense that the corresponding relations defining the crossed product are ‘non-degenerate’ and do not include any ideal of \(A\) (Theorem 6.6). This canonical construction is related to but as we argue differs from the similar result due to Katsura.

1 Covariant representations, orthogonal ideals, crossed product

1.1 Covariant representations and ideals of covariance

Let \((A, \alpha)\) be a pair consisting of a \(C^*\)-algebra \(A\), containing an identity and an endomorphism \(\alpha : A \to A\) (by a homomorphism between \(C^*\)-algebras we always mean a \(^*\)-homomorphism). Throughout the paper the pair \((A, \alpha)\) will be called a \(C^*\)-dynamical system.

**Definition 1.1.** Let \((A, \alpha)\) be a \(C^*\)-dynamical system. A representation of \((A, \alpha)\) is a triple \((\pi, U, H)\) consisting of a unital representation \(\pi : A \to L(H)\) on a Hilbert space \(H\) and an operator \(U \in L(H)\) satisfying the following relation

\[ U\pi(a)U^* = \pi(\alpha(a)), \quad a \in A. \quad (1) \]

If \(\pi\) is a faithful representation of \(A\), then \((\pi, U, H)\) is called a faithful representation.

Since \(\alpha^n(1)\) is a projection for every \(n\) it follows that the operator \(U\) in the above definition is necessarily a power partial isometry.

Note also that iterating \((1)\) we get

\[ U^n\pi(a)U^*n = \pi(\alpha^n(a)), \quad a \in A, \ n \in \mathbb{N} \quad (2) \]

which means that if \((\pi, U, H)\) is a representation of \((A, \alpha)\), then \((\pi, U^n, H)\) is a representation of \((A, \alpha^n)\) for every \(n \in \mathbb{N}\).

The next lemma shows that representations of \(C^*\)-dynamical systems possess one more important property which plays, in fact, a crucial role in the whole story.

**Lemma 1.2.** If \((\pi, U, H)\) is a representation of \((A, \alpha)\), then for every \(n \in \mathbb{N}\) we have

\[ U^nU^*n \in \pi(A)'. \quad (3) \]

**Proof.** Let \((\pi, U, H)\) consists of a unital representation \(\pi : A \to L(H)\) on a Hilbert space \(H\) and an operator \(U \in L(H)\) such that \((1)\) holds. In view of \((2)\) it suffice to prove \((3)\) only for \(n = 1\). As \(U\) is a partial isometry \(1 - U^*U\) is a projection, and for all \(a \in A\)

\[
\|U\pi(a)(1 - U^*U)\|^2 = \|U\pi(a)(1 - U^*U)\pi(a^*)U^*\| = \|U\pi(a)\pi(a^*)U^* - U\pi(a)U^*U\pi(a^*)U^*\| = \|\pi(\alpha(aa^*)) - \pi(\alpha(a)\alpha(a^*))\| = 0.
\]
Thus $U\pi(a) = U\pi(a)U^*U = \pi(\alpha(a))U$ and by passing to adjoints we also get $U^*\pi(\alpha(a)) = \pi(a)U^*$. Using these two relations we get

$$U^*U\pi(a) = U^*\pi(\alpha(a))U = \pi(a)U^*U,$$

which proves the assertion. ■

**Remark 1.3.**

1. The notion of a representation of a $C^*$-dynamical system appears in a similar or identical form, for instance, in [Sta93], [ALNR94], [Mur96], [Exel03], [LiR04], [LO04], [Kwa05], [Kwa07]. For isometric crossed products, cf. [Sta93], [ALNR94], [Mur96], it is assumed that $U$ is an isometry satisfying (1). In general, cf. [Exel03], [LiR04], [LO04], [Kwa05], [Kwa07], definitions of representations of $C^*$-dynamical systems contained conditions (1) and (3) (for $n = 1$) or a certain equivalent of (3). Lemma 1.2 shows that (3) is redundant.

2. In [LO04, Prop 2.2] and [LiR04, Lem. 4.3] it was shown that when (1) is assumed relation (3) is equivalent to the condition

$$U\pi(a) = \pi(\alpha(a))U, \quad a \in A. \quad (4)$$

In view of Lemma 1.2 the conditions (3) and (4) are not only equivalent in the presence of (1) but they actually follow from (1).

Relation (3) imply that for any representation $(\pi, U, H)$ of $(A, \alpha)$ the set

$$J := \{a \in A : U^*U\pi(a) = \pi(a)\}$$

is an ideal in $A$ (by which we always mean a closed two-sided ideal). This ideal will play one of the key roles in the paper. The next statement shows a certain property of $J$ which is important for the further analysis.

**Proposition 1.4.** Let $(\pi, U, H)$ be a representation of $(A, \alpha)$ and let

$$I = \{a \in A : (1 - U^*U)\pi(a) = \pi(a)\}, \quad J = \{a \in A : U^*U\pi(a) = \pi(a)\}. \quad (5)$$

Then

$$I = \ker(\pi \circ \alpha) \quad \text{and} \quad I \cap J = \ker \pi.$$ 

**Proof.** Observe that $U(U^*U\pi(a))U^* = U\pi(a)U^* \quad \text{and} \quad U^*(U\pi(a)U^*)U = U^*U\pi(a)$. Therefore the mappings $U(\cdot)U^* : U^*U\pi(a) \mapsto \pi(\alpha(a))$, $U^*(\cdot)U : \pi(\alpha(a)) \mapsto U^*U\pi(a)$ are each other inverses and in particular $U^*U\pi(A) \cong \pi(\alpha(A))$, where $U^*U\pi(a) \mapsto \pi(\alpha(a))$. Thus

$$\pi(\alpha(a)) = 0 \iff U^*U\pi(a) = 0 \iff (1 - U^*U)\pi(a) = \pi(a).$$

Which means that $I$ is the kernel of $\pi \circ \alpha$.

Clearly $\ker \pi \subset I \cap J$ and on the other hand

$$a \in I \cap J \Rightarrow (1 - U^*U)\pi(a) = U^*U\pi(a) \Rightarrow \pi(a) = 0,$$

that is $I \cap J \subset \ker \pi$. ■
Corollary 1.5. Let \((\pi, U, H)\) be a faithful representation of \((A, \alpha)\), and let \(I\) and \(J\) be ideals in \(A\) given by (5). Then
\[
I = \ker \alpha \quad \text{and} \quad I \cap J = \{0\}.
\]

Having in mind this observation we introduce the following

Definition 1.6. Let \(I\) and \(J\) be two ideals in \(A\). We say that \(J\) is orthogonal to \(I\) if
\[
I \cap J = \{0\}.
\]

There exists the biggest ideal orthogonal to \(I\) (in the sense that it contains all other ideals that are orthogonal to \(I\)) denoted by \(I^\perp\). This ideal could be defined explicitly as \(I^\perp = \{a \in A : aI = \{0\}\}\) and it is called the annihilator of the ideal \(I\) in \(A\).

Proposition 1.4 and Corollary 1.5 make it natural to introduce the following definition.

Definition 1.7. Let \((\pi, U, H)\) be a representation of \((A, \alpha)\) and let \(J\) be an ideal in \(A\). If \((\pi, U, H)\) and \(J\) are such that
\[
J = \{a \in A : U^*U\pi(a) = \pi(a)\},
\]
then we will say that \((\pi, U, H)\) is a \(J\)-covariant representation. In this situation we will also say that \(J\) is the ideal of covariance for the representation \((\pi, U, H)\). If \(J = (\ker \alpha)^\perp\), then we simply call \((\pi, U, H)\) a covariant representation.

Remark 1.8. For any \(C^*\)-dynamical system \((A, \alpha)\) one can always construct a faithful representation of \((A, \alpha)\) (see e.g. the Toeplitz representation defined in Subsection 1.2) such that \(U\) is not an isometry. However, if \((\pi, U, H)\) is a covariant representation of \((A, \alpha)\), then the following implications hold true:

1) \(\alpha\) is a monomorphism \(\Rightarrow U\) is an isometry,

2) \(\alpha\) is an automorphism \(\Rightarrow U\) is unitary.

Hence, in contrast to arbitrary representations, covariant representations as defined above involve the operators typically used in similar definitions for automorphisms and monomorphisms of \(C^*\)-algebras, cf. [Ped79], [KR86], [Pas80], [Sta93], [Mur96].

In connection with the above remark and for future applications we recall the following observation, which is a part a) of [Kwa07], Prop. 1.9, and follows immediately from Proposition 1.4.

Proposition 1.9. Suppose \((A, \alpha)\) is such that \(\ker \alpha\) is unital and let \((\pi, U, H)\) be a representation of \((A, \alpha)\). Then the following conditions are equivalent:

i) \((\pi, U, H)\) is a covariant representation

\(\Rightarrow\) \(U^*U \in \pi(A)\)

iii) \(U^*U \in \pi(Z(A))\) (\(Z(A)\) stands for the center of \(A\))

iv) \(U^*U\) is the unit in \(\pi((\ker \alpha)^\perp)\)

Though we have defined \(J\)-covariant representations, their existence for all appropriate ideals \(J\) is not established yet. Henceforth we present a construction resolving this problem.
1.2 Construction of a faithful $J$-covariant representation. Crossed product

Let us fix a $C^*$-dynamical system $(A, \alpha)$, an ideal $J \subset (\ker \alpha)^\perp$, and a faithful non-degenerate representation $\pi : A \to L(H)$. First we define a triple $(\tilde{\pi}, \tilde{U}, \mathcal{H})$ by the formulae

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \pi(\alpha^n(1))H, \quad (\tilde{\pi}(a)h)_n := \pi(\alpha^n(a))h_n, \quad (\tilde{U}h)_n := h_{n+1}.$$ 

One readily sees that $(\tilde{\pi}, \tilde{U}, \mathcal{H})$ is a faithful representation of $(A, \alpha)$. Actually $(\tilde{\pi}, \tilde{U}, \mathcal{H})$ is $\{0\}$-covariant and having in mind the classical associations one could call $(\tilde{\pi}, \tilde{U}, \mathcal{H})$ a Toeplitz representation of $(A, \alpha)$.

In order to obtain a $J$-covariant representation we introduce the following algebra of operators

$$c_0(\mathbb{N}, J) := \{a = \bigoplus_{n \in \mathbb{N}} \pi(a_n) : a_n \in \alpha^n(1)Ja^n(1), \lim_{n \to \infty} a_n = 0\} \subset L(\mathcal{H}),$$

and consider the $C^*$-algebra $C^*(c_0(\mathbb{N}, J), \tilde{U})$ generated by $c_0(\mathbb{N}, J)$ and $\tilde{U}$. One checks that

$$\tilde{U}c_0(\mathbb{N}, J)\tilde{U}^* \subset c_0(\mathbb{N}, J), \quad \tilde{U}^*c_0(\mathbb{N}, J)\tilde{U} \subset c_0(\mathbb{N}, J), \quad \tilde{U}^*\tilde{U} \in c_0(\mathbb{N}, J)^{\prime},$$

and hence, see [LO04] Prop 2.3], $C^*(c_0(\mathbb{N}, J), \tilde{U})$ is the closure of elements of the form

$$\tilde{U}^{*n}a^{(-n)} + \ldots + \tilde{U}^{*(-1)}a^{(1)} + a^{(0)} + a^{(1)}\tilde{U} + \ldots + a^{(n)}\tilde{U}^n,$$

where $a^{(k)} \in c_0(\mathbb{N}, J), k = 0, \pm 1, \ldots, \pm n$. Thus using the relations

$$\tilde{\pi}(A)c_0(\mathbb{N}, J) \subset c_0(\mathbb{N}, J), \quad \tilde{\pi}(A)\tilde{U}^* = \tilde{U}^*\tilde{\pi}(\alpha(A)), \quad \tilde{U}\tilde{\pi}(A) = \tilde{\pi}(\alpha(A))\tilde{U}$$

one sees that $C^*(c_0(\mathbb{N}, J), \tilde{U})$ is an ideal in $C^*(\tilde{\pi}(A), \tilde{U})$. Let us now take any faithful non-degenerate representation of the quotient algebra

$$\tilde{\pi}_J : C^*(\tilde{\pi}(A), \tilde{U})/C^*(c_0(\mathbb{N}, J), \tilde{U}) \to L(H_J)$$

and put

$$U_J := \tilde{\pi}_J([\tilde{U}]), \quad \pi_J(a) := \tilde{\pi}_J([\tilde{\pi}(a)]), \quad a \in A,$$

where $[\cdot] : C^*(\tilde{\pi}(A), \tilde{U}) \to C^*(\tilde{\pi}(A), \tilde{U})/C^*(c_0(\mathbb{N}, J), \tilde{U})$ is the quotient mapping.

**Proposition 1.10.** For any ideal $J \subset (\ker \alpha)^\perp$, the triple $(\pi_J, U_J, H_J)$ defined above is a faithful $J$-covariant representation of $(A, \alpha)$.

**Proof.** That $(\pi_J, U_J, H_J)$ is a representation of $(A, \alpha)$ is straightforward. What we need to verify is that it is faithful and $J$-covariant.
To see that $\pi_J$ is faithful let $a \in \ker \pi_J \subset c_0(\mathbb{N}, J)$. Then $\alpha^n(a) \in J$ for all $n = 0, 1, 2, \ldots$, and $\pi(\alpha^n(a)) \to 0$. Since $J \cap \ker \alpha = \{0\}$ the homomorphism $\alpha : J \to A$ is isometric, and as $\pi$ is a faithful representation of $A$ we get

$$\|a\| = \lim_{n \to \infty} \|\pi(\alpha^n(a))\| = 0.$$  

To see that $\pi_J$ is $J$-covariant note that for any $a \in A$

$$U^*\bar{U}\bar{\pi}(a) - \bar{\pi}(a) = \{\pi(a), 0, 0, \ldots\},$$

and so, by the definition of $\ker c_0(\mathbb{N}, J)$

$$[U^*\bar{U}\bar{\pi}(a) - \bar{\pi}(a)] = [0] \iff a \in J.$$  

\[\blacksquare\]

As an immediate corollary we have

**Theorem 1.11.** Let $(A, \alpha)$ be a $C^*$-dynamical system and $J$ be an ideal in $A$. Then there exists a faithful $J$-covariant representation iff $\ker \alpha \cap J = \{0\}$.

\[\text{Proof.}\] Sufficiency follows from Proposition [1.10] while Corollary [1.5] implies necessity. \[\blacksquare\]

The foregoing observations make the next definition of the crossed product natural.

**Definition 1.12.** Let $(A, \alpha)$ be a $C^*$-dynamical system and $J$ an ideal in $A$ such that $J \cap \ker \alpha = \{0\}$. The crossed product $C^*(A, \alpha, J)$ of $A$ by $\alpha$ associated with $J$ is a universal $C^*$-algebra generated by the the copy of the algebra $A$ and a partial isometry $u$ subject to relations

$$uau^* = \alpha(a), \quad J = \{a \in A : u^*ua = a\}.$$  

The aim of the paper is the analysis of the crossed product introduced. It will be shown that this construction covers the main known constructions of crossed products. In addition we present a thorough description of the internal structure of this crossed product (Sections 2, 3) and discuss its relation to the relative Cuntz-Pimsner algebras (Section 4). In particular, by means of the reduction procedure (Section 5) we show that all relative Cuntz-Pimsner algebras are associated with orthogonal ideals.

## 2 Crossed product and matrix calculus

This section presents an alternative definition of the crossed product which provides us with one more interesting, integral point of view leading to a transparent description of its internal structure. One can also consider the construction described here as a sort of regular representation of the crossed product.
2.1 Matrix calculus

We introduce a matrix calculus that will be an algebraic framework for our crossed product.

Let us denote by $\mathcal{M}(A)$ the set of infinite matrices $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ indexed by pairs of natural numbers with entries $a_{i,j}$ in $A$ such that

$$a_{i,j} \in \alpha^i(1)A\alpha^j(1), \quad i, j \in \mathbb{N},$$

and there is at most finite number of $a_{i,j}$ which are non-zero.

We will take advantage of this standard matrix notation when defining operations on $\mathcal{M}(A)$ and investigating a natural homomorphism from $\mathcal{M}(A)$ to the covariance algebras generated by representations of $(A, \alpha)$. However, when calculating norms of elements in covariance algebras, it is more handy to index the entries of an element in $\mathcal{M}(A)$ by a pair consisting of a natural number and an integer. Hence we will parallelly use two notations concerning matrices in $\mathcal{M}(A)$. Namely, we presume the following identifications

$$a_{i,j} = a_{\min(i,j)}^{(j-i)}, \quad i, j \in \mathbb{N},$$

$$a^{(k)}_n = \begin{cases} a_{n,k+n}, & k \geq 0 \\ a_{n-k,n}, & k < 0 \end{cases} \quad n \in \mathbb{N}, \ k \in \mathbb{Z},$$

under which we have two equivalent matrix presentations

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a^{(0)}_0 & a^{(1)}_0 & a^{(2)}_0 & \cdots \\ a^{(-1)}_0 & a^{(0)}_1 & a^{(1)}_1 & \cdots \\ a^{(-2)}_0 & a^{(-1)}_1 & a^{(0)}_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$ 

The use of each of these conventions will always be clear from the context.

We define the addition, multiplication by scalar, and involution on $\mathcal{M}(A)$ in a natural manner. Namely, for $a = \{a_{ij}\}_{i,j \in \mathbb{N}}$ and $b = \{b_{ij}\}_{i,j \in \mathbb{N}}$ in $\mathcal{M}(A)$ we put

$$(a + b)_{m,n} = a_{m,n} + b_{m,n}, \quad (\lambda a)_{m,n} = \lambda a_{m,n}, \quad (a^*)_{m,n} = a^*_{m,n}.$$ 

Moreover, we introduce a convolution multiplication $'*'$ on $\mathcal{M}(A)$, which is a reflection of the operator multiplication in covariance algebras. We set

$$a \ast b = a \cdot \sum_{j=0}^{\infty} \Lambda^j(b) + \sum_{j=1}^{\infty} \Lambda^j(a) \cdot b$$ 

where $\cdot$ is the standard multiplication of matrices and mapping $\Lambda : \mathcal{M}(A) \to \mathcal{M}(A)$ is defined to act as follows: $\Lambda(a)_{i,j} = \alpha(a_{i-1,j-1})$, for $i, j > 0$, and $\Lambda(a)_{i,j} = 0$ otherwise,
that is \( \Lambda \) assumes the following shape

\[
\Lambda(a) = \begin{pmatrix}
0 & 0 & \alpha(a_{0,0}) & 0 & \alpha(a_{0,1}) & \alpha(a_{0,2}) & \cdots \\
0 & \alpha(a_{1,0}) & \alpha(a_{1,1}) & \alpha(a_{1,2}) & \cdots \\
0 & \alpha(a_{2,0}) & \alpha(a_{2,1}) & \alpha(a_{2,2}) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(11)

**Proposition 2.1.** The set \( \mathcal{M}(A) \) with operations (7), (8), (9), (10) becomes an algebra with involution.

**Proof.** The only thing we show is associativity of multiplication (10), the rest is straightforward. For that purpose we note that \( \Lambda \) preserves the standard matrix multiplication and thus we have

\[
a \ast (b \ast c) = a \ast \left( b \cdot \sum_{j=0}^{\infty} \Lambda^j(c) + \sum_{j=1}^{\infty} \Lambda^j(b) \cdot c \right)
\]

\[
= a \sum_{k=0}^{\infty} \Lambda^k \left( b \sum_{j=0}^{\infty} \Lambda^j(c) + \sum_{j=1}^{\infty} \Lambda^j(b) c \right) + \sum_{k=1}^{\infty} \Lambda^k(a) \left( b \sum_{j=0}^{\infty} \Lambda^j(c) + \sum_{j=1}^{\infty} \Lambda^j(b) c \right)
\]

\[
= \sum_{k,j=0}^{\infty} a \Lambda^k(b) \Lambda^j(c) + \sum_{k=1}^{\infty} \Lambda^k(a) b \cdot \Lambda^j(c) + \sum_{k,j=1}^{\infty} \Lambda^k(a) \Lambda^j(b) c
\]

\[
= \left( a \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) b \right) \sum_{j=0}^{\infty} \Lambda^j(c) + \sum_{j=1}^{\infty} \Lambda^j \left( a \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) b \right) c
\]

\[
\left( a \cdot \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) \cdot b \right) \ast c = (a \ast b) \ast c.
\]

We embed \( A \) into \( \mathcal{M}(A) \) by identifying an element \( a \in A \) with the matrix \( \{a_{m,n}\}_{m,n \in \mathbb{N}} \) where \( a_{0,0} = a \) and \( a_{m,n} = 0 \) if \( (m,n) \neq (0,0) \). We also define a ‘partial isometry’ \( u = \{u_{m,n}\}_{m,n \in \mathbb{N}} \) in \( \mathcal{M}(A) \) such that \( u_{0,1} = \alpha(1) \) and \( u_{m,n} = 0 \) if \( (m,n) \neq (0,1) \). In other words, we adopt the following notation

\[
u = \begin{pmatrix}
0 & \alpha(1) & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and

\[
a = \begin{pmatrix}
a & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \text{for } a \in A.
\]

(12)

One can check that the *-algebra \( \mathcal{M}(A) \) is generated by \( u \) and \( A \). Furthermore, for every \( a \in A \) we have

\[
u \ast a \ast u^* = \alpha(a) \quad \text{and} \quad u^* \ast a \ast u = \begin{pmatrix}
0 & 0 & \alpha(1) \alpha(1) & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
Proposition 2.2. Let \((\pi, U, H)\) be a representation of \((A, \alpha)\). Then there exists a unique \(*\)-homomorphism \(\Psi_{(\pi, U)}\) from \(\mathcal{M}(A)\) onto a \(*\)-algebra \(C_0^*(\pi(A), U)\) generated by \(\pi(A)\) and \(U\), such that

\[
\Psi_{(\pi, U)}(a) = \pi(a), \quad a \in A, \quad \Psi_{(\pi, U)}(u) = U.
\]

Moreover, \(\Psi_{(\pi, U)}\) is given by the formula

\[
\Psi_{(\pi, U)}(\{a_{m,n}\}_{m,n \in \mathbb{N}}) = \sum_{m,n = 0}^{\infty} U^{m} \pi(a_{m,n}) U^n,
\]

and thus \(C_0^*(\pi(A), U) = \{\sum_{m,n = 0}^{\infty} U^{m} \pi(a_{m,n}) U^n : \{a_{m,n}\}_{m,n \in \mathbb{N}} \in \mathcal{M}(A)\}\).

Proof. It is clear that \(\Psi_{(\pi, U)}\) has to satisfy (13). Thus it is enough to check that \(\Psi_{(\pi, U)}\) is a \(*\)-homomorphism, and in fact we only need to show that \(\Psi_{(\pi, U)}\) is multiplicative as the rest is obvious. For that purpose let us fix two matrices \(a = \{a_{m,n}\}_{m,n \in \mathbb{N}}, b = \{b_{m,n}\}_{m,n \in \mathbb{N}} \in \mathcal{M}(A)\). We will examine the product

\[
c_{p,r,s,t} = U^{sp} \pi(a_{p,r}) U^{sr} U^{ss} \pi(b_{s,t}) U^t
\]

Depending on the relationship between \(r\) and \(s\) we have two cases.

1) If \(s \leq r\), then using equality \((U^{s} U^{ss}) \pi(b_{s,t}) = \pi(\alpha^s(1)b_{s,t}) = \pi(b_{s,t})\) and relation \(U^{sr-s} U^{r-s} \in \pi(A)'\), see Lemma 1.2 we get

\[
c_{p,r,s,t} = U^{sp} \pi(a_{p,r}) U^{sr-s} (U^{s} U^{ss}) \pi(b_{s,t}) U^t = U^{sp} \pi(a_{p,r}) U^{sr-s} U^{r-s} \pi(b_{s,t}) U^t
\]

Putting \(r - s = j, r = i, p = m\) and \(t + r - s = n\) we get

\[
c_{m,r,s-n+s} = c_{p,r,s,t} = U^{m} \pi(a_{m,i} \alpha^i(b_{i-j,n-j})) U^n
\]

and thus

\[
\sum_{s,r \in \mathbb{N}} c_{m,r,s,n-r+s} = \sum_{j=0}^{\infty} U^{m} \pi(a_{m,i} \alpha^i(b_{i-j,n-j})) U^n = U^{m} \pi((a \cdot \sum_{j=0}^{\infty} \Lambda^i(b))_{m,n}) U^n
\]

2) If \(r < s\), then analogously

\[
c_{p,r,s,t} = U^{sp} \pi(a_{p,r}) (U^{r} U^{sr}) U^{ss-r} \pi(b_{s,t}) U^t = U^{sp} \pi(a_{p,r}) (U^{ss-r} U^{r-s}) U^{sr-s} \pi(b_{s,t}) U^t
\]

Putting \(s - r = j, r = i, p + s - r = m\) and \(t = n\) we get

\[
c_{m-s+r,r,s,n} = c_{p,r,s,t} = U^{m} \pi(\alpha^i(a_{m-j,i-j})b_{i,n}) U^n
\]

and thus

\[
\sum_{s,r \in \mathbb{N}} c_{m-s+r,r,s,n} = \sum_{j=1}^{\infty} U^{m} \pi(\alpha^i(a_{m-j,i-j})b_{i,n}) U^n = U^{m} \pi((\sum_{j=1}^{\infty} \Lambda^i(a) \cdot b)_{m,n}) U^n
\]
Using the formulas obtained in 1) and 2) we have
\[ \Psi_{(\pi,U)}(a)\Psi_{(\pi,U)}(b) = \sum_{p,r,s,t \in \mathbb{N}} c_{p,r,s,t} = \sum_{p,r,s,t \in \mathbb{N}} c_{p,r,s,t} + \sum_{p,r,s,t \in \mathbb{N}} c_{p,r,s,t} \]

\[ = \sum_{m,r,s,n \in \mathbb{N}} c_{m,r,s,n-r+s} + \sum_{m,r,s,n \in \mathbb{N}} c_{m-r+r,s,n} = \sum_{m,n \in \mathbb{N}} U^{*m}(a \ast b)_{m,n}U^n = \Psi_{(\pi,U)}(a \ast b) \]

and the proof is complete. □

We now examine the structure of \( M(\mathcal{A}) \). We will say that a matrix \( \{a_{m,n}\}_{n \in \mathbb{N}, m \in \mathbb{Z}} \) in \( M(\mathcal{A}) \) is \( k \)-diagonal, where \( k \) is an integer, if it satisfies the condition

\[ a_{m,n} \neq 0 \implies m = k. \]

In other words \( k \)-diagonal matrix is the one of the form

\[
\begin{pmatrix}
  k & 0 \\
  0 & 0
\end{pmatrix}
\]

if \( k \geq 0 \), or

\[
\begin{pmatrix}
  |k| & 0 \\
  0 & 0
\end{pmatrix}
\]

if \( k < 0 \).

The linear space consisting of all \( k \)-diagonal matrices will be denoted by \( M_k \). These spaces will correspond to spectral subspaces, see Corollaries 2.3 and 3.2. We write \( M_k \ast M_l \) for the linear span of elements \( a \ast b, a \in M_k, b \in M_l \).

**Proposition 2.3.** The spaces \( M_k \) define a \( \mathbb{Z} \)-graded algebra structure on \( M(\mathcal{A}) \). Namely

\[ M(\mathcal{A}) = \bigoplus_{k \in \mathbb{Z}} M_k, \]

and for every \( k, l \in \mathbb{Z} \) we have the following relations

\[ M^*_k = M_{-k}, \quad M_k \ast M_l \subset M_{k+l}. \]

In particular, \( M_0 \) is a *-algebra, \( M_k \ast M_{-k} \) is a self-adjoint two sided ideal in \( M_0 \).

**Proof.** Relations \( M^*_k = M_{-k}, \quad M_k \ast M_l \subset M_{k+l} \) and \( (M_k \ast M^*_l)^* = (M_k \ast M^*_l) \) can be checked by means of an elementary matrix calculus. Using these relations we get

\[ M_0 \ast (M_k \ast M^*_l) = (M_0 \ast M_k) \ast M^*_l \subset M_k \ast M^*_l, \]

\[ (M_k \ast M^*_l) \ast M_0 = M_k \ast (M^*_l \ast M_0) \subset (M_k \ast M^*_l), \]

and thus \( M_k \ast M^*_l \) is an ideal in \( M_0 \). □

Proposition 2.3 indicates in particular that \( M_0 \) may be regarded as a coefficient algebra in the sense of [LO04] for \( M(\mathcal{A}) \). This will be shown explicitly in Proposition 2.6.
Corollary 2.4. Let \((\pi, U, H)\) be a representation of \((A, \alpha)\) and let \(B_k = \Psi(\pi, U)(M_k)\) be the linear space consisting of the elements of the form

\[
\sum_{n=0}^{N} U^{*n}\pi(a_n^{(k)})U^{n+k}, \quad \text{if} \ k \geq 0, \quad \text{or} \quad \sum_{n=0}^{N} U^{*n+|k|}\pi(a_n^{(k)})U^{n}, \quad \text{if} \ k < 0.
\]

Then for every \(k\) and \(l \in \mathbb{Z}\) we have the following relations

\[
B_{-k}^* = B_k, \quad B_k B_l \subset B_{k+l}.
\]

In particular, \(B_0\) is a \(C^*\)-algebra, \(B_k B_k^*\) is a self-adjoint two sided ideal in \(B_0\).

The importance of \(B_0\) was observed in \([LO04]\) and is clarified by the next proposition, see \([LO04\) Proposition 2.4].

Proposition 2.5. Let \((\pi, U, H)\) be a representation of \((A, \alpha)\) and adopt the notation from Proposition 2.2 and Corollary 2.4. Every element \(a \in C_0^*(\pi(A), U)\) can be presented in the form

\[
a = \sum_{k=1}^\infty u^*(a_{-k}) + \sum_{k=0}^\infty a_k u^*k
\]

where \(a_{-k} \in B_0\pi(\alpha^k(1))\), \(a_k \in \pi(\alpha^k(1))B_0\), \(k \in \mathbb{N}\), and only finite number of these coefficients are non-zero.

We will now formulate a similar result concerning \(\mathcal{M}(A)\). For each \(k \in \mathbb{Z}\) we define a mapping \(N_k : \mathcal{M}(A) \to \mathcal{M}_0\), \(k \in \mathbb{Z}\), that carries the \(k\)-diagonal onto a 0-diagonal and delete all the remaining ones. Namely, for \(a = \{a_n^{(k)}\}\) we set

\[
[N_k(a)]_m^{(k)} = \begin{cases} a_n^{(k)} & \text{if } m = 0, \\ 0 & \text{otherwise}, \quad k \in \mathbb{Z}. \end{cases}
\]

One readily checks that for \(k \geq 0\) we have \(N_k(\mathcal{M}_k) = \mathcal{M}_0 \ast \alpha^k(1)\), \(N_{-k}(\mathcal{M}_{-k}) = \alpha^k(1) \ast \mathcal{M}_0\). Thus the algebra \(\mathcal{M}_0\) consists of elements that play the role of Fourier coefficients in \(\mathcal{M}(A)\).

Proposition 2.6. Every element \(a\) of \(\mathcal{M}(A)\) is uniquely presented in the form

\[
a = \sum_{k=1}^\infty u^*k \ast a_{-k} + \sum_{k=0}^\infty a_k \ast u^*k
\]

where \(u\) is given by (12) and \(a_{-k} \in \mathcal{M}_0 \ast \alpha^k(1)\), \(a_k \in \alpha^k(1) \ast \mathcal{M}_0\), \(k \in \mathbb{N}\), and only finite number of these coefficients is non-zero. Namely, \(a_k = N_k(a)\) for \(k \in \mathbb{Z}\).

2.2 Norm evaluation of elements in \(C_0^*(\pi(A), U)\)

In this subsection we gather a number of technical results concerning norm evaluation of elements in \(C_0^*(\pi(A), U)\). We will make use of these results in the sequel.

The mappings \(\mathcal{N}_k : \mathcal{M}_k \to \mathcal{M}_0\) factor through \(\Phi(\pi, U)\) to the mappings \(N_k : B_k \to B_0\).
Proposition 2.7. Let $(\pi, U, H)$ be a representation of $(A, \alpha)$ and let $k \in \mathbb{Z}$. Then the norm $\|a\|$ of an element $a \in B_k$ corresponding to the matrix $\{a_n^{(m)}\}_{n \in \mathbb{N}, m \in \mathbb{Z}}$ in $\mathcal{M}_k$ is given by

$$
\lim_{n \to \infty} \max \left\{ \max_{i=1, \ldots, n} \left\| (1 - U^*U) \sum_{j=0}^{i} \pi(\alpha^{i-j}(a_j^{(k)})) \right\|, \left\| U^*U\pi(a_n^{(k)}) \right\| \right\}.
$$

In particular, the mapping $N_k : B_k \to B_0$ given by

$$
N_k(\Phi_{(\pi, U)}(a)) = \Phi_{(\pi, U)}(N_k(a)), \quad a \in \mathcal{M}_k,
$$

is a well defined linear isometry establishing the following isometric isomorphisms

$$
B_k \cong B_0 \alpha^k(1), \quad \text{if} \quad k \geq 0, \quad B_k \cong \alpha^{|k|}(1)B_0, \quad \text{if} \quad k < 0.
$$

Proof. Let us assume that $k \geq 0$. Let $N$ be such that $a_m^{(k)} = 0$ for $m > N$, that is

$$
a = \sum_{m=0}^{N} U^*m \pi(a_m^{(k)})U^{m+k}.
$$

Then, similarly as it was done in the proof of [Kwa07, Prop. 3.1], one sees that defining

$$
a_i = (1 - U^*U) \pi \left( \sum_{j=0}^{i} \alpha^{i-j}(a_j^{(k)}) \right), \quad i = 0, \ldots, N, \quad a_{N+1} = U^*U\pi(a_N^{(k)}),
$$

we have

$$
a = \left( a_0 + U^*a_1U + \ldots + U^*N(a_N + a_{N+1})U^N \right)U^k
$$

and

$$
a_i \in (1 - U^*U)\pi(\alpha^i(1)A\alpha^{i+k}(1)), \quad i = 0, \ldots, N, \quad a_{N+1} \in U^*U\pi(\alpha^N(1)A\alpha^{N+k}(1)),
$$

Hence it follows that

$$
U^{*i}a_iU^i \in (U^{*i}U^i - U^{*i+1}U^{i+1})\pi(A)U^kU^*k, \quad i = 0, \ldots, N,
$$

and

$$
U^{*N}a_{N+1}U^N \in U^{*N+1}U^{N+1}\pi(A)U^kU^*k.
$$

These relations and the fact that $U^{*i}U^i - U^{*i+1}U^{i+1}, i = 0, \ldots, N,$ and $U^{*N+1}U^{N+1}$ are pairwise orthogonal projections lying in $\pi(A)'$ (cf. Lemma 1.2 or better [LO04, Prop 3.6]), imply the following equalities

$$
\|a\| = \|(a_0 + U^*a_1U + \ldots + U^*N(a_N + a_{N+1})U^N)U^kU^*k\|
= \|a_0 + U^*a_1U + \ldots + U^*N(a_N + a_{N+1})U^N\|
= \max \left\{ \max_{i=0, \ldots, N} \|U^{*i}a_iU^i\|, \|U^{*N}(a_{N+1})U^N\| \right\}
= \max \left\{ \max_{i=0, \ldots, N} \|a_i\|, \|a_{N+1}\| \right\}
$$
where the final equality follows from that the linear mapping $a \to U^* a U$ is isometric on $\pi(\alpha^n(1) A \alpha^n(1)) = U^n U^* \pi(A) U^n U^*$, $n \in \mathbb{N}$. Since $N$ was arbitrary (sufficiently large) this proves the assertion in the case when $k \geq 0$.

In the case of negative $k$ one may apply the part of proposition proved above to the adjoint $a^*$ of the element $a$ and thus obtain the hypotheses. ■

We denote by

$$d(a, K) = \inf_{b \in K} \|a - b\|$$

the usual distance of an element $a$ from the set $K$. The definition of an ideal of covariance (Definition 1.7) and a known fact expressing quotient norms in terms of projections, see for instance, [KR86, Lemma 10.1.6] gives us the following

**Corollary 2.8.** If $(\pi, U, H)$ is a faithful $J$-covariant representation of $(A, \alpha)$ and $I$ denotes the kernel of $\alpha$, then the norm of an element $a \in B_k$ corresponding to a matrix $\{a_{mn}^{(m)}\}_{n \in \mathbb{N}, m \in \mathbb{Z}}$ in $\mathcal{M}_k$ is given by

$$\|a\| = \lim_{n \to \infty} \max \left\{ \max_{i=1, \ldots, n} \left\{ d\left( \sum_{j=0}^{i} \alpha^{i-j}(a_{j,j}^{(k)}), J \right), d(a_{n,n}^{(k)}, I) \right\} \right\}.$$  

In particular, if $J$ is a fixed ideal orthogonal to $I$ and $(\pi, U, H)$ is a faithful $J$-covariant representation, then the spaces $B_k$, $k \in \mathbb{Z}$ do not depend on its choice.

We showed in Proposition 2.7 that for an arbitrary representation $(\pi, U, H)$ of $(A, \alpha)$, the mappings $N_k$ factor through to the mappings $N_k$ acting on spaces $B_k$. In general, however, $\mathcal{N}_k : \mathcal{M}(A) \to B_0$ do not factor through $\Psi_{(\pi, U)}$ to the mappings acting on the algebra $C_0^*(\pi(A), U)$. In fact, this is the case if and only if the representation $(\pi, U, H)$ satisfies a certain property we are just about to introduce.

**Definition 2.9.** We will say that a faithful representation $(\pi, U, H)$ of $(A, \alpha)$ possesses property $(\ast)$ if for any $a \in C_0^*(\pi(A), U)$ given by a matrix $\{a_{mn}\}_{m,n \in \mathbb{N}} \in \mathcal{M}(A)$ the inequality

$$\| \sum_{m \in \mathbb{N}} U^* a_{m,m} U^m \| \leq \| \sum_{m,n \in \mathbb{N}} U^* \pi(a_{m,n}) U^m \|,$$  

holds. In view of Corollary 2.8 the above equality could be equivalently stated in the form

$$\lim_{n \to \infty} \max \left\{ \max_{i=1, \ldots, n} \left\{ d\left( \sum_{j=0}^{i} \alpha^{i-j}(a_{j,j}), J \right), d(a_{n,n}, I) \right\} \right\} \leq \|a\|, \quad (\ast)$$

where $I$ is the kernel of $\alpha$ and $J$ is the ideal of covariance of $(\pi, U, H)$.

The next result, which follow immediately from [LO04, Thm. 2.8], indicates that under the fulfillment of property $(\ast)$ elements of $B_0$ play the role of 'Fourier' coefficients in the algebra $C_0^*(\pi(A), U)$. 

**Theorem 2.10.** Let \((\pi, U, H)\) be a faithful representation of \((A, \alpha)\) possessing property 
\((\ast)\) then the mappings \(N_k : C^*_0(\pi(A), U) \to B_0, \ k \in \mathbb{Z}\), given by formulae
\[
N_k(\Phi_{(\pi,U)}(a)) = \sum_{n \in \mathbb{N}} U^{*n} \pi(a^{(k)}_n) U^n ,
\]
where \(\{a^{(m)}_n\}_{n \in \mathbb{N}, m \in \mathbb{Z}} \in \mathcal{M}(A)\), are well defined contractions and thus they extend 
uniquely to bounded operators on \(C^*_0(\pi(A), U)\). In particular, every element \(a \in C^*_0(\pi(A), U)\) can be uniquely presented in the form
\[
a = \sum_{k=-\infty}^{\infty} U^{sk} a_{-k} + \sum_{k=0}^{\infty} a_k U^{sk}
\]
where \(a_{-k} \in B_0 \pi(\alpha^k(1))\), \(a_k \in \pi(\alpha^k(1))B_0\), \(k \in \mathbb{N}\), namely, \(a_k = N_k(a), \ k \in \mathbb{Z}\).

Let us also recall [LO04, Thm. 2.11].

**Theorem 2.11.** If \((\pi, U, H)\) possesses property \((\ast)\), then for any element \(a \in C^*_0(\pi(A), U)\) we have
\[
\|a\| = \lim_{k \to \infty} \frac{1}{4k} \sqrt[k]{\|N_0[(aa^*)^{2k}]\|}
\]
where \(N_0\) is the mapping defined by \((14)\).

Using the above results one sees that in the presence of property \((\ast)\) the norm of an 
element \(a \in C^*_0(\pi(A), U)\) may be calculated only in terms of the elements of \(A\). Indeed, 
as \(N_0[(aa^*)^{2k}]\) belongs to \(B_0\) one can apply Corollary 2.8 to calculate \(\|N_0[(aa^*)^{2k}]\|\) 
in terms of the matrix from \(\mathcal{M}(A)\) corresponding to \(a\). However in practice, calculation 
of the matrix corresponding to the element \((aa^*)^{2k}\) starting from \(a\), see formula \((10)\), 
seems to be an extremely difficult task.

### 2.3 Crossed product defined by matrix calculus

The foregoing observations make it now possible to give one more 'internal' definition of the crossed product. This is the aim of the present subsection.

The set \(\mathcal{M}(A)\) with operations \((7), (8), (9), (10)\) is an algebra with involution. We define a seminorm on \(\mathcal{M}(A)\) that will depend on the choice of an orthogonal ideal. Let \(J\) be a fixed ideal in \(A\) having zero intersection with the kernel of \(\alpha\). Let \(J\) be a fixed ideal in \(A\) having zero intersection with the kernel of \(\alpha\). Let
\[
\|\|a\|\|_J := \sum_{k \in \mathbb{Z}} \lim_{n \to \infty} \max_{i=1,\ldots,n} \left\{ \max_{j=0,\ldots,i} \left\{ d\left( \sum_{j=0}^{i} \alpha^{i-j}(a^{(k)}_j, J) \right) , d(a^{(k)}_n, I) \right\} \right\}
\]
where \(a = \{a^{(k)}_n\}_{n \in \mathbb{N}, k \in \mathbb{Z}} \in \mathcal{M}(A)\).

**Proposition 2.12.** The function \(\|\cdot\|_J\) defined above is a seminorm on \(\mathcal{M}(A)\) which 
is \(\ast\)-invariant and submultiplicative.
Proof. Let \((\pi, U, H)\) be a faithful representation of \((A, \alpha)\) and \(J\) be its covariance ideal. Such representation does exist by Theorem \[1.11\]. Then in view of Corollary \[2.8\] for every \(a \in M_k, k \in \mathbb{Z}\), we have

\[\|a\| = \|\Phi(\pi, U)(a)\|\]

where \(\Phi(\pi, U) : M(A) \to L(H)\) is the \(*\)-homomorphism defined in Proposition \[2.2\]. Thus since every element \(a \in M(A)\) can be presented in the form \(a = \sum_{k \in \mathbb{Z}} a^{(k)}\) where \(a^{(k)} \in M_k\) one easily sees that \(\|\cdot\|\) is \(*\)-invariant seminorm. To show that it is submultiplicative take \(a = \sum_{k \in \mathbb{Z}} a^{(k)} \in M(A)\) and \(b = \sum_{k \in \mathbb{Z}} b^{(k)} \in M(A)\) such that \(a^{(k)}, b^{(k)} \in M_k\). Then

\[
\|a \ast b\| = \|\sum_{k \in \mathbb{Z}} a^{(k)} \ast \sum_{l \in \mathbb{Z}} b^{(l)}\| = \|\sum_{k \in \mathbb{Z}, l \in \mathbb{Z}} a^{(k)} \ast b^{(l)}\| \leq \sum_{k, l \in \mathbb{Z}} \|a^{(k)} \ast b^{(l)}\| \leq \sum_{k, l \in \mathbb{Z}} \|\Phi(\pi, U)(a^{(k)} \ast b^{(l)})\| \\
= \sum_{k, l \in \mathbb{Z}} \|a^{(k)}\| \cdot \|b^{(l)}\| = \sum_{k \in \mathbb{Z}} \|a^{(k)}\| \sum_{l \in \mathbb{Z}} \|b^{(l)}\| = \|a\| \cdot \|b\|.
\]

\[\square\]

Definition 2.13. Let \((A, \alpha)\) be a \(C^*\)-dynamical system and \(J\) an ideal in \(A\) having zero intersection with the kernel of \(\alpha\). The crossed product \(C^*(A, \alpha, J)\) of \(A\) by \(\alpha\) associated with \(J\) is the enveloping \(C^*\)-algebra of the quotient \(*\)-algebra \(M(A)/\|\cdot\|\cdot J\).

Regardless of \(J\), composing the quotient map with natural embedding of \(A\) into \(M(A)\) one has an embedding of \(A\) into \(C^*(A, \alpha, J)\). Moreover, denoting by \(\hat{u}\) an element of \(C^*(A, \alpha, J)\) corresponding to \(u \in M(A)\) (see \[12\]), one sees that \(C^*(A, \alpha, J)\) is generated by \(A\) and \(\hat{u}\).

The equivalence of this definition and that introduced previously (Definition \[1.12\]) will be established in the next section (Proposition \[3.7\]).

3 Isomorphism theorems and faithful representations

Once a universal object (the crossed product) is defined it is reasonable to have its faithful representation. This section is devoted to the description of the properties of such representations and, in particular, we establish the equivalence of two previously mentioned definitions of the crossed product. In addition we present one more alternative crossed product construction based on \[ABL11\] approach and prove faithfulness by means of the topologically free action on the arising coefficient algebras.

3.1 Isomorphism Theorem

Theorem 3.1 (Isomorphism Theorem). Let \(J\) be an ideal in \(A\) having zero intersection with the kernel \(I\) of \(\alpha\) and let \((\pi_i, U_i, H_i), i = 1, 2,\) be faithful \(J\)-covariant representations of \((A, \alpha)\) possessing property \((*)\). Then the relations

\[\Phi(\pi_1(a)) := \pi_2(a), \quad a \in A, \quad \Phi(U_1) := U_2\]
gives rise to an isomorphism between the $C^*$-algebras $C^*(\pi_1(A), U_1)$ and $C^*(\pi_2(A), U_2)$.

**Proof.** Let $B_{0,i}$ be a $*$-algebra consisting of elements of the form $\sum_{n=0}^N U_i^n \pi_i(a_n)U_i^n$, $i = 1, 2$. In view of Corollary 2.8, $\Phi$ extends to an isometric isomorphism from $B_{0,1}$ onto $B_{0,2}$. Moreover, we have

$$\Phi(U_1aU_1^*) = U_2(\Phi(a))U_2^*, \quad a \in B_{0,1}.$$  

Hence the assumptions of [LO04, Theorem 2.13] are satisfied and the hypotheses follows. ■

**Corollary 3.2.** If $(\pi, U, H)$ possesses property $(\ast)$, then we have a point-wise continuous action $\gamma$ of the group $S^1$ on $C^*(\pi(A), U)$ by automorphisms given by

$$\gamma_z(\pi(a)) := \pi(a), \quad a \in A, \quad \gamma_z(U) := zU, \quad z \in S^1.$$  

Moreover the spaces $\overline{B}_k$ are the spectral subspaces corresponding to this action, that is we have

$$\overline{B}_k = \{ a \in C^*(A, U) : \gamma_z(a) = z^k a \}.$$  

In particular, the $C^*$-algebra $\overline{B}_0$ is the fixed point algebra for $\gamma$.

**Proof.** Let $(\pi, U, H)$ be a $J$-covariant representation of $(A, \alpha)$ possessing property $(\ast)$ and let $z \in S^1$. It is clear that $(\pi, zU, H)$ is also a $J$-covariant representation of $(A, \alpha)$ and $(\pi, zU, H)$ possesses property $(\ast)$. Hence by virtue of Theorem 3.1 $\gamma_z$ extends to an isomorphism of $C^*(\pi(A), U) = C^*(\pi(A), zU)$. The remaining part of the statement is obvious. ■

The next theorem is an immediate corollary of the previous statements and [LO04, Thm. 2.15]. It is another manifestation of the fact that the elements $N_k(a), k \in \mathbb{Z}$, should be considered as Fourier coefficients for $a \in C^*(\pi(A), U)$.

**Theorem 3.3.** Let $(\pi, U, H)$ possess property $(\ast)$ and let

$$a \in C^*(\pi(A), U).$$

Then the following conditions are equivalent:

(i) $a = 0$;

(ii) $N_k(a) = 0, k \in \mathbb{Z}$;

(iii) $N_0(a^*a) = 0$.

The results presented above give us a possibility to write out a criterion for a representation of the crossed product to be faithful.

**Theorem 3.4.** Let $C^*(A, \alpha, J)$ be the crossed product given by Definition 2.13 and $(\pi, U, H)$ be a faithful $J$-covariant representations of $(A, \alpha)$. Then the relations

$$(\pi \times U)(a) = \pi(a), \quad a \in A; \quad (\pi \times U)(\hat{u}) = U$$

determine in a unique way an epimorphism $\pi \times U : C^*(A, \alpha, J) \to C^*(\pi(A), U)$. Moreover $\pi \times U$ is an isomorphism iff $(\pi, U, H)$ possesses property $(\ast)$. 

3.2 Construction of faithful representations

Given a faithful $J$-covariant representation of $(A, \alpha)$ one can construct a covariant representation of $(A, \alpha)$ possessing property (*) thus obtaining a faithful representation of $C^*(A, \alpha, J)$ (in view of Theorem 3.4). Actually, we exploit the standard argument cf. [ABL11].

**Proposition 3.5.** For any faithful $J$-covariant representation $(\pi, \tilde{U}, \tilde{H})$ of $(A, \alpha)$ the triple $(\pi, U, H)$ where $H := l^2(\mathbb{Z}, \tilde{H})$,

\[(\pi(a)\xi)_n := \tilde{\pi}(a)(\xi_n), \quad \text{and} \quad (U\xi)_n := \tilde{U}(\xi_{n-1})\]  

(17) for $a \in A$, $\xi = \{\xi_n\}_{n \in \mathbb{Z}} \in H = l^2(\mathbb{Z}, \tilde{H})$, is a faithful $J$-covariant representation which integrate via (16) to a faithful representation $(\pi \times U)$ of $C^*(A, \alpha, J)$.

**Proof.** Routine verification shows that $(\pi, U, H)$ is a faithful $J$-covariant representation of $(A, \alpha)$. By Theorem 3.4 it suffices to verify that $(\pi, U, H)$ possesses property (*). To this end take any $N \in \mathbb{N}$, $a_{m,n} \in A$, $n, m = 0, 1, ..., N$, and note that by the explicit form of (17) we have

\[
\| \sum_{m=0}^{N} \tilde{U}^* \pi(a_{m,m}) \tilde{U}^m \| = \| \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \| \]  

(18)

For a given $\varepsilon > 0$ there exists a vector $\eta \in \tilde{H}$ such that $\|\eta\| = 1$ and

\[
\| \left( \sum_{m=0}^{N} \tilde{U}^* \pi(a_{m,m}) \tilde{U}^m \right) \eta \| > \| \sum_{m=0}^{N} \tilde{U}^* \pi(a_{m,m}) \tilde{U}^m \| - \varepsilon. \]  

(19)

Set $\xi = \{\xi_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}, \tilde{H})$ by $\xi_n = \delta_{0,n}\eta$, where $\delta_{i,j}$ is the Kronecker symbol. We have that $\|\xi\| = 1$ and (17) along with (18) and (19) imply

\[
\| \left( \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \right) \xi \| > \| \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \| - \varepsilon \]  

(20)

Now the explicit form of $\left( \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \right) \xi$ and (20) imply

\[
\| \sum_{m, n=0}^{N} U^* \pi(a_{m,n}) U^m \|^2 \geq \left( \sum_{m, n=0}^{N} U^* \pi(a_{m,n}) U^m \right) \xi \| \geq \]  

\[
\geq \left( \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \right) \xi \|^2 > \left( \| \sum_{m=0}^{N} U^* \pi(a_{m,n}) U^m \| - \varepsilon \right)^2 \]  

(21)

which by the arbitrariness of $\varepsilon$ proves property (*) for $(\pi, U, H)$. ■

Following the foregoing construction one can also arrive at the next
Proposition 3.6. The crossed product $C^*(A, \alpha, J)$, given by Definition 1.12 possesses property $(\ast)$, that is the algebra $A$ is embedded in the crossed product $C^*(A, \alpha, J)$ and for any $N \in \mathbb{N}$ and $a_{m,n} \in A$, $n,m = 0,1,...,N$ the following inequality holds

$$\| \sum_{m=0}^{N} u^m a_{m,n} u^m \| \leq \| \sum_{m,n=0}^{N} u^m a_{m,n} u^n \|. \quad (22)$$

Proof. Let $C^*(A, \alpha, J)$ be given by Definition 1.12. Take any faithful representation $\pi : C^*(A, \alpha, J) \to L(\bar{H})$ of $C^*(A, \alpha, J)$ in a Hilbert space $\bar{H}$ and 'disintegrate' $\pi$ to $(\tilde{\pi}, \tilde{\bar{U}}, \tilde{\bar{H}})$, i.e. let $\tilde{\pi} := \pi|_A$ and $\tilde{\bar{U}} := \pi(u)$. Then $(\tilde{\pi}, \tilde{\bar{U}}, \tilde{\bar{H}})$ is a faithful $J$-covariant representation of $(A, \alpha)$ (note that Theorem 1.11 and Definition 1.12 imply that $\tilde{\pi}$ is a faithful representation of $A$). Consider the space $H = l^2(\mathbb{Z}, \tilde{\bar{H}})$ and representation $(\pi, U, H)$ given by (17). By the universality of $C^*(A, \alpha, J)$ (Definition 1.12) this representation give rise to a representation of $C^*(A, \alpha, J)$. And therefore for any $N \in \mathbb{N}$ and $a_{m,n} \in A$, $n,m = 0,1,...,N$ one has

$$\| \sum_{m=0}^{N} u^m a_{m,n} u^m \| = \| \sum_{m,n=0}^{N} U^m \pi(a_{m,n}) U^n \|. \quad (23)$$

Now (22) follows from (21). \[\blacksquare\]

The above results imply

Proposition 3.7. The crossed products given by Definitions 1.12 and 2.13 are canonically isomorphic.

Proof. Let $C^*(A, \alpha, J)$ be the crossed product given by Definition 1.12. Set $\pi_1 : A \to \hat{A}$, $\pi_1(a) := \hat{a}$, and $U_1 := \hat{u}$, where $\hat{a}$ and $\hat{u}$ are the universal elements, corresponding to $a \in A$ and $u$ according to Definition 1.12. Then

$$C^*(A, \alpha, J) = C^*(\pi_1(A), U_1).$$

By Definition 1.12 and Theorem 1.11 $\pi_1$ establishes an isomorphism between $A$ and $\hat{A}$. By Proposition 3.6 $\hat{C}^*(A, \alpha, J)$ possesses property $(\ast)$. Now, let $C^*(A, \alpha, J)$ be the crossed product given by Definition 2.13 and let $(\pi, U, H)$ be a faithful $J$-covariant representation of $(A, \alpha)$ possessing property $(\ast)$ (such representation does exist by Remark 3.2). Set $(\pi_2, U_2, H)$ by

$$\pi_2(a) := (\pi \times U)(a) = \pi(a), \quad a \in A; \quad U_2 := (\pi \times U)(\hat{u}) = U \quad (23)$$

(cf. (16)). Then by Theorem 3.4

$$C^*(\pi_2(A), U_2) \cong C^*(A, \alpha, J)$$

and $(\pi_2, U_2, H)$ possesses property $(\ast)$. Thus by Theorem 3.1

$$C^*(\pi_1(A), U_1) \cong C^*(\pi_2(A), U_2).$$

\[\blacksquare\]
3.3 Topological freeness

Crossed product $C^*(A, \alpha, J)$ can also be constructed by means of the crossed product introduced in [ABL11]. Here we present this construction. It can be considered as one more alternative definition of $C^*(A, \alpha, J)$.

Let $(A, \alpha)$ be a $C^*$-dynamical system, $J$ be an ideal orthogonal to the kernel of $\alpha$, and $(\pi, U, H)$ be a faithful $J$-covariant representation of $(A, \alpha)$. Consider the algebra

$$B := C^* \left( \bigcup_{n \in \mathbb{N}} U^n \pi(A) U^n \right).$$

(24)

By Corollary 2.8 algebra $B$ can be described in terms of $(A, \alpha)$ and $J$, and therefore it does not depend on the choice of a faithful $J$-covariant representation $(\pi, U, H)$. Routine calculation (cf. [LO04, Prop. 3.10.]) shows that

$$UBU^* \subset B, \quad U^* BU \subset B, \quad U^* U \in Z(B).$$

(25)

Thus $B$ is a coefficient algebra in the sense of [LO04]. In particular

$$\alpha : B \to B, \quad \alpha(\cdot) := U(\cdot) U^*$$

is an endomorphism of $B$ (identifying $\pi(A)$ with $A$ the mapping $U(\cdot) U^*$ extends the endomorphism $\alpha : A \to A$ and the notational collision disappears) and

$$\mathcal{L} : B \to B, \quad \mathcal{L}(\cdot) := U^*(\cdot) U$$

is a complete transfer operator in the sense of [BL05]. Note also that the mapping $\mathcal{L}$ is defined uniquely by the $C^*$-dynamical system $(B, \alpha)$ ([BL05, Theorem 2.8.]), and therefore it is uniquely defined by $(A, \alpha)$ and $J$. In particular, Proposition 3.6 and [ABL11, Theorem 3.5.] imply the following

**Proposition 3.8.** We have a natural isomorphism

$$C^*(A, \alpha, J) \cong B \times_{\alpha} \mathbb{Z},$$

(26)

where on the right hand side stands the crossed product of [ABL11, Def. 2.6.].

**Remark 3.9.** Thus the alternative construction of crossed product involve essentially two steps: 1) extending an irreversible system $(A, \alpha)$ up to a reversible system on $(B, \alpha)$ (here $\mathcal{L}$ plays the role of the inverse to $\alpha$), and then 2) attaching the crossed product from [ABL11] to the extended system.

We have to stress that the general procedure of extension of $(A, \alpha)$ up to $(B, \alpha)$ by means of (24) involves the ideal $J$ but ‘does not see’ it explicitly and namely the material of the present paper shows that by means of these orthogonal ideals all the possible extensions are parametrised (recall in this connection the discussion in [ABL11, Section 5]).
The isomorphism (26) and the results of [Kwa10] give us a possibility to obtain one more isomorphism theorem which can be written in terms of topological freeness of the action $\alpha$ on $B$. Indeed, it follows immediately from (25) that

$$\mathcal{L} : \alpha(B) \to \mathcal{L}(B) \quad \text{and} \quad \alpha : \mathcal{L}(B) \to \alpha(B)$$

are mutually inverse isomorphisms $\mathcal{L}(B) = \mathcal{L}(1)B$ is an ideal in $B$ and $\alpha(B) = \alpha(1)B\alpha(B)$ is a hereditary subalgebra of $B$. Therefore, cf. e.g. [Mur96 Thm 5.5.5], we may naturally identify the spectra $\widehat{\alpha(B)}$ and $\mathcal{L}(B)$ of $\alpha(B)$ and $\mathcal{L}(B)$ with open subsets of the spectrum $\widehat{B}$ of $B$, and then

$$\widehat{\alpha(B)} = \{ \pi \in \widehat{B} : \pi(\alpha(1)) \neq 0 \}, \quad \mathcal{L}(B) = \{ \pi \in \widehat{B} : \pi(\mathcal{L}(1)) \neq 0 \}. \quad (27)$$

Under the above identifications the dual $\widehat{\alpha} : \widehat{\alpha(B)} \to \mathcal{L}(B)$ to the isomorphism $\alpha : \mathcal{L}(B) \to \alpha(B)$ becomes a partial homeomorphism of $B$. More precisely, let $\pi : B \to L(H)$ be an irreducible representation. If $\pi(\alpha(1)) \neq 0$, then

$$\widehat{\alpha}(\pi) = \pi \circ \alpha : B \to L(\alpha(1)H) \quad (28)$$

is an irreducible representation such that $\widehat{\alpha}(\pi)((1)) \neq 0$. Conversely if $\pi(\mathcal{L}(1)) \neq 0$, then $\widehat{\alpha}^{-1}(\pi) = \mathcal{L}(\pi)$ is a unique (up to unitary equivalence) irreducible extension of the representation $\pi \circ \mathcal{L} : \alpha(1)B\alpha(1) \to L(H)$.

**Definition 3.10.** We say that $\widehat{\alpha}$ where $\widehat{\alpha} : \widehat{\alpha(B)} \to \mathcal{L}(B)$ is a homeomorphism given by (28), between the open subsets (27) of $\widehat{B}$, is a partial homeomorphism dual to the endomorphism $\alpha : B \to B$. We call the pair $(\widehat{B}, \widehat{\alpha})$ the partial dynamical system dual to the C*-dynamical system $(B, \alpha)$.

We recall that a partial homeomorphism of a topological space, i.e. a homeomorphism between open subsets, is *topologically free* if for any $n \in \mathbb{N}$ the set of periodic points of period $n$ has empty interior. Applying [Kwa10 Thm. 2.24] we arrive at the following result

**Theorem 3.11 (Isomorphism theorem and topologically free action).** Let $(A, \alpha)$ be a C*-dynamical system, $J$ an ideal orthogonal to the kernel of $\alpha$, and $(B, \alpha)$ a partially reversible system associated to the triple $(A, \alpha, J)$ as described above. If the partial homeomorphism $\widehat{\alpha}$ dual to $\alpha : B \to B$ is topologically free, then for any faithful $J$-covariant representations $(\pi, U, H)$ of $(A, \alpha)$ the epimorphism $\pi \times U : C^*(A, \alpha, J) \to C^*(\pi(A), U)$ given by (16) is an isomorphism:

$$C^*(A, \alpha, J) \cong C^*(\pi(A), U).$$

A crucial and (in the noncommutative case) still open question is the following.

**Problem:** How to express the topological freeness of the partial reversible dynamical system $(\widehat{B}, \widehat{\alpha})$ dual to $(B, \alpha)$ in terms of the initial C*-dynamical system $(A, \alpha)$ and the ideal $J$?
Crossed product by endomorphism

\begin{tabular}{|c|c|c|}
\hline
N. & endomorphism $\alpha : A \to A$ & $J \triangleleft A$ & $C^*(A, \alpha, J)$ \\
\hline
1. & automorphism & $J = (\ker \alpha) = A$ & classical unitary crossed product \\
2. & monomorphism & $J = (\ker \alpha)^{\perp} = A$ & isometric crossed product \\
3. & $\ker \alpha$ unital and $\alpha(A)$ hereditary in $A$ & $J = (\ker \alpha)^{\perp}$ & crossed product using complete transfer operator \\
4. & $\ker \alpha$ unital and $A$ commutative & $J = (\ker \alpha)^{\perp}$ & covariance algebra \\
5. & arbitrary & $J = \{0\}$ & partial-isometric crossed product \\
6. & arbitrary & $\{0\} \subset J \subset (\ker \alpha)^{\perp}$ & partial-isometric crossed product \\
\hline
\end{tabular}

Table 1: Different crossed products

3.4 Crossed product overview

One can see that the most popular crossed products by endomorphisms coincide with $C^*(A, \alpha, J)$ for certain $J$. Table 1 presents the corresponding juxtaposition of the objects chosen.

To see the coincidence in N.3 of Table 1 we refer the reader to [Kwa07, Prop. 2.6]. The crossed product N.6 (Definitions 1.12 and 2.13) is the most general in the sense that it gives all the remaining ones for an appropriate choice of $J$; namely $J = (\ker \alpha)^{\perp}$ for N.1-4 and $J = \{0\}$ for N.5.

As it is shown in [ABL11, Section 4] the crossed product N.3 of Table 1 covers a lot of most popular crossed product constructions, and in particular two kinds of crossed products introduced by R. Exel in [Exel94] and [Exel03] may be obtained from the crossed product N.3.

Note also that there are a number of crossed products that at first sight are not of type $C^*(A, \alpha, J)$ and are related to ideals that are not orthogonal to $\ker \alpha$. As an example let us mention Stacey’s (multiplicity one) crossed product [Sta93, Defn. 3.1]. For the sake of simplicity we state his definition in a unital setting, cf. [ALNR94].

**Definition 3.12.** Stacey’s crossed product for an endomorphism $\alpha$ of a unital $C^*$-algebra $A$ is a unital $C^*$-algebra $B$ together with a unital $*$-homomorphism $i_A : A \to B$ and an isometry $u \in B$ such that

i) $i_A(\alpha(a)) = u i_A(a) u^*$ for all $a \in A$

ii) $B$ is generated by $i_A(A)$ and $u$

iii) for every non-degenerate representation $\pi : A \to L(H)$ and an isometry $T \in L(H)$
there is a representation \( \pi \times T : B \to L(H) \) such that \( (\pi \times T) \circ i_A = \pi \) and \( (\pi \times T)(u) = T \).

The mapping \( a \to uau^* \) for an isometry \( u \) is injective. Therefore, in view of condition i), the homomorphism \( i_A \) can not be injective unless \( \alpha \) is a monomorphism. Hence in general \( A \) does not embeds into the Stacey’s crossed product. In particular, see [Sta93, Prop. 2.2], Stacey proved that \( B \) degenerates to \( \{0\} \) if and only if the inductive limit of the inductive sequence \( A \xrightarrow{\alpha} A \xrightarrow{\alpha} \ldots \) degenerates to zero. We will refine this result in Example 5.11 below. Actually we show that Stacey’s crossed product can also be presented in the form of \( C^*(A, \alpha, J) \) for an appropriate \( A, \alpha \) and \( J \), but to achieve this one first needs to ‘reduce’ the initial \( C^* \)-dynamical system. The general procedure of such a reduction is discussed in the forthcoming part of the paper where we also analyse the relation between the crossed products \( C^*(A, \alpha, J) \) and relative Cuntz-Pimsner algebras.

4 Crossed products and relative Cuntz-Pimsner algebras

In this section we start to discuss relations between the crossed products introduced and relative Cuntz-Pimsner algebras. This requires a description of a series of known objects and results and we do this job in Subsections 4.1, 4.2 and 4.3 while a presentation of the crossed products as relative Cuntz-Pimsner algebras is given in Subsection 4.4. To begin with we recall the basic necessary objects related to Hilbert \( C^* \)-modules and \( C^* \)-correspondences. A general information on Hilbert \( C^* \)-modules can be found, for example, in [Lan95]. The term \( C^* \)-correspondence was popularized, among the others, by T. Katsura [Kat03], [Kat04], [Kat07].

4.1 \( C^* \)-correspondences and their representations

Let \( A \) be a (not necessarily unital) \( C^* \)-algebra and \( X \) a right Hilbert \( A \)-module with an \( A \)-valued inner product \( \langle \cdot, \cdot \rangle_A \), see [Lan95, ch. 1]. We denote by \( \mathcal{L}(X) \) the \( C^* \)-algebra of adjointable operators on \( X \). For \( x, y \in X \), we let \( \Theta_{x,y} \in \mathcal{L}(X) \) be the 'one-dimensional operator': \( \Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A \), and we denote by \( \mathcal{K}(X) \) the ideal of 'compact operators' in \( \mathcal{L}(X) \) which is a closed linear span of the operators \( \Theta_{x,y}, x, y \in X \). We recall that any \( C^* \)-algebra \( A \) can be naturally treated as a right Hilbert \( A \)-module where \( \langle a, b \rangle_A = a^*b \) and then \( \mathcal{K}(A) = A \) and \( \mathcal{L}(A) = M(A) \) is the multiplier algebra of \( A \).

**Definition 4.1.** A \( C^* \)-correspondence \( X \) over a \( C^* \)-algebra \( A \) is a (right) Hilbert \( A \)-module equipped with a homomorphism \( \phi : A \to \mathcal{L}(X) \). We refer to \( \phi \) as the left action of a \( A \) on \( X \) and write

\[
a \cdot x := \phi(a)x.
\]

**Remark 4.2.** A \( C^* \)-correspondence is also sometimes called a Hilbert bimodule, see e.g. [Pim97], [FR99], [FMR03]. However, there are plenty of reasons, see e.g. [AEE98], [Kat03], [Kwa13] or [Kwa12], that the term Hilbert bimodule should be reserved for a
special sort of $C^*$-correspondence, namely a $C^*$-correspondence $X$ with an additional structure which is an $A$-valued sesqui-linear form $A\langle \cdot, \cdot \rangle$ such that
\[ x \cdot \langle y, z \rangle_A = A\langle x, y \rangle \cdot z, \quad \text{for all } x, y, z \in X. \]

Then, see [Kat03, Lem 3.4] or [Kwa13, Prop. 1.11], $X$ is both a left and a right Hilbert $A$-module and $\|\langle x, x \rangle_A\| = \|A\langle x, x \rangle\|$.

**Definition 4.3.** A representation $(\pi, t, B)$ of a $C^*$-correspondence $X$ in a $C^*$-algebra $B$ consists of a linear map $t : X \to B$ and a homomorphism $\pi : A \to B$ such that
\[ t(x \cdot a) = t(x)\pi(a), \quad t(x)^*t(y) = \pi(\langle x, y \rangle_A), \quad t(a \cdot x) = \pi(a)t(x), \quad (30) \]
for $x, y \in X$ and $a \in A$. If $\pi$ is faithful (then automatically $t$ is isometric, cf. [FR99, Rem 1.1]) we say that the representation $(\pi, t, B)$ is faithful. If $B = L(H)$ for a Hilbert space $H$ we say that $(\pi, t, L(H))$ is a representation of $X$ in $H$.

**Remark 4.4.** The above introduced notion is called in [FR99], [FMR03] a Toeplitz representation of $X$, and in [MS98] it is called an isometric covariant representation of $X$.

Any representation $(\pi, t, B)$ of a $C^*$-correspondence $X$ in a $C^*$-algebra $B$ naturally give rise to a representation $(\pi, t, B)^{(1)} : \mathcal{K}(X) \to B$ of the $C^*$-algebra $\mathcal{K}(X)$ of 'compact operators' on $X$ which is uniquely determined by the condition that
\[ (\pi, t, B)^{(1)}(\Theta_{x,y}) := t(x)t(y)^* \quad \text{for } x, y \in X, \quad (31) \]
see [FR99, Prop. 1.6] or [Kwa13, Prop. 3.13]. Moreover the left action $\phi : A \to \mathcal{L}(X)$ restricted to the ideal
\[ J(X) := \phi^{-1}(\mathcal{K}(X)) \]
is a representation of $J(X)$ in $\mathcal{K}(X)$ and it is of a particular interest to understand the relationship between $\pi : J(X) \to B$ and $(\pi, t, B)^{(1)} \circ \phi : J(X) \to B$.

**Definition 4.5.** For any ideal $J$ contained in $J(X)$ a representation $(\pi, t, B)$ of $X$ is said to be covariant on $J$ or $J$-covariant if
\[ (\pi, t, B)^{(1)}(\phi(a)) = \pi(a) \quad \text{for all } a \in J. \]
Actually, the set $\{a \in J(X) : (\pi, t, B)^{(1)}(\phi(a)) = \pi(a)\}$ is the biggest ideal on which $(\pi, t, B)$ is covariant and we will call this ideal an ideal of covariance for $(\pi, t, B)$.

**Remark 4.6.** What we call above covariant representations was originally called by Muhly and Solel coisometric representations, cf. [MS98], [FMR03]. This name was motivated by specific applications and examples, and therefore we choose, following Katsura [Kat03], [Kat04], more universal (neutral) name - covariance.

**Remark 4.7.** Plainly, for $J$-covariant representation $(\pi, t, B)$ the ideal $\ker \phi \cap J$ is contained in $\ker \pi$. Hence a necessary condition for the existence of a faithful $J$-covariant representation of $X$ is that $J$ is orthogonal to $\ker \phi$ (by Proposition 4.12 below this condition is also sufficient).
4.2 The Fock representation

The original idea of Pimsner [Pim97], see also [MS98], [FR99], [Kat04], is to construct representations of $C^*$-correspondences by a natural adaptation of the celebrated Hilbert space construction introduced by Fock. Namely, given a $C^*$-correspondence $X$ over $A$, for $n \geq 1$, the $n$-fold internal tensor product $X^\otimes n := X \otimes_A \cdots \otimes_A X$, see e.g. [Lan95, ch. 4], is a $C^*$-correspondence where $A$ acts on the left by

$$\phi(n)(a)(x_1 \otimes_A \cdots \otimes_A x_n) := (a \cdot x_1) \otimes_A \cdots \otimes_A x_n;$$

here $a \cdot x_1$ is given by (29). For $n = 0$, we take $X^\otimes 0$ to be the Hilbert module $A$ with left action $\phi(0)(a)b := ab$. Then the Hilbert-module direct sum, see [Lan95, p. 6],

$$\mathcal{F}(X) := \bigoplus_{n=0}^{\infty} X^\otimes n$$

carries a diagonal left action $\phi_\infty$ of $A$ in which $\phi_\infty(a)(x) := \phi(n)(a)x$ where $x \in X^\otimes n$. The $C^*$-correspondence $\mathcal{F}(X)$ is called the Fock space over the $C^*$-correspondence $X$. For each $x \in X$, we define a creation operator $T(x)$ on $\mathcal{F}(X)$ by

$$T(x)y = \begin{cases} x \cdot y & \text{if } y \in X^\otimes 0 = A \\ x \otimes_A y & \text{if } y \in X^\otimes n \text{ for some } n \geq 1; \end{cases}$$

routine calculations show that $T(x)$ is adjointable and its adjoint is the annihilation operator

$$T(x)^*z = \begin{cases} 0 & \text{if } z \in X^\otimes 0 = A \\ \langle x, x_1 \rangle_A \cdot y & \text{if } z = x_1 \otimes_A y \in X \otimes_A X^\otimes n-1 = X^\otimes n. \end{cases}$$

One sees that $T : X \to \mathcal{L}(\mathcal{F}(X))$ is an injective linear mapping and since $A$ is a summand of $\mathcal{F}(X)$, the map $\phi_\infty : A \to \mathcal{L}(\mathcal{F}(X))$ is injective as well. Actually, $(\phi_\infty, T, \mathcal{L}(\mathcal{F}(X)))$ is a faithful representation of $X$ whose ideal of covariance is $\{0\}$.

**Definition 4.8.** The representation $(\phi_\infty, T, \mathcal{L}(\mathcal{F}(X)))$ of $X$ in $\mathcal{L}(\mathcal{F}(X))$ defined above is called Fock representation and the Toeplitz $C^*$-algebra $\mathcal{T}(X)$ of $X$ is by definition the $C^*$-subalgebra of $\mathcal{L}(\mathcal{F}(X))$ generated by $\phi_\infty(A) \cup T(X)$, cf. [MS98, Def. 2.4], [Pim97, Def. 1.1].

4.3 Relative Cuntz-Pimsner algebras $\mathcal{O}(X, J)$

Composing the Fock representation $(\phi_\infty, T, \mathcal{L}(\mathcal{F}(X)))$ of $X$ with the quotient map one can get a representation of $X$ whose ideal of covariance is an arbitrarily chosen ideal contained in $J(X) = \phi^{-1}(\mathcal{K}(X))$. Namely, let $J$ be an ideal in $J(X)$ and let $P_0$ be the projection in $\mathcal{L}(\mathcal{F}(X))$ that maps $\mathcal{F}(X)$ onto the first summand $X^\otimes 0 = A$. One can show [MS98, Lem. 2.17] that $\phi_\infty(J)P_0$ is contained in $\mathcal{T}(X)$. We will write $\mathcal{J}(J)$ for the ideal in $\mathcal{T}(X)$ generated by $\phi_\infty(J)P_0$. 
**Definition 4.9** ([MS98], Def. 2.18). If \( X \) is a \( C^* \)-correspondence over a \( C^* \)-algebra \( A \), and if \( J \) is an ideal in \( J(X) = \phi^{-1}(K(X)) \) we denote by \( \mathcal{O}(J, X) \) the quotient algebra \( T(X)/J(J) \) and call it relative Cuntz-Pimsner algebra determined by \( J \).

The algebras \( \mathcal{O}(J, X) \) are characterized by the following universal property.

**Proposition 4.10** ([FMR03], Prop. 1.3). Let \( X \) be a \( C^* \)-correspondence over \( A \), and let \( J \) be an ideal in \( J(X) \). Let \( q : T(X) \to \mathcal{O}(J, X) \) be the quotient map and put

\[
\kappa_A = q \circ \phi_\infty \quad \text{and} \quad \kappa_X = q \circ T.
\]

Then \( (\kappa_A, \kappa_X, \mathcal{O}(J, X)) \) is a representation of \( X \) which is covariant on \( J \) and satisfies:

(i) for every representation \( (\pi, t) \) of \( X \) which is covariant on \( J \), there is a homomorphism \( \pi \times_J t \) of \( \mathcal{O}(J, X) \) such that

\[
(\pi \times_J t) \circ \kappa_A = \pi \quad \text{and} \quad (\pi \times_J t) \circ \kappa_X = t,
\]

(ii) \( \mathcal{O}(J, X) \) is generated as a \( C^* \)-algebra by \( \kappa_X(X) \cup \kappa_A(A) \).

The representation \( (\kappa_A, \kappa_X, \mathcal{O}(J, X)) \) is unique in the following sense: if \( (\kappa'_A, \kappa'_X, B) \) has similar properties, there is an isomorphism \( \theta : \mathcal{O}(J, X) \to B \) such that \( \theta \circ \kappa_X = \kappa'_X \) and \( \theta \circ \kappa_A = \kappa'_A \). In particular, there is a strongly continuous gauge action \( \gamma : \mathbb{T} \to \text{Aut} \mathcal{O}(J, X) \) where \( \gamma_z(\kappa_A(a)) = \kappa_A(a) \) and \( \gamma_z(\kappa_X(x)) = z\kappa_X(x) \) for \( a \in A, x \in X \).

**Remark 4.11.** One may show, cf. [Kwa13, Prop. 4.7], that the ideal of coisometricity for the universal representation \( (\kappa_X, \kappa_A, \mathcal{O}(J, X)) \) coincides with \( J \). Hence \( (\kappa_X, \kappa_A, \mathcal{O}(J, X)) \) could be equivalently defined as a universal triple with respect to representations whose ideal of covariance not only contains but actually equals \( J \).

| N. | \( \phi : A \to (X) \) | \( J \trianglelefteq J(X) \) | \( \mathcal{O}(J, X) \) |
|----|----------------|------------------|------------------|
| 1. | monomorphism | \( J = J(X) \) | Cuntz-Pimsner algebra of \( X \) [Pim97], [FMR03] |
| 2. | arbitrary | \( J = \{0\} \) | Toeplitz algebra of \( X \) [FR99], [FMR03] |
| 3. | arbitrary | \( J = (\ker \phi)^\perp \cap J(X) \) | Katsura’s algebra of \( X \) [Kat03], [Kat04], [Kat07] |
| 4. | \( \phi(J) = K(X) \) | \( J = (\ker \phi)^\perp \cap J(X) \) | crossed product by the Hilbert bimodule [AEE98] |

Table 2: Different relative Cuntz-Pimsner algebras

Table 2 presents a juxtaposition of various relative Cuntz-Pimsner algebras studied in the literature obtained from \( \mathcal{O}(J, X) \) for different choice of the ideal \( J \). To see the coincidence in N.4 of Table 2 we refer the reader, for instance, to [Kat03]. In view of the following proposition, the algebra \( A \) embeds into \( \mathcal{O}(J, X) \) for all the algebras presented in the Table 2.
The relations establish a one-to-one correspondence between representations \((\pi, \tau, L(H))\) of \((A, \alpha)\) and representations \((\pi, t, L(H))\) of \(X\), under this correspondence the ideal of covariance for \((\pi, t, L(H))\) coincides with the ideal of covariance for \((\pi, U, H)\).

**Proposition 4.13.** Let \(X\) be the \(C^*\)-correspondence of a \(C^*\)-dynamical system \((A, \alpha)\). The relations

\[
U := t(\alpha(1))^* \quad \text{and} \quad t(x) := U^* \pi(x)
\]

establish a one-to-one correspondence between representations \((\pi, U, H)\) of \((A, \alpha)\) and representations \((\pi, t, L(H))\) of \(X\), under this correspondence the ideal of covariance for \((\pi, t, L(H))\) coincides with the ideal of covariance for \((\pi, U, H)\).

**Proof.** Let \((\pi, t, L(H))\) be a representations of \(X\) and put \(U := t(\alpha(1))^*\). Then exploiting (30) and (33) one gets

\[
\pi(\alpha(a)) = \pi((\alpha(1), \alpha(a))_A) = t(\alpha(1))^* t(\alpha(a)) = Ut(a \cdot \alpha(1)) = U^* \pi(a) U^*.
\]

Thus, \((\pi, U, H)\) is a representation of \((A, \alpha)\). Moreover, observe that the operator \(\phi(a)\) is just \(\Theta_{\alpha(a), \alpha(1)}\) and since

\[
(\pi, t, L(H))^{(1)}(\Theta_{\alpha(a), \alpha(1)}) = t(\alpha(a)) t(\alpha(1))^* = U^* \pi(\alpha(a)) U = U^* U^* U = U^* U^* U
\]

it follows that the ideal of covariance for \((\pi, t, L(H))\) coincides with the ideal of covariance for \((\pi, U, H)\).

Conversely, for any covariant representation \((\pi, U, H)\) of \((A, \alpha)\) putting \(t(x) := U^* \pi(x)\), \(x \in X\), we have \(U = t(\alpha(1))^*\) and one easily checks conditions (30).

By the universality of \(O(J, X)\) and \(C^*(A, \alpha, J)\) (recall Definition 1.12), see also Remark 4.11, we get the following
Corollary 4.14. Let $X$ be the $C^*$-correspondence of a $C^*$-dynamical system $(A, \alpha)$ and let $J$ be an ideal in $(\ker \alpha)^\perp$. Then algebras $\mathcal{O}(J, X)$ and $C^*(A, \alpha, J)$ are canonically isomorphic. In particular,

(i) $\mathcal{O}(J, X)$ is generated as a $C^*$-algebra by the partial isometry $u = k_X(\alpha(1))^*$ and the $C^*$-algebra $k_A(A)$.

(ii) for every $J'$-covariant representation $(\pi, U, H)$ of $(A, \alpha)$ with $J' \supset J$, there is a homomorphism $\pi \times JU$ of $\mathcal{O}(J, X)$ uniquely determined by $$(\pi \times JU)(u) := U \quad \text{and} \quad (\pi \times JU) \circ k_A := \pi.$$ 

Corollary 4.15. Let $X$ be the $C^*$-correspondence of a $C^*$-dynamical system $(A, \alpha)$. Then the algebra $\mathcal{O}(A, X)$ together with the mapping $k_A$ and operator $u := k_X(\alpha(1))^*$, cf. Proposition 4.10, forms a Stacey’s crossed product for $(A, \alpha)$ (Definition 3.12).

Note that $\mathcal{O}(J, X)$ is defined for ideals $J$ that are not necessarily orthogonal to $\ker \alpha$. This along with Proposition 4.13 may make one guess that $\mathcal{O}(J, X)$ is a more general object than $C^*(A, \alpha, J)$. At the same time Proposition 4.12 shows that when $J$ is not orthogonal to $\ker \alpha$ the algebra $\mathcal{O}(J, X)$ possesses certain 'degeneracy'. All this stimulates us to take a closer look and provide a more thorough analysis of the structure of $\mathcal{O}(J, X)$ and its relation to $C^*(A, \alpha, J)$. This is the theme of the next section, where we present the procedure of the canonical reduction of $C^*$-correspondences, algebras and $C^*$-dynamical systems. In particular, as a result of this reduction we establish the coincidence of $\mathcal{O}(J, X)$ with appropriate crossed products introduced in the article.

5 Reductions of $C^*$-correspondences

5.1 Reduction of $C^*$-correspondences

Let us now fix a $C^*$-correspondence $X$ over $A$ and an ideal $J$ in $J(X)$. We will reduce $X$ by taking quotient to a certain 'smaller' $C^*$-correspondence satisfying (32) and yielding the same relative Cuntz-Pimsner algebra as $X$ and $J$.

To this end we note that $C^*$-correspondences behave nice under quotients. Namely, if $I$ is an ideal in $A$, then $XI := \text{span}\{xi : x \in X, i \in I\}$ is both a right Hilbert $A$-submodule of $X$ and a right Hilbert $I$-module, as we have

$$XI = \{xi : x \in X, i \in I\} = \{x \in X : (x, y)_A \in I \text{ for all } y \in X\},$$

(35) cf. [Kat07] Prop. 1.3. Moreover, we may consider the quotient space $X/XI$ as a right Hilbert $A/I$-module with an $A/I$-valued inner product and right action of $A/I$ given by

$$\langle q_{XI}(x), q_{XI}(y) \rangle_{A/I} := q_I(\langle x, y \rangle_A), \quad q_{XI}(x) \cdot q_I(a) := q_{XI}(x \cdot a),$$

(36) where $q_I : A \to A/I$ and $q_{XI} : X \to X/XI$ are the quotient maps, cf. [PMR03] Lem. 2.1. However, in order to define a left action on the quotient $X/XI$ we need to impose the following condition on an ideal $I$ in $A$:

$$\phi(I)X \subset XI.$$  

(37)
An ideal $I$ in $A$ satisfying (37) is called $X$-invariant and for such an ideal the quotient $A/I$-module $X/XI$ is equipped with quotient left action $\phi_{A/I}: A/I \to \mathcal{L}(X/XI)$ given by

$$\phi_{A/I}(q_I(a))q_{XI}(x) := q_{XI}(\phi(a)x), \quad x \in X, \ a \in A.$$ 

and hence $X/XI$ naturally becomes a $C^\ast$-correspondence over $A/I$, cf. [FMR03, Lem. 2.3], [Kat07]. We refer to it as to a quotient $C^\ast$-correspondence.

We will apply the following main result of [FMR03] to the reduction ideal introduced below.

**Theorem 5.1** ([FMR03], Thm. 3.1). Suppose $X$ is a $C^\ast$-correspondence over $A$, $J$ is an ideal in $J(X)$, and $I$ is an $X$-invariant ideal in $A$. If we denote by $\mathcal{I}(I)$ the ideal in $\mathcal{O}(J, X)$ generated by $k_A(I)$, then the quotient $\mathcal{O}(J, X)/\mathcal{I}(I)$ is canonically isomorphic to $\mathcal{O}(q_I(J), X/XI)$.

**Definition 5.2.** For any ideal $J$ in $J(X)$ we define recursively an increasing sequence of ideals

$$J_0 := \{0\} \quad \text{and} \quad J_{n+1} := \{a \in J : \phi(a)X \subseteq XJ_n\} \quad \text{for } n \geq 0,$$

and call the ideal

$$J_\infty := \bigcup_{n \in \mathbb{N}} J_n$$

the reduction ideal for $X$ and $J$.

**Remark 5.3.** Since $\phi(J_{n+1})X \subseteq XJ_n$ and $J_n \subseteq J_{n+1}$, the ideals $J_n$, $n \in \mathbb{N}$, and therefore also $J_\infty$ are $X$-invariant ideals in $A$. Actually, let us note that

$$a \in J \quad \text{and} \quad \phi(a)X \subseteq XJ_\infty \implies a \in J_\infty,$$

and this implication characterizes $J_\infty$ in the sense that $J_\infty$ is the smallest $X$-invariant ideal in $A$ satisfying (40). In particular, $J_\infty = \{0\}$ if and only if $\ker \phi \cap J = \{0\}$. Note also that $C^\ast$-correspondences $X/XJ_n$, $n \in \mathbb{N}$, may be considered as ‘approximations’ of $X/XJ_\infty$, and if $J_n = J_{n+1}$, for certain $n \in \mathbb{N}$, then $J_\infty = J_n$ and $X/XJ_n = X/XJ_\infty$.

The next result states, in particular, that the quotient $C^\ast$-correspondence $X/XJ_\infty$ and the quotient $C^\ast$-algebra $A/J_\infty$ may be identified with the image of the initial $C^\ast$-correspondence $X$ and $C^\ast$-algebra $A$ in the relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$.

**Theorem 5.4.** Let $X$ be a $C^\ast$-correspondence over $A$ and $J$ an ideal in $J(X)$. Then for $n \in \mathbb{N} \cup \{\infty\}$, we have a canonical isomorphism

$$\mathcal{O}(J, X) \cong \mathcal{O}(q_{J_n}(J), X/XJ_n).$$

Moreover, for $n = \infty$ we have

$$\ker \phi_{A/J_\infty} \cap q_{J_\infty}(J) = \{0\}$$

and thus we have the following (again canonical) isomorphisms

$$k_A(A) \cong A/J_\infty, \quad k_X(X) \cong X/XJ_\infty.$$
Proof. In view of Theorem 5.4 to prove the first part of theorem it is enough to show that for every ideal $J_n$, $n = 0, 1, ..., \infty$, we have $k_A(J_n) = 0$. It is clear that $k_A(J_0) = 0$. Assume that $k_A(J_n) = 0$ and let $a \in J_{n+1}$. Then for every $x \in X$ there exists $y(x) \in X$ and $i(x) \in J_n$ such that $\phi(a)x = y(x)i(x)$ and thus

$$k_A(a)k_X(x) = k_X(\phi(a)x) = k_X(y(x)i(x)) = k_X(y(x))k_A(i(x)) = 0,$$

that is $k_A(a)k_X(X) = \{0\}$. Moreover, since $J_{n+1} \subset J$, and the universal representation is $J$-covariant we have $k_A(a) = (k_A, k_X, \mathcal{O}(J, X))^{(1)}(\phi(a))$, for the mapping given by (31). By (31), relation $(k_A, k_X, \mathcal{O}(J, X))^{(1)}(\phi(a))k_X(X) = \{0\}$ imply $(k_A, k_X, \mathcal{O}(J, X))^{(1)}(\phi(a)) = 0$ and therefore $k_A(a) = 0$. Hence $k_A(J_{n+1}) = 0$, and it follows that $k_A(J_n) = 0$ for every $n = 0, 1, ..., \infty$.

To prove that $\ker \phi_{A/J_{\infty}} \cap q_{J_{\infty}}(J) = \{0\}$ take $a \in J$ and suppose that $\phi_{A/J_{\infty}}(q_{J_{\infty}}(a)) = 0$. Then by (38) we see that $\phi(a)X \subset XJ_{\infty}$ and by (40) we have $q_{J_{\infty}}(a) = 0$. Now it suffices to apply Proposition 4.12.

Remark 5.5. Theorem 5.4 shows, in particular, that the ideal $J_{\infty}$ plays the role of a certain ‘measure’ of the degree of degeneracy of $\mathcal{O}(J, X)$ – the bigger $J_{\infty}$ is the smaller $\mathcal{O}(J, X)$ is. In particular, $\mathcal{O}(J, X) = 0$ if and only if $J_{\infty} = A$. Obviously, $A$ embeds into $\mathcal{O}(J, X)$ if and only if $J_{\infty} = 0$ which is equivalent to $X = X/XJ_{\infty}$. Moreover, it follows that one may always restrict his interest only to the relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$ determined by ideals such that

$$J \subset (\ker \phi)^{\perp},$$

since otherwise one has to pass (either explicitly or implicitly) to the reduced $C^*$-correspondence $X/XJ_{\infty}$ over the reduced $C^*$-algebra $A/J_{\infty}$ and the reduced ideal $q_{J_{\infty}}(J) \subset (\ker \phi_{A/J_{\infty}})^{\perp}$.

5.2 Katsura’s canonical relations for relative Cuntz-Pimsner algebras

In [Kat07] T. Katsura associated with a $C^*$-correspondence $X$ a $C^*$-algebra $\mathcal{O}_X$ which is the relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ for $J = (\ker \phi)^{\perp} \cap J(X)$. By subtle analysis he proved that any relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ is isomorphic to $\mathcal{O}_{X_\omega}$ for certain $X_\omega$. In this subsection we apply the reduction procedure presented above to give an alternative definition of these $C^*$-correspondences.

The reduction ideal $J_{\infty}$ defined above coincides with the ideal $J_{-\infty}$ constructed in [Kat07], Sect. 11]. It is noted in [Kat07] that for the pair $\omega = (J_{\infty}, J)$ there is a natural (yet a bit sophisticated) $C^*$-correspondence structure on the following pair of ‘pullbacks’

$$A_\omega = \{(a, a') \in A/J_{\infty} \oplus A/J : q_{J(J_{\infty})}(a) = q_{J(J_{\infty})}(a')\}$$

$$X_\omega = \{(x, x') \in X/XJ_{\infty} \oplus X/XJ : q_{XJ(J_{\infty})}(x) = q_{XJ(J_{\infty})}(x')\}$$

where $J(J_{\infty}) := (q_{XJ_{\infty}})^{-1}(\ker \phi_{A/J_{\infty}})^{\perp} \cap J(X/XJ_{\infty})$ (then $J_{\infty} \subset J \subset J(J_{\infty})$) and $q_{J(J_{\infty})}$ denotes here the quotient map composed with a corresponding isomorphism.
\((A/J_\infty)/J(J_\infty) \cong A/J(J_\infty)\) or \((A/I)/J(J_\infty) \cong A/J(J_\infty)\) and similar convention is used for \(q_{X,J(J_\infty)}\). Let us note that the mappings
\[ A/J_\infty \ni a \mapsto a \oplus q_J(a) \in A_\omega, \quad X/XJ_\infty \ni x \mapsto x \oplus q_J(x) \in X_\omega \]
are injective and therefore the pair \((A_\omega, X_\omega)\) is an extension of the reduced pair \((A/J_\infty, X/XJ_\infty)\) (it is a nontrivial extension unless \(J_\infty = J(J_\infty)\)).

Thus Katsura’s construction in fact involves two steps – reduction (in the sense of Theorem 5.4 and an extension (which we describe precisely in Definition 5.6 below); and perhaps exposing them explicitly makes his construction a bit more transparent.

**Definition 5.6.** Let \(J\) be an ideal in \(J(X)\) and \(J_X = (\ker \phi)^\perp \cap J(X)\). If \(J\) is orthogonal to \(\ker \phi\), then **Katsura’s canonical \(C^*\)-correspondence for \(X\) and \(J\)** is a \(C^*\)-correspondence \(X_\omega\) over \(A_\omega\) where
\[ A_\omega = \{(a, a') \in A \oplus A/J : q_{J_X}(a) = q_{J_X}(a')\} \]
\[ X_\omega = \{(x, x') \in X \oplus X/XJ : q_{X,J_X}(x) = q_{X,J_X}(x')\} \]
and operations are defined simply by coordinates:
\[ (x, x') \cdot (a, a') = (xa, x'a'), \quad (a, a') \cdot (x, x') = (ax, a'x') \]
and
\[ \langle (x, x'), (y, y') \rangle_{A_\omega} = \langle (x, y)_A, (x', y')_{A/J} \rangle \in A_\omega. \]

Extending the procedure presented above to the general situation we naturally arrive at the following

**Definition 5.7.** If \(J\) is arbitrary (not necessarily orthogonal to \(\ker \phi\)), we define a pair \((A_\omega, X_\omega)\) to be Katsura’s canonical \(C^*\)-correspondence for the reduced \(C^*\)-correspondence \(X/XJ_\infty\) and the reduced ideal \(q^{J_\infty}(J)\) and also call it **Katsura’s canonical \(C^*\)-correspondence for \(X\) and \(J\)**.

**Remark 5.8.** By Remark 5.3 the coincidence of notation and terminology in Definitions 5.6 and 5.7 does not cause confusion: if \(J\) is orthogonal to \(\ker \phi\), then \((A_\omega, X_\omega)\) is an extension of \((A, X)\). Moreover \((A_\omega, X_\omega) \cong (A, X)\) if and only if \(J = (\ker \phi)^\perp \cap J(X)\).

One can see that the above Definition 5.7 coincides with Katsura’s construction and therefore by [Kat07, Prop. 11.3] we have

**Theorem 5.9.** If \(J\) is an ideal in \(J(X)\) and \((A_\omega, X_\omega)\) is Katsura’s canonical \(C^*\)-correspondence for \(X\) and \(J\), then we have an isomorphism
\[ \mathcal{O}(J, X) \cong \mathcal{O}((\ker \phi_\omega)^\perp \cap J(X_\omega), X_\omega) \]
where \(\phi_\omega\) denotes the left action of \(A_\omega\) on \(X_\omega\).

Additional discussion of Katsura’s canonical \(C^*\)-correspondence and its relation to the crossed product will be given in 6.2.
5.3 Reduction of C*-dynamical systems

Once we have noted the relation between crossed products and relative Cuntz-Pimsner algebras in Subsection 4.4 and established the reduction procedure for relative Cuntz-Pimsner algebras in Subsection 5.1 it is reasonable to apply this procedure to crossed products, and this is the theme of the present subsection. As a by product we also establish the coincidence of $\mathcal{O}(J, X)$ with appropriate crossed products introduced in the article.

Let $X$ be the $C^*$-correspondence of a $C^*$-dynamical system $(A, \alpha)$ and let $J$ be an ideal in $A$. We note that an ideal $I$ satisfies \( (37) \) if and only if $\alpha(I) \subset I$ hence $X$-invariance is equivalent to $\alpha$-invariance. In particular, the ideals $J_n, n = 0, 1, \ldots, \infty$, from Definition 5.2 are $\alpha$-invariant. Formulae $(39)$ mean that

\[
J_n = \alpha^{-1}(\alpha^{-1}(\ldots(\alpha^{-1}(\{0\}) \cap J)\ldots) \cap J) \cap J, \quad n \in \mathbb{N},
\]

that is

\[
J_n = (\ker \alpha^n) \cap \bigcap_{k=0}^{n-1} \alpha^{-k}(J).
\]

**Remark 5.10.** Recalling Remark 5.3 we note that $J_\infty = \bigcup_{n \in \mathbb{N}} J_n$ is the smallest ideal in $A$ such that

\[ a \in J \quad \text{and} \quad \alpha(a) \in J_\infty \implies a \in J_\infty, \]

and $J_\infty = \{0\}$ if and only if $\ker \alpha \cap J = \{0\}$.

We give an alternative description of $J_\infty$ in the following lemma.

**Lemma 5.11.** The sets $\{a \in A : \exists n \in \mathbb{N} \alpha^n(a) = 0\}$ and $\{a \in A : \lim_{n \to \infty} \alpha^n(a) = 0\}$ coincide and form the smallest $\alpha$-invariant ideal $I_\infty$ in $A$ such that $\alpha$ factors through to a monomorphism on the quotient algebra $A/I_\infty$. In particular,

\[
J_\infty = \{a \in A : \alpha^n(a) \in J \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \alpha^n(a) = 0\}
\]

is the largest $\alpha$-invariant ideal contained in $J \cap I_\infty$.

**Proof.** One sees that both $I_1 = \{a \in A : \exists n \in \mathbb{N} \alpha^n(a) = 0\}$ and $I_2 := \{a \in A : \lim_{n \to \infty} \alpha^n(a) = 0\}$ are $\alpha$-invariant ideals in $A$ such that $\alpha$ factors through to monomorphisms both on $A/I_1$ and $A/I_2$. Moreover, it is clear that $I_1$ is the minimal ideal with the aforementioned properties. In particular, $I_1 \subset I_2$ and to see the opposite inclusion note that since $a \mapsto \alpha(a)$ factors through to the monomorphism (isometric mapping) on $I_2/I_1$ and $\alpha^n(a) \to 0$ for all $a \in I_2$ it follows that $I_2/I_1 = \{0\}$.

For the second part of the assertion note that $a \in \bigcup_{n \in \mathbb{N}} J_n$ if and only if there is $n = 1, 2, \ldots, n - 1$, such that

\[ \alpha^n(a) = 0, \quad \alpha^k(a) \in J, \quad k = 1, 2, \ldots, n - 1. \]

Hence $\bigcup_{n \in \mathbb{N}} J_n = (\bigcup_{n=0}^\infty \ker \alpha^n) \cap \bigcap_{n=0}^\infty \alpha^{-n}(J)$ and thus $J_\infty = I_\infty \cap \bigcap_{n=0}^\infty \alpha^{-n}(J)$.
Let \( n \in \mathbb{N} \cup \{\infty\} \). Since the ideal \( J_n \) is \( \alpha \)-invariant, we have a quotient \( C^* \)-dynamical system \((A/J_n, \alpha_n)\) where \( \alpha_n : A/J_n \to A/J_n \) is given by \( \alpha_n \circ q_{J_n} = q_{J_n} \circ \alpha \) and the quotient \( C^* \)-correspondence \( X/XJ_n \) may be viewed as the \( C^* \)-correspondence of \((A/J_n, \alpha_n)\). In particular
\[
\alpha_\infty(a + J_\infty) := \alpha(a) + J_\infty \tag{46}
\]
is an endomorphism of \( A/J_\infty \) such that
\[
\ker \alpha_\infty \cap q_{J_\infty}(J) = \{0\}. \tag{47}
\]
Obviously one may apply Theorem 5.4 to each of the systems \((A/J_n, \alpha_n), n \in \mathbb{N} \cup \{\infty\}\), however, we focus on the case \( n = \infty \). Then by virtue of Theorem 5.4 and Corollary 4.14 along with (47) we get

**Proposition 5.12.** If \( X \) is the \( C^* \)-correspondence of a \( C^* \)-dynamical system \((A, \alpha)\) and \( J \) is an ideal in \( A \), then
\[
\mathcal{O}(J, X) = \mathcal{O}(q_{J_\infty}(J), X/XJ_\infty) = C^*(A/J_\infty, \alpha_\infty, q_{J_\infty}(J))
\]
is a universal algebra generated by a copy of the algebra \( A/J_\infty \) and a partial isometry \( u \) subject to relations
\[
u au^* = \alpha_\infty(a), \quad a \in A/J_\infty, \quad q_{J_\infty}(J) = \{a \in A/J_\infty : u^*ua = a\}.
\]

**Corollary 5.13.** If \((\pi, U, H)\) is a \( J \)-covariant representation of \((A, \alpha)\), then \( J_\infty \subset \ker \pi \).

**Example 5.14.** Let us apply the above results to Stacey’s crossed product (Definition 3.12), and in particular, obtain a refinement of [Sta93, Prop. 2.2]. Note that if \( J = A \), then
\[
J_\infty = \{a \in A : \lim_{n \to \infty} \alpha^n(a) = 0\} = \bigcup_{n=0}^{\infty} \ker \alpha^n, \tag{48}
\]
cf. Lemma 5.11. Accordingly, the system \((A_\infty, \alpha_\infty), A_\infty := A/J_\infty\), obtained by the quotient of \((A, \alpha)\) by \( J_\infty \) could be considered as the largest subsystem of \((A, \alpha)\) with the property that \( \alpha_\infty : A_\infty \to A_\infty \) is a monomorphism. Moreover, by Corollary 4.15 and Proposition 5.12 Stacey’s crossed product is a universal \( C^* \)-algebra generated by a copy of \( A_\infty \) and an isometry \( u \) such that \( uau^* = \alpha_\infty(a) \), for all \( a \in A_\infty \). In particular, the Stacey’s crossed product is the crossed product \( C^*(A_\infty, \alpha_\infty, A_\infty) \) studied in the present paper and it reduces to the zero algebra if and only if \( A = \{a \in A : \lim_{n \to \infty} \alpha^n(a) = 0\} \).

### 6 Canonical \( C^* \)-dynamical systems

Looking at Table 1 one can not help feeling that among the ideals satisfying \( \{0\} \subset J \subset (\ker \alpha)^\perp \) the ideal \( J = (\ker \alpha)^\perp \) is somewhat privileged. Taking into account Cuntz-Pimsner algebras and their established relation to crossed products it may seem completely natural as \( \mathcal{O}((\ker \alpha)^\perp, X) \) should be considered as ‘the smallest’ relative
Cuntz-Pimsner algebra containing all the information about the $C^*$-dynamical system $(A, \alpha)$. Now, much in the spirit of Katsura’s construction, cf. subsection 5.2 yet in a slightly different way we will show that for an arbitrary choice of $J$ the algebra $O(J, X)$ coincides with Cuntz-Pimsner algebra $O((\ker \alpha_J)^\perp, X_J)$ where $X_J$ is the $C^*$-correspondence of a canonically constructed $C^*$-dynamical system $(A_J, \alpha_J)$ which will be presented below (see Definition 6.4, Theorem 6.6).

### 6.1 Unitization of the kernel of an endomorphism

Let us fix a $C^*$-dynamical system $(A, \alpha)$ and an ideal $J$ in $A$. Above results show us how to reduce the investigation of crossed products to the case where $J$ is orthogonal to $\ker \alpha$, thereby let us assume for a while that $\ker \alpha \cap J = \{0\}$. The first named author described in [Kwa07] a procedure of extending $(A, \alpha)$ up to a $C^*$-dynamical system $(A +, \alpha +)$ with a property that the kernel of $\alpha +$ is unital. Moreover, the resulting system $(A +, \alpha +)$ is in a sense the smallest extension of $(A, \alpha)$ possessing that property, see [Kwa07].

Let us now slightly generalize this construction, which will be essential for our future purposes. Our extension construction depends on the choice of an ideal orthogonal to $I := \ker \alpha$ (recall Definition 1.6). Thus let us fix a certain ideal $J$ which satisfies $\{0\} \subset J \subset I^\perp$.

By $A_J$ we denote the direct sum of quotient algebras

$$A_J = (A/I) \oplus (A/J),$$

and we set $\alpha_J : A_J \rightarrow A_J$ by the formula

$$A_J \ni ((a + I) \oplus (b + J)) \xrightarrow{\alpha_J} (\alpha(a) + I) \oplus (\alpha(a) + J) \in A_J. \quad (48)$$

Since $I = \ker \alpha$ it follows that an element $\alpha(a)$ does not depend on the choice of a representative of $a + I$ and so the mapping $\alpha_J$ is well defined.

Note that, as $I \cap J = \{0\}$,

$$\alpha_J([a], [b]) = (0, 0) \iff \alpha(a) \in I \text{ and } \alpha(a) \in J \iff \alpha(a) = 0.$$ 

And therefore

$$\ker \alpha_J = (0, A/J) \quad \text{and} \quad (\ker \alpha_J)^\perp = (A/I, 0). \quad (49)$$

Clearly, $\alpha_J$ is an endomorphism and its kernel is unital with the unit of the form $(0 + I) \oplus (1 + J)$. Moreover, the $C^*$-algebra $A$ embeds into $C^*$-algebra $A_J$ via

$$A \ni a \mapsto (a + I) \oplus (a + J) \in A_J. \quad (50)$$

Since $I \cap J = \{0\}$ this mapping is injective and we will treat $A$ as the corresponding subalgebra of $A_J$. Under this identification $\alpha_J$ is an extension of $\alpha$, and in particular $\alpha_J(A_J) = \alpha(A) \subset A$. 
Remark 6.1. The larger ideal $J$ is the smaller $A_J$ is. Namely, since $\ker \alpha_J = (0, A/J)$ and $J$ varies from $\{0\}$ to $I^\perp$ it follows that the kernel of $\alpha_J$ varies from $(0, A)$ to $(0, A/I^\perp)$. On the other hand, the image of $\alpha_J$ always coincides with $\alpha(A) \cong A/I$ and thereby it does not depend on the choice of $J$. The case when $J = I^\perp$ was considered in [Kwa07].

The next statement presents a motivation of the preceding construction.

Proposition 6.2. Let $(\pi, U, H)$ be a faithful $J$-covariant representation of $(A, \alpha)$. Then $J$ is orthogonal to $I = \ker \alpha$ (cf. Corollary 1.4) and if $(A_J, \alpha_J)$ is the extension of $(A, \alpha)$ constructed above, then $\pi$ uniquely extends to the isomorphism $\pi : A_J \to C^*(\pi(A), U^*U)$ such that $(\pi, U, H)$ is a faithful covariant representation of $(A_J, \alpha_J)$. Namely $\pi$ is given by

$$\pi(a + I \oplus b + J) = U^*\pi(a) + (1 - U^*U)\pi(b), \quad a, b \in A. \quad (51)$$

Proof. If $\pi : A_J \to C^*(\pi(A), U^*U)$ is onto and $(\pi, U, H)$ is a faithful covariant representation of $(A_J, \alpha_J)$, then by Proposition 1.9 we have

$$U^*U = \pi(1 + I \oplus 0 + J).$$

Thus $\pi$ is of the form (51) where $\pi = \pi|_A$ and plainly $(\pi, U, H)$ is a faithful $J$-covariant representation of $(A, \alpha)$. Conversely, let $(\pi, U, H)$ be a faithful $J$-covariant representation of $(A, \alpha)$. Then by Corollary 1.5

$$\pi(I) = [(1 - U^*U)\pi(A)] \cap \pi(A), \quad \pi(J) = [U^*\pi(A)] \cap \pi(A),$$

and consequently, cf. for instance [KR86, Lem. 10.1.6], we have natural isomorphisms

$$A/I \cong U^*\pi(A), \quad A/J \cong [(1 - U^*U)\pi(A)].$$

Thus formula (51) defines an isomorphism $\pi : A_J \to C^*(\pi(A), U^*U)$. It is readily checked, cf. Proposition 1.9 that $(\pi, U, H)$ is a covariant representation of $(A_J, \alpha_J)$.

Theorem 6.3. Let $X$ be the $C^*$-correspondence of a $C^*$-dynamical system $(A, \alpha)$ and let $J$ be an ideal in $A$ orthogonal to the kernel of $\alpha$. If $X_J$ is the $C^*$-correspondence of the system $(A_J, \alpha_J)$ constructed above, then

$$\mathcal{O}(J, X) = \mathcal{O}((\ker \alpha_J)^\perp, X_J) = C^*(A, \alpha, J) = C^*(A_J, \alpha_J, (\ker \alpha_J)^\perp)$$

is a universal algebra generated by a copy of the algebra $A_J$ and a partial isometry $u$ subject to relations

$$uau^* = \alpha_J(a), \quad a \in A_J, \quad u^*u \in A_J \quad (52)$$

(relations (52) imply that $u^*u$ belongs to the center of $A_J$, cf. Proposition 1.9).

Proof. In view of Corollary 1.14 we have natural identifications $\mathcal{O}(J, X) = C^*(A, \alpha, J)$ and $\mathcal{O}((\ker \alpha_J)^\perp, X_J) = C^*(A_J, \alpha_J, (\ker \alpha_J)^\perp)$. In order to prove that $C^*(A, \alpha, J) = C^*(A_J, \alpha_J, (\ker \alpha_J)^\perp)$ it suffices to show that we have a one-to-one correspondence between faithful $J$-covariant representations $(\pi, U, H)$ of $(A, \alpha)$ and faithful $(\ker \alpha_J)^\perp$-covariant representations $(\tilde{\pi}, U, H)$ of $(A_J, \alpha_J)$, but this follows from Proposition 6.2 since by Proposition 1.9 a faithful covariant representation $(\tilde{\pi}, U, H)$ is $(\ker \alpha_J)^\perp$-covariant if and only if $U^*U \in \tilde{\pi}(A_J)$. \qed
6.2 Canonical $C^*$-dynamical systems

Proposition 5.12 describes the natural reduction of relations to the case when $J \subset (\ker \alpha)\perp$. This proposition along with the argument of Subsection 6.1 gives us a tool to achieve the goal of the present section; namely, to reduce the whole construction to the case when $\ker \alpha$ is unital and $J = (\ker \alpha)\perp$.

**Definition 6.4.** Let $(A, \alpha)$ be a $C^*$-dynamical system and $J$ an arbitrary ideal in $A$. Let $((A/J_\infty)_{q^J_\infty(J)}, (\alpha_\infty)_{q^J_\infty(J)})$ be the above constructed extension of the reduced $C^*$-dynamical system $(A/J_\infty, \alpha_\infty)$ given by (45), (46). We will write $(A_J, \alpha_J) := ((A/J_\infty)_{q^J_\infty(J)}, (\alpha_\infty)_{q^J_\infty(J)})$

and say that $(A_J, \alpha_J)$ is the canonical $C^*$-dynamical system associated with $(A, \alpha)$ and $J$.

**Remark 6.5.** By Remark 5.10 the above notation does not cause confusion (in the situation when $\{0\} \subset J \subset I\perp$ the pair $(A_J, \alpha_J)$ coincides with the corresponding pair introduced in 6.1) and therefore we keep the notation $(A_J, \alpha_J)$ in the general situation.

Combining Propositions 5.12, 6.2, see also [Kwa07, Cor. 1.7], we get

**Theorem 6.6.** Let $X$ be the $C^*$-correspondence of $(A, \alpha)$ and let $J$ be an ideal in $A$. If $X_J$ is the $C^*$-correspondence of the canonical system $(A_J, \alpha_J)$, then

$O(J, X) = O((\ker \alpha_J)^\perp, X_J) = C^*(A_J, \alpha_J, (\ker \alpha_J)^\perp)$

is a universal algebra generated by a copy of the algebra $A_J$ and a partial isometry $u$ subject to relations

$uau^* = \alpha_J(a), \ a \in A_J, \ u^*u \in A_J. \quad (53)$

**Remark 6.7.** The usefulness of canonical $C^*$-dynamical system $(A_J, \alpha_J)$ manifests in reducing relations (1), (2), that apart from endomorphism involve an ideal and which may degenerate, to the nondegenerated natural relations (53). In fact, one could go even further and use the construction from [Kwa07] to extend, the canonical system $(A_J, \alpha_J)$ up to a $C^*$-dynamical system $(B, \tilde{\alpha})$ possessing a complete transfer operator (cf. subsection 3.3). Then $B$ corresponds to the fixed point subalgebra of $O(J, X)$ for the gauge action $\gamma$ (Proposition 4.10), and by [Kwa12] Prop. 1.9 the $C^*$-correspondence $\tilde{X}$ of the $C^*$-dynamical system $(B, \tilde{\alpha})$ is actually a Hilbert bimodule (in the sense of Remark 3.3). Thus $O(J, X)$ can be modeled not only by the crossed product of $(B, \tilde{\alpha})$, N.3 in Table 1, cf. Proposition 3.8 but also by by the $C^*$-correspondence $\tilde{X}$, N.4 in Table 2. Hence the results of [ABL11], [AEE98] or isomorphism theorem [Kwa10] applied to $(B, \tilde{\alpha})$ can be exploited in the study of $O(J, X)$ in terms of ‘Fourier’ coefficients.

We end up by noting that Katsura’s ‘canonical relations’ for $C^*$-correspondences (see Definitions 5.6, 5.7) when applied to the $C^*$-correspondence $X$ of $(A, \alpha)$ also leads to a certain dynamical system, which however in general is slightly smaller than
\((A_J, \alpha_J)\) and is less natural in our context. Indeed, by passing if necessary to the reduced objects, we need to consider only the case when \(J \subset (\ker \alpha)^\perp\), and then
\[
A_\omega = \{ (a, q_J(a')) \in A \oplus A/J : q_{(\ker \alpha)^\perp}(a) = q_{(\ker \alpha)^\perp}(a') \}.
\]
In particular, the mapping
\[
A_\omega \ni (a, q_J(a')) \mapsto \alpha_\omega(a), q_J(\alpha(a)) \in A_\omega
\]
yields a well-defined endomorphism \(\alpha_\omega : A_\omega \to A_\omega\), and one sees that \(X_\omega\) coincides with the \(C^*\)-correspondence of the \(C^*\)-dynamical systems \((A_\omega, \alpha_\omega)\). Thus we have three \(C^*\)-dynamical systems \((A, \alpha), (A_\omega, \alpha_\omega), (A_J, \alpha_J)\), and each of them is an extension of the preceding one. Indeed, we have natural homomorphisms
\[
\begin{align*}
A & \ni a \mapsto a \oplus q_J(a) \in A_\omega, \\
A_\omega & \ni (a, q_J(a')) \mapsto q_{\ker \alpha}(a) \oplus q_J(a') \in A_J.
\end{align*}
\]
Clearly, \(\iota_1\) is injective and to see that \(\iota_2\) is injective note that \(\iota_2(a, q_J(a')) = 0\) means that \(a \in \ker \alpha\) and \(a' \in J \subset (\ker \alpha)^\perp\), and then relation \(q_{(\ker \alpha)^\perp}(a) = q_{(\ker \alpha)^\perp}(a') = 0\) imply that \(a = 0\). The monomorphisms \(\iota_1, \iota_2\) make the following diagram commute
\[
\begin{array}{ccc}
A & \xrightarrow{\iota_1} & A_\omega \\
\alpha \uparrow & & \downarrow \alpha_\omega \\
A & \xrightarrow{\iota_2} & A_J
\end{array}
\]
Moreover, \(\iota_1\) is an isomorphism iff \(J = (\ker \alpha)^\perp\) and \(\iota_2\) is an isomorphism iff \((\ker \alpha)^\perp\) is unital. In particular, in 'Katsura’s picture’, the crossed product \(C^*(A, \alpha, J)\) could be considered as a universal \(C^*\)-algebra subject to relations
\[
u a u^* = \alpha_\omega(a), \quad a \in A_\omega, \quad \{ a \in A_\omega : u^*ua = a \} = (\ker \alpha_\omega)^\perp,
\]
which apparently are more complicated than relations (53).

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B. K. Kwaśniewski, Institute of Mathematics, University of Bialystok, ul. Akademicka 2, PL-15-267 Bialystok, Poland
e-mail: bartoszk@math.uwb.edu.pl,
www: http://math.uwb.edu.pl/~zaf/kwasniewski

A. V. Lebedev, Department of Mechanics and Mathematics, Belarus State University, pr. Nezavisimosti, 4, 220050, Minsk, Belarus, & Institute of Mathematics, University of Bialystok, ul. Akademicka 2, PL-15-267 Bialystok, Poland, e-mail: lebedev@bsu.by