ON GENERIC IDENTIFIABILITY OF 3-TENSORS OF SMALL RANK

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ABSTRACT. We introduce an inductive method for the study of the uniqueness of decompositions of tensors, by means of tensors of rank 1. The method is based on the geometric notion of weak defectivity. For three-dimensional tensors of type \((a, b, c)\), \(a \leq b \leq c\), our method proves that the decomposition is unique (i.e. \(k\)-identifiability holds) for general tensors of rank \(k\), as soon as \(k \leq (a + 1)(b + 1)/16\). This improves considerably the known range for identifiability. The method applies also to tensors of higher dimension. For tensors of small size, we give a complete list of situations where identifiability does not hold. Among them, there are \(4 \times 4 \times 4\) tensors of rank 6, an interesting case because of its connection with the study of DNA strings.

1. Introduction

1.1. Statement of main results. Let \(A, B, C\) three complex vector spaces, of dimension \(a, b, c\) respectively. A tensor \(t \in A \otimes B \otimes C\) is said to have rank \(k\) if there is a decomposition

\[ t = \sum_{i=1}^{k} u_i \otimes v_i \otimes w_i \]

with \(u_i \in A, v_i \in B, w_i \in C\) and the number of summands \(k\) is minimal. Such a decomposition is said to be unique if for any other expression

\[ t = \sum_{i=1}^{k} u'_i \otimes v'_i \otimes w'_i \]

there is a permutation \(\sigma\) of \(\{1, \ldots, r\}\) such that

\[ u_i \otimes v_i \otimes w_i = u'_{\sigma(i)} \otimes v'_{\sigma(i)} \otimes w'_{\sigma(i)} \quad \forall i = 1, \ldots, k. \]

When \(t\) has a unique decomposition, the vectors \(u_i \in A, v_i \in B, w_i \in C\) can be identified uniquely from \(t\), up to scalars.

It is known that the set of tensors of rank \(k\) consists of a dense subset of an irreducible algebraic variety \(S_k(Y)\), which is called the \(k\)-th secant variety of the variety \(Y\) of tensors of rank one. This last variety is isomorphic to the (cone over the) Segre product \(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)\).

The main result of our paper determines a bound for the rank, in terms of the dimensions of the vector spaces, which implies identifiability.

**Theorem 1.1.** Let \(a \leq b \leq c\). Let \(\alpha, \beta\) be maximal such that \(2^\alpha \leq a\) and \(2^\beta \leq b\). The general tensor \(t \in A \otimes B \otimes C\) of rank \(k\) has a unique decomposition, if \(k \leq 2^\alpha + \beta - 2\).

So if \(a, b\) are both a power of 2, then the general tensor of rank \(k\) has a unique decomposition if \(k \leq \frac{ab}{2}\). In the general case, the inequality of the theorem can be
written as $k \leq 2^{\lceil \log_2 a \rceil + \lceil \log_2 b \rceil - 2}$. Since $\frac{a+1}{2} \leq 2^a$ and $\frac{b+1}{2} \leq 2^b$, one can say that the unique decomposition holds if $k \leq (a + 1)(b + 1)/16$.

In our terminology, when the unique decomposition holds for the general tensor of rank $k$, we will say that the variety of tensors of rank one is $k$-identifiable.

Here the meaning of “general” is that, among tensors of rank $k$, the ones which do not have a unique decomposition consist in a set of zero measure, more specifically in a proper subvariety of $S_k$.

In particular, the Theorem applies to “cubic” tensors. The general tensor $t \in A \otimes A \otimes A$ of rank $k$ has a unique decomposition if $k \leq (a+1)(b+1)/16$ (indeed, the Theorem provides a better bound, when $a$ is close to a power of 2).

Our bound is log-asymptotically sharp, in the following sense. As explained in Proposition 2.2, one cannot have a unique decomposition, when the rank exceeds a value $k_{\text{max}} = k(a,b,c)$, which depends on $a, b, c$. Then $\sup_c k(a,b,c)$ is finite. On the other hand, even for tensors of small size, the result is not sharp. In the first cases, with the help of a computer, we can improve Theorem 1.1.

Unique decomposition has been studied by several authors, and there is a huge amount of literature, on this theme. Let us remind that Strassen and Lickteig (Lick) proved that the general tensor $t \in A \otimes A \otimes A$ has rank $\lceil a^3/3 - 2 \rceil$ for $a \neq 3$ and rank 5 for $a = 3$ (indeed, the case $a = 3$ is known to be defective, meaning that the corresponding 4-secant variety has dimension smaller than the expected one). In this case, the aforementioned bound implies that, if $a \geq 3$, then the generic tensor of rank $k$ can have a unique decomposition only if $k \leq \lceil a^3/3 - 2 \rceil - 1$. The following theorem shows that this bound is almost always achieved, for small $a$.

**Theorem 1.2.** The general tensor $t \in A \otimes A \otimes A$ of rank $k$ has a unique decomposition if $k \leq k(a)$ where

| $a$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|----|
| $k(a)$ | 2 | 3 | 5 | 9 | 13 | 18 | 22 | 27 | 32 |

A more general list, which holds in the non cubic case, is given in section 5.

Comparing the previous table with the table of the general rank (for $a > 3$, the general rank $-1$ is the best possible achievement), and with Kruskal’s result (see Proposition 1.4), one can appreciate the improvement.

| $a$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|----|
| gen.rank ($a \neq 3$) | Lick | $\lceil a^3/3 - 2 \rceil$ | 2 | 4 | 7 | 10 | 14 | 19 | 24 | 30 | 36 |
| Kruskal bound | K | $\lceil 3a^2/2 \rceil$ | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |

The more evident lack of uniqueness is when $a = 4$ and $k = 6$. The case $a = 4$ is particularly interesting due to the models in phylogenetics [AR, ERSS], where a basis in $\mathbb{C}^4$ can be indexed by the nucleotids $\{A, C, G, T\}$.

**Theorem 1.3.** The general tensor $t \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ of rank 6 has exactly two decompositions.
It is interesting that the exception on uniqueness \((a = 4)\) holds very close to the defective case \(a = 3\). This phenomenon is quite general and it can be already encountered in the case of symmetric tensors.

1.2. A few historical remarks. In this subsection we sketch how our result fills in the literature.

The most celebrated result about uniqueness of decomposition of tensors is due to Kruskal \([K]\). It is often quoted in terms of Kruskal’s rank. A consequence of Kruskal’s criterion is the following statement, which applies to generic tensors (see Corollary 3 in \([AMR]\)).

**Proposition 1.4. (Kruskal’s criterion)** The generic tensor \(t \in A \otimes B \otimes C\) of rank \(k\) has a unique decomposition if

\[
k \leq \frac{1}{2} \left[ \min(a, k) + \min(b, k) + \min(c, k) - 2 \right]
\]

In the cubic case, the generic tensor \(t \in A \otimes A \otimes A\) of rank \(k\) has a unique decomposition if

\[
k \leq \frac{3a - 2}{2}
\]

Kruskal’s result is so important in the literature, that recently there have been published (at least!) three different proofs \([Land1, R, SS]\).

De Lathauwer (\([Lat]\)) proves that the generic tensor \(t \in A \otimes B \otimes C\) of rank \(k\) has a unique decomposition if \(k \leq c\) and \(k(k - 1) \leq a(a - 1)b(b - 1)/2\). Rhodes, in \([R]\) addresses explicitly, as a problem at the end of the introduction, the need of sufficient conditions, stronger than Kruskal’s, that guarantee the uniqueness of the decomposition, for generic tensors. Our Theorem \([1.1]\) gives a sufficient condition which improves both Kruskal’s and de Lauthawer’s bounds.

The tensor decomposition we are looking for are called also Candecomp or Parafac decompositions in the numerical literature. Among recent surveys on the topic, see §3.2 in \([KB]\) and Landsberg book \([Land0]\), which tries to use a language understandable by both the numerical and the geometrical communities. From this point of view, one should also consider section 2 of \([AMR]\), an interesting bridge between the two worlds.

1.3. Outline of the proof. In a line, our technique consists in putting together the inductive approach of \([AOP]\) with the tool of weak defectivity developed in \([CC1]\) and \([CC2]\).

We consider the projective space of tensors \(\mathbb{P}(A \otimes B \otimes C)\). In this space, the tensors of rank one give the Segre variety \(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)\).

Our geometric point of view consists in the use of the celebrated Terracini’s Lemma, which allows to study the identifiability of varieties, using properties of their tangent spaces. We refer to \([CC1]\) and \([CC2]\) for a more precise account of the theory behind.

A variety is called \(tangentially k weakly defective (k-twd, see Definition [2.6])\) if the span of the tangent spaces at \(k\) general points of \(X\), is tangent also in some other points.

It is a consequence of Terracini’s Lemma that, if \(X\) is \(k\)-not twd, then the general tensor of rank \(k\) has a unique decomposition.

So our aim is to prove the \(k\)-not twd of Segre varieties \(X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)\). The proof is performed by induction, by splitting \(A = A' \oplus A''\) and by specializing
some points on the lower dimensional Segre varieties $\mathbb{P}(A') \times \mathbb{P}(B) \times \mathbb{P}(C)$ and $\mathbb{P}(A'') \times \mathbb{P}(B) \times \mathbb{P}(C)$. It turns out that the induction works if we prove a stronger statement, concerning the so called $(k, p, q, r)$-weakly defectivity, which is defined in section 3.

1.4. Outline of the paper. In section 2 we develop the basic notations on Segre varieties and weak defectivity. At the end of this section we prove the cases $a \leq 7$ of the Theorem 1.2. Section 3 contains the definition of $(k, p, q, r)$-defectivity and the inductive step (Prop. 3.6). At the end of this section we prove the remaining cases of the Theorem 1.2. In the section 4 we prove the Theorem 1.1. In section 5 we prove the Theorem 1.3 and we give other examples of small dimension. Also we expose a list of all the examples of triple Segre product that we know when the uniqueness for general tensors of a given rank does not hold. In section 6 we show an extension of the previous results to products of many factors.

2. Preliminaries on Segre varieties

Let $A, B, C$ be complex vector spaces, of dimension $a, b, c$ respectively. Consider the product $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. $X$ is naturally embedded, by means of the Segre map, into $\mathbb{P}^N$, where $N = abc - 1$.

Sometimes, when there is no need to specify the vector spaces, we will refer to the variety $X$ also as $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$.

Call $S^k(X)$ the $k$-th secant variety of $X$, defined as the closure of the union of linear spans of $k$ general points in $X$.

**Definition 2.1.** $X$ is called $k$-identifiable if a general element in $S^k(X)$ has a unique expression as sum of $k$ elements in $X$.

From the tensorial point of view, this means that a general tensor of type $a \times b \times c$ and rank $k$, can be written uniquely (up to scalar multiplication) as a sum of $k$ decomposable tensors.

**Proposition 2.2.** There is a maximal rank for which the $k$-identifiability of tensors is possible, namely

$$k_{\text{max}} = \left\lfloor \frac{N + 1}{\dim(X) + 1} \right\rfloor = \left\lfloor \frac{abc}{a + b + c - 2} \right\rfloor.$$

**Proof.** For $k > k_{\text{max}}$, the abstract secant variety

$$\text{Ab}S^k(X) = \{(x_1, \ldots, x_k, u) \in X^k \times \mathbb{P}^N : u \in \langle x_1, \ldots, x_k \rangle\}$$

has dimension bigger than $N$, so that necessarily the general $u \in S^k(X)$ belongs to infinitely many $k$-secant spaces.

Our theoretical starting point is a criterion for $k$-identifiability, which follows from the Terracini’s Lemma, which we will use under the following form (see e.g. [CC1])

**Lemma 2.3. (Terracini)** Let $X$ be an irreducible variety and consider a general point $u \in S_k(X)$. If $u$ belongs to the span of points $x_1, \ldots, x_k \in X$, then the tangent space to $S_k$ at $u$ is the span of the tangent spaces to $X$ at the points $x_1, \ldots, x_k$.

Our criterion is the following:
Proposition 2.4. Let $X \subset \mathbb{P}^N$ be a non-degenerate, irreducible variety of dimension $n$. Consider the following statements:

(i) $X$ is $k$-identifiable

(ii) Given $k$ general points $x_1, \ldots, x_k \in X$, then the span $\langle T_{x_1}X, \ldots, T_{x_k}X \rangle$ contains $T_xX$ only if $x = x_i$ for some $i = 1, \ldots, k$.

(iii) There exists a set of $k$ particular points $x_1, \ldots, x_k \in X$, such that the span $\langle T_{x_1}X, \ldots, T_{x_k}X \rangle$ contains $T_xX$ only if $x = x_i$ for some $i = 1, \ldots, k$.

Then we have $(iii) \implies (ii) \implies (i)$.

Proof. $(iii) \implies (ii)$ follows at once by semicontinuity.

Let us prove that $(ii) \implies (i)$. Take a general point $u \in S_k(X)$ and assume that $u$ belongs to the span of points $x_1, \ldots, x_k \in X$. By the generality of $u$, we may assume that $x_1, \ldots, x_k$ are general points of $X$. If $u$ also belongs to the span of points $y_1, \ldots, y_k \in X$, with at least one of them, say $y_1$, not among the $x_i$’s, then, by Terracini’s Lemma, the span of the tangent spaces to $X$ at the points $x_i$’s, which is the tangent space to $S_k(X)$ at $u$, also contains the tangent space to $X$ at $y_1$. This contradicts $(ii)$. □

Condition $(ii)$ of the previous Proposition is strongly related with the notion of $k$-weak defectivity.

In [CC1], C. Ciliberto and the first author give the following definition: a variety $X$ is $k$-weakly defective if the general hyperplane which is tangent to $X$ at $k$ general points $x_1, \ldots, x_k$, is also tangent in some other point $y \neq x_1, \ldots, x_k$.

It is clear that a variety which does not satisfy condition $(ii)$ of the Proposition, is also $k$-weakly defective. However the converse does not hold.

Example 2.5. Consider the Segre product $X = \mathbb{P}^1 \times \mathbb{P}^2$. It is classical (see e.g. Zak’s Theorem on tangencies in [Z]) that the tangent space at one point to a smooth variety is not tangent elsewhere.

On the other hand, a general hyperplane tangent to $X$ at one point, is also tangent along a line. Indeed, it is well known that the dual variety of $X$ is not a hypersurface (see [E]). Thus $X$ is 1-weakly defective.

For maintaining the consistency with all the previous notation in this subject, we dare proposing the following:

Definition 2.6. If $X$ satisfies condition $(ii)$ of the previous Proposition, we will say that $X$ is $k$-not tangentially weakly defective. Otherwise, we say that $X$ is $k$-tangentially weakly defective ($k$-twd, for short).

We understand that the notation is becoming odd. However, the increasing number of definitions is a phenomenon which also occurs in the study of contact loci, which seems however helpful for applications to the Geometry of secant varieties (see e.g. [CC3]).

Weak defectivity has been intensively studied in [CC1]. Notice that when $X$ is $k$ weakly defective, then a general hyperplane tangent to $X$ at general points $x_1, \ldots, x_k$ is also tangent along a positive dimensional variety. We do not know if a similar phenomenon takes place also for $k$-twd.

Relations between $k$-weak defectivity and $k$-twd are probably stronger than expected, at least as far as one is interested in $k$-identifiability. We do not develop further this analysis.
Notice than, when we deal with inductive steps in the proofs, we will need an even more complicated notion of weak defectivity. Compare with Definition 3.1 below.

For our purposes, Proposition 2.4 establishes that $k$-not tangentially weakly defectivity implies $k$-identifiability, when $N \geq k(n+1)$.

Remark 2.7. Let us notice that, by Proposition 2.2, if $N+1 < (k+1)n$, then $k$-identifiability is excluded. Thus, the criterion of Proposition 2.4 cannot be applied only for at most one value of $k$, namely $k = (N+1)/(n+1)$, which occurs only when $N+1$ is a multiple of $n+1$. E.g., our criterion could not be applied to study the 2-identifiability of $P_1 \times P_1 \times P_1$.

Now we are already able to prove the first cases of Theorem 1.2.

Proof of the Theorem 1.2 in case $a \leq 7$.

The proof is a straightforward application of Proposition 2.4. A random choice of $k(a)$ points satisfies condition (iii) of Proposition 2.4. Then $X$ is $k$-identifiable. The Macaulay2 files which we used are available as ancillary files in the arXiv submission of this paper.

Remark 2.8. More powerful computers and/or better suited algorithms will allow eventually to check the condition (iii) for larger values of $a$, and we encourage experts in Numerical Algebraic Geometry in going further. We stopped at $a = 7$, because for $a = 8$ our algorithm on a common PC consumed too much time and memory. In the next section we show how the computation for larger values of $a$ can be reduced to other computations for smaller values of $a$.

3. The inductive statement

The inductive criterion makes use of the fact that if $x = u \otimes v \otimes w$ is a point of $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$, then the tangent space $T_xX$ is the projectification of the linear space $A \otimes v \otimes w + u \otimes B \otimes w + u \otimes v \otimes C$.

The idea is to fix two linear subspaces $A', A''$ of $A$, such that $A = A' \oplus A''$, then split the set of $k$ points in two subsets and specialize them to the two spaces $\mathbb{P}(A') \times \mathbb{P}(B) \times \mathbb{P}(C)$ and $\mathbb{P}(A'') \times \mathbb{P}(B) \times \mathbb{P}(C)$. Then, the implication (iii) $\implies$ (i) of Proposition 2.4 suggests that one could play induction.

Unfortunately, the situation is a little bit more complicated, since one cannot translate condition (ii) of Proposition 2.4 into the analogous condition on lower-dimensional spaces.

Instead, following the idea of [AOP] (Theorem 3.4) (suggested also from the Splitting Method of [BCS]), we need a more elaborated condition.

Definition 3.1. A triple product $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ is called $(k,p,q,r)$-not weakly defective if:

- for $k$ general points $x_1, \ldots, x_k \in X$,
- for $p$ general points $u_i \in \mathbb{P}(B) \times \mathbb{P}(C)$,
- for $q$ general points $v_i \in \mathbb{P}(A) \times \mathbb{P}(C)$,
- for $r$ general points $w_i \in \mathbb{P}(A) \times \mathbb{P}(B)$,

then the span of $T_{x_i}X$, $A \otimes u_i, B \otimes v_i, C \otimes w_i$ contains $T_xX$ if and only if $x = x_i$, for some $i = 1, \ldots, k$. Otherwise $X$ is called $(k,p,q,r)$-weakly defective.

Clearly, $(k,0,0,0)$ weak defectivity coincides with $k$-twd.
Remark 3.2. We will often use the computer algorithm, available in our arXiv submission, to prove that some triple Segre product is \((k, p, q, r)\)-not weakly defective.

For instance, the algorithm shows that \(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\) is \((1, 2, 1, 1)\)-not and \((2, 1, 1, 1)\)-not weakly defective. This is rather interesting, because \(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\) is 3-defective.

Example 3.3. Consider \(A, B, C\), all of dimension 2 with basis \(\{u_1, u_2\}, \{v_1, v_2\}, \{w_1, w_2\}\). Then \(T_{u_1v_1w_1} + u_2v_2C = T_{w_2v_2w_1} + u_1v_1C\). This shows that \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) is \((1, 0, 0, 1)\) weakly defective. Nevertheless, \(T_{u_1v_1w_1} + u_2v_2C\) has the expected (affine dimension) 6 and it does not fill the ambient space.

Remark 3.4. (a) With the previous notation, by semicontinuity it is clear that when \(X\) is \((k, p, q, r)\)-not weakly defective, then it is also \((k, p, q, r)\)-not weakly defective, whenever \((k', p', q', r') \leq (k, p, q, r)\), in the strict ordering.

(b) By semicontinuity, \(X\) is \((k, p, q, r)\)-not weakly defective whenever one gets that for particular sets of points \(\{x_i\}\), \(\{u_i\}\), \(\{v_i\}\) and \(\{w_i\}\) as above, then the span of \(T_{x_i}X\), \(A \otimes u_i, B \otimes v_i, C \otimes w_i\) contains \(T_{x_i}X\) if only if \(x = x_i\), for some \(i = 1, \ldots, k\).

(c) By Proposition 2.4 one gets soon that \((k, 0, 0, 0)\)-not weakly defective implies \(k\)-identifiable.

We will often apply the following reduction step:

**Lemma 3.5.** Assume that \(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}\) is \((k, p, q, r)\)-not weakly defective. Then \(\mathbb{P}^{a'} \times \mathbb{P}^{b'} \times \mathbb{P}^{c'}\) is \((k, p, q, r)\)-not weakly defective, for any triple \((a', b', c') > (a - 1, b - 1, c - 1)\) (in the strict ordering).

**Proof.** We need just to prove the statement for \((a', b', c') = (a, b - 1, c - 1)\). Write \(X = \mathbb{P}^a \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1} = \mathbb{P}(A') \otimes \mathbb{P}(B) \otimes \mathbb{P}(C)\) so that \(\dim(A') = a + 1\).

Assume that \(X\) is \((k, p, q, r)\)-weakly defective. Thus, for \(k\) general points \(x_1, \ldots, x_k \in X\), \(p\) general points \(u_i \in \mathbb{P}(B) \times \mathbb{P}(C)\), \(q\) general points \(v_i \in \mathbb{P}(A') \times \mathbb{P}(C)\), \(r\) general points \(w_i \in \mathbb{P}(A') \times \mathbb{P}(B)\), then the span \(\Lambda\) of the tangent spaces to \(X\) at the \(x_i\)‘s and the spaces \(A' \otimes u_i, B \otimes v_i, C \otimes w_i\), is also tangent in another point \(y\).

Take a general point \(P = (u, v, w) \in \mathbb{P}^a \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}\) and consider the projection \(\pi\) of \(X\) from \(L = u \otimes B \otimes C\). The image of the projection is \(Y = \mathbb{P}(A) \otimes \mathbb{P}(B) \otimes \mathbb{P}(C)\) where \(A \subset A'\) has codimension 1. Furthermore, by the generality of \(P\), \(L\) does not meet \(\Lambda\), as well as any line spanned by \(y, x_i\). It follows that the span of the tangent spaces to \(Y\) at the general points \(\pi(x_1), \ldots, \pi(x_k)\) and containing the spaces \(A \otimes \pi(u_i), B \otimes \pi(v_i), C \otimes \pi(w_i)\) is also tangent in another point \(\pi(y)\). Thus \(Y\) is \((k, p, q, r)\)-weakly defective. By induction, we get a contradiction. \(\square\)

Now we are ready to state and prove our inductive criterion.

Let \(X' = \mathbb{P}(A') \times \mathbb{P}(B) \times \mathbb{P}(C)\), \(X'' = \mathbb{P}(A'') \times \mathbb{P}(B) \times \mathbb{P}(C)\). Note that \(A \otimes B \otimes C = (A' \otimes B \otimes C) \oplus (A'' \otimes B \otimes C)\). Denote by \(\pi'\) and \(\pi''\) the two projections.

**Proposition 3.6.** (Inductive Step.) Assume that \(X'\) is \((k_1 + k_2, q_1, r_1)\)-not weakly defective and \(X''\) is \((k_2, p + k_1, q_2, r_2)\)-not weakly defective. Then \(X\) is \((k_1 + k_2 + q_1 + q_2, r_1 + r_2)\)-not weakly defective.

**Proof.** We specialize \(k_1 + k_2\) points on \(X\) in order that \(k_1\) of them belong to \(X_1\) and \(k_2\) of them belong to \(X_2\). Let \(x_1, \ldots, x_{k_1} \in X'\) and \(y_1, \ldots, y_{k_2} \in X''\).

Let \(A \otimes \tilde{v}_i \otimes \tilde{w}_i\) for \(i = 1, \ldots, p\), be subspaces.
We specialize \( q_1 + q_2 \) points in \( \mathbb{P}(A) \times \mathbb{P}(C) \) in order that the first \( q_1 \) of them belong to \( \mathbb{P}(A') \times \mathbb{P}(C) \) and the last \( q_2 \) of them belong to \( \mathbb{P}(A'') \times \mathbb{P}(C) \). Call \( Q_1 \) the span of the first \( q_1 \) spaces \( Bv_i \) and \( Q_2 \) the span of the last \( q_2 \) spaces \( Bv_i \).

We specialize \( r_1 + r_2 \) points in \( \mathbb{P}(A) \times \mathbb{P}(B) \) in order that the first \( r_1 \) of them belong to \( \mathbb{P}(A') \times \mathbb{P}(B) \) and the last \( r_2 \) of them belong to \( \mathbb{P}(A'') \times \mathbb{P}(B) \). Call \( R_1 \) the span of the first \( r_1 \) spaces \( Cw_i \) and \( R_2 \) the span of the last \( r_2 \) spaces \( Cw_i \).

We want to prove that \( T = T_{x_1}X + \ldots + T_{x_{k_1}}X + T_{y_1}X + \ldots + T_{y_{k_2}}X + Q_1 + Q_2 + R_1 + R_2 + A \otimes \hat{a}_1 \otimes \hat{a}_1 + \ldots + A \otimes \hat{a}_p \otimes \hat{a}_p \) is tangent to \( X \) only at \( x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2} \).

Let \( T_xX \subset T \), with \( x = u \otimes v \otimes w \). Then \( \pi_1(T_xX) \subset \pi_1(T) \). Let \( u = u' + u'' \), \( y_j = u''_j \otimes v''_j \otimes w''_j \), \( j = 1, \ldots, k_2 \). At least one among \( u' \) and \( u'' \) is non zero, so let's assume \( u' \neq 0 \). Then we get \( \pi_1(T_xX) = A' \otimes v \otimes w + u' \otimes B \otimes w + u' \otimes v \otimes C \) while \( \pi_1(T) = T_{x_1}X' + \ldots + T_{x_{k_1}}X' + A' \otimes v'' \otimes w'' + \ldots + A' \otimes \hat{a}_i \otimes \hat{a}_i + \ldots + Q_1 + R_1 \) (with \( i = 1, \ldots, p \)). By the assumption that \( X' \) is \((k_1, p + q_1, r_1)-\)not weakly defective it follows that \( u' \otimes v \otimes w \) is one among \( x_i \).

If also \( u'' \neq 0 \) the same argument shows that \( u'' \otimes v \otimes w \) is one among \( y_i \), which is a contradiction. Then \( u'' = 0 \), that is \( x = u' \otimes v \otimes w \) is one among \( x_i \). It follows that \( X \) is \((k_1 + k_2, p, q_1 + q_2, r_1 + r_2)-\)not weakly defective, as we wanted. \( \square \)

The inductive procedure stops eventually when we find some condition on weak defectivity, which does not hold. This does not means, in general, that our starting example was not \( k \)-identifiable, but merely that we specialized the points too much, in order to expect a meaningful answer.

**Proof of Theorem 1.2 in cases a = 8, 9, 10.**

In case \( a = 8 \) we start with 22 points and we want to apply iteratively the Proposition 3.6. Splitting one 8-dimensional vector space of the product in a direct sum of two 4-dimensional spaces, one sees that the \((22, 0, 0, 0)-\)not weak defectivity of \( \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \) follows if one knows that \( \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^7 \) is \((11, 11, 0, 0)-\)not weakly defective. Repeating the procedure with the second factor, everything reduces to prove that \( \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^7 \) is \((5, 7, 6, 0)-\)not weakly defective and \((6, 4, 5, 0)-\)not weakly defective. The first statement reduces to show that \( \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \) is \((3, 3, 3, 3)-\)not weakly defective and \((2, 4, 3, 3)-\)not weakly defective. These statements have finally a reasonable size and can be checked with a random choice of points with our Macaulay2 algorithm. The last statement reduces to show that \( \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \) is \((3, 2, 3, 3)-\)not weakly defective and \((3, 2, 2, 3)-\)not weakly defective, which follows from the above check and by the Remark 3.4 (a).

In the case \( a = 9 \) we start with 27 points and we split the 9 dimensional space in three 3-dimensional summands. The inductive step is better explained by the following table

|   | a | b | c | k | p | q | r |
|---|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 27| 0 | 0 | 0 |   |
| 3 | 9 | 9 | 9 | 18| 0 | 0 |   |
| 3 | 3 | 9 | 3 | 6 | 6 | 0 |   |
| 3 | 3 | 3 | 1 | 2 | 2 | 2 |   |

The last statement can be checked again with Macaulay2.
The $a = 10$ case starts as follows

\[
\begin{array}{cccccccc}
a & b & c & k & p & q & r \\
10 & 10 & 10 & 32 & 0 & 0 & 0 \\
5 & 10 & 10 & 16 & 16 & 0 & 0 \\
5 & 5 & 10 & 8 & 8 & 8 & 0 \\
5 & 5 & 5 & 4 & 4 & 4 & 4 \\
\end{array}
\]

The second statement reduces to show that $\mathbb{P}^1 \times \mathbb{P}^4 \times \mathbb{P}^4$ is $(1, 7, 2, 2)$-not weakly defective and $\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^4$ is $(3, 5, 2, 2)$-not weakly defective. Both these statements can be checked with Macaulay2. This concludes the proof.

4. Proof of Theorem 1.1

In order to use the inductive step, we need a starting point.

**Lemma 4.1.** $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $(1, 0, 0, 0)$-not and $(0, 1, 1, 1)$-not weakly defective.

**Proof.** The first fact is true for any smooth variety, see Example 2.5. For the second one, we consider $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ where $A, B, C$ have all dimension 2 and we choose basis $A = \langle a_0, a_1 \rangle$, $B = \langle b_0, b_1 \rangle$, $C = \langle c_0, c_1 \rangle$. Then, without loss of generality, we may consider the span $T = A \otimes b_0 \otimes c_0 + a_0 \otimes B \otimes c_1 + a_1 \otimes b_1 \otimes C$. In the monomial basis of $A \otimes B \otimes C$ this span contains all the monomials with the only exception of $a_0 \otimes b_1 \otimes c_0$ and $a_1 \otimes b_0 \otimes c_1$. Then, a vector $v = \sum x_{ijk} a_i \otimes b_j \otimes c_k$, belongs to $X \cap \mathbb{P}(T)$ if all the $2 \times 2$-minors of the two following flattening matrices vanish

\[
\begin{bmatrix}
x_{000} & x_{001} & x_{100} & 0 \\
0 & x_{011} & x_{110} & x_{111}
\end{bmatrix}
\quad \begin{bmatrix}
x_{000} & 0 & x_{100} & x_{110} \\
x_{001} & x_{011} & 0 & x_{111}
\end{bmatrix}
\]

A straightforward check on the minors shows that $X \cap \mathbb{P}(T)$ consists of the following six lines in the 5-dimensional space $\mathbb{P}(T) = \{x_{010} = x_{101} = 0\}$ (Macaulay2 can be helpful at this step)

- $r_0 = V(x_{001}, x_{000}, x_{100}, x_{110})$
- $r_1 = V(x_{000}, x_{100}, x_{110}, x_{111})$
- $r_2 = V(x_{100}, x_{110}, x_{111}, x_{011})$
- $r_3 = V(x_{110}, x_{111}, x_{011}, x_{001})$
- $r_4 = V(x_{111}, x_{011}, x_{001}, x_{000})$
- $r_5 = V(x_{011}, x_{001}, x_{000}, x_{100})$

which have the property that, for $i \neq j$

\[
\begin{array}{c}
\text{one point} \\
\emptyset
\end{array}
\text{if } i = j + 1, j - 1 \text{ mod } 6
\]

It follows that $\mathbb{P}(T)$ is not tangent anywhere, because the tangent space at a point meets $X$ in three concurrent lines. This proves that $X$ is $(0, 1, 1, 1)$-not weakly defective. □

**Remark 4.2.** We will use affine spaces whose dimension is a power of 2, as well as sets of points or subspaces whose number is expressed in terms of powers of 2, essentially because they allow the following recursive application of Lemma 3.6.

Assume we want to prove that $\mathbb{P}^{2^a-1} \times \mathbb{P}^{2^b-1} \times \mathbb{P}^{2^c-1} = \mathbb{P}(A) \otimes \mathbb{P}(B) \otimes \mathbb{P}(C)$ is $(2x, 2^u, 2^v, 2^w)$-not weakly defective. Then, by splitting the first linear space $A$ in a direct sum of two subspaces of dimension $2^{a-1}$ and balancing the splitting of the
number of points and linear spaces, by Proposition 3.6 it is sufficient to prove that \( \mathbb{P}^{2^u-1} \times \mathbb{P}^{2^v-1} \times \mathbb{P}^{2^w-1} \) is \((x, 2^u, 2^{v-1}, 2^{w-1})\)-not weakly defective.

We will use this trick so often, in the arguments below.

The final statement will be that, if we order the dimensions so that \( 1 \leq \alpha \leq \beta \leq \gamma \), then \( X = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1} \) is \((k, 0, 0, 0)\)-not weakly defective, for \( k \leq 2^{\alpha+\beta-2} \).

Before showing this fact, we need a series of lemmas.

**Proposition 4.3.** Assume that \( X = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1} \) is not \((k, 0, 0, 0)\)-not weakly defective. Then also \( X' = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma} \) is \((k, 0, 0, 0)\)-not weakly defective.

**Proof.** By Lemma 3.5. \qed

So, in order to prove Theorem 1.1 we can reduce ourselves to the case \( \beta = \gamma \), \( k = 2^{\alpha+\beta-2} \).

**Lemma 4.4.** Take \( X = \mathbb{P}^{2^{a_1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1} \), with \( a_1, a_2, a_3 \geq 1 \). Pick non-negative integers \( u_1, u_2, u_3 \) such that \( u_i \leq a_j + a_k - 2 \), whenever \( \{i, j, k\} = \{1, 2, 3\} \). Then \( X \) is \((0, 2^{u_1}, 2^{u_2}, 2^{u_3})\)-not weakly defective.

**Proof.** We make induction on the sum \( a_1 + a_2 + a_3 \).

If \( a_1 = a_2 = a_3 = 1 \), then the numerical conditions imply that \( u_1 = u_2 = u_3 = 0 \) and the conclusion follows from the fact that \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) is \((0, 1, 1, 1)\)-not weakly defective, which holds by Lemma 4.4.

Assume \( a_1 > 1 \) and split the first projective space in a sum of two spaces of dimension \( 2^{a_1-1} \). Then there are three possibilities:

1. Assume \( u_2 = u_3 = 0 \). Then, by using Lemma 3.6, the claim reduces to prove that \( \mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}} \) is \((0, 2^{u_1}, 1, 1)\)-not weakly defective and it is \((0, 2^{u_1}, 0, 0)\)-not weakly defective. The second condition is contained in the first. Since \( a_1 > 1 \), the six numbers \( a_1 - 1, a_2, a_3, u_1, 0, 0 \) fulfill the numerical inequalities of the statement. Hence the claim follows by induction, in this case.

2. Assume \( u_3 > u_2 = 0 \). Then the claim reduces to prove that \( \mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}} \) is \((0, 2^{u_1}, 1, 2^{u_3-1})\)-not weakly defective and it is \((0, 2^{u_1}, 0, 2^{u_3-1})\)-not weakly defective. The second condition is contained in the first. One checks that the six numbers \( a_1 - 1, a_2, a_3, u_1, 0, u_3 - 1 \) fulfill the numerical inequalities of the statement. Hence the claim follows by induction.

3. Assume \( u_2, u_3 > 0 \). Then the claim reduces to prove that \( \mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}} \) is \((0, 2^{u_1}, 2^{u_2-1}, 2^{u_3-1})\)-not weakly defective. One checks that the six numbers \( a_1 - 1, a_2, a_3, u_1, u_2 - 1, u_3 - 1 \) fulfill the numerical inequalities of the statement. Hence the claim follows by induction. \qed

**Lemma 4.5.** Take \( X = \mathbb{P}^{2^{a_1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1} \), with \( a_1, a_2, a_3 \geq 1 \). Pick non-negative integers \( u_1, u_2, u_3 \) such that \( u_i \leq a_j + a_k - 2 \), whenever \( \{i, j, k\} = \{1, 2, 3\} \). Then \( X \) is \((1, 2^{u_1} - 1, 2^{u_2} - 1, 2^{u_3} - 1)\)-not weakly defective.

**Proof.** We make induction on the sum \( a_1 + a_2 + a_3 \).

If \( a_1 = a_2 = a_3 = 1 \), then the numerical conditions imply that \( u_1 = u_2 = u_3 = 0 \) and the conclusion follows from the fact that \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) is \((1, 0, 0, 0)\)-not weakly defective (Lemma 4.4).

Assume \( a_1 > 1 \) and split the first projective space in a sum of two spaces of dimension \( 2^{a_1-1} \). Then there are three possibilities:
(1) Assume $u_2 = u_3 = 0$. Then, by using Lemma 4.6, the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}}$ is $(0, 2^{a_1}, 1, 1)$-not weakly defective and it is $(1, 2^{a_1} - 1, 0, 0)$-not weakly defective. The first condition follows by the previous Lemma 4.4. For the second condition, notice that since $a_1 > 1$, the six numbers $a_1 - 1, a_2, a_3, u_1, 0, 0$ fulfill the numerical inequalities of the statement $(0 = 2^0 - 1)$. Hence the claim follows by induction, in this case.

(2) Assume $u_3 > u_2 = 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}}$ is $(0, 2^{a_1}, 1, 2^{a_3-1})$-not weakly defective and it is $(1, 2^{a_1} - 1, 0, 2^{a_3-1} - 1)$-not weakly defective. The first condition follows by the previous Lemma. The second condition follows by induction, since one checks that the six numbers $a_1 - 1, a_2, a_3, u_1, 0, u_3 - 1$ fulfill the numerical inequalities of the statement.

(3) $u_2, u_3 > 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}} \times \mathbb{P}^{2^{a_2-1}} \times \mathbb{P}^{2^{a_3-1}}$ is $(0, 2^{a_1}, 2^{a_2-1}, 2^{a_3-1})$-not weakly defective and it is $(1, 2^{a_1} - 1, 2^{a_2-1} + 1, 2^{a_3-1} - 1)$-not weakly defective. One checks that the numerical conditions in the statement are still fulfilled, by the six numbers $a_1 - 1, a_2, a_3, u_1, u_2 - 1, u_3 - 1$. Hence the claim follows by induction. □

Now we are ready to prove:

**Theorem 4.6.** $X = \mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{b-1}} \times \mathbb{P}^{2^{c-1}}$ is $(k, 0, 0, 0)$-not weakly defective, for $k \leq 2^{a+b+c} - 2$.

**Proof.** Write $\alpha + \beta - 2 = 2p + e$, where $e$ is the remainder.

Now we start our reduction.

(A1) One can split the vector space in the middle as a sum of two spaces of dimension $2^\beta - 1$. By using Lemma 3.6 it turns out that $X$ is $(2^\alpha + 2, 0, 0, 0)$-not weakly defective when $\mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{\beta-1}} \times \mathbb{P}^{2^{\beta-1}}$ is $(2^\alpha + 2, 0, 2^{\alpha + \beta - 3}, 0)$-not weakly defective.

(A2) Splitting now the third vector space as a sum of two spaces of dimension $2^{\beta - 1}$, and using the Lemma, this reduces to prove that $\mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{\beta-1}} \times \mathbb{P}^{2^{\beta-1}}$ is $(2^\alpha + \beta - 4, 0, 2^\alpha + 2^{\beta - 4}, 0)$-not weakly defective.

(A3) Now repeat the procedure, splitting the space in the middle: Everything reduces to prove that $\mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{\beta-2}} \times \mathbb{P}^{2^{\beta-1}}$ is $(2^\alpha + 2^{\beta - 5}, 0, 2^\alpha + 2^{\beta - 4} + 2^\alpha + 2^{\beta - 5}, 2^\alpha + 2^{\beta - 5})$-not weakly defective.

Now split again the third vector space, and repeat the steps. At the end of the $(\alpha + \beta - 2)$-th step, after the computation, we find out that we need just to prove that $\mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{\beta - p - e - 1}} \times \mathbb{P}^{2^{\beta - p - 1}}$ is $(1, 0, \sum_{i=0}^{p+e-1} 2^i, \sum_{i=0}^{p-1} 2^i)$-not weakly defective.

Notice that all these steps can be performed, because $\beta - p \geq \beta - p - e \geq 1$. Indeed we have $\alpha \leq \beta$, thus $2\beta - 2 \geq 2p + e$, hence $2\beta \geq 2p + e + 2 > 2p + 2e$.

Now, $\sum_{i=0}^{p+e-1} 2^i = 2^{p+e} - 1$ while $\sum_{i=0}^{p-1} 2^i = 2^p - 1$. Moreover

\[ p + e \leq \alpha + (\beta - p) - 2 \quad \text{since} \quad 2p + e = \alpha + \beta - 2 \]

\[ p \leq \alpha + (\beta - p - e) - 2 \quad \text{since} \quad 2p = \alpha + \beta - e - 2. \]

Thus we may apply Lemma 4.5 and see that $\mathbb{P}^{2^{a-1}} \times \mathbb{P}^{2^{\beta - p - e - 1}} \times \mathbb{P}^{2^{\beta - p - 1}}$ is $(1, 0, 2^{p+e} - 1, 2^p - 1)$-not weakly defective. The result is settled. □

When $\alpha = \beta$, i.e. when the product is balanced, we find that $X$ is $k$-identifiable for $k \leq 2^{2a - 2}$. 

Proof of Theorem 4.7 Fix $\alpha, \beta$ maximal such that $2^\alpha \leq a$ and $2^\beta \leq b$. Then, by the previous Theorem, $\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\beta-1}$ is $(k, 0, 0, 0)$-not weakly defective, for $k \leq 2^{\alpha+\beta-2} = 2^{\alpha+2}/4$. Thus also $\mathbb{P}(A) \otimes \mathbb{P}(B) \otimes \mathbb{P}(C) = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ is $(k, 0, 0, 0)$-not weakly defective, for $k \leq 2^{a+2}/4$. The conclusion follows. □

Comparing our result with the maximal $k$ for which the identifiability of $\mathbb{P}(A) \otimes \mathbb{P}(B) \otimes \mathbb{P}(C)$ makes sense, i.e.

$$k_{\text{max}} = \left\lfloor \frac{abc}{a + b + c - 2} \right\rfloor.$$  
(see Proposition 2.2) and considering that $ab/3 \leq k_{\text{max}} \leq ab$, we see that the bound in the Theorem is, at least log-asymptotically, sharp, as explained in the Introduction.

In any events, it improves Kruskal’s bound for identifiability.

Remark 4.7. In principle, there are no obstructions in repeating the argument of Theorem 4.6, when we substitute powers of 2 with powers of any other integer $p > 1$. The final statement is:

$X = \mathbb{P}^{p^\alpha-1} \times \mathbb{P}^{p^\beta-1} \times \mathbb{P}^{p^\beta-1}$ is $(k, 0, 0, 0)$-not weakly defective, for $k \leq p^{\alpha+\beta-2}$.

The proof is achieved very similarly, by splitting, step by step, a vector space of dimension $p^n$ into $p$ spaces of dimension $p^{n-1}$. (see e.g. the case $a = 9$ in the proof of Theorem 1.2).

We can use this statement, instead of Theorem 4.6, in the proof of Theorem 1.1, obtaining another bound which implies $k$-identifiability.

In most cases, however, the new bound is weaker than the one of the Theorem 1.1.

On the other hand, in some specific case, typically when powers of 3 are involved, it can be stronger.

To give an example, let us consider $X = \mathbb{P}^{26} \times \mathbb{P}^{26} \times \mathbb{P}^{26}$. Using Theorem 1.1 we obtain $k$-identifiability for $k \leq 2^{4+4+2} = 64$. Using powers of $p = 3$, instead, we get $k$-identifiability for $k \leq 3^{3+3+2} = 81$. It is an improvement, but still a long way from $k_{\max} = 249$.

5. Some Examples in Low Dimension

In this section, we study the $k$-identifiability of Segre products $X = \mathbb{P}(A) \otimes \mathbb{P}(B) \otimes \mathbb{P}(C)$, when the dimensions $a, b, c$ are small. We also provide a proof for Theorem 1.3.

Proof of Theorem 1.3 Consider $X = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$. This product is 5-identifiable, by Kruskal’s criterion. On the other hand, accordingly with Proposition 2.2, one may ask about the 6-identifiability of $X$.

We are able to prove that $X$ is not 6-identifiable, and the general point in $S^6(X)$ sits in exactly two 6-secant, 5-planes. From the tensorial point of view, this means that a general $4 \times 4 \times 4$ tensor of rank 6, can be written as a sum of 6 decomposable tensors in exactly 2 ways (up to scalar multiplication and permutations).

The reason relies in the fact that through 6 general points $x_1, \ldots, x_6$ of $X = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$, one can draw an elliptic normal curve $\Gamma$ of degree 12, which spans a projective space $L = \mathbb{P}^{11}$, containing the linear span of $x_1, \ldots, x_6$. So, a general point $u \in S^6(X)$ lies in a linear space $L$ spanned by an elliptic normal curve $\Gamma \subset X$. By CC2, Proposition 5.2, it is known that $\Gamma$ has 6-secant order 2, i.e. there are exactly two 5-planes, 6-secant to $\Gamma$, inside $L$. By CC2 Proposition 2.4, if we prove that $\Gamma$ coincides with
the contact locus of a general 6-tangent hyperplane, also \(X\) must have 6-secant order equal to 2. This last fact can be checked by our Macaulay2 algorithm. Unfortunately, the existence of an elliptic normal curve, of degree 12, passing through 6 randomly chosen points of \(X\), gives only a probabilistic argument for the existence of such a curve passing through 6 general points of \(X\). To overcome this problem, we offer the following theoretical argument.

Consider the projections \(z_{1i}, \ldots, z_{6i}\) of \(x_1, \ldots, x_6\), into the \(i\)-th copy of \(\mathbb{P}^3\), so that \(z_{1i}, \ldots, z_{6i}\) are general points of \(\mathbb{P}^3\). Normal elliptic curves \(C\) passing through the 6 points of \(\mathbb{P}^3\) are given by pairs of quadrics through the points, so they are parametrized by the Grassmannian \(G\) of lines in the space \(\mathbb{P}^3\) of quadrics through \(z_{1i}, \ldots, z_{6i}\). In order that three normal elliptic curves \(C, C', C''\) in the three copies of \(\mathbb{P}^3\), correspond to the same abstract curve, they need to differ by an element of \(\text{PGL}(3)\). So, once we have \(C\) (4 parameters), we can choose \(\phi, \psi \in \text{PGL}(3)\) for the two remaining maps \(C \to \mathbb{P}^3\) (thus a total of \(4 + 15 + 15 = 34\) parameters). On the other hand, we need to impose that \(\phi(C) = C'\) (resp. \(\psi(C) = C''\)) pass through \(z_{21}, \ldots, z_{26}\) (resp. \(z_{31}, \ldots, z_{36}\)). Since each point imposes 2 conditions, we get a total of 24 algebraic conditions on the 34 parameters.

Moreover, if we want that after this correspondence, \(C, C', C''\) are projection of the same curve passing through \(x_1, \ldots, x_6\), we also need that the projectivity \(\phi : C \to C'\) (resp. \(\psi : C \to C''\)), composed with the automorphisms of the curves which sends \(z_{1i}\) to \(z_{2i}\) (resp. \(z_{1i}\) to \(z_{3i}\)), also sends any \(z_{1i} \) to \(z_{2i}\) (resp. \(z_{1i}\) to \(z_{3i}\)), for \(i \geq 2\). This gives 10 more conditions, which are algebraic on the coefficients of the two quadrics and the entries of the matrices of \(\phi, \psi\).

So, we have a total of 34 conditions, which are algebraic on the 34 parameters, i.e. on the projective coordinates of \(G \times \text{PGL}(3) \times \text{PGL}(3)\). Thus we get at least a finite number of curves passing through \(x_1, \ldots, x_6\), for a general choice of the points. \(\Box\)

**Remark 5.1.** In the previous example, notice that when the three projections of the points \(x_1, \ldots, x_i\) differ by a projectivity, then the number of conditions decreases, and we find infinitely many normal elliptic curves.

It is easy to see that this implies that a point in the secant variety \(S_6\) of any of these curves, belongs indeed to infinitely many 6-secant spaces.

The case of products of projective spaces of dimension 3 is particularly interesting, due to its applications to statistical studies on DNA strings.

If we have many substrings of DNA strings, each formed by three positions, and we record the occurrence of the four bases in each position, we get a distribution which can be arranged in a \(4 \times 4 \times 4\) tensor \(T\). The rank \(k\) of \(T\) suggests the existence of \(k\) different types of substrings, in the probe, such that for each type, the distribution of bases is independent. So \(T\) is the sum of \(k\) tensors \(T_1, \ldots, T_k\), of rank 1.

An obvious question concerns the possibility of recovering the \(k\) tensors \(T_i\), starting from \(T\). When \(k \geq 7\), this possibility is excluded, since 7 exceeds the maximum given in Proposition 2.2. For \(k \leq 5\), \(k\)-identifiability (by Kruskal’s criterion) tells us that, at least theoretically, the reconstruction is possible.

The amazing situation happens for \(k = 6\). Although one could expect 6-identifiability, Theorem 1.3 shows that there are exactly two sets of tensors of rank 1, whose sum...
is $T$. Hence, at least over the complex field, there are exactly two different sets of
distributions, in the 6 types, that produce the same distribution $T$.

In [AOP] 6.3 one finds the list of known Segre varieties $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C) = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ (with $a \leq b \leq c$) such that the dimension of $k$-th secant variety is smaller than the expected value. Recall that when the dimension of $S_k(X)$ is smaller than the expected value, i.e. when the variety $X$ is $k$-defective, then the $k$-identifiability necessarily fails.

A list of known Segre varieties $X$ which are not $k$-identifiable, i.e. such that the general tensor of rank $k$ in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ has not a unique decomposition, is the following (for $k < k_{\text{max}})$:

| $(a,b,c)$ | $k$ | notes |
|-----------|-----|-------|
| defective unbalanced | $c \geq (a-1)(b-1) + 3$ | $(a-1)(b-1) + 2 \leq k$ | [AOP] |
| defective | $(3,4,4)$ | 5 | [AOP] |
| defective | $(3,b,b)$ | $b$ odd | $3b-1$ | [S] |
| w. defective unbalanced | $3 \leq a$ | $(a-1)(b-1) + 1$ | $(a-1)(b-1)+1)$ decompositions where $d = \frac{a+b-2}{a-1}$ (Theorem 5.6) |
| w. defective | $(4,4,4)$ | 6 | 2 decompositions (Theorem 1.3) |
| w. defective | $(3,6,6)$ | 8 | $(**)$ |

A computer check shows that this list is complete for $c \leq 7$. In the last case marked with $(**)$, the contact variety is a 4-fold in $\mathbb{P}^{39}$ of degree 108. This case needs an "ad hoc" analysis which goes beyond the space of the present note and will be addressed in a forthcoming paper [CMO].

In the unbalanced case, the identifiability can be proved theoretically.

**Proposition 5.2.** The general tensor of rank $(a-1)(b-1)$ in $\mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ has a unique decomposition as sum of $(a-1)(b-1)$ summands in $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ for $c \geq (a-1)(b-1)$.

**Proof.** Let $\phi \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ be general of rank $(a-1)(b-1)$. It induces the flattening contraction operator

$$A_\phi: (\mathbb{C}^c)^\vee \to \mathbb{C}^a \otimes \mathbb{C}^b$$

which has still rank $(a-1)(b-1)$, by the assumption $c \geq (a-1)(b-1)$. Indeed, if $\phi = \sum_{i=1}^{(a-1)(b-1)} u_i \otimes v_i \otimes w_i$ with $u_i \in \mathbb{C}^a, v_i \in \mathbb{C}^b, w_i \in \mathbb{C}^c$, where $w_i$ can be chosen as part of a basis of $C$, then $\text{Im} A_\phi$ is the span of the representatives of $v_i \otimes w_i$ for $i = 1, \ldots, (a-1)(b-1)$. It is well known that the projectification of this span, whose
dimension is smaller than the codimension of the Segre variety \( Y = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \subset \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b) \), meets \( Y \) only in these \((a-1)(b-1)\) points (see for example the Theorem 2.6 in [CC1]). The claim follows.

**Proposition 5.3.** When \( c = (a-1)(b-1) \) or \( c = (a-1)(b-1) + 1 \), then the rank of a generic tensor in \( \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \) is \( ab - a - b + 2 \).

**Proof.** When \( c \geq (a-1)(b-1) + 1 \), we are in the unbalanced case, according to the definition 4.2 of [AOP]. In this case the generic rank is \( \min\{c, ab\} \) by (ii) of the Theorem 4.4 of [AOP].

Assume \( c = (a-1)(b-1) \). Using the same technique, we show that the secant variety \( S_k(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}) \) has the expected dimension, for \( k \leq (a-1)(b-1) \), and fills the ambient space, for \( k = (ab - a - b + 2) \).

Indeed, with the notations of [AOP], \( T(a-1, b-1, ab - a - b; (a-1)(b-1); 0, 0, 0) \) reduces to \( T(a-1, b-1, 0; 1; 0, 0, ab - a - b) \), which is true and subabundant, while \( T(a-1, b-1, ab - a - b; ab - a - b + 2; 0, 0, 0) \) reduces (for \( b \geq 3 \)) to \( T(a-1, b-1, 0; 1; 0, 0, ab - a - b + 1) \) and \( T(a-1, b-1, 0; 2; 0, 0, ab - a - b - 1) \) which are both superabundant and true. \( \square \)

**Proposition 5.4.** Assume \( c \geq (a-1)(b-1) + 2 \). Then the generic rank in \( \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \) is at least \((a-1)(b-1) + 2\), and it is equal to \((a-1)(b-1) + 2\) in the border case \( c = (a-1)(b-1) + 2 \). The number of different decomposition of a general tensor of rank \((a-1)(b-1)+1\) is \( \binom{d}{a-1(b-1)+1} \) where \( d = \deg(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}) = \binom{a+b-2}{a-1} \). This number is always bigger than 1, with the only exception \( a = b = 2 \).

**Proof.** We apply the same argument of the proof of the Proposition 5.2. The unique difference is that, now, the dimension of the projectification of \( \mathrm{Im} \, A_\phi \) equals the codimension of \( \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \). Thus we get \( d \) points of intersection. Any choice of \((a-1)(b-1)+1\) among these \( d \) points, yields a decomposition. \( \square \)

**Remark 5.5.** The case \( a = b = 3 \) of Prop. 5.4 is connected to the work of tenBerge, who showed in [LB], that there are six different decompositions of a general rank 5 tensor in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5 \), chosen taking 5 among 6 possible summands. Our argument, which we gave for \( c \geq 6 \), can be extended to the case \( c = 5 \) and \( k = k_{\text{max}} = 5 \), and it gives a geometric explanation of this phenomenon, indeed the six possible summands correspond to the six intersection points of \( \mathbb{P}^2 \times \mathbb{P}^2 \) with a general \( \mathbb{P}^4 \).

As a consequence of the two previous results, we get:

**Theorem 5.6.** Assume \( c \geq (a-1)(b-1) + 2 \). Then the general tensor of rank \( k \) in \( \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \) has a unique decomposition as sum of \( k \) summands in \( \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1} \) if and only if \( k \leq (a-1)(b-1) \).

6. Products with many factors

At the cost of the growth of the notation, we can generalize the statement of our main Theorem 4.1 to products of many vector spaces.

In this section, we simply list the corresponding definitions and results. The proofs are absolutely straightforward, following the pattern of the corresponding arguments in the previous sections. Only the initial step of the induction needs an extra argument, which is displayed in Lemma 6.5 below.
For a given set of complex vector spaces $A_1, \ldots, A_n$, with $n \geq 3$ and $\dim A_i \geq 2$, let us give the general:

**Definition 6.1.** A Segre product $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is called $(k, p_1, \ldots, p_n)$-not weakly defective if:

- for $k$ general points $x_1, \ldots, x_k \in X$,
- for $p_i$ general points $w_{ij} \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_i) \times \cdots \times \mathbb{P}(A_n)$,

the span of the spaces $T_{x_i}X$, $A_i \otimes w_{ij}$ contains $T_xX$ if and only if $x = x_i$, for some $i = 1, \ldots, k$. Otherwise $X$ is called $(k, p_1, \ldots, p_n)$-weakly defective.

**Remark 6.2.** (a) With the previous notation, by semicontinuity it is clear that when $X$ is $(k, p_1, \ldots, p_n)$-not weakly defective, then it is also $(k', p'_1, \ldots, p'_n)$-not weakly defective, whenever $(k', p'_1, \ldots, p'_n) \leq (k, p_1, \ldots, p_n)$, in the strict ordering.

(b) By semicontinuity, $X$ is $(k, p_1, \ldots, p_n)$-not weakly defective whenever one gets that for particular sets of points $\{x_i\}$, $\{w_{ij}\}$, as above, then the span of $T_{x_i}X$ and all $A_i \otimes w_{ij}$ contains $T_xX$ if only if $x = x_i$, for some $i = 1, \ldots, k$.

(c) By Proposition 2.3, one gets soon that $(k, 0, \ldots, 0)$-not weakly defective implies $k$-identifiable.

**Lemma 6.3.** Consider $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and assume that, for a choice of subspaces $A'_i \subset A_i$, the product $\mathbb{P}(A'_1) \times \cdots \times \mathbb{P}(A'_n)$ is $(k, p_1, \ldots, p_n)$-not weakly defective. Then $X$ is $(k, p_1, \ldots, p_n)$-not weakly defective.

The inductive criterion can be rephrased as follows, always following the lines in [AOP].

**Proposition 6.4.** Inductive Step Split the vector space $A_i$ in the sum of two spaces $A'_i$ and $A''_i$. Let $X' = \mathbb{P}(A_1) \times \cdots \times (\mathbb{P}(A'_1) \times \cdots \times \mathbb{P}(A_n))$, $X'' = \mathbb{P}(A_1) \times \cdots \times (\mathbb{P}(A''_1) \times \cdots \times \mathbb{P}(A_n))$.

Assume that the product $X'$ is $(k_1, p'_1, \ldots, p_i + k_2, \ldots, p'_n)$-not weakly defective and the product $X''$ is $(k_2, p'_1, \ldots, p_i + k_1, \ldots, p'_n)$-not weakly defective. Then, setting $p_j = p'_j + p''_j$ for $j \neq i$, we get that $X$ is $(k_1 + k_2, p_1, \ldots, p_i, \ldots, p_n)$-not weakly defective.

Now we use again the previous criterion, when the dimension of the vector spaces are powers of 2, i.e. when $\dim(A_i) = 2^{\alpha_i}$, for all $i$. We agree to order the spaces, so that

$$\alpha_1 \leq \cdots \leq \alpha_n.$$

The following numerical criterion is the exact generalization of Lemmas 4.3 and 4.5.

**Lemma 6.5.** Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = 2^{\alpha_i} \geq 2$. Pick non-negative integers $u_1, \ldots, u_n$ such that, for all $i$:

$$u_i \leq \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_n - (n - 1).$$

Then $X$ is $(0, 2^{u_1}, \ldots, 2^{u_n})$-not weakly defective and $(1, 2^{u_1} - 1, 2^{u_2} - 1, 2^{u_3} - 1)$-not weakly defective.

**Proof.** The proof goes by induction. For the inductive step, one can follow the proof of Lemmas 4.4 and 4.5 rephrased for products of many vector spaces. Thus we only need to check the starting points of the induction, namely that $Y_n = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is $(1, 0, \ldots, 0)$-not weakly defective and $(0, 1, \ldots, 1)$-not weakly defective.

The first fact follows soon, as $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is smooth, so that the general tangent hyperplane is not bitangent.
The second fact follows by induction on $n$. Namely it is true for $n = 3$, as observed in Lemma 4.5. For general $n$, write $Y_n = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $\dim(A_i) = 2$, and split $A_1$ in a direct sum of two 1-dimensional spaces $A', A''$. Using Lemma 6.3 one has thus to prove that $Y_{n-1} = \mathbb{P}^0 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is $(0, 1, 0, \ldots, 0)$-not weakly defective and $(0, 0, 1, \ldots, 1)$-not weakly defective. The former claim is obvious. The latter follows by induction. □

We get:

**Proposition 6.6.** Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = 2$ and $\alpha_i \geq 2$. Order the $\alpha_i$’s so that $\alpha_1 \leq \cdots \leq \alpha_n$. Then $X$ is not $k$-weakly defective, for $k \leq 2^{\alpha_1 + \cdots + \alpha_{n-1} - (n-1)}$.

It follows that:

**Theorem 6.7.** Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = a_i \geq 2$ and, for all $i$, take $\alpha_i$ maximal, such that $a_i \geq 2^{\alpha_i}$. Then $X$ is $k$-identifiable, for

$$k \leq 2^{\alpha_1 + \cdots + \alpha_{n-1} - (n-1)}.$$

Comparing our result with the maximal $k$ for which the identifiability of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ makes sense, which, in the case of a product of many factors, reads as:

$$k_{\text{max}} = \left\lfloor \frac{\prod_{i=1}^{n-1} a_i}{1 + \sum_{i=1}^{n-1} a_i - (n-1)} \right\rfloor$$

we see again that the bound in the Theorem is log-asymptotically sharp.

The inequality of the theorem can be written as

$$k \leq 2^{\sum_{i=1}^{n-1} \lfloor \log_2 a_i - 1 \rfloor}$$

Since $2^{\alpha_i} \geq \frac{a_i + 1}{2}$ we get the general tensor of rank $k$ is $k$-identifiable if

$$k \leq \frac{\prod_{i=1}^{n-1} (a_i + 1)}{2^{n-2}}$$

In [SB] Kruskal bound was extended to the case of $n$ factors. A sufficient condition for the $k$-identifiability of the general tensor of rank $k$ is

$$2k + n - 1 \leq \sum_{i=1}^{n} \min(k, a_i)$$

To compare with our condition, in the hypercubic case where $a_i = a$, the bound in [SB] is

$$k \leq \frac{n(a - 1) + 1}{2}$$

while our bound is

$$k \leq 2^{(n-1)(\lfloor \log_2 a - 1 \rfloor)}$$

For $a \geq 4$ we get also the weaker, but more handy, inequality

$$k \leq \left(\frac{a + 1}{4}\right)^{n-1}$$
Example 6.8. Instead of giving the proofs, which, we repeat, are analogue to the proofs of the statement of section 4, let us see how the reduction works in a concrete example.

Take $A_1 = \cdots = A_5 = C^{16}$ and consider $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_5)$. We want to prove that $X$ is $k$-not weakly defective for $k = 2^{4+4+4+4-4} = 4096$.

The reduction step starts as in the following table:

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $k$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|-------|-------|-------|-------|-------|-----|-------|-------|-------|-------|-------|
| 16    | 16    | 16    | 16    | 16    | 4096| 0     | 0     | 0     | 0     | 0     |
| 8     | 16    | 16    | 16    | 16    | 2048| 0     | 0     | 0     | 0     | 0     |
| 8     | 8     | 16    | 16    | 16    | 1024| 0     | 0     | 0     | 0     | 0     |
| 8     | 8     | 8     | 16    | 16    | 512 | 0     | 0     | 0     | 0     | 0     |
| 8     | 8     | 8     | 8     | 16    | 256 | 0     | 0     | 0     | 0     | 0     |
| 8     | 8     | 8     | 8     | 8     | 128 | 0     | 0     | 0     | 0     | 0     |
| 4     | 8     | 8     | 8     | 8     | 64  | 0     | 0     | 0     | 0     | 0     |
| 4     | 4     | 8     | 8     | 8     | 32  | 0     | 0     | 0     | 0     | 0     |
| 4     | 4     | 4     | 8     | 8     | 16  | 0     | 0     | 0     | 0     | 0     |
| 4     | 4     | 4     | 4     | 8     | 8   | 0     | 0     | 0     | 0     | 0     |
| 4     | 4     | 4     | 4     | 4     | 4   | 0     | 0     | 0     | 0     | 0     |
| 2     | 4     | 4     | 4     | 4     | 2   | 0     | 0     | 0     | 0     | 0     |
| 2     | 2     | 4     | 4     | 4     | 1   | 0     | 0     | 0     | 0     | 0     |

Then use Lemma 6.5 with $u_1 = u_2 = 3$, $u_3 = u_4 = u_5 = 2$.

Remark 6.9. As in the case of triple Segre products, in principle, there are no obstructions in repeating the argument, when we substitute powers of 2 with powers of 3 (see the proof of Theorem 1.2 we gave in case $a = 9$), or any other integer $p > 1$.

For some numerical cases, the bound for identifiability that we get using powers of numbers bigger than two, can be closer to the maximal value $k_{\text{max}}$.

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