ON THE CUBIC DIRAC EQUATION WITH POTENTIAL AND THE LOCHAK–MAJORANA CONDITION

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Abstract. We study a cubic Dirac equation on \( \mathbb{R} \times \mathbb{R}^3 \)

\[
i \partial_t u + D u + V(x) u = (\beta u, u) \beta u,
\]

perturbed by a large potential with almost critical regularity. We prove global existence and
scattering for small initial data in \( H^1 \) with additional angular regularity. The main tool is an
endpoint Strichartz estimate for the perturbed Dirac flow. In particular, the result covers the
case of spherically symmetric data with small \( H^1 \) norm.

When the potential \( V \) has a suitable structure, we prove global existence and scattering for
large initial data having a small chiral component, related to the Lochak–Majorana condition.

1. Introduction

We consider the Cauchy problem for a cubic Dirac equation with potential

\[
i \partial_t u + D u + V(x) u = (\beta u, u) \beta u, \quad u(0, x) = u_0(x).
\]

in an unknown function \( u = u(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4 \), with initial data \( u_0 : \mathbb{R}^3 \to \mathbb{C}^4 \). Here \( \langle \cdot, \cdot \rangle \) is
the \( \mathbb{C}^4 \) inner product, \( D \) is the Dirac operator defined by

\[
D = i^{-1} \sum_{j=1}^{4} \alpha_j \partial_j = i^{-1} \alpha \cdot \partial, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3)
\]

where \( \partial = (\partial_1, \partial_2, \partial_3) \) are the partial derivatives, and \( \beta, \alpha_j \) are the Dirac matrices

\[
\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

We recall the basic anticommuting relations

\[
\alpha_j^* = \alpha_j, \quad \beta^* = \beta, \quad \beta^2 = I_4, \quad \beta \alpha_j + \alpha_j \beta = 0 \quad \text{for} \ j = 1, 2, 3,
\]

\[
\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk} I_4 \quad \text{for} \ j, k = 1, 2, 3,
\]

where \( M^* \) is the conjugate transpose of the matrix \( M \), \( \delta_{jk} \) the Kronecker delta and \( I_4 \) the \( 4 \times 4 \)
identity matrix.

Concerning the potential \( V(x) : \mathbb{R}^3 \to M_4(\mathbb{C}) \), we decompose it in the form

\[
V = \sum_{j=1}^{3} A_j(x) \alpha_j + A_0(x) \beta + V_0(x) = A \cdot \alpha + A_0 \beta + V_0
\]

where the magnetic potential \( A \), the pseudoscalar potential \( A_0 \) and \( V_0 \) are such that

\[
A = (A_1, A_2, A_3) : \mathbb{R}^3 \to \mathbb{R}^3, \quad A_0 : \mathbb{R}^3 \to \mathbb{R}, \quad V_0 = V_0^* : \mathbb{R}^3 \to M_4(\mathbb{C}).
\]

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The magnetic field associated to the potential \( A \) will be denoted by
\[
B = [B_{jk}]_{j,k=1}^3, \quad B_{jk} = \partial_j A_k - \partial_k A_j, \quad j, k = 1, 2, 3. \tag{1.3}
\]

The first goal of the paper is to study the dispersive properties of the Dirac flow perturbed by a large potential, and to prove several smoothing and (endpoint) Strichartz estimates for it. We then apply the estimates to prove the global existence of small solutions for the nonlinear equation (1.1), for \( H^1 \) initial data with additional angular regularity, in the spirit of [18] and [7]. Moreover, if the potential has an additional structure, we are able to reduce the smallness assumption to smallness of the chiral component of the initial data; to this end we exploit the Lochak–Majorana condition.

A crucial but natural assumption concerns the absence of a resonance at 0 for the operator \( \mathcal{D} + V \). It is well known that in presence of a resonance the dispersive properties of the flow deteriorate. For the Dirac equation with potential, the natural notion is the following:

**Definition 1.1 (Resonance at 0)** We say that 0 is a resonance for the operator \( \mathcal{D} + V \) if there exists \( v \in H^1_{loc}(\mathbb{R}^3 \setminus 0) \) solution of \((\mathcal{D} + V)v = 0\) such that \(|x|^{-\frac{7}{3}}\sigma v \in L^2\) for all \( \sigma \) in a right nbd of 0; \( v \) is called a resonant state.

In order to state the results we introduce the dyadic norms
\[
\|v\|_{\ell^p L^q} := \left( \sum_{j \in \mathbb{Z}} \|v\|_{L^p(2^j \leq |x| < 2^{j+1})}^p \right)^{1/p}, \tag{1.4}
\]
with obvious modification when \( p = \infty \). More generally, we denote the mixed radial–angular \( L^q L^r \) norms on a spherical ring \( C = \{ R_1 \leq |x| \leq R_2 \} \) with
\[
\|v\|_{L^q L^r(C)} = \|v\|_{L^q L^r(C)} := (\int_{R_1}^{R_2} (\int_{|x|=\rho} |v|^q dS)^{1/q} \rho^{d-1} d\rho)^{1/q}.
\]
and we define for all \( p, q, r \) in \([1, \infty]\)
\[
\|v\|_{\ell^p L^q L^r(C)} := \left\{ \|v\|_{L^q L^r(2^j \leq |x| < 2^{j+1})} \right\}_{j \in \mathbb{Z}} \|v\|_{\ell^p L^q L^r}.
\tag{1.5}
\]
Clearly, when \( q = r \) we have simply \( \|v\|_{\ell^p L^q L^r} = \|v\|_{\ell^p L^q} \). In the following we shall also need mixed space–time norms, and to avoid confusion we shall always write \( L^q_t \) with an explicit index \( t \), the \( L^p \) norms with respect to time variable. Thus we write
\[
\|u\|_{L^q_t L^r_x} := \|u\|_{L^q_t L^r_x} := \left( \int \|u(t, x)|_{L^q_t L^r_x}^p dt \right)^{1/p}, \tag{1.6}
\]
Angular regularity will be expressed via fractional powers of the Laplace–Beltrami operator on the sphere \( S^2 \)
\[
\Lambda_\alpha := (1 - \Delta_{S^2})^{-\alpha/2}.
\]

We shall impose several decay and smoothness conditions on the potential \( V \). The minimal set of assumptions is the following.

**Condition (V)** The operator \( \mathcal{D} + V \) is selfadjoint on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) with domain \( H^1(\mathbb{R}^3) \), 0 is not an resonance, and \( V \) satisfies (see (1.2), (1.3)) \( A \in \ell^\infty L^3 \), and
\[
|||x||_{V_0}||_{L^\infty} < \sigma \tag{1.6'}
\]
\[
|V|^2 + |\nabla V| \lesssim |x|^{-2-\delta}, \quad |x||H| + |x|^2(|V|^2 + |\nabla V| + |V_0|) \in \ell^4 L^\infty, \tag{1.7}
\]
for some \( \delta > d \) (recall (1.3)).

The decay properties of the flow \( e^{it(\mathcal{D}+V)} \) are summarized in Theorem 1.2 below. For the following statement, we fix a radially symmetric weight function \( \rho \in \ell^2 L^\infty \) such that \( \rho^{-2} |x| \) is in the Muckenhoupt class \( A_2 \) on \( \mathbb{R}^3 \); possible explicit examples for \( \rho \) are
\[
\rho = |x|^{\gamma} \text{ for } |x| \leq 1 \quad \rho = |x|^{-\epsilon} \text{ for } |x| \geq 1
\]

\[\rho = |x|^\gamma \text{ for } |x| 
\]
for some $c > 0$ small, or also
\[ \rho = (\log |x|)^{-c} \]
for some $\nu > 1/2$. Recall that a locally integrable function $w > 0$ is in $A_2$ if its averages over arbitrary balls $B$ satisfy $\int_B w \cdot |f|^2 \leq C$.

**Theorem 1.2** (Linear decay estimates).

(i) *(Smoothing estimate)* If Condition (V) holds with $\sigma$ small enough, then
\[ \| \rho|x|^{-1/2} e^{it(\mathcal{D} + V)} f \|_{L^2_z L^2_t} \lesssim \| f \|_{L^2}. \]  
(ii) *(Endpoint Strichartz estimate)* If Condition (V) holds with $\sigma$ small enough, and in addition $\rho^{-2}|x|(|V| + |\partial V|) \in L^\infty$, then
\[ \| e^{it(\mathcal{D} + V)} f \|_{L^2_z L^\infty_t L^2_x} \lesssim \| f \|_{H^1}. \]  
(iii) *(Estimates with angular regularity)* Let $V$ be of the form $V = A_0 + V_0$, satisfying Condition (V). Assume in addition that for some $1 < s \leq 2$
\[ \rho^{-2}|x||\Lambda_s V_0(|x|\omega)\|_{L^2(S^2)} \leq \sigma, \quad \rho^{-2}|x||\Lambda_s \partial V(|x|\omega)\|_{L^2(S^2)} \in L^\infty, \]  
\[ \rho^{-2}|x||x \wedge \partial A_0(|x|\omega)\|_{L^\infty(S^2)} + \rho^{-2}|x||\Delta_\omega A_0(|x|\omega)\|_{L^2(S^2)} \in L^\infty \]
with $\sigma$ small enough. Then we have
\[ \| \Lambda_s^* e^{it(\mathcal{D} + V)} f \|_{L^2_z L^\infty_t L^2_x} + \| \Lambda_s^* e^{it(\mathcal{D} + V)} f \|_{L^2_t H^1_x} \lesssim \| \Lambda_s^* f \|_{H^1}. \]

Note that the estimates in (iii) require smallness of the magnetic potential $A$ (so that $A$ can be absorbed in $V_0$). On the other hand, the pseudoscalar potential $A_0$ can still be large. Note also that (1.11) is trivially satisfied if $A_0$ is a radially symmetric function. The estimates are proved, besides several others, in Theorem 2.1, Corollary 2.2 and Theorem 3.2 of Sections 2–3.

As an application of the previous estimates, we prove the global existence and scattering for initial data small in the $\Lambda^{-s}_{-1} H^1$ norm. In particular, the result applies to all spherically symmetric data with small $H^1$ norm. For simplicity we restrict ourselves to the standard nonlinearity (1.1), but it is clear that the same proof applies to more general cubic nonlinearities.

**Theorem 1.3** (Global existence, small data). If $V = A_0 + V_0$ satisfies the assumptions of Theorem 1.2–(iii), then there exists $\epsilon_0 > 0$ such that, for any initial data $u_0$ with $\| \Lambda_s^* u_0 \|_{H^1} \leq \epsilon_0$, Problem (1.1) has a unique global solution $u \in CH^1 \cap L^2 L^\infty$ with $\Lambda_s^* u \in L^\infty H^1$. Moreover $u$ scatters to a free solution, i.e., there exists $u_+ \in \Lambda^{-s}_{-1} H^1$ such that
\[ \lim_{t \to -\infty} \| \Lambda_s^* u(t) - \Lambda_s^* e^{it(\mathcal{D} + V)} u_+ \|_{H^1} = 0. \]
A similar result holds for $t \to +\infty$.

In our last result we construct a family of large global solutions to Equation (1.1), related to the so called *Lochak–Majorana condition* (see [17], [3]). To define the condition we introduce the subspace $E$ of $C^4$ defined by
\[ E := \{ z \in C^4 : z_1 = \overline{z}_4, z_2 = -\overline{z}_3 \} = \{ z \in C^4 : \gamma z = \overline{z} \}, \]  
where $\gamma$ is the matrix
\[ \gamma := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

Then we have:
Definition 1.4 (LM condition). We say that a function $f(x) \in L^2(\mathbb{R}^3; \mathbb{C}^4)$ satisfies the Lochak–Majorana condition if

$$f(x) \in E \quad \text{for a.e. } x$$

(or more generally, if $\exists \theta \in \mathbb{R}$ such that $e^{i\theta} f \in E$ for a.e. $x$.)

A few elementary facts will clarify the relevance of this definition:

- The LM condition is preserved by the free Dirac flow:
  $$\text{if } f \in E \text{ a.e.}, \text{ then } e^{i\theta} f \in E \text{ a.e. for all } t.$$  

- A function $f$ satisfies LM iff its chiral invariant $\rho(f)$ vanishes. The chiral invariant is the quantity
  $$\rho(f) := |\langle \beta f, f \rangle|^2 + |\langle \alpha_5 f, f \rangle|^2, \quad \alpha_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$  

- As a consequence of the previous two facts, if the initial data $f$ satisfy LM, then the free flow $e^{i\theta} f$ is also a solution of the cubic NLD
  $$iu_t + \mathcal{D} u = \langle \beta u, u \rangle \beta u \quad (\equiv 0).$$

Then a natural conjecture is that small perturbations of initial data satisfying LM give rise to global large solution of the cubic Dirac equation. This is indeed the case, as proved by Bachelot [3] for small perturbations in the $H^6$ norm. If we introduce the projection $P : \mathbb{C}^4 \to E$ given by

$$P \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 + z_4 \\ z_2 - z_3 \\ z_3 - z_2 \\ z_1 + z_4 \end{pmatrix}$$

(1.15)

then Bachelot’s condition on the initial data can be written simply

$$\|(I - P)f\|_{H^6} \ll 1.$$  

We shall prove that a similar situation occurs also in presence of a potential $V$, provided $V$ has a suitable structure. Denote by $\mathcal{V}$ the subspace of $4 \times 4$ complex matrices $M \in M_4(\mathbb{C})$ of the form

$$M = \begin{pmatrix} a & z & w & 0 \\ \bar{z} & b & 0 & w \\ \bar{w} & 0 & -b & z \\ 0 & \bar{w} & \bar{z} & -a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$ and $z, w \in \mathbb{C}$. The space $\mathcal{V}$ can be characterized in the following equivalent way:

$$M \in \mathcal{V} \iff M = M^* \quad \text{and} \quad \overline{M} \gamma = -\gamma M,$$

(1.16)

where $\gamma$ is defined in (1.14). Note that the Dirac matrix $\beta$ belongs to $\mathcal{V}$, thus if in the decomposition (1.2) we assume $A_1 = A_2 = A_3 = 0$ and $V_0(x) \in \mathcal{V}$, we have $V(x) \in \mathcal{V}$ for all $x$.

We are in position to state our final result:

Theorem 1.5 (Global existence, large data). Assume $V = A_0 \beta + V_0$ satisfies the conditions of Theorem 1.2–(iii) and in addition $V_0(x) \in \mathcal{V}$ for all $x$.

Then there exists $\epsilon_0 > 0$ such that, for any data $u_0 \in \Lambda_{-5}^* H^1$ with $\|(I - P)\Lambda_{+}^* u_0\|_{H^1} \leq \epsilon_0$, Problem (1.1) has a unique global solution $u \in CH^1 \cap L^2L^\infty$ with $\Lambda_{+}^* u \in L^\infty H^1$; moreover $u$ scatters to a free solution, i.e., there exists $u_+ \in \Lambda_{-5}^* H^1$ such that

$$\lim_{t \to \infty} \|\Lambda_{+}^* u(t) - \Lambda_{+}^* e^{it(D+V)} u_+\|_{H^1} = 0.$$  

A similar result holds for $t \to -\infty$.  

Since \( u_0 \) is not small, the Theorem implies the existence of global solutions and scattering for a suitable class of large data. Note that the result depends heavily on the special structure of the nonlinearity. Indeed, if we replace the nonlinear term \((\beta u, u)\beta u\) with \(|u|^3 I_4\), it is possible to construct data such that \( Pu_0 = 0 \) and the solution blows up in a finite time, even in the case \( V(x) = 0 \) (see [13]). Note also that the static potential \( A_0 \) can be large.

There are many results for the cubic Dirac equation when \( V(x) \) is the constant matrix \( m \beta \), \( m \geq 0 \) (see [9, 20, 14, 18, 4, 5, 6] and references therein). In particular, Machihara et al. [18] proved small data scattering in \( H^1(\mathbb{R}^3) \) with some additional regularity in the angular variables; our paper is in part an extension of theirs, and of [7], to the case of a large potential depending on \( x \). Note that in the massless case \( H^1(\mathbb{R}^3) \) is the critical space for scaling. The final results on the constant coefficient case are due to Bejenaru and Herr [4] and Bournaveas and Candy [6], who proved small data scattering in \( H^1 \).

Global existence for large data is a much more difficult problem, in part since the conserved Dirac energy is not positive definite. In the one dimensional case, Candy [9] proved the global well-posedness by using the conservation of the \( L^2 \) mass only. In the higher dimensional case, one does not expect local well posedness in time for \( L^2 \) data, since the critical norm is stronger.

As mentioned above, Bachelot [3] showed global existence of large amplitude solutions, by assuming smallness only for the Chiral invariant related to the Lochak-Majorana condition; taking \( V = 0 \) in Theorem 1.5 we reobtain his result and actually improve on his \( H^0 \) condition on the initial data. Indeed, the main tool in [3] was the commutating vector field method, which requires rather high regularity of the data to be applied. We finally recall that in [7] a result similar to Theorem 1.3 was proved, but only for a \( \text{small} \) potentials \( V \).

The outline of the paper is the following. Sections 2 and 3 are devoted to dispersive estimates for the linear flow. In Section 4 we prove global existence for small data, Theorem 1.3. In Section 5 we check that the chiral invariant is preserved by the perturbed flow if the potential has the appropriate structure, and we apply this result to prove global existence of large solutions, Theorem 1.5, in the concluding Section 6.

2. Smoothing estimates for the perturbed Dirac system

We prove here a smoothing estimate for the operator

\[
\mathcal{D} + V, \quad V = A \cdot \alpha + \beta V_0 + V_0
\]

where \( A = (A_1, A_2, A_3) : \mathbb{R}^3 \to \mathbb{R}^3, A_0 : \mathbb{R}^3 \to \mathbb{R} \) and \( V_0 = V_0^* : \mathbb{R}^3 \to M_4(\mathbb{C}) \). The relevant spaces are the Banach spaces \( X, \dot{Y}, \dot{Y}^* \) with norms

\[
\| v \|_X^2 := \sup_{R > 0} \frac{1}{R^4} \int_{|x| = R} |v|^2 dS \simeq \| x^{-1} v \|_{L^\infty L^2},
\]

\[
\| v \|_{\dot{Y}}^2 := \sup_{R > 0} \frac{1}{R} \int_{|x| \leq R} |v|^2 dx \simeq \| x^{-1/2} v \|_{L^\infty L^2}, \quad \| v \|_{\dot{Y}^*} \simeq \| x^{1/2} v \|_{L^2}.
\]

Note that \( \dot{Y}^* \) is the predual of \( \dot{Y} \) and an homogeneous version of the Agmon–Hörmander space \( B \) (see [2]). In the following statement, \( B_{jk} = \partial_j A_k - \partial_k A_j \).

**Theorem 2.1** (Smoothing estimates for Dirac). Assume Condition (V) is satisfied with \( \sigma \) small enough. Then the perturbed flow \( e^{it(\mathcal{D} + V)} \) satisfies: for any \( \rho \in \ell^2 L^\infty \),

\[
\| \rho |x|^{-1/2} e^{it(\mathcal{D} + V)} f \|_{L^2_x L^2_t} \lesssim \| \rho \|_{\ell^2 L^\infty} \| f \|_{L^2},
\]

\[
\| \rho |x|^{-1/2} \int_0^t e^{i(t-s)(\mathcal{D} + V)} F(s) ds \|_{L^2_x L^2_t} \lesssim \| \rho \|_{\ell^2 L^\infty} \| \rho^{-1} |x|^{1/2} F \|_{L^2_x L^2_t}.
\]

If in addition \( V \) satisfies

\[
\rho^{-2} |x| |\partial V| \leq K < \infty
\]
then we have also the estimate
\[ \| \rho|x|^{-1/2} \partial t e^{i(t(D + V)} f \|_{L_t^2 L_x^2} \lesssim \| \rho \|_{L^\infty} (1 + \| \rho \|_{L^\infty} K) \| f \|_{H^1}. \] (2.5)

Note that the additional condition (2.4) is implied by Condition (V) for large \( x \) and it only restricts the singularity of \( V \) near 0.

The proof of the Theorem is based on a resolvent estimate for the squared operator \( (D + V)^2 \). This produces a system of stationary Schrödinger equations with diagonal principal part, as detailed in the following sections. Two different methods are necessary in order to handle the large frequency and the short frequency regimes.

For the next result we need to assume that the magnetic potential \( A = (A_1, A_2, A_3) \) is small, while the scalar potential \( A_0 \) may still be large. By absorbing \( A \cdot \alpha \) in the term \( V_0 \), we see that it is sufficient to consider a potential \( V \) of the form
\[ V(x) = A_0 \beta + V_0. \]

**Corollary 2.2.** Assume \( V \) and \( D + V \) satisfy the conditions of the previous Theorem with \( V \) of the special form
\[ V(x) = A_0 \beta + V_0. \]

In addition, assume that for some \( s \in (1, 2] \) and some \( \rho \in \ell^2 L^\infty \)
\[ \rho^{-2} |x| \| \Lambda^s V_0(|x|) \|_{L_x^2 (\mathbb{R}^3)} \leq \epsilon, \quad \rho^{-2} |x| \| \Lambda^s \partial V(|x|) \|_{L_x^2 (\mathbb{R}^3)} \in L^\infty. \] (2.6)
\[ \rho^{-2} |x| \| \Lambda^s \partial A_0(|x|) \|_{L_x^2 (\mathbb{R}^3)} + \rho^{-2} |x| \| \Delta A_0(|x|) \|_{L_x^2 (\mathbb{R}^3)} \in L^\infty \] (2.7)

Then if \( \epsilon \) is sufficiently small, the following estimates hold:
\[ \| \rho|x|^{-1/2} \Lambda^s e^{i(t(D + V)} f \|_{L_t^2 L_x^2} \lesssim \| \Lambda^s f \|_{L^2}, \] (2.8)
\[ \| \rho|x|^{-1/2} \Lambda^s \int_0^t e^{i(t-s)(D + V)} F(s) ds \|_{L_t^2 L_x^2} \lesssim \| \rho^{-1} |x|^{1/2} \Lambda^s F \|_{L_t^2 L_x^2}, \] (2.9)
\[ \| \rho|x|^{-1/2} \partial \Lambda^s e^{i(t(D + V)} f \|_{L_t^2 L_x^2} \lesssim \| \Lambda^s f \|_{H^1}. \] (2.10)

Note that for a radial scalar potential \( A_0 = A_0(|x|) \) assumption (2.7) is trivially satisfied.

### 2.1. Large frequencies.
We consider a 4–dimensional system of stationary Schrödinger equations on \( \mathbb{R}^3 \)
\[ I_4 \Delta_A v + W(x) v + \sum_{j=1}^3 Z_j(x) \partial_j^4 v + z v = f, \quad z \in \mathbb{C} \] (2.11)
where \( v = (v_1, v_2, v_3, v_4), \ W(x), Z_j(x) : \mathbb{R}^3 \to M_4(\mathbb{C}) \) are square 4 \times 4 matrices, \( I_4 \) is the 4–dimensional identity matrix, \( \Delta_A \) the magnetic laplacian on \( \mathbb{R}^3 \)
\[ \Delta_A = \sum_{j=1}^3 (\partial_j + i A_j)^2, \quad \partial_j = \frac{\partial}{\partial x_j}, \]
and \( A(x) = (A_1(x), A_2(x), A_3(x)) \) is a vector of real valued functions. We also use the notations
\[ \partial_j^A = \partial_j + i A_j(x), \quad \partial = (\partial_1, \partial_2, \partial_3), \quad \partial^A = (\partial_1^A, \partial_2^A, \partial_3^A) \]
and, writing \( \hat{x}_j = \frac{x_j}{|x|} \) and \( \hat{x} = \frac{x}{|x|}, \)
\[ B_{jk} = \partial_j A_k - \partial_k A_j, \quad \hat{B}_j = B_{jk} \hat{x}_k, \quad \hat{B} = (\hat{B}_1, \hat{B}_2, \hat{B}_3). \]

Here and in the following we use the convention of implicit summation over repeated indices.

We begin by studying the case of large frequency \(|\Re z| \gg 1\). In this regime we use a direct approach, via the Morawetz multiplier approach.
Proposition 2.3 (Resolvent estimate for large frequencies). There exists a constant $\sigma_0$ such that the following holds.

Let $W_L(x), W_S(x), Z_j(x) : \mathbb{R}^3 \to M_4(\mathbb{C})$ be square $4 \times 4$ matrices, let $W = W_L + W_S$, and let $v, f : \mathbb{R}^3 \to \mathbb{C}^4$ satisfy (2.11). Assume that $|3z| \leq 1$ and

$$
\|[x]^{3/2}W_S\|_{L^2 L^\infty} + \|[x]Z\|_{L^\infty} \leq \sigma_0, \quad |\Re z| \geq \sigma_0^{-1} \left[ \|[x]B\|_{L^2 L^\infty} + \||x|W_L\|_{L^2 L^\infty} \right] + 2. \tag{2.12}
$$

Then the following estimate holds

$$
\|v\|_X^2 + |z|\|v\|_Y^2 + \|\partial^A v\|_Y^2 \lesssim \|f\|_Y^2. \tag{2.13}
$$

Remark 2.4. Under a weak additional assumption on $A$, the norm $\|\partial^A v\|_Y$ in (2.13) can be replaced by $\|\partial v\|_Y$, thanks to the following

Lemma 2.5. Assume $A \in L^\infty L^3$. Then the following estimate holds

$$
\|\partial v\|_Y \lesssim (1 + \|A\|_{L^\infty L^3}) \left[ \|\partial^A v\|_Y + \|v\|_X \right] \tag{2.14}
$$

with an implicit constant independent of $A$.

Proof. Let $C_j$ be the spherical shell $2^j \leq |x| \leq 2^{j+1}$ and $\tilde{C}_j = C_{j-1} \cup C_j \cup C_{j+1}$. Let $\phi$ be a nonnegative cutoff function equal to 1 on $C_j$ and vanishing outside $\tilde{C}_j$, and let $\phi_j(x) = \phi(2^{-j}x)$. Then we can write

$$
\|\partial v\|_{L^2(C_j)} \leq \|\phi_j \partial v\|_{L^2} \leq \|\phi_j \partial^A v\|_{L^2} + \|\phi_j A v\|_{L^2}
$$

By Hölder’s inequality and Sobolev embedding we have

$$
\|\phi_j A v\|_{L^2} \leq \|A\|_{L^3(C_j)} \|\phi_j v\|_{L^6} \lesssim \|A\|_{L^\infty L^3} \|\phi_j v\|_{L^6} \lesssim \|A\|_{L^\infty L^2} \|\partial (\phi_j v)\|_{L^2}.
$$

We expand the last term as

$$
\|\partial (\phi_j v)\|_{L^2} \leq \|\partial (\phi_j) v\|_{L^2} + \|\phi_j \partial (v)\|_{L^2}.
$$

We note that $|\partial \phi_j| \lesssim 2^{-j}$ and we recall the pointwise diamagnetic inequality

$$
|\partial v| \leq |\partial^A v|
$$

valid since $A \in L^2_{loc}$. Then we can write

$$
\|\partial v\|_{L^2(C_j)} \lesssim 2^{-j} \|\partial (\phi_j v)\|_{L^2(\tilde{C}_j)} + \|\partial^A v\|_{L^2(\tilde{C}_j)} \lesssim 2^{-j/2} \|\partial (\phi_j v)\|_{L^2(\tilde{C}_j)} + \|\partial^A v\|_{L^2(\tilde{C}_j)}.
$$

Summing up, we have proved

$$
\|\partial v\|_{L^2(C_j)} \lesssim (1 + \|A\|_{L^\infty L^3}) \left[ \|\partial^A v\|_{L^2(\tilde{C}_j)} + 2^{-j/2} \|\partial (\phi_j v)\|_{L^2(\tilde{C}_j)} \right].
$$

Multiplying both sides by $2^{-j/2}$ and taking the sup in $j \in \mathbb{Z}$ we get the claim. \(\square\)

2.2. Large frequencies: formal identities. In the course of the proof we shall reserve the symbols

$$
\lambda = \Re z, \quad \epsilon = \Im z
$$

for the components of the frequency $z = \lambda + i\epsilon$ in (2.11).

The main tools are a few Morawetz type identities, based on the two multipliers

$$
\overline{\Delta A, \psi w} = (\Delta \psi)\overline{w} + 2\partial \psi \cdot \partial^A w \quad \text{and} \quad \phi \overline{w}
$$

where $\phi(x), \psi(x)$ are real valued, spherically symmetric weight functions to be chosen in the following, and $w \in H^2_{loc}(\mathbb{R}^3)$ is complex valued. Define, with $c(x)$ a complex valued function,

$$
Q_j := \partial^A w [\Delta A, \psi] w - \frac{1}{2} \partial_j \Delta \psi |w|^2 - \partial_j \psi [c(x)|w|^2 + |\partial^A w|^2]
$$
and

\[ P_j := \partial_j^4 w \overline{\psi} - \frac{i}{2} \partial_j \phi |w|^2 \]

Then the following identities hold

\[
\Re \partial_j Q_j = \Re (\Delta_A w - cw) \overline{\Delta_A \psi_1} |w|^2 - \frac{1}{2} \Delta^2 \psi_1 |w|^2 + 2 \partial_j^4 w (\partial_j \partial_k \phi) \overline{\partial_k^4 w} \\
- \Re (\partial_j \psi \partial_j \phi) |w|^2 + 2 \Im (\overline{\psi} B_{jk} \partial_j^4 w \partial_k \phi) - 2(\Im \Delta (w \partial_j \psi \partial_j^4 w))
\]  

(2.15)

and

\[
\partial_j P_j = \overline{\psi} \Delta_A w \phi + |\partial^4 w|^2 \phi - \frac{i}{2} \Delta \phi |w|^2 + 4 \Im (w \partial_j \phi \partial_j^4 w) \]

(2.16)

These Morawetz type identities are well known (see e.g. [8] for the form used here), and are not difficult to check directly by expanding the derivatives of \(Q_j, P_j\) at the left hand side and keeping track of the resulting terms.

We need to apply the previous identities to a 4–tuple of functions \(\nu = (\nu_\alpha)_{\alpha=1}^4\). We shall use the notation \(|\nu|^2 = |\nu_1|^2 + \cdots + |\nu_4|^2\) and follow the convention of implicit summation over repeated index \(\alpha = 1, \ldots, 4\). If we define

\[ g_\alpha := \Delta_A \nu_\alpha + (\lambda + i\epsilon) \nu_\alpha, \quad \alpha = 1, \ldots, 4 \]

and denote by \(Q_\alpha^\alpha, P_\alpha^\alpha\) the quantities \(Q_j, P_j\) with \(w\) replaced by \(\nu_\alpha\), we obtain

\[
\Re \partial_j \{\sum_{\alpha} (Q_\alpha^\alpha + P_\alpha^\alpha)\} = I_{\nu\nu} + I_v + I_B + I_g
\]  

(2.17)

where

\[
I_{\nu\nu} = 2\partial_j^4 \nu_\alpha \Re (\partial_j \partial_k \psi) \overline{\partial_k^4 \nu_\alpha} + \phi |\partial^4 \nu|^2, \quad I_v = -\frac{i}{2} \Delta (\Delta \psi + \phi) |\nu|^2 - \lambda |\phi|^2 \]

\[
I_B = 2 \Im (\overline{\nu} B_{jk} \partial_j^4 \nu_\alpha \partial_k \phi), \quad I_e = 2 \epsilon \Im (\nu_\alpha \partial_j \psi \partial_j^4 \nu_\alpha), \quad I_g = \Re (g_\alpha \overline{\Delta_A \psi_1} \nu_\alpha + g_\alpha \overline{\psi}_1 \phi)
\]

2.3. Large frequencies: preliminary estimates. We begin with a few elementary estimates based on identity (2.16), with different choices of the radial weight \(\psi\). Writing (2.16) with \(\phi = 1\) and taking the imaginary part, we get

\[ \epsilon |\nu|^2 = \Im (g_\alpha \overline{\nu_\alpha}) - \Im \partial_j \{\overline{\nu_\alpha} \partial_j^4 \nu_\alpha\} \]

and after integration on \(\mathbb{R}^3\) we obtain

\[ \epsilon \|\nu\|^2_{L^2} = \Im \int g_\alpha \overline{\nu_\alpha}. \]

(2.18)

(Here and in the following we shall freely use the fact that the boundary term vanish after integration, as it is easy to check.) Taking instead the real part of the same identity (with \(\phi = 1\)) we obtain

\[ |\partial^4 \nu|^2 = \lambda |\nu|^2 - \Re (g_\alpha \overline{\nu_\alpha}) + \Re \partial_j \{\overline{\nu_\alpha} \partial_j^4 \nu_\alpha\} \]

and after integration

\[ \|\partial^4 \nu\|^2_{L^2} = \lambda \|\nu\|^2_{L^2} - \Re \int g_\alpha \overline{\nu_\alpha}. \]

(2.19)

In order to estimate the term \(I_e\) we use (2.18) and (2.19) as follows:

\[ \int I_e \leq 2 \epsilon \|\partial^4 \psi\|_{L^\infty} \|\nu\|_{L^2} \|\partial^4 \nu\|_{L^2} \leq C\epsilon^{1/2} \left( \int |g_\alpha \overline{\nu_\alpha}| \right)^{1/2} \left( \|\nu\|^2_{L^2} + \int |g_\alpha \overline{\nu_\alpha}| \right)^{1/2} \]

with \(C = 2 \|\partial^4 \psi\|_{L^\infty}\), then again by (2.18)

\[ \leq C \left( \int |g_\alpha \overline{\nu_\alpha}| \right)^{1/2} \left( |\lambda| \int |g_\alpha \overline{\nu_\alpha}| + |\epsilon| \int |g_\alpha \overline{\nu_\alpha}| \right)^{1/2} \]

and we arrive at the estimate

\[ \int I_e \leq 2 \|\partial^4 \psi\|_{L^\infty} (|\lambda| + |\epsilon|)^{1/2} \|g_\alpha \overline{\nu_\alpha}\|_{L^1}. \]

(2.20)
Another auxiliary estimate will cover the (easy) case of negative $\lambda = -\lambda_- \leq 0$. Write the real part of identity (2.16) in the form
\[
\lambda_- |v|^2 \phi + |\partial^A v|^2 \phi - \frac{1}{2} \Delta \phi |v|^2 = \sum_a \partial_j \Re P^a_j - \Re (g_\alpha \overline{\varphi}_\alpha) \phi
\]
and choose the radial weight
\[
\phi = \frac{1}{|x|} \implies \phi' = -\frac{1}{|x|^2} 1_{|x| > R}, \quad \phi'' = -\frac{1}{|x|^3} \delta_{|x| = R} + \frac{2}{|x|^2} 1_{|x| > R}.
\]
Note that
\[
-\Delta \phi = \frac{1}{|x|^3} \delta_{|x| = R}.
\]
Integrating over $\mathbb{R}^3$ and taking the supremum over $R > 0$ we obtain the estimate
\[
\lambda_- \|v\|_Y^2 + \|\partial^A v\|_Y^2 + \frac{1}{2} \|v\|_X^2 \leq \|\|2 g_\alpha \overline{\varphi}_\alpha\|_{L^1}.
\]

2.4. Large frequencies: the main terms. In the following we assume $|\epsilon| \leq 1$ and $\lambda \geq 2$. We choose in (2.17), for arbitrary $R > 0$,
\[
\psi = \frac{1}{2R} |x|^2 1_{|x| \leq R} + |x| 1_{|x| > R}, \quad \phi = \frac{1}{R} 1_{|x| \leq R}.
\]
We have then
\[
\psi' = \frac{|x|}{|x| \vee R}, \quad \psi'' = \frac{1}{R} 1_{|x| \leq R}, \quad \Delta \psi + \phi = \frac{2}{|x| \vee R},
\]
\[
\Delta (\Delta \psi + \phi) = -\frac{2}{|x|^3} \delta_{|x| = R}.
\]
This implies
\[
3 \sup_{R > 0} \int I_\psi \geq \|v\|_X^2 + \lambda \|v\|_Y^2.
\]

Next we can write, since $\psi$ is radial,
\[
2 \partial_y^2 v_\alpha (\partial_y \partial_y \psi) \partial_y \psi \overline{\varphi}_\alpha = 2 \psi' [\overline{\varphi} \cdot \partial^A v_\alpha] \cdot \partial_y^2 v_\alpha = 2 \psi' \left[ (\partial^A v_\alpha)^2 - [\overline{\varphi} \cdot \partial^A v_\alpha]^2 \right] \geq \frac{3}{2} 1_{|x| < R \partial^A v_\alpha}.
\]
This implies
\[
\sup_{R > 0} \int I_\psi \geq 2 \|\partial^A v\|_Y^2.
\]

Further we have, since $B_\ell k \partial_y \psi = B_\ell \partial_y \psi = \hat{B} \psi'$,
\[
|I_B| \leq \frac{2|x|}{|x| \vee R} \|v\| \|\partial^A v\| \hat{B} \leq 2 \|v\| \|\partial^A v\| \hat{B}
\]
which implies
\[
\int |I_B| \leq 2 \|\hat{B} v\|_{L^\infty} \|v\|_{L^2} \|v\|_{L^1} = 2 \|\hat{B} v\|_{L^\infty} \|v\|_{L^1}
\]
and by Cauchy–Schwarz, for any $\delta > 0$,
\[
\int |I_B| \leq \delta \|\partial^A v\|_Y^2 + \delta^{-1} \|\hat{B} v\|_Y^2 \|v\|_Y.
\]
Finally, since $|\Delta \psi + \phi| \leq 2 |x|^{-1}$ and $|\partial \psi| \leq 1$, we have
\[
\int |I_\psi| \leq 2 \|\hat{B} v\|_{L^\infty} \|v\|_{L^1} + 2 \|g_\alpha \partial^A v_\alpha\|_{L^1}.
\]

Summing up, by integrating identity (2.17) over $\mathbb{R}^3$ and using estimates (2.20) (2.24), (2.25), (2.26) and (2.27) we obtain (recall that $|\partial \psi| \leq 1$; recall also that $\lambda \geq 2$ and $|\epsilon| \leq 1$ so that $|\epsilon| + |\lambda| \leq \lambda$)
\[
\|v\|_X^2 + \lambda \|v\|_Y^2 + \|\partial^A v\|_Y^2 \leq \delta \|\partial^A v\|_Y^2 + \lambda^{-1} \|\hat{B} v\|_{L^\infty} \|v\|_Y^2 + \lambda^{1/2} \|g_\alpha \overline{\varphi}_\alpha\|_{L^1} + \|v\|_{L^1} \leq \|\partial^A v\|_Y^2 + \|g_\alpha \overline{\varphi}_\alpha\|_{L^1}.
where $\delta > 0$ is arbitrary and the implicit constant is a universal constant depending only on $n, N$. Note now that if $\delta$ is chosen small enough with respect to $n$ and we assume
\[
\lambda \geq c \|x|B\|_{1, L^\infty}^2
\]  
(2.28)
for a suitably large $c$, we can absorb two terms at the right and we get the estimate
\[
\|v\|_{X}^2 + \|\lambda v\|_{Y}^2 + \|\partial^A v\|_{Y}^2 \leq c_0 \left( \|2\alpha v\|_{1, L^1} + \|g_\alpha \partial^A v\|_{1, L^1} + \lambda^{\frac{1}{2}} \|g_\alpha v\|_{L^1} \right)
\]  
(2.29)
where $c_0 \geq 1$ is a universal constant.

2.5. Large frequencies: conclusion. We now define, for $v = (v_\alpha)_{\alpha=1}^4$,
\[
f := I_4(\Delta_A + (\lambda + i\epsilon))v + W(x)v + Z(x) \cdot \partial^A v
\]
where $Z = (Z_1, Z_2, Z_3)$ and $W(x), Z_j(x)$ are $4 \times 4$ matrices. We can apply estimate (2.29) by defining $g = (g_1, \ldots, g_4)$ as
\[
g = f - W(x)v - Z(x) \cdot \partial^A v
\]
We now estimate the terms at the right in (2.29), assuming that $W$ has a (small) short range component and a (large) long range component:
\[
W = W_S + W_L.
\]
We denote by $\gamma, \Gamma$ the quantities
\[
\gamma := |||x|||^{3/2}W_S \|e_{1, L^2, L^\infty} + |||x|||Z\|e_{1, L^2, L^\infty}, \quad \Gamma := |||x|||W_L \|e_{1, L^2, L^\infty}.
\]
Then we have (we omit for simplicity the index $\alpha$)
\[
||x|^{-1} g\|_{L^1} \leq |||x|||^{-1} W(x)v^2 \|_{L^1} + |||x|||^{-1} Z(x) \cdot \partial^A v \|_{L^1} + |||x|||^{-1} \mathcal{F}\|_{L^1},
\]
and, for any $\delta > 0$,
\[
|||x|||^{-1} W(x)v^2 \|_{L^1} \leq |||x|||^{1/2} W_S \|e_{1, L^2, L^\infty} \|v\|_X \|v\|_Y + |||x|||W_S \|e_{1, L^2, L^\infty} \|v\|_X \|\partial^A v\|_Y
\]
\[
\leq (\delta + \gamma) \|v\|_{X}^2 + \delta^{-1} \Gamma^2 \|v\|_{Y}^2,
\]
\[
|||x|||^{-1} Z(x) \cdot \partial^A v \|_{L^1} \leq |||x|||^{1/2} Z \|e_{1, L^2, L^\infty} \|v\|_X \|\partial^A v\|_Y \leq \gamma \|v\|_{X}^2 + \gamma \|\partial^A v\|_{Y}^2,
\]
\[
|||x|||^{-1} \mathcal{F}\|_{L^1} \leq ||\mathcal{F}\|_{Y} \|v\|_{X} \leq \delta \|v\|_{X}^2 + \delta^{-1} \|\mathcal{F}\|_{Y}^2.
\]
In a similar way we have
\[
\|g \partial^A v\|_{L^1} \leq \|W(x)v\|_X \|\partial^A v\|_{L^1} + \|Z(x)(\partial^A v)^2\|_{L^1} + \|f \mathcal{F}\|_{L^1},
\]
and
\[
\|W(x)v\|_{\partial^A v} \|_{L^1} \leq \||x||W_S \|e_{1, L^2, L^\infty} \|v\|_X \|\partial^A v\|_Y + \||x||^{1/2} W_S \|e_{1, L^2, L^\infty} \|v\|_X \|\partial^A v\|_Y
\]
\[
\leq \|\partial^A v\|_{X}^2 + \delta^{-1} \Gamma^2 \|v\|_{Y}^2 + \delta^{-1} \gamma \|v\|_{X}^2,
\]
\[
\|Z(x)(\partial^A v)^2\|_{L^1} \leq \|x||Z\|e_{1, L^2, L^\infty} \|\partial^A v\|_Y \|\partial^A v\|_Y \leq \gamma \|\partial^A v\|_{Y}^2,
\]
\[
\|f \mathcal{F}\|_{L^1} \leq \delta \|\partial^A v\|_{X}^2 + \delta^{-1} \|\mathcal{F}\|_{Y}^2.
\]
Finally we have
\[
\lambda^{1/2} \|g\|_{L^1} \leq \lambda^{1/2} \|W^2\|_{L^1} + \lambda^{1/2} \|Z v \partial^A v\|_{L^1} + \lambda^{1/2} \|f \mathcal{F}\|_{L^1},
\]
and
\[
\lambda^{1/2} \|W^2\|_{L^1} \leq \lambda^{1/2} \||x||W_S \|e_{1, L^2, L^\infty} \|v\|_X^2 + \lambda^{1/2} \||x||^{3/2} W_S \|e_{1, L^2, L^\infty} \|v\|_X \|v\|_Y
\]
\[
\leq \lambda^{1/2} \|\partial^A v\|_{X}^2 + \lambda \|v\|_{X}^2,
\]
\[
\lambda^{1/2} \|Z v \partial^A v\|_{L^1} \leq \lambda^{1/2} \||x||Z\|e_{1, L^2, L^\infty} \|v\|_Y \|\partial^A v\|_Y \leq \lambda \|v\|_X \|v\|_Y + \|\partial^A v\|_Y^2,
\]
\[
\lambda^{1/2} \|f \mathcal{F}\|_{L^1} \leq \delta \lambda \|v\|_{X}^2 + \delta^{-1} \|\mathcal{F}\|_{Y}^2.
\]
Summing up, we get
\[ \|x^{-1}gv\|_{L^1} + \|g\partial^4 v\|_{L^1} + \lambda^{1/2}\|g\mathcal{P}\|_{L^1} \leq (2\delta + 3\gamma + \delta^{-1}\gamma^2)\|v\|_{X}^2 + (2\delta^{-1}\Gamma^2 + \lambda^{1/2}\Gamma + 2\lambda\gamma + \delta\lambda)\|v\|_{Y}^2 + 3(\delta + \gamma)\|\partial^4 v\|_{Y}^2 + 3\delta^{-1}\|f\|_{Y}^2. \]

Recalling that \(c_0 \geq 1\) is the constant in (2.29), we require that
\[ \delta = \frac{1}{4c_0}, \quad \gamma \leq \frac{1}{2c_0}, \quad |\lambda| \geq 2^{3}c_0^2\Gamma^2 + 2 + c\|x|\hat{B}\|_{L^\infty}^2 \quad (2.30) \]
(note that this implies also (2.28) and \(\lambda \geq 2\)) and one checks that
\[ 2\delta + 3\gamma + \delta^{-1}\gamma^2 \leq \frac{1}{c_0}, \quad 3(\delta + \gamma) \leq \frac{1}{c_0}, \]
and
\[ 2\delta^{-1}\Gamma^2 + \lambda^{1/2}\Gamma + 2\lambda\gamma + \delta\lambda \leq \frac{1}{c_0}. \]
Thus with the choices (2.30) we have
\[ \|x^{-1}gv\|_{L^1} + \|g\partial^4 v\|_{L^1} + \lambda^{1/2}\|g\mathcal{P}\|_{L^1} \leq \frac{1}{2c_0}\|v\|_{X}^2 + \frac{1}{2c_0}\|v\|_{Y}^2 + \frac{1}{c_0}\|\partial^4 v\|_{Y}^2 + 3\delta^{-1}\|f\|_{Y}^2. \]
and plugging this into (2.29), and absorbing the first three terms at the right from the left side of the inequality, we conclude that
\[ \|v\|_{X}^2 + \lambda\|v\|_{Y}^2 + \|\partial^4 v\|_{Y}^2 \leq c_1\|f\|_{Y}^2. \quad (2.31) \]

Note that if we consider the case of negative \(\lambda\), starting from estimate (2.21) instead of (2.29) and applying the same argument, we obtain a similar estimate, provided \(\lambda\) satisfies (2.30). Recalling also that by assumption \(|\epsilon| \leq |\lambda|\), we see that the proof of Proposition 2.3 is concluded.

2.6. Small frequencies. We now consider the remaining case of small frequencies. In this region we shall follow an indirect approach. We consider an operator \(L\) defined by
\[ Lv := -i\Delta v - W(x)v - i\sum_{j=1}^{3} Z_j(x) \partial_j v - i\sum_{j=1}^{3} \partial_j(Z_j(x)v) \quad (2.32) \]
with \(W = W^*\), and we assume that \(L\) is selfadjoint on \(L^2(\mathbb{R}^3; \mathbb{C}^4)\). (Note that in the case of small frequencies it is not useful to handle the magnetic part \(A\) of the potential separately.) In order to estimate the resolvent operator of \(L\)
\[ R(z) := (L - z)^{-1} = (-i\Delta - W - iz^* \cdot \partial - iz \cdot Z - z)^{-1} \]
we use the (Lippmann–Schwinger) representation of \(R(z)\)
\[ R(z) = R_0(z)(I_4 - K(z))^{-1}, \quad K(z) := [W + iz^* \cdot \partial + iz \cdot Z]R_0(z) \quad (2.33) \]
in terms of the free resolvent
\[ R_0(z) = I_4(-\Delta - z)^{-1}. \]
We recall a few (more or less standard) facts on the free resolvent \(R_0(z)\). For \(z \in \mathbb{C} \setminus [0, +\infty)\), \(R_0(z)\) is a holomorphic map with values in the space of bounded operators \(L^2 \to H^2\) and satisfies an estimate
\[ \|R_0(z)f\|_{X} + |z|^2\|R_0(z)f\|_{Y} + \|\partial R_0(z)f\|_{Y} \lesssim \|f\|_{Y}. \quad (2.34) \]
with an implicit constant independent of \(z\) (a proof of this estimate is actually contained in the previous section since for vanishing potentials there is no restriction on \(\lambda\); for a detailed proof see e.g. [8]). When \(z\) approaches the spectrum of the Laplacian \(\sigma(-\Delta) = [0, +\infty)\), it is possible to define two limit operators
\[ R(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon), \quad \epsilon > 0, \lambda \geq 0 \]
but the two limits are different if \(\lambda > 0\). These limits exist in the norm of bounded operators from the weighted \(L^2_x\) space with norm \(\|x^s f\|_{L^2_x}\) to the weighted Sobolev space \(H^2_{x \cdot} \) with norm
we see that multiplication by \( \hat{Y}^* \) and \( \hat{Y} \) (or \( \hat{X} \)) respectively, and estimate (2.34) is uniform in \( z \), one obtains that (2.34) is valid also for the limit operators \( R_0(\lambda \pm i0) \). In the following we shall write simply \( R_0(z) \), \( z \in \mathbb{C}^\pm \), to denote either one of the extended operators \( R_0(\lambda \pm i\varepsilon) \) with \( \varepsilon > 0 \), defined on the closed upper (resp. lower) complex half-plane. Note also that the map \( z \mapsto R_0(z) \) is continuous with respect to the operator norm of bounded operators \( L^2_s \to H^2_{-s'} \), for every \( s, s' > 1/2 \), and from this fact one easily obtains that it is also continuous with respect to the operator norm of bounded operators from \( \hat{Y}^* \to H^2_{-s'} \).

Thus in particular

\[
R_0(z) : \hat{Y}^* \to \hat{X}, \quad \partial R_0(z) : \hat{Y}^* \to \hat{Y}
\]

are uniformly bounded operators for all \( z \in \mathbb{C}^\pm \); note also the formula

\[
\Delta R_0(z) = -L_4 - zR_0(z)
\]

Moreover, for any smooth cutoff \( \phi \in C^\infty_c(\mathbb{R}^3) \) and all \( z \in \mathbb{C}^\pm \), the map \( z \mapsto \phi R_0(z) \) is continuous w.r.t. the norm of bounded operators \( \hat{Y}^* \to H^2 \), and hence

\[
\phi R_0(z) : \hat{Y}^* \to L^2 \quad \text{and} \quad \phi \partial R_0(z) : \hat{Y}^* \to L^2 \quad \text{are compact operators.}
\]

Similarly one gets that \( z \mapsto \phi R_0(z) \) is continuous w.r.t. the norm of bounded operators \( \hat{Y}^* \to L^\infty_{|z|} L^2_w \) and

\[
\phi R_0(z) : \hat{Y}^* \to L^\infty_{|z|} L^2_w \quad \text{is a compact operator.}
\]

In order to invert the operator \( I - K(z) \) we shall apply Fredholm theory. An essential step is the following compactness result:

**Lemma 2.6.** Let \( z \in \mathbb{C}^\pm \) and assume \( W, Z \) satisfy

\[
N := \|x|^{3/2}(W + i(\partial \cdot Z))\|_{\ell^1 L^\infty} + \|x|Z\|_{\ell^1 L^\infty} < \infty. \tag{2.35}
\]

Then \( K(z) = (W + iZ^* \cdot \partial + i\partial \cdot Z)R_0(z) \) is a compact operator on \( \hat{Y}^* \), and the map \( z \mapsto K(z) \) is continuous with respect to the norm of bounded operators on \( \hat{Y}^* \).

**Proof.** We decompose \( K \) as follows. Let \( \chi \in C^\infty_c(\mathbb{R}^3) \) be a cutoff function equal to 1 for \( |x| \leq 1 \) and to 0 for \( |x| \geq 2 \). Define for \( r > 2 \)

\[
\chi_r(x) = \chi(x/r)(1 - \chi(rx))
\]

so that \( \chi_r \) vanishes for \( |x| \geq 2r \) and also for \( |x| \leq 1/r \), and equals 1 when \( 2r \leq |x| \leq r \). Then we split

\[
K = A_r + B_r
\]

where

\[
A_r(z) = \chi_r \cdot K(z), \quad B_r(z) = (1 - \chi_r) \cdot K(z).
\]

First we show that \( A_r \) is a compact operator on \( \hat{Y}^* \). Indeed, for \( s > 2r > 4 \) we have \( \chi_r \chi_s = \chi_r \) and we can write

\[
A_r = \chi_s A_r = \chi_s(W + i(\partial \cdot Z))\chi_r R_0(z) + i\chi_s(Z + Z^*) \cdot \chi_r \partial R_0(z).
\]

By the estimate

\[
\|(W + i(\partial \cdot Z))v\|_{\hat{Y}^*} \leq \|x|^{3/2}(W + i(\partial \cdot Z))\|_{\ell^1 L^\infty} \|v\|_{\hat{X}} \leq N \|v\|_{\hat{X}} \tag{2.36}
\]

we see that multiplication by \( W + i(\partial \cdot Z) \) is a bounded operator from \( \hat{X} \) to \( \hat{Y}^* \). Moreover, multiplication by \( \chi_s \) is a bounded operator \( L^\infty_{|z|} L^2_w \to \hat{X} \) and the operator \( \chi_r R_0 : \hat{Y}^* \to L^\infty_{|z|} L^2_w \) is compact as remarked above. A similar argument applies to the second term in \( A_r \), using the estimate

\[
\|Zv\|_{\hat{Y}^*} \leq \|x|Z\|_{\ell^1 L^\infty} \|v\|_{\hat{Y}} \leq N \|v\|_{\hat{Y}} \tag{2.37}
\]
and compactness of $\chi rR_0 : \dot{Y}^* \to L^2$. Summing up, we obtain that $A_r : \dot{Y}^* \to \dot{Y}^*$ is a compact operator. Similarly, we see that $z \mapsto A_r(z)$ is continuous with respect to the norm of bounded operators on $\dot{Y}^*$.

Then to conclude the proof it is sufficient to show that $B_r \to 0$ in the norm of bounded operators on $\dot{Y}^*$, uniformly in $z$. We have, as in (2.36)-(2.37),

$$\|B_r v\|_{Y^*} \leq N_r(\|R_0\|_{Y^* \to X} + \|\partial R_0\|_{Y^* \to Y^*})\|v\|_{Y^*} \, .$$

where

$$N_r := \|x^{3/2}(1 - \chi_r)(W + i(\partial \cdot Z))\|_{\ell^1 L^\infty} + 2\|\|x(1 - \chi_r)Z\|_{\ell^1 L^\infty}.$$ 

Since $N_r \to 0$ as $r \to \infty$, we obtain that $\|B_r\|_{Y^* \to Y^*} \to 0$. \hfill \Box

We now study the injectivity of $I - K(z) : \dot{Y}^* \to \dot{Y}^*$. Note that if $f \in \dot{Y}^*$ satisfies

$$(I_4 - K(z)) f = 0$$

then setting $v = R_0(z)f$ by the properties of $R_0(z)$ we have $v \in H^1_{loc} \cap \dot{X}$, $\nabla v \in \dot{Y}$, $v \in H^2_{loc}(R^3 \setminus 0)$, $\Delta v \in \dot{Y} + \dot{Y}^*$ (or $\Delta v \in \dot{Y}^*$ if $z = 0$) and if $z \neq 0$ we have also $v \in \dot{Y}$. In particular, $v$ is a solution of the equation

$$(L - z)v = 0.$$

For $z$ outside the spectrum of $L$ it is easy to check that this implies $v = f = 0$.

**Lemma 2.7.** Let $W, Z, K(z)$ be as in Lemma 2.6 and $L = -I_4\Delta - W - iZ^* \cdot \partial - i\partial \cdot Z$. If $f \in \dot{Y}^*$ satisfies

$$(I_4 - K(z)) f = 0$$

for some $z \not\in \sigma(L)$, then $f = 0$.

**Proof.** Let $v = R_0(z)f$, fix a compactly supported smooth function $\chi$ which is equal to 1 for $|x| \leq 1$, and for $M > 1$ consider $v_M := v(x)\chi(x/M)$. Then $v_M \in L^2$ and

$$(L - z)v_M = \frac{1}{M^2}\nabla \chi(\frac{x}{M})(2\nabla v + i(Z + Z^*)v) + \frac{1}{M^2}\Delta \chi(\frac{x}{M})v =: f_M.$$

We have, for $\delta \in (1, \frac{1}{2})$, using the estimate $|Z| \lesssim |x|^{-1}$,

$$\|f_M\|_{L^2} \lesssim M^{\delta - \frac{1}{2}} \left(\|x|^{-\frac{1}{2}} \|\nabla v\|_{L^2(|x| \geq M)} + \|x|^{-\frac{1}{2}} \|v\|_{L^2(|x| \geq M)}\right)$$

$$\lesssim M^{\delta - \frac{1}{2}} \left(\|\nabla v\|_{Y^*} + \|v\|_{X^*}\right)$$

uniformly in $M$, so that $f_M \to 0$ in $L^2$ as $M \to \infty$. Since $v_M = R_0(z)f_M$ and $R_0(z)$ is a bounded operator on $L^2$, we conclude that $v = f = 0$. \hfill \Box

The hard case is of course $z \in \sigma(L)$. Then we have the following result, in which we write simply

$$R_0(\lambda) \quad \text{instead of} \quad R_0(\lambda \pm i0)$$

since the computations for the two cases are identical.

**Lemma 2.8.** Assume $W = W^*$ and $Z$ satisfy for some $\delta > 0$

$$\|x\|^{2\delta} (W + i\partial \cdot Z)\|_{\ell^1 L^\infty} + \|x(x)^{\delta} Z\|_{\ell^1 L^\infty} < \infty$$

(2.38)

and $L = -\Delta I_4 - W - i\partial \cdot Z - iZ^* \cdot \partial$ is a non negative selfadjoint operator on $L^2$. Let $f \in \dot{Y}^*$ be such that, for some $\lambda \geq 0$,

$$(I_4 - K(\lambda)) f = 0, \quad K(\lambda) := (W + i\partial \cdot Z + iZ^* \cdot \partial)R_0(\lambda).$$

Then in the case $\lambda > 0$ we have $f = 0$, while in the case $\lambda = 0$ we have $|x|^{3/2} f \in L^2$ and the function $v = R_0(0)f$ belongs to $H^2_{loc}(\mathbb{R}^3 \setminus 0) \cap \dot{X}$ with $\partial v \in \dot{Y}$, solves $Lv = 0$ and satisfies $|x|^{-\frac{1}{2} - \delta} v \in L^2$ and $|x|^{\frac{1}{2} - \delta} \partial v \in L^2$ for any $\delta' > 0$. 
Proof. Defining as in the previous proof \( v = R_0(\lambda)f \), we see that \( v \) solves
\[
\Delta I_4 v + \lambda v + g = 0, \quad g := Wv + iZ^* \cdot \partial v + i\partial \cdot Zv. \tag{2.39}
\]
Then given a radial function \( \psi \geq 0 \) to be precised later, we apply again identities (2.15), (2.16) with the choices
\[
\psi' = \chi, \quad \phi = -\chi'
\]
so that in particular \( \Delta \psi + \phi = \frac{\partial}{\partial r} \chi \). We sum the two identities and integrate on a ball \( B(0, R) \); it is easy to check that the boundary terms tend to 0 as \( R \to \infty \), provided \( \chi \) does not grow to fast \( (\chi(x) \lesssim |x| \text{ is enough).} \) After straightforward computations (see Proposition 3.1 of [8] for a similar argument), we arrive at the following radiation estimate:
\[
\int \chi' |\partial_S v|^2 + 2(\frac{\chi}{|x|^2} - \chi') |(\partial_S v)^2| - \int \Delta (\frac{\chi}{|x|^2})|v|^2 = \Re \int \chi g(\frac{\partial}{|x|^2} + 2\tilde{x} \cdot \partial_S v) \tag{2.40}
\]
where we denoted the “Sommerfeld” gradient of \( v \) with
\[
\partial_S v := \partial v - i\sqrt{\lambda} \tilde{x} v, \quad \tilde{x} = x/|x|
\]
and the tangential component of \( \partial v \) with
\[
|(\partial v)_{T}|^2 := |\partial v|^2 - |\tilde{x} \cdot \partial v|^2.
\]
We now estimate the right hand side of (2.40). We have
\[
|\Re \int \chi g(\frac{\partial}{|x|^2} + 2\tilde{x} \cdot \partial_S v)| \leq \|\chi(W + i(\partial \cdot Z))v||\partial_S v||L^1 | + 2\|\chi Z|\partial v||x|^{-1} v||L^1 | \leq \|\chi |x| W + i(\partial \cdot Z))v||\partial_S v||L^1 | + 2\|\chi Z|\partial v||x|^{-1} v||L^1 |
\]
and similarly
\[
|\Re \int \chi g(\frac{\partial}{|x|^2} + 2\tilde{x} \cdot \partial_S v)| \leq \|\chi(W + i(\partial \cdot Z))v||\partial_S v||L^1 | + 2\|\chi Z|\partial v||x|^{-1} v||L^1 | \leq \|\chi |x|^{3/2} W + i(\partial \cdot Z))v||\partial_S v||L^1 | + 2\|\chi Z|\partial v||x|^{-1} v||L^1 |
\]
Since the quantities \( ||v||_{\tilde{X}}, ||\partial v||_{\tilde{Y}}, \) and \( ||\partial_S v||_{\tilde{Y}} \) are all estimated by \( ||f||_{\tilde{Y}'} \) (recall (2.34)), we conclude
\[
|\Re \int \chi g(\frac{\partial}{|x|^2} + 2\tilde{x} \cdot \partial_S v)| \lesssim N^2_{\chi} ||f||_{\tilde{Y}'}^2, \tag{2.41}
\]
where
\[N^2_{\chi} := \|\chi |x|^{3/2} W + i(\partial \cdot Z))||\partial_S v||_{L^\infty} + \|\chi |x| Z||\partial v||_{L^\infty} \leq \_ \quad \text{by (2.40) and (2.41)) we obtain, dropping a (nonnegative term at the left,}
\]
\[
||x|^{(\delta-1)/2} v||_{L^2} + ||x|^{(\delta-3)/2} v||_{L^2} \lesssim N_{\chi} ||f||_{\tilde{Y}'}, \tag{2.42}
\]
by assumption
\[
N^2_{\chi} := \|\chi |x|^{3/2+\delta} W + i(\partial \cdot Z))||\partial_S v||_{L^\infty} + \|\chi |x|^{1+\delta} Z||\partial v||_{L^\infty} < \infty.
\]
Consider now the following identity, obtained using the divergence formula:
\[
\int_{|x|=R} (|\partial v|^2 + \lambda |v|^2 - |\partial_S v|^2) d\sigma = 2\Re \int_{|x|\leq R} i\sqrt{\lambda} \partial \cdot (v, \partial v) = 2\Re \int_{|x|\leq R} i\sqrt{\lambda} \langle v, \Delta v \rangle
\]
for arbitrary \( R > 0 \). Substituting \( \Delta v = -\lambda v - g \) from (2.39) and dropping two pure imaginary terms, we get
\[
\int_{|x|=R} (|\partial v|^2 + \lambda |v|^2 - |\partial_S v|^2) d\sigma = 2\Re \int_{|x|\leq R} (Z^* \cdot \partial v + \partial \cdot Zv, v)
\]
The last term can be written, again by the divergence formula,
\[
= 2\Re \int_{|x|\leq R} \partial(Zv, v) = 2 \sum_j \int_{|x|=R} \tilde{x}_j (Z_j v, v) d\sigma, \quad \tilde{x}_j = x_j/|x|.
\]
By assumption $|Z| \lesssim |x|^{-1}$, hence for some $R_0 > 0$ we have $\lambda > 2|Z(x)|$ for all $|x| > R_0$, and the term in $Z$ can be absorbed by the left of the identity. Summing up, we have proved that

$$
\int_{|x| = R} (|\partial v|^2 + \lambda |v|^2) \, ds \leq 2 \int_{|x| = R} |\partial_S v|^2 \, ds, \quad R \geq R_0.
$$

(2.43)

Multiplying both sides by $|x|^{d-1}$, integrating in the radial direction from $R_0$ to $\infty$, and using (2.42), we conclude

$$
|||x|^{(d-1)/2} \partial v||_{L^2(|x| \geq R_0)} + \sqrt{\lambda} \|||x|^{(d-1)/2} v||_{L^2(|x| \geq R_0)} \lesssim \|f\|_{Y^*}.
$$

(2.44)

In the case $\lambda > 0$ we have proved that $|x|^{(d-1)/2} v \in L^2$ i.e., $\lambda$ is a resonance, and this is enough to conclude that $v = 0$ by applying one of the available results on the absence of embedded eigenvalues. For instance, we can apply the results from [16] which are particularly sharp. Note that in [16] a scalar operator is considered, but it is easy to check that the same proof covers also the case of an operator which is diagonal in the principal part and coupled only in lower order terms. We need to check the assumptions on the potentials required in [16]. The potential $V$ in [16] is simply $V = z$ in our case, which we are assuming real and $> 0$, thus condition A.1 is trivially satisfied. Concerning $W$ we have $W \in L^{3/2}$ and condition A.2 in [16] is satisfied. Concerning the potential $Z$, we have $Z \in \ell^\infty L^3$; moreover a similar computation applied to $1_{|x| > M} Z$ gives

$$
||1_{|x| > M} Z||_{\ell^\infty L^3} \leq |||x|^{-1}||_{\ell^\infty L^3} |||x||_{L^\infty} \lesssim \infty < \infty.
$$

Thus to check that $Z$ satisfies condition A.3 in [16] it remains to check that the low frequency part $S_{< R} Z$ of $Z$ satisfies A.2 for $R$ large enough. $S_{< R} Z$ is obviously smooth. Moreover, it is clear that $|x| Z \to 0$ as $|x| \to \infty$; in order to prove the same decay property for $S_{< R} Z$ we represent it as a convolution with a suitable Schwartz kernel $\phi$

$$
\phi \ast Z(x) = \int_{|y| \leq \frac{|x|}{M}} Z(y) \phi(x - y) + \int_{|y| \geq \frac{|x|}{M}} Z(y) \phi(x - y).
$$

The first integral is bounded by $C_k (x)^{-k}$ for all $k$. For the second one we write

$$
|x| \int_{|y| \geq \frac{|x|}{M}} Z(y) \phi(x - y) \leq \int_{|y| \geq \frac{|x|}{M}} |y| Z(y) \phi(x - y) = o(|x|).
$$

We have thus proved that $|x| S_{< R} Z \to 0$ as $|x| \to \infty$ (for any fixed $R$) and hence $Z$ satisfies condition A.3. Applying Theorem 8 of [16], we conclude that $v = 0$.

It remains to consider the case $\lambda = 0$. We denote by $\hat{L}^{2,s}$ the Hilbert space with norm

$$
\|v\|_{\hat{L}^{2,s}} := \||x|^s v\|_{L^2}.
$$

By the well known Stein–Weiss estimate for fractional integrals in weighted $L^p$ spaces, applied to $R_0(0)v = \Delta^{-1} v = c|x|^{-1} * v$, we see that $R_0(0)$ is a bounded operator

$$
R_0(0) : \hat{L}^{2,s} \to \hat{L}^{2,s-2} \quad \text{for all} \quad \frac{1}{2} < s < \frac{3}{2}
$$

while $\partial R_0(0) = c(x|x|^{-3}) * v$ is a bounded operator

$$
\partial R_0(0) : \hat{L}^{2,s} \to \hat{L}^{2,s-1} \quad \text{for all} \quad -\frac{1}{2} < s < \frac{3}{2}.
$$

Recall also that $R_0(0)$ is bounded from $\dot{Y}^*$ to $\dot{X}$ and $\partial R_0(0)$ is bounded from $\dot{Y}^*$ to $\dot{Y}$. Moreover from the assumption on $W, Z$ it follows that the corresponding multiplication operators are bounded operators

$$
W + i(\partial \cdot Z) : \dot{X} \to \dot{L}^{2,1/2+\delta}, \quad W + i(\partial \cdot Z) : \dot{L}^{2,s-2} \to \dot{L}^{2,s+\delta} \quad \forall s \in \mathbb{R},
$$
Combining all the previous properties we deduce that \( K(0) = (W + i\partial \cdot Z + iZ \cdot \partial)R_0(0) \) is a bounded operator

\[
K(0) : \dot{Y}^* \to \dot{L}^{2,1/2+\delta}, \quad Z : \dot{L}^{2,s-1} \to \dot{L}^{2,s+\delta}, \quad \forall s \in \mathbb{R}.
\]

Since we know that \( f \in \dot{Y}^* \) and that \( f = K(0)f \), applying (2.45) repeatedly, we obtain in a finite number of steps that \( f \in \dot{L}^{2,3/2} \), which in turn implies \( v = R_0(0)f \in \dot{L}^{2,s} \) for all \( s < -\frac{1}{2} \) and \( \partial v = \partial R_0(0)f \in \dot{L}^{2,s} \) for all \( s < \frac{3}{2} \). The proof is concluded.

\( \Box \)

Note that \( z \mapsto I - K(z) \) is trivially continuous (and actually holomorphic for \( z \notin \sigma(L) \)). Since \( K(z) \) is compact and \( I - K(z) \) is injective on \( \dot{Y}^* \), it follows from Fredholm theory that \((I - K(z))^{-1}\) is a bounded operator for all \( z \in \mathbb{C} \). However we need a bound uniform in \( z \), and to this end it is sufficient to prove that the map \( z \mapsto (I - K(z))^{-1} \) is continuous. This follows from a general well known result on Fredholm operators (a proof can be found e.g. in [12]):

**Lemma 2.9.** Let \( X_1, X_2 \) be two Banach spaces, \( K_j, K \) compact operators from \( X_1 \) to \( X_2 \), and assume \( K_j \to K \) in the operator norm as \( j \to \infty \). If \( I - K_j, I - K \) are invertible with bounded inverses, then \((I - K_j)^{-1} \to (I - K)^{-1} \) in the operator norm.

We finally sum up the previous results. We shall need to assume that \( 0 \) is not a resonance, in the following sense:

**Definition 2.10** (Resonance). We say that \( 0 \) is a resonance for the operator \( L \) if there exists a nonzero \( v \in H^2_{loc}(\mathbb{R}^3 \setminus \{0\}) \cap \dot{X} \) with \( \partial v \in \dot{Y} \), solution of \( Lv = 0 \) with the properties

\[
|x|^{\frac{1}{2} - \sigma} v \in L^2 \quad \text{and} \quad |x|^{\frac{1}{2} - \sigma} \partial v \in L^2 \quad \forall \sigma > 0.
\]

The function \( v \) is then called a resonant state at \( 0 \) for \( L \).

Note that in Lemma 2.8 we proved in particular that if \( f \in \dot{Y}^* \) satisfies \( f = K(0)f \), then \( v = R_0(0)f \) is a resonant state at \( 0 \).

**Proposition 2.11.** Assume the operator \( L \) defined in (2.32) is non negative and selfadjoint on \( L^2 \), with \( W = W^* \) and \( Z \) satisfying (2.38) for some \( \delta > 0 \). In addition, assume that \( 0 \) is not a resonance for \( L \), in the sense of (2.46).

Then \( I_4 - K(z) \) is a bounded invertible operator on \( \dot{Y}^* \), with \((I_4 - K(z))^{-1}\) bounded uniformly for \( z \) in bounded subsets of \( \mathbb{C}^+ \). Moreover, the resolvent operator \( R(z) = (L - z)^{-1} \) satisfies the estimate

\[
\|R(z)f\|_X + |z|^\frac{1}{2} \|R(z)f\|_Y + \|\partial R(z)f\|_Y \leq C(z)\|f\|_Y,
\]

for all \( z \in \mathbb{C}^+ \), where \( C(z) \) is a continuous function of \( z \).

**Proof.** It is sufficient to combine Lemmas 2.5, 2.6, 2.7, 2.8, 2.9 and apply Fredholm theory in conjunction with assumption (2.46), to prove the claims about \( I - K(z) \); note that (2.38) include the assumptions of Lemmas 2.6–2.9. Finally, using the representation (2.33) and the free estimate (2.34) we obtain (2.47).

\( \Box \)

2.7. **Proof of Theorem 2.1.** Squaring the operator \( D + V \) produces a non negative, selfadjoint operator with domain \( H^2(\mathbb{R}^3) \), of the form

\[
L := (D + V)^2 = -I_4\Delta + V^2 + DV + VD.
\]

We want to apply Propositions 2.11 and 2.3 to the operator \( L \). First of all we check the 0 resonance assumption:

**Lemma 2.12.** If \( 0 \) is a resonance for the operator \( L = (D + V)^2 \), in the sense of Definition 2.10, then \( 0 \) is a resonance for the operator \( D + V \) in the sense of Definition 1.1.
Proof. Let $v$ be the resonant state for $L$, with the properties listed in Definition 2.10, and let $w = (\mathcal{D} + V)v$. If $w = 0$ then $v$ is a resonant state at 0 for $\mathcal{D} + V$ and the proof is concluded, thus we can assume $w$ nonzero. By the properties of $v$ we have directly $w \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus 0) \cap \hat{Y}$ and $(\mathcal{D} + V)w = 0$, so in particular $w \in L^2_{\text{loc}}$. We now prove that $|x|^{\frac{3}{2} - \sigma}Dv$ and $|x|^{\frac{3}{2} - \sigma}Vv$ belong to $L^2$; thus $w \in L^2$ which means that 0 is an eigenvalue of $\mathcal{D} + V$. The first fact is also contained in the definition of the resonant state $v$, while the second one is an immediate consequence of the property $|V| \lesssim |x|^{-1}$ and of the following generalized Hardy inequality

$$
\| |x|^\sigma w\|_{L^2} \lesssim \| |x|^{\sigma+1/2}\partial w\|_{L^2}, \quad \sigma > -1.
$$

The proof of (2.49) is simple: for a compactly supported smooth function $\phi$, integrate on $\mathbb{R}^3$ the identity

$$
\partial \cdot \{\hat{x}|x|^{2\sigma}(|\phi|^2)\} = (2 + 2\sigma)|x|^{2\sigma-1}|\phi|^2 + 2\mathfrak{R}|x|^{2\sigma}\phi\phi_r, \quad \phi_r := \hat{x} \cdot \partial \phi
$$

and use Cauchy–Schwartz to obtain

$$(\sigma + 1) \int_{B_R} |x|^{2\sigma-1} |\phi|^2 \, dx \leq 2 \left( \int_{B_R} |x|^{2\sigma-1} |\phi|^2 \, dx \right)^{1/2} \left( \int_{B_R} |x|^{2\sigma+1} |\phi_r|^2 \, dx \right)^{1/2}.
$$

We next check that $L$ satisfies assumption (2.38). Following (2.48) we must choose

$$
W := -V^2, \quad Z := \alpha V.
$$

It is easy to check that conditions (2.38) are implied by

$$
|V|^2 + |Dv| \leq \frac{C}{|x|^{2+\delta}}, \quad |x|^2(|V|^2 + |Dv|) \in \ell^1 L^\infty
$$

for some $\delta > 0$ (compare with Condition (V)). Note that the first condition is effective for large $x$ while the second one restricts the singularity at 0 of the potential. Then we are in position to apply Proposition 2.11 and we obtain that the resolvent operator $R(z) = (L - z)^{-1}$ satisfies the estimate

$$
\|R(z)f\|_X + \|z^\delta R(z)f\|_Y + \|\partial R(z)f\|_Y \leq C(z)\|f\|_Y,
$$

for all $z \in \mathbb{C}^*$, with a constant $C(z)$ depending continuously on $z$.

Next, in order to apply Proposition 2.3, using the decomposition $V = \alpha \cdot A + A_0 \beta + V_0$, we may rewrite $L$ in the form

$$
L := -i4\Delta_A - i\{V_0, \alpha\} \partial^A v - i \sum_{j<k} B_{jk} \alpha_j \alpha_k + \mathcal{D}(V_0 + A_0 \beta) + (V_0 + A_0 \beta)^2,
$$

where

$$
\{V_0, \alpha\} = V_0 \alpha + \alpha V_0, \quad B_{jk} = \partial_j A_k - \partial_k A_j, \quad j, k = 1, 2, 3.
$$

By comparing with (2.11), we choose now

$$
Z := \{V_0, \alpha\}, \quad -W := -i \sum_{j<k} B_{jk} \alpha_j \alpha_k + \mathcal{D}(V_0 + A_0 \beta) + (V_0 + A_0 \beta)^2
$$

and we verify that assumption (2.12) is satisfied as soon as we impose on the coefficients, besides (2.50), the conditions

$$
|x||B| + |x|^2|\mathcal{D}V_0| \in \ell^1 L^\infty, \quad ||x|V_0||_{\ell^1 L^\infty} < \sigma_0
$$

with $\sigma_0 > 0$ as in Proposition 2.3 (compare with Condition (V)). From (2.52) it follows directly that $||\mathcal{D}V_0 + A_0 \beta||_{\ell^1 L^\infty} < \sigma_0$ and $|x|B \in \ell^1 L^\infty$ as required. Next we define

$$
W_S := -\left[ \mathcal{D}(V_0 + A_0 \beta) + (V_0 + A_0 \beta)^2 \right] 1_{|x|<R}
$$

where $1_{|x|<R}$ is the characteristic function of $\{ |x| < R \}$, and we remark that

$$
|x|^2\mathcal{D}(V_0 + A_0 \beta) \in \ell^1 L^\infty
$$
since both $V$ and $V_0$ satisfy a similar assumption (and hence also $A, A_0$, in view of the linear independence of Dirac matrices); on the other hand
\[ |x|V_0 \in \ell^1 L^\infty \Rightarrow \ell^2 L^\infty \quad \Rightarrow \quad |x|^2 V_0^2 \in \ell^1 L^\infty \]
and since $|x|^2 V^2 \in \ell^1 L^\infty$ we have also
\[ |x|^2 A_0^2 \in \ell^1 L^\infty. \]
All this implies that $|x|^2 W_S \in \ell^1 L^\infty$, and hence
\[ \lim_{R \to 0} \| |x|^2 W_S \|_{\ell^1 L^\infty} \to 0. \]
Picking $R$ sufficiently small we see that $W_S$ satisfies (2.12). It remains to check that $W_L := W - W_S$ satisfies $|x|W_L \in \ell^1 L^\infty$, and this follows from assumption (2.52) on $B$ and from the previous estimates on $V_0, A_0$ (thanks to the cutoff $1 - 1_{|x| < R}$ vanishing near 0).

Thus all the assumptions of Proposition 2.3 are satisfied and we have
\[ \|R(z)f\|_X + |z|^{1/2}\|R(z)f\|_X + \|\partial^A R(z)f\|_Y \leq C\|f\|_Y, \]
for all $z$ large enough in the strip $|3z| \leq 1$. Taking into account Remark 2.4 in order to replace $\partial^A$ with $\partial$, and the previous estimate for small $z$, we conclude that the estimate
\[ \|R(z)f\|_X + |z|^{1/2}\|R(z)f\|_X + \|\partial R(z)f\|_Y \leq C\|f\|_Y, \quad (2.53) \]
holds for all $z$ in the strip $|3z| \leq 1$, with a constant independent of $z$, provided (2.52), (2.50) hold and $A \in \ell^\infty L^3$.

Since $|V| \lesssim |x|^{-1}$ and $\||x|^{-1}v\|_Y \lesssim \|v\|_X$, this implies
\[ \|(\mathcal{D} + V)R(z)f\|_Y \leq C\|f\|_Y, \quad (2.54) \]
uniformly in the strip $|3z| \leq 1$. Moreover, for any positive function $\rho \in \ell^2 L^\infty$, using the inequalities
\[ \|\rho f\|_{L^2} \leq \|\rho\|_{\ell^2 L^\infty} \|f\|_{\ell^2 L^2}, \quad \|f\|_{\ell^2 L^2} \leq \|\rho\|_{\ell^2 L^\infty} \|\rho^{-1} f\|_{L^2}, \]
we deduce from (2.54) the estimate
\[ \|\rho x^{-1/2}(\mathcal{D} + V)R(z)x^{-1/2} \rho f\|_{L^2} \leq C \|\rho\|_{\ell^2 L^\infty} \|\rho^{-1} f\|_{L^2}. \quad (2.55) \]

We now introduce the spectral projections $P_+, P_-$ defined as
\[ P_+ = \int_0^{+\infty} dE_\lambda, \quad P_- = \int_{-\infty}^0 dE_\lambda \]
where $dE_\lambda$ is the spectral measure of the selfadjoint operator $\mathcal{D} + V$. We decompose $L^2$ accordingly as
\[ L^2 = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_\pm = P_\pm L^2 \]
and we denote with $L_\pm = L P_\pm$ the parts of $L$ in $\mathcal{H}_\pm$. Note that
\[ (\mathcal{D} + V)(L - z)^{-1} P_\pm = \pm L^{1/2}_\pm (L - z)^{-1}. \]
Then for $f \in \mathcal{H}_\pm$ we have from (2.55)
\[ \|\rho x^{-1/2} L^{1/4}_\pm (L - z)^{-1} \rho x^{-1/2} L^{1/4}_\pm f\|_{L^2} \leq C \|\rho\|_{\ell^2 L^\infty} \|f\|_{L^2}. \quad (2.56) \]
This means that the operator $\rho x^{-1/2} L^{1/4}_\pm$ (resp. the operator $\rho x^{-1/2} L^{1/4}_-$) is supersmoothing for the selfadjoint operator $L_\pm$ on the Hilbert space $\mathcal{H}_\pm$ (resp. for $L_-$ on $\mathcal{H}_-$) in the sense of Kato–Yajima [15]; see [11] for a detailed account of the theory. By the Kato smoothing theory, this implies the following smoothing estimate for the Schrödinger flow $e^{it L^\pm}$
\[ \|\rho x^{-1/2} L^{1/2}_\pm e^{it L^\pm} f\|_{L^2 L^2} \lesssim \|\rho\|_{\ell^2 L^\infty} \|f\|_{L^2}, \quad f \in \mathcal{H}_\pm \]
and an analogous nonhomogeneous estimate for \( \int_0^t e^{i(t-s)\Omega} F(s)ds \). However, by Theorem 2.4 in [11], a smoothing estimate holds also for the wave flows \( e^{it\Omega} \), with a \( L^{1/4}_\pm \) derivative loss:
\[
\|\rho|x|^{-1/2}e^{itL^{1/2}_\pm} f\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} \|L^{1/4}_\pm f\|_{L^2}, \quad f \in \mathcal{H}_\pm
\]
(and similarly for the nonhomogeneous flows \( \int_0^t e^{i(t-t')L^{1/2}_\pm} F(t')dt' \)) so that we have proved
\[
\|\rho|x|^{-1/2}e^{itL^{1/2}_\pm} f\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} \|f\|_{L^2}, \quad f \in \mathcal{H}_\pm.
\]
Since \( L^{1/2}_\pm = \pm (D + V)P_\pm \), we arrive at
\[
\|\rho|x|^{-1/2}e^{it(D+V)} P_\pm f\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} \|f\|_{L^2}, \quad f \in \mathcal{H}_\pm,
\]
and summing over \( \pm \) we obtain (2.2). The same argument gives the nonhomogeneous estimate (2.3).

Finally, let \( u = e^{it(D+V)} f \), and let \( u_j = \partial_t u, f_j = \partial_t f, V_j = \partial_t V \); by differentiating the equation \( iu_t + (D + V)u = 0 \) we have
\[
i\partial_t u_j + (D + V)u_j = -V_j u, \quad u_j(0) = f_j
\]
so that
\[
u_j = e^{it(D+V)} f_j + i \int_0^t e^{i(t-t')(D+V)} V_j u dt'\]
and by (2.2), (2.3)
\[
\|\rho|x|^{-1/2}u_j\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} \|f_j\|_{L^2} + \|\rho\|_{L^\infty} \|\rho\|_{L^\infty} \|\rho^{-1}|x|^{1/2}V_j u\|_{L^2_tL^2_\pm}.
\]
Then we can write
\[
\|\rho^{-1}|x|^{1/2}V_j u\|_{L^2_tL^2_\pm} \leq \|\rho^{-2}|x|V_j\|_{L^\infty} \|\rho|x|^{-1/2}u\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} \|\rho^{-2}|x|V_j\|_{L^\infty} \|f\|_{L^2}
\]
again by (2.3), and in conclusion
\[
\|\rho|x|^{-1/2}\partial u\|_{L^2_tL^2_\pm} \lesssim \|\rho\|_{L^\infty} (1 + \|\rho\|^2_{L^2} \|\rho^{-2}|x|V_j\|_{L^\infty}) \|f\|_{H^1}
\]
and this gives (2.5).

2.8. Proof of Corollary 2.2. The scalar operator \( \Lambda_\omega = (1 - \Delta^{S^2})^{1/2} \), used to define the Sobolev norms on the sphere, is not convenient when working with the Dirac equation since it does not commute with \( \mathcal{D} \). We shall use instead the \textit{spin–orbit operator} \( K \), defined on \( L^2(\mathbb{R}^3)^4 \) as
\[
K := \beta (2S \cdot \Omega + 1)
\]
where \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) are the tangential vector fields to \( S^2 \)
\[
\Omega = x \wedge \partial
\]
while \( S = (S_1, S_2, S_3) = -\frac{i}{4} \alpha \wedge \alpha \) are the constant matrices
\[
S_j = -\frac{i}{4} \alpha_k \alpha_\ell, \quad (j, k, \ell) \text{ a cyclic permutation of } (1, 2, 3).
\]
To describe the action of the Dirac operator it is necessary to recall the \textit{partial wave decomposition} of \( L^2(S^2)^4 \). See Section 4.6 of [22] for a complete account. Let \( Y^m_\ell \), \( \ell = 0, 1, 2, \ldots \), \( m = -\ell, -\ell + 1, \ldots, \ell \), the usual spherical harmonics on \( S^2 \), which are an orthonormal basis of \( L^2(S^2) \); then an orthonormal basis of \( L^2(S^2)^4 \) is given by the family of functions
\[
\Phi_{m,j,k}^\pm, \quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \quad m_j = -j, -j + 1, \ldots, j, \quad k_j = \pm (j + \frac{1}{2}) \quad (2.58)
\]
defined as follows: when \( k_j = j + 1/2 \) we have
\[
\Phi^+_{m_j, k_j} = \frac{i}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j + 1 - m_j} Y_{m_j, k_j}^{-1/2} \\ -\sqrt{j + 1 + m_j} Y_{m_j, k_j}^{1/2} \\ 0 \\ 0 \end{pmatrix},
\]
while when \( k_j = -(j + 1/2) \) we have
\[
\Phi^+_{m_j, k_j} = \frac{i}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j - m_j} Y_{m_j, k_j}^{-1/2} \\ \sqrt{j - m_j} Y_{m_j, k_j}^{1/2} \\ 0 \\ 0 \end{pmatrix},
\]
\[
\Phi^-_{m_j, k_j} = \frac{i}{\sqrt{2j+2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{j + 1 - m_j} Y_{m_j, k_j}^{-1/2} \\ -\sqrt{j + 1 + m_j} Y_{m_j, k_j}^{1/2} \end{pmatrix}.
\]

For each choice of \( j, m_j, k_j \) as in (2.58), the couple \( \{\Phi^+_{m_j, k_j}, \Phi^-_{m_j, k_j}\} \) generates a 2D subspace \( H_{m_j, k_j} \) of \( L^2(\mathbb{S}^2)^4 \), and we have the natural decomposition
\[
L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{3}{2}, \frac{5}{2}, \ldots} \bigoplus_{m_j=-\infty}^{\infty} \bigoplus_{k_j=\pm(j+1/2)} L^2(0, +\infty; dr) \otimes H_{m_j, k_j}.
\]
The isomorphism is expressed by the explicit expansion
\[
\Psi(x) = \sum_{\substack{j m_k}} \frac{1}{r} \psi^+_{m_j, k_j}(r) \Phi^+_{m_j, k_j} + \frac{1}{r} \psi^-_{m_j, k_j}(r) \Phi^-_{m_j, k_j}
\]
with
\[
\|\Psi\|_{L^2} = \sum_{\substack{j m_k}} \int_0^\infty \left( |\psi^+_{m_j, k_j}|^2 + |\psi^-_{m_j, k_j}|^2 \right) dr.
\]
Notice also that
\[
\|\Psi\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\substack{j m_k}} \frac{1}{r^2} |\psi^+_{m_j, k_j}|^2 + \frac{1}{r^2} |\psi^-_{m_j, k_j}|^2.
\]
Each summand \( L^2(0, +\infty; dr) \otimes H_{m_j, k_j} \) is an eigenspace of the Dirac operator \( \mathcal{D} = i^{-1} \sum \alpha_j \partial_j \) and the action of \( \mathcal{D} \) can be written, in terms of the expansion (2.59), as
\[
\mathcal{D}\Psi = \sum_{\substack{j m_k}} \left( -\frac{d}{dr} \psi^-_{m_j, k_j} + \frac{k_j}{r} \psi^-_{m_j, k_j} \right) \Phi^+_{m_j, k_j} + \left( \frac{d}{dr} \psi^+_{m_j, k_j} + \frac{k_j}{r} \psi^+_{m_j, k_j} \right) \Phi^-_{m_j, k_j}.
\]
Note that the \( \Phi^\pm_{m_j, k_j} \) are eigenvectors for \( \Lambda_\omega \) but with different eigenvalues (satisfying \( \simeq j \)), while \( \mathcal{D} \) swaps them, hence \( \mathcal{D} \) and \( \Lambda_\omega \) do not commute. On the other hand, the spin–orbit operator \( K \) satisfies
\[
K \Phi^\pm_{m_j, k_j} = -k_j \Phi^\pm_{m_j, k_j}.
\]
Since \( k_j \simeq \pm j \), we have obviously
\[
\|Kv\|_{L^2(\mathbb{S}^2)} \simeq \|\Lambda_\omega v\|_{L^2(\mathbb{S}^2)}
\]
and more generally, if we define \( |K|^\alpha \) via
\[
|K|^\alpha \Phi^\pm_{m_j, k_j} = k_j^\alpha \Phi^\pm_{m_j, k_j},
\]
we have also
\[
\|\|K|^\alpha v\|_{L^2(\mathbb{S}^2)} \simeq \|\Lambda_\omega^\alpha v\|_{L^2(\mathbb{S}^2)}.
\]
Thus the differential operator \( K \) can replace \( \Lambda_\omega \) to measure angular regularity of functions. Moreover \( K \) commutes with the Dirac matrix \( \beta \):
\[
[K, \beta] = 0.
\]
and as a consequence, the commutator \([K, A_0β]\) is a bounded operator on \(L^2(S^2)\):
\[
\|\{K, A_0β\}v\|_{L^2(S^2)} \lesssim \|ΩA_0\|_{L^∞(S^2)}\|v\|_{L^2(S^2)}.
\] (2.64)

We turn now to the proof of Corollary 2.2. Assume first \(V_0 = 0\) i.e. \(V = A_0β\) only. Then by applying \(K\) to the equation we get
\[
iut + (D + V)u = F \implies i(Ku)t + (D + V)(Ku) = KF + [K, A_0β]u.
\]
By estimates (2.2)–(2.3) we have then
\[
\|\rho|x|^{-1/2}Ku\|_{L^2_tL^2_x} \lesssim \|Ku(0)\|_{L^2} + \|\rho^{-1}|x|^1/2([K, A_0β]u + KF)\|_{L^2_tL^2_x}
\]
and using (2.64), (2.7) and the estimates (2.2), (2.3) already proved, we obtain
\[
\|\rho|x|^{-1/2}Ku\|_{L^2_tL^2_x} \lesssim \|u(0)||_{L^2} + \|Ku(0)||_{L^2} + \|\rho^{-1}|x|^1/2F||_{L^2_tL^2_x} + \|\rho^{-1}|x|^1/2KF||_{L^2_tL^2_x}.
\]
Using the equivalence (2.62) on the sphere, we obtain (2.8), (2.9) for \(s = 1\). By interpolation with the case \(s = 0\), we have proved (2.8), (2.9) for all \(0 \leq s \leq 1\) under the additional assumption \(V_0 = 0\). The same argument gives the estimate in the range \(1 \leq s \leq 2\), if \(V_0 = 0\).

Assume now \(V_0 \neq 0\). We have
\[
iut + (D + V)u = F \implies iut + (D + A_0β)u = F - V_0u
\]
and by the previous part of the proof
\[
\|\rho|x|^{-1/2}Λ^s_u\|_{L^2_tL^2_x} \lesssim \|Λ^s_u(0)\|_{L^2} + \|\rho^{-1}|x|^1/2Λ^s_u(F - V_0u)\|_{L^2_tL^2_x}.
\]
If \(s > 1\) we can use the product rule
\[
\|Λ^s_u(fg)\|_{L^2_tL^2_x} \lesssim \|Λ^s_uF\|_{L^2_tL^2_x}||Λ^s_u||_{L^2_tL^2_x} (2.65)
\]
(see (4.9) in [7]). Then we have
\[
\|Λ^s_u(V_0u)\|_{L^2_tL^2_x} \lesssim \|Λ^s_uV_0\|_{L^2_tL^2_x}||Λ^s_u||_{L^2_tL^2_x} \lesssim \epsilon \rho^2|x|^{-1}\|Λ^s_u||_{L^2_tL^2_x}
\]
where we used assumption (2.6), and if \(ε\) is sufficiently small the resulting term can be absorbed at the left hand side, proving (2.8), (2.9) also for nonzero \(V_0\).

To prove the last estimate (2.10) it is sufficient to differentiate the equation (with \(F = 0\))
\[
i(\partial_t u)_t + (D + V)(\partial_j u) = -(\partial_j V)u
\]
and apply (2.8), (2.9), using again the product estimate and assumption (2.6) as above in order to estimate the term \((\partial_j V)u\), and then estimate (2.8) already proved.

3. ENDPOINT STRICHartz ESTIMATES

Strichartz estimates for the free Dirac equation on \(R^3\) take the form
\[
\|e^{itD}f\|_{L^p_tL^q_x} \lesssim \|D^{\frac{3}{2}}f\|_{L^2},
\]
\[
\|D^{\frac{n}{2}} \int_0^t e^{i(t-t')D} Fdt'\|_{L^p_tL^q_x} \lesssim \|D^{\frac{n}{2}}F\|_{L^p_tL^q_x},
\]
where \((p, q)\) and \((\tilde{p}, \tilde{q})\) are unrelated couples of admissible indices, i.e., satisfying
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty, \quad \infty \geq p > 2.
\]
The estimates fail at the so–called endpoint \((p, q) = (2, \infty)\), however the following replacement is true:
\[
\|e^{itD}f\|_{L^2_tL^∞} \lesssim \|f\|_{H^1}.
\]
Moreover, we have the mixed Strichartz–smoothing endpoint estimate
\[
\|\int_0^t e^{i(t-t')D} Fdt'\|_{L^2_tL^∞} \lesssim \langle x \rangle^{\frac{n}{2} + 1}\|D\|_{L^2}.
\]
Both estimates are proved in [7]. Actually, by a minor modification in the arguments of [7], we can prove the following:

**Proposition 3.1.** Let $\rho \in \ell^2 L^\infty$, $\rho > 0$, $\rho$ radially symmetric. For all $s \geq 0$, the flow $e^{itD}$ satisfies the estimates

$$
\|A^n_\rho e^{itD} f\|_{L_t^2 L^\infty L^2} \lesssim \|A^n_\rho f\|_{H^1} \quad (3.3)
$$

and

$$
\|A^n_\rho \int_0^t e^{i(t-t')D} F dt'\|_{L_t^2 L^\infty L^2} \lesssim \|\rho^{-1}|x|^\frac{3}{2}|D|A^n_\rho F\|_{L_t^2 L^2}. \quad (3.4)
$$

**Proof.** The first estimate is precisely (2.36) of Corollary 2.4 in [7]. In order to prove (3.4), we argue exactly as in the proofs of Theorem 2.3 and Corollary 2.4 in [7], expanding the flow in spherical harmonics. The only modification is to replace the estimate after formula (2.30) in that paper with the following one:

$$
\int_0^t |\tilde{G}^n_k| ds \leq \int_{-\infty}^\infty w(\lambda + t - s)^{-1} |\tilde{G}^n_k(s, \lambda + t - s)| ds \leq \|w(r)^{-1}\|_{L^2} Q_k^n(\lambda + t)
$$

where the weight $w$ is now $w(r) = |r|^{1/2} \rho(|r|)^{-1}$ instead of $w(r) = (r)^{1/4}$, and

$$
Q_k^n(\mu) := \left(\int_{-\infty}^\infty w(\mu - s)^2 |\tilde{G}^n_k(s, \mu - s)|^2\right)^{\frac{1}{2}}.
$$

Since we have

$$
\|w(r)^{-1}\|_{L^2(R)} = \|\rho^{-\frac{1}{4}}\|_{L^2(R)} \leq \|\rho\|_{L^\infty} \|r|^{-\frac{1}{4}}\|_{L^\infty} < \infty,
$$

this implies

$$
\int_0^t |\tilde{G}^n_k(s, t - s + \lambda)| ds \lesssim Q_k^n(\lambda + t)
$$

as in [7]. The rest of the proof is unchanged. \qed

With the help of (3.3), (3.4) we can deduce from the smoothing estimates of Theorem 2.1 the endpoint Strichartz estimates for the perturbed flow:

**Theorem 3.2.** Let $\rho \in \ell^2 L^\infty$, radially symmetric, with $\rho^{-2}|x| \in A_2$. Assume Condition (V) holds with $\sigma$ small enough. If in addition we assume

$$
\rho^{-2}|x|((|V| + |\partial V|) \in L^\infty, \quad (3.5)
$$

then the perturbed flow satisfies

$$
\|e^{it(D+V)} f\|_{L_t^2 L^\infty L^2} \lesssim \|f\|_{H^1}. \quad (3.6)
$$

On the other hand, if $V$ has the special form

$$
V = A_0 \beta + V_0
$$

and satisfies (besides Condition (V)) the assumptions (2.6), (2.7) for some $s > 1$, then we have

$$
\|A^n_\rho e^{it(D+V)} f\|_{L_t^2 L^\infty L^2} + \|A^n_\rho e^{it(D+V)} f\|_{L_t^\infty H^s} \lesssim \|A^n_\rho f\|_{H^1}. \quad (3.7)
$$

**Proof.** By Duhamel’s formula we can write

$$
e^{it(D+V)} f = e^{itD} f - i \int_0^t e^{i(t-t')D} (V f) dt', \quad (3.8)
$$

where $u = e^{it(D+V)} f$. By (3.3), (3.4) we get

$$
\|A^n_\rho e^{it(D+V)} f\|_{L_t^2 L^\infty L^2} \lesssim \|A^n_\rho f\|_{H^1} + \|\rho^{-1}|x|^{1/2}\|D\|L_t^2 L^2.
$$

Since $\rho^{-2}|x| \in A_2$, we can replace $|D|$ by $\partial$ in the last term:

$$
\|\rho^{-1}|x|^{1/2} A^n_\rho |D|(V f)\|_{L_t^2 L^2} \simeq \|\rho^{-1}|x|^{1/2} A^n_\rho (|\partial V| f + V(\partial u))\|_{L_t^2 L^2}.
$$

In the case $s = 0$, we continue the estimate as follows

$$
\lesssim \|\rho^{-2}|x|((|V| + |\partial V|)\|_{L^\infty} \|\rho|x|^{-1/2}(|u| + |\partial u|)\|_{L_t^2 L^2}
$$
and using the smoothing estimates of Theorem 2.1 and assumption (3.5), we obtain (3.6). If instead \( s > 1 \), we estimate as follows
\[
\lesssim \|\rho^{-2}|x|(|A_s^u D V| + |A_s^u V|)\|_{L^\infty L^2} \leq \|\rho|x|^{-1/2}(|A_s^u u| + |A_s^u \partial u|)\|_{L^1 L^2}
\]
thanks to the product rule (2.65), and using the estimates of Corollary 2.2 we obtain the first part of (3.7).

It remains to prove the second part of (3.7), i.e., the energy estimate with angular regularity. First of all we note that the untruncated estimate for the free flow
\[
\| \int_0^{+\infty} e^{i(t-t')D} F(t')dt'\|_{L^\infty L^2} \lesssim \|\rho^{-1}|x|^{1/2} F\|_{L^1 L^2}
\]
can be proved by splitting the integral as
\[
e^{itD} \int_0^{+\infty} e^{-i\tau D} F(t')dt'
\]
and then using the conservation of \( L^2 \) norm for \( e^{i\tau D} \) in combination with the dual of the smoothing estimate (2.2) in the case \( V = 0 \). Then by a standard application of the Christ–Kiselev Lemma the same estimate holds for the truncated integral:
\[
\| \int_0^{+\infty} e^{i(t-t')D} F(t')dt'\|_{L^\infty L^2} \lesssim \|\rho^{-1}|x|^{1/2} F\|_{L^1 L^2}.
\]
The corresponding estimate with angular regularity
\[
\|A_s^u \int_0^{+\infty} e^{i(t-t')D} F(t')dt'\|_{L^\infty L^2} \lesssim \|\rho^{-1}|x|^{1/2} A_s^u F\|_{L^1 L^2}
\]
does not follow immediately since \( A_s^u \) does not commute with \( D \); however, to overcome this difficulty, it is sufficient to replace \( A_s^u \) with the operator \( |K|^s \) defined in (2.63), which commutes with \( D \) and generates equivalent Sobolev norms on \( S^2 \). With the same arguments one proves
\[
\|A_s^u \int_0^{+\infty} e^{i(t-t')D} |K|F(t')dt'\|_{L^\infty L^2} \lesssim \|\rho^{-1}|x|^{1/2} A_s^u |K|F\|_{L^1 L^2},
\]
Thus we see that, using again the representation (3.8), the previous computations give also the second part of (3.7) and the proof is concluded.

4. Global existence for small data

We now prove Theorem 1.3. The proof is based on a straightforward fixed point argument in the space \( X \) defined by the norm
\[
\|u\|_X := \|A_s^u u\|_{L^1 L^\infty [0,\infty]} + \|\Lambda_s^u u\|_{L^\infty H^1}. \tag{4.1}
\]
Notice that estimate (1.12) can be written simply
\[
\|e^{it(D+V)} f\|_X \lesssim \|\Lambda_s^u f\|_{H^1}. \tag{4.2}
\]
Define \( u = \Phi(v) \) for \( v \in X \) as the solution of the linear problem
\[
iu_t + Du + Vu = \langle \beta u, u \rangle \beta u, \quad u(0, x) = u_0(x) \tag{4.3}
\]
and represent \( u \) as
\[
u = \Phi(v) = e^{it(D+V)} u_0 - i \int_0^t e^{i(t-t')(D+V)} \langle \beta u, u \rangle \beta u dt'.
\]
Now by the product estimate (2.65) and by (4.2) we have
\[
\|u\|_X \lesssim \|A_s^u f\|_{H^1} + \int_0^\infty \|e^{i(t-t')D} \langle \beta u, u \rangle \beta u\|_X dt' \lesssim \|A_s^u f\|_{H^1} + \int_0^\infty \|\Lambda_s^u \langle \beta u, u \rangle \beta u\|_{H^1} dt' \equiv \|A_s^u f\|_{H^1} + \|\Lambda_s^u P(v, v)\|_{L^1 H^1}.
\]
Using again (2.65) we have
\[
\|\Lambda_s^u (v^3)\|_{L^2(S^2)} \lesssim \|\Lambda_s^u v\|_{L^2(S^2)}^3
\]
so that
\[ \| \Lambda^\omega_s(v^3) \|_{L^2} \lesssim \| \Lambda^\omega_s v \|_{L^2} \| \Lambda^\omega_s v \|_{L^2}^2 \]
and
\[ \| \Lambda^\omega_s(v^3) \|_{L^1 L^2} \lesssim \| \Lambda^\omega_s v \|_{L^\infty} \| \Lambda^\omega_s v \|_{L^2} \| \Lambda^\omega_s v \|_{L^2}^2 \lesssim \| v \|_{H^1}^3. \]  \hspace{1cm} (4.4)

In a similar way,
\[ \| \Lambda^\omega_s \nabla(v^3) \|_{L^2(G^2)} \lesssim \| \Lambda^\omega_s \nabla v \|_{L^2(G^2)} \| \Lambda^\omega_s v \|_{L^2}^2 \%_2 \]
so that
\[ \| \Lambda^\omega_s \nabla(v^3) \|_{L^2} \lesssim \| \Lambda^\omega_s \nabla v \|_{L^2} \| \Lambda^\omega_s v \|_{L^2} \| \Lambda^\omega_s v \|_{L^2}^2 \lesssim \| v \|_{X}^3. \]  \hspace{1cm} (4.5)

In conclusion, (4.4) and (4.5) imply
\[ \| \Lambda^\omega_s P(v, \nabla v) \|_{L^1 H^1} \lesssim \| v \|_X^3 \]
and the estimate for \( u = \Phi(v) \) is
\[ \| u \|_X \equiv \| \Phi(v) \|_X \lesssim \| \Lambda^\omega_s f \|_{H^1} + \| v \|_{X}^3. \]

An analogous computation gives the estimate
\[ \| \Phi(v) - \Phi(w) \|_X \lesssim \| v - w \|_X \cdot (\| v \|_X + \| w \|_X)^2 \]
and an application of the contraction mapping theorem gives the existence and uniqueness of a global solution. The proof of scattering is completely standard and is omitted.

5. Conserved quantities

We observe the conserved quantities for (1.1) (see [10, 19]).

**Lemma 5.1.** Let \( u \) be a solution to (1.1). Then,
\[ \int_{\mathbb{R}^3} |u(t, x)|^2 dx, \quad \int_{\mathbb{R}^3} (\gamma u(t, x), \overline{u(t, x)}) dx \]
are independent of \( t \).

**Proof.** Since \( V(x) \) is hermitian, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |u(t, x)|^2 dx \\
= \int_{\mathbb{R}^3} \{i(Du(t, x) + V(x)u(t, x) - (\beta u(t, x), u(t, x))\beta u(t, x), u(t, x)) \\
- i(u(t, x), Du(t, x) + V(x)u(t, x) - (\beta u(t, x), u(t, x))\beta u(t, x))\} dx \\
= \sum_{j=1}^3 \int_{\mathbb{R}^3} \partial_j(\alpha_j u(t, x), u(t, x)) dx = 0.
\]

From
\[ \beta \gamma = -\gamma \beta, \quad \alpha_j \gamma = \gamma \alpha_j, \quad V(x) \gamma = -\gamma V(x), \]
we have
\[\frac{d}{dt} \int_{\mathbb{R}^3} (\gamma u(t,x), \overline{u(t,x)}) dx = \int_{\mathbb{R}^3} \left\{ i(\gamma Du(t,x) + V(x)u(t,x) - (\beta u(t,x), u(t,x))\gamma \beta u(t,x), \overline{u(t,x)}) + i(\gamma u(t,x), Du(t,x) + V(x)u(t,x) - (\beta u(t,x), u(t,x))\beta u(t,x), \overline{u(t,x)}) \right\} dx \]
\[= \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_j(\gamma \alpha_j u(t,x), \overline{u(t,x)}) dx = 0. \]
\[\square\]

From these conserved quantities, \((\beta u, u) = 0\) for any \(t \in \mathbb{R}\) provided that \(\gamma u_0 = \overline{u_0}\); this is called the Lochak–Majorana condition [17, 3].

**Corollary 5.2.** Let \(u\) be a solution to (1.1) with \(\gamma u_0 = \overline{u_0}\). Then, \((\beta u, u) = 0\) for any \(t \in \mathbb{R}\).

**Proof.** From \(|\gamma u - \overline{u}|^2 = 2|u|^2 + 2\Re(\gamma u, \overline{u})\),
\[\int_{\mathbb{R}^3} |\gamma u(t,x) - u(t,x)|^2 dx \]
is also a conserved quantity. By the assumption, \(\gamma u = \overline{u}\) for any \(t \in \mathbb{R}\). Then,
\[\langle \beta u, u \rangle = (\beta \gamma, u) = -\langle \gamma \beta, u \rangle = -\langle \beta, u \rangle = -\langle \beta u, u \rangle.\]
Since \((\beta u, u)\) is real valued, we obtain \((\beta u, u) = 0\). \(\square\)

## 6. Global existence for large data

We now prove Theorem 1.5. Denote by \(\chi_0 = Pu_0\) the projection of the initial data on the subspace \(E\) (see (1.13)–(1.15)), and let \(\chi\) be a solution to
\[i\partial_t \chi + D\chi + V(x)\chi = (\beta \chi, \chi)\beta \chi, \quad \chi(0, x) = \chi_0(x).\]
From \(A\chi_0 = \overline{\chi_0}\) and Corollary 5.2, the nonlinear term vanishes. In particular, \(\chi\) is a solution to the linear problem
\[i\partial_t \chi + D\chi + V(x)\chi = 0, \quad \chi(0, x) = \chi_0(x)\]
that is to say, \(\chi = e^{it(D+V)}\chi_0\).

Setting \(v = u - \chi\), where \(u\) is the solution to be constructed, we consider the following Cauchy problem:
\[i\partial_t v + Dv + V(x)v = F(v, \chi), \quad v(0, x) = v_0(x) := u_0(x) - \chi_0(x),\]
(6.1)
where
\[F(v, \chi) := (\beta u, u)\beta u - (\beta \chi, \chi)\beta \chi = (\beta v, v) + (\beta \chi, v)v + (\beta v, \chi)v + (\beta \chi, \chi)\]
\[+ (\beta \chi, \chi)v + (\beta \chi, v)\chi + (\beta v, \chi)\chi.\]
Let
\[\|u\|_{X_I} := \|A^\gamma u\|_{L^\infty_t H^1} + \|A^\gamma u\|_{L^\infty_t L^2_{\alpha|x|}}\]
for an interval \(I \subset \mathbb{R}\). We define
\[\Phi(v)(t) := e^{it(D+V)}v_0 - i \int_0^t e^{i(t-t')(D+V)}F(v(t'), \chi(t'))dt'.\]
Since $\chi_0$ is not small, we shall divide the time interval into a finite number of subintervals such that the norm of $\chi$ is sufficiently small on each.

Let $C_0$ and $C_1$ be the absolute constants appearing in the estimates below. From Theorem 1.2, estimate (1.12), there exists $T^* > 0$ such that

$$\|A^\ast_\nu \chi\|_{L^2_{[T^*, \infty]} L^\infty_{[0, T^*]}} \leq \frac{1}{10(C_1\|A^\ast_\nu \chi_0\|_{H^1} + 1)}.$$  

In addition, we can take $T_\ast > 0$ satisfying

$$\sup_{0 \leq T \leq T_\ast} \|A^\ast_\nu \chi\|_{L^2_{[T, T+T_\ast]} L^\infty_{[0, T]}} \leq \frac{1}{10(C_1\|A^\ast_\nu \chi_0\|_{H^1} + 1)}.$$  

Let $k$ be a minimum natural number satisfying $kT_\ast > T^*$. We take sufficiently small $\epsilon > 0$ with

$$4(2C_0)^{2(k+1)} C_1 \epsilon^2 + 2(2C_0)^{k+1} C_1 (\|A^\ast_\nu \chi_0\|_{H^1} + 1) \epsilon < \frac{1}{10}. \quad (6.2)$$

We assume that $\|A^\ast_\nu v_0\|_{H^1} \leq \epsilon$. Again (1.12) yields

$$\|\Phi(v)\|_{X_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v_0\|_{H^1} + \|A^\ast_\nu F(v, \chi)\|_{L^1_{[0, T_\ast]} H^1}.$$  

For simplicity, we denote a cubic part with respect to $f$, $g$ and $h$ by $fg$, e.g., $v^2 \chi$ means $(\beta \chi, v) \beta v$ or $(\beta v, \chi) \beta v$ or $(\beta v, v) \beta \chi$. By (2.65), we have

$$\|A^\ast_\nu (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2}^2,$$

$$\|A^\ast_\nu (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2}^2 \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}}.$$  

Similarly, we have

$$\|A^\ast_\nu \nabla (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu \nabla v\|_{L^2} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu \nabla v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}}.$$  

The calculation used above gives

$$\|A^\ast_\nu (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} L^\infty_{[0, T_\ast]},$$

$$\|A^\ast_\nu \nabla (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} L^\infty_{[0, T_\ast]},$$

$$\|A^\ast_\nu \nabla (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} L^\infty_{[0, T_\ast]},$$

$$\|A^\ast_\nu \nabla (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} L^\infty_{[0, T_\ast]},$$

$$\|A^\ast_\nu \nabla (v^3)\|_{L^2_{[0, T_\ast]}} \lesssim \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu v\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]}} \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} L^\infty_{[0, T_\ast]},$$

Hence, we have

$$\|\Phi(v)\|_{X_{[0, T_\ast]}} \leq C_0 \|A^\ast_\nu v_0\|_{H^1} + C_1 \|v\|_{X_{[0, T_\ast]}},$$

$$\quad + C_1 \left( \|A^\ast_\nu \chi_0\|_{H^1} + \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} \right) \|v\|_{X_{[0, T_\ast]}},$$

$$\quad + C_1 \left( \|A^\ast_\nu \chi_0\|_{H^1} + \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} \right) \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} \|v\|_{X_{[0, T_\ast]}},$$

$$\quad \leq C_0 \|A^\ast_\nu v_0\|_{H^1} + C_1 \|v\|_{X_{[0, T_\ast]}},$$

$$\quad + C_1 \left( \|A^\ast_\nu \chi_0\|_{H^1} + \|A^\ast_\nu \chi\|_{L^2_{[0, T_\ast]} L^\infty_{[0, T_\ast]}} \right) \|v\|_{X_{[0, T_\ast]}},$$

Then, $\Phi$ is a mapping from $B_1 := \{ v \in X_{[0, T_\ast]} : \|v\|_{X_{[0, T_\ast]}} \leq 2C_0 \epsilon \}$ into itself because of (6.2).
Let $v_1$ and $v_2$ be solutions to (6.1). The difference $v_1 - v_2$ satisfies
\[ i \partial_t (v_1 - v_2) + D (v_1 - v_2) + V (x) \beta (v_1 - v_2) = F (v_1, \chi) - F (v_2, \chi), \quad (v_1 - v_2) (0, x) = 0. \]

Accordingly, for $v_1, v_2 \in B_1$, we have
\[
\| \Phi (v_1) - \Phi (v_2) \|_{X [0, T_*]} \leq C_1 \left( \| v_1 \|_{X [0, T_*]} + \| v_2 \|_{X [0, T_*]} + \| \Delta^4 w_2 \|_{L^{\infty} (0, T_*) L^2} + \| \Delta^4 w_2 \|_{L^{\infty} (0, T_*) L^2} \right) \\
+ \left( \| v_1 \|_{X [0, T_*]} + \| v_2 \|_{X [0, T_*]} \right) \left( \| \Delta^2 \chi \|_{H^1} + \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \\
+ \left( \| \Delta^2 \chi \|_{L^2} + \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \left( \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \| v_1 - v_2 \|_{X [0, T_*]} \\
\leq \frac{1}{2} \| v_1 - v_2 \|_{X [0, T_*]}.
\]

Therefore, $\Phi : B_1 \mapsto B_1$ is a contraction mapping, and we obtain a unique solution $v$ to (6.1).
Since the existence time $T_*$ depends only on $\chi_0$, we can extend the existence time to $[0, k T_*]$. Indeed, setting $I_n := [(n - 1) T_*, n T_*]$ for $n = 1, 2, \ldots, k$, we have
\[
\| \Phi (v) \|_{X_{I_n}} \leq C_0 \| \Delta^2 \chi \|_{((n - 1) T_*, T_*)} + C_1 \| v \|_{X_{I_n}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \| v \|_{X_{I_n}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \| v \|_{X_{I_n}}^2 \\
\leq C_0 \| \Delta^2 \chi \|_{((n - 1) T_*, n T_*)} + C_1 \| v \|_{X_{I_n}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \right) \| \Delta^2 \chi \|_{L^{\infty} (0, T_*) L^2} \| v \|_{X_{I_n}}^2.
\]

Then, $\Phi$ is a mapping from $B_1 := \{ v \in X_{I_n} : \| v \|_{X_{I_n}} \leq (2 C_0)^n \| v \|_{X_{I_n}} \}$ into itself because of (6.2).

The estimate for the difference is similarly handled. Hence, the existence of a unique solution $v$ to (6.1) follows from the contraction mapping theorem. Thus, we obtain the unique solution $u$ on the time interval $[0, k T_*]$. Similarly, we have
\[
\| \Phi (v) \|_{X_{[T^*, \infty)}} \leq C_0 \| \Delta^2 \chi \|_{(T^*, \infty)} + C_1 \| v \|_{X_{[T^*, \infty)}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T^*) L^2} \right) \| v \|_{X_{[T^*, \infty)}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T^*) L^2} \right) \| \Delta^2 \chi \|_{L^{\infty} (0, T^*) L^2} \| v \|_{X_{[T^*, \infty)}}^2 \\
\leq C_0 \| \Delta^2 \chi \|_{(T^*, \infty)} + C_1 \| v \|_{X_{[T^*, \infty)}}^2 \\
+ C_1 \left( \| \Delta^2 \chi \|_{L^{\infty} (0, T^*) L^2} \right) \| \Delta^2 \chi \|_{L^{\infty} (0, T^*) L^2} \| v \|_{X_{[T^*, \infty)}}^2.
\]

The estimate for the difference follows in the same manner. Then, $\Phi$ is a contraction mapping from $B_{\infty} := \{ v \in X_{[T^*, \infty)} : \| v \|_{X_{[T^*, \infty)}} \leq (2 C_0)^{k + 1} \| v \|_{X_{[T^*, \infty)}} \}$ into itself because of (6.2).

To show the scattering, we set
\[
v_+ := v_0 - i \int_0^\infty e^{-i t (D + V)} F (v_0', \chi (t')) dt',
\]
which satisfies $\Lambda_+^s v + e^{it(D+V)} v = \Lambda_+^s F(v, \chi) \in L^1(\mathbb{R}; H^1(\mathbb{R}^3))$. Then,

$$
\|\Lambda_+^s v(t) - \Lambda_+^s e^{it(D+V)} v_+\|_{H^1} \\
\lesssim \|\Lambda_+^s F(v, \chi)\|_{L^1(t, \infty; H^1)} + 2 \sum_{j=1}^{r(s-1)/2} \|\Lambda_+^s v\|_{L^2(t, \infty; L^2)} \|\Lambda_+^s \chi\|_{L^\infty(t, \infty; L^2)} \\
+ \|\Lambda_+^s \chi\|_{H^1} + \|\Lambda_+^s \chi\|_{H^1(t, \infty; L^2)} \|\Lambda_+^s v\|_{H^1(t, \infty; L^1)} \\
+ \|\Lambda_+^s \chi\|_{L^2(t, \infty; L^\infty)} \|\Lambda_+^s v\|_{H^1(t, \infty; L^1)}
$$

for any $t > 0$. Therefore,

$$
\lim_{t \to \infty} \|\Lambda_+^s v(t) - \Lambda_+^s e^{it(D+V)} v_+\|_{H^1} = 0.
$$

From $\chi(t) = e^{it(D+V)} \chi_0$, setting $u_+ := v_+ + \chi_0$, we obtain the desired result.

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