On an extension of the generalized BGW tau-function

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Abstract
For an arbitrary solution to the Burgers–KdV hierarchy, we define the tau-tuple $(\tau_1, \tau_2)$ of the solution. We show that the product $\tau_1 \tau_2$ admits Buryak’s residue formula. Therefore, according to Alexandrov’s theorem, $\tau_1 \tau_2$ is a tau-function of the KP hierarchy. We then derive a formula for the affine coordinates for the point of the Sato Grassmannian corresponding to the tau-function $\tau_1 \tau_2$ explicitly in terms of those for $\tau_1$. Applications to the analogous open extension of the generalized BGW tau-function and to the open partition function are given.

Keywords Burgers-KdV hierarchy · Dubrovin–Zhang type tau-function · KP tau-function · Analogous open extension · Schur polynomial · Generalized BGW tau-function

Mathematics Subject Classification 37K10 · 14H70 · 53D45

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Dedicated to the memory of Boris Anatol’evich Dubrovin, with admiration.

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1 Introduction

The Brézin–Gross–Witten (BGW) partition function was introduced in [11,24]; see also [3,16,20,32,33]. Recently, Norbury [34] constructed a cohomology class (called the Theta-class) on the moduli space of closed Riemann surfaces and indicated [34] that the partition function of the intersection numbers of the Theta-class coupling the well-known ψ-classes coincides with the BGW partition function. Recalling the construction about the moduli space of open Riemann surfaces [35] as well as the open extension of the partition function of the ψ-class intersection numbers [13,35], we consider in this paper the analogous open extension of the BGW partition function.

To illustrate more the motivations of the paper and of the terminology, let us provide some geometric ideas [17,21,38]. One important observation is due to Paolo Rossi. Rossi finds that the open WDVV equations given by Horev and Solomon [25] for \((n+1)\) observables could be viewed as the WDVV equations of an \((n+1)\)-dimensional generalized Frobenius manifold (with the existence-of-potential condition dropped, sometimes called flat F-manifold), which is a one-dimensional extension of an \(n\)-dimensional Frobenius manifold. Nevertheless, this type of generalized Frobenius manifold has a vector potential, from which the genus zero part of the abstract open-and-closed descendent invariants can be re-constructed [6]. For more details about Rossi’s observation as well as its outcome see [6]. Rossi’s observation gives rise to a tau-tuple \((e^{F_0}, e^{F_0^0})\) (in the notation of [6]) of the topological solution to the corresponding dispersionless open hierarchy. It also naturally invokes, following Witten and Dubrovin–Zhang [6,17–19,21,38], the tau-tuple \((e^F, e^{F_0})\) (in the notation of [6]) of the topological solution to the topological deformation of the dispersionless open hierarchy. This is non-trivial already when \(n = 1\). The structure of a tau-tuple (being referred to as the tau-structure) will capture the geometry for the open-and-closed descendent invariants and enables one, due to its differential-polynomial nature, to deal with an arbitrary solution to the open hierarchy (not just the topological solution giving the open-and-closed descendent invariants). Then, if the tau-function of some other solution to the integrable hierarchy of topological type of a Frobenius manifold has a topological expansion possessing enumerative meanings on the moduli space of closed Riemann surfaces, then its analogous open extension could be expected to have a meaning for the moduli space of open-and-closed Riemann surfaces. In this paper we will define the tau-structure for the Burgers–KdV hierarchy [12] and use it to give the analogous open extension for the BGW partition function. (The latter is proved to be a particular tau-function for the KdV hierarchy [32] (cf. also [3,16,34]) and possess enumerative meanings on the moduli space of closed Riemann surfaces [34].) Noting that there is a one-parameter generalization of the BGW partition function given in [3], we will actually study in detail the analogous open extension for an arbitrary value of the parameter, although we do not know if the generalized BGW partition function with the general parameter is connected to the moduli space of curves. We will also derive some general explicit formulae for the open extensions of an arbitrary solu-

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1 One of the authors D.Y. is grateful to Paolo Rossi and Sasha Buryak for sharing with him this interesting observation during a workshop at the University of Leeds in 2018.
tion using Buryak’s residue formula [13]. According to our knowledge, our explicit simplification is new even for the open partition function [13].

Denote by \( \mathcal{A}(w,\rho) \) the ring of polynomials of \( w_x, \rho_x, w_{xx}, \rho_{xx}, \ldots \) with coefficients being smooth functions of \( w, \rho \). The Burgers–KdV integrable hierarchy [12] is defined as the following pairwise commuting system of nonlinear PDEs:

\[
\frac{\partial w}{\partial q_n} = \partial_x K_n, \quad (1.1) \\
\frac{\partial \rho}{\partial q_n} = \partial_x R_n, \quad (1.2)
\]

where \( n \geq 1 \) and \( K_n, R_n \in \mathcal{A}(w,\rho) \) are defined recursively by

\[
K_1 = w, \quad K_2 = 0, \quad R_1 = \rho, \quad R_2 = \rho^2 + \rho_x + 2w, \quad (1.3)
\]

\[
\partial_x K_{2j+1} = \left( 2w \partial_x + w_x + \frac{1}{4} \partial_x^3 \right) K_{2j-1}, \quad K_{2j} = 0, \quad (1.4)
\]

\[
R_{2j+1} = \left( \partial_x^2 + 2\rho \partial_x + \rho^2 + \rho_x + 2w \right) R_{2j-1} + \left( \rho + \frac{3}{2} \partial_x \right) K_{2j-1}, \quad (1.5)
\]

\[
R_{2j+2} = \left( \partial_x^2 + 2\rho \partial_x + \rho^2 + \rho_x + 2w \right) R_{2j}. \quad (1.6)
\]

Here, \( j \geq 1 \), and the integration constants in the recursive procedure of solving \( K_{2j+1} \) are chosen to be zero. The \( n = 1 \) flow reads

\[
\frac{\partial w}{\partial q_1} = w_x, \quad \frac{\partial \rho}{\partial q_1} = \rho_x.
\]

Therefore, we can identify the variable \( q_1 \) with \( x \). Note that equations (1.1) are the KdV integrable hierarchy. Hence, the Burgers–KdV hierarchy can be viewed as an integrable extension of (1.1); for more details about the Burgers–KdV hierarchy see [12].

Recall that the canonical tau-structure [20,21] of the KdV hierarchy is defined as the unique family of elements \( (\Omega_{i,j})_{i,j \geq 1} \in \mathcal{A}_w \) satisfying

\[
\Omega_{1,1} = w, \quad \Omega_{i,j} = \Omega_{j,i}, \quad \frac{\partial \Omega_{i,j}}{\partial q_{2k-1}} = \frac{\partial \Omega_{i,k}}{\partial q_{2j-1}}, \quad \forall i, j, k \geq 1 \quad (1.7)
\]

along with the normalization condition \( \Omega_{i,j} |_{w_{x_k}=0,k \geq 0} = 0 \). Here, \( \mathcal{A}_w \subset \mathcal{A}(w,\rho) \) denotes the ring of polynomials of \( w_x, w_{xx}, \ldots \) with coefficients being smooth functions in \( w \). For the existence of \( \Omega_{i,j} \) see for example [20,21]. Inspired by the open and closed intersection numbers [9,12,13,35] and by the tau-structure of the Burgers hierarchy [19], we will define the tau-tuple for the Burgers–KdV hierarchy. It is shown in [12] that

\[
\frac{\partial R_i}{\partial q_j} = \frac{\partial R_j}{\partial q_i}, \quad \forall i, j \geq 1. \quad (1.8)
\]
By using these relations as well as relations (1.7) we arrive at the following lemma.

**Lemma 1.1** For an arbitrary solution \((w, \rho)\) in \(\mathbb{C}[[q]]^2\) to the Burgers–KdV hierarchy, there exist \(\tau_1, \tau_2 \in \mathbb{C}[[q]]^2\), such that

\[
\begin{align*}
\frac{\partial^2 \log \tau_1}{\partial q_{2i-1} \partial q_{2j-1}} &= \Omega_{i,j}, & \tau \frac{\partial^2 \log \tau_1}{\partial q_{2i} \partial q_{2j}} &= 0, & \forall \ i, j \geq 1, \\
\frac{\partial \log \tau_2}{\partial q_i} &= R_i, & \forall \ i \geq 1.
\end{align*}
\]

Moreover, the tuple \((\tau_1, \tau_2)\) is uniquely determined by \((w, \rho)\) up to the freedom described by

\[
(\tau_1, \tau_2) \mapsto \left( \tau_1 e^{\beta_0 + \sum_{n \geq 1} \beta_n q_n}, \tau_2 e^{\alpha_0} \right),
\]

where \(\alpha_0\) and \(\beta_n, n \geq 0\) are arbitrary constants.

We call \((\tau_1, \tau_2)\) the tau-tuple of the solution \((w, \rho)\) to the Burgers–KdV hierarchy and call the product \(\tau_1 \tau_2 =: \tau_E\) the tau-function for the Burgers–KdV hierarchy. Equations (1.9)–(1.10) for \(i = j = 1\) read

\[
\begin{align*}
w &= \partial_x^2 \log \tau_1, & \rho &= \partial_x \log \tau_2.
\end{align*}
\]

As it is proved in [7], the definition of a Dubrovin–Zhang-type tau-function \(\tau_1\) for the KdV hierarchy is equivalent to that of a Sato-type tau-function; see also [4,15]. We proceed to the study of the tau-tuple from the points of view of the Sato theory. Let us first recall some notations. Recall that a partition \(\lambda\) is a non-increasing sequence of non-negative integers \((\lambda_1, \lambda_2, \ldots)\) satisfying \(\sum_{i \geq 1} \lambda_i < +\infty\). The length of \(\lambda\) is defined as the number of the nonzero components of \(\lambda\), denoted by \(\ell(\lambda)\). The weight of \(\lambda\) is defined as \(\sum_{i \geq 1} \lambda_i\), denoted by \(|\lambda|\). Denote by \((m_1, \ldots, m_r | n_1, \ldots, n_r)\) the Frobenius notation (cf. [31]). The partition \((0, 0, \ldots)\) is denoted by \((0)\). By convention \(\ell((0)) := 1\). Denote by \(\mathcal{Y}\) the set of all partitions. The Schur polynomial associated with a partition \(\lambda\) is defined as follows:

\[
s_\lambda := \det(h_{\lambda_i-i+j})_{1 \leq i, j \leq \ell(\lambda)},
\]

where \(\{h_k\}_{k \geq 0}\) are polynomials in \(\mathbb{C}[q]\) defined by the generating function

\[
\sum_{k \geq 0} h_k z^k = \exp \left( \sum_{n \geq 1} q_n z^n \right).
\]

According to the Sato theory, the component \(\tau_1\) can be expressed as the linear combination of the Schur polynomials with coefficients being the Plücker coordinates, i.e.,

\[
\tau_1 = c_0 \sum_{\lambda \in \mathcal{Y}} \pi_\lambda s_\lambda.
\]
Here, \( c_0 \) is a nonzero constant often taken as 1, and

\[
\pi_{\lambda} := (-1)^{\sum_i n_i} \det(A_{m_i,n_j})_{1 \leq i,j \leq r}
\]  

(1.16)

with \( A_{m,n}, m,n \geq 0 \) being the affine coordinates \([5,22,39,40]\) for the point of Sato Grassmannian corresponding to \( \tau_1 \). In this paper, we will derive explicit expressions for \( \tau_1 \) and \( \tau_2 \).

Before presenting the results, we introduce further some notations. Note that the Schur polynomials form a basis of \( \mathbb{C}[[q]] \), and define a sequence of linear operators \( T_n : \mathbb{C}[[q]] \to \mathbb{C}[[q]] \) in the following way:

\[
T_n(s_{\lambda}) = \begin{cases} 
(-1)^i s_{\lambda_1-1,\ldots,\lambda_i-1,n+i,\lambda_{i+1},\ldots,\lambda_{\ell(\lambda)}), & \text{if } \lambda_i > n + i \geq \lambda_{i+1}, \text{ for some } i, \\
0, & \text{otherwise}.
\end{cases}
\]

(1.17)

Here we set \( \lambda_0 = +\infty \). The operator \( T_0 \) has the alternative expression:

\[
T_0(s_{m_1,\ldots,m_r|n_1,\ldots,n_r}) = \begin{cases} 
(-1)^r s_{m_1-1,\ldots,m_r-1|n_1+1,\ldots,n_r+1}, & \text{if } m_r \geq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(1.18)

Note that for each \( n \geq 0 \), the linear operator \( T_n \) induces the linear transformation on \( \text{Span}_\mathbb{C}(\mathcal{Y}) \), denoted again by \( T_n \). The first result of the paper is given by the following proposition.

**Proposition 1.2** Let \((w, \rho)\) be an arbitrary solution in \( \mathbb{C}[[q]]^2 \) to Burgers–KdV hierarchy (1.1)–(1.2), and \((\tau_1, \tau_2) \in \mathbb{C}[[q]]^2 \) its tau-tuple with \( \tau_2(0) \) chosen as 1. There exists a unique sequence of numbers \( b_1, b_2, b_3, \ldots \), such that

\[
\tau_1 \tau_2 = Q \circ T_0 \circ Q^{-1}(\tau_1),
\]

(1.19)

where \( Q = e^{\sum_{k \geq 1} b_k q_k} \) denotes the multiplication operator on \( \mathbb{C}[[q]] \).

The proof is in Sect. 3.

Noticing that \( Q \circ T_0 \circ Q^{-1} \) is a linear operator and using (1.15), we will simplify (1.19) via computing the action of \( Q \circ T_0 \circ Q^{-1} \) on each single Schur polynomial. Let us present the results here and leave the details of calculations to Sect. 3. For \( \lambda \in \mathcal{Y} \), define \( \pi_{-\lambda} := -\pi_{\lambda} \). Let \( c_{\lambda,\mu}^v, \lambda, \mu, v \in \mathcal{Y} \) denote the Littlewood–Richardson coefficients [30]:

\[
s_{\lambda} s_{\mu} := \sum_{v \in \mathcal{Y}} c_{\lambda,\mu}^v s_v.
\]

(1.20)

For \( \lambda, \mu, v \in \mathcal{Y} \), we set \( c_{\lambda,-\mu}^v := -c_{\lambda,\mu}^v \). Define a sequence of numbers \( g_0, g_1, g_2, g_3, \ldots \) by

\[
\sum_{n \geq 0} g_n z^{-n} := e^{\sum_{k \geq 1} b_k z^{-k}}.
\]

(1.21)
Proposition 1.3 The tau-function $\tau_E$ has the expression

$$\tau_E = \sum_{\lambda \in \mathbb{Y}} s_\lambda \sum_{1 \leq i \leq \ell(\lambda)} (-1)^{i-1} g_{\lambda_i - i + 1} \pi(\lambda_1 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots). \tag{1.22}$$

Alternatively, denoting $b = (b_1, b_2, b_3, \ldots)$, we have

$$\tau_E = \sum_{\lambda \in \mathbb{Y}} s_\lambda \sum_{\alpha, \mu, \nu, \beta \in \mathbb{Y} \setminus \text{Ker}(T_0)} \pi_{\alpha} c_{\mu \alpha}^\beta c_{\nu, T_0(\beta)}^\lambda (-b) s_\mu (b). \tag{1.23}$$

Remark 1.4 We note that in (1.23), for each $\lambda \in \mathbb{Y}$, the second summand is a finite sum, due to a simple degree argument ($|T_0(\beta)| = |\beta|$, and $c_{\lambda \mu}^\nu \neq 0$ only if $|\lambda| + |\mu| = |\nu|$, cf. [30]).

For a hook partition $\lambda$ with the Frobenius notation $(m|n)$, $m, n \geq 0$, we have

$$\{ i \mid 1 \leq i \leq \ell(\lambda), \lambda_i - i + 1 \geq 0 \} = \begin{cases} \{1\}, & n = 0, \\ \{1, 2\}, & n \geq 1. \end{cases} \tag{1.24}$$

Then from (1.22) of Proposition 1.3 we see that the coefficient of $s_{(m|n)}$ in $\tau_E$, denoted by $c_{m,n}$, has the more explicit expression:

$$c_{m,n} = \begin{cases} g_{m+1}, & n = 0, \\ (-1)^n \left(g_{m+1} A_{0,n-1} - A_{m+1,n-1}\right), & n \geq 1. \end{cases} \tag{1.25}$$

Combining with Theorem B.1 and the formula similar to (1.15) for the KP hierarchy (cf. [5,14,15,22,26,36,40]), we arrive at the following corollary.

Corollary 1.5 The tau-function $\tau_E$ has the following explicit expression:

$$\tau_E = \sum_{\lambda \in \mathbb{Y}} \pi'_\lambda s_\lambda. \tag{1.26}$$

Here for a partition $\lambda = (m_1, \ldots, m_r|n_1, \ldots, n_r)$, $\pi'_\lambda = \det(c_{m_i,n_j})_{1 \leq i, j \leq r}$ with $c_{m,n}$ defined by (1.25).

Both Proposition 1.3 and Corollary 1.5 can be used for efficient computations of $\tau_E$.

Remark 1.6 Propositions 1.2, 1.3 and Corollary 1.5 could be generalized to the KP hierarchy. In fact, let $\tau_1$ be an arbitrary tau-function of the KP hierarchy, and let $\tau_E$ be defined via Buryak’s residue formula:

$$\tau_E := -\text{res}_{z = \infty} \left(\sum_{n \geq 0} g_n z^{-n}\right) \tau_1 \left(q_1 - \frac{1}{z}, q_2 - \frac{1}{2z^2}, \ldots\right) e^{\sum_{k \geq 1} q_k z^k} \frac{dz}{z}. \tag{1.27}$$
where $g_0 = 1$ and $g_2, g_3, \ldots$ are arbitrary constants. Then $\tau_E$ is also a tau-function of the KP hierarchy [2] (see Theorem B.1 of “Appendix B” for the details and a new proof), and as a generalization of Proposition 1.2 this tau-function can be written into

$$\tau_E = Q \circ T_0 \circ Q^{-1}(\tau_1),$$

where the coefficients in $Q = e^{\sum_{k \geq 1} b_k q_k}$ are determined by the relation

$$e^{\sum_{k \geq 1} b_k z^{-k}} = \sum_{n \geq 0} g_n z^{-n}.$$  

Moreover, formulae (1.22), (1.23) and (1.26) (with $c_{m,n}$ defined by (1.25)) in Proposition 1.3 and Corollary 1.5 are still valid. We will give some combinatorial identities as applications of these results at the end of “Appendix B.”

We proceed with defining the extension of the generalized BGW tau-function. Consider the solution to Burgers–KdV hierarchy (1.1)–(1.2) specified by the initial condition

$$w|_{q_2 = q_3 = \ldots = 0} = \frac{C}{(1 - x)^2}, \quad \rho|_{q_2 = q_3 = \ldots = 0} = 0,$$  

(1.27)

where $C$ is an arbitrarily given constant ($N = \sqrt{1 + 2C}$ corresponds to the parameter in [3]). We call this solution the generalized BGW solution to the Burgers–KdV hierarchy, denoted by $(w/\Theta_1(C), \rho/\Theta_1(C))$. Let $(\tau_1^{\Theta(C)}, \tau_2^{\Theta(C)})$ be the tau-tuple of $(w^{\Theta(C)}, \rho^{\Theta(C)})$. As in [3,8,10,16,20] we normalize the first component $\tau_1^{\Theta(C)}$ by

$$\sum_{k \geq 0} (2k + 1)(q_{2k+1} - \delta_{k,0}) \frac{\partial \tau_1^{\Theta(C)}}{\partial q_{2k+1}} + C \tau_1^{\Theta(C)} = 0$$  

(1.28)

and by requiring $\partial \tau_1^{\Theta(C)}/\partial q_{2n} = 0$, call it the generalized BGW tau-function. (Note that $\tau_1^{\Theta(C)}$ is uniquely determined up to a constant factor that is irrelevant of the study.) We normalize the second component $\tau_2^{\Theta(C)}$ by requiring

$$\tau_2^{\Theta(C)}(0) = 1.$$  

(1.29)

We note that when the parameter $C$ is taken to be $1/8$, the first component $\tau_1^{\Theta(C)}$ becomes the BGW tau-function. Norbury [34] proves the following identity:

$$\tau_1^{\Theta(\frac{1}{8})} = \exp \left( \sum_{n \geq 0}^{n} \sum_{i_1, \ldots, i_n \geq 0} \frac{1}{n!} \prod_{k=1}^{n} (2i_k + 1)!! q_{2i_{k+1}} \int_{M_{g,n}} \Theta_{g,n} \psi_1^{i_1} \ldots \psi_n^{i_n} \right).$$  

(1.30)
Here, $\psi_i$ denotes the first Chern class of the $i$th tautological line bundle on $\overline{M}_{g,n}$, and $\Theta_{g,n}$ denotes the Theta-class [34]. Our next result is given by the following theorem.

**Theorem 1.7** Define a sequence of numbers $g_n^{(\Theta(C))}$ by

$$g_0^{(\Theta(C))} = 1, \quad g_n^{(\Theta(C))} = (n - 1)! \sum_{k=1}^{n} \frac{1}{k!(k-1)!} \prod_{i=1}^{k} \left( C + \frac{i(i-1)}{2} \right), \quad n \geq 1. \quad (1.31)$$

Define $b_1, b_2, b_3, \ldots$ via $\sum_{k \geq 1} b_k z^{-k} = \log \left( \sum_{n \geq 0} g_n^{(\Theta(C))} z^{-n} \right)$, and denote by $Q = e^{\sum_{k \geq 1} kb_k q_k}$ the multiplication operator. The following identity holds true:

$$\tau_2^{(\Theta(C))} = \frac{1}{\tau_1^{(\Theta(C))}} Q \circ T_0 \circ Q^{-1} (\tau_1^{(\Theta(C))}). \quad (1.32)$$

Moreover, $\tau_E^{(\Theta(C))} := \tau_1^{(\Theta(C))} \tau_2^{(\Theta(C))}$ satisfies the following linear equation:

$$\sum_{n \geq 1} n \bar{q}_n \frac{\partial \tau_E^{(\Theta(C))}}{\partial q_n} + C \tau_E^{(\Theta(C))} = 0, \quad \bar{q}_n = q_n - \delta_{n,1}. \quad (1.33)$$

Recall that based on [5,40] the explicit expression for the affine coordinates $A_m^{(\Theta(C))}$ of the point of the Sato Grassmannian corresponding to $\tau_1^{(\Theta(C))}$ was obtained in [20] (cf. also [23]):

$$A_m^{(\Theta(C))} = (-1)^m \sum_{r \geq s \geq 0, r + s = m + n + 1} \frac{r-s}{r+s} a_r a_s, \quad a_k := \frac{(-1)^k}{k!} \prod_{i=1}^{k} \left( C + \binom{i}{2} \right). \quad (1.34)$$

Define $\pi^{(\Theta(C))}_\lambda$ via (1.16) with $A_{m,n}$ replaced by $A_m^{(\Theta(C))}$. Combining with (1.22) we arrive at

**Corollary 1.8** The tau-function $\tau_E^{(\Theta(C))}$ has the form

$$\tau_E^{(\Theta(C))} = \sum_{\lambda \in \mathbb{Y}} \sum_{1 \leq i \leq \ell(\lambda)} (-1)^{i-1} g_{\lambda_i-i+1}^{(\Theta(C))} \prod_{1 \leq i \leq \ell(\lambda)} \pi^{(\Theta(C))}_{\lambda_i-i+1, \lambda_i-i+2, \ldots}. \quad (1.35)$$

For the reader’s convenience we list the first few coefficients of $\tau_E^{(\Theta(C))}$ in “Appendix A.”

Propositions 1.2 and 1.3 also apply to the topological solution to the Burgers–KdV hierarchy, denoted by $(w_{WK}, \rho_{WK})$, which is specified by the initial condition

$$w^{WK}|_{q_2=q_3=\ldots=0} = x, \quad \rho^{WK}|_{q_2=q_3=\ldots=0} = 0. \quad (1.36)$$

This solution governs the open and closed intersection numbers [12,13,35]. Let $\tau_1^{WK}$ be the tau-function of the solution $w^{WK}$ to the KdV hierarchy, normalized by the string equation:
\[
\sum_{n \geq 0} (2n + 1) \left( q_{2n+1} - \delta_{n,1} \right) \frac{\partial \tau_{WK}^{\lambda_1}}{\partial q_{2n+1}} + \frac{x^2}{2} \tau_{WK}^{\lambda_1} = 0. \tag{1.37}
\]

The explicit expression for the affine coordinates \( A_{WK}^{m,n} \) for the point of the Sato Grassmannian corresponding to \( \tau_{WK}^{\lambda_1} \) was first obtained in [39] and later re-proved in [5]:

\[
A_{WK}^{3m-1,3n} = A_{WK}^{3m-3,3n+2} = \frac{(-1)^n}{36^{m+n}} (6m + 1)!! \prod_{j=0}^{n-1} (m + j) \prod_{j=1}^{n} (2m + 2j - 1) \left( B(n, m) + \frac{2^n(6n + 1)!!}{(6m + 1)(2n)!} \right),
\tag{1.38}
\]

\[
A_{WK}^{3m-2,3n+1} = \frac{(-1)^{n+1}}{36^{m+n}} (6m + 1)!! \prod_{j=0}^{n-1} (m + j) \prod_{j=1}^{n} (2m + 2j - 1) \left( B(n, m) + \frac{2^n(6n + 1)!!}{(6m + 1)(2n)!} \right),
\tag{1.39}
\]

where \( n \geq 0, m \geq 1 \) and \( B(n, m) \) are defined as

\[
B(n, m) := \frac{1}{6} \sum_{j=1}^{n} 108^j \frac{2^n j (6n - 6j + 1)!!}{(2n - 2j)!} \frac{\Gamma(m + n + 1)}{\Gamma(m + n + 2 - j)}. \tag{1.40}
\]

Recall from [13] that the values \( g_n \) for the topological solution are given by

\[
g_n^{WK} = \begin{cases} 
\sum_{i=0}^{m} \frac{3^i (6m-6i)!}{888^{m-i} (2m-2i)! (3m-3i)!} \prod_{j=1}^{i} (m + \frac{1}{2} - j), & n = 3m, \\
0, & \text{otherwise.}
\end{cases} \tag{1.41}
\]

Define \( \pi_{\lambda}^{WK} \) via (1.16) with \( A_{m,n} \) replaced by \( A_{m,n}^{WK} \). Using (1.22) we then arrive at

**Corollary 1.9** The power series \( \tau_{WK}^{E} \) has the following expression:

\[
\tau_{WK}^{E} = \sum_{\lambda \in \Lambda'} \sum_{\substack{1 \leq i \leq \ell(\lambda) \\lambda_i \geq 0 \\ell_{\lambda_i-1} \geq 0}} (-1)^{i-1} g_{\lambda_{i-1}+1} \pi_{\lambda_{i+1}}^{WK} \tau_{E}^{WK}(\lambda_{i+1}, \ldots, \lambda_{i-1}+1, \lambda_i+1, \lambda_{i+2}, \ldots), \tag{1.42}
\]

**Organization of the paper** In Sect. 2, we will recall about some basics about the KdV hierarchy. In Sect. 3, we prove Propositions 1.2 and 1.3. In Sect. 4, we prove Theorem 1.7. Besides, we give a new proof of a result of Alexandrov in “Appendix B.”
2 A brief review of the KdV hierarchy

In this section, we recall some basic facts about KdV hierarchy (1.1). It is well known that this hierarchy can be written into the Lax form

$$\frac{\partial L}{\partial q_n} = \left[ \left( L_{\frac{n}{2}} \right)_+, L \right], \quad n \geq 1,$$  \hspace{1cm} (2.1)

where $L := \partial_x^2 + 2w(x)$. Here, for the meaning of $L_{\frac{n}{2}}$ see, e.g., [15]. For any solution $w \in \mathbb{C}[[q]]$ to the KdV hierarchy\(^2\), there exists [15,20] a pseudo-differential operator $P = \sum_{i \geq 0} \phi_i \partial_x^{-i} \in \mathbb{C}[[q]][[\partial_x^{-1}]]$, $\phi_0 := 1$ satisfying the followings:

$$L = P \circ \partial_x^2 \circ P^{-1},$$ \hspace{1cm} (2.2)

$$\frac{\partial P}{\partial q_n} = -(L_{\frac{n}{2}})_- \circ P, \quad \forall n \geq 1.$$ \hspace{1cm} (2.3)

The formal adjoint operator of $P$ is defined by

$$P^* := \sum_{i \geq 0} (-\partial_x)^{-i} \circ \phi_i =: \sum_{n \geq 1} \phi_i^* \partial_x^{-i}.$$ \hspace{1cm} (2.4)

For each solution $w$, define the wave function $\psi$ and its dual by

$$\psi := \psi(q; z) = \sum_{i \geq 0} \phi_i z^{-i} e^{\sum_{n \geq 1} q_n z^n},$$ \hspace{1cm} (2.5)

$$\psi^* := \psi^*(q; z) = \sum_{i \geq 0} \phi_i^* z^{-i} e^{-\sum_{n \geq 1} q_n z^n}.$$ \hspace{1cm} (2.6)

Then $\psi$ and $\psi^*$ satisfy the following linear equations:

$$L \psi = z^2 \psi, \quad L \psi^* = z^2 \psi^*,$$ \hspace{1cm} (2.7)

$$\frac{\partial \psi}{\partial q_n} = \left( L_{\frac{n}{2}} \right)_+ \psi, \quad \frac{\partial \psi^*}{\partial q_n} = \left( (L^*)_{\frac{n}{2}} \right)_+ \psi^*, \quad \forall n \geq 1.$$ \hspace{1cm} (2.8)

The following lemma was proven in, e.g., [14].

**Lemma 2.1** ([14]) The following bilinear identity holds true:

$$\text{res}_{z=\infty} \psi(q; z) \psi^*(q'; z) dz = 0, \quad \forall q = (q_1, q_2, \ldots), \quad q' = (q'_1, q'_2, \ldots).$$ \hspace{1cm} (2.9)

---

\(^2\) Usually the even flows are not included. Since these flows vanish and are therefore trivial, we have added them in (1.1). In other words, we understand in the way that (1.1) provides solutions for the KP hierarchy.
According to the Sato theory [14,15,36], (cf. also [27]) there exists a (unique up to a multiplicative constant) formal series \( \tau(q) \) satisfying
\[
\psi (q; z) = \frac{\tau(q - [z^{-1}])}{\tau(q)} e^{\sum_{n \geq 1} q_n z^n}, \\
\psi^* (q; z) = \frac{\tau(q + [z^{-1}])}{\tau(q)} e^{-\sum_{n \geq 1} q_n z^n},
\]
where \([z^{-1}] := (\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots)\). The series \( \tau(q) \) is called the Sato-type tau-function of the solution \( w \) for the KdV hierarchy, which will play an important role in the later sections. Observe that the above \( \phi_i \) and \( \phi_i^* \), \( i \geq 0 \) do not depend on \( q_2, q_4, q_6, \ldots \). Therefore, if we view \( q_2, q_4, q_6, \ldots \) as constants, then as it was proved in [20] the \( \psi \) and \( \psi^* \) defined by (2.5)–(2.6) satisfy the pair condition of [20]. This statement is applied in several places of this paper.

3 Proofs of Propositions 1.2 and 1.3

The goal of this section is to prove Propositions 1.2 and 1.3.

The following lemma that will play a fundamental role generalizes a result of Buryak [13].

**Lemma 3.1** Let \((w, \rho) \in \mathbb{C}[[q]]^2\) be an arbitrary solution to the Burgers–KdV hierarchy, and let \((\tau_1, \tau_2) \in \mathbb{C}[[q]]^2\) be its tau-tuple with fixed choice (1.11). There exists a unique series
\[
g(z) = \sum_{n \geq 0} g_n z^{-n} \in \mathbb{C}[[z^{-1}]], \\
g_n \in \mathbb{C}, \ g_0 \neq 0,
\]
such that the component \( \tau_2 \) has the expression
\[
\tau_2 = -\text{res}_{z=\infty} g(z) \frac{\tau_1(q - [z^{-1}])}{\tau_1(q)} e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}.
\]

**Proof** According to [7], the Dubrovin–Zhang-type tau-function \( \tau_1 \) of the solution \( w \) for the KdV hierarchy defined by (1.9) is also a Sato-type tau-function of \( w \), i.e., the \( \psi \) defined by
\[
\psi(q; z) := \frac{\tau_1(q - [z^{-1}])}{\tau_1(q)} e^{\sum_{n \geq 1} q_n z^n}
\]
a wave function (2.7), (2.8) of the KP hierarchy. Let us define a sequence of numbers \( g_n, n \geq 0 \):
\[
g_0 = \tau_2(0), \\
g_n = \partial^n_q \mid_{q=0}(\tau_2) + \sum_{k=0}^{n} \sum_{i=0}^{\min\{k, n-1\}} \text{res}_{z=\infty} \frac{g_i dz}{z^{k-i-1}} \mid_{x=0} \left( \frac{\tau_1(x - \frac{1}{z}, -\frac{1}{2z^2}, \ldots)}{\tau_1(x, 0, \ldots)} \right), \ n \geq 1.
\]
Define $g(z) = \sum_{n \geq 0} g_n z^{-n}$, and define $\tilde{\tau}_2 \in \mathbb{C}[[q]]$ by $\tilde{\tau}_2 = - \text{res}_{z=\infty} g(z) \psi(q; z) \frac{dz}{z}$.

From the definition of $g_n$ we have

$$\tilde{\tau}_2(x, 0, \ldots) = \tau_2(x, 0, \ldots).$$

(3.5)

Using (2.8) we know that $\forall n \geq 1$,

$$\frac{\partial \tilde{\tau}_2}{\partial q_n} = (L_2^{q})_+ (\tilde{\tau}_2),$$

(3.6)

where $L = \partial_x^2 + 2w$. According to (1.10) and [12], the second component $\tau_2$ satisfies

$$\frac{\partial \tau_2}{\partial q_n} = (L_2^{q})_+ (\tau_2), \; \forall n \geq 1.$$  

(3.7)

Combining with formulae (3.5)–(3.7), we obtain $\tau_2 = \tilde{\tau}_2$. The uniqueness follows from requirement (3.2) (in fact restricting to $q_2 = q_3 = \cdots = 0$ already implies the uniqueness of $g_n$). The lemma is proved.

The following lemma gives explicit computation on the maps $T_n$ in terms of the Schur basis.

**Lemma 3.2** For each $n \geq 0$ and $\lambda \in \mathbb{Y}$, we have

$$T_n(s_\lambda) = \det(h_{k_i-i+j})_{1 \leq i,j \leq \ell(\lambda)+1},$$

(3.8)

where $k_1 = n$, and $k_i = \lambda_{i-1} \; (2 \leq i \leq \ell(\lambda)+1)$.

The proof of this lemma is elementary, and we omit its details.

**Lemma 3.3** The transformation $T_0$ defined by (1.18) satisfies

$$T_0(f) = - \text{res}_{z=\infty} f(q-[z^{-1}]) e^{\sum_{n \geq 1} g_n z^n} \frac{dz}{z}, \; \forall f \in \mathbb{C}[[q]].$$

(3.9)

**Proof** Due to the linearity of $T_0$ and the fact that Schur polynomials form a basis of $\mathbb{C}[[q]]$, it suffices to show the validity of (3.9) for all $s_\lambda$, $\lambda \in \mathbb{Y}$. On the one hand, using Lemma 3.2, we find

$$T_0(s_\lambda) = \det(h_{k_i-i+j})_{1 \leq i,j \leq \ell(\lambda)+1},$$

(3.10)

where $k_1 = 0, k_i = \lambda_{i-1} \; (2 \leq i \leq \ell(\lambda)+1)$. On the other hand, observe, using (1.14), that

$$\sum_{k \geq 0} h_k(q-[z^{-1}]) y^k = \exp \left(\sum_{k \geq 1} \frac{1}{k z^k} y^k\right) = \exp \left(\sum_{k \geq 1} q_k y^k\right) \left(1 - \frac{y}{z}\right).$$

(3.11)
This implies that

\[ h_k(q - [z^{-1}]) = h_k(q) - \frac{1}{z} h_{k-1}(q), \quad \forall k \geq 1. \] (3.12)

Therefore,

\[ s_\lambda(q - [z^{-1}]) = \det \left( h_{\lambda_i-i+j} - \frac{1}{z} h_{\lambda_i-i+j-1} \right)_{1 \leq i,j \leq \ell(\lambda)} = \sum_{k=0}^{\ell(\lambda)} C[k] z^{-k}, \] (3.13)

where \( C[k] \) denotes the \((1, k+1)\) cofactor of the matrix \((h_{ki-i+j} - \frac{1}{z} h_{ki-i+j-1})_{1 \leq i,j \leq \ell(\lambda)}\). Hence,

\[ -\text{res}_{z=\infty} s_\lambda(q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z} = -\text{res}_{z=\infty} \left( \sum_{k=0}^{\ell(\lambda)} C[k] z^{-k} \right) \left( \sum_{k \geq 0} h_k z^k \right) \frac{dz}{z} = \sum_{k=0}^{\ell(\lambda)} h_k C[k] = \det \left( h_{ki-i+j} \right)_{1 \leq i,j \leq \ell(\lambda)+1} \cdot \] 

The lemma is proved.

We notice that similarly with (3.9) and (1.18) the following formula is true:

\[ -\text{res}_{z=\infty} s_{(m_1,...,m_r|n_1,...,n_r)}(q + [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z} = \begin{cases} \begin{array}{ll} (-1)^r s_{(m_1+1,...,m_r+1|n_1-1,...,n_r-1)}(q), & n_r \geq 1, \\ 0, & n_r = 0. \end{array} \end{cases} \] (3.14)

Now we are ready to prove Proposition 1.2.

**Proof of Proposition 1.2** From Lemma 3.1 we know that there exists a series \( g(z) \in \mathbb{C}[[z^{-1}]] \) such that

\[ \tau_1 \tau_2 = -\text{res}_{z=\infty} g(z) \tau_1(q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}. \] (3.15)

Consider the expansion

\[ \log g(z) =: b_0 + \sum_{k \geq 1} b_k z^{-k}, \] (3.16)

and denote \( Q := e^{\sum_{k \geq 1} k b_k q_k} \). Using the condition that \( \tau_2(0) \) equals 1, we have \( b_0 = 0 \). Then by using identity (3.15) we obtain

\[ \tau_1 \tau_2 = -\text{res}_{z=\infty} e^{\sum_{k \geq 1} b_k z^{-k}} \tau_1(q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}. \]
\[ e^{-\sum_{k \geq 1} b_k (q_k - \frac{1}{k})} = e^{-\sum_{k \geq 1} b_k q_k} \tau_1 (q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z} \]
\[ = - \text{res}_{z = \infty} \left( Q^{-1} \tau_1 (q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z} \right), \]
which, due to Lemma 3.3, gives
\[ Q^{-1} \tau_1 \tau_2 = T_0 \left( Q^{-1} \tau_1 \right). \]

The proposition is proved.

**Lemma 3.4** Let \( Q = e^{\sum_{k \geq 1} b_k q_k} \) denote the multiplication operator. The following formula holds:
\[ Q \circ T_0 \circ Q^{-1} (s_{\lambda}) = \sum_{n \geq 0} h_n (b) T_n (s_{\lambda}). \]  
(3.17)

**Proof** Using Lemma 3.3, we have
\[ Q \circ T_0 \circ Q^{-1} (s_{\lambda}) = - \text{res}_{z = \infty} e^{\sum_{k \geq 1} b_k z^{-k}} s_{\lambda} (q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}. \]
Then by substituting (3.13) into this equality we obtain that
\[ Q \circ T_0 \circ Q^{-1} (s_{\lambda}) = - \text{res}_{z = \infty} \sum_{n, m \geq 0} h_n (b) h_m (q) z^{m-n} C^{[k]} \frac{dz}{z}, \]
(3.18)
Here we recall that \( C^{[k]} \) is the \((1, k + 1)\) cofactor of the matrix \((h_{k,i-i+j})_{1 \leq i, j \leq \ell(\lambda)+1}\) with \(k_1 = 0, k_i = \lambda_{i-1} \) \(2 \leq i \leq \ell(\lambda) + 1\). Obviously, for any \(n \geq 1\), \(C^{[k]}\) is also the \((1, k + 1)\) cofactor of the matrix \((h_{k,0,i-i+j})_{1 \leq i, j \leq \ell(\lambda)+1}\) with \(k_{n,0} = n\) and \(k_{n,i} = \lambda_{i-1} \) \(2 \leq i \leq \ell(\lambda) + 1\). We therefore conclude that
\[ Q \circ T_0 \circ Q^{-1} (s_{\lambda}) = \sum_{n \geq 0} h_n (b) \det (h_{k,0,i-i+j})_{1 \leq i, j \leq \ell(\lambda)+1} \]
Combined with Lemma 3.2 the lemma is then proved.

We are now ready to prove Proposition 1.3.

**Proof of Proposition 1.3** By using Proposition 1.2 and Lemma 3.4, it is easy to see that
\[ \tau_E = \sum_{\lambda \in \Xi} \pi_{\lambda} \sum_{n \geq 0} h_n (b) T_n (s_{\lambda}) \]  
(3.19)
Here $T_k : \mathcal{Y} \cup (-\mathcal{Y}) \rightarrow \mathcal{Y} \cup (-\mathcal{Y})$ is defined as $T_k|_{\mathcal{Y} \cup (-\mathcal{Y})}$. We note that for a partition $\lambda \in \mathcal{Y}$, $T_k^{-1}((\lambda))$ denotes the set of pre-images of $\lambda$ in $\mathcal{Y} \cup (-\mathcal{Y})$, and this set only contains a finite number of elements (actually 0 or 1, so the third summand in (3.20) is a finite sum; of course, this can also be seen from (3.19) with the fact that $|T_n(\lambda)| = n + |\lambda|$). Then by a more careful but direct computation, we obtain (1.22).

Next we are to show (1.23). It is known [31] that the following identity holds true:

$$e^{\sum_{k \geq 1} \lambda \mu q_k q'_k} = \sum_{\lambda \in \mathcal{Y}} s_\lambda(q) s_\lambda(q').$$  \hspace{1cm} (3.21)

Then we have

$$Q \circ T_0 \circ Q^{-1}(s_\lambda) = \sum_{v \in \mathcal{Y}} s_v(b) s_v(q) T_0 \left( \sum_{\mu \in \mathcal{Y}} c_{\lambda,\mu}^{\beta} s_\mu(-b) s_\beta(q) \right) = \sum_{\alpha,\mu, v \in \mathcal{Y}} c_{\lambda,\mu}^{\beta} c_{v, T_0(\beta)}^{\alpha} s_\mu(-b) s_v(b) s_\alpha(q).$$

Formula (1.23) is then obtained in a similar way with (3.20). The proposition is proved.

For the reader’s convenience we provide in below a few Littlewood–Richardson coefficients $c_{\lambda,\mu}^{\beta}$ that appear in the above proof:

$$(s_3)(s_3) = s(6) + s(5,1) + s(4,2) + s(3,3),$$

$$(s_3)(s_{2,1}) = s(5,1) + s(4,2) + s(4,1,1) + s(3,2,1),$$

$$(s_3)(s_{1,1,1}) = s(4,1,1) + s(3,1,1,1),$$

$$(s_{2,1})(s_{2,1}) = s(4,2) + s(4,1,1) + s(3,3) + 2s(3,2,1) + s(3,1,1,1) + s(2,2,2) + s(2,2,1,1),$$

$$(s_{2,1})(s_{1,1,1}) = s(3,2,1) + s(3,1,1,1) + s(2,2,1,1) + s(2,1,1,1,1),$$

$$(s_{1,1,1})(s_{1,1,1}) = s(2,2,2) + s(2,2,1,1) + s(2,1,1,1,1) + s(1,1,1,1,1,1).$$

For the case of the topological solution to the Burgers–KdV hierarchy, it was conjectured by Witten [38] and proved by Kontsevich the following identity [29]:

$$\tau^{\text{WK}}_1 = \exp \left( \sum_{n \geq 1} \sum_{i_1, \ldots, i_n \geq 0} \frac{1}{n!} \prod_{k=1}^n (2i_k + 1)!! q^{2i_k+1} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \right),$$  \hspace{1cm} (3.22)

where $\tau^{\text{WK}}_1$ is defined by (1.36)–(1.37). The second component $\tau^{\text{WK}}_2$ in the tau-tuple of the topological solution plays the role of the partition function for the open intersection numbers [12,13,35]. From Corollary 1.9 and a straightforward computation, we obtain
the first few terms of $\tau_{WK}^E$ as follows:

$$\tau_{WK}^E = 1 + \frac{41}{24} s^{(3)} - \frac{5}{24} s^{(2,1)} - \frac{7}{24} s^{(1,1,1)} + \frac{9241}{152} s^{(6)} - \frac{385}{152} s^{(5,1)} - \frac{455}{576} s^{(3,2,1)} + \frac{25}{1152} s^{(3,1,1,1)} - \frac{35}{576} s^{(2,2,2)} - \frac{385}{1152} s^{(2,1,1,1,1)} - \frac{455}{1152} s^{(1,1,1,1,1,1)} + \cdots,$$

which agree with the computations in [28]. We hope that Corollary 1.9 could be helpful in understanding open and closed intersection numbers.

4 The extended generalized BGW tau-function

In this section, following [9,13] we prove Theorem 1.7. We also prove that the extended generalized BGW tau-function with some particularly chosen value of $C$ is a polynomial.

Let $(w^{\Theta(C)}, \rho^{\Theta(C)})$ and $(\tau_1^{\Theta(C)}, \tau_2^{\Theta(C)})$ be defined in Introduction (cf. (1.27)–(1.29)). According to [7], the series $\tau_1^{\Theta(C)}$ is also the Sato-type tau-function of the solution $(w^{\Theta(C)}, \rho^{\Theta(C)})$ for the KdV hierarchy. Therefore, as mentioned in Sect. 2, the $\psi^{\Theta(C)} := \psi^{\Theta(C)}(q; z)$ and $\psi^*^{\Theta(C)} := \psi^*^{\Theta(C)}(q; z)$ defined by

$$\psi^{\Theta(C)} = \frac{\tau_1^{\Theta(C)}(q - [z]^{-1})}{\tau_1^{\Theta(C)}(q)} e^{\sum_{n \geq 1} q_n z^n}, \quad \psi^*^{\Theta(C)} = \frac{\tau_1^{\Theta(C)}(q + [z]^{-1})}{\tau_1^{\Theta(C)}(q)} e^{-\sum_{n \geq 1} q_n z^n},$$

form a pair of wave and dual wave functions of $w^{\Theta(C)}$ for the KdV hierarchy. To proceed we introduce the notations:

$$K^{\Theta(C)} := \tau_1^{\Theta(C)} \psi^{\Theta(C)}, \quad f^{\Theta(C)}(x, z) := \psi^{\Theta(C)}|_{q_2 = q_3 = \cdots = 0},$$

$$K^*^{\Theta(C)} := \tau_1^{\Theta(C)} \psi^*^{\Theta(C)}, \quad f^*^{\Theta(C)}(x, z) := \psi^*^{\Theta(C)}|_{q_2 = q_3 = \cdots = 0},$$

$$L^{\Ext}_0 := \sum_{n \geq 1} n \tilde{q}_n \frac{\partial}{\partial q_n} + C, \quad \tilde{q}_n := q_n - \delta_{n,1}.$$

Lemma 4.1 The following formulae hold true:

$$L^{\Ext}_0 K^{\Theta(C)} = (z \partial_z - z) K^{\Theta(C)}, \quad (4.5)$$

$$L^{\Ext}_0 K^*^{\Theta(C)} = (z \partial_z + z) K^*^{\Theta(C)}.$$

\(\text{ Springer}\)
Proof Performing a shift $q \mapsto q - [z^{-1}]$ to identity (1.28), we obtain

$$
\sum_{k \geq 0} \left( (2k + 1) \tilde{q}_{2k+1} - \frac{1}{z^{2k+1}} \right) \frac{\partial \tau_1^{\Theta(C)}(q - [z^{-1}])}{\partial q_{2k+1}} = C \tau_1^{\Theta(C)}(q - [z^{-1}]) = 0.
$$

(4.7)

Then the left-hand side of (4.5) is equal to

$$
e^{\sum_{n \geq 1} q_n z^n} \left( \sum_{k \geq 0} (2k + 1) \tilde{q}_{2k+1} \frac{\partial \tau_1^{\Theta(C)}(q - [z^{-1}])}{\partial q_{2k+1}} + \tau_1^{\Theta(C)}(q - [z^{-1}]) \left( \sum_{n \geq 1} n \tilde{q}_n z^n + C \right) \right)
$$

$$= e^{\sum_{n \geq 1} q_n z^n} \sum_{k \geq 0} \tilde{z}^{-2k-1} \frac{\partial \tau_1^{\Theta(C)}(q - [z^{-1}])}{\partial q_{2k+1}} + \tau_1^{\Theta(C)}(q - [z^{-1}]) \left( \sum_{n \geq 1} n \tilde{q}_n z^n \right)
$$

$$= (z \partial_z - z) K_{\Theta(C)}.
$$

(4.8)

The proof for (4.6) is similar. The lemma is proved.

The next lemma gives the explicit expressions of $f_{\Theta(C)}$ and $f_{\Theta(C)}^*$.

Lemma 4.2 The following formulae are true:

$$f_{\Theta(C)} = e^{xz} \sum_{n \geq 0} \theta_n z^n (1 - x)^n, \quad f_{\Theta(C)}^* = e^{-xz} \sum_{n \geq 0} (-1)^n \theta_n z^n (1 - x)^n,
$$

(4.9)

where $\theta_n := (-1)^n \prod_{i=0}^{n-1} (C + \frac{i(i+1)}{2})/n!$, $n \geq 0$.

Proof Setting $q_2 = q_3 = \cdots = 0$ in (4.5) we have

$$(x - 1) \partial_x f_{\Theta(C)} = (z \partial_z - z) f_{\Theta(C)}. \quad (4.10)$$

It follows that there exists a formal series $\theta(z) \in \mathbb{C}[[z^{-1}]]$ such that $f_{\Theta(C)} = e^{xz} \theta(z (1 - x))$. Setting $q_2 = q_3 = \cdots = 0$ in formula (2.7) we have

$$\partial_x^2 f_{\Theta(C)} + \left( \frac{2C}{(1 - x)^2} - z^2 \right) f_{\Theta(C)} = 0.
$$

(4.11)

Therefore, we find that $\theta(z)$ satisfies the ODE:

$$\left( z^2 \partial_z^2 - 2z^2 \partial_z + 2C \right) \theta(z) = 0.
$$

(4.12)
Solving this ODE with $\theta(z) \sim 1$ as $z \to \infty$ we get the first equality. The proof of the second equality is similar.

We are ready to prove Theorem 1.7 using the method of [13] improved in [9].

**Proof of Theorem 1.7** Due to Lemma 3.1, there exists a series $g(z)$ of form (3.1) satisfying

$$
\tau_{\theta(C)}^2 = - \operatorname{res}_{z=\infty} g(z) \frac{\tau_{\theta(C)}(q - \left[ z^{-1} \right])}{\tau_{\theta(C)}(q)} e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}.
$$

(4.13)

Using (1.29) we find $g_0 = 1$. Using Lemma 4.1 we have

$$
L_{\theta(C)}^0 (g(z)) = \tau_{\theta(C)}^1 \operatorname{res}_{z=\infty} \left( g'(z) + g(z) \right) \psi_{\theta(C)} dz.
$$

(4.14)

It follows from (1.12), (1.27) and (1.28) that

$$
\left| L_{\theta(C)}^0 (g(z)) \right|_{q_2 = q_3 = \cdots = 0} \equiv 0.
$$

(4.15)

Thus taking $q_2 = q_3 = \cdots = 0$ on the both sides of (4.14) we find

$$
\operatorname{res}_{z=\infty} \left( g'(z) + g(z) \right) f_{\theta(C)} (x, z) dz \equiv 0.
$$

(4.16)

Since $\psi_{\theta(C)}$ satisfies equations (2.8), it follows from (4.18) that

$$
\operatorname{res}_{z=\infty} (g'(z) + g(z)) \psi_{\theta(C)} (q; z) dz \equiv 0.
$$

(4.17)

Following [9] we employ the bilinear equation (2.9) with $q = (x, 0), q' = (x', 0)$, and we find

$$
\operatorname{res}_{z=\infty} f_{\theta(C)}^* (x', z) f_{\theta(C)} (x, z) dz \equiv 0.
$$

(4.18)

Further taking $x' = 0$ and using Lemma 4.2 we find that

$$
\operatorname{res}_{z=\infty} \theta(-z) f_{\theta(C)} (x, z) dz \equiv 0.
$$

(4.19)

Comparing (4.16) and (4.19), and using the uniqueness nature we find that $g(z)$ must satisfy

$$
g'(z) + g(z) = \theta(-z).
$$

(4.20)

Obviously, solution to (4.20) of form (3.1) is unique. By solving (4.20) we finally obtain that

$$
g(z) = 1 + \sum_{n \geq 1} g_{\theta(C)}^{\Theta} z^{-n}
$$

with $g_{\theta(C)}^{\Theta}$ being given by (1.31). Identities (1.32) and (1.33) are proved.
It was shown in [3] that if $C = -m(m + 1)/2$ for some non-negative integer $m$, the generalized BGW tau-function $\tau^{(C)}_1$ is a polynomial. We find in the next proposition that when $m$ is even, the tau-function $\tau^{(C)}_E$ is also a polynomial.

**Proposition 4.3** For $C = -m(m + 1)/2$, $m \in \mathbb{Z}_{\geq 0}$, if $m$ is even, then $\tau^{(C)}_E$ is a polynomial.

**Proof** Define $g(z) = \sum_{k \geq 0} g^{(C)}_k z^{-k}$. Using Theorem 1.7 we find

$$g(z) = \sum_{k=0}^{m-1} \frac{g^{(C)}_k}{z^k} + a_m \sum_{k \geq m} \frac{(k - 1)!}{z^k}, \quad (4.21)$$

where

$$a_m = \sum_{k=1}^{m} \frac{\prod_{l=0}^{k-1} \left( -\frac{m(m+1)}{2} + \frac{i(i+1)}{2} \right)}{k!(k - 1)!}. \quad (4.22)$$

One can show that the number $a_m$ satisfies

$$a_m = \frac{1}{m!} f^{(m)}(0), \quad f(z) := -z \left(1 + z^2\right)^{-3/2}. \quad (4.23)$$

Since $m$ is even, we find $a_m$ is zero. Then from (4.21) it follows that $g(z)$ is a polynomial in $z^{-1}$. Combining this with the polynomiality property of the component $\tau^{(C)}_1$, one can see

$$g(z) \tau^{(C)}_1(q - [z^{-1}]) \in \mathbb{C}[q][z^{-1}]. \quad (4.24)$$

By using Lemma 3.1 we obtain that

$$\tau^{(C)}_E = - \text{res}_{z=\infty} g(z) \tau^{(C)}_1(q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}. \quad (4.25)$$

We conclude from (4.24) and (4.25) that $\tau^{(C)}_E$ is a polynomial.

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A The first few coefficients of $\tau^\Theta(C)$

| $\lambda$ | Coefficient of $s^\lambda$ |
|-----------|---------------------------|
| (1)       | $C$                       |
| (2)       | $C(C + 3)/2$              |
| (1, 1)    | $C(C - 1)/2$              |
| (3)       | $C(C + 3)(C + 7)/6$       |
| (2, 1)    | $C(C + 3)(2C - 1)/6$      |
| (1, 1, 1) | $C(C + 1)(C - 3)/6$       |
| (4)       | $C(C + 3)(C + 9)(C + 10)/24$ |
| (3, 1)    | $C(C + 3)(C^2 + 7C - 2)/8$ |
| (2, 2)    | $C^2(C + 1)(C + 3)/12$    |
| (2, 1, 1) | $C(C + 1)(C + 3)(C - 2)/8$ |
| (1, 1, 1) | $C(C + 1)(C + 3)(C - 6)/24$ |
| (5)       | $C(C + 3)(C + 10)(C^2 + 27C + 186)/120$ |
| (4, 1)    | $C(C + 3)(C + 10)(2C^2 - 3C - 30)/60$ |
| (3, 2)    | $C^2(C + 1)(C + 3)(C + 8)/24$ |
| (3, 1, 1) | $C(C + 1)(C + 3)(C^2 + 6C - 10)/20$ |
| (2, 2, 1) | $C^3(C + 1)(C + 3)/24$    |
| (2, 1, 1, 1)| $C(C + 1)(C + 3)(2C^2 - 3C - 30)/60$ |
| (1, 1, 1, 1)| $C(C + 1)(C + 3)(C + 6)(C - 10)/120$ |
| (6)       | $C(C + 3)(C + 10)(C + 21)(C^2 + 31C + 270)/720$ |
| (5, 1)    | $C(C + 3)(C + 10)(C^3 + 28C^2 + 201C - 18)/144$ |
| (4, 2)    | $C^2(C + 1)(C + 3)(C + 10)(C + 11)/80$ |
| (4, 1, 1) | $C(C + 1)(C + 3)(C + 10)(C^2 + 9C - 9)/72$ |
| (3, 3)    | $C^2(C + 1)(C + 3)^2(C + 10)/144$ |
| (3, 2, 1) | $C^2(C + 1)(C + 3)(4C^2 + 34C + 15)/180$ |
| (3, 1, 1, 1)| $C(C + 1)(C + 3)(C^3 + 7C^2 - 15C - 90)/72$ |
| (2, 2, 2) | $C^2(C + 1)(C + 3)(C^2 + C + 6)/144$ |
| (2, 2, 1, 1)| $C^2(C + 1)(C + 3)(C^2 + C - 10)/144$ |
| (2, 1, 1, 1, 1)| $C(C + 1)(C + 3)(C + 6)(C^2 - 5C - 30)/144$ |
| (1, 1, 1, 1, 1)| $C(C + 1)(C + 3)(C + 6)(C + 10)(C - 15)/720$ |

B A novel proof of Alexandrov’s theorem

In this section, we give a new proof of a result of Alexandrov [2] (cf. also [1]).

Let $\mathbb{C}((z^{-1}))$ be the linear space of formal series with finitely many terms of positive powers, and consider the Sato Grassmannian $GM$ as defined originally in [36]. A point $W \in GM$ is a subspace of $\mathbb{C}((z^{-1}))$ which can be written as a linear span of a set of basis vectors. A particular useful choice of basis of $W$ is of the following form:

$$W = \text{Span}_\mathbb{C}\left\{z^l + \sum_{k \geq 0} A_{k,l} z^{-k-1}\right\}_{l \geq 0}. \quad \text{(B.1)}$$

Here the collection $(A_{k,l})_{k,l \geq 0}$ is called the affine coordinate of $W$. Note that such a choice is not always possible: existence of basis of form (B.1) characterizes the big

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cell (see [37] for the details). Given a point \( W \in GM \), the affine coordinate, if exist, must be unique.

Define a family of numbers as follows:

\[
\pi(m_1, \ldots, m_r | n_1, \ldots, n_r)(W) := (-1)^r \sum_{i=1}^r n_i \det(A_{m_i, n_j})_{1 \leq i, j \leq r}.
\]  

(B.2)

Then according to [5, 22], the formal series defined by

\[
\tau_W := \sum_{\lambda \in \mathcal{Y}} \pi_{\lambda}(W) s_{\lambda}
\]  

(B.3)

is a Sato-type tau-function of the KP hierarchy, i.e., it satisfies the following bilinear identity:

\[
\text{res}_{z=\infty} \tau_W(q - [z^{-1}]) \tau_W(q' + [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} dz = 0.
\]  

(B.4)

**Theorem B.1** ([2]) Let \( f(z) = 1 + \sum_{n \geq 1} f_i z^{-n} \) be an arbitrary formal series in \( \mathbb{C}[[z^{-1}]] \). If \( \tau \) is a Sato-type tau-function of the KP hierarchy, so is

\[
\tilde{\tau} := - \text{res}_{z=\infty} f(z) \tau(q - [z^{-1}]) e^{\sum_{n \geq 1} q_n z^n} \frac{dz}{z}.
\]  

(B.5)

**Proof** By using Lemma 3.3, we can rewrite (B.5) as

\[
\tilde{\tau} = Q \circ T_0 \circ Q^{-1}(\tau),
\]  

(B.6)

where \( Q = e^{\sum_{k \geq 1} k b_k q_k} \) denotes the multiplication operator with \( b_1, b_2, \ldots \) being numbers such that \( \log f(z) = \sum_{k \geq 1} b_k z^{-k} \).

Let \( \tau \) be the Sato-type tau-function given by (B.3). We first show \( T_0(\tau) \) is a Sato-type tau-function. Define a point in \( GM \) as

\[
\tilde{W} := \text{Span}_\mathbb{C}\left\{z^l - \sum_{k \geq 0} A_{k+1, l-1} z^{-k-1}\right\}_{l \geq 1}.
\]  

(B.7)

It is obvious that the affine coordinates \( (\tilde{A}_{k, l})_{k, l \geq 0} \) of \( \tilde{W} \) are given by

\[
(\tilde{A}_{k, l})_{k, l \geq 0} = \begin{pmatrix}
\vdots & \vdots & \vdots \\
0 & -A_{20} & -A_{21} \\
0 & -A_{10} & -A_{11} \\
\end{pmatrix}.
\]  

(B.8)

Recall expression (1.18) of \( T_0 \). Then by using (B.3) and observing that

\[
\det(\tilde{A}_{m, n})_{1 \leq i, j \leq r} \begin{cases} (-1)^r \det(A_{m_i+1, n_j-1})_{1 \leq i, j \leq r}, & n_r \geq 1, \\
0, & n_r = 0,
\end{cases}
\]  

(B.9)
one can obtain that

\[ T_0(\tau) = \tau_\tilde{W}. \]

Hence \( T_0(\tau) \) is Sato-type tau-function of the KP hierarchy. In other words, the transformation \( T_0 \) has the property that it maps an arbitrary Sato-type tau-function of the KP hierarchy to another one. Obviously, the operator \( Q \) also has this property. Combining with (B.6), we therefore conclude that the formal series \( \tilde{\tau} \) is also a Sato-type tau-function. The theorem is proved.

Let us give a direct consequence of Proposition 1.3 and Corollary 1.5. For an arbitrary KP tau-function in the big cell with affine coordinates \( A_{m,n} \) \((m, n \geq 0)\) and for arbitrary \( g_k \) \((k \geq 0)\) with \( g_0 = 1 \), define \( c_{m,n} \) by (1.25) and \( \pi_\lambda \) by (1.16). Then for any partition \( \lambda = (m_1, \ldots, m_r | n_1, \ldots, n_r) \), the following identities are true:

\[
\det(c_{m_i,n_j})_{1 \leq i, j \leq r} = \sum_{1 \leq i < \ell(\lambda)} (-1)^{i-1} g_{\lambda_i - i + 1} \pi(\lambda_1 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots) = \sum_{\alpha, \mu, \nu} \pi_\alpha c_{\nu, T_0(\beta)} s_\alpha(-b) s_\nu(b), \tag{B.10}
\]

where \( b := (b_1, b_2, \ldots) \) is defined via \( e^\sum_{k \geq 1} b_k z^{-k} = \sum_{n \geq 0} g_n z^{-n} \).

References

1. Alexandrov, A.: Open intersection numbers, matrix models and MKP hierarchy. J. High Energy Phys. 3, 1–14 (2015)
2. Alexandrov, A.: Open intersection numbers, Kontsevich-Penner model and cut-and-join operators. J. High Energy Phys. 8, 1–25 (2015)
3. Alexandrov, A.: Cut-and-join description of generalized Brezin-Gross-Witten model. Adv. Theor. Math. Phys. 22, 1347–1399 (2018)
4. Babelon, O., Bernard, D., Talon, M.: Introduction to Classical Integrable Systems. Cambridge University Press, Cambridge (2003)
5. Balogh, F., Yang, D.: Geometric interpretation of Zhou’s explicit formula for the Witten-Kontsevich tau function. Lett. Math. Phys. 107, 1837–1857 (2017)
6. Basalaev, A., Buryak, A.: Open WDVV equations and Virasoro constraints. Arnold Math. J. 13, 827–883 (2019)
7. Bertola, M., Dubrovin, B., Yang, D.: Correlation functions of the KdV hierarchy and applications to intersection numbers over \( \overline{M}_{g,n} \). Physica D 327, 30–57 (2016)
8. Bertola, M., Ruzza, G.: Brezin-Gross-Witten tau-function and isomonodromic deformations. Commun. Number Theory Phys. 13, 827–883 (2019)
9. Bertola, M., Yang, D.: The partition function of the extended \( r \)-reduced Kadomtsev-Petviashvili hierarchy. J. Phys. A 48, 195205 (2015)
10. Borot, G., Bouchard V., Chidambaram N.K., Creutzig T., Noshchenko D.: Higher Airy structures, \( \mathcal{W} \) algebras and topological recursion. arXiv:1812.08738
11. Brezin, E., Gross, D.J.: The external field problem in the large \( N \) limit of QCD. Phys. Lett. B 97, 120–124 (1980)
12. Buryak, A.: Equivalence of the open KdV and the open Virasoro equations for the moduli space of Riemann surfaces with boundary. Lett. Math. Phys. 105, 1427–1448 (2015)
13. Buryak, A.: Open intersection numbers and the wave function of the KdV hierarchy. Mosc. Math. J. 16, 27–44 (2016)
14. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. In: Proceedings of RIMS (1981)
15. Dickey, L.A.: Soliton Equations and Hamiltonian Systems. World Scientific, Singapore (2003)
16. Do, N., Norbury, P.: Topological recursion on the Bessel curve. Commun. Number Theory Phys. 12, 53–73 (2018)
17. Dubrovin, B.: Geometry of 2D topological field theories. Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Fond. CIME/CIME Found. Subser. Springer, Berlin (1996)
18. Dubrovin, B., Liu, S.-Q., Yang, D., Zhang, Y.: Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs. Adv. Math. 293, 382–435 (2016)
19. Dubrovin, B., Yang, D.: Remarks on intersection numbers and integrable hierarchies. I. Quasi-triviality. Adv. Theor. Math. Phys. 24, 1055–1082 (2020)
20. Dubrovin, B., Yang, D., Zagier, D.: On tau-functions for the KdV hierarchy. Selecta Math. 27 (2021), Paper No. 12, p. 47
21. Dubrovin, B., Zhang, Y.: Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Grassmann–Witten invariants. arXiv:math/0108160
22. Enolski, V.Z., Harnad, J.: Schur function expansions of Kadomtsev-Petviashvili r-functions associated with algebraic curves. Uspekhi Mat. Nauk 66, 137–178 (2011)
23. Fu, Z.: Remark on the affine coordinates for KdV tau-functions (In preparation)
24. Gross, D.J., Witten, E.: Possible third-order phase transition in the large-N lattice gauge theory. Phys. Rev. D Part. Fields 21, 446–453 (1980)
25. Horev, A., Solomon, J.P.: The open Gromov–Witten–Welschinger theory of blowups of the projective plane. arXiv:1210.4034
26. Itzykson, C., Zuber, J.-B.: Combinatorics of the modular group. II. The Kontsevich integrals. Int. J. Modern Phys. A 7, 5661–5705 (1992)
27. Kac, V., Schwarz, A.: Geometric interpretation of the partition function of 2D gravity. Phys. Lett. B 257, 329–334 (1991)
28. Ke, H.-Z.: On a geometric solution to open KdV and Virasoro. Adv. Math. (China) 46, 91–96 (2017)
29. Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Commun. Math. Phys. 147, 1–23 (1992)
30. Littlewood, D.E., Richardson, A.R.: Group characters and algebra. Philos. Trans. R. Soc. Lond. Ser. A 233, 99–141 (1934)
31. Macdonald, I.G.: Symmetric Functions and Hall Polynomials. Oxford Classic Texts in the Physical Sciences. Clarendon Press, Oxford (1998)
32. Mironov, A., Morozov, A., Semenoff, G.W.: Unitary matrix integrals in the framework of generalized Kontsevich model. Int. J. Modern Phys. A 11, 5031–5080 (1996)
33. Morozov, AY u.: Unitary integrals and related matrix models. Theor. Math. Phys. 162, 1–33 (2010)
34. Norbury, P.: A new cohomology class on the moduli space of curves. arXiv:1712.03662
35. Pandharipande, R., Solomon, J.P., Tessler, R.J.: Intersection theory on moduli of disks, open KdV and Virasoro. arXiv:1409.2191
36. Sato, M.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds. Res. Inst. Math. Sci. Kyoto Univ. 39, 30–46 (1981)
37. Segal, G., Wilson, G.: Loop groups and equations of KdV type. Surv. Differ. Geom. 4, 403–466 (1998)
38. Witten, E.: Two-Dimensional Gravity and Intersection Theory on Moduli Space. Surveys in Differential Geometry (Cambridge, MA, 1990), pp. 243–310. Lehigh University, Bethlehem (1991)
39. Zhou, J.: Explicit formula for Witten-Kontsevich tau-function. arXiv:1306.5429
40. Zhou, J.: Emergent geometry and mirror symmetry of a point. arXiv:1507.01679