QUASIMODULAR MOONSHINE AND ARITHMETIC CONNECTIONS

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Abstract. We prove the existence of a module for the largest Mathieu group, whose trace functions are weight two quasimodular forms. Restricting to the subgroup fixing a point, we see that the integrality of these functions is equivalent to certain divisibility conditions on the number of $\mathbb{F}_p$ points on Jacobians of modular curves. Extending such expressions to arbitrary primes, we find trace functions for modules of cyclic groups of prime order with similar connections. Moreover, for cyclic groups, we give an explicit vertex operator algebra construction whose trace functions are given only in terms of weight two Eisenstein series.

1. Introduction

The idea of moonshine dates back to the 1970s when McKay noticed that 196884, the coefficient of $q$ in the expansion of the normalized $j$-invariant, $J(\tau)$, is the sum of 1 and 196883, which are dimensions of irreducible representations of the monster group $\mathbb{M}$. Thompson further conjectured that this is true for the rest of the coefficients of $J(\tau)$ — that there exists a graded infinite-dimensional $\mathbb{M}$-module $V^\natural = \bigoplus_n V^\natural_n$ such that

$$\sum_{n=1}^{\infty} \dim(V^\natural_n)q^n = J(\tau).$$

More generally, Thompson conjectured that there exists a module $V^\natural$ such that the graded trace functions of other group elements $g \in \mathbb{M}$, defined as $T_g(\tau) := \sum_{n=1}^{\infty} \text{tr}(g | V^\natural_n)q^n$, are distinguished functions [28,29]. In 1979, Conway and Norton conjectured further (the Monstrous Moonshine Conjecture) that for each $g \in \mathbb{M}$ the graded trace function $T_g$ is the unique modular function that generates the genus zero function field arising from a specific subgroup $\Gamma_g$ of $SL_2(\mathbb{R})$, normalized such that $T_g(\tau) = q^{-1} + O(q)$ [8]. In 1988, Frenkel, Lepowsky, and Meurman [17] constructed $V^\natural$ and in 1992, Borcherds proved the Monstrous Moonshine Conjecture, showing that $V^\natural$ is the moonshine module with properties conjectured by Conway and Norton [2].

Almost twenty years after the proof of the Monstrous Moonshine Conjecture, other sporadic simple groups appeared in the theory of moonshine. One of these groups, $M_{24}$, entered the story in 2010 when physicists Eguchi, Ooguri, and Tachikawa [14] noticed that the dimensions of representations of $M_{24}$ are the multiplicities (the first few coefficients of a function denoted $\Sigma(\tau)$) of superconformal algebra characters in the K3 elliptic genus. This led to the conjecture of the existence of an infinite-dimensional graded $M_{24}$-module whose existence was proven in 2012 by Gannon [20]. More precisely, Gannon showed that there exists a graded $M_{24}$-module $K = \bigoplus_n K_{n-1/8}$ whose graded dimension function, $H_e(\tau) := -2q^{-1/8} + \sum_{n=1}^{\infty} \text{tr}(e | K_{n-1/8})$ is equal to $\Sigma(\tau)$ from the expansion of the K3 elliptic genus in [14]. The graded trace functions for other $g \in M_{24}$, defined analogously and denoted $H_g(\tau)$, are mock modular forms of weight 1/2 and level $|g|$, the order of $g$. This is in contrast to monstrous moonshine, whose trace functions are weakly holomorphic modular functions.

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This $M_{24}$ moonshine belongs to a larger class of 23 moonshines, umbral moonshine, which was conjectured by Cheng, Duncan, and Harvey [1] and whose existence was proven by Duncan, Griffin, and Ono [10]. But there are some disparities between umbral moonshine and monstrous moonshine. In addition to the fact, already mentioned, that umbral moonshine involves (mock modular) forms that are not of integral weight, another difference compared to the monstrous case is that despite some recent progress by Anagiannis–Cheng–Harrison, Cheng–Duncan, Duncan–Harvey, and Duncan–O’Desky (see [11,11,12]), the umbral moonshine theory does not yet include module constructions in all of its cases.

An initial motivation for this work was to bring umbral moonshine in a closer context to monstrous moonshine. In carrying this out we discovered two new infinite families of similar phenomena: one with arithmetic content, and one for which we are able to give vertex algebra constructions.

We start with Mathieu moonshine and reframe $M_{24}$ moonshine in terms of trace functions that are weight two quasimodular forms instead of mock modular forms. Restricting to $M_{23}$ we find expressions for these forms that contain arithmetic information. We generalize this type of expression to $\mathbb{Z}/NZ$ for arbitrary $N$ prime, from which we can observe more connections of a similar kind. At the expense of arithmetic connections, we give a second set of quasimodular trace functions for another $\mathbb{Z}/NZ$-module which are given only in terms of Eisenstein series. We construct these modules explicitly as tensor products of Heisenberg and Clifford module vertex operator algebras.

The quasimodular forms that we give as trace functions of an $M_{24}$-module come from multiplying the functions $H_g(\tau)$, the completions of the mock modular forms $H_g(\tau)$, by $\eta^g(\tau)$ to bring the weight to 2, and then taking the holomorphic projection. In this way we define weight two quasimodular forms $Q_g(\tau)$ for every $g \in M_{24}$. The first example of a weight two quasimodular form is

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

the usual weight 2 Eisenstein series. In our setting, each coefficient of $-2E_2(\tau)$ has a natural interpretation as the dimension of a virtual $M_{24}$-module. This is because our $Q_g(\tau)$ is equal to $-2E_2(\tau)$ and we prove the following:

**Theorem 1.** There exists a virtual graded $M_{24}$-module $V = \bigoplus V_n$ such that

$$Q_g(\tau) = \sum_{n=0}^{\infty} \text{tr}(g | V_n) q^n.$$

Interestingly, if one restricts to the trace functions of $M_{23}$, a subgroup of $M_{24}$ whose group elements have fixed points in their permutation representations, the $Q_g(\tau)$ have convenient expressions in terms of Eisenstein series and cusp forms. These expressions are of the form

$$Q_g(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N}G_N(\tau)$$

where we use $N$ to denote the order of $g$, we let $n_N := \text{num} \left( \frac{N-1}{12} \right)$, $E_{2,N}(\tau)$ is an expression in terms of Eisenstein series (cf. [3,1]), and $G_N$ is a specific cusp form of weight 2 for $\Gamma_0(N)$ with integer coefficients.

The trace functions of this module involve weight two cusp forms and so they contain arithmetic information. As an example, for certain $N$, the Jacobian of $X_0(N)$, denoted $J_0(N)$, is an elliptic curve, and so the integrality of the trace functions is equivalent to certain divisibility conditions on the number of $\mathbb{F}_p$ points on these curves (this holds even when the dimension of $J_0(N)$ is greater than 1, but we restrict to elliptic curves here for simplicity). These results depend on the cooperation of Eisenstein series and cusp forms to sum to integral coefficients. For example we have the following (known, see Appendix A of [22]) divisibility conditions arising from $M_{23}$:

**Corollary 1.1.**
(1) For $p \neq 11$, we have $5 \mid \#J_0(11)(\mathbb{F}_p)$.
(2) For $p \neq 2, 7$ we have $3 \mid \#J_0(14)(\mathbb{F}_p)$.
(3) For $p \neq 3, 5$, we have $4 \mid \#J_0(15)(\mathbb{F}_p)$.

Moreover, the pattern we observe in the denominators of the cusp forms reflects a result of Mazur on a congruence between Eisenstein series and cusp forms in the cases where $N$ is prime [24]. The type of expression in (1.1) can be generalized, and in fact, the formula for $Q_g(\tau)$ does not depend on $M_{23}$ and can be defined for arbitrary $N$. We restrict our focus to $N$ prime and prove the existence of a $\mathbb{Z}/N\mathbb{Z}$-module with trace functions:

\begin{align*}
f_g^{(N)}(\tau) := \begin{cases} -\frac{\ell_N}{n_N} E_2(\tau) & \text{if } g = e, \\ -\frac{\ell_N}{n_N} E_{2,N}(\tau) - \frac{N}{n_N} G_N(\tau) & \text{if } g \neq e, \end{cases}
\end{align*}

where $\ell_N := \text{num} \left( \frac{N^2 - 1}{24} \right)$. The functions $f_g^{(N)}(\tau)$ are quasimodular forms of weight 2 with integral coefficients defined in terms of Eisenstein series and cusp forms. The $E_{2,N}(\tau)$ (cf. (3.4)) are again defined in terms of Eisenstein series and the $G_N(\tau)$ are certain cusp forms of level $N$ and weight 2 with integer coefficients (cf. Section 3, Proposition 3.1). With these definitions we state the following result:

**Theorem 2.** Let $N$ be a prime and $f_g^{(N)}(\tau)$ be as in (1.2). Then there exists a virtual graded $\mathbb{Z}/N\mathbb{Z}$-module $V^{(N)} = \bigoplus_n V_n^{(N)}$ such that

\begin{equation}
f_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \text{tr}(g | V_n^{(N)}) q^n.
\end{equation}

As a consequence, for $N$ prime, one can observe many more examples analogous to those in Corollary [4] arising from the trace functions $f_g^{(N)}(\tau)$.

In the trace function for $\mathbb{Z}/N\mathbb{Z}$ in Theorem 2, the multiple in front of $E_{2,N}(\tau)$ is $\frac{\ell_N}{n_N}$ (recall that $\ell_N = \text{num} \left( \frac{N^2 - 1}{24} \right)$ and $n_N = \text{num} \left( \frac{N^2 - 1}{12} \right)$). We have that $\ell_N$ is the minimal number which clears the denominators of $E_{2,N}(\tau)$, and we can further divide by $n_N$ because we find a cusp form that satisfies a congruence modulo $n_N$. If we restrict our functions to be only in terms of Eisenstein series and do not use cusp forms, we can no longer divide by $n_N$ and instead have trace functions as follows: for $N$ prime, let

\begin{align*}
F_g^{(N)}(\tau) := \begin{cases} -\ell_N E_2(\tau) & \text{if } g = e, \\ -\ell_N E_{2,N}(\tau) & \text{if } g \neq e, \end{cases}
\end{align*}

These are weight two purely Eisenstein quasimodular trace functions for a $\mathbb{Z}/N\mathbb{Z}$-module. Although this comes at the cost of arithmetic connections arising from cusp forms, we give a vertex operator algebra construction for this module. For each $N$, this module is $W^{(N)}_{tw}$, a twisted module for the vertex operator algebra which we denote $W^{(N)}$. The vertex operator algebras $W^{(N)}$ are defined as tensor products of Heisenberg and Clifford module vertex algebras. The precise construction of $W^{(N)}$ is given in Section 5.

**Theorem 3.** For $N$ prime, the canonically twisted module $W^{(N)}_{tw} = \bigoplus_n W^{(N)}_{tw,n}$ of the vertex operator algebra $W^{(N)}$ is an infinite dimensional virtual graded module for $\mathbb{Z}/N\mathbb{Z}$ such that

\begin{equation}
F_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \text{tr}(g | W^{(N)}_{tw,n}) q^n.
\end{equation}
This paper is organized as follows. Section 2 includes information on the composition of the quasimodular forms we associate to $M_{24}$ including a proof that they are quasimodular and a proof of the existence of the graded module for which they are the trace functions. Section 3 gives an alternative expression for the quasimodular forms for the cases $g \in M_{23}$, generalizes that expression to $\mathbb{Z}/N\mathbb{Z}$ for arbitrary $N$ prime, and includes the proof of the existence of the corresponding $\mathbb{Z}/N\mathbb{Z}$ module. Section 4 additionally includes an elementary proof of Mazur’s congruence between Eisenstein series and cusp forms. Section 5 explains the arithmetic connections of the trace functions for the modules in Section 3. Section 6 includes purely Eisenstein quasimodular trace functions for $\mathbb{Z}/N\mathbb{Z}$ and constructs the corresponding modules explicitly.

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2. Quasimodular $M_{24}$ Forms

The setup of the $M_{24}$ functions starts with building forms of weight 2 from the original Mathieu moonshine functions. More specifically, the forms come from multiplying the completions $\hat{H}_g(\tau)$ of the mock modular forms $H_g(\tau)$ from Mathieu moonshine by $\eta^3(\tau)$. Note that $\hat{H}_g(\tau)\eta^3(\tau)$ is a non holomorphic modular form of weight 2, and since it does not have singularities at cusps, we can apply holomorphic projection to extract something holomorphic. In weight 2, holomorphic projection results in a quasimodular form. For more information about mock modular forms, their completions, and holomorphic projection, we refer the reader to [3].

To give the expression for the holomorphic projection explicitly, we first define a function $F_2(\tau)$ as follows:

$$F_2(\tau) := \sum_{r>s>0, r-s \text{ odd}} sq^{r+s/2}.$$

The holomorphic projections of the $\hat{H}_g(\tau)\eta^3(\tau)$ are given in terms of $H_g(\tau)\eta^3(\tau)$ and some multiple of $F_2(\tau)$ for each $g \in M_{24}$. We will see that these expressions are quasimodular forms. For the first case, Dabholkar, Murthy, and Zagier give a formula in [9] that (when rearranged) says

(2.1) $H_\chi(\tau)\eta^3(\tau) - 48F_2(\tau) = -2E_2(\tau)$.

Remark. Dabholkar, Murthy, and Zagier also define a higher weight analogue of $F_2^{(k)}$ for $k \geq 2$. For our purposes $k = 2$.

A theorem of Mertens, when specialized to these functions, gives explicitly that the holomorphic projection of $\hat{H}_e(\tau)\eta^3(\tau)$ is equal to the left hand side of (2.1) and is a quasimodular form [23]. Mertens’ theorem can be applied to the other functions $H_g(\tau)$, and in fact, we have such a formula more generally, for any $g \in M_{24}$. Let $\chi(g)$ be the number of fixed points of $g$ in the 24-dimensional permutation representation of $M_{24}$. We define

$$Q_g(\tau) \quad := \quad H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau).$$

This is consistent with the $g = e$ case given above because the number of fixed points in that case is $\chi(e) = 24$. We will show that these functions are quasimodular.

To be precise in the description of the $Q_g(\tau)$, we first define $\rho_g$, a function from $\Gamma_0(|g|)$ to $\mathbb{C}$ given by $\rho_g(\gamma) := \exp \left( 2\pi i \left( -\frac{cd}{|g|h} \right) \right)$, where $h$ is the minimal length among cycles in the cycle shape of $g$ and $c, d$ are the entries of the lower row of a matrix $\gamma$ in $\Gamma_0(|g|)$.

Proposition 2.1. The $Q_g(\tau)$ are quasi-modular of weight 2 on $\Gamma_0(|g|)$, with multiplier system $\rho_g$. 

Proof. The following explicit formula

\[
H_g(\tau) = \frac{\chi(g)}{24} H_e(\tau) - \frac{T(g)}{\eta^3(\tau)}
\]

was obtained in [6,13,18,19] (see Section 3 of [7]). When rearranged, the above formula relates \(H_e(\tau)\eta^3(\tau)\) to each of the \(H_g(\tau)\eta^3(\tau)\). The \(T(g)\) are weight 2 forms on \(\Gamma_0(|g|)\) with multiplier \(p_g\) and their explicit expressions are given in Appendix B.3.1 of [10]. Combining these with the equation \(24\) for \(H_e(\tau)\eta^3(\tau)\) gives that the functions \(H_g(\tau)\eta^3(\tau)\) are quasimodular of weight 2. \(\square\)

Now that we have described our new functions \(Q_g(\tau)\) we show that there exists an \(M_{24}\)-module for which these are the graded trace functions.

**Theorem 2.2.** There exists a virtual graded \(M_{24}\)-module \(V = \bigoplus V_n\) such that

\[
Q_g(\tau) = \sum_{n=0}^{\infty} \text{tr}(g \mid V_n) q^n.
\]

**Proof.** First we show that the \(Q_g(\tau)\) have integral coefficients. Gannon [20] shows that the functions \(H_g(\tau)\) have integral coefficients. It is known that \(\eta^3(\tau)\) has integral coefficients, and \(F_2(\tau)\) also has integral coefficients. So we know that

\[
Q_g(\tau) := H_g(\tau)\eta^3(\tau) - 2\chi(g) F_2(\tau)
\]

must have integral coefficients.

Next we show that the multiplicities \(m_i(n)\) of the \(M_{24}\) irreducible representations in the class functions defined by the coefficients of \(Q_g(\tau)\) are integral.

Gannon shows that the multiplicity generating function

\[
\sum_{n>0} m_i(n) q^n = \frac{1}{|M_{24}|} \sum_{g \in M_{24}} H_g(\tau) \overline{\chi_i(g)}
\]

(with \(\chi_i\) an irreducible character of \(M_{24}\)) has integral coefficients. What we need to show is that the coefficients \(m_i(n)\) are integral, where

\[
\sum_{n>0} m_i(n) q^n = \frac{1}{|M_{24}|} \sum_{g \in M_{24}} \left[ H_g(\tau)\eta^3(\tau) - 2\chi(g) F_2(\tau) \right] \overline{\chi_i(g)}.
\]

To do this, we can split the right hand side of equation \(2.4\) into two parts. First consider

\[
\frac{1}{|M_{24}|} \sum_{g \in M_{24}} H_g(\tau)\eta^3(\tau) \overline{\chi_i(g)}.
\]

This differs from \(2.3\) only from multiplying by \(\eta^3(\tau)\), which does not change the integrality. So it suffices to show that

\[
\frac{1}{|M_{24}|} \sum_{g \in M_{24}} \chi(g) F_2(\tau) \overline{\chi_i(g)}
\]

has integral coefficients. We already know that \(F_2(\tau)\) has integral coefficients. The integrality of \(\langle \chi, \chi_i \rangle\) can be seen from the fact that \(\chi(g)\) is a character of a module, and so \(\langle \chi, \chi_i \rangle\) is the multiplicity of \(\chi_i\) in \(\chi\), which is necessarily integral. Thus the \(m_i(n)\) from \(2.4\) are integral. \(\square\)
3. More general framework

In this section we show that for certain conjugacy classes $[g]$, the $Q_g(\tau)$ have convenient expressions containing arithmetic information. Further, we show that this type of expression can be generalized to be in terms of an arbitrary prime $N$.

If we restrict to $M_{23}$, a subgroup of $M_{24}$ for which all $g$ have $\chi(g) \neq 0$, we can give an alternate expression for the corresponding $Q_g(\tau)$. First, let

$$E_{2,N}(\tau) := \frac{1}{i(N)\varphi(N)} \sum_{M \mid N} \mu \left( \frac{N}{M} \right) M^2 E_2(M \tau)$$

where $N$ is defined to be the order of $g$, $i(N)$ is the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$, and $\varphi$ is the Euler totient function. Note that these functions $E_{2,N}(\tau)$ are quasimodular of weight 2 on $\Gamma_0(N)$.

For each $N = |g|$ with $g \in M_{23}$, let $G_N(\tau)$ denote the specific cusp form of level $N$ given explicitly in the appendix. We also let $n_N := \text{num} \left( \frac{N - 1}{12} \right)$. Then we have the following formula:

$$H_g(\tau) q^3(\tau) - 2\chi(g) F_2(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N} G_N(\tau).$$

This can be easily checked by comparing case-by-case to formula (B.24) in [10]. From this it follows that for $g \in M_{23}$

$$Q_g(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N} G_N(\tau).$$

Note that the formula for $Q_g(\tau)$ above is defined in terms of $N = |g|$ but the expression does not depend on $M_{23}$ at all. We can define such functions $Q_N(\tau)$ for arbitrary prime $N$ with suitable cusp forms $G_N(\tau)$ that come from a result of Mazur. In what follows we show precisely how to define the $Q_N(\tau)$ including how to use the result of Mazur to determine the cusp form.

The multiples of the cusp forms $G_N(\tau)$ in the expressions (3.2) have denominators equal to $n_N$ (recall, $n_N = \text{num} \left( \frac{N - 1}{12} \right)$). In the trace functions where $N$ is prime, these denominators reflect a result of a congruence between Eisenstein series and cusp forms that is due to Mazur. Due to this result, we are able to define quasimodular forms as in (3.2) and show the existence of a $\mathbb{Z}/N\mathbb{Z}$-module such that these forms are its trace functions.

First, we give a more elementary argument for Mazur’s result (Proposition 5.12 of [24]) that there exists a cusp form congruent to the Eisenstein series of level $N$ for $N$ prime. Denote the $m$th coefficient of the normalized Eisenstein series $\frac{1}{24}(NE_2(N \tau) - E_2(\tau))$ by $\sigma_N(m)$. To prove the existence of a cusp form whose coefficients are congruent to $\sigma_N(m)$, we will use theta series, defined in [21]. Let $i, j \in \{1, \ldots, n\}$ where $n$ is the number of left ideal classes in the quaternion algebra over $\mathbb{Q}$ ramified at the two places $N$ and $\infty$. Then the theta series are defined as

$$f_{ij} := \frac{1}{2w_j} + \sum_{m \geq 1} B_{ij}(m)q^m,$$

where $B_{ij}(m)$ are entries of the Brandt matrix of degree $m$, and $w_j$ are integers that correspond to the cardinalities of certain groups. We will use the fact that these $f_{ij}$ are functions with integral coefficients (except the constant term) on the upper half plane. In fact, the $f_{ij}$ span $M_2(\Gamma_0(N))$ and a certain explicit linear combination of $f_{ij}$ recovers the normalized Eisenstein series. See [27] and [23] for related results using these theta series. We will use them to prove the following proposition.

**Proposition 3.1.** Let $N$ be prime. Then there exists a cusp form $g(\tau) = \sum_{m > 0} c_g(m)q^m$ of weight 2 on $\Gamma_0(N)$ with integer coefficients such that

$$c_g(m) \equiv \sigma_N(m) \pmod{n_N}$$

for all $m > 0$. 

The constant term of $2w_1 f_{11}$ is 1 and so the constant term of $-w_1 f_{11} n_N$ is $-n_N \frac{1}{2} = -\frac{N-1}{24}$ since $12 \mid N - 1$, $(N - 1, 12) = 12$. Thus we have:

$$g(\tau) = \frac{N - 1}{24} + \sum_{m > 0, j = 1}^{n} B_{1j}(m)q^m - \frac{N - 1}{24} - w_1 n_N \sum_{m > 0} B_{1j}(m)q^m,$$

and we can cancel the constant terms to get

$$g(\tau) = \sum_{m > 0, j = 1}^{n} B_{1j}(m)q^m - w_1 n_N \sum_{m > 0} B_{1j}(m)q^m \equiv \frac{3(N - 1)}{24} (\text{mod } n_N),$$

where the left hand side has integer coefficients and the right hand side is equal to the normalized Eisenstein series minus its constant term.

The next case is $N \equiv 5 \pmod{12}$ so $(N - 1, 12) = 4$. In particular, $N - 1$ is coprime to 3.

Here we take

$$g(\tau) = \left(3 \sum_{j = 1}^{n} f_{1j}\right) - w_1 f_{11} n_N.$$

The constant term of $2w_1 f_{11}$ is 1 and so the constant term of $-w_1 f_{11} n_N$ is $-n_N \frac{1}{2} = -\frac{3(N - 1)}{24}$ since $(N - 1, 12) = 4$. Thus we have:

$$g(\tau) = \frac{3(N - 1)}{24} + 3 \sum_{m > 0, j = 1}^{n} B_{1j}(m)q^m - \frac{3(N - 1)}{24} - w_1 n_N \sum_{m > 0} B_{1j}(m)q^m,$$

and we can cancel the constant terms to get

$$g(\tau) = 3 \sum_{m > 0, j = 1}^{n} B_{1j}(m)q^m - w_1 n_N \sum_{m > 0} B_{1j}(m)q^m \equiv 3 \sum_{m > 0, j = 1}^{n} B_{1j}(m)q^m \pmod{n_N}.$$

Since we mentioned that $3 \nmid N - 1$, then $3 \nmid n_N$ and so $(n_N, 3) = 1$. Multiplying both sides by the inverse of 3 (mod $n_N$) leaves us with a cusp form with integer coefficients congruent to the
normalized Eisenstein series (minus its constant term) modulo $n_N$.

The remaining cases, $N = 7, 11 \pmod{12}$ follow similarly.

Note that, for $N = 7 \pmod{12}$, we have $(N - 1, 12) = 6$ and we use

$$g(\tau) = \left( 2 \sum_{j=1}^{n} f_{1j} \right) - w_1 f_{11} n_N.$$ 

Again we will need that 2 is invertible modulo $n_N$ and since in this case $n_N = \frac{N - 1}{6} = 2k + 1$, we see that $(n_N, 2) = 1$ so 2 is invertible modulo $n_N$.

And finally, for $N = 11 \pmod{12}$, we have $(N - 1, 12) = 2$ and we use

$$g(\tau) = \left( 6 \sum_{j=1}^{n} f_{1j} \right) - w_1 f_{11} n_N.$$ 

Lastly we will need that 6 is invertible modulo $n_N$ and since in this case $n_N = \frac{N - 1}{2} = 6k + 5$, we see that $(n_N, 6) = 1$ so 6 is invertible modulo $n_N$. \hfill \Box

Now, we have that there exists a cusp form of level $N$ with integral coefficients which is congruent to $\frac{1}{24}(N E_2(N\tau) - E_2(\tau))$ modulo $n_N$ except for the constant term. We will use this result in the next proposition.

First we require some notation, let $\ell_N := \text{num} \left( \frac{N^2 - 1}{24} \right)$. This is the minimum positive integer that clears denominators of $E_{2,N}(\tau)$ because multiplying $E_{2,N}(\tau)$ by $N^2 - 1$ clears denominators and 24 is the largest number we can divide by that does not hurt integrality (all coefficients of $E_{2,N}$ except the constant term are divisible by 24). Note that for $N > 3$, $\ell_N = \left( \frac{N^2 - 1}{24} \right)$ and for $N = 2, 3$, $\ell_N = 1$.

**Remark.** Hanson Smith noted that $\ell_N$ is an upper bound for the genus of $X_1(N)$ ($N > 3$ prime).

With the notation defined above we prove the following proposition.

**Proposition 3.2.** Let $\ell_N$ and $n_N$ be as above. Then $\ell_N E_{2,N}$ has integral coefficients and there exists a cusp form $G_N(\tau)$ of level $N$ and weight 2 with integral coefficients such that

$$\ell_N E_{2,N}(\tau) \equiv -N G_N(\tau) \pmod{n_N}.$$ 

**Proof.** We first treat the cases $N = 2, 3$, since for these we have $\ell_N = 1$. We would like to show that $E_{2,2}(\tau)$ and $E_{2,3}(\tau)$ are congruent to 0 (mod $n_N$) because the space of weight 2 cusp forms of levels 2 and 3 are both empty. Since $n_2 = n_3 = 1$, this follows because the coefficients of $E_{2,2}(\tau)$ and $E_{2,3}(\tau)$ are integers.

Now we continue with the remaining cases (and can assume $N > 3$). When $N$ is prime, we have the following simplified expression for $E_{2,N}(\tau)$ (cf. (3.1)):

$$E_{2,N}(\tau) = \frac{1}{(N + 1)(N - 1)} \left( N^2 E_2(N\tau) - E_2(\tau) \right).$$ 

We can rearrange this to get a more convenient expression for $E_{2,N}$ as follows:

$$E_{2,N}(\tau) = \frac{N}{(N + 1)(N - 1)} \left( N E_2(N\tau) - E_2(\tau) \right) + \frac{1}{N + 1} E_2(\tau).$$
Then we can multiply $E_{2,N}$ by $\ell_N$ to get

$$\ell_N E_{2,N}(\tau) = \frac{N}{24} (NE_2(N\tau) - E_2(\tau)) + \frac{N-1}{24} E_2(\tau).$$

By the congruence in the previous proposition we have that there exists a cusp form $g(\tau)$ with integral coefficients of weight 2 and level $N$ such that

$$\left(\frac{1}{24} (NE_2(N\tau) - E_2(\tau))\right) - \frac{N-1}{24} = g(\tau) + K(\tau)n_N$$

where $K(\tau)$ is some $q$-series with integer coefficients. We subtract $\frac{N-1}{24}$ which is equal to the constant term of $\left(\frac{1}{24} (NE_2(N\tau) - E_2(\tau))\right)$ so that the left hand side of (3.5) has integer coefficients. Thus we have

$$\ell_N E_{2,N}(\tau) = N \left( g(\tau) + \frac{N-1}{24} + K(\tau)n_N \right) + \frac{N-1}{24} E_2(\tau).$$

We define $G_N(\tau) := -g(\tau)$, distribute $N$, and combine constant terms. Then we can write:

$$\ell_N E_{2,N}(\tau) = -NG_N(\tau) + \frac{N^2-1}{24} + NK(\tau)n_N + \frac{N-1}{24} (E_2(\tau) - 1).$$

Since $\frac{N^2-1}{24}$ is an integer multiple of $n_N$ it is $0 \pmod{n_N}$. Also, since $N-1$ is a multiple of $n_N$ and each coefficient of $\frac{E_2(\tau)-1}{24}$ is an integer, each coefficient of $\frac{N-1}{24} (E_2(\tau) - 1)$ is $0 \pmod{n_N}$, and similarly for $NK(\tau)n_N$. So we have shown that $\ell_N E_{2,N}(\tau) \equiv -NG_N(\tau) \pmod{n_N}$. \( \square \)

Now that we have shown this congruence between $\ell_N E_{2,N}(\tau)$ and $-NG_N(\tau)$ modulo $n_N$, we have that for $N$ prime, the expression $\frac{\ell_N E_{2,N}(\tau) - NG_N(\tau)}{n_N}$ has integer coefficients. We can call these our

$$Q_N^{(N)}(\tau) := -\ell_N E_{2,N}(\tau) - NG_N(\tau),$$

and we define

$$Q_1^{(N)}(\tau) := \frac{-\ell_N}{n_N} E_{2,1}(\tau).$$

Note that $Q_1^{(N)}(\tau)$ also has integer coefficients and that $E_{2,1}(\tau) = E_2(\tau)$. These functions (3.6) and (3.7) will be our trace functions for the $\mathbb{Z}/N\mathbb{Z}$ module. In order to prove the existence of this module we require the following lemma:

**Lemma 3.3.** Let $N$ be prime. Then

$$Q_N^{(N)}(\tau) \equiv Q_1^{(N)}(\tau) \pmod{N}.$$  

**Proof.** Again we start with the cases $N = 2, 3$:

For $N = 2$, we have $Q_2^{(2)}(\tau) = -E_{2,2}(\tau) = -\frac{1}{3} (4E_2(2\tau) - E_2(\tau))$ and $Q_1^{(2)}(\tau) = -E_2(\tau)$. To show that $Q_2^{(2)}(\tau) \equiv Q_1^{(2)}(\tau) \pmod{2}$, it suffices to show that $3Q_2^{(2)}(\tau) \equiv 3Q_1^{(2)}(\tau) \pmod{2}$ because $(2, 3) = 1$. We see the latter because $3Q_2^{(2)}(\tau) = -4E_2(2\tau) + E_2(\tau)$, $3Q_1^{(2)}(\tau) = -3E_2(\tau)$, and $-3 \equiv 1 \pmod{2}$.

The case $N = 3$ is similar, we have $Q_2^{(3)}(\tau) = E_{2,3}(\tau) = -\frac{1}{8} (9E_2(3\tau) - E_2(\tau))$ and $Q_1^{(3)}(\tau) = -E_2(\tau)$. Again, it suffices to show that $8Q_2^{(3)}(\tau) \equiv 8Q_1^{(3)}(\tau) \pmod{3}$, which can be seen using the fact that $-8 \equiv 1 \pmod{3}$. 


For $N > 3$, we will show that

$$Q_{N}^{(N)}(\tau)n_{N} \equiv Q_{1}^{(N)}(\tau)n_{N} \pmod{N}.$$  

If the above congruence is true then since $(n_{N}, N) = 1$, the congruence in the statement of the lemma will be true. Beginning with the left hand side, we have that

$$Q_{N}^{(N)}(\tau)n_{N} = -\frac{1}{24}(N^{2}E_{2}(N\tau) - E_{2}(\tau)) - NG_{N}(\tau).$$

We can rewrite this as

$$Q_{N}^{(N)}(\tau)n_{N} = -\frac{N^{2}}{24}(E_{2}(N\tau) - 1) - \frac{1}{24}(E_{2}(\tau) - 1) + \frac{1}{24} - NG_{N}(\tau),$$

or equivalently

$$Q_{N}^{(N)}(\tau)n_{N} = -\frac{N^{2}}{24}(E_{2}(N\tau) - 1) - \frac{N^{2} - 1}{24}.$$  

Next we look at the right hand side. We have

$$Q_{N}^{(N)}(\tau)n_{N} = \ell_{N}E_{2,1}(\tau) = -\frac{N^{2} - 1}{24}E_{2}(\tau) = \frac{N^{2}}{24}E_{2}(\tau) + \frac{1}{24}E_{2}(\tau) - \frac{N^{2}}{24}E_{2}(\tau) - \frac{1}{24}(E_{2}(\tau) - 1) - \frac{1}{24} - NG_{N}(\tau).$$

Reducing modulo $N$ gives

$$Q_{N}^{(N)}(\tau)n_{N} \equiv 1^{(N)}(\tau)n_{N} \pmod{N}.$$  

And thus

$$Q_{N}^{(N)}(\tau)n_{N} \equiv Q_{1}^{(N)}(\tau)n_{N} \pmod{N}.$$  

For $N$ prime, we give quasimodular weight 2 trace functions for $\mathbb{Z}/N\mathbb{Z}$ as follows:

$$f_{g}^{(N)}(\tau) := \begin{cases} Q_{1}^{(N)}(\tau) & \text{if } g = e, \\ Q_{N}^{(N)}(\tau) & \text{if } g \neq e. \end{cases}$$

**Theorem 3.4.** There exists a virtual graded $\mathbb{Z}/N\mathbb{Z}$-module $V = \bigoplus_{n=0}^{\infty} V_{n}$ such that

$$f_{g}^{(N)}(\tau) = \sum_{n=0}^{\infty} \text{tr}(g | V_{n})q^{n}$$

where $f_{g}^{(N)}(\tau)$ is the quasimodular form of weight 2 and level $N$ with integral coefficients as defined above.

**Proof.** Showing the existence of this virtual module amounts to showing that the multiplicities of the irreducible representations of $\mathbb{Z}/N\mathbb{Z}$ in the module are integral. Thus we need to show the integrality of $\frac{1}{N} \sum_{g \in \mathbb{Z}/N\mathbb{Z}} c_{g}(n)x_{i}(g)$, for each $i$, where $c_{g}(n)$ are the coefficients of the trace functions for $g \in \mathbb{Z}/N\mathbb{Z}$, and $x_{i}(g)$ are irreducible characters of $\mathbb{Z}/N\mathbb{Z}$. This is equivalent to
showing that \( \langle \chi_i, f^{(N)}_g(\tau) \rangle \) is integral, for \( i \in \{1 \ldots N\} \). For \( \chi_1 \), the character corresponding to the trivial representation, we have

\[
(3.8) \quad \langle \chi_1, f^{(N)}_g(\tau) \rangle = \frac{1}{N} (Q^{(N)}_1(\tau) + (N - 1)Q^{(N)}_N(\tau)) = Q^{(N)}_1(\tau) + \frac{1}{N} (Q^{(N)}_1(\tau) - Q^{(N)}_N(\tau)).
\]

In other words, for integrality of (3.8) we need that \( Q^{(N)}_1(\tau) \equiv Q^{(N)}_N(\tau) \pmod{N} \). This is true by Lemma 3.3.

For all other \( \chi_i \), we make use of the fact that, for \( \zeta \) a primitive \( N \)th root of unity, \( \zeta + \cdots + \zeta^{N-1} = -1 \) and in particular that \( \zeta^k + \cdots + \zeta^{(N-1)} = -1 \) for \( 1 \leq k \leq N - 1 \). So all other \( \chi_i \)'s give

\[
(3.9) \quad \langle \chi_i, f \rangle = \frac{1}{N} (Q^{(N)}_1(\tau) - Q^{(N)}_N(\tau)),
\]

which is again integral by the congruence \( Q^{(N)}_1(\tau) \equiv Q^{(N)}_N(\tau) \pmod{N} \).

4. Arithmetic/Geometric Connections

In this section, we describe some arithmetic connections between the trace functions of the \( M_{23} \)-module given in the previous section with expressions in (3.2) and the \( \mathbb{F}_p \) point counts on (Jacobians of) modular curves. The expressions for \( E_{2,N}(\tau) \) are not always integral on their own, and in the levels with cusp forms we saw that adding a multiple (with specific denominator) of a cusp form to \( E_{2,N}(\tau) \) gives an expression with integral coefficients. The choices of cusp forms we used in the previous section (given explicitly in the appendix) are such that we get integral coefficients when we add \( \frac{N}{n_N} G_N \) to \( E_{2,N}(\tau) \).

For \( M_{23} \) this is summarized below:

(a) \(-2E_{2,11}(\tau) + \frac{11}{5} G_{11}(\tau)\) has integral coefficients.

(b) \(-2E_{2,14}(\tau) + \frac{14}{3} G_{14}(\tau)\) has integral coefficients.

(c) \(-2E_{2,15}(\tau) + \frac{15}{4} G_{15}(\tau)\) has integral coefficients.

(d) \(-2E_{2,23}(\tau) + \frac{23}{11} G_{23}(\tau)\) has integral coefficients.

Because coefficients of weight 2 cusp forms admit a certain geometric interpretation, these expressions give divisor conditions on the number of \( \mathbb{F}_p \) points on Jacobians of modular curves. Let \( J_0(N) \) denote the Jacobian of the modular curve \( X_0(N) \). For \( N = 11, 14, 15 \), we have that \( J_0(N) \) is an elliptic curve. For \( N = 23 \) \( J_0(N) \) is an abelian surface, namely, the Jacobian of a genus 2 curve.

For simplicity, here, we restrict our attention to \( N \) such that \( X_0(N) \) are elliptic curves.

The result below relies on the relationship between cusp forms and point counts on elliptic curves. For an introductory reference for elliptic curves, their \( \mathbb{F}_p \) points, and the relationship to cusp forms of weight 2, see [20] (in particular, we apply Theorem 7.10 of loc. cit.).

Corollary 4.1. We have the following (known) divisibility conditions arising from \( M_{23} \)

1. \( p \neq 11 \), we have \( 5 \mid \#J_0(11)(\mathbb{F}_p) \).
2. \( p \neq 2, 7 \), we have \( 3 \mid \#J_0(14)(\mathbb{F}_p) \).
3. \( p \neq 3, 5 \), we have \( 4 \mid \#J_0(15)(\mathbb{F}_p) \).

Proof. (1) We have that

\[
\left(-\frac{21}{60} E_{2}(11\tau) + \frac{1}{60} E_{2}(\tau)\right) + \frac{11}{5} G_{11}(\tau) \in \mathbb{Z}[q].
\]
Using the definition of $E_2(\tau)$, we see that this is equal to

$$-2 + \frac{242}{5} \sum_{m=1}^{\infty} \sigma(m)q^{11m} - \frac{2}{5} \sum_{n=1}^{\infty} \sigma(n)q^n + \frac{11}{5} G_{11}(\tau).$$

Note that we defined $G_{11}(\tau) = 2\eta^2(\tau)\eta'(11\tau)$ so all of its coefficients are even, and in fact $G_{11}(\tau) = 2\tilde{G}_{11}(\tau)$ where the $\tilde{G}_{11}(\tau) = \sum_{n>0} c_{11}(n)$ is the normalized cusp form whose coefficients correspond to the number of $\mathbb{F}_p$ points on the Jacobian of $X_0(11)$, denoted $\#J_0(11)(\mathbb{F}_p)$. The correspondence is given as follows:

$$(4.1) \quad c_{11}(p) = p + 1 - \#J_0(11)(\mathbb{F}_p).$$

We can see that if $11 \nmid n$, then the $n$th coefficient of (a) is $-\frac{2}{5}\sigma(n) + \frac{22}{5}c_{11}(n) \in \mathbb{Z}$. The integrality of the coefficient of $p \neq 11$ then implies that $-2(p+1) + 22c_{11}(p) \equiv 0 \pmod{5}$. Substituting $c_{11}(p)$ for the right hand side of (4.1) gives us that $\#J_0(11)(\mathbb{F}_p) \equiv 0 \pmod{5}$.

We note that for $p = 11$, we can directly compute $\#J_0(11)(\mathbb{F}_{11})$ with the fact that $c_{11}(11) = 1$. Thus we have $\#J_0(11)(\mathbb{F}_{11}) = 11$.

(2) If $2, 7 \nmid n$, then the $n$th coefficient of (b) is $\frac{1}{3}\sigma(n) + \frac{14}{3}c_{14}(n) \in \mathbb{Z}$. The integrality of the coefficient of $p \neq 2, 7$ then implies that $p + 1 + 14c_{14}(p) \equiv 0 \pmod{3}$. We substitute $c_{14}(p)$ for the right hand side of the following:

$$(4.2) \quad c_{14}(p) = p + 1 - \#J_0(14)(\mathbb{F}_p),$$

and this gives us that $\#J_0(14)(\mathbb{F}_p) \equiv 0 \pmod{3}$.

We note that for $p = 2$ and $p = 7$, we can directly compute $\#J_0(14)(\mathbb{F}_2)$ and $\#J_0(14)(\mathbb{F}_7)$ with the facts that $c_{14}(2) = -1$ and $c_{14}(7) = 1$. Thus we have $\#J_0(14)(\mathbb{F}_2) = 4$ and $\#J_0(14)(\mathbb{F}_7) = 7$.

(3) If $3, 5 \nmid n$, then the $n$th coefficient of (c) is $\frac{1}{4}\sigma(n) + \frac{15}{4}c_{15}(n) \in \mathbb{Z}$. The integrality of the coefficient of $p \neq 3, 5$ then implies that $p + 1 + 15c_{15}(p) \equiv 0 \pmod{4}$. We substitute $c_{15}(p)$ for the right hand side of the following:

$$(4.3) \quad c_{15}(p) = p + 1 - \#J_0(15)(\mathbb{F}_p),$$

and this gives us that $\#J_0(15)(\mathbb{F}_p) \equiv 0 \pmod{4}$.

We note that for $p = 3$ and $p = 5$, we can directly compute $\#J_0(15)(\mathbb{F}_3)$ and $\#J_0(15)(\mathbb{F}_5)$ with the facts that $c_{15}(3) = -1$ and $c_{15}(5) = 1$. Thus we have $\#J_0(15)(\mathbb{F}_3) = 5$ and $\#J_0(15)(\mathbb{F}_5) = 5$. □

We expect infinitely many more such expressions (as in Corollary 1.1) arising from the trace functions of the $\mathbb{Z}/N\mathbb{Z}$-modules of Theorem 3.3. In the cases above, since the modular curves are elliptic curves and therefore (isomorphic to) their own Jacobians, any divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobians are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the modular curves themselves. In general, for any prime $N$, the integrality of trace functions from these $\mathbb{Z}/N\mathbb{Z}$-modules are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobians of $X_0(N)$ (cf. Theorem 7.10 of [36]).

Remark. The integrality conditions used in the proof or Corollary 1.1 implied congruences of the form $-2(p + 1) + 22c_{11}(p) \equiv 0 \pmod{5}$. These are equivalent to statements such as $(p + 1) \equiv 0 \pmod{5}$ iff $c_{11}(p) \equiv 0 \pmod{5}$. Jeffrey Lagarias noted that if one reframes these statements as, for example,

$$p = 4 \pmod{5} \text{ if and only if } 5 \mid c_{11}(p),$$

the expressions can then be written in the following way resembling the Ramanujan congruences:

For $p = 5n + 4$ we have $c_{11}(p) = 0 \pmod{5}$.
Remark. In this formulation of the trace functions of the $M_{23}$-module (3.2), we made a choice with the multiple of the cusp form. The denominator is fixed but the choices we made of the numerator are not unique. In fact, we could add any multiple of $NG_N(\tau)$ and still satisfy the congruences necessary for Mathieu moonshine. Therefore, the module here is one in an infinite family of possible modules one can consider. A similar statement holds for the modules of Theorem 3.4. It would be interesting to see if stronger results about point counts on modular Jacobians might be obtained by studying these families as a whole.

5. An explicit module construction

In the previous sections, the trace functions in Theorem 3.4 and those of the $M_{23}$ module (3.2) have involved cusp forms. The results of these were some arithmetic/geometric observations in addition to the minimality of the constant $\left(\frac{\ell_N}{N}\right)$ in front of $E_{2,N}(\tau)$ that guarantees integrality of trace functions’ coefficients. If we restrict our functions to only involve Eisenstein series, we remove the cusp form contribution (thus we can no longer divide by $n_N$). We define an alternative set of trace functions for $\mathbb{Z}/N\mathbb{Z}$ in this way. For $N$ prime, we give purely Eisenstein quasimodular weight 2 trace functions for a $\mathbb{Z}/N\mathbb{Z}$-module as follows:

$$F_g^{(N)}(\tau) := \begin{cases} -\ell_N E_2(\tau) & \text{if } g = e, \\ -\ell_N E_{2,N}(\tau) & \text{if } g \neq e. \end{cases}$$

It can be easily seen from the methods in Section 3 that there also exists a module for which these are the trace functions. Indeed, we will construct such a module explicitly in this section.

Although the modules with quasimodular trace functions involving cusp forms were arguably more interesting, the advantage of purely Eisenstein quasimodular trace functions is that we can actually give a construction of the module in terms of vertex operator algebras. Moreover, when the space of cusp forms of level $N$ is empty, the purely Eisenstein trace functions are equal to the trace functions in Theorem 3.4. Given this, it would be interesting to see if the method presented here may be modified so as to obtain the modules that do include cusp form contributions, discussed in Section 3.

To construct the purely Eisenstein modules we will find a vertex operator algebra that has the $F_g^{(N)}(\tau)$ as their trace functions. For this we will use two Heisenberg vertex algebras and a Clifford module vertex algebra. First, note that for $N > 3$ we have $\ell_N = \frac{N^2 - 1}{24}$ and

$$q \frac{d}{dq} \log \left( \frac{1}{\eta^{(N^2-1)}(\tau)} \right) = -\frac{N^2 - 1}{24} E_2(\tau), \text{ and}$$

$$q \frac{d}{dq} \log \left( \frac{\eta(\tau)}{\eta^N(\tau)} \right) = -\frac{N^2 - 1}{24} E_{2,N}(\tau).$$

For the remaining cases $N = 2, 3$, we have that $\ell_N = 1$.

When $N = 2$ we have

$$q \frac{d}{dq} \log \left( \frac{1}{\eta^{24}(\tau)} \right) = -E_2(\tau), \text{ and}$$

$$q \frac{d}{dq} \log \left( \frac{\eta^2(\tau)}{\eta^{16}(2\tau)} \right) = -E_{2,2}(\tau).$$

And when $N = 3$ we have

$$q \frac{d}{dq} \log \left( \frac{1}{\eta^{24}(\tau)} \right) = -E_2(\tau), \text{ and}$$

$$q \frac{d}{dq} \log \left( \frac{\eta^3(\tau)}{\eta^{9}(3\tau)} \right) = -E_{2,3}(\tau).$$

In what follows, we describe the module construction.
Let $D$ denote the derivative $D(\cdot) := \frac{dq}{dq}(\cdot)$. From the above equations we see that finding a module whose trace functions are equal to $F_g^{(N)}(\tau)$ with $N > 3$ is equivalent to finding a module whose trace functions are equal to:

\begin{align}
D \left( \frac{1}{\eta^{(N^2-1)}(\tau)} \right) \eta^{N^2}(\tau) \frac{1}{\eta(\tau)} & \text{ if } g = e, \\
D \left( \frac{\eta(\tau)}{\eta^N(\tau)} \right) \eta^N(N\tau) \frac{1}{\eta(\tau)} & \text{ if } g \neq e.
\end{align}

(5.1)

Similarly, we note that finding a module whose trace functions are $F_g^{(2)}(\tau)$ and $F_g^{(3)}(\tau)$ are equivalent to finding a module whose trace functions are equal to:

\begin{align}
D \left( \frac{1}{\eta^{24}(\tau)} \right) \eta^{32}(\tau) \frac{1}{\eta^8(\tau)} & \text{ if } g = e, \\
D \left( \frac{\eta^8(\tau)}{\eta^{16}(2\tau)} \right) \eta^{16}(2\tau) \frac{1}{\eta^8(\tau)} & \text{ if } g \neq e,
\end{align}

(5.2)

and

\begin{align}
D \left( \frac{1}{\eta^{24}(\tau)} \right) \eta^{27}(\tau) \frac{1}{\eta^9(\tau)} & \text{ if } g = e, \\
D \left( \frac{\eta^9(\tau)}{\eta^{9}(3\tau)} \right) \eta^9(3\tau) \frac{1}{\eta^9(\tau)} & \text{ if } g \neq e,
\end{align}

(5.3)

respectively. The next two lemmas indicate how to recover the first of the three factors in each of eqs. (5.1) to (5.3). To formulate it, let $\mathfrak{h} = \mathbb{C}^{24N}$ and let $\gamma$ be an automorphism of order $N$ of $\mathfrak{h}$ such that its characteristic polynomial is $\text{char}_\gamma(x) = \frac{(x^N - 1)^N}{(x - 1)}$ (for $N = 3, 2$, take the characteristic polynomials $\text{char}_\gamma(x) = \frac{(x^2 - 1)^{16}}{(x - 1)^8}$ and $\text{char}_\gamma(x) = \frac{(x^3 - 1)^9}{(x - 1)^3}$, respectively). Let $V := S(\mathfrak{h}(-n) \mid n > 0; i = 1, \ldots, 24N)$ (where $S(x_1, x_2 \ldots) := S(\oplus_{i=1}^{24N} \mathbb{C}x_i)$) be the Heisenberg vertex algebra for $\mathfrak{h}$ with non-degenerate symmetric bilinear form $(\cdot, \cdot)$ fixed by $\gamma$. The action of $\gamma$ on $\mathfrak{h}$ extends naturally to $V$. See [15] for more information on the Heisenberg vertex algebra construction. We denote the $L(0)$ operator for this Heisenberg vertex algebra by $L_1(0)$ and let $c_1$ be its central charge.

**Lemma 5.1.** For $N > 3$, we have graded trace functions for $V$ as follows:

\[ \text{tr} \left( q^{L_1(0) - \frac{c_1}{24}} V \right) = \frac{1}{\eta^{(N^2-1)}(\tau)} \]

and

\[ \text{tr} \left( \gamma q^{L_1(0) - \frac{c_1}{24}} V \right) = \frac{\eta(\tau)}{\eta^N(N\tau)}. \]

Since we instead need the derivatives of those functions, we can take the traces as follows:

**Lemma 5.2.** For $N > 3$, we have

\[ \text{tr} \left( \left( L_1(0) - \frac{c_1}{24} \right) q^{L_1(0) - \frac{c_1}{24}} V \right) = D \left( \frac{1}{\eta^{(N^2-1)}(\tau)} \right) \]

\[ \text{tr} \left( \left( L_1(0) - \frac{c_1}{24} \right) \gamma q^{L_1(0) - \frac{c_1}{24}} V \right) = D \left( \frac{\eta(\tau)}{\eta^N(N\tau)} \right). \]

Note that the two equations above correspond to the first factor of the $g = e$ and $g \neq e$ functions in [5.1] and the analogous statements hold for $N = 2, 3$ (eqs. 5.2 and 5.3).
Next, for the cases $N > 3$, in order to recover the second factor in each of the functions in eq. (5.1), we need a module with trace functions $\eta^{N^2}(\tau)$ and $\eta(N\tau)^N$, respectively. This can be done using a Clifford module vertex algebra. For this construction we follow Duncan and Harvey [13]. In this setting, let $p$ be a one dimensional complex vector space with a symmetric bilinear form. Let $a(r) := a \otimes t^r$ for $a \in p$. Let $p = p[t, t^{-1}]|_{t^{1/2}}$ and $\tilde{p}_w = p[t, t^{-1}]$ with the bilinear form extended so that $\langle a(r), b(s) \rangle = (a, b)\delta_{r+s, 0}$.

We define $\text{Cliff}(p)$ to be the Clifford algebra attached to $p$. Let $\hat{p}^+ := \langle \hat{p}^+ \rangle$ be a subalgebra of the Clifford algebra and let $Cl$ be a $\hat{p}^+$ module such that that $1v = v$ and $p(r)v = 0$ for $r > 0$. Then we define

$$A(p) := \text{Cliff}(\hat{p}) \otimes_{\langle \hat{p}^+ \rangle} Cl,$$

and $A(p)$ has the structure of a super vertex operator algebra with Virasoro element

$$\omega := p(-3/2)p(-1/2)v.$$

Let $\text{Cliff}(\tilde{p}_w)$ be the Clifford algebra attached to $\tilde{p}_w$, let $v_{tw}$ be such that $1v_{tw} = v_{tw}$, and let $a(r)v_{tw} = 0$ for $a \in p$ and $r > 0$. Then take $p \in p$ such that $\langle p, p \rangle = -2$ and $p(0)^2 = 1$. Define $v_{tw}^+ := (1 + p(0))v_{tw}$ so that $p(0)v_{tw}^+ = v_{tw}^+$ and let $\hat{p}_w^+ := p[t, t]$. Then we define

$$A(p)_w^+ := \text{Cliff}(\hat{p}_w) \otimes_{\langle \hat{p}_w^+ \rangle} Cl_{tw}^+,$$

so that $A(p)_w^+$ is isomorphic to $\bigwedge(p(-n) \mid n > 0)v_{tw}^+$ (where $\bigwedge(x_1, x_2 \ldots) := \bigwedge(\oplus_{i=1}^\infty Cx_i)$).

By the reconstruction theorem described in [15] we can see that $A(p)_w$ is a twisted module for $A(p)$ with fields $Y_{tw} : A(p) \otimes A(p)_w \to A(p)_w(\langle z^{1/2} \rangle)$ where

$$Y_{tw}(u(-1/2)v, z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1/2}$$

with $u \in p$. Since $A(p)_w^+$ is a submodule of $A(p)_w$ (generated by $v_{tw}^+$), it can be verified that $A(p)_w^+$ is a twisted module for $A(p)$ so that the above map can be restricted to $A(p)_w^+$.

Let $L_2(0)$ be the $L(0)$ operator for the Clifford module vertex algebra and $c_2$ its central charge. Then we can see that $\text{tr} \left(p(0)q^{L_2(0)} - \frac{c_2}{24} \bigm| A(p)_w^+ \right) = \eta(\tau)$. We would like a module with graded dimension equal to $\eta(\tau)^{N^2}$ so we will consider a tensor product of these $A(p)_w^+$ (we have from [16] that the tensor product of vertex algebras is naturally a vertex algebra).

To do this, we define

$$\tilde{A}(p) := A(p_1) \otimes \cdots \otimes A(p_{N^2})$$

and

$$\tilde{A}(p)_w := A(p_1)_w^+ + \cdots \otimes A(p_{N^2})_w^+$$

where each $A(p_i)_w^+$ is isomorphic to $\bigwedge(p_i(-n) \mid n > 0)v_{tw}^+$. Then we can define

$$\tilde{Y}_{tw} : \tilde{A}(p) \otimes \tilde{A}(p)_w \to \tilde{A}(p)_w(\langle z^{1/2} \rangle)$$

where

$$\tilde{Y}_{tw}(u(-1/2)v_1 \cdots u(-1/2)v_{N^2})v, z) = Y_1(u_1(-1/2)v_1, \cdots \otimes Y_{N^2}(u_{N^2}(-1/2)v_{N^2}, z)$$

$$= \sum_{n \in \mathbb{Z}^{N^2}} u_1(n_1) \cdots \otimes u_{N^2}(n_{N^2}) z^{-n_1-\cdots-n_{N^2} - \frac{N^2}{2}},$$

with $n = (n_1, \ldots, n_{N^2})$, and finally

$$\tilde{p}(0) := p_1(0) \otimes \cdots \otimes p_{N^2}(0).$$

Let $\sigma$ act on $\tilde{A}(p)_w$ by permuting tensor factors with cycle shape $N^N$. Now we have in the next lemma the second factor of each equation in (5.1).
Lemma 5.3.

\[ \text{tr} \left( \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^{N^2}(\tau), \]
\[ \text{tr} \left( \sigma \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^N(N\tau). \]

When \( N = 2, 3 \), the construction is similar, but we define \( \widetilde{A}(p) \), \( A(p)_{tw} \), and \( \overline{p}(0) \) to have 32 (resp. 27) tensor factors (instead of \( N^2 \)) and we take instead permutations \( \sigma \) with cycle shape \( 2^16 \) (resp. \( \sigma \) with cycle shape \( 3^3 \)).

Then for \( N = 2 \), with \( \widetilde{A}(p)_{tw} := A(p_1)_{tw} \otimes \cdots \otimes A(p_{32})_{tw} \), we have:
\[ \text{tr} \left( \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^{32}(\tau), \]
\[ \text{tr} \left( \sigma \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^{16}(2\tau). \]

And for \( N = 3 \) and \( \widetilde{A}(p)_{tw} := A(p_1)_{tw} \otimes \cdots \otimes A(p_{27})_{tw} \), we have:
\[ \text{tr} \left( \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^{27}(\tau), \]
\[ \text{tr} \left( \sigma \overline{p}(0) q^{L_1(0) - \frac{27}{24}} \mid \widetilde{A}(p)_{tw} \right) = \eta^{9}(3\tau). \]

Lastly, we recover the third factor in eqs. (5.1) to (5.3). For this we define \( \mathfrak{g} \) to be \( \mathbb{C} \) when \( N > 3 \), \( \mathbb{C}^8 \) when \( N = 2 \), and \( \mathbb{C}^3 \) when \( N = 3 \). Let \( U := S(k(-n) \mid n > 0) \) (suitably modified when \( N = 2, 3 \)) be the Heisenberg vertex algebra for \( \mathfrak{g} \). We denote the \( L(0) \) operator for this Heisenberg vertex algebra by \( L_3(0) \) and let \( c_3 \) be its central charge. Then we have the following lemma

**Lemma 5.4.** When \( N > 3 \), we have

\[ \text{tr} \left( q^{L_1(0) - \frac{27}{24}} \mid U \right) = \frac{1}{\eta(\tau)}. \]

And note that for \( N = 2, 3 \) we get graded dimension equal to \( \frac{1}{\eta^2(\tau)} \) and \( \frac{1}{\eta^3(\tau)} \), respectively.

To get a vertex algebra whose trace function is the desired product in eqs. (5.1) to (5.3), we let

\[ W(N) := V \otimes \widetilde{A}(p) \otimes U. \]

We take the following canonically twisted module for the vertex algebra \( W(N) \):

\[ W_{tw}^{(N)} := V \otimes \widetilde{A}(p)_{tw} \otimes U. \]

The actions of \( \gamma \) and \( \sigma \) naturally extend to \( W(N) \) and \( W_{tw}^{(N)} \), by letting them act trivially on the factors where they have not already been defined (i.e. \( \gamma \) acts as \( \gamma \otimes \text{id} \otimes \text{id} \) and \( \sigma \) acts as \( \text{id} \otimes \sigma \otimes \text{id} \)). Note that with this definition both \( \gamma \) and \( \sigma \) are automorphisms of \( W(N) \), and act equivariantly on the twisted module in the sense that we have \( Y_{tw}(gu, z)gv = gY_{tw}(u, z)v \) for \( u \in W(N) \) and \( v \in W_{tw}^{(N)} \), and \( g \) equal to \( \gamma \) or \( \sigma \).

Lastly, we define the operator \( L(0) := L_1(0) + L_2(0) + L_3(0) \) and the central charge \( c := c_1 + c_2 + c_3 \) where the subscripts indicate the operators and central charges for the Heisenberg and Clifford vertex algebras described above. Then we have that the following forms are equal to the forms in (5.1) when \( N > 3 \), (5.2) when \( N = 2 \), and (5.3) when \( N = 3 \):

\[
\begin{cases}
\text{tr} \left( \overline{p}(0) \left( L_1(0) - \frac{c_1}{24} \right) q^{L(0) - \frac{27}{24}} \mid W_{tw}^{(N)} \right) & \text{if } g = e, \\
\text{tr} \left( \sigma \overline{p}(0) \left( L_1(0) - \frac{c_1}{24} \right) q^{L(0) - \frac{27}{24}} \mid W_{tw}^{(N)} \right) & \text{if } g \neq e.
\end{cases}
\]

Thus we have constructed a vertex algebra for the \( \mathbb{Z}/N\mathbb{Z} \)-module with purely Eisenstein quasi-modular trace functions.
Theorem 5.5. Let $N$ be prime. Then $W_{tw}^{(N)} = \bigoplus_{n} W_{tw,n}^{(N)}$ is an infinite dimensional virtual graded module for $\mathbb{Z}/N\mathbb{Z}$ such that

$$F_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \text{tr}(g | W_{tw,n}^{(N)}) q^n.$$ 

APPENDIX A. CUSP FORMS

We give the cusp forms relevant for $M_{23}$ explicitly below:

$$G_{11}(\tau) = 2\eta^2(\tau)\eta^2(11\tau),$$
$$G_{14}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau),$$
$$G_{15}(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau),$$
$$G_{23a}(\tau) = \frac{\eta^3(\tau)\eta^3(23\tau)}{\eta(2\tau)\eta(46\tau)} + 3\eta^2(\tau)\eta^2(23\tau) + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta^2(2\tau)\eta^2(46\tau),$$
$$G_{23b}(\tau) = \eta^2(\tau)\eta^2(23\tau),$$
$$G_{23}(\tau) = G_{23a}(\tau) + 3G_{23b}(\tau).$$

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