The first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links and of virtual links

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We characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of ribbon surface-links in the 4–sphere fixing the number of components and the total genus, and then the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links in the 4–sphere fixing the number of components. Using the result of ribbon torus-links, we also characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links fixing the number of components. For a general surface-link, an estimate of the total genus is given in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module. We show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of all surface-links and then a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of classical links, surface-links and higher-dimensional manifold-links.

57M25; 57Q35, 57Q45

1 The first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of a surface-link

For every non-negative partition $g = g_1 + g_2 + ... + g_r$ of a non-negative integer $g$, we consider a closed oriented 2–manifold $F = F_g = F_{g_1,g_2,...,g_r}$ with $r$ components $F_i$ ($i = 1, 2, ..., r$) such that the genus $g(F_i)$ of $F_i$ is $g_i$. The integer $g$ is called the total genus of $F$ and denoted by $g(F)$. An $F$–link $L$ is the ambient isotopy class of a locally-flatly embedded image of $F$ into $S^4$, and for $r = 1$ it is also called an $F$–knot. The exterior of $L$ is the compact 4–manifold $E = S^4 \setminus \text{int}(L)$, where $N(L)$ denotes the tubular neighborhood of $L$ in $S^4$. Let $p: \tilde{E} \to E$ be the infinite cyclic covering associated with the epimorphism $\gamma: H_1(E) \to \mathbb{Z}$ sending every oriented meridian of $L$ in $H_1(E)$ to 1 $\in \mathbb{Z}$. An $F$–link $L$ is trivial if $L$ is the boundary of the union of disjoint handlebodies embedded locally-flatly in $S^4$. A ribbon $F$–link is an $F$–link obtained from a trivial $F_0$–link by surgeries along embedded 1–handles in $S^4$ (see Kawauchi, Shibuya and Suzuki [12, page 52]). When we put the trivial $F_0$–link in the equatorial 3–sphere $S^3 \subset S^4$, we can replace the 1–handles by mutually disjoint 1–handles embedded in the 3–sphere $S^3$ without changing the ambient isotopy class of the ribbon $F$–link by an argument of [12, Lemma 4.11] using a result of Hosokawa.
and Kawauchi [2, Lemma 1.4]. Thus, every ribbon $F$–link is described by a disk–arc presentation consisting of oriented disks and arcs intersecting the interiors of the disks transversely in $S^3$ (see Figure 1 for an illustration), where the oriented disks and the arcs represent the oriented trivial 2–spheres and the 1–handles, respectively.

![Figure 1: A ribbon $F^2_{1,1}$–link](image)

Let $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ be the integral Laurent polynomial ring. The homology $H_*(\tilde{E})$ is a finitely generated $\Lambda$–module. Specially, the first homology $H_1(\tilde{E})$ is called the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module, or simply the module of an $F$–link $L$ and denoted by $M(L)$. In this paper, we discuss the following problem:

**Problem 1.1** Characterize the modules $M(L)$ of $F^r_g$–links $L$ in a topologically meaningful class.

In Section 2, we discuss some homological properties of $F^r_g$–links. Fixing $r$ and $g$, we shall solve Problem 1.1 for the class of ribbon $F^r_g$–links in Section 3. We also solve Problem 1.1 for the class of all $F^r_g$–links not fixing $g$ as a corollary of the ribbon case in Section 3. In Section 4, we characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links by using the characterization of ribbon $F^r_{1,1,...,1}$–links. In Section 5, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of all $F^r_g$–links by establishing an estimate of the total genus $g$ in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of an $F^r_g$–link. In fact, we show that there is the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of an $F^r_g$–link which is not the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of any $F^r_g$–link for every $r$ and $g$. In Section 6, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of classical links, surface-links and higher-dimensional manifold-links. We mention here that most results of this paper are announced in [11] without proofs. A group version of this paper is given in [10].
2 Some homological properties on surface-links

The following computation on the homology $H_\ast(E)$ of the exterior $E$ of an $F^r_g$–link $L$ is done by using the Alexander duality for $(S^4, L)$:

**Lemma 2.1**

$$H_d(E) = \begin{cases} 
\mathbb{Z}^{r-1} & (d = 3) \\
\mathbb{Z}^2 & (d = 2) \\
\mathbb{Z}^r & (d = 1) \\
\mathbb{Z} & (d = 0) \\
0 & (d \neq 0, 1, 2, 3)
\end{cases}$$

For a finitely generated $\Lambda$–module $M$, let $TM$ be the $\Lambda$–torsion part, and $BM = M/TM$ the $\Lambda$–torsion-free part. Let $\beta(M)$ be the $\Lambda$–rank of the module $M$, namely the $Q(\Lambda)$–dimension of the $Q(\Lambda)$–vector space $M \otimes_\Lambda Q(\Lambda)$, where $Q(\Lambda)$ denotes the quotient field of $\Lambda$. Let

$$DM = \{ x \in M \mid \exists f_i \in \Lambda (i = 1, 2, ..., s \geq 2) \text{ with } (f_1, ..., f_s) = 1 \text{ and } f_i x = 0 \},$$

which is the maximal finite $\Lambda$–submodule of $M$ (cf Kawauchi [5, Section 3]), where the notation $(f_1, ..., f_s)$ denotes the greatest common divisor of the Laurent polynomials $f_1, ..., f_s$. We note that $DM$ contains all finite $\Lambda$–submodules of $M$, which is a consequence of $M$ being finitely generated over $\Lambda$. Let $TDM = TM/DM$, and $E^qM = \text{Ext}^q_\Lambda(M, \Lambda)$. The following proposition is more or less known (see J Levine [14] for $S^n$–knot modules and [5] in general):

**Proposition 2.2** We have the following properties (1)–(5) on a finitely generated $\Lambda$–module $M$.

1. $E^0M = \text{hom}_\Lambda(M, \Lambda) = \Lambda^{\beta(M)}$,
2. $E^1M = E^2M = 0$ if and only if $M$ is $\Lambda$–free,
3. there are natural $\Lambda$–exact sequences $0 \to E^1BM \to E^1M \to E^1TM \to 0$ and $0 \to BM \to E^0E^0BM \to E^2E^1BM \to 0$,
4. $E^1BM = DE^1M$,
5. $E^1TM = \text{hom}_\Lambda(TM, Q(\Lambda)/\Lambda)$ and $E^2M = E^2DM = \text{hom}_\mathbb{Z}(DM, Q/\mathbb{Z})$.

The $d$th $\Lambda$–rank of an $F^r_g$–link $L$ is the number $\beta_d(L) = \beta(H_d(\tilde{E}))$. We call the integer $\tau(L) = r - 1 - \beta_1(L)$ the torsion-corank of $L$, which is shown to be non-negative in Lemma 2.5. We use the following notion:
Definition 2.3 A finitely generated $\Lambda$–module $M$ is a cokernel-free $\Lambda$–module of corank $n$ if there is an isomorphism $M/(t-1)M \cong \mathbb{Z}^n$ as abelian groups.

The corank of a cokernel-free $\Lambda$–module $M$ is denoted by $cr(M)$. We shall show in Corollary 3.3 that a $\Lambda$–module $M$ is a cokernel-free $\Lambda$–module of corank $n$ if and only if there is an $F_g^{n+1}$–link $L$ for some $g$ such that $M(L) = M$. The following lemma implies that the cokernel-free $\Lambda$–modules appear naturally in the homology of an infinite cyclic covering:

Lemma 2.4 Let $p: \tilde{X} \to X$ be an infinite cyclic covering over a finite complex $X$. If $H_d(X)$ is free abelian, then the $\Lambda$–modules $H_d(\tilde{X})$, $TH_d(\tilde{X})$ and $T\partial H_d(\tilde{X})$ are cokernel-free $\Lambda$–modules. In particular, if $H_1(X) \cong \mathbb{Z}^r$ and $\tilde{X}$ is connected, then $H_1(\tilde{X})$ is cokernel-free of corank $r - 1$.

Proof By Wang exact sequence, the sequence

$$H_d(\tilde{X}) \xrightarrow{\partial} H_{d-1}(\tilde{X})$$

is exact, which also induces an exact sequence

$$TH_d(\tilde{X}) \xrightarrow{\partial} TH_{d-1}(\tilde{X}),$$

for $(t-1)TH_d(\tilde{X}) = TH_d(\tilde{X}) \cap (t-1)H_d(\tilde{X})$. Since $H_d(X)$ is free abelian, we have also the induced exact sequence

$$T\partial H_d(\tilde{X}) \xrightarrow{\partial} T\partial H_{d-1}(\tilde{X}),$$

obtaining the desired result of the first half. The second half follows from the calculation that

$$\text{im}[p_*: H_1(\tilde{X}) \to H_1(X)] = \ker[\partial: H_1(X) \to H_0(\tilde{X})] \cong \mathbb{Z}^{r-1}. \quad \Box$$

From Lemmas 2.1 and 2.4, we see that the $\Lambda$–modules $H_n(\tilde{E})$, $TH_n(\tilde{E})$ and $T\partial H_n(\tilde{E})$ are all cokernel-free $\Lambda$–modules for every $F_g^{r}$–link $L$. On these $\Lambda$–modules, we make the following calculations by using the dualities on the homology $H_n(\tilde{E})$ in [5]:

Lemma 2.5

(1) $\beta_1(L) = \beta_3(L) \leq r - 1$ and $\beta_2(L) = 2(g - \tau(L))$,

(2) $H_d(\tilde{E}) = 0$ for $d \neq 0, 1, 2, 3$, $H_0(\tilde{E}) \cong \Lambda/(t-1)\Lambda$ and $H_3(\tilde{E}) \cong \Lambda^{\beta_3(L)}$,

(3) $cr(M(L)) = r - 1$ and $cr(TM(L)) = cr(T\partial M(L)) = \tau(L)$.

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The following corollary follows directly from Lemma 2.5.

**Corollary 2.6** An $F^r_g$–link $L$ has $\beta_r(L) = 0$ if and only if $\beta_1(L) = 0$ and $g = r - 1$. 

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3 Characterizing the first Alexander \(\mathbb{Z}[\mathbb{Z}]\)–modules of ribbon surface-links

For a finitely generated \(\Lambda\)–module \(M\), let \(e(M)\) be the minimal number of \(\Lambda\)–generators of \(M\). The following estimate is given by Sekine [17] and Kawauchi [7] for the case \(r = 1\) where we have \(\tau(L) = 0\):

**Lemma 3.1** If \(L\) is a ribbon \(F^r_g\)–link, then we have

\[
g \geq e(E^2 M(L)) + \tau(L).
\]

**Proof** Since \(L\) is a ribbon \(F^r_g\)–link, there is a connected Seifert hypersurface \(V\) for \(L\) such that \(H_1(V)\) and \(H_1(V, \partial V)\) are torsion-free. In fact, we can take \(V\) to be a connected sum of \(r\) handlebodies and some copies, say \(n\) copies, of \(S^1 \times S^2\) (cf [12]). Then we have \(H_1(V) = \mathbb{Z}^{n+s}\) and \(H_2(V) = \mathbb{Z}^{n+r-1}\). Let \(E'\) be the compact 4–manifold obtained from \(E\) by splitting it along \(V\). Let \(\bar{V}\) and \(\bar{E}'\) be the lifts of \(V\) and \(E'\) by the infinite cyclic covering \(p: \bar{E} \to E\), respectively. By the Mayer-Vietoris exact sequence, we have the following exact sequence

\[
0 \to B \to H_1(\bar{V}) \to H_1(\bar{E}') \to H_1(\bar{E}) \to 0,
\]

where \(B\) denotes the image of the boundary operator \(\partial: H_2(\bar{E}) \to H_1(\bar{V})\). Since \(H_1(\bar{V}) \cong \mathbb{Z}^{n+s}\), we have \(H_1(\bar{V}) \cong \Lambda^{n+s}\). We note that

\[
H_1(\bar{E}') \cong H_1(S^4 - V) \cong H_2(S^4, S^4 - V) \cong H^2(V) \cong \mathbb{Z}^{n+r-1},
\]

so that \(H_1(\bar{E}') \cong \Lambda^{n+r-1}\). Using that \(\Lambda\) has the graded dimension 2, we see that \(B\) must be a free \(\Lambda\)–module whose \(\Lambda\)–rank is calculated from the exact sequence to be

\[
(n + g) - (n + r - 1 - \beta_1(L)) = g - \tau(L).
\]

Since by definition \(E^2 M(L) = E^2 H_1(\bar{E})\) is a quotient \(\Lambda\)–module of \(E^0 B \cong \Lambda^{\geq \tau(L)}\), we have \(e(E^2 M(L)) \leq g - \tau(L)\). \(\square\)

The following theorem is our first theorem, which shows that the estimate of Lemma 3.1 is best possible and generalizes [7, Theorem 1.1].

**Theorem 3.2** A finitely generated \(\Lambda\)–module \(M\) is the module \(M(L)\) of a ribbon \(F^r_g\)–link \(L\) if and only if \(M\) is a cokernel-free \(\Lambda\)–module of corank \(r - 1\) and \(g \geq e(E^2 M) + \tau(M)\). Further, if a non-negative partition \(g = g_1 + g_2 + \ldots + g_r\) is arbitrarily given, then we can take a ribbon \(F^r_g\)–link \(L\) with \(g(F_i) = g_i\) for all \(i\).
Proof The “only if” part is proved by Lemmas 2.5 and 3.1. We show the “if” part. Let $M/(t−1)M \cong \mathbb{Z}^n$. We construct a ribbon $F_{g}^{m+1}$–link $L$ with $M(L) = M$ and $g = e(E^2M) + \tau(M)$ and observe that the module $M(L)$ is independent of a choice of the partitions $g = g_1 + g_2 + ... + g_r$ in our construction. This will complete the proof, since an $F_{g}^{m+1}$–link $L'$ with $g' > g$ and $M(L') = M$ can be obtained from $L$ by taking suitable connected sums of $L$ with $g' − g$ trivial $F_1$–knots. The proof will be done by establishing the following three steps:

1. Finding a nice $\Lambda$–presentation matrix $B$ for $M$.

2. Constructing a finitely presented group $G$ and an epimorphism $\gamma: G \to \mathbb{Z}$ which induces a $\Lambda$–isomorphism $\ker \gamma / [\ker \gamma, \ker \gamma] \cong M$.

3. Applying T. Yajima’s construction to find a ribbon $F_{g}^{r}$–link $L$ with a prescribed disk–arc presentation such that $\pi_1(S^1 \setminus L) = G$.

In (2), recall that $\ker \gamma / [\ker \gamma, \ker \gamma]$ has a natural $\Lambda$–module structure with the $t$–action meant by the conjugation of any element $g \in G$ with $\gamma(g) = 1 \in \mathbb{Z}$. This $\Lambda$–module is calculable from the group presentation of $G$ by the Fox calculus (see Kawauchi [4] and H Zieschang [20]). We shall show how to construct a desired Wirtinger presented group $G$ from the $\Lambda$–presentation $B$ of $M$ by this inverse process, so that we can establish (3).

Let $m = e(E^2M)$ and $\beta = \beta(M)$. We take a $\Lambda$–exact sequence

$$0 \to \Lambda^k \to \Lambda^{m+k} \to \Lambda^m \to E^2M \to 0$$

for some $k \geq 0$, which induces a $\Lambda$–exact sequence

$$0 \to \Lambda^m \to \Lambda^{m+k} \to \Lambda^k \to E^2E^2M = DM \to 0.$$ 

On the other hand, using $D(M/DM) = 0$, we have $E^2(M/DM) = 0$ and hence we have a $\Lambda$–exact sequence

$$0 \to \Lambda^s \to \Lambda^{s+\beta} \to M/DM \to 0$$

for some $s \geq 0$. Thus, we have a $\Lambda$–exact sequence

$$0 \to \Lambda^m \to \Lambda^{m+k+s} \to \Lambda^{k+s+\beta} \to M \to 0.$$ 

Let $B = (b_{ij})$ be a $\Lambda$–matrix of size $(k + s + \beta, m + k + s)$ representing the $\Lambda$–homomorphism $\Lambda^{m+k+s} \to \Lambda^{k+s+\beta}$. Since $M/(t−1)M = \mathbb{Z}^n$, we can assume

$$B(1) = \begin{pmatrix} E^u & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

by base changes of $\Lambda^{m+k+s}$ and $\Lambda^{k+s+\beta}$, where $E^u$ is the unit matrix of size $u = k + s + \beta − n$, and $O_{12}, O_{21}, O_{22}$ are the zero matrices of sizes $(u, m − \beta + n), (n, u), (n, m − \beta + n)$.
respectively. Let $b_{ij} = -\sum_{i=1}^{k+s+\beta} b_{ij}$, and $B^+ = (b_{ij})$ ($0 \leq i \leq k + s + \beta$, $1 \leq j \leq m + k + s$) We take $c_{ij} \in \Lambda$ so that

$$b_{ij} = \begin{cases} (t - 1)c_{ij} & (j > u) \\ (t - 1)c_{ij} + \delta_{ij} & (i > 0, 1 \leq j \leq u) \\ (t - 1)c_{ij} - 1 & (i = 0, 1 \leq j \leq u) \end{cases}$$

Let $\gamma$ be the epimorphism from the free group $G_0 = \langle x_0, x_1, \ldots, x_{k+s+\beta} \rangle$ onto $\mathbb{Z}$ defined by $\gamma(x_i) = 1$, and $\gamma^+ : \mathbb{Z}[G_0] \to \mathbb{Z}[\mathbb{Z}] = \Lambda$ the group ring extension of $\gamma$ with $\gamma^+(x_i) = t$. Using that $\Sigma_{i=0}^{k+s+\beta} c_{ij} = 0$, an algorithm of A Pizer [15] enables us to find a word $w_j$ in $G_0$ such that $\gamma(w_j) = 0$ and the Fox derivative

$$\gamma^+(\partial w_j/\partial x_i) = c_{ij} (j = 1, \ldots, m + k + s)$$

for every $i$. Let

$$R_j = \begin{cases} x_jw_jx_0^{-1}w_j^{-1} & (1 \leq j \leq u) \\ x_hw_jx_h^{-1}w_j^{-1} & (u + 1 \leq j \leq m + k + s), \end{cases}$$

where we can take any $h$ for the $x_h$ in every $R_j$ with $u + 1 \leq j \leq m + k + s$. Then the finitely presented group $G = \langle x_0, x_1, \ldots, x_{k+s+\beta} \mid R_1, R_2, \ldots, R_{m+k+s} \rangle$ has the Fox derivative $\gamma^+(\partial R_j/\partial x_i) = b_{ij}$ for every $i, j$. We note that $G/[G, G] = \mathbb{Z}^{1+k+s+\beta} = \mathbb{Z}^{1+n}$. Let $\gamma_s : G \to \mathbb{Z}$ be the epimorphism induced from $\gamma$. Then $\text{Ker} \gamma_s / [\text{Ker} \gamma_s, \text{Ker} \gamma_s] \cong M$. By T Yajima’s construction in [19], there is a ribbon $F_{g+1}^m$–link $L$ with $\pi_1(S^1 \setminus L) = G$ (hence $M(L) = M$) so that, in terms of a disk–arc presentation of a ribbon surface-link, the generators $x_i$ ($i = 0, 1, \ldots, k + s + \beta$) correspond to the oriented disks $D_j$ ($i = 0, 1, \ldots, k + s + \beta$), respectively, and the relation $R_j : w_j^{-1}x_jw_j = x_0$ (or $w_j^{-1}x_hw_j = x_h$, respectively) corresponds to an oriented arc $\alpha_j$ which starts from a point of $\partial D_j$ (or $\partial D_h$, respectively), terminates at a point of $\partial D_0$ (or $\partial D_s$, respectively), and is described in the following manner: When $w_j$ is written as $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_l^{\varepsilon_l}$ ($\varepsilon_l = \pm 1$), the arc $\alpha_j$ should be described so that it first intersects the interior of the disk $D_j$ in a point with sign $\varepsilon_1$. Next, it intersects the interior of the disk $D_{j_2}$ in a point with sign $\varepsilon_2$. This process should be continued in the order of the letters $x_j$ appearing in $w_j$ until they are exhausted. Thus, the arc $\alpha_j$ is constructed. Then we have

$$g = m + k + s - u = m + (n - \beta) = e(E^2M) + \tau(M).$$

The arbitrariness of $h$ for the $x_h$ in $R_j$ with $u + 1 \leq j \leq m + k + s$ guarantees us to construct a 2–manifold $F_{g+1}^{m+1} = F_{g_1}^{m+1} \times \cdots \times F_{g_n}^{m+1}$ corresponding to any partition $g = g_1 + g_2 + \cdots + g_n + 1$.

The following corollary comes directly from Lemmas 2.4, 2.5 and Theorem 3.2.

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Corollary 3.3 A finitely generated $\Lambda$–module $M$ is a cokernel-free $\Lambda$–module of corank $n$ if and only if there is an $F_{g}^{n+1}$–link $L$ with $M(L) = M$ for some $g$.

The following corollary gives a characterization of the modules $M(L)$ of ribbon $F_{g}^{n+1}$–links $L$ with $\beta_{n}(L) = 0$.

Corollary 3.4 A cokernel-free $\Lambda$–module $M$ of corank $n$ is the module $M(L)$ of a ribbon $F_{g}^{n+1}$–link $L$ with $\beta_{n}(L) = 0$ (in this case, we have necessarily $g = n$) if and only if $\beta(M) = 0$ and $DM = 0$.

Proof For the proof of “if” part, we note that $E_{2}^{2}M = E_{2}^{2}DM = 0$ and hence $e(E_{2}^{2}M) + \tau(M) = n$. By Theorem 3.2, we have a ribbon $F_{g}^{n+1}$–link $L$ with $M(L) = M$. Since $\beta(M) = 0$, we see from Corollary 2.6 that $\beta_{n}(L) = 0$. For the proof of “only if” part, we note $g = n$ by Corollary 2.6. Hence by Lemma 3.1, $n \geq e(E_{2}^{2}M) + \tau(M)$. Since $\beta(M) = 0$ means $\tau(M) = n$, we have $e(E_{2}^{2}M) = 0$, so that $E_{2}^{2}M = 0$ which is equivalent to $DM = 0$.

Here are two examples which are not covered by Corollary 3.4.

Example 3.5 For a cokernel-free $\Lambda$–module $M$ of corank $n$ with $\beta(M) = 0$ (so that $\tau(M) = n$) and $DM = 0$, we have the following examples (1) and (2).

1. Let $M' = M \oplus \Lambda/(t+1,a)$ for an odd $a \geq 3$. Since $E_{2}^{2}M' \cong \Lambda/(t+1,a) \neq 0$, the $\Lambda$–module $M'$ is not the module $M(L)$ of a ribbon $F_{g}^{n+1}$–link $L$ with $\beta_{n}(L) = 0$. On the other hand, $\Lambda/(t+1,a)$ is well-known to be the module of a non-ribbon $F_{1}^{3}$–knot $K$ (for example, the 2–twist-spun knot of the 2–bridge knot of type $(a,1)$) and $M$ is the module $M(L)$ of a ribbon $F_{g}^{n+1}$–link $L$ with $\beta_{n}(L) = 0$ by Corollary 3.4. Hence $M'$ is the module $M(L')$ of a non-ribbon $F_{g}^{n+1}$–link $L'$ (taking a connected sum $L#K$) with $\beta_{n}(L') = 0$.

2. Let $M'' = M \oplus \Lambda/(2t-1,a)$ for an odd $a \geq 5$. Although $M''$ is cokernel-free of corank $n$ and $\beta(M'') = 0$, we can show that $M''$ is not the module $M(L)$ of any $F_{g}^{n+1}$–link $L$ with $\beta_{n}(L) = 0$. To see this, suppose $M'' = M(L)$ for an $F_{g}^{n+1}$–link $L$. Since $\Lambda/(2t-1,a)$ is not $\Lambda$–isomorphic to $\Lambda/(2t-1-1,a) = \Lambda/(t-2,a)$, the $\Lambda$–module $DM'' = \Lambda/(2t-1,a)$ is not $t$–anti isomorphic to the $\Lambda$–module $E_{2}^{2}DM'' = \text{hom}_{\mathbb{Z}}(DM'', \mathbb{Q}/\mathbb{Z}) \cong \Lambda/(2t-1,a)$ and hence by the second duality of [5] there is a $t$–anti isomorphism

$$\theta : DM'' \rightarrow E_{1}BH_{2}(\tilde{E}, \partial \tilde{E}).$$

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This implies that $\beta_2(L) = \beta(H_2(\tilde{E}, \partial\tilde{E})) \neq 0$. Thus, $M''$ is not the module $M(L)$ of any $F_{n+1}^r$–link $L$ with $\beta_1(L) = 0$. On the other hand, there is a ribbon $F_{n+1}^r$–link $L''$ with $M(L'') = M''$ by Theorem 3.2, because $e(E^2M'') = e(\Lambda/(2t - 1, a)) = 1$ and hence $e(E^2M'') + \tau(M'') = 1 + n$. In this case, we have $\beta_2(L'') = 2$ by Lemma 2.5.

4 A characterization of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links

Figure 2: A real or virtual crossing point

The notion of virtual links was introduced by L H Kauffman [3]. A virtual $r$–link diagram is a diagram $D$ of immersed oriented $r$ loops in $S^2$ with two kinds of crossing points given in Figure 2, where the left or right crossing point is called a real or virtual crossing point, respectively. A virtual $r$–link $\ell$ is the equivalence class of virtual $r$–link diagrams $D$ under the local moves given in Figure 3 which are called R-moves for the first three local moves and virtual R-moves for the other local moves. A virtual $r$–link is called a classical $r$–link if it is represented by a virtual link diagram without virtual crossing points. The group $\pi(\ell)$ of a virtual $r$–link $\ell$ is the group with Wirtinger presentation whose generators consist of the edges of a virtual link diagram $D$ of $\ell$. 

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and whose relations are obtained from \( D \) as they are indicated in Figure 4. It is easily checked that the Wirtinger group \( \pi(\ell) \) up to Tietze equivalences is unchanged under the R-moves and virtual R-moves. Figure 5 defines a map \( \sigma' \) from a virtual \( r \)–link diagram to a disk–arc presentation of a ribbon \( F_{1,1,\ldots,1} \)–link. S Satoh proved in \([16]\) that this

\[
\begin{align*}
  a &= d, b = a^{-1}ca & a &= d, b = c \\
  b & \quad c & b & \quad c
\end{align*}
\]

Figure 4: Relations

map \( \sigma' \) induces a (non-injective) surjective map \( \sigma \) from the set of virtual \( r \)–links onto the set of ribbon \( F_{1,1,\ldots,1} \)–links. For example, the map \( \sigma \) sends a nontrivial virtual knot into a trivial \( F_{1} \)–knot in Figure 6, where non-triviality of the virtual knot is shown by the Jones polynomial (see \([3]\)) and triviality of the \( F_{1} \)–knot is shown by an argument of \([2]\) on deforming a 1–handle. It would be an important problem to find a finite

Figure 6: A non-trivial virtual knot sent to the trivial \( F_{1} \)–knot
number of local moves generating the preimage of $\sigma$ (see [16]). Yajima in [19] gives
a Wirtinger presentation of the group $\pi_1(S^4 \setminus L)$ of a ribbon $F^r_g$–link $L$. From an analogy
of the constructions, we see that the map $\sigma$ induces the same Wirtinger presentation
of a virtual $r$–link diagram $D$ and the disk–arc presentation $\sigma'(D)$. Thus, we have
the following proposition which has been independently observed by S G Kim [13], S
Satoh [16], and D Silver and S Williams [18] in the case of virtual knots:

**Proposition 4.1** The set of the groups of virtual $r$–links is the same as the set of the
groups of ribbon $F^r_{1,1,\ldots,1}$–links.

For a virtual $r$–link $\ell$, let $\gamma : \pi(\ell) \to \mathbb{Z}$ be an epimorphism sending every generator of a
Wirtinger presentation to 1, which is independent of a choice of Wirtinger presentations.
The *first Alexander* $\mathbb{Z}[\mathbb{Z}]$–module, or simply the module of a virtual $r$–link $\ell$ is the
$\Lambda$–module $M(\ell) = \text{Ker}\gamma/\text{[Ker}\gamma, \text{Ker}\gamma]$. The following corollary comes directly from
4.1.

**Corollary 4.2** The set of the modules of virtual $r$–links is the same as the set of the
modules of ribbon $F^r_{1,1,\ldots,1}$–links.

The following theorem giving a characterization of the modules of virtual $r$–links comes
directly from *Theorem 3.2* and *Corollary 4.2*.

**Theorem 4.3** A finitely generated $\Lambda$–module $M$ is the module $M(\ell)$ of a virtual
$r$–link $\ell$ if and only if $M$ is a cokernel-free $\Lambda$–module of corank $r - 1$ and has
$e(E^2M) \leq 1 + \beta(M)$.

![Figure 7: A virtual 2–link sent to the ribbon $F^3_{1,1}$–link in Figure 1](image)
Example 4.4  The ribbon $F^2_{1,1}$–link in Figure 1 is the $\sigma$–image of a virtual 2–link $\ell$ illustrated in Figure 7 with group $\pi(\ell) = \langle x, y \mid x = (yx^{-1}y^{-1})x(yx^{-1}y^{-1})^{-1}, y = (x^{-1}yx^{-1})(x^{-1}yx^{-1})^{-1} \rangle$ and module $M(\ell) = \Lambda/((t - 1)^2, 2(t - 1))$. Since $DM(\ell) = \Lambda/((t - 1), 2) \neq 0$, the virtual 2–link $\ell$ is not any classical 2–link. In fact, if $\ell$ is a classical link with $M(\ell)$ a torsion $\Lambda$–module, then we must have $DM(\ell) = 0$ by the second duality of [5] (cf [6]). It is unknown whether there is a classical link $\ell$ such that $t - 1 : DM(\ell) \to DM(\ell)$ is not injective (cf [6]), but this example means that such a virtual link exists.

We see from Theorem 4.3 that $M$ is the module of a virtual knot (ie, a virtual 1–link) if and only if $M$ is a cokernel-free $\Lambda$–module of corank 0 and has $e(E^2M) \leq 1$, for we have $\beta(M) = 0$ for every cokernel-free $\Lambda$–module of corank 0. For a direct sum on the modules of virtual knots, we obtain the following observations.

Corollary 4.5

1. For the module $M$ of every virtual knot with $e(E^2M) = 1$, the $n(> 1)$–fold direct sum $M^n$ of $M$ is a cokernel-free $\Lambda$–module of corank 0, but not the module of any virtual knot.

2. For the module $M$ of every virtual knot and the module $M'$ of a virtual knot with $e(E^2M') = 0$, the direct sum $M \oplus M'$ is the module of a virtual knot.

Proof  The module $M^n$ is obviously cokernel-free of corank 0. Using that $E^2M^n = (E^2M)^n$, we see that $e(E^2M^n) \leq n$. If $E^2M$ has an element of a prime order $p$, then we consider the non-trivial $\Lambda_p$–module $(E^2M)_p = E^2M/pE^2M$, where $\Lambda_p = \mathbb{Z}_p[\mathbb{Z}] = \mathbb{Z}_p[t, t^{-1}]$ which is a principal ideal domain. Using $e((E^2M)_p) = 1$, we have

$$e(E^2M^n) = e((E^2M)^n) \geq e(((E^2M)_p)^n) = n$$

and hence $e(E^2M^n) = n > 1$. By Theorem 4.3, $M^n$ is not the module of any virtual knot, proving (1). For (2), the module $M \oplus M'$ is also cokernel-free of corank 0. Since $E^2M' = 0$, we have $E^2(M \oplus M') = E^2M$ and by Theorem 4.3 $M \oplus M'$ is the module of a virtual knot, proving (2).
5  A graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links

Let $\mathcal{A}_g^r$ be the set of the modules $M(L)$ of all $F^r_g$–links $L$, and $\mathcal{A}[2] = \bigcup_{r=0}^{+\infty} \mathcal{A}_g^r$. In this section, we show the properness of the inclusions

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{A}[2].$$

To see this, we establish an estimate of the total genus $g$ by the module of a general $F^r_g$–link. To state this estimate, we need some notions on a finite $t$–module. A finite $\Lambda$–module $D$ is symmetric if there is a $t$–anti isomorphism $D \cong E^2 D = \text{hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$, and nearly symmetric if there is an $\Lambda$–exact sequence

$$0 \to D_i \to D \to D^* \to D_0 \to 0$$

such that $D_i(i = 0, 1)$ are finite $\Lambda$–modules with $(t - 1)D_i = 0$ and $D^*$ is a finite symmetric $\Lambda$–module. For a general $F^r_g$–link $L$, we shall show the following theorem:

**Theorem 5.1** If $M$ is the module $M(L)$ of an $F^r_g$–link $L$, then we have a nearly symmetric finite $\Lambda$–submodule $D \subset DM$ such that $g \geq e(E^2(M/D))/2 + \tau(M)$.

**Proof** Let $F^r_g = F^r_{g_1, g_2, \ldots, g_r}$. Let $L_i$ be the $F^1_{g_i}$–component of $L$, and $\partial_i E$ the component of the boundary $\partial E$ corresponding to $L_i$. We parametrize $\partial_i E$ as $L_i \times S^1$ so that the natural composite

$$H_1(L_i \times 1) \to H_1(\partial_i E) \to H_1(E) \xrightarrow{\gamma} \mathbb{Z}$$

is trivial. Let $V_i$ be the handlebody of genus $g_i$. We construct a closed connected oriented 4–manifold $X = E \cup (\bigcup_{i=1}^{r} V_i \times S^1)$ obtained by pasting $\partial_i E$ to $L_i \times S^1 = (\partial V_i) \times S^1$. Then the infinite cyclic covering $p: \bar{E} \to E$ associated with $\gamma$ extends to an infinite cyclic covering $p_X: \bar{X} \to X$, so that $(p_X)^{-1}(V_i \times S^1) = V_i \times R^1$. Since $H_*(\bar{X}, \bar{E}) \cong \oplus_{i=1}^{r} H_*(\{V_i, \partial V_i\} \times R^1)$, the exact sequence of the pair $(\bar{X}, \bar{E})$ induces a $\Lambda$–exact sequence

$$0 \to T_1 \to H_1(\bar{E}) \xrightarrow{i} H_1(\bar{X}) \to 0$$

where $(t - 1)T_1 = 0$. This exact sequence induces a $\Lambda$–exact sequence

(5.1.1) $$0 \to D_1 \to DH_1(\bar{E}) \xrightarrow{i} DH_1(\bar{X}) \to D_0 \to 0$$

for some finite $\Lambda$–modules $D_i(i = 0, 1)$ with $(t - 1)D_i = 0$.

To see (5.1.1), it suffices to prove that the cokernel $D_0$ of the natural homomorphism $i^*_{\Lambda}: DH_1(\bar{E}) \to DH_1(\bar{X})$ has $(t - 1)D_0 = 0$. For an element $x \in DH_1(\bar{X})$, we take...
an element \( x' \in H_1(\tilde{E}) \) with \( i_*(x') = x \). Since there is a positive integer \( n \) such that \((t^n - 1)x = 0\), the element \((t^n - 1)x' \in H_1(\tilde{E})\) is the image of an element in \( T_1 \). Hence \((t^n - 1)(t - 1)x' = 0\). Also, since there is a positive integer \( m \) such that \( mx = 0\), we also see that \( m(t - 1)x' = 0\), so that \( (t - 1)x' \) is in \( DH_1(\tilde{E}) \) and \( D_0((t - 1)x') = (t - 1)x \). This means \((t - 1)D_0 = 0\), showing (5.1.1).

By the second duality in [5], there is a natural \( t \)-anti epimorphism \( \theta : DH_1(\tilde{X}) \to E^1BH_2(\tilde{X}) \) whose kernel \( D^* = DH_1(\tilde{X})^\theta \) is symmetric. Then

\[
e(E^2(DH_1(\tilde{X})/D^*)) = e(E^2E^1BH_2(\tilde{X})) \leq \beta BH_2(\tilde{X}),
\]

where the later inequality is obtained by using 2.2. Since \( H_*(\tilde{X}, \tilde{E}) \) is \( \Lambda \)-torsion, we see from Lemma 2.5 that

\[
\beta BH_2(\tilde{X}) = \beta_2(L) = 2(g - \tau(L)).
\]

In (5.1.1), the \( \Lambda \)-submodule \( D = (D^0)^{-1}(D^*) \subset DH_1(\tilde{E}) = DM(L) \) induces a \( \Lambda \)-exact sequence \( 0 \to D_1 \to D \to D^* \to D_0' \to 0 \) for a finite \( \Lambda \)-module \( D_0' \) with \((t - 1)D_0' = 0\), so that \( D \) is nearly symmetric. Using that \( i_*^D \) induces a \( \Lambda \)-monomorphism \( DM(L)/D \to DH_1(\tilde{X})/D^* \), we see that there is a \( \Lambda \)-epimorphism \( E^2(DH_1(\tilde{X})/D^*) \to E^2(DM(L)/D) \), so that

\[
e(E^2(DM(L)/D)) \leq e(E^2(DH_1(\tilde{X})/D^*)) \leq 2(g - \tau(L)).
\]

Thus, we have \( g \geq e(E^2(DM(L)/D))/2 + \tau(L) \). \( \square \)

For an application of this theorem, it is useful to note that every finite \( \Lambda \)-module \( D \) has a unique splitting \( D_{r-1} \oplus D_c \) (see [9, Lemma 2.7]), where \( D_{r-1} \) is the \( \Lambda \)-submodule consisting of an element annihilated by the multiplication of some power of \( t - 1 \) and \( D_c \) is a cokernel-free \( \Lambda \)-submodule of corank \( 0 \). As a direct consequence of this property, we see that if \( D \) is nearly symmetric, then \( D_c \) is symmetric. Then we can obtain the following result from Theorem 5.1.

**Corollary 5.2** For every \( r \geq 1 \), we have

\[
\mathcal{A}_0' \subseteq \mathcal{A}_1' \subseteq \mathcal{A}_2' \subseteq \cdots \subseteq \mathcal{A}_n' \subseteq \cdots \subseteq \mathcal{A}'[2]
\]

and the set \( \mathcal{A}'[2] \) is equal to the set of finitely generated cokernel-free \( \Lambda \)-modules of corank \( r - 1 \), so that \( \mathcal{A}'[2] \cap \mathcal{A}'[2] = \emptyset \) if \( r \neq r' \).

**Proof** We have \( \mathcal{A}_g' \subseteq \mathcal{A}_{g+1}' \) for every \( g \) by a connected sum of a trivial \( F^1_g \)-knot. Let \( L_0 \) be a trivial \( F^0_0 \)-link whose module \( M(L_0) = \Lambda^{-1} \). Let \( K \) be a ribbon \( F^1_1 \)-knot with

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We have a classical theorem. This means that among the modules \( M \) and higher-dimensional manifold-links with \( r \) knots with non-trivial Alexander polynomials, we see that the set \( A \) of the modules of virtual manifold-links. By Theorem 3.2 and Corollary 3.3, we have
\[ e(T^2 M_n) = n. \]
Hence \( g \geq n/2 \) by Theorem 5.1. This means that among the modules \( M_n(2g + 1 \leq n \leq 2g + 1) \) there is a member \( M_n \) in \( A_{g+1}' \) but not in \( A_g' \). In fact, if \( M_{g+1} \not\in A_g' \), then \( M_{g+1} \) is a desired member. If \( M_{g+1} \in A_g' \), then we take the largest \( n \geq g + 1 \) such that \( M_n \in A_g' \). Since \( M_{2g+1} \not\in A_g' \), we have \( n < 2g + 1 \). Let \( L' \) be an \( F_g' \)-link with \( M(L') = M_n \), and \( L'' \) an \( F_g' \)-knot which is a connected sum of \( L' \) and \( K \). Then \( M_{g+1} = M(L'') \) is in \( A_{g+1}' \), but not in \( A_g' \). The characterization of \( A'[2] \) follows directly from Corollary 3.3, so that if \( r \neq r' \), then \( A'[2] \cap A'[2] = \emptyset \).

\[ \square \]

6 A graded structure on the first Alexander \( \mathbb{Z}[\mathbb{Z}] \)-modules of classical links, surface-links and higher-dimensional manifold-links

An \( n \)-dimensional manifold-link with \( r \) components is the ambient isotopy class of a closed oriented \( n \)-manifold with \( r \) components embedded in the \((n + 2)\)-sphere \( S^{n+2} \) by a locally-flat embedding. A \( 1 \)-dimensional manifold-link with \( r \) components coincides with a classical \( r \)-link even when we regard it as a virtual link by a result of M Goussarov, M Polyak and O Viro [1]. Let \( E_Y = S^{n+2} \setminus \Int(N(Y)) \) for a tubular neighborhood \( N(Y) \) of \( Y \) in \( S^{n+2} \). Since \( H_1(E_Y) \cong \mathbb{Z}^r \) has a unique oriented meridian basis, we have a unique infinite cyclic covering \( p: \tilde{E}_Y \to E_Y \) associated with the epimorphism \( \gamma: H_1(E_Y) \to \mathbb{Z} \) sending every oriented meridian to \( 1 \). The first Alexander \( \mathbb{Z}[\mathbb{Z}] \)-module, or simply the module of the manifold-link \( Y \) is the \( \Lambda \)-module \( M(Y) = H_1(\tilde{E}_Y) \). Let \( A'[n] \) denote the set of the modules of \( n \)-dimensional manifold-links with \( r \) components by generalizing the case \( n = 2 \). Let \( R_{g}' \) be the set of the modules of ribbon \( F_g' \)-links. By Theorem 3.2 and Corollary 3.3, we have \( A'[2] = \bigcup_{g=0}^{\infty} R_{g}' \). Let \( V A'[1] \) denote the set of the modules of virtual \( r \)-links. By Theorem 3.2 and Corollary 4.2, we have \( V A'[1] = R_{A_g'} \). For the set \( A'[1] \), we further consider the subset \( A_{g+1}'[1] = A'[1] \cap A_{g}' \). We have \( A_{g}'[1] \subset A_{g+1}'[1] \subset A'[1] \) for every \( g \geq 0 \). Taking a split union of classical knots with non-trivial Alexander polynomials, we see that the set \( A_0[1] \) is infinite. We have the following comparison theorem on the modules of classical \( r \)-links, \( F_g' \)-links and higher-dimensional manifold-links with \( r \) components, which explains why we
consider the strictly nested class of classical and surface-links for the classification problem of the Alexander modules of general manifold-links.

**Theorem 6.1**

\[ A'_g[1] \subsetneq A'_g[2] \subsetneq \cdots \subsetneq A'_g[n] = A'_g[1] \subsetneq RA'_g[2] \subsetneq RA'_g = V A'_g[1] \]

\[ \subsetneq A'_1 \subsetneq \cdots \subsetneq A'_2 \subsetneq \cdots \subsetneq A'[2] = A'[3] = A'[4] = \cdots. \]

**Proof** By Lemma 2.4 and Corollary 3.3, we have \( A'[2] \supset A'[n] \) for every \( n \geq 1 \). To see that \( A'[n] \subset A'[n+1] \), we use a spinning construction. To explain it, let \( M(Y) \in A'[n] \) for a manifold-link \( Y \). We choose an \((n+2)\)-ball \( B^{n+2}_0 \subset S^{n+2} \) such that the pair \((B^{n+2}_0, Y_0)\) \((Y_0 = Y \cap B^{n+2}_0)\) is homeomorphic to the standard disk pair \((D^2 \times D^n, 0 \times D^n)\), where \( D^n \) denotes the \( n \)-disk and \( o \) denotes the origin of the \( 2 \)-disk \( D^2 \). Let \( B^{n+2} = \text{cl}(S^{n+2} \setminus B^{n+2}_0) \) and \( Y' = \text{cl}(Y \setminus Y_0) \). We construct an \((n+1)\)-dimensional manifold link \( Y^+ \subset S^{n+3} \) by

\[ Y^+ = Y' \times S^1 \cup (\partial Y') \times D^2 \subset B^{n+2} \times S^1 \cup (\partial B^{n+2}) \times D^2 = S^{n+3}. \]

Then the fundamental groups \( \pi_1(E_Y) \) and \( \pi_1(E_{Y^+}) \) are meridion-preservingly isomorphic by van Kampen theorem and hence \( M(Y) = M(Y^+) \). This implies that \( A'[1] \subset RA'[2] \) and \( A'[2] = A'[3] = A'[4] = \cdots \). Let \( g \) be an integer with \( 0 < g \leq r - 1 \). Let \( \ell \) be a classical \((g + 1)\)-link with \( M(\ell) \) a torsion \( \Lambda \)-module. Then \( M(\ell) = M(\ell) \) for a ribbon \( F^r_{g-1} \)–link \( L \) by the spinning construction. The \( \Lambda \)-module \( M' = M(\ell) \oplus \Lambda^{r-1-g} \) is in \( A'[1] \) as the module of a split union \( \ell^r_{g} \) of \( \ell \) and a trivial \((r-1-g)\)-link and in \( RA'_{g} \subset A'_{g} \) as the module of a split union \( L^r_{g} \) of \( L \) and a trivial \( F^r_{g-1-g} \)–link. Hence \( M' \) is in \( A'_g[1] \). If \( M' = M(L') \) for an \( F^r_{g-1} \)–link \( L' \), then we have \( \tau(L') = (r-1) - (r-1-g) = g \) and by Lemma 2.5 \( \beta_2(L') = 2(s - \tau(L')) = 2(s - g) \geq 0 \). Hence \( s \geq g \). Thus, \( M' \) is not in \( A'_g[1] \). This shows that \( A'_g[1] \subsetneq A'_g[2] \subsetneq RA'_g \subsetneq RA'_g \). This last proper inclusion also holds for every \( g \geq r \). In fact, by taking \( M = (\Lambda/(t-1))^r_{g-1} \oplus (\Lambda/(t+1,a))^g_{r-1} \) for an odd \( a \geq 3 \), we have \( (E^2_M) + \tau(M) = (g-r+1) + (r-1-g) = g \). Since \( M \) is cokernel-free and \( \text{cr}(M) = r-1 \), we have \( M \in RA'_g \) by Theorem 3.2. Next, let \( M = M(L) \in RA'_g \) have \( (E^2_M) + \tau(M) = g \) and \( pDM = 0 \) for an odd prime \( p \). Let \( K \) be an \( S^2 \)–knot with \( M(K) = \Lambda/(t+1,p) \) (see Example 3.5 (1)). Then we have \( M' = M \oplus \Lambda/(t+1,p) = M(LK) \in A'_g \) for a connected sum \( L#K \) of \( L \) and \( K \). Then we have \( E^2_M' + \tau(M') = g + 1 \) and \( M' \notin RA'_g \) by Theorem 3.2. Thus, \( RA'_g \subsetneq A'_g \) for every \( g \). The properness of \( A'[1] \subsetneq RA'_g \) follows by a reason that the torsion Alexander polynomial of every classical \( r \)–link in [8] is symmetric, but there is a ribbon \( S^2 \)–knot with non-symmetric Alexander polynomial (see [10] for the detail). □
On the inclusion $\mathcal{A}'[1] \subset \mathcal{A}'[2]$, we note that the invariant $\kappa_1(\ell)$ in [8] is equal to the torsion-corank $\tau(L)$ for every classical $r$–link $\ell$ and every $F_r^\ell$–link $L$ with $M(\ell) = M(L)$.

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Received: 6 October 2005 Revised: 9 March 2007