COSMOLOGIES FROM NONLINEAR MULTIDIMENSIONAL GRAVITY WITH ACCELERATION AND SLOWLY VARYING $G$

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We consider multidimensional gravity with a Lagrangian containing the Ricci tensor squared and the Kretschmann invariant. In a Kaluza-Klein approach with a single compact extra space of arbitrary dimension, with the aid of a slow-change approximation (as compared with the Planck scale), we build a class of spatially flat cosmological models in which both the observed scale factor $a(\tau)$ and the extra-dimensional one, $b(\tau)$, grow exponentially at large times, but $b(\tau)$ grows slowly enough to admit variations of the effective gravitational constant $G$ within observational limits. Such models predict a drastic change in the physical laws of our Universe in the remote future due to further growth of the extra dimensions.

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1 Introduction

Multidimensional models of gravity and cosmology as low-energy limits of unified theories of physical interactions [1] are a powerful tool for studies of the present challenges to modern physics [2–4], such as the dark energy and dark matter problems, possible variations of fundamental constants [5–7], the role of strong field objects (black holes, wormholes) etc.

It has been recently argued [8–10] that multidimensional gravity with curvature-nonlinear terms in the action can be a source of a great diversity of effective theories able to address a number of important problems of modern astrophysics and cosmology using a minimal set of postulates. Among such problems one can mention the essence of dark energy, early formation of supermassive black holes (which is a necessary stage in some scenarios of cosmic structure formation), and sufficient particle production at the end of inflation. In this approach, the particular value of the total space-time dimension $D > 4$ and the topological properties of space-time are supposed to be determined by quantum fluctuations and may vary from one space-time region to another, leading to drastically different universes. Different effective theories can take place even with fixed parameters of the original Lagrangian. It can be shown that this approach, without need for fields other than gravity, is able to produce such different structures as inflationary (or simply accelerating) universes, brane worlds [10], black holes etc. The role of scalar fields is played by the metric components of extra dimensions.

One of the challenging problems of modern physics and cosmology is that of possible time-, location-, and scale-dependent variations of the fundamental physical constants, in particular, of Newton’s gravitational constant $G$. Variable effective constants are a common feature of multidimensional cosmologies, where these constants depend on the properties of extra dimensions which can vary from one space-time point to another. This problem, among others, was addressed in [8]. A number of examples of cosmological models were built, where the size of the extra dimensions was stabilized at a minimum of the effective potential. In such models, the constants could vary at earlier stages while the effective scalar field only approached this minimum, but take on stable values together with the scalar when this minimum is achieved.

In the present paper, we discuss another type of multidimensional cosmologies (their possibility was also mentioned in [8]) in which the extra-dimensional scale factor can grow indefinitely at large times but is yet small enough at present. Such models are of interest since they predict rather an exotic, though sad, fate of the Universe we live in: sooner or later, its physical laws must drastically
change due to the growth of the internal space. We will show that such models can in principle be viable since they simultaneously predict an accelerated (de Sitter-like) expansion and, under certain conditions on the input parameters of the theory, a small enough variation of the gravitational constant in agreement with observations.

2 The multidimensional theory and its reduction

We consider a \((D = 4 + d_1)\)-dimensional manifold with the metric
\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{2\beta(x)}b_{ab}dx^ax^bd
\]
(1)
where the extra-dimensional metric components \(b_{ab}\) are independent of \(x^\mu\), the observable four space-time coordinates.

The \(D\)-dimensional Riemann tensor has the nonzero components
\[
R^{\mu\nu}_{\rho\sigma} = \bar{R}^{\mu\nu}_{\rho\sigma},
\]
\[
R^{\mu\nu}_{\mu\nu} = \delta^\rho_b B^\rho, \quad B^\rho := e^{-\beta} \nabla^\rho(e^\beta \partial^\mu),
\]
\[
R^{ab}_{cd} = e^{-2\beta} \bar{R}^{ab}_{cd} + \delta^{ab}_{cd} \delta^\rho_\mu \partial^\rho \partial^\mu,
\]
(2)
where capital Latin indices cover all \(D\) coordinates, the bar marks quantities obtained from \(g_{\mu\nu}\) and \(b_{ab}\) taken separately, \(\beta^\mu = \partial^\mu \beta\) and \(\delta^{ab}_{cd} = \delta^{a_1b_1}_{c_1d_1} \delta^{a_2b_2}_{c_2d_2}\).

The nonzero components of the Ricci tensor and the scalar curvature are
\[
R^\mu_\nu = \bar{R}^\mu_\nu + d_1 B^\mu_\nu,
\]
\[
R^b_\mu = e^{-2\beta} \bar{R}^b_\mu + \delta^b_\mu [\Box + d_1(\partial^2 \beta)],
\]
\[
R = \bar{R} + e^{-2\beta} \bar{R}[b] + 2d_1 \Box \beta + d_1(1 + d_1)(\partial^2 \beta),
\]
(3)
where \((\partial^2 \beta)^2 = \partial^\mu \partial^\mu \beta, \Box = \nabla^\mu \nabla_\mu\) is the d’Alembert operator while \(\bar{R}[g]\) and \(\bar{R}[b]\) are the Ricci scalars corresponding to \(g_{\mu\nu}\) and \(b_{ab}\), respectively. Let us also present, using similar notations, the expressions for two more curvature invariants, the Ricci tensor squared and the Kretschmann scalar \(\mathcal{K} = R^{ABC\mu}R_{ABC\nu}\):
\[
R_{AB}R^{AB} = \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + 2d_1 \bar{R}_{\mu\nu} B^{\mu\nu} + e^{-4\beta} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + 2e^{-2\beta} \bar{R}[\mu][\Box + d_1(\partial^2 \beta)] + d_1(1 + d_1)(\partial^2 \beta)^2
\]
(4)
\[
\mathcal{K} = \bar{R} + 4d_1 B_{\mu\nu} B^{\mu\nu} + e^{-4\beta} \bar{R}[h] + 4e^{-2\beta} \bar{R}[h](\partial^2 \beta)^2 + 2d_1(1 - d_1)(\partial^2 \beta)^2
\]
(5)
Suppose now that \(b_{ab}\) describes a compact \(d_1\)-dimensional space of nonzero constant curvature, i.e., a sphere \((k = 1)\) or a compact \(d_1\)-dimensional hyperbolic space \([11]\) \((k = -1)\) with a fixed curvature radius \(r_0\) normalized to the \(D\)-dimensional analogue \(m_D\) of the Planck mass, i.e., \(r_0 = 1/m_D\) (we use the natural units, with the speed of light \(c\) and Planck’s constant \(h\) equal to unity). We have
\[
\bar{R}_{\mu\nu} = k m_D^2 \delta_{\mu\nu}, \quad \bar{R}^{\mu\nu} = k m_D^2 (d_1 - 1) \delta^{\mu\nu}, \quad \bar{R}[b] = k m_D^2 d_1 (d_1 - 1) = R_b.
\]
(6)
The scale factor \(b(x) = e^{\beta}\) in (1) is thus kept dimensionless; \(R_b\) has the meaning of a characteristic curvature scale of the extra dimensions.

Consider, in the above geometry, a sufficiently general curvature-nonlinear theory of gravity with the action
\[
S = \frac{1}{2} m_D^{D-2} \int \sqrt{-g} d^Dx \ (L_g + L_m),
\]
\[
L_g = F(R) + c_1 R^{AB}R_{AB} + c_2 \mathcal{K},
\]
(7)
where \(F(R)\) is an arbitrary smooth function, \(c_1\) and \(c_2\) are constants, \(L_m\) is a matter Lagrangian and \(Dg = |\det(g_{MN})|\).

The extra coordinates are easily integrated out, reducing the action to four dimensions:
\[
S = \frac{1}{2} V[d_1] m_D^2 \int \sqrt{-g} d^4x \ e^{d_1 \beta} [L_g + L_m], \quad (8)
\]
where \(4g = |\det(g_{\mu\nu})|\) and \(V[d_1]\) is the volume of a compact \(d_1\)-dimensional space of unit curvature.

Eq. (8) describes a curvature-nonlinear theory with non-minimal coupling between the effective scalar field \(\beta\) and the curvature. Let us simplify it in the following way (putting, for convenience, \(m_D = 1\)):

(a) Express everything in terms of 4D variables and \(\beta(x)\); we have, in particular,
\[
R = R_4 + \phi + f_1,
\]
\[
R_4 = \bar{R}[g], \quad f_1 = 2d_1 \Box \beta + d_1(1 + d_1)(\partial^2 \beta)^2, \quad (9)
\]
where we have introduced the effective scalar field
\[
\phi(x) = R_b e^{-2\beta(x)} = kd_1 (d_1 - 1) e^{-2\beta(x)}.
\]
(10)
The sign of \(\phi\) coincides with \(k = \pm 1\), the sign of curvature in the \(d_1\) extra dimensions.

(b) Suppose that all quantities are slowly varying, i.e., consider each derivative \(\partial_\mu\) (including those in
the definition of $R$) as an expression containing a small parameter $\varepsilon$; neglect all quantities of orders higher than $O(\varepsilon^2)$ (see [8,12]).

(c) Perform a conformal mapping leading to the Einstein conformal frame, where the 4-curvature appears to be minimally coupled to the scalar $\phi$.

In the decomposition (9), both terms $f_1$ and $R_4$ are regarded small in our approach, which actually means that all quantities, including the 4D curvature, are small as compared with the D-dimensional Planck scale. The only term which is not small is $\phi$, and we can use a Taylor decomposition of the function $F(R) = F(\phi + R_4 + f_1)$:

$$F(R) = F(\phi + R_4 + f_1) \approx F(\phi) + F'(\phi) \cdot (R_4 + f_1) + \ldots,$$

with $F'(\phi) \equiv dF/d\phi$. Substituting this, and the corresponding decompositions of the expressions (4) and (5), into Eq. (8), we obtain, up to $O(\varepsilon^2)$, the following effective gravitational Lagrangian $L_g$ in Eq. (8):

$$L_g = F'\phi R_4 + F(\phi) + F'(\phi) f_1 + c_\phi \phi^2 + 2c_1 \phi \phi \beta + 2(c_1 d_1 + 2c_2) (\partial \phi)^2$$

(12)

with $c_\phi = c_1 d_1 + 2c_2 /[d_1 (d_1 - 1)].$

The action (8) with (12) is typical of a scalar-tensor theory (STT) of gravity in a Jordan frame. To study the dynamics of the system, it is helpful to pass on to the Einstein frame. Applying the conformal mapping

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = |f(\phi)|g_{\mu\nu}, \quad f(\phi) = e^{d_1 \beta} F'(\phi),$$

(13)

after a lengthy calculation, we obtain the action in the Einstein frame as

$$S = \frac{1}{2} \sqrt{g} \left[ \frac{1}{2} V[d_1] \int \sqrt{g} \left( \text{sign} F' \right) L, \right.$$

$$L = \tilde{R}_4 + \frac{1}{2} K_E(\phi)(\partial \phi)^2 - V_E(\phi) + \tilde{L}_m,$$

(14)

$$\tilde{L}_m = \left( \text{sign} F' \right) \frac{e^{-d_1 \beta}}{F'(\phi)^2} L_m;$$

(15)

$$K_E(\phi) = \frac{1}{2\phi^2} \left[ 6 \phi^2 \left( \frac{F''}{F} \right)^2 - 2d_1 \phi \frac{F''}{F'} \right.$$

$$+ \frac{1}{2}(c_1 + c_2) \phi^2 \right];$$

(16)

$$V_E(\phi) = \left( \text{sign} F' \right) \frac{e^{-d_1 \beta}}{F'(\phi)^2} \left[ F(\phi) + (\ partial \phi)^2 \right],$$

(17)

where the tilde marks quantities obtained from or with $\tilde{g}_{\mu\nu}$; everywhere $F = F(\phi)$ and $F' = dF/d\phi$; $e^\beta$ is expressed in terms of $\phi$ using (10).

3 The cosmological model

Depending on the choice of $F(R)$, the parameter $c_1$ and $c_2$ and the matter Lagrangian in the action (7), the theory under consideration can lead to a great variety of cosmological models. Many of them were discussed in [8], mostly those related to minima of the effective potential (17) at nonzero values of $\phi$. Such minima correspond to stationary states of the scalar $\phi$ and consequently of the scale factor $b = e^\beta$ of the extra dimensions (see also [13]). If the minimum value of the potential is positive, it can play the role of a cosmological constant that launches an accelerated expansion of the Universe.

Here, we would like to pay attention to one more minimum of the potential $V_{Ein}$, existing for generic choices of the function $F(R)$ with $F' > 0$ and located at the point $\phi = 0$. The asymptotic $\phi \rightarrow 0$ corresponds to growing rather than stabilized extra dimensions: $b = e^\beta \sim 1/\sqrt{|\phi|} \rightarrow \infty$. A model with such an asymptotic growth at late times may still be of interest if the growth is sufficiently slow and the size $b$ does not reach detectable values by now. Let us recall that the admissible range of such growth comprises as many as 16 orders of magnitudes if the $D$-dimensional Planck length $1/m_D$ coincides with the 4D one, i.e. about $10^{-33}$ cm. This estimate certainly changes if there is no such coincidence.

Let us check whether it is possible to describe the modern state of the Universe by an asymptotic form of the solution for $\phi \rightarrow 0$ as a spatially flat cosmology with the 4D Einstein-frame metric

$$d\tilde{s}_4^2 = dt^2 - a_E^2(t) d\vec{x}^2,$$

(18)

where $a_E$ is the Einstein-frame scale factor. At small $\phi$, assuming a smooth function $F(\phi)$, we can use its Taylor decomposition

$$F(\phi) = -2\Lambda + \phi + c_\phi \phi^2 + \ldots,$$

(19)

where the form of the first two terms is chosen to obtain in (7) multidimensional Einstein gravity in the first approximation in $R$; $\Lambda$ is the initial cosmological constant. For the kinetic and potential terms in the Lagrangian (14) we then have in the main approximation with respect to $\phi$:

$$K_E \approx K_0/\phi^2,$$

$$K_0 = \frac{1}{4} [d_1 (d_1 + 2) + 8(c_1 + c_2)];$$

$$V_E \approx \Lambda e^{-d_1 \beta} = \Lambda [d_1 (d_1 - 1) \beta^{d_1/2}]^{d_1/2}.$$
for $\beta(t)$ and $a_E(t)$ in the form

$$\dot{\beta} + 3\frac{\dot{a}_E}{a_E}\beta = \frac{\Lambda d_1}{4K_0} e^{-d_1\beta}, \quad (21)$$

$$3\frac{\dot{a}_E^2}{a_E^2} = 2K_0\beta^2 + 6\Lambda e^{-d_1\beta}. \quad (22)$$

It is hard to solve this set of equations exactly. Let us, however, notice that our equations are of the same type as those appearing in studies of inflationary cosmologies, and the field $\beta$ plays the role of an inflaton. Hence it seems possible to apply the slow-rolling approximation frequently used there: we suppose

$$|\dot{\beta}| \ll 3\frac{\dot{a}_E}{a_E}\beta, \quad K_0\beta^2 \ll 3\Lambda e^{-d_1\beta}. \quad (23)$$

and drop the corresponding terms in Eqs. (22) and (21). Then we can express $\dot{a}_E/a_E$ from (22) and insert it to (21), getting

$$\dot{\beta} = \frac{d_1^2\sqrt{2}\Lambda}{24K_0} e^{-d_1\beta/2}, \quad (24)$$

which is easily integrated to give

$$e^{d_1\beta/2} = \frac{d_1^2\sqrt{2}\Lambda}{48K_0}(t - t_0), \quad (25)$$

where $t_0$ is an integration constant which can be eliminated without loss of generality by changing the zero point of the coordinate $t$. For the scale factor $a_E$ we obtain

$$\dot{a}_E/a_E = p/t \quad \Rightarrow \quad a_E \propto t^p \quad (26)$$

with

$$p = 48K_0/d_1. \quad (27)$$

Substituting the solution to the slow-rolling conditions (23), we see that both of them hold if $3p \gg 1$, or, in terms of the input parameters of the theory,

$$3p = 36 \left[ \frac{d_1 + 2}{d_1} + \frac{8(c_1 + c_2)}{d_1^2} \right] \gg 1. \quad (28)$$

For $c_1 + c_2 \geq 0$, we have $3p > 36$. This verifies a sufficiently good precision of our solution.

Passing over to the Jordan frame with

$$ds_4^2 = d\tau^2 - a^2(\tau)d\vec{x}^2 = \frac{1}{f}[dt^2 - a_E^2(t)d\vec{x}^2] \quad (29)$$

(so that $\tau$ is the cosmic time in the Jordan frame), due to $F'(0) \approx 1$, we can put simply $f = e^{d_1\beta} \approx t^2$, to obtain

$$t = e^{\sqrt{2}\Lambda t/p}, \quad a(\tau) \propto e^{(p-1)\sqrt{2}\Lambda t/p},$$

$$b(\tau) = e^\beta(\tau) = \left(\frac{\sqrt{2}\Lambda}{p} e^{\sqrt{2}\Lambda t/p}\right)^{2/d_1}, \quad (30)$$

where we have fixed an integration constant by choosing the zero point of the time variable $\tau$.

A further interpretation of the results depends on which conformal frame is regarded physical (observational) [14, 15], and this in turn depends on the manner in which fermions appear in the (so far unknown) underlying unification theory involving all interactions. We here restrict ourselves to the simplest and maybe the most natural assumption, that the observational frame coincides with the fundamental (Jordan) one, in which the initial theory (7) has been formulated.

Then an immediate observation is that the external scale factor $a(\tau)$ grows exponentially, in a de Sitter manner, which conforms to modern observations if one properly chooses the constants, namely,

$$(p - 1)\sqrt{2}\Lambda/p = H_0 \approx 2.3 \times 10^{-18} \text{ s}^{-1}$$

$$\approx 7.25 \times 10^{-11} \text{ yr}^{-1}, \quad (31)$$

where $H_0$ is the modern value of the Hubble parameter.

The internal scale factor $b(\tau)$ grows much slower for sufficiently large $d_1$, while the volume factor $b^d$ behaves like $e^{2\sqrt{2}\Lambda t/p}$. The effective gravitational constant is known to change inversely proportionally to the volume factor, so that

$$\dot{G}/G = -2\sqrt{2}\Lambda/p. \quad (32)$$

(here and henceforth the dot means $d/d\tau$). The dimensionless parameter of $G$ variation is

$$\delta = \dot{G}/(GH_0) = -2/(p - 1). \quad (33)$$

The tightest experimental constraint on $G$ variation has been obtained, to our knowledge, from lunar laser ranging (LLR) data [16], namely,

$$\dot{G}/G = (2 \pm 7) \times 10^{-13} \text{ yr}^{-1}. \quad (34)$$

Thus a viable cosmology should predict $\delta \lesssim 10^{-2}$, which according to (33) constrains our model parameters by

$$p = 12 \left[ \frac{d_1 + 2}{d_1} + \frac{8(c_1 + c_2)}{d_1^2} \right] \gtrsim 100, \quad (35)$$
It follows that our initial field model (7) with only \( F(R) \) is unable to satisfy the constraint (35); to do so, it is necessary to invoke the Ricci tensor squared or/and the Kretschmann scalar, with the input constants \( c_1 \) and \( c_2 \) such that

\[
c_1 + c_2 \gtrsim d_1^2.
\]  

(36)

We conclude that, under the condition (36), our model with the asymptotic behavior (30) is potentially viable since it combines a de Sitter-like expansion of the observable Universe with a slow enough variation of the effective gravitational constant.

Three more observations can be added. First, comparing the constraints (35) and (28), we see that our approximation works so much the better, the smaller is variation of the observable Universe with a slow enough variation of the effective gravitational constant.

Second, according to (31), the constant \( \Lambda \) is determined by the modern value of the Hubble parameter and is thus approximately the same as the observed cosmological constant. So this model suffers the same fine-tuning problem for the cosmological constant value as the Standard model.

Third, the size \( b(\tau) \) of the extra dimensions, given by (30), remains uncertain since it depends on the arbitrarily chosen values of the time variable \( \tau \) (note that observable quantities depend on \( H_0 \) rather than \( \tau \)). Thus one can easily satisfy the conditions that \( b(\tau) \) must be much greater than the Planck length but still below the observational threshold (about \( 10^{-17} \) cm). We also see that even an exponential growth of the extra dimensions can be slow enough to conform to the observational bounds on the stability of fundamental constants.

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References

[1] V.N. Melnikov, Multidimensional Classical and Quantum Cosmology and Gravitation, Exact Solutions and Variations of Constants, CBPF-NF-051/93, Rio de Janeiro, 1993; in: Cosmology and Gravitation, ed. M. Novello (Editions Fronti\'eres, Singapore, 1994), p. 147; Multidimensional Cosmology and Gravitation, CBPF-MO-002/95, Rio de Janeiro, 1995, 210 p.; in: Cosmology and Gravitation, II, ed. M. Novello (Editions Fronti\'eres, Singapore, 1996), p. 465; Exact Solutions in Multidimensional Gravity and Cosmology III, CBPF-MO-03/02, Rio de Janeiro, 2002, 297 pp.

[2] V.N. Melnikov, Gravity as a key problem of the millennium. Proc. 2000 NASA/JPL Conference on Fundamental Physics in Microgravity, NASA Document D-21522, 2001, p. 4.1–4.17, Solvang, CA, USA.

[3] V.N. Melnikov, Gravity and cosmology as key problems of the millennium. In: Proc. Albert Einstein Century Int. Conf., eds. J.-M. Alimi and A. Fuzfa (AIP Conf. Proc., Melville–NY, 2006), v. 861, p. 109–126.

[4] V.N. Melnikov, Grav. Cosmol. 13, 81 (2007).

[5] S.A. Kononogov and V.N. Melnikov, Izmeritel'\'naya Tekhnika 6, 1 (2005).

[6] K.A. Bronnikov and S.A. Kononogov, Metrologia 43, R1 (2006).

[7] K.A. Bronnikov, S.A. Kononogov and V.N. Melnikov, Gen. Rel. Grav. 38, 1215 (2006).

[8] K.A. Bronnikov and S.G. Rubin, Phys. Rev. D 73, 124019 (2006).

[9] K.A. Bronnikov, R.V. Konoplich and S.G. Rubin, Class. Quantum Grav. 24, 1261 (2007).

[10] K.A. Bronnikov and S.G. Rubin, Grav. & Cosmol. 13, 191 (2007).

[11] Compact hyperbolic spaces of constant curvature on the basis of a usual open Lobachevsky space \( \mathbb{H}^d \) are isometric to quotient spaces \( \mathbb{H}^d/\Gamma \) where \( \Gamma \) is a nontrivial discrete group of isometries of \( \mathbb{H}^d \), see, e.g., B.A. Dubrovin, A.T. Fomenko and S.P. Novikov, Modern Geometry — Methods and Applications (Springer-Verlag, New York, 1992). On possible applications of such (3D) spaces in cosmology see, e.g., D. Mûller, H.V. Fagundes and R. Opher, Phys. Rev. D 66, 083507 (2002) and references therein.

[12] J.F. Donoghue, Phys. Rev. D 50, 3874 (1994).

[13] U. Günther, P. Moniz and A. Zhuk, Astrophys. Space Sci. 283, 679-684 (2003); gr-qc/0209045; U. Günther and A. Zhuk, Remarks on dimensional reduction in multidimensional cosmological models, gr-qc/0401003.

[14] K.A. Bronnikov and V.N. Melnikov, Gen. Rel. Grav. 33, 1549 (2001).

[15] K.A. Bronnikov and V.N. Melnikov, “Conformal frames and D-dimensional gravity”, gr-qc/0310112, in: Proc. 18th Course of the School on Cosmology and Gravitation: The Gravitational Constant. Generalized Gravitational Theories and Experiments (30 April–10 May 2003, Erice), Ed. G.T. Gillies, V.N. Melnikov and V. de Sabbata, (Kluwer, Dordrecht/Boston/London, 2004) pp. 39–64.

[16] J. Müller and L. Biskupek, Class. Quantum Grav. 24, 4533 (2007).