On an inverse problem for finite-difference operators of second order

Mikhail Kudryavtsev

November 22, 2021

The Institute for Low Temperature Physics and Engineering
of the National Academy of Science of Ukraine,
Ukraine, Kharkov, 61103, Lenine Ave., 47
E-mail: kudryavtsev@ilt.kharkov.ua

Abstract

The Jacobi matrices with bounded elements whose spectrum of multiplicity 2 is separated from its simple spectrum and contains an interval of absolutely continuous spectrum are considered. A new type of spectral data, which are analogous for scattering data, is introduced for this matrix. An integral equation that allows us to reconstruct the matrix from this spectral data is obtained. We use this equation to solve the Cauchy problem for the Toda lattice with the initial data that are not stabilized.

Introduction

1. The Cauchy problem for the equation of the oscillation of the doubly-infinite Toda lattice

\[ \frac{d^2 x_k}{dt^2} = e^{x_{k+1}} - e^{-x_{k-1}}, \quad k \in \mathbb{Z}, \]  

\[ x_k(0) = v_k, \quad \dot{x}_k(0) = w_k, \]  

with stabilized initial data

\[ \sum_{-\infty}^{\infty} |k||a_k| < \infty, \quad \sum_{-\infty}^{\infty} |k||b_k - 1| < \infty, \]  

where

\[ a_k = w_k, \quad b_{k-1} = \exp^{\frac{v_k - v_{k-1}}{2}}, \]  

is solved by Manakov and Flashka by the method of inverse scattering problem (see [1],[2]). Here the finite-difference operator of second order with the matrix

\[ J = \begin{pmatrix}
  \ddots & \ddots & \ddots & \\
  b_{-2} & a_{-1} & b_{-1} & 0 \\
  & b_{-1} & a_0 & b_0 \\
  & & b_0 & a_1 & b_1 \\
  & & & b_1 & a_2 & b_2 \\
  & & & & \ddots & \ddots & \ddots
\end{pmatrix}, \]  

where the coefficients \( a_k, b_k \) are defined by (0.3), plays the role of the L-operator.

However, we cannot apply the method of inverse scattering to solve the Cauchy problem with non-stabilized initial data due to the fact that the objects of with the inverse scattering problem (the Jost
solutions, the reflection coefficient, etc.) do not exist in this case. In this connection it is important to find a more general inverse spectral problem for the Jacobi matrices (0.4) so that this problem can be applied to solve the equation of the oscillation of the Toda lattice with quite a wide class of initial data.

Note, that in the case of the semi-infinite Toda lattices the corresponding Cauchy problem with arbitrary (bounded) initial data is solved by M. Kac [3] and Yu.M. Berezanski [4] by means of the inverse problem of the reconstruction of the semi-infinite Jacobi matrix by its spectral function. However, their method cannot be extended to the case of the doubly-infinite Toda lattices. Besides, the method of inverse problem is also used in the paper of N.V. Zhernakov [5], where the Cauchy problem for the doubly-infinite Toda lattice is solved in the case when the operator defined by (0.4) is the Hilbert-Schmidt operator. But this method cannot be used when the spectrum of the operator is not discrete. Our aim is to solve the inverse problem with more general type of spectrum.

We denote by $P_k(\lambda)$, $Q_k(\lambda)$ the solutions of the equation
\[
b_{k-1}\omega_{k-1} + (a_k - \lambda)\omega_k + b_k\omega_{k+1} = 0, \quad k \in \mathbb{Z},
\]with initial data $P_0(\lambda) = 1$, $P_{-1}(\lambda) = 0$, $Q_0(\lambda) = 0$, $Q_{-1}(\lambda) = 1$. It is evident that every solution of this equation is their linear combination.

It is known (see, for example, [5], [6]) that for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the equation (0.5) has two solutions (the Weyl solutions of the matrix $J$):
\[
\varphi^R(k,\lambda) = m^R(\lambda)P_k(\lambda) - \frac{Q_k(\lambda)}{b_{-1}}, \quad k \in \mathbb{Z},
\]
\[
\varphi^L(k,\lambda) = -\frac{P_k(\lambda)}{b_{-1}} + m^L(\lambda)Q_k(\lambda), \quad k \in \mathbb{Z},
\]such that $\sum_{k=N}^{\infty} |\varphi^R(k,\lambda)|^2 < \infty$, $\sum_{k=-\infty}^{N} |\varphi^L(k,\lambda)|^2 < \infty$ for any finite $N \in \mathbb{Z}$. In this notation $m^R(\lambda)$ and $m^L(\lambda)$ are the Weyl functions of the matrix $J$ represented in the form
\[
b_{-1}m^R(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho_R(\tau)}{\tau - \lambda}, \quad -\frac{1}{b_{-1}m^L(\lambda)} = \frac{\lambda}{b_{-1}} + \beta + \int_{-\infty}^{\infty} \frac{d\rho_L(\tau)}{\tau - \lambda},
\]where $\beta \in \mathbb{R}$, and $\rho_R(\lambda)$ and $\rho_L(\lambda)$ are nondecreasing functions. Here $d\rho_R(\lambda)$ is the spectral measure of the semi-infinite (to the right) matrix $J^R$ that is contained in $J$ and starts with the element $a_0$. The measure $d\rho_L(\lambda)$ is spectral for the semi-infinite (to the left) matrix $J^L$ starting with the element $a_{-2}$.

In this paper we consider a larger class of matrices (0.4) with bounded elements whose principal property is the existence of an interval $[a, b]$ of absolutely continuous spectrum of multiplicity 2, which is separated from the other parts of the spectrum of the matrix $J$. Without restriction of generality we can suppose that $[a, b] = [-2, 2]$ and $b_{-1} > 0$.

For such matrices a new inverse spectral problem is found and solved. In this problem the functions $m^R(\lambda)$ and $m^L(\lambda)$ and the number $b_{-1}$ play the role of spectral data from which the matrix $J$ is reconstructed. We obtain linear integral equation (3.46) of the inverse problem and prove its unique solvability. All the parameters contained in the equation are explicitly expressed by $b_{-1}$ and the functions $m^R(\lambda)$ and $m^L(\lambda)$.

In a forthcoming paper [8] we will apply this integral equation to solve the corresponding Cauchy problem. We also note that one can find a short and complete exposition of these results in papers [9], [10], resp.

In our case the functions $m^R(\lambda)$ and $m^L(\lambda)$ have nonreal limits on the interval $[-2, 2]$. Instead of two Weyl functions and two Weyl solutions we introduce the function
\[
n(z) = \begin{cases} 
-\frac{b_{-1}}{1} m^R(z + z^{-1}), & |z| < 1, \\
\frac{1}{b_{-1}} m^L(z + z^{-1}), & |z| > 1,
\end{cases}
\]
and the solution
\[
\psi(k, z) = n(z)P_k(z + z^{-1}) + Q_k(z + z^{-1}) = \begin{cases} 
-\frac{b_{-1}}{1} \varphi^R(k, z + z^{-1}), & |z| < 1, \\
\varphi^L(k, z + z^{-1}) \frac{1}{m^L(z + z^{-1})}, & |z| > 1,
\end{cases}
\]
which we will also call the Weyl function and the Weyl solution of the matrix $J$.  

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They are defined and holomorphic at such $z \in \mathbb{C}$, for which both functions $m^R$ and $\frac{1}{m^L}$ are regular at the point $z + z^{-1}$. At such points

$$\sum_{N} |\psi(k,z)|^2 < \infty, \quad |z| < 1, \quad \sum_{-\infty}^{N} |\psi(k,z)|^2 < \infty, \quad |z| > 1.$$  

Note, that in the papers of L.K. Maslov [11],[12] the class of matrices (0.4) was studied such that the limits of the Weyl functions $m^R(\lambda)$ and $\frac{1}{m^L(\lambda)}$ coincide on a certain interval $(a, b)$ of the real line:

$$b_{-1}m^R(\lambda + i0) = \frac{1}{b_{-1}m^L(\lambda - i0)}, \quad a < \lambda < b.$$

In this case $\psi(k,z)$ and $n(z)$ are holomorphic at the points of the unit circle. In our case the limit values of functions $\psi(k,z)$ and $n(z)$ from within and from the outside of the circle $|\xi| = 1$ are different.

2. Notation. In the presented work we use the following notations. Put the limit values of any function $f(z)$ on the unit circle and the real line:

$$f^\pm(e^{i\theta}) = \lim_{r \to 1 \mp 0} f(re^{i\theta}), \quad \theta \in (-\pi, \pi),$$

and

$$f^\pm(t) = f(t \pm i0), \quad t \in \mathbb{R}.$$

We will also denote by $\xi$ the points of the unit circle $\{\xi = 1\}$ in the complex plane.

3. In order to clarify the analogy between our case and the classical inverse scattering problem (ISP), we will remind some essential points of ISP (see, for example, [1],[2]). Like in the case of arbitrary Jacobi matrices, there exist two solutions of equation (0.5), one of which is square-summable for positive $k$ and the other one, for negative. In the case of stabilized initial data the following asymptotic formulas are true for them:

$$\tilde{\psi}(k,z) = z^k + o(z^k), \quad k \to +\infty, \quad |z| < 1, \quad (0.11)$$

$$\tilde{\psi}(k,z) = z^k + o(z^k), \quad k \to -\infty, \quad |z| > 1, \quad (0.12)$$

where $z + z^{-1} = \lambda$ is the spectral parameter ($\tilde{\psi}(k,z)$, $|z| < 1$, are square-summable for positive $k$, and $\tilde{\psi}(k,z)$, $|z| > 1$, are square-summable for negative $k$). The functions $\tilde{\psi}(k,z)$ are called the Jost solutions.

For a fixed point $\xi$ on the unit circle the limit values $\tilde{\psi}^+(k,\xi)$, $\tilde{\psi}^-(k,\xi)$, $\tilde{\psi}^-(k,\xi^{-1})$ are solutions (with respect to $k$) of the equation (0.5) at the point $\lambda = \xi + \xi^{-1} \in [-2, 2]$. These limit values exist because the segment $[-2, 2]$ belongs to the absolutely continuous spectrum; here $\tilde{\psi}^-(k,\xi) = \tilde{\psi}^-(k,\xi^{-1})$.) The solution $\tilde{\psi}^-(k,\xi)$ is represented in the form of a linear combination of two other solutions:

$$\tilde{\psi}^+(k,\xi) = \tilde{c}(\xi)\tilde{\psi}^-(k,\xi) + \tilde{d}(\xi)\tilde{\psi}^-(k,\xi^{-1}),$$

or

$$\frac{1}{\tilde{c}(\xi)}\tilde{\psi}^+(k,\xi) = \tilde{\psi}^-(k,\xi) + \frac{\tilde{d}(\xi)}{\tilde{c}(\xi)}\tilde{\psi}^-(k,\xi^{-1}), \quad (0.13)$$

where

$$\frac{1}{\tilde{c}(\xi)} \equiv \begin{cases} < \tilde{\psi}^-(k,\xi^{-1}), \tilde{\psi}^-(k,\xi) >, & < \psi^-(k,\xi^{-1}), \psi^+(k,\xi) > = \frac{\xi - \xi^{-1}}{< \tilde{\psi}^-(k,\xi^{-1}), \psi^-(k,\xi) >} \quad (0.14) \end{cases}$$

(ther the Wronskian $< \psi^-(k,\xi^{-1}), \psi^-(k,\xi) >$ is explicitly calculated in view of the asymptotic formulae (0.11), (0.12)). In this case the function $\frac{1}{\tilde{c}(\xi)} \equiv \frac{\xi - \xi^{-1}}{< \tilde{\psi}^-(k,\xi^{-1}), \psi^-(k,\xi) >}$ is meromorphic inside the unit disk and its limit at the point $\xi$ from within of the circle is equal to the coefficient $\frac{1}{\tilde{c}(\xi)}$ from (0.13). Hence, this coefficient can be analytically continued inside the circle. Therefore the function

$$\Phi_k(z) = \begin{cases} \frac{1}{\tilde{c}(\xi)} z^{-\xi-1}h_k\tilde{\psi}(k,z), & |z| < 1, \\ z^{-\xi(k+1)}h_k\tilde{\psi}(k,z), & |z| > 1, \end{cases}$$

where

$$h_k = \begin{cases} b_{-1} \ldots b_{k-1}, & k \geq 0, \\ 1, & k = -1, \\ \frac{1}{\xi_k}, & k \leq -2, \end{cases}$$

$$\Phi_k(z) = \begin{cases} \frac{1}{\tilde{c}(\xi)} z^{-\xi-1}h_k\tilde{\psi}(k,z), & |z| < 1, \\ z^{-\xi(k+1)}h_k\tilde{\psi}(k,z), & |z| > 1, \end{cases}$$

where

$$h_k = \begin{cases} b_{-1} \ldots b_{k-1}, & k \geq 0, \\ 1, & k = -1, \\ \frac{1}{\xi_k}, & k \leq -2, \end{cases}$$
is meromorphic inside the unit disk. It has, according to (0.13), the jump
\[ \xi^{2(k+1)} \left( \Phi^+_k(\xi) - \Phi^-_k(\xi) \right) = r(\xi) \Phi^-_k(\xi^{-1}), \quad (0.16) \]
where \( r(\xi) = \frac{d(\xi)}{e(\xi)} \) is the reflection coefficient, from which we reconstruct \( J \). The function \( \Phi_k(z) \) has inside the disk simple real poles \( z_n, \) \( 0 < |z_n| < 1, \) \( n = 1, 2, \ldots, N. \) In these points \( \tilde{\psi}(k, z_n) = p_n \tilde{\psi}(k, z_n^{-1}) \) (these are the components of an eigenvector), and the residue of the function \( \Phi_k(z) \) equals \( \frac{1}{e(z_n)} \Phi_k(z_n^{-1}). \)

Besides, the function \( \Phi_k(z) \) tends to the point \( z \rightarrow \infty \). According to the Cauchy theorem,
\[ \Phi_k(z) = 1 + \sum_{m=1}^{N} \frac{z_{m}^{-2(k+1)} p_\nu \Phi_k(z_{m}^{-1})}{e^\nu(z_{m})(z_{m} - z_{m})} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta^{-2(k+1)} r(\zeta) \Phi_k(\zeta^{-1})}{\zeta - z} d\zeta. \quad (0.17) \]

Let \( z \) tend to the point \( \xi^{-1}, |\xi^{-1}| = 1, \) from the outside of the circle. According to the formulas of Plemelj–Sokhotski,
\[ \Phi_k(\xi^{-1}) = 1 + \sum_{m=1}^{N} \frac{z_{m}^{-2(k+1)} p_\nu \Phi_k(z_{m}^{-1})}{e^\nu(z_{m})(\xi^{-1} - z_{m})} \]
\[ - \frac{1}{2} \xi^{2(k+1)} r(\xi^{-1}) \Phi_k^-(\xi) + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta^{-2(k+1)} r(\zeta) \Phi_k(\zeta^{-1})}{\zeta - \xi^{-1}} d\zeta. \quad (0.18) \]

(We observe that namely relation (0.16) let us replace \( \Phi^+_k(\zeta) - \Phi^-_k(\zeta) \) under the integral sign in (0.17) by \( \zeta^{-2(k+1)} r(\zeta) \Phi^-_k(\zeta^{-1}) \).

Thus we have one unknown function \( \Phi^-_k(\xi^{-1}) \) in both sides of the integral equation (0.18).) Letting in (0.17) \( z = z_n^{-1} \), we obtain \( N \) equations for \( \Phi_k(z_n^{-1}) \):
\[ \Phi_k(z_n^{-1}) = 1 + \sum_{m=1}^{N} \frac{z_{m}^{-2(k+1)} p_\nu \Phi_k(z_{m}^{-1})}{e^\nu(z_{m})(z_n^{-1} - z_{m})} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta^{-2(k+1)} r(\zeta) \Phi_k(\zeta^{-1})}{\zeta - z_n^{-1}} d\zeta. \quad (0.19) \]

Let us define the unknown function on the unit circle and at the points \( z_n \)
\[ u_k(\beta) := \begin{cases} \beta^{-2(k+1)} \Phi_k^-\beta^1, & |\beta| = 1, \\ z_n^{-2(k+1)} \Phi_k^-(z_n^{-1}), & \beta = z_n, \end{cases} \]
and define at the points \( z_n \) the point measure \( d\sigma(z_n) := \frac{1}{e(z_n)}. \) Let us also define the function \( \chi_0(\beta) \) as the indicator of the unit circle. Then the system (0.18), (0.19) can be rewritten in the form of one integral equation
\[ \beta^{2(k+1)} u_k(\beta) + \int_{-\infty}^{\infty} \frac{u_k(\alpha)}{\alpha - \beta^{-1}} d\sigma(\alpha) + \frac{1}{2} \chi_0(\beta) r(\beta) u_k(\beta^{-1}) - \frac{1}{2\pi i} \left. \int_{|\zeta|=1} \frac{r(\zeta) u_k(\zeta)}{\zeta - \beta^{-1}} d\zeta \right| = 1, \quad (0.21) \]
where the \textit{reflection coefficient} \( r(\xi) \) and the \textit{measure} \( d\sigma \) are the \textit{scattering data}, and \( u_k(\zeta) \) is the unknown function. Further, the solution \( \Phi_k(z) \) is represented in the form
\[ \Phi_k(z) = 1 - \int_{-\infty}^{\infty} \frac{u_k(\alpha)}{\alpha - z} d\sigma(\alpha) - \frac{1}{2\pi i} \left. \int_{|\zeta|=1} \frac{r(\zeta) u_k(\zeta)}{\zeta - z} d\zeta \right|, \quad (0.22) \]
where \( u_k(\beta) \) is the solution of equation (0.21). The elements \( a_k \) and \( b_k \) of the matrix \( J_N \) can be simply found from \( \Phi_k \).

As we can see, the principal step of the derivation of the ISP-equation (0.19) is to obtain the relation (0.16). To obtain this relation we multiply the Jost solution \( \tilde{\psi}(k, z) \) by the factor coefficient that equals \( \frac{1}{e(z)} \) inside the disk and 1 outside the disk (we also multiplied the Jost solution by \( z^{-2(k+1)} h_k \) in order to have \( \Phi_k(z) \rightarrow 1, z \rightarrow \infty \)).

However, in our case the Jost solutions, which are defined by asymptotic formulas (0.11), (0.12), do not exist. We consider the Weyl solutions \( \psi(k, z) \) defined by (0.10) (they exist for any Jacobi matrix). It is easy to obtain for these solutions the following relations on the unit circle \( |\xi| = 1 \):
\[ \frac{1}{a(\xi)} \psi^-(k, \xi) = \psi^+\psi(k, \xi) + \frac{b(\xi)}{a(\xi)} \psi^+(k, \xi^{-1}), \quad (0.23) \]
\[
\frac{1}{c(\xi)} \psi^+(k, \xi) = \psi^-(k, \xi) + \frac{d(\xi)}{c(\xi)} \psi^-(k, \xi^{-1}) \tag{0.24}
\]

(explicit expressions for \(a(\xi), b(\xi), c(\xi), d(\xi)\) are given in section 3). These relations are completely analogous to (0.13), but for the coefficient
\[
\frac{1}{c(\xi)} \equiv \frac{<\psi^-(k, \xi^{-1}), \psi^-(k, \xi)>}{<\psi^-(k, \xi^{-1}), \psi^+(k, \xi)>}
\]
the equality (0.14) is not fulfilled, because the Weyl solutions do not satisfy the asymptotic formulas (0.11), (0.12). From now on there is no more coincidence between our case and classical inverse scattering, because \(R^{-1}\) does not have the necessary properties.

Due to this reason, in order to obtain the equality, analogous to (0.16), we have to multiply the Weyl solutions \(\psi(k, z)\) by the specially chosen function \(R(z)\). The choice of the function \(R(z)\) and the subsequent reducing (0.23), (0.24) to a more symmetric look is done with the help of the following theorem, which is a key result for the considered inverse problem:

**Theorem 2 (factorization).** The function \((z - z^{-1})^{-1}(n(z) - n(z^{-1}))^{-1}\) can be represented in the domain, where it is holomorphic, in the form of the product of two functions \(R(z), R(z^{-1})\), which are holomorphic in this domain

\[
-\frac{z - z^{-1}}{n(z) - n(z^{-1})} = R(z)R(z^{-1}). \tag{0.25}
\]

The function \(R(z)\) may only have singularities (i.e. not to be holomorphic) in such points \(z\) that \(z + z^{-1}\) belongs to the spectrum of the matrix \(J\). In addition,

\[
R^+(\xi)R^-(\xi^{-1}) |\text{Im} n^+(\xi)| = R^-(\xi)R^+(\xi^{-1}) |\text{Im} n^-(\xi)| > 0, \quad |\xi| = 1.
\]

(The explicit definition of the function \(R(z)\) is performed in section 2.)

Further, we define the function

\[
g(k, z) = \frac{R(z)}{R(\infty)} z^{-(k+1)} h_k \psi(k, z), \tag{0.26}
\]

where \(h_k\) are defined in (0.15). Theorem 2 allows us to derive from equalities (0.23), (0.24) the following relation, which groups together the limit values of the function \(g(k, z)\) at the points \(\xi\) and \(\xi^{-1}\):

\[
\xi^{2(k+1)} (g^+(k, \xi) - g^-(k, \xi)) = \tilde{r}(\xi) (g^+(k, \xi^{-1}) + g^-(k, \xi^{-1})), \tag{0.27}
\]

where \(|\tilde{r}(\xi)| < C < 1\). (See explicit expression for \(\tilde{r}(\xi)\) in section 3.)

Precisely this relation is the analogue of the equality (0.16) and makes it possible to derive the integral equation of the inverse problem, which is appropriate for solving the Cauchy problem (0.1), (0.2).

4. We explain the principal steps of the derivation of the inverse problem equation. The transformation \(z + z^{-1} = \lambda\) maps the spectrum of the matrix \(J\) to the unit circle and a certain set on the real line. I turns out that the function \(g(k, z)\) is holomorphic outside of these sets. In view of theorem 1 (see section 1), \(g(k, z) \rightarrow 1, \ z \rightarrow \infty\). Hence, according to the Cauchy theorem,

\[
g(k, z) = 1 + \frac{1}{2\pi i} \int_{\Gamma} \frac{g(k, \zeta)}{\zeta - z} d\zeta, \tag{0.28}
\]

where \(\Gamma\) is a contour enclosing all the singularities of the function \(g(k, z)\). These singularities are concentrated on the real line and the unit circle and correspond to the different types of the spectrum of the matrix \(J\). Further we tighten the contour to the singularities of the function \(g(k, z)\) that are concentrated on the sets of each one of these types:

1) on the set \(\Omega_1 \cup \Phi\), corresponding to:
   a) disjoint parts of the spectra of the right semi-infinite matrix \(J^R\) and the left semi-infinite matrix \(J^L\),
   b) the eigenvalues of the matrix \(J\) that are not contained in the spectra of the two semi-infinite matrices;

2) on the set \(\Gamma_2\), corresponding to the eigenvalues of the matrix \(J\) that are also eigenvalues of both matrices \(J^R\) and \(J^L\);

3) on the set \(\Omega_3\), corresponding to the parts of the absolutely continuous spectrum of multiplicity 2 of the matrix \(J\) except the segment \([-2, 2]\);
4) on the unit circle, corresponding to the segment $[-2, 2]$ of the absolutely continuous spectrum of multiplicity 2 of the matrix $J$.

Therefore the spectrum of the matrix $J$ splits into four sets of different types. We calculate in the corresponding way the integrals along the contours (see (3.2)) enclosing each one of these sets. First, we remind that in the classical ISP there are parts of the spectrum of two types: the simple spectrum (the finite set of eigenvalues) and the segment of absolutely continuous spectrum of multiplicity 2. (The unit circle in $z$-plane corresponds to this segment.) At the points of the simple spectrum we expressed the residue of the function $\Phi_0(z)$ at the point $z_0$ by the value $\Phi_0(z^{-1})$, substituted the obtained values of the residues in the integrals along the contours, enclosing the eigenvalues, and obtained the sum in (0.17), which is represented in the form of the integral by the measure $d\sigma$. This measure is defined on the set of the eigenvalues (i.e. on the simple spectrum). In a resembling way, in lemmas 3.1 and 3.2 we calculate the integrals from (0.27) along the contours, enclosing the singularities on $\Omega_1 \cup \Phi$ and $\Gamma_2$, and represent these integrals in the form of integrals along the set $\Omega_1 \cup \Phi \subset \mathbb{R}$ and $\Gamma_2 \subset \mathbb{R}$ of the function $g_k(z^{-1})$. In lemma 3.5 we obtain the relation (3.20) for the limit values on the absolutely continuous spectrum (outside $[-2, 2]$), which is analogous to the relation (0.16) from ISP. Further we derive the main integral equation (3.46) from the relations (3.20), (0.27) and lemmas 3.1 and 3.2 (see theorem 3). Formulas (3.47), (3.46) and (3.44) of the theorem 3, are, evidently the analogs of the formulas (0.22), (0.21) and (0.20) of the classical ISP. The functions $g(k, z)$ are simply expressed from the solution of the integral equation (formula (3.47)).

It is easy to conclude from the theorem 1 (formulas (1.3) and (1.4)) that the elements $a_k$ and $b_k$ of the matrix $J$ can be found from $g(k, z)$ by formulas (b_k up to its sign):

$$b_{k-1}^2 = \frac{g(k, 0)}{g(k-1, 0)}, \quad a_k = \lim_{z \to \infty} \frac{z(g(k, z) - g(k + 1, z))}{g(k, z)}$$

(0.29)

So, the obtained inverse problem equation allows us to reconstruct the matrix $J$ by its spectral data.

5. Now we list the conditions on the spectrum for which the solution of the inverse problem is given. Put

$$\eta^R(\tau) = \lim \arg m^R(\tau + i\varepsilon), \quad \eta^L(\tau) = \lim \arg \frac{-1}{m^L(\tau + i\varepsilon)}.$$

Let $\tilde{\Omega}^R$ and $\tilde{\Omega}^L$ be the supports of measures $d\rho_R(\tau)$ and $d\rho_L(\tau)$, resp., defined by the non-decreasing functions $\rho_R(\tau)$ and $\rho_L(\tau)$ from (0.8),

$$\tilde{\Omega}_2 \equiv \tilde{\Omega}^R \cap \tilde{\Omega}^L, \quad \tilde{\Omega}_1 \equiv (\tilde{\Omega}^R \setminus \tilde{\Omega}^L) \cup (\tilde{\Omega}^L \setminus \tilde{\Omega}^R),$$

$\tilde{\Omega}_2^L \subset \tilde{\Omega}_2$ be the set of the common poles of the functions $m^R(\lambda)$ and $\frac{1}{m^L(\lambda)}$, and let $\tilde{\Omega}_2^R \equiv \tilde{\Omega}_2 \setminus \tilde{\Omega}_2^L$.

We assume that:

A) All the three sets $\tilde{\Omega}_1$, $\tilde{\Omega}_2^L$, $\tilde{\Omega}_2^R$ have positive mutual distances, $[-2, 2] \subset \tilde{\Omega}_2^L$, and the set $\tilde{\Omega}_2^R$ is finite or empty.

B) For some $\varepsilon > 0$ almost everywhere (with respect to Lebesque measure) on the set $\tilde{\Omega}_2^L$

$$0 < \varepsilon < \eta^R(\alpha) < \pi - \varepsilon, \quad 0 < \varepsilon < \eta^L(\alpha) < \pi - \varepsilon.$$

C) In some neighborhood of the set $\tilde{\Omega}_2^L$ the function $\eta^R(\alpha) - \eta^L(\alpha)$ satisfies the Hölder condition.

D) The set $\tilde{\Omega}_2^L \cap [-2, 2]$ can be covered with mutually disjoint intervals $\delta_i$ on each of which the following inequalities are true:

$$\ess \sup_{\alpha \in \delta_i} \eta^R(\alpha) - \ess \inf_{\alpha \in \delta_i} \eta^R(\alpha) < \pi, \quad \ess \sup_{\alpha \in \delta_i} \eta^L(\alpha) - \ess \inf_{\alpha \in \delta_i} \eta^L(\alpha) < \pi,$$

E) For some small $\varepsilon > 0$ and $0 < \alpha < \varepsilon$

$$\eta^R(2 + \alpha) = \eta^L(2 + \alpha) = 0, \quad \eta^R(-2 - \alpha) = \eta^L(-2 - \alpha) = \pi,$$

and the functions $\eta^R(\tau)$, $\eta^L(\tau)$ satisfy the Hölder condition on the interval $[-2, 2]$.

From A)–E) it follows that the set $\tilde{\Omega}_2^L$ is the absolutely continuous spectrum of multiplicity 2 of the matrix $J$ and that the segment $[-2, 2] \subset \tilde{\Omega}_2^L$ is divided from other parts of the spectrum. We will need the conditions B)–D) to prove that the limit values of the functions $g(k, z)$ on the absolutely continuous spectrum of multiplicity 2 belong to $L^2$. (We have to add the condition E) for the segment $[-2, 2]$.)
Remark. The conditions A)–E) can be exposed only in the terms of the functions $\eta^R(\tau)$ and $\eta^L(\tau)$. For this we have to define the sets $\hat{\Omega}^R$, $\hat{\Omega}^L$, $\hat{\Omega}_2$, $\hat{\Omega}_2^R$, $\hat{\Omega}_1^R$ directly from $\eta^R(\tau)$ and $\eta^L(\tau)$, without using the functions $\rho_R(\tau)$ and $\rho_L(\tau)$. We can do it in the following way. (The equivalence of the two definitions is a conclusion of the fact that $m^R(\lambda)$ and $\frac{1}{m^L(\lambda)}$ have a positive imaginary part in the upper half-plane and are holomorphic in it.)

Let $C\hat{\Omega}^R (C\hat{\Omega}^L)$ be the open set on the real axis that is the union of intervals such that the function $\eta^R(\tau)$ ($\eta^L(\tau)$) equals either 0 or $\pi$ almost everywhere on the interval and of the points such that $\eta^R(\tau)$ ($\eta^L(\tau)$) is equal to $\pi$ in the left neighborhood of the point and to 0 in the right neighborhood. (It means that $C\hat{\Omega}^R (C\hat{\Omega}^L)$ is the set on the real axis where $m^R(\lambda)$ ($\frac{1}{m^L(\lambda)}$) is holomorphic). Let $\hat{\Omega}^R (\hat{\Omega}^L)$ be the complement of the set $\hat{\Omega}^R (\hat{\Omega}^L)$,

$$\hat{\Omega}_2 \equiv \hat{\Omega}^R \cap \hat{\Omega}^L,$$

$$\hat{\Omega}_1 \equiv (\hat{\Omega}^R \setminus \hat{\Omega}^L) \cup (\hat{\Omega}^L \setminus \hat{\Omega}^R),$$

$\hat{\Omega}_2 \subset \hat{\Omega}_2$ be the set of points such that $\eta^R(\tau)$, $\eta^L(\tau)$ both equal 0 in the left neighborhood of the point and equal $\pi$ in the right one, $\hat{\Omega}_2^R \equiv \hat{\Omega}_2 \setminus \hat{\Omega}_2^R$.

We also refer to paper [13] from which we take the idea of the proofs of Lemmas 1.1, 1.2, 3.1–3.5, where this technique is used for a differential operator.

1. Asymptotics of the Weyl solutions and auxiliary results

Theorem 1. (L.K. Maslov). The Weyl solutions $\psi(k, z)$ and the Weyl functions $n(z)$ of the matrix $J$, defined by (0.9), (0.10), satisfy the asymptotic formulas

$$\frac{n(z)}{z} \to b_{-1}, \quad z \to 0,$$

$$\frac{n(z)}{z} \to \frac{1}{b_{-1}}, \quad z \to \infty,$$

$$\frac{z^{-(k+1)}}{h_k} \psi(k, z) \to 1, \quad z \to 0,$$

$$h_k z^{-(k+1)} \psi(k, z) \to 1, \quad z \to \infty,$$

where $h_k$ are defined by formula (0.15).

We give the sketch of the proof, obtained by L.K. Maslov in 1991.

1. For $N > 0$ we consider the matrix $J_N$ that is the matrix $J$ with changed right lower minor: instead of all $b_k$, $k \geq N - 1$, it has 1, and instead of $a_k$, $k \geq N$, it has zeroes:

$$J_N = \begin{pmatrix} \ddots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \ddots \\ b_{-1} & a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots \\ b_{N-3} & a_{N-2} & b_{N-2} & 0 & 1 \\ b_{N-2} & a_{N-1} & b_{N-1} & 1 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots \end{pmatrix}.$$  

It is clear that the orthogonal polynomials, corresponding to $J_N$, for $k \leq N - 1$ coincide with $P_k(\lambda)$, $Q_k(\lambda)$. Hence for these $k$ the Weyl solution of the matrix $J_N$, summable for positive $k$ (such that $\sum_{k=M}^{\infty} |y^R_k(k, \lambda)|^2 < \infty$), is representable in the form

$$y^R(k, \lambda) = m^R(\lambda) P_k(\lambda) - \frac{Q_k(\lambda)}{b_{-1}}, \quad k \leq N - 1.$$

On the other hand, the following statement is true:

Proposition. For $k \leq N - 2$ the solutions $y^R(k, \lambda)$ are equal to

$$\tilde{y}_k = \frac{z^k}{c_k} (1 + z R_k(z)), \quad k \leq N - 2,$$
to within a factor, not depending on \( k \), with \( c_k = b_kb_{k+1} \ldots b_{N-2} \), and \( R_k(z) \) is a polynomial in \( z \) (not of degree \( k \)). (We remind that we parametrized \( \lambda \) by the variable \( z \), connected with \( \lambda \) by relation \( z + z^{-1} = \lambda \).

**Proof.** We use the induction in \( k = N - 2, N - 3, N - 4, \ldots \). The function \( \check{y}_{N-2} \) satisfies the equation

\[
 b_{N-2} \check{y}_{N-2} + (a_{N-1} - (z + z^{-1})) \check{y}_{N-1} + \check{y}_N = 0,
\]

from where

\[
 \check{y}_{N-2} = \frac{1}{b_{N-2}} ((z + z^{-1}) z^{N-1} - a_N z^{N-1} - z^N) = \frac{z^{N-2}}{c_{N-2}} (1 + z R_{N-2}(z)),
\]

where the polynomial \( R_{N-2}(z) = -a_N \), and \( c_{N-2} = b_{N-2} \). Thus, for \( k = N - 2 \) the proposition is verified.

Having the statement for \( k \), let us prove it for \( k - 1, k \leq N - 2 \). For \( k \leq N - 2 \) the functions \( \check{y}_k \) satisfy the equation (0.5), so

\[
 \check{y}_{k-1} = \frac{1}{b_{k-1}} \{(z + z^{-1}) - ak \} \frac{k}{ck} (1 + z R_k(z)) - b_k \frac{k+1}{ck+1} (1 + z R_{k+1}(z)) \}
\]

where \( R_{k-1}(z) \) is a polynomial in \( z \), as was to be proved.

Now we find the normalized Weyl solution \( y_k = \alpha(z) \check{y}_k \), comparing the result of the proposition with (1.5) for \( k = 0 \) and \( k = -1 \):

\[
 y_R^0 (0, \lambda) = \alpha(z) \check{y}_0 = m_R P_0 - \frac{Q_0}{b_{-1}},
\]

\[
 y_R^1 (-1, \lambda) = \alpha(z) \check{y}_{-1} = m_R P_{-1} - \frac{Q_{-1}}{b_{-1}}.
\]

Taking into account the initial data for \( P_k \) and \( Q_k \) and the proposition above, we rewrite it in the form

\[
 \alpha(z) \frac{1}{c_0} (1 + z R_0(z)) = m_R,
\]

\[
 \alpha(z) \frac{1}{c_{-1}} (1 + z R_{-1}(z)) = -\frac{1}{b_{-1}}.
\]

Thus,

\[
 \alpha(z) = -\frac{c_{-1}}{b_{-1} (1 + z R_{-1}(z))},
\]

\[
 m_R^R(z + z^{-1}) = -\frac{c_{-1}}{b_{-1} c_0} \frac{1 + z R_0(z)}{1 + z R_{-1}(z)} = -\frac{1}{b_{-1}} \frac{1 + z R_0(z)}{1 + z R_{-1}(z)}
\]

and \( y_R^R(k, \lambda) \) satisfy for \( k \leq N - 2 \) the equality

\[
 y_R^R(k, \lambda) = -\frac{c_{-1} z^{k+1}}{b_{-1} c_k} \frac{1 + z R_k(z)}{1 + z R_{-1}(z)} = -\frac{h_k}{b_{-1}} z^{k+1} \frac{1 + z R_k(z)}{1 + z R_{-1}(z)}, \quad k \leq N - 2,
\]

where \( h_k = \frac{c_k}{b_{-1}} \) are defined by (0.15), and \( z + z^{-1} = \lambda, \ |z| < 1 \).

The functions of the form \( \frac{1 + z R_k(z)}{1 + z R_{-1}(z)} \) are holomorphic and equal to 1 at the point \( z = 0 \). Thus, the last equality implies the asymptotic formulas

\[
 -\frac{h_k}{b_{-1}} z^{-(k+1)} \check{y}_R^R(k, z + z^{-1}) \to 1, \quad z \to 0, \quad k \leq N - 2.
\]

(1.6)

2. Let \( J_N^R \) be the semi-infinite (to the right) matrix that is contained in \( J_N \) and starts with the element \( a_0 \). We denote the spectral functions of the matrices \( J_N^R \) and \( J_N^R \) by \( \rho^R(\tau) \) \( \rho_N^R(\tau) \). They are related with the Weyl functions \( m_R^R(\lambda) \) \( m_R^N(\lambda) \) of the matrices \( J_N^R \), \( J_N^N \) by the transformation

\[
 m_R^R(\lambda) = \int_{-\infty}^{\infty} \frac{dp^R(\tau)}{\tau - \lambda}, \quad m_R^N(\lambda) = \int_{-\infty}^{\infty} \frac{dp^N(\tau)}{\tau - \lambda}.
\]

(1.7)

Since the principal corner \( N \times N \) minors of the matrices \( J_N^R \) and \( J_N^R \) are the same, their spectral functions \( \rho^R(\tau) \) and \( \rho_N^R(\tau) \) have the same first \( 2N \) moments (the integrals of the functions \( 1, \tau, \tau^2, \ldots, \tau^{2N-1} \)). Taking this into account, after decomposing \( \frac{1}{1 - \rho^R(\tau)} \) into the sum of geometric progression, we estimate

\[
 |m_R^R(\lambda) - m_R^N(\lambda)| = \left| \int_{-\infty}^{\infty} \frac{d(\rho_R - \rho_N^R)(\tau)}{\tau - \lambda} \right|.
\]
for which \( z \) points all the integer the reason for which we had to prove theorem 1, which allowed us to obtain the asymptotics (1.3), (1.4) for same asymptotic behavior:

\[
\psi_k(z) = a_k z^k + \cdots
\]

is a simple consequence of the well-known equalities for the leading coefficients of the polynomials \( \psi_k \).

At the same time, it is not possible to obtain in this way formula (1.10) for positive conditions C) and D) that in some neighborhood of the closed set \( \Omega \) some auxiliary facts, which are consequence of conditions A)–E). First, we remark that it follows from the estimate (1.8), we have

\[
\max_{|z| \leq N-1} |\psi^R(k, z + z^{-1}) - \psi^R_N(k, z + z^{-1})| = \max_{|z| \leq N-1} |(m^R(z + z^{-1}) - m^R_R(z + z^{-1}))P_k(z + z^{-1})| \leq |z|^{2N} \left| \frac{1}{|z|^{N-2}} K_N(z) \right| = |z|^{N+2} K_N(z), \tag{1.9}\]

where \( K_N(z) \) is a certain function, bounded in the neighborhood of the point \( z = 0 \).

4. Comparing estimate (1.9) with asymptotics (1.6), we see that the functions \( \psi^R(k, z + z^{-1}) \) have the same asymptotic behavior:

\[
-\frac{b_{k-1}}{h_k} z^{-(k+1)} \psi^R(k, z + z^{-1}) \to 1, \quad z \to 0. \tag{1.10}\]

5. Analogously, it is easy to obtain the asymptotics of the solutions \( \phi^L(k, \lambda) \). After using transformation (0.10), we have for \( \psi(k, z) \) asymptotic formulas (1.3) and (1.4). Asymptotics (1.1) and (1.2) can be easily obtained, for example, from the integral representation (0.8) of the Weyl function.

We observe that for negative \( k \) asymptotic formula (1.10) for the increasing as \( \lambda \to \infty \) solution \( \varphi^R(k, \lambda) \) is a simple consequence of the well-known equalities for the leading coefficients of the polynomials \( Q_k(\lambda) \).

At the same time, it is not possible to obtain in this way formula (1.10) for positive \( k \), because the solutions \( \varphi^R(k, \lambda), k > 0 \), diminsh as \( \lambda \to \infty \). The same situation takes place with \( \varphi^L(k, \lambda) \) for negative \( k \). That is the reason for which we had to prove theorem 1, which allowed us to obtain the asymptotics (1.3), (1.4) for all the integer \( k \).

We also precize that the function \( \psi(k, z) \) and \( n(z) \) are holomorphic at such points inside the unit disk for which \( z + z^{-1} \) do not belong to the spectrum of the semi-infinite matrix \( J^R \) (besides \( z = 0 \)) and at such points \( z \) outside the unit disk for which \( z + z^{-1} \) do not belong to the spectrum of the semi-infinite matrix \( J^L \) (starting with the element \( a_{-2} \)). Further, the functions \( h_k z^{-(k+1)} \psi(k, z) \) and \( \psi(k, z) \) are holomorphic in a certain deleted neighborhood of the points \( z = 0, z = \infty \) and have a finite limit at these points. So, they are holomorphic at the points \( z = 0 \) and \( z = \infty \).

In order to obtain factorization (0.25) with the necessary properties of the function \( R(z) \) we will need some auxiliary facts, which are consequence of conditions A)–E). First, we remark that it follows from the conditions C) and D) that in some neighborhood of the closed set \( \Omega_2[-2, 2] \) we have the inequality

\[
-\pi < \eta^R(\alpha) - \eta^L(\alpha) < \pi.
\]
Hence, in condition D) we can constrict the intervals \( \delta_i \) so that this inequality holds under these intervals. Further, in condition D) we can select a finite subcovering from the covering of the compact set \( \tilde{\Omega} \setminus [-2, 2] \) with the open intervals \( \delta_i \). Thus, the condition D) can be replaced with a stronger one:

D') The set \( \tilde{\Omega} \setminus [-2, 2] \) can be covered with a finite system of mutually disjoint intervals \( \delta_i \), under each of which the following inequalities are true:

\[
-\pi < \eta^R(\alpha) - \eta^L(\alpha) < \pi, \tag{1.11}
\]

\[
\text{ess sup}_{\alpha \in \delta_i} \eta^R(\alpha) - \text{ess inf}_{\alpha \in \delta_i} \eta^R(\alpha) < \pi, \tag{1.12'}
\]

\[
\text{ess sup}_{\alpha \in \delta_i} \eta^L(\alpha) - \text{ess inf}_{\alpha \in \delta_i} \eta^L(\alpha) < \pi. \tag{1.12''}
\]

Let

\[
\tilde{\Delta} = R \setminus (\tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup [-2, 2]) = \cup_{k} \tilde{\Delta}_k,
\]

where \( \tilde{\Delta}_k = (\tilde{\alpha}_k, \tilde{\beta}_k) \) are mutually disjoint intervals.

Let us also define the function

\[
M(\lambda) = b_{-1} m^R(\lambda) - \frac{1}{b_{-1} m^L(\lambda)}.
\]

The two following lemmas clarify the behavior of the arguments

\[
\eta(\tau) := \arg M(\tau + i0),
\]

\[
\eta^R(\tau) := \arg m^R(\tau),
\]

\[
\eta^L(\tau) := \arg \left(-\frac{1}{b_{-1} m^L(\lambda)}\right).
\]

on the set \( \tilde{\Delta} \setminus \tilde{\Omega}^2 \).

**Lemma 1.1.** The arguments \( \eta^R(\tau), \eta(\tau) \) of the functions \( m^R(\tau + i0), M(\tau + i0) \) take on the same constant value, equal either to 0 or \( \pi \), on every set \( (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2 \).

**Proof.** We see that the set \( \tilde{\Delta} \) only include the part \( \tilde{\Omega}^2 \) of the set \( \tilde{\Omega} \). Remind, that the set \( \tilde{\Omega}^2 \) lies in the union of the intervals \( \delta_i \) (not to mix \( \delta_i \) with \( \tilde{\Delta}_k \)), i.e., \( \delta_i \setminus \tilde{\Omega}^2 \) is not empty. Since the set \( (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2 \) lies in the complement of the set \( \tilde{\Omega} \), the functions \( m^R(\lambda) \) and \( M(\lambda) \) are holomorphic and real in the points of this sets, and their arguments can only take value 0 or \( \pi \). Condition D') implies that the oscillation of the arguments \( \eta^R(\alpha) \) of \( m^R(\alpha + i0) \) on the interval \( \delta_i \) is less, then \( \pi \); and it follows that the range of the function \( \eta^R(\alpha) \) on the set \( (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2 \) cannot contain both 0 and \( \pi \). Therefore, the function \( \eta^R(\alpha) \) is constant, equal to 0 or \( \pi \) on the whole set \( (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2 \). Further, since the values of \( \eta^R(\alpha) - \eta^L(\alpha) \) on this set are multiples of \( \pi \) and because of the condition D'), the inequalities (1.11) hold there, we have

\[
\eta^R(\alpha) - \eta^L(\alpha) = 0, \quad \alpha \in (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2. \tag{1.13}
\]

According to the definition of the function \( M(\lambda) \),

\[
M(\lambda) = m^R(\lambda) \left(b_{-1} + \frac{-1}{b_{-1} m^R(\lambda) m^L(\lambda)}\right).
\]

So,

\[
\arg M(\tau + i0) = \arg m^R(\tau + i0) + \arg \left(b_{-1} + \frac{-1}{b_{-1} m^R(\lambda) m^L(\lambda)}\right).
\]

But the function \( b_{-1} m^R(\lambda) m^L(\lambda) \) is regular and real on the set \( (\delta_i \cap \tilde{\Delta}) \setminus \tilde{\Omega}^2 \); its argument equals \( \eta^L(\tau) - \eta^R(\tau) \) on this set. On the other hand, we have from (1.13) that \( \frac{-1}{b_{-1} m^R(\lambda) m^L(\lambda)} > 0 \) for \( P(\tau, \eta^L - \eta^R) > 0 \), and \( b_{-1} + \frac{-1}{b_{-1} m^R(\lambda) m^L(\lambda)} > 0 \), too. Hence, \( \eta^R(\tau) = \eta(\tau) \) on this set.

**Lemma 1.2.** Each interval \( (\tilde{\alpha}_k, \tilde{\beta}_k) = \tilde{\Delta}_k \) splits into two pieces

\[
\tilde{\Delta}_k = (\tilde{\alpha}_k, \tilde{\beta}_k), \quad \tilde{\Delta}^+_k = (\tilde{\varphi}_k, \tilde{\beta}_k),
\]

so that

\[
M(\alpha) < 0, \quad \alpha \in \tilde{\Delta}_k \setminus \tilde{\Omega}^2, \tag{1.15'}
\]

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\[ M(\alpha) > 0 \quad \alpha \in \tilde{\Delta}_k^1 \setminus \tilde{\Omega}_2^1. \]  

(One of the intervals \( \tilde{\Delta}_k^1, \tilde{\Delta}_k^+ \) may be empty.)

Remind that \( M(\lambda) \) is holomorphic on \( \tilde{\Delta} \tilde{\Omega}_2^1 \); so we can write \( M(\alpha) = \alpha \in \tilde{\Delta}_k^1 \setminus \tilde{\Omega}_2^1 \).

Proof. If the interval \( \tilde{\Delta}_k \) contains no point of \( \tilde{\Omega}_2^1 \), then it lies in the complement of \( \tilde{\Omega} = \tilde{\Omega}_R \cup \tilde{\Omega}_L \) and the function \( M(\alpha) \) is real and increasing. Hence in this interval there is at most one point, where \( M(\alpha) \) changes its sign from \(-\) to \(+\). Let us denote this point by \( \tilde{\varphi}_k \) (if \( M(\alpha) \) is positive, resp., negative, on the whole interval, we set \( \tilde{\varphi}_k = \tilde{\alpha}_k \) or \( \tilde{\beta}_k \)), and obtain the intervals (1.14), on which the inequalities (1.15) hold.

If the interval \( \tilde{\Delta}_k \) contains points of \( \tilde{\Omega}_2^1 \) then, according to condition \( D' \), the set \( \tilde{\Delta}_k \setminus \tilde{\Omega}_2^1 \) is covered by a finite number of mutually disjoint intervals \( \tilde{\delta}_i \cap \tilde{\Delta}_k = (\alpha_k^{(i)}, \beta_k^{(i)}) \), on which the inequalities (1.11), (1.12) are fulfilled. Obviously, these intervals can be labelled so that

\[ \tilde{\alpha}_k = \beta_k^{(0)} \leq \alpha_k^{(1)} < \beta_k^{(1)} \leq \beta_k^{(2)} < \ldots < \alpha_k^{(n)} \leq \beta_k^{(n)} \leq \alpha_k^{(n+1)} = \tilde{\beta}_k \]

(for the sake of convenience the endpoints \( \tilde{\alpha}_k, \tilde{\beta}_k \) of \( \tilde{\Delta}_k \) are denoted by \( \beta_k^{(0)}, \alpha_k^{(n+1)} \)). According to Lemma 1.1 the argument \( \eta(\alpha) \) of the function \( M(\alpha + i0) \) is constant, equal to 0 or to \( \pi \), on each set

\[ \Phi_p = (\alpha_k^{(p)} \beta_k^{(p)}) \setminus \tilde{\Omega}_2^1, \quad 1 \leq p \leq n. \]

Therefore the function \( M(\alpha + i0) = M(\alpha) \) is real, regular, of constant sign, on each set \( \Phi_p \). Let us denote by \( \Phi_{p^+} \) (resp. \( \Phi_{p^-} \)) the set with the smallest (resp. greatest) number \( p^+ \) (resp. \( p^- \)), at which the function \( M(\alpha) \) is positive (resp. negative) and \( \Phi_0 = (\beta_k^{(0)}, \alpha_k^{(1)}) \) (resp. \( \Phi_{n+1} = (\beta_k^{(n)}, \alpha_k^{(n+1)}) \)). If this function is positive (resp. negative) on the set \( \Phi_p \) \((1 \leq p \leq n)\),

Since the function \( M(\alpha) \) is regular and grows monotonically on the segment \([\beta_k^{(p+1)}, \alpha_k^{(p+1)}]\) (resp. \([\beta_k^{(p-1)}, \alpha_k^{(p-1)}]\)) and in its neighborhood, it remains positive (negative) on this segment and on the next (previous) set \( \Phi_{p+1} \) (resp. \( \Phi_{p-1} \)). Hence, \( p^+ = p^- + 1 \) and the function \( M(\alpha) \) is positive on the set \([\alpha_k^{(p^+)} \tilde{\beta}_k] \setminus \tilde{\Omega}_2^1 \) and negative on the set \([\tilde{\alpha}_k \beta_k^{(p^+)}]\). Due to the monotonicity on the segment \([\beta_k^{(p-1)}, \alpha_k^{(p+1)}]\) there is only one point \( \tilde{\varphi}_k \), at which the function \( M(\alpha) \) changes sign from \(-\) to \(+\). Setting \( \tilde{\varphi}_k = \tilde{\alpha}_k \) if \( p^+ = 0 \) (\( \tilde{\varphi}_k = \tilde{\beta}_k \), if \( p^- = n + 1 \)), we obtain the intervals (1.14), on which (1.15) are fulfilled.

2. Factorization and choice of the function \( R(z) \)

1. Let us define the function

\[ N(z) = n(z) - n(z^{-1}). \]

This function is connected with the resolvent matrix \( R(m, n, \lambda) \) of the matrix \( J \) in the following way (see, for example, [10]): the two diagonal elements of this matrix are equal to

\[ R(-1, -1, \lambda) |_{\lambda = z + z^{-1}} = R(-1, -1, z + z^{-1}) = \frac{1}{b_{-1} N(z)}, \]

\[ R(0, 0, \lambda) |_{\lambda = z + z^{-1}} = R(0, 0, z + z^{-1}) = \frac{n(z) n(z^{-1})}{b_{-1} N(z)}. \]

The goal of this section is to factorize in a suitable way the function \( \frac{\beta_{z^{-1}}}{N(z)} \) in form (0.25). Such a factorization is presented in more general case in [15].

The functions \( b_{-1} m^R(\lambda) \) and \( \frac{1}{b_{-1} m^L(\lambda)} \) are holomorphic in the upper half-plane and have positive imaginary part in it. Thus, (see, e.g., [14]), taking into account their asymptotic behavior (see (1.1), (1.2)), we can represent these functions in the form:

\[ b_{-1} m^R(\lambda) = \int_{-\infty}^{\infty} \frac{d \rho_R(\tau)}{\rho_R(\tau - \lambda)}, \quad -\frac{1}{b_{-1} m^L(\lambda)} = \frac{\lambda}{b_{-1}} + \beta + \int_{-\infty}^{\infty} \frac{d \rho_L(\tau)}{\rho_L(\tau - \lambda)}, \]

where, according to the Stieltjes-Perron inversion formula,

\[ \rho_R(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\lambda}{\rho_R(\tau + i \varepsilon)} d\tau, \quad \rho_L(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{-1}{\rho_L(\tau + i \varepsilon)} d\tau, \]

(2.4)
the number $\beta \in \mathbb{R}$, and $\rho_R(\lambda)$ and $\rho_L(\lambda)$ are constant in the infinity (the Weyl functions of the matrices $J^R$ and $J^L$ are regular out of their spectrum, which is bounded because of the boundedness of the matrices).

On the parts of the real line where the two functions $\rho_R(\tau)$ and $\rho_L(\tau)$ are absolutely continuous and $\rho_R'(\tau)$ and $\rho_L'(\tau)$ do not vanish (it is shown in the end of this section that it is the set $\Omega_2$), we introduce

$$
\mu(\tau) := \sqrt{\frac{\rho_R'(\tau)}{\rho_L'(\tau)}}.
$$

2. Notation 1. Let us define on the real line the involutive map $V$ of symmetry with respect to the unit circle:

$$
V(t) = t^{-1}, \quad t \in \mathbb{R}\setminus\{0\}.
$$

For a set $A \subset \mathbb{R}\setminus\{0\}$ and for a function $\rho(t)$, defined on $\mathbb{R}\setminus\{0\}$, we define

$$
V(A) = \{t \mid t^{-1} \in A\},
$$

$$
V(\rho)(t) = \rho(t^{-1}).
$$

Notation 2. Let us introduce

$$
P(\gamma, \gamma) = \exp\left\{\frac{1}{\pi} \int_{-\infty}^{\infty} \gamma(t)\left(\frac{1}{t - \gamma} - \frac{t}{1 + t^2}\right)dt\right\},
$$

$$
\hat{P}(\gamma, \hat{\gamma}) = \exp\left\{-\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\theta} + \gamma}{e^{\theta} - \hat{\gamma}} \hat{\gamma}(\theta)\,d\theta\right\},
$$

where $\gamma(t) = \gamma(t)$, $-\infty < t < \infty$, and $\hat{\gamma}(\theta) = \hat{\gamma}(\theta)$, $-\pi < \theta < \pi$, are bounded and measurable functions. The function $P(\gamma, \gamma)$ is defined and holomorphic, at least, for nonreal $z$, and $\hat{P}(\gamma, \hat{\gamma})$ is defined and holomorphic inside and outside the unit disk. It is easy to obtain from the formulas of Plemelj-Sokhotsky the following equalities for the limit values of the functions $P(\gamma, \gamma)$ and $\hat{P}(\gamma, \hat{\gamma})$ on the real line and the unit circle, resp.:

$$
\arg P^+(t, \gamma) = -\arg P^-(t, \gamma) = \gamma(t), \quad -\infty < t < \infty,
$$

$$
|P^+(t, \gamma)| = |P^-(t, \gamma)|, \quad -\infty < t < \infty,
$$

$$
\arg \hat{P}^+(e^{i\theta}, \hat{\gamma}) = -\arg \hat{P}^-(e^{i\theta}, \hat{\gamma}) = \hat{\gamma}(\theta), \quad -\pi < \theta < \pi,
$$

$$
|\hat{P}^+(e^{i\theta}, \hat{\gamma})| = |\hat{P}^-(e^{i\theta}, \hat{\gamma})|, \quad -\pi < \theta < \pi.
$$

Besides, it is easy to verify that the following lemma is true:

Lemma 2.1. The functions $P(z, \gamma)$ and $\hat{P}(z, \hat{\gamma})$ in their domain of holomorphy satisfy the equalities:

$$
P(z, \gamma_1 + \gamma_2) = P(z, \gamma_1)P(z, \gamma_2),
$$

$$
P(\overline{z}, \gamma) = \overline{P(z, \gamma)},
$$

$$
P(z^{-1}, \gamma) = P(z, -V(\gamma)),
$$

$$
\hat{P}(z, \gamma_1 + \gamma_2) = \hat{P}(z, \gamma_1)\hat{P}(z, \gamma_2),
$$

If the function $\hat{\gamma}(\theta) = -\gamma(-\theta)$ is odd, then

$$
\hat{P}(z^{-1}, \hat{\gamma}) = \hat{P}(z, \hat{\gamma}).
$$

The function $M(\lambda)$, being a function with positive imaginary part in the upper half-plane, can be represented in the multiplicative form

$$
M(\lambda) = b_{-1}m_R(\lambda) - \frac{1}{b_{-1}m_L(\lambda)} = C_1 P(\lambda, \eta),
$$

where

$$
\eta(\tau) = \lim_{\varepsilon \downarrow 0} \arg M(\tau + i\varepsilon)
$$
and the constant $C_1 > 0$. (It is easy to obtain this representation by considering the logarithm of these functions, which also have positive imaginary part in the upper half-plane, and then by representing the logarithm in the integral form of the type (2.3) (see, for example, [16]).)

3. We remind that we defined in the plane of the spectral parameter $\lambda$ the sets $\tilde{\Omega}^R, \tilde{\Omega}^L$ as the supports of the measures $d\rho_R(\tau), d\rho_L(\tau)$. This means that $\tilde{\Omega}^R, \tilde{\Omega}^L$ are the sets on which the functions $m^R(z), \frac{1}{m^R(z)}$ have singularities, i.e. are not regular. Then we introduced

$$\tilde{\Omega}_2 = \tilde{\Omega}^R \cap \tilde{\Omega}^L,$$

$$\tilde{\Omega}_1 = (\tilde{\Omega}^R \setminus \tilde{\Omega}_2) \cup (\tilde{\Omega}^L \setminus \tilde{\Omega}_2)$$

and divided $\tilde{\Omega}_2$ into $\tilde{\Omega}^2_2$ and $\tilde{\Omega}^2_1$. Here $\tilde{\Omega}^2_2$ is the finite or empty set of the common poles of the functions $m^R(z)$ and $\frac{1}{m^R(z)}$ ($\tilde{\Omega}^2_2 \subset \tilde{\Omega}_2$), and $[-2, 2] \subset \tilde{\Omega}^2_1 = \tilde{\Omega}_2 \setminus \tilde{\Omega}^2_2$. It will be seen from lemma 2.3 that the set $\tilde{\Omega}^2_2$ (including the segment $[-2, 2]$) belongs to the absolutely continuous spectrum of multiplicity 2 of the semi-infinite matrices $J^R$ and $J^L$.

Now we are going to find the equivalent of these sets in the $z$-plane ($z + z^{-1} = \lambda$). Let us define on the real line in the $z$-plane

$$\Omega = \{ t \in \mathbb{R} \mid |t| < 1, t + t^{-1} \in \tilde{\Omega}^R \} \cup \{ t \in \mathbb{R} \mid |t| > 1, t + t^{-1} \in \tilde{\Omega}^L \}.$$ 

According to this definition, $\Omega$ is the set of the points on the real line (except $\pm 1$) at which the function $n(z)$ is not holomorphic. Besides the set $\Omega$, the function $n(z)$ has singularities on the unit circle $T$.

Let us divide $\Omega$ into its symmetric and asymmetric (with respect to the unit circle) parts:

$$\Omega_2 = \Omega \cap V(\Omega) = \{ t \in \mathbb{R} \mid t \in \Omega, t^{-1} \in \Omega \},$$

$$\Omega_1 = \Omega \setminus V(\Omega) = \{ t \in \mathbb{R} \mid t \in \Omega, t^{-1} \notin \Omega \},$$

where the map $V$ is defined by formula (2.6). It may be seen that these sets are the preimages of the sets $\tilde{\Omega}_2$ and $\tilde{\Omega}_1$ with respect to the map $z + z^{-1} = \lambda$:

$$\Omega_2 = \{ t \in \mathbb{R} \mid t \neq \pm 1, t + t^{-1} \in \tilde{\Omega}_2 \},$$

$$\Omega_1 = \{ t \in \mathbb{R} \mid t + t^{-1} \in \tilde{\Omega}_1 \}.$$ 

We also define the sets $\Omega^2_2$ and $\Omega^2_1$ as the preimages of $\tilde{\Omega}^2_2$ and $\tilde{\Omega}^2_1$ with respect to the map $\lambda = z + z^{-1}$:

$$\Omega^2_2 = \{ t \in \mathbb{R} \mid t + t^{-1} \in \tilde{\Omega}^2_2 \},$$

$$\Omega^2_1 = \{ t \in \mathbb{R} \mid t \neq \pm 1, t + t^{-1} \in \tilde{\Omega}^2_1 \}.$$ 

Thus, the set $\Omega$ is divided into three sets:

$$\Omega = \Omega_2 \cup \Omega^2_2 \cup \Omega^2_1.$$ 

It follows from condition A) that the three sets have positive mutual distances and that $\Omega^2_2$ is the finite or empty set of the common poles of $n(z)$ and $n(z^{-1})$ (they are all simple). Since $\Omega_1$ is the asymmetric part of the set $\Omega$, the function $n(z)$ is holomorphic on the set $V(\Omega)$.

Let

$$\Delta = \mathbb{R} \setminus (-1, 1) \cup \Omega_2 \cup V(\Omega_1) \cup \Omega_2^2) = \cup_k \Delta_k,$$  

(2.20)

where $\Delta_k = (\alpha_k, \beta_k)$ are mutually disjoint intervals. (The set $\Omega_2^2$, which corresponds to the absolutely continuous spectrum except the segment $[-2, 2]$, lies in the intervals $\Delta_k$.) According to the definition, all $|\alpha_k| \geq 1$ and $|\beta_k| \geq 1$.

The following fact is an immediate consequence of lemma 1.2:

**Lemma 2.2.** Each interval $(\alpha_k, \beta_k) = \Delta_k$ splits into two pieces:

$$\Delta_k^{(1)} = (\alpha_k, \varphi_k), \quad \Delta_k^{(2)} = (\varphi_k, \beta_k),$$ 

(2.21)

so that

$$N(\alpha) < 0 \quad \alpha \in \Delta_k^{(1)} \setminus \Omega_2^2,$$  

(2.22')

$$N(\alpha) > 0 \quad \alpha \in \Delta_k^{(2)} \setminus \Omega_2^2.$$  

(2.22'')

(One of the intervals $\Delta_k^{(1)}, \Delta_k^{(2)}$ may be empty.)

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The points $\tilde{\alpha}_k = \alpha_k + \alpha_k^{-1}$ and $\tilde{\varphi}_k = \varphi_k + \varphi_k^{-1}$ from the plane of the spectral parameter correspond to the pairs of points $\{\alpha_k, \alpha_k^{-1}\}$ and $\{\varphi_k, \varphi_k^{-1}\}$ in $z$-plane. To make the factorization with the properties we will need later on, we have to choose one number $\alpha_k^*$ (resp. $\varphi_k^*$) in each pair, according to the following rule: put

$$\alpha_k^* = \begin{cases} \alpha_k, & \text{if } \alpha_k \notin V(\Omega_1), \\
\alpha_k^{-1}, & \text{if } \alpha_k \in V(\Omega_1). \end{cases}$$

(2.23)

If both endpoints of an interval $(\alpha_k, \beta_k)$ belong to the set $\Omega_1$ or $V(\Omega_1)$, we set

$$\varphi_k^* = \begin{cases} \varphi_k, & \text{if } \alpha_k, \beta_k \in \Omega_1, \\
\varphi_k^{-1}, & \text{if } \alpha_k, \beta_k \in V(\Omega_1). \end{cases}$$

(2.24')

and in all other cases (there are only finitely many, which is a corollary of the disjointedness of $\Omega_1$ and $V(\Omega_1)$) we set

$$\varphi_k^* = \begin{cases} \varphi_k, & \text{if } \varphi_k \notin V(\Omega_1), \\
\varphi_k^{-1}, & \text{if } \varphi_k \in V(\Omega_1). \end{cases}$$

(2.24'')

Remark that for some $k, j$ $\alpha_k = -\infty$ and (because of the condition $E$) $\beta_j = \varphi_j = -1$. We will suppose that the intervals $\Delta_k$ are numbered so that $\Delta_k^{(1)} = (-\infty, \varphi_0)$ and $\Delta_1 = (\alpha_1, -1)$. (If $\beta_0 = -1$, we put $\alpha_1^* = \varphi_0^* = \varphi_1^* = -1$, and than the final formula also cover this case.)

Let

$$\Delta^{(1)} = \cup_k \Delta_k^{(1)}.$$  

We denote by $\Phi = \{\varphi_k^*\}$ the set of all $\varphi_k^*$ except the point $-1$. We also denote by $\chi_1(t), \chi_2^*(t)$ and $\chi_0(t)$ the indicators of the sets $\Omega_1, \Omega_2^*$ and $\Delta^{(1)} \cup V(\Delta^{(1)})$.

The key point of the inverse problem is presented by

**Theorem 2 (factorization).** The function $(z - z^{-1})N(z)^{-1} = (z - z^{-1})(n(z) - n(z^{-1}))^{-1}$ in its domain of holomorphy is represented in the form of the product of two functions, which are holomorphic in this domain:

$$\frac{z - z^{-1}}{N(z)} = R(z) R(z^{-1}),$$

(2.25)

$$R(z) = CR_0(z) R_1(z) R_2(z) R_3(z) R_\mu(z),$$

(2.26)

where

$$R_0(z) = \frac{z - \alpha_1}{z - \varphi_0} \prod_{k \geq 2} \frac{z - \alpha_k^*}{z - \varphi_k^*},$$

(2.27)

$$R_1(z) = P(z, -\gamma_1), \quad R_2(z) = P(z, -\gamma_2),$$

(2.28)

$$\gamma_1(t) = \chi_1(t)s(t)\eta(t + t^{-1}),$$

(2.29)

$$\gamma_2(t) = \frac{1}{2} \chi_2^*(t)s(t)(\eta(t + t^{-1}) - \chi_0(t)\pi),$$

(2.30)

$$s(t) = \begin{cases} 1, & |t| > 1, \\
-1, & |t| < 1, \end{cases}$$

(2.31)

the constant $C > 0$, and the numbers $\alpha_k^*$ and $\varphi_k^*$ are defined by equalities (2.23), (2.24') and (2.24'') (with $\eta(\tau)$, defined by formula (2.19)),

$$R_3(z) = \tilde{P}(z, -\frac{\gamma_0}{2}),$$

(2.32)

$$\tilde{P}_3(\theta) = \begin{cases} -\eta(e^{i\theta} + e^{-i\theta}) + \frac{\pi}{2}, & 0 < \theta < \pi, \\
\eta(e^{i\theta} + e^{-i\theta}) - \frac{\pi}{2}, & -\pi < \theta < 0, \end{cases}$$

(2.33)

$$R_\mu(z) = \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha}}{e^{i\alpha} - z} \ln \mu(e^{i\alpha} + e^{-i\alpha}) \, d\alpha \right\}$$

(2.34)

(with $\mu(\tau)$, defined by formula (2.5)).

In addition, the limit values of the function $R(z)$ on the unit circle $T$ satisfy the equality

$$R^+ (\xi) R^+(\xi^{-1}) |\text{Im} n^+(\xi)| = R^- (\xi) R^- (\xi^{-1}) |\text{Im} n^- (\xi)|, \quad |\xi| = 1.$$

(2.35)
The function $R(z)$ may have singularities only at such points $z$, that $z + z^{-1}$ belongs to the spectrum of the matrix $J$. If $z + z^{-1}$ belongs to $\Omega^1$, then $R(z)$ may have singularities at both points $z, z^{-1}$, and if it belongs to the simple spectrum, then it only have a singularity in one of these points. Moreover, the functions $R_0(z)$, $R_1(z)$, $R_2(z)$ and $R_3(z)R_\mu(z)$ may have singularities only on the sets $\Phi = \{ \varphi^1 \}, \Omega_1, \Omega^2_2$ and $T$, resp. The sets

$$\Omega_1 \cup \Phi, \quad V(\Omega_1) \cup V(\Phi), \quad \Omega^2_2, \quad \Omega^3_2, \quad T$$

are at positive distances from one another.

Proof. The holomorphy of the functions $R_0(z)$, $R_1(z)$, $R_2(z)$ and $R_3(z)R_\mu(z)$ outside the sets $\Phi = \{ \varphi^1 \}, \Omega_1, \Omega^2_2$ and $T$ is an immediate consequence of their definition and properties (2.9)–(2.12). From property (2.12) we have $|R_{0123}(e^{i\theta})| = |R_{0123}(e^{-i\theta})|$, where

$$R_{0123}(z) := R_0(z)R_1(z)R_2(z)R_3(z).$$

So, in order to show (2.35), we only have to demonstrate that

$$R_\mu^+(\xi)R_\mu^-(\xi^{-1}) \rho_R(\xi + \xi^{-1}) = R_\mu^-(\xi)R_\mu^+(\xi^{-1}) \rho_L(\xi + \xi^{-1}), \quad |\xi| = 1.$$

(We remind that, according to (2.4), taking into account the absolute continuity of $\rho_R(\tau)$, $\rho_L(\tau)$ on the set $[-2, 2] \subset \Omega^2_2$, we have $\rho_R(\tau) = \frac{i}{\pi} \text{Im } m^R(\tau + i0)$, $\rho_L(\tau) = \frac{1}{\pi} \text{Im } m^L(\tau + i0)$.) In order to prove the last equality it is sufficient to show that

$$\frac{R_\mu^+(e^{i\theta})}{R_\mu^-(e^{i\theta})} = \mu(e^{i\theta} + e^{-i\theta}) = \sqrt{\frac{\rho_R'(e^{i\theta} + e^{-i\theta})}{\rho_L'(e^{i\theta} + e^{-i\theta})}},$$

which is a simple implication of the definition of $R_\mu(z)$ and the Plemelj-Sokhotsky formulas. So, (2.35) is proved. Further, taking into account the evenness of the ratio $\sqrt{\frac{\rho_R'(e^{i\theta} + e^{-i\theta})}{\rho_L'(e^{i\theta} + e^{-i\theta})}}$ (as a function on $\theta$, $-\pi < \theta < \pi$), it is easy to verify from the definition of $R_\mu(z)$, after some elementary transformations, that

$$R_\mu(z)R_\mu(z^{-1}) = \text{const} > 0, \quad |z| \neq 1.$$

Thus, to prove (2.25), it suffice to prove that in the domain if holomorphy of the function $\frac{z - z^{-1}}{N(z)}$, the following equality holds

$$\frac{z - z^{-1}}{N(z)} = 1 R_{0123}(z)R_{0123}(z^{-1}), \quad C_1 > 0.$$

Let us now consider the functions $\frac{z - z^{-1}}{N(z)}$ and $R_{0123}(z)R_{0123}(z^{-1})$. They are holomorphic and do not vanish outside the unit circle $T$ and a certain set of the real line, which is separated from the circle, zero and the infinity. (We concluded this from the asymptotics of $n(z)$ and the definition of $R(z)$.) Moreover, both functions are holomorphic and positive at the infinity and take on positive values under the real line in a neighborhood of zero. Thus, both functions are holomorphic and do not vanish in two simply connected domains on the Riemann sphere, one of which includes the nonreal points inside the unit disk and a neighborhood of zero, and the other one includes the nonreal points outside the disk and a neighborhood of the infinity. Hence, in each one of the two simply connected components we can take the logarithm of these function (so that on the real line in the neighborhood of zero and the infinity, where these functions are positive, it is the principal branch). Let us denote

$$f_1(z) = \ln \left( \frac{z - z^{-1}}{N(z)} \right), \quad f_2(z) = \ln \left( R_{0123}(z)R_{0123}(z^{-1}) \right)$$

and prove that the limit values of $\text{Im } f_1(z)$ and $\text{Im } f_2(z)$ (i.e. the arguments of $\frac{z - z^{-1}}{N(z)}$ and $R_{0123}(z)R_{0123}(z^{-1})$) are the same almost everywhere on the unit circle and the real line.

Generally speaking, the equality of the limits of $\text{Im } f_1^\pm(z)$ and $\text{Im } f_2^\pm(z)$ almost everywhere on the boundary of the domain does not imply that $f_1(z) = f_2(z) + \alpha, \alpha \in \mathbb{R}$. However, in our case, as it is shown in paper [15], the boundedness of these imaginary parts will allow us to use the maximum principle and conclude that $f_1(z)$ and $f_2(z)$ differ by a real constant. Hence, $\frac{z - z^{-1}}{N(z)}$ and $R_{0123}(z)R_{0123}(z^{-1})$ differ by a constant positive factor (it is evident that this factor is the same inside and outside the circle).

The last speculation is not completely rigorous. In paper [15] the factorization theorem is rigorously proved in another way and in more general form. So, now we will restrict ourselves by proving the equality
of \( \text{Im } f_1(z) \) and \( \text{Im } f_2(z) \) on the boundary of the domain of their holomorphy, omitting the consideration of some special cases.

Begin with the circle. First, the function

\[
R_{012}(z) = R_0(z)R_1(z)R_2(z)
\]

is regular and does not vanish on the circle. The expression

\[
R_{012}(\xi)R_{012}(\xi^{-1}) = R_{012}(\xi)R_{012}(\bar{\xi}) = R_{012}(\xi)\overline{R_{012}(\xi)}, \quad |\xi| = 1,
\]

is always real, its argument is precisely equal to zero (not to \(2k\pi\)). Really, for nonreal \( z \) outside the unit circle \( R_{012}(\overline{\xi}) = \overline{R_{012}(z)} \) because of (2.14). Linking with a curve an arbitrary point of the circle with, for example, a big positive \( x \), where \( R_{012}(x)R_{012}(x) > 0 \), we make sure that at any point of this curve \( R_{012}(\overline{\xi})R_{012}(z) > 0 \). Hence, on the unit circle we only have to show the equality of the arguments of the functions \( \frac{z - \overline{z}}{N(z)} \) and \( R_3(z)R_3(z^{-1}) \). But, as it may be easily seen,

\[
\lim_{r \uparrow 1} \arg \left( \frac{z - \overline{z}}{N(z)} \right)_{z = re^{i\theta}} = -\lim_{r \uparrow 1} \arg \left( \frac{z - \overline{z}}{N(z)} \right)_{z = re^{-i\theta}} = -\gamma_3(\theta).
\]

On the other hand, taking into account definition (2.32), property (2.11) and the oddness of \( \gamma_3(\theta) \), we see that the limit values of argument \( R_3(z) \) on the unit circle equal

\[
\arg R^+_3(e^{i\theta}) = -\arg R^-_3(e^{i\theta}) = -\arg R^+_3(e^{-i\theta}) = \arg R^-_3(e^{-i\theta}) = \frac{\gamma_3}{2},
\]

which proves the wanted equality of the arguments on the circle.

Let us now consider the real line. Here we prove the equality of the arguments only under the interval \((1, +\infty)\) (the proof for the intervals \((-\infty, 1)\), \((-1, 0)\) and \((0, 1)\) is analogous). Without loss of generality, we only consider the limit values from above of the real line. The function \( R_3(z) \) is holomorphic outside the circle, and for the real \( x \) we have \( R_3(x)R_3(x^{-1}) > 0 \). That is why, instead of \( R_{0123}(z) \), we only consider \( R_{012}(z) \). First, under the interval \((1, +\infty)\)

\[
\arg N^+(x) = \arg M^+(x + x^{-1}) = \eta(x + x^{-1}),
\]

\[
\arg \left( \frac{x + i0}{} \right)_{N(x + i0)} = -\eta(x + x^{-1}). \tag{2.36}
\]

Let us now consider the function \( R_0(z)R_0(z^{-1}) \). For the sake of simplicity we will assume that \( \alpha^*_1 = \varphi_0^* = -1 \), and that for \( k \geq 2 \) all \( \alpha^*_k > 0 \), \( \varphi_k^* > 0 \), and also either \( 0 < \alpha^*_k, \varphi_k^* < 1 \), or \( 1 < \alpha^*_k, \varphi_k^* \). (As it was already said there are finitely many cases when \( \alpha^*_k, \varphi_k^* \) lie on different sides of 1.) In the first case \( 0 < \varphi_k^* < \alpha_k^* < 1 \) and

\[
\arg \left( \frac{x - i0 - \alpha_k^*}{x + i0 - \varphi_k^*} \right) = \begin{cases} 0, & x < \varphi_k^* \text{ or } x > \alpha_k^*, \\ -\pi, & \varphi_k^* < x < \alpha_k^*. \end{cases}
\]

In the second case \( 1 < \alpha_k^* < \varphi_k^* \) and

\[
\arg \left( \frac{x + i0 - \alpha_k^*}{x + i0 - \varphi_k^*} \right) = \begin{cases} 0, & x < \alpha_k^* \text{ or } x > \varphi_k^*, \\ -\pi, & \alpha_k^* < x < \varphi_k^*. \end{cases}
\]

Hence, for all \( x > 1 \), except \( x = \alpha_k^*, \varphi_k^* \),

\[
\arg \left( R_0(x + i0)R_0((x + i0)^{-1}) \right) = -\pi\chi_0(x). \tag{2.37}
\]

It is also easy to see (from the definition of \( R_1(z) = P(z, -\gamma_1) \)) that on the set \( \Omega_1 \cup V(\Omega_1) \)

\[
\arg \left( R_1(x + i0)R_1((x + i0)^{-1}) \right) = -\gamma_1(x + \gamma_1(x^{-1}) = -\left(\chi_1(x) + \chi_1(x^{-1})\right)\eta(x + x^{-1}), \tag{2.38}
\]

and on the set \( \Omega_2^* \)

\[
\arg \left( R_2(x + i0)R_2((x + i0)^{-1}) \right) = -\gamma_2(x + \gamma_2(x^{-1}) = -\chi_2^*(x)(\eta(x + x^{-1}) - \chi_0(t)\pi). \tag{2.39}
\]
Thus, it follows from (2.36)–(2.39) that on the set
\[ \left( \Omega_1 \cup V(\Omega_1) \cup \Omega_2^0 \right) \cap (1, +\infty) \]
\[
\arg\left( R_{012}(x + i0)R_{012}((x + i0)^{-1}) \right) = \arg\left( \frac{-(x + i0) - (x + i0)^{-1}}{N(x + i0)} \right).
\]

Further, on the set \((1, \infty)\setminus\left( \Omega_1 \cup V(\Omega_1) \cup \Omega_2^0 \cup \Omega_2^\Phi \right)\), according to (2.22'), (2.22''),
\[
\arg\left( \frac{(x + i0) - (x + i0)^{-1}}{N(x + i0)} \right) = -\pi\chi_0(x)
\]
(this argument may not be of the form \(-\pi\chi_0(x) + 2k\pi, k \neq 0\), because in the upper half-plane except the upper half-disk the argument of the function \(\frac{-1}{N(x + i0)}\) is contained in \((-\pi, \pi))

So, we obtained that the imaginary parts of the functions
\[
f_1(z) = \ln \left( \frac{z - \frac{1}{\bar{z}}}{N(z)} \right),
\]
\[
f_2(z) = \ln \left( R_{0123}(z)R_{0123}(\bar{z})^{-1} \right),
\]
which are holomorphic in two domains, are equal almost everywhere on the boundaries of these domains, which completes the proof of the theorem.

\[ \Box \]

4. Let us establish some additional properties of the functions \(\rho_R(\tau)\) and \(\rho_L(\tau)\).

**Lemma 2.3.** The set \(\Omega_2^0\) belongs to the absolutely continuous component of the measures \(d\rho_R(\tau)\) and \(d\rho_L(\tau)\) (i.e. to the intersection of their absolutely continuous components).

**Proof.** We only prove the lemma for \(d\rho_R(\tau)\) (first on the set \(\Omega_2^0 \setminus [-2, 2]\)). For this we will use the following fact: if under the interval \(\delta_1\)
\[
\omega = \text{ess sup}_{\alpha \in \delta_1} \eta_R(\alpha) - \text{ess inf}_{\alpha \in \delta_1} \eta_R(\alpha) < \pi,
\]
then for every \(p < \pi\omega^{-1}\) the function \(m^R(\tau \pm ih) = P(\tau \pm ih, \eta^R)\) for \(h \downarrow 0\) converge in \(L^p\)-norm to a limit \(m^R(\tau \pm i0)\) on every compact subset of the interval \(\delta_1\). (This statement is included in lemma 3.4 in more general form.) Applying to our case this statement and taking into account (1.12), we have that, under a certain interval \(\delta_i \subset \delta_1\) (such that in \(\delta_i \setminus \delta_i\) there is no points of \(\Omega_2^0\)), the function
\[
\rho_R(\lambda) = \frac{b - 1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\lambda} \text{Im} m^R(\tau + i\varepsilon)d\tau,
\]
is absolutely continuous, it has almost everywhere the derivative
\[
\rho_R'(\lambda) = \frac{1}{\pi} \text{Im} m^R(\tau + i0),
\]
which belongs to \(L^1(\delta_i)\), and \(d\rho_R(\tau) = \rho_R'(\tau)d\tau\). Since all the set \(\Omega_2^0 \setminus [-2, 2]\) can be covered with such \(\delta_i\), this means that \(\rho_R(\tau)\) is absolutely continuous on \(\Omega_2^0 \setminus [-2, 2]\).

Let us now show that \(\rho_R(\tau)\) is absolutely continuous in some neighborhood of the set \([-2, 2]\). In fact, conditions B) and E) imply that, for some \(\varepsilon > 0\) the function \(\eta^R(\tau)\) satisfy the condition
\[
\omega = \text{ess sup}_{\alpha} \eta^R(\alpha) - \text{ess inf}_{\alpha} \eta^R(\alpha) < \pi,
\]
individually under the intervals \((-2 - \varepsilon, 0]\) and \([0, 2 + \varepsilon]\). This have as a conclusion that
\[
\rho_R'(\lambda) = \frac{1}{\pi} \text{Im} m^R(\tau + i0)
\]
belongs to \(L^1\) individually on \((-2 - \varepsilon, 0]\) and \([0, 2 + \varepsilon]\). Hence, it belongs to \(L^1\) under all \((-2 - \varepsilon, 2 + \varepsilon)\). \(\Box\)
Lemma 2.4. The expression $\sqrt{\frac{\rho^l_{\lambda}(\tau)}{\rho^R_{\lambda}(\tau)}}$ satisfies the Hölder condition and is bounded away from zero and the infinity under the segment $[-2,2]$.

Proof. Here we only prove the lemma for the right part $[-1,2]$ of the segment $[-2,2]$ (the proof for the left part is analogous). As it is seen from conditions B), C) and E), the functions $\eta^R(\tau)$, $\eta^L(\tau)$ have the same jump at the point 2, i.e.

$$\eta^R(\tau) = \eta^R(\tau) + \delta_2(\tau), \quad \eta^L(\tau) = \eta^L(\tau) + \delta_2(\tau),$$

where

$$\delta_2(\tau) = \begin{cases} \varepsilon_1, & \tau < 2, \\ 0, & \tau \geq 2, \end{cases}$$

with a number $\varepsilon_1 : 0 < \varepsilon_1 < \pi$. Also, $\tilde{\eta}^R(\tau)$, $\tilde{\eta}^L(\tau)$ satisfy the Hölder condition on the set $[-1,2+\varepsilon]$, $\varepsilon > 0$. So,

$$m^R(\lambda) = CP(\lambda, \tilde{\eta}^R(\tau))P(\lambda, \delta_2(\tau)) = CP(\lambda, \tilde{\eta}^R(\tau))(\lambda - 1)\frac{\varepsilon_1}{\pi},$$

(2.40')

$$-\frac{1}{m^R(\lambda)} = CP(\lambda, \tilde{\eta}^L(\tau))P(\lambda, \delta_2(\tau)) = CP(\lambda, \tilde{\eta}^L(\tau))(\lambda - 1)\frac{\varepsilon_1}{\pi},$$

(2.40'')

and $P(\tau + i0, \tilde{\eta}^R)$, $P(\tau + i0, \tilde{\eta}^L)$ are bounded functions satisfying the Hölder condition (see, for example, [17]), which are bounded away from the real line. Besides, it is seen from (2.40'), (2.40'') and the equalities

$$\rho^R_{\mu}(\lambda) = \frac{1}{\pi} \text{Im} \frac{m^R(\tau + i0)}{m^R(\tau)},$$

$$\rho^L_{\mu}(\lambda) = \frac{1}{\pi} \text{Im} \frac{-1}{m^L(\tau + i0)}$$

that

$$\frac{\rho^R_{\mu}(\tau)}{\rho^L_{\mu}(\tau)} = \frac{\text{Im} m^R(\tau + i0)}{\text{Im} m^L(\tau + i0)} = \frac{\text{Im} \left(P(\tau + i0, \tilde{\eta}^R(\tau - 1)\frac{\varepsilon_1}{\pi})\right)}{\text{Im} \left(P(\tau + i0, \tilde{\eta}^L(\tau - 1)\frac{\varepsilon_1}{\pi})\right)} = \frac{\text{Im} \left(P(\tau + i0, \tilde{\eta}^R)e^{\frac{\varepsilon_1}{\pi}}\right)}{\text{Im} \left(P(\tau + i0, \tilde{\eta}^L)e^{\frac{\varepsilon_1}{\pi}}\right)}$$

is a bounded function satisfying the Hölder condition.

In the end of this section we also observe one more property of $R_{\mu}(z)$. It follows from the lemma 2.6 that the expression

$$\text{ln } \mu(e^{i\theta} + e^{-i\theta}) = \text{ln } \sqrt{\frac{\rho^R_{\mu}(e^{i\theta})}{\rho^L_{\mu}(e^{i\theta})}}$$

is bounded and satisfies the Hölder condition. Hence, the following lemma is true:

Lemma 2.5. For $\varepsilon \downarrow 0$ the function

$$R_{\mu}((1 \mp \varepsilon)e^{i\theta}), \quad -\pi \leq \theta \leq \pi,$$

converges uniformly to its limit $R^\pm_{\mu}(e^{i\theta}) \equiv \lim_{\varepsilon \downarrow 0} R_{\mu}((1 \mp \varepsilon)e^{i\theta})$ and is bounded.

3. Deduction of the fundamental equation of the inverse problem

Let us consider for every $k$

$$g(k, z) \equiv \frac{R(z)}{R(\infty)} e^{-(k+1)h_k}\psi(k, z) = \frac{R(z)}{R(\infty)} e^{-(k+1)h_k}\left\{n(z)P_h(z + z^{-1}) + Q_h(z + z^{-1})\right\},$$

(3.1)

Since $\tilde{\eta}^R(\tau)$ and $\tilde{\eta}^L(\tau)$ satisfy the Hölder condition, the real part of the integral

$$\int_{-\infty}^{\infty} \tilde{\eta}^R(\tau)(\frac{1}{\tau - \lambda} - \frac{\tau}{1 + \tau^2})d\tau,$$

is bounded as $\lambda \to \lambda_0$, $\lambda_0 \in (-1, 2 + \varepsilon)$, i.e. $P(\tau + i0, \tilde{\eta}^R)$ is bounded away from zero and the infinity. Further, $\tilde{\eta}^R(\tau)$ is isolated from 0 and $\pi$, from where $P(\tau + i0, \tilde{\eta}^R)$ is isolated from the real line. So is $P(\tau + i0, \tilde{\eta}^L)$.
where \( h_k \) are defined in \((0.15)\). According to \((1.4)\), the function \( g(k, z) \) satisfies the asymptotic formula
\[
g(k, z) \to 1, \quad z \to \infty.
\]

This function is holomorphic everywhere but the unit circle and a certain set of the real line, which is separated from the circle, zero and the infinity. So, according to the Cauchy theorem
\[
g(k, z) = 1 + \frac{1}{2\pi i} \int_{\Gamma} \frac{g(k, \zeta)}{\zeta - z} d\zeta,
\]
where \( \Gamma \) is a contour, enclosing all the singularities of the function \( g(k, z) \), concentrated on the real line and the unit circle. In the part of the plane, bounded by this contour, the functions \( P_k(z + z^{-1}), Q_k(z + z^{-1}) \) are regular (we remind that they are polynomials in \( z + z^{-1} \)). The singularities of the function \( \eta(z) \) are concentrated on the set \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \) and the unit circle \( T \). The singularities of the function \( R(z) \) are concentrated on the set \( \Omega_1 \cup \Phi \cup \Omega_2 \) and on the unit circle. Hence, all the singularities of the function \( g(k, z) \) in this part of the plane lie in the union of the four sets \( \Omega_1 \cup \Phi, \Omega_2, \Omega_3, T \), which lie at positive distances one from another and from the set \( V(\Omega_1) \cup V(\Phi) \). This makes it possible to deform the contour \( \Gamma \) into a finite system of simple contours \( \Gamma_1, \Gamma_2, \Gamma_3 \) which enclose the sets \( \Omega_1 \cup \Phi, \Omega_2, \Omega_3, T \), respectively, and the contour \( \Gamma_3 \), formed by two concentric circles bounding an annulus of small area containing the unit circle. Here the contours \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3 \), can be chosen so close to the singularities in their interiors that they lie at positive distance one from another and from the set \( V(\Omega_1) \cup V(\Phi) \). Thus,
\[
g(k, z) = 1 + \frac{1}{2\pi i} \left\{ \int_{\Gamma_1} \frac{g(k, \zeta)}{\zeta - z} d\zeta + \int_{\Gamma_2} \frac{g(k, \zeta)}{\zeta - z} d\zeta + \int_{\Gamma_3} \frac{g(k, \zeta)}{\zeta - z} d\zeta \right\}.
\]

Let us first calculate the integrals in the right-hand side of \((3.2)\). We denote by \( O_1, O_2, O_3 \) the domains that lie inside the contours \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3 \), which are close to the sets \( \Omega_1 \cup \Phi, \Omega_2, \Omega_3, T \), so that the distances between any two of the domains \( O_1, O_2, O_2, O_3 \) and between these domains and the set \( V(\Phi) \cup \Omega_1 \) are positive.

Later in this section we will omit somewhere, for the sake of brevity, the index \( k \) of the functions \( g(k, z) \), \( P_k(z + z^{-1}), Q_k(z + z^{-1}) \) (although all of them, certainly, depend on \( k \)).

**Calculation of the integrals along the contours belonging to \( \Gamma_1 \)**

**Lemma 3.1.** The integral \( \int_{\Gamma_1} \) in \((3.2)\) is calculated by formula
\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - z} d\zeta = -\int_{\Phi \cup \Omega_1} \frac{\alpha^{-2(\alpha+1)} R(\alpha^{-1}) R^{-2} (\alpha^{-1} - 1)}{\alpha - z} g(\alpha^{-1}) d\rho_1(\alpha),
\]
where the measure \( d\rho_1(t) \) is defined by the nondecreasing function \( \rho_1(t) \) with
\[
\rho_1(t_2) - \rho_1(t_1) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \text{Im} \left( -N(t + i\epsilon)^{-1} \right) dt.
\]

We remark that the measure \( d\rho_1(t) \) is defined by formula \((3.4)\) on a larger set than \( \Phi \cup \Omega_1 \), and that for \( \pm 1 \in [t_1, t_2] \) formula \((3.4)\) is senseless. But, according to \((3.3)\), we only need this measure on \( \Phi \cup \Omega_1 \), so later on we suppose that \( d\rho_1(t) \) is only defined on \( \Phi \cup \Omega_1 \).

We also observe that the notation \( R(\alpha^{-1}), g(\alpha^{-1}) \) in formula \((3.3)\) makes sense, because \( R(z) \) and \( g(z) \) are holomorphic on the set \( V(\Phi) \cup V(\Omega_1) \), i.e. \( R(\alpha^{-1}) \) and \( g(\alpha^{-1}) \) are regular for \( \alpha \in \Phi \cup \Omega_1 \).

**Proof.** According to definition \((3.1)\),
\[
g(\zeta) = \frac{\zeta^{-(k+1)}}{R(\zeta)} h_k R(\zeta) \left( Q(\zeta + \zeta^{-1}) + n(\zeta) P(\zeta + \zeta^{-1}) \right),
\]
\[
g(\zeta^{-1}) = \frac{\zeta^{-(k+1)}}{R(\zeta^{-1})} h_k R(\zeta^{-1}) \left( Q(\zeta + \zeta^{-1}) + n(\zeta^{-1}) P(\zeta + \zeta^{-1}) \right).
\]
Let us exclude \( Q(\zeta + \zeta^{-1}) \) from these relations:

\[
Q(\zeta + \zeta^{-1}) = R(\infty)g(\zeta^{-1})\zeta^{-(k+1)h}\bar{R}(\zeta^{-1})^{-1} - n(\zeta^{-1})P(\zeta + \zeta^{-1}),
\]

\[
g(\zeta) = g(\zeta^{-1})\zeta^{2(k+1)}R(\zeta^{-1})^{-1}R(\zeta) + \frac{\zeta^{k+1}}{R(\infty)}h_kR(\zeta) \left( -n(\zeta^{-1})P(\zeta + \zeta^{-1}) + n(\zeta)P(\zeta + \zeta^{-1}) \right)
\]

\[
= g(\zeta^{-1})\zeta^{2(k+1)}R(\zeta^{-1})^{-1}R(\zeta) + P(\zeta + \zeta^{-1})\frac{\zeta^{-(k+1)}}{R(\infty)}h_kR(\zeta)N(\zeta)
\]

(we remind that \( N(\zeta) = n(\zeta) - n(\zeta^{-1}) \)). We find from this and from the factorization \( R(\zeta)R(\zeta^{-1})N(\zeta) = - (\zeta - \zeta^{-1}) \), obtained in theorem 2, that

\[
g(\zeta) = f_1(\zeta)(-N(\zeta))^{-1} + f_2(\zeta),
\]

where

\[
f_1(\zeta) = -g(\zeta^{-1})\zeta^{2(k+1)}R(\zeta^{-1})^{-2}(\zeta - \zeta^{-1}),
\]

\[
f_2(\zeta) = P(\zeta + \zeta^{-1})\frac{\zeta^{-(k+1)}}{R(\infty)}h_kR(\zeta^{-1})^{-1}(\zeta - \zeta^{-1}).
\]

The functions \( R(\zeta^{-1}) \), \( R(\zeta^{-1})^{-1} \), \( n(\zeta^{-1}) \), and so as the functions \( g(\zeta^{-1}) \), \( f_1(\zeta) \), \( f_2(\zeta) \) are holomorphic outside the unit circle and the compact set

\[
V(\Phi \cup \Omega_1) \cup \Omega_2 \subset \mathbb{R}.
\]

In particular, all these functions are holomorphic in the domain \( O_1 \) and, according to the Cauchy theorem, we have

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\zeta)}{\zeta - z} (-N(\zeta))^{-1} d\zeta,
\]

if \( z \notin O_1 \).

Let us consider the function \( \frac{1}{-N(\zeta)} \). According to (1.1), (1.2), at the zero and the infinity it vanishes and is holomorphic. In the upper half-plane, except the arc of the unit circle, \( \text{Im} N(\zeta) > 0 \), from where \( \text{Im} (-N(\zeta)^{-1}) > 0 \). All the singularities of the function are concentrated on the unit circle and the parts of the real line, separated from \(-1, 0, 1, \infty\). As it will be shown below,

\[
(-N(\zeta))^{-1} = \int_{\Omega_1 \cup \Phi} \frac{d\rho_1(\alpha)}{\alpha - \zeta} + \tilde{N}(\zeta),
\]

(3.5)

where the measure \( d\rho_1(t) \) is defined by (3.4), and \( \tilde{N}(\zeta) \) is holomorphic in the domain \( O_1 \).

Taking into account the regularity of \( \tilde{N}(\zeta) \) in the domain \( O_1 \),

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\zeta)\tilde{N}(\zeta)}{\zeta - z} d\zeta = 0,
\]

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\zeta)}{\zeta - z} \int_{\Phi \cup \Omega_1} \frac{d\rho_1(\alpha)}{\alpha - \zeta} d\zeta.
\]

The family of contours \( \Gamma_1 \) encloses the set \( \Phi \cup \Omega_1 \) and are at positive distance from it, which allows to change the order of integration. We do this and use theorem of residue to calculate the inner integral:

\[
-\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\zeta)d\zeta}{(\zeta - z)(\zeta - \alpha)} = f_1(\alpha) \frac{1}{\alpha - z},
\]

because the residue

\[
\text{Res}_{\zeta=\alpha} \frac{f_1(\zeta)}{(\zeta - z)(\zeta - \alpha)} = \lim_{\zeta \to \alpha} (\zeta - \alpha) \frac{f_1(\zeta)}{(\zeta - z)(\zeta - \alpha)} = f_1(\alpha) \frac{1}{\alpha - z}.
\]
Thus,

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - z} d\zeta = \int_{\Phi \cup \Omega_1} \frac{f_1(\alpha)}{\alpha - z} d\rho_1(\alpha).
\]

(Here we remind that the contour \(\Gamma_1\) is oriented clockwise.) According to the definition of \(f_1(\alpha)\), it follows that

\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - z} d\zeta = - \int_{\Phi \cup \Omega_1} \frac{\alpha^{-2(k+1)} R(\alpha^{-1})^{-2} (\alpha - \alpha^{-1})^{-2} g(\alpha^{-1})}{\alpha - z} d\rho_1(\alpha),
\]

as was to be proved.

Here we remind that we also have to prove the decomposition (3.4) for the function \((-N(\zeta))^{-1}\). (Note, that this is not the Stiltjes representation for the functions with positive imaginary part in the upper half-plane, because the function \((-N(\zeta))^{-1}\), which has singularities on the unit circle, is not holomorphic in the half-plane.) Let us denote by \(\Gamma_4\) the contour, enclosing the singularities \((-N(z))^{-1}\) on the real line (remind that \(\Gamma_3\) is a contour, consisting of two concentric circles, close to the unit circle). By the Cauchy theorem,

\[
(-N(\zeta))^{-1} = \frac{1}{2\pi i} \int_{\Gamma_4} \frac{(-N(\alpha))^{-1}}{\alpha - \zeta} d\alpha + \frac{1}{2\pi i} \int_{\Gamma_3} \frac{(-N(\alpha))^{-1}}{\alpha - \zeta} d\alpha.
\]

We denote the second one of these integrals by \(\hat{N}(\zeta)\). If the contour \(\Gamma_3\) is close enough to the unit circle, the function \(N(\zeta)\) is holomorphic in the domain \(O_1\). Then for little \(\varepsilon\) and great \(M > 0\)

\[
(-N(\zeta))^{-1} = \frac{1}{2\pi i} \lim_{\delta \to +0} \left( \int_{-M}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{1+\varepsilon} + \int_{1+\varepsilon}^{M} \right) \left( \frac{(-N(\alpha + i\delta))^{-1}}{\alpha + i\delta - \zeta} - \frac{(-N(\alpha - i\delta))^{-1}}{\alpha - i\delta - \zeta} \right) d\alpha + \hat{N}(\zeta)
\]

It can be shown, using the methods of [15], the function \(N(z)\) is decomposed into a product of two functions: \(N(z) = N_0(z)N_1(z)\), where \(N_0(z)\) is holomorphic outside the unit circle and is positive on the real line \((N_0(t) > 0, \ t \in \mathbb{R}\setminus\{-1,1\})\), and \(N_1(z)\) is holomorphic outside the real line and has positive imaginary part in the upper half-plane. Therefore, for the function \(-N_1(z)^{-1}\) the following representation is true:

\[
-N_1(z)^{-1} = az + b + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - z},
\]

where \(a > 0, \ b \in \mathbb{R}\) and \(d\sigma(t)\) is a certain bounded measure on \(\mathbb{R}\), concentrated on the union of the intervals \((-M, -1 - \varepsilon) \cup (-1 + \varepsilon, 1 - \varepsilon) \cup (1 + \varepsilon, M)\). Taking into account the holomorphy of the function \(N_0(z)(az + b)\) on this union of intervals,

\[
(-N(\zeta))^{-1} = \hat{N}(\zeta) + \frac{1}{2\pi i} \lim_{\delta \to +0} \left( \int_{-M}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{1+\varepsilon} + \int_{1+\varepsilon}^{M} \right) \left( \frac{N_0(\alpha + i\delta)^{-1}N_1(\alpha + i\delta)^{-1}}{\alpha + i\delta - \zeta} - \frac{N_0(\alpha - i\delta)^{-1}N_1(\alpha - i\delta)^{-1}}{\alpha - i\delta - \zeta} \right) d\alpha
\]

\[
\times \left( \frac{N_0(\alpha + i\delta)^{-1}N_1(\alpha + i\delta)^{-1}}{\alpha + i\delta - \zeta} - \frac{N_0(\alpha - i\delta)^{-1}N_1(\alpha - i\delta)^{-1}}{\alpha - i\delta - \zeta} \right) d\alpha.
\]

\[
\times \left( \frac{N_0(\alpha + i\delta)^{-1}}{\alpha + i\delta - \zeta} - \frac{N_0(\alpha - i\delta)^{-1}}{\alpha - i\delta - \zeta} \right) d\alpha.
\]
\[
\hat{N}(\zeta) + \frac{1}{2\pi i} \lim_{\delta \to +0} \int_{-\infty}^{\infty} d\sigma(t) \left( \int_{-\infty}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{1+\varepsilon} + \int_{1-\varepsilon}^{-\infty} \right) \frac{N_0(\alpha + i\delta)^{-1}}{(\alpha + i\delta - \zeta)(t - (\alpha + i\delta))} - \frac{N_0(\alpha - i\delta)^{-1}}{(\alpha - i\delta - \zeta)(t - (\alpha - i\delta))} d\alpha.
\]

The function \( \frac{N_0(\alpha)}{(\alpha - \zeta)(t - \alpha)} \) is holomorphic in the variable \( \alpha \) at the points of union \((-M, -1-\varepsilon) \cup (-1+\varepsilon, 1-\varepsilon) \cup (1+\varepsilon, M)\), except the point \( \alpha = t \), at which it has a simple pole. Thus,

\[
\frac{1}{2\pi i} \lim_{\delta \to +0} \left( \int_{-\infty}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{1+\varepsilon} + \int_{1-\varepsilon}^{-\infty} \right) \frac{N_0(\alpha + i\delta)^{-1}}{(\alpha + i\delta - \zeta)(t - (\alpha + i\delta))} - \frac{N_0(\alpha - i\delta)^{-1}}{(\alpha - i\delta - \zeta)(t - (\alpha - i\delta))} d\alpha
\]

Introducing the measure \( d\rho_1(t) \equiv N_0(t)^{-1}d\sigma(t) \), we obtain

\[
(-N(\zeta))^{-1} = \hat{N}(\zeta) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{N_0(t)^{-1}}{t - \zeta} d\sigma(t) = \hat{N}(\zeta) + \frac{1}{2\pi i} \int_{\partial \Omega \cup \Phi} N_0(t)^{-1} d\sigma(t),
\]

where

\[
\hat{N}(\zeta) = \frac{1}{2\pi i} \int_{R \setminus \{ \Omega \cup \Phi \}} N_0(t)^{-1} d\sigma(t),
\]

which was our goal.

Taking into account the definition of \( \rho_1(t) \) and \( \sigma(t) \), the fact that \( N_0(t) > 0 \) for \( t \in R \setminus [-1, 1] \), we have

\[
\rho_1(t_2) - \rho_1(t_1) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{t_1}^{t_2} N_0(t)^{-1} \text{Im} \left( -N_1(t + i\varepsilon)^{-1} \right) dt = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{t_1}^{t_2} \text{Im} \left( -N(t + i\varepsilon)^{-1} \right) dt.
\]

It is seen that \( \rho_1(t) \) is non-decreasing.

**Calculation of the integrals along the contours belonging to \( \Gamma_2 \)**

**Lemma 3.2.** The integral \( \int_{\Gamma_2} \) in (3.2) is calculated by formula

\[
\frac{1}{2\pi i} \int_{\Gamma_2} \frac{g(\zeta)}{g(\zeta - z)} d\zeta = \int_{\Omega_2 \cap (-1, 1)} \frac{\alpha^{-2(k+1)} R(\alpha)^2}{(\alpha - z)(\alpha^{-1} - \alpha)} g(\alpha^{-1}) d\rho_2(\alpha),
\]

where \( d\rho_2(\alpha) \) is the measure, defined on the point set \( \Omega_2 \) by

\[
\rho_2(\alpha_i) = 0, \quad \alpha_i \in \Omega_2 \setminus [-1, 1],
\]

\[
\rho_2(\alpha_i^{-1}) = \frac{d\rho^R(\alpha_i + \alpha_i^{-1})}{d\rho^B(\alpha_i + \alpha_i^{-1})} \left( d\rho^R(\alpha + \alpha_i^{-1}) + d\rho^B(\alpha + \alpha_i^{-1}) \right), \quad \alpha_i^{-1} \in \Omega_2 \cap (-1, 1).
\]

The integration is performed on the point set \( \Omega_2 \cap (-1, 1) \), in whose points the functions \( R(z) \) and \( g(z^{-1}) \) are regular, i.e. \( R(\alpha) \) and \( g(\alpha^{-1}) \) make sense.

**Proof.** According to the considerations of section 2, the part \( \Omega_2 \) of the set \( \Omega \) contains of a finite number of pairs of points \( (\alpha_i, \alpha_i^{-1}) \), and the positive masses

\[
\mu(\alpha_i) \equiv d\rho^B(\alpha_i + \alpha_i^{-1}), \quad \mu(\alpha_i^{-1}) \equiv d\rho^R(\alpha_i + \alpha_i^{-1})
\]

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Let us investigate the behavior of the function \( g(\zeta) \) in the neighborhood of \( \alpha_l \) and \( \alpha_l^{-1} \). It follows from the definition of \( n(z) \) and the representation (2.3) that

\[
N(\zeta) = \left\{ \frac{1}{\alpha_l - \zeta} + \frac{1}{\alpha_l^{-1} - \zeta} \right\} (\mu(\alpha_l) + \mu(\alpha_l^{-1})) + N_l(\zeta),
\]

where the functions \( n_l(\zeta), N_l(\zeta) \) are holomorphic in a neighborhood of the points \( \alpha_l \) and \( \alpha_l^{-1} \). According to (2.23) and theorem 2, \( \alpha_l^* = \alpha_l \), the function \( R(\zeta) \) is holomorphic in the points \( \alpha_l, \alpha_l^{-1} \) and \( R(\alpha_l) = 0 \), \( R(\alpha_l^{-1}) \neq 0 \). Hence, both functions \( R(\zeta), R(\zeta)n(\zeta) \) are holomorphic in the points \( \alpha_l \), and so is the function \( g(\zeta) \). The function \( R(\zeta) \) is holomorphic in the point \( \alpha_l^{-1} \), and the function \( R(\zeta)n(\zeta) \) has a simple pole in it with residue \(-\mu(\alpha_l^{-1})\) \( R(\alpha_l^{-1}) \). Let \( \Gamma_l, \Gamma_l^* \) be contours enclosing the points \( \alpha_l, \alpha_l^{-1} \) and close enough to them (it is seen that all the contour \( \Gamma_l^* \) consists of such \( \Gamma_l, \Gamma_l^* \)). Write

\[
\frac{1}{2\pi i} \int_{\Gamma_l} \frac{g(\zeta)}{\zeta - z} d\zeta = 0,
\]

\[
\frac{1}{2\pi i} \int_{\Gamma_l} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{\mu(\alpha_l^{-1}) R(\alpha_l^{-1}) \alpha_l^{k+1} h_k}{R(\zeta) (-\alpha_l^{-1} + z)}.
\]

Hence,

\[
\frac{1}{2\pi i} \int_{\Gamma_l} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{R(\zeta)} \sum_{\alpha_l \in \Omega_l^*} \frac{R(\alpha_l^{-1}) \alpha_l^{k+1} h_k}{\alpha_l^{-1} - z} \mu(\alpha_l^{-1}) P(\alpha_l + \alpha_l^{-1}).
\]

Since \( R(\alpha_l) = 0 \), we have from (3.1) and (3.8), that

\[
g(\alpha_l) = \alpha_l^{-(k+1)} \frac{h_k}{R(\zeta)} P(\alpha_l + \alpha_l^{-1}) \mu(\alpha_l) \lim_{\zeta \to \alpha_l} \left\{ (\alpha_l - \zeta)^{-1} R(\zeta) \right\}.
\]

On the one hand, we have from theorem 2 that

\[
\lim_{\zeta \to \alpha_l} R(\zeta) N(\zeta) = \frac{\alpha_l - \alpha_l^{-1}}{R(\alpha_l^{-1})},
\]

and, on the other hand, we have from (3.9) that

\[
\lim_{\zeta \to \alpha_l} R(\zeta) N(\zeta) = \left( \mu(\alpha_l) + \mu(\alpha_l^{-1}) \right) \lim_{\zeta \to \alpha_l} \left\{ (\alpha_l^{-1} - \zeta)^{-1} R(\zeta) \right\}.
\]

Therefore,

\[
\lim_{\zeta \to \alpha_l} (\alpha_l^{-1} - \zeta)^{-1} R(\zeta) = \frac{\alpha_l - \alpha_l^{-1}}{R(\alpha_l^{-1})} \left( \mu(\alpha_l) + \mu(\alpha_l^{-1}) \right)^{-1},
\]

\[
g(\alpha_l) = \alpha_l^{-(k+1)} \frac{h_k}{R(\zeta)} P(\alpha_l + \alpha_l^{-1}) \left( \frac{\mu(\alpha_l)}{\mu(\alpha_l) + \mu(\alpha_l^{-1})} \right) \left( \frac{\alpha_l - \alpha_l^{-1}}{R(\alpha_l^{-1})} \right),
\]

\[
P(\alpha_l + \alpha_l^{-1}) = \left( \frac{\mu(\alpha_l)}{\mu(\alpha_l) + \mu(\alpha_l^{-1})} \right) \left( \frac{R(\alpha_l^{-1})}{\alpha_l - \alpha_l^{-1}} \right) \frac{h_k}{\alpha_l^{-1} - z} \mu(\alpha_l) \mu(\alpha_l^{-1}) g(\alpha_l).
\]

Substituting this into the right-hand side of (3.10), we obtain

\[
\frac{1}{2\pi i} \int_{\Gamma_l} \frac{g(\zeta)}{\zeta - z} d\zeta = \sum_{\alpha_l \in \Omega_l^*} \frac{R(\alpha_l^{-1}) \alpha_l^{2(k+1)} h_k}{(\alpha_l^{-1} - z)(\alpha_l - \alpha_l^{-1})} \mu(\alpha_l^{-1}) \mu(\alpha_l) g(\alpha_l).
\]

Let \( dp_2(\alpha) \) be the measure, defined on the point set \( \Omega_l^* \) by (3.7):

\[
p_2(\alpha_l) = 0,
\]

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\[
\rho_2(\alpha_{i}^{-1}) = \frac{\mu(\alpha_{i}^{-1})}{\mu(\alpha_{i})}(\mu(\alpha_i) + \mu(\alpha_{i}^{-1})) = \frac{dp^R(\alpha + \alpha_{i}^{-1})}{dp^R(\alpha + \alpha_{i}^{-1})} \left( dp^R(\alpha + \alpha_{i}^{-1}) + dp^R(\alpha + \alpha_{i}^{-1}) \right).
\]

Then the latter integral can be rewritten as (3.7), as was to be proved.

**Calculation of the integrals along the contours belonging to \( \Gamma_2^{a} \)**

**Lemma 3.3.** The following equality is true for the integral along the contour \( \Gamma_2^{a} \):

\[
\frac{1}{2\pi i} \int_{\Gamma_2^{a}} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \lim_{\gamma(h) \downarrow 0} \left( \int_{\gamma(h) + \gamma^{-}(h)} \right) \frac{g(\zeta)}{\zeta - z} d\zeta
\]

(3.11)

The functions \( \chi_2(\alpha)g^{+}(\alpha, x) \) and \( \chi_2(\alpha)g^{-}(\alpha, x) \) belong to the Hilbert space \( L^2(\mathbb{R}) \).

**Proof.** The set \( \Omega_2^{a} \) lies in a finite union of intervals \( \Delta_1^{(1)}, \Delta_2^{(1)}, V(\Delta_2^{(1)}), V(\Delta_2^{(2)}) \) of the real line. The contour \( \Gamma_2^{a} \) consists of a finite number of contours each of which encloses the subsets \( \Delta_1^{(1)} \cap \Omega_2^{a}, \Delta_2^{(2)} \cap \Omega_2^{a}, V(\Delta_1^{(1)}) \cap \Omega_2^{a}, V(\Delta_2^{(2)}) \cap \Omega_2^{a} \) of the set \( \Omega_2^{a} \) and lies close enough to it. The calculations of the integrals along each of these contours are completely similar and we will examine only one of them, for example, the contour that encloses \( V(\Delta_2^{(2)}) \cap \Omega_2^{a} \). Let

\[
V(\Delta_k^{(2)}) = (\beta_k^{-1}, \varphi_k^{-1}), \quad \overline{\beta}_k = \inf(V(\Delta_k^{(2)}) \cap \Omega_2^{a}), \quad \overline{\varphi}_k = \sup(V(\Delta_k^{(2)}) \cap \Omega_2^{a}).
\]

Since the compact set \( V(\Delta_k^{(2)}) \cap \Omega_2^{a} \) is contained in the interval \( V(\Delta_2^{(2)}) \), we have \( \beta_k^{-1} < \overline{\beta}_k < \varphi_k < \varphi_k^{-1} \), and, without changing the value of the integral, the corresponding contour can be replaced by a contour \( \gamma(h) \), consisting of two intercepts

\[
\gamma^{\pm}(h) = \{ \zeta | \zeta = \alpha \pm ih, \beta_k < \alpha < \varphi_k \}
\]

and of two intercepts connecting the endpoints of \( \gamma^{\pm}(h) \). Here \( \alpha_k', \varphi_k' \) denote numbers, chosen so that they satisfy the inequalities

\[
\beta_k^{-1} < \beta_k < \alpha_k', \quad \varphi_k < \varphi_k' < \varphi_k^{-1},
\]

and \( h > 0 \) is a small number. On the linear segments of the contour \( \gamma(h) \) the function \( g(\zeta)(\zeta - z)^{-1} \) is continuous. So, the integrals along these intervals tend to zero when \( h \downarrow 0 \), and

\[
\frac{1}{2\pi i} \int_{\gamma(h)} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \lim_{h \downarrow 0} \left( \int_{\gamma^{+}(h) \cup \gamma^{-}(h)} \right) \frac{g(\zeta)}{\zeta - z} d\zeta
\]

(with the corresponding orientation of the integration along \( \gamma^{\pm}(h) \)). Let us assume that under the interval \( [\alpha_k', \varphi_k'] \), the function \( g(\alpha \pm ih) \) converges in \( L^p \)-norm \((p \geq 1)\) to a finite limite

\[
\lim_{h \downarrow 0} g(\alpha \pm ih) = g^{\pm}(\alpha).
\]

Then

\[
\lim_{h \downarrow 0} \int_{\gamma^{\pm}(h)} \frac{g(\zeta)}{\zeta - z} d\zeta = \int_{\beta_k}^{\varphi_k'} \frac{g^{\pm}(\alpha)}{\alpha - z} d\alpha
\]

and

\[
\frac{1}{2\pi i} \int_{\gamma(h)} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\beta_k}^{\varphi_k} \frac{g^{+}(\alpha) - g^{-}(\alpha)}{\alpha - z} d\alpha.
\]

Since the function \( g(\zeta) \) is holomorphic at the points \( V(\Delta_2^{(2)}) \cap \Omega_2^{a} \),

\[
g^{+}(\alpha) - g^{-}(\alpha) = 0, \quad \alpha \in V(\Delta_2^{(2)}) \cap \Omega_2^{a}.
\]

Therefore, we have

\[
\frac{1}{2\pi i} \int_{\gamma(h)} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{V(\Delta_2^{(2)}) \cap \Omega_2^{a}} \frac{g^{+}(\alpha) - g^{-}(\alpha)}{\alpha - z} d\alpha,
\]

(3.12)
or

$$\lim_{h \to 0} \|g(\alpha \pm ih) - g^\pm(\alpha)\|_{L^p[\beta'_k, \varphi'_k]} = 0, \quad p \geq 1.$$  \hspace{1cm} (3.13)

So, let us prove (3.13). It follows from the definition of the function \(R(\zeta)\) and the specifics of the function \(n(\zeta)\) that the following factorization is true:

$$R(\zeta) = R_k^{(1)}(\zeta) R_k^{(2)}(\zeta), \quad n(\zeta) = n_k^{(1)}(\zeta) n_k^{(2)}(\zeta),$$

where \(R_k^{(1)}(\zeta)\) and \(n_k^{(1)}(\zeta)\) are functions whose singularities are concentrated on the set \(V(\Delta_k^{(2)}) \cap \Omega_2\), and the functions \(R_k^{(2)}(\zeta), n_k^{(2)}(\zeta)\) are holomorphic on the set \(V(\Delta_k^{(2)})\). The functions \(R_k^{(1)}(\zeta), n_k^{(1)}(\zeta)\) are defined by formulas

$$R_k^{(1)}(\zeta) = P\left(\zeta, -J(\alpha) \gamma_2(\alpha)\right),$$

$$n_k^{(1)}(\zeta) = P\left(\zeta, \pi - J(\alpha) n(\alpha + \alpha^{-1})\right),$$

where \(J(\alpha)\) is the indicator of the interval \(V(\Delta_k^{(2)})\).

The representability of the function \(R(\zeta)\) in the form of such product is an immediate conclusion of its definition; the representability of \(n(\zeta)\) can be proved by decomposing \(n(z)\) in a product of two functions of the form \(P(z, \gamma)\) and \(P(z, \tilde{\gamma})\) with \(\gamma\) and \(\tilde{\gamma}\), defined by the limit values of the argument of \(n(z)\); see, for example, [15].

Hence,

$$g(\zeta) = \tilde{g}_k^{(1)}(\zeta) R_k^{(1)}(\zeta) + \tilde{g}_k^{(2)}(\zeta) R_k^{(2)}(\zeta) n_k^{(1)}(\zeta),$$

where the functions \(\tilde{g}_k^{(1)}(\zeta)\) and \(\tilde{g}_k^{(2)}(\zeta)\) are holomorphic in a neighborhood of \(V(\Delta_k^{(2)})\). Therefore, the functions \(\tilde{g}_k^{(1)}(\alpha \pm ih)\) and \(\tilde{g}_k^{(2)}(\alpha \pm ih)\) converge uniformly to their limits on the interval \([\beta'_k, \varphi'_k] \subset V(\Delta_k^{(2)})\) as \(h \downarrow 0\). So, to prove equalities (3.13) and (3.12), it is sufficient to check that the functions \(R_k^{(1)}(\alpha \pm ih), R_k^{(2)}(\alpha \pm ih) n_k^{(1)}(\alpha \pm h)\) converge to their limits in \(L^p\)-norm when \(h \downarrow 0\). This is a consequence of conditions B), C) and the following lemma.

**Lemma 3.4.** Let \(\delta(\alpha) = \delta_1(\alpha) + \delta_2(\alpha)\) and suppose that the function \(\delta_2(\alpha)\) satisfies the Hölder condition on an interval \((\alpha_1, \alpha_2) \subset \mathbb{R}\) and that on the same interval the function \(\delta_1(\alpha)\) satisfies the inequality

$$\omega = \text{ess sup}_{\alpha_1 < \alpha < \alpha_2} \delta_1(\alpha) - \text{ess inf}_{\alpha_1 < \alpha < \alpha_2} \delta_1(\alpha) < \pi.$$ 

Then for any \(p < \pi \omega^{-1}\) the function \(P(\alpha \pm ih, \delta)\) converges in \(L^p\)-norm to a limit \(P^\pm(\alpha, \delta)\) on all compact subsets of \((\alpha_1, \alpha_2)\).

**Proof.** In our conditions the role of the interval \((\alpha_1, \alpha_2)\) is played by \(V(\Delta_k^{(2)}) = (\beta_k^{-1}, \varphi_k^{-1})\), so in this lemma we continue denoting the indicator of the interval \((\alpha_1, \alpha_2)\) by \(J(\alpha)\). Let

$$d_1(\alpha) = J(\alpha) (\delta_1(\alpha) - C),$$

$$d_2(\alpha) = \delta(\alpha) - d_1(\alpha) = (1 - J(\alpha)) \delta_1(\alpha) + \delta_2(\alpha) + J(\alpha) C,$$

where

$$C = \frac{1}{2} \left( \text{ess sup}_{\alpha_1 < \alpha < \alpha_2} \delta_1(\alpha) + \text{ess inf}_{\alpha_1 < \alpha < \alpha_2} \delta_1(\alpha) \right).$$

Then \(\delta(\alpha) = d_1(\alpha) + d_2(\alpha)\),

$$P(\zeta, \delta) = P(\zeta, d_1) P(\zeta, d_2),$$

(3.16)

where the function \(d_1(\alpha)\) satisfies the inequality

$$\text{ess sup}_{\alpha_1 < \alpha < \alpha_2} |d_1(\alpha)| = \frac{\omega}{2} < \frac{\pi}{2},$$

(3.17)

and the function \(d_2(\alpha)\) satisfies the Hölder condition on the interval \((\alpha_1, \alpha_2)\). Thus, \(P(\zeta, d_2)\) converges uniformly to its limit on every segment \([\alpha'_1, \alpha'_2] \subset (\alpha_1, \alpha_2)\). Let us introduce

**Definition.** Let \(A\) be a set of real line, \(U_\delta(A)\) be its \(\delta\)-neighborhood, and \(f(z)\) be a holomorphic on \(U_\delta(A)\backslash A\) function. We say that the function \(f(z)\) belongs locally to the Hardy class \(H^p\) in a neighborhood of the set \(A\), if for a certain \(\delta_0 > 0\) the functions \(f(t \pm i\varepsilon), \ t \in \mathbb{R} \cap U_\delta(A)\), converge as \(\varepsilon \downarrow 0\) in \(L^p\)-norm.

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It follows from the inequality (3.17) that the function \( P(z, d_1) \) belongs locally to the Hardy space \( H^p \) for any \( p < \pi \omega^{-1} \) in a neighborhood of \( [\alpha_1', \alpha_2'] \subset (\alpha_1, \alpha_2) \) (see, for example, [18], [19]).

We explain how to demonstrate it. Let us decompose \( P(z, d_1) \) in the product of two functions one of which \( (P(z, Jd_1)) \) is holomorphic outside the segment \( [\alpha_1, \alpha_2] \), and the other one is holomorphic on \( (\alpha_1, \alpha_2) \). Then in the domain of holomorphy of \( P(z, Jd_1) \)

\[
\sup |\arg P(z, Jd_1)| < \frac{\omega}{2}. \tag{3.18}
\]

The problem is reduced to the existence of the limits of the function \( P(z, Jd_1) \) on the interval \( [\alpha_1, \alpha_2] \). With the help of a conformal mapping, we reduce the problem to the following: A holomorphic in the disk function, which satisfies in the disk condition (3.18), belongs to the space \( P(z) \). Smirnov’s theorem. If \( f(z) \) is analytic inside the disk \( |z| < 1 \) and \( \Re f(z) \geq 0 \), then \( f \in H^p \) for all \( p < 1 \).

After raising the function, which satisfy inside the disk property (3.18), to the power \( \pi \omega^{-1} \), we apply the last statement.

Thus,

\[
\lim h \int_0 \alpha' \max |P(\alpha \pm ih, d_2) - P^\pm(\alpha, d_2)| = 0,
\]

\[
\lim h \int_{\alpha} |P(\alpha \pm ih, d_1) - P^\pm(\alpha, d_1)|^p d\alpha = 0,
\]

and, together with (3.16), this means (3.13), which proves lemma 3.4.

It follows from conditions B) and C) that the function \( \eta^R(\tau) - \eta^L(\tau) \) satisfies the Hölder condition. Also, the inequality is true

\[-\pi < \eta^R(\alpha) - \eta^L(\alpha) < \pi \]

on the interval \( \Delta_k^{(2)} \) (which is the image of \( V(\Delta_k^{(2)}) \) under the transformation \( \tau = \alpha + \alpha^{-1} \)). So, the function

\( 1 + P(\lambda + i0, \eta^L(\tau) - \eta^R(\tau)) \)

satisfies the Hölder condition on this interval (see [17]), too, and is equal to zero almost everywhere. Hence, the function

\( \beta(\tau) = \arg \left( 1 + P(\tau + i0, \eta^L - \eta^R) \right) - \pi \)

satisfies the Hölder condition on the interval \( \Delta_k^{(2)} \). Since

\[
b_{-1}m^R(\lambda) = C_1 P(\lambda, \eta^R(\tau)); \quad -\frac{1}{b_{-1}m^R(\lambda)} = C_2 P(\lambda, \eta^L(\tau)),
\]

then

\[
M(\lambda) = b_{-1}m^R(\lambda) - \frac{1}{b_{-1}m^R(\lambda)} = C_1 P(\lambda, \eta^R) + C_2 P(\lambda, \eta^L) =
\]

\[
= C_1 P(\lambda, \eta^R)(1 + C_3 P(\lambda, \eta^R) P(\lambda, \eta^L)) = m^R(\lambda)(1 + C_3 P(\lambda, \eta^L - \eta^R)),
\]

where the constants \( C_i > 0 \), from where

\[
\arg M(\tau + i0) = \eta(\tau) = \eta^R(\tau) + \arg(1 + C_3 P(\tau + i0, \eta^L - \eta^R)),
\]

that is

\[
\eta(\alpha + \alpha^{-1}) = \eta^R(\alpha + \alpha^{-1}) + \beta(\alpha + \alpha^{-1}) + \pi.
\]

It follows from the definition of \( \gamma_2(\alpha) \) (formula (2.30)) and from what \( \eta(\alpha + \alpha^{-1}) = 0 \) when \( \alpha \in V(\Delta_k^{(2)}) \setminus \Omega^2_2 \) (see lemma 2.2) that we have the following equalities on the interval \( V(\Delta_k^{(2)}) \):

\[
-\gamma_2(t) = \chi^2(t) \left( \frac{1}{2} \eta^R(t + t^{-1}) + \frac{1}{2} \beta(t + t^{-1}) \right) = \frac{1}{2} \eta^R(t + t^{-1}) + \frac{1}{2} \beta(t + t^{-1}),
\]

\[
-\eta^R(t + t^{-1}) - \gamma_2(t) = -\frac{1}{2} \eta^R(t + t^{-1}) + \frac{1}{2} \beta(t + t^{-1}).
\]

Taking into account (3.14), (3.15), we conclude that

\[
R_k^{(1)}(\zeta) = P(\zeta, \delta_1 + \delta_2); \quad R_k^{(1)}(\zeta) n_k^{(1)}(\zeta) = - P(\zeta, -\delta_1 + \delta_2), \tag{3.19}
\]
Therefore where

\[ J(\alpha) = \frac{\beta(\alpha + \alpha^{-1})}{2}, \quad \delta_2(\alpha) = \frac{J(\alpha)}{2}, \quad \delta_1(\alpha) = \frac{\beta(\alpha + \alpha^{-1})}{2}, \]

and \( J(\alpha) \) is the indicator of \( V(\Delta_k^{(2)}) \). The function \( \delta_2(\alpha) \), evidently, satisfies the Hölder condition on the interval \( V(\Delta_k^{(2)}) \) and, according to condition B), the functions \( \pm \delta_1(\alpha) \) satisfy the inequality

\[ \operatorname{ess} \max_{\alpha \in V(\Delta_k^{(2)})} (\pm \delta_1(\alpha)) - \operatorname{ess} \min_{\alpha \in V(\Delta_k^{(2)})} (\pm \delta_1(\alpha)) = \frac{\omega}{2} < \frac{\pi}{2}. \]

In view of lemma 3.4 this means that functions (3.19) converge in \( L^p \)-norm to their limits on the segment \([\beta_k^\prime, \beta_k^\prime] \subset V(\Delta_k^{(2)})\) for all \( p < 2\pi \omega^{-1} \). Since \( \omega < \pi \), they also converge in \( L^2(\beta_k^\prime, \beta_k^\prime) \).

Thus, we proved equality (3.13) and formula (3.12). Analogous speculations can be applied to all the other components of the contour \( \Gamma^{(2)}_k \), which completes the proof of lemma 3.3.

Remark. It follows from the proof above that the functions \( \chi_2(\alpha)g^+(\alpha, x) \) and \( \chi_2(\alpha)g^-(\alpha, x) \) belong to the Hilbert space \( L^2(\mathbb{R}) \).

**Lemma 3.5.** The following equalities hold almost everywhere on the set \( \Omega_2^{(2)} \):

\[
\frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} \left( g^+(\alpha^{-1}) + g^-(\alpha^{-1}) \right) = \frac{2i\alpha^{2(k+1)}q(\alpha)\left( g^+(\alpha) - g^-(\alpha) \right) + im(\alpha)\alpha_{01} \left( g^+(\alpha^{-1}) - g^-(\alpha^{-1}) \right)}{p(\alpha)},
\]

where

\[
p(\alpha) = \left( \mu(\alpha + \alpha^{-1}) + \left( \mu(\alpha + \alpha^{-1})^{-1} \right) \right) \tan |\varphi_2(\alpha)| > 0,
\]

\[
q(\alpha) = \frac{|R(\alpha^{-1})|}{|R(\alpha)|} \left( \mu(\alpha + \alpha^{-1}) \right)^{s(\alpha)} > 0,
\]

\[
m(\alpha) = s(\alpha) \left( \mu(\alpha + \alpha^{-1}) - \left( \mu(\alpha + \alpha^{-1})^{-1} \right) \right) \cos \chi_0(\alpha)\pi.
\]

(The indicator \( \chi_0(\alpha) \) is defined above in theorem 2, and the functions \( \mu(\alpha), \gamma_2(\alpha) \) and \( s(\alpha) \) are defined by (2.5), (2.30) and (2.31), resp.)

Proof. It can be easily seen from the definition (2.5) of the function \( \mu(t) \) that (3.21), (3.22), (3.23) can be rewritten in the following way:

\[
p(\alpha) = \sqrt{\frac{\sigma(\alpha^{-1}) + \sigma(\alpha)}{\sigma(\alpha)^2}} \tan |\varphi_2(\alpha)| > 0,
\]

\[
q(\alpha) = \frac{|R(\alpha^{-1})|}{|R(\alpha)|} \left( \frac{\sigma(\alpha^{-1})}{\sigma(\alpha)} \right) > 0,
\]

\[
m(\alpha) = \left( \frac{\sigma(\alpha^{-1}) - \sigma(\alpha)}{\sigma(\alpha)} \right) \cos \chi_0(\alpha)\pi,
\]

where

\[
\sigma(\alpha) \equiv \begin{cases} \rho_\mu(\alpha + \alpha^{-1}), & |\alpha| < 1, \\ \rho_\mu^\prime(\alpha + \alpha^{-1}), & |\alpha| > 1. \end{cases}
\]

Thus, we will prove the lemma with this coefficients.

Since the functions \( P(\zeta + z^{-1}), Q(\zeta + z^{-1}) \) are holomorphic everywhere but 0 and \( \infty \), the limit values of the function \( g(z) \) on the set \( \Omega_2^{(2)} \) equal

\[
g^\pm(\alpha) = \alpha^{-(k+1)} \frac{R(\infty)}{\rho_\mu} \left( n^\pm(\alpha)R^\pm(\alpha)P(\alpha + \alpha^{-1}) + R^\pm(\alpha)Q(\alpha + \alpha^{-1}) \right).
\]

Therefore

\[
g^+(\alpha) - g^-(\alpha) = \alpha^{-(k+1)} \frac{R(\infty)}{\rho_\mu} \left( n^+(\alpha)R^+(\alpha) - n^-(\alpha)P^-(\alpha) \right) P(\alpha + \alpha^{-1}) =
\]
and

\[ \rho \tau (\alpha - 1) = \left( \begin{array}{c} R^+(\alpha - 1)P(\alpha + \alpha^{-1}) + R^-(\alpha^{-1})Q(\alpha^{-1} + \alpha^{-1}) \\ \end{array} \right). \] 

(3.24)

Let us now consider the limit values of the functions \( g(z) \) at the point \( \alpha^{-1} \):

\[ \alpha^{-(k+1)}h^{-1}_k R(\infty)g^+(\alpha^{-1}) = \left\{ n^+(\alpha^{-1})R^+(\alpha^{-1})P(\alpha + \alpha^{-1}) + R^-(\alpha^{-1})Q(\alpha^{-1} + \alpha^{-1}) \right\}. \]

The two latter equalities (for \( g^+(\alpha^{-1}) \) and \( g^-(\alpha^{-1}) \)) provide us with a system of equations for the components of the functions \( P(\alpha + \alpha^{-1}) \), \( Q(\alpha + \alpha^{-1}) \) with the matrix

\[ \left( \begin{array}{cc} R^+(\alpha^{-1})n^+(\alpha^{-1}) & R^+(\alpha^{-1}) \\ R^-(\alpha^{-1})n^-(\alpha^{-1}) & R^-(\alpha^{-1}) \end{array} \right), \]

whose determinant is

\[ D(\alpha) = R^+(\alpha^{-1})R^-(\alpha^{-1}) (n^+(\alpha^{-1}) - n^-(\alpha^{-1})) \]

Tackling into account the definition of \( n(z) \), formulas (2.3), (2.4) and the absolute continuity of the functions \( \rho \tau (\tau + 1) \), \( \rho L(\tau + 1) \) on \( \Omega_2 \), we have

\[ D(\alpha) = 2\pi i R(\alpha^{-1})^2 \sigma(\alpha^{-1}) \neq 0 \]

almost everywhere on the set \( \Omega_2 \) (with \( \sigma(\alpha) \), defined as above). Thus, the linear system is uniquely solvable and

\[ P(\alpha + \alpha^{-1}) = \alpha^{-(k+1)}h^{-1}_k R(\infty)D(\alpha)^{-1} \left\{ R^-(\alpha^{-1})g^+(\alpha^{-1}) - R^+(\alpha^{-1})g^-(\alpha^{-1}) \right\}, \]

\[ Q(\alpha + \alpha^{-1}) = \alpha^{-(k+1)}h^{-1}_k R(\infty)D(\alpha)^{-1} \left\{ -R^-(\alpha^{-1})n^-(\alpha^{-1})g^+(\alpha^{-1}) + R^+(\alpha^{-1})n^+(\alpha^{-1})g^-(\alpha^{-1}) \right\}. \]

Substituting these expressions into (3.24), we obtain

\[ g^+(\alpha) - g^-(\alpha) = \alpha^{-2(k+1)}D(\alpha)^{-1} \left\{ (n^+(\alpha)R^+(\alpha) - n^-(\alpha)R^-(\alpha)) \times \left( R^-(\alpha^{-1})g^+(\alpha^{-1}) - R^+(\alpha^{-1})g^-(\alpha^{-1}) \right) \\ + \left( R^+(\alpha) - R^-(\alpha) \right) \left( -R^-(\alpha^{-1})n^-(\alpha^{-1})g^+(\alpha^{-1}) + R^+(\alpha^{-1})n^+(\alpha^{-1})g^-(\alpha^{-1}) \right) \right\} \]

\[ = \alpha^{-2(k+1)}D(\alpha)^{-1} \left\{ n^+(\alpha)R^+(\alpha)R^-(\alpha^{-1}) - n^-(\alpha)R^-(\alpha)R^+(\alpha^{-1}) \\ - R^+(\alpha)R^-(\alpha^{-1})n^-(\alpha^{-1}) + R^-(\alpha)R^+(\alpha^{-1})n^-(\alpha^{-1})g^+(\alpha^{-1}) + \\ + \left[ -n^+(\alpha)R^+(\alpha)R^+(\alpha^{-1}) + n^-(\alpha)R^-(\alpha)R^+(\alpha^{-1}) \\ + R^+(\alpha)R^+(\alpha^{-1})n^+(\alpha^{-1}) - R^-(\alpha)R^+(\alpha^{-1})n^+(\alpha^{-1})g^-(\alpha^{-1}) \right] \right\} \]

\[ = \alpha^{-2(k+1)}D(\alpha)^{-1} \left\{ (R^+(\alpha)R^-(\alpha^{-1}) \left[ R^+(\alpha)R^+(\alpha^{-1})n^+(\alpha) - n^-(\alpha) - \frac{R^+(\alpha)}{R^-(\alpha)}n^-(\alpha^{-1}) + n^-(\alpha^{-1}) \right] g^+(\alpha^{-1}) \\ + R^+(\alpha)R^+(\alpha^{-1}) \left[ -n^+(\alpha) + R^+(\alpha)R^+(\alpha^{-1})n^+(\alpha) + \frac{R^+(\alpha)}{R^-(\alpha)}n^-(\alpha^{-1}) \right] g^-(\alpha^{-1}) \right\} \]

\[ = \alpha^{-2(k+1)}D(\alpha)^{-1} \left\{ A(\alpha)g^+(\alpha^{-1}) + B(\alpha)g^-(\alpha^{-1}) \right\}, \quad (3.25) \]

where

\[ A(\alpha) = R^-(\alpha)R^+(\alpha^{-1}) \times \left\{ R^+(\alpha) \left[ n^+(\alpha) - n^-(\alpha^{-1}) \right] + n^-(\alpha^{-1}) - n^-(\alpha) \right\}. \]
In view of theorem 2 and formulas (2.28), (2.30) we have the following equalities on $\Omega$:

$$\frac{R^{-}(\alpha)}{R^{+}(\alpha)} = e^{2i\varphi_{2}(\alpha)}; \quad R^{+}(\alpha)R^{-}(\alpha^{-1}) = \frac{\alpha - \alpha^{-1}}{|N(\alpha)|} e^{-i\nu(\alpha)},$$

with

$$\nu(\alpha) = \arg N^{+}(\alpha) = \begin{cases} \eta(\alpha + \alpha^{-1}), \ |\alpha| > 1, \\ \pi - \eta(\alpha + \alpha^{-1}), \ |\alpha| < 1. \end{cases}$$

(We have taken into account that $|N^{+}(\alpha)| = |N^{-}(\alpha)| = |N(\alpha)|$ for $\alpha \in \mathbb{R}$; further we also used $|R^{+}(\alpha)| = |R^{-}(\alpha)| \equiv |R(\alpha)|.$) It follows from these equalities that

$$R^{-}(\alpha)R^{-}(\alpha^{-1}) = R^{+}(\alpha)R^{-}(\alpha^{-1}) \frac{R^{+}(\alpha)}{R^{-}(\alpha)} = \frac{\alpha - \alpha^{-1}}{|N(\alpha)|} e^{-i\nu(\alpha)-2\varphi_{2}(\alpha)},$$

$$\left( \frac{n^{+}(\alpha) - n^{-}(\alpha^{-1})}{R^{+}(\alpha)} \right) \frac{R^{+}(\alpha)}{R^{-}(\alpha)} = N^{+}(\alpha)e^{-2i\varphi_{2}(\alpha)} = |N(\alpha)| e^{i\nu(\alpha)-2\varphi_{2}(\alpha)},$$

$$\frac{R^{-}(\alpha)}{R^{+}(\alpha)} = 1 = e^{2i\varphi_{2}(\alpha)} - 1 = \sin 2\varphi_{2}(\alpha) \left( i - \tan \varphi_{2}(\alpha) \right).$$

Since $\nu(\alpha) - 2\varphi_{2}(\alpha) = \left( 1 - s(\alpha) \right) \frac{\pi}{2} + s(\alpha) \chi_{0}(\alpha) \pi \geq 0,$ we have

$$e^{i\nu(\alpha)-2\varphi_{2}(\alpha)} = e^{i \left( 1-s(\alpha) \right) \frac{\pi}{2} } e^{is(\alpha)\chi_{0}(\alpha)\pi} = s(\alpha) \cos \chi_{0}(\alpha)\pi;$$

$$\sin 2\varphi_{2}(\alpha) = \sin \nu(\alpha) \cos \left( \left( 1 - s(\alpha) \right) \frac{\pi}{2} + s(\alpha) \chi_{0}(\alpha)\pi \right) - \cos \nu(\alpha) \sin \left( \left( 1 - s(\alpha) \right) \frac{\pi}{2} + s(\alpha) \chi_{0}(\alpha)\pi \right)$$

$$= \sin \nu(\alpha) \left\{ \cos \left( \left( 1 - s(\alpha) \right) \frac{\pi}{2} \right) \cos \left( s(\alpha) \chi_{0}(\alpha)\pi \right) - \sin \left( \left( 1 - s(\alpha) \right) \frac{\pi}{2} \right) \sin \left( s(\alpha) \chi_{0}(\alpha)\pi \right) \right\} + 0$$

$$= \sin \nu(\alpha) \cos \left( \left( 1 - s(\alpha) \right) \frac{\pi}{2} \right) \cos \left( s(\alpha) \chi_{0}(\alpha)\pi \right) + 0 + 0$$

$$= \sin \nu(\alpha) \cdot s(\alpha) \cos \left( \chi_{0}(\alpha)\pi \right) = s(\alpha) \cos \chi_{0}(\alpha)\pi \sin \nu(\alpha).$$

Hence,

$$R^{-}(\alpha)R^{-}(\alpha^{-1}) = \frac{\alpha - \alpha^{-1}}{|N(\alpha)|} s(\alpha) \cos \chi_{0}(\alpha)\pi;$$

$$\left( n^{+}(\alpha) - n^{-}(\alpha^{-1}) \right) \frac{R^{+}(\alpha)}{R^{-}(\alpha)} = |N(\alpha)| s(\alpha) \cos \chi_{0}(\alpha)\pi;$$

$$\frac{R^{-}(\alpha)}{R^{+}(\alpha)} - 1 = s(\alpha) \cos \chi_{0}(\alpha)\pi \sin \nu(\alpha) \left( i - \tan \varphi_{2}(\alpha) \right).$$

In fact, by the definition

$$\nu(t) = \begin{cases} \eta(\alpha + \alpha^{-1}), \ |\alpha| > 1, \\ \pi - \eta(\alpha + \alpha^{-1}), \ |\alpha| < 1, \end{cases}$$

$$2\varphi_{2}(\alpha) = \eta(\alpha + \alpha^{-1}) - \chi_{0}(\alpha)\pi.$$

for $|\alpha| > 1$ we have $\nu(\alpha) - 2\varphi_{2}(\alpha) = \chi_{0}(\alpha)\pi$, i.e. the formula is true, and for $|\alpha| < 1$ we have $\nu(\alpha) - 2\varphi_{2}(\alpha) = \pi - \chi_{0}(\alpha)\pi$, i.e. the formula is also true.
Therefore

\[ R^-(\alpha)R^-(\alpha^{-1}) = |R(\alpha)||R(\alpha^{-1})| \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} s(\alpha) \cos \chi_0(\alpha) \pi, \]

\( \left( n^+(\alpha) - n^-(\alpha^{-1}) \right) R^+(\alpha) \frac{R^-(\alpha)}{R^+(\alpha)} \left( R^-(\alpha) - 1 \right) \)

\[ = |N(\alpha)| s(\alpha) \cos \left( \chi_0(\alpha) \pi \right) s(\alpha) \cos \left( \chi_0(\alpha) \pi \right) \sin \nu(\alpha) \left( i - \tan \varphi_2(\alpha) \right) \]

\[ = |N(\alpha)| \sin \nu(\alpha) \left( i - \tan \varphi_2(\alpha) \right) = \left( i - \tan \varphi_2(\alpha) \right) \text{Im} N^+(\alpha). \]

Substituting these expressions into (3.26), we obtain

\[ A(\alpha) = |R(\alpha)||R(-\alpha)| \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} s(\alpha) \cos \chi_0(\alpha) \pi \left\{ - \left( i - \tan \varphi_2(\alpha) \right) \text{Im} N^+(\alpha) + 2i \text{Im} n^+(\alpha) \right\}. \]

Finally, from the definition of \( N(z) \), formulas (2.3), (2.4) and the absolute continuity of the functions \( \rho_R(\tau + \tau^{-1}), \rho_L(\tau + \tau^{-1}) \) on \( \Omega^2 \) it follows that

\[ \text{Im} N^+(\alpha) = \pi \left( \sigma(\alpha) + \sigma(\alpha^{-1}) \right), \quad \text{Im} n^+(\alpha) = \pi \sigma(\alpha), \]

and, since

\[ \tan \varphi_2(\alpha) = s(\alpha) \tan \frac{1}{2} \left( \eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi \right) = s(\alpha) \cos \chi_0(\alpha) \pi \tan |\varphi_2(\alpha)|, \]

we obtain the equality

\[ A(\alpha) = |R(\alpha)||R(\alpha^{-1})| \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} s(\alpha) \cos \chi_0(\alpha) \pi \left\{ - \left( i - \tan \varphi_2(\alpha) \right) \pi(\sigma(\alpha) + \sigma(\alpha^{-1})) + 2i \pi \sigma(\alpha) \right\} \]

\[ \times \left\{ s(\alpha) \cos \left( \chi_0(\alpha) \pi \right) \tan |\varphi_2(\alpha)| \left( \sigma(\alpha) + \sigma(\alpha^{-1}) \right) - i \left( \sigma(\alpha^{-1}) - \sigma(\alpha) \right) \right\} \]

\[ = |R(\alpha)||R(\alpha^{-1})| \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} s(\alpha) \pi \]

\[ \times \left\{ s(\alpha) \tan |\varphi_2(\alpha)| \left( \sigma(\alpha) + \sigma(\alpha^{-1}) \right) - i \left( \sigma(\alpha^{-1}) - \sigma(\alpha) \right) \cos \chi_0(\alpha) \pi \right\} \]

\[ = |R(\alpha)||R(\alpha^{-1})| \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} s(\alpha) \pi \sqrt{\sigma(\alpha) \sigma(\alpha^{-1})} \]

\[ \times \left\{ s(\alpha) \tan |\varphi_2(\alpha)| \left( \sqrt{\frac{\sigma(\alpha)}{\sigma(\alpha^{-1})}} + \sqrt{\frac{\sigma(\alpha^{-1})}{\sigma(\alpha)}} - i \left( \sqrt{\frac{\sigma(\alpha)}{\sigma(\alpha^{-1})}} - \sqrt{\frac{\sigma(\alpha^{-1})}{\sigma(\alpha)}} \right) \cos \chi_0(\alpha) \pi \right) \right\} \]

\[ = |R(\alpha)||R(\alpha^{-1})| \pi \sqrt{\sigma(\alpha) \sigma(\alpha^{-1})} \left\{ \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} p(\alpha) - i \frac{\alpha}{|\alpha|} m(\alpha) \right\}. \]

where the functions \( p(\alpha) \) and \( m(\alpha) \) are defined by (3.21'), (3.23').

3In fact, if \( \chi_0(\alpha) = 0 \), then the argument of \( \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) \) lies in \((0, \pi)\) and

\[ \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) = \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) = \frac{1}{2}(\eta(\alpha + \alpha^{-1})). \]

And if \( \chi_0(\alpha) = 1 \), then the argument of \( \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) \) lies in \((-\pi, 0)\) and

\[ \tan \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) = - \tan \left| \frac{1}{2}(\eta(\alpha + \alpha^{-1}) - \chi_0(\alpha) \pi) \right|. \]
Substitute the obtained expressions of $D(\alpha)$, $A(\alpha)$ and $B(\alpha) = A(\alpha)$ into (3.25):

\[ g^+(\alpha) - g^-(\alpha) = \alpha^{-2(k+1)} \cdot \frac{1}{2\pi i |R(\alpha^{-1})|^2 \sigma(\alpha^{-1})} \]

\[ \times \left\{ |R(\alpha)| |R(\alpha^{-1})| |\sqrt{\sigma(\alpha)\sigma(\alpha^{-1})} \left[ \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} p(\alpha) - i \frac{\alpha}{|\alpha|} m(\alpha) \right] g^+(\alpha^{-1}) + \right. \]

\[ + |R(\alpha)| |R(\alpha^{-1})| |\sqrt{\sigma(\alpha)\sigma(\alpha^{-1})} \left[ \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} p(\alpha) + i \frac{\alpha}{|\alpha|} m(\alpha) \right] g^-(\alpha^{-1}) \left. \right\} \]

\[ = - \frac{\alpha^{-2(k+1)}}{2i} \frac{|R(\alpha)|}{|R(\alpha^{-1})|} \sqrt{\sigma(\alpha)\sigma(\alpha^{-1})} \left[ \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} p(\alpha) \left( g^+(\alpha^{-1}) + g^-(\alpha^{-1}) \right) + i \frac{\alpha}{|\alpha|} m(\alpha) \left( g^+(\alpha^{-1}) - g^-(\alpha^{-1}) \right) \right], \]

or

\[ -2i \pi(\alpha)^2 \frac{\alpha^{-2(k+1)}}{2i} \left( g^+(\alpha) - g^-(\alpha) \right) \]

\[ = -p(\alpha) \frac{\alpha - \alpha^{-1}}{|\alpha - \alpha^{-1}|} \left( g^+(\alpha^{-1}) + g^-(\alpha^{-1}) \right) + i \frac{\alpha}{|\alpha|} m(\alpha) \left( g^+(\alpha^{-1}) - g^-(\alpha^{-1}) \right), \]

which is equivalent to (3.20).

Note that the function $\varphi_2(\alpha)$ on the set $\Omega_2$ is devided of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, from which it follows that $\tan \varphi_2(\alpha)$ is bounded and bounded away from zero. Since $\mu(\alpha + \alpha^{-1})$ is also bounded and bounded away from zero, on this set, the function $p(\alpha)$ is also bounded and bounded away from zero.

**Calculation of the integrals along the contour $\Gamma_3$.**

**Lemma 3.6.** For $\theta \in [-\pi, \pi)$ the functions $g \left( k, (1 + h) e^{i\theta} \right)$ have limits $g^\pm(k, e^{i\theta}) \in L^2[-\pi, \pi]$ as $h \downarrow 0$.

The following equality is true for the integral along the contour $\Gamma_3$ in (3.2):

\[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{g(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{g^+(\xi) - g^-(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g^+(e^{i\theta}) - g^-(e^{i\theta})}{e^{i\theta} - z} e^{i\theta} d\theta. \tag{3.27} \]

**Proof.** We remind that the contour $\Gamma_3$ consists of two concentric circles of radius $(1 - \delta)$ and $(1 + \delta)$ with the center at the point $0$. Let us deform the contour $\Gamma_3$ in the following way: we deform its part that lies in the $\varepsilon$-neighborhoods ($\varepsilon > 0$) of the points $-1, 1$ by the arcs of radius $\varepsilon$ with the center at these points. Tending $\delta \downarrow 0$ and using the methods analogous to those of the previous subsection we obtain that

\[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{g(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{g^+(\xi) - g^-(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{|\xi| = \varepsilon} \frac{g(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{g(\xi)}{\xi - z} d\xi, \]

if

\[ \lim_{h \downarrow 0} \| g \left( (1 + h) e^{i\theta} \right) - g^\pm(e^{i\theta}) \|_{L^p[|\alpha(\varepsilon), \pi - \alpha(\varepsilon)|]} = 0, \tag{3.28} \]

\[ \lim_{h \downarrow 0} \| g \left( (1 + h) e^{i\theta} \right) - g^\pm(e^{i\theta}) \|_{L^p[-\pi + \alpha(\varepsilon), -\alpha(\varepsilon)]} = 0, \tag{3.29} \]

with $p \geq 1$, where $\alpha(\varepsilon)$ is such a small number that $e^{i\alpha(\varepsilon)}$ lies in the $\varepsilon$-neighborhood of the point $1$. Tending $\varepsilon \downarrow 0$ in the last formula we obtain that

\[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{g(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{g^+(\xi) - g^-(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g^+(e^{i\theta}) - g^-(e^{i\theta})}{e^{i\theta} - z} e^{i\theta} d\theta, \]
if
\[
\lim_{\epsilon \downarrow 0} \int_{|\zeta| = \epsilon} \frac{g(\zeta)}{\zeta - z} d\zeta = 0, \\
\]
and also if for all small \( \epsilon > 0 \) the conditions (3.28) and (3.29) are fulfilled.

Without restriction of generality, we will prove these facts for the interior part of the disk, i.e., we will prove that
\[
\lim_{\epsilon \downarrow 0} \int_{|\zeta| = \epsilon, \, |\zeta| < 1} \frac{g(\zeta)}{\zeta - z} d\zeta = 0,
\]
\[
g^\pm(e^{i\theta}) \in L^p[-\pi, \pi], \quad p \geq 1,
\]
\[
\lim_{h \downarrow 0} \|g\left((1 - h)e^{i\theta}\right) - g^\pm(e^{i\theta})\|_{L^p[|\alpha(\epsilon)|, \pi - \alpha(\epsilon)]} = 0,
\]
(3.28')
\[
\lim_{h \downarrow 0} \|g\left((1 - h)e^{i\theta}\right) - g^\pm(e^{i\theta})\|_{L^p[-\pi + \alpha(\epsilon), -\alpha(\epsilon)]} = 0.
\]
(3.29')

As it is known from theorem 2, for the function \( R(z) \) the representation
\[
R(z) = C R_\nu(z) R_0(z) R_0(z) R_1(z) R_2(z) = \hat{R}(z) R^{(1)}(z) R^{(2)}(z),
\]
is true, where \( R^{(1)}(z) = R_0(z) = \hat{P}(z, \pm \frac{\mu}{2}) \), and \( R^{(2)}(z) = C R_0(z) R_1(z) R_2(z) \) is a holomorphic function on the unit circle. Let us derive an analogous decomposition for the function \( n(z) \). It can be shown (see this method, for example, in [15]) that inside the unit disk
\[
n(z) = n_0(z) n_1(z), \quad |z| < 1,
\]
where
\[
n_0(z) = \hat{P}(z, \hat{\eta}^R),
\]
\[
\hat{\eta}^R(\theta) = \begin{cases} 
-\eta^R(e^{i\theta} + e^{-i\theta}) + \frac{\pi}{2}, & 0 < \theta < \pi, \\
\eta^R(e^{i\theta} + e^{-i\theta}) - \frac{\pi}{2}, & -\pi < \theta < 0,
\end{cases}
\]
and \( n_1(z) \) is a holomorphic on the circle function, which vanishes at the points \(-1, 1\), i.e.
\[
n(z) = (z - 1)(z + 1)n^{(1)}(z)n^{(2)}(z),
\]
where \( n^{(1)}(z) = n_0(z) = \hat{P}(z, \hat{\eta}^R) \), and \( n^{(2)}(z) \) is holomorphic on the circle. Thus, in view of (3.1),
\[
g(\zeta) = g^{(1)}(\zeta) R_\nu(\zeta) R^{(1)}(\zeta) + g^{(2)}(\zeta) R_\nu(\zeta) R^{(1)}(\zeta) n^{(1)}(\zeta)(\zeta - 1)(\zeta + 1),
\]
(3.30)
where the functions \( g^{(1)}(\zeta), g^{(2)}(\zeta) \) are holomorphic on the unit circle. Since the functions \( \eta(\tau) \) and \( \hat{\eta}^R(\tau) \) satisfy the Hölder condition under \((-2, 2)\), the functions \( R^{(1)}((1 - h)e^{i\theta}) = \hat{P}((1 - h)e^{i\theta}, -\frac{\pi}{2}) \), \( R^{(1)}((1 - h)e^{i\theta}) R^{(1)}((1 - h)e^{i\theta})(e^{i\theta} - 1) = \hat{P}((1 - h)e^{i\theta}, -\frac{\pi}{2} + \hat{\eta}^R(e^{i\theta} - 1)) \), (as functions of \( \theta \)), converge uniformly when \( h \to 0 \) to their limits on the segments \([\alpha(\epsilon), \pi - \alpha(\epsilon)], [-\pi + \alpha(\epsilon), -\pi + \alpha(\epsilon)]\). All the more, they converge in \( L^2 \)-norm, and their limits also belong to \( L^2 \). Further, according to lemma 2.5, the function \( R_\nu((1 - h)e^{i\theta}) \) converges uniformly to its limit \( R_\nu^+(e^{i\theta}) \) as \( h \downarrow 0 \). That is why all the function \( g((1 - h)e^{i\theta}) \) converges uniformly as \( h \downarrow 0 \) to its limit \( g^+(e^{i\theta}) \) on segments \([\alpha(\epsilon), \pi - \alpha(\epsilon)], [-\pi + \alpha(\epsilon), -\pi + \alpha(\epsilon)]\). Without loss of generality, we will prove the two other properties only in a neighborhood of the point \( \theta = 0 \), i.e., we will prove that
\[
\lim_{\epsilon \downarrow 0} \int_{|\zeta| = \epsilon, \, |\zeta| < 1} \frac{g(\zeta)}{\zeta - z} d\zeta = 0, \quad g^+(e^{i\theta}) \in L^2[-\frac{\pi}{2}, \frac{\pi}{2}].
\]
(3.29)

We remind that the functions \( \hat{\eta}_{13}(\theta), \hat{\eta}_R^R(\theta) \) are odd and have a jump at the point \( \theta = 0 \). Let
\[
\hat{\eta}_1^R(\theta) = \begin{cases} 
\hat{\eta}_1^R(+0), & 0 < \theta < \pi, \\
\hat{\eta}_1^R(-0), & -\pi < \theta < 0,
\end{cases}
\]
\[
\hat{\eta}_2^R(\theta) = \hat{\eta}^R(\theta) - \hat{\eta}_1^R(\theta);
\]
and also if for all small \( \epsilon > 0 \) the conditions (3.28) and (3.29) are fulfilled.
\[ \hat{v}_1(\theta) = \begin{cases} -\frac{\gamma_3(0)}{2}, & 0 < \theta < \pi, \\ -\frac{\gamma_3(0)}{2}, & -\pi < \theta < 0, \end{cases} \]

\[ \hat{v}_2(\theta) = -\frac{\gamma_3(0)}{2} - \hat{v}_1(\theta). \]

The functions \( \hat{v}_2(\theta), \hat{v}_2^R(\theta) \) satisfy the Hölder condition under \(-\frac{\pi}{2}, \frac{\pi}{2}\). That is why the functions \( \hat{P}((1-h)e^{i\theta}), \hat{P}_2((1-h)e^{i\theta}), \hat{v}_2^R \) converge uniformly to their limits \( \hat{P}^+(e^{i\theta}), \hat{P}_2^+(e^{i\theta}), \hat{v}_2^R \) for \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), in particular, they are uniformly bounded in a neighborhood of \( \theta = 0 \). Further,

\[ \hat{P}(z, \hat{\eta}^R) = C \left( \frac{z-1}{z+1} \right)^{\frac{2\eta R(0)}{\pi}}, \quad \hat{P}(z, \hat{v}_1) = C \left( \frac{z-1}{z+1} \right)^{\frac{\gamma_3(0)}{2}}. \]

In fact, looking at the limit values of the arguments of the function \( \frac{z-1}{z+1} \) on the circle, it can be seen that, to within a constant factor,

\[ \hat{P}(z, \theta, \frac{\pi}{2}) = \frac{z-1}{z+1}, \]

and, since

\[ \hat{\eta}^R_1(\theta) = \frac{2\eta^R(\theta)}{\pi} \left( \frac{\theta}{|\theta|} \right), \quad \hat{v}_1(\theta) = -\frac{\gamma_3(0)}{\pi} \left( \frac{\theta}{|\theta|} \right), \]

then

\[ \hat{P}(z, \hat{\eta}^R) = \hat{P}(z, \frac{2\eta^R(\theta)}{\pi} \left( \frac{\theta}{|\theta|} \right)), \quad \hat{P}(z, \hat{v}_1) = \hat{P}(z, -\frac{\gamma_3(0)}{\pi} \left( \frac{\theta}{|\theta|} \right)), \]

from where we have the necessary identities.

Thus,

\[ n(1) \left( (1-h)e^{i\theta} \right) \left( (1-h)e^{i\theta} - 1 \right) = \hat{P}((1-h)e^{i\theta}, \hat{v}_1^R) \left( (1-h)e^{i\theta} - 1 \right) \left( (1-h)e^{i\theta} + 1 \right)^{\frac{2\eta R(0)}{\pi}}. \]

Since \(-\frac{\pi}{2} < \eta^R(0) < \frac{\pi}{2}\), the function \((1-h)e^{i\theta} - 1)^{1+\frac{2\eta^R(0)}{\pi}}\) is uniformly bounded in a neighborhood of \( \theta = 0 \) and converges uniformly as \( h \downarrow 0 \) to its limit \((e^{i\theta} - 1)^{1+\frac{2\eta^R(0)}{\pi}}\). It means that \( n(1) \left( (1-h)e^{i\theta} \right) \left( (1-h)e^{i\theta} - 1 \right)^{1+\frac{2\eta^R(0)}{\pi}} \) is uniformly bounded and uniformly converges to its limit (which belongs to \( L^2 \)). According to lemma 2.5, the function \( R_\nu((1-h)e^{i\theta}) \) has the same property. That is why, according to decomposition (3.30), to prove that \( g^+(e^{i\theta}) \in L^2[-\frac{\pi}{2}, \frac{\pi}{2}] \) we have to prove that \( R^{(1)}+(e^{i\theta}) \in L^2[-\frac{\pi}{2}, \frac{\pi}{2}] \). But this fact follows from what

\[ R^{(1)}+(1-h)e^{i\theta} = \hat{P}((1-h)e^{i\theta}, \hat{v}_1) \hat{P}((1-h)e^{i\theta}, \hat{v}_2) \]

\[ = \left( (1-h)e^{i\theta} - 1 \right) \left( (1-h)e^{i\theta} + 1 \right)^{\frac{\gamma_3(0)}{\pi}} \hat{P}((1-h)e^{i\theta}, \hat{v}_2). \]

As it was already said, the second one of the functions of the last product, \( \hat{P}((1-h)e^{i\theta}, \hat{v}_2) \), is uniformly bounded and converges uniformly as \( h \downarrow 0 \). As for the first function, \( \left( (1-h)e^{i\theta} - 1 \right) \left( (1-h)e^{i\theta} + 1 \right)^{\frac{\gamma_3(0)}{\pi}} \), in view of condition

B) we have \( 0 < \eta(2-\nu) < \pi \), from where \(-\frac{\pi}{2} < \gamma_3(0) < \frac{\pi}{2}\). Hence, the function \( \left( (1-h)e^{i\theta} - 1 \right) \left( (1-h)e^{i\theta} + 1 \right)^{\frac{\gamma_3(0)}{\pi}} \) is square-summable for \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), from where we deduce the belonging \( R^{(1)}+(e^{i\theta}) \in L^2[-\frac{\pi}{2}, \frac{\pi}{2}] \). Therefore, \( g^+(e^{i\theta}) \in L^2[-\frac{\pi}{2}, \frac{\pi}{2}] \).

Let us now pass to the property

\[ \lim_{\varepsilon \downarrow 0} \int_{|\zeta|=1} \frac{g(\zeta)}{\zeta - z} d\zeta = 0. \]
As it may be seen, 

\[ g(\zeta) = g^{(3)}(\zeta) \left( \frac{\zeta - 1}{\zeta + 1} \right)^{-\frac{\gamma_3(\mu)}{\delta}}, \]

where the function 

\[ g^{(3)}(\zeta) = (g^{(1)}(\zeta) + g^{(2)}(\zeta)n^{(1)}(\zeta)(\zeta - 1)(\zeta + 1))R_{\mu}(\zeta)P(\zeta, \nu_2) \]

is uniformly bounded in a neighborhood of 1 (we have just proved it individually for \(n^{(1)}(\zeta)(\zeta - 1), R_{\mu}(\zeta)\) and \(P(\zeta, \nu_2)\)). Hence,

\[
| \int_{|\zeta - 1| = \varepsilon, |\zeta| < 1} \frac{g(\zeta)}{\zeta - z} d\zeta | \leq \max_{|\zeta - 1| = \varepsilon, |\zeta| < 1} \left| \frac{g^{(3)}(\zeta)}{(\zeta - z)(\zeta + 1)} \right| \int_{|\zeta - 1| = \varepsilon, |\zeta| < 1} |\zeta - 1|^{-\frac{\gamma_3(\mu)}{\delta}} |d\zeta| \\
\leq \max_{|\zeta| < 1} \left| \frac{g^{(3)}(\zeta)}{(\zeta - z)(\zeta + 1)} \right| \varepsilon \left| \zeta - 1 \right|^{-\frac{\gamma_3(\mu)}{\delta}} \to 0, \quad \varepsilon \downarrow 0,
\]

because \(\left| \zeta - 1 \right|^{-\frac{\gamma_3(\mu)}{\delta}} < 1\), which complete the proof of the lemma. 

**Lemma 3.7.** Almost everywhere on the unit circle \(|\xi| = 1\) the following equality holds

\[ \xi^{2(k+1)}(g^+(k, \xi) - g^-(k, \xi)) = -\hat{r}(\xi)(g^+(k, \xi^{-1}) + g^-(k, \xi^{-1})) , \]

(3.31)

where

\[ \hat{r}(\xi) = -\frac{1}{1 + g(\xi)} \cdot \frac{n^+(\xi) - n^-(\xi)}{\xi - \xi^{-1}} R^+(\xi)R^+(\xi^{-1}) , \] \hspace{1cm} (3.25')

\[ q(\xi) = 2\pi \sqrt{\rho_r'(\xi + \xi^{-1})\rho_r'(\xi + \xi^{-1})} \frac{n^+(\xi) - n^-(\xi^{-1})}{n^+(\xi) - n^-(\xi^{-1})} . \] \hspace{1cm} (3.25'')

Moreover,

\[ |\hat{r}(\xi)| < C < 1 . \]

**Proof.** The functions \(\psi^-(k, \xi), \psi^+(k, \xi)\) (the limit values of the Weyl solutions on the unit circle) are the solutions (with respect to \(k\)) of finite-difference equation (0.5). Hence, they are linear combinations of the functions \(\psi^+(k, \xi), \psi^+(k, \xi^{-1}) = \psi^+(k, \xi)\) and \(\psi^-(k, \xi), \psi^-(k, \xi^{-1}) = \psi^-(k, \xi),\) respectively:

\[ \psi^-(k, \xi) = a(\xi)\psi^+(k, \xi) + b(\xi)\psi^+(k, \xi^{-1}) , \] \hspace{1cm} (3.34)

\[ \psi^+(k, \xi) = c(\xi)\psi^-(k, \xi) + d(\xi)\psi^-(k, \xi^{-1}) . \] \hspace{1cm} (3.35)

1. Let us first find the expression for the functions \(a(\xi), b(\xi), c(\xi), d(\xi)\) and show that \([\frac{b(\xi)}{a(\xi)}] \leq 1, [\frac{d(\xi)}{c(\xi)}] \leq 1\). We remind that

\[ \psi(k, z) = n(z)P_k(z + z^{-1}) + Q_k(z + z^{-1}) . \]

Substituting this expression in (3.34) and (3.35) for \(k = 0, k = -1\), and taking into account the initial data on \(Q_k(z)\) and \(P_k(z)\), we obtain two systems of linear equation. Solving these linear systems, we find

\[ a(\xi) = \frac{n^+(\xi^{-1}) - n^-}{} \frac{n^-(\xi) - n^+}{} , \quad b(\xi) = \frac{n^+(\xi) - n^-}{} \frac{n^-(\xi) - n^+}{} , \]

\[ c(\xi) = \frac{n^+(\xi^{-1}) - n^-}{} \frac{n^-(\xi) - n^+}{} , \quad d(\xi) = \frac{n^+(\xi) - n^-}{} \frac{n^-(\xi) - n^+}{} . \]

Further,

\[ \frac{b(\xi)}{a(\xi)} = \frac{n^+(\xi) - n^-}{} \frac{n^-(\xi) - n^+}{} , \quad \frac{n^+(\xi^{-1}) - n^-}{} \frac{n^-(\xi) - n^+}{} = -\frac{n^+(\xi) - n^-}{} \frac{n^-(\xi) - n^+}{} = \frac{n^+(\xi) - n^-}{} \frac{n^-(\xi) - n^+}{} . \]

The real parts of the numerator and the denominator in the last expression are equal. Thus, in order to verify that \([\frac{b(\xi)}{a(\xi)}] \leq 1\), it is sufficient to show that \(|\text{Im}(n^+(\xi) - n^-(\xi))| \leq |\text{Im}(n^+(\xi) - n^-(\xi))|\). But it is an
We also remind that the signs of the imaginary parts of $n^+(\xi)$ and $n^-(\xi)$, $|\xi| = 1$, are the same. In the same way we prove that the absolute value of the expression

$$\frac{d(\xi)}{c(\xi)} = \frac{n^-(\xi) - n^+(\xi)}{n^-(\xi^{-1}) - n^+(\xi)}$$

is less or equal than 1.

2. It is seen, that the coefficients $a(\xi), b(\xi), c(\xi), d(\xi)$ are defined by the limit values of the function $n(z)$ at the points $\xi, \xi^{-1}$ and are not independent. To clarify the connection between the coefficients in equalities (3.34) and (3.35), we introduce the notation:

$$v^\pm(\xi) = \text{Im} n^\pm(\xi) = \frac{n^\pm(\xi) - n^\pm(\xi^{-1})}{2i}.$$  

Taking into account that the signs of Im $n^+(\xi)$ and Im $n^-(\xi)$ are the same, we can rewrite the equality (3.28) in the form

$$v^+(\xi)R^+(\xi)R^+(\xi^{-1}) = v^-(\xi)R^-(\xi)R^-(\xi^{-1}).$$

In these notations

$$\frac{1}{a(\xi)} = \frac{n^+(\xi) - n^+\xi^{-1})}{n^-(\xi) - n^+(\xi^{-1})} = \frac{2i v^+(\xi)}{N^+(\xi)} = \frac{2i}{\xi - \xi^{-1}}v^+(\xi)$$

$$\frac{1}{c(\xi)} = \frac{n^-(\xi) - n^-(\xi^{-1})}{n^+(\xi) - n^-(\xi^{-1})} = \frac{2i v^-(\xi)}{N^-(\xi)} = \frac{2i}{\xi - \xi^{-1}}v^-(\xi).$$

In view of our factorization theorem the last expression can be rewritten in the form

$$\frac{1}{a(\xi)} = \frac{2i}{\xi - \xi^{-1}}v^+(\xi) \cdot R^-(\xi)R^+(\xi^{-1}) = \frac{2i}{\xi - \xi^{-1}}v^+(\xi) \cdot R^+(\xi)R^+(\xi^{-1}) = q(\xi)$$

$$\frac{1}{c(\xi)} = \frac{2i}{\xi - \xi^{-1}}v^-(\xi) \cdot R^+(\xi)R^-(\xi^{-1}) = \frac{2i}{\xi - \xi^{-1}}v^-(\xi) \cdot R^+(\xi)R^+(\xi^{-1}) = q(\xi),$$

with

$$q(\xi) = \frac{2i}{\xi - \xi^{-1}}v^+(\xi) \cdot R^-(\xi)R^+(\xi^{-1}) = \frac{2i}{\xi - \xi^{-1}}v^-(\xi) \cdot R^+(\xi)R^+(\xi^{-1}).$$

Let us analyze the coefficient $q(\xi)$. We remind, that, due to the definition of $R(z)$, we have $R^+(\xi^{-1}) = R^+(\xi)$. Besides, $v^+(\xi) > 0, \text{Im} \xi > 0,$ and $v^-(\xi) < 0, \text{Im} \xi < 0.$ That is why for all $\xi \neq \pm 1$

$$q(\xi) > 0.$$  

We also remind that $R(z) = R_{0123}(z)R_\mu(z)$, where $R^+_{0123}(\xi) = R^-_{0123}(\xi)$, and also $R^+_\mu(\xi) = \frac{1}{\sqrt{\rho(\xi)\rho(\xi)}}.$

Then,

$$q(\xi) = 2\left|\frac{v^+(\xi)}{\xi - \xi^{-1}}R^+_{0123}(\xi)R^\mu_\xi(\xi) R^+(\xi^{-1})\right|$$

$$= 2\sqrt{v^+(\xi)\xi^{-1}} R^+_{0123}(\xi)R^\mu_\xi(\xi) R^+(\xi^{-1}) = 2\sqrt{\text{Im} n^+(\xi)\xi^{-1}} R^+(\xi)$$

$$= 2\sqrt{\text{Im} n^+(\xi)\xi^{-1}} = 2\pi\sqrt{\rho(\xi)\rho(\xi^{-1})}.$$  

3. We obtain from equality (3.36)

$$\frac{1}{a(\xi)} \psi^-(k, \xi) = \psi^+(k, \xi) + \frac{b(\xi)}{a(\xi)} \psi^+(k, \xi^{-1}),$$

which is equivalent, as we have proved, to

$$q(\xi) R^-(\xi) \psi^-(k, \xi) = R^+(\xi) \psi^+(k, \xi) + \frac{b(\xi)}{a(\xi)} R^+(\xi^{-1})\psi^+(k, \xi^{-1}),$$

or

$$q(\xi) R^-(\xi) \psi^-(k, \xi) = R^+(\xi) \psi^+(k, \xi) + \frac{b(\xi)}{a(\xi)} R^+(\xi^{-1})\psi^+(k, \xi^{-1}).$$  

(3.39)
In the same way, we have from (3.35) that
\[ q(\xi) R^+(\xi) \psi^+(k, \xi) = R^- (\xi) \psi^- (k, \xi) + \frac{d(\xi)}{c(\xi)} \frac{R^+(\xi)}{R^- (\xi^{-1})} R^- (\xi^{-1}) \psi^- (k, \xi^{-1}). \] (3.40)

To reduce the two last equalities to the necessary form let us verify the following relation for the coefficients in the right-hand side:
\[ \frac{b(\xi)}{a(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} = \frac{d(\xi)}{c(\xi)} \cdot \frac{R^- (\xi)}{R^- (\xi^{-1})}. \] (3.41)

In fact,
\[ \frac{b(\xi)}{a(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} + \frac{d(\xi)}{c(\xi)} \cdot \frac{R^- (\xi)}{R^- (\xi^{-1})} = \frac{n^+(\xi) - n^-(\xi)}{n^+(\xi^{-1}) - n^-(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} - \frac{n^+(\xi) - n^-(\xi)}{n^+(\xi) - n^-(\xi)} \cdot \frac{R^- (\xi)}{R^- (\xi^{-1})} = \frac{\xi^2 - \xi^{-1}}{\xi - \xi^{-1}} \left\{ \frac{R^+(\xi)}{R^+(\xi^{-1})} + \frac{R^- (\xi)}{R^- (\xi^{-1})} \right\} = 0. \]

Having verified (3.41), substract (3.39) from (3.40). We obtain on the unit circle the equality
\[ (1 + q(\xi))(\psi^+(k, \xi) R^+(\xi) - \psi^- (k, \xi) R^- (\xi)) \]
\[ = \frac{b(\xi)}{a(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} \left( R^+(\xi^{-1}) \psi^+(k, \xi^{-1}) + R^- (\xi^{-1}) \psi^- (k, \xi^{-1}) \right). \]

Thus, for the function \( g(k, z) \), defined in (3.1), we have on the unit circle an equality, which establish connection between its limit values at the points \( \xi \) and \( \xi^{-1} \):
\[ \xi^{2(k+1)}(1 + q(\xi))(g^+(k, \xi) - g^-(k, \xi)) = -\frac{b(\xi)}{a(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} (g^+(k, \xi^{-1}) + g^-(k, \xi^{-1})). \] (3.42)

It remains to reduce the expression
\[ -\frac{b(\xi)}{a(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} = \frac{n^+(\xi) - n^-(\xi)}{N^+(\xi) - N^-(\xi)} \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} \]
\[ = -\frac{n^+(\xi) - n^-(\xi)}{\xi - \xi^{-1}} \cdot R^- (\xi) R^+(\xi^{-1}) \cdot \frac{R^+(\xi)}{R^+(\xi^{-1})} = -\frac{n^+(\xi) - n^-(\xi)}{\xi - \xi^{-1}} \cdot R^+(\xi) R^- (\xi). \]

Thus, equality (3.42) can be rewritten as (3.31) with the coefficient
\[ \hat{r}(\xi) = \frac{-1}{1 + q(\xi)} \cdot \frac{n^+(\xi) - n^-(\xi)}{\xi - \xi^{-1}} \cdot R^+(\xi) R^- (\xi), \]
as was to be proved.

4. To complete the proof of the lemma we have to make sure that \( \max_{\xi=1} |\hat{r}(\xi)| < 1 \). For this let us write \( \hat{r}(\xi) \) out in the form
\[ \hat{r}(\xi) = \frac{-1}{1 + q(\xi)} \cdot \left\{ \frac{n^+(\xi) - n^-(\xi)}{\xi - \xi^{-1}} \cdot R^+(\xi) R^+(\xi^{-1}) \right\}. \]
As it was already said, the absolute value of the factor in braces is less or equal than 1. (In fact, \( \left| \frac{b(\xi)}{a(\xi)} \right| < 1 \) and \( \left| \frac{R^+(\xi)}{R^+(\xi^{-1})} \right| = \left| \frac{R^+_{123}(\xi)}{R^+_{0123}(\xi^{-1})} \right| \left| \frac{R^+_{123}(\xi^{-1})}{R^+_{0123}(\xi)} \right| = 1 \).) Hence, it is sufficient to demonstrate that
\[ \max_{|\xi|=1} \frac{1}{1 + q(\xi)} < 1, \]
i.e. that the positive coefficient \( q(\xi) \) is bounded below from zero. But, from the definition (3.38),
\[ q(\xi) = 2\pi \sqrt{\rho^+_{0123}(\xi^{-1}) R^+_{0123}(\xi^{-1})} = \sqrt{\frac{\text{Im} m R^+ (\tau + i \theta) \text{Im} \frac{R_{\mu}^- (\tau + i \theta)}{|M(\tau + i \theta)|}}{|\tau - \xi + \xi^{-1}|}. \]
\[
\sqrt{\frac{|m^R(\tau + i0)| \sin \eta^R(\tau) | \frac{1}{m^L(\tau + i0)} | \sin \eta^L(\tau)}{|M(\tau + i0)|}}
\]

We remind, that, to within a constant number,
\[
b_{-1}m^R(\lambda) = P(\lambda, \eta^R), \quad \frac{-1}{b_{-1}m^L(\lambda)} = P(\lambda, \eta^L),
\]
\[
M(\lambda) = b_{-1}m^R(\lambda) - \frac{1}{b_{-1}m^L(\lambda)} = P(\lambda, \eta),
\]
where the functions \(\eta^R(\tau), \eta^L(\tau)\) and \(\eta(\tau)\) satisfy the Hőlder condition on the interval \((-2, 2)\) and have, according to the condition E), the same jumps at the points \(-2\) and \(2\). This means that
\[
\eta^R(\tau) = \eta_1^R(\tau) + \delta_1(\tau),
\]
\[
\eta^L(\tau) = \eta_1^L(\tau) + \delta_1(\tau),
\]
\[
\eta(\tau) = \eta_1(\tau) + \delta_1(\tau),
\]
where the functions \(\eta_1^R(\tau), \eta_1^L(\tau), \eta_1(\tau)\) satisfy the Hőlder condition in a certain neighborhood of the segment \([-2, 2]\), and \(\delta_1(\tau)\) is the function, which has jumps at the points \(-2\) and \(2\), and is constant at the other points. Hence,
\[
q(\xi) = \sqrt{\frac{|P(\tau + i0, \eta^R)|^2 |P(\tau + i0, \eta^-)|^2}{|P(\tau + i0, \eta)|^2}}
\]
\[
\frac{\sin \eta^R(\tau) |P(\tau + i0, \eta^-)|^2}{|P(\tau + i0, \eta)|} = \sqrt{\frac{|P(\tau + i0, \eta^R)|^2 |P(\tau + i0, \eta^-)|^2}{|P(\tau + i0, \eta)|^2}}
\]
\[
|\tau^\varepsilon + \xi^{-1}|.
\]

The functions \(|P(\tau + i0, \eta^R)|, |P(\tau + i0, \eta^-)|\) and \(|P(\tau + i0, \eta)|\) are bounded and bounded away from zero, because \(\eta_1^R(\tau), \eta_1^L(\tau), \) and \(\eta_1(\tau)\) satisfy the Hőlder condition. Further, \(\sin \eta^R(\tau)\) and \(\sin \eta^- (\tau)\) are bounded away from zero according to condition B). Thus,
\[
\min_{|\xi|=1} q(\xi) > 0,
\]
which complete the proof of the lemma.

**Deduction of the equation**

Let us denote by \(\chi_1(\beta), \chi_2^2(\beta)\) and \(\chi_2^2(\beta)\) the indicators of the sets \(\Phi \cup \Omega_1, (-1, 1) \cap \Omega_2^*\) and \(\Omega_2^*\) and define the measure \(d\sigma(\beta)\) and the function \(u(k, \beta)\) on the union
\[
\Omega_0 = \Phi \cup \Omega_1 \cup ((-1, 1) \cap \Omega_2^*) \cup \Omega_2^*
\]
of these sets and also on the unit circle by the equalities
\[
d\sigma(\beta) = \chi_1(\beta) \frac{1}{|\beta - \beta^{-1}|^2} dp_1(\beta) + \chi_2^2(\beta) \frac{1}{|\beta - \beta^{-1}|^2} dp_2(\beta) + \chi_2^2(\beta) \frac{1}{|\beta - \beta^{-1}|^2} p(\beta) d\beta,
\]
(3.43)
\[
u(k, \beta) = (\chi_1(\beta) + \chi_2^2(\beta)\beta^{-2(k+1)}) g(k, \beta^{-1})
\]
\[
-\chi_2^2(\beta) i p(\beta)^{-1} \frac{\beta - \beta^{-1}}{|\beta - \beta^{-1}|^2} (g^+(k, \beta) - g^-(k, \beta))
\]
\[
+ \chi_2^2(\beta) \frac{1}{2} \tilde{r}(\beta)^{-1} (g^+(k, \beta) - g^-(k, \beta)),
\]
(3.44)
where the measures \(dp_1(\tau), dp_2(\tau)\) and the functions \(p(\beta), q(\beta), \tilde{r}(\beta)\) are defined by equalities (3.3), (3.4) and (3.21), (3.22), (3.32)–(3.33), respectively.

According to the definition, the support of the measure \(d\sigma(\beta)\) is contained in \(\Omega_0\).

In view of (3.2) and lemmas 3.1–3.7
\[
g(k, z) = 1 + \frac{1}{2\pi i} \int_{\Gamma} \frac{g(k, \xi)}{\xi - z} d\xi
\]

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where, as it was already defined, 

$$s(\alpha) = \frac{\alpha}{|\alpha|} \cdot \frac{\alpha^{-1}}{|\alpha^{-1}|} = \begin{cases} 
-1, & \alpha \in (-1, 1) \\
1, & \alpha \in (-\infty, -1) \cup (1, +\infty) 
\end{cases}.$$ 

Hence, when 

$$\beta \in \Omega_0$$ 

or 

$$|\beta| = 1,$$ 

we have 

$$\beta^+ \frac{4}{\Omega_0, \Omega_0^2} + \beta^- \frac{4}{\Omega_0, \Omega_0^2} = 1 + \int_{\Omega_0, \Omega_0^2} \frac{u(k, \alpha) \alpha^{-1}}{|\alpha - \beta^{-1}|} \alpha d\sigma(\alpha) + v.p. \int_{\Omega_0^2} \frac{u(k, \alpha) \alpha^{-1}}{|\alpha - \beta^{-1}|} \alpha d\sigma(\alpha)$$

$$+ \frac{1}{\pi} \int_{|\xi|=1} \frac{1}{\hat{r}(\xi) \beta^{-1}(k, \xi) - g^{-1}(k, \xi) \hat{r}(\xi) d\xi,$$

where v.p. denotes the principal value of the integral. Let us denote, for the sake of brevity, the sum of the integrals in the right-hand side of (3.45) by v.p. \(\int\) (although not all of them are singular for \(\beta \notin \Omega_0 \setminus \Omega_0^2\)).

If \(\beta \in \Omega_0 \setminus \Omega_0^2\), then according to (3.44),

$$\beta^+ \frac{4}{\Omega_0, \Omega_0^2} + \beta^- \frac{4}{\Omega_0, \Omega_0^2} = -(\chi(\beta) + \chi^2(\beta)) \beta^{2(k+1)} u(k, \beta),$$

and if \(\beta \in \Omega_0^2\), then, in view of (3.44) and lemma 3.5,

$$\beta^+ \frac{4}{\Omega_0, \Omega_0^2} + \beta^- \frac{4}{\Omega_0, \Omega_0^2} = \beta \beta^{-1} i q(\beta) \beta^{2(k+1)} (g^+ (k, \beta) - g^{-1} (k, \beta))$$

$$+ \beta \beta^{-1} \frac{1}{|\beta|} \frac{1}{|\beta - \beta^{-1}|} \frac{i m(\beta)}{2 p(\beta)} (g^+ (k, \beta^*) - g^{-1} (k, \beta^*)) = -\beta^{2(k+1)} u(k, \beta) + \beta \frac{m(\beta)}{|\beta|} \frac{1}{2 q(\beta^{-1})} u(k, \beta^{-1})$$

(we remind that \(p(\beta) = p(\beta^{-1})\)).

Hence, for all \(\beta \in \Omega_0\)

$$\beta^+ \frac{4}{\Omega_0, \Omega_0^2} + \beta^- \frac{4}{\Omega_0, \Omega_0^2} = -\chi(\beta) \beta^{2(k+1)} u(k, \beta) + \chi^2(\beta) \beta \frac{m(\beta)}{|\beta|} \frac{1}{2 q(\beta^{-1})} u(k, \beta^{-1}).$$

Finally, if \(|\beta| = 1\), then

$$\beta^+ \frac{1}{\Omega_0, \Omega_0^2} + \beta^- \frac{1}{\Omega_0, \Omega_0^2} = -\frac{1}{2} \frac{1}{\hat{r}(\beta^{-1}) \beta^{2(k+1)} (g^+ (k, \beta) - g^{-1} (k, \beta))} = -\beta^{2(k+1)} u(k, \beta).$$
Thus, taking into account these equalities and (3.45), we conclude that \( u(k, \beta) \) satisfy the equation
\[
\beta^{2(k+1)} u(k, \beta) - \chi_{\Omega_2}(\beta) \frac{\beta}{2q(\beta^{-1})} m(\beta) u(k, \beta^{-1}) + \text{v.p.} \int_{\Omega_0} \frac{u(k, \alpha)}{1 - \beta^{-1} \alpha^{-1}} s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) u(k, e^{i\theta})}{1 - \beta^{-1} e^{-i\theta}} d\theta + 1 = 0,
\]
(3.46)
where \( \chi_{\Omega_0}(\beta) + \chi(\beta) = 1 \) on the set on which the equation is considered.

In this section we proved

**Theorem 3.** The function \( g(k, z) \) can be represented in the form
\[
g(k, z) = 1 + \int_{\Omega_0} \frac{u(k, \alpha) s(\alpha)}{1 - z \alpha^{-1}} d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) u(k, e^{i\theta})}{1 - ze^{-i\theta}} d\theta.
\]
(3.47)
for \( z \notin \Omega_0 \cup T \), where the function \( u(k, \alpha) \) is the solution of integral equation (3.46), and the function \( s(\alpha) \), the coefficients \( q(\beta), m(\beta), \hat{r}(e^{i\theta}) \) and the measure \( d\sigma(\alpha) \) are defined by equalities (2.31), (2.32), (2.33), (3.32)–(3.33) and (3.44), respectively.

Remark. Equation (3.46), as it is seen from its form, is considered on the unit circle \( T \) and on the support of the measure \( d\sigma(\alpha) \), which is a subset of the set \( \Omega_0 \) (let us denote this support by \( \Omega_0' \)). In fact, the set \( \Omega_0' \cup T \) exactly corresponds (in the parametrization \( z + z^{-1} \)) to the spectrum of the matrix \( J \). We outline a short explanation of this. (For the sake of convenience we take our considerations in \( z \)-plane, so in this remark by the word ”spectrum” we mean the image in the plane \( z \) of the honest spectrum in the plane \( \lambda = z + z^{-1} \).

Since \( J \) is a finite-dimensional perturbation of the direct sum of the operators \( J_R \) and \( J_L \), the continuous spectrum \( J \) is the same as the union of the continuous spectrum \( J_R \) and \( J_L \), i.e. the nonisolated points of \( \Omega_1 \cup \Omega_2' \cup T \). Besides, the spectrum of \( J \) includes its isolated eigenvalues. According to (2.1) and (2.2), these are zeroes of the function \( N(z) \) (a part of the set \( \Phi \)) and the common poles of the functions \( n(z) \) and \( n(z^{-1}) \) (i.e. the set \( \Omega_2' \)). Hence, the spectrum of \( J \) belongs to \( \Phi \cup \Omega_1 \cup \Omega_2' \cup \Omega_2'' \cup T = \Omega_0 \cup T \).

But what we need to prove is to prove that the spectrum of \( J \) is exactly \( \Omega_0' \cup T \), that is to prove that
(i) the parts of \( \Omega_0 \) that do not belong to the support of the measure \( d\sigma(\alpha) \), do not belong to the spectrum.
(ii) the whole of the set \( \Omega_0' \cup T \) (i.e. the circle \( T \) and the support \( \Omega_0' \) of the measure \( d\sigma(\alpha) \)) belongs to the spectrum.

Let us begin with the inverse inclusion (ii). Since all the points \( \Omega_0' \cup T \) are nonisolated (\( \Omega_2' \) entirely belongs to the support of the measure \( d\sigma(\alpha) \)), and \( d\rho_R(\tau) \), \( d\rho_L(\tau) \) are absolutely continuous on \( \Omega_2'' \cup T \), all this set belongs to the continuous spectrum of \( J_R \) and \( J_L \), from where it follows that \( \Omega_0' \cup T \) completely belongs to the spectrum of \( J \) (more precisely, to the absolutely continuous one). Further, according to (3.43) and (3.7), the support of the measure \( d\sigma(\alpha) \) contains the set \((-1, 1) \cap \Omega_2'' \). But, according to (2.2), this set completely belongs to the spectrum, because it consists of common poles of the functions \( n(z) \) and \( n(z^{-1}) \) and, hence, these are poles of the function \( \frac{n(z)n(z^{-1})}{2\pi i N(z)} \). To finish the proof of (ii) we have to show that the spectrum of \( J \) include the whole intersection of the support of \( d\sigma(\alpha) \) with the set \( \Phi \cup \Omega_1 \). But it is evident from the definition of \( d\sigma(\alpha) \) in terms of the measure \( d\rho_1(\alpha) \) on this set (formula (3.43)) and from the definition of the measure \( d\rho_1(\alpha) \) (formula (3.4)) by the element of the resolvent \( N(z)^{-1} \) (formula (2.1)); we remind that all the points of increase of \( \rho_1(t) \) are singularities of \( N(z)^{-1} \), and all the singularities of the resolvent belong to the spectrum.

The last speculation also prove (i). In fact, \( \Omega_0' \) and \((-1, 1) \cap \Omega_2'' \) completely belong to the support of the measure \( d\sigma(\alpha) \). That is, in the set \( \Omega_0 \) only a certain subset \((\Phi \cup \Omega_1) \setminus \Omega_0' \) of the set \( \Phi \cup \Omega_1 \) may not belong to the support of the measure \( d\sigma(\alpha) \); more precisely, some isolated points of this set. (In other terms, \( \Omega_0 \setminus (\Phi \cup \Omega_1) = (\Phi \cup \Omega_1) \setminus \Omega_0' \)). But, according to (3.43), if \( \sigma((\Phi \cup \Omega_1) \setminus \Omega_0') = 0 \), then also \( \rho_1((\Phi \cup \Omega_1) \setminus \Omega_0') = 0 \), from where, according to (3.4), it follows that the points of the set \((\Phi \cup \Omega_1) \setminus \Omega_0' \) are points of holomorphy of \( N(z)^{-1} \). Hence, according to (2.1), \( R(-1, -1, z + z^{-1}) \) is regular on this set. The function \( R(0, 0, z + z^{-1}) \) is also regular at these points. (Since the set \( \Omega_1 \) is asymmetric, either \( n(z) \), or \( n(z^{-1}) \) is regular at each point, so, in view of (2.2), every pole of \( n(z) \) of \( n(z^{-1}) \) is reduced with the same pole of \( N(z) \)). Thus, the resolvent is regular on the set \((\Phi \cup \Omega_1) \setminus \Omega_0' \). Therefore, this set does not belong to the spectrum.
Conclusion. Thus for the reconstruction of the infinite Jacobi matrix \( J \) by its spectral data \( n(z) \) one have to find the functions \( \hat{r}(e^{i\theta}) \), \( m(\beta) \), \( q(\beta) \) and the measure \( \sigma(\alpha) \), corresponding to the Weyl function \( n(z) \), then to solve the equation (3.46) for all \( k \in \mathbb{Z} \) and define from \( u(k, \beta) \) the function \( g(k, z) \) and the entries \( a_k, b_k \) of the Jacobi matrix by (3.47) and (0.29) (\( b_0 \) are reconstructed up to their sign). Here the points of increase of the function \( \sigma \), together with the points of the unit circle are exactly the set of such \( z \) for which \( z + z^{-1} \) belongs to the spectrum of matrix \( J \), and also both points \( z \in \mathbb{R} \) and \( z^{-1} \in \mathbb{R} \) are the points of increase of the function \( \sigma \) if and only if \( z + z^{-1} \) belongs to the absolutely continuous spectrum of multiplicity 2 of the matrix \( J \).

4. Solvability of the fundamental equation

In this section we will prove the solvability of the equation of more general form than (3.46), namely

\[
e^{\beta t} b^{2(k+1)} u(k, \beta) - e^{\beta^{-1} t} \chi_{\Omega}^2(\beta) \frac{\beta m(\beta)}{2q(\beta^{-1})} u(k, \beta^{-1}) + e^{\beta^{-1} t} \n\frac{u(k, \alpha)}{1 - \beta^{-1} \alpha^{-1}} s(\alpha) d\sigma(\alpha) + e^{\beta^{-1} t} \n v(k, \beta^{-1}) + \n v(k, \alpha) = 1 - \beta^{-1} e^{-i\theta} d\theta = -1, \tag{4.1}
\]

where \( t \geq 0 \) is a real parameter. It is evident that, when \( t = 0 \), (4.1) coincide with the equation of the inverse problem (3.46). After dividing (4.1) by \( e^{\beta^{-1} t} \) and denoting

\[
v(k, \beta) = e^{(\beta - \beta^{-1}) t} b^{2(k+1)} u(k, \beta),
\]

we rewrite the equation for \( v(k, \beta) \) in the form

\[
v(k, \beta) - \chi_{\Omega}^2(\beta) \frac{\beta m(\beta)}{2q(\beta^{-1})} v(k, \beta^{-1}) + v(k, \alpha) = 1 - \beta^{-1} e^{-i\theta} d\sigma(\alpha)
\]

\[
+ \n v(k, \alpha), \tag{4.2}
\]

Let us introduce for \( |\beta| = 1 \) the function

\[
\tilde{r}(\beta) = \frac{\hat{r}(\beta)}{e^{(\beta - \beta^{-1}) t} b^{2(k+1)}}
\]

and the measure on the real line

\[
d\tilde{\sigma}(\alpha) = \frac{d\sigma(\alpha)}{e^{(\alpha - \alpha^{-1}) t} b^{2(k+1)}}. \tag{4.3}
\]

Since on the circle we have \( |e^{(\beta - \beta^{-1}) t} b^{2(k+1)}| = 1 \) \( (\beta - \beta^{-1}) \) is a pure imaginary number), then like it was in lemma 3.7,

\[
\max_{|\beta| = 1} |\tilde{r}(\beta)| \leq 1.
\]

In new notations the equation (4.2) is replaced by

\[
v(k, \beta) - \chi_{\Omega}^2(\beta) \frac{\beta m(\beta)}{2q(\beta^{-1})} e^{(\beta - \beta^{-1}) t} b^{2(k+1)} v(k, \beta^{-1})
\]

\[
+ \n v(k, \alpha) + \frac{\beta}{1 - \beta^{-1} \alpha^{-1}} d\tilde{\sigma}(\alpha) + \n v(k, \alpha) = \frac{\hat{r}(e^{i\theta}) v(k, e^{i\theta})}{1 - \beta^{-1} e^{-i\theta}} d\theta = -1. \tag{4.4}
\]

We will seek the solution of this equation in the class \( L^2(\mathbb{R} \cup T_{\mathbb{R}}) \) \( (T \) is the unit circle and \( \frac{d\theta}{\pi} \) is the Lebesque measure on it, divided by \( \pi \).
So, we will prove the invertibility of the operator $L$:

$$(Lv)(\beta) = v(\beta) - \chi\Omega_2 (\beta) \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} e^{(\beta-\beta^{-1})t\beta^2(k+1)} v(\beta^{-1})$$

$$+ \text{v.p.} \int_{\Omega_0} \frac{v(\alpha)s(\alpha)}{1 - \beta^{-1}\alpha^{-1}} d\bar{\sigma}(\alpha) + \frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \tilde{r}(e^{i\theta}) v(e^{i\theta}) \frac{d\theta}{1 - \beta^{-1}e^{-i\theta}}$$

which is defined in the introduced space. Thus, $\text{Re}(\cdot)$ because it is a pure imaginary number whereas we only need to estimate $\text{Re}(\cdot)$.

Let us now extract the real part from the third term. We denote by $\lambda$ the operator $L$.

Lemma 4.1. /see [20], p. 107/ Let $A \in B(H)$ ($B(H)$ denotes the algebra of the bounded operators in the Hilbert space $H$) and let there exist such positive number $d$ that $\text{Re}(A\tilde{f}, \tilde{f}) \geq d\|\tilde{f}\|^2$ for all $\tilde{f} \in H$. Then the operator $A$ is invertible and $\|A^{-1}\| \leq d^{-1}$.

It is seen from the last lemma that we need to calculate the real part of the expression

$$(Lv, v) = \int_{\Omega_0} d\tilde{\sigma}(\beta)v(\beta) \left\{ v(\beta) - \chi\Omega_2 (\beta) \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} e^{(\beta-\beta^{-1})t\beta^2(k+1)} v(\beta^{-1}) \right\}$$

$$+ \text{v.p.} \int_{\Omega_0} \frac{v(\alpha)s(\alpha)}{1 - \beta^{-1}\alpha^{-1}} d\bar{\sigma}(\alpha) + \frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \tilde{r}(e^{i\theta}) v(e^{i\theta}) \frac{d\theta}{1 - \beta^{-1}e^{-i\theta}}$$

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta v(e^{i\theta}) \left\{ v(e^{i\theta}) + \text{v.p.} \int_{\Omega_0} \frac{v(\alpha)s(\alpha)}{1 - \alpha^{-1}e^{-i\theta}} d\bar{\sigma}(\alpha) + \frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \tilde{r}(e^{i\theta}) v(e^{i\theta}) \frac{d\theta}{1 - e^{-i\theta}e^{-i\theta}} \right\}$$

$$(I + (II) + (III) + (IV) + (VI) + (VII))$$

we decomposed $(Lv, v)$ in the sum of terms $(I) - (VII)$, which are obtained after removing the parentheses. Let us transform each one of them. First, we remark that the second term can be omitted from our consideration because it is a pure imaginary number whereas we only need to estimate $\text{Re}(Lv, v)$. In fact, taking into account that, according to (4.3) and (3.33), on $\Omega_0^2$ the measure $d\tilde{\sigma}(\beta) = \frac{1}{e^{(\beta-\beta^{-1})t\beta^2(k+1)}} \cdot \frac{d\sigma(\beta)}{2\pi|\beta|}$, and that, according to (3.21) and (3.23), we have $p(\alpha) = p(\alpha^{-1})$, $m(\alpha) = -m(\alpha^{-1})$, we obtain that

$$(II) = - \int_{\Omega_0^2} d\tilde{\sigma}(\beta)v(\beta) \cdot \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} e^{(\beta-\beta^{-1})t\beta^2(k+1)} v(\beta^{-1})$$

$$= - \int_{\Omega_0^2} d\tilde{\sigma}(\beta) v(\beta) \cdot \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} e^{(\beta-\beta^{-1})t\beta^2(k+1)} \cdot \frac{\beta}{|\beta|} \cdot \frac{m(\beta)}{2q(\beta^{-1})} v(\beta^{-1})$$

$$= - \int_{\Omega_0^2} d\tilde{\sigma}(\beta) v(\beta) \cdot \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} v(\beta^{-1})$$

$$= \left[ \alpha = \beta^{-1} \right] = - \int_{\Omega_0} \frac{d\alpha p(\alpha)}{2\pi} \frac{v(\alpha)v(\alpha^{-1})}{|\alpha|^{-1}q(\alpha^{-1})} \cdot \frac{\alpha}{|\alpha|} \frac{m(\alpha)}{2q(\alpha)}$$

$$= - \int_{\Omega_0} \frac{d\alpha p(\alpha)}{2\pi} \frac{v(\alpha)v(\alpha^{-1})}{|\alpha|^{-1}q(\alpha^{-1})} \cdot \frac{\alpha}{|\alpha|} \frac{m(\alpha)}{2q(\alpha)} = -(II).$$

Thus, $\text{Re}(II) = 0$, and later on we will not take $(II)$ into account.

Let us now extract the real part from the third term. We denote by $A$ the integral operator

$$(Av)(\beta) = \text{v.p.} \int_{\Omega_0} \frac{v(\alpha)s(\alpha)}{1 - \beta^{-1}\alpha^{-1}} d\bar{\sigma}(\alpha),$$
defined in $L^2_\beta(R)$. For this operator

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}.$$ 

The second term $\frac{A - A^*}{2}$ is an asymmetric operator in $L^2_\beta(R)$. Hence, $(\frac{A - A^*}{2}, v)_\alpha$ is a pure imaginary number. Further, $\frac{A + A^*}{2}$ is a symmetric operator of the form

$$(\frac{A + A^*}{2} v)(\beta) = v.p. \int_{\Omega_0} \frac{S(\alpha, \beta)}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha),$$

where $S(\alpha, \beta) = \frac{S(\alpha) + S(\beta)}{2}$. Hence,

$$\text{Re} (III) = \text{Re} (Av, v)_\alpha = (\frac{A + A^*}{2} v, v)_\alpha = \int_{\Omega_0} d\tilde{\sigma}(\beta) v(\beta) \int_{\Omega_0} \frac{S(\alpha, \beta)}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha).$$

Let us remark that the internal integral is no more singular because its singularities were the points for which $\alpha = \beta^{-1}$, and now for all these points $S(\alpha, \beta) = 0$.

Thus,

$$\text{Re} (Lv, v) = \int_{\Omega_0} d\tilde{\sigma}(\alpha)v(\alpha) v(\beta) \int_{\Omega_0} d\tilde{\sigma}(\beta) v(\beta)$$

$$+ \int_{\Omega_0} d\tilde{\sigma}(\beta) v(\beta) \int_{\Omega_0} \frac{S(\alpha, \beta)}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha) + \text{Re} \int_{\Omega_0} d\tilde{\sigma}(\beta) v(\beta) \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{r}(e^{i\theta}) v(e^{i\theta}) \frac{1}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha)$$

$$+ \text{Re} \left\{ \int_{\Omega_0 \cap (-1, 1)} + \int_{\Omega_0 \cap R \setminus (-1, 1)} \right\} d\tilde{\sigma}(\beta) v(\beta) \int_{\Omega_0 \cap (-1, 1)} \frac{1}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha) + \text{Re} \left\{ \int_{\Omega_0 \cap (-1, 1)} + \int_{\Omega_0 \cap R \setminus (-1, 1)} \right\} d\tilde{\sigma}(\beta) v(\beta) \int_{\Omega_0 \cap (-1, 1)} \frac{1}{1 - \beta^{-1} \alpha^{-1}} v(\alpha) d\tilde{\sigma}(\alpha)$$

$$+ \|v_t\|^2 + \text{Re} \int_{-\pi}^{\pi} d\phi \frac{v(e^{i\theta})}{1 - \alpha^{-1} e^{-i\phi}} d\tilde{\sigma}(\alpha) + \text{Re} \int_{-\pi}^{\pi} d\phi \frac{v(e^{i\theta})}{1 - \alpha^{-1} e^{-i\phi}} d\tilde{\sigma}(\alpha)$$

$$+ \|v_t\|^2 + \text{Re} \int_{-\pi}^{\pi} d\phi \frac{v(e^{i\theta})}{1 - \alpha^{-1} e^{-i\phi}} d\tilde{\sigma}(\alpha).$$

(4.5)
In the first six of the seven integrals obtained we decompose the kernels of the form \( \frac{1}{1-\alpha^{-1} e^{-i\phi}} \), \( \frac{1}{1-\beta^{-1} e^{-i\phi}} \), into sums of geometric progression:

1. \( \alpha \in (-1, 1), \beta \in (-1, 1): \)

\[
\frac{1}{1-\alpha^{-1}} = -\alpha \beta, \quad \frac{1}{1-\alpha} = -\alpha \beta \sum_{n=0}^{\infty} \alpha^n \beta^n = -\sum_{n=1}^{\infty} \alpha^n \beta^n;
\]

2. \( \alpha \in \mathbb{R}\setminus(-1, 1), \beta \in \mathbb{R}\setminus(-1, 1): \)

\[
\frac{1}{1-\alpha^{-1}} = \sum_{n=0}^{\infty} (\alpha^{-1})^n = \sum_{n=0}^{\infty} \alpha^n \beta^n;
\]

3. \( \theta \in (-\pi, \pi), \beta \in (-1, 1): \)

\[
\frac{1}{1-\beta^{-1} e^{-i\theta}} = -\beta e^{i\theta}, \quad \frac{1}{1-\beta e^{i\theta}} = -\beta e^{i\theta} \sum_{n=0}^{\infty} \beta^n e^{in\theta} = -\sum_{n=1}^{\infty} \beta^n e^{in\theta};
\]

4. \( \theta \in (-\pi, \pi), \beta \in \mathbb{R}\setminus(-1, 1): \)

\[
\frac{1}{1-\beta^{-1} e^{-i\theta}} = \sum_{n=0}^{\infty} (\beta^{-1} e^{-i\theta})^n = \sum_{n=0}^{\infty} \beta^n e^{in\theta};
\]

5. \( \phi \in (-\pi, \pi), \alpha \in (-1, 0) \) (3):

\[
\frac{1}{1-\alpha^{-1} e^{-i\phi}} = -\sum_{n=1}^{\infty} \alpha^n e^{in\phi};
\]

6. \( \phi \in (-\pi, \pi), \alpha \in \mathbb{R}\setminus(-1, 1) \) (analogously to (4):

\[
\frac{1}{1-\alpha^{-1} e^{-i\phi}} = \sum_{n=0}^{\infty} \frac{1}{\alpha^n e^{in\phi}}.
\]

Let us now calculate the seven integral in (4.5). First,

\[
\frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) \nu(e^{i\theta})}{1-e^{-i\theta} e^{-i\theta}} d\theta = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left\{ \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) \nu(e^{i\theta})}{1-e^{-i\theta} e^{-i\theta}(1-\varepsilon)} d\theta + \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) \nu(e^{i\theta})}{1-e^{-i\theta} e^{-i\theta}(1+\varepsilon)} d\theta \right\}
\]

so we once more decompose the kernels of the two last integrals in the sum of geometric progression:

1. \( |e^{-i\phi} e^{-i\theta}(1-\varepsilon)| < 1 \)

\[
\frac{1}{1-e^{-i\theta} e^{-i\theta}(1-\varepsilon)} = \sum_{n=0}^{\infty} e^{-in\phi} e^{-in\theta}(1-\varepsilon)^n;
\]

2. \( |e^{-i\phi} e^{-i\theta}(1+\varepsilon)| > 1 \)

\[
\frac{1}{1-e^{-i\theta} e^{-i\theta}(1+\varepsilon)} = -e^{i\phi} e^{i\theta}(1+\varepsilon), \quad \frac{1}{1-e^{i\theta} e^{i\theta}(1+\varepsilon)} = -\sum_{n=1}^{\infty} e^{in\phi} e^{in\theta}(1+\varepsilon)^{-n}.
\]

Hence,

\[
\frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) \nu(e^{i\theta})}{1-e^{-i\theta} e^{-i\theta}} d\theta
\]

\[
= \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left\{ \sum_{n=0}^{\infty} e^{-in\phi}(1-\varepsilon)^n \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) \nu(e^{i\theta}) e^{-in\theta} d\theta - \sum_{n=1}^{\infty} e^{in\phi}(1+\varepsilon)^{-n} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) \nu(e^{i\theta}) e^{in\theta} d\theta \right\}.
\]
Thus, the last integral in expression (4.5) is equal to

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} d\phi \, v(e^{i\phi}) \cdot \frac{1}{\pi} \cdot v.p. \cdot \frac{\hat{r}(e^{i\theta})v(e^{i\theta})}{1 - e^{-i\phi}e^{-i\theta}} d\theta
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi}v(e^{i\phi}) d\phi \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{-in\theta} d\theta
\]

\[-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{in\phi}v(e^{i\phi}) d\phi \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{in\theta} d\theta.
\]

After substituting the obtained expression into sum (4.5) and taking into account that the support of the measure \(d\sigma(\alpha)\) lies in \(I_0\), we obtain that

\[
\text{Re} \left( L v, v \right) = \|v_r\|^2 + \int_{(-1,1)} d\sigma(\beta) \overline{v(\beta)} \int_{(-1,1)} \left[ \sum_{n=1}^{\infty} \alpha^n \beta^n \right] v(\alpha) d\sigma(\alpha)
\]

\[+ \int_{\mathbb{R} \setminus (-1,1)} d\sigma(\beta) \overline{v(\beta)} \int_{\mathbb{R} \setminus (-1,1)} \left[ \sum_{n=0}^{\infty} \alpha^n \beta^n \right] v(\alpha) d\sigma(\alpha)
\]

\[-\text{Re} \int_{(-1,1)} d\sigma(\beta) \overline{v(\beta)} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) \left[ \sum_{n=1}^{\infty} \alpha^n e^{in\theta} \right] v(e^{i\theta}) d\theta
\]

\[+\text{Re} \int_{\mathbb{R} \setminus (-1,1)} d\sigma(\beta) \overline{v(\beta)} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) \left[ \sum_{n=0}^{\infty} \beta^n e^{in\theta} \right] v(e^{i\theta}) d\theta
\]

\[+\|v_r\|^2 + \text{Re} \int_{-\pi}^{\pi} d\phi \, v(e^{i\phi}) \int_{(-1,1)} \left[ \sum_{n=1}^{\infty} \alpha^n e^{in\phi} \right] v(\alpha) d\sigma(\alpha)
\]

\[+\text{Re} \int_{-\pi}^{\pi} d\phi \, v(e^{i\phi}) \int_{\mathbb{R} \setminus (-1,1)} \left[ \sum_{n=0}^{\infty} \alpha^n e^{in\phi} \right] v(\alpha) d\sigma(\alpha)
\]

\[+\text{Re} \int_{-\pi}^{\pi} d\phi \, v(e^{i\phi}) \int_{(-1,1)} \left[ \sum_{n=0}^{\infty} \alpha^n e^{in\phi} \right] v(\alpha) d\sigma(\alpha)
\]

\[+\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi}v(e^{i\phi}) d\phi \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{-in\theta} d\theta
\]

\[-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{in\phi}v(e^{i\phi}) d\phi \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{in\theta} d\theta
\]

\[= \|v_r\|^2 + \sum_{n=1}^{\infty} \int_{(-1,1)} \beta^n v(\beta) d\sigma(\beta) \int_{(-1,1)} \alpha^n v(\alpha) d\sigma(\alpha)
\]

\[+\sum_{n=0}^{\infty} \int_{\mathbb{R} \setminus (-1,1)} \beta^n v(\beta) d\sigma(\beta) \int_{\mathbb{R} \setminus (-1,1)} \alpha^n v(\alpha) d\sigma(\alpha)
\]

\[-\sum_{n=1}^{\infty} \int_{(-1,1)} \beta^n v(\beta) d\sigma(\beta) \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{in\theta} d\theta
\]

\[-\sum_{n=1}^{\infty} \int_{(-1,1)} \beta^n v(\beta) d\sigma(\beta) \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta})v(e^{i\theta}) e^{in\theta} d\theta
\]
\[
+ \text{Re} \sum_{n=0}^{+\infty} \int_{\mathbb{R} \setminus (-1,1)} \beta^{-n} v(\beta) d\bar{\sigma}(\beta) \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) v(e^{i\theta}) e^{-in\theta} d\theta \\
+ \|v_t\|^2 + \text{Re} \sum_{n=1}^{+\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi} v(e^{i\phi}) d\phi \int_{\mathbb{R} \setminus (-1,1)} \alpha^n v(\alpha) d\bar{\sigma}(\alpha) + \\
+ \text{Re} \sum_{n=0}^{+\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi} v(e^{i\phi}) d\phi \int_{-\pi}^{\pi} \alpha^{-n} v(\alpha) d\bar{\sigma}(\alpha) + \\
+ \text{Re} \sum_{n=0}^{+\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi} v(e^{i\phi}) d\phi \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) v(e^{i\theta}) e^{-in\theta} d\theta \\
- \text{Re} \sum_{n=1}^{+\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-in\phi} v(e^{i\phi}) d\phi \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) v(e^{i\theta}) e^{in\theta} d\theta \\
= \text{Re} \left\{ (\vec{a}, \vec{a}) + (\vec{a}, \vec{b}) + (\vec{c}, \vec{a}) + \frac{1}{2} (\vec{b}, \vec{c}) \right\} + \|v_t\|^2 + \|v_t\|^2,
\]
that is
\[
\text{Re} (Lv, v) = \text{Re} \left\{ (\vec{a}, \vec{a}) + (\vec{a}, \vec{b}) + (\vec{c}, \vec{a}) + \frac{1}{2} (\vec{b}, \vec{c}) \right\} + \|v_t\|^2 + \|v_t\|^2
\]
where the vectors \( \vec{a}, \vec{b}, \vec{c} \in l^2(\mathbb{Z}) \), whose coordinates are defined by the formulas
\[
a_n = \int_{\mathbb{R} \setminus (-1,1)} \alpha^n v(\alpha) d\bar{\sigma}(\alpha), \quad n \leq 0,
\]
\[
a_n = \int_{(-1,1)} \alpha^n v(\alpha) d\bar{\sigma}(\alpha), \quad n > 0,
\]
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) v(e^{i\theta}) e^{-in\theta} d\theta, \quad n \leq 0,
\]
\[
b_n = -\frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) v(e^{i\theta}) e^{in\theta} d\theta, \quad n > 0,
\]
\[
c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} v(e^{i\phi}) e^{-in\phi} d\phi, \quad n \in \mathbb{Z}.
\]
Here \( b_n \) and \( c_n \) are the Fourier coefficients in the expansion of the functions \( \sqrt{2} \hat{r}(e^{i\theta}) v(e^{i\theta}) \) and \( \sqrt{2} v(e^{i\theta}) \) resp., defined under \((\pi, \pi)\), by orthogonal systems \( \left\{ \frac{e^{i\phi}}{\sqrt{2}} \right\}_{n=\infty}^{-\infty} \) and \( \left\{ \frac{v}{\sqrt{2}} \right\}_{n=\infty}^{-\infty} \). That means,
\[
\|\vec{b}\|^2 \leq 2 \max \|\hat{r}(e^{i\theta})\|^2 \cdot \|v_t\|^2, \quad \|\vec{c}\|^2 = 2 \|v_t\|^2.
\]
Now we use the identity
\[
\text{Re} \left\{ (\vec{a}, \vec{a}) + (\vec{a}, \vec{b}) + (\vec{c}, \vec{a}) + \frac{1}{2} (\vec{b}, \vec{c}) \right\} = \|\vec{a}\|^2 + \frac{\|\vec{b} + \vec{c}\|^2}{2} - \frac{1}{4} \left( \|\vec{b}\|^2 + \|\vec{c}\|^2 \right),
\]
which is true for any scalar product (can be simply verified starting from the right-hand side). Applying this identity to our expression, we obtain
\[
\text{Re} (Lv, v) = \text{Re} \left\{ (\vec{a}, \vec{a}) + (\vec{a}, \vec{b}) + (\vec{c}, \vec{a}) + \frac{1}{2} (\vec{b}, \vec{c}) \right\} + \|v_t\|^2 + \|v_t\|^2 =
\]
\[ \| \vec{a} + \frac{\vec{b} + \vec{c}}{2} \|^2 \geq \| v_r \|^2 + \| v_t \|^2 - \frac{1}{4} \left(2 \max |\hat{r}(e^{i\theta})|^2 \| v_r \|^2 + 2 |v_t|^2\right) \geq d \| v \|^2 \]

where \( d = \frac{1}{2}(1 - \max |\hat{r}(e^{i\theta})|^2) > 0 \). Comparing this result with lemma 4.1, we see that the operator \( L \) is invertible and, moreover,

\[ \| L^{-1} \| \leq d^{-1}. \]

Thus, the solvability of the equation (4.4), as well as (4.1), is proved. So, we deduce

**Theorem 4.** Equations (3.46) and (4.1) are uniquely solvable for every \( k \), i.e. the operator in the left-hand side of (4.1) is invertible in the space \( L^2(\mathbb{R}_\sigma \cup T\hat{m}) \) and the inverse operator is bounded.

The collection \( \{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha-1)}, d\sigma(\alpha)\} \) consisting of the functions \( \hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha-1)} \) and the measure \( d\sigma(\alpha) \), which determine integral equation (3.46) and representation (3.47), is called the reduced spectral data of the Jacobi matrix \( J \). The map

\[ J \mapsto n(z) \mapsto \{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha-1)}, d\sigma(\alpha)\}, \]

described in the previous sections, is the solution of the direct spectral problem, and the map

\[ \{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha-1)}, d\sigma(\alpha)\} \mapsto J \]

solves the inverse spectral problem.

Summing up the results obtained in the preceding sections, we arrive to the following theorem:

**Theorem 5 (the main theorem).** The infinite Jacobi matrix \( J \) whose Weyl functions satisfy the conditions A)–E), is uniquely defined by its reduced spectral data. The equations (3.46), reconstructed according to these data, have a unique solution in the space \( L^2(\mathbb{R}_\sigma \cup T\hat{m}) \) for all \( k \in \mathbb{Z} \). To reconstruct the Jacobi matrix \( J \) according to given spectral data it is necessary to solve equation (3.46) and then use formulas (3.47) and (0.29).

Acknowledgments. This work is supported by INTAS 2000-272 and by the State Foundation of Fundamental Researches (Ukraine) 1.4/20.
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