Classifying Groups With a Small Number of Subgroups

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Abstract. We provide lower bounds on the number of subgroups of a group \( G \) as a function of the primes and exponents appearing in the prime factorization of \( |G| \). Using these bounds, we classify all abelian groups with 22 or fewer subgroups, and all nonabelian groups with 19 or fewer subgroups. This allows us to extend the integer sequence A274847 in the On-Line Encyclopedia of Integer Sequences introduced by Slattery.

1. INTRODUCTION. It is a classic problem in a first course in group theory to show that a group \( G \) has exactly two subgroups if and only if \( G \cong \mathbb{Z}_p \) for a prime \( p \). We let \( \text{Sub} \ G \) denote the set of all subgroups of \( G \). The main idea then is to observe that if \( |\text{Sub} \ G| = 2 \) or if \( G \cong \mathbb{Z}_p \), then every nonidentity element \( x \in G \) must by necessity generate all of \( G \), i.e., \( \langle x \rangle = G \) for all \( x \in G \setminus \{e\} \). Slightly less frequently, a course may follow up by considering groups \( G \) with exactly three or four subgroups. In those cases, it turns out we can again argue that \( G \) must be cyclic.

Indeed, if \( |\text{Sub} \ G| = 3 \) and \( H \leq G \) is the unique, nontrivial, proper subgroup of \( G \), then observe that for any \( x \in G \setminus H \), we must have \( \langle x \rangle = G \) as these elements are nontrivial, must generate a subgroup of \( G \), and cannot generate the trivial subgroup or \( H \). Since \( G \) must be cyclic, the fact that cyclic groups have exactly one subgroup for each positive divisor of \( |G| \) implies that \( |G| = p^3 \) for some prime \( p \) and thus \( G \cong \mathbb{Z}_{p^3} \) as cyclic groups of order \( |G| \) are unique up to isomorphism.

Similarly, if \( |\text{Sub} \ G| = 4 \) and \( H \neq K \) are the two nontrivial subgroups, then recall that \( H \cup K \leq G \) if and only if \( H \leq K \) or \( K \leq H \). It follows that \( H \cup K \neq G \), and thus there exists some \( x \in G \setminus (H \cup K) \). Once again, \( \langle x \rangle = G \) and \( G \) is cyclic. As before, \( G \) must have exactly one subgroup for each divisor of \( |G| \); hence it follows that \( G \cong \mathbb{Z}_{pq} \) or \( \mathbb{Z}_{p^3} \) for primes \( p \) and \( q \). We summarize these classic results below.

Classic Results.

1. If \( |\text{Sub} \ G| = 2 \), then \( G \cong \mathbb{Z}_p \) for some prime \( p \).
2. If \( |\text{Sub} \ G| = 3 \), then \( G \cong \mathbb{Z}_{p^2} \) for some prime \( p \).
3. If \( |\text{Sub} \ G| = 4 \), then \( G \cong \mathbb{Z}_{pq} \) or \( G \cong \mathbb{Z}_{p^3} \) for some primes \( p \) and \( q \).

These classic results bring us naturally to the question: Which (necessarily finite) groups \( G \) have exactly \( k \) subgroups when \( k \geq 5 \)? Fortunately this question becomes much more interesting moving forward as \( G \) need not be cyclic when \( |\text{Sub} \ G| \geq 5 \). Thus, from here on we will need a completely different approach.

Miller explored this topic previously in a series of papers \([7]–[11]\) in which he worked to classify the groups with 16 or fewer subgroups. Given the assertions within, it is clear that Miller is not applying the techniques we use here, however his results agree with ours, except in the case when \( |\text{Sub} \ G| = 14 \) where he seems to have skipped a case, causing him to miss \( S_3 \times \mathbb{Z}_3 \) and \( \mathbb{Z}_3 \times \mathbb{Z}_{32} \). More recently, Slattery \([13,14]\) explored this idea once more; reducing any group \( G \) by factoring out any cyclic central Sylow \( p \)-subgroups of \( G \) first. Using this method, he worked to classify groups with 12 or fewer subgroups up to similarity defined in the following sense:

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Definition ([14]). Let $G$ and $H$ be finite groups. Write $G = P_1 \times P_2 \times \cdots \times P_t \times \tilde{G}$ and $H = Q_1 \times \cdots \times Q_d \times \tilde{H}$, where $P_i$ (respectively, $Q_j$) are cyclic central Sylow subgroups within $G$ (respectively, $H$). Then $G$ is similar to $H$ if and only if the following conditions hold:

- $\tilde{G}$ is isomorphic to $\tilde{H}$.
- $c = d$
- $n_i = m_i$ for some reordering, where $|P_i| = p_i^{n_i}$ and $|Q_i| = q_i^{m_i}$.

Using this definition of similarity, groups that are similar will always have the same number of subgroups (see Theorem 1). Slattery was encouraged to submit a sequence (A274847 [15]) to the On-Line Encyclopedia of Integer Sequences that counts the number of similarity classes of groups with $k$ subgroups. As with Miller, his results—which run up to groups with at most twelve subgroups—agree with ours. As a further point of clarification, in what follows, we stick with GAP\(^1\) notation for the groups we list. As an example, for the extraspecial group of order 27, $\langle x, y \mid x^9 = y^3 = e, yxy^{-1} = x^4 \rangle$, which Slattery lists as $E_{27}$, we use the label $M_{27}$ provided by GAP.

Our approach is different from Slattery’s as well and takes us further. In Section 2, we first deal with the case when $G$ is abelian by exploring the number of subgroups of abelian $p$-groups and then considering products of these groups for different primes. In Section 3, we then approach nonabelian groups using the Sylow theorems and the orbit-stabilizer theorem to place a lower bound on the number of subgroups of $G$ as a function of the primes and exponents in the prime factorization of $|G|$. This allows us to greatly reduce the search space for nonabelian groups with 19 or fewer subgroups. Using the complete lists of similarity classes of abelian and nonabelian groups with 19 or fewer subgroups, we then give the first 19 terms in sequence A274847.

2. ABELIAN GROUPS. Cyclic groups are a straightforward case to begin with, as it is well known that each cyclic group $G$ has exactly one subgroup for each divisor of $|G|$. Thus, given a cyclic group $G$ of order $|G| = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}$, it follows that $|\text{Sub } G| = (a_1 + 1)(a_2 + 1) \cdots (a_n + 1)$. This implies the existence of at least one group with exactly $k$ subgroups for each $k \in \mathbb{N}$ (namely the group $\mathbb{Z}_{p^k}$). In addition, there is exactly one cyclic group of order $|G|$ up to isomorphism; thus we may work backwards to quickly find all cyclic groups with a fixed number of subgroups.

More generally, by the fundamental theorem of finite abelian groups, every such group can be written as a direct product of cyclic groups of prime power orders. Recall that such groups (whose order is a power of a prime $p$) are called $p$-groups. Observe that, for each prime $p$ dividing $|G|$, we may combine the cyclic $p$-groups in the product into a single component subgroup $H_{p^\mu}$, where $p^\mu$ is the highest power of $p$ that divides $|G|$. In this way, we can think of any finite abelian group as a direct product of abelian $p$-groups for different primes $p$—a useful perspective given the following key result:

**Theorem 1.** Let $G$ and $H$ be groups. If $\gcd(|G|, |H|) = 1$ then $|\text{Sub } G \times H| = |\text{Sub } G| \cdot |\text{Sub } H|$

**Proof.** Certainly $G' \times H' \leq G \times H$ for all $G' \leq G$ and $H' \leq H$, so it suffices to show that every subgroup $K \leq G \times H$ can be split as $K = G' \times H'$ for some $G'$ and $H'$. Observe that, since $G \times H = \{(g, h) \mid g \in G, h \in H\}$, if $K \leq G \times H$, then we may define

$$K_G = \{g \in G \mid (g, h) \in K \text{ for some } h \in H\}.$$
\[K_H = \{h \in H \mid (g, h) \in K \text{ for some } g \in G\}.\]

Our goal is to show that \(K = K_G \times K_H\). By assumption and Lagrange’s theorem, we know \(\sigma(g) \mid |G|\) and \(\gcd(\sigma(g), \sigma(h)) = 1\) for all \(g \in G, h \in H\). Consider \(g \in K_G\) with \((g, h_g) \in K\). Since \(g\) and \(h_g\) have coprime orders, it follows that \((g, h_g)\) will be the cyclic group \(\mathbb{Z}_{\sigma(g)\sigma(h_g)}\). Since \((g, h_g) \leq K\), it follows that \((g, e_H) \in K\) for all \(g \in K_G\). A similar argument will show that \((e_G, h) \in K\) for all \(h \in K_H\) as well.

Since \(K \leq G \times H\), by closure we have \((g, h) \in K\) for all \(g \in K_G, h \in K_H\). It follows that \(K_G \times K_H \leq K\) and thus, since \(K \leq K_G \times K_H\) by definition, we have \(K = K_G \times K_H\). The above argument also shows that \(K_G \cong K \cap (G \times \{e_H\})\); thus \(K_G \leq G\). Similarly, \(K_H \leq H\), which completes the proof.

Given this result and the perspective above, we may count the number of subgroups of any finite abelian group as long as we know the number of subgroups of its abelian \(p\)-group components.

**Corollary 2.** If \(G\) is an abelian group as defined above with \(|G| = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}\) then \(|\text{Sub } G| = |\text{Sub } H_{p_1}^{a_1}| \cdot |\text{Sub } H_{p_2}^{a_2}| \cdots |\text{Sub } H_{p_n}^{a_n}|\). Furthermore, this implies that if an abelian group \(G\) has a prime number of subgroups then \(G\) must be a \(p\)-group.

This leaves us to describe the number of subgroups in abelian \(p\)-groups. The case when an abelian \(p\)-group has exactly two cyclic factors was fully described by Ali in his Master’s thesis [1].

**Theorem 3.** [1, Theorem 4.2.1] If \(G \cong \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}\) where \(a \leq b\), then the number of subgroups of \(G\) is exactly

\[
\frac{1}{(p - 1)^2} \left[(b - a + 1) p^{a+2} - (b - a - 1) p^{a+1} - (b + a + 3) p + (b + a + 1)\right].
\]

In addition to this powerful result, we also wish to consider abelian \(p\)-groups (recall, these are groups of order a power of \(p\)) with more than two factors. We therefore address a few special cases which will be enough for our purposes.

**Proposition 4.** If \(G \cong (\mathbb{Z}_p)^n = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p\), then

\[
|\text{Sub } G| = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right)_p \text{, where } \left(\begin{array}{c} n \\ i \end{array}\right)_p = \frac{(1 - p^n)(1 - p^{n-1})\cdots(1 - p^{n-i+1})}{(1 - p)(1 - p^2)\cdots(1 - p^i)}.
\]

**Proof.** The group \((\mathbb{Z}_p)^n\) is also an \(n\)-dimensional vector space over \(\mathbb{Z}_p\) and each subgroup of order \(p^i\) in \(G\) corresponds to an \(i\)-dimensional subspace. It is well known that the Gaussian binomial coefficient \(\left(\begin{array}{c} n \\ i \end{array}\right)_p\) counts the number of \(i\)-dimensional subspaces of an \(n\)-dimensional vector space over \(\mathbb{Z}_p\).

In addition to these specific cases, a recent paper by Aivazidis and Müller [2] gives more general lower bounds on the number of subgroups in noncyclic \(p\)-groups. For example, the results below imply that any noncyclic abelian \(p\)-group with fewer than 23 subgroups must have order \(p^a\) for \(a \leq 7\).

**Theorem 5.** [2, Theorem A] Let \(G\) be a noncyclic group with \(|G| = p^a\) for a prime \(p \geq 3\). Then \(|\text{Sub } G| \geq (a - 1)(p + 1) + 2\), with equality if and only if \(G \cong \mathbb{Z}_{p^{a-1}} \times \mathbb{Z}_p\) or \(G \cong M_{p^a} = (\langle x, y \mid x^{p^{a-1}} = y^p = e, yxy^{-1} = x^{1+p^{a-2}}\rangle\).

\(^2\)Note: The statement of Theorem 4.2.1 in [1] has a typo, but the proof proves the statement given here and [16] confirms this result.

\(^3\)We thank Aivazidis for bringing this to our attention.
Theorem 6. [2, Theorem B] Let $G$ be a noncyclic group with $|G| = 2^a$. If $a = 3$, then $|\text{Sub } G| \geq 6$ with equality if and only if $G \cong Q_8$. And if $a \geq 4$, then $|\text{Sub } G| \geq 3a - 1$ with equality if and only if $G \cong Q_{16}$ or $G \cong \mathbb{Z}_{2a-1} \times \mathbb{Z}_2$, or $M_{2a} = \langle x, y \mid x^{2^{a-1}} = y^2 = e, yxy^{-1} = x^{1+2^{a-2}} \rangle$.

Classifying abelian groups with exactly $k$ subgroups is now a matter of finding all possible ways to combine abelian $p$-groups for different primes so that the product of their individual numbers of subgroups equals $k$. Recall also that, thanks to Theorem 1, an abelian group can only have a prime number of subgroups if it is a $p$-group.

Table 1. Similarity classes of abelian groups with fewer than 23 subgroups.

| $|\text{Sub } G|$ | Similarity Classes of Abelian Groups | Classes |
|------------------|--------------------------------------|--------|
| 1                | $\{e\}$                              | 1      |
| 2                | $\mathbb{Z}_p$                        | 1      |
| 3                | $\mathbb{Z}_2 \times \mathbb{Z}_2$   | 1      |
| 4                | $\mathbb{Z}_p \times \mathbb{Z}_2$   | 2      |
| 5                | $\mathbb{Z}_p^2 \times \mathbb{Z}_2$ | 2      |
| 6                | $\mathbb{Z}_p^3 \times \mathbb{Z}_2$ | 3      |
| 7                | $\mathbb{Z}_p^4$                      | 1      |
| 8                | $\mathbb{Z}_p^5 \times \mathbb{Z}_2$ | 2      |
| 9                | $\mathbb{Z}_p^6 \times \mathbb{Z}_2$ | 2      |
| 10               | $\mathbb{Z}_p^7 \times \mathbb{Z}_2$ | 5      |
| 11               | $\mathbb{Z}_p^8 \times \mathbb{Z}_2$ | 5      |
| 12               | $\mathbb{Z}_p^9 \times \mathbb{Z}_2$ | 5      |
| 13               | $\mathbb{Z}_p^{10} \times \mathbb{Z}_2$ | 5      |
| 14               | $\mathbb{Z}_p^{11} \times \mathbb{Z}_2$ | 5      |
| 15               | $\mathbb{Z}_p^{12} \times \mathbb{Z}_2$ | 5      |
| 16               | $\mathbb{Z}_p^{13} \times \mathbb{Z}_2$ | 5      |
| 17               | $\mathbb{Z}_p^{14} \times \mathbb{Z}_2$ | 5      |
| 18               | $\mathbb{Z}_p^{15} \times \mathbb{Z}_2$ | 5      |
| 19               | $\mathbb{Z}_p^{16} \times \mathbb{Z}_2$ | 5      |
| 20               | $\mathbb{Z}_p^{17} \times \mathbb{Z}_2$ | 5      |
| 21               | $\mathbb{Z}_p^{18} \times \mathbb{Z}_2$ | 5      |
| 22               | $\mathbb{Z}_p^{19} \times \mathbb{Z}_2$ | 5      |

As an example, suppose we wish to find all abelian groups with exactly 10 subgroups. We must consider abelian $p$-groups with exactly 10 subgroups themselves, or a product of an abelian $p$-group with an abelian $q$-group such that one has 5 subgroups and the other has 2 subgroups. Applying Theorem 3, we find that the only abelian groups (up to similarity) with 10 subgroups are $\mathbb{Z}_{p^5}, \mathbb{Z}_{p^4q}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$ ($p \neq 2$), $\mathbb{Z}_7 \times \mathbb{Z}_2$, and $\mathbb{Z}_5 \times \mathbb{Z}_3$. Continuing in this manner, Table 1 reports all similarity classes of abelian groups with fewer than 23 subgroups. Note that, just as in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, arbitrary primes are always assumed to be distinct from any others appearing.
One could continue to extend this process quite a bit further without running into too much resistance. [3] The much more interesting and challenging case lies in the discussion of nonabelian groups.

3. NONABELIAN GROUPS. For nonabelian groups, it would be nice if we could apply the same sorts of techniques to count the number of subgroups by understanding smaller components. The best analog available for decomposing a group into relatively prime $p$-group parts is the collection of Sylow theorems.

The Sylow Theorems. Let $G$ be a finite group, $p$ a prime divisor of $|G|$, and write $|G| = p^at$, where $a$ and $t$ are positive integers and $p$ does not divide $t$. Let $\text{Syl}_p(G) = \{\mathcal{P} \leq G \mid |\mathcal{P}| = p^a\}$. Then

1. $\text{Syl}_p(G) \neq \emptyset$.
2. If $\mathcal{P}, \mathcal{P}' \in \text{Syl}_p(G)$, then there exists a $g \in G$ with $\mathcal{P}' = g\mathcal{P}g^{-1}$.
3. Let $n_p = |\text{Syl}_p(G)|$. Then $n_p \mid t$ and $n_p = [G : N(\mathcal{P})] \equiv 1 \pmod{p}$, where $N(\mathcal{P})$ is the normalizer.

A subgroup $\mathcal{P} \in \text{Syl}_p(G)$ is called a Sylow $p$-subgroup of $G$. These famous results give us some information regarding the number of $p$-subgroups within a group $G$, but they do not directly tell us about how the different $p$-group components will interact with one another. If $G$ is especially nice—i.e., if each of its Sylow subgroups is unique and normal (making $G$ nilpotent)—then $G$ will decompose as a direct product of its Sylow subgroups (see, e.g., [12, Corollary 5.4.2]) and we may apply Theorem 1 to count the number of subgroups directly. Unfortunately, this is frequently not the case when $G$ is nonabelian. In addition, the Sylow $p$-subgroups themselves need not be abelian; thus we need a way to explore the subgroups of nonabelian $p$-groups as well.

As the Sylow theorems lend themselves to breaking a group into $p$-group components, they do not give us much information in the case when $G$ is itself a $p$-group (in which case $G = \mathcal{P}$ and $n_p = 1$). Thankfully, there is a generalization of part 3 of the Sylow theorems due to Wielandt [17] which places conditions on the number of $p$-subgroups for each power of $p$.

Theorem 7. [17] Let $G$ be a group with $|G| = p^a$ for a prime $p$. Then the number of subgroups of order $p^i$ ($i \leq a$) is congruent to 1 (mod $p$).

In addition, Theorems 5 and 6 also provide lower bounds on the number of subgroups of nonabelian $p$-groups. With these results in hand, we move on to nonabelian groups whose orders are divisible by multiple primes.

Nonabelian groups with $|G|$ divisible by multiple primes. When $|G|$ is divisible by multiple primes, the Sylow theorems allow us to explore the $p$-group components for each prime $p$ in the prime factorization of $|G|$; however, it is unclear exactly how those components will interact with one another. In the nicest situation, when $G$ is nilpotent, then $G$ can be written as a direct product of its Sylow subgroups$^4$ and we may apply Theorem 1 to directly count the number of subgroups. Slightly more generally, whenever $G$ can be expressed as a direct product of subgroups with coprime orders, then this avenue will be available to us—i.e., we can find all such groups with exactly $k$ subgroups by exploring the ways to factor $k$ (having already understood groups with fewer than $k$ subgroups).

$^4$Note that this situation exactly corresponds to $G$ having only normal Sylow subgroups.
When this is not the case, we need a different way to count subgroups. The Sylow theorems and Theorem 7 provide some information about the number of \( p \)-subgroups within \( G \), but we need to be able to count subgroups of composite orders as well. Thankfully (see, e.g., [12, Proposition 4.2.11]), whenever \( N \leq G \) and \( H \leq G \), then the set product \( NH \leq G \) and, in fact, \( NH \trianglelefteq G \) if \( H \trianglelefteq G \), too. Moreover, if \( N \) and \( H \) have relatively prime orders, then we must have \( N \cap H = \{e\} \) and it follows that \( |NH| = |N| \cdot |H| \). This will allow us to use normal \( p \)-subgroups, together with \( q \)-subgroups, to create subgroups of composite orders.

To demonstrate the effectiveness of this idea through an example, we need to set up some notation. Given a fixed group \( G \), in what follows we will use \( H_n \) to denote a subgroup of order \( n \) in \( G \). Now suppose that \( p^a \) and \( q^b \) are the highest powers of primes \( p \neq q \) that divide \( |G| \). By Theorem 7, there exists at least one subgroup \( H_{p^1} \leq G \) for each \( 1 \leq i \leq a \) and, similarly, we have at least one \( H_{q^j} \leq G \) for each \( 0 \leq j \leq b \). (This second collection includes the trivial subgroup \( \{e\} \).) Now, for each such prime power \( p^i \), observe that Theorem 7 also implies that either \( H_{p^i} \) is unique — in which case we can create at least \( b+1 \) distinct product subgroups \( H_{p^i} H_{q^j} \) for each \( 0 \leq j \leq b \) (including \( H_{p^i} \) itself) — or \( H_{p^i} \) is not unique, in which case there must be at least \( p+1 \) subgroups of order \( p^i \). Running through the \( a \) different prime powers, we have demonstrated the existence of at least \( a \cdot \min(b+1, p+1) \) distinct subgroups in \( G \) (possibly including \( G \) itself). Note that, in some situations it will be helpful to treat the Sylow subgroups themselves separately (i.e., not allowing \( i = a \) or \( j = b \)).

In what follows, we consider different cases based on the number of distinct primes that divide \( |G| \). In each case, we demonstrate a lower bound on the number of subgroups of nonnilpotent \( G \) as a function of the primes and exponents in the prime factorization of \( |G| \), thereby reducing the search space to a small finite number of cases.

Beginning with \( |G| \) being divisible by only two primes \( p < q \), it is helpful to recall (see, e.g., [12]) that there exists a nonnilpotent group \( G \) of order \( pq \) if and only if \( q \equiv 1 \pmod{p} \). Moreover, since subgroups of index \( p \), where \( p \) is the smallest prime dividing \( |G| \) must be normal (see [6, Theorem 1]), it follows that \( Q \in \text{Syl}_q(G) \) must be normal and thus part 3 of the Sylow theorems implies that \( |\text{Sub} \, G| = q + 3 \) for such a group. For the more general situation, we recall a classic consequence of Burnside’s normal \( p \)-complement theorem [4]:

**Lemma 8.** Let \( G \) be a group and let \( p \) be the smallest prime dividing \( |G| \). If \( P \in \text{Syl}_p(G) \) is cyclic, then \( P \) has a normal complement.

It follows immediately that if \( |G| \) is divisible by exactly two primes \( p < q \) and \( P \in \text{Syl}_p(G) \) is cyclic, then \( n_q = 1 \) as the complement of \( P \) is a Sylow \( q \)-subgroup. In addition, it is well known that \( G \) is nilpotent if and only if every maximal subgroup of \( G \) is normal. Thus, if \( G \) is nonnilpotent it follows that \( n_p \neq 1 \) and \( G \) must contain at least one nonnormal maximal subgroup. When \( G \) has cyclic Sylow subgroups though, the maximal subgroups of \( G \) must have prime index.

Indeed, let \( |G| = p^aq^b \) with \( \langle x \rangle = P \in \text{Syl}_p(G) \) and \( \langle y \rangle = Q \in \text{Syl}_q(G) \) both cyclic, and let \( H \leq G \) be a subgroup with \( |H| = p^aq^i \) for \( i < a \) and \( j < b \). Since \( Q \trianglelefteq G \), we have that \( HQ \) is a proper subgroup that contains \( H \) and has order \( p^aq^b \); hence \( H \) cannot be maximal. Similarly, if \( |H| = p^aq^j \) for \( j < b - 1 \), then \( H = P \langle y^{q^{b-j}} \rangle \) for some Sylow \( p \)-subgroup. Thus, the product of \( H \) with the unique normal subgroup \( \langle y^q \rangle \) of order \( q^{b-1} \) will create a proper subgroup containing \( H \) — again implying that \( H \) cannot be normal. Thus, maximal subgroups of \( G \) must have order \( p^aq^b \) or \( p^a \langle q \rangle \).
This lower bound can be further simplified to \( a > q \) such subgroups. Moreover, applying the orbit-stabilizer theorem, this implies that \( G \) must also contain (at least \( q \)) nonnormal subgroups of order \( p^aq^b \) for each \( 1 \leq j < b \). With these observations, we are now ready to describe bounds on \( |\text{Sub } G| \) when \( |G| \) is divisible by exactly two distinct primes.

**Theorem 9.** Let \( G \) be nonnilpotent with \( |G| = p^aq^b \) for primes \( p < q \). Then

\[
|\text{Sub } G| \geq \min \left\{ \begin{array}{ll}
bq + ab + a + 1, \\
b(q + 1) + 2a + (a - 1) \min(p, b), \\
b + q + 1 + (b - 1) \min(a, q) + m_p
\end{array} \right. ,
\]

where we only consider the second possibility if \( b > 1 \), and we only consider the third possibility if \( a > 1 \), in which case \( m_2 = 5 \) if \( a = 2 \), \( m_2 = 6 \) if \( a = 3 \), and if \( p \neq 2 \) or \( a \geq 4 \), then \( m_p = (a - 1)(p + 1) + 2 \).

**Proof.** Recall, the subgroups of cyclic \( p \)-groups form a single ascending chain (with a unique subgroup for each power of \( p \) dividing the group order). As such, it is helpful to separate into cases based on whether \( G \) and \( e \), we have \( |\text{Sub } G| \geq bq + ab + a + 1 \).

If instead, \( G \) is cyclic, but \( Q \) is not cyclic, then \( b > 1 \) and by **Theorem 5**, \( Q \) must contain at least \((b - 1)(q + 1) + 2\) subgroups (including \( \{e\} \)). Since \( Q \leq G \), as before there also exist at least \( a \) subgroups of orders \( p^aq^b \) \((1 \leq i \leq a)\) including \( G \) itself. In addition to the at least \( q \) Sylow \( p \)-subgroups, \( G \) must contain at least \((a - 1)\min(p + 1, b + 1)\) subgroups of orders \( p^aq^j \) \((1 \leq i < a, 0 \leq j < b)\). Collectively, we have

\[
|\text{Sub } G| \geq (b - 1)(q + 1) + 2 + a + q + (a - 1) \min(p + 1, b + 1).
\]

This lower bound can be further simplified to \( b(q + 1) + 2a + (a - 1) \min(p, b) \).

Finally, if \( G \) is not cyclic, then \( a > 1 \) and we may apply **Theorem 6** or **Theorem 3** (subtracting 1) to count the proper subgroups of \( G \). Since \( G \) is nonnilpotent, it contains at least one nonnormal Sylow subgroup. It follows that \( G \) contains at least \( q + 2 \) subgroups of orders \( p^a, q^b, \) or \( pq^b \). Indeed, if \( Q \leq G \) (so \( n_q = 1 \)), then \( G \notin G \), \( n_p \geq q \), and \( G \) contains at least one product \( H_{pq^b} \) (since \( a > 1 \)), while if \( Q \notin G \), then \( n_q \geq q + 1 \) and \( n_p \geq 1 \). Moreover, for each power \( q^j \) \((i < b)\), either \( G \) contains a unique subgroup \( H_{q^j} \) and we may create \( a + 1 \) product subgroups with the subgroups of \( G \), or \( G \) contains at least \( q + 1 \) such subgroups by **Theorem 7**. Thus, \( G \) contains at least \((b - 1)\min(a + 1, q + 1)\) subgroups of orders \( p^aq^j \) \((0 \leq i \leq a, 1 \leq j < b)\) as well. \( G \) itself makes up for the 1 we subtracted from the subgroups of \( G \); thus

\[
|\text{Sub } G| \geq q + 2 + (b - 1) \min(a + 1, q + 1) + m_p,
\]

where \( m_p \) is the number of subgroups of \( G \) coming from **Theorem 6** or **Theorem 3** as applicable. This can be simplified as \( b + q + 1 + (b - 1) \min(a, q) + m_p \).
decomposable as a nontrivial direct product of subgroups with coprime orders, we can apply Theorem 1; thus we again consider only groups that cannot be decomposed in this way.

**Theorem 10.** Let $G$ be a nonnilpotent group that is not decomposable as a nontrivial direct product of groups with coprime orders. If $|G| = p^aq^br^c$ with $p < q < r$, then

$$|\text{Sub } G| \geq a + b + c + (a - 1) \min(b + 1, p) + (b - 1) \min(c + 1, q)$$

$$+ (c - 1) \min(a + 1, r) + \min\left\{ \begin{array}{ll} p + q + r + 2, & \\
q + 2 + \min(p^i + 1, q^j, 2r + 1), & \\
\min(r + 1, 2q + 2) + \min(r + 1, 2q), & \end{array} \right\}, \quad (2)$$

where $i$ and $j$ are minimal such that $p^i, q^j \geq r + 1$.

**Proof.** Recall, since $G$ is nonnilpotent, it contains at least one nonnormal Sylow subgroup. First we explore subgroups of orders $p^a, q^b, r^c, p^aq^b, p^ar^c,$ and $q^br^c$ by considering the number of Sylow subgroups that are normal in $G$. The most straightforward situation is when all three are nonnormal, in which case part 3 of the Sylow theorems implies that $n_p + n_q + n_r \geq 2q + r + 2$. In this case, it is possible that $G$ contains zero subgroups of orders $p^aq^b$, $p^ar^c$, or $q^br^c$.

If instead exactly two of the Sylow subgroups are normal, then $G$ contains product subgroups $H_{p^aq^b}$, $H_{p^ar^c}$, and $H_{q^br^c}$. The two constructed as products with the lone nonnormal Sylow subgroup cannot be normal themselves however, as that would imply $G$ could be decomposed as a direct product of groups with coprime orders. In addition, since those product subgroups are self-stabilizing, the orbit-stabilizer theorem implies that the size of their orbits under conjugation must divide whatever prime power is missing in their order. For example, if $\mathcal{R} \in \text{Syl}_p(G)$ is the nonnormal one, then $G$ contains at least $p + q$ subgroups of orders $p^r$ or $q^r$ in addition to $r + 3$ total Sylow subgroups for at least $p + q + r + 3$ subgroups. Similarly, if $\mathcal{Q} \in \text{Syl}_q(G)$ is nonnormal, this count becomes $p + r + (q + 3)$, while if $\mathcal{P} \in \text{Syl}_p(G)$ is nonnormal, it is $q + r + (q + 2)$ (which is at least $p + q + r + 3$ for all primes $p < q < r$).

The most delicate case is when only one Sylow subgroup is normal, and we proceed based on whether $\mathcal{R}$ is normal or not. If $\mathcal{R} \not\leq G$, then $n_r \geq r + 1$ by part 3 of the Sylow theorems. Moreover, if $\mathcal{P} \not\leq G$ (respectively, $\mathcal{Q} \not\leq G$), then $n_p + n_q \geq q + 2$ (respectively, $q + 1$). In addition, $G$ contains product subgroups $H_{p^aq^b}$ and $H_{p^ar^c}$ (respectively, $H_{q^br^c}$). However, if any subgroup of order $p^a r^c$ (respectively, $q^b r^c$) contains multiple Sylow $r$-subgroups, then $n_r \geq p^i$ for some $i$ (respectively, $n_r \geq q^j$ for some $j$) as well. And if no subgroups of order $p^a r^c$ (respectively, $q^b r^c$) contain multiple Sylow $r$-subgroups, then there exist at least $n_r \geq r + 1$ of them. Thus, counting only Sylow subgroups and their products, $G$ contains at least $q + \min(p^i + 4, q^j + 3, 2r + 4)$ such subgroups, where $i$ and $j$ are minimal such that $p^i, q^j \geq r + 1$.

If instead $\mathcal{R} \leq G$, then $G$ contains product subgroups $H_{p^aq^b}$ and $H_{q^br^c}$. As in the previous case, if any subgroup of order $p^a r^c$ contains multiple Sylow $p$-subgroups, then $n_p \geq r$, and if none do, then there must be exactly $n_p \geq q$ subgroups of order $p^a r^c$. Hence, $G$ contains at least $\min(r + 1, 2q)$ subgroups of order $p^a$ or $p^a r^c$. Repeating the argument, if any subgroup of order $q^b r^c$ contains multiple Sylow $q$-subgroups, then $n_q \geq r$, and if not, then there must be $n_q \geq q + 1$ subgroups of order $q^b r^c$. Hence, $G$ contains at least $\min(r + 1, 2q + 2)$ subgroups of orders $q^b$ or $q^b r^c$. 

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Summarizing, if exactly one Sylow subgroup is normal, then $G$ contains at least

$$\min\{q + 3 + \min(p^i + 1, q^j, 2r + 1), \min(r + 1, 2q + 2) + \min(r + 1, 2q) + 1\}$$

subgroups of orders $p^a, q^b, r^c, p^aq^b, p^ar^c,$ or $q^br^c$.

Aside from $G$ and $\{e\}$, this leaves subgroups whose orders involve at least one lower prime power to account for. Starting with the prime powers $p^i$, for $1 \leq i < a$, as before these account for at least $(a - 1) \min(b + 2, p + 1)$ subgroups (by pairing them up with the $q$-subgroups and $\mathcal{R}$). Similarly, each prime power $q^j$, for $1 \leq j < b$, accounts for at least $(b - 1) \min(c + 2, q + 1)$ subgroups (by pairing them up with the $r$-subgroups and $\mathcal{P}$) and each power $r^\ell$, for $1 \leq \ell < c$, accounts for $(c - 1) \min(a + 2, r + 1)$ subgroups (by pairing them up with the $p$-subgroups and $\mathcal{Q}$). In total, since $p + q + r + 3 \leq 2q + r + 2$ for all primes $p < q < r$, with some minor arithmetic simplifications we have demonstrated that the inequality in (2) holds.

**Remark.** We could have taken multiple perspectives when counting the subgroups of lower orders (i.e., by pairing them up in different ways). Since all of these perspectives would be valid, the lower bound would be the maximum of each. However, for our purposes, this particular choice (which exhibits some level of symmetry in $a$, $b$, and $c$) was good enough. Indeed, as soon as at least one of $a$, $b$, or $c$ is greater than $1$, our bound shows that $|\text{Sub } G| \geq 18$ when $r = 5$ and $|\text{Sub } G| \geq 20$ when $r \geq 7$.

In the special case when $|G| = pqr$ it is known that $\mathcal{R} \in \text{Syl}_p(G)$ is normal (see, e.g., [5]). Moreover, since each Sylow subgroup in $G$ must be cyclic in this case, Lemma 8 implies that $G$ must be decomposable as a direct product of coprime parts whenever $\mathcal{P} \in \text{Syl}_q(G)$ is normal. Thus, when $G$ is nonnilpotent and not decomposable, we may use these facts to improve our bound slightly.

**Theorem 11.** Let $G$ be nonnilpotent with $|G| = pqr$ ($p < q < r$) that is not decomposable as a nontrivial product of subgroups with coprime orders. Then $|\text{Sub } G| \geq r + 4 + \min(r + 1, 2q)$.

**Proof.** By the above discussion, $\mathcal{R} \in \text{Syl}_p(G)$ is normal and $\mathcal{P} \in \text{Syl}_q(G)$ is not normal. This allows us to reduce to two cases based on whether $\mathcal{Q} \in \text{Syl}_q(G)$ is normal. In both cases, the normality of $\mathcal{R}$ implies that $G$ contains product subgroups $H_{pr}$ and $H_{qr}$, and we note that $H_{qr}$ is normal as it has index $p$ (again, see [6, Theorem 1]).

If $\mathcal{Q} \unlhd G$, then $G$ also contains a product subgroup $H_{pq}$ that cannot be normal (otherwise $G$ could be decomposed). Thus, in fact, $G$ contains $r$ subgroups of order $pq$ by the orbit-stabilizer theorem. Note, however, that with $\mathcal{Q}$ unique, there can only be $r$ subgroups of order $pq$ when $n_p \geq r$. Similarly, since $\mathcal{Q} \unlhd G$, the subgroup $H_{pr}$ cannot be normal either. Thus $G$ also contains $q$ subgroups of order $pr$. Accounting for $\mathcal{Q}, \mathcal{R}, H_{qr}, G,$ and $\{e\}$ as well, it follows that $|\text{Sub } G| \geq q + 2r + 5$.

If instead $\mathcal{Q} \not\unlhd G$, then the normal subgroup $H_{qr}$ must contain all Sylow $q$-subgroups of $G$—meaning that $n_q = r$ by part 3 of the Sylow theorems. In addition, since the product of any Sylow $p$-subgroup with $\mathcal{R}$ will result in a group of order $pr$, it follows that either a single subgroup $H_{pr}$ contains all of the Sylow $p$-subgroups—meaning $n_p = r$—or $G$ contains $q$ subgroups of order $pr$ by the orbit-stabilizer theorem. Accounting for $\mathcal{R}, G,$ and $\{e\}$, we have $|\text{Sub } G| \geq r + 4 + \min(r + 1, 2q)$. Note that this is strictly less than the count obtained in the previous case.

Together, Theorems 10 and 11 imply that if there exists a nonnilpotent group $G$, with $|G|$ divisible by three primes, that is not decomposable as a direct product of subgroups with coprime orders, and having fewer than 20 subgroups, then $|G| = 30$, March 2022] GROUPS WITH FEW SUBGROUPS 263
42, 60, 90, or 150. Next, we rule out all nonnilpotent groups whose orders are divisible by four or more primes. The argument is much more elementary than in the three-prime case as there are many more products involving only the various coprime Sylow subgroups themselves.

**Theorem 12.** Let \( G \) be a nonnilpotent group with \( |G| = p^aq^br^cs^dt \) for primes \( p < q < r < s \), where \( t \in \mathbb{N} \) is relatively prime to \( pqr \). Then \( |\text{Sub } G| \geq 20 \).

**Proof.** Just as in Theorem 10 and Theorem 11, the minimum number of distinct product subgroups involving two or more coprime Sylow subgroups which exist in \( G \) depends directly on the number of Sylow subgroups which are assumed to be normal. More specifically, for our purposes, we need only consider which Sylow subgroups are normal from within the collection of only \( \mathcal{P} \in \text{Syl}_p(G), \mathcal{Q} \in \text{Syl}_q(G), \mathcal{R} \in \text{Syl}_r(G), \) and \( \mathcal{S} \in \text{Syl}_s(G) \); this leaves us with five cases.

First, suppose all four Sylow subgroups, \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \) and \( \mathcal{S} \) are normal. Since the product of normal subgroups is again normal, it follows that \( G \) contains at least 10 product subgroups \( H_{p^ar^b}, H_{p^br^c}, H_{p^cr^d}, H_{q^br^c}, H_{q^cr^d}, H_{r^d}, H_{p^qr^rc}, H_{p^qr^rd}, H_{p^qr^rds}, \) and \( H_{q^br^c} \)—all of which are themselves normal in \( G \). Counting \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \) and \( \{e\} \), this already demonstrates that \( |\text{Sub } G| \geq 16 \). However, the fact that \( G \) is not nilpotent implies that \( t \neq 1 \), and thus \( G \) contains at least one other Sylow subgroup for a prime dividing \( t \) and at least 14 additional product subgroups.

Next, suppose exactly one of \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \) or \( \mathcal{S} \) is nonnormal. The 10 product subgroups described in the previous case all still exist, but may no longer be normal. In addition, \( \text{Sub } G \) contains \( \mathcal{G}, \{e\} \), and at least three normal Sylow subgroups. If \( \mathcal{S} \notin \mathcal{G} \), then \( n_s \geq s + 1 \) which implies \( |\text{Sub } G| \geq 16 + s \geq 23 \). If instead \( \mathcal{R} \notin \mathcal{G} \), then \( n_r \geq r + 1 \) and \( |\text{Sub } G| \geq 16 + r \geq 21 \). If \( \mathcal{Q} \notin \mathcal{G} \), then either a single product \( H_{p^br^c} \) contains multiple Sylow \( q \)-subgroups and \( n_q \geq s \), or there are at least \( p - 1 \) as yet uncounted subgroups of order \( q^b \cdot s^d \) and \( n_q \geq q + 1 \)—hence \( |\text{Sub } G| \geq 15 + \min(s, p + q) \geq 20 \). Similarly, if \( \mathcal{P} \notin \mathcal{G} \), then either a single product \( H_{p^qr^r} \) contains multiple Sylow \( p \)-subgroups and \( n_p \geq q \), or there are at least \( q - 1 \) additional subgroups of order \( p^a s^d \) and \( n_p \geq q \)—hence \( |\text{Sub } G| \geq 15 + \min(s, 2q - 1) \geq 20 \).

Suppose instead that exactly two of \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \) or \( \mathcal{S} \) are nonnormal. Under that assumption, part 3 of the Sylow theorems implies that \( n_p + n_q + n_r + n_s \geq 2q + 3 \). Now, without loss of generality, assume \( \mathcal{P} \) and \( \mathcal{Q} \) are nonnormal. The 7 product subgroups that do not involve both \( p \) and \( q \) must still exist. Note, however, that either \( H_{p^br^c} \) is normal—and \( G \) must also contain \( H_{p^qr^rc} \)—or it is not normal, in which case \( G \) contains multiple subgroups of order \( p^a r^c \). Thus, the product \( p^a r^c \) must account for at least two product subgroups. A similar argument shows that \( p^a s^d \) must also account for at least two product subgroups. Hence, in total, \( G \) contains at least 9 product subgroups involving two or more coprime Sylow subgroups. Together with \( G \) and \( \{e\} \), this demonstrates that \( |\text{Sub } G| \geq 2q + 14 \geq 20 \).

Continuing to build up, suppose that three of \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \) or \( \mathcal{S} \) are nonnormal. \( G \) still contains at least 3 product subgroups involving the lone normal Sylow subgroup. If \( \mathcal{S} \notin \mathcal{G} \), then part 3 of the Sylow theorems implies \( n_p + n_q + n_r + n_s \geq 2q + s + 2 \), which further implies that \( |\text{Sub } G| \geq 2q + s + 7 \geq 20 \). If instead \( \mathcal{S} \leq \mathcal{G} \), then either \( H_{p^br^c} \) contains all of the Sylow \( p \)-subgroups (implying that \( n_p \geq s \)), or \( G \) contains at least \( q \) subgroups of order \( p^a s^d \). Similarly, either \( n_q \geq s \) (respectively, \( n_r \geq s \)), or \( G \) contains at least \( p \) subgroups of order \( q^b r^c s^d \) (respectively, \( r^c s^d \)). Accounting for \( G \) and \( \{e\} \) as well, we minimally have \( |\text{Sub } G| \geq (2q + r + 2) + 9 \geq 22 \).

Finally, if all four of \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \) or \( \mathcal{S} \) are nonnormal, then part 3 of the Sylow theorems implies that \( |\text{Sub } G| \geq n_p + n_q + n_r + n_s \geq 2q + r + s + 3 \geq 21 \).
Classifying nonabelian groups with $|\text{Sub } G| \leq 19$. For each potential prime factorization of $|G|$ up to similarity, we can apply Theorem 9, 10, 11, or 12 as appropriate to determine the smallest primes that force $|\text{Sub } G| \geq 20$—thereby reducing the search space for nonabelian groups with 19 or fewer subgroups. Specifically, any nonabelian group $G$ with $|\text{Sub } G| \leq 19$ must either be a direct product of groups with coprime orders (whose individual subgroup counts multiply to $|\text{Sub } G|$), or $G$ must be non-nilpotent with $|G|$ satisfying one of the options listed in Table 2.

Table 2. Potential values for $|G|$ if $|\text{Sub } G| \leq 19$.

| $|\text{Sub } G|$ | Similarity Classes of Nonabelian Groups | # of Classes |
|---------------|----------------------------------------|--------------|
| 6             | $Q_8, S_3$                             | 2            |
| 8             | $D_{10}, Dic_{12}$                     | 2            |
| 10            | $Z_7 \times Z_3, Z_3 \times Z_7, D_8, D_{14}, M_{27}, Dic_{20}, A_4$ | 7            |
| 11            | $Q_{16}, M_{16}$                       | 2            |
| 12            | $Q_8 \times Z_p, S_3 \times Z_p, Z_3 \times Z_9, Z_3 \times Z_3, Z_3 \times Z_3$ | 6            |
| 14            | $M_{32}, S_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3$ | 11           |
| 15            | $SL(2, 3), SD_{16}, Z_4 \times Z_4, (Z_2 \times Z_2) \times Z_4$ | 4            |
| 16            | $Dic_{12} \times Z_p, D_{10} \times Z_p, D_{18}, D_{12}, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5$ | 13           |
| 17            | $Z_3 \times Z_3, Z_3 \times Z_3$      | 1            |
| 18            | $Q_8 \times Z_2, S_3 \times Z_2, Z_8 \times Z_8, Z_8 \times Z_8, Z_8 \times Z_8$ | 15           |
| 19            | $Z_2 \times Q_8, D_{16}, (Z_3 \times Z_3) \times Z_3, Dic_{36}$ | 4            |

With this reduced search space, we use GAP to search systematically beginning with smaller numbers of subgroups working up. (Of course, when $|\text{Sub } G| < 19$, we need not consider the entire search space.) Table 3 lists the similarity classes of all nonabelian groups with 19 or fewer subgroups. Note we omit empty rows and, as before, any arbitrary primes that appear are assumed to be distinct from the others.

Table 3. Nonabelian groups with 19 or fewer subgroups.

| $|\text{Sub } G|$ | Similarity Classes of Nonabelian Groups | # of Classes |
|---------------|----------------------------------------|--------------|
| 6             | $Q_8, S_3$                             | 2            |
| 8             | $D_{10}, Dic_{12}$                     | 2            |
| 10            | $Z_7 \times Z_3, Z_3 \times Z_7, D_8, D_{14}, M_{27}, Dic_{20}, A_4$ | 7            |
| 11            | $Q_{16}, M_{16}$                       | 2            |
| 12            | $Q_8 \times Z_p, S_3 \times Z_p, Z_3 \times Z_9, Z_3 \times Z_3, Z_3 \times Z_3$ | 6            |
| 14            | $M_{32}, S_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3, Z_3 \times Z_3$ | 11           |
| 15            | $SL(2, 3), SD_{16}, Z_4 \times Z_4, (Z_2 \times Z_2) \times Z_4$ | 4            |
| 16            | $Dic_{12} \times Z_p, D_{10} \times Z_p, D_{18}, D_{12}, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5, Z_5 \times Z_5$ | 13           |
| 17            | $Z_3 \times Z_3, Z_3 \times Z_3$      | 1            |
| 18            | $Q_8 \times Z_2, S_3 \times Z_2, Z_8 \times Z_8, Z_8 \times Z_8, Z_8 \times Z_8$ | 15           |
| 19            | $Z_2 \times Q_8, D_{16}, (Z_3 \times Z_3) \times Z_3, Dic_{36}$ | 4            |

Given the classification of both abelian and nonabelian groups with 19 or fewer subgroups, we can combine these results to give the first 19 terms in sequence A274847 [15]. Those terms are 1, 1, 1, 2, 2, 5, 1, 2, 12, 4, 11, 1, 17, 8, 22, 3, 22, 5. Further exploration using these techniques is possible, however, as we are already running into the limits of what GAP can check, it would require improving some of the bounding arguments made above. One place ripe for improvement is the case of $p^aq^b$ with $a, b \geq 2$. There appear to be no such non-nilpotent groups with fewer than 20 subgroups when $q > 3$ even though our bounds did not rule out some of those cases.
Remark (| Sub G| prime). At first, the authors suspected that a nonabelian group with a prime number of subgroups would have to be a $p$-group (as is true for abelian groups by **Corollary 2**). Discovering the counterexample of $A_5$—which has 59 subgroups—we adjusted this conjecture to perhaps solvable nonabelian groups. However, as seen above, $\text{Dic}_{36}$ has 19 subgroups despite being solvable and not a $p$-group.

**Remark (Clarifying Table 3).** To further clarify, recall that the named groups in Table 3 follow the notation from GAP. For example, $\text{Dic}_{12}$ represents the dicyclic group twice in Table 3. With 12 subgroups, $\mathbb{Z}_5 \ltimes \mathbb{Z}_8 = \langle x, y \mid x^5 = y^8 = e, yxy^{-1} = x^{-1} \rangle$. With 14 subgroups, $\mathbb{Z}_5 \ltimes \mathbb{Z}_{16} = \langle x, y \mid x^5 = y^{16} = e, yxy^{-1} = x^{-1} \rangle$ and $\mathbb{Z}_{25} \ltimes \mathbb{Z}_5 = \langle x, y \mid x^{25} = y^5 = e, yxy^{-1} = x^6 \rangle$. With 16 subgroups, $\mathbb{Z}_5 \ltimes \mathbb{Z}_8 = \langle x, y \mid x^5 = y^8 = e, yxy^{-1} = x^3 \rangle$ and $\mathbb{Z}_5 \ltimes \mathbb{Z}_{32} = \langle x, y \mid x^5 = y^{32} = e, yxy^{-1} = x^{-1} \rangle$. With 17 subgroups $\mathbb{Z}_{32} \ltimes \mathbb{Z}_2 = \langle x, y \mid x^{32} = y^2 = e, yxy^{-1} = x^{17} \rangle$. With 18 subgroups $\mathbb{Z}_8, \mathbb{Z}_4 = \langle x, y \mid x^8 = e, x^4 = y^4, yxy^{-1} = x^{-1} \rangle$, $\mathbb{Z}_3 \ltimes \mathbb{Z}_{128} = \langle x, y \mid x^3 = y^{128} = e, yxy^{-1} = x^{-1} \rangle$, $\mathbb{Z}_5 \ltimes \mathbb{Z}_{16} = (x, y \mid x^5 = y^{16} = e, yxy^{-1} = x^3 \rangle$, and $\mathbb{Z}_5 \ltimes \mathbb{Z}_{64} = \langle x, y \mid x^5 = y^{64} = e, yxy^{-1} = x^{-1} \rangle$.

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100 Years Ago This Month in The American Mathematical Monthly
Edited by Vadim Ponomarenko

As Babbage was leaving Paris he wrote to M. Bouvard a letter of appreciation and thanks, acknowledging the gift of a portrait of Laplace which had long been in Bouvard’s study. The letter, now in my collection, shows something of the personal side of Babbage and his natural courtesy on such an occasion, and is as follows:

(M. Alexis Bouvard.)
À Paris

My dear Sir:

I cannot leave Paris without thanking you for the very delightful day I spent with you in the society of Madame de Laplace. I shall preserve with the respect it deserves the portrait of her illustrious husband and if any circumstance could have rendered the gift more valuable it is the fact that it has so long adorned the study of that chosen friend by whose unremitted labors the Mecanique Coeleste received such valuable assistance.

I am My Dear Sir
with the greatest respect and Regard
Ever very sincerely Yours,
C. Babbage

Paris 2 Sep. 1840

P.S. I enclose two copies of the drawing of the Engine of Differences on [sic] to replace your own and one which I beg your nephew to do me the favour to accept.

—Excerpted from D. E. Smith “Among My Autographs” (1922), 29(3): 114–116.