THE KADISON-SINGER PROBLEM IN DISCREPANCY
THEORY, II

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Abstract. We apply Srivastava’s spectral sparsification technique to a vector balancing version of the Kadison-Singer problem. The result is a one-sided version of the conjectured solution.

The celebrated Kadison-Singer problem (KSP) is equivalent to a simple vector balancing question ([9], Theorem 3). Do there exist constants $N, r \in \mathbb{N}$ which make the following statement true?

If $k \in \mathbb{N}$ and $\{v_1, \ldots, v_m\}$ is a finite sequence of vectors in $\mathbb{C}^k$ satisfying $\|v_i\| = \frac{1}{\sqrt{N}}$ for all $i$ and

$$\sum_i |\langle u, v_i \rangle|^2 = 1$$

for all unit vectors $u$, then the index set $\{1, \ldots, m\}$ can be partitioned into subsets $S_1, \ldots, S_r$ such that

$$\sum_{i \in S_j} |\langle u, v_i \rangle|^2 \leq 1 - \frac{1}{\sqrt{N}}$$

for all unit vectors $u$ and all $j = 1, \ldots, r$.

Here $\|v\|$ is the euclidean norm of $v$. It is unclear whether allowing $r > 2$ makes the problem any easier. If we take $r = 2$ then we can equivalently ask whether it is always possible to find a subset $S \subseteq \{1, \ldots, m\}$ satisfying $\frac{1}{\sqrt{N}} \leq \sum_{i \in S} |\langle u, v_i \rangle|^2 \leq 1 - \frac{1}{\sqrt{N}}$ for all unit vectors $u$. The purpose of this note is to present a partial positive result in this direction: for any $n < m$ we can find a subset $S \subseteq \{1, \ldots, m\}$ with $|S| = n$ and such that

$$\sum_{i \in S} |\langle u, v_i \rangle|^2 \leq \frac{n}{m} + O \left( \frac{1}{\sqrt{N}} \right)$$

for all unit vectors $u$. This is a “one-sided” version of the desired result in the sense that we achieve an upper bound but not a lower bound.

Our theorem is a straightforward application of the spectral sparsification technique introduced in Srivastava’s thesis [8]. This technique was already related to KSP via Bourgain and Tzafriri’s restricted invertibility theorem [7, 8]. That result can be converted into one resembling ours as in the proof of Theorem 4.2 of [4], but with a substantially worse bound (on the order of $\frac{n}{m} + 2\sqrt{\frac{n}{m}}$).

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1. The Projection Version of KSP

The Kadison-Singer problem was first posed in [5] in the form of a C*-algebraic question relating pure states on $B(l^2)$ to pure states on its diagonal subalgebra. Since then it has been found to have numerous equivalent versions, and it is now considered a major open problem with relevance to topics ranging from Banach space theory to signal processing. We refer to [4] for general background and a survey of a variety of equivalent versions of the problem.

Our version is based on the approach of Akemann and Anderson [1] in terms of projection matrices. A complex $m \times m$ matrix is a projection if the associated linear map orthogonally projects vectors in $\mathbb{C}^m$ onto some linear subspace $E$. Note that a diagonal matrix is a projection if and only if its diagonal entries are all either zero or one. The Akemann-Anderson version of KSP asks whether there exist constants $\varepsilon, \delta > 0$ and $r \in \mathbb{N}$ which make the following statement true.

If $m \in \mathbb{N}$ and $P$ is a complex $m \times m$ projection matrix whose diagonal entries $p_{ii}$ satisfy $p_{ii} \leq \delta$, then there are diagonal projections $Q_1, \ldots, Q_r$ which sum to the identity matrix and satisfy $\|Q_jPQ_j\| \leq 1 - \varepsilon$ for all $j = 1, \ldots, r$.

That a positive answer to this question implies a positive solution to KSP is essentially proven in Propositions 7.6 and 7.7 of [1], and the reverse direction is shown in Theorem 1 of [9]. A more elementary approach to this reduction appears in [3].

The projection version of KSP is easily seen to be equivalent to a vector balancing question similar to the one stated in the introduction. Identify the range $E$ of $P$ with $\mathbb{C}^k$ where $k$ is the rank of $P$ and define $v_i = Pe_i$, $1 \leq i \leq m$, where $\{e_i\}$ is the standard basis of $\mathbb{C}^m$. Then for any vector $u \in \mathbb{C}^m$ we have

$$\sum_i |\langle u, v_i \rangle|^2 = \sum_i |\langle Pu, e_i \rangle|^2 = \|Pu\|^2;$$

in particular, if $u$ is a unit vector in $E$ then this sum equals 1. Also, $\|v_i\|^2 = \langle Pe_i, e_i \rangle = p_{ii} \leq \delta$, giving us a bound on the size of the vectors $v_i$. The diagonal projections $Q_j$ correspond to a partition of $S$ into $r$ pieces.

The version of KSP stated in the introduction, in which the vectors $v_i$ all have the same norm $\frac{1}{\sqrt{N}}$, can be achieved by adding extra dimensions to the space and augmenting the vectors $v_i$ with components in these extra dimensions. See Theorem 3 of [9] for details. In the language of projections, this corresponds to requiring that the diagonal entries of $P$ all equal $\frac{1}{N}$, and asking for $\|Q_jPQ_j\| \leq 1 - \frac{1}{\sqrt{N}}$.

2. Counterexamples

Let $k, N \in \mathbb{N}$ and suppose $\{v_1, \ldots, v_m\}$ is a finite sequence of vectors in $\mathbb{C}^k$ satisfying $\|v_i\| = \frac{1}{\sqrt{N}}$ for $1 \leq i \leq m$ and

$$\sum_i |\langle u, v_i \rangle|^2 = 1$$

for all unit vectors $u$. According to Proposition 4 of [9] we can find a subset $S \subseteq \{1, \ldots, m\}$ such that

$$\frac{1}{2} - \frac{1}{N} \leq \sum_{i \in S} |\langle e_j, v_i \rangle|^2 \leq \frac{1}{2} + \frac{1}{N}.$$
for all $1 \leq j \leq k$, where $\{e_j\}$ is the standard basis of $\mathbb{C}^k$. This follows by applying the continuous Beck-Fiala theorem \cite{2} to the vectors $\{(|e_1, v_i|)^2, \ldots, (|e_k, v_i|)^2\} \in \mathbb{R}^k$ for $1 \leq i \leq m$. Thus, on a fixed orthonormal basis a very tight bound can be achieved. However, this bound is too strong in general. We know from Example 7 of \cite{3} that there are configurations of vectors with $N, k \to \infty$ such that for any $S$ there is some unit vector $u$ for which the sum $\sum_{i \in S} |\langle u, v_i \rangle|^2$ lies outside an interval which is asymptotic to $(\frac{1}{2} - \frac{1}{\sqrt{2N}}, \frac{1}{2} + \frac{1}{\sqrt{2N}})$. Thus, the worst counterexamples we know of are $O(\frac{1}{\sqrt{N}})$ away from $\frac{1}{2}$. The obvious conjecture is that we can always find a set of indices $S$ for which

$$\frac{1}{2} - O\left(\frac{1}{\sqrt{N}}\right) \leq \sum_{i \in S} |\langle u, v_i \rangle|^2 \leq \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right)$$

for all unit vectors $u$. Or perhaps even for any $q \in (0, 1)$ we can always find an $S$ for which

$$q - O\left(\frac{1}{\sqrt{N}}\right) \leq \sum_{i \in S} |\langle u, v_i \rangle|^2 \leq q + O\left(\frac{1}{\sqrt{N}}\right)$$

for all unit vectors $u$.

What we are trying to accomplish is to build up a set of indices $S$ which makes $\sum_{i \in S} |\langle u, v_i \rangle|^2$ uniformly greater than 0 in all directions $u$, while preventing this sum from getting too close to 1 in any direction. An example due to Nets Katz \cite{4} shows why this may be difficult. Fix $N \in \mathbb{N}$ and let $X$ be the family of all subsets of $\{1, \ldots, 2N\}$ of size $N$. For each $1 \leq i \leq 2N$ let $f_i : X \to \mathbb{R}$ be the function satisfying $f_i(A) = 0$ if $i \not\in A$ and $f_i(A) = \frac{1}{N}$ if $i \in A$. Then $\sum_i f_i(A) = |A| \cdot \frac{1}{N} = 1$ for each $A \in X$, so the functions $f_i$ have sup norm $\frac{1}{N}$ and sum up to 1 at every point. Now for any $S \subseteq \{1, \ldots, 2N\}$, if $|S| \leq N$ then we can find $A \in X$ disjoint from $S$, so that $\sum_{i \in S} f_i(A) = 0$. But if $|S| \geq N$ then it contains some $A \in X$, and we then have $\sum_{i \in S} f_i(A) = 1$. So we cannot get away from 0 at all points of $X$ without summing to 1 at some point.

However, the result we prove in the next section shows that nothing like this can happen with KSP. In Katz’s example any set of at least half of the functions $f_i$ must sum to 1 at some point. Whereas our theorem achieves an upper bound only $O(\frac{1}{\sqrt{N}})$ higher than $\frac{1}{m}$.

### 3. An upper bound

As in the introduction, $\{v_1, \ldots, v_m\}$ will be a finite set of vectors in $\mathbb{C}^k$, each of norm $\frac{1}{\sqrt{N}}$, satisfying

$$\sum_i |\langle u, v_i \rangle|^2 = 1$$

for all unit vectors $u$. We work with the linear operators $v \otimes v : \mathbb{C}^k \to \mathbb{C}^k$ defined by $(v \otimes v)(u) = \langle u, v \rangle v$. For any $S \subseteq \{1, \ldots, m\}$ the operator $T = \sum_{i \in S} v_i \otimes v_i$ satisfies

$$\langle Tu, u \rangle = \sum_{i \in S} |\langle u, v_i \rangle|^2,$$

and the values $\langle Tu, u \rangle$, for $u$ a unit vector in $\mathbb{C}^k$, are all bounded above by $\|T\|$. Thus we are interested in choosing $S$ so as to minimize $\|T\|$. Note that $\text{Tr}(v_i \otimes v_i) = \|v_i\|^2 = \frac{1}{N}$ and $\sum_{i=1}^m v_i \otimes v_i = I$, which has trace $k$, so that $m = kN$. 


Let $n < m$. As in [8], we build the subset $S$ one vector at a time. Thus our procedure will select vectors $v_{i1}, \ldots, v_{in}$ with corresponding operators $T_j = \sum_{d=1}^j v_{id} \otimes v_{id}$. For any positive operator $T$ and any $a > \|T\|$, define the upper potential $\Phi^a(T)$ to be

$$\Phi^a(T) = \text{Tr}((aI - T)^{-1});$$

then having chosen the vectors $v_{i1}, \ldots, v_{i_{j-1}}$ we will select a new vector $v_{ij}$ so as to minimize $\Phi^a(T_j)$, where the $a_j$ are an increasing sequence of upper bounds. This potential function disproportionately penalizes eigenvalues which are close to $a_j$ and thereby controls the maximum eigenvalue, i.e., the norm, of $T_j$. The key fact about the upper potential is given in the following result.

**Lemma 3.1.** ([8, Lemma 3.4]) Let $T$ be a positive operator on $\mathbb{C}^k$, let $a, \delta > 0$, and let $v \in \mathbb{C}^k$. Suppose $\|T\| < a$. If

$$\frac{\langle (a + \delta)I - T \rangle^{-2} v, v \rangle}{\Phi^a(T) - \Phi^{a+\delta}(T)} + \frac{\langle (a + \delta)I - T \rangle^{-1} v, v \rangle}{\Phi^a(T)} \leq 1$$

then $\|T + v \otimes v\| < a + \delta$ and $\Phi^{a+\delta}(T + v \otimes v) \leq \Phi^a(T)$.

The proof relies on the Sherman-Morrison formula, which states that if $T$ is positive and invertible then $(T + v \otimes v)^{-1} = T^{-1} - \frac{T^{-1}(v \otimes v)T^{-1}}{1 + \langle v, v \rangle}$. We also require a simple inequality.

**Lemma 3.2.** Let $a_1 \leq \cdots \leq a_k$ and $b_1 \geq \cdots \geq b_k$ be sequences of positive real numbers, respectively increasing and decreasing. Then $\sum a_ib_i \leq \frac{1}{k} \sum a_i \sum b_i$.

**Proof.** Let $M = \frac{1}{k} \sum b_i$. We want to show that $\sum a_ib_i \leq \sum a_iM$, i.e., that $\sum a_i(b_i - M) \leq 0$. Since the sequence $(b_i)$ is decreasing, we can find $j$ such that $b_i \geq M$ for $i \leq j$ and $b_i < M$ for $i > j$. Then $\sum_{i=1}^j a_i(b_i - M) \leq a_j \sum_{i=1}^j (b_i - M)$ (since the $a_i$ are increasing and the values $b_i - M$ are positive) and $\sum_{i=j+1}^k a_i(b_i - M) \leq a_j \sum_{i=j+1}^k (b_i - M)$ (since the $a_i$ are increasing and the values $b_i - M$ are negative). So

$$\sum_{i=1}^k a_i(b_i - M) \leq a_j \sum_{i=1}^k (b_i - M) = 0,$$

as desired. \qed

**Theorem 3.3.** Let $k, N \in \mathbb{N}$ and let $\{v_1, \ldots, v_m\}$ be a finite sequence of vectors in $\mathbb{C}^k$ satisfying $\|v_i\| = \frac{1}{\sqrt{N}}$ for $1 \leq i \leq m$ and

$$\sum_{i} |\langle u, v_i \rangle|^2 = 1$$

for all unit vectors $u$. Then for any $n < m$ there is a set $S \subseteq \{1, \ldots, m\}$ with $|S| = n$ such that

$$\sum_{i \in S} |\langle u, v_i \rangle|^2 \leq \frac{n}{m} + O \left( \frac{1}{\sqrt{N}} \right)$$

for all unit vectors $u$.\[\]
Proof. Define \( a_i = \frac{1}{\sqrt{N}} + \left(1 + \frac{1}{\sqrt{N-1}}\right) \frac{1}{m} \) for \( 0 \leq i \leq n \). We will find a sequence of distinct indices \( i_1, \ldots, i_n \) such that the operators \( T_j = \sum_{d=1}^j v_{i_d} \otimes v_{i_d}, \ 0 \leq j \leq n \), satisfy \( \|T_j\| < a_j \) and \( \Phi^{a_0}(T_0) \geq \cdots \geq \Phi^{a_n}(T_n) \). Thus

\[
\|T_n\| < \frac{1}{\sqrt{N}} + \left(1 + \frac{1}{\sqrt{N-1}}\right) \frac{n}{m} = \frac{n}{m} + O\left(\frac{1}{\sqrt{N}}\right),
\]

yielding the desired conclusion. We start with \( T_0 = 0 \), so that \( \Phi^{a_0}(T_0) = \Phi^{1/\sqrt{N}}(0) = \text{Tr}(\frac{1}{\sqrt{N}} I^{-1}) = k \sqrt{N} \).

To carry out the induction step, suppose \( v_{i_1}, \ldots, v_{i_j} \) have been chosen. Let \( \lambda_1 \leq \cdots \leq \lambda_k \) be the eigenvalues of \( T_j \). Then the eigenvalues of \( I - T_j \) are \( 1 - \lambda_1 \geq \cdots \geq 1 - \lambda_k \) and the eigenvalues of \( (a_{j+1} I - T_j)^{-1} \) are \( \frac{1}{a_{j+1} - \lambda_i} \leq \cdots \leq \frac{1}{a_{j+1} - \lambda_k} \).

Thus by Lemma 3.2

\[
\text{Tr}(\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)) = \sum_{d=1}^k \frac{1}{a_{j+1} - \lambda_d} (1 - \lambda_d)
\]

\[
\leq \frac{1}{k} \sum_{d=1}^k \frac{1}{a_{j+1} - \lambda_d} \sum_{d=1}^k (1 - \lambda_d)
\]

\[
= \frac{1}{k} \text{Tr}((a_{j+1} I - T_j)^{-1} \text{Tr}(I - T_j))
\]

\[
= \frac{1}{k} \Phi^{a_{j+1}}(T_j) \text{Tr}(I - T_j)
\]

\[
\leq \frac{1}{k} \Phi^{a_j}(T_j) \text{Tr}(I - T_j)
\]

\[
\leq \frac{1}{k} \Phi^{a_0}(T_0) \text{Tr}(I - T_j)
\]

\[
= \sqrt{N} \text{Tr}(I - T_j).
\]

Next, \( a_{j+1} - a_j = (1 + \frac{1}{\sqrt{N-1}}) \frac{1}{m} \), so we can estimate

\[
\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j) = \text{Tr}((a_j I - T_j)^{-1} - (a_{j+1} I - T_j)^{-1})
\]

\[
= \left(1 + \frac{1}{\sqrt{N-1}}\right) \frac{1}{m} \text{Tr}((a_j I - T_j)^{-1} (a_{j+1} I - T_j)^{-1})
\]

\[
> \left(1 + \frac{1}{\sqrt{N-1}}\right) \frac{1}{m} \text{Tr}((a_{j+1} I - T_j)^{-2})
\]

since each of the eigenvalues \( \frac{1}{a_j - \lambda_d} \frac{1}{a_{j+1} - \lambda_d} \) of the operator \((a_j I - T_j)^{-1} (a_{j+1} I - T_j)^{-1}\) is greater than the corresponding eigenvalue \( \frac{1}{(a_{j+1} - \lambda_d)^2} \) of the operator \((a_{j+1} I - T_j)^{-2}\). Combining this with Lemma 3.2 yields

\[
\text{Tr}((a_{j+1} I - T_j)^{-2} (I - T_j)) \leq \frac{1}{k} \text{Tr}((a_{j+1} I - T_j)^{-2}) \text{Tr}(I - T_j)
\]

\[
< N \left(1 + \frac{1}{\sqrt{N}}\right) (\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)) \text{Tr}(I - T_j)
\]

since \((1 + \frac{1}{\sqrt{N-1}})^{-1} = 1 - \frac{1}{\sqrt{N}}\). Thus

\[
\frac{\text{Tr}((a_{j+1} I - T_j)^{-2} (I - T_j))}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} \leq N \left(1 + \frac{1}{\sqrt{N}}\right) \text{Tr}(I - T_j).
\]

Now let \( S' \subseteq \{1, \ldots, m\} \) be the set of indices which have not yet been used.

Observe that \( \langle T v, v \rangle = \text{Tr}(\Phi(T v \otimes v)) \) and that \( \sum_{i \in S'} v_i \otimes v_i = I - \sum_{d=1}^j v_{i_d} \otimes v_{i_d} = \)
Thus
\[
\sum_{i \in S'} \left( \frac{((a_{j+1}I - T_j)^{-2}v_i, v_i)}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} + \frac{((a_{j+1}I - T_j)^{-1}v_i, v_i)}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} \right) = \frac{\text{Tr}((a_{j+1}I - T_j)^{-2}(I - T_j)) - \text{Tr}((a_{j+1}I - T_j)^{-1}(I - T_j))}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} \\
\leq N \left( 1 - \frac{1}{\sqrt{N}} \right) \text{Tr}(I - T_j) + \sqrt{N} \text{Tr}(I - T_j) = N \text{Tr}(I - T_j)
\]

But
\[
N \text{Tr}(I - T_j) = N(k - \text{Tr}(T_j)) = m - j
\]
is exactly the number of elements of $S'$. So there must exist some $i \in S'$ for which
\[
\frac{((a_{j+1}I - T_j)^{-2}v_i, v_i)}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} + \frac{((a_{j+1}I - T_j)^{-1}v_i, v_i)}{\Phi^{a_j}(T_j) - \Phi^{a_{j+1}}(T_j)} \leq 1.
\]
Therefore, by Lemma 3.3, choosing $v_{i_{j+1}} = v_i$ allows the inductive construction to proceed. $\square$

In terms of projections, Theorem 3.3 states that if $k, N \in \mathbb{N}$ and $P$ is a projection acting on $\mathbb{C}^k$ whose diagonal entries $p_{ij}$ all equal $\frac{1}{N}$, then for each $n < kN$ there is a diagonal projection $Q$ with $\text{Tr}(Q) = \frac{n}{N}$ and such that $\|QPQ\| \leq \frac{1}{n} + O(\frac{1}{\sqrt{N}})$.

4. A LOWER BOUND

In order to produce a positive solution to the Kadison-Singer problem we would have to improve Theorem 3.3 to simultaneously include a lower bound on $\sum_{i \in S} |(u, v_i)|^2$. Now $\text{Tr}(T_n) = \frac{n}{N}$ and $T_n \leq a_n I$ where $a_n = \frac{n}{m} + O(\frac{1}{\sqrt{N}})$, and thus $\text{Tr}(a_n I) = \frac{n}{N} + O(\frac{k}{\sqrt{N}})$. So most of the eigenvalues of $T_n$ must be around $\frac{n}{m}$. The problem is that there could be a small fraction of eigenvalues at or near zero.

If we only want a lower bound, the simplest way to achieve this is to apply Theorem 3.3 and take the operator $I - T_n$. If $T_n = \sum_{i \in S} v_i \otimes v_i$ then $I - T_n = \sum_{i \in S'} v_i \otimes v_i$, so $I - T_n$ is obtained by summing over $m - n$ vectors. And the upper bound $T_n \leq a_n I$ translates to the lower bound $I - T_n \geq (1 - a_n)I = (\frac{m - n}{m} - O(\frac{1}{\sqrt{N}}))I$. Here the danger is that there could be a small fraction of eigenvalues of $I - T_n$ at or near one.

If one tries to run the argument of Theorem 3.3 in a way that simultaneously achieves both upper and lower bounds, one discovers that the two cases are not really symmetric. At each step the upper bound recedes, and we need to choose a new vector $v_{i_{j+1}}$ in a way that avoids overtaking the upper bound. By making the upper bound recede faster, i.e., by increasing the step size from $a_j$ to $a_{j+1}$, we can ensure that any desired fraction of the remaining vectors will accomplish this. The lower bound, on the other hand, is chasing the lower eigenvalues of $T_j$ and is in order to avoid increasing the lower potential we may have to choose a vector which is concentrated on a possibly small number of low eigenvalues. Slowing down the lower step size would only delay this.

In order to handle both upper and lower bounds simultaneously, we have to avoid falling into a situation where the lower bound is approaching a handful of small eigenvalues, and the only vectors available which have components among these small eigenvalues also have components among the largest eigenvalues, and thus...
cannot be selected without overtaking the upper bound. It does not seem possible that any greedy algorithm of the kind used in the proof of Theorem 3.3 could be sure to prevent such a situation from developing.

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