Non-Poisson dichotomous noise: higher-order correlation functions and aging

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We study a two-state symmetric noise, with a given waiting time distribution $\psi(\tau)$, and focus our attention on the connection between the four-time and the two-time correlation functions. The transition of $\psi(\tau)$ from the exponential to the non-exponential condition yields the breakdown of the usual factorization condition of high-order correlation functions, as well as the birth of aging effects.

I. INTRODUCTION

Dichotomous noise is one of the fundamental representations of stochastic processes. It is used in random walks, quantum two-state systems, as well as other mathematical models of physical and biological processes. This representation is used because it is simple enough to obtain analytic solutions to dynamical equations, yet rich enough to model a variety of complex physical and biological phenomena. The history of such two-state stochastic processes dates back more than a century to Markov representations of random telegraphic signals and yet such noise still finds application in models of contemporary complex phenomena. A few recent examples of complex phenomena modeled by dichotomous stochastic processes are disorder-induced spatial patterns [1]; first-passage [2] and thermally activated escape [2] processes; hypersensitive transport [1]; rocking rachets [3]; intermittent fluorescence [4]; stochastic resonance [5, 6]; quantum multifractality [7]; and blinking quantum dots [8, 9]. These and many other applications study the physical effects of dichotomous fluctuations, either Poisson or non-Poisson, without addressing, however, the consequences that relaxing the Poisson assumption might have on the high-order correlation functions.

In this paper we are interested in the high-order correlation properties of the dichotomous noise $\xi(t)$, that is, a symmetrical two-state statistical process with the values $+W$ and $-W$. Usually, for the purpose of making statistical calculations we focus on stationary noise and use the stationary correlation function,

$$\Phi_\xi(t_1, t_2) = \frac{\langle \xi(t_1)\xi(t_2) \rangle}{\langle \xi^2 \rangle},$$

where the brackets denote an average over an ensemble of realizations of the dichotomous noise. It is worth illustrating the difference between this dichotomous noise and a Gaussian noise with the same two-point correlation function. The difference between the two processes resides in the high-order correlation functions. Furthermore, because the noise is symmetric we only need to focus on even-time correlation functions. According to Ref. [10], for Gaussian noise the fourth-order correlation function is related to the second-order correlation function via the following expression:

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle = \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\xi(t_4) \rangle$$

$$+ \langle \xi(t_1)\xi(t_3) \rangle \langle \xi(t_2)\xi(t_4) \rangle + \langle \xi(t_1)\xi(t_4) \rangle \langle \xi(t_2)\xi(t_3) \rangle.$$  

The higher-order correlation functions are analogously defined. In the case where all times are identical, the definition [2] yields

$$\langle \xi^{2n} \rangle = (2n - 1)!\langle \xi^2 \rangle^n,$$

a property ensuring that the distribution of $\xi$ is a Gaussian function. By the same token, it seems natural to factor the fourth-order correlation function for the dichotomous symmetric noise as

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle = \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\xi(t_4) \rangle.$$  

with analogous prescriptions for the higher-order correlation functions. In the case of equal times, the definition [4] reduces to

$$\langle \xi^{2n} \rangle = \langle \xi^2 \rangle^n,$$

which is similar to, but not identical to [3]. Equation [4] is implied for the moments of a stochastic process with the equilibrium distribution function

$$p(\xi) = \frac{1}{2}[\delta(\xi - W) + \delta(\xi + W)].$$

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Hereafter, we refer to property (1) and the factorization of the corresponding higher-order correlation equations, as Dichotomic Factorization (DF).

The vast majority of papers dealing with dichotomous noise assume the statistics of the two-states to be Poisson, that is, the length of time the system remains in a given state has an exponential distribution. It is important to remark that the simplest physical phenomenon modeled by the stochastic variable \( \xi(t) \) is diffusion. This means that all the properties of the phenomenon can be determined by the solution to the stochastic equation

\[
\frac{dx}{dt} = \xi(t). \tag{7}
\]

Allegrini et al. \cite{14} found that the evolution of the probability density, corresponding to the dichotomous Langevin equation \cite{17}, is given by the Generalized Diffusion Equation (GDE)

\[
\frac{\partial p(x,t)}{\partial t} = (\xi^2) \int_0^t dt' \Phi_\xi(t-t') \frac{\partial^2}{\partial x^2} p(x,t'), \tag{8}
\]

where the two-point correlation function under the integral is arbitrary.

It is interesting to note that the same GDE emerges from the analysis of Cáceres \cite{15}, who studied the Langevin equation

\[
\frac{dx}{dt} = -\gamma x(t) + \xi(t), \tag{9}
\]

with \( \xi(t) \) being a dichotomous noise and \( \gamma \) a friction parameter of arbitrary intensity. This same equation was studied in an earlier paper by Annunziato et al. \cite{16}. It is evident that with \( \gamma = 0 \) Eq. \ref{eq:9} becomes equivalent to Eq. \ref{eq:7}. The equation for densities found by Cáceres \cite{15} is identical to that found by Annunziato et al. and both results for \( \gamma \to 0 \) reduce to Eq. \ref{eq:8}. These results are valid independently of the form of the correlation function \( \Phi_\xi(t) \). The fact that GDE is obtained using these different approaches is significant, since the work by Cáceres rests on van Kampen’s lemma \cite{17} and the Bourret-Frisch-Pouquet theorem \cite{18}, while the theory adopted by Annunziato et al. is the same as that used by Allegrini et al. \cite{14}, the Zwanzig’s projection method \cite{19}. In any event, both approaches adopt of a Liouville-like perspective.

Bologna et al. \cite{20} established that the GDE produces the same higher-order \( x \)-moments as those derived from the integration of the diffusion equation, supplemented with the assumption that the correlation functions of the dichotomous variable \( \xi(t) \) fit the prescription of DF. Bologna et al. also established that the exact solution of the GDE does not lead to the process of Lévy diffusion, a result previously obtained using stochastic trajectories, thereby suggesting a possible conflict between the adoption of stochastic trajectories obeying renewal theory in the continuous time random walk (CTRW) formalism and the adoption of a Liouville-like approach to the dynamics \cite{20}. The DF assumption is not explicitly made by Cáceres \cite{15}. However, the analysis of Bologna et al. indicate that the theory of Cáceres \cite{15} implies the DF property. Others have also assumed non-Poisson statistics, while still retaining the DF property \cite{21}.

We establish herein that the DF condition breaks down as a consequence of the non-Poisson condition. Furthermore, we show that the violation of the DF condition emerges from non-Poisson statistics in the same way as do aging properties. These results have the desirable effect of establishing the limits of validity of the elegant GDE, leaving aside for the present the analysis of the issue as to whether the density and Liouville-like formalism are compatible with the emergence of these properties.

\section{II. FOUR-TIME CORRELATION FUNCTION}

In this section we show that in the non-Poisson case, the four-time correlation function of the dichotomous noise departs from the DF prescription. It has to be pointed out that our arguments are based on examining a single sequence \( \xi \), and thus on time averages, rather than on ensemble averages. We assume that the theoretical sequence is built up by creating a sequence \( \{\tau_i\} \) of real positive numbers using the probability density

\[
\psi(\tau) = (\mu - 1) \frac{T^{(\mu-1)}}{(\tau + T)^\mu}. \tag{10}
\]

The choice of this analytical form is determined by simplicity, in which we obtain in the time asymptotic limit an inverse power law with index \( \mu \), while satisfying the normalization condition

\[
\int_0^\infty \psi(\tau) d\tau = 1. \tag{11}
\]

The parameter \( T > 0 \) insures the normalization condition, required by the fact that \( \psi(\tau) \) is a probability density and is related to the average time interval generated by the density. To generate a realization of the time series we split the time axis into many time intervals of length determined by the set of numbers \( \{\tau_i\} \). The first interval begins at time \( t = 0 \) and ends at \( t = \tau_1 \), the second begins at \( t = \tau_1 \) and ends at \( t = \tau_1 + \tau_2 \), the third begins at \( t = \tau_1 + \tau_2 \) and ends at \( t = \tau_1 + \tau_2 + \tau_3 \), and so on. We refer to this sequence of time intervals, which is not observable, as the theoretical sequence. The dichotomous sequence under study in this paper, which can be observed, is created as follows. At the beginning of any time interval we toss a coin, and fill the interval with either the value \( W \) or the value \( -W \), according to whether we get a head or a tail. Thus, if we move along the observable sequence, we meet large time portions of the sequence, within which the sequence retains the same value, either \( W \) or \( -W \). We refer to these time intervals with the same value of \( \xi \), as experimental laminar regions and to the corresponding distributions of time lengths as
We point out that \( \psi_{\text{exp}}(\tau) \) does not necessarily coincide with \( \psi(\tau) \). According to Ref. 22, the theoretical waiting time distribution \( \psi(t) \) is connected to the experimental waiting time distribution by the Laplace transform relation

\[
\hat{\psi}(u) = \frac{2\hat{\psi}_{\text{exp}}(u)}{1 + \hat{\psi}_{\text{exp}}(u)},
\]

(12)

where the Laplace transform of a function \( f(t) \) is denoted by \( \hat{f}(u) \). However, in the time asymptotic limit \( \psi_{\text{exp}}(\tau) \) has the same inverse power law form as does \( \psi(\tau) \), that being Eq. (10), with the same power-law index \( \mu \). In the special case of blinking quantum dots the experimental waiting time distribution is found to be an inverse power law with index \( \mu < 2 \). Here we consider the complementary case \( \mu > 2 \), so as to realize a condition compatible with the existence of a stationary correlation function for \( \xi(t) \).

Due to the theoretical prescription that we adopt to realize the dichotomic sequence under study, a given experimental laminar region, namely, a time interval where, as earlier pointed out, \( \xi(t) \) keeps the same sign, might correspond to an arbitrarily large number of theoretical time intervals, to which the coin tossing procedure assigns the same sign. We shall refer to these theoretical time intervals as theoretical laminar regions, or, more simply, as laminar regions. It is evident that the beginning of a laminar region corresponds to the occurrence of a random event, namely the coin tossing that determines its sign. The laminar regions are not observable, while the experimental laminar regions are observable, by definition, and begin and end with a random event. We cannot establish if other random events occur or not, and how many, between the beginning and the end of an experimental laminar region.

The theoretical approach that we adopt in this section rests on the same time average procedure as that adopted by Geisel et al. 22. Let us devote some attention to the prescription given by these authors to evaluate the two-point correlation function \( \Phi_\xi([t_2 - t_1]) \):

\[
\Phi_\xi(t_2 - t_1) = \frac{\int_0^\infty [\tau - (t_2 - t_1)] \psi(\tau) d\tau}{\int_0^\infty \tau \psi(\tau) d\tau},
\]

(13)

where we assume \( t_2 > t_1 \). This equation for the correlation function implies that, with a window of size \( \Delta = t_2 - t_1 \) we move along the entire (infinite) theoretical sequence of laminar regions and count how many window positions are compatible with the window being located within a theoretical laminar region, which must have a length larger than the window size. In addition we have to count the total number of window positions. In other words, the stationary correlation function of \( \xi(t) \) is nothing but the probability that the two times \( t_1 \) and \( t_2 \) are located within the same laminar region. If these two times are located in different laminar regions, the adoption of the coin tossing procedure for any contribution of a given sign to the correlation function would produce, with equal probability, a contribution with opposite sign, thereby providing a vanishing contribution. An attractive way to explain this procedure is through the concept of random events. First of all, the lengths of the laminar regions are determined by the random drawing of the numbers \( \tau \), with distribution \( \psi(\tau) \). At the border between one laminar region and the next we toss a coin to decide the sign of the next laminar region. This coin tossing is a random event and no random event can occur between two times located in the same laminar region. If the two times are located in different laminar regions, one or more random events must have occurred between them. Thus the correlation function \( \Phi_\xi([t_1 - t_2]) \) can also be interpreted as the probability that no random event occurs between times \( t_1 \) and \( t_2 \).

We evaluate the four-time correlation function, using the same arguments. Consider four times, ordered as \( t_3 < t_2 < t_3 < t_4 \). The corresponding correlation function exists, under the following conditions. The first condition is that all four times are located in the same laminar region. The second condition is compatible with the pairs \( (t_1, t_2) \) and \( (t_3, t_4) \) being located in distinct laminar regions. This means that the times \( t_1 \) and \( t_2 \) belong to a laminar region, denoted by \( T_{1,2} \), the times \( t_3 \) and \( t_4 \) belong to a laminar region denoted by \( T_{3,4} \), and \( T_{1,2} \neq T_{3,4} \). Using the random event concept, the second condition implies that no random event occurs between \( t_1 \) and \( t_2 \), or between \( t_3 \) and \( t_4 \), while at least one random event occurs between \( t_2 \) and \( t_3 \).

We use the notation \( p(ij) \) to denote the probability that \( t_i \) and \( t_j \) belong to the same laminar region. Thus the prescription for the correlation function given by Eq. (13) can be expressed as the probability function

\[
\Phi_\xi(t_2 - t_1) = p(12).
\]

(14)

We also use the notation

\[
p(ij) = 1 - p(ij)
\]

(15)

to denote the probability that at least one transition occurs between times \( t_i \) and \( t_j \). It is convenient to use the conditional probability concept, and the Bayesian notation (see, for instance, 24). We denote the joint probability of events \( A \) and \( B \) by \( p(A, B) \) and the conditional probability of occurrence of event \( A \) given event \( B \) with
\[ p(A|B) = \frac{p(A, B)}{p(B)}. \] (16)

We denote the conditional probability that event A occurs, given that event B does not, by \( p(A|\neg B) \). Using the prescription of Eq. (16), the latter conditional probability, \( p(A|\neg B) \), is expressed as follows

\[ p(A|\neg B) = \frac{p(A) - p(A, B)}{1 - p(B)}, \] (17)

where we have used the relation \( p(A) = p(A, B) + p(A, \neg B) \) for the numerator.

The probability that times \( t_i \) and \( t_j \) belong to the same laminar region \( T_{i, j} \) and that, simultaneously, times \( t_r \) and \( t_s \) belong to the same laminar region \( T_{r, s} \), regardless of whether \( T_{i, j} \) coincides with \( T_{r, s} \), or not, is a joint probability by the symbol \( p(ij, rs) \). Thus the four-time correlation function can be formally expressed as follows:

\[ \frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle}{\langle \xi^2 \rangle^2} = p(12, 34). \] (18)

On the other hand, using the notation introduced earlier, we have two contributions to the four-time correlation function. The first contributions is determined by all four times being in the same laminar region with no random event occurring between \( t_1 \) and \( t_4 \) (condition 1), whereas the second contribution corresponds to the probability that at least one random event occurs between \( t_2 \) and \( t_3 \), given the condition that no random event occurs between \( t_1 \) and \( t_2 \) and none between \( t_3 \) and \( t_4 \) (condition 2):

\[ p(12, 34) = p(14) + p(23)p(12) p(34). \] (19)

Eq. (19) corresponds to the superposition of independent contributions from condition 1 and condition 2. The contribution due to condition 1, \( p(14) \), according to the earlier definitions, is the probability that \( t_1 \) and \( t_4 \) belong to the same laminar region. The contribution due to condition 2 is given by the second term on the right hand side of Eq. (19). Again, according to the notation that we are using, see Eq. (19), \( p(23) \) is the probability that a random event occurs between \( t_2 \) and \( t_3 \), thereby disconnecting the two laminar regions. Consequently \( p(12) \) is the probability that \( t_1 \) and \( t_2 \) belong to the same laminar region given that at least one random event occurs between \( t_2 \) and \( t_3 \). Finally, \( p(34) \) is the probability that \( t_3 \) and \( t_4 \) belong to the same laminar region, given that at least one random event occurs between \( t_2 \) and \( t_3 \). Thus, the product of these three probabilities is the appropriate quantity corresponding to condition 2.

To transform the equality Eq. (19) into a relation involving correlation functions, we use Eq. (18), for the four-time correlation function. The two-time correlation functions emerge from the second term on the right hand side of Eq. (19) via the proper use of Eq. (14), Eq. (17) and Eq. (15). Thus, we obtain

\[ \frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle}{\langle \xi^2 \rangle^2} = \Phi_\xi(t_4 - t_1) \] (20)

Eq. (20) is a major result, being an exact expression for the four-time correlation function independently of the statistics of the dichotomous process. We stress that the general form of Eq. (20) is not factorable and is therefore distinct from DF.

Note that in the Poisson case, the waiting time distribution \( \psi(t) \) is exponential. Using the prescription given by Eq. (16) it is not difficult to show that the correlation function of \( \xi \) is also exponential. Then, after tedious but straightforward algebra, we establish that Eq. (20) reduces to

\[ \frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle}{\langle \xi^2 \rangle} = \langle \xi(t_1)\xi(t_2)\rangle \langle \xi(t_3)\xi(t_4) \rangle, \] (21)

which coincides with Eq. (4), that is, the process becomes distinct from DF. Therefore we replace the integrand in Eq. (22) with \( \Phi_\xi(t_4 - t_1) \), and using the inverse power-law form of the correlation function, we carry out the four time integrations and obtain \( \langle x^4(t) \rangle \propto t^{4-\mu} \). By extending...
this way of proceeding to the calculation of the $2n$-times correlation function, we derive the general result
\begin{equation}
\langle x^{2n}(t) \rangle \propto t^{2n-\mu+2},
\end{equation}
for $2 \leq \mu \leq 3$, and $\langle x^{2n}(t) \rangle \propto t^{2n-1}$ for $\mu > 3$, in agreement with the numerical results of Ref. 22.

The asymptotic result (23) establishes that the $2n$-moments do not have the scaling corresponding to the DF condition. If we assume that the condition of Eq. (4) applies, in keeping with the nature of the GDE, instead of (23) we would obtain $\langle x^{2n}(t) \rangle \propto t^{2n(4-\mu)/2}$, with one factor of $\mu$ occurring for each order of the moment. Consequently, the DF implies the existence of the scaling $x \propto t^\delta$, with the scaling index given by
\begin{equation}
\delta = \frac{4-\mu}{2} \quad \text{for} \quad 2 \leq \mu \leq 3,
\end{equation}
\begin{equation}
\delta = \frac{1}{2} \quad \text{for} \quad \mu > 3,
\end{equation}
where $\mu - 1$ is the Lévy index. This later result agrees with the scaling predicted by the GDE, as established in Ref. 20. Here the central fact to keep in mind is that Eq. (14) generates Lévy walks, rather than Lévy flights. A Lévy flight is a kind of random walk in which the step lengths have an inverse power-law distribution, so the second moment of the dynamical variable diverges. The Lévy walk, on the other hand, ties the length of a step to the time required to take the step, resulting in a finite second moment for the dynamical variable. Furthermore, it takes an infinite time for a Lévy walk to yield the same scaling as a corresponding Lévy flight, the latter scaling index being given by
\begin{equation}
\delta = \frac{1}{\mu - 1} \quad \text{for} \quad 2 \leq \mu \leq 3,
\end{equation}
\begin{equation}
\frac{1}{2} \quad \text{for} \quad \mu > 3.
\end{equation}

For this reason, the Lévy walk, introduced by Shlesinger et al. 22, can be considered to be a manifestation of the Living State of Matter (LSM) 26, in the sense described in some recent work 30, 51. The LSM is interpreted as the existence of a scaling condition intermediate between that of dynamics and thermodynamics and which can last forever.

### III. AGING

In this section we adopt the Bayesian formalism to evaluate the correlation functions in a non-stationary condition. This enables us to establish that the breakdown of the DF condition is closely related to aging.

Before proceeding with the formalism, we briefly review why non-Poisson statistics produces aging, as discussed in detail in Refs. 24, 32. Suppose that we create an infinite sequence of time intervals of length $\tau_i$, namely, the theoretical sequence discussed earlier. As mentioned, we create the observable sequence by filling the time intervals, called laminar regions, with either $W$ or $-W$, according to the coin tossing prescription, with the first laminar region beginning at time $t = t_0$. Let us imagine, to facilitate the discussion of this section, that the theoretical sequence is observable, even if in practice it is not. If we begin the observation process at the same time when the theoretical sequence is generated, the result of our observation yields the waiting time distribution of Eq. (11). If the observation of the theoretical sequence begins at a given time $t_1 > t_0$, the distribution of the waiting times before the first exit from the laminar region, denoted by $\psi_{1, t_0}(t)$, will not coincide with $\psi(t)$. This is a consequence of the first laminar region observed having begun at any time between $t_1$ and $t_0$. Thus, the resulting waiting time will be, in general, shorter than the real sojourn time generated by $\psi(t)$. In the Poisson case this shortening of the time does not have any effect on the shape of $\psi_{1, t_0}(t)$, which remains identical to $\psi(t)$. In the non-Poisson case, on the contrary, delaying the process of observation does influence the shape of $\psi_{1, t_0}(t)$ causing it to depart from the form of $\psi(t)$ 26, 32.

Let us now address the problem of building up the aging correlation function of $\xi(t)$. We study the correlation between $\xi(t_2)$ and $\xi(t_1)$, with the condition that $t_2 > t_1 > t_0$; $t_0$ being the time at which the laminar region begins. We solve this problem in two steps. In the first step we define the correlation function $A^{(t_0)}(t_2 - t_1)$, without requiring that the laminar region begins at $t = t_0$, but that it in fact begins at a time intermediate between $t_1$ and $t_0$. This corresponds to stating that $A^{(t_0)}(t_2 - t_1)$ is a correlation function of undefined age, younger, though, than the $(t_1 - t_0)$-old correlation function. In the second step we set the additional condition that the laminar regions begin at $t = t_0$, and we give the prescription to determine the correlation function $\Phi^{(t_0)}_\xi(t_2 - t_1)$, a notation denoting in fact the $(t_1 - t_0)$-old correlation function. The latter aging correlation function fits the earlier definition of $\psi_{1, t_0}(t)$. The corresponding analytical expression will make it possible to establish the effect of aging on the phenomenon, namely the effect of moving both $t_2$ and $t_1$ away from $t_0$ as well as the more traditional effect of increasing the distance between $t_1$ and $t_2$.

Note that the first correlation function is given by
\begin{equation}
A^{(t_0)}(t_2 - t_1) = p(A|\mathcal{F}).
\end{equation}

This identification of $A^{(t_0)}(t_2 - t_1)$ is consistent because we define $A$, by the condition that both $t_1$ and $t_2$ belong to the same laminar region, while $\mathcal{F}$ is defined by the condition that $t_0$ does not belong to the same laminar region as $t_1$. Of course, with this interpretation $B$ is defined by the condition that $t_0$ belongs to the same laminar region as $t_1$.

The conventional correlation function is the probability that $t_1$ and $t_2$ belong to the same laminar region, and thus is the probability that property $A$ occurs, so we can write the second equality
\begin{equation}
\Phi^{(t_0)}_\xi(t_2 - t_1) = p(A).
\end{equation}
The probability that $A$ and $B$ take place in the same laminar region allows us to write, in terms of the correlation function,

$$\Phi_\xi(t_2 - t_0) = p(A, B). \quad (28)$$

In fact, this is the probability that $t_2$ and $t_0$ belong to the same laminar region, and, thanks to the time ordering $t_2 > t_1 > t_0$, this is equal to the probability that both $A$ and $B$ occur. Finally, the probability that $t_1$ and $t_0$ belong to the same laminar region, namely, the probability that the property $B$ applies, enables us to write

$$\Phi_\xi(t_1 - t_0) = p(B). \quad (29)$$

At this stage, to express $A^{(t_0)}(t_2 - t_1)$ in terms of more familiar correlation functions we insert Eq. (17) into Eq. (26), which yields

$$A^{(t_0)}(t_2 - t_1) = \frac{\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_2 - t_0)}{1 - \Phi_\xi(t_1 - t_0)}. \quad (30)$$

It is easy to show that in the Poisson case Eq. (30) reduces to

$$A^{(t_0)}(t_2 - t_1) = \Phi_\xi(t_2 - t_1), \quad (31)$$

independently of $t_0$.

Now let us take the second step, and explicitly evaluate $\Phi_\xi^{(t_0)}(t_2 - t_1)$. This aging correlation function is the sum of two probabilities. The first contribution is the probability that no event occurs between $t_0$ and $t_2$, thereby ensuring that $t_1$ and $t_2$ belong to the same laminar region. The second contribution is the probability that an arbitrary number of events occurred between $t_0$ and $t_2$. Note that the laminar region beginning at $t = t_0$ implies that at this time a random event occurs, which is in fact, the beginning of the laminar region. As stated a number of time earlier, at the beginning of any laminar region, we toss a coin, to decide the sign of the laminar region. This is the random event that makes it possible for us to express $\Phi_\xi^{(t_0)}(t_2 - t_1)$ as follows

$$\Phi_\xi^{(t_0)}(t_2 - t_1) = \Psi(t_2 - t_0)$$

$$+(1 - \Psi(t_1 - t_0)) \frac{\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_2 - t_0)}{1 - \Phi_\xi(t_1 - t_0)}. \quad (32)$$

In Eq. (32) we have used the conventional notation of the CTRW formalism (33).

$$\Psi(t) \equiv \int_t^\infty dt' \psi(t'), \quad (33)$$

where $\psi(t)$ is the waiting time distribution of Eq. (10). Montroll and Weiss (35) make the implicit assumption that the laminar region begins at $t = 0$. Thus, $\Psi(t)$ is the probability that no event occurs up to time $t$, after the random event occurs at time $t = 0$. Here we replace the initiation time $t = 0$ with $t = t_0$. Thus, $\Psi(t_2 - t_0)$ is the probability that no random event occurs between $t_0$ and $t_2$, as required. The second term in Eq. (32) is the product of the probability that one or more events occurred between $t_1$ and $t_0$, given the fact that $t_2$ and $t_1$ are in the same laminar region and $t_0$ is not.

We note that Eq. (32) interrelates factorability and aging and consequently is the most relevant expression for our discussion. The importance of this result can be made transparent by going back to the discussion in Section 2. Eq. (20), the expression for the fourth-order correlation functions, can be reexpressed as, using Eq. (30),

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle}{\langle \xi^2 \rangle^2} = \Phi_\xi(t_4 - t_1)$$

$$+ (\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_3 - t_1)) A^{(t_2)}(t_4 - t_3). \quad (34)$$

As pointed earlier, in the Poisson case, see Eq. (31),

$$A^{(t_2)}(t_4 - t_3) = \Phi_\xi(t_4 - t_3), \quad (35)$$

independently of $t_2$. By inserting Eq. (34) into Eq. (31), and noting that $\Phi_\xi(t_4 - t_3) = \Phi_\xi(t_4 - t_3)\Phi_\xi(t_3 - t_1)$, we see immediately that the DF condition is recovered:

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle}{\langle \xi^2 \rangle^2} = \Phi_\xi(t_2 - t_1)\Phi_\xi(t_4 - t_3). \quad (36)$$

Thus, we have established that the breakdown of the DF condition and aging are interrelated. In fact, annihilating the aging property has the effect of reestablishing the DF property.

IV. CONCLUDING REMARK

The equivalence between the trajectory and density pictures of physical phenomena is one of the major tenants of modern physics. It therefore came as quite a surprise when Bologna et al. [26] discovered an inconsistency between these two pictures in the case of non-ordinary statistical mechanics. The form of the inconsistency had to do with the derivation of anomalous diffusion of the Lévy kind, using dichotomous noise and either CTRW or the generalized central limit theorem. Both of these approaches use trajectories and not the Liouville-like approach for densities, such as does GDE. It is a simple matter, using Eq. (8), to show that GDE yields a hierarchy of moments like approach for densities, such as does GDE. It is a simple matter, using Eq. (8), to show that GDE yields a hierarchy of moments.
this order is sufficient to identify the source of the inconsistency between the trajectory and density pictures as being due to the non-Poisson character of the statistics.

We have also shown that a departure from Poisson statistics has the effect of introducing a memory into the correlation functions that can last for an infinitely long time. For dichotomous noise the two-time correlation function, using either trajectories or densities is the same, however, higher-order correlations are not the same for non-Poisson statistics. The deviation from Poisson statistics is manifest in a dependence of correlations on the difference between the initiation time and the observation time, that is, on the age of the system. Age destroys the DF property and may represent a state of matter intermediate between the dynamic and thermodynamic condition, mentioned earlier, the Living State of Matter. This eternal state of nonequilibrium, in which a perturbed phenomena relaxes to, but never attains, equilibrium, should be contrasted with the Onsager Principle in which physical systems are assumed to be aged. An aged physical system is one that has reached equilibrium with a heat bath long before measurements are taken.

It is evident that to establish a density picture equivalent to the trajectory picture, in which the time averages and ensemble averages are the same, in the non-Poisson as well as in the Poisson case, we have to overcome the limitations of the Liouville-like approaches of Refs. [17, 18, 19]. This difficult issue calls for further research. Nevertheless, the merit of the present paper lies in the fact that it has revealed the violation of the DF property when the statistics of the underlying process are non-Poisson. DF is a factorization property assumed for dichotomous noise by researchers in multiple fields, often non-Poisson. DF is a factorization property when the statistics of the fluctuation ξ cannot be defined; not even in the non-stationary sense of Section 3. The discussion herein focuses on superdiffusion and addresses the problem of computing high-order correlations for renewal process with non-exponential waiting time distributions. The solution to this problem is given by Eq. (20), however this crucial property has not yet been obtained using Liouville-like methods [17, 18, 19].

In conclusion, by means of the conditional probability formalism, we have found the exact expression for the fourth-order correlation function, and we have shown that in the non-Poisson case, this expression violates the DF condition. We have also established a close connection between the DF breakdown and aging. In the case where μ > 2 the aged condition is possible. However, if μ < 3, the aging condition lasts forever. We see, in fact, from Eq. (13) that in this case the correlation function Φξ(t) is an inverse power law with index μ - 2. Thus, it takes an infinitely long time for the age-dependent correlation function Eq. (32) to become stationary. This is a remarkable result, which challenges the traditional treatments of such stochastic dynamical processes based on the generalized master equation (GME). The analysis of the GME based on these results will be taken up elsewhere.

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