Looking for Kähler- Einstein Structure on Cartan Spaces with Berwald connection

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Abstract

A Cartan manifold is a smooth manifold $M$ whose slit cotangent bundle $T^*M_0$ is endowed with a regular Hamiltonian $K$ which is positively homogeneous of degree 2 in momenta. The Hamiltonian $K$ defines a (pseudo)-Riemannian metric $g_{ij}$ in the vertical bundle over $T^*M_0$ and using it a Sasaki type metric on $T^*M_0$ is constructed. A natural almost complex structure is also defined by $K$ on $T^*M_0$ in such a way that pairing it with the Sasaki type metric an almost Kähler structure is obtained. In this paper we deform $g_{ij}$ to a pseudo-Riemannian metric $G_{ij}$ and we define a corresponding almost complex Kähler structure. We determine the Levi-Civita connection of $G$ and compute all the components of its curvature. Then we prove that if the structure $(T^*M_0, G, J)$ is Kähler- Einstein, then the Cartan structure given by $K$ reduce to a Riemannian one.

Keywords: Cartan space, Kähler structure, symmetric space, Einstein manifold, Laplace operator, Divergence, Gradient.

1 Introduction

É. Cartan has originally introduced a Cartan space, which is considered as dual of Finsler space [8]. Then H. Rund [24], F. Brickell [7] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([19], [20]). He proved that Cartan space is a particular case of Hamilton space. Indeed the geometry of regular Hamiltonians as smooth functions on the cotangent bundle is due to R. Miron and it is now systematically described in the monograph [15].

Let us denote the Hamiltonian structure on a manifold $M$ by $(M, H(x, p))$. If the fundamental function $H(x, p)$ is 2-homogeneous on the fibres of the cotangent bundle $(T^*M, M)$, then the notion of Cartan space is obtained. The modern formulation of the notion of Cartan spaces is due of the R. Miron [16], [17], [18]. Based on the studies of E. Cartan, A. Kawaguchi [11], R. Miron [19], [17], [18], S. Vacaru [30, 31, 32], D. Hrimiuc and H. Shimada [9], [10], P.L. Antonelli and M. Anastasiei [2], [3], [13], [14], etc., the geometry of Cartan spaces is today an important chapter of differential geometry.

Under Legendre transformation, the Cartan spaces appear as dual of the Finsler spaces [19]. It is remarkable that regular Lagrangian which is 2-homogeneous in velocities is nothing but the square of a fundamental Finsler function and its geometry is Finsler geometry. This geometry was developed since 1918 by P. Finsler, E. Cartan, L. Berwald, H. Akbar-Zadeh and many others, see [1, 2, 12, 25, 26]. Using this duality several important results in the Cartan spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection, the notion of $(\alpha, \beta)$-metrics, etc [21]. Therefore, the theory of Cartan spaces

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has the same symmetry and beauty like Finsler geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields.

Let \((M, K)\) be a Cartan space on a manifold \(M\) and put \(\tau := \frac{1}{2}\epsilon^2\). Let us define the symmetric \(M\)-tensor field \(G_{ij} := \frac{1}{4}g_{ij} + \frac{\epsilon(v)}{\epsilon^3}p_i p_j\) on slit cotangent bundle \(T^*M_0 := T^*M - \{0\}\), where \(v = v(\tau)\) is a real valued smooth function defined on \([0, \infty) \subset \mathbb{R}\) and \(\alpha\) and \(\beta\) are real constants. Using this, we can define a Riemannian metric and almost complex structure on \(T^*M_0\) as follows

\[
G = G_{ij}dx^i dx^j + G^{ij}\delta p_i \delta p_j, \\
J(\delta_i) = G_{ik}\delta^k, \quad J(\delta^i) = -G^{ik}\delta_k,
\]

where \(G^{ij}\) is the inverse of \(G_{ij}\).

In this paper, we prove that \((T^*M_0, G, J)\) is an almost Kählerian manifold. Then we show that the almost complex structure \(J\) on \(T^*M_0\) is integrable if and only if \(M\) has constant scalar curvature \(c\) and the function \(v\) is given by \(v = -c\alpha\beta\). We conclude that on a Cartan manifold \(M\) of negative constant flag curvature, \((T^*M_0, G, J)\) has a Kählerian structure. For Cartan manifolds of positive constant flag curvature, we show that the tube around the zero section has a Kählerian structure (see Theorem \([5, 6]\)).

Then we find the Levi-Civita connection \(\nabla\) of the metric \(G\). For the connection \(\nabla\), we compute all of components curvature. For a Cartan space \((M, K)\) of constant curvature \(c\), we prove that in the following cases \((M, K)\) reduce to a Riemannian space: (i) for \(c < 0\), \((T^*M_0, G, J)\) became a Kähler Einstein manifold, (ii) for \(c > 0\), \((T^*M_0, G, J)\) became a Kähler Einstein manifold, where \(T^*M_0\) the tube around the zero section in \(T^*M\), defined by the condition \(2\tau < \frac{1}{\epsilon^2}\). It result that, there is not any non-Riemannian Cartan structure such that \((T^*M_0, G, J)\) became an Einstein manifold.

Finally we define divergence, gradient and Laplace operators on the Cartan manifold \((M, K)\) with Berwald connection. Let \((M, K)\) be a Cartan space with Berwald connection, \(S = \epsilon p^i \delta_i\) is the geodesic spray of \((M, K)\), \(X = X^i \delta_i + \dot{X}^i \delta_i\) and \(g := \det(g_{ij})\). We show that \(\text{div}(X) = 0\) if and only if the mean Landsberg curvature of \(K\) satisfies \(J_i = g_i(\ln g_{ij}).\)

We define the gradient operator by \(G(\nabla \delta f, X) = Xf, \quad \forall X \in \chi(T^*M)\), \(f \in C^\infty(TM)\) and prove that the gradient operator is determined by \(\nabla \delta f = G^{ik}(\nabla_{\delta_i} f)\delta_k + G_{ik}(\nabla_{\delta_k} f)\delta_i\).

The Laplace operator of a scalar field \(f \in C^\infty(TM)\), is defined by \(\Delta f = \text{div} (\nabla \delta f)\). With vanishing the Laplace operator, we prove that \(\text{div}(S) = 0\) if and only if \(K\) is a mean Landsberg metric.

2 Preliminaries

Let \(M\) be an \(n\)-dimensional \(C^\infty\) manifold and \(\pi^* : T^*M \longrightarrow M\) its cotangent bundle. If \((x^i)\) are local coordinates on \(M\), then \((x^i, p_i)\) will be taken as local coordinates on \(T^*M\) with the momenta \((p_i)\) provided by \(p_i = p_i dx^i\) where \(p_i \in T^*_xM, x = (x^i)\) and \((dx^i)\) is the natural basis of \(T^*_xM\). The indices \(i, j, k, \ldots\) will run from 1 to \(n\) and the Einstein convention on summation will be used.

Put \(\partial_i := \frac{\partial}{\partial x^i}\) and \(\delta^i := \frac{\partial}{\partial p^i}\). Let \((\partial_i, \delta^i)\) be the natural basis in \(T_{(x^i, p_i)}T^*M\) and \((dx^i, dp_i)\) be the dual basis of it. The kernel \(V_{(x,p)}\) of the differential \(\pi^* : T_{(x^i, p_i)}T^*M \longrightarrow T^*_xM\) is called the vertical subspace of \(T_{(x^i, p_i)}T^*M\) and the mapping \((x, p) \rightarrow V_{(x,p)}\) is a regular distribution on \(T^*M\) called the vertical distribution. This is integrable with the leaves \(T^*_xM, x \in M\) and is locally spanned by \(\delta\). The vector field \(C^* = p_i \delta^i\) is called the Liouville vector field and \(\omega = p_i dx^i\) is called the Liouville 1-form on \(T^*M\). Then \(dx^i\) is the canonical symplectic structure on \(T^*M\). For an easier handling of the geometrical objects on \(T^*M\), it is usual to consider a supplementary distribution to the vertical distribution, \((x, p) \rightarrow N_{(x,p)}\), called the
The pieces produced by the decomposition (1) are called \( d \)-geometrical objects (\( d \) is for distinguished) since their local components behave like geometrical objects on \( M \), although they depend on \( x = (x^i) \) and momenta \( p = (p_i) \).

The horizontal distribution is taken as being locally spanned by the local vector fields

\[ \delta_i := \partial_i + N_{ij}(x,p)\dot{\partial}^j. \]  

(2)

The horizontal distribution is called also a nonlinear connection on \( T^*M \) and the functions \((N_{ij})\) are calledthe local coefficients of this nonlinear connection. It is important to note that any regular Hamiltonian on \( T^*M \) determines a nonlinear connection whose local coefficients verify \( N_{ij} = N_{ji} \). The basis \((\delta_i, \dot{\partial}^i)\) is adapted to the decomposition (1). The dual of it is \((dx^i, \delta P_i)\), for \( \delta p_i = dp_i - N_{ij}dx^j \).

A Cartan structure on \( M \) is a function \( K : T^*M \to [0, \infty) \) which has the following properties: (i) \( K \) is \( C^\infty \) on \( T^*M \setminus M \); (ii) \( K(x, \lambda p) = \lambda K(x, p) \) for all \( \lambda > 0 \) and (iii) the \( n \times n \) matrix \((g^{ij}(x,p))\), where \( g^{ij}(x,p) = \frac{1}{2}\dot{\partial}^i \dot{\partial}^j K^2(x,p) \), is positive definite at all points of \( T^*M_0 \). We notice that in fact \( K(x,p) > 0 \), whenever \( p \neq 0 \). The pair \((M,K)\) is called a Cartan space. Using this notations, let us define

\[ p^i = \frac{1}{2} \dot{\partial}^i K^2 \quad \text{and} \quad C^{ijk} = -\frac{1}{4} \delta^i \dot{\partial}^j \dot{\partial}^k K^2. \]

The properties of \( K \) imply that

\[ p^i = g^{ij} p_j, \quad p_i = g_{ij} p^j, \]

(3)

\[ g^{ij} p_i p_j = p^2 = K^2, \]

\[ C^{ijk} p_k = C^{ikj} p_k = C^{kij} p_k = 0. \]  

(4)

One considers the formal Christoffel symbols

\[ \gamma^j_{ik}(x,p) := \frac{1}{2} g^{js}(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{jk}), \]

(5)

and the contractions \( \gamma^i_{jk}(x,p) := \gamma^j_{ik}(x,p)p_i, \gamma^i_{ji}(x,p) := \gamma^i_{jk}(x,p)p^k \). Then the functions

\[ N_{ij}(x,p) = \gamma^i_{jk}(x,p) - \frac{1}{2} \gamma^i_{jk}(x,p) \dot{\partial}^k g_{ij}(x,p), \]

(6)

define a nonlinear connection on \( T^*M \). This nonlinear connection was discovered by R. Miron [16]. Thus a decomposition (1) holds. From now on, we shall use only the nonlinear connection given by (5).

A linear connection \( D \) on \( T^*M \) is said to be an \( N \)-linear connection if \( D \) preserves by parallelism the distribution \( N \) and \( V \), also we have \( D\theta = 0 \), for \( \theta = \delta p_i \wedge dx^i \). One proves that an \( N \)-linear connection can be represented in the adapted basis \((\delta_i, \dot{\partial}^i)\) in the form

\[ D_{\delta} \delta_i = B^i_{ij} \delta_j, \quad D_{\delta} \dot{\partial}^i = -B^i_{ij} \dot{\partial}^j, \]

(7)

\[ D_{\dot{\partial}^j} \delta_i = V^{kj}_{i} \delta_k, \quad D_{\dot{\partial}^j} \dot{\partial}^i = -V^{kj}_{i} \dot{\partial}^k, \]  

(8)

where \( V^{kj}_{i} \) is a \( d \)-tensor field and \( B^i_{ij}(x,p) \) behave like the coefficients of a linear connection on \( M \). The functions \( B^i_{ij} \) and \( V^{kj}_{i} \) define operators of \( h \)-covariant and \( v \)-covariant derivatives
in the algebra of \( d \)-tensor fields, denoted by \(|_k\) and \(|^k\), respectively. For \( g^{ij} \), these are given by following equation

\[
g^{ij}_k = \delta_k g^{ij} + g^{s j} B^i_{sk} + g^{is} B^j_{sk}, \tag{9}
g^{ij} |^k = \delta^k g^{ij} + g^{sj} V^i_{sk} + g^{is} V^j_{sk}. \tag{10}
\]

An \( N \)-linear connection given in the adapted basis \((\delta, \dot{\lambda})\) as \( D\Gamma(N) = (B^i_{jk}, V^i_{jk})\) is called Berwald connection if

\[
g^{ij}_k = -2 L^{ij}_k, \quad g^{ij} |^k = -2 C^{ij}_k, \tag{11}
\]

where \( L^{ij}_k = C_{k}^{ij} |^h \) are components of the Landsberg tensor on \( M \) (see [5][6][28][27]).

The Berwald connection \( B\Gamma(N) = (\dot{\lambda} N^i_{jk}, 0) \) of the Cartan spaces has the torsions \( d \)-tensors as follows

\[
T^i_{jk} = 0, \quad S^i_{jk} = 0, \quad P^i_{jk} = 0, \quad R^i_{jk} = 0, \tag{12}
\]

\[
R^i_{ijk} = \delta_k N^i_{jk} - \delta_j N^i_{ik}. \tag{13}
\]

The \( d \)-tensors of curvature of \( B\Gamma(N) \) are given by

\[
R^i_{ijkj} = \delta_h B^i_{jk} - \delta^k B^i_{jh} + B^s_{jk} B^i_{sh} - B^s_{jh} B^i_{sk}, \tag{14}
\]

\[
P^i_{jh} = \dot{\lambda}^h B^i_{jk}, \tag{15}
\]

\[
S^i_{jk} = 0. \tag{16}
\]

where \( B^i_{jk} = \dot{\lambda} N^i_{jk} \) are the coefficients of the \( B\Gamma(N) \)-connection. It has also the following properties

\[
K^2 = \delta_i K^2 = 0, \quad K^2 |^j = 2p^i, \tag{17}
\]

\[
p_{ij} = p^i_j = 0, \quad p^i |^j = \delta^i_j, \quad p |^j = \delta^j_i, \quad R_{ki} p^k, \tag{18}
\]

\[
\delta_i g_{jk} = B^s_{jk} g_{sk} + B^s_{ki} g_{sj}. \tag{19}
\]

### 3 Kähler Structures on Cotangent Bundle

Suppose that

\[
\tau := \frac{1}{2} K^2 = \frac{1}{2} g^{ij}(x,p)p_i p_j. \tag{20}
\]

We consider a real valued smooth function \( v \) defined on \([0,\infty) \subset \mathbb{R}\) and real constants \( \alpha \) and \( \beta \). We define the following symmetric \( M \)-tensor field of type (0,2) on \( T^* M_0 \) having the components

\[
G_{ij} := \frac{1}{\beta} g_{ij} + \frac{v(\tau)}{\alpha \beta} p_i p_j. \tag{21}
\]

It follows easily that the matrix \((G_{ij})\) is positive definite if and only if \( \alpha, \beta > 0, \quad \alpha + 2\tau v > 0 \). The inverse of this matrix has the entries

\[
G^{kl} = \beta g^{kl} - \frac{v\beta}{\alpha + 2\tau v} p^k p^l. \tag{22}
\]

The components \( G^{kl} \) define symmetric \( M \)-tensor field of type (0,2) on \( T^* M_0 \). It is easy to see that if the matrix \((G_{ij})\) is positive definite, then matrix \((G^{kl})\) is positive definite too.

Using \((G_{ij})\) and \((G^{ij})\), the following Riemannian metric on \( T^* M_0 \) is defined

\[
G = G_{ij} d\sigma^i d\sigma^j + G^{ij} dp_i dp_j. \tag{23}
\]
Now, we define an almost complex structure $J$ on $T^*M_0$ by
\[ J(\delta_i) = G_{ik} \dot{\partial}^k, \quad J(\dot{\partial}^i) = -G^{ik} \delta_k. \] (24)
It is easy to check that $J^2 = -I$.

**Theorem 3.1.** $(T^*M_0, G, J)$ is an almost Kählerian manifold.

**Proof.** Since the matrix $(G^{kl})$ is the inverse of the matrix $(G_{ij})$, then we have
\[ G(J\delta_i, J\delta_j) = G_{ik}G_{jr}G(\dot{\partial}^k, \dot{\partial}^r) = G_{ik}G_{jr}G^{kr} = G_{ij} = G(\delta_i, \delta_j). \]

The relations
\[ G(J\dot{\partial}^i, J\dot{\partial}^j) = G(\dot{\partial}^i, \dot{\partial}^j), \quad G(J\delta_i, J\dot{\partial}^j) = G(\delta_i, \dot{\partial}^j) = 0, \]
may be obtained in a similar way, thus
\[ G(JX, JY) = G(X, Y), \quad \forall X, Y \in \Gamma(T^*M_0). \]
It means that $G$ is almost Hermitian with respect to $J$. The fundamental 2-form associated by this almost Kähler structure is $\theta$, defined by
\[ \theta(X, Y) := G(X, JY), \quad \forall X, Y \in \Gamma(T^*M_0). \]
Then we get
\[ \theta(\dot{\partial}^i, \delta_j) = G(\dot{\partial}^i, J\delta_j) = G(\dot{\partial}^i, G_{jk}\dot{\partial}^k) = G^{ik}G_{jk} = \delta^i_j, \]
and
\[ \theta(\delta_i, \dot{\partial}^j) = \theta(\dot{\partial}^i, \dot{\partial}^j) = 0. \]
Hence, we have
\[ \theta = \delta p_i \wedge dx^i, \] (25)
that is the canonical symplectic form of $T^*M$.

Here, we study the integrability of the almost complex structure defined by $J$ on $TM$. To do this, we need the following lemma.

**Lemma 3.2.** ([15][22][23]) Let $(M, F)$ be a Finsler manifold. Then we have:
1. $[\delta_i, \delta_j] = R_{kij} \dot{\partial}^k$,
2. $[\delta_i, \dot{\partial}^j] = -(\dot{\partial}^j N_{ik}) \dot{\partial}^k$,
3. $[\dot{\partial}^i, \dot{\partial}^j] = 0$.

**Lemma 3.3.** Let $(M, K)$ be a Cartan space. Then $J$ is complex structure on $T^*M_0$ if and only if $A_{kij} = 0$ and
\[ R_{kij} = \frac{v}{\alpha_{j2}}(g_{ik}p_j - g_{jk}p_i), \] (26)
where $A_{kij} = \delta_i G_{jk} - \delta_j G_{ik} + G_{ir} \dot{\partial}^r N_{jk} - G_{jr} \dot{\partial}^r N_{ik}$.

**Proof.** Using the definition of the Nijenhuis tensor field $N_J$ of $J$, that is,
\[ N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \forall X, Y \in \Gamma(T^*M) \]
we get
\[ N_J(\delta_i, \delta_j) = A_{kij} G^{hk} \delta_k + (M_{kij} - R_{kij}) \dot{\partial}^k, \] (27)
where $M_{kij} = G_{ir} \tilde{\partial}^r G_{jk} - G_{jr} \tilde{\partial}^r G_{ik}$. Let $C^r_{jk} := g_{jk} g_{sk} C^{rs}$. Then we have

$$\tilde{\partial}^r g_{jk} = -g_{jk} g_{sk} \tilde{\partial}^s g^{ls} = 2 g_{jk} g_{sk} C^{rs} = 2 C^r_{jk}.$$  

By above equation, we obtain

$$G_{ir} \tilde{\partial}^r G_{jk} = \frac{2}{\beta^2} C_{ijk} + \frac{v}{\alpha \beta^2} (g_{jq} p_k + g_{ik} p_j) + \frac{v'}{\alpha \beta^2} + \frac{2 v^2}{\alpha^2 \beta^2} p_j p_k.$$  

where $C_{ijk} = g_{ir} C^r_{jk}$. From (28) we get

$$M_{kij} = \frac{v}{\alpha \beta^2} (g_{ik} p_j - g_{jk} p_i).$$  

By a straightforward computation, it follows that $N_j(\tilde{\partial}^i, \tilde{\partial}^j) = 0$, whenever $N_j(\delta_i, \delta_j) = 0$. Therefore, from relations (24) and (26), we conclude that the necessary and sufficient conditions for the Nijenhuis tensor field $N_j$ to vanish, so that $J$ is a complex structure, are that $A_{kij} = 0$ and (26) hold.

In equation (30), we put $-\frac{v}{\alpha \beta^2} = c$, where $c$ is constant. Then we get

$$R_{kij} = c (g_{ik} p_j - g_{jk} p_i).$$  

**Theorem 3.4.** Let $(M, K)$ be a Cartan space of dimension $n \geq 3$. Then the almost complex structure $J$ on $T^* M$ is integrable if and only if (29) is hold and the function $v$ is given by

$$v = -c \alpha \beta^2.$$  

**Proof.** From equation $p_{ijk} = 0$ of relation (15), we conclude that $\delta_i p_k = N_{ik}$. Hence we obtain

$$A_{kij} = \delta_j g_{jk} - \delta_j g_{ik} + g_{jr} \tilde{\partial}^r N_{jk} - g_{jr} \tilde{\partial}^r N_{ik}$$

$$= \delta_j g_{jk} - \delta_j g_{ik} + g_{jr} B^r_{jk} - g_{jr} B^r_{ik}$$

$$= g_{jk} - g_{ik}$$

$$= 2 L_{ikj} - 2 L_{jki} = 0.$$  

Now we suppose that $v = -c \alpha \beta^2$. Thus from equation $A_{kij} = 0$ and Lemma [33] we conclude that $J$ is integrable if and only if (30) is hold.

A Cartan space $K^n$ is of constant scalar curvature $c$ if

$$H_{hijk} p^i p^j X^h X^k = c (g_{ij} g_{ik} - g_{jk} g_{ij}) p^i p^j X^h X^k,$$  

for every $(x, p) \in T^*_0 M$ and $X = (X^i) \in T_x M$. Here $H_{hijk}$ is the (hh)h-curvature of the linear Cartan connection of $K^n$. We replace $H_{hijk}$ in (33) with $g_{is} H_{hjsk}$ and so it reduce to following

$$p_s H_{hjk} p^i X^h X^k = c (p_h p_k - K^2 g_{hk}) X^h X^k.$$  

By part (ii) of Proposition 5.1 in chapter 7 of [15], $p_s H_{hjk} = -R_{hjk}$, hence we get

$$R_{hjk} p^i X^h X^k = c (K^2 g_{hk} - p_h p_k) X^h X^k,$$  

or equivalently

$$R_{hjk} p^i = c (K^2 g_{hk} - p_h p_k),$$  

because $(X^h)$ and $X^k$ are arbitrary vector fields on $M$. It is easy to check that (35) follows from (30). Similarly can be shown that if Cartan space $K^n$ has the constant scalar curvature $c$, then the equation (30) is hold (see [12]).
Lemma 4.1. The Levi-Civita connection of the Levi-Civita connection field of $\nabla$ is characterized by the conditions $2\tau = K^2 < \frac{1}{c^2}$, is a Kähler manifold.

Proof. The function $v$ must satisfies in the following condition
\[ \alpha + 2\tau v = \alpha(1 + 2(-c)\beta^2\tau) > 0, \quad \alpha, \beta > 0. \] (36)
By using the above relation and Theorem 3.4, we complete the proof. \[ \square \]

By attention to the Theorem 3.5, the components of the Kähler metric $G$ on $T^*M_0$ are
\[ \begin{cases} G_{ij} = \frac{1}{\beta} g_{ij} - c \beta p_ip_j, \\ G^{ij} = \beta g_{ij} + \frac{1}{1 - 2c^2\tau} p_ip_j. \end{cases} \] (37)

4 A Kähler Einstein Structure on Cotangent Bundle

In this section, we study the property of $(T^*M_0, G)$ to be Einstein. We find the expression of the Levi-Civita connection $\nabla$ of the metric $G$ on $T^*M_0$, then we get the curvature tensor field of $\nabla$. Then, by computing the corresponding traces, we find the components of Ricci tensor field of $\nabla$.

4.1 The Levi-Civita Connection

Lemma 4.1. The Levi-Civita connection of the Kähler metric $G$ are given by following
\[ \begin{align*} 
\nabla_{\partial_i} \hat{\partial}^j &= (\beta^2 L^j|^i) \partial_s + (-C^j_i + c \beta G^{ij} p_j) \hat{\partial}^s, \\
\nabla_{\hat{\partial}_j} \hat{\partial}^i &= (C^i_j - c \beta G^{ij} p_j) \partial_s - (L^j_i + B^j_i) \hat{\partial}^s, \\
\nabla_i \partial_j &= (C^i_j - c \beta G^{ij} p_j) \partial_s - L^j_i \hat{\partial}^s, \\
\nabla_i \hat{\partial}_j &= (L^j_i + B^j_i) \partial_s + (-\frac{1}{\beta^2} C_{ij}s + c \beta G_{js}p_j) \hat{\partial}^s. 
\end{align*} \] (38) (39) (40) (41)

Proof. Recall that for Cartan space with Berwald connection, the relation $B^j_i = \hat{\partial}^i N_{ik}$ is hold, and so we have $[\partial_i, \hat{\partial}^j] = B^j_i \hat{\partial}^i$. Also the Levi-Civita connection $\nabla$ of the Riemannian manifold $(T^*M_0, G)$ is obtained from the formula
\[ 2G(\nabla_X Y, Z) = X(G(Y, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X), \quad \forall X, Y, Z \in \Gamma(T^*M_0), \] (42)
and is characterized by the conditions $\nabla G = 0$ and $T = 0$, where $T$ is the torsion tensor of $\nabla$. Let $\nabla_i \partial^j = \Gamma^j_{ih} \partial_h + \Gamma^j_{ki} \hat{\partial}^k$. Then we get
\[ \begin{align*} 
\Gamma^j_{is} &= \frac{1}{2}[-\delta_k(\beta g^{ij} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^k p^j) - B^j_{km}(\beta g^{mj} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^m p^j)] \\
&= B^j_{km}(\beta g^{mj} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^m p^j)(\beta g^{ks} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^k p^s) - \beta^2 g^{ks} L^j_i. 
\end{align*} \] (43)

Similarly we obtain
\[ \begin{align*} 
\Gamma^j_{si} &= \frac{1}{2} [\hat{\partial}^i(\beta g^{ik} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^i p^k) + \hat{\partial}^i(\beta g^{ik} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^i p^k)] \\
&= \hat{\partial}^i(\beta g^{ik} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^i p^k)\left(\frac{1}{\beta^2} g_{ks} - c \beta p_k p_s\right) - C_{sj} + c \beta G^{sj} p_s. 
\end{align*} \] (44)
Using two above equation, we have (38). By a similar way, we obtain (39), (40) and (41).

We say that the vertical distribution $VT^*M_0$ is totally geodesic (resp. minimal) in $TT^*M_0$ if $H\nabla_{\dot{\beta}^j} \dot{\beta}^j = 0$ (resp. $g_{ij}H\nabla_{\dot{\beta}^j} \dot{\beta}^j = 0$), where $H$ denotes the horizontal projection. Similarly, if we denote by $V$ the vertical projection, then we say that the horizontal distribution $HT^*M_0$ is totally geodesic (resp. minimal) in $TT^*M_0$ if $V\nabla_{\delta_j} \delta_j = 0$ (resp. $g^{ij}V\nabla_{\delta_j} \delta_j = 0$). By using (38), we obtain

$$H\nabla_{\dot{\beta}^j} \dot{\beta}^j = \beta^2 L^{ij}\delta_s,$$

and

$$g_{ij}H\nabla_{\dot{\beta}^j} \dot{\beta}^j = \beta^2 g_{ij}L^{ij}\delta_s = \beta^2 J^s\delta_s,$$

where $J^s$ is the mean Landsberg tensor. Hence, we have the following.

**Corollary 4.2.** Let $(M, K)$ be a Cartan space with Berwald connection. Then we have

(i) $K$ is Landsberg metric if and only if the vertical distribution $VT^*M_0$ is totally geodesic in $TT^*M_0$;

(ii) $K$ is weakly Landsberg metric if and only if the vertical distribution $VT^*M_0$ is minimal in $TT^*M_0$.

**Corollary 4.3.** The horizontal distribution $HT^*M_0$ can not be totally geodesic or minimal in $TT^*M_0$.

**Proof.** By (41), we have

$$V\nabla_{\delta_j} \delta_j = (-\frac{1}{\beta^2} C_{ij} + c\beta G_{ij} p^i) \dot{\beta}^j.$$

If $HT^*M$ is totally geodesic, then we have $c\beta G_{ij} p^i = 0$. Therefore, we obtain

$$cp_ip^i(1 - 2c\beta^2) = 0,$$

which can not be true. \hfill \Box

### 4.2 The Curvature Tensors

**Theorem 4.4.** The coefficients of the curvature tensor of Kähler metric $G$ as follows

$$K(\dot{\beta}^i, \dot{\beta}^j) \dot{\beta}^k = \left[\beta^2 (L^{ij}|^ih - L^{jk}|^ih)\right] \delta_h + \left[C^{|ih}_\beta \delta_h - C^{|ih}_\delta_h \delta_h + c\beta G_{ij} \delta_h + C^{|is}_\delta_h\right] \dot{\beta}^k,$$

$$K(\delta_i, \delta_j) \delta_k = \left[C^{|is}_\delta_h \delta_h + \beta^2 (L^{ij}|^ih - L^{jk}|^ih)\right] \delta_h + \left[C_{ij}s^k \delta_h + C_{ij}L^{jk}_h + C_{ij}L^{js}_h + C_{ij}L^{ik}_h\right] \dot{\beta}^k,$$

$$K(\delta_i, \delta_j) \delta_k = \left[\beta^2 (L^{ij}|^ih - L^{jk}|^ih)\right] \delta_h + \left[C^{|ij}_\delta_h \delta_h - C^{|ih}_\delta_h \delta_h + 2C^{|ij}_\delta_h \delta_h + 2C^{|ij}_\delta_h \delta_h\right] \dot{\beta}^k,$$

$$K(\delta_i, \delta_j) \delta_k = \left[C^{|ij}_\delta_h \delta_h - 2C^{|ij}_\delta_h \delta_h + C^{|ij}_\delta_h \delta_h + C^{|ij}_\delta_h \delta_h\right] \dot{\beta}^k,$$

$$K(\delta_i, \delta_j) \delta_k = \left[C_{ij}s^k \delta_h + C_{ij}L^{js}_h + C_{ij}L^{ik}_h\right] \dot{\beta}^k.$$
\[ K(\delta_i, \delta_j) \dot{\delta}^k = \left[ C_{j}^{kh} - C_{i}^{kh} \right]_j + c^2 (p_j L_j^{kh} - p_i L_i^{kh}) + C_{j}^{ks} L_s - C_{i}^{ks} L_s \]

\[ + C_{j}^{sh} L_s - C_{i}^{sh} L_s \right] \dot{\delta}_h + \left[ - R_{hji}^k + \frac{1}{\beta^2} (C_{i}^{sh} C_{jhs} - C_{j}^{sh} C_{iths}) + c^2 \beta^2 p_h (p_j \delta^k_i - p_i \delta^k_j) + L^k_{hi} - L^k_{hji} + L^k_{sh} L_s - L^k_{sh} L_s \right] \dot{\delta}^h, \]

\[ (50) \]

\[ K(\dot{\delta}^i, \dot{\delta}^j) \delta_k = \left[ C_{j}^{ih} - C_{i}^{ih} + C_{i}^{js} C_{j}^{sh} - C_{i}^{js} C_{j}^{sh} + c^2 (C_{i}^{sh} \delta^k_j - C_{i}^{sh} \delta^k_i) \right] + \beta^2 (L_{js} L_{si} - L_{s} L_{s} L_{s}) \delta^j \delta^j \]

\[ + \left[ C_{j}^{sh} L_s - C_{i}^{sh} L_s \right] \delta_h + \left[ L^i_{sh} \delta^j - L^j_{sh} \delta^j \right] \dot{\delta}^h, \]

\[ (51) \]

\[ K(\delta_i, \dot{\delta}^j) \delta_k = \left[ C_{j}^{ih} - C_{i}^{ih} + C_{i}^{js} C_{j}^{sh} - C_{i}^{js} C_{j}^{sh} + c^2 (C_{i}^{sh} \delta^k_i - C_{i}^{sh} \delta^k_j) \right] \delta_h + \left[ \frac{1}{\beta^2} (C_{i}^{sh} - C_{i}^{sh} C_{j}^{sh} - C_{j}^{sh} C_{j}^{sh}) + C_{i}^{sh} C_{j}^{sh} + C_{i}^{sh} C_{j}^{sh} \right] \]

\[ - c^2 \beta G_{hh} \delta^i \delta^j + L^i_{sk} L_s + L^j_{sh} L_s - L^i_{sh} L_s \right] \dot{\delta}^i \]

\[ (52) \]

**Proof.** Recall that the curvature \( K(\nabla) \) is obtained from the following

\[ K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad \forall X, Y, Z \in \Gamma(TM). \]

Using (53) we have

\[ K(\dot{\delta}^i, \dot{\delta}^j) \dot{\delta}^k = \nabla_{\dot{\delta}^i} \nabla_{\dot{\delta}^j} \dot{\delta}^k + \nabla_{\dot{\delta}^j} \nabla_{\dot{\delta}^i} \dot{\delta}^k. \]

\[ (54) \]

By (38), it follows that

\[ \nabla_{\dot{\delta}^i} \nabla_{\dot{\delta}^j} \dot{\delta}^k = \dot{\delta}^i \left( \beta^2 g^{ik} L^k_j \right) \delta s + (\beta^2 g^{ik} L^k_j) \nabla_{\dot{\delta}^i} \delta s \]

\[ + \dot{\delta}^j \left( -C^i_{ks} + c \beta G^j k p_s \right) \delta s + \left( -C^i_{ks} + c \beta G^j k p_s \right) \nabla_{\dot{\delta}^i} \delta s. \]

\[ (55) \]

Since \( p_h L^h_{i} = 0, p^j L^j_{i} = p^j L^j_{i} = 0 \) and \( \dot{\delta}^i g^{ij} = -2C^{kij} \), then by (55) the following relation yields

\[ \nabla_{\dot{\delta}^i} \nabla_{\dot{\delta}^j} \dot{\delta}^k = \left[ \beta^2 L^j_{k} - \beta^2 L^j_{k} \right] \delta s + \left( -C^i_{ks} + c \beta G^j k p_s \right) \delta s \]

\[ + \dot{\delta}^j \left( -C^i_{ks} + c \beta G^j k p_s \right) \delta s + \left( -C^i_{ks} + c \beta G^j k p_s \right) \nabla_{\dot{\delta}^i} \delta s, \]

\[ (56) \]

where \( L^j_{k} = \dot{\delta}^i \dot{L}^j_{k} \). Since \( \dot{\delta}^i = \dot{p}^i \) and \( \dot{\delta}^j = \dot{p}^j \), then we obtain

\[ c \beta G^j k p_s - c \beta G^j k p_s + c^2 \beta^2 G^j k G^j k p_s = c^2 \beta^4 \frac{1}{1 - 2c^2 \beta^2} \left( g^{ik} p^i - g^{ik} p^i + g^{ik} p^i - g^{ik} p^i \right) = 0. \]

\[ (57) \]

With replace \( i, j \) in (50) and setting these equations in (51), also by attention (57), we get

\[ K(\dot{\delta}^i, \dot{\delta}^j) \dot{\delta}^k = \left[ \beta^2 (L^{jk} i - L^{jk} i) \right] \delta s + \left[ C^i_{ks} - C^i_{ks} \right] + c \beta G^j k \delta s - c \beta G^j k \delta s + C^j k s \delta s \]

\[ - C^j k s \delta s + \beta^2 (L^j_{k} L^j_{k} - L^j_{k} L^j_{k}) \dot{\delta}^h. \]

\[ (58) \]

Similarly we can obtain the other components of curvature tensor.
Theorem 4.5. Let \((M, K)\) be a Cartan space of constant curvature \(c\) and the components of the metric \(G\) are given by \([37]\). Then the following are hold if and only if \((M, K)\) is reduce to a Riemannian space.

(i) for \(c < 0\), \((T^* M_0, G, J)\) is a Kähler Einstein manifold.

(ii) for \(c > 0\), \((T^* M_0, G, J)\) is a Kähler Einstein manifold, where \(T^* M_0\) the tube around the zero section in \(TM\), defined by the condition \(2r < \frac{1}{\sqrt{c}}\).

Proof. Let \((M, K)\) is a Riemannian space. Then \(C^h_k\) and \(P^h_{ik}\) are vanish and \(H^i_{jk}\) is a function of \((x^h)\). Therefore \([43]\) reduces to following

\[
K(\delta_i, \delta_j)\delta_k = [R^s_{kji} + c^2 \beta^2 (p_i \delta_j^s - p_j \delta_i^s) p_k] \delta_s. \tag{59}
\]

From Proposition 10.2 in chapter 4 of \([15]\), we have \(R_{kji} = -p_h R^h_{kji}\). Then we have

\[
p_h R^h_{kji} = c(g_{kj} \delta_i^h - g_{ki} \delta_j^h) p_k. \tag{60}
\]

Differentiating \(\tag{60}\) with respect to \(p_s\) and taking \(p = 0\), follows that

\[
R^s_{kji} = c(g_{kj} \delta_i^s - g_{ki} \delta_j^s). \tag{61}
\]

By putting \(\tag{61}\) in \(\tag{59}\), one can obtains

\[
K(\delta_i, \delta_j)\delta_k = c \beta \left( \frac{1}{\beta} g_{kj} - \frac{1}{\beta} g_{ki} \right) \delta_i^s - \frac{1}{\beta} g_{ki} \delta_j^s \delta_s,
\]

\[
= c \beta (G_{kj} \delta_i^s - G_{ki} \delta_j^s) \delta_s. \tag{62}
\]

Also from \(\tag{62}\), we get

\[
K(\delta_i^s, \delta_j)\delta_k = c \beta G_{sk} \delta_j^k \delta^s. \tag{63}
\]

From \(\tag{62}\) and \(\tag{63}\), we conclude that

\[
Ric(\delta_j, \delta_k) = G^{bi} G(K(\delta_i, \delta_j)\delta_k, \delta_h) + G_{hi} G(K(\delta_i, \delta_j)\delta_k, \delta^h),
\]

\[
= c \beta (G_{kj} \delta_i^s - G_{ki} \delta_j^s) G^{bi} G_{sh} + c \beta G_{sk} \delta_j^k G_{hs} G^{bi}.
\]

\[
= c n \beta G_{kj},
\]

\[
= c n \beta G (\delta_j, \delta_k). \tag{64}
\]

Similarly from \(\tag{68}\) and \(\tag{48}\), respectively, it follows that

\[
K(\delta^i, \delta^j)\delta^k = c \beta (G^{ik} \delta^j_s - G^{jk} \delta^i_s) \delta^k, \tag{65}
\]

and

\[
K(\delta_i, \delta^j)\delta^k = c \beta G^{ik} \delta^j_s \delta^s. \tag{66}
\]

By using \(\tag{65}\) and \(\tag{66}\), we obtain

\[
Ric(\delta^i, \delta^k) = G^{bi} G(K(\delta_i, \delta^j)\delta^k, \delta_h) + G_{hi} G(K(\delta_i, \delta^j)\delta^k, \delta^h),
\]

\[
= c \beta G^{ik} \delta_j^k G_{hs} + c \beta (G^{ik} \delta^j_s - G^{jk} \delta^i_s) G_{hs} G^{bi},
\]

\[
= c n \beta G_{jk},
\]

\[
= c n \beta G (\delta^j, \delta^k). \tag{67}
\]

From \(\tag{48}\) and \(\tag{60}\), we have, respectively

\[
K(\delta_i, \delta_j)\delta^k = (R^k_{hij} + c \beta R_{hij} G^{hk} p_s) \delta^s. \tag{68}
\]
From (64), (67), (70) and (72), it follows that
\[ K(\hat{\phi}^j, \hat{\phi}^k) = -c\beta G^{ks} \delta_j^s, \]  
(69)

By using (73)-(76), we obtain
\[ Ric(\hat{\phi}_j, \hat{\phi}^k) = G^{th}G(K(\hat{\phi}_i, \hat{\phi}_j)\hat{\phi}^k, \hat{\phi}_h) + G_{ih}G(K(\hat{\phi}^i, \hat{\phi}^j)\hat{\phi}_k, \hat{\phi}^h) = 0. \]  
(70)

From (51), we get
\[ K(\hat{\phi}^i, \hat{\phi}^j)\delta_k = c\beta(G^{ks} \delta_j^s - G^{js} \delta_k^s)\delta_s. \]  
(71)

By attention to (68) and (69), one can yields
\[ Ric(\hat{\phi}^i, \delta_k) = G^{th}G(K(\delta_i, \hat{\phi}^j)\delta_k, \delta_h) + G_{ih}G(K(\hat{\phi}^i, \delta_k)\delta_h, \delta^h) = 0. \]  
(72)

From (61), (67), (70) and (72), it follows that \( Ric(X, Y) = cn\beta G(X, Y), \forall X, Y \in \chi(T^*M). \)

This means that \((T^*M, G)\) is a Einstein manifold. Conversely, let (i), (ii) are hold. Then there exist constant \( \lambda \) such that \( Ric(X, Y) = \lambda G(X, Y). \) We consider following cases:

Case (1). If \( \lambda = 0 \) (i.e., \((T^*M, G)\) is Ricci flat), then we have \( Ric(\hat{\phi}^i, \hat{\phi}^k) = 0. \) By using (68) and (69) we get
\[ p_kG_{ih}(K(\hat{\phi}^i, \hat{\phi}^j)\hat{\phi}^k, \hat{\phi}^h) = p_kC^{hk} + (n - 1)c\beta p_kG^{jk} \]
and
\[ G^{th}G(K(\delta_i, \hat{\phi}^j)\delta_k, \delta_h) = c\beta p_kG^{jk} - p_kC^{hk} + \beta^2 p_kL^{hjk}_L |_h \]

By using two above equation, it results that
\[ 0 = p_kRic(\hat{\phi}^i, \hat{\phi}^k) = cn\beta p_kG^{jk} - p_kC^{jk}_L + \beta^2 p_kL^{hjk}_L |_h. \]  
(73)

With a simple calculation, one can obtains
\[ cn\beta p_kG^{jk} = cn\beta p_k(\beta g^{jk} + \frac{c\beta^2}{1 - 2c\beta^2} p^j p^k) = \frac{cn\beta^2}{1 - 2c\beta^2} p^j, \]  
(74)

and
\[ p_kC^{jk}_L = -p_kC^{jk}_G = -\delta^j_h - C^{jk}_h^G = -I^j, \]  
(75)
\[ p_kL^{hjk}_L |_h = -p_kL^{hjk}_L |_h = 0. \]  
(76)

By using (73)-(76), we obtain
\[ \frac{cn\beta^2}{1 - 2c\beta^2} p^j + I^j = 0. \]  
(77)

Since \( p_j I^j = 0 \), then by contracting (77) with \( p_j \), we have \( \frac{2cn\beta^2}{1 - 2c\beta^2} = 0. \) Thus we get \( \beta = 0 \), which is a contradiction.

Case (2). If \( \lambda \neq 0 \), then we have \( p_kRic(\hat{\phi}^i, \hat{\phi}^k) = \lambda G^{jk}p_k. \) By using (18), (58), (74) and (76), we obtain
\[ I^j = (\lambda - cn\beta) \frac{\beta}{1 - 2c\beta^2} p^j. \]  
(78)

Contracting (78) with \( p_j \) yields
\[ (\lambda - cn\beta) \frac{2\beta \tau}{1 - 2c\beta^2} = 0, \]  
(79)
i.e., \( \lambda = cn\beta. \) Thus by (78), we conclude that \( I^j = 0 \), i.e., \((M, K)\) reduces to a Riemannian space.

**Corollary 4.6.** There is not any non-Riemannian Cartan structure such that \((T^*M_0, G, J)\) became a Einstein manifold.
5 Divergence, Gradient and Laplace Operators

The divergence and Laplace operator have a number of applications for study various electromagnetic, gravitational and diffusion processes. For instance, in general relativity theory they are uniquely defined by the Levi-Civita connection for a corresponding fixed of local frames of coordinates. Such constructions are naturally generalized on Finsler spaces if we work with the Cartan distinguished connection because it is also metric compatible and completely defined by the metric structure. Even such a linear connection contains non-trivial torsion components (uniquely determined by some prescribed metric and nonlinear connection structures), the torsion contribution can be encoded into some divergence terms, for instance, in the case of stochastic/diffusion processes. How to define in a unique self-consistent form the divergence and Laplace operators, in Finsler-Lagrange and Hamilton-Cartan geometries with nonmetricity, without involving the Cartan distinguished connection (for instance, for the Chern and/or Berwald distinguished connections) it is an unsolved task in modern mathematical physics (see [29], [30]).

Proposition 5.1. Let \((M, K)\) be a Cartan space with Berwald connection. Then we have,
\[
\text{div}(X^V) = 0, \quad \text{div}(X^H) = X^i \delta_i (\ln \sqrt{g}) - X^i J_i,
\]
where \(X = X^i \hat{\delta}_i + \hat{X}_i \hat{\theta}^i\) and \(g := \det(g_{ij})\).

Proof. By a simple calculation, we have
\[
\text{div}(\hat{\theta}^i) = G^{ijl} G(\nabla_{\hat{\theta}^j} \hat{\delta}^l, \delta_i) + G_{jl} G(\nabla_{\hat{\theta}^j} \hat{\theta}^l, \hat{\theta}^i)
\]
\[
= G^{ijl} (C^s_{jl} - c \beta G^{is} p_s) G_{sl} + G_{jl} (-C^s_{jl} + c \beta G^{is} p_s) G^{sl}
\]
\[
= C^s_{sl} - c \beta G^{is} p_s + c \beta G^{is} p_s - C^s_{sl} = 0,
\]
\[
\text{div}(\delta_i) = G^{ijl} G(\nabla_\delta \delta_j, \delta_l) + G_{jl} G(\nabla^{\hat{\theta}}_l \delta_j, \hat{\theta}^i)
\]
\[
= G^{ijl} (L^s_{ij} + B^s_{ij}) G_{sl} + G_{jl} (-L^s_{ij}) G^{sl}
\]
\[
= L^s_{is} + B^s_{is} - L^s_{is} = B^s_{is}.
\]
Let \(H^i_{jk}\) are coefficients of Cartan connection. Since \(B^i_{jk} = H^i_{jk} = L^i_{jk}\), then we get \(\text{div}(X^H) = X^i \text{div}(\delta_i) = X^i H^i_{is} - X^i J_i\). Then it is easy to check that \(H^i_{as} = \frac{1}{\sqrt{g}} \delta_i (\ln \sqrt{g}) = \delta_i (\ln \sqrt{g})\).

Corollary 5.2. Let \((M, K)\) be a Cartan space with Berwald connection. Then \(\text{div}(X) = 0\) if and only if \(J_i = \delta_i (\ln \sqrt{g})\).

Let us define \(\text{grad} f\) by
\[
G(\text{grad}, X) = X f, \quad \forall X \in \chi(T^*M).
\]
Then in the adapted frames \(\{\delta_i, \hat{\theta}^i\}\), one can yields
\[
G(\text{grad}, \delta_i) = \delta_i f = \nabla_{\delta_i} f,
\]
\[
G(\text{grad}, \hat{\theta}^i) = \hat{\theta}^i f = \nabla_{\hat{\theta}^i} f.
\]
Put \(\text{grad} f := \alpha^i \delta_i + \beta_i \hat{\theta}^i\). Then from the above equations we have
\[
\alpha^i = G^{ih} \nabla_{\delta_h} f, \quad \beta_i = G_{ih} \nabla_{\hat{\theta}^h} f.
\]
Therefore we conclude the following.
Proposition 5.3. Let $(M, K)$ be a Cartan space with Berwald connection. Then we have
\[ \text{grad} f = G^{ih}(\nabla_{\delta_i} f)\delta_i + G_{ih}(\nabla_{\hat{\delta}_i} f)\hat{\delta}_i. \] (81)

The Laplace operator of a scalar field $f \in C^\infty(TM)$, is then defined as
\[ \Delta f = \text{div}(\text{grad} f). \] (82)

Then we have

Theorem 5.4. Let the Riemannian metric $G$ on $T^*M_0$ comes from a Cartan space $(M, K)$. Then the Laplace operator has the following form
\[ \Delta f = G^{ih}(\nabla_{\delta_i} f)(\frac{1}{\sqrt{g}}\delta_i(\sqrt{g}) - J_i), \quad f \in C^\infty(T^*M). \] (83)

By attention to Theorem 5.4, we conclude that if $f$ be a horizontally constant function, then $\Delta f = 0$. We have, also

Corollary 5.5. The Laplace operator $\Delta$ is vanish if and only if $J_i = \frac{1}{\sqrt{g}}\delta_i(\sqrt{g}) = \delta_i(\ln \sqrt{g})$.

Proposition 5.6. Let the Riemannian metric $G$ on $TM$ is the Kähler metric with components defined by (37), which is induced by the Cartan structure $K$ on $M$. Then we have
\[ \text{div}(C^*J^i) = 0, \quad \text{div}(S^i) = p^i\delta_i(\ln \sqrt{g}), \quad \Delta K^2 = 0, \]

where $S = p^i\delta_i$ is the geodesic spray of $(M, K)$.

Proof. Since $p^iJ_i = 0$ and $\nabla_{\delta_i}K^2 = \delta_iK^2 = 0$, then by using Proposition 5.1 and Theorem 5.4 the proof will be complete. \hfill \square

From above proposition we result that $\text{div}(S)$ is zero if and only if $\delta_i(\sqrt{g}) = 0$. Then by Corollary 5.5 we have the following.

Theorem 5.7. Let $(M, K)$ be a Cartan space with Berwald connection. Suppose that Laplace operator is vanishes. Then $\text{div}(S) = 0$ if and only if $K$ is a mean Landsberg metric.

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