Logical Pseudocode: Connecting Algorithms with Proofs

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SUMMARY
Proofs (sequent calculus, natural deduction) and imperative algorithms (pseudocodes) are two well-known co-existing concepts. Then what is their relationship? Our answer is that

\[ \text{imperative algorithms} = \text{proofs with cuts} \]

This observation leads to a generalization to pseudocodes which we call logical pseudocodes. It is similar to natural deduction proof of computability logic [1,2]. Each statement in it corresponds to a proof step in natural deduction. Therefore, the merit over pseudocode is that each statement is guaranteed to be correct and safe with respect to the initial specifications. It can also be seen as an extension to computability logic web (CoLweb) with forward reasoning capability.

key words: forward reasoning, logical imperative language.

1. Introduction

Pseudocode (and imperative languages) is low-level and unregulated. For example, the assignment statement \( x = E \) is not regulated at all and is considered harmful. As a consequence, it is difficult to tell whether a pseudocode is correct or safe.

Of course, it is possible to use high-level logic languages. These languages make code much safer and much easier to verify but often this requires some sacrifice in the performance. This is mainly because they are not able to utilize lemmas.

What are the remaining alternatives then? We believe CoLweb[3] is a best choice for expressing algorithms. It is a promising model of multi-agent programming where agents exchange services. It can be viewed as a high-level, bottom-up version of natural deduction proof of computability logic. In contrast to logic languages, the main feature of CoLweb is a full utilization of lemmas. In other words, the main idea of CoLweb is to create high-level, regulated, safe and lemma-based codes. However, while implementing CoLweb in a distributed setting poses little problems, implementing it in a non distributed setting is somewhat complicated.

For this reason, we introduce logical pseudocode – as a companion to CoLweb – which can be seen as a version of CoLweb with top-down, centralized control.

Logical pseudocode is much easier to implement than CoLweb in a non-distributed environment. It can be seen as a major evolution of pseudocode.

2. Problems with Pseudocode

Let us consider an array-based implementation of the fibonacci sequence. In the traditional imperative approach, it can be specified as:

\[
\begin{align*}
\text{fib}(n) &= \% \text{an array-based implementation} \\
a[0] &= 1; \\
a[1] &= 1; \\
\text{for } i = 0 \text{ to } n-2 & \quad a[i+2] = a[i+1] + a[i]
\end{align*}
\]

return \( a[n] \);

In the above, \( a[0] = 1 \), etc is called an assignment statement.

The above code has at least two weaknesses:

- It is not clear whether the code is correct, and
- It is not clear whether the code is safe.

That is, the machine always executes pseudocodes, regardless of their correctness and safety. This is a serious problem which leads to various researches such as proof-carrying code. As a different direction of this trend, we introduce logical assignment statements as a replacement of assignment statements.

3. Preliminaries

In this section a brief overview of CoL is given.

There are two players: the machine \( \top \) and the environment \( \bot \).

There are two sorts of atoms: elementary atoms \( p, q, \ldots \) to represent elementary games, and general atoms \( P, Q, \ldots \) to represent any, not-necessarily-elementary, games.

Constant elementary games \( \top \) is always a true proposition, and \( \bot \) is always a false proposition.

Negation \( \neg \) is a role-switch operation: For example, \( \neg (0 = 1) \) is true, while \( (0 = 1) \) is false.

Choice operations The choice group of operations: \( \sqcap, \sqcup, \forall \) and \( \exists \) are defined below.
\[ \forall x \mathcal{A}(x) \] is the game where, in the initial position, only \( \bot \) has a legal move which consists in choosing a value for \( x \). After \( \bot \) makes a move \( c \in \{0,1,\ldots\} \), the game continues as \( \mathcal{A}(c) \). \( \forall x \mathcal{A}(x) \) is similar, only here the value of \( x \) is invisible. \( \sqcup \) and \( \exists \) are symmetric to \( \sqcap \) and \( \forall \), with the difference that now it is \( \top \) who makes an initial move. 

**Parallel operations** Playing \( A_1 \land \ldots \land A_n \), means playing the \( n \) games concurrently. In order to win, \( \top \) needs to win in each of \( n \) games. Playing \( A_1 \lor \ldots \lor A_n \) also means playing the \( n \) games concurrently. In order to win, \( \top \) needs to win one of the games. To indicate that a given move is made in the \( i \)th component, the player should prefix it with the string “\( i \)”. The operations \( \land \mathcal{A} \) means an infinite parallel game \( A \land \ldots \land A \land \ldots \). To indicate that a given move is made in the \( (i(i > 1)) \)th component, we assume the player should first replicate \( A \) and then prefix it with the string “\( i \)”. 

**Reduction** \( A \to B \) is defined by \( \neg A \lor B \). Intuitively, \( A \to B \) is the problem of reducing \( B \) (consequent) to \( A \) (antecedent).

### 4. Introducing Directories

Logical formulas are inadequate for locating subformulas. Our approach to achieving this effect is through the use of **directories**. For example, consider the following directory definition.

\[ /m = p(a) \]

where \( /m \) is a directory name and \( p(a) \) is a formula. In this case, we call \( p(a) \) its “content”. Alternatively, we can view \( /m \) as an agent and \( p(a) \) as its knowledgebase. A directory is a sequence of subdirectories and formulas. For example,

\[ /m = (/m_1, \ldots, /m_n = \ldots, F_1, \ldots, F_n) \]

where \( /m \) is a directory, each \( /m_i \) is a subdirectory and each \( F_i \) is a file. We also introduce a class directory/agent which is a combination of the form

- \( \forall x /m(x) = F \), or
- \( \land /m = F \).

Our directory system is very flexible and is designed to represent both formulas and cirquents. For example, \( /n = /m \land /m \) represents that the directory \( /n \) contains \( p(a) \land p(a) \). Here \( !/m \) is intended to read as “a copy of the content of \( /m \). In contrast, \( /o = /m \land /m \) represents that \( /o \) contains a cirquent \( p(a) \land p(a) \) where two \( p(a) \)s in \( /o \) and \( p(a) \) in \( /m \) are shared.

As another example, consider the following recursive directory definition.

\[ /m(0) = q \]

Given this definition, \( p \land (p \land (p \land q)) \) can be represented simply as \( /m(s(s(0))) \). We assume in the above that \( s \) is the number-successor function.

Thus, we propose the notion of **directoryed formulas**. They are formulas enhanced with directories. These formulas are better-suited to structuring large formulas such as pigeonhole principle formulas. It is interesting to note that directories also play the role of global variables in imperative languages and much more.

### 5. Logical Assignment Statements

In the sequel, we focus on a single machine with local memories. Note that a single machine with local memories can be viewed as a multiagent system where each agent shares a single CPU.

To overcome the problems mentioned in the previous section, we introduce **logical assignment statements** of the form

\[ /x = F^{l_1,\ldots,l_n} \]

where \( /x \) is an agent/location, \( F \) is a logical formula and each \( l_i \) is an agent. We call \( F^{l_1,\ldots,l_n} \) querying knowledgebase. We call the above statement well-formed if \( F \) is a logical consequence of knowledgebases at \( l_1,\ldots,l_n \). Agents operate in three modes: idle, reactive and proactive. Agents in reactive mode means that they are busy processing service requests from other agents. Agents in proactive mode means that they invoke querying knowledgebase to other agents.

**Executing** \( /x \) means that \( /x \) is in proactive mode. That is, the machine tries to solve the query \( F \) using knowledgebases at \( l_1,\ldots,l_n \). If it succeeds, it binds \( /x \) to \( F' \) to which \( F \) evolves. Otherwise, it reports failure.

For example, consider the following:

\[ /x = p(0,1) \]
\[ /y = \sqcup \text{wp}(0, w)/x \]
\[ /z = p(0,5)/x \]

In the above, note that \( /y \) is well-formed but \( /z \) is not. Now executing \( /y \) means the machine tries to solve the goal using \( p(0,1) \). This will succeed and bind \( /y \) to \( p(0,1) \). On the contrary, executing \( /z \) will not be allowed. Note that destructive logical assignments are disallowed.

### 6. Forward Construct

In pseudocode, for example, \( x = 4; y = x \) imposes the following sequence between two statements: execute \( x = 4 \) and then execute \( y = x \). This forward ordering supports forward reasoning in theorem proving.
Note that forward chaining is often simple to implement, while implementing backward chaining requires a chain of stacks.

From this viewpoint, we add a similar construct of the form

\[ /x = F; /y = G. \]

The above means the following: to execute /y, execute /x first.

For example, consider the following:

\[
\begin{align*}
/x &= p(0,1) \\
/y &= \bigcup y p(0,y)/^x; \\
/z &= \bigcup z p(0,z)/^y
\end{align*}
\]

Executing /z requires to execute /y first. After execution, we obtain the following:

\[
\begin{align*}
/x &= p(0,1) \\
/y &= p(0,1) \\
/z &= p(0,1)
\end{align*}
\]

The forward relation can be generalized to the forward quantifier of the following form:

\[ \text{for } i_{m_{n}} \quad /x[i] = F(i) \]

which means

\[ /x[m] = F(m); \ldots; /x[n] = F(n). \]

To activate /x[k], the machine does the following:

(step 1) It generates the following:

\[
\begin{align*}
/x[m] &= F(m); \\
& \quad \vdots \\
/x[k] &= F(k); \\
\text{for } i_{k+1} \quad /x[i] &= F(i)
\end{align*}
\]

(step 2) The machine executes /x[m], \ldots, /x[k] in that order.

Finally, a well-formed logical pseudocode \( P \) is a set of well-formed logical assignment statements. Given \( P \), execution tries to solve a query using a mix of backward reasoning and forward reasoning.

7. Adding Induction

Let \( R \) be a set of initial assumptions or axioms. To deal with a query \( /q = (\bigcap x F(x))^R \), we need to introduce mathematical induction to logical pseudocode. There are several reasonable systems for dealing with induction. One simple example is the constructive induction [2] of the form

\[ F(1) \land \bigcap x (F(x) \rightarrow F(x+1)) \rightarrow \bigcap x F(x). \]

By assuming that \( F(X) \) is stored at a location \( /a[X] \), the above can be rewritten in a more economical form, which we call \( \text{IND} \). That is, we assume that the programmer generates the following code:

\[
\begin{align*}
/a[1] &= F(1)^R \\
/istep &= \text{for } i_{\infty} /a[i] = F(i)/a[i-1],^R \\
/q &= (\bigcap x F(x) )^{\text{IND},/a[1],/istep}
\end{align*}
\]

In the above, \( /istep \) denotes an agent which handles the inductive steps.

The above \( \text{IND} \) is called a special induction which is a restricted case of general induction. The general induction \( \text{GIND} \) can be similarly defined but is not shown here for simplicity. It, however, will be used in the sequel.

As an example, consider \( /fib = \bigcap x \bigcup y fib(x,y)^R \). Then it can be written as:

\[
\begin{align*}
/a[1] &= \bigcup y fib(1,y)^R \\
/a[2] &= \bigcup y fib(2,y)^R \\
/istep &= \text{for } i_{\infty} /a[i] = \bigcup y fib(i,y)/a[i-1],/a[i-2],^R \\
/fib &= (\bigcap x \bigcup w fib(x,w))^{\text{GIND},/a[1],/a[2],/istep}
\end{align*}
\]

In the above, \( R \) is a set of rules for the fibonacci sequence.

Note that the above code is well-formed. Now let us execute the query \( /fib \). This expression waits for an input. Suppose the \( /fib \) types in 4. The machine then invokes \( \bigcup y fib(4,w) \) to the agent \( /a[4] \). Due to the for-loop, the machine creates and executes two agents \( /a[3],/a[4] \) in that order. This in turn invokes executing \( /a[1],/a[2] \). Now the above code becomes the following:

\[
\begin{align*}
/r[1] &= fib(1,1) \\
/r[2] &= fib(2,1) \\
/r[3] &= \forall x, y, z (fib(x,y) \land fib(x+1,z)) \rightarrow fib(x+2,y+z) \\
/a[1] &= fib(1,1) \\
/a[2] &= fib(2,1) \\
/a[3] &= fib(3,2) \\
/a[4] &= fib(4,3) \\
/istep &= \text{for } i_{\infty} /a[i] = \bigcup y fib(i,y)/a[i-1],/a[i-2],^R \\
/fib &= \bigcup w fib(4,w)^/a[4]
\end{align*}
\]

Now the agent \( /fib \) stores \( fib(4,3) \) as desired and execution terminates.

8. An Example

As an example, we consider the fibonacci sequence. We assume that the user invokes queries to the agent \( /fib \).

Then \( fib \) can be written as:
Note that the above code is well-formed. In particular, \( /query \) is well-formed, as it is a logical consequence of \(/fib\). Now let us execute \(/query\). This expression waits for an input. Suppose \(/query\) types in 4 for \( x \). The machine then tries to solve it relative to the program in \(/fib\). Then \(/fib\) must select 4 for \( n \). The rest proceeds as explained in the preceding section.

The above code is very concise but has some interesting features:

1. It supports automatic memoization.
2. It supports a mix of backward reasoning and forward reasoning (via for-loop).

9. Thinking Theorem Proving in a Bigger Paradigm

A sequence of well-chosen lemmas plays a key role in classical theorem proving such as natural deduction (ND), sequent calculus with cuts (LK+cut) and Coq. Yet, finding lemmas can be extremely challenging. For this reason, effective strategies and tactics for finding lemmas have been studied but with little progress.

One way to approach this problem is to embed theorem proving in a bigger paradigm. For example, logical pseudocode (and CoLweb) – based on computability logic – provides some useful solutions in this regard. That is, it allows us to specify a set of lemma candidates instead of a single lemma. This makes life simpler. To be specific, it extends lemmas to include the following:

- \( (\text{lemma}_1 \sqcup \ldots \sqcup \text{lemma}_n)^{t_1, \ldots, t_k} \): In this case, the machine tries to find \( \text{lemma}_i \) which is a logical consequence of knowledgebases at \( t_1, \ldots, t_k \).
- \( \sqcup \text{lemma}(x)^{t_1, \ldots, t_k} \): In this case, the machine tries to find a value \( c \) of \( x \) such that \( \text{lemma}(c/x) \) is a logical consequence of knowledgebases at \( t_1, \ldots, t_k \).

In the above \( \text{lemma} \) is a first-order formula. For example, an infinite set of lemmas such as \( \sqcup x fib(100, x) \) is now allowed.

In conclusion, logical pseudocode (and CoLweb) can be seen as a nondeterministic version of ND and a better alternative to ND. We can apply this idea to the cut formula of LK or to the tactics of Coq to obtain a nondeterministic version of LK+cut or a nondeterministic version of tactics.

10. Conclusion

Our ultimate goal is to implement the computability logic web which is a promising approach to reaching general AI. New ideas in this paper – forward reasoning construct – will be useful for practical implementations.

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