Elements of Generalized Tsallis Relative Entropy in Classical Information Theory

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Abstract
In this article, we propose a modification in generalised Tsallis entropy which makes it efficient to be utilized in classical information theory. This modification offers the product rule
\[(xy)^{r+k} \ln_{(k,r)}(xy) = x^{r+k} \ln_{(k,r)}(x) + y^{r+k} \ln_{(k,r)}(y) + 2k x^{r+k} y^{r+k} \ln_{(k,r)}(x) \ln_{(k,r)}(y),\]
for the two-parameter deformed logarithm \[\ln_{(k,r)}(x) = x^{r+k} - \frac{x - k}{2k} \]. It assists us to derive a number of properties of the generalised Tsallis entropy, and related entropy for instance the sub-additive property, joint convexity, and information monotonicity. This article is an expository investigation on the information-theoretic, and information-geometric characteristics of generalised Tsallis entropy.

1 Introduction
Information geometry [1] has been developed in the field of statistics as a geometric way to analyse different order dependencies between random variables. The information geometry has a unique feature. It has a dualistic structures of affine connections. In this article, we study information geometry of a two parameter generalization of Tsallis entropy which also reduces to the Gibbs Shannon entropy [2, 3, 4, 5]. The Tsallis entropy which is followed by the \(\kappa\)-thermostatistics is a generalization of the thermostatistics based on \(\kappa\)-entropy [6]. It plays an important role in power law distribution. We have explained the information geometric structures associated to the generalized Tsallis entropy.

In literature, a number of variations of the Sharma-Mittal entropy [7, 8] is studied. We consider the expression of Sharma-Mittal entropy discussed in [9]. In this article, we propose a rectification in the definition of Sharma-Mittal entropy which leads to one of its nearest term forms. We call it generalised

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Tsallis entropy and denote it by $S_{\{k, r\}}(X)$. Similarly, we define the generalised Tsallis relative entropy or generalised Tsallis divergence $D_{\{k, r\}}$. The Tsallis and Shannon entropies, and divergences are obtained by specific parameter values of the corresponding terms of these new definitions. We have shown that the generalised Tsallis relative entropy satisfies a number of characteristics, such as positivity, symmetry, pseudo-additivity, joint convexity, etc. The significant results discussed in this article are listed below:

1. Sub-additivity of generalised Tsallis entropy:

$$S_{\{k, r\}}(X, Y) = S_{\{k, r\}}(X) + S_{\{k, r\}}(Y) - 2kS_{\{k, r\}}(X)S_{\{k, r\}}(y).$$  

2. Sub-additivity of generalised Tsallis relative entropy:

$$D_{\{k, r\}}(P^{(1)} \otimes P^{(2)} || Q^{(1)} \otimes Q^{(2)}) = D_{\{k, r\}}(P^{(1)} || Q^{(1)}) + D_{\{k, r\}}(P^{(2)} || Q^{(2)}) - 2kD_{\{k, r\}}(P^{(1)} || Q^{(1)})D_{\{k, r\}}(P^{(2)} || Q^{(2)}).$$  

3. Joint convexity of generalised Tsallis relative entropy:

$$D_{\{k, r\}}(P^{(1)} + \lambda P^{(2)} || Q^{(1)} + \lambda Q^{(2)}) \leq D_{\{k, r\}}(P^{(1)} || Q^{(1)}) + \lambda D_{\{k, r\}}(P^{(2)} || Q^{(1)}).$$  

4. Information monotonicity of generalised Tsallis relative entropy:

$$D_{\{k, r\}}(WP || WQ) \leq D_{\{k, r\}}(P || Q).$$

This article consists of six sections. The next section redefines the Sharma-Mittal logarithm and establishes a number of its characteristics required for the calculations in the remaining parts of the article. Section 3 discusses the modified generalised Tsallis entropy. The chain rule of joint generalised Tsallis entropy is discussed here. Section 4 is dedicated to generalised Tsallis relative entropy and its properties. We discuss the information geometric aspects of relative entropy in section 5. Then we conclude the article with a number of open problems.

## 2 Preliminary properties of a two parameter deformed logarithm

A function $f$ is convex [10] if $f(x_1 + \lambda x_2) \leq f(x_1) + \lambda f(x_2)$, for all $\lambda \in [0, 1]$. More generally,

$$f \left( \sum_{x \in X} t_x x \right) \leq \sum_{x \in X} t_x f(x), \text{ where } \sum_{x \in X} t_x = 1 \text{ and } 0 \leq t_x \leq 1.$$  

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It can be proved that, if \( f \) is a twice differentiable convex function then \( f''(x) \geq 0 \). The function \( f \) is concave if \( -f(x) \) is convex. Hence, a function \( f \) is said to be concave if

\[
f \left( \sum_{x \in X} t_i x_i \right) \geq \sum_{x \in X} t_i f(x_i), \quad \text{where} \quad \sum_{x \in X} t_i = 1 \quad \text{and} \quad 0 \leq t_i \leq 1.
\] (6)

Given probability distribution \( \mathcal{P} = \{p(x) : x \in X, p(x) \geq 0, \sum_x p(x) = 1\} \), the Sharma-Mittal entropy [9] of the random variable \( X \) is defined by

\[
S_{(k,r)}(X) = -\sum_{x \in X} p(x) \ln_{(k,r)}(p(x)), \quad \text{where} \quad \ln_{(k,r)}(x) = x^r \frac{x^k - x^{-k}}{2k},
\] (7)

and \((k,r) \in \mathcal{R} \subset \mathbb{R}^2\), such that,

\[
\mathcal{R} = \{(k,r) : -|k| \leq r \leq |k| \quad \text{when} \quad 0 \leq |k| < \frac{1}{2}\}
\]

\[
\cup \{(k,r) : |k| - 1 \leq r \leq 1 - |k| \quad \text{when} \quad \frac{1}{2} \leq |k| < 1\}.
\] (8)

Now, recall a few properties of natural logarithm useful in the literature of information theory. The function \( f(x) = -\log(x) > 0 \) for all \( x \in (0,1) \). Also, for all \( x > 0 \) we have \( f'(x) = -\frac{1}{x} < 0 \), that is \(-\log(x)\) is monotonically decreasing. Moreover, \( f''(x) = \frac{1}{x^2} > 0 \), which indicates \(-\log(x)\) is a convex function for all \( x \neq 0 \). Now, we restrict our discussion of the deformed logarithm \( \ln_{(k,r)} \) to the range of its parameters \( k \) and \( r \) such that \( \ln_{(k,r)} \) fulfils these characteristics.

**Theorem 1.** For \( r < 0 \), and \( 0 < k \leq \frac{1}{2} \), the function \( -\ln_{(k,r)}(x) = -x^r \frac{x^k - x^{-k}}{2k} \) is positive, convex, and monotonically decreasing for all \( x \in (0,1] \).

**Proof.** Define a function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( f(x) = x^r \). Note that, \( f(x) > 0 \) for all \( x > 0 \) and for all \( r < 0 \). If \( r < 0 \) we have \( r = -|r| \) and \( f(x) = \frac{1}{x^{|r|}} \). Differentiating we find \( f'(x) = -\frac{|r|}{x^{|r|+1}} < 0 \), that is \( f(x) \) is a monotonically decreasing function. Also, \( f''(x) = \frac{|r|(|r|+1)}{x^{|r|+2}} \geq 0 \). Hence, \( f(x) \) is a convex function.

For all \( k > 0 \), we have \( x^{-k} \geq x^k \) for all \( x \in (0,1] \). Therefore, \(-x^k - x^{-k} > 0 \) for all \( k \neq 0 \) and \( 0 < x \leq 1 \). Define another function \( g(x) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( g(x) = -\frac{x^k - x^{-k}}{2k} \). Differentiating we get \( g'(x) = -\frac{x^{k-1} + x^{-k-1}}{2} < 0 \), for all \( x > 0 \), which indicates \( g(x) \) is a monotonically decreasing function for all \( x > 0 \). Again, the double derivative \( g''(x) = -(k-1)x^{k-2} - (k+1)x^{-k-2} \). If \( k \leq \frac{1}{2} \) we have \( k-1 < 0 \). Also, \( x^{k-2}, (k+1) \) and \( x^{-k-2} > 0 \) for all \( x > 0 \). Combining we get \( g''(x) > 0 \) which is sufficient for convexity.

It can be proved that, if two given functions \( f, g : \mathbb{R} \rightarrow \mathbb{R}^+ \) are convex, and both monotonically non-decreasing (or non-increasing) functions on an interval, then \( fg(x) = f(x)g(x) \) is convex [10]. Hence, \( -\ln_{(k,r)}(x) \) is convex under the stated conditions.
Moreover, $f(x) > 0$ and $g(x) > 0$ for all $x \in (0, 1]$ and both are monotonically decreasing. Therefore, their product $-\ln_{(k,r)}$ is monotonically decreasing under the stated conditions.

The convexity of $-\ln_{(k,r)}(x)$ requires $r < 0$. Hence, we often write $\ln_{(k,r)}$ as

$$\ln_{(k,r)}(x) = \frac{x^k - x^{-k}}{2kx^r} = \frac{x^{2k} - 1}{2kx^r} + \frac{1}{2},$$

with $r > 0$ and $0 < k \leq \frac{1}{2}$. Clearly, $\ln_{(k,r)}(1) = 0$ and $\ln_{(k,r)}(0)$ is undefined.

The product rule of two parameter deformed logarithm $\ln_{(k,r)}(x)$ mentioned in equation (7) is given by

$$\ln_{(k,r)}(xy) = u_{(k,r)}(x) \ln_{(k,r)}(y) + \ln_{(k,r)}(x)u_{(k,r)}(y),$$

where $u_{(k,r)}(x) = x^r \left( \frac{x^k + x^{-k}}{2} \right)^4$. Now, for a joint random variables $(X, Y)$ we have $p(x, y) = p(x)p(y|x)$. The equation (7) suggests that the joint generalised Tsallis entropy is

$$S_{(k,r)}(X, Y) = -\sum_{x \in X} \sum_{y \in Y} p(x, y) \ln_{(k,r)}(p(x, y))$$

$$= -\sum_{x \in X} \sum_{y \in Y} p(x)p(y|x) \ln_{(k,r)}(p(x)p(y|x))$$

$$= -\sum_{x \in X} \sum_{y \in Y} p(x)p(y|x)u_{(k,r)}(p(x)) \ln_{(k,r)}(p(y|x))$$

$$- \sum_{x \in X} \sum_{y \in Y} p(x)p(y|x) \ln_{(k,r)}(p(x))u_{(k,r)}(p(y|x)).$$

This expression prevents us to derive the chain rule for generalised Tsallis entropy. It prevents the generalised Tsallis entropy to be used in classical information theory [11]. Now, we modify the product rule for this two parameter deformed logarithm as follows:

**Lemma 1.** Given any two real numbers $x, y \neq 0$ we have $(xy)^{r+k} \ln_{(k,r)}(xy) = x^{r+k} \ln_{(k,r)}(x) + y^{r+k} \ln_{(k,r)}(y) + 2kx^{r+k}y^{r+k} \ln_{(k,r)}(x) \ln_{(k,r)}(y)$.

**Proof.**

$$\ln_{(k,r)}(x) \ln_{(k,r)}(y) = \frac{x^{2k} - 1}{2kx^{r+k}} - \frac{y^{2k} - 1}{2ky^{r+k}} = \frac{x^{2k}y^{2k} - x^{2k} - y^{2k} + 1}{4k^2x^{r+k}y^{r+k}}$$

$$= \frac{x^{2k}y^{2k} - 1}{4k^2x^{r+k}y^{r+k}} - \frac{x^{2k} - 1}{4k^2x^{r+k}y^{r+k}} - \frac{y^{2k} - 1}{4k^2x^{r+k}y^{r+k}}$$

$$= \frac{\ln_{(k,r)}(xy)}{2k} - \frac{\ln_{(k,r)}(x)}{2kx^{r+k}} - \frac{\ln_{(k,r)}(y)}{2ky^{r+k}}.$$

It leads us to the result.
Lemma 2. Given any two real numbers \( x, y \neq 0 \) we have

\[
\ln_{(k,r)}(xy) = \frac{1}{x^{r-k}} \ln_{(k,r)}(x) + \frac{1}{y^{r+k}} \ln_{(k,r)}(y).
\]

**Proof.** Note that,

\[
\frac{(xy)^k - (xy)^{-k}}{2k} = \frac{x^ky^k - x^ky^{-k} + x^ky^{-k} - x^ky^k}{2k} = \frac{x^k(y^k - y^{-k})}{2k} + \frac{y^k(x^k - x^{-k})}{2k},
\]

or

\[
\frac{(xy)^k - (xy)^{-k}}{2k(xy)^r} = \frac{1}{x^{r-k}} \left( \frac{y^k}{2ky^r} \right) + \frac{1}{y^{r+k}} \left( \frac{x^k}{2kx^r} \right).
\]

Hence, we find the result. \( \square \)

**Corollary 1.** For any non-zero real number \( x \) we have

\[
\ln_{(k,r)} \left( \frac{1}{x} \right) = -\frac{1}{x^{2r}} \ln_{(k,r)}(x), \text{ or } \ln_{(k,r)}(x) = -x^{2r} \ln_{(k,r)} \left( \frac{1}{x} \right).
\]

**Proof.** Putting \( y = \frac{1}{x} \) in the lemma 2 we find

\[
\ln_{(k,r)}(1) = \ln_{(k,r)}(x, \frac{1}{x}) = x^{r+k} \ln_{(k,r)} \left( \frac{1}{x} \right) + \left( \frac{1}{x} \right)^{r-k} \ln_{(k,r)}(x)
\]

or

\[
x^{r+k} \ln_{(k,r)} \left( \frac{1}{x} \right) = -\left( \frac{1}{x} \right)^{r-k} \ln_{(k,r)}(x),
\]

which leads to the result. \( \square \)

**Corollary 2.** For any two non-zero real numbers \( x \) and \( y \) we have

\[
\ln_{(k,r)} \left( \frac{x}{y} \right) = \frac{1}{x^{r-k}} \ln_{(k,r)}(x) - \frac{1}{y^{r+k}} \ln_{(k,r)}(y).
\]

**Proof.** From the observation 2 we find that

\[
\ln_{(k,r)} \left( \frac{x}{y} \right) = \frac{1}{x^{r-k}} \ln_{(k,r)}(x) + y^{r+k} \ln_{(k,r)} \left( \frac{1}{y} \right)
\]

or

\[
\ln_{(k,r)} \left( \frac{x}{y} \right) = \frac{1}{x^{r-k}} \ln_{(k,r)}(x) - \frac{y^{r+k}}{y^{2r}} \ln_{(k,r)} \left( \frac{1}{y} \right)
\]

which leads to the result. \( \square \)

**Lemma 3.** For any non-zero real number \( x \) and any real number \( a \) we have

\[
\ln_{(k,r)}(x^a) = a \ln_{(ak,ar)}(x).
\]
Proof.

\[
\ln_{(k,r)}(x^a) = \frac{(x^a)^k - (x^a)^{-k}}{2k(x^a)^r} = a \frac{x^{ak} - x^{-ak}}{2akx^{ar}} = a \ln_{(ak,ar)}(x). \tag{16}
\]

\[\Box\]

**Lemma 4.** The function \( f(x) = -x^{r+k} \ln_{(k,r)}(x) \) is a convex function for \( 0 \leq k \leq \frac{1}{2} \), and \( r > 0 \).

**Proof.** \( f(x) = -x^{r+k} \ln_{(k,r)}(x) = \frac{1-x^{2k}}{2k} \). Therefore \( f''(x) = (1 - 2k)x^{2k-2} \). Now \( f''(x) \geq 0 \) if \( 1 - 2k \geq 0 \), that is \( 0 \leq k \leq \frac{1}{2} \). \[\Box\]

**Lemma 5.** For any real number \( x > 0 \) and for \( 0 < k \leq \frac{1}{2} \) the function \( f(x) = x^{r-k+1} \ln_{(k,r)}(x) \) is a convex function.

**Proof.** Simplifying we get \( f(x) = x^{r-k+1} \ln_{(k,r)}(x) = \frac{x-x^{-1-2k}}{2k} \). Therefore, \( f'(x) = \frac{1-(1-2k)x^{-2k}}{2k} \) and \( f''(x) = \frac{1-2k}{x^{1+2k}} \). For \( k \leq \frac{1}{2} \) we have \( 1-2k > 0 \). Hence, \( f''(x) \geq 0 \) for all \( x > 0 \), which indicates that \( f(x) \) is a convex function. \[\Box\]

**Theorem 2.** Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be non-negative numbers. In addition, \( a = \sum_{i=1}^{n} a_i \) and \( b = \sum_{i=1}^{n} b_i \). Then,

\[
\sum_{i=1}^{n} a_i \left( \frac{a_i}{b_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{a_i}{b_i} \right) \geq a \left( \frac{a}{b} \right)^{r-k} \ln_{(k,r)} \left( \frac{a}{b} \right) .
\]

**Proof.**

\[
\sum_{i=1}^{n} a_i \left( \frac{a_i}{b_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{a_i}{b_i} \right) = b \sum_{i=1}^{n} \frac{b_i a_i}{b b_i} \left( \frac{a_i}{b_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{a_i}{b_i} \right) = b \sum_{i=1}^{n} \frac{b_i}{b} f \left( \frac{a_i}{b_i} \right) ,
\]

where \( f(x) = x^{r-k+1} \ln_{(k,r)}(x) \), which is a convex function. Therefore,

\[
\sum_{i=1}^{n} a_i \left( \frac{a_i}{b_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{a_i}{b_i} \right) \geq b f \left( \sum_{i=1}^{n} \frac{b_i a_i}{b b_i} \right) = b f \left( \frac{1}{b} \sum_{i=1}^{n} a_i \right) = b f \left( \frac{a}{b} \right) = b \left( \frac{a}{b} \right)^{r-k+1} \ln_{(k,r)} \left( \frac{a}{b} \right) ,
\]

which indicates the proof. \[\Box\]
3 Modified generalised Tsallis entropy

To fulfil the chain rule of entropy, we modify the definition of the generalised Tsallis entropy mentioned in the equation (7) as follows:

**Definition 1.** We define the generalised Tsallis entropy for a random variable \( X \) with probability distribution \( \mathcal{P} = \{ p(x) \}_{x \in X} \) as

\[
S_{\{k,r\}}(X) = - \sum_{x \in X} (p(x))^{r+k+1} \ln_{\{k,r\}}(p(x)),
\]

where \( 0 < k \leq \frac{1}{2} \), \( r > 0 \) and \( \ln_{\{k,r\}} \) is defined in equation (9).

A simple calculation shows that if \( p(x) = 0 \) for some \( x \in X \) then we have \( (p(x))^{r+k+1} \ln_{\{k,r\}}(p(x)) = 0 \). For any non-zero probability \( p(x) \) the theorem 1 suggests that \( - \ln_{\{k,r\}}(p(x)) > 0 \), that is \( (p(x))^{r+k+1} \ln_{\{k,r\}}(p(x)) \geq 0 \) for any probability \( p(x) \). Therefore, for any random variable \( X \) we have \( S_{\{k,r\}}(X) > 0 \).

This definition of generalised Tsallis entropy is consistent to the Tsallis entropy for specific values of the parameter for instance, of the generalised Tsallis entropy. A number of similar expressions yields the Tsallis entropy for specific values of the parameter for instance,

\[
- \sum_{x \in X} (p(x))^{2k+1} \ln_{\{k,r\}}(p(x)), - \sum_{x \in X} (p(x))^{2r+1} \ln_{\{k,r\}}(p(x)),
\]

which is the Tsallis logarithm. Putting the same values of \( p \) and \( q \) in the expression of \( S_{\{k,r\}}(X) \) we find that

\[
\ln_{\{k,-k\}}(x) = x^{-k} \frac{x^k - x^{-k}}{2k} = \frac{1 - x^{-2k}}{2k}.
\]

Now putting \( q = 1 + 2k \) that is \( k = \frac{q-1}{2} \) in the expression of \( \ln_{\{k,-k\}}(x) \) we find that

\[
\ln_{\{\frac{q-1}{2}, \frac{q+1}{2}\}}(x) = \frac{1 - x^{1-q}}{q-1} = \frac{x^{1-q} - 1}{1 - q} = \ln_q(x),
\]

which is the Tsallis logarithm. Putting the same values of \( r \) and \( k \) in the expression of \( S_{\{k,r\}} \) mentioned in the definition 1 we find that

\[
S_{\{\frac{q-1}{2}, \frac{q+1}{2}\}}(X) = - \sum_{x \in X} (p(x))^{q} \frac{(p(x))^{1-q} - 1}{1 - q} = S_q(X),
\]

which is the Tsallis entropy discussed in the literature [6, 12].

**Definition 2.** (Joint entropy) Let \( \mathcal{P} = \{ p(x,y) \}_{(x,y) \in (X,Y)} \) be a probability distribution of the joint random variable \((X,Y)\). The generalised Tsallis joint entropy of the joint random variable \((X,Y)\) is defined by

\[
S_{\{k,r\}}(X,Y) = - \sum_{x \in X} \sum_{y \in Y} (p(x,y))^{k+r+1} \ln_{\{k,r\}}(p(x,y)).
\]

An interesting and ambiguous fact is non-uniqueness of the modification of the generalised Tsallis entropy. A number of similar expressions yields the Tsallis entropy for specific values of the parameter for instance,
and many others. Here we propose the definition 1 as an ideal expression. One may derive similar kinds of properties for any of the other expressions. But this ambiguity offers a benefit in the definition of generalised Tsallis conditional entropy discussed in the following paragraph.

Recall that the probability distribution of conditional random variable \( Y|x \) is given by \( p(Y|x) = \frac{p(X,Y)}{p(X=x)} \), that is \( p(x,y) = p(y|x)p(x) \). Now we define the conditional entropy as follows:

**Definition 3.** (Conditional entropy) We define the generalised Tsallis conditional entropy as

\[
S_{(k,r)}(Y|X) = - \sum_{x \in X} (p(x))^{2k+1} S_{(k,r)}(Y|X=x)
\]

\[
= - \sum_{x \in X} (p(x))^{2k+1} \sum_{y \in Y} (p(y|x))^{k+r+1} \ln_{(k,r)}(p(y|x))
\]

\[
= - \sum_{x \in X} \sum_{y \in Y} (p(x))^{2k+1} (p(y|x))^{k+r+1} \ln_{(k,r)}(p(y|x)).
\]

The definitions of the generalised Tsallis joint entropy and conditional entropy are consistent to the definitions of Tsallis joint and conditional entropy [12], respectively. Note that,

\[
S\{\frac{1}{q} - \frac{1}{q}\}(X, Y) = - \sum_{x \in X} \sum_{y \in Y} (p(x,y))^q \ln_q(p(x,y)) = S_q(X, Y),
\]

which is the Tsallis joint entropy. In addition,

\[
S\{\frac{1}{q} - \frac{1}{q}\}(X|Y) = - \sum_{x \in X} \sum_{y \in Y} (p(x,y))^q \ln_q(p(x|y)) = S_q(X|Y),
\]

which is the Tsallis conditional entropy.

**Lemma 6.** Given two independent random variables \( X \) and \( Y \) the generalised Tsallis conditional entropy can be expressed as

\[
S_{(k,r)}(Y|X) = S_{(k,r)}(Y) - 2kS_{(k,r)}(X)S_{(k,r)}(Y).
\]

**Proof.** From the definition of conditional entropy we find that

\[
S_{(k,r)}(Y|X) = - \sum_{x \in X} (p(x))^{2k+1} \sum_{y \in Y} (p(y|x))^{r+k+1} \ln_{(k,r)}(p(y|x)).
\]

Considering \( r < 0 \) in the expression of \( \ln_{(k,r)}(x) \) we can write \( \ln_{(k,r)}(p(x)) = \frac{(p(x))^{2k-1}}{2k(p(x))^{r+k}} \) that is \( (p(x))^{2k} = 1 + 2k(p(x))^{r+k} \ln_{(k,r)}(p(x)) \). Putting it in the above equation we construct \( S_{(k,r)}(Y|X) = \)

\[
- \sum_{x \in X} (p(x)) \left[ 1 + 2k(p(x))^{r+k} \ln_{(k,r)}(p(x)) \right] \sum_{y \in Y} (p(y|x))^{r+k+1} \ln_{(k,r)}(p(y|x)).
\]
For independent random variables $X$ and $Y$ we have $p(y|x) = p(y)$. Therefore, $S_{\{k,r\}}(Y|X) =$

$$ - \sum_{x \in X} (p(x)) \left[ 1 + 2kp(x)^{r+k} \ln_{\{k,r\}}(p(x)) \right] $$

$$ \times \sum_{y \in Y} (p(y))^{r+k+1} \ln_{\{k,r\}}(p(y)) $$

$$ = - \sum_{x \in X} (p(x)) \sum_{y \in Y} (p(y))^{r+k+1} \ln_{\{k,r\}}(p(y)) $$

$$ - \sum_{x \in X} 2kp(x)^{r+k+1} \ln_{\{k,r\}}(p(x)) \sum_{y \in Y} (p(y))^{r+k+1} \ln_{\{k,r\}}(p(y)) $$

$$ = S_{\{k,r\}}(Y) - 2kS_{\{k,r\}}(X)S_{\{k,r\}}(Y). \tag{27} $$

**Theorem 3.** *Chain rule for generalised Tsallis entropy* Given any two random variables $X$ and $Y$ the generalised Tsallis joint entropy can be expressed as

$$ S_{\{k,r\}}(X,Y) = S_{\{k,r\}}(X) + S_{\{k,r\}}(Y|X). $$

**Proof.** The probability of joint random variables can be expressed as $p(x, y) = p(x)p(y|x)$. The product rule of $\ln_{\{k,r\}}(x)$ mentioned in lemma 1 indicates that

$$ (p(x)p(y|x))^{r+k} \ln_{\{k,r\}}(p(x)p(y|x)) $$

$$ = p(x)^{r+k} \ln_{\{k,r\}}(p(x)) + p(y|x)^{r+k} \ln_{\{k,r\}}(p(y|x)) $$

$$ + 2kp(x)^{r+k}p(y|x)^{r+k} \ln_{\{k,r\}}(p(x)) \ln_{\{k,r\}}(p(y|x)). \tag{28} $$

Applying $p(x, y) = p(x)p(y|x)$ we find that

$$ (p(x,y))^{r+k} \ln_{\{k,r\}}(p(x,y)) $$

$$ = p(x)^{r+k} \ln_{\{k,r\}}(p(x)) + p(y|x)^{r+k} \ln_{\{k,r\}}(p(y|x)) $$

$$ + 2kp(x)^{r+k}p(y|x)^{r+k} \ln_{\{k,r\}}(p(x)) \ln_{\{k,r\}}(p(y|x)) \tag{29} $$

Considering $r < 0$ in the expression of $\ln_{\{k,r\}}(x)$ we find that $\ln_{\{k,r\}}(p(x)) = \frac{(p(x))^2k-1}{(2k)(p(x))^{r+k}}$ that is $(p(x))^{2k} = 1 + 2kp(x)^{r+k} \ln_{\{k,r\}}(p(x))$. Putting it in the above equation we construct

$$ (p(x,y))^{r+k} \ln_{\{k,r\}}(p(x,y)) $$

$$ = p(x)^{r+k} \ln_{\{k,r\}}(p(x)) + (p(x))^{2k}p(y|x)^{r+k} \ln_{\{k,r\}}(p(y|x)). \tag{30} $$


Multiplying both sides by $p(x, y)$ and summing over $X$ and $Y$ we get

$$- \sum_{x \in X} \sum_{y \in Y} p(x, y)^{r+k+1} \ln_{(k,r)}(p(x, y))$$

$$= - \sum_{x \in X} \sum_{y \in Y} p(x, y)p(x)^{r+k} \ln_{(k,r)}(p(x))$$

$$- \sum_{x \in X} \sum_{y \in Y} p(x, y)(p(x))^{2k}(p(y|x))^{r+k} \ln_{(k,r)}(p(y|x)).$$

Using definition 2 and 3 we find

$$S_{\{k,r\}}(X, Y) = - \left[ \sum_{x \in X} p(x)^{r+k+1} \ln_{(k,r)}(p(x)) \right] \left[ \sum_{y \in Y} p(y|x) \right]$$

$$- \sum_{x \in X} \sum_{y \in Y} (p(x))^{2k+1}(p(y|x))^{r+k+1} \ln_{(k,r)}(p(y|x))$$

or $S_{\{k,r\}}(X, Y) = S_{\{k,r\}}(X) + S_{\{k,r\}}(Y|X).$  \( \square \)

For two independent random variables $X$ and $Y$ the lemma 6 and theorem 3 together indicates the sub-additive property for generalised Tsallis entropy which is

$$S_{\{k,r\}}(X, Y) = S_{\{k,r\}}(X) + S_{\{k,r\}}(Y) - 2kS_{\{k,r\}}(X)S_{\{k,r\}}(Y).$$  \(33\)

Putting $k = \frac{q-1}{2}$, and $r = -\frac{q-1}{2}$ in the above equation we find

$$S_q(X, Y) = S_q(X) + S_q(Y) + (q-1)S_q(X)S_q(y),$$

which is the sub-additive property of Tsallis entropy [12].

### 4 Fundamental properties of generalised Tsallis relative entropy

In Shannon information theory, the relative entropy, or Kullback-Leibler divergence is a measure of difference between two probability distributions. Recall that given two probability distributions $P = \{p(x)\}_{x \in X}$ and $Q = \{q(x)\}_{x \in X}$ the Kullback-Leibler divergence [11] is defined by

$$D(P||Q) = \sum_{x \in X} p(x) \ln \left( \frac{p(x)}{q(x)} \right) = - \sum_{x \in X} p(x) \ln \left( \frac{q(x)}{p(x)} \right).$$  \(35\)

We generalize it in terms of generalised Tsallis entropy as follows:
Definition 4. (generalised Tsallis relative entropy) Given two probability distributions \( P = \{ p(x) \}_{x \in X} \) and \( Q = \{ q(x) \}_{x \in X} \) the generalised Tsallis relative entropy is given by

\[
D_{\{k, r\}}(P \| Q) = \sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{r+k} \ln_{\{k, r\}} \left( \frac{q(x)}{p(x)} \right)
= -\sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{r+k} \ln_{\{k, r\}} \left( \frac{q(x)}{p(x)} \right).
\]

The equivalence between two expressions of \( D_{\{k, r\}}(P \| Q) \) follows from corollary 1. Putting \( k = \frac{q-1}{r} \), and \( r = -\frac{q-1}{q} \ln - \sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{r+k} \ln_{\{k, r\}} \left( \frac{q(x)}{p(x)} \right) \) we find

\[
D_{\{\frac{q-1}{r}, \frac{q-1}{q}\}} = -\sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{q-1} - 1 = D_q(P \| Q), \quad (36)
\]

which is the Tsallis relative entropy defined in [13]. Now we discuss a few properties of the generalised Tsallis divergence.

Lemma 7. (Nonnegativity) For any two probability distribution \( P \) and \( Q \) the generalised Tsallis relative entropy \( D_{\{k, r\}}(P \| Q) \geq 0 \). The equality holds for \( P = Q \).

Proof. Lemma 4 suggests that \(-x^{k+r} \ln_{\{k, r\}}(x)\) is a convex function for all \( x \geq 0 \) and \( 0 \leq k \leq \frac{1}{2} \). Therefore,

\[
D_{\{k, r\}}(P \| Q) = -\sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{r+k} \ln_{\{k, r\}} \left( \frac{q(x)}{p(x)} \right)
\geq -\sum_{x \in X} p(x) \left( \frac{q(x)}{p(x)} \right)^{r+k} \ln_{\{k, r\}} \left( \sum_{x \in X} p(x) \right).
\]

Now, \( \ln_{\{k, r\}} \left( \sum_{x \in X} p(x) \frac{q(x)}{p(x)} \right) = \ln_{\{k, r\}} \left( \sum_{x \in X} q(x) \right) = \ln_{\{k, r\}}(1) = 0 \). Note that, if \( P = Q \) then

\[
D_{\{k, r\}}(P \| P) = -\sum_{x \in X} p(x) \frac{p(x)}{p(x)}^{r+k} \ln_{\{k, r\}} \left( \frac{p(x)}{p(x)} \right) = -\sum_{x \in X} p(x) \ln_{\{k, r\}}(1) = 0,
\]

as \( \ln_{\{k, r\}}(1) = 0 \). \qed

Lemma 8. (Symmetry) Let \( P' = \{ p'_i \} \) and \( Q' = \{ q'_i \} \) be two probability distributions, such that, \( p(x)' = p_{\pi(i)} \) and \( q(x)' = q_{\pi(i)} \) for a permutation \( \pi \) and probability distributions \( P = \{ p(x) \}_{x \in X} \) and \( Q = \{ q(x) \}_{x \in X} \). Then \( D_{\{k, r\}}(P' \| Q') = D_{\{k, r\}}(P \| Q) \).
Proof. The permutation \( \pi \) alters the position of \( p(x) \left( \frac{p(x)}{\eta(x)} \right)^{r-k} \ln \{k,r\} \left( \frac{p(x)}{\eta(x)} \right) \) under addition and keeps the sum \( D_{\{k,r\}}(P||Q) \), unaltered. Hence, the proof follows trivially. \( \square \)

**Lemma 9.** (Possibility of extension) Let \( \mathcal{P}' = \mathcal{P} \cup \{0\} \) and \( \mathcal{Q}' = \mathcal{Q} \cup \{0\} \), then \( D_{\{k,r\}}(\mathcal{P}'||\mathcal{Q}') = D_{\{k,r\}}(\mathcal{P}||\mathcal{Q}) \).

**Proof.** Define \( 0 \left( \frac{q}{p} \right)^{r+k} \ln \{k,r\} \left( \frac{q}{p} \right) = \lim_{x \to 0} x \left( \frac{q}{p} \right)^{r+k} \ln \{k,r\} \left( \frac{q}{p} \right) \). Expanding logarithm of \( x \left( \frac{q}{p} \right)^{r+k} \ln \{k,r\} \left( \frac{q}{p} \right) \) we find

\[
x \left( \frac{y}{x} \right)^{r+k} \ln \{k,r\} \left( \frac{y}{x} \right) = x \frac{y^{2k} - x^{2k}}{2kx^{2k}}.
\]  

(39)

Hence, we find \( \lim_{x \to 0} \lim_{y \to 0} x \left( \frac{y}{x} \right)^{r+k} \ln \{k,r\} \left( \frac{y}{x} \right) = 0 \). In addition, we have \( \lim_{y \to 0} \lim_{x \to 0} x \left( \frac{y}{x} \right)^{r+k} \ln \{k,r\} \left( \frac{y}{x} \right) = 0 \). Now applying Moore-Osgood theorem \([14]\) we find that \( \lim_{(x,y) \to (0,0)} x \left( \frac{y}{x} \right)^{r+k} \ln \{k,r\} \left( \frac{y}{x} \right) = 0 \). Therefore, \( 0 \ln \{k,r\} \left( \frac{0}{y} \right) = 0 \). Hence, \( D_{\{k,r\}}(\mathcal{P}'||\mathcal{Q}') = D_{\{k,r\}}(\mathcal{P}||\mathcal{Q}) \). \( \square \)

Given two probability distributions \( \mathcal{P} = \{p(x)\}_{x \in X} \) and \( \mathcal{Q} = \{q(y)\}_{y \in Y} \) we can define a joint probability distribution \( \mathcal{P} \otimes \mathcal{Q} = \{p(x)q(y)\}_{(x,y) \in X \times Y} \). Note that, for all \( x \in X \) and \( y \in Y \) we have \( 0 \leq p(x)q(y) \leq 1 \). In addition, \( \sum_{x \in X} \sum_{y \in Y} p(x)q(y) = 1 \). Now, we have the following theorem.

**Theorem 4.** (Pseudo-additivity) Given probability distributions \( \mathcal{P}^{(1)} = \{p^{(1)}(x)\}_{x \in X} \), \( \mathcal{Q}^{(1)} = \{q^{(1)}(x)\}_{x \in X} \), \( \mathcal{P}^{(2)} = \{p^{(2)}(y)\}_{y \in Y} \) and \( \mathcal{Q}^{(2)} = \{q^{(2)}(y)\}_{y \in Y} \) we have

\[
D_{\{k,r\}}(\mathcal{P}^{(1)} \otimes \mathcal{P}^{(2)}||\mathcal{Q}^{(1)} \otimes \mathcal{Q}^{(2)}) = D_{\{k,r\}}(\mathcal{P}^{(1)}||\mathcal{Q}^{(1)}) + D_{\{k,r\}}(\mathcal{P}^{(2)}||\mathcal{Q}^{(2)}) - 2kD_{\{k,r\}}(\mathcal{P}^{(1)}||\mathcal{Q}^{(1)})D_{\{k,r\}}(\mathcal{P}^{(2)}||\mathcal{Q}^{(2)}).
\]

**Proof.** Recall the product rule of \( \ln \{k,r\}(xy) \) mentioned in the lemma 1. Expanding the logarithm we find

\[
\ln \{k,r\} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)p^{(2)}(y)} \right)^{r+k} \ln \{k,r\} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right) + 2k \ln \{k,r\} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)^{r+k} \ln \{k,r\} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right).
\]

(40)
Multiplying $p^{(1)}(x)p^{(2)}(y)$ with both side we find

$$-p^{(1)}(x)p^{(2)}(y) \left( \frac{q^{(1)}(x)q^{(2)}(y)}{p^{(1)}(x)p^{(2)}(y)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(1)}(x)q^{(2)}(y)}{p^{(1)}(x)p^{(2)}(y)} \right)$$

$$= -p^{(1)}(x) \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right) p^{(2)}(y)$$

$$- p^{(2)}(y) \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right) p^{(1)}(x)$$

$$- 2k \times p^{(1)}(x) \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)$$

$$\times p^{(2)}(y) \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right).$$

Now, applying the definition 4 we find

$$D_{\{k,r\}}(P^{(1)} \otimes P^{(2)} || Q^{(1)} \otimes Q^{(2)})$$

$$= - \left[ \sum_{x \in X} p^{(1)}(x) \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right) \right] \left[ \sum_{y \in Y} p^{(2)}(y) \right]$$

$$- \left[ \sum_{y \in Y} p^{(2)}(y) \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right) \right] \left[ \sum_{x \in X} p^{(1)}(x) \right]$$

$$- 2k \times \left[ \sum_{x \in X} p^{(1)}(x) \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(1)}(x)}{p^{(1)}(x)} \right) \right]$$

$$\times \left[ \sum_{y \in Y} p^{(2)}(y) \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right)^{r+k} \ln_{k,r} \left( \frac{q^{(2)}(y)}{p^{(2)}(y)} \right) \right]$$

$$= D_{\{k,r\}}(P^{(1)} || Q^{(1)}) + D_{\{k,r\}}(P^{(2)} || Q^{(2)})$$

$$- 2k D_{\{k,r\}}(P^{(1)} || Q^{(1)}) D_{\{k,r\}}(P^{(2)} || Q^{(2)}).$$

Putting $k = \frac{q-1}{2}$ and $r = -\frac{q+1}{2}$ in the above result we find $D_{\phi}(P^{(1)} \otimes P^{(2)} || Q^{(1)} \otimes Q^{(2)})$.

$$D_{\phi}(P^{(1)} || Q^{(1)}) + D_{\phi}(P^{(2)} || Q^{(2)}) - (q-1)D_{\phi}(P^{(1)} || Q^{(1)}) D_{\phi}(P^{(2)} || Q^{(2)}),$$

which is the Pseudo-additive property of Tsallis relative entropy [12].

**Theorem 5. (Joint convexity) Let $P^{(k)} = \{p^{(k)}(x)\}_{x \in X}$ and $Q^{(k)} = \{q^{(k)}(x)\}_{x \in X}$ for $k = 1, 2$ are probability distributions. Construct new probability distributions $P^{(1)} + \lambda P^{(2)} = \{p^{(1)}(x) + \lambda p^{(2)}(x)\}_{x \in X}$, and $Q^{(1)} + \lambda Q^{(2)} = \{q^{(1)}(x) + \lambda q^{(2)}(x)\}_{x \in X}$ as convex combinations. Then

$$D_{\{k,r\}}(P^{(1)} + \lambda P^{(2)} || Q^{(1)} + \lambda Q^{(2)}) \leq D_{\{k,r\}}(P^{(1)} || Q^{(1)}) + \lambda D_{\{k,r\}}(P^{(1)} || Q^{(1)}).$$

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Proof. Note that, \( D_{\{k,r\}}(P^{(1)} + \lambda P^{(2)} \parallel Q^{(1)} + \lambda Q^{(2)}) = \)

\[
\sum_{x \in X} (p^{(1)}(x) + \lambda p^{(2)}(x)) \left( \frac{p^{(1)}(x) + \lambda p^{(2)}(x)}{q^{(1)}(x) + \lambda q^{(2)}(x)} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{p^{(1)}(x) + \lambda p^{(2)}(x)}{q^{(1)}(x) + \lambda q^{(2)}(x)} \right).
\]

(44)

Now applying the lemma 2 we find

\[
\begin{align*}
(p^{(1)}(x) + \lambda p^{(2)}(x)) & \left( \frac{p^{(1)}(x) + \lambda p^{(2)}(x)}{q^{(1)}(x) + \lambda q^{(2)}(x)} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{p^{(1)}(x) + \lambda p^{(2)}(x)}{q^{(1)}(x) + \lambda q^{(2)}(x)} \right) \\
\leq & p^{(1)}(x) \left( \frac{p^{(1)}(x)}{q^{(1)}(x)} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{p^{(1)}(x)}{q^{(1)}(x)} \right) \\
& + \lambda p^{(2)}(x) \left( \frac{p^{(2)}(x)}{q^{(2)}(x)} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{p^{(2)}(x)}{q^{(2)}(x)} \right).
\end{align*}
\]

(45)

Summing over \( x \) we find the result.

\[ \square \]

Before proceeding farther, we make a change of notations from now on. Let \( X \) be a random variable with outcomes \((x_1, x_2, \ldots, x_n)\). We represent a probability distribution \( P = \{p(x)\}_{x \in X} \) by a finite sequence as \( P = \{p_i : p_i = p(x_i), \sum_{i=1}^{n} p_i = 1, 0 \leq p_i \leq 1\} \). Now consider a transition probability matrix \( W = (w_{j,i})_{m \times n} \) such that \( \sum_{j=1}^{m} w_{j,i} = 1 \) for all \( i = 1, 2, \ldots, n \). Let \( P = \{p_i^{(in)}\}_{i=1}^{n} \) and \( Q = \{q_i^{(in)}\}_{i=1}^{n} \) be two probability distributions. After a transition with \( W \), the new probability distributions are \( WP = \{p_j^{(out)}\}_{j=1}^{m} \) and \( WQ = \{q_j^{(out)}\}_{j=1}^{m} \), where \( p_j^{(out)} = \sum_{i=1}^{n} w_{j,i} p_i^{(in)} \), and \( q_j^{(out)} = \sum_{i=1}^{n} w_{j,i} q_i^{(in)} \). Now, we have the following theorem.

Theorem 6. (Information monotonicity in general) Given probability distributions \( P, Q \) and transition probability matrix \( W \) we have \( D_{\{k,r\}}(WP \parallel WQ) \leq D_{\{k,r\}}(P \parallel Q) \).

Proof. Modifying the notations in definition 4 we find that \( D_{\{k,r\}}(WP \parallel WQ) = \)

\[
\sum_{j=1}^{m} p_j^{(out)} \left( \frac{p_j^{(out)}}{q_j^{(out)}} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{p_j^{(out)}}{q_j^{(out)}} \right) \\
= \sum_{j=1}^{m} \left[ \sum_{i=1}^{n} w_{j,i} p_i^{(in)} \left( \frac{\sum_{i=1}^{n} w_{j,i} p_i^{(in)}}{\sum_{i=1}^{n} w_{j,i} q_i^{(in)}} \right)^{r-k} \ln_{\{k,r\}} \left( \frac{\sum_{i=1}^{n} w_{j,i} p_i^{(in)}}{\sum_{i=1}^{n} w_{j,i} q_i^{(in)}} \right) \right].
\]

(46)
Now, theorem 2 suggests that $D_{\{k,r\}}(W|Q)\leq D_{\{k,r\}}(P|Q)$.

In theorem 6, if the probability transition matrix $W = (w_{ij})_{m \times n}$ has $m < n$, then $W$ partitions the random variable $X = (x_1, x_2, \ldots, x_n)$ into $m$ groups $G_1, G_2, \ldots, G_m$ such that $X = \bigcup_{j=1}^m G_j$, and $G_k \cap G_l = 0$. Then $p_j^{(\text{out})}(G_j) = \sum_{x_i \in G_j} p_{i}^{(\text{in})}$. Now the theorem 6 indicates $D(W|Q) \leq D(P|Q)$, which is formally mentioned as information monotonicity.

5 Information geometric aspects

Recall that, $X$ be a random variable with outputs $(x_1, x_2, \ldots, x_n)$ with probabilities $P = (p_1, p_2, \ldots, p_n)$. Note that, the set of all probability distributions defined on $X$ forms a manifold with local coordinate system $P = (p_1, p_2, \ldots, p_n)$. We call it statistical manifold which is precisely given by

$$S = \{P : P = (p_1, p_2, \ldots, p_n), 0 \leq p_i \leq 1, \sum_{i=1}^n p_i = 1\}. \quad (48)$$

In information geometry [15, 16], A function $D(P||Q)$ for $P, Q \in S$ is called divergence if it fulfills the following conditions:

1. $D(P||Q) \geq 0$.
2. $D(P||Q) \geq 0$ if and only if $P = Q$.
3. For small $dP$ we have $D(P + dP||P) \approx \frac{1}{2} \sum g_{ij} dp_i dp_j$, forms a positive-definite quadratic form.

Recall that, the manifold $S$ is Riemannian, when a symmetric positive-definite tensor $g_{ij}(P)$ is defined on $S$ such that the squared length of a small line element $dP$ is given by

$$ds^2 = \sum g_{ij} dp_i dp_j. \quad (49)$$

Hence, the Riemannian metric induced by a divergence $D$ is

$$g_{ij}(P) = \frac{\partial^2}{\partial p_i \partial p_j} D_{\{k,r\}}(P|Q)|_{Q=P}. \quad (50)$$
Therefore, the length of small line segment is given by
\[ ds^2 = \frac{1}{2} D(P||P + dP). \] (51)

In this article, we consider
\[ D_{(k,r)}(P||Q) = \sum_{i=1}^{n} p_i \left( \frac{p_i}{q_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{p_i}{q_i} \right). \] (52)

Now,
\[ \frac{\partial}{\partial p_i} D_{(k,r)}(P||Q) = \frac{\partial}{\partial p_i} \left[ p_i \left( \frac{p_i}{q_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{p_i}{q_i} \right) \right] \]
\[ = \left( (2r+1) \left( \left( \frac{p_i}{q_i} \right)^{2k} - 1 \right) + 2k \right) \left( \frac{p_i}{q_i} \right)^{2r-2k} \]
\[ \frac{\partial^2}{\partial^2 p_i} D_{(k,r)}(P||Q) = \frac{\partial^2}{\partial^2 p_i} \left[ p_i \left( \frac{p_i}{q_i} \right)^{r-k} \ln_{(k,r)} \left( \frac{p_i}{q_i} \right) \right] \]
\[ = \left( r(2r+1) \left( \left( \frac{p_i}{q_i} \right)^{2k} - 1 \right) - 2k^2 + 4kr + k \right) \left( \frac{p_i}{q_i} \right)^{2r-2k} \]
\[ \frac{\partial^2}{\partial p_j \partial p_i} D_{(k,r)}(P||Q) = 0. \] (53)

The above calculation indicates \( G = (g_{ij})_{n \times n} \) where
\[ g_{ij} = \begin{cases} \frac{-2k+4r+1}{p_i}, & \text{for } i = j \\ 0, & \text{for } i \neq j. \end{cases} \] (54)

The matrix \( G \) is also called the Fisher information matrix.

**Theorem 7.** The statistical manifold induced by the generalised Tsallis relative entropy is Hessian.

**Proof.** A manifold is called Hessian if there is a function \( \Psi(u) \) such that \( g_{ij}(P) = \partial_{ij}(\Psi) \). For \( i = j \) we have \( \partial_{ii}(\Psi) = g_{ii}(u) = \frac{1-2k+4r}{u} \). Integrating twice we find
\[ \Psi_{ii}(u) = c_2 + u(c_1 + 2k - 4r - 1) + (-2k + 4r + 1)u \log(u), \] (55)
where \( c_1 \) and \( c_2 \) are integrating constants. For \( i \neq j \) we have \( \partial_{ii} = g_{ij} = 0 \), that is \( \Psi(u) = c_1u + c_2 \). Hence, the statistical manifold is Hessian. \( \square \)

6 Conclusion and open problems

In recent years, the idea of entropy is generalized in the context of thermodynamics, information theory, and dynamical systems with the help of advanced
mathematical tools [17, 18]. It offers a broad scope of mathematical investigations. The Tsallis entropy has been widely utilized in different branches of science and engineering [19, 20, 21, 22, 23]. This article is a detailed description of the characteristics of generalised Tsallis relative entropy. Here, we propose a modification in the definition of the Sharma-Mittal entropy, such that, the new entropy fulfils the chain rule. Similarly, we modify the definitions of Sharma-Mittal joint entropy, conditional entropy, and relative entropy. We establish a number of characteristics of the generalised Tsallis divergence, which make it efficient to be utilized in classical information theory. Also, we justify that the statistical manifold induced by the generalised Tsallis relative entropy is Hessian. The following problems may be discussed in future:

1. In Shannon information theory, the mutual information of two random variables $X$ and $Y$ is defined by $I(X; Y) = D(p(x, y)|p(x)p(y))$, which is the Kullback-Leibler divergence between two probability distributions $p(x, y)$ and $p(x)p(y)$. In case of generalised Tsallis entropy, one may introduce the mutual information $I_{\{k,r\}}(X; Y) = D_{\{k,r\}}(p(x, y)|p(x)p(y))$ then investigates its properties. Moreover, the mutual information has a crucial role in the literature of data processing inequalities. Hence, two parameter deformation of data-processing inequalities will be very crucial in this direction.

2. In Shannon information theory, it is proved that

$$I(X; Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X, Y), \quad (56)$$

where $H$ denotes the Shannon entropy. As the generalised Tsallis mutual information is not well proposed we may define mutual entropy as

$$I_{\{k,r\}}(X; Y) = S_{\{k,r\}}(Y) - S_{\{k,r\}}(Y|X)$$

$$= S_{\{k,r\}}(X) + S_{\{k,r\}}(Y) - S_{\{k,r\}}(X, Y). \quad (57)$$

Note that, here we do not assign the term mutual information [24]. Although, it is used as relative entropy for various applications [25]. In quantum information theory, these identities generates quantum discord, which is a well known quantum correlation. There are a few works discussing the deformation of quantum discord in terms of Tsallis [26], Renyi [27], and generalised Tsallis entropy [28, 29]. There is a scope for further investigation in this direction.

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