Harmonic maps and black holes

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Abstract. We address two applications of existence and uniqueness properties of harmonic maps to the theory of stationary and axisymmetric electro-vacuum black holes. More specifically, we will consider: (1) The classification of such black hole space-times and (2) the proof of a Dain inequality with charge.

A map

$$\Phi : M \rightarrow N$$

between two Riemannian manifolds $(M,g)$ and $(N,h)$ is said to be harmonic if, given local coordinates, $\Phi = (\phi^1, \ldots, \phi^n)$ is a critical point of

$$\int_{\Omega} h_{ij} g^{\mu \nu} \partial_\mu \phi^i \partial_\nu \phi^j d\mu_g,$$

for every conditionally compact coordinate domain $\Omega$ of $M$. In these local coordinates the Euler-Lagrange equations read

$$\Delta_g \phi^i + \Gamma^i_{jk} g^{\mu \nu} \partial_\mu \phi^j \partial_\nu \phi^k = 0.$$

Finite energy harmonic maps were a focus of attention in the 1970s; here we will be using the results of Hildebrandt et al [8]. Later, motivated by the many black hole equilibrium problem in general relativity, Weinstein extensively studied the Dirichlet problem for harmonic maps with prescribed singularities; here we will consider the results in [10].

1. The classification of stationary space-times

Conjecture 1.1 Let $(M, g, F)$ be a stationary, asymptotically flat, electro-vacuum, four-dimensional regular space-time. Then the domain of outer communications $(\langle M_{\text{ext}} \rangle)$ is either isometric to the domain of outer communications of a Kerr-Newman space-time or to the domain of outer communications of a (standard) Majumdar-Papapetrou space-time.

When addressing the axisymmetric case it is possible to construct a global representation for the domain of outer communications, away from the axis of rotation, in terms of Weyl coordinates [2, 5]. In such privileged coordinates the metric takes the block diagonal form

$$g = -\rho^2 e^{2u} dt^2 + e^{-2u}(d\varphi - w dt)^2 + e^{2\lambda} (d\rho^2 + dz^2),$$

(1)

where $\eta = \partial_\varphi$ is the axial Killing vector, and is completely determined by an axisymmetric harmonic map

$$\Phi = (u, v, \chi, \psi) : \mathbb{R}^3 \setminus \{z = 0\} \rightarrow \mathbb{R}^2_C,$$
with energy
\[
H = \int_{\mathbb{R}^3} (Du)^2 + e^{4u} (Dv + \chi D\psi - \psi D\chi)^2 + e^{2u} ((D\chi)^2 + (D\psi)^2) ; \tag{2}
\]
the relevant electromagnetic potentials are defined by
\[
\partial_\alpha \chi = F_{\mu\alpha} \eta^\mu , \quad \partial_\alpha \psi = *F_{\mu\alpha} \eta^\mu , \quad 2dv = \chi d\psi - \psi d\chi - *(d\eta^\flat \wedge \eta^\flat) . \tag{3}
\]
Given any set of axis data, for instance prescribed Poincaré and Maxwell charges for each horizon component, we construct a reference axisymmetric (not necessarily harmonic) map \( \Phi_0 \) modelled on the local behavior of the relevant Kerr-Newman maps near each component of the horizon and at a neighborhood of infinity. Then, there exists a unique harmonic map \( \Phi \) whose hyperbolic distance satisfies \[10\]
\[
d(\Phi, \Phi_0) \in L^\infty(\mathbb{R}^3 \setminus \{ z = 0 \}) , \quad d(\Phi, \Phi_0) \to_{r \to +\infty} 0 . \tag{4}
\]
From such harmonic maps it is always possible to construct a stationary and axisymmetric solution of the source free Einstein-Maxwell field equations \[10\]. Such (not necessarily \( I^+ \)-regular) space-times will be referred to as Weinstein solutions. We then have the following partial realization of Conjecture 1.1 \[2, 5\]:

**Theorem 1.2** Let \( (M, g, F) \) be a stationary, asymptotically flat, \( I^+ \)-regular, electro-vacuum, four-dimensional analytic space-time. If each component of the event horizon is mean non-degenerate, then \( \langle \langle M_{\text{ext}} \rangle \rangle \) is isometric to the domain of outer communications of a Weinstein solution. In particular, if the event horizon is connected and mean non-degenerate, then \( \langle \langle M_{\text{ext}} \rangle \rangle \) is isometric to the domain of outer communications of a Kerr-Newman space-time.

2. A Dain inequality with charge

The formation of event horizons is expected to occur generically in large families of space-times, with the exterior region approaching a Kerr-Newman metric asymptotically with time. This is the essence of the “standard picture of gravitational collapse” \[9\]. Since mass \( m \) is non-increasing, Maxwell charges \( Q_E \) and \( Q_B \) are conserved quantities, and angular momentum \( \vec{J} \) is also conserved if one further assumes axisymmetry, the inequality
\[
m \geq \sqrt{\frac{\|\vec{J}\|^2}{m^2} + Q_E^2 + Q_B^2} , \tag{5}
\]
follows for initial data for such a collapse. Besides its intrinsic value, such inequalities provide some support for this “standard picture”, and in particular for weak cosmic censorship.

Dain \[7\] proved an upper bound for angular-momentum in terms of the mass for a class of maximal, vacuum, axisymmetric initial data sets. Recently a generalized Dain inequality including electric and magnetic charges was obtained in \[3, 6\]. More precisely it was proven that:

**Theorem 2.1** Let \( (M, g, K, E, B) \) be an axisymmetric electro-vacuum three-dimensional initial data set, which is maximal (\( \text{tr}_gK = 0 \)) and such that \( M \) is the union of a compact set and two asymptotically flat ends \( M_1 \). Let \( m, \vec{J}, Q_E \) and \( Q_B \) denote respectively the ADM mass, the ADM angular momentum and the total electric and magnetic charges of \( M_1 \). Then, if \( M/U(1) \) is simply connected inequality 5 holds.
From the purely geometric and topological assumptions we obtain \[ g = e^{-2U + 2\alpha} \left( d\rho^2 + dz^2 \right) + \rho^2 e^{-2U} \left( d\varphi + \rho W d\rho + Wz dz \right)^2 , \] (6)

where \( \eta = \partial_\varphi \) is again the generator of axial symmetry whose axis \( \{ \eta = 0 \} \supset \{ \rho = 0, z \neq 0 \} \). This is a global representation for one of the asymptotic flat ends, say \( M_1 \), with the remaining end \( M_2 \) represented by the “puncture” \( \{ \rho = z = 0 \} \). Moreover, the analysis of [1] provides detailed information concerning the asymptotic behavior of the fields involved. For instance, the existence of multiple ends manifests itself via

\[ U = 2 \log r + O(1) , \text{ for small } r = \sqrt{\rho^2 + z^2} . \] (7)

Note that for extreme Kerr-Newman, for which it is impossible to construct initial data with multiple ends, one has instead

\[ \tilde{U} = \log r + O(1) , \text{ for small } r = \sqrt{\rho^2 + z^2} . \] (8)

Using the Einstein-Maxwell constraints we are then able to show that \([3, 6]\)

\[ m \geq \frac{1}{8\pi} I(U, v, \chi, \psi) , \]

where

\[ I := \int_{\mathbb{R}^3} (DU)^2 + \frac{4U}{\rho^4} (Dv + \chi D\psi - \psi D\chi)^2 + \frac{e^{2U}}{\rho^2} ((D\chi)^2 + (D\psi)^2) ; \] (9)

the electromagnetic potentials are the same as in (3). It is important to note that for stationary-axisymmetric data: \( m = \frac{1}{8\pi} I(U, v, \chi, \psi) \).

There exist striking similarities between this mass functional and the harmonic map energy defined in (2). In fact, using the rescaling \( U = u + \ln \rho \) we see that

\[ I(U, v, \chi, \psi) = H(u, v, \chi, \psi) + B(U) . \] (10)

Since \( I \) and \( H \) differ by a boundary term \( B \) they satisfy the same variational equations. It is also well known that stationary-axisymmetric solutions of the Einstein-Maxwell equations provide critical points of \( H \) and consequently of \( I \) as well.

So, let \( (\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi}) \) be the harmonic map associated to the extreme Kerr-Newman satisfying

\[ J_z = (v_2 - v_1)/8 \quad \text{and} \quad Q_E = (\psi_2 - \psi_1)/2 . \]

The desired result will follow immediately if one is able to prove that:

\[ I = I(U, v, \chi, \psi) \geq I(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi}) = \tilde{I} . \] (11)

Now, using the results in [1] and in view of this new goal we see that: asymptotic flatness provides the necessary control to estimate both \( I \) and \( \tilde{I} \) when the defining integrals are taken near infinity; near the axis, and away from both infinity and the origin, axisymmetry suffices; also, the necessary control near the puncture \( \{ \rho = z = 0 \} \) follows from the referred controlled behavior of the fields near the origin and the know behavior of extreme Kerr-Newman near its horizon, here exemplified by (7) and (8), respectively. Consequently we are left with the task of establishing (11) for “large” compact sets not intersecting the singular set \( \{ \rho = 0 \} \).

1 It should be noted that the this \( \rho \)-coordinate does not, in general, agree with the Weyl \( \rho \)-coordinate of the previous section.
With this in mind let \( \{ A_{\sigma,\epsilon} \} \) be a family of such compact sets saturating \( \mathbb{R}^3 \) and construct associated auxiliary fields
\[
(U_{\sigma,\epsilon}, v_{\sigma,\epsilon}, \chi_{\sigma,\epsilon}, \psi_{\sigma,\epsilon}),
\]
converging pointwise to \( (U, v, \chi, \psi) \), as \( \sigma, \epsilon \to 0 \), and satisfying the same boundary conditions as the fixed extreme Kerr-Newman map, i.e.
\[
(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi}) = (U_{\sigma,\epsilon}, v_{\sigma,\epsilon}, \chi_{\sigma,\epsilon}, \psi_{\sigma,\epsilon}) \text{ on } \partial A_{\sigma,\epsilon}.
\]

Since \( \mathbb{H}^2_0 \) has negative sectional curvature, on the compact regions \( A_{\sigma,\epsilon} \) we see from [8] that: Minimizers of \( H \) with Dirichlet boundary conditions exist and are smooth solutions of the variational equations. It is also well known, see for instance [4, 10], that solutions of the boundary value problem are, in the case of negatively curved target manifolds, unique. Since the map associated to extreme Kerr-Newman is necessarily a solution we see that (recall (12))

\[
H_{A_{\sigma,\epsilon}}(u_{\sigma,\epsilon}, v_{\sigma,\epsilon}, \chi_{\sigma,\epsilon}, \psi_{\sigma,\epsilon}) > H_{A_{\sigma,\epsilon}}(\tilde{u}, \tilde{v}, \tilde{\chi}, \tilde{\psi}),
\]

where by \( H_{A_{\sigma,\epsilon}} \) we mean the integral (2) over the set \( A_{\sigma,\epsilon} \). From (10) and relying once more on (12) we finally obtain

\[
I_{A_{\sigma,\epsilon}}(U_{\sigma,\epsilon}, v_{\sigma,\epsilon}, \chi_{\sigma,\epsilon}, \psi_{\sigma,\epsilon}) > I_{A_{\sigma,\epsilon}}(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi}),
\]

which, together with a careful asymptotic analysis in the lines previously described, allows us to establish (11).

There seem to be relevant connections between the two addressed problems which go beyond the purely technical level. In fact: this charged Dain inequality provides strong evidence that extreme Kerr-Newman initial data gives rise to the unique minimum of mass for fixed angular momentum and charges, within a class of axisymmetric and asymptotically flat initial data for space-times containing at most one black-hole or two asymptotic flat ends. Since the Majumdar-Papapetrou metrics provide the existence of regular and extreme multiple black hole solutions we see that the last restriction concerning the number of black holes is not merely technical. On the other hand for pure vacuum it is expect to be superfluous [4]. Such improvements of the known results could help us remove the undesired degeneracy assumption from Theorem 2.1 or could provide new insights concerning the stationary-axisymmetric \( N \)-body problem in relativity.

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