Existence of positive solution of the Choquard equation

$$-\Delta u + a(x)u = (I_{\mu} * |u|^{2_{*}^\mu})|u|^{2_{*}^\mu - 2}u \text{ in } \mathbb{R}^N$$

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Abstract

In this paper we show existence of a positive solution to the problem

$$(P) \begin{cases} -\Delta u + a(x)u = (I_{\mu} * |u|^{2_{*}^\mu})|u|^{2_{*}^\mu - 2}u, \\ u > 0 \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $I_{\mu} = \frac{1}{|x|^\mu}$ is the Riesz potential, $0 < \mu < \min\{N, 4\}$ and $2_{*}^\mu = \frac{(2N - \mu)}{N - 2}$ with $N \geq 3$. In order to prove the main result, we used variational methods combined with a splitting theorem.

Key Words. Choquard equation, Variational methods, Critical exponents

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1 Introduction

In this paper we shall focus our attention on the existence of positive solutions for the following class of Choquard equation

$$(P) \begin{cases} -\Delta u + a(x)u = (I_{\mu} * |u|^{2_{*}^\mu})|u|^{2_{*}^\mu - 2}u \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $I_{\mu} = \frac{1}{|x|^\mu}$ is the Riesz potential, $0 < \mu < \min\{N, 4\}$, $2_{*}^\mu = \frac{(2N - \mu)}{N - 2}$ with $N \geq 3$ and $a(x)$ is a positive function satisfying some technical conditions.

The existence of solution for problem $(P)$ ensures the existence of standing waves solutions for a nonlinear Schrödinger equation of the kind

$$i\partial_t \Psi = -\Delta \Psi + W(x)\Psi - (I_{\mu} * |\Psi|^{2_{*}^\mu})|\Psi|^{2_{*}^\mu - 2}\Psi, \quad \text{in } \mathbb{R}^N, \quad \text{(1.1)}$$
where $W$ is the external potential and $I_\mu$ is the response function possesses information on the mutual interaction between the bosons. This type of nonlocal equation appears in a lot of physical applications, for instance in the study of propagation of electromagnetic waves in plasmas [9] and in the theory of Bose-Einstein condensation [13]. We recall that a standing wave solution is a solution of the type $\Psi(x,t) = u(x)e^{-iEt}$, which solves (1.1) if, and only if, $u$ solves the equation

$$-\Delta u + a(x)u = \left(\frac{1}{|x|^\mu} * |u|^{2^* - 2}\right)|u|^{2^* - 2}u \quad \text{in} \quad \mathbb{R}^N,$$

(1.2)

with $a(x) = W(x) - E$, which is a Choquard-Pekar equation.

After a bibliography review we have found only a paper related to (1.2) that is due to Du and Yang [14]. In that interesting paper, Du and Yang has considered only the case $a = 0$, and they showed that any positive solution of (1.2) with $a = 0$ must be of the form

$$\Psi_{\delta,y}(x) = C\left(\frac{\delta}{\delta^2 + |x-y|^2}\right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N,$$

for some $\delta > 0, y \in \mathbb{R}^N$, and $C > 0$ is a constant that only depends on $N$. Still related to (1.2), we would like to cite the paper due to Du, Gao and Yang [15] where the authors has studied existence and qualitative properties of solutions of the problem

$$-\Delta u = \frac{1}{|x|^\alpha} \left(\frac{1}{|x|^\mu} * |u|^{2^* - 2}\right)|u|^{2^* - 2}u \quad \text{in} \quad \mathbb{R}^N,$$

(1.3)

for some values of $\alpha$ and $\mu$. In that paper, the authors has proved an interesting version of the Concentration-Compactness principle due to Lions [23] that can be used for Choquard equations with critical growth, for more details see [15, Lemma 2.5]. On the other hand, there is a rich literature associated with Choquard-Pekar equation of the type

$$-\Delta u + V(x)u = K(x) \left(\frac{1}{|x|^\mu} * H(u)\right)h(u) \quad \text{in} \quad \mathbb{R}^N,$$

(1.4)

where $H(t) = \int_0^t h(s)ds$ with $V, K : \mathbb{R}^N \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ being continuous functions verifying some technical conditions, and reader can find some interesting results in [1, 2, 4, 5, 12, 20, 23, 24, 27, 28, 29, 30, 32] and their references.

The motivation of the present comes from of the seminal paper due to Benci and Cerami [7], where the authors studied the existence of solution for the following class of local critical problem

$$-\Delta u + a(x)u = |u|^{2^* - 2}u \quad \text{in} \quad \mathbb{R}^N,$$

(1.5)

by supposing that function $a : \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions below

(i) The function $a$ is positive in a set of positive measure.

(ii) $a \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$, with $p_2 < 3$ if $N = 3$.

(iii) $|a|_{L^{2^*/2}(\mathbb{R}^N)} < S(2^{2/N} - 1)$, where

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{2^*}^2},$$

with $2^* = \frac{2N}{N-2}$, $N \geq 3$, and

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{1}{2}}, \quad \forall u \in D^{1,2}(\mathbb{R}^N).$$
By using variational methods, the authors were able to prove the existence of a positive solution $u \in D^{1,2}(\mathbb{R}^N)$ with
\[ f(u) \in (S, 2^{2/N} S) \quad \text{and} \quad \int_{\mathbb{R}^N} |u|^2 \, dx = 1, \]
where $f : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is the functional given by
\[ f(u) = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + a(x)|u|^2 \right) \, dx. \]

The main difficulty to prove the existence of solution comes from the fact that the nonlinearity has a critical growth. To overcome this difficult, the authors used Variational methods, Deformation lemma, and the well known Concentration-Compactness principle due to Lions [23]. After the publication of [7], some authors studied related problem to (1.5), see for example, [3], [6], [8], [10], [25], [26], [31] and references therein.

As in the local case, see [7], in the present paper it was necessary to do a careful study of the energy functional $I_{\infty} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by
\[ I_{\infty}(u) = \frac{1}{2} \|u\|^2 - \frac{1}{22^\mu} \int_{\mathbb{R}^N} (I_{\mu} \ast |u|^{2^*_\mu})|u|^{2^*_\mu - 2} \, dx, \]
whose the critical points are weak solutions of the limit problem
\[ (P_{\infty}) \quad \left\{ \begin{array}{l} -\Delta u = (I_{\mu} \ast |u|^{2^*_\mu})|u|^{2^*_\mu - 2} \quad \text{in} \quad \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{array} \right. \]

Here, we show a nonlocal version of a technical lemma due to Struwe [33, Lemma 3.3], see Lemma 3.1 in Section 3, which permitted to use the Concentration-Compactness principal for limit case due to Lions [23] to establish a splitting theorem that is a key ingredient to deal with problems with critical exponent. We would like point out that to overcome the fact that we are working with a nonlocal problem, it was necessary to do some modifications in the proof of Lemma 3.1 for example, we developed a Cherrier type inequality that can be used for Choquard equation with critical growth and Neumann boundary condition, for more details see Lemma 2.2 in Section 3. The reader will observe that different from [15], we has used the original version of Concentration-Compactness principle due to Lions to show the existence of solution for (1.2). We believe that this is the first paper involving Choquard equation (1.2) with $a \neq 0$.

Before enunciating our theorem, we need to fix our conditions on function $a : \mathbb{R}^N \to [0, +\infty)$ and some notations:

1. The function $a$ is positive in a set of positive measure.
2. $a \in L^q([\mathbb{R}^N])$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N+2-\mu}{4N-2\mu} < p_2$ with $p_2 < \frac{N - \mu/2}{4 - N - \mu/2}$ if $N = 3$.
3. $|a|_{L^{(N-\mu)/2/(2-\mu/2)}([\mathbb{R}^N])} < S_{H,L}(2(4-\mu)/(2N-\mu) - 1)$, where
   \[ S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \frac{\|u\|^2}{\left( \int_{\mathbb{R}^N} (I_{\mu} \ast |u|^{2^*_\mu})|u|^{2^*_\mu - 2} \, dx \right)^{1/2^*_\mu}}. \]

We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of $(P)$, if $u \in D^{1,2}(\mathbb{R}^N)$ is a positive function such that for all $\varphi \in D^{1,2}(\mathbb{R}^N)$ we have
\[ \int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} a(x)u \varphi \, dx = \int_{\mathbb{R}^N} (I_{\mu} \ast |u|^{2^*_\mu})|u|^{2^*_\mu - 2}u \varphi \, dx. \]
In order to state the main result, we consider the functional of $C^1$ class $I : D^{1,2}({\mathbb{R}}^N) \to {\mathbb{R}}$ associated to problem $(P)$ given by

$$I(u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} a(x)u^2dx - \frac{1}{22^u} \int_{\mathbb{R}^N} (I_{\mu} * |u|^{2_0^*)}|u|^{2_0^*) dx$$

with

$$I'(u)v = \int_{\mathbb{R}^N} \nabla u \varphi dx + \int_{\mathbb{R}^N} a(x)u \varphi dx - \int_{\mathbb{R}^N} (I_{\mu} * |u|^{2_0^*)}|u|^{2_0^*) - 2u \varphi dx, \quad \forall \varphi \in D^{1,2}(\mathbb{R}^N).$$

Using the above notations, our main result has the following statement

**Theorem 1.1.** Assume that $(a_1) - (a_3)$ hold. Then, problem $(P)$ has a positive solution $u_0 \in D^{1,2}(\mathbb{R}^N)$ with

$$\frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} < I(u_0) < \frac{2(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

The paper is organized as follows: In Section 2, we prove some results involving the limit problem. In Section 3, we prove a splitting theorem and show some compactness results involving the energy functional associated with $(P)$. In Section 4, we make the proof of some technical lemmas that will be used in Section 5 to prove Theorem 1.1.

**2 Limit problem**

In this section, we will show important results involving the limit problem that are crucial in our approach. To begin with, we recall that to apply variational method, we must have

$$\left| \int_{\mathbb{R}^N} (I_{\mu} * |u|^{2_0^*)}|u|^{2_0^*) dx \right| < +\infty, \quad \forall u \in D^{1,2}(\mathbb{R}^N). \quad (2.1)$$

This fact is an immediate consequence of the Hardy-Littlewood-Sobolev inequality, which will be frequently used in this paper.

**Proposition 2.1.** [19] [Hardy – Littlewood – Sobolev inequality]: Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$. If $g \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(s, N, \mu, r)$, independent of $f, h$, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x - y|^\mu} \leq C(s, N, \mu, r) |g|_s |h|_r.$$

As a direct consequence of this inequality, we have that

$$\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_s}|u(y)|^{2_0^*)}{|x - y|^\mu} dxdy \right)^{\frac{2_{s}}{N-s}} \leq C(N, \mu)^{\frac{N-s}{2_{s}}} |u|_{2_s}^{2_s}, \quad \forall u \in D^{1,2}(\mathbb{R}^N). \quad (2.2)$$

The next result is a key point in our paper and its proof can be found in [19] Lemma 1.2

**Lemma 2.1.** The constant $S_{H,L}$ defined in (1.0) is achieved if, and only if,

$$u(x) = C \left( \frac{\delta}{\delta^2 + |x-y|^2} \right)^{N/2}, \quad \forall x \in \mathbb{R}^N,$$

where $C > 0$ is a fixed constant, $y \in \mathbb{R}^N$ and $\delta > 0.$
Our next result is a Cherrier type inequality involving the Riesz potential that will be a key point in the proof of Lemma 3.1, see Section 3.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^N \) a smooth bounded domain. For each \( \tau > 0 \), there is \( M_\tau > 0 \) such that

\[
\left( \frac{N}{2} - \tau \right) \int_{\Omega} (I_\mu * |v|^{2^*}) |v|^{2^*} \, dx \leq \left( C(N, \mu) |\nabla v|_{L^2(\Omega)}^2 + M_\tau |v|_{L^2(\Omega)}^2 \right)^{2^*}, \quad \forall v \in H^1(\Omega),
\]

where \( C(N, \mu) \) is given in \( \text{[23]} \).

**Proof.** From \( \text{[22]} \),

\[
\left( \frac{N}{2} - \tau \right) \int_{\Omega} (I_\mu * |v|^{2^*}) |v|^{2^*} \, dx \leq \left( C(N, \mu) \left( \frac{N}{2} - \tau \right) |v|_{L^{2^*}(\Omega)}^{2^*} \right)^{2^*}.
\]

Now, we apply the Cherrier’s inequality \( \text{[11]} \) to get the desired result. \( \square \)

Our first result establishes preliminary properties involving \((PS)\) sequences of \( I_\infty \).

**Lemma 2.3.** Let \( (u_n) \) be a sequence \((PS)_c\) for \( I_\infty \). Then

(i) The sequence \((u_n)\) is bounded in \( D^{1,2}(\mathbb{R}^N) \).

(ii) If \( u_n \rightharpoonup u \) in \( D^{1,2}(\mathbb{R}^N) \), then \( I'_\infty (u) = 0 \).

(iii) If \( c \in (-\infty, \frac{(N+2-\mu)(N+2-\mu)}{4N-2\mu}) \), then \( I_\infty \) satisfies the \((PS)_c\) condition, i.e., up to a subsequence, \( u_n \rightharpoonup u \) in \( D^{1,2}(\mathbb{R}^N) \).

**Proof.** By hypothesis \( I_\infty (u_n) \to c \) and \( I'_\infty (u_n) \to 0 \), then there exists \( K > 0 \) such that

\[
K + \|u_n\| \geq I_\infty (u_n) - \frac{1}{22^*} I'_\infty (u_n) u_n = \frac{(N+2-\mu)}{4N-2\mu} \|u_n\|^2, \quad \forall n \in \mathbb{N},
\]

proving (i).

Since \( u_n \rightharpoonup u \) in \( D^{1,2}(\mathbb{R}^N) \), up to a subsequence, we get

\[
 u_n \to u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N), \quad q \in (2, 2^*)
\]

and

\[
 u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.
\]

By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \) to \( L^{\frac{2N}{N-2}}(\mathbb{R}^N) \). Hence

\[
I_\mu * |u_n|^{2^*} \to I_\mu * |u|^{2^*} \text{ in } L^{\frac{2N}{N-2}}(\mathbb{R}^N).
\]

and for each \( \phi \in C^\infty_0(\mathbb{R}^N) \),

\[
|u_n|^{2^* - 2} u_n \phi \to |u|^{2^* - 2} u \phi \text{ in } L^{\frac{2N}{N-2}}(\mathbb{R}^N)
\]

The above limits implies that

\[
\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*}) |u_n|^{2^* - 2} u_n \phi dx \to \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*}) |u|^{2^* - 2} u \phi dx,
\]

and so, \( I'(u) \phi = 0 \) for all \( \phi \in C^\infty_0(\mathbb{R}^N) \). Now, (ii) follows using the density of \( C^\infty_0(\mathbb{R}^N) \) in \( D^{1,2}(\mathbb{R}^N) \).
In order to prove (iii), consider $v_n = u_n - u$ and note that employing \[21\] Lemma 4.6, we derive

$$
o_n(1) = I'_\infty(u_n)u_n = \|u_n\|^2 - \int_{\mathbb{R}_N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu}dx
= \|v_n\|^2 + \|u\|^2 - \int_{\mathbb{R}_N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu}dx - \int_{\mathbb{R}_N} (I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu}dx + o_n(1)
= \|v_n\|^2 - \int_{\mathbb{R}_N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu}dx + o_n(1). \tag{2.3}
$$

Thus, up to a subsequence, we can assume that

$$0 \leq \rho = \lim_{n \to \infty} \|v_n\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}_N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu}dx.$$

Suppose, by contradiction, that $\rho > 0$. From the inequality

$$S_{H,L} \left[ \int_{\mathbb{R}_N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu}dx \right]^{1/2^*_\mu} \leq \|v_n\|^2,$$

we obtain

$$\rho \geq S_{H,L} \mu^{1/2^*_\mu} \Rightarrow \rho \geq S_{H,L}^{(2N-\mu)/(N+2-\mu)}. \tag{2.4}
$$

As

$$I_\infty(u) = \left( \frac{1}{2} - \frac{1}{2 2^*_\mu} \right) \|u\|^2 = \frac{(N + 2 - \mu)}{4N - 2\mu} \|u\|^2 \geq 0$$

and

$$c = \frac{(N + 2 - \mu)}{4N - 2\mu} \|v_n\|^2 + I_\infty(u) + o_n(1), \tag{2.5}
$$

it follows that

$$c = \frac{(N + 2 - \mu)}{4N - 2\mu} \|v_n\|^2 + I_\infty(u) + o_n(1) \geq \frac{(N + 2 - \mu)}{4N - 2\mu} \|v_n\|^2 + o_n(1)
= \frac{(N + 2 - \mu)}{4N - 2\mu} \rho \geq \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

which is a contradiction. Hence $\rho = 0$ and

$$\|v_n\|^2 = \|u_n - u\|^2 \to 0,$$

showing the lemma. \qed

Before concluding this section, we will prove an important estimate involving the nodal solutions of the limit problem, which will be used later on.

**Lemma 2.4.** If $u \in D^{1,2}(\mathbb{R}_N)$ is a nodal solution of $(P)_\infty$, then

$$I_\infty(u) \geq 2^{\frac{4-\mu}{4+N+2-\mu}} \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} I_\infty(t^+ u^+) + I_\infty(t^- u^-) \frac{(N + 2 - \mu)}{4N - 2\mu} \leq I_\infty(u) \frac{(N + 2 - \mu)}{4N - 2\mu}.$$

**Proof.** Arguing as in the proof of \[17\] Proposition 3.2, for all $t^+, t^- > 0$ we see that

$$I_\infty(t^+ u^+) \frac{(N + 2 - \mu)}{4N - 2\mu} + I_\infty(t^- u^-) \frac{(N + 2 - \mu)}{4N - 2\mu} \leq I_\infty(u) \frac{(N + 2 - \mu)}{4N - 2\mu}.$$


Fixing $t^+, t^- > 0$ such that $I'_\infty (t^\pm u^\pm)(t^\pm u^\pm) = 0$, it follows that

$$I_\infty (t^\pm u^\pm) \geq \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H, L}^{(2N-\mu)/(N+2-\mu)}.$$ 

The last two inequalities combine to give

$$I_\infty (u) \geq 2\frac{4^{\frac{\mu}{N-\mu}}}{N - 2\mu} \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H, L}^{(2N-\mu)/(N+2-\mu)},$$

finishing the proof. \hfill \Box

## 3 A splitting theorem

We start this section by proving a technical lemma for $I_\infty$ that will be useful to prove our splitting theorem.

**Lemma 3.1.** (Main lemma) Let $(u_n)$ be a $(PS)_c$ sequence for the functional $I_\infty$ with $u_n \to 0$ and $u_n \rightharpoonup 0$. Then, there are sequences $(R_n) \subset \mathbb{R}^+$, $(x_n) \subset \mathbb{R}^N$ and $v_0 \in D^{1,2}(\mathbb{R}^N)$ nontrivial solution of $(P_\infty)$ such that, up to a subsequence of $(u_n)$, we have

$$w_n(x) = u_n(x) - R_n^{(N-2)/2}v_0(R_n(x - x_n)) + o_n(1),$$

where $(w_n)$ is a $(PS)_{c-I_\infty(v_0)}$ for the $I_\infty$.

**Proof.** Let $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional $I_\infty$, i.e,

$$I_\infty (u_n) \to c \text{ and } I_\infty' (u_n) \to 0. \quad (3.1)$$

From Lemma 2.3(i), we know that $(u_n)$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Since $u_n \to 0$ and $u_n \rightharpoonup 0$, it follows Lemma 2.3(iii) that

$$c \geq \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H, L}^{(2N-\mu)/(N+2-\mu)}.$$ 

Note that

$$c + o_n(1) = I_\infty (u_n) - \frac{1}{2^{2\mu}}I_\infty' (u_n) u_n = \frac{(N + 2 - \mu)}{4N - 2\mu} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

which leads to

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq S_{H, L}^{(2N-\mu)/(N+2-\mu)}. \quad (3.2)$$

Let $L$ be a number such that $B_2(0)$ is covered by $m$ balls of radius 1, $k \in \mathbb{N}^*$, $(R_n) \subset \mathbb{R}^+$, $(x_n) \subset \mathbb{R}^N$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B_{R_n^{-1}}(y)} |\nabla u_n|^2 dx = \int_{B_{R_n^{-1}}(x_n)} |\nabla u_n|^2 dx = \frac{S_{H, L}^{(2N-\mu)/(N+2-\mu)}}{km}$$

and

$$v_n(x) = R_n^{(2-N)/2} u_n \left( \frac{x}{R_n} + x_n \right).$$

Hereafter, we fix $k$ such way that

$$\frac{S_{H, L}^{(2N-\mu)/(N+2-\mu)}}{k} < \left( \frac{S}{2^N} \right)^{\frac{2N-\mu}{N+2-\mu}} \left( \frac{1}{C(N, \mu)} \right)^{\frac{N-2}{N+2-\mu}}. \quad (3.3)$$
Using a change of variable, we can prove that
\[
\int_{B_1(0)} |\nabla v_n| \mu = \frac{S_{H,L}^{(2N-1)/(N+2-\mu)}}{km} = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\nabla v_n| \mu.
\] (3.4)

Now, for each \( \Phi \in D^{1,2}(\mathbb{R}^N) \), we define the function
\[
\tilde{\Phi}_n(x) = R_n^{(N-2)/2} \Phi(R_n(x - x_n))
\]
that satisfies
\[
\int_{\mathbb{R}^N} \nabla u_n \nabla \tilde{\Phi}_n \mu = \int_{\mathbb{R}^N} \nabla v_n \nabla \Phi \mu.
\] (3.5)

and
\[
\int_{\mathbb{R}^N} (I_{j} \ast |u_n|^{p-2}u_n)|u_n|^{2^* - 2} \tilde{\Phi}_n \mu = \int_{\mathbb{R}^N} (I_{j} \ast |v_n|^{p-2}v_n)|v_n|^{2^* - 2} \Phi \mu.
\] (3.6)

These limits ensure that
\[
I_{\infty}(v_n) \to \infty \text{ and } I'_{\infty}(v_n) \to 0.
\]

From Lemma 2.3, there exists \( v_0 \in D^{1,2}(\mathbb{R}^N) \) such that, up to a subsequence, \( v_n \rightharpoonup v_0 \) in \( D^{1,2}(\mathbb{R}^N) \) and \( I'(v_0) = 0 \).

As a consequence of the well known Lions’ Lemma 2.3, we can assume that
\[
\int_{\mathbb{R}^N} |v_n|^{2^*} \Phi \mu \to \int_{\mathbb{R}^N} |v_0|^{2^*} \Phi \mu + \sum_{j \in J} \phi(x_j) \mu_j, \forall \phi \in C_0^\infty(\mathbb{R}^N)
\] (3.7)

and
\[
|\nabla v_n| \mu \to \mu \geq |\nabla v_0| \mu + \sum_{j \in J} \phi(x_j) \mu_j, \forall \phi \in C_0^\infty(\mathbb{R}^N),
\]

for some \( \{x_j\}_{j \in J} \subset \mathbb{R}^N, \{\nu_j\}_{j \in J}, \{\mu_j\}_{j \in J} \subset \mathbb{R}^+ \) with \( S_{\nu_j}^{2^*/2} \leq \mu_j \), where \( J \) is at most a countable set.

We are going to show that \( J \) is finite. Consider \( \phi \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \phi(x) \leq 1 \), for all \( x \in \mathbb{R}^N \), \( \phi(x) = 0 \) for all \( x \in B_2(0) \) and \( \phi_{\rho}(x) = 1 \) for all \( x \in B_1(0) \). Now fix \( x_j \in \mathbb{R}^N, j \in J \) and define \( \psi_{\rho}(x) = \phi(\frac{x-x_j}{\rho}) \), for each \( \rho > 0 \). Then \( 0 \leq \psi_{\rho}(x) \leq 1 \), for all \( x \in \mathbb{R}^N \), \( \psi_{\rho}(x) = 0 \) for all \( x \in B_2(\rho) \) and \( \psi(x) = 1 \) for all \( x \in B_1(\rho) \). We have that \( (v_n \psi_{\rho}) \) is bounded in \( D^{1,2}(\mathbb{R}^N) \) and \( I'(v_n) v_n \psi_{\rho} = o_n(1) \).

Hence,
\[
\int_{\mathbb{R}^N} |\nabla v_n| \mu \to \int_{\mathbb{R}^N} \nabla v_n \nabla \psi_{\rho} \mu \to \int_{\mathbb{R}^N} (I_{j} \ast |v_n|^{2^*}) |v_n|^{2^*} \psi_{\rho} \mu \mu + o_n(1). \] (3.8)

Using Proposition 2.1 and seeing that
\[
\lim_{\rho \to 0} \left[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla v_n \nabla \psi_{\rho} \mu \right] = 0,
\]
we find
\[
S_{\nu_j}^{2^*/2} \leq \mu_j \leq C_{\nu_j}^{2^*/2^*}.
\]
As \( 2^* \mu \) and \( \sum_{j \in J} \nu_j^{2^*/2^*} < +\infty \), we deduce that \( \nu_j \) does not converge to zero, which implies that \( J \) is finite. From now on, we denote by \( J = \{1, 2, ..., m\} \) and \( \Gamma \subset \mathbb{R}^N \) the set given by
\[ \Gamma = \{x_j \in \{x_j\}_{j \in J}; |x_j| > 1\} \text{, \( (x_j \text{ given by (3.7))}. \]
In the sequel, we are going to show that \(v_0 \neq 0\). Suppose, by contradiction that \(v_0 = 0\). Thereby, by (3.7),

\[
\int_{\mathbb{R}^N} |v_n|^2 \phi \, dx \to 0, \quad \forall \phi \in C^\infty_0 (\mathbb{R}^N \setminus \{x_1, x_2, \ldots, x_m\}).
\]  

(3.9)

Using again Proposition 2.1, we derive the inequality below

\[
\int_{\mathbb{R}^N} (I_\mu * |v_n|^2) |v_n|^2 \phi \, dx \leq C |v_n|_2^2 \left( \int |v_n|^2 \phi \, dx \right)^2 = o_n(1),
\]

which leads to

\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 \phi = o_n(1).
\]

(3.10)

Consequently, if \(\rho \in \mathbb{R}\) is a number that satisfies \(0 < \rho < \min\{\text{dist}(\Gamma, \overline{B}_1(0)), 1\}\), it follows that

\[
\int_{B_{1+\rho/3}(0) \setminus B_{1+2\rho}(0)} |\nabla v_n|^2 \, dx = o_n(1).
\]

(3.11)

In the sequel, let us consider the sequence \((\Phi_n)\) given by \(\Phi_n(x) = \Phi(x) v_n(x)\), where \(\Phi \in C^\infty_0 (\mathbb{R}^N)\) satisfies \(0 \leq \Phi(x) \leq 1\), \(\Phi(x) = 1\) if \(x \in B_{1+\rho/3}(0)\) and \(\Phi(x) = 0\) if \(x \in B_{1+2\rho/3}(0)\). Note that,

\[
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} |\nabla \Phi_n|^2 \, dx \leq C \left[ \int_{B_{1+\rho/3}(0) \setminus B_{1+2\rho}(0)} |\nabla v_n|^2 \, dx + \int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} |v_n|^2 \, dx \right],
\]

that is,

\[
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} |\nabla \Phi_n|^2 \, dx = o_n(1).
\]

(3.12)

Since \(I_\infty(v_n) \Phi_n = o_n(1)\), we have

\[
\begin{align*}
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} \nabla v_n \nabla \Phi_n \, dx + & \int_{B_{1+2\rho}(0)} \nabla v_n \nabla \Phi_n \, dx \\
- & \int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} (I_\mu * |v_n|^2) \Phi_n \, dx - \int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} (I_\mu * |v_n|^2) |v_n|^2 \Phi_n \, dx = o_n(1),
\end{align*}
\]

which implies

\[
\begin{align*}
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} \nabla v_n \nabla \Phi_n \, dx + & \int_{B_{1+2\rho}(0)} |\nabla v_n|^2 \, dx \\
- & \int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} (I_\mu * |v_n|^2) |v_n|^2 \Phi_n \, dx - \int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} (I_\mu * |v_n|^2) |v_n|^2 \Phi_n \, dx = o_n(1).
\end{align*}
\]

(3.13)

(3.14)

Note that from Hölder’s inequality and (3.12)

\[
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} \nabla v_n \nabla \Phi_n \, dx \to 0 \quad \text{when} \quad n \to \infty
\]

(3.15)

and that (3.9) together with Proposition 2.1 gives

\[
\int_{B_{1+\rho}(0) \setminus B_{1+2\rho}(0)} (I_\mu * |v_n|^2) |v_n|^2 \Phi_n \, dx = o_n(1).
\]

(3.16)
Thereby, from (3.14), (3.15) and (3.16),
\[
\int_{B_{1+\frac{\rho}{3}(0)}} |\nabla v_n|^2 dx - \int_{B_{1+\frac{\rho}{3}(0)}} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = o_n(1). \quad (3.17)
\]
The last equality together with the boundedness of \((v_n)\) and (3.4) implies that for some subsequence
\[
\lim_{n \to \infty} \int_{B_{1+\frac{\rho}{3}(0)}} |\nabla v_n|^2 dx = \int_{B_{1+\frac{\rho}{3}(0)}} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = A > 0.
\]
The last limits combine with Cherrier’s inequality ( see Lemma 2.2 ) to give
\[
A \geq \left( \frac{S}{2^\pi} \right)^{\frac{2N-\mu}{N+2-\mu}} \left( \frac{1}{C(N,\mu)} \right)^{\frac{N-2}{N+2-\mu}}.
\quad (3.18)
\]
Note that
\[
\|\Phi_n\|^2 = \int_{B_{1+\rho}(0) \setminus B_{1+\frac{\rho}{3}}(0)} |\nabla \Phi_n|^2 dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Phi_n|^2 dx = o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Phi_n|^2 dx.
\]
Since \(\Phi_n = v_n\) in \(B_{1+\frac{\rho}{3}}(0)\) and that \(B_{1+\frac{\rho}{3}}(0) \subset B_2(0)\), we obtain
\[
\|\Phi_n\|^2 \leq o_n(1) + \int_{B_2(0)} |\nabla v_n|^2 dx \\
\leq o_n(1) + \int_{\bigcup_{k=1}^m B_1(y_k)} |\nabla v_n|^2 dx \\
\leq o_n(1) + \sum_{k=1}^m \int_{B_1(y_k)} |\nabla v_n|^2 dx \\
\leq o_n(1) + m \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\nabla v_n|^2 dx \leq o_n(1) + \frac{S_{H, L}^{(2N-\mu)/(N+2-\mu)}}{k}.
\]
Then,
\[
\int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx \leq o_n(1) + \frac{S_{H, L}^{(2N-\mu)/(N+2-\mu)}}{k},
\quad (3.19)
\]
implicating that
\[
A \leq \frac{S_{H, L}^{(2N-\mu)/(N+2-\mu)}}{k}.
\]
Hence, by (3.3),
\[
A \leq \left( \frac{S}{2^\pi} \right)^{\frac{2N-\mu}{N+2-\mu}} \left( \frac{1}{C(N,\mu)} \right)^{\frac{N-2}{N+2-\mu}},
\]
which contradicts (3.18), and so, \(v_0 \neq 0\).

Now, we are going to show that there is \((w_n) \subset D^{1,2}(\mathbb{R}^N)\) such that \((w_n)\) is a \((PS)_{c-I_{\infty}(v_0)}\) sequence for \(I_{\infty}\) satisfying
\[
w_n(x) = u_n(x) - \frac{R_n^{(N-2)/2}v_0}{\|R_n(x - x_n)\|_2} + o_n(1),
\]
for some subsequence of \((u_n)\) that still denote by \((u_n)\). To this end, we fix \(\psi \in C_0^\infty(\mathbb{R}^N)\) such that \(0 \leq \psi(x) \leq 1\) for all \(x \in \mathbb{R}^N\) and

\[
\psi(x) = \begin{cases} 
1, & \text{if } x \in B_1(0), \\
0, & \text{if } x \in B_2^e(0)
\end{cases}
\]

and let be a sequence defined by

\[
w_n(x) = u_n(x) - R_n^{(N-2)/2}v_0(R_n(x-x_n))\psi(\tilde{R}_n(x-x_n)), \tag{3.20}
\]

where \((\tilde{R}_n)\) satisfies \(\tilde{R}_n = \frac{R_n}{R_n} \to \infty\). From (3.20),

\[
R_n^{(2s-N)/2}w_n(x) = R_n^{(2s-N)/2}u_n(x) - v_0(R_n(x-x_n))\psi(\tilde{R}_n(x-x_n)).
\]

Making change of variable, we arrive in

\[
R_n^{(2s-N)/2}w_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(2s-N)/2}u_n\left(\frac{z}{R_n} + x_n\right) - v_0\psi\left(\frac{z}{R_n}\right). \tag{3.21}
\]

Now we define

\[
\tilde{w}_n = R_n^{(2s-N)/2}w_n\left(\frac{z}{R_n} + x_n\right)
\]

and since

\[
v_n(x) = R_n^{(2s-N)/2}u_n\left(\frac{x}{R_n} + x_n\right),
\]

we get,

\[
\tilde{w}_n(z) = v_n(z) - v_0(z)\psi\left(\frac{z}{R_n}\right). \tag{3.22}
\]

If

\[
\psi_n(z) = \psi\left(\frac{z}{\tilde{R}_n}\right), \tag{3.22}
\]

we have that

\[
\psi_n(z) = \begin{cases} 
1, & \text{if } z \in B_{\tilde{R}_n}(0), \\
0, & \text{if } z \in B_{2\tilde{R}_n}^c(0).
\end{cases}
\]

From (3.21) and (3.22),

\[
\tilde{w}_n(z) = v_n(z) - v_0(z)\psi_n(z).
\]

The result is over if we show that \(v_0\psi_n \to v_0\) in \(D^{1,2}(\mathbb{R}^N)\) and that \((w_n)\) is a \((PS)_{c-I_\infty(v_0)}\) sequence for \(I_\infty\). By a straightforward computation, we get

\[
\|v_0\psi_n - v_0\|^2 = \int_{\mathbb{R}^N} |\nabla(v_0\psi_n - v_0)|^2dx \leq \int_{\mathbb{R}^N \setminus B_{\tilde{R}_n}} |\nabla v_0|^2dx + \int_{\mathbb{R}^N} |v_0|^2|\nabla(\psi_n - 1)|^2dx = o_n(1). \tag{3.23}
\]

Therefore

\[
v_0\psi_n \to v_0 \text{ in } D^{1,2}(\mathbb{R}^N),
\]

and then

\[
\tilde{w}_n = v_n - v_0 + o_n(1) \text{ in } D^{1,2}(\mathbb{R}^N).
\]
Note that
\[ \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \int_{\mathbb{R}^N} |\nabla v_0|^2 dx + \int_{\mathbb{R}^N} |\nabla (v_n - v_0)|^2 dx + o_n(1) \]
and
\[ \int_{\mathbb{R}^N} (I_\mu * |v_n|^{2^*_u}) |v_n|^{2^*_u} dx = \int_{\mathbb{R}^N} (I_\mu * |v_0|^{2^*_u}) |v_0|^{2^*_u} dx + \int_{\mathbb{R}^N} (I_\mu * |v_n - v_0|^{2^*_u}) |v_n - v_0|^{2^*_u} dx + o_n(1), \]
ensure that
\[ I_\infty(w_n) = I_\infty(v_n) - I_\infty(v_0) + o_n(1). \]
Hence,
\[ I_\infty(w_n) \to c - I_\infty(v_0) \quad \text{when} \quad n \to \infty. \]

Now, recalling that
\[ \| I'_\infty(\tilde{w}_n) - I'_\infty(v_n) + I'_\infty(v_0) \| \to 0, \] (3.24)
and the fact that \( v_0 \) is a critical point of \( I_\infty \), we can claim that
\[ I'_\infty(\tilde{w}_n) = I'_\infty(v_n) + o_n(1) = o_n(1). \]
As \( \| I'_\infty(w_n) \| \leq \| I'_\infty(\tilde{w}_n) \| \), it follows that \( I'_\infty(w_n) \to 0 \), and the proof of this lemma is over. \( \square \)

**Theorem 3.1.** (A splitting theorem) Let \( (u_n) \) be a \((PS)_c\) sequence for \( I \) with \( u_n \rightharpoonup u_0 \) in \( D^{1,2}(\mathbb{R}^N) \). Then, up to a subsequence, \( (u_n) \) satisfies either,

(a) \( u_n \to u_0 \) in \( D^{1,2}(\mathbb{R}^N) \) or,

(b) there exists \( k \in \mathbb{N} \) and nontrivial solutions \( z_0^1, z_0^2, \ldots, z_0^k \) for the problem \( (P_\infty) \), such that
\[ \|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^k \|z_0^j\|^2 \]
and
\[ I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(z_0^j). \]

**Proof.** From the weak convergence, we have that \( u_0 \) is a critical point of \( I \). Suppose that \( u_n \rightharpoonup u_0 \) in \( D^{1,2}(\mathbb{R}^N) \) and let \( (z_n^1) \subset D^{1,2}(\mathbb{R}^N) \) be the sequence given by \( z_n^1 = u_n - u_0 \). Then, \( z_n^1 \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \) and \( z_n^1 \rightharpoonup 0 \) in \( D^{1,2}(\mathbb{R}^N) \). Arguing as in [8, Lemma 3], we obtain
\[ I_\infty(z_n^1) = I(u_n) - I(u_0) + o_n(1) \] (3.25)
and
\[ I'_\infty(z_n^1) = I'(u_n) - I'(u_0) + o_n(1). \] (3.26)
Then, from (3.25) and (3.26) that \( (z_n^1) \) is a \((PS)_{c_1}\) sequence for \( I_\infty \). Hence, by Lemma 3.1 there are sequences \( (R_{n,1}) \subset \mathbb{R}, (x_{n,1}) \subset \mathbb{R}^N, z_0^1 \in D^{1,2}(\mathbb{R}^N) \) nontrivial solution for problem \( (P_\infty) \) and a \((PS)_{c_2}\) sequence \( (z_n^2) \subset D^{1,2}(\mathbb{R}^N) \) for \( I_\infty \) such that
\[ z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-2)/2} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1). \]
If we define
\[ u_n^1(x) = R_{n,1}^{(2-N)/2} z_n^1 \left( \frac{x}{R_{n,1}} + x_{n,1} \right) \] (3.27)
and
\[ \tilde{z}_n^2(x) = R_{n,1}^{(2-N)/2} \tilde{z}_n^2 \left( \frac{x}{R_{n,1}} + x_{n,1} \right), \]
we obtain
\[ \tilde{z}_n^2(x) = v_n^1(x) - z_0^1(x) + o_n(1) \quad (3.28) \]
and
\[ ||v_n^1|| = ||z_0^1|| \quad \text{and} \quad \int_{\mathbb{R}^N} (I_{I_n} + |v_n^1|^2) |v_n^1|^2 dx = \int_{\mathbb{R}^N} (I_{I_n} + |z_0^1|^2) |z_0^1|^2 dx. \quad (3.29) \]
Hence,
\[ I_\infty(v_n^1) = I_\infty(z_0^1) \quad \text{and} \quad I'_\infty(v_n^1) \to 0 \quad \text{in} \quad (D^{1,2}(\mathbb{R}^N))' \quad (3.30) \]
From (3.30) and Lemma 2.3(a), we see that \((v_n^1)\) is a bounded sequence in \(D^{1,2}(\mathbb{R}^N)\) and, up to a subsequence,
\[ v_n^1 \to z_0^1 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \quad (3.31) \]
Arguing again as in Lemma 3,
\[ I_\infty(\tilde{z}_n^2) = I_\infty(v_n^1) - I_\infty(z_0^1) + o_n(1) = I(u_n) - I(u_0) - I_\infty(z_0^1) + o_n(1). \quad (3.32) \]
and
\[ I'_\infty(\tilde{z}_n^2) = I'_\infty(v_n^1) - I'_\infty(z_0^1) + o_n(1). \quad (3.33) \]
If \(\tilde{z}_n^2 \to 0\) in \(D^{1,2}(\mathbb{R}^N)\), the proof is over for \(k = 1\), because in this case, we have
\[ ||u_n||^2 \to ||u_0||^2 + ||z_0^1||^2. \]
Moreover, from continuity of \(I_\infty\),
\[ I(u_n) \to I(u_0) + I_\infty(z_0^1). \]
If \(\tilde{z}_n^2 \not\to 0\), using (3.28) and (3.31) that \(\tilde{z}_n^2 \to 0\), by (3.32) and (3.33), we conclude that \((\tilde{z}_n^2)\) is a \((PS)_{c_2}\) sequence for \(I_\infty\). By Lemma 3.1 there are sequences \((R_{n,2}) \subset \mathbb{R}, (x_{n,2}) \subset \mathbb{R}^N, z_0^2 \in D^{1,2}(\mathbb{R}^N)\) nontrivial solution of problem \((P_\infty)\) and a \((PS)_{c_3}\) sequence \((z_n^3) \subset D^{1,2}(\mathbb{R}^N)\) for \(I_\infty\) such that
\[ z_n^3(x) = \tilde{z}_n^2(x) - R_{n,2}^{(N-2)/2} z_0^2(0, R_{n,2}(x - x_{n,2})) + o_n(1). \]
If
\[ u_n^2(x) = R_{n,2}^{(2-N)/2} \tilde{z}_n^2 \left( \frac{x}{R_{n,2}} + x_{n,2} \right), \]
and
\[ \tilde{z}_n^3(x) = R_{n,2}^{(2-N)/2} \tilde{z}_n^3 \left( \frac{x}{R_{n,2}} + x_{n,2} \right), \]
we have that
\[ \tilde{z}_n^3(x) = v_n^2(x) - z_0^2(x) + o_n(1). \quad (3.34) \]
Arguing of same way as before, we arrive in
\[ ||z_n^3||^2 = ||u_n||^2 - ||u_0||^2 - ||z_0^1||^2 - ||z_0^2||^2 + o_n(1), \quad (3.35) \]
\[ I_\infty(\tilde{z}_n^3) = I(u_n) - I(u_0) - I_\infty(z_0^1) - I_\infty(z_0^2) + o_n(1). \quad (3.36) \]
and

\[ I'_\infty(\tilde{z}_n^3) = I'_\infty(v_n^2) - I'_\infty(\tilde{z}_0^3) + o_n(1). \quad (3.37) \]

If \( \tilde{z}_n^3 \to 0 \) in \( \mathbb{D}^{1,2}(\mathbb{R}^N) \), the proof is over with \( k = 2 \), because \( \|\tilde{z}_n^3\|^2 \to 0 \), and by (3.35), we have

\[ \|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{2} \|z_j^0\|^2. \]

Moreover, from continuity of \( I_\infty \), we also have that \( I_\infty(\tilde{z}_n^3) \to 0 \). Thereby, by (3.36),

\[ I(u_n) \to I(u_0) + \sum_{j=1}^{2} I(\tilde{z}_j^0). \]

If \( \tilde{z}_n^3 \to 0 \), we can repeat the same arguments before and we will find \( z_0^1, z_0^2, \ldots, z_0^{k-1} \) nontrivial solutions for problem (\( P_\infty \)) satisfying

\[ \|\tilde{z}_n^k\|^2 = \|u_n\|^2 - \sum_{j=1}^{k-1} \|z_j^0\|^2 + o_n(1) \quad (3.38) \]

and

\[ I_\infty(\tilde{z}_n^k) = I(u_n) - I(u_0) - \sum_{j=1}^{k-1} I(\tilde{z}_j^0) + o_n(1). \quad (3.39) \]

From definition of \( S_{H,L} \),

\[ \left( \int_{\mathbb{R}^N} (I_\mu \ast |z_0^j|^{2^*_\mu}) |z_0^j|^{2^*_\mu} dx \right)^{1/2^*_\mu} \leq \|z_0^j\|^2, \quad j = 1, 2, \ldots, k - 1. \quad (3.40) \]

Since \( z_0^j \) is nontrivial solution of (\( P_\infty \)), for all \( j = 1, 2, \ldots, k - 1 \), we have

\[ \|z_0^j\|^2 = \int_{\mathbb{R}^N} (I_\mu \ast |z_0^j|^{2^*_\mu}) |z_0^j|^{2^*_\mu} dx. \]

Hence,

\[ \|z_0^j\|^2 \geq S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \quad j = 1, 2, \ldots, k - 1. \quad (3.41) \]

From (3.38) and (3.41),

\[ \|\tilde{z}_n^k\|^2 = \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|z_j^0\|^2 + o_n(1) \leq \|u_n\|^2 - \|u_0\|^2 - (k-1)S_{H,L}^{(2N-\mu)/(N+2-\mu)} + o_n(1). \quad (3.42) \]

Since \( (u_n) \) is bounded in \( \mathbb{D}^{1,2}(\mathbb{R}^N) \), for \( k \) sufficient large, we conclude that \( \tilde{z}_n^k \to 0 \) in \( \mathbb{D}^{1,2}(\mathbb{R}^N) \) and the proof is over.

An immediate consequence of the last theorem are the next two corollaries

**Corollary 3.1.** Let \( (u_n) \) be a (PS)\(_c\) sequence for \( I \) with \( c \in \left( 0, \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} \right) \). Then, up to a subsequence, \( (u_n) \) strong converges in \( \mathbb{D}^{1,2}(\mathbb{R}^N) \).
**Proof.** We have that \((u_n)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\), \(u_n \to u_0\) in \(D^{1,2}(\mathbb{R}^N)\) and \(I'(u_0) = 0\). Suppose, by contradiction, that

\[ u_n \rightharpoonup u_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \]

From Theorem 3.1 there are \(k \in \mathbb{N}\) and nontrivial solutions \(z_0^1, z_0^2, \ldots, z_0^k\) of problem \((P_\infty)\) such that,

\[ \|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2 \]

and

\[ I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j). \]

Note that

\[
I(u_0) = \frac{1}{2}\|u_0\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} a(x)u_0^2 dx - \frac{1}{22\mu} \int_{\mathbb{R}^N} (I_\mu * |u_0|^{2^*_\mu})|u_0|^{2^*_\mu} dx
\]

\[ = \frac{1}{2}\|u_0\|^2 + \frac{1}{2} \left( \int_{\mathbb{R}^N} (I_\mu * |u_0|^{2^*_\mu})|u_0|^{2^*_\mu} dx - \|u_0\|^2 \right) - \frac{1}{22\mu} \int_{\mathbb{R}^N} (I_\mu * |u_0|^{2^*_\mu})|u_0|^{2^*_\mu} dx
\]

\[ = \left( \frac{1}{2} - \frac{1}{22\mu} \right) \int_{\mathbb{R}^N} (I_\mu * |u_0|^{2^*_\mu})|u_0|^{2^*_\mu} dx \geq 0. \]

Then,

\[ c = I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j) \geq k \frac{(N+2-\mu)(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} \geq \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \]

which is a contradiction with \(c \in \left(0, \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}\right)\).

**Corollary 3.2.** The functional \(I : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}\) satisfies the Palais-Smale condition in

\[
\left( \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, 2^*_{\mu} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} \right).
\]

**Proof.** Let \((u_n)\) be a sequence in \(D^{1,2}(\mathbb{R}^N)\) that satisfies

\[ I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0. \]

Since \((u_n)\) is bounded, up to a subsequence, we have that \(u_n \to u_0\) in \(D^{1,2}(\mathbb{R}^N)\) and \(I(u_0) \geq 0\). Suppose by contradiction that

\[ u_n \rightharpoonup u_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \]

From Theorem 3.1 there are \(k \in \mathbb{N}\) and nontrivial solutions \(z_0^1, z_0^2, \ldots, z_0^k\) of problem \((P_\infty)\) such that

\[ \|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2 \]

and

\[ I(u_n) \to c = I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j). \]

The above information ensures that \(u_0 \neq 0\). Since \(I(u_0) \geq 0\), then \(k = 1\) and \(z_0^1\) cannot change of sign, because otherwise, by Lemma 2.4,

\[ I_\infty(z_0^1) \geq 2^*_{\mu} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \]

which is a contradiction with the functional \(I\) satisfying the Palais-Smale condition in the given interval.
which leads to a contradiction. On the other hand, if $z^1_0$ has definite sign, by [13 Theorem 1.3] and [22 Theorem 1.1],

$$I_\infty(z^1_0) = \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

On the other hand, by a direct computation,

$$I(u_0) \geq \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Hence,

$$c = I(u_0) + I_\infty(z^1_0) \geq 2 \frac{(N + 2 - \mu)}{4N - 2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} + \frac{2}{4N - 2\mu} (N + 2 - \mu) S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

obtaining again a contradiction. This proves the result. \[\square\]

From now on we consider the functional $f : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$f(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)|u|^2\right) dx$$

and the manifold $\mathcal{M} \subset D^{1,2}(\mathbb{R}^N)$ given by

$$\mathcal{M} = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \frac{1}{C(N, \mu)} \int_{\mathbb{R}^N} (I_\mu * |u|^2) |u|^2 \ dx = 1 \right\}.$$ 

A direct computation gives

**Lemma 3.2.** Let $(u_n) \subset \mathcal{M}$ be a sequence that satisfies

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then, the sequence $v_n = c^{(N-2)/(2N-2\mu+4)} u_n$ satisfies the following limits.

$$I(v_n) \to \frac{(N - \mu + 2)}{4N - 2\mu} c^{(2N-\mu)/(N+2-\mu)} \quad \text{and} \quad I'(v_n) \to 0.$$

The next results are direct consequence of the Corollaries above.

**Lemma 3.3.** Suppose that there are a sequence $(u_n) \subset \mathcal{M}$ and

$$c \in (S_{H,L}, 2^{\frac{(4-\mu)(N-\mu+2)}{4N-2\mu}} S_{H,L})$$

such that

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then, up to a subsequence, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$.

**Corollary 3.3.** Suppose that there is a sequence $(u_n) \subset \mathcal{M}$ and

$$c \in (S_{H,L}, 2^{\frac{(4-\mu)(N-\mu+2)}{4N-2\mu}} S_{H,L})$$

such that

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then $I$ has a critical point $v_0 \in D^{1,2}(\mathbb{R}^N)$ with $I(v_0) = \frac{(N + 2 - \mu)}{4N - 2\mu} c^{(2N-\mu)/(N+2-\mu)}$. 

16
4 Technical Lemmas

From now on, we consider the function $\Phi_{\delta,y} \in D^{1,2}(\mathbb{R}^N)$ given by

$$\Phi_{\delta,y}(x) = C_N \left( \frac{\delta}{\delta^2 + |x-y|^2} \right)^{(N-2)/2}, \quad x, y \in \mathbb{R}^N \text{ and } \delta > 0,$$

(4.1)

where $C_N$ is a positive constant. From [14] and [22], we know that every positive solution of $(P_\infty)$ is as (4.1). Moreover, a simple computation ensures that we can fix $C_N > 0$ in such a way that

$$\|\Phi_{\delta,y}\|_2 = S_{H,L} \text{ and } \frac{1}{C(N,\mu)} \int_{\mathbb{R}^N} (I_{\mu} * |\Phi_{\delta,y}|^2_\mu) |\Phi_{\delta,y}|^2_\mu = 1.$$  (4.2)

In this subsection we prove some properties of the family $(\Phi_{\delta,y})$ given by in (4.1). First of all, we recall

$$\Phi_{\delta,y} \in \Sigma = \left\{ u \in D^{1,2}(\mathbb{R}^N); u \geq 0 \right\}$$  (4.3)

and

$$\Phi_{\delta,y} \in L^q(\mathbb{R}^N) \text{ for } q \in \left( \frac{(N - \mu/2)}{N - 2}, 2^*_\mu \right), \quad \forall \delta > 0 \text{ and } \forall y \in \mathbb{R}^N.$$  (4.4)

The proof of the next two lemmas follow as in [3] and their proofs will be omitted.

**Lemma 4.1.** For each $y \in \mathbb{R}^N$, we have

(i) $\|\Phi_{\delta,y}\|_{H^1(\mathbb{R}^N)} \to 0$ when $\delta \to +\infty$.

(ii) $\|\Phi_{\delta,y}\|_{H^1(\mathbb{R}^N)} \to +\infty$ when $\delta \to 0$

(iii) $|\Phi_{\delta,y}|_q \to 0$ when $\delta \to 0$, $\forall q \in \left( \frac{(N - \mu/2)}{N - 2}, 2^*_\mu \right)$,

(iv) $|\Phi_{\delta,y}|_q \to +\infty$ when $\delta \to +\infty$, $\forall q \in \left( \frac{(N - \mu/2)}{N - 2}, 2^*_\mu \right)$.

**Lemma 4.2.** For each $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^N \setminus B_{\varepsilon}(0)} |\nabla \Phi_{\delta,0}|^2 dx \to 0 \text{ when } \delta \to 0.$$  

Next, we are showing to prove some technical lemmas that are crucial in the proof of Theorem 1.1.

**Lemma 4.3.** Suppose that $a \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N - \mu}{4 - \mu} < p_2$ with $p_2 < \frac{N - \mu/2}{4 - N - \mu/2}$ if $N = 3$. Then, for each $\varepsilon > 0$, there are $\delta = \delta(\varepsilon) > 0$ and $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that

$$\sup_{y \in \mathbb{R}^N} f(\Phi_{\delta,y}) < S_{H,L} + \varepsilon, \quad \delta \in (0, \delta] \cup [\overline{\delta}, \infty).$$

**Proof.** Consider $y \in \mathbb{R}^N$, $q \in \left( \frac{2N - \mu}{4 - \mu}, p_2 \right)$ and $t \in (1, +\infty)$ with $\frac{1}{q} + \frac{1}{t} = 1$. By a simple calculus,

$$\frac{N - \mu/2}{N - 2} < 2t < 2^*_\mu.$$  (4.5)
Since $\Phi_{\delta,y} \in L^d(\mathbb{R}^N), \forall d \in \left(\frac{N-\mu/2}{N-2}, 2\mu\right)$, we see that $|\Phi_{\delta,y}|^2 \in L'(\mathbb{R}^N)$. Then, by Hölder’s inequality,

$$
\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 \, dx \leq |a|_q |\Phi_{\delta,0}|^2, \forall y \in \mathbb{R}^N.
$$

From Lemma 4.1 (iii), given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$
\sup_{y \in \mathbb{R}^N} f(\Phi_{\delta,b}) \leq S_{H,L} + \frac{\varepsilon}{2} < S_{H,L} + \varepsilon, \forall \delta \in (0, \delta).
$$

Suppose that $q \in \left[p_1, \frac{2N-\mu}{4-\mu}\right]$ with $t \in (1, +\infty)$ and $\frac{1}{q} + \frac{1}{t} = 1$. Note that $2t - 2\mu > 0$ and $\delta > 1$,

$$
|\Phi_{\delta,y}| \in L^\infty(\mathbb{R}^N) \tag{4.6}
$$

and $|\Phi_{\delta,y}|^2 \in L^1(\mathbb{R}^N)$. Then, $|\Phi_{\delta,y}|^2 \in L'(\mathbb{R}^N)$. Applying again Hölder’s inequality with $q$ and $t$, we get

$$
\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 \, dx \leq |a|_q |\Phi_{\delta,0}|^{(2t-2\mu)/t} |\Phi_{\delta,0}|^{2\mu/t} \leq C|a|_q \delta^{((2N-\mu)/2)((2t-2\mu)/t)}, \forall y \in \mathbb{R}^N.
$$

Then, given $\varepsilon > 0$, there is $\overline{\delta} = \overline{\delta}(\varepsilon) > 1$ such that

$$
\delta^{((2N-\mu)/2)((2t-2\mu)/t)} < \frac{\varepsilon}{2|a|_q C}, \forall \delta \in [\overline{\delta}, \infty),
$$

which implies

$$
f(\Phi_{\delta,b}) = \int_{\mathbb{R}^N} |\nabla \Phi_{\delta,y}|^2 \, dx + \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 \, dx
\leq S_{H,L} + \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 \, dx
\leq S_{H,L} + \frac{\varepsilon}{2}
\leq S_{H,L} + \varepsilon, \forall y \in \mathbb{R}^N \text{ and } \forall \delta \in [\overline{\delta}, \infty).
$$

Lemma 4.4. Suppose that $|a|_{L((N-\mu/2)/(2-\mu))} < \frac{(4-\mu)(N-\mu+4)}{(2N+4-2\mu)(2N-\mu)} S_{H,L}$. Then,

$$
\sup_{y \in \mathbb{R}^N} f(\Phi_{\delta,y}) < 2^{\frac{(4-\mu)(N-\mu+4)}{(2N+4-2\mu)(2N-\mu)}} S_{H,L}.
$$

Proof. Employing Hölder’s inequality with $(N - \mu/2)/(N - 2)$ and $(N - \mu/2)/(N - 2)$, we find

$$
\sup_{y \in \mathbb{R}^N} f(\Phi_{\delta,y}) < S_{H,L} + S_{H,L}2^{\frac{(4-\mu)(N-\mu+4)}{(2N+4-2\mu)(2N-\mu)}} (2N+4-2\mu) S_{H,L}.
$$

In what follows, we set the functions

$$
\xi(x) = \begin{cases} 
0, & \text{if } |x| < 1 \\
1, & \text{if } |x| \geq 1
\end{cases}
$$

18
and $\alpha : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by
\[
\alpha(u) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \left( \frac{x}{|x|}, \xi(x) \right) |\nabla u|^2 \, dx = (\beta(u), \gamma(u)),
\]
where
\[
\beta(u) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx
\]
and
\[
\gamma(u) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \xi(x) |\nabla u|^2 \, dx.
\]

Lemma 4.5. If $|y| \geq \frac{1}{2}$, then
\[
\beta(\Phi_{\delta,y}) = \frac{y}{|y|} + o_{\delta}(1) \quad \text{when} \quad \delta \to 0.
\]

Proof. The proof follows can in \[3, Lemma 8\].

Now we define the set
\[
\mathcal{I} = \left\{ u \in \mathcal{M} : \alpha(u) = \left(0, \frac{1}{2}\right) \right\}.
\]
It is easy to see that $\mathcal{I}$ is not empty, because $\beta(\Phi_{\delta,0}) = 0$ and the limits below hold
\[
\gamma(\Phi_{\delta,0}) \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{and} \quad \gamma(\Phi_{\delta,0}) \to 1 \quad \text{as} \quad \delta \to +\infty.
\]

The next lemma establishes a first estimative from below for $c_0$.

Lemma 4.6. The number $c_0 = \inf_{u \in \mathcal{I}} f(u)$ satisfies the inequality $c_0 > S_{H,L}$.

Proof. Since $\mathcal{I} \subset \mathcal{M}$, we know that $S_{H,L} \leq c_0$.

Suppose, by contradiction, that $S_{H,L} = c_0$. By Ekeland’s variational principle \[34\], there exists $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ such that
\[
\frac{1}{C(N, \mu)} \int_{\mathbb{R}^N} (I_{\mu} * |u_n|^2 |u_n|^2) |u_n|^2 = 1, \quad \alpha(u_n) \to \left(0, \frac{1}{2}\right)
\]
and
\[
f(u_n) \to S_{H,L}, \quad f'|_{\mathcal{M}}(u_n) \to 0.
\]

Then, $(u_n)$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$.

If $v_n = e^{(N-2)/(2N-2\mu+4)} u_n$ and $v_0 = e^{(N-2)/(2N-2\mu+4)} u_0$, we have that $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$.

Moreover, from (4.8) and Lemma 3.2, we have
\[
I(v_n) \to \frac{(N-\mu+2)}{(4N-2\mu)} S^{(2N-\mu)/(N+2-\mu)}_{H,L} \quad \text{and} \quad I'(v_n) \to 0.
\]

We are going to show that $v_0 \equiv 0$. First of all, note that
\[
u_n \rightharpoonup u_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N),
\]
because otherwise, $u_0 \neq 0$ and
\[
S_{H,L} = \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx < \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx + \int_{\mathbb{R}^N} a(x) |u_0|^2 \, dx = S_{H,L},
\]
which is absurd. Hence, $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$, and since $(v_n)$ is a $(PS)_c$ sequence for $I$, by Theorem 3.1 we obtain that

$$I(v_n) \rightarrow I(v_0) + \sum_{j=1}^{k} I'_{\infty}(z_0^j) = \frac{(N - \mu + 2)}{(4N - 2\mu)} S_{H,L} (2N - \mu)/(N+2 - \mu).$$

Recalling that $I'_{\infty}(z_0^j) = 0$, we have that

$$I(v_0) = 0, \quad k = 1 \quad \text{and} \quad I'_{\infty}(z_0^1) = \frac{(N - \mu + 2)}{(4N - 2\mu)} S_{H,L} (2N - \mu)/(N+2 - \mu). \quad (4.10)$$

Since $v_0$ is weak solution to problem $(P)$, we also have

$$I(v_0) = \frac{(N - \mu + 2)}{(4N - 2\mu)} \int_{\mathbb{R}^N} (I_\mu * |v_0|^2')|v_0|^2 dx.$$

This fact combined with (4.10) yields $v_0 \equiv 0$. Then, $(v_n)$ is a $(PS)_c$ sequence for $I$ such that $v_n \rightharpoonup 0$, $v_n \rightarrow 0$ and $\int_{\mathbb{R}^N} a(x)|v_n|^2 dx = o_n(1)$. Therefore,

$$\frac{(N - \mu + 2)}{(4N - 2\mu)} S_{H,L} (2N - \mu)/(N+2 - \mu) + o_n(1) = I(v_n) = I'_{\infty}(v_n) + \int_{\mathbb{R}^N} a(x)|v_n|^2 dx = I_{\infty}(v_n) + o_n(1) \quad (4.11)$$

and

$$\|I'_{\infty}(v_n)\| = o_n(1). \quad (4.12)$$

From (4.11) and (4.12), $(v_n)$ is a $(PS)_c$ sequence for $I_{\infty}$, and by Lemma 3.1 there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$, $z_0^1$ nontrivial solution of $(P_{\infty})$ and $(w_n)$ a $(PS)_{c_0}$ sequence for $I_{\infty}$ such that

$$v_n(x) = w_n(x) + R_n^{(N-2)/2} z_0^1(R_n(x - x_n)) + o_n(1). \quad (4.13)$$

Setting

$$z_n(x) = R_n^{(N-2)/2} z_0^1(R_n(x - x_n)),$$

we have

$$I'_{\infty}(z_n)\varphi = I'_{\infty}(z_0^1)\varphi_n = 0, \quad \forall \varphi \in D^{1,2}(\mathbb{R}^N), \quad \forall n \in \mathbb{N},$$

i.e., $z_n$ a solution of $(P_{\infty})$, for all $n \in \mathbb{N}$. From (4.10), we know that $z_0 = \Phi_{\delta,y}$ for some $\delta > 0$ and $y \in \mathbb{R}^N$. Hence, there are $\delta_n > 0$ and $y_n \in \mathbb{R}^N$ such that

$$z_n(x) = \Phi_{\delta_n,y_n}(x) = C \left(\frac{\delta_n}{\delta_n^2 + |x - y_n|^2}\right)^{N-2/2}, \quad \forall x \in \mathbb{R}^N.$$

Thereby, by (4.13),

$$u_n(x) = \tilde{w}_n(x) + \Phi_{\delta_n,y_n}(x) + o_n(1),$$

where

$$\tilde{w}_n(x) = \frac{1}{S_{H,L} (2N - \mu)/(N+2 - \mu)} w_n(x).$$

Using (4.10), we derive that $w_n \rightarrow 0$, which implies that $\tilde{w}_n \rightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$. Thereby, from (4.7),

$$\left(0, \frac{1}{2}\right) + o_n(1) = \alpha(u_n) = \alpha(\tilde{w}_n + \Phi_{\delta_n,y_n} + o_n(1)) = \alpha(\Phi_{\delta_n,y_n})$$

implifying that

$$\begin{align*}
(i) & \quad \beta(\Phi_{\delta_n,y_n}) \rightarrow 0 \\
(ii) & \quad \gamma(\Phi_{\delta_n,y_n}) \rightarrow \frac{1}{2}.
\end{align*}$$

Passing to a subsequence, one of these cases below must occur:
(a) \( \delta_n \to +\infty \) when \( n \to +\infty \);
(b) \( \delta_n \to \tilde{\delta} \neq 0 \) when \( n \to +\infty \);
(c) \( \delta_n \to 0 \) and \( y_n \to \tilde{y} \) when \( n \to +\infty \) with \( |\tilde{y}| < \frac{1}{2} \);
(d) \( \delta_n \to 0 \) when \( n \to +\infty \) and \( |y_n| \geq \frac{1}{2} \) for \( n \) sufficient large.

Suppose that (a) is true. Then,
\[
\gamma(\Phi_{\delta_n,y_n}) = 1 - \frac{1}{S_{H,L}} \int_{B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx,
\]
together with Lemma 1 gives
\[
|\gamma(\Phi_{\delta_n,y_n}) - 1| = \frac{1}{S_{H,L}} \int_{B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx = o_n(1),
\]
which contradicts (ii).

Suppose that (b) is true. In this case we can suppose that \( |y_n| \to +\infty \), because if \( y_n \to \tilde{y} \), a direct computation shows that
\[
\Phi_{\delta_n,y_n} \to \Phi_{\tilde{\delta},\tilde{y}} \text{ in } D^{1,2}(\mathbb{R}^N).
\]
Since \( \tilde{w}_n \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \) and \( u_n = \tilde{w}_n + \Phi_{\delta_n,y_n} + o_n(1) \), \( (u_n) \) converges in \( D^{1,2}(\mathbb{R}^N) \) but this is a contradiction with (4.9). Therefore,
\[
\gamma(\Phi_{\delta_n,y_n}) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \xi(x)|\nabla \Phi_{\delta_n,y_n}|^2 dx = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx = 1 - \frac{1}{S_{H,L}} \int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dx.
\]
Applying Lebesgue Theorem we see that
\[
\int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dx \to 0,
\]
then by (4.14)
\[
\gamma(\Phi_{\delta_n,y_n}) \to 1 \text{ when } n \to +\infty,
\]
which is impossible by (ii).

Suppose that (c) is true. Note that
\[
\gamma(\Phi_{\delta_n,y_n}) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \xi(x)|\nabla \Phi_{\delta_n,y_n}|^2 dx = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx
\]
\[
= \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} |\nabla \Phi_{\delta_n,y_n}|^2 dx - \frac{1}{S_{H,L}} \int_{B_1(0)} |\nabla \Phi_{\delta_n,0}|^2 dx
\]
\[
= 1 - \frac{1}{S_{H,L}} \int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dx.
\]
Thereby, employing again Lebesgue Theorem, we find
\[
\lim_{n \to +\infty} \int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dx = S_{H,L},
\]
21
then by (4.15),
\[ \gamma(\Phi_{\delta_n, y_n}) \to 0, \]
which is a contradiction with (ii).

Suppose that (d) is true. Since \(|y_n| \geq \frac{1}{2}\) for \(n\) large, we have that \(y_n \not\to 0\) in \(\mathbb{R}^N\). From Lemma 4.5,
\[ \beta(\Phi_{\delta_n, y_n}) = \frac{y_n}{|y_n|} + o_n(1). \]
Hence,
\[ \beta(\Phi_{\delta_n, y_n}) \not\to 0, \]
which contradicts (i). Thus, \(S_{H,L} < c_0\) and the proof is over.
\[ \square \]

**Lemma 4.7.** There is \(\delta_1 \in (0, 1/2)\) such that

(a) \(f(\Phi_{\delta, y}) < \frac{S_{H,L} + c_0}{2}, \forall y \in \mathbb{R}^N;\)

(b) \(\gamma(\Phi_{\delta, y}) < \frac{1}{2}, \forall y \in \mathbb{R}^N\) such that \(|y| < \frac{1}{2};\)

(c) \(\left| \beta(\Phi_{\delta, y}) - \frac{y}{|y|} \right| < \frac{1}{4}, \forall y \in \mathbb{R}^N\) such that \(|y| \geq \frac{1}{2}.\)

**Proof.** Applying Lemma 4.3 with \(\varepsilon = \frac{c_0 - S_{H,L}}{2} > 0\) and \(\delta_2 < \min\{\delta, 1/2\},\) we find
\[ f(\Phi_{\delta, y}) \leq \sup_{y \in \mathbb{R}^N} f(\Phi_{\delta, y}) < S_{H,L} + \frac{c_0 - S_{H,L}}{2} = \frac{S_{H,L} + c_0}{2}, \forall y \in \mathbb{R}^N, \] showing that (i) holds. Now, by definition of \(\xi,\)
\[ \gamma(\Phi_{\delta, y}) = 1 - \frac{1}{S_{H,L}} \int_{B_1(-y)} |\nabla \Phi_{\delta, 0}|^2 dz. \]
By Lebesgue Theorem,
\[ \lim_{\delta \to 0} \int_{B_1(-y)} |\nabla \Phi_{\delta, 0}|^2 dz = S_{H,L} \]
proving (ii). Finally, note that from Lemma 4.5,
\[ \beta(\Phi_{\delta, y}) = \frac{y}{|y|} + o_\delta(1) \text{ when } \delta \to 0, \forall y \in \mathbb{R}^N; |y| \geq \frac{1}{2}, \]
and the proof is finished.
\[ \square \]

**Lemma 4.8.** There is \(\delta_2 > 1\) such that

(a) \(f(\Phi_{\delta_2, y}) < \frac{S_{H,L} + c_0}{2}, \forall y \in \mathbb{R}^N;\)

(b) \(\gamma(\Phi_{\delta_2, y}) > \frac{1}{2}, \forall y \in \mathbb{R}^N.\)

**Proof.** Applying again Lemma 4.3 with \(\varepsilon = \frac{c_0 - S_{H,L}}{2} > 0\) and \(\delta_3 > \max\{\delta, 1\},\) we derive
\[ f(\Phi_{\delta, y}) \leq \sup_{y \in \mathbb{R}^N} f(\Phi_{\delta, y}) < S_{H,L} + \frac{c_0 - S_{H,L}}{2} = \frac{S_{H,L} + c_0}{2}, \forall y \in \mathbb{R}^N. \]
Moreover, the definition of $\xi$ together with Lemma 4.1 leads to

$$\gamma(\Phi_{\delta,y}) \to 1 \text{ when } \delta \to +\infty,$$

and the proof is over.

\[\Box\]

**Lemma 4.9.** There is $R > 0$ such that

(a) $f(\Phi_{\delta,y}) < \frac{S_{H,L} + c_0}{2}$, \(\forall y; |y| \geq R \text{ and } \delta \in [\delta_1, \delta_2]\),

(b) $|\beta(\Phi_{\delta,y})|_{\mathbb{R}^N} > 0 \ \forall y; |y| \geq R \text{ and } \delta \in [\delta_1, \delta_2].$

**Proof.** The item (a) follows by employing Lemma 4.3 with $\varepsilon = \frac{c_0 - S_{H,L}}{2} > 0$. The item (b) can be done as in [7, Lemma 3.10].

\[\Box\]

## 5 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Have this in mind, we will fix some notations and prove more some technical lemmas. From now on, $\mathcal{V}$ denotes the following set

$$\mathcal{V} = \{(y, \delta) \in \mathbb{R}^N \times (0, \infty); |y| < R \text{ and } \delta \in (\delta_1, \delta_2)\},$$

where $\delta_1, \delta_2$ and $R$ are given in Lemmas 4.7, 4.8 and 4.9 respectively. Moreover, let us consider the continuous function $Q : \mathbb{R}^N \times (0, +\infty) \to D^{1,2}(\mathbb{R}^N)$ given by

$$Q(y, \delta) = \Phi_{\delta,y}.$$

Using the above notations, we also fix the sets

$$\Theta = \{Q(y, \delta); (y, \delta) \in \mathcal{V}\},$$

$$\mathcal{H} = \left\{h \in C(\Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M}); h(u) = u, \forall u \in (\Sigma \cap \mathcal{M}); f(u) < \frac{S_{H,L} + c_0}{2}\right\}$$

and

$$\Gamma = \{\mathcal{A} \subset (\Sigma \cap \mathcal{M}); \mathcal{A} = h(\Theta), h \in \mathcal{H}\}.$$

Note that $\Theta \subset (\Sigma \cap \mathcal{M}), \Theta = Q(\mathcal{V})$ is compact and $\mathcal{H} \neq \emptyset$, because the identity function is in $\mathcal{H}$.

**Lemma 5.1.** Let $\mathcal{F} : \mathcal{V} \to \mathbb{R}^{N+1}$ be a function given by

$$\mathcal{F}(y, \delta) = (\alpha \circ Q)(y, \delta) = \frac{1}{S_{H,L}} \int_{\mathbb{R}^N} \left(\frac{x}{|x|} \xi(x)\right) |\nabla \Phi_{\delta,y}|^2 dx.$$

Then,

$$d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = 1. \ (\text{Topological degree})$$

**Proof.** Let $\mathcal{Z} : [0, 1] \times \mathcal{V} \to \mathbb{R}^{N+1}$ be the homotopy given by

$$\mathcal{Z}(t, (y, \delta)) = t\mathcal{F}(y, \delta) + (1 - t)I_{\mathcal{V}}(y, \delta),$$

where $I_{\mathcal{V}}$ is the identity. We are going to show that $(0, 1/2) \notin \mathcal{Z}([0, 1] \times (\partial \mathcal{V}))$, i.e,

$$t\beta(\Phi_{\delta,y}) + (1 - t)y \neq 0, \ \forall t \in [0, 1] \text{ and } \forall (y, \delta) \in \partial \mathcal{V}$$

(5.1)
Note that \( \partial \mathcal{V} = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4 \), where

\[
\Lambda_1 = \{(y, \delta); |y| < 1/2\},
\]
\[
\Lambda_2 = \{(y, \delta); 1/2 \leq |y| \leq R\},
\]
\[
\Lambda_3 = \{(y, \delta); |y| \leq R\}
\]

and

\[
\Lambda_4 = \{(y, \delta); |y| = R \text{ and } \delta \in [\delta_1, \delta_2]\}.
\]

If \((y, \delta) \in \Lambda_1\), then \((y, \delta) = (y, \delta_1)\). From Lemma 4.7, \(\delta_1 < \frac{1}{2}\) and \(\gamma(\Phi_{\delta_1,y}) < \frac{1}{2}\). Hence,

\[
t\gamma(\Phi_{\delta,y}) + (1 - t)\delta = t\gamma(\Phi_{\delta_1,y}) + (1 - t)\delta_1 < \frac{t}{2} + (1 - t) \frac{1}{2} = \frac{1}{2}, \forall t \in [0, 1] \text{ and } \forall (y, \delta) \in \Lambda_1
\]

showing that (5.2) is true and \((0, 1/2) \notin \mathcal{Z}(\{0, 1\} \times \Lambda_1)\).

If \((y, \delta) \in \Lambda_2\), then \((y, \delta) = (y, \delta_1)\) and \(|y| \geq \frac{1}{2}\). Therefore,

\[
|t\beta(\Phi_{\delta,y}) - (1 - t)y| \geq |(1 - t)y + \frac{t}{|y|}y - \beta(\Phi_{\delta_1,y})| = \left|\frac{(1 - t)|y| + t}{|y|}y - \beta(\Phi_{\delta_1,y})\right|
\]
\[
= |(1 - t)|y| + t - \frac{y}{|y|} + \beta(\Phi_{\delta_1,y})|
\]
\[
= (1 - t)|y| + t - \frac{y}{|y|} - \beta(\Phi_{\delta_1,y})
\]

proving that (5.1) is true and \((0, 1/2) \notin \mathcal{Z}(\{0, 1\} \times \Lambda_2)\).

If \((y, \delta) \in \Lambda_3\), then \((y, \delta) = (y, \delta_2)\). By Lemma 1.8, \(\delta_2 > \frac{1}{2}\) and \(\gamma(\Phi_{\delta_2,y}) > \frac{1}{2}\), and so,

\[
t\gamma(\Phi_{\delta,y}) + (1 - t)\delta = t\gamma(\Phi_{\delta_2,y}) + (1 - t)\delta_2 < \frac{t}{2} + (1 - t) \frac{1}{2} = \frac{1}{2}, \forall t \in [0, 1] \text{ and } \forall (y, \delta) \in \Lambda_3,
\]

showing that (5.2) is true and \((0, 1/2) \notin \mathcal{Z}(\{0, 1\} \times \Lambda_3)\).

If \((y, \delta) \in \Lambda_4\), we must have \(|y| = R\). Thus, by Lemma 1.9, \((\beta(\Phi_{\delta,y})|y|)_{\mathbb{R}N} > 0\). From this,

\[
(t\beta(\Phi_{\delta,y}) + (1 - t)|y|)_{\mathbb{R}N} = t(\beta(\Phi_{\delta,y})|y|)_{\mathbb{R}N} + (1 - t)(|y|)_{\mathbb{R}N} > 0, \forall t \in [0, 1].
\]

Therefore (5.1) is true and \((0, 1/2) \notin \mathcal{Z}(\{0, 1\} \times \Lambda_4)\).
The previous analysis ensures that \((0, 1/2) \notin \mathcal{Z}([0, 1] \times \partial \mathcal{V})\), then by properties of the Topological degree
\[
d(F, \mathcal{V}, (0, 1/2)) = d(I_{\mathcal{V}}, \mathcal{V}, (0, 1/2)).
\]
Since \((0, 1/2) \in \mathcal{V}\), we derive that
\[
d(F, \mathcal{V}, (0, 1/2)) = d(I_{\mathcal{V}}, \mathcal{V}, (0, 1/2)) = 1.
\]

\[\square\]

**Lemma 5.2.** If \(A \in \Gamma\), then \(A \cap \mathcal{S} \neq \emptyset\).

**Proof.** It is sufficient to prove that for all \(h \in \mathcal{H}\), there exists \((y_0, \delta_0) \in \mathcal{V}\) such that
\[
(\alpha \circ \mathcal{H} \circ Q)(y_0, \delta_0) = \left(0, \frac{1}{2}\right).
\]
Given \(h \in \mathcal{H}\), let \(\mathcal{F}_h : \mathcal{V} \rightarrow \mathbb{R}^{N+1}\) be the continuous function given by
\[
\mathcal{F}_h(y, \delta) = (\alpha \circ h \circ Q)(y, \delta).
\]
We are going to show that \(\mathcal{F}_h = \mathcal{F}\) in \(\partial \mathcal{V}\). Note that
\[
\partial \mathcal{V} = \Pi_1 \cup \Pi_2 \cup \Pi_3,
\]
where
\[
\Pi_1 = \{(y, \delta_1); |y| \leq R\},
\]
\[
\Pi_2 = \{(y, \delta_2); |y| \leq R\}
\]
and
\[
\Pi_3 = \{(y, \delta); |y| = R \text{ and } \delta \in [\delta_1, \delta_2]\}.
\]
If \((y, \delta) \in \Pi_1\), then \((y, \delta) = (y, \delta_1)\), and by Lemma 4.7 (a),
\[
f(Q(y, \delta)) = f(Q(y, \delta_1)) = f(\Phi_{\delta_1}, y) < \frac{S_{H,L} + c_0}{2}, \quad \forall (y, \delta) \in \Pi_1.
\]
If \((y, \delta) \in \Pi_2\), then \((y, \delta) = (y, \delta_2)\), and by Lemma 4.8 (a),
\[
f(Q(y, \delta)) = f(Q(y, \delta_2)) = f(\Phi_{\delta_2}, y) < \frac{S_{H,L} + c_0}{2}, \quad \forall (y, \delta) \in \Pi_2.
\]
If \((y, \delta) \in \Pi_3\), then \(|y| = R\) and \(\delta \in [\delta_1, \delta_2]\), and by Lemma 4.9 (a),
\[
f(Q(y, \delta)) = f(\Phi_{\delta}, y) < \frac{S_{H,L} + c_0}{2}, \quad \forall (y, \delta) \in \Pi_3.
\]
From (5.4), (5.5), (5.6) and (5.7),
\[
f(Q(y, \delta)) < \frac{S_{H,L} + c_0}{2}, \quad \forall (y, \delta) \in \partial \mathcal{V}.
\]
Hence,
\[
\mathcal{F}_h(y, \delta) = (\alpha \circ h \circ Q)(y, \delta) = (\alpha \circ h)Q(y, \delta)
\]
\[
= \alpha(h(Q(y, \delta))) = \alpha(Q(y, \delta))
\]
\[
= (\alpha \circ Q)(y, \delta) = \mathcal{F}(y, \delta), \quad \forall (y, \delta) \in \partial \mathcal{V}.
\]
Since \((0, 1/2) \notin \mathcal{F}(\partial \mathcal{V})\), we have
\[ d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = d(\mathcal{F}_h, \mathcal{V}, (0, 1/2)). \]
Then by Lemma 5.1
\[ d(\mathcal{F}_h, \mathcal{V}, (0, 1/2)) = d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = 1, \]
and so, there exists \((y_0, \delta_0) \in \mathcal{V}\) such that
\[ \mathcal{F}_h(y_0, \delta_0) = (\alpha \circ h \circ Q)(y_0, \delta_0) = \left(0, \frac{1}{2}\right), \]
finishing the proof.

5.1 Proof of Theorem 1.1

Consider the number
\[ c = \inf_{A \in \Gamma} \max_{u \in A} f(u). \]
Our first step is to show that
\[ S_{H,L} < c < 2^{\frac{(4-\mu)(N-\mu+4)}{2N+4-2\mu(N-\mu)}} S_{H,L}. \]  
(5.8)
To this end, note that
\[ c = \inf_{A \in \Gamma} \max_{u \in A} f(u) \leq \max_{u \in \Theta} f(u) \leq \sup_{y \in \mathbb{R}^N} \Phi_{\delta,b} < 2^{\frac{(4-\mu)(N-\mu+4)}{2N+4-2\mu(N-\mu)}} S_{H,L}. \]
On the other hand, from Lemmas 5.2 and 4.6
\[ S_{H,L} < c_0 = \inf_{u \in \Sigma} f(u) \leq c, \]
from where it follows (5.8). In what follows, \(\lambda_\mu\) denotes the following real number
\[ \lambda_\mu = \frac{(4-\mu)(N-\mu+4)}{2N+4-2\mu(N-\mu)}. \]
Hence,
\[ S_{H,L} < c_0 \leq c < 2^{\lambda_\mu} S_{H,L}. \]  
(5.9)
Now, in order to conclude the proof of theorem, we are going to prove the following claim

Claim 5.1. \(K_c = \{u \in \Sigma \cap \mathcal{M}; f(u) = c \text{ and } f'|\mathcal{M}(\tilde{u}) = 0\} \neq \emptyset.\)

If the claim does not hold, by Deformation Lemma [13], there exists a continuous application \(\eta : [0, 1] \times (\Sigma \cap \mathcal{M}) \to \Sigma \cap \mathcal{M}\) and \(\varepsilon_0 > 0\) such that

1. \(\eta(0, u) = u;\)
2. \(\eta(t, u) = u, \forall u \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}, \forall t \in [0, 1];\)
3. \(\eta(1, f^{c+\varepsilon_0}) \subset f^{c-\varepsilon_0};\)
where for each $q \in \mathbb{R}$, $f^q$ denotes the set

$$f^q = \{ u \in \Sigma \cap M : f(u) \leq q \}.$$

Using the above notations, we have the following inclusion

$$f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0} \subset f^{2^{\lambda_S} S_{H,L}} \setminus f^{(S_{H,L} + c_0)/2}. \quad (5.10)$$

Indeed, given $u \in f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}$, we have

$$c - \varepsilon_0 < f(u) \leq c + \varepsilon_0.$$

By (5.10), for $\varepsilon_0$ sufficiently small, we get

$$c - \varepsilon_0 < f(u) \leq c + \varepsilon_0 < 2^{\lambda_S} S_{H,L}. \quad (5.11)$$

On the other hand, Lemma 4.6 together with (5.9) gives

$$\frac{S_{H,L} + c_0}{2} < c_0 - \varepsilon_0 \leq c - \varepsilon_0 < f(u). \quad (5.12)$$

Now, (5.10) follows from (5.11) and (5.12).

From definition of $c$, there exists $\tilde{A} \in \Gamma$ such that

$$c \leq \max_{u \in \tilde{A}} f(u) < c + \frac{\varepsilon_0}{2},$$

where

$$\tilde{A} \subset f^{c+\frac{\varepsilon_0}{4}}. \quad (5.13)$$

Since $\tilde{A} \in \Gamma$, $\tilde{A} \subset (\Sigma \cap M)$ and there exists $\tilde{h} \in \mathcal{H}$ such that

$$\tilde{h}(\Theta) = \tilde{A}. \quad (5.14)$$

From definition of $\eta$, we have

$$\eta(1, \tilde{A}) \subset \Sigma \cap M. \quad (5.15)$$

In the sequel, we fix the function $\hat{h} : \Sigma \cap M \to \Sigma \cap M$ given by $\hat{h}(u) = \eta(1, \tilde{h}(u))$ that belongs to $C(\Sigma \cap M, \Sigma \cap M)$. Consider $u \in \Sigma \cap M$ such that

$$f(u) < \frac{S_{H,L} + c_0}{2}. \quad (5.16)$$

Then,

$$\tilde{h}(u) = u$$

and from (5.15) and (5.16), $u \notin f^{2^{\lambda_S} S_{H,L} \setminus f^{(S_{H,L} + c_0)/2}}$, and so, $u \notin f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}$. Hence,

$$u \in f^{c-\varepsilon_0} \cup \{ (\Sigma \cap M) \setminus f^{c+\varepsilon_0} \}.$$

Thereby, by Deformation lemma,

$$\eta(1, u) = u,$$

and so,

$$\hat{h}(u) = \eta(1, \tilde{h}(u)) = \eta(1, u) = u.$$

This shows that $\hat{h} \in \mathcal{H}$, and so,

$$\hat{h}(\Theta) = \eta(1, \tilde{h}(\Theta)) = \eta(1, \tilde{A}) \in \Gamma. \quad (5.17)$$
Hence,

$$c = \inf_{A \in \Gamma} \max_{u \in A} f(u) \leq \max_{u \in \eta(1,\tilde{A})} f(u).$$  \hspace{1cm} (5.18)$$

On the other hand, from Deformation lemma and (5.13),

$$\eta(1,\tilde{A}) \subset \eta(1,f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}},$$

that is,

$$f(u) \leq c - \frac{\varepsilon_0}{2}, \quad \forall u \in \eta(1,\tilde{A}),$$

or yet,

$$\max_{u \in \eta(1,\tilde{A})} f(u) \leq c - \frac{\varepsilon_0}{2}.$$  \hspace{1cm} (5.15)$$

This together with (5.15) leads to

$$c \leq \max_{u \in \eta(1,\tilde{A})} f(u) \leq c - \frac{\varepsilon_0}{2},$$

which is absurd, showing Claim 5.1. Therefore, there is $\tilde{u}_0 \in \Sigma \cap M$, such that

$$f(\tilde{u}_0) = c \quad \text{and} \quad f'|_M(\tilde{u}_0) = 0.$$  \hspace{1cm} (5.16)$$

The positivity of $\tilde{u}_0$ is a consequence of the maximum principles. \hfill \Box

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