THE CRITICAL HEIGHT IS A MODULI HEIGHT

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Abstract. Silverman defined the critical height of a rational function $f(z)$ of degree $d \geq 2$ in terms of the asymptotic rate of growth of the Weil height along the critical orbits of $f$, and conjectured that this quantity was commensurate to an ample Weil height on the moduli space of rational functions degree $d$. We prove this conjecture.

If $f(z) \in \overline{\mathbb{Q}}(z)$ is a rational function of degree $d \geq 2$, the canonical height $\hat{h}_f$ associated to $f$ is uniquely determined by the first two of its three fundamental properties [16, p. 99]:

A. $\hat{h}_f(f(P)) = d\hat{h}(f)$,
B. $\hat{h}_f(P) = h_{\mathbb{P}^1}(P) + O_f(1)$, with $h_{\mathbb{P}^1}$ the usual Weil height, and
C. $\hat{h}_f(P) = 0$ if and only if $P$ is preperiodic.

These properties make the canonical height the natural choice of measure of arithmetic complexity on $\mathbb{P}^1$, relative to $f$, and indeed it has been ubiquitous in the study of arithmetic dynamics.

Moving from dynamical space to the moduli space $\mathcal{M}_d$ of all rational functions of degree $d$, Silverman proposed a natural measure of dynamical complexity based on the critical orbits. In particular, the critical height is defined by

$$\hat{h}_{\text{crit}}(f) = \sum_{P \in \mathbb{P}^1} (e_P(f) - 1)\hat{h}_f(P),$$

a definition which is independent of choice of coordinates. The orbits of critical points carry a great deal of information about a dynamical system, and so this is a natural candidate for a "canonical height" on moduli space, and indeed it enjoys two properties similar to properties above

A’. $\hat{h}_{\text{crit}}(f^n) = n\hat{h}_{\text{crit}}(f)$,
C’. $\hat{h}_{\text{crit}}(f) = 0$ if and only if $f$ is post-critically finite (PCF), that is, if and only if every critical orbit is finite.

One might hope that $\hat{h}_{\text{crit}}$ turns out to be an ample Weil height on $\mathcal{M}_d$, in analogy with the second property of $\hat{h}_f$, but the algebraic geometry of $\mathcal{M}_d$ is more complicated than that of $\mathbb{P}^1$, and besides the Lattès examples show that this is just not true. Indeed, $\hat{h}_f$ is constructed directly from $h_{\mathbb{P}^1}$, making property B above fairly unsurprising, while $\hat{h}_{\text{crit}}$ is defined pointwise in terms of the dynamics of each map, with no mention of, or obvious relation

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to, the arithmetic geometry of $M_d$. Nonetheless, Silverman conjectured [17, Conjecture 6.29, p. 101] that $\hat{h}_{\text{crit}}$ is commensurate to any ample Weil height on $M_d$, away from the Lattès examples, and our main result confirms this conjecture.

**Theorem 1.** For any ample Weil height $h_{M_d}$ on $M_d$,

$$\hat{h}_{\text{crit}} \asymp h_{M_d}$$

except at the Lattès examples. In other words there exist positive constants $c_1, c_2, c_3,$ and $c_4$ such that

$$c_1 h_{M_d}(f) - c_2 \leq \hat{h}_{\text{crit}}(f) \leq c_3 h_{M_d}(f) + c_4$$

for all non-Lattès $f \in M_d$.

Note that, since $\hat{h}_{\text{crit}}(f) = 0$ for any PCF map $f$, Theorem 1 contains as a special case the main result of [4], namely that the non-Lattès PCF locus is a set of bounded height in moduli space. We also note that the upper bound on $\hat{h}_{\text{crit}}(f)$ is relatively straightforward from estimates of the difference between the canonical height and the usual Weil height, and is contained already in [17, Theorem 6.31, p. 101]. The lower bound on $\hat{h}_{\text{crit}}(f)$ is the new piece added here.

The proof goes roughly as follows (with estimates made precise in Sections 1 and 2). It is known, classically over $\mathbb{C}$ and from [4] over non-archimedean fields, that any sufficiently attracting (but not super-attracting) fixed point linearly attracts a critical orbit. For each valuation $v$ and each fixed point $\gamma$ of multiplier $\lambda \neq 0$, there exists a critical point $\zeta$ for which this linear attraction gives an estimate of the form

$$\log^+ \left| \frac{1}{k(f^k(\zeta) - \gamma)} \right| \geq k \log^+ |\lambda^{-1}|_v - C_1(f, k, v),$$

adjusting the constant to make the statement true even when $|\lambda|_v$ is not small. Summing over all critical points, all fixed points, and all places, we obtain an estimate of the form

$$kh(\lambda_0, \ldots, \lambda_{d-1}) \leq \sum_{f^k(\zeta) = 0} h(f^k(\zeta)) + C_2(f, k),$$

where $\lambda_0, \ldots, \lambda_{d-1}$ are the multipliers at the fixed points (some of which may now vanish), and using an estimate $h(P) = \hat{h}_f(P) + O_f(1)$, we further obtain

$$kh(\lambda_0, \ldots, \lambda_{d-1}) \leq d^k \hat{h}_{\text{crit}}(f) + C_3(f, k).$$

This can be applied to some iterate to bound $\hat{h}_{\text{crit}}(f)$ below by the heights of multipliers of points of period dividing $n$, for any $n$, and this in turn may be bounded below in terms of $h_{M_d}(f)$ for some $n$ depending just on $d$ (an idea used already in [4], relying on McMullen’s Theorem on stable families [14], and made more precise below). Ultimately we end up with an estimate of the form

$$\varepsilon h_{M_d}(f) \leq d^k \hat{h}_{\text{crit}}(f) + C_4(f, k),$$
for all \( k \geq 1 \), with \( \varepsilon > 0 \) depending just on \( d \). Of course, with no information about the error term this estimate could easily be trivial, but it turns out that we may take
\[
C_4(f, k) = C_5 h_{M_d}(f) + C_6 k,
\]
with \( C_5 \) and \( C_6 \) depending just on \( d \). Once \( k \) is large enough, the contribution of \( h_{M_d}(f) \) in the lower bound exceeds that in the error term, giving a lower bound on \( \hat{h}_{\text{crit}}(f) \) of the sort claimed in Theorem 1.

The proof of Theorem 1 gives lower bounds on the critical height based on the height of the multiplier of any given periodic point, which is of interest in the study of certain fibrations of \( M_d \). In particular, the subvarieties \( \text{Per}_n(\lambda) \subseteq M_d \) of rational functions with an \( n \)-cycle of multiplier \( \lambda \) are well-studied \cite{6,12,13}, and of notable interest is the distribution of PCF points on these subvarieties. It follows from the main result of \cite{4} that \( \text{Per}_n(\lambda) \) contains no PCF points at all once \( h(\lambda) \) is large enough, and so for \( h(\lambda) \) large the function \( \hat{h}_{\text{crit}} \) is non-vanishing on \( \text{Per}_n(\lambda) \). It is natural to ask whether or not a Bogomolov-type phenomenon occurs, in which \( \hat{h}_{\text{crit}} \) has a positive infimum on \( \text{Per}_n(\lambda) \). Our next result gives a uniform asymptotic in this direction.

**Theorem 2.** For fixed \( n \geq 1 \) and \( d \geq 2 \), there exist constants \( \varepsilon > 0 \) and \( B \) such that
\[
\frac{\hat{h}_{\text{crit}}(f)}{h(\lambda)} \geq \varepsilon > 0
\]
for all \( f \in \text{Per}_n(\lambda) \), for all \( \lambda \) with \( h(\lambda) > B \).

It follows from this that if \( \hat{h}_{\text{crit}} \) takes arbitrarily small values on \( \text{Per}_n(\lambda) \), then \( h(\lambda) \) is bounded, and the proof gives such a bound. The \( m \)th multiplier spectrum is the morphism taking \( f \in M_d \) to (the symmetric functions in) the multipliers of the fixed points of \( f^m \), and by McMullen’s Theorem on stable families \cite[Corollary 2.3]{14} there exists an \( m \) for which this morphism is finite away from the Lattès locus. For this \( m \), if \( \hat{h}_{\text{crit}} \) takes arbitrarily small values on \( \text{Per}_1(\lambda) \), we have (from the proof of Theorem 2)
\[
h(\lambda) \leq \frac{1}{n} \left( 1.04d^{2mn} + d^{mn} \log 3 - \log 3 \right).
\]

Although Theorem 1 resolves the conjecture made by Silverman, it is by no means easy to recover explicit values for the claimed constants from the proof, and so it is not obvious how one could effectively list, say, all \( f(z) \in \mathbb{Q}(z) \) of degree 4 and critical height at most 10 (although the theorem certainly says that this is a finite list, up to conjugacy). Note that the more elementary methods employed for polynomials in \cite{10} did offer this level of information, and here there are two special cases in which the constants can be made increasingly concrete.

The first case is that of rational functions with a super-attracting fixed point (generalizing the case of polynomials \cite{10}). Note that if we are to obtain any sort of explicit lower bound on \( \hat{h}_{\text{crit}}(f) \) in terms of the coefficients
of \( f \), we will need to make some assumptions normalizing the choice of coordinate. Here \( h_{\text{Hom},d}(f) \) is the height of the tuple of coefficients of \( f \) as a point in \( \mathbb{P}^{2d+1} \).

**Theorem 3.** For any \( d \geq e \geq 2 \) there exists an explicit constant \( C_{d,e} \) such that if \( \deg(f) = d \), \( f(z) = z^e + O(z^{e+1}) \) formally at \( z = 0 \), and \( f(\infty) = \infty \), then

\[
\hat{h}_{\text{crit}}(f) \geq \left( \frac{1}{(d-1)d^2(4d^2 - 2d - 1)\log d/\log e} \right) h_{\text{Hom},d}(f) - C_{d,e}.
\]

Note that any rational function of degree \( d \) with a fixed point of local degree \( e \geq 2 \) can be put in this form, since a super-attracting fixed point cannot be the only fixed point of \( f \). On the other hand, the lower bound in the statement depends on the chosen form, since \( h_{\text{Hom},d} \) is by no means constant on conjugacy classes. An explicit value for \( C_{d,e} \) appears in equation (20) below. Examining the effect of change-of-coordinates on \( f \), we can establish an inequality of the sort in Theorem 3 for any rational function with a fixed point of local degree \( e \), but the error term will depend (linearly) on the height of this fixed point, the height of the first non-zero coefficient of the Taylor series of \( f \) at this point, and the height of one other fixed point (three data which suffice to fix a coordinate).

**Corollary 4.** For any \( d \geq 2 \), and any \( B \), let \( S \subseteq M_d \) be the set of conjugacy classes of PCF rational functions of degree \( d \) admitting a super-attracting fixed point, and admitting a model with coefficients of algebraic degree at most \( B \). Then there is a finite and effectively computable list representatives of the classes in \( S \).

The finiteness follows from [4, Theorem 1.1]; what is new in this corollary is the explicit nature of the bound.

We present one last case in which we can give completely precise (although probably not sharp) bounds. Milnor [13] explicitly described the moduli space of quadratic morphisms, and (except for a single one-parameter family that can be handled easily on its own) the family defined for \((\lambda_0, \lambda_\infty) \in \mathbb{A}^2\) by

\[
f_{\lambda_0, \lambda_\infty}(z) = \frac{\lambda_0 z + z^2}{\lambda_\infty z + 1}
\]

offers a double-cover of \( M_2 \). For computational purposes, it is preferable to work on this cover, and to replace \( h_{M_2}(f) \) by the more explicit, but comparable, \( h_{P^2}(\lambda_0, \lambda_\infty) \).

**Theorem 5.** For all \( \lambda_0, \lambda_\infty \in \overline{\mathbb{Q}} \), we have

\[
\hat{h}_{\text{crit}}(f_{\lambda_0, \lambda_\infty}) \geq \frac{1}{2048} h_{P^2}(\lambda_0, \lambda_\infty) - \frac{3}{256}.
\]

Unfortunately, the constants above, while less intimidating than those in Theorem 3, are still large enough that computing the smallest positive critical height on \( M_2(\mathbb{Q}) \), say, might still be out of reach.
Corollary 6. Let $\text{Per}_1(\lambda) \subseteq M_2$ be the collection of quadratic morphisms with a fixed point of multiplier $\lambda$, and suppose that

$$\inf_{f \in \text{Per}_1(\lambda)} \hat{h}_{\text{crit}}(f) = 0.$$  

Then $h(\lambda) \leq \log 12$.

The results above hold in number fields, but the proofs generally depend on a lot less, all carrying over for function fields of characteristic 0, and some for function fields of characteristic $p > d$, often with much improved constants. The observations have consequences for holomorphic families of dynamical systems.

In order to state the next theorem, let $X$ be a smooth, projective curve, and let $f : X \to \text{Rat}_d$ be a morphism not landing entirely in the resultant locus, that is, a family of rational functions of degree $d$. We may speak of $\deg(P)$ for any section $X \to \mathbb{P}^1$, and more generally for a pair of morphisms $\varphi : Y \to X$ and $Q : Y \to \mathbb{P}^1$, write

$$\deg_X(Q) = \deg_Y(Q) \deg(\varphi).$$

Theorem 7. Let $f$ be a family of rational functions of degree $d \geq 2$, with

$$f(z) = z^e + O(z^{e+1})$$

formally, for some $e \geq 2$, and $f(\infty) = \infty$. Then there exists a critical point $P$ of the generic fibre with

$$\deg_X(f^n(P)) \geq (\varepsilon d^n - C) \deg(f^* \text{Res}_d)$$

for all $n \geq 0$, for some explicit constants $\varepsilon > 0$ and $C$ depending just on $d$.

Note that if the generic fibre of the family is PCF, then this theorem immediately implies that the family is constant.

As another application of the main result, we give a refinement of McMullen’s Theorem on stable families. Let $f : X \to \text{Rat}_d$ as above, let $S_f = \text{Supp}(f^* \text{Res}_d)$, and let $U = X \setminus S_f$ (which we assume to be non-empty). Also, let

$$A_f = \{ n \in \mathbb{Z}^+ : f_t \text{ has an attracting } n\text{-cycle for some } t \in U(\mathbb{C}) \}.$$  

McMullen’s rigidity theorem for stable families may be phrased as follows.

Theorem 8 (McMullen [14 Theorem 2.2]). Given $f$ and $A_f$ as above, either

1. $f$ is isotrivial or Lattès, and $A_f$ is finite, or
2. $f$ is neither isotrivial nor Lattès, and $A_f$ is infinite.

In order to state our refinement of this, we say that the family $f$ is weakly minimal if there does not exist an extension of the base $\varphi : Y \to X$ and a $g : Y \to \text{Rat}_d$ in the same conjugacy class as $f \circ \varphi$ with

$$\deg g^* \text{Res}_d < \frac{1}{2} \deg(\varphi) \deg f^* \text{Res}_d.$$
Note that any $f$ is conjugate, over some extension, to a weakly minimal family.

**Theorem 9.** Given $f$ and $A_f$ as above, either

1. $f$ is isotrivial or Lattès, and $A_f$ is finite, or
2. $f$ is neither isotrivial nor Lattès, and $A_f$ is cofinite. In this case if $f$ is weakly minimal, then the the cardinality of $\mathbb{Z}^+ \setminus A_f$ is bounded in terms of $d$, $X$, and $\# \text{Supp} f^* \text{Res}_d$.

Given a family $f : X \to \text{Rat}_d$ without a generically attracting cycle, let $u \in \mathbb{C}(X)$ vanish on any $t$ such that $f_t$ has an attracting cycle of period up to $N$. Then $A_{u^{-1}fu} \cap \{1, \ldots, N\} = \emptyset$, simply because $\text{Res}(u^{-1}fu) = \text{Res}(f)u^{d(d-1)}$ vanishes on every fibre that had an attracting cycle. In other words, without some measure of how much unnecessary bad reduction had been introduced into the model, there is no hope for a bound as uniform as that in Theorem 9.

In Section 1, we gather the key estimates on local Arakelov-Greens’s functions associated to endomorphisms of $\mathbb{P}^1$. We sum these in Section 2 to prove Theorem 1 and in the process Theorem 2. In Section 3, we take up the problem of obtaining explicit bounds for rational functions admitting super-attracting fixed points, proving Theorems 3 and 7, and in Section 4 we treat Theorem 5. Finally, in Section 5, we present an application of Theorem 1 to a question of primitive divisors, resulting in Theorem 9.

Before proceeding to the technical details, we leave the reader with a conjecture. In some sense, Silverman’s conjecture that the non-Lattès PCF maps form a set of bounded height was justified by Thurston’s result that any non-Lattès family of PCF maps is isotrivial. As was pointed out to the author by Epstein, using McMullen’s results in [14], one can establish many other such rigidity statements. For instance, given roots of unity $\lambda_0, \ldots, \lambda_k$, any family of rational functions with $k$ fixed points of multiplier $\lambda_0, \ldots, \lambda_k$ and fewer than $k$ infinite critical orbits must be stable, and hence isotrivial. We propose a generalization of Theorem 1 motivated by this observation.

Write out the critical points of $f$ as $c_1, \ldots, c_{2d-2}$, and for each $0 \leq k \leq 2d-2$ define the $k$-depleted critical height of $f$ by

$$\hat{h}_{\text{crit}}^{(k)}(f) = \min_{I \subseteq \{1, \ldots, 2d-2\} \atop |I| = k} \sum_{i \notin I} \hat{h}(c_i).$$

Note that

$$0 = \hat{h}_{\text{crit}}^{(2d-2)}(f) \leq \cdots \leq \hat{h}_{\text{crit}}^{(k)}(f) \leq \cdots \leq \hat{h}_{\text{crit}}^{(0)}(f) = \hat{h}_{\text{crit}}(f).$$

Also, for any $\lambda_0, \ldots, \lambda_k \in \mathbb{C}$, let

$$\text{Per}_{n_1, \ldots, n_k}(\lambda_0, \ldots, \lambda_k) \subseteq M_d$$

be the subvariety of rational functions with (generically distinct) $n_i$ cycles of multiplier $\lambda_i$, for all $1 \leq i \leq k$.
Conjecture. For any non-zero $\lambda_0, \ldots, \lambda_k \in \overline{\mathbb{Q}}$, $h_{M_d}(f) \ll \hat{h}^{(k)}_{\text{crit}}(f)$ for non-Lattès $f \in \text{Per}_{n_1, \ldots, n_k}(\lambda_0, \ldots, \lambda_k)$, with implied constants depending on $n_1, \ldots, n_k$ and $h(\lambda_0), \ldots, h(\lambda_k)$. In particular, for fixed $n_1, \ldots, n_k$ and $\lambda_0, \ldots, \lambda_k$, the collection of non-Lattès $f \in \text{Per}_{n_1, \ldots, n_k}(\lambda_0, \ldots, \lambda_k)$ with at most $k$ infinite critical orbits is a set of bounded height.

The assumption that the $\lambda_i$ are non-zero is certainly necessary. Note that $\text{Per}_1(0) \subseteq M_2$ is the space of quadratic polynomials, on which we certainly have $\hat{h}^{(1)}_{\text{crit}} = 0$ with $h_{M_d}$ unbounded.

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1. Estimates for Green’s functions

In this section we let $K$ be an algebraically closed field, complete with respect to the absolute value $|\cdot|$, which might be archimedean or not. In the archimedean case, it will suffice to consider the case $K = \mathbb{C}$ (since every archimedean field is, up to scaling of the absolute value, a subfield of $\mathbb{C}$). Note that if $n$ is an integer,

$$\log^+ |n| = \begin{cases} \log n & \text{if } v \text{ is archimedean} \\ 0 & \text{otherwise,} \end{cases}$$

which we use to simplify notation in several places. Indeed, the triangle and ultametric inequalities appear often as

$$\log |x_1 + \cdots + x_n| \leq \log \max \{|x_1|, \ldots, |x_n|\} + \log^+ |n|.$$ 

We will also write $D(a, r)$ for the open disk of radius $r$ at $a$, that is, $D(a, r) = \{z \in K : |a - z| < r\}$.

Given a rational function $f(z) \in K(z)$ with $d = \deg(f) \geq 2$, we recall the construction of the dynamical Arakelov-Greens’ function $g_f : (\mathbb{P}^1_K)^2 \to \mathbb{R}$ associated to $f$ (see [3] for more details). We first choose a pair of homogeneous forms $F_1, F_2 \in K[x, y]$ with $f(x/y) = F_1(x, y)/F_2(x, y)$, and then define for $F = (F_1, F_2)$,

$$H_F(x, y) = \lim_{n \to \infty} d^{-n} \log \|F^n(x, y)\|,$$

where we always take $\|x_1, \ldots, x_m\| = \max\{|x_1|, \ldots, |x_m|\}$. We then set $g_f([x : y], [z : w]) = -\log |yz - xw| + H_F(x, y) + H_F(z, w) - r(F)$, for $r(F) = \frac{1}{d(d - 1)} \log |\text{Res}(F_0, F_1)|$. 

This is easily shown to be independent of choice of representative homogeneous coordinates, and even of the choice of the forms $F_0$ and $F_1$ (but we will in fact impose a particular choice below).

For the remainder of this section, we consider rational functions of the form

\[(1) \quad f(z) = \frac{\lambda z + \cdots + a_d z^d}{1 + b_1 z + \cdots + b_d z^d} = \frac{\lambda z \prod (1 - \alpha_i z)}{\prod (1 - \beta_j z)},\]

and after the proof of Lemma 10 we will assume that $\lambda \neq 0$ and that $b_d = 0$, ensuring that $f(\infty) = \infty$. We set

$$\|f\| = \max\{|\lambda|, |a_2|, \cdots, 1, |b_1|, \cdots, |b_d|\},$$

noting that this implies $\log \|f\| \geq 0$. For convenience of notation, we will also set $a_1 = \lambda$ and, in the early part of the next proof, refer to the constant term in the denominator simply as $b_0$ (it does not matter until later that $b_0 = 1$). In terms of the construction of $g_f$ above, we choose once and for all the obvious homogeneous lift

$$F(x, y) = (\lambda x y^{d-1} + \cdots + a_d x^d, y^d + \cdots + b_d x^d),$$

allowing us to speak unambiguously about $r(f) = r(F)$.

**Lemma 10.** For $f$ of the form (1) and for all $z \in \mathbb{P}_K^1$,

$$g_f(z, 0) \geq \log^+ |z^{-1}| - \frac{1}{d-1} \log^+ |d(2d - 1)!| - \frac{2d - 1}{d - 1} \log \|f\| + (d - 1)r(f).$$

**Proof.** This estimate is quite general (we do not assume $\lambda \neq 0$), and in large part standard, but for lack of a good reference with explicit constants we present a complete proof.

Let

$$F_1(x, y) = A_d x^d + \cdots + A_0 y^d$$
$$F_2(x, y) = B_d x^d + \cdots + B_0 y^d$$

with coefficients left indeterminate for now, and with no other hypotheses. If

$$G_1(x, y) = C_{d-1} x^{d-1} + \cdots + C_0 y^{d-1}$$
$$G_2(x, y) = D_{d-1} x^{d-1} + \cdots + D_0 y^{d-1},$$

then

$$G_1F_1 + G_2F_2 = \sum_{k=0}^{2d-1} x^k y^{2d-1-k} \sum_{i \geq 0, j \geq 0 \atop i + j = k} (A_i C_j + B_i D_j).$$
Attempting to solve $G_1 F_1 + G_2 F_2 = x^{2d-1}$ in $\mathbb{Q}(A_0, ..., B_d)$, then, amounts to solving the equation

$$
\begin{bmatrix}
A_0 & 0 & \cdots & 0 & B_0 & 0 & \cdots & 0 \\
A_1 & A_0 & \cdots & 0 & B_1 & B_0 & \cdots & 0 \\
A_2 & A_1 & \cdots & 0 & B_2 & B_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_d & 0 & 0 & \cdots & B_d \\
\end{bmatrix}
\begin{bmatrix}
C_0 \\
\vdots \\
C_{d-1} \\ \\
D_0 \\
\vdots \\
D_{d-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
\end{bmatrix},
\end{equation}
$$

which we may do using Cramer’s Rule. Note that the solution is

$$
C_i = \frac{\det(M_i)}{\det(M)} \quad \text{and} \quad D_j = \frac{\det(M'_j)}{\det(M)},
$$

where $M$ is the circulant matrix above, $M_i$ is its $(2d, i+1)$ minor, and $M'_j$ the $(2d, d+1+j)$ minor. We have $\text{Res}(F_1, F_2) = \det(M') = \det(M)$ by definition. Clearing denominators, we have (for $g_i = \text{Res}(F_1, F_2) G_i$)

$$
g_1 F_1 + g_2 F_2 = \text{Res}(F_1, F_2) x^{2d-1},
$$

where $g_1$ and $g_2$ are homogeneous forms of degree $d - 1$ in $x$ and $y$, with coefficients in $\mathbb{Z}[A_0, ..., B_d]$. Even more concretely, each coefficient of $g_i$ is the determinant of a $(2d-1) \times (2d-1)$ matrix with entries in the set $\{0, A_0, ..., B_d\}$, and is hence a sum of at most $(2d-1)!$ terms of the form $\pm A_0^{e_0} \cdots B_d^{e_d}$ of degree $2d - 1$. If we specialize the $A_i$ and $B_i$, such that $\text{Res}(F_1, F_2) \neq 0$, this gives a straightforward bound on the coefficients of $g_1$ and $g_2$. Construct in the same way forms $h_1, h_2$ satisfying

$$
h_1 F_1 + h_2 F_2 = \text{Res}(F_1, F_2) y^{2d-1}.
$$

Now, specializing $A_i = a_i \in K$ and $B_i = b_i \in K$, and letting $\xi_{i,j}$ and $\zeta_{i,j}$ be the coefficients of $g_i$ and $h_i$ respectively, we have for any $x, y \in K$

$$
(2d - 1) \log \|x, y\| \leq \log \|F(x, y)\| + \log \max\{|g_1(x, y)|, |g_2(x, y)|, |h_1(x, y)|, |h_2(x, y)|\} - \log |\text{Res}(F_1, F_2)|
$$

$$
\leq \log \|F(x, y)\| + \log^+ |d|
$$

$$
+ \log \|\xi_{i,j}, \zeta_{i,j}\|
$$

$$
+ (d - 1) \log \|x, y\| - \log |\text{Res}(F_1, F_2)|
$$

$$
\leq \log \|F(x, y)\| + \log^+ |d(2d - 1)!|
$$

$$
+ (2d - 1) \log \|a_0, ..., b_d\|
$$

$$
+ (d - 1) \log \|x, y\| - \log |\text{Res}(F_1, F_2)|,
$$

where $\log^+$ is the non-negative logarithm.
or
\[
\frac{1}{d} \log \|F(x, y)\| \geq \log \|x, y\| - \frac{2d - 1}{d} \log \|a_0, ..., b_d\| - \frac{1}{d} \log^+ |d(2d - 1)!| + \frac{1}{d} \log |\text{Res}(F_1, F_2)|.
\]

Iterating, and taking limits, we have
\[
H_F(x, y) \geq \log \|x, y\| - \frac{1}{d - 1} \log^+ |d(2d - 1)!| - \frac{2d - 1}{d - 1} \log \|F\| + dr(F).
\]

With \(\|F\| = \max\{|a_0|, ..., |b_d|\}\) and \(r(F) = \frac{1}{d(d-1)} \log |\text{Res}(F_1, F_2)|\).

Now, returning to our hypotheses, our lift of \(f\) satisfies \(F(0, 1) = (0, 1)\) so \(H_f(0, 1) = 0\). By definition, for \(z = [x : y]\) we have
\[
g_f(z, 0) = g_f([x : y], [0 : 1]) = - \log |x| + H_f(x, y) - r(f) \geq \log^+ |z^{-1}| - \frac{1}{d - 1} \log^+ |d(2d - 1)!| - \frac{2d - 1}{d - 1} \log \|F\| + (d - 1)r(f).
\]

\(\square\)

Our next lemma is essentially a classical result of Fatou, that any attracting cycle attracts a critical point, along with its \(p\)-adic analogue \([4]\), both given a slightly more explicit form. For the statement, we define two constants \(\varepsilon_v > 0\) and \(C_v\), depending on \(d\) and on the nature of the valuation \(v\). We set
\[
C_v = \begin{cases} 
3^{d-1} & \text{if } v \text{ is archimedean,} \\
1 & \text{otherwise,}
\end{cases}
\]
and
\[
\varepsilon_v = \begin{cases} 
\frac{1}{8} & \text{if } d = 2 \text{ and } v \text{ is archimedean,} \\
\frac{1}{C_v} & \text{if } d \geq 3 \text{ and } v \text{ is archimedean,} \\
\min_{1 \leq m \leq d} |m|^d & \text{otherwise.}
\end{cases}
\]

We note that if \(v\) is non-archimedean, and not \(p\)-adic for any \(p \leq d\), then \(\varepsilon_v = C_v = 1\). This is the case, for example, for any place of the function field \(\mathbb{C}(X)\) of a variety \(X/\mathbb{C}\).

**Lemma 11.** Let \(f\) be of the form (1), and suppose that \(0 < |\lambda| < \varepsilon_v\). Then there exists a branch point \(\beta\) of \(f\) with
\[
0 < |f^k(\beta)| \max\{|\alpha_1|, ..., |\beta_{d-1}|\} \leq (C_v |\lambda|)^k
\]
for all \(k \geq 1\), with \(\alpha_i\) and \(\beta_j\) as in (1).

In the case \(K = \mathbb{C}\), we use a more-or-less standard argument from complex dynamics.
Proof of Lemma \[ \ref{lem:main} \] for $K = \mathbb{C}$. Note that both sides of the inequality are left fixed by the conjugacy $f(z) \mapsto \xi^{-1} f(\xi z)$, and so without loss of generality we may assume that $\max\{|\alpha_1|, \ldots, |\beta_{d-1}|\} = 1$. Given this, if $0 < |z| \leq \frac{1}{2}$, then

$$0 \neq |f(z)| = |\lambda z| \frac{\prod_{i=1}^{d-1} (1 - \alpha_i z)}{\prod_{i=1}^{d-1} (1 - \beta_i z)} \leq |\lambda z| \frac{\left(\frac{2}{3}\right)^{d-1}}{\left(\frac{2}{3}\right)^{d-1}} = |\lambda z|^{3^{d-1} - 1} < |z|$$

by the triangle inequality, and by our choice of $\varepsilon_v$. Note that $f(z) \neq 0$ because every non-zero root $\alpha_i^{-1}$ of $f$ has $|\alpha_i^{-1}| \geq 1$. By induction,

$$0 \neq |f^k(z)| \leq (Cv|\lambda|)^k$$

for all $k \geq 1$ and all $z \in D(0, \frac{1}{2}) = \{ z \in \mathbb{C} : |z| < \frac{1}{2} \}$. It remains to show that $D(0, \frac{1}{2})$ contains a branch point of $f$.

Suppose that $f$ has no branch point in the disk $D(0, \frac{1}{2})$, and let $W$ be the connected component of $f^{-1}(D(0, \frac{1}{2}))$ containing $0$. Since $f(W) = D(0, \frac{1}{2})$ we know that $W$ contains no poles of $f$, and since $f : W \to D(0, \frac{1}{2})$ is unbranched we know that $W$ contains no zeros other than $z = 0$, and that $W$ is simply connected. But $f$ has at least one pole or non-zero root on the unit circle, and so $W$ does not contain the closed unit disk. By Koebe’s $\frac{1}{4}$ Theorem, $W$ has conformal radius no greater than 4 (relative to the origin). On the other hand, the map $f : W \to D(0, \frac{1}{2})$ witnesses that $W$ has conformal radius exactly $(2|\lambda|)^{-1}$, whereupon $|\lambda| \geq \frac{1}{8}$. This contradicts our hypothesis that $|\lambda| < \varepsilon_v$. \qed

Proof of Lemma \[ \ref{lem:main2} \] for $K \neq \mathbb{C}$. Here we use the main result of \[ \ref{finiteness} \]. Again scaling coordinates, we may take $\max\{|\alpha_1|, \ldots, |\beta_d|\} = 1$. By the ultrametric inequality, for any $z \in D(0, 1)$ we have

$$0 \neq |f^k(z)| = |\lambda f^{k-1}(z)| \frac{\prod_{i=1}^{d-1} (1 - \alpha_i f^{k-1}(z))}{\prod_{i=1}^{d-1} (1 - \beta_j f^{k-1}(z))} = |\lambda f^{k-1}(z)| = |\lambda|^{k}|z| \leq |\lambda|^k$$

for all $k \geq 1$. It is now enough to show that $D(0, 1)$ contains a branch point of $f$. This follows from \[ \ref{finiteness} \] Theorem 4.1. Specifically, as in the preamble to the proof of that theorem, we have chosen coordinates so that $z = 0$ is the fixed point with multiplier $\lambda$ and that $z = \infty$ is also a fixed point of $f$. We have also scaled the coordinate so that the smallest pole or non-zero root of $f$ has absolute value 1, noting that $\alpha_1, \ldots, \beta_{d-1}$ are the reciprocals of these roots and poles. By the proof of \[ \ref{finiteness} \] Theorem 4.1, there is now a branch point of $f$ in $D(0, 1)$ as long as

$$0 < |\lambda| < |\deg_{\zeta, \vec{w}} f|^d$$

for all $\zeta$ in the Berkovich analytic space $\mathbb{P}^d_{\text{Berk}}$ and all tangent directions $\vec{w}$. But the directional multiplicity $\deg_{\zeta, \vec{w}} f$ is an integer between 1 and $d$, and so this condition is ensured by $0 < |\lambda| < \varepsilon_v$. \qed
Now, for notational convenience, we extend $g_f(\cdot, 0)$ linearly to divisors. In other words, if $D = \sum m_P[P]$, then $g_f(D, 0) = \sum m_P g_f(P, 0)$. We write $B_f$ for the branch locus of $f$, and $f^*_k B_f$ for its $k$th iterated forward image (which is the $(k + 1)$th iterated forward image of the critical divisor). We write $B'_f$ for the part of the branch locus not consisting of iterated preimages of 0. In other words, if $e_P(f)$ is the index of ramification of $f$ at $P$, and $O_f^{-1}(Q)$ is the backward orbit of $Q$ under $f$, then

$$B'_f = \sum_{P \notin O_f^{-1}(0)} (e_P(f) - 1) [f(P)].$$

**Lemma 12.** For any $f$ of the form (1), we have for any $k \geq 1$,

$$g_f(f^*_k B'_f, 0) \geq (k - 1) \log^+ |\lambda^{-1}| + k \log \varepsilon_v + \log \|\alpha_1, ..., \beta_{d-1}\|$$

$$- \log \|f\| - \log^+ |2|$$

$$- \deg(B'_f) \left( \frac{2d - 1}{d - 1} \log \|f\| + \frac{1}{d - 1} \log^+ |d(2d - 1)!| - (d - 1)r(f) \right).$$

**Proof.** By Lemma 10, we have

$$g_f(f^*_k(\beta), 0) \geq \log^+ \left| \frac{1}{f^k(\beta)} \right|$$

$$- \frac{1}{d - 1} \log^+ |d(2d - 1)!| - \frac{2d - 1}{d - 1} \log \|f\| + (d - 1)r(f)$$

for every branch point $\beta$ with $f^k(\beta) \neq 0$.

We first assume that $0 < |\lambda| < \varepsilon_v \leq 1$, and so by Lemma 11 there exists a $\beta \in \text{Supp}(B'_f)$ with

$$|f^k(\beta)| \leq (C_v |\lambda|)^k / \max\{ |\alpha_1|, ..., |\beta_{d-1}| \}.$$

Note that $- \log C_v \geq \log \varepsilon_v$ for all $v$. For this branch point, we apply (3) along with the estimate $\log^+ |z| \geq \log |z|$ and the estimate in (4). For every other branch point, we apply (3) with the trivial estimate $\log^+ |z| \geq 0$ to obtain

$$g_f(f^*_k B'_f, 0) \geq k \log^+ |\lambda^{-1}| + k \log \varepsilon_v + \log \|\alpha_1, ..., \beta_{d-1}\|$$

$$- \deg(B'_f) \left( \frac{1}{d - 1} \log^+ |d(2d - 1)!| + \frac{2d - 1}{d - 1} \log \|f\| - (d - 1)r(f) \right).$$

To obtain (2) in this case, it is enough to note that

$$\log^+ |\lambda^{-1}| + \log \|f\| + \log^+ |2| \geq 0,$$

and that subtracting the left-hand-side of this from the lower bound in (5) gives (2).
It remains to show that (2) still holds when $|\lambda| \geq \varepsilon_v$. If we apply (3) for each branch point, with the trivial estimate $\log^+ |z| \geq 0$, we have

$$g_f(f^k B'_f, 0) \geq -\deg(B'_f) \left( \frac{1}{d-1} \log^+ |d(2d-1)!| + \frac{2d-1}{d-1} \log \|f\| - (d-1)r(f) \right).$$

Comparing this with the desired inequality (2), we note that it is enough to show that

$$0 \geq (k-1) \log^+ |\lambda^{-1}| + k \log \varepsilon_v + \log \|\alpha_1, \ldots, \beta_{d-1}\| - \log \|f\| - \log^+ |2|.$$ 

Our hypothesis in this case ensures that $k(\log^+ |\lambda^{-1}| + \log \varepsilon_v) \leq 0$, so we are left with showing

$$\log \|\alpha_1, \ldots, \beta_{d-1}\| \leq \log^+ |\lambda^{-1}| + \log \|f\| + \log^+ |2|.$$ 

In general, if

$$(z - e_1) \cdots (z - e_k) = z^k + A_{k-1}z^{k-1} + \cdots + A_0,$$

we have

$$\log \|e_1, \ldots, e_k\| \leq \log \|A_{k-1}, A_{k-2}^{1/2}, \ldots, A_0^{1/k}\| + \log^+ |2| \leq \log^+ \|A_{k-1}, \ldots, A_0\| + \log^+ |2|.$$ 

Since

$$(z - \beta_1) \cdots (z - \beta_{d-1}) = z^{d-1} + b_1 z^{d-2} + \cdots + b_{d-1}$$

$$(z - \alpha_1) \cdots (z - \alpha_{d-1}) = z^{d-1} + \frac{a_2}{\lambda} z^{d-2} + \cdots + \frac{a_d}{\lambda}$$

we obtain

$$\log \|\beta_1, \ldots, \beta_{d-1}\| \leq \log^+ \|b_1, \ldots, b_{d-1}\| + \log^+ |2|$$

$$\log \|\alpha_1, \ldots, \alpha_{d-1}\| \leq \log^+ \|a_2/\lambda, \ldots, a_d/\lambda\| + \log^+ |2| \leq \log^+ \|a_2, \ldots, a_d\| + \log^+ |\lambda^{-1}| + \log^+ |2|,$$

and so

$$\log \|\alpha_1, \ldots, \beta_{d-1}\| \leq \log \|f\| + \log^+ |\lambda^{-1}| + \log^+ |2|,$$

as claimed, noting that $\log^+ \|f\| = \log \|f\|$. 

Note that it is a priori possible that $B'_f$ is the zero divisor, but this does not present a problem for the previous lemma. In this case our definitions give $g_f(f^k B'_f, 0) = 0$, and the lower bound simplifies to a bound on $|\lambda^{-1}|$ for PCF maps, as proved in [4].
2. Heights on $M_d$

With local estimates in place, we turn out attention to global heights. In this section, we fix a number field $K$, although all estimates will remain unchanged after a finite extensions, and so we are in some sense always working over $\mathbb{Q}$. Let $M_K$ be the standard set of valuations on $K$, with the absolute value $| \cdot |_v$ normalized to restrict to $\mathbb{Q}$ as either the usual or one of the $p$-adic absolute values. Thus normalized, the usual Weil height is

$$h(\alpha) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\alpha|_v.$$ 

Heights on projective varieties are defined relative to line bundles, with an ample height being one defined relative to an ample bundle (see, e.g., [18]). Quantities from Section 1 which depended on the absolute value now acquire a subscript $v$.

Recall (from [17]) the space $\text{Rat}_d = \mathbb{P}^{2d+1}$ of rational functions of degree at most $d$, with

$$c = [c_0 : \cdots : c_{2d+1}]$$

corresponding to the rational function

$$f_c(z) = \frac{c_0 + c_1 z + \cdots + c_d z^d}{c_{d+1} + c_{d+2} z + \cdots + c_{2d+1} z^d}.$$ 

The resultant of the numerator and denominator of $f_c$ cuts out a hypersurface in $\mathbb{P}^{2d+1}$, and the complement of this is $\text{Hom}_d \subseteq \text{Rat}_d$ consisting of those rational functions of degree exactly $d$. There is a natural action of $\text{PGL}_2$ on $\text{Hom}_d$, by change of coordinates, and the quotient $M_d$ is an affine variety parametrizing coordinate-free dynamical systems of degree $d$. By some abuse of notation, we will use the same symbol $f$ to identify a rational function, the point representing it in $\text{Rat}_d$, and the point representing its conjugacy class in $M_d$. When we write $h_{M_d}$, we mean the height relative to some ample line bundle on $M_d$. Any two such functions will be commensurate (for instance, by Lemma [17]).

Much more concretely, $\text{Hom}_d$ as a subvariety of $\mathbb{P}^{2d+1}$ carries a natural height $h_{\text{Hom}_d}$, which is just the usual Weil height

$$h_{\text{Hom}_d}(f) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|f\|_v$$

where $\|f\|_v = \max\{|c_0|_v, \ldots, |c_{2d+1}|_v\}$ as in Section 1. Note that, even without the normalization in (1), this is well-defined by the product formula

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |\alpha|_v = 0$$

for $\alpha \neq 0$. 

Finally, we recall Silverman’s critical height. To each \( f \in \text{Hom}_d \) is associated a non-negative canonical height \( \hat{h}_f : \mathbb{P}^1 \to \mathbb{R} \) defined by

\[
\hat{h}_f(P) = \lim_{n \to \infty} \frac{h \circ f^n(P)}{d^n}.
\]

Note that we can also decompose the canonical height locally as

\[
\hat{h}_f(P) + \hat{h}_f(Q) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} g_{f,v}(P, Q)
\]

for \( P \neq Q \) [3, p. 310]. We extend \( \hat{h}_f \) linearly to divisors, and set, for \( C_f \) the critical divisor of \( f \),

\[
\hat{h}_{\text{crit}}(f) = \hat{h}_f(C_f) = \sum_{P \in \mathbb{P}^1} (e_P(f) - 1) \hat{h}_f(P).
\]

It is easy to show that \( \hat{h}_{\text{crit}} \) is invariant under change of coordinates, giving a well-defined non-negative function \( \hat{h}_{\text{crit}} : M_d \to \mathbb{R} \). Also, since \( h_f \circ f = dh_f \), we note that \( \hat{h}_f(f^*D) = dh_f(D) \) for any divisor \( D \), and so in particular the branch locus \( B_f \) satisfies

\[
\hat{h}_f(f^k B_f) = d^{k+1} \hat{h}_{\text{crit}}(f)
\]

for any \( k \geq 0 \).

Because the results in Section [1] depend on the way in which \( f \) is written, we must first show that we can change coordinates without changing the estimates too much.

**Lemma 13.** Suppose that \( f \in \text{Hom}_d \) has a fixed point with multiplier \( \lambda \neq 1 \). Then there exists a \( g \in \text{Hom}_d \), conjugate to \( f \), such that \( g(0) = 0 \) with multiplier \( \lambda \), \( g(\infty) = \infty \), and

\[
(6) \quad h_{\text{Hom}_d}(g) \leq (d+2)^2 h_{\text{Hom}_d}(f) + (d+3)(d^2 + 2d + 2)^2 \log 2 + (d+3) \log(d+1)!
\]

**Proof.** If the fixed point of multiplier \( \lambda \) is at \( z = \infty \), then let \( f_1 = 1/f(1/z) \). Then \( f_1 \) has the same coefficients as \( f \), only permuted, so \( h_{\text{Hom}_d}(f_1) = h_{\text{Hom}_d}(f) \), and now the fixed point of interest is at \( z = 0 \). Otherwise, suppose that \( f(\gamma) = \gamma \) with multiplier \( \lambda \), and let \( f_1(z) = f(z + \gamma) - \gamma \) (now with \( \gamma \neq \infty \)). By the triangle inequality,

\[
h_{\text{Hom}_d}(f_1) \leq h_{\text{Hom}_d}(f) + (d + 1) h(\gamma) + \log 2 + \log(d + 1)!
\]

(independent of the fact that \( \gamma \) is a fixed point). At the same time, \( \gamma \) is a root of \( f(z) - z \), and so by standard inequalities between heights of roots of polynomials and heights of coefficients, we have \( h(\gamma) \leq h_{\text{Hom}_d}(f) + (d + 1) \log 2 \), and so

\[
h_{\text{Hom}_d}(f_1) \leq (d + 2) h_{\text{Hom}_d}(f) + (d^2 + 2d + 2) \log 2 + \log(d + 1)!
\]

This inequality is weaker than the equality in the case where the fixed point was at \( \infty \), so it holds in any case.
Now, $f_1(0) = 0$ with multiplier $\lambda \neq 1$. Since $\gamma$ is a simple root of $f(z) - z$, there must be another fixed point, say $\gamma' \neq 0$. If $\gamma' = \infty$ then $f_1$ is a conjugate of $f$ with the claimed properties, satisfying a stronger estimate than (6).

If not, let $f_2(z) = 1/f_1(1/z)$, $f_3 = (z+1/\gamma')-1/\gamma'$, and $f_4(z) = 1/f_3(1/z)$. With this choice, $f_2$ has a fixed point at $z = \infty$ with multiplier $\lambda$, and another fixed point at $z = 1/\gamma'$, while $h_{\text{Hom}}(f_2) = h_{\text{Hom}}(f_1)$. Applying the estimates above,

$$h_{\text{Hom}}(f_3) \leq h_{\text{Hom}}(f_2) + (d+1)h(1/\gamma') + \log 2 + \log(d+1)!$$

$$\leq h_{\text{Hom}}(f_2) + (d+1)h_{\text{Hom}}(f_2) + (d+1)^2 \log 2 + \log 2 + \log(d+1)!$$

$$= (d+2)h_{\text{Hom}}(f_1) + (d+1)^2 \log 2 + \log 2 + \log(d+1)!$$

$$\leq (d+2)^2 h_{\text{Hom}}(f) + (d+3)(d^2 + 2d + 2)^2 \log 2 + (d+3)\log(d+1)!.$$  

Note that $f_3$ has a fixed point at $z = \infty$ with multiplier $\lambda$ and another at $z = 0$, so $f_4$ is the conjugate satisfying our conditions. And $h_{\text{Hom}}(f_4) = h_{\text{Hom}}(f_3)$. 

The next lemma is a global version of Lemma 12, and in some sense is the crux of the main result.

**Lemma 14.** Let $f \in \text{Hom}_d$ have a fixed point with multiplier $\lambda$. Then for any $k \geq 1$

$$d^{k+1} \hat{h}_{\text{crit}}(f) \geq (k-1)h(\lambda) - (4d-1)(d+2)^2 h_{\text{Hom}}(f) - c_0 k$$

for some explicit positive constant $c_0$ depending just on $d$.

**Proof.** First, note that the inequality certainly holds when $\lambda = 0$ or $\lambda = 1$, since $h(0) = h(1) = 0$ while $\hat{h}_{\text{crit}}$ and $h_{\text{Hom}}$ are both non-negative, so we will assume that $\lambda \neq 0, 1$. If it happens that $f$ is in the form (1), then an even stronger estimate follows from Lemma 12. We sum the estimate (2) from Lemma 12 over all places. Note that since $h_f(0) = 0$, and the same for
preimages of 0, we have
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} g_{f,v}(f_{x}^{k}B_{f}'_{x}, 0) = \hat{h}_{f}(f_{x}^{k}B_{f}'_{x}) + \deg(B_{f}')\hat{h}_{f}(0) = d^{k+1}\hat{h}_{\text{crit}}(f)
\]
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\lambda^{-1}|_v = h(\lambda^{-1}) = h(\lambda)
\]
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|\alpha_1, ..., \beta_{d-1}\|_v = h([\alpha_1 : \cdots : \beta_{d-1}]) \geq 0
\]
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|f\|_v = h_{\text{Hom}}(f)
\]
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \varepsilon_v = -d \log \text{lcm}(1, ..., d) - \log \max\{8, 3^{d-1}\}
\]
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |N|_v = \log N
\]
for any integer $N$, and
\[
\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} r_v(f) = \frac{1}{d(d-1)} \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |\text{Res}(F_1, F_2)|_v = 0,
\]
with this last equality following from the product formula and $\text{Res}(F_1, F_2) \neq 0$. Combining these identities with (2), and using $\deg(B_{f}') \leq 2d-2$, we have
\[
d^{k+1}\hat{h}_{\text{crit}}(f) \geq (k-1)h(\lambda) - (4d-1)h_{\text{Hom}}(f) - 2\log 2d(2d-1)! - kd\log \text{lcm}(1, ..., d) - k\log \max\{8, 2 \cdot 3^{d-1}\}.
\]
Again, this holds only in the case where $f$ is of the form $(1)$, but the estimate depends only on the coefficients of $f$ as a point in $\text{Hom}_d \subseteq \mathbb{P}^{2d+1}$, so the choice of representative of homogeneous coordinates in $(1)$ no longer matters. In obtaining $(8)$, then, we are using only that $f$ has a fixed point of multiplier $\lambda \neq 0$ at $z = 0$, and another fixed point at $z = \infty$.

But if $f$ has a fixed point anywhere of multiplier $\lambda \neq 0, 1$, let $g$ be the conjugate produced in Lemma $13$. Then applying $(8)$ to $g$, and then $(6)$ to bound $-h_{\text{Hom}}(g)$ below in terms of $-h_{\text{Hom}}(f)$, we have the estimate $(7)$.

Lemma $14$ has the virtue of being completely explicit, but it is a lower-bound on a conjugacy-class invariant with an error that can get arbitrarily bad within a conjugacy class. The following lemma, due to Silverman, relates $h_{\text{Hom}}$ to the height on $M_d$, allowing coordinate-free estimates.

**Lemma 15** (Silverman [17, p. 103]). For $f \in M_d$,
\[
h_{M_d}(f) \asymp \min_{g \sim f} h_{\text{Hom}}(g),
\]
Combining Lemma 15 with Lemma 14 gives an inequality in which no term is coordinate dependent, as we will see below.

We now construct, more explicitly, the height on $M_d$ used in [4]. Let $\text{Hom}^\text{Fix}_d$ be the space of rational functions with all fixed points $\gamma_1, \ldots, \gamma_{d+1}$ marked, and consider the map $\text{Hom}^\text{Fix}_d \to \mathbb{A}^{d+1} \subseteq (\mathbb{P}^1)^{d+1}$ by

$$(f, \gamma_1, \ldots, \gamma_{d+1}) \mapsto (\lambda_0, \ldots, \lambda_{d+1}),$$

where $\lambda_i$ is the multiplier at the fixed point $\gamma_i$. Since change of coordinates on $\text{Hom}^\text{Fix}_d$ acts by permuting the multipliers, this map induces a morphism $\sigma : M_d \to S^d \mathbb{P}^1$, where $S^d \mathbb{P}^1$ denotes the $d$th symmetric power of $\mathbb{P}^1$. Note that the image of the pull-back map $(\mathbb{P}^1)^{d+1} \to S^d \mathbb{P}^1$ on divisors is exactly the subgroup of the form $\sum \pi_i^*D$, where $\pi_i : (\mathbb{P}^1)^{d+1} \to \mathbb{P}^1$ is the $i$th coordinate projection. In particular,

$$h_{S^d \mathbb{P}^1}(\lambda_0, \ldots, \lambda_{d+1}) = h_{\mathbb{P}^1}(\lambda_0) + \cdots + h_{\mathbb{P}^1}(\lambda_{d+1})$$

in an ample height on $S^d \mathbb{P}^1$, and any ample Weil height is a scalar multiple of this (up to $O(1)$).

Now, for each $n$ we define a morphism $\sigma_n : M_d \to S^n \mathbb{P}^1$ by composing $\sigma$ with the iteration map $M_d \to M_d^n$. In other words, $\sigma_n(f) = \sigma(f^n)$. In [4], we used (more-or-less) $h_{S^n \mathbb{P}^1} \circ \sigma_n$ as a height on $M_d$, which we justify more explicitly in the following lemma.

**Lemma 16.** For some $n$ depending only on $d$, have have

$$h_{M_d}(f) \ll h_{S^n \mathbb{P}^1} \circ \sigma_n(f)$$

for all non-Lattès $f \in M_d$.

We deduce this lemma from a more general result.

**Lemma 17.** Let $X$ and $Y$ be irreducible projective varieties equipped with ample line bundles $L$ and $M$ (respectively), and let $U \subseteq X$ be a Zariski open subset with a morphism $F : U \to Y$ with finite fibres. Then for all $u \in U$,

$$h_{X,L}(u) \ll h_{Y,M}(F(u)).$$

**Proof.** First note that the statement is immediate when $X$ is a point. We proceed by induction on $\dim(X)$, assuming now that $\dim(X) \geq 1$ and that the claim is confirmed in every lower dimension. Also, since the claim is vacuously true otherwise, we will suppose that $U$ is non-empty.

Now, there is a variety $\tilde{X}$ with morphisms $\pi : \tilde{X} \to X$ and $\Phi : \tilde{X} \to Y$ such that $\pi$ is an isomorphism over $U$ and $\Phi = \pi \circ F$ for all $x \in \pi^{-1}(U)$. For example, we could take $\tilde{X}$ to be the Zariski closure in $X \times Y$ of the graph of $F$, with $\pi$ and $\Phi$ the coordinate projections.

Since $\Phi$ has finite fibres over the non-empty Zariski open set $\pi^{-1}(U)$, and since $M$ is ample, and hence big (the product of an ample and an effective bundle), it follows that $\Phi^*M$ is a big line bundle on $\tilde{X}$. By [18]
Proposition 9.13, p. 146], there is a proper Zariski-closed $\tilde{Z} \subseteq \tilde{X}$ such that $h_{\tilde{X}, \pi^* L}(x) \ll h_{\tilde{X}, \phi^* M}(x)$ for all $x \not\in \tilde{Z}$. So

$$h_{X, L}(\pi(x)) = h_{\tilde{X}, \pi^* L}(x) + O(1) \ll h_{\tilde{X}, \phi^* M}(x) = h_{Y, M}(\Phi(x)) + O(1) = h_{Y, M}(F(\pi(x))) + O(1)$$

for all $x \not\in \tilde{Z}$. Since $\pi$ is surjective, setting $Z = \pi(\tilde{Z})$, we have

$$h_{X, L}(x) \ll h_{Y, M}(F(x))$$

for all $x \in U \setminus Z$.

But $\pi$ is also proper, and so $Z = \pi(\tilde{Z}) \neq X$ is a Zariski-closed subset. Each irreducible component of $Z'$ of $Z$ has $\dim(Z') < \dim(X)$, and so we may apply the induction hypothesis to $F$ restricted to $U \cap Z'$. Since there are only finitely many irreducible components of $Z$, we may adjust the constants in (9) finitely many times so that the inequality holds for all $x \in U$.

Proof of Lemma 16. This is now a direct application of the previous lemma. We have chosen a projective $X \supseteq M_d$ and an ample line bundle $L$ relative to which we are defining $h_{M_d}$. Let $U \subseteq M_d$ be the complement of the Lattès locus. For some $n$, the morphism $\sigma_n : U \to S^{d^n + 1}P^1$ has finite fibres (see [14, Corollary 2.3], and the comments following the proof).

We are now in a position to prove the main result.

Proof Theorem 1. As mentioned in the introduction, the inequality $\hat{h}_{\text{crit}} \ll h_{M_d}$ is found in [17]. We are concerned only with the other direction.

Suppose that $f \in \text{Hom}_d$ has a fixed point with multiplier $\lambda$. By Lemma 14, there are constants $c_1$ and $c_2$ depending just on $d$ with

$$d^{k+1} \hat{h}_{\text{crit}}(f) \geq (k - 1)h(\lambda) - c_1 h_{\text{Hom},d}(f) - c_2 k$$

for any $k \geq 1$. Summing over all fixed points, we have

$$(d + 1)d^{k+1} \hat{h}_{\text{crit}}(f) \geq (k - 1)h_{S^{d+1}P^1} \circ \sigma(f) - c_1(d + 1)h_{\text{Hom},d}(f) - c_2(d + 1)k.$$ Applying Lemma 15 and assuming without loss of generality that $f$ is of minimal height in its conjugacy class (which we also denote $f$), we obtain

$$(d + 1)d^{k+1} \hat{h}_{\text{crit}}(f) \geq (k - 1)h_{S^{d+1}P^1} \circ \sigma(f) - c_3 h_{M_d}(f) - c_4 k,$$

for new constants $c_3$ and $c_4$ depending just on $d$, in particular since $h \circ \sigma(f)$ and $\hat{h}_{\text{crit}}(f)$ are constant on conjugacy classes.

Note that an inequality of this form holds for all $d$. From the triangle inequality and Lemma 15 we have $h_{M_d}(f^n) \ll h_{M_d}(f)$, with constants depending on $n$, while the chain rule gives $\hat{h}_{\text{crit}}(f^n) = n \hat{h}_{\text{crit}}(f)$. It follows that, applying (11) to $f^n$, we have (for any $n$)

$$(d^n + 1)d^{n(k+1)} n \hat{h}_{\text{crit}}(f) \geq (k - 1)h_{S^{d^n+1}P^1} \circ \sigma_n(f) - c_5 h_{M_d}(f) - c_6 k,$$
where $c_5$ and $c_6$ are constants now depending on both $d$ and $n$. By Lemma [10] there exists an $n$, which we now fix, and constants $\varepsilon > 0$ and $c_7$ such that

$$h_{Sd^m+1p_1} \circ \sigma_n(f) \geq \varepsilon h_{M_d}(f) - c_7.$$  

From this we have, for any $k \geq 1$,

$$\hat{h}_{\text{crit}}(f) \geq \left(\frac{(k-1)\varepsilon - c_5}{(d^n+1)d^n(n+1)n} \right) h_{M_d}(f) - c_8,$$

for some $c_8$ depending just on $k$ (and our fixed $d$ and $n$). Choosing $k > 1 + c_5/\varepsilon$ gives the inequality in the statement of Theorem 1.

The proof of Theorem 2 is now quite quick.

**Proof of Theorem 2.** Let $\text{Per}_n(\lambda) \subseteq M_d$ be the subvariety consisting of conjugacy classes of rational functions admitting an $n$-cycle of multiplier $\lambda$. By (10) combined with Lemma 15 and the estimate $h_{M_d}(f^n) \ll h_{M_d}(f)$, we have

$$(k-1)h(\lambda) \leq (d^k + 1)n\hat{h}_{\text{crit}}(f) + c_1h_{M_d}(f) + c_2,$$

for some constants depending on $d$ and $n$. On the other hand, Theorem 1 now gives $h_{M_d}(f) \ll \hat{h}_{\text{crit}}(f)$, and so taking $k = 2$ we have

$$h(\lambda) \leq c_3\hat{h}_{\text{crit}}(f) + c_4,$$

with $c_3$ and $c_4$ dependent on $d$ and $n$. This also gives

$$c_3^{-1} \leq \frac{\hat{h}_{\text{crit}}(f)}{h(\lambda)} + o(1)$$

with $o(1) \to 0$ as $h(\lambda) \to \infty$, proving the first claim.

For the second claim, we consider the constants somewhat more carefully. In particular, if $P$ is a point of period $n$ for $f$, with multiplier $\lambda$, then it is a fixed point of $f^{nm}$ with multiplier $\lambda^n$. Summing the value $\varepsilon_v$ above over all places, we see that we may take

$$c_2(d) = (d-1)\log 3 + d \log \text{lcm}(1, \ldots, d) \leq 1.04d^2 + (d-1)\log 3,$$

by an estimate of Rosser and Schoenfeld [15] (noting that $\log \text{lcm}(1, \ldots, d)$ is the second Tchebyshev function from the proof of the Prime Number Theorem). Thus, applying the estimates above to $f^{nm}$, we have

$$((k-1)h(\lambda^n) - kc_2(d^{nm})) \leq (d^{k+1}nm + c_3)\hat{h}_{\text{crit}}(f) + c_4,$$

where $c_3$ and $c_4$ depend on $n$, $m$, and $d$. Now fix $m$, let $\delta > 0$, and suppose that $\hat{h}_{\text{crit}}$ admits no positive lower bound on $\text{Per}_n(\lambda)$. Then we must in fact have

$$((k-1)h(\lambda^n) - kc_2(d^{nm})) \leq c_4,$$

for each $k$, which is possible only if

$$h(\lambda) \leq \frac{k}{m(k-1)c_2(d^{nm})}$$
for all \( k \), or in other words \( h(\lambda) \leq \frac{1}{m}c_2(d^{am}) \). This proves the second claim.

\[ \square \]

3. Super-attracting fixed points

The arguments in Section 2 make use of McMullen’s Theorem on the multiplier spectrum, as well as estimates relating \( h_{\text{Hom}} \) to \( h_{\mathcal{M}_d} \), both of which interfere with the presentation of explicit constants. If we are willing to restrict attention to rational functions with a super-attracting fixed point, we may avoid any inexplicit estimates. In Subsection 3.1 we present local estimates that play the role of those in Section 1 but that, in this context, make no reference to multipliers. In Subsection 3.2 we sum these over all places to obtain Theorem 3.

3.1. Local estimates. As in Section 1, we will assume that \( K \) is an algebraically closed field of characteristic 0 or \( p > d \), complete with respect to some absolute value \( |\cdot| \) corresponding to the valuation \( v \). In Subsection 3.2 below, quantities depending on \( v \) will acquire a subscript.

We restrict attention to \( f \) of the form

\[
(12) \quad f(z) = \frac{z^e + \cdots + a_d z^d}{1 + \cdots + b_{d-1} z^{d-1}} = \frac{z^e \prod_{i=1}^{d-e} (1 - \alpha_i z)}{\prod_{j=1}^{d-1} (1 - \beta_j z)}.
\]

Note that, unlike in the previous section, the normal form is not maintained under a scaling \( f(z) \mapsto \alpha^{-1} f(\alpha z) \), and so we will need to keep more careful track of the \( \alpha_i \) and \( \beta_j \). For such \( f \), define

\[
\rho_f = \frac{1}{\max\{|\alpha_1|, \ldots, |\alpha_{d-e}|, |\beta_1|, \ldots, |\beta_{d-1}|\}}
\]

and a constant \( C_v \) by

\[
C_v = \begin{cases} 
\left(\frac{2e + 1}{e}ight) \log 2 + \frac{d-e}{e-1} \log 3 & \text{if } v \text{ is archimedean} \\
\frac{d}{e-1} \log \max_{1 \leq k \leq d} |k^{-1}| & \text{if } v \text{ is non-archimedean}
\end{cases}
\]

Lemma 18. Suppose that \( \log \rho_f + C_v < 0 \). Then there is a branch point \( \beta \) of \( f \) satisfying

\[-\infty < \log |f^k(\beta)| < e^k \log \rho_f + \log^+ |2^{e-1} 3^{d-e} |^{1/(1-e)} \]

for all \( k \geq 1 \).

Proof of Lemma 18 over \( \mathbb{C} \). Note that for any \( 0 \neq |z| < \frac{1}{2} \rho_f \) and \( |z| < \exp(-C_v) \),

\[
0 \neq |f(z)| = |z| \frac{\prod_{i=1}^{d-e} |1 - \alpha_i z|}{\prod_{j=1}^{d-1} |1 - \beta_j z|} \leq |z|^{e(3/2)^{d-e}} \leq |z|^{e3^{d-e} 2^{e-1}} \leq |z|,
\]

and so by induction

\[
(13) \quad \log |f^k(z)| \leq e^k \log |z| + \frac{1 - e^k}{1 - e} \log |3^{d-e} 2^{e-1}| < e^k \log |z| + \frac{1}{1 - e} \log |3^{d-e} 2^{e-1}|.
\]
It suffices to show that $D(0, \frac{1}{2}\rho_f)$ contains a branch point of $f$ other than $z = 0$.

Suppose that $f$ has no branch points in $D(0, \frac{1}{2}\rho_f)$, other than at $z = 0$, and let $W$ be the connected component of $f^{-1}(D(0, \frac{1}{2}\rho_f))$ containing 0. Topologically, $W$ is a disk with some number of punctures, but since $f : W \setminus f^{-1}(0) \to D(0, \frac{1}{2}\rho_f) \setminus \{0\}$ is unbranched and $f(0) = 0$, we see that $W$ is simply connected, and $f^{-1}(0) \cap W = \{0\}$. Since $f(W) \subseteq D(0, \frac{1}{2}\rho_f)$, $W$ contains no poles of $f$. In particular, $W$ does not contain $\overline{D(0, \rho_f)}$, and so by Koebe’s $\frac{1}{4}$ Theorem, the conformal radius of $W$ is no greater than $4\rho_f$.

On the other hand, $f : W \to D(0, \frac{1}{2}\rho_f)$ factors through an analytic root $\beta : W \to D(0, (\frac{1}{2}\rho_f)^{1/e})$ given by $\beta(z) = z + O(z^2)$. This map witnesses $W$ having conformal radius exactly $(\frac{1}{2}\rho_f)^{1/e}$, and so

$$\left(\frac{1}{2}\rho_f\right)^{1/e} \leq 4\rho_f,$$

or $\rho_f \geq 2^{-(2e+1)/(e-1)}$. This contradicts our hypothesis that $\log \rho_f < -C_v$. \hfill $\square$

**Proof of Lemma 18 over non-archimedean fields.** The proof closely follows [4].

Let $U \subseteq \mathbb{P}^1_{\text{Berk}}$ be the open disk at 0 of radius $\rho_f$, and let $V \supseteq U$ be the connected component of $f^{-1}(U)$ containing 0. We know that $f : V \to U$ is $m$-to-1, for some $m \geq e = e_f(0)$ which we fix now. We also know that $V$ is an open affinoid, that is, $V = D(0, R) \setminus (W_1 \cup \cdots W_k)$ for some closed disks $W_i = \overline{D(b_i, R_i)} \subseteq D(0, R)$ with $W_i \cap U = \emptyset$.

For $\zeta \in \mathbb{P}^1_{\text{Berk}}$ we set (as in [4])

$$\text{rad}(\zeta) = \inf_{a \in \mathbb{P}^1(K)} \|z - a\|_{\zeta}$$

the distortion

$$\delta(f, \zeta) = \log (\zeta) + \log \|f'\|_{\zeta} - \log \|f\|_{\zeta}$$

and

$$G(\zeta) = m\delta(f, \zeta) + \log \|f\|_{\zeta}.$$

Also, for any point $\zeta_{a,t}$ corresponding to a disk, let

$$N^+(f, \zeta_{a,t}, b) = \#\{z \in \overline{D(a, t)} : f(z) = b\}$$

$$N^-(f, \zeta_{a,t}, b) = \#\{z \in D(a, t) : f(z) = b\}.$$

Note that $t \mapsto G(\zeta_{0,t})$ is continuous and piecewise linear in $\log t$, with slope (14)

$$m(1 + N^+(f', \zeta_{0,t}, 0) - N^-(f', \zeta_{0,t})) + (1 - m)(N^+(f, \zeta_{0,t}, 0) - N^-(f, \zeta_{0,t}, \infty))$$

except at the points where the slope is undefined (see [4] Proof of Theorem 4.1); note that the points at which the slope is undefined are exactly those at which there is a distinction between $N^+$ and $N^-$. 
Note that 
\[(15) \quad \log |N^\pm(f, \zeta, 0) - N^\pm(f, \zeta, \infty)| \leq \delta(f, \zeta) \leq 0,\]
by \([4, \text{Lemma 3.3}]\), and so we have 
\[G(\zeta_{0,R}) \leq \log \|f\|_{\zeta_{0,R}} = e \log \rho_f.\]

On the other hand, we have 
\[f(\zeta_{0,R}) = f(\zeta_{0,R}) = \zeta_{0,R},\]
and so 
\[\log \|f\|_{\zeta_{0,R}} = \log \|f\|_{\zeta_{0,R}} = \log \rho_f.\]

Again by \((15)\) 
\[G(\zeta_{0,R}) \geq m \min_{1 \leq k \leq d} \log |k| + \log r \geq \log r - \frac{m(e-1)}{d} C_v,\]
and hence 
\[G(\zeta_{0,R}) - G(\zeta_{0,R}) \geq (e-1) \log \rho_f^{-1} - \frac{m(e-1)}{d} C_v > 0\]
by our hypothesis on \(\rho_f\) (and since \(m \leq d\)). Since the function \(\log t \mapsto G(\zeta_{0,t})\) increases on average from \(t = \rho_f\) to \(t = R\), there exist \(t \in [\rho_f, R]\) where the graph has positive slope (given by \((14)\)). We take \(S\) to be the infimum of such \(t\). We have 
\[(16) m(1 + N^+(f', \zeta_{0,S}, 0) - N^+(f', \zeta_{0,S})) + (1 - m)(N^+(f, \zeta_{0,S}, 0) - N^+(f, \zeta_{0,S}, \infty)) \geq 1,\]
since \(t \mapsto N^+(g, \zeta_{0,t}, b)\) is upper semi-continuous for any \(g\) and \(b\), and therefore the quantity on the left is both positive and an integer. Note that 
\[G(\zeta_{0,S}) \leq G(\zeta_{0,R})\], or else the same argument again gives a \(t < S\) at which the graph of \(t \mapsto G(\zeta_{0,t})\) has positive slope, contradicting the construction of \(S\).

We now discard any \(W_i\) with \(b_i \notin \overline{D(0,S)}\), renumbering so that now \(W_1, ..., W_k\) remain. Note that for each \(i\) 
\[G(\zeta_{b_i,R}) - G(\zeta_{0,S}) \geq (e-1) \log \rho_f^{-1} - \frac{m(e-1)}{d} C_v > 0\]
just as above. We take \(S_i\) to be the supremum of the (nonempty) set of \(t\) on which \(\log t \mapsto G(\zeta_{b_i,t})\) is decreasing, so that 
\[(17) m(1 + N^-(f', \zeta_{b_i,S_i}, 0) - N^-(f', \zeta_{b_i,S_i}))) + (1 - m)(N^-(f, \zeta_{b_i,S_i}, 0) - N^-(f, \zeta_{b_i,S_i}, \infty)) \leq -1\]
by the lower-semicontinuity of \(N^-(g, \zeta_{b_i,t}, b)\) in \(t\).

Now let \(N(g, W, b)\) count solutions to \(g(z) = b\) in \(W\). Exactly as in \([4]\), we subtract from \((16)\) the sum of \((17)\) for \(1 \leq i \leq k\) to obtain 
\[m(1 - k) + N(f', W, 0)) + (1 - m)N(f, W, 0) \geq 1 + k,\]
since \(f\) and \(f'\) have no poles in \(W\). Exactly as in \([4]\), we now see that \(W\) contains a critical point \(\zeta\) with \(f(\zeta) \neq 0\). In particular, if \(W\) contains \(A\) critical points that are not roots of \(f\) and \(B\) distinct roots of \(f\), then 
\[A + N(f, W, 0) = N(f', W, 0) + B.\]
If \(k = 0\) then \(W = \overline{D(0,S)}\). Since \(W\)
contains no poles of \( f \), but \( \overline{D(0, S)} \supset D(0, \rho_f) \) contains either a root or a pole, it follows that in this case \( B \geq 2 \). From this,

\[
1 \leq m(2 + A - B)
\]

(since \( N(f, W, 0) \leq m \)), and so \( A \) must be positive.

If, on the other hand, \( k \geq 1 \), we have

\[
2 + m \leq 1 + k(1 + m) \leq m(1 + A - B),
\]

and so again \( A > 0 \).

We have shown that there is critical point \( \zeta \in W \) with \( f(\zeta) \neq 0 \), and so there is a branch point \( \beta = f(\zeta) \in U \setminus \{0\} \). But for any \( z \in U \setminus \{0\} \) we have

\[
\log |f^k(z)| = e^k \log |z| < e^k \log \rho_f
\]

for all \( k \geq 1 \), by the ultrametric inequality. \( \square \)

The following estimate plays the role of Lemma \([12]\) in the present case. As in Section \([11]\) take \( B'_f \) to be that part of the branch locus whose forward orbit does not contain 0, that is,

\[
B'_f = \sum_{P \notin \mathcal{O}_{\lambda}^{-1}(0)} (e_P(f) - 1) [f(P)].
\]

**Lemma 19.** For all \( f \),

\[
g_f(f^k B'_f, 0) + \deg(B'_f) \left( \frac{1}{d-1} \log^+ |2(2d-1)|! + \frac{2d-1}{d-1} \log \|f\| - (d-1) r(f) \right)
\]

\[
\geq \frac{e^k}{d-1} \log \|f\| - \frac{e^k}{d-1} \log^+ |d-1| - e^k C_v - \frac{1}{1-e} \log^+ |3^{d-2} e^{-1}|,
\]

**Proof.** We first remark that we may apply the estimates in Lemma \([10]\) which nowhere used the hypothesis that \( \lambda \neq 0 \) in the normal form \([11]\).

We will show that

\[
g_f(f^k B'_f, 0) + \deg(B'_f) \left( \frac{1}{d-1} \log^+ |2(2d-1)|! + \frac{2d-1}{d-1} \log \|f\| - (d-1) r(f) \right)
\]

\[
\geq e^k (\log^+ \rho_f^{-1} - C_v) - \frac{1}{1-e} \log^+ |3^{d-2} e^{-1}|.
\]

Note that by Lemma \([10]\) the left-hand side is non-negative, and so \([19]\) is immediate when \( \log^+ \rho_f^{-1} - C_v \leq 0 \). To treat the other case, we will assume that

\[
\log \rho_f^{-1} = \log^+ \rho_f^{-1} > C_v \geq 0.
\]
It follows from Lemma 11 that there is a branch point $\beta$ satisfying (13) for all $k \geq 1$. By Lemma 10, the left-hand-side of (19) is bounded below by

$$\log^+ |f^k(\beta)| \geq \log |f^k(\beta)| \geq e^k \log |\rho_f^{-1}| - \frac{1}{1-e} \log^+ |3^{d-e}2^{e-1}|.$$ 

Now (19) follows from the fact that $C_v \geq 0$.

We are left with deducing the estimate in the statement of the lemma from that in (19). For any polynomial 

$$(z - e_1) \cdots (z - e_k) = z^k + c_{k-1}z^{k-1} + \cdots + c_0,$$

we have

$$\log^+ \|c_{k-1}, ..., c_0\| \leq k \log^+ \|e_1, ..., e_k\| + \log^+ |k|,$$

by the triangle inequality. From this we have (in the notation above)

$$\log^+ \|a_i\| \leq (d - e) \log^+ \|\alpha_i\| + \log^+ |d - e|$$

and

$$\log^+ \|b_j\| \leq (d - 1) \log^+ \|\beta_j\| + \log^+ |d - 1|,$$

from which

$$\log \|f\| \leq (d - 1) \log \rho_f^{-1} + \log^+ |d - 1|$$

(recalling that $\rho_f < 1$ based on our assumptions at this point). Combining this with (19) gives the intended estimate. 

3.2. Global estimates. For the proof of Theorem 3 we let $K$ be a number field, and $M_K$ its standard set of absolute values, normalized as in Section 2. For each place $v$ we will apply the estimates in Subsection 3.1 over $C_v$, the completion of the algebraic closure of the completion of $K$ with respect to $v$, and resulting quantities will acquire a subscript $v$.

Proof of Theorem 3. We sum the estimate from Lemma 19 over all places. Note that

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} g_f, v(f^k B_f', 0) = d^{k+1} h_{\text{crit}}(f)$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|f\|_v = h_{\text{Hom}}(f)$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} r_v(f) = 0$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} C_v = \left(\frac{2e + 1}{e - 1}\right) \log 2 + \frac{d - e}{e - 1} \log 3 + \frac{d}{e - 1} \log \lcm(1, 2, ..., d)$$

and

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |N|_v = \log N.$$
for any integer \( N \geq 1 \). So summing [18] over all places, and using \( \deg(B'_f) \leq (2d - 2) \), gives

\[
d^{k+1} \hat{h}_{\text{crit}}(f) \geq \left( \frac{e^k}{d-1} - 4d + 2 \right) h_{\text{Hom}_d}(f) - E(d, e, k)
\]

for

\[
E(d, e, k) = 2 \log(2(2d - 1)!) + \frac{1}{e - 1} \log(3^{d-1}2^{e-1})
\]
\[+ e^k \left( \frac{\log(d - 1)}{d - 1} + \frac{2e}{e - 1} \log 2 + \frac{d - e}{e - 1} \log 3 + \frac{d}{e - 1} \log \text{lcm}(1, 2, ..., d) \right).
\]

We now fix \( k \) so that \( 4d^2 - 2d - 1 \leq e^k < e(4d^2 - 2d - 1) \), from which we get

\[
d^{k+1} < d \cdot d^{\log_d(4d^2 - 2d - 1) + 1} = d^2(4d^2 - 2d - 1) \log d / \log e.
\]

This value of \( k \) gives us

\[
\hat{h}_{\text{crit}}(f) \geq \frac{1}{(d - 1)d^2(4d^2 - 2d - 1) \log d / \log e} h_{\text{Hom}_d}(f) - C_{d,e},
\]

with

\[
(20) \quad C_{d,e} = \frac{1}{d^2(4d^2 - 2d - 1) \log d / \log e} \left( 2 \log(2(2d - 1)!) \right.
\]
\[+ \frac{(d - e)(e(4d^2 - 2d - 1) + 1)}{e - 1} \log 3 + \frac{e - 1 + 2e^2(4d^2 - 2d - 1)}{e - 1} \log 2
\]
\[+ \frac{e(4d^2 - 2d - 1) \log(d - 1)}{d - 1} + \frac{de(4d^2 - 2d - 1)}{e - 1} \log \text{lcm}(1, 2, ..., d) \).
\]

Note that \( \log \text{lcm}(1, ..., d) \) is the second Chebyshev function and by explicit estimates in the direction of the Prime Number Theorem by Rosser and Schoenfeld [15], we have \( \log \text{lcm}(1, ..., d) < 1.04d \). \( \square \)

We now turn our attention to function fields. Let \( k \) be an algebraically closed field of characteristic 0 or \( p > d \), and let \( X \) be a curve over \( k \).

Note that in the case of a non-archimedean absolute value which is not \( p \)-adic for any \( p \leq d \), many of our constants vanish. In particular, if \( K \) is the function field of a variety over an algebraically closed field of characteristic 0 or \( p > d \), the above estimate simplifies to

\[
(21) \quad \hat{h}_{\text{crit}}(f) \geq \frac{1}{(d - 1)d^2(4d^2 - 2d - 1) \log d / \log e} h_{\text{Hom}_d}(f).
\]

One immediate corollary of this is that any PCF family in the form (12) must have constant coefficients, whether in characteristic 0 or \( p > d \). This statement is closely related to Theorem [7] in the introduction.
**Proof of Theorem** 7. As noted above, in a function field (of characteristic 0 or \( p > d \)) we have \( \varepsilon_v = C_v = 1 \) at every place, and so the proof of Theorem 3 here gives (21). So there is a critical point \( Q \) of \( f \) satisfying
\[
\hat{h}_f(f^k(Q)) \geq \varepsilon(d, e) d^k h_{\text{Hom}}(f),
\]
for an explicit \( \varepsilon(d, e) > 0 \). Note that \( h(Q) = \deg_X(Q) \), while \( \hat{h}_f(Q) - h(Q) \ll h_{\text{Hom}}(f) \) (for instance by summing Lemma 10 over all places), and so we have
\[
\deg_X(f^k(Q)) \geq (\varepsilon d^k - C) h_{\text{Hom}}(f),
\]
for some constant \( C \). In general, for \( P \in \mathbb{P}^N(X) \) we have \( h(P) = \deg(P^*H) \), with \( H \) any hyperplane, and so \( h_{\text{Hom}}(f) = \frac{1}{d} \deg(f^* \text{Res}_d) \). \( \square \)

4. Quadratic morphisms

In this section we turn our attention to morphisms of the form
\[
f(z) = \frac{\lambda_0 z + z^2}{\lambda_\infty z + 1},
\]
with \( \lambda_0 \lambda_\infty \neq 1 \), and give explicit estimates. Note that, over an algebraically closed field, every quadratic morphism is conjugate to one of this form, or one of the form \( f(z) = z^{-1} + a + z \), the latter family being simpler to handle [16, Lemma 4.59, p. 190]. For convenience, estimates are relative to the fixed point \( z = \infty \) rather than \( z = 0 \).

4.1. Local estimates. Let \( K \) be an algebraically closed field, complete with respect to some absolute value \( | \cdot | \). We work with the lift
\[
F(x, y) = (F_1(x, y), F_2(x, y)) = (\lambda_0 xy + x^2, \lambda_\infty xy + y^2),
\]
noting that
\[
\text{Res}(F_1, F_2) = 1 - \lambda_0 \lambda_\infty.
\]

**Lemma 20.** For all \( z \),
\[
g_f(z, \infty) \geq \log^+ |z| - 2 \log \|1, \lambda_0, \lambda_\infty\| - 3r(F) - \log^+ |2|.
\]

**Proof.** Since
\[
\lambda_\infty^2 y F_1(x, y) + (- \lambda_\infty x + (1 - \lambda_0 \lambda_\infty) y) F_2(x, y) = (1 - \lambda_0 \lambda_\infty) y^3
\]
and
\[
\lambda_0^2 x F_2(y, x) + (- \lambda_0 y + (1 - \lambda_0 \lambda_\infty) x) F_1(x, y) = (1 - \lambda_0 \lambda_\infty) x^3
\]
We have that
\[
\log \|P\| \leq \frac{1}{2} \log \|F(P)\| + \frac{1}{2} \log^+ \max\{|1 - \lambda_0 \lambda_\infty|, |\lambda_0|^2, |\lambda_\infty|^2\} - \frac{1}{2} \log |1 - \lambda_0 \lambda_\infty|.
\]
So
\[
\log \|P\| \leq H_F(P) + 2 \log \|1, \lambda_0, \lambda_\infty\| + \log 2 - 2r(F).
\]
Since \( F(1, 0) = (1, 0) \), \( H_F(1, 0) = 0 \) and
\[
g_f(z, \infty) \geq \log^+ |z| - 2 \log \|1, \lambda_0, \lambda_\infty\| + r(F) - \log^+ |2|.
\]
Lemma 21. Let
\[ \varepsilon = \begin{cases} 
1/3 & \text{if } v \text{ is archimedean} \\
1/4 & \text{if } v \text{ is 2-adic} \\
1 & \text{otherwise.} 
\end{cases} \]

Then for \( f_{\lambda_0, \lambda_\infty} \) as above with \( 0 < |\lambda_\infty| < \varepsilon \), there is a branch point \( \beta \) with
\[ |f^k(\beta)| \geq (|\lambda_\infty^{-1}| \varepsilon)^k \]
for all \( k \geq 1 \).

Proof. We start with the case of \( v \) archimedean. Note that for any \( z \) with \( |z| \geq 2 \max\{ |\lambda_0|, |\lambda_\infty^{-1}| \} \) we have
\[ |f(z)| = |z| \cdot \left| \frac{\lambda_0 + 1}{z + \lambda_\infty} \right| \geq |z| \cdot \frac{|\lambda_\infty^{-1}|}{3} > |z|, \]
and so \( |f^k(z)| \geq |z|(|\lambda_\infty^{-1}| \varepsilon)^k \geq 6(|\lambda_\infty^{-1}| \varepsilon)^k \) for all \( k \) by induction. It remains to show that some branch point satisfies this hypothesis. Note that the branch points \( \beta_1 \) and \( \beta_2 \) satisfy
\[ \beta_1 + \beta_2 = \frac{2(\lambda_0 \lambda_\infty - 2)}{\lambda_\infty^2} \]
\[ \beta_1 \beta_2 = \left( \frac{\lambda_0}{\lambda_\infty} \right)^2. \]

Suppose, on the contrary, we have \( |\beta_1|, |\beta_2| \leq 2 \max\{ |\lambda_0|, |\lambda_\infty^{-1}| \} \). If \( |\lambda_0| \geq |\lambda_\infty^{-1}| \), then the second equation gives
\[ \left| \frac{\lambda_0}{\lambda_\infty} \right|^2 = |\beta_1 \beta_2| \leq (2|\lambda_0|)^2, \]
whereupon
\[ \frac{1}{2} \leq |\lambda_\infty| < \varepsilon = \frac{1}{3}, \]
a contradiction. So we have \( |\lambda_0| < |\lambda_\infty^{-1}| \). But then
\[ \left| \frac{\lambda_0}{\lambda_\infty} \right|^2 = |\beta_1 \beta_2| \leq (2|\lambda_\infty^{-1}|)^2 \]
implies \( |\lambda_0| \leq 2 \), and then
\[ \left| \frac{2(\lambda_0 \lambda_\infty - 2)}{\lambda_\infty^2} \right| = |\beta_1 + \beta_2| \leq 2|\lambda_\infty^{-1}| \]
implies (given \( |\lambda_\infty| \leq 1/3 \))
\[ \frac{4}{3} = 2 - \frac{2}{3} \leq |2 - \lambda_0 \lambda_\infty| \leq |\lambda_\infty| \leq \frac{1}{3}, \]
another contradiction.
For a non-archimedean $v$, note that

$$|f(z)| = |\lambda_{\infty}^{-1}z|$$

as soon as $|z| > \max\{|\lambda_0|, |\lambda_{\infty}^{-1}|\}$. Suppose both branch points fail this. If $|\lambda_0| \geq |\lambda_{\infty}^{-1}|$, we have

$$\left|\frac{\lambda_0}{\lambda_{\infty}}\right|^2 = |\beta_1\beta_2| \leq |\lambda_0|^2,$$

giving $|\lambda_{\infty}| \geq 1$, a contradiction. On the other hand, if $|\lambda_0| < |\lambda_{\infty}^{-1}|$ we have $|\lambda_0| \leq 1$ as above, and

$$\frac{|4|}{|\lambda_{\infty}^2|} \leq \left|\frac{2(\lambda_0\lambda_{\infty} - 2)}{\lambda_{\infty}^2}\right| = |\beta_1 + \beta_2| \leq |\lambda_{\infty}^{-1}|,$$

and hence $|\lambda_{\infty}| \geq |4|$. This contradicts our choice of $\varepsilon$. □

As before, let $B'_f$ be that part of the branch locus whose support does not contain iterated preimages of $\infty$.

**Lemma 22.** For every $k \geq 1$,

$$g_f(t^k B'_f, \infty) \geq k \log^+ |\lambda_{\infty}| + k \log \varepsilon$$

$$- \deg(B'_f) \left(2 \log \|1, \lambda_0, \lambda_{\infty}\| + \log^+ |2| - r(F)\right).$$

**Proof.** Combine the previous two lemmas. □

### 4.2. Global estimates

Now let $K$ be a number field, with its standard set of absolute values $M_K$. Quantities from the previous subsection gain a subscript.

**Proof of Theorem 4** Note that

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\lambda_{\infty}| = \log 2$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\lambda_{\infty}| = \log 2$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\lambda_{\infty}| = \log 2,$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |2| = \log 2,$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |2| = \log 2,$$

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |2| = \log 2.$$
and so for any $k \geq 1$, we have, using $\deg(B'_f) \leq 2$,

\[(22) \quad 2^{k+1} \hat{h}_{\text{crit}}(f) \geq k h(\lambda_{\infty}) - k(2 \log 2 + \log 3) - 4h_{P^2}(\lambda_0, \lambda_{\infty}) - 2 \log 2.\]

Adding the corresponding estimate for $\lambda_0$, and using the fact that

\[h_{P^2}(\lambda_0, \lambda_{\infty}) \leq h(\lambda_0) + h(\lambda_{\infty}),\]

we have

\[2^{k+2} \hat{h}_{\text{crit}}(f) \geq (k - 8)h_{P^2}(\lambda_0, \lambda_{\infty}) - 4 \log 2 - 2k(2 \log 2 + \log 3).\]

The estimate in Theorem 5 is obtained by taking $k = 9$.

Finally, note that the lower bound on $\hat{h}_{\text{crit}}$ in Theorem 5 combined with (22) gives

\[k(h(\lambda) - \log 12) \leq \left(2^{k+1} + 8192\right) \hat{h}_{\text{crit}}(f) + 98\]

for any $f \in \text{Per}_1(\lambda)$, and for any $k \geq 1$. If $\hat{h}_{\text{crit}}$ takes arbitrarily small values on $\text{Per}_1(\lambda)$, the left-hand-side cannot be positive. \qed

**Remark 23.** Note that we neglected to handle the case $f(z) = z + a + z^{-1}$, but in fact it is not hard to prove the restriction of Theorem 1 to any one-parameter family. In general, if $U$ is a curve, and $f \in \text{Hom}_d(U)$, we have by [5, Theorem 4.1] that

\[\hat{h}_f(P_t) = (\hat{h}_f(P) + o(1))h_U(t) + O(1)\]

for any $P \in \mathbb{P}^1(U)$, where $o(1) \to 0$ as $h(t) \to \infty$, where $h_U$ is an degree-one Weil height on $U$, and where $\hat{h}_f(P)$ is the canonical height computed on the generic fibre. If all of the critical points are in $\mathbb{P}^1(U)$, then we have

\[(23) \quad \hat{h}_{\text{crit}}(f_t) = (\hat{h}_{\text{crit}}(f) + o(1))h(t) + O(1)\]

by summing over them. It then follows from a theorem of Baker [1] that $\hat{h}_{\text{crit}}(f_t) \asymp h(t)$ as long as the generic fibre $f$ is not isotrivial and not PCF. By Thurston Rigidity, the the family can only be PCF if it is Lattès or isotrivial.

But even if the critical points are not $U$-rational, they are $V$-rational for some finite cover $\varphi : V \to U$, and by functoriality of heights $\deg(\varphi)h_V = h_U \circ \varphi + O(1)$, giving again (23).

Finally, if $f \in \text{Hom}_d(U)$ is a non-constant, non-Lattès family, then composing with the map $\text{Hom}_d \to M_d$ gives a map $[f] : U \to M_d$ with image $\Gamma$ and finite fibres. We have, if $L$ is the ample class relative to which we are computing heights,

\[h_{M_d}([f]_t) = h_{\Gamma, L_{\varphi}}([f]_t) = h_{U,F^*L_{\varphi}}(t) + O(1) = Ch_U(t) + O(1).\]
5. An application

We conclude by proving Theorem 8. Note that, up to this point our global field has always been a number field, but the proofs carry over verbatim in any field of characteristic 0 or \( p > d \) with a collection of absolute values satisfying a product formula. In particular, if \( X \) is a curve over \( \mathbb{C} \), then we may apply the estimates to \( K = \mathbb{C}(X) \). If \( X \) is also smooth and projective, then the absolute values \( M_K \) are in one-to-one correspondence with the points \( X(\mathbb{C}) \), with \(- \log |f|_v \) corresponding to the order of vanishing of \( f \) at \( v \in X(\mathbb{C}) \). It should be noted that the reverse is true of results in this section, namely that the same tools may be used to establish uniform estimates for primitive divisors in critical orbits for weakly minimal rational functions defined over number fields. We present the function field case as it is likely to be of broader interest.

Let \( d_v \) denote the usual chordal metric on \( \mathbb{P}^1 \), defined by
\[
0 \leq d_v(P, Q) = \frac{|P \wedge Q|_v}{\|P\|_v \cdot \|Q\|_v} \leq 1.
\]
For any \( f \) and critical point \( P \), let \( S \subseteq X(\mathbb{C}) \) be the support of \( f^* \) \( \text{Res}_{d_v} \), and define
\[
a_n = \sum_{v \notin S} \log^+ d_v(f^n(P), P)^{-1}[v] \in \text{Div}(X).
\]
Our aim will be to establish a primitive divisor result in the direction of work of Krieger [11] and others. In particular, we will attempt to say something about the set of \( n \) such that
\[
\text{Supp}(a_n) \subseteq \bigcup_{m<n} \text{Supp}(a_m),
\]
which in the classical language of recurrence sequences, is the the set of indices at which there is no primitive divisor.

Our next lemma highlights the significance of the divisors \( a_n \) to the problem at hand.

**Lemma 24.** Suppose that
\[
\text{Supp}(a_n) \not\subseteq \bigcup_{m<n} \text{Supp}(a_m).
\]
Then there are values \( t \in X(\mathbb{C}) \) such that \( f_t : \mathbb{P}^1_C \to \mathbb{P}^1_C \) has an attracting cycle of length \( n \).

**Proof.** Let \( v \in \text{Supp}(a_n) \), with \( v \not\in \text{Supp}(a_m) \) for \( m < n \). Let \( \lambda_n(t) \) be the product of the multipliers of the points of formal period \( n \) for \( f_t \), noting that \( \lambda_n \) is holomorphic on \( X \setminus S \). Since \( d_v(f^n(P), P) < 1 \), the function \( t \mapsto f^n_t(P_t) - P_t \) vanishes at \( t = v \). On the other hand, \( f^m(P) - P \) does not vanish at \( t = v \) for any \( m < n \), and consequently \( P_t \) is a point of exact period \( n \) for \( f_t \). This means that \( \lambda_n(v) = 0 \). It follows that \( |\lambda_n(t)| < 1 \) in a neighbourhood of \( v \), and for all of these values \( f_t \) has an attracting cycle of formal period, and hence exact period, \( n \). \( \square \)
Before proceeding, we establish some terminology in the present context. A point \( P_1 \) is a morphism \( P : X \to \mathbb{P}^1 \) over \( \mathbb{C} \), and so we may speak of \( \deg(P) \), which is the same thing as the Weil height on \( \mathbb{P}^1 \) over \( \mathbb{C}(X) \). We then define

\[
\hat{h}_f(P) = \lim_{n \to \infty} d^{-n} \deg_X(f^n(P)),
\]

with \( \deg_X(P) \) the degree of \( P \), noting that the same basic properties hold as over number fields. If \( \varphi : Y \to X \) is a finite morphism and \( P \in \mathbb{P}^1(Y) \), then the same definition holds with \( \deg_X(P) = \deg(P)/\deg(\varphi) \). Similarly, if \( P \in \mathbb{P}^1(Y) \) we define the divisor \( a_n \) on \( Y \) as above, and set \( \deg_X(a_n) = \deg(a_n)/\deg(\varphi) \).

Lemma 25. Fix \( n \), and suppose that

\[
\text{Supp}(a_n) \subseteq \bigcup_{m<n} \text{Supp}(a_m).
\]

Then

\[
\deg_X(a_n) \leq (d^{n/2} + \log_2(n)) \hat{h}_f(P) + O(h_{\text{Hom}_d}(f) \log n).
\]

Proof. We will assume that \( P \in \mathbb{P}^1(X) \). For \( P \in \mathbb{P}^1(Y) \), for some finite cover \( \varphi : Y \to X \), the argument is the same dividing both sides of the final inequality by \( \varphi \).

First, note that if \( v \in \text{Supp}(a_m) \) and \( v \in \text{Supp}(a_n) \) with \( m < n \), then \( m | n \) and \( \text{ord}_v(a_m) = \text{ord}_v(a_n) \). This second claim is true because \( P \) is a critical point. First, note that if \( v \in \text{Supp}(a_1) \) then \( d_v(f(P),P) < 1 \). Since \( d_v \) is preserved by the involution \( z \mapsto \frac{1}{z} \), we can assume without loss of generality that \( P \in \mathcal{D}_v(0,1) \), the \( v \)-adic unit disk. Since \( v \notin S \), \( f \) admits a series expansion

\[
f(Q) - f(P) = a_2(Q-P)^2 + \cdots
\]

with regular coefficients. In particular, \( |f(Q) - f(P)|_v < |Q-P|_v \) if the latter is strictly less than one. Now, since \( d_v(f(P),P) < 1 \), we must also have \( f(P) \in \mathcal{D}_v(0,1) \), and so \( d_v(f(P),P) = |f(P) - P|_v \). Applying the above observation with \( Q = f(P) \), we have \( |f(f(P)) - P|_v = |f(P) - P|_v \). By induction, \( d_v(f^n(P),P) = |f^n(P) - P|_v = |f(P) - P|_v \). For \( m > 1 \), we simply apply the same argument to \( f^m \).

So now, the hypothesis (24) implies

\[
a_n \leq \sum_{p|n} a_{n/p}.
\]
Now we estimate the degree of each term on the right. Note that
\[
\deg_X(a_m) = \sum_{v \notin S} -\log d_v(f^m(P), P)
\]
\[
= \sum_{v \in X(\mathbb{C})} \left( \log \|f^m(P)\|_v + \log \|P\|_v - \log |f^m(P) \wedge P|_v \right)
\]
\[
+ \sum_{v \in S} \log d_v(f^m(P), P)
\]
\[
= \deg_X(f^m(P)) + \deg_X(P) + \sum_{v \in S} \log d_v(f^m(P), P)
\]
\[
= (d^m + 1)\hat{h}_f(P) + O(h_{\text{Hom}, d}(f))
\]
using the fact that \(d_v(Q, P) \leq 1\) for all \(Q\) and \(P\), and \(\deg_X(Q) \leq \hat{h}_f(Q) + O(h_{\text{Hom}, d}(f))\).

It follows that
\[
\deg_X(a_n) \leq \sum_{p|n} \deg_X(a_{n/p}) \leq \sum_{p|n} (d^{n/p} + 1)\hat{h}_f(P) + O(h_{\text{Hom}, d}(f)) \leq (d^{n/2+1} + \log_2(n))\hat{h}_f(P) + O(h_{\text{Hom}, d}(f) \log n),
\]
since \(n\) has at most \(\log_2 n\) prime divisors, the smallest of which is 2. □

We next show that the inequality in the previous lemma is actually quite rare. The following is a direct consequence of a result of Hsia and Silverman [9]. Note that Hsia and Silverman present their result over number fields, but the key ingredient is a quantitative version of Roth’s Theorem which is also available over function fields of characteristic zero.

**Lemma 26.** For any \(\varepsilon > 0\) and \(\varphi : Y \to X\) there exists a constant \(\gamma\) depending just on \(\varepsilon, d, \) and \(\deg(\varphi)\) such that for any \(P \in \mathbb{P}^1(Y)\) the number of \(n\) not satisfying
\[
(1 - \varepsilon)d^m\hat{h}_f(P) \leq \deg_X(a_n) + O(h_{\text{Hom}, d}(f))
\]
is at most
\[
4^\#S \gamma + \log_d^+ \left( 1 + \frac{h_{\text{Hom}, d}(f)}{\hat{h}_f(P)} \right).
\]

**Proof.** This is an immediate consequence of [9, Theorem 11], once one notes that the argument works equally well over function fields of characteristic zero. (Over function fields of positive characteristic, the analogue of the requisite theorem of Roth is in fact false.) □
Proof of Theorem 8. Note that, by Theorem 1 and Lemma 15 (the proof of which carries over verbatim in function fields) there exists a \( \phi : Y \to X \) and an \( a \in \text{Hom}_d(Y) \) conjugate to \( f \) with
\[
h_{\text{Hom}_d}(g) \leq c_1 \hat{h}_{\text{crit}}(f) + c_2,
\]
with \( c_1 > 0 \) and \( c_2 \) depending just on \( d \). Our minimality hypothesis then gives
\[
h_{\text{Hom}_d}(f) \leq 2 \left( c_1 \hat{h}_{\text{crit}}(f) + c_2 \right) \leq c_3 \hat{h}_{\text{crit}}(f)
\]
for \( f \) not PCF, since \( \hat{h}_{\text{crit}} \) has a smallest positive value on \( M_d(X) \). We also have a critical point \( P \) satisfying \( \hat{h}_{\text{crit}}(P) \geq \frac{1}{2d-2} \hat{h}_{\text{crit}}(f) \), which we fix now. Note that \( P \) might not be defined over \( \mathbb{C}(X) \), but it is defined over an extension of degree at most \( (2d-2) \). Lemma 26 is uniform over extensions of bounded degree, and the number of places over \( \mathcal{S}_f \) in this extension is at most \( (2d-2) \# \mathcal{S}_f \). We can assume without loss of generality, then, that \( P \) is \( \mathbb{C}(X) \)-rational.

Now, applying Lemma 25 and Lemma 26 if \( \text{Supp}(a_n) \subseteq \bigcup_{m<n} \text{Supp}(a_m) \) then either
\[
(1 - \varepsilon)d^n \hat{h}_{f}(P) \leq (d^{n/2+1} + \log_2(n)) \hat{h}_{f}(P) + O(h_{\text{Hom}_d}(f) \log n)
\]
or else \( n \) is one of at most
\[
4^{\# \mathcal{S}} \gamma + \log^+ \left( 1 + \frac{h_{\text{Hom}_d}(f)}{h_{f}(P)} \right)
\]
extceptions. Note that
\[
\frac{h_{\text{Hom}_d}(f)}{h_{f}(P)} \leq \frac{c_3 \hat{h}_{\text{crit}}(f)}{\frac{1}{2d-2} \hat{h}_{\text{crit}}(f)} = c_3 (2d-2),
\]
and so the number of exceptions is bounded with dependence only on \( \# \mathcal{S} \). On the other hand, (26) along with (27) implies
\[
(1 - \varepsilon)d^n \leq (d^{n/2+1} + \log_2(n)) + O(\log n),
\]
with implied constants depending on \( d \) and \( \varepsilon > 0 \). Fixing \( \varepsilon = \frac{1}{4} \), this bounds \( n \) in terms of \( d \).

So, with a number of exceptions bounded in terms of \( d \), \( \# \mathcal{S} \), and \( X \), there is some \( v \in \text{Supp}(a_n) \) such that \( v \notin \text{Supp}(a_m) \) for any \( m < n \). By Lemma 24, the complement of \( A_f \) is contained within this set of exceptions.

References

[1] M. Baker. A finiteness theorem for canonical heights attached to rational maps over function fields, J. Reine Angew. Math 626 (2009), pp. 205–233.
[2] M. Baker and R. Rumely. Equidistribution of Small Points, Rational Dynamics, and Potential Theory, Ann. Inst. Fourier (Grenoble) 56 no. 3 (2006), pp. 625–688.
[3] M. Baker and R. Rumely. Potential Theory and Dynamics on the Berkovich Projective Line, AMS Mathematical Surveys and Monographs 159 (2010).
[4] R. L. Benedetto, P. Ingram, R. Jones, and A. Levy. Attracting cycles in $p$-adic dynamics and height bounds for post-critically finite maps, Duke Math. J. 163 no. 13 (2014), pp. 2325-2356.

[5] G. S. Call and J. H. Silverman, Canonical heights on varieties with morphisms, Compositio Math. 89 (1993), pp. 163–205.

[6] L. DeMarco, X. Wang, and H. Ye. Bifurcation measures and quadratic rational maps, Proc. London Math. Soc. 111 no. 1 (2015), pp. 149–180.

[7] P. Fatou. Sur les équations fonctionelles, Bull. de la S. M. F. 47 (1919), pp. 161–271.

[8] P. Fatou. Sur les équations fonctionelles, Bull. de la S. M. F. 48 (1920), pp. 208–314.

[9] L.C. Hsia and J.H. Silverman, A quantitative estimate for quasi-integral points in orbits, Pacific J. Math. 249 (2011), no. 2, pp. 321–342.

[10] P. Ingram. A finiteness result for post-critically finite polynomials, Int. Math. Res. Not. (2012) Issue 3, pp. 524–543.

[11] H. Krieger. Primitive divisors of the critical orbit of $z^d + c$. Int. Math. Res. Not. (2013) Issue 23, pp. 5498–5525.

[12] J. Milnor. Remarks on iterated cubic maps, Experiment. Math. 1 (1992) Issue 1, pp. 5–24.

[13] J. Milnor. Geometry and dynamics of quadratic rational maps, Experiment. Math. 2 (1993) Issue 1, pp. 37–83.

[14] C. McMullen. Families of rational maps and iterative root-finding algorithms, Ann. of Math. 125 (1987), pp. 467–493.

[15] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois Journal Math. 6 (1962), pp. 64–94.

[16] J. H. Silverman. The Arithmetic of Dynamical Systems, volume 241 of Graduate Texts in Mathematics. Springer, 2007.

[17] J. H. Silverman. Moduli Spaces and Arithmetic Dynamics, volume 30 of CRM Monograph Series. AMS, 2012.

[18] P. Vojta. Diophantine Approximation and Nevanlinna Theory, In P. Corvaja and C. Gasbarri (ed.s), Arithmetic Geometry: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, September 10-15, 2007, Springer, 2010.

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