A GENERALIZATION OF J-QUASIPOLAR RINGS

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Abstract. In this paper we introduce a class of quasipolar rings which is a generalization of J-quasipolar rings. Let \( R \) be a ring with identity. An element \( a \in R \) is called \( \delta \)-quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \) is contained in \( \delta(R) \), and the ring \( R \) is called \( \delta \)-quasipolar if every element of \( R \) is \( \delta \)-quasipolar. We use \( \delta \)-quasipolar rings to extend some results of \( J \)-quasipolar rings. Then some of the main results of \( J \)-quasipolar rings are special cases of our results for this general setting. We give many characterizations and investigate general properties of \( \delta \)-quasipolar rings.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity unless otherwise stated. Let \( R \) be a ring. According to Koliha and Patricio \([10]\), the commutant and double commutant of an element \( a \in R \) are defined by \( \text{comm}(a) = \{ x \in R \mid xa = ax \} \), \( \text{comm}^2(a) = \{ x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a) \} \), respectively. If \( R^{\text{qnil}} = \{ a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a) \} \) and \( a \in R^{\text{qnil}} \), then \( a \) is said to be quasinilpotent (see \([9]\)). The element \( a \) is called quasipolar if there exists \( p^2 = p \in R \) such that \( p \in \text{comm}^2(a) \), \( a + p \) is invertible in \( R \) and \( ap \in R^{\text{qnil}} \). Any idempotent \( p \) satisfying the above conditions is called a spectral idempotent of \( a \), and this term is borrowed from spectral theory in Banach algebra and it is unique for \( a \).

Quasipolar rings have been studied by many ring theorists (see \([1, 2, 5–7, 9, 10]\) and \([15]\)). In \([7]\), the element \( a \in R \) is called nil-quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \) is nilpotent, the idempotent \( p \) is called a nil-spectral idempotent of \( a \). The ring \( R \) is said to be nil-quasipolar if every element of \( R \) is nil-quasipolar. Recently, \( J \)-quasipolar rings are studied in \([4]\). The element \( a \) is called \( J \)-quasipolar if there exists \( p^2 = p \in R \) such that \( p \in \text{comm}^2(a) \) and \( a + p \in J(R) \), \( p \) is called a \( J \)-spectral idempotent of \( a \). The ring \( R \) is said to be \( J \)-quasipolar if

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every element of $R$ is $J$-quasipolar. Motivated by these, we introduce a new class of quasipolar rings which is a generalization of $J$-quasipolar rings. By using $\delta$-quasipolar rings, we extend some results of $J$-quasipolar rings.

An outline of the paper is as follows: Section 2 deals with $\delta$-quasipolar rings. We prove various basic characterizations and properties of $\delta$-quasipolar rings. It is proven that every $J$-quasipolar ring is $\delta$-quasipolar. We supply an example to show that all $\delta$-quasipolar rings need not be $J$-quasipolar. Among others the $\delta$-quasipolarity of Dorroh extensions and some classes of matrix rings are investigated. In Section 3, we introduce an upper class of $\delta$-quasipolar rings, namely, weakly $\delta$-quasipolar rings. We show that every direct summand of a weakly $\delta$-quasipolar ring is weakly $\delta$-quasipolar and every direct product of weakly $\delta$-quasipolar rings is weakly $\delta$-quasipolar, and we give some properties of such rings.

In what follows, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer $n$, $\mathbb{Z}/n\mathbb{Z}$ is the ring of integers modulo $n$. For a positive integer $n$, let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in $R$. We write $J(R)$ and $nil(R)$ for the Jacobson radical of $R$ and the set of nilpotent elements of $R$, respectively.

2. $\delta$-QUASIPOLAR RINGS

In this section we introduce the concept of $\delta$-quasipolar rings and investigate some properties of such rings. We show that every quasipolar ring need not be $\delta$-quasipolar (Example 2). It is proven that every $J$-quasipolar ring is $\delta$-quasipolar and the converse does not hold in general (see Example 3). Among others we extend some results of $J$-quasipolar rings for this general setting.

A right ideal $I$ of the ring $R$ is said to be $\delta$-small in $R$ if whenever $R = I + K$ with $R/K$ singular right $R$-module for any right ideal $K$ then $R = K$. In [16], the ideal $\delta(R)$ is introduced as a sum of $\delta$-small right ideals of $R$. We begin with the equivalent conditions for $\delta(R)$ which is proved in [16, Theorem 1.6] for an easy reference for the reader.

**Lemma 1.** Given a ring $R$, each of the following sets is equal to $\delta(R)$.

1. $R_1 = \text{the intersection of all essential maximal right ideals of } R$.
2. $R_2 = \text{the unique largest } \delta\text{-small right ideal of } R$.
3. $R_3 = \{x \in R \mid xR + K_R = R \text{ implies } K_R \text{ is a direct summand of } R_R\}$.
4. $R_4 = \bigcap \{\text{ideals } P \text{ of } R \mid R/P \text{ has a faithful singular simple module}\}$.
5. $R_5 = \{x \in R \mid \text{for all } y \in R \text{ there exists a semisimple right ideal } Y \text{ of } R \text{ such that } (1+xy)R \oplus Y = R_R\}$.

Now we give our main definition.
Definition 1. Let $R$ be a ring. An element $a \in R$ is called $\delta$-quasipolar if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$ and $p$ is called a $\delta$-spectral idempotent. The ring $R$ is called $\delta$-quasipolar if every element of $R$ is $\delta$-quasipolar.

The following are examples for $\delta$-quasipolar rings.

Example 1. (1) Every semisimple ring and every Boolean ring is $\delta$-quasipolar.
(2) Since $\delta(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is $\delta$-quasipolar. On the other hand, $\mathbb{Z}$ is not $\delta$-quasipolar since $\delta(\mathbb{Z}) = 0$.

One may suspects that every quasipolar ring is $\delta$-quasipolar. But the following example erases the possibility.

Example 2. Let $p$ be a prime integer with $p \geq 3$ and $R = \mathbb{Z}((p))$ the localization of $\mathbb{Z}$ at the ideal $(p)$. By [4, Example 2.8], $R$ is a quasipolar ring. Since $J(R) = \delta(R)$, it is not $\delta$-quasipolar.

Let $S_r$ denote the right socle of the ring $R$, that is, $S_r$ is the sum of minimal right ideals of $R$. We now prove that the class of $J$-quasipolar rings is a subclass of $\delta$-quasipolar rings.

Lemma 2. If $R$ is a $J$-quasipolar ring, then $R$ is $\delta$-quasipolar. The converse holds if $S_r \subseteq J(R)$.

Proof. The first assertion is clear since $J(R) \subseteq \delta(R)$. Assume that $R$ is $\delta$-quasipolar. If $S_r \subseteq J(R)$, then $J(R)/S_r = J(R/S_r) = \delta(R)/S_r$ by [16, Corollary 1.7] and we have $J(R) = \delta(R)$. Hence, $R$ is $J$-quasipolar.

The converse of Lemma 2 is not true in general as the following example shows.

Example 3. Let $F$ be a field and consider the ring $R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. Then $R$ is a semisimple ring and $R = \delta(R)$ and $J(R) = 0$. Hence $R$ is $\delta$-quasipolar and it is not $J$-quasipolar.

Lemma 3. Let $R$ be a ring. Then we have the following.

(1) If $a, u \in R$ and $u$ is invertible, then $a$ is $\delta$-quasipolar if and only if $u^{-1}au$ is $\delta$-quasipolar.
(2) The element $a \in R$ is $\delta$-quasipolar if and only $-1 - a$ is $\delta$-quasipolar.
(3) If $R$ is a $\delta$-quasipolar ring with $\delta(R) = J(R)$, then the spectral idempotent for any invertible element in $R$ is the identity of $R$.

Proof. (1) Assume that $a$ is $\delta$-quasipolar. Let $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$. Let $x \in \text{comm}(u^{-1}au)$. Then $(uxu^{-1})a = a(uxu^{-1})$. Since $p \in \text{comm}^2(a)$, $(uxu^{-1})p = p(uxu^{-1})$. Hence $(u^{-1}pu)^2 = u^{-1}pu \in \text{comm}^2(u^{-1}au)$. Since $\delta(R)$ is an ideal of $R$, $u^{-1}(a + p)u = u^{-1}au + u^{-1}pu \in \delta(R)$. Thus $u^{-1}au$ is $\delta$-quasipolar. Conversely, if $u^{-1}au$ is $\delta$-quasipolar, then by the preceding proof
(2) Assume that $a$ is a $\delta$-quasipolar. Let $p^2 = p \in \text{comm}^2(a)$ such that $a + p = r \in \delta(R)$. Then $-1 - a + (1 - p) = -r \in \delta(R)$. Then $1 - p \in \text{comm}^2(1 - a)$ and $1 - p$ is the spectral idempotent of $-1 - a$. Conversely, if $-1 - a$ is $\delta$-quasipolar, then from what we have proved that $-1 - (-1 - a) = a$ is quasipolar.

(3) Assume that $\delta(R) = J(R)$. Then $\delta$-quasipolarity of $R$ implies $J$-quasipolarity of $R$. So its proof can be directly obtained from [4, Example 2.2].

In [4, Corollary 2.3], it is proved that if $R$ is a $J$-quasipolar ring, then $2 \in J(R)$. In this direction we prove the following.

**Lemma 4.** If $R$ is a $\delta$-quasipolar ring, then $2 \in \delta(R)$.

*Proof.* For the identity $1$, there exists $p^2 = p \in R$ such that $1 + p \in \delta(R)$. Multiplying the latter by $p$, we have $2p \in \delta(R)$. So $2 = (1 + p) - 2p \in \delta(R)$.

Lemma 4 can be used to determine whether given rings are $\delta$-quasipolar.

**Example 4.** (1) The ring $\mathbb{Z}_3$ is a semisimple ring and $\delta$-quasipolar but the ring $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not $\delta$-quasipolar since $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ and $2$ does not contained in $\delta(R)$.

(2) Let $R = \{ (a_{ij}) \in T_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn} \}$. $\mathbb{Z}_3$ is $\delta$-quasipolar but $R$ is not since $\delta(R) = \{ (a_{ij}) \in T_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn} = 0 \}$ and $2$ does not contained in $\delta(R)$.

Recall that a ring $R$ is called *local* if it has only one maximal left ideal, equivalently, maximal right ideal.

**Proposition 1.** Let $R$ be a local ring. If $R/J(R) \cong \mathbb{Z}_2$, then $R$ is $\delta$-quasipolar.

*Proof.* Let $a \in R$. If $a \in J(R)$, it is clear. Assume that $a \notin J(R)$. Since $R$ is local, $a$ is invertible. Hence $a + 1 \in \delta(R)$ by $\delta(R) = J(R)$.

A ring $R$ is said to be *clean* [12] if for each $a \in R$ there exists $e^2 = e \in R$ such that $a - e$ is invertible, and $R$ is called *strongly clean* [13] provided that every element of $R$ can be written as the sum of an idempotent and an invertible element that commute.

**Example 5.** Let $R = \{ (q_1, q_2, q_3, \ldots, q_n, a, a, a, \ldots) \mid n \geq 1; q_i \in \mathbb{Q}; a \in \mathbb{Z}_{(2)} \}$. Then $R$ is strongly clean but not quasipolar (see [15, Example 3.4(3)]). Therefore $R$ is not $J$-quasipolar since every $J$-quasipolar ring is quasipolar. On the other hand, since $S_R = 0$ and $\delta(R)/S_R = J(R)/S_R$, $\delta(R) = J(R)$. Thus $R$ is not $\delta$-quasipolar.

In [4, Theorem 2.9], it is shown that if the ring $R$ is $J$-quasipolar, then $R/J(R)$ is Boolean and idempotents in $R/J(R)$ lift $R$. We have the following result for $\delta$-quasipolar rings.
Theorem 1. If \( R \) is a \( \delta \)-quasipolar ring, then \( R/\delta(R) \) is a Boolean ring and idempotents in \( R/\delta(R) \) lift \( R \).

Proof. Let \( \overline{a} \in R/\delta(R) \). There exists \( p^2 = p \in \text{comm}^2(-1 + a) \) such that \(-1 + a + p \in \delta(R) \). Hence \( \overline{a} = 1 - p \) is an idempotent in \( R/\delta(R) \) and \( R/\delta(R) \) is a Boolean ring. Let \( \overline{a^2} = \overline{a} \in R/\delta(R) \). Then there exists \( p^2 = p \in \text{comm}^2(-a) \) such that \(-a + p \in \delta(R) \). This yields \( \overline{a} = \overline{p} \), as asserted. \( \square \)

The concept of \( \delta_r \)-clean rings are defined in [8]. A ring \( R \) is called \( \delta_r \)-clean if for every element \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in \delta(R) \). A ring is abelian if all idempotents are central.

Lemma 5. If \( R \) is a \( \delta \)-quasipolar ring, then it is \( \delta_r \)-clean. The converse holds if \( R \) is abelian.

Proof. Let \( R \) be a \( \delta \)-quasipolar ring and \( a \in R \). There exists \( p^2 = p \in \text{comm}^2(-1 + a) \) such that \(-1 + a + p \in \delta(R) \). Then \( a - (1 - p) \in \delta(R) \). For the converse, assume that \( R \) is abelian. Let \( a \in R \). There exists an idempotent \( e \) such that \( 1 + a - e \in \delta(R) \). By assumption, \( 1 - e \) is a central idempotent and so \( 1 - e \in \text{comm}^2(a) \). \( \square \)

Recall that a ring \( R \) is exchange if for every \( a \in R \), there exists an idempotent \( e \in R \) such that \( 1 - e \in (1 - a)R \). Namely, von Neumann regular rings and clean rings are exchange.

Corollary 1. Let \( R \) be a \( \delta \)-quasipolar ring. Then

1. \( R \) is an exchange ring.
2. \( R/\delta(R) \) is a clean ring.

Proof. (1) Let \( R \) be a \( \delta \)-quasipolar ring. By Lemma 5, \( R \) is a \( \delta_r \)-clean ring. By [8, Theorem 2.2(2)], every \( \delta_r \)-clean ring is an exchange ring.

(2) By Theorem 1, \( R/\delta(R) \) is Boolean, therefore, it is clean. \( \square \)

Corollary 2. Consider following conditions for a ring \( R \).

1. \( R \) is \( \delta \)-quasipolar and \( \delta(R) = 0 \).
2. \( R \) is Boolean.
3. \( R \) is von Neumann regular and \( \delta \)-quasipolar.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

Proof. (1) \( \Rightarrow \) (2) Assume that \( R \) is \( \delta \)-quasipolar and \( \delta(R) = 0 \). By Theorem 1, \( R \) is Boolean.

(2) \( \Rightarrow \) (3) Assume that \( R \) is Boolean. Then it is commutative with characteristic 2 and \( a^2 + a = 0 \in \delta(R) \) and \( a^2 = a = a^3 \) for all \( a \in R \). Hence \( R \) is von Neumann regular and \( \delta \)-quasipolar. \( \square \)

Strongly \( J \)-clean rings were introduced by Chen in [3]. For a ring \( R \) the element \( a \in R \) is called \( J \)-clean if \( a \) is the sum of an idempotent and a radical element in
its Jacobson radical. The ring \( R \) is called \( J \)-clean if every element is a sum of an idempotent and a radical element.

**Theorem 2.** If \( R \) is an abelian \( J \)-clean ring, then it is \( \delta \)-quasipolar.

*Proof.* Let \( a \in R \). Then we have \(-a \in R \). Since \( R \) is \( J \)-clean, there exist \( e^2 = e \in R \) and \( j \in J(R) \) such that \(-a = e + j \). Hence \( a + e \in J(R) \). Since \( R \) is abelian, \( e^2 = e \in \text{comm}^2(a) \) and \( J(R) \subseteq \delta(R) \), \( R \) is \( \delta \)-quasipolar as asserted. \( \square \)

All \( \delta \)-quasipolar rings need not be Boolean and the converse statement of Theorem 2 is not true in general.

**Example 6.** The ring \( \mathbb{Z}_3 \) is semisimple and so \( \mathbb{Z}_3 = \delta(\mathbb{Z}_3) \). Therefore \( \mathbb{Z}_3 \) is \( \delta \)-quasipolar, but it is neither Boolean nor \( J \)-clean.

In \([4, \text{Proposition 2.11}]\), it is shown that a ring \( R \) is local and \( J \)-quasipolar if and only if \( R \) is \( J \)-quasipolar with only trivial idempotents if and only if \( R/J(R) \cong \mathbb{Z}_2 \).

We have the following for \( \delta \)-quasipolar rings.

**Proposition 2.** Let \( R \) be a ring with only trivial idempotents. Then \( R \) is \( \delta \)-quasipolar if and only if \( R/\delta(R) \cong \mathbb{Z}_2 \).

*Proof.* Assume that \( R \) is \( \delta \)-quasipolar. Let \( a \in R \). There exists an idempotent \( p \in \text{comm}^2(a) \) such that \(-a + p \in \delta(R) \). By hypothesis \( p = 1 \) or \( p = 0 \). If \( \delta(R) = 0 \), then \( R/\delta(R) \cong \mathbb{Z}_2 \). Suppose that \( \delta(R) \neq 0 \). For any \( a \in R \setminus \delta(R) \), \( \overline{a} = \overline{1} \in R/\delta(R) \). Hence \( R/\delta(R) \cong \mathbb{Z}_2 \). Conversely, suppose that \( R/\delta(R) \) is isomorphic to \( \mathbb{Z}_2 \) by isomorphism \( f \). Let \( a \in R \setminus \delta(R) \). Then \( f(-\overline{a}) = \overline{1} \in \mathbb{Z}_2 \). Then \( f(-\overline{a}) = f(\overline{1}) \) implies \(-\overline{a} - \overline{1} \in \text{ker} f = 0 \). Hence \(-\overline{a} = \overline{1} \). That is, \( a + 1 \in \delta(R) \). Thus \( R \) is \( \delta \)-quasipolar. \( \square \)

Recall that a ring \( R \) is called strongly \( \pi \)-regular if for every element \( a \) of \( R \) there exist a positive integer \( n \) (depending on \( a \)) and an element \( x \) of \( R \) such that \( a^n = a^{n+1}x \), equivalently, an element \( y \) of \( R \) such that \( a^n = ya^{n+1} \). In spite of the fact that \( J(R) \) is contained in both \( \delta(R) \) and \( R^{\text{qnil}} \), no comparings between \( \delta(R) \) and \( R^{\text{qnil}} \) exist. Strongly \( \pi \)-regular rings play crucial role in this direction.

**Proposition 3.** Let \( R \) be a \( \delta \)-quasipolar ring and \( \delta(R) = J(R) \). Then \( R \) is strongly \( \pi \)-regular if and only if \( J(R) = R^{\text{qnil}} = \text{nil}(R) = \delta(R) \).

*Proof.* Necessity. Let \( a \in R^{\text{qnil}} \). Then for any \( x \in \text{comm}(a) \), \( 1 - ax \) is invertible. By hypothesis, there exist a positive integer \( m \) and \( b \in R \) such that \( a^m = a^{m+1}b \).

Since \( b \in \text{comm}(a) \) by \([11, \text{Page 347, Exercise 23.6(1)}] \), \( a^m = 0 \). Hence \( a \in \text{nil}(R) \) and so \( R^{\text{qnil}} \subseteq \text{nil}(R) \). To prove \( \text{nil}(R) \subseteq \delta(R) \), let \( a \in \text{nil}(R) \). By hypothesis there exists \( p^2 = p \in \text{comm}^2(1 - a) \) such that \( 1 - a + p \in \delta(R) \). Since \( 1 - a \) is invertible, \( p = 1 \) by Lemma 3 (3). Hence \( 2 - a \in \delta(R) \). Also \( 2 \in \delta(R) \) by Lemma 4, we then have \( a \in \delta(R) \).
Sufficiency. Let \( a \in R \). There exists \( p^2 = p \in \text{com}m^2(-1 + a) \) such that \(-1 + a + p \in \delta(R)\). Set \( u = -1 + a + p \in \text{nil}(R) \). Then \( a + p \) is not nilpotent so that \( a^n p = 0 \) for some positive integer \( n \). So \( a^n = a^n(1 - p) = (u + (1 - p))^n(1 - p) = (u + 1)^n(1 - p) = (a + p)^n(1 - p) = (1 - p)(a + p)^n \). By [13, Proposition 1], \( a \) is strongly \( \pi \)-regular. This completes the proof. \( \square \)

Let \( R \) and \( V \) be rings and \( V \) be an \( (R, R) \)-bimodule that is also a ring with \( (vw)r = v(wr), (wr)v = v(rw), \) and \( (r)v = r(vw) \) for all \( v, w \in V \) and \( r \in R \). The Dorroh extension \( D(R, V) \) of \( R \) by \( V \) defined as the ring consisting of the additive abelian group \( R \oplus V \) with multiplication \( (r, v)(s, w) = (rs, rw + vs + vw) \) where \( r, s \in R \) and \( v, w \in V \).

Uniquely clean rings were introduced by Nicholson and Zhou in [14]. A ring \( R \) is uniquely clean in case for any \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in R \) is invertible. In [8], among others, uniquely \( \delta_r \)-clean rings are studied. A ring \( R \) is called uniquely \( \delta_r \)-clean if for every element \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in \delta(R) \). Uniquely clean Dorroh extensions in [14, Proposition 7] and uniquely \( \delta_r \)-clean Dorroh extensions in [8, Proposition 3.11] are considered. Now we consider \( \delta \)-quasipolar Dorroh extensions.

**Proposition 4.** Let \( R \) be a ring. Then we have the following.

1. If \( D(R, V) \) is \( \delta \)-quasipolar, then \( R \) is \( \delta \)-quasipolar.
2. If the following conditions are satisfied, then \( D(R, V) \) is \( \delta \)-quasipolar.
   1. \( R \) is \( \delta \)-quasipolar;
   2. \( e^2 = e \in R \), then \( ev = ve \) for all \( v \in V \);
   3. \( V = \delta(V) \).

**Proof.** (1) Let \( r \in R \). There exists \( e^2 = e \in D(R, V) \) such that \( e \in \text{com}m^2(r, 0) \) and \( (r, 0) + e \in \delta(D(R, V)) \). Since \( e \in D(R, V) \), \( e \) has the form such that \( (p, v)^2 = (p, v) \) and \( p^2 = p \). Then \( e = (p, v) \in \text{com}m^2(r, 0) \) implies that \( p \in \text{com}m^2(r) \) and \( r + p \in \delta(R) \) since \( (r + p, v) \in \delta(D(R, V)) \) and by [8, Proposition 3.11]. Hence \( R \) is \( \delta \)-quasipolar.

(2) Assume that (i), (ii) and (iii) hold. Let \((r, v) \in D(R, V) \). There exists \( p^2 = p \in \text{com}m^2(r) \) such that \( r + p \in \delta(R) \). By (iii), \((0, V) \subseteq \delta(D(R, V))\). Then \((r, v) + (p, 0) = (r + p, v) \in \delta(D(R, V)) \). To see that \((p, 0) \in \text{com}m^2((r, v))\), let \((a, b) \in D(R, V) \) and \((a, b)(r, v) = (r, v)(a, b) \). Then \( ar = ra \) and so \( ap = pa \) since \( p \in \text{com}m^2(r) \). Also \( pb = bp \) by (ii). Therefore we have \((p, 0)(a, b) = (a, b)(p, 0) \) that is \((p, 0) \in \text{com}m^2((r, v)) \). \( \square \)

As an application of Dorroh extensions we consider the following example. This example also shows that in Proposition 4 (2), the conditions (i), (ii) and (iii) are not superfluous.

**Example 7.** Consider the ring \( D(\mathbb{Z}, \mathbb{Q}) \). Then \( D(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Q} \). Then \( \delta(\mathbb{Z} \times \mathbb{Q}) = (0) \times \mathbb{Q} \). Since \( \mathbb{Z} \) is not \( \delta \)-quasipolar, \( D(\mathbb{Z}, \mathbb{Q}) \) is not \( \delta \)-quasipolar.
Let $R$ and $S$ be any ring and $M$ an $(R,S)$-bimodule. Consider the ring of the formal upper triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. It is well known that $\delta(T) \subseteq \begin{bmatrix} \delta(R) & M \\ 0 & \delta(S) \end{bmatrix}$. However, if $M = R = S = F$ is a field, then $\delta(T) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$.

The following example illustrates the $\delta$-quasipolarity of full matrix rings and upper triangular matrix rings depend on the coefficient ring.

**Example 8.**

1. Consider the ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. $R$ is $\delta$-quasipolar.

2. As noted in Example 4, the ring $\mathbb{Z}_3$ is semisimple and therefore $\delta$-quasipolar. However, the ring $\begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not $\delta$-quasipolar.

3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \text{Mat}_2(\mathbb{Z})$. For any $P^2 = P \in \text{comm}^2(A)$, the matrix $P$ has the form $P = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ with $x^2 = x$ and $2xy = y$ where $x, y \in \mathbb{Z}$. This would imply that $P$ is the zero matrix or the identity matrix. Since $\delta(\mathbb{Z}) = 0$, $\delta(\text{Mat}_2(\mathbb{Z})) = 0$. In consequence, $A + P$ can not be in $\delta(\text{Mat}_2(\mathbb{Z}))$. Therefore $\text{Mat}_2(\mathbb{Z})$ is not $\delta$-quasipolar.

4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T_2(\mathbb{Z})$. The idempotents of $T_2(\mathbb{Z})$ are zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix}$ where $y$ is an arbitrary integer. Since $A$ commutes with only zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, among these idempotents there is no idempotent $P$ such that $A + P \in \delta(T_2(\mathbb{Z}))$ since $\delta(T_2(\mathbb{Z})) = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Hence $T_2(\mathbb{Z})$ is not $\delta$-quasipolar.

### 3. Weakly $\delta$-quasipolar Rings

In this section, we introduce an upper class of $\delta$-quasipolar rings, namely, weakly $\delta$-quasipolar rings, and we give some properties of such rings.

**Definition 2.** Let $R$ be a ring and $a \in R$. The element $a$ is called weakly $\delta$-quasipolar if there exists $p^2 = p \in \text{comm}(a)$ such that $a + p \in \delta(R)$, and $p$ is called a weakly $\delta$-spectral idempotent. A ring $R$ is called weakly $\delta$-quasipolar if every element of $R$ is weakly $\delta$-quasipolar.
An element of a ring is called strongly $J$-clean [3] provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is strongly $J$-clean in case each of its elements is strongly $J$-clean.

Example 9. (1) Every semisimple ring and every Boolean ring is weakly $\delta$-quasipolar, since $\delta$-quasipolar rings are weakly $\delta$-quasipolar. 
(2) Every strongly $J$-clean ring is weakly $\delta$-quasipolar.

Proposition 5. Let $f : R \to S$ be a surjective ring homomorphism. If $R$ is weakly $\delta$-quasipolar, then $S$ is weakly $\delta$-quasipolar.

Proof. Let $s \in S$ with $s = f(r)$ where $r \in R$. There exists an idempotent $p \in \text{comm}(r)$ such that $r + p \in \delta(R)$. Let $q = f(p)$. Then $q^2 = q \in \text{comm}(f(r)) = \text{comm}(s)$. By [16], $f(\delta(R)) \subseteq \delta(S)$. Then $s + q = f(r) + f(p) = f(r + p) \in f(\delta(R)) \subseteq \delta(S)$. Hence $S$ is weakly $\delta$-quasipolar. □

Corollary 3. Every direct summand of a weakly $\delta$-quasipolar ring is weakly $\delta$-quasipolar.

Proposition 6. Let $R = \prod_{i=1}^{n} R_i$ be a finite direct product of rings. $R$ is weakly $\delta$-quasipolar if and only if each $R_i$ is weakly $\delta$-quasipolar for $(i = 1, 2, \ldots, n)$.

Proof. One way is clear from Corollary 3. We may assume that $n = 2$ and $R_1$ and $R_2$ are weakly $\delta$-quasipolar. Let $a = (x_1, x_2) \in R$. There exist idempotents $p_i \in \text{comm}(x_i)$ such that $x_i + p_i \in \delta(R_i)$ for $(i = 1, 2)$. Then $p = (p_1, p_2)$ is an idempotent in $R$ and $p \in \text{comm}(a)$ and $a + p \in \delta(R)$. Hence $R$ is weakly $\delta$-quasipolar. □

In [8], Gurgun and Ozcan introduce and investigate properties of $\delta_r$-clean rings. Motivated by this work strongly $\delta_r$-clean rings can be defined as follows.

Definition 3. An element $x \in R$ is called strongly $\delta_r$-clean provided that there exist an idempotent $e \in R$ and an element $w \in \delta_r$ such that $x = e + w$ and $ew = we$. A ring $R$ is called strongly $\delta_r$-clean in case every element in $R$ is strongly $\delta_r$-clean.

Any strongly $J$-clean ring is strongly $\delta_r$-clean. But the converse need not be true, for example any commutative semisimple ring which is not a Boolean ring is such a ring.

Note that in the following theorem it is proved that the notions of strongly $\delta_r$-clean rings and weakly $\delta$-quasipolar rings coincide.

Theorem 3. Let $R$ be a ring. Then $R$ is a weakly $\delta$-quasipolar ring if and only if it is strongly $\delta_r$-clean.

Proof. Let $R$ be a weakly $\delta$-quasipolar ring and $a \in R$. There exist $p^2 = p \in \text{comm}(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$ and $a(1 - p) = (1 - p)a$. Hence $R$ is a strongly $\delta_r$-clean ring. Conversely, assume that $R$ is a strongly
δ_r-clean ring. Let a ∈ R. Since −a ∈ R, by assumption there exists an idempotent p ∈ R such that −a − p ∈ δ(R) and (−a)p = p(−a). So R is a weakly δ-quasipolar ring.

Theorem 3 states that the weakly δ-quasipolarity of a ring is equivalent to the strongly δ_r-cleanness of this ring. The following example reveals that a weakly δ-quasipolar element is different from a strongly δ_r-clean element.

Example 10. Let R = Z and a = 1 ∈ R. There exists no idempotent p such that a + p ∈ δ(R). Then a is not weakly δ-quasipolar. Let p = 1 ∈ R. Since a − p ∈ δ(R), a is strongly δ_r-clean. On the other hand, if a = −1 ∈ R, then there exists no idempotent p such that a − p ∈ δ(R). Then a is not strongly δ_r-clean. Let p = 1 ∈ R. Since a + p ∈ δ(R), a is weakly δ-quasipolar.

Theorem 4. Let R be a local ring with non-zero maximal ideal. Then the following are equivalent.

1. R is weakly δ-quasipolar;
2. R is strongly J-clean;
3. R is uniquely clean;
4. \( R/J(R) \cong \mathbb{Z}_2 \);
5. \( R/\delta(R) \cong \mathbb{Z}_2 \).

Proof. Let R be a local ring with non-zero maximal ideal.

(1) \( \Leftrightarrow \) (2) Assume that R is weakly δ-quasipolar. Let a ∈ R. There exists \( p^2 = p \in \text{comm}(−1 + a) \) such that \( −1 + a + p ∈ δ(R) \). Then \( a − (1 − p) ∈ δ(R) \). Since \( p \in \text{comm}(−1 + a) \), \( pa = ap \). Hence R is strongly J-clean by \( J(R) = δ(R) \). Similarly, the rest is clear.

(2) \( \Leftrightarrow \) (3) follows from [3, Lemma 4.2].

(3) \( \Leftrightarrow \) (4) follows from [14, Theorem 15].

(1) \( \Rightarrow \) (5) Let R be weakly δ-quasipolar and \( \overline{0} \neq \overline{a} = a + δ(R) ∈ R/δ(R) \), we show that \( \overline{a} = \overline{1} \). Then there exists an idempotent \( p ∈ R \) such that \( −a + p ∈ δ(R) \) and \( p^2 = p ∈ \text{comm}(−a) \). Since R is a local, \( p = 0 \) or \( p = 1 \). If \( p = 0 \), this contradicts \( \overline{0} \neq \overline{a} \). Therefore \( p = 1 \). It follows that \( \overline{a} = \overline{1} \).

(5) \( \Rightarrow \) (1) It follows from Proposition 2.

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References

[1] M. B. Calci, S. Halicioglu, and A. Harmanci, “A Class of J-Quasipolar Rings,” J. Algebra Relat. Topics, vol. 3, no. 2, pp. 1–15, 2015.

[2] M. B. Calci, B. Ungor, and A. Harmanci, “Central Quasipolar Rings,” Rev. Colombiana Mat, vol. 49, no. 2, pp. 281–292, 2015.
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[3] H. Chen, “On strongly J-clean rings.” Comm. Algebra, vol. 38, no. 10, pp. 3790–3804, 2010, doi: 10.1080/00927870903286835.
[4] J. Cui and J. Chen, “A class of quasipolar rings.” Comm. Algebra, vol. 40, no. 12, pp. 4471–4482, 2012, doi: 10.1080/00927872.2011.610854.
[5] J. Cui and J. Chen, “Pseudopolar matrix rings over local rings.” J. Algebra Appl., vol. 13, no. 3, pp. 1350 109, 12, 2014, doi: 10.1142/S0219498813501090.
[6] O. Gurgun, S. Halicioglu, and A. Harmanci, “Quasipolar Subrings of 3 × 3 Matrix Rings.” An. St. Univ. Ovidius Constanta, vol. 21, no. 3, pp. 133–146, 2013, doi: 10.2478/auom-2013-0048.
[7] O. Gurgun, S. Halicioglu, and A. Harmanci, “Nil-quasipolar rings.” Bol. Soc. Mat. Mex., vol. 20, no. 1, pp. 29–38, 2014, doi: 10.1007/s40590-014-0005-y.
[8] O. Gurgun and A. C. Ozcan, “A class of uniquely (strongly) clean rings.” Turk. J. Math., vol. 38, pp. 40–51, 2014, doi: 10.3906/mat-1209-9.
[9] R. E. Harte, “On quasinilpotents in rings.” Panamer. Math. J., vol. 1, pp. 10–16, 1991.
[10] J. J. Koliha and P. Patricio, “Elements of rings with equal spectral idempotents.” J. Aust. Math. Soc., vol. 72, no. 1, pp. 137–152, 2002.
[11] T. Y. Lam, First course in noncommutative rings. New York: Springer, 2001.
[12] W. K. Nicholson, “Lifting idempotents and exchange rings.” Trans. Amer. Math. Soc., vol. 229, pp. 269–278, 1977.
[13] W. K. Nicholson, “Strongly clean rings and fitting’s lemma.” Comm. Algebra, vol. 27, no. 8, pp. 3583–3592, 1999, doi: 10.1080/00927879908826649.
[14] W. K. Nicholson and Y. Zhou, “Rings in which elements are uniquely the sum of an idempotent and a unit.” Glasg. Math. J., vol. 46, no. 2, pp. 227–236, 2004, doi: 10.1017/S0017089504001727.
[15] Z. Ying and J. Chen, “On quasipolar rings.” Algebra Colloq., vol. 19, no. 4, pp. 683–692, 2012, doi: 10.1142/S1005386712000557.
[16] Y. Zhou, “Generalizations of perfect, semiperfect and semiregular rings.” Algebra Colloq., vol. 7, no. 3, pp. 305–318, 2000, doi: 10.1007/s10011-000-0305-9.

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