COUPLED SYSTEMS OF HILFER FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

SAÏD Abbas
Laboratory of Mathematics, Geometry, Analysis
Control and Applications, Tahar Moulay University of Saida
P.O. Box 138, EN-Nasr, 20000 Saida, Algeria

MOUFFAK Benchahra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, 22000, Algeria

JOHN R. GRAEF*
Department of Mathematics, University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA

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Abstract. This paper deals with some existence results in Banach spaces for Hilfer and Hilfer-Hadamard fractional differential inclusions. The main tools used in the proofs are Mönch’s fixed point theorem and the concept of a measure of noncompactness.

1. Introduction. Fractional differential equations and inclusions appear in several areas such as engineering, mathematics, bio-engineering, physics, and other applied sciences [19, 32]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas et al. [4, 5], Kilbas et al. [23], Samko et al. [31], Zhou [35], as well as the papers by Abbas et al. [1, 2], Benchahra et al. [10], Lakshmikantham et al. [24, 25, 26] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [14, 15, 19, 21, 33, 34] and the references therein.

Recently, in [3, 8, 9, 11, 12, 16, 17] the authors applied the measure of noncompactness to some classes of Riemann-Liouville or Caputo fractional differential equations in Banach spaces. In this paper we discuss the existence of solutions to the coupled system of Hilfer fractional differential inclusions

\[
\begin{align*}
(D_0^{\alpha_1,\beta_1} u)(t) & \in F_1(t,u(t),v(t)), \\
(D_0^{\alpha_2,\beta_2} v)(t) & \in F_2(t,u(t),v(t)),
\end{align*}
\]

\( t \in I := [0,T], \) (1)

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* Corresponding author.
with the initial conditions
\[
\begin{align*}
(I_0^{1-\gamma_1} u)(0) &= \phi_1, \\
(I_0^{1-\gamma_2} v)(0) &= \phi_2,
\end{align*}
\] (2)
where \( T > 0, \alpha_i \in (0, 1), \beta_i \in [0, 1], \gamma_i = \alpha_i + \beta_i - \alpha_i \beta_i, \) \( E \) is a real (or complex) separable Banach space with a norm \( \| \cdot \| \), \( P(E) \) is the family of all nonempty subsets of \( E \), \( \phi_i \in E, F_i : I \times E \times E \to P(E), i = 1, 2, \) are given multivalued maps, \( I_0^{1-\gamma_i} \) is the left-sided mixed Riemann-Liouville integral of order \( 1 - \gamma_i \) with the initial conditions
\[\{(I_0^{1-\gamma_i} u)(0) = \phi_i, \} \] where \( T > E \) if
\[
\{H D_1^{\alpha_i, \beta_i} u(t) \in G_1(t, u(t), v(t)), \}
\]
\[
\{H D_1^{\alpha_i, \beta_i} v(t) \in G_2(t, u(t), v(t)), \}
\]
t \( \in [1, T], \) (3)
with the initial conditions
\[
\begin{align*}
(H I_1^{1-\gamma_i} u)(1) &= \psi_1, \\
(H I_1^{1-\gamma_i} v)(1) &= \psi_2,
\end{align*}
\] (4)
where \( T > 1, \alpha_i \in (0, 1), \beta_i \in [0, 1], \gamma_i = \alpha_i + \beta_i - \alpha_i \beta_i, \psi_i \in E, G_i : [1, T] \times E \times E \to P(E), i = 1, 2, \) are given multivalued maps, \( H I_1^{1-\gamma_i} \) is the left-hand mixed Hadamard integral of order \( 1 - \gamma_i \), and \( H D_1^{\alpha_i, \beta_i} \) is the Hilfer-Hadamard fractional derivative of order \( \alpha_i \) and type \( \beta_i \).

2. Preliminaries. Let \( C := C(I) \) be the Banach space of all continuous functions \( w \) from \( I \) into \( E \) with the supremum (uniform) norm
\[
\|w\|_\infty := \sup_{t \in I} \|w(t)\|.
\]
As usual, \( AC(I) \) denotes the space of absolutely continuous functions from \( I \) into \( E \). We denote by \( AC^1(I) \) the space defined by
\[
AC^1(I) := \{w : I \to E : \frac{d}{dt}w(t) \in AC(I)\}.
\]
By \( L^1(I) \), we denote the space of measurable functions \( v : I \to E \) that are Bochner integrable and normed by
\[
\|v\|_1 = \int_0^T \|v(t)\| dt.
\]
Let \( L^\infty(I) \) be the Banach space of measurable functions \( v : I \to \mathbb{R} \) that are essentially bounded and equipped with the norm
\[
\|v\|_{L^\infty} = \inf\{c > 0 : |v(t)| \leq c, \text{ a.e. } t \in I\}.
\]
By \( C_\gamma(I) \) and \( C_\gamma^1(I) \), we denote the weighted spaces of continuous functions defined by
\[
C_\gamma(I) = \{w : (0, T] \to E : t^{1-\gamma}w(t) \in C\},
\]
with the norm
\[
\|w\|_{C_\gamma} := \sup_{t \in I} \|t^{1-\gamma}w(t)\|,
\]
and
\[
C_\gamma^1(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},
\]
with the norm
\[ \|w\|_{C^2} := \|w\|_\infty + \|w'\|_{C}\. \]
Also, by \( C := C_{r1} \times C_{r2} \) we denote the product weighted space with the norm
\[ \|(u,v)\|_{C} = \|u\|_{C_{r1}} + \|v\|_{C_{r2}}. \]

Let \( P_{cl}(E) = \{ A \in \mathcal{P}(E) : A \text{ is closed} \} \), \( P_{xx}(E) = \{ A \in \mathcal{P}(E) : A \text{ is convex} \} \), and \( P_{cp,xx}(E) = \{ A \in \mathcal{P}(E) : A \text{ is compact and convex} \} \). If there exists \( x \in E \) such that \( x \in G(x) \) then the multivalued map \( G : E \to \mathcal{P}(E) \) has a fixed point. The symbol \( FixG \) stands for the set of fixed point of \( G \). If for every \( y \in E \), the function \( t \mapsto d(y,G(t)) = \inf \{|y-z| : z \in G(t)\} \), then a multivalued map \( G : J \to P_{cl}(E) \) is said to be measurable.

**Definition 2.1.** Let \( X \) and \( Y \) be two sets. The graph of a set-valued map \( N : X \to \mathcal{P}(Y) \) is defined by
\[
\text{graph}(N) = \{(x,y) : x \in X, y \in N(x)\}.
\]

For more details about multivalued functions, see for instance [6, 7, 13, 20].

**Definition 2.2.** If
1. \( t \to F(t,u) \) is measurable for each \( u \in E \), and
2. \( u \to F(t,u) \) is upper semicontinuous for a.e. \( t \in I \),
then the multivalued function \( F : I \times E \to \mathcal{P}(E) \) is Carathéodory.

For each \( u \in C(I) \), we defined the set of selections of \( F \) by
\[
S_{Fou} = \{ v \in L^1(I) : v(t) \in F(t,u(t)) \text{ a.e. } t \in I \}.
\]

We next give some results and properties of the fractional calculus.

**Definition 2.3 ([4, 23, 31]).** The left-sided mixed Riemann-Liouville integral of order \( r > 0 \) of a function \( w \in L^1(I) \) is defined by
\[
(I_0^\alpha w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds \text{ for a.e. } t \in I,
\]
where \( \Gamma(\cdot) \) is the (Euler's) Gamma function defined by
\[
\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t}dt, \quad \xi > 0.
\]

Notice that for all \( r, r_1, r_2 > 0 \) and each \( w \in C \), we have \( I_0^r w \in C \), and
\[
(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t) \text{ for a.e. } t \in I.
\]

**Definition 2.4 ([4, 23, 31]).** The Riemann-Liouville fractional derivative of order \( r \in (0,1] \) of a function \( w \in L^1(I) \) is defined by
\[
(D_0^\alpha w)(t) = \left( \frac{d}{dt} I_0^{1-r} w \right)(t) = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds \text{ for a.e. } t \in I.
\]

Let \( r \in (0,1], \gamma \in [0,1) \), and \( w \in C_{1-\gamma}(I) \). Then the next expression leads to the left inverse operator as follows:
\[
(D_0^\alpha I_0^\gamma w)(t) = w(t) \text{ for all } t \in (0,T].
\]
Moreover, if \( I_0^{1-r} w \in C_{1-\gamma}(I) \), then the following composition is proved in [31]:
\[
(I_0^r D_0^\alpha w)(t) = w(t) - \frac{I_0^{1-r} w(0^+)}{\Gamma(r)} t^{r-1} \text{ for all } t \in (0,T].
\]
Definition 2.5 ([4, 23, 31]). If \( w \in L^1(I) \),

\[
(\mathcal{D}^\alpha w)(t) = \left( \mathcal{I}^{1-\gamma}(\mathcal{I}^{\alpha-1}w) \right)(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s) ds \quad \text{for a.e. } t \in I.
\]

is the Caputo fractional derivative of the function \( w \) of order \( r \in (0, 1] \).

In [19], Hilfer studied applications of a generalized fractional operator with the Riemann-Liouville and the Caputo derivatives as specific cases (see also [21, 33]).

Definition 2.6 (Hilfer derivative). Let \( \alpha \in (0, 1) \), \( \beta \in [0, 1] \), \( w \in L^1(I) \), and \( \mathcal{I}_0^{(1-\alpha)(1-\beta)} \in AC^1(I) \). The Hilfer fractional derivative of order \( \alpha \) and type \( \beta \) of \( w \) is defined as

\[
(\mathcal{D}^\alpha_0 w)(t) = \left( \mathcal{I}^{\beta(1-\alpha)} \mathcal{I}^\alpha \mathcal{I}^{\gamma - 1} w \right)(t) \quad \text{for a.e. } t \in I.
\]

Properties. Let \( \alpha \in (0, 1) \), \( \beta \in [0, 1] \), \( \gamma = \alpha + \beta - \alpha \beta \), and \( w \in L^1(I) \).

1. The operator \( (D^\alpha_0 w)(t) \) can be written as

\[
(D^\alpha_0 w)(t) = \left( \mathcal{I}^{\beta(1-\alpha)} \frac{d}{dt} \mathcal{I}^\alpha \mathcal{I}^{\gamma - 1} w \right)(t) \quad \text{for a.e. } t \in I.
\]

Moreover, the parameter \( \gamma \) satisfies

\[
\gamma \in (0, 1], \quad \gamma \geq \alpha, \quad \gamma > \beta, \quad 1 - \gamma < 1 - \beta(1 - \alpha).
\]

2. The special case of (5) with \( \beta = 0 \) coincides with the Riemann-Liouville derivative, and with \( \beta = 1 \), it coincides with the Caputo derivative. In addition,

\[
D^\alpha_0 = D^\alpha_0 \quad \text{and} \quad D^{\alpha,1}_0 = \mathcal{D}^\alpha_0.
\]

3. If \( D_0^{\beta(1-\alpha)} w \) exists and is in \( L^1(I) \), then

\[
(D_0^{\beta} \mathcal{I}_0^{\beta(1-\alpha)} w)(t) = (\mathcal{I}_0^{\alpha} D_0^{\beta(1-\alpha)} w)(t) \quad \text{for a.e. } t \in I.
\]

Furthermore, if \( w \in C_\gamma(I) \) and \( \mathcal{I}_0^{\beta(1-\alpha)} w \in C_\gamma(I) \), then

\[
(D_0^{\alpha,\beta} \mathcal{I}_0^{\beta(1-\alpha)} w)(t) = w(t) \quad \text{for a.e. } t \in I.
\]

4. If \( D_0^\gamma w \) exists and is in \( L^1(I) \), then

\[
(\mathcal{I}_0^\gamma D_0^{\alpha,\beta} w)(t) = (\mathcal{I}_0^\gamma D_0^\gamma w)(t) = w(t) - \mathcal{I}_0^{1-\gamma(0+)} \frac{t^{\gamma-1}}{\Gamma(\gamma)} \quad \text{for a.e. } t \in I.
\]

Corollary 1. Let \( h \in C_\gamma(I) \). Then the linear Cauchy problem

\[
\begin{cases}
(D_0^{\alpha,\beta} u)(t) = h(t); \quad t \in I, \\
(\mathcal{I}_0^{1-\gamma} u)(t)|_{t=0} = \phi,
\end{cases}
\]

has a unique solution given by

\[
u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (\mathcal{I}_0^\gamma h)(t).
\]

From the above corollary, we have the following lemma.
Lemma 2.7. Consider the maps $F_i : I \times E \times E \to \mathcal{P}(E)$, $i = 1, 2$, such that $S_{F_i u} \subset C_{\gamma_1}$ for any $u \in C_{\gamma_1}$ and $S_{F_2 v} \subset C_{\gamma_2}$ for any $v \in C_{\gamma_2}$. Then solving the system (1)–(2) is equivalent to the finding the solutions of the system of integral equations
\[
\begin{align*}
u(t) &= \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + (I_0^{\alpha_1} w_1)(t), \\
u(t) &= \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + (I_0^{\alpha_2} w_2)(t),
\end{align*}
\]
where $w_1 \in S_{F_1 u}$ and $w_2 \in S_{F_2 v}$.

The symbol $\mathcal{M}_X$ will stand for the class of all bounded subsets of a metric space $X$.

Definition 2.8. Let $X$ be a complete metric space. A function $\mu : \mathcal{M}_X \to [0, \infty)$ is said to be a measure of noncompactness on $X$ if the following conditions are satisfied for all $B, B_1, B_2 \in \mathcal{M}_X$:
(a) Regularity, i.e., $\mu(B) = 0$ if and only if $B$ is precompact;
(b) Invariance under closure, i.e., $\mu(B) = \mu(\overline{B})$;
(c) Semi-additivity, i.e., $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.

Example 1 ([8], Example 1, p. 19). Let $X$ be a metric space. The map $\phi : \mathcal{M}_X \to [0, \infty)$ is a discrete measure of noncompactness if
\[
\phi(B) = \begin{cases} 0 & \text{if } B \text{ is relatively compact}, \\ 1 & \text{otherwise}. \end{cases}
\]

Definition 2.9 ([9]). Let $E$ be a Banach space and let $\Omega_E$ denote the family of bounded subsets of $E$. If
\[
\mu(M) = \inf\{\epsilon > 0 : M \subset \bigcup_{j=1}^n M_j, \text{diam}(M_j) \leq \epsilon\}, \quad M \in \Omega_E,
\]
then the map $\mu : \Omega_E \to [0, \infty)$ is called the Kuratowski measure of noncompactness.

Properties.
(1) $\mu(M) = 0$ if and only if $\overline{M}$ is compact ($M$ is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2$ implies $\mu(M_1) \leq \mu(M_2)$.
(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
(6) $\mu(\text{conv } M) = \mu(M)$.

Theorem 2.10 ([18]). Let $E$ be a Banach space. Let $C \subset L^1(I)$ be countable set with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(I, \mathbb{R}^+)$, Then $\phi(t) = \mu(C(t)) \in L^1(I, \mathbb{R}^+)$ and
\[
\alpha\left(\left\{\int_0^T u(s) \, ds : u \in C\right\}\right) \leq 2 \int_0^T \mu(C(s)) \, ds.
\]

Lemma 2.11 ([27]). Let $I$ be a compact real interval, let $F$ be a Carathéodory multivalued map, and let $\Theta$ be a continuous linear map from $L^1(I) \to C(I)$. Then the operator
\[
\Theta \circ S_{Fou} : C(I) \to \mathcal{P}_{cp,c}(C(I)), \quad u \mapsto (\Theta \circ S_{Fou})(u) = \Theta(S_{Fou})
\]
is a closed graph operator in $C(I) \times C(I)$.

We now recall the set-valued version of Mönch’s fixed point theorem.
Theorem 2.12 ([28]). Let $E$ be Banach space, $K \subset E$ be a closed and convex set, $U$ be a relatively open subset of $K$, and $N : \overline{U} \to \mathcal{P}(K)$. Assume that $N$ maps compact sets into relatively compact sets, $\text{graph}(N)$ is closed, and for some $x_0 \in U$, we have:

(i) $M \subset \overline{U}$, $M \subset \text{conv}(x_0 \cup N(M))$, and $\overline{M} = \overline{U}$ with $C$ a countable subset of $M$, implies $\overline{M}$ is compact;

(ii) $x \notin (1 - \lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U} \setminus U$ and $\lambda \in (0, 1)$.

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

3. Coupled system of Hilfer fractional differential inclusions. First, we define what we mean by a solution of the system (1)--(2).

Definition 3.1. By a solution of the system (1)--(2) we mean a pair of measurable functions $(u, v) \in \mathcal{C}$ that satisfy conditions (2) and the inclusions (1) on $I$.

In the sequel, we will need the following conditions.

(H$_1$) The multivalued maps $F_i : I \times E \times E \to \mathcal{P}_{sp,c}(E)$, $i = 1, 2$, are Carathéodory.

(H$_2$) There exist functions $p_i \in L^\infty(I, [0, \infty))$, $i = 1, 2$, such that

$$\|F_1(t, u, v)\|_P = \sup \{\|w_1\|_{C_{\gamma_1}} : w_1(t) \in F_1(t, u, v)\} \leq p_1(t)$$

and

$$\|F_2(t, u, v)\|_P = \sup \{\|w_2\|_{C_{\gamma_2}} : w_2(t) \in F_2(t, u, v)\} \leq p_2(t)$$

for a.e. $t \in I$ and $u, v \in E$.

(H$_3$) For each bounded and measurable set $B_i \subset C_{\gamma_i}$, $i = 1, 2$, and for each $t \in I$, we have

$$\mu(F_i(t, B_1(t), B_2(t))) \leq p_i(t)\mu(B_i(t)), \quad i = 1, 2,$$

where $B_i(t) = \{u(t) : u \in B_i\}$, $i = 1, 2$.

(H$_4$) The function $\Phi = (\phi_1, \phi_2) \equiv (0, 0)$ is the unique solution in $\mathcal{C}$ of the inequalities

$$\Phi_i(t) \leq 2p_i^*(I_{0}^{\alpha_i} \phi_i)(t),$$

where

$$p_i^* = \text{ess} \sup_{t \in I} p_i(t), \quad i = 1, 2.$$

We now prove our main result in this section on the existence of solutions to the system (1)--(2).

Theorem 3.2. Assume that (H$_1$)--(H$_4$) hold. Then the system (1)--(2) has at least one solution defined on $I$.

Proof. Define the multivalued operators $N_1 : C_{\gamma_1} \to \mathcal{P}(C_{\gamma_1})$ and $N_2 : C_{\gamma_2} \to \mathcal{P}(C_{\gamma_2})$ by

$$N_1(u) = \left\{ h_1 \in C_{\gamma_1} : h_1(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + \int_{0}^{t} (t - s)^{\alpha_1 - 1} \frac{w_1(s)}{\Gamma(\alpha_1)} ds, \quad w_1 \in S_{F_1, ou} \right\}$$

and

$$N_2(v) = \left\{ h_2 \in C_{\gamma_2} : h_2(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + \int_{0}^{t} (t - s)^{\alpha_2 - 1} \frac{w_2(s)}{\Gamma(\alpha_2)} ds, \quad w_2 \in S_{F_2, ov} \right\}.$$
Clearly, the fixed points of $N$ are solutions of the system (1)–(2). We shall show that the multivalued operator $N$ satisfies all the assumptions of Theorem 2.12. The proof will be given in several steps.

**Step 1.** $N(u,v)$ is convex for each $(u,v) \in C$.

If $(h_1,k_1), (h_2,k_2) \in N(u,v)$, then there exist $w_1, w_2 \in S_{F_{ou}}$ and $z_1, z_2 \in S_{F_{ov}}$ such that for each $t \in I$, we have

$$h_i(t) = \frac{\phi_{t_i}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-1} \frac{w_i(s)}{\Gamma(\alpha_i)} ds, \quad i = 1, 2,$$

and

$$k_i(t) = \frac{\phi_{t_i}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-2} \frac{z_i(s)}{\Gamma(\alpha_i)} ds, \quad i = 1, 2.$$

Let $0 \leq \lambda \leq 1$; then, for each $t \in I$,

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \frac{\phi_{t}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-1} \frac{\lambda w_1(s) + (1-\lambda)w_2(s)}{\Gamma(\alpha_i)} ds.$$

Since $S_{F_{ou}}$ is convex (because $F_1$ has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N_1(u)$. Also, for each $t \in I$, we have

$$(\lambda k_1 + (1-\lambda)k_2)(t) = \frac{\phi_{t_i}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-2} \frac{\lambda z_1(s) + (1-\lambda)z_2(s)}{\Gamma(\alpha_i)} ds.$$

Since $S_{F_{ov}}$ is convex (because $F_2$ has convex values), we have $\lambda k_1 + (1-\lambda)k_2 \in N_2(v)$. Hence, $\lambda(h_1,k_1) + (1-\lambda)(h_2,k_2) \in N(u,v)$.

**Step 2.** For each compact $M \subseteq C$, $N(M)$ is relatively compact.

Let $(h_n,k_n)$ be any sequence in $N(M)$ with $M \subseteq C$ and $M$ compact. To apply the Arzelà-Ascoli compactness criterion on $C$, we will show that $(h_n,k_n)$ has a convergent subsequence. Since $(h_n,k_n) \in N(M)$ there exist $(u_n,v_n) \in M$, $w_n \in S_{F_{ou}}$, and $z_n \in S_{F_{ov}}$ such that

$$h_n(t) = \frac{\phi_{t_i}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-1} \frac{w_n(s)}{\Gamma(\alpha_i)} ds$$

and

$$k_n(t) = \frac{\phi_{t_i}}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-2} \frac{z_n(s)}{\Gamma(\alpha_i)} ds.$$

Using Theorem 2.10 and the properties of the Kuratowski measure of noncompactness, we have

$$\mu(\{h_n(t)\}) \leq \frac{2}{\Gamma(\alpha_i)} \int_0^t \mu(\{(t-s)^{\alpha_i-1}w_n(s)\}) ds \quad (6)$$

and

$$\mu(\{k_n(t)\}) \leq \frac{2}{\Gamma(\alpha_i)} \int_0^t \mu(\{(t-s)^{\alpha_i-2}z_n(s)\}) ds. \quad (7)$$

On the other hand, since $M$ is compact, the sets $\{w_n(s) : n \geq 1\}$ and $\{z_n(s) : n \geq 1\}$ are compact. Consequently, $(\mu(\{w_n(s) : n \geq 1\}), \mu(\{w_n(s) : n \geq 1\})) = (0,0)$ for a.e. $s \in I$. Furthermore,

$$\mu(\{(t-s)^{\alpha_i-1}w_n(s)\}) = (t-s)^{\alpha_i-1} \mu(\{w_n(s) : n \geq 1\}) = 0$$

and

$$\mu(\{(t-s)^{\alpha_i-2}z_n(s)\}) = (t-s)^{\alpha_i-2} \mu(\{z_n(s) : n \geq 1\}) = 0$$

for a.e. $t, s \in I$. Now from (6) and (7) we obtain that $\{h_n(t) : n \geq 1\}$ and $\{k_n(t) : n \geq 1\}$ are relatively compact for each $t \in I$. 
For each $t_1, t_2 \in I$ with $t_1 < t_2$, we have
\[
\|t_2^{-\gamma}h_n(t_2) - t_1^{-\gamma}h_n(t_1)\|
\leq \|t_2^{-\gamma}\int_0^{t_2} (t_2-s)^{\alpha_1-1}w_n(s)\frac{d s}{\Gamma(\alpha_1)} - t_1^{-\gamma}\int_0^{t_1} (t_1-s)^{\alpha_1-1}w_n(s)\frac{d s}{\Gamma(\alpha_1)}\|
\leq T^{1-\gamma}\int_{t_1}^{t_2} (t_2-s)^{\alpha_1-1}\frac{p_1^*_s}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\]
\[
+ \int_0^{t_1} |t_2^{-\gamma}(t_2-s)^{\alpha_1-1} - t_1^{-\gamma}(t_1-s)^{\alpha_1-1}|\frac{p_1(s)}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\leq \frac{p_1^*T^{1-\gamma}}{\Gamma(\alpha_1)}\int_{t_1}^{t_2} (t_2-s)^{\alpha_1-1}ds
\]
\[
+ \int_0^{t_1} |t_2^{-\gamma}(t_2-s)^{\alpha_1-1} - t_1^{-\gamma}(t_1-s)^{\alpha_1-1}|\frac{p_1(s)}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\leq \frac{p_1^*T^{1-\gamma}}{\Gamma(\alpha_1)}(t_2 - t_1)^{\alpha_1}
\]
\[
+ \int_0^{t_1} |t_2^{-\gamma}(t_2-s)^{\alpha_1-1} - t_1^{-\gamma}(t_1-s)^{\alpha_1-1}|\frac{p_1(s)}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\tag{8}
\]
Similarly,
\[
\|t_2^{-\gamma_2}k_n(t_2) - t_1^{-\gamma_2}k_n(t_1)\|
\leq \frac{p_2^*T^{1-\gamma_2}}{\Gamma(1+\alpha_2)}(t_2 - t_1)^{\alpha_2}
\]
\[
+ \frac{p_2^*}{\Gamma(\alpha_2)}\int_0^{t_1} |t_2^{-\gamma_2}(t_2-s)^{\alpha_2-1} - t_1^{-\gamma_2}(t_1-s)^{\alpha_2-1}|ds.
\tag{9}
\]
As $t_1 \to t_2$, the right-hand sides of the inequalities (8) and (9) tend to zero. This shows that $\{(h_n, k_n): n \geq 1\}$ is equicontinuous. Consequently, $\{(h_n, k_n): n \geq 1\}$ is relatively compact in $C$.

**Step 3.** The graph of $N$ is closed.
Let $((u_n, v_n), (h_n, k_n)) \in \text{graph}(N)$, $n \geq 1$, with $\|(u_n, v_n) - (u, v)\|_C$, $\|(h_n, k_n) - (h, k)\|_C \to 0$ as $n \to \infty$. We must show that $((u, v), (h, k)) \in \text{graph}(N)$. Now $((u_n, v_n), (h_n, k_n)) \in \text{graph}(N)$ means that $(h_n, k_n) \in N(u_n, v_n)$, which in turn implies there exists $w_n \in S_{F_1\circ u_n}$ and $z_n \in S_{F_2\circ v_n}$ such that for each $t \in I$,
\[
h_n(t) = \frac{\phi_1}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{w_n(s)}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\]
and
\[
k_n(t) = \frac{\phi_2}{\Gamma(\gamma_2)}t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1}\frac{z_n(s)}{\Gamma(\alpha_2)}\frac{d s}{\Gamma(\alpha_2)}.
\]
Consider the continuous linear operators $\Theta_i : L^1(I) \to C_\gamma$, $i = 1, 2$, defined by
\[
\Theta_1(w)(t) \mapsto h_n(t) = \frac{\phi_1}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{w_n(s)}{\Gamma(\alpha_1)}\frac{d s}{\Gamma(\alpha_1)}
\]
and
\[
\Theta_2(z)(t) \mapsto k_n(t) = \frac{\phi_2}{\Gamma(\gamma_2)}t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1}\frac{z_n(s)}{\Gamma(\alpha_2)}\frac{d s}{\Gamma(\alpha_2)}.
\]
Clearly, $||(h_n(t), k_n(t)) - (h(t), k(t))||_C \to 0$ as $n \to \infty$. From Lemma 2.11 it follows that $\Theta_i \circ S_{F_i}, i = 1, 2$, are closed graph operators. Moreover, $h_n(t) \in \Theta_1(S_{F_1\circ u_n})$, $k_n(t) \in \Theta_2(S_{F_2\circ v_n})$. 
and \(k_n(t) \in \Theta_2(S_{F_{2\cup n}})\). Since \((u_n, v_n) \to (u, v)\), Lemma 2.11 implies that

\[
h(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{w(s)}{\Gamma(\alpha_1)} ds
\]

for some \(w \in S_{F_1\cup u}\), and

\[
k(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1} \frac{z(s)}{\Gamma(\alpha_2)} ds
\]

for some \(z \in S_{F_2\cup v}\).

**Step 4.** \(M = M_1 \times M_2\) is relatively compact in \(C\).

Let \(M \subseteq \overline{U}\), where \(M \subseteq \text{conv}\{0\} \cup N(M)\), and for some countable set \(C \subseteq M\), let \(\overline{M} = C\). In view of (8), it is easy to see that \(N(M)\) is equicontinuous. Therefore, \(M \subseteq \text{conv}\{0\} \cup N(M)\) implies \(M\) is equicontinuous. It remains to apply the Arzelà-Ascoli theorem to show that for each \(t \in I\) the set \(M(t)\) is relatively compact. Taking into account that \(C\) is countable and \(C \subseteq \text{conv}\{0\} \cup \{0\}\), we can find a countable set \(H = \{(h_n, k_n) : n \geq 1\} \subseteq N(M)\) such that \(C \subseteq \text{conv}\{0\} \cup H\).

Then, there exist \((u_n, v_n) \in M\) and \((w_n, z_n) \in S_{F_{1\cup u}} \times S_{F_{2\cup v}}\) with

\[
h_n(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} w_n(s) \frac{1}{\Gamma(\alpha_1)} ds
\]

and

\[
k_n(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1} z_n(s) \frac{1}{\Gamma(\alpha_2)} ds.
\]

Taking into account Theorem 2.10 and the fact that \(M \subseteq \overline{C} \subseteq \text{conv}\{0\} \cup H\), we obtain

\[
\mu(M(t)) \leq \mu(\overline{C}(t)) \leq \mu(H(t)) = \mu(\{(h_n(t), k_n(t)) : n \geq 1\}).
\]

Using (6), we obtain

\[
\mu(t^{1-\gamma_1} M_1(t)) \leq \frac{2}{\Gamma(\alpha_1)} \int_0^t \mu(t^{1-\gamma_1}(t-s)^{\alpha_1-1} w_n(s)) ds
\]

and

\[
\mu(t^{1-\gamma_2} M_2(t)) \leq \frac{2}{\Gamma(\alpha_2)} \int_0^t \mu(t^{1-\gamma_2}(t-s)^{\alpha_2-1} z_n(s)) ds.
\]

Now, since \((w_n, z_n) \in S_{F_{1\cup u}} \times S_{F_{2\cup v}}\) and \((u_n(s), v_n(s)) \in M(s) := M_1(s) \times M_2(s)\), we have

\[
\mu(t^{1-\gamma_1} M_1(t)) \leq \frac{2}{\Gamma(\alpha_1)} \int_0^t \mu(t^{1-\gamma_1}(t-s)^{\alpha_1-1} w_n(s) : n \geq 1) ds
\]

and

\[
\mu(t^{1-\gamma_2} M_2(t)) \leq \frac{2}{\Gamma(\alpha_2)} \int_0^t \mu(t^{1-\gamma_2}(t-s)^{\alpha_2-1} z_n(s) : n \geq 1) ds.
\]

Also, since \((w_n, z_n) \in S_{F_{1\cup u}} \times S_{F_{2\cup v}}\) and \((u_n(s), v_n(s)) \in M(s)\), from \(H_3\) we have

\[
\mu(t^{1-\gamma_1}(t-s)^{\alpha_1-1} w_n(s) : n \geq 1) = t^{1-\gamma_1}(t-s)^{\alpha_1-1} p_1(s) \mu(M_1(s))
\]

and

\[
\mu(t^{1-\gamma_2}(t-s)^{\alpha_2-1} z_n(s) : n \geq 1) = t^{1-\gamma_2}(t-s)^{\alpha_2-1} p_2(s) \mu(M_2(s)).
\]
It follows that
\[ \mu(M(t)) \leq \left( \frac{2p_1^1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} + \frac{2p_2^2}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \right) \mu(M(s))ds. \]

Consequently by (H4), the function $\Phi$ given by $\Phi(t) = \mu(M(t))$ satisfies $\Phi \equiv (0,0)$; that is, $\mu(M(t)) = 0$ for all $t \in I$. Now, by the Arzelà-Ascoli theorem, $M$ is relatively compact in $C$.

**Step 5. A priori estimate.**

Let $(u,v) \in C$ be such that $(u,v) \in \lambda N(u,v)$ for some $\lambda \in (0,1)$. Then for each $t \in I$, we have
\[
\begin{align*}
    u(t) &= \frac{\lambda \phi_1}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \frac{\lambda}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} w(s)ds \\
    v(t) &= \frac{\lambda \phi_2}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} z(s)ds
\end{align*}
\]
for some $w \in S_{F_{1,ou}}$, and
\[
\begin{align*}
    v(t) &= \frac{\lambda \phi_2}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} z(s)ds
\end{align*}
\]
for some $z \in S_{F_{2,ov}}$. On the other hand,
\[
\begin{align*}
    t^{1-\gamma_1} \|u(t)\| &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + t^{1-\gamma_1} \|w(t)\| ds \\
    &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + T^{1-\gamma_1} \|w(\gamma_1)\| + \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{p_1^1 T^{1-\gamma_1 + \gamma_1}}{\Gamma(1 + \alpha_1)},
\end{align*}
\]
and similarly,
\[
\begin{align*}
    t^{1-\gamma_2} \|v(t)\| &\leq \frac{\|\phi_2\|}{\Gamma(\gamma_2)} + \frac{p_2^2 T^{1-\gamma_2 + \gamma_2}}{\Gamma(1 + \alpha_2)}.
\end{align*}
\]
Thus,
\[
\begin{align*}
    \|(u,v)\| &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{p_1^1 T^{1-\gamma_1 + \alpha_1}}{\Gamma(1 + \alpha_1)} + \frac{\|\phi_2\|}{\Gamma(\gamma_2)} + \frac{p_2^2 T^{1-\gamma_2 + \alpha_2}}{\Gamma(1 + \alpha_2)} := d.
\end{align*}
\]
Set
\[
U = \{(u,v) \in C : \|(u,v)\| < d + 1\}.
\]
Condition (ii) in Theorem 2.12 is satisfied by our choice of the open set $U$. From Steps 1–5 and Theorem 2.12, we conclude that $N$ has at least one fixed point $(u,v) \in C$ which in turn is a solution of the system (1)–(2).

4. **Coupled system of Hilfer-Hadamard fractional differential inclusions.**

In this section, we study the existence of solutions for the system (3)–(4).

Set $C := C([1, T])$. Denote by
\[
C_{\gamma_1, in}([1, T]) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\}
\]
the weighted space of continuous functions equipped with the norm
\[
\|w\|_{C_{\gamma_1, in}} := \sup_{t \in [1, T]} \|w(t)\|.
\]
Let $C' := C_{\gamma_1, in} \times C_{\gamma_2, in}$ be the product weighted space with the norm
\[
\|(u, v)\|_{C'} = \|u\|_{C_{\gamma_1, in}} + \|v\|_{C_{\gamma_2, in}}.
\]

We recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [23] for more detailed analysis.
Definition 4.1 ([23], Hadamard fractional integral). The Hadamard fractional integral of order \( q > 0 \) of the function \( g \in L^1([1,T]) \) is defined as
\[
(\mathcal{H}I^q_1 g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left( \ln \frac{x}{s} \right)^{q-1} g(s) \frac{ds}{s},
\]
provided the integral exists.

Example 2. Let \( 0 < q < 1 \). Then
\[
\mathcal{H}I^q_1 \ln t = \frac{1}{\Gamma(2 + q)}(\ln t)^{1+q}, \text{ for a.e. } t \in [0,e].
\]

Set
\[
\delta = x \frac{d}{dx}, \quad q > 0, \quad n = \lfloor q \rfloor + 1,
\]
where \( \lfloor q \rfloor \) denotes the greatest integer less than or equal to \( q \), and
\[
AC^n_\delta := \{ u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I) \}.
\]

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way.

Definition 4.2 ([23], Hadamard fractional derivative). The Hadamard fractional derivative of order \( q > 0 \) of the function \( w \in AC^n_\delta \) is defined as
\[
(\mathcal{H}D^q_1 w)(x) = \delta^n (\mathcal{H}I^{n-q}_1 w)(x).
\]

In particular, if \( q \in (0,1] \), then
\[
(\mathcal{H}D^q_1 w)(x) = \delta (\mathcal{H}I^{1-q}_1 w)(x).
\]

Example 3. Let \( 0 < q < 1 \). Then
\[
\mathcal{H}D^q_1 \ln t = \frac{1}{\Gamma(2 - q)}(\ln t)^{1-q}, \text{ for a.e. } t \in [0,e].
\]

It is known (see, e.g., Kilbas [22, Theorem 4.8]) that in the space \( L^1(I,E) \), the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,
\[
(\mathcal{H}D^q_1)(\mathcal{H}I^q_1 w)(x) = w(x).
\]

From [23, Theorem 2.3], we have
\[
(\mathcal{H}I^q_1)(\mathcal{H}D^q_1 w)(x) = w(x) - \frac{(\mathcal{H}I^{1-q}_1 w)(1)}{\Gamma(q)}(\ln x)^{q-1}.
\]

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way.

Definition 4.3 (Caputo-Hadamard fractional derivative). The Caputo-Hadamard fractional derivative of order \( q > 0 \) of the function \( w \in AC^n_\delta \) is defined as
\[
(\mathcal{H}_c D^q_1 w)(x) = (\mathcal{H}I^{n-q}_1 \delta^q w)(x).
\]

In particular, if \( q \in (0,1] \), then
\[
(\mathcal{H}_c D^q_1 w)(x) = (\mathcal{H}I^{1-q}_1 \delta w)(x).
\]

Based on the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [29]) is defined in the following way.
Assume that the following conditions hold:

**Theorem 4.6.**

The Hilfer-Hadamard fractional derivative is defined as

\[
(H D_1^{\alpha,\beta} w)(t) = \left( H I_1^{\beta(1-\alpha)} (H D_1^\alpha w) \right)(t) = \left( H I_1^{\beta(1-\alpha)} \delta (H I_1^{\gamma-\gamma} w) \right)(t) \text{ for a.e. } t \in [1, T].
\] (10)

This new fractional derivative (10) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for \( \beta = 0 \), this derivative reduces to the Hadamard fractional derivative, and if \( \beta = 1 \), we recover the Caputo-Hadamard fractional derivative, i.e.,

\[ H D_1^{\alpha,0} = H D_1^\alpha \text{ and } H D_1^{\alpha,1} = H c D_1^\alpha. \]

From [30, Theorem 21], we conclude the following lemma.

**Lemma 4.5.** Let \( F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) be such that \( S_{Fou} \subset C_{\gamma,ln}([1, T]) \) for any \( u \in C_{\gamma,ln}([1, T]) \). Then the problem (3) is equivalent to the Volterra integral equation

\[ u(t) = \frac{\phi_0}{\Gamma(\gamma)} (\ln t)^{\gamma-1} + (H I_1^\alpha v)(t), \]

where \( v \in S_{Fou} \).

We now give without proof an existence result for system (3)–(4).

**Theorem 4.6.** Assume that the following conditions hold:

*(H1)* The multivalued maps \( G_i : [1, T] \times E \times E \to \mathcal{P}_{cp,c}(E) \), \( i = 1, 2 \), are Carathéodory.

*(H2)* There exist two functions \( q_i \in L^\infty([1, T], [0, \infty)) \), \( i = 1, 2 \), such that

\[ \| G_1(t, u, v) \|_P = \sup \{ \| w \|_{C_{\gamma_1,ln}} : w(t) \in G_1(t, u, v) \} \leq q_1(t) \]

and

\[ \| G_2(t, u, v) \|_P = \sup \{ \| z \|_{C_{\gamma_2,ln}} : z(t) \in G_2(t, u, v) \} \leq q_2(t) \]

for a.e. \( t \in [1, T] \) and \( u, v \in E \).

*(H3)* For each bounded and measurable set \( B_i \subset C_{\gamma,ln} \) and for each \( t \in [1, T] \), we have

\[ \mu(G_1(t, B_1(t), B_2(t))) \leq q_i(t) \mu(B_i(t)), \quad i = 1, 2. \]

*(H4)* The function \( \Lambda = (\Lambda_1, \Lambda_2) \equiv (0, 0) \) is the unique solution in \( C' \) of the inequalities

\[ \Lambda_i(t) \leq 2q_i^* (H I_1^{\alpha_i} \Lambda_i)(t), \]

where

\[ q_i^* = \text{ess sup}_{t \in [1, T]} q_i(t); \quad i = 1, 2. \]

Then the system (3)–(4) has at least one solution defined on \([1, T]\).
5. **Example.** To illustrate our results, let

\[ E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\} \]

be the Banach space with the norm

\[ \|w\|_E = \sum_{n=1}^{\infty} |w_n|. \]

Consider the coupled system of Hilfer fractional differential inclusions

\[
\begin{aligned}
\left\{ \begin{array}{l}
(H D_{0^+}^{\alpha_1} u_n)(t) \in F_n(t, u(t), v(t)), \quad t \in [0, e], \\
(H D_{0^+}^{1+\beta_1} v_n)(t) \in G_n(t, u(t), v(t)), \quad t \in [0, e], \\
(H I_{0^+}^{\gamma_1} u)(t)|_{t=0} = (1, 0, \ldots, 0, \ldots), \\
(H I_{0^+}^{\gamma_1} v)(t)|_{t=0} = (1, 0, \ldots, 0, \ldots),
\end{array} \right.
\end{aligned}
\]

where

\[
F_n(t, u(t), v(t)) = \frac{ce^{-2}}{1 + \|u(t)\|_E + \|v(t)\|_E} [u_n(t) - 1, u_n(t)], \quad t \in [0, e],
\]

and

\[
G_n(t, u(t), v(t)) = \frac{ct^2e^{-4-t}}{1 + \|u(t)\|_E + \|v(t)\|_E} [v_n(t), 1 + v_n(t)], \quad t \in [0, e],
\]

with

\[ u = (u_1, u_2, \ldots, u_n, \ldots), \quad v = (v_1, v_2, \ldots, v_n, \ldots), \quad \text{and} \quad c := \frac{e^3}{8} \Gamma \left( \frac{1}{2} \right). \]

Set

\[ F = (F_1, F_2, \ldots, F_n, \ldots), \quad G = (G_1, G_2, \ldots, G_n, \ldots), \]

and \( \alpha_i = \beta_i = \frac{1}{2}, \quad i = 1, 2, \) so that \( \gamma_i = \frac{3}{4}. \) We assume that \( F \) and \( G \) are closed and convex valued. For each \( u, v \in E \) and \( t \in [0, e], \) we have

\[ \|F(t, u, v)\|_p \leq ce^{-2} \]

and

\[ \|G(t, u, v)\|_p \leq ct^2e^{-t-4}. \]

Hence, condition (H2) is satisfied with \( p_1^* = p_2^* = ce^{-2}. \) Simple computations show that the remaining hypotheses of Theorem 3.2 are satisfied. Hence, the system (11) has at least one solution defined on \([0, e].\)

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E-mail address: said.abbas@univ-saida.dz, abbasmsaid@yahoo.fr
E-mail address: benchohra@univ-sba.dz
E-mail address: John-Graef@utc.edu