Impact Representation of Generalized Distribution Amplitudes

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Abstract

We develop an impact representation for the generalized distribution amplitude which describes the exclusive hadronization of a quark-antiquark pair to a pair of mesons. Experiments such as $\gamma^* \gamma \rightarrow \pi\pi$ and $\gamma^* N \rightarrow \pi\pi N'$ are shown to probe the transverse size of the hadronization region of the quark antiquark pair that one can interpret as the transverse overlap of the two emerging mesons. An astonishing feature of this description is that low energy $\pi\pi$ phase shift analysis can be used for understanding some properties of quark hadronization process.

1. A considerable amount of theoretical and experimental work is currently being devoted to the study of generalized parton distributions \cite{1,2}, which are defined as Fourier transforms of matrix elements between different hadron states, such as:

$$\int d\lambda e^{i\lambda z(p+p')\cdot n} \langle N'(p', s')|\bar{\psi}(-\lambda n)(\gamma \cdot n)\psi(\lambda n)|N(p, s)\rangle$$

and their crossed version which describe the exclusive hadronization of a $q\bar{q}$ or $gg$ pair in a pair of hadrons, a pair of $\pi$ mesons for instance. These generalized distribution amplitudes (GDA) \cite{3}, defined in the quark-antiquark case, as

$$\Phi_q^{\pi\pi}(z, \zeta, W) = \frac{d\lambda}{2\pi} e^{-i\lambda z(p+p')\cdot n^*} \langle \pi(p')\pi(p)|\bar{\psi}(\lambda n^*)(\gamma \cdot n^*)\psi(0)|0\rangle$$

where $W$ is the squared energy of the $\pi\pi$ system, $W^2 = (p + p')^2$, are non perturbative parts entering in a factorized way \cite{4} the amplitudes of the light cone dominated processes $\gamma^* \gamma \rightarrow \pi\pi$.

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which may be measured \[^5\] in electron positron colliders of high luminosity and

\[\gamma^* N \rightarrow \pi \pi N'\]

which is under intense study at HERA and JLab \[^6\] and has been the subject of some recent theoretical works \[^7\ 8\]. This new QCD object allows to treat in a consistent way the final state interactions of the meson pair. Its phase is related to the phase of \(\pi \pi\) scattering amplitude, and thus contains information on the resonances which may decay in this channel.

The measurements of GPDs and GDAs are expected to yield important contributions to our understanding of how quarks and gluons assemble themselves into hadrons.

A peculiar feature of GPDs has recently been the subject of intense investigation \[^9\ 10\]. Apart from longitudinal momentum fraction variables, GPDs also depend on the momentum transfer \(t\) between the initial and final hadrons. Fourier transforming this transverse momentum information leads to information on the transverse location of quarks and gluons in the hadron. Real-space images of the target can thus be obtained. Spatial resolution is determined by the virtuality of the incoming photon.

In this note, we show that a similar argument can be developed for the \(W^2\) dependence of the GDAs. By developing essentialy the same techniques as in Ref. \[^10\] we Fourier transform the 2–dimensional transverse momentum behavior of the GDAs into a transverse impact coordinate \(\vec{b}\) representation. We then give the physical interpretation of the \(\vec{b}\) behaviour of this new function. We show that general theorems allow us to get some insight on this \(\vec{b}\) behaviour; we discuss the consequences of such representation in a simple analytical model and briefly comment on future phenomenology.

Throughout this paper we restrict our study to the quark-antiquark initial state and to the two pion case, but gluon contributions may be studied along similar lines. The remarks on possible other channels we present in the discussion. Also we do not include the QCD evolution of the GDAs, which doesn’t influence their dependence on \(W\), hence the \(\vec{b}\)-dependence of their Fourier transform.

2. Let us describe the kinematics of the \(q \bar{q} \rightarrow \pi \pi\) process under consideration. We parametrize all momenta by means of the Sudakov decomposition with two light-like vectors \(n\) and \(n^*\) satisfying the conditions \(n \cdot n^* = 1\). The parametrization of pions momenta \(p\) and \(p'\) reads

\[p = \zeta \, n + \frac{\vec{p}^2 + m_{\pi}^2}{2\zeta} n^* + p_\perp\]

\[p' = \bar{\zeta} \, n + \frac{\vec{p'}^2 + m_{\pi}^2}{2\bar{\zeta}} n^* + p'_\perp\]

\[p_\perp^2 = -\vec{p}^2, \quad \bar{\zeta} = 1 - \zeta, \quad (2)\]

where \(m_{\pi}\) is the pion mass. The invariant mass squared of the two pion system equals then

\[W^2 = \frac{m_{\pi}^2}{\zeta \bar{\zeta}} + \zeta \bar{\zeta} \left( \frac{\vec{p}}{\zeta} - \frac{\vec{p}'}{\bar{\zeta}} \right)^2, \quad (3)\]
i.e. its dependence on transverse momenta of pions enters via the modulus squared of the two dimensional vector \( \vec{D} \)

\[
\vec{D} = \frac{\vec{p}}{\zeta} - \frac{\vec{p}'}{\bar{\zeta}}.
\]

The crucial property of this vector is its invariance under transverse boosts [11]

\[
\zeta \rightarrow \zeta, \quad \vec{p} \rightarrow \vec{p} - \zeta \vec{v}, \quad \vec{p}' \rightarrow \vec{p}' - \bar{\zeta} \vec{v},
\]

\( \vec{D} \rightarrow \vec{D} \).

The definition of such a vector \( \vec{D} \) is of course not unique: all vectors which differ from the one given by Eq. (4) by a function of \( \zeta \) are equally good. Nevertheless, the physical conclusions which we draw below do not depend on this arbitrariness.

3. The description of hadrons in coordinate space is done through the use of wave packets with a finite width to avoid the appearance of infinities in the intermediate stages. We define first these wave packets in momentum space as

\[
|p^+, \vec{p}\rangle_n = \Phi(p^+, \vec{p}) |p\rangle,
\]

where \( \Phi(p^+, \vec{p}) \) is a function describing the shape of the wave packet. In analogy with Ref. [10] we use a simple Gaussian ansatz for this shape function, which permits to perform the calculations analytically. We define

\[
\Phi(p) = p^+ \delta(p^+ - p_0^+) \Phi_{\perp}(\vec{p}),
\]

\[
\Phi_{\perp}(\vec{p}) = G(\vec{p}, \frac{1}{\sigma^2}),
\]

where

\[
G(\vec{x}, \sigma^2) = e^{-\frac{\vec{x}^2}{2\sigma^2}},
\]

\( \sigma \) being the transverse width of the shape function. We can now Fourier-transform in the 2-dimensional transverse space. Thus we define the wave packet as

\[
|p^+, \vec{b}_0\rangle = \int \frac{d^2 \vec{p}}{16\pi^3} e^{-i\vec{p}\vec{b}_0} G(\vec{p}, \frac{1}{\sigma^2}) |p\rangle,
\]

which describes a state with a definite light-cone momentum fraction, but localized around the transverse position \( \vec{b}_0 \) up to the width \( \sigma \); its normalization is given by

\[
\sigma \langle p^+, \vec{b}' | p^+, \vec{b} \rangle = \frac{1}{16\pi^2\sigma^2} G(\vec{b} - \vec{b}', \sigma^2) \left. \right|_{\sigma \rightarrow 0} \frac{1}{4\pi} p^+ \delta(p^+ - p^{' +}) \delta^2(\vec{b} - \vec{b}').
\]

4. Let us now define the 2-dimensional Fourier transform of the GDAs which will allow us to interpret experimental data in a new way. For the operator defining the GDA in Eq. (1), we get

\[
\hat{0}(z, \vec{r}) = \int \frac{d\lambda}{2\pi} e^{-i\lambda z(p^+ + p^{' +})} \bar{\psi}(\lambda n^*, \vec{r})(\gamma \cdot n^*) \psi(0, \vec{r}),
\]
Figure 1: Representation of GDA in the impact parameter space for \( \zeta \geq 1/2 \). The pions are characterized by large light-cone components and transverse impact representation coordinates.

which allows to define the GDA for the wave packets defined above as:

\[
\tilde{\Phi}^{\pi\pi}_{\sigma}(z, \zeta, \vec{b}, \vec{b}') = \sigma \langle \pi(p^+, \vec{b}') \pi(p^+, \vec{b}) | \hat{O}(z, \vec{0}) | 0 \rangle_{\sigma} = \int \frac{d^2 \vec{p}^'}{16\pi^3} \frac{d^2 \vec{p}}{16\pi^3} e^{i\vec{p}' \cdot \vec{b}'} + i\vec{p} \cdot \vec{b} G(\vec{p}', \frac{1}{\sigma^2}) G(\vec{p}, \frac{1}{\sigma^2}) \Phi^{\pi\pi}(z, \zeta, W(\vec{D})) .
\]  

(12)

A simple change of variables and straightforward gaussian integral lead to the result

\[
\tilde{\Phi}^{\pi\pi}_{\sigma}(z, \zeta, \vec{b}, \vec{b}') = \int \frac{d^2 \vec{D}}{16\pi^3} \frac{\zeta^2 \bar{\zeta}^2}{8\pi^2 \sigma^2 (\zeta^2 + \bar{\zeta}^2)} e^{-i \vec{b} \vec{D} + i\vec{D} \delta \vec{b} \delta \vec{b} / \zeta \bar{\zeta}} G(\vec{D}, \frac{\zeta^2 + \bar{\zeta}^2}{\sigma^2 \zeta^2 \bar{\zeta}^2}) G(\delta \vec{B}, \sigma^2 (\zeta^2 + \bar{\zeta}^2)) \Phi^{\pi\pi}(z, \zeta, W(\vec{D})) ,
\]

(13)

where the transverse positions of the two mesons are given by

\[
\vec{b} = \frac{1}{\zeta} (\delta \vec{B} - \vec{B}) , \quad \vec{b}' = \frac{1}{\bar{\zeta}} \vec{B} .
\]

(14)

The first observation coming from Eqs. (13, 14) is that, analogously as in the GPD case studied in [10], the transverse positions of the two produced mesons \((\vec{b}, \vec{b}')\) depend on \(\zeta\). In particular, in the limit of vanishing width \(\sigma\), we have

\[
\tilde{\Phi}^{\pi\pi}_{\sigma}(z, \zeta, B) \overset{\sigma \rightarrow 0}{=} \frac{\zeta^2 \bar{\zeta}^2}{4\pi} \delta^2(\delta \vec{B}) \int \frac{d^2 \vec{D}}{16\pi^3} e^{-i \vec{b} \vec{D}} \Phi^{\pi\pi}(z, \zeta, W(\vec{D})) .
\]

(15)
Rewriting Eq. (13) with vanishing smearing function, $\delta \vec{B} = 0$, we obtain

$$
\int \frac{d^2 \vec{D}}{(2\pi)^2} e^{-i \vec{B} \cdot \vec{D}} G(\vec{D}, \mathcal{D}_\sigma^2) \Phi_{\pi\pi}(z, \zeta, W(\vec{D}))
$$

$$
= \frac{32\pi^3 \sigma^2 (\zeta^2 + \zeta'^2)}{\zeta^2 \zeta'^2} \sigma \langle \pi(p^+, \vec{B}) \pi(p^+, -\vec{B}) | \hat{O}(z, \vec{0}) | 0 \rangle
$$

where

$$
\mathcal{D}_\sigma^2 = \frac{\zeta^2 + \zeta'^2}{\sigma^2 \zeta^2 \zeta'^2}.
$$

The result given by Eq. (13) is the GDA analog of relations appearing in studies of GPDs, see Eqs. (16, 17) in Ref. [10]. One can perform the angular integration in Eq. (16) since the Gauss function and the GDA only depend on $\vec{D}^2$. One gets

$$
\int \frac{d \vec{D}^2}{4\pi} J_0(|\vec{B}||\vec{D}|) G(\vec{D}, \mathcal{D}_\sigma^2) \Phi_{\pi\pi}(z, \zeta, W(\vec{D})) = \frac{32\pi^3 \sigma^2 (\zeta^2 + \zeta'^2)}{\zeta^2 \zeta'^2} \Phi_{\pi\pi}(z, \zeta, \vec{b}, \vec{b}'),
$$

where $J_0$ is the Bessel function, and

$$
\vec{b} = -\frac{\vec{B}}{\zeta}, \quad \vec{b}' = \frac{\vec{B}}{1-\zeta}.
$$

Let us also note now that due to the normalization of wave packets [10], $\sim 1/\sigma^2$, the right hand side of Eq. (18) has a finite limit for vanishing width $\sigma$.

5. The physical picture which emerges from Eq. (13) is presented in Fig. 1. The operator $\hat{O}(z, \vec{0})$ probes the partons at the transverse position $\vec{0}$ in the two pion system. Since the natural partonic description of this process is in the infinite momentum frame, the two emerging pions will be quickly separated in longitudinal space, at least for a non-symmetric configuration ($\zeta \neq \zeta'$) and we may consider that their transverse separation is frozen just after production. This allows to identify the transverse locations of outgoing mesons as the transverse coordinates derived above. In order to be more specific, let us visualize the information in a simplistic quark-antiquark toy model for the pion structure. A $u\bar{u}$ initial state (produced by the $\gamma^*\gamma$ collision for instance) begins to hadronize at impact parameter $\vec{0}$. The QCD vacuum then produces a $d\bar{d}$ pair in order to allow color rearrangement and the formation of the two final pions. This production process proceeds in a non-local way, i.e. the transverse positions of the emerging $d$ and $\bar{d}$ quarks are not at the same point. The final state pions are then produced at transverse localizations around $2\vec{B}$ and $-2\vec{B}$ (for almost symmetric configuration of the produced pions i.e. with $\zeta \sim 1/2$). If $\zeta$ increases, which is the case shown in Fig. 1, the transverse localizations of pions change by amounts proportional to the difference $(\zeta - 1/2)$, namely $2(\zeta - 1/2)/(1-\zeta)$ and $2(\zeta - 1/2)/\zeta$, respectively, directed towards the top of the figure. If $\zeta$ decreases the shifts of the transverse positions of pions occur in the opposite direction. Note however that the values of these shifts are different for both produced pions. One should note also that this behaviour doesn’t depend on the longitudinal variable $z$ of quarks entering the operator.
The $\vec{b}$—dependence measures the transverse size of the hadronization process. If the transverse distance $D$ between the pions transverse positions $\vec{b}$ and $\vec{b}'$ (the interaction point being at position $\vec{0}$) is larger than the typical pion radius then the GDA of the two pion system describes a highly non-local (in transverse space) phenomenon which depends much on the long distance structure of the QCD vacuum. If this transverse distance $D$ is very small (which corresponds to a large invariant mass of two pion system), a perturbative treatment can be applied and the two meson distribution amplitude can be expressed in terms of the distribution amplitudes (DA) of each separate pion. In the case when the distance $D$ is smaller than the typical pion radius, but still large enough for non-perturbative physics to dominate, we have to deal with a complicated two pion system (at least in the infinite momentum frame) without reference to the DAs of each separate pion. The impact representation of GDAs serves as a tool for studying consistently these three regimes.

6. The phenomenology of this impact representation is quite rich. Without developing it in this short note, let us point out a very peculiar feature, namely the fact that phase shift analysis at low energy can lead to the understanding of quark hadronization, although the concept of quarks is not suitable for the description of $\pi\pi$ elastic scattering in this energy domain. This astonishing conclusion comes from the following argument. The Watson theorem allows to derive the low $W^2$ behavior of the phase of the GDA. Indeed this theorem based on the requirement of the unitarity of the $S$—matrix implies that the phase of the GDA equals the phase of the $\pi\pi$ scattering matrix, provided there is no inelastic channel open. The phase of $\pi\pi$ scattering has been the subject of intense study in the last 50 years (for instance see Ref. [12]) and quite a consensus has been achieved at least at energies below 1 GeV. Through a dispersion relation of the Omnès Muskhelishvili type, one is thus able to derive a plausible model of the $W^2$ dependence of the GDA. For instance in the P-wave channel, a single Breit Wigner representation with a $\rho$ meson pole is quite reasonable for the low $W^2$ region. If we assume that this region gives the dominant contribution to the Fourier integral defining $\Phi_{\pi\pi}(z,\zeta, B)$, one understands that the $\pi\pi$ scattering data collected at low energies convey information on the transverse picture of the hadronization process $q\bar{q} \to \pi \pi$. This information is not complete since the $W^2$ behaviour of $\Phi_{\pi\pi}$ in the medium energy region cannot be derived from $\pi\pi$ scattering data. However, standard arguments on the Fourier transformation allow to trust the transverse location information derived from data with $W^2$ up to 1 GeV$^2$ as a reliable picture with a resolution of the order of 1 GeV$^{-1}$ i.e. one fifth of a femtometer. Finally, the large $W^2$ region is of very difficult experimental access, but only leads to further refinement of the knowledge of the impact picture. One may however trust a perturbative estimate of the GDA in this domain and implement it in the Fourier transform.

7. As an example we consider the simple model with the outgoing pions in the $p-$wave, being described as a $\rho$ meson. The $W$—dependence of the GDA enters in this case via a
Figure 2: Impact parameter dependence of the function $F$ defined in Eq. (21) for $\zeta = 0.4$ and $z = 0.5$ with the Breit-Wigner ansatz. $|\vec{B}|$ units are GeV$^{-1}$.

Figure 3: The same as in Fig. 2 for $\zeta = 0.1$.

simple Breit-Wigner form

$$\Phi^{\pi\pi}(z, \zeta, W) = 6z\bar{z}(2\zeta - 1)F_\pi(W)$$

$$F_\pi(W) = \frac{m_\rho^2}{m_\rho^2 - W^2 - i\Gamma_\rho m_\rho}, \quad (20)$$

with $m_\rho = 0.773$ GeV, $\Gamma_\rho = 0.145$ GeV. We define the function $F$ as

$$F(z, \zeta, |\vec{B}|) = \int_0^\infty \frac{d|\vec{D}|^2}{4\pi} J_0(|\vec{B}||\vec{D}|) \Phi^{\pi\pi}(z, \zeta, W)$$

$$\quad = -\frac{3z\bar{z}(2\zeta - 1)m_\rho^2}{\pi \zeta \zeta} K_0 \left( |\vec{B}| \sqrt{\frac{1}{\zeta \zeta} \left( \frac{m_\rho^2}{\zeta \zeta} - m_\rho^2 + i\Gamma_\rho m_\rho \right) } \right), \quad (21)$$

which corresponds to Eq. (16) in the limit of vanishing $\sigma$. On Figs. 2 and 3 we plot the real and imaginary parts of the function $F$ for $(z = 0.5, \zeta = 0.4)$ and for $(z = 0.5, \zeta = 0.1)$, respectively, as a function of $|\vec{B}|$ in units of GeV$^{-1}$. The value of $|\vec{B}| = 1$ GeV$^{-1}$ corresponds (see Fig. 1) to a transverse distance between the two pions of the order of 0.83 fm (for $\zeta = 0.4$) and 2.2 fm (for $\zeta = 0.1$).
In summary, we have proposed in this note a new way to analyse data on two pion production. For this aim, we have introduced the impact picture of \( q \bar{q} \) hadronization in a pair of mesons. We emphasize the connection of the phase shift analysis of low energy reactions to this new approach. More data are needed in \( \gamma^* \gamma \) and \( \gamma^* p \) reactions to get a precise transverse space picture of exclusive hadronization processes. As in the case of the impact picture of GPDs, the ultimate spatial resolution is determined by the value of the virtuality of the photon.

Let us also emphasize possible application of our analysis to other channels like \( K^+ K^- \) or \( \rho \rho \), which can be studied along similar lines. Final states with more particles, e.g. \( \pi \pi \pi \) in \( \gamma^* \gamma \) collisions [14], or the reaction \( \gamma^* N \rightarrow \gamma \pi \pi^0 N' \) [15] will require more study.

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