Harmonic analysis on a Galois field and its subfields

A. Vourdas
Department of Computing,
University of Bradford,
Bradford BD7 1DP, United Kingdom

Complex functions $\chi(m)$ where $m$ belongs to a Galois field $GF(p^\ell)$, are considered. Fourier transforms, displacements in the $GF(p^\ell) \times GF(p^\ell)$ phase space and symplectic $Sp(2, GF(p^\ell))$ transforms of these functions are studied. It is shown that the formalism inherits many features from the theory of Galois fields. For example, Frobenius transformations are defined which leave fixed all functions $h(n)$ where $n$ belongs to a subfield $GF(p^d)$ of the $GF(p^\ell)$. The relationship between harmonic analysis (or quantum mechanics) on $GF(p^\ell)$ and harmonic analysis on its subfields, is studied.

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I. INTRODUCTION

Quantum mechanics with complex wavefunctions $\chi(m)$ where $m$ belongs to the ring $\mathbb{Z}_d$ (the integers modulo $d$) has been studied by Weyl [30] and Schwinger [21, 22] and later by many authors (a review with the relevant literature has been presented in [24]). Fourier transforms and displacements in phase space (which is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$) have been studied extensively.

When $d$ is equal to a prime number number $p$, we get stronger results. This is due to the fact that the $\mathbb{Z}_p$ is a field and the corresponding phase space $\mathbb{Z}_p \times \mathbb{Z}_p$ is a finite geometry [9, 14, 23]. In this case there is nothing special with the ‘position-momentum’ (or ‘time-frequency’) directions in phase space; but all results can be proved with respect to the other directions in the finite geometry. Symplectic transformations are well defined and form the $Sp(2, \mathbb{Z}_p)$ group.

In recent work [25, 26, 27] we have studied in a quantum mechanical context how we can go from $\mathbb{Z}_p$ to a bigger Galois field $GF(p^\ell)$. Related work has also been reported in [6, 7, 15, 28].

With a very different motivation there has been a lot of related work in the context of mutually unbiased bases in quantum systems with $d$-dimensional Hilbert space. It is known that the number of such bases is less or equal to $d + 1$; and that when $d$ is the power of a prime, it is equal to $d + 1$. This problem has recently been studied extensively in the quantum mechanical literature [1, 3, 4, 8, 10, 11, 12, 13, 14, 15, 16, 20, 31, 32].

In this paper we continue the work of [25, 26, 27] with emphasis on the mathematical aspects. We use ideas from field extension to go from harmonic analysis (or quantum mechanics) on $\mathbb{Z}_p$, to harmonic analysis on $GF(p^\ell)$. This mathematical structure inherits many features from the theory of field extension. Frobenius transformations and Galois groups in the context of finite fields, have counterparts in the present context. For example, we define Frobenius transforms which map the values of a function at Galois conjugates, $(\chi(m), \chi(m^p), \chi(m^{p^2}), \text{etc})$ to each other. We also discuss Galois groups which in the present context are transformations in the Hilbert space $H$ of functions $\chi(m)$, which leave fixed all functions in the subspace $\mathcal{H}_d$ comprised of functions $h(n)$ where $n$ belongs to a subfield $GF(p^d)$ of $GF(p^\ell)$.

In section II we consider complex functions on a Galois field and we discuss the basic formalism. In section III we present Fourier transforms in both the space $H$ and its subspace $\mathcal{H}_d$. In section IV we discuss Frobenius transforms and the corresponding Galois groups, in the present context. In section V we study displacements and the Heisenberg-Weyl group, in both the space $H$ and its subspace $\mathcal{H}_d$. In
section VI we consider symplectic transformations and the \( Sp(2, GF(p^\ell)) \) group. We conclude in section VII with a discussion of our results.

A. Notation

The elements of the Galois field \( GF(p^\ell) \) can be written as polynomials

\[
m = m_0 + m_1 \epsilon + \ldots + m_{\ell-1} \epsilon^{\ell-1}; \quad m_0, m_1, \ldots, m_{\ell-1} \in \mathbb{Z}_p
\]

(1)

They are defined modulo an irreducible polynomial of degree \( \ell \):

\[
P(\epsilon) \equiv c_0 + c_1 \epsilon + \ldots + c_{\ell-1} \epsilon^{\ell-1} + \epsilon^\ell; \quad c_0, c_1, \ldots, c_{\ell-1} \in \mathbb{Z}_p
\]

(2)

Different irreducible polynomials of the same degree \( \ell \) lead to isomorphic finite fields. We refer to the \( m_0, m_1, \ldots, m_{\ell-1} \) as the Galois components of \( m \) in the basis \( 1, \epsilon, \ldots, \epsilon^{\ell-1} \).

The \( m, mp, \ldots, m^{p-1} \) are Galois conjugates. The Frobenius map is

\[
\sigma(m) = m^p; \quad \sigma^\ell = 1
\]

(3)

The

\[
\text{Gal}[GF(p^\ell)/\mathbb{Z}_p] = \{1, \sigma, \ldots, \sigma^{\ell-1}\}
\]

(4)

is a cyclic group of order \( \ell \). It comprises of all automorphisms of \( GF(p^\ell) \) which leave the elements of the subfield \( \mathbb{Z}_p \) fixed.

If \( d \) is a divisor of \( \ell \) (which we denote as \( d|\ell \)) the \( GF(p^d) \) is a subfield of \( GF(p^\ell) \). The

\[
\text{Gal}[GF(p^\ell)/GF(p^d)] = \{1, \sigma^d, \ldots, \sigma^{\ell-d}\}
\]

(5)

is a cyclic group of order \( \ell/d \) and is a subgroup of \( \text{Gal}[GF(p^\ell)/\mathbb{Z}_p] \). It comprises of all automorphisms of \( GF(p^\ell) \) which leave the elements of the subfield \( GF(p^d) \) fixed.

The trace of \( m \in GF(p^\ell) \) is defined as:

\[
\text{Tr}(m) = m + m^p + \ldots + m^{p^{\ell-1}}; \quad \text{Tr}(m) \in \mathbb{Z}_p; \quad m \in GF(p^\ell)
\]

(6)

When \( m \) belongs to the subfield \( GF(p^d) \) we make the distinction between the trace with regard to the extension from \( \mathbb{Z}_p \) to \( GF(p^\ell) \) given in Eq.(6) and the trace with regard to the extension from \( \mathbb{Z}_p \) to \( GF(p^d) \) given by

\[
\mathbb{I}r(m) = m + m^p + \ldots + m^{p^{d-1}}; \quad \mathbb{I}r(m) \in \mathbb{Z}_p; \quad m \in GF(p^d)
\]

(7)

Indices in the notation of the trace which indicate the extension, are sometimes used in the literature. For simplicity we use \( \mathbb{I}r \) in Eq.(6) and Tr in Eq.(6). It is easily seen that

\[
\text{Tr}(m) = \ell/d \mathbb{I}r(m); \quad m \in GF(p^d)
\]

(8)

In the special case that \( \ell/d \) is a multiple of the prime \( p \), all \( m \) in \( GF(p^d) \) have \( \text{Tr}(m) = 0 \). We note that this does not contradict a theorem which states that in every finite field there exists at least one element with non-zero trace. According to this theorem it is impossible to have \( \mathbb{I}r(m) = 0 \) for all \( m \) in \( GF(p^d) \). But it is possible to have \( \text{Tr}(m) = 0 \) for all \( m \) in \( GF(p^d) \) because this is a different trace.
Following [2], we introduce the $\ell \times \ell$ matrices
\begin{equation}
\begin{aligned}
g_{\lambda\kappa} & \equiv \text{Tr}(\epsilon^{\lambda+\kappa}); & G & \equiv g^{-1}; & g_{\lambda\kappa}, G_{\lambda\kappa} & \in \mathbb{Z}_p
\end{aligned}
\end{equation}
where $\lambda$ and $\kappa$ take values from 0 to $\ell - 1$. Using them we define the dual basis $E_0, E_1, \ldots, E_{\ell-1}$, as:
\begin{equation}
E_\kappa = \sum_{\lambda} G_{\kappa\lambda} \epsilon^\lambda; \quad \text{Tr}(\epsilon^\kappa E_\lambda) = \delta_{\kappa\lambda}
\end{equation}

A number $m \in GF(p^\ell)$ can be expressed in the two bases as:
\begin{equation}
\begin{aligned}
m &= \sum_{\lambda=0}^{\ell-1} m_\lambda \epsilon^\lambda = \sum_{\lambda=0}^{\ell-1} \overline{m}_\lambda E_\lambda \\
m_\lambda &= \text{Tr}[mE_\lambda]; \quad \overline{m}_\lambda = \text{Tr}[mc^\lambda] \\
m_\lambda &= \sum_{\kappa} G_{\lambda\kappa} \overline{m}_\kappa; \quad \overline{m}_\lambda = \sum_{\kappa} g_{\lambda\kappa} m_\kappa
\end{aligned}
\end{equation}

We refer to $\overline{m}_\lambda$ as the dual Galois components of $m$. The trace of the product $\alpha\beta$ is given in terms of the components of these numbers as
\begin{equation}
\begin{aligned}
\text{Tr}(mn) &= \sum_{\lambda,\kappa} g_{\lambda\kappa} m_\lambda n_\kappa = \sum_{\lambda,\kappa} G_{\lambda\kappa} \overline{m}_\lambda \overline{n}_\kappa \\
&= \sum_{\lambda} m_\lambda \overline{n}_\lambda = \sum_{\lambda} \overline{m}_\lambda n_\lambda
\end{aligned}
\end{equation}

II. COMPLEX FUNCTIONS ON A GALOIS FIELD AND ITS SUBFIELDS

We consider the $p^\ell$-dimensional Hilbert space $H$ of complex vectors $\chi(m)$ where $m \in GF(p^\ell)$. The scalar product $\langle \chi, h \rangle$ is defined as
\begin{equation}
\langle \chi, h \rangle = \sum_{m} [\chi(m)]^* h(m)
\end{equation}
where the summation is over all $m \in GF(p^\ell)$. We define the projection operators $Q_k$
\begin{equation}
Q_k(m,n) = 1; \quad \text{if} \quad m = n = k \\
Q_k(m,n) = 0; \quad \text{otherwise}
\end{equation}

They obey the relations
\begin{equation}
Q_kQ_m = \delta(k,m)Q_k; \quad \sum_k Q_k = 1
\end{equation}

where $\delta$ is the Kronecker delta.

We have explained that when $d$ is a divisor of $\ell$ the $GF(p^d)$ is a subfield of $GF(p^\ell)$. The complex vectors $\chi(m)$ where $m \in GF(p^d)$ belong to a subspace of $H$ which we call $\mathcal{H}_d$. For example, the subspace $\mathcal{H}_1$ contains complex functions $f(m)$ where $m \in \mathbb{Z}_p$, and the subspace $\mathcal{H}_\ell = H$. The projection operator $\Pi_d(m,n)$ from $H$ to $\mathcal{H}_d$ is
\begin{equation}
\Pi_d = \sum_{k \in GF(p^d)} Q_k
\end{equation}
If \( c \) is a divisor of \( d \) then
\[
c|d \rightarrow \Pi_c \Pi_d = \Pi_c
\] (17)
The projection of the vector \( \chi(m) \) to the space \( \mathcal{H}_d \) is \( \sum_n \Pi_d(m, n) \chi(n) \). We note that the subspaces \( \mathcal{H}_d \) depend on the basis that we choose. With a unitary transformation \( U \) we can go to a different basis and then we get different subspaces which we denote as \( U\mathcal{H}_d \) (the corresponding projection operator is \( U\Pi_d U^\dagger \)).

An orthonormal basis in the Hilbert space \( H \) is the functions
\[
\phi_n(m) = (p^\ell)^{-1/2} \omega[-\text{Tr}(nm)]
\] (18)
where \( n \in GF(p^\ell) \) and
\[
\omega = \exp\left(\frac{2\pi i}{p}\right); \quad \omega(\alpha) \equiv \omega^\alpha; \quad \alpha \in \mathbb{Z}_p
\] (19)
Indeed we can easily show that for \( k, n, m, r \in GF(p^\ell) \):
\[
(\phi_n, \phi_r) = \delta(n, r)
\]
\[
\sum_n [\phi_n(m)]^* \phi_n(k) = \delta(m, k)
\] (20)
The properties of the trace lead to the relation
\[
\phi_n(m) = \phi_{n^p}(m^p) = \ldots = \phi_{n^{p^\ell-1}}(m^{p^\ell-1})
\] (21)
The Hilbert space \( H \) can be written as the tensor product \( H \otimes \ldots \otimes H \) where \( H \) are \( p \)-dimensional ‘component Hilbert spaces’ of complex functions \( g(m_i) \) where \( m_i \in \mathbb{Z}_p \). Indeed, the general function \( \chi(m) \) can be written in terms of the Galois components of \( m \) as \( \chi(m_0, \ldots, m_{\ell-1}) \). As an example of this, we use Eq. (12) and show that
\[
\phi_n(m) = \varphi_{n^0}(m_0) \ldots \varphi_{n^{\ell-1}}(m_{\ell-1})
\]
where
\[
\varphi_\alpha(\beta) = p^{-1/2} \omega(-\alpha \beta); \quad \alpha, \beta \in \mathbb{Z}_p
\] (23)
belongs to the Hilbert space \( H \).

### III. FOURIER TRANSFORM

The Fourier matrix \( F \) is the \( p^\ell \times p^\ell \) matrix:
\[
F(n, m) = (p^\ell)^{-1/2} \omega[\text{Tr}(nm)]; \quad F^4 = 1; \quad FF^\dagger = 1
\] (24)
The Fourier transform of a function \( \chi(m) \) in \( H \), is given by
\[
\tilde{\chi}(n) = (\phi_n, \chi) = \sum_m F(n, m) \chi(m)
\] (25)
Using Eq. (12) we show that
\[ F(n, m) = F(\overline{n}_0, m_0) \ldots F(\overline{n}_{t-1}, m_{t-1}) \]
\[ = F(n_0, \overline{m}_0) \ldots F(n_{t-1}, \overline{m}_{t-1}) \]  
(26)
where
\[ F(\alpha, \beta) = p^{-1/2} \omega(\alpha \beta); \quad \alpha, \beta \in \mathbb{Z}_p \]  
(27)
are Fourier transforms in the ‘component spaces’ \( \mathcal{H} \). It is seen that the dual components of \( m \) and the components of \( n \) appear in Eq. (26) (or vice-versa). Therefore in general
\[ F(n, m) \neq F(n_0, m_0) \ldots F(n_{t-1}, m_{t-1}) \]  
(28)
This equation shows one of the differences between harmonic analysis on \( GF(p^t) \) and harmonic analysis on \( \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p \). The former uses the Fourier transform \( F(n, m) \) and the latter the Fourier transform \( F(n_0, m_0) \ldots F(n_{t-1}, m_{t-1}) \).

The properties of the trace lead to the relation
\[ F(n, m) = F(n^p, m^p) = \ldots = F(n^{p^{t-1}}, m^{p^{t-1}}) \]  
(29)

A. Fourier transforms in \( \delta_d \)

We consider the space \( \delta_d \) (where \( d \) is a divisor of \( \ell \)) and in analogy with Eq. (24), we introduce the \( p^d \times p^d \) Fourier matrix
\[ \mathfrak{F}(n, m) = (p^d)^{-1/2} \omega(nm); \quad n, m \in GF(p^d) \]
\[ \mathfrak{F}^\dagger = \Pi_d; \quad \mathfrak{F} \mathfrak{F}^\dagger = \Pi_d \]  
(30)
We note that the trace of Eq. (14) is used here, in contrast to Eq. (24) where the trace of Eq. (6) is used. We compare \( \mathfrak{F}(n, m) \) with the matrix \( \Pi_d F \Pi_d \). The \( \Pi_d F \Pi_d \) is a \( p^d \times p^d \) matrix but only \( p^d \times p^d \) elements are non-zero. Taking into account Eq. (3) we show that for \( n, m \in GF(p^d) \)
\[ (\Pi_d F \Pi_d)(n, m) = [\mathfrak{F}(n, m)]^{\ell/d} \]  
(31)
This shows that the elements of \( F \) with indices in \( GF(p^d) \) are powers of the corresponding elements of \( \mathfrak{F} \). Of course, the matrix \( F \) has other elements also, with indices in the set \( GF(p^t) - GF(p^d) \). In the special case that \( \ell/d \) is a multiple of the prime \( p \), Eq. (31) shows that the \( p^d \times p^d \) submatrix of \( F(n, m) \) with indices in \( GF(p^d) \), has all its elements equal to \( p^{-\ell/2} \).

B. Spectrum of Fourier transforms

From Eq. (24) it follows that the eigenvalues of \( F \) are \( 1, i, -1, -i \). We express \( F \) in terms of its eigenvalues and eigenvectors as:
\[ F = \pi_0 + i \pi_1 - \pi_2 - i \pi_3 \]
\[ \pi_r \pi_s = \pi_r \delta(r, s); \quad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1; \quad r, s = 0, 1, 2, 3 \]  
(32)
Here \( \pi_\lambda \) are orthogonal projectors to the eigenspaces corresponding to the various eigenvalues of \( F \). From Eq. (24) it follows that:
\[ \pi_r = \frac{1}{4} [1 + (i^{-r} F) + (i^{-r} F)^2 + (i^{-r} F)^3]; \quad r = 0, 1, 2, 3 \]  
(33)
IV. FROBENIUS TRANSFORM

The Frobenius map of Eq. (3) leads in the present context to the Frobenius transform. We define the Frobenius matrix as

\[ G(n, m) = \delta(n, m^p); \quad G^\ell = 1; \quad GG^\dagger = 1 \]  \hspace{1cm} (34)

The Frobenius transform of a function \( \chi(m) \) in \( H \), is given by

\[ (G\chi)(n) = \sum_m G(n, m)\chi(m) = \chi(n^{p^\ell-1}) \]  \hspace{1cm} (35)

We note that \( G \) depends on the basis that we choose. With a unitary transformation \( U \) we can go to a different basis and then the Frobenius transform becomes \( UGU^\dagger \). The operator \( G \) commutes with the projection operators \( \Pi_d \) (where \( d \) is a divisor of \( \ell \)):

\[ [G, \Pi_d] = 0 \]  \hspace{1cm} (36)

A direct consequence of our comments about the group \( \text{Gal}(GF(p^\ell)/\mathbb{Z}_p) \) in Eq. (4), is that the

\[ \text{Gal}(H/\mathfrak{H}_1) = \{1, G, ..., G^{\ell-1}\} \]  \hspace{1cm} (37)

is a cyclic group of order \( \ell \). These transformations leave all the functions in \( \mathfrak{H}_1 \) fixed:

\[ G\Pi_1 = \Pi_1 \]  \hspace{1cm} (38)

The Frobenius transform commutes with the Fourier transform and also with the projection operators \( \pi_r \):

\[ [F, G] = [\pi_r, G] = 0; \quad r = 0, 1, 2, 3 \]  \hspace{1cm} (39)

This can be proved using Eqs. (29), (33).

A. Frobenius transform in \( \mathfrak{H}_d \)

The Frobenius operator in the space \( \mathfrak{H}_d \) (where \( d \) is a divisor of \( \ell \)) is \( G\Pi_d \). The analogue of \( G^\ell = 1 \) in Eq. (34) is here

\[ G^d\Pi_d = \Pi_d \]  \hspace{1cm} (40)

The

\[ \text{Gal}(H/\mathfrak{H}_d) = \{1, G^d, ..., G^{\ell-d}\} \]  \hspace{1cm} (41)

form a cyclic group of order \( \ell/d \) which is a subgroup of \( \text{Gal}(H/\mathfrak{H}_1) \). These transformations leave all the functions in \( \mathfrak{H}_d \) fixed.

We have explained earlier that with a unitary transformation \( U \) we go to a different basis and the subspace \( \mathfrak{H}_d \) becomes \( U\mathfrak{H}_d \) and the Frobenius transform becomes \( UGU^\dagger \). In this case

\[ \text{Gal}(H/U\mathfrak{H}_d) = \{1, UG^dU^\dagger, ..., UG^{\ell-d}U^\dagger\} \]  \hspace{1cm} (42)
B. Spectrum of Frobenius transforms

A direct consequence of Eq. (34) is that we can express $G$ in terms of its eigenvalues and eigenvectors as:

$$G = \omega_0 + \Omega \omega_1 + \ldots + \Omega^{\ell-1} \omega_{\ell-1}; \quad \Omega = \exp\left(i \frac{2\pi}{\ell}\right)$$

$$\omega_\lambda \omega_\mu = \omega_\lambda \delta(\lambda, \mu); \quad \sum_\lambda \omega_\lambda = 1; \quad \lambda, \mu = 0, \ldots, \ell - 1 \quad (43)$$

Here $\omega_\lambda$ are orthogonal projectors to the eigenspaces corresponding to the various eigenvalues of $G$. From Eq. (32) it follows that:

$$\omega_\lambda = \frac{1}{\ell} \left[ 1 + (\Omega^{-\lambda} G) + (\Omega^{-\lambda} G)^2 + \ldots + (\Omega^{-\lambda} G)^{\ell-1} \right] \quad (44)$$

and using Eq. (39) we show that

$$[F, \omega_\lambda] = [\pi_r, \omega_\lambda] = 0; \quad r = 0, 1, 2, 3; \quad \lambda = 0, \ldots, \ell - 1 \quad (45)$$

Using Eq. (38) we prove that

$$\Pi_1 \omega_0 = \omega_0 \Pi_1 = \Pi_1 \quad (46)$$

It is seen that the space $H_1$ is a subspace of the eigenspace of $G$ corresponding to the eigenvalue 1.

More generally, for $d$ which is a divisor of $\ell$, we express $G^d$ as

$$G^d = [\omega_0 + \omega_{\ell/d} + \ldots + \omega_{(d-1)\ell/d}] + \Omega^d[\omega_1 + \omega_{1+\ell/d} + \ldots + \omega_{1+(d-1)\ell/d}] + \ldots$$

$$+ \Omega^{\ell-d}[\omega_{(\ell-d)/d} + \omega_{(\ell-d)/d} + \ldots + \omega_{\ell-1}] \quad (47)$$

and using Eq. (40) we prove that

$$\Pi_d [\omega_0 + \omega_{\ell/d} + \ldots + \omega_{(d-1)\ell/d}] = [\omega_0 + \omega_{\ell/d} + \ldots + \omega_{(d-1)\ell/d}] \Pi_d = \Pi_d \quad (48)$$

It is seen that the space $H_d$ is a subspace of the combined eigenspace of $G$ corresponding to the eigenvalues $1, \Omega^{\ell/d}, \ldots, \Omega^{(d-1)\ell/d}$.

In the appendix we present the matrices $\omega_\lambda$ for an example.

V. DISPLACEMENTS AND THE HEISENBERG-WEYL GROUP

The displacement matrices are defined as

$$Z^\alpha(n, m) = \omega[\text{Tr}(\alpha m)] \delta(n, m); \quad X^\beta(n, m) = \delta(n, m + \beta)$$

$$Z_\alpha X^\beta = X^\beta Z_\alpha \omega[\text{Tr}(\alpha \beta)]; \quad \alpha, \beta \in GF(p^\ell) \quad (49)$$

Acting with them on functions $\chi(m)$ in the space $H$ we get

$$(Z^\alpha \chi)(n) = \omega[\text{Tr}(\alpha m)] \chi(n); \quad (X^\beta \chi)(n) = \chi(n - \beta) \quad (50)$$

Both $Z$ and $X$ are $p^\ell \times p^\ell$ matrices with elements which are complex numbers. We have explained in that powers of these matrices to elements of a Galois field, are defined through the above equations and they form the Heisenberg-Weyl group.
General displacement operators are given by
\[ D(\alpha, \beta) = Z^\alpha X^\beta \omega \left[ -\frac{\omega}{2} \text{Tr}(\alpha \beta) \right] ; \quad [D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta) \]
\[ D(\alpha, \beta)D(\gamma, \delta) = D(\alpha + \gamma, \beta + \delta) \omega \left[ \frac{\omega}{2} \text{Tr}(\alpha \delta - \beta \gamma) \right] \]

(51)

We use the notation \([D(\alpha, \beta)](n, m)\) for the \((n, m)\) element of the matrix \(D(\alpha, \beta)\). They are given by:
\[ [D(\alpha, \beta)](n, m) = \omega \left[ \text{Tr}(2^{-1/2} \alpha \beta + \alpha m) \right] \delta(n, m + \beta) \]

(52)

We can show that
\[ FD(\alpha, \beta)F^\dagger = D(\beta, -\alpha) \]
\[ G^\lambda D(\alpha, \beta)(G^\dagger)^\lambda = D(\alpha^\lambda, \beta^\lambda); \quad \lambda = 0, \ldots, \ell - 1 \]

(53) (54)

Eq. (53) has no analogue in harmonic analysis on the field of real numbers or on \(\mathbb{Z}_p\). It is tempting to interpret it as magnification of the phase space, where the 'coordinates' are replaced by their powers. We stress however that we are in a finite field and magnification is simply reordering.

If \(\alpha, \beta\) belongs to the subfield \(\mathbb{Z}_p\) then
\[ \alpha, \beta \in \mathbb{Z}_p \rightarrow GD(\alpha, \beta)G^\dagger = D(\alpha, \beta) \]

(55)

It is seen that \(G\) commutes with the matrices \(X\) and \(Z\). Normally when two matrices commute, their powers also commute. \(G\) commutes with 'ordinary powers' of \(X\) and \(Z\) which belong in \(\mathbb{Z}_p\); but it does not commute with 'extraordinary powers' of \(X\) and \(Z\) which belong in \(GF(p^\ell) - \mathbb{Z}_p\) and which are are defined through Eq. (49). We discuss further this point below in connection with the spectrum of the displacement operators.

A more general result than Eq. (55) is that if \(\alpha, \beta\) belongs to the subfield \(GF(p^\ell)\) then
\[ \alpha, \beta \in GF(p^\ell) \rightarrow G^d D(\alpha, \beta)(G^\dagger)^d = D(\alpha, \beta) \]

(56)

We have shown in [27] that the displacement operators acting on \(H\) are expressed in terms of the displacement operators
\[ D(\alpha, \beta) = Z^\alpha X^\beta \omega \left[ -\frac{\omega}{2} \alpha \beta \right] ; \quad \alpha, \beta \in \mathbb{Z}_p \]

(57)

acting on the various component spaces \(H\) as:
\[ D(\alpha, \beta) = D(\alpha_0, \beta_0) \otimes \ldots \otimes D(\alpha_{\ell-1}, \beta_{\ell-1}) \]

(58)

The dual components of \(\alpha\) and the components of \(\beta\), enter in this equation. We see here another difference between harmonic analysis on \(GF(p^\ell)\) and harmonic analysis on \(\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p\). Displacements in the latter will be \(D(\alpha_0, \beta_0) \otimes \ldots \otimes D(\alpha_{\ell-1}, \beta_{\ell-1})\).

Special cases of Eq. (58) are
\[ Z^\alpha = Z^{\alpha_0} \otimes \ldots \otimes Z^{\alpha_{\ell-1}}; \quad X^\beta = X^{\beta_0} \otimes \ldots \otimes X^{\beta_{\ell-1}} \]

(59)
A. Properties of displacement operators

An arbitrary operator $\Theta$ acting on $H$ can be expanded in terms of the displacement operators as

$$\Theta = \frac{1}{p^l} \sum_{\alpha, \beta} D(\alpha, \beta) W(-\alpha, -\beta); \quad W(\alpha, \beta) = \text{tr}[\Theta D(\alpha, \beta)]$$

(60)

where $\text{tr}$ denotes the usual trace of a matrix. This is proved using Eq. (52). $W(\alpha, \beta)$ is the Weyl (or ambiguity) function. An important special case is the $SU(p^l)$ unitary transformations. For infinitesimal $SU(p^l)$ transformations $\theta$

$$\theta = 1 + \sum_{\alpha, \beta} D(\alpha, \beta) e(\alpha, \beta)$$

(61)

where $e(\alpha, \beta)$ are infinitesimal coefficients. In this case the $D(\alpha, \beta)$ (with $(\alpha, \beta) \neq (0,0)$) play the role of the $p^{2l} - 1$ generators of the $SU(p^l)$. Finite $SU(p^l)$ transformations are given by Eq. (60).

Another property is the ‘generalized resolution of the identity’. For an arbitrary operator $\Theta$

$$\frac{1}{p^l} \sum_{\alpha, \beta} D(\alpha, \beta) \frac{\Theta}{\text{tr}\Theta} [D(\alpha, \beta)]^\dagger = 1$$

(62)

This is also proved using Eq. (52). There is analogous property in the theory of coherent states for the harmonic oscillator (e.g., [29]). As special case we use

$$\Theta(m, n) = \psi(m)\psi(n)^*; \quad \sum_m |\psi(m)|^2 = 1$$

(63)

in Eq. (62), where $\psi(m)$ is an arbitrary normalized vector in $H$. In this case we show that the $p^{2l}$ vectors $D(\alpha, \beta)\psi$ (for all $\alpha, \beta \in GF(p^l)$) form an ‘overcomplete basis’ of vectors in the $p^l$ dimensional space $H$. Indeed Eq. (62) shows that we can expand an arbitrary vector $\chi(m)$ as:

$$\chi(m) = \frac{1}{p^l} \sum_{\alpha, \beta} u(\alpha, \beta) D(\alpha, \beta)\psi(m); \quad u(\alpha, \beta) = (D(\alpha, \beta)\psi, \chi)$$

(64)

Other important properties of the displacement operators are the marginal properties [24]:

$$\frac{1}{p^l} \sum_{\alpha \in GF(p^l)} D(\alpha, \beta) = Q_{-2^{-1}\beta}$$

$$\frac{1}{p^l} \sum_{\beta \in GF(p^l)} D(\alpha, \beta) = \tilde{Q}_{-2^{-1}\alpha}$$

(65)

where $\tilde{Q}_k$ are projection operators which are the Fourier transforms of $Q_k$:

$$\tilde{Q}_k = F Q_k F^\dagger$$

(66)
In analogy with Eq. (49) introduce the displacement matrices $Z$ and $X$ in the space $\mathcal{H}_d$:

$$Z^\alpha_{(n,m)} = \omega^{|\text{Tr}(\alpha m)|} \delta_{(n,m)}; \quad X^\beta_{(n,m)} = \delta_{(n,m)+\beta}$$

$$Z^\alpha X^\beta = X^\beta Z^\alpha \omega^{|\text{Tr}(\alpha \beta)|}; \quad \alpha, \beta \in GF(p^d)$$

We note that the trace of eq. (7) is used here. General displacements are defined as

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega^{|\text{Tr}(\alpha \beta)|}; \quad \alpha, \beta \in GF(p^d)$$

$D(\alpha, \beta)$ is a $p^d \times p^d$ matrix with elements which are complex numbers. We compare the matrices $D(\alpha, \beta)$ with their counterparts $D(\alpha, \beta)$.

$$[\Pi_d D(\alpha, \beta) \Pi_d](n,m) = \{ D(\alpha, \beta)(n,m) \}^{\ell/d}$$

This shows that the elements of $D(\alpha, \beta)$ (with $\alpha, \beta \in GF(p^d)$) which have indices $(n,m)$ in $GF(p^d)$ are powers of the corresponding elements of $D(\alpha, \beta)$. Of course, the matrices $D(\alpha, \beta)$ (with $\alpha, \beta \in GF(p^d)$) have other elements also, with indices $(n,m)$ in the set $GF(p^d) - GF(p^d)$. Furthermore, there are more matrices $D(\alpha, \beta)$ with $\alpha, \beta$ in the set $GF(p^d) - GF(p^d)$, which do not enter in Eq. (69).

**C. Spectrum of displacement operators: an example**

$Z$ and $X$ are $p^d \times p^d$ matrices with $p$ distinct eigenvalues (the powers of $\omega$). Therefore there is a large degeneracy. For simplicity we discuss the spectrum of these matrices for the example of the field $GF(9)$ ($p = 3$ and $\ell = 2$). The elements of this field are $m_0 + m_1 \epsilon$ where $m_0, m_1 \in \mathbb{Z}_3$. They are defined modulo an irreducible polynomial which we choose to be $\epsilon^2 + \epsilon + 2$. Below we present matrices using the following order for their indices which are elements of $GF(9)$:

$$\{0, 1, 2, \epsilon, 1 + \epsilon, 2 + \epsilon, 1 + 2\epsilon, 2 + 2\epsilon\}$$

We consider as an example, the operator $Q_2$ defined in Eq. (14) and we use Eq. (60) to expand it as

$$Q_2 = \frac{1}{9} \sum_{\alpha \in GF(9)} \omega^{-2 \text{Tr}(\alpha)} Z^\alpha$$

$Z$ is the $9 \times 9$ diagonal matrix

$$Z = \text{diag}(1, \omega^2, \omega^2, \omega, 1, \omega, 1, \omega^2)$$

$$\omega = \exp\left(\frac{2\pi i}{3}\right)$$

Its eigenvalues are $1$, $\omega$ and $\omega^2$ and we call $q_0$, $q_1$ and $q_2$ the projection operators to the 3-dimensional eigenspaces corresponding to these eigenvalues. Then

$$Z = q_0 + \omega q_1 + \omega^2 q_2$$

$$q_0 = Q_0 + Q_{2+\epsilon} + Q_{1+2\epsilon}$$

$$q_1 = Q_2 + Q_{1+\epsilon} + Q_{2\epsilon}$$

$$q_2 = Q_1 + Q_\epsilon + Q_{2+2\epsilon}$$
It is clear from this that sums of ‘ordinary powers’ of $Z$ (i.e., powers in the field $\mathbb{Z}_3$) will give a combination of $q_0$, $q_1$ and $q_2$ and they will never give $Q_2$. In general, the purpose of going to larger fields is to overcome restrictions which we get in smaller fields. In the present context, powers of $Z$ in $GF(9)$ (defined through Eq. (49)), are able to give $Q_2$, as seen in Eq. (71). In the same spirit, we write the matrix $Z$ as

$$Z = \Omega_0 + \omega \Omega_1 + \omega^2 \Omega_2$$

$$\Omega_0 = Q_0 + Q_2 + Q_{2+e}$$

$$\Omega_1 = Q_2 + Q_{2+e} + Q_{2+2e}$$

$$\Omega_2 = Q_1 + Q_{1+e} + Q_{1+2e}$$

It is seen that $Z'$ is a combination of the projectors $\Omega_0$, $\Omega_1$, $\Omega_2$ which are different from the projectors $q_0$, $q_1$, $q_2$. Therefore the degeneracy in the matrices $Z$ and $X$ which might be viewed as an obstacle for expansions like Eq. (60), is not obstacle when we work with powers of these matrices in a larger field.

VI. SYMPLECTIC TRANSFORMATIONS AND THE $Sp(2,GF(p^\ell))$ GROUP

General symplectic transformations $S(r,s,t)$ perform by definition, the following unitary transformations:

$$S(r,s,t)Z^\alpha[S(r,s,t)]^\dagger = D(u\alpha, t\alpha) \equiv (Z')^\alpha$$

$$S(r,s,t)X^\beta[S(r,s,t)]^\dagger = D(s\beta, r\beta) \equiv (X')^\beta$$

$$ru - st = 1; \quad r,s,t,u \in GF(p^\ell)$$

These transformations preserve Eqs (49):

$$(X')^\beta(Z')^\alpha = (Z')^\alpha(X')^\beta \omega[-Tr(\alpha\beta)]; \quad \alpha, \beta \in GF(p^\ell)$$

The $X'$ and $Z'$ play the same role as the $X$ and $Z$ but in different directions in the $GF(p^\ell) \times GF(p^\ell)$ phase space (which is a finite geometry).

More generally, the displacement operators $D(\alpha, \beta)$ are transformed as follows:

$$S(r,s,t)D(\alpha, \beta)[S(r,s,t)]^\dagger = D(u\alpha + s\beta, t\alpha + r\beta) \equiv D'(\alpha, \beta)$$

It is seen that displacements by $(u\alpha + s\beta, t\alpha + r\beta)$ in the ‘old frame’ are also displacements by $(\alpha, \beta)$ in the ‘new frame’.

The transformations contain three independent variables; and the fourth variable is defined through the constraint. Since the variables belong to a field, for a given triplet $r,s,t$ (with $r \neq 0$) there exist $u = r^{-1}(st + 1)$ which satisfies the constraint. These transformations form the $Sp(2,GF(p^\ell))$ group.

Following we present the symplectic operators $S(r,s,t)$. We first give the matrix elements of three important special cases of symplectic operators. We also explain briefly some of their properties.

The elements of the first one are:

$$[S(\xi, 0, 0)](n,m) = \delta(\xi^{-1}n, m); \quad S(\xi_1, 0, 0)S(\xi_2, 0, 0) = S(\xi_1\xi_2, 0, 0)$$

$$[S(\xi, 0, 0)]^{p^\ell} = [S(\xi, 0, 0)]$$

These operators form a subgroup of $Sp(2,GF(p^\ell))$. 
The second special case of symplectic operators is
\[
[S(1, \xi, 0)](n, m) = \omega[\text{Tr}(2^{-1} \xi m^2)]\delta(n, m); \quad S(1, \xi_1, 0)S(1, \xi_2, 0) = S(1, \xi_1 + \xi_2, 0)
\]
\[
[S(1, \xi, 0)]^p = 1
\]
These operators also form a subgroup of \(Sp(2, GF(p^\ell))\).

The third special case of symplectic operators is
\[
S(1, 0, \xi) = FS(1, \xi, 0)F^\dagger; \quad [S(1, 0, \xi)](n, m) = p^{-\ell} \sum_n \omega[\text{Tr}(2^{-1} \xi \kappa^2 + \kappa n - knm)]
\]
\[
S(1, 0, \xi_1)S(1, 0, \xi_2) = S(1, 0, \xi_1 + \xi_2); \quad [S(1, 0, \xi)]^p = 1
\]
These operators also form a subgroup of \(Sp(2, GF(p^\ell))\).

The general symplectic operator \(S(r, s, t)\) is given by
\[
S(r, s, t) = S(1, 0, \xi_1)S(1, \xi_2, 0)S(\xi_3, 0, 0)
\]
\[
\xi_1 = rt(1 + st)^{-1}
\]
\[
\xi_2 = sr^{-1}(1 + st)
\]
\[
\xi_3 = r(1 + st)^{-1}
\]
Taking into account Eqs. (77), (78), (79) we calculate the matrix elements of \(S(r, s, t)\):
\[
[S(r, s, t)](n, m) = p^{-\ell} G(A) \omega[\text{Tr} B]; \quad G(A) = \sum_{k \in GF(p^\ell)} \omega[\text{Tr}(Ak^2)]
\]
\[
A = -2^{-1}(1 + st)^{-1}rt; \quad B = (2rt)^{-1}[(1 + st)n^2 - 2nmr + m^2r^2]
\]
\(G(A)\) is the Gauss sum related to \(GF(p^\ell)\)².

We can show that
\[
G^\lambda S(r, s, t)(G^\dagger)^\lambda = S(r^\lambda, s^\lambda, t^\lambda); \quad \lambda = 0, ..., \ell - 1
\]
If \(\alpha, \beta\) belong to the subfield \(GF(p^{\ell})\) then
\[
\alpha, \beta \in GF(p^{\ell}) \rightarrow G^d S(r, s, t)(G^\dagger)^d = S(r, s, t)
\]
We have seen in Eq. (12) that there is a simple relation between Fourier transforms in \(H\) and Fourier transforms in the component spaces \(\mathcal{H}\). The same is true about displacements in Eq. (58). There is no simple relation between symplectic transforms in \(H\) and symplectic transforms in the component spaces \(\mathcal{H}\). In order to explain this, we point out that Fourier transforms and displacements involve addition in \(GF(p^\ell)\) and the trace of a product. Addition in \(GF(p^\ell)\) is simply addition of \(\ell\)-dimensional vectors; and the trace of a product is expressed in a simple way in terms of the dual components in Eq. (12). Symplectic transformations involve multiplication in \(GF(p^\ell)\) (the products \(u \alpha, t \alpha, s \beta\) and \(r \beta\) in Eq. (13)), which is more complicated than addition and is defined modulo the irreducible polynomial of Eq. (12). This confirms again the difference between harmonic analysis on \(\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p\) and harmonic analysis on \(GF(p^\ell)\). The two are very different from each other, especially in situations where the product of two Galois numbers enters (like the symplectic transformations).

We consider an example which shows clearly that there is no simple relation between symplectic transforms in \(H\) and symplectic transforms in the component spaces \(\mathcal{H}\). It is the example with \(GF(9)\) considered earlier. As above polynomials are defined modulo \(\epsilon^2 + \epsilon + 2\). In this case we show that
\[
S(1, 1 + \epsilon, \epsilon)X^\ell[S(1, 1 + \epsilon, \epsilon)]^\dagger = D(2\epsilon, 1 + 2\epsilon)
\]
Taking into account that $X^\epsilon = 1 \otimes X$ and also that in the dual basis $2\epsilon = E_0$ we show that

$$D(2\epsilon, 1 + 2\epsilon) = D(1, 1) \otimes D(0, 2) \quad (85)$$

Therefore the $S(1, 1 + \epsilon, \epsilon)$ can not be the tensor product of two symplectic transformations $S_1 \otimes S_2$ acting on $\mathcal{H} \otimes \mathcal{H}$ (the $S_1 S_1^\dagger$ cannot give the $D(1, 1)$).

We note that results like Eq. (65) have been derived with respect to the two particular directions in phase space (which we might call position-momentum or time-frequency). Acting with $S(r, s, t)$ on both sides of Eq. (65) we transform them to other directions of the $GF(p^f) \times GF(p^f)$ phase space which is a finite geometry:

$$\frac{1}{p^f} \sum_{\alpha \in GF(p^f)} D'(\alpha, \beta) = Q'_{-2^{-1}\beta}$$

$$\frac{1}{p^f} \sum_{\beta \in GF(p^f)} D'(\alpha, \beta) = \tilde{Q}'_{2^{-1}\alpha} \quad (86)$$

Here

$$Q'_{-2^{-1}\beta} = S(r, s, t) Q_{-2^{-1}\beta} S(r, s, t)^\dagger$$

$$\tilde{Q}'_{2^{-1}\alpha} = S(r, s, t) \tilde{Q}_{2^{-1}\alpha} S(r, s, t)^\dagger \quad (87)$$

are projection operators as in Eq. (14) but with respect to the ‘new frame’. This property shows that the phase space is isotropic. This isotropy is of course discrete in the present context.

VII. DISCUSSION

An important aspect of the theory of Galois fields is the relationship of the ‘large’ field with its subfields. In the present context, we have studied various aspects of harmonic analysis (and quantum mechanics) on $GF(p^f)$ and its relationship to harmonic analysis on the subfield $GF(p^d)$. The Frobenius transformations of Eqs. (34), (35) play a central role in our study and lead to the Galois groups of Eqs. (37), (42).

We first studied Fourier transforms and expressed in Eq. (26) the Fourier transform in $H$ in terms of Fourier transforms in the component spaces $\mathcal{H}$. We have also discussed Fourier transforms in $H_d$ in Eq. (33) and the spectrum of Fourier transforms in Eq. (37).

We next studied displacements and expressed in Eq. (58) displacements in $H$ in terms of displacements in the component spaces $\mathcal{H}$. The action of Frobenius transformations on displacements has been discussed in Eqs. (54), (55), (56). An expansion of an arbitrary operator in terms of displacement operators with the Weyl (or ambiguity) functions as coefficients, has been given in Eq. (60). A generalized resolution of the identity that involves displacements has been given in Eq. (62). The marginal properties of displacements have been given in Eq. (65). We have also discussed displacements in $H_d$ in Eq. (67). The spectrum of displacements has been discussed for a particular example which however exemplifies the general features.

General symplectic transformations have been given in Eq. (80), (81). The action of Frobenius transformations on symplectic transformations has been discussed in Eqs. (82), (83). The phase space is isotropic and acting with symplectic operators on various properties we get analogous properties in a different frame. We have seen an example of this in Eqs. (65), (86).

The work uses algebraic concepts from field extension in the context of quantum mechanics and harmonic analysis.
VIII. APPENDIX

We consider the example with $GF(9)$ discussed earlier. As above polynomials are defined modulo $\epsilon^2 + \epsilon + 2$. We calculate the matrix $G(n,m)$ and its eigenvalues and eigenvectors. There are six eigenvectors corresponding to the eigenvalue 1 and we call $\varpi_0$ the corresponding projection operator. There are three eigenvectors corresponding to the eigenvalue $-1$ and we call $\varpi_1$ the corresponding projection operator. They are:

\[
\varpi_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
\end{pmatrix}
\]

\[
\varpi_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & -0.5 \\
0 & 0 & 0 & -0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\
0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
\end{pmatrix}
\]

According to Eq (82)

\[
G = \varpi_0 - \varpi_1; \quad \varpi_0 + \varpi_1 = 1; \quad \varpi_0 \varpi_1 = 0
\]

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