ON EXTENSIONS OF REPRESENTATIONS FOR COMPACT LIE GROUPS

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ABSTRACT. Let $H$ be a closed normal subgroup of a compact Lie group $G$ such that $G/H$ is connected. This paper provides a necessary and sufficient condition for every complex representation of $H$ to be extendible to $G$, and also for every complex $G$-vector bundle over the homogeneous space $G/H$ to be trivial. In particular, we show that the condition holds when the fundamental group of $G/H$ is torsion free.

1. INTRODUCTION

One of the classical problems in finite group theory is to characterize extensions of representations. We mean an extension of a representation in the following way: Given a normal subgroup $H$ of a group $G$, a (complex) representation $\rho: H \to \text{GL}(n, \mathbb{C})$ is called extendible to $G$ if there exists a representation $\tilde{\rho}: G \to \text{GL}(n, \mathbb{C})$ (called a $G$-extension) such that $\rho = \tilde{\rho}$ on $H$. It is to be noted that the dimension $n$ is not changed, since $\rho$ as a sub-representation is always contained in the restriction of the induced representation of $\rho$ to $H$.

In the case of finite $G$, it is well known that every complex irreducible representation of $H$, which is $G$-invariant under conjugation (see Section 2 for the definition), is extendible to $G$ if the second group cohomology $H^2(G/H, \mathbb{C}^*)$ vanishes \cite{Isa76, Theorem 11.7}. On the other hand the extension problem for infinite groups has not been extensively studied. In this article we study the problem for compact Lie groups when $G/H$ is connected. Our main result is a necessary and sufficient condition for every complex representation of $H$ to be extendible to $G$. It is also shown that the condition is related to a topological invariant, the fundamental group of $G/H$.

For any group $G$, let $G'$ denote the commutator subgroup of $G$.

Theorem 1.1. Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is connected. Then every complex representation of $H$ is extendible to $G$ if and only if $H$ is a direct summand of $G'/H$.

Corollary 1.2. Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is connected. Then every complex representation of $H$ is extendible to $G$.
if the fundamental group $\pi_1(G/H)$ is torsion free, or equivalently if $(G/H)'$ is simply connected.

Our theorem provides a complete characterization of the triviality of complex $G$-vector bundles over the homogeneous space $G/H$. Let $E$ be a complex $G$-vector bundle over $G/H$. We recall that $E$ is trivial if it is isomorphic to the product bundle $G/H \times V$ for some complex $G$-module $V$. Since $E$ is uniquely determined by the fiber at the identity element of $G/H$ (say $E_0$), the bundle $E$ is trivial if and only if $E_0$ as a complex representation of $H$ is extendible to $G$. Theorem \cite{Seg68} leads us to the following corollary.

**Corollary 1.3.** Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is connected. Then every complex $G$-vector bundle over the homogeneous space $G/H$ is trivial if and only if $H$ is a direct summand of $G'H$. □

The existence of $G$-extensions plays an important role even in equivariant $K$-theory. Let $X$ be a connected topological space with a compact Lie group $G$ action. Let $H$ be the normal subgroup of $G$ which consists of all elements of $G$ acting trivially on $X$. Then the projection $G \to G/H$ induces the canonical homomorphism $\phi: K_{G/H}(X) \to K_G(X)$ which sends a $G/H$-vector bundle over $X$ to the same bundle viewed as a $G$-vector bundle with the trivial $H$-action.

On the other hand, suppose that every complex irreducible representation of $H$ is extendible to $G$. Then there is an injective group homomorphism $e: R(H) \to R(G)$ between two representation rings defined as follows. For each irreducible complex $H$-module $U$ choose a $G$-extension $U_G$, and define $e([U]) = [U_G]$ where $[\ ]$ denote the classes in the representation rings. Then extend the definition of $e$ to $R(H)$ so that it defines a homomorphism $R(H) \to R(G)$. For each complex $G$-module $V$ we can associate the trivial complex $G$-vector bundle $V' = X \times V$, which defines the natural homomorphism $t: R(G) \to K_G(X)$. We now define a group homomorphism

$$\mu: R(H) \otimes K_{G/H}(X) \to K_G(X), \quad (V, \xi) \mapsto t \circ e(V) \otimes \phi(\xi). \tag{1}$$

This homomorphism is an isomorphism. Indeed, the inverse is given as follows. Let $\text{Irr}(H)$ denote the set of all isomorphism classes of complex irreducible representations of $H$. For each $[\chi] \in \text{Irr}(H)$ choose a $G$-extension of $\chi$, and let $V_\chi$ be the corresponding $G$-module to the chosen $G$-extension. For a complex $G$-vector bundle $E$ over $X$, the canonical isomorphism

$$E \overset{\cong}{\rightarrow} \bigoplus_{[\chi] \in \text{Irr}(H)} V_\chi \otimes \text{Hom}_H(V_\chi, E)$$

induces a group homomorphism $K_G(X) \to R(H) \otimes K_{G/H}(X)$ which is the desired inverse (see \cite{CKMS93} Section 2 for more general arguments). Therefore we have a generalization of Proposition 2.2 in \cite{Seg68} which deals with the extreme case when $G$ acts trivially on $X$.

**Corollary 1.4.** Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is connected. Let $X$ be a connected $G$-space such that $H$ acts trivially on $X$. If $H$ is a direct summand of $G'H$, then the map $\mu: R(H) \otimes K_{G/H}(X) \to K_G(X)$ in \cite{Seg68} can be defined, and it is a group isomorphism. □

This article is organized as follows. In Section 2 we shall give some basic notions and then show that a complex irreducible representation of $H$, which is $G$-invariant under
conjugation, induces an associated projective representation of $G$ which may be viewed as a $G$-extension in the projective representation level. Section 3 is devoted to prove that every complex representation of $H$ has a $G$-extension when $G/H$ is connected and abelian. In Section 4 we shall proceed the study in the case that $G/H$ is semisimple and connected. After showing that the extension problem can be reduced to this case, we shall prove Theorem 1.1.

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2. Associated projective representations

Let $G$ be a topological group and $H$ a closed normal subgroup of $G$. By a (complex) representation of $G$ we shall mean a continuous homomorphism of $G$ into the general linear group $GL(n, \mathbb{C})$ of nonsingular $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. A representation $\rho: H \rightarrow GL(n, \mathbb{C})$ is called extendible to $G$ if there exists a representation $\tilde{\rho}: G \rightarrow GL(n, \mathbb{C})$ (called a $G$-extension of $\rho$) such that $\rho(h) = \tilde{\rho}(h)$ for all $h \in H$.

Moreover, it is enough to get a $G$-extension of $\rho$ that there is a representation $\tilde{\rho}: G \rightarrow GL(n, \mathbb{C})$ such that its restriction to $H$ is isomorphic (or similar) to $\rho$, i.e., there exists a matrix $M \in GL(n, \mathbb{C})$ such that $M^{-1}\tilde{\rho}(h)M = \rho(h)$ for all $h \in H$.

Given a representation $\rho: H \rightarrow GL(n, \mathbb{C})$ the map $^g\chi: H \rightarrow \mathbb{C}$ defined by the conjugation $^g\chi(h) = \chi(g^{-1}hg)$ becomes a representation of $H$ for each $g \in G$. We say that $\rho$ is $G$-invariant if it is isomorphic to the conjugate representation $^g\rho$ for all $g \in G$, which is a necessary condition of $\rho$ to be extendible to $G$.

In the following we assume that a representation $\rho: H \rightarrow GL(n, \mathbb{C})$ is irreducible and $G$-invariant. Then there exists a matrix $M_g \in GL(n, \mathbb{C})$ for each $g \in G$ such that $M_g^{-1}\rho(h)M_g = ^g\rho(h) = \rho(g^{-1}hg)$ for all $h \in H$. Since $\rho$ is irreducible, the Schur’s lemma implies that $M_g$ is unique up to multiplication by nonzero constant in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. So we are able to define a function $\rho^*: G \rightarrow GL(n, \mathbb{C})/\mathbb{C}^*$ by $\rho^*(g) = [M_g]$ for each $g \in G$, where $[M_g]$ denotes the image of $M_g$ by the canonical projection $\pi: GL(n, \mathbb{C}) \rightarrow PGL(n, \mathbb{C})$.

Let $G$ be a topological group and $H$ a compact normal subgroup of $G$. Given a complex irreducible representation $\rho: H \rightarrow GL(n, \mathbb{C})$ which is $G$-invariant, the function $\rho^*: G \rightarrow PGL(n, \mathbb{C})$ defined above is a continuous homomorphism, called the projective representation of $G$ associated with $\rho$. Moreover, the image of $\rho^*$ is contained in $U(n)/S^1 \subset PGL(n, \mathbb{C})$ if $\rho$ is a unitary representation of $H$.

Lemma 2.1.
Proof. It is immediate that $\rho^*$ is a homomorphism. Since $H$ is compact we may assume that $\rho$ is a unitary representation of $H$, i.e., the image of $\rho$ is contained in the unitary group $U(n)$. Then $M_g$ is a constant multiple of a matrix in $U(n)$ so that $\rho^*(g)$ is contained in $U(n)/S^1$ for all $g \in G$. For the continuity of $\rho^*$ it suffices to show that the graph of $\rho^*$ in $G \times \text{PGL}(n, \mathbb{C})$ is closed, since $U(n)/S^1$ is a compact Hausdorff space.

Consider the family of continuous maps $\Phi_h: G \times \text{GL}(n, \mathbb{C}) \to \text{GL}(n, \mathbb{C})$ for each $h \in H$ given by $(g, M) \mapsto \rho(h)M\rho(g^{-1}hg)^{-1}M^{-1}$. Then the set

$$\bigcap_{h \in H} \Phi_h^{-1}(I) = \bigcup_{g \in G} \{(g, M) \in G \times \text{GL}(n, \mathbb{C}) \mid M \in \pi^{-1}(\rho^*(g))\},$$

is the inverse image of the graph of $\rho^*$ in $G \times \text{PGL}(n, \mathbb{C})$ by the canonical projection $1 \times \pi: G \times \text{GL}(n, \mathbb{C}) \to G \times \text{PGL}(n, \mathbb{C})$, which is obviously closed in $G \times \text{GL}(n, \mathbb{C})$. Therefore the graph of $\rho^*$ is also closed in $G \times \text{PGL}(n, \mathbb{C})$.

We may say that $\rho$ is extendible to $G$ in the projective representation level, since $\rho^*(h) = [\rho(h)]$ for all $h \in H$, i.e., $\rho^* = \pi \circ \rho$ on $H$.

$$\begin{array}{ccc}
H & \xrightarrow{\rho} & \text{GL}(n, \mathbb{C}) \\
\downarrow & & \downarrow \pi \\
G & \xrightarrow{\rho^*} & \text{PGL}(n, \mathbb{C})
\end{array}$$

Note that any $G$-extension (if exists) $\tilde{\rho}$ of $\rho$ is a lifting homomorphism of $\rho^*$, i.e., $\rho^* = \pi \circ \tilde{\rho}$, since $\rho^*(g) = [\tilde{\rho}(g)]$ for all $g \in G$.

Remark. In case that $G$ is finite, choose a transversal $T$ containing $e$ for $H$ in $G$ and set $M_e = I$, the identity matrix in $\text{GL}(n, \mathbb{C})$. For each $t \in T$ and $h \in H$, the map $\rho': G \to \text{GL}(n, \mathbb{C})$ sending $th \mapsto M_t\rho(h)$ is a lifting (not necessarily homomorphism) of $\rho^*$, i.e., $\pi \circ \rho' = \rho^*$, and it determines a cocycle $\beta$ in the second group cohomology $H^2(G/H, \mathbb{C}^*)$, which depends only on $\rho$. Moreover, $\rho$ is extendible to $G$ if and only if $\beta$ is trivial, see [Isa76] Theorem 11.7 for more details.

3. Extensions when $G/H$ is connected abelian

In this section we shall prove that every complex representation of $H$ is extendible to $G$ when $G/H$ is compact, connected, and abelian, that is a torus. We begin with a general result on extensions of representations in the special case when $G = SH$ for some closed subgroup $S$ of $G$.

Lemma 3.1. Let $G$ be a compact topological group such that $G = SH$ for a closed subgroup $S$ and a closed normal subgroup $H$ of $G$. Then a complex representation $\rho: H \to \text{GL}(n, \mathbb{C})$ is extendible to $G$ if and only if there exists a representation $\varphi: S \to \text{GL}(n, \mathbb{C})$ such that

1. $\varphi = \rho$ on $S \cap H$, and
2. $\varphi(s^{-1}\rho(h))\varphi(s) = \rho(s^{-1}hs)$ for all $s \in S$ and $h \in H$.

Proof. The necessity is obvious so we prove the sufficiency. Define a function $\tilde{\rho}: G \to \text{GL}(n, \mathbb{C})$ by $\tilde{\rho}(sh) = \varphi(s)\rho(h)$ for $s \in S$ and $h \in H$. It is immediate that $\tilde{\rho} = \rho$ on $H$. In this proof we shall use the symbols $s, s'$ and $h, h'$ for elements in $S$ and $H$, respectively.
for each $g \in S$ every connected component of $G$ contains an element of $S$. Then the path $g_t$ induces a continuous family of conjugate representations $g^\ast \rho$ so that all representations $g^\ast \rho$ are isomorphic (see [CR63] Lemma 38.1 for more general result). In particular, $g^\ast \rho = g_0^\ast \rho$ and $\rho = h^\ast \rho = g_0^\ast \rho$ are isomorphic.

Let $\rho$ be a complex irreducible representation of $H$. Since $\rho$ is always $G$-invariant, the associated projective representation $\rho^\ast$ exists by Lemma 2.1. To get a $G$-extension of $\rho$ we shall first find a closed subgroup $S$ of $G$ such that $G = SH$, and then construct a lifting homomorphism $\varphi$ of $\rho^\ast$ over $S$ (so that the condition (2) is satisfied). Finally modifying $\varphi$ a little to satisfy the condition (1) we may get a $G$-extension of $\rho$.

**Lemma 3.2.** Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H \cong S^1$. Then there exists a circle subgroup $S$ of $G$ such that $G = SH$ and $S \cap H$ is finite cyclic.

**Proof.** Let $G_0$ denote the identity component of $G$. Since the canonical projection $p: G \to G/H$ is open and closed, $p(G_0)$ is a connected component of $G/H$ so that $p(G_0) = G/H$. It is well known in Lie group theory [HM98, Theorem 6.15] that $G_0 = Z_0 G'_0$, where $Z_0$ is the identity component of the center of $G_0$, which is a torus and $G'_0$ is the commutator subgroup of $G_0$. Then $G'_0 \subset G_0 \cap H \subset H$ since $G/H = G_0/(G_0 \cap H)$
is abelian, and thus $p(Z_0) = G/H$. Using the isomorphism $G/H \cong U(1)$ we may view $p|_{Z_0}$ as a one-dimensional unitary representation of the torus $Z_0$. It is elementary in representation theory that there exists a circle subgroup $S \subset Z_0$ such that $p(S) = G/H$. Therefore $G = SH$ and, furthermore, the proper subgroup $S \cap H$ of the circle group $S$ is finite cyclic.

Lemma 3.3. Let $T$ be a maximal torus in $U(n)$. Then the exact sequence $0 \to S^1 \to T \to T/S^1 \to 0$ splits. Here $S^1$ is identified with the subgroup of $U(n)$ consisting of constant multiples $zI$ for $z \in S^1 \subset \mathbb{C}$ where $I$ denotes the identity matrix.

Proof. Since any maximal torus $T$ in $U(n)$ is conjugate to the subgroup $\Delta(n) \subset U(n)$ of diagonal matrices $D(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}, \quad z_i \in S^1,$ it suffices to show that the exact sequence $0 \to S^1 \to \Delta(n) \to \Delta(n)/S^1 \to 0$ splits. But the splitting is immediate because of the homomorphism $\Delta(n) \to S^1$ mapping a diagonal matrix $D(z_1, \ldots, z_n)$ to the constant multiple $z_1 I \in S^1$. □

Proposition 3.4. Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H \cong S^1$. Then every complex representation of $H$ is extendible to $G$.

Proof. Let $\rho: H \to \text{GL}(n, \mathbb{C})$ be a given representation. Since $H$ is compact, we may assume that all the images of $\rho$ are contained in $U(n) \subset \text{GL}(n, \mathbb{C})$. Moreover, it is enough to prove the case that $\rho$ is irreducible. Since $G/H \cong S^1$ is connected, $\rho$ is $G$-invariant so that the associated projective representation $\rho^*: G \to U(n)/S^1 \subset \text{PGL}(n, \mathbb{C}) = \text{GL}(n, \mathbb{C})/\mathbb{C}^*$ exists by Lemma 2.1. From Lemma 3.2 we can choose a circle subgroup $S$ of $G$ such that $G = SH$ and $S \cap H$ is finite cyclic.

We shall find a lifting homomorphism $\varphi_0: S \to U(n)$ of $\rho^*$ over $S$. Since $\rho^*(S)$ is compact, connected, and abelian, it is a torus in $U(n)/S^1$. Note that every maximal torus in $U(n)/S^1$ has the form $T/S^1$ for some maximal torus $T$ of $U(n)$ [BtDS5, Theorem 2.9, Chapter IV]. Choose a maximal torus $T$ of $U(n)$ such that $\rho^*(S) \subset T/S^1$. By Lemma 3.3 the exact sequence $0 \to S^1 \to T \to T/S^1 \to 0$ splits, i.e., the canonical projection $\pi: T \to T/S^1$ has a continuous section (homomorphism) $s: T/S^1 \to T$ such that the composition $\pi \circ s$ is the identity map of $T/S^1$. Then $\varphi_0 = s \circ \rho^*|_{S}$ is a desired lifting homomorphism of $\rho^*$ over $S$.

Let $t_0$ denote a generator of the finite cyclic group $S \cap H$. Since $\pi \circ \varphi_0 = \rho^* = \pi \circ \rho$ on $S \cap H$, $\varphi_0(t_0) = \xi \rho(t_0)$ for some constant $\xi \in S^1 \subset \mathbb{C}^*$. Note that $\xi$ is an $n$-th root of unity, where $n$ is the order of $S \cap H$. So it is possible to choose a one-dimensional unitary representation $\tau$ of the circle group $S$ such that $\tau(t_0) = \xi^{-1}$. Then the unitary representation $\varphi = \tau \otimes \varphi_0$ satisfies the conditions (1) and (2) in Lemma 3.1. □
Corollary 3.5. Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is connected and abelian. Then every complex representation of $H$ is extendible to $G$.

Proof. Since $G/H$ is compact, connected, and abelian, it is isomorphic to a torus. So we have a finite chain of subgroups

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_{n-1} \lhd H_n = G$$

such that $H_i$ is normal in $H_{i+1}$ and $H_{i+1}/H_i \cong S^1$. Applying Proposition 3.4 inductively, any representation of $H$ is extendible to $G$. □

4. Extensions when $G/H$ is connected

In this section we consider the general case, so $G/H$ will be assumed to be connected (not necessarily abelian). In this case the commutator subgroup $(G/H)' = G'H/H$ of $G/H$ is semisimple connected [HM98, Theorem 6.18]. The following proposition reduces the extension problem to the case that $G/H$ is semisimple and connected.

Proposition 4.1. Let $G$ be a compact Lie group and $H$ a closed normal subgroup of $G$ such that $G/H$ is connected. A complex representation of $H$ is extendible to $G$ if and only if it is extendible to $G'H$.

Proof. The necessity is obvious, and the sufficiency follows from Corollary 3.5 since the factor group $G/G'H \cong (G/H)/(G'H/H) = (G/H)/(G/H)'$ is compact, connected, and abelian, that is a torus. □

In the case that $G/H$ is semisimple connected, the following result is well known in Lie group theory (see for instance, [HM98, Proposition 6.14]).

Lemma 4.2. Let $G$ be a compact Lie group and $H$ a closed normal subgroup such that $G/H$ is semisimple and connected. Then there is a semisimple connected closed normal subgroup $S$ in $G$ such that $G = SH$ and the map $S \times H \to G$ sending $(s,h) \mapsto sh$ is a homomorphism with a discrete kernel isomorphic to $S \cap H$. □

Remark. Proposition 6.14 in [HM98] deals with the case when $G$ is connected. However, the same proof holds even if $G$ is not connected, since $G/H$ is connected. Moreover, we can find the fact in the proof that $S$ is semisimple and connected.

The following result implies that the existence of a $G$-extension when $G/H$ is semisimple and connected is completely determined by the restriction of a given representation to $S \cap H$.

Proposition 4.3. Under the hypotheses of Lemma 4.2, a complex irreducible representation $\rho$ of $H$ is extendible to $G$ if and only if $\rho$ is trivial on $S \cap H$, i.e., $\rho(g) = I$, the identity matrix, for all $g \in S \cap H$.

Proof. It is immediate that $S$ commutes with $H$, since the map $S \times H \to G$ sending $(s,h) \mapsto sh$ is a homomorphism. To prove the sufficiency, it is enough to choose the trivial representation $\varphi$ of $S$, i.e., $\varphi(s) = I$ for all $s \in S$. Since $S$ commutes with $H$, the two conditions (1) and (2) in Lemma 3.1 are satisfied immediately.

On the other hand, suppose $\tilde{\rho}$ is a $G$-extension of $\rho$. Since $S$ commutes with $H$, we have $\tilde{\rho}(s)^{-1}\rho(h)\tilde{\rho}(s) = \rho(h)$ for all $s \in S$ and $h \in H$. Then the Schur’s lemma implies that $\tilde{\rho}(s)$ is constant for all $s \in S$, so we may view the restriction $\tilde{\rho}|_S$ as a
one-dimensional complex representation of \( S \). Since semisimple Lie groups have no nontrivial abelian factor group, the trivial representation is the unique one-dimensional complex representation of \( S \). Therefore, \( \tilde{\rho} \) is trivial on \( S \), in particular, on \( S \cap H \). \( \square \)

**Remark.** Note that the number of \( G \)-extensions (if exist) is exactly one, since every \( G \)-extension should be trivial on \( S \).

**Corollary 4.4.** Let \( G \) be a compact Lie group and \( H \) a closed normal subgroup such that \( G/H \) is semisimple and connected. Every complex representation of \( H \) is extendible to \( G \) if and only if \( H \) is a direct summand of \( G \), i.e., \( G \cong S \times H \) for some subgroup \( S \) of \( G \).

**Proof.** The sufficiency is obvious so we prove the necessity. If \( H \) is not a direct summand of \( G \), then \( S \cap H \) in Lemma 4.2 contains a nontrivial element, say \( s_0 \). Since a faithful representation of \( H \) always exists [BtD85, Theorem 4.1, Chapter III], we can choose an irreducible sub-representation \( \rho \) of \( H \) such that \( \rho(s_0) \) is not trivial. Then \( \rho \) does not extend to a representation of \( G \) by Proposition 4.3. \( \square \)

We shall now prove the main result in this paper. For the second statement of Theorem 1.1, we need the following lemma giving a relation between the normal subgroup \( S \cap H \) in Lemma 4.2 and the fundamental group of \( G/H \).

**Lemma 4.5.** Under the hypotheses of Lemma 4.2, there exists a surjective homomorphism \( \pi_1(G/H) \to S \cap H \).

**Proof.** Since \( S/(S \cap H) = G/H \), the restriction of the canonical projection \( p: G \to G/H \) on \( S \) is surjective and its kernel \( S \cap H \) is discrete. It follows that \( p|_S \) is a covering homomorphism of \( G/H \). From the uniqueness of the universal covering homomorphism \( \tilde{q}: \tilde{G}/H \to G/H \), there exists a covering homomorphism \( q: \tilde{G}/H \to S \) such that the diagram

\[
\begin{array}{ccc}
\tilde{G}/H & \xrightarrow{q} & S \\
\downarrow{\tilde{q}} & & \downarrow{p|_S} \\
G/H & & \end{array}
\]

commutes (compare with [HM98, Proposition 9.12]). Since \( S \cap H = \ker p|_S = q(\ker \tilde{q}) \) and \( \ker \tilde{q} \) is isomorphic to \( \pi_1(G/H) \), we have a surjective homomorphism of \( \pi_1(G/H) \) onto \( S \cap H \). \( \square \)

**Theorem 1.1** (rephrased). Let \( G \) be a compact Lie group and \( H \) a closed normal subgroup such that \( G/H \) is connected. Then every complex representation of \( H \) is extendible to \( G \) if and only if \( H \) is a direct summand of \( G'H \).

**Proof.** Since the factor group \( G'H/H = (G/H)' \) is semisimple and connected, the theorem follows immediately from Proposition 4.1 and Corollary 4.3. \( \square \)

**Proof of Corollary 1.2.** We claim that \( \text{Tor}(\pi_1(G/H)) \), the torsion subgroup of \( \pi_1(G/H) \), is isomorphic to \( \pi_1((G/H)') \). Denote by \( T \) the torus \( (G/H)/(G/H)' \). Then the homotopy exact sequence of the fibration \( (G/H)' \to G/H \to T \) implies that \( \pi_1(G/H) \cong \pi_1((G/H)') \oplus \pi_1(T) \), since the second homotopy group of a compact Lie group vanishes, see [BtD85, Proposition 7.5, Chapter V]. Since \( (G/H)' \) is semisimple, \( \pi_1((G/H)') \) is finite [BtD85, Remark 7.13, Chapter V] so that it is isomorphic to \( \text{Tor}(\pi_1(G/H)) \) as...
we claimed. Therefore, the condition of $\pi_1(G/H)$ being torsion free is equivalent to $(G/H)'$ being simply connected.

By Lemma 4.2 and 4.5, $G'H = SH$ for some semisimple connected closed normal subgroup $S$ in $G'H$ and there is a surjective homomorphism $\pi_1(G'H/H) = \pi_1((G/H)') \to S \cap H$. Therefore, if $(G/H)'$ is simply connected, then $\pi_1((G/H)') = S \cap H = \{e\}$ so that $H$ is a direct summand of $G'H$. □

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