THE EDGE-ISOPERIMETRIC PROBLEM ON SIERPINSKI GRAPHS

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Abstract. Some families of graphs, such as the $n$-cubes and Sierpinski gaskets are self-similar. In this paper we show how such recursive structure can be used systematically to prove isoperimetric theorems.

1. Introduction

1.1. Background & Motivation. The Sierpinski gasket is a topological curiosity (fractal and self-similar). According to Wikipedia, ”It is named after the Polish mathematician Wacław Sierpiński but appeared as a decorative pattern many centuries prior to the work of Sierpiński”. To construct it recursively,

0: Start with an equilateral triangle of side 1 (the interior as well as the boundary).

$n+1$: After round $n \geq 0$, there remain $3^n$ congruent equilateral triangles of side $1/2^n$. In each triangle subdivide each edge at its midpoint and connect the midpoints to make 4 triangles of side $1/2^{n+1}$. Remove the interior of the central one, leaving its boundary. The Sierpinski gasket is the limit (set) of this process as $n \to \infty$ (see Wikipedia).

The Sierpinski Gasket Graph, $SG_n$, is the boundary of the set remaining after $n$ rounds of the Sierpinski gasket construction (above). The vertices of $SG_n$ are the vertices of the constituent triangles and its edges are the edges of those triangles. $SG_n$ may be defined recursively as follows: $SG_0 = K_3$, the complete graph on 3 vertices. If $SG_n$ has been defined for $n \geq 0$, then $SG_{n+1}$ may be constructed from 3 copies of $SG_n$, each copy sharing one corner vertex with each of the other two copies. In most papers on the Sierpinski gasket graph it is denoted by $S_n$. We use $SG_n$ because in our context $S$ is already used for several other structures.

$SP_n$, a 3-dimensional analog of $SG_n$, is proposed in [14] as a connection architecture for multiprocessing computers. They call it the "Sierpinski gasket pyramid network" but it is also known as the Sierpinski sponge. The best-known multiprocessor architecture is $Q_n$, the graph of the $n$-dimensional cube. The properties of $Q_n$ relevant to its employment in computer architecture have been well studied. [14] begins the analysis of $SP_n$ by showing that (among other things)

1. $SP_n$ has diameter $2^{n-1}$.
2. $SP_n$ has chromatic number 4.
3. $SP_n$ is hamiltonian.

Date: December 9, 20015.

2000 Mathematics Subject Classification. Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Sierpinski graphs, fractal, self-similar.
In their conclusion the authors propose studying $SP_n$ for its "message routing and broadcasting" properties. This paper is following up on that suggestion. The Edge-Isoperimetric Problem (EIP) (see [3]) is of interest for connection graphs of multiprocessoring computers because it has implications for message routing and broadcasting. Other authors ([5]) had previously proposed graphs related to $SG_n$ & $SP_n$ for computer architecture but did not consider their EIP.

1.2. Definitions & Examples.

1.2.1. Graphs.

**Definition 1.** An ordinary graph, $G = (V, E)$ consists of a set $V$, of vertices and a set $E \subseteq \binom{V}{2} = \{\{v, w\} : v, w \in V, v \neq w\}$, of pairs of vertices called edges.

**Example 1.** $K_n$, the complete graph on $n$ vertices has $V_{K_n} = \{0, 1, 2, ..., n - 1\}$ and $E_{K_n} = \binom{V_{K_n}}{2}$.

**Example 2.** The (disjunctive) product, $K_m \times K_m \times ... \times K_m = K_m^n$ is called the Hamming graph. $V_{K_m^n} = \{0, 1, 2, ..., n - 1\}^n$. Two vertices ($n$-tuples of vertices of $K_m$) have an edge between them if they differ in exactly one coordinate (i.e. are at Hamming distance 1). Note that $K_2^n = Q_n$, the graph of the $n$-dimensional cube.

1.2.2. The Edge-Isoperimetric Problem. The Edge-Isoperimetric Problem (EIP) is a combinatorial analog of the classical isoperimetric problem: Given a graph, $G = (V, E)$ and $S \subseteq V$, $$\Theta(S) = \{\{v, w\} \in E : v \in S \& w \notin S\}$$ is called the edge-boundary of $S$. Then the EIP is to calculate $|\Theta|(G; \ell) = \min \{|\Theta(S)| : S \subseteq V, |S| = \ell\}$ for every integer $\ell$, $0 \leq \ell \leq |V|$, and identify sets that achieve the minimum. The function $|\Theta|(G; \ell)$ is called the (edge-)isoperimetric profile of $G$.

**Example 3.** For $K_m$, the complete graph on $m$ vertices, any $S \subseteq V_{K_m}$ with $|S| = \ell$ has $|\Theta(S)| = \ell (m - \ell)$. Thus every $\ell$-set is a solution of the EIP for $K_m$. Also, its isoperimetric profile is $|\Theta|(K_m; \ell) = \ell (m - \ell)$.

**Example 4.** Initial $\ell$-segments of $V_{K_m^n}$ in Lexicographic order,

$$\{0^n, 0^{n-1}, ..., \ell_1 \ell_2 ... \ell_m\}$$

where $\ell = 1 + \sum_{i=1}^{m} \ell_i m^{n-i}$, are solutions of the EIP on $K_m^n$ (proved for $m = 2$ by the author in 1962 and for $m > 2$ by John Lindsay in 1963). Ching Guu [3] pointed out that if we divide the isoperimetric profile of the $n$-cube, $Q_n = K_2^n$ by $2^n$ we have

$$|\Theta|(Q_n; \ell) / 2^n = T(\ell/2^n),$$

$T : [0, 1] \rightarrow [0, 1]$ being the celebrated Takagi function (see Lagarias’s survey [10]).

The property of having a numbering, $\eta : V \rightarrow \{1, 2, ..., |V|\}$, 1-1 & onto, whose initial $\ell$-segments, $\eta^{-1}(\{1, 2, ..., \ell\})$, are solutions of the EIP is called nested solutions.
Example 5. The classical isoperimetric problem in the plane has nested solutions, concentric discs. \( \eta : \mathbb{R}^2 \to \mathbb{R}^+ \) is defined by \( \eta(x, y) = \pi (x^2 + y^2). \) For every \( a \geq 0, \eta^{-1}([0, a]) \) is a disc of area \( a \) and radius \( r = \sqrt{\frac{a}{\pi}} \) centered at \((0, 0)\). The length of the boundary of the disc is \( \lambda = 2\pi r = 2\sqrt{\pi a}. \) This function, \( \lambda(a) \), giving the minimum length of the boundary of any set, \( S \subseteq \mathbb{R}^2 \), of area \( a \), is the isoperimetric profile of \( \mathbb{R}^2 \) (wrt the Euclidean metric).

2. Results on \( SG_n \)

2.1. \( SG_1 \) and \( SG_2 \) have Nested Solutions for \( EIP \). For \( n = 1 \) the result is trivial since \( SG_1 = K_3 \) all numberings are equivalent under symmetry and all \( \ell \)-sets achieve \( \min \{ |\Theta(S)| : S \subseteq V_{K_3}, |S| = \ell \} \).

For \( n = 2 \) we apply stabilization to simplify the problem (See [4], Chapter 3): Figure 1 shows a diagram of \( SG_2 \) with basic reflections \( R_0, R_1. \)

![Figure 1-SG2 with basic reflections R0, R1](image-url)

Vertices \( a, b \) lie in the fundamental chamber (the sextant containing the point, \( p \)). These are the minimal elements of the components of the stabilization-order, \( S-O(S(2, 3)) \). Coxeter theory tells us that \( S-O(S(n, m)) \) may be constructed recursively from its minimal elements, extending each component from rank \( r \) to rank \( r + 1 \) by applying the adjacent transpositions, \( i (i + 1) \) to \( v \) (in rank \( r \)) if \( v_{j_0} = i \), where \( j_0 = \min \{ j : v_j = i \text{ or } i + 1 \} \). The resulting vector \( v' \) (\( v \) with entries \( i \) and \( i + 1 \) transposed) is then in rank \( r + 1 \). The Hasse diagram of \( S-O(S(2, 3)) \) is in
Figure 2.

Figure 2-The stabilization order of $SG_2$

An *ideal*, $\iota$, of a partially ordered set (poset), $\mathcal{P}$, is a subset of the poset that is downwardly closed. I.e. if $x \leq y \land y \in \iota$ then $x \in \iota$. The *ideal transform*, $\mathcal{I}(\mathcal{P})$, of a poset, $\mathcal{P}$, is the set of all ideals of $\mathcal{P}$, partially ordered by $\subseteq$. The *derived network* of a $\text{StOp}$-order is the Hasse diagram of the ideal transform of the $\text{StOp}$-order, with weight $|\Theta(S)|$ for each ideal, $S$. The derived network of the stabilization-order of $SG_2$ is shown in Figure 3.

Figure 3-The derived network of $\mathcal{S-O}(SG_2)$

The large vertices represent solutions of the EIP for $SG_2$. The darkened edges, tracing a path from $\emptyset$ to \{a, b, c, d, e, f\} through solution sets, shows that $SG_2$ has nested solutions,

$$
\emptyset \subset \{a\} \subset \{a, b\} \subset \{a, b, c\} \subset \{a, b, c, d\} \subset \{a, b, c, d, e\} \subset \{a, b, c, d, e, f\}
$$
Indra Rajasingh and collaborators has shown that if $\emptyset \subset S \subsetneq V_{SG_n}$ then $|\Theta(S)| \geq 2$ and this bound is sharp (achieved by the subgraphs of $SG_n$ isomorphic to $SG_{n'}$) if $|S| = |SG_{n'}| = \left(3^{n'} + 3\right)/2$, $0 \leq n' < n$.

2.2. $SG_3$ does not have Nested Solutions.

2.2.1. A Necessary Condition. In order for a graph, $G$, to have nested solutions for $EIP$, initial segments of the numbering must be sequentially optimal. I.e. the additional vertex in each successive initial segment must minimize its marginal contribution to edge-boundary. This observation gives us an easy way to generate candidates for an optimal numbering. It also leads to a necessary condition for nested solutions because the derived network (See [14], p. 24) of $G$ is self-dual (by complementation). The dual of a sequentially optimal s-t path in the derived network must also be sequentially optimal so we can start at both ends and meet in the middle. If no sequentially optimal paths meet in the middle then $G$ cannot have nested solutions.

The intuition about the $EIP$ on $SG_n$ is that the Sierpinski gasket subgraph, $SG_m$ for $m < n$, has greater edge-density than $SG_n$, so once a numbering includes one vertex of $SG_m$ it should continue numbering in that $SG_m$ until it is completely numbered. Unfortunately that strategy contains the seeds of its own destruction and $SG_3$ fails the above test: $SG_2$ is the unique (up to isomorphism) sequential minimizer of size 6. Its complement is then the unique dual sequential minimizer of size $15 - 6 = 9$. However, since $SG_2$ has a common vertex with every other copy of itself in $SG_3$, $SG_3 - SG_2$ cannot contain a copy of $SG_2$ and there is no way sequentially optimal paths can meet in the middle. Another way to say the same thing is that the function $|\Theta(G; \ell)| = \min \{|\Theta(S)| : S \subseteq V_G, |S| = \ell\}$ is symmetric about $n/2$ (since $|\Theta(S)| = |\Theta(V_G - S)|$). Therefore any numbering, $\mu$, for which $|\Theta(\mu^{-1}(\ell))|$ is not symmetric about $n/2$ cannot give nested solution for $EIP$ (even if it is sequentially optimal).

However, there are graphs closely related to the Sierpinski gasket graphs, namely the extended Sierpinski graphs, which pass this test for nested solutions for $EIP$.

3. Generalized & Expanded Sierpinski Graphs

The generalized & expanded Sierpinski graph, $S(n, m)$, $n \geq 1, m \geq 2$, was defined in 1944 by Scorer, Grundy and Smith [13]: $V_{S(n, m)} = \{0, 1, \ldots, m - 1\}^n$. For $\{u, v\} \in \binom{V}{2}$, $\{u, v\} \in E_{S(n, m)}$ iff $\exists h \in \{1, 2, \ldots, n\}$ such that following 3 conditions hold:

1. $u_i = v_i$ for $i = 1, 2, \ldots, h - 1$;
2. $u_h \neq v_h$; and
3. $u_j = v_h$ and $v_j = u_h$ for $j = h + 1, \ldots, n$.

The motivation for defining $S(n, m)$ was that $S(n, 3)$ is (isomorphic to) the graph of the 3-peg Towers of Hanoi puzzle with $n$ disks [13]. Scorer, Grundy and Smith pointed out that $SG_n$ is a quotient of $S(n, 3)$ where every edge of $S(n, 3)$ not contained in a triangle ($K_3$) is contracted to a vertex. Also, the Sierpinski sponge, $SP_n = S[n, 4]$ is a similar quotient of $S(n, 4)$. Jakovac [9] generalized the construction to $S[n, m]$, the quotient of $S(n, m)$ in which every edge of $S(n, m)$ not contained in a $K_3$ is contracted to a vertex. He showed that $S[n, m]$ is hamiltonian and its chromatic number is $m$. 
3.1. Structure of $S(n, m)$.

3.1.1. The Basics. $|V_{S(n,m)}| = m^n$. All $v \in V_{S(n,m)}$ have $m-1$ ”interior” neighbors. These are the $n$-tuples that agree with $v$ in all coordinates except the $n^\text{th}$ (the case $h = n$ in the definition of $S(n,m)$). If $v \neq i^n$ then $v$ has one other (”exterior”) neighbor: If $v \neq i^n \exists h, 1 \leq h < n$, such that $v_h \neq v_{h+1} = v_{h+2} = \ldots = v_n$ and by definition the exterior neighbor of $v$ is $u = v_1v_2\ldots v_{h-1}v_{h+1}v_h v_{h+2}v_{h+3}\ldots v_n$. Thus $i^n$, with $i = 0, 1, 2, \ldots, m - 1$, has degree $m - 1$ and every other vertex has degree $m$. Summing the degrees of all vertices we get $m (m - 1) + (m^n - m) = m^{n+1} - m$. Since each edge is incident to two vertices, $|E_{S(n,m)}| = (m^{n+1} - m)/2$. $K_n^m$ also has $m^n$ vertices (we take $V_{K_n^m}$ to be the same set, $\{0, 1, \ldots, m - 1\}^n$) and $m^n (m - 1) n/2$ edges. Thus the density of edges of $S(n,m)$ relative to $K_n^m$ is $((m^{n+1} - m)/2)/(m^n (m - 1) n/2) = 1/n$ which is decreasing in $n$.

The vertices that agree in all except the last coordinate induce a complete subgraph, $K_n$. These $K_n$s are maximal, nonoverlapping and contain all the vertices of $S(n,m)$. There are $m^n$ of them, constituting a $K_n$-decomposition of $S(n,m)$. Since any vertex is incident to at most one exterior edge, any triangle ($K_3$) must contain at least two internal edges. But then the third edge would also be internal to the same $K_n$, so this $K_n$-decomposition is unique.

The vertices $i^n$, for $i = 0, 1, 2, \ldots, m - 1$, are called corner vertices of $S(n,m)$. We can use them to characterize $S(n,m)$ (up to isomorphism) recursively: Let $S(n,m)|_{v_i=i}$ be the subgraph of $S(n,m)$ induced by the vertices whose first coordinate is $i$. It is easy to see that $S(n,m)|_{v_i=i} \simeq S(n-1,m)$. Again, these copies of $S(n-1,m)$ partition the vertices of $S(n,m)$. The edges of $S(n,m)$ not induced by $S(n,m)|_{v_i=i}$ for some $i$, connect a corner of $S(n,m)|_{v_i=i}$ to a corner of $S(n,m)|_{v_j=j}$, $i \neq j$. The rule for such a connection is that $u \in S(n,m)|_{v_i=i}$ is connected to $v \in S(n,m)|_{v_j=j}$ iff $u = ij^{n-1}$ and $v = ji^{n-1}$. The initial step of the recursion is to take $S(1,m) = K_m$ with $V_{K_m} = \{0, 1, 2, \ldots, m - 1\}$. Given $S(n - 1,m)$, $n \geq 2$, we construct $S(n,m)$ from $\{0, 1, \ldots, m - 1\} \times S(n - 1,m)$ by connecting $ij^{n-1}$ to $ji^{n-1}$ ($\forall i \neq j$) as above.

Theorem 1. The symmetry group of $S(n,m)$ is $S_m$, the symmetric group on $m$ generators. $S_m$ acts on the coordinates of $V_{S(n,m)} = \{0, 1, 2, \ldots, m - 1\}^n$, i.e.

$$\pi (v_1, v_2, \ldots, v_n) = (\pi (v_1), \pi (v_2), \ldots, \pi (v_n)).$$

For a proof see [7] (Theorem 1) or [5] (Theorem 4.14).

3.1.2. Recursive Definition of $S(n,m)$. The Sierpinski graph, $S(n,m)$, has been defined analytically at the beginning of Section 3. $S(n,m)$ may also be characterized recursively: $S(1,m) = K_m$ and given $S(n - 1,m)$ for $n - 1 \geq 1$,

$$V_{S(n,m)} = \{0, 1, \ldots, m - 1\} \times V_{S(n-1,m)}$$

and

$$E_{S(n,m)} = \{0, 1, \ldots, m - 1\} \times E_{S(n,m)} + \{ij^{n-1}, ji^{n-1} : 0 \leq i, j \leq m - 1, i \neq j\}.$$

The key property here is that the edge, $\{ij^{n-1}, ji^{n-1}\}$ connects $(i, S(n - 1,m))$ to $(j, S(n - 1,m))$ at unique corner vertices $ij^{n-1}, ji^{n-1}$. That is to say every copy of $S(n - 1,m)$ is connected to every other copy and the edges form a complete matching. However, any such correspondence between vertices of $(i, S(n - 1,m))$
for \( i = 0, 1, \ldots, m - 1 \) determines a graph isomorphic to \( S(n, m) \). Reducing the relationship to its essence, \( \{ \{ i, j \} : 0 \leq i, j \leq m - 1, i \neq j \} \) is a complete matching of \( K_{m,m} - \{ \{ i^2 : 0 \leq i < m \} \} \). Actually, any complete matching of \( K_{m,m} \) will determine a graph isomorphic to \( S(n, m) \). Another such correspondence is

\[
\{ \{ ik, jk : 0 \leq i, j \leq m - 1, i \neq j, i + j = k (\text{mod} \ m) \} \}.
\]

In [7] it is shown that with the latter coordinates, \( S(n, m) \) is a subgraph of \( K_{m,m}^n \).

3.1.3. Linear Coordinates for \( S(n, m) \). If \( v \in V_{S(n,m)} \), so \( v = (v_1, v_2, \ldots, v_n) \) where \( v_i \in \{ 0, 1, \ldots, m - 1 \} \), then its representation in \( \mathbb{R}^m \) is

\[
y(v) = \left( \sum_{v_i=0}^{m} 2^i, \sum_{v_i=1}^{v_n} 2^i, \ldots, \sum_{v_i=m-1}^{m} 2^i \right).
\]

Note that

\[
\sum_{j=1}^{n} y_j(v) = \sum_{i=0}^{m-1} 2^i = 2^n - 1.
\]

So the vertices of \( S(n, m) \) are actually lying in the hyperplane,

\[
\sum_{j=1}^{n} y_j = 2^n - 1.
\]

The coordinates of \( y(v) \) are integral, non-negative and characterized by the fact that each power of 2 \( (2^i \text{ for } 0 \leq i < m) \) occurs in exactly one of the base two representations of the \( y_j \)'s. Also, two vertices are connected by an edge iff the Euclidean distance between them is 1.

3.2. The EIP on \( S(n, m) \). The intuition (Section 2.2.1, second paragraph) suggesting that \( SG_n = S[n, 3] \) might have nested solutions for EIP applies more generally to \( S[n, m] \). However, the same counterexample works for \( S[n, m] \) except when \( n = 1 \) (all \( m \)) and \( n = 2, m = 3 \). On the other hand, since the density of edges in \( S(n, m) \) decreases with \( n \) (see Section 3.1.1), the intuition also applies to \( S(n, m) \). It suggests that for \( \ell = (3^n - 1)/2 \), the disjoint union \( \{0\} \times S(n-1, 3) + \{10\} \times S(n-2, 3) + \ldots + \{1^{n-1}0\} \), which is sequentially optimal, should be optimal (minimize \( \Theta(S) \)) for \( S \) having cardinality \( 3^{n-1} + 3^{n-2} + \ldots + 1 = \frac{3^n - 1}{3-1} = \frac{3^n - 1}{2} \). The transposition 02 of \( \{0, 1, 2\} \) gives another sequentially optimal set, \( \{2\} \times S(n-1, 3) + \{12\} \times S(n-2, 3) + \ldots + \{1^{n-1}2\} \). Both of these sets are optimally extended by adding \( 1^n \). The complement of the first set is the extension of the second (and vice versa). Thus the two sequentially optimal paths meet in the middle satisfying our necessary condition (Section 2.2.1). A sequentially optimal numbering for \( S(n, m) \) is given by lexicographic order on \( \{0, 1, \ldots, m - 1\}^n \),

\[
\eta(v) = \text{Lex}(v) = 1 + \sum_{i=1}^{n} v_i m^{n-i}.
\]

Note that \( \sum_{i=1}^{n} v_i m^{n-i} \) is the base \( m \) representation of an integer between 0 and \( m^n - 1 \).

The analog of the theorem of Rajasingh et al (see the last paragraph of Section 2.11) holds for \( S(n, m) \).
**Theorem 2.** If \( \emptyset \subsetneq S \subsetneq V_{S(n,m)} \) then \( |\Theta(S)| \geq m - 1 \) and this bound is sharp (achieved by the subgraphs of \( S(n,m) \) isomorphic to \( S(n',m) \)) if \( |S| = m^\nu \), \( 0 \leq n' < n \).

**Proof.** This follows from the fact that any two distinct points in \( S(n,m) \) may be connected by \( m - 1 \) disjoint paths. So if \( v \in S \) and \( w \notin S \) there must be \( m - 1 \) disjoint paths from \( v \) to \( w \). Each such path will have a first vertex, \( w' \), not in \( S \) and the edge, \( \{v',w'\} \) from the previous vertex, \( v' \in S \), to \( w' \notin S \), will be in the edge-boundary of \( S \). Since the paths are disjoint, \( |\Theta(S)| \geq m - 1 \). Also, \( \text{Lex}^{-1}(\{1,2,\ldots,m^\nu\}) \simeq S(n',m) \) so those initial segments are solutions. \( \square \)

However, being sequentially optimal and satisfying the necessary condition does not constitute a proof that initial \( \ell \)-segments of \( \text{Lex} \) actually minimize \( |\Theta(S)| \) for all cardinalities \( \ell \). It could still be possible that some \( \ell \neq m^\nu \) there is a strange collection of \( \ell \) vertices in \( S(n,m) \) that has smaller edge-boundary than \( \text{Lex}^{-1}(\{1,2,\ldots,\ell\}) \).

**Conjecture 1.** The generalized \( \ell \)-expanded Serpinski graph, \( S(n,m) \), has nested solutions for the EIP. \( \text{Lex} \) (lexicographic order on \( V_{S(n,m)} = \{0,1,\ldots,m-1\}^n \)) is not only sequentially optimal but initial \( \ell \)-segments of \( \text{Lex} \) minimize \( |\Theta(S)| \) for \( S \) having cardinality \( \ell \). In other words, \( |\Theta(\text{Lex}^{-1}(\{0,1,\ldots,\ell\}))| = |\Theta(S(n,m);\ell)| \) \( \forall \ell, 0 \leq \ell \leq m^n \).

3.2.1. **The Case \( m = 2 \), all \( n \).** In \([7]\) It is shown that \( S(n,2) \) is a path of length \( 2^n - 1 \). The endpoints are the corner vertices \( 0^n \) and \( 1^n \). In between, the vertices (\( V_{S(n,2)} = \{0,1\}^n \)) appear in lexicographic order. So, starting from \( 0^n \), the initial segment, \( \text{Lex}^{-1}(\{0,1,\ldots,\ell\}) \), of cardinality \( \ell \), is a solution of the EIP on \( S(n,2) \) for \( |S| = \ell \). This proves Conjecture 1 for \( m = 2 \).

**4. Preliminaries to Proving Conjecture 1 for \( m > 2 \)**

4.1. **Our Strategy.** Our basic logical strategy for proving that \( S(n,m) \) has Lex-nested solutions is induction on \( n \). The inductive step, reducing the conjecture for \( S(n+1,m) \) to that for \( S(n,m) \) is accomplished through a series of morphisms for EIP called ”Steiner operations”.

4.2. **Steiner Operations on \( S(n,m) \).** A **Steiner operation** on a graph, \( G = (V,E) \), is a function, \( \text{StOp} : 2^V \to 2^V \), mapping subsets of \( V \) to subsets of \( V \) such that

1. \( \forall S \subsetneq V, |\text{StOp}(S)| = |S| \),
2. \( \forall S \subsetneq V, |\Theta(\text{StOp}(S))| \leq |\Theta(S)| \) and
3. \( \forall S \subsetneq T \subsetneq V, \text{StOp}(S) \subsetneq \text{StOp}(T) \).

This definition is taken from Chapter 2 of \([4]\) where the theory of StOps is developed with applications. Properties 1 & 2 are essential for a mapping to preserve the EIP. If they hold we can say that the \( \text{StOp} \) represents a simplification of the EIP on \( G \) since we need only consider sets in the range of the \( \text{StOp} \). However, to make that simplification effective, we must be able to pick out those subsets of \( V \) that are in the range of \( \text{StOp} \) and do it efficiently. Originally this was done for
each StOp considered by finding a partial order on \( V \) (the StOp-order, \( S-O \ (\text{StOp}) \)) such that \( S \) is in the range of StOp iff \( S \) is an ideal (If \( x \leq_{S-O} y \in S \) then \( x \in S \) of \( S-O \ (\text{StOp}) \)). More recently [5] we showed that Properties 1 & 3 imply that every StOp has such a StOp-order, \( S-O \ (\text{StOp}) \) (characterizing its range). Property 3 is called monotonicity.

4.3. Stabilization. Stabilization is a StOp that utilizes reflective symmetry of \( G \) in a systematic way to achieve its simplification. The theory of stabilization is presented in [4], Chapters 3 & 6. The most important fact about stabilization (besides being a Steiner operation) is that cyclic compositions of \( \text{Stab}_{i,i+1} \) eventually become constant. We denote the resulting "limit" as \( \text{Stab}_\infty \).

4.3.1. The Stabilization-Order of \( S(n,m) \). The symmetry group of \( S(n,m) \) is \( \mathfrak{S}_m \), the symmetric group on \( \{0,1,...,m-1\} \), acting on the components of \( V_{S(n,m)} = \{0,1,...,m-1\}^n \) (See [7] or [8], Theorem 4.14). \( \mathfrak{S}_m \), with the generating set \( W = \{01,12,..., (m-2)(m-1)\} \) is a Coxeter group, i.e. it is generated by elements of order 2 (See [4] for Coxeter theory). When we consider \( S(n,m) \) as embedded in \( \mathbb{R}^m \) (see Section 3.1.3), the transpositions, \( ij \), correspond to reflections so they define stabilization operations (see Section 3.2.4 of [4]). In particular, the fixed hyperplanes of the reflections induced by the (adjacent) transpositions of \( W \) surround the fundamental chamber, \( C_0 = \{y \in \mathbb{R}^m : y_1 \geq y_2 \geq ... \geq y_m \geq 0\} \).

There are \( m! \) chambers altogether, one for each member of \( \mathfrak{S}_m \).

The stabilization-order, \( S-O(S(n,m)) \), is a disjoint union of components, each of which has a unique minimum element in \( C_0' = y^{-1}(C_0) \) (Theorems 5.3-5 of [4]).

**Example 6.** The corner vertex, \( 0^n \), is in \( C_0' \). Its connected component (of \( S-O(S(n,m)) \)) is \( \{0^n \leq ^1 \leq ^2 \leq ..., \leq (m-1)^n\} \). \( \leq \) represents the covering relation in \( S-O \) (given by a basic transposition, \( i(i+1) \)).

**Example 7.** The vertex, \( 01^{n-1} \), is in \( C_0' \). For \( m = 3 \) its component consists of
\[
\{01^{n-1}, 10^{n-1}, 02^{n-1}, 20^{n-1}, 12^{n-1}, 21^{n-1}\}
\]

with basic covering relations (given by adjacent transpositions, \( i(i+1) \)) \( 01^{n-1} \leq_{01} 10^{n-1} \leq_{12} 20^{n-1} \leq_{01} 21^{n-1} \) and \( 01^{n-1} \leq_{12} 02^{n-1} \leq_{01} 12^{n-1} \leq_{12} 21^{n-1} \). Also it has nonbasic covering relations \( 10^{n-1} \leq_{02} 12^{n-1} \) and \( 02^{n-1} \leq_{02} 20^{n-1} \). Its Hasse diagram is shown in Figure 4. The heavy lines represent actual edges of \( S(n,3) \). Note that the component of \( 01^{n-1} \) is also \( 21^{n-1} \downarrow \), the ideal (of \( S-O S(n,3) \)) generated by (below) \( 21^{n-1} \).

![Figure 4-The component of 01^n for m = 3](image-url)
**Example 8.** For \( m = 4 \) the corresponding Hasse diagram is Figure 5.

![Hasse diagram](image)

*Figure 5-The component of \( 01^n \) for \( m = 4 \)*

**Example 9.** The stabilization-order of \( S(2,3) \) has just two components, those given in the Examples 6 & 7. Compare to \( S-O(S[2,3]) \) in Figure 2.

Recall that an ideal, \( \iota \), of a partially ordered set (poset) is subset of the poset that is downward closed, *i.e.* if \( x \leq y \) & \( y \in \iota \) then \( x \in \iota \). The **ideal transform**, \( I(P) \) of a poset \( P \), is the set of all ideals of \( P \), partially ordered by containment.

**Example 10.** Figure 6 shows the ideal transform of the component of \( 01^n \) for \( m = 3 \) (Figure 4).

![Ideal transform diagram](image)

*Figure 6-The ideal transform of \( 21 \downarrow \)*

Each point in the diagram represents an ideal, the set of elements below it. So the (unlabeled) midpoint of the diagram represents the ideal \( \{01^n, 02^n, 10^n\} \).
4.3.2. Three Key Observations. Given \( S \subseteq V_{S(n,m)} = \{0, 1, \ldots, m-1\}^n \), with \(|S| = \ell\), let \(|S \cap (\{k\} \times S(n-1, m))| = \ell_k\), so \(\sum_{k=0}^{m-1} \ell_k = \ell\). Also let \(\ell(S) = (\ell_0, \ell_1, \ldots, \ell_{m-1})\) and totally order the \(m\)-tuples \(\ell(S)\) lexicographically. Then

(1) \(\ell(\text{Stab}_{ij}(S)) \geq \ell(S)\).
(2) If \(S\) is stabilized wrt all the generators of \(\mathcal{E}_m\) (i.e. \(\text{Stab}_{ij}(S) = S \forall ij\)), then \(\ell_0 \geq \ell_1 \geq \ldots \geq \ell_{m-1}\).
(3) \(\forall i,j\), if \(i, j \in \mathcal{E}_m\) and \(\text{Lex}_\pi(v) = 1 + \sum_{i=1}^n v_{\pi(i)} m^{n-i}\), then

\[
\text{Stab}_{ij} (\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})) = \begin{cases} 
\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\}) & \text{if } \pi(i) < \pi(j) \\
\text{Lex}_\pi^{ij} (\{1, 2, \ldots, \ell\}) & \text{if } \pi(i) > \pi(j)
\end{cases}
\]

So if we apply basic stabilizations sufficiently many times to \(\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})\) we will get \(\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})\), \(\pi\) being the identity permutation and \(\text{Lex}_\pi = \text{Lex}\), the standard lexicographic order. \(\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})\) is stable, i.e.

\(\forall i, j\), \(\text{Stab}_{ij} (\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})) = \text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\})\).

4.4. Compression. In \cite{4} compression is a Steiner operation on a product of graphs, \(G \times H\), based on at least one of the factors (say \(G\)) having nested solutions (See the Appendix (Section 7.2) for more about compression on \(G \times H\)). The fact that \(\text{Comp}_{\mu,G \times H}\) is a Steiner operation is key to proving nested solutions for a product, such as \(K_n^m\), recursively. Compression has been the single most powerful tool in proving combinatorial isoperimetric theorems. In \cite{6} we were able to extend compression to a self-similar structure (not a product, but still having nested solutions) and thereby induct on the depth of self-similarity. Applying the strategy to \(S(n,m)\) has been problematic, however, in that we were unable to incorporate Property 3 (monotonicity) into our extended definition of compression for \(S(n,m)\).

Our ultimate goal is to prove Conjecture 1, that \(\forall n, m, \ell\), \(|\Theta(S(n,m);\ell)| = |\Theta(\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell\}))|\). Now suppose Conjecture 1 is true for some \(n > 1\) and we wish to prove it for \(n+1\) using some form of compression: Then \(\{h\} \times S(n,m)\) is a sub-Sierpinski graph of \(S(n+1,m)\). \(\{h\} \times S(n,m)\) has \(\text{Lex}\) order on it, induced by restricting \(\text{Lex}\) on \(S(n+1,m)\) to \(\{h\} \times S(n,m)\), which is the same as \(\text{Lex}\) order on \(S(n,m)\). That is, the numbers are not the same but their relative order is.

If \(S \subseteq V_{S(n+1,m)}\) is stabilized, \(|S| = \ell\) and \(|S \cap V_{\{j\} \times S(n,m)}| = \ell_j\), then \(\ell_0 \geq \ell_1 \geq \ldots \geq \ell_{m-1} \geq 0\) and \(\sum_{j=0}^{m-1} \ell_j = \ell\). Emulating the definition of \(\text{Comp}_{\mu,G \times H}\), we should define the compression operation, \(\text{Comp}_{\text{Lex},j} : V_{S(n+1,m)} \rightarrow V_{S(n+1,m)}\), by \(\text{Comp}_{\text{Lex},j} (S) = S - (S \cap V_{\{j\} \times S(n,m)}) + \text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell_j\})\). That is, \(\text{Comp}_{\text{Lex},j}\) should remove those elements of \(S\) in \(V_{\{j\} \times S(n,m)}\) and replace them by the initial \(\ell_j\)-segment of \(\text{Lex}_\pi\). And we then want \(\text{Comp}_{\text{Lex},j}\) to be a Steiner operation, utilizing the fact that \(\text{Lex}_\pi^{-1} (\{1, 2, \ldots, \ell_j\})\) minimizes \(|\Theta(S \cap V_{\{j\} \times S(n,m)})|\). However, there is a problem with the exterior edges of \(\{j\} \times S(n,m)\) (the ones of the form \(\{j^n, ij^n\}, i \neq j\)). Those edges can effect the contribution of \(S \cap V_{\{j\} \times S(n,m)}\) to \(|\Theta(S)|\) and \(|\Theta(\text{Comp}_{\text{Lex},j} (S))|\) so that \(\text{Comp}_{\text{Lex},j}\) need not satisfy Property 2 of a Steiner operation. To remedy the situation we define \(I_j = \{i \neq j : ij^n \in S\}\) and reorder \(\{0, 1, \ldots, m-1\}\) so that the members of \(I_j\) appear in their relative order, \(i_0 < i_1 < \ldots < i_{|I_j|-1}\), then \(i_{|I_j|} = j\) (note that \(j^n+1\) is the corner vertex in \(\{j\} \times S(n,m)\) and not connected to any vertex outside of \(\{j\} \times S(n,m)\) followed by the members of \(K_j = \{0, 1, \ldots, m-1\} - I_j - \{j\}\). Then in our amended definition of \(\text{Comp}_{\text{Lex},j}\), \(\text{Lex}_\pi\) will be \(\text{Lex}_\pi\) order on \(\{j\} \times S(n,m)\).
We could then try to prove

Proof. Let

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by induction. Unsuccessful attempts to do so led us to extend Conjecture 1 even further: Let




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are not) does not increase that number either. Anyway, the sum of their cardinalities is not increased so $|\Theta_{s,t}(\text{CompLex}_{h}(S))| \leq |\Theta_{s,t}(S)|$.

Unfortunately, $\text{CompLex}_{h}$ is not monotonic (Property 3 of a Steiner operation) as the following example shows.

**Example 11.**

$$\text{CompLex}_{0}(\{01^n\}) = \{0^{n+1}\},$$

and

$$\text{CompLex}_{0}(\{01^n, 10^n\}) = \{01^n, 10^n\}.$$  

So

$$\{01^n\} \subseteq \{01^n, 10^n\},$$

but

$$\text{CompLex}_{0}(\{01^n\}) \not\subseteq \text{CompLex}_{0}(\{01^n, 10^n\}).$$

Thus $\text{CompLex}_{0}$ is not a full Steiner operation. This creates technical difficulties but (as we remarked after the definition of a Steiner operation) properties 1) & 2) are the essential ones for solving isoperimetric problems and technical difficulties can be overcome. We call $\text{CompLex}_{h}$ a nonmonotone Steiner operation.

As noted in the previous section, the cyclic composition of stabilizations are eventually constant (defining $\text{Stab}_{\infty}$). In fact the cyclic compositions of any finite set of Steiner operations, as long as they are consistent (their StOp-Orders have a common total extension) will eventually become constant functions. This “limit” of cyclic compositions is, in category theory, the pushout of the constituent Steiner operations. This also applies to compression in [4] because in [4], compression is defined only for products and is a full Steiner operation. But what about our nonmonotone compressions? Do cyclic compositions of $\text{CompLex}_{0}$, $\text{CompLex}_{1}$, ..., $\text{CompLex}_{m-1}$ eventually become constant? This is the "technical problem" referred to earlier as caused by the nonmonotonicity of our extended compression operation. Since the domain, $2^{V_G}$, (which is also the codomain) of any StOp is finite, cyclic compositions must eventually cycle, so the question is whether that cycle is necessarily of length 1?

For $S \subseteq V_{S_{s,t}(n,m)}$, let

$$\tau(S; ij^n) = \begin{cases} 0 & \text{if } ij^n \in (S \cap I_i) \cup (S \cap J_i) \cup (S^c \cap K_i), \\ 1 & \text{if } ij^n \in (S^c \cap I_i) \cup (S \cap K_i), \end{cases}$$

and

$$\tau(S) = \sum_{i,j} \tau(S; ij^n).$$

Then $\tau(S)$ counts the number of external edges of the sub-Sierpinski graph, $\{i\} \times S(n-1,m), 0 \leq i < m$, cut by $S$. Also let

$$\rho(S; ij^n) = \begin{cases} 0 & \text{if } ij^n \in (S \cap I_i) \cup (S \cap J_i) \cup (S^c \cap K_i), \\ m-j & \text{if } ij^n \in (S^c \cap I_i) \\ j & \text{if } ij^n \in (S \cap K_i). \end{cases}$$

and

$$\rho(S) = \sum_{i,j} \rho(S; ij^n).$$
If \( \forall h, S \cap V_{(h)} \times S_{s,t}(n,m) = Lex^{-1}_{\pi} \{1, 2, \ldots, \ell_h\} \) for some \( \pi \in \mathcal{S}_m \) (which can be achieved with any \( S \) by one round of \( h \)-compression, \( 0 \leq h < m \)), then the number of exterior vertices in \( S \), \( \ell'_h = \left| \{h^{j^n} \in S \cap V_{(h)} \times S_{s,t}(n,m)\} \right| \), is determined by \( \ell_h \):

Given the base \( m \) representation of \( \ell_h \):

\[
\ell_h = \sum_{i=1}^{n-1} \ell_{h,i} m^{n-1-i},
\]

then

\[
\ell'_h = 1 + \max \left\{ j : j^{n-1} \leq_{Lex} (\ell_{h,1}, \ell_{h,2}, \ldots, \ell_{h,n-1}) \right\}.
\]

\( |\Theta_{s,t}(S)| \geq 0 \) is integer valued and nonincreasing under cyclic compression operations and so must eventually be constant (it cannot decrease forever). Also, the number of internal edges cut by \( S \) will be constant after one cycle of compressions. Therefore the difference, \( \tau(S) \), must also eventually be constant.

**Lemma 1.** If \( \forall h, S \cap V_{(h)} \times S_{s,t}(n,m) = Lex^{-1}_{\pi} \{1, 2, \ldots, \ell_h\} \) for some \( \pi \in \mathcal{S}_m \), then \( \rho(Comp_{Lex_h}(S)) \leq \rho(S) \) with equality iff \( Comp_{Lex_h}(S) = S \).

**Proof.** \( S' = Comp_{Lex_h}(S) \) alters \( S \) in \( \{h\} \times S_{s,t}(n,m) \) so as to minimize \( |\Theta(S')| \) over all such alterations while maintaining \( |S' \cap V_{(h)} \times S_{s,t}(n,m)| = \ell_h \). It is also the unique minimizer of \( \rho(S') \) over all such alterations. The sum for \( i \neq h \) (in the definition of \( \rho(S) \)) remains unchanged. \( \square \)

**Theorem 4.** Cyclic compositions of \( Comp_{Lex_{h(mod\ m)}}(S) \), \( h = 0, 1, \ldots \), will eventually be constant, defining a nonmonotone Steiner operation, \( Comp_{\infty} \), on \( S_{s,t}(n+1,m) \).

**Proof.** The cyclic compositions, \( Comp_{Lex_0}(S), Comp_{Lex_1}(Comp_{Lex_0}(S)), \ldots \) have nonincreasing values of \( \rho \). If those values do not decrease through a full cycle of \( m \) consecutive applications of \( Comp_{Lex_h} \), then by Lemma 1 \( Comp_{Lex_{h+1(mod\ m)}}(S') = S' \), \( Comp_{Lex_{h+i(mod\ m)}}(S') = S' \ldots, Comp_{Lex_{h+m-1(mod\ m)}}(S') = S' \) and we are then repeating the same compressions that already left \( S' \) unchanged. In that case \( Comp_{\infty}(S) = S' \). This must happen after some finite number of compositions because \( \rho(S) \) is a (finite) nonnegative integer and each change will decrease that integer by at least 1. Since \( \rho(S') \) must remain a nonnegative integer, the number of rounds of cyclic composition required is at most \( 1 + m\rho(S) \). \( \square \)

**Corollary 1.** If \( Comp_{\infty}(S) = S' \), then \( S' \) is \( h \)-compressed for every \( h = 0, 1, \ldots, m-1 \).

4.5. **Stabilization Redux.** In extending the solution of the EIP from \( K_n^m \) to \( S(n,m) \), compression is the most essential of the three basic Steiner operations. To make compression work we had to modify the concept of a graph. We added sets of “external” vertices, \( V_I = \{v_i : i \in I\} \), \( V_K = \{v_i : i \in K\} \), to \( S(n,m) \) so that \( S_{s,t}(n,m) \) is a generalization of the subSierpinski graph, \( \{h\} \times S(n,m) \). So, can stabilization also be extended to \( S_{s,t}(n,m) \)? In terms of the definition of stabilization in [4], the symmetry group of \( S_{s,t}(n,m) \) is only \( \mathcal{S}_s \times \mathcal{S}_t \times \mathcal{S}_n \), so the number of stabilizing reflections is much less \((\binom{n}{2} + \binom{m}{2} + \binom{m-2}{t})\) than for \( S(n,m) \) (where it is \( \binom{m}{2} \)). However, every stabilization operation on \( S(n,m) \) still acts on the subsets of vertices of \( S_{s,t}(n,m) \). Does that action induce a Steiner operation?
Theorem 5. Any stabilization, \( \text{Stab}_{i,j} : 2^{V_G(n,m)} \to 2^{V_G(n,m)} \), is still a Steiner operation on \( S_{x,t}(n,m) \).

Proof. Properties 1) & 3) are not effected by the "exterior" vertices. Property 2) could be destroyed but is not because, whatever \( S \) is, if \( i < j \) replacing \( j^n \) in \( S \) by \( i^n \) in \( \text{Stab}_{i,j}(S) \) cannot increase \( |\Theta_s,t(S)| \): Either \( (i,j \in I \text{ or } J \text{ or } K) \) or \( (i \in I \& j \in J) \) or \( (i \in J \& j \in K) \). Whichever of the 6 possibilities is manifested, \( \text{Stab}_{i,j}(S) \), will not cut more exterior edges than \( S \) (and will cut fewer in the last 3 cases listed). \( \square \)

Corollary 2.

\[
\text{Stab}_{i,j}(\text{Lex}_{x}^{-1}([1, 2, ..., \ell])) = \begin{cases} 
\text{Lex}_{\pi}^{-1}([1, 2, ..., \ell]) & \text{if } \pi(i) < \pi(j), \\
\text{Lex}_{\pi(i)}^{-1}([1, 2, ..., \ell]) & \text{if } \pi(i) > \pi(j).
\end{cases}
\]

Proof. This is the extension of 4.3.2(3) to \( S_{x,t}(n,m) \). \( \square \)

Corollary 3. \( \text{Lex}^{-1}([1, 2, ..., \ell]) \) is stable (wrt all \( i < j \)).

Corollary 4. The stabilization-order of \( S_{x,t}(n,m) \) is the same as that of \( S(n,m) \).

After compressing \( S \subseteq V_{S_{x,t}(n+1,m)} \) in every sub-Sierpinski graph, \( \{h\} \times S_{x,t}(n,m) \) with \( 0 \leq h < m \), \( S \) need not be stable. If we then apply the (extended) stabilization, \( \text{Stab}_{i,j} \), will \( \text{Stab}_{i,j}(S) \) still be compressed?

Lemma 2. If \( S \subseteq V_{S_{x,t}(n+1,m)} \) is \( h \)-compressed (i.e. \( \text{Comp}_{\text{Lex}_h}(S) = S \)), then either

1. \( \ell(\text{Stab}_{i,j}(S)) > \ell(S) \) or
2. \( \text{Stab}_{i,j}(S) \) is \( h \)-compressed.

Proof. In calculating \( \text{Comp}_{\text{Lex}_h}(S) \) we first identify

\[
I_h = \{i \in \{0, 1, ..., m-1\} : ((i \neq h) \& (ih \in S)) \text{ or } ((i = h) \& (h \in I))\} \\
K_h = \{i \in \{0, 1, ..., m-1\} : (i \neq h) \& (ih \notin S) \text{ or } ((i = h) \& (h \in K))\}, \\
J_h = \{0, 1, ..., m-1\} - I_h - K_h.
\]

(Note that \( J_h = \emptyset \) unless \( h \in J \) which means that \( J_h = \{h\} \)). So \( I_hJ_hK_h = \pi \in \mathfrak{S}_m \) defines \( \text{Lex}_h \), the lexicographic order on \( V_{S_{x,t}(n+1,m)} \) with components in the order \( I_hJ_hK_h \). Now \( \ell(\text{Stab}_{i,j}(S)) \geq \ell(S) \), so either \( \ell(\text{Stab}_{i,j}(S)) > \ell(S) \) or \( \ell(\text{Stab}_{i,j}(S)) = \ell(S) \). The latter implies that \( \ell_h(\text{Stab}_{i,j}(S)) = \ell_h(S) \) for \( 0 \leq h < m \) which means that \( \text{Stab}_{i,j} \) maps \( S \cap \{h\} \times S_{x,t}(n,m) \) into \( \{h\} \times S_{x,t}(n,m) \). Since \( S \) is \( h \)-compressed, \( S \cap \{h\} \times S_{x,t}(n,m) = \text{Lex}_{x}^{-1}([1, 2, ..., \ell_h]) \) and by Corollary 2

\[
\text{Stab}_{i,j}(S) \cap \{h\} \times S_{x,t}(n,m) = \begin{cases} 
\text{Lex}_{x}^{-1}([1, 2, ..., \ell_h]) & \text{if } \pi(i) < \pi(j), \\
\text{Lex}_{\pi(i)}^{-1}([1, 2, ..., \ell_h]) & \text{if } \pi(i) > \pi(j).
\end{cases}
\]

In either case, \( \text{Stab}_{i,j}(S) \) is \( h \)-compressed. \( \square \)

Theorem 6. If \( S \subseteq V_{S_{x,t}(n+1,m)} \) is compressed (\( h \)-compressed for \( h = 0, 1, ..., m-1 \)), then either

1. \( \ell(\text{Stab}_\infty(S)) > \ell(S) \) or
2. \( \text{Stab}_\infty(S) \) is compressed.
Proof. Apply stabilizations \( \text{Stab}_{i,j} \) cyclically. Eventually the composition are constant, defining \( \text{Stab}_{\infty} \). Since stabilization does not decrease \( \ell(S) \), either some \( \text{Stab}_{i,j} \) increases it or it remains constant \( (= \ell(S)) \) until \( \text{Stab}_{\infty}(S) \) is reached. In the latter case, Lemma 2 says that \( \text{Stab}_{\infty}(S) \) is \( h \)-compressed for \( h = 0, 1, \ldots, m-1 \).

4.6. Subadditivity.

**Definition 2.** A function, \( f : \mathbb{Z}_N \to \mathbb{Z} \), is called subadditive \( \pmod{N} \) if

1. \( f(0) = 0 \),
2. \( \forall x, y \in \mathbb{Z}_N, f((x + y) \pmod{N}) \leq f(x) + f(y) \).

**Example 12.** Bernstein’s Lemma (Lemma 1.3 of [4]) is equivalent to \( |\Theta \left( \text{Lex}^{-1} \{1, 2, \ldots, \ell\} \right)| \) on \( Q_n \) being subadditive, i.e. that

1. \( |\Theta| (Q_n; 0) = 0 \),
2. \( \forall k, \ell \in \mathbb{Z}_{2^n}, |\Theta|(Q_n; (k + \ell) \pmod{2^n}) \leq |\Theta|(Q_n; k) + |\Theta|(Q_n; \ell) \).

Note that if \( x \) is 0 then \( f(x + y) = f(0 + y) = f(y) = 0 + f(y) = f(x) + f(y) \) and similarly if \( y = 0 \).

A subadditive function, \( f : \mathbb{Z}_N \to \mathbb{Z} \), is called strongly subadditive if

\( \forall x, y \neq 0, f((x + y) \pmod{N}) < f(x) + f(y) \).

From this point on, due to the complexity of our proof technique for \( m > 3 \), we must restrict some theorems to \( m = 3 \). Our goal is to prove that

\[ |\Theta \left( \text{Lex}^{-1} \{1, 2, \ldots, \ell\} \right)| : \mathbb{Z}_{m^n} \to \mathbb{Z}_{m^n} \]

is the edge-isoperimetric profile of \( S(m, n) \). We cannot yet assert that

\[ |\Theta|(S(m, n); \ell) = |\Theta \left( \text{Lex}^{-1} \{1, 2, \ldots, \ell\} \right)| \]

so we need a notation for the latter that is more compact yet fully descriptive. To this end we let

\[ |\Theta \left( L^{-1} \right)| (n, m; \ell) = |\Theta \left( \text{Lex}^{-1} \{1, 2, \ldots, \ell\} \right)| \]
The reader might compare these with the diagrams of $S(n, 3)$, $n = 1, 2, 3$. They are in Figure 2 of [7].

**Theorem 7.** $\forall n, |\Theta(L^{-1})|(n, 3; \ell)$ is strongly subadditive (as a function of $\ell \in \mathbb{Z}_{3^n}$).

Before the proof of Theorem 7, we need some technical lemmas. Let

$$|\Theta_0|(n, 3; \ell) = \begin{cases} 0 & \text{if } \ell = 0, \\ 1 & \text{if } 0 < \ell < 3^{n-1}/2 \text{ or } 3^n - 3^{n-1}/2 < \ell < 3^n, \\ 2 & \text{if } 3^{n-1}/2 < \ell < 3^n - 3^{n-1}/2. \end{cases}$$

and

$$|\Theta_1|(n, 3; \ell) = \begin{cases} 0 & \text{if } \ell = 0 \mod 3^{n-1}, \\ |\Theta(L^{-1})|(n - 1, 3; \ell \mod 3^{n-1}) - 1 & \text{if } \ell \neq 0 \mod 3^{n-1}. \end{cases}$$

**Lemma 3.** $|\Theta_0|(n, 3; \ell) + |\Theta_1|(n, 3; \ell) = |\Theta(L^{-1})|(n, 3; \ell)$. 

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Figure 7-Graphs of $|\Theta(L^{-1})|(n, 3; \ell)$, $n = 1, 2, 3$. 

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Proof. \(|\Theta_1|(n, 3; \ell)\) is essentially the contribution of the edges interior to \(\{i\} \times S(n - 1, 3), \ i = 0, 1, 2\) to \(|\Theta(L^{-1})|(n, 3; \ell)\). We say "essentially" because when \(10^{n-1}\) is numbered, \(\text{Lex } (10^{n-1}) = 3^{n-1} + 1\), \(|\Theta|\) increases by 1 from its predecessor \(02^{n-1}\) (and similarly with \(20^{n-1}\)). However at \(0^n\) the increase is from 0 to 2, but this is compensated for by adding \(|\Theta_0|(n, 3; 1)\) to \(|\Theta_1|(n, 3; 1)\). At \(\ell = (3^{n-1} + 1)/2\) there is another external edge, \(\{00^{n-1}, 10^{n-1}\}\), added to the edge-boundary of \(\text{Lex}^{-1} (\{1, 2, \ldots, \ell\})\), so \(|\Theta_0|\) increases from 1 to 2 at that point. The decreases on the other side of \(3^n/2\) are symmetric.

Lemma 2 gives a recursive procedure for computing the isoperimetric profile \(|\Theta|(S(n, 3); \ell)\). There is also a formula that holds for all \(m\): Given the base \(m\) expansion of \(\ell - 1\),

\[
\ell - 1 = \sum_{i=1}^{n} \ell_{m,i}m^{n-i},
\]

let

\[
\ell'_{m,j} = 1 + \max \{j: j^{n-i} \leq \text{Lex } (\ell_{m,i+1}, \ell_{m,i+2}, \ldots, \ell_{m,n})\}
\]

then

**Proposition 1.** \(|\Theta(L^{-1})|(n, m; \ell) = \sum_{h=1}^{n} \ell_{m,h} (m - \ell_{m,h}) + |\ell'_{m,h} - \ell_{m,h}| - \ell_{m,h} .

Proof. Note that

\[
|\ell'_{m,h} - \ell_{m,h}| - \ell_{m,h} = \begin{cases} 
-\ell'_{m,h} & \text{if } \ell'_{m,h} \leq \ell_{m,h} \\
\ell'_{m,h} - 2\ell_{m,h} & \text{if } \ell'_{m,h} \geq \ell_{m,h} 
\end{cases} .
\]

Lex numbering proceeds through the Sierpinski subgraphs of level \(h\), \(v^{(h)} \times S(n - h, m), 1 \leq h \leq n\), in Lex order (wrt \(v^{(h)}\)), completely numbering each one before moving on to the next. The definition of \(S(n, m)\) says that the edges at level \(h\) connect a corner of some \(v^{(h)} \times S(n - h, m)\) to a corner of \(u^{(h)} \times S(n - h, m)\) where \(v_i = u_i\) for \(1 \leq i < h\), but \(v_h = j \neq k = u_h\). The edge then connects \(v^{(h)}k^{n-h}\) to \(u^{(h)}j^{n-h}\). Since \(\ell_{m,h}\) is the number of copies of \(S(n-h, m)\) contained in some \(S(n-h+1, m)\) that are completely contained in \(\text{Lex}^{-1} (\{1, 2, \ldots, \ell\})\), \(\ell_{m,h} (m - \ell_{m,h})\) is the number of those edges cut by \(\text{Lex}^{-1} (\{1, 2, \ldots, \ell\})\). The next copy of \(S(n-h, m)\) is only partially contained in \(\text{Lex}^{-1} (\{1, 2, \ldots, \ell\})\) and \(\ell'_{m,h}\) is the number of its corner vertices that are in \(\text{Lex}^{-1} (\{1, 2, \ldots, \ell\})\). If \(\ell'_{m,h} < \ell_{m,h}\), each of those corner vertices cover up the other end of an edge that had been counted by \(\ell_{m,h} (m - \ell_{m,h})\), so its marginal contribution to the edge-boundary is \(-1\). If \(\ell'_{m,h} > \ell_{m,h}\), those corner vertices are cutting new edges so its marginal contribution is \(+1\). Since all edges are at some level, \(h\), and the sets of edges involved at different levels are disjoint, the contributions of each level add up so \(|\Theta(L^{-1})|(n, 3; \ell) = \sum_{h=1}^{n} \ell_{m,h} (m - \ell_{m,h}) + |\ell'_{m,h} - \ell_{m,h}| - \ell_{m,h} .

Proof. If \(f, g : \mathbb{Z}_N \rightarrow \mathbb{Z}\) are subadditive and \(a, b \in \mathbb{Z}^+\), then \(af + bg\) is subadditive.

Proof.

\[
(af + bg)(x + y) = af(x + y) + bg(x + y)
\leq (af(x) + af(y)) + (bg(x) + bg(y))
= (af + bg)(x) + (af + bg)(y) .
\]
Lemma 5. If $j < n$ and $3^{j-1}/2 < \ell < 3^j/2$, then $|\Theta_1|(n,3;\ell) = |\Theta_1|(j,3;\ell) + 1$.

Proof. If $3^{j-1}/2 < \ell < 3^j/2$ then
$$|\Theta_1|(j,3;\ell) = |\Theta(L^{-1})|(j,3;\ell \mod 3^j) - 1$$
by the definition of $|\Theta_1|$, 
$$= |\Theta_0|(j,3;\ell) + |\Theta_1|(j,3;\ell) - 1$$
by Lemma 2, 
$$= 2 + |\Theta_1|(j,3;\ell) - 1$$
by the definition of $|\Theta_0|$, 
$$= |\Theta_1|(j,3;\ell) + 1.$$

Similarly, for $n > j + 1$,
$$|\Theta_1|(n,3;\ell) = |\Theta(L^{-1})|(n-1,3;\ell \mod 3^{n-1}) - 1$$
$$= |\Theta_0|(n-1,3;\ell) + |\Theta_1|(n-1,3;\ell) - 1$$
$$= 1 + |\Theta_1|(n-1,3;\ell)$$
$$= |\Theta_1|(n,3;\ell) + 1.$$

Lemma 6. For $j = 0,1,...,n$, if $3^n/2 - 3^j/2 \leq \ell < 3^n/2$ then
$$|\Theta_1|(n,3;\ell) = |\Theta_1|(j,3;\ell - (3^n/2 - 3^j/2)) + (n-j)$$
$$= |\Theta_1|(j,3;\ell \mod 3^j) + (n-j).$$

Proof. If $j = n$ the result says that $|\Theta_1|(n,3;\ell) = |\Theta_1|(n,3;\ell - 0) + 0$ which is true. If $j < n$ then we proceed by induction on $n$. For $n = 1$, the only value of $j$ we need to consider is $j = 0$. In that case, $3^n/2 - 3^j/2 = 3^j/2 - 3^0/2 = 1$ and $1 \leq \ell < 3^j/2$ implies $\ell = 1$. Then $|\Theta_1|(n,3;\ell) = |\Theta_1|(1,3;1) = 1$ and $|\Theta_1|(j,3;\ell - (3^n/2 - 3^j/2)) + (n-j) = |\Theta_1|(0,3;1 - (3^j/2 - 3^n/2)) + (1-0) = |\Theta_1|(0,3;0) + 1 = 1$. So we have established the base case. Assume the theorem is true for some $n-1 \geq 1$. We may assume that $j < n$, so $(3^n/2 - 3^j/2) - 3^{n-1} = 3^{n-1}/2 - 3^j/2 \leq \ell - 3^{n-1} < 3^n/2$. Now from the definition of $|\Theta_1|$, $|\Theta_1|(n,3;\ell) = |\Theta_1|(n-1,3;\ell - 3^n-1) + 1$, and by the inductive hypothesis, $|\Theta_1|(n-1,3;\ell - 3^n-1) = |\Theta_1|(j,3;\ell - 3^n-1 - (3^{n-1}/2 - 3^j/2)) + (n-1) - j$. Therefore $|\Theta_1|(n,3;\ell) = |\Theta_1|(j,3;\ell - (3^n/2 - 3^j/2)) + (n-j)$.

Lemma 7. $|\Theta_0|(n,3;\ell)$ is subadditive.

Proof. By direct computation from the definition of $|\Theta_0|$.

Definition 3. A function $g : [mN] \rightarrow \mathbb{Z}$ is the $m$-replicate of $f : [N] \rightarrow \mathbb{Z}$ if $g(\ell) = f(\ell \mod mN)$.

Example 13. $|\Theta_1|(n,3;\ell)$ is the $3$-replicate of $|\Theta(L^{-1})|(n-1,3;\ell \mod 3^{n-1}) - 1$ except that the value of $-1$ at $\ell = 0,3^{n-1}$ is replaced by $0$.

Lemma 8. If $f : \mathbb{Z}_N \rightarrow \mathbb{Z}$ is subadditive, then its $m$-replicate, $g : \mathbb{Z}_{mN} \rightarrow \mathbb{Z}$, is also subadditive.
Proof. 
\[ g((k + \ell) \mod mN) = f((k + \ell) \mod N) \]
\[ \leq f((k) \mod N) + f((\ell) \mod N) \]
\[ = g(k) + g(\ell). \]

\[ \square \]

**Lemma 9.** If \(|\Theta (L^{-1})| (n, 3; \ell)\) is strongly subadditive, then \(|\Theta_1| (n + 1, 3; \ell)\) is subadditive.

Proof. If \(|\Theta (L^{-1})| (n, m; \ell)\) is strongly subadditive, then the function
\[ |\Theta_1|'(n, 3; \ell) = \begin{cases} 0 & \text{if } \ell = 0 \\ |\Theta (L^{-1})| (n, m; \ell) - 1 & \text{if } \ell \neq 0 \end{cases} \]
is subadditive. \(|\Theta_1| (n + 1, 3; \ell)\) is just the 3-replicate of \(|\Theta_1|' (n, 3; \ell)\), so by Lemma 7, \(|\Theta_1| (n + 1, 3; \ell)\) is subadditive. \(\square\)

Proof. (of theorem 7) It follows from Lemmas 3, 4, 7 & 9 that if \(|\Theta (L^{-1})| (n, 3; \ell)\) is strongly subadditive, then \(|\Theta (L^{-1})| (n + 1, 3; \ell)\) is subadditive. However, to complete the induction step for theorem 7, we must show that \(|\Theta (L^{-1})| (n + 1, 3; \ell)\) is strongly subadditive. For that we argue by contradiction: If \(|\Theta (L^{-1})| (n + 1, 3; \ell)\) is not strongly subadditive, \(3k, \ell > 0\) such that
\[ |\Theta (L^{-1})| (n + 1, 3; k + \ell) - (|\Theta (L^{-1})| (n + 1, 3; k) + |\Theta (L^{-1})| (n + 1, 3; \ell)) = 0. \]
However \(\Delta = \Delta_0 + \Delta_1\) and since \(|\Theta_0| (S(n+1, 3); \ell)\) and \(|\Theta_1| (S(n+1, 3); \ell)\) are both subadditive, \(\Delta_0, \Delta_1 \leq 0\), so the only way that \(\Delta\) can be 0 is if both \(\Delta_0, \Delta_1 = 0\).

The zeroes of \(\Delta_0\) are easy to identify: If \(0 < k \leq \ell < 3^{n+1}/2, \) only \(\ell < 3^n/2\) can give a zero and then \(|\Theta_0| (n + 1, 3; k) = 1 = |\Theta_0| (n + 1, 3; \ell)\) & \(|\Theta_0| (n + 1, 3; k + \ell) = 2\). This happens if \(k, \ell < 3^n/2\) and \(k + \ell > 3^n/2\). Since \(k \leq \ell, \ell > 3^n/4\). For \(3^n+1/2 < k, \ell < 3^{n+1}\) there are symmetric conditions.

To show that none of these values of \(k, \ell\) that give a zero of \(\Delta_0\) also give a zero of \(\Delta_1\), we partition the values of \(k\) by \(k_j = \sum_{j=0}^{j} 3^j = (3^j+1 - 1)/2, j = 0, 1, ..., n - 1. k_0 = 1, \) the minimum possible value of \(k\). If \(k = k_0 = 1, \) then \(\ell < 3^n/2\) and \(1 + \ell > 3^n/2\) has only one solution, \(\ell = (3^n - 1)/2\). But then \(|\Theta_1| (n + 1, 3; 1) = 1\) whereas
\[ |\Theta_1| (n + 1, 3; (3^n + 1)/2) - (|\Theta_1| (n + 1, 3; (3^n - 1)/2)) \]
\[ = |\Theta (L^{-1})| (n, 3; (3^n + 1)/2) - |\Theta (L^{-1})| (n, 3; (3^n - 1)/2) \]
\[ = (n + 1) - (n + 1) = 0, \]
and we are done with \(j = 0. \) If \(j > 0\) and \(k_{j-1} < k \leq k_j\) then \(3^n+1/2 - k_j = 3^n+1/2 - (3^{j+1} - 1)/2\) so \(3^n+1/2 - 3^{j+1}/2 < \ell < 3^{n+1}/2. \) By Lemma 6 then
\[ |\Theta_1| (n + 1, 3; \ell) = |\Theta_1| (j + 1, 3; \ell - (3^{n+1}/2 - 3^{j+1}/2)) + ((n + 1) - (j + 1)). \] Also, for \((3^j - 1)/2 = k_{j-1} < k \leq k_j = (3^{j+1} - 1)/2 < 3^{j+1}/2, \) by Lemma 5 \(|\Theta_1| (n + 1, 3; k_j) = |\Theta_1| (j + 1, 3; k_j) + 1. \) Therefore if \(k_{j-1} < k \leq k_j, \)
\[ \Delta_1 = |\Theta_1| (n + 1, 3; k + \ell) - (|\Theta_1| (n + 1, 3; k) + |\Theta_1| (n + 1, 3; \ell)) \\
= |\Theta_1| (j + 1, 3; (k + \ell) - (3^{n+1}/2 - 3^{j+1}/2)) + ((n + 1) - (j + 1)) \\
= ((\Theta_1) (j + 1, 3; k + \ell) + 1) \\
= ((\Theta_1) (j + 1, 3; \ell - (3^{n+1}/2 - 3^{j+1}/2)) + ((n + 1) - (j + 1))) \\
\leq 1 + |\Theta_1| (j + 1, 3; (k + \ell) \mod 3^{j+1}) \\
\leq -1 + 0 \] by the inductive hypothesis and Lemma 9.

This covers all the values of \( k \) (and the corresponding values of \( \ell \)). \( \square \)

**Lemma 10.** If \( 0 < k \leq \ell < 3^n/2 \) & \( k + \ell > 3^n/2 \), then
\[ |\Theta (L^{-1})| (n, 3; k) + |\Theta (L^{-1})| (n, 3; \ell) \geq |\Theta (L^{-1})| (n, 3; k + \ell) + 2. \]

**Proof.** If \( 0 < k < 3^n/2 \), then
\[ |\Theta (L^{-1})| (n, 3; k) = |\Theta_0| (n, 3; k) + |\Theta_1| (n, 3; k) \] by Lemma 3,
\[ = 1 + (|\Theta (L^{-1})| (n - 1, 3; k) - 1) \] by the definitions of \( \Theta_0 \) & \( \Theta_1 \),
\[ = |\Theta (L^{-1})| (n - 1, 3; k). \]

Also since \( \ell > 3^n/2 - 3^{n-1}/2 = 3^{n-1} \), by the same reasoning,
\[ |\Theta (L^{-1})| (n, 3; \ell) = |\Theta_0| (n, 3; \ell) + |\Theta_1| (n, 3; \ell) \\
= 2 + (|\Theta (L^{-1})| (n - 1, 3; \ell - 3^{n-1}) - 1) \\
= |\Theta (L^{-1})| (n - 1, 3; \ell - 3^{n-1}) + 1. \]

Therefore
\[ |\Theta (L^{-1})| (n, 3; k) + |\Theta (L^{-1})| (n, 3; \ell) \\
= |\Theta (L^{-1})| (n - 1, 3; k) + |\Theta (L^{-1})| (n - 1, 3; \ell) + 1. \]

Also, \( k + \ell - 3^{n-1} > 3^n/2 - 3^{n-1} = 3^{n-1}/2 \), so by induction on \( n \),
\[ |\Theta (L^{-1})| (n - 1, 3; k) + |\Theta (L^{-1})| (n - 1, 3; \ell - 3^{n-1}) \\
\geq |\Theta (L^{-1})| (n - 1, 3; k + (\ell - 3^{n-1})) + 2, \]
and we have
\[ |\Theta (L^{-1})| (n, 3; k) + |\Theta (L^{-1})| (n, 3; \ell) \Theta| \\
\geq (|\Theta (L^{-1})| (n, 3; k + \ell) - 1) + 1 + 2 \\
= |\Theta (L^{-1})| (n, 3; k + \ell) + 2. \]

If, on the other hand, \( k > 3^n/2 \), then \( \ell > 3^n/2 \) and
\[ |\Theta (L^{-1})| (n, 3; k) = |\Theta (L^{-1})| (n - 1, 3; k \mod 3^{n-1}) + 1, \]
\[ |\Theta (L^{-1})| (n, 3; \ell) = |\Theta (L^{-1})| (n - 1, 3; \ell \mod 3^{n-1}) + 1, \]
\[ |\Theta (L^{-1})| (n, 3; k + \ell) \leq |\Theta (L^{-1})| (n - 1, 3; (k + \ell) \mod 3^{n-1}) + 1. \]
Therefore
\[
|\Theta(L^{-1})|(n, 3; k) + |\Theta(L^{-1})|(n, 3; \ell) \\
= (|\Theta(L^{-1})|(n - 1, 3; \ell \mod 3^n) + 1) \\
+ (|\Theta(L^{-1})|(n - 1, 3; \ell \mod 3^n) + 1), \\
\geq (|\Theta(L^{-1})|(n - 1, 3; (k + \ell) \mod 3^n)) \\
+ 1) + 2, \text{ by strong subadditivity (Theorem 7),} \\
= (|\Theta(L^{-1})|(n, 3; k + \ell) - 1) + 3, \\
= |\Theta(L^{-1})|(n, 3; k + \ell) + 2.
\]

\[\square\]

**Corollary 5.** If \(3^n/2 < k \leq \ell < 3^n\) & \(k + \ell < 3^n + 1/2\), then
\[
|\Theta(L^{-1})|(n, 3; k) + |\Theta(L^{-1})|(n, 3; \ell) \geq |\Theta(L^{-1})|(n, 3; (k + \ell) - 3^n + 2).
\]

**Proof.**
\[
|\Theta(L^{-1})|(n, 3; k) + |\Theta(L^{-1})|(n, 3; \ell) \\
= |\Theta(L^{-1})|(n, 3; 3^n - k) + |\Theta(L^{-1})|(n, 3; 3^n - \ell) \\
\quad \text{by the duality of } \Theta, \\
\geq |\Theta(L^{-1})|(n, 3; 2 \cdot 3^n - (k + \ell)) + 2 \\
\quad \text{by Lemma 10,} \\
= |\Theta(L^{-1})|(n, 3; (k + \ell) - 3^n) + 2 \\
\quad \text{by the duality of } \Theta \text{ again.}
\]

\[\square\]

5. **The Main Theorem**

**Theorem 8.** \(S(n, 3)\) has Lex nested solutions for EIP

**Proof.** (We actually prove Conjecture 2 for \(m = 3\), which implies Conjecture 1 for \(m = 3\)) By induction on \(n\):

**Initial Step:** It is true for \(n = 1\), since in Corollary 1 (of Section 4.4) we noted that the stabilization-order of \(S_{s,t}(1, m)\) is the same as that of \(S(1, m)\) which is the natural total order \(0 < 1 < ... < m - 1\).

**Inductive Step:** Assume the theorem is true for \(n \geq 1\) and that
\[
S \subseteq V_{S_{s,t}(n+1,m)} = \{0, 1, ..., m - 1\}^{n+1}
\]
with \(|S| = \ell\). We shall use the following three Steiner operations to reduce any such \(S\) to \(Lex^{-1}IJK\) (\(\{1, 2, ..., \ell\}\) (since \(IJK = t\), the identity permutation, \(Lex_{IJK} = Lex\)):

1. Apply stabilization, so we need only consider \(S\) that are stable,
2. Apply compression, utilizing the inductive hypothesis. Then we need only consider \(S\) that are compressed and stable,
3. Apply subadditivation (a StOp based on the subadditivity of \(|\Theta|(S(n, m); \ell)\) reducing \(S\) to \(Lex^{-1}(\{1, ..., \ell\})\).
StOp 1-Stabilization: See Section 4.4. \( S \) is stable iff it is an ideal in \( S-O(S_{s,t}(n+1,m)) = S-O(S(n+1,m)) \). The structure of \( S-O(S(n+1,m)) \) was discussed in Sections 4.2.1 & 4.2.2. In particular, with \( \ell_h = |S \cap ((h \times S_{s,t}(n,3))| \) we have \( \ell_0 \geq \ell_1 \geq \ldots \geq \ell_{m-1} \) and \( \sum_{h=0}^{m-1} \ell_h = \ell \). We may assume, not only that \( S \) is a stable optimal \( \ell \)-set, but that \( \ell(S) = (\ell_0, \ell_1, ..., \ell_{m-1}) \) is Lex-last for such a set. Given \( |S| = \ell = \sum_{h=1}^{m} v_h m^{n-h} \), the Lex-last \( \ell(S) \) for all \( \ell \)-sets is

\[
\ell \left( \text{Lex}^{-1}(\{1,2,\ldots,\ell\}) \right)_h = \begin{cases} 
  m^{n-1} & \text{if } h < v_0 \\
  \ell - v_0 m^{n-1} & \text{if } h = v_0 \\
  0 & \text{if } h > v_0 
\end{cases}
\]

If our optimal \( S \) is \( \text{Lex}^{-1}(\{1,2,\ldots,\ell\}) \) we are done. We may assume then that

\[ S \neq \text{Lex}^{-1}(\{1,2,\ldots,\ell\}), \]

so \( \ell(S) < \ell \left( \text{Lex}^{-1}(\{1,2,\ldots,\ell\}) \right) \).

StOp 2-Compression: See Section 4.3. Remember that the vertices in \( V_I \) are not actually in \( S \) and not counted as such, even though they can contribute to \( |\Theta_s(I)| \). Similarly for those in \( V_K \). For each \( h, 0 \leq h \leq m-1 \), we apply \( \text{Comp}_h \) to \( \text{Comp}_{h-1}(\text{Comp}_{h-2}(\ldots(\text{Comp}_0(S)))) \). After the application of \( \text{Comp}_{h-1} \), every section of \( S \cap ((h \times S_{s,t}(n,3)) \) has been compressed \( \approx \text{Lex}^{-1}(\{1,2,\ldots,\ell_{h-1}\}) \) where \( \pi = I_h J_h K_h \). We then repeat the cycle of compressions as many times as necessary. By Theorem 4 the result will eventually be constant, \( \text{Comp}_\infty(S) \). Also, \( \ell(\text{Comp}_\infty(S)) = \ell(S) \). However, the compressed set may no longer be stable. If we re-stabilize it, according to Theorem 6, either \( \ell(\text{Stab}_\infty(\text{Comp}_\infty(S))) > \ell(\text{Comp}_\infty(S)) \) or \( \text{Stab}_\infty(\text{Comp}_\infty(S)) \) is compressed. We may assume then that \( \text{Stab}_\infty(\text{Comp}_\infty(S)) \) is not only stable but compressed.

StOp 3-Subadditivation: Subadditivation is a Steiner operation based on the fact that \( |\Theta_s(\{n,3\};\ell) \) is subadditive. From StOps 1 & 2 we may assume that our \( \ell \)-set \( S \), which minimizes \( |\Theta(\ell) \) over all \( S \subseteq V_s,t(n+1,3) \) with \( |S| = \ell \), is stabilized, compressed and has a 3-tuple \( \ell(S) = (\ell_0, \ell_1, \ell_2) \) that is Lex-last over all such sets.

If \( S \) is an initial segment of Lex order we are done. If not, \( S \) is still an ideal of \( S-O(S_{s,t}(n,1,3)) \) and its intersection, \( S \cap ((\{h\} \times S_{s,t}(n,3)) \), is an initial \( \ell_h \)-segment of Lex order. Let \( h_{\min} = \min \{ h : \ell_h < 3^n \} \) and \( h_{\max} = \max \{ h : \ell_h > 0 \} \). So since \( S \neq \text{Lex}^{-1}(\{1,2,\ldots,\ell\}) \), we must have \( h_{\min} < h_{\max} \). Then let \( S' = S - S \cap \{ (h_{\min}) \times S_{s,t}(n,3) \} - S \cap \{ h_{\min} \} \times S_{s,t}(n,3) \) and we have

\[
\text{SubAdd}(S) = \left\{ \begin{array}{ll}
  S' + (h_{\min}) \times \text{Lex}_{h_{\min}}^{-1}(\ell_{\min} + \ell_{\max}) & \text{if } \ell_{\min} + \ell_{\max} \leq 3^n, \\
  S' + (h_{\min}) \times S_{s,t}(n,3) + (h_{\max}) \times \text{Lex}_{h_{\max}}^{-1}(\ell_{\max} + \ell_{\min} - 3^n) & \text{if } \ell_{\min} + \ell_{\max} > 3^n.
\end{array} \right.
\]

In either case \( |\text{SubAdd}(S)| = |S| = \ell \), so \( \text{SubAdd} \) has property 1 of a StOp. Contributions to the difference, \( \Delta = |\Theta(\text{SubAdd}(S))| - |\Theta(S)| \), come from 3 sources:

\[ \Delta_1: \] The difference in the number of interior edges (within some \( \{h\} \times S(n,3) \)) cut by \( S \) and \( \text{SubAdd}(S) \),

\[ \Delta_2: \] The difference in the number of exterior edges (from \( \{h_1\} \times S(n,3) \) to \( \{h_2\} \times S(n,3), h_1 \neq h_2 \)) cut by \( S \) and \( \text{SubAdd}(S) \),

\[ \Delta_3: \] The difference in the number of corner edges (\( \{v_h, h^{n+1}\} \)) cut by \( S \) and \( \text{SubAdd}(S) \).
And then
\[ \Delta = \Delta_I + \Delta_E + \Delta_C \]

To prove that \( \text{SubAdd} \) has Property 2 of a StOp (\( |\Theta(\text{SubAdd}(S))| \leq |\Theta(S)| \)) or equivalently, \( \Delta \leq 0 \), we reduce to cases: First we consider the possibilities for \( i = S \cap (21^n \downarrow) \), where \( 21^n \downarrow \) denotes the ideal (in \( \mathcal{S} \mathcal{O} \)) generated by \( \{i, e \} \), i.e., below \( 21^n \). Since \( 21^n \) is the maximal element of the component whose minimum element is \( 01^n \), \( 21^n \downarrow \) is exactly that same component. Since \( S \) is stable and therefore an ideal in \( \mathcal{S} \mathcal{O} \( S_{s,t}(n+1,3)) \), \( i = S \cap (21^n \downarrow) \) is an ideal of that component. As we saw in Example 10, \( 21^n \downarrow \) has 9 ideals so there are just 9 possible intersections. Next we consider the possible values of \( (s, t) \) with \( s, t \geq 0 \) \& \( s + t \leq m \). In general there are \( \binom{m+2}{2} = (m+2)(m+1)/2 \) such ordered pairs. For \( m = 3 \) then there are then 10 pairs. Since the endpoints of all exterior edges of \( S_{s,t}(n+1,3) \) are in \( 21^n \downarrow \), and the other ends of corner edges are in \( I \cup J, S \cap (21^n \downarrow) \) and \( (s, t) \) determine \( I_h, J_h, K_h \) which determine the relative order of the digits, 0, 1 \& 2, in the definition of \( \text{Lex}_h \). Altogether these possibilities give \( 9 \times 10 = 90 \) cases to be considered. However, duality reduces that number by almost half (to 46, two of the cases being self-dual) and many of those cases are trivial. Some are not, however and even have several subcases. For each case a range of values of \( \ell_0, \ell_1, \ell_2 \) are possible. In each case we show that if \( S \) is not already \( \text{Lex}^{-1}\{1, 2, \ldots, \ell\} \), then subadditivation will nontrivially reduce it \( (\ell(\text{SubAdd}(S)) > \text{Lex}(\ell(S))) \).

For the first four cases below \((1a - 1d)\), we proceed step-by-step. After that we leave out routine steps. In order to verify the proof, the reader should fill in the missing steps.

\( \Delta = 0 \)

1. \( i = \emptyset \)
   (a) \( (s, t) = (0,0) \): Then
   \[ I_0 = \emptyset, \quad J_0 = \emptyset \quad \text{so} \quad K_0 = \{0,1,2\} \quad \text{and} \quad 0 <_0 0 <_0 2, \]
   \[ I_1 = \emptyset, \quad J_1 = \emptyset \quad \text{so} \quad K_1 = \{0,1,2\} \quad \text{and} \quad 0 <_1 1 <_1 2, \]
   \[ I_2 = \emptyset, \quad J_2 = \emptyset \quad \text{so} \quad K_2 = \{0,1,2\} \quad \text{and} \quad 0 <_2 1 <_2 2. \]
   \( 10^n \notin S \) \& \( 0 <_1 0 <_1 2 \Rightarrow \ell_1 = 0 \Rightarrow h_{\text{max}} = 0 \) so the conclusion \( (\Delta = 0) \) is trivial.
   (b) \( (s, t) = (1,0) \): Then
   \[ I_0 = \{1\}, \quad J_0 = \emptyset \quad \text{so} \quad K_0 = \{1,2\} \quad \text{and} \quad 0 <_0 1 <_0 2, \]
   \[ I_1 = \emptyset, \quad J_1 = \emptyset \quad \text{so} \quad K_1 = \{0,1,2\} \quad \text{and} \quad 0 <_1 1 <_1 2, \]
   \[ I_2 = \emptyset, \quad J_2 = \emptyset \quad \text{so} \quad K_2 = \{0,1,2\} \quad \text{and} \quad 0 <_2 1 <_2 2. \]
   \( 10^n \notin S \) \& \( 0 <_1 1 <_1 2 \Rightarrow \ell_1 = 0 \Rightarrow h_{\text{max}} = 0 \) so the conclusion \( (\Delta = 0) \) is trivial.
   (c) \( (s, t) = (2,0) \): Then
   \[ I_0 = \{0\}, \quad J_0 = \emptyset \quad \text{so} \quad K_0 = \{1,2\} \quad \text{and} \quad 0 <_0 1 <_0 2, \]
   \[ I_1 = \{1\}, \quad J_1 = \emptyset \quad \text{so} \quad K_1 = \{0,2\} \quad \text{and} \quad 1 <_1 0 <_1 2, \]
   \[ I_2 = \emptyset, \quad J_2 = \emptyset \quad \text{so} \quad K_2 = \{0,1,2\} \quad \text{and} \quad 0 <_2 1 <_2 2. \]
   \( 20^n \notin S \) \& \( 0 <_1 1 <_1 2 \Rightarrow \ell_2 = 0, \) so \( h_{\text{max}} = 1. \) \( 01^n \notin S \) \& \( 0 <_1 1 <_1 2 \Rightarrow \ell_0 < 3^n/2. \)
   (i) If \( \ell_0 + \ell_1 < 3^n/2 \) then \( \Delta_E = 0 \& \Delta_C = 1 \) so \( \Delta \leq 0 \) by Theorem 7.
(ii) If $\ell_0 + \ell_1 > 3^n/2$ then $\Delta_E = 1$ & $\Delta_C = 1$ but $\Delta \leq 0$ by Lemma 10.

(d) $(s, t) = (3, 0)$: Then

$I_0 = \{0\}, J_0 = \emptyset$ so $K_0 = \{1, 2\}$ and $0 <_0 1 <_0 2$.
$I_1 = \{1\}, J_1 = \emptyset$ so $K_1 = \{0, 2\}$ and $1 <_1 0 <_1 2$.
$I_2 = \{2\}, J_2 = \emptyset$ so $K_2 = \{0, 2\}$ and $2 <_2 0 <_2 1$.

$01^n \notin S$ & $0 <_0 1 <_0 2 \Rightarrow \ell_0 < 3^n/2$. Also, $\ell_2 \leq \ell_1 \leq \ell_0$.

(i) If $\ell_2 = 0$, the result ($\Delta = 0$) follows as in Case 1c.

(ii) If $\ell_2 > 0$, $h_{\text{min}} = 0$ & $h_{\text{max}} = 2$.

(A) If $\ell_0 + \ell_2 < 3^n/2$ then $\Delta_E = 0$ & $\Delta_C = 1$ so $\Delta \leq 0$ by Theorem 7.

(B) If $\ell_0 + \ell_2 > 3^n/2$ then $\Delta_E = 1$ & $\Delta_C = 1$ but $\Delta \leq 0$ by Lemma 10.

(e) $(s, t) = (0, 1)$: Then $0 < h_1 < h_2$, for $h = 0, 1, 2$ and the result ($\Delta = 0$) follows as in Case 1a.

(f) $(s, t) = (1, 1)$: Then $0 < h_1 < h_2$, for $h = 0, 2$ but $1 <_1 0 <_1 2$.

$0 < \ell_1 \leq \ell_0 < 3^n/2$ & $\ell_2 = 0$. $h_{\text{min}} = 0$ & $h_{\text{max}} = 1$. $\Delta_E \leq 1$ & $\Delta_C = 0$, so by Theorem 7, $\Delta \leq 0$.

(g) $(s, t) = (2, 1)$: Then $0 <_0 1 <_0 2$, $1 <_1 0 <_1 2$ and $2 <_2 0 <_2 1$. The result ($\Delta \leq 0$) follows as in Case 1d.

(h) $(s, t) = (0, 2)$: Then $0 < h_1 < h_2$, for $h = 0, 2$ but $1 <_1 0 <_1 2$. The result ($\Delta \leq 0$) follows as in Case 1f.

(i) $(s, t) = (1, 2)$: Then $0 <_0 1 <_0 2$, $1 <_1 0 <_1 2$ and $2 <_2 0 <_2 1$. The result ($\Delta \leq 0$) follows as in Case 1d.

(j) $(s, t) = (0, 3)$: Then $0 <_0 1 <_0 2$, $1 <_1 0 <_1 2$ and $2 <_2 0 <_2 1$. The result ($\Delta \leq 0$) follows as in Case 1d.

(2) $i = \{01^n\}$: Since $01^n \in S$, $10^n \notin I_1$ and so $Lex_{1}^{-1}(1) = 10^n$ no matter what $(s, t)$ is. Since $10^n \notin S$, $\ell_1 = 0 = \ell_2$ and our result is trivial ($\Delta = 0$).

(3) $i = \{01^n, 02^n\}$: The same reasoning as for Case 2 applies and $\Delta = 0$.

(4) $i = \{01^n, 10^n\}$:

(a) $(s, t) = (0, 0)$: Then $1 <_0 0 <_0 2$ & for $h = 1, 2$, $0 < h < h_2$. Therefore $0 < \ell_1 \leq \ell_0 < 3^n$ & $\ell_2 = 0$. Therefore $h_{\text{min}} = 0$ & $h_{\text{max}} = 1$.

(i) If $\ell_0 < 3^n/2$ $(\Rightarrow \ell_1 < 3^n/2 & \ell_0 + \ell_1 < 3^n)$,

(A) If $\ell_0 + \ell_1 < 3^n/2$, $\Delta_E = 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$.

(B) If $\ell_0 + \ell_1 > 3^n/2$, $\Delta_E = 1$ & $\Delta_C = 1$ but Lemma 10 implies that $\Delta \leq 0$

(ii) If $\ell_0 > 3^n/2$,

(A) If $\ell_0 + \ell_1 < 3^n$ $(\Rightarrow \ell_1 < 3^n/2)$, $\Delta_E = 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$.

(B) If $\ell_0 + \ell_1 = 3^n$, $\Delta_E = 2$ & $\Delta_C = 0$, but by Theorem 2,

$\Delta_I \leq 0 - (2 + 2) = -4$, so

$\Delta = \Delta_I + \Delta_E + \Delta_C$

$\leq (-4) + 2 + 0$

$= -2 < 0$.

(C) If $\ell_0 + \ell_1 > 3^n$, then $\Delta_E = 1$ & $\Delta_C \leq 0$, so Theorem 7
implies that $\Delta \leq 0$.

(b) $(s,t) = (1,0)$: Then $0 < h_1 < h_2$, for $h = 0, 1, 2$. Therefore $3^n/2 < \ell_0 < 3^n$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 = 0$, so $h_{\text{min}} = 0$ and $h_{\text{max}} = 1$.

(i) If $\ell_1 < 3^n/2$

(A) If $\ell_0 + \ell_1 < 3^n$, $\Delta_E = 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$.

(B) If $\ell_0 + \ell_1 = 3^n$, $\Delta_E = 2$ & $\Delta_C = 0$ but by Theorem 2, $\Delta \leq 0$.

(C) If $\ell_0 + \ell_1 > 3^n$, $\Delta_E = 1$ & $\Delta_C \leq 0$ so Theorem 7 implies that $\Delta \leq 0$.

(ii) If $\ell_1 > 3^n/2$ ($\Rightarrow \ell_0 + \ell_1 > 3^n$), then $\Delta_E = 1$ & $\Delta_C \leq 0$ so by Theorem 7, $\Delta \leq 0$.

(c) $(s,t) = (2,0)$: Then $0 < h_1 < h_2$, for $h = 0, 1, 2$. Therefore $3^n/2 < \ell_0 < 3^n$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 = 0$, so $h_{\text{min}} = 0$ and $h_{\text{max}} = 1$.

(i) If $\ell_1 < 3^n/2$ the result follows essentially as for (4b).

(ii) If $\ell_1 > 3^n/2$ ($\Rightarrow \ell_0 + \ell_1 > 3^n$),

(A) If $\ell_0 + \ell_1 < 3^{n+1}/2$, $\Delta_E = 1$ & $\Delta_C = 1$ but by the dual of Lemma 10 (Corollary 5), $\Delta \leq 0$.

(B) If $\ell_0 + \ell_1 > 3^{n+1}/2$, $\Delta_E = 1$ & $\Delta_C = 0$ so by Theorem 7, $\Delta \leq 0$.

(d) $(s,t) = (3,0)$: Then $0 < h_1 < h_2$, for $h = 0, 1 & 2 < 2 < 2$. Therefore $3^n/2 < \ell_0 < 3^n$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 \leq \min \{\ell_1, 3^n/2\}$.

(i) If $\ell_2 = 0$ then the result follows as for (4c).

(ii) If $\ell_2 > 0$ then $h_{\text{min}} = 0$ & $h_{\text{max}} = 2$.

(A) If $\ell_0 + \ell_2 < 3^n$, then $\Delta_E = 0$ & $\Delta_C = 1$ so by Theorem 7, $\Delta \leq 0$.

(B) If $\ell_0 + \ell_2 = 3^n$, then $\Delta_E = 1$ & $\Delta_C = 1$ so by Theorem 2, $\Delta \leq 0$.

(C) If $\ell_0 + \ell_2 > 3^n$, then $\Delta_E = 1$ & $\Delta_C = 0$ so by Theorem 7, $\Delta \leq 0$.

(e) $(s,t) = (0,1)$: Then $1 < 0 < 0 < 2$ & for $h = 0, 1, 0 < h_1 < 2$. Therefore $0 < \ell_0 < 3^n$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 = 0$, so $h_{\text{min}} = 0$ and $h_{\text{max}} = 1$.

(i) If $\ell_0 + \ell_1 < 3^n$ then $\Delta_E = 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$.

(ii) If $\ell_0 + \ell_1 = 3^n$, then $\Delta_E = 2$ & $\Delta_C = 0$ but by Theorem 2, $\Delta \leq 0$.

(iii) If $\ell_0 + \ell_1 > 3^n$, then $\Delta_E = 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$.

(f) $(s,t) = (1,1)$: Then $0 < h_1 < h_2$, for $h = 0, 1, 2$. Therefore $3^{n+1}/2 < \ell_0 < 3^{n+1}$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 = 0$. It then follows as in Case 4c that $\Delta \leq 0$.

(g) $(s,t) = (2,1)$: Then for $h = 0, 1$, $0 < h_1 < h_2$ & $2 < 2 < 2$. Therefore, $3^n/2 < \ell_0 < 3^n$, $0 < \ell_1 \leq \ell_0$ and $\ell_2 \leq \min \{\ell_1, 3^n/2\}$. Therefore, $h_{\text{min}} = 0$.

(i) If $\ell_2 = 0$ then the result, $\Delta \leq 0$, follows as in 4c.

(ii) If $\ell_2 > 0$ then $h_{\text{max}} = 2$. Whatever $\ell_0 + \ell_2$ is, $\Delta_E \leq 1$ & $\Delta_C = 0$ so Theorem 7 implies that $\Delta \leq 0$. 

(h) \((s, t) = (0, 2)\): Then \(1 < 0 < 0 < 2\) and for \(h = 0, 1, 0 < h 1 < 2\).
Therefore \(0 < \ell_0 < 3^n, 0 < \ell_1 \leq \ell_0 \& \ell_2 = 0\), so \(h_{min} = 0 \& h_{max} = 1\).
The subsequent cases are the same as Case 4e.
(i) \((s, t) = (1, 2)\): Then for \(h = 0, 1, 0 < h 1 < 2 & 2 < 2 0 < 2 1\).
Therefore, \(3^n/2 < \ell_0 < 3^n, 0 < \ell_1 \leq \ell_0 \& \ell_2 \leq \min \{\ell_1, 3^n/2\}\).
Therefore, \(h_{min} = 0\).
(i) If \(\ell_2 = 0\) \(\Rightarrow h_{max} = 1\), the result, \(\Delta \leq 0\), follows as in 4c.
(ii) If \(\ell_2 > 0\), \(\Rightarrow h_{max} = 2\), then \(\Delta_E \leq 1 \& \Delta_C = 0\) so Theorem 7 implies that \(\Delta \leq 0\).
(j) \((s, t) = (0, 3)\): Then \(1 < 0 0 < 0 1 < 1 2 & 2 < 2 0 < 2 \).
Therefore \(3^n/2 < \ell_0 < 3^n, 0 < \ell_1 \leq \ell_0 \& \ell_2 \leq \min \{\ell_1, 3^n/2\}\) so \(h_{min} = 0\).
The result, \(\Delta \leq 0\), then follows as in Case 4i.
(5) \(i = \{01^n, 10^n, 02^n\}\): All these cases are trivial \((\Delta = 0)\) because
\[(0 \in I_2 \& 20^n \notin S) \Rightarrow (\ell_2 = 0)\]
and
\[(2 \in K_0 \& 02^n \in S) \Rightarrow (\ell_0 = 3^n).\]

All other stable \& compressed sets are dual to one of those we have considered above, so our proof is complete. \(\square\)

6. Conclusions \& Comments

6.1. Possibilities for \(m > 3\). The complexity of the final phase of the argument (Section 5) prohibited us from carrying it through for \(m > 3\). For \(m = 3\) the initial estimate of the number of cases was 90 (the actual number of cases (last part of Section 5) was 41). The corresponding estimate for \(m = 4\) is the product of 28 (= \(|S-S\cdot S\cdot (32 \downarrow)|\), see Figure 5) by 15 (= \(4 + 2\)) which is 420. \(m = 3\) already pushed our limits for hand computation so we are hoping to do \(m \geq 4\) by computer.

6.2. Our Logical Strategy (3 StOps).

(1) This strategy is modeled on that of the first proof of Theorem 1.1 in [4].
That proof for had its origins in our first paper (1962) but was only included in the monograph [4] to show how complicated proofs of such theorems could be. It was sufficiently complicated that in the original paper we missed a case. Fortunately A. J. Bernstein noticed the oversight and filled in the missing argument (Lemma 1.3 of [4]). After developing the theory of Steiner operations in Chapters 2 \& 3 of [4], stabilization and compression were used to give a relatively short (and easily verified) proof of Theorem 1.1 (Sec. 3.3.5 of [4]). In this paper Bernstein’s lemma developed into the Steiner operation subaddification.

More recently, in [3], we showed that initial \(\ell\)-segments of Hales order (see [4], p. 56) maximize Type over all stable \(\ell\)-sets of vertices of the \(n\)-cube, \(Q_n\). That proof, based on the self-similarity of the stabilization order of \(Q_n\) (also known as its Bruhat order (see [4], Sect. 5.2)) seemed novel at first, until we realized that the first proof of Theorem 1.1 in [4], by not using the full power inherent in compression, treated \(Q_n\) as a self-similar structure. This suggested the possibility of proving isoperimetric theorems for other self-similar structures. The first target for our project to solve isoperimetric problems on other self-similar structures was the EIP on the Sierpinski gasket graph. \(SG_n\) is self-similar but has no product
decomposition and relatively little symmetry (so a proof technique beyond compression and stabilization is required).

(2) The proof of our Main Theorem is by induction, but the reduction of $S_{s,t}(n+1,3)$ to $S_{s,t}(n,3)$ is based on three different Steiner operations, stabilization, compression & subadditivation. Each reduces the number of possible solution $\ell$-sets, the last reducing it to just one, $\text{Lex}^{-1}(\{1, 2, ..., \ell\})$. This logic, which follows naturally from the notion of morphism, is essentially the same as that of Steiner’s ”proof” of the classical isoperimetric theorem in the Euclidean plane (c. 1840): If $S$ is any closed set other than a disk, it can be transformed by symmetrization (the original Steiner operation) to a set having the same area and smaller boundary. Therefore the only possible solution set (up to isomorphism) is a disk.

Steiner’s symmetrization was (as far as we know) the first noninvertible morphism in mathematics. Symmetry and similarity were known to Euclid, of course, and Galois made use of symmetry. However, as Weierstrass pointed out, Steiner’s epoch-making insight was incomplete: It was still logically possible that the isoperimetric problem had no solution; that the greatest lower bound of all boundary lengths might not be achieved by any set. Weierstrass was a pioneer in functional analysis but it took another 40 years to clear up this last detail. Combinatorial StOps do not suffer from this problem. By finiteness, if there is only one set fixed by a StOp, it has to be a solution.

(3) The flexibility and adaptability of Steiner operations continues to be amazing & gratifying. The 3 StOps in the proof of our theorem, though based on those in the proof of Theorem 1.1 of [4], had to be substantially modified to accomplish the purpose. They held up well. However, the additional complications made the complexity of its role model, the proof of Theorem 1.1 of [4], seem insignificant by comparison.

6.3. A Coincidence? $K^m_n$ and $S(n, m)$ have the same set of vertices, $\{0, ..., m-1\}^n$ but very different sets of edges. It is curious that their EIPs have a common solution, initial segments of Lex order.

6.4. The Nested Solutions Property. The nested solutions property is shared by most isoperimetric problems, finite and continuous, that have been solved. However, continuous isoperimetric problems may also be solved by variational means, even if they do not have nested solutions. Variational methods do not work well for combinatorial isoperimetric problems without nested solutions because the solution sets become chaotic near the break. What has worked on several combinatorial isoperimetric problems lacking nested solutions is passage to a continuous limit and solving the resulting continuous isoperimetric problem (see Chapter 10 of [4]). With Sierpinski graphs however, we have (surprisingly) the opposite situation: $S(n, m)$, the generalized & expanded Sierpinski graph, is fractal but solutions of its isoperimetric problems are nested. The main challenge was to adapt compression to the recursive structure of $S(n, m)$.

6.5. The Isoperimetric Problem on $SG_\infty$. The history of continuous variational problems (such as the classical isoperimetric problem in the plane and the brachystochrone problem) shows that they present two challenging questions:

(1) What is the solution?
(2) (Assuming we "know" the solution) Can we prove it?

Galileo erred on question 1, guessing that the solution of the brachystochrone problem was the arc of a circle. And Archimedes knew that the solution of the isoperimetric problem was the disk but a logically rigorous proof eluded mathematicians until the late 19th century. So what are the answers to these questions for the isoperimetric problem on the Sierpinski gasket?

For Question 1 we claim that $SG_\infty$ has nested solutions given by the function, $$\eta^{-1} : [0, 1] \to SG_\infty,$$
defined by

$$\eta^{-1}(a) = \left( \sum_{a_n=0}^{\infty} 2^{-i}, \sum_{a_n=1}^{\infty} 2^{-i}, \sum_{a_n=2}^{\infty} 2^{-i} \right),$$

where $a = \sum_{i=1}^{\infty} a_i 3^{-i}$ is the base 3 representation of $a \in [0, 1]$ and $SG_\infty$ is constructed in $\mathbb{R}^3$ starting with the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$ & $(0, 0, 1)$. If $a$ is a triadic rational then it has two base 3 representations (such as $1/3 = \sum_{i=1}^{\infty} 3^{-i} = \sum_{i=2}^{\infty} 2 \cdot 3^{-i}$). In that case use the infinite one in calculating $\eta^{-1}(a)$.

Example 14. $\eta^{-1}(1/3)$

$$= \eta^{-1} \left( \sum_{i=2}^{\infty} 2 \cdot 3^{-i} \right)$$

$$= \left( \sum_{i=1}^{1} 2^{-i}, 0, \sum_{i=2}^{\infty} 2^{-i} \right)$$

$$= (1/2, 0, 1/2).$$

Example 15. $\eta^{-1}(1/2)$

$$= \eta^{-1} \left( \sum_{i=1}^{\infty} 1 \cdot 3^{-i} \right)$$

$$= (0, \sum_{i=1}^{\infty} 2^{-i}, 0)$$

$$= (0, 1, 0).$$

Example 16. $\eta^{-1}(1/6)$

$$= \eta^{-1} \left( \sum_{i=2}^{\infty} 1 \cdot 3^{-i} \right)$$

$$= (\sum_{i=1}^{1} 2^{-i}, \sum_{i=2}^{\infty} 2^{-i}, 0)$$

$$= (1/2, 1/2, 0).$$

This function, $\eta^{-1}$, is the limit, as $n \to \infty$, of the composition of $Lex^{-1} : \{1, 2, ..., 3^n\} \to S(n, 3)$ with the embedding, $y : S(n, 3) \to \mathbb{R}^3$, of Section 3.1.3. The insight behind our claim is that as $n \to \infty$, the EIP on $S(n, 3)$ converges to the natural isoperimetric problem on the Sierpinski gasket. Intuitively, as $n \to \infty$, and
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$S(n, 3) \to SG_\infty$, the edges, of length $1/2^n$, are shrinking to 0. So that, in the limit, the edge-boundary becomes the topological boundary. The isoperimetric profile of the Sierpinski gasket is then the limit of the edge-isoperimetric profile of $S(n, 3)$ as $n \to \infty$: If $\lambda(a)$ denotes the minimum "length" of the topological boundary of any closed set in $SG_\infty$ of "area" $a$, then

$$\lambda(a) = \begin{cases} 
|\Theta(S(n, 3); \ell)| & \text{if } a = \ell/3^n, \text{ a triadic rational}, \\
\omega & \text{(countable } \infty \text{) otherwise.}
\end{cases}$$

For a triadic rational, $\ell/3^n$, $0 < \ell < 3^n$, $\lambda(a)$ is actually given by the formula of Proposition 1 with $m = 3$ :

$$\lambda(\ell/3^n) = \sum_{h=1}^{n} \ell_h (3 - \ell_h) + |\ell'_h - \ell_h| - \ell_h,$$

where $0 < \ell < 3^n$,

$$\ell + 1 = \sum_{h=1}^{n} \ell_h 3^{n-h} \text{ and }$$

$$\ell'_h = 1 + \max \{ j : j^{n-h} \leq \text{Lex}(\ell_{h+1}, \ell_{h+2}, ..., \ell_n) \}.$$

Note that $\lambda(a)$ is countable infinity except at a countably infinite set (which is necessarily of measure zero)!

This result, the solution of a continuous isoperimetric problem by combinatorial means, is the realization of a longheld fantasy of the author. Having appropriated so much from the classical analytic theory of isoperimetric problems, it is gratifying to be able to give something back. Also, we are grateful to Michel Lapidus, UCR colleague and expert on fractal geometry, for his encouragement and intellectual support of this project.

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7. Appendix

7.1. How Many Components in $S\cdot O(S(n,m))$? A vertex $v \in \{0, 1, \ldots, m-1\}^n$ may be thought of as an ordered partition of $\{1, \ldots, n\}$ into $m$ blocks: $v = (v_1, v_2, \ldots, v_n)$ corresponds to $p(v) = (v^{-1}(0), v^{-1}(1), \ldots, v^{-1}(m-1))$. Note that if $i$ does not appear in $v$, then $v^{-1}(i) = \emptyset$, the empty set. The covering relations, $v \prec v'$, of $S\cdot O(S(n,m))$ are given by transpositions of consecutive integers, $i(i+1)$ such that $\min \{j : v_j = i\} < \min \{j : v_j = i+1\}$ operating on the coordinates of $v$ to give $v'$. Those transpositions generate all permutations of the components of $p(v)$ and components are characterized by their common unordered partition of $\{1, \ldots, n\}$ into $m$ or fewer blocks. Every component has a unique minimum element in $C'_0$, so the number of components is the same as $|C'_0|$. This number is $\sum_{k=1}^{m} S_{n,k}$, $S_{n,k}$ being the Sterling number of the second kind. The $S_{n,k}$'s are well known (see The On-line Encyclopedia of Integer Sequences).

7.2. The Derivation of Compression for Products. In [4] compression is a Steiner operation on a product of graphs, $G \times H$, based on at least one of the factors (say $G$) having nested solutions. If $|S \cap (G \times \{w\})| = \ell_w$, then

Question: What lower bound can be inferred on $|\Theta(S)|$ for $S \subseteq V_{G \times H}$, $|S| = \ell$?

Answer: $|\Theta(S)| \geq \sum_{w \in V_H} |\Theta(G; \ell_w)| + \sum_{\{w_1, w_2\} \in E_H} |\ell_{w_1} - \ell_{w_2}|$.

This lower bound can be achieved if $\eta : V_G \to \{1, 2, \ldots, |V_G|\}$ is a numbering of the vertices of $G$ such that $|\Theta(\eta^{-1}(\{1, 2, \ldots, \ell\}))| = |\Theta(G; \ell)|$. The existence of such a numbering is the definition of nested solutions. See Section 1.2.2.

We define compression by

$$\text{Comp}_{\eta, G \times H}(S) = \bigcup_{w \in V_H} (\eta^{-1}(\{1, 2, \ldots, \ell_w\}) \times \{w\})$$

and then

Theorem 9. (Theorem 3.4 of [4]) $\text{Comp}_{\eta, G \times H}$ is a Steiner operation.

Property 1 is trivial but the crucial Property 2 (of a StOp) is also easy to verify: $\eta^{-1}(\{1, 2, \ldots, \ell_w\}) \times \{w\}$ minimizes edges cut within $G \times \{w\}$ by a set of cardinality $\ell_w$ and all edges between $G \times \{w_1\}$ and $G \times \{w_2\}$ are of the form $\{(v, w_1), (v, w_2)\}$. The obvious lower bound on the number of such edges cut by any $S$ given $\ell_{w_1}, \ell_{w_2}$ is $|\ell_{w_1} - \ell_{w_2}|$, which is achieved by $\text{Comp}_{\eta, G \times H}(S)$. 

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