Horn maps of holomorphic functions locally
pseudo-conjuguated on their local parabolic basin

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Abstract

The lifted horn map $h$ of a holomorphic function $f$ with a simple parabolic point at $z_0$
is well known to be a local conjugacy complete invariant at $z_0$; this is a classical result
proved independently by Écalle [É75], Voronin [Vor81], Martinet and Ramis [MR83].
Landford and Yampolski have shown in [LY] that, if two functions $f_1, f_2$ with simple
parabolic points at $z_1, z_2$ are globally conjugated over their immediate parabolic basins,
with the conjugacy and its inverse continuous at $z_1$, resp. $z_2$, their horn maps must be
equal as analytic ramified coverings. In this article, we introduce a notion of local con-
jugacy over immediate parabolic basins, and show that two functions $f_1, f_2$ with simple
parabolic points are locally conjugated over parabolic basins (without the hypothesis of
continuity) if and only if the non-lifted horn maps $h_i$ are equivalent as ramified cov-
erings. This result is a first step to understand better invariant classes by parabolic
renormalization.
1 Introduction

Let \( f \) be a holomorphic function with a parabolic point at 0 of multiplier 1 and one petal. This amounts to the fact that \( f \) admits a Taylor expansion at 0 of the form \( f(z) = z + az^2 + o(z^2) \) where \( a \in \mathbb{C}^* \). The pair of germs\(^1\)

\[(h_g^+, h_g^-)\]

(here the index \( g \) denotes the initial of the word "germ") at \( \pm i\infty \) of the lifted horn map \( h \) of \( f \) is well known to be a local conjugacy complete invariant; this is a classical result proved independently by Écalle [É75], Voronin [Vor81], Martinet and Ramis [MR83].

**Theorem 1.1.** Let \( f_1, f_2 \) be holomorphic functions from an open neighborhood of 0 \( \in \mathbb{C} \) to \( \mathbb{C} \), with the following Taylor expansion at 0:

\[ f_i(z) = z + a_i z^2 + o(z^2) \]

where \( a_i \neq 0 \).

Suppose that \( f_1, f_2 \) are conjugated in a neighborhood of 0 via a local biholomorphism \( \phi \). Then the germs at \( \pm i\infty, h_1^{+g}, h_2^{-g} \) of the horn maps \( h_1, h_2 \) are equivalent as analytic ramified coverings\(^2\) via translations of \( \mathbb{C} \) at the domain and at the range. More precisely, there exists \( \sigma, \sigma' \in \mathbb{C} \) (independent of the sign \( \pm \)) such that:

\[ h_1^{\pm} = T\sigma \circ h_2^{\pm} \circ T\sigma' \]

where \( T\sigma \) denotes the translation of the cylinder \( \mathbb{C}/\mathbb{Z} \) given by the formula \( z \rightarrow z + \sigma \).

Conversely, if there exists \( \sigma, \sigma' \in \mathbb{C} \) (independent of the sign \( \pm \)) such that

\[ h_1^{\pm} = T\sigma \circ h_2^{\pm} \circ T\sigma' \]

then \( f_1, f_2 \) are analytically conjugated in a neighborhood 0.

The aim of this paper is to build up a complete invariant with a more flexible covering equivalence. We will introduce a notion of local pseudo-conjugacy designed for immediate parabolic basins. We no longer work with germs of horn maps, but with bigger definition domains. Let \( D^{\pm} \subset \mathbb{C}/\mathbb{Z} \) be the connected components of the definition domain of the horn map \( h \) containing a punctured neighborhood of \( \pm i\infty \). Let \( h^{\pm} \) be the map \( h : D^{\pm} \rightarrow h(D^{\pm}) \).

For each open neighborhood \( U_1 \) of 0 \( \in \mathbb{C} \), define \( U_1^0 \) the connected component of \( U_1 \cap B_{0}^{\pm} \) containing a germ of the attractive axis of \( f_1 \).

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\(^1\)Let \( z \in S, S' \) where \( S, S' \) are Riemann surfaces, and \( U \) an open subset of \( S \) containing \( z \). The germ of the holomorphic function \( f \in \mathcal{O}(U, S') \) (holomorphic function from \( U \) to \( S' \)) at \( z \) is given by the equivalence class of \( f \) for the equivalence relation \( \sim \) over \( \mathcal{O}(U, S') \) defined by \( g \sim h \) if and only if there exists an open set \( V \) containing \( z \) such that \( g|_V = h|_V \).

\(^2\)This theorem is usually formulated in this way: \( h_1 \) and \( h_2 \) are equal modulo post-composition and pre-composition by translations. We use there this terminology of analytic ramified covering equivalence because it allows an analogy with the theorem 1.4 given below.
**Definition 1.2.** A holomorphic function \( \phi \) is said to be a local semi-conjugacy of \( f_1, f_2 \) on their immediate basins if there exists \( U_1 \) an open neighborhood of \( 0 \in \mathbb{C} \) such that \( \phi : U_1^0 \to B_{f_2}^0 \) is a semi-conjugacy, namely:

\[
\phi \circ f_1(z) = f_2 \circ \phi(z) \text{ for } z \in U_1^0 \text{ such that } f_1(z) \in U_1^0.
\]

It is important to note that the range of \( \phi \) is not required to be a small neighborhood of \( 0 \), i.e. \( \phi \) is allowed to map points close to \( 0 \) anywhere in the immediate basin of \( f_2 \).

A semi-conjugacy \( \phi \) induces a map \( \overline{\phi} : B^{f_1} / f_1 \to B^{f_2} / f_2 \) in the following way: for \( z = \overline{z} \) mod \( f_1 \in B^{f_1} / f_1 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( f_1^n(z) \in U_1^0 \). The element \( \phi(f_1^n(z)) \mod f_2 \in B_{f_2}^0 \) is independent of the representative \( z \) of \( \overline{z} \) and of the chosen integer \( n \geq N \). We will see that \( \overline{\phi} \) is necessarily a biholomorphism in proposition 5.17.

**Definition 1.3.** Let \( (\phi, \phi') \) denote a pair of local semi-conjugacies, i.e. \( \phi : U_1^0 \to B_{f_2}^0 \) local semi-conjugacy of \( f_1, f_2 \) and \( \phi' : U_2^0 \to B_{f_1}^0 \) local semi-conjugacy of \( f_2, f_1 \). We say that \( (\phi, \phi') \) is a local pseudo-conjugacy of \( f_1, f_2 \) if \( \overline{\phi} : B^{f_1} / f_1 \to B^{f_2} / f_2 \) has for inverse \( \overline{\phi} : (B^{f_2} / f_2) \to (B^{f_1} / f_1) \).

For two maps \( f_1, f_2 \) to be locally pseudo-conjugated at 0, it is sufficient that they are globally conjugated on their respective parabolic immediate basins.

Here is the principal theorem of the paper. It is proven in part 6 in two separated propositions (4 propositions if one counts the complements):

**Theorem 1.4.** Let \( f_1, f_2 \) denote holomorphic maps from an open neighborhood of \( 0 \in \mathbb{C} \) to \( \mathbb{C} \), with Taylor expansion at 0: \( f_i(z) = z + a_i z^2 + o(z^2) \) where \( a_i \neq 0 \).

Suppose \( f_1, f_2 \) locally semi-conjugated at 0 on \( B_{f_1}^0 \). Then there exists \( \sigma \in \mathbb{C} \) and a pair of holomorphic maps \( \psi = (\psi^+, \psi^-) \), where \( \psi^\pm : D_{1}^\pm \to D_2^\pm \), such that:

\[
h_{1}^{\pm} = \sigma \circ h_{2}^{\pm} \circ \psi^\pm
\]

and such that \( \psi^\pm \) have removable singularities at \( \pm i \infty \).

Conversely, if there exists \( \sigma \in \mathbb{C} \) and a pair of holomorphic maps \( \psi = (\psi^+, \psi^-) \), where \( \psi^\pm : D_{1}^\pm \to D_2^\pm \), with removable singularities at \( \pm i \infty \) such that:

\[
h_{1}^{\pm} = \sigma \circ h_{2}^{\pm} \circ \psi^\pm
\]

then \( f_1, f_2 \) are locally semi-conjugated at 0 on their immediate basins.

Furthermore, the maps \( \psi^\pm \) have at \( \pm i \infty \) an expansion of the form \( \psi^\pm(w) = w + \rho^\pm + o(1) \).

**Remark 1.5.** Theorem 1.1 involves the germs \( h_{1g}^{\pm}, h_{2g}^{\pm} \) of the lifted horn maps \( h_1, h_2 \) defined on the subsets \( D_i = \pi^{-1}(D_i) \) of \( \mathbb{C} \), where \( \pi : \mathbb{C} \to \mathbb{C} / \mathbb{Z} \) is the canonical projection, whereas theorem 1.4 involves the maps \( h_{1}^{\pm}, h_{2}^{\pm} \) defined on the subsets \( D_{1}^{\pm} \) of the cylinder \( \mathbb{C} / \mathbb{Z} \).
Complement 1.6. The theorem is still valid by replacing each occurrence of “semi-conjugacy” by “pseudo-conjugacy”, and each occurrence of “ψ = (ψ⁺, ψ⁻)” pair of holomorphic maps ψ⁺ : \( D₁^+ \rightarrow D₂^+ \) by “ψ = (ψ⁺, ψ⁻)” pair of biholomorphisms ψ± : \( D₁^± \rightarrow D₂^± \). In this case, \( h₁^+ \), \( h₂^± \) are equivalent as analytic ramified coverings via biholomorphism between \( D₁^± \) and \( D₂^± \) (more precisely via \( ψ^± \)), and a translation of \( \mathbb{C}/\mathbb{Z} \) (more precisely via \( T_σ \)).

More explicitly. Let \( f_1, f_2 \) denote holomorphic maps from an open neighborhood of 0 ∈ \( \mathbb{C} \) to \( \mathbb{C} \), with Taylor expansion at 0: \( f_i(z) = z + a_i z^2 + o(z^2) \) where \( a_i \neq 0 \).

Suppose \( f_1, f_2 \) locally pseudo-conjugated at 0 on \( B₀^f \). Then there exists \( σ \in \mathbb{C} \) and a pair of biholomorphisms \( ψ = (ψ⁺, ψ⁻) \), where \( ψ^± : D₁^± \rightarrow D₂^± \), such that:

\[
h₁^± = T_σ \circ h₂^± \circ ψ^±
\]

and such that the maps \( ψ^± \) have removable singularities at ±i∞.

Conversely, if there exists \( σ \in \mathbb{C} \) and a pair of biholomorphisms \( ψ = (ψ⁺, ψ⁻) \), where \( ψ^± : D₁^± \rightarrow D₂^± \), with removable singularities at ±i∞ such that:

\[
h₁^± = T_σ \circ h₂^± \circ ψ^±
\]

then \( f₁, f₂ \) are locally pseudo-conjugated at 0 on their immediate basins.

Furthermore, the maps \( ψ^± \) have at ±i∞ an expansion of the form \( ψ^±(w) = w + ρ^± + o(1) \).

Remark 1.7. Let us make the following remark whose demonstration we will omit. If we replaced in theorem 1.4 the maps \( (h^+, h^-) \) by the germs \( (h^+_g, h^-_g) \), we would obtain a statement that is a logical equivalence between the two following statements, which are easily seen to be always true. The germs are equivalent as analytic ramified coverings (this is true since the germs are inversible) if and only if \( f_1, f_2 \) conjugated on the very large attractive petals\(^3\) included in a small neighborhood of 0 (this is true since \( f_1, f_2 \) are conjugated to translations). Whence the interest to work with \( (h^+, h^-) \) for the study of this more flexible covering equivalence.

The book of Landford and Yampolsky [LY] proves the direct implication of theorem 1.4 with some supplementary hypotheses. Namely: \( φ \) is a global biholomorphism on the immediate basins \( B₁^f, B₀^f \), and the conjugacy \( φ \) and its reciprocal \( φ' = φ⁻¹ \) are continuous at 0. The construction of \( φ \) which will be exposed here becomes, without these continuity hypotheses, more difficult.

It is expected that the theorem 1.4 may be naturally adapted for maps \( f_1, f_2 \) with parabolic points with any number (possibly distinct for \( f_1, f_2 \)) of petals and cycles of petals and for each of their horn maps. This conjectural generalisation of the theorem will be explicitly formulated in part 7.

Theorem 1.1 up to renormalizing the Fatou coordinates, gives an equality between \( h₁^+_g, h₂^+_g \). This enables to study the dynamic of \( h₂^-_g \) independently of the choice of the representative of \( f \) modulo local conjugacy. The theorem 1.4 up to renormalizing the

\(^3\)See definition 2.1
Fatou coordinates, gives an equivalence modulo post-composition by $\psi^{\pm}$. This implies a preservation of ramified covering properties of $h$. The local degree preservation of $h$ implies the following corollary:

**Corollary 1.8.** Let $f_1, f_2$ be locally pseudo-conjugated at 0 on their respective immediate basins. Suppose that:

$$U_i = P^i_A \cup P^i_R \cup \{0\}$$

where $P^i_A, P^i_R$ attractive, repulsive petals, of $f_i$ and $P^2_A = \phi(P^1_A)$. Let $z_1 \in U^0_1$, $z_2 = \phi(z_1)$. Then the product of local degrees of $f_1$ at the elements of the forward orbit of $z_1$ is equal to the product of local degrees of $f_2$ at the elements of the forward orbit of $z_2$ (in each of these forward orbits, there is only a finite numbers of terms out of the union of an attractive and repulsive petal, in such a way that the products contains only a finite number of terms distinct of 1). In particular, the forward orbit of $z_1$ contains a critical point if and only if the forward orbit of $z_2$ contains a critical point.

Remark that a local pseudo-conjugacy of immediate basins does not only preserves local properties. That is why one may wonder about the extension properties of a local semi-conjugacy.

If the two functions $f_i : B^f_0 \to B^f_0$ are proper and $B^f_0$ are simply connected, it is shown in [Mor17] that a local conjugacy on all a neighborhood of 0 extends, even if it means post-composing by $f^n_2$ where $n \in \mathbb{N}$, into a semi-conjugacy of the immediate basins. Thus the following question is natural:

**Question 1.9.** Let $\phi$ be a local semi-conjugacy of $f_1, f_2$ over their immediate basins. Does there exist $n \in \mathbb{N}$ such that $f^n_2 \circ \phi$ extends into a global semi-conjugacy over the immediate basin ?

Unlike a local conjugacy over a whole neighborhood of 0, a local semi-conjugacy over the immediate basin has on one hand a rigidity introduced by the preservation of immediate basins, on other hand a flexibility introduced by the definition over smaller open sets and by the absence of continuity hypothesis on the border of the parabolic basins. A local semi-conjugacy can modify topologically these borders, for instance by turning a non locally connected border into a locally connected border. Such a semi-conjugacy, in a generalized definition presented in part 7, may also modify the number of petals of $f$.

For illustrative purposes, we give an auxiliary lemma of classification which enables to apply theorem 1.4 to functions that are unisingular on their simply connected immediate basin.

This following lemma is taken from [Che14].

**Lemma 1.10.** Let $f : B^f_0 \to B^f_0$ be holomorphic with a simple parabolic point at 0, restricted to its simply connected immediate basin $B^f_0$, with a unique singular value $v$ in $B^f_0$.

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*When $f_1, f_2$ are pseudo-conjugated, we can always choose open sets $U_i$ of this form. This is shown in the proof of the reciprocal of theorem 1.4.*
For integer $d \geq 2$, denote the following holomorphic functions from $\mathbb{D}$ to $\mathbb{D}$:

$$B_d(z) = \left( \frac{z + a}{1 + az} \right)^d \quad \text{where} \quad a = a_d = \frac{d-1}{d+1}$$

and:

$$B_\infty(z) = \exp \left( \frac{2z - 1}{z + 1} \right)$$

Then $f$ is conjugated to $B_d$, where $d \geq 2$ (possibly infinite) is equal to the degree of the ramified covering $f : B_0^f \rightarrow B_0^f$.

This gives the following conjugacies on the immediate parabolic basins, of which only the first two can be treated by the work of Landford and Yampolsky [LY]. Those which follow cannot be treated by this work anymore due to the absence of continuity, and require theorem [1.4]

- The parabolic basin of the map $\tan : \mathbb{C} \rightarrow \mathbb{C}$ is $\mathbb{C} \setminus \mathbb{R}$. The map $\tan$ is conjugated on each of its immediate basins $\pm \mathbb{H}$ to $B_\infty$. More precisely, the map $\tan$ is conjugated to $f : z \mapsto \exp(2 \frac{z - 1}{z + 1})$ over all $\hat{\mathbb{C}}$ via the homography $\mu : z \mapsto \frac{i(z - 1)}{z + 1}$: $\tan = \mu \circ f \circ \mu^{-1}$.

- The cauliflower map $f_0 : z \mapsto z + z^2$ is conjugated on its immediate basin to the parabolic Blaschke product $B_2$ (passing from an immediate basin with fractal boundary to $\mathbb{D}$).

- $z \mapsto \exp(z) - 1$ and $z \mapsto z \exp(z)$ are respectively analytically conjugated over their immediate basins to $B_\infty$, $B_2$ (passing from an immediate basin with non locally connected boundary to $\mathbb{D}$).

- Let $f$ be a rational map which has only the following non-repulsive cycles: a fixed parabolic point with only one petal and a Cremer cycle of period 2. Suppose that the immediate parabolic basin is simply connected and $f$ unisingular of degree $d$ over this immediate parabolic basin. Then $f$ is conjugated to $B_d$ over its immediate basin, of which the border is non locally connected (passing from an immediate basin with non locally connected boundary to $\mathbb{D}$).

Theorem [1.4] allows in each case to reduce the study of the horn map modulo analytic ramified covering equivalence of the aforementioned functions ($\tan$, $f_0$, etc.) to horn maps of the functions $B_d$, $d \in [2, +\infty]$ ($B_\infty$, $B_2$, etc.). These last functions have the advantage to have a simple definition domain: the unit disk.

Remark 1.11. We observe in these examples that the understanding of covering properties of a function $f : B_0^f \rightarrow B_0^f$ on its immediate parabolic basin leads to the understanding of $f : B_0^f \rightarrow B_0^f$ modulo conjugacy; it is a more general property, valid for all parabolic Blaschke products.

We now consider the parabolic renormalization operator, consisting in associating to $f$ its normalized horn map (see definition 2.11 and 4.1). By theorem [1.4] two conjugated maps have two equivalent parabolic renormalization as coverings, and the preceding
discussion enables to get in appropriate circumstances a conjugacy between these two renormalizations on their parabolic basin, this gives sufficient hypothesis to apply the theorem again. This allows to build an invariant space by parabolic renormalization, and to think about the dynamic of this operator. Its fixed points have been studied. See the work of Inou and Shishikura [IS08], Landford and Yampolsky [LY] for the quadratic case, and of Chéritat [Ché14] for the unisingular case.

Plan of the paper: In part 2, we recall a definition of Fatou coordinate and petals which is not fully compliant with [Mil], and their classical consequences. In part 3, we define the Fatou coordinate extensions and the attractive, repulsive cylinders, isomorphic via Fatou coordinates to $B^f/f$ and $P_R/f$ (these objects will be defined in this part 3). In part 4, we define the horn map, seen as a map from the repulsive cylinder $C_R$ to the attractive cylinder $C_A$, using the natural map $P_R/f \to B^f/f$. In part 5, we define the pseudo-conjugacies and we prove some properties, which will allow in part 6 to show the main theorem (theorem 1.4). We give in part 7 the definition of generalized pseudo-conjugacy for parabolic points with several petal cycles, and the expected generalization of the theorem in this setting.

2 Fatou coordinates, petals

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ a holomorphic function defined on a open neighborhood $U$ of 0. We assume that $f$ has a parabolic point 0 with multiplier 1 with one petal, in other words $f$ has the Taylor expansion:

$$f(z) = z + az^2 + O(z^3)$$

with $a \in \mathbb{C}^*$. The attractive axis of $f$ has argument $-\arg(a) + \pi$, and the repulsive axis of $f$ has argument $-\arg(a)$.

Here is a definition of attractive petal that comes from [LY] with slight modifications. Among others, the condition $f(P_A) \setminus \{0\} \subset P_A$ is replaced by $f(P_A) \subset P_A$. This modified definition will prove to be useful to show that a local conjugation of the immediate basin, not assumed continuous at 0, sends an attractive petal on an attractive petal, see proposition 5.27. This is not a standard definition, it differs for instance from the more classical definition of Milnor [Mil]. Nevertheless, a petal in our sense is a petal in the sense of Milnor. The classical existence results are hence still valid concerning our definition.

Definition 2.1. An attractive petal $P_A$ of $f$ is an open subset of $\mathbb{C}$ with a holomorphic function $\Phi_A : P_A \to \mathbb{C}$ named Fatou coordinate such that:

- $P_A$ is connected, simply connected (i.e. $P_A$ is homeomorphic to $\mathbb{D}$).
- $f(P_A) \subset P_A$.
- $\forall z \in P_A$, $\lim_{n \to \infty} f^n(z) = 0$. 


Conversely, if $f^n(z)$ converges to 0, it may be stationary at 0, or else $f^n(z)$ must belongs to $P_A$ after a certain rank.

- We have the following conjugacy: $\forall z \in P_A, \Phi_A \circ f(z) = T_1 \circ \Phi_A(z)$, with $T_1(z) = z + 1$ for all $z \in \mathbb{C}$.
- $\Phi_A$ is injective on $P_A$.

We call (petal) the following proposition, for a better comparison with the definitions of $\alpha$-petal, large petal, very large petal which will be soon defined.

**Proposition 2.2** (petal). An attractive petal $P_A$ satisfies the following properties:

- for all $w \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $w + n \in \Phi_A(P_A)$
- for any compact subset $K$ of $\mathbb{C}$, there exists $n \in \mathbb{N}$ such that $K + n \subset \Phi_A(P_A)$.

**Proof.** The first point is already shown in [Mil]. The proof does not use the property $f(P_A) \subset P_A \cup \{0\}$.

The set $\Phi_A(P_A)$ is open, so the map which from $z$ associates the least integer $\tau(z)$ such that $z + \tau(z) \in \Phi_A(P_A)$ is upper semi-continuous. This implies that $\tau(z)$ admits a maximum $n$ in the compact set $K$. As $\Phi_A(P_A)$ is stable by the translation $T_1$, we have $K + n \subset \Phi_A(P_A)$.

**Definition 2.3.** Let $\alpha \in ]0, \pi[$, and $P_A$ an attractive petal. We say that $P_A$ is an attractive $\alpha$-petal, resp. large petal, very large, if:

- (petal) The image of $P_A$ by $\Phi_A$ equals a sector, more precisely:

  $\exists w_0 \in \mathbb{C}, \exists R > 0, \{w \in \mathbb{C}, |w - w_0| > R \text{ and } |\arg(w)| < \alpha\} = \Phi_A(P_A)$

- (large) For all $\alpha > \pi/2$, $P_A$ contains an $\alpha$-petal.

- (very large) The image of $P_A$ by $\Phi_A$ contains an upper half-plane, a lower half-plane, and a right half-plane.

To define repulsive petals of $f$: it is an attractive petal for a local inverse $f^{-1}$ of $f$ which sends 0 on 0. But concerning the Fatou coordinate, we post-compose it by $z \to -z$, in such a way that it conjugates $f$ (and not $f^{-1}$) to $T_1$.

**Lemma 2.4.** [equivalent] Let $w_0 \in \mathbb{C}$ and $\beta \in ]0, \pi[$. Let a sector $S = \{w \in \mathbb{C}, |w - w_0| > R \text{ and } |\arg(w)| < \beta\}$ be included in $\Phi_A(P_A)$, and $P'_A = \Phi_A^{-1}(S)$. We have $\Phi_A(z) \sim -\frac{1}{az}$ when $z \to 0, z \in P'_A$.

**Proof.** This is a classical result presented in [Che08].
Consequence 2.5. Recall that \( f(z) = z + a z^2 + O(z^3) \) where \( a \in \mathbb{C}^* \). The fact that an attractive petal is large is equivalent to the fact that it contains an \( \alpha' \)-sector for all \( 0 < \alpha' < \pi \), in the following sense: there exists \( R > 0 \) such that \( \{ z \in \mathbb{C}, 0 < |z| < R, |\arg(-a z)| < \alpha' \} \subset P_A \). More geometrically, the petal \( P_A \) contains a punctured sector of width \( 2\alpha' \) and small radius centered on the attractive axis.

The fact that an attractive petal is an \( \alpha \)-petal implies that it contains an \( \alpha' \)-sector for all \( 0 < \alpha' < \alpha \), in the same sense as before.

Proposition 2.6. Let \( P_A \) be an attractive petal containing a \( \frac{\pi}{2} \)-petal, \( D \) a fundamental domain of \( P_A \) (i.e. a set containing exactly once each grand orbit of \( f \) restricted to \( P_A \)) inverse image of a strip \( \{ z \in \mathbb{C}, r \leq \Re(z) < r + 1 \} \) by \( \Phi_A \), and \( P_R \) a large repulsive petal. Let \( M > 0 \). Then there exists \( M' > 0 \) satisfying the following property: if \( z \in D \) is such that \( \Im(\Phi_A(z)) > M' \), then \( z \in P_R \) and \( \Im(\Phi_R(z)) > M \). There also exists \( M' \) satisfying the property: if \( z \in D \) is such that \( \Im(\Phi_A(z)) < -M' \), then \( z \in P_R \) and \( \Im(\Phi_R(z)) < -M \).

The property is also available if one permutes simultaneously each occurrence of repulsive by attractive and vice versa. In particular, two large petals, one of which is attractive and the other repulsive, always share a non-empty intersection.

Remark 2.7. The constant \( M' \) depends on the choice of \( P_A, P_R \), and the choice of \( D \).

Proof. Let \( w \in \mathbb{C} \) such that \( r \leq w < r + 1 \). The equivalent of \( \Phi_A \) shows that:

\[ \Phi_A^{-1}(w) \sim -\frac{1}{aw} \]

when \( |\Im(w)| \to +\infty \).

When \( \Im(w) \) is large enough (resp. large enough with negative sign), then we have \( |\Phi_A^{-1}(w)| \) small and \( \arg(\Phi_A^{-1}(w)) \) close of \( -\arg(a) - \pi/2 \) (resp. close of \( -\arg(a) + \pi/2 \)).

Recall that the attractive axis admits as argument \( -\arg(a) + \pi \), the repulsive axis as argument \( -\arg(a) \). The quantity \( \Phi_A^{-1}(w) \) is close to 0, in a little cone containing a germ of the line \( \mathcal{X} \), where \( \mathcal{X} \) is the line passing through 0 perpendicular to both attractive and repulsive axis. Since \( P_R \) is large, we may apply remark \( 2.5 \) to deduce that \( \Phi_A^{-1}(w) \) must belongs to \( P_R \).

Now, if \( z \in D \) is such that the quantity \( \Im(\Phi_A(z)) \) is big, resp. big with negative sign, we may apply the preceding argument to deduce that \( z \in P_R \). Since \( z \) is close to 0 and avoids a sector centered on the attractive axis and a sector centered on the repulsive axis, we have the equivalents \( \Phi_A(z) \sim -\frac{1}{az} \sim \Phi_R(z) \), hence \( \Im(\Phi_R(z)) \) must be big (resp. big with negative sign) when \( \Im(\Phi_A(z)) \) is big (resp. big with negative sign). This concludes.

The reasoning obtained inversing the role of attractive and repulsive petals is identical. \( \square \)

Proposition 2.8. \( f \) admits attractive \( \alpha \)-petals, large petals and very large petals.

Proof. Although it is not expressed in the same terminology, it is shown in [Mil] that \( f \) admits attractive \( \alpha \)-petals, where \( 0 < \alpha < \pi \). It is sufficient to prove the existence of a very large attractive petal, since it is a particular case of a large petal.
Existence of very large attractive petals: Let $P_A$ a attractive $\pi/2$-petal of $f$. By definition, $D = \Phi_A(P_A)$ is a right half-plane: $D = \{z \in \mathbb{C}, a < \Re(z)\}$. Let $B = \{z \in \mathbb{C}, a < \Re(z) \leq a + 1\}$. For $M > 0$ we note $\Delta = \{z \in \mathbb{C}, |\Im(z)| \leq M\}$, $P^M_A = \Phi_A^{-1}(D\setminus\Delta)$ and $X^M = \Phi_A^{-1}(B\setminus\Delta)$. If $M$ is big enough, $X^M$ is included in a large repulsive petal $P_R$ fixed by proposition 2.6. We note $f^{-1} : P_R \to f^{-1}(P_R)$ a local inverse of $f$ at $0$.

Let $X^M_n = f^{-n}(X^M)$ for all $n \in \mathbb{N}$, and

$$
\begin{align*}
P^L_A &= P_A \cup \bigcup_{n \in \mathbb{N}^*} X^M_{-n} \\
D^L &= D \cup (\mathbb{C}\setminus\Delta) = D \cup \bigcup_{n \in \mathbb{N}^*} ((B - n)\setminus\Delta)
\end{align*}
$$

Let $n \in \mathbb{N}^*$. Note that $X^M_{-n}$ is disjoint of $P_A$. Indeed, $\Phi_A(f^n(X^M_{-n})) \subset B$ and $\Phi_A(f^n(P_A)) \subset D + n$, and $B$, $D + n$ are disjoint. By a similar reasoning, $X^M_{-n}$ and $X^M_{n+1}$ are also disjoint.

Let for $z \in P^L_A$, $\Phi_A(z) = \Phi_A(f^n(z)) - n$ where $n$ is such that $f^n(z) \in P_A$ (the value of the expression does not depend on the integer $n$ chosen). Note that the map $\Phi_A$ is a biholomorphism from $P^L_A$ to $D^L$. We check this point. It is holomorphic since $n$ can be chosen bounded when $z$ is in a compact subset of $B^L \supset P^L_A$. It is surjective by construction: it remains to show injectivity. The map $\Phi_A$ is injective on $P_A$, and on $X^M_{n+1}$ for $n \in \mathbb{N}^*$. These sets are a partition of $P^L_A$ and have disjoint images by $\Phi_A$, this finally implies that $\Phi_A$ is injective.

It remains to check that $P^L_A$ has the desired properties.

- $P^L_A$ is homeomorphic via $\Phi_A$ to $D^L$, which is homeomorphic to the unit disk.
- $f(P^L_A) \subset P^L_A$, because $f(P_A) \subset P_A$ and $f(X^M_{-n}) = X^M_{-n+1} \subset P^L_A$ for $n \in \mathbb{N}^*$.
- $\forall z \in P^L_A$, $\lim_{n \to \infty} f^n(z) = 0$. This property is indeed valid if $z \in P_A$, and if $z \in X^M_{-n}$, then $f^n(z) \in P_A$.
- Conversely, if $f^n(z)$ converges to $0$, either it is stationary at $0$, or $f^n(z)$ belongs to $P^L_A$ from a certain rank. It results from $P^L_A \supset P_A$.
- $\Phi_A$ is injective on $P^L_A$.
- (very large) The image of $P^L_A$ by $\Phi_A$ contains an upper half-plane, a lower half-plane, and a right half-plane.
- We have the following conjugacy: $\forall z \in P_A, \Phi_A \circ f(z) = T_1 \circ \Phi_A(z)$, with $T_1(z) = z + 1$ for all $z \in \mathbb{C}$.

\[\square\]

**Proposition 2.9** (Uniqueness of Fatou coordinates). Let $P^1_A$, $P^2_A$ be two petals of $f$, and $\Phi^1_A$, $\Phi^2_A$ their Fatou coordinates. Then $\Phi^1_A, \Phi^2_A$ differ by a constant on $P^1_A \cap P^2_A$. 

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Proof. We work on $P = P_A^1 \cap P_A^2$. This set is not necessarily homeomorphic to the unit disk. However $P$ satisfies all the other points of the definition of petals, definition [2.1] with Fatou coordinates restricted to $P$, namely $\Phi_A^1|P$, $\Phi_A^2|P$. Note that $P$ still satisfies proposition (petal). Let $w \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $w + n \in \Phi_A^1(P_A^1)$. Let $z = \Phi_A^1(w + n) \in P_A^1$. There exists $m \in \mathbb{N}$ such that $f^m(z) \in P_A^2$. Since $P_A^1$ is stable under the action of $f$, we have $f^m(z) \in P_A^1 \cap P_A^2 = P$, this enables to say that $w + n + m \in \Phi_A^1(P)$.

Let $P/f$, the quotient set of $P$ obtained from the equivalence relation whose equivalence classes are the grand orbits of $f|_P$.

Since $\Phi_A^1$ conjugates $f$ to the translation $T_1$ and $\Phi_A^1(P)$ is stable by $T_1$, we have $P/f \sim \Phi_A^1(P)/\mathbb{Z}$. By proposition (petal), for $z \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $z + n \in \Phi_A^1(P)$. Thus $P/f \sim \mathbb{C}/\mathbb{Z}$.

The function $\Phi_A^1 \circ (\Phi_A^2)^{-1}: \Phi_A^1(P) \to \Phi_A^2(P)$ commutes with the translation $T_1$, since $T_1 \circ \Phi_A^1 = \Phi_A^1 \circ f|_{P_A^1}$ and $f \circ (\Phi_A^2)^{-1} = (\Phi_A^2)^{-1} \circ T_1|_{\Phi_A^2(P_A^2)}$. The map $\Phi_A^1 \circ (\Phi_A^2)^{-1}$ induces therefore a holomorphic function

$$\alpha: \Phi_A^1(P)/\mathbb{Z} = \mathbb{C}/\mathbb{Z} \to \Phi_A^1(P)/\mathbb{Z} = \mathbb{C}/\mathbb{Z}$$

Similarly for $\Phi_A^1 \circ (\Phi_A^2)^{-1}$, which induces a map $\beta: \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. The equalities

$$\text{id}_{\Phi_A^1(P)} = \left(\Phi_A^1 \circ (\Phi_A^2)^{-1}\right) \circ \left(\Phi_A^1 \circ (\Phi_A^2)^{-1}\right)$$

$$\text{id}_{\Phi_A^1(P)} = \left(\Phi_A^1 \circ (\Phi_A^2)^{-1}\right) \circ \left(\Phi_A^1 \circ (\Phi_A^2)^{-1}\right)$$

become after quotient $\beta \circ \alpha = \alpha \circ \beta = \text{id}_{\mathbb{C}/\mathbb{Z}}$. The map $\alpha$ is a cylinder automorphism. Therefore $\alpha(w) = w + \sigma$ or $\alpha(w) = -w + \sigma$, where $\sigma \in \mathbb{C}/\mathbb{Z}$. Let $X := \Phi_A^1(P) = \bigcup_{i \in I} X_i$, where the $X_i$ are the connected components of $X$, with the set $I$ at most countable. Even thought $X$ might be not connected, this set is stable by $T_1: w \to w + 1$. Lifting the map $\alpha$ to a map from $\mathbb{C}$ to $\mathbb{C}$ we distinguish two cases.

- $\Phi_A^1 \circ (\Phi_A^2)^{-1}(w) = w + \tilde{\sigma}_i$ for all $w \in X_i$, where $\tilde{\sigma}_i \in \mathbb{C}$ and $\tilde{\sigma}_i$ mod $\mathbb{Z} = \sigma$. Let $w \in X_i$, $w' \in X_j$ and $K$ a connected compact subset of $\mathbb{C}$ containing $\{w, w'\}$. By proposition (petal), there exists $n \in \mathbb{N}$ such that $K + n \subset X$. For such an integer $n$, the complex numbers $w + n, w' + n$ belongs to a connected component $X_k$ of $X$. So $\Phi_A^1 \circ (\Phi_A^2)^{-1}(w + n) = w + n + \tilde{\sigma}_k = \Phi_A^2 \circ (\Phi_A^1)^{-1}(w + n) = w + n + \tilde{\sigma}_i$, that is why $\tilde{\sigma}_i = \tilde{\sigma}_k$. In the same way, $\tilde{\sigma}_j = \tilde{\sigma}_j$ for all $i, j \in I$. By noting $\tilde{\sigma}$ the common value of $\tilde{\sigma}_i$ we get $\Phi_A^1(w) = \Phi_A^1(w) + \tilde{\sigma}$ for all $w \in P$. Which was to be proven.

- $\Phi_A^1 \circ (\Phi_A^2)^{-1}(w) = -w + \tilde{\sigma}_i$ for all $w \in X_i$. We apply the preceding reasoning by choosing an integer $n \geq 1$. We get $\Phi_A^1 \circ (\Phi_A^2)^{-1}(w + n) = -w - n + \tilde{\sigma}_k = \Phi_A^2 \circ (\Phi_A^1)^{-1}(w + n) = -w + n + \tilde{\sigma}_i$, so $\tilde{\sigma}_i = -2n + \tilde{\sigma}_k$, and in the same way $\tilde{\sigma}_k = -2n + \tilde{\sigma}_k$. We get the following contradiction: $\tilde{\sigma}_i = \tilde{\sigma}_j$ for all $i, j \in I$, but $\tilde{\sigma}_i \neq \tilde{\sigma}_k$. This case is thus impossible.
Which completes the proof.

Proposition 2.10. We assume that \( f(z) = z + az^2 + bz^3 + o(z^3) \) where \( a \in \mathbb{C}^* \), \( b \in \mathbb{C} \).

Let \( \gamma = 1 - \frac{b}{a} \) be the iterative residue of \( f \). This quantity \( \gamma \) is the residue at 0 of the meromorphic form \( \frac{dz}{f(z)-z} + \frac{dz}{z} \).

Let \( P_A, P_R \) be attractive, repulsive petals of \( f \), and \( \Phi_A, \Phi_R \) be Fatou coordinates defined on these petals. We have the following asymptotic expansion, when \( z \to 0 \) avoiding a sector containing the repulsive resp. attractive axis:

\[
\Phi_A(z) = -\frac{1}{az} - \gamma \log \frac{1}{az} + \text{cst} + O(z) \\
\Phi_R(z) = -\frac{1}{az} - \gamma \log \frac{1}{az} + \text{cst} + O(z)
\]

where \( \log \) denotes the principal branch of the logarithm.

Elements of proof. This is a classical property presented in [Mil]. Nevertheless, we give some elements of proof, from a more general demonstration presentend in the first chapter of [Che08]. We will not detail the precision of the approximations. We consider only the attractive Fatou coordinate (for the repulsive coordinate, the reasoning is done with the local section \( f^{-1} \) of \( f \)). We consider the two following vector fields: \( \frac{f(z)-z}{dz} \) and \( \frac{\log(f'(z))(f(z)-z)}{(f'(z)-1)dz} \). The second one is well defined in a neighborhood of \( z = 0 \), via the principal branch of the logarithm (since \( f'(0) = 1 \) and the analytic continuation at 0.

The flow associated to these vector fields will constitute approximations of the Fatou coordinate. We will not give a proof concerning the precision of the approximation. Nevertheless, let motivate the choice of these vector fields. Where \( z \) is close of the parabolic point, the distance between \( f(z) \) and \( z \) is insignificant with respect to the distance between \( z \) and the parabolic point 0. Thus the quantity \( f(z) - z \) must be almost constant along an integral curve of the vector field \( \frac{f(z)-z}{dz} \) between \( t = 0 \) and \( t = 1 \) close to the parabolic point, which suggests that its flow is a good candidate to be an approximation of the Fatou coordinate. Note that \( \frac{\log(f'(z))}{f(z)-1} \) is close to 1: it is a corrective factor which enables to precise the approximation. The introduction of this corrective term, which is an idea of Buff, is motivated by the following fact. When we perturb \( f \) to get a function \( f_n \) close to \( f \), the new function \( f_n \) has two fixed points which are in general distinct in a neighborhood of 0. The vector field \( \frac{f_n(z)-z}{dz} \) has for eigenvalues at these fixed points \( f_n'(z) - 1 \), so its flow admits for multiplier at these points \( \exp(f_n'(z) - 1) \). Furthermore, the fixed points of \( f_n \) has multiplier \( f_n'(z) \). The corrective factor ensures that the vector field \( \frac{\log(f'(z))(f(z)-z)}{(f'(z)-1)dz} \) has for eigenvalue \( \ln(f_n'(z)) \), so the associated flow has for multiplier \( f_n'(z) \) at these fixed points.

The flow of a vector field is obtained by integrating the differential form associated to this vector field, which is the inverse of this vector field (the complex dimension is 1). The anti-derivatives of the differential forms \( \frac{dz}{f(z)-z} \) and \( \frac{f'(z)-1}{\log(f'(z))(f(z)-z)} \) will constitute good approximations of Fatou coordinates. The expansion:

\[
\frac{f'(z)-1}{\log(f'(z))(f(z)-z)} = \frac{1}{f(z)-z} + \frac{1}{z} + O(1)
\]
suggests that an anti-derivative of the differential form \( \frac{dz}{f(z) - z} + \frac{dz}{z} \) will give a better approximation of \( f \) than an antiderivative of \( \frac{dz}{f(z) - z} \).

We have:

\[
\frac{1}{f(z) - z} = \frac{1}{az} \left( 1 + \frac{b}{a}z \right) + O(1)
= \frac{1}{az^2} \left( 1 - \frac{b}{a}z \right) + O(1)
= \frac{1}{az^2} - \frac{b}{a^2z} + O(1)
\]

This shows that \( \gamma = 1 - \frac{b}{a^2} \) is the residue at 0 of the meromorphic form \( \frac{dz}{f(z) - z} + \frac{dz}{z} \). We obtain by integrating:

\[
\int \frac{dz}{f(z) - z} + \frac{dz}{z} = \frac{-1}{az} - \gamma \log \frac{-1}{az} + \text{cst} + O(z)
\]

Where \( \log \) is the principal branch of the logarithm, and \( z \) close to 0 and avoiding a sector containing the repulsive axis. This corresponds to the terms appearing in the asymptotic expansion of the attractive Fatou coordinate.

**Definition 2.11.** By proposition 2.9 there exists a unique attractive Fatou coordinate where the constant of its asymptotic expansion is 0, and the same holds for the repulsive Fatou coordinate. We call these Fatou coordinates the normalized Fatou coordinates of \( f \).

**Remark 2.12.** Assume that \( \Phi_A, \Phi_R \) are normalized. Let \( S \subset \mathbb{C} \) be an open nonempty cone, symmetrical with respect to 0, avoiding the attractive and repulsive axis. See figure 1. Let \( S^+ \) be the connected component of \( S \) included in \( -\frac{1}{a} \mathbb{H} \), \( S^- \) the component included in \( \frac{1}{a} \mathbb{H} \). The sets \( S^+, S^- \) could also be defined as the connected components of \( S \) satisfying the following property: each neighborhood of 0 intersected with \( S^+ \) is mapped by \( \Phi_A \mod \mathbb{Z} \) (or by \( \Phi_R \mod \mathbb{Z} \), this is immaterial) onto a punctured neighborhood of \(+i\infty\), and all neighborhood of 0 intersected with \( S^- \) is mapped onto a punctured neighborhood of \(-i\infty\).

By computing the difference between the two asymptotic expansions, one gets:

\[
\Phi_A(z) = \Phi_R(z) + \epsilon i \pi \gamma + O(z)
\]

(1)

with \( \epsilon = \mp 1 \), where \( z \to 0, z \in S^\pm \).

The maps \( \Phi_A, \Phi_R \) share indeed the same asymptotic expansion up to the logarithmic term; the logarithmic terms are equal up to a constant depending on whether \( z \) belongs to \( -\frac{1}{a} \mathbb{H} \) or to \( \frac{1}{a} \mathbb{H} \).
Figure 1: Taylor expansion of $\Phi_A - \Phi_R$ over $S^+$ and $S^-$

The cone $S^+ \cup S^-$ is delimited by green lines. The set $U$ is a neighborhood of 0. The grey parts represent $S^+ \cap U$, $S^- \cap U$ and their images by $\Phi_A$ and $\Phi_R$. 
Remark 2.13. More generally, the Fatou coordinates admits a complete asymptotic expansion of the form:

\[ \Phi_A(z) = -\frac{1}{az} - \gamma \log \frac{1}{az} + \sum_{k=0}^{n} b_n z^n + o(z^n) \]

\[ \Phi_R(z) = -\frac{1}{az} - \gamma \log \frac{1}{az} + \sum_{k=0}^{n} b_n z^n + o(z^n) \]

where only the terms in log differs. This is shown for instance in [LY].

Equation (1) is also true by writing \( O(z^n) \) (and more precisely \( O(\exp(\pm \frac{2\pi}{n\pi})) \) when \( z \in S^\pm \)) instead of \( O(z) \), with arbitrary \( n \in \mathbb{N} \). The neglected term is nonzero in general, simply negligible with respect to \( z^n \) for all \( n \in \mathbb{N} \).

The series \( \sum b_n z^n \) admits generically a convergence radius of zero.

3 Extension of Fatou coordinates and attractive, repulsive cylinders

The results of this section are already known, although the presentation might be original.

Let \( f : U \to \mathbb{C} \) a holomorphic function, where \( U \subset \mathbb{C} \) is an open neighborhood of 0, with a simple parabolic point at 0.

Definition 3.1. Define the extended attractive Fatou coordinate of \( f \) in the following way.

For \( z \in B_f \), there exists \( n \in \mathbb{N} \) such that \( f^n(z) \in P_A \). The quantity \( w := \Phi_A(f^n(z)) - n \) does not depend on the chosen integer \( n \). Let \( \Phi^\text{ext}_A(z) = w \) be the extended attractive Fatou coordinate.

Remark 3.2. Let \( P^1_A, P^2_A \) be two attractive petals of \( f \), whose Fatou coordinates \( \Phi^1_A, \Phi^2_A \) coincide on their intersection. Then these two Fatou coordinates gives the same extended Fatou coordinate. This comes from the fact that \( \Phi^\text{ext}_A(z) \) equals to \( \Phi_A(f^n(z)) - n \) for all \( n \) such that \( f^n(z) \in P^1_A \cap P^2_A \). In this sense, the extended Fatou coordinate does not depend on the choice of the petals. It is a holomorphic function on \( B^f \). Two extended Fatou coordinates differs by a constant (post-composition by a translation).

Definition 3.3. Define the extended repulsive Fatou parametrisation of \( f \) in the following way. Let \( P_R \) be a repulsive petal of \( f \). Let \( w \in \mathbb{C} \). By proposition (petal), there exists \( n \in \mathbb{N} \) such that \( w - n \in \Phi_R(P_R) \). Then \( z = f^n(\Phi^{-1}_R(w - n)) \) (if this quantity does exist) is independent of the chosen integer \( n \). We note \( \Psi^\text{ext}_R(w) = z \).

Remark 3.4. In the same way as the preceding paragraph, the extended repulsive Fatou parametrisation does not depend on the chosen repulsive petal. If \( f \) is defined on \( \mathbb{C} \), the holomorphic map \( \Psi^\text{ext}_R \) is defined on \( \mathbb{C} \). Else, \( \Psi^\text{ext}_R \) is holomorphic on its domain of definition, which is open. In all other cases, two extended Fatou parametrisations differ by pre-composition by a translation.
We have $\text{Im}(\Psi^\text{ext}_R) = \bigcup_{n \in \mathbb{N}} f^n(P_R)$, independent of the chosen petal $P_R$. If $f$ is defined on $\mathbb{C}$, the fact that $\Psi^\text{ext}_R$ is a nonconstant holomorphic function on $\mathbb{C}$ (since it is injective on the sets of the form $\Phi_R(P_R)$) implies that $\text{Im}(\Psi^\text{ext}_R)$ avoids at most one point of $\mathbb{C}$.

**Definition 3.5.** Let $E, F$ be two sets (whose intersection will be generally nonempty), and $\phi : E \to F$. We call grand orbit of $\phi$ a nonempty subset $X$ of $E$ which is minimal for the inclusion satisfying the following properties:

- $\phi(X) \cap E \subset X$
- $\phi^{-1}(X) \subset X$

**Remark 3.6.** The grand orbits of $\phi$ might be characterized as the equivalence classes of the following equivalence relation:

$$x \sim y \iff \exists n, m \in \mathbb{N}, \phi^n(x) = \phi^m(y) \in E$$

where it is implied that the intermediate iterates are also in $E$. We note $GO(z)$ the grand orbit of $z$, which is by definition the equivalence class of $z$.

The set $E$ is not supposed stable by $\phi$. In this setting, it is important to suppose that $\phi^n(x) = \phi^m(y)$ belongs to $E$.

**Proposition 3.7.** The map $i$ is open and continuous. Furthermore, it is:

- a surjection if and only if every grand orbit of $f$ in $B$ intersects $A$.
- an injection if and only if every grand orbit of $f$ in $B$ intersects at most one grand orbit in $A$. This last condition is automatically verified if $f(A) \cap B \subset A$. To satisfy $f(A) \cap B \subset A$, it is sufficient that $A$ is stable by $f$.
- a homeomorphism if and only if $i$ is bijective. In this case, $D \subset A$ is a fundamental domain of $A$ (i.e. it intersects exactly once each grand orbit of $A$) if and only if it is a fundamental domain of $B$. 

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Proof. We check the continuity of \( i \). Let \( \pi_B(U) \) be an open subset of \( B/f \), where \( U \) is a saturated open subset of \( B \). Remark that \( U \cap A \) is a saturated open subset of \( A \), and that \( i^{-1}(\pi_B(U)) = \pi_A(U \cap A) \), that is why this set is open and \( i \) is continuous. We check that \( i \) is open. Let \( \pi_A(U) \) be an open subset of \( A/f \) where \( U \) is a saturated open subset of \( A \). Remark that \( i(\pi_A(U)) = \pi_B(U) \) is open, this implies the openness of \( i \).

The map \( i \) is open and continuous, \( i \) is a homeomorphism if and only if it is bijective. The necessary and sufficient conditions of surjectivity, injectivity are clear.

We check that the injectivity of \( i \) is satisfied as soon as \( f(A) \cap B \subset A \). Let two grand orbits of \( A \) be included in the same grand orbit of \( B \). Let \( z_1 \in A \) be an element of the first grand orbit, and \( z_2 \in A \) an element of the second one. There exists \( n_1, n_2 \in \mathbb{N} \) such that \( f^{n_1}_{|B}(z_1) = f^{n_2}_{|B}(z_2) \in B \), where it is implied that the intermediate iterates are all in \( B \). Since \( f(A) \cap B \subset A \) and \( z_1, z_2 \in A \), this implies that the intermediate iterates are also in \( A \), and finally that \( f^{n_1}_{|A}(z_1) = f^{n_2}_{|A}(z_2) \in A \), hence \( z_1, z_2 \) are in the same grand orbit of \( A \). Hence \( i \) is injective.

\[ \square \]

**Proposition 3.8.** Let \( P_A \) an attractive petal of \( f \). Then the induced map by the inclusion \( P_A \subset B^f \), \( i_1 : P_A/f \rightarrow B^f/f \) is a homeomorphism.

Let \( P^1_R, P^2_R \) two repulsive petals of \( f \). Then \( P^1_R/f \) and \( P^2_R/f \) are homeomorphic. During the proof, these two sets both appear to be homeomorphic to \((P^1_R \cap P^2_R)/f) \) via the induced maps from the inclusions. Note that a grand orbit of \( P^1_R \) intersects exactly one grand orbit of \( P^2_R \) and conversely, the homeomorphism constructed between \( P^1_R/f \) and \( P^2_R/f \) is formed by the association of these two grand orbits.

**Proof.** Let us use proposition [3.7] Each grand orbit of \( B^f \) intersects \( P_A \), this implies that \( i_1 \) is surjective. The petal \( P_A \) being stable by \( f \), this implies that \( i_1 \) is injective. Hence \( i_1 \) is a homeomorphism.

The inclusion \( P^1_R \cap P^2_R \subset P^1_R \) induces a homeomorphism \( i_2 : (P^1_R \cap P^2_R)/f \rightarrow P^1_R/f \). Indeed, each grand orbit of \( P^1_R \) intersects \( P^1_R \cap P^2_R \), so \( i_2 \) is injective. Let \( f^{-1} \) be the inverse of \( f : P^1_R \rightarrow f(P^1_R) \). The grand orbits of \( f \) on \( P^1_R \), resp. \( P^1_R \cap P^2_R \) coincides with those of \( f^{-1} \). The set \( P^1_R \cap P^2_R \) is stable by \( f^{-1} \), this implies that \( i_2 \) is injective. Hence \( i_2 \) is a homeomorphism.

\[ \square \]

**Remark 3.9.** We will consider with a convenient abuse that the constructed homeomorphisms of the proposition are identifications: we will hence write \( P_A/f = B^f/f \), and \( P^1_R/f = P^2_R/f \). More generally, we will identify \( A/f = B/f \) when the inclusions \( A \cap B \subset A, B \) both induce homeomorphisms from \((A \cap B)/f \) to \( A/f, B/f \); the identified orbits of \( A \) and \( B \) (the identified elements of \( A/f \cap B/f \)) are those which intersect together.

**Proposition 3.10.** Let \( P_A \) be an attractive petal of \( f \). Passing to the quotient, the map \( \Phi_A : P_A \rightarrow \mathbb{C} \) becomes a homeomorphism \( \tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z} \).

Let \( P_R \) be a repulsive petal of \( f \). Passing to the quotient, the map \( \Phi_R|P_R : P_R \rightarrow \mathbb{C} \) becomes a homeomorphism \( \tilde{\Phi}_R : P_R/f \rightarrow \mathbb{C}/\mathbb{Z} \).

**Proof.** Passing to the quotient, the map \( \Phi_A \) becomes a homeomorphism \( \tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z} \) for the following reasons:

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• \(\Phi_A\) conjugates \(f\) to the translation, so it induces a map \(\tilde{\Phi}_A : P_A/f \to \mathbb{C}/\mathbb{Z}\).

• Proposition (petal) implies that \(\tilde{\Phi}_A : P_A/f \to \mathbb{C}/\mathbb{Z}\) is surjective.

• Recall that \(\Phi_A\) is injective on \(P_A\). Let \(z, z' \in P_A\) be such that \(\Phi_A(z) = \Phi_A(z')\) mod \(\mathbb{Z}\). If \(\Phi_A(z) = \Phi_A(z') + n\) where \(n \in \mathbb{N}\), we get \(\Phi_A(z) = \Phi_A(f^n(z'))\) and \(z = f^n(z')\). If the integer \(n\) is negative, note that \(\Phi_A(z) + (-n) = \Phi_A(z')\) which enables to apply the same argument. Hence \(\Phi_A(z) = \Phi_A(z')\) mod \(\mathbb{Z}\) implies \(z = z'\) mod \(f\). Hence \(\tilde{\Phi}_A : P_A/f \to \mathbb{C}/\mathbb{Z}\) is injective.

• \(\Phi_A\) is a nonconstant holomorphic function, hence it is open. So \(\tilde{\Phi}_A\) is open.

\[\tilde{\Phi}_A : P_A/f \to \mathbb{C}/\mathbb{Z}\] is an open continuous bijection, that is why it is a homeomorphism.

The reasoning for \(\tilde{\Phi}_R\) is identical.

**Proposition-Definition 3.11.** We consider the space of the grand orbits of \(f\) on \(B^f : B^f/f\).

The map \(\Phi_A^{ext} : B^f \to \mathbb{C}\) goes to quotient in a homeomorphism \(\Phi_A : B^f/f \to \mathbb{C}/\mathbb{Z}\). By transport of structure, this gives to \(B^f/f\) a Riemann surface structure. Let \(C_A = \mathbb{C}/\mathbb{Z}\) be the codomain of \(\Phi_A\), which we label the attractive cylinder of \(f\). Via the identification \(B^f/f = P_A/f\), we have \(\Phi_A = \tilde{\Phi}_A\) (in particular \(\Phi_A\) does not depend on the chosen petal \(P_A\), but only on the normalisation).

**Proof.** The map \(\Phi_A^{ext}\) goes to quotient since it semi-conjugates \(f\) to \(T_1\). The fact that \(\tilde{\Phi}_A\) is a homeomorphism results from \(\Phi_A = \tilde{\Phi}_A\).

**Proposition-Definition 3.12.** Let \(P_R\) be a repulsive petal of \(f\). The topological space \(P_R/f\) is independent of the chosen repulsive petal. Furthermore, the map \(\Phi_R : P_R/f \to \mathbb{C}/\mathbb{Z}\) is independent of the chosen repulsive petal \(P_R\). By transport of structure, this gives to \(P_R/f\) a Riemann surface structure. We note \(C_R = \mathbb{C}/\mathbb{Z}\) the codomain of \(\Phi_R\), which we name repulsive cylinder of \(f\). We also note \(\Psi_R = \Phi_{R}^{-1} : C_R \to P_R/f\) the map independent of the choice of the petal that \(\Psi_R : \Phi_R(P_R) \to P_R\) induces by quotient by \(\mathbb{Z}\) on the domain and by \(f\) on the codomain.

**Proof.** The space \(P_R/f\) is independent of the repulsive petal by proposition 3.3. If we note \(P_R^1, P_R^2\) two repulsive petals of \(f\) and \(\Phi_R^1, \Phi_R^2\) their respective Fatou coordinates, chosen in such a way that they coincide on \(P_R^1 \cap P_R^2\), we have via the identification \(P_R^1/f = P_R^2/f = (P_R^1 \cap P_R^2)/f\) the equality \(\Phi_R^1 = \Phi_R^2\). This concludes.

**Remark 3.13.** Let \(D_A, D_R\) be fundamental domains of \(f\) restricted respectively to an attractive petal \(P_A\), to a repulsive petal \(P_R\) (for instance the inverse images by the Fatou coordinates of a semi-open strip of width 1). The cylinder \(C_A\) is homeomorphic to \(B^f/f\), which is in natural bijection with \(D_A\) (each grand orbit has a unique element in \(D_A\)), by the same process \(C_R\) is in natural bijection with \(D_R\).

If \(D_A, D_R\) are inverse images by \(\Phi_A, \Phi_R\) of two strips of the form \(\{z \in \mathbb{C}, a \leq \Re(z) < a + 1\}\) (we call \(\overline{D}_A \backslash \{0\}, \overline{D}_R \backslash \{0\}\) crescents), one may also remark that \(B^f/f = \)
(\overline{D_A}\setminus\{0\})/f$, and $P_R/f = (\overline{D_R}\setminus\{0\})/f$, where the quotient consists exactly to glue the inverse images of the extremal vertical lines of the strips by the map $f$. This phenomenon enables to see $B^f$ and $P_R/f$ as topological cylinders via gluing the two boundaries of a crescent.

4 Horn maps

In this section, we remind the elements of a standard theory, with a presentation which might be original. We suppose in this section and those that follow that the Fatou coordinates $\Phi_A, \Phi_R$ are normalized (see definition 2.11).

Let $P_R$ be a repulsive petal. The grand orbits of $P_R \cap B^f$ are equal to grand orbits of $P_R$, in such a way that by 3.7, the inclusion $P_R \cap B^f \subset P_R$ induces an injection of $(P_R \cap B^f)/f$ in $P_R/f$; we assimilate the first set to a subset of the second one. These sets and this inclusion do not depend on the choice of $P_R$.

On the other hand, the map $i : (P_R \cap B^f)/f \rightarrow B^f/f$ induced by the inclusion $P_R \cap B^f \subset B^f$ is noninjective. This map $i$ and its domain do not depend on the chosen repulsive petal.

**Definition 4.1.** Set

$$h = \overline{\Phi}_A \circ i \circ \overline{\Psi}_R : \overline{\Psi}_R^{-1}((P_R \cap B^f)/f) \subset \mathcal{C}_R \rightarrow \mathcal{C}_A$$

where $\overline{\Phi}_A, \overline{\Psi}_R$ are defined in proposition-definition 3.11 and 3.12. We name this map the horn map associated to $f$. We note

$$\mathcal{D} = \mathcal{D}_h = \overline{\Psi}_R^{-1}((P_R \cap B^f)/f)$$

the definition domain of $h$. Note that:

$$\overline{\Psi}_R^{-1}((P_R \cap B^f)/f) = (\overline{\Psi}_R^{\text{ext}})^{-1}(P_R \cap B^f)/\mathbb{Z} = \overline{\Phi}_R((P_R \cap B^f)/f) = \Phi_R((P_R \cap B^f)/\mathbb{Z})$$

The horn map is said to be normalized if both attractive and repulsive Fatou coordinates are normalized.

**Remark 4.2.** Let $\Phi_A$ be the normalized extended attractive Fatou coordinate, and $\Psi_R^{\text{ext}}$ be the normalized extended repulsive Fatou parametrisation. Then

$$h(w) = \Phi_A(\Psi_R^{\text{ext}}(\tilde{w})) \mod \mathbb{Z}$$

for all $\tilde{w} \in (\Psi_R^{\text{ext}})^{-1}(B^f)$ such that $w = \tilde{w} \mod \mathbb{Z}$.

**Remark 4.3.** Let $D_A, D_R$ be fundamental domains of $f$ restricted to an attractive petal $P_A$, resp. restricted to a repulsive petal $P_R$. We associate a point $z \in \mathcal{C}_R$ to the corresponding element of $D_R$, then we associate, if it does exist, the unique element of $D_A$ which belongs to the same grand orbit for $f$ (this element exists if and only if
\( \Psi_R(z) \in (P_R \cap B^f)/f \), this explains the definition domain of \( \mathfrak{h} \), then we finally associate the corresponding element of \( \mathcal{C}_A \).

In a certain sense, \( \mathfrak{h} \) may be assimilated to an iterate with nonconstant exponent of \( f \). More precisely, we associate \( z_0 \in D_R \cap B^f \) first to \( f^n(z_0) \in P_A \) where \( n \in \mathbb{N} \), then by applying the right number of times \( f \) or \( f^{-1} : f(P_A) \to P_A \) we get \( (f|_{P_A})^k \circ f^n(z_0) \in D_A \) where \( k \in \mathbb{Z} \), the obtained result is independent of \( n, k \) chosen as soon as we respect the conditions \( f^n(z_0) \in P_A \) and \( (f|_{P_A})^k \circ f^n(z_0) \in D_A \). Remark that even with \( n \) minimal, it is still possible that \( k \) is negative. The issue of the negativity of \( k \) might be suppressed locally: there exists a neighborhood \( V \) of \( z_0 \) and \( n \in \mathbb{N} \) such that \( f^n(V) \subset P_A \); provided we choose \( V \) small enough and consider another fundamental domain \( D_A' \) of \( P_A \), we can assume that \( f^n(V) \subset D_A' \). The map \( f^n : z \in V \to f^n(z) \in D_A' \) gives after quotient \( \mathfrak{h} \Phi_R(V) : \Phi_R(V) \to \Phi_A^{-1}(D_A') = \mathbb{C}/\mathbb{Z} \), is the first passing map for \( f \) to \( D_A' \) from a \( z \in V \subset D_R \).

**Remark 4.4.** The definition domain \( \mathfrak{h} \) contains \( \Phi_R(P_A \cap P_R)/\mathbb{Z} \), where \( P_A, P_R \) are attractive, repulsive petals of \( f \). Let \( D_R \) be a fundamental domain of \( f \) restricted to a repulsive petal \( P_R \). The definition domain of \( \mathfrak{h} \) is exactly equal to \( \Phi_R ((B^f \cap D_R)/f) = \Phi_R(B^f \cap D_R)/\mathbb{Z} \).

Recall that \( D \) is the domain of \( \mathfrak{h} \), and that \( D^\pm \) is the connected component of \( D \) containing a neighborhood of \( \pm \infty \) (which exists by proposition 4.6).

**Proposition-Definition 4.5.** Let \( \pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z} \) be the canonical projection. Set

\[
\tilde{D} = \tilde{D}_h = \pi^{-1}(D)
\]

The extended Fatou parametrisation \( \Psi_R^{ext} \) is defined at least on \( \tilde{D} \) (and in particular on \( D^\pm \)) and \( \Psi_R^{ext}(\tilde{D}) \subset B^f \).

More precisely, for \( z \) in the definition domain of \( \Psi_R^{ext} \), \( z \in \tilde{D} \) if and only if \( \Psi_R^{ext}(z) \in B^f \). This enables to set \( h = \Phi_A \circ \Psi_R^{ext} \), the lifted horn map, defined on \( \tilde{D} \) (\( \Phi_A \) denotes the extended attractive Fatou coordinate).

**Proof.** Let \( P_R \) be a repulsive petal of \( f \). Recall that by definition 4.4 of the horn map, we have \( D = \Phi_R(P_R \cap B^f) \mod \mathbb{Z} \), in such a way that \( D = \Phi_R(P_R \cap B^f) + \mathbb{N} \) given that \( \Phi_R(P_R \cap B^f) \) is already stable by \( T_{-1} \). The extended Fatou parametrisation \( \Psi_R^{ext} \) is defined at least on \( \Phi_R(P_R \cap B^f) \), with values on this set in \( B^f \), it indeed coincides with \( \Phi_A^{-1} \). Since \( B^f \) is included in the definition domain of \( f \) and stable by \( f \), this implies by definition 3.1 of the extended repulsive Fatou parametrisation that \( \Psi_R^{ext} \) is also defined on \( \Phi_R(P_R \cap B^f) + \mathbb{N} \) (use the formula \( \Psi_R^{ext}(w + n) = f^n(\Psi_R^{ext}(w)) \), well defined for \( w \in \Phi_R(P_R \cap B^f) \)). Thus the definition domain of \( \Psi_R^{ext} \) contains \( \tilde{D} \) and \( \Psi_R^{ext}(\tilde{D}) \subset B^f \).

Conversely, let \( w \in \text{Dom}(\Psi_R^{ext}) \) be such that \( z = \Psi_R^{ext}(w) \in B^f \). Since \( w \in \text{Dom}(\Psi_R^{ext}) \), there exists \( z_0 \in P_R \) and \( n \in \mathbb{N} \) such that \( f^n(z_0) = z \). Since \( f^{-1}(B^f) \subset B^f \), we also have \( z_0 \in B^f \). Note \( w_0 = \Phi_R(z_0) = w - n \). We have

\[
w \mod \mathbb{Z} = w_0 \mod \mathbb{Z} = \Phi_R(z_0) \mod \mathbb{Z} \in \Phi_R(B^f \cap P_R) = D
\]

hence \( w \in \tilde{D} \), which completes the proof.
Proposition 4.6. The definition domain of $h$ contains a punctured neighborhood $V_+$ of $+i\infty$, and a punctured neighborhood $V_-$ of $-i\infty$. More precisely, if $P_A$, $P_R$ are attractive, repulsive large petals of $f$, take the following punctured neighborhoods (which are non connected in general):

$$V_+ = \Phi_A\left(P_A \cap P_R \cap \left(-\frac{1}{a} \mathbb{H}\right)\right)$$

$$V_- = \Phi_A\left(P_A \cap P_R \cap \left(\frac{1}{a} \mathbb{H}\right)\right)$$

We denote $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$ the Poincare half-plane. For $\lambda \in \mathbb{C}^*$, the set $\lambda \mathbb{H}$ is the image of $\mathbb{H}$ by the map $z \to \lambda z$.

Let $D_+^h$ and $D_-^h$ be the connected components of $D_h$, the definition domain of $h$ appearing in definition 4.1, punctured neighborhood of $+i\infty$ and $-i\infty$ respectively (these connected components might possibly by equal).

Proof. Let $P_A$, $P_R$ be $\alpha$-petals of $f$, where $\frac{\pi}{2} < \alpha < \pi$.

When $z$, belonging to the intersection $P_A \cap P_R$, tends to 0:

$$\Phi_A(z) \sim -\frac{1}{az} \sim \Phi_R(z)$$

Set $0 < \delta < \alpha - \frac{\pi}{2}$. Then, by remark 2.5 there exists $r > 0$ such that

$$\{z \in \mathbb{C}, 0 < |z| < r, \pi/2 - \delta < \arg\left(\frac{z}{a}\right) < \pi/2 + \delta\} \subset P_A \cap P_R$$

By the preceding equivalent, and applying a winding number argument, we have an inclusion of the form $\Phi_R(P_A \cap P_R \cap (-1/a \mathbb{H})) \supset \{x + iy \in \mathbb{C}, 0 \leq x \leq 1, y > R\}$ where $R > 0$ is large enough.

The set:

$$V_+ = \Phi_R(P_A \cap P_R \cap \left(-\frac{1}{a} \mathbb{H}\right))$$

is then a punctured neighborhood of $+i\infty$.

The proof is analogous for $V_-$. \hfill \Box

Remark 4.7. One may replace $-\frac{1}{a} \mathbb{H}$ by any nonempty open sector centered at 0 contained in this set. The new set formed $V_+$ will also be a neighborhood of $+i\infty$. The same holds for $V_-$ by replacing $\frac{1}{a} \mathbb{H}$ by an nonempty open sector centered at 0 contained in this set.

Proposition 4.8. Set $\tilde{D}^\pm = \pi^{-1}(D^\pm)$. Then we have $\tilde{D}^\pm \subset \text{Dom}(\Psi_R^{\text{ext}})$ and $\Psi_R^{\text{ext}}(\tilde{D}^\pm) \subset B^f_0$, where $B^f_0$ is the immediate basin of $f$ at 0.

Proof. By proposition-definition 4.5, we have already the inclusions $\tilde{D}^\pm \subset \text{Dom}(\Psi_R^{\text{ext}})$ and $\Psi_R^{\text{ext}}(\tilde{D}^\pm) \subset B^f_0$. The set $\Psi_R^{\text{ext}}(\tilde{D}^\pm)$ is connected. Moreover, let $G$ be a left half plane such that $P_R = \Psi_R^{\text{ext}}(G)$ is a repulsive petal and let $P_A$ be a large attractive petal. By proposition 2.6, $P_R \cap P_A \neq \emptyset$, hence $P_R$ intersects $B^f_0$. The desired inclusion is proved. \hfill \Box
Proposition 4.9. The map $h$ has a removable singularity at $+i\infty$. More precisely, $h$ admits the following expansion:

$$h(w) = w - i\pi \gamma + O\left(\exp \left(-2i\pi|\Im(w)\right)\right)$$

Likewise, $h$ has a removable singularity at $-i\infty$. More precisely, $h$ admits the following expansion:

$$h(w) = w + i\pi \gamma + O\left(\exp \left(-2i\pi|\Im(w)\right)\right)$$

The expansion of $h$ implies in particular:

- For all $M > 0$, there exists $M' > 0$ such that $\Im(w) > M'$ implies $\Im(h(w)) > M$.
- For all $M > 0$, there exists $M' > 0$ such that $\Im(w) < -M'$ implies $\Im(h(w)) < -M$.

These property are summarized in the following way: if $w$ has a high imaginary part (positively, negatively) then $h(w)$ also have a high imaginary part (positively, negatively).

Proof. This is a consequence of the asymptotic expansion of Fatou coordinates, more precisely of remark 2.12. Recall that the maps $\Phi_A, \Phi_R$ are normalized; this remark (and consequently the expansion of $h$) would not be true without this assumption: there would be an additional constant term.

Let $S, S^\pm$ as in remark 2.12 Since $\Phi_A(z) = \Phi_R(z) + \epsilon \pi \gamma + O(z)$ when $z \to S^\pm$, with $\epsilon = \mp 1$, we get $\Phi_A \circ \Phi_R^\text{ext}(w) = w + \epsilon \pi \gamma + O(w)$ when $w \to \pm i\infty$. The result is thus obtained by passing the domain and the codomain to quotient by $\mathbb{Z}$: $\lim_{w \to \pm i\infty} h(w) = \pm i\infty$ and the singularity of $h$ at $\pm i\infty$ is removable. Since a neighborhood of $\pm i\infty$ in $\mathbb{C}/\mathbb{Z}$ is isomorphic to a neighborhood of $0$ in $\mathbb{C}$ via the map $z \to \exp(\pm i\pi w)$, the neglected term $O(w)$ must in reality be a $O\left(\exp (\pm 2i\pi w)\right) = O\left(\exp (-2i\pi|\Im(w)|)\right)$.

Consequence 4.10. Let $P_A, P_R$ be attractive, repulsive petals of $f$ included in a neighborhood $W$ of $0$ where $f$ admits a local section $f^{-1}$. We note $\Phi_A, \Phi_R$ Fatou coordinates on these petals. Then for $z \in P_A \cap P_R$, $\Im(\Phi_A(z))$ is high (resp. low) if and only if $\Im(\Phi_R(z))$ is high (resp. low).

Nevertheless for $z \in B^f \cap P_R$ we have in general only one implication: if $\Im(\Phi_R(z))$ is high (resp. low) then $\Im(\Phi_A^\text{ext}(z))$ is high (resp. low), where $\Phi_A^\text{ext}$ is the extended Fatou coordinate.

Proof. To show the equivalence, we use the horn maps $h, h'$ of $f$ and $f^{-1}$. The formulas, when $z \in P_A \cap P_R$:

$$\Phi_A(z) \mod \mathbb{Z} = h(\Phi_R(z) \mod \mathbb{Z})$$

$$-\Phi_R(z) \mod \mathbb{Z} = h'(\Phi_A(z) \mod \mathbb{Z})$$

enable easily to conclude by applying the expansions of $h$ and $h'$.

To show the implication, we use the fact that the first formula is still available when $z \in B^f \cap P_R$, by replacing $\Phi_A$ by $\Phi_A^\text{ext}$. Note nonetheless that the second formula would
be only available for \( z \in B^f \cap P_A \) where \( P_A \) is an attractive petal of \( f \) (and consequently a repulsive petal of \( f^{-1} \)), and this is not enough to show the reciprocal implication.

This reciprocal implication is false. Let us give a counter-example. Note \( f : z \to z + z^2 \) the cauliflower map. An attractive petal \( P_A \) admits iterated inverse images \( P' \) by \( f \) which accumulate densely over the boundary of \( B^f \), with Euclidean diameters converging to 0. Since a repulsive petal \( P_R \) contains in its interior boundary elements of \( B^f \), it must contain the closure of one of the \( P' \) totally. That is why \( \Im(\Phi_R) \) is bounded on \( P' \), although it is not the case for \( \Im(\Phi_A) \). This concludes: the forementioned reciprocal is false.

\[ \square \]

**Remark 4.11.** We have shown that the reciprocal of the implication of the consequence \ref{4.10} is false, i.e. for \( z \in B^f \cap P_R \), if \( \Im(\Phi_A^{\text{ext}}(z)) \) is high (resp. low), it is not necessary that \( \Im(\Phi_R(z)) \) is high (resp. low), where \( \Phi^{\text{ext}} \) is the extended Fatou coordinate. This implies that the reciprocal of proposition \ref{4.9} is also false: for \( z \in D \), if \( h(w) \) has a high imaginary part (positively, resp. negatively), it is not necessary that \( w \) also have a high imaginary part (positively, resp. negatively).

We show some properties concerning the very large attractive petals:

**Proposition 4.12.** Let \( f \) be a holomorphic function admitting an expansion at 0 of the form \( f(z) = z + az^2 + o(z^2) \) where \( a \in \mathbb{C}^* \). Let \( P_A \) be a very large attractive petal of \( f \), and \( P_R \) be a repulsive petal of \( f \). Then there exists an upper half-plane \( H \) and a lower half-plane \( B \) such that \( \Phi^{-1}_R(H \cup B) = \Phi^{-1}_R((H \cup B) \cap \Phi_R(P_R)) \subset P_A \). The same holds by interchanging the roles of \( P_A, P_R \) (this amounts to applying the result to \( f^{-1} \)).

**Proof.** Let \( P_A \) be a very large attractive petal. Let \( H', B' \) be upper, lower half-planes such that \( \Phi_A(P_A) \supset H' \cup B' \). Proposition \ref{4.9} shows that there exists two half-planes \( H, B \) such that \( \Phi_A(\Psi^{\text{ext}}_R(H \cup B)) \subset H' \cup B' \), so \( \Psi^{\text{ext}}_R(H \cup B) \subset P_A \). \[ \square \]

**Proposition 4.13.** Let \( f \) be a holomorphic function admitting an expansion at 0 of the form \( f(z) = z + az^2 + o(z^2) \) where \( a \in \mathbb{C}^* \).

Let \( P_A \) be a very large attractive petal, contained in a small neighborhood \( f(W) \) of 0 where \( f \) admits a local section \( f^{-1} : f(W) \to W \). Let \( P_R \) be a repulsive petal containing an \( \alpha \)-petal, where \( \pi/2 < \alpha < \pi \) (it is sufficient that \( P_R \) is large). Then there exists an upper half-plane \( H \), a lower half-plane \( B \) both included in \( \Phi_A(P_A) \), and a left half-plane \( G \) such that \( L = L_{AR} := \Phi_A^{-1}((H \cup B) \cap G) \) is included in \( P_R \).

Let \( P_R \) be a very large repulsive petal, contained in a small neighborhood of 0 where \( f \) admits a local section \( f^{-1} \). By applying the preceding statement (formed by the first two paragraphs of the present proposition) to \( f^{-1} \), we get the following statement: let \( P_A \) be an attractive petal containing an \( \alpha \)-petal, where \( \pi/2 < \alpha < \pi \). Then there exists an upper half-plane \( H \), a lower half-plane \( B \) both included in \( \Phi_R(P_R) \), and a right half-plane \( D \) such that \( L = L_{RA} := \Phi_R^{-1}((H \cup B) \cap D) \) is included in \( P_A \).

In both cases, we call \( L \) a petal pair of \( f \).

See figure \ref{figure2} next page.
Figure 2: Lepal pairs of $f$
Proof. Without loss of generality we suppose (up to taking a conjugate of $f$) that $f(z) = z + z^2 + o(z^2)$ at 0. We only show the statement for the lepal pair $L_{AR}$, the other statement admits a similar proof. Let $R > 0$ be such that $S_{R, a'} := \{ z \in \mathbb{C}, |z| > R \text{ and } \pi - a' < \arg(z) < a' \} \subset \Phi_A(P_R)$, where $a' \in [\pi/2, \alpha]$. We also note $a'' \in ]a', \alpha[$, which will be used later. Set $P'_A = \Phi_A^{-1}(S_{R, a'}) \subset P_R$.

The Fatou coordinate $\Phi_A$ admits within $P'_A$ the equivalent:

$$\Phi_A(z) \sim -\frac{1}{z}$$

Set $\epsilon > 0$. The Fatou coordinate equivalent implies that there exists a neighborhood $V = V_\epsilon$ of infinity in $S_{R, a'} = \Phi_A(P'_A)$ such that $\Phi_A^{-1}(V)$ is included in a cone $\{ z \in \mathbb{C}, |z| < \epsilon \text{ and } \pi - a'' < |\arg(z)| < a'' \}$. We choose $\epsilon > 0$ in such a way that this last cone is included in a repulsive $a$-petal $P_R$ of $f$. The existence of such an $\epsilon$ is shown likewise by applying the equivalent of $\Phi_R$ instead of the one of $\Phi_A$. This finally implies that for a neighborhood $V$ of infinity in $S_{R, a'} = \Phi_A^{-1}(P'_A)$, the image of $\Phi_A(V)$ is included in a repulsive $a$-petal $P_R$ of $f$. Let $H$ be an upper half-plane, chosen in such a way that $H'_0 := H \cap \{ z \in \mathbb{C}, a - 1 \leq \Re(z) < a \} \subset V$ for some $a \in \mathbb{R}$.

Set for $n \in \mathbb{N}$, $H'_n = H \cap \{ z \in \mathbb{C}, a - n - 1 \leq \Re(z) < a - n \}$. We show by induction that for all $n \in \mathbb{N}$, $\Phi_A^{-1}(H'_n) \subset P_R$. This property is indeed true for $n = 0$, since $H'_0 \subset V$. Now, let $n \in \mathbb{N}$ be such that $\Phi_A^{-1}(H'_n) \subset P_R$. So $f(\Phi_A^{-1}(H'_{n+1})) = \Phi_A^{-1}(H'_n) \subset P_R$. The very large attractive $a$-petal $P_A$ is included in $W$. This implies that $\Phi_A^{-1}(H'_{n+1})$ is also included in $W$, and then $\Phi_A^{-1}(H'_{n+1}) = f^{-1}(\Phi_A^{-1}(H'_n))$ where $f^{-1} : W \to W$ is the local section of $f$. Hence $\Phi_A^{-1}(H'_{n+1}) \subset P_R$.

This finally implies, denoting $G' = \{ z \in \mathbb{C}, \Re(z) < a \}$ that we have $\Phi_A^{-1}(H \cap G') \subset P_R$. Likewise, there exists a lower half-plane $B$ and a left half-plane $G''$ such that $\Phi_A^{-1}(B \cap G'') \subset P_R$. This concludes by setting $G = G \cap G''$.

5 Definition of local pseudo-conjugacy and immediate consequences

5.1 Intermediate statement

We show proposition 5.2. It will be useful in the proofs of proposition 5.17 and proposition 5.27, which are also treated in this section.

Notation 5.1. Let $S$ be the set of (infinite) punctured closed sectors at 0 of $\mathbb{C} \setminus \mathbb{R}_+$. Note for $A \subset \mathbb{C}$ and $r > 0$: $A[r] = A \cap r + \mathbb{D}$

Proposition 5.2. Let $U, V$ be two simply connected open sets of $\mathbb{C}$ such that $\lambda U \to \mathbb{C} \setminus \mathbb{R}_+$, $\lambda V \to \mathbb{C} \setminus \mathbb{R}_+$, the limits are taken in the Caratheodory sense, when $\lambda \to +\infty$. This implies that 0 belongs to the boundary of $U$ and $V$.

Let $X = \{ x_n, n \in \mathbb{N} \}$ be a subset of $U$ such that $(x_n)$ converges in the Euclidean sense to 0 and the sequence of hyperbolic distances in $U$, $(d_U(x_n, x_{n+1}))$, converges to 0. We also assume that $X$ is included in a sector $S_X \in S$. 

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Let $\phi : U \rightarrow V$ be a holomorphic function. Suppose that there exists $\lambda_0 \in \mathbb{R}_+^*$ such that $\phi(z) \sim \lambda_0 z$ when $z \rightarrow 0$, $z \in X$ (this amounts to saying that $\phi(x_n) \sim \lambda_0 x_n$ when $n \rightarrow \infty$). Then:

- for all sector $S \in \mathcal{S}$ we have $\phi(z) \sim \lambda_0 z$ when $z \rightarrow 0$, $z \in S$
- for all sector $S' \in \mathcal{S}$, and all sector $S$ such that $S' \cap \partial \mathbb{D} \subset \text{int}(S)$ there exists $r,r' > 0$ such that $S[r] \subset U$, $S'[r'] \subset V$, and $\phi(S[r]) \supset S'[r']$. The set $\text{int}(S)$ denotes the interior of $S$.

**Remark 5.3.** One may replace $X$ by any set which contains such a sequence $(x_n)$, for instance any curve included in a sector $S_X \in \mathcal{S}$ converging towards the origin, or $X = S_X \in \mathcal{S}$.

**Remark 5.4.** The proposition is still valid with the following modifications:

- replace $\mathbb{C}\setminus \mathbb{R}_+^*$ by $\mathbb{C}\setminus S_0$, where $S_0 = \{0\} \cup \{z \in \mathbb{C}^*, \alpha_0 \leq \text{arg}(z) \leq \beta_0\}$ nonpunctured closed sector at the origin
- replace the definition of $\mathcal{S}$ by: the set of punctured closed sectors at the origin such that $S \cap S_0 = \emptyset$.

The given proof of the proposition readapts in this more general case with slight adaptations.

Discussion about the technical hypothesis done on the sequence $(x_n)$:

**Lemma 5.5.** The hypothesis “$d_U(x_n,x_{n+1})$ converges to 0” is equivalent (if all the other hypothesis of the proposition are assumed) to $\frac{x_{n+1}}{x_n}$ converges to 1.”
Proof. Note that \( d_U(x_n, x_{n+1}) = \frac{1}{|x_n|} \| \phi(x_n) - \phi(x_{n+1}) \|_U \). Let us show that \( \frac{x_n}{|x_n|} \) and \( \frac{x_{n+1}}{|x_n|} \) are for \( n \) big enough in a fixed compact set \( K \) of \( \mathbb{C} \setminus \mathbb{R}_+ \) as soon as \( \frac{x_{n+1}}{|x_n|} \) converges to 1 or \( d_U(x_n, x_{n+1}) \) converges to 0. This will allow in both cases to compare the hyperbolic metric of \( \frac{1}{|x_n|} U \) and the one of \( \mathbb{C} \setminus \mathbb{R}_+ \).

- Suppose that \( \left( \frac{x_{n+1}}{|x_n|} \right) \) converges to 1. Then choose for \( K \) any \( h \)-neighborhood of \( S_X \cap \partial \mathbb{D} \) where \( 0 < h < 1 \). The element \( \frac{x_n}{|x_n|} \) is in \( S_X \cap \partial \mathbb{D} \) by definition of \( S_X \), thus the element \( \frac{x_{n+1}}{|x_n|} = \frac{x_n}{|x_n|} \frac{x_{n+1}}{|x_n|} \) is in \( K \) for \( n \) big enough.

- Suppose that \( d_U(x_n, x_{n+1}) = \frac{1}{|x_n|} \| \phi(x_n) - \phi(x_{n+1}) \|_U \) converges to 0. Then choose for \( K \) any \( h' \)-hyperbolic neighborhood of \( S_X \cap \partial \mathbb{D} \) for the hyperbolic metric of \( \mathbb{C} \setminus \mathbb{R}_+ \) where \( h' > 0 \). The element \( \frac{x_{n+1}}{|x_n|} \) is in \( K \). Using the Carathéodory convergence of \( \left( \frac{1}{|x_n|} U \right) \) to \( \mathbb{C} \setminus \mathbb{R}_+ \), \( K \) is included in \( \frac{1}{|x_n|} U \) for \( n \) big enough and the Poincaré coefficient of the hyperbolic metric of \( \frac{1}{|x_n|} U \) converges uniformly over \( K \) to the Poincaré coefficient of the hyperbolic metric of \( \mathbb{C} \setminus \mathbb{R}_+ \). Thus for \( n \) big enough, the hyperbolic ball of \( \frac{1}{|x_n|} U \) of center \( \frac{x_n}{|x_n|} \) and radius \( h'/2 \) is included in \( K \). Since \( d_U(x_n, x_{n+1}) \) converges to 0, this implies that \( \frac{x_{n+1}}{|x_n|} \in K \) for \( n \) big enough.

This implies in both cases, by Carathéodory convergence of \( \frac{1}{|x_n|} U \) to \( \mathbb{C} \setminus \mathbb{R}_+ \) (and thus the uniform convergence of the Poincaré coefficient over the compact set \( K \ni \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \)):

\[
A_n \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \leq d_U \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right) \leq B_n \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \leq d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right)
\]

where \( \lim A_n = \lim B_n = 1 \). That is why in both cases, \( d_U(x_n, x_{n+1}) = d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right) \) is equivalent to \( d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right) = d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right) \).

We write:

\[
d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right) = d \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|} \right)
\]

where for all \( n \in \mathbb{N} \), \( \alpha_n = \frac{x_n}{|x_n|} \in S_X \cap \partial \mathbb{D} \).

Since \( \alpha_n \) belongs to a fixed compact subset of \( \mathbb{C} \setminus \mathbb{R}_+ \), the property \( d \left( \alpha_n, \frac{x_{n+1}}{|x_n|} \right) \) converges to 0 is equivalent to say that the Euclidean distance \( |\alpha_n - \frac{x_{n+1}}{|x_n|}| = |\frac{x_n}{|x_n|} - \frac{x_{n+1}}{|x_n|}| \) converges to 0, i.e. \( |x_{n+1} - x_n| = o(x_n) \), which is equivalent to \( x_{n+1} \sim x_n \). This concludes the proof.

Let us show proposition 5.2

Proof. Denote for \( \lambda \geq 1 \), \( \phi_\lambda : \lambda U \to \lambda V \) defined by \( \phi_\lambda(z) = \lambda \phi(z) \). We show that \( (\phi_\lambda)_{\lambda \geq 1} \) is normal on \( G = \text{int}(S[2r]) \) where \( S \in S \) and \( r > 0 \) is chosen such that \( S[2r] \subset U \).

Set \( Y = \phi(X) \). Remark that \( \lambda Y = \phi_\lambda(\lambda X) \). Since \( \lambda V \to \mathbb{C} \setminus \mathbb{R}_+ \) when \( \lambda \to \infty \), we have that for all \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), there exists \( a > 0 \) such that for all \( \lambda \geq a \), \( z \in \lambda V \).

We also have that for all \( z_0 \in \mathbb{R}_+ \), \( \epsilon > 0 \), there exists \( a > 0 \) such that for \( \lambda \geq a \),
\(B(z_0, \epsilon) \cap (\lambda V)^c \neq \emptyset\). Taking the values \(z_0 = 1, \epsilon = \frac{1}{2}\), we get for \(\lambda \geq a\) that there exists \(z(\lambda) \in B(1, \frac{1}{2}) \cap (\lambda V)^c\). Note that the function \(z\) is not necessarily continuous. The family \((\phi_\lambda)_{\lambda \geq a}\) is normal since \(\phi_\lambda(\lambda U)\) avoids three points, 0, \(z(\lambda), \infty\) where \(z(\lambda)\) belongs to a fixed compact set avoiding 0, \(\infty\).

Let \(\psi\) be an accumulation point of \((\phi_\lambda)\) for the uniform local convergence. Let \((\lambda_n)\) be such that \((\phi_{\lambda_n})\) converges to \(\psi\). Set \(F\) be an accumulation point of the sequence \(\lambda_n X\) for the Hausdorff metric. Note that \(F\) is a closed set included in \(S_X\). Up to extraction of a subsequence, we may assume that \(F\) is the limit of the sequence \(\lambda_n X\).

**Lemma 5.6.** For all \(r > 0\), the set \(F\) contains an element \(z\) of modulus \(r\). In particular, \(F\) admits accumulation points.

**Proof.** Set \(\epsilon > 0\). Let \(A\) be the annulus delimited by circles of center 0 and respective radii \(r, r + \epsilon\).

Since \(\lim_{n \to +\infty} \frac{x_n + 1}{x_n} = 1\), there exists \(k_0 \geq 0\) such that for \(k \geq k_0\) we have \(\frac{r}{r+\epsilon} \leq \frac{x_k + 1}{x_k} \leq \frac{r}{r+\epsilon} + \epsilon\). Let \(N \in \mathbb{N}\) be such that for all \(n \geq N\) we have \(|\lambda_n x_k| \geq r\). Then \(\lambda_n X \cap A\) is nonempty for all \(n \geq N\). When \(n \to \infty\), this set converges in the Hausdorff sense to a set included in \(F \cap A\). This set must be nonempty as a limit of nonempty sets included in a compact subset of \(C\). The property \(F \cap A\) nonempty must be true for all \(\epsilon > 0\). The set \(F\) is closed, it thus intersects \(r\partial\mathbb{D}\). This concludes.

Set \(z \in F\). There exists \(z'_{n} \in \lambda_n X\) such that \(|z - z'_{n}|\) tends to 0 when \(n\) tends to infinity. Note that the \(z'_{n}\) belong to a compact neighborhood \(K\) of \(z\). We have:

\[
\phi_{\lambda_n}(z'_{n}) = \lambda_n \phi\left(\frac{z'_{n}}{\lambda_n}\right) \to_{n \to \infty} \lambda_0 z
\]

where \(\lambda_0\) is given in the hypothesis of proposition 5.2. The elements of the sequence \((z'_{n})\) belongs to a fixed compact set \(K\), and \(\phi_{\lambda_n}\) converges uniformly to \(\psi\) over \(K\). This gives the existence of the limit:

\[
\lim_{n \to \infty} \phi_{\lambda_n}(z'_{n}) = \psi(z) = \lambda_0 z
\]

Hence \(\psi(z) = \lambda_0 z\) for all \(z \in F\). By the preceding lemma and the principle of isolated zeros, we must have \(\psi(z) = \lambda_0 z\) for all \(z \in C \setminus \mathbb{R}_+\). Since the family \((\phi_\lambda)\) is normal and has only one accumulation point, it follows that \((\phi_\lambda)\) converges to \(\lambda_0 \id\).

Let \(K\) be a compact subset of \(U\). Set \(\hat{K} = \cup_{\lambda \geq 1} \frac{1}{\lambda} K\). We show that, when \(z \in \hat{K}\) and \(z \to 0\), we have the equivalent \(\phi(z) \sim_{\lambda \to \infty} \lambda_0 z\). Set \(z \in \hat{K}\). We write \(z = \frac{1}{\lambda} z_0\) where \(z_0 \in K\). The element \(z\) converges to 0 if and only if \(\lambda\) converges to infinity (although the decomposition \(z = \frac{1}{\lambda} z_0\) is not unique). We have \(\phi(z) = \frac{1}{\lambda} \phi_\lambda(\lambda z) = \frac{1}{\lambda} \phi_\lambda(z_0) \sim_{\lambda \to \infty} \frac{1}{\lambda} \lambda_0 z_0 = \lambda_0 z\). We note that the equivalent is uniform in \(z_0 \in K\) when \(\lambda \to \infty\) by uniformity of the limit of \(\phi_\lambda\) on the compact set \(K\). This gives an equivalent when \(z \to 0, z \in \hat{K}\).

Note that for all \(S \in \mathcal{S}\), we have \(S \cap \partial = \hat{K}\), with \(K = S \cap \partial\mathbb{D}\). In particular:

\[
\phi(z) \sim_{z \to 0} \lambda_0 z
\]

\[29\]
This concludes the proof of the first part of the proposition.

Concerning the proof of the second part, let us assume, even if it means to post-compose by a homothety centered at 0, that $\lambda_0 = 1$. The result to be shown is a direct consequence of the following lemma, applied to $g = \ln_S \circ \phi \circ \exp_B$, where $\ln_S : S \to \mathbb{C}$ is a holomorphic branch of the logarithm on the sector $S$, of image $B$. The hypothesis $\phi(z) \sim_{z \to 0} z$ is equivalent to the Taylor expansion $g(z) = z + o(1)$ when $\text{Re}(z) \to -\infty$.

**Lemma 5.7.** Let $B = \{ z \in \mathbb{C}, \text{Re}(z) \leq a \text{ and } b \leq \text{Im}(z) \leq c \}$ be a half strip of the plane $\mathbb{C}$. Let $0 < h < \frac{c-b}{2}$ and $g : B \to \mathbb{C}$ be a continuous functions such that $|g(z) - z| \leq h$ for all $z \in B$. Then the strip $B' = \{ z \in \mathbb{C}, \text{Re}(z) \leq a-h \text{ and } b+h \leq \text{Im}(z) \leq c-h \}$ is included in $g(B)$.

See figure 4.

**Proof.** Note $X$ the open $h$-neighborhood of $\partial B$. We write $\mathbb{C} \setminus X = C_0 \cup C_1$ the partition in connected components, where $C_0$ is contained in $B$. Remark that $C_0 = B'$.

Set $g(\infty) = \infty$, in such a way that $g$ defines a continuous function of $B \cup \{\infty\}$ to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The closed loops of $\hat{\mathbb{C}}$ given by $\gamma = \partial B \cup \{\infty\}$ and $\gamma' = g \circ \gamma = g(\partial B) \cup \{\infty\}$ are homotopic in $X \cup \{\infty\}$. The homotopy is indeed given by:

$$\phi(s,t) = (1-t)\gamma(s) + t\gamma'(s)$$

Since $\gamma$ is contractile in $B$, by direct image by $g$, the curve $\gamma' = g(\partial B) \cup \{\infty\}$ is contractile in $g(B) \cup \{\infty\}$. Since $\gamma$ is homotopic to $\gamma'$ in $X$, this implies that $\gamma$ is contractile in $g(B) \cup X \cup \{\infty\}$. This is possible only if $g(B)$ contains $C_0$ or $C_1$ (see figure 4). Indeed, if $g(B)$ avoids one point $z_0$ of $C_0$ and one point $z_1$ on $C_1$, the homotopy class of $\gamma$ in $\mathbb{C} \cup \{\infty\} \setminus \{z_0, z_1\}$ would be nontrivial, thus $\gamma$ cannot be contractile in $g(B) \cup X \cup \{\infty\} \subset$
\( \mathbb{C} \cup \{ \infty \} \setminus \{ z_0, z_1 \} \), which is absurd. The set \( g(B) \) is included in \( B \cup X \) so it cannot contain \( C_1 \). It follows that \( g(B) \) contains \( C_0 \). This concludes the proof. \( \square \)

This concludes the proof of the second part of the proposition. \( \square \)

### 5.2 Local semi-conjugacy on immediate basins

For all open set \( U_1 \) of \( \mathbb{C} \) containing 0, note \( U^0_1 \) the connected component of \( U_1 \cap B^i_0 \) containing a germ of the attractive axis of \( f_1 \).

**Definition 5.8.** A holomorphic map \( \phi \) is said to be a local semi-conjugacy of \( f_1, f_2 \) on their immediate basins if \( \phi : U^0_1 \rightarrow B^2_0 \) where \( U_1 \) open neighborhood of 0 in \( \mathbb{C} \), where \( \phi \) semi-conjugacy, in other words:

\[
\phi \circ f_1(z) = f_2 \circ \phi(z) \quad \text{for } z \in U^0_1, \text{ such that } f_1(z) \in U^0_1.
\]

**Remark 5.9.** We require neither that \( \phi \) admits a continuous extension at 0, nor that \( \phi(U^0_1) \) is included in a small neighborhood of 0.

To motivate the choice of the set \( U^0_1 \), let us give some properties of it.

**Remark 5.10.** The attractive petals are connected and contain a germ of the attractive axis. Thus the set \( U^0_1 \) might be understood as the connected component of \( B^i_0 \cap U_1 \) containing a given attractive petal (resp. all the attractive petals) of \( f_1 \) included in \( U_i \). In particular, \( U^0_1 \) contains all attractive petals of \( f_i \) included in \( U_i \).

**Remark 5.11.** Let \( P^i_A \) be a very large attractive petal of \( f_i \) included in \( U_i \) containing \( \Phi^i_R \text{ext}(H \cup B) \) where \( H, B \) are upper, lower half planes such that \( H \cup B = \{ z \in \mathbb{C}, |\text{Im}(z)| > M \} \). Let \( P^i_R \) be a repulsive petal included in \( U_i \). Then the open set \( U^0_i \) contains all the connected components \( C \) of \( (\Phi^i_R)^{-1}(\tilde{D}^i \cap \Phi^i_R(P^i_R)) \) such that \( |\text{Im}(\Phi^i_R)| \) is non majorized by \( M \), where \( P^i_R \) is a repulsive petal included in \( U_i \). Recall that \( \tilde{D}^i \) is the connected component of the domain \( D^i \) of \( h_i \) containing a neighborhood of \( \pm i \infty \), which exists by proposition 4.6, and that \( \tilde{D}^i = \pi^{-1}(D^i) \), where \( \pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z} \) is the canonical projection. These sets \( C \) are connected, included in \( U_i \cap B^i_0 \), and intersect \( P^i_A \) by proposition 4.12, that is why they must be included in \( U^0_i \). See figure 5.

This remark implies the following proposition:

**Proposition 5.12.** Let \( P^i_R \) be a repulsive petal of \( f_i \). Then

\[
\Phi^i_R(U^0_1 \cap P^i_R) \mod \mathbb{Z} \supset \tilde{D}^i \cup \tilde{D}^i_*
\]

**Proof.** We consider \( z \in \tilde{D}^i \) and a path \( \gamma \) joining \( z \) with an element \( z_0 \in \tilde{D}^i_* \) such that \( |\text{Im}(z_0)| > M \). We lift this path by \( \pi \) to obtain a path \( \tilde{\gamma} \) of \( \tilde{D}^i \cup \tilde{D}^i_* \) joining \( \tilde{z}, \tilde{z}_0 \in \mathbb{C} \) such that \( \pi(\tilde{z}) = z \) and \( \pi(\tilde{z}_0) = z_0 \). The set \( \tilde{\gamma} \) is compact, so by proposition ( petal) there exists \( n \in \mathbb{N} \) such that \( \tilde{\gamma} - n \) is included in \( \Phi^i_R(P^i_R) \). This shows that \( \tilde{z} - n \) is in a connected component of \( \tilde{D}^i \cap \Phi^i_R(P^i_R) \) whose imaginary part has an absolute value non majorized by \( M \). Thus \( \tilde{z} - n \in \Phi^i_R(U^0_1 \cap P^i_R) \). Hence \( z \in \Phi^i_R(U^0_1 \cap P^i_R) \mod \mathbb{Z} \). \( \square \)
Figure 5: Some connected components of \((\Phi_R^i)^{-1}(\tilde{D}^\pm_i \cap \Phi_R^i(P_R^i))\) included in \(U_i^0\). On the figure, \(U_i = P_A^i \cup P_R^i \cup \{0\}\) and the set \(\Phi_R^i(P_R^i)\) is a left half-plane. The grey sets represent the connected components \(C\) of \((\Phi_R^i)^{-1}(\tilde{D}^\pm_i \cap \Phi_R^i(P_R^i))\) such that \(\Phi_R^i(C)\) intersects \(H \cup B\), and their image by \(\Phi_R^i\). The fuchsia sets represents the other connected components \(C\) of \((\Phi_R^i)^{-1}(\tilde{D}^\pm_i \cap \Phi_R^i(P_R^i))\) and their images by \(\Phi_R^i\). One can see that the grey components are included in \(U_i^0\).
Proposition 5.13. If $U_i$ is small enough, we have the converse inclusion. More precisely, set $\alpha \in [\pi/2, \pi]$ and let $P_A^i, P_R^i$ be attractive, repulsive $\alpha$-petals of $f_i$ such that $P_A^i \cap P_R^i$ has two connected components $C^+_i, C^-_i$. This is possible by the Leau-Fatou flower theorem. In this case, we must have (even if it means permuting $C^+_i$ and $C^-_i$):
\[ \Phi_R^i(C^+_i) \mod Z \subset D^+_i \]

In particular:
\[ \Phi_R^i(P_A^i \cap P_R^i) \mod Z \subset D^+_i \cup D^-_i \]

Suppose that $U_i \subset P_A^i \cup P_R^i \cup \{0\}$. Then:
\[ \Phi_R^i(U_i \cap P_R^i) \mod Z \subset D^+_i \cup D^-_i \]

Proof. It is sufficient to show this inclusion when $U_i = P_A^i \cup P_R^i \cup \{0\}$. Without loss of generality, suppose $i = 1$. We set
\[ (P_R^1)^0 = (\Phi_R^1)^{-1}(\hat{D}_1^+ \cup \hat{D}_1^-) \]

and
\[ V_1^0 = P_A^1 \cup (P_R^1)^0 \]

We show the following lemma:

Lemma 5.14. $U_1^0 \subset V_1^0$.

If this lemma is shown, we obtain by applying $\Phi_R^1$ the desired inclusion, which concludes the proof. Indeed, $\Phi_R^1(U_1^0) = \Phi_R^1(U_1^0 \cap P_R^1)$ and $\Phi_R^1(V_1^0) = \Phi_R^1(P_A^1 \cap P_R^1) \cup \Phi_R^1((P_R^1)^0) \subset D^+_1 \cup D^-_1$. Let us show this lemma.

Proof. Since $U_1 = P_A^1 \cup P_R^1 \cup \{0\}$, we have $U_1 \cap B_0^i = P_A^1 \cup (P_R^1 \cap B_0^i)$. The set $U_1^0$ is the connected component of $U_1 \cap B_0^i$ containing $P_A^1$. It is then the union of $P_A^1$ with the connected components $C$ of $P_R^1 \cap B_0^i$ intersecting $P_A^1$.

To show the lemma, it is thus sufficient to show that each connected component $C$ of $P_R^1 \cap B_0^i$ intersecting $P_A^1$ (and thus intersecting $P_A^1 \cap P_R^1 = C^+_i \cup C^-_i$) is contained in $(P_R^1)^0$. The nonempty set $(C^+_i \cup C^-_i) \cap C$ is included in the definition domain of $\Phi_R^1$, this implies that $C' = \Phi_R^1(C) \mod Z$ (connected set included in $D_1$, the definition domain of $h_1$), intersects
\[ \Phi_R^1(C^+_i \cup C^-_i) \mod Z \subset D^+_i \cup D^-_i \]

Thus $C' \subset D^+_i \cup D^-_i$. Hence $\Phi_R^1(C) \subset \hat{D}^+_i \cup \hat{D}^-_i$, and $C \subset (P_R^1)^0$, this concludes the proof. \(\square\)

Remark 5.15. We do not know if proposition [5.13] is still true for general $U_i$, which would be equivalent to say that:

\[ \Phi_R^i(B_0^i \cap P_R^i) \mod Z \subset D^+_i \cup D^-_i \]
The attractive petal is still represented by a cardioid, and the repulsive petal by a circle. The set $\Phi^1_R(P^1_R)$ is represented by a left half plane. The dark grey region at the left represents $U^0_1$, while the union of the dark grey and the light grey is equal to $V^0_1$. The colored regions at the right represent the images by $\Phi^1_R$ of the zones colored at the left.

**Remark 5.16.** The equality valid for small enough $U_i$

$$\Phi^1_R(U^0_i \cap P^1_R) \mod Z = D^+_i \cup D^-_i$$

is the motivation of the definition of $U^0_i$: restricting $f_i$ to $U^0_i$ corresponds to restricting its horn map to $D^+_i \cup D^-_i$. See definition 4.1 of the horn map $h$ of a function $f$, especially its definition domain $D_h$, which depends on the definition domain of $f$, since the immediate basin of $f|_U$ is smaller than the immediate basin of $f$.

The map $\phi$ induces a map

$$\bar{\phi}: B^{f_1}/f_1 \to B^{f_2}/f_2$$

in the following way: for $z = z \mod f_1 \in B^{f_1}/f_1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $f^n(z) \in U^0_1$. The element $\phi(f^n(z)) \mod f_2 \in B^{f_2}/f_2$ is independent of the representative $z$ of $\bar{z}$ and of the chosen integer $n \geq N$.

**Proposition 5.17.** Let $\phi$ be a local semi-conjugacy. Let $P^1_A$ be an attractive petal included in $U_1$, with a Fatou coordinate denoted $\Phi^1_A$. Then $\bar{\phi}: B^{f_1}/f_1 \to B^{f_2}/f_2$ is a biholomorphism preserving each end of the cylinders, where we recall that the ends of the cylinder $B^{f_1}/f_1$ are given by the biholomorphism induced by the Fatou coordinate $\Phi_A: B^{f_1}/f_1 \to \mathbb{C}/\mathbb{Z}$. This amounts to saying that the map $\alpha = \Phi^2_A \circ \bar{\phi} \circ (\Phi^1_A)^{-1}: \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ is a translation.
Proof. Set \( z \in P^1_A \). We note
\[
X = \{f^n_1(z), n \in \mathbb{N}\} \subset P^1_A
\]
and:
\[
Y = \phi(X) = \{f^n_2(\phi(z)), n \in \mathbb{N}\} \subset B^{f_2}
\]
We remark that, by the equivalent of the inverses of Fatou coordinates, \( f^n_1(z) = (\Phi^1_A)^{-1}(\Phi^1_A(z) + n) \sim -\frac{C_1}{n} \) and \( f^n_2(z) = (\Phi^2_A)^{-1}(\Phi^2_A(z) + n) \sim -\frac{C_2}{n} \) where \( C_1, C_2 > 0 \) (recall that the \( z^2 \) coefficient of \( f_i \) is supposed positive real), in such a way that \( \phi(z) \sim \frac{C_2}{C_1}z \) when \( z \in X, \ z \to 0 \). Also note that \( \lim_{n \to \infty} \frac{f^{n+1}_1(z)}{f^n_1(z)} = 1 \), thus by lemma 5.5 \( d_{U^n_1}(f^n_1(z), f^{n+1}_1(z)) \)
converges to 0. We may apply proposition 5.2 with \( U = U^n_1 \supset P^1_A, V = B^{f_2}, \lambda_0 = \frac{C_2}{C_1} \).
Hence for all sector \( S \in S \) (recall that \( S \) is the set of infinite punctured closed sectors at 0 of \( \mathbb{C}\setminus\mathbb{R}_+ \)):
\[
\phi(z) \sim_{z \in S} \frac{C_2}{C_1}z
\]
Let \( P^2_A \) be a large attractive petal of \( f_2 \) and denote \( \tilde{\alpha} = \Phi^2_A \circ \phi \circ (\Phi^1_A)^{-1} \). Let \( S \subset \Phi^1_A(P^1_A) \) be a nonempty open sector containing no horizontal half-line such that \( \phi((\Phi^1_A)^{-1}(S)) \subset P^2_A \). This last inclusion is possible by the equivalents of \( \phi \) and \( (\Phi^1_A)^{-1} \).
Remark that \( \text{Im}(z) \) is unbounded on \( S \).
The map \( \tilde{\alpha} = \Phi^2_A \circ \phi \circ (\Phi^1_A)^{-1} : S \to \mathbb{C} \) satisfies, by exploiting the Fatou coordinates equivalents and their inverse, and the equivalent of \( \phi \) given by proposition 5.2
\[
\tilde{\alpha}(w) \sim (-\frac{1}{C_2}) \frac{C_2}{C_1}(-C_1)w = w
\]
where the equivalent is taken when \( w \to \infty, w \in S \).
In particular, since \( S \) contains no horizontal ray, \( \text{Im}(\tilde{\alpha}(w)) \to \text{Im}(w) \to \pm \infty \pm \infty \).
Remark that \( S \mod \mathbb{Z} \) is a neighborhood of \( \pm i\infty \), the sign depending on the choice of \( S \). Thus the map \( \alpha = \Phi^2_A \circ \phi \circ (\Phi^1_A)^{-1} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} \) satisfies :
\[
\text{Im}(\alpha(w)) \to w \to \pm \infty \pm i\infty
\]
The singularities at \( \pm i\infty \) are thus removable. Via the biholomorphism \( \mathbb{C}/\mathbb{Z}\cup\{\pm i\infty\} \to \hat{\mathbb{C}} \) given by \( w \to \exp(2\pi i w) \), the map \( \alpha \) induces a map \( \hat{\alpha} \) from \( \hat{\mathbb{C}} \) to itself admitting \( \{0\} \) and \( \{\infty\} \) as completely invariant sets (i.e. invariant by direct and inverse images). The map \( \hat{\alpha} \) is thus of the form \( z \to Cz^n \) with \( C \in \mathbb{C}^* \) and \( n \in \mathbb{N}^* \). Hence the map \( \alpha \) must be of the form \( z \to nz + \sigma \). Since \( \alpha \) commutes with \( z \to z + 1 \), we must have \( n = 1 \), thus \( \alpha \) is a translation. This allows us to conclude: \( \hat{\phi} \) is a biholomorphism of the cylinder preserving the ends.

Let \( \phi \) be a local semi-conjugacy from \( f_1 \) to \( f_2 \). Let \( \Phi^1_A, \Phi^2_A \) be Fatou coordinates of \( f_1, f_2 \), associated to the attractive petals \( P^1_A, P^2_A \) included in \( U_1, U_2 \). The map \( \alpha = \Phi^2_A \circ \phi \circ (\Phi^1_A)^{-1} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} \) is a biholomorphism preserving each end of the cylinder, in such a way that it is a translation \( T_\sigma \). We call \( \sigma \) the phase shift of \( \phi \) with respect to the Fatou coordinates, and denote it \( \sigma(\phi) \).

\[35\]
Proposition 5.18. The map $\tilde{\sigma} = \Phi_{A}^{2,ext} \circ \phi \circ (\Phi_{A}^{1})^{-1} : \Phi_{A}^{1}(P_{A}^{1}) \to \mathbb{C}$ is a translation $T_{\tilde{\sigma}}$.

We call $\tilde{\sigma}$ the lifted phase shift of $\phi$ with respect to the attractive Fatou coordinates, and denote it $\tilde{\sigma}(\phi)$. We have $\tilde{\sigma} \mod \mathbb{Z} = \sigma$. This phase shift depends only on the normalizations of Fatou coordinates, and not on their definition domains.

Proof. In a composition of functions, it is equivalent to go to quotient at each step and to go to quotient only at the beginning and at the end. Then, the map $\tilde{\sigma} : \Phi_{A}^{1}(P_{A}^{1}) \to \mathbb{C}$ is a lift of $\alpha : \Phi_{A}^{1}(P_{A}^{1})/\mathbb{Z} = \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. That is why $\tilde{\sigma}$ is a translation $T_{\tilde{\sigma}}$ where $\tilde{\sigma} \mod \mathbb{Z} = \sigma$. It is clear that $\tilde{\sigma}$ neither depends on the choice of the definition domain of $\Phi_{A}^{2}$, nor the one of $\Phi_{A}^{1}$.

Corollary 5.19. $\phi$ is injective on $P_{A}^{1}$.

Proof. The injectivity of $\phi$ on $P_{A}^{1}$ follow from the equality $T_{\tilde{\sigma} \circ \Phi_{A}^{1}(P_{A}^{1})} = \Phi_{A}^{2,ext} \circ \phi \circ (\Phi_{A}^{1})^{-1}$ and of the injectivity of $T_{\tilde{\sigma} \circ \Phi_{A}^{1}(P_{A}^{1})}$.

Remark 5.20. The post-composition of $\phi$ by positive powers of $f_{2}$ leaves the definition domain invariant. It is not the case for pre-composition by integer powers of $f_{1}$.

Proposition 5.21. Let $\phi$ be a local semi-conjugacy of $f_{1}, f_{2}$. For $m \in \mathbb{N}$, the map $f_{2}^{m} \circ \phi$ is also a local semi-conjugacy. Furthermore:

$$\tilde{\sigma}(f_{2}^{m} \circ \phi) = \tilde{\sigma}(\phi) + m$$

Let $m \in \mathbb{N}$ and $W_{1}, W_{1}'$ be open neighborhoods of $0$ such that $f_{1}^{m} : W_{1} \to W_{1}'$ is a biholomorphism. We note $f_{1}^{-m} : W_{1}' \to W_{1}$ the inverse of this biholomorphism. Set $V_{1} = f_{1}^{-m}(U_{1} \cap W_{1}')$ the direct image of $U_{1} \cap W_{1}'$ by the biholomorphism $f_{1}^{-m}$ and $V_{1}' = f_{1}^{m}(U_{1} \cap W_{1})$. The sets $V_{1}, V_{1}'$ are an open neighborhoods of $0$. Let finally $V_{1}^{0}$ resp. $V_{1}'^{0}$ be the connected component of $V_{1} \cap B^{i}$ resp. $V_{1}' \cap B^{i}$ containing a germ of the attractive axis. The maps $\phi \circ f_{1}^{m} |_{V_{1}^{0}}, \phi \circ f_{1}^{-m} |_{V_{1}'^{0}}$ are also a local semi-conjugacies. Furthermore:

$$\tilde{\sigma}(\phi \circ f_{1}^{m} |_{V_{1}^{0}}) = \tilde{\sigma}(\phi) + m$$
$$\tilde{\sigma}(\phi \circ f_{1}^{-m} |_{V_{1}'^{0}}) = \tilde{\sigma}(\phi) - m$$

Proof. It is clear that the indicated maps are also semi-conjugacies, since $f_{2}^{m}$ commutes with $f_{2}$, and $f_{1}^{m}$ commutes with $f_{1}$. For the equalities with the lifted phase shift, use $\Phi_{A}^{2} \circ f_{2}^{m} = T_{m} \circ \Phi_{A}^{2}$ and $f_{1}^{m} \circ (\Phi_{A}^{1})^{-1} = (\Phi_{A}^{1})^{-1} \circ T_{x_{m}}$ where $T_{u} = z \to z + a$.

Proposition 5.22. Let $P_{A}^{1}, P_{A}^{2}$ be attractive petals of $f_{1}, f_{2}$. Then:

- $\phi(P_{A}^{1}) \subset P_{A}^{2}$ is equivalent to $T_{\tilde{\sigma}}(\Phi_{A}^{1}(P_{A}^{1})) \subset \Phi_{A}^{2}(P_{A}^{2})$
- $\phi(P_{A}^{2}) \supset P_{A}^{1}$ is equivalent to $T_{\tilde{\sigma}}(\Phi_{A}^{2}(P_{A}^{2})) \supset \Phi_{A}^{2}(P_{A}^{2})$

In particular, there exists a very large attractive petal $P_{A}^{1} \subset U_{1}$ such that $\phi(P_{A}^{1}) \subset U_{2}$.
See figure 7.

**Proof.** We know that $T_{\tilde{\sigma}}|_{\Phi_1^1(P_1^1)} = \Phi_2^2,\text{ext} \circ \phi \circ (\Phi_1^1)^{-1}$. So:

$$T_{\sigma}(\Phi_1^1(P_1^1)) = \Phi_2^2,\text{ext}(\phi(P_1^1))$$

and $\Phi_2^2,\text{ext}$ induces a biholomorphism from $\phi(P_1^1)$ to $\Phi_2^2,\text{ext}(\phi(P_1^1))$. Furthermore:

**Lemma 5.23.** Let $X, Y$ be two topological spaces, $U, V$ two non-disjoint open connected subsets of $X$. Let $\Phi : X \to Y$ be a map such that $\Phi : V \to \Phi(V)$ is open and proper. Then $U \subset V$ is equivalent to $\Phi(U) \subset \Phi(V)$.

**Proof.** The direct implication is clear. For the other implication, we use contraposition. Since $U$ is connected and intersects $V$, the fact that $U$ is not included in $V$ implies that $U$ intersects the boundary of $V$. Set $z \in U \cap \partial V$. Then $\Phi(z) \in \Phi(U)$ and $\Phi(z) \in \partial \Phi(V)$ since $\Phi : V \to \Phi(V)$ is proper, so $\Phi(z) \notin \Phi(V)$ since $\Phi(V)$ is open.

We apply the preceding lemma with $U = \phi(P_1^1)$, $V = P_2^2$ and $\Phi = \Phi_2^2,\text{ext}$. We apply then the same lemma, permuting the role of $U$ and $V$. Since $U$ contains a forward orbit of $f_2$, it is not disjoint from $V$. We also note that $\Phi_2^2,\text{ext} : U \to \Phi_2^2,\text{ext}(U)$ and $\Phi_2^2,\text{ext} : V \to \Phi_2^2(V)$ are biholomorphisms (since, for the first one, $\Phi_2^2,\text{ext} \circ \phi \circ (\Phi_1^1)^{-1}|_{\Phi_1^1(P_1^1)}$ is a translation, and for the second one the extended Fatou coordinates coincides with Fatou coordinates on petals), hence open and proper.

We thus obtain:
• \( \phi(P^1_A) \subset P^2_A \) if and only if \( T_\sigma(\Phi^1_A(P^1_A)) \subset \Phi^2_A(P^2_A) \)
• \( \phi(P^1_A) \supset P^2_A \) if and only if \( T_\sigma(\Phi^1_A(P^1_A)) \supset \Phi^2_A(P^2_A) \)

To conclude, let \( P^2_A \subset U_2 \) be a very large attractive petal for \( f_2 \). There exists a very large attractive petal \( P^1_A \) for \( f_1 \) such that \( \Phi^1_A(P^1_A) \subset T_\delta(\Phi^2_A(P^2_A)) \), that is why \( \phi(P^1_A) \subset U_2 \).

Proposition 5.24. If \( P^1_A, P^2_A \) are chosen such that \( \phi(P^1_A) \subset P^2_A \), we have for \( z \in P^1_A \):

\[
\phi(z) = (\Phi^2_A)^{-1} \circ T_\delta \circ \Phi^1_A(z)
\]

Proof. This follows from the equality:

\[
T_\delta|\Phi^1_A(P^1_A) = \Phi^2_A|^{\text{ext}} \circ \phi \circ (\Phi^1_A)^{-1} = \Phi^2_A \circ \phi \circ (\Phi^1_A)^{-1}
\]

Let \( \phi \) be a local semi-conjugacy between \( f_1, f_2 \). The expression of \( \phi \) on attractive petals, cf. proposition 5.24, implies by uniqueness of analytic extension that \( \phi \) is uniquely determined by \( \delta \). More precisely:

Proposition 5.25 (Uniqueness of the local semi-conjugacy). Let \( \Phi^1_A, \Phi^2_A \) be Fatou coordinates of \( f_1, f_2 \). Let \( \phi, \psi \) be local semi-conjugacies of \( f_1, f_2 \) admitting the same lifted phase shift. Denote \( U_1 \) the open set corresponding to \( \phi \) and \( V_1 \) the one corresponding to \( \psi \). Set \( W_1 = U_1 \cap V_1 \) and let \( W^0_1 \) be the connected component of \( W_1 \) containing a germ of the attractive axis of \( f_1 \). Then:

\[
\phi|_{W^0_1} = \psi|_{W^0_1}
\]

Corollary 5.26. Let \( \phi, \psi \) be two local semi-conjugacies such that \( \overline{\phi} = \overline{\psi} \) (recall that \( \overline{\phi}, \overline{\psi} : U^0_1/f_1 \to B^0_2/f_2 \) are the quotient maps induced by \( \phi, \psi \)), i.e. having the same phase shift, recall that the phase shift is defined modulo \( \mathbb{Z} \). Let \( U_1 \) be the open set corresponding to \( \phi \) and \( V_1 \) the one corresponding to \( \psi \). Set \( W_i = U_1 \cap V_1 \) and let \( W^0_i \) be the connected component of \( W_1 \) containing a germ of the attractive axis of \( f_i \). Then there exists \( m \in \mathbb{N} \) such that:

\[
f^m_2 \circ \phi|_{W^0_i} = \psi|_{W^0_i} \quad \text{or} \quad \phi|_{W^0_i} = f^m_2 \circ \psi|_{W^0_i}
\]

Proof. We use the preceding proposition 5.24 and the lifted phase shift calculus given by proposition 5.21.

Proposition 5.27. Let \( \phi \) be a local semi-conjugacy of \( f_1, f_2 \) on their immediate basin. Let \( P^1_A \) be an attractive petal (resp. large, very large petal or \( \alpha \)-petal) of \( f_1 \) included in \( U_1 \), we note \( P^2_A = \phi(P^1_A) \). Suppose that \( P^1_A \) contains an \( \alpha \)-petal, where \( 0 < \alpha < \pi \). Then \( P^2_A \) is an attractive petal (resp. large, very large petal or \( \alpha \)-petal) of \( f_2 \) for the map \( \Phi^2_A = \Phi^1_A \circ (\phi P^1_A)^{-1} \).

Furthermore, let \( U_2 \) be an open neighborhood of \( 0 \). If \( \Phi^1_A(P^1_A) \) is included in \( H \cup B \cup D \) where \( H, B, D \) are upper, lower, right half-plane small enough in the sense of inclusion, we have \( \phi(P^1_A) \subset U_2 \).
Proof. Note that the attractive petal $P_A$ is included in $U^0_1$. Set $G = \Phi_A \circ (\phi p_A)^{-1}$. We show that $Y_2 = \phi(p_A)$ is an attractive petal of $f_2$ with Fatou coordinate $G$.

- $Y_2 = \phi(p_A)$ is homeomorphic to $P_A$, since $\phi p_A = (\phi : P_A \rightarrow \phi(p_A))$ is a biholomorphism by corollary 5.19. So $Y_2$ is homeomorphic to the unit disk.

- The local semi-conjugacy $\phi$ and the stability of $P_A$ by $f_1$ show that $f_2(Y_2) \subset Y_2$.

- The inclusion of $Y_2$ in the immediate parabolic basin of $f_2$ shows that: $\forall z \in Y_2$, $\lim_{n \to \infty} f_2^n(z) = 0$.

- Since $G = \Phi_A \circ (\phi p_A)^{-1}$, the map $G$ is injective.

- By exploiting the conjugacy $\phi p_A$ between $f_1 : P_A \rightarrow f_1(P_A)$ and $f_2 : Y_2 \rightarrow f_2(Y_2)$, we show that for all $z \in Bf_2$, $f_2^n(z) \in Y_2$ for $n$ big enough. Set $z \in Bf_2$.

Recall that $(f_2^n(z))$ converges to 0 tangentially to the attractive axis, and $U^0_b$ is the connected component of $U^b \cap Bf_2$ containing a germ of the attractive axis. There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $f_2^n(z) \in \phi(p_A)$ (since by proposition 5.2, $\phi(p_A)$ contains a set of the form $S[\epsilon]$; it follows from the fact that $p_A$ contains a set of the form $S[\epsilon]$).

- The condition large, very large, $\alpha$-petal, for $Y_2$ (if $p_A$ is a petal of this type) follows from $G(Y_2) = (\Phi_A \circ (\phi p_A)^{-1})(Y_2) = \Phi_A(p_A)$.

- We have the following conjugacy: $\forall z \in Y_2, G \circ f_2(z) = T_1 \circ G(z)$, where $T_1(z) = z + 1$ for all $z \in \mathbb{C}$.

Concerning the proof of the second point, note $\Phi_A^2$ the Fatou coordinate associated to $P_A = \phi(p_A)$. Suppose that $P_A$ is very large such that $\Phi_A(p_A) = H_1 \cup B_1 \cup D_1$ where $H_1, B_1, D_1$ upper, lower, right half planes. We have that $P_A$ is also very large, with $\Phi_A(p_A) = \Phi_A(p_A)$. Let $Q_A^2$ be a very large attractive petal included in $U_2$ with Fatou coordinate $G$ such that $G^\text{ext} = \Phi_A^2$ (such a $G$ exists by uniqueness of Fatou coordinates and uniqueness of holomorphic extension) and $\Phi_A^2(Q_A^2)$ can be written $H_2 \cup B_2 \cup D_2$ with $H_2, B_2, D_2$ upper, lower and right half planes (note that $\Phi_A^2$ is injective on $Q_A^2$). We note $H = H_1 \cap H_2$, $B = B_1 \cap B_2$, $D = D_1 \cap D_2$. Set $P_A^1 \subset (\Phi_A^1)^{-1}(H \cup B \cup D)$ and $P_A^2 \subset (\Phi_A^2)^{-1}(H \cup B \cup D)$. We have $\Phi_A^1(p_A^1) \subset \Phi_A^2(Q_A^2)$ and $P_A^2 \subset Q_A^2 \neq \emptyset$ since two attractive petals must intersect. By lemma 5.23 applied with $\Phi = \Phi_A^2$, we get $P_A^1 \subset U_1$. More generally, for all petal $P_A$ included in $P_A^1$ (this amounts to saying that $\Phi_A(p_A^1) \subset H \cup B \cup D$ by proposition 5.22), we have $P_A^2 : = \Phi_A(p_A^1) \subset P_A^2 \subset U_2$. \hfill \Box

We give an essential corollary to prove proposition 6.1

Corollary 5.28. Let $\phi$ be a local semi-conjugacy of $f_1, f_2$ on their immediate basin. The petals of $f_1$ are assumed to be included in $U_1$. 39
Figure 8: A local semi-conjugacy preserves some type of backward orbits

Let $P_R^1$ be a repulsive petal of $f_1$ and $P_R^2$ be a repulsive petal of $f_2$. Denote $\Phi_R^1, \Phi_R^2$ their Fatou coordinates. Then there exists $M > 0$ satisfying the following property. Let $z_n \in B_0^{f_1}$ be such that for all $n \in \mathbb{N}^*$, $f_1(z_n) = z_{n+1}$. Suppose that $z_n \in P_R$ for all $n \in \mathbb{N}$ and $|\Im(\Phi_R^1(z_0))| > M$. Then $z_n' = \phi(z_n)$ ($z_n$ is in $U_1^0$ by proposition 5.12). Then $z_n'$ belongs to $P_R^2$ after a certain rank.

Proof. See figure 8. It can be assumed that $P_R^1$ is a $\frac{\pi}{2}$-petal as small as desired, and that $P_R^2$ is large. Indeed, by proposition-definition 3.12, two repulsive petals $P_R^1, Q_R$ of a function $f$ have the same quotient by $f$, so a backwards orbit is after a certain rank in $P_R$ if and only if it is after a certain rank in $Q_R$. Let $P_A^1$ be a very large attractive petal of $f_1$ with Fatou coordinate $\Phi_A^1$, small enough for the inclusion. Then, by proposition 5.27, the set $P_A^2 := \phi(P_A^1)$ is a very large attractive petal of $f_2$ with Fatou coordinate $\Phi_A^2 = \Phi_A^1 \circ \phi$. Up to taking a petal $P_A^1$ small enough, $P_A^2$ satisfies the hypothesis of proposition 4.13: it is included in a small enough neighborhood $0$, this follows from $\Phi_A^2(P_A^2) = T_{\phi}(\Phi_A^1(P_A^1))$.

Let $M' > 0$ be big enough, and $a \in \mathbb{R}$ be big enough negatively in such a way that the half planes $H = \{ z \in \mathbb{C}, \Im(z) > M' \}$, $B = \{ z \in \mathbb{C}, \Im(z) < -M' \}$, and $G = \{ z \in \mathbb{C}, \Re(z) < a \}$ satisfies $L_{AR}^2 := (\Phi_A^2)^{-1}\left((H \cup B) \cap G \right) \subset P_R$ (see proposition 4.13). Set $X := (\Phi_A^1)^{-1}\left((H \cup B) \cap G \right) = \phi^{-1}(L_{AR}^2)$. Note that the set $X$ is a lepal pair AR for $f_1$.

Even if it means choosing the left half plane $\Phi_R^1(P_R^1) = G$ small enough for the inclusion, by proposition 4.13 there exists $M > 0$ such that $z \in P_R^1$ and $|\Im(\Phi_R^1(z))| > M$.
remains that \( \phi \) is a local semi-conjugacy of \( f_1, f_2 \),
- \( \phi' \) is a local semi-conjugacy of \( f_2, f_1 \)

Definition 5.30. Let \((\phi, \phi')\) be a pair of local semi-conjugacies. The pair \((\phi, \phi')\) is said to be a local pseudo-conjugacy of \( f_1, f_2 \) if \( \overline{\phi} : B^{f_1}/f_1 \to B^{f_2}/f_2 \) (we already know by proposition 5.17 that \( \overline{\phi} \) is a biholomorphism) has for inverse \( \overline{\phi}' : B^{f_2}/f_2 \to B^{f_1}/f_1 \).

The term pseudo-conjugacy is motivated by the fact that the maps \( \phi, \phi' \) are mutually inverse only after quotient.

Remark 5.31. The fact that \( \overline{\phi} \circ \overline{\phi} = \text{id} \) may be reformulated in the following way. For all \( z \in U^0_1 \), there exists \( n_0 \in \mathbb{Z} \) and \( N \geq \max(0, -n_0) \) such that for all \( n \geq N \) we have the three following conditions (the first two are automatically true for big enough \( n \)):

1. \( f^n_1(z) \in U^0_1 \)
2. \( \phi(f^n_1(z)) \in U^2_1 \)
3. \( \phi' \circ \phi(f^n_1(z)) = f^{n+n_0}_1(z) \)

Proof. Let \( N_0 \) such that for all \( n \geq N_0 \) the points 1. and 2. are true. Remark that for \( n \geq N_0 \) we have by definition of \( \overline{\phi} \) and \( \overline{\phi}' \):

\[
\overline{\phi} \circ \overline{\phi}(z \mod f_1) = \phi' \circ \phi(f^n_1(z)) \mod f_1 \tag{2}
\]

If \( \overline{\phi} \circ \overline{\phi} = \text{id} \), we must have \( z \mod f_1 = \phi' \circ \phi(f^{N_0}_1(z)) \mod f_1 \). Hence there exists \( a, b \in \mathbb{N} \) such that \( f^a(z) = f^b(\phi' \circ \phi(f^{N_0}_1(z))) = \phi' \circ \phi(f^{N_0+b}_1(z)) \). Let \( N = N_0 + b \). We have \( \phi' \circ \phi(f^N_1(z)) = f^a(z) \), hence for all \( n \geq N \) we get \( \phi' \circ \phi(f^n_1(z)) = f^{n+a-N}(z) \). Thus we take \( n_0 = a - N \) and this concludes the proof since \( N \geq -n_0 \).

Let us show the converse implication. Let \( z \in U^0_1 \). The equality \( \phi' \circ \phi(f^n_1(z)) = f^{n+n_0}_1(z) \) for all integer \( n \) bigger than \( N \) implies that \( \overline{\phi} \circ \overline{\phi}(z \mod f_1) = z \mod f_1 \). This is valid for all \( z \in U^0_1 \), hence \( \overline{\phi} \circ \overline{\phi} = \text{id} \). \( \square \)
We will not use this property, but one can show that \( n_0 \) may be chosen independently of \( z \). Note that the first \( N \) terms of the orbit \( z, f_1(z), \ldots, f_1^{N-1}(z) \) might go out of \( U_1^0 \). The same for \( f(z), f_2(f(z)), \ldots, f_2^{N-1}(f(z)) \) and \( U_2^0 \). This prevents in general to write the equality \( f_1^n \circ \phi' \circ \phi(z) = f_1^{n+n_0}(z) \). Nevertheless, under the assumption that \( \phi, \phi' \) can be extended into maps defined on the whole basin \( \phi : B_0^{f_1} \rightarrow B_0^{f_2}, \phi' : B_0^{f_1} \rightarrow B_0^{f_2} \), it is possible to write \( f_1^n \circ \phi' \circ \phi(z) = f_1^{n+n_0}(z) \) without any restriction. This last equality has the advantage to let only a finite number of possibilities (depending on \( n \)) for the value of \( \phi' \circ \phi(z) \), under the condition that \( f : B_1^{f_1} \rightarrow B_1^{f_1} \) is a ramified covering of finite degree.

The fact that \( \overline{\phi} \circ \overline{\phi'} = \text{id} \) can be reformulated in a symetrical way.

Recall that \( \sigma(\phi), \sigma(\phi') \), the phase shifts of \( \phi, \phi' \), are defined in \( \mathbb{C}/\mathbb{Z} \), and \( \overline{\sigma}(\phi), \overline{\sigma}(\phi') \), the lifted phase shifts, are defined in \( \mathbb{C} \) by proposition \([5.18]\).

**Remark 5.32.** Let \((\phi, \phi')\) be a pair of local semi-conjugacies. The pair \((\phi, \phi')\) is a pseudo-conjugacy if and only if \( \sigma(\phi) + \sigma(\phi') = 0 \). Note that \( \sigma(\phi) \) and \( \sigma(\phi') \) depends on the normalisation of the Fatou coordinates, but that by proposition \([5.35]\) \( \sigma(\phi) + \sigma(\phi') \) is independant of the normalisation.

**Definition 5.33.** A local pseudo-conjugacy is said to be **synchronous** if \( \overline{\sigma}(\phi) + \overline{\sigma}(\phi') = 0 \).

**Example 5.34.** A semi-conjugacy \( \phi \) inducing a biholomorphism \( \phi : U_1^0 \rightarrow U_2^0 \) is a particular case of synchronous local pseudo-conjugacy, with \( \phi' = \phi^{-1} \). In this case, the map \( \phi \) is said to be a local conjugacy of \( f_1, f_2 \) on their immediate basins.

**Proposition 5.35.** The value \( \overline{\sigma} + \overline{\sigma}' \) does not depend on the choice of the normalization of Fatou coordinates.

**Proof.** We have by definition of \( \sigma, \sigma' \):

\[
\begin{align*}
T_{\overline{\sigma} | \Phi_A^1(P_A^1)} &= \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1} \\
T_{\sigma' | \Phi_A^2(P_A^2)} &= \Phi_A^1 \circ \phi' \circ (\Phi_A^2)^{-1}
\end{align*}
\]

choose \( P_A^1, P_A^2 \) such that \( \phi(P_A^1) \subset P_A^2 \). This is possible by proposition \([5.22]\). By composition, we get that \( T_{\overline{\sigma} + \overline{\sigma}' | \Phi_A^1(P_A^1)} = \Phi_A^1 \circ \phi' \circ (\Phi_A^1)^{-1} \). This map does not depend on the normalisation of Fatou coordinates. More precisely, if we had chosen an other Fatou coordinate \( T_\alpha \circ \Phi_A^1 \), the resulting function would be the same (up to restriction) since \( T_{\overline{\sigma} + \overline{\sigma}'} \) commutes with \( T_\alpha \). This is equivalent to say that \( \overline{\sigma} + \overline{\sigma}' \) is independent of the petal choice. \( \square \)

There is a consequence of proposition \([5.21]\)

**Proposition 5.36.** The existence of a local pseudo-conjugacy between \( f_1, f_2 \) is equivalent to the existence of a synchronous local pseudo-conjugacy.

**Proof.** Let \((\phi, \phi')\) be a local pseudo-conjugacy. Since \( \sigma(\phi') = -\sigma(\phi) \), we obtain \( \overline{\sigma}(\phi') = -\overline{\sigma}(\phi) \) up to post-composing or pre-composing \( \phi, \phi' \) by the adapted integer powers of \( f_1, f_2 \). \( \square \)
Proposition 5.37. Let \((\phi, \phi')\) a synchronous pseudo-conjugacy. Let \(P_A^1, P_A^2\) be attractive petals of \(f_1, f_2\) included in \(U_1, U_2\). Then \(\phi(P_A^1) = P_A^2\) is equivalent to \(P_A^1 = \phi'(P_A^2)\).

Proof. This is an immediate consequence of proposition 5.22.

Proposition 5.38. A pseudo-conjugacy is synchronous if and only if there exists an attractive \(P_A^1\) such that \(\phi : P_A^1 \rightarrow \phi(P_A^1)\) is a biholomorphism with inverse \(\phi' : \phi(P_A^1) \rightarrow P_A^1\), where it is implied that \(P_A^1 \subset U_1\) and \(\phi(P_A^1) \subset U_2\). This amounts to saying that for all attractive petal \(P_A^1\) such that \(P_A^1 \subset U_1\) and \(\phi(P_A^1) \subset U_2\), the map \(\phi : P_A^1 \rightarrow \phi(P_A^1)\) is a biholomorphism of inverse \(\phi' : \phi(P_A^1) \rightarrow P_A^1\).

Proof. Let \((\phi, \phi')\) be a synchronous pseudo-conjugacy. By proposition 5.22, there exists a petal \(P_A^1 \subset U_1^0\) such that \(\phi(P_A^1) \subset U_2^0\). Denote \(P_A^2 := \phi(P_A^1)\), which is an attractive petal for \(f_2\) by proposition 5.27. Suppose that \(\phi'(P_A^2) \subset P_A^1\).

By proposition 5.24, we have:

\[
\forall z \in P_A^1, \quad \phi' \circ \phi(z) = (\Phi_A^1)^{-1} \circ T_{\theta + \sigma'} \circ \Phi_A^1(z) \quad (3)
\]

\[
\forall z \in P_A^2, \quad \phi \circ \phi'(z) = (\Phi_A^2)^{-1} \circ T_{\theta + \sigma'} \circ \Phi_A^2(z) \quad (4)
\]

Now, \(P_A^1\) is any petal such that \(P_A^1 \subset U_1^0\) and \(\phi(P_A^1) \subset U_2^0\) and \(P_A^2 := P_A^1\). It is sufficient to show the following equivalence: \(\phi\) is synchronous if and only if \(\phi : P_A^1 \rightarrow \phi(P_A^1)\) is a biholomorphism with inverse \(\phi' : \phi(P_A^1) \rightarrow P_A^1\). Indeed, since this equivalence does not depend on the choice of the petal \(P_A^1\) such that \(\phi(P_A^1) \subset U_2^0\), this will conclude.

Suppose that \(\phi : P_A^1 \rightarrow P_A^2\) is a biholomorphism of inverse \(\phi' : P_A^2 \rightarrow P_A^1\). We can use equation (3) since \(\phi'(P_A^2) = P_A^1\). By this equation, the map of \((\phi, \phi')\) is synchronous. Conversely, suppose that \((\phi, \phi')\) is synchronous. Then by proposition 5.37, \(P_A^1 = \phi'(P_A^2)\), so we can use equations (3) and (4). Thus we get that \(\phi : P_A^1 \rightarrow P_A^2\) is a biholomorphism of inverse \(\phi' : P_A^2 \rightarrow P_A^1\).

Let \(\phi\) be a local semi-conjugacy on the immediate basins. We now show a lemma which will be called the lemma of pseudo-inversibility of \(\phi\). Before invoquing the lemma, let us introduce some notations.

Let \(W\) a neighborhood of 0 in which \(f_1\) admits a local section \(f_1^{-1} : W \rightarrow f_1^{-1}(W)\). Suppose that \(P_R^1 \subset U_1 \subset W\) and that \(\Phi_R^1(P_R^1)\) contains a left half-plane \(G\). Let \(H\) an upper half-plane included in \(D_1^+\) an \(C_2\) a lower half-plane included in \(D_1^-\). Let \(X = C_1 \cup C_2\) where \(C_1\) is the connected component of \((D_1^+ \cup D_1^-) \cap \Phi_R^1(P_R^1)\) containing \(G \cap H\), and \(C_2\) the connected component containing \(G \cap \partial B\). Notice that \(X\) is stable by the left translation \(T_{-1}\), since \(G, B, H\) and \(D_1^\pm, \Phi_R^1(P_R^1)\) are stable by \(T_{-1}\). Let \(Y := (\Phi_R^1)^{-1}(X)\) : notice that \(Y \subset U_1^0 \cap P_R\) is stable by the local section \(f_1^{-1}\) of \(f_1\), that is why we introduced \(Y\).

Let \(A_0\) be a set of the form \(f_1^n(A_{-n_0})\) where \(n_0\) is a possibly big natural integer and \(A_{-n_0}\) a connected open set relatively compact in \(Y \subset U_1 \cap P_R^1\). Note that \(A_{-n_0} \subset U_1^0 \subset B_{1^0}^1\), in such a way that \(A_{-n_0}\) is included in the definition domain of \(f_1^n\) for all \(n \in \mathbb{N}\), and in particular for \(n = n_0\).

Such a set \(A_0\) may be caracterized by the following property. There exists a backward orbit \((A_{-n})\) for \(f_1\) such that:
• for all $n \in \mathbb{N}$, $f_1(A_{-n-1}) = A_{-n}$ (backward orbit definition)

• $A_{-n}$ is a connected open set relatively compact in $Y$ from a certain rank $N$

Let us show this equivalence.

**Proof.** If there exists a backward orbit $(A_{-n})$ for $f_1$ as mentionned before, take $n_0 = N$. Thus $A_0 = f_1^{n_0} (A_{-n_0})$ and $A_{-n_0}$ is a connected open set relatively compact in $U_1^0 \cap P_R^1$.

If $A_0$ is a set of the form $f_1^{n_0}(A_{-n_0})$ with $A_{-n_0}$ a connected open set relatively compact in $Y \subset U_1^0 \cap P_R^1$, take for $n \in \mathbb{N}$:

$$A_{-n} = f_1^{n_0 - n}(A_{n_0})$$

where for $n \in \mathbb{N}^*$, $f_1^{-n}$ is the local section restricted to $P_R^1$ of $f_1^n$. For all $n \in \mathbb{N}$ we have $f_1(A_{-n-1}) = A_{-n}$.

Notice that $f_1^{-1}$ from $Y \subset P_R^1$ to its image, included in $Y$, is open and continuous. Hence if $A_{-n}$ is a connected open set relatively compact in $Y$, then $A_{-n-1}$ satisfies the same properties. Since $A_{-n_0}$ satisfy this property, the property is also true for all integer $n$ bigger than $n_0$. Thus we take $N = n_0$, and this concludes the proof. 

**Remark 5.39.** For $n \geq N$, the map $f_1 : A_{-n-1} \to A_{-n}$ defines a biholomorphism. More precisely, $f_1^{-1} : A_{-n} \to A_{-n-1}$ is a restriction of the local section of $f_1$ defined on $P_R^1$, this
implies that $A_{-n-1}$ is defined uniquely from $A_{-n}$. Remark that given $A_0$, the backward orbit $(A_{-n})$ is not unique, since $f_1 : A_{-n-1} \rightarrow A_{-n}$ is in general only surjective, in such a way that there is several definitions of $A_{-n-1}$ from $A_{-n}$.

Let for $n \geq N$, $B_{-n} = \phi(A_{-n})$. We prove that the sequence $(B_{-n})$ satisfies the following properties:

- for all $n \geq N$, $f_2(B_{-n-1}) = B_{-n}$ (backward orbit definition)
- $B_{-n}$ is a connected open set compactly contained in $P^2_R$ from a certain rank
- $B_{-n}$ is compactly contained in $U^0_2$ from a certain rank

**Proof.** We prove successively the three points.

- Since $\phi$ is a semi-conjugacy, it is clear that $(B_{-n})$ is also a backward orbit.
- $B_{-n}$ is connected as the direct image of a connected set by a continuous map; it is open as the direct image of an open set by a nonconstant holomorphic map. The orbit $(\tilde{A}_{-n})$ is compactly contained in an orbit $(A'_{-n})$ satisfying the same properties. To build such an orbit $(A'_{-n})$, denote $A'_{-N}$ an open neighborhood of $\overline{A_{-N}}$. Then apply $f_1$ to build $A'_{-n}$ for $n < N$ and apply the local section of $f_1$ on $P^1_R$ to build $A'_{-n}$ for $n > N$. Suppose without loss of generality that $N = 0$. Consider the map $\omega$ which at $z \in A'_0$ associates the least integer $n \in \mathbb{N}$ such that $\phi(f_1^{-n}(z)) \in P^2_R$, such an integer exists by the construction of $\psi : D_1^+ \rightarrow D_2^+$. Since the set $P^2_R$ is open and the maps $\phi$, $f_1$ are continuous, the map $\omega$ is upper semi-continuous. Since the set $A_0$ is relatively compact in $A'_0$, the map $\omega$ is majorized over $A_0$. Thus there exists $N' \in \mathbb{N}$ such that for all $z \in A_0$, $\phi(f_1^{-n}(z)) \in P^2_R$, namely such that $B_{-n} \subset P^2_R$. Even if it means choosing $P^2_R$ small enough, one may suppose that $P^2_R \subset U_2$.

- See figure 9. Since $A_0 \subset P^1_R \cap U^0_1$, the set $A_0$ is of the form $A_0 = \Psi^1_R(\Gamma_0)$ where $\Gamma_0 = \Phi^1_R(A_0)$ open set included in $\overline{D^+_1} \cup \overline{D^-_1}$ by the inclusion $\Phi^1_R(P^1_R \cap U^0_1)$ mod $\mathbb{Z} \subset \overline{D^+_1} \cup \overline{D^-_1}$ stated in proposition 5.13. Let $H, B$ be upper, lower half-planes such that $\Psi^{1, \text{ext}}(H \cup B)$ is included in an very large attractive petal $P^1_A$. There exists a connected open set $\Gamma'_0$ relatively compact in $\overline{D^+_1} \cup \overline{D^-_1}$ containing $\Gamma_0$ and intersecting $H \cup B$. To prove this, join $\Gamma'_0$ with $H \cup B$ by a path included in $\overline{D^+_1} \cup \overline{D^-_1}$, and consider an open neighborhood of the path which is relatively compact in $\overline{D^+_1} \cup \overline{D^-_1}$. By noting for all $n \in \mathbb{N}$, $A'_{-n} = \Psi^1_R(\Gamma'_0 - n)$, it follows that $A'_{-n}$ is connected, that it intersects $P^1_A$, and that for $n$ big enough it is relatively compact in $U_1$, and thus in $U^0_1$. Its image $B'_{-n}$ by $\phi$ is then a connected set relatively compact in $U_2$ intersecting $P^2_A$, it is thus relatively compact in $U^0_2$. It follows that $B_{-n}$ is relatively compact in $U^0_2$.

This enables to state the lemma of pseudo-inversibility of $\phi$:

\[ \square \]
Lemma 5.40. For all $n$ big enough, the map $\phi : A_{-n} \to B_{-n}$ is a biholomorphism of inverse $\phi' : B_{-n} \to A_{-n}$.

Proof. Take back the notations of the last point of the preceding proof, see figure. Even if it means removing the first terms of the orbit $(A_{-n})$, one may suppose that $A'_{-n}$ is relatively compact in $U_1^n$ for all $n \in \mathbb{N}$. Write $B'_{-n} = \phi(A'_{-n})$. We apply the second point of the preceding proof to $(A'_{-n})$: there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, $B'_{-n} \subset P_R^1 \subset U_2$. There exists also $N'' \in \mathbb{N}$ such that for all $n \geq N''$, $\phi'(B'_{-n}) \subset P_R^1$. Thus for $n \geq \max(N', N'')$, $\phi' \circ \phi$ is defined on the connected set $A'_{-n}$ and $\phi \circ \phi'$ is defined on the connected set $B'_{-n}$. Since $\phi$ is a bijection from $P_1^1$ to $P_2^1 := \phi(P_1^1)$ of inverse $\phi' : P_2^1 \to P_1^1$, we obtain that $\phi' \circ \phi$ equals the identity over $P_1^1 \cap A_{-n} \neq \emptyset$, and thus equals identity over the connected set $A_{-n}$. Likewise $\phi' \circ \phi$ equals the identity over the connected set $B_{-n}$. Thus $\phi : A_{-n} \to B_{-n}$ is a biholomorphism of inverse $\phi' : B_{-n} \to A_{-n}$. \hfill \Box

The reader may wisely consult figure by replacing mentally the paths $\gamma, \gamma'$ by open sets $\Gamma, \Gamma'$ relatively compact, and the paths $\alpha_{-n}, \beta_{-n}$ by open sets $A_{-n}, B_{-n}$ relatively compact.

Remark that the pseudo-inversibility lemma may be reformulated in this way.

Proposition 5.41. There exists open sets $V \subset \hat{D}_1^+, W \subset \hat{D}_2^+$ stable by the translation $T_{-1} : z \to z - 1$ such that $V \equiv \mathbb{Z} = \hat{D}_1^+$, $W \equiv \mathbb{Z} = \hat{D}_2^+$ and $\phi : \Psi_R^{1, ext}(V) \to \Psi_R^{2, ext}(W)$ is a biholomorphism of inverse $\phi' : \Psi_R^{2, ext}(W) \to \Psi_R^{1, ext}(V)$. The same holds by replacing $+$ with $-$. The pseudo-inversibility is therefore retranslated in a concrete inversibility on a fixed domain.

Proof. The pseudo-inversibility lemma allows to say that for all open set $V'$ relatively compact in $\hat{D}_1^+$, there exists an open set $W'$ relatively compact in $\hat{D}_2^+$ such that $\phi : \Psi_R^{1, ext}(V' - n) \to \Psi_R^{2, ext}(W' - n)$ is a biholomorphism of inverse $\phi'$ for all $n$ big enough, say $n \geq n_{V'}$ where $n_{V'}$ is chosen minimal. The roles of $V'$ and $W'$ are symmetrical, therefore we may also denote the integer corresponding to $W'$: $n_{W'}$. Writing $V = \bigcup_{n \geq n_{V'}} V' - n$ and $W = \bigcup_{m \geq m_{W'}} W' - m$, where the unions are taken under all the open sets $V'$, $W'$ relatively compact in $\hat{D}_1^+, \hat{D}_2^+$, we get the result. \hfill \Box

Here is a last variant where the inversibility domain is bigger (we add an attractive petal).

Proposition 5.42. Let $P_1^1 := \phi(P_1^1)$ be attractive petals included in $U_1^0, U_2^0$. There exists open sets $A \subset U_1^0$ and $B \subset U_2^0$ such that $\phi : A \to B$ biholomorphism of inverse $\phi' : B \to A$ satisfying the following properties.

- $A$ and $B$ contains the attractive petals $P_1^1, P_2^1$
- $A \cap P_R^1$ resp. $B \cap P_R^2$ is invariant by the local section of $f_1$ resp. $f_2$ defined over $P_R^1, P_R^2$.
• \((A \cap P^1_B)/f_1 = (U_1^0 \cap P^1_B)/f_1\) and \(B/f_2 = (U_2^0 \cap P^1_B)/f_2\).

**Remark 5.43.** The last proposition gives a characterisation of being a pseudo-conjugacy: \(\phi\) is a pseudo-conjugacy if and only if it satisfies this property, with inversibility on domains \(A, B\) before quotient.

**Proof.** Remark that the last point is equivalent to \(\Phi_R^1(A) \mod \mathbb{Z} = D_1^+ \cup D_1^-\) and \(\Phi_R^2(B) \mod \mathbb{Z} = D_2^+ \cup D_2^-\) by the equality \(\Phi_R^1(U_i^0 \cap P^1_B) \mod \mathbb{Z} = D_i^+ \cup D_i^-\) shown in propositions 5.12 and 5.13.

Consider \(A = P_A^1 \cup \Psi_R^1(V^+) \cup \Psi_R^1(V^-)\) where \(V^+, V^-\) are the open sets built by the preceding proposition, and in the same way \(B = P_A^2 \cup \Psi_R^2(W^+) \cup \Psi_R^2(W^-)\). The three points are shown easily. This concludes the proof. \(\square\)

### 6 Proof of the theorem 1.4

We first show the direct implication of the theorem 1.4. Recall that the sets \(D_1, D_2\) are the definition domains of \(h_1, h_2\), and that \(D_1^+, D_2^+\) are the connected components of this definition domains containing a punctured neighborhood of \(\pm i\infty\) as in proposition 4.6. The maps \(h_i^\pm\) are the horn maps of \(h_i\) restricted to \(D_i^\pm\). Here is the direct implication of the theorem 1.4.

**Proposition 6.1.** Let \(f_1, f_2\) be two holomorphic maps with parabolic point at 0, with Taylor expansion at 0 of the form \(f_i(z) = z + a_i z^2 + O(z^3)\) with \(a_i \neq 0\). Assume that \(f_1, f_2\) are locally semi-conjugated at 0 on \(B_0^0\). Denote \(\phi\) a local semi-conjugacy (see definition 5.8).

Then there exists \(\sigma \in \mathbb{C}\), and a pair of holomorphic maps \(\psi = (\psi^+, \psi^-)\), where \(\psi^+ : D_1^+ \to D_2^+\) such that:

\[h_1^\pm = T_\sigma \circ h_2^\pm \circ \psi^\pm\]

where \(T_\sigma\) is the translation of the cylinder \(\mathbb{C}/\mathbb{Z}\) given by the formula \(z \to z + \sigma\).

Besides, the constructed maps \(\psi^\pm\) admits at \(\pm i\infty\) removable singularities, and an expansion of the form:

\[\psi^\pm(w) = w + \rho^\pm + o(1)\]

**Complement 6.2.** We add the following assumption: \(\phi\) is a local pseudo-conjugacy, i.e. there exists a semi-conjugacy \(\phi' : U_2^0 \to B_0^1\) such that the induced maps \(\bar{\phi} : B_0^1/f_1 \to B_0^2/f_2\) and \(\bar{\phi}' : B_0^1/f_1 \to B_0^2/f_2\) are mutually inverses (see definition 5.30). Then the horn maps \(h_1^\pm, h_2^\pm\) of \(f_1, f_2\) are equivalent as ramified analytic coverings via biholomorphisms from \(D_1^+\) to \(D_2^+\) and translation of \(\mathbb{C}/\mathbb{Z}\). More precisely, there exists \(\sigma \in \mathbb{C}\), and a pair of biholomorphisms \(\psi = (\psi^+, \psi^-)\), where \(\psi^+ : D_1^+ \to D_2^+\) such that:

\[h_1^\pm = T_\sigma \circ h_2^\pm \circ \psi^\pm\]

where \(T_\sigma\) is the translation of the cylinder \(\mathbb{C}/\mathbb{Z}\) given by the formula \(z \to z + \sigma\).
Proof. We simultaneously prove the proposition and the complement. Let \( \phi \) be a local semi-conjugacy, which might be completed in a local pseudo-conjugacy \( (\phi, \phi') \). We will clearly identify the parts of the proof devoted to the complement, in which the local pseudo-conjugacy \( (\phi, \phi') \) will be exploited. We give first the main elements of the proof, and we will give in the following parts the details.

We will build \( T_\sigma \) and \( \psi \) satisfying the relation \( T_\sigma \circ \mathbf{h}^+_R = \mathbf{h}^+_\sigma \circ \psi^\pm \) instead of the announced equality (to get the exact relation of the proposition, replace \( \sigma \) by \( -\sigma \)). The map \( \phi \) will induce a map \( T_\sigma \) from \( C^l_{\phi} \) to \( C^l_{\psi} \), and a partial map \( \psi \) from \( C^l_{\phi} \) to \( C^l_{\psi} \) that can be restricted to the maps \( \psi^\pm : D^+_R \to D^+_\psi \). These maps \( T_\sigma \) and \( \psi^\pm \) will enable to show the desired equality \( T_\sigma \circ \mathbf{h}^+_R = \mathbf{h}^+_\psi \circ \psi^\pm \).

Concerning the proof of the complement, we will follow the same construction with \( \phi' \); it induces the map \( T_{-\sigma} \) from \( C^l_{\phi} \) to \( C^l_{\psi} \) and maps \( \psi^{\pm} \) from \( D^+_\phi \) to \( D^+_\psi \) (thus \( T_{-\sigma} \circ \mathbf{h}^+_R = \mathbf{h}^+_\phi \circ \psi^{\pm} \)). We will prove that the maps \( \psi^{\pm} : D^+_R \to D^+_\psi \) is a biholomorphism of inverse \( \psi^{\pm} \). We will see that the maps \( T_\sigma \) and \( \psi^{\pm} \) built after quotient remain identical when we post-compose, pre-compose \( \phi \) (or \( \phi' \)) by powers of \( f_1, f_2 \). Thus, if \( \phi \) is a local pseudo-conjugacy, we may suppose without lost of generality that the pair \( (\phi, \phi ') \) is synchronous.

During the proof, we will assume that all the petals of \( f_1 \), resp. \( f_2 \) are included in \( U_1 \), resp. \( U_2 \), this will enable to apply remark \( \ref{remark}:5.10 \) and proposition \( \ref{proposition}:5.12 \) concerning attractive and repulsive petals. We will use in a remark (which is useful for a better understanding, but not for the constitution itself) proposition \( \ref{proposition}:5.13 \) which is valid only when the open sets \( U_1, U_2 \) are supposed small enough.

**Construction of \( T_\sigma \):**

The quotient map \( T_\sigma : C^l_{\phi} \to C^l_{\psi} \) induced by \( \phi : P^l_A \to B^l_2 \) is built in proposition \( \ref{proposition}:5.17 \).

**Construction of \( \psi \):**

The construction of the induced map on the repulsive petals is harder. We give in a first paragraph the main elements of the construction; we will give the details in the subsequent paragraphs.

**Summary of the construction:** First, set \( \tilde{z} = \overline{\mathcal{V}}_R(w) \in P^l_R/f_1 \). Recall that \( \overline{\mathcal{V}}_R : \mathbb{C}/\mathbb{Z} \to P^l_R/f_1 \) is the map induced by the Fatou parametrisation \( \Psi^l_R : \Phi^l_R(P^l_R) \to P^l_R \) after quotient by \( \mathbb{Z} \) on the domain and by \( f_1 \) on the codomain. The element \( \tilde{z} \) is a backward orbit for \( f_1 \). When \( \phi(\tilde{z}) \) belongs to \( P^l_R \) after a certain rank, choose the following element of \( C^l_{\psi} \):

\[
\psi(w) = \Phi^l_R(\phi(\tilde{z} \text{ FACR })) \mod \mathbb{Z}
\]

where FACR means from a certain rank, and the rank is chosen such that \( \phi \) is defined on \( \tilde{z} \) FACR and the backward orbit \( \phi(\tilde{z} \text{ FACR}) \) is completely included in \( P^l_R \).

We check in the next paragraphs that for all initial choice of \( w \in D^+_R \cup D^-_R \), the backward orbit \( \phi(\tilde{z}) \) is contained in \( P^l_R \) after a certain rank. This point follows first from the corollary \( \ref{corollary}:5.28 \) when \( |\Im(w)| \) big; we then show that it is valid for all \( w \in D^+_R \), by
exploiting the arc connectedness of this set and by controlling the hyperbolic length of
the paths formed at each step of the construction.

This will allow to quotient the map \( \phi \), to get a map \( \psi^{\pm} \) from \( D_1^+ \) to \( D_2^\pm \).

Why \( \psi(w) = \Phi^1_R(\phi(\hat{z} \text{ FACR})) \mod Z \) is the natural choice: The map \( \phi \) induces by
quotient a partial map \( \psi \) from \( C^1_R \) to \( C_2^J \). We want to define an element of \( C^2_J \) from an
element of \( C^1_R \). Set \( \hat{z} = \Phi^1_R(w) \in \hat{P}^1_R / f_1 \). It is possible to apply \( \phi \) at the elements of
this orbit \( \hat{z} \) after a certain rank if and only if \( \hat{z} \) is included after a certain rank in \( U^0_1 \).

To satisfy this property, it is sufficient that \( w \in D_1^+ \cup D_1^- \) by proposition 5.12.

**Remark 6.3.** If the open sets \( U_1, U_2 \) are small enough to apply proposition 5.13, \( \hat{z} \) is included after a certain rank in \( U^0_1 \) if and only if \( w \in D_1^+ \cup D_1^- \). In this setting, there is no reasonable definition of \( \psi \) on a bigger set than \( D_1^+ \cup D_1^- \).

Consider the backward orbit after a certain rank \( \phi(\hat{z} \text{ FACR}) \) for \( f_2 \), where FACR
means from a certain rank ; the rank is first chosen such that \( \phi \) is defined. To get at
least one element of \( C^2_J \), the backward orbit \( \phi(\hat{z} \text{ FACR}) \) must intersect a repulsive petal
\( P^2_R \). However, such an intersection with \( P^2_R \) is a priori not ensured, since the map \( \phi \) is
not supposed continuous at 0. Also, note that the backward orbit \( \phi(\hat{z} \text{ FACR}) \) for \( f_2 \)
might enter several times in the repulsive petal \( P^2_R \), in such a way that the intersection
of the orbit and the repulsive petal may in general define distincts elements in \( P^2_R / f_2 \) at
each step. Each of these values are candidates for the value of \( \Phi^1_R \) applied on each of
these orbits of \( P^2_R \). Doing a choice independent of the petal \( P^1_R \) is equivalent to doing
a choice independent of the backward orbit \( \hat{z} \) of \( f_1 \) only after arbitrarily high ranks.
The identification \( P^1_R / f_1 = P^2_R / f_1 \) remove indeed the first terms of the backward orbit,
the removed initial segment may be chosen arbitrarily long. For this purpose, choose
\( P^1_R = f_1^{-N}(P^1_R) \) to remove the first \( N \) elements of the orbit.

Let us now begin the construction.

**Construction of \( \psi^{\pm}(w) \) when \( \Im(w) \) is big:** An element \( w \in D_1 \) such that \( \Im(w) \) is
big, using proposition 4.9 is send by the horn map \( h_1 \) on an element \( h_1(w) \) such that
\( \Im(h_1(w)) \) is big. Let \( \hat{P}^1_A \) be a very large attractive petal of \( f_1 \) and \( P^2_R \) be a large
repulsive petal of \( f_2 \). With the aim of applying the consequence 4.10, we suppose that
\( \hat{P}^2_A := \phi(\hat{P}^1_A) \), \( P^2_R \) are included in a neighborhood \( W_2 \) of 0 in which \( f_2 \) admits a local
section \( f_2^{-1} \). It is possible if \( P^1_A \) is small enough, see proposition 5.27 with \( U_2 \) of the
proposition equal to \( W_2 \) here ; note that \( P^2_A \) is very large since \( \Phi^2_A(\hat{P}^1_A) = \Phi^2_A(P^1_A) \) (recall
that \( \Phi^1_A = \Phi^2_A \circ \phi_{P^1_A} \) with \( \phi_{P^1_A} : P^1_A \rightarrow P^2_A \) biholomorphism). Let also \( P^1_A \) be a repulsive
petal of \( f_1 \).

Set \( w \in C^1_R \), with a high imaginary part. Let \( z \in P^1_R \) be a pre-image of \( w \) by \( \Phi^1_R \)
mod \( Z \). Note \( z_{-n} = f_1^{-n}(z) \) where \( f_1^{-1} : f(P^1_R) \rightarrow P^1_R \), and \( z'_{-n} = \phi(z_{-n}) \) \((z_{-n} \) belongs
to \( U^0_1 \) by proposition 5.12). If we choose \( w \) with big enough imaginary part, \( z'_{-n} \) will
necessarily belong to \( P^1_R \) after a certain rank \( N \in \mathbb{N} \) by corollary 5.28. This enables to
note \( \psi(w) \) the common value of the \( \Phi^2_R(z'_{-n}) \) mod \( Z \in C^1_R \) when \( n \geq N \). Note that by
consequence 4.10, since \( \Phi^1_R(z_{-n}) \) has a high imaginary part (independent of \( n \), \( z_{-n} \) must
be in \( P^1_A \) and \( \Phi^1_A(z_{-n}) = \Phi^2_A(z'_{-n}) \) also have a high imaginary part (which is independent

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modulo \(Z\) of \(n \geq N\). Thus, exploiting this consequence \ref{fig:construction} again, \(\psi(w) = \Phi_R^2(z'_{-n})\) mod \(Z\) must have a high imaginary part for all \(n \geq N\).

This finally implies that if \(w\) has a big enough imaginary part, we must have \(w \in D_1^+\) and \(\psi(w) \in D_2^+\); define for all such \(w\) \(\hat{\gamma}_n := \psi(w)\). Then \(\lim_{w \to +i\infty} \psi(w) = +i\infty\), and \(\psi^+\) admits a continuous extension at \(+i\infty\). This will imply, when it will be shown that \(\psi^+\) is holomorphic, that its singularity at \(+i\infty\) is removable.

**Construction of \(\psi^+ : D_1^+ \to D_2^+\).** See figure \ref{fig:construction}.

Set \(w \in D_1^+\). Note \(w_0 \in D_1^+\) with high imaginary part as in the preceding paragraph, in such a way that \(\psi^+(w_0) := \psi(w_0) \in D_2^+\). There exists a path \(\gamma\) included in \(D_1^+\) such that \(\gamma(0) = w_0, \gamma(1) = w\). Recall that \(D_1^+ \subset \mathcal{C}_R^f = \mathbb{C}/\mathbb{Z}\). Note \(\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}\) the canonical projection. Note \(\hat{\gamma}_0\) a lift of \(\gamma\) by \(\pi\), and for \(n \in \mathbb{N}\), \(\hat{\gamma}_{-n} = \hat{\gamma}_0 - n\). We choose \(\hat{\gamma}_0\) sufficiently to the left to guarantee the existence of a repulsive \(\pi/2\)-petal \(P^l_R\) and a Fatou coordinate \(\Phi_R^1\) such that the left half plane \(G := \Phi_R^1(P^l_R)\) contains \(\hat{\gamma}_0\). Then \(G\) contains all the \(\hat{\gamma}_{-n}\). Remark that \(D_1^+\) contains a upper half-plane \(\text{Im}(z) > M\) where \(M \in \mathbb{R}\). We choose \(w_0\) satisfying furthermore \(\text{Im}(z) > M\) (which is possible), hence we have for \(n \in \mathbb{N}\), \(\hat{\gamma}_{-n} \subset X\) where \(X\) is the connected component of \(\pi^{-1}(D_1^+) \cap G\) whose imaginary part is nonmajorized. We have by proposition \ref{prop:holomorphicity} \(\Psi_{f_1}(X) \subset U_0\). The map
\( \Psi^1_R : X \to \Psi^1_R(X) \) is a biholomorphism. Note \( \alpha_{-n} = \Psi^1_R \circ \gamma_{-n} \) and \( \beta_{-n} = \phi \circ \alpha_{-n} \) for all \( n \in \mathbb{N} \).

We give a general notation: for all hyperbolic Riemann surface \( S \) and \( \gamma \) included in \( S \), note \( L_S(\gamma) \) the hyperbolic length of the path \( \gamma \).

We have for \( n \in \mathbb{N} \):

\[
L_{\phi(U^n_1)}(\beta_{-n}) \leq L_{U^n_1}(\alpha_{-n}) \leq L_{\Psi^1_R(X)}(\alpha_{-n}) = L_X(\gamma_{-n}) \leq L_X(\gamma_0)
\]

we use at each equality the fact that the biholomorphisms (more generally the nonramified holomorphic coverings, although it is not used here) preserve the hyperbolic length, and at each inequality that the holomorphic maps (especially the inclusions) reduce nonstrictly the hyperbolic length. The successive inequalities, equalities, are justified by the following inclusions, holomorphic maps:

\[
\begin{align*}
\phi : U^n_1 &\to \phi(U^n_1) \\
\Psi^1_R(X) &\subset U^n_1 \\
\Psi^1_R &\to \Psi^1_R(X) \quad \text{(biholomorphism)} \\
T_{-n} : X &\to X
\end{align*}
\]

Remark that \( T_{-n} \) sends \( X \to X \). Indeed, let \( M \) such that the upper half plane \( H \) of equation \( \text{Im}(z) > M \) is included in \( \hat{D}^+_1 \). The set \( X \) is the union of the connected components of \( \hat{D}^+_1 \cap G \) intersecting \( H \). Since \( T_{-n} \) sends \( \hat{D}^+_1 \) in \( \hat{D}^+_1 \) and \( H \) in \( H \), it must send a connected component \( C \) of \( \hat{D}^+_1 \cap G \) intersecting \( H \) in a connected component \( C \) of \( \hat{D}^+_1 \cap G \) intersecting \( H \).

The sequence of hyperbolic lengths \( \left( L_{\phi(U^n_1)}(\beta_{-n}) \right) \) is thus majorized.

Note for \( n \in \mathbb{N} \), \( z_0^n = \alpha_{-n}(0) \) and \( z_1^n = \alpha_{-n}(1) \). The sequence \( (z_0^n) \) constitutes a backward orbit of \( f_1 \) each term of which is a pre-image of \( w_0 \) by \( \Phi^1_R \) mod \( \mathbb{Z} \). In a similar way, \( (z_1^n) \) is a backward orbit each term of which is a pre-image of \( w \) by \( \Phi^1_R \) mod \( \mathbb{Z} \). Recall that the two orbits are included in the repulsive petal \( P^1_R = \Psi^1_R(G) \).

Note for \( n \in \mathbb{N} \), \( z_0^n = \phi(z_0^n) = \beta_{-n}(0) \) and \( z_1^n = \phi(z_1^n) = \beta_{-n}(1) \). The element \( w_0 \) has a high imaginary part. By the first paragraph, the element \( z_0^n \) must belong to \( P^2_R \) after a certain rank, where \( P^2_R \) is an unspecified repulsive petal. Thus \( (z_0^n) \) converges to 0. Since \( (\beta_{-n}) \) is a sequence of paths of \( B^0_{f_2} \) with one extremity converging in the Euclidean sense to 0 in \( \partial B^0_{f_2} \) and having a bounded hyperbolic length in \( \phi(U^n_1) \subset B^0_{f_2} \), this implies that \( (\beta_{-n}) \) has a Euclidean diameter converging to 0. Thus \( (\beta_{-n}) \) converges uniformly to 0 in the Euclidean sense.

Choose \( P^2_R \) as a large repulsive petal of \( f_2 \) included in \( U_2 \). Let \( P^2_A \) be an attractive petal of \( f_2 \). We choose \( P^2_A \) as the inverse image of a right half plane. By Taylor expansion of Fatou coordinates (proposition 2.10), we must have that \( P^2_A \cup P^2_R \cup \{0\} \) is a neighborhood of 0. Note for \( n \in \mathbb{N} \), \( F_n = \{ t \in [0,1], \beta_{-n}(t) \in f_2(P^2_A) \} \). The sequence \( (F_n) \) is a sequence of compacta of \([0,1] \), decreasing for the inclusion, this is shown by the stability of \( f_2(P^2_A) \) by \( f_2 \) and the fact that \( f_2 \circ \beta_{-n-1} = \beta_{-n} \). Note that their intersection is empty. If we had a \( t \) in their intersection, then the equality \( \Phi^2_A(\beta_{-n}(t)) = \Phi^2_A(\beta_0(t)) - n \) valid

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for all \( n \in \mathbb{N} \) would give a contradiction: the image of \( \Phi^2_R \), defined on \( P^2_A \), is supposed to be a right half plane. The sequence \( \{F_n\}_n \) is a sequence of compacta decreasing for the inclusion, and its intersection is empty, thus there exists an integer \( n \) such that \( F_n = \emptyset \). Since \( \beta_{-n} \) converges uniformly in the Euclidean sense to 0, this finally proves that \( \beta_{-n} \) is included in \( P^2_R \) after a certain rank \( N \) (recall that \( P^2_A \cup P^2_R \cup \{0\} \) is a neighborhood of 0). Since \( P^2_R \subset U_2 \), we get \( \beta_{-n} \subset U_2 \cap P^2_R \) if \( \beta_{-n}(0) \in U^0_2 \).

Let us show that \( \beta_{-n}(0) \in U^0_2 \). The point \( \beta_{-n}(0) \) corresponds to the point \( z'_{-n} \) in the construction of \( \psi^+(w) \) when \( \text{Im}(w) \) is big. It has been shown in this part that \( z'_{-n} \in P^2_R \).

Since we know that for \( n \) big enough, \( z'_{-n} \in U_2 \), it follows that \( z'_{-n} \in U^0_2 \).

Let us define a path \( \gamma' \) as follows. For \( t \in [0,1] \), \( \gamma'(t) \) is equal to the common value of the \( \Phi^2_R(\beta_{-n}(t)) \mod \mathbb{Z} \) when \( n \geq N \). Finally, set

\[
\psi^+(w) = \gamma'(1)
\]

The path \( \gamma' = (\Phi^2_R \mod \mathbb{Z}) \circ \beta_{-N} \) links \( \psi^+(w_0) \) to \( \psi^+(w) \) and is included in \( \Phi^2_R(U^0_2 \cap P^2_R) \mod \mathbb{Z} \subset D_2 \). Since \( \psi^+(w_0) \in D^+_2 \), this shows that \( \psi^+(w) \in D^+_2 \).

The quantity \( \psi^+(w) \) depends a priori of the choice of the path \( \gamma \) made at the begining.

The map \( \psi^+ \) is well-defined from \( D^+_1 \) to \( D^+_2 \): The element \( \psi^+(w) \) may be expressed without any paths. This will prove that it is independent of the chosen path. Let \( \Phi^1_R : P^1_R \to \Phi^1_R(P^1_R) \) and \( \Phi^2_R : P^2_R \to \Phi^2_R(P^2_R) \) be normalized Fatou coordinates, with arbitrary repulsive petals \( P^1_R, P^2_R \). We have for all \( w \in D^+_1 \):

\[
\psi^+(w) = \Phi^2_R(\phi(\overline{\Psi}_R(w)) \text{ FACR}) \mod \mathbb{Z}
\]  

(5)

We recall that \( \overline{\Psi}_R(w) \) is the grand orbit of \( P^1_R/f_1 \) (hence a backward orbit) associated to \( w \in C^f_R \), and the notation \( \phi(\overline{\Psi}_R(w) \text{ FACR}) \) means that we keep only the elements of the orbit with high enough ranks (to be more precise we talk about the absolute value of the rank, since it is a backward orbit). The rank from which the formula is valid is the one from which the backward orbit \( \phi(\overline{\Psi}_R(w)) \) falls definitively in the repulsive petal \( P^2_R \).

Since the petal is open, this minimal rank can only decrease locally, this corresponds to an upper semi-continuity of the minimal rank as a function of \( w \).

In this paragraph, the notations are inspired by those of figure 10, replacing letters by capital letters and paths by relatively compact open sets. Given an relatively compact open set \( \Gamma \subset D^+_1 \), one can get a uniform upper bound \( N \) for this minimal rank on \( \Gamma \). This allows to get an expression of \( \psi^+ \) as a composition of holomorphic maps in \( \Gamma \). More generally, set \( \Gamma_0 = \{w \in C, a \leq \text{Re}(z) \leq a + 1 \text{ and } w \mod \mathbb{Z} \in \Gamma \} \) where \( a \in \mathbb{R} \) is chosen such that \( \Gamma_0 \subset \Phi^1_R(P^1_R) \) and for \( n \in \mathbb{N} \), \( \Gamma_{-n} = \Gamma_0 - n \). We have \( A_{-n} := \Psi^1_R(\Gamma_{-n}) \subset P^1_R \) and there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have \( B_{-n} := \phi(\Psi^1_R(\Gamma_{-n})) \subset P^2_R \). By equation (5), we get for all \( n \geq N \) and all \( w \in \Gamma_{-n} \):

\[
\psi^+(w \mod \mathbb{Z}) = \Phi^2_R(\phi(\Psi^1_R(w - n))) \mod \mathbb{Z}
\]  

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Hence \( \psi^+ \) is holomorphic on \( \Gamma \). This holds for all \( \Gamma \in D_1^+ \). That is why \( \psi^+ \) is holomorphic on its definition domain \( D_1^+ \).

The map \( \psi^+ \) is a biholomorphism from \( D_1^+ \) to \( D_2^+ \); this paragraph is only for the proof of the complement 5.2.

We take again the construction, in the other sense. More precisely, we define a map from \( D_2^+ \) to \( D_1^+ \):

\[
\psi^+(w) = \Phi_R^1(\phi'(\Psi_R^2(w))) \mod \mathbb{Z} \tag{6}
\]

Let \( \Gamma, \Gamma' \) be connected open relatively compact subsets of \( D_1^+, D_2^+ \). We apply the lemma 5.40 of pseudo-inversibility to show that \( \psi^+ : \Gamma \rightarrow \Gamma' \) is a biholomorphism of inverse \( \psi'^{-1} : \Gamma' \rightarrow \Gamma \). This is independent of the choice of \( \Gamma \) and \( \Gamma' \), hence \( \psi^+ : D_1^+ \rightarrow D_2^+ \) is a biholomorphism of inverse \( \psi'^+ : D_2^+ \rightarrow D_1^+ \).

Construction (and possible biholomorphy) of \( \psi^- \) from \( D_1^- \) to \( D_2^- \): A reasoning identical to the preceding paragraphs enables to build \( \psi^- \) in the general case, and to show its biholomorphy when \( \phi \) is a pseudo-conjugacy.

Conclusion: Finally, it remains to show that \( T_\sigma \circ \mathbf{h}_1^\pm = \mathbf{h}_2^\pm \circ \psi^\pm \) (recall that \( \mathbf{h}_1^\pm \) and \( \psi^\pm \) are defined on the same domain \( D_1^\pm \)) and to establish the Taylor expansion of \( \psi^\pm \).

Recall that \( T_\sigma = \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1} \) where \( \phi \) is the map that \( \phi \) induces from \( B^f_1 / f = P_A^1 / f = B^f_2 / f = P_A^2 / f \) by quotient (it may also be though as the map that \( \phi \) induces by direct image if \( P_A^1 = \phi(P_A^1) \)). Recall also that \( \mathbf{h}_1^\pm = \Phi_A^1 \circ i \circ \Psi_R^1 \) where \( i \) is the quotient of the inclusion map \( P_R^1 \cap B^f_1 \rightarrow B^f_1 \) from \( (P_R^1 \cap B^f_1) / f_1 \) to \( B^f_1 / f_1 = P_A^1 / f_1 \), see definition 4.1. We get:

\[
T_\sigma \circ \mathbf{h}_1^\pm = \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1} \circ \Phi_A^1 \circ i \circ \Psi_R^1 = \Phi_A^2 \circ \phi \circ i \circ \Psi_R^1 \tag{7}
\]

Use the fact that in a composite function, passing to quotient at each step is equivalent to passing to quotient on the range and the target. Recall that \( \pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z} \) is the canonical projection, that \( \Phi_R^1 \) is defined on \( P_R^1 \), and that \( \Phi_A^1 \) is defined over \( P_A^1 \). Let \( P_R^1 \) be a repulsive petal containing a \( \pi/2 \)-petal, and \( G \) be a left half plane such that \( \Phi_R^1(P_R^1) \supset G \). Let \( H = \{ z \in \mathbb{C}, \text{Im}(z) > M \} \subset D_1^+ \) be an upper half-plane such that \( \Psi_R^1(H \cap G) \subset P_A^1 \), where \( P_A^1 \) is a very large attractive petal, and such that \( M \) satisfies the corollary 5.28. For all \( w \in H \cap G \), there exists \( n(w) \in \mathbb{N} \) such that \( \phi(\Psi_R^1(w)) \in P_R^1 \) by corollary 5.28. Let

\[
X = \bigcup_{w \in H \cap G} \{ w - n(w) \} \subset H \cap G
\]

This enables to calculate the composite functions before the quotient (note that \( \pi(X) = \pi(H) \) :)

- By equation (7), the composite \( (T_\sigma \circ \mathbf{h}_1^\pm)|_{\pi(H)} \) is equal to \( \Phi_A^2 \circ \phi \circ (\Psi_R^1)|_X \) after passing to quotient by \( \mathbb{Z} \) at domain and range.

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\begin{itemize}
  \item \((h_2^+ \circ \psi^+)_{|\pi(H)}\) is equal to \(\Phi_2^A \circ \psi_{R,2}^{2,ext} \circ \phi_{R}^2 \circ (\Psi_1^1)_{|X} = \Phi_2^A \circ \phi \circ (\Psi_1^1)_{|X}\) after passing to quotient by \(\mathbb{Z}\) at domain and range. Note that the term \(\Phi_2^A \circ \phi \circ (\Psi_1^1)_{|X}\) corresponds to \(\psi^+\), and the term \(\Phi_2^A \circ \psi_{R,2}^{2,ext}\) to \(h_2^+\).

  We thus obtain that \((T_{\sigma} \circ h_1^+)_{|\pi(H)}\) and \((h_2^+ \circ \psi^+)_{|\pi(H)}\) are both equal to \(\Phi_2^A \circ \phi \circ (\Psi_1^1)_{|X}\) after passing to quotient by \(\mathbb{Z}\) at domain and range. This implies by uniqueness of analytic extension (\(\pi(H)\) is open in the connected set \(D_1^+\)) that \(T_{\sigma} \circ h_1^+ = h_2^+ \circ \psi^+\). We would show symmetrically that \(T_{\sigma} \circ h_1^- = h_2^- \circ \psi^-.\)

  The singularity of \(\psi^+\) at \(\pm i\infty\) being removable, we already know that the Taylor expansion of \(\psi^+\) exists and is of the form \(\psi^+(w) = dw + c + o(1)\). We have to show that \(d = 1\). We may write locally at \(\pm i\infty\):

  \[
  \psi^+ = (h_2^+)^{-1} \circ T_{-\sigma} \circ h_1^+
  \]

  where \((h_2^+)^{-1}\) is the local section of \(h_2^+\) at \(\pm i\infty\). We conclude by inversion and composition of Taylor expansions (the \(h_2^+\) are in \(\pm i\infty\) approximated to \(o(1)\) by translations, and an inverse or composite of translations is a translation). \(\Box\)

\textbf{Remark 6.4.} Suppose that \(\phi : U_1 \cap B_0^{|f_1} \rightarrow U_2 \cap B_0^{|f_2}\) defines a biholomorphism of inverse \(\phi^{-1} = \phi'\), where \(U_1, U_2\) neighborhood of 0, conjugating \(f_1|_{U_1 \cap B_0^{|f_1}}, f_2|_{U_2 \cap B_0^{|f_2}}\) (to simplify, one might suppose \(U_1 = U_2 = \mathbb{C}\), in such a way that \(\phi\) is a global biholomorphism on immediate basins); it is a more restrictive hypothesis than being a local conjugacy on immediate basins, see example 5.34. Note that the definition domains are not supposed connected. In this setting, one might define \(\psi\) over a bigger set than \(D_1^+ \cup D_1^-\). More precisely, we may rewrite the formula:

  \[
  \psi(w) = \Phi_2^A(\phi(\Psi_1^1(w))) \text{ FACR } \mod \mathbb{Z}
  \]

  which is defined when \(w \in D_1\) is such that the backward orbit \(\phi(\Psi_1^1(w))\) is included after a certain rank in the repulsive petal \(P_2^2\) of \(f\). The condition \(w \in D_1\) is equivalent to the backward orbit \(\Psi_1^1(w) \in P_2^2/\ker f\) being included in \(U_1 \cap B_0^{|f_1}\). Recall that we choose \(P_2^2 \subset U_1\). The rank - i.e. the absolute value of the rank - from which the orbit is definitively in \(P_2^2\) being an upper semi-continuous function of \(w\), the definition domain of \(\psi\) is open. For the same reason, the map \(\psi\) is holomorphic, since it can be written locally as a composite of holomorphic functions.

  The proof of proposition 6.1 defines \(\psi^+\) on the connected set \(D_1^+\) from the definition of \(\psi^+\) at a single point \(w_0\) of this set, using paths. By following to the letter this proof, we may define \(\psi\) over a whole connected component of \(D_1\) from the definition of \(\psi\) on a single point of this component; this shows that the definition domain \(\psi\) must be a union of connected components of \(D_1\). Considering \(\psi^+\) as in the proof in the case \(\phi\) is a pseudo-conjugacy (substituting \(w \in D_2\) for the restriction \(w \in D_2^+\)), the arguments of the proof do not apply to show that \(\psi\) is a biholomorphism over its image of inverse \(\psi^+\), even though one might expect it is true.
We will not think anymore about this extension question in this article, our aim being to build a local classification (cf. theorem 1.4), which only needs to exploit the components $D_i^\pm$.

Let us show now the reciprocal of theorem 1.4:

**Proposition 6.5.** Let $f_1, f_2$ be two holomorphic maps with parabolic point at 0, with Taylor expansion at 0 of the form $f_i(z) = z + a_iz^2 + O(z^3)$ with $a_i \neq 0$.

Suppose that there exists $\sigma \in \mathbb{C}$ and a pair of holomorphic maps $\psi = (\psi^+, \psi^-)$, where $\psi^\pm : D_1^\pm \to D_2^\pm$, with removable singularities at $\pm i\infty$ such that:

$$h_1^\pm(w) = T_\sigma \circ h_2^\pm \circ \psi^\pm(w)$$

for all $w \in D_1^\pm$.

Then there exists a local semi-conjugacy $\phi$ over the immediate basins of $f_1, f_2$ (see definition 5.8). Moreover, the maps $\psi^\pm$ admit an expansion at $\pm i\infty$ of the form $\psi^\pm(w) = w + \rho^\pm + o(1)$.

**Complement 6.6.** Same hypothesis as the preceding proposition. Suppose furthermore that $\psi = (\psi^+, \psi^-)$ is a pair of biholomorphisms $\psi^\pm : D_1^\pm \to D_2^\pm$.

Then there exists a local pseudo-conjugacy $(\phi, \phi')$ over the immediate basins of $f_1, f_2$ (see definition 5.30). We may choose this local pseudo-conjugacy synchronous.

**Proof.** We show simultaneously the proposition and its complement. Let $T_\sigma$ and $\psi^\pm$ satisfy the hypothesis of the proposition. We will suppose that $\psi^\pm$ are biholomorphisms during the paragraphs referring to the demonstration of the complement, paragraphs which will be explicitly mentioned.

Taylor expansions of $\psi^\pm$ at $\pm i\infty$: The Taylor expansions of $\psi^\pm$ are shown by applying the end of the proof of proposition 6.1, last paragraph. Note $\rho^\pm \in \mathbb{C}$ the corresponding constant terms.

Choice of good lifts $\tilde{\psi}^\pm$ of $\psi^\pm$ by $\pi$: Note $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ the canonical projection and:

$$\tilde{D}_i = \pi^{-1}(D_i)$$
$$\tilde{D}_i^\pm = \pi^{-1}(D_i^\pm)$$

for $i = 1, 2$. Let $\tilde{\psi}^\pm : \tilde{D}_1^\pm \to \tilde{D}_2^\pm$ be a lift of $\psi^\pm$ by $\pi$. Since $\tilde{\psi}^\pm(w + 1) = \tilde{\psi}^\pm(w) \mod \mathbb{Z}$, we get by continuity that there exists $n \in \mathbb{Z}$ such that $T_n \circ \tilde{\psi}^\pm = \tilde{\psi}^\pm \circ T_1$. The Taylor expansion of $\psi^\pm$ at $\pm i\infty$ gives by continuity of the lift $\psi$ an expansion of $\tilde{\psi}^\pm$ at $\pm i\infty$ over a vertical band $D$ of width 1 (fundamental domain for the translation $T_1$)

$$\tilde{\psi}^\pm(w) = w + \rho^\pm + o(1)$$
The attractive petals $P_1^A, P_2^A$ are represented by cardioids, the repulsive petals $P_1^R, P_2^R$ by circles. We have chosen $U_1 = P_1^A \cup P_2^A \cup \{0\}$. The set $U_1^0$ is represented in dark grey, in the central part at the left of the diagram. The set $U_1 \cap B_{f_1}^1 \setminus U_1^0$ is represented in light grey. We represent also in dark grey, light grey their diverse images by the maps of the diagram. The map $\phi$ will be build as the unique function completing this commutative diagram. Note that a priori, $\phi(U_1^0)$ is not a set of the form $U_2^0$, and that $\phi(P_1^A)$ is not a set of the form $(P_2^R)^0$ where $P_2^R$ repulsive petal of $f_2$. Notice that $Y_2^0$ is the direct image.
even if it means adding fixed integers to $\rho^\pm$. This last Taylor expansion implies that $n = 1$. Hence $\tilde{\psi}^\pm$ commutes with $T_1$. Note that $h_1^\pm = T_\sigma \circ h_2^\pm \circ \psi^\pm$. This implies:

$$\Phi_A^1 \circ \Psi_R^{1,\text{ext}}(w) = T_\sigma \circ \Phi_A^2 \circ \Psi_R^2 \circ \tilde{\psi}^\pm(w) \mod Z$$

for $w \in \tilde{D}_1^+$, connected set.

Recall that $T_\sigma$ and $\Phi_A^2 \circ \Psi_R^2$ commute with $T_1$, this implies that $T_\sigma \circ \Phi_A^2 \circ \Psi_R^2$ commutes with $T_1$. Chosing the right lifts of $\psi^\pm$, we get the equality

$$\Phi_A^1 \circ \Psi_R^1(w) = T_\sigma \circ \Phi_A^2 \circ \Psi_R^2 \circ \tilde{\psi}^\pm(w)$$

without the modulo. Note that the reasoning is also valid when $D_1^+ = D_1^-$ i.e. $\psi^+ = \psi^-$. We choose these lifts $\tilde{\psi}^\pm$ for the rest of the argument. Note $\tilde{\psi} = \tilde{\psi}^+ \cup \tilde{\psi}^-.$

\textit{Construction of the petals $P_A^1, P_A^2$, and of the set $Y_R^2$:} We choose the attractive, repulsive petals $P_A^1, P_R^1$ of $f_1$, such that

$$U_1 = P_A^1 \cup P_R^1 \cup \{0\}$$

is an open neighborhood of 0 included in a neighborhood $W_1$ of 0 where $f_1$ admits an inverse branch $f_1^{-1} : W \to f_1^{-1}(W_1)$ sending 0 to 0. This allows to define $h_1^1$ the horn map of $f^{-1}$. We choose $P_A^1, P_R^1$ such that $P_A^1 \cap P_R^1$ contains two connected components $C^\pm$. The component $C^\pm$ is characterized by the fact that its image by $\Phi_A^1 \mod Z$ (or equivalently by $\Phi_A^1 \mod Z$) is a punctured neighborhood of $\pm i\infty$.

Write

$$(P_R^1)^0 = (\Phi_R^1)^{-1}(\tilde{D}_1^+ \cup \tilde{D}_1^-) = (\Phi_R^1)^{-1}(\tilde{D}_1^+ \cup \tilde{D}_1^- \cap \Phi_R^1(P_R^1))$$

and

$$V_1^0 = P_A^1 \cup (P_R^1)^0$$

Let $U_1^0$ be the connected component of $U_1 \cap B_0^1$ containing a germ of the attractive axis.

Recall that by lemma 5.14, we have $U_1^0 \subset V_1^0$ (the reader might look at the proof of the lemma again and see figure 6). Remark that choosing $P_A^1$ small enough, which is equivalent to choosing $\Phi_A^1(P_A^1)$ small, there exists a petal $P_2^2$ of $f_2$ with a Fatou coordinate $\Phi_2^1$ such that $\Phi_A^1(P_2^2) = \Phi_2^1(P_A^1) - \sigma$, i.e.

$$P_2^2 = (\Phi_2^1)^{-1}(\Phi_A^1(P_A^1) - \sigma)$$

the petal $P_2^2$ may be arbitrarily small.

Let $\Psi_R^{2,\text{ext}}$ be the extended Fatou parametrisation of $f_2$. By proposition-definition 4.8, $\text{Im}(\tilde{\psi}) = \tilde{D}_2^+ \cup \tilde{D}_2^-$ is included in the definition domain of $\Psi_R^{2}$ and $\Psi_R^{2}(\text{Im}(\tilde{\psi})) \subset B_0^{f_2}$. This allows to note the following open subset of the attractive immediate basin of $f_2$:

$$Y_R^2 = \Psi_R^{2,\text{ext}}(\tilde{\psi}(\Phi_R^1(V_1^0))) \subset B_0^{f_2}$$

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Remark 6.7. Since \( \tilde{\psi} \) does not admit a continuous extension to the border of \( \tilde{D}_1^+ \cup \tilde{D}_1^- \), it seems not obvious that \( Y_R^2 \) is included in a set of the form \((P_R^2)^0 = (\Phi_R^2)^{-1}(\tilde{D}_2^+ \cup \tilde{D}_2^-)\) where \( P_R^2 \) repulsive petal of \( f_2 \). Figure 11 shows how a counter-example might look like. That is why we used the extended repulsive parametrisation to define \( Y_R = \Psi^{2,ext}_R(\tilde{\psi}(\Phi_R^2(V_1^0))) \), since it seems difficult to use \((\Phi_R^2)^{-1} \) where \( \Phi_R^2 \) repulsive Fatou coordinate of a well chosen repulsive petal \( P_R^2 \).

The continuity problem of \( \tilde{\psi} \) at the border leads also to the following issue: it is not obvious that \( Y_R^2 \) contains a set of the form \( \tilde{D}_2^+ \cup \tilde{D}_2^- \) which \( \tilde{\psi} \) is included in a set of the form \((P_R^2)^0 \), where \( P_R^2 \) repulsive petal of \( f_2 \), see again the same figure to understand how a counter-example might look like.

\[ \text{Construction of } \phi : V_1^0 \to P_R^2 \cup Y_R^2 : \text{The relation valid over } (\tilde{D}_1^+ \cup \tilde{D}_1^-) \cap \Phi_R^1(P_R^1) \]

\[ \Phi_A^1 \circ (\Phi_R^1)^{-1} = T_\sigma \circ \Phi_A^1 \circ (\Phi_R^2)^{-1} \circ \tilde{\psi} \]

enables to show the following lemma:

Lemma 6.8.

\[ (\Phi_A^2)^{-1} \circ T_\sigma \circ \Phi_A^1(z) = \Psi_R^{2,ext} \circ \tilde{\psi} \circ \Phi_R^1(z) \]

for \( z \in P_A^1 \cap P_R^1 \).

Proof. The two members of the equality are indeed well defined by construction of \( P_A^1, P_R^1 \) and \( P_R^2 \). More explicitely, the left member is defined over \( P_A^1 \) since \( T_\sigma(\Phi_A^1(P_A^1)) = P_A^2 \), and the right member is defined over \( P_A^1 \cap P_R^1 \) since \( \Phi_R^1(P_A^1 \cap P_R^1) = \Phi_R^1(C^+) \cup \Phi_R^1(C^-) \subset \tilde{D}_1^+ \cup \tilde{D}_1^- \). Recall we chose during the \( P_A^1, P_R^1 \) construction the petals \( P_A^1, P_R^1 \) such that \( P_A^1 \cap P_R^1 \) contains two connected components \( C^+, C^- \), and that the component \( C^\pm \) is caracterized by the fact that its image by \( \Phi_A^1 \mod \mathbb{Z} \) (or equivalently by \( \Phi_R^1 \mod \mathbb{Z} \)) is a punctured neighborhood of \( \pm i\infty \).

Let \( P_R^2 \) be a repulsive petal for \( f_2 \) with Fatou coordinate \( \Phi_R^2 \). Denote

\[
\begin{align*}
S_R & = \Phi_R^1(P_A^1) \\
S_R^0 & = S_R \cap (\tilde{D}_1^+ \cup \tilde{D}_1^-) \\
T_R & = \Phi_R^2(P_R^2) \\
T_R^0 & = T_R \cap (\tilde{D}_2^+ \cup \tilde{D}_2^-)
\end{align*}
\]

Let \( \tilde{\psi}^{-1} \) be a local section of \( \tilde{\psi} \) at \( \pm i\infty \). Let \( W^\pm \) be a neighborhood of \( \pm i\infty \) in \( T_R^0 \). By continuity of \( \tilde{\psi}^{-1} \) at \( \pm i\infty \), we choose \( W^\pm \) small enough such that, \( \tilde{\psi}^{-1}(W^\pm) \subset S_R \cap \tilde{D}_1^\pm \).

We have immediately:

\[ (\Phi_A^2)^{-1} \circ T_\sigma \circ \Phi_A^1(z) = (\Phi_R^2)^{-1} \circ \tilde{\psi} \circ \Phi_R^1(z) \]

for \( z \in P_A^1 \cap P_R^1 \cap \Phi_R^1(\tilde{\psi}^{-1}(W^\pm)) \), open set of non-empty intersection with \( C^\pm \) since \( \tilde{\psi}^{-1}(W^\pm) \). Hence by uniqueness of analytic extension, we have for \( z \in P_A^1 \cap P_R^1 = C^+ \cup C^- \):

\[ (\Phi_A^2)^{-1} \circ T_\sigma \circ \Phi_A^1(z) = \Psi_R^{2,ext} \circ \tilde{\psi} \circ \Phi_R^1(z) \]
We may now denote the holomorphic map \( \phi : V_1^0 \to P_2^1 \cup Y_2^1 \) defined by:

\[
\phi(z) = \begin{cases} 
(\Phi_A^2)^{-1} \circ T_\sigma \circ \Phi_A^1(z) & \text{if } z \in P_A^1 \\
\psi_R^2 \circ \tilde{\psi} \circ \Phi_R^1(z) & \text{if } z \in (P_R^1)^0 
\end{cases}
\]

By construction of \( P_A^2 \) and \( Y_R^2 \):

- \( \phi \) induces a biholomorphism from \( P_A^1 \) to \( P_A^2 \)

- \( \phi \) induces a surjection from \( (P_R^1)^0 \) to \( Y_R^2 \).

In particular \( \phi \) is surjective.

Note that \( \phi \) induces a conjugacy between \((f_1)|_{P_A^1}\) and \((f_2)|_{P_A^2}\), and a semi-conjugacy between \((f_1)|_{(P_R^1)^0}\) and \((f_2)|_{Y_R^2}\) (for the semi-conjugacy, exploit that \( \tilde{\psi} \) commutes with \( T_1 \)). The map \( \phi \) then constitutes a semi-conjugacy between \((f_1)|_{V_1^0}\) and \((f_2)|_{P_A^2 \cup Y_R^2}\).

The map \( \phi \) is then a local semi-conjugacy in the sense of definition 5.8.

Construction of \( \phi^* : V_0^0 \to P_A^1 \cup Y_R^1 \): This paragraph is only for the proof of the complement. We take again the elements of the construction of \( \phi \), in the opposite direction: we build in the same way \( V_2, V_2^0, (P_R^2)^0 \) and \( Y_R^1 \).

Notice that \( P_A^1, P_R^1, P_R^2, P_R^2 \) were already built in the preceding construction, that is why we will note the new petals \( Q_A^2, Q_R^2, Q_A^1, Q_R^1 \).

Let us check that we can take \( Q_A^1 = P_A^1 \) and \( Q_R^1 = P_R^1 \) with all the conditions satisfied on the petals. Recall the conditions.

1. \( P_A^1 \cup P_R^1 \) is a punctured neighborhood of 0 containing two connected components, included in a neighborhood \( W_1 \) of 0 where \( f_1 \) admits an inverse branch \( f_1^{-1} : W_1 \to f_1^{-1}(W_1) \) sending 0 to 0.

2. \( \Phi_A^2(P_A^2) = \Phi_A^1(P_A^1) - \sigma \)

3. \( Q_A^2 \cup Q_R^2 \) is a punctured neighborhood of 0 containing two connected components, included in a neighborhood \( W_2 \) of 0 where \( f_2 \) admits an inverse branch \( f_2^{-1} : W_2 \to f_2^{-1}(W_2) \) sending 0 to 0.

4. \( \Phi_A^1(Q_A^1) = \Phi_A^2(Q_A^2) + \sigma \)

5. \( Q_A^1 = P_A^1 \) and \( Q_R^2 = P_R^2 \)

Let us take \( P_A^1, P_R^1, P_A^2 \) such that the conditions 1 and 2 are true, as in the construction of \( \phi \). Note that \( P_A^1 \) can be taken arbitrarily small, this implies, by condition 1, that \( P_A^2 \) can be taken arbitrary small. Take \( Q_A^2 := P_A^2 \). Since \( Q_A^2 \) is arbitrarily small, it is possible to build \( Q_R^1 \) such that condition 3 is true. Take \( Q_A^1 := P_A^1 \), this imply condition 4. Since \( \Phi_A^1(Q_A^1) = \Phi_A^2(P_A^1) = \Phi_A^2(P_A^2) + \sigma = \Phi_A^2(Q_A^2) + \sigma \) by condition 2. Finally, define \( Q_R^2 := P_R^1 \) and \( P_R^2 := Q_R^2 \) to have condition 5.

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This enables to build $\phi^\prime : V_2^0 \rightarrow P_1^A \cup Y_R^1$, the formula being the same for $\phi$ with the roles of $f_1, f_2$ inverted:

$$\phi^\prime(z) = \begin{cases} (\Phi_A^1)^{-1} \circ T_\sigma \circ \Phi_A^2(z) & \text{if } z \in P_2^A \\ \Psi_{A,ext}^1 \circ \tilde{\psi}^{-1} \circ \Phi_R^2(z) & \text{if } z \in (P_R^2)^0 \end{cases}$$

Notice that the biholomorphism induced by $\phi^\prime$ from $P_2^A$ to $P_1^A$ turns out to be exactly the inverse of the biholomorphism induced by $\phi$ from $P_1^A$ to $P_2^A$. This enables to say that $(\phi, \phi^\prime)$ is a synchronous pseudo-conjugacy.

\[\Box\]

**Remark 6.9.** There is no reason a priori for a local semi-conjugacy, even if it is a synchronous local pseudo-conjugacy, to be a local conjugacy over the immediate basins (see the definitions 5.34, 5.30). Recall that by the example 5.34, a local conjugacy over the immediate basins is a biholomorphism of the form $\phi : U_i^0 \rightarrow U_j^0$ which conjugates $f_1$ and $f_2$, it can be interpreted as a local pseudo-conjugacy $(\phi, \phi^\prime)$ where $\phi^\prime$ is the inverse of $\phi$.

### 7 Parabolic point with several petals

#### 7.1 Generalized definitions of semi-conjugacy, pseudo-conjugacy and reviewed statements

Let $f_1, f_2$ be two holomorphic functions from an open neighborhood of $0 \in \mathbb{C}$ to $\mathbb{C}$ with a general parabolic point at $0$:

$$f_i(z) = e^{2i\pi p_i/q_i}z + o(z)$$

where $p_i \in \mathbb{Z}, q_i \in \mathbb{N}^*$ and $p_i \wedge q_i = 1$.

Denote $A_1, A_2$ attractive axes of $f_1, f_2$, and $R_i^-, R_i^+$ the adjacent repulsive axes of $A_i$ such that $R_i^-, A_i, R_i^+$ are in the trigonometrical order (if $q_i = 1$ then $R_i^- = R_i^+$). Let $B_i^1, B_i^2$ be the parabolic basins of $f_1, f_2$ and $C^1, C^2$ the connected components of $B_i^1, B_i^2$ containing a germ of $A_1, A_2$. Notice that $q_i$ is the least positive integer such that $f_i^{q_i}(C^i) = C^i$.

For $U_1, U_2$ open neighborhoods of $0 \in \mathbb{C}$, denote $U_i^0$ the connected component of $U_i \cap B_i^k$ containing a germ of the attractive axis $A_i$. For all attractive petal $P_i^m$ of $f_i$ associated to the axis $A_i$ and included in $U_i$, we have $P_A^m \subset U_i^0$. See figure 12 for the next two definitions.

**Definition 7.1.** A holomorphic map $\phi$ is said to be a local semi-conjugacy of $(f_1, A_1), (f_2, A_2)$ over their immediate basins if $\phi : U_1^0 \rightarrow C^2$ where $U_1, U_2$ open neighborhood of $0 \in \mathbb{C}$, and satisfies the property of semi-conjugacy: $\phi \circ f_1^{q_1}(z) = f_2^{q_2} \circ \phi(z)$ for all $z \in U_1^0$ such that $f_1^{q_1}(z) \in U_1^0$. 

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The zones represented in gray are respectively $U_1^0$ at the left and $U_2^0$ at the right. Note these sets are not image the one by the other by $\phi$, $\phi'$. Remark also that the angles between the attractive axis and the adjacent repulsive axis are not necessarily the same, which is equivalent to say that $f_1$ and $f_2$ do not necessarily have the same number of petals.

**Definition 7.2.** A pair of local semi-conjugacies $(\phi, \phi')$ of $(f_1, A_1), (f_2, A_2)$ is a local pseudo-conjugacy of $(f_1, A_1), (f_2, A_2)$ over their immediate basins if $\phi : C_1^{f_1} \rightarrow C_2^{f_2}$ has for inverse $\phi' : C_2^{f_2} \rightarrow C_1^{f_1}$.

**Remark 7.3.** Suppose that $f_1, f_2$ are pseudo-conjugated, with sets $U_1, U_2$ small enough such that $f_i$ admits a local inverse at 0:

$$f_i^{-1} : f_i(U_i) \rightarrow U_i$$

Write $V_i = f_i(U_i)$, and $V_i^0$ the connected component of $V_i \cap B^{f_i}$ containing a germ of the attractive axis $T f_i(A_i)$, where $T f_i$ denotes the tangent map of $f_i$ at 0. Saying that $(f_1, A_1)$ and $(f_2, A_2)$ are pseudo-conjugated by $(\phi, \phi')$ is equivalent to saying that $(f_1, T f_1(A_1))$ and $(f_2, T f_2(A_2))$ are pseudo-conjugated by $(\phi \circ f_1^{-1}, \phi' \circ f_2^{-1})$, where $\phi \circ f_1^{-1}$ and $\phi' \circ f_2^{-1}$ are defined over $V_i^0, V_0^0$.

Note $\Phi^\text{ext}_A[A_i] : C^i \rightarrow C$ an extended Fatou coordinate associated to the attractive axis $A_i$, and $\Psi^\text{ext}_R[R_i^\pm]$ the extended repulsive Fatou parametrisation associated to the repulsive axis $R_i^\pm$. Note $D_i^\pm$ the connected component containing $\pm i \infty$ of the definition domain of the map

$$\Phi^\text{ext}_A[A_i] \circ \Psi^\text{ext}_R[R_i^\pm]$$

after quotient at the domain and range by $\mathbb{Z}$. Denote $h_i^\pm$ this map $\Phi^\text{ext}_A[A_i] \circ \Psi^\text{ext}_R[R_i^\pm]$ after quotient and restricted to $D_i^\pm$. Call $h_i^\pm$ the horn maps of $(f_i, A_i)$. Remark $h_i^+$ and $h_i^-$ both correspond to the same attractive axis $A_i$, but to distinct repulsive axis $R_i^+$ and $R_i^-$.

---

It would be possible to show, as in the case of simple parabolic points, that the sets $C^i/f_i^q$ are cylinders, and the map $\overline{\phi}$ is a translation. This would enable to define the concept of synchronous pseudo-conjugacy.
We believe that the proof of theorem 1.4 may be followed in this more general case, where the functions \( f \) have a general parabolic point at 0. Here is the statement we should obtain:

**Statement 7.4.** Suppose that \((f_1, A_1), (f_2, A_2)\) are locally pseudo-conjugated at 0 over \( B_0^f \). Then \( h_1^\pm, h_2^\pm \) are equivalent as analytic ramified coverings via biholomorphism between \( D_1^\pm \) and \( D_2^\pm \), and a translation of \( \mathbb{C}/\mathbb{Z} \). More precisely, there exists \( \sigma \in \mathbb{C} \), and a biholomorphism pair \( \psi = (\psi^+, \psi^-) \), where \( \psi^\pm : D_1^\pm \to D_2^\pm \), such that:

\[
    h_1^\pm = T \circ h_2^\pm \circ \psi^\pm
\]

Furthermore, the built maps \( \psi^\pm \) admit at \( \pm i\infty \) removable singularities.

Conversely, if there exists \( \sigma \in \mathbb{C} \) and a pair of biholomorphisms \( \psi = (\psi^+, \psi^-) \), where \( \psi^\pm : D_1^\pm \to D_2^\pm \), with removable singularities at \( \pm i\infty \) such that:

\[
    h_1^\pm = T \circ h_2^\pm \circ \psi^\pm
\]

then \( f_1, f_2 \) are locally pseudo-conjugated at 0 on their immediate basins.

Furthermore, the maps \( \psi^\pm \) admits at \( \pm i\infty \) an expansion of the form \( \psi^\pm(w) = w + \rho^\pm + o(1) \).

### 7.2 A class of pseudo-conjugacies examples

We give here a last class of pseudo-conjugacies examples, and we apply the conjectural statement 7.4. Let \( A, B : \mathbb{D} \to \mathbb{D} \) be two Blaschke products. Suppose there exists two holomorphic functions \( f_1 : D_{f_1} \to \mathbb{C} \), \( f_2 : D_{f_2} \to \mathbb{C} \) satisfying the following properties. The maps \( f_1, f_2 \) have a parabolic point at 0, and their immediate basins a cycle of two connected components \( B_0^f = C_0 \cup C_1, B_0^f = C_0' \cup C_1' \). Notice that the \( \partial C_i, \partial C_i' \) are not necessarily locally connected. Suppose that we have the following commutative diagram:

\[
\begin{array}{ccc}
    C_0 & \xrightarrow{f_1} & C_1 \\
\downarrow{\alpha_0} & & \downarrow{\alpha_0} \\
    \mathbb{D} & \xrightarrow{A} & \mathbb{D} \\
\downarrow{\beta_0} & & \downarrow{\beta_0} \\
    C_0' & \xrightarrow{f_2} & C_1' \\
\end{array}
\]

where \( \alpha_i : C_i \to \mathbb{D} \) and \( \beta_i : C_i' \to \mathbb{D} \) biholomorphisms. See figure 13.

Set \( g_1 = B \circ A \), \( g_2 = A \circ B \). The functions \( g_1, g_2 \) both have a parabolic point at \( z = 1 \), as composition of functions admitting fixed points at \( z = 1 \) of which the product of the multipliers is 1. Let \( X = [0,1] \) be their common attractive axis. Notice that \( g_1, g_2 \) are semi-conjugated by \( A \), and \( g_2, g_1 \) are semi-conjugated by \( B \). The pair of maps \((A, B)\) constitutes a non-synchronous pseudo-conjugacy between \((g_1, X)\) and \((g_2, X)\). The pseudo-conjugacy is non-synchronous since \( B \) is not a local section of \( A \) locally at \( z = 1 \); indeed, \( B \) is not a global section of \( A \) and \( B \) is holomorphic.
Figure 13: Pseudo-conjugated maps illustration

Figure 14: Blaschke products
Write $X_1 = \alpha^{-1}_1(X)$, $X_2 = \beta^{-1}_2(X)$. The maps $(f_1, X_1)$ and $(f_2, X_2)$ are then pseudo-conjugated by $(\beta^{-1}_2 \circ A \circ \alpha_1, \alpha^{-1}_1 \circ B \circ \beta_2)$. If true, the statement 7.4 would be applied, by noting $h_1^\pm, h_2^\pm$ the horn maps of $(f_1, X_1)$ and $(f_2, X_2)$. There would exist $\sigma \in \mathbb{C}$ and $\psi^\pm : D_1^\pm \rightarrow D_2^\pm$ with removable singularities at $\pm i\infty$ such that:

$$h_1^\pm = T_\sigma \circ h_2^\pm \circ \psi^\pm$$

Insofar as $f_1$ and $f_2$ exist, this enables to build a class of pseudo-conjugacies examples. If $\partial C_0$ is locally connected at 0 and $\partial C_i'$ non locally connected at 0, this would allow to get a pseudo-conjugacy which is not continuous at 0.

This examples motivates a general question of existence. Let us fix some notations.

Write $\frac{p}{q} \in \mathbb{Q}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$, $p \wedge q = 1$. Let $\mathcal{P}$ be the set of functions $f$ satisfying the following properties:

- $f : U \rightarrow \mathbb{C}$ is holomorphic on an open neighborhood of $0 \in \mathbb{C}$
- $f$ has a parabolic point at 0 of multiplier $e^{2\pi i p/q}$ with a cycle of attractive axis, i.e. a single cycle $B^i_0 = C = C_0 \cup \cdots \cup C_{q-1}$ of connected components of the immediate parabolic basin, with $f(C_i) = C_{i+1}$ (the index $i$ being defined modulo $q$).
- Each of the $C_i$ is simply connected.

**Question 7.5.** (see figure 14 for an illustration where $q = 4$) Let $B_0, \cdots, B_{q-1}$ be Blaschke products from $\mathbb{D}$ to $\mathbb{D}$. On what condition does there exist $f \in \mathcal{P}$ and biholomorphisms $\chi_i : C_i \rightarrow \mathbb{D}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
C_0 & \xrightarrow{f} & C_1 & \xrightarrow{f} & \cdots & \xrightarrow{f} & C_q \\
\chi_0 & \downarrow & \chi_1 & & & & \chi_q \\
\mathbb{D} & \xrightarrow{B_0} & \mathbb{D} & \xrightarrow{B_1} & \cdots & \xrightarrow{B_{q-1}} & \mathbb{D}
\end{array}
$$

where we recall that $C_q = C_0$ ?

Can it be further imposed that some specific $\partial C_i$ are locally connected, and the others non locally connected ?

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