Landau Criterion in a Bose–Condensed Sodium Gas

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The first experimental evidence concerning the phenomenon of superfluidity can be tracked down to the work of Kamerlingh Onnes in the year of 1911 with helium when it was found that if cooled below 2.2 °K He did not contract but rather expand [1]. Since then the amount of theoretical and experimental work has been able to provide a coherent picture to the subjacent Physics [2–4]. Another phenomenon, which also emerged in the last century, is the Bose–Einstein Condensation (BEC) the one, as in the case of superfluidity, is inherently related to the presence of very low temperatures. The need for low temperatures for the appearance of these two effects leads us to the obvious question concerning a possible relation between them. It was Fritz London [5] who put forward the idea of a connection between these two effects stating that the transition from He I (the high temperature phase of liquid helium) and He II (the low temperature phase) should be considered an example of a BEC. Even more, he suggested that in He II a macroscopic quantum current of matter could be present, i.e., he introduced the idea that BEC and superfluidity could appear, simultaneously, in a system.

It has to be stressed that this understanding cannot be considered a closed issue [6], and the answer to this aforementioned interrogant is that, though there is a close relationship, it is not a unique one. Indeed, we may state that BEC is neither necessary nor sufficient for the presence of superfluidity. For instance, and ideal Bose–Einstein condensate shows no superfluidity, and, as a counterpart a two–dimensional superfluid cannot condensate [7].

Landau (within the two–fluid model proposed by Tisza [8]) introduced the concept of elementary excitations [9] as a fundamental element in the description of the behavior of He II. Landau argued [10] that the normal fluid (the non–superfluid component) could be regarded as a dilute gas whose components are weakly–interacting elementary excitations which move in a background defined by the superfluid component. Accordingly, the phenomenon of superfluidity appears if the velocity of the corresponding flow lies below a certain threshold value given by

\[
v_{(\text{crit})} = \min \left( \frac{\epsilon(p)}{p} \right).
\]

Here \(\epsilon(p)\) denotes the energy of an elementary excitation and \(p\) its corresponding momentum. If the velocity is larger, then the microscopic rugosities of the walls of the container will scatter the particles of the fluid entailing the loss of kinetic energy of the fluid, i.e., viscosity appears.

In the experimental realm the quest for this critical velocity has been carried out in a sodium–BEC, and the results show a possible velocity threshold located around the value of 1.6mm/s [11]. The use of sodium–condensed gas in the experimental context is related not only to the aforementioned case but also to the excitation of phonons by light scattering [12] and the propagation of sound [13, 14].

The main purpose of the present work is to obtain a prediction for the critical velocity for a BEC. The deduced model will be compared against the reported measurement readouts [11], and, in addition, the speed of sound for a sodium–condensed gas will be found and compared with the current experimental results [13, 14].

From a fundamental point of view our mathematical model can be defined by an \(N\)–particle Hamiltonian the one in the formalism of second quantization is [7]

\[
\hat{H} = \int d\vec{r} \left[ -\hat{\psi}^\dagger(\vec{r},t) \frac{\hbar^2}{2m} \nabla^2 \hat{\psi}(\vec{r},t) + V(\vec{r}) \hat{\psi}^\dagger(\vec{r},t) \hat{\psi}(\vec{r},t) + \frac{U_0}{2} \hat{\psi}^\dagger(\vec{r},t) \hat{\psi}^\dagger(\vec{r},t) \hat{\psi}(\vec{r},t) \hat{\psi}(\vec{r},t) \right].
\]

In this Hamiltonian \(\hat{\psi}^\dagger(\vec{r},t)\) and \(\hat{\psi}(\vec{r},t)\) represent bosonic creation and annihilation operators, respectively. It is restricted to low energies and momenta and implies, as a consequence of the aforementioned conditions, that the interaction among the particles is, as usual, codified by the scattering length parameter \(a\), i.e., \(U_0 = \frac{4\pi\hbar^2}{m}\). The trapping potential \(V(\vec{r})\), for our case, corresponds to an isotropic harmonic oscillator whose frequency reads \(\omega\). In addition, there are \(N\) particles in the gas, each of them with mass \(m\), the volume occupied by the system is \(V\).
Our mathematical assumptions are:

(i) Only the ground and the first excited states are populated. This condition can be justified recalling that for a bosonic system, with chemical potential $\mu$ and energy levels of single–particle $\epsilon$, the occupation number in thermal equilibrium is given by [15] $(\beta = 1/(kT))$

$$< n_\epsilon > = \frac{1}{e^{(\epsilon - \mu)/\beta} - 1}. \quad (3)$$

It is readily seen that we deal with a monotonic decreasing function of $\epsilon$, and this feature justifies the present assumption.

(ii) The mathematical description of the two occupied states will be done resorting to the Hartree approximation, in which the ground state of the interacting system is deduced by a Ginzburg–Pitaevski–Gross energy functional [16], and it entails that the ground state wavefunction corresponds to the case of a harmonic oscillator situation but the frequency is modified due to the fact that the system has a non–vanishing scattering length [17], such that the fundamental length parameter reads.

$$R = \left(\frac{2}{\pi}\right)^{1/10} \left(\frac{N a}{T}\right)^{4/5} l. \quad (4)$$

Here $l$ is the radius related to the trap given by the isotropic harmonic oscillator, namely,

$$l = \sqrt{\frac{\hbar}{m\omega}}. \quad (5)$$

Clearly this condition implies an effective frequency

$$\tilde{\omega} = \frac{\hbar}{mR^2}. \quad (6)$$

Usually the experimental conditions entail $R > l$ [11] and, in consequence, $\tilde{\omega} < \omega$.

In other words, the order parameter related to the particles in the ground state is provided by

$$\psi_0(\vec{r}) = \sqrt{\frac{\mathcal{N}_0}{R\sqrt{\pi}}^3} \exp\left[-\frac{\vec{r}^2}{2R^2}\right]. \quad (7)$$

In this last expression $\mathcal{N}_0$ denotes the number of particles in the lowest energy state. The presence of a non–vanishing scattering length entails that in the ground state not all the particles can have zero–momentum, the reason for this lies in the fact that the two–body interaction mixes in components with atoms in other states [15] and

$$\mathcal{N}_0 = N\left[1 - \frac{8}{3}\sqrt{\frac{Na}{\pi V}}\right]. \quad (8)$$

Clearly,

$$\mathcal{N}_{(0)} = \int (\psi_0(\vec{r}))^2 d^3r, \quad (9)$$

$$V = \frac{4\pi}{3} R^3. \quad (10)$$

The wavefunction of the first excited state will be considered as the first excited state of an isotropic oscillator related to frequency given by (6) and, due to our symmetry, it has three possibilities, all with the same mathematical structure, namely,

$$\psi^{(i)}_1(\vec{r}) = \frac{8}{\sqrt{27\pi}} \sqrt{\frac{N a}{\pi V}} x^{(i)} \exp\left[-\frac{x^2}{2R^2}\right]. \quad (11)$$

Here $x^{(1)} = x$, $x^{(2)} = y$, and $x^{(3)} = z$.

Of course, (11) must be related to the total number of particles in excited states ($\mathcal{N}_{(e)} = \frac{8}{3}\sqrt{\frac{Na}{\pi V}}$), a condition that becomes [15]

$$\mathcal{N}_{(e)} = \int \left[\sum_{i=1}^{3} (\psi_1^{(i)}(\vec{r}))^2\right] d^3r. \quad (12)$$

Having stated our assumptions we proceed to compute the speed of sound and the critical velocity. The energy of the ground state in its three possibilities, i.e., kinetic, due to the trap, and interaction are [15, 17]

$$\frac{\hbar^2}{2m} \int (\nabla \psi_0)(\vec{r})^2 d^3r = \frac{3\hbar^2}{4mR^2} \mathcal{N}_0. \quad (13)$$

$$\int V(\vec{r})\psi_0(\vec{r}) d^3r = \frac{3}{4} m\omega^2 R^2 \mathcal{N}_0. \quad (14)$$

$$\frac{U_0}{2} \int (\psi_0(\vec{r}))^4 d^3r = \frac{U_0}{\sqrt{32\pi^3} R^3} \mathcal{N}_0^2. \quad (15)$$

The energy of the ground state (here denoted by $E_0$), no elementary excitations are present, is the sum of the last three expressions. The corresponding pressure ($P_0 = -\partial E_0/\partial V$) is

$$P_0 = \frac{4\pi\hbar^2 N}{mV^{5/3}} \left\{ \frac{1}{8\pi} \frac{4\pi}{3} \frac{2/3}{3} \left[1 + \frac{3}{4} \frac{\mathcal{N}_0}{\mathcal{N}_0} \right] \right\}$$
$$+ \left[ \frac{1}{18\pi} (1 - 2\mathcal{N}_0)(1 - \mathcal{N}_0) \right]$$
$$+ \frac{\sqrt{2\pi}}{4\pi} \left(1 - \frac{3}{2} \frac{\mathcal{N}_0}{\mathcal{N}_0} \right) \mathcal{N}_0 \frac{\mathcal{N}_0}{V^{1/3}}. \quad (16)$$
If \( v_{(0)} \) denotes the speed of sound related to the last expression \((v_{(0)}^2 = -(V^2/mN)\partial P_{(0)}/\partial V)\) we are led to
\[
v_{(0)}^2 = \frac{\hbar^2}{m^2} \frac{4\pi N a}{V} \left\{ \sqrt{\frac{2}{9\pi}} \left[ 1 - \frac{15}{8} N_{(0)} + 4N_{(e)}^2 \right] + \frac{1}{2} \sqrt{2} \left[ 1 - \frac{15}{8} N_{(e)} \right] + \frac{5}{2\pi} \left( \frac{4\pi}{3} \right)^{2/3} V^{1/3} \frac{2}{Na} (1 + \frac{39}{40} N_{(e)}) \right\}. \tag{17}
\]

For the case of sodium [13, 14] the physical parameters are: \( m = 36.8 \times 10^{-27} kg, a = 2.75 \times 10^{-9} m, \) \( N = 5 \times 10^6 \). Considering one of the reported peak densities of the condensate, namely, \( 1 \times 10^{14} cm^{-3} \), we obtain, in the roughest approximation from our calculations, 6.59 mm/s. Clearly, our prediction is in good agreement with the experimental result [13].

We now proceed to compute the lowest energy and momentum of the elementary excitations, physical parameters required for the deduction of the critical velocity [3]. The deduction of the energy of an elementary excitation and of its corresponding momentum requires the knowledge of the energy of a single–particle in the first excited state [7]. Our assumptions entail that the thermal cloud contains particles subject to an isotropic harmonic oscillator whose frequency is (6) therefore the energy of an excited particle is given by this assumption and easily calculated as a function of the effective frequency of our variational procedure
\[
\tilde{\epsilon} = \frac{5}{2} \tilde{\omega}. \tag{18}
\]

According to Bogoliubov [7, 18] the energy of an elementary excitation, here denoted by \( \epsilon \), is a function of the energy of the excited particles of the BEC, namely,
\[
\epsilon = \sum \sqrt{\tilde{\epsilon}^2 + \frac{2NU_{(0)}}{V} \tilde{\epsilon}}. \tag{19}
\]

The energy of all the elementary excitations turns out to be [7, 18]
\[
\tilde{E} = \sum \sqrt{\tilde{\epsilon}^2 + \frac{2NU_{(0)}}{V} \tilde{\epsilon}} < \tilde{n}_e > . \tag{20}
\]

Here \( < \tilde{n}_e > \) denotes the occupation number of the elementary excitations with energy \( \epsilon \). The relation between the occupation numbers of particles and elementary excitations is [7]
\[
< \tilde{n}_e > = \frac{< n_e >}{1 + < n_e >} . \tag{21}
\]

At this point, for the sake of simplicity, we resort to the experimental values related to the detection of a critical velocity in a sodium condensed gas [11] in which the occupation number of the particles in the first excited state fulfill the condition \( N_{(e)} \sim 10^2 > 1 \), and, in consequence, \( < \tilde{n}_{(1)} > = 1 \). Our assumptions imply \( < \tilde{n}_{(i)} > = 0, \quad \forall i > 1 \). Indeed, we have considered that the thermal cloud is comprised by particles which occupy only the first excited state, in other words, \( < \tilde{n}_{(i)} > = 0, \quad \forall i > 1 \). Introducing this condition into (21) leads us to the aforementioned result for the occupation number of the elementary excitations.

Casting (19) in terms of the effective volume \( V = 4\pi R^3/3 \), and using (4), (5), and (6), we have that the energy of our elementary excitation is
\[
\epsilon = \left( \frac{4\pi}{3} \right)^{1/3} \frac{\hbar^2}{mV^{2/3}} \sqrt{\frac{25}{4} \left( \frac{4\pi}{3} \right)^{2/3} + \frac{20\pi Na}{V^{1/3}}}. \tag{22}
\]

We must now find the momentum of this elementary excitation. Elementary excitations, which define the normal component of the fluid fluid, can be regarded as a bosonic gas whose components are weakly–interacting and moving in a region in which a constant potential exists, and this potential is defined by a mean field approach [7]. According to this interpretation we may rewrite (22) in the same form as in the case in which our BEC is a homogeneous one [7]. In other words, we cast our last expression in the following form
\[
\epsilon = \frac{\hbar^2}{2m} k \sqrt{k^2 + \frac{16\pi Na}{V}}. \tag{23}
\]

Clearly, (23) allows us to deduce the wavenumber related to our elementary excitation and, in consequence, its momentum. Indeed, we have for these two physical variables, respectively, that
\[
k = \left( \frac{4\pi}{3} \right)^{1/3} \sqrt{\frac{1}{5} \frac{1}{V^{1/3}}}, \tag{24}
\]
\[
p = \left( \frac{4\pi}{3} \right)^{1/3} \frac{\hbar}{V^{1/3}}. \tag{25}
\]

Resorting to Landau criterion (1) we obtain that the critical velocity is given by
\[
v_{(crit)} = \frac{1}{\sqrt{5}} \frac{\hbar}{mV^{1/3}} \sqrt{\frac{25}{4} \left( \frac{4\pi}{3} \right)^{2/3} + \frac{20\pi Na}{V^{1/3}}}. \tag{26}
\]

The experimental parameters [11] are a critical speed of \( v_{(crit)} = 1.6 \text{ mm/s} \). In addition, the number of particles in this experiment has a minimum of \( N = 3 \times 10^6 \) and a maximum of \( N = 12 \times 10^6 \), and for the evaluation of our expression we will take the arithmetic average, i.e., \( N = 7.5 \times 10^6 \). The effective volume is that of an ellipsoid whose axes are \( l_1 = 45 \times 10^{-6} m \) and \( l_1 = 150 \times 10^{-6} m \) such that \( V = 7.5 \times 10^6 \).
Introducing these values into (26) entails

\[ v^{(m)} = 1.95 \text{ mm/s}. \]  

(27)

The reported critical speed is [11]

\[ v^{(e)} = 1.6 \text{ mm/s}. \]  

(28)

The ensuing error is less that 18 percent

\[ \frac{|v^{(e)} - v^{(m)}|}{v^{(m)}} = 0.179. \]  

(29)

Since the number of particles in the corresponding experiment varies from \( N = 3 \times 10^6 \) to \( N = 12 \times 10^6 \) [11] the associated values for the critical speed go from 1.24 mm/s to 2.48 mm/s. If \( N = 5 \times 10^6 \), then \( v^{(m)} = v^{(e)} \).

In conclusion, we have put forward a theoretical model for the deduction of the critical velocity in a sodium–condensed gas. This threshold speed has been computed and compared against the extant experimental results, having a good agreement between them. A more precise evaluation of the present idea requires a better knowledge of the value of \( N \) employed in the experiment. Previous works offer larger critical velocities which have a bigger error than the one here deduced [11, 19], when compared to the experimental result.

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