Free Energy of an $SU(2)$ Model of (2+1)-dimensional QCD in the Constant Condensate Background

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Abstract

Gluon and quark contributions to the thermodynamic potential (free energy) of a (2+1)-dimensional QCD model at finite temperature in the background of a constant homogeneous chromomagnetic field $H$ combined with $A_0$ condensate is calculated. The role of the tachyonic mode in the gluon energy spectrum is discussed. A possibility of the free energy global minimum generation at nonzero values of $H$ and $A_0$ condensates is investigated.

Introduction

Quantum field theories in low dimensional space-time have recently excited a considerable interest [1, 2, 3] due to a close relation with their (3+1)-dimensional analogues [4, 5, 6], as well as to a possibility of explaining certain natural phenomena with their help. For example, the method of dimensional reduction [7] allows one to study the QCD$_{3+1}$ quark-gluon plasma in the high temperature limit above the critical temperature of the confinement–deconfinement phase transition. As another example, we mention the two-dimensional model of electrons interacting with external electromagnetic fields that was employed in the explanation of the quantum Hall effect [8, 9].

In the last years, much attention has been paid to investigation of the effective potential in the background field both at zero and at high temperatures in the one loop approximation [10, 11, 12, 13, 14, 15], and also with account of higher loop contributions [16, 17, 18, 19, 20]. The assumed $A_0$-condensate formation in QCD (see [10]) leads to important physical consequences, such as spontaneous breaking of global gauge symmetry, stabilization of the effective potential, elimination of the infrared divergence etc. New results for the two-loop gauge field effective potential in the presence of an external chromomagnetic field have been obtained recently in [19]. At the same time, a problem of gauge dependence of the results arises [18] in the case of nonzero $A_0$ potential, when higher orders of the perturbation expansion are taken into account.
In the present paper, we study the one-loop gauge field effective potential (thermodynamic potential) in the (2+1)-dimensional QCD model in the presence of a $A_0$ potential and a chromomagnetic field at finite temperatures. In the first section, some basic formulas that describe quantum field systems at finite temperatures are reminded, and the relation between the effective potential and the energy spectrum of one-particle excitations is demonstrated. In the second section, the gluon field contribution to the free energy density is calculated in the (2+1)-dimensional model of QCD. The tachyonic mode in the energy spectrum of gluons is demonstrated to lead to a nonanalytic dependence of the thermodynamic potential on the condensate field strength. A possible way of removing this nonanalytic behavior is presented. A series of nontrivial minima of the free energy, as well as phases of confinement and deconfinement are shown to exist when temperatures are below the critical value. In the third section, the quark contribution to free energy density is obtained. Calculations show that, in contrast to the gluon sector, its minimum exists in the absence of the condensate fields.

1 Energy spectrum of excitations and free energy

Consider a QCD$_{2+1}$ model with a gluon gauge field $A^a_{\mu}$ in the adjoint representation of the $SU(2)$ color group. It can be written as the sum $A^a_{\mu} = \overline{A}^a_{\mu} + a^a_{\mu}$, where $\overline{A}^a_{\mu}$ is the background potential and $a^a_{\mu}$ are the gluon quantum fluctuations. The gauge field Lagrangian in the Lorentz gauge in the Euclidean space-time has the form

$$L_g = \frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D^a_{\mu}A^b_{\mu})^2 + \chi^a(D^2)^{ab}\chi^b,$$

(1)

where $D^a_{\mu} = \delta^{ab}\partial_{\mu} - gf^{abc}\overline{A}^c_{\mu}$ is the covariant derivative, $\chi, \overline{\chi}$ are ghost fields, and $(D^2)^{ab} = D^a_{\mu}D^b_{\mu}$. In passing to the Euclidean space-time, we retained the subscript 0 to denote those components of vectors that are time-like, so that $x_0 = it$ and $\mu, \nu = 0, 1, 2$. The following representation of the Clifford algebra can be chosen in the 3-dimensional space-time:

$$\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2,$$

(2)

where $\sigma^1, \sigma^2, \sigma^3$ are Pauli matrices. Then the Lagrangian for quarks with $N_f$ flavors takes the form

$$L_q = \sum_{j=1}^{N_f} \overline{\psi}_j \left[ \gamma_{\mu}(\partial_{\mu} - ig\lambda^a A^a_{\mu}) + m_j \right] \psi_j.$$

(3)
Here $\lambda_a$ matrices are the color group generators. The generating functional
\[
Z[A, j, \eta, \bar{\eta}] = \int da^a \d \bar{\chi} d\chi d\bar{\psi} \psi \exp \left[ - \int d^3x (L + j^a_{\mu} a^a_{\mu} + \bar{\psi} \eta + \bar{\eta} \psi) \right], 
\]
where $L = L_g + L_q$ is the QCD Lagrangian, can be calculated in the one loop approximation. To this end, it is sufficient to retain in $L_g$ only the terms quadratic in gluon fluctuations $a^a_{\mu}$. Namely,
\[
L_g = L^{(0)}_g + L^{(2)}_g, 
\]
\[
L^{(2)}_g = - \frac{1}{2} a^a_{\mu} \left[ (D^2)^{ab}_{\mu \nu} \delta_{\mu \nu} + 2g F^{c}_{\mu \nu} f^{abc} a^b_{\nu} \right] a^a_{\nu},
\]
where $F^{a}_{\mu \nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + gf^{abc} A^{b}_{\mu} A^{c}_{\nu}$. In this case the path integrals in (4) take the Gaussian form and can be easily calculated:
\[
Z[A] = \exp \left\{ - \frac{1}{4} \int d^3x (F^{a}_{\mu \nu})^2 \right\} \left[ \det (-D^2) \delta_{\mu \nu} - 2g F^{c}_{\mu \nu} f^{abc} \right]^{-\frac{1}{2}} \times \det (-D^2) \prod_{j=1}^{N_f} \det \left[ \gamma_{\mu} (\partial_{\mu} - ig \lambda_a \frac{1}{2} A^{a}_{\mu}) + m_j \right].
\]
Using $Z = \exp(W_E)$, one obtains the effective Euclidean action
\[
W_E = - \frac{1}{2} \int \frac{dq_0}{2\pi} \sum_r \ln (q_0^2 + (\varepsilon_r^g)^2) + \sum_{j=1}^{N_f} \int \frac{dp_0}{2\pi} \sum_k \ln (p_0^2 + (\varepsilon_{jk}^q)^2). 
\]
We have taken into account here that contributions of the gauge field longitudinal components and those of the ghost fields cancel each other out. Summation over the quantum numbers $r$ and $k$ of the gluon $\varepsilon_r^g$ and quark $\varepsilon_{jk}^q$ energy spectra should include only physical degrees of freedom. Formally, we will consider our field system in a three-dimensional cube with volume $L^3$ and periodic boundary conditions. In this case, the effective potential $V$ is determined as follows:
\[
V = - \frac{W_E}{L^3}. 
\]
The effective potential $V$ in the one-loop approximation can be represented as the sum of two terms: $V = V^{(0)} + v$, where $V^{(0)} = (F^{a}_{\mu \nu})^2/4$ is the energy density of the background field, $v = v^g + v^q$ is the sum of the one-loop contributions of gluon and quark fluctuations. The limit of an infinite space-time volume is obtained, when $L \to \infty$. Since the explicit expression for the effective potential
does not depend on $L$, this limiting procedure will not be mentioned in what follows.

The temperature $T \equiv 1/\beta$ is introduced by imposing periodicity and antiperiodicity conditions on the boson and the fermion fields respectively in variable $x_0$ with the period equal to $\beta$. According to the well known prescriptions, this is achieved by introducing the Matsubara frequencies instead of $q_0, p_0$ and making the following substitutions: $q_0 \to 2\pi l/\beta - i\varkappa_1$ for bosons and $p_0 \to 2\pi (l + \frac{1}{2})/\beta - i\varkappa_2$ for fermions, where $\mu_1, \mu_2$ are the gluon and quark chemical potentials respectively, $\varkappa = \pm 1$ determines the particles and antiparticles charge signs, $l \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers). The one-loop effective potential, determined in (8), is proportional to the free energy density

$$V = \frac{-T \ln(Z)}{L^2} = \frac{\Omega}{L^2}. \quad (9)$$

Upon the above substitution, the quark and gluon contributions to the one loop effective potential take the form

$$v = v^g + v^q = \frac{1}{2\beta L^2} \sum_{l=-\infty}^{+\infty} \sum_{r, \varkappa = \pm 1} \ln \left[ \left( \frac{2\pi l}{\beta} - i\varkappa_1 \right)^2 + (\varepsilon^g_r)^2 \right] -$$

$$- \frac{1}{\beta L^2} \sum_{j=1}^{N_t} \sum_{l=-\infty}^{+\infty} \sum_{k, \varkappa = \pm 1} \ln \left[ \left( \frac{2\pi (l + 1/2)}{\beta} - i\varkappa_2 \right)^2 + (\varepsilon^q_{jk})^2 \right]. \quad (10)$$

We introduce the background field as a superposition of the constant homogeneous chromomagnetic field $H$, directed along the third coordinate axis, and the potential $A_0$, both pointing in the third direction of the group space. Namely,

$$\overline{A}_\mu^a = \delta_{\mu 2} \delta_{a3} H x_1 + \delta_{\mu 0} \delta_{a3} A_0 = \delta_{a3} \overline{A}_\mu. \quad (11)$$

In order to account for the $A_0$-condensate, it is sufficient to perform the substitutions $i\mu_1 \to gA_0$ and $i\mu_2 \to gA_0/2$.

In the (2+1)-dimensional space-time, fermions are expressed in terms of two-component spinors (see (2)), and thus they have no spin degrees of freedom. Moreover, the Lorentz gauge fixing conditions $a_0^a = 0$, $D^i a_i^b = 0$ lead to linear interdependence of the positive- and negative-frequency solutions of the Lagrange equations

$$\left[ (D^2)^{ab} \delta_{\mu \nu} + 2g F_{\mu \nu}^c f^{abc} \right] a_\nu^b = 0, \quad (12)$$

derived from (3). In fact, an arbitrary solution of (12) can be represented as an eigenvector-series expansion:

$$a^\pm = a_1 \pm i a_2 = \sum_n N_n^\pm f_n(x), \quad (13)$$
where $N_n$ are constant coefficients, $f_n(x)$ are eigenvectors of equation (12). The gauge fixing condition $D^-a^- + D^+a^+ = 0$ can be shown to lead to the following restrictions, imposed on the coefficients (see [21]): \( N_n^- = 2(n + 2)N_{n+2}^+ \), \( n = 0, 1, 2, \ldots \); \( N_1^+ = 0 \). No restrictions are imposed on the tachyonic mode coefficient \( N_0^+ \). Thus, by setting chemical potentials $\mu_1$ and $\mu_2$ equal to zero, we obtain the energy spectra of quark and gluon one-particle excitations in the chromomagnetic field

\[
\varepsilon^2_g = 2gH(n - \frac{1}{2}) - i\epsilon, \ n = 0, 2, 3, 4, \ldots ; \tag{14}
\]

\[
\varepsilon^2_q = gHn + m^2 - i\epsilon, \ n = 0, 1, 2, 3, \ldots, \ \epsilon > 0. \tag{15}
\]

Infinitely small negative imaginary term $-i\epsilon$ gives a prescription for handling the poles, as well as allows to determine a correct limit for the tachyonic mode at $T \to 0$ (see (22)).

2 Gluon contribution to the free energy density

In order to obtain an explicit expression for the effective potential, one needs to substitute (14) and (15) in (10), with account for the Landau levels degeneracy in a homogeneous magnetic field. In the case of the (2+1)-dimensional space-time this degeneracy is equal to $gHL^2/(2\pi)$. Then the contribution of the charged gluon loop $v^g$ to the thermodynamic potential $v$ can be represented in the form

\[
v^g = \frac{gH}{2\pi\beta} \sum_{l=-\infty}^{+\infty} \left\{ \ln \left[ \left( \frac{2\pi l}{\beta} + gA_0 \right)^2 - gH - i\epsilon \right] + \sum_{n=2}^{+\infty} \ln \left[ \left( \frac{2\pi l}{\beta} + gA_0 \right)^2 + 2gH(n - \frac{1}{2}) - i\epsilon \right] \right\}, \tag{16}
\]

where the first term corresponds to the contribution of the tachyonic mode (whose energy squared is negative). Limits of summation over $l$ are infinite, and hence, expression (13) is periodic in $gA_0$ with the period equal to $2\pi/\beta$. This fact is in close relation with the gauge invariance property. When $T = 0$, one can perform the following transformation:

\[
A'_0 = U A_0 U^+ + \frac{i}{g} U \partial_0 U^+, \quad A'_i = U A_i U^+, \quad \text{where} \quad U = \exp \left( -igx_0 A_0^a \frac{\lambda^a}{2} \right). \tag{17}
\]

Then $A'_0 = 0$, and so $A_0$ is not a physical parameter. When $T \neq 0$, the boundary conditions $A_\mu(x_0, 0) = A_\mu(x_0 + \beta, 0)$ lead to subsidiary conditions $[U, \lambda^a] = 0$ for all $\lambda^a$. By definition, it means that $U$ lies in the center of the gauge group.
Therefore, in the case of the $SU(2)$ group, only those gauge transformations are allowed that preserve the $Z_2$ symmetry:

$$A_0 \rightarrow A'_0 = A_0 + \frac{2\pi n}{\beta g}, \quad \text{where } n \in \mathbb{Z}. \quad (18)$$

The domain of gauge nonequivalent values of potential $A_0$ is given by $gA_0 \in [0, 2\pi T)$.

It follows from (16) that $v^g$ is real, when the argument of the first logarithm in (16) is positive, i.e. under the condition

$$\sqrt{gH} < gA_0 < 2\pi T - \sqrt{gH}, \quad (19)$$

otherwise the vacuum is unstable. As calculations demonstrate, the one-loop effective potential in the (3+1)-case has a finite nontrivial minimum, which however proves to be unstable [10, 11]. According to [19], consideration for higher loops (ring diagrams) also leads to appearance of an unstable minimum in the limit of high temperatures for fields of the order of $(gH)^{1/2} \sim g^{4/3}T$, which in the case of small $\alpha_s$ exceeds the one-loop estimates for the field. Returning to our general formula (16), we see, that under the condition $gA_0 = \sqrt{gH}$, the free energy $v^g$ becomes negative infinite. Such singularity does not arise in the (3+1)-dimensional space, because of the smoothing effect of integration over the momentum third component, absent in the (2+1)-dimensional case.

In our case, the divergence can be eliminated by accounting for radiative corrections, i.e. due to a nonvanishing imaginary part of the gluon polarization operator (PO). We are interested in the energy radiative shift for the tachyonic mode only, as it is responsible for the singular behavior of the effective potential. In order to make qualitative estimates without any detailed calculations, we write down the gluon PO in the form: $\Pi(\varepsilon, T) = \alpha_s \varepsilon (\Pi_1 + i\Pi_2)$, where $\Pi_1$ and $\Pi_2$ are some functions of the field and temperature, whose explicit form has no essential influence on the further considerations and qualitative results. Note, that $\Pi_2$ accounts for the gluon decay. Contribution of the PO is of principle significance only in the vicinity of the effective potential singularity. Consider the dispersion equation $\varepsilon^2 = \varepsilon^2_g + \langle \Pi(\varepsilon, T) \rangle$, and make a reasonable assumption, justified by further numerical calculations, that the behavior of the free energy near the singular point only weakly depends on the values of the functions $\Pi_1 \sim \Pi_2 \sim 1$. Then, the energy squared of the tachyonic mode, for $\alpha_s < (gH)^{1/2}$, with account for the radiative corrections and for the equality $\varepsilon^2_g = -gH$, can be approximately found to be:

$$\varepsilon^2_{tach} \simeq -gH - iC\alpha_s \sqrt{gH}, \quad (20)$$
where $C$ is a coefficient of the order unity. As it is seen, the nonvanishing imaginary part of $\varepsilon_{\text{tach}}^2$ guarantees the nonzero argument of the first logarithm in (16) at $gH \neq 0$, though its imaginary part is also nonvanishing.

Applying the known identity [22]

$$\prod_l \left[ 1 + \left( \frac{x}{2\pi l - a} \right)^2 \right] = \frac{\text{ch}(x) - \cos(a)}{1 - \cos(a)}$$

(21)
to (16) and discarding an irrelevant additive constant, similarly to [11], one obtains

$$v_{\text{tach}} = \frac{gH}{2\pi} \left\{ \omega_0 + \frac{1}{\beta} \ln \left[ 1 + e^{-2\beta\omega_0} - 2 e^{-\beta\omega_0} \cos(\beta gA_0) \right] \right\} =$$

$$= \frac{gH}{2\pi\beta} \ln \left\{ 2 \cos \left[ \beta(\sqrt{gH} + iC\alpha_s) \right] - 2 \cos(\beta gA_0) \right\},$$

(22)

$$v^g = v_{\text{tach}} + \frac{gH}{2\pi} \sum_{n=2}^{\infty} \left\{ \omega_n + \frac{1}{\beta} \ln \left[ 1 + e^{-2\beta\omega_n} - 2 e^{-\beta\omega_n} \cos(\beta gA_0) \right] \right\},$$

(23)

where

$$\omega_0 = -i\sqrt{gH} + \frac{1}{2} C\alpha_s, \quad \omega_n^2 = 2gH(n - \frac{1}{2}).$$

(24)

An expression for $v^g$ can be received with the help of the Fock-Schwinger method. It is useful in studying the effective potential in the region of small $gH$.

As the first step, recall the standard integral representation

$$\ln A = - \int_0^\infty \frac{ds}{s} \exp(-sA),$$

(25)

valid up to an additive infinite constant, and apply it to the second term in (16):

$$v^g = v_{\text{tach}} - \frac{gH}{2\pi\beta} \sum_{n=2}^{\infty} \sum_{l=-\infty}^{+\infty} \int_0^\infty \frac{ds}{s} \exp \left\{ -s \left[ \left( \frac{2\pi l}{\beta} + gA_0 \right)^2 + 2gH(n - \frac{1}{2}) \right] \right\}. \quad (26)$$

Then, let us separate the function of temperature $v^g_T$ in the effective potential $v^g = v^g_T + v^g_{T=0}$ (the zero temperature part $v^g_{T=0}$ is independent of $A_0$ and will be considered below, see (35)). To this end, we transform the summation over the Matsubara frequencies in (26) with the help of the following identity (see [11] or [22]):

$$\sum_{l=-\infty}^{+\infty} \exp \left[ -s \left( \frac{2\pi l}{\beta} + gA_0 \right)^2 \right] = \frac{\beta}{2\sqrt{\pi s}} \sum_{l=-\infty}^{+\infty} \exp(-\frac{\beta^2l^2}{4s}) \cos(\beta gA_0 l).$$

(27)
After performing summation over \( n \) in (26) we obtain the final result:

\[
v_{T}^{g} = v_{T}^{\text{tach}} - \frac{g H}{2\pi^{3/2}} \int_{0}^{\infty} \frac{ds}{s^{3/2}} \left[ \frac{1}{2 \cdot s \cdot \text{sh}(s g H)} - e^{-s g H} \right] \sum_{l=1}^{\infty} \exp\left( -\frac{\beta^{2} l^{2}}{4s} \right) \cos(\beta g A_{0} l). \tag{28}\n\]

Numerical estimations confirm that the above expression (28) coincides with the temperature part of (23). The advantage of expression (28) is that it has the evident limit at \( g H \to 0 \):

\[
v_{T,H=0}^{g} = -\frac{1}{\pi \beta^{3}} \sum_{l=1}^{\infty} \frac{\cos(\beta g A_{0} l)}{l^{3}}. \tag{29}\n\]

In the high temperature limit, \( T \gg g A_{0} \), the following approximate expression, demonstrating the nonanalytical dependence of the effective potential on the background field, can be obtained:

\[
v_{T,H=0}^{g} \approx -\frac{1}{\pi \beta^{3}} \left\{ \zeta(3) + \frac{(\beta g A_{0})^{2}}{2} \left[ \ln(\beta g A_{0}) - \frac{3}{2} \right] \right\}. \tag{30}\n\]

By setting \( A_{0} = 0 \) in (29) one obtains twice the effective potential of the uncharged gluons (\( v_{g}^{g} \) contains the contribution of gluons of two opposite color charges)

\[
2 v_{g}^{0} = -\frac{\zeta(3) T^{3}}{\pi}. \tag{31}\n\]

As has to be expected, we have obtained an analogue of the Planck law for the black body radiation in the (2+1)-dimensional space-time. The total free energy density can be obtained by adding \( v_{g}^{0} \) to (23). However, \( v_{g}^{0} \) is the function of temperature only, and is independent of the condensate field, and, hence, it is of no interest for us.

In order to obtain numerical estimates of the results, it is convenient to pass to dimensionless variables

\[
x = \beta \sqrt{g H}, \quad y = \beta g A_{0}. \tag{32}\n\]

At \( T = 0 \), this substitution is evidently incorrect. Nevertheless, in order to make our notations universal, we will employ \( x \) and \( y \) even at zero temperature, assuming that in this case \( \beta \) takes on a certain finite numerical value in the definition (32). Let us add a constant to \( v \) such that the dimensionless effective potential \( u \) vanishes when the condensate field goes to zero:

\[
u(x, y, T) = v(H, A_{0}, T) \frac{T^{3}}{T^{3}}, \quad u(0, 0) = 0. \tag{33}\n\]
We have to find the minimum of the real part of the function
\[ U(x, y, T) = U^{(0)}(x, T) + u(x, y, T), \quad U^{(0)}(x, T) = \frac{x^4 T}{2 g^2}, \] (34)
where the quantity \( T/g^2 \) provides the scale of the temperature with respect to the coupling constant \( g \). Along with the notations \( u^g \) and \( u^q \) for the gluon and quark contributions to the dimensionless effective potential \( u \), let us also introduce the following one:
\[ U^g = U^{(0)} + u^g. \]

Assuming \( C = 1 \), the zero temperature contribution (\( \beta \to \infty \)) can be obtained from (23):
\[ v^g_{T=0} = \frac{gH}{2\pi} \sum_{n=0,2}^{\infty} \omega_n = -\frac{(gH)^{3/2}}{2\pi} \left[ 1 - \frac{\sqrt{2} - 1}{4\pi} \zeta(\frac{3}{2}) \right] - i \frac{(gH)^{3/2}}{2\pi} + \frac{gH}{4\pi} \alpha_s. \] (35)

The real part of this expression coincides with the result obtained in [21] (see also [23]). The contribution, provided by the gluon tachyonic mode in the effective potential, was not obtained in [21]. Global minimum of \( U^g \) (Fig. 1) is achieved at \( \sqrt{gH} \approx 0.185g^2 \). There is also a local minimum of \( U^g_{T=0} = U^{(0)} + u^g_{T=0} \).

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The author of [21] claims, that the condensate evaporates, when the temperature is above some critical value \( T > T_{cr} \). The gluon tachyonic mode was considered unphysical and omitted in that paper. However, \( v^{tach} \) gives nontrivial contributions both to the imaginary and real parts of the free energy density at nonzero temperature, and hence, it has to be taken into account. Moreover, it is the tachyonic mode that can generate a minimum of \( v^g \) at \( A_0 \neq 0 \) [11].

Examination of the real part of \( U^g(x, y = \text{const}) \) as a function of \( x \) reveals a nontrivial minimum at \( x = x_{\text{min}} \) that exists at the temperature \( T \) lower than a certain critical value \( T_{cr} : T < T_{cr} \approx 0.15g^2 \), when the temperature part \( u^g_{T} \) is small with respect to the zero temperature part \( u^g_{T=0} \). As an artefact of the normalization chosen, the temperature contribution becomes weakly dependent on temperature, while the zero temperature part becomes a function of temperature. In this way, oscillating contribution \( u^g_{T} \) (Fig. 2) modulates the zero-temperature potential \( U^g_{T=0} \), possessing a nontrivial minimum (Fig. 1). The value of \( x_{\text{min}} \) is close to discrete points \( n\pi, n \in \mathbb{N} \) and increases with growing temperature. Moreover, a sequence of second-order phase transitions between phases with minima of \( U^g(x_{\text{min}}, y) \) either at \( y_{\text{min}} = \pi \) or at \( y_{\text{min}} = 0 \) takes place. When temperature tends to zero, there appears an infinite number of those phases. For
example, at \( T/g^2 = 0.1 \), a global minimum of the effective potential occurs for \( x = 3.03, \ y = \pi \) (Figs. 3, 4). However, presence of an imaginary part of \( v^g \) means instability of this condensate configuration, at least at the one-loop level (cf. [19]). When the temperature increases above \( T_{cr} \), the condensate values \( y_{min} \) and \( x_{min} \) decrease from \( \pi \) to 0, keeping correlation \( x_{min} \approx y_{min} \) unchanged.

As is well known, the system is in the confinement phase, if the trace of the Polyakov loop [24, 25] in the fundamental representation vanishes, \( \text{Tr}_F(\mathcal{P}) = 0 \). The Polyakov loop is determined as follows:

\[
\mathcal{P} = \mathcal{T} \exp \left[ i \int_0^\beta dt A_0^a \lambda_a^2/2 \right]. \tag{36}
\]

In our case (11) potential \( A_0 \) is directed along the third axis in the color space. Hence, \( \text{Tr}_F(\mathcal{P}) = 2 \cos(\beta g A_0/2) \). It is evident, that the condition \( \text{Tr}_F(\mathcal{P}) = 0 \) is satisfied, when \( \beta g A_0 = \pi \). Thus, the minima of the effective potential at temperatures below the critical value corresponds to the confinement and deconfinement phases.

### 3 Quark contribution to the free energy density

Let us consider the quark contribution to the free energy density in the same manner, as it has been done for gluons in section 2. The quark energy levels degeneracy in the external chromomagnetic field is proportional to the quark color charge, \( \pm 1/2 \), and is equal to \( gH L^2/(4\pi) \), which is one half that of the gluon case. Upon substitution (13) in (11), the expression for the quark and antiquark effective potential is obtained:

\[
N_f^{-1} v^q = -\frac{gH}{4\pi^2} \sum_{l=-\infty}^{+\infty} \sum_{n=0}^{\infty} \sum_{\lambda=\pm1} \ln \left\{ \left( \frac{\pi(2l+1)}{\beta} + \frac{\lambda}{2} g A_0 \right)^2 + gH n + m^2 \right\}. \tag{37}
\]

Here \( \lambda = \pm 1 \) corresponds to the quark color charge values. The essential difference from the nonabelian gauge field contribution in the chromomagnetic background is the absence of the tachyonic mode in the quark energy spectrum. Therefore, \( v^q \) is a well defined function of the condensate field in the whole domain \( 0 < gH < \infty, \ 0 < g A_0 < \infty \):

\[
N_f^{-1} v^q = -\frac{gH}{2\pi} \sum_{n=0}^{\infty} \left\{ \omega_n + \frac{1}{\beta} \ln \left[ 1 + e^{-2\beta \omega_n} + 2 e^{-\beta \omega_n} \cos\left( \frac{\beta g A_0}{2} \right) \right] \right\}, \tag{38}
\]

where

\[
\omega_n^2 = gH n + m^2. \tag{39}
\]
The quark field at finite temperature is subject to the fermionic antiperiodicity condition \( \psi(x_0, x) = -\psi(x_0 + \beta, x) \). Therefore, the period \( 4\pi T \) of (38) in variable \( gA_0 \) is twice the period of (23). This implies that the residual gauge \( \mathbb{Z}_2 \) symmetry is violated (\( \mathbb{Z}_N \) in the case of the \( SU(N) \) gauge group). In what follows, we restrict ourselves to consideration of the quark field in the chiral limit, \( m = 0 \). In this case the zero temperature quark contribution is equal to

\[
N_f^{-1}v_{T=0}^q = -\frac{gH}{2\pi} \sum_{n=0}^{\infty} \sqrt{gHn} = \frac{(gH)^{3/2}}{8\pi^2} \zeta\left(\frac{3}{2}\right),
\]

which coincides with the result of [21]. In contrast to the case of the gluon potential \( v_{T=0}^g \), the sign of \( v_{T=0}^q \) is always positive, and this does not allow a nontrivial minimum to be formed. The temperature dependent part of the quark effective potential reads

\[
N_f^{-1}v_{T}^q = -\frac{gH}{2\pi\beta} \sum_{n=0}^{\infty} \ln \left[ 1 + e^{-2\beta\omega_n} + 2e^{-\beta\omega_n}\cos\left(\frac{\beta gA_0}{2}\right) \right].
\]

By using the proper time method one can obtain an alternative representation, which evidently demonstrates continuous behavior of \( v_{T}^q \) at \( gH \to 0 \):

\[
N_f^{-1}v_{T}^q = \frac{gH}{2\pi^{3/2}} \int_{0}^{\infty} \frac{ds}{s^{3/2}} \sum_{l=1}^{\infty} \exp\left( -\frac{\beta^2 l^2}{4s} \right) \frac{\cos(\beta gA_0 l/2 + \pi l)}{1 - e^{-sgH}}.
\]

In the limit \( gH \to 0 \) we obtain

\[
N_f^{-1}v_{T,H=0}^q = \frac{2T^3}{\pi} \sum_{l=1}^{\infty} (-1)^l \frac{\cos(\beta gA_0 l/2)}{l^3}.
\]

The dimensionless effective potential \( u^q(x, y) \), defined in (33), does not depend on temperature \( T \). The family of curves for \( u^q \) with fixed values \( x = \text{const} \) and \( y = \text{const} \) are depicted in figs. 3, 4 respectively. It is evident that for any \( x = \text{const} \) the minimum of \( u^q(x, y) \) is reached at \( y_{\text{min}} = 0 \). Nevertheless, there is a nontrivial minimum at \( y = \text{const} < 3.5 \) for \( x_{\text{min}} \neq 0 \). Global minimum of the gluon effective potential is reached at \( x = 4.30, y = 0 \).

The quark contribution to the free energy is greater than the gluon one at \( y > 4 \), when the quark zero mode plays a dominant role. Therefore, our conclusions about the existence of the field condensate and of the confinement-deconfinement phase transitions at \( 0 \leq y \leq \pi \) remain unchanged even when the total free energy is considered.
Conclusions

Thus, in the present paper, contributions of gluons and quarks to the thermodynamic potential (free energy) in the (2+1)-dimensional space-time are calculated in the one-loop approximation at finite temperature in the background of the superposition of the uniform constant chromomagnetic field $H$ and the $A_0$-condensate. Consideration of the role of the tachyonic mode in the gluon energy spectrum leads us to a conclusion that it cannot be neglected, as opposed to what has been done in [21]. Moreover, accounting for the one-loop radiative correction to the gluon energy spectrum enables a nonanalytic behavior of the effective potential, related to the presence of the zero modes in the energy spectrum, to be cured. The free energy minimum is studied and a possibility of its formation is demonstrated at certain nonzero values of the chromomagnetic field strength $H$ and of the potential $A_0$. The analysis of the temperature dependence of the results demonstrates that below a certain critical value of temperature a number of subdomains in the domain of the parameters exist where the system is either in the confinement or in the deconfinement phases. This behavior is explained by the oscillating contribution of the tachyonic mode to the free energy density. Unfortunately, the imaginary part of the effective potential does not vanish at points $x_{\text{min}} \simeq \pi n, n \in \mathbb{N}$, where $V(x, y)$ reaches its minimum. Hence the nontrivial minimum of the effective potential generated by the condensate fields $A_0$ and $H$ turns out to be unstable. This instability is related to the choice of the homogeneous vacuum field, and it is reasonable to assume, that it may be removed in the realistic situation of a vacuum field inhomogeneous at large distances, where confinement is formed (see also the arguments of [11]). With regard for what has been said, the states found in the present paper may be considered as quasistable. More rigorous justifications for the assumptions made above might be obtained in the study of inhomogeneous vacuum field models (see, e.g. [26]), and, following [19], in elaboration of higher loop contributions.

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Figure 1: $\text{Re} U^g_{T=0}(x) \ T/g^2 = 0.01$.

Figure 2: $\text{Re} U^g_T(x) \ T/g^2 = 0.01$. 
Figure 3: \( \text{Re} U^g(x) \ T/g^2 = 0.1 \).

Figure 4: \( \text{Re} U^g(y) \ T/g^2 = 0.1 \).
Figure 5: Re $u^q(x)$.

Figure 6: Re $u^q(y)$. 