1 Introduction

Every modal interpretation of quantum mechanics has the following distinctive feature:

Given the (pure or mixed) quantum state $W$ of a system with Hilbert space $H$, the interpretation specifies those self-adjoint operators on $H$ which correspond to observables with definite values in state $W$.

We are not asserting that all realist interpretations of quantum mechanics must necessarily do this, nor are we asserting that doing this in itself counts as giving an interpretation. But certainly the central task of modal interpretations is to provide an ontology of the properties of quantum systems that circumvents the measurement problem, without falling prey to the various ‘no-hidden-variables’ theorems. So, to accomplish that task, modal interpretations must tell us which observables of a system we can and should be realists about. Moreover, this must at least involve specifying which of a system’s discrete-valued observables can be said to possess definite values statistically distributed in conformity with the statistics prescribed by the density operator $W$ of the system. Our main aim in this paper is to take a
detailed look at some of the mathematical issues that arise naturally in the context of such a specification.

For continuous-valued observables, the notion of ‘possessing a definite value’ may need to be replaced by something like ‘possessing a value lying in (or restricted to) a definite interval.’ Furthermore, although our mathematical analysis will indeed apply when \( H \) is infinite-dimensional, a few of our results remain sensitive to the difference between discrete- and continuous-spectrum observables on \( H \). Thus our analysis (both conceptual and mathematical) will be complete only with respect to the notion of ‘possessing a definite value’ appropriate to observables with a discrete spectrum. Of course, since modal interpretations have so far only been rigorously developed for such observables, this will not hamper the application of our results to them. But there is clearly more work to be done (for recent progress in this connection, see Clifton [1997]).

If at a certain instant of time the state of a system is \( W \), then we shall denote the set of definite-valued observables of the system by \( D(W) \), or simply \( D \). In purely mathematical discussions of \( D \), we shall take as given that its observables are represented by self-adjoint operators, and we shall refer to \( D \) as the system’s set of definite-valued operators.

For our purposes it will prove useful to ask the following question: a priori, what sort of mathematical structure, if any, is it natural to attribute to \( D \)? Is \( D \) a (real) vector space, in which case real linear combinations of definite-valued operators are necessarily definite-valued? An algebra of some kind, in which case polynomials involving definite-valued operators are definite-valued? Does it matter if the operators in question commute? And finally, is it helpful to view “functional closure” properties like these as normative requirements on possible modal interpretations?

In section 2 we shall define a few of these functional closure properties more carefully, drawing attention to some the mathematical issues that come into play when we prescribe them for \( D \). Such functional closure issues figure prominently, for example, in von Neumann’s [1955] ‘no-hidden-variables’ theorem – where it is assumed that any real linear combination of operators in \( D \) will itself be in \( D \), regardless of whether these operators are compatible.

Apart from making this ‘structural’ requirement, von Neumann’s theorem also contains an assumption about the values possessed by the observables in \( D \); specifically, it assumes that these possessed values must obey the same polynomial relationships as do the corresponding observables – again, regardless of whether these observables are compatible. The received view,
first clearly articulated by Bell [1966], is that the acceptability of the theorem as a ‘no-go’ result is undercut at this point by the lack of attention von Neumann paid to compatibility. Thus Bell argued that in the case of incompatibile observables, it is not reasonable to require of any hidden-variable theory that its value assignments necessarily reflect the observables’ algebraic relationships.

The received view, then, is that von Neumann’s functional requirement for possessed values is so strong that the theorem fails to rule out hidden variables in any convincing way. However, our own diagnosis of what makes the theorem unacceptable will be somewhat different. In fact, in most of what follows, we shall take the bold step of adopting functional requirements that are (in a sense) even stronger than von Neumann’s polynomial ones.

In the first place, we shall require that any self-adjoint function of observables in \( \mathbf{D} \) must itself be in \( \mathbf{D} \) – again, irrespective of the compatibility of the observables. Having adopted this requirement, the latter part of section 2 will be devoted to isolating a simple necessary and sufficient condition on the projection operators in \( \mathbf{D} \) for \( \mathbf{D} \) to be functionally closed in this strong sense. Interestingly, the projection sets specified as definite-valued by a number of proposed modal interpretations all meet this condition; hence we are able to show that according to all of those interpretations, arbitrary functions of definite-valued operators are themselves definite-valued.

Then in section 3 we turn to the issue of the values of the observables in \( \mathbf{D} \). This is where von Neumann’s no-go theorem packs its punch. If, for example, one assumes that \( \mathbf{D} \) is the set of all self-adjoint operators on \( \mathbf{H} \), then it is easy to show, as von Neumann did, that no assignment of values to the observables in \( \mathbf{D} \) can respect their polynomial functional relations. But modal interpretations are not so liberal about what they take \( \mathbf{D} \) to be. Because they take their sets of definite-valued observables to be a certain kind of subalgebra of the set of all self-adjoint operators on \( \mathbf{H} \), we shall show that there do indeed exist valuations on their definite-valued sets \( \mathbf{D} \) which respect polynomial relationships among the observables in \( \mathbf{D} \). Moreover – and here is where we make the second of our strengthened functional requirements – we shall show that even if we require that the valuations respect arbitrary functional relationships among the observables in \( \mathbf{D} \) (again regardless of whether the observables commute), then there are still enough of them to represent the statistics prescribed by quantum mechanics for observables in \( \mathbf{D} \), as measures over the available ‘functional’ valuations. Thus we locate the fault in von Neumann’s theorem, not directly in his assumption that
valuations must always respect this or that type of functional relationship, but rather in his tacit assumption that every self-adjoint operator may be considered a candidate for an element of D.

Section 3 ends with the primary mathematical result of the paper: a simple condition on the projections in a functionally closed set D which is necessary and sufficient for D to support enough functional valuations to represent quantum statistics.

In sections 2 and 3, which form the main part of the paper, a number of mathematical concepts will need to be invoked. Section 2 draws on the theory of von Neumann algebras, and section 3 draws on the lattice-theoretic idea of a quasiBoolean algebra (first introduced in Bell and Clifton [1995]). But our exposition will be self-contained, all of the mathematics needed (most of it well-known) will be introduced en route, and the theorems we prove will be understandable by anyone who has followed our mathematical definitions and terminology (most of it standard).

In section 4 we bring things to a close by amplifying the above remarks on the relevance of our results to von Neumann’s theorem. One point to be made in this respect is that since modal interpretations can recover quantum statistics, they provide an existence proof that all the explicitly stated demands placed by von Neumann on ‘hidden-variable theories’ can be met (save his tacit, and by no means compelling, assumption that every observable has a value). And having thereby circumvented von Neumann’s theorem, modal interpretations also automatically circumvent all ‘no-go’ theorems that attempt to strengthen the case against ‘hidden variables’ by making weaker assumptions than von Neumann did – most notably the theorems of Jauch and Piron [1963] and Kochen and Specker [1967].

2  Functional Closure Properties for Sets of Definite-valued Observables

2.1 Degrees of Functional Closure

Here are four properties that interpreters might consider attributing to the set of definite-valued self-adjoint operators D on a Hilbert space H. (Note that we shall always assume that D contains the identity operator.)

- Compatible polynomial *-closure. We will say that D has compatible
polynomial ∗-closure if, whenever the commuting operators $Q$ and $S$ are in $D$, the operators $aQ + S$ and $QS$ are also in $D$, for all real $a$. (To put it another way, $D$ has compatible polynomial ∗-closure if any real polynomial function of commuting operators in $D$ is also in $D$. In this case one might call $D$ a partial real algebra.)

- Compatible ∗-closure. We will say that $D$ has compatible ∗-closure if, whenever the commuting operators $\{Q_\alpha\}$ are in $D$, any self-adjoint operator that is a (not necessarily polynomial) function of the $Q_\alpha$ is in $D$. (For finite-dimensional $H$, this is equivalent to compatible polynomial ∗-closure. Note that a function is self-adjoint if it maps a set of self-adjoint operators to a self-adjoint operator.)

- Polynomial ∗-closure. We will say that $D$ has polynomial ∗-closure if any self-adjoint polynomial function of operators in $D$ is also in $D$. (In this case one might call $D$ a real algebra.)

- ∗-Closure. We will say that $D$ has ∗-closure if, whenever the operators $\{Q_\alpha\}$ are in $D$, any self-adjoint operator that is a (not necessarily polynomial) function of the $Q_\alpha$ is in $D$.

A brief word on the star in ∗-closure. Generally speaking, we are considering what it means for a set of operators to be closed under functional operations. When we come to spelling out how an arbitrary (not necessarily polynomial or self-adjoint) function of a set of operators is defined (i.e. in the next subsection), it will turn out that the question of the functional closure of a set of operators has everything to do with the question of whether the set is topologically closed, in an appropriate topology. We will need to have a different notation for these two closure concepts in order to discuss their relationship.

There are grounds to think that in any reasonable interpretation, the set of definite-valued observables ought, at least, to have compatible polynomial ∗-closure. The orthodox (Dirac-von Neumann) interpretation, for example, is certainly one in which the set of definite-valued observables has this property. This is because to an orthodox interpreter, if $\{Q_\alpha\}$ is a set of definite-valued operators, then the state vector must be an eigenvector of each $Q_\alpha$ in the set. But in that case, the state vector will clearly also be an eigenvector of any polynomial function of the $Q_\alpha$. Hence according to the orthodox interpretation, any self-adjoint polynomial function of definite-valued operators is itself definite-valued.
To refuse to attribute compatible polynomial $^\ast$-closure to the set of definite valued operators, one would have to believe something like the following: that in some situations a particle could, for example, have a definite value of energy without having a definite value of energy-squared. One way to believe this would be to deny that operators like “energy-squared” represent physical quantities in the first place, though it is not clear what extra insights on the problem that would bring. But in any case, it would seem that in order to dispute the a priori reasonableness of compatible polynomial $^\ast$-closure, one would have to adopt what is in some ways an extremely conservative viewpoint.

On the other end of the spectrum, an extremely liberal interpreter might be unsatisfied with a condition as weak as compatible, polynomial $^\ast$-closure. Such an interpreter might even be willing to entertain the idea that in any reasonable interpretation, the set of definite-valued operators should be nothing less than $^\ast$-closed (e.g., see Clifton [1995a,b]). Perhaps this goes too far. But for those who are tempted to consider $^\ast$-closure to be an outlandish requirement, we shall be showing that a large group of modal interpretations do in fact satisfy it, along with the orthodox interpretation and, of course, the naive realist interpretation (‘every observable has a definite value’).

We shall henceforth be adopting $^\ast$-closure as a requirement on $D$, partly because $^\ast$-closure is compatible with so many proposed interpretations, and partly because the requirement of $^\ast$-closure places a number of useful mathematical tools at our disposal. Using these tools, we shall translate the condition of $^\ast$-closure on $D$ into a simple equivalent condition on the set of projections in $D$. This condition will doubtless prove useful for generating new modal interpretations that, by construction, are manifestly functionally closed. (For a further discussion of the issues raised by various functional closure requirements, see Zimba [1998].)

As outlined in the introduction, another reason for focusing on $^\ast$-closed sets of definite-valued observables is that, by leading us to a class of modal interpretations that easily circumvent von Neumann’s ‘no-hidden-variables’ theorem, they allow us to stress that the difficulty with this theorem does not have to be seen as stemming solely from concerns about the functionality of valuations for incompatible observables.
2.2 Von Neumann Algebras and $^*$-Closure

We begin by summarizing some elementary notions concerning ‘functions of
operators’ which will elucidate the concept of $^*$-closure. We consider only
bounded linear operators on the Hilbert space $H$.

- **Strong limit of a sequence of operators.** Consider a sequence \( \{G_n\} \) of
operators. Suppose that for each vector $x$ there exists a vector $y_x$ such
that

$$
\lim_{n \to \infty} \|G_n x - y_x\| = 0.
$$

Then the map $x \mapsto G_x$ is said to be the strong limit of the sequence
$\{G_n\}$:

$$
\lim_{n \to \infty} G_n = G.
$$

It follows that if the $G$ defined above exists, then it is unique, and
linear if the $G_n$ are. (These facts are easy to prove using the triangle
inequality.)

(There are two other common notions of the limit of a sequence of oper-
ators: a stronger notion, called the uniform limit, and a weaker notion,
called the weak limit. We shall not be explicitly considering either
of these, though in all the cases we are concerned with the weak and
strong limits coincide. For a fuller discussion of some of the conceptual
issues at stake here, see Clifton [1997] and Zimba [1998].)

- **Polynomial function.** A polynomial function of the operators in
$\{Q_\alpha\}$ is a finite linear combination of products of powers of the
$Q_\alpha$, with complex coefficients.

- **Operator-valued function of operators.** An operator $G$ is said to be a
function of the operators in $\{Q_\alpha\}$ if it is the strong limit of a sequence of
polynomial functions of the $Q_\alpha$. (This recalls the approach of ordinary
analysis, in which functions are often defined as infinite series – or, in
other words, as limits of sequences of polynomials.)

In the hope that it will make the mathematics easier to read, we shall use
the following font conventions:
• Calligraphic capital: A set of operators. For example, $\mathcal{B}$.

• Bold-face capital: A set of specifically self-adjoint operators. For example, $\mathbf{D}$.

• Capital: An operator. For example, $Q$.

• Lower-case italics: A complex scalar or vector, depending on context. For example, $a$ or $x$.

More definitions:

• Self-adjoint set. If a set of operators $\mathcal{B}$ contains $Q^\dagger$ whenever it contains $Q$, then it is called a self-adjoint set. (We use $Q^\dagger$ for the adjoint instead of $Q^*$ to avoid confusing a "*-closed set" with a 'self-adjoint set.' Note also the distinction between the phrases “a set of self-adjoint operators” and “a self-adjoint set of operators”!)

• *-algebra. A self-adjoint set $\mathcal{B}$ is called a *-algebra if it contains $aQ + T$ and $QT$, where $a$ is any complex scalar, whenever it contains $Q$ and $T$. (In other words, a self-adjoint set is a *-algebra if it contains all polynomial functions of its members.)

• von Neumann algebra. A *-algebra $\mathcal{A}$ is called a von Neumann algebra if it contains the identity and is closed in the strong operator topology – that is, if strongly convergent sequences of operators in $\mathcal{A}$ converge to operators in $\mathcal{A}$. To put it another way, a *-algebra $\mathcal{A}$ is a von Neumann algebra if it contains the identity and if any function of operators in $\mathcal{A}$ is also in $\mathcal{A}$. (We have required that $\mathcal{A}$ contain the identity in order to simplify our presentation, but this requirement is not part of the standard definition.)

• Commutant. The commutant of a set of operators $\mathcal{B}$ is the set of all operators on $\mathcal{H}$ that commute with all operators in $\mathcal{B}$. We use a prime to denote the commutant:

$$\mathcal{B}' = \{T : TB = BT \text{ for all } B \in \mathcal{B}\}.$$  

It follows that $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{B}' \subseteq \mathcal{A}'$ and that $(\mathcal{A} \cup \mathcal{B})' = \mathcal{A}' \cap \mathcal{B}'$. Furthermore, $\mathcal{B}'$ will be a *-algebra whenever $\mathcal{B}$ is a self-adjoint set.
We write the second commutant \((B')'\) as \(B''\) (So: an operator \(Q\) is in \(B''\) if it commutes with any operator that commutes with every operator in \(B\).) It is then elementary to show that \(B \subseteq B''\) and \(B' = B'''\) for any operator set \(B\).

This last notion of the commutant of a set of operators is especially useful for elucidating \(*\)-closure. Given a set of operators \(B\), ask yourself what kinds of operators \(B''\) contains (apart from those in \(B\) itself). Well, suppose an operator \(T\) commutes with everything in \(B\). Then \(T\) certainly commutes with any polynomial function of operators in \(B\). So any polynomial function of operators in \(B\) commutes with any operator \(T\) that commutes with every operator in \(B\). In other words, any polynomial function of operators in \(B\) is contained in \(B''\). (Note that these polynomial functions need not be self-adjoint.) Hence \(B''\) is an algebra.

What’s more, if \(B\) is a self-adjoint set, then \(B''\) will also be a self-adjoint set. This follows as a result of the fact that self-adjointness of sets is preserved under the operation of taking the commutant. For suppose that \(B\) is a self-adjoint set, and consider any \(T\) in \(B'\). Then for any \(B\) in \(B\), we have \(B^\dagger \in B\), so \([T, B^\dagger] = 0\). Taking adjoints, we have \([T^\dagger, B] = 0\). Since \(B\) was arbitrary, we conclude that \(T^\dagger \in B'\). And since \(T\) was arbitrary, we conclude that \(B'\) is self-adjoint. Hence \(B'\) is self-adjoint whenever \(B\) is, which was to be shown. Together with the fact that \(B''\) is always an algebra, we see that if \(B\) is a self-adjoint set, then \(B''\) will be a \(*\)-algebra.

Summarizing then, for a self-adjoint set \(B\), the set \(B''\) is a \(*\)-algebra generated by \(B\), containing, for example, all polynomial functions of operators in \(B\). What’s more, the following remarkable theorem of von Neumann shows that \(B''\) contains all functions of operators in \(B\):

- **The Double Commutant Theorem** (von Neumann).

  Let \(\mathcal{A}\) be a \(*\)-algebra. Then \(\mathcal{A}\) is a von Neumann algebra (closed in the strong operator topology and containing the identity) if and only if \(\mathcal{A} = \mathcal{A}''\). (Topping [1971])

Since \(B' = B'''\), we have \(B'' = (B'')''\) from which it follows that for a self-adjoint set \(B\), the set \(B''\) is a von Neumann algebra. In fact, \(B''\) is the smallest von Neumann algebra containing \(B\). To see this, suppose that \(\mathcal{A}\) is a von Neumann algebra containing \(\mathcal{B}\); so \(B \subseteq \mathcal{A}\). Then \(\mathcal{A}' \subseteq B'\), whence \(B'' \subseteq \mathcal{A}''\). But since \(\mathcal{A}\) is a von Neumann algebra, we have \(\mathcal{A} = \mathcal{A}''\), and we may
therefore conclude $\mathcal{B}'' \subseteq \mathcal{A}$. Thus any von Neumann algebra containing $\mathcal{B}$ also contains $\mathcal{B}''$, so that $\mathcal{B}''$ is the smallest von Neumann algebra containing $\mathcal{B}$ – which is to say, $\mathcal{B}''$ is the smallest $\ast$-algebra containing $\mathcal{B}$ and strong limits of sequences of operators in $\mathcal{B}$. Hence:

*The von Neumann algebra $\mathcal{B}''$ generated by a self-adjoint set $\mathcal{B}$ is the set of all functions of operators in $\mathcal{B}$.*

Now let’s return to $\ast$-closure. We said that a set of self-adjoint operators $\mathcal{D}$ is $\ast$-closed if it contains all self-adjoint functions of operators in $\mathcal{D}$. So the important difference between $\ast$-closure and topological closure in the strong operator topology is that $\ast$-closure refers only to self-adjoint functions of operators. (Hence the star.) To relate the two notions precisely, we make the following definition:

- **Self-adjoint part.** The self-adjoint part of a $\ast$-algebra $\mathcal{A}$ is the set $S(\mathcal{A}) = \{Q \in \mathcal{A} : Q = Q^\dagger\}$.

With this, we can relate $\ast$-closure to topological closure in the strong operator topology in a way that should now be obvious:

**Theorem 1.** Let $\mathcal{D}$ be a set of self-adjoint operators. Then $\mathcal{D}$ is $\ast$-closed if and only if it is the self-adjoint part of a von Neumann algebra $\mathcal{A}$. (Symbolically, $\mathcal{D} = S(\mathcal{A})$.)

**Proof.** $(\Rightarrow)$ This direction says that if $\mathcal{D}$ is $\ast$-closed, then $\mathcal{D}$ is the self-adjoint part of a von Neumann algebra. To prove this, first recall what has just been said: that the von Neumann algebra $\mathcal{D}''$ is the set of all functions of operators in $\mathcal{D}$. In particular then, the self-adjoint functions of operators in $\mathcal{D}$ are just the operators in $S(\mathcal{D}'')$. If these are assumed to be contained in $\mathcal{D}$, then we must have $\mathcal{D} \supseteq S(\mathcal{D}'')$. And it is obvious that for a set of self-adjoint operators, $S(\mathcal{D}'') \supseteq \mathcal{D}$. So we see that if $\mathcal{D}$ is $\ast$-closed, then $\mathcal{D} = S(\mathcal{D}'')$. In other words, a $\ast$-closed set of self-adjoint operators is the self-adjoint part of the von Neumann algebra it generates.

$(\Leftarrow)$ This direction says that if $\mathcal{A}$ is a von Neumann algebra, then its self-adjoint part $S(\mathcal{A})$ is $\ast$-closed. To prove this, we need to show that strongly convergent sequences of self-adjoint operators in $\mathcal{A}$ converge to self-adjoint operators. To this end, suppose $\mathcal{A}$ is a von Neumann algebra, and consider the set $\mathcal{D} = S(\mathcal{A})$. If $\{Q_n\}$ is a strongly convergent sequence of operators...
in $\mathbf{D}$, then $\{Q_n\}$ is also a strongly convergent sequence of operators in the closed set $\mathcal{A}$; hence $\{Q_n\}$ converges strongly to some $Q \in \mathcal{A}$. We need to show that the limit operator $Q$ is self-adjoint, so that it lies in $\mathbf{D}$.

For any vector $x$, define a sequence of real numbers $\{q_n(x)\}$ by $q_n(x) = \langle x, Q_n x \rangle$. Also, define $q(x)$ to $\langle x, Qx \rangle$. Then, making use of the Schwarz Inequality, we have

$$|q_n(x) - q(x)| = |\langle x, Q_n x \rangle - \langle x, Qx \rangle|$$
$$= |\langle x, (Q_n - Q)x \rangle|$$
$$\leq \|x\| \cdot \|(Q_n - Q)x\|$$
$$\to 0$$

by strong convergence of the sequence $\{Q_n\}$ to $Q$. This shows that $q_n(x) \to q(x)$. But it is elementary to show that if a sequence of complex numbers $q_n$ converges in modulus to a complex number $q$, then the real and imaginary parts of $q_n$ converge separately to the real and imaginary parts of $q$. Since each $q_n$ is real, this means that the limit of our sequence, $q(x) \equiv \langle x, Qx \rangle$, must also be real. Hence the limit operator $Q$ has real expectation values on every vector $x$, from which it follows that $Q$ is self-adjoint. Thus $\mathbf{D}$ is $^\ast$-closed. QED.

### 2.3 Projection Operators and $^\ast$-Closure

With Thm. 1 we have a criterion for deciding when a definite-valued set $\mathbf{D}$ is $^\ast$-closed. But in most of the literature, modal interpretations are defined from the perspective of idempotent observables, i.e. projections. In this approach

1. We specify a set $\mathbf{d}$ of projections with definite values in the state $\mathcal{W}$;

2. We adopt the condition that a self-adjoint operator is in $\mathbf{D}$ if and only if all the spectral projections of the operator are in $\mathbf{d}$.

For now, we want to allow $\mathbf{D}$ to contain observables with continuous spectra. So if a self-adjoint operator has a continuous spectrum, we shall extend standard terminology and take the ‘spectral projections’ of the operator to be the set of all projections of the form $P - Q$, where $PQ = Q$ and both $P$ and $Q$ are in the spectral family of the operator. (Thus the ‘spectral projections’ are the projections associated with the various ranges of values the observable can take up.)
Rule 2 is often tacit in the literature, but it is usually what is intended. In fact, 2 follows from requiring \(*\)-closure of \(D\). For, by the spectral theorem, a self-adjoint operator can be approximated as closely as one likes by an appropriate real linear combination of its spectral projections, and conversely, each such projection is a characteristic function of the operator.

The procedure of specifying \(D\) by specifying the subset \(d\) of its projections and adopting rule 2 is at the heart of what one might call “the projection operator approach to the problem of definiteness.” Using this approach, a number of modal interpretations, along with the naive realist and orthodox interpretations, can be characterized as follows. Let the density matrix for the system be \(W\), with spectral resolution \(X = \{X_i\}\). So \(X_iX_j = \delta_{ij}X_j\), \(\sum X_i = I\), and \(\sum w_i X_i = W\). (And let \(X_0\) denote the projection onto the null space of \(W\), if it has a non-trivial null space.) In the special case where \(W\) is a pure state represented by a unit vector \(\psi\), let \(\{P_{\psi_{R_j}}\}\) be the projection operators associated with the one-dimensional subspaces generated by the (non-zero) components of \(\psi\) that lie in the eigenspaces \(\{R_j\}\) of an observable \(R\) (with discrete spectrum). In this notation, the definite-valued projections of a number of different modal interpretations are given by

\[
\begin{align*}
d_{NR} &= \{P^2 = P = P^\dagger\} \\
d_B &= \{P^2 = P = P^\dagger : PP_{\psi_{R_j}} = P_{\psi_{R_j}}\ or\ 0\ for\ all\ j\} \\
d_C &= \{P^2 = P = P^\dagger : PX_i = X_i\ or\ 0\ for\ all\ i \neq 0\} \\
d_{K,D} &= \{P^2 = P = P^\dagger : PX_i = X_i\ or\ 0\ for\ all\ i\} \\
d_O &= \{P^2 = P = P^\dagger : PW = W\ or\ 0\}.
\end{align*}
\]

Roughly speaking, we have ordered these projection sets from ‘largest’ to ‘smallest’. At the top of the list is the naive realist, who considers every projection to have a definite value. The three proposals in the middle, due to Bub [1997], Clifton [1995a], and Kochen [1985] and Dieks [1995], are more discriminating. They consider a projection \(P\) to be definite-valued whenever it “resolves” the projections in a certain orthogonal set into two classes: those whose ranges are contained in that of \(P\), and those whose ranges are orthogonal to that of \(P\). (\(d_C\) is closely related to \(d_{K,D}\) and, in fact, is called the ‘Kochen-Dieks’ interpretation by Clifton [1995a]. The difference is that since \(d_{K,D}\) includes \(X_0\) in its definition, it must form a Boolean algebra of projections – the Boolean algebra generated by the \(\{X_i\}\), which sum to the identity operator.) Most parsimonious is the orthodox interpreter, who does
not permit the projections in the spectral resolution of $W$ to be “resolved” in this way. According to the orthodox view, in order for a projection to have a definite value, the projection must either annihilate $W$ or preserve it.

In each of these interpretations, the set of projections is expressed in terms of a smaller set $X$. We generalize this notion as follows:

- **$X$-form set.** We shall say that a set of projections $d$ is an $X$-form set if there is a mutually orthogonal set of projections $X$, not containing the zero projection, in terms of which $d$ can be written as

$$d = \{P^2 = P = P^\dagger : PX = X \text{ or } 0 \text{ for all } X \in X\}.$$  

Equivalently, one may say that $d$ is an $X$-form set if there is a subset $X \subseteq d$, not containing the zero projection, in terms of which $d$ can be expressed as above.

Except for $d_{NR}$, all the projection sets above are $X$-form sets (noting that $X_O = \{\sum_{i \neq 0} X_i\}$). $X$-form sets are what Dickson [1995a,b] calls Faux-Boolean algebras, and he shows that they have desirable properties in addition to those we shall stress here. Our focus will be on the fact that an $X$-form set generates a $^*$-closed set of definite-valued observables, and that an $X$-form set guarantees the existence of sufficiently many valuations on that $^*$-closed set to justify the name ‘definite-valued.’

In order to talk about the sets of definite-valued operators $D_{NR}, D_B, D_C$, etc. corresponding to $d_{NR}, d_B, d_C$, etc., we make two natural definitions:

- **Restriction.** Given a set of self-adjoint operators $D$, define the restriction of $D$ to be the set of idempotent members of $D$. We denote the restriction by $\overline{D}$. (We shall also use the notation $\mathcal{B}$ for the set of all projections in an arbitrary set of operators $\mathcal{B}$.)

- **Extension.** Given a set of projections $d$, define the extension of $d$ as follows. A self-adjoint operator is in the extension if and only if all its spectral projections lie in $d$. Denote the extension of $d$ by $\overline{d}$.

Note that the extension is not defined to include only discrete observables with spectral projectors in $d$ (recall our earlier generalization of the terminology ‘spectral projections’ to cover the continuous case). When we need to confine ourselves to sets $d$ with extensions containing only discrete observables (and two of our main results below are, so far, limited to that case), we shall say so explicitly.
Let $d$ be a set of projections, with $\overline{d}$ its extension. With Thm. 1 we have a test of whether $\overline{d}$ is $^*$-closed. We now convert that into a test given directly in terms of projections and $d$ itself:

**Theorem 2.** Given a set of projections $d$, its extension $\overline{d}$ is $^*$-closed if and only if $d$ is the restriction of the commutant of some set of projections $P$. (Symbolically, $\overline{d}$ is $^*$-closed iff $d = P'$.)

**Proof.** We saw in Thm. 1 that $\overline{d}$ is $^*$-closed if and only if $\overline{d} = S(A)$ for some von Neumann algebra $A$. We first show that $\overline{d} = S(A)$ is equivalent to $d = S(A)$. It is easy to see that $(\overline{d}) = d$, so it suffices to show $\overline{S(A)} = S(A)$.

Let $Q$ be an operator in $\overline{S(A)}$. Then by definition the spectral projections of $Q$ are contained in $\overline{S(A)}$, and hence in $A$. Now, $Q$ may be approximated as closely as desired by an appropriate linear combination of these spectral projections; in other words, $Q$ is the strong limit of a sequence of operators in $A$. But since $A$ is a von Neumann algebra, $A$ contains its (strong) limits; this means that $Q$ itself must be in $A$. Moreover, since $Q$ is self-adjoint, $Q$ must actually be in $S(A)$. This shows that $\overline{S(A)} \subseteq S(A)$.

Conversely, let $Q$ be an operator in $S(A)$. Then of course $Q$ is in $A$. Therefore, since each projection in the spectral family of $Q$ is a characteristic function of $Q$, and since $A$ is a von Neumann algebra (hence functionally closed), each spectral projection of $Q$ must also be in $A$. Clearly then each spectral projection must be in $S(A)$. Thus all of the spectral projections of $Q$ are in $S(A)$, which is to say that $Q$ is in $\overline{S(A)}$. This shows that $S(A) \subseteq \overline{S(A)}$, so that $\overline{S(A)} = S(A)$ as claimed.

So $\overline{d}$ is $^*$-closed if and only if there is a von Neumann algebra $A$ for which $d = S(A) = A$. We now show that there exists such a von Neumann algebra if and only if there is a set of projections $P$ for which $d = \overline{P'}$.

If $P$ is a set of projections then it is self-adjoint, in which case $P'$ is a $^*$-algebra containing the identity. And since $P' = P''$, $P'$ is therefore a von Neumann algebra (by the Double Commutant Theorem). This establishes that if there is a set of projections $P$ for which $d = \overline{P'}$, then $d$ is the restriction of a von Neumann algebra.

Conversely, if there is a von Neumann algebra $A$ for which $d = \overline{A}$, then define $P = \overline{A'}$. We show that whenever $A$ is a von Neumann algebra $(A')' = A$ so that $P' = d$. 

14
Let $T$ be any operator (not necessarily self-adjoint) in $\mathcal{A}''$, where $\mathcal{A}$ is any set of operators. Then $T$ commutes with everything in $\mathcal{A}'$, so of course $T$ commutes with everything in $\mathcal{A}'$. By definition then $T$ is in $(\mathcal{A}')'$, and we have established (for any set of operators $\mathcal{A}$) that $\mathcal{A}'' \subseteq (\mathcal{A}')'$.

Next, let $T$ be any operator (not necessarily self-adjoint) in $(\mathcal{A}')'$, where $\mathcal{A}$ now is any self-adjoint set of operators. This means that $T$ commutes with all the projections in $\mathcal{A}'$. Consider then an arbitrary self-adjoint operator $Q$ in $\mathcal{A}'$. Since $\mathcal{A}$ is a self-adjoint set, $\mathcal{A}'$ is a von Neumann algebra, so all of the spectral projections of $Q$ must be contained in $\mathcal{A}'$. Since $T$ must commute with each of these spectral projections, $T$ must therefore commute with $Q$ itself. In other words, from the fact that $T$ is in $(\mathcal{A}')'$ we may conclude that $T$ commutes with everything in $\mathcal{A}'$.

Next, since $\mathcal{A}'$ is a $\ast$-algebra, any operator $V \in \mathcal{A}'$ can be written as $V = V_R + iV_I$, where $V_R = (V + V^+)/2 \in \mathcal{S}(\mathcal{A}')$ and $V_I = -i(V - V^+)/2 \in \mathcal{S}(\mathcal{A}')$. So if $T$ commutes with everything in $\mathcal{S}(\mathcal{A}')$, then in fact $T$ commutes with everything in $\mathcal{A}'$. Thus $T$ is in $\mathcal{A}''$, so that $(\mathcal{A}')' \subseteq \mathcal{A}''$, and we have finally shown (for any self-adjoint set of operators $\mathcal{A}$) that $(\mathcal{A}')' = \mathcal{A}''$. Consequently, for any von Neumann algebra $\mathcal{A}$, we will clearly have $(\mathcal{A}')' = \mathcal{A}$. And this of course implies $(\mathcal{A}')' = \mathcal{A}$. So if $d$ is of the form $d = \mathcal{A}$ for some von Neumann algebra $\mathcal{A}$, then there is a set of projections $P$ (namely $\mathcal{A}'$) for which $d = P'$. QED.

With this theorem we can quickly show that under modal interpretations, as well as under the orthodox interpretation, arbitrary self-adjoint functions of definite-valued operators are themselves definite-valued. (This is trivially true for the naive realist interpretation.)

**Corollary.** If $d$ is of $X$-form, then $\overline{d}$ is $\ast$-closed.

**Proof.** Suppose $d$ is of $X$-form for some set $X$, and define

$$P \equiv \{ P^2 = P = P^\dagger : XP = P \text{ for some } X \in X \}.$$ 

We show that $d$ coincides with $P'$, hence by the previous theorem $\overline{d}$ is $\ast$-closed.

Consider any projections $P$ in $P$ and $Q$ in $d$; so for some $X$ in $X$, $XP = P$ and $QX = X$ or 0. It follows that $QP = P$ or 0, so that $Q$ commutes with $P$. Therefore every projection in $d$ commutes with every element of $P$, and we have $d \subseteq P'$.
Conversely, suppose a projection \( Q \) commutes with every \( P \) in \( P \), i.e. suppose \( Q \) is in \( P' \). Since \( X \) is a subset of \( P \), \( Q \) commutes with every \( X \). To conclude \( P' \subseteq d \) we must show more, namely that \( QX = X \) or 0 (for any \( X \)).

Therefore consider the operators \( QX \) and \((I - Q)X \). Since \([Q, X] = 0\), these are orthogonal projections that sum to \( X \). If they are both non-zero, then there are normalized, orthogonal vectors \( v \) and \( w \) with \( v \) in the range of \( QX \) and \( w \) in the range of \((I - Q)X \). Write this as \( v \in \text{ran}(QX) \) and \( w \in \text{ran}((I - Q)X) \). Now consider the vector \( z = v + w \) and its associated one-dimensional projection \( Z \). Clearly \( z \in \text{ran}(X) \), so \( XZ = Z \); consequently \( Z \in P \). But note also that \([Z, Q] \neq 0\), since \( z \) is not in \( \text{ran}(Q) \) or \( \text{ran}(I - Q) \). This contradicts the initial assumption that \( Q \) is in \( P' \). Hence at least one of \( QX \) or \((I - Q)X \) must be 0. \( \text{QED} \).

Thm. 2 allows us to say something more specific about the structure of a set \( d \) of projections with *-closed extension, viz. about its lattice-theoretic structure. We first recall the relevant aspects of lattice theory.

- **Lattice.** A lattice is a partially ordered set \( L \) in which each pair of elements \( x, y \in L \) has a supremum or join – denoted by \( x \lor y \) – and an infimum or meet – denoted by \( x \land y \). (We shall be dealing only with lattices which have a maximum element 1, and a minimum element 0.)

- **Completeness.** A lattice \( L \) is complete if every subset of \( L \) has both a join and a meet in \( L \).

- **Ortholattice.** A lattice \( L \) is orthocomplemented, or an ortholattice, if every \( x \in L \) has a complement \( x^\perp \in L \) satisfying:
  \[
  \begin{align*}
  x \lor x^\perp &= 1 \\
  x \land x^\perp &= 0 \\
  x \leq y &\Rightarrow y^\perp \leq x^\perp \\
  (x^\perp)^\perp &= x.
  \end{align*}
  \]

- **Orthomodular lattice.** An ortholattice is orthomodular if in addition it satisfies:
  \[
  x \leq y \Rightarrow y = x \lor (y \land x^\perp).
  \]
• **Atom.** An *atom* of a lattice $L$ is a minimal non-zero element. That is, $x$ is an atom of $L$ if $x \neq 0$ and if, for all $y \in L$, $y \leq x$ implies $y = x$ or $y = 0$.

• **Atomic.** A lattice $L$ is *atomic* if for all non-zero $y \in L$ there is an atom $x \in L$ where $x \leq y$.

Note that if $L$ is a complete, atomic, orthomodular lattice, then every element of $L$ is the join of the atoms contained in that element, i.e. for any $y \in L$, $y = \bigvee A$ where $A = \{x \in L : x \leq y$ and $x$ is an atom\}. The proof is straightforward: for any $y \in L$, clearly $\bigvee A \leq y$ (noting $\bigvee A \in L$, by completeness). So by orthomodularity, $y = (\bigvee A) \lor [y \land (\bigvee A)']$. But $y \land (\bigvee A)' = 0$, otherwise, by atomicity of $L$, the set $A$ would not exhaust the atoms contained in $y$. (In the proof of Thm. 4, later on, we shall invoke this result without comment.)

The set of all projections on a Hilbert space $H$ forms a lattice $L(H)$. Since the projections $P$ are in one-to-one correspondence with the closed subspaces $\text{ran}(P)$ onto which they project, the projections may be ordered by ordering their ranges by inclusion.

Given two projections $P$ and $Q$ on $H$, their meet $P \land Q$ is defined to be the projection onto $\text{ran}(P) \cap \text{ran}(Q)$ – a well-defined notion since the intersection of closed subspaces is itself a closed subspace. And their join $P \lor Q$ is defined to be the projection onto the norm-closed span of $\text{ran}(P) \cup \text{ran}(Q)$. For arbitrary sets of projections $\{P_\alpha\}$, the existence of $\land \{P_\alpha\}$ follows from the fact that for any set of closed subspaces $\{\Pi_\alpha\}$, there is a largest closed subspace $\Pi$ contained in each $\Pi_\alpha$ (Topping [1971]). Then $\land \{P_\alpha\}$ is defined to be the projection onto $\Pi$. Similar remarks hold for $\lor \{P_\alpha\}$, and $L(H)$ is thus a *complete* lattice.

Identifying 1 with the identity operator and 0 with the zero operator, and associating to each projection $P$ the projection $P'^\perp$ onto the orthocomplement of $\text{ran}(P)$ – which is a closed subspace – $L(H)$ becomes a complete orthomodular lattice. $L(H)$ is also atomic, with its atoms being the projections onto the one-dimensional subspaces of $H$.

As we show next, much the same is true for any subset of projections on $H$, provided the subset has a *-$closed extension – such a subset always picks out a complete, orthomodular sublattice of $L(H)$. (Atomicity will be discussed shortly.)

**Theorem 3.** Given a subset of projections $d \subseteq L(H)$, its extension $\overline{d}$ is
*-closed only if \( \mathbf{d} \) forms a complete, orthomodular sublattice of \( L(H) \).

**Proof.** If \( \overline{\mathbf{d}} \) is *-closed, then Thm. 2 says that \( \mathbf{d} \) is given by \( \mathbf{P}' \), where \( \mathbf{P} \) is some set of projections on \( H \). So we must show that \( \mathbf{P}' \) forms a complete orthomodular lattice.

Let \( \mathbf{Q} \) be any subset of projections in \( \mathbf{P}' \) and let \( \wedge \mathbf{Q} \) be the meet (in \( L(H) \)) of all the elements in \( \mathbf{Q} \).

**Claim:** If an operator \( A \) commutes with every projection in \( \mathbf{Q} \), i.e. if \( A \in \mathbf{Q}' \), then \( A \) commutes with \( \wedge \mathbf{Q} \).

To see this, let \( r \) be a vector in \( \text{ran}(\wedge \mathbf{Q}) \). Then for any \( P \in \mathbf{Q} \) we have

\[
PAr = APr
\]

(by assumption)

\[
\Rightarrow P(Ar) = (Ar)
\]

\([r \in \text{ran}(\wedge \mathbf{Q}) \subseteq \text{ran}(P)]\)

\[
\Rightarrow Ar \in \text{ran}(P).
\]

But since this holds for all \( P \in \mathbf{Q} \), we must therefore also have

\[
Ar \in \text{ran}(\wedge \mathbf{Q}).
\]

Thus whenever \( r \in \text{ran}(\wedge \mathbf{Q}) \), we have also \( Ar \in \text{ran}(\wedge \mathbf{Q}) \). This is equivalent to the statement

\[
(\wedge \mathbf{Q})A(\wedge \mathbf{Q}) = A(\wedge \mathbf{Q}).
\]

Next, note that for any projection \( P \), \([A, P] = 0\) if and only if \([A^\dagger, P] = 0\). So by repeating the above argument with \( A^\dagger \) in place of \( A \), we find

\[
(\wedge \mathbf{Q})A^\dagger(\wedge \mathbf{Q}) = A^\dagger(\wedge \mathbf{Q}).
\]

Taking adjoints, this becomes

\[
(\wedge \mathbf{Q})A(\wedge \mathbf{Q}) = (\wedge \mathbf{Q})A.
\]

Comparing with the earlier result \((\wedge \mathbf{Q})A(\wedge \mathbf{Q}) = A(\wedge \mathbf{Q})\), we conclude that \((\wedge \mathbf{Q})A = A(\wedge \mathbf{Q})\), and the claim is proved.
The claim shows that given a set of projections \( Q \), the meet of its elements, \( \wedge Q \), commutes with any operator that commutes with every projection in \( Q \). In other words, \( \wedge Q \in Q'' \). But if \( Q \subseteq P' \subseteq P'' \), then \( P'' \subseteq Q' \) which in turn implies \( Q'' \subseteq P''' = P' \). So \( \wedge Q \in P' \) and \( P' \) is closed under taking arbitrary meets of its elements.

An argument similar to the above establishes that if an operator commutes with every projection in \( Q \), then it commutes with their join \( \vee Q \). Hence \( P' \) is closed under arbitrary joins and is a complete lattice.

The rest is trivial. Clearly \( P' \) contains 1 and 0, and if \( P \) is in \( P' \) then so is \( 1 - P \). So the orthocomplement on \( P' \) is just the restriction of the orthocomplement operation on \( L(H) \), and ipso facto satisfies the orthomodular identity. \( \text{QED} \).

Generally, sublattices of \( L(H) \) need not be atomic if \( H \) is infinite-dimensional – just consider the Boolean algebra generated by the spectral projections of an observable with a continuous spectrum. Under what circumstances, then, can we be assured that a sublattice \( d \) with \( * \)-closed extension will be atomic? The following corollary offers a sufficient condition.

**Discrete operator.** Call a self-adjoint operator discrete if there exists \( \varepsilon > 0 \) such that no two elements of its spectrum are closer than \( \varepsilon \) to one another.

Then we have the following:

**Corollary.** Given a subset of projections \( d \subseteq L(H) \), if \( d \) is \( * \)-closed and contains only discrete observables, then \( d \) is atomic.

*Idea behind proof:* If \( d \) is not atomic, then we show by explicit construction that \( d \) contains a non-discrete observable.

**Proof.** (In the following, the indices \( n \) and \( N \) run over the positive integers 1, 2, 3, \ldots)

**Step 1.** In any non-atomic lattice \( L \), there is a countable family of distinct elements \( \{ x_n \} \subseteq L \) for which

\[
0 < x_1 < x_2 < x_3 < \ldots.
\]

**Proof.** Observe first that if every non-zero, non-atomic element of \( L \) contained an atom, then in fact every non-zero element of \( L \) would contain
an atom, and $L$ would be atomic. So if $L$ is non-atomic, then there must be a non-zero, non-atomic element $x_1$ which does not contain an atom.

Since this element $x_1$ is non-zero and non-atomic, there must be a non-zero element $x_2$ distinct from $x_1$ with

$$0 < x_2 < x_1.$$ 

But $x_1$ does not contain an atom; so $x_2$ cannot be an atom; so $x_2$ must contain a non-zero element $x_3$ distinct from $x_2$:

$$0 < x_3 < x_2 < x_1.$$ 

Accordingly, it is clear that whenever a non-atomic lattice $L$ contains the distinct non-zero elements $x_n < \ldots < x_2 < x_1$, where $x_1$ does not contain an atom, then it also contains a non-zero element $x_{n+1}$, distinct from $x_n$, with $x_{n+1} < x_n < \ldots < x_2 < x_1$. Therefore, as $L$ does in fact contain such an element $x_1$, Step 1 follows by induction.

Applying Step 1 to the lattice $d$, we have

$$0 < \ldots < P_3 < P_2 < P_1$$

for some family of distinct projections $\{P_n\} \subseteq d$.

Step 2. There is a family of mutually orthogonal non-zero projections $\{M_n\} \subseteq d$ and a projection $P_\infty \in d$ which, together with $P_1^\perp \in d$, form a mutually orthogonal, complete set, that is:

$$\sum_{n=1}^{\infty} M_n + P_\infty + P_1^\perp = 1.$$ 

(The limit in the sum is a strong limit.)

Proof. First we define $P_\infty$. Since (by the theorem) $d$ is a complete lattice, the family of projections $\{P_n\} \subseteq d$ has an infimum $\wedge\{P_n\} \equiv P_\infty \in d$. (Alternatively, it is not hard to see that the infimum $P_\infty$ is none other than the strong limit of the sequence $\{P_n\}$:

$$\lim_{n \to \infty} P_n = P_\infty.$$ 

Thus, by $*$-closure, $P_\infty$ is in $\overline{d}$, hence in $d$.)
Next, define the mutually orthogonal projections
\[ M_1 = P_1 \land P_2^\perp = P_1 - P_2 \]
\[ M_2 = P_2 \land P_3^\perp = P_2 - P_3 \]
\[ \vdots \]
\[ M_n = P_n \land P_{n+1}^\perp = P_n - P_{n+1} \]
\[ \vdots \]

(Since \( d \) is an ortholattice, each \( M_n \) is in \( d \).) Then
\[
\sum_{n=1}^{\infty} M_n = \lim_{n \to \infty} (M_1 + M_2 + \cdots + M_n)
= \lim_{n \to \infty} (P_1 - P_2 + P_2 - P_3 + \cdots + P_n - P_{n+1})
= \lim_{n \to \infty} (P_1 - P_{n+1})
= P_1 - \lim_{n \to \infty} P_{n+1}
= P_1 - P_\infty.
\]

So \( \sum_{n=1}^{\infty} M_n \) exists and satisfies
\[
\sum_{n=1}^{\infty} M_n + P_\infty + P_1^\perp = 1
\]
as claimed. (Note \( P_1^\perp \) is in \( d \) since \( d \) is an ortholattice.)

Finally, note that since \( P_\infty \leq P_n \) for all \( n \), \( P_\infty \) is orthogonal to each \( M_n \):
\[ P_\infty M_n = P_\infty (P_n - P_{n+1}) = P_\infty - P_\infty = 0. \]
Thus one sees that \( \{M_n\}, P_\infty \), and \( P_1^\perp \) form a mutually orthogonal, complete set within \( d \), as claimed.

**Step 3.** There is a non-discrete observable \( Q \in \mathbf{\overline{d}} \).

**Proof.** For each \( N \), define
\[ Q_N = P_1^\perp + e^{-1} M_1 + e^{-2} M_2 + \ldots + e^{-N} M_N. \]
Let \( v \) be a non-zero but otherwise arbitrary vector in the Hilbert space, and consider the sequence \( \{Q_N v\} \). It is elementary to show that this is a Cauchy sequence. (Hint: Given \( \varepsilon > 0 \), let \( N_\varepsilon \) be greater than \( \log(2\|v\|/\varepsilon) \); note also that \( \|M_n v\| \leq \|v\| \).) By completeness of the Hilbert space, \( \{Q_N v\} \) therefore converges in norm. Denoting the limit vector by \( q(v) \), we have then
\[
\lim_{N \to \infty} \|Q_N v - q(v)\| = 0
\]
whenever $v \neq 0$. Further, if $v = 0$, then obviously
\[ \lim_{N \to \infty} \|Q v - 0\| = 0 \]
so define $q(0) = 0$. Then for each $v$ we have shown that there is a $q(v)$ such that
\[ \lim_{N \to \infty} \|Q v - q(v)\| = 0. \]
By definition therefore the map $v \mapsto q(v)$ defines the strong limit operator $Q = \lim_{n \to \infty} Q_n$. So we may write
\[ Q = P_{1}^{\perp} + \sum_{n=1}^{\infty} e^{-n} M_n. \]
Clearly this operator has spectrum $\{1, e^{-1}, e^{-2}, \ldots, e^{-n}, \ldots, 0\}$ (with $Q v = 0$ for $v \in \text{ran}(P_\infty)$), so it is non-discrete. Yet its spectral projections $P_{1}^{\perp}$, $\{M_n\}$, and $P_\infty$ are in $\mathfrak{d}$; so $Q$ is in $\mathfrak{f}$. This establishes Step 3, and the corollary is proved. QED.

It should be noted that the converse to Thm. 3 fails. Consider a two-dimensional Hilbert space $H_2$ and take the (trivially) complete, atomic and orthomodular lattice of projections $\mathfrak{d}$ generated by two distinct pairs of orthogonal, one-dimensional projections in the plane. There can be no $\mathfrak{P}$ satisfying $\mathfrak{d} = \mathfrak{P}'$ in this case. For since $\mathfrak{d} \neq L(H_2)$, $\mathfrak{P}$ must contain something other than 0 or 1. So $\mathfrak{P}$ must contain a one-dimensional projection. But there is no such projection that all four one-dimensional projections in $\mathfrak{d}$ commute with. In fact, it is easy to see that a (proper) subortholattice of $L(H_2)$ extends to a $^*$-closed set of observables only if it is a Boolean algebra (i.e. distributive ortholattice).

## 3 Valuations on Sets of Definite-Valued Observables
3.1 Homomorphisms and Valuations

To this point we have been focusing on structural questions regarding the set of definite-valued operators. The subject one might naturally wish to consider next is that of value assignments on the set of definite-valued operators. After all, for such a set to be dubbed ‘definite-valued,’ it must admit valuations! In this section we analyze valuations from a structural perspective.

As usual, we will need to introduce some definitions.

- **Two-Valued (Ortholattice) Homomorphism.** Given an ortholattice $L$, a two-valued ortholattice homomorphism is a map $[\cdot] : L \to \{0, 1\} \subset \mathbb{R}$ which respects the operations of orthocomplement, meet and join:

  \[
  [x^\perp] = 1 - [x] \\
  [x \wedge y] = [x] \cdot [y] \\
  [x \vee y] = [x] + [y] - [x] \cdot [y].
  \]

  (The right-hand sides of these equations are arithmetic operations in $\mathbb{R}$ involving the numbers 0 and 1.)

- **Faithful Valuation.** Consider a set $D$ of self-adjoint operators with polynomial $^\ast$-closure. We use the term faithful valuation to refer to a real-valued map $\langle \cdot \rangle : D \to \mathbb{R}$ which assigns to each operator $Q$ a value in its spectrum, and which satisfies

  \[
  \langle aQ + S \rangle = a\langle Q \rangle + \langle S \rangle \\
  \langle Q^2 \rangle = \langle Q \rangle^2.
  \]

  (Here $a$ is any real scalar.)

- **Functional Valuation.** Consider a set $D$ of self-adjoint operators with $^\ast$-closure. We use the term functional valuation to refer to a faithful valuation $\langle \cdot \rangle : D \to \mathbb{R}$ which satisfies

  \[
  \lim_{n \to \infty} \langle F_n \rangle = \langle F \rangle
  \]

  whenever the sequence $\{ F_n \} \subseteq D$ converges strongly to $F \in D$.

  A faithful valuation respects the polynomial relationships among the operators in a set with polynomial $^\ast$-closure. To be precise, if $F(Q_1, \ldots, Q_k) \in D$
is a polynomial function of some operators $Q_i \in D$, then consider the natural corresponding real-valued polynomial $f(x_1, \ldots, x_k)$, where each $x_i$ takes on values from the spectrum of $Q_i$. In this case, a faithful valuation $\langle . \rangle$ will satisfy

$$\langle F(Q_1, \ldots, Q_k) \rangle = f(\langle Q_1 \rangle, \ldots, \langle Q_k \rangle).$$

A functional valuation, on the other hand, respects arbitrary functional relationships among the operators in a $\ast$-closed set. To be precise, suppose that a sequence of polynomial functions $\{F_n(Q_1, \ldots, Q_k)\}$ approaches an operator $F$ more and more closely in the strong operator topology. Then for a functional valuation $\langle . \rangle$ the numbers

$$\langle F_n(Q_1, \ldots, Q_k) \rangle = f_n(\langle Q_1 \rangle, \ldots, \langle Q_k \rangle)$$

must approach the number $\langle F \rangle$ more and more closely. In other words, the number assigned to the strong limit of a sequence $\{F_n\}$ is the limit of the sequence of numbers assigned to the $F_n$'s. And each of these numbers is obtained in the natural way from the numbers assigned to the $Q_i$. This is what is meant by the phrase “the mapping $\langle . \rangle$ respects arbitrary functional relationships.”

It is worth emphasizing that “arbitrary functional relationships” means arbitrary functional relationships. To say that $F$ is a function of operators $Q_1$, $\ldots$, $Q_k$ means no more than that $F$ is the limit of a sequence of polynomial functions of the $Q_i$. According to this definition, the operator $F$ need not be representable as a series expansion in the $Q_i$, nor need it be in any sense a ‘continuous’ function of the $Q_i$.

### 3.2 QuasiBoolean Algebras and Homomorphisms

We are aiming to analyze the requirement that a $\ast$-closed set of observables admit enough functional valuations that the statistics prescribed by quantum mechanics for observables within the set can be represented as measures over the set of functional valuations on the set. In this section we lay the groundwork for showing that whether or not this requirement can be met has everything to do with whether the projections in the $\ast$-closed set form a certain kind of ortholattice, dubbed a quasiBoolean algebra by Bell and Clifton [1995]. Just as von Neumann algebras capture the structure required of a $\ast$-closed set for it to be functionally closed, quasiBoolean algebras capture
the structure required of the projections in a ∗-closed set in order for it to admit enough functional valuations to satisfy quantum statistics.

Here is one last round of definitions leading up to the concept of a quasiBoolean algebra.

- **Ideal.** An ideal $I$ of a lattice $L$ is a (non-empty) subset of $L$ such that:
  
  $$x \in I, y \leq x \Rightarrow y \in I$$
  $$x, y \in I \Rightarrow x \lor y \in I$$
  $$1 \notin I.$$

- **Principal ideal.** For any $x \neq 1$ in a lattice $L$, the set $x \downarrow \equiv \{y \in L : y \leq x\}$ is an ideal, called the principal ideal generated by $x$.

- **$I$-quasiBoolean algebra.** An ortholattice $L$ containing an ideal $I$ is called an $I$-quasidistributive ortholattice, or an $I$-quasiBoolean algebra, if for any $x \notin I$ there is a two-valued (ortholattice) homomorphism $[.] : L \rightarrow \{0, 1\}$ for which $[x] = 1$.

  (See Bell and Clifton [1995]. The terminology derives from the fact that distributive ortholattices, i.e. Boolean algebras, satisfy the stronger condition that for any $x \neq y$ there is a two-valued homomorphism $[.] : L \rightarrow \{0, 1\}$ for which $[x] \neq [y]$.)

How does the abstract lattice-theoretic concept of a quasiBoolean algebra connect with our problem? Well, we are interested in characterizing ∗-closed sets of observables that support enough functional valuations to satisfy quantum statistics. Imagine then that we have a ∗-closed set $D(W)$, and we want to know if it fits the bill. Considering only the projections in $D$, notice that we would certainly run into trouble if there were some projection $P$ in $D$ for which $\text{Prob}_W(P = 1) \neq 0$, but for which no functional valuation on $D$ allowed $P$ to take the value 1. For then the measure of the set of functional valuations sending $P$ to 1 would have to be zero, and our hidden-variable theory would be doomed to ‘statistical failure.’

Now let’s put the same argument somewhat differently. Suppose that $D$ does not form an $I$-quasiBoolean algebra with respect to the ideal $I = \{P \in D : \text{Prob}_W(P = 1) = 0\}$. Then there is some projection $P$ with $\text{Prob}_W(P = 1) \neq 0$ for which no homomorphism assigns $P$ the value 1. Now, it is intuitively plausible that if there were a functional valuation on $D$
sending \( P \) to 1, then by considering the restriction, there would also be a homomorphism on the underlying ortholattice \( \mathbb{D} \) sending \( P \) to 1. (We shall be addressing this and related issues in section 3.3 below.) Taking this on faith for the time being, then since in our scenario there is no homomorphism sending \( P \) to 1, there could not be a functional valuation sending \( P \) to 1. Hence the measure of the functional valuations sending \( P \) to 1 would have to be zero, in conflict with the quantum mechanical probability \( \text{Prob}_W(P = 1) \neq 0 \).

In short, a functionally closed modal interpretation is doomed to statistical failure unless \( \mathbb{D} \) forms an \( I \)-quasiBoolean algebra with respect to the ideal \( I = \{ P \in \mathbb{D} : \text{Prob}_W(P = 1) = 0 \} \). Hence a mathematically appropriate object to seek for a modal interpretation that does satisfy quantum statistics is a \(^*\)-closed set \( \mathbb{D} \) whose projections form an \( I \)-quasiBoolean algebra with respect to the ideal \( I = \{ P \in \mathbb{D} : \text{Prob}_W(P = 1) = 0 \} \).

As we have said, the primary aim of this paper is to describe a sense in which \( I \)-quasiBoolean algebras are both necessary and sufficient to generate functionally closed modal interpretations with enough functional valuations to satisfy quantum statistics. But before we can reach the goal, we need a tractable characterization of their lattice structure. (This will make it easy to check that, for example, projection sets of \( X \)-form have the required properties.) At present we only have a clean characterization for complete, orthomodular, atomic \( I \)-quasiBoolean algebras of projections, so we are forced to depend on the Corollary to Thm. 3; and in its present form, this Corollary dictates that we confine the rest of our results to \(^*\)-closed sets of discrete observables. This does not mean that we are specializing to the case of finite-dimensional Hilbert spaces; but it does mean that from this point forward our results are only complete for that case.

Since our characterization of \( I \)-quasiBoolean algebras does not make any use of Hilbert space beyond its ortholattice structure, we shall present a purely lattice-theoretic result (which extends the results of Bell and Clifton [1995]).

**Theorem 4.** Let \( L \) be a complete, atomic, orthomodular lattice with ideal \( I \). Then \( L \) is an \( I \)-quasiBoolean algebra if and only if there is a non-empty, mutually orthogonal subset \( A \) of \( L \), not containing 0, such that:

1. For any \( y \in L \), \( a \leq y \) or \( a \leq y^\perp \) for all \( a \in A \); and
2. \( I = (\vee A)^\perp \downarrow \).
Proof. \((⇐)\) Let \(y \in L\) be such that \(y \notin I = (\forall A)\perp\) (by (2)). We must show that there is a two-valued homomorphism on \(L\) sending \(y\) to 1. Since \(y \notin (\forall A)\perp\), there must be an element \(b \in A\) such that \(b \leq y\). (For if not, then by (1) \(a \leq y\perp\) for all \(a \in A\), which implies \(\forall A \leq y\perp\), i.e. \(y \leq (\forall A)^{⊥}\). But then \(y \in (\forall A)^{⊥}\), contradicting our hypothesis.) Invoking (1) (together with the fact that \(b \neq 0\)), construct the well-defined mapping \([.]^b : L \rightarrow \{0, 1\}\) by

\[
[x]^b = 1 \quad \text{if} \quad b \leq x ;
\]

\[
[x]^b = 0 \quad \text{if} \quad b \leq x^⊥ .
\]

By definition then, \([y]^b = 1\). To complete the argument we verify that \([.]^b\) is an ortholattice homomorphism. First, observe that \([x \land x^2]^b = 1\) if and only if \([x \land x^2]^b = 1\) and \([x \land x^2]^b = 1\), which is a contradiction. (It now follows from (1) that the elements of \(A\) must be mutually orthogonal.) To show (2), note that all atoms in \((\forall A)^⊥\) are in \(I\) (otherwise, by the definition of \(A\) there would be an element \(b \in A\) such that \(b \leq (\forall A)^{⊥}\), implying \(b \leq b^⊥\) and hence the contradiction \(b = 0\)). Since \((\forall A)^{⊥}\) is the join of its atoms and \(I\) is an ideal, \((\forall A)^{⊥} \subseteq I\) which implies \((\forall A)^{⊥} \subseteq I\). For equality, suppose that for some
\[ y \in I, \ y \not\in (\exists A)^\perp; \] that is, \( y \not\in (\exists A)^\perp \). By (1) (just proved) there must be an element \( b \in A \) such that \( b \leq y \). But then since \( y \in I \), \( b \in I \) contradicting \( A \cap I = \emptyset \). Thus \( I = (\exists A)^\perp \). \( \text{QED} \).

Returning now to our favorite example, sets of \( X \)-form, we get what we were after:

**Corollary.** If a set of projections \( d \) is of \( X \)-form, then it is an \( I \)-quasiBoolean algebra where

\[ I = \{ P \in d : P \sum_{X \in X} X = 0 \}. \]

**Proof.** The Corollary to Thm. 2 establishes that \( \overline{d} \) is *-closed. So Thm. 3 establishes that \( d \) is a complete orthomodular lattice. Since \( X \)-form lattices are clearly atomic, with the atoms being the \( X \in X \) and all one-dimensional projections orthogonal to all the \( X \in X \), the conclusion follows immediately from Thm. 4 (with \( X \) playing the role of \( A \)). \( \text{QED} \).

### 3.3 Projections and Functional Valuations

We are now in a position to fill in the last piece of our puzzle before taking a look at exactly how these technical results sidestep von Neumann’s no-hidden-variables theorem. Our final theorem simplifies the task of deciding whether a given *-closed set will support enough functional valuations to satisfy quantum statistics, by substituting the simpler question of whether its underlying set of projections forms the appropriate quasiBoolean algebra.

**Theorem 5.** Let \( d \) be a set of projections with \( \overline{d} \) a *-closed set of definite-valued operators having discrete spectra, and let \( W \) be a density operator. Then the following are equivalent:

1. There is a probability measure \( \mu \) on the set of all functional valuations \( \langle . \rangle: \overline{d} \to \mathbb{R} \) such that for any mutually commuting subset \( \{ A, B, C, \ldots \} \) of \( \overline{d} \) and corresponding sets of eigenvalues \( \{ \alpha, \beta, \gamma, \ldots \} \):

\[
\text{Prob}_W(A \in \alpha, B \in \beta, C \in \gamma, \ldots) = \mu\{ \langle . \rangle: \langle A \rangle \in \alpha, \langle B \rangle \in \beta, \langle C \rangle \in \gamma, \ldots \}.
\]

2. \( d \) is an \( I \)-quasiBoolean algebra, where \( I = \{ P \in d : PW = 0 \} \).
Proof.

\( (1) \implies (2) \)

Since \( \overline{d} \) is *-closed, \( d \) is a complete ortholattice (by Thm. 3). Assuming the existence of a probability measure \( \mu \) satisfying (1), we must show that \( d \) forms an \( I \)-quasiBoolean algebra. So let \( P \) be any element of \( d \) such that \( PW \neq 0 \) (i.e. \( P \not\in I \)). Then \( P \) is in \( \overline{d} \) and so, by (1), there exists a probability measure \( \mu \) such that:

\[
\text{Prob}_W(P = 1) = \mu(\langle . \rangle : \langle P \rangle = 1).
\]

But since \( PW \neq 0 \), \( \text{Prob}_W(P = 1) = \text{Tr}(PW) \neq 0 \), therefore \( \mu(\langle . \rangle : \langle P \rangle = 1) \neq 0 \). So there exists a functional valuation on \( \overline{d} \) sending \( P \) to 1. Since we seek a homomorphism sending \( P \) to 1, it suffices to complete the proof if we can show that every functional valuation on \( \overline{d} \) restricts to an ortholattice homomorphism on \( d \).

Let \( \langle . \rangle : \overline{d} \to \mathbb{R} \) be a functional valuation. Consider a projection \( P \) in \( d \) and its complement \( P^\perp \in d \). Then, \( P, P^\perp \in \overline{d} \) satisfy \( P + P^\perp = 1 \), so from \( \langle aQ + S \rangle = a\langle Q \rangle + \langle S \rangle \) we have \( \langle P \rangle + \langle P^\perp \rangle = \langle 1 \rangle = 1 \), or

\[
\langle P^\perp \rangle = 1 - \langle P \rangle.
\]

Next, let \( P_1 \) and \( P_2 \) be two projections in \( d \), with \( P_1 \land P_2 \in d \) their meet. It is easily verified that \( P_1 \land P_2 = \lim_{n \to \infty} (\frac{1}{2}[P_1P_2 + P_2P_1])^n \), and both \( P_1 \land P_2 \) and \( (\frac{1}{2}[P_1P_2 + P_2P_1])^n \) lie in \( \overline{d} \) (by *-closure). So by functionality of \( \langle . \rangle \) we must have

\[
\langle P_1 \land P_2 \rangle = \lim_{n \to \infty} \langle (\frac{1}{2}[P_1P_2 + P_2P_1])^n \rangle.
\]

Now, by faithfulness, \( \langle (\frac{1}{2}[P_1P_2 + P_2P_1])^n \rangle = \langle (\frac{1}{2}[P_1P_2 + P_2P_1])^n \rangle \). And again by faithfulness, for any \( Q, S \in \overline{d} \) we have \( \langle \frac{1}{2}(QS + SQ) \rangle = \langle Q \rangle \cdot \langle S \rangle \). (For the proof, use \( \frac{1}{2}(QS + SQ) = \frac{1}{2}(Q+S)^2 - \frac{1}{4}(Q-S)^2 \), and note that \( \langle 0 \rangle = 0 \).) Thus \( \langle (\frac{1}{2}[P_1P_2 + P_2P_1])^n \rangle = \langle (P_1) \cdot (P_2) \rangle^n = \langle P_1 \rangle^n \langle P_2 \rangle^n \). But since \( \langle . \rangle \) is a valuation, it assigns to \( P_1 \) and \( P_2 \) the values 0 or 1, so in either case \( \langle P_1 \rangle^n = \langle P_1 \rangle \). Hence \( \langle P_1 \rangle^n \langle P_2 \rangle^n = \langle P_1 \rangle \cdot \langle P_2 \rangle \) for each \( n \), and so we have

\[
\langle P_1 \land P_2 \rangle = \langle P_1 \rangle \cdot \langle P_2 \rangle.
\]

Finally, \( \langle P_1 \lor P_2 \rangle = \langle P_1 \rangle + \langle P_2 \rangle - \langle P_1 \rangle \cdot \langle P_2 \rangle \) follows by de Morgan’s law. So we have established that \( \langle . \rangle \) restricted to \( d \) is an ortholattice homomorphism.
This completes the proof that $d$ is an $I$-quasiBoolean algebra with respect to $I = \{ P \in d : PW = 0 \}$.

(2) $\Rightarrow$ (1)

Now suppose $d$ is $I$-quasiBoolean, where $I = \{ P \in d : PW = 0 \}$. We must exhibit a probability measure $\mu$ satisfying (1). As discussed earlier, a necessary condition for the existence of such a $\mu$ is that the following claim hold:

**Claim**: For any $P \in d$ such that $\text{Prob}_W(P = 1) \neq 0$, there exists a functional valuation $\langle \cdot \rangle : d \to \mathbb{R}$ sending $P$ to 1.

To establish that this is in fact the case, we make use of Thm. 4 and the corollary to Thm. 3. According to these results, since $d$ is an $I$-quasiBoolean algebra of projections with a $*$-closed extension, and since $d$ is assumed to contain only discrete observables, it follows that there is a set of mutually orthogonal projections $X \subseteq d$ such that:

$$d \subseteq \{ P : PX = X \text{ or } 0 \text{ for all } X \in X \},$$

$$I = \{ P \in d : P \sum_{X \in X} X = 0 \} = \{ P \in d : PX = 0 \text{ for all } X \in X \}.$$  

Since $I = \{ P \in d : PW = 0 \}$, it follows that for $P \in d$, $PW \neq 0$ is equivalent to $PX \neq 0$ for some $X \in X$, which is in turn equivalent to $PY = Y$ for some $Y \in X$.

Now consider any $R \in d$ such that $\text{Prob}_W(R = 1) = \text{Tr}(RW) \neq 0$. Then $RW \neq 0$, so $RY = Y$ for some $Y \in X$. The mapping $[\cdot] : d \to \{0, 1\}$ given by

$$[P] = 1 \text{ if } PY = Y$$
$$[P] = 0 \text{ if } PY = 0$$

is easily verified (as in the first part of Thm. 4) to be an ortholattice homomorphism which sends both $R$ and $Y$ to 1. So, to complete the proof of the claim, we need to show that the homomorphism $[\cdot]$ on $d$ extends to a functional valuation $\langle \cdot \rangle$ on $\overline{d}$. (For this we will eventually have to recall that $[\cdot]$ has been defined so that $[Y] = 1$, and that $Y \in X$.)

Define a map $\langle \cdot \rangle : \overline{d} \to \mathbb{R}$ as follows. For an operator $Q \in \overline{d}$, let $Q = \sum q_i Q_i$ be its spectral resolution (remember $\overline{d}$ consists of only discrete spectra.
observables), and define

\[ \langle Q \rangle \equiv \sum q_i [Q_i]. \]

It is clear that \( \langle . \rangle \) agrees with [.] on \( d \), since for a projection \( P \in d \) we have

\[ \langle P \rangle \equiv \sum p_i [P_i] = 1 \cdot [P]. \]

We argue next that \( \langle . \rangle \) is a faithful valuation on \( d \).

First of all, since \( \sum Q_i = 1 \), it is easy to show that [.] must assign the value 1 to exactly one of the projections \( Q_i \). One sees therefore that \( \langle . \rangle \) assigns to \( Q \) a value in its spectrum.

Second, \( \langle . \rangle \) has the property that \( \langle aQ + S \rangle = a \langle Q \rangle + \langle S \rangle \). To see this, let \( C = aQ + S \), which, phrased in terms of spectral resolutions, reads

\[ \sum c_i C_i = a \sum q_j Q_j + \sum s_k S_k. \]

Since [.] is an ortholattice homomorphism, there exist unique \( i' \), \( j' \), and \( k' \) such that \([C_{i'}] = [Q_{j'}] = [S_{k'}] = 1\). For these projections we will therefore have \([C_{i'} \wedge Q_{j'} \wedge S_{k'}] = 1 \cdot 1 \cdot 1 = 1\). It follows that \( C_{i'} \wedge Q_{j'} \wedge S_{k'} \) is a non-zero projection, hence there is a non-zero vector \( v \) in the range of \( C_{i'} \wedge Q_{j'} \wedge S_{k'} \).

Applying both sides of the above spectral resolution equation to this vector \( v \), we find

\[ c_{i'} = a q_{j'} + s_{k'}. \]

Since \( \langle C \rangle \) is none other than the eigenvalue \( c_{i'} \) for which \([C_{i'}] = 1\), and similarly for \( \langle Q \rangle \) and \( \langle S \rangle \), this just says that \( \langle C \rangle = a \langle Q \rangle + \langle S \rangle \). Thus \( \langle aQ + S \rangle = a \langle Q \rangle + \langle S \rangle \), as was to be shown.

Third, \( \langle . \rangle \) has the property that \( \langle Q^2 \rangle = \langle Q \rangle^2 \). To see this, let \( C = Q^2 \), which, phrased in terms of spectral resolutions, reads

\[ \sum c_i C_i = \sum q_j^2 Q_j. \]

Imitating the above reasoning, we find

\[ c_{i'} = q_{j'}^2 \]

which says that \( \langle C \rangle = \langle Q \rangle^2 \). Thus \( \langle Q^2 \rangle = \langle Q \rangle^2 \), as was to be shown.

These three arguments establish that \( \langle . \rangle : d \rightarrow \mathbb{R} \) is a faithful valuation.

We show next that \( \langle . \rangle \) is a functional valuation. For let \( Q_1, \ldots, Q_k \) be operators in \( d \), with \( \{F_n(Q_1, \ldots, Q_k)\} \) a sequence of self-adjoint polynomials
in the $Q_i$ converging strongly to $F$. Since $\overline{d}$ is $\ast$-closed, each $F_n$ belongs to $\overline{d}$, and $F$ is in $\overline{d}$ as well. By definition then, in the spectral resolutions $F_n = \sum f^n_i F^n_i$ and $F = \sum f_j F_j$, the projections $F^n_i$ and $F_j$ are all in $d \subseteq \{ P : PX = X \text{ or } 0 \text{ for all } X \in X \}$. Therefore, since $Y$ is an element of $X$, we have

$$F_n Y = \sum f^n_i F^n_i Y \equiv q_n Y$$

where $q_n$ is a real scalar. Similarly

$$FY = \sum f_j F_j Y \equiv qY$$

where $q$ is another real scalar. Furthermore, from the fact that $\{F_n\} \to F$ strongly, it follows easily that $\{q_n\} \to q$ in modulus. For let $w$ be a unit vector in the range of $Y$. (Recall that such a vector exists since $[Y] = 1$.) Then we have

$$|q_n - q| = \| (q_n - q) w \|$$

$$= \| (q_n - q) Y \| w \|$$

$$= \| (F_n - F) \| w \|$$

$$\to 0$$

by strong convergence of $\{F_n\}$ to $F$.

Next, from

$$q_n \to q$$

we have

$$q_n \langle Y \rangle \to q \langle Y \rangle$$

(recall that $\langle Y \rangle = 1$)

$$\Rightarrow \quad \langle q_n Y \rangle \to \langle qY \rangle$$

(since $\langle . \rangle$ is faithful)

$$\Rightarrow \quad \langle F_n Y \rangle \to \langle FY \rangle$$

$$\Rightarrow \quad \langle \frac{1}{2}(F_n Y + Y F_n) \rangle \to \langle \frac{1}{2}(FY + YF) \rangle$$
(since $F_n$ and $F$ both commute with $Y$)

$$\Rightarrow \langle F_n \rangle \cdot \langle Y \rangle \to \langle F \rangle \cdot \langle Y \rangle$$

(since $\langle \cdot \rangle$ is faithful). But since $\langle Y \rangle = 1$, this last statement requires

$$\langle F_n \rangle \to \langle F \rangle,$$

so that $\langle \cdot \rangle$ is functional as promised. This finally establishes the claim: For any $P \in d$ such that $\text{Prob}_W(P = 1) \neq 0$, there exists a functional valuation $\langle \cdot \rangle : \overline{d} \to \mathbb{R}$ sending $P$ to 1.

Having established this, we can now easily define a probability measure satisfying (1) as follows. Let our measure space consist of the set $\mathcal{F}$ of all functional valuations on $\overline{d}$; let the measurable sets $\mathcal{M}$ be sets of the form $S_P = \{\langle \cdot \rangle \in \mathcal{F} : \langle P \rangle = 1\}$ for some $P$ in $\overline{d}$; and let the measure be defined by

$$\mu \{\langle \cdot \rangle \in \mathcal{F} : \langle P \rangle = 1\} \equiv \text{Prob}_W(P = 1).$$

In order to show that everything is well-defined, we first show that $\langle \mathcal{F}, \mathcal{M}, \mu \rangle$ is a probability space.

$\mathcal{M}$ constitutes a sigma field on $\mathcal{F}$. For $\emptyset = S_0 \in \mathcal{M}$, $\mathcal{F} = S_1 \in \mathcal{M}$, and $(S_P)^c = S_1 - P \in \mathcal{M}$. Furthermore, $\cap_i S_{P_i} = S_{\wedge_i P_i} \in \mathcal{M}$ since $d$ is a complete lattice (Thm. 3), and

$$\cap_i^n P_i \to \wedge_i P_i$$

$$\Rightarrow \langle \cap_i^n P_i \rangle \to \langle \wedge_i P_i \rangle$$

$$\Rightarrow \Pi_i^n \langle P_i \rangle \to \langle \wedge_i P_i \rangle$$

which implies that $\langle \wedge_i P_i \rangle = 1$ exactly when $\langle P_i \rangle = 1$ for all $i$. It follows from de Morgan’s law that $\cup_i S_{P_i} = S_{\vee_i P_i} \in \mathcal{M}$.

The map $\mu$ is also a probability measure. It takes values in the interval $[0, 1]$; it satisfies $\mu(\emptyset) = 0$ (thanks to the claim); and it satisfies $\mu(\mathcal{F}) = 1$. To show that $\mu$ is countably additive, suppose we have mutually disjoint $\{S_{P_i}\}$, so the meet of any two projections in the set $\{P_i\} \subseteq d$ is zero. Recall that since $d$ is $I$-quasiBoolean, for all projections in $d \subseteq \{P : PX = X \text{ or } 0 \text{ for all } X \in X\}$, $PX = 0$ for all $X \in X$ if and only if $PW = 0$. This latter condition implies that the ranges corresponding to the $X \in X$ span the image space of $W$, i.e. the subspace generated by its non-zero eigenvalue.
eigenspaces. Now let \( x \) be a vector in the range of one of the \( X \in X \). It follows that \((\vee_i P_i)x = (\sum_i P_i)x\). For each \( P_i \) must satisfy \( P_ix = x \) or 0; and since any two projections in the set \( \{P_i\} \) have meet 0, this implies that either \( P_ix = 0 \) for all \( i \) (in which case \((\vee_i P_i)x = (\sum_i P_i)x = 0\)) or \( P_ix = x \) for exactly one \( i \) (in which case \((\vee_i P_i)x = (\sum_i P_i)x = x\)). Since every vector in the image space of \( W \) is a linear combination of vectors in the subspaces corresponding to the \( X \in X \), the claim implies \((\vee_i P_i)W = (\sum_i P_i)W\). This yields countable additivity:

\[
\mu(\bigcup_i S_{P_i}) = \mu(S_{\vee_i P_i}) \equiv \text{Prob}_W(\vee_i P_i = 1) = \text{Tr}(\vee_i P_i)W = \text{Tr}(\sum_i P_i)W = \sum_i \text{Tr}(P_iW) = \sum_i \text{Prob}_W(P_i = 1) = \sum_i \mu(S_{P_i}).
\]

Finally (!), we prove that \( \mu \) satisfies (1).

Consider any mutually commuting subset \( \{A, B, C, \ldots\} \) of \( \overline{d} \) with corresponding sets of eigenvalues \( \{\alpha, \beta, \gamma, \ldots\} \). We have:

\[
\text{Prob}_W(A \in \alpha, B \in \beta, C \in \gamma, \ldots) = \text{Tr}(P_\alpha P_\beta P_\gamma \ldots W) = \text{Prob}_W(P_\alpha P_\beta P_\gamma \ldots = 1).
\]

But since \( P_\alpha, P_\beta, P_\gamma, \ldots \) commute and \( \overline{d} \) is \(*\)-closed, \( P_\alpha P_\beta P_\gamma \ldots \) is a projection in \( \overline{d} \). So by definition \( \text{Prob}_W(P_\alpha P_\beta P_\gamma \ldots = 1) \) is equal to \( \mu(\{\langle , \rangle \in \mathcal{F} : \langle P_\alpha P_\beta P_\gamma \ldots \rangle = 1\}) \), and we have

\[
\text{Prob}_W(A \in \alpha, B \in \beta, C \in \gamma, \ldots) = \mu(\{\langle , \rangle \in \mathcal{F} : \langle P_\alpha P_\beta P_\gamma \ldots \rangle = 1\}),
\]

which, using the fact that \( \langle , \rangle \) is a functional valuation, gives:

\[
\text{Prob}_W(A \in \alpha, B \in \beta, C \in \gamma, \ldots) = \mu(\{\langle , \rangle \in \mathcal{F} : \langle P_\alpha \rangle \langle P_\beta \rangle \langle P_\gamma \rangle \ldots = 1\}) = \mu(\{\langle , \rangle \in \mathcal{F} : \langle P_\alpha \rangle = \langle P_\beta \rangle = \langle P_\gamma \rangle = \ldots = 1\}) = \mu(\{\langle , \rangle \in \mathcal{F} : \langle A \rangle \in \alpha, \langle B \rangle \in \beta, \langle C \rangle \in \gamma, \ldots\}).
\]

and (1) is proved. \( QED. \)

It is probably worthwhile to summarize as plainly and briefly as possible
what has happened. Together, Thms. 2 and 5 show that a set of projection operators \( d \) will serve as the basis of a set of (discrete) definite-valued observables that is \(*\)-closed and admits enough functional valuations to represent the quantum statistics for the observables in the set, if and only if two (logically independent) conditions are satisfied:

1. \( d = P' \) for some set of projections \( P \), and
2. \( d \) is an \( I \)-quasiBoolean algebra, where \( I = \{ P \in d : \text{Prob}_W(P = 1) = 0 \} \).

Furthermore, the Corollaries to Thms. 2 and 4 offer a direct method for constructing such a projection set: simply specify a set \( X \) of mutually orthogonal projections that span a subspace of \( H \) containing the image space of \( W \). The resulting \( X \)-form lattice will then satisfy both (1) and (2). And notice once more that all the concrete proposals for sets of definite-valued projections considered in Section 2.3 (save the naive realist’s!) are constructed in exactly this way. In view of this, it would be nice to know whether all sets of projections satisfying (1) and (2) arise as \( X \)-form lattices with the span of the mutually orthogonal projections in \( X \) containing \( W \)’s image space.

4 Von Neumann’s No-Hidden-Variables Theorem

At last, we arrive at the infamous theorem. The essential assumption of the theorem (the one without which the theorem would not follow, and from which the theorem does follow quite apart from von Neumann’s other assumptions) is stated by Bell [1966] to be:

Any real linear combination of any two self-adjoint operators represents an observable, and the same linear combination of expectation values is the expectation value of the combination.

Let us call this von Neumann’s Principle.

If the expectation values in question are those prescribed by the quantum state \( W \) of the system, then this principle is unobjectionable. But von Neumann’s proof came under fire by Bell (and most others following him) because von Neumann also required his principle to hold for the dispersion-free states postulated by hidden-variable theories.
A dispersion-free state is one in which there is no statistical spread in the values of observables, and hence the expectation value of any (discrete) observable in a dispersion-free state must equal one of its eigenvalues. Given that, it is trivial to show that von Neumann’s principle must fail. Consider a spin-1/2 particle and the linear combination of spin observables \((\sigma_x + \sigma_y)/\sqrt{2}\) which is, itself, the operator which corresponds to the particle’s spin component along the direction bisecting the \(x\) and \(y\) directions. (This example is due to Jammer [1974, 274].) If the expectations of dispersion-free states are to satisfy von Neumann’s principle, then we must have \(\pm 1 = (\pm 1 + \pm 1)/\sqrt{2}\) which is absurd.

Bell’s own reason for finding the application of von Neumann’s principle to dispersion-free states implausible places the blame on the incompatibility of \(\sigma_x\) and \(\sigma_y\). Their incompatibility implies that \(\sigma_x\), \(\sigma_y\), and \((\sigma_x + \sigma_y)/\sqrt{2}\) all require differently oriented Stern-Gerlach apparatus to be measured, and so there is no logical reason to require the values of these three spin components, only one of which can be measured at any one time (while the others’ values have to be inferred counterfactually) to conform to von Neumann’s principle. The only constraint is that for empirical adequacy of a hidden-variable theory, that principle – which is a “quite peculiar property of quantum states” (Bell [1966, 449]) – needs to be reproduced on averaging over its dispersion-free states. To make his point, Bell constructs a simple hidden-variable theory with just that property but with dispersion-free states that do not satisfy von Neumann’s principle.

In our terminology, what Bell questioned was von Neumann’s assumption of faithfulness for the definite values of incompatible observables. But now we see that there is another way around the ‘no-go’ theorem – one which does not focus exclusive attention on issues of compatibility, and which avoids making von Neumann’s principle out to be merely a peculiarity of quantum states. To circumvent the theorem in this sense, we simply drop von Neumann’s tacit assumption that every observable must receive a definite value. Then Thms. 2 and 5 show that one can actually strengthen the functional requirements on \(D\) and its valuations, and there will still be enough valuations to recover quantum statistics.

Why then did the above spin example go wrong? Well, if we require that the set of definite-valued projections \(d\) include the spectral projections of \(\sigma_x\) and \(\sigma_y\), and if we require that its extension be \(*\)-closed, then \(d\) will be a subortholattice of projections in \(L(H_2)\) containing one-dimensional projections \(P\) and \(Q\) that are neither parallel nor orthogonal (because of the
incompatibility of $\sigma_x$ and $\sigma_y$). But such a subortholattice cannot possibly be an $I$-quasiBoolean algebra, for any $I$, since (using Thm. 4’s characterization of such algebras) there is no non-zero projection in $L(H_2)$ contained in or orthogonal to each of $P$ and $Q$.

The conclusion is not that von Neumann’s principle (or a stronger principle based on *-closure rather than just polynomial *-closure) must fail for all ‘hidden-variable’ theories. Rather, all one can conclude is that the choice of $d$ that led to the difficulty above must be rejected. And, although in this simple example involving $\sigma_x$ and $\sigma_y$ their incompatibility again plays a direct role in defeating the satisfaction of von Neumann’s principle, this is only an artifact of the two-dimensional case $H_2$. Projection sets of $X$-form generate definite-valued sets of observables that satisfy von Neumann’s principle (even with respect to *-closure), yet in dimensions higher than 2 they can contain plenty of incompatible projections. By our characterization theorem for quasiBoolean algebras (Thm. 4), the issue is not compatibility per se, but rather a somewhat ‘finer’ notion: whether the projections in $d$ have sufficiently many common eigenvectors – the vectors in the ranges of the projections in $X$.

Once again, what we learn from von Neumann’s theorem is not that ‘hidden-variable’ theories must give up functional valuations for non-commuting observables, but that they must be more discriminating in what observables they count as ‘definite-valued’ (i.e. having dispersion-free values). And since both requirements (1) and (2) are satisfied by $X$-form projection sets, examples of which include a number of modal interpretations and the orthodox interpretation, these interpretations provide a clear existence proof that ‘hidden-variable’ theories can indeed be more discriminating while conforming to von Neumann’s principle – they do not have to simply adopt a naive realism whereby every observable has a definite-value.

The same conclusion spells the demise of the no-go theorems of Jauch and Piron [1963] and of Kochen and Specker [1967]. In particular, the latter’s theorem weakens von Neumann’s principle so that it only carries commitment to the idea that the set of definite-valued projections is compatible polynomial *-closed and its dispersion-free states prescribe faithful valuations respecting only the polynomial functional relations between compatible observables. But since we have shown that there is plenty of room for an interpretation to endorse even a stronger version of von Neumann’s original principle based on *-closure, and we have seen that eschewing incompatibility is not the ultimate reason for an interpretation’s success in that endeavour, strengthening von
Neumann’s (alleged) no-go theorem so that it is sensitive to issues about compatibility loses its point.

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