Large Sets in Boolean and Non-Boolean Groups and Topology

Ol’ga V. Sipacheva

Various notions of large sets in groups and semigroups naturally arise in dynamics and combinatorial number theory. Most familiar are those of syndetic, thick (or replete), and piecewise syndetic sets. Apparently, the term “syndetic” was introduced by Gottschalk and Hedlund in their 1955 book [1] in the context of topological groups, although syndetic sets of integers have been studied long before (they appear, e.g., in Khintchine’s 1934 ergodic theorem). During the past decades, large sets in \( \mathbb{Z} \) and in abstract semigroups have been extensively studied. It has turned out that, e.g., piecewise syndetic sets in \( \mathbb{N} \) have many attractive properties: they are partition regular (i.e., given any partition of \( \mathbb{N} \) into finitely many subsets, at least one of the subsets is piecewise syndetic), contain arbitrarily long arithmetic progressions, and are characterized in terms of ultrafilters on \( \mathbb{N} \) (namely, a set is piecewise syndetic if and only if it belongs to an ultrafilter contained in the minimal two-sided ideal of \( \beta \mathbb{N} \)). Large sets of other kinds are no less interesting, and they have numerous applications to dynamics, Ramsey theory, the ultrafilter semigroup on \( \mathbb{N} \), the Bohr compactification, and so on.

Quite recently Reznichenko and the author have found yet another application of large sets. Namely, we introduced special large sets in groups, which we called fat, and applied them to construct a discrete set with precisely one limit point in any countable nondiscrete topological group in which the identity element has nonrapid filter of neighborhoods. Using this technique and special features of Boolean groups, we proved, in particular, the nonexistence of a countable nondiscrete extremally disconnected group in ZFC (see [2]).

In this paper, we study right and left thick, syndetic, piecewise syndetic, and fat sets in groups (although they can be defined for arbitrary semigroups). Our main concern is the interplay between such sets in Boolean groups. We also consider natural topologies closely related to fat sets, which leads to interesting relations between fat sets and ultrafilters.

1. Basic Definitions and Notation

We use the standard notation \( \mathbb{Z} \) for the group of integers, \( \mathbb{N} \) for the set (or semigroup, depending on the context) of positive integers, and \( \omega \) for the set of nonnegative integers or the first infinite cardinal; we identify cardinals with the corresponding initial ordinals. Given a set \( X \), by |\( X \)| we denote its cardinality, by |\( X \)|\( k \) for \( k \in \mathbb{N} \), the \( k \)th symmetric power of \( X \) (i.e., the set of all \( k \)-element subsets of \( X \)), and by \( |X|^{\leq \omega} \), the set of all finite subsets of \( X \).

Definition 1 (see [3]). Let \( G \) be a group. A set \( A \subset G \) is said to be

This work was financially supported by the Russian Foundation for Basic Research (Project No. 15-01-05369).
(a) right thick, or simply thick if, for every finite $F \subset S$, there exists a $g \in G$ (or, equivalently, $g \in A$ [3 Lemma 2.2]) such that $Fg \subset A$;
(b) right syndetic, or simply syndetic, if there exists a finite $F \subset G$ such that $G = FA$;
(c) right piecewise syndetic, or simply piecewise syndetic, if there exists a finite $F \subset G$ such that $FA$ is thick.

Left thick, left syndetic, and left piecewise syndetic sets are defined by analogy; in what follows, we consider only right versions and omit the word “right.”

**Definition 2.** Given a subset $A$ of a group $G$, we shall refer to the least cardinality of a set $F \subset G$ for which $G = FA$ as the syndeticity index, or simply index (by analogy with subgroups) of $A$ in $G$. Thus, a set is syndetic if and only if it is of finite index. We also define the thickness index of $A$ as the least cardinality of $F \subset G$ for which $FA$ is thick.

A set $A \subset \mathbb{Z}$ is syndetic if and only if the gaps between neighboring elements of $A$ are bounded, and $B \subset \mathbb{Z}$ is thick if and only if it contains arbitrarily long intervals of consecutive integers. The intersection of any such sets $A$ and $B$ is piecewise syndetic; clearly, such a set is not necessarily syndetic or thick (although it may as well be both syndetic and thick). The simplest general example of a syndetic set in a group is a coset of a finite-index subgroup.

In what follows, when dealing with general groups, we use multiplicative notation, and when dealing with Abelian ones, we use additive notation.

Given a set $A$ in a group $G$, by $\langle A \rangle$ we denote the subgroup of $G$ generated by $A$.

As mentioned, we are particularly interested in Boolean groups, i.e., groups in which all elements are self-inverse. All such groups are Abelian. Moreover, any Boolean group $G$ can be treated as a countable-dimensional vector space over the two-element field $\mathbb{Z}_2$; therefore, for some set $X$ (basis), $G$ can be represented as the free Boolean group $B(X)$ on $X$, i.e., as $[X]^<\omega$ with zero $\emptyset$, which we denote by $0$, and the operation of symmetric difference, or Boolean sum, which we denote by $\triangle$: $A \triangle B = (A \cup B) \setminus A \cap B$. The elements of $B(X)$ (i.e., finite subsets of $X$) are called words. The length of a word equals its cardinality. The basis $X$ is embedded in $B(X)$ as the set of words of length 1. Given $n \in \omega$, we use the standard notation $B_n(X)$ for the set of words of length at most $n$; thus, $B_0(X) = \{0\}$, $B_1(X) = X \cup \{0\}$, and $B(X) = \bigcup_{n \in \omega} B_n(X)$. For the set of words of length precisely $n$, where $n \in \mathbb{N}$, we use the notation $B_n(X)$; we have $B_n(X) = B_n(X) \setminus B_{n-1}(X)$.

Any free filter $\mathcal{F}$ on an infinite set $X$ determines a topological space $X_{\mathcal{F}} = X \cup \{\ast\}$ with one nonisolated point $\ast$; the neighborhoods of this point are $A \cup \{\ast\}$ for $A \in \mathcal{F}$. The topology of the free Boolean topological group $B(X_{\mathcal{F}}) = [X \cup \{\ast\}]^<\omega$ on this space, that is, the strongest group topology that induces the topology of $X_{\mathcal{F}}$ on $X \cup \{\ast\}$, is described in detail in [3]. One of the possible descriptions is as follows. For each $n \in \mathbb{N}$, we fix an arbitrary sequence of neighborhoods $V_n$ of $\ast$, that is, of $A_n \cup \{\ast\}$, where $A_n \in \mathcal{F}$, and set

$$U(V_n) = \{x \triangle y: x, y \in V_n\} \quad \text{for } n \in \mathbb{N}$$
and
\[ U \left( (V_n)_{n \in \mathbb{N}} \right) = \bigcup_{n \in \mathbb{N}} \left( U(V_1) \triangle U(V_2) \triangle \ldots \triangle U(V_n) \right) \]
\[ = \bigcup_{n \in \mathbb{N}} \{ x_1 \triangle y_1 \ldots \triangle x_n \triangle y_n : x_i, y_i \in A_i \text{ for } i \leq n \}. \]

In particular, the subgroup generated by \((A \cup \{\ast\}) \triangle (A \cup \{\ast\})\) is a neighborhood of zero for any \(A \in \mathcal{F}\). Clearly, for \(n \in \omega\), a set \(Y \subset B_{=2n}(X_{\mathcal{F}})\) is a trace on \(B_{=2n}(X_{\mathcal{F}})\) of a neighborhood of zero in \(B(X_{\mathcal{F}})\) if and only if it contains a set of the form \(\underbrace{((A \cup \{\ast\}) \triangle \ldots \triangle (A \cup \{\ast\}))}_{2n \text{ times}} \cap B_{=2n}(X_{\mathcal{F}}) = [A \cup \{\ast\}]^{2n}\), and a set \(Y \subset B_{=2n}(X) \subset B_{=2n}(X_{\mathcal{F}})\) is a trace on \(B_{=2n}(X)\) of a neighborhood of zero in \(B(X_{\mathcal{F}})\) if and only if it contains a set of the form \(\underbrace{(A \triangle \ldots \triangle A)}_{2n \text{ times}} \cap B_{=2n}(X) = [A]^{2n}\).

The intersection of a neighborhood of zero with \(B_{=k}(X_{\mathcal{F}})\) may be empty for all odd \(k\).

In what follows, we deal with rapid, \(\kappa\)-arrow, and Ramsey filters and ultrafilters.

**Definition 3 (\[5\]).** A filter \(\mathcal{F}\) on \(\omega\) is said to be rapid if every function \(\omega \to \omega\) is majorized by the increasing enumeration of some element of \(\mathcal{F}\).

Clearly, any filter containing a rapid filter is rapid as well; thus, the existence of rapid filters is equivalent to that of rapid ultrafilters. Rapid ultrafilters are also known as semi-\(Q\)-point, or weak \(Q\)-point, ultrafilters. Both the existence and nonexistence of rapid ultrafilters is consistent with ZFC (see, e.g., \[6\] and \[7\]).

The notions of \(\kappa\)-arrow and Ramsey filters are closely related to Ramsey theory, more specifically, to the notion of homogeneity with respect to a coloring, or partition. Given a set \(X\) and positive integers \(m\) and \(n\), by an \(m\)-coloring of \([X]^n\) we mean any map \(c : X \to Y\) of \(X\) to a set \(Y\) of cardinality \(m\). Any such coloring determines a partition of \(X\) into \(m\) disjoint pieces, each of which is assigned a color \(y \in Y\). A set \(A \subset X\) is said to be homogeneous with respect to \(c\), or \(c\)-homogeneous, if \(c\) is constant on \([A]^n\). The celebrated Ramsey theorem (finite version) asserts that, given any positive integers \(k, l, m\), there exists a positive integer \(N\) such that, for any \(k\)-coloring \(c : [X]^l \to Y\), where \(|X| \geq N\) and \(|Y| = k\), there exists a \(c\)-homogeneous set \(A \subset X\) of size \(m\).

We consider \(\kappa\)-arrow and Ramsey filters on any, not necessarily countable, infinite sets. For convenience, we require these filters to be uniform, i.e., nondegenerate in the sense that all of their elements have the same cardinality (equal to that of the underlying set).

**Definition 4.** Let \(\kappa\) be an infinite cardinal, and let \(\mathcal{F}\) be a uniform filter on a set \(X\) of cardinality \(\kappa\).

(i) We say that \(\mathcal{F}\) is a Ramsey filter if, for any 2-coloring \(c : [X]^2 \to \{0, 1\}\), there exists a \(c\)-homogeneous set \(A \in \mathcal{F}\).

(ii) Given an arbitrary cardinal \(\lambda \leq \kappa\), we say that \(\mathcal{F}\) is a \(\lambda\)-arrow filter if, for any 2-coloring \(c : [X]^2 \to \{0, 1\}\), there exists either a set \(A \in \mathcal{F}\) such that \(c([A]^2) = \{0\}\) or a set \(S \subset X\) with \(|S| \geq \lambda\) such that \(c([S]^2) = \{1\}\).

\(^1\)Recall that \(X \subset X_{\mathcal{F}} = X \cup \{\ast\}\), and, therefore, \(B(X)\) (without topology) is naturally embedded in \(B(X_{\mathcal{F}})\) as a subgroup.
Any filter $\mathcal{F}$ on $X$ which is Ramsey or $\lambda$-arrow for $\lambda \geq 3$ is an ultrafilter. Indeed, let $S \subset X$ and consider the coloring $c: |X|^2 \to \{0, 1\}$ defined by
\[
c(\{x, y\}) = \begin{cases} 
0 & \text{if } x, y \in S \text{ or } x, y \in X \setminus S, \\
1 & \text{otherwise.}
\end{cases}
\]
Clearly, any $c$-homogeneous set containing more than two points is contained entirely in $S$ or in $X \setminus S$; therefore, either $S$ or $X \setminus S$ belongs to $\mathcal{F}$, so that $\mathcal{F}$ is an ultrafilter.

According to Theorem 9.6 in [8], if $\mathcal{U}$ is a Ramsey ultrafilter on $X$, then, for any $n < \omega$ and any 2-coloring $c: |X|^n \to \{0, 1\}$, there exists a $c$-homogeneous set $A \in \mathcal{U}$.

It is easy to see that if $\mathcal{F}$ is $\lambda$-arrow, then, for any $A \in \mathcal{F}$ and any $c: |A|^2 \to \{0, 1\}$, there exists either a set $B \in \mathcal{F}$ such that $B \subset A$ and $c([B]^2) = \{0\}$ or a set $S \subset A$ with $|S| \geq \lambda$ such that $c([S]^2) = \{1\}$.

In [9], where $k$-arrow ultrafilters for finite $k$ were introduced, it was shown that the existence of a 3-arrow (ultra)filter on $\omega$ implies that of a $P$-point ultrafilter; therefore, the nonexistence of $\kappa$-arrow ultrafilters for any $\kappa \geq 3$ is consistent with ZFC (see [10]).

On the other hand, the continuum hypothesis implies the existence of $k$-arrow ultrafilters on $\omega$ for any $k \leq \omega$. To formulate a more delicate assumption under which $k$-arrow ultrafilters exist, we need more definitions. Given a uniform filter $\mathcal{F}$ on $\omega$, a set $B \subset \omega$ is called a pseudointersection of $\mathcal{F}$ if the complement $\omega \setminus B$ is finite for all $A \in \mathcal{F}$. The pseudointersection number $p$ is the smallest size of a uniform filter on $\omega$ which has no infinite pseudointersection. It is easy to show that $\omega_1 \leq p \leq 2^\omega$, so that, under the continuum hypothesis, $p = 2^\omega$. It is also consistent with ZFC that, for any regular cardinals $\kappa$ and $\lambda$ such that $\omega_1 \leq \kappa \leq \lambda$, $2^\omega = \lambda$ and $p = \kappa$ (see [11] Theorem 5.1). It was proved in [9] that, under the assumption $p = 2^\omega$ (which is referred to as $P(c)$ in [9]), there exist $\kappa$-arrow ultrafilters on $\omega$ for all $\kappa \leq \omega$. Moreover, for each $k \in \mathbb{N}$, there exists a $k$-arrow ultrafilter on $\omega$ which is not $(k + 1)$-arrow, and there exists an ultrafilter which is $k$-arrow for each $k \in \mathbb{N}$ but is not Ramsey and hence not $\omega$-arrow [11] Theorems 2.1 and 4.10.

In addition to the free group topology of Boolean groups on spaces generated by filters, we consider the Bohr topology on arbitrary abstract and topological groups. This is the weakest group topology with respect to which all homomorphisms to compact topological groups are continuous, or the strongest totally bounded group topology; the Bohr topology on an abstract group (without topology) is defined as the Bohr topology on this group endowed with the discrete topology.

Finally, we need the definition of a minimal dynamical system.

**Definition 5.** Let $G$ be a monoid with identity element $e$. A pair $(X, (T_g)_{g \in G})$, where $X$ is a topological space and $(T_g)_{g \in G}$ is a family of continuous maps $X \to X$ such that $T_e$ is the identity map and $T_{gh} = T_g \circ T_h$ for any $g, h \in G$, is called a topological dynamical system. Such a system is said to be minimal if no proper closed subset of $X$ is $T_g$-invariant for all $g \in G$.

We sometimes identify sequences with their ranges.

All groups considered in this paper are assumed to be infinite, and all filters are assumed to have empty intersection, i.e., to contain the Fréchet filter of all cofinite subsets (and hence be free).
2. Properties of Large Sets

We begin with well-known general properties of large sets defined above. Let $G$ be a group.

1. A set $A \subset G$ is thick if and only if the family $\{gA : g \in G\}$ of all translates of $A$ has the finite intersection property.

   Indeed, this property means that, for every finite subset $F$ of $G$, there exists an $h \in \bigcap_{g \in F} g^{-1}A$, and this, in turn, means that $gh \in A$ for each $g \in F$, i.e., $Fh \subset A$.

2. [3, Theorem 2.4]. A set $A$ is syndetic if and only if $A$ intersects every thick set nontrivially, or, equivalently, if its complement $G \setminus A$ is not thick.

3. A set $A$ is thick if and only if $A$ intersects every syndetic set nontrivially, or, equivalently, if its complement $G \setminus A$ is not syndetic.

4. [3, Theorem 2.4]. A set $A$ is piecewise syndetic if and only if there exists a syndetic set $B$ and a thick set $C$ such that $A = B \cap C$.

5. [12, Theorem 4.48]. A set $A$ is thick if and only if $A^{-G} = \{p \in \beta G : A \in p\}$ (the closure of $A$ in the Stone–Čech compactification $\beta G$ of $G$ with the discrete topology) contains a left ideal of the semigroup $\beta G$.

6. [12, Theorem 4.48]. A set $A$ is syndetic if and only if every left ideal of $\beta G$ intersects $A^{-G}$.

7. The families of thick, syndetic, and piecewise syndetic sets are closed with respect to taking supersets.

8. Thickness, syndeticity, and piecewise syndeticity are translation invariant.

9. [3, Theorem 2.5]. Piecewise syndeticity is partition regular, i.e., whenever a piecewise syndetic set is partitioned into finitely many subsets, one of these subsets is piecewise syndetic.

10. [3, Theorem 2.4]. For any thick set $A \subset G$, there exists an infinite sequence $B = (b_n)_{n \in \mathbb{N}}$ in $G$ such that

    $$\text{FP}(B) = \{x_{n_1}x_{n_2}\ldots x_{n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N}, n_1 < n_2 < \cdots < n_k\}$$

    is contained in $A$.

11. Any IP*-set in $G$, i.e., a set intersecting any infinite set of the form FP($B$), is syndetic. This immediately follows from properties 2 and 10.

3. Fat Sets

As mentioned at the beginning of this section, in [2], Reznichenko and the author introduced a new class of large sets, which we called fat; they have played the key role in our construction of nonclosed discrete subsets in topological groups.

**Definition 6.** We say that a subset $A$ of a group $G$ is fat in $G$ if there exists a positive integer $m$ such that any $m$-element set $F$ in $G$ contains a two-element subset $D$ for which $D^{-1}D \subset A$. The least number $m$ with this property is called the fatness of $A$.

We shall refer to fat sets of fatness $m$ as $m$-fat sets.

In a similar manner, $\kappa$-fat sets for any cardinal $\kappa$ can be defined.

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[Later, we have found out that similar subsets of $\mathbb{Z}$ had already been used in [13]: the $\Delta^*_n$-sets considered there and $n$-fat subsets of $\mathbb{Z}$ are very much alike.]
Theorem 2 is valid.

Definition 7. Given a cardinal $\kappa$, we say that a subset $A$ of a group $G$ is $\kappa$-fat in $G$ if any set $S \subset G$ with $|S| = \kappa$ contains a two-element subset $D$ for which $D^{-1}D \subset A$.

The notions of an $\omega$-fat and a $k$-fat set are very similar to but different from those of $\Delta^*$- and $\Delta^*_k$-sets. $\Delta^*$-Sets were introduced and studied in [3] for arbitrary semigroups, and $\Delta^*_k$-sets with $k \in \mathbb{N}$ were defined in [13] for the case of $\mathbb{Z}$.

Definition 8. Given a finite of countable cardinal $\kappa$ and a sequence $(g_n)_{n \in \kappa}$ in a group $G$, we set

$$\Delta((g_n)_{n \in \kappa}) = \{ x \in G : \text{there exist } m < n < \kappa \text{ such that } x = g_m^{-1}g_n \}$$

and

$$\Delta_D((g_n)_{n \in \kappa}) = \{ x \in G : \text{there exist } m < n < \kappa \text{ such that } x = g_ng_m^{-1} \}.$$

A subset of a group $G$ is called a right (left) $\Delta^*_k$-set if it intersects $\Delta_I((g_n)_{n \in \kappa})$ (respectively, $\Delta_D((g_n)_{n \in \kappa})$) for any one-to-one sequence $(g_n)_{n \in \kappa}$ in $G$. $\Delta^*_\omega$-sets are referred to as $\Delta^*$-sets.

Remark. For any one-to-one sequence $S = (g_n)_{n \in \kappa}$ in a Boolean group with zero $0$, we have $\Delta_I(S) = \Delta_D(S) = (S \setminus S) \setminus \{ 0 \}$. Hence any $\kappa$-fat set in such a group is a right and left $\Delta^*_\omega$-set. Moreover, the only difference between $\Delta^*_\omega$- and $\kappa$-fat sets in a Boolean group is in that the latter must contain $0$.

The most obvious feature distinguishing fatness among other notions of largeness is symmetry (fatness has no natural right and left versions). In return, translation invariance is sacrificed. Thus, in studying fat sets, it makes sense to consider also their translates.

Clearly, a 2-fat set in a group must coincide with this group. The simplest nontrivial example of a fat set is a subgroup of finite index $n$; its fatness equals $n + 1$ (any $(n + 1)$-element subset has two elements $x$ and $y$ in the same coset, and both $x^{-1}y$ and $y^{-1}x$ belong to the subgroup).

It seems natural to refine the definition of fat sets by requiring $A \cap F^{-1}F$ to be of prescribed size rather than merely nontrivial. However, this (and even a formally stronger) requirement does not introduce anything new.

Proposition 1 ([2 Proposition 1.1]). For any fat set $A$ in a group $G$ and any positive integer $n$, there exists a positive integer $m$ such that any $m$-element set $F$ in $G$ contains an $n$-element subset $F'$ for which $F^{-1}F' \subset A$.

Indeed, considering the coloring $c : [G]^2 \rightarrow \{ 0, 1 \}$ defined by $c\{ x, y \} = 1 \iff x^{-1}y, y^{-1}x \in A$ and applying the finite Ramsey theorem, we find a $c$-homogeneous set $F$ of size $n$ (provided that $m$ is large enough). If $n$ is no smaller than the fatness of $A$ (which we can assume without loss of generality), then $c([F]^2) = \{ 1 \}$.

There is yet another important distinguishing feature of fat sets, namely, the finite intersection property. Neither thick, syndetic, nor piecewise syndetic sets have this property. (Indeed, the disjoint sets of even and odd numbers are syndetic in $\mathbb{Z}$, and $\bigcup_{i \geq 0}[2^i, 2^{i+1}] \cap \mathbb{Z}$ and $\bigcup_{i \geq 1}[2^{i-1}, 2^i]$ are thick.) The following theorem is valid.

Theorem 2 ([2]). Let $G$ be a group.

(i) If $A \subset G$ is fat, then so is $A^{-1}$. 


If $A \subset B \subset G$ and $A$ is fat, then so is $B$.

(iii) If $A \subset G$ and $B \subset G$ are fat, then so is $A \cap B$.

Assertions (i) and (ii) are obvious, and (iii) follows from Proposition 1.

**Proposition 3.** If $G$ is a group, $S \subset G$, and $S \cap (SS \cup S^{-1}S^{-1}) = \emptyset$, then $G \setminus S$ is 3-fat.

**Proof.** Take any three different elements $a, b, c \in G$. We must show that the identity element $e$ belongs to $G \setminus S$ (which is true by assumption) and either $(a^{-1}b)^{\pm 1} \in G \setminus S$, $(b^{-1}c)^{\pm 1} \in G \setminus S$, or $(c^{-1}a)^{\pm 1} \in G \setminus S$. Assume that, on the contrary, $(a^{-1}b)^{\epsilon} \in S$ (i.e., $a^{-1}b \in S^\epsilon$), $b^{-1}c \in S^\delta$, and $c^{-1}a \in S^\gamma$ for some $\epsilon, \delta, \gamma \in \{-1, 1\}$. At least two of the three numbers $\epsilon, \delta$, and $\gamma$ are equal. Suppose for definiteness that $\epsilon = \delta = \gamma$. Then we have $c^{-1}a = c^{-1}bb^{-1}a \in S^{-\epsilon}S^{-\epsilon}$, which contradicts the assumption $S \cap (S^2 \cup S^{-2}) = \emptyset$.

We see that the family of fat sets in a group resembles, in some respects, a base of neighborhoods of the identity element for a group topology. However, as we shall see in the next section, it does not generate a group topology even in a Boolean group: any Boolean group has a 3-fat subset $A$ containing no set of the form $B \triangle B$ for fat $B$. On the other hand, very many groups admit of group topologies in which all neighborhoods of the identity element are fat; for example, such are topologies generated by normal subgroups of finite index. A more precise statement is given in the next section. Before turning to related questions, we consider how fat sets fit into the company of other large sets.

We begin with a comparison of fat and syndetic sets.

**Proposition 4** (see [2, Proposition 1.7]). Let $G$ be any group with identity element $e$. Any fat set $A$ in $G$ is syndetic, and its syndeticity index is less than its fatness.

**Proof.** Let $n$ denote the fatness of $A$. Take a finite set $F \subset G$ with $|F| = n - 1$ such that $x^{-1}y \notin A$ or $y^{-1}x \notin A$ for any different $x, y \in F$. Pick any $g \in G \setminus F$. Since $|F \cup \{g\}| = n$, it follows that $x^{-1}g \in A$ and $g^{-1}x \in A$ for some $x \in F$, whence $g \in xA$, i.e., $G \setminus F \subset FA$. By definition, the identity element of $G$ belongs to $A$, and we finally obtain $G = FA$.

Examples of nonfat syndetic sets are easy to construct: any coset of a finite-index subgroup in a group is syndetic, while only one of them (the subgroup itself) is fat. However, the existence of syndetic sets with nonfat translates is not so obvious. An example of such a set in $\mathbb{Z}$ can be extracted from [13].

**Example 1.** There exists a syndetic set in $\mathbb{Z}$ such that none of its translates is fat. This is, e.g., the set constructed in [13, Theorem 4.3]. Namely, let $C = \{0, 1\}^\mathbb{Z}$, and let $\tau : C \to C$ be the shift, i.e., the map defined by $\tau(f)(n) = f(n + 1)$ for $f \in C$. It was proved in [13, Theorem 4.3] that if $M \subset C$ is a minimal closed $\tau$-invariant subset and the dynamical system $(M, (\tau^n)_{n \in \mathbb{Z}})$ satisfies a certain condition, then the support of any $f \in M$ is syndetic but not piecewise Bohr; the latter means that it cannot be represented as the intersection of a thick set and a set having nonempty interior in the Bohr topology on $\mathbb{Z}$. Clearly, any translate of $\text{supp } f$ has

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3Then the support of each $f \in M$ is syndetic in $\mathbb{Z}$ (see, e.g., [14]).
4Namely, is weakly mixing; see, e.g., [13].
Suppose that there exist different $x, y$ and of all $s$ where $s$ contains precisely one of the letters $\pm k_1, \ldots, \pm k_{n-1}, k_n$. Therefore, are canceled with letters of $w$. Obviously, the set on the right-hand side of this inclusion is syndetic; therefore, so is $S'$. If $w = 0$, then any translate of $S_0$ is syndetic. Since each $n$-tuple $(k_1, \ldots, k_n)$ of different integers is piecewise Bohr. Since every $n$-fat set is a $\Delta^*_n$-set, it follows that the translates of $\text{supp} f$ cannot be fat.

Bearing in mind our particular interest in Boolean groups, we also give a similar example for a Boolean group.

**Example 2.** We construct a syndetic set in the Boolean group $B(\mathbb{Z})$ with nonfat translates. Let $S$ be a syndetic set in $\mathbb{Z}$ all of whose translates are not $\Delta^*_n$-sets for all $n$ (see Example 1). By definition, $\mathbb{Z} = \bigcup_{k \leq r} (s_k + S)$ for some $r \in \mathbb{N}$ and different $s_1, \ldots, s_r \in \mathbb{Z}$. We set

$$S' = \bigcup_{k \leq r} S_k.$$ 

We have

$$s_k \triangle S_k = \{x_1 \triangle \ldots \triangle x_n : n \in \mathbb{N}, x_i \in \mathbb{Z} \text{ for } i \leq n, x_i \neq x_j \text{ for } i \neq j, \{x_1, \ldots, x_n\} \cap \{s_1, \ldots, s_r\} = \emptyset, \sum_{i \leq n} x_i \in s_k + S\}, \quad k \leq r.$$ 

Since $\bigcup_{k \leq r} (s_k + S) = \mathbb{Z}$, it follows that

$$\bigcup_{k \leq r} (s_k \triangle S') \subset \{x_1 \triangle \ldots \triangle x_n : n \in \mathbb{N}, x_i \in \mathbb{Z} \text{ for } i \leq n, \{x_1, \ldots, x_n\} \cap \{s_1, \ldots, s_r\} = \emptyset\}.$$ 

Obviously, the set on the right-hand side of this inclusion is syndetic; therefore, so is $S'$.

Let us show that no translate of $S'$ is fat. Suppose that, on the contrary, $k, n \in \mathbb{N}$, $z_1, \ldots, z_k \in \mathbb{Z}$, $w = z_1 \triangle \ldots \triangle z_k$ and $w \triangle S'$ is $n$-fat. Take any different $k_1, \ldots, k_n \in \mathbb{Z}$ larger than the absolute values of all elements of $w$ (which is a finite subset of $\mathbb{Z}$) and of all $s_i$, $i \leq r$. We set

$$F = \{k_1, k_1 \triangle (-k_1) \triangle k_2, k_1 \triangle (-k_1) \triangle k_2 \triangle (-k_2) \triangle k_3, \ldots, k_1 \triangle (-k_1) \triangle k_2 \triangle (-k_2) \triangle \ldots \triangle k_{n-1} \triangle (-k_{n-1}) \triangle k_n\}.$$ 

Suppose that there exist different $x, y \in F$ for which $x \triangle y \in w \triangle S'$, i.e., there exist $i, j \leq n$ for which $i < j$ and

$$k_1 \triangle (-k_1) \triangle \ldots \triangle k_{i-1} \triangle (-k_{i-1}) \triangle k_i \triangle (-k_i) \triangle \ldots \triangle k_{j-1} \triangle (-k_{j-1}) \triangle k_j$$

$$= k_1 \triangle k_i \triangle (-k_i) \triangle k_{i+1} \triangle (-k_{i+1}) \triangle \ldots \triangle k_{j-1} \triangle (-k_{j-1}) \triangle k_j = w \triangle s \in w \triangle S',$$

where $s$ is an element of $S'$ and hence belongs to $S'_l$ for some $l \leq r$, which means, in particular, that $s$ contains precisely one of the letters $s_1, \ldots, s_r$, namely, $s_l$. There are no such letters among $\pm k_1, \ldots, \pm k_{j-1}, k_j$. Therefore, one of the letters $z_m$ (say $z_1$) is $s_l$. The other letters of $w$ do not equal $\pm k_1, \ldots, \pm k_{j-1}, k_j$ either and, therefore, are canceled with letters of $s \in S'$ in the word $w + s$. By the definition of
the set $S'$ containing $s$, one letter of the word $w$ (namely, $z_1 = s_1$) belongs to the set 
\{s_1, \ldots, s_r\} and the other letters do not. Since the sum (in $\mathbb{Z}$) of the integer-letters of $s$ belongs to $2s_1 + S$ (by the definition of $S'$) and $s_1 = z_1$, it follows that the sum of letters of $w + s$ belongs to $S + z_1 - z_2 - \cdots - z_k$ and the letter $z_1$ is determined uniquely for the given word $w$. To obtain a contradiction, it remains to recall that the translates of $S$ (in particular, $S + z_1 - z_2 - \cdots - z_k$) are not $\Delta^*_n$-sets in $\mathbb{Z}$ and choose $k_1, \ldots, k_n$ so that $\{k_j - k_i : i < j \leq n\} \cap (S + z_1 - z_2 - \cdots - z_k) = \emptyset$.

**Example 3.** There exist fat sets which are not thick and thick sets which are not fat. Indeed, as mentioned, any proper finite-index group is fat, but it cannot be thick by the first property in the list of properties of large sets given above.

An example of a nonfat thick set is, e.g., any thick nonsyndetic set. In an infinite Boolean group $G$, such a set can be constructed as follows. Take any basis $X$ in $G$ (so that $G = B(X)$), fix any nonsyndetic thick set $T$ in $\mathbb{N}$ (say $T = \bigcup_n ([a_n, b_n]\cap \mathbb{N})$, where the $a_n$ and $b_n$ are numbers such that the $b_n - a_n$ and the $a_{n+1} - b_n$ increase without bound), and consider the set

$$A = \{x_1 \triangle \cdots \triangle x_n \in B(X) : n \in T, x_i \in X \text{ for } i \leq n, x_i \neq x_j \text{ for } i \neq j\}$$

of all words in $B(X)$ whose lengths belong to $T$. The thickness of this set is obvious (by the same property 1), because the translate of $A$ by any word $g \in B(X)$ of any length $l$ surely contains all words whose lengths belong to $\bigcup_n ([a_n + l, b_n - l]\cap \mathbb{N}) \subset T$ and, therefore, intersects $A$. However, $A$ is not fat, because it misses all words whose lengths belong to the set $\bigcup_n ((b_n, a_{n+1}) \cap \mathbb{N})$. The last set contains at least one even positive integer $2k$. It remains to choose different points $x_1, x_2, \ldots$ in $X$, set $B = \{x_{kn+1} \triangle x_{kn+2} \triangle \cdots \triangle x_{kn+k} : n \in \omega\}$, and note that all nonempty words in $B \triangle A$ have length $2k$. Therefore, $A$ is disjoint from $B \triangle A$ (much more from $F \triangle A$ for any finite $F \subset B$). Note that the translates of $A$ are not fat either, because both thickness and (non)syndeticity are translation invariant.

**Proposition 5.** Let $G$ be any group with identity element $e$.

(i) If a set $A$ in $G$ is 3-fat, then $(G \setminus A)^{-1}(G \setminus A) \subset A$.

(ii) If a set $A$ in $G$ is 3-fat, then either $AA^{-1} = G$ or $A$ is a subgroup of index 2.

**Proof.** (i) Suppose that $A$ is a 3-fat subset of a group $G$ with identity element $e$. Take any different $x, y \notin A$ (if there exist no such elements, then there is nothing to prove). By definition, the set $\{x, y, e\}$ contains a two-element subset $D$ for which $D^{-1}D \subset A$. Clearly, $D \neq \{x, e\}$ and $D \neq \{y, e\}$. Therefore, $x^{-1}y \in A$ and $y^{-1}x \in A$ (and $e \in A$, too), whence $(G \setminus A)^{-1}(G \setminus A) \subset A$.

(ii) If $AA^{-1} \neq G$, then there exists a $g \in G$ for which $gA \cap A = \emptyset$. If $A$ is, in addition, 3-fat, then (ii) implies $(gA)^{-1}gA = A^{-1}A \subset A$, which means that $A$ is a subgroup of $G$. According to (i), $A$ is syndetic of index at most 2; in fact, its index is precisely 2, because $A$ does not coincide with $G$. \hfill \Box

**4. Quotient sets**

In \cite{k} sets of the form $AA^{-1}$ or $A^{-1}A$ were naturally called quotient sets. We shall refer to the former as right quotient sets and to the latter as left quotient sets. Thus, a set in a group $G$ is $m$-fat if it intersects nontrivially the left quotient set of any $m$-element subset of $G$. Quotient sets play a very important role in combinatorics, and their interplay with large sets is quite amazing.
First, the passage to right quotient sets annihilates the difference between syndetic and piecewise syndetic sets.

**Theorem 6** (see [3 Theorem 3.9]). For each piecewise syndetic subset \( A \) of a group \( G \), there exists a syndetic subset \( B \) of \( G \) such that \( BB^{-1} \subset AA^{-1} \) and the syndeticity index of \( B \) does not exceed the thickness index of \( A \).

Briefly, the construction of \( B \) given in [3] is as follows: we take a finite set \( T \) such that \( TA \) is thick and, for each finite \( F \subset G \), let \( \Phi_F = \{ \varphi \in T^G : \bigcap_{x \in F} x^{-1}\varphi(x)A \neq \emptyset \} \). Then we pick \( \varphi^* \) in the intersection of all \( \Phi_F \) (which exists since the product space \( T^G \) is compact) and let \( B = \{ \varphi^*(x)^{-1}x : x \in G \} \). Since \( \varphi^*(G) \subset T \), it follows that \( TB = G \), which means that \( B \) is syndetic and its index does not exceed \( |T| = t \). Moreover, for any finite \( F \subset B \), there exists a \( g \in G \) such that \( Fg \subset A \), and this implies \( BB^{-1} \subset AA^{-1} \).

In Theorem 6, right quotient sets cannot be replaced by left ones: there are examples of piecewise syndetic sets \( A \) such that \( A^{-1}A \) does not contain \( B^{-1}B \) for any syndetic \( B \). One of such examples is provided by the following theorem.

**Theorem 7.**

(i) If a subset \( A \) of a group \( G \) is syndetic of index \( s \), then \( A^{-1}A \) is fat, and its fatness does not exceed \( s + 1 \).

(ii) If a subset \( A \) of an Abelian group \( G \) is piecewise syndetic of thickness index \( t \), then \( A - A \) is fat, and its fatness does not exceed \( t + 1 \).

(iii) There exists a group \( G \) and a thick (in particular, piecewise syndetic) set \( A \subset G \) such that \( A^{-1}A \) is not fat and, therefore, does not contain \( B^{-1}B \) for any syndetic set.

(iv) If a subset \( A \) of a group \( G \) is thick, then \( AA^{-1} = G \).

**Proof.** (i) Suppose that \( FA = G \), where \( F = \{g_1, \ldots, g_s\} \). Any \((s+1)\)-element subset of \( G \) has at least two points \( x \) and \( y \) in the same “coset” \( gA \). We have \( x = g_1a \) and \( y = g_2a' \), where \( a, a' \in A \). Thus, \( x^{-1}y, y^{-1}x \in A^{-1}A \).

Assertion (ii) follows immediately from (i) and Theorem 6.

Let us prove (iii). Consider the free group \( G \) on two generators \( a \) and \( b \) and let \( A \) be the set of all words in \( G \) whose last letter is \( a \). Then \( A \) is thick (given any finite \( F \subset G \), we have \(FAa^n \subset A \) for sufficiently large \( n \)). Clearly, all nonidentity words in \( A^{-1}A \) contain \( a \) or \( a^{-1} \). Therefore, if \( F \subset G \) consists of words of the form \( b^n \), then the intersection \( F^{-1}F \cap A^{-1}A \) is trivial, so that \( A^{-1}A \) is not fat.

Finally, to prove (iv), take any \( g \in G \). We have \( A \cap gA \neq \emptyset \) (by property 1 in our list of properties of large sets). This means that \( g \in AA^{-1} \).

We see that the right quotient sets \( AA^{-1} \) of thick sets \( A \) are utmostly fat, while the left quotient sets \( A^{-1}A \) may be rather slim. In the Abelian case, the difference sets of all thick sets coincide with the whole group.

It is natural to ask whether condition (i) in Theorem 6 characterizes fat sets in groups. In other words, given any fat set \( A \) in a group, does there exist a syndetic (or, equivalently, piecewise syndetic) set \( B \) such that \( B^{-1}B \subset A \) (or \( BB^{-1} \subset A \))? The answer is no, even for thick 3-fat sets in Boolean groups. The idea of the following example was suggested by arguments in paper [13] and in John Griesmer’s note [15], where the group \( Z \) was considered.

**Example 4.** Let \( G \) be a countable Boolean group with zero \( 0 \). Any such group can be treated as the free Boolean group on \( Z \). We set

\[
A = G \setminus \{m \triangle n = \{m, n\} : m, n \in \mathbb{Z}, m < n, n - m = k^3 \text{ for some } k \in \mathbb{N}\}.
\]
Clearly, $A$ is thick (if $F \subset G$ is finite and a word $g \in G$ is sufficiently long, then all words in the set $F \triangle g$ have more than two letters and, therefore, belong to $A$). Let us prove that $A$ is 3-fat. Take any different $a, b, c \in G$. We must show that $a \Delta b \in A$, $b \Delta c \in A$, or $a \Delta c \in A$. We can assume that $c = 0$; otherwise, we translate $a$, $b$, and $c$ by $c$, which does not affect the Boolean sums. Thus, it suffices to show that, given any different nonzero $x, y \notin G$, we have $x \Delta y \in A$. The condition $x, y \notin G$ means that $x = \{k, l\}$, where $k < l$ and $l - k = r^3$ for some $r \in \mathbb{Z}$, and $y = \{m, n\}$, where $m < n$ and $n - m = s^3$ for some $s \in \mathbb{Z}$. Suppose for definiteness that $n > l$ or $n = l$ and $m > k$. If $x \Delta y \notin A$, then either $k = m$ and $l - n = t^3$ for some $t \in \mathbb{N}$, $l = m$ and $n - k = t^3$ for some $t \in \mathbb{N}$, or $l = n$ and $m - k = t^3$ for some $t \in \mathbb{N}$. In the first case, we have $l - k = l - n + n - m$, i.e., $r^3 = t^3 + s^3$; in the second, we have $n - k = n - m + l - k$, i.e., $t^3 = s^3 + r^3$; and in the third, we have $l - k = n - m + m - k$, i.e., $r^3 = s^3 + t^3$. In any case, we obtain a contradiction with Fermat’s theorem.

It remains to prove that there exists no syndetic (and hence no piecewise syndetic) $B \subset G$ for which $B \triangle B \subset A$. Consider any syndetic set $B$. Let $F = \{f_1, \ldots, f_k\} \subset G$ be a finite set for which $FB = G$, and let $m$ be the maximum absolute value of all letters of words in $F$ (recall that all letters are integers). To each $n \in \mathbb{Z}$ with $|n| > m$ we assign a word $f_i \in F$ for which $n \in f_i \Delta B$; if there are several such words, then we choose any of them. Thereby, we divide the set of all integers with absolute value larger than $m$ into $k$ pieces $I_1, \ldots, I_k$. To accomplish our goal, it suffices to show that there is a piece $I_i$ containing two integers $r$ and $s$ such that $r - s = z^3$ for some $z \in \mathbb{Z}$. Indeed, in this case, we have $r \in f_i \Delta B$ and $s \in f_i \Delta B$, so that $r \Delta s \in B \Delta B$. On the other hand, $r \Delta s \notin A$.

From now on, we treat the pieces $I_1, \ldots, I_k$ as subsets of $\mathbb{Z}$. We have $\mathbb{Z} = \{-m, -m + 1, \ldots, 0, 1, \ldots, m\} \cup I_1 \cup \cdots \cup I_k$. Since piecewise syndeticity is partition regular (see property 9 of large sets), one of the sets $I_i$, say $I_i$, is piecewise syndetic. Therefore, by Theorem 5, $I_i \cap S \cap S - S$ for some syndetic set $S \subset \mathbb{Z}$.

Let $d^*(S)$ denote the upper Banach density of $S$, i.e.,

$$d^*(S) = \lim_{n \to \infty} \limsup_{d \to \infty} \frac{|S \cap \{n, n+1, \ldots, n+d\}|}{d}.$$ 

The syndeticity of $S$ in $\mathbb{Z}$ implies the existence of an $N \in \mathbb{N}$ such that every interval of integers longer than $N$ intersects $S$. Clearly, we have $d^*(S) \geq 1/N$. Proposition 3.19 in [14] asserts that if $X$ is a set in $\mathbb{Z}$ of positive upper Banach density and $p(t)$ is a polynomial taking on integer values at the integers and including 0 in its range on the integers, then there exist $x, y \in X$, $x \neq y$, and $z \in \mathbb{Z}$ such that $x - y = p(z)$ (as mentioned in [14], this was proved independently by Sárközy). Thus, there exist different $x, y \in S$ and a $z \in \mathbb{Z}$ for which $x - y = z^3$. Since $S - S \subset I_i - I_i$, it follows that $z^3 = r - s$ for some $r, s \in I_i$, as desired.

5. Large Sets and Topology

In the context of topological groups quotient sets arise again, because for each neighborhood $U$ of the identity element, there must exist a neighborhood $V$ such that $V^{-1}V \subset U$ and $VV^{-1} \subset U$. Thus, if we know that a group topology consists of piecewise syndetic sets, then, in view of Theorem 6, we can assert that all open sets are syndetic, and so on. Example 4 shows that if $G$ is any countable Boolean topological group and all 3-fat sets are open in $G$, then some nonempty open sets in this group are not piecewise syndetic. Thus, all syndetic or piecewise syndetic
subsets of a group $G$ do not generally form a group topology. Even their quotient (difference in the Abelian case) sets are insufficient; however, it is known that double difference sets of syndetic (and hence piecewise syndetic) sets in Abelian groups are neighborhoods of zero in the Bohr topology. These and many other interesting results concerning a relationship between Bohr open and large subsets of abstract and topological groups can be found in [16, 17]. As to group topologies in which all open sets are large, the situation is very simple.

**Theorem 8.** For any topological group $G$ with identity element $e$, the following conditions are equivalent:

(i) all neighborhoods of $e$ in $G$ are piecewise syndetic;
(ii) all open sets in $G$ are piecewise syndetic;
(iii) all neighborhoods of $e$ in $G$ are syndetic;
(iv) all open sets in $G$ are syndetic;
(v) all neighborhoods of $e$ in $G$ are fat;
(vi) $G$ is totally bounded.

Proof. The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) follow from the obvious translation invariance of piecewise syndeticity and syndeticity. Theorem 6 implies (i) $\iff$ (iii), Theorem 7 (i) implies (iii) $\Rightarrow$ (v), and Proposition 4 implies (v) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is trivial. Finally, (vi) $\iff$ (iii) by the definition of total boundedness. \hfill $\square$

Thus, the Bohr topology on a (discrete) group is the strongest group topology in which all open sets are syndetic (or, equivalently, piecewise syndetic, or fat).

For completeness, we also mention the following corollary of Theorem 7 and Theorem 3.12 in [3], which relates fat sets to topological dynamics.

**Corollary 9.** If $G$ is an Abelian group with zero $0$, $X$ is a compact Hausdorff space, and $(X, (T_g)_{g \in G})$ is a minimal dynamical system, then the set $\{g \in G : U \cap T_g^{-1}U \neq \emptyset\}$ is fat for every nonempty open subset $U$ of $X$.

6. Fat and Discrete Sets in Topological Groups

As mentioned above, fat sets were introduced in [2] to construct discrete sets in topological groups. Namely, given a countable topological group $G$ whose identity element $e$ has nonrapid filter $\mathcal{F}$ of neighborhoods, we can construct a discrete set with precisely one limit point in this group as follows. The nonrapidity of $\mathcal{F}$ means that, given any sequence $(m_n)_{n \in \mathbb{N}}$ of positive integers, there exist finite sets $F_n \subset G$, $n \in \mathbb{N}$, such that each neighborhood of $e$ intersects some $F_n$ in at least $m_n$ points (see [7, Theorem 3 (3)]). Thus, if we have a decreasing sequence of closed $m_n$-fat sets $A_n$ in $G$ such that $\bigcap A_n = \{e\}$, then the set

$$D = \bigcup_{n \in \mathbb{N}} \{a^{-1}b : a \neq b, a, b \in F_n, a^{-1}b \in A_n\}$$

is discrete (because $e \notin D$ and each $g \in G \setminus \{e\}$ has a neighborhood of the form $G \setminus A_n$ which contains only finitely many elements of $D$), and $e$ is the only limit point of $D$ (because, given any neighborhood $U$ of $e$, we can take a neighborhood $V$ such that

---

5It follows, in particular, that, given any piecewise syndetic set $A$ in an Abelian group, there exists an infinite sequence of fat sets $A_1, A_2, \ldots$ such that $A_1 - A_1 \subset A + A - A - A$ and $A_{n+1} - A_{n+1} \subset A_n$ for all $n$ (because all Bohr open sets are syndetic).
V^{-1}V \subset U; we have \( |V \cap F_n| \geq m_n \) for some \( n \), and hence \((V \cap F_n)^{-1}(V \cap F_n) \cap A_n \neq \emptyset\), so that \( U \cap D \neq \emptyset \). It remains to find a family of closed fat sets with trivial intersection and make it decreasing.

The former task is easy to accomplish in any topological group: by Proposition 3 in any topological group \( G \), the complements to open neighborhoods \( gU \) of all \( g \in G \) satisfying the condition \( gU \cap (U^2 \cup (U^{-1})^2) = \emptyset \) form a family of closed 3-fat sets with trivial intersection. In countable groups, this family can be made decreasing by using Theorem 2, according to which the family of fat sets has the finite intersection property. Unfortunately, no similar argument applies in the uncountable case, because countable intersections of fat sets may be very small. Thus, in \( Z^\omega \), the intersection of the 3-fat sets \( H_n = \{ f \in Z^\omega : f(n) = 0 \} \) (each of which is a subgroup of index 2 open in the product topology) is trivial.

7. Large Sets in Boolean Groups

In the case of Boolean groups, many assertions concerning large sets can be refined. For example, properties 10 and 11 of large sets are stated as follows.

**Proposition 10.** (i) For any thick set \( T \) in a Boolean group \( G \) with zero \( 0 \), there exists an infinite subgroup \( H \) of \( G \) for which \( T \cup \{ 0 \} \supseteq H \).
(ii) Any set which intersects nontrivially all infinite subgroups in a Boolean group \( G \) is syndetic.

Note that this is not so in non-Boolean groups: the set \( \{ n! : n \in \mathbb{N} \} \) intersects any infinite subgroup in \( Z \), but it is not syndetic, because the gaps between neighboring elements are not bounded. The complement of this set contains no infinite subgroups, and it is thick by property 2 of large sets.

Another specific feature of thick sets in Boolean groups is given by the following proposition.

**Proposition 11.** For any thick set \( T \) in a countable Boolean group \( G \) with zero \( 0 \), there exists a set \( A \subset G \) such that \( T \cup \{ 0 \} = A \triangle A \) (and \( A \triangle A \triangle A \triangle A = G \) by Theorem 1(iv)).

**Proof.** First, note that \( |A| = |G| \). Any Boolean group is algebraically free; therefore, we can assume that \( G = B(X) \) for a set \( X \) with \( |X| = |A| \). Let \( A_2 = A \cap B_{=2}(X) = \{ (x, y) : x \triangle y \in A, x, y \in X \} \) be the intersection of \( A \) with the set of words of length 2. We have \( |A_2| = |X| \), because \( A \) must intersect nontrivially each countable set of the form \( Y \triangle Y \) for
Y ⊂ X. Consider the coloring \( c: [X]^2 \to \{0, 1\} \) defined by
\[
c(\{x, y\}) = \begin{cases} 
0 & \text{if } \{x, y\} \in A_2, \\
1 & \text{otherwise.}
\end{cases}
\]

According to the well-known Erdős–Dushnik–Miller theorem \( \kappa \to (\kappa, \omega)_2 \) (see, e.g., [18]), there exists either an infinite set \( Y \subset X \) for which \( [Y]^2 \cap A_2 = \emptyset \) or a set \( Y \subset X \) of cardinality \( |X| \) for which \( [Y]^2 \subset A_2 \). The former case cannot occur, because \( [Y]^2 = Y \triangle Y \) in \( B(X) \), \( [Y]^2 \subset B_{=2}(X) \), and \( A_2 \) is a \( \Delta^* \)-set. Thus, the latter case occurs, and we set \( B = Y \). \( \square \)

We have already distinguished between fat sets and translates of syndetic sets in Boolean groups (see Example 2). For completeness, we give the following example.

**Example 5.** The countable Boolean group \( B(\mathbb{Z}) \) contains an IP*-set (see Property 11 of large sets) which is not a \( \Delta^* \)-set. An example of such a set is constructed from the corresponding example in \( \mathbb{Z} \) (see [14] p. 177) in precisely the same way as Example 2.

8. LARGE SETS IN FREE BOOLEAN TOPOLOGICAL GROUPS

As shown in Section 5, given any Boolean group \( G \), the filter of fat sets in \( G \) cannot be the filter of neighborhoods of zero for a group topology, because not all fat and even 3-fat sets are neighborhoods of zero in the Bohr topology. Moreover, if we fix any basis \( X \) in \( G \), so that \( G = B(X) \), then not all traces of 3-fat sets on the set \( B_{=3}(X) \) of two-letter words contain those of Bohr neighborhoods of zero. However, there are natural group topologies on \( B(X) \) such that the topologies which they induce on \( B_2(X) \) contain those generated by \( n \)-fat sets.

**Theorem 13.** Let \( k \in \mathbb{N} \), and let \( \mathcal{F} \) be a filter on an infinite set \( X \). Then the following assertions hold.

(i) For \( k \neq 4 \), the trace of any \( k \)-fat subset of \( B(X) \) on \( B_2(X) \subset B_2(X, \mathcal{F}) \) contains that of a neighborhood of zero in the free group topology of \( B(X, \mathcal{F}) \) if and only if \( \mathcal{F} \) is a \( k \)-arrow filter.

(ii) If the trace of any 4-fat set on \( B_2(X) \) contains that of a neighborhood of zero in the free group topology of \( B(X, \mathcal{F}) \), then \( \mathcal{F} \) is a 4-arrow filter, and if \( \mathcal{F} \) is a 4-arrow filter, then the trace of any 3-fat set on \( B_2(X, \mathcal{F}) \) contains that of a neighborhood of zero in the free group topology of \( B(X, \mathcal{F}) \).

(iii) The trace of any \( \omega \)-fat set on \( B_2(X) \) contains that of a neighborhood of zero in the free group topology of \( B(X, \mathcal{F}) \) if and only if \( \mathcal{F} \) is an \( \omega \)-arrow ultrafilter.

The proof of this theorem uses the following lemma.

**Lemma 14.** (i) If \( k \neq 4 \), \( w_1, \ldots, w_k \in B(X) \), and \( w_i \triangle w_j \in B_{=2}(X) \) for any \( i < j \leq k \), then there exist \( x_1, \ldots, x_k \in X \) such that \( w_i \triangle w_j = x_i \triangle x_j \) for any \( i < j \leq k \).

(ii) If \( k = 4 \), \( w_1, w_2, w_3, w_4 \in B(X) \), and \( w_i \triangle w_j \in B_{=2}(X) \) for any \( i < j \leq 4 \), then there exist either

(a) \( x_1, x_2, x_3, x_4 \in X \) such that \( w_i \triangle w_j = x_i \triangle x_j \) for any \( i < j \leq 4 \) or
(b) \(x_1, x_2, x_3 \in X\) such that
\[
\begin{align*}
\triangle &\triangle \triangle = \triangle \triangle x_3, \\
\triangle &\triangle \triangle = \triangle \triangle x_3, \\
\triangle &\triangle \triangle = \triangle \triangle x_3.
\end{align*}
\]

(iii) If \(w_1, w_2, \cdots \in B(X)\) and \(w_i \triangle w_j \in B_2(X)\) for any \(i < j\), then there exist \(x_1, x_2, \cdots \in X\) such that \(w_i \triangle w_j = x_i \triangle x_j\) for any \(i < j\).

Proof. We prove the lemma by induction on \(k\). There is nothing to prove for \(k = 1\), and for \(k = 2\), assertion (i) obviously holds.

Suppose that \(k = 3\). For some \(y_1, y_2, y_3, y_4 \in X\), we have \(w_1 \triangle w_2 = y_1 \triangle y_2\) and \(w_2 \triangle w_3 = y_3 \triangle y_4\). Since \(w_1 \triangle w_3 = w_1 \triangle w_2 \triangle w_2 \triangle w_3 \in B_{=2}(X)\), it follows that either \(y_1 = y_3, y_1 = y_4, y_2 = y_3,\) or \(y_2 = y_4\). If \(y_1 = y_3\), then \(w_1 \triangle w_3 = y_1 \triangle y_3\) and \(w_2 \triangle w_3 = y_2 \triangle y_3\), and we set \(w_1 \triangle w_3 = x_1, x_2 = y_1,\) and \(x_3 = y_3\). The remaining cases are treated similarly.

Suppose that \(k = 4\) and let \(x_1, x_2, x_3 \in X\) be such that \(w_1 \triangle w_j = x_i \triangle x_j\) for \(i = 1, 2, 3\). There exist \(y, z \in X\) for which \(w_1 \triangle w_4 = y \triangle z\). We have \(w_2 \triangle w_4 = w_1 \triangle w_2 \triangle w_1 \triangle w_4 = x_1 \triangle x_2 \triangle y \triangle z \in B_2(X)\). Therefore, either \(x_1 = y, x_2 = y, x_1 = z,\) or \(x_2 = z\).

If \(x_1 = y\) or \(x_1 = z\), then the condition in (ii) (a) holds for \(x_4 = z\) in the former case and \(x_4 = y\) in the latter.

Suppose that \(x_1 \neq y\) and \(x_1 \neq z\). Then \(x_2 = y\) or \(x_2 = z\). Let \(x_2 = y\). Then \(w_1 \triangle w_4 = x_2 \triangle z,\) and we have \(w_3 \triangle w_4 = w_1 \triangle w_3 \triangle w_1 \triangle w_4 = x_1 \triangle x_3 \triangle x_2 \triangle z \in B_2(X)\), whence \(x_3 = z\) (because \(x_1, x_2 \neq z\)), so that \(w_1 \triangle w_4 = x_2 \triangle x_3 = w_2 \triangle w_3,\) \(w_2 \triangle w_4 = w_1 \triangle w_2 \triangle w_1 \triangle w_4 = x_1 \triangle x_3 = w_1 \triangle w_3,\) and \(w_3 \triangle w_4 = x_1 \triangle x_3 = w_1 \triangle w_3,\) i.e., assertion (ii) (b) holds. The case \(x_2 = z\) is similar. Note for what follows that, in both cases \(x_2 = y\) and \(x_2 = z\), we have \(w_4 = w_1 \triangle w_2 \triangle w_3\).

Let \(k > 4\). Consider the words \(w_1, w_2, w_3,\) and \(w_4\). Let \(x_1, x_2, x_3 \in X\) be such that \(w_1 \triangle w_3 = x_i \triangle x_j\) for \(i = 1, 2, 3\). As previously, there exist \(y, z \in X\) for which \(w_1 \triangle w_4 = y \triangle z\) and either \(x_1 = y, x_2 = y, x_1 = z,\) or \(x_2 = z\).

Suppose that \(x_1 \neq y\) and \(x_1 \neq z\); then \(w_4 = w_1 \triangle w_2 \triangle w_3\). In this case, we consider \(w_5\) instead of \(w_4\). Again, there exist \(y', z' \in X\) for which \(w_1 \triangle w_5 = y \triangle z\) and either \(x_1 = y', x_2 = y', x_1 = z',\) or \(x_2 = z'\). Since \(w_5 \neq w_4\), it follows that \(w_5 \neq w_1 \triangle w_2 \triangle w_3,\) and we have \(x_1 = y'\) or \(x_1 = z'\). In the former case, we set \(x_5 = y'\) and in the latter, \(x_5 = z'\). Consider again \(w_4\); recall that \(w_1 \triangle w_4 = y \triangle z\).

We have \(w_1 \triangle w_4 = w_1 \triangle w_1 \triangle w_1 \triangle w_4 = x_1 \triangle x_1 \triangle y \triangle z \in B_{=2}(X)\) for \(i \in \{2, 3, 5\}\). Since \(x_2 \neq x_5\) and \(x_3 \neq x_5\), it follows that \(x_1 = y,\) which contradicts the assumption.

Thus, \(x_1 = y\) or \(x_1 = z\). As above, we set \(x_4 = z\) in the former case and \(x_4 = y\) in the latter; then the condition in (ii) (a) holds.

Suppose that we have already found the required \(x_1, \ldots, x_{k-1} \in X\) for \(w_1, \ldots, w_{k-1}\). There exist \(y, z \in X\) for which \(w_1 \triangle w_k = y \triangle z\). We have \(w_i \triangle w_k = w_1 \triangle w_i \triangle w_k = x_1 \triangle x_1 \triangle y \triangle z \in B_{=2}(X)\) for \(i \leq k - 1\). If \(x_1 \neq y\) and \(x_1 \neq z,\) then we have \(x_1 \in \{y, z\}\) for \(2 \leq i \leq k - 1,\) which is impossible, because \(k > 4\). Thus, either \(x_1 = y\) or \(x_1 = z\). In the former case, we set \(x_k = z\) and in the latter, \(x_k = y\).

Then \(w_1 \triangle w_k = x_1 \triangle x_k\) and, for any \(i \leq k - 1,\) \(w_i \triangle w_k = w_1 \triangle w_i \triangle w_k = x_1 \triangle x_1 \triangle x_1 \triangle x_k = x_1 \triangle x_k\).

The infinite case is proved by the same inductive argument. \(\square\)
Proof of Theorem 13 (i) Suppose that \( \mathcal{F} \) is a \( k \)-arrow filter on \( X \). Let \( C \) be a \( k \)-fat set in \( B(X) \). Consider the 2-coloring of \( [X]^2 \) defined by

\[
c\{x, y\} = \begin{cases} 0 & \text{if } \{x, y\} = x \triangle y \in C, \\ 1 & \text{otherwise.} \end{cases}
\]

Since \( \mathcal{F} \) is \( k \)-arrow, there exists either an \( A \in \mathcal{F} \) for which \( c([A]^2) = \{0\} \) and hence \( [A]^2 \subset C \cap B_2(X_\mathcal{F}) \) or a \( k \)-element set \( F \subset X \) for which \( c([F]^2) = \{1\} \) and hence \( [F]^2 \cap C = [F]^2 \cap C \cap B_{=2}(X_\mathcal{F}) = \emptyset \). The latter case cannot occur, because \( C \) is \( k \)-fat. Therefore, \( C \cap B_2(X_\mathcal{F}) \) contains the trace \( [A]^2 \cap \{0\} = ((A \cup \{\ast\}) \triangle (A \cup \{\ast\})) \)

of the subgroup \( \langle A \cup \{\ast\} \rangle \), which is an open neighborhood of zero in \( B(X_\mathcal{F}) \).

Now suppose that \( k \neq 4 \) and the trace of each \( k \)-fat set on \( B_2(X) \) contains the trace on \( B_2(X) \) of a neighborhood of zero in \( B(X_\mathcal{F}) \), i.e., a set of the form \( A \triangle A \) for some \( A \in \mathcal{F} \). Let us show that \( \mathcal{F} \) is \( k \)-arrow. Given any \( c : [X]^2 \to \{0, 1\} \), we set

\[
C = \{x \triangle y : c\{x, y\} = 1\} \quad \text{and} \quad C' = B(X_\mathcal{F}) \setminus C.
\]

If \( C' \) is not \( k \)-fat, then there exist \( w_1, \ldots, w_k \in B(X) \) such that \( w_i \triangle w_j \in C \)

for \( i < j \leq k \). By Lemma 13(i) we can find \( x_1, \ldots, x_k \in X \) such that \( x_i \triangle x_j \in C \)

(hence \( x_i \not\in \{\ast\} \)) for \( i < j \leq k \). This means that, for \( F = \{x_1, \ldots, x_k\} \), we have \( c([F]^2) = \{1\} \).

If \( C' \) is \( k \)-fat, then, by assumption, there exists an \( A \in \mathcal{F} \) for which \( A \triangle A \subset C' \cap B_2(X) = C \), which means that \( c([A]^2) = \{0\} \).

The same argument proves (ii); the only difference is that assertion (ii) of Lemma 13 is used instead of (i).

The proof of (iii) is similar. \( \square \)

Let \( R_r(s) \) denote the least number \( n \) such that, for any \( r \)-coloring \( c : [X]^2 \to Y \), where \( |X| \geq n \) and \( |Y| = r \), there exists an \( s \)-element \( c \)-homogeneous set. By the finite Ramsey theorem, such a number exists for any positive integers \( r \) and \( s \).

Theorem 15. There exists a positive integer \( N \) (namely, \( N = R_{36}(R_6(3)) + 1 \)) such that, for any uniform ultrafilter \( \mathcal{U} \) on a set \( X \) of infinite cardinality \( \kappa \), the following conditions are equivalent:

(i) the trace of any \( N \)-fat subset of \( B(X) \) on \( B_4(X_\mathcal{U}) \subset B_4(X_\mathcal{U}) \) contains that of a neighborhood of zero in the free group topology of \( B(X_\mathcal{U}) \);

(ii) all \( \kappa \)-fat sets in \( B(X) \) are neighborhoods of zero in the topology induced from the free topological group \( B(X_\mathcal{U}) \);

(iii) \( \mathcal{U} \) is a Ramsey ultrafilter.

Proof. Without loss of generality, we assume that \( X = \kappa \).

(i) \( \Rightarrow \) (iii) Suppose that \( N \) is as large as we need and the trace of each \( N \)-fat set on \( B_4(\kappa) \) contains the trace on \( B_4(\kappa) \) of a neighborhood of zero in \( B(X_\mathcal{U}) \), which, in turn, contains a set of the form \( (A \triangle A \triangle A) \cap B_{=4}(\kappa) \) for some \( A \in \mathcal{U} \). Let us show that \( \mathcal{U} \) is a Ramsey ultrafilter. Consider any 2-coloring \( c : [\kappa]^2 \to \{0, 1\} \).

We set

\[
C = \{ \alpha_1 \triangle \alpha_2 \triangle \alpha_3 \triangle \alpha_4 : \alpha_i \in \kappa \text{ for } i \leq 4, \ \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4, \\
c\{\alpha_1, \alpha_2\} \neq c\{\alpha_3, \alpha_4\}, \ c\{\alpha_1, \alpha_3\} \neq c\{\alpha_2, \alpha_4\}, \\
c\{\alpha_1, \alpha_4\} \neq c\{\alpha_2, \alpha_3\} \}
\]

and

\[
C' = B(X) \setminus C.
\]
If $C'$ is not $N$-fat, then there exist $w_1, \ldots, w_N \in B(\kappa)$ such that $w_i \triangle w_j \in C$ for $i < j \leq N$. We can assume that $w_N = 0$ (otherwise, we translate all $w_i$ by $w_N$). Then $w_i \in C \subset B_2(\kappa)$, $i < j < N$. Let $w_i = \alpha_1^i \Delta \alpha_2^i \Delta \alpha_3^i \Delta \alpha_4^i$ for $i < N$ and consider the 36-coloring of all pairs $\{w_i, w_j\}$, $i < j < N$, defined as follows. Since $w_i \triangle w_j$ is a four-letter word, it follows that $w_i \triangle w_j = \beta_1 \triangle \beta_2 \Delta \beta_3 \Delta \beta_4$, where $\beta_i \in \kappa$. Two letters among $\beta_1, \beta_2, \beta_3, \beta_4$ (say $\beta_1$ and $\beta_2$) occur in the word $w_i$ and the remaining two ($\beta_3$ and $\beta_4$) occur in $w_j$. We assume that $\beta_1 < \beta_2$ and $\beta_3 < \beta_4$. Let us denote the numbers of the letters $\beta_1$ and $\beta_2$ in $w_i$ (recall that the letters in $w_i$ are numbered in increasing order) by $i'$ and $i''$, respectively, and the numbers of the letters $\beta_3$ and $\beta_4$ in $w_j$ by $j'$ and $j''$. To the pair $\{w_i, w_j\}$ we assign the quadruple $(i', i'', j', j'')$. The number of all possible quadruples is 36, so that this assignment is a 36-coloring. We choose $N \geq R_{36}(N') + 1$ for $N'$ as large as we need. Then there exist two pairs $i_0', i_0''$ and $j_0', j_0''$ and $N'$ words $w_{i_n}$, where $n \leq N'$ and $i_n < i_n$ for $s < t$, such that $i' = i_0', i'' = i_0'', j' = j_0'$, and $j'' = j_0''$ for any pair $\{w_i, w_j\}$ with $i, j \in \{i_1, \ldots, i_{N'}\}$ and $i < j$. Clearly, if $N' \geq 3$, then we also have $j_0' = i_0'$ and $j_0'' = i_0''$. In the same manner, we can fix the position of the letters coming from $w_i$ and $w_j$ in the sum $w_i \triangle w_j$: to each pair $\{w_i, w_j\}$, $s, t \in \{1, \ldots, N'\}$, $s < t$, we assign the numbers of the $i_t$th and $i_{t'}$th letters of $w_i$, in the word $w_i \triangle w_j$, (recall that the letters of all words are in increasing order): the positions of the letters of $w_i$, in $w_i \triangle w_j$, are then determined automatically. There are six possible arrangements: 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, and 3, 4. Thus, we have a 6-coloring of the symmetric square of the $N'$-element set $\{w_1, \ldots, w_{i_{N'}}\}$, and if $N' \geq R_6(3)$ (which we assume), then there exists a 3-element set $\{w_k, w_l, w_m\}$ homogeneous with respect to this coloring, i.e., such that all pairs of words from this set are assigned the same color. For definiteness, suppose that this is the color 1, 2; suppose also that $i_0' = 1$, $i_0'' = 2$, $k < l < m$, and $w_k = \alpha_1^k \Delta \alpha_2^k \Delta \alpha_3^k \Delta \alpha_4^k$ for $t = k, l, m$. Then $w_k, w_l, w_m \in C$, $w_k \triangle w_l = \alpha_1^k \Delta \alpha_2^k \Delta \alpha_3^k \Delta \alpha_4^k \in C$, $w_k \triangle w_m = \alpha_1^k \Delta \alpha_2^m \alpha_3^k \Delta \alpha_4^m \in C$, and $w_k \triangle w_m = \alpha_1^k \Delta \alpha_2^m \alpha_3^k \Delta \alpha_4^m \in C$. By the definition of $C$ we have $c(\alpha_1^k \Delta \alpha_2^k) \neq c(\alpha_1^k \Delta \alpha_2^k)$, $c(\alpha_1^k \Delta \alpha_2^m) \neq c(\alpha_1^m \Delta \alpha_2^m)$, and $c(\alpha_1^m \Delta \alpha_2^m) \neq c(\alpha_1^m \Delta \alpha_2^m)$, which is impossible, because $c$ takes only two values. The cases of other colors and other numbers $i_0'$ and $i_0''$ are treated in a similar way.

Thus, $C'$ is $N$-fat and, therefore, contains $(A \Delta A \Delta A \Delta A) \cap B_2(\kappa)$ for some $A \in \mathcal{Z}$. Take any $\alpha \in A$ and consider the sets $A' = \{\beta > \alpha : c(\{\alpha, \beta\}) = \{0\}\}$ and $A'' = \{\beta > \alpha : c(\{\alpha, \beta\}) = \{1\}\}$. One of these sets belongs to $\mathcal{Z}$, because $\mathcal{Z}$ is uniform. For definiteness, suppose that this is $A'$. By Theorem 13 $\mathcal{Z}$ is 3-arrow. Therefore, there exists either an $A'' \subset A'$ for which $c([A'']^2) = \{0\}$ or $\beta, \gamma, \delta \in A'$, $\beta < \gamma < \delta$, for which $c([[\beta, \gamma, \delta]^2]) = \{1\}$. In the former case, we are done. In the latter case, we have $\alpha, \beta, \gamma, \delta \in A$, $\alpha < \beta < \gamma < \delta$, and $c(\{\alpha, \beta, \gamma, \delta\}) = c(\{\alpha, \gamma\}) = 1$, and $c(\{\alpha, \beta\}) = c(\{\alpha, \gamma\}) = c(\{\alpha, \delta\}) = 0$ (by the definition of $A'\Delta A''$). Therefore, $\alpha \Delta \delta \Delta \gamma \Delta \delta \in C$, which contradicts the definition of $A$.

(iii) $\Rightarrow$ (ii) Suppose that $\mathcal{Z}$ is a Ramsey ultrafilter on $X$ and $C$ is a $\kappa$-fat set in $B(X)$. Take any $n \in N$ and consider the coloring $c \colon [X]^{2n} \to \{0, 1\}$ defined by

$$c(\{x_1, \ldots, x_{2n}\}) = \begin{cases} 0 & \text{if } \{x_1, \ldots, x_{2n}\} = x_1 \Delta \ldots \Delta x_{2n} \in C, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathcal{Z}$ is Ramsey, there exists either a set $A_n \in \mathcal{Z}$ for which $[A]^{2n} \subset C$ or a set $Y \subset X$ of cardinality $\kappa$ for which $[Y]^{2n} \cap C = \emptyset$. In the latter case, for $Z = [Y]^n \subset B(X)$, we have $(Z \Delta Z) \cap C \subset \{0\}$, which contradicts $C$ being $\kappa$-fat.
Hence the former case occurs, and \( C \cap B_{2n}(X) \) contains the trace \([A_n]^{2n} \cap B_{2n}(X)\) of the open subgroup \( \langle (A_n \cup \{\ast\}) \rangle \) of \( B(X) \).

Thus, for each \( n \in \mathbb{N} \), we have found \( A_1, A_2, \ldots, A_n \in \mathcal{F} \) such that \([A_i]^{2i} \cap B_{2i}(X) \subset C\). Let \( A = \bigcap_{i \leq n} A_i \). Then \( A \in \mathcal{U} \) and \([A]^{2i} \cap B_{2i}(X) \subset C\) for all \( i \leq n \). Hence \( C \cap B_{2n}(X) \) contains the trace on \( B_{2n}(X) \) of the open subgroup \( \langle (A \cup \{\ast\}) \rangle \) of \( B(X) \) (recall that \( 0 \in C \)). This means that, for each \( n \), \( C \cap B_{2n}(X) \) is a neighborhood of zero in the topology induced from \( B(X) \).

If \( \kappa = \omega \), then \( B(X) \) has the inductive limit topology with respect to the decomposition \( B(X) = \bigcup_{\omega \in \omega} B_n(X) \), because \( \mathcal{F} \) is Ramsey (see [4]). Therefore, in this case, \( C \cap B(X) \) is a neighborhood of zero in the induced topology.

If \( \kappa > \omega \), then the ultrafilter \( \mathcal{U} \) is countably complete \( \mathbb{R} \) Lemma 9.5 and Theorem 9.6, i.e., any countable intersection of elements of \( \mathcal{U} \) belongs to \( \mathcal{U} \). Hence \( A = \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{U} \), and \( \langle (A \cup \{\ast\}) \rangle \cap (\bigcup_{n \in \omega} B_n(X)) \subset C \). Thus, \( C \cap B(X) \) is a neighborhood of zero in the induced topology in this case, too.

The implication (ii) \( \Rightarrow \) (i) is obvious.

Theorem 13 has the following purely algebraic corollary.

**Corollary 16** (\( p = \mathfrak{c} \)). Any Boolean group contains \( \omega \)-fat sets which are not fat and \( \Delta^* \)-sets which are \( \Delta^*_k \)-sets for no \( k \).

**Proof.** Theorem 4.10 of [9] asserts that if \( p = \mathfrak{c} \), then there exists an ultrafilter \( \mathcal{U} \) on \( \omega \) which is \( k \)-arrow for all \( k \in \mathbb{N} \) but not Ramsey and, therefore, not \( \omega \)-arrow [9, Theorem 2.1]. By Theorem 13 the traces of all fat sets on \( B_2(\omega) \) contain those of neighborhoods of zero in \( B(\omega) \), and thus exist \( \omega \)-fat sets whose traces do not. This proves the required assertion for the countable Boolean group. The case of a group \( B(X) \) of uncountable cardinality \( \kappa \) reduces to the countable case by representing \( B(X) \) as \( B(\kappa) = B(\omega) \times B(\kappa) \); it suffices to note that a set of the form \( C \times B(\kappa) \), where \( C \subset B(\omega) \), is \( \lambda \)-fat in \( B(\omega) \times B(\kappa) \) for \( \lambda \leq \omega \) if and only if so is \( C \) in \( B(\omega) \).

The author is unaware of where there exist ZFC examples of such sets in any groups.

**Acknowledgments**

The author is very grateful to Evgenii Reznichenko and Anton Klyachko for useful discussions.

**References**

[1] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics* (Amer. Math. Soc., Providence, R.I., 1955).

[2] E. Reznichenko and O. Sipacheva *Discrete Subsets in Topological Groups and Countable Extremally Disconnected Groups*, arXiv:1608.03546v2 [math.GN].

[3] V. Bergelson, N. Hindman, and R. McCutcheon, “Notions of size and combinatorial properties of quotient sets in semigroups,” Topology Proc. 23, 23–60 (1998).

[4] O. Sipacheva, “Free Boolean topological groups,” Axioms 4 (4), 492–517 (2015).

[5] G. Mokobodzki, “Ultrafiltres rapides sur \( N \). Construction d’une densité relative de deux potentiels comparables,” Séminaire Brelot–Choquet–Deny. Théorie du potentiel 12, 1–22 (1967–1968).

[6] A. R. D. Mathias, “A remark on rare filters,” in *Infinite and finite sets* (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday) (North-Holland, Amsterdam, 1975), Vol. 3, pp. 1095–1097.
[7] A. W. Miller, “There are no $q$-points in Laver’s model for the borel conjecture,” Proc. Amer. Math. Soc. 78 (1980) 103–106.
[8] W. W. Comfort and S. Negrepontis, The Theory of Ultrafilters (Springer-Verlag, Berlin, 1974).
[9] J. E. Baumgartner and A. D. Taylor, “Partition theorems and ultrafilters,” Trans. Amer. Math. Soc. 241, 283–309 (1978).
[10] S. Shelah, Proper and Improper Forcing (Springer-Verlag, Berlin, 1998).
[11] E. K. van Douwen, “The integers and topology,” in Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984), pp. 111–167.
[12] N. Hindman and D. Strauss, Algebra in the Stone–Čech Compactification, 2nd ed. (De Gryuter, Berlin/Boston, 2012).
[13] V. Bergelson, H. Furstenberg, and B. Weiss, “Piecewise-Bohr sets of integers and combinatorial number theory,” in Topics in Discrete Mathematics, Ed. by M. Klazar, J. Kratochvíl, M. Loebl, J. Matousek, P. Valtr, and R. Thomas (Springer, Berlin, Heidelberg, 2006), pp. 13–37.
[14] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory (Princeton Univ. Press, Princeton, N.J., 1981).
[15] John T. Griesmer, “A remark on intersectivity and difference sets,” private communication.
[16] R. Ellis and H. B. Keynes, “Bohr compactifications and a result of Følner,” Israel J. Math. 12 (3), 314–330 (1972).
[17] M. Beiglböck, V. Bergelson, and Alexander Fish, “Sunset phenomenon in countable amenable groups,” Adv. Math. 223, 416–432 (2010).
[18] T. Jech, Set Theory, 3rd millennium (revised) ed. (Springer, 2003).

Department of General Topology and Geometry, Lomonosov Moscow State University, Leninskie Gory 1, Moscow 119991, Russia
E-mail address: o-sipa@yandex.ru