On disjoint matchings in cubic graphs: maximum 3-edge-colorable subgraphs

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We show that any 2−factor of a cubic graph can be extended to a maximum 3−edge-colorable subgraph. We also show that the sum of sizes of maximum 2− and 3−edge-colorable subgraphs of a cubic graph is at least twice of its number of vertices.

1. Introduction

We consider finite undirected graphs that do not contain loops. For a graph $G$ and a positive integer $k$ define

$$B_k(G) \equiv \{(H_1, ..., H_k) : H_1, ..., H_k \text{ are pairwise edge-disjoint matchings of } G\},$$

and let

$$\nu_k(G) \equiv \max\{|H_1| + ... + |H_k| : (H_1, ..., H_k) \in B_k(G)\}.$$

A subgraph $H$ of $G$ is called maximum $k$-edge-colorable, if it is $k$-edge-colorable and contains exactly $\nu_k(G)$ edges.

Terms and concepts that we do not define can be found in [1, 5].

2. The main results

The first theorem allows us to ”embed” maximum 3−edge-colorable subgraphs of a cubic graphs into minimum 4−edge-colorings. Having done this, we can apply the results of [3, 4] for the investigation of our problems.

\textbf{Theorem 1} Let $H$ be a maximum 3−edge-colorable subgraph of a cubic graph $G$. Then $E(G)\setminus E(H)$ is a matching.

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Proof. First of all let us show that the graph \( G \setminus E(H) \) contains no path of length three. On the opposite assumption, consider a path \( u_0, u_1, u_2, u_3 \) of the graph \( G \setminus E(H) \), and let us assume that \( H \) is properly colored with colors \( \{1, 2, 3\} \). Define the sets \( C(u_0), C(u_1), C(u_2) \) as the colors of the edges incident to the vertices \( u_0, u_1, u_2 \), respectively.

Note that

\[
|C(u_0)| \leq 2.
\]

If \( |C(u_0)| < 2 \), then there is \( \alpha \in \{1, 2, 3\} \) with \( \alpha \notin C(u_0) \cup C(u_1) \). Now, if we color the edge \((u_0, u_1)\) with color \( \alpha \), then we would get a proper 3–edge-coloring of the subgraph \( H \cup \{(u_0, u_1)\} \), contradicting the maximality of \( H \). Thus:

\[
|C(u_0)| = 2.
\]

Now, let us show that

\[
|C(u_1)| = 1 \text{ and } C(u_0) \cup C(u_1) = \{1, 2, 3\}.
\] (1)

Assume the contrary, that is \( C(u_1) \subseteq C(u_0) \). There is \( \alpha \in \{1, 2, 3\} \) with \( \alpha \notin C(u_0) \cup C(u_1) \). Now, if we color the edge \((u_0, u_1)\) with color \( \alpha \), then we would get a proper 3–edge-coloring of the subgraph \( H \cup \{(u_0, u_1)\} \), contradicting the maximality of \( H \).

Thus, (1) must be true. Since \( |C(u_1) \cup C(u_2)| \leq 2 \), then there is \( \alpha \in \{1, 2, 3\} \) with \( \alpha \notin C(u_1) \cup C(u_2) \). Now, if we color the edge \((u_1, u_2)\) with color \( \alpha \), then we would get a proper 3–edge-coloring of the subgraph \( H \cup \{(u_1, u_2)\} \), contradicting the maximality of \( H \).

Thus, the graph \( G \setminus E(H) \) contains no path of length three. To complete the proof of the theorem, we need to verify the absence of adjacent edges in \( G \setminus E(H) \).

Suppose that \((u_0, u_1), (u_1, u_2) \in E(G) \setminus E(H)\). Again, let \( C(u_0), C(u_1), C(u_2) \) denote the colors of the edges incident to the vertices \( u_0, u_1, u_2 \), respectively. We need to consider two cases:

Case 1: \( u_0 = u_2 \), that is, \((u_0, u_1)\) is a multiple edge. Note that \( |C(u_0)| \leq 1, |C(u_1)| \leq 1 \), thus there is \( \alpha \in \{1, 2, 3\} \) with \( \alpha \notin C(u_0) \cup C(u_1) \). Now, if we color one of edges connecting \( u_0 \) and \( u_1 \) with color \( \alpha \), then we would get a proper 3–edge-coloring of the subgraph \( H \cup \{(u_0, u_1)\} \), contradicting the maximality of \( H \).

Case 2: \( u_0 \neq u_2 \). Note that \( |C(u_0)| \leq 2, |C(u_1)| \leq 1, |C(u_2)| \leq 2 \). Like in the proof of (1), one can easily show that

\[
C(u_0) \cup C(u_1) = \{1, 2, 3\}, \text{ and } C(u_1) \cup C(u_2) = \{1, 2, 3\},
\]

thus \( C(u_0) = C(u_2) \). Suppose that \( C(u_0) = C(u_2) = \{\alpha, \beta\} \) and \( C(u_1) = \{\gamma\} \). Consider the maximal \( \alpha - \gamma \) alternating paths \( P_0, P_1, P_2 \), starting from vertices \( u_0, u_1, u_2 \), respectively. Note that there is \( i \in \{0, 2\} \) such that \( u_1 \notin V(P_i) \). Now, shift the colors on the path \( P_i \) to obtain a new coloring of the maximum 3–edge-colorable subgraph \( H \), where the color \( \alpha \) is absence in both of vertices \( u_i \) and \( u_1 \). Now, if we color the edge \((u_1, u_i)\) with color \( \alpha \), then we would get a proper 3–edge-coloring of the subgraph \( H \cup \{(u_1, u_i)\} \), contradicting the maximality of \( H \). The proof of the theorem [1] is completed.
It is not always possible to extend a 1-factor into a maximum 2-edge-colorable subgraph of a cubic graph. Nevertheless, the following is true:

**Theorem 2** Any 1-factor of a cubic graph $G$ can be extended to a maximum 3-edge-colorable subgraph of $G$.

**Proof.** For a 1-factor $F$ of $G$, choose a maximum 3-edge-colorable subgraph $H$ of $G$ with

$$|E(F) \cap E(H)| \to \text{max.}$$

Let us show that $E(F) \subseteq E(H)$. On the opposite assumption, consider an edge $e = (u, v) \in E(F) \setminus E(H)$ and let us assume that $H$ is properly colored with colors $\{1, 2, 3\}$. Due to theorem 1, the edges adjacent to $e$ are belong to $H$. Let $C(u)$ and $C(v)$ denote the colors of edges that are incident to $u$ and $v$, respectively. Note that the maximality of $H$ implies that

$$|C(u) \cap C(v)| = 1 \text{ and } C(u) \cup C(v) = \{1, 2, 3\}.$$  

Choose $\alpha \in C(u) \setminus C(v)$. Consider the subgraph $H' = (H \setminus \{e'\}) \cup \{e\}$, where $e'$ is the edge that is incident to $u$ and is colored by $\alpha$. Note that $H'$ is a maximum 3-edge-colorable subgraph of $G$ with

$$|E(F) \cap E(H)| < |E(F) \cap E(H')|,$$

contradicting the choice of $H$. The proof of the theorem 2 is completed.

Next, we prove a result which claims that the uncolored edges with respect to a maximum 3-edge-colorable subgraph of $G$ always can be ”left” in a given 1-factor, or, equivalently, any 2-factor of a cubic graph $G$ can also be extended to a maximum 3-edge-colorable subgraph of $G$.

**Theorem 3** Let $F$ be any 1-factor of a cubic graph $G$, and let $\bar{F}$ be the complementary 2-factor of $F$. Then there is a maximum 3-edge-colorable subgraph $H$ of $G$, such that:

(a) $E(H) \cup E(F) = E(G)$;

(b) $E(\bar{F}) \subseteq E(H)$.

**Proof.** Note that (b) follows from (a), thus we will stop only on the proof of (a).

For a given 1-factor $F$ of a cubic graph $G$, consider a maximum 3-edge-colorable subgraph $H$ of $G$ with

$$|E(F) \cap E(H)| \to \text{min.}$$

Let us show that $E(H) \cup E(F) = E(G)$. We only need to verify that $E(F) \cup E(H) \supseteq E(G)$. Assume that there is $e \in E(G)$ such that $e$ belongs to neither of $F$ and $H$, and let us assume that $H$ is properly colored with colors $\{1, 2, 3\}$. 
Consider the edges adjacent to $e$. Due to theorem \[1\] these edges belong to $H$. If $C(u)$ and $C(v)$ denote the colors of edges that are incident to $u$ and $v$, respectively, then the maximality of $H$ implies that

$$|C(u) \cap C(v)| = 1 \text{ and } C(u) \cup C(v) = \{1, 2, 3\}.$$ 

Suppose that

$$\{\alpha\} = C(u) \cap C(v), C(u) = \{\alpha, \beta\} \text{ and } C(v) = \{\alpha, \gamma\}.$$ 

Due to [3, 4], there is an odd cycle $C_e$ containing the edge $e$, whose all other edges are colored $\beta$ and $\gamma$, alternatively, while the edges that are incident to a vertex of $C_e$ and do not lie on $C_e$, are colored by $\alpha$. We need to consider two cases:

Case 1: $E(C_e) \cap E(F) \neq \emptyset$.

Let $f \in E(C_e) \cap E(F)$. Consider a proper partial 3-edge-coloring of the graph $G$ obtained from the coloring of $H$ as follows: $f$ is left uncolored, the edges of the even path $C_e - f$ are colored $\beta$ and $\gamma$, alternatively, the colors of the rest of edges are left unchanged. Note that the new partial 3-edge-coloring corresponds to a maximum 3-edge-colorable subgraph $H'$ of $G$ with

$$|E(F) \cap E(H')| < |E(F) \cap E(H)|$$

contradicting the choice of $H$.

Case 2: $E(C_e) \cap E(F) = \emptyset$.

Note that in this case,

(I) the edges that are incident to a vertex of $C_e$, and do not lie on $C_e$, are colored by $\alpha$ and belong to $F$, which and $E(C_e) \cap E(F) = \emptyset$ imply that:

(II) all maximum 3-edge-colorable subgraphs $H'$ of $G$, which can be obtained from the coloring of $H$, by leaving any edge $g \in E(C_e)$ uncolored, by coloring the edges of the even path $C_e - g$ $\beta$ and $\gamma$, alternatively, and leaving the colors of the rest of edges unchanged, satisfy the condition

$$|E(F) \cap E(H')| = |E(F) \cap E(H)| \rightarrow \min.$$ 

Now, we consider a proper partial 3-edge-coloring $\theta$ of the graph $G$ obtained from the coloring of $H$ by deleting the colors of the all edges lying on $C_e$. Since $C_e$ is an odd cycle, we imply that there is an $\alpha - \gamma$ alternating path $P_w$ in the 3-edge-coloring $\theta$ that starts from a vertex $w \in V(C_e)$ and does not end on $C_e$. Choose an edge $x = (w, z) \in E(C_e)$, and let $y$ be the other edge of $C_e$ that is incident to $w$.

Consider a proper partial 3-edge-coloring of $G$ obtained from $\theta$ as follows:

- shift the colors on the path $P_w$, and clear the color of the edge of $P_e$ that is incident to $w$;
- color $x$ by $\alpha$, and color the edges of the even path $C_e - x$ by $\beta - \gamma$ alternatively, starting from the edge $y$. 

It is not hard to see that the new partial 3-edge-coloring of $G$ corresponds to a maximum 3-edge-colorable subgraph $H'$ of $G$, which due to (II) satisfies

$$|E(F) \cap E(H')| < |E(F) \cap E(H)|$$

contradicting the choice of $H$. The proof of the theorem 3 is completed.

We suspect that the theorem 3 can be generalized as follows:

**Conjecture 1** Let $F$ be any maximal (not necessarily, maximum) matching of a cubic graph $G$. Then there is a maximum 3-edge-colorable subgraph $H$ of $G$, such that $E(H) \cup F = E(G)$.

Recently we have proved the following:

**Theorem 4** ([2]): For every cubic graph $G$:

$$\nu_2(G) \geq \frac{4}{5}|V(G)|, \nu_3(G) \geq \frac{7}{6}|V(G)|.$$ 

There are graphs attaining bounds of the theorem 4. The graph from figure 1a attains the first bound and the graph from figure 1b the second bound.

![Figure 1](image.png)

**Figure 1. Examples attaining the bounds of the theorem 4**

Note that if there were a cubic graph $G$ attaining the two bounds at the same time, then we would have:

$$\nu_2(G) + \nu_3(G) = \frac{4}{5}|V(G)| + \frac{7}{6}|V(G)| = \frac{59}{30}|V(G)| < 2|V(G)|.$$ 

Thus, to show the absence of a cubic graph attaining all the bounds of theorem 4 at the same time, it suffices to show the following
Theorem 5 For every cubic graph $G$

$$\nu_2(G) + \nu_3(G) \geq 2|V(G)|.$$  

Proof. Suppose that the statement is false, and consider a minimum counter-example $G$, that is, a cubic graph $G$ with $\nu_2(G) + \nu_3(G) < 2|V(G)|$ and $|V(G)| \rightarrow \min$. Note that $G$ is connected.

Claim 1 Any triangle of $G$ contains a multiple edge.

Proof. Suppose that $G$ contains a triangle $T$ that has no multiple edge. Consider a cubic graph $G'$ obtained from $G$ by contracting the triangle $T$ to a new vertex. Since $|V(G')| = |V(G)| - 2 < |V(G)|$, we imply $\nu_2(G') + \nu_3(G') \geq 2|V(G')|$. On the other hand, it can be easily checked that

$$\nu_2(G) \geq \nu_2(G') + 2 \text{ and } \nu_3(G) \geq \nu_3(G') + 3,$$

(in fact, with a bit more work it can be shown that one always has equalities here), therefore

$$\nu_2(G) + \nu_3(G) \geq 5 + \nu_2(G') + \nu_3(G') \geq 5 + 2|V(G')| > 2|V(G)|,$$

contradicting the choice of $G$. The proof of the claim is completed.

Before me move to the proof of the next claim, let us also note that $G$ contains no edge that connects two triangles, since if there were such an edge, then taking into account that $G$ is connected, we would have that $G$ is isomorphic to the second graph from figure which clearly satisfies the statement of the theorem.

Claim 2 Let $x, y, z$ be vertices of a triangle of $G$, and let $(x, y)$ be its multiple edge (claim [1]). Assume that $u$ is the neighbour of $z$ lying outside the triangle. Then $u$ is not incident to a multiple edge.

Proof. Suppose that $u$ is incident to a multiple edge and consider the cubic graph $G'$ (figure [2]).

Note that $|V(G)| = |V(G')| + 2$, hence $\nu_2(G') + \nu_3(G') \geq 2|V(G')|$. It can be easily seen that

$$\nu_2(G) \geq \nu_2(G') + 2 \text{ and } \nu_3 \geq \nu_3(G') + 3,$$

therefore

$$\nu_2(G) + \nu_3(G) \geq 5 + \nu_2(G') + \nu_3(G') \geq 5 + 2|V(G')| > 2|V(G)|,$$

contradicting the choice of $G$. The proof of the claim is completed.
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Claim 3 Let \( x_1, y_1, z_1 \) and \( x_2, y_2, z_2 \) be vertices of two triangles of \( G \), and let \((x_1, y_1), (x_2, y_2)\) be their multiple edges (claim \( \Box \)). Then, there is no vertex \( u \) that is adjacent to both \( z_1 \) and \( z_2 \).

Proof. Suppose that there is such a vertex \( u \) and let us consider the cubic graph \( G' \) (figure 3).

Note that \(|V(G)| = |V(G')| + 4\), hence \( \nu_2(G') + \nu_3(G') \geq 2|V(G')| \). It can be easily seen that

\[
\nu_2(G) \geq \nu_2(G') - 2 + 1 + 4 = \nu_2(G') + 3 \quad \text{and} \quad \nu_3(G) \geq \nu_3(G') - 1 + 6 = \nu_3(G') + 5,
\]

therefore

\[
\nu_2(G) + \nu_3(G) \geq 8 + \nu_2(G') + \nu_3(G') \geq 8 + 2|V(G')| = 2|V(G)|,
\]

contradicting the choice of \( G \). The proof of the claim \( \Box \) is completed.

Claim 4 Let \( x_1, y_1, z_1 \) and \( x_2, y_2, z_2 \) be vertices of two triangles of \( G \), and let \((x_1, y_1), (x_2, y_2)\) be their multiple edges (claim \( \Box \)). Let \( u_1 \) and \( u_2 \) be the neighbours of \( z_1 \) and \( z_2 \), respectively, that lie outside the triangles. Then \( u_1 \) and \( u_2 \) are not connected by an edge.

Proof. First of all note that, due to claim \( \Box \), \( u_1 \neq u_2 \). Suppose that \( u_1 \) and \( u_2 \) are adjacent, and let us consider the cubic graph \( G' \) (figure 4).
Figure 3. Reducing $G$ to $G'$ in claim 3

Note that $|V(G)| = |V(G')| + 6$, hence $\nu_2(G') + \nu_3(G') \geq 2|V(G')|$. It can be easily seen that

\[
\nu_2(G) \geq \nu_2(G') - 1 + 6 = \nu_2(G') + 5 \quad \text{and} \quad \nu_3(G) \geq \nu_3(G') - 1 + 8 = \nu_3(G') + 7,
\]

therefore

\[
\nu_2(G) + \nu_3(G) \geq 12 + \nu_2(G') + \nu_3(G') \geq 12 + 2|V(G')| = 2|V(G)|,
\]

contradicting the choice of $G$. The proof of the claim 4 is completed.

**Claim 5** Let $x_1, y_1, z_1$ and $x_2, y_2, z_2$ be vertices of two triangles of $G$, and let $(x_1, y_1), (x_2, y_2)$ be their multiple edges (claim 1). Let $u_1$ and $u_2$ be the neighbours of $z_1$ and $z_2$, respectively, that lie outside the triangles. Then, there is no vertex $w$, that is adjacent to both $u_1$ and $u_2$.

**Proof.** Suppose that $w$ is adjacent to both $u_1$ and $u_2$, and let us consider the cubic graph $G'$ (figure 5).

Note that $|V(G)| = |V(G')| + 6$, hence $\nu_2(G') + \nu_3(G') \geq 2|V(G')|$. It can be easily seen that

\[
\nu_2(G) \geq \nu_2(G') - 1 + 6 = \nu_2(G') + 5 \quad \text{and} \quad \nu_3(G) \geq \nu_3(G') - 1 + 8 = \nu_3(G') + 7,
\]

therefore

\[
\nu_2(G) + \nu_3(G) \geq 12 + \nu_2(G') + \nu_3(G') \geq 12 + 2|V(G')| = 2|V(G')|,
\]

contradicting the choice of $G$. The proof of the claim 5 is completed.
The proved claims imply that the triangles of $G$ are placed "sufficiently far", and this allows us to complete the proof by using the odd cycles corresponding to edges that do not lie in some maximum 3-edge-colorable subgraph $[3, 4]$.

Let $H$ be any maximum 3-edge-colorable subgraph of $G$, and let $A, B, C$ be its color classes, which are pairwise disjoint matchings with $|A| + |B| + |C| = |E(H)| = \nu_3(G)$. Consider the 5-tuple $(A, B, C, A, B)$, and for any $v \in V$ define $\delta(v)$ as the number of edges that $v$ is incident from this 5-tuple (edges of $A$ and $B$ contribute to $\delta(v)$ by two).

Note that

$$2(\nu_2(G) + \nu_3(G)) \geq 2(|A| + |B| + |C| + |A| + |B|) = \sum_{v \in V} \delta(v).$$

Thus, to get the desired contradiction, it suffices to show, that

$$\sum_{v \in V} \delta(v) \geq 4|V(G)|. \quad (2)$$

Consider the cycles of the uncolored edges of $H$ $[3, 4]$. These cycles are triangles from figure 6 whose one edge is a multiple edge (claim 1),

or are odd cycles from figure 7 having length at least five.

Note that

- if a vertex $v \in V$ lies outside these cycles, then it contributes to the sum (2) by five;
the total contribution of vertices of triangles from figure 6 (a) and (b) to the sum (2) is twelve;

• the total contribution of vertices of triangles from figure 6 (c) to the sum (2) is eleven;

• the total contribution of vertices of cycles from figure 7 (a) and (b) to the sum (2) is \(4(2l + 1) + 2l - 2\), where the cycle is assumed to be of length \(2l + 1, l \geq 2\);

• the total contribution of vertices of cycles from figure 7 (c) to the sum (2) is \(4(2l + 1) + 2l - 3\), where the cycle is assumed to be of length \(2l + 1, l \geq 2\).

Thus, if we have a triangle with vertices \(x, y, z\), whose multiple edge is \((x, y)\) (claim 1), and \(u\) is the neighbour of \(z\), lying outside this triangle, then if \(u\) does not lie on either of these odd cycles, then

\[\delta(x) + \delta(y) + \delta(z) + \delta(u) \geq 16 = 4 \cdot 4.\]

On the other hand, if a cycle of length \(2l + 1, l \geq 2\) from figure 7 contains \(r\) vertices, which are adjacent to a vertex of a triangle from figure 6 (c), then the proved claims imply that

\[r \leq \frac{2l + 1}{3},\]

therefore the total contribution of vertices of these \(r\) triangles and the odd cycle of length

$\text{Figure 5. Reducing } G \text{ to } G' \text{ in claim 5}$
2l + 1 to the sum from (2) is at least

\[ \geq 4(2l + 1) + 4 \cdot 3r + (2l - 3) - r \geq 4(2l + 1) + 4 \cdot 3r + (2l - 3) - \frac{2l + 1}{3} = \\
4(2l + 1) + 4 \cdot 3r + \frac{4l - 10}{3} \geq 4(2l + 1) + 4 \cdot 3r - \frac{2}{3} \]

as \( l \geq 2 \).

Since \( \delta(v) \) is integral, we have that the mentioned contribution is at least

\[ \geq 4(2l + 1) + 4 \cdot 3r. \]

It is not hard to see that this implies that the inequality (2) holds. The proof of the theorem is completed.

**Remark 1** The graphs from figure 7 attain the bounds of the theorem, and we suspect that they are the only connected cubic graphs with this property.

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Figure 7. Cycles of length at least five corresponding to an edge lying outside a maximum 3-edge-colorable subgraph