Density Perturbations in the Early Universe

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We propose a way to construct manifestly gauge independent quantities out of the gauge dependent quantities occurring in the linearized Einstein equations. Thereupon, we show that these gauge-invariant combinations can be identified with measurable perturbations to the particle and energy densities. In order to follow the evolution of these quantities in time, we rewrite the full set of linearized Einstein equations as a set of four first order, ordinary differential equations for the four physical quantities in question, namely 1. the perturbations to the energy density, 2. the particle number density, 3. the divergence of the spatial fluid velocity and 4. the local spatial curvature due to a density perturbation.

Next, we recombine the linearized perturbation equations, which still contain gauge functions, in such a way that the resulting equations contain only gauge-invariant quantities; as it should be, all gauge functions have disappeared. In this way, we have arrived at a formulation of the equations for cosmological perturbations that is manifestly gauge-invariant. This set of linearized and manifestly gauge-invariant Einstein equations for the gauge-invariant combinations constitute the main result of this paper.

The set of equations for the new gauge-invariant quantities, however complicated it may seem, is more tractable than those of earlier treatments. This lucky circumstance entails that, unlike the situation encountered in former treatments, we do not need to have recourse to any approximation. Therefore, we obtain results that, sometimes, differ slightly from those found so far. As an illustration, we consider the equations for two cases that can be treated analytically, namely the radiation-dominated and matter-dominated eras of a flat Friedmann-Lemaître-Robertson-Walker universe.

In the radiation-dominated era we find, for small-scale perturbations, acoustic waves with an increasing amplitude, while standard treatments predict acoustic waves with a decaying amplitude. For large-scale perturbations we find exactly the same growth rates as in the standard literature.

Finally we mention that, when considering the non-relativistic limit of the linearized Einstein equations we find — not surprisingly — the Poisson equation. Unlike what has been done so far, however, we did not need to limit the discussion to a static universe, i.e., we found the Poisson equation to be valid not only in a static, but also in an expanding universe.

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I. INTRODUCTION

In order to link the density perturbations at the end of the period of inflation and the observed temperature perturbations in the cosmic 2.7 K microwave background radiation, one needs the linearized Einstein equations. These equations, which determine the growth of densities in the expanding universe after the era of inflation, are the main subject of this article. Our final results are contained in the Eqs. (6)–(8) with (203) and (207).

Mathematically, the problem is very simple, in principle. First, all quantities relevant to the problem are split up into two parts: ‘a background part’ and ‘a perturbation part’. The background parts are chosen to satisfy the Einstein equations for an isotropic universe, i.e., one chooses for the background quantities the Friedmann-Lemaître-Robertson-Walker-solution. Because of the isotropy, the background quantities depend on the time coordinate $t$ only. The perturbation parts are supposed to be small compared to their background counterparts, and to depend on the space-time coordinate $x = (ct, \mathbf{x})$. The background and perturbations are often referred to as ‘zero order’ and ‘first order’ quantities, respectively, and we will use this terminology also in this article. After substituting the sum of the

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zero order and first order parts of all relevant quantities into the Einstein equations, all terms that are products of two or more quantities of first order are neglected. This procedure leads, by construction, to a set of linear differential equations for the quantities of first order. The solution of this set of linear differential equations is then reduced to a standard problem of the theory of ordinary, linear differential equations.

A. Historical embedding

The first systematic and extensive study of cosmological perturbations is due to Lifshitz [1, 2, 3]. Almost half a century later, Mukhanov, Feldman and Brandenberger, in their 1992 review article [4] entitled ‘Theory of Cosmological Perturbations’ mention or discuss more than 60 articles on the subject, and, thereupon, suggest their own approach to the problem. An article of Chung-Pei Ma and Bertschinger [5], entitled ‘Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges’ critically reviews the existing literature on the subject, including the work of Mukhanov et al.

For a solid overview of the literature we refer to the references by the aforementioned authors [4, 5]. For a nice historical overview of gauge theories we refer to O’Raifeartaigh and Straumann [6]. An actual linking of inflationary perturbations and observable radiation background anisotropies has been performed by van Tent [7].

The fact that so many studies are devoted to a problem that is nothing but obtaining the solution of a set of ordinary, linear differential equations is due to the fact that there are several complicating factors, not regarding the mathematics involved, but with respect to the physical interpretation of the solutions. In this article we will try and explain what has been done so far, and we will define, at the same time, gauge-invariant expressions for the physical perturbations to the energy and particle number densities, which differ from gauge-invariant combinations encountered in the literature. In the limit of low velocities, but not necessarily in the limit of slow expansion of the universe, our first order equations reduce, not surprisingly, to one single equation, the well-known Poisson equation. We thus find that the Poisson equation is valid in an expanding universe. This result, which has not been found earlier, is an immediate result from our choice of gauge-invariant quantities. See Sec. IX.

Let us now try and explain in some detail what exactly is the complicating factor in any treatment of the linearized Einstein equations.

B. Origin of the problem

At the very moment that one has split up a physical quantity into a zero order and a first order part, one introduces an ambiguity. To be explicit, let us consider the two quantities that play the leading roles in the theory of cosmological perturbations. They are: the energy density of the universe, $\varepsilon(x)$, and the particle number density of the universe, $n(x)$. The linearized Einstein equations contain as known functions the zero order functions $\varepsilon(0)(t)$ and $n(0)(t)$, which describe the evolution of the background, i.e., they describe the evolution of the unperturbed universe and they obey the unperturbed Einstein equations, and as unknown functions the perturbations $\varepsilon(1)(x)$ and $n(1)(x)$. The latter are the solutions to be obtained from the linearized Einstein equations. The subindexes 0 and 1, which indicate the order, have been put between round brackets, in order to distinguish them from tensor indices. In all calculations, products of a zero order and a first order quantity are considered to be of first order, and are retained, whereas products of first order quantities are neglected. Having said all this, we can say where and how the ambiguity mentioned above arises.

The linearized Einstein equations, which, as stated above already, are the equations which determine the first order quantities $\varepsilon(1)(x)$ and $n_{(1)}(x)$, do not fix these quantities uniquely. In fact, it turns out that next to any solution for $\varepsilon(1)$ and $n(1)$ of the linearized Einstein equations, there exist solutions of the form

$$\dot{\varepsilon}(1)(x) = \varepsilon(1)(x) + \psi(x)\partial_{0}\varepsilon(0)(t), \quad (1a)$$

$$\dot{n}_{(1)}(x) = n_{(1)}(x) + \psi(x)\partial_{0}n_{(0)}(t), \quad (1b)$$

which also satisfy the linearized Einstein equations. Here the symbol $\partial_{0}$ stands for the derivative with respect to $x^{0} = ct$. In the equations (1) the function $\psi(x)$ is an arbitrary but ‘small’ function of the space-time coordinate $x = (ct, x)$, i.e., we consider $\psi(x)$ to be of first order. We will derive the equations (1) at a later point in this article; here it is sufficient to note that, as (1a) and (1b) show, the perturbations $\varepsilon(1)$ and $n_{(1)}$ are fixed by the linearized Einstein equations up to terms that are proportional to an arbitrary, small function $\psi(x)$, usually called a gauge function in this context. Since a physical quantity, i.e., a directly measurable property of a system, may not depend on an arbitrary function, the quantities $\varepsilon(1)$ and $n_{(1)}$ cannot be interpreted as the real, physical, values of the perturbations in the energy density or the particle number density. But, if $\varepsilon(1)$ and $n_{(1)}$ are not the physical
perturbations, what are the real perturbations? This is the notorious ‘gauge problem’ encountered in any treatment of cosmological perturbations. Many different answers to this question can be found in the literature, none of which is completely satisfactory, a fact which explains the ongoing discussion on this subject. It is our hope that the answer given in the present article will turn out to be the definitive one.

C. Gauge-invariant combinations

What we will do in this article is leave $\psi(x)$ undetermined, i.e., we do not choose any particular gauge, but we eliminate the gauge function $\psi(x)$ from the theory altogether. This is not a new idea: many articles have been written in which gauge-invariant quantities, i.e., quantities independent of a gauge function are used. What one essentially does in any gauge-invariant approach is to try and construct expressions for the perturbations [like we do in Eqs. (2)], which are such that $\psi(x)$ disappears from the defining expressions of the physical quantities. These quantities, the perturbations in energy density and particle number density in our case, are then shown to obey a set of linear equations, not containing the gauge function $\psi(x)$ anymore [see Eqs. (6) and (7) with (205) and (207)]. These equations follow, by elimination of the gauge dependent quantities in favor of the gauge-invariant ones, in a straightforward way from the usual linearized Einstein equations, which did contain $\psi(x)$. In this way, the theory is no longer plagued by the gauge freedom that is inherent to the original equations and their solutions: $\psi(x)$ has disappeared altogether, as it should, not with brute force, but as a natural consequence of the definitions (2) of the perturbations to the energy and the particle number densities.

As noted above, our approach also belongs to the class of gauge-invariant cosmological theories, where terms are added to the perturbations in such a way that they become independent of the particular choice of coordinates. However, this can be done, in principle, in infinitely many ways, since any linear combination of gauge-invariant quantities is gauge-invariant also. Our treatment distinguishes itself from earlier treatments by the fact that our set of first order equations reduces to the usual non-relativistic theory in the limit that the three-part of the cosmological fluid velocity four-vector $U^\mu$ is small compared to the velocity of light. Consequently, our splitting up of the energy density and the particle number density in a zero order and a first order part is such that the first order part reduces to the non-relativistic expression. In other words, our treatment of perturbations is ‘around’ the classical, Newtonian theory of gravity. By the way, this is not a conditio sine qua non for a treatment of cosmological perturbations to be true, valid or even useful for some particular purpose, but it is a very desirable property.

We will come back to this point in Sec. IX.

D. Structure of the article

This article being quite extended, it might be useful to outline shortly its structure.

In Sec. I E we define, in Eqs. (2), the physical perturbations to the energy density, $\varepsilon_{gi}^{(1)}$, and the particle number density $n_{gi}^{(1)}$ that we propose in this article. In Sec. I G, we explain how the gauge problem actually arises in cosmology once one chooses to solve the Einstein equations perturbatively, i.e., by means of a series expansion. Furthermore, we show in this section that the quantities $\varepsilon_{gi}^{(1)}$ and $n_{gi}^{(1)}$, defined in Sec. I E, are gauge-invariant.

In Sec. I H we further motivate our choice of these physical perturbations $\varepsilon_{gi}^{(1)}$ and $n_{gi}^{(1)}$ by noting that they reduce to the perturbations as they occur in Newton’s theory of gravitation, contrarily to what is the case for the perturbations defined in earlier treatments. The proof of this statement is postponed to Sec. IX. In Sec. X we apply our perturbation theory to the cases of a radiation-dominated and the matter-dominated universe and we show that our approach yields in the radiation-dominated era small-scale growing density perturbations, in contrast to the decaying density perturbations found in earlier literature.

In Sec. II we introduce a particular system of coordinates, the well-known synchronous coordinates, and rewrite the Einstein equations with respect to this particular coordinates. In Sec. III we restrict the problem of obtaining a solution of the perturbed Einstein equations to universes for which the zero order solution is homogeneous and isotropic, the so-called Friedmann-Robertson-Walker or Friedmann-Lemaître-Robertson-Walker universes. This section is rather technical: we treat zero order and first order quantities in Secs. III A and III B, zero order Einstein equations and conservation laws in Secs. III C 1 and III C 2, and, finally, first order equations and conservation laws in Secs. III D 1 and III D 2.

In Sec. IV the first order equations are split up according to their tensorial character, i.e., whether they are a scalar, a vector or a second rank tensor. The tensorial, the vectorial and the scalar parts of the perturbations, and the equations they should obey, are derived in Secs. IV A, IV B and IV C, respectively. Since, in this article, we are only interested in the energy density and the particle number density, which both are scalar quantities, we only need the scalar equations. In Sec. V, the linearized Einstein equations for scalar perturbations found in Sec. IV C are rewritten.
and simplified: in this section we show that the set of conservation laws and constraint equations can be rewritten as a set of four first order ordinary differential equations and one algebraic equation. It is found that only three scalars play a role in the theory of density perturbations, implying that there is very little choice for the construction of gauge-invariant quantities.

A summary of all relevant equations is given in Sec. VI. Sec. VII is devoted to the perturbations in some thermodynamical quantities related to the energy density and particle number density perturbations. In Sec. VIII the perturbation equations derived in Sec. V are rewritten in a manifestly gauge-invariant form. In Sec. IX we consider the Newtonian limit and in Sec. X we apply the evolution equations (209) to the two main eras of a flat FLRW universe. Finally, in Sec. XI we relate our results to results of earlier work.

In Appendix A some useful thermodynamic relations are collected. Appendix B is devoted to gauge transformations. In Appendix C we show that in the non-relativistic limit there remains some gauge freedom, left over from the general theory relativity. Details to the derivations in Sec. VIII are given in Appendix D. In Appendix E we derive the standard equations using the perturbation equations (202). Finally, Appendix F gives a list of the symbols and notations used in the main text.

E. Gauge-invariant quantities

In the existing literature on cosmological perturbations, one has attempted to solve the problem that corresponds to the non-uniqueness of the perturbations $\varepsilon^{(1)}$ and $n^{(1)}$ (1) in two, essentially different, ways. The first way is to impose an extra condition on the gauge field $\psi(x)$ [8, 9, 10, 11, 12]. Another way to get rid of the gauge field $\psi(x)$ is to take linear combinations of the matter variables and components of the perturbed metric tensor $g_{\mu\nu}$ to construct gauge-invariant quantities (Bardeen [13], Mukhanov, et al. [4]). The latter approach is generally considered better than the one where one fixes a gauge, because it not only leads to quantities that are independent of an undetermined function, as should be the case for a physical quantity, but it also does not rely on any particular choice for the gauge function, and, therefore, has less arbitrariness.

In this article we approach the gauge problem in a Bardeen- or Mukhanov-like way. However, instead of adding terms containing perturbations of the metric tensor field $g_{\mu\nu}$, we add a term that is proportional to (minus) the divergence of the (normalized) cosmological four velocity $U^\mu$: $\varepsilon_{(i)}^\xi := \varepsilon_{(i)} - \frac{\partial_0 \varepsilon_{(0)} }{ \partial_0 \theta_{(0)} } \theta_{(1)}$, $n_{(i)}^\xi := n_{(i)} - \frac{\partial_0 n_{(0)} }{ \partial_0 \theta_{(0)} } \theta_{(1)}$, (2a)

where $\theta_{(0)}$ and $\theta_{(1)}$ are the background and perturbation part of the covariant four-divergence $c^{-1}U^\mu_{,\delta}$ of the cosmological fluid velocity field $U^\mu(x)$. At first sight, these additional terms look quite artificial. However, in Sec. 1G we will show how they arise very naturally.

In Sec. IG, we will show that the quantities $\varepsilon_{(i)}^\xi$ and $n_{(i)}^\xi$ do not change if we switch from the old coordinates $x^\mu$ to new coordinates $\hat{x}^\mu$ according to $\hat{x}^\mu = x^\mu - \xi^\mu(x)$, (3)

where the $\xi^\mu(x)$ ($\mu = 0, \ldots, 3$) are four arbitrary functions, considered to be of first order, of the old coordinates $x^\mu$. In other words, we will first show that $\varepsilon_{(i)}^\xi(x) = \varepsilon_{(i)}^\xi(x)$, $n_{(i)}^\xi(x) = n_{(i)}^\xi(x)$, (4a)
i.e., the perturbations (2) are independent of $\xi^\mu(x)$, i.e., gauge-invariant. The combinations (2) are not completely obvious indeed, but the proof in Sec. IG that they will be gauge-invariant will take away the mystery completely: the above gauge-invariant combinations (2) will then even turn out the ‘obvious’ gauge-invariant quantities to study. Before showing this, we round off the present reasoning.

If we transform the linearized Einstein equations with $\varepsilon_{(i)}(x)$ and $n_{(i)}(x)$ to the new coordinates $\hat{x}$, (3), it will turn out that we find equations in which only the zero component of the gauge functions $\xi^\mu(x)$, ($\mu = 0, 1, 2, 3$), occurs. We will call it $\psi(x)$: $\psi(x) := \xi^0(x)$, (5)
In cosmology, the gauge function $\psi(x)$ is to be treated as a first order quantity, i.e., as a small (or ‘infinitesimal’) change of the coordinates. If we eliminate —with the help of (2)— the quantities $\varepsilon_{(1)}$ and $n_{(1)}$ from the linearized Einstein equations in favor of the gauge-invariant quantities $\varepsilon_{(1)}^g(x)$ and $n_{(1)}^g(x)$, we obtain equations in which the gauge function $\psi(x)$ is absent altogether. The disappearance of the gauge function $\psi(x)$ is due entirely to the introduction of the gauge-invariant quantities $\varepsilon_{(1)}^g(x)$ and $n_{(1)}^g(x)$ rather than $\varepsilon_{(1)}$ and $n_{(1)}$.

The gauge-invariant quantities $\varepsilon_{(1)}^g$ and $n_{(1)}^g$ could be the physical perturbations of the energy density and the particle number density we are after, since they are independent of the gauge field $\psi(x)$, but this is not true a priori, since any linear combination of gauge-invariant quantities is gauge-invariant. Hence, it is not evident that the particular combinations proposed in the equations (2) correspond to physical, i.e., measurable quantities. However, as we will show in section IX, their non-relativistic limit is just what one would expect.

**F. Final results**

The key to the solution of the gauge problem of cosmology are the equations (173)–(174), which are derived in Sec. V. We obtain from these equations (see Sec. VIII) a new set of evolution equations for the gauge-invariant perturbations $\varepsilon_{(1)}^g$ and $n_{(1)}^g$. The equation for $\varepsilon_{(1)}^g$ is a simple second order differential equation of the form:

$$\varepsilon_{(1)}^g + a_1 \varepsilon_{(1)}^g + a_2 \varepsilon_{(1)}^g = a_3 \sigma_{(1)}^g,$$

where $\sigma_{(1)}^g$ is given by the function (207). A dot denotes the derivative with respect to $x^0 = ct$. The coefficients $a_1$, $a_2$, and $a_3$ are complicated combinations of $\varepsilon_{(0)}$, $n_{(0)}$, the spatial curvature $^3R_{(0)}$, the Hubble function $H$ and the equation of state $p_{(0)} = p(\varepsilon_{(0)}, n_{(0)})$ and its partial derivatives [see Eq. (205)]. After one has solved this equation for $\varepsilon_{(1)}^g$, one can calculate the gauge-invariant perturbation to the particle number density with the help of the formula [see Eq. (189)]

$$n_{(1)}^g = \sigma_{(1)}^g + \frac{n_{(0)}}{\varepsilon_{(0)}(1 + w)} \varepsilon_{(1)}^g,$$

where $w := p_{(0)}/\varepsilon_{(0)}$. The second order differential equation (6) for the first order energy density perturbation $\varepsilon_{(1)}^g$ and the expression (7) for the first order particle density perturbation $n_{(1)}^g$ are the main results of this paper. They are obtained without any approximation.

With their help one can relate the (measurable) fluctuation in the cosmic background temperature [see Eq. (200)]

$$T_{(1)}^g = \frac{\varepsilon_{(1)}^g - \left(\frac{\partial \varepsilon}{\partial n}\right)_{T} n_{(1)}^g}{\left(\frac{\partial \varepsilon}{\partial T}\right)_{n}},$$

(8)

to fluctuations $T_{(1)}^g(t_0, x)$ at a time $t_0$ at the end of the era of inflation.

In order to get some insight into what these equations imply, we consider two particular cases: the flat FLRW universe in the radiation-dominated and the matter-dominated eras. As is common usage, we consider the so-called ‘contrast functions’ $\delta_{\varepsilon} := \varepsilon_{(1)}^g/\varepsilon_{(0)}$ and $\delta_{n} := n_{(1)}^g/n_{(0)}$ [see Eqs. (209)] rather than the fluctuations $\varepsilon_{(1)}^g$ and $n_{(1)}^g$ themselves. In the radiation-dominated era we find for these contrast functions the system of equations (235) with the exact solutions (236) and (244). For large-scale perturbations these solutions are in complete agreement with the solutions found in the standard theory. For small-scale perturbations, however, Eqs. (235) yield solutions different from those found in the standard theory: we find an oscillating solution with an increasing amplitude, whereas the standard result is an oscillating solution with a decaying amplitude.

In the matter-dominated era we find the equation (265), which is slightly different from the standard equation (279). For the density contrast function, $\delta_{\varepsilon}$, we find the solution (270) rather than the solution (280) found in the standard literature.

The purpose of this article is to present a logical and straightforward derivation of the linearized Einstein equations, culminating in the equations (6), (7) and (8). It is not our purpose to solve and discuss these equations in detail in the various stages of the evolution of the universe, and analyze the possible cosmological consequences that might be contained in our equations.
G. Gauge-invariant first order perturbations

We now proceed with the proof that \( \varepsilon^{(i)}_{i} \) and \( n^{(i)}_{i} \) are gauge-invariant. To that end, we start by recalling the defining expression for the Lie derivative of an arbitrary tensor field \( A^{\alpha \beta \cdots \mu \nu} \) with respect to a vector field \( \xi^{\tau}(x) \). It reads

\[
\left( \mathcal{L}_{\xi} A \right)^{\alpha \beta \cdots \mu \nu} = A^{\alpha \beta \cdots \mu \nu} + \left\{ \xi^{\tau} \right\} A^{\alpha \beta \cdots \mu \nu \tau} + \left\{ \xi^{\mu} \right\} A^{\alpha \beta \cdots \nu \tau \mu} + \cdots + \left\{ \xi^{\nu} \right\} A^{\alpha \beta \cdots \mu \tau \nu} + \cdots
\]

where the semi-colon denotes the covariant derivative. At the right-hand side, there is a term with a plus sign for each lower index and a term with a minus sign for each upper index. Recall also, that the covariant derivative in the expression for the Lie derivative may be replaced by an ordinary derivative, since the Lie derivative is, by definition, independent of the connection. This fact simplifies some of the calculations below.

We denote by \( \hat{A}^{\alpha \beta \cdots \mu \nu}(x) \) the components of the tensor \( A^{\alpha \beta \cdots \mu \nu} \) defined with respect to the old coordinates \( \{ x^{\mu} \} \) and \( \{ \hat{x}^{\mu} = x^{\mu} - \xi^{\mu}(x) \} \) be two sets of coordinate systems, where \( \xi^{\mu}(x) \) is an arbitrary —but infinitesimal, i.e., in this article, of first order— vector field. Then the components \( \hat{A}^{\alpha \beta \cdots \mu \nu}(x) \) of the tensor \( A \) with respect to the new coordinates \( \hat{x}^{\mu} \) can be related to the components of the tensor \( A^{\alpha \beta \cdots \mu \nu}(x) \), defined with respect to the old coordinates \( \{ x^{\mu} \} \) with the help of the Lie derivative. Up to and including terms containing first order derivatives one has

\[
\hat{A}^{\alpha \beta \cdots \mu \nu}(x) = A^{\alpha \beta \cdots \mu \nu}(x) + \left( \mathcal{L}_{\xi} A \right)^{\alpha \beta \cdots \mu \nu}(x) + \cdots. \tag{10}
\]

For a derivation of this equation, see Weinberg [14], Chap. 10, Sec. 9.

Now, let \( \{ x^{\mu} \} \) and \( \{ \hat{x}^{\mu} = x^{\mu} - \xi^{\mu}(x) \} \) be two sets of coordinate systems, where \( \xi^{\mu}(x) \) is an arbitrary —but infinitesimal, i.e., in this article, of first order— vector field. Then the components \( \hat{A}^{\alpha \beta \cdots \mu \nu}(x) \) of the tensor \( A \) with respect to the new coordinates \( \hat{x}^{\mu} \) can be related to the components of the tensor \( A^{\alpha \beta \cdots \mu \nu}(x) \), defined with respect to the old coordinates \( \{ x^{\mu} \} \) with the help of the Lie derivative. Up to and including terms containing first order derivatives one has

\[
\hat{A}^{\alpha \beta \cdots \mu \nu}(x) = A^{\alpha \beta \cdots \mu \nu}(x) + \left( \mathcal{L}_{\xi} A \right)^{\alpha \beta \cdots \mu \nu}(x) + \cdots. \tag{10}
\]

For a derivation of this equation, see Weinberg [14], Chap. 10, Sec. 9.

Note that \( x \) in the left-hand side corresponds to a point, \( P \) say, of space-time with coordinates \( x^{\mu} \) in the coordinate frame \( \{ x \} \), while in the right-hand side \( x \) corresponds to another point, \( Q \) say, with exactly the same coordinates \( x^{\mu} \), but now with respect to the coordinate frame \( \{ \hat{x} \} \). Thus, equation (10) is an expression that relates one tensor field \( A \) at two different points of space-time, points that are related via the relation (3).

The following observation is crucial. Because of the general covariance of the Einstein equations, they are invariant under general coordinate transformations \( x \to \hat{x} \) and, in particular, under coordinate transformations given by (3). Hence, if some tensorial quantity \( A(x) \) of rank \( n \) \( (n = 0, 1, \ldots) \) satisfies the Einstein equations with as source term the energy-momentum tensor \( T \), the quantity \( \hat{A}(x) = A(x) + \mathcal{L}_{\xi} A(x) \) satisfies the Einstein equations with source term \( \hat{T}(x) = T(x) + \mathcal{L}_{\xi} T(x) \), for a universe with precisely the same physical content. Because of the linearity of the linearized Einstein equations, a linear combination of any two solutions is also a solution. In particular, \( \mathcal{L}_{\xi} A \), being the difference of \( A \) and \( \hat{A} \), is a solution of the linearized Einstein equations with source term \( \mathcal{L}_{\xi} T \). In first order, \( \mathcal{L}_{\xi} A(x) \) may be replaced by \( \mathcal{L}_{\xi} A_{(0)}(t) \), where \( A_{(0)}(t) \) is the solution for \( A(t) \) of the zero order Einstein equations. The freedom to add a term of the form \( \mathcal{L}_{\xi} A_{(0)}(t) \), with \( \xi^{\mu} \) \( (\mu = 0, 1, 2, 3) \) four arbitrary functions of first order, to any solution of the Einstein equations of the first order, is the reason that none of the first order solutions is uniquely defined, and, hence, does not correspond in a unique way to a measurable property of the universe. This is the notorious gauge problem referred to in the introduction of this article. The additional terms \( \mathcal{L}_{\xi} A_{(0)}(t) \) are called ‘gauge modes’.

Combining (9) and (10) we have

\[
\hat{A}^{\alpha \beta \cdots \mu \nu}(x) = A^{\alpha \beta \cdots \mu \nu}(x) + \left( \mathcal{L}_{\xi} A \right)^{\alpha \beta \cdots \mu \nu}(x) + \cdots \tag{11}
\]

We now apply the equation (11) to the case that \( A \) is a scalar \( \sigma \), a four-vector \( V^{\mu} \) and a tensor \( A_{\mu \nu} \), respectively,

\[
\hat{\sigma}(x) = \sigma(x) + \xi^{\tau}(x) \partial_{\tau} \sigma(x), \quad \hat{V}^{\mu} = V^{\mu} + \xi^{\tau} \partial_{\tau} V^{\mu}, \quad \hat{A}_{\mu \nu} = A_{\mu \nu} + \xi^{\tau} \partial_{\tau} A_{\mu \nu} \tag{12a}\]

For the metric tensor, \( g_{\mu \nu} \), we find in particular, from Eq. (12c),

\[
\hat{g}_{\mu \nu} = g_{\mu \nu} + \xi_{\mu ; \nu} + \xi_{\nu ; \mu}, \tag{13}
\]

where we have used that the covariant derivative of the metric vanishes.
Our construction of gauge-invariant perturbations totally rest upon these equations for hatted quantities. In case $\sigma(x)$ is some scalar quantity obeying the Einstein equations, $\sigma(x)$ can be split up in the usual way in a zero order and a first order part:

$$\sigma(x) := \sigma_{(0)}(t) + \sigma_{(1)}(x),$$

where $\sigma_{(0)}(t)$ is some background quantity, and hence, not dependent on the spatial coordinates. Then (12a) becomes

$$\dot{\sigma}(x) = \sigma_{(0)}(t) + \sigma_{(1)}(x) + \xi^0(x)\partial_0\sigma_{(0)}(t) + \xi^\mu(x)\partial_\mu\sigma_{(1)}(x).$$

The last term, being a product of the first order quantity $\xi^\mu(x)$ and the first order quantity $\partial_\mu\sigma_{(1)}$, will be neglected. We thus find

$$\dot{\sigma}(x) = \sigma_{(0)}(t) + \dot{\sigma}_{(1)}(x),$$

with

$$\dot{\sigma}_{(1)}(x) := \sigma_{(1)}(x) + \psi(x)\partial_0\sigma_{(0)}(t),$$

where we used (5). Similarly, we find from (12b) and (13)

$$\dot{V}_{(1)}^\mu(x) = V_{(1)}^\mu + V_{(0)}^\mu_\tau\xi^\tau - V_{(0)}^\tau_\tau\xi^\mu,$$

and

$$\dot{\check{g}}_{(1)}^{\mu\nu}(x) = g_{(1)}^{\mu\nu}(x) + \xi^{\mu,\nu} + \xi^{\nu,\mu}.$$ (19)

The latter two equations will be used later.

We are now in a position that we can conclude the proof of the statement that $\varepsilon_{(1)}^{\hat{\mu}}$ and $n_{(1)}^{\hat{\mu}}$ are gauge-invariant. To that end, we now write down the equation (17) once again, for another arbitrary scalar quantity $\omega(x)$ obeying the Einstein equations. We then find the analogue of Eq. (17)

$$\dot{\omega}_{(1)}(x) = \omega_{(1)}(x) + \psi(x)\partial_0\omega_{(0)}(t).$$ (20)

The left-hand sides of (17) and (20) give the value of the perturbation at the point with coordinates $x$ with respect to the old coordinate system $\{x\}$; the right-hand sides of (17) and (20) contains quantities with the same values of the coordinates, $x$, but now with respect to the new coordinate system $\{\hat{x}\}$. Eliminating the function $\psi(x)$ from Eqs. (17) and (20) yields

$$\dot{\sigma}_{(1)}(x) - \partial_0\sigma_{(0)}(t)\dot{\omega}_{(1)}(x) = \sigma_{(1)}(x) - \partial_0\sigma_{(0)}(t)\omega_{(1)}(x).$$ (21)

In other words, the particular linear combination occurring in the right-hand side of (21) of any two quantities $\omega$ and $\sigma$ transforming as scalars under an arbitrary space-time transformation —of first order, cf. Eq. (3)— is gauge-invariant, and, hence, a possible candidate for a physical quantity.

The equation (21) is the key equation of this article as far as the scalar quantities $\varepsilon_{(1)}$ and $n_{(1)}$ are concerned. It tells us how to combine the scalar quantities occurring in the linearized Einstein equations in such a way that they become gauge independent. The equation (21) can be used to immediately derive the expressions (2) for the gauge-invariant energy and particle number densities.

In fact, let $U^\mu(x)$ be the four-velocity of the cosmological fluid. In the theory of cosmological perturbations [see Eqs. (170)-(171) and (173)-(174)] there will turn out to be only three scalars, namely

$$\varepsilon(x) = c^{-2}T^{\mu\nu}(x)U_\mu(x)U_\nu(x),$$

$$n(x) = c^{-2}N^\mu(x)U_\mu(x),$$

$$\theta(x) = c^{-1}U^\mu_{\mu}(x),$$

where

$$N^\mu := nU^\mu.$$ (23)
is the cosmological particle current four-vector normalized according to $U^\mu U_\mu = c^2$. These scalars are split up according to
\begin{align}
\varepsilon(x) &= \varepsilon_{(0)}(t) + \varepsilon_{(1)}(x), \tag{24a} \\
n(x) &= n_{(0)}(t) + n_{(1)}(x), \tag{24b} \\
\theta(x) &= \theta_{(0)}(t) + \theta_{(1)}(x), \tag{24c}
\end{align}
where the background quantities $\varepsilon_{(0)}(t)$, $n_{(0)}(t)$ and $\theta_{(0)}(t)$ are solutions of the unperturbed Einstein equations. They depend on the time coordinate $t$ only. The relation (21) inspires us to consider the gauge-invariant combinations
\begin{align}
\varepsilon_{(1)}^{gi}(x) &:= \varepsilon_{(1)}(x) - \frac{\partial \varepsilon_{(0)}(t)}{\partial \omega_{(0)}(t)} \omega_{(1)}(x), \tag{25a} \\
n_{(1)}^{gi}(x) &:= n_{(1)}(x) - \frac{\partial n_{(0)}(t)}{\partial \omega_{(0)}(t)} \omega_{(1)}(x), \tag{25b} \\
\theta_{(1)}^{gi}(x) &:= \theta_{(1)}(x) - \frac{\partial \theta_{(0)}(t)}{\partial \omega_{(0)}(t)} \omega_{(1)}(x). \tag{25c}
\end{align}

The question remains what to choose for $\omega$ in these three cases. In principle, for $\omega$ we could choose any of the following three scalar functions available in the theory, i.e., we could choose $\varepsilon$, $n$ or $\theta$. As follows from (25a) and (25b), the choices $\omega = \varepsilon$ and $\omega = n$ would lead to $\varepsilon_{(1)}^{gi}(x) = 0$ or $n_{(1)}^{gi}(x) = 0$, respectively. This would mean that perturbations in the energy or the particle number density would show up only in the second order. Apparently, these choices would not be suitable since the perturbations of precisely these quantities are the object of study. We therefore are left with the choice

$$\omega = \theta,$$ \tag{26}
which implies the equations (2a) and (2b) for the energy and particle number density perturbations, as was to be shown, and, moreover,

$$\theta_{(1)}^{gi} = 0. \tag{27}$$

The latter equation implies that perturbations in the divergence of the cosmological velocity field $\theta(x)$ will only show up in second order. This constitutes no problem, since we are not interested in this quantity. Inserting (26) into (25a) and (25b) we obtain the expressions (2). Hence, it now has been shown that $\varepsilon_{(1)}^{gi}$ and $n_{(1)}^{gi}$ are indeed invariant under the infinitesimal coordinate transformation (3), i.e., that they are gauge-invariant.

The linearized Einstein equations we are about to study contain the gauge function $\psi(x)$. If we eliminate $\varepsilon_{(1)}(x)$ and $n_{(1)}(x)$ from these equations in favor of $\varepsilon_{(1)}^{gi}(x)$ and $n_{(1)}^{gi}(x)$, with the help of the expressions (2), we obtain equations, which do not depend on $\psi(x)$ at all: all terms with $\psi(x)$ cancel automatically. This could be expected, because the equations should have solutions for $\varepsilon_{(1)}^{gi}(x)$ and $n_{(1)}^{gi}(x)$ which are, by construction, independent of the diffeomorphism (3). Stated differently, the fact that $\psi(x)$ disappears from the linearized perturbation theory is in line with the earlier observation that (2) are gauge independent combinations for arbitrary diffeomorphisms (3). In Sec. VIII we will show in detail how this will happen.

H. Non-relativistic limit of the gauge-invariant combinations

The following question may arise. It is clear now that $\varepsilon_{(1)}(x)$ and $n_{(1)}(x)$ are not gauge-invariant, but that the combinations $\varepsilon_{(1)}^{gi}(x)$ and $n_{(1)}^{gi}(x)$ are so. Hence, $\varepsilon_{(1)}(x)$ and $n_{(1)}(x)$ cannot be physical perturbations, while $\varepsilon_{(1)}^{gi}(x)$ and $n_{(1)}^{gi}(x)$ could be physical. But how can one be sure that the particular combinations (2) are indeed the real physical perturbations? The fact that they are gauge-invariant is a necessary, but not a sufficient reason. Since, moreover, any linear combination of gauge-invariant quantities is a new gauge-invariant quantity, it is not clear at all, yet, that the particular combinations (2) are the right ones.

This issue can be settled by considering the non-relativistic limit, i.e., the limit of low spatial velocities with respect to the speed of light. In this particular case the linearized Einstein equations (202)–(203) imply, see Eqs. (228) and (229),

$$\nabla^2 \varphi(x) = 4\pi G \frac{\varepsilon_{(0)}^{gi}(x^\mu t^\mu \cdot \mathbf{x})}{c^2}, \tag{28}$$
where \( t_p \) indicates the present time, \( c \) is the speed of light, \( G \) Newton’s gravitational constant and \( \nabla^2 \) is the usual Laplace operator. This is the well-known Poisson equation of the Newtonian theory of gravity. Hence, in the non-relativistic regime, the mathematical combination \( \varepsilon^{gi}_{(i)} \) (2a), divided by \( c^2 \), is to be interpreted as the normal, or ‘right’ physical perturbation of the mass density \( g_{(i)} \), implying that \( \varepsilon^{gi}_{(i)} \) may be interpreted as the ‘right’ or physical perturbation in the energy density indeed. Since the non-relativistic limit of another Einstein equation, Eq. (220b), will turn out to imply \( n^{gi}_{(i)}(t_p, x) = \varepsilon^{gi}_{(i)}(t_p, x)/(mc^2) \), Eq. (220b), we see that also \( n^{gi}_{(i)} \) is the right physical perturbation. This answers the question raised at the beginning of this section with respect to \( \varepsilon^{gi}_{(i)} \) and \( n^{gi}_{(i)} \): these quantities are, indeed, the physical perturbations we were looking for.

II. THE EINSTEIN EQUATIONS REWRITTEN IN SYNCHRONOUS COORDINATES

As most authors do, we choose a synchronous system of reference. A synchronous system of reference is a system in which the line element for the metric has the form:

\[
\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - g_{ij}(t, x) \mathrm{d}x^i \mathrm{d}x^j. \tag{29}
\]

The name synchronous stems from the fact that surfaces with \( t = \text{constant} \) are surfaces of simultaneity for observers at rest with respect to the synchronous coordinates, i.e., observers for which the three coordinates \( x^i \) \((i = 1, 2, 3)\) remain constant. A synchronous system can be used for an arbitrary space-time manifold, not necessarily a homogeneous or isotropic one. In a synchronous system, the coordinate \( t \) measures the proper time along lines of constant \( x^i \). From (29) we can read off that \((x^0 = ct)\):

\[
g_{00}(t, x) = 1, \quad g_{0i}(t, x) = 0. \tag{30}
\]

From the form of the line element in four-space, Eq. (29), it follows that minus \( g_{ij}(t, x) \), \((i = 1, 2, 3)\), is the metric of a three-dimensional subspaces with constant \( t \). Because of (30), knowing the three-geometry in all hypersurfaces, is equivalent to knowing the geometry of space-time. The following abbreviations will prove useful when we rewrite the Einstein equations with respect to synchronous coordinates:

\[
\begin{align*}
\varkappa_{ij} \&:= -\frac{1}{2}g_{ij}, \\
\varkappa^i \&:= g^{ik}\varkappa_{kj}, \\
\varkappa^i_j \&:= +\frac{1}{2}g^{ij},
\end{align*}
\tag{31}
\]

where a dot denotes differentiation with respect to \( x^0 = ct \). From Eqs. (30)–(31) it follows that the connection coefficients of (four-dimensional) space-time

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(g_{\mu\nu,\kappa} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}),
\tag{32}
\]

in synchronous coordinates are given by

\[
\begin{align*}
\Gamma^0_{00} &= \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^0_{ij} = 0, \\
\Gamma^0_{ij} &= \varkappa_{ij}, \\
\Gamma^0_{ij} &= \Gamma^i_{0j} = -\varkappa^i_j, \\
\Gamma^k_{ij} &= \frac{1}{2}g^{kl}(g_{li,j} + g_{lj,i} - g_{ij,l}).
\end{align*}
\tag{33}
\]

From Eq. (33c) it follows that the \( \Gamma^k_{ij} \) are also the connection coefficients of (three-dimensional) subspaces of constant time.

The Ricci tensor \( R_{\mu\nu} := R^\lambda_{\mu\lambda\nu} \) is, in terms of the connection coefficients, given by

\[
R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\sigma_{\mu\nu,\lambda} \Gamma^\lambda_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu,\sigma}. \tag{34}
\]

Upon substituting Eqs. (33) into Eq. (34) one finds for the components of the Ricci tensor

\[
\begin{align*}
R_{00} &= \varkappa^k_k - \varkappa^j_j - \varkappa^i_i, \\
R_{0i} &= \varkappa^k_{kij} - \varkappa^j_{ijk}, \tag{35a} \\
R_{ij} &= \varkappa_{ij} - \varkappa_{ij} + 2\varkappa_{ik}\varkappa^k_j + 3R_{ij}, \tag{35b}
\end{align*}
\]

where the vertical bar in Eq. (35b) denotes covariant differentiation with respect to the metric \( g_{ij} \) of a three-dimensional subspace:

\[
\varkappa^i_j[k] := \varkappa^i_{j,k} + \Gamma^i_{hk}\varkappa^h_j - \Gamma^i_{jk}\varkappa^j_t. \tag{36}
\]
The quantities $^3R_{ij}$ in Eq. (35c) are found to be given by

$$^3R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij} \Gamma^k_{kl} - \Gamma^l_{ik} \Gamma^k_{jl}. \quad (37)$$

Hence, $^3R_{ij}$ is the Ricci tensor of the three-dimensional subspaces of constant time. For the components $R^\mu_\nu = g^{\mu\tau} R_{\tau\nu}$ of the Ricci tensor (35), we get

$$
\begin{align*}
R^0_0 &= \dot{x}^k_k - \dot{x}^i_i x^k_i, \\
R^0_i &= \dot{x}^k_{k|i} - \dot{x}^k_{i|k}, \\
R^i_j &= \dot{x}^j_j - \dot{x}^j_{i} x^k_k + ^3R^i_j, \quad (38) \end{align*}
$$

where we have used Eqs. (30)–(31).

The Einstein equations read

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}, \quad (39)$$

where $G^{\mu\nu}$, the Einstein tensor, is given by

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}. \quad (40)$$

In (39) $\Lambda$ is a positive constant, the well-known cosmological constant. The constant $\kappa$ is given by

$$\kappa := \frac{8\pi G}{c^4}, \quad (41)$$

with $G$ Newton’s gravitational constant and $c$ the speed of light. In view of the Bianchi identities one has

$$G^{\mu\nu,\nu} = 0, \quad (42)$$

hence, since $g^{\mu\nu,\nu} = 0$, the source term $T^{\mu\nu}$ of the Einstein equations must fulfill the properties

$$T^{\mu\nu,\nu} = 0. \quad (43)$$

These equations are the energy-momentum conservation laws. An alternative way to write the Einstein equations (39) reads

$$R^\mu_\nu = \kappa(T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T^\alpha_\alpha) - \Lambda \delta^\mu_\nu. \quad (44)$$

Upon substituting the components (38) into the Einstein equations (44), and eliminating the time derivative of $x^k_k$ from the $R^0_0$-equation with the help of the $R^i_j$-equations, the Einstein equations can be cast in the form

$$
\begin{align*}
(x^k_k)^2 - 3R - x^k_i x^i_k &= 2(\kappa T^0_0 + \Lambda), \\
x^k_{k|i} - x^k_{i|k} &= \kappa T^0_i, \quad (45a) \\
\dot{x}^i_j - \dot{x}^i_j x^k_k + ^3R^i_j &= \kappa(T^i_j - \frac{1}{2} \delta^i_j T^\mu_\mu) - \Lambda \delta^i_j, \quad (45b) \\
3R &:= g^{ij} ^3R_{ij} = g^{ik} 3R_{ik}, \quad (46)
\end{align*}
$$

is the curvature scalar of the three-dimensional subspaces of constant time. The (differential) equations (45c) are the so-called dynamical Einstein equations: they define the evolution (of the time derivative) of the (spatial part of the) metric. The (algebraic) equations (45a) and (45b) are constraint equations: they relate the initial conditions, and, once these are satisfied at one time, they are satisfied automatically at all times.

The right-hand side of Eqs. (45) contain the components of the energy momentum tensor $T^{\mu\nu}$, which, for a perfect fluid, are given by

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p \delta^{\mu\nu}, \quad (47)$$

where $u^\mu(x) = e^{-1} U^\mu(x)$ is the hydrodynamic fluid four-velocity normalized to unity: $(u^\mu u^\mu = 1)$, $\varepsilon(x)$ the energy density and $p(x)$ the pressure at a point $x$ in space-time. In this expression we neglect terms containing the shear and volume viscosity, and other terms related to irreversible processes. The equation of state for the pressure

$$p = p(n, \varepsilon), \quad (48)$$
where \( n(x) \) is the particle number density at a point \( x \) in space-time, is supposed to be a given function of \( n \) and \( \varepsilon \) (see also Appendix A for equations of state in alternative forms).

As stated above already, the Einstein equations (45a) and (45b) are constraint equations to the Einstein equations (45c) only: they tell us what relations should exist between the initial values of the various unknown functions, in order that the Einstein equations be solvable. In the following, we shall suppose that these conditions are satisfied. Thus we are left with the nine equations (45c), of which, because of the symmetry of \( g_{ij} \), only six are independent. These six equations, together with the four equations (43) constitute a set of ten equations for the eleven \((6+3+1+1)\) independent quantities \( g_{ij}, u^i, \varepsilon \) and \( n \). The eleventh equation needed to close the system of equations is the particle number conservation law \( N^\nu_{;\mu} = 0 \), i.e.,

\[
(nu^\mu)_{;\mu} = 0, \tag{49}
\]

[see Eq. (23)] where a semicolon denotes covariant differentiation with respect to the metric tensor \( g_{\mu\nu} \). This equation can be rewritten in terms of the fluid expansion scalar defined by Eq. (22c). Using Eqs. (33), we can rewrite the four-divergence (22c) in the form

\[
\theta = u^0 - \kappa^k u^0 + \vartheta, \tag{50}
\]

where the three-divergence \( \vartheta \) is given by

\[
\vartheta := u^k_{;k}. \tag{51}
\]

Using now Eqs. (22c), (33), (36) and (50), the four energy-momentum conservation laws (43) and the particle number conservation law (49) can be rewritten as

\[
T^{00} + T^{0k}_{;k} + \kappa^k T^l_k - \kappa^k \kappa T^{00} = 0, \tag{52a}
\]

\[
\dot{T}^{00} + T^{0k}_{;k} - 2\kappa^l T^{k0} - \kappa^k T^{00} = 0, \tag{52b}
\]

and

\[
\dot{n} u^0 + n_k u^k + n\vartheta = 0, \tag{53}
\]

respectively. Since \( T^{00} \) is a vector and \( T^{ij} \) is a tensor with respect to coordinate transformations in a subspace of constant time, and, hence, are tensorial quantities in this three-dimensional subspace, we could use in (52) a bar to denote covariant differentiation with respect to the metric \( g_{ij}(t, x) \) of such a subspace of constant time \( t \).

The Einstein equations (45) and conservation laws (52) and (53) describe a universe filled with a perfect fluid and with a positive cosmological constant. The fluid pressure \( p \) is described by an equation of state of the form (48): in this stage we only need that it is some function of the particle number density \( n \) and the energy density \( \varepsilon \).

It is now time to actually derive the zero and first order Einstein equations. To that end, we expand all quantities in the form of series, and derive, recursively, equations for the successive terms of these series. Furthermore, we will now limit the discussion to a particular class of universes, namely the collection of universes that, apart from a small, local perturbation in space-time, are homogeneous and isotropic, the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) universes.

### III. ZERO AND FIRST ORDER EQUATIONS FOR THE FLRW UNIVERSE

Consider space-times which are a continuous collection (foliation) of three-dimensional, space-like slices of space-time, each of which is maximally symmetric. Expressed in synchronous coordinates, this statement means that we consider spaces with metric \( g_{\mu\nu} \) given by

\[
g_{00}(t, x) = 1, \quad g_{0i}(t, x) = 0, \tag{54a}
\]

\[
g_{ij}(t, x) = -a^2(t) \tilde{g}_{ij}(x), \tag{54b}
\]

where \( \tilde{g}_{ij}(x) \) is the metric of such a three-dimensional maximally symmetric subspace. The minus sign in (54b) has been introduced in order to switch from the conventional four-dimensional space-time with signature \((+, -, -, -)\) to the conventional three-dimensional spatial metric with signature \((+, +, +)\). We write \( a^2(t) \) rather than \( a(t) \) for the time dependent proportionality factor in (54b), the so-called scale factor or expansion factor of the universe, for reasons that will become clear later: the scale factor \( a(t) \) will turn out to be identifiable in some cases, as the ‘radius of the universe’, i.e., the slices of the foliation are three-dimensional surfaces in four-space, with radius \( a(t) \); see Eq. (77).
Essentially, however, we only suppose that the three-part of the metric can be factorized according to (54b), also in case the slices are no surfaces of hyperspheres.

We now expand all quantities concerned in series. We will distinguish the successive terms of a series by a subindex between brackets.

\[ \varepsilon = \varepsilon^{(0)} + \eta \varepsilon^{(1)} + \eta^2 \varepsilon^{(2)} + \cdots, \quad (55a) \]
\[ n = n^{(0)} + \eta n^{(1)} + \eta^2 n^{(2)} + \cdots, \quad (55b) \]
\[ p = p^{(0)} + \eta p^{(1)} + \eta^2 p^{(2)} + \cdots, \quad (55c) \]
\[ \theta = \theta^{(0)} + \eta \theta^{(1)} + \eta^2 \theta^{(2)} + \cdots, \quad (55d) \]
\[ \vartheta = \vartheta^{(0)} + \eta \vartheta^{(1)} + \eta^2 \vartheta^{(2)} + \cdots, \quad (55e) \]
\[ 3R = 3R^{(0)} + \eta 3R^{(1)} + \eta^2 3R^{(2)} + \cdots, \quad (55f) \]

where the subindex zero refers to quantities of the unperturbed, homogeneous and isotropic FLRW universe. In order to derive the background and first order Einstein equations, we need ancillary quantities, which will also be expanded in series:

\[ u^\mu = u_{(0)}^\mu + \eta u_{(1)}^\mu + \eta^2 u_{(2)}^\mu + \cdots, \quad (56a) \]
\[ g_{ij} = g_{(0)ij} + \eta g_{(1)ij} + \eta^2 g_{(2)ij} + \cdots, \quad (56b) \]
\[ \kappa_{ij} = \kappa_{(0)ij} + \eta \kappa_{(1)ij} + \eta^2 \kappa_{(2)ij} + \cdots, \quad (56c) \]
\[ T^\mu_{\nu} = T^\mu_{(0)\nu} + \eta T^\mu_{(1)\nu} + \eta^2 T^\mu_{(2)\nu} + \cdots, \quad (56d) \]
\[ 3R_{ij} = 3R_{(0)ij} + \eta 3R_{(1)ij} + \eta^2 3R_{(2)ij} + \cdots, \quad (56e) \]
\[ \Gamma^k_{ij} = \Gamma^k_{(0)ij} + \eta \Gamma^k_{(1)ij} + \eta^2 \Gamma^k_{(2)ij} + \cdots. \quad (56f) \]

In Eqs. (55) and (56) \( \eta = 1 \) is a bookkeeping parameter, the function of which is to enable us in actual calculations to easily distinguish between the terms of different orders.

### A. Zero order quantities

This section is concerned with the background or zero order quantities occurring in the Einstein equations. All results of this section are totally standard, and given here only to fix the notation unambiguously.

A tensor in a maximally symmetric space is called maximally form-invariant if its Lie-derivatives with respect to all Killing vectors of the maximally symmetric space vanish. Essentially, this means that the tensor is ‘the same’ in all directions and at all places, just as the metric of a maximally symmetric space is ‘the same’ in all directions and all places. It can be proved that the only maximally form-invariant scalar function which exists in a maximally symmetric space is a constant, that the only maximally form-invariant vector in a maximally symmetric space is the null vector, and that the only maximally form-invariant second rank tensor in a maximally symmetric space of dimension \( N \geq 3 \) is proportional to the metric tensor (see Weinberg [14], Chap. 13). In our case, of a four-dimensional space-time which is a foliation of three-dimensional maximally symmetric spaces, this leads to the following, general, conclusions. A scalar can only be a function of time and the spatial part \( A^i \) of a vector \( A^\mu \) is the null vector. Furthermore, the spatial components \( F^{ij} \) of a tensor \( F^{\mu\nu} \) must be proportional to the metric \( \tilde{g}^{ij} \) and the components \( F^{0j} \) are identically zero.

In particular, the background energy density \( \varepsilon_{(0)}(t, \mathbf{x}) \), Eq. (55a), a space-time scalar, is independent of the coordinates \( \mathbf{x} = (x^1, x^2, x^3) \) of the maximally symmetric subspace. Similarly, the background particle number density \( n_{(0)} \), Eq. (55b), the background pressure \( p_{(0)} \), Eq. (55c), and the fluid expansion scalar \( \theta_{(0)} \), Eq. (55d), all space-time scalars, are functions of time only. Thus we have, with respect to our particular coordinates,

\[ \varepsilon_{(0)}(t, \mathbf{x}) = \varepsilon_{(0)}(t), \quad n_{(0)}(t, \mathbf{x}) = n_{(0)}(t), \quad p_{(0)}(t, \mathbf{x}) = p_{(0)}(t), \quad \theta_{(0)}(t, \mathbf{x}) = \theta_{(0)}(t). \quad (57) \]

The three-divergence \( \vartheta_{(0)} \), Eq. (55e), and the curvature of the three-dimensional subspaces \( 3R_{(0)} \), Eq. (55f), which are scalars only with respect to spatial coordinate transformations, are also functions of time only, i.e.,

\[ \vartheta_{(0)}(t, \mathbf{x}) = \vartheta_{(0)}(t), \quad 3R_{(0)}(t, \mathbf{x}) = 3R_{(0)}(t). \quad (58) \]

Furthermore, the component of the four-vector \( u^\mu \), Eq. (56a), tangent to a maximally symmetric space is a three-vector in that maximally symmetric space. The only maximally form-invariant vector in a maximally symmetric space is the null vector. Hence,

\[ u^i_{(0)} = 0. \quad (59) \]
Consequently, $u^\mu$ is proportional to $\delta^{\mu}_{0}$. Moreover, since $u^\mu$ is a unit vector, we have

$$u_{(0)}^\mu = \delta_{0}^{\mu}.$$ \hspace{1cm} (60)

The zero order background metric tensor, occurring in Eq. (56b), has been supposed to be that of a maximally symmetric space, i.e.,

$$g_{(0)ij}(t, x) = -a^2(t)\tilde{g}_{ij}(x),$$ \hspace{1cm} (61)

compare Eq. (54b). The time derivative of the three-part of the metric $g_{(0)ij}$, $\kappa_{(0)ij}$, Eq. (56c), may be expressed in the usual Hubble function $H(t) := (da/dt)/a(t)$. We prefer to use a function $H(t) = c^{-1}H(t)$, which we will call Hubble function also. Recalling that a dot denotes differentiation with respect to $ct$, we have

$$H := \dot{a}/a.$$ \hspace{1cm} (62)

Substituting the expansion (56b) into the definitions (31), we obtain

$$\kappa_{(0)ij} = -Hg_{(0)ij}, \quad \kappa_{(0)i} = -H\delta_{ij}, \quad \kappa_{(0)j} = -Hg_{(0)i}.$$ \hspace{1cm} (63)

where we considered only terms up to the zero order in the bookkeeping parameter $\eta$.

Similarly, with Eqs. (50), (51), (55d), (55e), (60) and (63) we find for the background fluid expansion scalar, $\theta_{(0)}$, and the three-divergence, $\vartheta_{(0)}$,

$$\theta_{(0)} = 3H, \quad \vartheta_{(0)} = 0.$$ \hspace{1cm} (64)

Using Eqs. (47), (55a), (55c), (56b), (60) and (61) we find for the components of the energy momentum tensor, Eq. (56d),

$$T_{(0)0}^0 = \varepsilon_{(0)}, \quad T_{(0)i}^0 = 0, \quad T_{(0)ij} = -p_{(0)}\delta_{ij},$$ \hspace{1cm} (65)

where the background pressure $p_{(0)}$ is given by the equation of state (48), which, for the background pressure, is defined by

$$p_{(0)} = p(n_{(0)}, \varepsilon_{(0)}).$$ \hspace{1cm} (66)

i.e., the subindex zero in $p_{(0)}$ refers to the zero order quantities it depends on; it is not a different function of its arguments.

Finally, the Ricci tensor of the three-dimensional maximally symmetric subspaces is proportional to the metric tensor of that subspace, i.e.,

$$^{3}R_{(0)ij} = K\tilde{g}_{ij},$$ \hspace{1cm} (67)

(see Weinberg [14], Chap. 15, Sec. 1). The quantity $K$ is time independent, as may be seen as follows. As follows from Eqs. (56e) and (56f), the background three-dimensional Ricci tensor, (37), is given by

$$^{3}R_{(0)ij} = \Gamma_{(0)ij,k}^{\ k} - \Gamma_{(0)ik,j}^{\ k} + \Gamma_{(0)j}^{\ l}\Gamma_{(0)kl} - \Gamma_{(0)ik}\Gamma_{(0)jl},$$ \hspace{1cm} (68)

where the connection coefficients $\Gamma_{(0)ij}^{\ k}$ are given by

$$\Gamma_{(0)ij}^{\ k} = \frac{1}{2}g_{(0)}^{kl}(g_{(0)li,j} + g_{(0)lj,i} - g_{(0)lj,i}),$$ \hspace{1cm} (69)

where $g_{(0)}^{ij}$ and $g_{(0)ij}$ depend on time. From Eq. (61), we have

$$g_{(0)}^{ij}(t, x) = -\frac{1}{a^2(t)}\tilde{g}_{ij}(x).$$ \hspace{1cm} (70)

Hence, the connection coefficients $\Gamma_{(0)ij}^{\ k}$ are equal to the connection coefficients $\tilde{\Gamma}_{ij}^{\ k}$ of the metric $\tilde{g}_{ij}$:

$$\Gamma_{(0)ij}^{\ k} = \tilde{\Gamma}_{ij}^{\ k} := \frac{1}{2}\tilde{g}_{ij}^{kl}(\tilde{g}_{li,j} + \tilde{g}_{lj,i} - \tilde{g}_{lj,i}).$$ \hspace{1cm} (71)
Therefore, they do not depend on time. As a consequence, $3R_{(0)ij}$ is time independent, implying that $K$ is a constant. Maximally symmetric spaces may have curvature $K$ that is positive, negative or zero. In the latter case the space is called flat.

We now choose coordinates $(r, \theta, \phi)$ such that the metric coefficients $g_{(0)ij}$ get the well-known Robertson-Walker form

$$g_{(0)ij} = -a^2(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - kr^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (72)$$

where $k = 0, \pm 1$. Comparing (61) and (72) we see that for this choice of coordinates we have

$$\tilde{g}_{ij} = \text{diag} \left( \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right). \quad (73)$$

Substituting (72) into (68), combined with (71), we find

$$3R_{(0)ij} = 2k \tilde{g}_{ij}. \quad (74)$$

We thus find, for the chosen RW coordinates

$$K = 2k. \quad (75)$$

From Eqs. (74) and (75) we have

$$3R_{(0)ij}(t) = -\frac{2k}{a^2(t)} \delta_{ij}, \quad (76)$$

implying that the zero order curvature scalar $3R_{(0)} = g_{(0)}^{ij} 3R_{(0)ij}$ is given by

$$3R_{(0)}(t) = -\frac{6k}{a^2(t)}, \quad (77)$$

where we have used Eqs. (70) and (74). Note, that in view of our choice of the metric $(+, -, -, -)$, spaces of positive curvature $k$ (such as spheres) have a negative curvature scalar $3R_{(0)}$.

The results (61), (63), (65) and (74) are in agreement with the general observation that maximally form-invariant tensors of the second rank are proportional to the metric tensor of the space concerned.

Thus we have found all background quantities: they are either space independent or proportional to $g_{(0)ij}(t, \mathbf{x})$, Eq. (61). The latter is proportional to $\tilde{g}_{ij}(\mathbf{x})$, the metric characteristic for spatial sections of constant time.

### B. First order quantities

In this quite technical section we express all quantities occurring in the Einstein equations in terms of zero and first order quantities. The equations of state for the energy and pressure, $\varepsilon(n, T)$ and $p(n, T)$, are not specified yet.

Upon substituting the series (56a) into the normalization condition $u^\mu u_\mu = 1$, one finds, equating equal powers of the bookkeeping parameter $\eta$,

$$u^0_{(1)} = 0, \quad (78)$$

for the first order perturbation to the four-velocity. Writing the inverse of (56b) as

$$g^{kl} = g_{(0)}^{kl} + \eta g_{(1)}^{kl} + \cdots, \quad (79)$$

where $g_{(0)}^{kl}$ is the inverse, (70), of $g_{(0)kl}$, (72), we find

$$g_{(1)}^{kl} = -g_{(0)}^{kl} g_{(0)i} g_{(1)ij}, \quad (80)$$

and

$$g_{(1)k}^{i} = -g_{(0)}^{kl} g_{(1)li}. \quad (81)$$
It is convenient to introduce
\[ h_{ij} := -g_{(1)ij}, \] (82)
so that
\[ h_{ij} = g_{(1)ij}, \quad h^i_j = g^{ik}_{(0)h_{kj}}. \] (83)

For the time derivative of the first order perturbations to the metric, \( \kappa_{(1)ij} \), Eq. (31), we get
\[ \kappa_{(1)ij} = \frac{1}{2} \dot{h}_{ij}, \quad \kappa^i_j = \frac{1}{2} \dot{h}^i_j, \quad \kappa^i_{(1)j} = \frac{1}{2} \dot{h}^i_j. \] (84)

The first order perturbation \( \theta_{(1)} \) to the fluid expansion scalar \( \theta \), Eq. (50), can be found in the same way. Using (55d) and (60) one arrives at
\[ \theta_{(1)} = \dot{\vartheta}_{(1)} - \frac{1}{2} \dot{h}^k_k, \] (85)
where we used Eqs. (78) and (84). The first order perturbation \( \vartheta_{(1)} \) to the three-divergence \( \vartheta \), Eq. (51), is
\[ \vartheta_{(1)} = u^k_{(1)k}, \] (86)
where we have used that
\[ (u^k_{[k]}(1) = u^k_{(1)k}, \] (87)
which is a consequence of
\[ \Gamma^k_{(1)lk} u^l_{(0)} = 0. \] (88)

The latter equality follows from Eq. (59).

Upon substituting the series expansion (55a), and (55c)–(56b) into Eq. (47) and equating equal powers of \( \eta \), one finds for the first order perturbation to the energy-momentum tensor
\begin{align*}
T^0_{(1)0} &= \varepsilon_{(1)}, \\
T^i_{(1)0} &= (\varepsilon_{(0)} + p_{(0)}) u^i_{(1)}, \\
T^i_{(1)j} &= -p_{(1)} \delta^i_j,
\end{align*} (89)
where we have used Eqs. (60) and (78). The first order perturbation to the pressure is related to \( \varepsilon_{(1)} \) and \( n_{(1)} \) by the first order perturbation to the equation of state (48). We have
\[ p_{(1)} = p_n n_{(1)} + p_\varepsilon \varepsilon_{(1)}, \] (90)
where \( p_n \) and \( p_\varepsilon \) are the partial derivatives of \( p(n, \varepsilon) \) with respect to \( n \) and \( \varepsilon \),
\[ p_n := \left( \frac{\partial p}{\partial n} \right)_\varepsilon, \quad p_\varepsilon := \left( \frac{\partial p}{\partial \varepsilon} \right)_n. \] (91)

Since we consider only first order quantities, the partial derivatives are functions of the background quantities only, i.e.,
\[ p_n = p_n(n_{(0)}, \varepsilon_{(0)}), \quad p_\varepsilon = p_\varepsilon(n_{(0)}, \varepsilon_{(0)}). \] (92)

Using Eq. (33c) and the expansions (56b) and (56f), we find for the first order perturbations of the connection coefficients
\[ \Gamma^k_{(1)ij} = -g^{kl}_{(0)} g_{(1)lm} \Gamma^m_{(0)ij} + \frac{1}{2} g^{kl}_{(0)} (g_{(1)ik,j} + g_{(1)lj,i} - g_{(1)ij,k}). \] (93)

The first order perturbation \( \Gamma^k_{(1)ij} \), Eq. (93), occurring in the non-tensor \( \Gamma^k_{ij} \), Eq. (56f), happen to be expressible as a tensor. Indeed, using Eq. (82), one can rewrite Eq. (93) in the form
\[ \Gamma^k_{(1)ij} = -\frac{1}{2} g^{kl}_{(0)} (h_{tiij} + h_{lij} - h_{ijlj}). \] (94)
Using the expansion for $3R_{ij}$, (56e), and $\Gamma^k_{ij}$, (56f), one finds for the first order perturbation to the Ricci tensor (37)

$$3R_{(1)ij} = \Gamma^k_{(1)ij,k} - \Gamma^k_{(1)ik,j} + \Gamma^l_{(1)ij,l} + \Gamma^l_{(1)ij,l} + \Gamma^l_{(1)ij,l} - \Gamma^l_{(1)ik,l} \Gamma^k_{(0)lj},$$

(95)

which can be rewritten in the compact form

$$3R_{(1)ij} = \Gamma^k_{(1)ij,k} - \Gamma^k_{(1)ik,j}.$$  

(96)

By substituting Eq. (94) into Eq. (96), one can express the first order perturbation to the Ricci tensor of the three-dimensional subspace in terms of the perturbation to the metric and its covariant derivatives

$$3R_{(1)ij} = -\frac{1}{2} g^{k(0)}_{(0)lj}(h_{lij}|k + h_{lij}|k - h_{lij}|k - h_{lij}|k).$$

(97)

The perturbation $3R_{(1)ij}$ is given by

$$3R_{(1)ij} := (g^{ij}3R_{pj})_{(1)} = g^{ij}3R_{ij} + \frac{1}{3}3R_{(0)}h^i_j,$$

(98)

where we have used Eqs. (61), (74), (77) and (83). Upon substituting Eq. (97) into Eq. (98) we get

$$3R_{(1)ij} = -\frac{1}{2} g^{ij(0)}(h^k_{|pij}|k + h^k_{|pij}|k - h^k_{|p(ij)|}) + \frac{1}{2} g^{ij(0)}h^k_{k|l} + \frac{1}{3}3R_{(0)}h^i_j.$$  

(99)

Taking $i = j$ in Eq. (99) and summing over the repeated index, we find for the first order perturbation to the curvature scalar of the three-dimensional spaces

$$3R_{(1)} = g^{ij}3R_{(0)}h^k_{k|ij} + \frac{1}{3}3R_{(0)}h^k_{k}.$$  

(100)

We thus have expressed all quantities occurring in the relevant dynamical equations, i.e., the system of equations formed by the Einstein equations combined with the conservation laws, in terms of zero and first order quantities to be solved from these equations. In the Secs. III C and III D below we derive the background and first order evolution equations, respectively. To that end we substitute the series (55) and (56) into the Einstein equations (45) and conservation laws (52) and (53). By equating the powers of $\eta^0$, $\eta^1$, ..., we obtain the zero order, the first order and higher order dynamical equations, constraint equations and conservation laws. We will carry out this scheme for the zero and first order equations only.

\section{C. Zero order equations}

With the help of Sec. III A and the expansions (55) and (56) we now can find from the Einstein equations (45) and conservation laws (52) and (53) the zero order Einstein equations and the conservation laws. Furthermore, in view of the symmetry induced by the isotropy, it is possible to switch from the six quantities $g_{ij}$ and the six quantities $\kappa_{ij}$ to the curvature $3R_{(0)}(t)$ and the Hubble function $H(t)$ only.

\subsection{1. Einstein equations}

Upon substituting Eqs. (63) and (65) into the $(0,0)$-component of the Einstein equations, Eq. (45a), one finds

$$3H^2 - \frac{1}{3}3R_{(0)} = \kappa \varepsilon_{(0)} + \Lambda.$$  

(101)

The $(0,i)$-components of the Einstein equations, Eqs. (45b), are identically fulfilled, as follows from Eqs. (63) and (65). We thus are left with the six $(i,j)$-components of the Einstein equations, Eqs. (45c). In view of Eqs. (63), (65) and (76) we find that $\sigma_{(0)1} = \sigma_{(0)2} = \sigma_{(0)3}$, $T_{(0)1} = T_{(0)2} = T_{(0)3}$ and $3R_{(0)1} = 3R_{(0)2} = 3R_{(0)3}$, whereas for $i \neq j$ these quantities vanish. Hence, the six $(i,j)$-components reduce to one equation,

$$\dot{H} = -3H^2 + \frac{1}{3}3R_{(0)} + \frac{1}{2}(\varepsilon_{(0)} - p_{(0)}) + \Lambda.$$  

(102)

In Eqs. (101) and (102) the background curvature $3R_{(0)}$ is given by Eq. (77). It is, however, of convenience to determine this quantity from a differential equation. Eliminating $a(t)$ from Eqs. (62) and (77) we obtain

$$3\dot{R}_{(0)} + 2H 3R_{(0)} = 0,$$

(103)

where the initial value $3R_{(0)}(t_0)$ is given by

$$3R_{(0)}(t_0) = -\frac{6k}{a^2(t_0)},$$  

(104)

in accordance with Eq. (77). It should be emphasized that Eq. (103) is not an Einstein equation, since it is equivalent to Eq. (77). It will be used here as an ancillary relation.
2. Conservation laws

Upon substituting Eqs. (63) and (65) into the 0-component of the conservation law, Eq. (52a), one finds

\[ \dot{\varepsilon}_{(0)} + 3H(\varepsilon_{(0)} + p_{(0)}) = 0, \]  

(105)

which is the relativistic background continuity equation. The background momentum conservation laws (i.e., the background relativistic Euler equations) are identically satisfied, as follows by substituting (63) and (65) into the spatial components of the conservation laws, Eq. (52b).

The background particle number density conservation law can be found by substituting (60) and (64) into Eq. (53). One gets

\[ \dot{n}_{(0)} + 3H n_{(0)} = 0. \]  

(106)

This concludes the derivation of the background equations.

What we have found is that for the Robertson-Walker metric (72) the Einstein equations (45) and conservation laws (52) and (53) reduce to the initial value condition (101) and four differential equations (102), (103), (105) and (106) for the four unknown functions \( H, 3R_{(0)}, \varepsilon_{(0)} \) and \( n_{(0)} \). If we use Eqs. (64) and (66), we can state that we have found equations for all unknown background quantities (55) in zero order approximation, namely \( \theta_{(0)}, \vartheta_{(0)}, 3R_{(0)}, \varepsilon_{(0)} \) and \( n_{(0)} \). In Sec. V we derive equations for the corresponding first order quantities \( \theta_{(1)}, \vartheta_{(1)}, 3R_{(1)}, \varepsilon_{(1)} \) and \( n_{(1)} \).

D. First order equations

In this section we derive the first order perturbation equations from the Einstein equations (45) and conservation laws (52) and (53). The procedure is, by now, completely standard. We use the series expansion in \( \eta \) for the various quantities occurring in the Einstein equations and conservation laws of energy-momentum, and we equate the coefficients linear in \( \eta \) to obtain the ‘linearized’ or first order equations.

1. Einstein equations

Using the series expansions for \( 3R, (55f), \varepsilon^j, (56c), \) and \( T^0_0, (56d) \), in the \((0,0)\)-component of the constraint equation (45a), one finds

\[ 2\kappa \varepsilon_{(0)k} \varepsilon^l_{(1)i} - 3R^l_{(1)i} - 2\kappa \varepsilon_{(0)j} \varepsilon^j_{(1)k} = 2\kappa T^0_{(1)0}. \]  

(107)

With the zero order equation (63), the abbreviation (84) and the expression for \( T^0_{(1)0}, (89) \), we may rewrite this equation in the form

\[ H\dot{h}_k + \frac{1}{2} 3R^l_{(1)i} = -\kappa \varepsilon_{(1)}. \]  

(108)

Employing the series expansions for \( \varepsilon^j, (56c), \) and \( T^0_0, (56d) \), we find for the \((0,i)\)-components of the constraint equations (45b)

\[ \varepsilon^k_{(1)k[i]} - \varepsilon^k_{(1)ij} = \kappa T^0_{(1)i}, \]  

(109)

where we noted that

\[ (\varepsilon^j_{(1)k})_{(1)} = \varepsilon^j_{(1)ij}, \]  

(110)

which is a consequence of

\[ \Gamma^i_{(1)jk} \varepsilon^j_{(1)i} = -\Gamma^l_{(1)jk} \varepsilon^l_{(0)i} = 0, \]  

(111)

which, in turn, is a direct consequence of Eq. (63). From Eqs. (84) and (89) we find

\[ \dot{\varepsilon}^k_{(1)i} - \dot{\varepsilon}^k_{(1)ij} = 2\kappa (\varepsilon_{(0)} + p_{(0)}) u_{(1)i}. \]  

(112)

Finally, we consider the \((i,j)\)-components of the Einstein equations (45c). Using the series expansions for \( \varepsilon^j, (56c), T^i_j, (56d), \) and \( 3R^i_j, (56e) \), we find

\[ \dot{\varepsilon}^i_{(1)j} - \dot{\varepsilon}^i_{(1)j} \varepsilon^k_{(0)k} - \dot{\varepsilon}^i_{(1)j} \varepsilon^j_{(1)k} + 3R^i_{(1)j} = \kappa (T^i_{(1)j} - \frac{1}{2} \delta^i_j T^0_{(1)0}). \]  

(113)
With Eqs. (63), (84) and (89), we get

\[ \ddot{h}^i_j + 3H\dot{h}^i_j + \delta^i_j H\dot{h}^k_k + 2\dot{3}R^i_{(1)j} = -\kappa\delta^i_j(\varepsilon_{(1)} - p_{(1)}), \]  

(114)

where \( 3R^i_{(1)j} \) is given by Eq. (99).

Note that the first order equations (108) and (114) are independent of the cosmological constant \( \Lambda \): the effect of the non-zero cosmological constant is accounted for by the zero order quantities [cf. Eqs. (101) and (102)].

2. Conservation laws

We now consider the energy conservation law (52a). With the help of the series expansions for \( \varepsilon^i_j \), (56c), and \( T^\mu_\nu \), (56d), one finds for the first order equation

\[ \dot{T}^{00} + T^{0k}_{(1)k} + \dot{\varepsilon}^{i}_j T^{i}_{(1)k} + \varepsilon^{i}_j T^{i}_{(0)k} - \varepsilon^{i}_j T^{i}_{(1)0} = 0, \]  

(115)

where we have used that for a three-vector \( T^{0k} \) we have

\[ (T^{0k})_{(1)k} = T^{0k}_{(1)k}, \]  

(116)

see Eq. (87). Employing Eqs. (63), (65), (84), (86) and (89) we arrive at the first order energy conservation law

\[ \dot{\varepsilon}^{i}_j + 3H(\varepsilon^{i}_j + p^{i}_j) + (\varepsilon^{i}_j + p^{i}_j)(u^k_{(1)k} - \frac{1}{2}\dot{h}^k_k) = 0, \]

(117)

Next, we consider the momentum conservation laws (52b). With the series expansions for \( \varepsilon^i_j \), (56c) and \( T^{\mu\nu} \), (56d), we find for the first order momentum conservation law

\[ \dot{T}^{ik}_{(1)k} - 2\varepsilon^{i}_j T^{k0}_{(0)} - 2\varepsilon^{i}_j T^{k0}_{(1)} - \varepsilon^{i}_j T^{k0}_{(1)} - \varepsilon^{i}_j T^{k0}_{(0)} = 0. \]  

(118)

Using that

\[ (T^{ik})_{(1)k} = -g^{ij}_{(1)k}p_{(1)k}, \]  

(119)

and Eqs. (63), (65), (84) and (89) we arrive at

\[ \frac{1}{c} \frac{d}{dt} \left[ (\varepsilon^{i}_j + p^{i}_j)u^i_{(1)} \right] - g^{ij}_{(1)k}p^{k}_{(1)k} + 5H(\varepsilon^{i}_j + p^{i}_j)u^{i}_{(1)} = 0, \]

(120)

where we have also used that the covariant derivative of \( g^{ij}_{(1)k} \) vanishes: \( g^{ij}_{(1)k} = 0 \).

Finally, we consider the particle number density conservation law (53). With the expansions for \( n \), (55b), \( \theta \), (55d), and \( u^\mu \), (56a), it follows that the first order equation reads

\[ \dot{n}_{(0)} u^0_{(1)} + \dot{n}_{(1)} n^0_{(0)} + n_{(0)} ku^k_{(1)} + n_{(1)} ku^k_{(0)} + n_{(0)} \theta_{(1)} + n_{(1)} \theta_{(0)} = 0. \]

(121)

With the help of Eqs. (60), (64) and (78) we find for the first order particle number conservation law

\[ \dot{n}^{i}_{(1)} + 3Hn^{i}_{(1)} + n^{i}_{(0)}(u^k_{(1)k} - \frac{1}{2}\dot{h}^k_k) = 0, \]

(122)

where we employed Eq. (85) to eliminate \( \theta_{(1)} \).

3. Summary

In the preceding two Secs. III D 1 and III D 2 we have found the equations which, basically, describe the perturbations in a FLRW universe, in first approximation. They are Eqs. (108), (112), (114), (117), (120) and (122). For convenience we repeat them here
where $3R^i_{t(1)j}$ and $3R^i_{t(1)}$ are given by Eqs. (99) and (100), respectively. Hence, the equations (123) essentially are fifteen equations for the eleven $(6 + 3 + 1 + 1)$ unknown functions $h^i_{|j}$, $u^i_{|j}$, $\varepsilon_{(1)}$, and $n^{(1)}_{t(1)}$. The pressure $p_{(0)}$ is given by an equation of state (66), and the perturbation to the pressure, $p_{(1)}$, is given by Eq. (90). The system of equations is not overdetermined, however, since the four equations (123a) and (123b) are only conditions on the initial values. These initial value conditions are fulfilled for all times $t$ automatically if they are satisfied at some (initial) time $t = t_0$.

IV. CLASSIFICATION OF THE SOLUTIONS OF FIRST ORDER

The set of equations (123), which are linear in their (eleven) unknown functions, can be split up into three sets of equations, which, together, are equivalent to the original set. We will refer to these sets by their usual names of scalar, vector and tensor perturbations. We will show that the vector and tensor perturbations do not, in first order, contribute to the physical perturbations $\varepsilon^g_{(1)}$, and $n^{(1)}_{t(1)}$. As a consequence, we only need, for our problem, the set of equations which are related to the scalar perturbations. By considering only the scalar part of the full set of perturbation equations we are able to cast the perturbation equations into a set which is directly related to the physical perturbations $\varepsilon^g_{(1)}$ and $n^{(1)}_{t(1)}$. This is the subject of the next section, Sec. V.

At the basis of the replacement of one set (123) by three sets of equations stands a theorem proved by Stewart [15, 16], which states that a symmetric second rank tensor can be split up into three irreducible pieces, and that a vector can be split up into two irreducible pieces. Here, we will use this general theorem to obtain equations for the scalar irreducible parts of the tensors $h^i_{|j}$ and $3R^i_{t(1)j}$ and the vector $u^i_{(1)}$, namely $h^i_{|j}$, $3R^i_{t(1)j}$ and $u^i_{(1)}$.

For the perturbation to the metric, a symmetric second rank tensor, we have in particular

$$h^i_{|j} = h^i_{||j} + h^i_{\perp j} + h^i_{*|j},$$

(124)

where, according to the theorem of Stewart [15], the irreducible constituents $h^i_{|||j}$, $h^i_{\perp j}$ and $h^i_{*|j}$ have the properties

$$h^i_{|||j} = \frac{2}{c^2}(\phi\delta^i_{|j} + \zeta^i_{|j}),$$

(125a)

$$h^i_{\perp j} = 0,$$

(125b)

$$h^i_{*|j} = 0, \quad h^i_{*|j} = 0,$$

(125c)

with $\phi(t, x)$ and $\zeta(t, x)$ arbitrary functions. The contravariant derivative $A^i_{|j}$ is defined as $g^{ij}_{(0)}A_{|j}$. The functions $h^i_{|||j}$, $h^i_{\perp j}$ and $h^i_{*|j}$ correspond to scalar, vector and tensor perturbations, respectively.

In the same way, the perturbation to the Ricci tensor can be decomposed into irreducible components, i.e.,

$$3R^i_{t(1)j} = 3R^i_{t(1)||j} + 3R^i_{t(1)\perp j} + 3R^i_{t(1)*|j},$$

(126)

The tensors $3R^i_{t(1)||j}$, $3R^i_{t(1)\perp j}$ and $3R^i_{t(1)*|j}$ have the properties comparable to (125), i.e.,

$$3R^i_{t(1)||j} = \frac{2}{c^2}(\gamma\delta^i_{|j} + \pi^i_{|j}),$$

(127a)

$$3R^i_{t(1)\perp j} = 0,$$

(127b)

$$3R^i_{t(1)*|j} = 0, \quad 3R^i_{t(1)*|j} = 0,$$

(127c)

where $\gamma(t, x)$ and $\pi(t, x)$ are two arbitrary functions. By now using Eq. (99) for each of the irreducible parts we find

$$3R^i_{t(1)||j} = \frac{1}{c^2} \left[ \phi^{ij}_{|k} - \phi^{ij}_{|k} - \phi^{ij}_{|k} \right] + \frac{1}{c^2} \left[ \phi_{(0)}^{ij}_{|k} - \phi_{(0)}^{ij}_{|k} + \phi_{(0)}^{ij}_{|k} \right] + \frac{3}{c^2} 3R_{(0)}^{ij}_{|k} \delta^i_{|j} + \zeta^i_{|j}],$$

(128a)

$$3R^i_{t(1)\perp j} = -\frac{1}{2} g^{ij}_{(0)} (h^k_{\perp i|j} + h^k_{\perp j|i}) + \frac{1}{2} (h^k_{(0)} h^i_{\perp j} + h^k_{(0)} h^i_{\perp j}),$$

(128b)

$$3R^i_{t(1)*|j} = -\frac{1}{2} g^{ij}_{(0)} (h^k_{* i|j} + h^k_{* j|i}) + \frac{1}{2} (h^k_{(0)} h^i_{* j} + h^k_{(0)} h^i_{* j}).$$

(128c)

Combining the expressions (127) and (128), we can derive relations, to be obeyed by $\gamma$, $\pi$, $h^i_{\perp j}$, and $h^i_{*|j}$. Firstly, from property (127a) and (128a) it follows that

$$\gamma = \frac{1}{2} (d^k_{|k} + \frac{3}{c^2} 3R_{(0)}^{ij}_{|k}),$$

(129a)

$$\pi^i_{|j} = \frac{1}{2} (\phi^{ij}_{|k} - \phi^{ij}_{|k}) + \frac{1}{2} (\phi^{ij}_{(0)} + \phi^{ij}_{(0)}),$$

(129b)
In a flat FLRW universe, Eqs. (129) reduce to
\[ \gamma = \frac{1}{2} \phi^{[k |k]}, \]  
\[ \pi^{|i| j} = \frac{1}{2} \phi^{|j |i}, \]  
(130a)
(130b)

implying that, for a flat FLRW universe,
\[ \gamma = \pi^{|i| |i} \]  
(131)

whereas there is no such restriction on the functions \( \phi \) and \( \zeta \) in a flat FLRW universe.

Secondly, combining expressions (127b) and (128b) it follows that \( h^i_{|k|l} \) has the property
\[ h^i_{|k|l} = 0, \]  
(132)
in addition to the property (125b). In Sec. IV B we show that this additional condition is needed to allow for the decomposition (141).

Finally, combining Eqs. (127c) and (128c), we find that \( h^i_{|j|} \) must obey
\[ g^{kl}(h^m_{|k|l|m} + h^m_{|i|k|m} - A^m_{|k|l|m}) = 0, \]  
(133)
in addition to (125c). The relations (133) are, however, fulfilled identically for FLRW universes. This can easily be shown. First, we recall the well-known relation that the difference of the covariant derivatives \( A^{i_{r...j}}_{k...l} |p|q \) and \( A^{i_{r...j}}_{k...l} |p|q \), of an arbitrary tensor can be expressed in terms of the curvature and the tensor itself (Weinberg [14], Chap. 6, Sec. 5)
\[ A^{i_{r...j}}_{k...l} |p|q - A^{i_{r...j}}_{k...l} |p|q = + A^{i_{r...j}}_{s...t} 3R^{s}_{(0)kpq} + \cdots + A^{i_{r...j}}_{k...s} 3R^{s}_{(0)lpq} - A^{i_{r...j}}_{k...l} 3R^{s}_{(0)spq} - \cdots - A^{i_{r...j}}_{k...l} 3R^{s}_{(0)spq}, \]  
(134)

where \( 3R^{s}_{(0)jk} \) is the Riemann tensor for the spaces of constant time. At the right-hand side, there is a term with a plus sign for each lower index and a term with a minus sign for each upper index.

We apply this identity taking for \( A \) the second rank tensor \( h_{|i|} \) to obtain
\[ h^m_{|k|l|m} - h^m_{|i|k|m} = h^m_{|s|} 3R^{s}_{(0)kim} - h^m_{|k|} 3R^{s}_{(0)sim}, \]  
(135)

Now note that \( h^m_{|k|l|m} \) vanishes in view of (125c). Next, we take the covariant derivative of (135) with respect to \( x^l \), and contract with \( g^{kl} \)
\[ g^{kl}h^m_{|k|l|m} = g^{kl}(h^m_{|s|} 3R^{s}_{(0)kim} - h^m_{|k|} 3R^{s}_{(0)sim})|l|. \]  
(136)

Next, using the expression which one has for the Riemann tensor of a maximally symmetric three-space,
\[ 3R^{a}_{(0)bcd} = k (\delta^a_{bc} g_{bd} - \delta^a_{bd} g_{bc}), \]  
(137)

(where \( k = 0, \pm 1 \) is the curvature constant) we find
\[ g^{kl}h^m_{|k|l|m} = 0, \]  
(138)
i.e., the first term of (133) vanishes. The second and third term can similarly be expressed in the curvature
\[ h^m_{|i|k|m} - h^m_{|i|k|m} = h^m_{|s|} 3R^{s}_{(0)sim} + h^m_{|s|} 3R^{s}_{(0)kml} - h^m_{|s|} 3R^{s}_{(0)smi}, \]  
(139)

where the general property (134) has been used. Upon substituting the Riemann tensor (137) and contracting with \( g^{kl} \), we then arrive at
\[ g^{kl}(h^m_{|i|k|m} - h^m_{|i|k|m}) = 0, \]  
(140)
i.e., the second and third term of (133) together vanish. Hence, for FLRW universes, Eq. (133) is identically fulfilled. Consequently, the decomposition (125c) imposes no additional condition on the irreducible part \( h^i_{|j|} \) of the perturbation \( h^{ij} \).
The three-vector \( u_{(1)} \) can be uniquely split up according to [15]

\[
\mathbf{u}_{(1)} = \mathbf{u}_{(1)}^\parallel + \mathbf{u}_{(1)}^\perp,
\]

where \( \mathbf{u}_{(1)}^\parallel \) is the *longitudinal* part of \( \mathbf{u}_{(1)} \), with the properties

\[
\hat{\nabla} \wedge (\mathbf{u}_{(1)}^\parallel) = 0, \quad \hat{\nabla} \cdot \mathbf{u}_{(1)} = \hat{\nabla} \cdot \mathbf{u}_{(1)}^\parallel,
\]

and \( \mathbf{u}_{(1)}^\perp \) is the *transverse* part of \( \mathbf{u}_{(1)} \), with the properties

\[
\hat{\nabla} \cdot \mathbf{u}_{(1)}^\perp = 0, \quad \hat{\nabla} \wedge \mathbf{u}_{(1)} = \hat{\nabla} \wedge \mathbf{u}_{(1)}^\perp,
\]

where the divergence of the vector \( \mathbf{u}_{(1)} \) is defined by, Eq. (51),

\[
\hat{\nabla} \cdot \mathbf{u}_{(1)} := u^k_{(1)}|k = \vartheta_{(1)},
\]

and the rotation of the vector \( \mathbf{u}_{(1)} \) is defined by

\[
(\hat{\nabla} \wedge \mathbf{u}_{(1)})_i := \varepsilon^{ijk} u^j_{(1)}|k = \varepsilon^{ijk} u^j_{(1),k},
\]

where \( \varepsilon^{ijk} \) is the Levi-Civita tensor with \( \varepsilon^{123} = +1 \). In Eq. (145) we could replace the covariant derivative by the ordinary partial derivative because of the symmetry of \( \Gamma^i_{jk} \).

Having decomposed the tensors \( h^i_{(1)j} \), \( \dot{R}^i_{(1)j} \) and \( u^i_{(1)} \) in a scalar \( \parallel \), a vector \( \perp \) and a tensor part \( \ast \), we can now decompose the set of equations (123) into three independent sets. The recipe is simple: all we have to do is to append a subindex \( \parallel, \perp \) or \( \ast \) to the relevant tensorial quantities in equations (123). This will be the subject of the Secs. IV A, IV B and IV C below.

### A. Tensor perturbations

We will show that tensor perturbations are not coupled to, i.e., do not give rise to, density perturbations. Upon substituting \( h^i_{(1)j} = \delta^i_{(1)j} \) and \( \dot{3}R^i_{(1)j} = \dot{3}R^i_{(1)j} \ast \) into the perturbation equations (123) and using the properties (125c) and (127c), we find from equations (123a), (123b) and (123d)

\[
\varepsilon_{(1)} = 0, \quad p_{(1)} = 0, \quad n_{(1)} = 0, \quad \mathbf{u}_{(1)} = 0,
\]

where we have also used Eq. (90). With (146), Eqs. (123e) and (123f) are identically satisfied. The only surviving equation is (123c), which now reads

\[
\dot{h}^i_{(1)j} + 3Hh^i_{(1)j} + 2\dot{3}R^i_{(1)j} = 0,
\]

where \( \dot{3}R^i_{(1)j} \) is given by Eq. (128c). Using Eqs. (85), (86), (125c) and (146) it follows from Eqs. (2) that

\[
\varepsilon^{(1)} = 0, \quad p^{(1)} = 0, \quad n^{(1)} = 0,
\]

so that tensor perturbations do not, in first order, contribute to physical energy density and particle number density perturbations. Hence, the Eqs. (147) do not play a role in this context, where we are interested in energy density and particle number density perturbations only.

The equations (147) have a wave equation like form with an extra term. The extra term \( 3H\dot{h}^i_{(1)j} \) in these equations is due to the expansion of the universe. Therefore, these tensor perturbations are sometimes called *gravitational waves*. The six components \( h^i_{(1)j} \) satisfy the four equations (125c), leaving us with two independent functions \( h^i_{(1)j} \). They are related to linearly and circularly polarized waves.

### B. Vector perturbations

We will show that, just like tensor perturbations, vector perturbations are not coupled to density perturbations. Upon replacing \( h^i_{(1)j} \) by \( h^i_{(1)j} \) and \( \dot{3}R^i_{(1)j} \) by \( \dot{3}R^i_{(1)j} \) in the perturbation equations (123), and using the expressions (125b) and (127b), we find from Eq. (123a) and the trace of Eq. (123c)

\[
\varepsilon_{(1)} = 0, \quad p_{(1)} = 0, \quad n_{(1)} = 0,
\]
where we have also used Eq. (90).

Since $h^i_{\perp j}$ is traceless and raising the index with $g^{ij}_{(0)}$ in Eq. (123b) we get

$$h^k_{\perp |k} + 2H h^k_{\perp |k} = 2\kappa (\varepsilon_{(0)} + p_{(0)}) u^j_{(1)} ,$$

(150)

where we have used Eqs. (31) and (63). We now calculate the covariant derivative of (150) with respect to $x^j$, and use (132) to obtain

$$\tilde{\nabla} \cdot u_{(1)} = 0,$$

(151)

where we made use of the fact that the time derivative and the covariant derivative commute. With Eqs. (141)–(143) we see that only the transverse part of $u_{(1)}$, namely $u_{(1)\perp}$, plays a role in vector perturbations. From (125b) and (149) it follows that the equations (123d) and (123f) are identically satisfied. The only surviving equations are (123b), (123c) and (123e), which now read

$$h^k_{\perp |i|k} = -2\kappa (\varepsilon_{(0)} + p_{(0)}) u_{(1)\perp i},$$

(152a)

$$h^k_{\perp |j} + 3H h^k_{\perp |j} + 2\, \varepsilon^k_{i|j} = 0,$$

(152b)

$$\frac{1}{c} \frac{d}{dt} \left[ (\varepsilon_{(0)} + p_{(0)}) u^i_{(1)\perp} \right] + 5H (\varepsilon_{(0)} + p_{(0)}) u^i_{(1)\perp} = 0,$$

(152c)

where $\varepsilon^k_{i|j}$ is given by Eq. (128b).

Using Eqs. (85), (86), (149) and (151) we get from Eqs. (2)

$$\varepsilon^{\parallel}_{(1)} = 0, \quad n^{\parallel}_{(1)} = 0,$$

(153)

implying that also vector perturbations do not, in first order, contribute to physical energy density and particle number density perturbations. Hence, the equations (152) do not play a role when we are interested in energy density and particle number density perturbations, as we are here. Vector perturbations are also called vortices.

Since vector perturbations obey $\nabla \cdot u_{(1)\perp} = 0$, they have two degrees of freedom. As a consequence, the tensor $h^i_{\perp j}$ has also two degrees of freedom. These degrees of freedom are related to clockwise and counter-clockwise rotation of matter.

### C. Scalar perturbations

Differentiation of Eqs. (123b) covariantly with respect to $x^j$ we obtain

$$h^k_{||i|j} - h^k_{||i|j} = 2\kappa (\varepsilon_{(0)} + p_{(0)}) u_{(1)\perp i}.$$

(154)

Interchanging $i$ and $j$ in this equation, and subtracting the resulting equation from (154) we get

$$h^k_{||i|j} - h^k_{||j|i} = -2\kappa (\varepsilon_{(0)} + p_{(0)}) (u_{(1)\perp ij} - u_{(1)\perp ji}),$$

(155)

where we have used that $h^k_{||i|j} = h^k_{||j|i}$. Using that $\nabla \wedge u_{(1)\perp} = 0$, we find from Eq. (125a) that the function $\zeta$ must obey the equations

$$\zeta^{\parallel}_{i|j} - \zeta^{\parallel}_{j|i} = 0.$$

(156)

These equations are fulfilled identically in FLRW universes. This can be seen as follows. We first rewrite these equations by interchanging the covariant derivatives in the form

$$\zeta^{\parallel}_{i} - \zeta^{\parallel}_{j} - (\zeta^{\parallel}_{i} - \zeta^{\parallel}_{j}) = 0.$$

(157)

Next, we use Eq. (134) and substitute the Riemann tensor (137) into the resulting expression. Using that $\zeta^{\parallel}_{ij} = \zeta^{\parallel}_{ij}$, we find that the left-hand sides of the Eqs. (157) vanish. As a consequence, the Eqs. (156) are identities. Therefore, the decomposition (125a) imposes no additional condition on the irreducible part $h^i_{\perp j}$ of the perturbation $h^j$. 
The evolution equations (123) for scalar perturbations read

\begin{align}
Hh_{i|k}^k + \frac{1}{2} R_{(1)i}^j &= - \kappa \varepsilon_{(1)}, \\
\dot{h}_{i|k}^i - \dot{h}_{i|k}^i &= 2\kappa (\varepsilon_{(0)} + p_{(0)}) u_{(1)i}, \\
\dot{h}_{i|k}^j + 3H \dot{h}_{i|j}^j + \delta^j_j Hh_{i|k}^k + 2^3 R_{(1)i}^j &= - \kappa \delta^j_j (\varepsilon_{(1)} - p_{(1)}), \\
\dot{\varepsilon}_{(1)} + 3H (\varepsilon_{(1)} + p_{(1)}) + (\varepsilon_{(0)} + p_{(0)}) [u_{(1)i}^i]_{j} - \frac{1}{2} \dot{h}_{i|j}^i &= 0, \\
\frac{1}{c} \frac{d}{dt} \left[(\varepsilon_{(0)} + p_{(0)}) u_{(1)i}^i - g_{ik}^j p_{(j)}^j + 5H (\varepsilon_{(0)} + p_{(0)}) u_{(1)i}^i \right] &= 0, \\
\dot{n}_{(1)} + 3H n_{(1)} + n_{(0)} [u_{(1)i}^i]_{j} - \frac{1}{2} \dot{h}_{i|j}^i &= 0, 
\end{align}

where the perturbations to the metric and the Ricci tensor are given by Eqs. (125a) and (128a), respectively. In the tensorial and vectorial case we found \( \varepsilon_{(1)} = 0 \) and \( n_{(1)} = 0 \), implying that \( \varepsilon_{(1)} = 0 \) and \( n_{(1)} = 0 \), which made the tensorial and vectorial equations irrelevant for our purpose. Such a conclusion cannot be drawn from the equations (158). Perturbations with \( \varepsilon_{(1)} \neq 0 \) and \( n_{(1)} \neq 0 \) are usually referred to as scalar perturbations.

In Sec. V we rewrite the set of equations (158) in such a way that they determine the evolution of the quantities (2). Since the perturbation equations contain only the components \( h_{i|j}^k \), it follows that relativistic energy density and particle number density perturbations are characterized by two potentials, \( \phi \) and \( \zeta \).

### D. Summary

In the foregoing three Secs. IV A–IV C we have written down, using the decompositions (124), (126) and (141), the perturbation equations for gravitational waves, Eqs. (147), vortex perturbations, Eqs. (152) and scalar perturbations, Eqs. (158). For gravitational waves and vortex it was shown that \( \varepsilon_{(1)} = 0 \) and \( n_{(1)} = 0 \). Therefore, we consider from now on only the Eqs. (158) for the scalar perturbations.

An important consequence of the theorem of Stewart, Eqs. (124)–(125) and (126)–(127), is that we need only two potentials \( \phi \) and \( \zeta \) (instead of six potentials \( h_{i|j}^k \)) to describe the evolution of the scalar perturbations. In Sec. IX we will show that, in a flat FLRW universe, the potential \( \zeta \) automatically disappears from the perturbed Einstein equations and that \( \phi(t, x) \) can be related to the well-known, time-independent potential \( \varphi(x) \) encountered in the Poisson equation \( \nabla^2 \varphi(x) = 4\pi G \varrho_{(1)}(x) \), where \( \varrho_{(1)}(x) = \varepsilon_{(1)}(t_p, x)/c^2 \) is the mass density at \( x \). In fact, we will show that in the non-relativistic limit we have \( \varphi(x) = \phi(x)/a^2(t_p) \), where \( a(t) \) is the scale factor occurring in (54b). In these equations \( t_p \) stands for the present time.

In the definitions (2) of the physical perturbations \( \varepsilon_{(1)} \) and \( n_{(1)} \), the background quantities \( \varepsilon_{(0)} \), \( n_{(0)} \) and \( \theta_{(0)} = 3H \) occur. The evolution of these quantities is governed by the background Einstein equations given in Sec. III C. Of the first order quantities \( \varepsilon_{(1)} \), \( n_{(1)} \) and \( \theta_{(1)} \) in (2), only the first two do explicitly show up in the first order equations (158). In Sec. V we rewrite this set of equations in such a way that also the first order quantity \( \theta_{(1)} \) does explicitly occur in the set of perturbation equations. The result is that the evolution of the physical quantities (2) is completely determined by the background- and first order equations.

### V. SCALAR FIRST ORDER EQUATIONS

In this quite technical section we rewrite the scalar perturbation equations (158) in terms of quantities \( \theta_{(1)} \), \( 3R_{(1)i}^j \), \( \dot{\theta}_{(1)} \), \( \varepsilon_{(1)} \) and \( n_{(1)} \), which are suitable to describe exclusively the scalar perturbations. The result is that the first order quantities occurring in the definitions (2) do explicitly occur in the set of equations.

Eliminating the quantity \( h_{i|k}^k \) from Eq. (158a) with the help of Eq. (85) yields

\[ 2H \theta_{(1)} - \dot{\theta}_{(1)} - \frac{1}{2} 3R_{(1)i}^0 = \kappa \varepsilon_{(1)}, \]

Thus the \( (0,0) \)-component of the constraint equations becomes an algebraic equation which relates the first order quantities \( \theta_{(1)} \), \( \dot{\theta}_{(1)} \), \( 3R_{(1)i}^j \) and \( \varepsilon_{(1)} \).

It now takes some steps to rewrite the three constraint equations (158b) in a suitable form. Firstly, multiplying both sides by \( g_{ij}^{(0)} \) and taking the covariant divergence with respect to the index \( j \) we find

\[ g_{ij}^{(0)} (\dot{h}_{i|k}^k - \dot{h}_{i|k}^k) = 2\kappa (\varepsilon_{(0)} + p_{(0)}) \dot{\theta}_{(1)}, \]

(160)
where we have also used Eq. (86). The left-hand side will turn up as a part of the first order derivative of the curvature $^3 R_{(1)\|}$. In fact, differentiating Eq. (100) with respect to $ct$ and recalling the fact that the connection coefficients $\Gamma_{(0)ij}^k$, Eqs. (71), are independent of time, one gets

$$3^\prime \dot{R}_{(1)\|} = -2H 3^\prime R_{(1)\|} + g_{ij}^o \dot{h}_{ij\|} + h_k^{ij\|} i j + 3^\prime R_{(0)\|} h_k^{ij\|},$$  \hspace{1cm} (161)

where we have used Eqs. (31), (63), (103) and

$$g_{ij}^o h_k^{ij\|} k = g_{ij}^o h_k^{ij\|} k,$$  \hspace{1cm} (162)

which is a consequence of $g_{ij}^o k = 0$ and the symmetry of $h_k^{ij\|}$. Next, combining Eqs. (160) and (161), and, finally, eliminating $\dot{h}_k^{ij\|}$ with the help of Eq. (85), one arrives at

$$3^\prime \dot{R}_{(1)\|} + 2H 3^\prime R_{(1)\|} - 2\kappa (\varepsilon (0) + p (0)) \theta (1) + \frac{\dot{\varepsilon} (1)}{\varepsilon (0) + p (0)} 3^\prime R_{(0)\|} (\theta (1) - \dot{\varepsilon} (1)) = 0.$$  \hspace{1cm} (163)

In this way we managed to recast the three $(0, i)$-components of the constraint equations in the form of one ordinary differential equation for the local perturbation, $3^\prime R_{(1)\|}$, to the spatial curvature.

We now consider the dynamical equations (158e). Taking the trace of these equations and using Eq. (85) to eliminate the quantity $\dot{h}_k^{ij\|}$, we arrive at

$$\dot{\varepsilon} (1) - \dot{\varepsilon} (1) + 6H (\dot{\theta} (1) - \dot{\varepsilon} (1)) - 3^\prime R_{(1)\|} = \frac{3}{2} \kappa (\dot{\varepsilon} (1) - p (1)).$$  \hspace{1cm} (164)

Thus, for scalar perturbations, the three dynamical Einstein equations (158c) with $i = j$ reduce to one ordinary differential equation for the difference $\dot{\theta} (1) - \dot{\varepsilon} (1)$. For $i \neq j$ the dynamical Einstein equations are not coupled to scalar perturbations. As a consequence, they need not be considered.

Taking the covariant derivative of Eq. (158e) with respect to the metric $g_{(0)ij}$ and using Eq. (86), we get

$$\frac{1}{3} \frac{d}{d\tau} \left[ (\varepsilon (0) + p (0)) \theta (1) - g_{ij}^o p (1) \dot{h}_k^{ij\|} k + 5H (\varepsilon (0) + p (0)) \dot{\theta} (1) = 0,$$  \hspace{1cm} (165)

where we have used that the operations of taking the time derivative and the covariant derivative commute, since the connection coefficients $\Gamma_{(0)ij}^k$, (71), are independent of time. With Eq. (105), we can rewrite Eq. (165) in the form

$$\dot{\theta} (1) + H \left( 2 - \frac{\dot{p} (0)}{\varepsilon (0)} \right) \theta (1) + \frac{1}{\varepsilon (0) + p (0)} \tilde{\nabla}^2 \frac{p (1)}{a^2} = 0,$$  \hspace{1cm} (166)

where $\tilde{\nabla}^2$ is the generalized Laplace operator which, for an arbitrary function $f(t, x)$ and with respect to an arbitrary three-dimensional metric $\tilde{g}^{ij}(x)$, is defined by

$$\tilde{\nabla}^2 f := \tilde{g}^{ij} f_{ij}.$$  \hspace{1cm} (167)

Thus, the three first order momentum conservation laws (158e) reduce to one ordinary differential equation for the divergence $\dot{\theta} (1)$.

Finally, we consider the conservation laws (158d) and (158f). Eliminating the quantity $\dot{h}_k^{ij\|}$ from these equations with the help of Eq. (85), we get

$$\dot{\varepsilon} (1) + 3H (\varepsilon (1) + p (1)) + (\varepsilon (0) + p (0)) \theta (1) = 0,$$  \hspace{1cm} (168)

and

$$n (1) + 3H n (1) + n (0) \theta (1) = 0.$$  \hspace{1cm} (169)

The algebraic equation (159) and the five ordinary differential equations (163), (164), (166), (168) and (169) is a system of six equations for the five quantities $\theta (1), 3^\prime R_{(1)\|}, \dot{\varepsilon} (1), \varepsilon (1)$ and $n (1)$, respectively. This system is, however, not overdetermined, since the constraint equation (159) is only an initial value condition. It can easily be shown by differentiation of Eq. (159) with respect to time and eliminating the time derivatives with the help of Eqs. (101), (102), (163), (164) and (168) that the general solution of the system (159), (163), (166), (168) and (169) is also a solution of Eq. (164). Therefore, we do not consider Eq. (164) anymore and we take the algebraic equation (159) and the four ordinary differential equations (163), (166), (168) and (169) for the five quantities $\theta (1), 3^\prime R_{(1)\|}, \dot{\varepsilon} (1), \varepsilon (1)$ and $n (1)$, respectively, as the basis of our perturbation theory. For scalar perturbations this system of equations is equivalent to the full set (123) of linear Einstein equations and conservation laws.
VI. SUMMARY QUANTITIES AND EQUATIONS

A. Zero order equations

The Einstein equations and conservation laws for the background of an FLRW universe are given by (102), (105), (106) and (103):

\[
\begin{align*}
\dot{H} &= -\frac{1}{6} R_{(0)} - \frac{1}{2} \kappa \varepsilon_{(0)} (1 + w), \\
\dot{\varepsilon}_{(0)} &= -3 H \varepsilon_{(0)} (1 + w), \\
\dot{n}_{(0)} &= -3 H n_{(0)}, \\
3 \dot{R}_{(0)} &= -2 H^3 R_{(0)},
\end{align*}
\]

and the constraint equation (101)

\[3H^2 = \frac{1}{2} 3R_{(0)} + \kappa \varepsilon_{(0)} + \Lambda.\]

In order to arrive at Eq. (170a) we eliminated the term $3H^2$ from Eq. (102) with the help of Eq. (101). Furthermore, we introduced the abbreviation

\[w(t) := \frac{p_{(0)}(t)}{\varepsilon_{(0)}(t)}.\]

The set (170) consists of four differential equations with respect to time for the four unknown quantities $\varepsilon_{(0)}(t)$, $n_{(0)}(t)$, $\theta_{(0)}(t)$, and $3R_{(0)}(t)$ Recall $v_{(0)} = 0$, Eq. (64). The pressure $p_{(0)}(t)$ is related to the energy density $\varepsilon_{(0)}(t)$ and the particle number density $n_{(0)}(t)$ via the equation of state (66). The algebraic equation (171) is a constraint on the initial values.

B. First order equations

The first order equations describing density perturbations are given by the set of four differential equations (168), (169), (166) and (163)

\[
\begin{align*}
\dot{\varepsilon}_{(1)} + 3H (\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)} (1 + w) \theta_{(1)} &= 0, \\
\dot{n}_{(1)} + 3H n_{(1)} + n_{(0)} \theta_{(1)} &= 0, \\
\dot{\theta}_{(1)} + H (2 - 3 \beta ^2) \theta_{(1)} + \frac{1}{\varepsilon_{(0)} (1 + w)} \nabla^2 p_{(1)} &= 0, \\
3 \dot{R}_{(1)} + 2H^3 R_{(1)} - 2 \kappa \varepsilon_{(0)} (1 + w) \theta_{(1)} + \frac{2}{3} 3R_{(0)} (\theta_{(1)} - \theta_{(1)}) &= 0,
\end{align*}
\]

together with one constraint equation, Eq. (159)

\[2H (\theta_{(1)} - \theta_{(1)}) - \frac{1}{2} 3R_{(1)} = \kappa \varepsilon_{(1)},\]

for the five unknown functions $\varepsilon_{(1)}$, $n_{(1)}$, $p_{(1)}$, $3R_{(1)}$, and $\theta_{(1)}$, respectively. These are the first order perturbations to the background quantities $\varepsilon_{(0)}$, $n_{(0)}$, $\theta_{(0)} = 0$, $3R_{(0)}$, and $\theta_{(0)} = 3H$. The first order perturbation to the pressure is given by the perturbed equation of state (90).

As has been shown in Sec. V, the equations (173)–(174) comprise the conservation laws and the constraint equations. As a consequence, the general solution of these equations is also a solution of the dynamical Einstein equation (164). Therefore, we may, in the study of scalar perturbations, replace the full set of perturbation equations (158) by the set (173)–(174). Note that only three scalars $\varepsilon$, $n$, and $\theta := u^{\mu} u_{\mu}$ play a role in a density perturbation theory. The only non-trivial gauge-invariant combinations which can be constructed from these scalars and their first order perturbations are the combinations (2). These combinations do always exist, since, in a non-static universe, we have $\theta_{(0)}(t) \neq 0$ for all times $t$, as follows from Eq. (170a).

The operator $\nabla^2$, occurring in Eq. (173c), is the generalized Laplace operator defined by Eq. (167). The quantity $\beta(t)$ occurring in Eq. (173c) is defined by

\[\beta(t) := \sqrt{\frac{\theta_{(0)}(t)}{\varepsilon_{(0)}(t)}}.\]
Eliminating the time derivatives of $\varepsilon_{(0)}$ and $n_{(0)}$ from Eqs. (2) with the help of the background equations (170), we arrive at

\begin{align}
\varepsilon_{(1)}^{\text{gi}} &= \varepsilon_{(1)} - \frac{6H \varepsilon_{(0)}(1 + w)}{3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)} \theta_{(1)}, \\
n_{(1)}^{\text{gi}} &= n_{(1)} - \frac{6H n_{(0)}}{3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)} \theta_{(1)},
\end{align}

where we have used that $\theta_{(0)} = 3H$, Eq. (64). We have achieved now that the set of perturbation equations (173) and (174), together with the background equations (170) and (171) determine the evolution of the physical quantities (2), or, equivalently, (176). In principle, we are ready. However, the solution of the set of equations (173) and (174) is gauge dependent (see Appendix B). Indeed, one may easily check, by a direct calculation, that these equations are invariant under the transformation

\begin{align}
\varepsilon_{(1)} &\rightarrow \hat{\varepsilon}_{(1)} = \varepsilon_{(1)} + \psi \dot{\varepsilon}_{(0)}, \\
n_{(1)} &\rightarrow \hat{n}_{(1)} = n_{(1)} + \psi \dot{n}_{(0)}, \\
\theta_{(1)} &\rightarrow \hat{\theta}_{(1)} = \theta_{(1)} - \frac{\nabla^2 \psi}{a^2}, \\
3R_{(1)||} &\rightarrow \hat{3R}_{(1)||} = 3R_{(1)||} + 4H \left( \frac{\nabla^2 \psi}{a^2} - \frac{1}{2} \frac{3R_{(0)} \psi}{a^2} \right), \\
\theta_{(1)} &\rightarrow \hat{\theta}_{(1)} = \theta_{(1)} + \psi \dot{\theta}_{(0)},
\end{align}

where $\psi$ is time independent in synchronous coordinates, see Eq. (B2). By switching from the variables $\varepsilon_{(1)}$, $n_{(1)}$, $\theta_{(1)}$, $3R_{(1)||}$ and $\theta_{(1)}$ to the variables $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$, we will arrive at a set of equations for $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$, with a unique, i.e., gauge-invariant solution. This will be the subject of Sec. VIII. First, however, we derive some auxiliary equations related to the entropy, pressure and temperature.

### VII. ENTROPY, PRESSURE AND TEMPERATURE

#### A. Gauge-invariant entropy perturbations

The second law of thermodynamics reads

\[ dE = T \, dS - p \, dV + \mu \, dN, \tag{178} \]

where $E$, $S$ and $N$ are the energy, the entropy and the number of particles of a system with volume $V$, and where $\mu$, the thermodynamic—or chemical—potential, is the energy needed to add one particle to the system. In terms of the energy per particle $E/N = \varepsilon/n$ and the entropy per particle $s = S/N$ the law (178) can be rewritten

\[ d\left( \frac{\varepsilon}{n} N \right) = T \, d(sN) - p \, d\left( \frac{N}{n} \right) + \mu \, dN, \tag{179} \]

where $\varepsilon$ and $n$ are the energy and particle number densities. With the Euler relation $\mu = (\varepsilon + p)n^{-1} - Ts$, the second law can be cast in a form without $\mu$ and $N$. In fact, using the rule $d(fg) = f \, dg + g \, df$ we immediately find from Eq. (179)

\[ T \, ds = d\left( \frac{\varepsilon}{n} \right) + p \, d\left( \frac{1}{n} \right), \tag{180} \]

where $\mu$ and $N$ have canceled, indeed. The thermodynamic relation (180) is true for a system in thermodynamic equilibrium. For a non-equilibrium system that is ‘not too far’ from equilibrium, the equation (180) may be replaced by

\[ T \frac{ds}{dt} = \frac{d}{dt} \left( \frac{\varepsilon}{n} \right) + p \frac{d}{dt} \left( \frac{1}{n} \right), \tag{181} \]
where \( \frac{d}{dt} \) is the time derivative in a local comoving Lorentz system. Now, using \( \varepsilon = \varepsilon \left(0\right) + \varepsilon \left(1\right), \: s = s \left(0\right) + s \left(1\right), \: p = p \left(0\right) + p \left(1\right) \) and \( n = n \left(0\right) + n \left(1\right), \) we find from Eq. (181)

\[
T \left(0\right) \frac{d s \left(0\right)}{d t} = \frac{d}{d t} \left( \frac{\varepsilon \left(0\right)}{n \left(0\right)} \right) + p \left(0\right) \frac{d}{d t} \left( \frac{1}{n \left(0\right)} \right),
\]

(182)

where we neglected time derivatives of first order quantities. With the help of Eqs. (170b), (170c) and (172) we find that the right-hand side of Eq. (182) vanishes. Hence,

\[
\frac{d s \left(0\right)}{d t} = 0,
\]

(183)

implying that, in zero order, the expansion takes place without generating entropy: \( s \left(0\right) \) is constant in time. Hence, in view of Eq. (17), which is valid for any scalar, and Eq. (183), the first order perturbation \( s \left(1\right) \) is automatically a gauge-invariant quantity, i.e., \( s \left(1\right) = s \left(1\right) \), in contrast to \( \varepsilon \left(1\right) \) and \( n \left(1\right) \), which had to be redefined according to Eqs. (2). Apparently, the entropy per particle \( s \left(1\right) \) is such a combination of \( \varepsilon \left(1\right) \) and \( n \left(1\right) \) that it need not be redefined. This can be made explicit by noting that in the linear approximation we are considering in this article, the second law of thermodynamics (180) should hold for zero order and first order quantities separately. In particular, Eq. (180) implies

\[
T \left(0\right) s \left(1\right) = \frac{1}{n \left(0\right)} \left( \varepsilon \left(1\right) - \frac{\varepsilon \left(0\right) + p \left(0\right)}{n \left(0\right)} n \left(1\right) \right),
\]

(184)

where we neglected products of differentials and first order quantities, and where we replaced \( \varepsilon \left(1\right) \) and \( n \left(1\right) \), respectively. We now note that the linear combination in the right-hand side of Eq. (184) has the property

\[
\varepsilon \left(1\right) - \frac{\varepsilon \left(0\right) + p \left(0\right)}{n \left(0\right)} n \left(1\right) = \varepsilon \left(1\right) - \frac{\varepsilon \left(0\right) + p \left(0\right)}{n \left(0\right)} n \left(1\right) g \left(1\right),
\]

(185)

as may immediately be verified with the help of Eqs. (2), (170b) and (170c). The right-hand side of Eq. (185) being gauge-invariant, the left-hand side must be gauge-invariant. This observation makes explicit the gauge invariance of the first order approximation to the entropy per particle, \( s \left(1\right) \). In order to stress the gauge invariance of the correction \( s \left(1\right) \) to the (constant) entropy per particle, \( s \left(0\right) \), we will write \( s \left(1\right) \), rather than \( s \left(1\right) \):

\[
s \left(1\right) := s \left(1\right).
\]

(186)

From Eqs. (184)–(186) we then get

\[
T \left(0\right) s \left(1\right) g \left(1\right) = \frac{1}{n \left(0\right)} \left( \varepsilon \left(1\right) - \frac{\varepsilon \left(0\right) \left(1 + w\right)}{n \left(0\right)} n \left(1\right) g \left(1\right) \right),
\]

(187)

where \( w \) is the quotient of zero order pressure and zero order energy density defined by Eq. (172). Notice that for the internal logic of our reasoning it is not essential at all to use the second law of thermodynamics. One may simply consider (187) as the defining expression for a certain linear combination of \( \varepsilon \left(1\right) \) and \( n \left(1\right) g \left(1\right) \), and replace everywhere in the equations below the product \( T \left(0\right) s \left(1\right) g \left(1\right) \) by the right-hand side of Eq. (187), without changing anything. However, the second law of thermodynamics yields a physical interpretation of the particular linear combination of \( \varepsilon \left(1\right) \) and \( n \left(1\right) g \left(1\right) \) that we will encounter in the equations of the next section. In fact, we rewrite Eq. (187) in the form

\[
T \left(0\right) s \left(1\right) g \left(1\right) = - \frac{\varepsilon \left(0\right) \left(1 + w\right)}{n \left(1\right) g \left(1\right)} \sigma \left(1\right),
\]

(188)

where the gauge independent, entropy related quantity \( \sigma \left(1\right) \) is given by

\[
\sigma \left(1\right) := n \left(1\right) - \frac{n \left(0\right)}{\varepsilon \left(0\right) \left(1 + w\right)} \varepsilon \left(1\right).
\]

(189)

The quantity \( \sigma \left(1\right) \) occurs as the source term in the gauge-invariant evolution equations (204) below. Equation (189) implies (7).
B. Gauge-invariant pressure perturbations

We will now derive a gauge-invariant expression for the physical pressure perturbations. To that end, we first calculate the time derivative of the background pressure. From Eq. (66) we have

$$\dot{p}_{(0)} = p_n \dot{n}_{(0)} + p_\varepsilon \dot{\varepsilon}_{(0)},$$

(190)

where $p_\varepsilon$ and $p_n$ are the partial derivatives given by Eqs. (91) and (92). Multiplying both sides of this expressions by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from Eq. (90) we get

$$p_{(1)} - \frac{\dot{p}_{(0)}}{\theta_{(0)}} \theta_{(1)} = p_n n_{gi}^{(1)} + p_\varepsilon \varepsilon_{gi}^{(1)},$$

(191)

where we have used Eqs. (2) to rewrite the right-hand side. Since $p_n$ and $p_\varepsilon$ depend on the background quantities $\varepsilon_{(0)}$ and $n_{(0)}$ only, the right-hand side is gauge-invariant. Hence, the quantity $p_{gi}^{(1)}$ defined by

$$p_{gi}^{(1)} := p_{(1)} - \frac{\dot{p}_{(0)}}{\theta_{(0)}} \theta_{(1)},$$

(192)

is gauge-invariant. We thus obtain the gauge-invariant counterpart of Eq. (90)

$$p_{gi}^{(1)} = p_\varepsilon \varepsilon_{gi}^{(1)} + p_n n_{gi}^{(1)},$$

(193)

We will now rewrite this equation in a slightly different form. From Eqs. (175) and (190) we obtain $\beta^2 = p_\varepsilon + p_n (\dot{n}_{(0)}/\dot{\varepsilon}_{(0)})$. Using Eqs. (170b) and (170c) we find

$$\beta = \sqrt{p_\varepsilon + \frac{n_{(0)} p_n}{\varepsilon_{(0)}(1 + w)}}.$$  

(194)

With this expression and Eqs. (189) and (194) we can express the gauge-invariant pressure (193) in terms of the energy density perturbation $\varepsilon_{gi}^{(1)}$ and the entropy related quantity $\sigma_{gi}^{(1)}$, rather than $\varepsilon_{gi}^{(1)}$ and the particle number density perturbation $n_{gi}^{(1)}$

$$p_{gi}^{(1)} = \beta^2 \varepsilon_{gi}^{(1)} + p_n \sigma_{gi}^{(1)},$$

(195)

a nice result which we will not use, however. Compare this relation with the one derived by Mukhanov et al. [4], their equation (5.3).

C. Gauge-invariant temperature perturbations

Finally, we will derive an expression for the gauge-invariant temperature perturbation $T_{gi}^{(1)}$ with the help of Eq. (A2a). For the time derivative of the energy density $\varepsilon_{(0)}(n_{(0)}, T_{(0)})$ we have

$$\dot{\varepsilon}_{(0)} = \left( \frac{\partial \varepsilon}{\partial n} \right)_T \dot{n}_{(0)} + \left( \frac{\partial \varepsilon}{\partial T} \right)_n \dot{T}_{(0)},$$

(196)

Replacing the infinitesimal quantities in Eq. (A2a) by perturbations, we find

$$\varepsilon_{(1)} = \left( \frac{\partial \varepsilon}{\partial n} \right)_T n_{(1)} + \left( \frac{\partial \varepsilon}{\partial T} \right)_n T_{(1)},$$

(197)

Multiplying both sides of Eq. (196) by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from Eq. (197) we get

$$\dot{\varepsilon}_{gi}^{(1)} = \left( \frac{\partial \varepsilon}{\partial n} \right)_T n_{gi}^{(1)} + \left( \frac{\partial \varepsilon}{\partial T} \right)_n \left( T_{(1)} - \frac{\dot{T}_{(0)}}{\theta_{(0)}} \theta_{(1)} \right),$$

(198)

where we have used Eqs. (2). Hence, the quantity

$$T_{gi}^{(1)} := T_{(1)} - \frac{\dot{T}_{(0)}}{\theta_{(0)}} \theta_{(1)},$$

(199)
is gauge-invariant. Thus, Eq. (198) can be written as

\[
\varepsilon^g_{(1)} = \left( \frac{\partial \varepsilon}{\partial n} \right)_T n^g_{(1)} + \left( \frac{\partial \varepsilon}{\partial T} \right)_n T^g_{(1)},
\]

(200)

implying that \( T^g_{(1)} \) can be interpreted as the gauge-invariant temperature perturbation. An equivalent form of (200) is given by Eq. (8). We thus have expressed the perturbation in the absolute temperature as a function of the perturbations in the energy density and particle number density for a given equation of state of the form \( \varepsilon = \varepsilon(n, T) \) and \( p = p(n, T) \). This equation will be used in Sec. VIII to derive an expression for the fluctuations in the background temperature, \( \delta_T \), a measurable quantity.

Finally, we give the evolution equation for the background temperature \( T_{(0)}(t) \). From Eq. (196) it follows

\[
\dot{T}_{(0)} = -3H \left[ \varepsilon_{(0)}(1 + w) - \left( \frac{\partial \varepsilon}{\partial n} \right)_T n_{(0)} \left( \frac{\partial \varepsilon}{\partial T} \right)_n \right],
\]

(201)

where we have used Eqs. (170b) and (170c). This equation will be used to follow the time development of the background temperature once \( \varepsilon_{(0)}(t) \) and \( n_{(0)}(t) \) are found from the zero order Einstein equations.

VIII. MANIFESTLY GAUGE-IN Variant FIRST ORDER EQUATIONS

The five perturbation equations (173) and (174) form a set of five equations for the five unknown quantities \( \varepsilon_{(1)}, n_{(1)}, \theta_{(1)}, 3R_{(1)} \parallel \) and \( \theta_{(1)} \). This system of equations can be reduced in the following way. In order to arrive at non-zero expressions for the physical quantities \( \varepsilon^g_{(1)} \) and \( n^g_{(1)} \) we have chosen in Eqs. (25) \( \omega = \theta \), Eq. (26), implying that \( \theta^g_{(1)} = 0 \), Eq. (27). As a consequence, we do not need the gauge dependent quantity \( \theta_{(1)} \). Eliminating the quantity \( \theta_{(1)} \) from equations (173) with the help of Eq. (174), we arrive at the set of four first order differential equations

\[
\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)}(1 + w) \left[ \dot{\theta}_{(1)} + \frac{1}{2H} \left( \kappa \varepsilon_{(1)} + \frac{1}{2} 3R_{(1)} \parallel \right) \right] = 0,
\]

(202a)

\[
\dot{n}_{(1)} + 3H n_{(1)} + n_{(0)} \left[ \dot{\theta}_{(1)} + \frac{1}{2H} \left( \kappa \varepsilon_{(1)} + \frac{1}{2} 3R_{(1)} \parallel \right) \right] = 0,
\]

(202b)

\[
\dot{\theta}_{(1)} + H(2 - 3\beta^2) \dot{\theta}_{(1)} + \frac{1}{\varepsilon_{(0)}(1 + w)} \nabla^2 p_{(1)} = 0,
\]

(202c)

\[
3\dot{R}_{(1)} \parallel + 2H^2 R_{(1)} \parallel - 2\kappa \varepsilon_{(0)}(1 + w) \dot{\theta}_{(1)} + \frac{3R_{(0)}}{3H} \left( \kappa \varepsilon_{(1)} + \frac{1}{2} 3R_{(1)} \parallel \right) = 0,
\]

(202d)

for the four quantities \( \varepsilon_{(1)}, n_{(1)}, \theta_{(1)} \), and \( 3R_{(1)} \parallel \).

The system of equations (202) is now cast in a suitable form to arrive at a system of manifestly gauge-invariant equations for the physical quantities \( \varepsilon^g_{(1)} \) and \( n^g_{(1)} \), since we then can immediately calculate these quantities. Indeed, eliminating the quantity \( \theta_{(1)} \) from Eqs. (176) with the help of Eq. (174), we get

\[
\varepsilon^g_{(1)} = \frac{\varepsilon_{(1)} 3R_{(0)} - 3\varepsilon_{(0)}(1 + w)\left( 2H \dot{\theta}_{(1)} + \frac{1}{2} 3R_{(1)} \parallel \right)}{3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)},
\]

(203a)

\[
n^g_{(1)} = n_{(1)} - \frac{3n_{(0)}(\kappa \varepsilon_{(1)} + 2H \dot{\theta}_{(1)} + \frac{1}{2} 3R_{(1)} \parallel)}{3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)}.
\]

(203b)

The quantities \( \varepsilon^g_{(1)} \) and \( n^g_{(1)} \) are now completely determined by the system of background equations (170)–(171) and the first order equations (202).

A. Evolution equations for density perturbations

Instead of calculating \( \varepsilon^g_{(1)} \) and \( n^g_{(1)} \) in the way described above, we proceed by first making explicit the gauge invariance of the theory. To that end, we rewrite the system of the four differential equations (202) for the gauge
dependent variables $\varepsilon_{(i)}$, $n_{(i)}$, $\theta_{(i)}$, and $\mathcal{R}_{(i)}$ into a system of equations for the gauge-invariant variables $\varepsilon_{(i)}^g$ and $n_{(i)}^g$. It is, however, of convenience to use the entropy related perturbation $\sigma_{(i)}^g$, defined by Eq. (189), rather than the particle number density perturbation $n_{(i)}^g$. The result is

$$\dot{\varepsilon}_{(i)}^g + a_1 \varepsilon_{(i)}^g + a_2 \dot{\varepsilon}_{(i)}^g = a_3 \sigma_{(i)}^g,$$

$$\dot{\sigma}_{(i)}^g = a_4 \sigma_{(i)}^g,$$

(204a, 204b)

where $\sigma_{(i)}^g$ is a short hand notation for the combination of $\varepsilon_{(i)}^g$ and $n_{(i)}^g$ given in Eq. (189). The derivation of these equations is given in detail in Appendix D. The coefficients $a_1, \ldots, a_4$ occurring in Eqs. (204) are given by

$$a_1 = \frac{\kappa \varepsilon_{(0)}(1 + w)}{H} - 2\frac{\dot{\beta}}{\beta} + H(4 - 3\beta^2) + 3\mathcal{R}_{(0)} \left( \frac{1}{3H} + \frac{2H(1 + 3\beta^2)}{3\mathcal{R}_{(0)} + 3k\varepsilon_{(0)}(1 + w)} \right),$$

$$a_2 = \kappa \varepsilon_{(0)}(1 + w) - 4H\frac{\dot{\beta}}{\beta} + 2H^2(2 - 3\beta^2) + 3\mathcal{R}_{(0)} \left( \frac{1}{2} + \frac{5H^2(1 + 3\beta^2) - 2H\dot{\beta}}{3\mathcal{R}_{(0)} + 3k\varepsilon_{(0)}(1 + w)} \right) - \beta^2 \left( \frac{\nabla^2}{a^2} - \frac{1}{2} \frac{3\mathcal{R}_{(0)}}{3\mathcal{R}_{(0)}} \right),$$

$$a_3 = \left\{ \begin{array}{l}
-18H^2 \\
3\mathcal{R}_{(0)} + 3k\varepsilon_{(0)}(1 + w)
\end{array} \right\} \left[ \varepsilon_{(0)} p_{cn}(1 + w) + \frac{2p_n \dot{\beta}}{3H \beta} - \beta^2 p_n + p_\varepsilon p_n + n_{(0)} p_{nn} \right] + p_n \left\{ \frac{\nabla^2}{a^2} - \frac{1}{2} \frac{3\mathcal{R}_{(0)}}{3\mathcal{R}_{(0)}} \right\},$$

$$a_4 = -3H \left( 1 - \frac{n_{(0)} p_n}{\varepsilon_{(0)}(1 + w)} \right),$$

(205a, 205b, 205c, 205d)

where the functions $w(t)$ and $\beta(t)$ are given by Eqs. (172) and (175), respectively. In the derivation of the above results, we used Eqs. (170) and (171). The abbreviations $p_n$ and $p_\varepsilon$ are given by Eq. (91). Furthermore, we used the abbreviations

$$p_{nn} := \frac{\partial^2 p}{\partial n^2}, \quad p_{cn} := \frac{\partial^2 p}{\partial \varepsilon \partial n}.$$

(206)

The equations (204) contain only gauge-invariant quantities and the coefficients are scalar functions. Thus, these equations are manifestly gauge-invariant.

The equations (204) are equivalent to one equation of the third order, whereas one would expect that the four first order equations (202) would be equivalent to one equation of the fourth order. This observation reflects the fact that the solutions of the first order equations are gauge dependent, while the solutions $\varepsilon_{(i)}^g$ and $\sigma_{(i)}^g$ of Eq. (204) are gauge independent. One ‘degree of freedom’, say, the gauge function $\psi$ has disappeared from the scene altogether.

The Eq. (204b) can be solved

$$\sigma_{(i)}^g(t, x) = \sigma_{(i)}^g(t_0, x) \exp \left\{ -3 \int_{t_0}^{t} H(\tau) \left( 1 - \frac{n_{(0)}(\tau) p_n(\tau)}{\varepsilon_{(0)}(\tau)[1 + w(\tau)]} \right) d\tau \right\}.$$  

(207)

Inserting this expression into Eq. (204a) we obtain the final equation for the perturbation (6).

The equations (204) constitute the main result of this article. In view of Eq. (189), they essentially are two differential equations for the perturbations $\varepsilon_{(i)}^g$ and $n_{(i)}^g$ to the energy density $\varepsilon_{(0)}(t)$ and the particle number density $n_{(0)}(t)$, respectively, for FLRW universes with $k = -1, 0, +1$. They describe the evolution of the energy density perturbation $\varepsilon_{(i)}^g$ and the particle number density perturbation $n_{(i)}^g$ for FLRW universes filled with a fluid which is described by an equation of state of the form $p = \rho(n, \varepsilon)$, the precise form of which is left unspecified.

### B. Evolution equations for contrast functions

In the study of the evolution of density perturbations it is of convenience to use a quantity which measures the perturbation to the density relative to the background densities. To that end we define the gauge-invariant contrast functions $\delta_\varepsilon$ and $\delta_n$ by

$$\delta_\varepsilon(t, x) := \frac{\varepsilon_{(i)}^g(t, x)}{\varepsilon_{(0)}(t)}, \quad \delta_n(t, x) := \frac{n_{(i)}^g(t, x)}{n_{(0)}(t)}.$$  

(208)
Using these quantities, Eqs. (204) can be rewritten as (see Appendix D 3)

$$\delta_{\varepsilon} + b_1 \delta_{\varepsilon} + b_2 \delta_{\varepsilon} = b_3 \left( \delta_n - \frac{\delta_{\varepsilon}}{1 + w} \right), \quad (209a)$$

$$\frac{1}{c} \frac{d}{dt} \left( \delta_n - \frac{\delta_{\varepsilon}}{1 + w} \right) = \frac{3Hn_{(0)p n}}{\varepsilon_{(0)}(1 + w)} \left( \delta_n - \frac{\delta_{\varepsilon}}{1 + w} \right), \quad (209b)$$

where the coefficients $b_1$, $b_2$ and $b_3$ are given by

$$b_1 = \frac{\kappa\varepsilon_{(0)}(1 + w)}{H} - 2 \frac{3}{\beta} + H \left( 2 + 6w + 3\beta^2 \right) + 3 \epsilon_{(0)} \left( \frac{1}{3H} + \frac{2H(1 + 3\beta^2)}{3R_{(0)} + 3\kappa\varepsilon_{(0)}(1 + w)} \right), \quad (210a)$$

$$b_2 = -\frac{4\kappa\varepsilon_{(0)}(1 + w)(1 + 3w) + H^2(1 - 3w + 6\beta^2(2 + 3w))}{3R_{(0)} + 3\kappa\varepsilon_{(0)}(1 + w)}$$

$$+ 6H \frac{\beta}{\beta} \left( w + \frac{\kappa\varepsilon_{(0)}(1 + w)}{3R_{(0)} + 3\kappa\varepsilon_{(0)}(1 + w)} \right) - 3 \epsilon_{(0)} \left( \frac{1}{2w} + H^2(1 + 6w)(1 + 3\beta^2) \right) - \beta^2 \left( \frac{\nabla^2}{a^2} - \frac{1}{2} 3 \epsilon_{(0)} \right), \quad (210b)$$

$$b_3 = \left\{ \frac{-18H^2}{3R_{(0)} + 3\kappa\varepsilon_{(0)}(1 + w)} \epsilon_{(0)p n}(1 + w) + 2 p_n \beta - \beta^2 p_n + p_z p_n + n_{(0)p n n} \right\} + \epsilon_{(0)} \frac{H^2}{3H \beta} \left( \frac{\nabla^2}{a^2} - \frac{1}{2} 3 \epsilon_{(0)} \right). \quad (210c)$$

In Sec. X we use the Eqs. (209) to study the evolution of small energy density perturbations and particle number perturbations in FLRW universes.

The entropy perturbation (187) reads, in terms of the contrast functions (208),

$$s_{(1)}^{gi} = -\frac{\epsilon_{(0)}(1 + w)}{n_{(0)} T_{(0)}} \left( \delta_n - \frac{\delta_{\varepsilon}}{1 + w} \right). \quad (211)$$

Finally, if we define the relative temperature perturbation $\delta_T$ by

$$\delta_T(t, x) := \frac{T_{(1)}(t, x)}{T_{(0)}(t)}, \quad (212)$$

then the relative temperature perturbation (8) is given by

$$\delta_T = \frac{\epsilon_{(0)} \delta_{\varepsilon} - \left( \frac{\partial \varepsilon_{(0)} \delta_{\varepsilon}}{\partial n} \right) n_{(0)} \delta_n}{T_{(0)} \left( \frac{\partial \varepsilon_{(0)}}{\partial T} \right)_n}. \quad (213)$$

We thus have found the relative temperature perturbation as a function of the relative perturbations in the energy density and particle number density for an equation of state of the form $\varepsilon = \varepsilon(n, T)$ and $p = p(n, T)$ (see Appendix A). The quantity $\delta_T(t, x)$ is a measurable quantity in the cosmic background radiation.

IX. NON-RELATIVISTIC LIMIT IN AN EXPANDING UNIVERSE

It is well known that if the gravitational field is weak and velocities are small with respect to the velocity of light ($v/c \to 0$), the system of Einstein equations and conservation laws may reduce to the single field equation of the Newtonian theory of gravity, namely the Poisson equation (28). Since in the Newtonian theory the gravitational field is described by only one, time-independent, potential $\phi(x)$, one cannot obtain the Newtonian limit by simply taking the limit $v/c \to 0$, since in a relativistic theory the gravitational field is described, in general, by six potentials, namely the components $h_{ij}(x)$ of the metric. In this article we have used the decomposition (124). Moreover, we have shown in Sec. IV that $h_{ij}$ given by (125a) describes the evolution of density perturbations. By using this decomposition we have reduced the number of potentials to only two, namely $\phi(x)$ and $\zeta(x)$. At first sight, it would follow from the evolution equations (158) for scalar perturbations that a further reduction of the number of potentials by one is impossible. However, we have rewritten the system of equations (158) for scalar perturbations into an equivalent system (202). As a result, the perturbation to the metric, $h_{ij}$, enters the system (202) only via the trace

$$3R_{(1)} = \frac{2}{c^2} \left[ 2\phi_{[k}^{|l|} - \zeta_{[k}^{|l|} |_{[l]|_{[k]} + \gamma_{[k}^{|l|} |_{[l]} + \frac{1}{3} 3 \epsilon_{(0)}(3\phi + \zeta_{[k}^{|l|}) \right], \quad (214)$$
of the perturbation to the Ricci tensor (128a). This shows explicitly that density perturbations are, in a non-flat FLRW universe, described by two potentials \( \phi(x) \) and \( \zeta(x) \). Hence, in non-flat FLRW universes the perturbation equations (202) do not reduce to the Newtonian theory of gravity. However, for a flat FLRW universe, i.e., a universe characterized by \( k = 0 \), implying, in view of (77), that

\[
3R_{(0)} = 0, \tag{215}
\]

the perturbation to the Ricci scalar, Eq. (214), reduces to

\[
3R_{(1)} = -\frac{4}{c^2} \nabla^2 \phi, \tag{216}
\]

where \( \nabla^2 \) is the usual Laplace operator. Hence, in a flat FLRW universe we need only one potential \( \phi \) to describe density perturbations. Therefore, we consider the only candidate FLRW universe in which the Newtonian limit could be obtained, namely a flat FLRW universe.

Usually, one considers the non-relativistic limit in the case of a static gravitational field. In this section we will show that our treatment of perturbations leads to the Poisson equation also for a non-static gravitational field. Thus, in the approach presented in this article there is no necessity to take the static limit at all. In fact, we will show that the set of perturbation equations (202)–(203) reduce to Eq. (28) without the assumption that the expansion is negligibly slow or absent.

Let us first consider the background equations. For a flat FLRW universe, the zero order equations (170) and (171) reduce to

\[
\dot{H} = -\frac{1}{2} \kappa \varepsilon_{(0)} (1 + w), \tag{217a}
\]

\[
\dot{\varepsilon}_{(0)} = -3H \varepsilon_{(0)} (1 + w), \tag{217b}
\]

\[
\dot{n}_{(0)} = -3H n_{(0)}, \tag{217c}
\]

and the constraint equation

\[
3H^2 = \kappa \varepsilon_{(0)} + \Lambda. \tag{218}
\]

We now consider the perturbation equations. Upon substituting Eq. (216) into Eqs. (202) and putting \( 3R_{(0)} = 0 \), we arrive at the set of perturbation equations

\[
\dot{\varepsilon}_{(1)} + 3H (\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)} (1 + w) \left[ \vartheta_{(1)} + \frac{1}{2H} \left( \kappa \varepsilon_{(1)} - \frac{2}{c^2} \nabla^2 \phi \right) \right] = 0, \tag{219a}
\]

\[
\dot{n}_{(1)} + 3H n_{(1)} + n_{(0)} \left[ \vartheta_{(1)} + \frac{1}{2H} \left( \kappa \varepsilon_{(1)} - \frac{2}{c^2} \nabla^2 \phi \right) \right] = 0, \tag{219b}
\]

\[
\dot{\vartheta}_{(1)} + H (2 - 3\beta^2) \vartheta_{(1)} + \frac{1}{\varepsilon_{(0)}} (1 + w) \frac{\nabla^2 p_{(1)}}{a^2} = 0, \tag{219c}
\]

\[
\nabla^2 \varphi + \frac{4\pi G}{c^4} a^2 \varepsilon_{(0)} (1 + w) \vartheta_{(1)} = 0, \tag{219d}
\]

for the four quantities \( \varepsilon_{(1)}, n_{(1)}, \vartheta_{(1)} \) and \( \phi \). Similarly, we obtain from Eqs. (203)

\[
\nabla^2 \left( \phi + \frac{2H \dot{\phi}}{(3H^2 - \Lambda)(1 + w)} \right) = \frac{4\pi G}{c^2} a^2 \varepsilon_{(1)}^{\parallel}, \tag{220a}
\]

\[
n_{(1)}^{\parallel} = n_{(1)} - \frac{n_{(0)}}{\varepsilon_{(0)}} (1 + w) (\varepsilon_{(1)} - \varepsilon_{(1)}^{\parallel}), \tag{220b}
\]

where we have eliminated \( \vartheta_{(1)} \) with the help of Eq. (219d) and \( \varepsilon_{(0)} \) with the help of Eq. (218). The scale factor of the universe \( a(t) \) follows from the Einstein equations via \( H := \dot{a} / a \).

We now consider the limits (see Appendix C)

\[
u_{(1)}^{\parallel} := \frac{U_{(1)}^{\parallel}}{c} = 0, \quad p = 0. \tag{221}
\]

In this limit, the kinetic energy is small compared to the rest energy of a particle [see, for example, Eq. (250)], so that the energy density of the universe in the limit (221) is

\[
\varepsilon = nmc^2, \tag{222}
\]
where \( mc^2 \) is the rest energy of a particle with mass \( m \) and \( n \) the particle number density. Thus, in the limit (221), the background equations (217)–(218) take the simple form

\[
\begin{align*}
\dot{H} &= -\frac{1}{2} \kappa \varepsilon_{(0)}, \\
\dot{\varepsilon}_{(0)} &= -3H \varepsilon_{(0)}, \\
\dot{n}_{(0)} &= -3Hn_{(0)},
\end{align*}
\]

while the constraint equation (218) reads

\[
3H^2 = \kappa \varepsilon_{(0)} + \Lambda.
\]

Note that in the limit (221), Eqs. (223b) and (223c) are identical, since then \( \varepsilon_{(0)} = n_{(0)} mc^2 \), in view of Eq. (222).

We now consider the perturbation equations (219) and (220) in the limit (221). Substituting \( \vartheta_{(1)} := (u_k^{(1)})_{k=0} \), \( w := p_{(0)}/\varepsilon_{(0)} = 0 \) and \( p_{(1)} = 0 \) into these equations we find that Eq. (219c) is identically satisfied, whereas the remaining equations (219) reduce to

\[
\begin{align*}
\dot{\varepsilon}_{(1)} + 3H\varepsilon_{(1)} + \frac{\varepsilon_{(0)}}{2H} \left( \kappa \varepsilon_{(1)} - \frac{2}{c^2} \nabla^2 \varphi \right) &= 0, \\
\dot{n}_{(1)} + 3Hn_{(1)} + \frac{n_{(0)}}{2H} \left( \kappa \varepsilon_{(1)} - \frac{2}{c^2} \nabla^2 \varphi \right) &= 0,
\end{align*}
\]

and Eqs. (220) become

\[
\begin{align*}
\nabla^2 \left( \varphi + \frac{2H\dot{\varphi}}{3H^2 - \Lambda} \right) &= \frac{4\pi G}{c^2} a^2 \varepsilon_{(1)}^{(i)}, \\
n_{(1)}^{(i)} &= \frac{\varepsilon_{(1)}^{(i)}}{mc^2}.
\end{align*}
\]

Note that in the limit (221), Eqs. (225a) and (225b) are identical, since then \( \varepsilon_{(1)} = n_{(1)} mc^2 \), in view of Eq. (222). In Eqs. (225c)–(226) the quantities \( \varepsilon_{(1)} \) and \( n_{(1)} \) do not occur. As a consequence, these equations can be solved for the quantities \( \varphi, \varepsilon_{(1)}^{(i)} \) and \( n_{(1)}^{(i)} \). In Appendix C we show that \( \varepsilon_{(1)} \) and \( n_{(1)} \) are gauge dependent also in the limit (221), so that Eqs. (225a) and (225b) need not be considered.

We now consider Eqs. (225c) and (226a) in some detail. By substituting Eq. (225c) into Eq. (226a) we obtain

\[
\nabla^2 \varphi(x) = \frac{4\pi G}{c^2} a^2(t) \varepsilon_{(1)}^{(i)}(t, x).
\]

Since \( \nabla^2 \varphi \) is independent of time, Eq. (225c), this equation is equivalent to

\[
\nabla^2 \varphi(x) = \frac{4\pi G}{c^2} a^2(t_p) \varepsilon_{(1)}^{(i)}(t_p, x),
\]

where \( t_p \) indicates the present time. This Einstein equation can be rewritten in a form that closely resembles the Poisson equation, by introducing the potential \( \varphi \)

\[
\varphi(x) := \frac{\varphi(x)}{a^2(t_p)}.
\]

Inserting (229) into (228) we obtain the result (28): the Einstein equation (227) for the time dependent perturbation \( \varphi_{(1)}^{(i)}(t, x) \) is, at fixed \( t = t_p \), identical to the usual, time independent, Poisson equation. Hence, as argued in Sec. 1H, the perturbations \( \varepsilon_{(1)}^{(i)} \) and \( n_{(1)}^{(i)} \) may indeed be interpreted as physical perturbations.

X. ANALYTICAL EXAMPLES OF THE PERTURBATION THEORY

In this section we consider two particular simple cases as an example: a flat FLRW universe with a vanishing cosmological constant in its radiation-dominated and matter-dominated stage. In order to keep this article self-contained, we start with the zero order equations and their solutions, although nothing is new here. We need these solutions to obtain explicit forms for the first order equations.
A. Radiation-dominated phase of the flat universe

In the radiation-dominated era we have \( p = \frac{1}{3} \varepsilon \), so that, according to (172), \( w = \frac{1}{3} \). Next to this the cosmological constant is put equal to zero: \( \Lambda = 0 \). Furthermore, a flat universe \( (k = 0) \) is considered, implying, with (77), that \( \frac{3}{a^2} R_{(0)}(t) = 0 \).

1. Zero-order equations and their solutions

The zero-order equations (170) then reduce to

\[
\begin{align*}
\dot{H} &= -\frac{2}{3} \kappa \varepsilon, \\
\dot{\varepsilon}_{(0)} &= -4H \varepsilon_{(0)}, \\
\dot{n}_{(0)} &= -3H n_{(0)},
\end{align*}
\]

while the constraint equation (171) becomes

\[
H^2 = \frac{1}{3} \kappa \varepsilon_{(0)}.
\]

The general solutions of these equations are

\[
\begin{align*}
H(t) &= \frac{1}{H} (ct)^{-1} = H(t_0) \left( \frac{t}{t_0} \right)^{-1}, \\
\varepsilon_{(0)}(t) &= \frac{3}{4\kappa} (ct)^{-2} = \varepsilon_{(0)}(t_0) \left( \frac{t}{t_0} \right)^{-2}, \\
n_{(0)}(t) &= n_{(0)}(t_0) \left( \frac{a(t)}{a(t_0)} \right)^{-3}.
\end{align*}
\]

The initial values \( H(t_0) \) and \( \varepsilon_{(0)}(t_0) \) are related by the initial value condition (231).

Using the definition of the Hubble parameter \( H := \dot{a}/a \) we find from (232a) that

\[
a(t) = a(t_0) \left( \frac{t}{t_0} \right)^{\frac{1}{3}},
\]

where \( t_0 \) is the time at which the radiation-dominated era sets in.

2. First-order equations and their solutions

The zero-order solutions (232) can now be substituted into the coefficients (210) of the equations (209). Recalling that \( p = \frac{1}{3} \varepsilon \) in the radiation-dominated regime, \( p_n = 0 \) and \( p_c = \frac{1}{3} \), so that \( \beta = 1/\sqrt{3} \), see (194). Since \( \tilde{\nabla}^2 = \nabla^2 \) for a flat universe, the coefficients \( b_1, b_2 \) and \( b_3 \) reduce to

\[
\begin{align*}
b_1 &= -H, \\
b_2 &= -\frac{1}{3} \frac{\nabla^2}{a^2} + \frac{2}{3} \kappa \varepsilon_{(0)}, \\
b_3 &= 0
\end{align*}
\]

where we have used (231). For the first-order equations (209) this yields the simple forms

\[
\begin{align*}
\ddot{\delta}_\varepsilon - H \dot{\delta}_\varepsilon + \left( -\frac{1}{3} \frac{\nabla^2}{a^2} + \frac{2}{3} \kappa \varepsilon_{(0)} \right) \delta_\varepsilon &= 0, \\
\frac{1}{c} \frac{d}{dt} (\delta_n - \frac{3}{4} \delta_\varepsilon) &= 0,
\end{align*}
\]

where \( H := \dot{a}/a \), (62). The solution of equation (235b) expresses the fact that particle number density perturbations are coupled to perturbations in the radiation, i.e.,

\[
\delta_n(t, \mathbf{x}) = \frac{3}{4} \delta_\varepsilon(t, \mathbf{x}).
\]
This relation is, in the extreme case of radiation-domination, independent of both the nature of the particles and the scale of the perturbation.

Equation (235a) may be solved by Fourier analysis of the function \( \delta_\epsilon \). Writing

\[
\delta_\epsilon(t, x) = \delta_\epsilon(t, q)e^{iq\cdot x}, \quad \delta_n(t, x) = \delta_n(t, q)e^{iq\cdot x},
\]

with \( q = |q| = 2\pi/\lambda \), where \( \lambda \) is the wavelength of the perturbation and \( i^2 = -1 \), yielding

\[
\nabla^2 \delta_\epsilon(t, x) = -q^2 \delta_\epsilon(t, q), \quad \nabla^2 \delta_n(t, x) = -q^2 \delta_n(t, q),
\]

so that the evolution equation (235a) for the amplitude \( \delta_\epsilon(t, q) \) reads

\[
\frac{\dot{\delta}_\epsilon}{H(t_0)} \left( \frac{t}{t_0} \right)^{-1} \delta_\epsilon + \left[ \frac{1}{3} \frac{q^2}{a^2(t_0)} \left( \frac{t}{t_0} \right)^{-1} + 2H^2(t_0) \left( \frac{t}{t_0} \right)^{-2} \right] \delta_\epsilon = 0,
\]

where we have used (231)–(232). This equation will be rewritten in such a way that the coefficients become dimensionless. To that end a dimensionless time variable is introduced, defined by

\[
\tau := \frac{t}{t_0}, \quad t \geq t_0.
\]

This definition implies

\[
\frac{d^n}{c^n dt^n} = \left( \frac{1}{ct_0} \right)^n \frac{d^n}{dt^n} = [2H(t_0)]^n \frac{d^n}{d\tau^n}, \quad n = 1, 2, \ldots ,
\]

where we have used (232a). Using (232a), (240) and (241), equation (239) for the density contrast \( \delta_\epsilon(\tau, q) \) can be written as

\[
\delta_\epsilon'' - \frac{1}{2\tau} \delta_\epsilon' + \left( \frac{\mu_\tau^2}{4\tau} + \frac{1}{2\tau^2} \right) \delta_\epsilon = 0,
\]

where a prime denotes differentiation with respect to \( \tau \). The constant \( \mu_\tau \) is given by

\[
\mu_\tau := \frac{q}{a(t_0)} \frac{1}{H(t_0)} \frac{1}{\sqrt{3}}.
\]

The general solution of equation (242) is a linear combination of the functions \( J_{\pm \frac{1}{2}}(\mu_\tau \sqrt{\tau})\tau^{3/4} \), where \( J_{\pm \frac{1}{2}}(x) = \sqrt{2/(\pi x)} \sin x \) and \( J_{-\frac{1}{2}}(x) = \sqrt{2/(\pi x)} \cos x \) are Bessel Functions of the first kind:

\[
\delta_\epsilon(\tau, q) = \left[ A_1(q) \sin (\mu_\tau \sqrt{\tau}) + A_2(q) \cos (\mu_\tau \sqrt{\tau}) \right] \sqrt{\tau}, \quad (244)
\]

where the functions \( A_1(q) \) and \( A_2(q) \) are given by

\[
A_1(q) = \delta_\epsilon(t_0, q) \sin \mu_\tau - \frac{\cos \mu_\tau}{\mu_\tau} \left[ \delta_\epsilon(t_0, q) - \frac{\dot{\delta}_\epsilon(t_0, q)}{H(t_0)} \right], \quad (245a)
\]

\[
A_2(q) = \delta_\epsilon(t_0, q) \cos \mu_\tau + \frac{\sin \mu_\tau}{\mu_\tau} \left[ \delta_\epsilon(t_0, q) - \frac{\dot{\delta}_\epsilon(t_0, q)}{H(t_0)} \right], \quad (245b)
\]

where we have used that

\[
\delta_\epsilon(t_0, q) = \delta_\epsilon(\tau = 1, q), \quad \dot{\delta}_\epsilon(t_0, q) = 2\delta_\epsilon'(\tau = 1, q)H(t_0),
\]

as follows from (241).

For large-scale perturbations, \( \lambda \to \infty \), the magnitude of the wave vector \( |q| = 2\pi/\lambda \) vanishes. Writing \( \delta_\epsilon(t) \equiv \delta_\epsilon(t, q = 0) \) and \( \dot{\delta}_\epsilon(t) \equiv \dot{\delta}_\epsilon(t, q = 0) \), we find from (243)–(245) that, for \( t \geq t_0 \),

\[
\delta_\epsilon(t) = -\left[ \delta_\epsilon(t_0) - \frac{\dot{\delta}_\epsilon(t_0)}{H(t_0)} \right] \frac{t}{t_0} + \left[ 2\delta_\epsilon(t_0) - \frac{\dot{\delta}_\epsilon(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{-\frac{1}{2}}.
\]
The growth rates proportional to $t$ and $t^{1/2}$ have been derived from the full set of linearized Einstein equations and conservation laws by a large number of researchers. See Lifshitz and Khalatnikov [2], (8.11), Adams and Camuto [17], (4.5b), Olson [18], page 329, Peebles [19], (86.20), and Kolb and Turner [20], (9.121). The complete solution (247), however, has never been presented in literature. From this solution it follows that large-scale perturbations grow only if the initial growth rate is large enough, i.e.,

$$\delta_\epsilon(t_0) \geq \delta_\epsilon(t_0)H(t_0),$$

(248)

otherwise the perturbations are decaying.

In the small-scale limit $\lambda \to 0$ (or, equivalently, $|q| \to \infty$) we find, using (243)–(245), that

$$\delta_\epsilon(t, q) \approx \delta_\epsilon(t_0, q) \left( \frac{t}{t_0} \right)^{1/2} \cos \left[ \mu_r - \mu_r \left( \frac{t}{t_0} \right)^{1/2} \right].$$

(249)

Due to the pressure gradients, which play a role only on small scales, a smaller growth rate than in the case of large-scale perturbations is found.

In contrast to the solution which results from the ‘standard equation’, Eq. (273), which yields oscillating density perturbations (274) with a decreasing amplitude, our equation leads to a solution which shows that small-scale perturbations oscillate with an increasing amplitude.

Although the growth rates $t$ and $t^{1/2}$ occurring in the large-scale solution (247) are well-known, the general solution (244), and, in particular, the large-scale solution (247) and the small-scale solution (249), have not been found earlier.

The approach presented in this article yields an evolution equation (235a) which differs from the standard equation (273). This difference will be explained in detail in Sec. XI.

B. Matter-dominated phase of the flat universe

Once protons and electrons recombine to yield hydrogen at a temperature around 4000 K, the radiation pressure becomes negligible, and the equations of state reduce to those of a non-relativistic monatomic perfect gas [Weinberg [14], equations (15.8.20) and (15.8.21)]

$$\varepsilon(n, T) = n m_H c^2 + \frac{4}{3} n k_B T, \quad p(n, T) = n k_B T,$$

(250)

where $k_B$ is Boltzmann’s constant, $m_H$ the proton mass, and $T$ the temperature of the matter.

1. Zero-order equations and their solutions

The maximum temperature in the matter-dominated era occurs around time $t_0$ of the decoupling of matter and radiation: $T(t_0) \approx 4000$ K. Hence, from equation (250) it follows that the pressure is negligible with respect to the energy density, i.e.,

$$\frac{p}{\varepsilon} \approx \frac{k_B T}{m_H c^2} \leq 3.7 \times 10^{-10}.$$

(251)

This implies that, to a good approximation, $\varepsilon_{(0)} \pm p_{(0)} \approx \varepsilon_{(0)}$ and $\varepsilon_{(0)} \approx n_{(0)} m_H c^2$. Hence, in an unperturbed flat FLRW universe, the pressure can be neglected with respect to the energy density. The above facts, yield that the Einstein equations and conservation laws (170)–(171) for a flat FLRW universe reduce to

$$\dot{H} = -\frac{1}{3} \kappa \varepsilon_{(0)},$$

(252a)

$$\dot{\varepsilon}_{(0)} = -3 H \varepsilon_{(0)},$$

(252b)

$$\dot{n}_{(0)} = -3 H n_{(0)},$$

(252c)

and the constraint equation

$$H^2 = \frac{1}{3} \kappa \varepsilon_{(0)},$$

(253)
where we have put the cosmological constant $\Lambda$ equal to zero. The general solutions of the zero-order Einstein equations and conservation laws are

$$H(t) = \frac{2}{3}(ct)^{-1} = H(t_0) \left(\frac{t}{t_0}\right)^{-1},$$  \hspace{0.5cm} (254a)  

$$\varepsilon_{(o)}(t) = \frac{4}{3\kappa} (ct)^{-2} = \varepsilon_{(o)}(t_0) \left(\frac{t}{t_0}\right)^{-2},$$  \hspace{0.5cm} (254b)  

$$n_{(o)}(t) = n_{(o)}(t_0) \left(\frac{t}{t_0}\right)^{-2}.$$  \hspace{0.5cm} (254c)

The initial values $H(t_0)$ and $\varepsilon_{(o)}(t_0)$ are related by the initial value condition (253).

Using the definition of the Hubble parameter $H := \dot{a}/a$, it is found from (254a) for the scale factor that

$$a(t) = a(t_0) \left(\frac{t}{t_0}\right)^{\frac{2}{3}},$$  \hspace{0.5cm} (255)  

where $t_0$ is the time at which the matter-dominated era sets in.

2. First-order equations and their solutions

Using Eqs. (A5) we find from the equations of state (250)

$$p_c = \frac{2}{3}, \quad p_n = -\frac{2}{3}n_{Hc}^2,$$  \hspace{0.5cm} (256)  

so that, to a good approximation, using (194)

$$\beta(t) \approx \frac{v_s(t)}{c} = \sqrt{\frac{5k_B T_{(o)}(t)}{3m_{Hc^2}}},$$  \hspace{0.5cm} (257)  

where $v_s$ is the speed of sound. Differentiating (257) with respect to time yields

$$\frac{\dot{\beta}}{\beta} = \frac{T_{(o)}}{2T_{(o)}}.$$  \hspace{0.5cm} (258)  

For the time development of the background temperature it is found from Eqs. (201) and (250), this time not neglecting the pressure with respect to the energy density, that

$$\dot{T}_{(o)} = -2HT_{(o)}.$$  \hspace{0.5cm} (259)  

(Note that putting $w = 0$ in Eq. (201) yields the incorrect result $\dot{T}_{(o)} = 0$.) Combining (258) and (259) results in

$$\frac{\dot{\beta}}{\beta} = -H.$$  \hspace{0.5cm} (260)  

For the evolution of the background temperature it is found that

$$T_{(o)}(t) = T_{(o)}(t_0) \left(\frac{a(t)}{a(t_0)}\right)^{-2},$$  \hspace{0.5cm} (261)  

where we have used that $H := \dot{a}/a$. Using (256) it is found for equation (209b)

$$\frac{1}{c} \frac{d}{dt} (\delta_n - \delta_\varepsilon) = -2H (\delta_n - \delta_\varepsilon).$$  \hspace{0.5cm} (262)  

A solution to this equation is

$$\delta_n(t, x) = \delta_\varepsilon(t, x),$$  \hspace{0.5cm} (263)
where we have used that in the matter-dominated phase nearly all energy resides in the rest mass of the particles. After substituting (256), (257) and (260) into the coefficients (210) it is found that

$$b_1 = 3H, \quad b_2 = -\frac{5}{6} \kappa \varepsilon_0 - \frac{v_s^2}{c^2} \frac{\nabla^2}{a^2}, \quad b_3 = -\frac{2}{3} \frac{\nabla^2}{a^2},$$

(264)

considering that for a flat universe $\nabla^2 = \nabla^2$. For the evolution equation for density perturbations, (209a), this results in the simple form

$$\ddot{\delta}_c + 3H \dot{\delta}_c - \left( \frac{v_s^2}{c^2} \frac{\nabla^2}{a^2} + \frac{5}{6} \kappa \varepsilon_0 \right) \delta_c = 0,$$

(265)

where we have used the result (263). Using (237) and (238) the evolution equation for the amplitude $\delta_c(t, q)$ can be rewritten as

$$\ddot{\delta}_c + 3H(t_0) \left( \frac{t}{t_0} \right)^{-1} \dot{\delta}_c + H^2(t_0) \left[ \mu_m^2 \left( \frac{t}{t_0} \right)^{-\frac{5}{3}} + \frac{5}{2} \left( \frac{t}{t_0} \right)^{-2} \right] \delta_c = 0,$$

(266)

having incorporated (253)–(255), (257) and (261). The constant $\mu_m$ is given by

$$\mu_m = \frac{q}{a(t_0) H(t_0)} \frac{v_s(t_0)}{c}, \quad v_s(t_0) = \sqrt{\frac{5 k_B T_0(t_0)}{m_H}}.$$

(267)

Using the dimensionless time variable (240), it is found from (254a) that

$$\frac{d^n}{c^n dt^n} = \left( \frac{1}{ct_0} \right)^n \frac{d^n}{d\tau^n} = \left[ \frac{3}{2} H(t_0) \right]^n \frac{d^n}{d\tau^n}, \quad n = 1, 2, \ldots.$$

(268)

Using this expression, equation (266) can be rewritten in the form

$$\ddot{\delta}_c + \frac{2}{\tau} \dot{\delta}_c + \left( \frac{4 \mu_m^2}{9 \tau^{4/3}} - \frac{10}{9 \tau^2} \right) \delta_c = 0,$$

(269)

where a prime denotes differentiation with respect to $\tau$. Hence, the general solution of equation (269) is

$$\delta_c(\tau, q) = \left[ B_1(q) J_{\pm \frac{2}{3}} \right] \left( 2 \mu_m \tau^{-1/3} \right) + B_2(q) J_{\pm \frac{2}{3}} \left( 2 \mu_m \tau^{-1/3} \right) \tau^{-1/2},$$

(270)

where $B_1(q)$ and $B_2(q)$ are arbitrary functions and $J_{\pm \nu}(x)$ is the Bessel function of the first kind. In other words, in the matter-dominated era density perturbations oscillate with a slowly decaying amplitude.

In the large-scale limit ($\langle q \rangle \to 0$), it is found that, transforming back from $\tau$ to $t$,

$$\delta_c(t) = \frac{1}{7} \left[ 5 \delta_c(t_0) + \frac{2 \delta_c(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{\frac{2}{3}} + \frac{2}{7} \left[ \delta_c(t_0) - \frac{\dot{\delta}_c(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{-\frac{5}{3}},$$

(271)

having used that

$$\delta_c(t_0, q) = \delta_c(\tau = 1, q), \quad \dot{\delta}_c(t_0, q) = \frac{2}{3} \delta_c(\tau = 1, q) H(t_0),$$

(272)

as follows from (268). The solution proportional to $t^{2/3}$ is a standard result. Since $\delta_c$ is gauge-invariant, the standard non-physical gauge mode proportional to $t^{-1}$ is absent from our theory. Instead, a physical mode proportional to $t^{-5/3}$ is found.

The approach presented in this article yields an evolution equation (265) which differs from the standard equation (279). This difference will be explained in detail in the next section.

**XI. RELATION TO EARLIER WORK**

In this section it is shown that, in contrast to the approach developed in this article, the standard theory of cosmological density perturbations results in *decaying* small-scale density perturbations in a radiation-dominated flat FLRW universe. Furthermore, we consider the standard Newtonian treatment for density perturbations in a flat FLRW matter-dominated universe in the light of our equations, which are refined compared to the usual equations of earlier treatments.

The difference between our treatment and the standard approach is that the solutions of the equations of the standard approach contain the gauge function $\psi(x)$, whereas our solutions are gauge-invariant.
A. Radiation-dominated universe

The standard equation for the density contrast function $\delta$ which can be found e.g., in the textbook of Peacock [21], equation (15.25), is given by

$$\ddot{\delta} + 2H\dot{\delta} - \left(\frac{1}{3}\frac{\nabla^2}{a^2} + \frac{4\kappa\varepsilon}{(0)}\right)\delta = 0.$$  \hfill (273)

This equation is derived by using special relativistic fluid mechanics and the Newtonian theory of gravity with a relativistic source term. In agreement with the text under equation (15.25) of this textbook, the term $-\frac{4}{3}\nabla^2\delta/a^2$ has been added. The same result, equation (273), can be found in Weinberg’s classic [14], equation (15.10.57) with $p = \frac{1}{3}\rho$ and $\kappa = 1/\sqrt{3}$. Using (237), it is found for the general solution of equation (273) that

$$\delta(\tau, q) = \frac{8C_1(q)}{\mu_r^2}J_2(\mu_r\sqrt{\pi}) + \psi(q)\pi\mu_r^2H(t_0)Y_2(\mu_r\sqrt{\pi}),$$  \hfill (274)

where $C_1(q)$ and $\psi(q)$ are arbitrary functions (the integration ‘constants’), $\tau$ is given by (240), the constant $\mu_r$ is given by (243) and $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind, respectively. The factors $8/\mu_r^2$ and $\pi\mu_r^2H(t_0)$ have been inserted for convenience. The derivation of the solution (274) runs along the same lines as the derivation of (244). Thus, the standard equation (273) yields oscillating density perturbations with a decaying amplitude.

For large-scale perturbations ($|q| \to 0$, or, equivalently, $\mu_r \to 0$), the asymptotic expressions for the Bessel functions $J_2$ and $Y_2$ are given by

$$J_2(\mu_r\sqrt{\pi}) = \frac{\mu_r^2}{8}\pi, \quad Y_2(\mu_r\sqrt{\pi}) \approx -\frac{4}{\pi\mu_r^2}\tau^{-1}.$$  \hfill (275)

Substituting these expressions into (274), it is found for large-scale perturbations that

$$\delta(\tau, q) = C_1(q)\tau - 4H(t_0)\psi(q)\tau^{-1}.$$  \hfill (276)

Large-scale perturbations can also be obtained from the standard equation (273) by substituting $\nabla^2\delta = 0$, i.e.,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{4\kappa\varepsilon}{(0)}\delta = 0.$$  \hfill (277)

The general solution of this equation is, using Eqs. (232), given by (276). Thus far, the functions $C_1(q)$ and $\psi(q)$ are the integration ‘constants’ which can be determined by the initial values $\delta(t_0, q)$ and $\dot{\delta}(t_0, q)$. However, equation (277) can also be derived from the linearized Einstein equations and conservation laws for scalar perturbations (202): see the derivation in Appendix E. As a consequence, equation (277) is found to be also a relativistic equation and the quantity $\delta = \varepsilon_1(1)/\varepsilon_0(0)$ is gauge-dependent. Therefore, the second term in the solution (276) is not a physical mode, but equal to the gauge mode

$$\delta_{\text{gauge}}(\tau, q) = \psi(q)\frac{\dot{\varepsilon}_0(0)}{\varepsilon_0(0)} = -4H(t_0)\psi(q)\tau^{-1},$$  \hfill (278)

as follows from (177a), (230b) and (232a). Consequently, $\psi(q)$ should not be interpreted as an integration constant, but as a gauge function, which cannot be determined by imposing initial value conditions. Thus, the general solution (274) of the standard equation (273) depends on the gauge function $\psi(q)$ and has, as a consequence, no physical significance. This, in turn, implies that the standard equation (273) does not describe the evolution of density perturbations. Here the negative effect of the gauge function is clearly seen: up till now it was commonly accepted that small-scale perturbations in the radiation-dominated era of a flat FLRW universe oscillate with a decaying amplitude, according to (274). The approach presented in this article reveals, however, that small-scale density perturbations oscillate with an increasing amplitude, according to (249).

B. Matter-dominated universe

The standard perturbation equation of the Newtonian theory of gravity is derived from approximate, non-relativistic equations. It reads

$$\ddot{\delta} + 2H\dot{\delta} - \left(\frac{\kappa\varepsilon}{c^2 a^2} + \frac{4\kappa\varepsilon}{(0)}\right)\delta = 0.$$  \hfill (279)
where \( v_s \) is the speed of sound. (See, e.g., Weinberg [14], Sec. 15.9, or Peacock [21], Sec. 15.2.) Using (237), the general solution of equation (279) is found to be

\[
\delta(\tau, q) = \left[ \frac{15}{8} \sqrt{\frac{\pi}{\mu_m^3}} D_1(q) J_{+\frac{\tau}{2}}(2\mu_m^{1/3}) - 4\psi(q) \sqrt{\frac{\pi \mu_m}{T}} H(t_0) J_{-\frac{\tau}{2}}(2\mu_m^{1/3}) \right] \tau^{-1/6},
\]

(280)

where \( D_1(q) \) and \( \psi(q) \) are arbitrary functions (the ‘constants’ of integration) and \( J_{+\nu}(x) \) is the Bessel function of the first kind. The factors \( \frac{15}{8} \sqrt{\pi/\mu_m^3} \) and \( 4H(t_0) \sqrt{\pi \mu_m} \) have been inserted for convenience. The constant \( \mu_m \) is given by (267) and \( \tau \) is given by (240). The derivation of the solution (280) runs along the same lines as the derivation of (270).

We now consider large-scale perturbations characterized by \( \nabla^2 \delta = 0 \) (i.e., \(|q| \to 0\)) or perturbations of all scales in the non-relativistic limit (i.e., \( v_s/c \to 0 \)). Both limits imply \( \mu_m \to 0 \), as follows from (267). The asymptotic expressions for the Bessel functions in the limit \( \mu_m \to 0 \) are given by

\[
J_{+\frac{\tau}{2}}(2\mu_m^{1/3}) \approx \frac{8}{15} \sqrt{\frac{\mu_m^3}{\pi}} \tau^{5/6}, \quad J_{-\frac{\tau}{2}}(2\mu_m^{1/3}) \approx \frac{3}{4} \sqrt{\frac{\pi \mu_m}{\tau}} \tau^{-5/6}.
\]

(281)

Substituting these expressions into the general solution (280), results in

\[
\delta(\tau, q) = D_1(q)\tau^{2/3} - 3H(t_0)\psi(q)\tau^{-1}.
\]

(282)

In the limit \( \mu_m \to 0 \), equation (279) reduces to

\[
\ddot{\delta} + 2H\dot{\delta} - \frac{5}{6} \kappa \varepsilon(0) \delta = 0.
\]

(283)

Using (254a)–(254c) we find the general solution (282) of this equation. Thus far, the functions \( D_1(q) \) and \( \psi(q) \) are the integration ‘constants’ which can be determined by the initial values \( \delta(t_0, q) \) and \( \dot{\delta}(t_0, q) \). However, equation (283) can also be derived from the general theory of relativity, and is, as a consequence, a relativistic equation. In fact, this equation follows for large-scale perturbations from equations (202) (see Appendix E for a derivation), whereas in the non-relativistic limit it follows from equations (225a) and (225c). In both cases, however, it is based on the gauge-dependent quantity \( \delta = \varepsilon(\nu)/\varepsilon(0) \). (In Appendix C it is shown that if a variable, e.g., \( \varepsilon(\nu) \), is gauge-dependent, it is also gauge-dependent in the non-relativistic limit.) As a consequence, the second term of (282) is equal to the gauge mode

\[
\delta_{\text{gauge}}(\tau, q) = \psi(q) \frac{\dot{\varepsilon}(0)}{\varepsilon(0)} = -3H(t_0)\psi(q)\tau^{-1},
\]

(284)

as follows from (177a), (252b) and (254a). Therefore, \( \psi(q) \) should not be interpreted as an integration constant, but as a gauge function, which cannot be determined by imposing initial value conditions. Since the solution (280) of equation (279) depends on the gauge function \( \psi(q) \) it has no physical significance. Consequently, the standard equation (279) does not describe the evolution of density perturbations. Again we encounter the negative effect of the gauge function: up till now it was commonly accepted that for small-scale density perturbations (i.e., density perturbations with wave lengths much smaller than the particle horizon) a Newtonian treatment suffices and gauge ambiguities do not occur and that the evolution of density perturbations in the Newtonian regime is described by the standard equation (279). The approach presented in this article reveals, however, that in a fluid with an equation of state (250), density perturbations are described by the relativistic equation (265), independent of the scale of the perturbations.

### Appendix A: EQUATIONS OF STATE FOR THE ENERGY DENSITY AND PRESSURE

We have used an equation of state for the pressure of the form \( p = p(n, \varepsilon) \). In general, however, this equation of state is given in the form of two equations for the energy density \( \varepsilon \) and the pressure \( p \) which contain also the absolute temperature \( T \):

\[
\varepsilon = \varepsilon(n, T), \quad p = p(n, T).
\]

(A1)

In principle it is possible to eliminate \( T \) from the two equations (A1) to get \( p = p(n, \varepsilon) \), so that our choice of the form \( p = p(n, \varepsilon) \) is justified. In practice, however, it may in general be difficult to eliminate the temperature \( T \) from the
equations (A1). However, this is not necessary, since the partial derivatives $p_\varepsilon$ and $p_n$ (91), the only quantities that are actually needed, can be found in an alternative way. From Eq. (A1) it follows

$$\begin{align*}
\mathrm{d}\varepsilon &= \left(\frac{\partial \varepsilon}{\partial n}\right)_T \mathrm{d}n + \left(\frac{\partial \varepsilon}{\partial T}\right)_n \mathrm{d}T, \\
\mathrm{d}p &= \left(\frac{\partial p}{\partial n}\right)_T \mathrm{d}n + \left(\frac{\partial p}{\partial T}\right)_n \mathrm{d}T.
\end{align*}$$

(A2a)

(A2b)

From Eq. (A2b) it follows that the partial derivatives (91) are

$$\begin{align*}
p_n &= \left(\frac{\partial p}{\partial n}\right)_T + \left(\frac{\partial p}{\partial T}\right)_n \left(\frac{\partial T}{\partial \varepsilon}\right)_n, \\
p_\varepsilon &= \left(\frac{\partial p}{\partial T}\right)_n \left(\frac{\partial T}{\partial \varepsilon}\right)_n.
\end{align*}$$

(A3a)

(A3b)

From Eq. (A2a) it follows

$$\begin{align*}
\left(\frac{\partial T}{\partial n}\right)_\varepsilon &= -\left(\frac{\partial \varepsilon}{\partial n}\right)_T \left(\frac{\partial \varepsilon}{\partial T}\right)_n^{-1}, \\
\left(\frac{\partial T}{\partial \varepsilon}\right)_n &= \left(\frac{\partial \varepsilon}{\partial T}\right)_n^{-1}. 
\end{align*}$$

(A4a)

(A4b)

Upon substituting the expressions (A4) into Eqs. (A3), we find for the partial derivatives defined by Eq. (91)

$$\begin{align*}
p_n &= \left(\frac{\partial p}{\partial n}\right)_T - \left(\frac{\partial p}{\partial T}\right)_n \left(\frac{\partial \varepsilon}{\partial n}\right)_T \left(\frac{\partial \varepsilon}{\partial T}\right)_n^{-1}, \\
p_\varepsilon &= \left(\frac{\partial p}{\partial T}\right)_n \left(\frac{\partial \varepsilon}{\partial T}\right)_n^{-1},
\end{align*}$$

(A5a)

(A5b)

where $\varepsilon$ and $p$ are given by (A1). In order to calculate the second order derivative $p_{nn}$ replace $p$ in Eq. (A5a) by $p_n$. For $p_{\varepsilon\varepsilon}$ replace $p$ in Eq. (A5b) by $p_\varepsilon$. Finally, for $p_{\varepsilon n} \equiv p_{n\varepsilon}$, replace $p$ in Eq. (A5a) by $p_\varepsilon$ or, equivalently, replace $p$ in Eq. (A5b) by $p_n$.

Appendix B: GAUGE INVARIANCE OF THE FIRST ORDER EQUATIONS

If we go over from one synchronous system of reference with coordinates $x$ to another synchronous system of reference with coordinates $\hat{x}$ given by Eq. (3), we have

$$\xi_{\mu;0} + \xi_{0;\mu} = 0,$$

(B1)

as follows from the transformation rule (13) and the synchronicity condition (30). From this equation we find, using Eqs. (31), (33a), (33b) and (63) that $\xi^\mu(t, x)$ must be of the form

$$\xi^0 = \psi(x), \quad \xi^i = \tilde{g}^{ik} \partial_k \psi(x) \int_\mathbb{C} \frac{\mathrm{d}\tau}{a^2(\tau)} + \chi^i(x),$$

(B2)

where $\psi(x)$ and $\chi^i(x)$ are arbitrary functions —of the first order— of the spatial coordinates $x$. The fact that the gauge function $\psi$ does not depend on the coordinate $t$ anymore, as it did in general coordinates, see Eq. (5), is a consequence of the choice of synchronous coordinates for the original coordinates as well as for the transformed system of reference. It does not imply that the gauge-invariant quantities $\xi^{\mbox{gi}}$ and $n^{\mbox{gi}}$ are independent of transformations within a plane of synchronicity only. In fact, they have been shown to be gauge-invariant, i.e., invariant under arbitrary infinitesimal coordinate transformations (3), and the fact that we are considering, from now on, only transformations that transform a synchronous system of reference to another synchronous system of reference does not take away, of course, this more general property of being invariant under arbitrary transformations (3).
The energy density perturbation transforms according to (1a), where \( \varepsilon_{(0)} \) is a solution of Eq. (170b). Similarly, the particle number density transforms according to (1b) where \( n_{(0)} \) is a solution of Eq. (170c). Finally, as follows from (17), the fluid expansion scalar \( \theta \), Eq. (22c), transforms as

\[
\dot{\theta}_{(1)}(x) = \theta_{(1)}(x) + \psi(x) \dot{\theta}_{(0)}(t),
\]

(B3)

where \( \theta_{(0)} = 3H \) is a solution of Eq. (170a).

From (17) with \( \sigma = p, \varepsilon, \) or \( n \) and (90) we find for the transformation rule for the first order perturbations to the pressure

\[
\dot{p}_{(1)} = p_{E} \dot{\varepsilon}_{(1)} + p_{n} \dot{n}_{(1)}.
\]

(B4)

The transformation rule (18) with \( V^\mu \) the four-velocity \( u^\mu \) implies

\[
\dot{u}^\mu_{(1)} = u^\mu_{(1)} - \xi^\mu_{(0)},
\]

(B5)

where we have used that \( u^\mu_{(0)} = \delta^\mu_{0} \), Eq. (60). From the transformation rule (B5) it follows that \( u^\mu_{(1)} \) transforms under synchronicity preserving transformations (B2) as

\[
\begin{align*}
\hat{u}^0_{(1)}(x) &= u^0_{(1)}(x) = 0, \\
\hat{u}^i_{(1)}(x) &= u^i_{(1)}(x) - \frac{1}{a^2(t)} \tilde{\gamma}^k_{ij}(x) \partial_k \psi(x).
\end{align*}
\]

(B6a, B6b)

We want to determine the transformation rules for \( \hat{\theta}_{(1)} \) and \( \hat{3}R_{(1)k} \). Since the quantities \( \hat{3}R, \) (46), and \( \hat{\theta}, \) (51), are both non-scalars under general space-time transformations, the transformation rule (17) is not applicable to determine the transformation of their first order perturbations under infinitesimal space-time transformations \( x^\mu \to \tilde{x}^\mu, \) (3). Since \( u^i_{(1)} \) satisfies Eq. (158e), and since \( u^i_{(1)} \) transforms according to (B6), and since we know that \( \hat{u}^i_{(1)} \) satisfies Eq. (158e) with hats, one may verify, using (B4), that

\[
\hat{\theta}_{(1)} := \theta_{(1)}(x) - \frac{\nabla^2 \psi(x)}{a^2(t)},
\]

(B7)

satisfies Eq. (173c) with hatted quantities. The quantity \( \hat{\theta}_{(1)} \) is defined in analogy to \( \theta_{(1)} \) in (86)

\[
\hat{\theta}_{(1)} = (\hat{u}^k_{(1)})_{|k}.
\]

(B8)

Apparently, \( \hat{\theta}_{(1)} \) transforms according to (B7) under arbitrary infinitesimal synchronicity preserving space-time transformations. Similarly, one may verify that

\[
\hat{3}R_{(1)k}(x) := 3R_{(1)k}(x) + 4H(t) \left( \frac{\nabla^2 \psi(x)}{a^2(t)} - \frac{1}{2} 3R_{(0)}(t) \psi(x) \right),
\]

(B9)

satisfies Eq. (174). Apparently, (B9) is the transformation rule under arbitrary infinitesimal synchronicity preserving space-time transformations. An alternative way to find the results (B7) and (B9) is to write \( \hat{\theta}_{(1)} = \theta_{(1)} - f \) and \( \hat{3}R_{(1)k} = 3R_{(1)k} - g \), where \( f \) and \( g \) are unknown functions, to substitute, thereupon, \( \hat{\theta}_{(1)} \) and \( \hat{3}R_{(1)k} \) into Eqs. (173c) and (174), and to determine \( f \) and \( g \) such that the old equations (173c) and (174) reappear. In fact, our approach to define \( \hat{\theta}_{(1)}, \) (B7), and \( \hat{3}R_{(1)k}, \) (B9), is nothing but a shortcut to this procedure.

It may now easily be verified by substitution that if \( \varepsilon_{(1)}, n_{(1)}, \theta_{(1)}, \psi_{(1)} \) and \( \hat{3}R_{(1)k} \) are solutions of the system (173)–(174), then the quantities \( \varepsilon_{(1)}, \) (1a), \( n_{(1)}, \) (1b), \( \dot{\theta}_{(1)}, \) (B3), \( \dot{\psi}_{(1)}, \) (B7), and \( \hat{3}R_{(1)k}, \) (B9), are, for an arbitrary function \( \psi(x) \), also solutions of this system.

Appendix C: GAUGE DEPENDENCE IN THE NON-RELATIVISTIC LIMIT

If we require that the limit (221) holds true before and after a gauge transformation, we find from the transformation rule (B6b) for \( u^i_{(1)} \) that \( \partial_k \psi(x) = 0 \), or, equivalently,

\[
\psi(x) = C,
\]

(C1)
where $C$ is an arbitrary, small, constant. In view of Eq. (C1), the gauge dependent functions $\varepsilon_{(1)}$ and $n_{(1)}$ transform under a gauge transformation in the non-relativistic limit according to \[\text{cf. Eqs. (1)}\]

$$
\varepsilon_{(1)} \rightarrow \tilde{\varepsilon}_{(1)} = \varepsilon_{(1)} + C\tilde{\varepsilon}_{(0)} ,
$$

$$(C2a)$$

$$
n_{(1)} \rightarrow \tilde{n}_{(1)} = n_{(1)} + C\tilde{n}_{(0)},
$$

$$(C2b)$$

Since in an expanding universe the time derivatives $\varepsilon_{(0)}(t) \neq 0$, (223b), and $\tilde{n}_{(0)}(t) \neq 0$, (223c), the functions $\varepsilon_{(1)}(x)$ and $n_{(1)}(x)$ do still depend upon the gauge, so that these quantities have, also in the non-relativistic limit, no physical significance. Therefore, Eqs. (225a) and (225b) need not be considered in the non-relativistic limit: these equations are not part of the Newtonian theory of gravity. The only remaining physical equations are Eqs. (225c)–(226).

Finally, it follows from equations (B2) and (C1) that the transformation (3) reduces in the non-relativistic limit to

$$
x^0 \rightarrow x^0 - C, \quad x^i \rightarrow x^i - \chi^i(x),
$$

$$(C3)$$

In other words, and not surprisingly, time and space transformations are decoupled: time coordinates may be shifted, whereas spatial coordinates may be chosen arbitrarily.

### Appendix D: DERIVATION OF THE MANIFESTLY GAUGE-IN Variant PERTURBATION EQUATIONS

In this appendix we derive the perturbation equations (204) and the evolution equations (209).

#### 1. Derivation of the evolution equation for the entropy

With the help of Eqs. (202a) and (202b) and Eqs. (170)–(172) one may verify that

$$
\frac{1}{c} \frac{d}{dt} \left( n_{(1)} - \frac{n_{(0)}}{\varepsilon_{(0)}(1 + w)} \varepsilon_{(1)} \right) = -3H \left( 1 - \frac{n_{(0)} p_n}{\varepsilon_{(0)}(1 + w)} \right) \left( n_{(1)} - \frac{n_{(0)}}{\varepsilon_{(0)}(1 + w)} \varepsilon_{(1)} \right).
$$

$$(D1)$$

In view of Eq. (185) one may replace $\varepsilon_{(1)}$ and $n_{(1)}$ by $\varepsilon_{(1)}^g$ and $n_{(1)}^g$. Using Eq. (189) yields Eq. (204b) with coefficient (205d) of the main text.

#### 2. Derivation of the evolution equation for the energy density perturbation

We will now derive Eq. (204a). To that end, we rewrite the system (202) and Eq. (203a) in the form

$$
\dot{\varepsilon}_{(1)} + \alpha_{11} \varepsilon_{(1)} + \alpha_{12} n_{(1)} + \alpha_{13} \dot{\varepsilon}_{(1)} + \alpha_{14} 3\dot{R}_{(1)} = 0,
$$

$$(D2a)$$

$$
\dot{n}_{(1)} + \alpha_{21} \varepsilon_{(1)} + \alpha_{22} n_{(1)} + \alpha_{23} \dot{\varepsilon}_{(1)} + \alpha_{24} 3\dot{R}_{(1)} = 0,
$$

$$(D2b)$$

$$
\dot{\varepsilon}_{(1)} + \alpha_{31} \varepsilon_{(1)} + \alpha_{32} n_{(1)} + \alpha_{33} \dot{\varepsilon}_{(1)} + \alpha_{34} 3\dot{R}_{(1)} = 0,
$$

$$(D2c)$$

$$
3\dot{R}_{(1)} + \alpha_{41} \varepsilon_{(1)} + \alpha_{42} n_{(1)} + \alpha_{43} \dot{\varepsilon}_{(1)} + \alpha_{44} 3\dot{R}_{(1)} = 0,
$$

$$(D2d)$$

$$
\varepsilon_{(1)}^g + \alpha_{51} \varepsilon_{(1)} + \alpha_{52} n_{(1)} + \alpha_{53} \dot{\varepsilon}_{(1)} + \alpha_{54} 3\dot{R}_{(1)} = 0.
$$

$$(D2e)$$

### Table I: The coefficients $\alpha_{ij}$ figuring in the equations D2.

| $\alpha_{ij}$ | Equation Form |
|---------------|--------------|
| $\alpha_{11}$ | $3H(1 + p_c) + \frac{\kappa \varepsilon_{(0)}(1 + w)}{2H}$ |
| $\alpha_{12}$ | $3Hp_n$ |
| $\alpha_{13}$ | $\varepsilon_{(0)}(1 + w)$ |
| $\alpha_{14}$ | $\frac{\varepsilon_{(0)}(1 + w)}{4H}$ |
| $\alpha_{21}$ | $\frac{\kappa n_{(0)}}{2H}$ |
| $\alpha_{22}$ | $3H$ |
| $\alpha_{23}$ | $n_{(0)}$ |
| $\alpha_{24}$ | $\frac{n_{(0)}}{4H}$ |
| $\alpha_{31}$ | $\frac{p_c}{\varepsilon_{(0)}(1 + w)} \frac{\varepsilon_{(0)}(1 + w) a^2}{a^2}$ |
| $\alpha_{32}$ | $\frac{p_n}{\varepsilon_{(0)}(1 + w)} \frac{\varepsilon_{(0)}(1 + w) a^2}{a^2}$ |
| $\alpha_{33}$ | $H(2 - 3\beta^2)$ |
| $\alpha_{34}$ | $0$ |
| $\alpha_{41}$ | $\frac{3R_{(0)}}{3H}$ |
| $\alpha_{42}$ | $0$ |
| $\alpha_{43}$ | $-2\varepsilon_{(0)}(1 + w)$ |
| $\alpha_{44}$ | $2H + \frac{3R_{(0)}}{6H}$ |
| $\alpha_{51}$ | $-3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)$ |
| $\alpha_{52}$ | $0$ |
| $\alpha_{53}$ | $\frac{6\varepsilon_{(0)}(1 + w)}{R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)}$ |
| $\alpha_{54}$ | $\frac{3\varepsilon_{(0)}(1 + w)}{3R_{(0)} + 3\kappa \varepsilon_{(0)}(1 + w)}$ |
where the coefficients $\alpha_{ij}(t)$ are given in Table I.

In calculating the coefficients $a_1$, $a_2$ and $a_3$, (205) in the main text, we use that the time derivative of the quotient $w$, defined by Eq. (172) is given by

$$\dot{w} = 3H(1+w)(w-\beta^2),$$  \hfill (D3)

as follows from Eqs. (170b) and (175). Moreover, it is of convenience not to expand the function $\beta(t)$ defined by Eq. (175) since this will complicate substantially the expressions for the coefficients $a_1$, $a_2$ and $a_3$.

**Step 1.** We first eliminate the quantity $3R_{(3)}$ from Eqs. (D2). Differentiating Eq. (D2e) with respect to time and eliminating the time derivatives $\dot{\varepsilon}_{(1)}$, $\dot{n}_{(1)}$, $\dot{\vartheta}_{(1)}$ and $3\dot{R}_{(1)}$ with the help of Eqs. (D2a)–(D2d), we arrive at the equation

$$\dot{\varepsilon}_{(1)} + p_1\dot{\varepsilon}_{(1)} + p_2n_{(1)} + p_3\dot{\vartheta}_{(1)} + p_43\dot{R}_{(1)} = 0,$$  \hfill (D4)

where the coefficients $p_1(t), \ldots, p_4(t)$ are given by

$$p_i = \alpha_{5i} - \alpha_{51}\alpha_{1i} - \alpha_{52}\alpha_{2i} - \alpha_{53}\alpha_{3i} - \alpha_{54}\alpha_{4i}. \hfill (D5)$$

From Eq. (D4) it follows that

$$3\dot{R}_{(1)} = -\frac{1}{p_4}\dot{\varepsilon}_{(1)} - \frac{p_1}{p_4}\varepsilon_{(1)} - \frac{p_2}{p_4}n_{(1)} - \frac{p_3}{p_4}\dot{\vartheta}_{(1)}. \hfill (D6)$$

In this way we have expressed the quantity $3\dot{R}_{(1)}$ as a linear combination of the quantities $\dot{\varepsilon}_{(1)}$, $\varepsilon_{(1)}$, $n_{(1)}$ and $\dot{\vartheta}_{(1)}$. Upon replacing $3\dot{R}_{(1)}$ given by (D6), in Eqs. (D2), we arrive at the system of equations

$$\dot{\varepsilon}_{(1)} + q_1\dot{\varepsilon}_{(1)} + \beta_{11}\varepsilon_{(1)} + \beta_{12}n_{(1)} + \beta_{13}\dot{\vartheta}_{(1)} = 0,$$  \hfill (D7a)

$$\dot{n}_{(1)} + q_2\dot{\varepsilon}_{(1)} + \beta_{21}\varepsilon_{(1)} + \beta_{22}n_{(1)} + \beta_{23}\dot{\vartheta}_{(1)} = 0,$$  \hfill (D7b)

$$\dot{\vartheta}_{(1)} + q_3\dot{\varepsilon}_{(1)} + \beta_{31}\varepsilon_{(1)} + \beta_{32}n_{(1)} + \beta_{33}\dot{\vartheta}_{(1)} = 0,$$  \hfill (D7c)

$$3\dot{R}_{(1)} + q_4\dot{\varepsilon}_{(1)} + \beta_{41}\varepsilon_{(1)} + \beta_{42}n_{(1)} + \beta_{43}\dot{\vartheta}_{(1)} = 0,$$  \hfill (D7d)

$$\dot{\varepsilon}_{(1)} + q_5\dot{\varepsilon}_{(1)} + \beta_{51}\varepsilon_{(1)} + \beta_{52}n_{(1)} + \beta_{53}\dot{\vartheta}_{(1)} = 0,$$ \hfill (D7e)

where the coefficients $q_i(t)$ and $\beta_{ij}(t)$ are given by

$$q_i = -\frac{\alpha_{i4}}{p_4}, \quad \beta_{ij} = \alpha_{ij} + q_ip_j. \hfill (D8)$$

We now have achieved that the quantity $3\dot{R}_{(1)}$ occurs only in Eq. (D7d). Since we are not interested in the non-physical quantity $3\dot{R}_{(1)}$, we do not need this equation any more.

**Step 2.** We now proceed in the same way as in step 1: eliminating this time the quantity $\dot{\vartheta}_{(1)}$ from the system of equations (D7). Differentiating Eq. (D7e) with respect to time and eliminating the time derivatives $\dot{\varepsilon}_{(1)}$, $\dot{n}_{(1)}$ and $\dot{\vartheta}_{(1)}$ with the help of Eqs. (D7a)–(D7c), we arrive at

$$q_5\dot{\varepsilon}_{(1)} + r\dot{\varepsilon}_{(1)} + s_1\varepsilon_{(1)} + s_2n_{(1)} + s_3\dot{\vartheta}_{(1)} = 0,$$ \hfill (D9)

where the coefficients $r(t)$ and $s_i(t)$ are given by

$$s_i = \beta_{5i} - \beta_{52}\beta_{2i} - \beta_{53}\beta_{3i}, \hfill (D10a)$$

$$r = 1 + q_5 - \beta_{52}q_1 - \beta_{53}q_2 - \beta_{53}q_3. \hfill (D10b)$$

From Eq. (D9) it follows that

$$\dot{\vartheta}_{(1)} = -\frac{q_5}{s_3}\dot{\varepsilon}_{(1)} - \frac{r}{s_3}\dot{\varepsilon}_{(1)} - \frac{s_1}{s_3}\varepsilon_{(1)} - \frac{s_2}{s_3}n_{(1)}.$$ \hfill (D11)
In this way we have expressed the quantity \( \vartheta \), as a linear combination of the quantities \( \varepsilon_{i(1)}, \varepsilon_{i(2)}, \varepsilon_{i(3)} \), and \( n_{(i)} \). Upon replacing \( \vartheta \) given by (D11) in Eqs. (D7), we arrive at the system of equations

\[
\begin{align*}
\dot{\varepsilon}_{(1)} &- \beta_{13} \frac{q_5}{s_3} \varepsilon_{i(1)} + \left( q_1 - \beta_{13} \frac{r}{s_3} \right) \varepsilon_{i(1)} + \left( \beta_{11} - \beta_{13} \frac{s_1}{s_3} \right) \varepsilon_{(i)} + \left( \beta_{12} - \beta_{13} \frac{s_2}{s_3} \right) n_{(i)} = 0, \\
\dot{n}_{(1)} &- \beta_{23} \frac{q_5}{s_3} \varepsilon_{i(1)} + \left( q_2 - \beta_{23} \frac{r}{s_3} \right) \varepsilon_{i(1)} + \left( \beta_{21} - \beta_{23} \frac{s_1}{s_3} \right) \varepsilon_{(i)} + \left( \beta_{22} - \beta_{23} \frac{s_2}{s_3} \right) n_{(i)} = 0, \\
\dot{\varepsilon}_{(1)} &- \beta_{33} \frac{q_5}{s_3} \varepsilon_{i(1)} + \left( q_3 - \beta_{33} \frac{r}{s_3} \right) \varepsilon_{i(1)} + \left( \beta_{31} - \beta_{33} \frac{s_1}{s_3} \right) \varepsilon_{(i)} + \left( \beta_{32} - \beta_{33} \frac{s_2}{s_3} \right) n_{(i)} = 0, \\
3\dot{R}_{(1)} &- \beta_{43} \frac{q_5}{s_3} \varepsilon_{i(1)} + \left( q_4 - \beta_{43} \frac{r}{s_3} \right) \varepsilon_{i(1)} + \left( \beta_{41} - \beta_{43} \frac{s_1}{s_3} \right) \varepsilon_{(i)} + \left( \beta_{42} - \beta_{43} \frac{s_2}{s_3} \right) n_{(i)} = 0, \\
\varepsilon_{i(1)} &- \beta_{53} \frac{q_5}{s_3} \varepsilon_{i(1)} + \left( q_5 - \beta_{53} \frac{r}{s_3} \right) \varepsilon_{i(1)} + \left( \beta_{51} - \beta_{53} \frac{s_1}{s_3} \right) \varepsilon_{(i)} + \left( \beta_{52} - \beta_{53} \frac{s_2}{s_3} \right) n_{(i)} = 0. 
\end{align*}
\]

We have achieved now that the quantities \( \vartheta_{(1)} \) and \( 3\vartheta_{(1)} \) occur only in Eqs. (D12c) and (D12d), so that these equations will not be needed anymore. We are left, in principle, with Eqs. (D12a), (D12b) and (D12e) for the three unknown quantities \( \varepsilon_{(1)}, n_{(1)} \), and \( \varepsilon_{i(1)} \), but we first proceed with all five equations.

**Step 3.** At first sight, the next steps would be to eliminate, successively, the quantities \( \varepsilon_{(1)} \) and \( n_{(1)} \) from Eqs. (D12a) with the help of Eqs. (D12a) and (D12b). We then would end up with a fourth order differential equation for the unknown quantity \( \varepsilon_{i(1)} \).

However, it is possible to extract a second order equation for the gauge-invariant energy density from the equations (D12). This will now be shown. Eq. (D12c) can be rewritten

\[
\varepsilon_{i(1)} + a_1 \varepsilon_{i(1)} + a_2 \varepsilon_{i(1)} = a_3 \left( n_{(1)} + \frac{\beta_{51} s_4 - \beta_{53} s_1}{\beta_{52} s_3 - \beta_{53} s_2} \varepsilon_{(1)} \right),
\]

where the coefficients \( a_1(t), a_2(t) \) and \( a_3(t) \) are given by

\[
\begin{align*}
a_1 &= -\frac{s_3}{\beta_{53}} + \frac{r}{q_5}, \\
a_2 &= -\frac{s_3}{\beta_{53} q_5}, \\
a_3 &= \frac{\beta_{52} s_3 - \beta_{53} s_2}{\beta_{53} q_5}. 
\end{align*}
\]

These are precisely the coefficients (205a)–(205c) of the main text. Furthermore, we find

\[
\frac{\beta_{51} s_4 - \beta_{53} s_1}{\beta_{52} s_3 - \beta_{53} s_2} = -\frac{n_{(0)}}{\varepsilon_{(0)}(1 + w)}.
\]

Hence,

\[
n_{(1)} + \frac{\beta_{51} s_3 - \beta_{53} s_1}{\beta_{52} s_3 - \beta_{53} s_2} \varepsilon_{(1)} = n_{(1)} - \frac{n_{(0)}}{\varepsilon_{(0)}(1 + w)} \varepsilon_{(1)}.
\]

With the help of this expression and Eq. (185) we can rewrite Eq. (D13) in the form (204a).

The derivation of the expressions (205) from (D14) and the proof of the equality (D15) is straightforward, but extremely complicated. We used MAPLE V [22] to perform this algebraic task.

### 3. Evolution equations for the contrast functions

In this section we derive Eqs. (209). We start off with Eq. (209b). From Eq. (189) and the definitions (208) it follows that

\[
\sigma_{i(1)} = n_{(0)} \left( \delta_n - \frac{\delta_{i}}{1 + w} \right).
\]


Differentiating this equation with respect to $ct$ yields
\[ a_4 \sigma_{(1)}^i = \dot{n}_{(0)} \left( \delta_n - \frac{\delta_c}{1 + w} \right) + n_{(0)} \frac{1}{c} \frac{d}{dt} \left( \delta_n - \frac{\delta_c}{1 + w} \right), \] (D18)
where we have used Eq. (204b). Using Eqs. (170c), (205d) and (D17) to eliminate $\sigma_{(1)}^i$, we arrive at Eq. (209b) of the main text.

Finally, we derive Eq. (209a). Upon substituting the expressions
\[ \varepsilon_{(1)}^i = \varepsilon_{(0)} \delta_c, \] (D19a)
\[ \varepsilon_{(1)}^c = \dot{\varepsilon}_{(0)} \delta_x + \varepsilon_{(0)} \dot{\delta}_x, \] (D19b)
\[ \varepsilon_{(1)}^{a3} = \ddot{\varepsilon}_{(0)} \delta_x + 2 \dot{\varepsilon}_{(0)} \ddot{\delta}_x + \varepsilon_{(0)} \dddot{\delta}_x, \] (D19c)
into Eq. (204a), and dividing by $\varepsilon_{(0)}$, we find
\[ b_1 = 2 \frac{\dot{\varepsilon}_{(0)}}{\varepsilon_{(0)}} + a_1, \] (D20a)
\[ b_2 = \frac{\varepsilon_{(0)}}{\varepsilon_{(0)}} + a_2, \] (D20b)
\[ b_3 = a_3 \frac{n_{(0)}}{\varepsilon_{(0)}}, \] (D20c)
where we have also used Eq. (D17). These are the coefficients (210) of the main text.

**Appendix E: DERIVATION OF THE STANDARD EQUATION FOR DENSITY PERTURBATIONS**

In this appendix equations (277) and (283) of the main text are derived, for a flat FLRW universe, $\vartheta_{(0)} = 0$, with vanishing cosmological constant, $\Lambda = 0$, using the background equations (170)–(171) and the linearized Einstein equations and conservation laws for scalar perturbations (202).

From (D3) it follows that $w$ is constant if and only if $w = \beta^2$ for all times. Using (194) it is found for constant $w$ that $p_n = 0$ and $p_x = w$, i.e., the pressure does not depend on the particle number density. Consequently, in the derivation of equations (277) and (283) the equations (170c) for $n_{(0)}(t)$ and (202b) for $n_{(1)}(t, x)$ are not needed. In this case, the equation of state is given by
\[ p = w \varepsilon. \] (E1)

To derive the standard equations (277) and (283), it is required that $u^i_{(1)\parallel} = 0$, implying that
\[ \vartheta_{(1)}(t, x) = 0. \] (E2)

Since $\nabla^2 p_{(1)} = 0$ for large-scale perturbations, equation (202c) is identically satisfied. After substituting $\varepsilon_{(1)} = \varepsilon_{(0)} \delta$ into equation (202a) and eliminating $\dot{\varepsilon}_{(0)}$ with the help of equation (170b), it is found that
\[ \dot{\delta} + \frac{1 + w}{2H} (\kappa \varepsilon_{(0)} \delta + \frac{1}{2} \vartheta_{(1)}^i) = 0. \] (E3)

Differentiating equation (E3) with respect to $x^0 = ct$ and using equations (170a), (170b), (171) and (202d), yields
\[ \ddot{\delta} + \frac{1}{2H} (1 + w) H \dot{\delta} - \frac{1}{4} (1 + w)^2 \kappa \varepsilon_{(0)} \delta - \frac{1}{4} (1 + w) (1 - 3w) \dot{\vartheta}_{(1)}^i = 0. \] (E4)
Eliminating $\vartheta_{(1)}^i$ from equation (E4) with the help of equation (E3), yields the standard equation for large-scale perturbations in a flat FLRW universe:
\[ \ddot{\delta} + 2H \dot{\delta} - \frac{1}{2} \kappa \varepsilon_{(0)} \delta (1 + w) (1 + 3w) = 0. \] (E5)

This equation has been derived by Weinberg [14], Eq. (15.10.57) and Peebles [19], Eq. (86.11). For $w = \frac{1}{3}$ (the radiation-dominated era) equation (277) is found, whereas for $w = 0$ (the matter-dominated era) equation (283) applies.
Using that the general solution of the background equations (170)–(171) for $3R_{(0)} = 0$, $\Lambda = 0$ and constant $w$ is given by

$$H(t) = \frac{2}{3(1 + w)}(ct)^{-1} = H(t_0)\left(\frac{t}{t_0}\right)^{-1},$$

(E6a)

$$\varepsilon_{(0)}(t) = \frac{4}{3\kappa(1 + w)^2}(ct)^{-2} = \varepsilon_{(0)}(t_0)\left(\frac{t}{t_0}\right)^{-2},$$

(E6b)

we find for the general solution of Eq. (E5)

$$\delta(r, q) = E_1(q)\tau^{(2+6w)/(3+3w)} - 3(1 + w)H(t_0)\psi(q)\tau^{-1},$$

(E7)

where $E_1(q)$ is an arbitrary function (the integration ‘constant’) and $\psi(q)$ is the gauge function [see (278) with $w = \frac{1}{3}$ or (284) with $w = 0$]. The solution (E7) is exactly equal to the result found by Peebles [19, §86, Eq. (86.12)].

Finally, we note that in the derivation of (277) Peebles, §86 uses Eq. (E2) (in his notation: $\theta = 0$). In this case, Peebles’ approach yields a physical mode $\delta \propto \tau$ and a gauge mode $\delta \propto \tau^{-1}$. For $\vartheta_{(1)} \neq 0$ Peebles finds a physical mode $\delta \propto \tau^{1/2}$. However, in the approach presented in this article both physical modes $\delta \propto \tau$ and $\delta \propto \tau^{1/2}$ [see Eq. (247)] follow from one second-order differential equation (235a), without taking an explicit value for $\vartheta_{(1)}$.

### Appendix F: SYMBOLS AND THEIR MEANING

Table II: Symbols and their meaning of all quantities, except for those occurring in the appendices.

| Symbol | Meaning | Reference Equation |
|--------|---------|--------------------|
| $\nabla$ | $(\partial_1, \partial_2, \partial_3)$ | — |
| $\nabla^2 f$ | $\nabla \cdot (\nabla f) = \delta^{ij} f_{,ij}$ | (28) |
| $(\nabla f)^i$ | $\bar{g}^{ij} f_{ij}$ | — |
| $\nabla \cdot v$ | $v^k|^{(k)} = \epsilon^{ijk} u_{(1)jk}$ | (142), (143) |
| $(\nabla \wedge u_{(1)}i)_i$ | $\epsilon^{ijk} u_{(1)jk} = \epsilon^{ijk} u_{(1)jk}$ | (145) |
| $\nabla^2 f$ | $\nabla \cdot (\nabla f = \bar{g}^{ij} f_{ij})$ | (167) |
| $\partial_i$ | derivative with respect to $x^i = \tau$ | — |
| $\hat{x}$ | computed with respect to $\hat{x}$ | (1) |
| dot: $\cdot$ | derivative with respect to $x^0 = \tau$ | (31) |
| tilde: $\tilde{\cdot}$ | computed with respect to three-metric $\tilde{g}_{ij}$ | (54b), (71) |
| superindex: $^g$ | gauge-invariant | (2) |
| superindex: $^{|k}$ | contravariant derivative with respect to $x^k$: $\zeta^{[k} = \delta_{(0)}^{ij} \zeta_{ij}$ | (125a) |
| subindex: $(0)$ | background quantity | (24) |
| subindex: $(1)$ | perturbation of first order | (24) |
| subindex: $(2)$ | perturbation of second order | (55) – (56) |
| subindex: $\lambda$ | covariant derivative with respect to $x^\lambda$ | (9) |
| subindex: $|k$ | covariant derivative with respect to $x^k$ | (36) |
| subindex: $|^\mu$ | derivative with respect to $x^\mu$ | (32) |
| subindex: $\parallel$ | longitudinal part of a vector or tensor | (124) |
| subindex: $\perp$ | perpendicular part of a vector or tensor | (124) |
| subindex: $*$ | transverse and traceless part of a tensor | (124) |
| $\beta$ | $c^{-1}$ times speed of sound: $\sqrt{f_{(0)}/\varepsilon_{(0)}}$ | (175) |
| $\Gamma^\lambda_{\rho\nu}$ | connection coefficients | (32) |
| $\gamma$ | arbitrary function | (127a) |
| $\delta_{\rho\nu}$ | energy contrast function (gauge-invariant) | (208) |
| $\delta_{\rho\nu}^{\parallel}$ | Kronecker delta | (60) |
| $\delta_{\rho\nu}^{\parallel}$ | particle number contrast function (gauge-invariant) | (208) |
| $\delta_T$ | temperature contrast function (gauge-invariant) | (212) |
| $\epsilon$ | energy density | (22a) |
| $\epsilon_{s}^{jk}$ | Levi-Civita tensor, $\epsilon_{s}^{23} = +1$ | (145) |
| $\zeta, \phi$ | potentials due to relativistic density perturbations | (125a), (214) |
| $\eta$ | bookkeeping parameter equal to 1 | (55) – (56) |
| $\theta$ | expansion scalar in four-space | (22c) |
| $\vartheta$ | expansion scalar in three-space | (51) |
| $\kappa$ | $8\pi G/c^4$ | (41) |
| $\kappa_{ij}$ | time derivative metric coefficients | (31) |
### Table II: (continued)

| Symbol | Definition                                                                 | Page |
|--------|---------------------------------------------------------------------------|------|
| $\Lambda$ | cosmological constant                                                  | (39) |
| $\lambda$ | wavelength                                                              | (237) |
| $\mu$ | thermodynamic potential                                                  | (178) |
| $\mu_m$ | reduced wave length (matter)                                             | (267) |
| $\mu_r$ | reduced wave length (radiation)                                          | (243) |
| $\xi^\mu$ | first order space-time translation                                        | (3)  |
| $\pi$ | arbitrary function                                                      | (127a) |
| $g_{(1)}$ | $\varepsilon_{(1)}^{\mu}/c^2$                                          |      |
| $\sigma$ | arbitrary scalar                                                        | (12a) |
| $\sigma_{(1)}$ | abbreviation for $n_{(1)}^{\mu} - n_{(0)}^{\mu}\varepsilon_{(1)}^{\mu}/[\varepsilon_{(0)}(1 + w)]$ | (188)–(189), (207) |
| $\tau$ | dimensionless time                                                       | (240) |
| $\phi, \xi$ | potentials due to relativistic density perturbations                    | (125a), (214) |
| $\varphi^\mu$ | Newtonian potential                                                      | (28), (229) |
| $\chi^\mu$ | arbitrary three-vector                                                  | (82)  |
| $\psi$ | first order time translation (gauge function)                           | (1), (3), (5) |
| $\omega$ | arbitrary scalar                                                        | (20)  |
| $A^{\alpha...\beta...\mu...\nu}$ | arbitrary tensor                                                      | (9)   |
| $A_{\mu\nu}$ | arbitrary rank two tensor                                               | (12c) |
| $a$ | scale factor or radius of universe                                      | (54b) |
| $a_1, a_2, a_3, a_4$ | coefficients in perturbation equations                                  | (205) |
| $b_1, b_2, b_3$ | coefficients in perturbation equations                                  | (210) |
| $C$ | arbitrary first order constant                                           | (C1)  |
| $c$ | speed of light                                                          |       |
| $ds^2$ | line element in four-space                                              | (29)  |
| $E$ | energy within volume $V$                                                | (178) |
| $G$ | Newton’s gravitation constant                                            |       |
| $g_{\mu\nu}$ | metric tensor                                                          | (13)  |
| $\tilde{g}_{\mu\nu}$ | metric with respect to $\hat{\xi}^\mu$                                | (13)  |
| $g_{ij}$ | time-independent metric of three-space                                  | (54b) |
| $H$ | $c^{-1}\mathcal{H}$                                                    | (62)  |
| $\mathcal{H}$ | Hubble function: $\mathcal{H} = (da/dt)/a$                             |       |
| $h_{ij}$ | minus first order perturbation of metric                                | (82)  |
| $J_{\mu\nu}$ | Bessel function of the first kind                                       | (274), (280) |
| $K$ | constant relating Ricci and metric tensors                              | (67)  |
| $k$ | $k = -1$ (open), 0 (flat), +1 (closed)                                  | (72)  |
| $k_B$ | Boltzmann’s constant                                                    | (250) |
| $L_\xi$ | Lie derivative with respect to $\xi^\mu$                               | (9)   |
| $m$ | rest mass of particle of cosmological fluid                            | (222) |
| $m_H$ | proton rest mass                                                       | (250) |
| $N$ | number of particles within volume $V$                                   | (178) |
| $N^\mu$ | particle density four-flow                                              | (23)  |
| $n$ | particle number density                                                | (22b) |
| $p$ | pressure                                                                | (48)  |
| $p_e, p_n$ | partial derivatives of pressure                                        | (91)  |
| $p_{en}, p_{nn}$ | partial derivatives of pressure                                        | (206) |
| $q = |q|$ | magnitude of wave vector: $2\pi/\lambda$                               | (237) |
| $q$ | wave vector                                                            | (237) |
| $3R$ | Ricci scalar in three-space                                             | (46)  |
| $R_{\mu\nu}$ | Ricci tensor in four-space                                             | (34)  |
| $3R_{ij}$ | Ricci tensor in three-space                                             | (37)  |
| $S$ | entropy within a volume $V$                                             | (178) |
| $s$ | entropy per particle $s := S/N$                                         | (179) |
| $T$ | absolute temperature                                                   | (178) |
| $T_{\mu\nu}$ | energy momentum tensor                                                | (47)  |
| $t$ | cosmological time                                                       |       |
| $t_0$ | initial cosmological time                                               |       |
| $t_p$ | present cosmological time                                               |       |
| $U_{\mu}$ | cosmological four-velocity $U^\mu U_\mu = c^2$                         | (22)  |
| $u^\mu$ | $c^{-1}U^\mu$                                                          | (221) |
| $\mathcal{U}$ | spatial velocity                                                       | (141) |
| $V_{\mu}$ | arbitrary four-vector                                                  | (12b) |
| $w$ | pressure divided by energy                                              | (172) |
| $x$ | space-time point $x^\mu = (ct, \mathbf{x})$                           |       |
| $x^\mu$ | space-time point $\mathbf{x} = (x^1, x^2, x^3)$                       | (3)   |
| $Y_{\pm\nu}$ | Bessel function of the second kind                                      | (274) |
Table II: (continued)

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