Effects of Dynamical Compactification on $D$-Dimensional Gauss-Bonnet FRW Cosmology

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Abstract

We examine the effect on cosmological evolution of adding a string motivated Gauss-Bonnet term to the traditional Einstein-Hilbert action for a $(1 + 3) + d$ dimensional Friedman-Robertson-Walker (FRW) metric. By assuming that the additional dimensions compactify as the usual 3 spatial dimensions expand, we find that the Gauss Bonnet terms give perturbative corrections to the FRW equations. We find corrections that appear in the calculation of both the Hubble constant, $H_0$, and the acceleration parameter, $q_0$, for a variety of cases that are consistent with a dark energy equation of state.

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I. INTRODUCTION

During the last decade observational cosmology has advanced to the point of placing a number of constraints on our theoretical understanding of the universe. Observations of Type Ia Supernovae (SNe Ia) have become a critical probe of the expansion history of the universe \[1\] and recent measurements of the cosmic microwave background indicate an accelerated expansion of the universe occurred in an early inflationary epoch \[2\]. Current efforts to understand these observations within a mathematically consistent framework often start with the ten or eleven dimensional spacetime manifolds motivated by superstring or M theory \[3\] \[4\]. These theories often require a compactification phase to achieve the four dimensional spacetime manifold associated with the current universe and a means for describing the six dimensional manifold in terms of current observational constraints, i.e. they are not obviously visible. The observations indicate that the expansion of the universe is accelerating, at critical density, has a nonzero cosmological constant and is flat. These observations also estimate that some 70% of the universe is composed of dark energy, another 25% dark matter and the rest baryonic matter. The dark energy is often characterized by a negative pressure equation of state \( p = w \rho \) where the equation of state parameter for the exotic matter, \( w \), lies near -1. One candidate for the exotic matter is a type of dark energy associated with a cosmological constant arising from zero point vacuum energy fluctuations \[5\] \[6\] that give a constant value of \( w = -1 \) or quintessence arising from a time varying scalar field which gives a dark energy that decreases with time \(-1 < w < -1/3\) where dark energy density decreases with scale factor \( a(t) \) as \( \rho \propto a(-3(1+w)) \) or phantom energy corresponding to cases of exotic matter with \( w < -1 \) \[7\] \[8\].

An understanding of dark energy can be searched for in string/M-theory and the resulting generalizations of low energy gravity that are motivated by such investigations. It is possible to utilize unusual couplings to matter fields \[9\] or a dilaton field \[10\], or coupling constants \[11\] extra dimensions \[12\] or Kaluza Klein compactifications \[14\], or a time varying cosmological constant \[15\] or dimensional reductions \[16\] or corrections to the pure gravity sector of the theory through interesting functions of the curvature \[17\] or Gauss-Bonnet motivated couplings \[18\] or couplings to branes \[19\]. The use of phantom fluids where the pressure and energy density are related by an equation of state given by \( p < -\rho \); which for a scalar field can be achieved by using a negative kinetic energy term in the Lagrangian density, is another promising avenue of development \[20\] \[21\]. One demands that such theories give reasonable descriptions of the large scale four dimensional universe described by the Friedmann-Roberston-Walker (FRW) equations arrived at from the standard Einstein field equations. An insightful way to approach the dark energy problem is to utilize modifications of the Einstein field equations; to search for ways to accommodate the dark energy observations within the context of higher dimensional or higher order modifications to the theory or to use noncommutative geometries and Casimir \[22\] energy or by equivalent scalar-tensor theories or moduli fields \[23\]. Nojiri and Carrol examine positive and negative powers of the scalar curvature in order to describe a stable inflationary phase and an accelerating late phase in geometric terms \[24\] \[25\] \[26\]. Within the framework of General Relativity the cosmic acceleration can be accommodated for with a phantom equation of state. Lovelock modifications consist of dimensionally extended Euler densities which include higher order curvature invariants in such a way that they lead to second order gravitational field equations but are always total derivatives in four dimensions \[27\]. The quadratic curvature terms are referred to as Gauss-Bonnet terms which come about in the low energy limit of heterotic
Here we consider a dynamic compactification of a \((4+d)\)-dimensional manifold to a maximally symmetric manifold of dimension \(d\) and an expanding FRW space of dimension 4. We modify the gravity Lagrangian to first order in string tension \(\alpha\) by a Gauss-Bonnet term that is quadratic in the curvature\([30]\). Here we have considered these terms explicitly coupled to matter fields obeying some equation of state with the strength of each term treated as a parameter to be fixed by observations. It is known that when the internal space is a non-singular time independent compact manifold without boundary then there is no accelerated expansion unless the strong energy condition is violated and if the higher dimensional stress energy tensor obeys the strong energy condition than so does the lower dimensional stress energy tensor \([31]\). Here we will consider the time dependent solutions of arbitrary dimension on a maximally symmetric manifold thereby avoiding the time independent nature of the compact space. In this paper we will investigate the Einstein-Gauss-Bonnet field equations on a manifold \(M^n\) that undergoes dynamical compactification to \(M^n = (R^1 \times M^3) \times S^d\). We will include matter by specifying an equation of state and applying the constraint equation on the energy momentum tensor \(\partial_\mu T^{\mu \nu} = 0\). Within this model we will extract the Friedmann-Robertson-Walker equations with the Gauss-Bonnet terms as correction terms to the standard 4-d equations. These equations are then solved pertubatively and analyzed within the context of the observational constraints and an equation of state. To understand the impact of higher order curvature terms on the observed cosmic evolution we will solve the Gauss-Bonnet corrected FRW equations that undergo a dynamical \(d\)-dimensional compactification to a maximally symmetric space. This paper is organized as follows. In Section II we present a general Einstein plus Gauss-Bonnet action in \(d\)-dimensions where the field equations are calculated and the correction to the FRW equation is given. In section III we prepare a perturbative solution for special cases including with and without the cosmological constant and calculate the corrected values of the Hubble constant and the deceleration parameter. In Section IV we summarize our results.

II. FRAMEWORK FOR FIELD EQUATIONS

In this paper we will follow the notation of Paul and Mukherjee\([13]\) and Mohmmedi \([14]\) to express the Einstein-Hilbert action with an extra Gauss-Bonnet term as

\[
S = \int d^Dx \sqrt{-g} [R - 2\lambda - \epsilon \mathcal{G}] \tag{2.1}
\]

where \(\mathcal{G}\) is a Gauss-Bonnet term \((\mathcal{G} \equiv R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2)\). A variation of the action \((2.1)\) with respect to \(g_{AB}\) produces the equation;

\[
G_A^C + \lambda g_A^C + \epsilon \mathcal{G}_A^C = \frac{1}{2\Upsilon} T_A^C \tag{2.2}
\]

where \(\Upsilon\) is the \(4 + d\) dimensional coupling constant and the Einstein and Gauss-Bonnet tensors, respectively, are

\[
G_A^C = R_A^C - \frac{1}{2} g_A^C R
\]

\[
\mathcal{G}_A^C = \frac{1}{2} \left( R_{BDEF} R^{BDEF} - 4 R_{BD} R^{BD} + R^2 \right) \delta_A^C - (2 R_{BDEA} R^{BDEC} + 2 R R_A^C - 4 R_B^D R_{BA}^{DC} - 4 R_B^C R_A^B). \tag{2.4}
\]
Furthermore we will assume that the stress-energy tensor will be that of a perfect fluid, thus it is of the form

\[ T_{\mu \nu} = \text{diag} [\rho(t), p(t), p(t), p(t), p_d(t), ... p_d(t)] \] (2.5)

where \( p_d(t) \) is the pressure of the higher dimensional compact manifold.

We will choose a metric ansatz:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] + b^2(t) \gamma_{mn}(y)dy^m dy^n \] (2.6)

where the extra dimension is defined to be maximally symmetric such that the Riemann tensor for \( \gamma \) has the form \( R_{abcd} = \kappa(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}) \). In agreement with current observations we will consider the usual 3 spatial dimensions to be flat (\( K = 0 \)) and also demand that the extra dimensions be flat as well (\( \kappa = 0 \)) in agreement with the finding that one finds unphysical properties for \( \rho \) and \( p \) if \( \kappa \neq 0 \) [14].

The metric leads to Riemann tensors of the following form (where both the dimensions are flat)

\[ R_{\alpha\beta\alpha\beta} = a \ddot{a}, \quad R_{\alpha\beta\alpha\gamma} = a^2 \dot{a}^2, \quad R_{\alpha\delta\alpha\delta} = b \ddot{b}, \quad R_{\gamma\gamma\gamma\gamma} = a \dot{a} \dot{b}, \quad R_{\alpha\beta\beta\alpha} = b^2 \dot{b}^2 \] (2.7)

where \( a, b \) are indices which run from 1...3 and \( m, n \) are indices which are in the extra dimensions. (obviously the symmetries as well \( R_{abcd} = R_{badc} = -R_{abdc} = -R_{bacd} \)) The Ricci Tensor and Ricci Scalar are

\[ R_{00} = 3 \ddot{a}^a + \frac{1}{2} \left[ 6 + d n(d n - n - 6) \right] \dot{a}^2 a^2 \]
\[ R_{aa} = 2 \ddot{a}^2 + a \ddot{a} + d \frac{\dot{a} \dot{b}}{b}, \quad R_{mm} = 3 \frac{b \dot{b}}{a} + (d - 1) \ddot{b} + \dot{b} b \] (2.8)

\[ R = 6 \ddot{a}^a + 2 \dot{b}^b + 6d \dot{a} \dot{b} b + 6 \frac{\dot{a}^2}{a^2} + d(d - 1) \frac{\dot{b}^2}{b^2} \] (2.9)

where \( d \) is the number of extra dimensions.

The full Einstein tensor may be expressed as having an Einstein term \( G_{\mu \nu} \) and a Gauss-Bonnet term \( G_{\mu \nu} \). We make the assumption that the extra dimensions compactify as the 3 spatial dimensions expand as [14]

\[ b(t) \sim \frac{1}{a^n(t)} \] (2.10)

where \( n > 0 \) in order to insure that the scale factor of the compact manifold is both dynamical and compactifies as a function of time. With this ansatz, the non-zero elements of the Einstein tensor takes the form

\[ G_{00} = \frac{1}{2} \left[ 6 + d n(d n - n - 6) \right] \dot{a}^2 a^2 \]
\[ = \eta_1 \frac{\dot{a}^2}{a^2} \] (2.11)

\[ G_{aa} = (dn - 2) \frac{\dot{a}}{a} - \frac{1}{2} [2 + d n(d n + n - 2)] \frac{\dot{a}^2}{a^2} \]
\[ = \eta_2 \frac{\dot{a}}{a} + \eta_3 \frac{\dot{a}^2}{a^2} \] (2.12)

\[ G_{mm} = (dn - n - 3) \frac{\ddot{a}}{a} - \frac{1}{2} [6 + n(d - 1)(d n - 4)] \frac{\dot{a}^2}{a^2} \]
\[ = \eta_4 \frac{\ddot{a}}{a} + \eta_5 \frac{\dot{a}^2}{a^2}. \] (2.13)
Note that when \( d = 4 \) the equations become the well known FRW equations in four dimensions. In \( 4 + d \) dimensions, the Gauss-Bonnet terms become

\[
\mathcal{G}_0^0 = d n \left[ (d - 1) n \left[ \frac{1}{2} (d - 2) n \left[ (d - 3) n - 12 \right] + 18 \right] - 12 \right] \frac{\dot{a}^4}{a^4}
\]

\[
= \xi_1 \frac{\dot{a}^4}{a^4}
\]

\[
\mathcal{G}_a^a = \frac{1}{2} d n \left[ n (d + 1)(d - 1)(d - 2) n^3 + 4 (2 - d (d + 1)) n^2 - 4 (d - 3) n + 8 \right] \frac{\dot{a}^4}{a^4}
\]

\[
- 2 d n \left[ (d - 1) n [(d - 2) n - 6] + 6 \right] \frac{\ddot{a} \dot{a}^2}{a^3}
\]

\[
= \xi_2 \frac{\dot{a}^4}{a^4} + \xi_3 \frac{\ddot{a} \dot{a}^2}{a^3}
\]

\[
\mathcal{G}_m^m = \frac{1}{2} (d - 1) d n^2 \left[ (d - 2) n \left[ (d - 3) n - 8 \right] + 12 \right] \frac{\dot{a}^4}{a^4}
\]

\[
- 2 (d - 1) n \left[ (d - 2) n \left[ (d - 3) n - 9 \right] + 18 \right] - 12 \frac{\ddot{a} \dot{a}^2}{a^3}
\]

\[
= \xi_4 \frac{\dot{a}^4}{a^4} + \xi_5 \frac{\ddot{a} \dot{a}^2}{a^3}
\]

where the constants \( \eta_i \) and \( \xi_i \) depend upon the values of \( n \) and \( d \) as defined above. Note that if we allow \( d \to 0 \) then the Gauss-Bonnet terms vanish for \( \mathcal{G}_0^0 \) and \( \mathcal{G}_a^a \) as one would expect in four dimensions.

III. DYNAMIC COMPACTIFICATION

For the case of both spaces having flat curvature, the \( D \)-dimensional Friedmann-Robertson-Walker (FRW) equations (eqn 2.2) take the form

\[
\frac{\dot{a}}{a} \frac{d}{dt} (\rho) = \frac{3 \dot{a}^2}{a^2} - 3 \frac{\lambda}{\eta_1} + \epsilon \xi_1 \frac{\dot{a}^4}{a^4}
\]

(3.1)

\[
\frac{\dot{a}}{a} \frac{d}{dt} (p) = \left[ \eta_2 \frac{\dot{a}}{a} + \eta_3 \frac{\dot{a}^2}{a^2} + \lambda \right] + \epsilon \left( \xi_2 \frac{\dot{a}^4}{a^4} + \xi_3 \frac{\ddot{a} \dot{a}^2}{a^3} \right)
\]

(3.2)

\[
\frac{\dot{a}}{a} \frac{d}{dt} (p_d) = \left[ \eta_4 \frac{\dot{a}}{a} + \eta_5 \frac{\dot{a}^2}{a^2} + \lambda \right] + \epsilon \left( \xi_4 \frac{\dot{a}^4}{a^4} + \xi_5 \frac{\ddot{a} \dot{a}^2}{a^3} \right)
\]

(3.3)

As the scale factor of the compact manifold tends to zero with time, we will make the further assumption that the pressure in the extra \( d \) dimensions is zero (\( p_d = 0 \)). Together with these Einstein equations, we also demand that the conservation equation hold for the stress-energy tensor (\( \nabla_\mu T^\mu_\nu = 0 \)) or

\[
\left\{ \frac{d}{dt} (a^3 \rho) + p \frac{d}{dt} (a^3) \right\} + d a^3 \frac{\dot{b}}{b} (\rho + p_d) = 0
\]

(3.4)

Using the assumption that \( b = 1/a^n \), this becomes

\[
\frac{d}{dt} (a^3 \rho) + \tilde{p} \frac{d}{dt} (a^3) = 0
\]

(3.5)
which by simple algebra may be written in the more familiar form

\[ \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + \tilde{p}) = 0. \]  

(3.6)

Note that this is simply a statement that \( dE = -P \, dV \) where we have defined an “effective” pressure \( \tilde{p} \) [14] which an observer constrained to exist only upon the “usual” 3 spatial dimensions would see as

\[ \tilde{p} = p - \frac{1}{3} dn \, (\rho + p_d). \]  

(3.7)

This effective pressure can be easily computed from the \( d \)-dimensional FRW equations (3.1)-(3.3) and is given by the relation

\[ \frac{\tilde{p}}{2Y} = \frac{1}{3} \eta_4 \left[ -\left( \frac{2 \dddot{a}}{a} + \frac{\dddot{a}^2}{a^2} \right) + 3 \frac{\lambda}{\eta_1} \right] + \epsilon \left( \gamma_1 \frac{\ddot{a}^2}{a^3} + \gamma_2 \frac{\dot{a}^4}{a^4} \right) \]  

(3.8)

where we’ve defined the constants

\[ \gamma_1 = \xi_3 - \frac{1}{3} dn \, \xi_5 , \quad \gamma_2 = \xi_2 - \frac{1}{3} dn \, (\xi_1 + \xi_4). \]  

(3.9)

Note that the above field equations (3.1)-(3.3), (3.8) become the same as Mohammedi [14] in the limit where \( \epsilon \to 0 \) (ie. No Gauss-Bonnet terms). By redefining the coupling and cosmological constant in eqns. (3.1) and (3.8), one recovers standard 4-D FRW cosmology for arbitrary values of \( n \) and \( d \) in the \( \epsilon \to 0 \) limit.

A. Asymptotic solution when \( \lambda = 0 \)

We will first examine the case when the Gauss-Bonnet terms are small with respect to the Einstein terms so they may be treated as perturbations. We take the pressure in the extra dimensions, \( p_d \), to be zero so equation (3.3) becomes

\[ \frac{p_d}{2Y} = 0 = \eta_4 \frac{\dddot{a}}{a} + \eta_5 \frac{\dddot{a}^2}{a^2} + \epsilon \left( \xi_4 \frac{\ddot{a}^4}{a^4} + \xi_5 \frac{\dot{a}^2}{a^4} \right) \]  

(3.10)

This equation may be solved perturbatively by assuming that \( a(t) \simeq a_0(t) + \epsilon \, a_1(t) \). Collecting terms in \( \epsilon \), we get a zeroth order equation

\[ \frac{\dddot{a}_0}{a_0} + \frac{\eta_5}{\eta_4} \frac{\dddot{a}_0^2}{a_0^2} = 0. \]  

(3.11)

This equation yields the solution

\[ a_0(t) = (\mu t + \nu)^{\eta_4/(\eta_4 + \eta_5)} \]  

(3.12)

where \( \mu \) and \( \nu \) are constants of integration and \( \eta_4 + \eta_5 \neq 0 \).¹ This solution can be used to find an expression for the zeroth-order energy density and effective pressure. We find

\[ \frac{\rho_0}{2Y} = \frac{\mu^2}{(\eta_4 + \eta_5)^2} \cdot \frac{\eta_4^{\eta_4^2}}{\eta_4^{2(\eta_4 + \eta_5)/\eta_4}} \]  

(3.13)

\[ \frac{\tilde{p}_0}{2Y} = \frac{1}{2} \mu^2 \eta_4 \eta_5 (\eta_4 - 2\eta_5) \cdot \frac{1}{\eta_4^{2(\eta_4 + \eta_5)/\eta_4}} \]  

(3.14)

¹ \( \eta_4 + \eta_5 = 0 \) demands a complex value of \( n \) which will not be considered here.
Combining (3.13) and (3.14), we can eliminate \( a_0 \) and obtain an equation relating \( \ddot{p} \) and \( \rho \) of the form

\[
\ddot{p}_0 = -\frac{1}{3} \left( 1 - 2 \frac{\eta_5}{\eta_4} \right) \rho_0 = w \rho_0. \tag{3.15}
\]

This equation gives a constraint on the possible values of \( n \) and \( d \) by choosing a value for \( w \) and so it mimics an equation of state (EOS). At first glance it seems odd that the dynamical equations themselves impose a relation between \( p \) and \( \rho \), but this happened because of our choice that the extra dimensions compactify as \( b(t) \sim 1/a(t)^n \). Equation (3.15) may be solved for \( w \) as

\[
w = \frac{1}{3} \left( 3 + (d - 1)(dn - 3)n \right) \frac{n}{n(d - 1) + 3}. \tag{3.16}
\]

Note, in the limit where \( n \to 0 \), \( w = 1/3 \) which is the relationship one would expect for a radiation dominated universe. Thus, the geometrical terms in the compactification are playing the same role as matter. Thus, by demanding that \( w \) have a physical value; one may use this relationship to restrict the choices of \( n \) and \( d \). For instance if \( d = 7 \), then \( n \) must be less than 1/2 if \( w \) is demanded to have a physically reasonable value of between 1 and \(-2\).

The first order perturbation is of the form

\[
\ddot{a}_1(t) + \left( \frac{2 \eta_5}{\eta_4 a_0} \right) \dot{a}_1(t) + \left( \frac{3 a_0}{a_0} + 2 \frac{\eta_5}{\eta_4 a_0^2} \right) a_1(t) = -\frac{1}{\eta_4} \left( \frac{\beta^4 \xi_5}{\eta_4} + \xi_4 \frac{\eta_4}{\xi_5} \right) \frac{\mu^4}{(\mu t + \nu)^4 - \beta}. \tag{3.17}
\]

Using the solution for \( a_0 \) as found above, the solution for \( a_1 \) can be found and is of the form

\[
a_1(t) = a_{1, \text{hom}}(t) + a_{1, \text{non}}(t). \tag{3.18}
\]

The homogeneous solution is given by

\[
a_{1, \text{hom}}(t) = A(\mu t + \nu)^{\eta_4/(\eta_4 + \eta_5)} + B(\mu t + \nu)^{-\eta_5/(\eta_4 + \eta_5)} \tag{3.19}
\]

where \( A \) and \( B \) are constants of integration and \( a_{1, \text{non}} \) is the non-homogeneous solution. The resulting non-homogeneous differential equation is of the Cauchy-Euler form (if you plug in for \( a_0 \)):

\[
\ddot{a}_{1, \text{non}}(t) - \frac{2\mu(\beta - 1)}{(\mu t + \nu)} \dot{a}_{1, \text{non}}(t) + \frac{\mu^2 \beta(\beta - 1)}{(\mu t + \nu)^2} a_{1, \text{non}}(t) = \frac{\beta^4 \xi_5}{\eta_4} \frac{\eta_4}{\xi_4} \frac{\mu^4}{(\mu t + \nu)^4 - \beta}. \tag{3.20}
\]

where we’ve defined the quantity

\[
\beta = \frac{\eta_4}{(\eta_4 + \eta_5)} \tag{3.21}
\]

The non-homogeneous solution is \( a_{1, \text{non}}(t) = C(\mu t + \nu)^{\beta - 2} \) where

\[
C = \frac{\mu^2 \beta^4 \xi_5}{2 \eta_4} \left[ \frac{\eta_4}{\eta_4} - \frac{\xi_4}{\xi_5} \right]. \tag{3.22}
\]

Thus the general solution for \( a(t) \) to first order is

\[
a(t) = (\mu t + \nu)^\beta \left[ 1 + \epsilon \left( A + \frac{B}{(\mu t + \nu)} + \frac{C}{(\mu t + \nu)^2} \right) \right]. \tag{3.23}
\]
We may use this form of $a(t)$ to find the Hubble parameter and the acceleration for small $\epsilon$. We find

\[
H = \frac{\dot{a}}{a} \simeq H_0 \left[ 1 - \frac{\epsilon}{\beta} \left( \frac{B}{(\mu t + \nu)} + \frac{2C}{(\mu t + \nu)^2} \right) \right]
\]

(3.24)

\[
qH^2 = \frac{\ddot{a}}{a} \simeq q_0 H_0^2 \left[ 1 - \frac{2\epsilon}{\beta} \left( \frac{B}{(\mu t + \nu)} + \frac{(2\beta - 3)}{(\beta - 1)} \frac{C}{(\mu t + \nu)^2} \right) \right]
\]

(3.25)

where $H_0$ and $q_0 H_0^2$ are the zeroth-order values of these parameters. Note that in the large time limit ($t \to \infty$) these terms tend to their zeroth-order values. Hence, we see that the Gauss-Bonnet contributions become vanishingly insignificant for late cosmological times as one would expect.

**B. Exact Solution when $\eta_5/\eta_4 = \xi_4/\xi_5$**

We will now examine the equation for $p_d$ (eqn 3.3) when the ratio of the coefficients of the Gauss-Bonnet terms is equal to the ratio of the coefficients of the Einstein terms or

\[
\frac{\eta_5}{\eta_4} = \frac{\xi_4}{\xi_5}
\]

(3.26)

This equation holds for a wide variety of $n$ and $d$. Notice that for the special case of parameters the equation for the $d$-dimensional pressure (eqn 3.10) can be factored as

\[
0 = \left[ \frac{\dot{a}}{a} + \frac{\eta_5}{\eta_4} \frac{\dot{a}^2}{a^2} \right] \left[ \eta_4 + \epsilon \xi_5 \frac{\dot{a}^2}{a^2} \right].
\]

(3.27)

This equation can be solved exactly and has a solution

\[
a(t) = (\mu t + \nu)^{\eta_4/(\eta_4 + \eta_5)}
\]

(3.28)

This solution can be used to find an expression for the energy density and effective pressure from the remaining field equations. As was done above, we can eliminate $a(t)$ and obtain an expression relating $\tilde{p}$ and $\rho$.

\[
\left[ 1 - \epsilon \frac{\sigma}{\zeta (1 - w/\zeta)} (\tilde{p} - \zeta \rho) \right]^2 = 1 + 2\epsilon \sigma \rho
\]

(3.29)

where we defined the constants

\[
\sigma = \frac{\xi_1}{\eta_4^2 \beta}, \quad \zeta = \frac{1}{\xi_1} \left( \gamma_2 - \gamma_1 \frac{\eta_5}{\eta_4} \right)
\]

(3.30)

with $w$ given by (3.15). This equation relating $\rho$ and $\tilde{p}$ is inherently non-linear as one might expect from an exact solution to Gauss-Bonnet FRW cosmology. For late cosmological times, we expect a small Gauss-Bonnet contribution to the field equations (2.2). In this small $\epsilon$ regime, (3.29) reduces to

\[
(\tilde{p} - w \rho) - \frac{\epsilon}{2 \zeta (1 - w/\zeta)} (\tilde{p} - \zeta \rho)^2 \simeq 0
\]

(3.31)
Notice that in the $\epsilon \to 0$ limit (no Gauss-Bonnet terms), this expression yields a 4D equation of state (3.15). For a small but non-zero $\epsilon$, we have a small correction term to the FRW equation of state. This equation can be expressed as

$$\tilde{p} = \rho w + \epsilon \rho^2 w'$$

where $w' = (((8(w + 3)(d^2 + d - 3))/(d(d + 2)))$ for the special case of $n = 2/d$.

We can also examine the regime where the Gauss-Bonnet term dominates over that of the Einstein term for early cosmological times. To examine this regime, we take the large $\epsilon$ limit of (3.29) and obtain

$$(\tilde{p} - \zeta \rho) - \frac{2}{\epsilon \sigma} \zeta (1 - w/\zeta)(\tilde{p} - \rho w) \simeq 0$$ (3.33)

Notice that in the $\epsilon \to \infty$ limit, we obtain a linear relation between the density and pressure as in (3.15) but with the parameter $w$ being replaced by $\zeta$.

### C. Asymptotic solution when $\lambda = \text{constant}$

In the previous subsection, we examined Gauss-Bonnet FRW Cosmology for a vanishing cosmological constant. For a non-zero value of $\lambda$, the higher dimensional pressure equation is of the form

$$\frac{p_d}{2\Upsilon} = 0 = \eta_4 \ddot{a} + \eta_5 \frac{\ddot{a}^2}{a} + \lambda + \epsilon \left( \xi_4 \frac{\dot{a}^4}{a^4} + \xi_5 \frac{\ddot{a} a^2}{a^2} \right)$$ (3.34)

Again, we may solve this equation perturbatively. The zeroth-order equation yields the solution

$$a_0(t) = [A \exp (\delta t) + B \exp (-\delta t)]^{4\beta}$$

where $\delta^2 = \lambda(\eta_4 + \eta_5)/\eta_1^2$. As was done in the previous subsection, we may use the remaining zeroth-order field equations to obtain an expression relating $\tilde{p}$ and $\rho$. We find the relation

$$\tilde{p}_0 = w\rho_0 + 4\beta(1 + 2w)\lambda$$ (3.36)

The first-order perturbation is of the form

$$\ddot{a}_1(t) + \left( \frac{2\eta_5}{\eta_4} \frac{\ddot{a}_0}{a_0} \right) \dot{a}_1(t) + \left( \frac{\ddot{a}_0}{a_0} + \frac{2\eta_5}{\eta_4} \frac{\dot{a}_0^2}{a_0^2} + 4\frac{\lambda}{\eta_4} \right) a_1(t) = -\frac{1}{\eta_4} \left( \xi_4 \frac{\dot{a}_0^4}{a_0^4} + \xi_5 \frac{\dot{a}_0 \ddot{a}_0^2}{a_0^2} \right) a_0$$ (3.37)

Using the solution for $a_0(t)$, (3.37) becomes

$$\ddot{a}_1(t) + \left( \frac{2\eta_5}{\eta_4} \frac{\ddot{a}_0}{a_0} \right) \dot{a}_1(t) + \frac{\beta^2 a_1(t)}{a_0(t)} = -\frac{\beta^4}{\eta_4} \left[ \xi_4 - \xi_5 \left( 1 + \frac{2\eta_5}{\eta_4} \right) \right] A \exp (\beta t)$$ (3.38)

where we set $B = 0$ as we are interested in an expanding universe and defined the constant $\beta^2 = \lambda/(\eta_4 + \eta_5)$. The solution for $a_1(t)$ can be found and is

$$a_1(t) = a_{1,\text{hom}}(t) + a_{1,\text{non}}(t)$$ (3.39)

where the homogeneous and transient solutions are given by

$$a_{1,\text{hom}}(t) = a_0(t) \left[ c_1 e^{\tilde{\beta}(\gamma - 1/\beta) t} + c_2 e^{-\tilde{\beta}(\gamma + 1/\beta) t} \right]$$

$$a_{1,\text{non}}(t) = -\chi a_0(t)$$ (3.40)
where \( c_1 \) and \( c_2 \) are arbitrary constants and

\[
\gamma^2 = \left( \frac{\eta_5^2}{\eta_4^2} - 1 \right),
\]

\[
\chi = \frac{1}{2 \eta_5 (\eta_4 + \eta_5)} \left[ \xi_4 - \xi_5 \left( 1 + 2 \frac{\eta_5}{\eta_4} \right) \right]. \tag{3.41}
\]

Notice that the sign of the quantity \( \gamma^2 \) determines whether the scale factor experiences oscillatory motion or exponential growth and decay. Explicitly, exponential growth and decay occur when \( \eta_5/\eta_4 > 1 \) or \( \eta_5/\eta_4 < -1 \). In terms of \( n \) and \( d \), when

\[
(d - 1) \left( n - \frac{3}{d} \right)^2 + \frac{3(d + 3)}{d^2} > 0 \tag{3.42}
\]

\[
n(d - 1)(2 - d n) > 0 \tag{3.43}
\]

Notice that (3.42) is trivially satisfied for all permissible values of \( n \) and \( d \). However, (3.43) is only satisfied when \( n < 2/d \); thus \( n = 2/d \) represents a critical point of the system for which value the scale factor behaves as if it is critically damped. For this critical point, we find that the coefficients dramatically simplify to

\[
\eta_1 = \eta_3 = \eta_4 = \eta_5 = -\frac{2 + d}{d}, \quad \eta_2 = 0.
\]

As in subsection III A, we may use this value of \( a(t) \) to calculate the Hubble and acceleration parameters in the limit where the Gauss-Bonnet terms are small with respect to the Einstein terms:

\[
H = \frac{\dot{a}}{a} \simeq H_0 \left[ 1 + \epsilon \left( (\gamma - 1/\beta)c_1 e^{\beta(\gamma-1/\beta)t} - (\gamma + 1/\beta)c_2 e^{-\beta(\gamma+1/\beta)t} \right) \right] \tag{3.44}
\]

\[
qH^2 = \frac{\ddot{a}}{a} \simeq q_0 H_0^2 \left[ 1 - \epsilon \left\{ (1 - [\gamma + (1 - 1/\beta)]^2) c_1 e^{\beta(\gamma-1/\beta)t} - \right. \right.
\]

\[
\left. (1 - [\gamma - (1 - 1/\beta)]^2) c_2 e^{-\beta(\gamma+1/\beta)t} \right\} \right] \tag{3.45}
\]

Note that these terms are either exponentially decaying or growing.

\textbf{IV. CONCLUSION}

To summarize, we have found the form of the conformal factor, \( a(t) \), within the context of an action that is the sum of Einstein and Gauss Bonnet terms in \( 4 + d \) dimensions with a compactifying \( d \)-dimensional flat Kaluza-Klein space for a variety of cases. By assuming that the extra dimensions compactify as \( a(t) \sim b(t)^{-n} \), we showed that restrictions on the possible form of the equation of state are imposed. Using perturbative methods to solve the field equations, a general form for \( a(t) \) to order \( \epsilon \) was found for both a zero and a non-zero cosmological constant. We used these values of \( a(t) \) to calculate values for both the Hubble and the acceleration parameters \( (H(t), q(t)) \) and found a first order correction due to the Gauss Bonnet terms which can play the role of additional source terms. These solutions are consistent with having a dark energy era for reasonable values of \( n \) and \( d \). An
interesting subcase is when $\eta_5/\eta_4 = \xi_4/\xi_5$, for these values of $n$ and $d$ we are able to solve the 
$d$ dimensional pressure equation exactly. This is consistent with and generalizes the results
of both Paul and Mukherjee [13] and Mohammedi [14]. A special case happens if $n = 2/d$,
when this is true then the equations simplify considerably as detailed above and may be
indicative of an underlying symmetry.

The Gauss Bonnet terms can only contribute significantly during the early phase of the
universe as one would surmise from dimensional arguments since the Einstein terms go
as $R \sim t^{-1/2}$ but $\mathcal{G} \sim R^2 \sim t^{-1}$ etc. It is apparent that the Gauss-Bonnet term does
not contribute significantly to any Big Rip scenario and that the current EoS with exotic
matter dominating over baryonic matter still gives rise to this singularity in the future. A
scenario where a scalar field coupled to a Gauss-Bonnet term that avoids a Big Rip has been
investigated by Nojiri, Odinstov and Sasaki [33]. However in the early universe the phantom
and/or quintessence energies need not dominate over the Gauss-Bonnet contributions and
the EoS in this regime can still be radiation dominated.

This paper leads to several questions. What explicit mechanism might drive cementifi-
cation and on what timescale does this occur? First how does the coupling constant $(\Upsilon)$
rise as the extra dimensions compactify? A second issue is how the addition of these Gauss
Bonnet terms would change the semi-classical states of FRW solutions to the Wheeler-de
Witt equation. The general relation of time scales is also important since it is not uncommon
to have all of the critical compactification processes complete on the order of a Planck time.
These issues will be reported on in future papers.

[1] Knop, R. A., et al., 2003 New Constraints on $\Omega_M$, $\Omega_{\Lambda}$, and $w$ from an Independent Set of
Eleven High-Redshift Supernovae Observed with HST Astrophys.J. 598, 102, (Preprint astro-ph/0309368)
Riess, A.G. et al., 2004 Type Ia Supernova Discoveries at $z > 1$ From the Hubble Space
Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution
Astrophys.J. 607, 665 (Preprint astro-ph/0402512).
Virey,J. M. et al., On the determination of the deceleration parameter from the SNe data,
Phys.Rev. D72 061302 (Preprint astro-ph/0502163v1)
[2] Eriksen,H. K., 2006 A re-analysis of the three-year WMAP temperature power spectrum and
likelihood (Preprint astro-ph/0606088)
Lewis, B., 2002 Cosmological parameters from CMB and other data: a Monte-Carlo approach
Phys.Rev. D 66, 103511 (Preprint astro-ph/0205436),
Bennett,et al. 2003 First Year Wilkinson Microwave Anisotropy Probe (WMAP) Obser-
vations: Preliminary Maps and Basic Results, Astrophys.J.Suppl. 148 1 (Preprint
astro-ph/0302207)
[3] Narain, K.S., Sarmadi M., Witten E., 1986 A Note On Toroidal Compactification Of Heterotic
String Theory. Nucl. Phys.B 268, 253
[4] Ohta, N., 2005 Accelerating Cosmologies and Inflation from M/Superstring Theories
Int.J.Mod.Phys. A20 1-40, (Preprint hep-th/0411230)
[5] Kantowski, R., Milton, K. A., 1987 Casimir Energies In M(4) X S(N) For Even N. Green’s
Function And Zeta Function Techniques. Phys. Rev. D 35, 549
[6] Birmingham,D., Kantowski, R., Milton, K. A., 1988 Scalar And Spinor Casimir Energies In
Even Dimensional Kaluza-Klein Spaces Of The Form $M(4) \times S(N_1) \times S(N_2) \times ...$ \textit{Phys. Rev. D} 38, 1809

[7] Ratra, B, Peebles, P.J.E., 1987 Cosmological Consequences Of A Rolling Homogeneous Scalar Field \textit{Phys. Rev. D} 37, 3406-3427

[8] Calwell, R.R., Kamionkowski, M., Weinberg, N.N., 2003 Phantom Energy and Cosmic Doomsday \textit{Phys.Rev.Lett.} 91 071301 (Preprint astro-ph/0302506 v1)

[9] Mukohyama, S., Randall, L. 2003 A Dynamical Approach to the Cosmological Constant \textit{Phys.Rev.Lett.} 92 211302 (Preprint hep-th/0306108)

[10] Fang, W., Lu, H.Q., Huang, Z.G. Exponential Potentials and Attractor Solution of Dilatonic Cosmology (Preprint hep-th/0606032)

[11] Vereshchagin, G. V., Yegorian, G., 2006 Cosmological models with Gurzadyan-Xue Dark Energy \textit{Class.Quant.Grav.} 23 (2006) 5049: (Preprint astro-ph/0601073)

[12] Zhu, Z. H. and Alcaniz J. S., 2005, Accelerating universe from gravitational leakage into extra dimensions: confrontation with SNela \textit{Astrophys.J.} 620, 7. (Preprint astro-ph/0404201)

[13] Paul, B. S., Mukherjee S., 1990, Higher-dimensional cosmology with Gauss-Bonnet terms and the cosmological-constant problem, \textit{Phys. Rev. D}, 42.2595

[14] Mohammed, N., 2002 Dynamical Compactification, Standard Cosmology and the Accelerating Universe \textit{Phys.Rev. D} 65 104018 (Preprint hep-th/0202119)

[15] Sussman, R. A., Quiros, I., Gonzalez, O. M., Inhomogeneous models of interacting dark matter and dark energy \textit{Gen.Rel.Grav.} 37 (2005) 2117-2143 (Preprint astro-ph/503609 v1)

[16] Filippov, A.T., 2006 Some Unusual Dimensional Reductions of Gravity: Geometric Potentials, Separation of Variables, and Static - Cosmological Duality (Preprint hep-th/0605276)

[17] Weinberg, S., 1989 The Cosmological Constant Problem \textit{Rev. Mod. Phys.} 61, 1-23

[18] Calcagni, G., Tsujikawa, S., Sami, M. 2005 Dark energy and cosmological solutions in second-order string gravity \textit{Class.Quant.Grav.} 22 3977-4006 (Preprint hep-th/0505193)

[19] Padilla, A., 2005 Cosmic acceleration from asymmetric branes \textit{Class.Quant.Grav.} 22 681-694 (Preprint hep-th/0406157)

[20] Caldwell, A., 2002 Phantom Menace? Cosmological consequences of a dark energy component with super-negative equation of state, \textit{Phys.Lett.} B545 23-29(Preprint astro-ph/9908168)

[21] Kujat, J, Scherrer, R. J., Sen, A.A., 2006 Phantom Dark Energy Models with Negative Kinetic Term (Preprint astro-ph/0504052)

[22] Fabi S., Harms B., Karatheodoris G. 2006 Dark Energy from Casimir Energy on Noncommutative Extra Dimensions (Preprint hep-th/0607153 v1)

[23] Cavaglia, M, Moniz, P.V., FRW Cosmological Solutions in M-theory proceedings of the Ninth Marcel Grossmann Meeting (Preprint gr-qc/0011098 v1)

[24] Nojiri, S., Odintsov, S., Sasaki, M., 2005 Gauss-Bonnet dark energy, \textit{Phys.Rev. D}71 123509 (Preprint hep-th/0509126)

[25] Sotiriou, T., 2005 Constraining f(R) gravity in the Palatini formalism \textit{Class.Quant.Grav.} 23 (2006) 5117-5128 (Preprint gr-qc/0512017)

[26] Carrol, S., Sawicki, L.,Silvestri, A., Trodden, M., Modified source Gravity and Cosological Structure Formation, (Preprint astro-ph/0607458)
[27] Zumino, B., 1986 Gravity Theories In More Than Four-Dimensions Phys Rep. 137, 109
[28] Gross, D. J., Witten, E. 1986 Superstring Modifications Of Einstein’s Equations Nucl. Phys. B277
  Bento, M.C., Bertolmi, O. 1996 Maximally Symmetric Cosmological Solutions of higher cur-
vature string effective theories with dilatons Phys. Lett. B368, 198
[29] Deser, S., Redlich, A.N., 1986 String Induced Gravity And Ghost Freedom Phys. Lett. B
176, 350
[30] Nojiri, S., Odintsov, S., Introduction to Modified Gravity and Gravitational Alternative
  for Dark Energy, lectures for 42 Karpacz Winter School on Theor Physics (Preprint
  hep-th/0601213)
[31] Townsend, P., 2003 Accelerating cosmologies from compactification Phys. Rev. Lett., 91,
061302 (Preprint hep-th/0303097)
[32] Nojiri, S., Odintsov, S., Tsujikawa S., 2005 Properties of singularities in (phantom) dark
  energy universe” Phys.Rev. D71 063004 (Preprint hep-th/0501025)
[33] Nojiri, Odintsov, Sasaki, 2005 Gauss Bonnet Dark energy, Phys.Rev. D71 (2005) 123509,
(Preprint hep-th/0504052)