Exact Phase Diagram of a model with Aggregation and Chipping

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We revisit a simple lattice model of aggregation in which masses diffuse and coalesce upon contact with rate 1 and every nonzero mass chips off a single unit of mass to a randomly chosen neighbour with rate $w$. The dynamics conserves the average mass density $\rho$ and in the stationary state the system undergoes a nonequilibrium phase transition in the $(\rho-w)$ plane across a critical line $\rho_c(w)$. In this paper, we show analytically that in arbitrary spatial dimensions, $\rho_c(w) = \sqrt{w + 1} - 1$ exactly and hence, remarkably, independent of dimension. We also provide direct and indirect numerical evidence that strongly suggest that the mean field asymptotic answer for the single site mass distribution function and the associated critical exponents are super-universal, i.e., independent of dimension.

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I. INTRODUCTION

Nonequilibrium phase transitions [1] occur in various systems including heterogeneous catalysis [2], chemical reaction models [3], polymolecular growth models [4], monomer-dimer models [5], models of fungal growth [6], non-equilibrium kinetic Ising models [7], and branching annihilating random walks [8]. A common feature in all the above is that the transition is from a state that has no activity to one that has continued activity. Such transitions are well characterized by the critical exponents of Directed Percolation, Parity Conserving, or the DP2 universality classes. There are other nonequilibrium models that undergo phase transitions which do not belong to the above universality classes. These include boundary driven phase transitions [9] and models whose steady states undergo a ‘de-pinning’ or ‘unbinding’ transitions [10–13]. Recently a simple lattice model where masses diffuse, aggregate on contact, and also chips off a single unit of mass was studied [14,15]. This ‘chipping’ model (CM) exhibits a nonequilibrium phase transition in its steady state from a phase in which an infinite aggregate is present to one that has none. This nonequilibrium phase transition is in a completely different universality class compared to the other models studied in the literature and mentioned above. The mathematical mechanism giving rise to the formation of the infinite aggregate at the onset of phase transition was found to be very similar to that of the equilibrium Bose-Einstein condensation in an ideal Bose gas. The difference is that in CM, the infinite aggregate or the condensate forms in real space as opposed to the Bose gas where the condensation takes place in the momentum space. Besides, the phase transition in the CM occurs even in 1-dimension as opposed to the Bose gas where the condensation occurs in 2 and higher dimensions. A slightly different off-lattice version of the CM was studied earlier within rate equation approach in the context of aggregation in dry environments [16]. A directed version of the CM where masses move asymmetrically only along one direction was studied in Ref. [17] and its critical properties were found to belong to a different universality class from that of the undirected CM. This directed CM also appeared recently in the context of a traffic model with passing [17].

The undirected CM is defined on a $d$-dimensional hyper cubic lattice with periodic boundary conditions. Starting from a random distribution of nonnegative integer masses at each lattice site, the system evolves in continuous time via the following microscopic processes. In an infinitesimal time interval $\Delta t$, (i) with probability $\Delta t$, the mass at each site hops to one of its neighboring sites, chosen at random, (ii) with probability $w\Delta t$, a unit mass is chipped off from an already existing mass at each site and added to one of the neighboring sites, again chosen at random, and (iii) with probability $(1 - (1 + w)\Delta t)$, each mass stays at its original site. Following the steps (i)-(iii), the masses at any given site add up. Since this system is closed to the environment (periodic boundary condition), the total mass is conserved by the dynamics. Thus, there are two parameters in the problem, the average mass per site $\rho$, and the ratio of the chipping rate to the rate of hopping as a whole, $w$. In the long time limit, the system evolves into a time independent steady state. The steady state single site mass distribution function $P(m)$, i.e., the probability that a site has mass $m$ when $t \to \infty$, was shown to undergo a phase transition in the $\rho$-$w$ plane [14,15]. There is a critical line $\rho_c(w)$ in the $\rho$-$w$ plane that separates two types of asymptotic behaviors of $P(m)$. For fixed $w$, as $\rho$ is varied across the critical value $\rho_c(w)$, the large $m$ behaviour of $P(m)$ was found to be,
Thus, the tail of the mass distribution changes from having an exponential decay to an algebraic decay as $\rho$ approaches $\rho_c$ from below. As one increases $\rho$ beyond $\rho_c$, this asymptotic algebraic part of the critical distribution remains unchanged but in addition an infinite aggregate forms. This means that all the additional mass ($\rho - \rho_c)V$ (where $V$ is the volume of the system) condenses onto a single site and does not disturb the background critical distribution.

Mathematically this means that for $\rho > \rho_c(w)$,

$$P(m) \sim \begin{cases} e^{-m/m^*} & \text{if } \rho < \rho_c(w), \\ m^{-\tau} & \text{if } \rho = \rho_c(w), \\ m^{-\tau} + \text{infinite aggregate} & \text{if } \rho > \rho_c(w). \end{cases}$$

(1)

These results were found both analytically within a mean field approximation which involved ignoring all correlations between masses, and numerically in one dimension. Within the mean field approximation, the locus of the critical line was found to be $\rho_c(w) = \sqrt{w+1} - 1$ and the exponent $\tau = 5/2$. In one dimension, the numerically obtained critical line was found to be close to the mean field phase curve. Even the exponent $\tau$, determined from a simple linear fit on the log-log plot of data from relatively smaller size lattices, was found to be $\tau = 2.33$, rather close to the mean field exponent $2.5$. This raises the question whether or not the mean field answers for asymptotic behaviors of $P(m)$ are exact even in one dimension. On the other hand, one can show explicitly (see section II) that there exist nonzero correlations between masses in any finite dimension, even in the thermodynamic limit. Thus, one is confronted with a puzzle.

The purpose of this paper is to shed new light on this puzzling issue. We first prove analytically a remarkable result that the mean field phase boundary, $\rho_c(w) = \sqrt{w+1} - 1$, is indeed exact and independent of the spatial dimension $d$. This, of course, still does not prove, but hints, that the exponent $\tau$ may also be independent of $d$. However, we provide rather unambiguous numerical evidence, in conjunction with several direct and indirect checks, which suggest that even the exponent $\tau = 5/2$ is super-universal and independent of $d$. Thus, our results seem to suggest strongly that even though there are nonzero correlations between masses in finite dimensions, these correlations do not affect the asymptotic behavior of the single site steady state mass distribution $P(m)$. However, the possibility remains that other higher order correlation functions, such as the joint distribution of two masses $P(m_1, m_2)$ will depend on the spatial dimension $d$.

The paper is organized as follows. In section II, we show analytically that in arbitrary dimensions $d$, the locus of the phase boundary is independent of $d$ and is given by the mean field expression, $\rho_c(w) = \sqrt{w+1} - 1$. In section-III, we derive analytically the mean field expression for the mass distribution $P(m)$ by a method different from that used in [3] and compare these analytical expression with the numerical results obtained in one and two dimensions. In section IV, we do a finite size scaling analysis that provides additional evidence that the exponent $\tau$ is super-universal. Finally, we conclude with a summary and discussion in section V.

II. EXACT PHASE DIAGRAM IN ARBITRARY DIMENSIONS

As mentioned in the introduction, the steady state of the CM undergoes a phase transition in its parameter space ($\rho$-$w$ plane) across a critical line $\rho_c(w)$ in all dimensions. In this section, we compute $\rho_c(w)$ exactly in arbitrary dimensions by analyzing the two point equal time mass-mass correlation function, $C(x, t) = \langle m(x, t)m(0, t) \rangle$. Let $\eta(x, x', t)$ denote the mass transferred from a site $x$ to a neighboring site $x'$ in the time interval between $t$ and $t + \Delta t$. Clearly, $\eta(x, x', t)$ is a random variable that takes the following values,

$$\eta(x, x', t) = \begin{cases} m(x), & \text{with prob. } \frac{1}{2d}w\Delta t, \\ 1 - \delta_{m(x), 0}, & \text{with prob. } \frac{1}{2d}w\Delta t, \\ 0, & \text{with prob. } (1 - \frac{1}{2d}w\Delta t), \end{cases}$$

(2)

where $2d$ is the number of neighbors of any given site. The Kronecker delta function in the second line of Eq. (2) indicates that a chipping of a unit mass can take place provided the mass $m(x)$ is a positive integer bigger than 0. Then, the evolution of mass at site $x$ can be written as,

$$m(x, t + \Delta t) = m(x, t) - \sum_{x'} \eta(x, x', t) + \sum_{x'} \eta(x', x, t),$$

(3)
where the sum is over the neighbors \( x' \) of the site \( x \). The second term on the right hand side of Eq. (3) describes the outflow of mass from \( x \) while the third term accounts for the inflow of mass into \( x \) from neighboring sites. It is quite straightforward to write down the 2 point correlator for \( \eta \) to order \( \Delta t \). Suppressing the explicit \( t \) dependence in \( \eta \) we find,

\[
\langle \eta(x_1,x'_1)\eta(x_2,x'_2) \rangle = \frac{1}{2d} \left( m^2(x_1) + w(1 - \delta_{m(x_1),0}) \right) \delta_{x_1,x'_1} \delta_{x_2,x'_2} \Delta t.
\] (4)

Using Eqs. (3) and (4), the evolution equations for the two-point equal time correlation function, \( C(x,t) = \langle m(x,t)m(0,t) \rangle \) can be written down. Multiplying Eq. (3) by \( m(0,t + dt) \) and taking average on both sides, and putting all time derivatives to zero in the steady state, we get,

\[
-C(x) + \frac{1}{2d} \sum_{x'} C(x') = w \left( -D(x) + \frac{1}{2d} \sum_x D(x') \right)
- \left( C(0) + ws \right) \left( \delta_{x,0} - \frac{1}{2d} \sum_{x_0} \delta_{x,x_0} \right),
\] (5)

where \( x_0 \) denotes the neighbors of the site \( 0 \). Also, \( D(x,t) = \langle m(x,t)m(0,t) \rangle \delta_{m(0,t),0} \), and \( s = 1 - \langle \delta_{m(0),0} \rangle \), is the probability of a site having non zero mass. Thus, in the CM, the two point correlation functions do not form a closed set of equations making it difficult to solve for \( C(x,t) \) exactly. This is unlike many other models where two point correlations do form a closed set of equations and hence are solvable. A few examples include the 1-d Glauber model \[18\], asymmetric random average process \[19,20\], the Takayasu model of aggregation \[21\] and the \( q \) model of force fluctuations in bead packs \[22\].

Remarkably, however, Eq. (4) allows for the solution,

\[
C(x) = w[D(x) - s], \quad x \neq 0.
\] (6)

This solution is also the unique solution. To see this, observe that the homogeneous part of Eq. (3) is the Laplace’s equation \( \nabla^2 (C(x) - wD(x)) = 0 \), with the boundary condition that \( (C(x) - wD(x)) \) is a constant as \( |x| \to \infty \). Since the solution Eq. (4) satisfies the inhomogeneous part too, as well as the boundary conditions, it is the unique solution.

Note that the above solution Eq. (4) is also valid on a finite lattice. Summing Eq. (4) over all \( x \neq 0 \) and using the fact that the conserved total mass is given by, \( \sum_x m(x) = \rho V \) (where \( V = L^d \) is the volume of the system), we get the following exact equation,

\[
\rho^2 - \frac{(m^2)}{V} = wp(1 - s) - ws.
\] (7)

This equation is reminiscent of Bose-Einstein condensation in ideal Bose gas. In the low density phase, we expect that the system reaches a stationary state in which \( \langle m^2 \rangle \) is a finite number of order \( O(1) \). Therefore, in the thermodynamic limit \( V \to \infty \), the second term on the left hand side of Eq. (5) drops out and we get,

\[
\rho^2 = wp(1 - s) - ws.
\] (8)

Note that the Eq. (8) could have been obtained from Eq. (4) if we had assumed that the two point correlation functions decouple, i.e., \( C(x) = \langle m(0)m(x) \rangle = \langle m(0) \rangle \langle m(x) \rangle = \rho^2 \) and \( D(x) = \langle m(x) \delta_{m(0),0} \rangle = \rho(1 - s) \). However, there is apriori no reason for the two point correlations to decouple. Our derivation of Eq. (8) does not rely on this decoupling.

From Eq. (8) we get,

\[
s = \frac{wp - \rho^2}{w(p + 1)}. \tag{9}
\]

According to Eq. (8), as one increases the density \( \rho \) keeping \( w \) fixed, the occupation probability \( s \) first increases with \( \rho \), attains a maximum at \( \rho = \sqrt{w + 1} - 1 \) (obtained by setting \( \frac{dp}{d\rho} = 0 \) in Eq. (5)) and then starts decreasing with increasing \( \rho \). However, it is clear that \( s \), the probability that a site has nonzero mass, must be a monotonically non decreasing function of \( \rho \). Hence we conclude that Eq. (8) is valid as long as \( \rho \leq \rho_c = \sqrt{w + 1} - 1 \). For \( \rho > \rho_c \), the basic assumption \( \langle m^2 \rangle/V \to 0 \) as \( V \to \infty \) breaks down and Eq. (8) ceases to be valid.

Thus, the critical density is given by,
\[ \rho_c(w) = \sqrt{1+w-1}, \]  

(10)

and remarkably, it is independent of \( d \) and not surprisingly, therefore, coincides with the mean field expression \([14,15]\).

For \( \rho \leq \rho_c \), \( s \) is given by Eq. (9). As \( \rho \) increases from 0 to \( \rho_c(w) \) (for fixed \( w \)), \( s \) increases monotonically according to Eq. (11) up to the value \( s_c \) given by,

\[ s_c = \frac{\sqrt{1+w-1}}{\sqrt{1+w+1}}. \]  

(11)

For \( \rho > \rho_c(w) \), \( s \) does not increase any further and sticks to its value \( s_c \). Putting \( s = s_c \) in Eq. (11) and using the expression of \( s_c \) from Eq. (11), we get for \( \rho > \rho_c(w) \),

\[ \lim_{V \to \infty} \frac{m^2}{V} = (\rho - \rho_c)^2. \]  

(12)

Thus, for \( \rho > \rho_c \), \( \langle m^2 \rangle \) becomes macroscopic, i.e., proportional to volume. Since \( s \), the fraction of occupied sites, does not increase anymore for \( \rho > \rho_c \), this indicates that all the extra mass \( (\rho - \rho_c)V \) condenses onto a thermodynamically negligible number of sites (indeed, a single site only) with density \( \sim \frac{1}{V} \), leading to the macroscopic behavior of \( \langle m^2 \rangle \sim \frac{1}{V} [(\rho - \rho_c)V]^2 \sim (\rho - \rho_c)^2V \). This is similar in spirit, though not in details, to the Bose-Einstein condensation where below a certain temperature, the number of particles in the \( k = 0 \) mode also become macroscopic.

Let us conclude this section by stressing on an important point. We note that the mean field solution (assuming decoupling) for the stationary two point correlation function, \( C(x) = \rho^2 \) for \( x \neq 0 \) and \( D(x) = \rho (1 - s) \) for \( x \neq 0 \) with \( s \) satisfying Eq. (9), is indeed an exact solution of Eqs. (5) and (6). However, this need not be the only stationary solution. Besides, even if the mean field solution for the two point stationary correlation function is the correct one, it still does not prove that the mean field theory is exact. For example, one can show \([24]\) that indeed 3 and higher point correlation functions do not decouple. In any case, the main result of this section, namely the derivation of the exact phase boundary, does not rely on whether the correlation functions decouple or not.

### III. COMPARISON WITH MEAN FIELD THEORY

In the previous section, we proved that the mean field phase diagram is exact in any dimension. This, of course, does not prove but suggests that, perhaps, even the mean field expression for the distribution \( P(m) \) may also be asymptotically exact in all dimensions. In this section, we try to provide evidence in favor of this hypothesis. For this we first derive the mean field expression for \( P(m) \) and compare it with the numerical results obtained in 1 and 2 dimensions.

In Ref. [13], the steady state single site mass distribution function \( P(m) \) was computed analytically by assuming that the joint distribution \( P(m_1, m_2) \), the probability that two consecutive sites have masses \( m_1 \) and \( m_2 \) respectively, factorises, i.e., \( P(m_1, m_2) = P(m_1)P(m_2) \). With this assumption, \( P(m) \) was shown to satisfy a closed set of equations which were then solved via generating function method yielding results mentioned in Eq. (11) with \( \tau = 5/2 \) and \( \rho_c(w) = \sqrt{w+1}-1 \). In this section, we first derive the mean field results by a different method that requires lesser restrictions than the product measure used in Ref. [14].

Here we use a technique used before for solving the mass distribution function in other models of aggregation \([24]\) as well as the \( w = 0 \) limit of the CM \([24]\). We consider the CM on a 1-d lattice. Let \( P(m_1, m_2, \ldots, m_n) \) denote the joint probability that \( n \) consecutive sites on the lattice have masses \( m_1, m_2, \ldots, m_n \) respectively in the stationary state. We define two generating functions,

\[ Z_n = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} x^{m_1+\cdots+m_n} P(m_1, \ldots, m_n), \]  

(13)

\[ Y_n = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} x^{m_1+\cdots+m_n} P(m_1, \ldots, m_n, 0). \]  

(14)

Here \( Z_n = \langle x^{m_1+\cdots+m_n} \rangle \) is an unconditional average but \( Y_n = \langle x^{m_1+\cdots+m_n} \rangle_0 \) is a conditional average where the \((n+1)\)-th site is conditioned to have 0 mass. Using the dynamics of \( m_i \)'s (as given by Eq. (3)) and following steps similar to those used in Refs. \([23,24]\), one can write down the evolution equations for \( Z_n \)'s. In the steady state, when all time derivatives go to zero, we get, after some algebra,
with the boundary conditions, \( Z_0 = 1 \) and \( Y_0 = P(0) = 1 - s \), the probability of having no mass at any given site. If we now make the assumption that \( P(m_1 + \ldots + m_n = m, 0) = P(m_1 + \ldots + m_n = m)P(0) \), i.e., \( Y_n = P(0)Z_n \), then we get equations that contain only the \( Z_n \)'s. As mentioned before, this assumption is less strict than the product measure approximation as it requires the factorization of only a special conditional probability where a site at the beginning of a string is empty. In order to determine \( P(m) \), we need to compute \( Z_1(x) \) which, by definition in Eq. \( (14) \), is the generating function for \( P(m) \),

\[
Z_{n+1} - 2Z_n + Z_{n-1} + w \left( \frac{1-x^2}{x} Z_n - \frac{1-x}{x} Y_{n-1} + (1-x)Y_n \right) = 0,
\]

(15)

Since \( Z_1 \) depends on other \( Z_n \)'s, we need to solve the full Eq. \( (15) \) for all \( n \). Eq. \( (15) \) can be solved by standard generating function method. Let \( G(x, y) = \sum_{n=1}^{\infty} Z_n(x)y^n \). Multiplying Eq. \( (15) \) by \( y^n \) and summing over \( n \), we get, after straightforward algebra,

\[
G(x, y) = \frac{y[wP(0)y(1-x) - xy + xZ_1(x)]}{x(1-y)^2 + wy(1-x)^2 + wP(0)y(1-x)(x-y)}.
\]

(17)

For a fixed \( x \), when considered as a function of \( y \) only, \( G(x, y) \) has two poles,

\[
y_{1,2} = \frac{-2x + w(1-x)(1-sx) \pm (1-x)\sqrt{w(1-sx)^2 - 4sx}}{2((1-s)w(1-x) - x)}.
\]

(18)

For a fixed \( w \) and \( s \), \( |y_{1,2}| < 1 \) for small values of \( x \). This implies that \( Z_n \sim |y_{1,2}|^{-n} \) for large \( n \). However, we cannot have a diverging probability for large \( n \). Hence this pole must be canceled off by the numerator of \( G \). This ‘pole canceling’ mechanism was also useful in deriving exact results in other recently studied aggregation models \[20,21\]. Demanding \( G(x, y_2) = 0 \), we get the following expression for \( Z_1(x) \),

\[
Z_1(x) = \frac{2x - w(1-x)(1-sx) + (1-x)\sqrt{w(1-sx)^2 - 4sx}}{2x}.
\]

(19)

The coefficient of \( x^n \) on the right hand side of Eq. \( (14) \) will then give the desired distribution \( P(m) \).

The expression for the generating function \( Z_1(x) = \sum_{m=0}^{\infty} x^m P(m) \) in Eq. \( (14) \) is identical to the one derived in \[14\] using the approximations of product measure. Thus, the two methods, though different in details, yield the same \( P(m) \) for any \( m \) and not just for large \( m \). Thus, this result for mean field \( P(m) \) seems to be extremely robust and does not depend on the details of how the mean field assumption is incorporated.

The asymptotic properties of \( P(m) \) for large \( m \) can be derived by analyzing the behaviour of \( Z_1(x) \) near \( x = 1 \) \[14\] and one recovers the results in Eq. \( (1) \) with \( \tau = 5/2 \) and \( \rho_c(w) = \sqrt{w+1} - 1 \). However, by expanding the expression for \( Z_1(x) \) in Eq. \( (19) \) in powers of \( x \) using Mathematica, we have determined \( P(m) \) for all \( m \). In the aggregate phase \( (\rho > \rho_c) \), we set \( s = s_c \) in Eq. \( (14) \) and calculate the distribution \( P(m) \) by expanding in powers of \( x \). In Fig. \( (1) \), we compare this analytical mean field answer for all \( m \) with the numerical results obtained in 1 and 2 dimensions in the aggregate phase \( (\rho > \rho_c) \). Note that Fig. \( (1) \) shows only the power law part of the spectrum. The numerical data is for \( V = 900 \) in 1-dimensional and for a 30X30 lattice in 2-dimensions. The two curves for 1 and 2-d are almost indistinguishable from each other. While the numerical data matches excellently with the mean field answer for small \( m \), it has a small deviation for larger masses. This deviation at large \( m \) is due to finite size effects. To confirm this, we also did simulations for larger lattice sizes up to \( V = 2000 \) in 1-dimension. For a fixed mass, we confirmed that the deviation from the mean field decreases with increasing \( V \) (see the panel inside Fig. \( (1) \)). We also compared the mass distribution in the exponential phase \( (\rho < \rho_c) \) with the mean field prediction. Again excellent agreement is seen (see Fig. \( (2) \)). These results thus provide strong evidence that the mean field expression for \( P(m) \) is exact in all dimensions and therefore the exponent \( \tau = 5/2 \) is also super-universal, i.e., independent of \( d \).

IV. FINITE SIZE SCALING AND INDIRECT NUMERICAL CHECKS

In this section, we provide further indirect numerical checks which again strongly suggests that the mean field answer for \( P(m) \) is indeed exact. We start by making a reasonable finite size scaling ansatz for \( P(m, V) \) (where \( V \) is the volume of the system) in the aggregate phase \( \rho > \rho_c \) for large \( m \),
\[ P(m, V) \approx \frac{1}{m^\phi} \int \left( m - (\rho - \rho_c) V \right) + \frac{1}{V} \delta(m - (\rho - \rho_c) V), \]  
\hspace{1cm} (20)

where the exponent \( \phi \) is a crossover exponent and the delta function peaked at \((\rho - \rho_c) V \) indicates the aggregate containing a macroscopic amount of mass. The power-law part of the mass distribution gets cut off at \( \sim V^\phi \) for finite \( V \). Since the largest mass inside the power law part is much less compared to that in the aggregate, we get the inequality, \( \phi < 1 \).

Also, since the mass density, \( f m P(m)dm = \rho \) is finite. Substituting Eq. (20) in this integral, it is evident that for large \( V \), the integral will converge provided \( \tau > 2 \). Also, from Eq. (20), one finds that the second moment, \( \left< m^2 \right> \approx \frac{1}{(\rho - \rho_c)^2} + O(V^{\phi(3-\tau)-1}) \). The first term is from the aggregate and the second term is from the power-law part. This is consistent with the exact result Eq. (12) provided, \( \phi(3-\tau) < 1 \). This provides an upper bound for \( \tau \), i.e., \( \tau < 3 \). Thus, we have the exact bounds for \( \tau, 2 < \tau < 3 \). The mean field results as well as numerical simulations in 1 and 2-d are consistent with these bounds.

Next we derive a scaling relation between \( \tau \) and \( \phi \). This is obtained by demanding that \( P(m, V) \) is normalized, \( \int P(m, V)dm = 1 \). Substituting Eq. (20) in this integral, we get,

\[ \int_{m_i}^{\infty} \frac{1}{m^\phi} f(m)dm = c - \frac{1}{V}, \]  
\hspace{1cm} (21)

where \( c \) is a constant of \( O(1) \), and \( m_i \sim O(1) \), is the mass beyond which the scaling starts holding. Differentiating Eq. (21) twice with respect to \( V \) and making a change of variable, we arrive at the relation,

\[ \int_{m_i V^{-\phi}}^{\infty} f''(y) y^{2-\tau} dy = -(1 - \phi) V^{\phi \tau - 1 - 1}, \]  
\hspace{1cm} (22)

where \( f''(y) = \frac{\phi f(y)}{V^{\phi \tau}} \). The reason for twice differentiation is as follows. Since \( 2 < \tau < 3 \), one can replace the lower limit of the integral in Eq. (22) by 0 in the \( V \to \infty \) limit and there is no divergence from the lower cut off. Once the lower cut off is replaced by 0, the integral on the left hand side of Eq. (22) is of \( O(1) \). Comparing this with the right hand side of Eq. (22), we immediately arrive at the scaling relation,

\[ \phi(\tau - 1) = 1. \]  
\hspace{1cm} (23)

If \( \tau = 5/2 \), Eq. (23) would indicate \( \phi = 2/3 \).

We found that the cleanest way to measure the exponent \( \phi \) is via the following indirect finite size method. On a finite lattice, the critical value of the occupation probability of a site, \( s_c(V) \) will differ from the \( V = \infty \), \( s_c(\infty) \). One can assume a reasonable finite size correction of the form,

\[ s_c(V) = s_c^{\infty} - \frac{a}{V^\theta}, \]  
\hspace{1cm} (24)

where \( a \) is a constant and \( \theta \) is a new exponent. Using this ansatz along with the expression of \( \left< m^2 \right> \) obtained from Eq. (24) in Eq. (7), we find \( V^{\phi(3-\tau)-1} \sim V^{-\theta} \), implying another scaling relation,

\[ \theta = 1 - \phi(3-\tau). \]  
\hspace{1cm} (25)

If \( \tau = 5/2 \) and \( \phi = 2/3 \), we get from Eq. (23), \( \theta = 2/3 \). In Fig. (3), we plot, in one dimension, \( s_c(V) \) for various values of \( V \) and indeed find that \( \theta = 2/3 \). This provides an indirect numerical check on these scaling assumptions as well as suggests, once again, that \( \tau = 5/2 \) is a super-universal exponent.

Another indirect check can be done by mapping the one dimensional CM to a fluctuating interface model. We outline the procedure in brief. The first step is to map the CM on to a system of hard core particles moving on a ring. For this, interpret \( m_i \), the mass at site \( i \), to be the gap between the \((i-1)th\) and \(i\)th particle. Then the chipping move corresponds to a particle jumping forward one step provided the target site is empty, while the aggregation move corresponds to a particle making a long range jump to the site adjacent to the particle nearest to it. There is a standard procedure to map a lattice gas configuration to an interface configuration. Let \( n_j = 1(-1) \) if a particle is present (absent) at site \( i \). Then \( h_i = \sum_{j=1}^{l} n_j \). While the chipping move corresponds to local moves of the interface, the aggregation move corresponds to nonlocal moves of the interface. The width of this interface was monitored numerically as a function of time \( t \). On the critical line \( \rho_c(w) \) of the CM, the width was found to have the scaling form.\( [\text{Eq. 2}] \)
\[ w(V, t) \approx V^\chi f \left( \frac{t}{V^z} \right), \]

with \( \chi \approx 0.67 \) and \( z \approx 2.0 \).

We show here that the roughness exponent \( \chi \) can be related to the exponents \( \tau \) and \( \phi \) of the CM. We first map the CM in 1-d to an interface model with the height field \( h_i = \sum_{j=1}^n m_j \). Then \( \langle h_i \rangle = i \rho \) and \( \langle h_i^2 \rangle \approx i \langle m^2 \rangle + i(i-1)\rho^2 \). If we approximate \( \langle m_i m_j \rangle \) by \( \rho^2 \) (assuming that the mean field \( P(m) \) is exact), the width is then simply \( w^2 = 1/V \sum_{i=1}^V (h_i - \langle h_i \rangle)^2 \sim V^{1+\phi(3-\tau)} = V^{2\chi} \). This implies that provided \( \langle m_i m_j \rangle \approx \rho^2 \), one gets the scaling relation,

\[ \chi = \frac{1}{2} [1 + \phi(3 - \tau)]. \]

If \( \tau = 5/2 \) and \( \phi = 2/3 \), we would get from Eq. (27), \( \chi = 2/3 \), in excellent agreement with the numerical value \( \chi \approx 0.67 \). This is further evidence for \( \tau = 5/2 \) and \( \phi = 2/3 \).

**V. SUMMARY AND OUTLOOK**

In this paper, we have studied a simple stochastic lattice model where masses diffuse as a whole and coalesce upon contact with rate 1 and every nonzero mass chips off a single unit of mass to a neighbour with rate \( w \). The mass density \( \rho \) is conserved by the dynamics. This model undergoes a nonequilibrium phase transition in the \((\rho - w)\) plane across a critical line \( \rho_c(w) \). We have shown analytically that \( \rho_c(w) = \sqrt{w + 1} - 1 \) in all dimensions. We also provided both direct and indirect numerical evidence that strongly suggest that the mean field answer for the single site mass distribution function \( P(m) \) might be exact in all dimensions and that the exponent \( \tau = 5/2 \) is super-universal.

However, we would like to stress one important point. Even though the single site distribution \( P(m) \) may be given exactly by the mean field answer, that does not prove that mean field theory or product measure is the exact stationary state in all dimensions. In this sense, the CM is different from the recently studied \( q \)-model of force fluctuations \[20\] where the product measure is exact in the stationary state. More precisely, in the \( q \)-model, the evolution equation for single site mass distribution \( P(m) \) involves the 2-point distribution \( P(m_1, m_2) \). Similarly the equation for 2-point involves the 3-point function \( P(m_1, m_2, m_3) \) and so on. However, if one assumes the product measure, i.e., \( P(m_1, m_2, \ldots m_n) = P(m_1)P(m_2) \ldots P(m_n) \) for all \( n \), then it was shown \[20\] that this ansatz satisfies all the equations of the hierarchy. In the CM, in a similar way, one can write down the full hierarchy of equations satisfied by the \( n \)-point distribution functions. However, unlike the \( q \)-model, the product measure ansatz does not satisfy all the equations of this hierarchy \[20\]. Nevertheless, the expression of \( P(m) \) obtained from the first equation of this hierarchy (the equation that involves only \( P(m) \) and \( P(m_1, m_2) \)) seems to be extremely close to the numerical answer. This suggests that in the stationary state of CM, correlations between masses appear only in the 3 or higher order correlations but seem to be absent at the 2-point level. This remarkable fact was also noticed recently in another aggregation model namely the asymmetric random average process with sequential updates \[19,20\], suggesting that such unusual correlations may be more generic and less exceptional.

In this paper we have studied the undirected CM. As mentioned in the introduction, the directed CM also has a qualitatively similar phase transition in the steady state though the associated critical exponents are entirely different from the undirected one. Also, the phase boundary in the \( \rho - w \) plane of the directed CM was found be quite different from the mean field phase boundary \[17\]. A proper understanding of the directed model remains an outstanding challenging problem.

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FIG. 1. The power law part of the steady state mass distributions $P(m)$ in 1- and 2-dimensions are compared with the mean field answer in the aggregate phase $\rho > \rho_c$. The data is for $w = 3$ and $\rho = 10$. The critical value is $\rho_c = 1$ for $w = 3$. In the inset panel, we show the convergence of the probability distribution to its mean field value as the system size is increased in one dimension.
FIG. 2. The steady state mass distributions in 1 and 2 dimensions in the exponential phase ($\rho < \rho_c$) is compared with the mean field answer. The data is for $w = 8$ and $\rho = 1$. The critical value is $\rho_c = 2$ for $w = 8$. 
FIG. 3. The probability of a site being occupied, $s_c(V)$, converges to its asymptotic value, $s_c(\infty)$, as a power law, i.e., $s_c(\infty) - s_c(V) \sim 1/V^{0.66}$. The data is for lattices in 1-dimension.

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