Solitons on Noncommutative Orbifold $T^2/Z_N$

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Abstract

Following the construction of the projection operators on $T^2$ presented by Gopakumar, Headrick and Spradlin, we construct a set of projection operators on the integral noncommutative orbifold $T^2/G(G = Z_N, N = 2, 3, 4, 6)$ which correspond to a set of solitons on $T^2/Z_N$ in noncommutative field theory. In this way, we derive an explicit form of projector on $T^2/Z_6$ as an example. We also construct a complete set of projectors on $T^2/Z_N$ by series expansions for integral case.

Keywords: Soliton, Projection operators, Noncommutative orbifold.

1 Introduction

Noncommutative geometry is originally an interesting topic in mathematics[1][2][3]. In the last few years, noncommutative field theories have renewed the physicist’s interest primarily due to the discovery that non-commutative gauge theories naturally arise from the low energy dynamics of D-branes in the presence of a background $B$ field and as various limits of M-theory compactification[4, 5, 6]. Quantum field theory on a noncommutative space is useful to understand various physical phenomena, such as string behaviors and D-brane dynamics. They also appear as theories describing the behavior of the electron gas in the presence of a strong, external magnetic field, the quantum Hall effect[7]. Recently Susskind and Hu, Zhang[8] proposed that noncommutative Chern-Simons theory on the plane may provide a description of (fractionally filled) quantum Hall fluid. Being nonlocal, noncommutative field theory may help to understand nonlocality at short distant in quantum gravity.

After the connection between string theory and noncommutative field theories was unraveled, the study of solitons in noncommutative space have attracted much attention[9, 10, 11, 12, 13]. Soliton solutions in field theory and string theory often shed light on the nonperturbative and strong coupling behavior of the theory, thus it is interesting to investigate these solutions in noncommutative fields theories. Gopakumar, Minwalla and Strominger found that soliton solution of noncommutative flat space can be exactly given in terms of projection operators[12]. Harvey et al set up a new method to investigate the soliton solution and M. Hamanaka and S. Terashima generalised this "Solution Generating

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integral. Next we introduce a linear transformation $R$ and discussed the relation between their trace and the commutator $q$ of the operators $U$ and $V$ on noncommutative torus ($UV = VUe^{2\pi i q}$). He proved the existence of nontrivial projection operators and explicitly presented an example with trace $1/q$ for $q$ an integer in $Z_4$ case [14]. Boca expressed the projection operators in terms of the $\theta$ function depend respectively on $U$ and $V$. Konechny, Schwartz and Walters [16, 17] have also given some $Z_2, Z_4$ invariant projection operators. Martinec and Moore pointed out that no explicit expressions for the projection operator on noncommutative integral torus with generic parameter. We find that if the vacuum state $|0>$ in their paper is replaced by any state vector $|\phi>$, their construction still works. We notice that if the state vector $|\phi>$ has some symmetries, the operators are just the projection operators on orbifold $T^2/G$ ($G = Z_N$ is a symmetry group). We then give a set of projection operators in $T^2/Z_N$. As an example, an explicit projection operator in $T^2/Z_6$ is obtained by this approach.

This paper is organized as following. We introduce the noncommutative orbifold $T^2/Z_N$ in section 2 and in next section we review the construction proposed by Gopakumar et al on the integral torus $T^2$. We show how this approach can be used to construct the projection operators which is invariant under the transformation group $Z_N$ in section 4. In section 5, we presented an explicit form of projector on $T^2/Z_6$ as an example, using theta functions with $\hat{y}_1$ and $\hat{y}_2$ as variables. In section 6, we provided a complete set of projectors on $T^2/Z_N$ by series expansions for integral case.

## 2 Noncommutative Orbifold $T^2/Z_N$

In this section, we introduce operators on the noncommutative orbifold $T^2/Z_N$. First we introduce two operators $\hat{y}_1$ and $\hat{y}_2$ on noncommutative $R^2$ which satisfy the following commutation relation:

$$[\hat{y}_1, \hat{y}_2] = i. \quad (1)$$

Define operators

$$U_1 = e^{-il\hat{y}_2}, \quad U_2 = e^{il(\tau_2\hat{y}_1 - \tau_1\hat{y}_2)}, \quad (2)$$

where $l, \tau_1, \tau_2$ are all real numbers and $l, \tau_2 > 0$. All operators on $R^2$ which commute with $U_1$ and $U_2$ constitute the operators defined on noncommutative torus $T^2$. This torus is formed as manifold which identify two points $(\hat{y}_1, \hat{y}_2) \sim (\hat{y}_1, \hat{y}_2) + r$ with $r = ml_1 + nl_2$ on noncommutative plane $R^2$, where $l_1 = (l,0), l_2 = (l\tau_1, l\tau_2)$. Thus we have

$$U_1^{-1}\hat{y}_1U_1 = \hat{y}_1 + l, \quad U_2^{-1}\hat{y}_1U_2 = \hat{y}_1 + l\tau_1, \quad (3)$$

$$U_1^{-1}\hat{y}_2U_1 = \hat{y}_2, \quad U_2^{-1}\hat{y}_2U_2 = \hat{y}_2 + l\tau_2.$$ 

The operators $U_1$ and $U_2$ are two different wrapping operators around the noncommutative torus and their commutation relation is $U_1U_2 = U_2U_1e^{-2\pi i l^2/2\tau}$. When $A = \frac{l^2\tau}{2\pi}$ is an integer, we call the torus integral. Next we introduce a linear transformation $R$, which gives

$$R^{-1}\hat{y}_1R = a\hat{y}_1 + b\hat{y}_2, \quad R^{-1}\hat{y}_2R = c\hat{y}_1 + d\hat{y}_2. \quad (4)$$
Setting \( U_1 = U, U_2 = V \), Under \( R \), if \( U \) and \( V \) change as \([10]\)

\[
Z_2: \quad U \rightarrow U^{-1}, \quad V \rightarrow V^{-1}, \\
Z_3: \quad U \rightarrow V, \quad V \rightarrow U^{-1}V^{-1}, \\
Z_4: \quad U \rightarrow V, \quad V \rightarrow U^{-1}, \\
Z_6: \quad U \rightarrow V, \quad V \rightarrow U^{-1}V, \\
\]

then we refer \( R \) as a \( Z_N \) symmetry rotation of the torus \( T^2 \). The operators on noncommutative orbifold \( T^2/Z_N \) are the operators of \( T^2 \) which are invariant under transformation \( R \).

If we define operators \( \hat{y}'_1 \) and \( \hat{y}'_2 \) as

\[
\hat{y}_1 = ay'_1 + b\hat{y}'_2, \quad \hat{y}_2 = \frac{1}{a}\hat{y}'_2, \\
\]

\( a = \sqrt{\frac{\tau_2}{\tau_2}}, \quad b = -\frac{\tau_1 + \tau_1'}{\sqrt{\tau_2\tau_2'}}, \quad l' = \frac{l}{a}, \)

then we get

\[
[\hat{y}'_1, \hat{y}'_2] = i, \quad U_1 = e^{-il'\hat{y}'_2}, \\
U_2 = e^{il'(\tau'_2\hat{y}_1 - \tau'_1\hat{y}_2)}. \\
\]

From the above result, we notice that the noncommutative torus is invariant after taking a suitable module parameter \( \tau = \tau_1 + i\tau_2 \). Now we consider the rotations in symmetric orbifolds \( T^2/Z_N(\tau_N = Z_2, Z_3, Z_4, Z_6) \). Let \( \tau = e^{\frac{2\pi i}{N}} = e^{i\theta} \), then the transformation

\[
R^{-1}\hat{y}_1 R = \cos \theta \hat{y}_1 + \sin \theta \hat{y}_2, \quad R^{-1}\hat{y}_2 R = \cos \theta \hat{y}_2 - \sin \theta \hat{y}_1. \\
\]

will give corresponding transformation of the operators \( U_1 \) and \( U_2 \) as in equation (5). Such \( R \) can be realized by

\[
R = e^{-i\theta\frac{\hat{y}_1^2 + \hat{y}_2^2}{2} + i\frac{\theta}{2}}. \\
\]

Next we take \( Z_6 \) and \( Z_4 \) as examples.

(1) \( \theta = \frac{\pi}{3} \)

\[
\tau_1 = \frac{1}{2}, \quad \tau_2 = \frac{\sqrt{3}}{2}, \quad N = 6, \\
U_1 = e^{-il\hat{y}_2}, \quad U_2 = e^{il(\frac{\sqrt{3}}{2}\hat{y}_1 - \frac{1}{2}\hat{y}_2)}, \\
R^{-1}U_1 R = U_2, \quad R^{-1}U_2 R = e^{-\pi iA}U_1^{-1}U_2. \\
\]

From the above result, we find that the lattice remain invariant under rotation.

(2) \( \theta = \frac{\pi}{2} \)

\[
\tau_1 = 0, \quad \tau_2 = 1, \quad N = 4, \\
U_1 = e^{-il\hat{y}_2}, \quad U_2 = e^{il\hat{y}_1}, \\
R^{-1}U_1 R = U_2, \quad R^{-1}U_2 R = U_1^{-1}. \\
\]

\( ^1 \)These relations may include some phase factors. It will not change the derivation in the text, see equation (10).
This shows that the whole lattice remain invariant. We can realize the operators \( \hat{y}_1 \) and \( \hat{y}_2 \) as the operators in Fock space. Introducing

\[
a = \frac{\hat{y}_2 - i\hat{y}_1}{\sqrt{2}}, \quad a^+ = \frac{\hat{y}_2 + i\hat{y}_1}{\sqrt{2}}
\]

and we have \([a, a^+] = 1\). The rotation \( R \) can be expressed by \( a, a^+ \) via

\[
R = e^{-i\theta a^+ a}.
\]

### 3 Review GHS Construction for Soliton

In this section, we review the results in paper [9]. A noncommutative space \( R^2 \) has been orbifolded to a torus \( T^2 \) with double periods \( l \) and \( \tau l \). The generators are

\[
U_1 = e^{-il\hat{y}_2}, \quad U_2 = e^{i(\tau_2\hat{y}_1 - \tau_1\hat{y}_2)}
\]

where \([\hat{y}_1, \hat{y}_2] = i\), here we just consider the case when \( A = \frac{m^2}{2\pi} \) is an integer. Introduce a state vector

\[
|\psi\rangle = \sum_{j_1,j_2} C_{j_1,j_2} U_{j_1}^{j_2} |\Omega\rangle > (j_1,j_2 \in Z)
\]

that satisfies

\[
<\psi|U_{j_1}^{j_2} |\psi\rangle = \delta_{j_1,0}\delta_{j_2,0}.
\]

The state \(|\Omega\rangle > \) will be specified later. Then a projection operator on \( T^2 \) can be constructed as

\[
P = \sum_{j_1,j_2} U_{j_1}^{j_2} |\psi\rangle <\psi|U_{j_1}^{j_2} |\psi\rangle.
\]

The power series of \( \hat{y}_1 \) and \( \hat{y}_2 \) can be made up of the power series of \( a \) and \( a^+ \). Moreover the formula \(|0 > < 0| =: e^{-a^+ a} : \) indicates that any \(|\psi > < \psi|\) can be constituted by the power series of \( a \) and \( a^+ \). The projection operator is therefore spanned by the operators \( \hat{y}_1 \) and \( \hat{y}_2 \). It is easy to check \( P^2 = P \) and \( U_1^{-1} PU_1 = P \). So \( P \) is an projection operator on noncommutative \( T^2 \). The \( kq \) representation[18][19] provides a basis of common eigenstate of \( U_1 \) and \( U_2 \):

\[
|k,q\rangle = \sqrt{\frac{1}{2\pi}} e^{-i\tau_1 \hat{y}_2^2/2\tau_2} \sum_j e^{ijkl} |q+jl\rangle
\]

where the ket on the right is a \( \hat{y}_1 \) eigenstate. We have

\[
U_1 |k,q\rangle = e^{-ik}|k,q\rangle, \quad U_2 |k,q\rangle = e^{ij\tau_2 q}|k,q\rangle,
\]

\[
id = \int_a^{2\pi+a} dk \int_b^{l+b} dq|k,q\rangle <k,q| \]

where \( a \) and \( b \) are real numbers and \(|k,q\rangle = |k + \frac{2\pi}{\tau_2}, q\rangle = e^{ik} |k,q + l\rangle\). In terms of wave functions in the \( kq \) representation, \(|\psi\rangle \) becomes

\[
C_{\psi}(k,q) \equiv <k,q|\psi\rangle = \sum_{j_1,j_2} C_{j_1,j_2} e^{-ij_1 \hat{y}_1 + ij_2 \hat{y}_2 \tau_2 q} <k,q|\Omega\rangle > = \tilde{c}(k,q) C_0(k,q)
\]
where \( \tilde{c}(k, q) = \sum_{j_1, j_2} C_{j_1, j_2} e^{-i j_1 l k + i j_2 l r_2 q}, C_0(k, q) = < k, q|\Omega >. \) Note that \( \tilde{c}(k, q) \) is doubly periodic:

\[
\tilde{c}(k + \frac{2\pi}{l}, q) = \tilde{c}(k, q + \frac{l}{A}) = \tilde{c}(k, q).
\]

The orthonormality condition (16) becomes

\[
\delta_{j_1,0}\delta_{j_2,0} = \int_0^{2\pi} dk \int_0^l dq e^{-i j_1 l k + i j_2 l r_2 q}|\tilde{c}(k, q)|^2 |C_0(k, q)|^2.
\]

(21)

The coefficient \( C_{j_1, j_2} \) can be obtained if and only if \( \tilde{c}(k, q) \) is double periodic function with periods \( 2\pi/l \) and \( l/A \) for \( k \) and \( q \) respectively. So we rewrite the above equation as

\[
\delta_{j_1,0}\delta_{j_2,0} = \int_0^{2\pi} dk \int_0^{l/A} dq e^{-i j_1 l k + i j_2 l r_2 q}|\tilde{c}(k, q)|^2 \sum_{n=0}^{A-1} |C_0(k, q + \frac{ln}{A})|^2.
\]

(22)

This hold for any \( j_1 \) and \( j_2 \) if and only if \( |\tilde{c}(k, q)|^2 \sum_{n=0}^{A-1} |C_0(k, q + \frac{ln}{A})|^2 = \frac{1}{2\pi} \). Then we have

\[
|\tilde{c}(k, q)| = \frac{1}{\sqrt{2\pi \sum_{n=0}^{A-1} |C_0(k, q + \frac{ln}{A})|^2}}.
\]

(23)

Setting \( e^{i\beta} \) as phase factor of \( \tilde{c}(k, q) \), we have

\[
C_{\psi}(k, q) = \frac{C_0(k, q)e^{i\beta}}{\sqrt{2\pi \sum_{n=0}^{A-1} |C_0(k, q + \frac{ln}{A})|^2}}.
\]

(24)

4 The Projection Operator on \( T^2/Z_N \)

In the last section, we reviewed how to construct the projection operators on noncommutative torus. In this section, we will discuss how to construct the projection operator on the noncommutative orbifold \( T^2/Z_N \) following the result of the last section. Recall the projection operator

\[
P = \sum_{j_1, j_2} U_1^{j_1} U_2^{j_2} |\psi >= < \psi|U_2^{-j_2} U_1^{-j_1}
\]

(25)

and transform it by rotation \( R \)

\[
R^{-1}PR = \sum_{j_1, j_2} (U_1')^{j_1} (U_2')^{j_2} R^{-1} |\psi >= < \psi|R(U_2')^{-j_2} (U_1')^{-j_1}
\]

(26)

where \( U_i' = R^{-1}U_iR \). Considering the transformation group \( G = Z_N \), we get

\[
R^{-1}PR = \sum_{j_1', j_2'} U_1^{j_1'} U_2^{j_2'} R^{-1} |\psi >= < \psi|R U_2^{-j_2'} U_1^{-j_1'}
\]

(27)

where \( j_1' = -j_1, j_2' = -j_2 \) for \( Z_2 \) case, \( j_1' = -j_2, j_2' = j_1 - j_2 \) for \( Z_3 \) case, \( j_1' = -j_2, j_2' = j_1 \) for \( Z_4 \) case and \( j_1' = -j_2, j_2' = j_1 + j_2 \) for \( Z_6 \) case. Then we can obtain \( R^{-1}PR = P \) as long as

\[
R|\psi >= e^{i\alpha}|\psi >.
\]

(28)
We can show that this can be satisfied if \( R|\Omega >= e^{i\alpha}|\Omega >\). In the next step, we take \( G = Z_6\) as an example to prove this (it is easy to generalize this to other cases). Assume

\[
|\psi >= \sum_{j_1,j_2} C_{j_1,j_2} U_1^{j_1} U_2^{j_2} |\Omega > (j_1, j_2 \in Z)
\]  

(29)
satisfies

\[
< \psi| U_1^{j_1} U_2^{j_2} |\psi > = \delta_{j_1,0}\delta_{j_2,0}.
\]  

(30)
Setting \( R|\Omega >= e^{i\alpha}|\Omega >\), we have

\[
R|\psi > = \sum_{j_1,j_2} C_{j_1,j_2} U_1^{j_2} U_2^{j_1+j_2} R|\Omega >
\]

\[
= \sum_{j_1,j_2} C'_{j_1,j_2} U_1^{j_2} U_2^{j_1} |\Omega >
\]

(31)
where

\[
C'_{j_1,j_2} = C_{j_1+j_2,-j_1} e^{i\alpha}
\]

(32)
and

\[
< \psi|R^{-1} U_1^{j_1} U_2^{j_2} R|\psi > = < \psi|U_2^{j_1} (U_1^{-j_2} U_2^{j_2})|\psi >
\]

\[
= \delta_{j_1,j_2,0} \delta_{-j_2,0} = \delta_{j_1,0}\delta_{j_2,0}.
\]

(33)
Adding a condition \( C_{j_1,j_2}^* = C_{-j_1,-j_2} e^{-2i\beta}\), we have

\[
\tilde{c}(k,q) = \tilde{c}(k,q) e^{-2i\beta}.
\]

(34)
The unique solution satisfying the above equations is

\[
\tilde{c}(k,q) = e^{i\beta} \frac{e^{i\beta}}{\sqrt{\frac{2\pi}{A} \sum_{n=0}^{A-1} |C_0(k,q + \frac{ln}{A})|^2}}.
\]

(35)
In brief, The \( C_{j_1,j_2} \) which satisfies the condition(29, 30, 34) is uniquely determined as demonstrated in section 3. On the other hand, from equation (32) we find

\[
(C'_{j_1,j_2})^* = C''_{-j_1,-j_2} e^{-2i(\alpha+\beta)},
\]

(36)
giving

\[
\tilde{c}^*(k,q) = \tilde{c}(k,q) e^{-2i(\alpha+\beta)}.
\]

(37)
Equations (31)(33) and (37) also uniquely determine \( \tilde{c}(k,q) \) as

\[
\tilde{c}(k,q) = e^{i(\alpha+\beta)} \frac{e^{i(\alpha+\beta)}}{\sqrt{\frac{2\pi}{A} \sum_{n=0}^{A-1} |C_0(k,q + \frac{ln}{A})|^2}} = \tilde{c}(k,q) e^{i\alpha}
\]

(38)
for \( R|\psi >\). We then have \( |\psi' >= R|\psi > = e^{i\alpha}|\psi >\). In conclusion, if the vector \(|\Omega >\) satisfies \( R|\Omega >= e^{i\alpha}|\Omega >\), the state \( |\psi >\) will satisfy equation (28) and the projection operator by GHS construction will be a projection operator on noncommutative orbifold \( T^2/G \).

In the above discussion, we know that the crucial point is that the state vector \(|\Omega >\) must be invariant under the rotation \( R\), namely \( R|\Omega >= e^{i\alpha}|\Omega >\). Now we show how to construct such state vector. The operator for rotation is

\[
R = e^{-i\theta a^a}.
\]

(39)
We can set
\[ |\Omega> = \sum_{j=0}^{N-1} R^j e^{\frac{2\pi}{N}js} |\phi> \]  
(40)

where \(|\phi>\) is an arbitrary state vector, \(R = e^{-\frac{2\pi}{N}a^+a}, s = 0, \ldots, N - 1\). Since
\[ R^N |\phi> = e^{-\frac{2\pi}{N}a^+a} |\phi> = \sum_{n=0}^{N-1} c_n (a^+)^n |0> \]
(41)

thus
\[ R|\Omega> = \sum_{j=0}^{N-1} R^{j+1} e^{\frac{2\pi}{N}js} |\phi> \]
\[ = e^{-is\frac{2\pi}{N}} |\Omega> \]
\[ = e^{i\alpha} |\Omega> \]  
(42)

If we obtain the expression for \(C_0(k,q)\), it is easy to write the expression for the field configuration for the projection operator by equation (24) \[9\] or get the Fourier expansion by equations (55)(56). In next step, we take an example to show how to construct \(C_0(k,q) = <k,q|\Omega>\). Introduce coherent states
\[ |z> = e^{-\frac{1}{2}zz^*} e^{a^+a} |0> \]  
(43)

where \(z = x + iy, \bar{z} = x - iy\), which satisfies
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy |z><z| = \text{identity}. \]
(44)

Thus from equation(39), we get
\[ R|z> = e^{-\frac{1}{2}zz^*} e^{a^+a} |\omega z> \]
(45)

where \(\omega = e^{-i\frac{2\pi}{N}}\), we can employ \(<y_2|0> = \frac{1}{\pi^2} e^{-y_2^2}/2\), \(|y_2>\) is the eigenstate of the operator \(\hat{y}_2\), to obtain
\[ <y_2|z> = \frac{1}{\pi^4} e^{-z_2^2/2-z\bar{z}/2} e^{-y_2^2/2+\sqrt{2}zy_2}. \]  
(46)

We have
\[ <k,q|z> = \int <k,q|y_2><y_2|z> dy_2 \]
\[ = \frac{1}{\sqrt{\pi^4 l}} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left( \frac{q + \frac{\pi}{2}\sqrt{2}z}{l}, \frac{\tau}{A} \right) e^{-\frac{\pi}{2\tau^2}k^2+ikq+\sqrt{2}kz-(z^2+z\bar{z})/2}. \]
(47)

Letting
\[ |\phi> = \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy |z><z|\phi> = \int_{-\infty}^{\infty} dx dy F(z)|z>, \]
(48)
we can get

$$C_0(k, q) = \langle k, q|\Omega > = \frac{1}{\pi} \int < k, q| \sum_{j=0}^{N-1} R_j e^{i \frac{2\pi j}{N} s} |z > < z|\phi > dxdy$$

$$= \int \sum_{j=0}^{N-1} e^{i \frac{2\pi j}{N} s} < k, q|\omega^j z > |F(z)dx dy \quad (49)$$

where $F(z)$ is an arbitrary function. After obtaining $C_0(k, q)$, we can then compute $< k, q|\psi >$ by equation (24) and the Fourier coefficient for the projection operator $P$ on noncommutative orbifold $T^2/Z_N$ by equation (56).

5 An Example for $T^2/Z_6$ case

In this section, we give an example for $T^2/Z_6$ case. We first review the Weyl-Moyal transformation on torus and then present the explicit expression for the projection operator by the Fourier series of the operator $\hat{y}_1$ and $\hat{y}_2$. Define

$$A(\hat{p}) = \sum_{j_1, j_2} U_{j_1}^{j_2} b(\hat{p}) U_{j_2}^{j_1} (j_1, j_2 \in Z) \quad (50)$$

where $U_1 = e^{is_1 \hat{p}_1}, U_2 = e^{is_2 \hat{p}_2}$ and $\hat{p}_1, \hat{p}_2$ are linear combinations of $\hat{y}_1$ and $\hat{y}_2, [\hat{p}_1, \hat{p}_2] = i$. It is easy to see that $U_i^{-1} A(\hat{p}) U_i = A(\hat{p})$, namely $A(\hat{p})$ is an operator on noncommutative torus $T^2$. The field configuration for $A(\hat{p})$ is

$$\Phi_A(p) = \frac{(2\pi)^2}{s_1 s_2} \sum_{mn} tr\{ e^{2\pi i [(\hat{y}_1 - p_1) \frac{m}{s_1} + (\hat{y}_2 - p_2) \frac{n}{s_2}] } b(\hat{p}) \}. \quad (51)$$

We can also reobtain $A(\hat{p})$ by the Weyl-Moyal transformation from $\Phi_A(p)$,

$$A(\hat{p}) = \sum_{mn} \frac{1}{2\pi s_1 s_2} \int_{0}^{s_1} dp_1 \int_{0}^{s_2} dp_2 \Phi_A(p) e^{2\pi i [(\hat{y}_1 - p_1) \frac{m}{s_1} + (\hat{y}_2 - p_2) \frac{n}{s_2}]} . \quad (52)$$

We now set $\hat{p}_1 = -\hat{y}_2, \hat{p}_2 = \hat{y}_1 - \frac{\pi}{s_2} \hat{y}_2, s_1 = l \tau_2, s_2 = l, b(\hat{p}) = |\psi > < \psi|$, and have

$$U_1 = e^{-il \hat{y}_2}, \quad U_2 = e^{il(\tau_2 \hat{y}_1 - \tau_1 \hat{y}_2)}. \quad (53)$$

Then the operator $A(\hat{p})$ becomes the projection operator $P$. The field configuration for the projection operator $P$ is

$$\Phi_P(y) = \frac{(2\pi)^2}{l^2 \tau_2} \sum_{j_1, j_2} < \psi | e^{2\pi i (j_1 \hat{y}_1 - y_1) + \frac{2\pi i j_1}{\tau_2} (j_2 - y_2)} |\psi > . \quad (54)$$

The Fourier expansion for the projection operator is obtained by Weyl-Moyal transformation (52),

$$P = \sum_{j_1, j_2} D_{j_1, j_2} e^{-2\pi i (j_1 \hat{y}_1 + \frac{2\pi i j_1}{\tau_2} \hat{y}_2)}, \quad (55)$$

$$D_{j_1, j_2} = \frac{1}{A} < \psi | e^{2\pi i (j_1 \hat{y}_1 + \frac{2\pi i j_1}{\tau_2} \hat{y}_2)} |\psi > \quad (56)$$

$$= \frac{1}{A} \int_{0}^{2\pi} dk \int_{0}^{l} dq < k, q|\psi > |\psi|k, q - \frac{l s}{A} > e^{2\pi i j_1 (q/l - s/A)} e^{im k e^{\pi ij_1 j_2/A}}$$

8
where \( j_2 = mA + s, s = 0, \ldots, A - 1 \).

Rewrite the above equations as

\[
P = \sum_{j_1j_2} D_{j_1j_2} e^{-2\pi i j_2 \frac{y_1}{l' r_2}} e^{-2\pi i j_1 \frac{y_1}{l' r_2}} e^{-\pi i j_1j_2/A}
\]

\[
= \sum_s (e^{-2\pi i \frac{y_1}{l' r_2}})^s \sum_{j_1m} (e^{-2\pi i A \frac{y_1}{l' r_2}})^m (e^{-2\pi i \frac{y_1}{l' r_2}})^{j_1} D_{j_1m s}
\]

\[
= \sum_s (u_1)^s \sum_{j_1m} D_{j_1m s} (u_1^A)^m (u_2)^{j_1}
\]

(57)

where \( u_1 = e^{-2\pi i \frac{y_1}{l' r_2}}, u_2 = e^{-2\pi i \frac{y_1}{l' r_2}} \) and

\[
D_{j_1m s} = D_{j_1j_2} e^{-\pi i j_1j_2/A} |_{j_2=mA+s}
\]

\[
= \frac{1}{A} \int_0^{2\pi} dk \int_0^l dq' <k, q'+\frac{ls}{A}|\psi><|k, q'> e^{2\pi i j_1(q')} e^{2\pi i m(q')}
\]

(58)

where \( q' = q - \frac{ls}{A} \). This is a calculation for the Fourier coefficient of periodic (in \( k \) and \( q' \)) function \(<k, q'+\frac{ls}{A}|\psi><|k, q'> \equiv f_s \), that is

\[
f_s = \frac{A}{2\pi} \sum_{j_1m} D_{j_1m s} e^{-2\pi i j_1(q')} e^{-2\pi i m(q')}
\]

(59)

Notice that in equation (57), \( u_1^A \) and \( u_2 \) commute with each other. Thus when we obtain an explicit expression of \( f_s(K, Q) \) in terms of \( Q = e^{-2\pi i (\frac{q}{T})} \) and \( K = e^{-2\pi i (\frac{q}{A})} \) for real \( \frac{q}{T} \) and \( \frac{q}{A} \), we can immediately write \( P \) as

\[
P = \frac{2\pi}{A} \sum_s u_1^s f_s(u_1^A, u_2).
\]

(60)

We calculate a projector \( P \) as an example in the following. Let \( |\Omega| = |0> \), which is obviously \( R \) invariant. We have

\[
<kq|\psi> = C\psi(k, q) = \frac{C_0(k, q)}{\sqrt{\frac{2\pi}{A} \sum_{n=0}^{A-1} |C_0(k, q + \frac{ln}{A})|^2}}
\]

(61)

with \( C_0(k, q) = <k, q|0> \) (ref. [9]). Then the corresponding \( f_s \) is

\[
f_s = \frac{C_0(k, q + \frac{ls}{A})C_0^*(k, q)}{\frac{2\pi}{A} \sum_{n=0}^{A-1} C_0(k, q + \frac{ln}{A})C_0^*(k, q + \frac{ln}{A})}
\]

(62)

where

\[
C_0(k, q) = \frac{1}{\sqrt{l\pi 1/4}} \begin{bmatrix} q & 0 \\ l & \tau \end{bmatrix} e^{-\frac{\pi i}{l' r_2} k^2 + ikq}
\]

\[
= \sqrt{\frac{A_1}{l' r_2 \sqrt{\pi}}} \begin{bmatrix} 0 & 0 \\ 0 & \frac{lk}{2\pi} + \frac{Aq}{l' \pi} \end{bmatrix} e^{-\frac{\pi i Aq^2}{2\sqrt{\pi}}}.\quad (63)
\]
Thus we have for real \( x = u \) and \( \frac{y}{2\pi} = v \),

\[
g(u, v)_{ss'} = C_0(k, q + \frac{ls}{A})C_0^*(k, q + \frac{ls'}{A})
\]

\[
= \frac{1}{l\sqrt{\pi}} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \left( u + \frac{s}{A} + \frac{sv}{A}, \frac{\tau v}{A} \right) \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u + \frac{s'}{A} + \frac{s'v}{A}, -\frac{\tau'^*}{A})
\]

\[
e^{-\frac{i\pi}{A}sv^2 + 2\pi i \frac{s'}{2\pi}v^2},
\]

\[
= \frac{A}{l|\tau|\sqrt{\pi}} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \left( v + \frac{A}{\tau}(u + \frac{s}{A}), -\frac{A}{\tau} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (v + \frac{A}{\tau^*}(u + \frac{s'}{A}), \frac{1}{\tau^*})
\]

\[
e^{-\frac{i\pi}{A}(u + \frac{s}{2\pi} + s'T)^2 + 2\pi i \frac{s'}{2\pi}(u + \frac{s'}{2\pi})^2},
\]

due to

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z, \tau)^* = \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z^*, -\tau^*).
\]

In the \( T^2/Z_6 \) case, \( \tau = e^{\frac{i\pi}{3}} \), so we have

\[
g(u + 1, v) = g(u, v + 1) = g(u, v),
\]

\[
g(u + \tau, v) = e^{-2\pi i A(2u + \frac{s}{2\pi} + x)}g(u, v),
\]

\[
g(u, v + A\tau) = e^{-2\pi i (2v + Au + y)}g(u, v)
\]

where \( x = \tau - \frac{1}{2} + \frac{s + s'}{A} \) and \( y = A\tau - \frac{A}{2} + \frac{s}{\tau} + \frac{s'}{\tau^*} \). From the above equation, one can prove

\[
g(u, v) = \{ \sum_{i=1}^{2} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (Au + v - vi + \frac{1}{2}, \frac{\tau A}{2}) \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (v - vi + \frac{A\tau}{2}, \tau A)
\]

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u + \frac{s}{A} + \frac{\tau v}{A}, \frac{\tau A}{2}) \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u + \frac{s'}{A} + \frac{\tau' v}{A}, \frac{\tau'^*}{A})
\]

\[
X(v_i)/[\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (Au + \frac{1}{2}, \tau A) \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{\tau A}{2}, \tau A)] \equiv \{G_{ss'}(u, v)/[v(u)]
\]

where

\[
v_1 = \frac{1}{2} \left( \frac{A}{2} + \frac{A\tau}{2} + \frac{1}{2} - \frac{As}{\tau} - \frac{As'}{\tau^*} \right),
\]

\[
v_2 = \frac{1}{2} \left( \frac{A}{2} + \frac{A\tau}{2} + \frac{1}{2} - \frac{As}{\tau} - \frac{As'}{\tau^*} \right),
\]

\[
X(v_i) = \frac{1}{l\sqrt{\pi}} e^{-\frac{2\pi i}{A}vu^2 + 2\pi ivu(A - s')},
\]

Thus

\[
f_s = \frac{G_{ss}(u, v)}{2\pi \sum_{n=0}^{A-1} G_{nn}(u, v)}.
\]

The proof is as follows. Because of equations (67) and (69), the entire function \( g \) of \( v \) belongs to a two dimensional function space of \( v \). Properly choose two functions as a base. Then fix the coefficients (they are functions of \( u \)) at two special values of \( v \) \( (v_1 \) and \( v_2) \), we finally obtain equation (70). Let

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z, \tau) \equiv \Theta(Z, \tau) = \sum_m e^{\pi im^2 \tau} Z^m, \quad Z = e^{-2\pi iz}.
\]
Then

\[ G_{ss'}(u,v) = \sum_{i=1}^{2} \Theta(KQ^Ae^{2\pi i(v_i-\frac{\pi}{4})},A)\Theta(Ke^{2\pi i(v_i-\frac{A\pi}{4})},A)\Theta(Qe^{-2\pi i(\frac{v}{A}+\frac{\pi}{4})},\frac{T}{A}) \]

\[ \Theta(Qe^{-2\pi i(\frac{v}{A}+\frac{\pi}{4})},\frac{T}{A})X(v_i) \equiv \Phi_{ss'}(K,Q), \quad (76) \]

giving

\[ P = \sum_{s=0}^{A-1} u_1^A \frac{\Phi_{s0}(u_1^A,u_2)}{\sum_{n=0}^{A-1} \Phi_{nn}(u_1^A,u_2)}. \quad (77) \]

This is an explicit \( T^2/Z_6 \) projector.

## 6 Complete set of projections

We assume that all operators in \( T^2 \) can be expressed as

\[ \hat{B} = \sum_{j_1,j_2} U_{1}^{j_1} U_{2}^{j_2} \hat{b} U_{2}^{-j_2} U_{1}^{-j_1} \quad (78) \]

for some operators \( \hat{b} \) in noncommutative space \( R^2 \). Reorganize the complete set \(|k,q>\) as \(|k,q,s>\) = \(|k,q_0,s>\), \( q = q_0 + \frac{s}{A} \), where \( k \in [0,\frac{2\pi}{A}), q_0 \in [0,\frac{\pi}{A}), s = 0,\ldots,A-1 \). Equation (19) becomes

\[ id = \sum_{s=0}^{A-1} \int_{0}^{2\pi} \int_{0}^{\pi} dq_0 |k,q_0,s><k,q_0,s|. \quad (79) \]

Combining the above equation and

\[ \sum_{j} e^{ijx} = \sum_{m} 2\pi \delta(x + 2\pi m). \quad (80) \]

We can get

\[ <k,q_0 + \frac{l}{A} n |\hat{B} |k',q_0' + \frac{l}{A} n' > = \frac{(2\pi)}{A} \delta(k' - k) \delta(q_0' - q_0) <k,q_0 + \frac{l}{A} n |\hat{b} |k,q_0 + \frac{l}{A} n' >. \quad (81) \]

So we have

\[ <k,q_0 + \frac{l}{A} n |\hat{A}\hat{B} |k',q_0' + \frac{l}{A} n' > \]

\[ = <k,q_0 + \frac{l}{A} n |\hat{A}(id)\hat{B} |k',q_0' + \frac{l}{A} n' > \]

\[ = \int_{0}^{2\pi} \int_{0}^{\pi} \sum_{n''} <k,q_0 + \frac{l}{A} n |\hat{b} |k'',q_0'' + \frac{l}{A} n'' > <k'',q_0'' + \frac{l}{A} n'' |\hat{b} |k',q_0' + \frac{l}{A} n' > dk'' dq''_0 \]

\[ = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{(2\pi)^{2}}{A^2} \sum_{n''} \delta(k'' - k) \delta(k' - k'') \delta(q_0'' - q_0') \delta(q_0'' - q_0) \]

\[ <k,q_0 + \frac{l}{A} n |\hat{a} |k,q_0 + \frac{l}{A} n'' > <k'',q_0'' + \frac{l}{A} n'' |\hat{b} |k'',q_0'' + \frac{l}{A} n' > dk'' dq''_0 \]

\[ = \frac{(2\pi)^{2}}{A^2} \sum_{n''} \delta(k'' - k) \delta(q_0'' - q_0) <k,q_0 + \frac{l}{A} n |\hat{a} |k,q_0 + \frac{l}{A} n'' > <k,q_0 + \frac{l}{A} n'' |\hat{b} |k,q_0 + \frac{l}{A} n' >. \quad (82) \]
From equation (81) and (82), one concludes that the necessary and sufficient condition of a projection operator in $T^2$ is
\[
\sum_{n''} M_b(k, q_0)_{nn''} M_b(k, q_0)_{n''n'} = M_b(k, q_0)_{nn'}
\]  
(83)
at each point $(k, q_0)$, where $M_b(k, q_0)_{nn''} = \frac{2\pi}{A} < k, q_0 + \frac{l}{A} n'' | b | k, q_0 + \frac{l}{A} n'' >$, The matrix $M(k, q_0)$ satisfying the above equation is always diagonalizable. Thus
\[
M_b(k, q_0) = S(k, q_0)^{-1} \tilde{M}_b(k, q_0) S(k, q_0)
\]  
(84)gives the complete set of projections of the form (78) in $T^2$, where $S$ is an arbitrary invertible matrix and $\tilde{M}$ is a diagonized matrix with entry 0 and 1, that is to say
\[
\tilde{M}_b(k, q_0)_{nn'} = \delta_{nn'} \epsilon_n(k, q_0),
\]  
(85)
\[
\epsilon_n(k, q_0) = \{0, 1\}.
\]  
(86)
We next study the conditions for $P$ being invariant under rotation $R$,
\[
R^{-1} P R = P.
\]  
(87)
First notice that from equation (5), the common eigenvectors $|k, q_0 + \frac{l}{A} s >$ of $U_1$ and $U_2$ are still eigenvectors of
\[
U'_1 = R^{-1} U_1 R, \quad U'_2 = R^{-1} U_2 R.
\]  
(88)
We have
\[
U'_1 |k, q_0 + \frac{l}{A} s > = \lambda_1 |k, q_0 + \frac{l}{A} s >,
\]  
(89)
\[
U'_2 |k, q_0 + \frac{l}{A} s > = \lambda_2 |k, q_0 + \frac{l}{A} s >,
\]  
(90)giving
\[
U_1 R |k, q_0 + \frac{l}{A} s > = \lambda_1 R |k, q_0 + \frac{l}{A} s >,
\]  
(91)
\[
U_2 R |k, q_0 + \frac{l}{A} s > = \lambda_2 R |k, q_0 + \frac{l}{A} s >.
\]  
(92)Thus $R |k, q_0 + \frac{l}{A} s >$ is still a common eigenvector of $U_1$ and $U_2$. Since the eigenvalue of $U_2$ is $A$ fold degenerate, we conclude from equation (19)
\[
R |k, q_0 + \frac{l}{A} n > = \sum_{n'} A(k, q_0)n' | k', q_0 + \frac{l}{A} n' >
\]  
(93)for a definite $(k', q_0')$. One can derive explicit relations for $(k', q_0')$ and $(k, q_0)$ for all $Z_N$ cases, which is essentially linear relations $W : (k, q_0) \rightarrow (k', q_0')$ and $(W)^N = id$. Since $R$ is unitary, the matrix $A(k, q_0)$ is also unitary, namely
\[
A^*(k, q_0)_{nn'} = A^{-1}(k, q_0)_{n'n}. 
\]  
(94)
One can show that the map $W$ is an area preserving map, thus
\[
\delta(k_1 - k_2) \delta(q_{01} - q_{02}) = \delta(k'_{01} - k'_{02}) \delta(q'_{01} - q'_{02}).
\]  
(95)
Then we have

\[<k_1, q_01 + \frac{1}{A}n_1|R^{-1}PR|k_2, q_02 + \frac{1}{A}n_2>\]

\[= \delta(k'_1 - k'_2)\delta(q'_01 - q'_02) <k'_1, q'_01 + \frac{1}{A}n'_1|\hat{b}|k'_2, q'_02 + \frac{1}{A}n'_2>A^* (k_1, q_01)n_1n'_1A(k_2, q_02)n_2n'_2\]

\[= \delta(k_1 - k_2)\delta(q_01 - q_02) <k_1, q_01 + \frac{1}{A}n_1|\hat{b}|k_2, q_02 + \frac{1}{A}n_2> . \quad (96)\]

At last, we obtain

\[M(kq_0)_{n_1n_2} = \sum_{n'_1n'_2} A^* (k, q_0)_{n_1n'_1}M(k'q'_0)_{n'_1n'_2}A(k, q_0)n_2n'_2, \quad (97)\]

\[M(k'q'_0) = A^* (k, q_0)M(kq_0)A^* (k, q_0). \quad (98)\]

That is, from the matrix \(M\) of a giving point \((k, q_0)\), we can get a definite \(M\) for the point \((k', q'_0) = W(k, q_0)\), if the corresponding operator is \(R\) rotation invariant. From the explicit expression of \(W\), we can show that one can always divide the area \(\sigma : (k \in [0, \frac{2\pi}{A}], q_0 \in [0, \frac{2\pi}{A}])\) into \(N\) pieces \(\sigma_1, \sigma_2, \ldots, \sigma_N\) for \(T^2/Z_N\), where \(W : \sigma_i \to \sigma_{i+1}, \sigma_N \to \sigma_1\). We can arbitrarily choose \(M(k, q_0)\) in \(\sigma_1\) by equation (84) and get \(M(k, q_0)\) in \(\sigma_2, \ldots, \sigma_N\) by equation (98). In such way we obtain all projective operators in \(T^2/Z_N\) by series expansion

\[P = \sum_{j_1,j_2} D_{j_1j_2} e^{-\frac{2i\pi}{A}(j_1y_1 + j_2y_2) + \frac{1}{2}\frac{j_1^2 + j_2^2}{a^2}}\]

\[D_{j_1j_2} = \frac{1}{A}tr(\hat{b}e^{\frac{2i\pi}{A}(j_1y_1 + j_2y_2) + \frac{1}{2}\frac{j_1^2 + j_2^2}{a^2}})\]

\[= \frac{1}{A} \sum_{nn'} \int dk dq k' dq' <k, q_0 + \frac{1}{A}n|\hat{b}|k', q_0 + \frac{1}{A}n' > \]

\[<k', q'_0 + \frac{1}{A}n'|e^{\frac{2i\pi}{A}(j_1y_1 + j_2y_2) + \frac{1}{2}\frac{j_1^2 + j_2^2}{a^2}}|k, q_0 + \frac{1}{A}n>\]

\[= \frac{1}{A} \sum_{nn'} \int dk dq k' dq' <k, q_0 + \frac{1}{A}n|\hat{b}|k', q_0 + \frac{1}{A}n' > \]

\[\delta(k' - k')\delta(q'_0 - q'_0)E(k, q_0, j_1, j_2)n'n\]

\[= \sum_{nn'} \int dk dq (2\pi)^{-1}M(k, q_0)_{nn'}E(k, q_0, j_1, j_2)n'n\]

\[= \int dk dq (2\pi)^{-1}trM(k, q_0)E(k, q_0, j_1, j_2) \quad (99)\]

where

\[E(k, q_0, j_1, j_2)_{n'n} = e^{2\pi i j_1(q_0 + \frac{1}{A}n')/l}e^{\pi ij_2j^2/A}e^{i(j_2 + n' - n)2\pi A/\lambda} \sum_j \delta_{j_2 + n' - n,jA}. \quad (100)\]

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