Criterion for robust global asymptotic stability of the linear time-varying systems

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Abstract

In this paper we prove a new criterion for robust global asymptotic stability of the zero solution of LTV system \( \dot{x} = A(t)x \). To prove the result, a logarithmic norm of the system matrix \( A(t) \) will be used under which the stability becomes a topological notion depending on the chosen vector norm in \( \mathbb{R}^n \).

Keywords: Linear time-varying system, perturbation, robust global asymptotic stability, logarithmic norm.

1. Introduction

Let us consider \( x = 0 \) being asymptotically stable equilibrium point of the linear time-varying (LTV) nominal system \( \dot{x} = A(t)x, \ t \geq 0 \). What can we say about the asymptotic stability its perturbation \( \dot{x} = A(t)x + w(x,t) \)? This question represents one of the fundamental problems in the area of robust stability and robustness of the systems in general and so the effect of (known or unknown) perturbations on the solutions of nominal system as a potential source of instability attracts the attention and interest of scientific community for a long time in the various contexts. A comprehensive overview of the most significant results on robust control theory and its history is presented in [11].

In this paper, we specifically prove that asymptotically stable zero solution of unperturbed system remains "attractive" for a wide class of perturbations \( w(x,t) \) in the sense of convergence of all solutions of perturbed system to the origin \( x = 0 \) as \( t \to \infty \) provided the perturbing term satisfies some
growth constraints - the sufficient conditions are summarized in Theorem 6. Note that \( x = 0 \) may not be solution of perturbed system. The (potentially unknown) perturbation \( w \) represents modeling errors, uncertainties, external disturbances, etc. Throughout this paper we assume that \( x \) and \( w \) are \( n \)-dimensional column vectors and \( A(t) \) is a square matrix of the same dimension. We will always assume that \( A(t) \) is continuous for all \( t \geq 0 \), and \( w \) is continuous in \((x, t)\) for \( \|x\| < \infty \) and \( 0 \leq t < \infty \).

Stability analysis for time-varying linear systems is of constant interest in the control community. One reason is the growing importance of adaptive controllers for which underlying closed-loop adaptive system is time-varying and linear \([7], [14]\). The second one is that LTV systems naturally arise when one linearizes nonlinear systems about a non-constant nominal trajectory. In contrast the linear time-invariant (LTI) cases which have been thoroughly understood in the analysis and synthesis, many properties of the LTV systems are still not clear and not resolved. Here, the system (robust) stability analysis can serve as an appropriate example.

There are some papers providing some sufficient conditions for exponential, see e. g., \([6, 18, 19]\) or/and asymptotic stability \([13, 20]\) of LTV systems but none of those do not deal with the robustness of systems and the developed techniques are not directly applicable for perturbed systems.

We will derive the results for unspecified vector norm, \( \|\cdot\| \). For the matrices, as an operator norm is always used the induced norm, \( \|A\| = \max_{\|x\|=1} \|Ax\| \).

For example, the Frobenius norm of the matrix is not induced norm because \( \|I_n\| = 1 \) for any induced norm, but \( \|I_n\|_F = \sqrt{n} \). We use for both vector norm and matrix operator norm the same notation but it will always be clear from the context that norm is just being used. In particular cases we will consider the three most common vector norm:

\[
\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.
\]

We denote by \( \mu[A(t)], \ t \geq 0 \), the logarithmic norm of matrix \( A(t) \). The classical definition is

\[
\mu[A(t)] \triangleq \lim_{h \to 0^+} \frac{\|I_n + hA(t)\| - 1}{h},
\]

where \( I_n \) denotes the identity on \( \mathbb{R}^n \), \([3, 15, 16]\). Because it is assumed that \( A(t) \) is continuous for all \( t \geq 0 \), the function \( \mu[A(t)] \) is a continuous function.
of $t$, by virtue of the inequality $|\mu[A(t_1)] - \mu[A(t_2)]| \leq \|A(t_1) - A(t_2)\|$, $t_1, t_2 \geq 0$, see e. g. [2]. While the matrix norm $\|A\|$ is always positive if $A \neq 0$, the logarithmic norm $\mu(A)$ may also take negative values, e. g. for the Euclidean vector norm $\|\cdot\|_2$ and when $A$ is negative definite because $\frac{1}{2}(A + A^T)$ is also negative definite, [5, Corollary 14.2.7.] and Lemma 1. Therefore, the logarithmic norm does not satisfy the axioms of a norm.

Now, in Lemmas 1 and 2, we summarize important and interesting properties of the logarithmic norm useful for stability analysis of dynamical systems.

**Lemma 1.** For the norms (1) we have, see e. g. [1, p. 54], [4, p. 33]:

| Norm of vector $\|x\|_i$, $i = 1, 2, \infty$ | Induced norm of matrix $A$ $\|A\|_i$ | Logarithmic norm $\mu_i[A]$ |
|-------------------------------------------------|-----------------------------------|------------------------|
| $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ | $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$ (column sum) | $\mu_1[A] = \max_{1 \leq j \leq n} (a_{jj} + \sum_{i \neq j} |a_{ij}|)$ |
| $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ | $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ | $\mu_2[A] = \frac{1}{2}\lambda_{\max}(A + A^T)$ |
| $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ | $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$ (row sum) | $\mu_\infty[A] = \max_{1 \leq i \leq n} (a_{ii} + \sum_{j \neq i} |a_{ij}|)$ |

In Table 1 and elsewhere in the paper, the superscript ‘$T$’ denotes transposition, the number $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of the matrix $A^T A$.

The fundamental advantage of approach based on the use of logarithmic norm is the fact that to estimate the norm of fundamental matrix solution $\Phi(t)$ for unperturbed system $\dot{x} = A(t)x$ and transition matrix $\Phi(t)\Phi^{-1}(\tau)$ we do not need to know its solutions explicitly.

**Lemma 2** ([4, 15, 16]).

$p1)$ $\max_{1 \leq i \leq n} \Re(\lambda_i) \leq \mu[A]$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, and $\Re(\lambda)$ is a real part of $\lambda$;
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p2) \(-\|A\| \leq \mu[A] \leq \|A\|\);

p3) Let \(\Phi(t)\) is a fundamental matrix solution for \(\dot{x} = A(t)x\), \(t \geq 0\). Then

\[
\left\|\Phi(t)\Phi^{-1}(\tau)\right\| \leq e^{\int_{\tau}^{t} \mu[A(s)]ds}
\]

for all \(t \geq \tau \geq 0\).

This lemma, Item p3, allows to estimate the norm of state-transition matrix without knowing the fundamental matrix solution, purely on the basis of the matrix \(A(t)\) entries, which can be especially important if \(A\) is a non-constant matrix.

The following example demonstrates that the value \(\mu[A]\) may depend on the used vector norm.

Example 1. [1, p. 56]

a) \(A_1 = \begin{bmatrix} -11 & 10 \\ 2 & -3 \end{bmatrix} \Rightarrow \mu_1[A_1] = 7, \mu_2[A_1] = 0.2111\) and \(\mu_{\infty}[A_1] = -1\);

b) \(A_2 = \begin{bmatrix} -11 & 2 \\ 10 & -3 \end{bmatrix} \Rightarrow \mu_1[A_2] = -1, \mu_2[A_2] = 0.2111\) and \(\mu_{\infty}[A_2] = 7\);

c) \(A_3 = \begin{bmatrix} -1 & 3 \\ -3 & -2 \end{bmatrix} \Rightarrow \mu_1[A_3] = 2, \mu_2[A_3] = -1\) and \(\mu_{\infty}[A_3] = 2\).

Thus, we can verify whether the system \(\dot{x} = A_ix, i = 1, 2, 3\) is stable or not by means of the vector norm with negative value of \(\mu[A_i]\). In general, we obtain such logarithmic norm for any Hurwitz matrix \(A\) [10, p. 135] using a vector norm \(\|x\|_H = \sqrt{x^THx}\), where the symmetric positive definite matrix \(H\) satisfies the Lyapunov equation \(A^TH + HA = -2I_n\). The corresponding logarithmic norm \(\mu_H[A] = -1/\lambda_{\max}(H)\), see Lemma 2.3 in [8]. It turns out that the stability analysis based on the logarithmic norm becomes a topological notion unlike the spectrum of matrices which is topological invariant.

2. Robust global asymptotic stability of LTV systems

Let us define the following classes of functions [17].
Definition 3. Let \( h : [0, \infty) \to \mathbb{R}^n \) be continuous. Define

\[
\mathcal{V} \triangleq \{ h : \|h(t)\| \to 0 \; \text{as} \; t \to \infty \},
\]

\[
\mathcal{A} \mathcal{D} \triangleq \left\{ h : \int_t^{t+1} \|h(s)\| \, ds \to 0 \; \text{as} \; t \to \infty \right\},
\]

\[
\mathcal{D} \triangleq \left\{ h : \sup_{0 \leq \eta \leq 1} \left\| \int_t^{t+\eta} h(s) \, ds \right\| \to 0 \; \text{as} \; t \to \infty \right\}.
\]

Lemma 4. \( \mathcal{V} \subsetneq \mathcal{A} \mathcal{D} \subsetneq \mathcal{D} \).

Proof. Assume that \( \|h(t)\| \to 0 \) as \( t \to \infty \). Using the monotonicity of the function \( H(t) = \int_t^0 \|h(s)\| \, ds \) and Lagrange’s Mean Value Theorem we obtain the chain of inequalities from which immediately follows the claim of lemma,

\[
\sup_{0 \leq \eta \leq 1} \left\| \int_t^{t+\eta} h(s) \, ds \right\| \leq \sup_{0 \leq \eta \leq 1} \int_t^{t+\eta} \|h(s)\| \, ds = \int_t^{t+1} \|h(s)\| \, ds = \|h(\xi)\|,
\]

where \( \xi \in (t, t+1) \), \( \xi \to \infty \) as \( t \to \infty \). Now we need only to show that \( \mathcal{V} \neq \mathcal{A} \mathcal{D} \neq \mathcal{D} \).

Example 2.

i) Let \( h(t) = [\sin(e^t), \cos(e^t), 0, \ldots, 0]^T \). Then \( h \in \mathcal{D} \); in fact, for any \( \eta \geq 0 \),

\[
\left| \int_t^{t+\eta} \sin(e^\tau) \, d\tau \right| \leq 2e^{-t}(1 + e^{-\eta}) \leq 4e^{-t}.
\]

The same inequality holds for the second component of \( h \), and thus

\[
\left\| \int_t^{t+\eta} h(s) \, ds \right\|_2 \leq \sqrt{32}e^{-t} \to 0 \; \text{as} \; t \to \infty,
\]

but

\[
\int_t^{t+1} \|h(s)\|_2 \, ds = 1 \not\to 0 \; \text{as} \; t \to \infty.
\]
i. e., \( h \in \mathcal{D} \), but \( h \notin \mathcal{AD} \). Recall that the use of the Euclidean norm in the last two steps does not impair the generality of analysis because all norms on finite-dimensional vector space are equivalent. Specifically, there exists a pair of real numbers \( 0 < C_1 \leq C_2 \) such that, for all \( x \in \mathbb{R}^n \), the following inequality holds:

\[
C_1 \|x\|_a \leq \|x\|_b \leq C_2 \|x\|_a.
\]

In particular,

\[
\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \leq n \|x\|_\infty.
\]

ii) Let \( h(t) = [h_1(t), 0, \ldots, 0]^T \), where \( h_1(t) = \sum_{n=1}^{\infty} h_{1,n}(t) \) is ”needle-like” function with \( h_{1,n} \) defined as follows:

\[
h_{1,n}(t) = \begin{cases} 
2n(t - n + 1) & \text{for } t \in \left[ n - 1, n - 1 + \frac{1}{2n} \right) \\
2(-nt + n^2 - n + 1) & \text{for } t \in \left[ n - 1 + \frac{1}{2n}, n - 1 + \frac{1}{n} \right) \\
0 & \text{elsewhere}
\end{cases}
\]

Then

\[
\int_t^{t+1} h_1(s)ds \leq \frac{1}{2n} \to 0 \quad \text{as} \quad t \to \infty,
\]

but \( \|h(t)\| \not\to 0 \) as \( t \to \infty \) because for all \( n \in \mathbb{N} \), the value of \( h_1 \) at the points \( n - 1 + 1/2n, n \in \mathbb{N} \) is equal to 1; \( h \in \mathcal{AD} \), but \( h \notin \mathcal{V} \).

Graphs of the first components of vector functions \( h \) from the example above are shown in Figure 1.

![Figure 1: The functions \( h_1(t) \) from Example 2, i) on the left and ii) on the right.](image)

This completes the proof of Lemma 4.

As corollary of Theorem A and Theorem B in [17], for the linear systems we have the following theorem.
Theorem 5. Let $x = 0$ is globally asymptotically stable equilibrium point of the LTV system $\dot{x} = A(t)x$ and $h$ is continuous. Then all solutions of

$$\dot{x} = A(t)x + h(t) \tag{2}$$

converge to 0 as $t \to \infty$ if $h \in \mathcal{AD}$.

Moreover, if $A(t)$ is entry-wise bounded, then all solutions of (2) converge to 0 as $t \to \infty$ if and only if $h \in \mathcal{D}$.

Because only the sufficient condition for LTV systems with unbounded $A(t)$ was established, for such systems the origin $x = 0$ could be globally attractive also for the perturbations $w \not\in \mathcal{AD}$, even monotonically unbounded ones as it also turns out in Example 3. First, however, we formulate the following theorem on robustness of asymptotically stable LTV systems, which seems to be a completely new result.

Theorem 6. If for some vector norm in $\mathbb{R}^n$

1) $\int_0^\infty \mu[A(s)]ds = -\infty$,

then $x = 0 \in \mathbb{R}^n$ is globally asymptotically stable equilibrium point of the nominal system $\dot{x} = A(t)x$. Moreover, if

2) $\mu[A(t)] < 0$ in some left neighborhood of $\infty$, and

3) for all $x \in \mathbb{R}^n$ and all $t \geq 0$ is $\|w(x, t)\| \leq \|\tilde{w}(t)\|$ with

$$\lim_{t \to \infty} \left(\frac{\|\tilde{w}(t)\|}{\mu[A(t)]}\right) = 0 \quad \text{(that is, } \|\tilde{w}(t)\| = o(\mu[A(t)]) \text{ as } t \to \infty),$$

then all solutions of perturbed system $\dot{x} = A(t)x + w(x, t)$ converge to 0 as $t \to \infty$.

Proof. Using the variation constant formula and Lemma 2 (Item p3) for $\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$ we have that

$$x(t) = \Phi(t) \left[\Phi^{-1}(0)x(0) + \int_0^t \Phi^{-1}(\tau)w(x(\tau), \tau) \right]d\tau,$$
that is,
\[ \|x(t)\| \leq \|x(0)\| e^{\int_0^t \mu[A(s)] ds} + \int_0^t e^{\int_\tau^t \mu[A(s)] ds} \|\tilde{w}(\tau)\| d\tau. \]

Obviously, from the Assumption 1,
\[ \|x(0)\| e^{\int_0^t \mu[A(s)] ds} \to 0 \text{ for } t \to \infty \]
for an arbitrary \( x(0) \), proving the global asymptotic stability of the equilibrium \( x = 0 \) of unperturbed system. It remains to analyze the second term on the left-hand side of the above inequality. We have that
\[ \int_0^t e^{\int_0^\tau \mu[A(s)] ds} \|\tilde{w}(\tau)\| d\tau = \int_0^t e^{\int_0^\tau \mu[A(s)] ds} d\tau \]
\[ = \int_0^t e^{\int_0^\tau \mu[A(s)] ds} \|\tilde{w}(\tau)\| d\tau + \int_0^t \frac{e^{\int_0^\tau \mu[A(s)] ds}}{e^{\int_0^\tau \mu[A(s)] ds}} \|\tilde{w}(\tau)\| d\tau \]
\[ = \int_0^t \frac{d}{dt} e^{\int_0^\tau \mu[A(s)] ds} \|\tilde{w}(\tau)\| d\tau \]
and the L’Hospital rule yields
\[ \lim_{t \to \infty} \frac{d}{dt} \int_0^t e^{\int_0^\tau \mu[A(s)] ds} \|\tilde{w}(\tau)\| d\tau = \lim_{t \to \infty} \frac{-\int_0^t \mu[A(s)] ds}{\int_0^t e^{\int_0^\tau \mu[A(s)] ds} d\tau} \]
\[ = \lim_{t \to \infty} \frac{-\int_0^t \mu[A(s)] ds}{\int_0^t e^{\int_0^\tau \mu[A(s)] ds} (-\mu[A(t)])} \]
which, together with Assumption 3, gives the claim of Theorem 6.

Remark 1. For the LTI systems \( \dot{x} = Ax \) with a Hurwitz matrix \( A \), from the end of Example 1, \( \mu[H[A]] = -1/\lambda_{\max}(H) \) is negative constant and the Assumption 1 and 2 of Theorem 6 are trivially fulfilled. Assumption 3 is satisfied if \( \|w(x, t)\| \leq \|\tilde{w}(t)\| \to 0 \) as \( t \to \infty \).
Example 3. To illustrate the Theorem 6 by an example, let us consider the system

\[
\dot{x} = \begin{bmatrix} -a_1(t) & \beta(t) \\ -\beta(t) & -a_2(t) \end{bmatrix} x + \begin{bmatrix} t^{7/8} \\ 100 \cos(t) \end{bmatrix}, \ t \geq 0,
\]

where \( a_1(t) = (t+1), \ a_2(t) = (3+t+\sin t) \) and \( \beta(t) \) is an arbitrary continuous function on \([0, \infty)\). Using the Euclidean norm,

\[
\mu_2[A(t)] = \frac{1}{2} \lambda_{\text{max}}(A(t) + A^T(t)) = \max \{ -a_1(t), -a_2(t) \} = -(t+1)
\]

by Lemma 1 and

\[
\int_0^\infty \mu_2[A(s)] ds = - \int_0^\infty (s+1) ds = -\infty \quad \text{(Assumption 1 of Theorem 6)},
\]

\[
\mu_2[A(t)] = -(1+t) < 0 \quad \text{for all } t \geq 0 \quad \text{(Assumption 2), and}
\]

\[
\lim_{t \to \infty} \left( \frac{\|w(t)\|_2}{\mu_2[A(t)]} \right) = \lim_{t \to \infty} \left( \frac{(t^{7/8})^2 + (100 \cos(t))^2)^{1/2}}{-(t+1)} \right) = 0 \quad \text{(Assumption 3)}.
\]

Thus all solutions of the system (3) converge to 0 for \( t \to \infty \) on the basis of Theorem 6. The result of one simulation in the MATLAB environment is shown in Figure 2.

![Figure 2: Solution \( x(t) = [x_1(t), x_2(t)]^T \) of the perturbed system (3) for \( \beta(t) = t^4 \) and initial state \( x(0) = [-5, 2]^T \).](image)

Note that for \( \beta(t) = t^4 \) and \( a_i(t), \ i = 1, 2 \) as above, the logarithmic norms \( \mu_1[A(t)] = \mu_\infty[A(t)] = t^4 - a_1(t) \), and so, the vector norms \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \) are not suitable for stability analysis in this particular case because Assumption 1 of Theorem 6 is not satisfied.

We now show that the perturbation \( w(t) \) does not belong to the class \( \mathcal{D} \), and hence also to the class \( \mathcal{AD} \). For the first component of \( w \) and each \( \eta > 0 \) we have

\[
\int_t^{t+\eta} s^{7/8} ds = \frac{8}{15} \left( (t+\eta)^{15/8} - t^{15/8} \right) = \frac{8}{15} \left( \frac{t^{-15/8} - (t + \eta)^{-15/8}}{t^{-15/8}(t + \eta)^{-15/8}} \right)
\]
Then the L'Hospital rule yields
\[
\lim_{t \to \infty} \frac{8}{15} \left[ 1 - \frac{(1 + \eta/t)^{-15/8}}{(t + \eta)^{-15/8}} \right] = \lim_{t \to \infty} \frac{8}{15} \left[ \frac{\eta t^{7/8}}{8} \right] \to \infty \text{ as } t \to \infty.
\]

**Example 4.** In this example we show, that for perturbation
\[
h(t) = \lambda \begin{bmatrix} \sin(e^t), \cos(e^t) \end{bmatrix}^T \in \mathcal{D} \setminus \mathcal{AD}, \quad \lambda > 0,
\]
there exists matrix \( A(t) \) (necessarily unbounded, of course) such that 0 is globally (uniformly) asymptotically stable for \( \dot{x} = A(t)x \) but solution \( x^*(t) \) of \( \dot{x} = A(t)x + h(t) \) with \( x^*(0) = 0 \) does not converge to 0 as \( t \to \infty \). For
\[
A(t) = \begin{bmatrix} -\lambda & e^t \\ -e^t & -\lambda \end{bmatrix}, \quad \lambda > 0,
\]
(4)
obviously, \( \mu_2[A(t)] = -\lambda < 0 \) for all \( t \geq 0 \) proving global uniform asymptotic (⇔ uniform exponential) stability of zero solution for unperturbed system, see [2, 10, 12, 19] and Item p3 of Lemma 2. Note that perturbation \( h(t) \) does not satisfy Assumption 3 of Theorem 6.

The fundamental matrix for \( \dot{x} = A(t)x \) satisfies
\[
\Phi(t) = e^{-\lambda t} \begin{bmatrix} \sin(e^t) & -\cos(e^t) \\ \cos(e^t) & \sin(e^t) \end{bmatrix}, \quad \Phi^{-1}(t) = e^{\lambda t} \begin{bmatrix} \sin(e^t) & \cos(e^t) \\ -\cos(e^t) & \sin(e^t) \end{bmatrix}
\]
(5)
and so
\[
x^*(t) = \Phi(t) \int_0^t \Phi^{-1}(\tau) h(\tau) d\tau = \begin{bmatrix} \sin(e^t) \\ \cos(e^t) \end{bmatrix} \not\to 0 \text{ as } t \to \infty
\]
because
\[
\|x^*(t)\|_2 = (1 - e^{-\lambda t}) \to 1.
\]
This example, together with the previous one, showed that there is no relationship between the admissible perturbations \( h(t) \) for LTV systems with bounded and unbounded \( A(t) \), respectively. This result also indicates that, in Theorem 6 Assumption 3 regarding the asymptotics of perturbing term cannot be weakened too much and in the last example cannot be achieved better results of the form \( \|w(t)\| = o(t^\alpha) \) as \( t \to \infty \) with \( \alpha > 0 \) for the admissible range of perturbations preserving convergence to zero of all solutions of perturbed system as those we have obtained using the Euclidean norm, \( \|w(t)\|_2 = o(1) \).
Conclusions

In this paper we derived the novel and relatively easy-to-use criterion for robust global asymptotic stability of the nominal system $\dot{x} = A(t)x$ being affected by the disturbances and/or system uncertainties. Roughly speaking, all solutions of its perturbation $\dot{x} = A(t)x + w(x, t)$ converge to 0 if $\|w(x, t)\| \leq \|\tilde{w}(t)\|$ and $(\|\tilde{w}(t)\|/\mu[A(t)]) \to 0$ as $t \to \infty$, where by $\mu[A(t)]$ we denote the logarithmic norm of the system matrix $A(t)$.

The fundamental advantage of the approach based on the use of logarithmic norm is the fact that to estimate the norm of transition matrix $\Phi(t)\Phi^{-1}(\tau)$ for unperturbed system $\dot{x} = A(t)x$ we do not need to know the fundamental matrix solution $\Phi(t)$ and all necessary estimates are based purely on the matrix $A(t)$ entries.

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