Effective scalar-tensor description of regularized Lovelock gravity in four dimensions

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We reformulate the recently proposed regularized version of Lovelock gravity in four dimensions as a scalar-tensor theory. By promoting the warp factor of the internal space to a scalar degree of freedom, we show that regularized Lovelock gravity is effectively described by a particular subclass of the Horndeski theory. Cosmological aspects of regularized Einstein-Gauss-Bonnet gravity are studied based on this scalar-tensor reformulation. It is found that the background with a scalar charge is generically allowed. The consequences of this scalar charge is briefly discussed.

\textbf{I. INTRODUCTION}

Lovelock gravity \cite{1} is the most general metric theory of gravity in higher dimensions retaining the second-order nature of field equations for the metric. The action for Lovelock gravity in \(D\) dimensions is given by

\[ S = \int d^Dx \sqrt{-g_D} \sum_{p=0}^{[D/2]} \alpha_p \mathcal{L}_p, \]

where \(\alpha_p\) is the coupling constant and

\[ \mathcal{L}_p := \frac{1}{2^p} \delta^{A_1 B_1 \cdots A_p B_p} \mathcal{R} \mathcal{R}_{A_1 B_1} \cdots \mathcal{R}_{A_p B_p} \mathcal{C}_1 \cdots \mathcal{C}_p D_1 \cdots D_p. \]

In \(D = 4\) dimensions, the Lovelock Lagrangian uniquely reduces to the Einstein-Hilbert term (\(\mathcal{L}_1\)) plus a cosmological constant (\(\mathcal{L}_0\)). This puts strong limitations on constructing the metric theory of gravity other than Einstein in four dimensions.

Recently, a trick has been proposed to circumvent this limitation \cite{2,3}. (See also \cite{4,5} for earlier works.) The trick amounts to rescaling the coupling constants as Eq. (4),

\[ \alpha_p = \frac{\alpha_p'}{D-4} \]

and then taking the \(D \to 4\) limit. This procedure leaves nonvanishing contributions in the gravitational field equations, and thus one ends up with a seemingly novel theory of gravity in four dimensions. Although how this “regularization” works is not so evident at the level of the action or the covariant field equations, an explicit analysis of cosmological solutions, black hole solutions, and perturbations \cite{2,3,6,7,8,9,10,11} shows that the Lovelock terms yield the factor of \((D-4)\) in the field equations to cancel \((D-4)\) in the denominator. See also Refs. \cite{12,13,14,15} for aspects of black hole solutions in this regularized version of Lovelock gravity.

Let us take a look at the cosmological spacetime studied in \cite{2,3}. The \(D\)-dimensional cosmological metric is assumed to take the form

\[ g_{AB} dx^A dx^B = -dt^2 + a^2(t) \left( \delta_{ij} dx^i dx^j + \delta_{ab} dx^a dx^b \right), \]

where \(i, j = 1, 2, 3\) and indices \(a, b\) run through \(4, 5, \cdots , 3+n\) with \(n = D-4\). Substituting this metric to the gravitational field equations and then taking the \(D \to 4\) limit along with rescaling the coupling constants as Eq. (4), one obtains the modified background equations in four dimensions \cite{2,3}. Here some questions arise. What happens if one assumes a different scale factor of the \(n\)-dimensional internal space, \(b^2(t)\)? And then, what is the role of the \((a, b)\) (i.e., the internal-space) components of the field equations in the \(D \to 4\) limit?

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To address the above questions, in this paper we study the dynamics of regularized Lovelock gravity by assuming a general warp factor of the internal space. We show that taking the $D \to 4$ limit after the Kaluza-Klein reduction leaves a dynamical scalar degree of freedom in four dimensions, yielding a particular class of scalar-tensor theories within the Horndeski family. Employing this scalar-tensor reformulation, we revisit the dynamics of a cosmological spacetime in regularized Einstein-Gauss-Bonnet gravity in four dimensions.

The rest of the paper is organized as follows. In the next section we clarify the relation between the Horndeski theory and the $D \to 4$ limit of the Lovelock theory to provide the scalar-tensor reformulation of the latter theory. We then study the background dynamics and linear perturbations of a cosmological spacetime based on the scalar-tensor reformulation in Sec. III. A brief summary of the paper is presented in Sec. IV.

II. HORNDESKI AND REGULARIZED LOVELOCK

For simplicity, we focus on the case of Einstein-Gauss-Bonnet gravity for the moment. Let us consider the $(4+n)$-dimensional metric of the form

$$g_{AB}dx^A dx^B = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\chi(x)}d\sigma^2_K,$$

where $\mu, \nu = 0,1,2,3$ and $d\sigma^2_K$ is the line element of a $n$-dimensional maximally symmetric space with constant curvature $K$. We perform a Kaluza-Klein reduction starting from the metric (9). Substituting Eq. (6) to the Gauss-Bonnet term and doing integration by parts, we obtain

$$\sqrt{-g} n \mathcal{L}_2 = n \sqrt{-g} \left(-6K^2 e^{-4\chi} + 24K e^{-2\chi} X - 2K e^{-2\chi} R + 8X^2 + 8X \Box X + 4G^\mu_\nu \chi_\mu \chi_\nu + \chi \mathcal{G} + O(n^2)\right),$$

where $\chi$ is the Gauss-Bonnet combination of the four-dimensional curvature tensors, $\chi = \nabla \mu \chi$ and $X = -\chi_\mu \chi^\mu/2$. Here, $R$ and $G^\mu_\nu$ are the four-dimensional Ricci scalar and Einstein tensor, respectively, and $\mathcal{G} = R^2 - 4R^\mu_\nu R^\nu_\rho R^\rho_\sigma$. By rescaling the coupling constant $\alpha_2$ as $\alpha_2 = \alpha_2' n$ and taking the $n \to 0$ limit, we thus arrive at the following alternative description of regularized Gauss-Bonnet gravity,

$$\mathcal{L} = \alpha_0 + \left( \alpha_1 - 2\alpha_2' K e^{-2\chi} \right) R + \alpha_2' \left[ -6K^2 e^{-4\chi} + 24K e^{-2\chi} X + 8X^2 + 8X \Box X + 4G^\mu_\nu \chi_\mu \chi_\nu + \chi \mathcal{G} \right].$$

In the case of $K = 0$, the theory has the invariance under $\chi \to \chi + \text{const}$.

This theory can be viewed as a particular subclass of the Horndeski theory \cite{19} (see \cite{20} for a review). The Lagrangian of the Horndeski theory is the sum of the following four Lagrangians \cite{21, 22},

$$\mathcal{L}_2^H \{ G_2 \} := G_2(\chi, X),$$

$$\mathcal{L}_3^H \{ G_3 \} := -G_3(\chi, X) \Box \chi,$$

$$\mathcal{L}_4^H \{ G_4 \} := G_4(\chi, X) R + G_4 X \delta^\mu_\nu \chi_\mu \chi_\nu,$$

$$\mathcal{L}_5^H \{ G_5 \} := G_5(\chi, X) G^\mu_\nu \chi_\mu \chi_\nu - \frac{1}{6} G_5 X \delta^\mu_\nu \chi_\mu \chi_\nu X \delta^\mu_\nu \chi_\mu \chi_\nu.$$

The Lagrangian \cite{8} corresponds to the case with

$$G_2 = \alpha_0 + \alpha_2' \left[ -6K^2 e^{-4\chi} + 24K e^{-2\chi} X + 8X^2 \right],$$

$$G_3 = -8\alpha_2' X,$$

$$G_4 = \alpha_1 + \alpha_2' \left[ -2K e^{-2\chi} + 4X \right],$$

$$G_5 = -4\alpha_2' X \ln X.$$

Here, we used the fact that $G^\mu_\nu \chi_\mu \chi_\nu$ can be written equivalently as $XR + \delta^\mu_\nu \chi_\mu \chi_\nu$ up to total derivatives. Note that the nonminimal coupling to the Gauss-Bonnet term can be reproduced from the Horndeski functions including the $\ln X$ terms \cite{22}.

Rewriting explicitly the $p \geq 3$ Lovelock terms as the Horndeski theory is much more involved, though they must reduce to the form of the second-order scalar-tensor theory anyway \cite{17}. For example, substituting the metric (9) with $K = 0$ to $L_3$ yields

$$\mathcal{L}_3 = -6X \mathcal{G} - 6\delta^\mu_\nu \chi_\mu \chi_\nu R_{\mu_1 \mu_2 \nu_1 \nu_2} \chi_{\mu_3} \chi_{\nu_3}$$

$$- 12\delta^\mu_\nu \chi_\mu \chi_\nu R_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \chi_{\lambda_1} \chi_{\lambda_2} \chi_{\lambda_3} \chi_{\lambda_4}$$

$$+ 192X^3 - 288X \nabla \mu X \chi^\mu + \mathcal{L}_4 \left[ -72X^2 \right] + \mathcal{L}_5 \{ 96X \}$$

$$+ 96X \left( G^\mu_\nu \chi_\mu \chi_\nu + \delta^\mu_\nu \chi_\mu \chi_\nu \right) - 96 \Box X \chi \chi - 96 \nabla \mu X \chi^\mu + O(n).$$

(17)
The first line is of the form of the generalized Galileon [21],
\[
\mathcal{L}^{\text{gal}}_6 = \delta^\mu_{\nu_1 \ldots \nu_4} \left[ \frac{3}{4} G_6(\chi, X) R_{\mu_1 \mu_2} \nu_1 \nu_2 R_{\mu_3 \mu_4} \nu_3 \nu_4 + 3 G_6, X R_{\mu_1 \mu_2} \nu_1 \nu_2 \chi_{\mu_3} \chi_{\mu_4} + G_6, X \chi_{\mu_1} \ldots \chi_{\mu_4} \right],
\]
which is a total derivative in four dimensions. The second line, the derivative coupling to the double dual Riemann tensor, can be written equivalently as \( \mathcal{L}_5 \{ -48X \} \) [23]. It is easy to rearrange the other terms and we finally obtain, in the \( n \to 0 \) limit,
\[
\alpha_3 \mathcal{L}_3 \to \alpha_3^3 \left[ L^H_2 \{ 192X^3 \} + \mathcal{L}_4^H \{ -144X^2 \} + \mathcal{L}_5^H \{ 24X^2 \} + \mathcal{L}_8^H \{ 48X \} \right].
\]
Similarly, the \( p \)-th Lovelock term yields \( \mathcal{L}_p^H \{ X^p \} \), \( \mathcal{L}_3^H \{ X^{p-1} \} \), \( \mathcal{L}_5^H \{ X^{p-1} \} \), and \( \mathcal{L}_2^H \{ X^{p-2} \} \) with particular coefficients. It is straightforward to include the curvature \( K \).

### III. REVISITING COSMOLOGY IN REGULARIZED EINSTEIN-GAUSS-BONNET GRAVITY

Let us study the cosmological dynamics, focusing again on the simplest case of regularized Einstein-Gauss-Bonnet gravity and its scalar-tensor reformulation [8]. Since \( \chi \) is promoted to be a dynamical field in our scalar-tensor reformulation, we will emphasize its consequences on the background and perturbation dynamics.

#### A. Background Cosmology

The field equations derived from [8] take the form
\[
-\frac{\alpha_0}{2} \delta^\mu_\nu + \alpha_1 G^\mu_\nu + \alpha_2^2 H^\mu_\nu = \frac{1}{2} T^\mu_\nu,
\]
where \( T^\mu_\nu \) is the energy-momentum tensor of matter. For the flat Friedmann-Lemaître-Robertson-Walker metric, \( ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \), we have
\[
H_0^2 = 3 \chi^4 - 12 H \dot{\chi}^3 + 18 H^2 \chi^2 - 12 H^3 \chi,
\]
\[
H_i^j = \left( -\ddot{\chi}^2 + 6 H^2 \chi^2 - 8 H^3 \chi - 4 \dot{\chi} \ddot{\chi} - 4 H \dddot{\chi} - 8 \dot{H} \dddot{\chi} - 8 H \dddot{\chi} \right) \delta_{ij},
\]
where \( H = \dot{a}/a \) is the Hubble parameter and a dot stands for differentiation with respect to \( t \). The \( \chi \)-field equation, or, equivalently, \( \alpha_2^2 \nabla_\nu H^\mu_\nu = 0 \), reduces to
\[
\frac{\alpha_2^2}{a^3} \frac{d}{dt} \left[ a^3 (\dot{\chi} - H)^3 \right] = 0.
\]
This can be integrated to give
\[
\dot{\chi} = H + \frac{C}{a},
\]
where the integration constant \( C \) is the scalar charge associated with the shift symmetry \( \chi \to \chi + \text{const} \). The case with \( C = 0 \) corresponds to the isotropically expanding solution, \( ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j + a^2 \delta_{ab} dx^a dx^b \), which is assumed from the beginning in [2]. The scalar-tensor reformulation reveals that the background with the nonvanishing scalar charge is in fact allowed. In an accelerating universe, the second term in Eq. (24) decays quickly compared to the first, so that \( \dot{\chi} = H \) is a dynamical attractor. However, this is not the case in a decelerating universe.

Substituting the solution (24) into Eqs. (21) and (22), we now have the following background cosmological equations,
\[
\alpha_0 + 6(\alpha_1 H^2 + \alpha_2^2 H^4) = \rho + \frac{6 \alpha_2^2 C^4}{a^4},
\]
\[
-4 \alpha_1 \dot{H} = \rho + P + \frac{8 \alpha_2^2 C^4}{a^4},
\]
where
\[
\Gamma := 1 + \frac{2 \alpha_2^2 H^2}{\alpha_1},
\]
and \( \rho \) and \( P \) are the energy density and pressure of matter, respectively. It can be seen that the nonvanishing scalar charge gives rise to an extra radiation-like component in the background equations. In the case of \( C = 0 \) the background equations derived in [2] are reproduced correctly.
B. Cosmological Perturbations

Let us move to the perturbation dynamics. As a matter field we simply add a canonical scalar field described by the Lagrangian \( \mathcal{L}_\phi = -\phi_\mu \phi^\mu / 2 - V(\phi) \). We fix the temporal gauge degree of freedom by imposing that \( \phi(t, x^i) = \phi(t) \). The remaining spatial gauge degrees of freedom can be used to write the metric as

\[
\text{ds}^2 = -N^2 \text{dt}^2 + \gamma_{ij}(\text{dx}^i + N^i \text{dt})(\text{dx}^j + N^j \text{dt}),
\]

where

\[
N = 1 + \delta N, \quad N_i = \partial_i \psi, \quad \gamma_{ij} = a^2 e^{2\zeta}(e^h)_{ij},
\]

with \((e^h)_{ij} = \delta_{ij} + h_{ij} + h_{ik}h_{kj}/2 + \cdots \). The \( \chi \)-field also fluctuates,

\[
\chi = \ln a + G \int \frac{\text{d}t}{a} + \delta \chi.
\]

Since all the gauge degrees of freedom have already been fixed, we cannot gauge away \( \delta \chi \). One would therefore expect that there are two dynamical modes in the scalar sector.

It is straightforward to compute the quadratic Lagrangian for the tensor modes \( h_{ij} \):

\[
\sqrt{-g} \mathcal{L} = 2a^3 \left[ \left( \alpha_1 \Gamma - \frac{2\alpha'_1 C^2}{a^2} \right) \dot{h}_{ij}^2 - \left( \alpha_1 \Gamma + 4\alpha'_2 \dot{H} + \frac{2\alpha'_1 C^2}{a^2} \right) (\partial h_{ij})^2 \right].
\]

Clearly, this Lagrangian reproduces the equation of motion presented in [2] for \( C = 0 \). One might worry about ghost instabilities for a sufficiently large scalar charge. However, even if the scalar charge is as large as \( C/a \geq H \), we have, from the background equations, that \( \alpha_1 \Gamma H^2 \sim \alpha'_2 C^4/a^4 \gtrsim \alpha'_2 H^2 C^2/a^2 \), which implies that ghost instabilities in the tensor sector are generically avoided.

The quadratic Lagrangian for the scalar perturbations reads

\[
\sqrt{-g} \mathcal{L} = a^3 \left[ \left( \frac{\dot{\phi}^2}{2} - 6\alpha_1 \Gamma H^2 + \frac{12\alpha'_1 C^4}{a^2} \right) \delta N^2 - \frac{2AH}{a^2} \delta N \partial^2 \psi + \frac{2B}{a^2} \delta \zeta \phi^2 + 6AH\delta \zeta \right.
\]

\[
- \frac{2B}{a^2} \delta N \partial^2 \zeta - 3B \dot{\zeta}^2 + \frac{2}{a^2} \left( \alpha_1 \Gamma + 4\alpha'_2 \dot{H} + \frac{2\alpha'_1 C^2}{a^2} \right) (\partial \zeta)^2
\]

\[
+ \frac{4\alpha'_2 C^2}{a^2} \left( \frac{2}{a^2} \delta \chi \partial^2 \psi + 3\delta \chi^2 - \frac{1}{a^2} (\partial \delta \chi)^2 - \frac{2}{a^2} \delta N \partial^2 \delta \chi - 6\delta \chi \dot{\zeta} \right)
\]

\[
+ \frac{8\alpha'_2 C^3}{a^3} \left( \frac{1}{a^2} \delta \chi \partial^2 \psi - 3\delta N \delta \chi + 3\zeta \delta \chi \right) \right],
\]

where

\[
A := 2\alpha_1 \Gamma + \frac{4\alpha'_2 C^3}{a^3 H}, \quad B := 2\alpha_1 \Gamma - \frac{4\alpha'_2 C^2}{a^2}.
\]

The variation with respect to \( \psi \) gives

\[
\delta N = \frac{B}{A H} \dot{\zeta} \partial^2 \psi + \frac{4\alpha'_2 C^2}{a H} \delta \chi + \frac{4\alpha'_2 C^3}{a^3 H} \delta \chi.
\]

Substituting this back to the above Lagrangian, we obtain the quadratic Lagrangian written in terms of \( \zeta \) and \( \delta \chi \). The general expression is messy, and hence we expand the Lagrangian in terms of \( C \) assuming that the contribution of the scalar charge to the background is subdominant, leading to

\[
\sqrt{-g} \mathcal{L} \simeq 2\alpha_1 a^3 \Gamma \epsilon \left[ \dot{\zeta}^2 - \frac{(\partial \zeta)^2}{a^2} \right] + 12\alpha'_2 a \left[ \delta \chi^2 - \frac{(\partial \delta \chi)^2}{3a^2} \right],
\]

where \( \epsilon := -\dot{H}/H^2 \). One sees that the sound speed of \( \delta \chi \) is given by \( 1/\sqrt{3} \), signaling its radiation-like nature. On the \( C = 0 \) background, \( \delta \chi \) apparently drops off from the Lagrangian and thereby the equation of motion in [2] is reproduced from our Lagrangian. Note, however, that the \( C \to 0 \) limit must be taken with care because the disappearance of the kinetic term of \( \delta \chi \) would indicate a strong coupling. To look into this issue, one may use the rescaled variable \( \tilde{\delta \chi} = C \delta \chi \) and expand the Lagrangian to higher order in perturbations. It would thus be interesting to explore the nonlinear dynamics of the \( \chi \) mode, which is however beyond the scope of this paper.
IV. SUMMARY

In this paper, we have proposed an effective scalar-tensor description of the recently proposed regularized version of Lovelock gravity in four dimensions. The effective scalar-tensor theory is obtained by promoting the warp factor of the internal $(D - 4)$-dimensional space to a dynamical scalar field, which survives after taking the $D \to 4$ limit while keeping the rescaled coupling constants $\alpha'_p = (D - 4)\alpha_p$ finite. The resultant theory resides in a particular subclass of the Horndeski theory.

Employing our scalar-tensor reformulation, we have studied cosmological aspects of regularized Einstein-Gauss-Bonnet gravity. We have found that cosmological solutions generically admit a scalar charge $C$, which yields a radiation-like component. All the previous results [2] are reproduced at the level of the Lagrangian by taking $C \to 0$. However, in this limit the gravitational scalar degree of freedom might be strongly coupled. This point needs further investigation.

Note added: While we were in the final stage of this work, the paper by Lu and Pang [24] appeared in the arXiv, where the same idea of reformulating regularized Einstein-Gauss-Bonnet gravity is presented. Their main focus is on black hole solutions, while ours is on cosmological aspects. Our conclusion agrees with them where we overlap.

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[1] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498.
[2] D. Glavan and C. Lin, Einstein-Gauss-Bonnet gravity in 4-dimensional space-time, Phys. Rev. Lett. 124 (2020) 081301 [1905.03601].
[3] A. Casalino, A. Colleaux, M. Rinaldi and S. Vicentini, Regularized Lovelock gravity, 2003.07068.
[4] Y. Tomozawa, Quantum corrections to gravity, 1107.1424.
[5] G. Cognola, R. Myrzakulov, L. Sebastiani and S. Zerbini, Einstein gravity with Gauss-Bonnet entropic corrections, Phys. Rev. D 88 (2013) 024006 [1304.1873].
[6] P. G. S. Fernandes, Charged Black Holes in AdS Spaces in 4D Einstein Gauss-Bonnet Gravity, 2003.05491.
[7] R. A. Konoplya and A. Zhidenko, Black holes in the four-dimensional Einstein-Lovelock gravity, 2003.07788.
[8] S.-W. Wei and Y.-X. Liu, Testing the nature of Gauss-Bonnet gravity by four-dimensional rotating black hole shadow, 2003.07769.
[9] R. Kumar and S. G. Ghosh, Rotating black holes in the novel 4D Einstein-Gauss-Bonnet gravity, 2003.08927.
[10] D. D. Doneva and S. S. Yazadjiev, Relativistic stars in 4D Einstein-Gauss-Bonnet gravity, 2003.10284.
[11] S. G. Ghosh and S. D. Maharaj, Radiating black holes in the novel 4D Einstein-Gauss-Bonnet gravity, 2003.09841.
[12] R. A. Konoplya and A. F. Zinhailo, Quasinormal modes, stability and shadows of a black hole in the novel 4D Einstein-Gauss-Bonnet gravity, 2003.01188.
[13] M. Guo and P.-C. Li, The innermost stable circular orbit and shadow in the novel 4D Einstein-Gauss-Bonnet gravity, 2003.02523.
[14] K. Hegde, A. N. Kumara, C. L. A. Rizwan, A. K. M. and M. S. Ali, Thermodynamics, Phase Transition and Joule Thomson Expansion of novel 4-D Gauss Bonnet AdS Black Hole, 2003.08778.
[15] Y.-P. Zhang, S.-W. Wei and Y.-X. Liu, Spinning test particle in four-dimensional Einstein-Gauss-Bonnet Black Hole, 2003.10960.
[16] D. V. Singh and S. Siwach, Thermodynamics and P-v criticality of Bardeen-AdS Black Hole in 4-D Einstein-Gauss-Bonnet Gravity, 2003.11754.
[17] K. Van Acoleyen and J. Van Doorsselaere, Galileons from Lovelock actions, Phys. Rev. D83 (2011) 084025 [1102.0487].
[18] C. Charmousis, B. Gouteraux and E. Kiritsis, Higher-derivative scalar-vector-tensor theories: black holes, Galileons, singularity cloaking and holography, JHEP 09 (2012) 011 [1206.1499].
[19] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. 10 (1974) 363.
[20] T. Kobayashi, Horndeski theory and beyond: a review, Rept. Prog. Phys. 82 (2019) 086901 [1901.07183].
[21] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, From k-essence to generalised Galileons, Phys. Rev. D84 (2011) 064039 [1103.3260].
[22] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. 126 (2011) 511 [1105.5723].
[23] T. Kobayashi, N. Tanahashi and M. Yamaguchi, *Multifield extension of G inflation*, Phys. Rev. D **88** (2013) 083504 [1308.4798].

[24] H. Lu and Y. Pang, *Horndeski Gravity as D → 4 Limit of Gauss-Bonnet*, [2003.11552].