Dynamical deformations of three-dimensional Lie algebras in Bianchi classification over the harmonic oscillator

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Abstract

Operadic Lax representations for the harmonic oscillator are used to construct the dynamical deformations of three-dimensional (3D) real Lie algebras in the Bianchi classification. It is shown that the energy conservation of the harmonic oscillator is related to the Jacobi identities of the dynamically deformed algebras. Based on this observation, it is proved that the dynamical deformations of 3D real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras.

1 Introduction

In Hamiltonian formalism, a mechanical system is described by the canonical variables $q^i, p_i$ and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$ (1.1)

By a Lax representation [4, 1] of a mechanical system one means such a pair $(L, M)$ of matrices (linear operators) $L, M$ that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = ML - LM$$ (1.2)

Thus, from the algebraic point of view, mechanical systems can be described by linear operators, i.e by linear maps $V \to V$ of a vector space $V$. As a generalization of this one can pose the following question [6]: how can the time evolution of the linear operations (multiplications) $V^\otimes n \to V$ be described?

The algebraic operations (multiplications) can be seen as an example of the operadic variables [2]. If an operadic system depends on time one can speak about operadic dynamics [6]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of the operadic variables may be given by the operadic Lax equation. In Refs. [7, 8, 9], low-dimensional binary operadic Lax representations for the harmonic oscillator were constructed.

In the present paper, the operadic Lax representations for the harmonic oscillator are used to construct the dynamical deformations of three-dimensional (3D) real Lie algebras in the Bianchi classification. It is shown that the energy conservation of the harmonic oscillator is related to the Jacobi identities of the dynamically deformed algebras. Based on this observation, it is proved
that the dynamical deformations of 3D real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras.

It turns out that four Lie algebras (I, VII, IX, VIII from the Bianchi classification) remain undeformed (rigid) and all other ones are deformed. However, it is interesting to note that these remain to be Lie algebras over canonical variables \( q, p \). Namely, four of them (II, IV, t, V, VI) lead straightforwardly to the Jacobi identity, while in other cases (VII, III, V) satisfy the Jacobi identity with the energy conservation law.

## 2 Endomorphism operad and Gerstenhaber brackets

Let \( K \) be a unital associative commutative ring, \( V \) be a unital \( K \)-module, and \( \mathcal{E}_V^n := \mathcal{E}nd_V^n := \text{Hom}(V^\otimes n, V) \ (n \in \mathbb{N}) \). For an operation \( f \in \mathcal{E}_V^n \), we refer to \( n \) as the degree of \( f \) and often write (when it does not cause confusion) \( f \) instead of \( \text{deg} \ f \). For example, \((-1)^f := (-1)^n, \mathcal{E}^f_V := \mathcal{E}_V^n \) and \( \circ f := \circ_n \). Also, it is convenient to use the reduced degree \( |f| := n - 1 \). Throughout this paper, we assume that \( \otimes := \otimes_K \).

**Definition 2.1** (endomorphism operad [2]). For \( f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g \) define the partial compositions

\[
\circ_i g := (-1)^{|g|} f \circ (\text{id}_V^i \otimes g \otimes \text{id}_V^{(|f|-i)}) \in \mathcal{E}_V^{f+|g|}, \quad 0 \leq i \leq |f|
\]

The sequence \( \mathcal{E}_V := \{\mathcal{E}_V^n\}_{n \in \mathbb{N}} \), equipped with the partial compositions \( \circ_i \), is called the endomorphism operad of \( V \).

**Definition 2.2** (total composition [2]). The total composition \( \bullet : \mathcal{E}_V^f \otimes \mathcal{E}_V^g \to \mathcal{E}_V^{f+|g|} \) is defined by

\[
f \bullet g := \sum_{i=0}^{|f|} f \circ_i g \in \mathcal{E}_V^{f+|g|}, \quad |\bullet| = 0
\]

The pair \( \text{Com} \mathcal{E}_V := \{\mathcal{E}_V, \bullet\} \) is called the composition algebra of \( \mathcal{E}_V \).

**Definition 2.3** (Gerstenhaber brackets [2]). The Gerstenhaber brackets \( [\cdot, \cdot] \) are defined in \( \text{Com} \mathcal{E}_V \) as a graded commutator by

\[
[f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f = -(-1)^{|f||g|}[g, f], \quad ||[\cdot, \cdot]| = 0
\]

The commutator algebra of \( \text{Com} \mathcal{E}_V \) is denoted as \( \text{Com}^{-1} \mathcal{E}_V := \{\mathcal{E}_V, [\cdot, \cdot]\} \). One can prove (e.g [2]) that \( \text{Com}^{-1} \mathcal{E}_V \) is a graded Lie algebra. The Jacobi identity reads

\[
(-1)^{|f||h|}[f, [g, h]] + (-1)^{|g||f|}[g, [h, f]] + (-1)^{|h||g|}[h, [f, g]] = 0
\]

## 3 Operadic Lax equation and harmonic oscillator

Assume that \( K := \mathbb{R} \) or \( K := \mathbb{C} \) and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by
**Definition 3.1** (operadic Lax pair). Allow a classical dynamical system to be described by the Hamiltonian system (1.1). An **operadic Lax pair** is a pair \((\mu, M)\) of operations \(\mu, M \in E_V\), such that the Hamiltonian system (1.1) may be represented as the **operadic Lax equation**

\[
\frac{d\mu}{dt} = [M, \mu] := M \bullet \mu - (-1)^{|M||\mu|} \mu \bullet M
\]

The pair \((\mu, M)\) is also called an **operadic Lax representations** of/for Hamiltonian system (1.1). In this paper we assume that \(|M| = 0\).

**Remark 3.2.** Evidently the degree constraints \(|M| = |\mu| = 0\) give rise to the ordinary Lax equation (1.2) [4, 1]. If \(|\mu| \neq 0\) then the Gerstenhaber brackets do not coincide with the usual commutator (see Section 4 for details).

The Hamiltonian of the harmonic oscillator is

\[
H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2)
\]

Thus, the Hamiltonian system of the harmonic oscillator reads

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q
\]

(3.1)

If \(\mu\) is a linear algebraic operation we can use the above Hamilton equations to obtain

\[
\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu]
\]

Therefore, we get the following linear partial differential equation for \(\mu(q, p)\):

\[
\frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu]
\]

(3.2)

By integrating (3.2) one can get collections of operations called the **operadic (Lax representations for) harmonic oscillator**. Since the general solution of the partial differential equations depends on arbitrary functions, these representations are not uniquely determined.

**4 Evolution of binary algebras**

Let \(A := \{V, \mu\}\) be a binary algebra with an operation \(xy := \mu(x \otimes y)\), i.e \(|\mu| = 1\). Assume that \(|M| = 0\). We require that \(\mu = \mu(q, p)\) so that \((\mu, M)\) is an operadic Lax pair, i.e the Hamiltonian system (3.1) of the harmonic oscillator may be written as the operadic Lax equation

\[
\dot{\mu} = [M, \mu] := M \bullet \mu - \mu \bullet M, \quad |\mu| = 1, \quad |M| = 0
\]

Let \(x, y \in V\). Assuming that \(|M| = 0\) and \(|\mu| = 1\) we have

\[
M \bullet \mu = \sum_{i=0}^{0} M \circ_i \mu = M \circ_0 \mu = M \circ \mu
\]

\[
\mu \bullet M = \sum_{i=0}^{1} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M = \mu \circ (M \otimes \text{id}_V) + \mu \circ (\text{id}_V \otimes M)
\]
Thus we can see that since $|\mu| = 1$ the Gerstenhaber brackets of $\mu$ and $M$ do not coincide with the common commutator bracketing that is used in the case of the ordinary Lax representations. Using the above formulae, we have

$$\frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My)$$

Let $\dim V = n$. In a basis $\{e_1, \ldots, e_n\}$ of $V$, the structure constants $\mu^i_{jk}$ of $A$ are defined by

$$\mu(e_j \otimes e_k) := \mu^i_{jk}e_i, \quad j, k = 1, \ldots, n$$

In particular,

$$\frac{d}{dt}(e_j e_k) = M(e_j e_k) - (M e_j)e_k - e_j(M e_k)$$

By denoting $M e_i := M^i_s e_s$, it follows that

$$\dot{\mu}^i_{jk} = \mu^i_{jk} M^i_s - M^i_s \mu^i_{jk} - M^i_s \mu^i_{jk}, \quad i, j, k = 1, \ldots, n$$

## 5 3D binary anti-commutative operadic Lax representations for harmonic oscillator

**Lemma 5.1.** Matrices

$$L := \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \frac{\omega}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

give a 3-dimensional Lax representation for the harmonic oscillator.

**Definition 5.2.** For the harmonic oscillator, define its auxiliary functions $A_{\pm}$ of canonical variables $p, q$ by

$$A^2_{+} + A^2_{-} = 2\sqrt{2H}, \quad A^2_{+} - A^2_{-} = 2p, \quad A_{+}A_{-} = \omega q$$

(5.1)

**Remark 5.3.** Note that the second and the third relations imply the first one in (5.1).

**Theorem 5.4** ([9]). Let $C_\nu \in \mathbb{R}$ ($\nu = 1, \ldots, 9$) be arbitrary real–valued parameters, such that

$$C^2_2 + C^2_3 + C^2_5 + C^2_6 + C^2_7 + C^2_8 \neq 0$$

(5.2)

Let $M$ be defined as in Lemma 5.1 and the binary anti-commutative multiplication $\mu : V \otimes V \to V$ is given by coordinates

$$\mu^1_{11} = \mu^1_{22} = \mu^1_{33} = \mu^2_{11} = \mu^2_{22} = \mu^2_{33} = \mu^3_{11} = \mu^3_{22} = \mu^3_{33} = 0$$

$$\mu^1_{23} = -\mu^1_{32} = C_2 p - C_3 \omega q - C_4$$

$$\mu^2_{13} = -\mu^2_{31} = C_2 p - C_3 \omega q + C_4$$

$$\mu^3_{31} = -\mu^3_{13} = C_2 \omega q + C_3 p - C_1$$

$$\mu^2_{23} = -\mu^2_{32} = C_2 \omega q + C_3 p + C_1$$

$$\mu^1_{12} = -\mu^1_{21} = C_5 A_+ + C_6 A_-$$

$$\mu^2_{12} = -\mu^2_{21} = C_5 A_- - C_6 A_+$$

$$\mu^3_{31} = -\mu^3_{31} = C_7 A_+ + C_8 A_-$$

$$\mu^3_{23} = -\mu^3_{32} = C_7 A_- - C_8 A_+$$

$$\mu^3_{12} = -\mu^3_{21} = C_9$$

(5.3)
6 Initial conditions and dynamical deformations

It seems attractive to specify the coefficients $C_{\nu}$ in Theorem 5.4 by the initial conditions

$$\mu\big|_{t=0} = \tilde{\mu}, \quad p\big|_{t=0} = p_0, \quad q\big|_{t=0} = 0$$

The latter together with (5.1) yield the initial conditions for $A_\pm$:

$$\begin{align*}
(A_+^2 + A_-^2)\big|_{t=0} &= 2|p_0| \\
(A_+^2 - A_-^2)\big|_{t=0} &= 2p_0 \\
A_+A_-\big|_{t=0} &= 0
\end{align*}$$

$$\iff \begin{cases} p_0 > 0 & A_+\big|_{t=0} = \pm\sqrt{2p_0} \\
A_-\big|_{t=0} = 0 & A_+\big|_{t=0} = 0 \\
A_-\big|_{t=0} = 0 & A_-\big|_{t=0} = \pm\sqrt{-2p_0} \end{cases}$$

In what follows assume that $p_0 > 0$ and $A_+\big|_{t=0} > 0$. Other cases can be treated similarly. Note that then $p_0 = \sqrt{2E}$, where $E > 0$ is the total energy of the harmonic oscillator, $H = H\big|_{t=0} = E$.

From (5.3) we get the following linear system:

$$\begin{align*}
\tilde{\mu}_{23} &= C_2 p_0 - C_4, \quad \tilde{\mu}_{31} = C_3 p_0 - C_1, \quad \tilde{\mu}_{12} = C_5 \sqrt{2p_0} \\
\tilde{\mu}_{13} &= C_2 p_0 + C_4, \quad \tilde{\mu}_{12} = -C_6 \sqrt{2p_0}, \quad \tilde{\mu}_{23} = C_3 p_0 + C_1 \\
\tilde{\mu}_{13} &= C_7 \sqrt{2p_0}, \quad \tilde{\mu}_{23} = -C_8 \sqrt{2p_0}, \quad \tilde{\mu}_{12} = C_9
\end{align*}$$

(6.1)

One can easily check that the unique solution of the latter system with respect to $C_{\nu}$ ($\nu = 1, \ldots, 9$) is

$$\begin{align*}
C_1 &= \frac{1}{2} \left( \tilde{\mu}_{23} - \tilde{\mu}_{31} \right), \quad C_2 = \frac{1}{2p_0} \left( \tilde{\mu}_{13} + \tilde{\mu}_{23} \right), \quad C_3 = \frac{1}{2p_0} \left( \tilde{\mu}_{23} + \tilde{\mu}_{31} \right) \\
C_4 &= \frac{1}{2} \left( \tilde{\mu}_{13} - \tilde{\mu}_{23} \right), \quad C_5 = \frac{1}{2p_0} \tilde{\mu}_{12}, \quad C_6 = -\frac{1}{2p_0} \tilde{\mu}_{12} \\
C_7 &= \frac{1}{\sqrt{2p_0}} \tilde{\mu}_{13}, \quad C_8 = -\frac{1}{\sqrt{2p_0}} \tilde{\mu}_{23}, \quad C_9 = \tilde{\mu}_{12}
\end{align*}$$

Remark 6.1. Note that the parameters $C_{\nu}$ have to satisfy condition (5.2) to get the operadic Lax representations.

Definition 6.2. If $\tilde{\mu} = \mu$, then the multiplication $\tilde{\mu}$ is called dynamical rigid. If $\mu \neq \tilde{\mu}$, then the multiplication $\mu$ is called a dynamical deformation of $\tilde{\mu}$ (over the harmonic oscillator).

7 Bianchi classification of 3d real Lie algebras

We use the Bianchi classification of the 3d real Lie algebras given in [3]. The structure equations of the 3d real Lie algebras can be presented as follows:

$$[e_1, e_2] = -ae_2 + n^3 e_3, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + ae_3$$

The values of the parameters $a, n^1, n^2, n^3$ and the corresponding structure constants are presented in Table 7.1.

The Bianchi classification is for instance used to describe the spatially homogeneous space-times of dimension 3+1. In particular, the Lie algebra VIIo is very interesting for the cosmological applications, because it is related to the Friedmann-Robertson-Walker metric. One can find more details in Refs. [3][5].
### Table 7.1: 3d real Lie algebras in Bianchi classification. Here $a > 0$. 

| Bianchi type | $a$ | $n^1$ | $n^2$ | $n^3$ | $\text{O}_1$ | $\text{O}_2$ | $\text{O}_3$ | $\text{O}_1^2$ | $\text{O}_2^2$ | $\text{O}_3^2$ | $\text{O}_1^3$ | $\text{O}_2^3$ | $\text{O}_3^3$ |
|--------------|-----|-------|-------|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| I            | 0   | 0     | 0     | 0     | 0          | 0          | 0          | 0           | 0           | 0           | 0           | 0           | 0           |
| II           | 0   | 1     | 0     | 0     | 0          | 0          | 0          | 0           | 0           | 0           | 0           | 0           | 0           |
| VII          | 0   | 1     | 1     | 0     | 0          | 0          | 0          | 0           | 0           | 0           | 1           | 0           | 0           |
| VI           | 0   | 1     | -1    | 0     | 0          | 0          | 1          | 0           | 0           | 0           | 0           | -1          | 0           |
| IX           | 0   | 1     | 1     | 1     | 0          | 0          | 1          | 0           | 0           | 0           | 0           | 0           | 0           |
| VIII         | 0   | 1     | 1     | -1    | 0          | 0          | -1         | 1           | 0           | 0           | 0           | 1           | 0           |
| V            | 1   | 0     | 0     | 0     | 0          | -1         | 0          | 0           | 0           | 0           | 0           | 0           | 1           |
| IV           | 1   | 0     | 0     | 1     | 0          | -1         | 1          | 0           | 0           | 0           | 0           | 0           | 1           |
| VII$_a$      | a   | 0     | 1     | 1     | 0          | -a         | 1          | 0           | 0           | 0           | 0           | 0           | 1           |
| III$_{a=1}$  | 1   | 0     | 1     | -1    | 0          | -1         | -1         | 0           | 0           | 0           | 0           | 0           | 1           |
| VI$_{a\neq1}$ | a   | 0     | 1     | -1    | 0          | -a         | -1         | 0           | 0           | 0           | 0           | 0           | 1           |

8 Dynamical deformations of 3d real Lie algebras

By using the structure constants of the 3-dimensional Lie algebras in the Bianchi classification, Theorem 5.4 and relations (6.1) we can find evolution of these algebras generated by the harmonic oscillator (see Table 8.1).

**Theorem 8.1** (dynamically rigid algebras). The algebras I, VII, VIII, and IX are dynamically rigid over the harmonic oscillator.

**Proof.** This is evident from Tables 7.1 and 8.1.

**Theorem 8.2** (dynamical Lie algebras). The algebras II$^t$, IV$^t$, VII$^t$, VI$^t$, III$_{a=1}^t$, VI$_{a\neq1}^t$, and VII$^t_a$ are Lie algebras.

**Proof.** Denoting $\mu := [\cdot, \cdot]_t$, one has to calculate the Jacobiator defined by

$$J_t(x; y; z) := [x, [y, z]]_t + [y, [z, x]]_t + [z, [x, y]]_t$$

$$J_t^1(x; y; z)e_1 + J_t^2(x; y; z)e_2 + J_t^3(x; y; z)e_3$$

At first, by direct calculations one can check that for the algebras II$^t$, IV$^t$, VII$^t$, VI$^t$ the Jacobiator identically vanishes, i.e $J_t = 0$.

Denote the scalar triple product of the vectors $x, y, z$ by

$$(x, y, z) := \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$$

Then, by direct calculations one can check that for the algebras VI$_{a\neq1}^t$ and VII$^t_a$ the Jacobiator coordinates are

$$J_t^1(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0}} [A_+ \omega q + A_+ (p - p_0)]$$

$$J_t^2(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0}} [A_+ \omega q - A_- (p + p_0)]$$

$$J_t^3(x; y; z) = 0$$

(8.1)
Table 8.1: Evolution of 3d real Lie algebras. Here $p_0 = \sqrt{2E}$.

and for the algebra $\text{III}^{t}_{a=1}$ one has the same formulae with $a = 1$.

Now, by using relations (5.1) calculate:

$A_-\omega q + A_+(p - p_0) = A_+ A_-^2 + A_+(p - p_0)$

$= A_+ (A_-^2 + p - p_0)$

$= A_+ \left( A_-^2 + \frac{1}{2} A_+^2 - \frac{1}{2} A_-^2 - p_0 \right)$

$= A_+ \left( \frac{1}{2} A_+^2 + \frac{1}{2} A_-^2 - p_0 \right)$

$= A_+ (\sqrt{2H} - p_0)$

$H = E = A_+ (\sqrt{2E} - p_0)$

$= A_+ 0$

$= 0$

Here we used the fact that the Hamiltonian $H$ is a conserved observable, i.e

$H = H|_{t=0} = E = p_0^2/2$

Thus, we have proved that $J^1_t = 0$. In the same way one can check that $J^2_t = 0$. \qed
9 Energy conservation from Jacobi identities

When proving Theorem 8.2 we observed how the conservation of energy \( H = E \) implies the Jacobi identities \( J_1^t = J_2^t = 0 \) of the dynamically deformed algebras. Now let us show vice versa, i.e.

Theorem 9.1. The Jacobi identity \( J_t = 0 \) implies conservation of energy \( H = E \).

Proof. By setting in (8.1) \( J_1^t = J_2^t = 0 \), we obtain the following system:

\[
\begin{align*}
A_- \omega q + A_+ p &= A_+ p_0 \\
A_+ \omega q - A_- p &= A_- p_0
\end{align*}
\]  

(9.1)

Now use the the defining relations (5.1) of \( A_{\pm} \) and the Cramer formulae to express the canonical variables \( q, p \) via \( A_{\pm} \). First calculate

\[
\Delta := \begin{vmatrix} A_- & A_+ \\ A_+ & -A_- \end{vmatrix} = -A_-^2 - A_+^2 = -2 \sqrt{2H} \neq 0
\]

\[
\Delta_{\omega q} := \begin{vmatrix} A_+ p_0 & A_+ \\ A_- p_0 & -A_- \end{vmatrix} = -2 A_+ A_- p_0 = -2 \omega qp_0
\]

\[
\Delta_{p} := \begin{vmatrix} A_- & A_+ p_0 \\ A_+ & A_- p_0 \end{vmatrix} = A_-^2 p_0 - A_+^2 p_0 = -2 p p_0
\]

Thus we have

\[
\omega q = \frac{\Delta_{\omega q}}{\Delta} = - \frac{2 \omega qp_0}{-2 \sqrt{2H}} \quad \Rightarrow \quad \frac{p_0}{\sqrt{2H}} = 1 \quad \Rightarrow \quad H = \frac{p_0^2}{2} = E
\]

\[
p = \frac{\Delta_{p}}{\Delta} = - \frac{2 p p_0}{-2 \sqrt{2H}} = \frac{p p_0}{\sqrt{2H}} \quad \Rightarrow \quad \frac{p_0}{\sqrt{2H}} = 1 \quad \Rightarrow \quad H = \frac{p_0^2}{2} = E
\]

Actually, the last implications are possible only at the time moments when \( q \neq 0 \) and \( p \neq 0 \), respectively. But \( q \) and \( p \) can not be simultaneously zero, thus really \( H = E \) for all \( t \).

10 Concluding remarks

In the present paper, the operadic Lax representations for the harmonic oscillator were used to construct the dynamical deformations of 3d real Lie algebras in the Bianchi classification. It was shown that the energy conservation of the harmonic oscillator is related to the Jacobi identities of the dynamically deformed algebras. Based on this observation, it was proved that the dynamical deformations of 3D real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras.

It turned out that four Lie algebras (I, VII, IX, VIII) remain undeformed (rigid) and all other ones are deformed. However, it is interesting to note that these remain to be Lie algebras over canonical variables \( q, p \). Namely, four of them (II\( ^t \), IV\( ^t \), V\( ^t \), VI\( ^t \)) lead straightforwardly to the Jacobi identity, while in other cases (VII\( _{ai=1}^t \), III\( _{a=1}^t \), VI\( _{ai\neq1}^t \)) satisfy the Jacobi identity with the energy conservation law.
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