The deformed Hermitian-Yang-Mills equation on almost Hermitian manifolds

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Abstract In this paper, we consider the deformed Hermitian-Yang-Mills equation on closed almost Hermitian manifolds. In the case of the hypercritical phase, we derive a priori estimates under the existence of an admissible $C$-subsolution. As an application, we prove the existence of solutions for the deformed Hermitian-Yang-Mills equation under the condition of existence of a supersolution.

Keywords deformed Hermitian-Yang-Mills equation, almost Hermitian manifold, maximum principle, a priori estimates

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1 Introduction

Motivated by mirror symmetry and mathematical physics, on a Kähler manifold $(M, \chi)$ of complex dimension $\dim \mathbb{C} M = n$, the deformed Hermitian-Yang-Mills equation, which has been studied extensively, can be written as the following form:

$$\Im (\chi + \sqrt{-1} \omega_u)^n = \tan(\hat{\theta}) \Re (\chi + \sqrt{-1} \omega_u)^n,$$

where $\hat{\theta}$ is a constant, $\omega$ is a smooth real $(1,1)$-form and $\omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u$. Assume that $(\lambda_1(u), \ldots, \lambda_n(u))$ are the eigenvalues of $\omega_u$ with respect to $\chi$. Without confusion, we also denote $\lambda_i(u)$ by $\lambda_i$, $i = 1, \ldots, n$. The equation (1.1) can be rewritten as

$$\sum_i \arctan \lambda_i = \hat{\theta}.$$

By solving this equation, we can find a Hermitian metric on the line bundle over $M$ such that the argument of Chern curvature is constant (see [19]).

For the dimension $n = 2$, one only needs to solve a Monge-Ampère equation (see [19]). In addition, Jacob and Yau [19] used a parabolic flow to prove the existence of the solution when $(M, \chi)$ has non-negative orthogonal bisectional curvature and $\hat{\theta}$ satisfies the hypercritical phase condition, i.e., $n \pi \frac{2}{f} >$
satisfies the hypercritical phase condition, i.e., equation (1.2) in the non-Kähler case. Motivated by these works, we prove this result.

In fact, the deformed Hermitian-Yang-Mills equation on Calabi-Yau manifolds. Generalized complex geometry, proposed by Hitchin [18], is closely related to that Lau et al. [20] studied SYZ (Strominger-Yau-Zaslow) mirror symmetry in the context of non-Kähler bundles over almost Hermitian manifolds was researched in [11,30,32] and the references therein. Note that Lau et al. [20] studied SYZ (Strominger-Yau-Zaslow) mirror symmetry in the context of non-Kähler manifolds.

In order to prove the theorem above, we will use the maximum principle. A crucial ingredient of the proof is the concavity of the equation (see Lemma 5.5). In the proof, we give a positive lower bound for the third order terms. We can obtain these properties when $\hat{\theta} > (n-1)\frac{\pi}{2}$ for general dimensions. Pingali [21,22] proved the existence of the solution when $n = 3$. Collins et al. [6] gave the existence theorem of (1.1) under the condition of existence of a subsolution in general dimensions. Takahashi [27] introduced the tangent Lagrangian phase flow and used it to prove the existence of the solution of the deformed Hermitian-Yang-Mills equation, assuming the existence of a C-subsolution. For more details, we refer to [2,3,7–10] and the references therein.

In this paper, we give a priori estimates on an almost Hermitian manifold $(M, \chi, J)$ with real dimension $2n$ for the deformed Hermitian-Yang-Mills equation. Consider the following general equation:

$$F(\omega_u) = f(\lambda_1, \ldots, \lambda_n) = \sum_{i} \arctan \lambda_i = h, \quad (1.2)$$

where $h : M \to ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$ is a given function on $M$ and $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $\omega_u$ with respect to $\chi$.

We now state our main result.

**Theorem 1.1.** Assume that $h : M \to ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$ is a smooth function and $u : M \to \mathbb{R}$ is a smooth $C$-subsolution (see Definition 2.1). Suppose that $\omega$ is a smooth (1,1)-form. Let $u$ be a solution of (1.2). For each $0 < \beta < 1$, then we have

$$\|u\|_{C^{0,\beta}} \leq C, \quad (1.3)$$

where $C$ depends on $\|h\|_{C^{k+1}(M)}$, $\inf_M h$, $u$, $k$, $\beta$, $(M, \chi, J)$ and $\omega$.

Almost Hermitian manifolds have been studied extensively motivated by differential geometry and mathematical physics (see [12,16–18,24] and the references therein). The theory of fully nonlinear elliptic equations was developed, such as in [4,5,31]. On the other hand, the geometry of complex vector bundles over almost Hermitian manifolds was researched in [11,30,32] and the references therein. Note that Lau et al. [20] studied SYZ (Strominger-Yau-Zaslow) mirror symmetry in the context of non-Kähler Calabi-Yau manifolds. Generalized complex geometry, proposed by Hitchin [18], is closely related to flux compactifications in string theory (see [15]). In fact, the deformed Hermitian-Yang-Mills equation plays an important role in mirror symmetry and string theory (see [8]). It is natural to consider the equation (1.2) in the non-Kähler case. Motivated by these works, we prove this result.

In order to prove the theorem above, we will use the maximum principle. A crucial ingredient of the proof is that the equation (1.2) is concave. To prove that the equation is concave, we need to assume that $h$ satisfies the hypercritical phase condition, i.e., $n\frac{\pi}{2} > \hat{\theta} > (n-1)\frac{\pi}{2}$. In addition, to use Proposition 2.4 provided by the $C$-subsolution, we have to let the equation satisfy some properties in Lemma 2.3. We can obtain these properties when $h$ satisfies the hypercritical phase condition.

In the proof of the second order estimates, we apply the maximum principle. We will use the arguments of [5,31]. To deal with the bad third order terms, we need to give a lower bound for the third order terms from the concavity of the equation (see Lemma 5.5). In the proof, we give a positive lower bound for the complex eigenvalues $\omega_u$ provided by $\inf_M h > \frac{(n-1)\pi}{2}$.

When $h \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$ is a constant, we can prove the existence under the condition of existence of a supersolution.

**Theorem 1.2.** Assume that there exists a supersolution $\hat{u}$ of (1.2), i.e., $F(\omega_{\hat{u}}) \leq h$. Suppose that $F(\omega_{\hat{u}}) > \frac{(n-1)\pi}{2}$ and $h \in ((n-1)\frac{\pi}{2}, \frac{n\pi}{2})$ is a constant. If there exists a $C$-subsolution $u$ for (1.2), we have a function $u$ and a constant $c$ such that $\sum_i \arctan \lambda_i = h + c$, where $h + c > \frac{(n-1)\pi}{2}$.

**Remark 1.3.** If $(M, \chi)$ is a Kähler manifold, then the supersolution $\hat{u}$ is a solution of the equation (1.2), by using $\hat{\theta}$ is an invariant. However, $\hat{\theta}$ is not an invariant on almost Hermitian manifolds.

Let $\pi : L \to M$ be a complex line bundle on $M$ and $\varpi$ be a Hermitian metric on $L$ (for more details, see [11, Section 2] or [30]). There exists a unique $(1,0)$ type Hermitian connection $D_{\varpi}$ which is called the canonical Hermitian connection. Let $F(\varpi)$ be the curvature form of the connection $D_{\varpi}, F^{1,1}(\varpi)$ be the $(1,1)$ part of the curvature of $\varpi$, and $\varpi(u) = e^{-u} \varpi$. We have (see [11] or [30, (2.5)])

$$F^{1,1}(\varpi(u)) = F^{1,1}(\varpi) + \partial \bar{\partial} u.$$
We assume $\int_M \chi^n = 1$. Set $\omega = \sqrt{-1} F^{1,1}(\varpi)$ and

$$\hat{\theta}(\varpi(u)) = \text{Arg} \int_M (\chi + \sqrt{-1} (\omega + \sqrt{-1} \partial \bar{\partial} u))^n.$$

Here, $\text{Arg} \varphi$ means the argument of a complex function $\varphi$. Note that (1.1) is equivalent to (1.2). We immediately obtain the following corollary.

**Corollary 1.4.** Suppose that there exists a supersolution $\dot{u}$ of (1.2). Assume that $F(\omega_0) > \frac{(n-1)\pi}{2}$ and $h \in (\frac{(n-1)\pi}{2}, \frac{n\pi}{2})$ is a constant. Suppose that there exists $\varpi(u) = e^{-\varpi} \varpi$ such that $u$ is a $C$-subsolution of (1.2). There exists a Hermitian metric $\varpi(u)$ on the line bundle $L$ such that the argument of $\chi^{\frac{1}{\sqrt{2}}(\varpi(u))}$ is constant.

In the Kähler case, the proof in [6] relied on the argument $\hat{\theta}$ of the integral to be independent of the choice of $\omega_1$. They used this fact to prove the $C$-subsolution is preserved along the family of equations used in the continuity method. However, $\hat{\theta}(\varpi(u))$ depends on $u$ in our case. Under the existence of the supersolution, we prove the $C$-subsolution and hypercritical condition are preserved when we use the continuity method, by using the arguments of [6, 25].

The rest of this paper is organized as follows. In Section 2, we recall the definition of the $C$-subsolution and some properties of the equation (1.2). In Section 3, we give the $C^0$ estimates. We use the argument of Székelyhidi [26] (see also [1]). In Section 4, the gradient estimates are proved. In Section 5, we give the second order estimates and complete the proof of Theorem 1.1. In Section 6, by the continuity method, we prove Theorem 1.2 under the condition of existence of a supersolution.

### 2 Preliminaries

On an almost Hermitian manifold $(M, \chi, J)$ with real dimension $2n$, for any $(p, q)$-form $\beta$, we can define $\partial$ and $\bar{\partial}$ operators (see [5, 17]). Denote by $A^{1,1}(M)$ the space of smooth real $(1,1)$ forms on $(M, \chi, J)$. Then, for any $\varphi \in C^2(M)$, $\sqrt{-1} \partial \bar{\partial} \varphi = \frac{1}{2} (d J d \varphi)^{(1,1)}$ is a real $(1,1)$-form in $A^{1,1}(M)$. Let $\{e_i\}_{i=1}^n$ be a local frame for $\Omega_c^{(1,0)} M$ and $\{\theta^1, \ldots, \theta^n\}$ be a dual coframe associated with the metric $\chi$ on $(M, \chi, J)$ (see [17, (2.5)]). Let $\chi J = \chi(e_i, \tau_j)$ and $g_J = \omega(e_i, \tau_j)$. Then $\chi = \chi J \sqrt{-1} \theta^i \wedge \bar{\theta}^j$ and $\omega = g_J \sqrt{-1} \theta^i \wedge \bar{\theta}^j$. We have

$$\varphi J = (\sqrt{-1} \partial \bar{\partial} \varphi)(e_i, \tau_j) = e_i \tau_j(\varphi) - [e_i, \tau_j]^{(0,1)}(\varphi),$$

where $[e_i, \tau_j]^{(0,1)}$ is the $(0,1)$ part of the Lie bracket $[e_i, \tau_j]$. We use the following notation:

$$F^i J = \partial \sum_k \arctan \lambda_k(\bar{g}_J) \frac{\partial}{\partial \bar{g}_J} \lambda_k,$$

where $\bar{g}_J = g_J + \nu J$. For any point $x_0 \in M$, let $\{e_i\}_{i=1}^n$ be a local unitary frame (with respect to $\chi$) such that $\bar{g}_J(x_0) = \delta_{ij} g_{\bar{J}}(x_0)$. We denote $\bar{g}_{\bar{J}}(x_0)$ by $\lambda_i$. It is useful to order $\{\lambda_i\}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then at $x_0$, we have

$$F^i J = F^i \delta_{ij} = \frac{1}{1 + \lambda_i} \delta_{ij}. \quad (2.1)$$

By [26, (66)] or [13], we deduce

$$F^{i k J} = f_{i j} \delta_{i k} \delta_{j l} + \frac{f_i - f_j}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{il} \delta_{jk}.$$

It follows that, at $x_0$,

$$F^{i k J} = \begin{cases} F^i \delta_{ij}, & \text{if } i = j = k = l, \\ F^i \delta_{ik}, & \text{if } i = l, \; k = j, \; i \neq k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$
Moreover, at \( x_0 \),
\[
F_{i,i} = -\frac{2\lambda_i}{(1 + \lambda_i^2)^2},
\]
\[
F_{k,k} = -\frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)}.
\]  
(2.3)

The linearization operator of (1.2) is
\[
L := \sum_{i,j} F_{i,j} (e_i \bar{e}_j - [e_i, \bar{e}_j])^{0.1}.
\]  
(2.4)

Note that \([e_i, \bar{e}_j]^{0.1}\) are first order differential operators. By (2.1), \( L \) is a second order elliptic operator.

2.1 The \( C \)-subsolution

Now we recall the definition of the \( C \)-subsolution of (1.2) (see [6,26]). Let
\[
\Gamma_n = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \lambda_i > 0, 1 \leq i \leq n \},
\]
\[
\Gamma = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \sum_i \arctan(\lambda_i) > (n - 1)\frac{\pi}{2} \right\}
\]
and
\[
\Gamma^\sigma = \left\{ \lambda \in \Gamma, \sum_i \arctan(\lambda_i) > \sigma \right\},
\]
where \( \sigma \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2}) \).

**Definition 2.1** (See [6,26]). We say that a smooth function \( u : M \rightarrow \mathbb{R} \) is a \( C \)-subsolution of (1.2) if at each point \( x \in M \), we have
\[
\left\{ \lambda \in \Gamma : \sum_{i=1}^n \arctan(\lambda_i) = h(x) \text{ and } \lambda - \lambda(u) \in \Gamma_n \right\}
\]
is bounded.

Collins et al. [6] gave an explicit description of the \( C \)-subsolution.

**Lemma 2.2** (See [6, Lemma 3.3]). A smooth function \( u : M \rightarrow \mathbb{R} \) is a \( C \)-subsolution of (1.2) if and only if at each point \( x \in M \), for all \( j = 1, \ldots, n \), we have
\[
\sum_{i \neq j} \arctan(\lambda_i(u)) > h(x) - \frac{\pi}{2},
\]
where \( \lambda_1(u), \ldots, \lambda_n(u) \) are the eigenvalues of \( \omega_u \) with respect to \( \chi \).

Therefore, there are uniform constants \( \delta, R > 0 \) such that at each \( x \in M \) we have
\[
(\lambda(u) - \delta 1 + \Gamma_n) \cap \partial \Gamma^{h(x)} \subset B_R(0),
\]
(2.5)

where \( B_R(0) \) is an \( R \)-radius ball in \( \mathbb{R}^n \) with center 0, and \( 1 = (1, 1, \ldots, 1) \).

We now prove the following lemma.

**Lemma 2.3.** Suppose \( h \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2}) \). Then we have the following properties:
(1) \( f_i = \frac{df}{\partial \lambda_i} > 0 \) for all \( i \), and the equation (1.2) is concave;
(2) \( \sup_{\partial M} f < \inf_M h \);
(3) for any \( \sigma < \sup_{\Gamma} f \) and \( \lambda \in \Gamma \) we have \( \lim_{t \rightarrow \infty} f(t\lambda) > \sigma \).
Proof. Note \( f_i = \frac{1}{1+\lambda_i} \). It is obvious that \( f_i > 0 \) and \( \sup_{\partial M} f < \inf_M h \), if \( h > (n-1)\frac{\pi}{2} \).

When \( h > (n-1)\frac{\pi}{2} \), we have
\[
\lambda_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, n.
\] (2.6)

In fact, if there is \( \lambda_i \leq 0 \), then we must have \( \sum_j \arctan \lambda_j \leq (n-1)\frac{\pi}{2} \) and this leads to a contradiction. Hence, by (2.2), \( f \) is concave and \( \lim_{t \to \infty} f(t\lambda) = n\frac{\pi}{2} > \sigma \).

Using the above lemma and [26, Proposition 6 and Lemma 9], we have the following proposition. It plays an important role in the proof.

**Proposition 2.4.** Let \([a, b] \subset ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})\) and \( \delta, R > 0 \). There exists \( \theta > 0 \), depending on \( \sigma \) and the set in (2.7) with the following property: suppose that \( \lambda, B \) is a Hermitian matrix such that
\[
(\lambda(B) - 2\delta 1 + \Gamma_n) \cap \partial \Gamma^\sigma \subset B_R(0),
\] (2.7)

and then for any Hermitian matrix \( A \) with eigenvalues \( \lambda(A) \in \partial \Gamma^\sigma \) and \( |\lambda(A)| > R \), we either have
\[
\sum_{p,q} F^\varphi(A)[B_{pq} - A_{pq}] > \theta \sum_p F^\varphi(A)
\] (2.8)

or
\[
F^\varphi(A) > \theta \sum_p F^\varphi(A)
\] (2.9)

for all \( i \). In addition, there exists a constant \( K \) depending on \( \sigma \) such that
\[
\sum_i F^{ii} > K.
\] (2.10)

**Corollary 2.5.** Suppose \( h \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}) \). Assume that \( \bar{u} \) is an admissible \( C \)-subsolution and \( u \) is the smooth solution of (1.2). Then there exists a constant \( \theta > 0 \) (depending only on \( h \) and \( \bar{u} \)) such that either
\[
L(\bar{u} - u) \geq \theta \sum_i F^{ii}(\bar{g})
\] (2.11)

or
\[
F^{kk}(\bar{g}) \geq \theta \sum_i F^{ii}(\bar{g}) \quad \text{for} \quad k = 1, 2, \ldots, n,
\] (2.12)

if \( \lambda(\omega_u) \in (\lambda(\omega_u) - \delta 1 + \Gamma_n) \cap \partial \Gamma^h \).

In addition, there is a constant \( K > 0 \) depending on \( h \) and \( \bar{u} \) such that
\[
F := \sum_i F^{ii}(\bar{g}) > K, \quad \text{if} \quad \lambda(\omega_u) \in \partial \Gamma^h.
\] (2.13)

**Proof.** By Definition 2.1, there are uniform constants \( \delta, R > 0 \) such that at each \( x \in M \) we have
\[
(\lambda(u) - \delta 1 + \Gamma_n) \cap \partial \Gamma^h(x) \subset B_R(0).
\]

If \( |\lambda(u)| > R \), by Proposition 2.4, the results follow. If \( |\lambda(u)| \leq R \), then \( 1 \geq F^{ii} \geq \frac{1}{1+R^2}, \ i = 1, 2, \ldots, n \), which implies (2.12) and (2.13) hold.

### 3 Zero order estimates

In this section, we prove the \( C^0 \) estimates. We need the following proposition provided by [5, Proposition 2.3].

**Proposition 3.1.** Let \((M, \chi, J)\) be a compact almost Hermitian manifold. Suppose that \( \psi \) satisfies
\[
\omega + \sqrt{-1} \partial \bar{\partial} \psi > 0, \quad \sup_M \psi = 0.
\]

Then there exists a constant \( C \) depending only on \((M, \chi, J)\) and \( \omega \) such that \( \int_M (\psi) \chi^n \leq C. \)
Proof. There exists a constant $C_0$ such that $C_0 \chi \geq \omega$. Therefore, by (2.6), $C_0 \chi + \sqrt{-1} \partial \bar{\partial} \psi > 0$. Then by [5, Proposition 2.3], we have
\[
\int_M (-\psi)^n \leq C.
\]
This completes the proof. \hfill \Box

Indeed, by (2.6), the assumption in Proposition 3.1 is satisfied in our paper. The following variant of the Alexandroff-Bakelman-Pucci maximum principle [26, Proposition 11], similar to Gilbarg and Trudinger [14, Lemma 9.2], is used to prove the $C^0$ estimates.

**Proposition 3.2.** Let $\varphi : B_1(0) \to \mathbb{R}$ be a smooth function, such that $\varphi(0) + \varepsilon \leq \inf_{\partial B_1(0)} \varphi$, where $\varepsilon > 0$ and $B_1(0) \subset \mathbb{R}^{2n}$. Define the set
\[
P = \left\{ x \in B_1(0) : |D\varphi(x)| < \frac{\varepsilon}{2} \text{ and } \varphi(y) \geq \varphi(x) + D\varphi(x) \cdot (y - x) \text{ for all } y \in B_1(0) \right\}.
\]
Then there exists a constant $c_0$ depending only on $n$ such that $c_0 \varepsilon^{2n} \leq \int_P \det(D^2 \varphi)$.

The $C^0$ estimates follow the argument of [26, Proposition 10] or [5, Proposition 3.1]. It is similar to [31, Proposition 3.2]. For the reader’s convenience, we include the proof.

**Proposition 3.3.** Let $u$ be the solution of (1.2) with $\sup_M (u - \bar{u}) = 0$. Then
\[
\|u\|_{L^\infty} \leq C, \tag{3.1}
\]
for some constant $C > 0$ depending on $(M, \chi, J)$, $\omega$, $h$ and $\bar{u}$.

Proof. From the hypothesis, it suffices to estimate the infimum $m_0 = \inf_M (u - \bar{u})$. We may assume that $m_0$ is attained at $x_0$. Choose a local coordinate chart $(x^1, \ldots, x^{2n})$ in a neighborhood of $x_0$ containing the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ such that the coordinates of $x_0$ are the origin $0 \in \mathbb{R}^{2n}$.

Consider the test function
\[
v := u - \bar{u} + \varepsilon \sum_{i=1}^{2n} (x^i)^2
\]
for a small $\varepsilon > 0$ determined later. Then we have
\[
v(0) = m_0 \quad \text{and} \quad v \geq m_0 + \varepsilon \quad \text{on } \partial B_1(0).
\]
We define the lower contact set of $v$ by
\[
P := \left\{ x \in B_1(0) : |Dv(x)| \leq \frac{\varepsilon}{2}, \ v(y) \geq v(x) + Dv(x) \cdot (y - x) \text{ for all } y \in B_1(0) \right\}. \tag{3.2}
\]
By Proposition 3.2, we have
\[
c_0 \varepsilon^{2n} \leq \int_P \det(D^2 v). \tag{3.3}
\]
Let $(D^2(u - \bar{u}))^J$ be the $J$-invariant part of $(D^2(u - \bar{u}))$, i.e.,
\[
(D^2(u - \bar{u}))^J = \frac{1}{2} (D^2(u - \bar{u}) + J^T D^2(u - \bar{u}) J), \tag{3.4}
\]
where $J^T$ is the transpose of $J$. Note that $0 \in P$ and $D^2 v \geq 0$ on $P$. Then we deduce
\[
(D^2(u - \bar{u}))^J(x) \geq (D^2 v)^J(x) - C \varepsilon I \geq -C \varepsilon I d \quad \text{for } x \in P. \tag{3.5}
\]
Consider the bilinear form $H(v)(X, Y) = \sqrt{-1} \partial \bar{\partial} v(X, JY)$. In fact, we obtain
\[
H(v)(X, Y)(x) = \frac{1}{2} (D^2 v)^J(x) + E(v)(x), \quad x \in M,
\]
where $E(v)(x)$ is an error matrix which depends linearly on $Dv(x)$ (see, e.g., [29, p.443]). By using $|D(u - \bar{u})| \leq \frac{\nu}{2}$ on $P$ and (3.5), it follows that

$$H(u) - H(\bar{u}) = (D^2u + E(Du))^T - (D^2\bar{u} + E(D\bar{u}))^T = (D^2(u - \bar{u}))^T + (E(D(u - \bar{u})))^T \geq -C\varepsilon \text{Id}.$$ 

Hence $\omega_u - \omega_{\bar{u}} \geq -C\varepsilon \chi$. Therefore, if we choose $\varepsilon$ sufficiently small such that $C\varepsilon \leq \delta$, then

$$\lambda(u) = \lambda(\bar{u}) - \delta 1 + \Gamma_n.$$ 

On the other hand, the equation (1.2) implies $\lambda(u) \in \partial\Gamma^h$. Consequently,

$$\lambda(u) = (\lambda(u) - \delta 1 + \Gamma_n) \cap \partial\Gamma^h \subset B_R(0)$$

for some $R > 0$ by the argument (2.5). This gives an upper bound for $H(u)$ and hence also for $H(v) - E(Dv)$ on $P$.

Note $\det(A + B) \geq \det(A) + \det(B)$ for positive definite Hermitian matrices $A$ and $B$. Recall the definition of $(D^2v)^T$ in (3.4). Then on $P$, we have

$$\det(D^2v) \leq 2^{2n-1} \det((D^2v)^T) = 2^{2n-1} \det(H(v) - E(Dv)) \leq C. \tag{3.6}$$

Plugging (3.6) into (3.3), we obtain

$$c_0\varepsilon^{2n} \leq C|P|. \tag{3.7}$$

For each $x \in P$, choosing $y = 0$ in (3.2), we have

$$m_0 = v(0) \geq v(x) - |Dv(x)||x| \geq v(x) - \frac{\varepsilon}{2}.$$ 

We may and do assume $m_0 + \varepsilon \leq 0$ (otherwise we are done). Then, on $P$, $-v \geq |m_0 + \varepsilon|$. Integrating it on $P$, we get

$$|P| \leq \int_P (-v)^n \frac{\chi^n}{|m_0 + \varepsilon|} \leq C \frac{C}{|m_0 + \varepsilon|},$$

where in the last inequality we used Proposition 3.1. By (3.7), we get a uniform lower bound for $m_0$. \hfill \Box

## 4 First order estimates

In this section, we give the proof of the $C^1$ estimates. Let $|\nabla u|_\chi$ be the norm of the gradient $u$ with respect to $\chi$. For convenience, we use $|\nabla u|$ to denote $|\nabla u|_\chi$. Let $\mathcal{F} = \sum_i F^i$.

### Proposition 4.1.

It holds that

$$|\nabla u|_\chi \leq C \tag{4.1}$$

for some constant $C$ depending on $(M, \chi, J)$, $\omega$, $\|h\|_{C^1}$ and $\bar{u}$.

**Proof.** Let $\zeta = Ae^{B_1\eta}$, where

$$\eta = u - u - \inf_M (u - \bar{u}) \geq 0,$$

and $A$ and $B_1$ are positive constants to be chosen later. Consider the test function

$$\tilde{Q} := e^\zeta |\nabla u|^2.$$ 

Suppose that $\tilde{Q}$ achieves its maximum at the point $x_0$. Near $x_0$, there exists a local unitary frame $\{e_i\}_{i=1}^n$ (with respect to $\chi$) such that at $x_0$, we have

$$\chi_{ij} = \delta_{ij}, \quad \tilde{g}_{ij} = \delta_{ij} \tilde{g}_{ij} \quad \text{and} \quad \tilde{g}_{11} \geq \tilde{g}_{22} \geq \cdots \geq \tilde{g}_{nn}.$$
From now on, we will use the Einstein summation convention, and all the following calculations are done at $x_0$. The $C$ below in this section denote the constants that may change from line to line, where $C$ depends on all the allowed data that we determine later.

Recall that $L$ is defined in (2.4). By the maximum principle, it follows that

$$0 \geq \frac{L(\hat{Q})}{B_1 \zeta \zeta |\nabla u|^2} = \frac{L(|\nabla u|^2)}{B_1 \zeta \zeta |\nabla u|^2} + \frac{L(\hat{u})}{B_1 \zeta \zeta} + 2F^{\bar{i}}\Re\left\{\hat{e}_i(\zeta) \bar{\hat{e}}_i(|\nabla u|^2)\right\}$$

$$= \frac{L(|\nabla u|^2)}{B_1 \zeta \zeta |\nabla u|^2} + L(\eta) + B_1 (1 + \zeta) F^{\bar{i}}|e_i(\eta)|^2$$

$$+ \frac{2}{|\nabla u|^2} \sum_j F^{\bar{i}}\Re\left\{e_i(\eta) \bar{\hat{e}}_i \bar{e}_j(u) + e_i(\eta) \bar{\hat{e}}_j(u) \hat{e}_j(u)\right\}. \quad (4.2)$$

This completes the proof.

Now we deal with these terms in turn. First we have the following lemma.

**Lemma 4.2.** It holds that

$$L(|\nabla u|^2) \geq 2 \sum_j \Re\{e_j(h) \bar{e}_j u\} + (1 - \varepsilon) \sum_j F^{\bar{i}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} F |\nabla u|^2.$$  

**Proof.** By a direct calculation,

$$L(|\nabla u|^2) = F^{\bar{i}}(e_i e_j (|\nabla u|^2) - [e_i, \bar{e}_j]^{0,1}(|\nabla u|^2)) =: I + II + III, \quad (4.3)$$

where

$$I = \sum_j F^{\bar{i}}(e_i \bar{e}_j \bar{e}_j u - [e_i, \bar{e}_j]^{0,1} e_j u),$$

$$II = \sum_j F^{\bar{i}}(e_i \bar{\hat{e}}_j \bar{e}_j u - [e_i, \bar{\hat{e}}_j]^{0,1} \bar{e}_j u),$$

$$III = \sum_j F^{\bar{i}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2).$$

Differentiating (1.2) along $e_j$ without summation, we have

$$F^{\bar{i}} \bar{e}_j(g_{\bar{i}j}) + F^{\bar{i}}(e_j e_i \bar{e}_j u - e_j [e_i, \bar{e}_j]^{0,1} u) = e_j(h).$$

Recall the definition of the Lie bracket $[e_i, e_j] = e_i e_j - e_j e_i$. Then we have

$$I + II = 2 \sum_j F^{\bar{i}} \Re\{(e_i \bar{e}_j \bar{e}_j u - [e_i, \bar{e}_j]^{0,1} e_j u)\bar{e}_j(u)\}$$

$$= 2 \sum_j F^{\bar{i}} \Re\{(e_j e_i \bar{e}_i u + e_i \bar{e}_j \bar{e}_j u + [e_i, e_j] \bar{e}_u - [e_i, \bar{e}_j]^{0,1} e_j u)\bar{e}_j(u)\}$$

$$= 2 \sum_j \Re\{e_j h \bar{e}_j(u)\} - 2 \sum_j F^{\bar{i}} \Re\{e_j(g_{\bar{i}j})\bar{e}_j(u)\} + 2 \sum_j \Re\{F^{\bar{i}} e_i\bar{e}_j^{0,1} \bar{e}_j(u)\}$$

$$+ 2 \Re\sum_j \{F^{\bar{i}}(e_i \bar{e}_j \bar{e}_j u + [e_i, e_j] \bar{e}_u - [e_i, \bar{e}_j]^{0,1} e_j u)\bar{e}_j(u)\}. \quad (4.4)$$

We may assume $|\nabla u| > 1$. It follows that

$$I + II \geq 2 \sum_j \Re\{e_j h \bar{e}_j(u)\} - C |\nabla u| \sum_j F^{\bar{i}}(|e_i e_j u| + |e_i \bar{e}_j u|) - C |\nabla u|^2 F$$

$$\geq 2 \sum_j \Re\{e_j h \bar{e}_j(u)\} - \frac{C}{\varepsilon} |\nabla u|^2 F - \varepsilon \sum_j F^{\bar{i}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2). \quad (4.5)$$
Combining (4.5) with (4.3), we obtain
\[
L(|\nabla u|^2) \geq 2 \sum_j \text{Re}\{e_j(h)e_ju\} + (1 - \varepsilon) \sum_j F^{i\bar{i}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} F|\nabla u|^2. \tag{4.6}
\]
This completes the proof. \( \Box \)

Using the above lemma, it follows that
\[
\frac{L(|\nabla u|^2)}{B_1 \zeta_1|\nabla u|^2} \geq \frac{2}{B_1 \zeta_1|\nabla u|^2} \sum_j \text{Re}\{e_j(h)e_ju\} + (1 - \varepsilon) \sum_j F^{i\bar{i}}|e_i e_j u|^2 + |e_i \bar{e}_j u|^2 - \frac{C}{B_1 \zeta_1 \varepsilon} F. \tag{4.7}
\]
Now we estimate the last term of (4.2). By the Cauchy-Schwarz inequality, for \(0 < \varepsilon \leq \frac{1}{2}\), we have
\[
2 \sum_j F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} = 2 \sum_j F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} \geq 2 \sum_j F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)g_{ij}\}
\]
where in the last inequality we used \(|\nabla u| > 1\). When \(0 < \varepsilon \leq \frac{1}{2}\), we have \(1 \leq (1 - \varepsilon)(1 + 2\varepsilon)\). Using the Cauchy-Schwarz inequality again, we obtain
\[
2 \sum_j F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} \geq \frac{2(1 - \varepsilon)}{B_1 \zeta_1} \sum_j F^{i\bar{i}}|\bar{e}_j(u)|^2 - (1 + 2\varepsilon)B_1 \zeta_1|\nabla u|^2 F^{i\bar{i}}|e_i(\eta)|^2.
\]
Therefore,
\[
\frac{2}{|\nabla u|^2} \sum_j F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} \geq \frac{2 F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\}}{|\nabla u|^2} - (1 + 3\varepsilon)B_1 \zeta_1 F^{i\bar{i}}|e_i(\eta)|^2 - \frac{C}{B_1 \zeta_1 \varepsilon} F - (1 - \varepsilon) \sum_j F^{i\bar{i}}|\bar{e}_j(u)|^2 \tag{4.8}
\]
Then, using (4.2), (4.7) and (4.8), we obtain
\[
0 \geq L(\eta) + B_1(1 + \varepsilon) F^{i\bar{i}}|e_i(\eta)|^2 - \frac{2C}{B_1 \zeta_1 \varepsilon} F + \frac{2}{B_1 \zeta_1|\nabla u|^2} \sum_j \text{Re}\{e_j(h)\bar{e}_j u\}
\]
\[
+ 2 F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} \geq 0 \geq L(\eta) + B_1(1 - 3\varepsilon)F^{i\bar{i}}|e_i(\eta)|^2 - \frac{2C}{B_1 \zeta_1 \varepsilon} F + \frac{C}{B_1 \zeta_1|\nabla u|} + 2 F^{i\bar{i}} \text{Re}\{e_i(\eta)\bar{e}_j(u)\} \tag{4.9}
\]
We have \(\varepsilon = \frac{1}{12 \sup_{x \in M} \zeta(\xi)} \leq \frac{1}{2}\) if \(A\) is big enough. It follows that
\[
B_1(1 - 3\varepsilon)F^{i\bar{i}}|e_i(\eta)|^2 \geq \frac{1}{2}B_1 F^{i\bar{i}}|e_i(\eta)|^2. \tag{4.10}
\]
We use the Cauchy-Schwarz inequality to obtain
\[ F^i \lambda_i \frac{2 \operatorname{Re} \{e_i(u) \partial \bar{e}_i(\eta) \}}{|\nabla u|^2} \geq -\frac{B_1}{4} F^i |e_i(\eta)|^2 - \frac{4}{B_1 |\nabla u|^2} F^i \lambda_i^2. \] (4.11)

Combining (4.9)–(4.11), we have
\[ \frac{B_1}{4} F^i |e_i(\eta)|^2 + L(\eta) \leq \frac{C}{B_1 |\nabla u|^2} + \frac{C}{B_1 |\nabla u|^2} F + \frac{4}{B_1 |\nabla u|^2} F^i \lambda_i^2 \]
\[ \leq \frac{C}{B_1 |\nabla u|^2} + \frac{C}{B_1 |\nabla u|^2} F + \frac{4n}{B_1 |\nabla u|^2}, \] (4.12)
where in the last inequality we used \( F^i \lambda_i^2 = \frac{\lambda_i^2}{1+\lambda_i^2} < 1 \) provided by (2.1) for each \( i = 1, \ldots, n \).

The proof is divided into two cases, where \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

Case (a) First, suppose that (2.11) holds, i.e.,
\[ L(\eta) \geq \theta F. \] (4.13)

Therefore, using (2.10) and (4.12), we have
\[ \frac{1}{2} \theta K + \frac{\theta}{2} F \leq \theta F \leq \frac{C}{B_1 |\nabla u|^2} + \frac{C}{B_1 |\nabla u|^2} F + \frac{4n}{B_1 |\nabla u|^2}. \]

Note that the terms involving \( F \) can be discarded if \( B_1 \) is big enough. Then we have
\[ \frac{1}{2} \theta K \leq \frac{C}{B_1 |\nabla u|^2} + \frac{4n}{B_1 |\nabla u|^2}. \]
It follows that \( |\nabla u| \leq C \).

Case (b) Second, suppose that (2.12) holds. Then, by (2.10), we have
\[ 1 \geq F^i \geq \theta F \geq \theta K \quad \text{for} \quad i = 1, 2, \ldots, n. \] (4.14)

Hence, by (2.1), we have \( |\lambda_i| \leq C \) and \( F^i \leq 1 \). It follows that
\[ L(\eta) = F^i (g_{ii} + u_{ii} - \lambda_i) \geq -C. \] (4.15)

By (4.14), (4.15) and (4.12), we obtain
\[ -C + \frac{1}{C} |\nabla \eta|^2 \leq \frac{C}{B_1 |\nabla u|^2} + \frac{4n}{B_1 |\nabla u|^2} + C. \]
We can assume that \( |\nabla u| \geq 2|\nabla \eta| \). Hence \( |\nabla \eta| \geq \frac{1}{2} |\nabla u| \). It follows that
\[ -C + \frac{1}{C} |\nabla u|^2 \leq \frac{C}{B_1 |\nabla u|^2} + \frac{4n}{B_1 |\nabla u|^2} + C. \]
Therefore, \( |\nabla u| \leq C \).

5 Second order estimates

In this section, we prove the following second order estimates.

Theorem 5.1. There exists a constant \( C_0 > 0 \) such that
\[ \| \nabla^2 u \|_{C^0(M)} \leq C_0, \] (5.1)
where \( C_0 \) depends on \( (M, \chi, J), \omega, \|h\|_{C^2}, \inf_M h \) and \( u \) and \( \nabla \) is the Levi-Civita connection of \( \chi \).
Let $\mu_1(\nabla^2 u) \geq \cdots \geq \mu_{2n}(\nabla^2 u)$ be the eigenvalues of $\nabla^2 u$ with respect to $\chi$. By (2.6), we have

$$\sum_{\beta=1}^{2n} \mu_\beta = \Delta u = \Delta^C u + \tau(du) = \sum \lambda_i + \tau(du) \geq \tau(du) \geq -C$$

(see [5, (2.5)]). Then $\mu_{2n} \geq -C \mu_1 - C$, which implies

$$|\nabla^2 u|_g \leq C \mu_1(\nabla^2 u) + C$$

for a uniform constant $C$. Hence, it suffices to give an upper bound for $\mu_1$. First, we consider the function

$$\tilde{Q} := \log \mu_1(\nabla^2 u) + \phi(|\nabla u|^2) + \varphi(\tilde{u})$$

on $\Omega := \{ \mu_1(\nabla^2 u) > 0 \} \subset M$. Here, $\varphi$ is a function defined by

$$\varphi(\tilde{u}) := e^{B\tilde{u}}, \quad \tilde{u} := \tilde{u} - u + \sup_{M} (u - \tilde{u}) + 1$$

for a real constant $B > 0$ to be determined later, and $\phi$ is defined by

$$\phi(s) := -\frac{1}{2} \log \left( 1 + \sup_M |\nabla u|^2 - s \right).$$

Set $K = 1 + \sup_M |\nabla u|^2$. Note that

$$\frac{1}{2K} \leq \phi'(|\nabla u|^2) \leq \frac{1}{2}, \quad \phi'' = 2(\phi')^2.$$  \hfill (5.2)

We may assume that $\Omega$ is a nonempty open set (otherwise we are done). Note that when $z$ approaches $\partial \Omega$, then $\tilde{Q}(z) \to -\infty$. Suppose that $\tilde{Q}$ achieves a maximum at $x_0$ in $\Omega$. Near $x_0$, choose a local unitary frame $\{ e_i \}_{i=1}^n$ (with respect to $\chi$) such that at $x_0$,

$$\chi_{ij} = \delta_{ij}, \quad \tilde{g}_{ij} = \delta_{ij} \tilde{g}_{ij} \quad \text{and} \quad \tilde{g}_{11} \geq \tilde{g}_{22} \geq \cdots \geq \tilde{g}_{nn}.$$  \hfill (5.3)

For convenience, we denote $\tilde{g}_{ij}(x_0)$ by $\lambda_i$. In addition, there exists a normal coordinate system $(U, \{ x^\alpha \}_{\alpha=1}^{2n})$ in a neighbourhood of $x_0$ such that

$$e_i = \frac{1}{\sqrt{2}} (\partial_{2i-1} - \sqrt{-1} \partial_{2i}) \quad \text{for} \quad i = 1, 2, \ldots, n$$  \hfill (5.4)

and

$$\frac{\partial \chi_{\alpha\beta}}{\partial x^\gamma} = 0 \quad \text{for} \quad \alpha, \beta, \gamma = 1, 2, \ldots, 2n$$  \hfill (5.5)

at $x_0$, where $\chi_{\alpha\beta} = \chi(\partial_\alpha, \partial_\beta)$.

Suppose that $V_1, \ldots, V_{2n}$ are $\chi$-unit eigenvectors for $\Phi$ at $x_0$ with eigenvalues

$$\mu_1(\nabla^2 u) \geq \cdots \geq \mu_{2n}(\nabla^2 u),$$

respectively.

Assume $V_\alpha = V_\alpha^{\beta} \partial_\beta$ at $x_0$ and extend vectors $V_\alpha$ to vector fields on $U$ by taking the components $V_\alpha^{\beta}$ to be constant. Since $\mu_1(\nabla^2 u)$ may not be smooth, we apply a perturbation argument as in [5, 26]. On $U$, define

$$\Phi = \Phi_\alpha^{\beta} \frac{\partial}{\partial x^\alpha} \otimes dx^\beta$$

$$= (g^{\alpha\gamma} u_{\gamma\beta} - g^{\alpha\gamma} B_{\gamma\beta}) \frac{\partial}{\partial x^\alpha} \otimes dx^\beta,$$  \hfill (5.6)

where $B_{\gamma\beta} = \delta_{\gamma\beta} - V_1^\gamma V_1^{\beta}$. Assume that $\mu_1(\Phi) \geq \mu_2(\Phi) \geq \cdots \geq \mu_{2n}(\Phi)$ are the eigenvalues of $\Phi$. Then $V_1, V_2, \ldots, V_{2n}$ are still eigenvectors of $\Phi$, corresponding to eigenvalues $\mu_1(\Phi), \mu_2(\Phi), \ldots, \mu_{2n}(\Phi)$ at $x_0$. 
Note that $\mu_1(\Phi)(x_0) > \mu_2(\Phi)(x_0)$, which implies $\mu_1(\Phi)$ is smooth near $x_0$. On $U$, we replace $\tilde{Q}$ by the following smooth quantity:

$$Q = \log \mu_1(\Phi) + \phi(|\nabla u|^2) + \varphi(\bar{\eta}).$$

Since $\mu_1(\nabla^2 u)(x_0) = \mu_1(\Phi)(x_0)$ and $\mu_1(\nabla^2 u) \geq \mu_1(\Phi)$, $x_0$ is still the maximum point of $Q$. For convenience, we denote $\mu_\alpha(\Phi)$ by $\mu_\alpha$, for $\alpha = 1, 2, \ldots, 2n$.

The proof needs the first and second derivatives of the first eigenvalue $\mu_1$ at $x_0$ (see [5, Lemma 5.2] or [23, 26]).

**Lemma 5.2.** At $x_0$, we have

$$\frac{\partial \mu_1}{\partial \Phi_\beta} = V_1^\alpha V_1^\beta,$$

$$\frac{\partial^2 \mu_1}{\partial \Phi_\beta \partial \Phi_\delta} = \sum_{\kappa = 1}^{n} \frac{1}{\mu_\kappa} (V_1^\kappa V_1^\beta V_\kappa^\gamma V_1^\gamma + V_\kappa^\alpha V_1^\beta V_\kappa^\gamma V_\kappa^\delta).$$

(5.7)

At $x_0$, for each $i = 1, 2, \ldots, n$, we have

$$\frac{1}{\mu_1} e_i(\mu_1) = -\phi' e_i(|\nabla u|^2) - B e_{\beta} e_i(\bar{\eta})$$

(5.8)

and

$$0 \geq L(Q) = \frac{L(\mu_1)}{\mu_1} - F^{ii} \frac{|e_i(\mu_1)|^2}{\mu_i^2} + \phi'' F^{ii} |e_i(|\nabla u|^2)|^2$$

$$+ \phi' L(|\nabla u|^2) + B e_{\beta} e_i(\bar{\eta}) + B^2 e_{\beta} e_i e_i(\bar{\eta})^2.$$  

(5.9)

### 5.1 The lower bound for $L(Q)$

In this subsection, we calculate $L(Q)$.

**Lemma 5.3.** For $\varepsilon \in (0, \frac{1}{2}]$, at $x_0$, we have

$$L(Q) \geq (2 - \varepsilon) \sum_{\beta > 1} \frac{F^{ii} |e_i(u_{\gamma\alpha} V_1)|^2}{\mu_1(\mu_1 - \mu_\beta)} - \frac{1}{\mu_1} F^{i\kappa, \beta} V_1(g_{i\kappa}) V_1(g_{j\beta})$$

$$- (1 + \varepsilon) F^{ii} \frac{|e_i(\mu_1)|^2}{\mu_i^2} - \frac{C}{\varepsilon} F + \phi' \frac{\varepsilon}{2} F^{ii} |e_i(\mu_1)|^2 + |e_i(\mu_1)|^2$$

$$+ \phi'' F^{ii} |e_i(|\nabla u|^2)|^2 + B e_{\beta} e_i(\bar{\eta}) + B^2 e_{\beta} e_i e_i(\bar{\eta})^2.$$  

(5.10)

First, we calculate $L(\mu_1)$. Let $u_{ij} = e_i e_j u - (\nabla e_i e_j) u$ and $u_{\gamma\alpha} V_1 = u_{\gamma\alpha} V_1^\gamma V_1^\alpha$.

By Lemma 5.2 and (5.5), we have

$$L(\mu_1) = F^{ii} \frac{\partial^2 \mu_1}{\partial \Phi_\beta \partial \Phi_\delta} e_i(\Phi_\delta^{\beta}) e_i(\Phi_\beta^{\delta}) + F^{ii} \frac{\partial \mu_1}{\partial \Phi_\beta} (e_i e_i - |e_i|) (\Phi_\beta^{\alpha})$$

$$= F^{ii} \frac{\partial^2 \mu_1}{\partial \Phi_\beta \partial \Phi_\delta} e_i(u_{\gamma\alpha}) e_i(u_{\alpha\beta}) + F^{ii} \frac{\partial \mu_1}{\partial \Phi_\beta} (e_i e_i - |e_i|) (u_{\alpha\beta})$$

$$+ F^{ii} \frac{\partial \mu_1}{\partial \Phi_\beta} u_{\gamma\beta e_i} e_i(\chi^{\alpha\gamma})$$

$$\geq 2 \sum_{\beta > 1} \frac{F^{ii} |e_i(u_{\gamma\alpha} V_1)|^2}{\mu_1 - \mu_\beta} + F^{ii} (e_i e_i - |e_i|) (u_{\gamma\alpha} V_1) - C \mu_1 F,$$  

(5.11)

where $(\chi^{\alpha\beta})$ is the inverse of the matrix $(\chi_{\alpha\beta})$. Let $W$ be a vector field. Differentiating the equation (1.2), we obtain

$$F^{ii} W(\tilde{g}_{ii}) = W(h)$$

(5.12)
and
\[ F^{ij}V_1 V_1 (\tilde{g}_{ij}) = - F^{ik,j}V_1 (\tilde{g}_{ik}) V_1 (\tilde{g}_{jl}) + V_1 V_1 (b). \] (5.13)

Commuting the derivatives and using Proposition 4.1, we obtain, for any vector field \( W \),
\[ |L(W(u))| \leq C + C \mu_1 F. \] (5.14)

**Claim 1.** If \( \mu_1 \gg 1 \), then
\[ F^{ij}(e_i e_i - [e_i, e_i]^{0,1})(u V_i V_j) \geq - F^{ik,j}V_1 (\tilde{g}_{ik}) V_1 (\tilde{g}_{jl}) + \mu_1 F - 2 F^{ij} \{ [V_1, e_i] V_1 e_i (u) + [V_1, e_i] V_1 e_i (u) \}. \] (5.15)

**Proof.** By (5.14), we have
\[ F^{ij}(e_i e_i - [e_i, e_i]^{0,1})(u V_i V_j) = F^{ij} e_i e_i (V_1 V_1 (u) - (\nabla V_i V_j) V_1 (u) - (\nabla V_i V_1) V_1 (u) - (\nabla V_i V_1) V_1 (u)) \geq F^{ij} e_i e_i (V_1 V_1 (u) - (\nabla V_i V_1) V_1 (u) - (\nabla V_i V_1) V_1 (u) - (\nabla V_i V_1) V_1 (u)) \geq \mu_1 F - C. \] (5.16)

Recall the definition of the Lie bracket \([e_i, e_j] = e_i e_j - e_j e_i\). Then we get (for more details, see [5, (5.12)])
\[ F^{ij} e_i e_i (V_1 (\varphi) - F^{ij} e_i e_i (V_1 (\varphi)) \geq F^{ij} e_i e_i (V_1 (\varphi) + [e_i, V_1] e_i V_1 (\varphi) - [V_1, e_i] e_i V_1 (\varphi) - V_1 e_i e_i (\varphi)) - \mu_1 F \] (5.17)

By combining (2.13), (5.13) and (5.16), Claim 1 follows if \( \mu_1 \gg 1 \).

Combining the equalities (5.11) and (5.15) together, we have
\[ L(\mu_1) \geq 2 \sum_{\beta > 1} \frac{F^{ij} [e_i (u V_i V_j)]^2}{\mu_1 - \mu_\beta} - F^{ik,j}V_1 (\tilde{g}_{ik}) V_1 (\tilde{g}_{jl}) + \mu_1 F \] (5.18)

By (2.13), Lemma 4.2 and Proposition 4.1, we have
\[ L(|\nabla u|^2) \geq \frac{1}{2} \sum_j F^{ij} [e_i e_j u]^2 + [e_i e_j u]^2 - C F. \] (5.19)

Substituting (5.17) and (5.18) into (5.9), we obtain
\[ L(Q) \geq 2 \sum_{\beta > 1} \frac{F^{ij} [e_i (u V_i V_j)]^2}{\mu_1 (\mu_1 - \mu_\beta)} - \frac{F^{ik,j}V_1 (\tilde{g}_{ik}) V_1 (\tilde{g}_{jl}) + B^2 e^{B \eta} F^{ij} [e_i (\tilde{\eta})]^2}{\mu_1} + B e^{B \eta} L(\tilde{\eta}) - 2 F^{ij} \frac{Re\{[V_1, e_i] V_1 e_i (u) + [V_1, e_i] V_1 e_i (u)\}}{\mu_1} - C F \] (5.20)

Now we deal with the third order derivatives of the right-hand side of (5.19).

**Claim 2.** For any \( \varepsilon \in (0, \frac{1}{2}] \), we have
\[ 2 F^{ij} \frac{Re\{[V_1, e_i] V_1 e_i (u) + [V_1, e_i] V_1 e_i (u)\}}{\mu_1} \leq \varepsilon F^{ij} [e_i (u V_i V_j)]^2 + \varepsilon \sum_{\beta > 1} \frac{F^{ij} [e_i (u V_i V_j)]^2}{\mu_1 (\mu_1 - \mu_\beta)} + C F. \]
Proof. Assume
\[ [V_1, e_i] = \sum_{\beta=1}^{2n} \mu_{i\beta} V_\beta, \quad [V_1, \bar{e}_i] = \sum_{\beta=1}^{2n} \bar{\mu}_{i\beta} V_\beta, \]
where \( \mu_{i\beta} \in \mathbb{C} \) are constants. Thus,
\[
\text{Re} \{ [V_1, e_i] V_1 \bar{e}_i(u) + [V_1, \bar{e}_i] V_1 e_i(u) \} \leq C \sum_{\beta=1}^{2n} |V_\beta V_1 e_i(u)|.
\]
Then we are reduced to estimating \( \sum_{\beta} F^{\bar{i}} \frac{|V_\beta V_1 e_i(u)|}{\mu_1} \). Using the definition of the Lie bracket \( e_i e_j - e_j e_i = [e_i, e_j] \), we have
\[
|V_\beta V_1 e_i(u)| = |e_i V_\beta V_1(u) + V_\beta [V_1, e_i](u)| + |V_\beta [V_1, e_i](u)|
\leq |e_i(u V_\beta V_1)| + |V_\beta [V_1, e_i](u)| + |V_\beta V_1 e_i(u)|.
\]
Therefore,
\[
\sum_{\beta} F^{\bar{i}} \frac{|V_\beta V_1 e_i(u)|}{\mu_1} \leq \sum_{\beta} F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + CF
= F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + \sum_{\beta > 1} F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + CF.
\]
By the Cauchy-Schwarz inequality, for \( \varepsilon \in (0, \frac{1}{2}] \), we derive
\[
F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} \leq \varepsilon F^{\bar{i}} \left| \frac{e_i(u V_\beta V_1)}{\mu_1} \right|^2 + \frac{C}{\varepsilon} F
\]
and
\[
\sum_{\beta > 1} F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} \leq \varepsilon F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + \sum_{\beta > 1} \frac{\mu_1 - \mu_\beta}{\varepsilon} F
\leq \varepsilon \sum_{\beta > 1} F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + \frac{C}{\varepsilon} F,
\]
where in the last inequality we used \( \sum_{\beta=1}^{2n} \mu_\beta = \Delta u = \Delta \xi u + \tau(du) \geq -C + \tau(du) \geq -C \) (see [5, (2.5)]).

Here, \( \tau \) is the torsion vector field of \( (\chi, J) \) (the dual vector field of Lee form for \( (\chi, J) \), see, e.g., [28, Lemma 3.2]). Combining the above three inequalities, we have
\[
\sum_{\beta} F^{\bar{i}} \frac{|V_\beta V_1 e_i(u)|}{\mu_1} \leq \varepsilon F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + \varepsilon \sum_{\beta > 1} F^{\bar{i}} \frac{|e_i(u V_\beta V_1)|}{\mu_1} + \frac{C}{\varepsilon} F.
\]
Then by (5.21), (5.20) follows. □

Consequently, Lemma 5.3 follows from (5.19) and (5.20). Now we continue to prove Theorem 5.1.

5.2 Proof of Theorem 5.1

The proof can be divided into three cases.

Case 1. It holds that
\[
F^{mn} \leq B^3 e^{2B \bar{m}} F^{11}.
\]
In this case, we can choose \( \varepsilon = \frac{1}{2} \). Using the elemental inequality \( |a + b|^2 \leq 4|a|^2 + \frac{2^2}{3}|b|^2 \) for (5.8), we get
\[
- (1 + \varepsilon) F^{\bar{i}} \frac{|e_i(\mu_1)|}{\mu_1^2} \geq -6 \sup_M (|\nabla e_i|^2) B^2 e^{2B \bar{m}} F - 2(\phi')^2 F^{\bar{i}} |e_i(\nabla u)|^2.
\]
Plugging (5.24) and (5.2) into (5.10), we get

\[
L(Q) \geq (2 - \varepsilon) \sum_{\beta > 1} \frac{F^{\bar{\beta}}}{\mu_1 (\mu_1 - \mu_\beta)} \frac{|e_i (uv_\beta V_\beta)|^2}{\mu_1} - \frac{1}{\mu_1} F^{i k, j l} V_1 (\tilde{g}_{i k}) V_1 (\tilde{g}_{j l})
\]

\[
- \left( \frac{C}{\varepsilon} + 6 \sup_M (|\nabla \tilde{m}|^2) B^2 e^{2B\tilde{m}} \right) \mathcal{F} + \frac{\delta'}{2} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2)
\]

\[
+ B e^{B\tilde{m}} L(\tilde{m}) + B^2 e^{2B\tilde{m}} F^{\bar{i} i} |e_i \tilde{m}|^2
\]

\[
\geq - \left( \frac{C}{\varepsilon} + 6 \sup_M (|\nabla \tilde{m}|^2) B^2 e^{2B\tilde{m}} \right) \mathcal{F} + \frac{\delta'}{2} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + B e^{B\tilde{m}} L(\tilde{m}),
\]

(5.25)

where in the last inequality we used the concavity of \( F \). Note \( F^{\bar{i} i} \leq 1 \). We have

\[
L(\tilde{m}) = \sum_i F^{\bar{i} i} (g_{i i} - u_{i i}) = \sum_i F^{\bar{i} i} (g_{i i} + g_{i i} - \tilde{g}_{i i})
\]

\[
\geq - C - \sum_i F^{\bar{i} i} |g_{i i}| = - C - \sum_i \lambda_i \frac{|\lambda_i|}{1 + \lambda_i^2} \geq - C.
\]

(5.26)

Then, by (2.12), we have

\[
0 \geq L(Q) \geq \frac{\delta'}{2} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B \mathcal{F}.
\]

(5.27)

Here, \( C_B \) are positive constants depending on \( B \) which might be different from line to line. By (5.23), we have

\[
\sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_B.
\]

Then the complex covariant derivatives

\[
u_{ij} = e_i e_j u - (\nabla e_i e_j) u \quad \text{and} \quad \bar{v}_{ij} = e_i \bar{e}_j u - (\nabla e_i \bar{e}_j) u
\]

satisfy

\[
\sum_{i,j} (|u_{ij}|^2 + |v_{ij}|^2) \leq C_B.
\]

**Case 2.** It holds that

\[
\frac{\delta'}{4} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \geq 6 \sup_M (|\nabla \tilde{m}|^2) B^2 e^{2B\tilde{m}} \mathcal{F}.
\]

(5.28)

Note that (5.25) is still true. By (5.28) and (5.25), we have

\[
L(Q) \geq \frac{\delta'}{4} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} \mathcal{F} + B e^{B\tilde{m}} L(\tilde{m}).
\]

(5.29)

(a) If (2.11) holds, \( L(\tilde{m}) \geq \theta \mathcal{F} \). Then, by (5.29), we have

\[
0 \geq \frac{\delta'}{4} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \left( \theta B e^{B\tilde{m}} - \frac{C}{\varepsilon} \right) \mathcal{F}.
\]

This yields a contradiction if we further assume that \( B \) is large enough.

(b) If (2.12) holds, then \( F^{kk} \geq \theta \mathcal{F}, \ k = 1, 2, \ldots, n \). Then by (5.29) and (5.26) we obtain

\[
0 \geq \frac{\delta'}{4} \sum_j F^{\bar{i} i} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B \mathcal{F}
\]

\[
\geq \frac{\delta'}{4} \theta \mathcal{F} \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B \mathcal{F}.
\]
Proof. By (2.2), we have
\[ \sum_{i,j} |(c_i c_j u|^2 + |c_i \bar{c}_j u|^2) \leq C_B. \]

Case 3. If Cases 1 and 2 do not hold, define the index set
\[ I := \{1 \leq i \leq n : F^{i,n} \geq B^3 e^{2B\bar{\eta}} \}. \]
Clearly, we have \( 1 \in I \) and \( n \not\in I \). Hence we may write \( I = \{1,2,\ldots,p\} \) for some positive integer \( p < n \). Now we deal with the third order terms.

**Lemma 5.4.** Assume \( B \geq 6n \sup_M |\nabla \bar{\eta}|^2 \). At \( x_0 \), we have
\[ - (1 + \varepsilon) \sum_{i \in I} F^{ii} |e_i| \frac{e_i(\mu_i)}{\mu_i^2} \geq -\mathcal{F} - 2(\phi')^2 \sum_{i \in I} |e_i| (|\nabla u|^2)|. \]

Proof. Using (5.8) and the inequality \( |a + b|^2 \leq 4|a|^2 + \frac{4}{3}|b|^2 \), we obtain
\[ - (1 + \varepsilon) \sum_{i \in I} F^{ii} |e_i| \frac{e_i(\mu_i)}{\mu_i^2} = - (1 + \varepsilon) \sum_{i \in I} F^{ii} |e_i| \frac{e_i(\mu_i)}{\mu_i^2} + Be^{B\bar{\eta}} e_i(\bar{\eta})^2 \]
\[ \geq - 6 \sup_M |\nabla \bar{\eta}|^2 B^2 e^{2B\bar{\eta}} \sum_{i \in I} F^{ii} - 2(\phi')^2 \sum_{i \in I} F^{ii} |e_i| (|\nabla u|^2)| \]
\[ \geq - 6n \sup_M |\nabla \bar{\eta}|^2 B^{-1} F^{i,n} - 2(\phi')^2 \sum_{i \in I} F^{ii} |e_i| (|\nabla u|^2)|^2 \]
\[ \geq - \mathcal{F} - 2(\phi')^2 \sum_{i \in I} F^{ii} |e_i| (|\nabla u|^2)|^2, \]
where we used the hypothesis \( B \geq 6n \sup_M |\nabla \bar{\eta}|^2 \) in the penultimate inequality. \( \square \)

To deal with the bad third order terms, we need to give a lower bound for the good third order terms from the concavity of the equation (1.2).

**Lemma 5.5.** We have
\[ - \frac{1}{\mu_1} F^{ik,j} |V_i(\bar{g}_{ik}) V_i(\bar{g}_{jk}) | \geq \frac{2}{\mu_1} \sum_{i \in I, k \in I} F^{ik} \bar{g}^{jj} |V_i(\bar{g}_{ik})|^2 + \frac{C_B}{\varepsilon \mu_1} \sum_{i \in I, k \in I} F^{ik} |V_i(\bar{g}_{ik})|^2, \]
where \( (\bar{g}^{ik}) \) is the inverse of the matrix \( (\bar{g}_{ij}) \).

Proof. By (2.2), we have
\[ - F^{ik,j} |V_i(\bar{g}_{ik}) V_i(\bar{g}_{jk}) | = \sum_i \frac{2 \lambda_i}{(1 + \lambda_i^2)} |V_i(\bar{g}_{ik})|^2 + \sum_{i \neq k} \frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_i(\bar{g}_{ik})|^2. \]
Now we calculate these two terms. We claim that
\[ \sum_{i \neq k} \frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_i(\bar{g}_{ik})|^2 \]
\[ \geq 2 \sum_{i \in I, k \in I} \frac{1}{(1 + \lambda_i^2) \lambda_k} |V_i(\bar{g}_{ik})|^2 + \frac{C_B}{\varepsilon \mu_1} \sum_{i \in I, k \in I, i \neq k} \frac{1}{(1 + \lambda_i^2)} |V_i(\bar{g}_{ik})|^2. \]

Let
\[ S_1 := \{(i, k) : i \not\in I, k \in I, 1 \leq i, k \leq n\}, \]
\[ S_2 := \{(i, k) : i \neq k, i \not\in I, k \not\in I, 1 \leq i, k \leq n\}, \]
\[ S_3 := \{(i, k) : i \in I, k \not\in I, 1 \leq i, k \leq n\}. \]
Note $S_j \subset \{(i, k) : i \neq k, 1 \leq i, k \leq n\}$ and $S_j \cap S_l = \emptyset$, for $1 \leq j, l \leq 3$ and $j \neq l$. By the symmetry of $i$ and $k$, we have

$$\sum_{i \neq k} \frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_1(\tilde{g}_{ik})|^2 \geq 2 \sum_{i \neq k} \frac{\lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_1(\tilde{g}_{ik})|^2$$

$$\geq 2 \sum_{i=1}^{3} \sum_{S_i} \frac{\lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_1(\tilde{g}_{ik})|^2$$

$$= 2 \sum_{S_1} \frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_1(\tilde{g}_{ik})|^2 \quad \text{(reverse } i \text{ and } k \text{ in } S_3)$$

$$+ 2 \sum_{S_2} \frac{\lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} |V_1(\tilde{g}_{ik})|^2.$$

Hence, to prove (5.34), we only need to prove

$$\frac{\lambda_i + \lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} \geq \frac{1}{(1 + \lambda_i^2)\lambda_k}, \quad (i, k) \in S_1 \quad (5.35)$$

and

$$\frac{\lambda_k}{(1 + \lambda_i^2)(1 + \lambda_k^2)} \geq \frac{C_B}{\varepsilon \mu_1^2 \frac{1}{1 + \lambda_k^2}}, \quad (i, k) \in S_2. \quad (5.36)$$

Now we give the proof of the inequality (5.35). Note

$$\arctan \lambda_j + \arctan \lambda_n = h - \sum_{i \neq j, n} \arctan \lambda_i$$

$$\geq (n - 1)\frac{\pi}{2} - (n - 2)\frac{\pi}{2} = \frac{\pi}{2}. \quad (5.37)$$

Hence

$$\arctan \lambda_j \geq \frac{\pi}{2} - \arctan \lambda_n = \arctan \frac{1}{\lambda_n}. \quad (5.38)$$

By the monotonicity of $\arctan x$, we have

$$\lambda_j \lambda_n \geq 1, \quad j = 1, 2, \ldots, n - 1, \quad (5.39)$$

which implies (5.35).

Next, for (5.36), it suffices to prove

$$\frac{\lambda_k}{1 + \lambda_k^2} \geq \frac{C_B}{\varepsilon \mu_1^2}, \quad k = 1, 2, \ldots, n. \quad (5.40)$$

Since $h > (n - 1)\frac{\pi}{2}$, we have

$$\arctan \lambda_n \geq h - \sum_{1 \leq i \leq n - 1} \arctan \lambda_i \geq h - (n - 1)\frac{\pi}{2} \geq C_1^{-1} > 0$$

for some uniform constant $C_1$ depending on $\inf_M h$. This implies

$$\lambda_k \geq \lambda_n \geq C_1^{-1}. \quad (5.42)$$

Therefore, when $\lambda_1 \gg 1$, we have

$$\frac{\lambda_k}{1 + \lambda_k^2} \geq \frac{1}{C_1(1 + \lambda_k^2)} \geq \frac{1}{C_1 \mu_1^2}.$$

Now if $\mu_1 \geq C_B / \varepsilon$, then we obtain (5.39). (5.36) follows. We complete the proof of (5.34).
Now we deal with the first term on the right-hand side in (5.33). By (5.39), we have
\[ \sum_{i} \frac{2\lambda_i}{(1 + \lambda_i^2)^{\frac{3}{2}}} |V_1(\tilde{g}_i)|^2 \geq \frac{C_B}{\varepsilon \mu_1} \sum_{i \in I_l} \frac{1}{1 + \lambda_i^2} |V_1(\tilde{g}_i)|^2. \]
Therefore, by (5.33) and (5.34), we have
\[ -\frac{1}{\mu_1} F_{\mu_1}^k j V_1(\tilde{g}_i k) V_1(\tilde{g}_j l) \geq \frac{2}{\mu_1} \sum_{i \in I_l, k \in I} F_{\mu_1}^i \tilde{g}_{i k} |V_1(\tilde{g}_i k)|^2 + \frac{C_B}{\varepsilon \mu_1} \sum_{i \in I_l, k \in I} F_{\mu_1}^i |V_1(\tilde{g}_i k)|^2. \]
This completes the proof. \(\square\)

Define a new \((1, 0)\) vector field by
\[ \tilde{e}_1 = \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} J V_1). \]
At \(x_0\), we can find a sequence of complex numbers \(\nu_1, \ldots, \nu_n\) such that
\[ \tilde{e}_1 := \sum_{k=1}^{n} \nu_k e_k, \quad \sum_{k=1}^{n} |\nu_k|^2 = 1. \]

**Lemma 5.6.** We have
\[ |\nu_k| \leq \frac{C_B}{\mu_1} \quad \text{for all } k \not\in I. \]

**Proof.** The idea of the proof is similar to the argument of [5, Lemma 5.6]. Since Case 2 does not hold, we obtain
\[ \frac{1}{4} \sum_{i \in I} \sum_{j \in I} F_{\mu_1}^i (|e_i e_j u|^2 + |e_i e_j u|^2) \leq \left( 6n^2 \sup_{\mathcal{M}} |\nabla \tilde{g}|^2 \right) B^2 e^{2B^3} F^n. \]
While \(F^n \leq B^3 e^{2B^3} F^n\) for each \(i \not\in I\), it follows that
\[ \sum_{\gamma = 2p + 1}^{2n} \sum_{\beta = 1}^{2n} |\nabla \gamma \beta u| \leq C_B. \]
Therefore, \(|\Phi_{\gamma}^\beta| \leq C_B\) for \(2p + 1 \leq \gamma \leq 2n, 1 \leq \beta \leq 2n\). Since \(\Phi(V_1) = \mu_1 V_1\), we have
\[ |V_1^\gamma| = \left| \frac{1}{\mu_1} (\Phi(V_1))^\gamma \right| = \frac{1}{\mu_1} \sum_{\beta} \Phi_{\gamma}^\beta V_1^\beta \leq \frac{C_B}{\mu_1} \quad 2p + 1 \leq \gamma \leq 2n. \]
Then, by (5.43), \(|\nu_k| \leq |V_1^{2k-1}| + |V_1^{2k}| \leq \frac{C_B}{\mu_1}, k \not\in I. \)

Now we can estimate the first three terms in Lemma 5.3. Since \(J V_1\) is \(\chi\)-unit and \(\chi\)-orthogonal to \(V_1\), we can find real numbers \(\xi_2, \ldots, \xi_{2n}\) such that
\[ J V_1 = \sum_{\beta > 1} \xi_\beta V_\beta, \quad \sum_{\beta > 1} \xi_\beta^2 = 1 \quad \text{at } x_0. \]

**Lemma 5.7.** For any constant \(\tau > 0\), we have
\[ (2 - \varepsilon) \sum_{\beta > 1} F_{\mu_1}^i |e_i (u_{V_\beta V_\beta})|^2 - \frac{1}{\mu_1} F_{\mu_1}^i j V_1(\tilde{g}_i k) V_1(\tilde{g}_j l) - (1 + \varepsilon) \sum_{i \in I_l} F_{\mu_1}^i |e_i (\mu_1)|^2 \mu_1^2 \]
\[ \geq (2 - \varepsilon) \sum_{i \in I_l} \sum_{\beta > 1} F_{\mu_1}^i |e_i (u_{V_\beta V_\beta})|^2 \mu_1^2 (\mu_1 - \mu_\beta) + \sum_{k \in I_l} \sum_{i \in I_l} \frac{2}{\mu_1} F_{\mu_1}^i g_{i k} |V_1(\tilde{g}_i k)|^2 \]
\[ - 3 \varepsilon \sum_{i \in I_l} \frac{F_{\mu_1}^i |e_i (\mu_1)|^2}{\mu_1^2} - 2(1 - \varepsilon)(1 + \tau) \tilde{g}_{11} \sum_{k \in I_l} \sum_{i \in I_l} F_{\mu_1}^i g_{i k} |V_1(\tilde{g}_i k)|^2 \mu_1^2 \]
\[ - \frac{C_B}{\varepsilon} J - (1 - \varepsilon) \left( \frac{1}{\mu_1} \sum_{\beta > 1} \mu_\beta^2 \right) \sum_{i \in I_l} \sum_{\beta > 1} \frac{F_{\mu_1}^i |e_i (u_{V_\beta V_\beta})|^2}{\mu_1^2 (\mu_1 - \mu_\beta)}, \]
if we assume \(\mu_1 \geq \frac{C_B}{\varepsilon}\), where \(\tilde{g}_{11} = \sum \tilde{g}_{i, i} |\nu_i|^2\).
Proof. First, we can prove

$$e_i(u_{V_1 V_1}) = \sqrt{2} \sum_k \bar{\nu}_k V_1(\bar{g}_i) - \sqrt{-1} \sum_{\beta > 1} \xi_\beta e_i(u_{V_1 V_3}) + O(\mu_1), \tag{5.45}$$

where $O(\mu_1)$ denotes the terms which can be controlled by $\mu_1$. Indeed, since

$$\bar{e}_1 = \frac{1}{\sqrt{2}}(V_1 + \sqrt{-1}J V_1),$$

we have

$$e_i(u_{V_1 V_1}) = \sqrt{2} e_i(u_{V_1 \bar{e}_1}) - \sqrt{-1} e_i(u_{V_1 J V_1}).$$

For the first term, using $\bar{g}_i = g_i + u_i$, we have

$$e_i(u_{V_1 \bar{e}_1}) = e_i(V_1 \bar{e}_1 u - (\nabla_{V_1 \bar{e}_1})u) = \bar{e}_1 e_i V_1 u + O(\mu_1) = \sum_k \bar{\pi}_k V_1(\bar{g}_i) + O(\mu_1). \tag{5.46}$$

For the second term, by (5.44),

$$e_i(u_{V_1 J V_1}) = e_iV_1 J V_1(u) + O(\mu_1) = J V_1 e_i V_1(u) + O(\mu_1) = \sum_{\beta > 1} \xi_\beta e_i(u_{V_1 V_3}) + O(\mu_1). \tag{5.47}$$

Thus, (5.45) follows from (5.46) and (5.47). Hence, by (5.45), Lemma 5.6 and the Cauchy-Schwarz inequality, we have

$$- (1 + \varepsilon) \sum_{i \in I} F^{ii} \frac{|e_i(\mu_1)|^2}{\mu_1^2} \leq -(1 - 2\varepsilon) \sum_{i \in I} F^{ii} \frac{|e_i(\sqrt{2} \sum_k \bar{\nu}_k V_1(\bar{g}_i)) - \sqrt{-1} \sum_{\beta > 1} \xi_\beta e_i(u_{V_1 V_3}) + O(\mu_1))|^2}{\mu_1^2}$$

$$- 3\varepsilon \sum_{i \in I} F^{ii} \frac{|e_i(\mu_1)|^2}{\mu_1^2} \geq -(1 - \varepsilon) \sum_{i \in I} F^{ii} \frac{|\sqrt{2} \sum_{k \in I} \pi_k V_1(\bar{g}_i) - \sqrt{-1} \sum_{\beta > 1} \xi_\beta e_i(u_{V_1 V_3})|^2}{\mu_1^2}$$

$$- 3\varepsilon \sum_{i \in I} F^{ii} \frac{|e_i(\mu_1)|^2}{\mu_1^2} - C_B \sum_{i \in I, k \in I} F^{ii} \frac{|V_1(\bar{g}_i)|^2}{\mu_1^2} - \frac{C}{\varepsilon} F. \tag{5.48}$$

In addition, using the Cauchy-Schwarz inequality, we have

$$\left| \sum_{\beta > 1} \xi_\beta e_i(u_{V_1 V_3}) \right|^2 \leq \sum_{\beta > 1} (\mu_1 - \mu_\beta \xi_\beta^2) \sum_{\beta > 1} \frac{|e_i(u_{V_1 V_3})|^2}{\mu_1 - \mu_\beta}$$

and

$$\left| \sum_{k \in I} \pi_k V_1(\bar{g}_i) \right|^2 \leq \left( \sum_i g_{ii} |\nu_i|^2 \right) \sum_{k \in I} g^{kk} |V_1(\bar{g}_i)|^2 = \tilde{g}_{ii} \sum_{k \in I} g^{kk} |V_1(\bar{g}_i)|^2. \tag{5.49}$$
Then for each $\gamma > 0$, using the Cauchy-Schwarz inequality again, we get
\[
(1 - \varepsilon) \sum_{i,g,l} F_{i}^{\mu} |\sqrt{2} \sum_{k} \overline{r}_{k} V_{i}(\tilde{g}_{ik}) - \sqrt{-1} \sum_{\beta > 1} \xi_{\beta} e_{i}(u_{V_{i}V_{k}})|^{2} \mu_{i}^{2} \\
\leq 2(1 - \varepsilon)(1 + \tau) \sum_{i,g,l} F_{i}^{\mu} |\sum_{k} \overline{r}_{k} V_{i}(\tilde{g}_{ik})|^{2} \mu_{i}^{2} \\
+ (1 - \varepsilon) \left(1 + \frac{1}{\tau}\right) \sum_{i,g,l} F_{i}^{\mu} |\sum_{\beta > 1} \xi_{\beta} e_{i}(u_{V_{i}V_{k}})|^{2} \mu_{i}^{2} \\
\leq 2(1 - \varepsilon)(1 + \tau) \sum_{i,g,l} \sum_{k} \frac{F_{i}^{\mu}}{\mu_{i}^{2}} \tilde{g}_{kk} |V_{i}(\tilde{g}_{ik})|^{2} \\
+ (1 - \varepsilon) \left(1 + \frac{1}{\tau}\right) \left(\mu_{1} - \sum_{\beta > 1} \mu_{\beta} \xi_{\beta}^{2}\right) \sum_{i,g,l} \sum_{\beta > 1} \frac{F_{i}^{\mu}}{\mu_{i}^{2}} |e_{i}(u_{V_{i}V_{k}})|^{2} \mu_{1} - \mu_{\beta}.
\]
Combining (5.48), we have
\[
-(1 + \varepsilon) \sum_{i,g,l} F_{i}^{\mu} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{i}^{2}} \\
\geq -2(1 - \varepsilon)(1 + \tau) \sum_{i,g,l} \sum_{k} \frac{F_{i}^{\mu}}{\mu_{i}^{2}} \tilde{g}_{kk} |V_{i}(\tilde{g}_{ik})|^{2} \\
- (1 - \varepsilon) \left(1 + \frac{1}{\tau}\right) \left(\mu_{1} - \sum_{\beta > 1} \mu_{\beta} \xi_{\beta}^{2}\right) \sum_{i,g,l} \sum_{\beta > 1} \frac{F_{i}^{\mu}}{\mu_{i}^{2}} |e_{i}(u_{V_{i}V_{k}})|^{2} \mu_{1} - \mu_{\beta} \\
- 3 \varepsilon \sum_{i,g,l} F_{i}^{\mu} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{i}^{2}} - \frac{C_{B}}{\varepsilon} \sum_{i,g,l} \sum_{k} \frac{F_{i}^{\mu}}{\mu_{i}^{2}} |V_{i}(\tilde{g}_{ik})|^{2} - \frac{C}{\varepsilon} F.
\]
Then the lemma follows from it and Lemma 5.5.

Lemma 5.8. If we assume $\mu_{1} \geq C/\varepsilon^{3}$, then
\[
(2 - \varepsilon) \sum_{\beta > 1} F_{i}^{\mu} \frac{|e_{i}(u_{V_{i}V_{k}})|^{2}}{\mu_{1}(\mu_{1} - \mu_{\beta})} - \frac{1}{\mu_{1}} F_{i}^{\mu_{1}} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{1}^{2}} \\
\geq -6 \varepsilon B^{2} \varepsilon^{2B} \sum_{i,g,l} F_{i}^{\mu} |e_{i}(\tilde{u})|^{2} - 6 \varepsilon (\phi')^{2} \sum_{i,g,l} F_{i}^{\mu} |e_{i}(|\nabla u|^{2})|^{2} - \frac{C}{\varepsilon} F.
\]
Proof. By (5.8), it suffices to prove
\[
(2 - \varepsilon) \sum_{\beta > 1} F_{i}^{\mu} \frac{|e_{i}(u_{V_{i}V_{k}})|^{2}}{\mu_{1}(\mu_{1} - \mu_{\beta})} - \frac{1}{\mu_{1}} F_{i}^{\mu_{1}} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{1}^{2}} \\
\geq -3 \varepsilon \sum_{i,g,l} F_{i}^{\mu} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{i}^{2}} - \frac{C}{\varepsilon} F.
\]  
(5.50)
We divide the proof into two cases.

Case I. Assume that
\[
\mu_{1} + \sum_{\beta > 1} \mu_{\beta} \xi_{\beta}^{2} \geq 2(1 - \varepsilon) \tilde{g}_{11} > 0.
\]  
(5.51)
It follows from Lemma 5.7 and (5.51) that
\[
(2 - \varepsilon) \sum_{\beta > 1} F_{i}^{\mu} \frac{|e_{i}(u_{V_{i}V_{k}})|^{2}}{\mu_{1}(\mu_{1} - \mu_{\beta})} - \frac{1}{\mu_{1}} F_{i}^{\mu_{1}} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{1}^{2}} \\
\geq -3 \varepsilon \sum_{i,g,l} F_{i}^{\mu} \frac{|e_{i}(\mu_{1})|^{2}}{\mu_{i}^{2}} - \frac{C}{\varepsilon} F.
\]
Therefore, provided by (5.52).

Choose

\[ \tau = \frac{\mu_1 - \sum_{\beta > 1} \mu_\beta \xi_\beta^2}{\mu_1 + \sum_{\beta > 1} \mu_\beta \xi_\beta^2}. \]

Therefore,

\[
(1 + \tau) \left( \mu_1 + \sum_{\beta > 1} \mu_\beta \xi_\beta^2 \right) \sum_{k \in I} \sum_{l \in I} \widetilde{F}^{ii} \tilde{g}^{kk} \left| V_1 (\tilde{g}_{kk}) \right|^2 \\
+ (1 - \varepsilon) \left( 1 + \frac{1}{\tau} \right) \left( \mu_1 - \sum_{\beta > 1} \mu_\beta \xi_\beta^2 \right) \sum_{k \in I} \sum_{l \beta > 1} \widetilde{F}^{ii} \left| e_i (u_{V_3 V_1}) \right|^2 \\
= 2 \sum_{k \in I} \sum_{l \in I} \widetilde{F}^{ii} \tilde{g}^{kk} \left| V_1 (\tilde{g}_{kk}) \right|^2 + 2 (1 - \varepsilon) \sum_{k \in I} \sum_{l \beta > 1} \widetilde{F}^{ii} \left| e_i (u_{V_3 V_1}) \right|^2.
\]

Then (5.50) follows from (5.52).

**Case II.** Assume that

\[ \mu_1 + \sum_{\beta > 1} \mu_\beta \xi_\beta^2 < 2 (1 - \varepsilon) \tilde{g}_{11}. \]  

(5.53)

By a directly calculation, we have

\[
\tilde{g}(\tilde{e}, \tilde{e}) = g(\tilde{e}, \tilde{e}) + \tilde{e}^T (u) - [\tilde{e}, \tilde{e}]^{(0,1)} (u) \]

\[
= g(\tilde{e}, \tilde{e}) + \frac{1}{2} (V_1 V_1 (u) + (J V_1) (J V_1) (u) + \sqrt{-1} [V_1, J V_1] (u)) - [\tilde{e}, \tilde{e}]^{(0,1)} (u) \]

\[
= \frac{1}{2} \left( \mu_1 + \sum_{\alpha > 1} \mu_\alpha \xi_\alpha^2 \right) + g(\tilde{e}, \tilde{e}) + (\nabla V_1 V_1) (u) + (\nabla J V_1, J V_1) (u) \]

\[
+ \sqrt{-1} [V_1, J V_1] (u) - [\tilde{e}, \tilde{e}]^{(0,1)} (u) \]

\[
\leq \frac{1}{2} \left( \mu_1 + \sum_{\alpha > 1} \mu_\alpha \xi_\alpha^2 \right) + C. \]

(5.54)

Plugging (5.53) into (5.54), we obtain

\[ \tilde{g}_{11} \leq C / \varepsilon. \]  

(5.55)

By (5.54) we have \( \mu_1 + \sum_{\beta > 1} \mu_\beta \xi_\beta^2 \geq -C. \) Hence,

\[ 0 < \mu_1 - \sum_{\beta > 1} \mu_\beta \xi_\beta^2 \leq 2 \mu_1 + C \leq (2 + 2 \varepsilon^2) \mu_1 \]

provided by \( \mu_1 \geq C / \varepsilon^2. \) Choose \( \tau = 1 / \varepsilon^2. \) It follows that

\[
(1 - \varepsilon) \left( 1 + \frac{1}{\tau} \right) \left( \mu_1 - \sum_{\beta > 1} \mu_\beta \xi_\beta^2 \right) \leq 2 (1 - \varepsilon) (1 + \varepsilon^2) \mu_1 \leq (2 - \varepsilon) \mu_1,
\]
when $\varepsilon$ is small enough. Then, by Lemma 5.7, we have

$$
(2 - \varepsilon) \sum_{\beta > 1} F^{\alpha \beta} |e_i(u_\beta V_\alpha V_\beta)|^2 - \frac{1}{\mu_1} F^{\alpha \beta} V_\alpha (\tilde{V}_\beta) V_\beta (\tilde{V}_\beta) - (1 + \varepsilon) \sum_{i \in I} F^{\alpha \beta} |e_i(\mu_1)|^2 \mu_1^2 \\
\geq 2 \sum_{k \in I} \sum_{i \in I} F^{\alpha \beta} \tilde{g}^{kk} |V_\alpha (\tilde{g}_{kk})|^2 \frac{1}{\mu_1} - 3 \sum_{i \in I} F^{\alpha \beta} |e_i(\mu_1)|^2 \mu_1^2 \\
- 2(1 - \varepsilon) \left(1 + \frac{1}{\varepsilon^2}\right) \tilde{g}_{ii} \sum_{k \in I} \sum_{i \in I} F^{\alpha \beta} \tilde{g}^{kk} |V_\alpha (\tilde{g}_{kk})|^2 - C \varepsilon F \\
\geq 2 \sum_{k \in I} \sum_{i \in I} F^{\alpha \beta} \tilde{g}^{kk} |V_\alpha (\tilde{g}_{kk})|^2 \frac{1}{\mu_1} - 3 \sum_{i \in I} F^{\alpha \beta} |e_i(\mu_1)|^2 \mu_1^2 \\
- (1 - \varepsilon) \left(1 + \frac{1}{\varepsilon^2}\right) \frac{C}{\varepsilon} \sum_{k \in I} \sum_{i \in I} F^{\alpha \beta} \tilde{g}^{kk} |V_\alpha (\tilde{g}_{kk})|^2 - C \varepsilon F \\
\geq -3 \varepsilon \sum_{i \in I} F^{\alpha \beta} |e_i(\mu_1)|^2 \mu_1^2 - \frac{C}{\varepsilon} F,
$$

if we assume $\mu_1 \geq C/\varepsilon$ in the last inequality. This proves (5.50).

We now complete the proof of the second order estimates. By Lemmas 5.8 and 5.4 and (5.10), we have

$$
L(Q) \geq -6 \varepsilon B^2 \varepsilon^2 B^2 \tilde{g} F^{\alpha \beta} |e_i(\tilde{\eta})|^2 - 6 \varepsilon (\phi')^2 \sum_{i \in I} F^{\alpha \beta} |e_i(\nabla u|^2|^2) - \frac{C}{\varepsilon} F \\
+ \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + B^2 \varepsilon^2 B^2 \tilde{g} F^{\alpha \beta} |e_i(\tilde{\eta})|^2 + B e^{B^2 \tilde{g}} L(\tilde{\eta}) \\
+ \phi'' F^{\alpha \beta} |e_i(\nabla u|^2|^2) - 2 (\phi')^2 \sum_{i \in I} F^{\alpha \beta} |e_i(\nabla u|^2|^2).
$$

Choose $\varepsilon < \min\{\frac{1}{6\theta}, \theta/6\}$ such that $e^{B^2 \tilde{g}(0)} = \frac{1}{6}$. By $\phi'' = 2(\phi')^2$, we have

$$
0 \geq -\frac{C}{\varepsilon} F + \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \\
+ (B^2 \varepsilon^2 B^2 \tilde{g} - 6 \varepsilon B^2 \varepsilon^2 B^2 \tilde{g}) \sum_{i \in I} F^{\alpha \beta} |e_i(\tilde{\eta})|^2 + B e^{B^2 \tilde{g}} L(\tilde{\eta}) \\
= -\frac{C}{\varepsilon} F + \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + B e^{B^2 \tilde{g}} L(\tilde{\eta}).
$$

In other words,

$$
\frac{B^2 \varepsilon}{6 \varepsilon} L(\tilde{\eta}) - \frac{C}{\varepsilon} F + \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq 0.
$$

(a) Suppose that (2.11) holds. Then we have

$$
\left(\frac{B \theta}{6 \varepsilon} - \frac{C}{\varepsilon}\right) F + \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq 0.
$$

Choose $B$ sufficiently large and $\varepsilon < \theta/6$ small enough such that $B \theta/6 - C \geq B \varepsilon$. Then at $x_0$ we have

$$
0 \geq B F + \frac{\phi'}{2} \sum_{j} F^{\alpha \beta} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2).
$$

This leads to a contradiction.
(b) Suppose that (2.12) holds, i.e.,
\[ 1 \geq F_i \geq \theta F \geq \theta K \quad \text{for } i = 1, 2, \ldots, n. \]
Combining (5.26) with (5.57), we have
\[
\frac{\theta \phi'}{2} K \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq nB + \frac{C}{\varepsilon}.
\]
Therefore,
\[
\sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq CB. \tag{5.58}
\]
Then this proves Case 3. In conclusion, we obtain the second order estimates.

By (2.1) and Theorem 5.1, the equation (1.2) is uniformly elliptic. By the \(C^{2,\alpha}\) estimates (see, e.g., [29, Theorem 1.1]), it follows that \(\|u\|_{C^{2,\alpha}} \leq C\). Then, by a standard bootstrapping argument, we complete the proof of Theorem 1.1.

## 6 Proof of Theorem 1.2

We use the arguments of [6,25] to prove Theorem 1.2. We will use the continuity method. First we give the openness.

### 6.1 Openness

Consider the family of equations
\[
\sum_i \arctan \lambda_i(u_t) = (1 - t) h_0 + th_1 + c_t, \tag{6.1}
\]
where \(h_0, h_1 \in ((n - 1)\pi, n\pi)\) are smooth functions on \(M\).

**Proposition 6.1.** Suppose that \(u_{t_0}\) satisfies
\[
\sum_i \arctan \lambda_i(u_{t_0}) = (1 - t_0) h_0 + t_0 h_1 + c_{t_0}.
\]
Then there exists \(\varepsilon > 0\) such that when \(|t - t_0| \leq \varepsilon\), we can find \((u_t, c_t)\) solving (6.1).

**Proof.** Set \(L = P^i (e_i \bar{e}_j - [e_i, \bar{e}_j])\). Since the operator is homotopic to the canonical Laplacian operator \(\Delta^C\), the index of \(L\) is zero, where \(\Delta^C \phi = \frac{2n + 1}{n - 1} \omega \partial_\bar{\psi} \phi\). By the maximum principle,
\[
\ker(L) = \{\text{Constants}\}. \tag{6.2}
\]
Denote by \(L^*\) the \(L^2\)-adjoint operator of \(L\). By the Fredholm theorem, there is a smooth function \(\varphi_0\) such that
\[
\ker(L^*) = \text{Span}\{\varphi_0\}. \tag{6.3}
\]
Now we prove that \(\varphi_0\) does not change the sign. If \(\varphi_0\) changes sign, there is a positive function \(\tilde{f}\) which is perpendicular to it, but cannot be the image of \(L\) by the maximum principle. This leads to a contradiction. Without loss of generality, assume that \(\varphi_0\) is non-negative. By the strong maximum principle for the elliptic operator \(L^*\), the function \(\varphi_0 > 0\). Let
\[
\tilde{B} = \left\{ \psi \in C^{2,\alpha}(M) : \int_M \psi \varphi_0 \chi^n = 0 \text{ and } \omega + \sqrt{-1} \partial \bar{\partial} \psi > 0 \right\} \times \mathbb{R}.
\]
Consider the map \(G\), from \(\tilde{B}\) to \(C^\alpha(M)\), where
\[
G(\psi, c) := \sum_i \arctan \lambda_i(\psi) - c.
\]
Then the linear operator of $G$ at $(u_0, 0)$ is
\[ L - c : \left\{ \xi \in C^{2, \alpha}(M) : \int_M \xi \varphi_0 \chi^n = 0 \right\} \times \mathbb{R} \to C^\alpha(M). \tag{6.4} \]

For any $\hat{h} \in C^\alpha(M)$, there exists a unique constant $c$ such that
\[ \int_M (\hat{h} + c) \varphi_0 \chi^n = 0. \]

Then by (6.3) and the Fredholm theorem, there exists $\xi$ such that $L(\xi) - c = \hat{h}$. Hence $L - c$ is surjective. Let $(\xi_0, \hat{c})$ be the solution of $L - c = 0$. By (6.3) and the Fredholm theorem, $\hat{c} = 0$. Using (6.2) and (6.4), we obtain $\xi_0 = 0$. Hence, $L - c$ is injective. Then by the implicit function theory, when $|t - t_0|$ is small enough, there exist $u_t$ and a constant $c_t$ satisfying \( \sum_i \arctan \lambda_i(u_t) = (1 - t)h_0 + th_1 + c_t. \)

\[ \begin{array}{c}
6.2 \text{ Existence} \\
\end{array} \]

Suppose that $\bar{u}$ is a $C$-subsolution to the deformed Hermitian-Yang-Mills equation (1.2). Recall that $\lambda_i(\bar{u})$ are the eigenvalues of $\omega_{\bar{u}}$. Let $\theta_0 = \sum_i \arctan \lambda_i(\bar{u})$. Now we use the continuity method to prove that there exists a solution when the right-hand side $h$ of (1.2) is a constant.

**Proposition 6.2.** Under the assumption of Theorem 1.2, there exist a function $u$ on $M$ and a constant $c$ such that
\[ \sum_i \arctan \lambda_i = h + c, \]

where $h + c > \frac{(n-1)\pi}{2}$.

**Proof.** Consider the family of equations
\[ \sum_i \arctan \lambda_i(u_t) = (1 - t)\theta_0 + th + c_t. \tag{6.5} \]

Define
\[ I = \{ t \in [0, 1] : \text{there exists } (u_t, c_t) \in \hat{B} \text{ solving (6.5)} \}. \]

Note that $(\bar{u}, 0)$ is the solution of (6.5) at $t = 0$. Then $I$ is non-empty. By Proposition 6.1, $I$ is open. To prove $I$ is closed, by Theorem 1.1, it suffices to prove $\bar{u}$ is still a $C$-subsolution of (6.5) for any $t \in [0, 1]$ and
\[ (1 - t)\theta_0 + th + c_t \geq \inf_M \theta_0 > (n - 1)\frac{\pi}{2}. \tag{6.6} \]

First, assume that $u_t - \bar{u}$ achieves its maximum at the point $q$. Then
\[ \sqrt{-1} \partial \overline{\partial} (u_t - \bar{u})(q) \leq 0. \]

It follows that
\[ F(\omega_{u_t})(q) - F(\omega_{\bar{u}})(q) = \int_0^1 \int_{\mathbb{S}^{n-1}} F(s\omega_{u_t} + \sqrt{-1} \partial \overline{\partial} (u_t - \bar{u})) ds(u_t - \bar{u})_{-\gamma}(q) \leq 0. \]

Then at $q$,
\[ \theta_0(q) \geq \sum_i \arctan \lambda_i(u_t)(q) = (1 - t)\theta_0(q) + th + c_t, \]

which implies
\[ c_t \leq t(\theta_0(q) - h) \leq 0. \tag{6.7} \]

Here, we used $h \geq \theta_0$. Then, we have
\[ \sum_{i \neq j} \arctan(\lambda_i(\bar{u})) \geq h - \frac{\pi}{2} \geq (1 - t)\theta_0 + th + c_t - \frac{\pi}{2}. \]
By Lemma 2.2, \( u \) is a \( C \)-subsolution for \( t \in [0, 1] \). Assume that \( u_t - \hat{u} \) achieves its minimum at the point \( q' \). Similar to (6.7), we have
\[
c_t \geq -t(h - \theta_0(q')) \geq -\sup_M t(h - \theta_0). \tag{6.8}
\]
Assume that \( \theta_0 \) achieves its minimum at the point \( p \). Note that \( h \) is a constant. Then, we obtain
\[
\inf_M ((1 - t)\theta_0 + th + c_t) = (1 - t)\theta_0(p) + th + c_t \\
= \theta_0(p) + t(\theta_0(p)) + c_t \\
= \theta_0(p) + t\sup_M (h - \theta_0) + c_t \\
\geq \theta_0(p) = \inf_M \theta_0 > (n - 1)\frac{\pi}{2}. \tag{6.9}
\]
By Theorem 1.1, we conclude that \( I \) is closed.

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