Synthesis and characterization of entangled mesoscopic superpositions for a trapped electron

Michel Massini,1 Mauro Fortunato,1 Stefano Mancini,2 and Paolo Tombesi1
1INFM and Dipartimento di Matematica e Fisica, Università di Camerino, I–62032 Camerino, Italy
2INFM and Dipartimento di Fisica, Università di Milano, Via Celoria 16, I–20133 Milano, Italy
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We propose a scheme for the generation and reconstruction of entangled states between the internal and external (motional) degrees of freedom of a trapped electron. Such states also exhibit quantum coherence at a mesoscopic level.

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A single electron trapped in a Penning trap is one of the most fundamental quantum systems. Among its peculiar features, it allows the measurement of fundamental physical constants with striking accuracy. Recently, for instance, the electron cyclotron degree of freedom has been cooled to its ground state, where the electron may stay for hours, and quantum jumps between adjacent Fock states have been observed. It is therefore evident that the manipulation and the characterization of the state of a trapped electron is an important issue, with implications in the very foundations of quantum mechanics. Earlier works have dealt with this problem for one (motional) degree of freedom. On the other hand, entanglement has been recognized as one of the most puzzling features of quantum mechanics, being also the basis of quantum information processing. A striking achievement in this rapidly expanding field has been the recent entanglement of four trapped ions. However, it is also possible (and conceptually equivalent) to entangle different degrees of freedom of the same particle.

In the present work we propose to generate entangled states (combined cyclotron and spin states) of an electron in a Penning trap by using suitable applied fields. The complete structure of the cyclotron-spin quantum state is then obtained with the help of a tomographic reconstruction from the measured data.

In a Penning trap an electron is confined by the combination of a homogeneous magnetic field along the positive z axis and an electrostatic quadrupole potential in the xy plane. The spatial part of the electronic wave function consists of three degrees of freedom, but neglecting the slow magnetron motion (whose characteristic frequency lies in the kHz region), here we only consider the axial and cyclotron motions, which are two harmonic oscillators radiating in the MHz and GHz regions, respectively. On the other hand, the spin dynamics results from the interaction between the magnetic moment of the electron and the static magnetic field, so that the free Hamiltonian reads

\[
\hat{H}_{\text{free}} = \hbar \omega_z \hat{a}_z^\dagger \hat{a}_z + \hbar \omega_c \hat{a}_c^\dagger \hat{a}_c + \hbar \omega_s \hat{\sigma}_z / 2, \tag{1}
\]

where the indices \(z, c, \) and \(s\) refer to the axial, cyclotron and spin motions, respectively.

Here, in addition to the usual trapping fields, we consider an external radiation field as a standing wave along the z direction and rotating, i.e. circularly polarized, in the xy plane with frequency \(\Omega\). To be more specific, we consider a standing wave within the cylindrical cavity configuration with the (dimensionless) wave vector \(\kappa\). Then, the interaction Hamiltonian reads

\[
\hat{H}_{\text{int}} = \hbar \left[ \hat{a}_e e^{i\Omega t} + \hat{a}_e^\dagger e^{-i\Omega t} \right] \cos(\kappa \hat{z} + \phi) + \hbar \zeta \left[ \hat{\sigma}_- e^{i\Omega t} + \hat{\sigma}_+ e^{-i\Omega t} \right] \sin(\kappa \hat{z} + \phi), \tag{2}
\]

where \(\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2\), and \(\hat{z} = \hat{a}_z + \hat{a}_z^\dagger\). The phase \(\phi\) defines the position of the center of the axial motion with respect to the wave. Depending on its value the electron can be positioned in any place between a node (\(\phi = 0\)) and an antinode (\(\phi = \pm \pi/2\)). The two coupling constants \(\epsilon\) and \(\zeta\) are proportional to the amplitude of the applied radiation field. For our purposes we also consider the possibility to introduce pulsed standing waves through the microwave inlet so that \(\epsilon\) and \(\zeta\) become time dependent. The duration of the pulse is assumed to be much shorter than the characteristic axial period, which is of the order of microseconds. Depending on \(\Omega\) and \(\phi\), the interaction Hamiltonian gives rise to different contributions at leading order in the Taylor expansion of \(\sin(\kappa \hat{z} + \phi)\) and \(\cos(\kappa \hat{z} + \phi)\).

Now, nonclassical cyclotron states can be entangled with the spin states through the following steps. First, we consider \(\phi = 0\), \(\Omega = \omega_c\), and a pulsed standing wave of duration \(\Delta t_1 = t_1 - t_0 = t_1\). In the following we shall work in a frame rotating at the frequency \(\omega_s\). Then, the total Hamiltonian \((\hat{H}_{\text{free}} + \hat{H}_{\text{int}})\) can be written as

\[
\hat{H} = \hbar \omega_z \hat{a}_z^\dagger \hat{a}_z + \hbar (\omega_c - \omega_s) \hat{a}_c^\dagger \hat{a}_c + \hbar \zeta (t) \hat{\sigma}_x \kappa (\hat{a}_z + \hat{a}_z^\dagger). \tag{1}
\]

The first two terms can be neglected during the pulse duration. In fact, the latter is assumed to be much smaller than \(\omega_c^{-1}\) and \((\omega_c - \omega_s)^{-1} \approx \omega_c^{-1}\). Its effect on the axial degree of freedom can be described by means of the relation 

\[
\hat{p}_z(t_1) = \hat{p}_z(0) - \hat{\zeta} \hat{\sigma}_x, \tag{2}
\]

where \(\hat{\zeta} = \kappa \int_0^{t_1} \! dt \zeta(t)\), and \(\hat{p}_z = -i (\hat{a}_z - \hat{a}_z^\dagger)\). Subsequently, we allow a free evolution for a time \(\Delta t_2 = t_2 - t_1 = \pi/(2\omega_z)\). That amounts to having 

\[
\hat{z}(t_2) = \hat{p}_z(t_1) = \hat{p}_z(0) - \hat{\zeta} \hat{\sigma}_x. \tag{3}
\]

Finally, we consider the action of another pulsed standing wave with \(\phi = -\pi/2\), \(\Omega = \omega_c\), for a time \(\Delta t_3 = \pi/(2\omega_z)\)
\[ t_3 - t_2. \] In such a case the effective Hamiltonian (in the frame rotating at the frequency \( \omega_z \)) becomes \( \hat{H} = \hbar \omega_z \hat{a}_z^\dagger \hat{a}_z + \hbar (\omega_c - \omega_z) \hat{a}_c^\dagger \hat{a}_c + \hbar(t) \left( \hat{a}_c^\dagger + \hat{a}_c \right) \kappa \hat{z}. \) Again, since \( \omega_z - \omega_c \) is of the order of \( \omega, \) we can neglect the first two terms in the previous Hamiltonian. Then, the time evolution operator is equivalent to a displacement operator \( \hat{U}(\Delta t_3) = \hat{D}(-i\hat{z}), \) with \( \hat{D}_{ij} = \int_{\tau}^{t_3} dt \epsilon(t). \) Since \( \hat{z} \) is not affected by the time evolution under the previous Hamiltonian, it remains unaltered during \( \Delta t_3, \) that is \( \hat{U}(t_3 - t_0) = \exp \left\{-i\hat{z} \left( \hat{a}_c + \hat{a}_c^\dagger \right) \hat{p}_z - \hat{\zeta} \hat{\sigma}_z \right\}. \) This means that the state evolution for the spin-cyclotron system is given by

\[
\hat{\rho}(t_3) = \text{Tr}_z \left\{ \hat{D}(-i\hat{p}_z) \hat{D}^\dagger(i\hat{z} \hat{\sigma}_z) \times \hat{\rho}(0) \hat{D}^\dagger(i\hat{z} \hat{\sigma}_z) \hat{D}(-i\hat{p}_z) \exp(-\hat{p}_z^2/d^2) \right\},
\]

where \( \hat{R}(0) \) is the initial density operator for the whole system. If we consider for instance the initial axial state as a Gaussian state with momentum width \( d, \) the above equation can be rewritten as

\[
\hat{\rho}(t_3) = \frac{1}{\pi^{1/4}d} \int d\hat{p}_z \hat{D}(-i\hat{p}_z) \hat{D}(i\hat{z} \hat{\sigma}_z) \times \hat{\rho}(0) \hat{D}^\dagger(i\hat{z} \hat{\sigma}_z) \hat{D}(-i\hat{p}_z) \exp(-\hat{p}_z^2/d^2).
\]

Assuming \( \hat{\zeta} \gg d, \) which is easily obtained in the case of the ground state of the axial oscillator, we can approximate the evolution of an initial pure state \( \hat{\rho}(0) = |\Phi_0\rangle \langle \Phi_0 | \) into \( |\Phi = \hat{D}(\alpha \hat{\sigma}_z) |\Phi_0\rangle, \) where \( \alpha = i\hat{z} \hat{\zeta}. \)

It is now immediate to see that an initial state \( |\Phi_0\rangle = |0\rangle \uparrow \) evolves into

\[
|\Phi = (|\uparrow\rangle |\alpha_+\rangle + |\downarrow\rangle |\alpha_-\rangle) / \sqrt{2},
\]

where \( |\uparrow\rangle \) and \( |\downarrow\rangle \) denote spin eigenstates, while \( |\alpha_\pm\rangle = N(|\alpha \rangle \pm | - \alpha \rangle) \) are the even-odd coherent states \( |\alpha\rangle \) of the cyclotron mode with \( N \) a normalization factor. Since the latter are orthogonal states, Eq. (3) represents a maximally entangled state \( |\alpha\rangle \). On the other hand, even and odd coherent states may represent the basis of unconventional quantum bits \( |\alpha\rangle \), hence states of the form of Eq. (3) could be of importance to encoding and manipulating quantum information \( |\alpha\rangle \). Furthermore, a simple spin rotation (as it can be seen below) is sufficient to realize the transformation

\[
|\Phi \rangle \rightarrow (|\alpha\rangle |\uparrow\rangle + | - \alpha\rangle |\downarrow\rangle) / \sqrt{2},
\]

a state already discussed in Refs. [3][13]. The electronic state \( |\alpha\rangle \) possesses two very interesting features: first, if \( |\alpha\rangle \), i.e. the product \( \epsilon \zeta \) is much larger than 1, \( |\alpha\rangle \) is a typical example of Schrödinger-cat state \( |\Psi\rangle \). Second, the full state of the trapped electron is an entangled state between the spin and cyclotron degrees of freedom. It is worth noting that states like \( |\Psi\rangle \) or \( |\alpha\rangle \) persist for many cycles since the decoherence (i.e. the rapid destruction of superposition states due to the entanglement with the environment \( |\Psi\rangle \)) in such a system is quite small \( \langle \Psi | \hat{\rho} \langle \Psi | \rangle \) (in contrast, in Ref. [4] the mesoscopic superposition appears only cyclically).

The most general pure state of the trapped electron can be cast in the form \( |\Psi\rangle = c_1 |\psi_1\rangle \uparrow + c_2 |\psi_2\rangle \downarrow \), \( |\psi_1\rangle \) and \( |\psi_2\rangle \) being two unknown cyclotron states, and the complex coefficients \( c_1 \) and \( c_2 \) satisfying the normalization condition \( |c_1|^2 + |c_2|^2 = 1 \). The density operator \( \hat{\rho} = |\Psi\rangle \langle \Psi | \) associated to the pure state \( |\Psi\rangle \) can be expressed in the form

\[
\hat{\rho} = \left( \begin{array}{cc} |c_1|^2 |\psi_1\rangle \langle \psi_1 | & c_1 c_2^* |\psi_1\rangle \langle \psi_2 | \\ c_2 c_1^* |\psi_2\rangle \langle \psi_1 | & |c_2|^2 |\psi_2\rangle \langle \psi_2 | \end{array} \right),
\]

whose elements \( \rho_{ij} = c_i c_j^* \) are operators in the cyclotron Hilbert space. The phase-space description corresponding to \( |\Psi\rangle \) is given by the Wigner-function matrix \( |\Psi\rangle \langle \Psi | \) whose elements are

\[
\hat{W}_{ij}(\alpha) = \langle \delta_{ij} (\alpha - \tilde{\alpha}) \rangle = \text{Tr}\left[ \hat{\rho} \hat{\delta}_{ij} (\alpha - \tilde{\alpha}) \right],
\]

where \( \delta_{ij} (\alpha - \tilde{\alpha}) \) is the Fourier transform of the displacement operator with \( i, j = 1, 2 \) denoting the spin components.

In order to characterize the generic state \( |\Psi\rangle \) we use a simple reconstruction procedure: Adding a particular inhomogeneous magnetic field—known as the “magnetic bottle” field \( |\Psi\rangle \)—that already present in the trap, it is possible to perform a simultaneous measurement of both the spin and the cyclotron excitation numbers. The useful interaction Hamiltonian for the measurement process is then

\[
\hat{H}_{\text{bottle}} = \hbar \omega_c \left[ \hat{a}_c^\dagger \hat{a}_c + \frac{g \hat{\sigma}_z}{2} \right] \hat{\zeta}^2,
\]

where the angular frequency \( \omega_c \) is directly related to the strength of the magnetic bottle field.

Eq. (3) describes the fact that the axial angular frequency is affected both by the number of cyclotron excitations \( \hat{n}_c = \hat{a}_c^\dagger \hat{a}_c \) and by the eigenvalue of \( \hat{\sigma}_z \). The modified (shifted) axial frequency can be experimentally measured \( |\Psi\rangle \) after the application of the inhomogeneous magnetic bottle field. One immediately sees that it assumes different values for every pair of eigenvalues of \( \hat{n}_c \) and \( \hat{\sigma}_z \), due to the fact that the electron \( g \) factor is slightly (but measurably \( |\Psi\rangle \)) different from 2. Then, repeated measurements of this type allow us to recover the probability amplitudes associated to the two possible spin states and the cyclotron probability distribution in the Fock basis. The reconstruction of the density matrices \( |\psi_i\rangle \langle \psi_i | \) in the Fock basis is then possible by employing a technique similar to the Photon Number Tomography (PNT) \( |\Psi\rangle \langle \Psi | \) which exploits a phase-sensitive reference field that displaces in the phase space the particular state one wants to reconstruct \( |\Psi\rangle \).
Following Ref. [4], immediately before the measurement, we apply a pulsed standing wave (2) tuned to \( \Omega = \omega_c \) with \( \phi = 0 \) in order to get a displacement \( \gamma = -i\epsilon/\kappa \) on the cyclotron. Thus we can interpret the quantity

\[
P^{(i)}(n, \gamma) = \langle n|\hat{D}^\dagger(\gamma)\hat{\rho}^{(ii)}\hat{D}(\gamma)|n\rangle = \langle n, \gamma|\hat{\rho}^{(ii)}|n, \gamma\rangle ,
\]

as the probability of finding the cyclotron state \( |\psi_i\rangle \) in a displaced number state \( |n, \gamma\rangle \) [22]. It should be noted at this stage that the probability distribution (11) is not normalized to unity. Instead, one has

\[
P^{(i)}(n, \gamma) = \sum_{m=0}^\infty P^{(i)}(n, \gamma) = |c_i|^2 .
\]

Eq. (11) allows us to retrieve in a simple way the moduli of the coefficients \( c_1 \) and \( c_2 \) in Eq. (1) from the measured data. Fixing a particular value of \( \gamma \), it is then possible to recover the probability distribution (10) performing many identical experiments.

Expanding the density operator \( \hat{\rho}^{(ii)} \) in the Fock basis, and defining \( N_c \) as an approximate estimate of the maximum number of cyclotron excitations (cut-off), we have

\[
P^{(i)}(n, \gamma) = \sum_{k,m=0}^{N_c} \langle n, \gamma|k\rangle\langle k|\hat{\rho}^{(ii)}|m\rangle\langle m|n, \gamma\rangle .
\]

The projection of the displaced number state \( |n, \gamma\rangle \) onto the Fock state \( |m\rangle \) can be obtained generalizing the result derived in Ref. [23].

Let us now consider, for a given value of \( |\gamma| \), \( P^{(i)}(n, \gamma) \) as a function of \( \varphi = \arg[\gamma] \) [24] and calculate the coefficients of the Fourier expansion

\[
P^{(i)}(n, s) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi P^{(i)}(n, \varphi)e^{i\varphi} ,
\]

for \( s = 0, 1, 2, \ldots \). Combining Eqs. (12) and (13), we get

\[
P^{(i)}(n, s) = \sum_{m=0}^{N_c-s} G^{(s)}_{n,m}(|\gamma|)\langle m|s\hat{\rho}^{(ii)}|m\rangle ,
\]

where the explicit expression of the matrices \( G \) is given in Ref. [24].

We may now note that if the distribution \( P^{(i)}(n, \gamma) \) is measured for \( n \in [0, N] \) with \( N \geq N_c \), then Eq. (14) represents for each value of \( s \) a system of \( N + 1 \) linear equations between the \( N + 1 \) measured quantities and \( N_c + 1 - s \) unknown density matrix elements. Therefore, in order to obtain the latter, we only need to invert the system

\[
\langle m + s|\hat{\rho}^{(ii)}|m\rangle = \sum_{n=0}^{N} M_{m,n}^{(s)}(|\gamma|)P^{(i)}(n, s) ,
\]

where the matrices \( M \) are given by \( M = (G^T G)^{-1}G^T \). Since the overdetermined system (14) is inverted using the method of least squares, we are sure that when the measured probabilities are slightly inaccurate, the quantities calculated from the reconstructed density matrix best fit the measured ones [24].

However, this kind of measurement does not allow to retrieve the relative phase \( \theta \) between the complex coefficients \( c_1 \) and \( c_2 \). We can then again use the Hamiltonian (3) tuned to have a spin rotation, i.e. \( \Omega = \omega_s \) and \( \phi = \pi/2 \). After a \( \pi/2 \) spin rotation we have

\[
|\Psi\rangle \rightarrow |\overline{\Psi}\rangle = \frac{\sqrt{2}}{2}[(c_1|\psi_1\rangle - ic_2|\psi_2\rangle)]\uparrow + (-ic_1|\psi_1\rangle + c_2|\psi_2\rangle)|\downarrow\rangle .
\]

We can now repeat the spin measurements just as we have described above in the case of the unknown initial state \( |\Psi\rangle \). Repeating this procedure over and over again (with the same unknown initial state) for a large number of times and tracing out the cyclotron degree of freedom (no drive in this case is required), it is possible to recover the probabilities \( \bar{P}^{(i)} \) associated to the two spin eigenvalues for the state \( |\overline{\Psi}\rangle \). Without loss of generality, we can assume \( c_1 \in \mathbb{R} \), \( c_2 = |c_2|e^{i\beta} \), and \( \langle \psi_1|\psi_2\rangle = re^{i\beta} \), which yield

\[
\bar{P}^{(1)} = \frac{1}{2}[1 + 2r|c_1||c_2| \sin(\theta + \beta)]
\]

\[
\bar{P}^{(2)} = \frac{1}{2}[1 - 2r|c_1||c_2| \sin(\theta + \beta)] .
\]

It is important to note that the probabilities \( \bar{P}^{(i)} \) can be experimentally sampled and that the modulus \( r \) and the phase \( \beta \) of the scalar product \( \langle \psi_1|\psi_2\rangle \) can be both derived from the reconstruction of the cyclotron density matrices \( \hat{\rho}^{(1)} \) and \( \hat{\rho}^{(2)} \). Thus we are able to find the relative phase \( \theta \) by simply inverting one of the two Eqs. (17), e.g.

\[
\theta = \arcsin\left[\frac{2\bar{P}^{(1)} - 1}{2r|c_1||c_2|}\right] - \beta ,
\]

where the ambiguity in the arcsin function can be removed by repeating the procedure above using a second spin rotation [24] with an angle different from \( \pi/2 \). As in any tomographic scheme, our reconstruction procedure yields the state to be measured up to an uninteresting factor which, however, does not affect the results.

As an example of the states that can be generated with the method outlined above, and of the application of the proposed reconstruction procedure, we show in Fig. [5] the results of numerical Monte-Carlo simulations of the construction of the state (3). In this simulation we have used the value \( |\alpha| = 1.5 \) which is experimentally accessible [5] and gives mesoscopic entangled superpositions of cyclotron coherent states with opposite phase. In order to account for actual experimental conditions, we have
also considered the effects of a non-unit quantum efficiency $\eta$ in the counting of cyclotron excitations. When $\eta < 1$, the actually measured distribution is related to the ideal distribution by a binomial convolution $26$. As it can be seen from Fig. 1, the reconstructed distributions turn out to be quite faithful. We also would like to emphasize that the particular shape of $W_{12}$ is due to the quantum interference given by the entanglement between the two degrees of freedom: in fact, in absence of entanglement $\rho^{(12)}$ would just be a replica of the diagonal parts $\rho^{(11)}$ and $\rho^{(22)}$.

To conclude, we have proposed here a method which is able to synthesize and characterize highly nonclassical, maximally entangled states for a single trapped electron. The method is based on simple operations (switching on and off standing waves) which are currently realized in the present trap technology $21$. An experimental implementation of the proposed method might yield new insight in the foundations of quantum mechanics and allow further progress in the field of quantum information.

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**FIG. 1.** Simulated tomographic reconstruction of the Wigner matrix $W_{ij} = \tilde{W}_{ij}/c_i c_j^{*}$ for the state of Eq. (10) with $\alpha = 3i/2$. The quantum efficiency is $\eta = 0.9$ and $10^5$ data per phase have been simulated. In this simulation an amplitude $|\gamma| = 1.2$ of the applied reference field has been used. The theoretical distributions are plotted on the left for comparison.

We have performed a large number of simulations with different states and several values of the parameters, which confirm that the present method is quite stable and accurate. In addition, and for all the cases considered, the values of the parameters $c_1$, $c_2$, and $\theta$ are very well recovered, with a relative error of the order of $10^{-5}$.

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