Common Fixed Point Results for Generalized Symmetric Meir-Keeler Contraction

Bhavana Deshpande*, Amrish Handa

Department of Mathematics Govt. P. G. Arts & Science College Ratlam (M. P.) India

*Corresponding author: bhavnadeshpande@yahoo.com

Received November 13, 2014; Revised December 16, 2014; Accepted January 14, 2015

Abstract We introduce the concept of generalized weakly compatibility for the pair \( F,G \) of mappings \( F,G : X \times X \to X \) and also introduce the concept of common fixed point of the mappings \( F,G : X \times X \to X \). We establish a common fixed point theorem for generalized weakly compatible pair of mappings \( F,G : X \times X \to X \) without mixed monotone property of any mapping under generalized symmetric Meir-Keeler contraction on a non complete metric space, which is not partially ordered. An example supporting to our result has also been cited. We improve, extend and generalize several known results.

Keywords: common fixed point, generalized symmetric meir-keeler contraction, generalized compatibility, generalized weakly compatibility, commuting mapping

Cite This Article: Bhavana Deshpande, and Amrish Handa, “Common Fixed Point Results for Generalized Symmetric Meir-Keeler Contraction.” Turkish Journal of Analysis and Number Theory, vol. 3, no. 1 (2015): 7-11. doi: 10.12691/tjant-3-1-2.

1. Introduction and Preliminaries

The Banach contraction mapping principle has been generalized in several directions. One of these generalizations, known as the Meir-Keeler fixed point theorem [11], has been obtained by the following more general assumption: for all \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that

\[
x, y \in X, \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon.
\]

Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point, mixed monotone mappings in the setting of single-valued mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces.

In [3], Bhaskar and Lakshmikantham introduced the following.

Definition 1. Let \( (X, \preceq) \) be a partially ordered set and endow the product space \( X \times X \) with the following partial order:

\[
(u,v) \preceq (x,y) \iff x \succeq u \text{ and } y \succeq v,
\]

\[\forall (u,v), (x,y) \in X \times X.\]  

Definition 2. An element \((x,y) \in X \times X\) is called a coupled fixed point of the mapping \( F : X \times X \to X \) if

\[F(x,y) = x \text{ and } F(y,x) = y.\]

Definition 3. Let \( (X, \preceq) \) be a partially ordered set. Suppose \( F : X \times X \to X \) be a given mapping. We say that \( F \) has the mixed monotone property if for all \( x, y \in X \), we have

\[x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1,y) \preceq F(x_2,y)\]  

and

\[y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x,y_1) \preceq F(x,y_2).\]

Lakshmikantham and Ciric [10] extended the notion of mixed monotone property to mixed g-monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Bhaskar and Lakshmikantham [3].

In [10], Lakshmikantham and Ciric introduced the following:

Definition 4. An element \((x,y) \in X \times X\) is called a coupled coincidence point of the mappings \( F : X \times X \to X \) and \( g : X \to X \) if

\[x = F(x, y) = g(x) \text{ and } F(y,x) = g(y).\]

Definition 5. An element \((x,y) \in X \times X\) is called a common coupled fixed point of the mappings \( F : X \times X \to X \) and \( g : X \to X \) if

\[x = F(x, y) = g(x) \text{ and } y = F(y,x) = g(y).\]

Definition 6. An element \( x \in X \) is called a common fixed point of the mappings \( F : X \times X \to X \) and \( g : X \to X \) if

\[x = g(x) = F(x,x).\]

Definition 7. The mappings \( F : X \times X \to X \) and \( g : X \to X \) are said to be commutative if
\[ g(F(x, y)) = F(g(x), g(y)), \text{ for all } (x, y) \in X \times X. \] (9)

**Definition 8.** Let \((X, \preceq)\) be a partially ordered set. Suppose \(F: X \times X \to X\) and \(g: X \to X\) are given mappings. We say that \(F\) has the mixed g-monotone property if for all \(x, y \in X\) we have
\[ x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \] (10)
and
\[ y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \preceq F(x, y_2) \] (11)

If \(g\) is the identity mapping on \(X\); then \(F\) satisfies the mixed monotone property.

**Example 10.** Let \(F, G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) be defined by \(F(x, y) = x + y\) and \(G(x, y) = 2/3 (x + y)\) for all \((x, y) \in X \times X\). Note that \((0,0), (1,2)\) and \((2,1)\) are coupled coincidence points of \(F\) and \(G\).

**Example 15.** Let \((X, d)\) be a usual metric space where \(X = \{0, 1, 2, \ldots, \frac{1}{n}, \ldots\}\). Define \(F, G: X \times X \to X\) by
\[ F(x, y) = \begin{cases} \frac{1}{2n+1} & \text{if } (x, y) = (0, y) \\ 0 & \text{otherwise} \end{cases} \]
and
\[ G(x, y) = \begin{cases} \frac{1}{2n+1} & \text{if } (x, y) = (y, 0) \\ 0 & \text{otherwise} \end{cases} \]

Let \(x_0 = y_0 = \frac{1}{2n}\). Then, we have
\[ G(x_n, y_n) = \frac{1}{2n+1} \to 0, \quad F(x_n, y_n) = \frac{1}{(2n+1)^2} \to 0 \]
as \(n \to \infty\), but
\[ \lim_{n \to \infty} d\left( F(G(x_n, y_n), G(y_n, x_n)), \right) = d(0, 1) \neq 0. \]
be two generalized weakly compatible mappings and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$d\left(\frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2}\right) \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2} \leq \varepsilon$$

(14)

for all $x, y, u, v \in X$. Suppose that for any $x, y \in X$, there exist $u, v \in X$ such that

$$F(x,y) = G(u,v) \text{ and } F(y,x) = G(v,u)$$

(15)

Suppose that $F(X \times X)$ or $G(X \times X)$ is complete. Then there exists a unique $x \in X$ such that $x = G(x,x) = F(x,x)$.

**Proof.** Let $x_0, y_0$ be two arbitrary points in $X$. From (15); we can choose $x_1, y_1 \in X$ such that

$$G(x_1, y_1) = F(x_0, y_0)$$

and

$$G(y_1, x_1) = F(y_0, x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$G(x_{n+1}, y_{n+1}) = F(x_n, y_n)$$

and

$$G(y_{n+1}, x_{n+1}) = F(y_n, x_n),$$

for all $n \geq 0$.

The proof is divided into four steps.

**Step 1.** Prove that $\{G(x_n, y_n)\}$ and $\{G(y_n, x_n)\}$ are Cauchy sequences.

Now, by (14), for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2} \leq \varepsilon$$

(17)

Condition (17) implies the strict contractive condition

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2} < \frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2},$$

(18)

for $G(x,y) \leq G(u,v)$ and $G(y,x) \geq G(v,u)$. Thus, by (18), we have

$$d\left(G(x_{n+1}, y_{n+1}), G(x_n, y_n)\right) + d\left(G(y_{n+1}, x_{n+1}), G(y_n, x_n)\right)$$

$$< \frac{d\left(G(x_{n+1}, y_{n+1}), G(x_{n-1}, y_{n-1})\right)}{2} + \frac{d\left(G(y_{n+1}, x_{n+1}), G(y_{n-1}, x_{n-1})\right)}{2}\leq \frac{d\left(G(x_n, y_n), G(x_{n-1}, y_{n-1})\right)}{2} + \frac{d\left(G(y_n, x_n), G(y_{n-1}, x_{n-1})\right)}{2}$$

which shows that the sequence of nonnegative numbers $\{A_n\}_{n=0}^{\infty}$ given by

$$A_n = \frac{d\left(G(x_n, y_n), G(x_{n-1}, y_{n-1})\right)}{2} + \frac{d\left(G(y_n, x_n), G(y_{n-1}, x_{n-1})\right)}{2},$$

(19)

is non-increasing. Therefore, there exists some $\varepsilon \geq 0$ such that

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} \left[d\left(G(x_n, y_n), G(x_{n-1}, y_{n-1})\right) + d\left(G(y_n, x_n), G(y_{n-1}, x_{n-1})\right)\right] = \varepsilon.$$

We shall prove that $\varepsilon = 0$. Suppose, to the contrary, that $\varepsilon > 0$. Then there exists a positive integer $p$ such that

$$\varepsilon < A_p < \varepsilon + \delta(\varepsilon),$$

which, by (17); implies

$$\frac{d\left(F(x_p, y_p), F(x_{p-1}, y_{p-1})\right)}{2} + \frac{d\left(F(y_p, x_p), F(y_{p-1}, x_{p-1})\right)}{2} < \varepsilon$$

it follows, by (16) and (19), that

$$\frac{d\left(G(x_{p+1}, y_{p+1}), G(x_p, y_p)\right)}{2} + \frac{d\left(G(y_{p+1}, x_{p+1}), G(y_p, x_p)\right)}{2}$$

which is a contradiction. Thus $\varepsilon = 0$ and hence

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} \left[d\left(G(x_n, y_n), G(x_{n-1}, y_{n-1})\right) + d\left(G(y_n, x_n), G(y_{n-1}, x_{n-1})\right)\right] = 0.$$

Let now $\varepsilon > 0$ be arbitrary and $\delta(\varepsilon)$ the corresponding value from the hypothesis of our theorem. By (20), there exists a positive integer $k$ such that

$$\Delta_{k+1} = \frac{1}{2} \left[d\left(G(x_{k+1}, y_{k+1}), G(x_k, y_k)\right) + d\left(G(y_{k+1}, x_{k+1}), G(y_k, x_k)\right)\right] < \delta(\varepsilon).$$

(21)

For this fixed number $k$, consider now the set $A_k = \{G(x_k, y_k), G(y_k, x_k)\} \subseteq G(x_k, y_k), G(y_k, x_k) \subseteq \{G(x_n, y_n)\}_{n=0}^{\infty}$, based on $\delta(\varepsilon)$ and $\delta(\varepsilon) = \delta(\varepsilon)$. By (21), $A_k \neq \emptyset$. We claim that

$$\left\{G(x,y), G(y,x)\right\} \in A_k \Rightarrow \left\{F(x,y), F(y,x)\right\} \in A_k.$$
Let \( (G(x, y), G(y, x)) \in A_k \). Then
\[
d\left( G(x_k, y_k), G(x, y) \right) + d\left( G(y_k, x_k), G(y, x) \right) < \varepsilon, \quad (23)
\]
which, by (14), implies
\[
d\left( F(x_k, y_k), F(x, y) \right) + d\left( F(y_k, x_k), F(y, x) \right) < \varepsilon, \quad (24)
\]
Now, by (21) and (24), we have
\[
d\left( G(x_k, y_k), G(x, y) \right) + d\left( G(y_k, x_k), G(y, x) \right) < \varepsilon + \delta(\varepsilon).
\]
Thus \( (F(x, y), F(x, y)) \in A_k \). Again
\[
d\left( G(x_k, y_k), G(x_k+1, y_k+1) \right) + d\left( G(y_k, x_k), G(y_k+1, x_k+1) \right) < \varepsilon + \delta(\varepsilon).
\]
Thus \( (G(x_k+1, y_k+1), G(y_k+1, x_k+1)) \in A_k \) and by induction,
\[
(G(x_{k+1}, y_{k+1}), G(y_{k+1}, x_{k+1})) \in A_k,
\]
for all \( n > k \).

This implies that for all \( n, m > k \), we have
\[
d\left( G(x_n, y_n), G(x_m, y_m) \right) + d\left( G(y_n, x_n), G(y_m, x_m) \right) < 4\varepsilon.
\]

Step 2. Prove that \( G \) and \( F \) have a coupled coincidence point.
Since \( G(X \times X) \) is complete, then there exist \( x, y \in G(X \times X) \) and \( (a, b) \in X \times X \) such that
\[
\lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} F(x_n, y_n) = G(a, b) = x,
\]
\[
\lim_{n \to \infty} G(y_n, x_n) = \lim_{n \to \infty} F(y_n, x_n) = G(b, a) = y. \quad (25)
\]
Now, by (18), we have
\[
d\left( F(x_n, y_n), F(a, b) \right) + d\left( F(y_n, x_n), F(b, a) \right) < \varepsilon/2, \quad (26)
\]
Taking limit as \( n \to 1 \) in the above inequality and using (25), we have
\[
d\left( F(a, b), F(a, b) \right) = 0 \quad \text{and} \quad d\left( G(b, a), F(b, a) \right) = 0,
\]
which implies that
\[
F(a, b) = G(a, b) = x \quad \text{and} \quad F(b, a) = G(b, a) = y.
\]
Since \( F \) and \( G \) are generalized weakly compatible, we get
\[
G(F(a, b), F(b, a)) = G(a, b), \quad G(b, a, a, b) = G(b, a), \quad G(a, b),
\]
which implies that
\[
G(x, y) = F(x, y) \quad \text{and} \quad G(y, x) = F(y, x),
\]
that is, \((x, y)\) is a coupled coincidence point of \( F \) and \( G \).

Step 3. Prove that \( x = y \) and \( G(y, x) = x \).
If, not, then by (18), we have
\[
d\left( F(x, y), F(y, x) \right) + d\left( F(x, y), F(x, y) \right) < \varepsilon/2, \quad (27)
\]
\[
= \left( G(x, y, y), G(y, x, x) \right) + \left( G(x, y, y), G(y, x, x) \right) < 2\varepsilon.
\]
Letting \( n \to \infty \) in the above inequality and using (25), we have
\[
d\left( G(x, y), G(x, y) \right) + d\left( G(y, x), G(y, x) \right) < \varepsilon/2,
\]
which is a contradiction. Thus we must have \( G(x, y) = y \) and \( G(y, x) = x \).

Step 4. Prove that \( x = y \).
If, not, then by (18), we have
\[
d\left( F(x, y), F(y, x) \right) + d\left( F(x, y), F(x, y) \right) < \varepsilon/2, \quad (28)
\]
\[
= \left( G(x, y, y), G(y, x, x) \right) + \left( G(x, y, y), G(y, x, x) \right) < 2\varepsilon.
\]
Letting \( n \to \infty \) in the above inequality and using (25), we get
\[
d\left( x, y \right) + d\left( y, x \right) < \varepsilon, \quad (29)
\]
which is a contradiction. Thus \( x = y \).
Example 17. Suppose that $X = \mathbb{R}$, equipped with the usual metric $d : X \times X \to [0, +\infty)$. Let $F, G : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2 - y^2}{3}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

and

$$G(x, y) = \begin{cases} \frac{y^2 - x^2}{3}, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

From $F(x, y) = G(x, y)$, $F(y, x) = G(y, x)$, we can get $(x, y) = (0, 0)$ and we have $G(F(0, 0), F(0, 0)) = F(G(0, 0), G(0, 0))$, which implies that $F$ and $G$ are generalized weakly compatible.

Now, we prove that for any $x, y \in X$, there exist $u, v \in X$ such that

$$F(x, y) = G(u, v) \text{ and } F(y, x) = G(v, u).$$

Let $(x, y)(u, v) \in X \times X$ be fixed. We consider the following cases:

Case 1: If $x = y$, then we have $F(x, y) = 0 = G(y, x)$.

Case 2: If $x > y$, then we have

$$F(x, y) = \frac{x^2 - y^2}{3} = G\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}\right)$$

and

$$F(y, x) = 0 = G\left(\frac{y}{\sqrt{3}}, \frac{x}{\sqrt{3}}\right).$$

Case 3: If $x < y$, then we have

$$F(x, y) = \frac{y^2 - x^2}{3} = G\left(\frac{y}{\sqrt{3}}, \frac{x}{\sqrt{3}}\right)$$

and

$$F(y, x) = 0 = G\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}\right).$$

Now, we shall show that the mappings $F$ and $G$ satisfy the condition (14): For each $x, y, u, v \in X \times X$, we have

$$\varepsilon \leq \frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2} \leq \varepsilon + \delta(\varepsilon).$$

Then

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \frac{1}{2} \left[\frac{x^2 - y^2}{3} - \frac{u^2 - v^2}{3} + \frac{y^2 - x^2}{3} - \frac{v^2 - u^2}{3}\right] = \frac{1}{2} \left[\frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2}\right].$$

Thus the contractive condition (14) is satisfied for all $x, y, u, v \in X$. In addition, all the other conditions of Theorem 16 are satisfied and $0$ is a unique common fixed point of $F$ and $G$.

Corollary 18. Let $(X, d)$ be a metric space. Assume $F, G : X \times X \to X$ be two generalized compatible mappings satisfying (14), (15) and $F(X \times X)$ or $G(X \times X)$ is complete. Then there exists a unique $x \in X$ such that $x = G(x, x) = F(x, x)$.

Corollary 19. Let $(X, d)$ be a metric space. Assume $F, G : X \times X \to X$ be two commuting mappings satisfying (14), (15) and $F(X \times X)$ or $G(X \times X)$ is complete. Then there exists a unique $x \in X$ such that $x = G(x, x) = F(x, x)$.

References

[1] V. Berinde, Coupled fixed point theorems for $\varphi$-contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 75 (2012), 3218-3228.
[2] V. Berinde and M. Piceara, Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces, Fixed Point Theory Appl. 2012, 2012: 115.
[3] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 68 (2006), 1379-1393.
[4] L. Ciric, B. Djranovic, M. Jleli and B. Samet, Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications, Fixed Point Theory Appl. 2012, 2012:51.
[5] B. Deshpande and A. Handa, Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations, Aft. Mat.
[6] B. Deshpande and A. Handa, Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces, Adv. Fuzzy Syst. Volume 2014, Article ID: 348696, 11 pages. 10
[7] H. S. Ding, L. Li and S. Radenovic, Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces, Fixed Point Theory Appl. 2012, 2012:96.
[8] M. E. Gordji, E. Akbartabar, Y. J. Cho and M. Ramezani, Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2012, 2012:95.
[9] N. Hussain, M. Abbas, A. Azam and J. Ahmad, Coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl. 2014, 2014: 62.
[10] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.
[11] A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326-329.
[12] M. Mursaleen, S. A. Mohiuddine and R. P. Agarwal, Coupled fixed point theorems for alpha-psi contractive type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2012, 2012:228.
[13] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508-4517.
[14] B. Samet, E. Karapinar, H. Aydi and V. C. Rajic, Discussion on some coupled fixed point theorems, Fixed Point Theory Appl. 2013, 2013:50.
[15] W. Sintunavarat, P. Kumam and Y. J. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, Fixed Point Theory Appl. 2012, 2012: 170.