QUANTUM MECHANICS ON DISCRETE SPACE AND TIME

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We propose the assumption of quantum mechanics on a discrete space and time, which implies the modification of mathematical expressions for some postulates of quantum mechanics. In particular we have a Hilbert space where the vectors are complex functions of discrete variable.

As a concrete example we develop a discrete analog of the one-dimensional quantum harmonic oscillator, using the dependence of the Wigner functions in terms of Kravchuk polynomials. In this model the position operator has a discrete spectrum given by one index of the Wigner functions, in the same way that the energy eigenvalues are given by the other matricial index.

Similar picture can be made for other models where the differential equation and their solutions correspond to the continuous limit of some difference operator and orthogonal polynomial of discrete variable.

1. INTRODUCTION

In the last Symposium on Fundamental Problems in Quantum Physics [1] I explored the hypothesis of a realistic interpretation of lattice theories based on some ontological model that presupposes some fundamental network previous to the concept of space and time. According to this model the structure of space-time is a consequence of the relations among these fundamental entities, and gives raise to a discrete character of the space and time variables.

Even the standard interpretation of quantum mechanics in our model is conserved the assumption of a discrete space and time introduces some drastic changes in the mathematical formulation of quantum mechanics. These consequences have been also shared by recent authors who use lattice models as a mathematical tool. In particular non-commutative geometry
leads in a natural way to difference operators [2]. Quantum groups and q-
analysis implies a deformation of space-time groups with a q-casimir defined
on a space or time lattice [3]. Lattice field theories are widely spread in an
attempt to overcome infinities in perturbation theories [4]. Special functions
and orthogonal polynomials on continuous variable are studied together with
those of discrete variable [5]. Non-euclidean crystallography requires the use
of discrete groups of hyperbolie type that has been recently developped [6].
Recent literature advocates for the application of discrete models to unify
general relativity and quantum mechanics as Wheeler, Ponzano and Regge
and others have proposed [7]. For this purpose some modern tools are used,
such as discrete topology and partition functions defined over symplicial
networks.

2. THE POSTULATES OF QUANTUM MECHANICS ON $Z^4$

The assumption of discrete space and time imposes some restrictions
on the mathematical expressions for the postulates of Quantum Mechan-
ics. The question is now whether this assumption keeps the analogy with
standard formulation in the continuum case (in the limit both formulations
coincide) and at the same time avoids the unwanted infinities. The answer
is, generally speaking, in the affirmative.

In order to be more explicit, we start with the Hilbert space: this must
be defined over the gaussian numbers (complex of integer components) with
scalar products constructed with summation instead of integrals

$$\sum_{j=a}^{b} f_j^* g_j$$

where we use the notation $f_j \equiv f(j\varepsilon)$, $\varepsilon$ being the fundamental length of
the one-dimensional lattice, $j$ integer numbers.

As an example, we take the following orthonormal basis

$$f_j(k_m) = \frac{1}{\sqrt{N}} \left(1 + \frac{j \varepsilon k_m}{1 - \frac{j \varepsilon k_m}{2}}\right)^j, \quad j = 0,1,\cdots N - 1$$

where

$$k_m \equiv \frac{2}{\varepsilon} \tan \frac{\pi m}{N}, \quad m = 0,1,\cdots N - 1$$

satisfying

$$\sum_{j=0}^{N-1} f_j^* (k_m) f_j (k_{m'}) = \delta_{mm'}$$
with respect to which a finite Fourier transform (equivalent to a Fourier series) can be defined

\[ F_j = \sum_{m=0}^{N-1} a_m f_j(k_m), \quad a_m = \sum_{j=0}^{N-1} f_j^*(k_m) F_j \]

Notice that although the space-time variables are discrete the functions can be continuous.

If we consider observables, the corresponding operators must be self adjoint with respect to the scalar product mentioned before, and the spectrum is always discrete. As an example, we give the eigenvalues and eigenfunctions of the position and momentum operators.

\[ X : \quad j \delta_{j \ell} = \ell \delta_{j \ell}, \quad \ell \text{ fix} ; \quad j = 0, 1, \cdots N - 1 \]

\[ P : \quad -i \frac{1}{\varepsilon} \Delta_j f_j(k_m) = \frac{k_m}{1 - \frac{1}{2} \varepsilon k_m} f_j(k_m) \]

where \( \delta_{j \ell} \) is the Kronecker function and \( f_j(k_m) \) is defined as before.

With the help of the scalar product we can defined espection values, projection operators, density matrix, uncertainties or mean values of some operators.

Suppose now that a physical system is represented by a state vector \( \psi(t) \) depending on discrete time \( (t = n \tau) \). If \( H \) is the operator corresponding to the hamiltonian of the system (for simplicity we take \( H \) constant in time) the Schrödinger equation for that system is

\[ \frac{i}{\tau} \Delta_n \psi_n = H \tilde{\Delta}_n \psi_n \]

the solution of which is

\[ \psi_n = \left( \frac{1 - \frac{1}{2} i \tau H}{1 + \frac{1}{2} i \tau H} \right)^n \psi_0 \]

with the initial condition \( \psi_0 \).

Here \( \Delta_n \) is the forward difference operator \( \Delta_n \psi_n = \psi_{n+1} - \psi_n \) and \( \tilde{\Delta}_n \) the mean operator \( \tilde{\Delta}_n \psi_n = \frac{1}{2} (\psi_{n+1} + \psi_n) \)

As in the continuous case we can define an unitary evolution operator

\[ U_n \equiv \left( \frac{1 - \frac{1}{2} i \tau H}{1 + \frac{1}{2} i \tau H} \right)^n \]

which is unitary (because \( H \) is self adjoint) and satisfies the difference equation

\[ \frac{i}{\tau} \Delta_n U_n = H \tilde{\Delta}_n U_n \]
If we use the Heisenberg picture the evolution of some operators in time is given by

\[ A_n = U_n^+ A_0 U_n \]  \hspace{1cm} (1)

where \( A_n \) is some operators depending on discrete time and \( U_n \) is the evolution operator defined before. The Heisenberg equation now reads [8]

\[ \frac{i}{\tau} \Delta A_n = \frac{1}{1 - \frac{i}{2} \tau H} [A_n, H_n] \frac{1}{1 + \frac{i}{2} \tau H} \]

the solution of which is (1).

An other scheme may be used if we take the symmetric difference operator

\[ \delta_n A_n \equiv \left( A_{n+\frac{1}{2}} - A_{n-\frac{1}{2}} \right) , \]

then

\[ \frac{i}{\tau} \delta_n A_n = \frac{1}{\left( 1 + \frac{1}{4} \tau^2 H^2 \right)^{\frac{3}{2}}} [A_n, H_n] \frac{1}{\left( 1 + \frac{1}{4} \tau^2 H^2 \right)^{\frac{3}{2}}} \]

Also we have

\[ \frac{i}{\tau} (A_{n+1} - A_{n-1}) = \frac{1}{1 + \frac{1}{4} \tau^2 H^2} \left[ \left( 1 - \frac{\tau^2}{4} \right) [A_n, H_n] \frac{1}{1 + \frac{1}{4} \tau^2 H^2} \right] \]

Some simplification can be achieved if we take the particular case \( H^2 = 1 \). It can be easily proved the following equation for the operator in the Heisenberg picture:

\[ \frac{i}{\tau} \Delta_n A_n = \frac{1}{\left( 1 + \frac{\tau^2}{4} \right)} \left( \frac{1 - \frac{i}{2} \tau H}{1 + \frac{3}{2} \tau H} \right) [A_n, H] \]

\[ \frac{i}{\tau} \nabla_n A_n = \frac{1}{\left( 1 + \frac{\tau^2}{4} \right)} \left( \frac{1 + \frac{i}{2} \tau H}{1 - \frac{3}{2} \tau H} \right) [A_n, H] \]

\[ \left( \frac{i}{\tau} \right)^2 \Delta_n \nabla_n A_n = \frac{1}{\left( 1 + \frac{\tau^2}{4} \right)} \left[ [A_n, H], H \right] \]

\[ \frac{i}{\tau} \delta_n A_n = \frac{1}{\left( 1 + \frac{\tau^2}{4} \right)^2} [A_n, H] \]

\[ \frac{i}{\tau} (A_{n+1} - A_{n-1}) = \frac{2 \left( 1 - \frac{\tau^2}{4} \right)}{\left( 1 + \frac{\tau^2}{4} \right)^2} [A_n, H] \]
In the last three equations the dependence on the Hamiltonian operator $H$ is linear as in the continuous case.

The realization of operators in the coordinate or position representation is given by the substitution

$$X \rightarrow j \varepsilon, \quad P \rightarrow -\frac{i}{\varepsilon} \Delta_j$$

The substitution is not unique. We will discuss in the next section some different realization for the position and momentum operator of the harmonic oscillator.

3. QUANTUM HARMONIC OSCILLATOR OF DISCRETE VARIABLE

The quantum harmonic oscillator is described by the Schrödinger equation

$$\frac{\hbar \omega}{2} \left[ -\frac{d^2}{d\xi^2} + \xi^2 \right] \psi(\xi) = \lambda \psi(\xi)$$

with $\omega \equiv \sqrt{\frac{k}{M}}$, $\xi = \alpha s$, $\alpha \equiv \sqrt{\frac{M\omega}{\hbar}}$, $\lambda = \frac{2E}{\hbar \omega}$

For simplicity, we take $\alpha = 1$

The normalized solutions are

$$\psi_n(s) = \left( \frac{n^{1/2} 2^n n!}{\pi^{1/2}} \right)^{-1/2} e^{-s^2/2} H_n(s), \quad n = 0, 1, 2, \cdots$$

where $H_n(s)$ are the Hermite polynomials.

The $\psi_n(s)$ are eigenfunctions corresponding to the eigenvalues $\lambda = 2n + 1$, and they satisfy the following recurrence relations:

i) $2s \psi_n(s) = \sqrt{2(n + 1)} \psi_{n+1}(s) + \sqrt{2n} \psi_{n-1}(s)$

ii) $\frac{d}{ds} \psi_n(s) = -\sqrt{2(n + 1)} \psi_{n+1}(s) + \sqrt{2n} \psi_{n-1}(s)$

From these two relations one defines the creation and annihilation operators:

$$a^\dagger \psi_n(s) \equiv \frac{1}{\sqrt{2}} \left( s - \frac{d}{ds} \right) \psi_n(s) = \sqrt{n + 1} \psi_{n+1}(s)$$

$$a \psi_n(s) \equiv \frac{1}{\sqrt{2}} \left( s + \frac{d}{ds} \right) \psi_n(s) = \sqrt{n} \psi_{n-1}(s)$$
It is well known that the Hermite polynomials are the continuous limit of the Kravchuk polynomials of discrete variables $k_n(x)$ and the weight function of the Hermite polynomials is the continuous limit of the binomial distribution $\rho(x)$ which in turns is the weight function of the Kravchuk polynomial [9]. But the product of the Kravchuk polynomials time their weight function is proportional to the Wigner functions $d^j_{mm'}(\beta)$ that appear in the generalized spherical functions

$$D^j_{mm'}(\alpha, \beta, \gamma) = \exp(-im\alpha)d^j_{mm'}(\beta)\exp(-im'\gamma),$$

namely,

$$d^j_{mm'}(\beta) = (-1)^{m-m'}\frac{\rho(x)}{d_n}k_n^{(p)}(x, N)$$

where $d_n$ is some normalization constant, and $n = j-m$, $x = j-m'$, $p = \sin^2(\beta/2)$.

This connection between the functions of discrete and continuous variable suggests that the solution of the quantum harmonic oscillator are the continuous limit, up to a factor, of the Wigner functions. In order to prove this Ansatz we compare the recurrence relations of the two types of functions as it is done for the orthogonal polynomials of discrete variable. In our case we take the differential equation for the Wigner function [10]

$$\pm \frac{d}{d\beta}d^j_{mm'}(\beta) + \frac{m' - m \cos \beta}{\sin \beta}d^j_{mm'}(\beta) =$$

$$= \sqrt{(j+m)(j+m+1)}d^j_{m\pm1,m'}(\beta)$$

From this we deduce two recurrence relations:

i) $2\frac{m' - m \cos \beta}{\sin \beta}d^j_{mm'}(\beta) =$

$$= \sqrt{(j-m)(j+m+1)}d^j_{m+1,m'}(\beta) +$$

$$+ \sqrt{(j+m)(j-m+1)}d^j_{m-1,m'}(\beta)$$

ii) $\sqrt{(j+m')(j-m'+1)}d^j_{m,m'-1}(\beta) -$

$$- \sqrt{(j-m)(j+m+1)}d^j_{m,m'+1}(\beta) =$$

$$= \sqrt{(j-m)(j+m+1)}d^j_{m+1,m'} -$$

$$- \sqrt{(j+m)(j-m+1)}d^j_{m-1,m'}$$
Note that the last expression has been obtained with the help of the well known property of Wigner functions

$$d_{jm}^{m'} (\beta) = (-1)^{m-m'} d_{jm}^{m'} (\beta)$$

We suppose that

$$\lim_{N \to \infty} C_n (N) d_{jm}^{m'} (\beta) = \psi_n (s) \quad (13)$$

where we take $m = j - n$, $m' = j - x$, $N = 2j$, $x = Np + \sqrt{2Npq}$ $s$, and $C_n (N)$ some normalization constant to be determined.

We compare the recurrence relations i) that is to say, formulas (3) and (11). We divide the second one by $\sqrt{j}$ and substitute $d_{jm}^{m'} (\beta)$ by $v_n (x)$, with $v_n (x) \equiv C_n (N) d_{j-n,j-x}^{j} (\beta)$.

The result is:

$$2^{j-x-(j-n)\cos \beta} \frac{v_n (x)}{C_n (N)} = \sqrt{\frac{2n (N-n+1)}{2j}} \frac{v_{n-1} (x)}{C_{n-1} (x)} +$$

$$+ \sqrt{\frac{2 (N-n) (n+1)}{2j}} \frac{v_{n+1} (x)}{C_{n+1} (x)}$$

or

$$2\left(s + \frac{(2p-1)n}{\sqrt{2Npq}}\right) \frac{v_n (x)}{C_n (N)} = \sqrt{2(n+1)} \sqrt{1 - \frac{n}{N} \frac{v_{n-1} (x)}{C_{n-1} (N)}} +$$

$$+ \sqrt{2n} \sqrt{1 - \frac{n}{N} \frac{v_{n+1} (x)}{C_{n+1} (N)}}$$

In the limit $N \to \infty$ this expression goes to the recurrence relation (3) provided $\frac{C_n (N)}{C_{n+1} (N)} = \frac{C_n (N)}{C_{n-1} (N)} = 1$, or $C_n (N) = \text{const} = 1$.

The recurrence relations ii) formulas (4) and (12) can be compared by the same method. We substitute

$$v_n (x) \equiv d_{j-n, j-x}^{j} (\beta) \quad (14)$$

in (12) and divide both sides by $\sqrt{N}$.

The result is

$$\sqrt{\frac{(N-x)}{N}} \frac{(x+1)v_n (x+1)}{N} - \sqrt{\frac{(N-x+1)}{N}} v_n (x-1) =$$

$$= \sqrt{\frac{n (N-n+1)}{N}} v_{n-1} (x) - \sqrt{\frac{(N-n)}{N}} (n+1)v_{n+1} (x)$$
Substituting \( x = Np + \sqrt{2Npq} \), and extracting \( \sqrt{2Np} \equiv \frac{1}{\hbar} \) in the left side, we obtain

\[
\frac{1}{\hbar} \left\{ \sqrt{\left(1 - \frac{x}{N}\right) \left(1 + \frac{2q}{Np}s + \frac{1}{Np}\right)} v_n(x+1) - \sqrt{\left(1 + \sqrt{2q}\right) \left(1 - \frac{x - 1}{N}\right)} v_n(x-1) \right\} =
\]

\[
= \sqrt{2n \left(1 - \frac{n - 1}{N}\right)} v_{n-1}(x) - \sqrt{2(n + 1) \left(1 - \frac{n}{N}\right)} v_{n+1}(x)
\]

In the limit \( N \to \infty, \hbar \to 0 \) this expression goes to

\[
\lim_{\hbar \to 0} \frac{1}{\hbar} \left\{ \psi_n(s + \hbar) - \psi_n(s - \hbar) \right\} = -\sqrt{2(n + 1)} \psi_n(s) + \sqrt{2n} \psi_{n-1}(s)
\]

that coincides with (4).

We can use these results to construct creation and annihilation operators for the Wigner functions. We define

\[
A^d_{mm'}(\beta) \equiv \frac{1}{\sqrt{2}} \left\{ \frac{m' - m \cos \beta}{\sqrt{j \sin \beta}} d^l_{mm'}(\beta) + \right.
\]

\[
\sqrt{\left(j + m'\right) \left(j - m' + 1\right)} d^l_{m,m' - 1}(\beta) - \sqrt{\left(j - m'\right) \left(j + m' + 1\right)} d^l_{m,m' + 1}(\beta) \right\} =
\]

\[
= \sqrt{\frac{j - m}{j + m + 1}} d^l_{m+1,m'}(\beta) \tag{15}
\]

\[
A^l_{mm'}(\beta) \equiv \frac{1}{\sqrt{2}} \left\{ \frac{m' - m \cos \beta}{\sqrt{j \sin \beta}} d^d_{mm'}(\beta) - \right.
\]

\[
\sqrt{\left(j + m'\right) \left(j - m' + 1\right)} d^d_{m,m' - 1}(\beta) + \sqrt{\left(j - m'\right) \left(j + m' + 1\right)} d^d_{m,m' + 1}(\beta) \right\} =
\]

\[
= \sqrt{\frac{j + m}{j - m + 1}} d^d_{m-1,m'}(\beta) \tag{16}
\]

Using the limit of the recurrence relations we obtain
\[
\lim_{N \to \infty} A d_{mm'}^j(\beta) = \frac{1}{\sqrt{2}} \left( s + \frac{d}{ds} \right) \psi_n(s) \equiv a \psi_n(s) \quad (17)
\]

\[
\lim_{N \to \infty} A^\dagger d_{mm'}^j(\beta) = \frac{1}{\sqrt{2}} \left( s - \frac{d}{ds} \right) \psi_n(s) \equiv a^\dagger \psi_n(s) \quad (18)
\]

Relations (15) and (16) suggest that the creation and annihilation operators are connected with the raising and lowering operators for the spherical harmonics \( Y_{jm} \). In fact, multiplying (15) and (16) by \( Y_{jm'} \) and adding for \( m' \) we have

\[
A \sum_{m'} d_{mm'}^j(\beta) Y_{jm'} = \sqrt{(j - m)(j + m + 1)} \sum_{m'} d_{m+1,m'}^j(\beta) Y_{jm'}
\]

or

\[
AY_{jm} = \frac{1}{\sqrt{2j}} \sqrt{(j - m)(j + m + 1)} Y_{j,m+1} = \frac{1}{\sqrt{2j}} J_+ Y_{jm} \quad (19)
\]

similarly,

\[
A^\dagger \sum_{m'} d_{mm'}^j(\beta) Y_{jm'} = \sqrt{(j + m)(j - m + 1)} \sum_{m'} d_{m-1,m'}^j(\beta) Y_{jm'}
\]

or

\[
A^\dagger Y_{jm} = \frac{1}{\sqrt{2j}} \sqrt{(j + m)(j - m + 1)} Y_{j,m-1} = \frac{1}{\sqrt{2j}} J_- Y_{jm} \quad (20)
\]

In order to make more transparent the connection between the creation and annihilation operators with the raising and lowering operators of the spherical harmonics, we take the commutation and anticommutation relations of the former operators.

\[
\left( AA^\dagger - A^\dagger A \right) Y_{jm} = \frac{1}{2j} (J_+ J_- - J_- J_+) Y_{jm} = \frac{1}{2j} 2 J_z Y_{jm} = \frac{m}{j} Y_{jm} = \left( 1 - \frac{n}{j} \right) Y_{jm}
\]

Substituting \( Y_{jm} = \sum_{m'} d_{mm'}^j(\beta) Y_{jm'} \) we get

\[
[A, A^\dagger] d_{mm'}^j(\beta) = \left( 1 - \frac{n}{j} \right) d_{mm'}^j(\beta) \quad (21)
\]

which in the limit \( j \to \infty \) goes to
\[ [a_a^\dagger] \psi_n(s) = \psi_n(s) \]

Similarly

\[
(AA^\dagger + A^\dagger A) \ Y_{jm} = \frac{1}{2j} (j^2 - J_0^2) \ Y_{jm} = \frac{1}{j} (j(j+1) - m^2) \ Y_{jm} = \left\{ (2n + 1) - \frac{n^2}{j} \right\} \ Y_{jm}
\]

or

\[
(AA^\dagger + A^\dagger A) \ d^j_{mm'}(\beta) = \left\{ (2n + 1) - \frac{n^2}{j} \right\} \ d^j_{mm'}(\beta) \quad (22)
\]

which in the limit \( j \to \infty \) goes to

\[
(aa^\dagger + a^\dagger a) \ Y_{jm} = \ Y_{jm} (2n + 1) \ Y_{jm} = \psi_n(s) \]

If we multiply both sides by \( \hbar \omega / 2 \) we obtain the eigenvalue equation for the hamiltonian.

The interpretation of this model can be taken from the quantum harmonic oscillator of continuous variable. The energy levels are equally distant by the amount \( \hbar \omega \) and are labelled by \( n = 0, 1, 2, \ldots \infty \). In the quantum harmonic oscillator of discrete variable we have also the discrete eigenvalues of the hamiltonian connected with the index \( m = j - n \) of the Wigner function \( d^j_{mm'}(\beta) \). These values are equally separated but finite \( (m = -j, \ldots, +j) \).

Similarly the eigenvalue of the position operator \( A + A^\dagger \) are also discrete and connected to the index \( m' = j - x \) of the Wigner functions but finite \( (m' = -j, \ldots, +j) \).

The integer numbers \( x = 0, 1, \ldots, 2j \) are related to the quantity \( x = \alpha s \) where \( s \) is the continuous variable and \( \alpha = \sqrt{M \omega / \hbar} \). Since \( x \) is a pure number and \( s \) has the dimension of a length, the spacing of the one-dimensional lattice is equal to \( 1/\alpha = \sqrt{\hbar / M \omega} \). Therefore the Planck’s constant \( \hbar \) play an role with respect to discrete space similar to the role with respect to discrete energy values.

24. CONCLUDING REMARKS

The analysis we have made for the quantum harmonic oscillator of discrete variable can be applied to other cases, where the functions involved are orthogonal polynomials of continuous variable the limit of which are some orthogonal polynomial of discrete variables; we give some examples:
1. The function \( f_j (k_m) \) described in section 2 are polynomials of discrete variable the continuous limit of which are the exponential function. We have developed a new scheme for the Klein-Gordon and Dirac field equation that can be extended to lattice gauge theories [11].

2. The solution of the Schrödinger equation for the hydrogen atom is given in term of the orthogonal Laguerre polynomials and the spherical harmonics. The radical equation can be translated into the difference equation for the Meixner polynomial of discrete variable.

3. The quantification of the electromagnetic fields leads to the D’Alambert equation the solution of which are given in terms of the Bessel spherical functions that are related to the trigonometric functions. These functions suggest the connection with the orthogonal polynomials of discrete variable, that are solutions of difference equations of the hypergeometric type. General speaking a parallel study of discrete and continuous model can be made similar to that made by the russian school of mathematicians with respect to orthogonal polynomials.

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