ROTATIONAL LINEAR WEINGARTEN SURFACES
INTO THE EUCLIDEAN SPHERE

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Abstract. The aim of this paper is to present a complete description of all rotational linear Weingarten surface into the Euclidean sphere $S^3$. These surfaces are characterized by a linear relation $aH + bK = c$, where $H$ and $K$ stand for their mean and Gaussian curvatures, respectively, whereas $a, b$ and $c$ are real constants.

Key words: Rotational surfaces, Linear Weingarten surfaces.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The study of Weingarten’s surface $M^2$ into the Euclidean space $\mathbb{R}^3$ remount to classical works developed around the middle of nineteenth century by Weingarten contained in the papers [9] and [10]. Essentially these surfaces are a natural generalization of one with constant curvature, more precisely, they satisfy a relation $W(k_1, k_2) = 0$, where $k_1$ and $k_2$ stand for the principal curvatures of the surface while $W$ is a smooth function defined over the Euclidean space $\mathbb{R}^2$, distinguishing when $W(k_1, k_2) = f(H^2 - K)$, where $H$ and $K$ denote, respectively, the mean and the Gaussian curvatures of $M^2$. We point out that replacing the Euclidean space $\mathbb{R}^3$ either by the Euclidean sphere $S^3$ or by the hyperbolic space $H^3(-1)$ we have the same definition. In the later case the work due to Bryant [2] when $f(H^2 - K) = c$, for a constant $c$, retake this subject after a long delay, as well as works due to Rosenberg and Sa Earp [8]. When the ambient space is the Euclidean sphere $S^3$ the case of rotational surfaces was described by Dajczer and do Carmo [3] only for constant mean curvature. In recent works Almeida et al. [1] and Li et al. [4] obtained some results for Weingarten surfaces of the Euclidean three sphere. For the special case $U(H, K) = 0$, where $U$ is an affine function, Lopez [5] described such surfaces into the Euclidean space $\mathbb{R}^3$ with an additional requirement on the discriminant of $U$. Our purpose here is to extend this later description for a class of rotational surfaces into the Euclidean sphere $S^3$. Indeed, we shall give a special attention for such surfaces satisfying $U(H, K) = 0$, where the function satisfies

(1) \[ U(H, K) = aH + bK - c, \]
being \(a, b, c \in \mathbb{R}\). Let us call such class of surfaces as Rotational Linear Weingarten Surfaces or shortly by RLWS.

We can assume, without loss of generality, that \(c \geq 0\). Moreover, we choose \(a \neq 0\) and \(b \neq 0\), since the cases \(a = 0\) and \(b = 0\) were analyzed by Palmas in [6] and [7].

One fundamental ingredient to understand the behavior of a RLWS as well as its qualitative properties is the sign of the its discriminant which is defined according to \(\Delta = a^2 + 4bc\). In the quoted paper López [5] described RLWS of hyperbolic type \((\Delta < 0)\) in the Euclidean space \(\mathbb{R}^3\) under a suitable assumption.

Following Dajczer and do Carmo [3] we shall use the terminology of rotational surface into \(S^3\) as a surface invariant by the orthogonal group \(O(2)\) consider as a subgroup of the isometries group of \(S^3\). Hence we can consider a profile curve \(\gamma\) to describe the desired surface. Initially let us parametrize the profile curve \(\gamma\) by \(\gamma(s) = (x(s), y(s), z(s))\), with \(x(s) \geq 0\). If we choose \(\varphi(t) = (\cos t, \sin t)\) as an element in \(O(2)\) the rotational surface generated by \(\gamma\) is parametrized as follows

\[
\psi : M^2 \hookrightarrow S^3 \subset \mathbb{R}^4 \\
(s, t) \mapsto (x(s) \cos t, x(s) \sin t, y(s), z(s)).
\]

Moreover, we can choose the parameter \(s\) to be the arc length of \(\gamma\). Then using this parameter we obtain

\[
x^2(s) + y^2(s) + z^2(s) = 1, \quad \dot{x}^2(s) + \dot{y}^2(s) + \dot{z}^2(s) = 1.
\]

In order to compute the principal curvatures of a rotational surface \(M^2 \subset S^3\) we remember a fundamental lemma due to Dajczer and do Carmo [3].

**Lemma 1** (Dajczer-do Carmo). Let \(M^2\) be a rotational surface of \(S^3\) under the above choices. Then its principal curvatures \(k_1\) and \(k_2\) are given by

\[
k_1 = -\frac{\sqrt{1 - x^2 - \dot{x}^2}}{x} \quad \text{and} \quad k_2 = \frac{\dot{x} + x}{\sqrt{1 - x^2 - \dot{x}^2}}.
\]

With this setting we present the fundamental relation which characterizes a RLWS in the Euclidean sphere \(S^3\):

\[
(2) \quad \frac{a}{2} x \sqrt{1 - x^2 - \dot{x}^2} + \frac{b}{2} \left(x^2 + \dot{x}^2\right) + \frac{c}{2} x^2 = \alpha,
\]

where \(\alpha\) is a constant.

Let us denote by \(M_\alpha\) the RLWS associated with the function \(x\), solution of the equation (2) and the parameter \(\alpha\). Moreover, let us consider the special value

\[
\alpha_0 = \frac{\sqrt{a^2 + (b + c)^2}}{4} + \frac{b + c}{4}.
\]

**Theorem 1.** Let \(M_\alpha\) be a RLWS with \(a > 0\) and \(\Delta \neq 0\). Then we have:

1. \(\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]\);
2. There are no complete immersed RLWS \(M_\alpha \subset S^3\) that such

\[
\alpha \in \left(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\right) \cup \left(\frac{b}{2}, \frac{b + c}{2}\right);
\]
3. For any \(\alpha \in (\max\{0, \frac{b + c}{2}\}, \alpha_0)\), \(M_\alpha\) is a complete immersed RLWS in \(S^3\);
4. There is only one complete immersed RLWS (Clifford torus) in \(S^3\) that such \(\alpha = \alpha_0\).

Finally we prove the the following result.
Theorem 2. There is a family of complete immersed RLWS in $\mathbb{S}^3$ that does not contain isoparametric surfaces.

2. Preliminaries and basic results

From now on we shall choose the discriminant $\Delta \neq 0$ and $a > 0$. An analogous analysis can be made for the case $a < 0$. First of all we begin this section by proving a lemma that establishes the fundamental relation (2).

Lemma 2. A surface $M^2 \subset S^3$ is RLWS if, and only if, the function $x$ satisfies the following differential equation:

$$\frac{a}{2} x \sqrt{1 - x^2 - \dot{x}^2 + \frac{b}{2}(x^2 + \dot{x}^2)} + \frac{c}{2} x^2 = \alpha,$$

where $\alpha$ is a constant.

Proof. Taking into account that $aH + bK = c$ we use Lemma 1 to arrive at

$$\frac{a}{2} \left(\frac{\ddot{x} + x}{\sqrt{1 - x^2 - \dot{x}^2}} - \sqrt{1 - x^2 - \dot{x}^2} \frac{x}{x^2} - b \cdot \frac{\dot{x} + x}{x} \right) = c.$$

Now, note that

$$- \frac{d}{ds} \left(\frac{a}{2} x \sqrt{1 - x^2 - \dot{x}^2 + \frac{b}{2}(x^2 + \dot{x}^2)}\right) =$$

$$x \ddot{x} \left[\frac{a}{2} \left(\frac{\ddot{x} + x}{\sqrt{1 - x^2 - \dot{x}^2}} - \sqrt{1 - x^2 - \dot{x}^2} \frac{x}{x^2} - b \cdot \frac{\dot{x} + x}{x} \right) - b \cdot \frac{\dot{x} + x}{x} \right].$$

Therefore, the function $x$ satisfies the equation (3) if, and only if,

$$\frac{a}{2} x \sqrt{1 - x^2 - \dot{x}^2 + \frac{b}{2}(x^2 + \dot{x}^2)} + \frac{c}{2} x^2 = \alpha,$$

where $\alpha \in \mathbb{R}$ finishing the proof of the lemma.

Definition 1. A solution of (2) is complete if either $x$ is defined for all $s \in \mathbb{R}$ or if the pair $(x, \dot{x})$ admits only $(0, \pm 1)$ as limit values.

When $(x, \dot{x})$ has $(0, 1)$ or $(0, -1)$ as limit value, we deduce that the profile curve meets orthogonally the axis of rotation. Therefore, complete solutions of the equation (2) give rise to a complete RLWS.

In order to describe the behavior of a solution of equation (2) we follow the techniques contained in the next paper [6] due to Palmas. Initially we note that a local solution $x$ of the equation (2) paired with its first derivative $(x, \dot{x})$, is contained on a level curve of the function $F : D \rightarrow \mathbb{R}$ defined by

$$F(u, v) = \frac{a}{2} u \sqrt{1 - u^2 - v^2} + \frac{b}{2}(u^2 + v^2) + \frac{c}{2} u^2,$$

where $D = \{(u, v) \in \mathbb{R}^2 : u \geq 0 \text{ and } u^2 + v^2 \leq 1\}$.

Lemma 3. Let $\mathcal{P} := \{(u, v) \in \text{int}(D) : \frac{\partial F}{\partial u}(u, v) = \frac{\partial F}{\partial v}(u, v) = 0\}$ be the set of critical points of $F$ contained in the interior of $D$. Then we have:

(i) $\mathcal{P} = \{(u_+, 0)\} \iff b + c \geq 0$;
(ii) $\mathcal{P} = \{(u_-, 0)\} \iff b + c \leq 0$,

where $u_\pm^2 = \frac{1}{2} \left(1 \pm \sqrt{\frac{(b+c)^2}{a^2+\frac{(b+c)^2}{a^2}}} \right)$.
Proof. Straightforward calculations yield
\[
\frac{\partial F}{\partial u} = \frac{a}{2} \frac{1 - u^2 - v^2}{\sqrt{1 - u^2 - v^2}} - a \frac{u^2}{2\sqrt{1 - u^2 - v^2}} + (b + c)u; \\
\frac{\partial F}{\partial v} = \frac{-a}{2} \frac{uv}{\sqrt{1 - u^2 - v^2}} + bv = \left( -a \frac{u}{2\sqrt{1 - u^2 - v^2}} + b \right) v.
\]

For \((u, v) \in \mathcal{P}\) we affirm that \(-a \frac{u}{2\sqrt{1 - u^2 - v^2}} + b \neq 0\). Otherwise from \(\frac{\partial F}{\partial u} = 0\) we have
\[
\frac{a}{2} \frac{1 - u^2 - v^2}{\sqrt{1 - u^2 - v^2}} + cu = 0.
\]

Hence we conclude that \((a^2 + 4bc)u = \Delta \cdot u = 0\). Since \(\Delta \neq 0\) and \((u, v) \in int(D)\) we arrive at a contradiction. Therefore, \(v = 0\) and
\[
\frac{a}{2} \frac{1 - u^2 - v^2}{\sqrt{1 - u^2 - v^2}} + (b + c)u = 0.
\]

This is equivalent to
\[
a(1 - 2u^2) = -2(b + c)\sqrt{1 - u^2}.
\]

Moreover, the solutions of the equation (4) are also solutions of the equation below
\[
u^4 - u^2 + \frac{a^2}{4[a^2 + (b + c)^2]} = 0.
\]

The solutions of equation (5) are \(u^2_\pm = \frac{1}{2} \left( 1 \pm \sqrt{\frac{4}{3} \frac{(b+c)^2}{a^2 + (b+c)^2}} \right)\). Taking into account that \(\frac{1}{u_+} \cdot \frac{\partial F}{\partial \alpha}(u_+, 0) = (b + c) - |b + c|\) and \(\frac{1}{u_-} \cdot \frac{\partial F}{\partial \alpha}(u_-, 0) = (b + c) + |b + c|\), we conclude:
- \(\frac{\partial F}{\partial \alpha}(u_+, 0) = 0 \iff b + c \geq 0;\)
- \(\frac{\partial F}{\partial \alpha}(u_-, 0) = 0 \iff b + c \leq 0.\)

This completes the proof of the lemma. \(\square\)

In what follows, let us denote by \(C_\alpha = \{(u, v) \in D : F(u, v) = \alpha\}\) the level curves of the function \(F\) as well as \(\alpha_\pm := F(u_\pm, 0)\). The next lemma enables us to determine the minimum level as well as the maximum level of \(F\).

Lemma 4. Under the previous assumptions the following results hold:

(i) If \(b + c \leq 0\), then \(\alpha \in [\frac{b}{2}, \alpha_0]\) and \(F^{-1}(\alpha_0) = \{(u_-, 0)\};\)
(ii) If \(b + c \geq 0\), then \(\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]\) and \(F^{-1}(\alpha_0) = \{(u_+, 0)\}.\)

Proof. We start analyzing the function \(F\) on the sets \(X = D \cap \{u = 0\}\) and \(Y = D \cap S^1\). On the former case we have \(F(u, v) = \frac{b}{2}v^2\) while on the later one \(F(u, v) = \frac{b}{2} + \frac{c}{2}u^2\). Now, if \(b + c \leq 0\) we get \(b < 0 \leq c\), then \(\min F = \frac{b}{2}\) and \(\max F = 0 < \alpha_- = \alpha_0\). Therefore \(\min F = \frac{b}{2}\), \(\max F = \alpha_0\) and \(F^{-1}(\alpha_0) = \{(u_-, 0)\}\) because \((u_-, 0)\) is the only critical point of \(F\) in \(int(D)\). Now, if \(b + c \geq 0\) Lemma 3 yields that \((u_+, 0)\) is the only critical point of \(F\) in \(int(D)\). Thereby, we have two possibilities to consider:
\[\frac{\partial F}{\partial u}\] vanishes on the set

\[\Gamma \equiv \{(u, v) \in \text{int}(D) : 1 - u^2 - v^2 = \frac{r^2}{a^2}u^2\},\]

where \(r = \sqrt{a^2 + (b+c)^2} - (b+c)\).

**Proof.** From the expression of the partial derivatives found in the proof of the Lemma

we deduce that \(\frac{\partial F}{\partial u} = 0\) if, and only if,

\[\frac{a}{2} \sqrt{1 - u^2 - v^2} - a \frac{u^2}{2\sqrt{1 - u^2 - v^2}} + (b+c)u = 0.\]

We can suppose that \(u \neq 0\), since \((u,v) \in \text{int}(D)\). Then \((u,v)\) satisfies the relation (7) if, and only if,

\[at^2 + 2(b+c)t - a = 0,\]

where \(t = \sqrt{1 - u^2 - v^2} \frac{u}{a}\). Since its roots are \(t_\pm = \frac{-(b+c)\pm \sqrt{a^2 + (b+c)^2}}{a}\) and \(t_- < 0\)
we deduce \(t_+ = \frac{\sqrt{1 - u^2 - v^2} u}{a} = \frac{r}{a}\), which is equivalent to \((u,v) \in \Gamma\). \(\square\)

Geometrically, the points of the curve \(\Gamma\) are the points where the level curves have tangent vector parallel to the axis \(u\).

**Remark 1.** Analyzing the cases \(b+c \leq 0\) and \(b+c \geq 0\) we conclude that: \(b+c \leq 0 \Rightarrow (u_-, 0) \in \Gamma\) whereas \(b+c \geq 0 \Rightarrow (u_+, 0) \in \Gamma\).

**Lemma 6.** Under the previous notations the items below are valid.

(i) \(C_\alpha \cap \Gamma \neq \emptyset \Leftrightarrow \frac{b}{2} < \alpha \leq \alpha_0\). Moreover, if \(\alpha \in (\frac{b}{2}, \alpha_0)\) then \(C_\alpha \cap \Gamma\) has only two elements;

(ii) \((u,v) \in C_\alpha \cap \{u = 0\} \Leftrightarrow b \cdot v^2 = 2\alpha;\)

(iii) \((u,v) \in C_\alpha \cap S^1 \Leftrightarrow c \cdot u^2 = 2\alpha - b\).

**Proof.** By Lemma 5 it follows that, \((u,v) \in C_\alpha \cap \Gamma\) if, and only if,

\[\frac{a}{2} \sqrt{1 - u^2 - v^2} + \frac{b}{2} \left(1 - \frac{\tau^2}{a^2}\right) u^2 + \frac{c}{2} u^2 = \alpha \Leftrightarrow\]

\[\left(\tau - \frac{b}{a^2}\tau^2 + c\right) u^2 = 2\alpha - b.\]

Since \(\tau = \frac{a^2}{\sqrt{a^2 + (b+c)^2} + (b+c)}\) we deduce \(\tau > \frac{b}{a^2}\tau^2\) otherwise

\[b\tau < \tau \left(\sqrt{a^2 + (b+c)^2} + (b+c)\right) = a^2 \leq b\tau.\]
Then \((u, v) \in C_\alpha \cap \Gamma\) if, and only if, \(2\alpha > b\) which yields the first item. While the second one is an immediate consequence of the equality \(F(0, v) = \alpha\). Now observe that \((u, v) \in C_\alpha \cap S^1\) if, and only if, \(u^2 + v^2 = 1\) and \(F(u, v) = \alpha\). Using the function \(F\) we conclude the item (iii).

\[\square\]

3. Main result

Next we characterize the level curves of the function \(F\).

**Proposition 1.** (Level Curves) The level curves \(C_\alpha\) of the function \(F\) satisfy:

1. If \(\alpha \in (\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\})\), then \(C_\alpha\) intersects \(\{(0, v) : -1 < v < 1\}\) at two different points. Moreover, \(C_\alpha \cap \{u = 0\} = \{(0, \pm 1)\}\), \(C_\alpha \cap \{u = 0\} = \{(0, 0)\}\), \(b > 0\) implies \(C_\alpha = \{(0, 0)\}\) and \(b < 0\) implies \(C_\alpha = \{(0, \pm 1)\}\);
2. If \(\left(\frac{b}{2}, \frac{b+c}{2}\right)\), then the level curve \(C_\alpha\) intersects \(S^1_+ = \{(u, v) : u^2 + v^2 = 1\}\) at two different points. Moreover, \(c = 0\) implies \(C_\alpha = S^1_+\) and \(C_\alpha \cap \{u = 0\} = \{(0, \pm 1)\}\) and \(C_\alpha \cap \{u = 0\} = \{(1, 0)\}\);
3. For any \(\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)\), we get \(C_\alpha \cap \{u = 0\} = \emptyset\) and \(C_\alpha \cap S^1_+ = \emptyset\);
4. If \(|b+c| = \pm(b+c)\), then \(C_\alpha = \{(u, 0)\}\).

**Proof.** We note that items 1, 2 and 3 are a direct consequence of item 2 and 3 of Lemma [6]. The item (4) follows directly from Lemma [4] which completes the proof of the proposition. \(\square\)

**Corollary 1.** Under the previous assumptions the following results hold:

1. If \(\alpha \in (\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}) \cup \left(\frac{b}{2}, \frac{b+c}{2}\right)\), then the level curve \(C_\alpha\) is not complete;
2. If \(\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)\), then \(C_\alpha\) is a smooth, simple closed curve.

**Proof.** If \(\alpha \in (\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}) \cup \left(\frac{b}{2}, \frac{b+c}{2}\right)\), we get by Proposition [1] that the level curve \(C_\alpha\) is not defined for all \(s \in \mathbb{R}\). Therefore, \(C_\alpha\) is not complete. This proof the item (1). Follows directly from Proposition [1][item (3)] that \(C_\alpha\) is a smooth, simple closed curve. \(\square\)

**Proof of the Theorem [7]** We follow the numbering is accordance with the statements of the theorem.

1. Follows directly from Lemma 4 that \(\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]\);
2. If the function \(x\) satisfies \(F(x, \dot{x}) = \alpha\) and \(\alpha \in (\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}) \cup \left(\frac{b}{2}, \frac{b+c}{2}\right)\), we get by Corollary [1] that \(x\) is not defined for all \(s \in \mathbb{R}\). Therefore, the RLWS associated is not complete;
3. Next we note that item 2 of Corollary [1] yield: if \(F(x, \dot{x}) = \alpha\) and \(\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)\) then \(x\) is defined for all \(s \in \mathbb{R}\). Thereby, the RLWS associated is complete;
4. If \(x\) is such that \(F(x, \dot{x}) = \alpha_0\), then \(\dot{x} = 0\) and \(x = u_\pm\). Therefore, the RLWS associated is a Clifford torus, which completes the proof of the desired theorem. \(\square\)
Figure 1. $b + c < 0$ and $b + c = 0$, respectively.

Figure 2. $b + c > 0$: $b < 0$ and $b > 0$, respectively.

Figure 3. $c = 0$: $b < 0$ and $b > 0$, respectively.
In order to prove Theorem 2, we shall need the following lemma.

**Lemma 7.** Let $x$ be the solution of equation (2) such that $x(s) \neq 0$ and $\dot{x}(s) \neq 0$, $\forall s \in \mathbb{R}$. If $c = 0$ and $k_1$ is constant, then $\alpha = \frac{a}{b}$. 

**Proof.** By Lemma 1 we get $\alpha = \frac{a}{b}$. Let $x$ be the solution of equation (2) such that $x(s) \neq 0$ and $\dot{x}(s) \neq 0$, $\forall s \in \mathbb{R}$. If $k_1 = 0$, we have that $x^2 + \dot{x}^2 = 1$. It follows that $F(x, \dot{x}) = \frac{b}{2}$. Now suppose $k_1 \neq 0$. In this case, $-ak_1x^2 + b(x^2 + \dot{x}^2) = 2\alpha$. Differentiating this equality we obtain
\[
-2ak_1x\ddot{x} + 2b(x + \dot{x})\dot{x} = 0 \iff -2ak_1x + 2b(x + \dot{x}) = 0
\]
\[
\iff -a + b\frac{x + \ddot{x}}{k_1x} = 0 \iff -a - bk_2 = 0 \iff k_2 = -\frac{a}{b}.
\]

It follows from the expression of $k_2$ that $\sqrt{1 - x^2 - \dot{x}^2} = \frac{b}{2}x + \beta$, where $\beta \in \mathbb{R}$. Thus, $k_1 = -\frac{a}{b} - \frac{\beta}{2}$. As $k_1$ is constant, we deduce that $\beta = 0$ as well as $k_1 = k_2 = -\frac{a}{b}$. Therefore,
\[
F(x, \dot{x}) = \frac{a}{2}x\sqrt{1 - x^2 - \dot{x}^2} + \frac{b}{2}(x^2 + \dot{x}^2)
\]
\[
= -\frac{a}{2}k_1x^2 + \frac{b}{2}(1 - k_1^2x^2) = \frac{b}{2},
\]
which finishes the proof of lemma. $\square$

Finally, we shall prove the Theorem 2.

**Proof of the Theorem 2** If $x$ is solution of equation (2) such that $F(x, \dot{x}) = \alpha$ and $\alpha \in (\max\{0, \frac{b}{2}\}, \alpha_0)$, it follows from Proposition 1 that $(x, \dot{x})$ is a smooth, simple closed curve and $x(s) \neq 0$ $\forall s \in \mathbb{R}$. Thereby, Lemma 6 enables us to suppose, without loss of generality, that $\dot{x}(s) \neq 0$ $\forall s \in \mathbb{R}$. Therefore, it follows from Lemma 7 that when $c = 0$ the RLWS associated with $x$ is not isoparametric. Moreover, by Theorem 1 we deduce that such surfaces are complete and immersed. This completes the proof of the theorem. $\square$

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