Renormalization of chiral two pion exchange NN interactions with \(\Delta\)-excitations: correlations in the partial wave expansion

M. Pavon Valderrama
Institut für Kernphysik and Jülich Center for Hadron Physics, Forschungszentrum Jülich, 52425 Jülich, Germany

E. Ruiz Arriola
Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, E-18071 Granada, Spain.

(Dated: April 26, 2011)

In this work we consider the renormalization of the chiral two-pion exchange potential with explicit \(\Delta\)-excitations for nucleon-nucleon scattering at next-to-leading (NLO) and next-to-next-to-leading order (\(N^2\)LO). Due to the singular nature of the chiral potentials, correlations between different partial waves are generated. In particular we show that two-body scattering by a short distance power like singular attractive interaction can be renormalized in all partial waves with a single counterterm, provided the singularities are identical. A parallel statement holds in the presence of tensor interactions when the eigenpotentials in the coupled channel problem also coincide. While this construction reduces the total number of counterterms to eleven in the case of nucleon-nucleon scattering with chiral two-pion exchange interactions with \(\Delta\) degrees of freedom, the differences in the scattering phases as compared to the case with the uncorrelated partial wave renormalization become smaller as the angular momentum is increased in the elastic scattering region.

PACS numbers: 03.65.Nk,11.10.Gh,13.75.Cs,21.30.Fe,21.45.+v
Keywords: Potential Scattering, Renormalization, Two Pion Exchange, Chiral symmetry, Singular Potential

I. INTRODUCTION

The basic and fundamental problem of nuclear physics is the determination of the nucleon-nucleon (NN) interaction \([1]\). Field theoretical approaches to the nuclear force state that the NN potential can be expressed as a sum of increasingly heavy meson exchange contributions (for reviews see e.g. \([2,3]\) and references therein). The resulting high-quality potentials describe neutron-proton and proton-proton scattering data with a \(\chi^2/d.o.f. \leq 1\) \([4–7]\). They all include at large distances the charge dependent one pion exchange (OPE) potential and typically need of the order of 40 parameters for parametrizing the shorter range components of the interaction. On the contrary, quantum chromodynamics (QCD), the underlying fundamental theory of the strong interaction, requires only two parameters in the isospin symmetric limit: \(\Lambda_{\text{QCD}}\) and the average up and down quark masses. These fundamental QCD parameters can be traded for experimentally accessible observables such as \(f_\pi\), the pion weak decay constant, and \(m_\pi\), the averaged pion mass. Obviously, the large number of parameters needed in phenomenological approaches arises within very specific schemes and functional forms. It is not clear whether this number of parameters can effectively be reduced by invoking relevant QCD features while maintaining the quality of the description at the same time. Ultimately, lattice \textit{ab initio} calculations of the NN potential, for which incipient results already exist \([8,9]\), will eventually solve the problem.

In the present paper we deal with a situation where a reduction of parameters arises within the context of the renormalization of the chiral potentials deduced from the effective field theory (EFT) approach suggested by Weinberg \([10]\) two decades ago (for comprehensive reviews see Refs. \([11,12]\)). These chiral potentials turn out to be singular interactions which exhibit an inverse power law behaviour \(\sim 1/r^n\) at distances below the pion Compton wavelength, \(m_\pi r \ll 1\). In case they are attractive the resulting amplitudes are sensitive to short distance physics and require renormalization. As will be shown, the renormalizability of singular attractive potentials can be translated into a mathematical short distance constraint on the scattering amplitude for different partial waves. In the simplifying case of two-body scattering by a central attractive singular potential only one counterterm is needed in order to renormalize all the partial waves. In the more complex case of NN scattering, where spin, isospin dependence and tensor forces are present, this number can rise to eleven counterterms provided that certain conditions are met. We review below the EFT approach from the perspective of the number of parameters in a way that our results can easily be displayed.

The main appeal of the EFT idea in nuclear physics lies in the promise of a model independent approach where the long and short range contributions to observables can be disentangled with the aid of a sensible hierarchy, eluding the ubiquitous problem of fine tunings. This
feature is explicitly displayed through the introduction of counterterms in the effective Lagrangian, which encode the underlying but unresolved short distance physics and are organized according to a power counting. The standard EFT formulation of the nuclear forces exploits the spontaneous breakdown of chiral symmetry, which requires derivative couplings for the pion. This implies that higher pion exchanges are power suppressed at momenta \( p \sim (4 \pi f_{\pi}, M_N) \sim 1 \text{GeV} \). On the contrary, at high virtual momenta, pion exchanges become large, eventually requiring a suitable renormalization through the introduction of counterterms. Furthermore, the non-perturbative nature of the NN interaction makes using a sensible resummation of diagrams mandatory. A simple and effective method already suggested by Weinberg \([10,13]\) and implemented for the first time by Ray, Ordóñez and van Kolck \([14,15]\) proceeds in two steps: first, one deduces a chiral nuclear potential and afterwards solves the corresponding Lippmann-Schwinger equation in momentum space, or equivalently, the Schrödinger equation in coordinate space. This scheme complies to the familiar and widely accepted concept of nuclear potential, which in the chiral case can be organized as an expansion in powers of \( Q \). \[ V(r) = V_X^{(0)}(r) + V_X^{(2)}(r) + V_X^{(3)}(r) + V_X^{(4)}(r) + \mathcal{O}(Q^5) \], \[ V(r) = \frac{1}{\Lambda^2 r^{2r+3}} \], \[ V(r) = \frac{1}{\Lambda^2 r^{2r+3}} \]

where \( r \) represents the order in the chiral expansion \([16,18,26]\).

A stringent constraint follows from the natural requirement of short distance insensitivity: physics not explicitly taken into account should be under control by fixing a sufficient amount of low energy parameters. Such a condition represents the basis of the renormalization process as understood in the present work. To fulfill this goal it is necessary to achieve approximate cut-off independence over a certain cut-off region. This immediately raises the question of what cut-off values can be regarded as natural and how much the a priori arbitrary cut-off can be varied. The shortest de Broglie wavelength probed in elastic NN scattering below pion production threshold is \( \lambda_{\text{min}} \sim 0.5 \text{fm} \), and thus we might expect stable results for similar short distance cut-offs. Otherwise the cut-off becomes an essential parameter of the theory. This particularly applies when the cut-off must be fine tuned to physical observables, a situation which actually takes place for specific power counting schemes.

By naive power counting one expects chiral potentials to be singular for \( r \ll 1/m_r \),

\[ V(r) = \frac{1}{\Lambda^2 r^{2r+3}} \]

where \( r \) represents the order in the chiral expansion \([16,18,26]\). While the resummations implied by solving the wave equation mix up the chiral power counting, they also enable finding non-perturbative new features when the short distance cut-off \( r \) becomes much smaller than any other long distance length scale. There arises the possibility of finding the adequate number of counterterms which is compatible with the power counting of the long range potential and of obtaining a finite and unique limit for the corresponding scattering amplitude. In this regard several studies have found that the original Weinberg power counting is inconsistent with renormalizability \([29,35,36]\). This unexpected result has suggested several alternative approaches and heated debates questioning the particular power counting, the renormalization process itself or the correctness of the non-perturbative resummation. We will not ponder on the pros and cons of any particular approach as this has been already done from several viewpoints in Refs. \([22,31,32,33,37]\). At present it is unclear what aspects of the original EFT framework will ultimately prevail or be universally accepted by the nuclear physics community (in this regard, see the related discussion in Ref. \([38]\)). Rather than considering the problem solved, we think that further work is still needed to settle the issue.

In previous works \([33,36,39,41]\), we have shown that the necessary minimum number of counterterms renormalizing a singular interaction can in fact be determined a priori from the behaviour of the configuration-space potential near the origin (for an earlier coordinate space treatment, see Ref. \([42]\)). On a more numerical basis, similar observations have been made in momentum space either using finite cut-offs \([24,41]\) or subtractive methods \([43,46]\). The main result is that in any uncoupled

---

1 By renormalization we specifically mean the existence of well-defined scattering amplitudes when the cut-off is removed. This condition allows to identify all the short distance operators needed to remove the cut-off dependence, as done, for example, in Ref. \([24]\) for the OPE case. Once these counterterms are included in the computation approximate cut-off independence is assured and consequently there is no problem in keeping a finite cut-off. For a different view on renormalization within an effective field theory context, see Lepage \([31]\) and the related discussions of Refs. \([31,32]\).

2 The operator product expansion for six quark operators predict the functional form of the NN potential at short distance, which turns out to be a little weaker than \( 1/r^2 \) and repulsive \([33,34]\). This shows that the NN potential computed on the lattice is regular and hence might predict uniquely the NN scattering data as well as the deuteron properties and also that this short distance dependence is quite different from the chiral potentials.

3 The inclusion of static degrees of freedom, such as the \( \Delta \) isobar excitation in the small scale expansion, can change the expected power law behaviour of the potential.
partial wave where an attractive singular potential is present, a counterterm is needed in order to renormalize this particular partial wave. An interesting corollary is that finite cut-off effects are less important the more singular the potential. Indeed, at order $\nu$ in the chiral counting the potential behaves as $1/\Lambda^{\nu+2}r^{\nu+3}$, generating finite cut-off corrections $\delta A(k)$ to the renormalized phase shifts $\delta_\infty(k)$ which scale as

$$\delta_\infty(k) - \delta A(k) = O(\Lambda^{-5/2-\nu/2}),$$

for large enough cut-offs $40\,41$, meaning in particular that cut-off independence is achieved in this case. On the contrary, if the potential is singular and repulsive the effect of counterterms becomes negligible for small enough cut-off radii.

A particularly problematic consequence of the non-perturbative treatment of singular potentials is that the renormalization of two body scattering by an attractive singular central interaction requires an infinite number of counterterms, one per each partial wave. The unlimited proliferation of counterterms when renormalizing singular interactions has been, among others, an argument against removing the cut-off, as the resulting effective field theory will be unable to predict observables $34\,31$. This problem can be cured in perturbative power countings, like the one proposed by Kaplan, Savage and Wise $47\,48$, where the proliferation of counterterms is naturally limited by the order of the approximation. Unfortunately the singularity in the tensor forces makes the previous proposal poorly convergent in the low momentum $\langle k \rangle$ (see however $50$ for a renewed formulation). In the modified Weinberg proposal of Nogga, Timmermans and van Kolck $29$, OPE is iterated in low angular momentum waves while treated perturbatively in sufficiently peripheral waves (usually $l > 2$). This choice naturally limits the number of necessary counterterms and, although it has has been criticized as arbitrary in Ref. $31$, it is sustained by the perturbative analysis of Ref. $47$. Higher order corrections are treated in perturbation theory and the corresponding (finite) number of counterterms is determined by imposing cut-off independence on the results, generating convergent amplitudes for the central waves with the NLO and N$^2$LO chiral potentials $51$. However, as we will show, there is a non-perturbative way of also obtaining a finite number of counterterms.

In a recent paper $52$ we have analyzed the role of $\Delta$ degrees of freedom for the central waves and the deuteron with the chiral $\Delta$ potentials of Refs. $17\,53$ with a reasonable phenomenological success. This particular potential furnishes simultaneously the theoretical requirements of renormalizability $4$ and power counting. Actually, convergence is achieved for reasonable cut-offs of the order of $r_c \sim 0.5\text{fm}$, that is, scales comparable with the shortest wavelength probed in NN elastic scattering below pion production threshold $5$. Moreover, as discussed in Ref. $50$, it may provide a good starting point for nuclear matter calculations as it has a rather small $D$-state probability, implying a sufficiently small wound integral which insures better convergence properties for the few body correlations and the nuclear many body problem.

In the present work we analyze further the TPE potential with $\Delta$-excitations $17\,53$, addressing the calculation of non-central partial waves. We show how the number of counterterms can be made finite by implementing a renormalization prescription correlating an infinite number of partial waves. Thus, this is a compelling example where, contrary to naive expectations, singular potentials may be consistently renormalized with a single common counterterm for all partial waves. The idea behind such a procedure is quite simple: if the potential has an inverse power law behaviour at short distances, $V(r) \sim C/r^n$, with a coefficient $C$ independent of energy and angular momentum, we expect all the reduced wave functions of the system to behave the same way at small enough radii, regardless of the energy or the angular momentum, as the contribution from these two factors will become negligible in comparison with the strength of the potential. As a trivial consequence, all partial waves can be related to the zero energy s-wave. The issue is analyzed in detail in Section III both for regular and singular potentials. While this becomes a relevant observation for uncoupled channels, the tensor force requires a suitable generalization of the result for coupled channels, which is discussed in Section III. Surprisingly, the potentials computed in Refs. $17\,53$ including TPE with $\Delta$ excitation do fulfill the necessary mathematical conditions to link partial waves with different angular momenta (see Section IV). Actually, we can estimate the finite cut-off error induced by these angular momentum correlations and which are exclusive of singular potentials. Other potentials do not automatically comply with these requirements, so the question on the consistency of the partial waves correlations is not independent of the potential and indirectly on the power counting invoked to compute it. Of course, mathematical consistency does not necessarily mean phenomenological success, and we test our proposal against the widely accepted partial wave analysis (PWA) of the Nijmegen group in Section IV. We see that actually there is no big difference between using the finite number of counterterms or renormalizing independently wave by wave, suggesting that improvements might be sought in

---

4 The divergence structure of this potential is identical to that of the chiral quark model in the Born-Oppenheimer approximation $53$, which being second order perturbation theory provides only attractive and singular potentials.

5 This is a purely coordinate space argument where the cut-off in the potential has been removed. In momentum space the corresponding cut-off $\Lambda \sim m_M/M_N \sim 350\text{MeV}$ implies also a regularization of the potential and an effective quenching of the $g_{nNN}$ coupling constant. This might be one of the reasons why momentum space calculations renormalizing the LS equation $41\,42$ require much larger cut-offs $\Lambda \sim 1-4\text{GeV}$ than naively expected.
II. CENTRAL DELTA-SHELL POTENTIALS AND THE PARTIAL WAVE EXPANSION

As a preparation we will consider first the simplest two-body scattering problem described by a central potential $V$, which for the $l$-wave reads

$$-u_{k,l}'' + \left[ 2\mu V(r) + \frac{l(l+1)}{r^2} \right] u_{k,l}(r) = k^2 u_{k,l}(r),$$

where $u_{k,l}$ is the reduced wave function, $\mu$ the reduced mass of the system, and $k = \sqrt{2\mu E}$ is the center of mass momentum. The asymptotic long distance boundary condition is taken as

$$u_{k,l}(r) \to \sin \left[ kr - \frac{l\pi}{2} + \delta_l(k) \right],$$

where $\delta_l(k)$ is the corresponding phase shift. We assume that $V(r)$ can be decomposed as the sum of a finite range potential $V_F$ (bounded by an exponential fall-off $\sim e^{-mr}$) and a contact range interaction $V_C$

$$V(r) = V_F(r; r_c) + V_C(r; r_c),$$

where we have added the auxiliary cut-off scale $r_c$, which will be needed in order to regularize the contact range interaction. For convenience we have also regularized the finite range potential in the following way

$$V_F(r; r_c) = V_F(r) \theta(r - r_c),$$

which means that the short range components of the finite range potential are effectively absorbed in the contact potential $V_C$. For the contact potential we only consider for definiteness the case in which $V_C$ is a delta-shell interaction

$$V_C(r; r_c) = \frac{C_0(r_c)}{4\pi r^2} \delta(r - r_c),$$

where $C_0$ does not depend on energy, and no higher derivatives of the delta function are considered. This is the simplest possible contact interaction and it actually becomes equivalent to a short distance boundary condition. We analyze below what can be obtained with such an interaction when the cut-off $r_c$ is removed, both in the case of regular and singular interactions.

A. Delta-Shell Potentials and Regular Interactions

As mentioned, the two-body scattering problem can be described by the corresponding reduced Schrödinger equation, Eq. (1). For radii below the cut-off $r_c$, there is no potential (due to the specific regularization employed for the finite range piece of the potential, see Eq. (7)) and the solution for the wave function is simply

$$u_{k,l}(r) = \text{const} \times r^{l+1} \text{ for } r < r_c,$$

where the regular solution has been chosen.

The solution for radii above the cut-off depends (i) on the size of the cut-off with respect to the range of $V_F$ and (ii) on whether $V_F$ is a regular or singular interaction. For the present discussion, we will assume that the cut-off is much smaller than the range of $V_F$, which we will call $a_F$, $r_c \ll a_F$, and that the finite range potential is a regular one, $\lim_{r \to 0} r^2 V_F(r) = 0$. Under these conditions the reduced wave function can be written as a linear combination of a regular and irregular solution for $r > r_c$, i.e.

$$u_{k,l}(r) = a_i u_{k,l}^{(\text{reg})}(r) + b_i u_{k,l}^{(\text{irr})}(r) \text{ for } r > r_c,$$

where the superscripts (reg) and (irr) denote the regular and irregular solutions, respectively.

For small radii, say $r_c < r \ll a_F$, the behaviour of the regular and irregular wave functions is given by

$$u_{k,l}^{(\text{reg})}(r) \sim r^{l+1}, \quad u_{k,l}^{(\text{irr})}(r) \sim \frac{1}{r},$$

where corrections depending on the presence of the potential $V_F$ or the finite momentum $k$ do not appear until higher relative powers of $r$ are considered.

The effect of the delta shell potential in the Schrödinger equation, Eq. (1), is to generate a discontinuity in the first derivative of the reduced wave function at $r = r_c$. The previous statement can be summarized in the following relation between the logarithmic derivatives of the wave functions for $r < r_c$ and $r > r_c$

$$\frac{2\mu C_0(r_c)}{4\pi r_c^2} = \frac{a_i(r_c) u_{k,l}^{(\text{reg})}'(r_c) + b_i(r_c) u_{k,l}^{(\text{irr})}'(r_c)}{a_i(r_c) u_{k,l}^{(\text{reg})}(r_c) + b_i(r_c) u_{k,l}^{(\text{irr})}(r_c)} - \frac{l+1}{r_c}.$$
\[ C_0(r_c) \] to be a running coupling constant. In fact, for a constant value of \( C_0 \) one finds in the \( r_c \to 0 \) limit that
\[
\frac{b_l(r_c)}{a_l(r_c)} \to \frac{u_{k,l}^{(w)}(r_c)}{u_{k,l}^{(w)}(r_c)} \sim -r_c^{2l+1} \to 0 ,
\]
meaning that the regular solution is effectively chosen as \(|b_l| \ll |a_l|\). Therefore, in order to avoid a trivial or irrelevant contact interaction one needs that \( C_0(r_c) \) evolves with \( r_c \) in a very specific way, a dependence that can be obtained by solving Eq. (13) for a given \( b_l/a_l \) value.

The running of \( C_0(r_c) \) is so strongly determined by the scaling properties of the regular and irregular wave functions near the origin that, if \( C_0(r_c) \) is set to be non-trivial in a given partial wave, it will become trivial in all the other waves. This can be checked as follows. If we fit \( C_0(r_c) \) to reproduce \( b_l/a_l \) in the partial wave \( l = l_1 \) and call this counterterm \( C_0^{(l_1)}(r_c) \), its exact value can be obtained from solving Eq. (13) for \( l = l_1 \). Using now the counterterm \( C_0^{(l_1)}(r_c) \) for computing the linear combination of solutions for \( l = l_2(\neq l_1) \), we get the following
\[
\frac{b_{l_2}}{a_{l_2}} \sim r_c^{2l_2+1} \quad \text{for} \quad l_2 \neq l_1 ,
\]
which is just the same scaling as the corresponding one for a constant counterterm, Eq. (14). Therefore we can take the simplification that a given counterterm only acts on a determined partial wave when the cut-off is removed, as is usually assumed. This means de facto a total independence of non-trivial counterterms for any other partial wave. Note that only a trivial counterterm produces a short distance interaction common to all partial waves.

B. Delta Shell Potentials and Attractive Singular Interactions

As we have seen, in order to have a non-trivial effect, the running of the counterterm depends on the scaling properties of the wave function near the origin, \( r \to 0 \). Given the fact that for regular potentials the scaling is different for each partial wave, the result is that only one partial wave will be affected by a given counterterm. On the contrary, as we will see, for attractive singular potentials the scaling does not depend on the angular momentum. Therefore, the scaling is independent of the partial wave chosen, and a given counterterm will affect all the partial waves simultaneously. This is our key observation which we will extend to tensor forces in Section III and put forward below for the relevant case of TPE chiral NN interactions with \( \Delta \)-excitations in Section IV.

Indeed, if we consider the behaviour of the reduced wave function for a power-law attractive singular potential which for short enough distances behaves as
\[
2\mu V_F(r) \to -\frac{R_F^{n-2}}{r^n} ,
\]
where \( n > 2 \) and \( R_F \) is some given length scale which sets the range of the power-law behaviour of \( V_F \). This new scale \( R_F \) may not coincide with the generic range \( a_F \) of the potential as several lower energy scales may be present in the system\(^6\). For a potential like the previous one and for distances \( r_c < r \ll R_F \), the reduced wave function can be described by the WKB approximation since the de Broglie wavelength is slowly varying,
\[
R_F \frac{d}{dr} \frac{1}{\sqrt{2\mu(E-V_F(r))}} \sim \frac{n}{2}(r/R_F)^{n/2-1} \ll 1 ,
\]
yielding the short distance behaviour
\[
 u_{k,l}(r) \approx A_l \left( \frac{r}{R_F} \right)^{n/4} \sin \left[ \frac{1}{n-2} \left( \frac{R_F}{r} \right)^{n/2-1} + \varphi_l(k) \right]
\]
for \( R_F \gg r > r_c \),
where \( A_l \) is some normalization constant and \( \varphi_l(k) \) is a short distance phase which in principle depends on the angular momentum and the energy. For \( r < r_c \), the reduced wave function \( u_{k,l} \) will show the expected \( r^{l+1} \) behaviour, see Eq. (D).

Taking into account the behaviour of the wave function around the cut-off, we can rewrite the equation that describes the running of \( C_0 \) for the case of singular interactions as
\[
\frac{2\mu C_0(r_c)}{4\pi r_c^2} \to \frac{2}{R} \left( \frac{R_F}{r_c} \right)^{n/2} \times \cot \left[ \frac{2}{n-2} \left( \frac{R_F}{r_c} \right)^{n/2-1} + \varphi_l(k) \right]
\]
\[
- \frac{l+1}{r_c} ,
\]
As can be immediately realized, for \( r_c \to 0 \) the explicit \( l \)-dependent term stemming from the behaviour of the wave function for \( r < r_c \) can be dropped, leading to the following ultraviolet behaviour for \( C_0 \)
\[
\frac{2\mu C_0(r_c)}{4\pi r_c^2} \to \frac{2}{R} \left( \frac{R_F}{r_c} \right)^{n/2} \times \cot \left[ \frac{2}{n-2} \left( \frac{R_F}{r_c} \right)^{n/2-1} + \varphi_l(k) \right] ,
\]
which does not depend explicitly on angular momentum. In particular the previous equation means that for \( r_c \to 0 \) we have the following identifications
\[
\varphi_l(k_1) = \varphi_l(k_2) \quad \text{and} \quad \varphi_l(k) = \varphi_l(k_c) .
\]
In other words, the short distance phase is independent of angular momentum or energy. In the more general
case that we accept energy dependent counterterms \( C_k \) or higher derivatives of the delta, we can obtain an energy dependent semi-classical phase \( \varphi_1(k_1) \neq \varphi_1(k_2) \), but the angular momentum independence will hold. The only way in which one can break the condition \( \varphi_1(k) = \varphi_2(k) \) is by accepting terms explicitly depending on the angular momentum in the contact range interaction.

The previous results can be efficiently cast in the language of short distance boundary conditions as follows

\[
\left. \frac{u_{k,l}}{u_{k,l}} \right|_{r=r_c+\varepsilon} = \left. \frac{u_{k,l}}{u_{k,l}} \right|_{r=r_c+\varepsilon},
\]

that is, the logarithmic derivative at the cut-off radius of the reduced wave function does not depend on the angular momentum. This is the form in which we will effectively implement the condition \( \varphi_1(k) = \varphi_2(k) \).

It is important to notice that the angular momentum independence of the behaviour of the \( l \)-wave reduced wave function near the origin is only realized when the cut-off is small enough, so the behaviour described in Eq. (18) is valid. In general Eq. (18) will be applicable for a radius such that the WKB approximation holds, and this radius will be smaller for higher partial waves. If we denote the previous radius as \( r_W \), we will generally have \( r_W > r_c \) for \( l_1 < l_2 \). This means that for a given cut-off radius there will be a critical value of the angular momentum for which \( r_W > r_c \), and therefore the condition of angular momentum independence should only be used for \( l < r_c \), i.e.

\[
\left. \frac{u_{k,0}}{u_{k,0}} \right|_{r=r_c+\varepsilon} = \left. \frac{u_{k,1}}{u_{k,1}} \right|_{r=r_c+\varepsilon} = \ldots = \left. \frac{u_{k,l-1}}{u_{k,l-1}} \right|_{r=r_c+\varepsilon},
\]

while for higher angular momenta the behaviour of the reduced wave function roughly corresponds to what is to be expected for a regular potential, and it may be possible to make a perturbative treatment as suggested in Refs. [57, 58].

### III. THE INCLUSION OF THE TENSOR FORCE

The previous analysis about the connection between the short range physics in different partial waves is only valid for the uncoupled channel case. If a tensor force is present, as in the nucleon-nucleon interaction, we will need to account for the induced coupled channel structure happening in spin triplet channels in our analysis. For simplicity, we will assume a two-body system in which the finite range piece of the potential only contains a non-tensor (central) and a tensor piece

\[
V_F(\vec{r}) = V_{NT}(r) + S_{12}(\hat{r}) V_T(r).
\]

with \( S_{12}(\hat{r}) = 3 \hat{r}_1 \cdot \hat{r}_2 \cdot \hat{r} - \hat{r}_1 \cdot \hat{r}_2 \) (\( \hat{r}_1 \) and \( \hat{r}_2 \) are the spin operators of particle 1 and 2), and where we do not specify any additional operator structure of the tensor and non-tensor pieces of the potential (like, for example, spin or isospin dependence). The behaviour of the non-tensor and tensor piece at short enough distances is given by

\[
V_{NT}(r) \to \frac{C_{NT}}{r^n},
\]

\[
V_T(r) \to \frac{C_T}{r^n},
\]

where we have assumed that they have the same power-law divergent behaviour at short distances. We have not yet determined whether the potentials are attractive or repulsive.

The tensor force will couple spin triplet channels for which \( l = j \pm 1 \), being the corresponding reduced Schrödinger equation for \( r > r_c \)

\[
-u''_j + U_{11} u_j + U_{12} w_j = k^2 u_j, \quad -w''_j + U_{12} u_j + U_{22} w_j = k^2 w_j.
\]

where

\[
U_{11} = 2\mu \frac{V_{NT}}{2j+1} - 2\mu \frac{2j-2}{2j+1} \frac{V_T}{r^2} + \frac{j(j-1)}{r^2}, \quad U_{12} = 6 \frac{\sqrt{j(j+1)}}{2j+1}, \quad U_{22} = 2\mu \frac{V_{NT}}{2j+1} - 2\mu \frac{2j+4}{2j+1} \frac{V_T}{r^2} + \frac{j(j+1)(j+2)}{r^2}.
\]

The previous Schrödinger equation has four linearly independent solutions, of which only two of them are regular and therefore physically acceptable. The coupled channel equations can be efficiently cast into the following compact notation

\[
-u'' + \left[ 2\mu \mathbf{V} + \frac{\mathbf{L}^2}{r^2} \right] \mathbf{u} = k^2 \mathbf{u}
\]

where the wave function is now a matrix

\[
\mathbf{u} = \begin{pmatrix} u^{(a)}_j & u^{(b)}_j \\ u^{(a)}_w & u^{(b)}_w \end{pmatrix},
\]

with the \( ^{(a)} \) and \( ^{(b)} \) superscripts representing the two linearly independent asymptotic \( (r \to \infty) \) solutions of the
In the previous definition of the potential in matrix form, quantities are defined in the diagonal basis. In this way, the potential and the angular momentum may be attractive-attractive, attractive-repulsive or repulsive-repulsive. Only two of these situations, namely the attractive-attractive and attractive-repulsive case, admit counterterms [36, 53], and are therefore of interest from the point of view of renormalization.

### A. Attractive-Attractive Case

Situations where both eigenchannels are attractive are the easiest to handle. If a singular power-law potential is assumed, the behaviour of the reduced wave function for \( r < r_c \) can be safely ignored, yielding the following relation

\[
v_j''(r_c)v_j(r_c)^{-1} = \frac{2\mu}{4\pi r_c^2} C_D(r_c),
\]

meaning that there are three free parameters in this case (due to \( C_D \) being real and symmetric), in agreement with previous analysis of singular potentials in coupled channels [36, 53]. The relation between the short range wave functions of channels with different total angular momentum is therefore

\[
v_{j_2}^T v_{j_1}' = v_{j_2}' v_{j_1},
\]

where we have made use of \( C_D = C_D^T \). This relation is invariant with respect to the relative normalization of the two linearly independent solutions of each partial wave, which were previously denoted with the \( (a) \) and \( (b) \) superscripts, and also on the set of linearly independent solutions chosen, as can be easily checked. It should be noted too that the previous relation is reminiscent of the coupled-channel version of the two-potential formula of Ref. [59].

A problem arises in relating the \( j = 0 \) triplet state with other coupled channels, as the \( 3P_0 \) wave is effectively an uncoupled state,

\[
v_{j=0}(r) = \begin{pmatrix} 0 \\ 0 \\ v_{3P_0}(r) \end{pmatrix}.
\]

Therefore, the previous representation of the short distance potential, i.e. the coupled-channel delta-shell of Eq. (43), cannot be a correct representation of the short range physics of the \( 3P_0 \) channel. As a consequence the \( 3P_0 \) channel cannot be unambiguously related with the other coupled channels. It is possible however to obtain convergent amplitudes by relating the \( 3P_0 \) wave function with any of the “lower” eigen wave functions of the reference wave function \( v_j \),

\[
\frac{v_j'(r_c)}{v_{3P_0}(r_c)} = \frac{(v_j)_{21}^i(r_c)}{(v_j)_{21}(r_c)}, \quad \text{or} \quad \frac{v_j'(r_c)}{v_{3P_0}(r_c)} = \frac{(v_j)_{22}^i(r_c)}{(v_j)_{22}(r_c)},
\]

where \( 21 \) and \( 22 \) are the corresponding indices in the rotated wave function matrix \( v_2 \). Both possibilities yield a renormalizable phase for the \( 3P_0 \) wave, but not the same one: the phase shift depends on which of the previous two equations is used, indicating the presence of model dependence in any of these choises. Therefore, the only way to avoid model dependence is to treat the \( 3P_0 \) wave as an independent wave in the attractive-attractive case.
B. Attractive-Repulsive Case

If one of the eigenchannels is attractive and the other is repulsive, then we have that Eq. (43) can only be applied in the attractive eigenchannel. In particular, the delta-shell coupling matrix $C_D(r_c)$ takes the simplifying form

$$C_D(r_c)_{BC} = C_A(r_c) \delta_{AB} \delta_{AC}, \quad (47)$$

where the labels $B, C = 1, 2$ are matrix indices, and $A$ represents the index associated with the attractive solution. That is, only one counterterm is needed in order to renormalize an attractive-repulsive coupled channel, as delta-shell contributions become trivial in the $r_c \rightarrow 0$ limit, except if they happen in the $AA$ subchannel. Combining the previous result with the boundary condition induced by the delta-shell potential

$$v_j'(r_c) = \frac{2\mu}{4\pi r_c^2} C_D(r_c) v_j(r_c), \quad (48)$$

the following renormalization conditions are obtained

$$\left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{AA} = \left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{AR}, \quad (49)$$

$$\left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{RA} = \left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{AR}, \quad (50)$$

$$\left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{RA} = 0, \quad (51)$$

$$\left(\frac{v_j'(r_c)}{v_j(r_c)}\right)_{RR} = 0, \quad (52)$$

implies the equivalence of Eqs. (46) and (47). The last two equations, (51) and (52), are just regularization conditions for the repulsive eigenchannels. Contrary to what happened in the attractive-attractive case, in the attractive-repulsive case the $j = 0 \quad ^3P_0$ wave (if attractive) can be directly related with the other coupled triplet waves.

IV. APPLICATION TO NEUTRON-PROTON SCATTERING

The previous results can be applied to the case neutron-proton (np) scattering in nuclear effective field theory, where the resulting potentials are in many cases singular and attractive. In principle we can relate any two partial waves for which the potential diverges in the same way, i.e. the same power and the same coefficient. This means that correlations will emerge only between channels with the same spin and isospin values. For the spin triplet channels it is also necessary to consider whether the channels are coupled ($l = j \pm 1$) or uncoupled ($l = j$). Therefore, we obtain a total of six sets of correlated waves, namely (i) singlet isovectors, (ii) singlet isoscalars, (iii) uncoupled triplet isovectors, (iv) uncoupled triplet isoscalars, (v) coupled triplet isovectors, and (vi) coupled triplet isoscalars. We limit ourselves to the $j \leq 5$ partial waves. The $^3P_0$ wave remains uncorrelated in our current scheme and it is not further considered in this work.

A. The Chiral Neutron-Proton Potential

The finite range piece of the nucleon-nucleon (NN) potential in chiral perturbation theory is expressed as an expansion in powers of $Q$

$$V_{NN}(r) = V_0 + V^{(2)}(r) + V^{(3)}(r) + O(Q^4), \quad (54)$$

where $Q$ represents either the pion mass, the nucleon-$\Delta$ splitting or the momentum of the nucleons. We only consider here chiral potentials in which the $\Delta$ isobar has been explicitly included $^{17,53}$. The reasons for this decision are that (i) they have better convergence properties than their $\Delta$-less counterparts, and (ii) they are attractive-attractive singular interactions in all coupled channels at orders $Q^2$ (NLO) and $Q^3$ (N2LO), leading to a simpler analysis in general. For the finite range piece of the interaction we adopt the original Weinberg counting $^{11}$, in which $1/M_N$ corrections are treated as higher order as it is done in Refs. $^{12,60,61}$. At the orders considered in this work, the potential can be decomposed as a central, spin-spin and a tensor component which in coordinate space reads

$$V_{NN}(\vec{r}) = V_C(r) + \tau W_C(r) + \sigma (V_S(r) + \tau W_S(r)) + S_{12}(\vec{r}) (V_T(r) + \tau W_T(r)), \quad (55)$$

where spin-orbit and quadratic spin-orbit terms have been ignored as they do not appear up to higher orders. The use of previous counting rule for the $1/M_N$ corrections is necessary if we plan to correlate at short distance the behaviour of the different waves, as it generates a spin-orbit term which is less singular than the other components of the interaction. The operators $\tau, \sigma$ and $S_{12}$ are given by

$$\tau = \vec{\tau}_1 \cdot \vec{\tau}_2 = 2t(t + 1) - 3,$$

$$\sigma = \vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2s(s + 1) - 3,$$

$$S_{12}(\vec{r}) = 3 \vec{\sigma}_1 \cdot \vec{r} \vec{\sigma}_2 . \vec{r} - \vec{\sigma}_1 . \vec{\sigma}_2, \quad (56)$$

where $\vec{\tau}_1(2)$ and $\vec{\sigma}_1(2)$ are the proton(neutron) isospin and spin operators; $t$ and $s$ represent the total isospin $t = 0, 1$ and total spin $s = 0, 1$ of the np system. The precise form of the chiral $\Delta$ potential is taken from Ref. $^{53}$.

Note that in the singlet channel cases ($s = 0$) the tensor force operator does not contribute. For symmetry reasons (Fermi-Dirac statistics) we have $(-1)^{J^P + T + l} = -1$, where $l$ is the orbital angular momentum. This means in
particular that even partial waves are isovectors \((t = 1)\) and odd partial waves are isoscalars \((t = 0)\). The NN potential reads for the singlet channels

\[
\begin{align*}
V_{S_0}(r) & = V_{\Delta S_0}(r) = V_{G_4}(r) \\
& = V_C(r) + W_C(r) - 3V_S(r) - 3W_S(r), \\
V_{P_3}(r) & = V_{F_3}(r) = V_{H_5}(r) \\
& = V_C(r) - 3W_C(r) - 3V_S(r) + 9W_S(r),
\end{align*}
\]

(57)

that is, all the singlet channels can be described with two different potentials depending on whether we are in the isoscalar or isovector case (or equivalently, on whether even or odd partial waves are considered).

In the spin triplet channels \((s = 1)\) we must distinguish between uncoupled \((l = j)\) and coupled waves \((l = j \pm 1)\). In the uncoupled waves, we can again distinguish between the potential in the isoscalar \((3D_2, 3G_4)\) and isovector \((3P_1, 3D_2)\) waves

\[
\begin{align*}
V_{D_2}(r) & = V_{G_4}(r), \\
V_{P_1}(r) & = V_{F_3}(r) = V_{H_5}(r),
\end{align*}
\]

(58)

where the explicit expressions of the previous potentials in terms of the central, spin-spin and spin-tensor components is given by

\[
\begin{align*}
V_{D_2}(r) & = V_C(r) - 3W_C(r) + V_S(r) - 3W_S(r) + 2V_T(r) - 6W_T(r), \\
V_{P_1}(r) & = V_C(r) + W_C(r) + V_S(r) + W_S(r) + 2V_T(r) + 2W_T(r).
\end{align*}
\]

(59) (60)

Equivalently, for the coupled waves we have

\[
\begin{align*}
R_1 V_{3C_1}(r) R_1^T & = R_3 V_{3C_3}(r) R_3^T = R_5 V_{3C_5}(r) R_5^T, \\
R_2 V_{3C_2}(r) R_2^T & = R_4 V_{3C_4}(r) R_4^T,
\end{align*}
\]

(61)

for the isoscalar \((3C_1, 3C_3, 3C_5)\) and isovector \((3C_2, 3C_4)\) waves, and where the expressions are more cumbersome as they involve matrices. In the expressions above, \(R_j\) are the rotation matrices defined in Eq. 38. The notation \(3C_1, 3C_2, \ldots\) is a short hand for \(3S_1 - 3D_1, 3P_2 - 3F_2, \ldots\), and the explicit form of the rotated potentials is given by

\[
\begin{align*}
R_1 V_{3C_1}(r) R_1^T & = 1 \left( V_C(r) - 3W_C(r) + V_S(r) - 3W_S(r) \right) + S_{12D}^I \left( V_T(r) - 3W_T(r) \right), \\
R_2 V_{3C_2}(r) R_2^T & = 1 \left( V_C(r) + W_C(r) + V_S(r) + W_S(r) \right) + S_{12D}^I \left( V_T(r) + W_T(r) \right),
\end{align*}
\]

(62)

with \(1\) the \(2\times2\) identity matrix, and \(S_{12D}^I\) the diagonalized tensor matrix represented by Eq. 39.

### B. Van der Waals Behaviour of the Chiral Potentials

At distances below the pion Compton wavelength, \(m_\pi r < 1\), the chiral potentials exhibit at orders \(Q^2\) and \(Q^3\) a power-law behaviour of the type

\[
V^{(\nu)}(r) \rightarrow C_n^{(\nu)} \frac{r^n}{r^6},
\]

(63)

with \(n = 2, 3\) and where the value of \(C_n^{(\nu)}\) depends on the particular component of the potential considered. These coefficients were computed in Ref. 52 based on the spectral function representation of the potentials of Krebs, Epelbaum and Meißner. It should be noted though that the exact behaviour of the potential at short distances is inessential for the angular momentum correlations. What really matters is (i) that the potential is a singular attractive interaction and (ii) that it is much stronger than the centrifugal barrier at the chosen cut-off radius \(r_c\). In the case of the order \(Q^2\) and \(Q^3\) chiral \(\Delta\) full potentials these conditions are fulfilled in all partial waves with \(j \leq 5\) for cut-off radii as big as 1.0 fm.

### C. Correlated Renormalization of the Uncoupled Waves

We describe the scattering states in the uncoupled waves by solving the following reduced Schrödinger equation for \(r > r_c\)

\[
-u''_{k,l} + \left[ M_N V_{NN}(r) + \frac{l(l+1)}{r^2} \right] u_{k,l}(r) = k^2 u_{k,l}(r),
\]

(64)

where \(u_{k,l}\) is the reduced wave function, \(V_{NN}\) the corresponding chiral potential for the particular partial wave considered, \(k\) the center-of-mass momentum, \(l\) the angular momentum, and \(M_N\) is twice the neutron-proton reduced mass, i.e. \(M_N = 2M_nM_p/\left(M_p + M_n\right)\). The reduced wave function is asymptotically normalized to

\[
u_{k,l}(r) \rightarrow k^l \left( \cot \delta_l \tilde{j}_l(kr) - \tilde{y}_l(kr) \right),
\]

(65)

for \(r \rightarrow \infty\), with \(\delta_l\) the phase shift, and \(\tilde{j}_l(x) = xj_l(x)\) and \(\tilde{y}_l(x) = xy_l(x)\) the reduced spherical Bessel functions. The normalization factor \(k^l\) is added in order to have a well-defined normalization of the wave function in the \(k \rightarrow 0\) limit. At \(r = r_c\) the wave function can be determined by several means. One is by solving Eq. 19 for some value of the counterterm \(C_0\), which can be later fitted to reproduce some observable, like for example, the \(1S_0\) or \(3P_1/3P_3/3D_2\) scattering length. A different way is to construct an asymptotic wave function \((r \rightarrow \infty)\) reproducing the desired scattering length. In the case of the \(1S_0\) channel, this wave function is given by

\[
u_{0,1S_0}(r) \rightarrow 1 - \frac{r}{a_0},
\]

(66)
with $a_0$ the $^1S_0$ scattering length, and then integrate the reduced Schrödinger equation, Eq. (64), downwards from $r \to \infty$ to $r = r_c$. Then we use the different relations derived previously to obtain the logarithmic boundary condition at $r = r_c$ for the different energies and partial waves considered. For the particular case of the $^1S_0$ channel and its correlated channels $^1D_2$ and $^1G_4$, the relation takes the form

$$
\frac{u'_{0,1}S_0}{u_{0,1}S_0} \bigg|_{r=r_c} = \frac{u'_{k,1}S_0}{u_{k,1}S_0} \bigg|_{r=r_c},
$$

(67)

$$
\frac{u'_{k,1}S_0}{u_{k,1}S_0} \bigg|_{r=r_c} = \frac{u'_{k,1}D_2}{u_{k,1}D_2} \bigg|_{r=r_c} = \frac{u'_{k,1}G_4}{u_{k,1}G_4} \bigg|_{r=r_c},
$$

(68)

where the first equation relates the zero and finite energy states of the $^1S_0$ wave, and the second one represents the partial wave correlations. For the other correlated channels, we have the correlation conditions

$$
\frac{u'_{k,1}P_1}{u_{k,1}P_1} \bigg|_{r=r_c} = \frac{u'_{k,3}P_3}{u_{k,3}P_3} \bigg|_{r=r_c} = \frac{u'_{k,1}H_5}{u_{k,1}H_5} \bigg|_{r=r_c},
$$

(69)

$$
\frac{u'_{k,3}P_3}{u_{k,3}P_3} \bigg|_{r=r_c} = \frac{u'_{k,3}F_3}{u_{k,3}F_3} \bigg|_{r=r_c} = \frac{u'_{k,3}H_5}{u_{k,3}H_5} \bigg|_{r=r_c},
$$

(70)

$$
\frac{u'_{k,3}D_2}{u_{k,3}D_2} \bigg|_{r=r_c} = \frac{u'_{k,3}G_4}{u_{k,3}G_4} \bigg|_{r=r_c},
$$

(71)

which are to be supplemented with the regularization conditions for the base waves

$$
\frac{u'_{0,1}P_1}{u_{0,1}P_1} \bigg|_{r=r_c} = \frac{u'_{k,1}P_1}{u_{k,1}P_1} \bigg|_{r=r_c},
$$

(72)

$$
\frac{u'_{0,3}P_3}{u_{0,3}P_3} \bigg|_{r=r_c} = \frac{u'_{k,3}P_3}{u_{k,3}P_3} \bigg|_{r=r_c},
$$

(73)

$$
\frac{u'_{0,1}D_2}{u_{0,1}D_2} \bigg|_{r=r_c} = \frac{u'_{k,1}D_2}{u_{k,1}D_2} \bigg|_{r=r_c},
$$

(74)

These boundary equation conditions can be used as initial integration conditions for the corresponding Schrödinger equation Eq. (64). After integrating upwards from $r = r_c$ to $r \to \infty$, we match to the asymptotic behaviour of the wave functions, Eq. (64), in order to obtain the phase shifts. The equivalent value for the counterterm coupling $C_0(r_c)$ can be obtained from Eq. (19), giving in the $r_c \to 0$ limit

$$
\frac{M_N C_1S_0(r_c)}{4\pi r_c^2} \Bigg|_{r=r_c} \approx \frac{u'_{0,1}S_0(r_c)}{u_{0,1}S_0(r_c)},
$$

(75)

plus the corresponding expressions for the other base waves.

D. Correlated Renormalization of the Coupled Waves

For the coupled channels we solve the coupled reduced Schrödinger equation in its matrix form

$$
- \mathbf{u}_{k,j}'' + \left[ 2\mu \mathbf{V}_{NN}(r) + \frac{L^2}{r^2} \right] \mathbf{u}_{k,j}(r) = k^2 \mathbf{u}_{k,j}(r),
$$

(76)

where we now use the notation of Sect. [11] in which $\mathbf{u}_{k,j}$, $\mathbf{V}_{NN}$ and $\mathbf{L}^2$ are matrices. The reduced wave function is asymptotically ($r \to \infty$) normalized to

$$
\mathbf{u}_{k,j}(r) = \begin{pmatrix} j_{j-1}(kr) & j_{j+1}(kr) \\ 0 & 0 \end{pmatrix},
$$

(77)

$$
\mathbf{y}_{j}(kr) = \begin{pmatrix} y_{j-1}(kr) & y_{j+1}(kr) \\ 0 & 0 \end{pmatrix},
$$

(78)

$$
\mathbf{F}(k) = \begin{pmatrix} k^2 & 0 \\ 0 & k^2+1 \end{pmatrix},
$$

(79)

with $j \mathbf{u}(x) = xj \mathbf{u}(x)$ and $\mathbf{y}(x) = x \mathbf{y}(x)$ the reduced spherical Bessel functions. The normalization factor $\mathbf{F}(k)$ is included in order to have a well-defined asymptotic ($r \to \infty$) wave function at $k = 0$. The $\mathbf{M}(k)$ matrix is the coupled-channel equivalent of cot $\delta(k)$ and is related to the $S$-matrix $\mathbf{S}(k)$ by $\mathbf{M}(k) = i(\mathbf{S}(k) + 1)/\mathbf{S}(k)$ with the 2x2 identity matrix. For the chiral $\Delta$-full potential of Ref. [52] all the coupled channels are attractive-attractive singular potentials at distances below the pion Compton wavelength. Thus three renormalization conditions or counterterms are needed in order to obtain well-defined results. The usual procedure is to fix the asymptotic ($r \to \infty$) behaviour of the wave function at $k = 0$. That is, we fix the three scattering lengths of the coupled system. Then we integrate the Schrödinger equation, Eq. (76), downwards from $r \to \infty$ to $r = r_c$. If we define $\mathbf{L}_{k,j}(r)$ as

$$
\mathbf{L}_{k,j}(r) = \mathbf{u}_{k,j}^{-1}(r) \mathbf{u}_{k,j}(r),
$$

(80)

then, the finite energy solution is constructed from the following boundary condition at $r = r_c$

$$
\mathbf{L}_{k,j}(r_c) = \mathbf{L}_{0,0}(r_c).
$$

(81)

The procedure for correlating the different partial waves considered is similar to the one employed in constructing the finite energy solutions, the only difference being the rotation to the basis in which the tensor force is diagonal. For the two sets of correlated coupled channels, those with $j = 1, 3, 5$ and those with $j = 2, 4$, we have the boundary conditions

$$
\mathbf{R}_1 \mathbf{L}_{k,1}(r_c) \mathbf{R}_1^T = \mathbf{R}_3 \mathbf{L}_{k,3}(r_c) \mathbf{R}_3^T = \mathbf{R}_5 \mathbf{L}_{k,5}(r_c) \mathbf{R}_5^T,
$$

(82)

$$
\mathbf{R}_2 \mathbf{L}_{k,2}(r_c) \mathbf{R}_2^T = \mathbf{R}_4 \mathbf{L}_{k,4}(r_c) \mathbf{R}_4^T,
$$

(83)

from which the $\mathbf{M}_{k,j}(k)$ matrix (and the corresponding phase shifts) can be obtained.
E. Cut-off Dependence of the Phase Shifts

The cut-off dependence of the phase shifts in the correlated renormalization procedure can be easily estimated by making use of the renormalization group analysis of boundary condition regularization of Ref. [40]. For simplicity, we only consider in detail the uncoupled channel case. According to Ref. [40], the cut-off dependence of the phase shift for an uncoupled channel is given by

\[
\frac{d\delta_{l}(k; r_{c})}{dr_{c}} = \left[ M_{N} V_{NN}(r_{c}) - k^{2} + \frac{l(l+1)}{r_{c}^{2}} \right] u_{k,l}^{2}(r_{c}),
\]

where \(\delta_{l}(k)\) is the phase shift, \(r_{c}\) the cut-off radius, and with \(u_{k,l}, V_{NN}, k, l\) and \(M_{N}\) as defined in Eq. (63). In the previous formula \(L_{k,l}(r_{c})\) is the logarithmic derivative of the \(u_{k,l}\) reduced wave function at the cut-off radius, i.e.

\[
L_{k,l}(r_{c}) = \frac{u'_{k,l}(r_{c})}{u_{k,l}(r_{c})}.
\]

If we are correlating the \(l_{0}\)- and \(l\)-waves, we have for the logarithmic derivatives at the cut-off radius \(r_{c}\) that

\[
L_{k_{0},l_{0}}(r_{c}) = L_{k,l}(r_{c}),
\]

where the partial wave \(l_{0}\) is taken to be the base wave, i.e. the wave for which we have fixed the value of the phase shift at \(k = k_{0}\) (or the scattering length if \(k_{0} = 0\)). By taking into account that the reduced wave function \(u_{k_{0},l_{0}}(r)\) obeys the following Schrödinger equation

\[
-u''_{k_{0},l_{0}}(r) + \left[ M_{N} V_{NN}(r) + \frac{l_{0}(l_{0}+1)}{r_{c}^{2}} \right] u_{k_{0},l_{0}}(r) = k_{0}^{2} u_{k_{0},l_{0}}(r),
\]

it is trivial to check that the logarithmic boundary condition for \(k = k_{0}, l = l_{0}\) fulfills the differential equation

\[
M_{N} V_{NN}(r_{c}) + \frac{l_{0}(l_{0}+1)}{r_{c}^{2}} - k_{0}^{2}
\]

\[
+ L'_{k_{0},l_{0}}(r_{c}) + L_{k_{0},l_{0}}^{2}(r_{c}) = 0,
\]

which is also the differential equation obeyed by \(L_{k,l}(r_{c})\). In particular, the previous means that the cut-off dependence of the phase shift simplifies to

\[
\frac{d\delta_{l}(k; r_{c})}{dr_{c}} = \left[ \frac{l(l+1) - l_{0}(l_{0}+1)}{r_{c}^{2}} \right] u_{k,l}^{2}(r_{c}).
\]

For cut-off radii such that \(2m_{\pi}r_{c} \ll 1\), the behaviour of the wave functions will be determined by the van der

FIG. 1. (Color online) (Upper panel) \(^{1}S_{0}\), \(^{1}D_{2}\) and \(^{1}G_{4}\) phase shifts computed from Eq. (67), using the \(^{1}S_{0}\) scattering length as an input parameter with a coordinate space cut-off \(r_{c} = 0.3\) fm. (Lower panel) \(^{1}P_{1}\), \(^{1}F_{3}\) and \(^{1}H_{5}\) phase shifts computed from Eq. (69), using the \(^{1}P_{1}\) scattering length as an input parameter with a coordinate space cut-off \(r_{c} = 0.3\) fm. The light blue band is generated by varying the cut-off radius within the range \(r_{c} = 0.6 - 0.8\) fm.
Waals piece of the interaction, i.e. \( u_{\text{ Waals}}^2(r_c) \sim r_c^3 \), up to oscillations (see Eq. (15)) for the chiral NLO- and N^2LO-\( \Delta \) potentials of Ref. [53]. This implies that the cut-off dependence of the phase shifts can be approximated by

\[
\delta_{l_0}(k, r_c) - \delta_{l_0}(k, 0) \propto -(k - k_0^2) r_c^4,
\]

for \( l = l_0 \) (that is, the base wave), and

\[
\delta_{l}(k, r_c) - \delta_{l}(k, 0) \propto [(l + 1) - l_0(l_0 + 1)] r_c^2,
\]

for \( l \neq l_0 \). At the end of Section V, we illustrate these expectations for the chiral \( \Delta \)-potentials of Ref. [53]. It should be noted that Eq. (50) implies that the correlated renormalization procedure only generates converging phase shifts if the potential \( V_{\text{NN}} \) is singular, as expected from the discussion in Section II. The extension of the previous results to coupled channels is straightforward and leads to the same conclusion and cut-off dependence as the uncoupled channel case.

V. NUMERICAL RESULTS

As we have shown we can relate the phase shifts in different partial waves using the short range relation described by Eqs. (67-71). We take in our numerical computations \( f_\pi = 92.4 \text{MeV} \), \( m_\pi = 138.03 \text{MeV} \), \( 2m_{\text{np}} = M_\pi = 2M_pM_\pi/(M_p + M_\pi) = 938.918 \text{MeV} \), \( g_A = 1.29 \) in the OPE piece to account for the Goldberger-Treiman discrepancy and \( g_A = 1.26 \) in the TPE piece of the potential. The discussion of the standard OPE potential corresponds to the attractive-repulsive case and is relegated to Appendix A. We discuss here the TPE chiral potential with \( \Delta \) excitations as obtained from Ref. [53] (however with the spectral cut-off removed). For \( h_A \), the chiral couplings \( c_1 \), \( c_3 \) and \( c_4 \) and \( b = b_1 + b_8 \) we take the values corresponding to “Fit 1” of Ref. [53] (see table I inside the previous reference).

All the partial waves are renormalized at a cut-off radius \( r_c = 0.3 \text{fm} \). This cut-off is small enough by far: in most partial waves the phase shifts have already converged in the range \( r_c = 0.6 - 0.8 \text{fm} \). Smaller cut-off radii are in principle possible, but require too much computing time for the higher partial waves, while cut-off radii larger than \( 0.8 - 1.0 \text{fm} \) yield amplitudes which depend linearly on the cut-off for low partial waves. The appearance of the first deeply bound state usually happens in the \( 0.5 - 1.0 \text{fm} \) region, the exact location depending on the particular partial wave considered. Even in the case of G- and H-waves there are usually between two and three deeply bound states at \( r_c = 0.3 \text{fm} \). At these distances the wave functions are dominated by the van der Waals behaviour of the NLO- \( \Delta \) and N^2LO- \( \Delta \) potentials, meaning that the correlated renormalization procedure is guaranteed to work. This does not imply however that the low energy phase shifts are dominated by the singular structure of the chiral potentials at distances below the pion Compton wavelength. In fact, as will be commented in the following paragraphs, the results for peripheral waves do not significantly differ from those computed in first order perturbation theory [53]. The explicit cut-off dependence of the phase shifts is discussed in more detail for selected partial waves at the end of this section.

In Fig. (11), we show the results for the singlet waves. For the \( ^1S_0 \rightarrow ^3P_2, D_2 \rightarrow ^3P_2, G_2 \rightarrow ^3P_2, H_2 \rightarrow ^3P_2 \) correlation we have taken as input parameter the \( ^1S_0 \) \((^1P_1)\) scattering length from the Nijmegen II potential [5], which was computed in Ref. [62] yielding the result \( a_{^1S_0} = -23.727 \text{fm} \) \( (a_{^1P_1} = 2.797 \text{fm}^3) \). For the \( G_2 \) phase in the isovector channels and the \( F_3 \) wave in the isosinglet, the phase shifts do not differ much from those obtained in Ref. [52] in the Born approximation (in the previous reference only waves with \( l = 2, 3, 4 \) were considered). The \( H_2 \) is also very similar to the phases obtained in Refs. [16 17] for the NLO-\( \Delta \) potential. These waves are also quite similar to those obtained in Ref. [32] by renormalizing the N^2LO potential for the \( \Delta \)-less theory in a wave-by-wave basis, that is, by fixing the scattering lengths separately in each of the channels to their Nijmegen II values. In general, peripheral partial waves will not notice too much the inclusion of the two pion exchange interaction or the \( \Delta \) excitation and will behave very similarly as in first order perturbation theory. In this regard, the partial wave correlation is useful mainly as a way to renormalize all the peripheral waves with a minimum number of counterterms, but not necessarily as a real correlation. The only wave in which it can be effectively noticed is in the \( ^1D_2 \) one, in which the \( ^1S_0 \rightarrow ^1D_2 \) correlation predicts a scattering length of \( a_{^1D_2} = -1.728 \text{fm}^5 \) for the \( ^1D_2 \) wave [9] to be compared with an optimal scattering length of \( a_{^1D_2} = -1.686 \text{fm}^5 \) for which the N^2LO-\( \Delta \) potential effectively reproduces the Nijmegen II results for \( E_{\text{LAB}} \leq 150 \text{MeV} \). The previous observation indicates the necessity of the specific \( ^1D_2 \) wave N^2LO counterterm in order to reproduce the results in this partial wave. The predicted value greatly differs from the one corresponding to the Nijmegen II or Reid93 potentials [7], namely \( a_{^1D_2,\text{Nijm2}} = -1.389 \text{fm}^5 \) and \( a_{^1D_2,\text{Reid93}} = -1.377 \text{fm}^5 \), which were computed in Ref. [62]. This discrepancy is however common in most effective field theory computations in which the scattering length is fixed, see for example Ref. [33], or the related comments in Refs. [14 46], where a subtractive regularization approach is employed. This inconsistency between the Nijmegen low energy pa-
parameters and the chiral $\Delta$ potentials is explained by the fact that the Nijmegen phenomenological potentials do not contain either two pion exchange contributions or $\Delta$ excitations, and was briefly commented in the previous references.

In Fig. (2), we show the results for the uncoupled triplet waves. We have taken as input parameter for the $^3P_1$ ($^3D_2$) correlation the Nijmegen II scattering length [62], namely $a_{^3P_1} = 1.529 \text{ fm}^3$ ($a_{^3D_2} = -7.405 \text{ fm}^3$). Taking these values does not yield the better possible results for the $^3P_1$ ($^3D_1$) wave, but generates renormalized results for the $^3F_3 - ^3H_5$ ($^3G_4$) waves. As it happened in the singlet case, the phase shifts for the higher partial waves are very similar to the values obtained in first order perturbation theory either in the $\Delta$-less [16] and $\Delta$-full [17, 53] cases.

In Figs. (3) and (4) we show the results for the coupled triplet waves. For the $^3S_1 - ^3D_1$ correlation, we have taken as input parameters the deuteron binding energy $E_D = 2.224575 \text{ MeV}$ and $D/S$ asymptotic ratio $\eta = 0.0256$, and the $^3S_1$ scattering length $a_{^3S_1} = 5.419 \text{ fm}$. The scattering solutions are then constructed by orthogonality with respect to the deuteron wave function and the $^3S_1$ scattering state. The procedure is described in detail in Ref. [37], and was already used in Ref. [52] to construct the scattering solutions in the $^3S_1 - ^3D_1$ channel for the $\Delta$ potentials of Ref. [53]. For $r_c = 0.3 \text{ fm}$, we obtain the values $a_{^3S_1} = 1.953 \text{ fm}^3$ and $a_{^3D_1} = 5.034 \text{ fm}^3$ for the scattering lengths. The previous low energy information yields much better results than the use of the Nijmegen scattering length for this channel ($a_{^3S_1} = 1.647 \text{ fm}^3$ and $a_{^3D_1} = 6.505 \text{ fm}^3$), which induce a spurious resonance at $k_{cm} \simeq 100 \text{ MeV}$ when the $N^2\text{LO-}\Delta$ potentials are employed. This behaviour can also happen when using the standard chiral potentials without explicit $\Delta$ degrees of freedom, as was discussed in Refs. [44][46]. The corresponding phase shift for the $^3D_1$ is slightly better than the one obtained in Ref. [53], while the $^3G_3$ phase and the $\epsilon_3$ mixing parameter are quite similar to the ones obtained in the previous reference. The results for the $^3G_5 - ^3I_3$ coupled channel are good in general with the exception of the $^3G_3$ phase in which only the threshold behaviour is correctly reproduced.

In the case of the $^3P_2 - ^3F_2$ waves, we renormalize these waves by fixing the scattering lengths to the values $a_{^3P_2} = -0.320 \text{ fm}^3$, $a_{^3F_2} = 1.936 \text{ fm}^3$ and $a_{^3F_2} = -1.289 \text{ fm}^3$ which provide an acceptable description of the phase shifts for this channel, see Fig. (4). As happened in the $^3S_1 - ^3D_1$ channel, the Nijmegen II values for the scattering lengths [62] do not yield good results with the $NLO - \Delta$ and $N^2\text{LO} - \Delta$ potentials. However, in the $\Delta$-less theory the Nijmegen II scattering lengths generated good results at $N^2\text{LO}$ [30], meaning that the discrepancy is due to the long range physics introduced by the $\Delta$ excitations. The $^3F_2$ phases show a strong relative cut-off dependence in the range $r_c = 0.6 - 0.8 \text{ fm}$, although this is partly due to the small value of this phase. As in the previous cases, the $^3F_4 - ^3H_4$ waves are very similar to those of Ref. [53].

Finally, in Fig. (5) we consider the cut-off dependence of the phase shifts in the form of a logarithmic (or Lepage’s [20]) plot. For simplicity, we only consider two sets of correlated partial waves in detail. $^1S_0 - ^1D_2 - ^1G_4$ and $^3P_1 - ^3F_3 - ^3H_5$, a singlet and a triplet. For the other partial waves the cut-off dependence follows a similar pattern. According to Sect. IV E, for small enough cut-offs ($2m_v r_c \ll 1$) the convergence of the phase shift in the lower partial wave of the correlation (i.e. $^1S_0 - ^1P_1$ in this case) is given by

$$\log |\delta_A(k, r_c) - \delta_A(k, 0)| \simeq 4 \log r_c + C_A + f_A(r_c),$$

(93)

where $A = ^1S_0(^1P_1)$, $\delta_A(k, r_c)$ is the phase shift computed at the cut-off radius $r_c$, $\delta_A(k, 0)$ the phase shift in the $r_c \to 0$ limit, $C_A$ a constant, and $f_A(x)$ a small oscillatory contribution which takes into account the sine factor of the reduced wave function at short distances, see Eq. (13). As can be seen in Fig. (5), this behaviour is indeed fulfilled up to $r_c \sim 0.8 - 0.9 \text{ fm}$. In particular, the numerical factors multiplying the logarithms in the fits of Fig. (5) are very close to 4, indicating that the van der Waals contribution to the chiral potential dominates the behaviour of the wave functions at short distances. For the higher partial waves in the correlation, the expected scaling is

$$\log |\delta_B(k, r_c) - \delta_B(k, 0)| \simeq 2 \log r_c + C_B + f_B(r_c),$$

(94)

with $B = ^1D_2(^1G_4)$ and $B = ^3D_1(^3F_3 - ^3H_5)$. In these waves the van der Waals dominance is apparent for cut-off radii below $r_c \simeq 0.5 - 0.8 \text{ fm}$, with the lower bound corresponding to the most peripheral partial waves. It should be noted however that the appearance of van der Waals scaling in the renormalization group (RG) flow of the phase shifts for the higher partial waves does not imply that the phase shifts themselves are dominated by the $1/r^6$ piece of the interaction. The region in which the RG flow is driven by the chiral van der Waals force only amounts for a tiny contribution to the total phase shifts of the peripheral waves, as can be deduced from the large negative values of $\log |\delta(k, r_c) - \delta(k, 0)|$ in the case of the $^1G_4$ and $^3H_5$ waves, see Fig. (5). This feature fully agrees with the expectations of the renormalization approach.

VI. CONCLUSIONS

In the present paper we have considered the relation between the renormalization of attractive singular potentials and the partial wave expansion. Given that attractive singular interactions can be renormalized by including one counterterm per partial wave, each counterterm stabilizes the cut-off dependence in each one of the channels separately. While this is a sufficient condition for renormalizability it is actually not necessary. We have
shown that if the finite range (attractive singular) interaction is central, then it can be renormalized by means of a single delta-shell central potential in coordinate space, in contrast with the previous situation in which the predictive power is lost as there are an infinite number of partial waves. Of course, this result depends on the assumption that the unknown short range potential which is represented by a single delta-shell counterterm is central. Phenomenological potentials do depend on the orbital angular momentum at short distances [4–7]. For this more general situation in which nothing can be assumed about the short range interaction, the usual result of one counterterm per channel will be recovered.

Our analysis has been carried out in coordinate space, which on the other hand has been proven to be equivalent to momentum space calculations [41, 55]. The particularly interesting issue of extending the correlated renormalization method to momentum space, not addressed in the present work, is left for future research. A possible clue might be provided by the observation that the high momentum behaviour of the chiral potentials ought to reflect the partial wave independence observed and exploited in the present paper at short distances, suggesting a common subtraction perhaps along the lines of Refs. [43–45].

We have extended the previous result to the case of a finite range potential containing a tensor piece, which is of great interest for the renormalization of nuclear forces in the effective field theory approach. In that case, the number of counterterms depends on the sign of the eigenvalues of the coupled channel potential. The application to the chiral NN potentials with Δ excitations is possible and straightforward, and only requires to take into account the additional spin and isospin structure of the NN interaction. We stress that this is based on taking a counterterm structure based on the longer range OPE and TPE components of the interaction. For the order $Q^2$ and $Q^3$ Δ-potentials of Ref. [53] a total of eleven counterterms is found to be needed to completely renormalize the interaction in all channels. This is only two more counterterms than what Weinberg’s dimensional power counting dictates for the contact range interaction at the considered orders.

ACKNOWLEDGMENTS

We thank Alvaro Calle Cordón for discussions, Evgeny Epelbaum for a critical and careful reading of the manuscript, and Ulf-G. Meißner for correcting some references. M.P.V. is supported by the Helmholtz Association fund provided to the young investigator group “Few-
FIG. 3. (Color online) $^3S_1 - ^3D_1$, $^3D_3 - ^3G_3$ and $^3G_5 - ^3I_5$ coupled channel phase shifts. The $^3S_1 - ^3D_1$ wave is computed from orthogonality to the deuteron bound state and from the triplet scattering length $a_{0,t} = 5.419$ fm. The $^3D_3 - ^3G_3$ and $^3G_5 - ^3I_5$ coupled channels are computed from the partial wave correlation given by Eq. (83) without introducing new counterterms. We use the same cut-off values as in Figs. (1) and (2).

Nucleon Systems in Chiral Effective Field Theory” (grant VH-NG-222) and the virtual institute “Spin and strong QCD” (VH-VI-231). The work of E.R.A. is supported in part by funds provided by the Spanish DGI and FEDER funds with grant no. FIS2008-01143/FIS, and the Junta de Andalucía grant no. FQM225-05. M.P.V. and E.R.A. are supported by the EU HadronPhysics2 Project.

Appendix A: Partial Wave Correlations with the One Pion Exchange Potential

In this appendix we review the correlated renormalization for the one pion exchange potential case, which corresponds to the leading order piece of the chiral potential. The OPE potential can be decomposed as

$$V_{\text{OPE}}(\vec{r}) = \sigma \tau W_S(\vec{r}) + S_{12}(\vec{r}) \tau W_T(\vec{r}), \quad (A1)$$
where the operators $\sigma$, $\tau$ and $S_{12}$ were defined in Eq. (56) and $W_S(r)$ and $W_T(r)$ are given by

$$W_S(r) = \frac{m^2 g_A^2}{48 \pi f_0^2} e^{-m_{\pi} r} , \quad (A2)$$

$$W_T(r) = \frac{m^2 g_A^2}{48 \pi f_0^2} \left( 1 + \frac{3}{m_{\pi} r} + \frac{3}{(m_{\pi} r)^2} \right) e^{-m_{\pi} r} \quad (A3)$$

As can be seen, the only singular component of the OPE potential is the tensor piece. Therefore partial wave correlations only arise between attractive triplet partial waves. Specifically, there are three sets of correlated waves: (i) the $3^P_0 - 3^C_3 - 3^C_5$ case, which happens between coupled waves, (ii) the $3^P_0 - 3^C_2 - 3^C_3$ case, in which there is one uncoupled wave (the $3^P_0$) and the rest are coupled, and (iii) $3^D_2 - 3^G_4$ in which all waves are uncoupled triplets. All the coupled waves are of the attractive-repulsive type, and in total only three counterterms are needed in order to obtain finite scattering amplitudes for the OPE potential. Nonetheless it should be noted that in usual EFT computations a fourth counterterm will be added to renormalize the $1^S_0$ wave. In any case, we will only consider those waves which can be related.

Contrary to the NLO $- \Delta$ and N$^2$LO $- \Delta$ cases, the cut-off radius must be quite small in order for the partial wave correlations to converge (specially between the $3^P_0$
especially good; these waves improve noticeably with the
3. Below 0
5. As can be seen, the description of the renormalization conditions given in Eqs. (49-52). As can
3. The approximation confirms the dominance of the chiral van de Waals component (≈ 1/r^6) of the interaction for cut-off radii
3. In Fig. (7) we show the resulting phase shifts, which have been obtained by
3. The approximation confirms the dominance of the chiral van de Waals component (≈ 1/r^6) of the interaction for cut-off radii
3. The remaining j = 3 and j = 5 phases do not differ too much from their NLO-∆ and N^2LO-∆ counterparts, as expected from the fact that peripheral waves are OPE dominated.
3. In Fig. 7 we show the resulting phase shifts, which have been obtained by using the ^3P_0 wave as the base wave, and where the ^3P_0 scattering length has been taken to be a^3P_0 = -2.71 fm. We can see that the ^3F_2, ^3P_2 and ^3F_4 waves are not well
3. and ^3P_2 - ^3F_2 waves). In particular we take r_c = 0.15 fm.
3. The ^3C_1, ^3C_3 and ^3C_5 correlation is shown in Fig. 6. For this case, the ^3S_1 - ^3D_1 wave function is renormalized by reproducing the triplet ^3S_1 scattering length, a^3S_1 = 5.419 fm, a procedure which was described in detail in Ref. 39. The other partial waves are generated by the renormalization conditions given in Eqs. 101,102. As can be seen, the description of the E_1 and ^3D_3 wave is not especially good; these waves improve noticeably with the inclusion of two-pion exchange and the ∆. The remaining j = 3 and j = 5 phases do not differ too much from their NLO-∆ and N^2LO-∆ counterparts, as expected from the fact that peripheral waves are OPE dominated.
3. In Fig. 7 we show the resulting ^3P_0, ^3P_2 - ^3F_2 and ^3F_4 - ^3H_4 phase shifts, which have been obtained by using the ^3P_0 wave as the base wave, and where the ^3P_0 scattering length has been taken to be a^3P_0 = -2.71 fm. We can see that the ^3F_2, ^3P_2 and ^3F_4 waves are not well
reproduced with OPE alone and need the inclusion of the higher orders of the potential.

Finally, in Fig. 8 the $^3D_2$ and $^3G_4$ phase shifts are shown. The $^3D_2$ phase has been renormalized to reproduce the Nijmegen II value of the scattering length, $a_{^3D_2} = -7.405\,\text{fm}^5$. The OPE results for the $^3D_2$ are worse than those of NLO-Δ and N2LO-Δ at moderate energies of the order of $E_{\text{LAB}} > 150\,\text{MeV}$. The $^3G_4$ phase is nicely reproduced with OPE alone.

[1] G. Brown and A. D. Jackson, *The Nucleon-Nucleon Interaction* (North-Holland Publishing Company, Amsterdam, 1976)
[2] R. Machleidt, K. Holinde, and C. Elster, Phys. Rept. 149, 1 (1987)
[3] R. Machleidt, Adv. Nucl. Phys. 19, 189 (1989)
[4] V. G. J. Stoks, R. A. M. Kompl, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. C48, 792 (1993)
[5] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C49, 2950 (1994)
FIG. 7. $^3P_0$, $^3P_2 - ^3F_2$ and $^3F_4 - ^3H_4$ OPE phase shifts. The $^3P_0$ wave is constructed by fixing the scattering length to the value $a_{^3P_0} = -2.71$ fm$^3$, while the $^3P_2 - ^3F_2$ and $^3F_4 - ^3H_4$ wave are obtained from the partial wave correlation described in Eqs. (49-52) without introducing any new counterterm. We take the coordinate space cut-off $r_c = 0.15$ fm.

[6] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C51, 38 (1995), arXiv:nucl-th/9408016
[7] R. Machleidt, Phys. Rev. C63, 024001 (2001), arXiv:nucl-th/0006014
[8] N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007), arXiv:nucl-th/0611096
[9] S. Aoki, T. Hatsuda, and N. Ishii, Prog. Theor. Phys. 123, 89 (2010), arXiv:0909.5585 [hep-lat]
[10] S. Weinberg, Phys. Lett. B251, 288 (1990)
[11] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, 339 (2002), nucl-th/0203055
[12] E. Epelbaum, H.-W. Hammer, and U.-G. Meißner, Rev. Mod. Phys. 81, 1773 (2009), arXiv:0811.1338 [nucl-th]
[13] S. Weinberg, Nucl. Phys. B363, 3 (1991)
[14] C. Ordoñez, L. Ray, and U. van Kolck, Phys. Rev. Lett. 72, 1982 (1994)
[15] C. Ordoñez, L. Ray, and U. van Kolck, Phys. Rev. C53, 2086 (1996), hep-ph/9511380
[16] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. A625, 758 (1997), nucl-th/9706045
[17] N. Kaiser, S. Gerstendorfer, and W. Weise, Nucl. Phys. A637, 395 (1998), nucl-th/9802071
[18] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. A637, 107 (1998), nucl-th/9801064
[19] M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar, and J. J. de Swart, Phys. Rev. Lett. 82, 4992 (1999), nucl-th/9901054
[20] J. L. Friar, Phys. Rev. C60, 034002 (1999), nucl-th/9901082
FIG. 8. $^3D_2$ and $^3G_4$ phase shifts computed from Eq. (61), using the $^3D_2$ scattering length as an input parameter and the OPE potential with a coordinate space cut-off $r_c = 0.15$ fm.

[21] N. Kaiser, Phys. Rev. C61, 014003 (2000), arXiv:nucl-th/9910044
[22] N. Kaiser, Phys. Rev. C62, 024001 (2000), arXiv:nucl-th/9912054
[23] N. Kaiser, Phys. Rev. C65, 017001 (2002), arXiv:nucl-th/0109071
[24] N. Kaiser, Phys. Rev. C64, 057001 (2001), arXiv:nucl-th/0107064
[25] N. Kaiser, Phys. Rev. C63, 044010 (2001), arXiv:nucl-th/0101052
[26] D. R. Entem and R. Machleidt, Phys. Rev. C66, 014002 (2002), arXiv:nucl-th/0202039
[27] D. R. Entem and R. Machleidt, Phys. Rev. C68, 041001 (2003), arXiv:nucl-th/0304018
[28] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. A747, 362 (2005), arXiv:nucl-th/0405048
[29] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C72, 054006 (2005), arXiv:nucl-th/0506005
[30] G. P. Lepage (1997), arXiv:nucl-th/9706029
[31] E. Epelbaum and U. G. Meißner (2006), arXiv:nucl-th/0609037
[32] E. Epelbaum and J. Gegelia, Eur. Phys. J. A41, 341 (2009), arXiv:0906.3822 [nucl-th]
[33] S. Aoki, J. Balog, and P. Weisz, PoS LAT2009, 132 (2009), arXiv:0910.4255 [hep-lat]
[34] S. Aoki, J. Balog, and P. Weisz, JHEP 1005, 008 (2010), arXiv:1002.0977 [hep-lat]
[35] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C74, 054001 (2006), arXiv:nucl-th/0506047
[36] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C74, 064004 (2006), arXiv:nucl-th/0507075
[37] M. C. Birse, Phys. Rev. C74, 014003 (2006), arXiv:nucl-th/0507077
[38] R. Machleidt and D. Entem, J.Phys.G G37, 064041 (2010), arXiv:1001.0966 [nucl-th]
[39] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C72, 054002 (2005), arXiv:nucl-th/0504067
[40] M. Pavon Valderrama and E. Ruiz Arriola, Annals Phys. 323, 1037 (2008), arXiv:0705.2952 [nucl-th]
[41] D. R. Entem, E. Ruiz Arriola, M. Pavon Valderrama, and R. Machleidt, Phys. Rev. C77, 044006 (2008), arXiv:0709.2770 [nucl-th]
[42] S. R. Beane et al., Phys. Rev. A64, 042103 (2001), quant-ph/0010073
[43] C.-J. Yang, C. Elster, and D. R. Phillips, Phys.Rev. C77, 014002 (2008), arXiv:0706.1242 [nucl-th]
[44] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. C80, 034002 (2009), arXiv:0901.2663 [nucl-th]
[45] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. C80, 044002 (2009), arXiv:0905.4943 [nucl-th]
[46] C.-J. Yang, C. Elster, and D. Phillips, PoS CD09, 064 (2009), arXiv:0909.5414 [nucl-th]
[47] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998), nucl-th/9801034
[48] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B478, 629 (1999), nucl-th/9805002
[49] S. Fleming, T. Mehen, and I. W. Stewart, Nucl. Phys. A677, 313 (2000), nucl-th/9911001
[50] S. R. Beane, D. B. Kaplan, and A. Vuorinen, Phys.Rev. C80, 011001 (2009), arXiv:0812.3938 [nucl-th]
[51] M. Pavon Valderrama, Phys. Rev. C83, 024003 (2011), arXiv:0912.0699 [nucl-th]
[52] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C79, 044001 (2009), arXiv:0809.3186 [nucl-th]
[53] H. Krebs, E. Epelbaum, and U.-G. Meißner, Eur. Phys. J. A32, 127 (2007), arXiv:nucl-th/0703087
[54] E. Ruiz Arriola and A. Calle Cordon (2009), arXiv:0910.1333
[55] M. Pavon Valderrama, A. Nogga, E. Ruiz Arriola, and D. R. Phillips, Eur. Phys. J. A36, 315 (2008), arXiv:0711.4785 [nucl-th]
[56] R. Machleidt, P. Liu, D. R. Entem, and E. Ruiz Arriola, Phys.Rev. C81, 024001 (2010), arXiv:0910.3942 [nucl-th]
[57] B. Long and U. van Kolck, Annals Phys. 323, 1304 (2008), arXiv:0707.4325 [quant-ph]
[58] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C74, 054001 (2006), arXiv:nucl-th/0506047
[59] M. Pavon Valderrama and E. Ruiz Arriola, Annals Phys. 323, 1037 (2008), arXiv:0705.2952 [nucl-th]
[60] D. R. Entem, E. Ruiz Arriola, M. Pavon Valderrama, and R. Machleidt, Phys. Rev. C77, 044006 (2008), arXiv:0709.2770 [nucl-th]
[61] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Eur. Phys. J. A19, 125 (2004), nucl-th/0304037
[62] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Eur. Phys. J. A19, 401 (2004), nucl-th/0308010
[63] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C72, 044007 (2005)