Self-similar extrapolation of nonlinear problems from small-variable to large-variable limit

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Abstract

Complicated physical problems usually are solved by resorting to perturbation theory leading to solutions in the form of asymptotic series in powers of small parameters. However, finite, and even large values of the parameters often are of the main physical interest. A method is described for predicting the large-variable behaviour of solutions to nonlinear problems from the knowledge of only their small-variable expansions. The method is based on self-similar approximation theory resulting in self-similar factor approximants. The latter can well approximate a large class of functions, rational, irrational, and transcendental. The method is illustrated by several examples from statistical and condensed matter physics, where the self-similar predictions can be compared with the available large-variable behaviour. It is shown that the method allows for finding the behavior of solutions at large variables when knowing just a few terms of small-variable expansions. Numerical convergence of approximants is demonstrated.

\textit{Keywords:} Nonlinear physical problems; asymptotic series; self-similar extrapolation; large-variable behaviour; numerical convergence

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1 Introduction

There exist numerous problems in statistical physics, condensed-matter theory, and atomic physics, where the sought characteristic is defined by so complicated equations that it can be found only in the region of small parameters, being given by an asymptotic expansion. However, one may need this characteristic in the region of finite parameters and, moreover, one often needs the knowledge of the characteristic behaviour at very large parameters tending to infinity. In the present paper, we concentrate on the latter, the most difficult, problem of defining the large-variable behaviour of functions from the knowledge of only their small-variable asymptotic expansions.

It is clear that Padé approximants cannot describe the large-variable behaviour of the sought quantity for which the large-variable limit is not known [1]. This is because a Padé approximant \( P_{M/N}(x) \) describes the large-variable behaviour as proportional to \( x^{\frac{M-N}{N}} \), with an integer power, while, generally, this power can be irrational. Moreover, for the same problem one can define the whole table of Padé approximants, with different exponents \( M - N \) in the term \( x^{M-N} \), so that no concrete prediction can be made.

In addition, there exists the well known problem of incompatibility of small-variable and large-variable expansions. This is when the small-variable expansion is in powers of \( x^\alpha \), while the large-variable expansion is in powers of \( x^\beta \), such that \( \alpha/\beta \) is an irrational number, because of which there is no such a change of the variable that could allow for sewing the small-variable and large-variable limits. Several other problems in using Padé approximants have been discussed in [1–4].

The other way that could make it possible to guess the large-variable behaviour is the method of Borel transforms [5, 6]. But this method requires the knowledge of high-order terms of a small-variable expansion, which is rarely available. And it is not applicable to the cases where just a few terms of the small-variable expansion are provided. Several other methods of constructing approximate solutions to nonlinear problems are reviewed in Ref. [7].

In the present paper, we describe an original method that makes it straightforward to get reasonably accurate estimates for the large-variable limits of the sought solutions from the knowledge of only their small-variable expansions. The method is based on self-similar approximation theory that was advanced in Refs. [8] and described in all details in the previous publications [9–13]. The efficiency of the self-similar approximation theory stems from its use of the ideas of dynamic theory and renormalization-group theory [14–17], as well as of optimal control theory [18,19]. The use of renormalization-group theory is known to provide physically reasonable solutions to problems with divergencies [20,21].

In the frame of the self-similar approximation theory, the small-variable asymptotic expansions can be transformed into self-similar factor approximants, extrapolating the given asymptotic expansions to the finite values of their variables [22,24]. Here we demonstrate that the method allows us to extrapolate small-variable series, not merely to finite variables, but it makes it possible to predict the large-variable limits of the sought solutions.

Since the justification for the self-similar approximation theory has been expounded in our previous papers, we do not repeat here the published earlier material. A brief account of the basic points of the self-similar approximation theory can be found in the recent article [25]. Here we only briefly recall the main steps in using the method of self-similar factor approximants [22,24].
By treating the cases, where many terms of the asymptotic expansions are known, we prove that the method is numerically convergent. Moreover, we demonstrate that it predicts rather good estimates even for the cases, where just a few terms are available. We show this by several examples from statistical physics, condensed-matter theory, and atomic physics, in which the large-variable limits are known, thus, providing us the possibility of estimating the accuracy of our predictions.

2 Self-similar factor approximants

Let the bare approximants, valid at asymptotically small $x \to 0$, have the form

$$f_k(x) = f_0(x) \left(1 + \sum_{n=1}^{k} a_n x^n\right), \quad (1)$$

where $f_0(x)$ is a given function. The sum in the brackets is usually divergent for finite values of $x$ and makes no sense for such $x$. Moreover, in many cases it is important to know the behaviour of the sought function $f(x)$ at asymptotically large $x \to \infty$. This is exactly the problem we shall concentrate on.

First, by the fundamental theorem of algebra \[26\], it is known that a polynomial of any degree of one real variable over the field of real numbers can be split in a unique way into a product of irreducible first-degree polynomials over the field of complex numbers. This means that the finite series (1) can be represented as the product

$$f_k(x) = f_0(x) \prod_j (1 + b_j x), \quad (2)$$

with $b_j$ expressed through $a_n$.

Self-similar properties can be explicitly incorporated into a map employing fractal transforms \[27,28\]. Keeping this in mind, we introduce control functions $s_k$ and $u_k$ by the fractal transform

$$\hat{T}[s,u] f_k(x) = x^{s_k} f_k(x) + u_k. \quad (3)$$

Then following the scheme, described in Refs. \[22,23\], we obtain the self-similar factor approximants

$$f_k^*(x) = f_0(x) \prod_{j=1}^{N_k} (1 + A_j x)^{n_j}, \quad (4)$$

with $A_j$ and $n_j$ playing the role of control parameters. The number of factors $N_k$ equals $k/2$ for even $k$ and $(k+1)/2$ for odd $k$. If the factor approximant (4) represents the sought function, then their asymptotic expansions should coincide. Therefore, the parameters $A_j$ and $n_j$ have to be chosen so that the asymptotic expansion of (4), of order $k$, be equal to the asymptotic form (1), that is,

$$f_k^*(x) \simeq f_k(x) \quad (x \to 0). \quad (5)$$
This condition yields the equations
\[
\sum_{j=1}^{N_k} n_j A_j^n = D_n \quad (n = 1, 2, \ldots, k),
\] (6)

where
\[
D_n \equiv (-1)^{n-1} \lim_{x \to 0} \frac{d^n}{dx^n} \ln \left( 1 + \sum_{m=1}^{n} a_m x^m \right).
\]

When \( k \) is even, and \( N_k = k/2 \), we have \( k \) equations for \( k \) unknown parameters \( A_j \) and \( n_j \), uniquely defining these parameters \[29\]. But if \( k \) is odd, and \( N_k = (k + 1)/2 \), we have \( k \) equations for \( k + 1 \) parameters. Then, to make the system of equations complete, it is required to add one more condition. One possibility, based on scaling arguments \[29\], could be to set one of \( A_j \) to one, say fixing \( A_1 = 1 \). Although this condition is acceptable and gives the results close to the nearest even-order approximants, we shall not use it below, dealing mainly with uniquely defined even orders.

It is important to emphasize that there exists a large class of functions that are exactly reproducible by the factor approximants \[22, 24\]. This class consists of the functions that can be reduced to the form
\[
R_M(x) = \prod_{j=1}^{M} P_{m_j}^{\alpha_j}(x),
\] (7)

where \( P_m \) are polynomials of degree \( m \) and \( \alpha_j \) are any real numbers or complex-valued numbers entering by pairs with their complex-conjugate, which makes the function \( R_M(x) \) real. A function from this class is exactly reproducible by factor approximants of order
\[
k \geq \sum_{j=1}^{M} m_j.
\]

Thus the class of exactly reproducible functions includes rational, as well as irrational functions. Moreover, it also includes transcendental functions that are the limits of form (7). For instance, the exponential function is exactly reproducible for any order \( k \geq 2 \), due to the limit
\[
\lim_{b \to 0} P_1^{1/b}(x) = a e^{x/a} \quad P_1(x) = a + bx.
\]

As is stressed above, in the present paper, we are interested in the possibility of predicting the large-variable behaviour. Let the given function \( f_0 \) behave as
\[
f_0(x) \simeq A x^\alpha \quad (x \to \infty).
\] (8)

Then the self-similar factor approximant \[4\] for large \( x \) is
\[
f^*_x(x) \simeq B_k x^{\gamma_k} \quad (x \to \infty),
\] (9)

with the amplitude
\[
B_k = A \prod_{j=1}^{N_k} A_j^{n_j}.
\] (10)
and the exponent
\[ \gamma_k = \alpha + \sum_{j=1}^{N_k} n_j. \]  
(11)

In those cases, where the large-variable asymptotic behaviour of the sought function is known, say being
\[ f(x) \simeq B x^\gamma \quad (x \to \infty), \]  
(12)
we can determine the accuracy of our prediction (9) by calculating the percentage errors
\[ \varepsilon(B_k) \equiv \frac{B_k - B}{B} \times 100\%, \quad \varepsilon(\gamma_k) \equiv \frac{\gamma_k - \gamma}{\gamma} \times 100\%. \]  
(13)

And if the exact large-variable asymptotic behaviour is not available, then the accuracy can be estimated by the difference between the subsequent values of the sought quantity.

3 Convergence of factor approximants

In order to show that self-similar factor approximants provide convergent series, we need to consider the problems, where a number of terms are available. Below we study several such cases.

3.1 Partition function of zero-dimensional oscillator

The standard problem that one almost always considers as a probe of numerical procedures is the partition function of zero-dimensional oscillator
\[ Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\varphi^2 - g \varphi^4} d\varphi. \]  
(14)
Expanding this function in powers of the coupling parameter yields the series
\[ Z_k(g) = \sum_{n=0}^{k} a_n g^n \quad (g \to 0), \]  
(15)
with the coefficients
\[ a_n = \frac{(-1)^n}{\sqrt{\pi} n!} \Gamma \left( 2n + \frac{1}{2} \right). \]

The obtained series have the typical structure of many problems in condensed-matter theory and statistical physics, with factorially growing coefficients signaling divergence of the series for any finite value of \( g \). The strong-coupling limit is known to be
\[ Z(g) \simeq 1.022765 g^{-1/4} \quad (g \to \infty). \]  
(16)

Defining factor approximants (4) for series (15), we consider the limit
\[ Z^*_k(g) \simeq B_k g^{\gamma_k} \quad (g \to \infty). \]  
(17)

The results for the amplitude \( B_k \) and exponent \( \gamma_k \), together with their percentage errors (13) are listed in Table 1. We see the monotonic decrease of the errors, which means the evident numerical convergence to the exact values for both the amplitude and exponent.
Table 1: Amplitudes and exponents, with their percentage errors, for the strong-coupling limit of the zero-dimensional partition function, predicted by self-similar factor approximants.

| $k$ | $B_k$  | $\varepsilon(B_k)\%$ | $\gamma_k$  | $\varepsilon(\gamma_k)\%$ |
|-----|--------|-----------------------|-------------|-----------------------|
| 2   | 0.823  | -19.5                 | -0.090      | -62.5                 |
| 4   | 0.806  | -21.2                 | -0.129      | -48.4                 |
| 6   | 0.806  | -21.2                 | -0.148      | -40.6                 |
| 8   | 0.810  | -20.8                 | -0.161      | -35.6                 |
| 10  | 0.814  | -20.4                 | -0.170      | -32.0                 |
| 12  | 0.819  | -19.9                 | -0.178      | -29.3                 |
| 14  | 0.824  | -19.4                 | -0.182      | -27.1                 |
| 16  | 0.828  | -19.0                 | -0.187      | -25.4                 |

3.2 Ground-state energy of anharmonic oscillator

The other typical touchstone for analyzing numerical methods is the anharmonic oscillator with the Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + gx^4, \quad (18)$$

in which $x \in (-\infty, \infty)$ and $g \in [0, \infty)$. The ground-state energy of this Hamiltonian can be found [30, 31] as an asymptotic expansion in powers of the coupling parameter $g$,

$$E(g) \simeq \frac{1}{2} + \sum_{n=1}^{k} a_n g^n \quad (g \to 0). \quad (19)$$

The coefficients $a_n$ can be found in [30,31], with the first of them being

$$a_1 = 0.75, \quad a_2 = -2.625, \quad a_3 = 20.8125, \quad a_4 = -241.2890625, \quad (20)$$

$$a_5 = 3580.98046875, \quad a_6 = -63982.8134766, \quad a_7 = 1329733.72705,$$

$$a_8 = -31448214.6928, \quad a_9 = 833541603.263, \quad a_{10} = -24478940702.8 \quad (21).$$

The strong-coupling limit is

$$E(g) \simeq 0.667986 g^{1/3} \quad (g \to \infty). \quad (20)$$

Constructing factor approximants and considering their strong-coupling limit

$$E_k^*(g) \simeq B_k g^{\gamma_k} \quad (g \to \infty), \quad (21)$$

we find the amplitudes and exponents, with their percentage errors, listed in Table 2. Again we observe the monotonic decrease of the errors, implying the evident numerical convergence.
Table 2: Amplitudes and exponents, with the related percentage errors, for the strong-coupling limit of the one-dimensional anharmonic oscillator ground-state energy predicted by self-similar factor approximants.

| $k$ | $B_k$ | $\varepsilon(B_k)\%$ | $\gamma_k$ | $\varepsilon(\gamma_k)\%$ |
|-----|-------|----------------------|------------|----------------------|
| 2   | 0.729 | 9.2                  | 0.176      | -47.0                |
| 4   | 0.755 | 13.1                 | 0.231      | -30.6                |
| 6   | 0.756 | 13.2                 | 0.257      | -22.9                |
| 8   | 0.752 | 12.6                 | 0.272      | -18.4                |
| 10  | 0.748 | 11.9                 | 0.282      | -15.5                |
| 12  | 0.743 | 11.2                 | 0.289      | -13.4                |
| 14  | 0.739 | 10.7                 | 0.294      | -11.9                |
| 16  | 0.736 | 10.2                 | 0.298      | -10.7                |

3.3 Mittag-Leffler function in kinetic equations

Mittag-Leffler functions appear naturally in the solutions of fractional order integral and differential equations describing fractional generalizations of kinetic equations, random walks, Lévy flights, and super-diffusive transport. Here we consider the Mittag-Leffler function

$$E(x) = e^{x^2} \text{erfc}(x) = e^{x^2} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt . \quad (22)$$

It is straightforward to find the function expansion of arbitrary order in powers of $x$, with the low orders being

$$E(x) \simeq 1 - \frac{2}{\sqrt{\pi}} x + x^2 - \frac{4}{3\sqrt{\pi}} x^3 + \frac{x^4}{2} \quad (x \to 0) . \quad (23)$$

And the large-variable limit is

$$E(x) \simeq \frac{1}{\sqrt{\pi}} x^{-1} \quad (x \to \infty) . \quad (24)$$

This function is not easy to approximate, since it consists of two different functions each producing its own expansion. To catch at least a couple of the expansion terms of each of these two functions, one needs to consider at least the fourth-order expansion of the whole product.

Constructing factor approximants, we find the amplitudes and exponents at large $x$,

$$E_k(x) \simeq B_k x^{\gamma_k} \quad (x \to \infty) , \quad (25)$$

predicted by these expressions, which are shown in Table 3, together with their percentage errors. At the beginning, there are oscillations in the values of the errors for the amplitude, but from sixth order, the errors monotonically decrease. The convergence is quite fast, reaching in the twelfth order the error of only 3% for the amplitude $B_k$ and 0.7% for the exponent.
Table 3: Amplitudes and exponents, with their percentage errors, for the large-variable limit of the Mittag-Leffler function, predicted by self-similar factor approximants.

| $k$ | $B_k$   | $\varepsilon(B_k)\%$ | $\gamma_k$ | $\varepsilon(\gamma_k)\%$ |
|-----|---------|-----------------------|------------|--------------------------|
| 4   | 0.209   | −62.9                 | −0.618     | −38.2                    |
| 6   | 0.926   | 64.1                  | −1.160     | 16.0                     |
| 8   | 0.457   | −18.9                 | −0.939     | −6.1                     |
| 10  | 0.612   | 8.4                   | −1.02      | 2.2                      |
| 12  | 0.548   | −2.9                  | −0.993     | −0.7                     |

3.4 Massive Schwinger model in Hamiltonian lattice theory

Massive Schwinger model \[32\] is the lattice model imitating quantum theory in 1 + 1 space-time dimensions. This model exhibits several phenomena that are typical of quantum chromodynamics, such as confinement, chiral symmetry breaking with an axial anomaly, and a topological vacuum, which makes it, probably, the simplest nontrivial gauge theory \[33–37\]. This is why the Schwinger model is considered as a standard testbed for the trial of new techniques. Calculating the spectrum of excited states for a finite lattice, one meets \[38\] the functions of the type

$$f(z) \simeq 1 + \sum_{k=1}^{k} a_n z^n \quad (z \to 0)$$

(26)

expanded in powers of the variable $z \equiv 1/(ga)^4$, where $g$ is the coupling parameter and $a$ is the lattice spacing. For the vector boson, one has the fastly growing coefficients

$$a_1 = 2, \quad a_2 = -10, \quad a_3 = 78.66667, \quad a_4 = -7.362222 \times 10^2,$$

$$a_5 = 7.572929 \times 10^3, \quad a_6 = -8.273669 \times 10^4, \quad a_7 = 9.428034 \times 10^5,$$

$$a_8 = -1.108358 \times 10^7, \quad a_9 = 1.334636 \times 10^8, \quad a_{10} = -1.637996 \times 10^9.$$

The large $z$ limit is

$$f(z) \simeq 1.1284z^{1/4} \quad (z \to \infty).$$

(27)

Defining the factor approximants, we find the limiting behaviour

$$f_k^*(z) \simeq B_k z^{\gamma_k} \quad (z \to \infty).$$

(28)

The results for the amplitude and exponent are shown in Table 4. Again numerical convergence is clearly seen.

4 Extrapolation with small number of terms

It is important that self-similar factor approximants can extrapolate the series, derived for asymptotically small variables, to the large-variable limit, achieving good accuracy already for low-order approximants.
Table 4: Amplitudes and exponents, with the related errors, for the large-variable limit of the function \( f(z) \) for the finite-lattice Schwinger model, predicted by self-similar factor approximants.

| \( k \) | \( B_k \) | \( \varepsilon(B_k)\% \) | \( \gamma_k \) | \( \varepsilon(\gamma_k)\% \) |
|-------|-------|-------------|-------|-------------|
| 2     | 1.513 | 34.1        | 0.167 | −33.0       |
| 4     | 1.532 | 35.8        | 0.185 | −26.2       |
| 6     | 1.530 | 35.6        | 0.193 | −22.7       |
| 8     | 1.523 | 35.0        | 0.198 | −20.7       |
| 10    | 1.519 | 34.6        | 0.200 | −19.8       |

4.1 Ground-state energy of Schwinger model in continuum limit

The continuum limit for the ground-state energy of the Schwinger model, corresponding to a vector boson, can be found \([34–37]\) as an expansion in powers of the dimensionless variable \( x = m/g \), where \( m \) is the electron mass and \( g \) is the coupling parameter,

\[
\frac{E(x)}{g} \simeq 0.5642 - 0.219x + 0.1907x^2 \quad (x \to 0). \tag{29}
\]

A small \( x \to 0 \) implies strong coupling \( g \to \infty \). In the opposite limit of weak coupling, hence large \( x \), one has \([37–40]\)

\[
\frac{E(x)}{g} \simeq 0.6417x^{-1/3} \quad (x \to \infty). \tag{30}
\]

The number of small \( x \) terms is sufficient for constructing the sole factor approximant

\[
\frac{E^*_2(x)}{g} = \frac{0.5642}{(1 + 1.35339x^{0.286805})}. \tag{31}
\]

The large-\( x \) limit of the latter becomes

\[
\frac{E^*_2(x)}{g} \simeq 0.5173x^{-0.287}. \tag{32}
\]

The related percentage errors are

\[
\varepsilon(B_2) = -17.8\%, \quad \varepsilon(\gamma_2) = -13.9\%.
\]

4.2 Ground-state energy of harmonium atom

There exist finite quantum systems of different nature, but allowing for similar mathematical modeling \([41]\). One of such models is the model of harmonium atom described by the Hamiltonian

\[
H = \sum_{i=1}^{N} \left( -\frac{\nabla_i^2}{2} + \frac{\omega_i^2}{2} r_i^2 \right) + \frac{1}{2} \sum_{i \neq j}^{N} \frac{1}{|r_i - r_j|}. \tag{33}
\]
This model provides a reasonable description for several finite quantum systems, such as quantum dots, trapped ions, atomic nuclei, and metallic grains [41], because of which it has been intensively studied [42–46].

At a shallow harmonic potential, the ground-state energy of a two-electron harmonium atom can be presented [42] as an expansion in powers of the frequency \( \omega \),

\[
E(\omega) \simeq \frac{3}{2} \omega^{2/3} + \frac{3 + \sqrt{3}}{2} \omega + \frac{7}{36 \cdot 2^{4/3}} \omega^{4/3} \quad (\omega \to 0). \tag{34}
\]

For a rigid potential, one has

\[
E(\omega) \simeq 3\omega \quad (\omega \to \infty). \tag{35}
\]

Again, there are enough terms only for the construction of the second-order factor approximant

\[
E_2^*(\omega) = \frac{3}{2} \omega^{2/3} \left(1 + 1.88379\omega^{1/3}\right)^{1.05496}. \tag{36}
\]

The predicted large-\( \omega \) behaviour is

\[
E_2^*(\omega) \simeq 2.322\omega^{1.018} \quad (\omega \to \infty), \tag{37}
\]

which gives the errors

\[
\varepsilon(B_2) = -22.6\% , \quad \varepsilon(\gamma_2) = 1.8\%. \]

### 4.3 Anomalous dimension in supersymmetric theory

In the \( N = 4 \) supersymmetric Yang-Mills theory, the planar casp anomalous dimension of a lightlike Wilson loop \( \Gamma(g) \) is a function of the coupling parameter \( g \), whose behaviour is known for asymptotically small \( g \) and in the strong-coupling limit [47–52]. The weak-coupling expansion is

\[
\Gamma(g) \simeq 4g^2 - \frac{4\pi^2}{3} g^4 + \frac{44\pi^4}{45} g^6 \quad (g \to 0), \tag{38}
\]

while the strong-coupling limit gives

\[
\Gamma(g) \simeq 2g \quad (g \to \infty). \tag{39}
\]

The factor approximant

\[
\Gamma_2^*(g) = \frac{4g^2}{(1 + 11.1856g^2)^{0.294118}} \tag{40}
\]

predicts the strong-coupling limit

\[
\Gamma_2^*(g) \simeq 1.966g^{1.412} \quad (g \to \infty). \tag{41}
\]

This corresponds to the percentage errors

\[
\varepsilon(B_2) = -1.7\% , \quad \varepsilon(\gamma_2) = 41.2\%. \]
Table 5: Amplitudes and exponents, with the related errors, for the strong-coupling limit of the polymer chain expansion factor, predicted by self-similar factor approximants.

| $k$ | $B_k$ | $\varepsilon(B_k)\%$ | $\gamma_k$ | $\varepsilon(\gamma_k)\%$ |
|-----|------|-----------------|-----------|-----------------|
| 2   | 1.564 | 2.2             | 0.300     | -15.4           |
| 4   | 1.560 | 1.9             | 0.340     | -4.1            |
| 6   | 1.551 | 1.3             | 0.348     | -1.9            |

4.4 Expansion factor of polymer chain

Polymers have an extremely wide region of applicability, because of which their study has always been a rather hot topic [53]. The expansion factor of a polymer chain, as a function of the coupling parameter [54, 55] has the weak-coupling expansion

$$\alpha(g) \simeq 1 + \sum_{n=1}^{6} a_n g^n \quad (g \to 0),$$

with the fast growing coefficients

$$a_1 = \frac{4}{3}, \quad a_2 = -2.075385396, \quad a_3 = 6.296879676,$$

$$a_4 = -25.05725072, \quad a_5 = 116.134785, \quad a_6 = -594.71663.$$

The strong-coupling limit [54,55] is

$$\alpha(g) \simeq 1.531g^{0.3544} \quad (g \to \infty).$$

Composing the factor approximants, we find the predicted strong-coupling limit

$$\alpha^*_k(g) \simeq B_k g^{\gamma_k} \quad (g \to \infty),$$

with the amplitudes and exponents presented in Table 5, together with the related percentage errors. Again we observe good numerical convergence.

5 Conclusion

We have presented a method that makes it possible to extrapolate small-variable asymptotic expansions to the region of asymptotically large variables. This region is interesting because of two reasons. First, mathematically it is the most difficult for extrapolations that use only the coefficients of small-variable expansions. Second, the behaviour of the sought quantities in this region is often of high physical interest.

The described method is based on self-similar approximation theory that combines the ideas of dynamical theory, optimal control theory, and renormalization-group theory. It is the use of these techniques that makes the self-similar approximation theory a powerful tool for extrapolating asymptotic series over small variables to the region of large variables.
By several examples, where a number of expansion terms are available, we have demonstrated the numerical convergence for the sequences of approximations for large-variable amplitudes and exponents. The method is shown to give reasonable approximations even when extrapolating small-variable expansions with just a few terms.

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