Polyphase Golay Complementary Arrays: The Possible Sizes and New Constructions
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Abstract—Motivated by the recent application of two-dimensional Golay complementary arrays (2-D GCA) in uniform rectangular array (URA)-based omnidirectional precoding for the next generation of cellular communications, we propose in this paper new constructions of GCA pairs and GCA quads, which allow for more possible sizes. These constructions are facilitated by four identities over a commutative ring, of which two are established results of quaternions and octonions and the others are novel. We prove that the size of a 4-phase GCA pair is possible if the product of the array sizes in all dimensions is a 4-phase Golay number with an additional constraint on the factorization of the product. For 2-phase 2-D GCA quads, all sizes no greater than 78 × 78 can be covered. For 4-phase GCA quads, all the positive integers within 1000 can be covered for the size in one dimension. Besides producing GCAs that can accommodate to antenna arrays of various sizes, this study also yields new Hadamard matrices of sizes previously unavailable.

Index Terms—Golay complementary array pair, GCA quad, Golay number, omnidirectional precoding.

I. INTRODUCTION

The Golay complementary sequence pair is a pair of sequences whose respective periodic autocorrelations add to be a (scaled) $\delta$-function [1]. By the Wiener-Khinchin theorem, their power spectrums add to be flat everywhere. Owing to this property, the Golay sequence pair and its variants have been considered in at least two engineering applications: generation of OFDM signals with reduced peak-to-average power ratio (PAPR) [2], [3], and omnidirectional precoding using an antenna arrays [4], [5]. For the former application, it is interesting to find as many Golay sequences as possible of some particular length [2], [6], [7], [8], [9]; for the latter, it is important to construct Golay arrays of flexible sizes for accommodating to antenna arrays of various sizes. This paper is dedicated to the construction of Golay arrays and to reveal the possible sizes.

The 2-phase Golay sequence pair with entries $\{1, -1\}$ was first constructed via Boolean functions as well as some recursive methods by Golay [1], [10]. Turyn improved the recursive construction to generate 2-phase Golay sequence pairs of length $2^n10^k26^m$ where $a, b, c \in \mathbb{Z}^+ \cup \{0\}$ [11]. The above lengths are referred to as 2-phase Golay numbers in this paper. It was verified by exhaustive computational search that the 2-phase Golay numbers cover all the 14 possible lengths within 100 [12].

For obtaining more possible lengths, the 2-phase Golay sequence set was considered in [13], which allows for $L \geq 2$ sequences whose respective autocorrelation functions add to be a scaled $\delta$-function. With more degrees of freedom, the 2-phase Golay sequence quads (i.e., $L = 4$) was conjectured to exist for arbitrary lengths [11], which, if proved, can lead to an immediate validation of the celebrated Hadamard conjecture [14]. In [15], a systematic construction of Golay sequence quad, more sophisticated than the method proposed by Turyn [11] for Golay sequence pair, was proposed to cover many more possible lengths. In [16], with the aid of computational search of base sequence quads, this construction can cover all the sizes within 78 for 2-phase Golay sequence quads.

Another variant of more possible sizes is the polyphase Golay sequence set discussed in [17], [18], and [19], whose entries are the $N$-th unit roots where $N \geq 2$. For 4-phase Golay sequence quads with entries $\{1, -1, j, -j\}$ where $j$ is the imaginary unit, the possible lengths obtained by recursive constructions are $2^n+3^e5^d11^c13^b$ where $a, b, c, d, e, u \in \mathbb{Z}^+ \cup \{0\}$, $b + c + d + e \leq a + 2u + 1$, $u \leq c + e$ [19]. The above lengths are referred to as 4-phase Golay numbers in this paper. Obviously, the 4-phase Golay numbers are much more abundant than the 2-phase Golay numbers. And recently, it has been verified via exhaustive computational search that the 4-phase Golay numbers cover all the 17 possible sizes within 28 [20]. For 4-phase Golay sequence quads, the possible sizes can cover 827 integers within 1000 [19].

The Golay sequences set can be generalized to the multi-dimensional Golay complementary array (GCA) set [4], [21], [22], [23], [24], [25], [26], of which the arrays’ respective multi-dimensional autocorrelations add to be a scaled multi-dimensional $\delta$-function. For 2-phase GCA pairs, the possible sizes are similar to the one-dimensional counterpart: the array size in each dimension is a 2-phase Golay number [22]. More recently, it has been established that there exist 4-phase 2-D GCA pairs of size $2s_1 \times s_2$ or $s_1 \times 2s_2$ and 4-phase 2-D GCA quads of size $s_1 \times s_2$ [4], [5], where $s_1$ and $s_2$ are 4-phase Golay numbers. More possible sizes of the 3-phase GCA triads, whose entries are $\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}j, -\frac{1}{2} - \frac{\sqrt{3}}{2}j\}$, have been discovered in [27] and [26].

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In this paper, we focus on the possible sizes of the polyphase GCA pairs and quads. The constructions are based on four identities over a commutative ring. Our contributions in this paper are two-fold. First, we propose a generalized construction of 4-phase GCA pairs and discover some new sizes that has not been reported in the literature. Second, we construct GCA quads whose possible sizes are much more abundant than the known literatures. Specifically, for 2-phase 2-D GCA quads we cover all the sizes within $78 \times 78$, and for 4-phase GCA quads we cover all the sizes within 1000 in one dimension. As an interesting application, we construct 4-phase Golay sequence quads of new sizes, e.g., the length 87 raised as an open problem in [28], and use them to construct Hadamard matrices of sizes that were previously unknown.

It may be worth mentioning the complete complementary codes (CCC) [29], [30], which represent a different line of efforts made on top of the Golay sequences. A set of CCC consists of $L$ Golay sequences sets with the additional constraint that the cross-correlations of two arrays from any two distinct sets add to be 0 everywhere. Because of this constraint, it is not surprising that the known possible lengths of CCC [30] is rather limited.

The remainder of this paper is organized as follows. Section II provides the preliminaries. In Section II-A, we introduce three equivalent definitions of GCA set. In Section II-B, we provide an interpretation in the sight of commutative ring and its involutary automorphism. In Section II-C, we review the constructions of 2-phase Golay sequence pairs [11] and 2-phase GCA pairs [22]. In Section II-D, we present four lemmas on the identities over a commutative ring as the foundation of the new constructions proposed in this paper. In Section III, first we review the construction of 4-phase Golay sequence pair in [19], and then establish Theorem 11 to construct 4-phase GCA pairs of more possible sizes than the previous literature [4], [5]. The possible sizes are derived in Corollary 1. In Section IV, first we construct GCA quads from GCA pairs by using Lemma 3; second we generalize the construction of Golay sequence quads in [19, Theorem 11] to construct multi-dimensional GCA quads by using Lemma 2; third we propose another two constructions of GCA quads based on Lemma 4; fourth, we show some new results of 4-phase Golay sequence quads and Hadamard matrices as a special case of our constructions; finally, we compared the known constructions with our new ones. The possible sizes in this section are much more fruitful than Section III. The conclusions are given in Section V.

II. PRELIMINARIES: DEFINITIONS, KNOWN CONSTRUCTIONS AND FOUR LEMMAS

Let us first introduce three different but equivalent definitions [23], [25] of the Golay complementary array (GCA) set, of which the last two, i.e., Definition 2 and 3, are based on polynomial and will be used in this paper.

A. Three Equivalent Definitions of GCA Set

For an $r$-dimensional complex-valued array $A$ of size $s_1 \times \cdots \times s_r$, its aperiodic autocorrelation function $R_A$ is defined as an array of size $(2s_1 - 1) \times \cdots \times (2s_r - 1)$ with entries [23]:

$$R_A[\delta_1, \cdots, \delta_r] = \sum_{i_1} \cdots \sum_{i_r} A[i_1, \cdots, i_r] \overline{A[i_1 - \delta_1, \cdots, i_r - \delta_r]}, \quad (1)$$

where $i_1, \cdots, i_r$ and $\delta_1, \cdots, \delta_r$ are array indices, $A[i_1, \cdots, i_r] = 0$ if $i_k < 0$ or $i_k \geq s_k$ for any $k \in \{1, 2, \cdots, r\}$, and the overbar represents the complex conjugation. The array $A$ is indexed from $[0, \cdots, 0]$, while the index $[0, \cdots, 0]$ corresponds to the center of the array $R_A$.

Given the above notations, we present the first definition of the GCA set.

Definition 1: A set of arrays $\{A_1, A_2, \cdots, A_L\}$ with unimodular or zero entries is called a Golay complementary array (GCA) set with cardinality $L$ if

$$\sum_{i=1}^{L} R_{A_i} = \sum_{i=1}^{L} w(A_i) \cdot \Delta, \quad (2)$$

where $\Delta = \mathbb{C}^{(2s_1 - 1) \times \cdots \times (2s_r - 1)}$ is an $r$-dimensional unit pulse function, i.e., the entries of $\Delta$ are zeros except that the center $\Delta[0, \cdots, 0] = 1$, and

$$w(A) = \sum_{i_1} \cdots \sum_{i_r} |A[i_1, \cdots, i_r]|^2 \quad (3)$$

is the weight of an array $A$.

In particular, the GCA set is referred to as GCA pair for cardinality $L = 2$ and GCA quad for $L = 4$. It is referred to as an $N$-phase (or polyphase) array if each of its entries is an $N$-th unit root. Specifically, a 2-phase array has entries $\{1, -1\}$ and a 4-phase array has entries $\{1, -1, j, -j\}$ where $j = \sqrt{-1}$. As an example of a 4-phase 2-D GCA pair of size $2 \times 3$:

$$A_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & j & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -j & -1 \end{bmatrix}. \quad (4)$$

And their autocorrelations are

$$R_{A_1} = \begin{bmatrix} -1 & -1+j & j & -1-j & 1 \\ 0 & 0 & 6 & 0 & 0 \\ 1 & -1+j & -j & 1-j & 1 \end{bmatrix},$$

$$R_{A_2} = \begin{bmatrix} 1 & 1-j & -j & 1+j & -1 \\ 0 & 0 & 6 & 0 & 0 \\ -1 & 1-j & j & 1+j & 1 \end{bmatrix}, \quad (5)$$

whose summation is a scaled two-dimensional $\delta$-function.

The array is sparse if it has zero-entries. An array is trivial in the $i$-th dimension if $s_i = 1$, and when it’s trivial in all dimensions, we say the array is trivial. A GCA set degenerates into a Golay sequence set [13] when $r = 1$.

The multi-variable polynomial $A(z)$ of an array $A$ is defined as [23]

$$A(z) \triangleq A(z_1, \cdots, z_r) = \sum_{i_1} \cdots \sum_{i_r} A[i_1, \cdots, i_r] z_1^{i_1} \cdots z_r^{i_r}, \quad (6)$$

where $z_1, \cdots, z_r$ are indeterminates. Here the notation $(\cdot)$ rather than $[\cdot]$ is used to distinguish the variables from the array indices.
From a digital signal processing perspective, the polynomial \( A(z) \) is essentially the Z-transform of the array signal. By the Wiener-Khinchin theorem, the Z-transform of the autocorrelation of an array is equal to its power spectrum, i.e.,
\[
R_A(z) = A(z)A(z^{-1}),
\]
where
\[
A(z^{-1}) \triangleq \sum_{i_1} \cdots \sum_{i_r} A[i_1, \ldots, i_r]z_{i_1}^{i_1} \cdots z_{i_r}^{i_r},
\]
with \( A[i_1, \ldots, i_r] \) representing the conjugate of the entry. Since the Z-transform of the \( \delta \)-function equals 1, by combining (2) and (7), it’s straightforward to give the second definition of the GCA set [25].

**Definition 2**: A set of arrays \( \{A_1, A_2, \ldots, A_L\} \) with unimodular or zero entries is called a GCA set if
\[
\sum_{i=1}^{L} A_i(z)A_i(z^{-1}) = \sum_{i=1}^{L} w(A_i).
\]

Denote \( A^* \) as the flipped and conjugate of \( A \), i.e.,
\[
A^*[i_1, \ldots, i_r] \triangleq A[s_1 - 1 - i_1, \ldots, s_r - 1 - i_r].
\]

Then the polynomial of \( A^* \) is defined as
\[
A^*(z) \triangleq A^*(z_1, \ldots, z_r) = \sum_{i_1} \cdots \sum_{i_r} A[s_1 - 1 - i_1, \ldots, s_r - 1 - i_r]z_{i_1}^{s_1} \cdots z_{i_r}^{s_r}.
\]
Comparing (8) with (11), we see that
\[
A^*(z) = \sum_{i_1} \cdots \sum_{i_r} A[s_1 - 1 - i_1, \ldots, s_r - 1 - i_r]z_{i_1}^{s_1} \cdots z_{i_r}^{s_r}.
\]
Then by (9) and (12), we have the third definition [25]:

**Definition 3**: A set of arrays \( \{A_1, A_2, \ldots, A_L\} \) with unimodular or zero entries is called a GCA set if
\[
\sum_{i=1}^{L} A_i(z)A_i^*(z) = \sum_{i=1}^{L} w(A_i)z_{i_1}^{s_1-1} \cdots z_{i_r}^{s_r-1}.
\]

The above definition will turn out to be very useful for deriving the new constructions of the GCA sets. Definition 3 can be better understood using the concept of commutative ring and its involutive automorphism. Although these concepts are basic and can be found in any textbook of abstract algebra, e.g., [31], we introduce them in the below for readers’ convenience.

**B. Polynomials of Arrays and Commutative Ring**

**Definition 4**: A commutative ring is a triple \((\mathcal{R}, +, \cdot)\), where \( \mathcal{R} \) is a non-empty set (do not confuse it with the field of real number), + and \( \cdot \) are two compositions for elements in \( \mathcal{R} \), satisfying the following conditions: For \( \forall a, b, c \in \mathcal{R} \)
1) Closeness: \( a + b \in \mathcal{R} \) and \( a \cdot b \in \mathcal{R} \).
2) Associativity: \( (a + b) + c = a + (b + c) \) and \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \).
3) Commutativity: \( a + b = b + a \) and \( a \cdot b = b \cdot a \).
4) Distributivity: \( a \cdot (b + c) = a \cdot b + a \cdot c \) and \( (b + c) \cdot a = b \cdot a + c \cdot a \).
5) Identity elements: there exist elements 0, 1 \( \in \mathcal{R} \) such that \( a + 0 = 0 + a = a \) and \( a \cdot 1 = 1 \cdot a = a \).
6) Invertibility: there exists an element \( a^{-1} \in \mathcal{R} \) such that \( a + a^{-1} = a^{-1} + a = 0 \).

For ease of notation, \( b + a^{-1} \) is written as \( b - a \) and \( a \cdot b \) is written as \( ab \).

Polynomial ring is an example of commutative ring. Let \( \mathcal{R} = \{A(z)\} \) be the set of the multi-variable polynomials defined in (6). The composition + and \( \cdot \) are defined as the conventional polynomial summation and multiplication respectively. The identity elements 0 and 1 are the integers 0 and 1 respectively. It’s straightforward to verify that the polynomial ring defined above satisfy all the six conditions in Definition 4.

One may also define the compositions + and \( \cdot \) of two arrays as the conventional addition and convolution (flip and cross correlation) respectively, then the set of arrays forms a ring. It is direct to verify that the map \( A \mapsto A(z) \) is an isomorphism between the array ring and the polynomial ring, i.e., the bijective map is identity preserving, addition preserving and multiplication preserving. Indeed, the equivalence of the three definitions of GCA is a natural consequence of this isomorphism.

As mentioned above, the multiplication of two polynomials amounts to the convolution of two arrays. But the convolution of two polyphase arrays may not preserve the polyphase-ness. A well-known trick for maintaining the polyphase-ness is to adopt Kronecker product: given an array \( A \) of size \( s_1 \times \cdots \times s_r \), and another array \( B \) of size \( t_1 \times \cdots \times t_r \), suppose \( G = A \otimes B \), then \( G(z) = A(z^s)B(z) \) where \( A(z^s) \) is an abbreviation of \( A[z_1^{s_1}, \ldots, z_r^{s_r}] \). Indeed, the Kronecker product of two polyphase array is the convolution of a sparse array and a polyphase array. Given the above insights, one will find it natural that Kronecker products occur frequently in the constructions of the GCA through this paper.

**Definition 5**: An involutive automorphism of a commutative ring \( \mathcal{R} \) is a bijective map \( * \) of \( \mathcal{R} \) into itself, which maps an element \( a \in \mathcal{R} \) to its image \( a^* \in \mathcal{R} \), such that for any \( a, b \in \mathcal{R} \),
\[
(a + b)^* = a^* + b^*, \quad (a \cdot b)^* = a^* \cdot b^*,
\]
\[
1^* = 1, \quad (a^*)^* = a. \quad (14)
\]

It’s straightforward to verify that the map from \( A(z) \) to \( A^*(z) \) defined in (11) is an involutive automorphism. Thus (13) can be rewritten as \( \sum_{i=1}^{L} a_i a_i^* = x \) where \( a_i = A_i(z) \in \mathcal{R} \) and \( x = \sum_{i=1}^{L} w(A_i)z_{i_1}^{s_1-1} \cdots z_{i_r}^{s_r-1} \in \mathcal{R} \).

**Remark 1**: Using the concept of commutative ring and its involutive automorphism, we can treat a multi-variable polynomial as an entity rather than a tedious expansion, which lays the ground for the recursive constructions of GCAs proposed later in this paper. This approach is totally different from the generalized Boolean function-based one used in the direct construction [2], [3], [8].

**C. Construction and Possible Sizes of 2-Phase GCA Pairs**

The following reviews the construction of 2-phase Golay sequence pairs [11] and its generalization to 2-phase GCA pairs [22].
Proposition 6 [11]: Given a 2-phase Golay sequence pair \(\{a, b\}\) of length \(s\), and a 2-phase Golay sequence pair \(\{c, d\}\) of length \(t\), then a 2-phase Golay sequence pair \(\{e, f\}\) of length \(st\) can be constructed as

\[
e = \frac{1}{2} [a \otimes (c + d) + b \otimes (c - d)],
\]

\[
f = \frac{1}{2} [b^* \otimes (c + d) - a^* \otimes (c - d)],
\]

(15)

where * denotes flipping and conjugating a sequence, \(\otimes\) denotes the Kronecker product.

Proposition 7 [22]: Given a 2-phase GCA pair \(\{A, B\}\) of size \(s_1 \times s_2 \times \cdots \times s_r\), and a 2-phase GCA pair \(\{C, D\}\) of size \(t_1 \times t_2 \times \cdots \times t_r\), then a 2-phase GCA pair \(\{E, F\}\) of size \(s_1t_1 \times s_2t_2 \times \cdots \times s_rt_r\) can be constructed as

\[
E = \frac{1}{2} [A \otimes (C + D) + B \otimes (C - D)],
\]

\[
F = \frac{1}{2} [B^* \otimes (C + D) - A^* \otimes (C - D)]
\]

(16)

where * denotes flipping and conjugating a sequence, \(\otimes\) denotes the Kronecker product.

The generalization of Proposition 7 from 6 is possible owing to a simple yet important observation: if an entry of \(C + D\) is 0, then the corresponding entry of \(C - D\) is \(\pm 2\), and vice versa. In other words, \(C + D\) and \(C - D\) are disjoint. Therefore, the GCA can maintain its 2-phase-ness in the recursive construction.

Based on Proposition 6, one can use some seed sequences to construct recursively longer Golay sequence pairs. The seed sequences are discovered by computational search [1]. Their lengths are 2, 10 and 26, referred to as basic 2-phase Golay numbers. Then the sizes of the sequences constructed by Proposition 6 are 2-phase Golay numbers, which form the set

\[
\{2^2 10^{b/26^c} | a, b, c \in \mathbb{Z}^+ \cup \{0\}\}.
\]

(17)

It is also easy to see that the size in each dimension of the GCAs constructed by Proposition 7 must be a 2-phase Golay number. But the set of the feasible sizes of binary GCAs are rather sparse.

This paper focuses on constructing GCAs of more feasible sizes through two extensions: i) to go beyond binary GCAs to polyphase ones and ii) to go beyond GCA pairs to GCA quads, as we shall detail in Section III and IV, respectively.

D. Four Lemmas

Next we present four lemmas on the identities over a commutative ring, which are the cornerstones of the new constructions of the polyphase GCA proposed in this paper. Although the identities in Lemma 1 and Lemma 2 are essentially restatements of the established results from the literature [25] and [15], respectively, Lemma 3 and Lemma 4 are new.

**Lemma 1**: Given a commutative ring \(\mathcal{R}\) with an involutive automorphism *, and \(a, b, c, d \in \mathcal{R}\), suppose

\[
e = ac + bd, \quad f = b^*c - a^*d,
\]

(18)

then

\[
ee^* + ff^* = (aa^* + bb^*) (cc^* + dd^*).
\]

(19)

Lemma 1 is essentially a restatement of [25, Theorem 1], which can be dated back to the invention of quaternions by Hamilton in 1843 [32]. The straightforward proof of Lemma 1 given in [25] exploits the properties of a commutative ring with an involutive automorphism mentioned in Section II.

By identifying \(c + d\) and \(c - d\) in Proposition 6 (and \(C + D\) and \(C - D\) in Proposition 7) as ring elements \(c\) and \(d\) respectively, one can recognize that Proposition 6 and 7 can be inferred from Lemma 1.

**Lemma 2**: Given a commutative ring \(\mathcal{R}\) with an involutive automorphism *, and \(a, b, c, d, e, f, g, h \in \mathcal{R}\), suppose

\[
p = af^* - b^*e + cg + dh,
\]

\[
q = a^*e + bf^* - ch^* + dg^*,
\]

\[
r = c^*e - df + ah^* + bg^*,
\]

\[
s = -cf - d^*e + ag^* - bh^*.
\]

(20)

then

\[
pp^* + qq^* + rr^* + ss^* = (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*).
\]

(21)

Lemma 2 is a reproduction of the so-called Lagrange identity [15, Theorem L], which can be dated back to the invention of quaternions by John T. Graves (who was Hamilton’s friend) in 1843 [32]. The verification of Lemma 2 is also straightforward, albeit more laborious than Lemma 1.

**Remark 2**: In Lemma 1, if \(a, b \in \mathcal{R}\), i.e., \(a^* = a\) and \(b^* = b\), then by (18) we have

\[
(e, f) = (ac + bd, -ad + bc)
\]

(22)

which is exactly the multiplication rule of two complex number \(a + bj\) and \(c - dj\) if we separate the real part and the imaginary part in the above tuple. Then (19) follows naturally from the norm of a complex number. In this sense, Lemma 1 can be viewed as a generalization of the complex number system. More generally, according to [32], the pair \((e, f)\) in Lemma 1 and the quad \((p, q, r, s)\) in Lemma 2 can be viewed as a quaternion and an octonion respectively, which are normed division algebras constructed by the well-known Cayley-Dickson process. It is well-known that the real numbers, complex numbers, quaternions and octonions are the only normed division algebras [32, Theorem 1]. This may explain why most of the recursive constructions of Golay sequence set and its variants have long been restricted to cardinality 2 and 4, except for some sporadic work [27], [26].

**Lemma 3**: Given a commutative ring \(\mathcal{R}\) with an involutive automorphism *, and \(a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{R}\), suppose

\[
c_{ij} = a_i b_j, \quad 1 \leq i \leq m, 1 \leq j \leq n,
\]

(23)

then

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} c_{ij}^* = \left( \sum_{i=1}^{m} a_i a_i^* \right) \left( \sum_{j=1}^{n} b_j b_j^* \right).
\]

(24)
Based on a property of autocorrelation of a rank-one matrix, we proposed a construction of 2-D GCA set in the previous work [33, Lemma III.4]. That construction is delineated more succinctly by the identity in Lemma 3.

Compared with (18) and (20), the expression (23) involves no addition; hence the unimodularity of the constructed GCA is naturally maintained, although this benefit comes with the increase of the cardinality – the cardinality of $c_{ij}$’s is $mn$.

Lemma 4: Given a commutative ring $\mathcal{R}$ with an involutive automorphism $\ast$, and $a_1, \cdots, a_m, b_1, \cdots, b_n \in \mathcal{R}$, $m, n$ are even numbers. For $1 \leq i < \frac{1}{2}m, 1 \leq j < \frac{1}{2}n$, suppose

$$c_{ij} = a_{2i-1}b_{2j-1} + a_{2i}b_{2j},$$

$$d_{ij} = a_{2i-1}b_{2j} - a_{2i}b_{2j-1},$$

then

$$\sum_{i=1}^{\frac{1}{2}m} \sum_{j=1}^{\frac{1}{2}n} (c_{ij}c_{ij}^* + d_{ij}d_{ij}^*) = \left( \sum_{i=1}^{m} a_i a_i^* \right) \left( \sum_{j=1}^{n} b_j b_j^* \right).$$

**Proof:** According to Lemma 1,

$$c_{ij}^* c_{ij} + d_{ij}^* d_{ij} = (a_{2i-1}a_{2i-1}^* + a_{2i}a_{2i}^*)(b_{2j-1}b_{2j-1}^* + b_{2j}b_{2j}^*),$$

from which and Lemma 3 we can obtain (26) immediately.

III. ON POLYPHASE GCA PAIRS

In this section, we first review the construction of 4-phase Golay sequence pairs proposed in [19], and then generalize it to multi-dimensional polyphase GCA before deriving the possible sizes.

A. Construction and Possible Lengths of 4-Phase Golay Sequences

The recursive construction of polyphase Golay sequence pair was proposed by Craigen [19].

**Proposition 8:** [19]: Given a nontrivial 2-phase Golay sequence pair $\{a, b\}$ of length $s$, and two polyphase Golay sequence pairs $\{c, d\}$ and $\{e, f\}$ of length $t$ and $u$, respectively, for (recall that $a^\ast$ denotes the flipped and conjugate of $a$)

$$p = \frac{1}{4}[a + b + (b^* - a^*)], \quad q = \frac{1}{4}[a + b - (b^* - a^*)],$$

$$x = p \otimes c \otimes d, \quad y = q \otimes c \otimes d,$$

$$g = x \otimes e \otimes f, \quad h = y^* \otimes e \otimes f,$$

$$\{g, h\}$$ is a polyphase Golay sequence pair of length $stu$.

Note that the construction of $(x, y)$ and $(g, h)$ in (27) stems from Lemma 1, and that Craigen’s construction relies on such a property: for any given index $i$, only one element of $\{p[i], q[i], p^*[i], q^*[i]\}$ equals $\pm 1$ while the other three elements equal $0$; thus, $x$ and $y$ are disjoint and hence $\{g, h\}$ is still polyphase.

Given the 4-phase seed sequences obtained from computational searches [19], whose lengths are $3, 5, 11, 13$ and are referred to as basic 4-phase Golay numbers,

$$\{a^* + u b^* c^* d^* e^* f^* \}_{a, b, c, d, e, u \in Z^+ \cup \{0\}},$$

which is referred to as 4-phase Golay numbers.

Motivated by Dymond’s generalized construction [22] from Turyn’s [11], it is natural to raise the question: can the construction of 4-phase Golay sequence pairs also be generalized to multi-dimensional cases? The answer is yes, as we show in the below.

B. Construction of Polyphase GCA Pairs and The Possible Sizes

Some notations:

$$A[i] \triangleq A[i_1, \cdots, i_r]$$

$$A[i + j] \triangleq A[i_1 + j_1, \cdots, i_r + j_r]$$

$$i + 1 \triangleq i^\prime \text{ where } \sum_{k=1}^{r} i_k^\prime \prod_{l=1}^{k-1} s_l = 1 + \sum_{k=1}^{r} i_k \prod_{l=1}^{k-1} s_l$$

$$\sum_{j=0}^{r} \prod_{j=0}^{r} i_j \prod_{j=0}^{r} i_j \prod_{j=0}^{r} j = \prod_{k=1}^{r} (i_k + 1), \quad z^i \triangleq z_1^{i_1} \cdots z_r^{i_r}$$

$$i < j \triangleq \sum_{k=1}^{r} i_k \prod_{l=1}^{k-1} s_l < \sum_{k=1}^{r} j_k \prod_{l=1}^{k-1} s_l$$

**Proposition 9:** Given a nontrivial 2-phase GCA pair $\{A, B\}$ of size $s_1 \times \cdots \times s_r$, for any given index $i$ satisfying $0 \leq i \leq s - 1$,

$$A[s - 1 - i] A[i] B[s - 1 - i] B[i] = -1.$$  \hfill (35)

**Proof:** According to the definition of GCA,

$$\sum_{j=0}^{i} A[s - 1 - i + j] A[j] + B[s - 1 - i + j] B[j] = 0,$$  \hfill (36)
for any $0 \leq i < s - 1$. The left side of (36) must be a summation of $N_1$ “1”s and $N_1$ “−1”s. Therefore,

$$\prod_{j=0}^{i} A[s - 1 - i + j] A[j] B [s - 1 - i + j] B[j]$$

$$= \prod_{j=0}^{i} A[s - 1 - j] A[j] B [s - 1 - j] B[j] = (-1)^{N_1}. \quad (37)$$

By (37), (35) holds when $i = [0, 0, \cdots, 0]$. Suppose (35) holds for $\forall 1 \leq i$, then by (37), for $i = k + 1$,

$$\prod_{j=0}^{k+1} A[s - 1 - j] A[j] B [s - 1 - j] B[j] = (-1)^{N_{k+1}}. \quad (38)$$

The indices $j$ in (38) satisfy $j \leq k$ except for $j = k + 1$. Hence

$$A[s - 1 - k - 1] A[k + 1] B [s - 1 - k - 1] B[k + 1] = (-1)^{N_{k+1}} = -1. \quad (39)$$

Hence (35) holds for any $0 \leq i < s - 1$. Since (35) is identical when $i = 0$ or $i = s - 1$, (35) holds for any $0 \leq i < s - 1$. 

Based on Lemma 1 and Proposition 9, we have the following method to obtain a sparse GCA pair.

**Proposition 10:** Let $\{P, Q\}$ be a sparse GCA pair whose entries $\{P[i], Q[i], P^*[i], Q^*[i]\}$ consist of three zeros and one ±1 for any index $i$. Such a pair $\{P, Q\}$ exists if and only if there exists a pair of 2-phase GCA $\{A, B\}$ of the same size.

**Proof:** The “if” part: Set

$$P = \frac{1}{4}[A + B + (B^* - A^*)],$$

$$Q = \frac{1}{4}[A + B - (B^* - A^*)]. \quad (40)$$

Suppose a trivial GCA pair $C = D = 1$. Set

$$E = A \odot C + B \odot D, \quad F = B^* \odot C - A^* \odot D. \quad (41)$$

By Lemma 1, set $a = A(z)$, $b = B(z)$, $c = C(z)$, $d = D(z)$, $e = E(z)$, $f = F(z)$, then we have

$$ee^* + ff^* = (aa^* + bb^*)(cc^* + dd^*) = 4 \prod_{i=1}^{r} s_i z_i^{s_i-1}. \quad (42)$$

Note from (40) and (41) that

$$P = \frac{1}{4}(C \odot E + D \odot F), \quad Q = \frac{1}{4}(D^* \odot E - C^* \odot F). \quad (43)$$

Denoting $a = C(z)$, $b = D(z)$, $c = \frac{1}{2}E(z)$, $d = \frac{1}{2}F(z)$, $e = P(z)$, $f = Q(z)$ and using Lemma 1 again, we have

$$ee^* + ff^* = (aa^* + bb^*)(cc^* + dd^*) = \frac{1}{2} \prod_{i=1}^{r} s_i z_i^{s_i-1}. \quad (44)$$

By Definition 3, $\{P, Q\}$ is a sparse GCA pair. And by Proposition 9, without loss of generality, suppose $A[i] = -1$ and $B[i] = A^*[i] = B^*[i] = 1$ for a given index $i$, then $\{P[i], Q[i], P^*[i], Q^*[i]\}$ consist of three zeros and one ±1.

The “only if” part: Set

$$X = P + Q, \quad Y = Q^* - P^*, \quad (45)$$

then $\{X, Y\}$ is not only a sparse GCA pair, but disjoint by the structure of $\{P, Q\}$, i.e., for any index $i$, one element of $\{X[i], Y[i]\}$ is ±1 while the other element is 0. Therefore,

$$A = X + Y, \quad B = X - Y, \quad (46)$$

is a 2-phase GCA pair.

Based on the sparse GCA pair $\{P, Q\}$, we can establish the following construction of the polyphase GCA pairs as a generalization of Proposition 8.

**Theorem 11:** Given a nontrivial 2-phase GCA pair $\{A, B\}$ of size $s_1 \times \cdots \times s_r$, two polyphase GCA pairs $\{D, E\}$ and $\{P, Q\}$ of size $t_1 \times \cdots \times t_r$ and $u_1 \times \cdots \times u_r$ respectively, suppose

$$P = \frac{1}{4}[A + B + (B^* - A^*)],$$

$$Q = \frac{1}{4}[A + B - (B^* - A^*)],$$

$$X = P \otimes C + Q \otimes D,$$

$$Y = Q^* \otimes C - P^* \otimes D,$$

$$G = X \otimes E + Y \otimes F, \quad (47)$$

then $\{G, H\}$ is a polyphase GCA pair of size $s_1 t_1 u_1 \cdots s_r t_r u_r$.

**Proof:** Set $a = P(z^t), \quad b = Q(z^t), \quad c = C(z), \quad d = D(z), \quad e = X(z), \quad f = Y(z)$.

Then we have by Lemma 1 that

$$ee^* + ff^* = (aa^* + bb^*)(cc^* + dd^*) = \left(\prod_{i=1}^{r} s_i z_i^{(s_i-1) t_i}\right) \left(\prod_{i=1}^{r} t_i z_i^{t_i - 1}\right) = \prod_{i=1}^{r} s_i t_i z_i^{s_i t_i - 1}, \quad (48)$$

where $\prod_{i=1}^{r} s_i t_i z_i^{s_i t_i - 1}$ holds because $P$ and $Q$ are a pair of sparse GCA according to Proposition 10. Setting $a = X(z^t), \quad b = Y(z^t), \quad c = C(z), \quad d = D(z), \quad e = G(z), \quad f = H(z)$ and using Lemma 1 again, we have

$$ee^* + ff^* = (aa^* + bb^*)(cc^* + dd^*) = \left(\prod_{i=1}^{r} s_i t_i z_i^{(s_i t_i - 1) u_i}\right) \left(\prod_{i=1}^{r} u_i z_i^{u_i - 1}\right) = 2 \prod_{i=1}^{r} s_i t_i u_i z_i^{s_i t_i u_i - 1}. \quad (49)$$

That is, $G$ and $H$ form a GCA pair according to Definition 3. They are also polyphase because the GCA pair $X$ and $Y$ are disjoint owing to the sparse structure of $P, Q$ as specified in Proposition 10.

**For example,** let:

$$A = [1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1],$$

$$B = [1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1],$$

$$C = E = [1, 1, -1], \quad D = F = [1, j, 1]. \quad (50)$$
we construct a 4-phase GCA pair \( \{ G, H \} \) of size \( 10 \times 9 \) as follows:

\[
G = \begin{bmatrix}
1 & 1 & -1 & j & j & -j & 1 & 1 & -1 \\
-1 & -1 & 1 & -j & -j & j & -1 & -1 & 1 \\
-1 & -1 & 1 & -j & -j & j & -1 & -1 & 1 \\
1 & j & 1 & j & -1 & j & 1 & j & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\
-1 & j & 1 & j & -1 & j & 1 & j & 1 \\
-1 & -j & -1 & 1 & -1 & j & -1 & 1 & j \\
1 & -j & -1 & 1 & -1 & j & -1 & 1 & j \\
1 & j & 1 & j & 1 & -j & 1 & -j & 1 \\
-1 & -j & -1 & j & -1 & j & -1 & j & 1
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
1 & 1 & -1 & j & j & -j & 1 & 1 & -1 \\
-1 & -1 & 1 & -j & -j & j & -1 & -1 & 1 \\
-1 & -1 & 1 & -j & -j & j & -1 & -1 & 1 \\
1 & j & 1 & j & -1 & j & 1 & j & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\
-1 & j & 1 & j & -1 & j & 1 & j & 1 \\
-1 & -j & -1 & 1 & -1 & j & -1 & 1 & j \\
1 & -j & -1 & 1 & -1 & j & -1 & 1 & j \\
1 & j & 1 & j & 1 & -j & 1 & -j & 1 \\
-1 & -j & -1 & j & -1 & j & -1 & j & 1
\end{bmatrix}
\]

We construct a 4-phase GCA pair \( \{ G, H \} \) of size \( 10 \times 9 \) as follows:

In Theorem 11, the 2-phase GCA pair is like a binder to glue two polyphase GCA pairs together. From this point of view, we delineate the possible sizes of the 4-phase GCA pairs in the following corollary.

**Corollary 1:** There exist 4-phase GCA pairs of size \( s_1 \times \cdots \times s_r \) if \( \prod_{i=1}^r s_i = 2^{a+u}3^{b}5^{c}11^{d}13^{e} \), \( a, b, c, d, e, u \in \mathbb{Z}^+ \cup \{0\} \), \( b+c+d+e \leq a+2u+1 \), \( u \leq c+e \), and each of the \( u \) factors is odd.

**Proof:** Consider feeding into Theorem 11 the seed Golay sequences, whose lengths are basic 2-phase Golay numbers or basic 4-phase Golay numbers, i.e., \( \{2, 10, 26\} \) or \( \{3, 5, 11, 13\} \). Noting that \( 10 = 2 \times 5 \) and \( 26 = 2 \times 13 \), we set \( \prod_{i=1}^r s_i \) as a 4-phase basic Golay number.

It is verified by exhaustive computational search that there does not exist a 2-phase GCA pair of size \( 2 \times 5 \) or \( 2 \times 13 \). Therefore, each of \( u \) factors 2 should be from either 10 or 26 as a factor of some \( s_i \).

Here is an example illustrating the second constraint: for \( s_1s_2 = 90 = 2^{1+u}3^{2}5^{1}11^{0}13^{0} \), we construct a 4-phase 2-D GCA pair of size \( 10 \times 9 \) by Theorem 11 since a factor 2 is obtained from \( s_1 = 10 \), but we construct a 4-phase 2-D GCA pair of size \( 5 \times 18 \) since a factor 2 is obtained from \( s_2 = 18 \).

It is verified by exhaustive computational search that there does not exist a 2-phase GCA pair of size \( 2 \times 5 \) or \( 2 \times 13 \). Therefore, each of \( u \) factors 2 should be from either 10 or 26 as a factor of some \( s_i \).

Here is an example illustrating the second constraint: for \( s_1s_2 = 90 = 2^{1+u}3^{2}5^{1}11^{0}13^{0} \), we construct a 4-phase 2-D GCA pair of size \( 10 \times 9 \) by Theorem 11 since a factor 2 is obtained from \( s_1 = 10 \), but we construct a 4-phase 2-D GCA pair of size \( 5 \times 18 \) since a factor 2 is obtained from \( s_2 = 18 \).

Note that the generalization of 4-phase Golay sequence pair of 4-phase GCA pairs achieves more flexible sizes beyond 4-phase Golay number in some dimensions. For example, in the above example of GCA of size \( 10 \times 9 \), it is not a 4-phase Golay number. In contrast, for any known 2-phase GCA pair of size \( s_1 \times \cdots \times s_r \), \( s_1, \ldots, s_r \) must be 2-phase Golay numbers. The flexible sizes of 4-phase GCA pairs would play an important role in the next section where some 4-phase GCA pairs are used to construct 4-phase GCA quads, as detailed in Remark 4.

The construction in Theorem 11 is more general than the existing constructions in the literature [4, 5], and [24]. In Theorem 11 if \( A \) and \( B \) are vectors \( [1, 1] \) and \( [1, -1] \) in the \( i \)-th dimension respectively and trivial in the other \( r - 1 \) dimensions, then

\[
G = (D \otimes E) \vert (C \otimes F), \quad H = (C^* \otimes E) \vert (-D^* \otimes F)
\]

where \( \vert \) denotes concatenating two arrays in the \( i \)-th dimension. This is in fact the two-part concatenation methods used in [4], [5], and [24]. Theorem 11 provides an \( s_1 \times \cdots \times s_r \)-part concatenation method in \( r \) dimensions where \( s_1 \times \cdots \times s_r \) is the possible sizes of 2-phase GCA pairs. For example, for \( A = [1, -1, -1, 1, -1, 1, 1] \) and \( B = [1, -1, -1, 1, 1, 1, 1, 1] \), a 10-part concatenation is given as follows:

\[
(C \otimes E) \vert (-C \otimes E) \vert (-C \otimes E) \vert (-C \otimes F) \vert (-D \otimes E) \vert (-C \otimes F) \vert (-D \otimes E) \vert (D \otimes F) \vert (-D \otimes F) \vert (D^* \otimes E) \vert (-C^* \otimes E) \vert (-C^* \otimes F) \vert (C^* \otimes F) \vert (C^* \otimes F)
\]

It is this flexible multi-part concatenation that enables us to construct 4-phase GCA pairs with more possible sizes than the two-part construction in [4], [5], and [24]. For example, the 2-D GCA pair of size \( 10 \times 9 \) as obtained in (51) cannot be obtained by using the two-part concatenation. Indeed, a 10 \( \times \) 9 GCA can be constructed using the two-part concatenation of two 4-phase Golay sequence pairs of size 1 \( \times \) 9 and 5 \( \times \) 1, or two 2-D 4-phase GCA pairs of size 5 \( \times \) 3 and 1 \( \times \) 3. But there does not exist GCA's of size 1 \( \times \) 9 or 5 \( \times \) 3, since neither 9 nor 15 is a 4-phase Golay number.

**IV. ON POLYPHASE GCA QUADS**

The possible sizes of the aforementioned constructions of polyphase GCA pairs, albeit more flexible than the binary ones as given in Proposition 7 [cf. (17)], are still rather limited. In this section, we explore the construction of GCA quads, whose possible sizes are much denser than those of the GCA pairs. As an interesting byproduct, we also find some new Hadamard matrices unreported in the literature.

**A. Construction of GCA Quads Based on Lemma 3**

Theorem 12: Given two polyphase GCA sets \( \{ A_1, \cdots, A_m \} \) and \( \{ B_1, \cdots, B_n \} \) of size \( s_1 \times \cdots \times s_r \) and


\[ t_1 \times \cdots \times t_r \] respectively, the set

\[ \{ C_{ij} = A_i \otimes B_j \mid 1 \leq i \leq m, 1 \leq j \leq n \} \]

is a polyphase GCA of size \( s_1 t_1 \times s_2 t_2 \times \cdots \times s_r t_r \).

**Proof:** By Lemma 3, let \( a_i = A_i(z^t), b_j = B_j(z), c_{ij} = C_{ij}(z) \), and the involutive automorphism \( * \) is defined as in (11), then

\[
\sum_{i,j} c_{ij} c_{ij}^* = \left( \sum_i a_i a_i^* \right) \left( \sum_j b_j b_j^* \right) = \left( m \prod_{k=1}^{r} s_k z_k^{-1} \right) \left( n \prod_{k=1}^{r} t_k z_k^{-1} \right) = mn \prod_{k=1}^{r} s_k t_k z_k^{s_k t_k - 1}. \]

(55)

According to Definition 3, \( \{ C_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \} \) is a polyphase GCA set.

When \( m = n = 2 \), the possible sizes of 4-phase GCA quads constructed by Theorem 12 is summarized in the following corollary.

**Corollary 2:** If \( s_1 \times \cdots \times s_r \) and \( t_1 \times \cdots \times t_r \) are the possible sizes of 4-phase GCA pairs as specified in Corollary 1, then there exist 4-phase GCA quads of size \( s_1 t_1 \times s_2 t_2 \times \cdots \times s_r t_r \).

The existing construction given in [33, Lemma III.4] allows for \( s \times t \) 4-phase GCA quads, where \( s \) and \( t \) are two 4-phase Golay numbers. Theorem 12 is an improvement over [33, Lemma III.4]. For example, feeding into Theorem 12 a 4-phase 2-D GCA pair of size \( 6 \times 3 \) (constructed by Theorem 11):

\[
A_1 = \begin{bmatrix} 1 & j & 1 & 1 & 1 & -1 \\ 1 & j & 1 & j & j & -j \\ -1 & -j & -1 & 1 & 1 & -1 \end{bmatrix}^T \]

(56)

\[
A_2 = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 & j & -1 \\ -1 & 1 & 1 & -1 & -1 & j & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}^T \]

(57)

and a 4-phase 2-D GCA pairs of size \( 1 \times 3 \) \( \{ B_1 = [1, j, 1], B_2 = [1, j, 1] \} \), we can construct a 4-phase 2-D GCA quad of size \( 6 \times 9 \) as shown in (58)–(61), which cannot be constructed by [33, Lemma III.4], since 9 is not a 4-phase Golay number.

\[
C_{11} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & -1 & -1 \hspace{1cm} j & j \hspace{1cm} j & -j \hspace{1cm} -j & -j \hspace{1cm} j & j & -j \hspace{1cm} j & -j \hspace{1cm} -j & -j \hspace{1cm} 1 & 1 & -1 & 1 & 1 & -1 & -1 \hspace{1cm} -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \]

(58)

\[
C_{12} = \begin{bmatrix} 1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} j & 1 & j & -1 & -1 & j & -1 \hspace{1cm} 1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} 1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} -1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} -1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} -1 & j & 1 & j & 1 & -1 & j & -1 \hspace{1cm} -1 & j & 1 & j & 1 & -1 & j & -1 \end{bmatrix} \]

(59)



\[
\begin{aligned}
C_{21} &= \begin{bmatrix} -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & j & -j & j & -1 & -1 \\ -1 & -1 & 1 & -j & -j & j & -1 & -1 \\ j & j & j & -1 & -1 & 1 & j & j \\ j & j & j & j & j & j & j & j \\ -1 & -1 & 1 & -j & -j & j & -1 & -1 \end{bmatrix} \\
C_{22} &= \begin{bmatrix} -1 & j & 1 & -1 & -j & 1 & -1 & 1 \\ 1 & j & 1 & j & 1 & 1 & j & 1 \\ 1 & j & 1 & j & 1 & -1 & j & 1 \\ -1 & -j & -j & 1 & -j & 1 & -1 & 1 \\ j & -1 & j & -j & -j & 1 & j & -1 \\ j & -1 & j & -j & -j & 1 & j & -1 \\ -1 & -j & -j & 1 & -j & 1 & -1 & 1 \end{bmatrix}
\end{aligned}
\]

(60)

(61)

Note that Theorem 11 and Theorem 12 use different methods for maintaining the polyphase property. The construction of Theorem 11 places zeros at disjoint positions to avoid the summation of unit roots, at the expense of multiplying the size in some dimension with a 2-phase Golay number. The arrays constructed by Theorem 12 are intrinsically polyphase, since no “+” is involved in (23) of Lemma 3. But this construction increases the cardinality of the array set.

### B. Construction of GCA Quads Based on Lemma 2

The main result of this subsection is Theorem 15. But to prove it we need to first establish Proposition 13 and Proposition 14 as follows.

**Proposition 13:** Given a polyphase GCA quad \( \{ A, B, C, D \} \), where the size of \( A \) and \( B \) is \( s_1 \times \cdots \times (s_1+1) \times \cdots \times s_r \) and the size of \( C \) and \( D \) is \( s_1 \times \cdots \times s_i \times \cdots \times s_r \), suppose

\[
\begin{aligned}
E &= A \otimes 0_{s_i} \\
F &= B \otimes 0_{s_i} \\
G &= 0_{s_i+1} \otimes C \\
H &= 0_{s_i+1} \otimes D,
\end{aligned}
\]

(62)

where \( / \) denotes interleaving two arrays in the \( i \)-th dimension, \( 0_{s_i+1} \) denotes the zero array of the same size as \( C \) \( (A) \). Then \( \{ E, F, G, H \} \) is a sparse GCA quad of size \( s_1 \times \cdots \times (2s_i+1) \times \cdots \times s_r \).

**Proof:** Note that

\[
\begin{aligned}
e &\equiv E(z) = A(z_1, \cdots, z_i^2, \cdots, z_r) \equiv a, \\
g &\equiv F(z) = B(z_1, \cdots, z_i^2, \cdots, z_r) \equiv b, \\
f &\equiv F(z) = z_i C(z_1, \cdots, z_i^2, \cdots, z_r) \equiv z_i c, \\
h &\equiv H(z) = z_i D(z_1, \cdots, z_i^2, \cdots, z_r) \equiv z_i d.
\end{aligned}
\]

(63)

Define a polynomial ring including the polynomials of both positive and negative powers, and define its involutive automorphism \( \ast \) as a map from \( A(z) \) to \( A(z^{-1}) \). [Do not confuse it with the involutive automorphism \( \ast \) defined in (11)]. Then

\[
e e^\ast + f f^\ast + g g^\ast + h h^\ast \overset{(1)}{=} a a^\ast + b b^\ast + c c^\ast + d d^\ast = 2(2s_i+1) \prod_{j=1,j \neq i}^r s_j.
\]

(64)

where \( \overset{(1)}{=} \) holds because \( z_i z_i^\ast = 1 \). According to Definition 2, we have completed the proof.
There are at least two ways to obtain \{A, B, C, D\} of sizes as prescribed in Proposition 13: i) \{A, B\} and \{C, D\} are two GCA pairs, e.g., the size of A and B is \(2 \times 9\) and the size of C and D is \(2 \times 8\); ii) the base sequences [34] denoted as \(BS(m+1, m)\), where the lengths of the two sequences are \(m + 1\) while those of the other two are \(m\), is also a (degenerated) example of the array quad \{A, B, C, D\}. \(BS(m+1, m)\) can be constructed from a 2-phase or 4-phase Golay sequence pair \(\{a, b\}\) of length \(m\) [11]: \(A = [a, 1], B = [a, -1], C = b, D = b\). Besides, they can be searched by a computer and it is conjectured in [34] that there exist 2-phase \(BS(m+1, m)\) for any integer \(m \geq 1\), which has been verified for \(m \leq 38\) by computational search [16]. An example of \(BS(8, 7)\) \(\{A = [-1, 1, 1, 1, 1, -1, -1, 1], B = [1, 1, 1, -1, -1, 1, 1, 1], C = [-1, 1, 1, 1, 1, 1, -1, 1], D = [1, -1, 1, 1, 1, 1, -1, -1]\}\), and their autocorrelations add to be \(0, 0, 0, 0, 0, 0, 0, 0\). More base sequences searched by a computer are provided online: https://github.com/csrlab-fudan/ACM/blob/main/baseSequence.mat.

While Proposition 13 addresses interleaving Golay matrices with zeros, the following proposition considers concatenating zeros, a technique used in [19, Theorem 11].

**Proposition 14:** Given four polyphase GCA pairs \{A, B\}, \{C, D\}, \{I, J\}, \{K, L\}, where the sizes of \(A \otimes I\) and \(C \otimes K\) are \(s_1 \times \cdots \times s_j \times \cdots \times s_r\) and \(s_1 \times \cdots \times s_j' \times \cdots \times s_r\) respectively, suppose

\[
E = (A \otimes I) | 0_{s_{j'}} | 0_{s_j} | (B \otimes J) \\
G = (B \otimes I) | 0_{s_{j'}} | 0_{s_j} | (-A^* \otimes J) \\
F = 0_{s_j} | (C \otimes K) | (D \otimes L) | 0_{s_{j'}} \\
H = 0_{s_j} | (D^* \otimes K) | (-C^* \otimes L) | 0_{s_{j'}}
\]

where \(|\) denotes concatenating two arrays in the \(j\)th dimension. Here \(0_{s_j}\) and \(0_{s_{j'}}\) denote the zero arrays of the same size as \(A \otimes I\) and \(C \otimes K\), respectively. Then \(\{E, F, G, H\}\) is a sparse GCA quad of size \(s_1 \times \cdots \times 2(s_j + s_j') \times \cdots \times s_r\).

**Proof:** As a special case of Theorem 13,

\[
\{(A \otimes I) | (B \otimes J), (B^* \otimes I) | (-A^* \otimes J)\}
\]

and

\[
\{(C \otimes K) | (D \otimes L), (D^* \otimes K) | (-C^* \otimes L)\}
\]

are two polyphase GCA pairs. Concatenating zeros at the both ends of the matrices in (66), which leads to F and H, does not change the complementarity either. Inserting zeros in the middle of the GCAs in (66), which leads to E and G, does not change the complementarity either. The reason is as follows. Suppose the size of I and J is \(n_1 \times \cdots \times n_r\). Note that

\[
E(z) = \sum_{i=1}^{2s_j+s_{j'}} A(z^{n_i}) I(z) + B(z^{n_i}) J(z), \quad G(z) = \sum_{i=1}^{2s_j+s_{j'}} B^*(z^{n_i}) I(z) - A^*(z^{n_i}) J(z).
\]

(68) can be represented in commutative ring as follows:

\[
e = xac + bd, \quad f = xbc - a^* d.
\]

then by Lemma 1,

\[
e e^* + f f^* = (aa^* + bb^*) ((xc)(xc)^* + dd^*).
\]

Hence \(\{E, G\}\) is a sparse GCA pair. Then \(\{E, F, G, H\}\) is a sparse GCA quad.

The above treatment is a nontrivial generalization of the concatenation technique used in [19, Theorem 11] even in the one-dimensional case.

**Remark 3:** Note that when \(s_{j'} = m s_j, m \in \mathbb{Z}^+\), the construction of E, G can be also explained by a slight modification of the two-part concatenation derived from Theorem 11: the 2-phase Golay sequence pair \{1, 1, [1, -1]\} is replaced by a sparse Golay sequence pair \{1, 0_m, 0_{m-1}, 1, 0_m, 0_{m-1}\} where \(0_m\) represents zero sequence of length \(m\). However, the complementarity of 2-phase Golay sequence pair of length 10 or 26 would not hold if concatenating zeros in the middle. Hence one should not expect to concatenate zeros in the middle of multi-part concatenations while maintaining complementarity.

In Proposition 13 and Proposition 14, E, F, G and H are quasi-symmetric, i.e., the zeros of the flipped arrays occur in the same positions as the original arrays; E and G (F and H) are conjoint, i.e., zeros occur in the same positions; E and F (G and H) are disjoint. The above structure is sufficient to avoid the summation of unit roots when they are combined in the following theorem.

**Theorem 15:** Construct two sparse GCA quads of size \(m_1 \times \cdots \times m_r\) and \(n_1 \times \cdots \times n_r\) according to Proposition 13 or Proposition 14, which are denoted by \(\{E_1, F_1, G_1, H_1\}\) and \(\{E_2, F_2, G_2, H_2\}\), respectively. Suppose

\[
P = E_1 \otimes F_2^* - G_1^* \otimes E_2 + F_1 \otimes G_2 + H_1 \otimes H_2 \\
Q = E_1^* \otimes E_2 + G_1 \otimes F_2^* - F_1 \otimes H_2^* + H_1 \otimes G_2^* \\
R = F_1^* \otimes E_2 - H_1 \otimes F_2 + E_1 \otimes H_2^* + G_1 \otimes G_2 \\
S = -F_1 \otimes F_2 - H_1^* \otimes E_2 + E_1 \otimes G_2^* - G_1 \otimes H_2, \quad (71)
\]

then \(\{P, Q, R, S\}\) is a polyphase GCA quad of size \(m_1 n_1 \times \cdots \times m_r n_r\).

**Proof:** By Lemma 2, let \(a = E_1(z^n), c = F_1(z^n), b = G_1(z^n), d = H_1(z^n), e = E_2(z), f = F_2(z), g = G_2(z), h = H_2(z), p = P(z), q = Q(z), r = R(z), s = S(z),\) and \(a^*\) is as defined in (11), then

\[
pp^* + qq^* + rr^* + ss^* = (aa^* + bb^* + cc^* + dd^*) (ee^* + ff^* + gg^* + hh^*)
\]

\[
= \left( \prod_{i=1}^{r} m_i z_i^{m_i(n_i - 1)} \right) \left( \prod_{i=1}^{r} n_i z_i^{n_i - 1} \right)
\]

\[
= \prod_{i=1}^{r} m_i n_i z_i^{m_i n_i - 1}
\]

(72)

By Definition 3, \(\{P, Q, R, S\}\) is a GCA quad. Their entries are polyphase due to the quasi-symmetric, conjoint, and disjoint structure of the input arrays.

It should be pointed out that \(\{E_1, F_1, G_1, H_1\}\) and \(\{E_2, F_2, G_2, H_2\}\) can be from Proposition 13 and Proposition 14, respectively, or from a same one. For example, using base sequences \(\{A = [1, 1], B = [1, -1], C = [1], D = [1]\}\) as seed sequences in Proposition 13, one may construct two sparse GCA quads of size 1 \(\times 3 \times 3 \times 1\) respectively: \(\{E_1 = [1, 0, 1], E_2 = [1, 1, 1]\}\) and \(\{F_1 = [1, -1, 1], F_2 = [1, 1, -1]\}\).
Next let $a_1 = P(z^1)$, $a_2 = Q(z^1)$, $a_3 = R(z^1)$, $a_4 = S(z^1)$, $b_1 = I(z)$, $b_2 = J(z)$, $c_{11} = P'(z)$, $d_{11} = Q'(z)$, $c_{21} = R'(z)$, $d_{21} = S'(z)$, and $s$ is as defined in (11). By some algebraic manipulations, we have

$$\sum_{i=1}^{2} (c_{i1}^{*} c_{11} + d_{i1}^{*} d_{11}) = 4 \prod_{i=1}^{r} s t_{i} z_{i} s_{i} t_{i}^{-1}. \quad (77)$$

By Definition 3, $\{P', Q', R', S'\}$ is a GCA quad. Their entries are polyphase due to the disjoint structure of $\{I, J\}$.

The disjoint GCA pair in the above theorem can be the intermediate product $\{\frac{1}{2}(C + D), \frac{1}{2}(C - D)\}$ in Proposition 7 or $\{X, Y\}$ in Theorem 11. As an illustrative example, feeding into Theorem 16 the GCA quad of size $3 \times 3$ in (73) and a disjoint golay sequence pair $\{I = [1, 0], J = [0, -1]\}$, we can construct a 2-phase 2-D GCA quad of size $3 \times 6$ as follows:

$$P' = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}, \quad Q' = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}, \quad R' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}, \quad S' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}. \quad (78)$$

Feeding into Theorem 16 the 2-phase 2-D GCA quad constructed by Theorem 15 and the intermediate product $\{\frac{1}{2}(C + D), \frac{1}{2}(C - D)\}$ in Proposition 7, we have the following corollary about the possible sizes of 2-phase GCA quads:

**Corollary 3:** Their exist 2-phase GCA quads of size $m b_1 \times n b_2 \times b_3 \times \ldots \times b_r$ or $m b_1 b_2 \times \ldots \times b_r$ where $b_1, \ldots, b_r \in B \triangleq \{2^a 10^b 26^c | a, b, c \in \mathbb{Z} \}$ and $m, n \in \{2s + 1 | 0 \leq s \leq 38, or \ s \in B\} \cup \{2(b_1 + b_2) | b_1, b_2 \in B\}$. Specifically, for 2-phase 2-D GCA quads, all the sizes within $78 \times 78$ can be covered.

For 4-phase GCA quads constructed by Theorem 15 and Theorem 16, the possible sizes are more abundant. The following corollary provides an insight on the possible sizes of a 4-phase GCA quads.

**Corollary 4:** There exist 4-phase GCA quads of size $2s_1 t_1 + t_1' \times 2s_2 t_2 + s_2' t_2' \times \ldots \times s_r t_r$ where $s_1 \times s_2 \times \ldots \times s_r$, $t_1 \times t_2 \times \ldots \times t_r$ and $t_1' \times \ldots \times t_r$ are the possible sizes of 4-phase GCA pairs.

**Proof:** The proof is based on a three-step construction.

1) Construct four 4-phase GCA pairs: $\{A_1, B_1\}$ of size $s_1 \times s_2 \times \ldots \times s_r$, $\{C_1, D_1\}$ of size $s_1 \times s_2 \times \ldots \times s_r$, $\{A_2, B_2\}$ of size $t_1 \times \ldots \times t_r$ and $\{C_2, D_2\}$ of size $t_1' \times \ldots \times t_r$ by Theorem 11.

2) By Proposition 14, set $I = J = K = L = 1$ and use $\{A_1, B_1, C_1, D_1\}$ and $\{A_2, B_2, C_2, D_2\}$ to construct two quasi-symmetric, disjoint GCA quads $\{E_1, F_1, G_1, H_1\}$ of size $s_1 \times s_2 \times \ldots \times s_r$, and $\{E_2, F_2, G_2, H_2\}$ of size $2(t_1 + t_1') \times t_2 \times t_3 \times \ldots \times t_r$ respectively.
3) Feed \( \{E_1, F_1, G_1, H_1\} \) and \( \{E_2, F_2, G_2, H_2\} \) into Theorem 15 to construct a 4-phase GCA quad \( \{P, Q, R, S\} \) of size \( 2s_1(t_1 + t'_1) \times 2t_2(s_2 + s'_2) \times s_3t_3 \times \cdots \times s_r t_r \).

Nevertheless, the size has to be doubled [i.e., \( 2s_1(t_1 + t'_1) \times 2t_2(s_2 + s'_2) \)] to satisfy the quasi-symmetric condition induced by the complicated expression in Lemma 2. But according to the following theorem, we do not have to double the size in one dimension.

**Theorem 17:** Given a polyphase GCA quad \( \{A, B, C, D\} \), where the size of \( A \) and \( B \) is \( s_1 \times \cdots \times s_1 \times \cdots \times s_n \), and the size of \( C \) and \( D \) is \( s_1 \times \cdots \times s^r \times s_2 \times \cdots \times s_r \), and a polyphase GCA pair \( \{I, J\} \) of size \( t_1 \times \cdots \times t_r \), suppose

\[
A' = A|0_{s_1}, \quad B' = B|0_{s_1}, \\
C' = 0_{s_1}|C, \quad D' = 0_{s_1}|D, \quad (79)
\]

where \( | \) denotes concatenating two arrays in the \( i \)-th dimension, \( 0_{s_1}, (0_{s_1}') \) denotes the zero array of the same size as \( A \) (C). Suppose

\[
E = A' \otimes I + C' \otimes J,
F = A' \otimes J - C' \otimes I^*,
G = B' \otimes I + D' \otimes J,
H = B' \otimes J - D' \otimes I^*.
\quad (80)
\]

Then \( \{E, F, G, H\} \) is a polyphase GCA quad of size \( s_1 t_1 \times \cdots \times (s_1 + s'_1) t_2 \times \cdots \times s_r t_r \).

**Proof:** The proof of Theorem 17 is similar to the proof of Theorem 16, except that it is the GCA quad rather than the GCA pair that is disjoint. 

**Corollary 5:** There exist 4-phase GCA quads of size \( s_1 t_1 \times (s_2 + s'_2) t_2 \times \cdots \times s_r t_r \) where \( s_1 \times s_2 \times \cdots \times s_r \) and \( t_1 \times \cdots \times t_r \) are the possible sizes of 4-phase GCA pairs.

**Remark 4:** According to Corollary 1, \( s_2 + s'_2 \) can cover all the positive integers within 1000 except 799 and 959. By contrast, 173 integers within 1000 are not covered by [19, Theorem 10] for the possible lengths. The higher density of the possible sizes of the new construction owes to the more flexible array size in one dimension, since by Corollary 1, it is the product of the sizes in all dimensions that must be a 4-phase Golay number, rather than the size in each dimension. For example, in the 2-D case, to cover the number \( s_2 + s'_2 = 87 \), we only need two 2-D GCA pairs of sizes \( n \times 9 \) and \( n \times 78 \), respectively, where \( n \) is indeterminate. If \( n = 1 \) (i.e., in the degenerated 1-D case), there does not exist a 4-phase Golay sequence pair of length 9 since 9 is not a 4-phase Golay number. But a 4-phase GCA pair of size \( 2 \times 9 \) actually exists, because 18 is a 4-phase Golay number.

And by Corollary 1, there must be at least a basic 2-phase Golay number in the factorization of \( n \) since there are two basic 4-phase Golay numbers in 9.

Specifically, we may construct a 4-phase GCA quad of size

\[
s_1 t_1 \times (s_2 + s'_2) \times \cdots \times s_r t_r \quad (79)
\]

where \( t_1 \) is a basic 4-phase Golay number and \( s_2 + s'_2 \) can cover all the positive integers within 1000 except 799 and 959, with an additional restriction on \( \{s_1, s_3, \ldots, s_r\} \) if necessarily, as mentioned in the above remark. We note that the restriction of one additional basic 2-phase Golay number in the factorizations of \( s_1, s_3, \ldots, s_r \) can be compensated in the sense of gluing the basic 4-phase Golay number \( t_1 \), due to the disjoint structure of (79). E.g., we can construct a 4-phase 2-D GCA quad of size 36 \times 87: since \( 87 = 9 + 78 \), by Theorem 17 we only need to feed in three 4-phase 2-D GCA pairs of sizes \( 12 \times 9 \), \( 12 \times 78 \) and \( 3 \times 1 \) respectively, which exist according to Corollary 1. Otherwise, without gluing \( t_1 \), we need a 4-phase 2-D GCA pair of size 36 \times 9 which is unknown.

To cover the number 799, noting that 799 = \((2 \times 8 + 1) \times (2 \times 23 + 1)\), we first construct two sparse 2-D GCA quads of size \( 1 \times 17 \) and \( 1 \times 47 \) from the base sequences \( BS(9, 8) \) and \( BS(24, 23) \) by Proposition 13 respectively. Then feed them into Theorem 15 to construct a 2-D GCA quad of size \( 1 \times 799 \). Finally we may enlarge the other \( r - 1 \) dimensions via Theorem 16.

Noting that 959 = \((2 \times 3 + 1) \times (5^2 \times 3 \times 11)\), we cover the number 959 in the following steps (the other \( r - 1 \) dimensions may be enlarged by Theorem 16):

1) Construct two 4-phase 2-D GCA pairs \( \{A_1, B_1\} \) and \( \{C_1, D_1\} \) of size \( 1 \times 5 \) and \( 1 \times 132 \) respectively.
2) Feed \( \{A_1, B_1, C_1, D_1\} \) and a 4-phase 2-D GCA pair \( \{I_1, J_1\} \) of size \( t_1 \times 1 \) where \( t_1 \) is a 4-phase Golay number, into Theorem 17 to construct a 4-phase 2-D GCA quad \( \{A_2, B_2, C_2, D_2\} \) of size \( t_1 \times 137 \).
3) Feed \( \{A_2, B_2, C_2, D_2\} \) into Proposition 14 and set \( I = J = K = L = 1 \) to construct a sparse 2-D GCA quad \( \{E, F, G, H\} \) of size \( 4t_1 \times 137 \).

\(^2\)One may verify that in this case the constraint that \( \{A, B\} \) and \( \{C, D\} \) be two GCA pairs can be relaxed to the constraint that \( \{A, B, C, D\} \) be a GCA quad.
m, n \in \mathbb{F} \cup \{2(g_1 + g_2) | g_1, g_2 \in \mathcal{G}\}

\{g_1 + g_2, g_1, g_2 \in \mathcal{G}\}

\{2(g_1g_3 + g_2g_4) | g_1, g_2, g_3, g_4 \in \mathcal{G}\}

Theorem 15

Proposition 13

Proposition 14

TABLE I

LENGTHS OF 4-PHASE GOLAY SEQUENCE QUADS

| Construction | Length\$^1\$ |
|--------------|-------------|
| [19, Theorem 11] | $mn, m', n' \in \mathbb{F} \cup \{2(g_1 + g_2)|g_1, g_2 \in \mathcal{G}\}$ |
| [19, Theorem 10] | $g_1 + g_2, g_1, g_2 \in \mathcal{G}$ |
| Theorem 15 | $\{2(g_1g_3 + g_2g_4)|g_1, g_2, g_3, g_4 \in \mathcal{G}\}$ |
| Theorem 17 | $g_3(g_1 + g_2), g_1, g_2, g_3 \in \mathcal{G}$ |

\$^1\mathcal{F} \triangleq \{2s + 1|0 \leq s \leq 38, or s \in \mathcal{G}\}$ where \$\mathcal{G} \triangleq \{2^s + 3^x5^y11^z|a, b, c, d, e, u \in \mathbb{Z}^\ast \cup \{0\}, b + c + d + e \leq a + 2u + 1, u \leq c + e\}$.

4) Construct a sparse 2-D GCA quad \{A, B, C, D\} of size 1 x 7 from BS\{4, 3\} by Proposition 13.

5) Feed \{A, B, C, D\} and \{E, F, G, H\} into Theorem 15 to construct a 4-phase 2-D GCA quad \{P, Q, R, S\} of size 41 x 959.

Hence we have the following corollary.

Corollary 6: There exist 4-phase GCA quads whose sizes in one dimension can cover all the positive integers within 1000.

D. New Results on 4-Phase Golay Sequence Quad and Hadamard Matrix

Some constructions of 4-phase GCA quad are proposed in this section to achieve more possible sizes of multidimensional arrays. As a special case, these constructions result in new 4-phase Golay sequence quads. The forms of possible lengths are summarized in Table I, where $g_1, \cdots, g_5$ are 4-phase Golay numbers.

First, 4-phase Golay sequence quads of length $mmn$ can be constructed by Theorem 15 where $m, n \in \mathbb{F} \cup \{2(g_1g_3 + g_2g_4) | g_1, g_2, g_3, g_4 \in \mathcal{G}\}$ with \$\mathcal{F} \triangleq \{2s + 1|0 \leq s \leq 38, or s \in \mathcal{G}\}$ and \$\mathcal{G}$ being the set of all 4-phase Golay numbers. By contrast, the length $mn'$ can be covered by [19, Theorem 11] where $m', n' \in \mathbb{F} \cup \{2(g_1 + g_2) | g_1, g_2 \in \mathcal{G}\}$.

As listed in Table II, for lengths within 2000, there are 532 integers that cannot be covered by [19, Theorem 11], while only 479 integers cannot be covered by Theorem 15, which is an improvement due to Proposition 14. For example, the length 254 can be covered by Theorem 15 since 254 = 2 x (6 x 1 + 11 x 11), but 254 cannot be covered by [19, Theorem 11] since 254 = 2 x 127 and 127 is a prime number that cannot be written as a summation of two 4-phase Golay numbers.

Second, 4-phase Golay sequence quads of length $g_5(g_1 + g_2)$ can be constructed by Theorem 17 where $g_1, \cdots, g_3$ are 4-phase Golay numbers. By contrast, the length $g_1 + g_2$ can be covered by [19, Theorem 10]. For lengths within 2000, there are 504 integers that cannot be covered by [19, Theorem 10], while only 294 integers cannot be covered by Theorem 17. For example, the length 87 can be covered by Theorem 17 since 87 = 3 x (3 + 26), but 87 cannot be covered by [19, Theorem 10] since 87 cannot be written as a summation of two 4-phase Golay numbers.

Being combined, [19, Theorem 10] and [19, Theorem 11] leave out 314 integers within 2000. In contrast, Theorem 15 and Theorem 17 combined only leave out 217 integers. More lengths beyond 2000 can be discovered as well.

Interestingly, a Hadamard matrix of size $8n \times 8n$ can be constructed from a 4-phase Golay sequence quad of length $n$ [19, Theorem 15]. As some of these Hadamard matrices can also be constructed by other methods developed in the context of Hadamard design [35], the known sizes of Hadamard matrices were provided in [35, Table 1.51] (the results in [19] are not included though). Excluding these known sizes, we can still discover 3 new sizes of Hadamard matrices from the 97 new 4-phase Golay sequence quads.

E. Relationships Between Our New Constructions and the Existing Ones

Fig. 1 illustrates the logical connections between the known constructions of GCA and the new constructions of GCA proposed in this paper. The constructions in the same box are equivalent. The boxed theorems and propositions on the left side are known constructions while those on the right are our new contributions. The arrowed solid lines in red indicate that the corresponding known constructions are special cases of our constructions. The arrays constructed by a proposition or theorem may be fed into another proposition or theorem to construct larger arrays, as illustrated by the arrowed dashed line in blue. More specifically, the 4-phase GCA pairs constructed by Theorem 11 can be directly used by Theorem 12 and Theorem 17 to construct 4-phase GCA quads; with the aid of Proposition 13 and Proposition 14, Theorem 15 constructs 4-phase GCA quads from the 4-phase GCA pairs constructed by Theorem 11; the disjoint GCA pairs
Hadamard matrices.

new 4-phase Golay sequence quads and hence discover new positive integers within 1000. As a special case, we construct GCA quads whose sizes in one dimension can cover all the pairs, the sizes of 4-phase GCA quads can be
\[s \times \cdots \times t\]
dant. Given covered.

some 2-D 2-phase GCA quads, all the sizes within 78
\[s \leq nb\]
together with another two well-known identities as an insight into GCA, we propose some constructions of polyphase GCA pairs and polyphase GCA quads.

V. Conclusion

In this paper, we propose two identities over a commutative ring, and together with another two well-known identities as an insight into GCA, we propose some constructions of polyphase GCA pairs and polyphase GCA quads.

The possible sizes of the 4-phase GCA pairs are \(s_1 \times \cdots \times s_r\), where \(s_1 \cdots s_r\) are \(2^{2+u}3^b 5^c 11^d 13^e\), \(a, b, c, d, e, u \in \mathbb{Z}_+ \cup \{0\}\), \(b + c + d + e \leq a + 2u + 1\), \(u \leq c + e\), and each of the \(u\) factors 2 should reside in either 10 or 26, which is a factor of some \(s_i\).

The possible sizes of the 2-D 2-phase GCA quads are \(mb_1 \times nb_2 \times b_3 \times \cdots \times b_n\) or \(mn b_1 \times b_2 \times b_3 \times \cdots \times b_n\). Specifically, for 2-D 2-phase GCA quads, all the sizes within 78 \(\times 78\) can be covered.

The possible sizes of 4-phase GCA quads are more abundant. Given \(s_1 \times s_2 \times \cdots \times s_r, s_1 \times s_2 \times \cdots \times s_r, t_1 \times t_2 \times \cdots \times t_r\) and \(t'_1 \times t_2 \times \cdots \times t_r\) as the possible sizes of 4-phase GCA pairs, the sizes of 4-phase GCA quads can be \(s_1 t_1 \times s_2 t_2 \times \cdots \times s_r t_r\), \(2s_1 (t_1 + t'_1) \times 2t_2 (s_2 + s'_2) \times s_3 t_3 \times \cdots \times s_r t_r\), or \(s_1 t_1 \times (s_2 + s'_2) t_2 \times \cdots \times s_r t_r\). Besides, there exist 4-phase GCA quads whose sizes in one dimension can cover all the positive integers within 1000. As a special case, we construct new 4-phase Golay sequence quads and hence discover new Hadamard matrices.

Our Matlab\textsuperscript{TM} codes that can generate 2-D GCAs of known possible sizes in Corollary 1, 3 and 6 are available online: https://github.com/csrlab-fudan/ACM.

| Construction | Array Size | Cardinality | Polyphaseness |
|--------------|------------|-------------|---------------|
| [3, Theorem 16] | \(2^a \times 2^b, a, b \geq 0, a + b > 2\) | \(2^c, 1 \leq k \leq a + b\) | \(2p, p \in \mathbb{Z}^+\) |
| [4, Theorem III.3] | \(g_1 \times g_2, g_1, g_2 \in G\) \(1\) | \(2\) | \(4\) |
| [4, Theorem III.5] | \(g_1 \times g_2, g_1, g_2 \in G\) | \(4\) | \(4\) |
| Corollary 3 | \(N^m, \times \cdots \times N^m, m_1, \ldots, m_r \geq 0^+\) | \(N^2\) | \(q^2\) |
| Theorem 11 | \(s_1 \times \cdots \times s_r^4\) | \(2\) | \(4\) |
| Theorem 12 | \(s_1 t_1 \times \cdots \times s_r t_r^5\) | \(4\) | \(4\) |
| Corollary 4 | \(2s_1(t_1 + t'_1) \times 2t_2(s_2 + s'_2) \times s_3 t_3 \times \cdots \times s_r t_r^5\) | \(4\) | \(4\) |
| Theorem 17 | \(s_1 t_1 \times (s_2 + s'_2) t_2 \times s_3 t_3 \times \cdots \times s_r t_r^5\) | \(4\) | \(4\) |

1 \(G = \{2^{a+u}3^b 5^c 11^d 13^e\} | a, b, c, d, e, u \in \mathbb{Z}_+ \cup \{0\}\), \(b + c + d + e \leq a + 2u + 1\), \(u \leq c + e\).

2 \(N\) and \(q\) are the matrix size and the polyphaseness of known Butson-type Hadamard matrices respectively, e.g., \(N = q = \in \mathbb{Z}^+\).

3 \(b_1, \cdots, b_r \in B = \{2^{2+u}3^b 5^c 11^d 13^e\} | a, b, c, d, e, u \in \mathbb{Z}_+ \cup \{0\}\} \) and \(m, n \in \{2s + 1|0 \leq s \leq 38\), or \(s \in B\} \cup \{2(b'_1 + b'_2)|b'_1, b'_2 \in B\} \).

4 \(\prod_{r=1}^s s_r \in G\), and each of the \(u\) factors 2 resides in either 10 or 26, which is a factor of some \(s_i\).

5 \(s_1 \times s_2 \times \cdots \times s_r, s_1 \times s'_2 \times \cdots \times s_r, t_1 \times \cdots \times t_r\) and \(t'_1 \times \cdots \times t_r\) are array sizes of GCA pairs constructed by Theorem 11.

TABLE III

COMPARISONS OF GCAS OBTAINED BY DIFFERENT CONSTRUCTIONS

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