Non-Archimedean Integrals and Stringy Euler Numbers of Log Terminal Pairs

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Abstract

Using non-Archimedean integration over spaces of arcs of algebraic varieties, we define stringy Euler numbers associated with arbitrary Kawamata log terminal pairs. There is a natural Kawamata log terminal pair corresponding to an algebraic variety $V$ having a regular action of a finite group $G$. In this situation we show that the stringy Euler number of this pair coincides with the physicists’ orbifold Euler number defined by the Dixon-Harvey-Vafa-Witten formula. As an application, we prove a conjecture of Miles Reid on the Euler numbers of crepant desingularizations of Gorenstein quotient singularities.

1 Introduction

Let $X$ be a normal irreducible algebraic variety of dimension $n$ over $\mathbb{C}$, $Z_{n-1}(X)$ the group of Weil divisors on $X$, $\text{Div}(X) \subset Z_{n-1}(X)$ the subgroup of Cartier divisors on $X$, $Z_{n-1}(X) \otimes \mathbb{Q}$ the group of Weil divisors on $X$ with coefficients in $\mathbb{Q}$, $K_X \in Z_{n-1}(X)$ a canonical divisor of $X$.

Recall several definitions from the Minimal Model Program [14, 15, 16] (see also [17, 18]):

Definition 1.1 Let $\Delta_X \in Z_{n-1}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on a normal irreducible algebraic variety $X$. A resolution of singularities $\rho : Y \to X$ is called a log resolution of $(X, \Delta_X)$ if the union of the $\rho$-birational transform $\rho^{-1}(\Delta_X)$ of $\Delta_X$ with the exceptional locus of $\rho$ is a divisor $D$ consisting of smooth irreducible components $D_1, \ldots, D_m$ having only normal crossings.

Definition 1.2 Let $\rho : Y \to X$ be a log resolution of a pair $(X, \Delta_X)$. We assume that $K_X + \Delta_X$ is a $\mathbb{Q}$-Cartier divisor and write

$$K_Y = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X)D_i,$$
where $D_i$ runs through all irreducible components of $D$ and $a(D_i, \Delta_X) = -d_j$ if $D_i$ is a $\rho$-birational transform of an irreducible component $\Delta_j$ of $\text{Supp} \Delta_X$. Then the number rational number $a(D_i, \Delta_X)$ (resp. $a_l(D_i, \Delta_X) := a(D_i, \Delta_X) + 1$) is called the \textbf{discrepancy} (resp. \textbf{log discrepancy}) of $D_i$.

\textbf{Definition 1.3} A pair $(X, \Delta_X)$ is called \textbf{Kawamata log terminal} if the following conditions are satisfied:

(i) $\Delta_X = d_1\Delta_1 + \cdots + d_k\Delta_k$, where $\Delta_1, \ldots, \Delta_k$ are irreducible Weil divisors and $d_i < 1$ for all $i \in \{1, \ldots, k\}$;

(ii) $K_X + \Delta_X$ is a $\mathbb{Q}$-Cartier divisor;

(iii) for any log resolution of singularities $\rho : Y \to X$, we have $a_l(D_i, \Delta_X) > 0$ for all $i \in \{1, \ldots, m\}$,

Now we introduce a new invariant of Kawamata log terminal pairs:

\textbf{Definition 1.4} Let $(X, \Delta_X)$ be a Kawamata log terminal pair, $\rho : Y \to X$ a log resolution of singularities as above. We put $I = \{1, \ldots, m\}$ and set for any subset $J \subset I$

$$D_J := \begin{cases} \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases}, \quad D_J^0 := D_J \setminus \bigcup_{j \in (I \setminus J)} D_j,$$

$$e(D_J^0) := \text{(topological Euler number of } D_J^0).$$

We call the rational number

$$e_{\text{st}}(X, \Delta_X) := \sum_{J \subset I} e(D_J^0) \prod_{j \in J} a_l(D_j, \Delta_X)^{-1}$$

the \textbf{stringy Euler number} of the Kawamata log terminal pair $(X, \Delta_X)$ (in the above formula, we assume $\prod_{j \in J} = 1$ if $J = \emptyset$).

Using non-Archimedian integrals, we show that the stringy Euler number $e_{\text{st}}(X, \Delta_X)$ is well-defined:

\textbf{Theorem 1.5} \textit{In the above definition, $e_{\text{st}}(X, \Delta_X)$ does not depend on the choice of a log resolution $\rho : Y \to X$.}

We expect that the stringy Euler numbers have the following natural connections with log flips in dimension 3 (see [21, 22]):

\textbf{Conjecture 1.6} \textit{Let $X$ be a normal 3-dimensional variety and $\Delta_X$ is an effective $\mathbb{Q}$-divisor such that $(X, \Delta)$ is Kawamata log terminal, and $\varphi : (X, \Delta_X) \dashrightarrow (X^+, \Delta_{X^+})$ a log $(K_X + \Delta_X)$-flip. Then one has the following inequality:}

$$e_{\text{st}}(X, \Delta_X) > e_{\text{st}}(X^+, \Delta_{X^+}).$$
Remark 1.7 In [11] we show that the above conjecture is true for toric log flips in arbitrary dimension $n$.

Recall now a definition from the string theory [10] (see also [20]):

**Definition 1.8** Let $V$ be a smooth complex algebraic variety together with a regular action of a finite group $G$: $G \times V \to V$. For any element $g \in G$ we set

$$V^g := \{ x \in V : gx = x \}.$$

Then the number

$$e(V, G) := \frac{1}{|G|} \sum_{(g,h) \in G \times G, \ gh = hg} e(V^g \cap V^h)$$

is called the **physicists’ orbifold Euler number** of $V$.

Our main result in this paper is the following:

**Theorem 1.9** Let $V$ be as in 1.8, $X := V/G$ the geometric quotient, $\Delta_1, \ldots, \Delta_k \subseteq V/G$ the set of all irreducible components of codimension 1 in the ramification locus of the Galois covering $\phi : V \to X$. We denote by $\nu_i$ the order of a cyclic inertia subgroup $G_i \subseteq G$ corresponding to $\Delta_i$ and set

$$\Delta_X := \sum_{i=1}^{k} \left( \frac{\nu_i - 1}{\nu_i} \right) \Delta_i.$$

Then the pair $(X, \Delta_X)$ is Kawamata log terminal and the following equality holds

$$e_{st}(X, \Delta_X) = e(V,G).$$

As an corollary of 1.9, we obtain the following statement conjectured by Miles Reid in [20]:

**Theorem 1.10** Let $G \subseteq SL(n, \mathbb{C})$ be a finite subgroup acting on $V := \mathbb{C}^n$. Assume that there exists a crepant desingularization of $X := V/G$, i.e., a smooth variety $Y$ together with a projective birational morphism $\rho : Y \to X$ such that the canonical class $K_Y$ is trivial. Then the Euler number of $Y$ equals the number of conjugacy classes in $G$.

The paper is organized as follows. In Section 2 we review a construction of a non-Archimedean measure on the space of arcs $J_\infty(X)$ of a smooth algebraic variety $X$ over $\mathbb{C}$. This measure associate to a measurable subset $C \subseteq J_\infty(X)$ an element $Vol_X(X)$ of a 2-dimensional notherian ring $\hat{A}_1$ which is complete with respect to a non-Archimedean topology defined by powers of a principal ideal $(\theta) \subset$
In Section 3 we define exponentially integrable measurable functions and their exponential non-Archimedean integrals. Our main interest are measurable functions \( F_D \) associated with \( \mathbb{Q} \)-divisors \( D \in \text{Div}(X) \otimes \mathbb{Q} \). We prove a transformation formula for the exponential integral under a birational proper morphism.

In Section 4 we consider Kawamata log terminal pairs \( (X, \Delta_X) \), where \( X \) is a toric variety and \( \Delta_X \) is a torus invariant \( \mathbb{Q} \)-divisor. We give an explicit formula for \( e_{st}(X, \Delta_X) \) using a \( \Sigma \)-piecewise linear function \( \varphi_{K, \Delta} \) corresponding to the torus invariant \( \mathbb{Q} \)-Cartier divisor \( K_X + \Delta_X \). In Section 5 we investigate quotients of smooth algebraic varieties \( V \) modulo regular actions of finite groups \( G \). We define canonical sequences of blow ups of smooth \( G \)-invariant subvarieties in \( V \) which allow us to construct in a canonical way a smooth \( G \)-variety \( V' \) such that stabilizers of all points in \( V' \) are abelian. This construction is used in Section 6 where we prove our main theorem. In Section 7 we apply our results to a cohomological McKay correspondence in arbitrary dimension (this extends our \( p \)-adic ideas from [2]).

We note that Sections 2 and 3 are strongly influenced by the idea of “motivic integral” proposed by Kontsevich [19]. Its different versions are contained in the papers of Denef and Loeser [6, 7, 8, 9]. The case of divisors on surfaces was considered by Veys in [23, 24].

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2 Non-Archimedean measure on spaces of arcs

Recall definitions of jets and spaces of arcs (see [11], Part A).

Definition 2.1 Let \( X \) be a smooth \( n \)-dimensional complex manifold, \( x \in X \) an arbitrary point. A germ of a holomorphic curve at \( x \) is a germ of a holomorphic map \( \gamma \) of a small ball \( \{ |z| < \varepsilon \} \subset \mathbb{C} \) to \( X \) such that \( \gamma(0) = x \).

Let \( l \) be a nonnegative integer. Two germs \( \gamma_1, \gamma_2 \) of holomorphic curves at \( x \) are called \( l \)-equivalent if the derivatives of \( \gamma_1 \) and \( \gamma_2 \) at \( 0 \) coincide up to order \( l \). The set of \( l \)-equivalent germs of holomorphic curves is denoted by \( J_l(X, x) \) and called the jet space of order \( l \) at \( x \). The union

\[
J_l(X) = \bigcup_{x \in X} J_l(X, x)
\]

is a complex manifold of dimension \((l + 1)n\) which is a holomorphic affine bundle over \( X \). The complex manifold \( J_l(X) \) is called the jet space of order \( l \) of \( X \).

Definition 2.2 Consider canonical mappings \( j_l : J_{l+1}(X) \rightarrow J_l(X) \) \((l \geq 0)\) whose fibers are isomorphic to affine spaces \( \mathbb{C}^n \). We denote by \( J_\infty(X) \) the projective limit of \( J_l(X) \) and by \( \pi_l \) the canonical projection \( J_\infty(X) \rightarrow J_l(X) \). The space \( J_\infty(X) \) is called the space of arcs of \( X \).
Remark 2.3 Let $R$ be the formal power series ring $\mathbb{C}[t]$ considered as the inverse limit of finite dimensional $\mathbb{C}$-algebras $R_l := \mathbb{C}[t]/(t^{l+1})$. If $X$ is $n$-dimensional smooth quasi-projective algebraic variety over $\mathbb{C}$, then the set of points in $J_\infty(X)$ (resp. $J_l(X)$) coincides with the set of $R$-valued (resp. $R_l$-valued) points of $X$.

From now on we shall consider only the spaces $J_\infty(X)$, where $X$ is a smooth algebraic variety. In this case, $J_l(X)$ is a smooth algebraic variety for all $l \geq 0$.

Definition 2.4 A set $C \subset J_\infty(X)$ is called cylinder set if there exists a positive integer $l$ such that $C = \pi_l^{-1}(B_l(C))$ for some constructible subset $B_l(C) \subset J_l(X)$. Such a constructible subset $B_l(C)$ will be called the $l$-base of $C$. By definition, the empty set $\emptyset \subset J_\infty(X)$ is a cylinder set and its $l$-base in $J_l(X)$ is assumed to be empty for all $l \geq 0$.

Remark 2.5 Let $C \subset J_\infty(X)$ be a cylinder set with an $l$-base $B_l(X)$.

(i) It is clear that $B_{l+1}(C) := j_{l}^{-1}(B_l(C)) \subset J_{l+1}(X)$ is the $(l + 1)$-base of $C$ and $B_{l+1}(X)$ is a Zariski locally trivial affine bundle over $B_l(C)$ whose fibers are isomorphic to $\mathbb{C}^n$.

(ii) Using (i), it is a standard exercise to show that finite unions, intersections and complements of cylinder sets are again cylinder sets.

The following property of cylinder sets will be important:

Theorem 2.6 Assume that a cylinder set $C \subset J_\infty(X)$ is contained in a countable union $\bigcup_{i=1}^{\infty} C_i$ of cylinder sets $C_i$. Then there exists a positive integer $m$ such that $C \subset \bigcup_{i=1}^{m} C_i$.

Proof. The proof of theorem 2.6 is based on a classical property of constructible sets (see [12], Cor. 7.2.6). For details see Theorem 6.6 in [3]. Another version of the same statement is contained in [8] (see Lemma 2.4). \qed

Definition 2.7 Let $\mathbb{Z}[\tau^\pm1]$ be the Laurent polynomial ring in variable $\tau$ with coefficients in $\mathbb{Z}$, $A$ the group algebra of $(\mathbb{Q}, +)$ with coefficients in $\mathbb{Z}[\tau^\pm1]$. We denote by $\theta^s \in A$ the image of $s \in \mathbb{Q}$ under the natural homomorphism $(\mathbb{Q}, +) \rightarrow (A^*, \cdot)$, where $A^*$ is the multiplicative group of invertible elements in $A$ (the element $\theta \in A$ is transcendental over $\mathbb{Z}[\tau^\pm1]$). For this reason, we write

$$A := \mathbb{Z}[\tau^\pm1][\theta^\mathbb{Q}]$$

and identify $A$ with the direct limit of the subrings $A_N := \mathbb{Z}[\tau^\pm1][\theta^\mathbb{N}/\mathbb{Z}] \subset A$, where $N$ runs over all positive integers.
Definition 2.8 We consider a topology on $A$ defined by the non-Archimedean norm
\[ \| \cdot \| : A \rightarrow \mathbb{R}_{\geq 0} \]
which is uniquely characterised by the properties:
1. $\|ab\| = \|a\| \cdot \|b\|$, $\forall a, b \in A$;
2. $\|a + b\| = \max\{\|a\|, \|b\|\}$, $\forall a, b \in A$ if $\|a\| \neq \|b\|$;
3. $\|a\| = 1$, $\forall a \in \mathbb{Z}[\tau^{\pm 1}] \setminus \{0\}$;
4. $\|\theta^s\| = e^{-s}$ if $s \in \mathbb{Q}$.

The completion of $A$ (resp. of $A_N$) with respect to the norm $\| \cdot \|$ will be denoted by $\hat{A}$ (resp. by $\hat{A}_N$). We set
\[ \hat{A}_\infty := \bigcup_{N \in \mathbb{N}} \hat{A}_N \subset \hat{A}. \]

Remark 2.9 The noetherian ring $\hat{A}_N$ consists of Laurent power series in variable $\theta^{1/N}$ with coefficients in $\mathbb{Z}[\tau^{\pm 1}]$. The ring $\hat{A}$ consists of formal infinite sums
\[ \sum_{i=1}^{\infty} a_i \theta^{s_i}, \quad a_i \in \mathbb{Z}[\tau^{\pm 1}], \]
where $s_1 < s_2 < \cdots$ is an ascending sequence of rational numbers having the property $\lim_{i \rightarrow \infty} s_i = +\infty$.

Definition 2.10 Let $W$ be an arbitrary algebraic variety. Using a natural mixed Hodge structure in cohomology groups $H^i_c(W, \mathbb{C})$, $(0 \leq i \leq 2d)$, we define the number $h^{p,q}(H^i_c(W, \mathbb{C}))$ to be the dimension of the $(p, q)$-type Hodge component in $H^i_c(W, \mathbb{C})$. We set
\[ e^{p,q}(W) := \sum_{i \geq 0} (-1)^i h^{p,q}(H^i_c(W, \mathbb{C})) \]
and call
\[ E(W; u, v) := \sum_{p,q} e^{p,q}(W) u^p v^q, \]
the $E$-polynomial of $W$. By the usual Euler number of $W$ we always mean $e(W) := E(W; 1, 1)$.

Remark 2.11 For our purpose, it will be very important that $E$-polynomials have properties which are very similar to the ones of usual Euler numbers:
1. if $W = W_1 \cup \cdots \cup W_k$ is a disjoint union of Zariski locally closed subsets $W_1, \ldots, W_k$, then
\[ E(W; u, v) = \sum_{i=1}^{k} E(W_i; u, v); \]
(ii) if \( W = W_1 \times W_2 \) is a product of two algebraic varieties \( W_1 \) and \( W_2 \), then

\[
E(W; u, v) = E(W_1; u, v) \cdot E(W_2; u, v);
\]

(iii) if \( W \) admits a fibering over \( Z \) which is locally trivial in Zariski topology such that each fiber of the morphism \( f : W \to Z \) is isomorphic to the affine space \( \mathbb{C}^n \), then

\[
E(W; u, v) = E(\mathbb{C}^n; u, v) \cdot E(Z; u, v) = (uv)^n E(Z; u, v).
\]

**Definition 2.12** Let \( V \subset W \) is a constructible subset in a complex algebraic variety \( V \). We write \( V \) as a union

\[
V = W_1 \cup \cdots \cup W_k
\]

of pairwise nonintersecting Zariski locally closed subsets \( W_1, \ldots, W_k \). Then the **E-polynomial of** \( V \) is defined as follows:

\[
E(V; u, v) := \sum_{i=1}^{k} E(W_i; u, v).
\]

**Remark 2.13** Using 2.11(i), it is easy to check that the above definition does not depend on the choice of the decomposion of \( V \) into a finite union of pairwise nonintersecting Zariski locally closed subsets.

Now we define a **non-Archimedean cylinder set measure** on \( J_\infty(X) \).

**Definition 2.14** \( C \subset J_\infty(X) \) be a cylinder set. We define the **non-Archimedean volume** \( \text{Vol}_X(C) \in A_1 \) of \( C \) by the following formula:

\[
\text{Vol}_X(C) := E(B_l(C); \tau \theta^{-1}, \tau^{-1} \theta^{-1}) \theta^{(l+1)n} \in A_1,
\]

where \( C = \pi^{-1}_l(B_l(C)) \) and \( E(B_l(C); u, v) \) is the \( E \)-polynomial of the \( l \)-base \( B_l(C) \subset J_l(X) \). If \( C = \emptyset \), we set \( \text{Vol}_X(C) := 0 \).

**Remark 2.15** Using 2.3(i) and 2.11, one immediately obtains that \( \text{Vol}_X(C) \) does not depend on the choice of an \( l \)-base \( B_l(C) \) and

\[
\|\text{Vol}_X(C)\| = e^{2\dim B_l(C) - 2(l+1)n}.
\]

In particular, one has the following properties

(i) If \( C_1 \) and \( C_2 \) are two cylinder sets such that \( C_1 \subset C_2 \), then

\[
\|\text{Vol}_X(C_1)\| \leq \|\text{Vol}_X(C_2)\|.
\]
(ii) If $C_1, \ldots, C_k$ are cylinder sets, then
\[
\|Vol_X(C_1 \cup \cdots \cup C_k)\| = \max_{i=1}^k \|Vol_X(C_i)\|.
\]

(iii) If a cylinder set $C$ is a finite disjoint union of cylinder sets $C_1, \ldots, C_k$, then
\[
Vol_X(C) = Vol_X(C_1) + \cdots + Vol_X(C_k).
\]

**Definition 2.16** We say that a subset $C \subset J_\infty(X)$ is **measurable** if for any positive real number $\varepsilon$ there exists a sequence of cylinder sets $C_0(\varepsilon), C_1(\varepsilon), C_2(\varepsilon), \ldots$ such that
\[
(C \cup C_0(\varepsilon)) \setminus (C \cap C_0(\varepsilon)) \subset \bigcup_{i \geq 1} C_i(\varepsilon)
\]
and $\|Vol_X(C_i(\varepsilon))\| < \varepsilon$ for all $i \geq 1$. If $C$ is measurable, then the element
\[
Vol_X(C) \coloneqq \lim_{\varepsilon \to 0} C_0(\varepsilon) \in \hat{A}_1
\]
will be called the **non-Archimedean volume** of $C$.

**Theorem 2.17** If $C \subset J_\infty(X)$ is measurable, then $\lim_{\varepsilon \to 0} C_0(\varepsilon)$ exists and does not depend on the choice of sequences $C_0(\varepsilon), C_1(\varepsilon), C_2(\varepsilon), \ldots$.

**Proof.** The property 2.6 plays a crucial role in the proof of this theorem. For details see [3], Theorem 6.18. \qed

The proof of the following statement is a standard exercise:

**Proposition 2.18** Measurable sets possess the following properties:

(i) Finite unions, finite intersections of measurable sets are measurable.

(ii) If $C$ is a disjoint union of nonintersecting measurable sets $C_1, \ldots, C_m$, then
\[
Vol_X(C) = Vol_X(C_1) + \cdots + Vol_X(C_m).
\]

(iii) If $C$ is measurable, then the complement $\overline{C} := J_\infty(X) \setminus C$ is measurable.

(iv) If $C_1, C_2, \ldots, C_m, \ldots$ is an infinite sequence of nonintersecting measurable sets having the property
\[
\lim_{i \to \infty} \|Vol_X(C_i)\| = 0,
\]
then
\[
C = \bigcup_{i=1}^\infty C_i
\]
is measurable and
\[
Vol_X(C) = \sum_{i=1}^\infty Vol_X(C_i).
\]
The next example shows that our non-Archimedian measure does not have all properties of the standard Lebesgue measure:

**Example 2.19** Let \( C \subset R = C[[t]] \) be the set consisting of all power series \( \sum_{i \geq 0} a_i t^i \) such that \( a_i \neq 0 \) for all \( i \geq 0 \). For any \( k \in \mathbb{Z}_{\geq 0} \), we define \( C_k \subset R \) to be the set consisting of all power series \( \sum_{i \geq 0} a_i t^i \) such that \( a_i \neq 0 \) for all \( 0 \leq i \leq k \). We identify \( R \) with \( J_\infty(C) \). Then every \( C_k \subset J_\infty(C) \) is a cylinder set and \( \text{Vol}_C(C_k) = (1 - \theta^2)^{k + 1} \).

Moreover, we have

\[
C_0 \supset C_1 \supset C_2 \supset \cdots, \quad \text{and} \quad C = \bigcap_{k \geq 0} C_k.
\]

However, the sequence

\[
\text{Vol}_C(C_0), \, \text{Vol}_C(C_1), \, \text{Vol}_C(C_2), \, \ldots
\]

does not converge in \( \hat{A}_1 \).

**Definition 2.20** We shall say that a subset \( C \subset J_\infty(X) \) has **measure zero** if for any positive real number \( \varepsilon \) there exists a sequence of cylinder sets \( C_1(\varepsilon), C_2(\varepsilon), \ldots \) such that \( C \subset \bigcup_{i \geq 1} C_i(\varepsilon) \) and \( \| \text{Vol}_X(C_i(\varepsilon)) \| < \varepsilon \) for all \( i \geq 1 \).

**Definition 2.21** Let \( Z \subset X \) be a Zariski closed subvariety. For any point \( x \in Z \), we denote by \( \mathcal{O}_{X,x} \) the ring of germs of holomorphic functions at \( x \). Let \( I_{Z,x} \subset \mathcal{O}_{X,x} \) be the ideal of germs of holomorphic functions vanishing on \( Z \). We set

\[
J_l(Z, x) := \{ y \in J_l(X, x) : g(y) = 0 \quad \forall \, g \in I_{Z,x} \}, \quad l \geq 1,
\]

\[
J_\infty(Z, x) := \{ y \in J_\infty(X, x) : g(y) = 0 \quad \forall \, g \in I_{Z,x} \}
\]

and

\[
J_\infty(Z) := \bigcup_{x \in Z} J_\infty(Z, x).
\]

The space \( J_\infty(Z) \subset J_\infty(X) \) will be called **space of arcs with values in** \( Z \).

**Proposition 2.22** Let \( Z \) be an arbitrary Zariski closed subset in a smooth irreducible algebraic variety \( X \). Then \( J_\infty(X, Z) \subset J_\infty(X) \) is measurable. Moreover, one has

\[
\text{Vol}_X(J_\infty(Z)) = \begin{cases} 
0 & \text{if } Z \neq X \\
\text{Vol}_X(J_\infty(X)) & \text{if } Z = X.
\end{cases}
\]

**Proof.** If \( Z \neq X \), then the set \( J_\infty(Z) \) can be obtained as an intersection of cylinder sets \( C_k \) such that \( \| \text{Vol}_X(C_k) \| \leq e^{-2k} \) (see Theorem 6.22 in [2] and 3.2.2 in [3]). \( \square \)
3 Non-Archimedian integrals

Definition 3.1 By a measurable function $F$ on $J_{\infty}(X)$ we mean a function $F : M \to \mathbb{Q}$, where $M \subset J_{\infty}(X)$ is a subset such that $J_{\infty}(X) \setminus M$ has measure zero and $F^{-1}(s)$ is measurable for all $s \in \mathbb{Q}$. Two measurable functions $F_i : M_i \to \mathbb{Q}$ ($i = 1, 2$) on $J_{\infty}(X)$ are called equal if $F_1(\gamma) = F_2(\gamma)$ for all $\gamma \in M_1 \cap M_2$.

Definition 3.2 A measurable function $F : M \to \mathbb{Q}$ is called exponentially integrable if the series
\[ \sum_{s \in \mathbb{Q}} \|Vol_X(F^{-1}(s))\|e^{-2s} \]
converges. If $F$ is exponentially integrable, then the sum
\[ \int_{J_{\infty}(X)} e^{-F} := \sum_{s \in \mathbb{Q}} Vol_X(F^{-1}(s))\theta^{2s} \in \hat{A} \]
will be called the exponential integral of $F$ over $J_{\infty}(X)$.

Definition 3.3 Let $D \subset Div(X)$ be a subvariety of codimension 1, $x \in D$ a point, and $g \in \mathcal{O}_{X,x}$ the local equation for $D$ at $x$. We set $M(D) := J_{\infty}(X) \setminus J_{\infty}(Supp D)$. For any $\gamma \in M(D)$, we denote by $\langle D, \gamma \rangle_x$ the order of the holomorphic function $g(\gamma(t))$ at $t = 0$. The number $\langle D, \gamma \rangle_x$ will be called the intersection number of $D$ and $\gamma$ at $x \in X$. We define the function $F_D : M(D) \to \mathbb{Z}$ as follows:
\[ F_D(\gamma) = \begin{cases} 0 & \text{if } \pi_0(\gamma) = x \notin D \\ \langle D, \gamma \rangle_x & \text{if } \pi_0(\gamma) \in D \end{cases} \]

Remark 3.4 Using the property $\langle D' + D'', \gamma \rangle_x = \langle D', \gamma \rangle_x + \langle D'', \gamma \rangle_x$, we extend the definition of $F_D$ to an arbitrary $\mathbb{Q}$-Cartier divisor $D$: if $D = \sum_{i=1}^{m} a_i D_i \in Div(X) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-linear combination of irreducible subvarieties $D_1, \ldots, D_m$, then we set
\[ F_D := \sum_{i=1}^{m} a_i F_{D_i}. \]

It is easy to show that measurable functions form a $\mathbb{Q}$-vector space and $D \subset Div(X) \otimes \mathbb{Q}$ can be identified with its $\mathbb{Q}$-subspace, since $F_D : M(D) \to \mathbb{Q}$ is measurable for all $D \subset Div(X) \otimes \mathbb{Q}$.

The following theorem describes a transformation law for exponential integrals under proper birational morphisms:
Theorem 3.5  Let \( \rho : Y \to X \) be a proper birational morphism of smooth complex algebraic varieties, \( D = \sum_{i=1}^{r} d_i D_i \in \text{Div}(Y) \) the Cartier divisor defined by the equality
\[
K_Y = \rho^* K_X + \sum_{i=1}^{r} d_i D_i.
\]
Denote by \( \rho_\infty : J_\infty(Y) \to J_\infty(X) \) the mapping of spaces of arcs induced by \( \rho \). Then a measurable function \( F \) is exponentially integrable if and only if \( F \circ \rho_\infty + F_D \) is exponentially integrable. Moreover, if the latter holds, then
\[
\int_{J_\infty(X)} e^{-F} = \int_{J_\infty(Y)} e^{-F \circ \rho_\infty - F_D}.
\]

Proof. The proof of theorem 3.5 is based on the equality \( \text{Vol}_Y(C) = \text{Vol}_X(\rho_\infty(C)) \theta^2 \), where \( C \) is a cylinder set in \( J_\infty(Y) \) such that \( F_D(\gamma) = a \) for all \( \gamma \in C \) (see for details Theorem 6.27 in [3] and Lemma 3.3 in [8]). \( \square \)

Theorem 3.6  Let \( D := a_1 D_1 + \cdots + a_m D_m \in \text{Div}(X) \otimes \mathbb{Q} \) be a \( \mathbb{Q} \)-divisor. Assume \( \text{Supp} D \) is a normal crossing divisor. Then \( F_D \) is exponentially integrable if and only \( a_i > -1 \) for all \( i \in \{1, \ldots, m\} \). Moreover, if the latter holds, then
\[
\int_{J_\infty(X)} e^{-F_D} = \sum_{J \subset I} E(D^j_\gamma; \tau \theta^{-1}, \tau^{-1} \theta^{-1})(\theta^{-2} - 1)|J| \prod_{j \in J} \frac{1}{1 - \theta^{2(1 + a_j)}}
\]

Proof. The set \( M(D) \subset J_\infty(X) \) splits into a countable union of pairwise nonintersecting cylinder sets whose non-Archimedean volume can be computed via \( E \)-polynomials of the starta \( D^j_\gamma \) (see for details Theorem 6.28 in [3] and Theorem 5.1 in [8]). \( \square \)

Definition 3.7  Let \( (X, \Delta_X) \) be a Kawamata log terminal pair. Consider a log resolution \( \rho_1 : Y \to X \) and write
\[
K_Y = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X) D_i.
\]
Using the notations from [4], we define
\[
E_{st}(X, \Delta_X) := \sum_{J \subset I} E(D^j_\gamma; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_1(D_j, \Delta_X)} - 1}.
\]
The function \( E_{st}(X, \Delta_X; u, v) \) will be called stringy \( E \)-function of \( (X, \Delta_X) \).
Theorem 3.8 Let \((X, \Delta_X)\) be a Kawamata log terminal pair. Then the \(E\)-function of \((X, \Delta)\) does not depend on the choice of a log resolution.

Proof. Let \(\rho_1 : Y_1 \to X\) and \(\rho_2 : Y_2 \to X\) be two log resolutions of singularities such that
\[
K_{Y_1} = \rho_1^*(K_X + \Delta_X) + D_1, \quad K_{Y_2} = \rho_2^*(K_X + \Delta_X) + D_2
\]
where
\[
D_1 = \sum_{i=1}^{r_1} a(D'_i, \Delta_X)D'_i \quad \text{and} \quad D_2 = \sum_{i=1}^{r_2} a(D''_i, \Delta_X)D''_i
\]
and all discrepancies \(a(D'_i, \Delta_X), a(D''_i, \Delta_X)\) are \(> -1\). Choosing a resolution of singularities \(\rho_0 : Y_0 \to X\) which dominates both resolutions \(\rho_1\) and \(\rho_2\), we obtain two morphisms \(\alpha_1 : Y_0 \to Y_1\) and \(\alpha_2 : Y_0 \to Y_2\) such that \(\rho_0 = \rho_1 \circ \alpha_1 = \rho_2 \circ \alpha_2\). We set \(F := F_{D_0}\), where \(D_0 = K_{Y_0} - \rho_0^*(K_X + \Delta_X)\). Since
\[
K_{Y_0} - \rho_0^*(K_X + \Delta_X) = (K_{Y_0} - \alpha_1^*K_{Y_1}) + \alpha_1^*D_1, \quad (i = 1, 2),
\]
we obtain
\[
\int_{J_{\infty}(Y_1)} e^{-F_{D_1}} = \int_{J_{\infty}(Y_0)} e^{-F_{D_0}} = \int_{J_{\infty}(Y_2)} e^{-F_{D_2}} \quad \text{(see Theorem 3.8).}
\]
It follows from Theorem 3.6 that
\[
\int_{J_{\infty}(Y_1)} e^{-F_{D_1}} = E_{st}(X, \Delta_X; \tau\theta^{-1}, \tau^{-1}\theta^{-1}), \quad i \in \{0, 1, 2\}.
\]
Making the substitutions \(u = \tau\theta^{-1}, v = \tau^{-1}\theta^{-1}\), we obtain that the definition of the stringy \(E\)-function \(E_{st}(X, \Delta_X; u, v)\) does not depend on the choice of log resolutions \(\rho_1\) and \(\rho_2\).  

Proof of Theorem 1.5: The statement immediately follows from Theorem 3.8 using the equality
\[
e_{st}(X, \Delta_X) = \lim_{u,v \to 1} E_{st}(X, \Delta_X; u, v).
\]

4 Log pairs on toric varieties

Let \(X\) be a normal toric variety of dimension \(n\) associated with a rational polyhedral fan \(\Sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}\), where \(N\) is a free abelian group of rank \(n\). Denote by \(X(\sigma)\) the torus orbit in \(X\) corresponding to a cone \(\sigma \in \Sigma\) \((\text{codim}_XX_\sigma = \dim \sigma)\). Let \(\overline{X}(\sigma)\) be the Zariski closure of \(X(\sigma)\). Then the torus invariant \(\mathbb{Q}\)-divisors are \(\mathbb{Q}\)-linear combinations of the closed strata \(\overline{X}(\sigma_1^{(1)}), \ldots, \overline{X}(\sigma_k^{(1)})\), where \(\Sigma^{(1)} := \{\sigma_1^{(1)}, \ldots, \sigma_k^{(1)}\}\) is the set of all 1-dimensional cones in \(\Sigma\). We denote by \(e_1, \ldots, e_k\) the primitive lattice generators of the cones \(\sigma_1^{(1)}, \ldots, \sigma_k^{(1)}\) and set \(\Delta := \overline{X}(\sigma_i^{(1)}) \; i \in \{1, \ldots, k\}\).
Definition 4.1 Let \( \varphi_{K, \Delta} : \mathbb{N}_\mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) be a continuous function satisfying the conditions

(i) \( \varphi_{K, \Delta}(N) \subset \mathbb{Q} \);
(ii) \( \varphi_{K, \Delta} \) is linear on each cone \( \sigma \in \Sigma \);
(iii) \( \varphi_{K, \Delta}(p) > 0 \) for all \( p \in \mathbb{N} \setminus \{0\} \).

Then we define a \( \mathbb{Q} \)-divisor \( \Delta_X \in \mathbb{Z}^{n-1}(X) \) associated with \( \varphi_{K, \Delta} \) as follows:

\[
\Delta_X := \sum_{i=1}^k \left(1 - \varphi_{K, \Delta}(e_i)\right) \Delta_i.
\]

Remark 4.2 It is well-known that the canonical class \( K_X \) of a toric variety \( X \) is equal to \( -(\Delta_1 + \cdots + \Delta_k) \). The above definition of \( \Delta_X \) implies that \( K_X + \Delta_X \) is a \( \mathbb{Q} \)-Cartier divisor on \( X \) corresponding to the \( \Sigma \)-piecewise linear function \( -\varphi_{K, \Delta} \).

The following statement is well-known in toric geometry (see e.g. [10] §5-2):

Proposition 4.3 Let \( \rho : X' \rightarrow X \) be a toric desingularization of \( X \), which is defined by a subdivision \( \Sigma' \) of the fan \( \Sigma \). Denote by \( \{D_1, \ldots, D_m\} \) the set of all irreducible torus invariant strata on \( Y \) corresponding to primitive lattice generators \( e'_1, \ldots, e'_m \) of 1-dimensional cones \( \sigma' \in \Sigma' \). Then \( \sum_{i=1}^m D_i \) is a normal crossing divisor and one has

\[
K_{X'} = \rho^*(K_X + \Delta_X) + \sum_{i=1}^m a(D_i, \Delta_X)D_i,
\]

where \( a(D_i, \Delta_X) = \varphi_{K, \Delta}(e'_i) - 1 \forall i \in \{1, \ldots, m\} \).

Corollary 4.4 Let \( \varphi_{K, \Delta} \) be a \( \Sigma \)-piecewise linear function as in 4.1. Then the pair \( (X, \Delta_X) \) is Kawamata log terminal.

Denote by \( \sigma^\circ \) the relative interior of \( \sigma \) (we put \( \sigma^\circ = 0 \), if \( \sigma = 0 \)). We give the following explicit formula for the function \( E_{st}(X, \Delta_X; u, v) \):

Theorem 4.5

\[
E_{st}(X, \Delta_X; u, v) = (uv - 1)^n \sum_{\sigma \in \Sigma} \sum_{p \in \sigma^\circ \cap \mathbb{N}} (uv)^{-\varphi_{K, \Delta}(p)} = (uv - 1)^n \sum_{p \in \mathbb{N}} (uv)^{-\varphi_{K, \Delta}(p)}.
\]

Proof. Let \( T \subset X \) be an algebraic torus acting on \( X \), \( \partial X := X \setminus T \) its complement. Choose an isomorphism \( N \cong \mathbb{Z}^n \) and write \( p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \). Denote by \( K := \mathbb{C}((t)) \) the field of Laurent power series and define a cylinder subset \( C_p \subset J_\infty(X) \) as follows:

\[
C_p := \{(x_1(t), \ldots, x_n(t)) \in K^n : \text{Ord}_{t=0} x_i(t) = p_i, \ 1 \leq i \leq n\}.
\]
Consider the subset \( M(\partial X) \subset J_\infty(X) \) consisting of all arcs which are not contained in \( J_\infty(\partial X) \). Then \( M(\partial X) \) splits into a disjoint union

\[
M(\partial X) = \bigcup_{p \in N} C_p.
\]

Let \( \rho : X' \to X \) be a toric desingularization of \( X \), and

\[
K_{X'} = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X)D_i.
\]

By definition, we have

\[
E_{st}(X, \Delta_X; \tau\theta^{-1}, \tau^{-1}\theta^{-1}) = \int_{J_\infty(X')} e^{-F_D},
\]

where

\[
D = \sum_{i=1}^{m} a(D_i, \Delta_X)D_i.
\]

Now we notice that \( F_D \) is constant on each cylinder set \( C_p \) \((p \in N)\) and

\[
Vol(C_p) \theta^{2F(C_p)} = (\theta^{-2} - 1)^n \theta^{2\varphi_{K,\Delta}(p)}.
\]

Summing over \( p \in N \) and making the substitution \( u = \tau\theta^{-1}, v = \tau^{-1}\theta^{-1} \), we come to the required formula.

\[\Box\]

**Definition 4.6** Let \( X \) be an arbitrary \( n \)-dimensional normal toric variety defined by a fan \( \Sigma \), and \( X + \Delta_X \) a torus invariant \( \mathbb{Q} \)-Cartier divisor corresponding to a \( \Sigma \)-piecewise linear function \( \varphi_{K,\Delta} \). Denote by \( \Sigma^{(n)} \) the set of all \( n \)-dimensional cones in \( \Sigma \). Let \( \sigma \in \Sigma^{(n)} \) be a cone. Define the **\( \Delta \)-shed of \( \sigma \)** to be the pyramid

\[
\text{shed}_{\Delta}\sigma = \sigma \cap \{ y \in N \otimes \mathbb{R} : \varphi_{K,\Delta}(y) \leq 1 \}.
\]

Furthermore, define the **\( \Delta \)-shed of \( \Sigma \)** to be

\[
\text{shed}_{\Delta}\Sigma = \bigcup_{\sigma \in \Sigma^{(n)}} \text{shed}_{\Delta}\sigma.
\]

**Definition 4.7** Let \( \sigma \in \Sigma^{(n)} \) be an arbitrary cone. Define \( \text{vol}_{\Delta}(\sigma) \) to be the volume of \( \text{shed}_{\Delta}\sigma \) with respect to the lattice \( N \subset N_{\mathbb{R}} \) multiplied by \( n! \). We set

\[
\text{vol}_{\Delta}(\Sigma) := \sum_{\sigma \in \Sigma^{(n)}} \text{vol}_{\Delta}(\sigma).
\]
Definition 4.8 Let $X_0$, $X$, $X^+$ be $n$-dimensional normal projective toric varieties. Denote by $\Sigma$ (resp. by $\Sigma^+$) the fan defining $X$ (resp. $X^+$). Let $(X, \Delta_X)$ (resp. $(X^+, \Delta_{X^+})$) be a torus invariant Kawamata log terminal pair defined by a $\Sigma$-piecewise linear (resp. $\Sigma^+$-piecewise linear) function $\varphi_{K,\Delta}$ (resp. $\varphi_{K,\Delta}^+$). Assume that we are given two equivariant projective birational toric morphisms $\alpha : X \to X_0$ and $\beta : X^+ \to X_0$ such that $-(K_X + \Delta_X)$ is $\alpha$-ample, $K_X^+ + \Delta_X^+$ is $\beta$-ample, and both $\alpha$ and $\beta$ are isomorphisms in codimension 1. Then the birational rational map $\psi := \beta^{-1} \circ \alpha : (X, \Delta_X) \dasharrow (X^+, \Delta_{X^+})$ is called a toric log flip with respect to a $\mathbb{Q}$-Cartier divisor $K_X + \Delta_X$.

Proposition 4.9 Let $\psi : (X, \Delta_X) \dasharrow (X^+, \Delta_{X^+})$ be a toric log $(K_X + \Delta_X)$-flip as above. Then

$$vol_{\Delta}(\Sigma) > vol_{\Delta}(\Sigma^+).$$

Proof. Using a toric interpretation of ampleness via a combinatorial convexity, one obtains from the definition of toric log flips that $\varphi_{K,\Delta}(p) \leq \varphi_{K,\Delta}^+(p)$ for all $p \in N$ and there exists a $n$-dimensional cone $\sigma \in \Sigma^{(n)}$ such that $\varphi_{K,\Delta}(p) < \varphi_{K,\Delta}(p)$ for all interior lattice points $p \in \sigma \cap N$. This implies the statement (cf. [3], Prop. 4.9).

Proposition 4.10 Let $X$ be an arbitrary $n$-dimensional normal toric variety defined by a fan $\Sigma$, and $X + \Delta_X$ a torus invariant $\mathbb{Q}$-Cartier divisor corresponding to a $\Sigma$-piecewise linear function $\varphi_{K,\Delta}$. Then

$$e_{st}(X, \Delta_X) = vol_{\Delta}(\Sigma).$$

Proof. The statement follows from the formula in [4,3] using the same arguments as in the proof of Prop. 4.10 in [3].

Corollary 4.11 Let $(X, \Delta) \dasharrow (X^+, \Delta_{X^+})$ be a toric log flip. Then

$$e_{st}(X, \Delta_X) > e_{st}(X^+, \Delta_{X^+}).$$

5 Canonical abelianization

Let $G$ be a finite group, $V$ a smooth $n$-dimensional algebraic variety over $\mathbb{C}$ having a regular effective action of $G$. If $x \in V$ is an arbitrary point, then by $St_G(x)$ we denote the stabilizer of $x$ in $G$. For any element $g \in G$ we set $V^g := \{x \in V : gx = x\}$.

Definition 5.1 Let $D = \sum_{i=1}^{m} d_i D_i$ be a divisor on a $G$-manifold $V$. A pair $(V, D)$ will be called $G$-normal if the following conditions are satisfied:

(i) $Supp D$ is a union of normal crossing divisors $D_1, \ldots, D_m$;

(ii) for any element $g \in G$ and any irreducible component $D_i$ of $D$, the divisor $D_i$ is $St_G(x)$-invariant for all $x \in V^g \cap D_i$ (i.e., $h(D_i) = D_i \ \forall h \in St_G(x)$, but the $St_G(x)$-action on $D_i$ itself may be nontrivial).
Theorem 5.2 Let \((V, D)\) be a \(G\)-normal pair. Then, using a canonically determined sequence of blow ups of \(G\)-invariant submanifolds, one obtains a \(G\)-normal pair \((V^{ab}, D^{ab})\) and a projective birational \(G\)-morphism \(\psi : V^{ab} \to V\) having the properties:

(i) \(D^{ab} = (K_{V^{ab}} - \psi^* K_V) + \psi^* D;\)
(ii) for any point \(x \in V^{ab}\) the stabilizer \(St_G(x)\) is an abelian subgroup in \(G\).

Proof. Let \(Z(V, G) \subset V\) be the set of all points \(x \in V\) such that \(St_G(x)\) is not abelian. If \(Z(V, G)\) is empty, then we are done. Assume that \(Z(V, G) \neq \emptyset\). We set

\[s(V, G) := \max_{x \in Z(V, G)} |St_G(x)|.\]

Consider a Zariski closed subset

\[Z_{\text{max}}(V, G) := \{x \in Z(V, G) : |St_G(x)| = s(V, G)\} \subset Z(V, G).

We claim that the set \(Z_{\text{max}}(V, G) \subset V\) is a smooth \(G\)-invariant subvariety of codimension at least 2. By definition, \(Z_{\text{max}}(V, G)\) is a union of smooth subvarieties

\[F(H) := \{x \in V : gx = x \quad \forall g \in H\},\]

where \(H\) runs over all nonabelian subgroups of \(G\) such that \(|H| = s(V, G)|\). This implies that \(Z_{\text{max}}(V, G)\) is \(G\)-invariant. Since the \(G\)-action is effective and \(\dim F(H) = n - 1\) is possible only for cyclic subgroups \(H \subset G\), we obtain \(\dim Z_{\text{max}}(V, G) \leq n - 2\). It remains to observe that any two subvarieties \(F(H_1), F(H_2) \subset V\) must either coincide, or have empty intersection. Indeed, if \(x \in F(H_1) \cap F(H_2)\), then \(H_1, H_2 \subset St_G(x)\). Since \(|H_1|, |H_2|\) are maximal, we obtain \(H_1 = H_2 = St_G(x);\) i.e., \(F(H_1) = F(H_2)\).

We set \(V_0 := V, D_0 := D\) and define \(V_1\) to be the \(G\)-equivariant blow-up of \(V_0\) with center \(Z_{\text{max}}(V, G)\). Denote by \(\varphi_1 : V_1 \to V_0\) the corresponding projective birational \(G\)-morphism. It is obvious that the support of \(D_1 = K_{V_1} - \varphi_1^*(K_V - D)\) is a normal crossing divisor. If \(x \in V_1^g \cap E\), where \(E\) is a connected component of an \(\varphi_1\)-exceptional divisor, then \(St_G(x) \subset St_G(\varphi(x))\). Since \(\varphi(E)\) is a connected component of a smooth subvariety \(Z_{\text{max}}(V, G)\), \(\varphi(E)\) must be \(St_G(\varphi(x))-\)invariant. Hence, we conclude that \((V_1, D_1)\) is a \(G\)-normal pair. If \(Z(V_1, G) = \emptyset\), then we are done. Otherwise we apply the same procedure to the \(G\)-normal pair \((V_1, D_1)\), where \(D_1 = \varphi_1^* D_0\), and construct in the same way a next \(G\)-equivariant blow-up \(\varphi_2 : V_2 \to V_1\) etc.

It remains to show that the above procedure terminates. For this purpose, it suffices to show that \(s(V_i, G) < s(V_0, G)\) for some \(i > 0\). Assume that \(s(V_0, G) = s(V_i, G)\) for all \(i > 0\). Then there exist points \(x_i \in V_i (i \geq 0)\) such that \(\varphi_i(x_i) = x_{i-1}\) and \(St_G(x_i) = St_G(x_{i-1})\) \((i \geq 1)\). Let \(S(x_i)\) be the set of those irreducible components of \(\text{Supp } D_i\) which are \(St_G(x_i)-\)invariant and contain \(x_i\). We denote by \(n(x_i)\) the cardinality of \(S(x_i)\) and denote by \(D(x_i) \subset V_i\) the intersection of all divisors from \(S(x_i)\). Then \(F(St_G(x_i)) \subset D(x_i)\). If \(F(St_G(x_i)) \neq D(x_i)\), then the point \(n(x_{i+1}) = n(x_i) + 1\) (we obtain one more component from the \(\varphi_1\)-exceptional
divisor over $F(St_G(x_i))$. Since $n(x_i) \leq n$ for all $i \geq 0$, there exists a positive number $k$ such that $n(x_k) = n(x_{k+j})$ for all $j \geq 0$. So we obtain $F(St_G(x_{k+j})) = D(x_{k+j})$ for all $j \geq 0$. The latter means that the action of $St_G(x_k)$ on the tangent space to $x_k$ in $V_k$ splits into a direct sum of $n(x_k)$ 1-dimensional representations and a $(n - n(x_k))$-dimensional trivial representation. Since the action of $St_G(x_k)$ is effective, the group $St_G(x_k)$ must be abelian. Contradiction. \end{proof}

**Definition 5.3** Let $(V, D)$ be a $G$-normal pair. Then the $G$-normal pair $(V^{ab}, D^{ab})$ obtained in $\mathbb{P}^2$ will be called **canonical abelianization** of a $G$-normal pair $(V, D)$.

**Remark 5.4** If the stabiliser $St_G(x) \subset G$ of every point $x \in V$ is already abelian, then one can’t expect that $G$-equivariant blow ups of smooth subvarieties $Z \subset V$ could simplify singularities of the quotient-space $V/G$.

Here is the following simplest example: Let $V := \mathbb{C}^2$ and $G = \langle g \rangle$ is a cyclic group of order 5 whose generator $g$ acts by the diagonal matrix with the eigenvalues $e^{2\pi \sqrt{-1}/5}, e^{4\pi \sqrt{-1}/5}$. Let $V'$ be the blow up of $\mathbb{C}^2$ at 0. Then $V'$ has a natural covering by two open subsets $V'_1$ and $V'_2$ such that $V'_1 \cong V'_2 \cong \mathbb{C}^2$ and the $G$-action on one of these subsets coincides with the original $G$-action on $V$.

6 **Orbifold $E$-functions**

**Definition 6.1** Let $D = \sum_{j=1}^m d_j D_j$ be a $G$-invariant effective divisor on a smooth $G$-variety $V$ such that $(V, G)$ is a $G$-normal pair. Take an arbitrary element $g \in G$ and a connected component $W$ of $V^g$. Choose a point $x \in W$ and local $g$-invariant coordinates $z_1, \ldots, z_n$ at $x$ so that irreducible components of $Supp D$ containing $x$ are defined by local equations $z_i = 0$ for some $i \in \{1, \ldots, n\}$. Let $\delta_i (1 \leq i \leq n)$ be the multiplicity of $D$ along $\{z_i = 0\}$ ($\{\delta_1, \ldots, \delta_n\} \subset \{0, d_1, \ldots, d_m\}$), and $e^{2\pi \sqrt{-1} \alpha_i}$ ($1 \leq i \leq n$) the eigenvalue of the $g$-action on $z_i$ ($\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Q} \cap [0, 1]$). We define the **$D$-weight** of $g$ at $W$ as

$$wt(g, W, D) := \sum_{i=1}^n \alpha_i (\delta_i + 1).$$

If $D = 0$, then

$$wt(g, W) := wt(g, W, 0) = \sum_{i=1}^n \alpha_i$$

will be called simply the **weight** of $g$ at $W$. Let $I^g$ be the subset of $g$-fixed elements in $I := \{1, \ldots, m\}$. For any subset $J \subset I^g$ we set

$$F(g, W, D^g_J; u, v) := \prod_{j \in J} \frac{uv - 1}{(uv)^{d_{j+1}} - 1} E(W_J; u, v),$$

where $W_J$ is the geometric quotient of $W \cap D^g_J$ modulo the subgroup $C(g, W, J) \subset C(g)$ consisting of those elements in the centralizer of $g$ which leave the component $W \subset V^g$ and the subset $J \subset I^g$ invariant.
Remark 6.2 We note that \( wt(g, W, D) \) does not depend on the choice of a point \( x \in W \). Moreover, if \( h \in C(g) \) is an element in the centralizer of \( g \) and \( W' = hW \) is another connected component of \( V^g \), then \( wt(g, W', D) = wt(g, W, D) \).

Definition 6.3 We define the orbifold \( E \)-function of a \( G \)-normal pair \((V, D)\) by the formula:

\[
E_{\text{orb}}(V, D, G; u, v) = \sum_{\{g\}} \sum_{\{W\}} (uv)^{wt(g, W, D)} \sum_{J \subseteq I^g} F(g, W, D_J; u, v),
\]

where \( \{g\} \) runs over all conjugacy classes in \( G \), and \( \{W\} \) runs over the set of representatives of all \( C(g) \)-orbits in the set of connected components of \( V^g \).

In the case \( D = 0 \), we call

\[
E_{\text{orb}}(V, G; u, v) := E_{\text{orb}}(V, 0, G; u, v) = \sum_{\{g\}} \sum_{\{W\}} (uv)^{wt(g, W_i)} E(W/C(g, W); u, v),
\]

the orbifold \( E \)-function of a \( G \)-manifold \( V \) (here \( C(g, W) \) is the subgroup of all elements in \( C(g) \) which leave the component \( W \subset V^g \) invariant).

Remark 6.4 Using the equalities

\[
\frac{1}{|G|} \sum_{g \in G} \sum_{h \in C(g)} e(V^g \cap V^h) = \sum_{\{g\} \subseteq G} \frac{1}{|C(g)|} \sum_{h \in C(g)} e(V^g \cap V^h) = \sum_{\{g\} \subseteq G} e(V^g / C(g)),
\]

one immediately obtains that \( E_{\text{orb}}(V, G; 1, 1) \) equals the physicists’ orbifold Euler number \( e(V, G) \) (see \[13\]).

Example 6.5 Let \( G := \mu_d \) a cyclic group of order \( d \) acting by roots of unity on \( V := \mathbb{C} \). Then the corresponding orbifold \( E \)-function equals

\[
E_{\text{orb}}(V, G; u, v) = uv + \sum_{k=1}^{d-1} (uv)^{k/d} = (uv)^{1/d} + (uv)^{2/d} + \cdots + (uv)^{d-1/d} + uv.
\]

Lemma 6.6 Let \( V := \mathbb{C}^r \) and \( g \in \text{GL}(r, \mathbb{C}) \) a linear authomorphism of finite order. Denote by \( V' \) the blow up of \( V \) at 0. Let \( D \cong \mathbb{P}^{r-1} \) be the exceptional divisor in \( V' \) and \( \{W_1, \ldots, W_s\} \) the set of connected components of \( D^g \). Then

\[
\sum_{i=1}^{s} (uv)^{wt(g, W_i, D)} \frac{uv - 1}{(uv)^{r} - 1} E(W_i; u, v) = (uv)^{wt(g, V^g)}.
\]
Proof. Let \( \{e^{2\pi \sqrt{-1} \alpha_i}\} (1 \leq i \leq n) \) be the set of the eigenvalues of \( g \)-action. Without loss of generality, we assume \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1 \). We write the number \( r \) as a sum of \( s \) positive integers \( k_1 + \cdots + k_s \) where the numbers \( k_1, \ldots, k_s \) are defined by the conditions

\[
\alpha_i = \alpha_{i+1} \iff \exists j \in \{1, \ldots, s\} : k_1 + \cdots + k_j \leq i < k_1 + \cdots + k_j + k_{j+1}
\]

and

\[
\alpha_i < \alpha_{i+1} \iff \exists j \in \{1, \ldots, s\} : i + 1 = k_1 + \cdots + k_j.
\]

Then \( D^g \) is a union of \( s \) projectives linear subspaces \( W_1, \ldots, W_s \), where \( W_j \cong \mathbb{P}^{k_j-1} (j \in \{1, \ldots, s\}) \). By definition, we have \( wt(g, V^g) = \sum_{i=1}^n \alpha_i \). By direct computations, one obtains \( wt(g, W_j, D) = k_1 + \cdots + k_{j-1} + \sum_{i=1}^r \alpha_i \). Hence,

\[
\sum_{W_i \subset D^g} (uv)^{wt(g, W_i, D)} E(W_i; u, v) = (uv)^{wt(g, V^g)} \sum_{j=1}^s (uv)^{k_1 + \cdots + k_{j-1}} E(\mathbb{P}^{k_j-1}; u, v) = (uv)^{wt(g, V^g)} \sum_{j=1}^s (uv)^{k_1 + \cdots + k_{j-1}} (1 + (uv) + \cdots + (uv)^{k_j-1}) =
\]

\[
(uv)^{wt(g, V^g)} \sum_{l=0}^{r-1} (uv)^l = (uv)^{wt(g, V^g)} \frac{(uv)^r - 1}{uv - 1}.
\]

This completes the proof. \( \square \)

**Lemma 6.7** Let \( V \) and \( W \) be two smooth algebraic varieties having a regular action of a finite group \( G \). Assume that \( V \) is a Zariski locally trivial \( \mathbb{P}^r \)-bundle over \( W \) such that the canonical projection \( \pi : V \to W \) is \( G \)-equivariant. Then

\[
E(V/G; u, v) = \frac{(uv)^r - 1}{uv - 1} E(W/G; u, v).
\]

Proof. Let \( H \subset G \) be a subgroup and \( W(H) := \{ x \in W : St_G(x) = H \} \). Then \( W \subset W \) is a locally closed subvariety, and \( W \) admits a \( G \)-invariant stratification by locally closed strata

\[
W = \bigcup_{\{H\}} W(\{H\}),
\]

where \( \{H\} \) runs over the conjugacy classes of all subgroups in \( G \) and \( W(\{H\}) := \bigcup_{H' \in \{H\}} W(H') \). Denote \( V(\{H\}) := \pi^{-1}(W(\{H\})) \). Then \( V(\{H\}) \) is a \( G \)-equivariant \( \mathbb{P}^r \)-bundle over \( W(\{H\}) \) and we have isomorphisms \( V(\{H\})/G \cong V(H)/N(H) \), \( W(\{H\})/G \cong W(H)/N(H) \), where \( W(H) := \pi^{-1}(W(H)) \) and \( N(H) \) is the normalizer of \( H \) in \( G \). Since \( V(H) \) is a \( N(H) \)-equivariant \( \mathbb{P}^r \)-bundle over \( W(H) \), it suffices to prove our statement for the case \( G = N(H) \), \( W = W(H) \), and \( V = V(H) \). Furthermore, we can restrict ourselves to the case when \( W \) is irreducible and \( N(H) \) leaves \( W \) invariant. The last conditions imply \( N(H) = H \).
Therefore, \( W/G = W \) and the \( H \)-action on leaves each fiber of \( \pi \) invariant. Hence, \( E(V/G; u, v) = E(\mathbb{P}^r/H; u, v)E(W; u, v) \). Since all cohomology groups of \( \mathbb{P}^r \) have rank 1 and they are generated by an effective algebraic cycle, we get \( E(\mathbb{P}^r/H; u, v) = E(\mathbb{P}^r; u, v) \). Thus, we have obtained the required formula for \( E(\mathbb{P}^r/H; u, v) \).

\[ \square \]

**Theorem 6.8** Let \((V, G)\) be a \( G \)-normal pair, \( Z \subset V \) a smooth \( G \)-invariant subvariety such that after the \( G \)-equivariant blow up \( \psi : V' \to V \) with center in \( Z \) one obtains a \( G \)-normal pair \((V', D')\), where \( D' \) the effective divisor defined by the equality

\[ K_{V'} = \psi^*(K_V - D) + D'. \]

Then

\[ E_{\text{orb}}(V, D, G; u, v) = E_{\text{orb}}(V', D', G; u, v). \]

**Proof.** Let \( Z_1, \ldots, Z_k \) be the set of connected components of \( Z \) and \( D_1, \ldots, D_m \) the set of irreducible components of \( \text{Supp} D \). Then \( \text{Supp} D' = \psi^{-1}(\text{Supp} D) \cup D_{m+1} \cup \cdots \cup D_{m+k} \), where \( D_{m+1}, \ldots, D_{m+k} \) are irreducible \( \psi \)-exceptional divisors over \( Z_1, \ldots, Z_k \). It suffices to prove the equality

\[ \forall g \in G : \sum_{\{W\}} (uv)^{wt(g, W, D)} \sum_{J \subseteq I^g} F(g, W, D^g_j; u, v) = \sum_{\{W'\}} (uv)^{wt(g, W', D')} \sum_{J' \subseteq (I')^g} F(g, W', (D')^g_j; u, v), \]

where \( I' = I \cup \{m + 1, \ldots, m + k\} \). We note that the \( G \)-equivariant mapping \( \psi(V')^g \to V^g \) is surjective. Therefore, it suffices to prove the equality

\[ (uv)^{wt(g, W, D)} \sum_{J \subseteq I^g} F(g, W, D^g_j; u, v) = \sum_{i=1}^l (uv)^{wt(g, W'_i, D')} \sum_{J' \subseteq (I')^g} F(g, W'_i, (D')^g_j; u, v), \]

where \( W \) is a given connected component of \( V^g \) and \( W'_1, \ldots, W'_l \) are all connected components of \((V')^g\) such that \( \psi(W'_i) \subset W \) \((1 \leq i \leq k)\). Since \( \psi \) is an isomorphism over \( W \setminus W \cap Z \) and the \( \psi \)-exceptional divisors \( D_{m+1}, \ldots, D_{m+k} \) are pairwise nonintersecting, it suffices to prove the equality

\[ (uv)^{wt(g, W, D)} F(g, W \cap Z_j, D^g_j; u, v) = \sum_{i=1}^l (uv)^{wt(g, W'_i, D')} F(g, W'_i \cap D_{m+j}, (D')^g_j; u, v), \]

where \( j \in I \) and \( J' = J \cup \{j + m\} \). The last equality follows from Lemmas \([6.6]\) and \([6.7]\) using the fact that each \( W'_i \cap D_{m+j} \) is a locally trivial \( \mathbb{P}^{k_i} \)-bundle over \( W \cap Z_j \).

\[ \square \]
7 Main theorems

Let $V$ be a smooth $n$-dimensional algebraic variety, $G$ a finite group acting by regular automorphism on $V$, $X := V/G$ its geometric quotient, and $\phi : V \to X$ the corresponding finite morphism. Then $G$ acts on the set of irreducible components of the ramification divisor $\Lambda$ on $V$. Denote by $\{\Lambda_1, \ldots, \Lambda_k\}$ the set of representatives of $G$-orbits in the set of irreducible components of $\text{Supp} \Lambda$. Let $\nu_1 - 1, \ldots, \nu_k - 1$ be the multiplicities of $\Lambda_1, \ldots, \Lambda_k$ in $\Lambda$ (the number $\nu_i$ equals the order of the cyclic inertia subgroup $\text{St}_G(\Lambda_i) \subset G$). Since $\phi : V \to X$ is a Galois covering, the multiplicity $\nu_i - 1$ of $\Lambda_i$ depends only on the $G$-orbit of $\Lambda_i$ in $\text{Supp} \Lambda$. We set $\Delta_i := \phi(\Lambda_i)$ ($1 \leq i \leq k$) and consider the pair $(X, \Delta_X)$, where

$$\Delta_X := \sum_{i=1}^k \left( \frac{\nu_i - 1}{\nu_i} \right) \Delta_i \in Z_{n-1}(X) \otimes \mathbb{Q}.$$ 

By the ramification formula, we have

$$\phi^*(K_X + \Delta_X) = \phi^*K_X + \Lambda = K_V.$$

**Proposition 7.1** The pair $(X, \Delta_X)$ is Kawamata log terminal.

**Proof.** Let $\rho : Y \to X$ be a log resolution of singularities of $(X, \Delta_X)$ and

$$K_Y = \rho^*(K_X + \Delta_X) + \sum_i a(D_i, \Delta_X)D_i.$$ 

We consider the fiber product $V_1 := V \times_X Y$. Then $V_1$ has a natural finite Galois morphism $\phi_1 : V_1 \to Y$ and a natural birational $G$-morphism $\rho_1 : V_1 \to V$. We write

$$K_{V_1} = \rho_1^*K_V + \sum_{j=1}^m a(E_j, 0)E_j,$$

where $E_j$ runs over irreducible exceptional divisors of $\rho_1$.

By definition, the multiplicity of any irreducible component $\Delta_i$ of $\Delta$ is equal to $(\nu_i - 1)/\nu_i < 1$. Therefore, $a(D_i, \Delta_X) = -(\nu_j - 1)/\nu_j > -1$ if $\rho(D_i)$ coincides with an irreducible component $\Delta_j$ of $\text{Supp} \Delta$. Now consider the case when $\rho(D_i)$ is not an irreducible component of $\text{Supp} \Delta$. Denote by $E_j$ an irreducible divisor on $V_1$ such that $\phi_1(E_j) = D_i \subset Y$. Let $r_j$ be the ramification index of $\phi_1$ along $E_j$. By the ramification formula, one has $a(E_j, 0) + 1 = r_j(a(D_i, \Delta_X) + 1)$. Since $V$ is smooth, we have $a(E_j, 0) \geq 1$ for all $j \in \{1, \ldots, m\}$. Therefore, $a(D_i, \Delta_X) = a(E_j, 0) + 1/r_j - 1 > -1$. □

**Definition 7.2** Let $V$ be a smooth algebraic variety having a regular action of a finite group $G$, and $(X, \Delta_X)$ the pair constructed above. Then we call $(X, \Delta_X)$ the **Kawamata log terminal pair associated with** $(V, G)$. 

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Example 7.3 Let $G := \mu_d$ a cyclic group of order $d$ acting by roots of unity on $V := \mathbb{C}$. Then $X = V/G \cong \mathbb{C}$ and $\Delta_X = \frac{d-1}{d}x_0$, where $x_0 \in X$ is the zero point. The stringy $E$-function of $(X, \Delta_X)$ equals

$$E_{st}(X, \Delta_X; u, v) = (uv - 1) + \frac{uv - 1}{(uv)^{1/d} - 1} = (uv)^{1/d} + (uv)^{2/d} + \cdots + (uv)^{d-1/d} + uv.$$ 

Thus, it coincides with the orbifold $E$-function from Example 6.3.

Our next statements show the last phenomenon in more general situations:

Lemma 7.4 Let $G \subset \text{GL}(n, \mathbb{C})$ be a finite abelian subgroup acting by diagonal matrices on $V := \mathbb{C}^n$, and $(X, \Delta)$ the Kawamata log terminal pair associated with $(V, G)$. Then

$$E_{st}(X, \Delta_X; u, v) = E_{orb}(V, G; u, v).$$

Proof. First, we remark that the ramification locus $\text{Supp} \Lambda$ is contained in the union of the coordinate hyperplanes $\Lambda_i := \{z_i = 0\} \subset \mathbb{C}^n$ ($1 \leq i \leq n$). Therefore, we can write $\Lambda = \sum_{i=1}^n \nu_i \Lambda_i$, where $\nu_i \geq 1$ ($1 \leq i \leq n$). Second, we note that $X$ is a normal affine toric variety corresponding to the cone $\sigma := \mathbb{R}_{\geq 0}^n$ and the lattice

$$N := \mathbb{Z}^n + \sum_{g \in G} \mathbb{Z}(\alpha_1(g), \ldots, \alpha_n(g)),$$

where $e^{2\pi\sqrt{-1}\alpha_1(g)}, \ldots, e^{2\pi\sqrt{-1}\alpha_n(g)}$ are the eigenvalues of $g$, $\{\alpha_1(g), \ldots, \alpha_n(g)\} \in \mathbb{Q} \cap [0, 1)$. Moreover, $\Delta_X$ is a torus invariant divisor on $X$. Let us denote by $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{Z}^n$. Then the $\mathbb{Q}$-divisor $K_X + \Delta_X$ corresponds to a linear function $\varphi_{K, \Delta}$ which has value 1 on each $e_i$ ($1 \leq i \leq n$). By 4.4, $(X, \Delta_X)$ is a torus invariant Kawamata log terminal pair. By 4.5, we obtain

$$E_{st}(X, \Delta_X; u, v) = (uv - 1)^n \sum_{p \in N \cap \sigma} (uv)^{-\varphi_{K, \Delta}}.$$

We set $f_i := (1/\nu_i)e_i$ ($1 \leq i \leq n$). Then the system of vectors $\{f_1, \ldots, f_n\} \subset N$ generates a sublattice $N' \subset N$ containing $\mathbb{Z}^n$. Denote by $\mathcal{R} := \{v_1, \ldots, v_n\} \subset N$ the set of representatives of $N/N'$, where each element $v \in \mathcal{R}$ has a form $v = \sum_{i=1}^n \lambda_i(v)f_i$ ($0 \leq \lambda_i < 1$). Then, by summing a multidimensional geometric series, we obtain

$$E_{st}(X, \Delta_X; u, v) = \left(\frac{(uv - 1)^n}{1 - (uv)^{-1/\nu_i}}\right)^n \prod_{i=1}^n \frac{1}{1 - (uv)^{-1/\nu_i}}$$

(we used the property $\varphi_{K, \Delta}(f_i) = 1/\nu_i$, $1 \leq i \leq n$). Thus, we have

$$E_{st}(X, \Delta_X; u, v) = \left(\prod_{v \in \mathcal{R}} (uv)^{-\sum_{i=1}^n \lambda_i(v)/\nu_i}\right) \prod_{i=1}^n \frac{(uv - 1)}{1 - (uv)^{-1/\nu_i}}.$$
\[(uv)^n \left( \sum_{g \in R} (uv)^{-\sum_{i=1}^{n} \lambda_i(v)/\nu_i} \right) \prod_{i=1}^{n} (1 + (uv)^{-1/\nu_i} + \cdots + (uv)^{-(\nu - 1)/\nu_i}) =
\]
\[= (uv)^n \sum_{g \in G} (uv)^{-\sum_{i=1}^{n} \alpha_i(g)} = E_{\text{orb}}(V, G; u, v).\]

Now we come to our main theorem:

**Theorem 7.5** Let \( G \) be a finite group acting regularly on a smooth algebraic variety \( V \) and \((X, \Delta)\) the Kawamata log terminal pair associated with \((V, G)\). Then
\[ E_{\text{st}}(X, \Delta_X; u, v) = E_{\text{orb}}(V, G; u, v). \]

**Proof.** Let \((V^{ab}, D)\) be the canonical abelianization of the \( G \)-normal pair \((V, 0)\), \( D = K_{V^{ab}} - \psi^* K_V = \sum_{i=1}^{m} d_i D_i \). Denote by \( \phi^{ab} \) the finite morphism \( V^{ab} \to Y := V^{ab}/G \). Then \( \psi \) induces a birational proper morphism \( \overline{\psi} : Y \to X \) which can be considered as a partial desingularization of \( X \). Let \( W_1, \ldots, W_l \) be representatives of \( G \)-orbits in the set \( \{D_1, \ldots, D_m\} \) \((l \leq m)\), and \( \overline{W}_1, \ldots, \overline{W}_l \) their \( \phi^{ab} \)-images in \( Y \). By the ramification formula, we have
\[ K_Y = (\overline{\psi})^*(K_X + \Delta_X) + \sum_{j=1}^{l} \left( \frac{d_j + 1}{r_j} - 1 \right) \overline{W}_j + \sum_{i=l+1}^{l+k} \left( \frac{1}{\nu_i} - 1 \right) W_i, \]
where \( \overline{W}_{i+l} \) is the \( \phi^{ab} \)-image of \( \psi^{-1}(\Lambda_i) \subset V^{ab} \) in \( Y \), and \( r_j \) is the order of the ramification of \( W_j \) over \( \overline{W}_j \). We set \( I_1 := \{1, \ldots, l\} \), \( I_2 := \{l + 1, \ldots, l + k\} \) and \( I := I_1 \cup I_2 \). For any subset \( J \subset I \) we set \( J_1 = I_1 \cap J \) and \( J_2 = I_2 \cap J \). Denote by \( G(J) \) the \( G \)-stabilizer of a point \( x \in V^{ab} \) such that \( \phi^{ab}(x) \in \overline{W}_j \) and set
\[ S(J; u, v) := \sum_{g \in G(J)} (uv)^{wt(g, x, D)}. \]
It is easy to see that if \( x' \in V^{ab} \) is another point such that \( \phi^{ab}(x') \in \overline{W}_j \), then \( St_G(x') \) is conjugate to \( St_G(x) \), i.e., \( G(J) \) depends only on \( J \), but not on the choice of a point \( x \in (\phi^{ab})^{-1}(\overline{W}_j) \). Let \( G'(J) \) be the subgroup in \( G(J) \) generated by the cyclic inertia subgroups \( St_G(W_j) \) \((j \in J_1)\) and \( St_G(\psi^{-1}(\Lambda_{j-1})) \) \((j \in J_2)\); i.e., we have \( G'(J) \cong \prod_{j \in J_1} \mu_{r_j} \prod_{j \in J_2} \mu_{\nu_j} \), and
\[ S'(J; u, v) := \sum_{g \in G'(J)} (uv)^{wt(g, x, D)} = \prod_{j \in J_1} (uv)^{d_j + 1}/(uv)^{d_j + 1/r_j} - 1 \prod_{j \in J_2} (uv)^{1/\nu_j} - 1 \]
(1)
By \( \text{SS} \), we have
\[ E_{\text{orb}}(V, G; u, v) = E_{\text{orb}}(V^{ab}, D, G; u, v) = \sum_{J \subset I} S(J; u, v) \prod_{j \in J_1} (uv)^{d_j + 1}/(uv)^{d_j + 1} E(\overline{W}_j; u, v). \]
Since the singularities along $W_J$ are toroidal (cf. [5]), it follows from 7.4 that

$$E_{st}(X, \Delta_X; u, v) = \sum_{J \subset \Gamma} S(J; u, v) \prod_{j \in J_1} \frac{uv - 1}{(uv)^{(d_j+1)/c_j} - 1} \prod_{j \in J_2} \frac{uv - 1}{(uv)^{1/v_j} - 1} E(W_J; u, v),$$

where $S(J; u, v) = S'(J; u, v) = S(J; u, v)$. It remains to apply (1).

Proof of Theorem 1.9: The statement immediately follows from 6.4 and 7.5 by taking limits:

$$e_{st}(X, \Delta_X) = \lim_{u,v \to 1} E_{st}(X, \Delta_X; u, v) = \lim_{u,v \to 1} E_{orb}(V, G; u, v) = e(V, G).$$

Corollary 7.6 Let $X$ be a normal complex algebraic surface with at worst log terminal singularities. Then

$$e_{st}(X) = e(X \setminus X_{sing}) + \sum_{x \in X_{sing}} c_x,$$

where $c_x$ is the number of conjugacy classes in the local fundamental group of $X \setminus \{x\}$. In particular, $e_{st}(X)$ is always an integer.

Proof. It is well-known that a germ of a singular point $x \in X_{sing}$ is isomorphic to a germ of 0 in $\mathbb{C}^2/G_x$ where $G_x \subset \text{GL}(2, \mathbb{C})$ is a finite subgroup ($G_x$ is isomorphic to the local fundamental group of $X \setminus \{x\}$). Therefore, we have $J_\infty(X, x) \cong J_\infty(\mathbb{C}^2/G_x, 0)$. Let $\rho : Y \to X$ be a resolution of singularities, $D_1, \ldots, D_m$ are exceptional divisors over $x \in X$, $\{a_1, \ldots, a_m\}$ their discrepancies, and $I = \{1, \ldots, m\}$. By 1.5 and 1.9, the number

$$e_{st}(x) := \sum_{J \subset I} e(D_J^x) \prod_{j \in J} \frac{1}{a_j + 1}$$

does not depend on the choice of a resolution and equals $c_x$. □

8 Cohomological McKay correspondence

Definition 8.1 Let $G \subset \text{SL}(n, \mathbb{C})$ be a finite subgroup acting linearly on $V := \mathbb{C}^n$ and $X := V/G$. A resolution of singularities $\rho : Y \to X$ is called crepant if the canonical class $K_Y$ is trivial.

Proposition 8.2 Let $\mathbb{C}^* \times X \to X$ be the regular $\mathbb{C}^*$-action on $X$ induced by the action of scalar matrices on $\mathbb{C}^n$. Assume that there exists a crepant resolution of singularities $\rho : Y \to X$. Then the $\mathbb{C}^*$-action on $X$ extends uniquely to a regular $\mathbb{C}^*$-action on $Y$. □
Proof. Since $Y$ is birational to $X$, the $\mathbb{C}^*$-action on $X$ extends uniquely to a rational $\mathbb{C}^*$-action $\mathbb{C}^* \times Y \to Y$. It remains to show that it is regular. Let $\{D_1, \ldots, D_m\}$ be the set of all irreducible divisors on $Y$ in the exceptional locus of $\rho$. It was shown in [13] that the corresponding discrete valuations $V_{D_1}, \ldots, V_{D_m}$ of the field of rational functions on $Y$ are determined uniquely. Since the algebraic group $\mathbb{C}^*$ is connected, every such a valuation $V_{D_1}, \ldots, V_{D_m}$ must be invariant under the rational $\mathbb{C}^*$-action on $Y$. Therefore, the rational $\mathbb{C}^*$-action on $Y$ can be extended to a regular action on some Zariski dense open subsets $U_j \subset D_j (j = 1, \ldots, m)$, i.e., the rational $\mathbb{C}^*$-action on $Y$ is regular outside some Zariski closed subset

$$Z := \bigcup_{j=1}^m (D_j \setminus U_j) \subset Y, \quad \text{codim}_Y Z \geq 2.$$ 

Let $TY$ be the tangent vector bundle over $Y$. By the extension theorem of Hartogs, the restriction mapping on global sections $\Gamma(Y, TY) \to \Gamma(Y \setminus Z, TY)$ is bijective. Hence, the regular vector field $\eta \in \Gamma(Y \setminus Z, TY)$ corresponding to the regular $\mathbb{C}^*$-action on $Y \setminus Z$ extends to a regular vector field on the whole variety $Y$. The latter shows that the $\mathbb{C}^*$-action on $Y \setminus Z$ extends to a regular action on the whole $Y$. □

Lemma 8.3 Let $V$ be a smooth algebraic variety, and $W = \bigcup_j W_j$ a stratification of $W$ by locally closed irreducible subvarieties. Assume that the Hodge structure in the cohomology with compact supports $H^i_c(W_j, \mathbb{Q})$ is pure for all $i, j$. Then the Hodge structure in $H^i_c(W, \mathbb{Q})$ is pure for all $i$.

Proof. The statement follows by induction using the fact that for any closed subvariety $W' \subset W$ the long exact cohomology sequence

$$\to H^{i-1}_c(W') \to H^i_c(W \setminus W') \to H^i_c(W) \to H^i_c(W') \to H^{i+1}_c(W \setminus W') \to$$

respects the Hodge structure. □

The following statement was conjectured in [3] (see also [13]):

Theorem 8.4 Let $G \subset \text{SL}(n, \mathbb{C})$ be a finite subgroup. Assume that there exists a crepant desingularization $\rho : Y \to X := \mathbb{C}^n/G$. Then the Hodge structure in the cohomology $H^*(Y, \mathbb{C})$ is pure. Moreover, $H^{2i+1}(Y, \mathbb{C}) = 0$, $H^{2i}(Y, \mathbb{C})$ has the Hodge type $(i, i)$ for all $i$, and the dimension of $H^{2i}(Y, \mathbb{C})$ is equal to the number of conjugacy classes $\{g\} \subset G$ having the weight $\text{wt}(g) = i$.

Proof. Let $Y^{\mathbb{C}^*}$ be the fixed point set of the $\mathbb{C}^*$-action on $Y$, $Y^{\mathbb{C}^*} = \bigcup_{j=1}^k Y_j$ a decomposition of $Y^{\mathbb{C}^*}$ in its connected components, $Y_0 := \rho^{-1}(x_0) \subset X$, where $x_0$ is the image of $0 \in \mathbb{C}^n$ modulo $G$. Since $Y_0$ is the fiber over the unique $\mathbb{C}^*$-fixed point $x_0 \in X$, we have $Y^{\mathbb{C}^*} \subset Y_0$. Therefore $Y^{\mathbb{C}^*}$ is compact. Since the fixed point subvariety $Y^{\mathbb{C}^*}$ is smooth and compact, the cohomology of every connected component $Y_1, \ldots, Y_k$ of $Y^{\mathbb{C}^*}$ have pure Hodge structure. Consider the Bialynicki-Birula cellular decomposition [4]: $Y = \bigcup_{j=1}^k W_j$, where $W_j = \{y \in Y : \lim_{z \to 0} z(y) \in Y_j, z \in \mathbb{C}^*\}$. 

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Since every $W_j$ is a vector bundle over $Y_j$, the groups $H^i_c(W_j, \mathbb{C})$ have pure Hodge structures for all $i, j$. By $\S 3$, the Hodge structure in $H^i_c(Y, \mathbb{C})$ is pure for all $i$. By $8.3$, the Hodge structure in $H^i_c(Y, \mathbb{C})$ is pure for all $i$. 

Denote by $C_i(G)$ the number of conjugacy classes $\{g\} \subset G$ having the weight $\text{wt}(g) = i$. Since $G$ is contained in $\text{SL}(n, \mathbb{C})$, the ramification divisors $\Lambda$ and $\Delta_X$ are zero. By $3.6$ and $7.5$, we have

$$E(Y; u, v) = E_{st}(X, 0; u, v) = E_{\text{orb}}(\mathbb{C}^n, G; u, v).$$

Using the purity of $H^i_c(Y, \mathbb{C})$ and the fact that the Poincaré duality

$$H^{2n-i}_c(Y, \mathbb{C}) \otimes H^i_c(Y, \mathbb{C}) \to H^{2n}_c(Y, \mathbb{C}) \cong \mathbb{C}(n)$$

respects the Hodge structure, it remains to show that

$$E_{\text{orb}}(\mathbb{C}^n, G; u, v) = \sum_{\{g\}} C_i(G)(uv)^{n-i}. \quad (2)$$

Indeed, we have $E_{\text{orb}}(V, G; u, v) = \sum_{\{g\}} (uv)^{\text{wt}(g,V)}E(V^g/C(g); u, v)$, where $V := \mathbb{C}^n$. Since $V^g$ is a linear subspace of dimension $k(g) := \dim \text{Ker}(g - \text{id})$, we obtain $E(V^g/C(g); u, v) = (uv)^{k(g)}$. Hence,

$$(uv)^{\text{wt}(g,V)}E(V^g/C(g); u, v) = (uv)^{n-\text{wt}(g^{-1},V^g)}.$$

The summing over all conjugacy classes $\{g^{-1}\}$ implies (2). \hfill \square

**Proof of Theorem 1.10**: Now it follows immediately from $8.4$. \hfill \square

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