Research Article

Fekete–Szegö Problems for Certain Classes of Meromorphic Functions Involving $q$-Al-Oboudi Differential Operator

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Received 9 January 2022; Revised 20 February 2022; Accepted 21 February 2022; Published 28 March 2022

Academic Editor: Xiaolong Qin

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In this paper, we introduce a new derivative operator involving $q$-Al-Oboudi differential operator for meromorphic functions. By using this new operator, we define a new subclass of meromorphic functions and obtain the Fekete–Szegö inequalities.

1. Introduction

For two analytic functions $f$ and $g$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$, if there is a Schwarz function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(w(z))$, ($z \in \mathbb{U}$). If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. (1)

(see [8, 24]). A function $f$ is meromorphic if it is analytic throughout a domain $D$, except possibly for poles in $D$ (see [40]).

Let $\Sigma$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

(2)

which are analytic in the open punctured unit disc $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}/\{0\}$. A function $f \in \Sigma$ is said to be a meromorphic starlike function of order $\zeta$, denoted by $\Sigma^* (\zeta)$, if

$$-\Re \left[ \frac{zf'(z)}{f(z)} \right] > \zeta, 0 \leq \zeta < 1.$$

(3)

The class $\Sigma^* (\zeta)$ was studied by Pommerenke [29], Miller [23], and many others (see [9, 25]).

Let $\varphi(z)$ be an analytic function with a positive real part on $\mathbb{U}$ satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Let $\Sigma_b^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$1 - \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} + 1 \right] \varphi(z), \quad b \in \mathbb{C}^* = \mathbb{C}/\{0\}.$$

(4)

The class $\Sigma_b^*(\varphi)$ was introduced and studied by Mohammed and Darus [26] (see also Reddy and Sharma [30], with $\gamma = 1$).

We note that for suitable choices of $b$ and $\varphi(z)$, we obtain the following subclasses:

1. $\Sigma_1^* (\varphi) = \Sigma^* (\varphi)$ (see [4], with $\alpha = 1$ and [33]);
2. $\Sigma_b^* (1+z)/(1-z) = F^* (b)$ (see [6]);
3. $\Sigma_b^* ((1 + (1 - 2\zeta)z)/(1 - z)) = \Sigma^* (\zeta), \quad 0 \leq \zeta < 1$ (see [29]);
4. $\Sigma_1^* (1 + Az)/(1 - Az) = K_1 (A, B) \quad (0 \leq B \leq 1, -B \leq A < B)$ (see [17]);
5. $\Sigma_b^* (1 - p e^{-it} \cos \theta (1 + z)/(1 - z) = \Sigma_b^* (p), \quad 0 \leq p < 1, \{\theta \leq (\pi/2)\}$ (see [16, 31]).
Let $M^*_b (\varphi)$ be the class of functions $f \in \Sigma$ for which
\[ 1 - \frac{1}{b} \left[ \frac{z f''(z)}{f'(z)} + 2 \right] \varphi(z), \ b \in \mathbb{C}^*. \] (5)

We note that

1. $M^*_b (1 + z)/(1 - z) = G^* (b)$ (see Aouf [6]);
2. $M^*_{(1-\rho)e^{-i\theta}} (\varphi) = M^*_{\rho, \theta} (\varphi), \ 0 < \rho < 1, |\theta| < (\pi/2)$

In geometric function theory, operators play an important role. Many authors present differential and integral operators, for example ([1, 20, 32, 37]). For a function $f \in \Sigma$ given by (2), the $\delta$-derivative of a function $f(z)$ is defined by [3, 11] (see also [14, 15])

\[ D^*_\delta f(z) = \frac{f(z) - f(z - \delta)}{\delta} \] (6)

As $\delta \to 1^-$, we have $[k]_\delta \to k$ and $\lim_{\delta \to 1^-} D^*_\delta f(z) = f'(z)$.

Due to its use in numerous fields of mathematics and physics, the $\delta$-derivative operator $D^*_\delta$ has fascinated and inspired many researchers. Jackson [14] was among the key contributors of all the scientists who introduced and developed the $\delta$-calculus theory. In 1991, Ismail [13] was the first to demonstrate a crucial link between geometric function theory and the $\delta$-derivative operator, but a solid and comprehensive foundation was provided in 1989 in a book chapter by Srivastava [34]. Several recent works on this operator can be found in ([7, 18, 19, 35, 36]).

For $f(z) \in \Sigma, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}, \lambda \geq 0$ and $0 < \delta < 1$, we define the following operator $D^{*n}_\lambda f(z)$ as follows:

\[ D^{*n}_\lambda f(z) = (1 - \lambda) f(z) + \frac{\lambda}{z} D^{*n}_\lambda (z^2 f(z)) \] (8)

From (2) and (8), we obtain

\[ D^{*n}_\lambda f(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left( 1 + \lambda \left( (k + 2) \delta + 1 \right) \right) a_k z^k, \quad n \in \mathbb{N}_0. \] (9)

From (9), it is easy to see that, for $f(z) \in \Sigma,

\[ \lambda \delta^2 z D^*_\delta \left( D^{*n}_\lambda f(z) \right) = D^{*n+1}_\lambda f(z) - (\lambda \delta + 1) D^{*n}_\lambda f(z), \] \quad \lambda > 0. \] (10)

We note that

(i) $\lim_{\delta \to 1^-} D^{*n}_\lambda f(z) = D^{*n}_\lambda f(z) = (1/\lambda) + \sum_{k=0}^{\infty} \left[ 1 + \lambda \left( (k + 1) \delta \right) \right] a_k z^k \quad \text{[5]}, \quad \text{with} \ p = 1.$

(ii) $\lim_{\delta \to 1^-} D^{*n+1}_\lambda f(z) = D^{*n+1}_\lambda f(z) = (1/\lambda) + \sum_{k=0}^{\infty} \left[ (k + 2) \right] a_k z^k \quad \text{[21, 38]}, \quad \text{with} \ p = 1.$

Making use of $D^{*n}_\lambda f(z)$, we define the following class $\Sigma^{*n}_\lambda (b, \varphi)$ as follows:

**Definition 1.** For $n \in \mathbb{N}_0, \lambda > 0, 0 < \delta < 1, b \in \mathbb{C}^*$, and $0 \leq \alpha < (\delta \delta + 1)$, we say that a function $f \in \Sigma$ is in the class $\Sigma^{*n}_\lambda (b, \varphi)$ if and only if

\[ \delta D^*_\delta \left( f(z) \right) \in \Sigma^{*n}_\lambda (b, \varphi), \quad z \in \mathbb{U}^*. \]

Noting that

1. $\Sigma^{*n}_\lambda (b, \varphi) = \Sigma^{*n}_\lambda (b, \varphi) = \{ f \in \Sigma : 1 - (1/\beta) \varphi(z), \ z \in \mathbb{U}^* \};$
2. $\lim_{\delta \to 1^-} \Sigma^{*n}_\lambda (b, \varphi) = \Sigma^{*n}_\lambda (b, \varphi) = \{ f \in \Sigma : 1 - (1/\beta) \varphi(z), \ z \in \mathbb{U}^* \};$
3. $\alpha_0^{(1)} (b, \varphi) = \Sigma^{(1)} (b, \varphi) = \{ f \in \Sigma : 1 - (1/\beta) \varphi(z), \ z \in \mathbb{U}^* \}$.
4. $\Sigma^{*n}_8 (b, \varphi) = \Sigma^{*n}_8 (b, \varphi) = \{ f \in \Sigma : 1 - (1/\beta) \varphi(z), \ z \in \mathbb{U}^* \};$
5. $\alpha_0^{(1)} (b, \varphi) = \Sigma^{*n}_8 (b, \varphi) = \{ f \in \Sigma : 1 - (1/\beta) \varphi(z), \ z \in \mathbb{U}^* \};$
The result is sharp for the functions given by
\[
\begin{align*}
p(z) &= \frac{1+z^2}{1-z^2}, \\
p(z) &= \frac{1+z}{1-z}.
\end{align*}
\]

**Lemma 2** (see [22]). If \( h(z) = 1 + c_1z + c_2z^2 + \cdots \) is a function with positive real part in \( U \), then
\[
|c_2 - \nu c_1^2| \leq \begin{cases}
4\nu + 2, & \text{if } \nu \leq 0, \\
2, & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2, & \text{if } \nu \geq 1.
\end{cases}
\]
Proof. If \( f(z) \in \sum_{\lambda, \delta, \alpha}^{\text{c}} \), then there is a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \(|w(z)| < 1\) in \( U \), such that

\[
1 + \frac{1}{b} \left[ \frac{-(1 - (a/\delta))\delta z \frac{d}{dz} \left( D_{\alpha, \delta, \beta}^{\alpha \beta} f(z) \right) + a \delta z \frac{d}{dz} \left( D_{\alpha, \delta, \beta}^{\alpha \beta} f(z) \right) - 1}{1 - (a/\delta) D_{\alpha, \delta, \beta}^{\alpha \beta} f(z) - a z D_{\alpha, \delta, \beta}^{\alpha \beta} f(z)} \right] = \varphi(w(z)).
\]

(21)

If we set

\[
h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.
\]

(22)

Then, from (2) and (23), we see that \( b_1 = -(1/b)(1 - (a/\delta)) \) and \( b_2 = (1/b)(1 - (a/\delta))^2 \) \( \alpha \lambda^n \). Substituting these values into (23), we obtain

\[
p(z) = 1 + b_1 z + b_2 z^2 + \cdots.
\]

(23)

Using (21)–(23), we get

\[
p(z) = \varphi\left(\frac{h(z) - 1}{h(z) + 1}\right).
\]

(24)

Since

\[
\frac{h(z) - 1}{h(z) + 1} = \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 + \frac{c_1 c_2}{4} - c_1 c_2 \right) z^3 + \cdots \right].
\]

(25)

Then,

\[
\varphi\left(\frac{h(z) - 1}{h(z) + 1}\right) = 1 + \frac{1}{2}d_1 c_1 z + \left[ \frac{1}{2}d_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}d_2 c_3 \right] z^2 + \cdots.
\]

(26)

From (24) and (26), we have

\[
b_1 = \frac{1}{2}d_1 c_1,
\]

\[
b_2 = \frac{1}{2}d_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}d_2 c_3.
\]

(27)

Applying Lemma 1, we obtain the result (19). Also, if \( d_1 = 0 \), then

\[
a_1 - \mu a_0^2 = -\frac{\delta d_1 b}{2(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \left[ c_2 - \varphi_{1 c}^2 \right],
\]

where \( \varphi = \frac{1}{2} \left[ 1 - \frac{d_1}{d_1} + d_1 b \left( 1 - \mu \frac{\delta (\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n}{(\delta - \alpha)[1 + \lambda \delta]^{2n}} \right) \right].
\]

(29)

\[
a_0 = 0,
\]

\[
a_1 = -\frac{\delta d_1 c_1^2 b}{4(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n}.
\]

(30)
which completes the proof of Theorem 1.

Taking $\alpha = 0$ in Theorem 1, we get

\[
|a_1 - \mu a_0^2| \leq \frac{d_1|b|}{(\delta + 1)(1 + \lambda \delta (\delta + 1))^n} \cdot \max \left\{ 1, \left| \frac{d_2}{d_1} - \left[ 1 - \mu \left( \delta + 1 \right) \frac{1 + \lambda \delta (\delta + 1)^n}{1 + \lambda \delta (\delta + 1)^n} \right] \right| d_1 \right\}, \quad d_1 \neq 0,
\]

\[
|a_1| \leq \frac{d_2 |b|}{(\delta + 1)(1 + \lambda \delta (\delta + 1))^n}, \quad d_1 = 0.
\]

**Remark 1.**

(1) For $b = 1$ in Corollary 2, we get the result obtained by [12], [Theorem 2.3].

(2) For $\delta \longrightarrow 1^-$ in Corollary 2, we get the results obtained by [26, 30].

(3) For $\delta \longrightarrow 1^-$ and $b = 1$ in Corollary 2, we get the results obtained by [33] and [4], [Theorem 5.2].

Taking $n = 0$ in Theorem 1, we get

**Corollary 3.** Let $f(z)$ be defined by (2) and $\varphi(z) = 1 + d_1z + z^2 + \cdots (d_1 \geq 0)$. If $f(z) \in \sum_{\lambda \delta}^n (b, \varphi)$ and $\mu \in \mathbb{C}$, then

\[
|a_1 - \mu a_0^2| \leq \frac{d_1 \delta |b|}{(\delta + 1)(\delta - \alpha (\delta + 1))} \cdot \max \left\{ 1, \left| \frac{d_2}{d_1} - \left[ 1 - \mu \left( \delta + 1 \right) \delta \frac{1 + \lambda \delta (\delta + 1)^n}{(\delta - \alpha)^2} \right] \right| d_1 \right\}, \quad d_1 \neq 0,
\]

\[
|a_1| \leq \frac{d_2 |b|}{(\delta + 1)(\delta - \alpha (\delta + 1))}, \quad d_1 = 0.
\]
Remark 2.

(1) Taking \( b = 1 \) in Corollary 3, we get the result obtained by [12], [Theorem 2.8].

(2) Letting \( \delta \rightarrow 1^- \) in Corollary 3, we get the result obtained by [30].

\[
|a_1 - \mu a_0^2| \leq \frac{d_1 \delta}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^n} \left| \frac{1}{\delta - \alpha(\delta + 1)} \right|^\pi
\]

\[
\max \left\{ 1, \frac{|d_2|}{|d_1|}, 1 - \mu \frac{\delta(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n}{(\delta - \alpha)^2[1 + \lambda \delta]^n} \right\} d_1, \quad d_1 \neq 0.
\]

\[
|a_1| \leq \frac{\delta}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^n} \left| \frac{1}{\delta - \alpha(\delta + 1)} \right|^\pi
\]

The result is sharp.

Taking \( \alpha = 0 \) and \( b = (1 - \rho) e^{-i \theta} \cos \theta (0 \leq \rho < 1, |\theta| < (\pi/2)) \) in Theorem 1, we get

\[
|a_1 - \mu a_0^2| \leq \frac{d_1 (1 - \rho) \cos \theta}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^n} \left| \frac{1}{\delta - \alpha(\delta + 1)} \right|^\pi
\]

\[
\max \left\{ 1, \frac{|d_2|}{|d_1|}, 1 - \mu \frac{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n}{[1 + \lambda \delta]^n} \right\} d_1 (1 - \rho) \cos \theta, \quad d_1 \neq 0.
\]

The result is sharp.

Remark 3. Letting \( \delta \rightarrow 1^- \) and taking \( n = 0 \) and \( \varphi(z) = (1 + z)/(1 - z) \) in Corollary 5, we get the result obtained by [27], [Example 1.1].

\[
|a_1 - \mu a_0^2| \leq \frac{\delta}{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \left| \frac{1}{\delta - \alpha(\delta + 1)} \right|^\pi
\]

\[
\left\{ \begin{array}{l}
\frac{d_1}{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \delta d_1, \quad \text{if } \mu \leq \sigma_1 \\
\frac{\delta}{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \delta d_1, \quad \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{\delta}{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \delta d_1, \quad \text{if } \mu \geq \sigma_2,
\end{array} \right.
\]

Taking \( b = 1 \) in Theorem 1, we get

**Corollary 4.** Let \( f(z) \) be defined by (2) and \( \varphi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_i \geq 0) \). If \( f(z) \in \sum^{n}_{\lambda, \delta, \alpha}(\varphi) \) and \( \mu \in \mathbb{C} \), then

**Theorem 2.** For real \( \mu \), let \( \varphi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_i \geq 0) \). If \( f(z) \) given by (2) belongs to the class \( \sum^{n}_{\lambda, \delta, \alpha}(1, \varphi) = \sum^{n}_{\lambda, \delta, \alpha}(\varphi) \), then

By using Lemma 2, we can obtain the following theorem.
where

\[
\sigma_1 = \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [-d_1 - d_2 + d_1^2]}{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n d_1^2},
\]

\[
\sigma_2 = \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [d_1 - d_2 + d_1^2]}{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n d_1^2},
\]

The result is sharp. Further, let \( \sigma_3 = ((\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [-d_2 + d_1^2]/\delta (\delta + 1))[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n d_1^2) \).

(i) If \( \sigma_1 \leq \mu < \sigma_3 \), then

\[
|a_1 - \mu a_0| - \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} \delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n d_1^2}{d_1} 
\times \left\{ (d_1 + d_2) - \left[ 1 - \frac{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} |a_0|^2
\]

\[
\leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}
\]

(ii) If \( \sigma_3 \leq \mu < \sigma_2 \), then

\[
|a_1 - \mu a_0| - \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} \delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n d_1^2}{d_1} 
\times \left\{ (d_1 - d_2) + \left[ 1 - \frac{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} |a_0|^2
\]

\[
\leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}
\]

**Proof.** First, let \( \mu \leq \sigma_1 \). Then

\[
|a_1 - \mu a_0| \leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \left\{ -d_2 + \left[ 1 - \frac{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1 \right\}
\]

\[
= \frac{\delta}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \left\{ -d_2 + \left[ 1 - \frac{\delta (\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1 \right\}
\]

\[
|a_1 - \mu a_0| \leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}
\]

Let, \( \sigma_1 \leq \mu \leq \sigma_2 \). Then, we obtain
Finally, if $\mu \geq \sigma_2$. Then

$$|a_1 - \mu a_0^2| \leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}\left\{\frac{d_2}{d_1} - \left[1 - \frac{\delta(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta (\delta + 1)]^n}{(\delta - \alpha)^2[1 + \lambda \delta ]^{2n}}\right]d_1\right\}.$$  

(43)

The sharpness is an immediate consequence of Lemma 2. This completes the proof of Theorem 2.  

□

Remark 4. Taking $n = 0$ in Theorem 2, we get the result obtained by [12], [Theorem 2.10].

Taking $\alpha = 0$ in Theorem 2, we get

**Corollary 6.** For real $\mu$, let $\varphi(z) = 1 + d_1z + d_2z^2 + \cdots (d_i > 0, i = 1, 2)$. If $f(z)$ given by (2) belongs to the class $\Sigma_{\lambda, \delta, \alpha}^n(1, \varphi)$, then

$$|a_1 - \mu a_0^2| \leq \frac{1}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n}\left\{\frac{-d_2 + \left[1 - \frac{(\delta + 1)[\delta + \lambda \delta (\delta + 1)]^n}{[1 + \lambda \delta ]^{2n}}\right]d_1^2}{d_1}\right\},$$  

if $\mu \leq \sigma_4$

$$\frac{d_1}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n},$$  

if $\sigma_4 \leq \mu \leq \sigma_5$

$$\frac{1}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n}\left\{d_2 - \left[1 - \frac{(\delta + 1)[\delta + \lambda \delta (\delta + 1)]^n}{[1 + \lambda \delta ]^{2n}}\right]d_1\right\},$$  

if $\mu \geq \sigma_5$,  

(44)

where

$$\sigma_4 = \frac{[1 + \lambda \delta ]^{2n}[d_1 - d_2 + d_1^2]}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n d_1^4},$$

$$\sigma_5 = \frac{[1 + \lambda \delta ]^{2n}[d_1 - d_2 + d_1^2]}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n d_1^4}.$$  

The result is sharp. Further, let $\sigma_6 = ([1 + \lambda \delta ]^{2n}[d_1 - d_2 + d_1^2]),$

(i) If $\sigma_4 \leq \mu < \sigma_6$, then

$$|a_1 - \mu a_0^2| \leq \frac{[1 + \lambda \delta ]^{2n}}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n d_1^2} \times \left\{(d_1 + d_2) - \left[1 - \frac{(\delta + 1)[\delta + \lambda \delta (\delta + 1)]^n}{[1 + \lambda \delta ]^{2n}}\right]d_1\right\}|a_0|^2 \leq \frac{d_1}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n}.$$  

(45)
(ii) If $\sigma_6 \leq \mu < \sigma_5$, then
\[
|a_1 - \mu a_0^2| + \frac{[1 + \lambda\delta]^{2n}}{(\delta + 1)[1 + \lambda\delta(\delta + 1)]^n}d_i^n \leq \frac{d_i}{(\delta + 1)[1 + \lambda\delta(\delta + 1)]^n} \left\{ \left( d_1 - d_2 + \frac{1 - \mu}{1 + \lambda\delta(\delta + 1)} \right) \right\} \left| a_0 \right|^2 .
\]
\[(46)\]

**Remark 5.**

(1) Taking $n = 0$ in Corollary 6, we get the result obtained by [12].

(2) Letting $\delta \to 1^-$ and taking $n = 0$ in Corollary 6, we get the result obtained by [4].

### 3. Conclusion

In the fields of combinatorics and quantum calculus, the $\delta$-derivative introduced by Frank Hilton Jackson [14] plays an important role in the theory of functions of a complex variable and other fields of mathematics. In this paper, we define a new differential operator for meromorphic functions. By using this new operator, we define and study a new family of meromorphic functions. Several properties of the abovementioned family of functions are investigated, including coefficient inequalities and Fekete–Szegö functionals.

### Data Availability

No data were used to support this study

### Conflicts of Interest

The authors declare that there are no conflicts of interest.

### References

[1] M. I. Abbas, “Existence results and the Ulam stability for fractional differential equations with hybrid proportional-Caputo derivatives,” J. Nonlinear Funct. Anal. vol. 2020, Article ID 48, 2020.

[2] H. R. Abdel-Gawad and D. K. Thomas, “The Fekete-Szegö problem for strongly close-to-convex functions,” Proceedings of the American Mathematical Society, vol. 114, no. 2, pp. 345–349, 1992.

[3] M. H. Abu Risha, M. H. Annaby, Z. S. Ismail, and Z. S. Mansour, “Linear Q-difference equations,” Zeitschrift für Analysis und ihre Anwendungen, vol. 26, no. 4, pp. 481–494, 2007.

[4] R. M. Ali and V. Ravichandran, “Classes of meromorphic $a$-convex functions,” Taiwanese Journal of Mathematics, vol. 14, no. 4, pp. 1479–1490, 2010.

[5] F. M. Al-Oboudi and H. A. Al-Zkeri, “Applications of Briot-Bouquet differential subordination to some classes of meromorphic functions,” Arab J. Math. Sci. vol. 12, no. 1, pp. 17–30, 2006.

[6] M. K. Aouf, “Coefficient results for some classes of meromorphic functions,” J. Nat. Sci. Math. vol. 27, no. 2, pp. 81–97, 1987.

[7] M. Arif and B. Ahmad, “New subfamily of meromorphic multivalent starlike functions in circular domain involving q-derivative operator,” Mathematica Slovaca, vol. 68, no. 5, pp. 1049–1056, 2018.

[8] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[9] N. E. Cho, S. Lee, and S. Owa, “A class of meromorphic univalent functions with positive coefficients,” Kobe journal of mathematics, vol. 4, pp. 43–50, 1987.

[10] M. Fekete and G. Szegö, “Eine bemerkung über ungerade schlichte funktionen,” Journal of the London Mathematical Society, vol. s1-8, no. 2, pp. 85–89, 1933.

[11] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[12] T. Huo, H. M. Zayed, A. O. Mostafa, and M. K. Aouf, “Fekete-szego problems for certain classes of meromorphic functions using q-derivative operator,” J. Math Research Appl. vol. 38, no. 3, pp. 236–246, 2018.

[13] M. E. H. Ismail, E. Merkes, and D. Styer, “A generalization of starlike functions,” Complex Variables, Theory and Application: An International Journal, vol. 14, no. 1-4, pp. 77–84, 1990.

[14] F. H. Jackson, “The application of basic numbers to Bessel’s and Legendre’s functions,” Proceedings of the London Mathematical Society, vol. 3, no. 4, pp. 1–23, 1905.

[15] V. G. Kac and P. Cheung, Quantum Calculus, Universitext, Springer-Verlag, New York, 2002.

[16] J. Kaczmarski, “On the coefficients of some classes of starlike functions,” Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. vol. 17, pp. 495–501, 1969.

[17] V. Karunakaran, “On a class of meromorphic starlike functions in the unit disc,” Math. Chronicle, vol. 4, no. 2–3, pp. 112–121, 1976.

[18] B. Khan, Z.-G. Liu, H. M. Srivastava, N. Khan, and M. Tahir, “Applications of higher-order derivatives to subclasses of multivalent $q$-starlike functions,” Maejo Int. J. Sci. Technol. vol. 15, pp. 61–72, 2021.

[19] N. Khan, H. M. Srivastava, A. Rafiq, M. Arif, and S. Arijika, “Some applications of $q$-difference operator involving a family of meromorphic harmonic functions,” Advances in Difference Equations, vol. 2021, no. 1, p. 471, 2021.

[20] R. J. Libera, “Some classes of regular univalent functions,” Proceedings of the American Mathematical Society, vol. 16, no. 4, pp. 755–758, 1965.

[21] J.-L. Liu and S. Owa, “Certain meromorphic $p$-valent functions,” Taiwanese Journal of Mathematics, vol. 2, no. 1, pp. 107–110, 1998.

[22] W. Ma and D. Minda, “A unified treatment of some special classes of univalent functions,” in Proceedings of the Conference on Complex Analysis 157–169, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., Int. Press, 1994.

[23] J. Miller, “Convex meromorphic mappings and related functions,” Proceedings of the American Mathematical Society, vol. 25, no. 2, pp. 220–228, 1970.
[24] S. S. Miller and P. T. Mocanu, “Differential subordinations and univalent functions,” *Michigan Mathematical Journal*, vol. 28, pp. 157–171, 1981.

[25] M. L. Mogra, T. R. Reddy, and O. P. Juneja, “Meromorphic univalent functions with positive coefficients,” *Bulletin of the Australian Mathematical Society*, vol. 32, no. 2, pp. 161–176, 1985.

[26] A. Mohammed and M. Darus, “Bounded coefficient for certain subclass of meromorphic function,” in *Proceedings of the International Conference on Quality, Productivity and Performance Measurement*, p. 8, Palm Garden Hotel, Putrajaya, 2009.

[27] A. Mohammed and M. Darus, *On the class of starlike meromorphic function of complex order*, vol. 31, pp. 53–61, Rend. Mat., Ser. VII, Roma, 2011.

[28] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[29] C. Pommerenke, “On meromorphic starlike functions,” *Pacific Journal of Mathematics*, vol. 13, no. 1, pp. 221–235, 1963.

[30] T. Ram Reddy and R. B. Sharma, “Fekete–Szegő inequality for certain subclasses of meromorphic functions,” *Int. Math. Forum*, vol. 5, no. 68, pp. 3399–3412, 2010.

[31] V. Ravichandran, S. S. Kumar, and K. G. Subramanian, “Convolution conditions for spiral-likeness and convex spiral-likeness of certain p-valent meromorphic functions,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 1, pp. 1–7, 2004.

[32] S. Ruscheweyh, “New criteria for univalent functions,” *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.

[33] H. Silverman, K. Suchithra, B. Adolf Stephen, and A. Gangadharan, “Coefficient bounds for certain classes of meromorphic functions,” *Journal of Inequalities and Applications*, vol. 2008, pp. 1–9, 2008.

[34] H. M. Srivastava, *Univalent functions; fractional calculus; and associated generalized hypergeometric functions Univalent Functions, Fractional Calculus, and their Applications*, H. M. Srivastava and S. Owa, Eds., pp. 329–354, John Wiley and Sons, New York, Chichester, 1989.

[35] H. M. Srivastava, M. K. Aouf, and A. O. Mostafa, “Some properties of analytic functions associated with fractional q-calculus operators,” *Misskolc Mathematical Notes*, vol. 20, no. 2, pp. 1245–1260, 2019.

[36] H. M. Srivastava, M. Arif, M. Arif, and M. Raza, “Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator,” *AIMS Mathematics*, vol. 6, no. 6, pp. 5869–5885, 2021.

[37] H. M. Srivastava, M. I. Qureshi, and S. H. Malik, “Some hypergeometric transformations and reduction formulas for the Gauss function and their applications involving the Clausen function,” *J. Nonlinear Var. Anal.*, vol. 5, pp. 981–987, 2021.

[38] B. A. Uralegaddi and C. Somanatha, “New criteria for meromorphic starlike univalent functions,” *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 137–140, 1991.

[39] A. K. Wanas and H. Y. Althoby, “Fekete-szego problem for certain new family of Bi-univalent functions,” *Earthline Journal of Mathematical Sciences*, vol. 8, no. 2, pp. 263–272, 2022.

[40] D. G. Zill and P. D. Shanahan, *A First Course in Complex Analysis with Applications*, Jones and Bartlett Publishers, 2nd edition, 2009.