NORMALIZATION OF THE WAVEFUNCTION FOR THE CALOGERO-SUTHERLAND MODEL WITH INTERNAL DEGREES OF FREEDOM

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The exact normalization of a multicomponent generalization of the ground state wavefunction of the Calogero-Sutherland model is conjectured. This result is obtained from a conjectured generalization of Selberg’s $N$-dimensional extension of the Euler beta integral, written as a trigonometric integral. A new proof of the Selberg integral is given, and the method is used to provide a proof of the multicomponent generalization in a special two-component case.

1. INTRODUCTION

The $1/r^2$ quantum many body system (Calogero-Sutherland model) is the subject of much present day interest due to its connection with quantum chaos [1] and fractional statistics [2,3]. The development of these applications has been greatly assisted by the recent discovery [4,3] of mathematical methods which provide the exact evaluation of ground state correlations, both static and dynamic. These exact calculations rely on $N$-dimensional integration formulas, which are generalizations of Selberg’s [5] $N$-dimensional extension of the beta integral:

$$S(N, \lambda_1, \lambda_2; \lambda) := \left( \prod_{l=1}^{N} \int_{0}^{1} dt_l t_l^{\lambda_1} (1 - t_l)^{\lambda_2} \right) \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2\lambda} = \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1 + 1 + j\lambda)\Gamma(\lambda_2 + 1 + j\lambda)\Gamma(1 + j\lambda)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (N + j - 1)\lambda)\Gamma(1 + \lambda)}$$

(1.1)

Generalizations of the Calogero-Sutherland Hamiltonian to include internal degrees of freedom of the particles have recently been formulated [6]. These models are of interest in condensed matter physics because of their relationship with quantum lattice models, notably the $1/r^2$ exchange $t - J$ Hamiltonian [7]. However, from the viewpoint of exact calculations, the theoretical development of these models is not as advanced as that of the original model. In particular, the analogue of the Selberg integral (1.1) for the exact multicomponent ground state wavefunction [6]

$$|\psi_0(\{z_j^{(a)}\}_{j=1,\ldots,N_\alpha}, \{w_j\}_{j=1,\ldots,N_0})|^2 = \prod_{a=1}^{p} \prod_{1 \leq j < k \leq N_\alpha} |z_j^{(a)} - z_k^{(a)}|^{2(\lambda+1)} \prod_{1 \leq j' < k' \leq N_0} |w_{k'} - w_{j'}|^{2\lambda}$$

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analogue of the Selberg integral. We first transform the Selberg integral into an equivalent form due to Morris [8]:

\[
D(N; a, b, c) := \text{CT} \prod_{l=1}^{N} (1-t_l)^a (1-t_l^{-1})^b \prod_{1 \leq j < k \leq N} (1-t_k/t_j)^c (1-t_j/t_k)^c
\]

where

\[
\text{CT} := \alpha \beta \gamma \rho \lambda \theta \pi \iota \nu \omega \eta \zeta \xi \gamma \kappa \theta \iota \omicron \nu \xi \zeta \eta \omega \alpha \beta \gamma \rho \lambda \theta \iota \nu \xi \zeta \eta \omega
\]

is not known in the existing literature.

This deficiency has motivated us to pursue the task of formulating the appropriate analogue of the Selberg integral. We first transform the Selberg integral into an equivalent form due to Morris [8]:

\[
D(N; a, b, c)
\]

\[
:= \text{CT} \prod_{l=1}^{N} (1-t_l)^a (1-t_l^{-1})^b \prod_{1 \leq j < k \leq N} (1-t_k/t_j)^c (1-t_j/t_k)^c
\]

\[
= \left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} d\theta_l e^{\pi i \theta_l (a-b)} |1 + e^{2\pi i \theta_l}|^{a+b} \right) \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2c}
\]

\[
= \frac{\prod_{l=0}^{N-1} \Gamma(a + b + 1 + lc) \Gamma(1 + (l + 1)c)}{\prod_{l=0}^{N-1} \Gamma(a + b + 1 + lc) \Gamma(1 + (l + 1)c)}\]

These checks suffice for the system in a harmonic well.

2. EQUIVALENCE BETWEEN SELBERG-TYPE INTEGRALS AND SOME FOURIER INTEGRALS
2.1 The inter-relationship
To perform a numerical investigation of the value of the Selberg integral (1.1) or any generalizations, it is convenient to first transform the Selberg-type integral into an equivalent Fourier integral. For this purpose we use the following lemma.

Lemma 1
Let \( f(t_1, \ldots, t_N; \{ p \}) \) be a Laurent polynomial in \( t_1, \ldots, t_N \), with \( \{ p \} \) as parameters. For \( \Re(\epsilon) \) large enough so that the r.h.s. exists

\[
\left( \frac{\pi}{\sin \pi \epsilon} \right)^N \left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} d\theta_l e^{2\pi i \theta_l \epsilon} \right) f(-e^{2\pi i \theta_1}, \ldots, -e^{2\pi i \theta_N}; \{ p \}) = \left( \prod_{l=1}^{N} \int_{0}^{1} dt_l t_l^{-1+\epsilon} \right) f(t_1, \ldots, t_N; \{ p \})
\] (2.1)

This result follows immediately from term-by-term integration of the Laurent polynomial. Note that for \( \epsilon \) an integer the Fourier integral is equal to

\[
\text{CT}_{\{t_1, \ldots, t_N\}} \prod_{l=1}^{N} t_l \ f(-t_1, \ldots, -t_N; \{ p \})
\] (2.2)

From this lemma we see that the Selberg integral with \( \lambda_1 \) arbitrary and \( \lambda_2, \lambda \) non-negative integers is equivalent to the Fourier integral in Morris’s integral (1.3) with \( a-b \) arbitrary and \( a+b, c \) non-negative integers.

2.2 Numerical evaluation
For \( \epsilon \) an integer the Fourier integral in (2.1) can be computed by exact numerical integration. Thus, whenever

\[
g(x) = \sum_{n_1=-p_1}^{p_1} \ldots \sum_{n_N=-p_N}^{p_N} a_{n_1, \ldots, n_N} e^{2\pi i (x_1 n_1 + \ldots x_N n_N)}
\]

we have

\[
\int_{0}^{1} dx_1 \ldots \int_{0}^{1} dx_N g(x_1, \ldots, x_N) = \frac{1}{M_1} \sum_{n_1=0}^{M_1} \ldots \frac{1}{M_N} \sum_{n_N=0}^{M_N} g(n_1/M_1, \ldots, n_N/M_N)
\] (2.3)

provided \( M_l > p_l \) (\( l = 1, \ldots, N \)). This result follows by term-by-term integration and summation of the Fourier series for \( g(x) \). In both cases only the \( n_j = 0 \) term (\( j = 1, \ldots, N \)) of \( g(x) \) remains.

We will use the formula (2.3) below to provide numerical data on the evaluation of some generalizations of (1.3).

2.3 Analytic properties of the Fourier integrals
Denote the Fourier integral on the l.h.s. of (2.1) by \( I(\epsilon; \{ p \}) \). Suppose furthermore that \( f \) is real when each \( \theta_j \) is real so that

\[
f(-e^{2\pi i \theta_1}, \ldots, -e^{2\pi i \theta_N}; \{ p \}) = f(-e^{-2\pi i \theta_1}, \ldots, -e^{-2\pi i \theta_N}; \{ p \})
\] (2.4)

Some immediate properties of \( I(\epsilon; \{ p \}) \) are

(i) \( I(\epsilon; \{ p \}) \) is an entire function of \( \epsilon \).
(ii) $I(\epsilon; \{p\}) = I(-\epsilon; \{p\})$ (this property requires (2.4)).

(iii) $I(\epsilon; \{p\}) = \left((\sin \pi \epsilon) / \pi \right)^N J(\epsilon; \{p\})$, where $J(\epsilon; \{p\})$, which is given by the r.h.s. of (2.1) for $\text{Re}(\epsilon)$ large enough, is a rational function of $\epsilon$.

Consider further property (iii). Suppose in fact that $J(\epsilon; \{p\})$ is the reciprocal of a polynomial:

$$J(\epsilon; \{p\}) = \frac{k(\{p\})}{\prod_{j=1}^{M(\{p\})}(\epsilon + n_j(\{p\}))^{q_j(\{p\})}}$$

(2.5)

Then $q_j(\{p\})$ is the maximum number of times the integer $n_j(\{p\})$ occurs as a power in a single term of the Laurent expansion of $f$. We will see below that the Selberg-type integrals related to the wavefunction (1.2) have the property (2.5).

In the cases that (2.5) holds, a conjectured evaluation of (2.1) of the form

$$\left((\sin \pi \epsilon) / \pi \right)^N \alpha(\{p\}) A_N(\epsilon; \{p\})$$

(2.6)

can be proved to be correct up to the multiplicative function of the parameters $\alpha(\{p\})$ by an application of Liouville’s theorem. Thus we need to show that $1/A_N(\epsilon; \{p\})$ is a polynomial in $\epsilon$ and to specify the positions and orders of its zeros. Then we need to show that the positions of the poles of $J_N(\epsilon; \{p\})$ coincide with these zeros, and their order is no greater than that of the corresponding zero. Finally, we need to check that $J_N(\epsilon; \{p\})$ is the reciprocal of a polynomial by calculating its $|\epsilon| \to \infty$ behaviour.

Let us illustrate this method to prove that for $a + b := 2r, c \in \mathbb{Z}_{\geq 0}$, the Fourier integral in (1.3) as a function of $a - b := 2\epsilon$ is equal to the product of gamma functions in (1.3) up to a multiplicative function of $N, c$ and $r$. For this problem we have

$$f(t_1, \ldots, t_N; \{p\}) = \prod_{l=1}^{N}(1 - t_l)^r(1 - t_l)^r \prod_{1 \leq j < k \leq N}(1 - t_k/t_j)^c(1 - t_j/t_k)^c$$

(2.7)

and using the functional equation for the gamma function

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

we see that the product of gamma functions in (1.3) can be written in the form of (2.6) with

$$A_N(\epsilon; \{p\}) = \prod_{l=0}^{N-1} \frac{\Gamma(\epsilon - r - lc)}{\Gamma(\epsilon + r + 1 + lc)}$$

(2.8)

Consider (2.8). Since $r$ and $c$ are integers we see from the recurrence of the gamma function

$$\Gamma(z + 1) = \Gamma(z)$$

that $A_N(\epsilon; \{p\})$ is the reciprocal of a polynomial, with poles of order $j$ ($j = 1, \ldots, N - 1$) at

$$|\epsilon| = r + 1 + (N - j - 1)c, r + 2 + (N - j - 1)c, \ldots, r + c + (N - j - 1)c$$

(2.9)

and poles of order $N$ at

$$|\epsilon| = 0, \ldots, r$$

(2.10)
The order of the polynomial is equal to

\[ \sum_{n=1}^{N} l_n n \]

where \( l_n \) is the number of poles of order \( n \). Since \( l_n = 2c (n = 1, \ldots, N - 1) \) and \( l_N = 2r + 1 \) the order is thus

\[ c_N(N - 1) + N(2r + 1) \quad (2.11) \]

Now consider the r.h.s. of (2.1) with \( f \) given by (2.7). Then (2.1) can be written in the form (2.5), where in general \( k(\{p\}) \) may be a polynomial in \( \epsilon \). To prove that \( A_N (\epsilon; \{p\}) \) is given by (2.8) we need to show three features:

(a) \( |n_j(\{p\})| \) in (2.5) is no greater than \( r + c(N - 1) \).

(b) For \( |n_j(\{p\})| \) given by the r.h.s. of (2.9) and (2.10), \( q_j(\{p\}) \) is less than or equal to the order of the corresponding pole in (2.9) and (2.10).

(c) For large-\( |\epsilon| \), \( J_N (\epsilon; \{p\}) \) has an inverse power law decay with exponent (2.11).

To check (a), we see from (2.7) that the largest power of say \( t_1 \) is \( r \) in the product over \( l \) and \( c(N - 1) \) in the product over \( j \) and \( k \). It is thus \( r + c(N - 1) \) in total, which proves (a) for \( n_j(\{p\}) \) positive. Since (2.7) is also unchanged by the replacement \( t_j \mapsto 1/t_j \) \((j = 1, \ldots, N)\), the same is true for \( n_j(\{p\}) \) negative.

Consider statement (b). With \( q^{(N)}(n) \) denoting the maximum number of times the exponent \( n \) occurs in any term of the Laurent expansion of (2.7), statement (b) says

\[ q^{(N)}(n) \leq N \quad \text{for} \quad |n| \leq r \quad (2.12a) \]

\[ q^{(N)}(n) \leq N - j \quad \text{for} \quad |n| = r + (j - 1)c + \nu \quad (2.12b) \]

where \( j = 1, \ldots, N - 1 \) and \( \nu = 1, \ldots, c \). Since there are only \( N \) variables \( q^{(N)}(n) \leq N \) for any \( n \) so (2.12a) is true. To prove (2.12b) consider the explicit formula (2.7) for \( f \). Since the first two products can create all powers \( |n| \leq r \) in each variable independently, we see (2.12b) is equivalent to saying

\[ \hat{q}^{(N)}(n) \leq N - j \quad \text{for} \quad |n| = (j - 1)c + \nu \quad (2.13) \]

where \( \hat{q}^{(N)}(n) \) denotes the maximum number of times the exponent \( n \) occurs in the Laurent expansion of

\[ \prod_{1 \leq j < k \leq N} (1 - t_k/t_j)^c (1 - t_j/t_k)^c \]

\[ = (-1)^{c(N-1)/2} \prod_{j=1}^{N} t_j^{-c(N+1)+2cj} \prod_{1 \leq j < k \leq N} (1 - t_k/t_j)^{2c} \quad (2.14) \]

Writing

\[ \frac{t_j}{t_k} = \frac{t_j}{t_{j+1}} \frac{t_{j+1}}{t_{j+2}} \ldots \frac{t_{k-1}}{t_k}, \quad j < k \quad (2.15) \]

we see by expanding the last product in (2.14) that the terms in the Laurent expansion of (2.14) are of the form [9]

\[ \prod_{j=1}^{N} t_j^{-c(N+1)+2cj+n_j+1-n_j}, \quad (2.16) \]
where \( n_1 = n_{N+1} = 0 \) and \( n_j \geq 0 \) for each \( j = 2, \ldots, N \). Equivalently, setting

\[
n_{p+1} = c(N-p)p + m_{p+1}, \quad m_{p+1} \geq -c(N-p)p
\]

for each \( p = 1, \ldots, N-1 \) we have that all terms in the Laurent expansion of (2.16) are of the form

\[
t_N^{-m_N} t_{N-1}^{m_{N-1}} \cdots t_p^{m_p - m_{p+1}} \cdots t_1^{m_2}
\]

(2.18)

We want to determine the maximum number of exponents in (2.18) which can take the value \( (j-1)c + \nu \). Since (2.14) is a symmetrical function of all the variables we can suppose that the \( k \) variables \( t_N, t_{N-1}, \ldots, t_{N+k} \) have exponent \( (j-1)c + \nu \). Then from (2.18) we require

\[
m_{p+1} = -(j-1)c + \nu)(N-p), \quad p = N-1, \ldots, N-k
\]

(2.19)

Combining (2.19) with the inequality in (2.16) gives

\[
(j-1)c + \nu \leq cp \leq c(N-k)
\]

(2.20)

which, since \( 1 \leq \nu \leq c \) implies

\[
k \leq N-j
\]

(2.21)

The inequality (2.21) is precisely the statement (2.13) with \( n = (j-1)c + \nu \). Also, since (2.14) is unchanged by replacing all variables by their reciprocals, (2.21) also establishes (2.13) for the remaining case \( n = -(j-1)c + \nu \).

To check (c) we note from (2.7) and (1.3) that

\[
J_N(\epsilon; \{p\}) = (-1)^{rN+cN(N-1)/2} \left( \prod_{l=1}^{N} \int_{0}^{1} dt_l t_l^{-1+1+\epsilon-r-(N-1)c}(1-t_l)^{2r} \right)
\]

\[
\times \prod_{1 \leq j < k \leq N} (t_k - t_j)^{2c}
\]

(2.22)

The change of variables

\[
t_l = e^{-s_l} \quad \text{then} \quad s_l \mapsto s_l/\epsilon
\]

gives the large-\(|\epsilon|\) asymptotic behaviour

\[
J_N(\epsilon; \{p\}) \sim \frac{h_N(\{p\})}{e^{cN(N-1)+N(2r+1)}}
\]

(2.23)

with

\[
h_N(\{p\}) := (-1)^{rN+cN(N-1)/2} \left( \prod_{l=1}^{N} \int_{0}^{\infty} ds_l s_l^{2r} e^{-s_l} \right) \prod_{1 \leq j < k \leq N} (s_k - s_j)^{2c}
\]

which is precisely the inverse power law decay with exponent (2.11) required by (c).

By checking (a)-(c) we have shown, by Liouville’s theorem that the Fourier integral in (1.3) as a function of \( a-b := 2\epsilon \) is evaluated by the product of gamma functions in (1.3), up to multiplicative terms independent of \( \epsilon \).

Remark: The multiplicative function of \( N, c \) and \( r = (a+b)/2 \) in (1.3) undetermined by the above can readily be calculated (see the final paragraphs of section 3.1 for the method). We have thus provided a new proof of Morris’s integral and consequently of the Selberg integral.
2.4 Notation
In the remainder of this paper we will consider the generalization of (1.3)

\[ D_p(N_1, \ldots, N_p; N_0; a, b, \lambda) := \left( \prod_{\alpha=1}^{p} \prod_{j=1}^{N_\alpha} \int_{-1/2}^{1/2} dx_j^{(\alpha)} e^{\pi i x_j^{(\alpha)}(a-b)} |1 + e^{2\pi i x_j^{(\alpha)}}|^{a+b} \right) \times \left( \prod_{l=1}^{N_0} \int_{-1/2}^{1/2} dy_l e^{\pi i (a-b)y_l} |1 + e^{2\pi i y_l}|^{a+b} \right) \psi_0(\{e^{2\pi i x_j^{(\alpha)}}\}_{\alpha=1,\ldots,p}, \{e^{2\pi i y_j}\}_{j=1,\ldots,N_0}) \]

(2.24)

where \( \psi_0 \) is given by (1.2). We notice that

\[ D_1(N_1; 0; a, b, c - 1) = D_0(N; a, b, c) = D(N; a, b, c) \]

(2.25)

where \( D(N; a, b, c) \) is the integral in (1.3).

3. THE CASE \( p = 1 \)
In the case \( p = 1 \), \( a = b = 0 \), the exact evaluation of (2.24) is known analytically [10]:

\[ D_1(N_1; N_0; 0, 0, \lambda) = \frac{\Gamma(\lambda N_0 + (\lambda + 1) N_1 + 1) \Gamma(N_1 + 1)}{\Gamma(1 + \lambda) N_0 \Gamma(2 + \lambda) N_1 (1 + \frac{\lambda N_0}{\lambda + 1}) N_1} \]

(3.1)

where \( (a)_n := a(a+1)\ldots(a+n-1) \).

For general \( a \) and \( b \) the method used in [10] to prove (3.1) does not appear to be applicable. We thus resorted to the numerical approach outlined in Section 2.2.

3.1 The case \( \lambda = 1 \)
With \( \lambda = 1, N_0 = 1 \) and \( 2 \), and various values of \( a \) and \( b \), we found by sequentially increasing \( N_1 \) that our data fitted the following forms:

\[ D_1(N_1; 1; 1, 1, 1) = 2 \prod_{j=0}^{N_1-1} (j+1)(2j+3) \]
\[ D_1(N_1; 1; 2, 1, 1) = 3 \prod_{j=0}^{N_1-1} (j+1)(2j+4) \]
\[ D_1(N_1; 1; 2, 2, 1) = 6 \prod_{j=0}^{N_1-1} \frac{(j+1)(2j+4)(2j+5)}{2j+3} \]
\[ D_1(N_1; 2; 1, 1, 1) = 6 \prod_{j=0}^{N_1-1} (j+1)(2j+4) \]
\[ D_1(N_1; 2; 2, 1, 1) = 12 \prod_{j=0}^{N_1-1} (j+1)(2j+5) \]
\[ D_1(N_1; 2; 2, 2, 1) = 40 \prod_{j=0}^{N_1-1} \frac{(j+1)(2j+5)(2j+6)}{2j+4} \]
valid for $N_1 \geq 0$ (when $N_1 = 0$ the products are taken as unity).

To help fit this data into an analytic form we note from Morris’s integral (1.3) in the
case $c = 2$ and $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}^+$ that

$$D_1(N_1; 0; a, b, 1) = \prod_{j=0}^{N_1-1} \frac{(j+1)(2j+a+1)(2j+a+2)\ldots(2j+a+b)}{(2j+2)(2j+3)\ldots(2j+b)}$$

(3.2)

We see that the above data fits a similar form:

$$D_1(N_1; N_0; a, b, 1)$$

$$= f(N_0, a, b) \prod_{j=0}^{N_1-1} \frac{(j+1)(2j+a+N_0+1)(2j+a+N_0+2)\ldots(2j+a+b+N_0)}{(2j+N_0+2)(2j+N_0+3)\ldots(2j+N_0+b)}$$

$$= f(N_0, a, b) \prod_{j=0}^{N_1-1} \frac{(j+1)\Gamma(2j+a+b+N_0+1)\Gamma(2j+1+N_0)}{\Gamma(2j+a+N_0+1)\Gamma(2j+b+N_0+1)}$$

(3.3)

To evaluate $f(N_0, a, b)$ we set $N_1 = 0$ in (3.3) (the product over $j$ is then taken as unity)
to obtain

$$f(N_0, a, b) = D_1(0; N_0; a, b, 2) = D(N_0; a, b, 1)$$

(3.4)

where $D(N_0; a, b, 1)$ is given by (1.3).

The resulting conjecture for $D_1$ can be proved using the method of Section 2.3. Thus
we consider $D_1(N_1; N_0; a, b, 1)$ as a function of $a - b := 2\epsilon$. We have the following result.

**Theorem 1**
Suppose $r \in \mathbb{Z}_{\geq 0}$ and let

$$f(t_1, \ldots, t_{N_0}; s_1, \ldots, s_{N_1}; r)$$

$$:= \prod_{l=1}^{N_0} (1-t_l)^r (1-1/t_l)^r \prod_{l=0}^{N_1} (1-s_l)^r (1-1/s_l)^r$$

$$\times \prod_{1 \leq j < k \leq N_0} (1-t_k/t_j)(1-t_j/t_k) \prod_{1 \leq j < k \leq N_1} (1-s_k/s_j)^2(1-s_j/s_k)^2$$

$$\times \prod_{j=1}^{N_0} \prod_{k=1}^{N_1} (1-s_k/t_j)(1-t_j/s_k)$$

(3.5)

Then for $\text{Re}(\epsilon)$ large enough so that the l.h.s. converges,

$$\left(\prod_{l=1}^{N_0} \int_0^1 dt_l t_l^{-\epsilon} \right) \left(\prod_{l=1}^{N_1} \int_0^1 ds_l s_l^{-\epsilon} \right) f(t_1, \ldots, t_{N_0}; s_1, \ldots, s_{N_1}; r)$$

$$= A(N_0, N_1, r) \prod_{l=0}^{N_0-1} \frac{\Gamma(\epsilon-r-l)}{\Gamma(\epsilon+r+l+1)} \prod_{l=0}^{N_1-1} \frac{\Gamma(\epsilon-r-2l-N_0)}{\Gamma(\epsilon+r+1+2l+N_0)}$$

(3.6)

where $A(N_0, N_1, r)$ is independent of $\epsilon$.

**Proof**
Since $r \in \mathbb{Z}_{\geq 0}$ we see that the r.h.s. of (3.6) is the reciprocal of a polynomial in $\epsilon$.
Furthermore this polynomial is naturally factored as two polynomials, one for each of
the products. The first reciprocal polynomial factor is precisely (2.8) with $c = 1$ and
$N = N_0$. This factor therefore has poles at (2.9) and (2.10) with the order given therein. The second factor has poles of order $j$ ($j = 1, \ldots , N_1 - 1$) at integers $\epsilon$ satisfying

$$r + N_0 + 2(N_1 - j) \leq |\epsilon| \leq r + N_0 + 2(N_1 - j) + 1$$

(3.7)

and poles of order $N_1$ at

$$|\epsilon| = 0, 1, \ldots , r + N_0$$

(3.8)

The poles at (3.7) and (3.8) occur at the same points as the poles of the first factor, plus some additional points. Thus if we let $Q^{(N_0,N_1)}(n)$ denote the order of the pole of (3.5) at $n$, ($n \in \mathbb{Z}$), we see that

$$Q^{(N_0,N_1)}(n) = N_0 + N_1, \quad |n| \leq r$$

(3.9a)

$$Q^{(N_0,N_1)}(n) = N_1 + N_0 - j_1, \quad |n| = r + j_1$$

(3.9b)

where $j_1 = 1, \ldots , N_0$,

$$Q^{(N_0,N_1)}(n) = N_1 - j_2, \quad |n| = r + N_0 + 2(j_2 - 1) + \nu_2$$

(3.9c)

where $j_2 = 1, \ldots , N_1 - 1$, $\nu_2 = 1, 2$, and

$$Q^{(N_0,N_1)}(n) = 0, \quad \text{otherwise.}$$

(3.9d)

From (3.9) we see that the reciprocal of the r.h.s. of (3.6) is a polynomial of order

$$(2r + 1)(N_0 + N_1) + 2 \sum_{j=0}^{N_0-1} (j + N_1) + 4 \sum_{j=1}^{N_1-1} j$$

$$= (2r + 1)(N_0 + N_1) + 2N_0N_1 + (N_0 - 1)N_0 + 2(N_1 - 1)N_1$$

(3.10)

Let $q^{(N_0,N_1)}(n)$ denote the maximum number of times the exponent $n$ can occur in a term of the Laurent expansion of (3.5). In accordance with the method of Section 2.3 we want to show that

$$q^{(N_0,N_1)}(n) \leq Q^{(N_0,N_1)}(n)$$

(3.11)

First, for $|n| \leq r$ there is nothing to prove as (3.11) reads

$$q^{(N_0,N_1)}(n) \leq N_0 + N_1$$

which is true by definition. Next consider the statement (3.11) for $|n|$ and $Q^{(N_0,N_1)}(n)$ given by (3.9b). Since the products over $l$ in (3.5) give all exponents $n$ of $t_1, \ldots , t_{N_0}, s_1, \ldots , s_{N_1}$, with $|n| \leq r$ in each variable independently, and (3.5) is unchanged by replacing each variable by its reciprocal, we see that in these cases (3.11) is equivalent to proving

$$\tilde{q}^{(N_0,N_1)}(n) \leq N_1 + N_0 - j_1, \quad n = j_1$$

(3.12a)

and

$$\tilde{q}^{(N_0,N_1)}(n) \leq N_1 - j_2, \quad n = N_0 + 2(j_2 - 1) + \nu_2$$

(3.12b)

where $\tilde{q}^{(N_0,N_1)}(n)$ denotes the maximum number of times the exponent $n$ can occur in the Laurent expansion of

$$\prod_{1 \leq j < k \leq N_0} (1 - t_k/t_j)(1 - t_j/t_k) \prod_{1 \leq j < k \leq N_1} (1 - s_k/s_j)^2(1 - s_j/s_k)^2$$
\[
\times \prod_{j=1}^{N_0} \prod_{k=1}^{N_1} \frac{1 - s_k/t_j}{1 - t_j/s_k} \quad (3.13)
\]

Using a confluent form of the Vandermonde determinant expansion, we have previously shown [11] that (3.13) is equal to (up to an unimportant ± sign)

\[
\sum_{P(2t) > P(2t-1)} \epsilon(P) \sum_{Q=1}^{N!} \epsilon(Q) \prod_{j=1}^{N_1} s_j^{P(2j)+P(2j-1)-2N_0-N_1-1} (P(2j) - P(2j-1))
\]

\[
\times \prod_{k=1}^{N_0} t_k^{P(2N_1+k)+Q(k)-N_0-N_1-1} \quad (3.14)
\]

where \(P\) is a permutation of \(\{1, 2, \ldots, 2N_1 + N_0\}\) with parity \(\epsilon(P)\) and \(Q\) is a permutation of \(\{1, \ldots, N_0\}\) with parity \(\epsilon(Q)\).

Let \(j_1, k \in \{1, \ldots, N_0\}\) and \(j \in \{1, \ldots, N_1\}\). We see from (3.14) that an exponent of \(j_1\) in the variable \(t_k\) requires

\[
P(2N_1 + k) = j_1 + N_0 + N_1 + 1 - Q(k) \quad (3.15a)
\]

and thus

\[
P(2N_1 + k) \in \{j_1 + N_1 + 1, j_1 + N_1 + 2, \ldots, \min(2N_1 + N_0, j_1 + N_1 + N_0)\} \quad (3.15b)
\]

For an exponent of \(j_1\) in the variable \(s_j\) we see from (3.14) that we require

\[
P(2j) + P(2j - 1) = 2N_1 + N_0 + 1 + j_1, \quad P(2j) > P(2j - 1). \quad (3.16)
\]

Consider the case \(j_1 \leq N_1\). Then

\[
\min(2N_1 + N_0, j_1 + N_1 + N_0) = j_1 + N_1 + N_0
\]

and so the maximum number of solutions of (3.15a) is \(N_0\). When (3.15a) has this maximum number of solutions, (3.16) has solutions for

\[
(P(2j), P(2j-1)) = (j_1 + 1 + N_0 + N_1, N_1), (j_1 + 2 + N_0 + N_1, N_1 - 1), \ldots, (2N_1 + N_0, j_1 + 1),
\]

thus giving a maximum of \(N_1 - j_1\) exponents \(j_1\) to the variable \(t_j\), and thus a total of \(N_0 + N_1 - j_1\) exponents \(j_1\) in (3.14). Since decreasing the number of solutions of (3.15a) by say \(a_1\) can give no more than \(a_1\) new solutions to (3.16), we thus have shown that for \(j_1 \leq N_1\)

\[
\bar{q}^{(N_0, N_1)}(j_1) \leq N_1 + N_0 - j_1 \quad (3.17)
\]

In the cases that \(j_1 > N_1\) (which requires \(N_0 > N_1\))

\[
\min(2N_1 + N_0, j_1 + N_1 + N_0) = 2N_1 + N_0
\]

and so the maximum number of solutions of (3.15a) is \(N_1 + N_0 - j_1\). When (3.15a) has this maximum number of solutions, there are no solutions to (3.16). Arguing as in the sentence including (3.17), we conclude that (3.17) also holds for \(j_1 > N_1\) and thus (3.12a) is true.
For an exponent of \( n = N_0 + 2(j_2 - 1) + \nu_2 \) in the variable \( s_j \), (3.14) gives that we require (3.16) with \( j_1 \) replaced by \( n \). The maximum number of solutions occurs with

\[
(P(2j), P(2j-1)) = (2N_1 + N_0, 1 + n), (2N_1 + N_0 - 1, 2 + n), \ldots, (N_1 + N_0 + j_2 + 1, N_1 - j_2 + n)
\]

(note that \( N_1 - j_2 + n = N_1 + N_0 + j_2 + \nu_2 - 2 \) and is thus equal to \( N_1 - j_2 \). For an exponent of \( n = N_0 + 2(j_2 - 1) + \nu_2 \) in the variable \( t_k \), (3.14) gives that we require (3.15a) with \( j_1 \) replaced by \( n \). When (3.16) has its maximum number of solutions we see that (3.15a) doesn’t have any solutions. Furthermore, by decreasing the number of solutions of (3.16) by say \( a_1 \), we see that the number of solutions of (3.15a) can increase by no more than \( a_1 \) (since for (3.15a) to have a solution we require \( P(2N_1 + k) \geq n + N_1 + 1 \) and so we conclude that (3.12b) is true.

The validity of (3.12) implies that as a function of \( \epsilon \), the l.h.s. of (3.6) divided by the r.h.s. is bounded in the finite plane. Furthermore, using the method given in the paragraph including (2.22) above, it is straightforward to show that for large-\( |\epsilon| \) the r.h.s. of (3.6) has a reciprocal power law decay with exponent (3.10), which is the same as the large-\( |\epsilon| \) behaviour of the l.h.s. (recall the sentence including (3.10)). Hence, by Liouville’s theorem, both sides of (3.6) are the same functions of \( \epsilon \), up to a multiplicative function independent of \( \epsilon \). This is the required result.

Let us now specify the dependence on \( r \) of the function \( A(N_0, N_1, r) \) in (3.6). For this purpose we observe that when \( \epsilon = r \), the l.h.s. of (3.6) is independent of \( r \). More explicitly,

\[
CT \prod_{l=1}^{N_0} t_l^{r} \prod_{l=1}^{N_1} s_l^r f(-t_1, \ldots, -t_{N_0}; -s_1, \ldots, -s_{N_1}; r) = CT \prod_{l=1}^{N_0} (1 + t_l)^{2r} \prod_{l=1}^{N_1} (1 + s_l)^{2r} f(-t_1, \ldots, -t_{N_0}; -s_1, \ldots, -s_{N_1}; 0) = CT f(-t_1, \ldots, -t_{N_0}; -s_1, \ldots, -s_{N_1}; 0)
\]

where the last line follows from the second last line after noting \( f \) is a homogeneous function of order 0. For the r.h.s. of (3.6) to have this property, we require

\[
A(N_0, N_1, r) = B(N_0, N_1) \prod_{l=0}^{N_0-1} \Gamma(2r + 1 + l) \prod_{l=0}^{N_1-1} \Gamma(2r + 1 + 2l + N_0) \tag{3.18}
\]

The remaining unknown function \( B(N_0, N_1) \) can be specified immediately from the analytic result (3.1), or alternatively by using the general relationship

\[
D_1(N_1; N_0; \lambda, \lambda, \lambda) = D_1(N_1, N_0 + 1; 0, 0, \lambda) \tag{3.19}
\]

together with (2.25). We find

\[
B(N_0, N_1) = \prod_{l=0}^{N_0-1} \Gamma(l + 2) \prod_{j=0}^{N_1-1} \Gamma(j + 1) \Gamma(2(j + 1) + N_0) \tag{3.20}
\]

Substituting (3.20) in (3.18), then substituting the resulting expression in the r.h.s. of (3.6), we obtain the exact evaluation of the integral in (3.6). This exact evaluation agrees with the conjecture (3.3), and thus proves the conjecture.
3.2 General \( \lambda \)

Using Morris’s integral (1.3), the result (3.4) for \( \lambda = 1 \), and the analytic result (3.1) as guides, we conjecture that for general \( \lambda \)

\[
D_1(N_1; N_0; a, b, \lambda) = D(N_0; a, b, \lambda) \prod_{j=0}^{N_1-1} \frac{(j + 1)\Gamma((\lambda + 1)j + a + b + \lambda N_0 + 1)\Gamma((\lambda + 1)(j + 1) + \lambda N_0)}{\Gamma(1 + \lambda)\Gamma((\lambda + 1)j + a + \lambda N_0 + 1)\Gamma((\lambda + 1)(j + 1) + b + \lambda N_0 + 1)}
\]  

(3.21)

where \( D(N_0; a, b, \lambda) \) is given by (1.3). As well as being consistent with theorems used in its formulation, (3.21) satisfies the general relationship (3.19), and the consistency of the large-\( |\epsilon| \) behaviour of both sides (recall Section 2.3) can be checked as can the independence on \( a \) when \( b = 0 \) (recall the paragraph above (3.18)).

3.3 Normalization of the harmonic well wavefunction for \( p = 1 \)

From the conjecture (3.21), it is possible to deduce the value of the integral

\[
G_1(N_1; \ldots; N_p; N_0; \lambda) := \left( \prod_{\alpha=1}^{p} \prod_{j=1}^{N_0} \int_{-\infty}^{\infty} dx_j^{(\alpha)} \right) \left( \prod_{j'=1}^{N_0} \int_{-\infty}^{\infty} dy_{j'} \right) |\psi_0^{(b)}(\{x_j^{(\alpha)}\}_{\alpha=1,\ldots,p}, \{y_{j'}\}_{j'=1,\ldots,N_0})|^2
\]

(3.22)

where \( \psi_0^{(b)} \) is given by (1.4), in the case \( p = 1 \).

By setting \( a = b \) in (2.24) and changing variables \( x_j^{(1)} \mapsto x_j^{(1)} / 2\pi a, y_l \mapsto y_l / 2\pi a \), we see that for \( a \to \infty \)

\[
D_1(N_1; N_0; a, a, \lambda) \sim \left( \frac{1}{2\pi} \right)^{N_1+N_0} \left( \frac{1}{a} \right)^{N_1(N_1-1)+\lambda N_0 N_1+\lambda N_0(N_0-1)} 2^{2a(N_1+N_0)} G_1(N_1; N_0; \lambda)
\]

(3.23)

On the other hand, with \( a = b \) it is straightforward to obtain the large-\( a \) behaviour of the conjectured evaluation (3.21) of \( D_1 \) by using Stirling’s formula. Comparison with (3.23) then gives

\[
G_1(N_1; N_0; \lambda) = (2\pi)^{N_1+N_0}/2 \prod_{j=0}^{N_0-1} \frac{\Gamma((j+1)(\lambda+1))}{\Gamma(1+\lambda)} \prod_{k=0}^{N_1-1} \frac{\Gamma((\lambda+1)(k+1)+\lambda N_0)}{\Gamma(1+\lambda)}
\]

(3.24)

as the conjectured evaluation of \( G_1 \) (in the case \( \lambda = 1 \), since we have proved (3.21), we also have proved (3.24)).

4. THE GENERAL \( p \) cases

4.1 The case \( p = 2 \)

With \( p = 2 \) we have the general relations

\[
D_2(0, N_2; 0; a, b, \lambda) = D(N_2; a, b, \lambda + 1)
\]

(4.1)

where the r.h.s. is given by Morris’s integral (2.3), and

\[
D_2(1, N_2; 0; 0, 0, \lambda) = D_1(N_2; 1; 0, 0\lambda)
\]

(4.2)
where the r.h.s. is given by the conjecture (3.21).

These formulas provide analytic data for the cases \(N_0 = 0\) and 1. For \(N_1 = 2\) and 3, and with \(\lambda = 1, a = b = 0\), numerical data was obtained. By sequentially increasing \(N_2\), the data was seen to fit the forms

\[
D_2(2, N_2; 0; 0, 0, 1) = \frac{16}{3} \prod_{j=1}^{N_2} j(2j + 1), \quad N_2 \geq 1 \tag{4.3}
\]

\[
D_2(3, N_2; 0; 0, 0, 1) = 70 \prod_{j=1}^{N_2} j(2j + 2), \quad N_2 \geq 2 \tag{4.3}
\]

The results (4.1)-(4.4) suggest that

\[
D_2(N_1, N_2; 0; 0, 0, 1) = g(N_1) \prod_{j=1}^{N_2} j(2j - 1 + N_1) = g(N_1) \prod_{j=0}^{N_2-1} (j+1)\Gamma(2(j+1) + N_1) / \Gamma(2j + 1 + N_1), \quad N_2 \geq N_1 - 1 \tag{4.5}
\]

For general \(a, b\) and \(\lambda\) inspection of (4.5) and use of (4.1) and (1.3) suggest the same ansatz used in (3.3) for \(D_1(N_1; N_0; a, b, 1)\):

\[
D_2(N_1, N_2; 0; a, b, \lambda) = f(N_1, a, b) A(N_1, N_2; a, b, \lambda) \tag{4.6a}
\]

where

\[
A(N_1, N_2; a, b, \lambda) = \prod_{j=0}^{N_2-1} \frac{(j+1)\Gamma((\lambda+1)j + a + b + N_1 + 1)\Gamma((\lambda+1)(j+1) + N_1)}{\Gamma((\lambda+1)j + a + N_1 + 1)\Gamma((\lambda+1)(j+1) + b + N_1 + 1)} \tag{4.6b}
\]

valid for

\[N_2 \geq N_1 - 1 \tag{4.6c}\]

The restriction (4.6c) is the key distinguishing feature between (3.3) and (4.6a). The function \(f(N_1, a, b)\) can be determined from the symmetry relation

\[
D_2(N, N-1; 0; a, b, \lambda) = D_2(N-1, N; 0; a, b, \lambda)
\]

which gives the difference equation

\[
f(k; a, b, \lambda) A(k, k-1; a, b, \lambda) = f(k-1; a, b, \lambda) A(k-1, k; a, b, \lambda)
\]

This difference equation has solution

\[
f(N_1; a, b, \lambda) = \prod_{k=1}^{N_1} \frac{A(k-1, k; a, b, \lambda)}{A(k, k-1; a, b, \lambda)} \tag{4.7}
\]

where we have used the fact that

\[f(0; a, b, \lambda) = 1\]

which follows by choosing \(N_1 = N_2 = 0\) in (4.6a). Substituting (4.7) in (4.6a) gives the conjectured evaluation of \(D_2(N_1, N_2; 0; a, b, \lambda)\).
4.2 The general case
Guided by (4.6a) and (3.3), for general \( p \) and \( N_0 \) we conjecture that

\[
D_p(N_1, \ldots, N_p; N_0; a, b, \lambda) = f_{p-1}(N_1, \ldots, N_{p-1}; N_0; a, b, \lambda) A_p(N_1, \ldots, N_p; N_0; a, b, \lambda)
\]

where

\[
A_p(N_1, \ldots, N_p; N_0; a, b, \lambda)
= \frac{\prod_{j=0}^{N_p-1} (j + 1)\Gamma((\lambda + 1)j + a + b + \lambda \sum_{j=0}^{p-1} N_j + 1)\Gamma((\lambda + 1)(j + 1) + \lambda \sum_{j=0}^{p-1} N_j)}{\Gamma(1 + \lambda)\Gamma((\lambda + 1)j + a + \lambda \sum_{j=0}^{p-1} N_j + 1)\Gamma((\lambda + 1)(j + 1) + b + \lambda \sum_{j=0}^{p-1} N_j + 1)}
\]

and \( N_p \geq N_j - 1 \) (\( j = 1, \ldots, p - 1 \)). To calculate \( f_{p-1} \) we make the ordering

\[
N_j \geq N_{j-1} \quad (j = 2, \ldots, p)
\]

and use the symmetry relation

\[
D_p(N_1, \ldots, N_{k-2}, N-1, N, N_{k+1}, \ldots, N_p; N_0; a, b, \lambda)
= D_p(N_1, \ldots, N_{k-2}, N, N_{k+1}, \ldots, N_p; N_0; a, b, \lambda)
\]

for \( k = 2, \ldots, p \). From (4.9) and the initial condition

\[
D_p(0, \ldots, 0; N_0; a, b, \lambda) = D(N_0; a, b, \lambda)
\]

where \( D(N_0; a, b, \lambda) \) is given by (2.3), we obtain the recurrence equations

\[
f_{k-1}(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda)
= A_{k-1}(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda) f_{k-2}(N_1, \ldots, N_{k-2}; N_0; a, b, \lambda)
\]

where

\[
A_{k-1}(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda) := \prod_{j=1}^{N_{k-1}} \frac{A_k(N_1, \ldots, N_{k-2}, j - 1, j; N_0; a, b, \lambda)}{A_k(N_1, \ldots, N_{k-2}, j, j - 1; N_0; a, b, \lambda)}
\]

and

\[
f_0(N_0; a, b, \lambda) = D(N_0; a, b, \lambda)
\]

Taken in the order \( k = p, p - 1, \ldots, 2 \) these equations explicitly determine \( f_{p-1} \) and thus \( D_p \).

For example, with \( p = 3 \) we obtain

\[
D_3(N_1, N_2, N_3; N_0; a, b, \lambda)
= D(N_0; a, b, \lambda) A_3(N_1, N_2, N_3; N_0; a, b, \lambda) \prod_{j=1}^{N_2} \frac{A_3(N_1, j - 1, j; N_0; a, b, \lambda)}{A_3(N_1, j, j - 1; N_0; a, b, \lambda)}
\]

\[
\times \prod_{k=1}^{N_1} \frac{A_3(k - 1, k - 1, k; N_0; a, b, \lambda)}{A_3(k - 1, k, k - 1; N_0; a, b, \lambda)} \prod_{j=1}^{k-1} \frac{A_3(k - 1, j - 1, j; N_0; a, b, \lambda) A_3(k, j - 1; N_0; a, b, \lambda)}{A_3(k - 1, j, j - 1; N_0; a, b, \lambda) A_3(k, j - 1; a, b, \lambda)}
\]

\[
(4.11a)
\]
where
\[ N_p \geq N_{p-1} - 1, \quad p = 2, 3 \] (4.11b)
Note that this agrees with (4.6a) when \( N_1 = N_0 = 0 \). Also, we have made the exact numerical evaluations
\[ D_3(1, 2, 2; 0, 0, 1) = 720 \quad \text{and} \quad D_3(2, 2, 2; 0, 0, 1) = 10,080 \]
and found agreement with (4.11).

4.3 Normalization of the harmonic well wavefunction in the general case

Analogous to (4.6a), for the integral (3.22) we conjecture
\[ G_p(N_1, \ldots, N_p; N_0; \lambda) = g_{p-1}(N_1, \ldots, N_{p-1}; N_0; \lambda)B_p(N_1, \ldots, N_p; N_0; \lambda) \] (4.12a)
where
\[ B_p(N_1, \ldots, N_p; N_0; \lambda) := (2\pi)^{N_p/2} \prod_{j=0}^{N_p-1} \frac{(j+1)\Gamma((\lambda+1)(j+1) + \lambda \sum_{j=0}^{p-1} N_j)}{(1+\lambda)} \] (4.12b)
With the ordering (4.8c), and assuming the analogue of the symmetry relation (4.9), the conjecture (4.11a) gives the recurrence equations
\[ g_{k-1}(N_1, \ldots, N_{k-1}; N_0; \lambda) = B_{k-1}(N_1, \ldots, N_{k-1}; N_0; \lambda)g_{k-2}(N_1, \ldots, N_{k-2}; N_0; \lambda) \] (4.13a)
where
\[ B_{k-1}(N_1, \ldots, N_{k-1}; N_0; \lambda) := \prod_{j=1}^{N_{k-1}} \frac{B_k(N_1, \ldots, N_{k-2}, j-1; N_0; \lambda)}{B_k(N_1, \ldots, N_{k-2}, j, j-1; N_0; \lambda)} \] (4.13b)
and
\[ g_0(N_0; \lambda) = (2\pi)^{N_0/2} \prod_{j=0}^{N_0-1} \frac{(j+1) \Gamma(\lambda(1+j))}{\Gamma(1+\lambda)} \] (4.13c)
which when taken in the order \( k = p, p-1, \ldots, 2 \) explicitly determine \( g_{p-1} \) and thus \( G_p \).

5. SUMMARY

The objective of this paper has been to provide the exact evaluation of the trigonometric integral (2.24). This integral is a generalization of Morris’s integral (1.3) (which is equivalent to Selberg’s integral (1.1)), and includes as a special case the normalization of the multicomponent wavefunction (1.2). We have been partially successful in this task in that (4.8) and (4.10) provide the conjectured exact evaluation of (2.24) expressed in the forms of recurrence equations. Moreover (4.8) provides a conjecture for a specific functional property of the integrals (2.24), from which their exact evaluation follows.

In Section 2.3 we have also provided a new proof of Morris’s integral, which was used in Section 3.1 to prove the conjectured evaluation of \( D_1(N_1; N_0; a, b, 1) \). However we have not been successful in providing a proof in the general case, which we leave as an open problem.
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