Corporate Domination Number of the Cartesian Product of Cycle and Path

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Abstract  Domination in graphs is to dominate the graph G by a set of vertices D (⊆ V, vertex set of G) when each vertex in G is either in D or adjoining to a vertex in D. D is called a perfect dominating set if for each vertex v is not in D, which is adjacent to exactly one vertex of D. We consider the subset C which consists of both vertices and edges. Let S=(V∪E) denote the set of all vertices V and the edges E of the graph G. Then C ⊆ S is said to be a corporate dominating set if every vertex v not in P ∪ Q is adjacent to exactly one vertex of P ∪ Q, where the set P consists of all vertices in the vertex set of an edge induced sub graph G[E1], (E1, a subset of E) such that there should be maximum one vertex common to any two open neighborhood of different vertices in V(G[E1]) and Q, the set consists of all vertices in the vertex set V1, a subset of V such that there exists no vertex common to any two open neighborhood of different vertices in V1. The corporate domination number of G, denoted by \( \gamma_{cor}(G) \), is the minimum cardinality of elements in C. In this paper, we intend to determine the exact value of corporate domination number for the Cartesian product of the Cycle \( C_{4k} \) (k ≥ 1) and Path \( P_n \) (n ≥ 2).

Keywords  Cartesian Product, Domination, Perfect Dominating Set, Edge-Induced Sub Graph, Corporate Dominating Set, Corporate Domination Number

1. Introduction

For graph-theoretic terminology, we have referred to G. Chartrand, L. Lesniak, and Ping Zhang[2] and Harray[5]. In this paper, all graphs G=(V,E) are considered as simple and undirected with vertex set V and edge set E. The order of G is the number of vertices in the vertex set V and the size of G is the number of edges in the edge set E. Vertices u and v of G are neighbors if \( uv \in E \). The open neighborhood of \( v \), denoted by \( N(v) \), is the set which consists of all the neighbors of \( v \). The closed neighborhood of \( v \), denoted by \( N[v] \), consists of all the neighbors of \( v \) including the vertex \( v \). The complement of the vertex set \( S \) is denoted by \( S^c \). A detailed study of the dominating set and its algorithm of the Cartesian product of paths and cycles have been established by Polana Palvic, Janez Zerovnik [8]. The Cartesian product \( G_1 \Box G_2 \) of graphs \( G_1 \) and \( G_2 \) is a graph with \( V(G_1 \Box G_2) = V(G_1) \times V(G_2) \) and \( (x_1, y_1)(x_2, y_2) \in E(G_1 \Box G_2) \) if and only if either \( x_1 = y_1 \) and \( x_2 \) adjacent to \( y_2 \) in \( G_2 \) or \( x_2 = y_2 \) and \( x_1 \) adjacent \( y_1 \) in \( G_1 \).

In [6], the concepts of the variety of domination parameters such as total, perfect, mixed, and many more, have been studied by various authors. Domination with its variations is well studied in [7]. A set \( S \subseteq V \) is said to be a dominating set of \( G \) if \( N[S] = V \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum number of vertices of any dominating set of \( G \). In[4] Gayathri A., Abdul. Muneera, Nageswara Rao T., Srinivas Rao T. have explained the significance of domination in various fields and expressed some real-life applications where dominations in a graph are used. In [9] Shobha Shukla, Vikas Singh Thakur have attempted to categorize domination concepts into Some categories. In [1] D. Bange, A. Barkauskas and P. Slater have obtained some results of the efficient dominating set A set \( S \subseteq V \) is called an efficient and total efficient dominating set if
Perfect domination number conditions to dominate the graph $G$. Compared with as a subset of $V$ domination, the corporate dominating set $C$ is considered approval for a proposal from its board of directors. The following problem. The CEO of the company has to get corporate dominating set, graph $C$ is dominated and hence the proposal is approved with an optimum solution.

Corporate domination can be applied practically for the corporate domination number for the Cartesian product of cycle and path $k \geq 1, n \geq 2$.

We begin by stating definitions, examples, and results.

**Definition 1.1.** A subset of the edges of a graph $G$ together with any vertices that are their endpoints is said to be an edge-induced subgraph of $G$ and is denoted by $[E]$. The graph $G$ and $[G[E_1]]$, an edge induced subgraph of $E_1$ where $E_1 = \{v_1v_5, v_4v_5, v_2v_3\}$, are shown in Figure 2.

**Figure 2.** $G$ and $[G[E_1]]$

**Definition 1.2.** Let $G(= V, E)$ be a graph. Let $C = V_1 \cup E_1 (\subseteq V \cup E)$. Take $P = \{u \in V(G[E_1]) / |N(u) \cap N(w)| \leq 1 \}$ for all $w(\neq u) \in V(G[E_1])$ where $V(G[E_1])$ denote the vertex set of an edge induced subgraph $G[E_1]$ and $Q = \{v \in V_1 / N(v) \cap N(w) = \emptyset \}$ for all $w(\neq v) \in V_1$. A subset $C$ is said to be a corporate dominating set if every vertex $v \notin P \cup Q$ is adjacent to exactly one vertex of $P \cup Q$. The corporate domination number of $G$, denoted by $\gamma_{cor}(G)$, is the minimum cardinality of elements in $C$. For a graph $G$ which is given in Figure 3, $\gamma_{cor}(G) = 2$.

Here $E_i = \{v_2v_3\}$, $V(G[E_1]) = \{v_2, v_3\}$ and $C = \{v_2v_3, v_6\}$

**Proposition 1.3.** Let $G$ be a graph. Then $\gamma_{cor}(G) = 1$ if and only if one of the following holds.

i. There exists a full degree vertex in $G$.

ii. There exists an edge $uv$ in $G$ such that $uv$ does not lie on any triangle and $d(u) + d(v) = n$.

**Remark 1.4.** Corporate domination need not exist for all graphs.

**Example 1.5.** The 4-regular graph $G$ in Figure 4 does not have a corporate dominating set.

**Figure 1.** $C = \{v_2v_1, v_4\}$ and $\gamma_{cor}(C_7) = 2$
If we take $C = \{v_1\}, 1 \leq i \leq 6$, or $C = \{v_jv_i\}, 1 \leq i, j \leq 6, i \neq j$, then by using Proposition 1.3, the graph $G$ does not have a corporate dominating set. Let $C$ be the corporate dominating set with $|C| \geq 2$. Then $|P \cup Q| \geq 2$. But any two vertices in this graph have a common neighbor, which is a contradiction.

**Proposition 1.6** Let $C$ be a corporate dominating set with $C = V_1$. Then
i. Every corporate dominating set is the dominating set as well as the perfect dominating set.
ii. Both dominating and perfect dominating set need not be the corporate dominating set.

**Example 1.7** Consider the cycle $C_6 = V(C_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then the corporate dominating set $C = \{v_2, v_5\}$ is also the dominating set as well as the perfect dominating set.

**Proposition 1.8**

i. For any cycle $C_m$ with $m \geq 3$, $\gamma_{cor}(C_m) = \left\lceil \frac{m}{4} \right\rceil$.
ii. For any Path $P_n$ with $n \geq 2$, $\gamma_{cor}(P_n) = \left\lceil \frac{n}{4} \right\rceil$.
iii. For any complete graph $K_n(n \geq 3)$, $\gamma_{cor}(K_n) = 1$.
iv. For any $n \geq 2$, $\gamma_{cor}(K_{1,n}) = 1$.
v. For any wheel graph $W_n(n \geq 3)$, $\gamma_{cor}(W_n) = 1$.

## 2. Results on Corporate Domination

In this section, we establish $\gamma_{cor}(C_{4k} \square P_n)$ with illustrations.

**Theorem 2.1** Let $C_{4k}(k \geq 2)$ be any cycle and $P_{2n+1}(n \geq 1)$ be any path. Then

$$\gamma_{cor}(C_{4k} \square P_{2n+1}) = \left\lfloor \frac{2n+1}{2} \right\rfloor.$$

**Proof:** Let $C_{4k}(k \geq 2)$ be any cycle and $P_{2n+1}(n \geq 1)$ be any path and let $m = 4k$. Consider the following cases.

**Case 1:** Let $n \equiv 0(\text{mod}2)$.

For $0 \leq s \leq 2n$ and $2 \leq t \leq 2(n-1)$ where $s \equiv 0(\text{mod}4)$ and $t \equiv 2(\text{mod}4)$ and $0 \leq i \leq \frac{m}{4}-1$ and $1 \leq j \leq \frac{m}{4}-1$, let $C = \{v_{sm+4i+2}, v_{sm+4i+3}, v_{tm+1}, v_{tm+j} \}$, $V_{tm+4j} \{v_{tm+4j+1}\}$.

Here, $P = \{v_{sm+4i+2}, v_{sm+4i+3}, v_{tm+1}, v_{tm+j), v_{tm+4j}, v_{tm+4j+1}\}$.

Figure 4. $\gamma_{cor}(G)$ does not exist.

To prove $C$ is minimum, let $C'$ be any other corporate dominating set and $P', Q'$ be the sets corresponding to $C'$ such that every vertex in $(P' \cup Q')^c$ is adjacent to exactly one vertex in $P' \cup Q'$. Furthermore, the set $C'$ will be in one of the following forms.

i. $C' = V_1$
ii. $C' = E_1$
iii. $C' = V_1 \cup E_1^c$

If $C' = V_1$, then $P' = \varnothing$ and $Q' \neq \varnothing$. Since for any $u \in V_1 \setminus N(u) \cap N(w) = \varnothing$ for some $w \in V_1$, which is a contradiction.

If $C' = E_1$ holds, then $P' \neq \varnothing$ and $Q' = \varnothing$. Let $|P'| \geq |P|$ and $|P'| \leq m \left\lceil \frac{2n+1}{3} \right\rceil$. Thus $|C'| \leq m \left\lceil \frac{2n+1}{3} \right\rceil$ and hence $|C'| \geq |C|$.

If $C' = V_1 \cup E_1^c$ holds, then $P' \neq \varnothing$ and $Q' \neq \varnothing$. Suppose $|P' \cup Q'| < |P \cup Q|$. Then there exists at least one vertex $u \in (P' \cup Q')^c$ which is not adjacent to any one of the vertices in $P' \cup Q'$. This is a contradiction. Hence $|P' \cup Q'| \geq |P \cup Q|$. Therefore

a. Let $|P'| \leq |P|$ and $|Q'| > |Q|$ with $4 \leq |P'| \leq \left(\frac{m}{2}\right) \left\lceil \frac{2n+1}{2} \right\rceil$ and $2 \leq |Q'| \leq \left(\frac{2n+1}{3}\right) \left\lceil \frac{2n+1}{3} \right\rceil$. Then $E'$ contains at most $\left(\frac{m}{2} - 3\right) \left\lceil \frac{2n+1}{3} \right\rceil$ edges and $V'$ contains at most $2 \left\lceil \frac{2n+1}{3} \right\rceil$ vertices.

Hence $|C'| \leq \left(\frac{m}{2} - 3\right) \left\lceil \frac{2n+1}{3} \right\rceil + 2 \left\lceil \frac{2n+1}{3} \right\rceil$.

b. Suppose $|P'| > |P|$ and $|Q'| > |Q|$ with $\left(\frac{m}{2}\right) \left\lceil \frac{2n+1}{3} \right\rceil + 1 \leq |P'| \leq m \left\lceil \frac{2n+1}{3} \right\rceil - 1$ and $1 \leq |Q'| \leq \left(\frac{2n+1}{3}\right) \left\lceil \frac{2n+1}{3} \right\rceil$. Then

Hence $|C'| \leq m \left\lceil \frac{2n+1}{3} \right\rceil - 2 + 2 \left\lceil \frac{2n+1}{3} \right\rceil = (m + 2) \left\lceil \frac{2n+1}{3} \right\rceil - 2$. Thus, $|C'| \geq |C|$.

**Case 2:** Let $n \equiv 1(\text{mod}2)$ and $m = 4k$.

For $0 \leq s \leq 2(n - 1)$ and $2 \leq t \leq 2n$ where $s \equiv 0(\text{mod}4)$ and $t \equiv 2(\text{mod}4)$ and $0 \leq i \leq \frac{m}{4}-1$ and $1 \leq j \leq \frac{m}{4}-1$, let

$C = \{v_{sm+4i+2}, v_{sm+4i+3}, v_{tm+1}, v_{tm+j} \}$.

Here, $P = \{v_{sm+4i+2}, v_{sm+4i+3}, v_{tm+1}, v_{tm+j} \}$.
$Q = \varnothing$.

Since for any $v \in (P \cup Q)^c, N(v) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, $C$ is the corporate dominating set. Proceed as in Case 1, $C$ is the minimum corporate dominating set and $|C| = k \left[ \frac{2n+1}{2} \right]$.

In particular, for $k = 1$, $\gamma_{cor}(C_4 \square P_{2n+1}) = \left[ \frac{2n+1}{2} \right]$. As the proof for proving minimum in Theorem 2.1 is different from the proof of Theorem 2.2, we provide a separate theorem for $k = 1$.

**Theorem 2.2** Let $C_4$ be a cycle and $P_{2n+1}$ be any path. Then $\gamma_{cor}(C_4 \square P_{2n+1}) = \left[ \frac{2n+1}{2} \right]$.

**Proof:** Let $C_4$ be a cycle and $P_{2n+1}$ be any path.

For $0 \leq i \leq \frac{n}{2}$ and $0 \leq j \leq \frac{n}{2} - 1$, let $C = \{v_{16i+2}, v_{16j+3}, v_{16i+9}v_{16j+12}\}$.

By using theorem 2.1, $|C| = \left[ \frac{2n+1}{2} \right]$.

To prove $C$ is minimum, let $C'$ be any other corporate dominating set and $P', Q'$ be the sets corresponding to $C'$ such that every vertex in $(P' \cup Q')$ is adjacent to exactly one vertex in $P' \cup Q'$. Furthermore, the set $C'$ will be in one of the following forms.

- $C' = V'_1$ holds, then $P' = \varnothing$ and $Q' \neq \varnothing$. Since for any $u \in V'_1$, $|N(u) \cap N(w)| = \varnothing$ for some $w \in V'_1$ which is a contradiction.
- $C' = E'_1$ holds, then $P' \neq \varnothing$ and $Q' = \varnothing$. Since for any $v \in P'$, $|N(v) \cap N(w)| = \varnothing$ for some $w \in P'$ which is a contradiction.
- $C' = V'_1 \cup E'_1$ holds, then $P' \neq \varnothing$ and $Q' \neq \varnothing$. If $|P'| > |P \cup Q|$, then $|N(v_i) \cap (P' \cup Q')| > 1$ for some $v_i \in (P' \cup Q')^c$, a contradiction. Let $|P'| < |P \cup Q|$. Then there exists at least one vertex $u \in (P' \cup Q')^c$ which is not adjacent to any one of the vertices in $P' \cup Q'$. This is a contradiction. Hence $|P'| = |P \cup Q|$.

Let $|P'| < |P|$ and $|Q'| > |Q|$ with $2 \leq |P'| \leq 2n$ and $2 \leq |Q'| \leq 2 \left[ \frac{2n+1}{3} \right]$. Then $E'_1$ contains at most $\frac{2n}{3}$ edges and $V'_1$ contains at most $\frac{2n+1}{3}$ vertices. Hence $|C'| \leq \frac{2n}{3} + 2 \left[ \frac{2n+1}{3} \right]$. Thus $|C'| \geq |C|$.

Suppose $|P'| \geq |P|$ and $|Q'| > |Q|$. This is impossible, since $|P' \cup Q'| = |P \cup Q|$.

Hence $C$ is the minimum corporate dominating set and $\gamma_{cor}(C_4 \square P_{2n+1}) = \left[ \frac{2n+1}{2} \right]$.

**Illustration 2.3**

In Figure 5, let $C = \{v_1, v_3, v_5, v_7, v_9, v_{17}, v_{19}, v_{25}, v_{28}\}$. Since for any $u \in (P \cup Q)^c, N(u) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, $C$ is the corporate dominating set and $\gamma_{cor}(C_4 \square P_7) = 4$.

**Theorem 2.4** Let $C_4k$ be a cycle and $P_{2n}$ be any path. Then $\gamma_{cor}(C_4k \square P_{2n}) = \left\{ \begin{array}{ll} \frac{4kn}{3} & \text{if } k \equiv 0 \pmod{3} \\ \frac{2kn}{3} & \text{otherwise} \end{array} \right.$

**Proof.**

Let $C_4k$ be a cycle and $P_{2n}$ be any path. Let $m=4k$. Consider the following cases.

**Case 1:** Let $k \equiv 0 \pmod{3}$.

For $2 \leq i \leq m - 1$ where $i \equiv 2 \pmod{3}$, let $C = \{v_i, v_{m+i}, v_{2m+i}, v_{3m+i}, \ldots, v_{mn-4m+i}, v_{mn-3m+i}, v_{mn-2m+i}, v_{mn-m+i}\}$ and $Q = \varnothing$.

Since for any $w \in (P \cup Q)^c, N(u) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, $C$ is the corporate dominating set. Since $|Q| = 0$ and every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c, |P| = |P \cup Q| = (m^3 - 1)2n = \frac{2mn}{3}$.

Hence $C$ contains $\frac{mn}{3}$ edges. Thus, $|C| = \frac{mn}{3} = \frac{4km}{3}$.

**Case 2:** Let $k \equiv 1 \pmod{3}$.

We shall prove that $C$ is the minimum. Let $C'$ be any other corporate dominating set and $P', Q'$ be the sets corresponding to $C'$ such that every vertex in $(P' \cup Q')$ is adjacent to exactly one vertex in $P' \cup Q'$. Furthermore, the set $C'$ will be in one of the following forms.

- $C' = V'_1$ holds, then $P' = \varnothing$ and $Q' \neq \varnothing$.
- $C' = E'_1$ holds, then $P' \neq \varnothing$ and $Q' = \varnothing$.
- $C' = V'_1 \cup E'_1$ holds, then $P' \neq \varnothing$ and $Q' \neq \varnothing$.

If $C' = V'_1$ holds, then $P' = \varnothing$ and $Q' \neq \varnothing$. This happens only if $n = 1$.

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**Figure 5.** $C_4 \square P_7$
Clearly, |C'| = $\frac{4k}{3}$. Hence |C'| ≥ |C|.

If $C' = E'_1$ holds, then $P' \neq \varphi$ and $Q' = \varphi$. Let |P'| ≥ |P| with |P'| ≤ $\frac{2n}{3}$ and |Q'| = 0. Thus |C'| ≤ $\frac{2n}{3}$ and hence |C'| ≥ |C|.

If $C' = V'_1 \cup E'_1$ holds, then $P' \neq \varphi$ and $Q' \neq \varphi$. Suppose |P' ∪ Q'| < $\frac{2mn}{3}$. Then there exists at least one vertex $v_i \in (P' ∪ Q')^c$ which is not adjacent to any one of the vertices in $P' ∪ Q'$. This is a contradiction.

Hence |P' ∪ Q'| ≥ |P| and |Q| = (m $\frac{2mn}{3}$).

a) Let |P'| < |P| and |Q'| > |Q| with $\frac{2mn - 2m}{3}$ ≤ |P'| ≤ $\frac{2m}{3}$ and |V'_1| ≤ $\frac{2m}{3}$.

Hence |C'| ≤ $\frac{2mn - 2m}{3}$.

b) Suppose |P'| ≥ |P| and |Q'| > |Q| with $\frac{2mn}{3}$ ≤ |P'| ≤ m $\frac{2n}{3}$ - 1 and 1 ≤ |Q'| ≤ $\frac{2n}{3}$. Then |C'| ≤ m $\frac{2n}{3}$ - 1 + $\frac{2mn}{3}$.

Hence |C'| ≥ |C|.

Case 2: Let k ≠ 0(mod3) and let m = 4k.

Subcase 2.1: Let 2n ≡ 0(mod3).

For 1 ≤ t ≤ 2n - 2 where t ≡ 1(mod3), let C = $\{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{t+1)m-1}v_{(t+1)m}\}$.

Here P = $\{v_{tm+1}, v_{tm+2}, v_{tm+3}, v_{tm+4}, \ldots, v_{(t+1)m-1}, v_{(t+1)m}\}$ and Q = $\varphi$.

Since every vertex not in P ∪ Q is adjacent to exactly one vertex in P ∪ Q, C is the corporate dominating set.

As every vertex in P ∪ Q is adjacent to exactly two vertices in (P ∪ Q)^c, |P ∪ Q| = m $\frac{2n}{3}$ + 1 = m $\frac{2n}{3}$ + $\frac{2n}{3}$). Hence C contains m $\frac{2n}{3}$ edges.

To prove C is minimum, let C' be any other corporate dominating set and P', Q be the sets corresponding to Q such that |N(u) ∩ (P' ∪ Q')| = 1, ∀ u ∈ (P' ∪ Q')^c. Furthermore, the set C' will be in one of the following forms.

i). C' = V'_1

ii). C' = E'_1

iii). C' = V'_1 ∪ E'_1

If (i) holds, then P' = $\varphi$ and Q' ≠ $\varphi$. Since for any u ∈ V'_1, N(u) ∩ N(w) ≠ $\varphi$ for some w ∈ V'_1, which is a contradiction.

If (ii) holds, then P' ≠ $\varphi$ and Q' = $\varphi$. Let |P'| ≥ |P| with |P'| = $\frac{2mn}{3}$ - 1. Then |C'| ≤ $\frac{2mn}{3}$ and hence |C'| ≥ |C|.

If (iii) holds, then P' ≠ $\varphi$ and Q' ≠ $\varphi$. If |P' ∪ Q'| > |P ∪ Q|, then N(v_i) ∩ (P' ∪ Q') > 1 for some v_i ∈ (P' ∪ Q')^c, a contradiction. Suppose |P' ∪ Q'| < |P ∪ Q|. Then there exists at least one vertex v_i ∈ (P' ∪ Q')^c which is not adjacent to any one of the vertices in P' ∪ Q' which is a contradiction. Hence |P' ∪ Q'| = |P ∪ Q| = $\frac{2mn}{3}$.

a) Let |P'| < |P| and |Q'| > |Q| with (m - 2) $\left(\frac{2n}{3}\right)$ ≤ |P'| ≤ $\frac{2mn - 1}{3}$ and 1 ≤ |Q'| ≤ $\frac{2n}{3}$. Then C contains at most $\frac{2mn}{3} - 2$ and 4 $\left(\frac{2n}{3}\right)$ vertices. Hence |C'| ≤ (m + 2)$\left(\frac{2n}{3}\right)$ - 2. Thus |C'| ≥ |C|.

b) Suppose |P'| ≥ |P| and |Q'| > |Q|. This is impossible, since |P' ∪ Q'| = |P ∪ Q|.

Subcase 2.2: Let 2n ≡ 1(mod3).

For 0 ≤ t ≤ 2n - 1 where t ≡ 1(mod3), let C = $\{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-1}v_{(t+1)m}\}$.

Proceeding the similar argument which is used in the Subcase 2.1, we can prove that C is the corporate dominating set and |P ∪ Q| = $\left(\frac{2n}{3} - 1\right)m$ = m $\frac{2n}{3}$.

Hence |C| = $\left(\frac{m}{2}\right)\left(\frac{2n}{3}\right) + 2k\left(\frac{2n}{3}\right)$. We shall prove that C is minimum.

Replace (m $\frac{2n}{3}$) by (m $\frac{2n}{3}$) in subcase 2.1, |C'| ≥ |C|.

Subcase 2.3: Let 2n ≡ 2(mod3).

For 0 ≤ t ≤ 2n - 2 where t ≡ 2(mod3), let C = $\{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-1}v_{(t+1)m}\}$.

By the similar argument which is used in Subcase 2.2, we can prove that C is the corporate dominating set and |C'| ≥ |C|.

From all the above cases, $\gamma_{cor}(C_k \sqcap P_{2n}) = 2k\left(\frac{2n}{3}\right)$.

Illustration 2.5.

In the Figure 6, $k = 2(mod3)$ and $2n = 1(mod3)$.

C = $\{v_1v_2, v_2v_3v_4, v_5v_6, v_7v_8, v_9v_{10}, v_{15}v_{16}, v_{17}v_{18}, v_{19}v_{20}\}$ and $\gamma_{cor}(C_k \sqcap P_4) = 8$.  

Figure 6. $C_6 \sqcap P_4$
Proposition 2.6. Let $C_4$ be a cycle and $P_{2n}$ $(n \geq 1)$ be any path. Then $\gamma_{cor} (C_4 \square P_{2n}) = 3 \left(\frac{2n}{3}\right)$.

Proof. Let $C_4$ be a cycle and $P_{2n}$ $(n \geq 1)$ be any path.

Consider the following cases.

Case 1: Let $2n \equiv 0 \left(\text{mod } 3\right)$.

For $1 \leq t \leq 2n - 2$ where $t \equiv 1 \left(\text{mod } 3\right)$, let $C = \{v_4t+1,v_4t+2,v_4t+3,v_4t+4\}$. Clearly, $\left|C\right| = 2 \left(\frac{2n-3}{3} + 1\right) = 2 \left(\frac{2n}{3}\right)$.

Since every vertex not in $P \cup Q$ is adjacent to exactly one vertex in $P \cup Q$, $C$ is the corporate dominating set.

As every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P \cup Q| = 2 \left(\frac{2n-3}{3} + 1\right) + 2 \left(\frac{2n}{3}\right) = 4 \left(\frac{2n}{3}\right)$ and $|C| = \left(\frac{2n}{3}\right) + 4 \left(\frac{n}{3}\right)$. Hence $C$ contains $6 \left(\frac{n}{3}\right) (= 2n)$ edges.

We shall prove that $C$ is the minimum. Let $C'$ be any other corporate dominating set. Proceed as in Theorem 2.2. $C' = V'_t$ and $C' = E'_t$ are impossible. If $C' = V'_t \cup E'_t$ holds, then $P' \neq \phi$ and $Q' \neq \phi$. We have to prove that $|P'| = |P|$ and $|Q'| = |Q|$. If not, then $|N(v_i) \cap (P' \cup Q')| > 1$ for some $v_i \in (P' \cup Q')^c$ or there exists at least one vertex $u \in (P' \cup Q')^c$ which is not adjacent to any one of the vertices in $P' \cup Q'$. This is a contradiction. Hence $|C'| = |C| = 3 \left(\frac{2n}{3}\right)$ and $C$ is the minimum corporate dominating set.

Case 2: Let $2n \equiv 1 \left(\text{mod } 3\right)$.

For $0 \leq t \leq 2n - 1$ where $t \equiv 0 \left(\text{mod } 3\right)$, let $C = \{v_{4t+1},v_{4t+2},v_{4t+3},v_{4t+4}\}$.

As every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P \cup Q| = 2 \left(\frac{2n-1}{3} + 1\right) + 2 \left(\frac{2n-1}{3} + 1\right) = 4 \left(\frac{2n}{3}\right)$. Proceed as in Case 1, $C$ is the corporate dominating set and $|C| = 3 \left(\frac{2n}{3}\right)$. We shall prove that $C$ is the minimum. Replace $\left(\frac{2n}{3}\right)$ in Case 1, $C$ is the minimum.

Case 3: Let $2n \equiv 2 \left(\text{mod } 3\right)$.

For $0 \leq t \leq 2n - 2$ where $t \equiv 0 \left(\text{mod } 3\right)$, let $C = \{v_{4t+1},v_{4t+2},v_{4t+3},v_{4t+4}\}$.

By the similar argument in Case 2, $C$ is the minimum corporate dominating set and $|C| = 3 \left(\frac{2n}{3}\right)$. From all the above cases, $\gamma_{cor} (C_4 \square P_{2n}) = 3 \left(\frac{2n}{3}\right)$.

3. Conclusions

We have studied the new domination parameter namely, corporate domination, and explained the concepts with illustrations. The corporate domination number for some standard classes of graphs such as Path, Cycle, Wheel, Star, and Complete graphs has been computed. Also, we have found out that the exact value of the corporate domination number for the Cartesian product of $C_m$ and $P_n$ where $k \geq 1$ and $n \geq 2$ and illustrated the same with examples. Due to the minimum cardinality of the corporate dominating set, it can be beneficial when applied practically than perfect and perfect ev-domination. Further the general case for getting the corporate domination number for $C_m \square P_n$ where $m \neq 4k$ and $n \geq 2$, $m \equiv n \left(\text{mod } 3\right)$, and $P_m \square P_n$ ($m, n \geq 2$) can be investigated.

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