Lifshitz black holes in the Hořava-Lifshitz gravity

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Abstract

We investigate the Lifshitz black holes from the Hořava-Lifshitz gravity by comparing with the Lifshitz black hole from the 3D new massive gravity. We note that these solutions all have single horizons. These black holes are very similar to each other when studying their thermodynamics. It is shown that a second order phase transition is unlikely possible to occur between $z = 3, 2$ Lifshitz black holes and $z = 1$ Hořava black hole.

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1 Introduction

Horava has proposed a renormalizable theory of gravity at a Lifshitz point \[1, 2\], which may be regarded as a UV complete candidate for general relativity. At short distances the theory of Hořava-Lifshitz (HL) gravity with a flow parameter \( \lambda \) describes interacting non-relativistic gravitons and is supposed to be power counting renormalizable in (1+3) dimensions. Recently, its black hole solutions have been intensively investigated in \[3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17\].

There are two classes of Hořava-Lifshitz gravity in the literature: the projectable and nonprojectable theories where the former (latter) implies that the lapse function depends on time (time and space). A main issue of the Hořava-Lifshitz gravity is still to answer to the question of whether it can accommodate the Hořava scalar \( \psi \), in addition to two degrees of freedom (DOF) for a massless graviton. To this end, we would like to mention relevant works. The authors \[19\] have shown that in the nonprojectable theories, the Hořava scalar \( \psi \) is related to a scalar degree of freedom appeared in the massless limit of a massive graviton. Especially for the Hamiltonian approach to the HL gravity, the authors \[20\] did not consider the Hamiltonian constraint as a second class constraint, which leads to a strange result that there are no DOF left when imposing the constraints of the theory. Moreover, the authors \[21\] have claimed that there are no solution of the lapse function which satisfies the constraints. Unfortunately, it implies a surprising conclusion that there is no evolution at all for any observable. However, more recently, it was shown that the IR version of HL gravity (\( \lambda R \)-model) is completely equivalent to the general relativity for any \( \lambda \) when employing a consistent Hamiltonian formalism based on Dirac algorithm \[22\].

In the projectable theories, the authors \[23, 24\] have argued that \( \psi \) is propagating around the Minkowski space but it has a negative kinetic term, showing a ghost instability. In this case, the Hořava scalar becomes ghost if the sound speed square \( (c_\psi^2) \) is positive. In order to make this scalar healthy, the sound speed square must be negative, but it is inevitably unstable. Thus, one way to avoid this is to choose the case that the sound speed square is close to zero, which implies the limit of \( \lambda \to 1 \). However, in this limit, the cubic interactions are important at very low energies, called the strong coupled problem \[25\]. This invalidates any linearized analysis and any predictability of quantum gravity is lost due to unsuppressed loop corrections. This casts serious doubts on the UV completeness of the theory. The authors \[26\] tried to extend the theory to make a healthy HL gravity, but there has been some debate as to whether this theory is really healthy \[27, 28, 29\]. The projectability condition from condensed matter physics may not be appropriate for describing the (quantum) gravity.
Instead, if one does not impose the projectability condition, the HL gravity leads to general relativity without the strong coupling problem in the IR limit.

On the other hand, the Lifshitz-type black holes [30, 31, 32, 33, 34, 35, 36] have received considerable attentions since these may provide a model of generalizing AdS/CFT correspondence to non-relativistic condensed matter physics [37, 38, 39]. However, even though their asymptotic spacetimes are apparently simple as Lifshitz, the problem of obtaining an analytic solution seems to be a nontrivial task. Some examples include a 4D topological black hole which is asymptotically Lifshitz with the dynamical exponent \( z = 2 \) [40]. An analytic black hole solution with \( z = 2 \) that asymptotes a planer Lifshitz spacetime was found in 4D spacetimes [41], and numerical solutions were also explored in [42, 43]. The analytic examples of Lifshitz black holes in higher dimensions were reported in [44]. Interestingly, the \( z = 3 \) Lifshitz black hole [45] was derived from the new massive gravity (NMG) in 3D spacetimes [46]. It was claimed that there is a subtle issue to define thermodynamic quantities of this black hole because of negative mass and entropy [36]. However, a thermodynamic study for the \( z = 3 \) Lifshitz black hole was performed by using the 2D dilaton gravity approach [47], showing that its thermodynamics is rather simple and is consistently defined. Also, a boundary stress-tensor approach to this black hole has confirmed that the wrong (negative)-sign Einstein-Hilbert term provides really a consistent thermodynamics of the \( z = 3 \) Lifshitz black hole [48].

Concerning a static spherically symmetric solution, Lü-Mei-Pope (LMP) have obtained the black hole solution with \( \lambda \) [3] and topological black holes were found in [4]. We remind the reader that these black hole solutions were obtained from the Hořava-Lifshitz gravity without imposing the projectability condition (nonprojectable theories). Within the projectable theories, their black hole solutions are less interesting [49]. Its thermodynamics was studied in [7, 8], but there remain unclear issues in obtaining the ADM mass and the entropy because for \( 1/3 \leq \lambda \leq 1/2 \), the LMP solution belongs to Lifshitz black holes with \( 2 \leq z \leq 4 \). In this case, the entropy may take a very unusual form as \( S = A/4 - (\pi/\Lambda_W) \ln[A/4] \) with \( A \) the area of horizon [14]. It was well known that many different kinds of black holes from string theories have the Bekenstein-Hawking entropy of \( S_{BH} = A/4 \) [50]. Thus, one has to explain why a logarithmic term \( -\pi/\Lambda_W \ln[A/4] \) appears as a part of the entropy of Lifshitz black hole in the HL gravity [51, 52]. This term arises because one has used the first law of \( dS = dm/T_H \) to derive the entropy, provided that the Hawking temperature \( T_H \) and the mass \( m \) have been known. Indeed, the mass \( m \) was not clearly defined by either the condition of the zero metric function \( f = 0 \) [7] or a Hamiltonian approach [8]. Until now,
there is no definite way to calculate the Arnowitt-Deser-Misner (ADM) mass $M_{\text{ADM}}$ for the Lifshitz black hole, if one insists that the ADM mass should be evaluated at asymptotic Lifshitz. Hence, it would be better to use the Bekenstein-Hawking entropy to derive the horizon mass for Lifshitz black holes when applying the first law of $dM_h = T_HdS_{BH}$ [53, 43].

In this work, we obtain the horizon mass of the Lifshitz black holes in the nonprojectable HL gravity. In deriving this mass, we use the first law of thermodynamics and the Bekenstein-Hawking entropy. We investigate thermodynamics of $z = 3, 2$ Lifshitz black holes in the nonprojectable Hořava-Lifshitz gravity by comparing with the $z = 3$ Lifshitz black hole in the NMG. Finally, we discuss a second order phase transition between the $z = 3, 2$ Lifshitz black holes and the $z = 1$ Hořava black hole.

2 HL gravity

Introducing the ADM formalism where the metric is parameterized

$$ds_{\text{ADM}}^2 = -N^2dt^2 + g_{ij}(dx^i - N_i dt)(dx^j - N_j dt),$$  \hspace{1cm} (1)

the Einstein-Hilbert action can be expressed as

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{gN} \left[ K_{ij}K^{ij} - K^2 + R - 2\Lambda \right],$$  \hspace{1cm} (2)

where $G$ is Newton’s constant and extrinsic curvature $K_{ij}$ takes the form

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_f N_i \right).$$  \hspace{1cm} (3)

Here, a dot denotes a derivative with respect to $t$. The $Z = 3$ HL action of a non-relativistic gravitational theory is given by [1]

$$S_{\text{HL}} = \int dt d^3x \left[ \mathcal{L}_K + \mathcal{L}_V \right],$$  \hspace{1cm} (4)

where the kinetic Lagrangian is given by

$$\mathcal{L}_K = \frac{2}{\kappa^2} \sqrt{gN} \left( K_{ij}K^{ij} - \lambda K^2 \right).$$  \hspace{1cm} (5)

The potential Lagrangian is determined by the detailed balance condition as

$$\mathcal{L}_V = \sqrt{gN} \left[ \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1}{4} R^2 + \Lambda_\infty R - 3\Lambda_\infty^2 \right) - \frac{\kappa^2}{2w^4} \left( C_{ij} - \frac{\mu_0^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu_0^2}{2} R^{ij} \right) \right].$$  \hspace{1cm} (6)
In the IR limit, comparing (4) with (2) of general relativity, the speed of light, Newton’s constant and the cosmological constant are given by

$$c = \frac{\kappa^2 \mu}{4 \sqrt{\Lambda_W}} , \quad G = \frac{\kappa^2}{32\pi c} , \quad \Lambda_{cc} = \frac{3}{2} \Lambda_W .$$

The equations of motion were derived in [54] and [3]. In order to have a black hole solution, it requires that $\lambda > 1/3$ and $\Lambda_W < 0$ because the speed of light $c$ blows up at $\lambda = 1/3$.

### 3D Lifshitz black holes in the NMG

The NMG action [46] composed of the Einstein-Hilbert action with a cosmological constant $\Lambda$ and higher order curvature terms is given by

$$S_{NMG} = S_{EH} + S_{FH} ,$$

$$S_{EH} = -\frac{1}{16\pi G_3} \int d^3x \sqrt{-G} \left( \mathcal{R} - 2\Lambda \right) ,$$

$$S_{HC} = \frac{1}{16\pi G_3 u^2} \int d^3x \sqrt{-G} \left( \mathcal{R}_{MN}\mathcal{R}^{MN} - \frac{3}{8} \mathcal{R}^2 \right) ,$$

where $G_3$ is a 3D Newton constant and $u^2$ a parameter with mass dimension 2. We note that the wrong-sign appears in the Einstein-Hilbert term (EH) when comparing to the original action [46]. This means that we keep a relative ($-$) sign of higher order curvature term fixed with respect to the EH.

The field equation is given by

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} + \Lambda g_{MN} - \frac{1}{2u^2} K_{MN} = 0 ,$$

where

$$K_{MN} = 2\Box \mathcal{R}_{MN} - \frac{1}{2} \nabla_M \nabla_N \mathcal{R} - \frac{1}{2} \Box g_{MN}$$

$$+ 4\mathcal{R}_{MNPQ} \mathcal{R}^{PQ} - \frac{3}{2} \mathcal{R} \mathcal{R}_{MN} - \mathcal{R}_{PQ} \mathcal{R}^{PQ} g_{MN} + \frac{3}{8} \mathcal{R}^2 g_{MN} .$$

In order to have the Lifshitz black hole solution with dynamical exponent $z$, it is convenient to introduce dimensionless parameters

$$y = u^2 \ell^2 , \quad w = \Lambda \ell^2 ,$$

where $y$ and $w$ are proposed to take

$$y = -\frac{z^2 - 3z + 1}{2} , \quad w = -\frac{z^2 + z + 1}{2} .$$
For the $z = 1$ non-rotating BTZ black hole, one has $y = \frac{1}{2}$ and $w = -\frac{3}{2}$, while $y = -\frac{1}{2}$ and $w = -\frac{13}{2}$ are chosen for the $z = 3$ Lifshitz black hole. For $z = 1$ and $3$, the black hole solutions are given by

$$ds^2_{3D} = -x^2 F(r) dt^2 + \frac{1}{x^2} H(r) dr^2 + r^2 d\theta^2,$$

where

$$x = \frac{r}{\ell}, \quad F(r) = \frac{1}{H(r)} = 1 - \frac{\mathcal{M} \ell^2}{r^2} = 1 - \frac{r_+^2}{r^2}. \quad (16)$$

We emphasize that $\mathcal{M}$ is a mass parameter related to the horizon mass $M$. A naive condition of $F(r) = 0$ could not determine the horizon mass $M$ in the $z = 3$ Lifshitz black holes, as contrasts to $M = r_+^2/\ell^2$ for the $z = 1$ non-rotating BTZ black hole. The metric (15) implies that a curvature singularity appears at $r = 0$ as is shown [45]

$$R_{MN} R^{MN} = \frac{260}{\ell^4} - \frac{152 \mathcal{M}}{\ell^2 r^2} + \frac{24 \mathcal{M}^2}{r^4}, \quad (17)$$

and a single event horizon is located at $r = r_+ = \ell \sqrt{\mathcal{M}}$. For the $z = 3$ Lifshitz black hole, its thermodynamic quantities of Hawking temperature $T_H$, horizon mass $M \simeq \mathcal{M}^2$, heat capacity $C = \frac{dM}{dT_H}$, Bekenstein-Hawking entropy $S_{BH}$, and free energy $F = M - T_H S_{BH}$ are given by [17]

$$T_H = \frac{x_+^3}{2 \pi \ell}, \quad M = \frac{x_+^4}{2}, \quad C = 4 \pi x_+, \quad S_{BH} = 4 \pi r_+, \quad F = -\frac{3}{2} x_+^4 \quad (18)$$

with $x_+ = r_+ / \ell$. We check that the above quantities satisfy the first law of thermodynamics

$$dM = T_H dS_{BH}. \quad (19)$$

Now we are in a position to explain why we start with the wrong-sign EH term in (9). When making a replacement of $G_3 \rightarrow -G_3$ to go back the original NMG [46], the temperature is the same, but mass and entropy are negative [36]. Negative mass and entropy are not permissible for black hole physicists and thus, this problem should be resolved. One way to resolve it was to replace the Newton’s constant $G_3$ by $-G_3$, leading to our action (8). It was shown that this replacement is indeed necessary to regard the NMG as a unitary massive gravity [55]. The NMG is equivalent to the Fierz-Pauli massive gravity within the linearized theory. In three dimensions, a massless graviton has no DOF, while a massive graviton is a physically propagating mode with two helicities. In constructing the NMG with higher order curvature term, the important thing was that one can neglect a massless graviton whatever its norm is positive or negative, in favor of a massive graviton without ghost from [10] [56]. In this sense, our action (8) is a reliable one to derive the correct
thermodynamic quantities. Importantly, we note that the higher order curvature term \(10\) of the NMG is not a perturbative correction to the Einstein-Hilbert action, but the main term to obtain the \(z = 3\) Lifshitz black hole. Recently, a boundary stress-tensor approach to this black hole \([48]\) has confirmed that the wrong (negative) sign Einstein-Hilbert term provides really a thermodynamics of the \(z = 3\) Lifshitz black hole which is consistent with \([18]\). Especially, a proper definition for the mass of the \(z = 3\) Lifshitz black hole was introduced.

For \(z = 1\) and 3, their quantities could be rewritten by the compact forms

\[
T_H^z = \frac{x^z_+}{2\pi \ell}, \quad M^z = \frac{2}{1 + z} x_+^{z+1}, \quad C = 4\pi x_+, \quad S = 4\pi r_+, \quad F^z = -\frac{2z}{1 + z} x_+^{z+1}, \quad (20)
\]

where the heat capacity and Bekenstein-Hawking entropy remain unchanged.

Considering the free energy of \(F^z\) and the coordinate matching (see Fig. 1), there is a crossing point at \(x_+ = x_c = 0.82\) where \(F^{z=1}(x_+)\) is equal to \(F^{z=3}(x_+)\). This implies that for \(0 \leq x_+ \leq x_c\), \(F^{z=3} \geq F^{z=1}\), while for \(x_+ \geq x_c\), \(F^{z=3} \leq F^{z=1}\). That is, for \(0 \leq x_+ \leq x_c\), the \(z = 1\) non-rotating BTZ black hole is more favorable than the \(z = 3\) Lifshitz black hole, while for \(x_+ \geq x_c\), the \(z = 3\) Lifshitz black hole is more favorable than the \(z = 1\) non-rotating BTZ black hole. This may imply a second order phase transition between two black holes \([57]\). However, we note that two black holes have different asymptotes: Lifshitz and AdS\(_3\) spacetimes. Hence, the phase transition occurs unlikely between two black holes. In order to see it explicitly, we use the other called the temperature matching.

When expressing their free energy in terms of their Hawking temperatures, one has

\[
F^{z=3}(T_H) = -\frac{3}{2}(2\pi \ell T_H)^2, \quad F^{z=1}(T_H) = -(2\pi \ell T_H)^2, \quad (21)
\]

which shows that for \(0 \leq T_H \leq T_c\), \(F^{z=3}(T_H) \leq F^{z=1}(T_H)\), while for \(T_H \geq T_c\), \(F^{z=3}(T_H) \geq F^{z=1}(T_H)\) with \(T_c = \frac{1.82}{2\pi\ell}\). In Fig. 3, we use the notation of \(2\pi T_H \rightarrow T_H\) with \(\ell = 1\) for simplicity. This means that the temperature matching provides the result which is opposite to the coordinate matching.

Consequently, a second order phase transition is unlikely possible to occur between two black holes.

## 4 4D Lifshitz black holes in the nonprojectable HL gravity

A static spherically symmetric (SSS) solution to the nonprojectable HL gravity was obtained by considering the line element

\[
ds_{4D}^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (22)
\]
Figure 1: Graphs of 3D free energy $F^z(x_+)$ and $F^z(T_H)$. Left: two free energies (solid curve: $F^z=3$; dashed curve: $F^z=1$) with coordinate matching cross at $x_+ = x_c = 0.82$. Right: two free energies with temperature matching cross at $T_H = T_c = 1.82$. Two are opposite to each other around the crossing points.

Here the “nonprojectable” notion is clear because the lapse function $N$ depends on space coordinate $r$ only in the static solution of the black hole. For this purpose, choosing $K_{ij} = 0$ and $C_{ij} = 0$, the Lagrangian reduces to the $\mathcal{Z} = 2$ potential Lagrangian as

$$\mathcal{L}_V = \sqrt{g}N\frac{\kappa^2 \mu^2 (-\Lambda_W)}{8(3\lambda - 1)} \left[ R - 3\Lambda_W - \frac{4\lambda - 1}{4\Lambda_W} R^2 + \frac{(3\lambda - 1) R_{ij}^2}{\Lambda_W} \right]. \quad (23)$$

Substituting the metric ansatz (22) into $\mathcal{L}_V$, one has the reduced Lagrangian

$$\mathcal{L}^{\text{SSS}}_V = \frac{\kappa^2 \mu^2 (-\Lambda_W) N}{8(1 - 3\lambda)} \sqrt{f} \left[ - 2(1 - f - rf') + 3\Lambda_W r^2 \right. \quad (24)$$

$$\left. - \frac{(2\lambda - 1)(f - 1)^2}{\Lambda_W r^2} - \frac{\lambda - 1}{2\Lambda_W} f'^2 + \frac{2\lambda(f - 1)}{\Lambda_W r} f' \right],$$

where $'$ denotes the differentiation with respect to $r$. We note that $\mathcal{L}^{\text{SSS}}_V$ is not appropriately defined at $\lambda = 1/3$ because it blows up at this point. Hereafter, we exclude $\lambda = 1/3$ from our consideration. The Lü-Mei-Pope (LMP) solution for the HL gravity is given by

$$f(x) = 1 + x^2 - \alpha x^{p\pm(\lambda)}, \quad N(x) = x^{q\pm(\lambda)} \sqrt{f(x)}, \quad (25)$$

where $\alpha$ is an integration constant related to the horizon mass of the black hole and

$$x = \sqrt{-\Lambda_W r}, \quad p\pm(\lambda) = \frac{2\lambda \pm \sqrt{6\lambda - 2}}{\lambda - 1}, \quad q\pm(\lambda) = - \frac{1 + 3\lambda \pm 2\sqrt{6\lambda - 2}}{\lambda - 1}. \quad (26)$$

In this work, we choose $p\pm(\lambda) = p$ and $q\pm(\lambda) = q$ and thus, $2p + q = 1$. In this case, it was shown that for $p < 2$, the LMP solution is singular at $r = 0$ [58] as

$$R = 6\Lambda_W + \frac{2\alpha(1 + p)x^p}{r^2}. \quad (27)$$
Here we note that for $p = -1(\lambda = 1/3)$, 3D Ricci scalar $R$ is regular at $r = 0$, which implies that the $\lambda = 1/3$ does not correspond to a Lifshitz black hole. Its extremal black hole with $f(x_e) = 0$ and $f'(x_e) = 0$ are located as

$$x_e = 0, \text{ for } \frac{1}{3} < \lambda \leq \frac{1}{2}; \quad x_e = \sqrt{\frac{p}{2 - p}} = \sqrt{\frac{2\lambda - \sqrt{6\lambda - 2}}{-2 + \sqrt{6\lambda - 2}}}, \text{ for } \lambda > \frac{1}{2}. \quad (28)$$

However, assuming the near-horizon geometry of AdS$_2 \times S^2$, the radius $v_2$ of $S^2$ is negative for $1/3 < \lambda \leq 1/2$, which means that the near-horizon geometry of extremal black hole is ill-defined and the corresponding Bekenstein-Hawking entropy is zero [16]. For $\lambda > 1/2$, the near-horizon geometry of extremal black holes are AdS$_2 \times S^2$ with different radii, depending on the HL gravity. This shows clearly that for $1/3 < \lambda \leq 1/2$, the horizon at $x_e = 0$ is a single horizon but not a degenerate horizon. Actually, these correspond to the Lifshitz black holes with $2 \leq z < 4$ [7].

In order to understand this branch of the LMP black holes fully, it is necessary to introduce 4D Lifshitz black holes. Their line element takes the form with dynamical exponent \( z [31, 32, 40, 45] \)

$$ds^2_{Lif} = -x^{2z}F(r)dt^2 + \frac{1}{x^2}H(r)dr^2 + r^2d\Omega^2_2, \quad (29)$$

where $F(r)$ and $H(r)$ are functions of a radial coordinate $r$ with

$$\lim_{r \to \infty} F(r) = \lim_{r \to \infty} H(r) = 1. \quad (30)$$

Comparing the LMP black holes (22) with the Lifshitz black holes (29) leads to the correspondence

$$N^2 = \tilde{N}^2 f = x^{2(q+1)}f \rightarrow x^{2z}F(r), \quad f \rightarrow \frac{x^2}{H(r)} \quad (31)$$

which imply the two relations

$$z = q + 1, \quad F(r) = \frac{1}{H(r)} = \frac{f}{x^2}. \quad (32)$$

This indicates how the 4D Lifshitz black holes originate from a non-relativistic theory of the HL gravity. We note that the $z = 3, 2$ Lifshitz black holes are obtained from either $Z=2$ or 3 HL action, which means that a relevant quantity to determine the exponent $z$ is not the dynamical scaling dimension $Z$ but the flow parameter $\lambda$ through $q$ ($\lambda = 0.36 \rightarrow z = 3, \lambda = 1/2 \rightarrow z = 2$). As far as the LMP solution is concerned, there is no distinction between $Z = 2$ and $Z = 3$ HL gravities because of $C_{ij} = 0$. It seems that this is a feature of the HL gravity, compared with the NMG.
Figure 2: Graphs of Hawking temperature $T_H^z(x_+)$ and horizon mass $M^z_h(x_+)$ with $\Lambda_W = -1(x_+ = r_+)$. Left: two dashed curves represent the Hawking temperatures for $z = 3, 2$ from top to bottom and the solid curve denotes the $z = 1$ Hawking temperature which is positive for $x_+ > 1/\sqrt{3}$. Right: two dashed curves represent the horizon masses for $z = 3, 2$ from top to bottom and the solid curve denotes the $z = 1$ horizon mass which is positive only for $x_+ > 1$.

5 Thermodynamics of 4D Lifshitz black holes

In order to explore properties of the 4D Lifshitz back holes, let us first study the Hawking temperature because it is derived from the the surface gravity $\kappa$ defined at the horizon and thus, is really independent of the mass parameter $\alpha$. The Hawking temperature is defined by

$$T_H^z(x_+) = \frac{\kappa}{2\pi} = \sqrt{-\Lambda_W} \frac{1}{8\pi} \left[ (z+2)x^z_+ + (z-2)x^{z-2}_+ \right],$$

(33)

where we use the relations of $q = z + 1$ and $p = \frac{2-z}{2}$ in deriving the last expression. As is shown in Fig. 1, for $2 \leq z < 4$, it is a monotonically increasing function like $T \simeq x^z_+$ for large $x_+$. In order to find the horizon mass, we may use the first law of thermodynamics

$$dM_h = T_H dS_{BH},$$

(34)

where the Bekenstein-Hawking entropy $^2$ satisfies the area-law of

$$S_{BH} = \pi r_+^2.$$  

$^2$ The other entropy of $S = \pi r_+^2 - \frac{\Lambda_W}{8\pi} \ln[r^2_+]$ could be obtained from the first law of $dS = \frac{4m}{\kappa}$ provided the mass of $m \simeq \alpha^2$ and the temperature were known. However, it is hard to accept this entropy because one could not have a logarithmic term unless either thermal correction or quantum correction is considered. In the IL gravity, higher order curvature term plays an essential role to make the Lifshitz black hole. This is not a correction to the wrong-sign EH term. Hence, the Bekenstein-Hawking entropy is more natural to derive the horizon mass.
Figure 3: Graphs of heat capacity $C^z(x_+)$ and free energy $F^z(x_+)$ with $\Lambda_W = -1(x_+ = r_+)$. Left: two dashed curves represent the heat capacity for $z = 2, 3$ from top to bottom and the solid curve denotes the $z = 1$ heat capacity which is negative for $x_+ < 1/\sqrt{3}$. Right: two dashed curves of $z = 2, 3$ from top to bottom cross the solid curve of the $z = 1$ free energy at $x_+ = x_c = 1, 0.84$, respectively which are greater than $x_+ = x_e = 1/\sqrt{3}$.

Then, the horizon mass is obtained as

$$M_h^z(x_+) = \int_0^{x_+} T_H dS_{BH} = \frac{x_+^2}{4\sqrt{-\Lambda_W}} \left[ x_+^2 + \frac{z - 2}{z} \right]. \quad (36)$$

For $z = 3, 2$, the horizon masses are given by, respectively,

$$M_h^{z=3} = \frac{x_+^3}{4\sqrt{-\Lambda_W}} \left[ x_+^2 + \frac{1}{3} \right], \quad M_h^{z=2} = \frac{x_+^4}{4\sqrt{-\Lambda_W}}. \quad (37)$$

We could not determine the $z = 0(\lambda = 3)$ horizon mass because the last term in (36) blows up at $z = 0$.

As is depicted in Fig. 2, the horizon mass is a monotonically increasing function like $M_h^z \simeq x_+^{z+2}$ for $2 \leq z < 4$ and large $x_+$. We note that the horizon mass is different from the Komar charge defined as

$$M_H^z = 2T_H S_{BH} = \frac{x_+^z}{4\sqrt{-\Lambda_W}} \left[ (z + 2)x_+^2 + (z - 2) \right]. \quad (38)$$

The heat capacity is an important quantity to test the thermal stability: for $C > 0$ the system is stable against thermal perturbations, while for $C < 0$, it is unstable thermodynamically. This is defined by

$$C^z(x_+) = \frac{dM_h^z}{dT_H^z} = 2S_{BH} \left[ \frac{(z + 2)x_+^{z-1} + (z - 2)x_+^{z-3}}{z(z + 2)x_+^{z-1} + (z - 2)^2 x_+^{z-3}} \right]. \quad (39)$$

We observe from Fig. 3 that the heat capacity is a monotonically increasing function like $C \simeq S_{BH} \simeq x_+^2$ for $2 \leq z < 4$ and large $x_+$, which means that all Lifshitz black holes are
thermodynamically stable because of $C \geq 0$. The free energy is necessary to study a phase transition to other configuration. The free energy is defined by

$$F = M_h^z - T_H^z S_{BH} = - \frac{x_+^z}{8\sqrt{-\Lambda_W}} \left[ 2x_+^2 + \frac{(z-2)^2}{z} \right].$$

(40)

As is shown in Fig. 3, all free energies are always negative.

On the other hand, the case of $z = 1$ leads to different thermodynamic quantities as

$$T_H^{z=1} = \frac{3x_+^2 - 1}{8\pi r_+}, \quad M_h^{z=1} = \frac{r_+}{4} \left[ x_+^2 - 1 \right], \quad C^{z=1} = 2\pi r_+^2 \left[ \frac{3x_+^2 - 1}{3x_+^2 + 1} \right], \quad F^{z=1} = -\frac{r_+ (x_+^2 + 1)}{8}$$

(41)

whose asymptotic forms are given by

$$T_H^{z=1} \simeq x_+, \quad M_h^{z=1} \simeq x_+^3, \quad C^{z=1} \simeq x_+^2, \quad F^{z=1} \simeq -x_+^3.$$  (42)

This means that $\lambda = 1$ LMP black hole is just the $z = 1$ Hořava black hole. We call this the $z = 1$ Hořava black hole because its near-horizon geometry of the extremal black hole is $\text{AdS}_2 \times S^2$ and its asymptote is $\text{AdS}_4$ spacetimes [3], implying that it is basically different from the $z = 3, 2$ Lifshitz black holes. At the extremal point of $x_+ = x_e = 1/\sqrt{3}$, we have thermodynamic properties of $T_H^{z=1}(x_e) = C^{z=1}(x_e) = 0$ and $(dM_h^{z=1}/dx_+)|_{x_+ = x_e} = 0$. If one uses the entropy of $S = A/4 - (\pi/\Lambda_W) \ln[A/4]$ to derive quasinormal modes of this black hole [59], the area spacing is not equidistant. Here, we have equidistant area spacing because of the Bekenstein-Hawking entropy.

All solid curves in Figs. 2 and 3 represent thermodynamic quantities for the $z = 1$ Hořava black hole.

For large Lifshitz black holes with $x_+ \gg 1$, their forms of thermodynamic quantities are given by

$$T_H^z \simeq x_+^z, \quad M_h^z \simeq x_+^{z+2}, \quad C \simeq x_+^2, \quad F^z \simeq x_+^{z+2},$$

(43)

while the Bekenstein-Hawking entropy (35) remains unchanged.

Finally, comparing (43) with (20), their forms are very similar to each other, but the difference is exponents of $M_h$, $C$, $S_{BH}$, and $F$ arisen from the dimensionality.

### 6 Phase transitions

We discuss a possible phase transition by considering the coordinate matching. We note that the free energy of $F^{z=1}$ in Eq.(41) is available only for $x_+ > x_e = 1/\sqrt{3} = 0.58$ because one could not define the positive temperature for $x_+ < x_e = 1/\sqrt{3}$. In this case, as is shown
Fig. 3, there are two crossing points at $x_+ = x_c = 0.84, 1$ where $F_{z=1}(x_+) = F_{z=3,2}(x_+)$. This implies that for $x_c \leq x_+ \leq x_c$, $F_{z=3,2} \geq F_{z=1}$, while for $x_+ \geq x_c$, $F_{z=3,2} \leq F_{z=1}$. In other words, for $x_c \leq x_+ \leq x_c$, the $z = 1$ black hole is more favorable than $z = 3, 2$ Lifshitz black holes, while for $x_+ \geq x_c$, the $z = 3, 2$ Lifshitz black holes are more favorable than the $z = 1$ Hořava black hole. This may imply a second order phase transition between Lifshitz black holes and $z = 1$ Hořava black hole. It seems that this phase transition is related to the second order phase transition between black hole with scalar hair (Martinez-Troncoso-Zanelli black hole [60]) and topological black hole with $k = -1$ [61, 57], which have the same AdS$_4$ asymptotes. However, the coordinate matching is not an appropriate choice for investigating a second order phase transition. The appropriate one is the temperature matching. Unfortunately, we could not make a second order phase transition between $z = 3, 2$ Lifshitz black holes and $z = 1$ Hořava black hole when using the temperature matching because they have different asymptotes. It is clear from Fig. 2 that there is no crossing point between $T_{H_{z=3,2}}(x_+)$ and $T_{H_{z=1}}(x_+)$ to make the temperature matching.

7 Discussions

First of all, we mention that 4D Lifshitz black holes came from the nonprojectable HL gravity, not the projectable theory. The projectability condition from condensed matter physics may not be appropriate for describing the (quantum) gravity. Especially for static spherically symmetric solutions, the non-projectability condition is necessary to obtain 4D Lifshitz black holes.

It is important to note that 3D Lifshitz and 4D Lifshitz black holes are obtained only when including higher order curvature terms, even though the former action is Lorentz-invariant combination with wrong-sign Einstein-Hilbert action and the latter is Lorentz-violating combination. This mean that these Lifshitz black holes have purely gravity origin without introducing any matter field. We emphasize that the role of higher order curvature terms is essential for obtaining these black holes, where these terms are not considered simply as perturbative corrections to the Einstein-Hilbert action. If these terms are absent, one finds the $z = 1$ non-rotating BTZ black hole from the 3D Einstein gravity and the Schwarzschild-AdS black hole from the 4D Einstein gravity.

These Lifshitz black holes have much similarities. Their horizons are non-degenerate. Their thermodynamic quantities are derived from the Hawking temperature and the Bekenstein-Hawking entropy when using the first law of thermodynamics. In this approach, we cannot
derive the horizon mass unless the Bekenstein-Hawking entropy is used. We may insist that the area-law of the black hole entropy and the first law of thermodynamics are valid for Lifshitz black holes [36, 48]. In this direction, the dynamical evolution of a massless scalar perturbation was investigated in the 4D Lifshitz black hole spacetimes with the dynamical exponent \( z = 4(\lambda = 1/3) \), \( z = 2(\lambda = 1/2) \) and \( z = 0(\lambda = 3) \), respectively [17]. It has shown that scalar perturbations decay without any oscillation in which the decay rate may imprint thermodynamic quantities of the 4D Lifshitz black holes. These purely damped modes are different from those in the (small) Schwarzschild-AdS black hole but are similar to those in the 3D charged black hole [62, 63], supporting that thermodynamic nature of 4D Lifshitz black holes is simple. However, we are still lacking for deriving the ADM mass of the Lifshitz-type black holes using the Hamiltonian formalism because we do not know Lifshitz asymptotes precisely.

These Lifshitz black hole spacetimes have curvature singularities at \( r = 0 \), and their asymptotes are Lifshitz. Also there is no phase transition between Lifshitz black holes and \( z = 1 \) black hole, although there are crossing points between two free energies when considering the coordinate matching. This is mainly because their asymptotes are different: Lifshitz and anti-de Sitter spacetimes.

Consequently, we have understood thermodynamics of Lifshitz black holes and have discussed possible second order phase transitions between Lifshitz black holes and \( z = 1 \) black holes.

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