Quantum Moduli Spaces of Linear and Ring Mooses

Girma Hailu

Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138

Abstract

Quantum moduli spaces of four dimensional $SU(2)^r$ linear and ring moose theories with $\mathcal{N} = 1$ supersymmetry and link chiral superfields in the fundamental representation are produced starting from simple pure gauge theories of disconnected nodes.

* Research supported in part by the National Science Foundation under grant number NSF-PHY/98-02709.

† hailu@feynman.harvard.edu
1 Introduction

Moose diagrams give succinct graphical representations of the transformations of matter fields under gauge (and global) symmetries. The gauge symmetries are represented by nodes and the matter fields by links. The recent interest in moose theories is because a class of moose diagrams has been shown to transform into a description of extra dimensions. Furthermore, the “theory space” of mooses has been used to build models and investigate various fundamental issues such as electroweak symmetry breaking and accelerated grand unification. The transformation of a moose diagram into a description of extra dimensions occurs when the link fields develop vacuum expectation value (VEV) and “hop” between the nodes. It is well known that supersymmetric gauge theories have larger moduli spaces of vacua than non-supersymmetric gauge theories. Therefore, supersymmetric mooses could provide richer theory spaces for model building.

Our interest in this note is the construction of the quantum moduli spaces of \( N = 1 \) supersymmetric \( SU(2) \) linear and ring mooses where the gauge group at each node is \( SU(2) \) and the links are chiral superfields that transform as fundamentals under the nearest gauge groups and as singlets under the rest. We will obtain nontrivial quantum moduli spaces by starting from pure gauge theories of disconnected nodes and exploiting simple and efficient integrating in and out procedures. Explicit parameterization of the vacua in terms of gauge invariant objects constructed out of the chiral superfields will be found. A generic point in the moduli space of the ring moose has an unbroken \( U(1) \) gauge symmetry and the ring moose is in the Coulomb phase. We will find two singular submanifolds with modulus that is a nontrivial function of all the independent gauge invariants needed to parameterize the moduli space of the ring moose. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose will follow from our computation.

Seiberg-Witten elliptic curves of the ring moose were computed in using a different method where it was started with the curve for a ring with two nodes given in and various asymptotic limits and symmetry arguments were used to obtain the curve for a ring moose with three nodes. The result was then generalized to the curve for a ring moose with arbitrary number of nodes. Here we will directly and explicitly compute the singularities of the quantum moduli space and the corresponding Seiberg-Witten elliptic curve for a ring moose with arbitrary number of nodes. Our results agree with for a ring with two nodes and with for a ring with three nodes. We believe that the curve given in is incorrect for ring mooses with four or more nodes.

An expanded version of this note that also includes more topics and results is presented in [10].

2 Quantum moduli space of the linear moose

Consider a four dimensional \( N = 1 \) supersymmetric \( SU(2) \) linear moose theory with matter content shown in Table 1. The equivalent moose diagram representation is shown in Figure 1. An internal

\[ ^1 \text{Quantum moduli space constraint relations for a linear moose with two and more nodes were first shown to us by Howard Georgi. Many results in this section overlap with results in [11].} \]
chiral superfield $Q_i$ links the $i^{th}$ and $(i+1)^{th}$ nodes. The internal link $Q_i$ is a doublet that transforms as $(\Box, \Box)$ under $SU(2)_i \times SU(2)_{i+1}$ and as singlet under all the other gauge groups. One of the external links $Q_0$ transforms as $\Box$ under $SU(2)_1$ and the second external link $Q_r$ transforms as $\Box$ under $SU(2)_r$. Each external link is two doublets with $SU(2)$ subflavor symmetry.

|       | $SU(2)_1$ | $SU(2)_2$ | $SU(2)_3$ | $\cdots$ | $SU(2)_r$ |
|-------|-----------|-----------|-----------|-----------|-----------|
| $Q_0$ | $\Box$    | 1         | 1         | $\ldots$ | 1         |
| $Q_1$ | $\Box$    | $\Box$    | 1         | $\ldots$ | 1         |
| $Q_2$ | 1         | $\Box$    | $\Box$    | $\ldots$ | 1         |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $Q_r$ | 1         | 1         | 1         | $\ldots$ | $\Box$    |

Table 1: Matter content of the linear moose.

![Figure 1: Linear moose with $r$ nodes and $r+1$ links. The external links $Q_0$ and $Q_r$ each have one color and one subflavor indices and each internal link has two color indices.](image)

We will compute the quantum moduli space of this theory starting from pure disconnected gauge groups and integrating in the link fields. Gaugino condensation in the pure gauge theory gives a nonperturbative superpotential,

$$W_d = \sum_{i=1}^{r} 2\epsilon_i \Lambda^3_{0i},$$

(2.1)

where each $\epsilon_i = \pm 1$ labels the two vacua due to the breaking of the $Z_4 R$ symmetry to $Z_2$ and $\Lambda_{0i}$ is the nonperturbative dynamical scale of $SU(2)_i$. Our notation for the dynamical scales is $\Lambda_{0i}$ for the scale of $SU(2)_i$ with no link, $\Lambda_{id}$ when there is one link, and $\Lambda_i$ when there are two links attached. The scale $\Lambda_i$ is related to $\Lambda_{0i}$ by threshold matching of the gauge coupling running at the masses $m_{i-1}$ and $m_i$ of $Q_{i-1}$ and $Q_i$ respectively,

$$\Lambda^6_{0i} = \Lambda^4_i m_{i-1} m_i.$$  

(2.2)

There are $r+1$ link chiral superfields each with four complex degrees of freedom. We can construct a total of $\frac{1}{2}(r^2 + 3r + 8)$ gauge singlets given by determinants of products of one to $r$ consecutive link superfields, and the product of all the chiral superfields:

$$\det (Q_i),$$

(2.3)
\[
\det (Q_1 Q_{i+1}), \ldots, \det (Q_0 Q_1 \cdots Q_{r-1}), \det (Q_1 Q_2 \cdots Q_r),
\]  \tag{2.4}

and \( Q_0 Q_1 \cdots Q_r. \) \tag{2.5}

For a generic linear moose, the gauge symmetry is completely broken and 3r of the complex degrees of freedom become massive or are eaten by the super Higgs mechanism. Consequently, there are only \( 4(r + 1) - 3r = r + 4 \) massless complex degrees of freedom left. Because we have \( \frac{1}{2} (r^2 + 3r + 8) \) gauge singlets, there must be \( \frac{1}{2} (r^2 + 3r + 8) - (r + 4) = r(r + 1)/2 \) constraints. We claim that a constraint involving the determinants of only subsegments of the moose chain given in (2.4) are not modified by the extra links and nodes. We can see that as follows: Consider the determinant of a subsegment, \( \det (Q_i Q_{i+1} \cdots Q_j) \). The color indices from the gauge groups \( SU(2)_i \) and \( SU(2)_j \) are not contracted with the colors of \( SU(2)_{i-1} \) and \( SU(2)_{j+1} \) respectively. Consequently, these adjoining gauge groups behave like global subflavor symmetries. This amounts to saying that as far as \( \det (Q_i Q_{i+1} \cdots Q_j) \) is concerned, the moose chain is cut off at the \( (i - 1)^{\text{th}} \) and \( (j + 1)^{\text{th}} \) nodes. Therefore, finding a constraint for \( \det (Q_i Q_{i+1} \cdots Q_j) \) is not an independent problem. Thus all the \( r(r + 1)/2 \) moduli space constraints can be easily deduced from the one constraint which can be parameterized by the \( r + 5 \) independent gauge singlets:

\[
M_i = \frac{1}{2} (Q_1)_{\alpha_i \beta_i} (Q_1)_{\alpha'_i \beta'_i} \epsilon^{\alpha_i \alpha'_i} \epsilon^{\beta_i \beta'_i} = \det (Q_i)
\]  \tag{2.6}

and

\[
T_{fg} = \frac{1}{2} (Q_0)_{f \beta_0} (Q_1)_{\alpha_1 \beta_1} (Q_2)_{\alpha_2 \beta_2} \cdots (Q_r)_{\alpha_r \beta_r} \epsilon^{\beta_0 \alpha_1} \epsilon^{\beta_1 \alpha_2} \cdots \epsilon^{\beta_r \alpha_r}.
\]  \tag{2.7}

where \( \alpha_i, \beta_i \) are color indices and \( f, g \) are subflavor indices. For \( M_0 \) and \( M_r \) one of the indices in \( Q_0 \) and \( Q_r \) is for subflavor.

In order to integrate in the link fields to the pure gauge theory of disconnected nodes, first we use (2.2) to replace \( \Lambda_{\beta_i}^3 \rightarrow (\Lambda_i^4 m_{i-1} m_i)_{\frac{1}{2}} \) in (2.1). We then write

\[
W = W_d (\text{with } \Lambda_{\beta_i}^3 \rightarrow (\Lambda_i^4 m_{i-1} m_i)_{\frac{1}{2}}) + W_{\text{tree,d}} - W_{\text{tree}}.
\]  \tag{2.8}

\( W_{\text{tree}} \) is a tree level superpotential that contains couplings to all the independent gauge invariants,

\[
W_{\text{tree}} = \text{tr} (c T) + \sum_{i=0}^{r} m_i M_i,
\]  \tag{2.9}

where \( c \) is a constant \( 2 \times 2 \) matrix. The term \( W_{\text{tree,d}} \) is computed by picking up any one link chiral superfield \( Q_k \) and integrating out \( Q_k \) in the gauge and flavor invariant tree level superpotential\(^2\) \( \text{tr} (c T) + m_k M_k. \) This gives

\[
W_{\text{tree,d}} = -\frac{\det (c)}{m_k} \det (Q_0 Q_1 \cdots Q_{k-1}) \det (Q_{k+1} Q_{k+2} \cdots Q_r).
\]  \tag{2.10}

\(^2\)This point is discussed in detail in [11].
We will see that the final result on the moduli space constraint does not depend on \( k \). For simplicity of notation, we introduce a more general way of representing products of consecutive link chiral superfields and define

\[
T_{(i,j)} \equiv Q_i Q_{i+1} \cdots Q_j. \tag{2.11}
\]

Note that \( T_{(i,j)} \) is a \( 2 \times 2 \) matrix with hidden indices. The superpotential we need for integrating in all the independent gauge singlets is then, putting (2.1), (2.9) and (2.10) in (2.8),

\[
W = 2 \sum_{i=1}^{r} \epsilon_i (\Lambda_i^4 m_{i-1} m_i)^{\frac{1}{2}} - \frac{\text{det}(c)}{m_k} \text{det} T_{(0,k-1)} \text{det} T_{(k+1,r)} - \text{tr} (c T_{(0,r)}) - \sum_{i=0}^{r} m_i M_i. \tag{2.12}
\]

The integrating in procedure is completed by minimizing (2.12) with the coupling constants \( m_i \) and \( c \).

Integrating out \( m_i \) and \( c \) in (2.12), recursively solving for \( m_i \) and \( c \), and putting into (2.12) gives \( W = 0 \) and a quantum moduli space constrained by

\[
\text{det} T_{(0,r)} - \frac{\text{det} T_{(0,k-1)} \text{det} T_{(k+1,r)}}{\Omega_{(0,k-1)} \Omega_{(k+1,r)}} \Omega_{(0,r)} = 0, \tag{2.13}
\]

where we have introduced \( \Omega_{(i,j)} \) functions to simplify our notation. The \( \Omega \) functions are defined by

\[
\Omega_{(i,j)} \equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j} \left( \Lambda_p^4 \prod_{q \neq p-1,p} M_q \right) + \sum_{p=i+1}^{j-2} \sum_{l=0}^{j-p-2} \left( \Lambda_p^4 \Lambda_{p+l+2}^4 \prod_{q \neq p-1,p,p+l+1,p+l+2} M_q \right)
- \cdots + (-1)^{(j-i+1)/2} \prod_{p=1}^{(j-i+1)/2} \Lambda_{i+2p-1}^4, \tag{2.14}
\]

if \( j - i \) is odd, and

\[
\Omega_{(i,j)} \equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j} \left( \Lambda_p^4 \prod_{q \neq p-1,p} M_q \right) + \sum_{p=i+1}^{j-2} \sum_{l=0}^{j-p-2} \left( \Lambda_p^4 \Lambda_{p+l+2}^4 \prod_{q \neq p-1,p,p+l+1,p+l+2} M_q \right)
- \cdots + (-1)^{(j-i)/2} \sum_{q=0}^{(j-i)/2} \left( M_{i+2q} \prod_{p=0}^{q-1} \Lambda_{i+2q-2p-1}^4 \prod_{l=1}^{(j-i)/2-q} \Lambda_{i+2q+2l}^4 \right), \tag{2.15}
\]

if \( j - i \) is even. We take \( j > i \) unless explicitly stated. When \( i = j \), we have \( \Omega_{(i,i)} = \text{det} M_i \). Some recursion relations among the \( \Omega \) functions are given in [10].

Thus the quantum moduli space is constrained by the recursion relations given by (2.13). Note that \( k \) in (2.13) is arbitrary and could take any value from 0 to \( r \). As we have argued earlier in this section, a similar relation as (2.13) should hold for a subset of the linear chain, and we write a more general form of the moduli space constraints as

\[
\text{det} T_{(i,j)} - \frac{\text{det} T_{(i,k-1)} \text{det} T_{(k+1,j)}}{\Omega_{(i,k-1)} \Omega_{(k+1,j)}} \Omega_{(i,j)} = 0. \tag{2.16}
\]
Now we can easily prove that the result (2.16) is independent of \( k \), since we can repeatedly use the same recursion relations to simplify the fractional factor in the second term, and (2.16) gives

\[
\det T_{(i,j)} - \Omega_{(i,j)} = 0.
\] (2.17)

Note that (2.17) gives \( r(r + 1)/2 \) constraints that completely remove all the redundancy in the set of gauge singlets.

The first few \( \Omega \) functions are

\[
\Omega(i, i+1) = M_i M_{i+1} - \Lambda_i^4,
\] (2.18)

\[
\Omega(i, i+2) = M_i M_{i+1} M_{i+2} - \Lambda_i^4 M_{i+1} - \Lambda_i^4 M_i,
\] (2.19)

\[
\Omega(i, i+3) = M_i M_{i+1} M_{i+2} M_{i+3} - \Lambda_i^4 M_{i+1} M_{i+2} M_{i+3} \\
- \Lambda_i^4 M_i M_{i+3} M_{i+1} + \Lambda_i^4 \Lambda_i^4,
\] (2.20)

\[
\Omega(i, i+4) = M_i M_{i+1} M_{i+2} M_{i+3} M_{i+4} - \Lambda_i^4 M_{i+1} M_{i+2} M_{i+3} M_{i+4} \\
- \Lambda_i^4 M_i M_{i+3} M_{i+4} - \Lambda_i^4 M_{i+1} M_{i+2} M_{i+4} \\
- \Lambda_i^4 M_{i+1} M_{i+2} + \Lambda_i^4 \Lambda_i^4 M_{i+4} \\
+ \Lambda_i^4 \Lambda_i^4 M_{i+2} + \Lambda_i^4 \Lambda_i^4 M_i.
\] (2.21)

### 3 Quantum moduli space of the ring moose

Now we can construct the quantum moduli space of the ring moose starting from the linear moose. The matter content of the ring moose is shown in Table 3. The equivalent moose diagram is shown in Figure 2. There are \( r + 1 \) independent gauge singlets given by \( M_i \) defined in (2.6), where \( 0 \leq i \leq r - 1 \) now, and

\[
U_{(0, r-1)} = \frac{1}{2} (Q_0)_{\alpha_0 \beta_0} (Q_1)_{\alpha_1 \beta_1} (Q_2)_{\alpha_2 \beta_2} \cdots (Q_{r-1})_{\alpha_{r-1} \beta_{r-1}} \epsilon^{\beta_0 \alpha_1} \epsilon^{\beta_1 \alpha_2} \cdots \epsilon^{\beta_{r-1} \alpha_0}.
\] (3.1)
We will start with the quantum moduli space constraint of the linear moose we found in Section 2. We will then integrate out the external links. Finally, a link field that transforms as ($\Box$, $\Box$) under $SU(2) \times SU(2)_1$ will be integrated in to build the ring moose shown in Figure 2. Since we can at the same time obtain the superpotential for a linear moose with only one external link, let us first integrate out $Q_r$. The superpotential for the linear moose without $Q_r$ is obtained by integrating out $M_r, T_{(0,r)}$ and $A$ in

$$W = A \left( \det T_{(0,r)} - \Omega_{(0,r)} \right) + m_r M_r. \quad (3.2)$$

The resulting superpotential is

$$W = \frac{\Lambda_{rd}^5 \Omega_{(0,r-2)}}{\Omega_{(0,r-1)}}, \quad (3.3)$$

where $\Lambda_{rd}^5 = \Lambda_{rd}^4 m_r$. Next we integrate out $Q_0$ by adding $m_0 M_0$ to (3.3) and minimizing with $M_0$ which gives the superpotential of the linear moose without the external links,

$$W = \frac{\Lambda_{1d}^5 \Omega_{(2,r-1)}}{\Omega_{(1,r-1)}} + \frac{\Lambda_{1d}^5 \Omega_{(1,r-2)}}{\Omega_{(1,r-1)}} \pm 2 \frac{(\Lambda_{1d}^5 \Lambda^5_{rd} \prod_{i=2}^{r-1} \Lambda_i^4)^{1/2}}{\Omega_{(1,r-1)}}. \quad (3.4)$$

The superpotential (3.4) can be interpreted as follows: For the moose chain with only internal links, the original $SU(2)^r$ gauge symmetry is completely broken and there is a new unbroken diagonal $SU(2)_D$. The first term comes from a single instanton in the broken $SU(2)_1$ and infinite series of multi-instantons from the broken $SU(2)_2$ to $SU(2)_{r-2}$. Similarly, the second term comes from a single instanton in the broken $SU(2)_r$ and an infinite series of multi-instantons from the broken $SU(2)_2$ to $SU(2)_{r-2}$. These can be seen by using the explicit form of the $\Omega$ functions and making an expansion of $\Omega_{(1,r-1)}^{-1}$ in powers of the scales of $SU(2)_2$ to $SU(2)_{r-2}$. The last term comes from gaugino condensation in the unbroken diagonal $SU(2)_D$. In fact, we can read off from (3.4) that the scale of the diagonal $SU(2)_D$ is

$$\Lambda_D = \left( \frac{(\Lambda_{1d}^5 \Lambda^5_{rd} \prod_{i=2}^{r-1} \Lambda_i^4)^{1/2}}{\Omega_{(1,r-1)}} \right)^{1/4}. \quad (3.5)$$
Finally, we can construct the quantum moduli space of the ring moose shown in Figure 2 by integrating in $Q_o$. The new gauge singlets that appear in the ring moose which were not in the linear moose with only internal links are $M_0$ and $U_{(0,r-1)}$ and the tree level superpotential we need is

$$W_{\text{tree}} = b U_{(0,r-1)} + m_0 M_0,$$  

(3.6)

where $b$ and $m_0$ are constants. The $W_{\text{tree},d}$ we need, because of the non-quadratic gauge singlet $U_{(0,r-1)}$, is obtained by minimizing $b U_{(0,r-1)} + m_0 M_0$ with $Q_o$ which gives $W_{\text{tree},d} = - \frac{b^2}{4m_0} \Omega_{(1,r-1)}$.

The quantum moduli space constraint of the ring moose is then obtained by minimizing

$$W = m_0 \frac{\Omega_{(2,r-1)}}{\Omega_{(1,r-1)}} + m_0 \frac{\Lambda_1^4 \Omega_{(1,r-2)}}{\Omega_{(1,r-1)}} \pm 2 m_0 \left( \prod_{i=1}^{r} \Lambda_i^4 \right)^{1/2}$$

$$- \frac{b^2}{4m_0} \Omega_{(1,r-1)} - m_0 M_0 - b U_{(0,r-1)}$$  

(3.7)

with $m_0$ and $b$ which gives $W = 0$ and

$$U_{(0,r-1)}^2 + \Lambda_1^4 \Omega_{(2,r-1)} + \Lambda_1^4 \Omega_{(1,r-2)} - M_0 \Omega_{(1,r-1)} \pm 2 \left( \prod_{i=1}^{r} \Lambda_i^4 \right)^{1/2} = 0.$$  

(3.8)

This is symmetric in all links and scales.

Before we interpret (3.8), let us first recall the Seiberg-Witten hypothesis on the elliptic curve of an SU(2) gauge theory. According to the Seiberg-Witten hypothesis [12], the quantum moduli space of an SU(2) gauge theory coincides with the moduli space of the elliptic curve $y^2 = (x^2 - u)^2 - \Lambda^4$, where $u$ is a gauge invariant coordinate and $\Lambda$ is the dynamical scale of the theory. The singularities of this curve are given by the zeros of the discriminant $\Delta_\Lambda = (u^2 - \Lambda^4)(2\Lambda)^8$. This occurs at $u = \pm \Lambda^2$ and $u = \infty$. The first two singularities at $u = \pm \Lambda^2$ are in the strong coupling region, and there is a massless monopole at one and a massless dyon at the other of these singularities. The singularity at $u = \infty$ is in the semi-classical region.

Now let us rewrite (3.8) as

$$u_r = \pm \Lambda_{(1,r)}^2,$$  

(3.9)

where

$$u_r \equiv U_{(0,r-1)}^2 + \Lambda_1^4 \Omega_{(2,r-1)} + \Lambda_1^4 \Omega_{(1,r-2)} - M_0 \Omega_{(1,r-1)}$$  

(3.10)

and

$$\Lambda_{(1,r)}^2 \equiv 2 \left( \prod_{i=1}^{r} \Lambda_i^4 \right)^{1/2}$$  

(3.11)

Note that the modulus $u_r$ contains all the independent gauge invariants we needed to parameterize the moduli space of the ring. What (3.9) is telling us is that the function $u_r$ is locked at $\pm \Lambda_{(1,r)}^2$. In other words, (3.9) gives two $r$-complex dimensional singular submanifolds in the $r+1$-complex dimensional moduli space spanned by all the independent gauge invariants. Giving large VEVs to the link fields breaks the original SU(2)$^r$ gauge symmetry into a diagonal SU(2)D with matter in the adjoint representation. The two singularities given by (3.9) on the $u_r$ plane can be
nothing but the two singularities in the strong coupling region of the \( SU(2)_D \) gauge theory with \( \mathcal{N} = 2 \) supersymmetry. The monodromies around these singularities on the \( u_r \) plane must be the same as in Seiberg-Witten and the charge at the singularity \( u_r = +\Lambda^2_{(1,r)} \) is that of a monopole and the charge at \( u_r = -\Lambda^2_{(1,r)} \) is that of a dyon. A generic point in the moduli space of the ring moose has unbroken \( U(1) \) gauge symmetry and the ring moose is in the Coulomb phase. Having obtained these singularities and because the \( U(1) \) coupling coefficient is holomorphic, we have determined the elliptic curve that parameterizes the Coulomb phase of the ring moose. Thus the quantum moduli space of the ring moose can be parameterized by the elliptic curve

\[
y^2 = \left(x^2 - [U^2_{(0,r-1)} + \Lambda_1^4 \Omega_{(2,r-1)} + \Lambda_1^4 \Omega_{(1,r-2)} - M_0 \Omega_{(1,r-1)}]\right)^2 - 4 \prod_{i=1}^r \Lambda_i^4.
\] (3.12)

Note that although the singularities look the same as in Seiberg-Witten on the \( u_r \) plane, they are \( r \)-complex dimensional submanifolds with very nontrivial modulus given by (3.10). Using the definition (3.10) for \( u_r \) and the \( \Omega \) functions given in Section 2, the first few \( u \) functions are

\[
u_2 = U^2_{(0,1)} + \Lambda_1^4 + \Lambda_2^4 - M_0 M_1,
\]

(3.13)

\[
u_3 = U^2_{(0,2)} + \Lambda_1^4 M_2 + \Lambda_2^4 M_0 + \Lambda_3^4 M_1 M_2,
\]

(3.14)

\[
u_4 = U^2_{(0,3)} + \Lambda_1^4 M_2 M_3 + \Lambda_2^4 M_0 M_3 + \Lambda_3^4 M_0 M_1 M_2
\]

\[+ \Lambda_4^4 M_1 M_2 - \Lambda_1^4 \Lambda_3^4 - \Lambda_2^4 \Lambda_4^4 - M_0 M_1 M_2 M_3,
\]

(3.15)

\[
u_5 = U^2_{(0,4)} + \Lambda_1^4 M_2 M_3 M_4 + \Lambda_2^4 M_0 M_3 M_4 + \Lambda_3^4 M_0 M_1 M_4
\]

\[+ \Lambda_4^4 M_0 M_2 + \Lambda_5^4 M_1 M_2 M_3 - \Lambda_1^4 \Lambda_3^4 M_4 - \Lambda_2^4 \Lambda_4^4 M_2
\]

\[- \Lambda_2^4 \Lambda_5^4 M_3 - \Lambda_5^4 \Lambda_5^4 M_1 - \Lambda_2^4 \Lambda_5^4 M_0 - M_0 M_1 M_2 M_3 M_4.
\]

(3.16)

Seiberg-Witten curves for the ring moose were computed in \[8\] using a different method. A method used in \[8\] to obtain the curve for the \( r = 2 \) ring was continued in \[8\] to compute the curve for \( r = 3 \). The idea was as follows: Because giving large VEVs to the link fields breaks the \( SU(2)_r \) gauge symmetry into a diagonal \( SU(2)_D \) with matter in the adjoint representation, the theory in effect becomes that of a single \( SU(2) \) with \( \mathcal{N} = 2 \) supersymmetry. The curve for \( r = 3 \) was obtained by taking various asymptotic limits of the gauge singlet fields and the nonperturbative scales, comparing with the \( \mathcal{N} = 2 \ SU(2) \) curve and imposing symmetries. The result for \( r = 3 \) was then generalized to the curve for a ring moose with arbitrary \( r \). Our results agree with \[9\] for \( r = 2 \) and with \[8\] for \( r = 3 \). However, we do not agree with the curves in \[8\] for \( r \geq 4 \). Only few terms in \( u_r \) were obtained in \[8\], which would give incorrect singular submanifolds in moduli space. We are not suggesting that the method used in \[8\] is incorrect. Here we have obtained the quantum moduli space directly by integrating in all the independent link fields starting from a pure gauge theory of disconnected nodes and building the ring moose via the linear moose. This is done for a ring with arbitrary number of nodes without any need of imposing symmetries in the nodes or links and without taking asymptotic limits; and the result is automatically symmetric in all nodes and links.
4 Summary

We have produced nontrivial quantum moduli spaces for $\mathcal{N} = 1$ supersymmetric $SU(2)^r$ linear and ring mooses starting from simple pure gauge theories of disconnected nodes by integrating in all matter link fields. For the ring moose, we obtained two singular submanifolds with modulus that is a function of all the independent gauge singlets we needed to parameterize the quantum moduli space. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose followed from our computation. More details and results are presented in [10].

Acknowledgements

I am deeply grateful to Howard Georgi for various useful suggestions, comments and discussions. I am also very thankful to him for sharing his original quantum moduli space relations of the linear moose with me which initiated this work. This research is supported in part by the National Science Foundation under grant number NSF-PHY/98-02709.

References

[1] H. Georgi, “A tool kit for builders of composite models,” Nucl. Phys. B266 (1986) 274.

[2] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “(De)constructing dimensions,” Phys. Rev. Lett., 86, 4757 (2001), hep-th/0104005.

[3] C. T. Hill, S. Pokorski, and J. Wang, “Gauge invariant effective Lagrangians for Kaluza-Klein modes,” Phys. Rev. D64 (2001) 105005, hep-th/0104035.

[4] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “Electroweak symmetry breaking from dimensional deconstruction,” Phys. Lett., B513, 232(2001), hep-ph/0105239. “Accelerated unification,” hep-ph/0108089.

[5] N. Seiberg, “Exact results on the space of vacua of four dimensional susy gauge theories,” Phys. Rev. D49 (1994) 6857, hep-th/9402004.

[6] K. Intriligator, R. G. Leigh and N. Seiberg, “Exact superpotentials in four dimensions,” Phys. Rev. D50 (1994) 1092, hep-th/9403198.

[7] K. Intriligator, “Integrating in” and exact superpotentials in 4d,” Phys. Lett. b336 (1994) 409, hep-th/9407106.

[8] C. Csaki, J. Erlich, D. Freedman and W. Skiba, “$\mathcal{N} = 1$ supersymmetric product group theories in the coulomb phase,” Phys. Rev. D56 (1997) 5209, hep-th/9704067.
[9] K. Intriligator and N. Seiberg, “Phases of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions,” *Nucl. Phys.* B431 (1994) 551, hep-th/9408155.

[10] G. Hailu, “$\mathcal{N} = 1$ supersymmetric $SU(2)^r$ moose theories,” hep-th/0209266.

[11] S. Chang and H. Georgi, “Quantum modified mooses,” hep-th/0209033.

[12] N. Seiberg and E. Witten, “Monopole condensation and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory,” *Nucl. Phys.* B426 (1994) 19, hep-th/9407081. “Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD,” *Nucl. Phys.* B431 (1994) 484, hep-th/9408099.