ON THE EXTENSION OF WHITNEY ULTRAJETS, II

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Abstract. We characterize the validity of the Whitney extension theorem in the ultradifferentiable Roumieu setting with controlled loss of regularity. Specifically, we show that in the main Theorem 1.3 of [16] condition (1.3) can be dropped. Moreover, we clarify some questions that remained open in [16].

1. Introduction

The main goal of this paper is to prove:

Theorem 1. Let $\omega$ be a non-quasianalytic concave weight function. Let $\sigma$ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$. Then the following conditions are equivalent:

(i) For every compact $E \subseteq \mathbb{R}^n$ we have $j^\infty_E (\mathcal{B}(\omega)(\mathbb{R}^n)) \supseteq \mathcal{B}(\sigma)(E)$, where $j^\infty_E$ assigns to each $f \in C^\infty(\mathbb{R}^n)$ its infinite jet $(f^{(\alpha)}|_E)_{\alpha \in \mathbb{N}^n}$ on $E$.

(ii) There is $C > 0$ such that $\int_1^\infty \frac{\omega(\alpha u)}{u^2} \, du \leq C \sigma(t) + C$ for all $t > 0$.

(Here $\mathcal{B}(\omega)$ denotes the Roumieu class defined by the weight function $\omega$; we use the symbol $\mathcal{B}$ to emphasize that the defining estimates are global, cf. [16, 2.2 and 2.6].) It means that Theorem 1.3 of [16] holds without the assumption (1.3) that the associated weight matrix $S$ of $\sigma$ satisfies

$$\forall S \in \mathcal{S} \exists T \in \mathcal{S} \exists C \geq 1 \forall 1 \leq j \leq k : \frac{S_j}{jS_{j-1}} \leq C \frac{T_k}{kTk^{-1}}. \tag{1}$$

Theorem 1 is proved in Section 2. In Section 3 we clarify some questions that remained open in [16] and obtain several characterizations of concave weight functions. For an overview of the background of Theorem 1 we refer to the introduction in [16]. We use the notation and the definitions of said paper; the concept of weight matrices is recalled in the appendix at the end of this paper.

Note that in the special case that $\omega$ and $\sigma$ coincide we recover the result of [1]:

Corollary 2. Let $\omega$ be a weight function. The following conditions are equivalent:

(i') For every compact $E \subseteq \mathbb{R}^n$ we have $j^\infty_E (\mathcal{B}(\omega)(\mathbb{R}^n)) = \mathcal{B}(\omega)(E)$.

(ii') There is $C > 0$ such that $\int_1^\infty \frac{\omega(\alpha u)}{u^2} \, du \leq C \omega(t) + C$ for all $t > 0$.
Indeed, if \( \omega \) satisfies (ii’) then it is non-quasianalytic, equivalent to a concave weight function \([9, \text{Proposition 1.3}]\), and \( \omega(t) = o(t) \) as \( t \to \infty \) [2, Remark 3.20]. That (ii’) is a necessary condition for (i’) is well-known. Note that also (i’) implies that \( \omega \) is non-quasianalytic. Indeed, if \( \omega \) is quasianalytic, then the Borel map \( j_0^\infty : \mathcal{B}^{(\omega)}(\mathbb{R}^n) \to \mathcal{B}^{(\omega)}(\{0\}) \) is never surjective. For \( t \neq O(\omega(t)) \) as \( t \to \infty \) this follows from [15], for \( t = O(\omega(t)) \) as \( t \to \infty \) consider e.g. the formal series \( \sum_{k=0}^{\infty} x^k \) which converges to the unbounded real analytic function \( 1/(1 - x) \) function for \( |x| < 1 \).

2. Proof of Theorem (I)

**Preparations.** First we recall a few definitions and facts. Let \( m = (m_k) \) be a positive sequence satisfying \( m_0 = 1 \) and \( m_k^{1/k} \to \infty \). The **log-convex minorant** of \( m \) is given by

\[
m_k := \sup_{t > 0} \frac{t^k}{\exp(\omega_m(t))}, \quad k \in \mathbb{N},
\]

where

\[
\omega_m(t) := \sup_{k \in \mathbb{N}} \log \left( \frac{t^k}{m_k} \right), \quad t > 0.
\]

The function \( \omega_m \) is increasing, convex in \( \log t \), and zero for sufficiently small \( t > 0 \). Related is the function \( h_m(t) := \inf_{k \in \mathbb{N}} m_k t^k \), for \( t > 0 \), and \( h_m(0) := 0 \). It is increasing, continuous, positive for \( t > 0 \), and equals 1 for large \( t \).

Let \( m = (m_k) \) be a positive log-convex sequence (i.e., \( m = m \)) such that \( m_0 = 1 \) and \( m_k^{1/k} \to \infty \). Then the functions \( \Gamma_m \) and \( \Gamma_m \) defined in [16, Definition 3.1] coincide, we simply write \( \Gamma_m \) in this case; note that log-convexity and \( m_k^{1/k} \to \infty \) imply \( m_k/m_{k-1} \to \infty \). Thus

\[
\Gamma_m(t) := \min\{k : h_m(t) = m_k t^k\} = \min\left\{k : \frac{m_{k+1}}{m_k} \geq \frac{1}{t}\right\}, \quad t > 0.
\]

By [16, Lemma 3.2], \( \Gamma_m \) is decreasing, tending to \( \infty \) as \( t \to 0 \), and

\[
k \to m_k t^k \text{ is decreasing for } k \leq \Gamma_m(t).
\]

Recall that with every weight function \( \sigma \) (always understood as defined in [16, Section 2.1]) is associated a weight matrix \( \Sigma = \{S_k\}_{k > 0} \), where

\[
S_k := \exp(\frac{1}{|\xi|} \varphi^*(\xi k)), \quad \text{(here } \varphi = \sigma \circ \exp \text{ and } \varphi^* \text{ is its Young conjugate),}
\]

such that \( \mathcal{B}(\sigma) = \mathcal{B}(\Sigma) \) and \( \mathcal{B}(\sigma) = \mathcal{B}(\Sigma) \) algebraically and topologically; cf. [16, 2.5] and [12]. In the following we set \( S_k^\xi := S_k^\xi / k! \).

The next proposition shows that for a weight function \( \sigma \) which is equivalent to a concave weight function and satisfies \( \sigma(t) = o(t) \) as \( t \to \infty \) we additionally have \( \mathcal{B}(\sigma) = \mathcal{B}(\Sigma) \) and \( \mathcal{B}(\sigma) = \mathcal{B}(\Sigma) \), where \( \Sigma = \{S_k^\xi\}_{k > 0} \) and

\[
S_k^\xi := k! S_k^\xi.
\]

In particular, \( \Sigma \) satisfies (I). We say that \( \Sigma \) is strongly log-convex meaning that \( S_k^\xi = S_k^\xi / k! \) is log-convex. (Note the abuse of notation: \( S_k^\xi \) is not necessarily the log-convex minorant of \( S_k^\xi \); this will cause no confusion.) Recall that two weight functions \( \omega \) and \( \sigma \) are called equivalent if \( \omega(t) = O(\sigma(t)) \) and \( \sigma(t) = O(\omega(t)) \) as \( t \to \infty \); this means that they define the same ultradifferentiable class.
Proposition 3. Let $\sigma$ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$ which is equivalent to a concave weight function. For each $\xi > 0$ there exist constants $A, B, C > 0$ such that

$$A^{-1} s_k^{\xi/B} \leq s_k^\xi \leq C s_k^B$$

for all $k \in \mathbb{N}$. (4)

Moreover, there is a constant $H \geq 1$ such that $s_j^{\xi+k} \leq H^{j+k} s_{j+k}^{2\xi}$, for all $\xi > 0$ and all $j, k \in \mathbb{N}$, and thus $h_{\omega}(t) \leq h_{\omega}(Ht)^2$, for all $\xi > 0$ and all $t > 0$.

Proof. Clearly, $s_b^\xi \leq s^\xi$. Let $S_k^\xi := k! s_k^\xi$. By [14] Lemma 3.6, $\omega_{S\xi}$ and $\omega_{S\xi}$ are equivalent, in particular, there exists $C \geq 1$ such that

$$\omega_{S\xi} \leq C \omega_{S\xi} + C.$$ (5)

By [16] Lemma 2.4(3) and [14] Remark 2.5, we have

$$2 \omega_{S\xi} \leq \omega_{S\xi}, \quad \text{for all } \xi > 0.$$ (6)

If $n$ is an integer such that $B := 2^n \geq C$, then $\omega_{S\xi} \leq \omega_{S\xi+n} + C$ and hence

$$S_k^\xi = \sup_{t > 0} \frac{t^k}{\exp(\omega_{S\xi}(t))} \geq e^{-C} \sup_{t > 0} \frac{t^k}{\exp(\omega_{S\xi+n}(t))} = e^{-C} S_k^{\xi/B}.$$ (7)

This shows the first inequality in (4).

By [16] Lemma 3.13, there exists $D \geq 1$ such that for all $\xi > 0$,

$$2 \omega_{S\xi}(t) \leq \omega_{S\xi}(Dt), \quad \text{for } t > 0$$ (8)

and therefore

$$S_k^\xi \leq D^{2k} \sup_{t > 0} \frac{(Dt)^{2k}}{\exp(\omega_{S\xi}(Dt))} = D^{2k} \omega_{S\xi}(Dt)^2.$$ (9)

Thus, by [17] Theorem 9.5.1 (which is a generalization of [8]), there exists a constant $H \geq 1$ such that $s_j^{\xi+k} \leq H^{j+k} s_{j+k}^{2\xi}$, for all $j, k$. That $h_{\omega}(t) \leq h_{\omega}(Ht)^2$, for all $\xi > 0$ and all $t > 0$, follows from [16] Lemma 3.12. By [18] Proposition 3.6,

$$2 \omega_{S\xi}(t) \leq \omega_{S\xi}(Ht), \quad \text{for } t > 0,$$ (10)

for some (possibly different) $H \geq 1$. As above, using (4), we find $\omega_{S\xi}(bt) \leq \omega_{S\xi}(t) + 1$ for some constant $0 < b \leq 1$. Then

$$S_k^{B^\xi} = \sup_{t > 0} \frac{(bt)^k}{\exp(\omega_{S\xi}(bt))} \geq e^{-1} b^k \sup_{t > 0} \frac{t^k}{\exp(\omega_{S\xi}(t))} = e^{-1} b^k S_k^\xi.$$ (11)

The last inequality of (4) follows. \hfill \square

Proposition [1] alone is not enough to get rid of the assumption (1). It is not clear that $\mathfrak{S}$ has the property that for all $S \in \mathfrak{S}$ there is a $T \in \mathfrak{S}$ such that $S_{2k}/S_{2k-1} \leq T_{k}/T_{k-1}$. Note that $\mathfrak{S}$ has this property (see [10] Lemma 2.4(4)) and it enters crucially in Lemma 3.4 and Proposition 3.7 of [16].

We deal with this problem by introducing another intimately related weight matrix $\mathfrak{V} := \{ V_{\xi} \} \xi > 0$. For each $\xi > 0$ we define $V_{\xi}^k := k! v_k^\xi$ by setting

$$v_k^\xi := \min_{0 \leq j \leq k} \frac{2\xi}{2^{k-j}}, \quad k \in \mathbb{N}.$$ (12)
That means that for the sequence of quotients $v_2^\xi / v_{k-1}^\xi$ we have (cf. [7, Lemma 3.5])
\[
\left( \frac{v_2^\xi}{v_0^\xi}, \frac{v_2^\xi}{v_1^\xi}, \frac{v_2^\xi}{v_2^\xi}, \frac{v_3^\xi}{v_3^\xi}, \ldots \right) = \left( \frac{2^2}{2^2}, \frac{2^2}{2^2}, \frac{2^2}{2^2}, \frac{2^2}{2^2}, \frac{2^2}{2^2}, \frac{2^2}{2^2}, \frac{2^2}{2^2}, \ldots \right).
\]
Thus the sequence $v^\xi = (v_k^\xi)$ is log-convex and satisfies
\[
\frac{v_{2k}^\xi}{v_{2k-2}^\xi} = \frac{v_{2k-2}^\xi}{v_{2k-2}^\xi} = \frac{2^2}{2^2}, \quad \text{for all } k \geq 1. \tag{7}
\]
So, in view of (2),
\[
2\Gamma_2(t) = \Gamma_\sigma(t), \quad \text{for all } t > 0.
\tag{8}
\]
By Proposition 9 there is $H \geq 1$ such that for all $\xi > 0$
\[
\kappa^\xi_k \leq H^k \kappa^\xi_k \leq H^k \kappa^\xi_k, \quad \text{for all } k \in \mathbb{N}. \tag{9}
\]
Thus, we also have $B(\sigma) = B(\mathfrak{B})$ and $B(\sigma) = B(\mathfrak{B})$ algebraically and topologically.

**Proof of Theorem 1** The implication (i) $\Rightarrow$ (ii) follows from [2]. So we only prove the converse implication. Condition (ii) means that the weight function
\[
\kappa(t) := \int_1^\infty \frac{\omega(tu)}{u^2} \, du \tag{10}
\]
satisfies $\kappa(t) = O(\sigma(t))$ as $t \to \infty$, i.e., $B(\sigma) \subseteq B(\kappa)$. Now $\kappa$ is concave and $\kappa(t) = o(t)$ as $t \to \infty$, see [9, Proposition 1.3]. We will show that Whitney ultrajets of class $B(\kappa)$ admit extensions of class $B(\sigma)$. Thus from now on we assume without loss of generality that $\sigma = \kappa$ is concave. Since $\omega$ is increasing we have $\sigma = \kappa \geq \omega$ and hence, if $\mathfrak{W} = \{W^\xi\}_{\xi > 0}$ denotes the weight matrix associated with $\omega$,
\[
S^\xi \leq S^\xi \leq W^\xi, \quad \text{for all } \xi > 0.
\tag{11}
\]
Moreover, Proposition 3 as well as (3) and (4) apply. Let us now indicate the necessary changes in the proof of [16, Theorem 1.3]. The changes also lead to some simplifications. We provide details in the hope that this contributes to a better understanding.

- Every Whitney ultrajet $F = (F^\alpha)$ of class $B(\sigma)$ on the compact set $E \subseteq \mathbb{R}^n$ is an element of $B(\mathfrak{W})$ for some $\xi > 0$, i.e., there exist $C > 0$ and $\rho \geq 1$ such that
\[
|F^\alpha(a)| \leq C\rho^{k^\alpha} \nu_\alpha^\xi, \quad \alpha \in \mathbb{N}^n, \ a \in E, \tag{12}
\]
\[
|(R_a^k F)^\alpha(b)| \leq C\rho^{k^\alpha + 1} |\alpha|! \nu_{k+1}^\xi |b - a|^{k^\alpha + 1 - |\alpha|}, \quad k \in \mathbb{N}, |\alpha| \leq k, \ a, b \in E. \tag{13}
\]

Let $p \in \mathbb{N}$ be fixed (and to be specified later). Let $\{\varphi_{i,p}\}_{i \in \mathbb{N}}$ be the partition of unity provided by [16, Proposition 4.9], relative to the family of cubes $\{Q_i\}_{i \in \mathbb{N}}$ from [16, Lemma 4.7], and let $r_0 = r_0(p)$ be the constant appearing in this proposition. The center of $Q_i$ is denoted by $x_i$. We claim that an extension of class $B(\omega)$ of $F$ to a suitable neighborhood of $E$ in $\mathbb{R}^n$ is provided by
\[
f(x) := \begin{cases} \sum_{i \in \mathbb{N}} \varphi_{i,p}(x) T_{\hat{x}_i}^p(x_i) F(x), & \text{if } x \in \mathbb{R}^n \setminus E, \\
F_0(x), & \text{if } x \in E,
\end{cases}
\]
where, given $x \in \mathbb{R}^n \setminus E$, $\hat{x}$ is any point in $E$ with $d(x) := d(x, E) = |x - \hat{x}|$ and
\[
p(x) := \max \{2\Gamma_\nu \omega(Ld(x)) - 1, 0\}.
\]
Here \( L \) is a positive constant to be specified below. Recall that \( Q_i^* \) is the closed cube with the same center as \( Q_i \) expanded by the factor \( 9/8 \). By [16 Corollary 4.8],
\[
\frac{1}{2} d(x) \leq d(x_i) \leq 3d(x), \quad \text{for all } x \in Q_i^*.
\]
Then \( d(x) < 1/(3L^{2\xi}) \) guarantees that both \( \Gamma_{\mathcal{Z}1} (Ld(x_i)) \) and \( \Gamma_{\mathcal{Z}1} (Ld(x)) \) are \( \geq 1 \), by [23], thus \( p(x_i) = 2\Gamma_{\mathcal{Z}1} (Ld(x_i)) - 1 \) and \( p(x) = 2\Gamma_{\mathcal{Z}1} (Ld(x)) - 1 \).

- Replace [16 Lemma 5.2] by the following lemma. The only difference in the proof is that one uses [16 (5.4)].

**Lemma 4.** There is a constant \( C_0 = C_0(n) > 1 \) such that, for all Whitney ultrajets \( F = (F^\alpha)_\alpha \) of class \( \mathcal{B}^{(V^\xi)} \) that satisfy (12) and (13), all \( L \geq C_0 \rho, \) all \( x \in \mathbb{R}^n, \) and all \( \alpha \in \mathbb{N}^n, \)
\[
| (T_{\mathcal{Z}}^p(x))^{(\alpha)} (x) | \leq C(2L) |\alpha| + 1 V^\xi,
\]
and, if \( |\alpha| < p(x), \)
\[
| (T_{\mathcal{Z}}^p(x))^{(\alpha)} (x) - F^{\alpha} (\hat{x}) | \leq C(2L) |\alpha| + 1 \nu^\xi_{|\alpha| + 1} d(x).
\]

We remark that (here and below) by \( (T_{\mathcal{Z}}^p(x))^{(\alpha)} (x) \) we mean the \( \alpha \)-th partial derivative of the polynomial \( y \mapsto T_{\mathcal{Z}}^p(x) F(y) \) evaluated at \( y = x. \)

- Replace [16 Lemma 5.3] by:

**Lemma 5.** There is a constant \( C_1 = C_1(n) > 0 \) such that for all \( L > C_1 \rho, \) all \( \beta \in \mathbb{N}^n, \) and all \( x \in Q_i^* \) with \( d(x) < 1/(3L^{2\xi}), \)
\[
| \partial^\beta (T_{\mathcal{Z}}^p(x_i)) F - T_{\mathcal{Z}}^p(x_i)) F(x) | \leq CL |\beta| + 1 \sum_{|\beta|} h_{\mathcal{Z}1} (Ld(x_i)).
\]

**Proof.** It suffices to consider \( |\beta| \leq p(x_i) = 2\Gamma_{\mathcal{Z}1} (Ld(x_i)) - 1 =: 2q - 1 \). Let \( H_1 \) denote the left-hand side of (17). By [16 Lemma 5.1 and Corollary 4.8] and (6),
\[
H_1 \leq C(2n^2 \rho)^{2q |\beta|} \nu^\xi_{2q} (6d(x_i))^{2q - |\beta|} \leq C(2n^2 \rho)^{2q |\beta|} \nu^\xi_{2q} (6d(x_i))^{2q - |\beta|}.
\]
By the definition of \( q, \) \( h_{\mathcal{Z}1} (Ld(x_i)) = \sum_{|\beta|} h_{\mathcal{Z}1} (Ld(x_i))^q \leq \sum_{|\beta|} h_{\mathcal{Z}1} (Ld(x_i))^k \) for all \( k. \) Thus
\[
H_1 \leq C \left( \frac{12n^2 \rho}{L} \right)^{2q} L |\beta| \nu^\xi_{2q} h_{\mathcal{Z}1} (Ld(x_i)).
\]
If \( L > 12n^2 \rho, \) then (17) follows. \( \square \)

- Replace [16 Lemma 5.4] by:

**Lemma 6.** There is a constant \( C_2 = C_2(n) > 0 \) such that for all \( L > C_2 \rho, \) all \( \beta \in \mathbb{N}^n, \) and all \( x \in Q_i^* \) with \( d(x) < 1/(3L^{2\xi}), \)
\[
| \partial^\beta (T_{\mathcal{Z}}^p(x)) F - T_{\mathcal{Z}}^p(x)) F(x) | \leq C(\frac{3L}{\nu}) |\beta| + 1 \sum_{|\beta|} h_{\mathcal{Z}1} (3Ld(x)).
\]

**Proof.** Both \( p(x_i) \) and \( p(x) \) are majorized by \( \Gamma_{\mathcal{Z}1} (Ld(x)/2) \), indeed, by (8), (13), and since \( \Gamma_{\mathcal{Z}1} \) is decreasing,
\[
p(x_i) = 2\Gamma_{\mathcal{Z}1} (Ld(x_i)) - 1 \leq 2\Gamma_{\mathcal{Z}1} (Ld(x_i)) = \Gamma_{\mathcal{Z}1} (Ld(x_i)) \leq \Gamma_{\mathcal{Z}1} (Ld(x)/2).
\]
So the degree of the polynomial \( T_{\mathcal{Z}}^p(x)) F - T_{\mathcal{Z}}^p(x) F \) is at most \( \Gamma_{\mathcal{Z}1} (Ld(x)/2). \) The valuation of the polynomial is equal to \( \min \{ p(x_i), p(x) \} + 1 \) (unless \( p(x_i) = p(x) \) in
which case (18) is trivial) and so at least $2\Gamma_{\frac{2}{\varepsilon}}(3Ld(x)) = 2q$, by (14). So if $H_2$

denotes the left-hand side of (18), then (see the calculation in [16, (5.7)])

$$H_2 \leq \frac{C[\beta]!}{(nd(x))^{[\beta]}} \Gamma_{\varepsilon}(Ld(x)/2) \sum_{j=2q}^{\Gamma_{\varepsilon}(Ld(x)/2)} (2n^2 \rho d(x))^j v_j^\varepsilon.$$ 

By (5), $v_j^\varepsilon(Ld(x)/2)^j \leq v_{2q}^\varepsilon(Ld(x)/2)^{2q}$ for $2q \leq j \leq \Gamma_{\varepsilon}(Ld(x)/2)$. By the definition of $q$, $h_{\frac{2}{\varepsilon}}(3Ld(x)) = \frac{2}{\varepsilon} q \leq 2^k(3Ld(x))^k$ for all $k$. With (6) this leads to

$$H_2 \leq \frac{C[\beta]!}{(nd(x))^{[\beta]}} \Gamma_{\varepsilon}(Ld(x)/2) \sum_{j=2q}^{\Gamma_{\varepsilon}(Ld(x)/2)} \left( \frac{4n^2 \rho}{L} \right)^j v_{2q}^\varepsilon \left( \frac{Ld(x)}{2} \right)^{2q}$$

$$\leq \frac{C[\beta]!}{(nd(x))^{[\beta]}} \Gamma_{\varepsilon}(Ld(x)/2) \sum_{j=2q}^{\Gamma_{\varepsilon}(Ld(x)/2)} \left( \frac{4n^2 \rho}{L} \right)^j \left( \frac{Ld(x)}{2} \right)^{2q}$$

$$\leq C \left( \frac{3L}{n} \right)^{[\beta]} \frac{[\beta]!}{2^{[\beta]}} \sum_{j=2q}^{\Gamma_{\varepsilon}(Ld(x)/2)} \left( \frac{4n^2 \rho}{L} \right)^j.$$

If we choose $L \geq 8n^2 \rho$, then the sum is bounded by 2, and (18) follows. \qed

- Assume that $L$ is chosen such that

$$L > \max\{C_0, C_1, C_2\} \rho \quad (19)$$

so that [15], [16], [17], and [18] are valid. Recall that $\mathfrak{W}$ denotes the weight matrix associated with $\omega$. The next lemma is a substitute for the claim in the proof of Theorem 5.5 in [16].

**Lemma 7.** There exist constants $K_j = K_j(n, \omega)$, $j = 1, 2, 3$, such that the following holds. If $p = K_1L$ and $L > K_2p$, then there exist a weight sequence $W \in \mathfrak{W}$ and a constant $M_1 = M_1(n, \omega, L) > 0$ such that for all $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L^2\varepsilon)\}$ and all $\alpha \in \mathbb{N}^n$,

$$|\partial^\alpha(f - T^{p(x)}F)(x)| \leq C M_1^{[\alpha]+1} W_{[\alpha]} h_{\frac{2}{\varepsilon} \varepsilon}(K_3Ld(x)), \quad (20)$$

where $C$ and $\rho$ are the constants from [12] and [13] (and $B_1$ is the universal constant from [16, Lemma 4.7]).

**Proof.** By the Leibniz rule,

$$\partial^\alpha(f - T^{p(x)}F)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_i \varphi_{i,p}^{(\alpha-\beta)}(x) \partial^\beta(T_{\frac{p(x)}{p}}F - T^{p(x)}F)(x). \quad (21)$$

Now (17) and (18) imply, that for $x \in Q_t^\varepsilon$ with $d(x) < 1/(3L^2\varepsilon)$,

$$|\partial^\beta(T_{\frac{p(x)}{p}}F - T^{p(x)}F)(x)| \leq C (6L)^{[\beta]+1} \sum_{[\beta]} h_{\frac{2}{\varepsilon} \varepsilon}(3Ld(x)). \quad (22)$$

As in [16] we conclude (using [16, Proposition 4.9]) that there exist $W = W(p) \in \mathfrak{W}$ and $M = M(p) > 0$ such that for all $i \in \mathbb{N}$, all $x \in \mathbb{R}^n \setminus E$ with $d(x) < r_0/(3B_1)$, and all $\beta \in \mathbb{N}^n$,

$$|\varphi_{i,p}^{(\beta)}(x)| \leq M W_{[\beta]} \Pi(p,x) \quad (23)$$
where, by \cite{19} Corollary 3.11,
\[
\Pi(p, x) = \exp \left( \frac{A_1(n)}{p} \sigma^\ast \left( \frac{b_1p}{9A_2(n)} d(x) \right) \right) \leq \left( \frac{e}{h_{\alpha,n}^{\ast}(b_1p/d(\alpha))} \right)^\frac{A_1(n)\beta}{p}, \text{ for some } B \geq 1 \text{ and all } \eta > 0. \quad (24)
\]

($b_1$ is the universal constant from \cite{19} Lemma 4.7) and $A_1(n) \leq A_2(n)$ are constants depending only on $n$. By \cite{20}, we may assume that $S^{2\xi} \leq W$. Then, by \cite{21, 22, 23}, and \cite{19} Lemma 4.7, for $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L \Delta^{2\xi})\}$,
\[
|\partial^\alpha (f - T^{\frac{p}{2}}_x F)(x)|
\leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \cdot 12^n \cdot MW_{|\alpha| - |\beta|} \Pi(p, x) \cdot C(6L)^{\beta + 1} S^{2\xi} \Pi(p, x) h_{\alpha,\beta}(3Ld(x))
\leq 12^n \cdot CM \left( \sum_{j=0}^{\alpha-\beta} \frac{\alpha!}{j!(\alpha - j)!} (6L)^{j+1} W_{|\alpha| - j} S^{2\xi} \right) \Pi(p, x) h_{\alpha,\beta}(3Ld(x))
\leq 6 \cdot 12^n \cdot LCM n^{\alpha} W_{\alpha} \left( \sum_{j=0}^{\alpha-1} \frac{\alpha!}{j!(\alpha - j)!} (6L)^{j+1} \right) \Pi(p, x) h_{\alpha,\beta}(3Ld(x))
= 6 \cdot 12^n \cdot LCM (n + 6Ln)^{\alpha} W_{\alpha} \Pi(p, x) h_{\alpha,\beta}(3Ld(x)).
\]

By Proposition \cite{24} there is $H \geq 1$ (independent of $\xi$) such that $h_{\alpha,\beta}(t) \leq h_{\alpha,\beta}(Ht)^2$ for $t > 0$. Let us choose $L$ according to \cite{25} and such that $p := 2A_2(n)B \prod L/b_1 \geq A_1(n)B$ is an integer. Then, by \cite{26} and since $h_{\alpha,\beta} \leq 1$,
\[
\Pi(p, x) h_{\alpha,\beta}(3Ld(x)) \leq e \frac{h_{\alpha,\beta}(3Ld(x))}{h_{\alpha,\beta}(3HLd(x))} \leq e h_{\alpha,\beta}(3HLd(x))
\]
and we obtain (24). (Note that $M$ depends on $p$, and hence on $L$, which results in the non-explicit dependence of $M_1$.) \hfill \square

- Let us finish the proof of Theorem \cite{27} By \cite{19} and (20), for all $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L \Delta^{2\xi})\}$ and all $\alpha \in \mathbb{N}^n$,
\[
|f^{(\alpha)}(x)| \leq |(T^{\frac{p}{2}}_x F)^{(\alpha)}(x)| + |\partial^\alpha (f - T^{\frac{p}{2}}_x F)(x)| \leq CM^{\alpha+1} W_{\alpha}
\]
for a suitable constant $M = M(n, \omega, L)$.

Let us fix a point $a \in E$ and $\alpha \in \mathbb{N}^n$. Since $\Gamma_x^\xi(t) \to \infty$ as $t \to 0$, we have $|\alpha| < p(x)$ if $x \in \mathbb{R}^n \setminus E$ is sufficiently close to $a$. Thus, as $x \to a$,
\[
|f^{(\alpha)}(x) - F^{\alpha}(a)|
\leq |\partial^\alpha (f - T^{\frac{p}{2}}_x F)(x)| + |(T^{\frac{p}{2}}_x F)^{(\alpha)}(x) - F^{\alpha}(\hat{x})| + |F^{\alpha}(\hat{x}) - F^{\alpha}(a)|
\leq O(h_{\alpha,\beta}(K_3Ld(x))) + O(d(x)) + O(|\hat{x} - a|),
\]
by \cite{13, 14, 20}. Hence $f^{(\alpha)}(x) \to F^{\alpha}(a)$ as $x \to a$. We may conclude that $f \in C^{\infty}(\mathbb{R}^n)$ and extends $F$. After multiplication with a suitable cut-off function of class $\mathcal{B}^{(\omega)}$ with support in $\{x : d(x) < \min\{r_0/(3B_1), 1/(3L \Delta^{2\xi})\}\}$, we find that $f \in \mathcal{B}^{(\omega)}(\mathbb{R}^n)$ thanks to \cite{12, 28, 19} Lemma 2.4(5). The proof of Theorem \cite{27} is complete.
3. Concave, good, and strong weight functions

In [16, Definition 3.5] we called a weight function $\sigma$ good if its associated weight matrix $\mathcal{S}$ satisfies (1). A non-quasianalytic weight function $\omega$ is called strong if there is a constant $C > 0$ such that
\[
\int_1^\infty \frac{\omega(tu)}{u^2} \, du \leq C \omega(t) + C, \quad \text{for all } t > 0.
\]
Otherwise put, $\omega$ is strong if and only if it is equivalent to the concave weight function $\kappa = \kappa(\omega)$ defined in (10). In [16] we asked the following questions:

**Question 3.21:** Is every concave weight function equivalent to a good one?

**Question 5.11:** Is every strong weight function equivalent to a good one?

We will give partial answers to these questions and reveal some related connections in Theorem 11 below.

In [16] it was important that the associated weight matrix itself satisfies (1) as explained after the proof of Proposition 3. Since we could overcome this problem (by introducing $\mathfrak{M} = \{V^f\}$), it is more natural to allow for a wider concept of goodness. For completeness we will also treat the Beurling case. A weight function $\omega$ is called $R$-good if there exists a weight matrix $M$ satisfying
\[
\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} \exists C \geq 1 \forall 1 \leq j \leq k : \frac{\mu_j}{j} \leq C \frac{\nu_k}{k},
\]
such that $B(\omega) = B(\mathfrak{M})$. Recall that $\mu_k := M_k/M_{k-1}$ and $\nu_k := N_k/N_{k-1}$. Similarly, $\omega$ is called $B$-good if there exists a weight matrix $\mathfrak{M}$ satisfying
\[
\forall N \in \mathfrak{M} \exists M \in \mathfrak{M} \exists C \geq 1 \forall 1 \leq j \leq k : \frac{\mu_j}{j} \leq C \frac{\nu_k}{k},
\]
such that $B(\omega) = B(\mathfrak{M})$.

The next lemma, which is inspired by [5, Proposition 4.15], implies that for any weight matrix $\mathfrak{M}$ satisfying (26) (resp. (27)) there is a weight matrix $\mathcal{S}$ consisting of strongly log-convex weight sequences such that $B(\mathfrak{M}) = B(\mathcal{S})$ (resp. $B(\mathfrak{M}) = B(\mathcal{S})$).

**Lemma 8.** Assume that $1 = \mu_0 \leq \mu_1 \leq \cdots$ and $1 = \nu_0 \leq \nu_1 \leq \cdots$ satisfy
\[
\exists C > 0 : \frac{\mu_j}{j} \leq C \frac{\nu_k}{k}, \quad \text{for all } j \leq k.
\]

Then the sequence $\hat{\nu}$ defined by
\[
\frac{\hat{\nu}_k}{k} := \inf_{l \geq k} \frac{\nu_l}{l}, \quad \hat{\nu}_0 := 1,
\]
is such that $\hat{\nu}_k/k$ is increasing and $C^{-1} \mu \leq \hat{\nu} \leq \nu$. \qed

The next two corollaries are immediate from Lemma 8 and results of [12], [13], and [14].

**Corollary 9.** Let $\mathfrak{M}$ be a weight matrix with the property that for all $M \in \mathfrak{M}$ there is $N \in \mathfrak{M}$ such that $(M_{k+1}/N_k)^{1/k}$ is bounded. Consider the following conditions:

(a) $\mathfrak{M}$ satisfies (26).

(b) There is a weight matrix $\mathcal{S}$ consisting of strongly log-convex weight sequences such that $B(\mathfrak{M}) = B(\mathcal{S})$.

(c) $B(\mathfrak{M})$ is stable under composition.

(d) $\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} \exists C > 0 \forall j \leq k : m_{1/j}^{1/j} \leq C n_k^{1/k}$. 

Then (a) ⇔ (b) ⇒ (c) ⇔ (d). If additionally \( \mathfrak{M} \) satisfies
\[
\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} : \mu_k \lesssim N_k^{1/k},
\]
then all four conditions are equivalent.

**Corollary 10.** Let \( \mathfrak{M} \) be a weight matrix with the property that for all \( N \in \mathfrak{M} \) there is \( M \in \mathfrak{M} \) such that \((M_{k+1}/N_k)^{1/k}\) is bounded. Consider the following conditions:

(a) \( \mathfrak{M} \) satisfies \((27)\).

(b) There is a weight matrix \( \mathfrak{S} \) consisting of strongly log-convex weight sequences such that \( B(\mathfrak{M}) = B(\mathfrak{S}) \).

(c) \( B(\mathfrak{M}) \) is stable under composition.

(d) \( \forall N \in \mathfrak{M} \exists M \in \mathfrak{M} \exists C > 0 \forall j \leq k : m_j^{1/j} \leq C n_k^{1/k} \).

Then (a) ⇔ (b) ⇒ (c) ⇔ (d). If additionally \( \mathfrak{M} \) satisfies
\[
\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} : \mu_k \lesssim N_k^{1/k},
\]
then all four conditions are equivalent.

In general, (c) \( \not\Rightarrow \) (b) in neither of the corollaries which follows from [12, Example 3.6]. Note that if \( M = N \) then \((28)\) and \((29)\) reduce to a condition which is usually called moderate growth or \( M \).

For weight functions \( \omega \) we get a full characterization.

**Theorem 11.** Let \( \omega \) be a weight function satisfying \( \omega(t) = o(t) \) as \( t \to \infty \). Then the following are equivalent.

(a) \( \omega \) is equivalent to a concave weight function.

(b) \( \exists C > 0 \exists \lambda_0 > 0 \forall \lambda \geq 1 \forall t \geq t_0 : \omega(\lambda t) \leq C \lambda \omega(t) \).

(c) \( B(\omega) \) is stable under composition.

(d) \( B(\omega) \) is stable under composition.

(e) There is a weight matrix \( \mathfrak{S} \) consisting of strongly log-convex weight sequences such that \( B(\omega) = B(\mathfrak{S}) \).

(f) There is a weight matrix \( \mathfrak{S} \) consisting of strongly log-convex weight sequences such that \( B(\omega) = B(\mathfrak{S}) \).

(g) \( \omega \) is \( R \)-good.

(h) \( \omega \) is \( B \)-good.

Notice that the conditions in the theorem are furthermore equivalent to the classes \( B(\omega) \) and \( B(\omega) \) to be stable under inverse/implicit functions and solving ODEs, and, in terms of the associated weight matrix \( \mathfrak{M} = \{ W^{\xi}_\eta \}_{\xi, \eta > 0} \), to
\[
\forall \xi > 0 \exists \eta > 0 : (w_j^{\xi})^{1/j} \leq C (w_k^{\eta})^{1/k} \quad \text{for } j \leq k,
\]
as well as
\[
\forall \eta > 0 \exists \xi > 0 : (w_j^{\xi})^{1/j} \leq C (w_k^{\eta})^{1/k} \quad \text{for } j \leq k,
\]
see [13]. In the forthcoming paper [4] we shall see that they are also equivalent to the property that \( B(\omega) \), resp. \( B(\omega) \), can be described by almost analytic extensions; see also [11].

**Proof.** The equivalence of the first four conditions (a)–(d) is well-known, see e.g. [13], which is based on [10, Lemma 1] and [3]. That (a) implies (e) and (f) follows from Proposition [4] (e) ⇒ (c) and (f) ⇒ (d) are clear; cf. [12]. The equivalences (e) ⇔ (g) and (f) ⇔ (h) follow from Lemma [8].
Appendix A. Weight matrices

By a weight matrix we mean a family $\mathcal{M}$ of weight sequences $M \geq (k!)^k$ which is totally ordered with respect to the pointwise order relation on sequences, i.e.,

- $\mathcal{M} \subseteq \mathbb{R}^n$,
- each $M \in \mathcal{M}$ is a weight sequence, which means that $M_0 = 1$, $M_1^{1/k} \to \infty$, and $M$ is log-convex,
- each $M \in \mathcal{M}$ satisfies $k! \leq M_k$ for all $k$,
- for all $M, N \in \mathcal{M}$ we have $M \leq N$ or $M \geq N$.

For a weight matrix $\mathcal{M}$ and an open $U \subseteq \mathbb{R}^n$ we consider the Roumieu class

$$B^{(\mathcal{M})}(U) := \text{ind}_{M \in \mathcal{M}} B^{(M)}(U),$$

and the Beurling class

$$B^{(\mathcal{M})}(U) := \text{proj}_{M \in \mathcal{M}} B^{(M)}(U).$$

For weight matrices $\mathcal{M}, \mathcal{N}$ we have (cf. [12])

$$B^{(\mathcal{M})} \subseteq B^{(\mathcal{N})} \iff \forall M \in \mathcal{M} \exists N \in \mathcal{N} : (M_k/N_k)^{1/k} \text{ is bounded},$$

and

$$B^{(\mathcal{N})} \subseteq B^{(\mathcal{M})} \iff \forall N \in \mathcal{N} \exists M \in \mathcal{M} : (M_k/N_k)^{1/k} \text{ is bounded}.$$

Analogous equivalences hold for the local classes

$$E^{(\mathcal{M})}(U) := \text{proj}_{V \subseteq U} B^{(\mathcal{M})}(V) \quad \text{and} \quad E^{(\mathcal{N})}(U) := \text{proj}_{V \subseteq U} B^{(\mathcal{N})}(V).$$

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