Analytical study of fermion determinant and chiral condensate behavior at finite temperatures in toy model approximation.

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Abstract

Fermion determinant is computed analytically on extremely large lattices $N_\tau \to \infty$ in the toy model approximation in which action is truncated so that in the Hamiltonian limit of $a_\tau \to 0$ all terms of order $a_\tau/a_\sigma$ are discarded. Chiral condensate is studied in the area of small ($m \ll T$) quark masses.

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1 Introduction

Finite-temperature studies of the fermion contribution in LGT have been on the agenda over a number of years and both experiments and analytical models were tried to challenge the problem. A crucial issue for such studies is the fate of chiral symmetry in continuum limit. Urgent need to perform calculations in full QCD motivated substantial efforts made to overcome technical problems caused by full dynamical treatment of fermions. Recent developments in the technology of lattice gauge theory made the computation of fermion contribution feasible. Calculations with dynamical fermions and small lattice spacing, however, are still almost unthinkably expensive. Therefore even crude models that allow an analytical solution might be useful.

Unfortunately we can hardly attack the full problem, hence the approximations which hopefully capture some of the essential features of the physics may be considered. Here we attempt to study such a problem in 'electric toy' model approximation [1], where all terms proportional to 1/ξ has been removed \(^1\). Several of the tools we employ exist in some form in the earlier literature on the subject. For example in pure gluodynamics this approximation permits to discard the magnetic part of the action [2, 3]. In this aspect such approximation is similar to strong coupling approximation. Moreover, effective action formally coincides with obtained by [4, 5] in \(g^2 \gg 1\) limit. On highly anisotropic lattice \(\xi \gg 1\), however, the condition \(g^2 \gg 1\) is superfluous and for \(N_\tau \gg 1\) weak coupling region is penetrable which allows to study continuum limit in this model.

In our previous article [1] pure QCD was considered in such approximation. On extremely large lattices ( \(N_\tau \to \infty\) ) the model can be solved analytically and allows to study running coupling behavior in the continuum limit. Results obtained for the Callan-Symanzik beta function are not too surprising, but hardly close to reality since they predict trivial asymptotic freedom \(g^2 \sim a_\tau\). The simplest way to move toward a more realistic case of interacting quarks and gluons is to study dynamical fermions in the same pattern. Although it scarcely makes the model entirely realistic, it seems not incurious to study the chiral condensate behavior at finite temperatures within mentioned approximation.

\(^1\)The anisotropy parameter ξ is defined as ξ \(\sim a_\sigma/a_\tau\), where \(a_\tau\) and \(a_\sigma\) are temporal and spatial lattice spacings, respectively.
In this paper we focus our attention on lattice formulation of QCD using Wilson fermions, only for convenience reasons. As it is known both Wilson and Kogut-Susskind formulations have their advantages (comprehensive analysis of pros and contras is given in [6, 7]) and, of course, the results should agree in continuum limit. Due to well known "no-go" theorem [8] there is no straightforward way to remove fermion spectrum degeneracy without breaking chiral invariance. For Wilson fermions chiral symmetry is broken explicitly by the Wilson term needed to remove the unwanted doublers on the lattice. This makes it difficult to study the spontaneous breaking of chiral symmetry using Wilson fermions. Staggered fermions, indeed, provide massless fermions on the lattice but the continuum notion of chirality is not well defined for staggered fermions due to their inherent lattice construction [9].

For the fermionic part of action we choose the form suggested in [10] (see also [11])

\[- S = n_f \left( a^3 \sum_x \bar{\psi}_x D^0_{x,x'} \psi_{x'} + \frac{a^3}{\xi} \sum_{n=1}^d \sum_x \bar{\psi}_x D^n_{x,x'} \psi_x \right) \]

(1)

where \( n_f \) is the number of flavors and

\[ D^n_{x',x} = \frac{r - \gamma_n}{2} U_n(x) \delta_{x,x'-n} + \frac{r + \gamma_n}{2} U^\dagger_n(x') \delta_{x,x'+n} - r \delta_{x',x} \]

(2)

with

\[ D^0_{x',x} = \frac{r - \gamma_0}{2} U_0(x) \delta_{x,x'-0} + \frac{r + \gamma_0}{2} U^\dagger_0(x') \delta_{x,x'+0} - (m_{\tau} + r) \delta_{x',x} \]

(3)

Extra term with finite Wilson parameter \( r \) (0 < \( r \) ≤ 1 to guarantee positivity) becomes an irrelevant operator for ordinary fermions in the infrared limit, while being a relevant operator for mirror fermions in the high-energy regime, in fact, generating an effective mass for mirror fermions \( m_{\text{eff}} \).

Inasmuch as in suggested approximation we omit the terms of 1/\( \xi \) order, the 'toy' action will be simply \( a^3 \sum_x \bar{\psi}_x D^0_{x,x'} \psi_{x'} \). Since 'toy' fermions propagation in space direction is suppressed by factor 1/\( \xi \), they may hardly claim

\(^2\)Later (unless expressly specified otherwise) we shall consider \( n_f = 1 \), because the generalization is evident.

\(^3\)As it is shown below, in a given approximation it is equal to \( m_{\text{eff}} = m + \frac{2}{a_{\tau}} \arctanh r \).
to be dynamic in the proper sense of the word, nevertheless they bring a nontrivial term into the effective action, which we try to calculate.

Fermion fields $\psi_x$ can be presented as a linear combination of creation $a^\dagger_{p,\sigma}$ (annihilation $a_{p,\sigma}$) operators of the particles and antiparticles $b^\dagger_{-p,-\sigma}$ ($b_{-p,-\sigma}$) with four momenta $p$ and projection of spin $\sigma$

$$
\psi_x = \sum_{p,\sigma} \left( u_{p,\sigma} a_{p,\sigma} + u_{-p,-\sigma} b_{-p,-\sigma}^\dagger \right);
\bar{\psi}_x = \sum_{p,\sigma} \left( \bar{u}_{p,\sigma} a_{p,\sigma}^\dagger + \bar{u}_{-p,-\sigma} b_{-p,-\sigma} \right).
$$

In a standard quantization procedure (see, e.g. [12]) such operators serve as integration variables, however, their combinations may be used as well, on condition that corresponding Jacobian does not turn to zero. To choose a convenient combinations we remind that in a standard representation

$$
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma_n = \begin{pmatrix} 0 & \sigma_n \\ -\sigma_n & 0 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
$$

projectors

$$
\begin{array}{c}
\frac{1 + \gamma_0}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \\
\frac{1 - \gamma_0}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
$$

divide bispinors $\psi$ into two components $\psi(\pm)$, each including only one two-component spinor

$$
\psi(+) = \frac{1 + \gamma_0}{2} \psi = \begin{pmatrix} \psi^+(+) \\ 0 \end{pmatrix}; \quad \psi(-) = \frac{1 - \gamma_0}{2} \psi = \begin{pmatrix} 0 \\ \psi^-(\pm) \end{pmatrix}
$$

(similar to free particles and antiparticles at rest) and are the eigenvalue of inversion operator

$$
P \psi(\pm) = \pm i \psi(\pm)
$$

Taking into account that $(\delta_{x-\mu,x'})^\dagger_x = \delta_{x,x-\mu} = \delta_{x,\mu,x'}$, and presenting

$$
\frac{r \pm \gamma_0}{2} = \frac{1 + \gamma_0}{2} \frac{r \pm 1}{2} + \frac{r \mp 1}{2} \frac{1 - \gamma_0}{2}
$$

we may rewrite (1) in $\psi(\pm)$ terms as

$$
-S_F = \sqrt{1 - r^2} \left( \bar{\psi}^\dagger_{x'} \Delta_{x'x} \psi^{(+)}_x + \bar{\psi}^\dagger_{x'} \Delta_{x'x} \psi^{(-)}_x \right);
$$
where
\[
\Delta_{xx'} = \delta_{xx'} \left( \frac{e^{a_{\tau} m_r} U_0 (x, t)}{2} \delta_{t', t-1} - \frac{e^{-a_{\tau} m_r} U_0^\dagger (x, t')}{2} \delta_{t', t+1} - m' \delta_{t', t} \right) \quad (11)
\]
with
\[
a_{\tau} m_r = \text{arctanh } r \quad (12)
\]
and
\[
a_{\tau} m' = \frac{ma_{\tau} + r}{\sqrt{1 - r^2}} = ma_{\tau} \cosh (a_{\tau} m_r) + \sinh (a_{\tau} m_r) = (m + m_r) a_{\tau} + O \left( a_{\tau}^3 \right) \quad (13)
\]

By gauge transformation all $U_0 (x, t)_{\alpha\nu}$ matrices may be diagonalized simultaneously: $U_0 (x, t)_{\alpha\nu} = \delta_{\alpha\nu} U_0 (x, t)_{\alpha\alpha}$, therefore $\Delta_{xx'}$ in (11) is simply a set of $N$ matrices $N \times N$. If we fix the static gauge $U_0 (x, t)_{\alpha\alpha} = \omega (x)_{\alpha}$, all such matrices will depend only on $t - t'$ therefore they may be diagonalized by the discrete Fourier transformation $\exp (i 2\pi k N_{\tau} t)$ with integer $k$ in for periodic border conditions on fermion fields at time direction, and half-integer $k$ in for antiperiodic ones, demanded by Osterwalder-Schrader [13] positivity condition\(^4\). Formally a solution may be obtain with the help of the simple equation
\[
\frac{1}{N_{\tau}} \sum_k \left( \omega e^{-i \frac{2\pi k}{N_{\tau}}} - \frac{1}{\omega} \exp (i \frac{2\pi k}{N_{\tau}}) \right)^n e^{-i \frac{2\pi k}{N_{\tau}} l} = \sum_{l=-\infty}^{\infty} \left( \frac{n}{n-1+lN_{\tau}} \right) (-1)^l B \omega^{-l+1N_{\tau}} \quad (14)
\]
where $B = 0$ for periodic border conditions and $B = 1$ for aperiodic ones. The summing over $l$ is the tricky procedure and regularization is evidently needed. If, however, we simply change $\exp \left( -i \frac{2\pi k}{N_{\tau}} \right) \rightarrow \exp ( -i \phi )$ and $\frac{1}{N_{\tau}} \sum_k \rightarrow \int_0^{2\pi} \frac{d\phi}{2\pi}$, it will mean $l = 0$ in (14) so that the difference between periodic and antiperiodic border conditions is lost, which is regarded as very essential in

\(^4\)Osterwalder and Schrader [14] developed a mathematical procedure that allows the reconstruction of Hamiltonian and physical Hilbert space from continuum field theory defined in Euclidean space. For Wilson formulation of lattice gauge theory this condition was shown to hold by Osterwalder and Seiler [13].
this case. Hopefully, the computation of determinant
\[
\Delta = \begin{pmatrix}
m'a_r & \lambda U_0^\dagger(x, 0) & 0 & \ldots & -\lambda (-1)^B U_0(x, N_r - 1) \\
-\lambda U_0(x, 0) & m'a_r & \lambda U_0^\dagger(x, 1) & \ldots & 0 \\
0 & -\lambda U_0(x, 1) & m'a_r & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \lambda U_0^\dagger(x, N_r - 2) \\
\lambda (-1)^B U_0^\dagger(x, N_r - 1) & 0 & 0 & \ldots & m'a_r
\end{pmatrix}
\]  
(15)

where \( \lambda \equiv e^{-m'a_r} \), in the considered extremely simple case can be done straightforwardly and, incidentally, this does not even need the gauge fixing (see Appendix). After the integration over \( \psi_{x}^{(\pm1)} \) fields we get
\[
-S_{F}^{\text{eff}} = \ln \det \Delta^\dagger \Delta = \sum_{\alpha} \ln \det \Delta_{\alpha}^\dagger \Delta_{\alpha} = \sum_{\alpha} \Xi_{\alpha}^{\ast} \Xi_{\alpha}
\]  
(16)

with
\[
\Xi_{\alpha}(m') = M_{N_r} + (-1)^{N_r} \frac{1}{2} e^{\frac{m'}{T}} \Omega_{\alpha} + \frac{1}{2} e^{-\frac{m'}{T}} \Omega_{\alpha}^{\ast}
\]  
(17)

where
\[
M_{N_r} = \begin{cases} 
\cosh (N_r \arcsinh m'a_r) \approx \cosh \left(\frac{m'}{T}\right) + O\left(a_{r}^{2}\right); & N_r = 2k; \\
\sinh (N_r \arcsinh m'a_r) \approx \sinh \left(\frac{m'}{T}\right) + O\left(a_{r}^{2}\right); & N_r = 2k + 1.
\end{cases}
\]  
(18)

Therefore for even \( N_r \) we may write
\[
-S_{F}^{\text{eff}} = \sum_{\alpha=1}^{N} \ln \left(\rho^{2} + \cos^{2} \varphi_{\alpha} + (-1)^{B} \left(\cosh \left(\frac{m + 2m_r}{T}\right) + \cosh \left(\frac{m}{T}\right)\right) \cos \varphi_{\alpha}\right)
\]  
(19)

and , respectively, for odd \( N_r \)
\[
-S_{F}^{\text{eff}} = \sum_{\alpha=1}^{N} \ln \left(\rho^{2} - \cos^{2} \varphi_{\alpha} + (-1)^{B} \left(\cosh \left(\frac{m}{T}\right) - \cosh \left(\frac{m + 2m_r}{T}\right)\right) \cos \varphi_{\alpha}\right)
\]  
(20)

where
\[
\rho^{2} = 1 + \sinh^{2} \frac{m + m_r}{T} + \sinh^{2} \frac{m_r}{T}.
\]  
(21)

Both (19) and (20) can be nicely expressed as
\[
-S_{F}^{\text{eff}} = \ln \prod_{\alpha=1}^{N} (z_{r} - \cos \varphi_{\alpha}) \left(z - (-1)^{N_r} \cos \varphi_{\alpha}\right);
\]  
(22)
with\footnote{The parameter \( r \) was introduced to shift the mass of mirror fermion. As it follows from definitions \((12)\) and \((13)\) \( m' - m_r = m \) and \( m' + m_r = m + 2m_r \), so the suggested parametrization \( r = \tanh a \tau m_r \), makes the machinery of such shift more transparent. By increasing \( m_r \) one can make the doublers as heavy as the cutoff. It is easy to check that if \( \text{Im} r = 0 \) it does not introduce any imaginary part into the action.} \( z = (-1)^B \cosh \frac{m}{T}; \) \( z_r = (-1)^B \cosh \frac{m + 2m_r}{T} \) \( (23) \)

In case of \( SU(2) \) gauge group \( \varphi_1 = -\varphi_2 = \frac{\chi}{2} = \frac{1}{2} \arccos \frac{\chi}{2} \), where \( \chi \) is the character of fundamental representation, so \((22)\) can be easily expressed in invariant form

\[- S_{\text{eff}}^F = \ln \prod_{\alpha=1}^N \left( \cosh \frac{m}{T} + (-1)^{B+N_r} \frac{\chi}{2} \right) \left( \cosh \frac{m + 2m_r}{T} + (-1)^B \frac{\chi}{2} \right) ; \] \( (24) \)

In case of \( SU(3) \) gauge group we can express \( S_{\text{eff}}^F \) through the characters of fundamental representation \( \chi = \sum_{\alpha=1}^3 e^{i\varphi_\alpha} \) with the help of simple relation

\[
3 \prod_{\alpha=1}^3 \left( 1 + pe^{i\varphi_\alpha} + qe^{-i\varphi_\alpha} \right) = 1 + p^3 + q^3 - 3pq + \left( p + q^2 - 2p^2 q \right) \chi + (q + p^2 - 2pq^2) \chi^* + pq \chi^2 + p^2 q \chi^* \chi + pq \chi \chi^* ; \] \( (25) \)

that, e.g. in case of even \( N_r \), leads to

\[- S_{\text{eff}}^F = \ln f \left( \cosh \frac{m}{T} \right) + \ln f \left( \cosh \frac{m + 2m_r}{T} \right) ; \] \( (26) \)

with

\[
f (x) = x^3 PQ^2 + \frac{\chi + \chi^*}{2} x^2 PQ + (-1)^B \frac{\chi^2 + (\chi^*)^2}{8} + \frac{\chi \chi^*}{4} x ; \] \( (27) \)

and

\[
P(x) = 1 + \frac{(-1)^B}{x} ; \quad Q(x) = (-1)^B - \frac{1}{2x} \] \( (28) \)

As it can be seen from \((26)\) the effective action \( S_{\text{eff}}^F \) looses \( Z(N) \) invariance, already in zero order in \( 1/\xi \). Quark effects lift \( Z(N) \) degeneracy and only the phase in which Polyakov loop is real is stable. Other phases are
metastable. As it can be seen from (26), the strength of $Z(N)$ symmetry breaking decreases with increasing mass

$$- S_{\text{eff}}^{\text{f}} \simeq 2N \left( \frac{m + m_r}{T} - \ln 2 \right) + (-1)^B \left( (-1)^{N_r} e^{-\frac{m}{T}} + e^{-\frac{m + 2m_r}{T}} \right) \frac{\text{Re} \chi}{2},$$

(29)
in agreement with the reasons given in [13]. This will modify the thermodynamic properties of metastable phases, the lifetime of which increases with decreasing strength of $Z(N)$–symmetry breaking.

It is noteworthy to note that, if the integrand of partition function is highly peaked in proximity to $\chi_{\text{min}} = \sum_{\alpha} \cos \varphi_{\text{min}}^{\alpha}$, with fixed $\varphi_{\text{min}}^{\alpha}$, the partition function has zeroes in complex $m$-plane at

$$m_{\alpha} = \pm i (\varphi_{\alpha} + \pi n) T; \quad m'_{\alpha} = \pm i (\varphi_{\alpha} + \pi n) T + 2m_r; \quad n = \text{mod} (B + N_r + 1)$$

(30)

As it can be seen from (13), $M_{N_r}$ functions are polynomials in $m' a_r$ of order $N_r$, therefore, at finite $a_r$ the equation $\Xi_{\alpha} (m') = 0$ has $N_r$ solutions $\{m'_t\}$ which are, generally speaking, different and formally can be written as:

$$\frac{m'}{T} = \begin{cases} N_r \sinh \left( \frac{1}{N_r} \arccosh \left( \frac{(-1)^{\frac{B+1}{2}}}{2} \left( e^{\frac{m_r}{T} \Omega_{\alpha}} + e^{-\frac{m_r}{T} \Omega_{\alpha}} \right) \right) \right); & N_r = 2k; \\ N_r \sinh \left( \frac{1}{N_r} \arcsinh \left( \frac{(-1)^{\frac{B+1}{2}}}{2} \left( -e^{\frac{m_r}{T} \Omega_{\alpha}} + e^{-\frac{m_r}{T} \Omega_{\alpha}} \right) \right) \right); & N_r = 2k + 1. \end{cases}$$

(31)

For finite $a_r$ and $N_r$ the zeroes of $\text{det} \Delta \Delta^\dagger$ dispersed in $m$-plane, but with $a_r \to 0$, and $N_r \to \infty$ they steadily concentrate around $m_{\alpha}$ and $m'_{\alpha}$.

From (24) it is evident that fermion loop contribution increases in the parameter area where such zeroes approach the origin of coordinates in $m$-plane (while $T$ is fixed) and simultaneously $|\chi_{\text{min}}| \to 1$.

As it is known (see e.g., [16] and references therein), the chiral condensate can be expressed in terms of its zeros $m_t = \frac{m'_t}{\cosh(m_r a_r)} - \frac{\tanh(m_r a_r)}{a_r}$ as:

$$\langle \bar{\psi} \psi \rangle = \frac{T}{V_{\sigma}} \sum_{m_t} \frac{1}{m - m_t},$$

(32)

where $V_{\sigma}$ is spatial lattice volume. The behavior of chiral condensate in QCD can be investigated [16] having studied the distribution of partition

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\footnote{\textit{M}_{N_r} functions are very similar to Chebyshev polynomial, e.g., $\cosh (n \arcsinh \sqrt{z^2 + 1}) = T_n (x) = \cos (n \arccos x)$}
function zeros in the complex quark mass plane. Zeroes at (30), certainly, don’t exhaust the whole set of \( \{ m_r \} \), but seem likely to represent a very important part of it.

To finish with this issue we want to note that in a toy model approximation the fermion contribution is simply a ’potential’-like term and can be totally incorporated into the measure. For example in case of \( SU(2) \) gauge group the modified measure has the following form

\[
d\tilde{\mu} = \left( \cosh \frac{m}{T} + (-1)^{B+N_r} \cos \frac{\varphi}{2} \right) \left( \cosh \frac{m + 2m_r}{T} + (-1)^B \cos \frac{\varphi}{2} \right) \sin^2 \frac{\varphi}{2} d\varphi
\]

(33)

For arbitrary flavor number \( n_f \) it can be written as

\[
d\tilde{\mu}_{n_f} = \left( \cosh \frac{m}{T} + (-1)^{B+N_r} \cos \frac{\varphi}{2} \right)^{n_f} \left( \cosh \frac{m + 2m_r}{T} + (-1)^B \cos \frac{\varphi}{2} \right)^{n_f} \sin^2 \frac{\varphi}{2} d\varphi
\]

(34)

Therefore, in a toy model approximation the fermion contribution does not seriously complicate the partition function \( Z \) and thereby the computation of any average values. Here we concentrate on the evaluation of \( \langle \bar{\psi} \psi \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m} \).

2 Chiral condensate

In case of \( SU(3) \) gauge group in the area \( \frac{m}{T} \gg 1 \) one may write

\[
\langle \bar{\psi} \psi \rangle \simeq \frac{1}{z} \frac{\partial z}{\partial m} + \frac{1}{z_r} \frac{\partial z_r}{\partial m} + \left( \frac{(-1)^{N_r}}{z^2} \frac{\partial z}{\partial m} + \frac{1}{z_r^2} \frac{\partial z_r}{\partial m} \right) \langle \chi \rangle + ...
\]

(35)

\[
\simeq \left( \tanh \frac{m}{T} + \tanh \frac{m + 2m_r}{T} \right) \left( 1 + (-1)^{B} \left( \frac{(-1)^{N_r}}{\cosh \frac{m}{T}} + \frac{1}{\cosh \frac{m + 2m_r}{T}} \right) \langle \chi \rangle \right)
\]

which means strong breakdown of chiral symmetry. Therefore, gluon environment influence on \( \langle \bar{\psi} \psi \rangle \) value became inappreciable when \( \frac{m}{T} \) infinitely increases.

To compute \( \langle \bar{\psi} \psi \rangle \) in the area \( \frac{m}{T} \ll 1; \frac{m_r}{T} \ll 1 \), we use ’mean spin method’ \([7]\) which in fact is a modification of the method of quasiaverages \([8]\)

\[
\langle A \rangle = \int A \delta \left( \frac{\bar{N}}{2} N_\sigma - \sum_x \cos \varphi_x \right) \prod_x d\tilde{\mu}_x
\]

(36)
The effective action $\bar{S} = \bar{S}(\bar{\chi})$ can be computed as

$$e^{-S(\bar{\chi})} \equiv \left\langle e^{-\bar{S} \bar{\chi}} \right\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle e^{-\bar{S}} \right\rangle_n \left\langle 1 \right\rangle_{\bar{\chi}} \left\langle 1 \right\rangle_{\bar{\chi}} = \infty \sum_{n=0}^{\infty} \left( \frac{-1}{n!} \right)^n \left\langle S \right\rangle_n \left\langle \chi \right\rangle_{\chi+n} \left\langle \chi \right\rangle_{\chi+n} = \left\langle \chi \right\rangle_{\chi+n} \left\langle \chi \right\rangle_{\chi+n}$$

and in the lowest order can be obtained simply by substitution $2 \cos \varphi_\chi \to \bar{\chi}$. In such approximation the results coincide with a mean field method.

Let us consider for simplicity the case of $SU(2)$ gauge group were the
gluonic part of action for a ‘toy’ model \[1\] can be written as

$$-S = 4\beta \sum_{x,n} \cos \frac{\varphi_\chi}{2} \cos \frac{\varphi_{\chi+n}}{2} = \beta \sum_{x,n} \chi_{\chi+n}$$

and fermion contribution is incorporated in the measure $d\mu$ given by (33).

Since we introduced the collective variable $\bar{\chi} = \frac{2}{N_\sigma} \sum_{x} \cos \varphi_\chi$ the corresponding Jacobian $\left\langle 1 \right\rangle_{\bar{\chi}}$ must be computed. Having written

$$\delta \left( \frac{\bar{\chi}}{2} N^d_\sigma - \sum_{x} \cos \varphi_\chi \right) = \int_{c-i\infty}^{c+i\infty} \exp \left\{ s \left( \frac{\bar{\chi}}{2} N^d_\sigma - \sum_{x} \cos \varphi_\chi \right) \right\} \frac{ds}{2\pi i}$$

we find

$$\left\langle 1 \right\rangle_{\bar{\chi}} = \exp \left( -N^d_\sigma \bar{L} \right) = \int_{c-i\infty}^{c+i\infty} \exp \left\{ N^d_\sigma \left( \frac{\bar{\chi}}{2} s - L(s) \right) \right\} \frac{ds}{2\pi i}$$

where

$$e^{-L(s)} = \int_0^{4\pi} \exp \left\{ s \cos \frac{\varphi}{2} \right\} d\mu = \left( z_r - \frac{\partial}{\partial s} \right) \left( z - \frac{\partial}{\partial s} \right) I_1(s) s$$

or

$$-L(s) = \ln \left( \left( z_r z + 2 \frac{2}{s} (z_r + z) + 6 \right) \frac{I_1(s)}{s} - \left( z_r + z + \frac{3}{s} \right) \frac{I_0(s)}{s} \right)$$

As we have already mentioned above it coincides with the action suggested in \[4, 5\] (see also \[3\]).

Here we take $(B = 0; N_r = 2k)$ as an example, other cases can be easily obtained after the substitution of $z \to -z$ or $z_r \to -z_r$. 

10
Due to the fact that $N^d \rightarrow \infty$, the main contribution into $\mathcal{L}$ comes from $s \sim 0$, so

$$-L(s) \simeq \ln \left( \frac{z_r z + \frac{1}{2}}{2} \right) - \frac{3\eta s}{4} + \frac{s^2}{16\beta_0} + O\left(s^3\right)$$  \hspace{1cm} (43)$$

with

$$\beta_0 = \frac{(z_r z + \frac{1}{2})^2}{(z_r - \frac{1}{2}) (z^2 - \frac{1}{2}) + z_r z (z_r z + \frac{1}{2})};$$  \hspace{1cm} (44)$$

and

$$\eta = \frac{1}{3} \frac{z_r + z}{z_r z + \frac{1}{4}}.$$  \hspace{1cm} (45)$$

As far as $N^d \rightarrow \infty$, the function $\mathcal{L}$ may be computed by the steepest descent method, with good accuracy. The saddle point $s_0 = s_0(\bar{\chi})$ is given by the equation

$$\frac{\partial L(s_0)}{\partial s_0} \simeq \frac{3}{4} \eta - \frac{s_0}{8\beta_0} + O\left(s_0^2\right) = \bar{\chi}$$  \hspace{1cm} (46)$$

so

$$s_0 \simeq 4\beta_0 \left( \frac{3}{2} \eta - \bar{\chi} \right)$$  \hspace{1cm} (47)$$

and we obtain

$$-\mathcal{L}(s_0) \simeq -L(s_0) + s_0 \bar{\chi} \simeq -3\beta_0 \left( \bar{\chi} - \frac{3}{2} \eta \right) \left( \bar{\chi} - \frac{1}{2} \eta \right).$$  \hspace{1cm} (48)$$

Therefore we can finally write

$$-\bar{S}(\bar{\chi}) = 3 \left( \bar{\beta} - \beta_0 \right) \bar{\chi}^2 + 6\eta \beta_0 \bar{\chi} + \text{const}$$  \hspace{1cm} (49)$$

From the conditions

$$-\bar{S}(\bar{\chi})' = 6 \left( \bar{\beta} - \beta_0 \right) \bar{\chi} + 6\eta \beta_0; \quad -\bar{S}(\bar{\chi})'' = 6 \left( \bar{\beta} - \beta_0 \right);$$  \hspace{1cm} (50)$$

we see that $\bar{S}(\bar{\chi})$ has the minimum for $\bar{\beta} < \beta_0$ at

$$\bar{\chi}_{\text{min}} = \bar{\chi}_0 \equiv \frac{\eta}{1 - \bar{\beta}/\beta_0}.$$  \hspace{1cm} (51)$$

This solution (51) dominate for $\bar{\chi}_0^2 < 4$, but at points $\bar{\chi}_0 = \pm 2$ corresponding to

$$\bar{\beta} = \beta_c \equiv \beta_0 \left( 1 - \frac{1}{2} |\eta| \right);$$  \hspace{1cm} (52)$$

11
the regime changes. For $\tilde{\beta} > \beta_c$ the value $\bar{\chi}^2_{\text{min}}$ keeps to 4, so

$$\bar{\chi}^2_{\text{min}} = \min \{ 4; \bar{\chi}^2_0 \}; \quad \text{sign} (\bar{\chi}_{\text{min}}) = \text{sign} (\eta) \quad (53)$$

As it is seen from (52), 'toy' fermions contribution leads to a shift of the effective coupling. For all $m_r$, both $\beta_0$ and $|\eta|$ increase with increasing $m$. This leads to decreasing $\beta_c = \beta_c (\frac{m}{T})$ at small $m$, but for $m \gtrsim T$ it starts to slowly increase to the asymptotic value $\beta_c (\infty) = \frac{1}{2}$.

An average value of $\langle Q \rangle$ of some $Q = Q (\cos \frac{\varphi}{2})$ can be computed simply as

$$\langle Q \rangle = \left( \frac{\int_0^{4\pi} Q (\cos \frac{\varphi}{2}) \exp \left\{ s \cos \frac{\varphi}{2} \right\} d\tilde{\mu}}{\int_0^{4\pi} \exp \left\{ s \cos \frac{\varphi}{2} \right\} d\tilde{\mu}} \right)_{s=s_0} = \left( e^{L(s)} Q \left( \frac{\partial}{\partial s} \right) e^{-L(s)} \right)_{s=s_0} \quad (54)$$

with

$$s_0 \simeq 4 \beta_0 \left( \frac{3}{2} \eta - \bar{\chi}_{\text{min}} \right) \quad (55)$$

where $\bar{\chi}_{\text{min}}$ is defined by (53).

In particular, taking into account that

$$\left\langle \frac{1}{z_r - \cos \frac{\varphi}{2}} \right\rangle = e^{L(s)} \left( z - \frac{\partial}{\partial s} \right) \frac{I_1 (s)}{s} = \frac{z}{z_r z + \frac{1}{4} - \beta_0 \left( \bar{\chi}_{\text{min}} - \frac{3}{2} \eta \right)} \frac{z - \frac{1}{4}}{(z_r z + \frac{1}{4})^2} \quad (56)$$

we obtain for $\langle \bar{\psi} \psi \rangle$

$$\langle \bar{\psi} \psi \rangle = \frac{\partial z}{\partial m} \left( \frac{1}{z + (-1)^B z_r \cos \frac{\varphi}{2}} \right) + \frac{\partial z_r}{\partial m} \left( \frac{1}{z_r + (-1)^B \cos \frac{\varphi}{2}} \right) \quad (57)$$

which for the case considered (57) gives for the even $N_r$

$$\langle \bar{\psi} \psi \rangle = \frac{24 m + m_r}{11} \left( 1 - \frac{|\bar{\chi}_{\text{min}}|}{3} \right) \quad (58)$$

and for odd $N_r$

$$\langle \bar{\psi} \psi \rangle \simeq \frac{8 m}{3} \left( 1 - \frac{|\bar{\chi}_{\text{min}}|}{2} \right) \quad (59)$$

It can be shown that such shift becomes a bit larger for a number of flavours $n_f = 2$, but leaves $\beta_c > 0$, therefore does not wash out the phase transition.
It is easy to see that at critical point (where $|\bar{\chi}_{\text{min}}| \simeq 2$) both (58) and (59) became comparatively small but still remain finite.

A very important feature of the chiral condensate is the behavior of $\langle \bar{\psi} \psi \rangle$ with decreasing quark mass $m$. Excellent fit of experimental data (in the symmetric phase of QED$_4$) in the fermion mass range $0.01 - 0.06$ is given in [19] by the relation

$$\langle \bar{\psi} \psi \rangle \times \ln \frac{1}{\langle \bar{\psi} \psi \rangle} \sim m; \quad \varepsilon \sim \frac{1}{4}$$

(60)

The above roughly agrees with (58) that predicts that $\langle \bar{\psi} \psi \rangle \sim \frac{m}{T}$. It easy to check that the data in [20] (obtained for $SU(3)$ group with flavors number $n_f = 2$ on lattice $N_r = 6$ ) are consistent with $\langle \bar{\psi} \psi \rangle \simeq 6am = \frac{m}{T}$, which gives a reason to believe that (58) indeed provides a reasonable description of the data.

3 Discussion

It is commonly assumed that, quenched lattice QCD studies using Wilson-Dirac fermions bring about large statistical errors in calculations involving very light quarks. Indeed, in quenched simulations, using Wilson fermions, lattice spacing effects were shown to be a major problem [21]. Fortunately, in the parameter area available for modern computers, sea quark contribution does not introduce substantial changes in MC data, except on occasions. For example, comparison of spectroscopy results obtained with dynamical and quenched fermions does not reveal any dramatic differences. Apparently at the parameter values of the simulation the sea quarks simply do not affect spectroscopy above the five to ten per cent level [20]. The measurements of valence observables in an almost-quenched system show the physics with light dynamical quarks to be very similar to quenched physics with the same masses [22]. The comparison between Wilson and clover actions showed that the results were agreeable [23] and no significant impact of unquenching was observed [24] on the hyperfine splitting as well. Therefore, the major part of MC data helps to trace the sea quark contribution, rather than obligatory makes us take the latter into account.

Nevertheless, when the area of small quark masses and lattice spacing will be accessible in MC simulations, the situation presumably may change. Even with current limited computer capacity, there are some parameter areas
where the sea quark contribution is considerable. Indeed, in [25] effects of dynamical fermions were singled out within the confinement phase, by comparison of the quenched and full formulations of compact QED with Wilson fermions. It was established, that in the strong coupling limit (\( \beta = 0 \)) the quenched theory was a good approximation of the full one but appeared to be in sharp contrast with it at \( \beta = 0.8 \). At such \( \beta \) the physics changes significantly and the formation of metastable states was observed [25] presumably due to the presence of dynamical fermions.

We hope that suggested toy model approximation will help not only to estimate the sea quark contribution at lattices with size and spacings unaccessible for MC experiment, but may help to qualify the parameter area where the presence of dynamical fermions leads to particularly appreciable effects.

4 Conclusions

Suggested approach to the problem of dynamical fermion effect estimation is based on the discard of all terms of \( 1/\xi \) order in the action. Therefore, only quark lines winding the lattice in the time direction are possible. The model allows an analytical solution for small values of lattice spacings and light quark masses on infinitely large lattices, where MC simulations easily become prohibitively costly. For the vanishing chemical potential \( \mu \), the fermion determinant remains real and can be incorporated into the measure. Although such contribution is almost trivial, it significantly changes the phase structure of the model. It is believed that chiral symmetry violated by Wilson term is restored in the continuum limit, however, no such restoration is revealed in the model at finite \( \frac{m}{T} \) and \( \frac{\mu}{T} \).

In fact, two issues, which are fundamental to understand the consequences of such approximation, have not been fully clarified: a remarkable difference between the results obtained on lattices with odd and even \( N_\tau \) (see also [26]) as well as very marginal alterity between periodic and antiperiodic border conditions.

The first direct calculation of QCD properties for small quark masses was fulfilled long ago [27]. Substantial diversity between periodic and antiperiodic border conditions was established. However, it was also pointed out that such difference might smear out with increasing lattice volume. As it is seen from (24) and (26), alterity between periodic and antiperiodic bor-
der conditions formally does not disappear at any large $N_T$ and hardly will be still unobservable at higher orders in $\frac{m}{\tau}$ and $\frac{m_T}{m}$. As it follows from (22) $(-1)^B$ factor that reflects the difference between periodic and antiperiodic conditions for even $N$ may be absorbed into $Z(N)$-transformation. Therefore, there is still a chance that even broken $Z(2)$-invariance may smear the mentioned difference, so an additional consideration of $N = 3$ is desirable.

Another problem is that for large Wilson parameter $r \rightarrow 1$ (or $m_r \rightarrow \infty$) the condensate $\langle \bar{\psi} \psi \rangle$ does not vanish at any finite temperatures. Moreover, at given approximation (in contrast to suggestion made in [25]) chiral symmetry broken by the Wilson term can hardly be recovered by any fine-tuning of the bare parameters in the continuum limit.

5 Appendix

Expanding $F_{n+1}$

$$F_{n+1} = \begin{pmatrix}
q_{n+1} & b_n & 0 & \ldots & 0 & c_{n+1} \\
c_n & q_n & b_{n-1} & \ldots & 0 & 0 \\
0 & c_{n-1} & q_{n-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & q_2 & b_1 \\
b_{n+1} & 0 & 0 & \ldots & c_1 & q_1
\end{pmatrix}, \quad (61)$$

in elements of the first row

$$F_{n+1} = q_{n+1}f_n + b_n \frac{\partial}{\partial b_n} F_{n+1} + c_{n+1} \frac{\partial}{\partial c_{n+1}} F_{n+1}, \quad (62)$$

we obtain:

$$b_n \frac{\partial}{\partial b_n} F_{n+1} = -b_n c_n f_{n-1} - \prod_{j=1}^{n+1} (-b_j) \quad (63)$$

and

$$c_{n+1} \frac{\partial}{\partial c_{n+1}} F_{n+1} = -c_{n+1} b_{n+1} f_n^{(2)} - \sum_{j=1}^{n+1} (-c_j) \quad (64)$$

where

$$f_n = q_n f_{n-1} - b_{n-1} c_{n-1} f_{n-2} = F_n (b_n = c_n = 0) \quad (65)$$
and
\[ f^{(2)}_n = \frac{\partial}{\partial q_{n+1}} \frac{\partial}{\partial q_1} F_{n+1} \]  
(66)

In a simple case of \( q_j \) and the product \( b_j c_j \) independent of \( j \),
\[ q_j = q; \quad b_n = \frac{\lambda}{\theta_n}; \quad c_n = -\lambda \theta_n \]  
(67)

we can write
\[ f^{(2)}_j = f_{j-1} \]  
(68)
and
\[ f_{j+1} - q f_j - \lambda^2 f_{j-1} = 0 \]  
(69)

Equation (69) can be easily solved by introducing the generating function
\[ f(z) = \sum_{j=n_0}^{\infty} f_j z^{j-1} \]  
(70)

This leads to
\[ \sum_{j=n_0}^{\infty} \left( \frac{1}{z} f_{j+1} z^{j+1} - q f_j z^j - \lambda^2 z f_{j-1} z^{j-1} \right) = \left( \frac{1}{z} - q - \lambda^2 z \right) f(z) = 0 \]  
(71)

where \( n_0 \) is an arbitrary fixed number. For example for \( n_0 = 2 \) or \( n_0 = 3 \) from (61) we easily find
\[ f_2 = q^2 + \lambda^2 \quad f_3 = q^3 + 2\lambda^2 q \]  
(72)

therefore, general solution may be written as
\[ f_n = \frac{f_{n_0}}{2\pi i} \oint \frac{z^{-n-1}dz}{z - q - \lambda^2 z} = \left\{ \begin{array}{ll}
2\lambda^n \cosh \left( n \arcsinh \frac{q}{2\lambda} \right) & ; \quad n = 2k; \\
2\lambda^n \sinh \left( n \arcsinh \frac{q}{2\lambda} \right) & ; \quad n = 2k + 1.
\end{array} \right. \]  
(73)

Having collected everything we may finally write for \( F_n \)
\[ \frac{F_n}{2\lambda^n} = \left\{ \begin{array}{ll}
\cosh \left( n \arcsinh \frac{q}{2\lambda} \right) - \frac{1+\Theta}{2} & ; \quad n = 2k; \\
\sinh \left( n \arcsinh \frac{q}{2\lambda} \right) - \frac{1-\Theta}{2} & ; \quad n = 2k + 1,
\end{array} \right. \]  
(74)

where
\[ \Theta = \prod_{j=1}^{n} \theta_j \]  
(75)
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