Abstract

I consider magnetic Schrödinger operator in dimension $d = 2$ assuming that coefficients are smooth and magnetic field is non-degenerating. Then I extend the remainder estimate $O(\mu^{-1}h^{-1} + 1)$ derived in [Ivr1] for the case when $V/F$ has no stationary points to the case when it has non-degenerating stationary points. If some of them are saddles and $\mu^3 h \geq 2$ then asymptotics contains correction terms of magnitude $\mu^{-1}h^{-1}\log\mu^3h$.

0 Introduction

I consider spectral asymptotics of the magnetic Schrödinger operator

\begin{equation}
A = \frac{1}{2} \left( \sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = D_j - \mu V_j
\end{equation}

where $g^{jk}, V_j, V$ are smooth real-valued functions of $x \in \mathbb{R}^2$ and $(g^{jk})$ is positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a coupling parameter. I assume that $A$ is a self-adjoint operator and all the conditions are satisfied in the ball $B(0, 1)$.

In contrast to my recent papers [Ivr3, Ivr4, Ivr5] I assume that all the coefficients are very smooth; in contrast to [Ivr4] I consider only two-dimensional case here and in contrast to [Ivr6] I assume that magnetic field is non-degenerate. So I am completely in frames of

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section 6 [Ivr1] where I just forgot to consider the case of \( V/F \) having non-degenerating stationary points. My analysis will be sketchy, more details I will publish in the future. Thus this note together with Chapter 6 of [Ivr1] and with [Ivr6] completely covers generic 2-dimensional smooth case. One can generalize these results to non-smooth case using approach of [Ivr3].

Let \( g = \det(g^{jk})^{-1} \), \( F_{12} = \partial x_1 V_2 - \partial x_2 V_1 \) and \( F = |F_{12}g^{-\frac{1}{2}}| \) which is a scalar intensity of the magnetic field, \( g = \det(g^{jk})^{-2} \). I assume that both \( V \) and \( F \) are disjoint from 0:

\[
\sum_{jk} g^{jk} \xi_j \xi_k \geq \varepsilon |\xi|^2 \quad \forall \xi \in \mathbb{R}^2, 
\]

\[
V \geq \varepsilon_0, 
\]

\[
F \geq \varepsilon_0. 
\]

In this note I am going to consider the case when \( V/F \) has non-degenerate critical points and I will recover the same asymptotics and remainder estimate as either \( \mu \leq Ch^{-\frac{4}{3}} \) or \( V/F \) has no saddle points in the domain in question and there will be correction terms of magnitude \( \mu^{-1}h^{-1}|\log(\mu^3h)| \) associated with saddle points as \( \mu \geq 2h^{-\frac{1}{3}} \).

I am interested in asymptotics of \( \int e(x,x,0)\psi(x)\,dx \) as \( \mu \to +\infty, h \to 0 \) where \( e(x,y,\tau) \) is the Schwartz kernel of the spectral projector of \( A \) and \( \psi \in C_0^\infty(B(0, \frac{1}{2})) \).

**Theorem 1.** Let operator \( A \) defined by (1) with real-valued \( g^{jk}, V_j, V \) be self-adjoint in \( L^2(X) \). Further \( g^{jk}, V_j, V, \psi \) be smooth enough in \( B(0,1) \) and conditions (2) – (4) be fulfilled and there, let \( B(0,1) \subset X \). Finally, let all critical points of \( V/F \) in \( B(0,1) \) be non-degenerate. Then

(i) As \( 1 \leq \mu \leq h^{-\frac{1}{2}} \) the standard asymptotics holds (i.e. (5) – (6) without correction terms);

(ii) As \( h^{-\frac{1}{2}} \leq \mu \leq Ch^{-1} \) the following asymptotics holds

\[
|\int \left( e(x,x,0) - \mathcal{E}^{MW}(x,0) \right)\psi(x)\,dx - \sum_j \mathcal{E}^{MW}_{\text{corr}}(x_j)\psi(x_j)| \leq C\mu^{-1}h^{-1} + C 
\]

with summation over all saddle points \( x_j \) of \( V/F \) where

\[
\mathcal{E}^{MW}(x,0) = \frac{1}{2\pi} \sum_{n \geq 0} \theta \left( \tau - V(x) - (2n + 1)F\mu h \right) F\mu h^{-1} 
\]

is magnetic Weyl expression, and

\[
\mathcal{E}^{MW}_{\text{corr}} = \kappa \log \left( (\sigma + \mu^{-2})(1 + \mu^{-1}h^{-1}) \right) 
\]
where

$$\sigma(x) = \min_{n \in \mathbb{Z}^+} |V + (2n + 1)F\mu h|$$

and $\kappa$ is defined by (13); further, as $C(h|\log h|)^{-1} \leq \mu \leq \epsilon h^{-1}$ one must include in $E_{\text{corr}}^{\text{MW}}$

$$E_{\text{corr}}^{\text{MW}}(x) = \kappa_2 \mu h \log \left((\sigma + h^2)(1 + \mu^{-1}h^{-1})\right)$$

again associated with saddle points.

**Theorem 2.** Let operator $A$ defined by (1) with real-valued $g_{jk}, V_j$ be self-adjoint in $L^2(X)$. Further $g_{jk}, V_j, V, \psi$ be smooth enough in $B(0, 1)$ and conditions (2), (4) be fulfilled and there, let $B(0, 1) \subset X$. Further, let $h^{-1} \leq \mu$ and $V = -(2n + 1)\mu h F + W$ with smooth bounded $W$. Finally, let each critical point of $W/F$ in $B(0, 1)$ be either non-degenerate or satisfy $|W| \geq \epsilon_0$. Then asymptotics (5) holds with extra correction term $\mu h \int \psi(x) dx$ as $\mu \leq C\mu^{-3}|\log h|^{-1}$; for larger $\mu$ correction term contains also more complicated $O(\mu h^3|\log h|)$ terms.

**Remark 3.** One can drop condition (3) by rescaling arguments after main theorem 1 is established.

## 1 Ideas of the proof: weak magnetic field case

As $\mu \leq h^{-1+\delta}$ in zone $\{|\nabla V| \geq \rho = C(\mu h)^{\frac{1}{2}}h^{-\delta}\}$ one can apply weak magnetic field approach (see section 6.3 of [Ivr1]) and derive remainder estimate $O(\mu^{-1}h^{-1} + \rho^2\mu^{-1})$; furthermore, with logarithmic uncertainty principle replacing the standard microlocal uncertainty principle (see [BrIvr, Ivr3]) one can derive this remainder estimate with $\rho = C(\mu h)^{\frac{1}{2}}|\log h|$. This leads to the proof of the standard asymptotics with the remainder estimate $O(\mu^{-1}h^{-1})$ as $\mu \leq C(h|\log h|)^{-1}$. Furthermore, based on the canonical form (10) (see next section) one can prove the same asymptotics and the remainder estimate with $\rho = C(\mu h)^{\frac{1}{2}}$ and therefore achieve remainder estimate $O(\mu^{-1}h^{-1})$ as $\mu \leq C\mu^{-\frac{1}{2}}$, thus proving Theorem 1(i).

1) Where here and below $\delta, \delta', \ldots$ denote arbitrarily small positive exponents.
2 Ideas of the proof: intermediate and strong magnetic field cases

To prove Theorem 1(ii) and calculate correction term let me remind that according to section 6.4 of [Ivr1] one can reduce microlocally operator (1) to the canonical form

$$\sim \sum_{m,l,k:m+l \geq 1} a_{mnk}(x_2, hD_2) \left( h^2 D_1^2 + \mu^2 x_1^2 \right)^{m} \mu^{2-2m-2l} (\mu^{-1} h)^{2k}, \quad \hbar = \mu^{-1} h. $$

Then replacing harmonic oscillator \( \left( h^2 D_1^2 + \mu^2 x_1^2 \right) \) by its eigenvalues \( (2n + 1)\mu h \) \( n \in \mathbb{Z}^+ \) one arrives to the family of 1-dimensional \( \hbar \)-pdos \( A_n(x_2, hD_2; \mu^{-2}, \hbar) \) with symbols which modulo \( O(\mu^{-2} + \mu^{-1} h) \) are \( \left( V + (2n + 1)\mu h \right) \circ \Psi \) where \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a map with \( | \det D \Psi | = F^{-1} \).

Since I am interested in the energy level 0, I am most interested in the operator \( A_n \) which is not elliptic in the point in question i.e. in operator with \( n = \bar{n} \) delivering minimum to \( |V + (2n + 1)\mu h F| \) (which I have already denoted by \( \sigma \)).

Furthermore, according to formula (6.6.24) of [Ivr1] symbol of \( A_n \) with \( n = \bar{n} \) is equal modulo \( O(\mu^{-4} + h^2) \) to

$$F \left( -(VF^{-1}) + (2n + 1)\mu h + \mu^{-2} \omega_1 \right) \circ \Psi, $$

$$\omega_1 = \frac{1}{8} \kappa V^2 F^{-2} - \frac{1}{4} VF^{-1} L(VF^{-1})$$

where \( \kappa \) and \( L \) are scalar curvature and the Laplace-Beltrami operator associated with the metric \( F^{-1} g^{jk} \).

Then according to the theory of 1-dimensional operators the standard Weyl spectral asymptotics holds for each of them with the remainder estimate \( O(1) \) and thus the remainder estimate for the original problem is \( O(\mu^{-1} h^{-1}) \); however the principal part of such asymptotics includes the full symbol of operator, including terms of magnitude \( \mu^{-2} \) and \( h^{-2} \); however as \( \mu \geq C h^{-\frac{1}{2}} \) one can skip terms \( O(\mu^{-4}) \) and \( O(\mu^{-2} h^2) \) in \( A_n \) without penalty; further, as \( \mu \leq C (h|\log h|)^{-1} \) one can skip terms \( O(h^2) \) in \( A_n \) without penalty as well.

However to preserve remainder estimate one must compensate skipping \( O(\mu^{-2}) \) terms in \( A_n \) by the corresponding correction term and one can see easily that this correction term is equal to \( \kappa_0 \mu^{-2} h^{-2} \) plus the correction term associated with 1-dimensional operator

$$x_2 hD_2 + k^{-1}(w + \mu^{-2} \omega_1)$$

in zone \( \{|x_2| + |\xi_2| \leq \rho = C(\mu h)^{\frac{1}{2}}\} \) where

$$k = | \det \text{Hess}(V/F)|^{\frac{1}{2}} \quad w = \left( -\frac{V}{F} + (2\bar{n} + 1)\mu h \right), \quad \sigma = |w|.$$
and $k, w, \sigma, \omega_1$ are calculated in the critical point in question; this latter correction term is $O(\mu^{-1} h^{-1} |\log \mu^3 h|)$ for saddle points and $O(\mu^{-1} h^{-1})$ for maxima and minima and therefore only saddle points should be considered (i.e. critical points with $\text{det Hess}(V/F) < 0$).

Since this asymptotics should be consistent with one obtained by weak magnetic field approach $\kappa_0 = 0$ and the correction term in question is associated with perturbation $\mu^{-2} k^{-1} \omega_1$ in zone $|x_2| + |\xi_2| \leq \rho$ and thus modulo $O(\mu^{-1} h^{-1})$ it is
\[
(2\pi)^{-1} \mu h^{-1} F \sqrt{g} \times \omega_1 k^{-1} \mu^{-2} \times \log \left( \frac{\rho}{|w|^2 + \mu^{-1}} \right)
\]
which can be rewritten in (7) with
\[
(14) \quad \kappa = -\left(4\pi\right)^{-1} \left( \frac{1}{8} V^2 F^{-1} - \frac{1}{4} V L (VF^{-1}) \right) |\text{det Hess}(V/F)|^{-\frac{3}{2}} \sqrt{g}
\]
calculated at this point.

Actually, this is correct only as $\mu \leq C(h|\log h|)^{-1}$; for $C(h|\log h|)^{-1} \leq \mu \leq C h^{-1}$ one should not discard an extra term $\omega h^2$ in $A_n$ but this term will contribute above $O(\mu^{-1} h^{-1})$ only as $n = \bar{n}$ and it generates $\mathcal{E}^{\text{MW corr}}_2$. This leads to the proof of Theorem 1(ii).

3 Ideas of the proof: superstrong magnetic field case

As $\mu \geq \epsilon h^{-1}$ the same approach works but now only a single $n = \bar{n}$ produces non-trivial contribution while contribution of every $n < \bar{n}$ is $(2\pi)^{-1} \mu h^{-1} \int F \psi dx$ and contribution of every $n > \bar{n}$ is 0 (modulo negligible terms). So one should just repeat the same analysis where now $\rho = \epsilon$. One should not discard $\omega h^2$ in $A_n$ even if there are no critical points and this term produces extra correction term. This leads to the proof of Theorem 2.

References

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