CONSTRUCTING SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS

JAMES ISENBERG
Department of Mathematics
and
Institute of Theoretical Science
University of Oregon
Eugene, OR 97403 USA
E-mail: jim@newton.uoregon.edu

The first step in the building of a spacetime solution of Einstein's gravitational field equations via the initial value formulation is finding a solution of the Einstein constraint equations. We recall the conformal method for constructing solutions of the constraints and we recall what it tells us about the parameterization of the space of such solutions. One would like to know how to construct solutions which model particular physical phenomena. One useful step towards this goal is learning how to glue together known solutions of the constraint equations. We discuss recent results concerning such gluing.

1 Introduction

The constraint equations of Einstein’s theory of the gravitational field are familiar to most of those who work in general relativity: If we choose a set of appropriate local spacetime coordinates \((x^a, t)\) and if we write Einstein’s field equations for the gravitational metric field \(g\) together with a coupled matter source field \(\Psi\) in the tensorial form\(^a\)

\[
G_{\mu\nu}(g) = \kappa T_{\mu\nu}(g, \Psi),
\]

then the four equations

\[
G_{t\nu}(g) = \kappa T_{t\nu}(g, \Psi),
\]

involve no time derivatives of higher than first order. These are the Einstein constraint equations.

What makes the constraint equations important in the study of the physics of the gravitational field is their role in the initial value formulation of

\(^a\)We presume that the matter source field is not derivative-coupled, and that the coupled field equations constitute a well-posed PDE system.
\(^b\)We use MTW conventions on signs of the curvature and on indices: Greek letters run from 0 to 3 (spacetime) while Latin letters run from 1 to 3 (space). Here \(\kappa\) is a coupling constant.
Einstein’s equations. We recall how the initial value formulation works: Say we want to construct a spacetime solution \(\{g, \Psi\}\) of the Einstein equations on a manifold \(M^4 = \Sigma^3 \times \mathbb{R}\), where \(\Sigma^3\) is a smooth three-dimensional manifold. The first step in doing this is to find a set of \textit{initial data} consisting of

- \(\gamma_{ab}\) - a Riemannian metric on \(\Sigma^3\)
- \(K^{cd}\) - a symmetric tensor field on \(\Sigma^3\)
- \(\psi\) - space covariant pieces of the matter field \(\Psi\) on \(\Sigma^3\)

with \(\{\gamma, K, \psi\}\) satisfying the constraint equations (2). In terms of the data \(\{\gamma, K, \psi\}\) the constraint equations take the form

\[
R + (\text{tr}K)^2 - K^{cd}K_{cd} = 2\kappa \rho(\gamma, \psi) 
\]

\[
\nabla_a K^a_b - \nabla_b (\text{tr}K) = \kappa J_b(\gamma, \psi)
\]

where \(\nabla_a\) is the Levi-Civita covariant derivative corresponding to \(\gamma\), \(R\) is its scalar curvature, \(\rho\) is a scalar function of \(\gamma\) and \(\psi\), and \(J_b\) is a vector field function of \(\gamma\) and \(\psi\).

Once we have obtained constraint-satisfying initial data \(\{\gamma, K, \psi\}\), we may evolve this data in time by choosing everywhere on \(\Sigma^3\) a scalar field \(N\) (the “lapse function”) and a vector field \(M^a\) (the “shift vector”) and then solving the Einstein evolution equations in the form

\[
\frac{d}{dt}\gamma_{ab} = -2NK_{ab} + \mathcal{L}_M \gamma_{ab}
\]

and

\[
\frac{d}{dt}K^a_d = N(R^a_d + \text{tr}KK^a_d) - \nabla^c\nabla_d N + \kappa T^a_d + \mathcal{L}_M K^a_d
\]

where \(T^a_d\) are the spatial components of the stress-energy tensor corresponding to the matter field \(\Psi\). We note that the time evolution of the lapse and shift are not determined by the Einstein equations; they may be chosen freely, and they must be chosen for all \(t\) if one wishes to evolve \(\gamma\) and \(K\) in time. The role of \(N\) and \(M\) is to determine the coordinates relative to the evolving spacetime.

---

\(^1\) i.e., metric compatible and torsion-free

\(^2\) If, for example, the “matter source field” is Maxwell’s electromagnetic field, then the spacetime covariant field \(\Psi\) is the spacetime one-form field \(A_a\), the space covariant fields \(\psi\) may be chosen to be the magnetic spatial one-form field \(B_a\) and the electric spatial vector field \(E^a\), and one calculates \(\rho = \frac{1}{2}(E^a E^b \gamma_{ab} + B^a B^b \gamma^{ab})\) and \(J_c = \eta_{c[mn]} E^m B^n\), where \(\eta_{c[mn]}\) is the alternating symbol \(\eta_{123} = 1\), etc. In addition, the data must satisfy the Maxwell constraint equations \(\nabla^a B_a = 0\) and \(\nabla_a E^a = 0\).
After $N(x, t)$ and $M_c(x, t)$ have been chosen and the evolved data $\gamma_{ab}(x, t)$ and $K^{cd}(x, t)$ and $\psi(x, t)$ have been determined, we may reconstruct the spacetime metric (on the manifold $M^4 = \Sigma^3 \times I$, where $I$ is the interval for which the evolution can be carried out) via the formula
\[ g(x, t) = -N^2 dt^2 + \gamma_{ab}(dx^a + M^a dt)(dx^b + M^b dt). \] (7)

We similarly reconstruct $\Psi$. It then follows (from the Gauss-Codazzi-Mainardi $3+1$ decomposition equations for the spacetime curvature) that $g$ and $\Psi$ satisfy the Einstein equation on $M^4$.

Does the initial value formulation work? This general question is best considered by breaking it up into a number of more specific questions:

1. Does every choice of initial data $\{\gamma, K, \psi\}$ which satisfies the constraint equations evolve into a spacetime solution $\{g, \Psi\}$?

2. How do we find data $\{\gamma, K, \psi\}$ which satisfy the constraint equations?

3. What is the space of solutions of the constraint equations?

4. If we wish to use Einstein’s equations to model the physical behavior of a particular physical situation (e.g., a black hole collision), how do we build that physical situation into the choice of initial data?

5. What criteria should we use in making a choice of the lapse and shift?

6. What do we know about the long time behavior of solutions?

7. Is the initial value formulation a practical way to construct and study solutions of Einstein’s equations?

In this essay, we shall focus our attention on questions 2, 3, and 4, which concern finding and understanding solutions to the constraint equations. We start in Section 2 by discussing the conformal method for constructing solutions to the constraints. After recalling the basic steps of the conformal method, we state the results which tell us the extent to which the method is known to work. Based on these results, we discuss (also in Section 2) what is known about the function spaces which parameterize the set of solutions.

Knowing how to construct constraint-satisfying initial data sets from conformal data sets, and knowing how to parameterize the space of all such initial data sets, does not tell us much about how to use solutions constructed via the

\(^e\)The coupled Einstein-matter source field equations determine evolution equations for $\psi$. 

initial value formulation for studying physical questions. With observational
data from LIGO and other gravitational radiation detectors expected to start
arriving within the next few years, it is crucial that we learn how to do this.
Although progress to date on this front has been fairly limited, recent results
on the gluing of initial data sets could be useful. We discuss some of these
results and the ideas involved in Section 3.

Only if initial data can be evolved is it worthwhile to construct it. We
comment briefly on evolution in Section 4, touching on questions 1, 5, 6, and
7. We make some concluding remarks in Section 5.

2 The Conformal Method and the Space of Solutions of the
Constraint Equations

For the past thirty years, the conformal method has been the most success-
ful procedure for producing and studying solutions of the Einstein constraint
equations, both numerically and analytically. The key idea of this method,
which was developed primarily by Lichnerowicz, Choquet-Bruhat and York
is to split the initial data \{γ, K, ψ\} into two parts: The first part is to be cho-
sen freely, while the second part is to be determined by the constraints. As
a consequence of the split, the constraint equations take the form of a deter-
mined PDE system of four equations for four unknowns (equations (12)-(13)
below), rather than the form of an underdetermined system of four equations
for twelve unknowns (equations (3)-(4)).

To illustrate how this method works explicitly with matter source fields
as well as gravitational fields, let us consider the Einstein-Maxwell equa-
tions, for which the initial data consists of \{γ_{ab}, K^{cd}, B^a, E^a\} and for which the full
set of constraint equations takes the form (presuming no charges present)

\[
R + (\text{tr} K)^2 - K^{cd}K_{cd} = \kappa(B^a B_a + E^a E_a) \tag{8}
\]

\[
\nabla_a K^a_b - \nabla_b(\text{tr} K) = \kappa\eta_{mn}B^m E^n \tag{9}
\]

\[
\nabla^a B_a = 0 \tag{10}
\]

\[
\nabla_a E^a = 0 \tag{11}
\]

For this system, the split of the data is as follows:

**Free ("Conformal") Data:**

\[\lambda_{ab} - \text{a Riemannian metric (specified up to conformal factor)}\]
$\sigma^{cd}$ - a divergence-free and trace-free symmetric tensor field
\[(\nabla_c \sigma^{cd} = 0) \quad (\lambda_{cd} \sigma^{cd} = 0)\]
$\tau$ - a scalar field
$\beta^a$ - a divergence-free vector field
$\varepsilon^a$ - a divergence-free vector field

**Determined Data**

$\phi$ - a positive definite scalar field
$W^a$ - a vector field

**Determining Equations**

\[
\nabla_a (LW)_b^a = \frac{2}{3} \phi^6 \nabla_b \tau + \kappa \eta_{bmn} \beta^m \varepsilon^n \tag{12}
\]

\[
\nabla^2 \phi = \frac{1}{8} R \phi - \frac{1}{8} (\sigma^{ab} + LW_{ab})(\sigma_{ab} + LW_{ab}) \phi^{-7} - \frac{\kappa}{8} (\beta^a \beta_a + \varepsilon^a \varepsilon_a) \phi^{-3} + \frac{1}{12} \tau^2 \phi^5 \tag{13}
\]

Here the covariant derivative, the Laplacian, the scalar curvature, and the index manipulations all correspond to the metric $\lambda_{ab}$. Note that $L$ is the conformal Killing operator, defined by

\[
LW_{ab} := \nabla_a W_b + \nabla_b W_a - \frac{2}{3} \lambda_{ab} \nabla_c W^c. \tag{14}
\]

The determining equations (12)-(13) are to be solved for $\phi$ and $W^a$. If, for a given set of conformal data $\{\lambda_{ab}, \sigma^{cd}, \tau, \beta^a, \varepsilon^a\}$ one can indeed solve (12)-(13), then the reconstructed data

\[
\gamma_{ab} = \phi^4 \lambda_{ab} \tag{15}
\]

\[
K^{cd} = \phi^{-10} (\sigma^{cd} + LW^{cd}) + \frac{1}{3} \phi^{-4} \lambda^{cd} \tau \tag{16}
\]

\[
B_a = \phi^{-2} \lambda_{ab} \beta^b \tag{17}
\]

\[
E^a = \phi^{-6} \varepsilon^a \tag{18}
\]

---

\*In the conformal data, the divergence-free condition is defined using the Levi-Civita covariant derivative compatible with the conformal metric $\lambda_{cd}$.\*
constitute a solution of the constraint equations (8)-(11).

Does the conformal method always work in the sense that, for any choice of conformal data \( \{ \lambda_{ab}, \sigma^{cd}, \tau, \beta^a, \varepsilon^a \} \) one can always solve (12)-(13) for \( \{ \phi, W^a \} \) and thereby obtain a solution \( \{ \lambda_{ab}, K^{cd}, B, E^a \} \) for the constraints (8)-(11)? It is easy to see that this is not the case, as illustrated by the following simple example:

Let us choose the manifold \( \Sigma^3 \) to be the three-dimensional sphere \( S^3 \), and let us choose the conformal data to consist of the round sphere metric \( \lambda = g(\text{round}) \) with \( R = 1 \), together with \( \sigma^{cd} = 0, \varepsilon^a = 0, \beta^a = 0, \) and \( \tau = 1 \). With \( \tau \) constant and with \( \eta_{bmn} \varepsilon^m \beta^n \) equal to zero, the right hand side of equation (12) vanishes. Since the operator \( \nabla \cdot L \) is self-adjoint and injective (up to conformal Killing vector fields), it readily follows that the solutions \( W^a \) to (12) for this choice of conformal data all satisfy the condition \( LW_{ab} = 0 \) (which is the conformal Killing equation). Using this result, we find that the remaining equation (13) takes the form

\[
\nabla^2 \phi = \frac{1}{8} \phi + \frac{1}{12} \phi^5.
\]

Since we seek solutions \( \phi \) which are positive definite, for any such solution the right hand side of equation (19) is positive definite. But the maximum principle tells us that there are no functions \( \phi \) on a compact manifold for which \( \nabla^2 \phi > 0 \) (or \( < 0 \)). Hence, for this choice of conformal data, there is no solution \( \{ \phi, W^a \} \) to equations (12)-(13), and therefore no corresponding solution to the Einstein-Maxwell constraint equations.

This example shows that there are choices of conformal data which are not mapped to solutions of the constraint equations by the conformal method. What happens generally? This question is difficult to address in any complete sense because there is a very wide variety of cases of physical and mathematical interest, and each must be handled separately. The various cases may be classified using the following criteria:

**Manifold and Asymptotic Geometry**

- Closed \( \Sigma^3 \)
- Asymptotically Euclidean
- Asymptotically hyperbolic
- Compact \( \Sigma^3 \) with boundary conditions

**Mean Curvature**

- Maximal
• Constant non maximal (CMC)
• Non Constant (non CMC)

Fields
• Einstein vacuum
• Einstein - standard matter source field

Function Spaces
• $C^\alpha$ (analytic)
• $C^\infty$ (smooth)
• $C^{k+\beta}$ (Hölder)
• $H^p_k$ (Sobolev)
• $H^p_{k,\delta}$ (weighted Sobolev)

For a number of cases (labeled using these criteria), we have a fairly complete understanding of which sets of conformal data map to solutions and which do not. This is true for the following:

1. Closed $\Sigma^3$; CMC; Einstein vacuum or Einstein-standard matter source field; $\lambda \in C^3$ and $\sigma, \varepsilon, \beta \in H^p_{2}(p > 3)$.

2. Asymptotically Euclidean; Maximal; Einstein vacuum or Einstein-standard matter source field; Data in weighted Sobolev spaces.

3. Asymptotically hyperbolic: CMC; Einstein vacuum or Einstein-standard matter source field; Data in weighted Sobolev spaces.

For these three cases, one finds that all sets of conformal data which are not readily disqualified by simple sign criteria or by the maximal principle are mapped to solutions by the conformal method. To explain the sense in which this holds, let us focus for the moment on the first of these cases; that with $\Sigma^3$ closed:

The Yamabe classification of metric conformal classes is a key tool for determining which sets of conformal data on a closed manifold $\Sigma^3$ are mapped

---

9We label as “standard matter source field” any field theory which couples to Einstein’s theory without derivative coupling. Included are Einstein-Maxwell, Einstein-Yang-Mills, Einstein-Dirac, Einstein-Klein-Gordon, Einstein-Cartan, and Einstein fluids. For all these theories, there are conformal splittings such that, if $\nabla \tau = 0$, the constraint equations semi-decouple. See Isenberg and Nester [4].
to solutions of the constraints. This classification is based on the Yamabe Theorem\(^3\) which says that every metric \(\lambda_{ab}\) on a closed manifold \(\Sigma^3\) is conformally related to another which has constant scalar curvature either +1, 0, or −1; and moreover, the sign of \(R\) for constant scalar curvature metrics conformally related to a given \(\lambda_{ab}\) is unique. Thus the set of all metrics (and metric conformal classes) on a closed \(\Sigma^3\) is partitioned into the three Yamabe classes \(Y^+(\Sigma^3)\), \(Y^0(\Sigma^3)\), and \(Y^-(\Sigma^3)\) depending on this sign.

The Yamabe theorem and Yamabe classification are important for CMC conformal data on a closed \(\Sigma^3\) because solubility of (12)-(13) is invariant under the conformal transformation \(\{\lambda_{ab}, \sigma^{cd}, \tau, \beta^a, \varepsilon^a\} \rightarrow \{\theta^4\lambda_{ab}, \theta^{-10}\sigma^{cd}, \tau, \theta^{-6}\beta^a, \theta^{-6}\varepsilon^a\}\) for any positive \(\theta\). (This is not true if \(\nabla\tau \neq 0\)). Thus, to determine for which sets of conformal data (12)-(13) admit a solution, one may without loss of generality restrict attention to conformal metrics with \(R(\lambda) = -1, 0,\) or +1.

As a consequence of this conformal invariance and the resulting simplification, one immediately sees from (13) and the maximum principle that the following categories of conformal data do not lead to solutions\(^4\): 

\[
\{Y^+; (\sigma, \rho) \equiv 0; \tau = 0\}, \{Y^-; (\sigma, \rho) \equiv 0; \tau \neq 0\}, \{Y^0; (\sigma, \rho) \equiv 0; \tau = 0\}, \{Y^-; (\sigma, \rho) \neq 0; \tau = 0\}, \{Y^0; (\sigma, \rho) \neq 0; \tau = 0\}.
\]

Much less evident is the fact that for all other sets of conformal data on closed \(\Sigma^3\) (12)-(13) can be solved, and we obtain a solution of the constraints. The proof of this result (with one case left out) is discussed in the review paper of Choquet-Bruhat and York\(^2\); the complete proof appears in the work of the author\(^5\).

Analogous results, with a few important modifications, hold for the other two cases mentioned above. Specifically, for the asymptotically Euclidean case with \(\tau = 0\), we first note that there is a Yamabe-type partition of asymptotically Euclidean metrics into two classes \(Z^+\) and \(Z^-\), with the \(Z^+\) conformal geometries admitting a conformal transformation to metrics with \(R = 0\), and the \(Z^-\) conformal geometries not admitting such a transformation. It follows from work of Brill and Cantor\(^6\) that asymptotically Euclidean metrics with non-negative scalar curvature, as well as those satisfying a certain integral condition, are contained in \(Z^+\). The work of Cantor\(^7\) (See also\(^8\)) shows that a set of asymptotically Euclidean conformal data \(\{\lambda_{ab}, \sigma^{cd}, \tau, \beta^a, \varepsilon^a\}\) (in appropriate function spaces) is mapped to a solution by the conformal method if and only if \(\lambda \in Z^+\) (The behavior of the other fields is irrelevant.).

For the asymptotically hyperbolic case (with \(\tau \neq 0\)), there are important

---

\(^h\) Here \((\sigma, \rho) \equiv 0\) means that \(\sigma^{cd}\) and \(\rho\) are identically zero everywhere on \(\Sigma^3\), while \((\sigma, \rho) \neq 0\) means that there exists at least one point on \(\Sigma^3\) at which \(\sigma^{cd}\) or \(\rho\) is not zero.
questions regarding the asymptotic behavior of solutions; but ignoring those, the results are very simple: For every set of asymptotically hyperbolic conformal data, equations (12)-(13) can be solved, and we obtain a solution to the constraints.

These results together tell us that if we are interested in constant mean curvature (or maximal) initial data, the conformal method works, apart from a few readily identified special cases. This is useful for two reasons. The first is that since equations of the form of the determining equations (12)-(13) can be handled by modern computer algorithms (at least when they are semi-decoupled, which holds so long as we restrict to conformal data with constant $\tau$), the conformal method is an effective way to produce CMC initial data sets for numerical relativity. The second is that, as a consequence of uniqueness theorems relating conformal data and solutions of the constraints, together with the results noted above, certain function spaces of conformal data provide an effective parameterization of the set of solutions of the constraints.

In light of the uniqueness of CMC foliations of spacetime solutions of Einstein’s equations, one may use these function spaces of conformal data to parameterize the set of spacetime solutions as well (ignoring those solutions which do not admit CMC or maximal foliations).

These results are very useful, but they are limited in a number of ways, and so there is a lot more that needs to be done. First, one would like to know which non CMC conformal data sets lead to solutions. While there are some results concerning this question, they all presume that $|\nabla \tau|$ is small. Indeed, for small $|\nabla \tau|$ (with certain restrictions on the zeros of $\tau$), results show that solutions to the coupled system (12)-(13) always exist. Unfortunately, the iteration methods we have used to prove the results just cited do not appear to be able to handle general $|\nabla \tau|$, so new ideas are likely needed.

Next, we would like to be able to generate solutions to the constraint equations on manifolds with boundaries. The initial value-boundary value problem has been studied for Einstein’s theory in Friedrich’s formulation, but there has not been much mathematical work on it in the standard $\{\gamma, K\}$ formulation discussed here. In view of the recent numerical work on spacetimes with excised black holes, a mathematical treatment, in terms of the $\{\gamma, K\}$ formulation, of solutions with boundaries would be useful.

A third issue regarding the construction of solutions of the constraint equations which should be addressed further is the loosening of differentiability requirements for solutions, including weak solutions. The motivation for doing this comes from the progress that has recently been made by Klainerman and Rodnianski and by Smith and Tataru in proving local well-posedness...
for the Einstein equations for data of lower differentiability. The goal (based
on energy considerations) is $H^2_{3/2}$: so far $H^2_{2+\epsilon}$ has been obtained. This leads
one to try to construct initial data sets in $H^2_{2+\epsilon}$ or even $H^2_{3/2}$. The results
to date produce initial data sets which are considerably more restricted
(i.e., smoother) than this, but work is proceeding toward obtaining solutions
of the constraint equations with this low differentiability.

The question that remains is how to construct initial data sets which can
be used to model physical phenomena of interest. We discuss this in the next
section.

3 Physical Modeling and the Gluing of Solutions of the
Constraint Equations

Say we want to study the gravitational radiation produced by the inspiral
collision of a neutron star and a black hole. The first step of a numerical
modeling of this phenomenon is to find initial data $\{\gamma, K, \psi\}$ which constitutes
a snapshot of the two objects and their ambient spacetime at some moment
prior to the collision. It is important that the initial data accurately represent
the two objects and their surroundings in a relatively quiet pre-collision state,
without any extraneous unrelated radiation, or the modeling will be of little
use.

The electromagnetic analogue of this modeling problem is familiar and
easily handled, since the background space is fixed and independent of the
electromagnetic fields in Newtonian-Maxwell theory, and since the represen-
tation of electromagnetic radiation is well understood in this theory as well.
So, too, if we work with a post-Newtonian approximation to Einstein’s theory,
the choice of initial data for phenomena like the inspiral collision is understood
to some extent.

However, attempts to find physically accurate initial data sets for phe-



nenomenon such as this in terms of the full Einstein theory have been stymied,
despite many years of significant effort, both numerical and analytical. The
conformal method can of course be used to generate candidate sets of data,
and such sets are used in most numerical modeling studies. The lack of a priori
control of the conformal factor and therefore of the ambient space, together
with the difficulty in representing and recognizing gravitational radiation in
the gravitational radiation in the conformal data $\{\lambda, \sigma, \tau, \psi\}$ for massive relativistic objects renders these sets somewhat suspect, however.

One approach that has been discussed to try to control the choice of
initial data is to work with sets of post-Newtonian initial data in the very
early pre-radiation stages of the collision and then try to evolve these into the later, more relativistic stage. This approach has some promise, but can be very expensive in numerical evolution time.

Another possible approach that could be tried relies on “gluing.” The idea here is to start with well-understood initial data sets for separate portions of the spacetime—say, one for the black hole and one for the neutron star—and develop a procedure for joining these together into one set. Mathematically, the idea is to join two or more solutions of the constraint equations into a single solution in such a way that, away from the joining region, the data is largely unchanged. Physically, the challenge is to understand and control the physical effects of the joining region.

There has been substantial progress during the past couple of years on the mathematical aspects of gluing solutions of the Einstein constraint equations. Mazzeo, Pollack, and the author (IMP) have proven theorems which prescribe a procedure for gluing connected sums of sets of initial data, while Corvino and Schoen have shown that rather general asymptotically flat initial data sets can be glued to exact Schwarzschild or exact Kerr exteriors outside a transition region. We first discuss the IMP work, and then briefly describe the results of Corvino and Schoen.

3.1 Connected Sum Gluing

We start with a pair of constant mean curvature solutions \( \{ \Sigma^3_1, \gamma_1, K_1 \} \) and \( \{ \Sigma^3_2, \gamma_2, K_2 \} \) of the vacuum constraint equations (3)-(4). (We presume for now that there are no matter source fields \( \psi \), and so \( \rho = 0 \) and \( J = 0 \)). Each of these solutions may be asymptotically Euclidean, asymptotically hyperbolic, or one may have either or both of \( \Sigma^3_1 \) and \( \Sigma^3_2 \) closed. The two solutions need not have matching asymptotic properties, but they do need to have the same constant mean curvature \( \tau \).

If we now pick a pair of points \( p_1 \in \Sigma^3_1, p_2 \in \Sigma^3_2 \) on each manifold, then the gluing procedure produces a one-parameter family of solutions \( \{ \tilde{\Sigma}^3, \tilde{\gamma}_s, \tilde{K}_s \} \) with the following properties: (i) \( \tilde{\Sigma}^3 = \Sigma^3_1 \# \Sigma^3_2 \), with \( \Sigma^3_1 \) joined to \( \Sigma^3_2 \) by a “neck” \( (S^2 \times R) \) connecting a neighborhood of \( p_1 \) to a neighborhood of \( p_2 \) (both now excised); (ii) the data \( \{ \tilde{\gamma}_s, \tilde{K}_s \} \) can be made to be arbitrarily close to the original data \( \{ \gamma_1, K_1 \} \) and \( \{ \gamma_2, K_2 \} \) on appropriate regions of \( \tilde{\Sigma}^3 \); (iii) \( tr\tilde{K} = \tau \); and (iv) the geometry on the neck (i.e., lengths, curvature) can degenerate for \( s \to \infty \), but its behavior is controlled by exponential bounds in \( s \).

This gluing procedure works for quite general classes of data. The only restrictions on \( \{ \Sigma^3_1, \gamma_1, K_1 \} \) and \( \{ \Sigma^3_2, \gamma_2, K_2 \} \) besides the CMC matching con-
dition $trK_1 = trK_2$ and certain regularity restrictions (stated in terms of weighted Hölder spaces) are that if either of the sets of data has $\Sigma^3$ closed, that set of data must have $K^{cd}$ not vanishing identically, and must be free of nontrivial conformal Killing vector fields. Neither of these latter conditions are needed for asymptotically Euclidean or asymptotically hyperbolic data, and we note that if a set of data $\{\Sigma^3, \gamma, K\}$ fails to satisfy either condition, it will generally satisfy them both after a small perturbation is made.

The procedure allows a number of interesting sets of initial data to be produced. One can, for example, add a sequence of small black holes to any asymptotically flat spacetime $\{M^4, g\}$ by gluing a sequence of copies of Minkowski data to asymptotically Euclidean data for $\{M^4, g\}$ and then evolving the glued data. These spacetimes (somewhat reminiscent of the Misner multi-black hole solutions, but far more general) may be constructed with the interiors either independent or connected. Another set of spacetimes one obtains via this gluing procedure allows one to study the question of black holes in cosmological spacetimes: One can glue asymptotically Euclidean or asymptotically hyperbolic data to cosmological data on closed $\Sigma^3$ and then see if black hole-like physics develops. This gluing procedure also permits us to add one or more wormholes to most solutions of Einstein’s equations: If one chooses the pair of points $p_1$ and $p_2$ on a single connected initial data set $\{\Sigma^3, \gamma, K\}$ and carries out the gluing procedure, one obtains initial data much like $\{\Sigma^3, \gamma, K\}$ except with an $S^2 \times R$ wormhole glued on.

To what extent can one carry out these gluings in practice, say, numerically? To answer this question, we shall now describe the basic steps of the connected sum gluing procedure:

**Step 1:** One starts by performing a conformal transformation of the data $\gamma_{ab} \to \psi^{-4} \gamma_{ab}, \sigma^{cd} \to \psi^{10} \sigma^{cd}$, (Here $\sigma^{cd}$ is the trace-free part of $K^{cd}$.) which is trivial ($\psi = 1$) away from a neighborhood of each of the points $p_1$ and $p_2$, and which is singular at each of the points (with a specified type of singular behavior). The effect of this conformal blowup is to replace a ball surrounding $p_1$ with an $S^2 \times R$ infinite length half-cylinder which is asymptotically a standard round half-cylinder, and to do the same in a neighborhood of $p_2$. Note that there are standard explicit formulas one can choose for $\psi$ to carry out this first step.

**Step 2:** Each of the newly added half cylinders extends out infinitely far. If one cuts each at a distance $s/2$ (measured using the local conformally

---

1Actually, a Killing vector field can be present, so long as it does not vanish at the chosen points $p_1$ or $p_2$.

2Note that there is no accepted definition of a black hole in a spacetime without some sort "asymptotic infinity" structure.
transformed metric) out from the neighborhoods of $p_1$ and $p_2$, and identifies the two cut half cylinders on the joining sphere, then one has the manifold $\tilde{\Sigma}^3 = \Sigma_1^3 \# \Sigma_2^3$, with its connecting $s$-parametrized "neck". Then, using a pair of cut-off functions $\chi_1$ and $\chi_2$, one may patch together the metric $\tilde{\gamma}_s = \gamma_1 \chi_1 + \gamma_2 \chi_2$ and the traceless apart of the second fundamental form $\tilde{\sigma}_s = \sigma_1 \chi_1 + \sigma_2 \chi_2$. Both $\tilde{\gamma}_s$ and $\tilde{\sigma}_s$ are well-defined on $\tilde{\Sigma}^3$, and both depend on the gluing distance parameter $s$. One also glues together the conformal function $\tilde{\psi}_s = \psi_1 \xi_1 + \psi_2 \xi_2$, using a different set of cut-off functions. (For details, see IMP\textsuperscript{25}). This step, like the first, can be done using standard explicit formulas for $\chi_1$, $\chi_2$, $\xi_1$, and $\xi_2$.

**Step 3:** A careful choice of the cut-off functions results in $\tilde{\sigma}_s$ being everywhere traceless with respect to $\tilde{\gamma}_s$. The conformal transformation $\sigma^{cd} \rightarrow \psi_1^{10} \sigma^{cd}$ guarantees that away from the neck, $\tilde{\sigma}_s$ is divergence-free with respect to $\tilde{\gamma}_s$. However, in the neck, $\tilde{\nabla}_c \tilde{\sigma}_s^{cd} \neq 0$. So one replaces $\tilde{\sigma}_s^{cd}$ by $\tilde{\mu}_s^{cd}$, which is obtained by solving

$$\tilde{\nabla}_c (\tilde{L} Y^c)_s^{cd} = \tilde{\nabla}_c \tilde{\sigma}_s^{cd}$$

(20)

for the vector field $Y^c_s$, and setting

$$\tilde{\mu}_s^{cd} = \tilde{\sigma}_s^{cd} - (\tilde{L} Y_s^c)^{cd}.$$  

(21)

The tensor field $\tilde{\mu}_s^{cd}$ is divergence-free everywhere on $\tilde{\Sigma}^3$ with respect to $\tilde{\gamma}_s$. This step is straightforward to carry out in practice. However, to be able to proceed further with the gluing procedure, one needs to know not only that the divergence-free field $\tilde{\mu}_s^{cd}$ exists, but also that $\| \tilde{\mu}_s - \tilde{\sigma}_s \|_{C^{k+\beta}}$ is very small for sufficiently large $s$. It is shown in IMP that this estimate always holds.

**Step 4:** $\{\tilde{\gamma}_s, \tilde{\mu}_s, \tau\}$ is a standard set of conformal data on $\tilde{\Sigma}^3$. So one may set up and attempt to solve

$$\tilde{\nabla}^2 \phi_s = \frac{1}{8} \tilde{R}_s \phi_s - \frac{1}{8} \tilde{\mu}_{scd} \tilde{\mu}_s^{cd} \phi_s^{-7} + \frac{1}{12} \tau^2 \phi_s^5$$

(22)

for $\phi_s$. Although the Yamabe class of the metric $\tilde{\gamma}_s$ constructed in Step 2 is not evident, the estimates obtained in Step 3 allow one to show that (22) has a unique solution. Further, with a considerable amount of work, one shows that $\| \phi_s - \psi_s \|_{C^{k+\beta}}$ is very small for sufficiently large $s$. As noted above, once one knows that a solution to (22) exists, one can readily obtain that solution numerically.

**Step 5:** Using formulas analogous to (15)-(16), one constructs $\tilde{\gamma}_s$ and $\tilde{K}_s$ from $\phi_s$ and $\{\tilde{\gamma}_s, \tilde{\mu}_s, \tau\}$. One immediately verifies that for each value of the parameter $s$, $\{\Sigma^3, \tilde{\gamma}_s, \tilde{K}_s\}$ solves the vacuum constraint equations. As a consequence of the estimates of Steps 3 and 4, one verifies that away from the neck, $\tilde{\gamma}_s$ and $\tilde{K}_s$ approach the original data for sufficiently large $s$. 

**GR16Final: submitted to World Scientific on November 30, 2018**
We see from the description of these five steps that in practice, this connected sum gluing procedure involves making a few choices of standard cut off functions, plus solving familiar elliptic equations. It should be a useful procedure for studying solutions of Einstein’s equations numerically.

We note that, to date, the IMP connected sum gluing procedure has been proven to work for the vacuum Einstein case only. It is expected that it can be implemented for Einstein-Maxwell, Einstein-Yang-Mills, and the other Einstein-standard matter source fields as well; work is under way to show this. It would be nice to also extend the procedure to non CMC data, but this is likely to be difficult.

3.2 Exterior Schwarzschild and Kerr Gluing

The starting point for the exterior Schwarzschild gluing procedure is any asymptotically Euclidean, time symmetric \((K^{cd} = 0)\) initial data set \(\{\Sigma^3, \gamma\}\). One chooses a compact region \(\Lambda^3\) in \(\Sigma^3\). One then shows \(\gamma\) that there is an asymptotically Euclidean time symmetric solution \(\{\Sigma^3, \tilde{\gamma}\}\) such that (i) inside a ball \(B_r\) which contains \(\Lambda^3\) as a subset, \(\tilde{\gamma}|_{B_r} = \gamma\); and (ii) outside the ball \(B_{2r}\), we have \(\tilde{\gamma}|_{\Sigma^3 \setminus B_{2r}} = \gamma_{Sch}(m)\), where \(\gamma_{Sch}(m)\) is the spatial Schwarzschild metric for some positive mass \(m\).

We note three important features of this type of gluing: First, when the gluing is completed, the region inside \(B_r\) as well as the exterior region outside \(B_{2r}\) are unchanged. This is not the case for the connected sum gluing, in which the original solutions are slightly changed (away from the neck) by the gluing procedure. Second, the analysis used to prove this result does not rely in any way on the conformal method. That is, one does not construct the solution \(\{\Sigma^3, \tilde{\gamma}\}\) by first specifying a conformal metric everywhere and then solving for the conformal factor. Indeed, one can carry out exterior Schwarzschild gluing only if one works with the constraints as an underdetermined system to be solved for \(\tilde{\gamma}\), rather than as a determined system to be solved for \(\phi\). Third, the proof that this gluing works does not readily translate into a step by step procedure that one can carry out numerically. The proof guarantees existence, but does not prescribe a procedure for constructing \(\{\Sigma^3, \tilde{\gamma}\}\).

The result is quite surprising and remarkable. It says that for any compact portion of a time symmetric asymptotically Euclidean solution, one can arrange the gravitational fields in an annular region around it so that outside this region, the gravitational field is exactly Schwarzschild. Note that the analogous result does not hold true for Newtonian gravity. Note also that, combining this result with the work of Friedrich, one may be able to produce a very large family of solutions of Einstein’s equations with a complete...
I (null asymptotic infinity).

One large but unsurprising restriction on the type of data which can be glued to an exterior Schwarzschild spatial slice is that the data have $K^{cd} = 0$. More recent results of Corvino and Schoen allow one to remove this restriction provided that one replaces the Schwarzschild exterior by a slice of the more general Kerr (rotating black hole) They do impose restrictions on the data $\{\Sigma^3, \gamma, K\}$ which can be patched to an exterior Kerr slice: They require certain decay behavior in $\gamma$ and $K$ which are essentially just enough so that the momentum and angular momentum integrals at infinity are well-defined. These restrictions appear to be minor, in view of the result. Again, one consequence of these results might be an even larger family of solutions with complete $I$.

4 Some Comments on Evolution

Of the seven questions raised in the introduction, we have focussed in this essay on those three – 2, 3, and 4 – which deal with the constraint equations. We shall now briefly comment on questions 1, 5, and 6, which mostly concern evolving sets of initial data into spacetime solutions.

The first question addresses the issue of local existence and well-posedness of the initial value formulation of the Einstein equations. To a large extent, this issue was settled fifty years ago by Choquet-Bruhat who proved (using harmonic coordinates) that for smooth initial data ($C^l$, with large $l$) the Einstein system is well-posed. While this result is sufficient for most purposes, there have been strong attempts in recent years to prove well-posedness for more general classes of data. The goal is to prove it for data in the Sobolev space $H^{l+2}$. The recent work of Klainerman and Rodnianski, and of Smith and Tataru achieves it for $H^{l+2+\epsilon}(\epsilon > 0)$ data.

Well-posedness theorems show (among other things) that data in the specified function space can always be evolved to a spacetime solution for a sufficiently small time duration into the future and into the past. They say little about how long a spacetime solution will last, or what its behavior will be far into the future and far into the past. While these issues are very difficult and far from resolved, there has been substantial progress on them in recent years. We note in particular the work of Christodouloou and Klainerman, which verifies the stability of Minkowski spacetime (and thereby proves that there is a nontrivial family of asymptotically flat spacetimes which extend for infinite proper time into the future and into the past); that of Anderson and Moncrief, which does the same for the stability of the expanding
$k = -1$ vacuum Friedman-Robertson-Walker cosmological spacetimes; and the work of Christodoulou\(^k\) which verifies weak cosmic censorship\(^l\) for spherical collapsing Einstein-scalar field spacetimes. There has also been a collection of works by Berger, Chrusciel, Garfinkle, Kichenassamy, Moncrief, Rendall, Ringstrom, Wainwright, Weaver, and the author which provide increasing evidence for the presence of asymptotically velocity dominated behavior and mixmaster behavior in cosmological spacetimes, and further tends to support the validity of the strong cosmic censorship conjecture\(^l\) in these spacetimes. (See, for example, the review papers by Berger\(^k\) and Rendall\(^l\).)

The construction or study of spacetime solutions of Einstein’s equations via the initial value formulation requires one to make a choice of spacetime foliation (i.e., a specification of the $t = \text{constant}$ hypersurfaces which fill the spacetime) and of spacetime threading (i.e., a specification of the $x^a = \text{constant}$ observer world lines which are everywhere transverse to the foliation, and fill the spacetime). The choice of foliation is controlled infinitesimally by the lapse function $N$, and the choice of threading is similarly controlled by the shift vector $M^a$.

In constructing a spacetime from specified initial data, one may always make the simple choice $N = 1$ and $M^a = 0$ (“Gaussian normal coordinates”). However, this choice generally leads to a nonphysical and premature breakdown in the evolution (a coordinate singularity). Consequently, the maximal development\(^k\) of the initial data may not be obtained. So, one criterion for a good choice of the lapse and shift is that the choice effectively avoids such coordinate singularities. Also important is that the choice be relatively easy to implement in practice, and that it not obscure the gravitational physics of the spacetime (e.g., by simulating gravitational radiation which is not physically present).

Constant mean curvature or maximal slicing, coupled with some sort of coordinate shear minimizer, is often cited as a choice of foliation and threading which avoids coordinate singularities and clarifies the physics. However, implementing this choice requires that one solve a set of elliptic equations on each time slice. This is a large expense in computer time for numerical constructions, and it precludes explicit forms for the lapse and shift in analytical studies. For special families of solutions such as the Gowdy space-time,\(^k\) the weak cosmic censorship conjecture says that in asymptotically Euclidean spacetime solutions, the singularities which develop during gravitational collapse are generically shielded from the view of observers at infinity by the development of an event horizon.\(^l\) The strong cosmic censorship conjecture\(^l\) says that in generic solutions developed from Cauchy data on a compact Cauchy surface, a Cauchy horizon (with its attendant causality difficulties) does not form.

\(^k\) The weak cosmic censorship conjecture says that in asymptotically Euclidean spacetime solutions, the singularities which develop during gravitational collapse are generically shielded from the view of observers at infinity by the development of an event horizon.

\(^l\) The strong cosmic censorship conjecture says that in generic solutions developed from Cauchy data on a compact Cauchy surface, a Cauchy horizon (with its attendant causality difficulties) does not form.
times or the $U(1)$ Symmetric solutions, there are certain choices of lapse and shift picked out by the geometry (areal for Gowdy, harmonic time for $U(1)$ Symmetric); but these are special cases. More generally, the choice of lapse and shift remains a difficult issue.

5 Concluding Remarks

We have discussed a number of the challenges that one encounters in constructing and studying spacetimes via the initial value formulation. These occur both in finding sets of initial data which satisfy the Einstein constraint equation, and in evolving these sets of data. Some are fairly difficult. However, paraphrasing Winston Churchill’s description of democracy, one finds that the initial value formulation is the most impractical way to work with solutions of Einstein’s equations, except all of those other ways which have been tried from time to time. The fact that most numerical studies of solutions are carried out using the initial value formulation attests to the relative practicality of this approach.

6 Acknowledgments

I thank the Scientific Committee of GR16 for inviting me to speak, and I thank the local Organizing Committee and the local South African hosts for running a very fine conference. Portions of the work discussed have been supported by NSF grant PHY 0099373.

References

1. C. Misner, K. Thorne and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
2. Y. Choquet-Bruhat and J. York in *General Relativity and Gravitation*, ed. A. Held (Plenum, New York, 1980).
3. R. Schoen, *J. Diff. Geom.* **20**, 479 (1984).
4. J. Isenberg and J. Nester, *Ann. Phys.* **107**, 368 (1977).
5. J. Isenberg, *Class. Qm. Grav.* **12**, 2249 (1995).
6. D. Brill and M. Cantor, *Comput. Math.* **43**, 317 (1981).
7. M. Cantor, *Comm. Math. Phys.* **57**, 83 (1977).
8. Y. Choquet-Bruhat, J. Isenberg and J. York, *Phys. Rev.* D **61**, 084034 (2000).
9. L. Andersson, P. Chrusciel and H. Friedrich, *Comm. Math. Phys.* **149**, 587 (1992).
10. L. Andersson and P. Chrusciel, *Dissert. Math.* 355, 1 (1996).
11. J. Isenberg, *Phys. Rev. Lett.* 59, 2389 (1987).
12. D. Brill and F. Flaherty, *Ann. Inst. H. Poincare* 28, 335 (1978).
13. D. Brill in *Proc Third Marcel Grossman Meeting*, ed. H. Ning (North-Holland, Amsterdam, 1982).
14. R. Bartnik, *Comm. Math. Phys.* 117, 615 (1998).
15. Y. Choquet-Bruhat, J. Isenberg and V. Moncrief, *C. R. Acad. Sci. Paris* 315, 349 (1992).
16. J. Isenberg and V. Moncrief, *Class. Qutm. Grav.* 13, 1819 (1996).
17. J. Isenberg, *Fields Inst. Comm.* 15, 59 (1997).
18. J. Isenberg and J. Park, *Class. Qutm. Grav.* 14, A189 (1997).
19. H. Friedrich and G. Nagy, *Comm. Math. Phys.* 210, 619 (1999).
20. S. Klainerman and I Rodnianski, preprint: math.AP/0109173
21. H. Smith and D. Tataru, preprint: Sharp local well-posedness results for nonlinear wave equation.
22. T. Damour in *300 Years of Gravitation*, eds. S. Hawking and W. Israel (Cambridge Univ Press, New York, 1987).
23. A. Buonanno and T. Damour, *Phys. Rev. D* 62, 06401 (2000).
24. K. Alvi, *Phys. Rev. D* 64, 104020 (2001).
25. J. Isenberg, R. Mazzeo, and D. Pollack, preprint: gr-qc/0109043.
26. J. Corvino, *Comm. Math. Phys.* 214, 137 (2000).
27. J. Corvino and R. Schoen, unpublished.
28. C. Misner, *Ann. Phys.* 24, 102 (1963).
29. H. Friedrich in *Gravitation and Relativity at the Turn of the Millennium*, eds. N. Dadhich and J. Narlikar (Inter-University Centre, Pune, 1998).
30. Y. Bruhat, *Acta Math* 88, 141 (1952).
31. D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton Univ Press, Princeton, 1993).
32. L. Andersson and V. Moncrief, unpublished.
33. D. Christodoulou, *Ann. Math.* 149, 183 (1999).
34. B. Berger, *Living Reviews* 5, 2002-1 (2002).
35. A. Rendall, *Living Reviews* 3, 2000-1 (2000).
36. R. Penrose in *General Relativity-An Einstein Centenary Survey*, eds. S. Hawking and W. Israel (Cambridge Univ Press, New York, 1979).
37. Y. Choquet-Bruhat and R. Geroch, *Comm. Math. Phys.* 14, 329 (1969).
38. R. Gowdy, *Ann. Phys.* 83, 203 (1974).
39. V. Moncrief, *Ann. Phys.* 167, 118 (1986)
40. W. Churchill, House of Commons, 11 Nov., 1947.