Three-leg correlations in the two-component spanning tree on the upper half-plane

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Abstract. We present a detailed asymptotic analysis of correlation functions for the two-component spanning tree on the two-dimensional lattice when one component contains three paths connecting vicinities of two fixed lattice sites at large distance \(s\) apart. We extend the known result for correlations on the plane to the case of the upper half-plane with closed and open boundary conditions. We found asymptotics of correlations for distance \(r\) from the boundary to one of the fixed lattice sites for the cases \(r \gg s \gg 1\) and \(s \gg r \gg 1\).

Keywords: conformal field theory (theory), correlation functions (theory), sandpile models (theory)

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1. Introduction

In recent years the logarithmic conformal field theories (LCFT) [1, 2] and their relation to lattice models of statistical physics like dense polymers [3]–[5], the sandpile model [6]–[9], dimer models [10] and percolation [5, 11] have been the subject of active research. Among all these models, the Abelian sandpile model (ASM) [12] is one of the most interesting and fruitful, because correlation functions containing logarithmic corrections can be found explicitly using combinatorial methods. In this way, the full correspondence between the lattice model and the logarithmic conformal field theory can be transparently tested. A lot of successful checks have been made in [6]–[9], including various calculations for correlations in the bulk and on the boundaries, the determination of boundary changing fields, the insertion of isolated dissipation and the evaluation of some finite-size effects.

One of the most popular questions and checks was about the origin of the logarithmic corrections to the correlations. There are known just two causes of that: the insertion of the dissipation at isolated sites [7], that follows from the logarithmic behavior of the inverse Laplacian at a large distance, and non-locality of height variables for \( h \geq 2 \) in the ASM [9]. The last case is more important and difficult, because sandpile configurations with height variables \( h \geq 2 \) are mapped onto an infinite set of non-local configurations of spanning trees. This non-locality arises due to the presence of a specific three-leg subgraph, the so-called \( \Theta \)-graph [16]. The \( \Theta \)-graph is a subconfiguration of the spanning tree, consisting of three paths, that connect the vicinity of vertex \( j_0 \) with that of vertex \( t_0 \) (figure 1). The paths with additional branches attached to them form one component, which is surrounded by another component of the spanning tree. A generalization of the \( \Theta \)-graph is an odd ‘\( k \)-leg’ subgraph, which has been considered by Ivashkevich and Hu in [13] for the infinite square lattice. They obtained the asymptotic dependence \( P(r) \sim \ln r/r^{(k^2-1)/2} \), for \( k = 1, 3, 5, \ldots \) and concluded that it is the presence of the second component that is responsible for the logarithmic correction to the correlation function. Indeed, Saleur and Duplantier considered correlations of ‘\( k \)-leg operators’ by mapping the two-dimensional percolation problem onto a Coulomb gas and found that the asymptotics of correlation functions for a one-component spanning tree has a pure power-law decay, \( r^{-(k^2-1)/2} \) [14].

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Later on, Piroux and Ruelle calculated the height probabilities for \( h \geq 2 \) in the ASM on the upper half-plane with closed and open boundaries [9]. They enumerated the spanning tree configurations with a \( \Theta \)-graph, having one fixed site at distance \( r \) apart from the boundary and another site running over the whole upper half-plane. The summation over positions of the running site leads to cumbersome estimations of integrals, so it is difficult to follow details of correlations between different parts of a \( \Theta \)-graph explicitly. Calculations of two-point correlations \( P_{1h} - P_{1P} \) in the ASM on the plane for \( h \geq 2 \) also lead to the same difficulties. The \( \Theta \)-graph arising for \( h \geq 2 \) consists of three paths connecting one fixed site with height \( h \geq 2 \) with another site at the distance \( s \), running over the whole plane except the site with the height variable ‘1’. It was shown in [15], that evaluation of the logarithmic corrections to the two-point correlations does not need the summation over \( s \) over the whole plane. Instead, it is enough to take into account only those \( \Theta \)-graph configurations when the running site of the \( \Theta \)-graph is situated in the vicinity of the height variable ‘1’. The latter approach, being much simpler, is not so transparent, and an additional analysis of correlations of different parts of the \( \Theta \)-graph is desirable. In this work, we find the asymptotic behavior of three-leg correlations for the case of the upper half-plane. We test the validity of the method described in [15] for the upper half-plane, examining the order of expansion, where we can obtain a disagreement.

2. The model

We consider the labeled graph \( G = (V, E) \) with vertex set \( V \) and set of bonds \( E \). The vertices are sites of the square lattice and an additional point which is the root ‘\( \star \)’: \( V \equiv \{s_{x,y}, (x, y) \in \mathbb{Z}^2, |x| \leq M, |y| \leq N\} \cup \{\star\} \). The bonds of \( E \) connect only neighboring sites. Vertices \( i_{x_1,y_1}, j_{x_2,y_2} \in V \) are neighbors if \( (x_2 - x_1)^2 + (y_2 - y_1)^2 = 1 \). Also all boundary vertices \( \{s_{x,\pm N}, x \in \mathbb{Z}\} \) and \( \{s_{\pm M,y}, y \in \mathbb{Z}\} \) are neighbors of the root \( \star \). The graph \( G \) represents a finite square lattice of \( 2N + 1 \times 2M + 1 \). In thermodynamical limit, \( N, M \to \infty \), the lattice \( s_{x,y} \) covers the whole two-dimensional plane. We consider also the upper half-plane with closed and open boundary conditions at the lattice sites \( V_0 = \{s_{x,1}, x \in \mathbb{Z}, |x| \leq M\} \). The corresponding graphs are \( G_{\text{cl}} = (V_+, E_{\text{cl}}) \) and \( G_{\text{op}} = (V_+, E_{\text{op}}) \), where \( V_+ = \{s_{x,y}, x \in \mathbb{Z}, y \in \mathbb{N}, |x| \leq M, y \leq N\} \cup \{\star\} \), with the
Three-leg correlations in the two-component spanning tree on the upper half-plane

Figure 2. Component I consists of tree paths and branches attached to them; component II is another spanning tree surrounding component I.

sets of bonds $E_{\text{cl}} \equiv E \cap \{(i, j), \ i \in V_+, \ j \in V_4\}$ and $E_{\text{op}} \equiv E_{\text{cl}} \cup \{(i, \star), \ i \in V_0\}$. We construct the desired spanning tree configurations on the above graphs by using the arrow representation; see e.g. [17]. Accordingly, we attach to each vertex $i \in V \setminus \{\star\}$ an arrow directed along one of bonds $(i, i') \in E$ incident to it. Each arrow defines a directed bond $(i \to i')$ and each configuration of arrows $A$ on $G$ defines a spanning directed graph (digraph) $G_{\text{dir}}(A)$ with set of bonds $E_{\text{dir}}(A) = \{(i \to i'): i \in V \setminus \{\star\}, \ i' \in V, \ (i, i') \in E\}$ depending on $A$. Similarly, the arrow configurations on $G_{\text{cl}}$ and $G_{\text{op}}$ define spanning digraphs $G_{\text{cl,dir}}(A)$, $G_{\text{op,dir}}(A)$ with corresponding sets of bonds. Note that no arrow is attached to the root $\star$, so it has out-degree zero. A sequence of directed bonds $(i_1, i_2), (i_2, i_3), (i_3, i_4), \ldots, (i_{m-1}, i_m)$ is called the path of length $m$ from the site $i_1$ to the site $i_m$. This path forms a loop if $i_m = i_1$. The spanning tree is a spanning digraph without any loops. Our aim is to construct a two-component spanning tree, with one component containing three paths connecting neighboring sites $j_0, j_0 - \hat{x}, j_0 + \hat{y}$ with sites $t_0, t_0 - \hat{y}, t_0 + \hat{x}$, where $j_0$ and $t_0$ have coordinates $(0, r)$ and $(k, l + r)$ respectively, and $\hat{x} = (1, 0), \ \hat{y} = (0, 1)$ are unit vectors (figure 2). The relevant configurations will be investigated with the aid of the determinant expansion of the Laplace matrix. This technique is described in detail in [9, 12, 16]. Let the vertices of the set $V$ be labeled in arbitrary order from 1 to $n = |V \setminus \{\star\}| = (2M + 1)(2N + 1)$. Then the Laplacian $\Delta$ of size $n \times n$ has the elements

$$\Delta_{ij} = \begin{cases} z_i & \text{if } i = j, \\ -1 & \text{if } i, j \text{ are nearest neighbors}, \\ 0 & \text{otherwise,} \end{cases}$$

where $z_i$ is the degree of vertex $i \in V \setminus \{\star\}$. The determinant of the Laplace matrix is equal to the number of spanning trees on $G$ with the root $\star$. Laplace matrices $\Delta_{\text{op}}, \Delta_{\text{cl}}$ for the upper half-plane have size $n = (V_+ \setminus \{\star\}) = (2M + 1)N$ and are defined in the same way as $\Delta$, but for graphs $G_{\text{op}}, G_{\text{cl}}$ respectively. The determinant of $\Delta$ is a sum over all

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Three-leg correlations in the two-component spanning tree on the upper half-plane

permutations $\sigma$ of the set $\{1, 2, \ldots, n\}$:

$$
\det \Delta = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Delta_{1,\sigma(1)} \Delta_{2,\sigma(2)} \cdots \Delta_{n,\sigma(n)}
$$

(2)

where $S_n$ is the symmetric group, $\text{sgn} = \pm 1$ is the signature of permutation $\sigma$. In general, each permutation $\sigma \in S_n$ can be factorized into a composition of disjoint cyclic permutations, say, $\sigma = c_1 \circ c_2 \circ \cdots \circ c_k$. This representation partitions the set of vertices $V \setminus \{\ast\}$ into non-empty disjoint subsets which are orbits $O = \{v_{i,1}, v_{i,2}, \ldots, v_{i,l_i}\} \subset V$ of the corresponding cycles $c_i$, $i = 1, \ldots, k$, such that $\bigcup_{i=1}^{k} O_i = V \setminus \{\ast\}$ and $\sum_{i=1}^{k} l_i = n$, where $l_i$ is the length of cycle $c_i$. The orbits consisting of just one element, if any, constitute the set $S_{\text{fp}}(\sigma)$ of fixed points of the permutation: $S_{\text{fp}}(\sigma) = \{v = \sigma(v), v \in V \setminus \{\ast\}\}$. A cycle $c_i$ of length $|c_i| = l_i \geq 2$ is called a proper cycle. The proper cycles on $G$ are of even length only; hence, the number of proper cycles $p$ defines the signature of the permutation $\sigma$, that is $\text{sgn}(\sigma) = (-1)^p$. Thus equation (2) can be written as follows:

$$
\det \Delta = \prod_{i=1}^{n} z_i + \sum_{p=1}^{\lfloor n/2 \rfloor} (-1)^p \prod_{\sigma = c_1 \circ \cdots \circ c_p} \prod_{i=1}^{p} \Delta_{c_i(v_i), c_i(v_i) \circ \cdots \circ c_i(v_i)} \Delta_{v_i, c_i(v_i) \circ \cdots \circ c_i(v_i)} \prod_{j \in S_{\text{fp}}(\sigma)} z_j
$$

(3)

where $c_i^k$ is the $k$-fold composition of the cyclic permutation $c_i$ of even length $l_i$, $v_i \in O_i(\sigma)$, and so $c_i^{k-1}(v_i) \neq c_i^{k}(v_i)$ and $c_i^{l_i}(v_i) = v_i$. The term $\prod_{i=1}^{n} z_i$ equals the number of all spanning digraphs $G_{\text{dir}}(A)$, having the root $\ast$. Each of the other terms on the right-hand side of equation (3) having a non-zero set of fixed points $S_{\text{fp}} \neq \emptyset$ up to a sign equals $\prod_{j \in S_{\text{fp}}(\sigma)} z_j$, because all non-diagonal elements are equal to $-1$. That product represents the number of distinct spanning digraphs which have in common the specified cycles $c_1, \ldots, c_p$, and differ in the oriented edges outgoing from vertices $j \in S_{\text{fp}}(\sigma)$. These oriented edges may form cycles on their own which do not enter the list $c_1, \ldots, c_p$. The proper cycles formed by the oriented bonds incident to fixed points of a given permutation $\sigma = c_1 \circ c_2 \circ \cdots \circ c_p$ should enter the enlarged list of cycles $c_1 \circ c_2 \circ \cdots \circ c_p \circ \cdots \circ c_p$, $p' > p$, corresponding to another permutation $\sigma'$. The expansion (3) can be interpreted in the form of an inclusion–exclusion principle [16]. Let $c_1, c_2, \ldots, c_m$ be the list of all possible proper cycles on $G$. We define $A_i$, $i = 1, 2, \ldots, m$, as the set of all spanning digraphs on $G$ containing the particular cycle $c_i$ and $A_0$ is the set of all spanning digraphs $G_{\text{dir}}(A)$. Let $A_{\text{ST}}$ be the set of spanning trees on $G$. Then we can write of equation (3) in the form

$$
\det \Delta = |A_{\text{ST}}| = |A_0| - \sum_{i=1}^{m} |A_i| + \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \cdots + (-1)^m |A_1 \cap \cdots \cap A_m|
$$

(4)

where $|A|$ is cardinality of the set $A$. Equation (4) is the Kirchhoff theorem for the number of spanning tree subgraphs of a given graph [17].

3. Three-leg correlations

Now we modify the Laplace matrix by changing three non-diagonal elements:

$$
\Delta'_{ij} = \begin{cases} 
z_i & \text{if } i = j, 
-1 & \text{if } i, j \text{ are nearest neighbors}, 
-\varepsilon & \text{if } (i, j) \in \mathcal{B} \equiv \{(j_0, t_0), (j_0 - \hat{x}, t_0 - \hat{y}), (j_0 + \hat{y}, t_0 + \hat{x})\}, 
0 & \text{otherwise}.
\end{cases}
$$

(5)

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Three-leg correlations in the two-component spanning tree on the upper half-plane

In the same determinant expansion as for equation (3), only the terms containing the product \( \Delta_{j_0,t_0} \Delta_{j_0,\hat{x}} \Delta_{j_0+\hat{y},t_0+\hat{x}} = (-\varepsilon)^3 \) survive in the limit \( \lim_{\varepsilon \to \infty} \det \Delta'/\varepsilon^3 \). Permutations \( \sigma = c_1 \circ c_2 \circ \cdots \circ c_p \) corresponding to these terms contain cycles \( c_{p_1} \circ \cdots \circ c_{p_k} \) with directed bonds \( \mathcal{B} \). Since sites from \( \mathcal{B} \) form angles on the lattice, topologically we can draw only one or three cycle(s) containing these bonds. Thus, the expression \( \lim_{\varepsilon \to \infty} \det \Delta'/\varepsilon^3 \) equals the number of configurations with following features: (i) each configuration is a two-component spanning graph on the plane; (ii) one component consists of three paths connecting the vicinity of site \( j_0 \) with the vicinity of site \( \ell_0 \) and branches of the spanning tree attached to these paths; (iii) another component is the spanning tree having the root *, and surrounding the first component. The quotient of such configurations and all one-component spanning trees is

\[
\lim_{\varepsilon \to \infty} \frac{\det \Delta'}{(-\varepsilon)^3 \det \Delta} = - \lim_{\varepsilon \to \infty} \frac{\det(I + \delta G)}{\varepsilon^3} \equiv \det(\Lambda) \quad (6)
\]

where \( \delta = \Delta' - \Delta \) and \( G = \Delta^{-1} = G[(n_1, m_1); (n_2, m_2)] \) is the Green function, which is defined in the thermodynamical limit \( M, N \to \infty \) on the plane as

\[
G[(n_1, m_1); (n_2, m_2)]_{\text{plane}} = G(n_1 - n_2, m_1 - m_2) = G_{0,0} + \int_\pi \int_{-\pi} \frac{\cos \delta G \pi}{8\pi^2} \frac{e^{i(n_1-n_2)\alpha + i(m_1-m_2)\beta}}{2 - \cos \alpha - \cos \beta}. \quad (7)
\]

In the appendix we give more details about this function, including its asymptotics on a long distance for arbitrary direction of the vector between \((n_1, m_1) \) and \((n_2, m_2) \). For the case of the upper half-plane, the presence of the boundary changes the Green function:

\[
G_{\text{cl}} = \Delta_{\text{cl}}^{-1}[(n_1, m_1); (n_2, m_2)] = G(n_1 - n_2, m_1 - m_2) + G(n_1 - n_2, m_1 + m_2 - 1) \quad (8)
\]

\[
G_{\text{op}} = \Delta_{\text{op}}^{-1}[(n_1, m_1); (n_2, m_2)] = G(n_1 - n_2, m_1 - m_2) - G(n_1 - n_2, m_1 + m_2). \quad (9)
\]

Matrices in equation (6) are of size \( n \times n \) (or \( n_+ \times n_+ \) for the upper half-plane), but matrix \( \delta \) has only three non-zero elements:

\[
\begin{pmatrix}
  j_0 & j_0 - \hat{x} & j_0 + \hat{y} \\
  -\varepsilon & 0 & 0 \\
  0 & -\varepsilon & 0 \\
  0 & 0 & -\varepsilon \\
\end{pmatrix} \quad (10)
\]

so we obtain due to (6) matrices of size \( 3 \times 3 \) only:

\[
\Lambda_{\text{plane}} = \begin{pmatrix}
  G_{k,l} & G_{k+1,l} & G_{k,l-1} \\
  G_{k,l-1} & G_{k+1,l-1} & G_{k,l-2} \\
  G_{k+1,l} & G_{k+2,l} & G_{k+1,l-1}
\end{pmatrix}. \quad (11)
\]

\[
\Lambda_{\text{cl}} = \begin{pmatrix}
  G_{k,l} + G_{k+1,l-2r-1} & G_{k+1,l} + G_{k+1,l+2r-1} & G_{k,l-1} + G_{k,l+2r} \\
  G_{k+1,l} + G_{k+2,l-2r} & G_{k+1,l-1} + G_{k+1,l+2r-2} & G_{k+1,l-2} + G_{k+l+2r-1} \\
  G_{k+l+2r-1} & G_{k+2,l} + G_{k+2,l+2r-1} & G_{k+l+2r}
\end{pmatrix}. \quad (12)
\]

\[
\Lambda_{\text{op}} = \begin{pmatrix}
  G_{k,l} - G_{k,l+2r} & G_{k+1,l} - G_{k+1,l+2r} & G_{k,l-1} - G_{k+l+2r+1} \\
  G_{k+l-1} - G_{k+l+2r-1} & G_{k+1,l-1} - G_{k+1,l+2r-1} & G_{k+1,l-2} - G_{k+l+2r} \\
  G_{k+l+2r-1} & G_{k+2,l} - G_{k+2,l+2r} & G_{k+1,l-1} - G_{k+1,l+2r+1}
\end{pmatrix}. \quad (13)
\]
For further analysis we find it convenient to split determinant expansions into four groups:

\[
\det \Lambda_{cl}(k, l, r) = F^{(3,0)}(k, l) - F^{(2,1)}(k, l, z) + F^{(1,2)}(k, l, z) - F^{(0,3)}(k, z) 
\]

(14)

\[
\det \Lambda_{op}(k, l, r) = F^{(3,0)}(k, l) + F^{(2,1)}(k, l, z) + F^{(1,2)}(k, l, z) + F^{(0,3)}(k, z) 
\]

(15)

where \( z = l + 2r \) for an open boundary and \( z = l + 2r - 1 \) for a closed one.

The functions \( F^{(\mu,\nu)} \) are

\[
F^{(3,0)}(k, l) \equiv \det \Lambda_{\text{plane}}(k, l) 
\]

(16)

\[
F^{(2,1)}(k, l, z) \equiv -G_{k,z}G_{k+1,l-1} + G_{k,z}G_{k,l-2}G_{k+2,l} + G_{k,z-1}G_{k+1,l} \nonumber 
\]

\[
- G_{k,z-1}G_{k,l-1}G_{k+2,l} - 2G_{k+1,z}G_{k+1,l}G_{k+2,l} + 2G_{k+1,z}G_{k,l-1}G_{k+1,l-1} \nonumber 
\]

\[
- G_{k+1,z-1}G_{k,l}G_{k+1,l-1} + G_{k+2,z}G_{k,l-2} - G_{k+2,z}G_{k,l-1} \nonumber 
\]

\[
+ G_{k+1,z-1}G_{k+1,l}G_{k,l-1} - G_{k+1,z+1}G_{k+1,l-1} - G_{k,z}G_{k+1,l} \nonumber 
\]

\[
+ G_{k+1,z+1}G_{k+1,l}G_{k,l-1} - G_{k,z+1}G_{k,l+1}G_{k+2,l} - G_{k,z}G_{k+1,l} \nonumber 
\]

\[
+ G_{k,z}G_{k+1,l}G_{k+1,l-1} \nonumber 
\]

(17)

\[
F^{(1,2)}(k, l, z) \equiv -G_{k,z}G_{k+1,z}G_{k+1,l} - G_{k,z}G_{k+1,z}G_{k+1,l-1} - G_{k,z-1}G_{k+1,z}G_{k+1,l-1} \nonumber 
\]

\[
+ G_{k,z}G_{k+1,z}G_{k+1,l} - G_{k,z}G_{k+1,z}G_{k+1,l-1} + G_{k,z}G_{k+1,z}G_{k+1,l-1} + G_{k,z}G_{k+1,z}G_{k+1,l-1} \nonumber 
\]

\[
- G_{k,z}G_{k+1,z}G_{k+1,l} + G_{k,z}G_{k+1,z}G_{k+1,l-1} + G_{k,z}G_{k+1,z}G_{k+1,l-1} \nonumber 
\]

\[
- G_{k,z}G_{k+1,z}G_{k+1,l} + G_{k,z}G_{k+1,z}G_{k+1,l-1} \nonumber 
\]

\[
- G_{k,z}G_{k+1,z}G_{k+1,l} - G_{k,z}G_{k+1,z}G_{k+1,l} \nonumber 
\]

(18)

\[
F^{(0,3)}(k, z) \equiv -G_{k,z}G_{k+1,z}G_{k+1,z+1} + G_{k,z}G_{k+1,z}G_{k+1,z+1} \nonumber 
\]

\[
- G_{k,z}G_{k+1,z}G_{k+1,z} + G_{k,z}G_{k+1,z}G_{k+1,z} \nonumber 
\]

\[
- G_{k,z}G_{k+1,z}G_{k+1,z} + G_{k,z}G_{k+1,z}G_{k+1,z} - G_{k,z}G_{k+1,z}G_{k+1,z} \nonumber 
\]

(19)

The meaning of the functions \( F^{(\mu,\nu)} \) is shown in figure 3. The index \( \mu = 1, 2, 3 \) shows numbers of ‘short’ links between the vicinity of site \( j_0 \) and the vicinity of site \( t_0 \), and index \( \nu = 1, 2, 3 \) shows the number of ‘long’ links between the vicinity of the mirror image \( j_0^1 \) of site \( j_0 \) and the vicinity of \( t_0 \).

Using the asymptotic formula for the Green function (appendix (A.18)) we can analyze the behavior of functions \( F^{(\mu,\nu)} \) for \( s \gg |\bar{s}| \) and \( |\bar{s} + d| \gg 1 \), where \( \bar{s} = (k, l) \), \( d = (0, d) \); \( d = 2r \) for the case of UHP with the open boundary and \( d = 2r - 1 \) for the closed one:

\[
F^{(3,0)}(k, l) = -\frac{1}{8\pi^3} \frac{1 + \ln s + c_0}{s^4} + \ldots
\]

(20)

\[
F^{(2,1)}(k, l, l + d) = -\frac{1}{8\pi^3} \frac{2 \cos^2 \beta - 1}{s^2 |\bar{s} + d|^2} + \frac{1}{4\pi^3} \frac{\cos \alpha \cos \beta}{s^3 |\bar{s} + d|} + \frac{1}{8\pi^3} \frac{\ln |\bar{s} + d| + c_0}{s^4} + \ldots
\]

(21)

\[
F^{(1,2)}(k, l, l + d) = -\frac{1}{8\pi^3} \frac{2 \cos^2 \alpha - 1}{s^2 |\bar{s} + d|^2} + \frac{1}{4\pi^3} \frac{\cos \alpha \cos \beta}{s^3 |\bar{s} + d|^2 s} + \frac{1}{8\pi^3} \frac{\ln s + c_0}{|\bar{s} + d|^4} + \ldots
\]

(22)

\[
F^{(0,3)}(k, l + d) = -\frac{1}{8\pi^3} \frac{1 + \ln |\bar{s} + d| + c_0}{|\bar{s} + d|^4} + \ldots
\]

(23)
where $\alpha$ and $\beta$ are angles between horizontal axis and vectors $\vec{s}$, $\vec{d} + \vec{s}$ respectively. The constant is $c_0 \equiv -2\pi G_0 \gamma + \frac{3}{2}\ln 2$.

The function $F^{(1,2)}(k, l, l + d)$ has the leading term $1/d^2$ for $d \gg s \gg 1$, and its contribution tends to zero as $d \to \infty$. This function describes configurations with two ‘long’ links and one ‘short’ link. We can see from figure 4 that such configurations define two kinds of spanning tree subgraphs with opposite sign (this sign changed because the number of loops containing ‘long’ links is changed by 1). Paths on the square lattice in these two kinds of subgraphs are topologically almost identical, and differ only in the vicinity of the site $j_0'$. The difference disappears when $d \to \infty$.

Now we consider expansions (20)–(23) of (14) and (15) in two ways: first we expand these expressions in $r$ assuming $r \gg s \gg 1$; second we expand them in $s$ assuming $s \gg r \gg 1$. In both cases we take only $\alpha = 0$ or $\pi/2$ for simplicity. As a result, we get for $r \gg s \gg 1$, $\alpha = 0$

$$
\det \Lambda_{cl}(s, r) = -\left( \frac{1 + \ln(2r) + \ln s + 2c_0}{8\pi^3 s^4} + O\left(\frac{1}{s^5}\right) \right) + \frac{1}{16\pi^3 r} \left( \frac{1}{s^4} + O\left(\frac{1}{s^5}\right) \right)
- \frac{3}{8\pi^3 r^2} \left[ \frac{3}{8s^2} - \frac{2 + \ln(2r) + \ln s + 2c_0}{6s^3} + \frac{(41/72) + \ln(2r) + \ln s + 2c_0}{2s^4} \right]
+ O\left(\frac{1}{s^5}\right) + O\left(\frac{1}{r^{3/2}}\right)
$$

(24)

and for $\alpha = \pi/2$

$$
\det \Lambda_{cl}(s, r) = -\left( \frac{1 + \ln(2r) + \ln s + 2c_0}{8\pi^3 s^4} + O\left(\frac{1}{s^5}\right) \right) - \frac{1}{16\pi^3 r} \left( \frac{1}{s^3} + \frac{1}{s^4} + O\left(\frac{1}{s^5}\right) \right)
+ \frac{3}{8\pi^3 r^2} \left[ \frac{1}{24s^2} - \frac{1 + \ln(2r) + \ln s + 2c_0}{6s^3} - \frac{(53/72) + \ln(2r) + \ln s + 2c_0}{2s^4} \right]
+ O\left(\frac{1}{s^5}\right) + O\left(\frac{1}{r^{3/2}}\right).
$$

(25)
For large $d \gg s \gg 1$ two configurations are canceled due to the asymptotical similarity of the paths.

For $s \gg r \gg 1$ and $\alpha = 0$ we obtain

$$\det \Lambda_{\text{cl}}(s, r) = -\frac{3 + 2c_0 + 2 \ln s}{\pi^3 s^6} (r^2 - r)$$
$$+ \frac{2}{\pi^3 s^6} \left( (c_0 + \ln s)(6r^4 - 12r^3 + r^2 + 5r) + \frac{10}{3} (3r^4 - 6r^3 + r^2 + 2r) \right)$$
$$+ O \left( \frac{1}{s^7} \right) \quad (26)$$

and for $\alpha = \pi/2$

$$\det \Lambda_{\text{cl}}(s, r) = -\frac{r}{\pi^3 s^5} (1 + 2c_0 + 2 \ln s)$$
$$+ \frac{1}{\pi^3 s^6} \left( 2(c_0 + \ln s)(5r^2 - 4r) + (3r^2 - 2r) \right) + O \left( \frac{1}{s^7} \right). \quad (27)$$

The asymptotics of $\det \Lambda_{\text{op}}(s, r)$ for open boundary conditions for $r \gg s \gg 1$, $\alpha = 0$ is given by

$$\det \Lambda_{\text{op}}(s, r) = -\left( \frac{1 - \ln(2r) + \ln s}{8\pi^3 s^4} + O \left( \frac{1}{s^5} \right) \right)$$
$$+ \frac{1}{16\pi^3 r^2} \left( \frac{5}{4s^2} + \frac{\ln(2r) - \ln s}{s^3} - \frac{49/24}{s^4} + 3 \ln(2r) - 3 \ln s \right) + O \left( \frac{1}{s^5} \right)$$
$$+ O \left( \frac{1}{r^3} \right). \quad (28)$$

For $\alpha = \pi/2$ we have

$$\det \Lambda_{\text{op}}(s, r) = -\left( \frac{1 - \ln(2r) + \ln s}{8\pi^3 s^4} + O \left( \frac{1}{s^5} \right) \right) + \frac{1}{8\pi^3 r} \left( \frac{1}{2s^3} + \frac{1}{s^4} + O \left( \frac{1}{s^7} \right) \right)$$
$$+ \frac{1}{16\pi^3 r^2} \left( \frac{3}{4s^2} + \frac{(1/2) - \ln(2r) + \ln s}{s^3} - \frac{13/24}{s^4} + 3 \ln(2r) - 3 \ln s \right)$$
$$+ O \left( \frac{1}{s^5} \right) + O \left( \frac{1}{r^3} \right). \quad (29)$$
In the range of \( s \gg r \gg 1 \) we have for \( \alpha = 0 \)
\[
\det \Lambda_{op}(s, r) = \frac{2(r^4 - 5r^6 + 4r^8)}{3\pi^3 s^{12}} + \frac{74r^4 - 394r^6 + 416r^8 - 96r^{10}}{3\pi^3 s^{14}} + O\left(\frac{1}{s^{15}}\right)
\]  
and for \( \alpha = \pi/2 \)
\[
\det \Lambda_{op}(s, r) = \frac{4(r^2 + 3r^3 + 2r^4)}{3\pi^3 s^8} + \frac{10r^2 - 2r^3 - 76r^4 - 64r^5}{3\pi^3 s^9} + O\left(\frac{1}{s^{10}}\right).
\]

4. Discussion

As was noted above, the \( \Theta \)-graph is the key object in calculations of height variables \( h_i \geq 2 \) in the ASM. The previous analysis \([6]–[9]\) shows that the ASM belongs to the class of \( c = -2 \) minimal models of logarithmic conformal field theory. Height variable \( h = 1 \) is associated with a primary field \( \phi(z, \bar{z}) \) with conformal weights \((1,1)\), and the heights \( h = 2, 3, 4 \) behave like its logarithmic partner \( \psi(z, \bar{z}) \) with scaling dimension 2. Let \( P_h^{UHP}(r) \) be the probability of height \( h = 1, 2, 3, 4 \) at distance \( r \) apart from the boundary of the upper half-plane. In fact this is a two-point correlation function, and through point correlations, operator product expansion (OPE) and to conjecture logarithmic behavior of the two-point correlations \( w_{h_i, h_j}(r) = P_{h_i} P_{h_j} \) of the height variables \( h_i \geq 1 \) on the plane at site \( i \) and height variables \( h_j \geq 2 \) at site \( j \) at distance \( r \) apart from the site \( j \). Correlations \( P_h^{UHP}(r) \) have been obtained combinatorially, by mapping the ASM onto the spanning trees model \([9]\).

Due to the similarity between the \( \Theta \)-graph and the three-leg correlations, we may compare calculations in \([9]\) with the present ones. The ‘head’ of the \( \Theta \)-graph located at site \( i \) with \( h_i = 2 \) corresponds to one of the fixed points of the three-leg correlations; the running point of the \( \Theta \)-graph corresponds to another point at distance \( s \) apart. The dependence of \( \det \Lambda(s, r) \) on \( s \) for fixed \( r \) is shown in figure 5 for open and closed boundary conditions. We see that the function has a strong peak at \((k, l) = (0, 0)\). An essential part of the computational work in \([9]\) is the summation over all positions of the running point. Instead, we may try to use the expansions (20)–(23) for integration in the vicinity of the peak using the fact that \( \det \Lambda(s, r) \) decays as \( \ln s/s^4 \) with \( s \).

In the case of two-point correlations \( w_{1,2}(r) \), this method leads to drastic simplification of the calculations due to the rapid convergence of the integrals in the \( s < s_0 \) vicinity of the peak for \( s_0 \ll r \) \([15]\). In the case of boundary correlations \( P_2^{UHP}(r) \), the leading term of the asymptotics in \( r \) in (25) and (29) is \( \ln r/r^2 \) and its integration over \( s \) in the vicinity of the peak is not sufficient for obtaining a coefficient at \( \ln r/r^2 \). Indeed, the first terms of expansions (25) and (29) contain \( \ln s/s^2 \) which also gives \( \ln r/r^2 \) upon summation over the half-plane:
\[
\sum_{k=-\infty}^{\infty} \sum_{l=-r}^{\infty} \frac{\ln |s|}{|s|^4} \sim \frac{\ln r}{r^2}.
\]
But expansions (25) and (29) are not valid for \( s \sim r \) and therefore the method \([15]\) fails in the case of calculations of \( P_2^{UHP}(r) \).

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The failure with the description of the $\Theta$-graph near the boundary, the expansions (24)–(31) are still useful for a LCFT treatment. The previous attempts to describe the logarithmic correlations in both lattice and field theories were undertaken solely for the ASM. In this case, the logarithmic partner of the primary field is associated with the height variables $h_i > 1$ having a non-local representation in the spanning tree model. Moreover, the non-local representation is the infinite sum over positions of the running point of the $\Theta$-graph. But two branching points of the $\Theta$-graph are in turn correlating objects of the spanning tree, which can be considered in the framework of the LCFT independently of the ASM problem. Thus the collection of asymptotics (24)–(31) for the three-leg correlations near closed and open boundaries, together with the bulk asymptotics (20), should be found within the LCFT provided that one finds a proper identification for the three-leg branching points.

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Appendix

We consider here the Green function $G_{p,q}$ for a two-dimensional square lattice and derive its asymptotic expansion for large distances ($r = \sqrt{p^2 + q^2} \gg 1$) using the methods proposed in [9] and in [18]. We denote the angle between the vector $\vec{r} = (p, q)$ and the horizontal axis by $\varphi$, so

\begin{align}
    p &= r \cos \varphi, \\
    q &= r \sin \varphi.
\end{align}

Figure 5. $\det \Lambda_{\text{op}}(\vec{s}, \vec{r})$ and $\det \Lambda_{\text{cl}}(\vec{s}, \vec{r})$ for $\vec{s} = (k, l)$, $r = 50$, $-50 \leq k \leq 50$, $-50 \leq l \leq 50$. 

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Three-leg correlations in the two-component spanning tree on the upper half-plane

Since the Green function contains a singular part \( G_{0,0} \), we consider a function \( g_{p,q} = G_{p,q} - G_{0,0} \), which has an integral representation

\[
g_{p,q} = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{ip\alpha + iq\beta} - 1}{2 - \cos \alpha - \cos \beta} \, d\alpha \, d\beta \quad (A.3)
\]

and obeys the symmetry relations

\[
g_{p,q} = g_{-p,q} = g_{p,-q} = g_{q,p} \quad (A.4)
\]

so we can put \( p > q \geq 0 \) without loss of generality. After the integration over \( \alpha \) and symmetrization using \( \beta \) we come to the expression

\[
g_{p,q} = \frac{1}{2\pi} \int_{0}^{\pi} \frac{(2 - \cos \beta - \sqrt{(2 - \cos \beta)^2 - 1})^p \cos q\beta - 1}{\sqrt{(2 - \cos \beta)^2 - 1}} \, d\beta \quad (A.5)
\]

Consider the Taylor expansion of the function

\[
e^{p\beta} \left(2 - \cos \beta - \sqrt{(2 - \cos \beta)^2 - 1}\right)^p \quad (A.6)
\]

for small positive \( \beta \) up to order 14 and denote it as \( Q(\beta) \):

\[
Q_p(\beta) = 1 + \frac{p\beta}{12} - \frac{p^3\beta^3}{96} + \frac{p^2\beta^6}{288} + \frac{79p^7\beta^7}{40320} - \frac{p^2\beta^8}{1152} + \frac{(112p^3 - 493p)\beta^9}{116116} - \frac{421p^2\beta^{10}}{1935360} + \frac{(127741p - 46200p^2)\beta^{11}}{1277337600} + \frac{(140p^4 - 3887p^2)\beta^{12}}{69672960} + \frac{(69432p^3 - 152461p)\beta^{13}}{6131220480} + \frac{(629861p^2 - 43120p^4)\beta^{14}}{4291854360}. \quad (A.7)
\]

The function

\[
R_p(\beta) = |(2 - \cos \beta - \sqrt{(2 - \cos \beta)^2 - 1})^p - e^{-p\beta}Q(\beta)| \quad (A.8)
\]

vanishes at \( \beta = 0 \), and has a unique maximum over \( \beta \) for all \( p \). The location of the maximum, \( \beta^* \), can be found as a series in \( 1/p \) if we construct the series expansion of the derivative \( (dR_p(\beta)/d\beta) \) and recursively equate coefficients to 0. Calculations give

\[
\beta^* = \frac{15}{p} + \frac{795}{28p^3} + \frac{14503977}{17248p^5} + \cdots \quad (A.9)
\]

and

\[
R_p(\beta) \leq \frac{120135498046875}{8192 e^{15p^2}} + O(p^{-12}). \quad (A.10)
\]

It is easy to show that

\[
\int_{0}^{\pi} \beta^n e^{-p\beta} \cos(q\beta) \frac{d\beta}{2\pi} \sim \frac{1}{r^{n+1}}, \quad n \geq 0. \quad (A.11)
\]

This means that we can replace the function in the integral \( (A.5) \) by its Taylor expansion in \( \beta \) up to the 14th order and get an expression for the Green function with accuracy \( O(p^{-10}) \). It is convenient to express the Green function as a sum of three integrals

\[
g_{p,q} = I_1 + I_2 + I_3, \quad (A.12)
\]

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where

\[ I_1 = \frac{1}{2\pi} \int_0^\pi \left( \frac{e^{-\rho \beta} \cos(q\beta)}{\beta} - \frac{1}{\beta} \right) d\beta \quad (A.13) \]

\[ I_2 = \frac{1}{2\pi} \int_0^\pi \left( \frac{1}{\beta} - \frac{1}{\sqrt{(2 - \cos \beta)^2 - 1}} \right) d\beta \quad (A.14) \]

\[ I_3 = \int_0^\pi \left( \frac{e^{-\rho \beta} Q_p(\beta) \cos(q\beta)}{\sqrt{(2 - \cos \beta)^2 - 1}} - \frac{e^{-\rho \beta} \cos(q\beta)}{\beta} \right) \frac{d\beta}{2\pi} + O \left( \frac{1}{r^{16}} \right) . \quad (A.15) \]

The expressions give

\[ I_1 = -\frac{1}{2\pi} (\ln r + \gamma + \ln \pi) + \text{exponentially small terms} \quad (A.16) \]

\[ I_2 = \frac{1}{2\pi} \left( \ln \pi - \frac{3}{2} \ln 2 \right) . \quad (A.17) \]

Finally we obtain

\[ g_{p,q} = -\frac{\ln r + \gamma + (3 \ln 2/2)}{2\pi} + \cos(4\varphi) + \frac{1}{r^4} \left( \frac{3 \cos(4\varphi)}{80\pi} + \frac{5 \cos(8\varphi)}{96\pi} \right) \]
\[ + \frac{1}{r^6} \left( \frac{51 \cos(8\varphi)}{224\pi} + \frac{35 \cos(12\varphi)}{144\pi} \right) \]
\[ + \frac{1}{r^8} \left( \frac{217 \cos(8\varphi)}{640\pi} + \frac{45 \cos(12\varphi)}{16\pi} + \frac{1925 \cos(16\varphi)}{768\pi} \right) + O \left( \frac{1}{r^{10}} \right) \quad (A.18) \]

where the expansion

\[ \frac{1}{\sqrt{(2 - \cos \beta)^2 - 1}} = \frac{1}{\beta} - \beta - \frac{\beta^3}{12} + \frac{43\beta^3}{1440} - \frac{949\beta^5}{120960} + \frac{21727\beta^7}{9676800} - \frac{501451\beta^9}{766402560} \]
\[ + \frac{8112267073\beta^{11}}{41845579776000} - \frac{277899049\beta^{13}}{4782351974400} + O \left( \beta^{15} \right) \quad (A.19) \]

is used for the calculation of \((A.15)\).

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