BLOCH AND KATO’S EXPONENTIAL MAP: THREE EXPLICIT FORMULAS

by

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Abstract. — The purpose of this article is to give formulas for Bloch-Kato’s exponential map and its dual for an absolutely crystalline $p$-adic representation $V$, in terms of the $(\varphi, \Gamma)$-module associated to that representation. As a corollary of these computations, we can give a very simple (and slightly improved) description of Perrin-Riou’s exponential map (which interpolates Bloch-Kato’s exponentials for $V(k)$). This new description directly implies Perrin-Riou’s reciprocity formula.

Contents

Introduction ................................................................. 2
I. Periods of $p$-adic representations .................................... 3
   I.1. Notations .................................................... 3
   I.2. $p$-adic Hodge theory ....................................... 4
   I.3. $(\varphi, \Gamma)$-modules ..................................... 6
   I.4. $p$-adic representations and differential equations .......... 7
   I.5. Construction of cocycles .................................... 8
II. Explicit formulas for exponential maps ............................. 10
   II.1. Bloch-Kato’s exponential map ............................... 10
   II.2. Bloch-Kato’s dual exponential map ......................... 12
   II.3. Iwasawa theory of $p$-adic representations ............... 14
   II.4. Perrin-Riou’s exponential map ............................. 15
   II.5. The explicit reciprocity formula ............................ 17
Appendix A. The structure of $D(T)^{\psi=1}$ .......................... 19
References ..................................................................... 21

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Introduction

Let $p$ be a prime number, and let $V$ be a $p$-adic representation of $\text{Gal}(\overline{K}/K)$ where $K$ is a finite extension of $\mathbb{Q}_p$. Such objects arise (for example) as the étale cohomology of algebraic varieties, hence their interest in arithmetic algebraic geometry.

Let $B_{\text{cris}}$ and $B_{\text{dR}}$ be the rings of periods of Fontaine, and let $D_{\text{cris}}(V)$ and $D_{\text{dR}}(V)$ be the invariants attached to $V$ by Fontaine’s construction. Bloch and Kato have defined (in [4]), for a de Rham representation $V$, an “exponential” map, $\exp_{K,V} : D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V) \to H^1(K, V)$. It is obtained by tensoring the so-called fundamental exact sequence:

$$0 \to \mathbb{Q}_p \to B_{\text{cris}}^{\varphi=1} \to B_{\text{dR}}/B_{\text{dR}}^+ \to 0$$

with $V$ and taking the invariants under the action of $G_K$. The exponential map is then the connecting homomorphism $D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V) \to H^1(K, V)$.

The reason for their terminology is the following (cf [4, 3.10.1]): if $G$ is a formal Lie group of finite height over $\mathcal{O}_K$, and $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ where $T$ is the $p$-adic Tate module of $G$, then $V$ is a de Rham representation and $D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V)$ is identified with the tangent space $\tan(G(K))$ of $G(K)$. In this case, we have a commutative diagram:

$$\begin{array}{ccc}
\tan(G(K)) & \xrightarrow{\exp_C} & \mathbb{Q} \otimes_{\mathbb{Z}} G(\mathcal{O}_K) \\
\downarrow \hspace{1cm} & & \downarrow \delta_G \\
D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V) & \xrightarrow{\exp_{K,V}} & H^1(K, V),
\end{array}$$

where $\delta_G$ is the Kummer map, the upper $\exp_C$ is the usual exponential map, and the lower $\exp_{K,V}$ is Bloch-Kato’s exponential map.

The cup product $\cup : H^1(K, V) \times H^1(K, V^\ast(1)) \to H^2(K, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$ defines a perfect pairing, which we can use (by dualizing twice) to define Bloch and Kato’s dual exponential map $\exp_{K,V^\ast(1)} : H^1(K, V) \to \text{Fil}^0 D_{\text{dR}}(V)$.

Finally, Perrin-Riou has constructed in [12] a period map $\Omega_{V,h}$ which interpolates the $\exp_{K,V(k)}$ as $k$ runs over the positive integers. It is a crucial ingredient in the construction of $p$-adic $L$ functions, and is a vast generalization of Coleman’s map.

The goal of this article is to give formulas for $\exp_{K,V}$, $\exp_{K,V^\ast(1)}$, and $\Omega_{V,h}$ in terms of the $(\varphi, \Gamma)$-module associated to $V$ by Fontaine. As a corollary, we recover a theorem of Colmez which states that Perrin-Riou’s map interpolates the $\exp_{K,V^\ast(1-k)}$ as $k$ runs over the negative integers. This is equivalent to Perrin-Riou’s conjectured reciprocity formula. Our construction of $\Omega_{V,h}$ is actually a slight improvement over Perrin-Riou’s (one does not have to kill the Λ-torsion, see remark [11,2]).
We refer the reader to the text itself for a statement of the actual formulas (theorems I.2, I.6 and I.11) which are rather technical.

This article does not really contain any new results, and it is mostly a re-interpretation of formulas of Cherbonnier-Colmez (for the dual exponential map), and of Colmez and Benois (for Perrin-Riou’s map) in the language of the author’s article “p-adic representations and differential equations”.

I. Periods of p-adic representations

I.1. Notations

Throughout this article, \( k \) will denote a finite field of characteristic \( p > 0 \), so that \( F = W(k)[1/p] \) is a finite unramified extension of \( \mathbb{Q}_p \). Let \( \overline{\mathbb{Q}}_p \) be the algebraic closure of \( \mathbb{Q}_p \), let \( K \) be a finite totally ramified extension of \( F \), and let \( G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K) \) be the absolute Galois group of \( K \). Let \( \mu_p^n \) be the group of \( p^n \)th roots of unity; for every \( n \), we will choose a generator \( \varepsilon^{(n)} \) of \( \mu_p^n \), with the additional requirement that \( (\varepsilon^{(n)})^p = \varepsilon^{(n-1)} \). This makes \( \lim_{\varepsilon^{(n)}} \varepsilon^{(n)} \) into a generator of \( \lim_{\varepsilon^{(n)}} \mu_p^n \simeq \mathbb{Z}_p(1) \). We set \( K_n = K(\mu_p^n) \) and \( K_\infty = \bigcup_{n=0}^{+\infty} K_n \). Recall that the cyclotomic character \( \chi : G_K \to \mathbb{Z}_p^* \) is defined by the relation: \( g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)} \) for all \( g \in G_K \). The kernel of the cyclotomic character is \( H_K = \text{Gal}(\overline{\mathbb{Q}}_p/K_\infty) \), and \( \chi \) therefore identifies \( \Gamma_K = G_K/H_K \) with an open subgroup of \( \mathbb{Z}_p^* \). The torsion subgroup of \( \Gamma_K \) will be denoted by \( \Delta_K \). We also set \( \Gamma_n = \text{Gal}(K_\infty/K_n) \). When \( p \neq 2 \) and \( n \geq 1 \) (or \( p = 2 \) and \( n \geq 2 \), \( \Gamma_n \) is torsion free. If \( x \in 1 + p\mathbb{Z}_p \), then there exists \( k \geq 1 \) such that \( \log_p(x) \in p^k\mathbb{Z}_p^* \) and we’ll write \( \log^0_p(x) = \log_p(x)/p^k \).

The completed group algebra of \( \Gamma_K \) is \( \Lambda = \mathbb{Z}_p[[\Gamma_K]] \simeq \mathbb{Z}_p[\Delta_K] \otimes \mathbb{Z}_p \mathbb{Z}_p[[\Gamma_1]] \), and we set \( \mathcal{H}(\Gamma_K) = \mathbb{Q}_p[\Delta_K] \otimes \mathbb{Q}_p \mathcal{H}(\Gamma_1) \) where \( \mathcal{H}(\Gamma_1) \) is the set of \( f(\gamma - 1) \) with \( \gamma \in \Gamma_1 \) and where \( f(X) \in \mathbb{Q}_p[[X]] \) is convergent on the \( p \)-adic open unit disk. Examples of elements of \( \mathcal{H}(\Gamma_K) \) are the \( \nabla_i \) (which are Perrin-Riou’s \( \ell_i \)'s), defined by \( \nabla_i = \log(\gamma)/\log_p(\chi(\gamma)) - i \). We will also use the operator \( \nabla_0//(\gamma_n - 1) \), where \( \gamma_n \) is a topological generator of \( \Gamma_n \) (see [2, 4.1]). It is defined by the formula:

\[
\frac{\nabla_0}{\gamma_n - 1} = \frac{\log(\gamma_n)}{\log_p(\chi(\gamma_n))(\gamma_n - 1)} = \frac{1}{\log_p(\chi(\gamma_n))} \sum_{n \geq 1} \frac{(1 - \gamma_n)^{n-1}}{n},
\]

or equivalently by

\[
\frac{\nabla_0}{\gamma_n - 1} = \lim_{r \rightarrow n} \frac{\eta - 1}{\gamma_n - 1} \frac{1}{\log_p(\chi(\eta))}.
\]

It is easy to see that \( \nabla_0/(\gamma_n - 1) \) acts on \( F_n \) by \( 1/\log_p(\chi(\gamma_n)) \).
The algebra acts $\mathcal{H}(\Gamma_K)$ on $B^+_{\text{rig},F}$ and one can check that
\[
\nabla_i = t \frac{d}{dt} - i = \log(1 + \pi) \partial - i, \quad \text{where} \quad \partial = (1 + \pi) \frac{d}{d\pi}.
\]
In particular, $\nabla_0 B^+_{\text{rig},F} \subset tB^+_{\text{rig},F}$ and if $i \geq 1$, then $\nabla_{i-1} \circ \ldots \circ \nabla_0 B^+_{\text{rig},F} \subset t^i B^+_{\text{rig},F}$.

A $p$-adic representation $V$ is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of $G_K$. It is easy to see that there is always a $\mathbb{Z}_p$-lattice of $V$ which is stable by the action of $G_K$, and such lattices will be denoted by $T$. The main strategy (due to Fontaine, see for example [10]) for studying $p$-adic representations of a group $G$ is to construct topological $\mathbb{Q}_p$-algebras $B$, endowed with an action of $G$ and some additional structures so that if $V$ is a $p$-adic representation, then $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^G$ is a $B^G$-module which inherits these structures, and so that the functor $V \mapsto D_B(V)$ gives interesting invariants of $V$. We say that a $p$-adic representation $V$ of $G$ is $B$-admissible if we have $B \otimes_{\mathbb{Q}_p} V \simeq B^d$ as $B[G]$-modules.

I.2. $p$-adic Hodge theory

In this paragraph, we will recall the definitions of Fontaine’s rings of periods. One can find some of these constructions in [3]. Let
\[
\tilde{E} = \varprojlim_{x \rightarrow x^p} C_p = \{(x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}\},
\]
and let $\tilde{E}^+$ be the set of $x \in \tilde{E}$ such that $x^{(0)} \in \mathcal{O}_{C_p}$. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\tilde{E}$, we define their sum $x + y$ and their product $xy$ by:
\[
(x + y)^{(i)} = \lim_{j \rightarrow \infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \quad \text{and} \quad (xy)^{(i)} = x^{(i)}y^{(i)},
\]
which makes $\tilde{E}$ into an algebraically closed field of characteristic $p$. If $x = (x^{(n)})_{n \geq 0} \in \tilde{E}$, let $v_{\tilde{E}}(x) = v_p(x^{(0)})$. This is a valuation on $\tilde{E}$ for which $\tilde{E}$ is complete; the ring of integers of $\tilde{E}$ is $\tilde{E}_+$. Let $\tilde{A}^+$ be the ring $W(\tilde{E}^+)$ of Witt vectors with coefficients in $\tilde{E}^+$ and let $\tilde{B}^+ = \tilde{A}^+[1/p] = \{\sum_{k \gg -\infty} p^k [x_k], \ x_k \in \tilde{E}^+\}$ where $[x] \in \tilde{A}^+$ is the Teichmüller lift of $x \in \tilde{E}^+$. This ring is endowed with a map $\theta : \tilde{B}^+ \rightarrow C_p$ defined by the formula $\theta(\sum_{k \gg -\infty} p^k [x_k]) = \sum_{k \gg -\infty} p^k x_k^{(0)}$. Let $\varepsilon = (\varepsilon^{(i)}) \in \tilde{E}^+$ where $\varepsilon^{(n)}$ is defined above, and define $\pi = [\varepsilon] - 1$, $\pi_1 = \varepsilon^{1/p} - 1$, $\omega = \pi/\pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. One can show that $\ker(\theta : \tilde{A}^+ \rightarrow \tilde{A}^+)$ is the principal ideal generated by $\omega$.

Notice that $\varepsilon \in \tilde{E}^+$ and that $v_{\tilde{E}}(\varepsilon - 1) = p/(p - 1)$. We define $E_F = k((\varepsilon - 1))$ and $E$ as the separable closure of $E_F$ in $\tilde{E}$, as well as $E^+ = E \cap \tilde{E}^+$, the ring of integers of $E$. By definition, $E$ is separably closed and $\tilde{E}$ is the completion of its purely inseparable closure.
The ring $B_{dR}^+$ is defined to be the completion of $\tilde{B}^+$ for the $\ker(\theta)$-adic topology:

$$B_{dR}^+ = \lim_{\rightarrow}^{\text{proj}} \tilde{B}^+ / (\ker(\theta)^n).$$

It is a discrete valuation ring, whose maximal ideal is $\ker(\theta)$; the series which defines $\log([\varepsilon])$ converges in $B_{dR}^+$ to an element $t$, which is a generator of the maximal ideal, so that $B_{dR} = B_{dR}^+[1/t]$ is a field, endowed with an action of $G_F$ and a filtration defined by $\text{Fil}^i(B_{dR}) = t^iB_{dR}^+$ for $i \in \mathbb{Z}$.

We say that a representation $V$ of $G_K$ is de Rham if it is $B_{dR}$-admissible which is equivalent to the fact that the filtered $K$-vector space $D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is of dimension $d = \dim_{\mathbb{Q}_p}(V)$.

The ring $B_{\max}^+$ is defined as being

$$B_{\max}^+ = \{ \sum_{n \geq 0} a_n \frac{\omega^n}{p^n} \mid a_n \in \tilde{B}^+ \text{ is sequence converging to } 0 \},$$

and $B_{\max} = B_{\max}^+[1/t]$. One could replace $\omega$ by any generator of $\ker(\theta)$. The ring $B_{\max}^+$ injects canonically into $B_{dR}^+$ and, in particular, it is endowed with the induced Galois action and filtration, as well as with a continuous Frobenius $\varphi$, extending the map $\varphi : \tilde{\mathbb{A}}^+ \rightarrow \tilde{\mathbb{A}}^+$ coming from $x \mapsto x^p$ in $\tilde{E}^+$. Let us point out that $\varphi$ does not extend continuously to $B_{dR}$. One also sets $\tilde{B}_{\text{rig}}^+ = \cap_{n=0}^{+\infty} p^n(B_{\max}^+)$.

We say that a representation $V$ of $G_K$ is crystalline if it is $B_{\max}$-admissible or (which is the same) $\tilde{B}_{\text{rig}}^+[1/t]$-admissible (the periods of crystalline representations live in finite dimensional $F$-vector subspaces of $B_{\max}$, stable by $\varphi$, and so in fact in $\cap_{n=0}^{+\infty} \varphi^n(B_{\max}^+)[1/t]$); this is equivalent to requiring that the $F$-vector space

$$D_{\text{cris}}(V) = (B_{\max} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\tilde{B}_{\text{rig}}^+[1/t] \otimes_{\mathbb{Q}_p} V)^{G_K}$$

be of dimension $d = \dim_{\mathbb{Q}_p}(V)$. Then $D_{\text{cris}}(V)$ is endowed with a Frobenius $\varphi$ induced by that of $B_{\max}$, and $(B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K} = D_{dR}(V) = K \otimes_F D_{\text{cris}}(V)$ so that a crystalline representation is also de Rham and $K \otimes_F D_{\text{cris}}(V)$ is a filtered $K$-vector space.

If $V$ is a $p$-adic representation, we say that $V$ is Hodge-Tate, with Hodge-Tate weights $h_1, \ldots, h_d$, if we have a decomposition $C_p \otimes_{\mathbb{Q}_p} V \simeq \oplus_{j=1}^d C_p(h_j)$. In this case, we see that $(C_p \otimes_{\mathbb{Q}_p} V)^{H_K} \simeq \oplus_{j=1}^d \widehat{K}_\infty(h_j)$ and one can show that $D_{\text{Sen}}(V) = (C_p \otimes_{\mathbb{Q}_p} V)^{H_K}$, which is by definition the union of the finite dimensional $K_\infty$-vector subspaces of $(C_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ which are stable by $\Gamma_K$, is equal to $\oplus_{j=1}^d K_\infty(h_j)$.
The $K_\infty$-vector space $D_{\text{Sen}}(V)$ is endowed with a residual action of $\Gamma_K$ and if $\gamma \in \Gamma_K$ is sufficiently close to 1, then the series of operators $\log(\gamma)/\log_p(\chi(\gamma))$ converges to a $K_\infty$-linear operator $\nabla_V : D_{\text{Sen}}(V) \to D_{\text{Sen}}(V)$ which does not depend on the choice of $\gamma$, and which is diagonalizable with eigenvalues $h_1, \cdots, h_d$. We will say that $V$ is positive if its Hodge-Tate weights are negative (the definition of the sign of the Hodge-Tate weights is unfortunate; some people change the sign and talk about geometrical weights). By using the map $\theta : B_{\text{dR}}^+ \to C_p$, it is easy to see that the Hodge-Tate weights of $V$ are those integers $h$ such that $\text{Fil}^{-h}D_{\text{dR}}(V) \neq \text{Fil}^{-h+1}D_{\text{dR}}(V)$.

I.3. $(\varphi, \Gamma)$-modules

See [11] for a reference. Let $\tilde{A}$ be the ring of Witt vectors with coefficients in $\tilde{E}$ and $\tilde{B} = \tilde{A}[1/p]$. Let $A_F$ be the completion of $O_F[[\pi, \pi^{-1}]]$ in $\tilde{A}$ for this ring’s topology, which is also the completion of $O_F[[\pi]]$ for the $p$-adic topology (\(\pi\) being small in $\tilde{A}$). This is a discrete valuation ring whose residual field is $E_F$. Let $B$ be the completion for the $p$-adic topology of the maximal unramified extension of $B_F = A_F[1/p]$ in $\tilde{B}$. We then define $A = B \cap \tilde{A}$ and $A^+ = A \cap \tilde{A}^+$. These rings are endowed with an action of Galois and a Frobenius deduced from those on $\tilde{E}$. We set $A_K = A_{H_K}$ and $B_K = A_K[1/p]$. When $K = F$, the two definitions are the same. One also sets $B^+ = A^+[1/p]$, and $B_{\text{dR}}^+ = (B^+)_{H_F}$ as well as $A_{\text{dR}}^+ = (A^+)_{H_F}$.

If $V$ is a $p$-adic representation of $G_K$, let $D(V) = (B \otimes_{Q_p} V)^{H_K}$. We know by [11] that $D(V)$ is a $d$-dimensional $B_K$-vector space with a Frobenius and a residual action of $\Gamma_K$ which commute (it is a $(\varphi, \Gamma_K)$-module) and that one can recover $V$ by the formula $V = (D(V) \otimes_{B_K} B)^{\varphi = 1}$.

The field $B$ is a totally ramified extension (because the residual extension is purely inseparable) of degree $p$ of $\varphi(B)$. The Frobenius map $\varphi : B \to B$ is injective but therefore not surjective, but we can define a left inverse for $\varphi$, which will play a major role in the sequel. We set: $\psi(x) = \varphi^{-1}(p^{-1} \text{Tr}_{B/F}(\varphi(B))(x))$.

Let us now set $K = F$. We say that a $p$-adic representation $V$ of $G_F$ is of finite height if $D(V)$ has a basis over $B_F$ made up of elements of $D^+(V) = (B^+ \otimes_{Q_p} V)^{H_F}$. A result of Fontaine ([11] or [8, III.2]) shows that $V$ is of finite height if and only if $D(V)$ has a sub-$B_{\text{dR}}^+$-module which is free of finite rank $d$, and stable by $\varphi$. Let us recall the main result (due to Colmez, see [8] or also [2, theorem 3.10]) regarding crystalline representations of $G_F$: if $V$ is a crystalline representation of $G_F$, then $V$ is of finite height.
If $K \neq F$ or if $V$ is no longer crystalline, then it is no longer true in general that $V$ is of finite height, but it is still possible to say something about the periods of $V$. Every element $x \in \tilde{B}$ can be written in a unique way as $x = \sum_{k \geq -\infty} p^k [x_k]$, where $x_k \in \tilde{E}$. For $r > 0$, let us set:

$$\tilde{B}^{t,r} = \left\{ x \in \tilde{B}, \lim_{k \to +\infty} v_E(x_k) + \frac{pr}{p-1} k = +\infty \right\}.$$ 

This makes $\tilde{B}^{t,r}$ into an intermediate ring between $\tilde{B}^+$ and $\tilde{B}$. Let us set $B^{t,r} = B \cap \tilde{B}^{t,r}$, $\tilde{B}^t = \cup_{r \geq 0} \tilde{B}^{t,r}$, and $B^t = \cup_{r \geq 0} B^{t,r}$. If $R$ is any of the above rings, then by definition $R_K = R_{t,K}$.

We say that a $p$-adic representation $V$ is overconvergent if $D(V)$ has a basis over $B_K$ made up of elements of $D^t(V) = \left( B^t \otimes \mathbb{Q}_p \right)^{H_K}$. The main result on the overconvergence of $p$-adic representations of $G_K$ is the following (cf [3, 3]):

**Theorem I.1.** — Every $p$-adic representation $V$ of $G_K$ is overconvergent, that is there exists $r(V)$ such that $D(V) = B_K \otimes_{B_K^{t,r(V)}} D^{t,r(V)}(V)$.

The terminology “overconvergent” can be explained by the following proposition, which describes the rings $B_K^{t,r}$. Let $e_K = [K_{\infty} : F_{\infty}]$:

**Proposition I.2.** — Let $B_F^{0}$ be the set of power series $f(X) = \sum_{k \in \mathbb{Z}} a_k X^k$ such that $a_k$ is a bounded sequence of elements of $F$, and such that $f(X)$ is holomorphic on the $p$-adic annulus $\{ p^{-1/\alpha} \leq |T| < 1 \}$.

There exists $r(K)$ and $\pi_K \in B_K^{t,r(K)}$ such that if $r \geq r(K)$, then the map $f \mapsto f(\pi_K)$ from $B_F^{t,r}$ to $B_K^{t,r}$ is an isomorphism. If $K = F$, then one can take $\pi_F = \pi$.

### I.4. $p$-adic representations and differential equations

We shall now recall some of the results of [2], which allow us to recover $D_{\text{cris}}(V)$ from the $(\varphi, \Gamma)$-module of $V$. Let $\mathcal{H}_F^{\alpha}$ be the set of power series $f(X) = \sum_{k \in \mathbb{Z}} a_k X^k$ such that $a_k$ is a sequence (not necessarily bounded) of elements of $F$, and such that $f(X)$ is holomorphic on the $p$-adic annulus $\{ p^{-1/\alpha} \leq |T| < 1 \}$.

For $r \geq r(K)$, define $B_{\text{rig}}^{t,r,K}$ as the set of $f(\pi_K)$ where $f(X) \in \mathcal{H}_F^{t,r,K}$. Obviously, $B_K^{t,r} \subset B_{\text{rig}}^{t,r,K}$. If $V$ is a $p$-adic representation, let $D_{\text{rig}}^{t,r}(V) = B_{\text{rig}}^{t,r,K} \otimes_{B_K^{t,r}} D^{t,r}(V)$.

One of the main technical tools of [2] is the construction of a large ring $\tilde{B}_{\text{rig}}^{t}$, which contains $\tilde{B}_{\text{rig}}^{t}$ and $\tilde{B}^{t}$, so that $D_{\text{cris}}(V) \subset (\tilde{B}_{\text{rig}}^{t,1/t} \otimes \mathbb{Q}_p V)^{G_K}$ and $D_{\text{rig}}^{t}(V)[1/t] \subset (\tilde{B}_{\text{rig}}^{t,1/t} \otimes \mathbb{Q}_p V)^{H_K}$. We then have (cf [2, theorem 3.6]): $D_{\text{cris}}(V) = (D_{\text{rig}}^{t}(V)[1/t])^{T_F}$. 


Let us now return to the case when $K = F$ and $V$ is a crystalline representation of $G_F$. In this case, one can give a more precise result. Let $B^{+}_{\text{rig},F}$ be the set of $f(\pi)$ where $f(X) = \sum_{k \geq 0} a_k X^k$ where $a_k \in F$, and $f(X)$ is holomorphic on the $p$-adic open unit disk. Set $D^+_{\text{rig}}(V) = B^{+}_{\text{rig},F} \otimes_{B^F} D^+(V)$. One can then show (cf [3]): $D_{\text{cris}}(V) = (D^+_{\text{rig}}(V)[1/t])^{\Gamma}.$

I.5. Construction of cocycles

The purpose of this paragraph is to recall the constructions of [6 I.5] and extend them a little bit. In this paragraph and in the next, $V$ will be an arbitrary $p$-adic representation of $G_K$. Recall that in loc. cit., a map $h^1_{K,V} : D(V)^{\psi=1} \to H^1(K,V)$ was constructed, and that (when $\Gamma_K$ is torsion free at least) it gives rise to an exact sequence:

$$0 \longrightarrow D(V)^{\psi=1}_{K} \overset{h_{K,V}}{\longrightarrow} H^1(K,V) \longrightarrow \left(\frac{D(V)}{\psi-1}\right)^{\Gamma_K} \longrightarrow 0.$$ 

We shall extend $h^1_{K,V}$ to a map $h^1_{K,V} : D^+_{\text{rig}}(V)^{\psi=1} \to H^1(K,V)$. We will first need a few facts about the ring of periods $\tilde{B}^+_{\text{rig}}$ and the modules $D^+_{\text{rig}}(V)^{\psi=1}$.

Lemma I.3. — If $\gamma \in \Gamma_K$, then $1 - \gamma : D^+_{\text{rig}}(V)^{\psi=1} \to D^+_{\text{rig}}(V)$ is an isomorphism.

Proof. — Recall that $B^{+}_{\text{rig},K}$ is the completion of $B^{+}_{K}$ for the Fréchet topology (see [2, 2.6]), so that $D^+_{\text{rig}}(V)$ is the completion of $D^+(V)$ for the Fréchet topology. The lemma then follows from the fact that by [5 II.6.1], if $\gamma \in \Gamma_K$, then $1 - \gamma : D^+_{\text{rig}}(V) \to D^+_{\text{rig}}(V)$ is a bi-continuous isomorphism for the Fréchet topology.

Lemma I.4. — There is an exact sequence:

$$0 \longrightarrow Q_p \longrightarrow \tilde{B}^+_{\text{rig}} \overset{1-\varphi}{\longrightarrow} \tilde{B}^+_{\text{rig}} \longrightarrow 0$$

Proof. — We’ll start with the easiest part, namely the fact that $(\tilde{B}^+_{\text{rig}})^{\varphi=1} = Q_p$. If $x \in (\tilde{B}^+_{\text{rig}})^{\varphi=1}$, then [2, prop 3.2] shows that actually $x \in (\tilde{B}^+_{\text{rig}})^{\varphi=1}$, and the latter space is well-known to be $Q_p$.

Let us now show that if $\alpha \in \tilde{B}^+_{\text{rig}}$, then there exists $\beta \in \tilde{B}^+_{\text{rig}}$ such that $(1 - \varphi)\beta = \alpha$. By [2 lemma 2.18], one can write $\alpha = \alpha^+ + \alpha^-$ with $\alpha^+ \in \tilde{B}^+_{\text{rig}}$ and $\alpha^- \in \tilde{B}^+$. It is therefore enough to show that $1 - \varphi : \tilde{B}^+_{\text{rig}} \to \tilde{B}^+_{\text{rig}}$ and $1 - \varphi : \tilde{B}^+ \to \tilde{B}^+$ are surjective.

The first assertion follows from the facts that $1 - \varphi : B^+_{\text{cris}} \to B^+_{\text{cris}}$ is surjective (see [9 thm 5.3.7, ii]) and that $\tilde{B}^+_{\text{rig}} = \cap_{n=0}^{\infty} \varphi^n(B^+_{\text{cris}})$.

The second assertion follows from the facts that $1 - \varphi : \tilde{B} \to \tilde{B}$ is surjective and that if $\beta \in \tilde{B}$ is such that $(1 - \varphi)\beta \in \tilde{B}^+$, then $\beta \in \tilde{B}^+$. 

$\square$
If $K$ and $n$ are such that $\Gamma_n$ is torsion-free, then we will construct maps $h_{K_n,V}^1$ such that \( \text{cor}_{K_{n+1}/K_n} \circ h_{K_{n+1},V}^1 = h_{K_n,V}^1 \). If $\Gamma_n$ is no longer torsion free, we’ll therefore define $h_{K_n,V}^1$ by the relation $h_{K_n,V}^1 = \text{cor}_{K_{n+1}/K_n} \circ h_{K_{n+1},V}^1$. In the following proposition, we therefore assume that $\Gamma_K$ is torsion free (and therefore procyclic), and we let $\gamma$ denote a topological generator of $\Gamma_K$. If $M$ is a $\Gamma_K$-module, it is customary to write $M_{\Gamma_K}$ for $M/\im(\gamma - 1)$.

**Proposition I.5.** — If $y \in D_{\text{rig}}^1(V)^{\psi=1}$, and $b \in \tilde{B}_{\text{rig}}^1 \otimes_{\mathbb{Q}_p} V$ is a solution of the equation $(\gamma - 1)(\varphi - 1)b = (\varphi - 1)y$, then the formula

$$h_{K,V}^1(y) = \log^0_p(\chi(\gamma)) \left[ \sigma \mapsto \frac{\sigma - 1}{\gamma - 1}y - (\sigma - 1)b \right]$$

defines a map $h_{K,V}^1 : D_{\text{rig}}^1(V)^{\psi=1}_{\Gamma_K} \to H^1(K,V)$ which does not depend on the choice of a generator $\gamma$ of $\Gamma_K$, and if $y \in D(V)^{\psi=1} \subset D_{\text{rig}}^1(V)^{\psi=1}$, then $h_{K,V}^1(y)$ coincides with the cocycle constructed in [B, I.5].

**Proof.** — Our construction closely follows [B I.5]; to simplify the proof, we can assume that $\log^0_p(\chi(\gamma)) = 1$. The fact that if we start from a different $\gamma$, then the two $h_{K,V}^1$ we get are the same is left as an easy exercise for the reader.

We will check that the formula makes sense and that everything is well-defined. If $y \in D_{\text{rig}}^1(V)^{\psi=1}$, then $(\varphi - 1)y \in D_{\text{rig}}^1(V)^{\psi=0}$. By lemma [I.3], there exists $x \in D_{\text{rig}}^1(V)^{\psi=0}$ such that $(\gamma - 1)x = (\varphi - 1)y$. By lemma [I.4], there exists $b \in \tilde{B}_{\text{rig}}^1 \otimes V$ such that $(\varphi - 1)b = x$. We then define $h_{K,V}^1(y) \in H^1(K,V)$ by the formula:

$$h_{K,V}^1(y)(\sigma) = \frac{\sigma - 1}{\gamma - 1}y - (\sigma - 1)b.$$ 

Notice that, a priori, $h_{K,V}^1(y) \in H^1(K,\tilde{B}_{\text{rig}}^1 \otimes_{\mathbb{Q}_p} V)$, but

$$(\varphi - 1)h_{K,V}^1(y)(\sigma) = \frac{\sigma - 1}{\gamma - 1}(\varphi - 1)y - (\sigma - 1)(\varphi - 1)b = \frac{\sigma - 1}{\gamma - 1}(\gamma - 1)x - (\sigma - 1)x = 0,$$

so that $h_{K,V}^1(y)(\sigma) \in (\tilde{B}_{\text{rig}}^1)^{\psi=1} \otimes_{\mathbb{Q}_p} V = V$. In addition, two different choices of $b$ differ by an element of $(\tilde{B}_{\text{rig}}^1)^{\psi=1} \otimes_{\mathbb{Q}_p} V = V$, and therefore give rise to two cohomologous cocycles.

It is clear that if $y \in D(V)^{\psi=1} \subset D_{\text{rig}}^1(V)^{\psi=1}$, then $h_{K,V}^1(y)$ coincides with the cocycle constructed in [B I.5], and that if $y \in (\gamma - 1)D_{\text{rig}}^1(V)$, then $h_{K,V}^1(y) = 0$. \(\square\)
II. Explicit formulas for exponential maps

The goal of this chapter is to give explicit formulas for Bloch-Kato’s maps for a $p$-adic representation $V$, in terms of the $(\varphi, \Gamma)$-module $D(V)$ attached to $V$. Throughout, $V$ will be assumed to be crystalline.

Recall that (cf [3, III.2] or [2, 2.4] for example) we have maps $\varphi^{-n} : B_{\rig,F}^+ \to F_n[[t]]$ which are characterized by the fact that $\pi$ maps to $\varepsilon^{(m)} \varepsilon^{t/p^n} - 1$. If $z \in F_n((t)) \otimes_F D_{\cris}(V)$, then the constant coefficient (i.e. the coefficient of $t^0$) of $z$ will be denoted by $\partial_V(z) \in D_{\cris}(V)$. This notation should not be confused with that for the derivation map $\partial$.

We will make frequent use of the following fact:

**Lemma II.1.** — If $y \in B_{\rig,F}^+[1/t] \otimes_F D_{\cris}(V)$, then for any $m \geq n \geq 0$, the element $p^{-m} \Tr_{F_m/F_n} \partial_V(\varphi^{-m}(y)) \in F_n \otimes_F D_{\cris}(V)$ does not depend on $m$ and we have:

$$p^{-m} \Tr_{F_m/F_n} \partial_V(\varphi^{-m}(y)) = \begin{cases} p^{-n} \partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}) \partial_V(y) & \text{if } n \geq 0. \end{cases}$$

**Proof.** — Recall that $\varphi^{-m}(y) = (1 \otimes \varphi^{-m})y(\varepsilon^{(m)} \varepsilon^{t/p^n} - 1)$, and that $\psi(y) = y$ means that

$$(1 \otimes \varphi)y((1 + T)^p - 1) = \frac{1}{p} \sum_{\eta \equiv 1} y(\eta(1 + T) - 1).$$

The lemma then follows from the fact that if $m \geq 2$, then the conjugates of $\varepsilon^{(m)}$ under $\Gal(F_m/F_{m-1})$ are the $\eta \varepsilon^{(m)}$, where $\eta^p = 1$, while if $m = 1$, then the conjugates of $\varepsilon^{(1)}$ under $\Gal(F_1/F)$ are the $\eta$, where $\eta^p = 1$ but $\eta \neq 1$.

We will henceforth assume that $\log_p(\chi(\gamma_n)) = p^n$, so that $\log_p^0(\chi(\gamma_n)) = 1$, and in addition $\nabla_0/(\gamma_n - 1)$ acts on $F_n$ by $p^{-n}$.

II.1. Bloch-Kato’s exponential map

The goal of this paragraph is to show how to compute Bloch-Kato’s map in terms of the $(\varphi, \Gamma)$-module of $V$. Let $h \geq 1$ be an integer such that $\ Fil^{-h} D_{\cris}(V) = D_{\cris}(V)$. For $i \in \mathbb{Z}$, let $\nabla_i$ be the operator acting on $F_n((t))$ and $B_{\rig,F}^+$ defined in [3]. If $y \in B_{\rig,F}^+ \otimes_F D_{\cris}(V)$, then the fact that $\ Fil^{-h} D_{\cris}(V) = D_{\cris}(V)$ implies (cf [3] for example) that $t^h y \in D_{\cris}^+(V)$, so that if $y = \sum_{i=0}^d y_i \otimes d_i \in (B_{\rig,F}^+ \otimes_F D_{\cris}(V))^{\psi=1}$, then

$$\nabla_{h-1} \circ \cdots \circ \nabla_0(y) = \sum_{i=0}^d t^h \partial^h y_i \otimes d_i \in D_{\rig}^+(V)^{\psi=1}.$$  

One can then apply the operator $h_{F_n,V}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$, and the main result of this paragraph is:
Theorem II.2. — If \( y \in (B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V))^\psi=1 \), then
\[
h^{1}_{F_{n,V}}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} 
\exp_{F_{n,V}}(p^{-n}\partial_V(\varphi^{-n}(y))) & \text{if } n \geq 1 \\
\exp_{F,V}((1-p^{-1})\partial_V(y)) & \text{if } n = 0.
\end{cases}
\]

Proof. — Because the diagram
\[
\begin{array}{ccc}
F_{n+1} \otimes_F D_{\text{cris}}(V) & \overset{\exp_{F_{n+1},V}}{\longrightarrow} & H^1(F_{n+1},V) \\
\downarrow \text{Tr}_{F_{n+1}/F_n} & & \downarrow \text{cor}_{F_{n+1}/F_n} \\
F_n \otimes_F D_{\text{cris}}(V) & \overset{\exp_{F_{n},V}}{\longrightarrow} & H^1(F_n,V)
\end{array}
\]
is commutative, it is enough to prove the theorem under the further assumption that \( \Gamma_n \) is torsion free. Let us then set \( y_h = \nabla_{h-1} \circ \cdots \circ \nabla_0(y) \). The cocycle \( h^{1}_{F_{n,V}}(y_h) \) is defined by
\[
h^{1}_{F_{n,V}}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1)b_{n,h}
\]
where \( b_{n,h} \) is a solution of the equation \((\gamma_n - 1)(\varphi - 1)b_{n,h} = (\varphi - 1)y_h\). In lemma [13] below, we will prove that:
\[
\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1}(B_{\text{rig},F}^+)^\psi=0 \subset \left( \frac{t}{\varphi^n(\pi)} \right)^i (B_{\text{rig},F}^+)^\psi=0.
\]
It is then clear that if one sets
\[
z_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1}(\varphi - 1)y,
\]
then \( z_{n,h} \in (t/\varphi^n(\pi))^h(B_{\text{rig},F}^+)^\psi=0 \otimes_F D_{\text{cris}}(V) \subset \varphi^n(\pi^{-h})D_{\text{rig}}^+(V)^\psi=0 \subset D_{\text{rig}}^1(V)^\psi=0 \).

Let \( q_h = \varphi^{-1}(q) = \varphi^h(\pi)/\varphi^{-1}(\pi) \); by lemma [14] (which will be stated and proved below), there exists an element \( b_{n,h} \in \varphi^{-1}(\pi^{-h})B_{\text{rig}}^+ \otimes_{\mathbb{Q}_p} V \) such that \((\varphi - q^n_{h})(\varphi^{-1}(\pi^{-h})b_{n,h}) = \varphi^n(\pi^{-h})z_{n,h} \), so that \((1 - \varphi)b_{n,h} = z_{n,h} \) with \( b_{n,h} \in \varphi^{-1}(\pi^{-h})B_{\text{rig}}^+ \otimes_{\mathbb{Q}_p} V \).

If we set
\[
w_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1}y,
\]
then \( w_{n,h} \) and \( b_{n,h} \in B_{\text{cris}} \otimes_{\mathbb{Q}_p} V \) and the cocycle \( h^{1}_{F_{n,V}}(y_h) \) is then given by the formula
\[
h^{1}_{F_{n,V}}(y_h)(\sigma) = (\sigma - 1)(w_{n,h} - b_{n,h}).
\]
Now, \((\varphi - 1)b_{n,h} = z_{n,h} \) and \((\varphi - 1)w_{n,h} = z_{n,h} \) as well, so that \( w_{n,h} - b_{n,h} \in B_{\text{cris}}^\psi=1 \otimes_{\mathbb{Q}_p} V \).

We can also write \( h^{1}_{F_{n,V}}(y_h)(\sigma) = (\sigma - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})) \). Since we know that \( b_{n,h} \in \varphi^{-1}(\pi^{-h})B_{\text{cris}}^+ \otimes_{\mathbb{Q}_p} V \), we have \( \varphi^{-n}(b_{n,h}) \in B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V \). By definition of the Bloch-Kato exponential, the theorem will follow from the fact that:
\[
\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\partial_V(\varphi^{-n}(y)) \in B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V.
\]
In order to show this, first notice that \( \varphi^{-n}(y) - \partial_V(\varphi^{-n}(y)) \in tF_n[[t]] \otimes_F D_{\text{cris}}(V) \). We can therefore write \( \sum_{n=1}^{\infty} \varphi^{-n}(y) = p^{-n} \partial_V(\varphi^{-n}(y)) + tz_1 \) and a simple recurrence shows that
\[
\nabla_{i-1} \circ \cdots \circ \nabla_0 \varphi^{-n}(y) = (-1)^{i-1}(i-1)!p^{-n} \partial_V(\varphi^{-n}(y)) + t^iz_i,
\]
with \( z_i \in F_n[[t]] \otimes_F D_{\text{cris}}(V) \). By taking \( i = h \), we see that \( \varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n} \partial_V(\varphi^{-n}(y)) \in B_{d\mathrm{r}}^+ \otimes_{Q_{p}} V \), since we chose \( h \) such that \( t^hD_{\text{cris}}(V) \subset B_{d\mathrm{r}}^+ \otimes_{Q_{p}} V \).

We will now prove the following two technical lemmas which were used above:

**Lemma II.3.** — If \( n \geq 1 \), then \( \nabla_0/(\gamma_n-1)(B_{\text{rig},F})^{\psi=0} \subset (t/\varphi^n(\pi))(B_{\text{rig},F})^{\psi=0} \) so that if \( i \geq 1 \), then:
\[
\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1} (B_{\text{rig},F})^{\psi=0} \subset \left( \frac{t}{\varphi^n(\pi)} \right)^i (B_{\text{rig},F})^{\psi=0}.
\]

**Proof.** — Since \( \nabla_i = t \cdot d/dt - i \), the second claim follows easily from the first one. By the standard properties of \( p \)-adic holomorphic functions, what we need to do is to show that if \( x \in (B_{\text{rig},F})^{\psi=0} \), then \( (\nabla_0/(\gamma_n-1)x)(\varepsilon^{(m)} - 1) = 0 \) for all \( m \geq n + 1 \).

On the one hand, up to a scalar factor, one has for \( m \geq n + 1 \): \( (\nabla_0/(\gamma_n-1)x)(\varepsilon^{(m)} - 1) = \text{Tr}_{F_{m}/F_n} x(\varepsilon^{(m)} - 1) \) as can be seen from the fact that \( \nabla_0/(\gamma_n-1) \lim_{\gamma \to 1} (\varepsilon^{(m)} - 1) \) \( \log_p^{-1}(\chi(\eta)) \). On the other hand, the fact that \( \psi(x) = 0 \) implies that for every \( m \geq 2 \), \( \text{Tr}_{F_{m}/F_{m-1}} x(\varepsilon^{(m)} - 1) = 0 \). This completes the proof.

**Lemma II.4.** — If \( \alpha \in \tilde{B}_{\text{rig}}^+ \), then there exists \( \beta \in \tilde{B}_{\text{rig}}^+ \) such that \( (\varphi - q_{n_0}^h)\beta = \alpha \).

**Proof.** — By [2, prop 2.19] applied to the case \( r = 0 \), the ring \( \tilde{B}_+ \) is dense in \( \tilde{B}_{\text{rig}}^+ \) for the Fréchet topology. Hence, if \( \alpha \in \tilde{B}_{\text{rig}}^+ \), then there exists \( \alpha_0 \in \tilde{B}_+ \) such that \( \alpha - \alpha_0 = \varphi^n(\pi^h)\alpha_1 \) with \( \alpha_1 \in \tilde{B}_{\text{rig}}^+ \). The map \( \varphi - q_{n_0}^h : \tilde{A}_+ \to \tilde{A}_+ \) is surjective, because \( \varphi - q_{n_0}^h : \tilde{A}_+ \to \tilde{A}_+ \) is surjective, as can be seen by reducing modulo \( p \). One can therefore write \( \alpha_0 = (\varphi - q_{n_0}^h)\beta_0 \). Finally (see the proof of lemma [2,3]), there exists \( \beta_1 \in \tilde{B}_{\text{rig}}^+ \) such that \( \alpha_1 = (\varphi - 1)\beta_1 \), so that \( \varphi^n(\pi^h)\alpha_1 = (\varphi - q_{n_0}^h)(\varphi^{n-1}(\pi^h)\beta_1) \).

**II.2. Bloch-Kato’s dual exponential map**

In the previous paragraph, we showed how to compute Bloch-Kato’s exponential map for \( V \). We shall now do the same for the dual exponential map. The starting point is Kato’s formula, which we recall below:
**Proposition II.5.** — If $V$ is a de Rham representation, then the map from $D_{\text{dr}}(V)$ to $H^1(F, B_{\text{dr}} \otimes_{Q_p} V)$ defined by $x \mapsto [g \mapsto \log(\chi(\overline{g}))x]$ is an isomorphism, and the dual exponential map $\exp^{*}_{V^*} : H^1(F, V) \to D_{\text{dr}}(V)$ is equal to the composition of the map $H^1(F, V) \to H^1(F, B_{\text{dr}} \otimes_{Q_p} V)$ with the inverse of this isomorphism.

Let us point out that the image of $\exp^{*}_{V^*}$ is included in $\text{Fil}^0 D_{\text{dr}}(V)$ and that its kernel is $H^1_g(F, V)$. Let us return to a crystalline representation $V$ of $G_F$. We then have the following formula, which is to be found in [3, IV.2.1]:

**Theorem II.6.** — If $y \in D_{\text{rig}}^+(V)^{\psi=1}$, then

$$\exp^{*}_{F_n, V^*}(h_{F_n, V}(y)) = \begin{cases} p^{-n} \partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1 - p^{-1} \varphi^{-1}) \partial_V(y) & \text{if } n = 0. \end{cases}$$

**Proof.** — Since the following diagram

$$
\begin{array}{ccc}
H^1(F_{n+1}, V) & \xrightarrow{\exp^{*}_{F_{n+1}, V^*}} & F_{n+1} \otimes_F D_{\text{cris}}(V) \\
\downarrow \text{cok}_{F_{n+1}/F_n} & & \downarrow \text{Tr}_{F_{n+1}/F_n} \\
H^1(F_n, V) & \xrightarrow{\exp^{*}_{F_n, V^*}} & F_n \otimes_F D_{\text{cris}}(V)
\end{array}
$$

is commutative, we only need to prove the theorem when $\Gamma_n$ is torsion free. We then have

$$h_{F_n, V}(y)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y - (\sigma - 1)b,$$

where $(\gamma_n - 1)(\varphi - 1)b = (\varphi - 1)y$. Since $b \in B_{\text{rig}}^+$, there exists $m \gg 0$ such that $b \in B_{\text{rig}}^m \otimes_{Q_p} V$. Recall that (cf [2, 2.4]) the map $\varphi^{-m}$ embeds $B_{\text{rig}}^m$ into $B_{\text{dr}}$. We can then write

$$h^1(y)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} \varphi^{-m}(y) - (\sigma - 1) \varphi^{-m}(b),$$

and $\varphi^{-m}(b) \in B_{\text{dr}} \otimes_{Q_p} V$. In addition, $\varphi^{-m}(y) \in F_m((t)) \otimes_F D_{\text{cris}}(V)$ and $\gamma_n - 1$ is invertible on $t^k F_m \otimes_F D_{\text{cris}}(V)$ for every $k \neq 0$. This shows that the cocycle $h^1_{F_n, V}(y)$ is cohomologous in $H^1(F_n, B_{\text{dr}} \otimes_{Q_p} V)$ to

$$\sigma \mapsto \frac{\sigma - 1}{\gamma_n - 1}(\partial_V \varphi^{-m}(y))$$

which is itself cohomologous (since $\gamma_n - 1$ is invertible on $F_{m^{\text{Tr}_{F_m/F_n}=0}}$) to

$$\sigma \mapsto \frac{\sigma - 1}{\gamma_n - 1} \left( p^{n-1} \text{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) \right) = \sigma \mapsto p^{-n} \log(\chi(\overline{\sigma})) p^{n-1} \text{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)).$$

It follows from this and Kato’s formula (proposition II.3) that

$$\exp^{*}_{F_n, V^*}(h^1_{F_n, V}(y)) = p^{-m} \text{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) = \begin{cases} p^{-n} \partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1 - p^{-1} \varphi^{-1}) \partial_V(y) & \text{if } n = 0. \end{cases}$$
II.3. Iwasawa theory of $p$-adic representations

In this specific paragraph, $V$ can be taken to be an arbitrary representation of $G_K$. Recall that the Iwasawa cohomology groups $H^i_{Iw}(K, V)$ are defined by $H^i_{Iw}(K, V) = \mathbb{Q}_p \otimes \mathbb{Z}_p H^i_{Iw}(K, T)$ where $T$ is any $G_K$-stable lattice of $V$, and

$$H^i_{Iw}(K, T) = \lim_{\text{cor}} H^i(K_n, T)$$

Each of the $H^i(K_n, T)$ is a $\mathbb{Z}_p[\Gamma/\Gamma_n]$-module, and $H^i_{Iw}(K_n, T)$ is then endowed with the structure of a $\Lambda$-module where $\Lambda = \mathbb{Z}_p[\Delta] \otimes \mathbb{Z}_p[\Gamma_1]$. The $H^i_{Iw}(K, V)$ have been studied in detail by Perrin-Riou, who proved the following:

**Proposition II.7.** — If $V$ is a $p$-adic representation of $G_K$, then $H^i_{Iw}(K, V) = 0$ whenever $i \neq 1, 2$. In addition:

1. the torsion sub-module of $H^1_{Iw}(K, V)$ is a $\mathbb{Q}_p \otimes \mathbb{Z}_p \Lambda$-module isomorphic to $V^{H_K}$, and then $H^1_{Iw}(K, V)/V^{H_K}$ is a free $\mathbb{Q}_p \otimes \mathbb{Z}_p \Lambda$-module of rank $[K : \mathbb{Q}_p]d$;
2. $H^2_{Iw}(K, V) = V(-1)^{H_K}$.

If $y \in \mathcal{D}(T)^{\psi=1}$, then the sequence of $\{h_{F_n,V}(y)\}_n$ is compatible for the corestriction maps, and therefore defines an element of $H^1_{Iw}(K, T)$. The following theorem is due to Fontaine and is proved in [3, II.1]:

**Theorem II.8.** — The map $y \mapsto \lim_{\text{cor}} h_{F_n,V}(y)$ defines an isomorphism from $\mathcal{D}(T)^{\psi=1}$ to $H^1_{Iw}(K, T)$ and from $\mathcal{D}(V)^{\psi=1}$ to $H^1_{Iw}(K, V)$.

Notice that $V^{H_K} \subset \mathcal{D}(V)^{\psi=1}$, and it its $\mathbb{Q}_p \otimes \mathbb{Z}_p \Lambda$-torsion submodule. In addition, it is shown in [3, II.3] that the modules $\mathcal{D}(V)/(\psi - 1)$ and $H^2_{Iw}(K, V)$ are naturally isomorphic. One can nicely summarize the results of this paragraph as follows:

**Corollary II.9.** — The complex of $\mathbb{Q}_p \otimes \mathbb{Z}_p \Lambda$-modules

$$0 \longrightarrow \mathcal{D}(V) \xrightarrow{1-\psi} \mathcal{D}(V) \longrightarrow 0$$

computes the Iwasawa cohomology of $V$. 

\[ \square \]
II.4. Perrin-Riou’s exponential map

By using the results of the previous paragraphs, we can give a “uniform” formula for the image of \( y \in (\mathbf{B}_{rig,F}^+ \otimes_F D_{cris}(V))^{\psi=1} \) in \( H^1(F_n, V(j)) \) under the composition of the following maps:

\[
\begin{array}{c}
(\mathbf{B}_{rig,F}^+ \otimes_F D_{cris}(V))^{\psi=1} \\
\xrightarrow{\nabla_{h-1} \circ \cdots \circ \nabla_0} \\
\xrightarrow{\otimes e_j} \\
D_{rig}^+(V)^{\psi=1} \\
\end{array}
\xrightarrow{\nabla_{h+j-1} \circ \cdots \circ \nabla_0}
\begin{array}{c}
D_{rig}^+(V(j))^{\psi=1} \\
\xrightarrow{h_{F_n,V(j)}^1} \\
H^1(F_n, V(j)).
\end{array}
\]

**Theorem II.10.** — If \( y \in (\mathbf{B}_{rig,F}^+ \otimes_F D_{cris}(V))^{\psi=1} \), and \( h \geq 1 \) is such that \( \text{Fil}^{-h} D_{cris}(V) = D_{cris}(V) \), then for all \( j \) with \( h + j \geq 1 \), we have:

\[
h_{F_n,V(j)}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j) = (-1)^{h+j-1}(h + j - 1)! \begin{cases} 
\exp_{F_n,V(j)}(p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j))) & \text{if } n \geq 1 \\
\exp_{F,V(j)}((1 - p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y \otimes t^{-j}e_j)) & \text{if } n = 0,
\end{cases}
\]

while if \( h + j \leq 0 \), then we have:

\[
\exp_{F_n,V^{(1-j)}}^*(h_{F_n,V(j)}^{1})(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j) = (-h - j)!^{-1} \begin{cases} 
p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j)) & \text{if } n \geq 1 \\
(1 - p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y \otimes t^{-j}e_j) & \text{if } n = 0,
\end{cases}
\]

**Proof.** — If \( h + j \geq 1 \), then the following diagram is commutative:

\[
\begin{array}{c}
D_{rig}^+(V)^{\psi=1} \\
\xrightarrow{\nabla_{h-1} \circ \cdots \circ \nabla_0} \\
\xrightarrow{\otimes e_j} \\
D_{rig}^+(V(j))^{\psi=1} \\
\end{array}
\xrightarrow{\nabla_{h+j-1} \circ \cdots \circ \nabla_0}
\begin{array}{c}
(\mathbf{B}_{rig,F}^+ \otimes_F D_{cris}(V))^{\psi=1} \\
\xrightarrow{\partial^{-j} \otimes t^{-j}e_j} \\
(\mathbf{B}_{rig,F}^+ \otimes_F D_{cris}(V(j)))^{\psi=1}.
\end{array}
\]

and the theorem is then a straightforward consequence of theorem II.2 applied to \( \partial^{-j}y \otimes t^{-j}e_j \), \( h + j \) and \( V(j) \) (which are the \( j \)-th twists of \( y \), \( h \) and \( V \)).

If on the other hand \( h + j \leq 0 \), and \( \Gamma_n \) is torsion free, then theorem II.6 shows that

\[
\exp_{F_n,V^{(1-j)}}^*(h_{F_n,V(j)}^{1})(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j) = p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j))
\]

in \( D_{cris}(V(j)) \), and a short computation involving Taylor series shows that

\[
p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j)) = (-h - j)!^{-1}p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j)).
\]

Finally, to get the case \( n = 0 \), one just needs to corestrict. \( \square \)
We shall now use the above result to construct Perrin-Riou’s exponential map. One has an exact sequence

$$0 \longrightarrow \bigoplus_{k=0}^{h} t^{k} D_{\mathrm{cris}}(V)^{\varphi = p^{-k}} \longrightarrow \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\psi = 1} \xrightarrow{1 - \varphi} \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\psi = 0} \xrightarrow{\Delta} \bigoplus_{k=0}^{h} \left( D_{\mathrm{cris}}(V) \right)^{\Delta = 0} \xrightarrow{\partial} 0,$$

where $\Delta(f) = \bigoplus_{k=0}^{h} \partial^{k}(f)(0)$. If $f \in \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0}$, then there exists $y \in \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\psi = 1}$ such that $f = (1 - \varphi)y$, and $\nabla_{h-1} \circ \cdots \circ \nabla_{0}(y)$ does not depend upon the choice of such a $y$ (unless $Q_{p}(h) \subset V$), and one deduces from this a well-defined map (if $Q_{p}(k) \subset V$, we therefore require that $h \geq k + 1$):

$$\Omega_{V,h} : \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0} \rightarrow D_{\mathrm{rig}}^{+}(V)^{\psi = 1},$$

given by $\Omega_{V,h}(f) = \nabla_{h-1} \circ \cdots \circ \nabla_{0}(y)$.

**Theorem II.11.** — If $V$ is a crystalline representation and $h \geq 1$ is such that we have $\operatorname{Fil}^{-h} D_{\mathrm{cris}}(V) = D_{\mathrm{cris}}(V)$, then the map

$$\Omega_{V,h} : \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0} \rightarrow D_{\mathrm{rig}}^{+}(V)^{\psi = 1}/V^{H_{F}}$$

which takes $x \in \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_{0}((1 - \varphi)^{-1}(x))$ is well-defined and coincides with Perrin-Riou’s exponential map.

**Proof.** — The map $\Omega_{V,h}$ is well defined because the kernel of $1 - \varphi$ is killed by $\nabla_{h-1} \circ \cdots \circ \nabla_{0}$, except for $t^{h} \otimes (t^{-h}e_{h})$, which is mapped to $Q_{p}(h) \subset V^{H_{F}}$.

The fact that $\Omega_{V,h}$ coincides with Perrin-Riou’s exponential map follows directly from theorem II.10 above applied to those $j$’s for which $h + j \geq 1$, compared with [12, 3.2.3] (see remark II.15 however). □

**Remark II.12.** — By the above remarks, if $V$ is a crystalline representation and $h \geq 1$ is such that we have $\operatorname{Fil}^{-h} D_{\mathrm{cris}}(V) = D_{\mathrm{cris}}(V)$ and $Q_{p}(h) \not\subset V$, then the map

$$\Omega_{V,h} : \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0} \rightarrow D_{\mathrm{rig}}^{+}(V)^{\psi = 1}$$

which takes $x \in \left( B_{\mathrm{rig},F}^{+} \otimes_{F} D_{\mathrm{cris}}(V) \right)^{\Delta = 0}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_{0}((1 - \varphi)^{-1}(x))$ is well-defined, without having to kill the $\Lambda$-torsion of $H_{Iw}^{1}(F,V)$.

**Remark II.13.** — It is clear from theorem II.10 that we have:

$$\Omega_{V,h}(x) \otimes e_{j} = \Omega_{V(j),h+j}(\partial^{-j}x \otimes t^{-j}e_{j}) \quad \text{and} \quad \Omega_{V,h}(\nabla_{h}(x)) = \Omega_{V,h+1}(x),$$

and following Perrin-Riou, one can use these formulas to extend the definition of $\Omega_{V,h}$ to all $h \in \mathbb{Z}$. 
II.5. The explicit reciprocity formula

In this paragraph, we shall recall Perrin-Riou’s explicit reciprocity formula. There is a map $\mathcal{H} (\Gamma_F) \rightarrow (\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0$ which sends $f(\gamma)$ to $f(\gamma)(1 + \pi)$. This map is a bijection and its inverse is the Mellin transform so that if $g(\pi) \in (\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0$, then $g(\pi) = \text{Mel}(g)(1 + \pi)$. See [14] B.2.8 for a reference, where Perrin-Riou has also extended Mel to $(\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0$. If $f, g \in \mathbf{B}^+_{\text{rig}, \mathbb{Q}_p}$ then we define $f \ast g$ by the formula $\text{Mel}(f \ast g) = \text{Mel}(f) \text{Mel}(g)$. Let $[1] \in \Gamma_F$ be the element such that $\chi([-1]) = -1$, and let $\iota$ be the involution of $\Gamma_F$ which sends $\gamma$ to $\gamma^{-1}$. The operator $\partial^j$ on $(\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0$ corresponds to $\text{Tw}_j$ on $\Gamma_F$. For instance, it is a bijection. We will make use of the facts that $\iota \circ \partial^j = \partial^{-j} \circ \iota$ and that $[1] \circ \partial^j = (-1)^j \partial^j \circ [1]$.

If $V$ is a crystalline representation, then the natural maps

$$
\mathbf{D}_{\text{cris}}(V) \otimes_F \mathbf{D}_{\text{cris}}(V^*(1)) \longrightarrow \mathbf{D}_{\text{cris}}(V \otimes \mathbb{Q}_p, V^*(1)) \longrightarrow \mathbf{D}_{\text{cris}}(\mathbb{Q}_p(1)) \xrightarrow{\text{Tr}_{F/\mathbb{Q}_p}} \mathbb{Q}_p
$$

allow us to define a perfect pairing $[\cdot, \cdot]_V : \mathbf{D}_{\text{cris}}(V) \times \mathbf{D}_{\text{cris}}(V^*(1))$ which we extend by linearity to

$$
[\cdot, \cdot]_V : (\mathbf{B}^+_{\text{rig}, F} \otimes_F \mathbf{D}_{\text{cris}}(V))^\psi = 0 \times (\mathbf{B}^+_{\text{rig}, F} \otimes_F \mathbf{D}_{\text{cris}}(V^*(1)))^\psi = 0 \to (\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0
$$

by the formula $[f(\pi) \otimes d_1, g(\pi) \otimes d_2]_V = (f \ast g)(\pi)[d_1, d_2]_V$.

We can also define a semi-linear (with respect to $\iota$) pairing

$$
\langle \cdot, \cdot \rangle_V : \mathbf{D}^+_{\text{rig}}(V)^\psi = 1 \times \mathbf{D}^+_{\text{rig}}(V^*(1))^\psi = 1 \to (\mathbf{B}^+_{\text{rig}, \mathbb{Q}_p})^\psi = 0
$$

by the formula

$$
\langle y_1, y_2 \rangle_V = \lim_{n \to \infty} \sum_{\tau \in \Gamma_F / T_n} \langle \tau^{-1}(h^{1}_{F_n, V}(y_1)), h^{1}_{F_n, V^*(1)}(y_2) \rangle_{F_n, V} \cdot \tau(1 + \pi)
$$

where the pairing $\langle \cdot, \cdot \rangle_{F_n, V}$ is given by the cup product:

$$
\langle \cdot, \cdot \rangle_{F_n, V} : H^1(F_n, V) \times H^1(F_n, V^*(1)) \to H^2(F_n, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p.
$$

The pairing $\langle \cdot, \cdot \rangle_V$ satisfies the relation $\langle \gamma_1 x_1, \gamma_2 x_2 \rangle_V = \gamma_1 \iota(\gamma_2) \langle x_1, x_2 \rangle_V$. Perrin-Riou’s explicit reciprocity formula (proved by Colmez [7], Benois [4] and Kato-Kurihara-Tsuji (unpublished)) is then:

**Theorem II.14.** — If $x_1 \in (\mathbf{B}^+_{\text{rig}, F} \otimes_F \mathbf{D}_{\text{cris}}(V))^\psi = 0$ and $x_2 \in (\mathbf{B}^+_{\text{rig}, F} \otimes_F \mathbf{D}_{\text{cris}}(V^*(1)))^\psi = 0$, then for every $h$, we have:

$$
(-1)^h \langle \Omega_{V, h}(x_1), [-1] \cdot \Omega_{V^*(1), 1-h}(x_2) \rangle_V = -[x_1, \iota(x_2)]_V.
$$
Proof. — By the theory of $p$-adic interpolation, it is enough to prove that if $x_i = (1 - \varphi)y_i$ with $y_1 \in (\mathcal{B}_{\text{rig}, F}^+ \otimes_F \mathcal{D}_{\text{crist}}(V))^{\psi = 1}$ and $y_2 \in (\mathcal{B}_{\text{rig}, F}^+ \otimes_F \mathcal{D}_{\text{crist}}(V^*(1)))^{\psi = 1}$ then for all $j \gg 0$:

$$(\partial^{-j}(-1)^{h}\langle \Omega_{V,h}(x_1), [-1] \cdot \Omega_{V^*(1),1-h}(x_2) \rangle_V) \,(0) = - (\partial^{-j} [x_1, \iota(x_2)]_V) \,(0).$$

The above formula is equivalent to (a):

$$(-1)^{h}h_{F,V(j)}^{1} \Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j), (-1)^{j}h_{F,V^*(1-j)}^{1} \Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_j))_{F,V(j)}$$

$$= - [\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j), \partial_{V^*(1-j)}(\partial^{j}x_2 \otimes t^{j}e_j)]_{V(j)}.$$ 

By combining theorems [1.10] and [1.11] with remark [1.13] we see that for $j \gg 0$:

$$h_{F,V(j)}^{1} \Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j)$$

$$= (-1)^{h+j-1} \exp_{F,V(j)}((h + j - 1)! (1 - p^{-1} \varphi^{-1}) \partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j)),$$

and that

$$h_{F,V^*(1-j)}^{1} \Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_j)$$

$$= ([\exp_{F,V^*(1-j)}^*]^{-1}((1 - p^{-1} \varphi^{-1}) \partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_j))$$

Using the fact that by definition, if $x \in \mathcal{D}_{\text{crist}}(V(j))$ and $y \in H^1(F, V(j))$ then

$$[x, \exp_{F,V^*(1-j)}^* y]_{V(j)} = (\exp_{F,V(j)} x, y)_{F,V(j)},$$

we see that (b):

$$h_{F,V(j)}^{1} \Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j), h_{F,V^*(1-j)}^{1} \Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_j))_{F,V(j)}$$

$$= (-1)^{h+j-1}((1 - p^{-1} \varphi^{-1}) \partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j), (1 - p^{-1} \varphi^{-1}) \partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_j))_{V(j)}.$$ 

It is easy to see that under $[\cdot, \cdot]$, the adjoint of $(1 - p^{-1} \varphi^{-1})$ is $1 - \varphi$, and that if $x_i = (1 - \varphi)y_i$, then

$$\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j) = (1 - \varphi)\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j),$$

$$\partial_{V^*(1-j)}(\partial^{j}x_2 \otimes t^{j}e_j) = (1 - \varphi)\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_j),$$

so that (b) implies (a), and this proves the theorem. \hfill \square

Remark II.15. — One should be careful with all the signs involved in those formulas. Perrin-Riou has changed the definition of the $\ell_i$ operators from [12] to [13] (the new $\ell_i$ is minus the old $\ell_i$). The reciprocity formula which is stated in [13], 4.2.3] does not seem (to me) to have the correct sign. On the other hand, the formulas of [1, 7] do seem to give the correct signs, but one should be careful that [7] IX.4.5] uses a different definition for one of
the pairings, and that the signs in [3, IV.3.1] and [4, VII.1.1] disagree. Our definitions of \( \Omega_{V,h} \) and of the pairing agree with Perrin-Riou’s ones (as they are given in [13]).

Appendix A

The structure of \( D(T)^\psi=1 \)

The goal of this paragraph is to prove a theorem which says that for a crystalline representation \( V \), \( D(V)^\psi=1 \) is quite “small”. See theorem A.3 for a precise statement.

Let \( V \) be a crystalline representation of \( G_F \) and let \( T \) denote a \( G_F \)-stable lattice of \( V \). The following proposition, which improves slightly upon the results of N. Wach [15], is proved in detail in [3]:

**Proposition A.1.** — If \( T \) is a lattice in a positive crystalline representation \( V \), then there exists a unique sub \( A_F^+ \)-module \( N(T) \) of \( D^+(T) \), which satisfies the following conditions:

1. \( N(T) \) is free of rank \( d \);
2. \( N(T) \) is stable under the action of \( \Gamma_F \);
3. the action of \( \Gamma_F \) is trivial on \( N(T)/\pi N(T) \);
4. there exists an integer \( s \geq 0 \) such that \( \pi^s D^+(T) \subset N(T) \).

In this case, \( N(T) \) is stable by \( \varphi \), and the sub \( B_F^+ \)-module \( N(V) = N(T) \otimes_{A_F^+} B_F^+ \) of \( D^+(V) \) satisfies the corresponding conditions.

Notice that \( N(T(-1)) = \pi N(T) \otimes e_{-1} \). When \( V \) is no longer positive, we therefore define \( N(T) \) as \( \pi^{-h} N(T(-h)) \otimes e_h \), for \( h \) large enough so that \( V(-h) \) is positive.

**Proposition A.2.** — If \( T \) is a lattice in a crystalline representation \( V \) of \( G_F \), whose Hodge-Tate weights are in \([a;b] \), then \( N(T) \) is the unique sub-\( A_F^+ \)-module of \( D^+(T)[1/\pi] \) which is free of rank \( d \), stable by \( \Gamma_F \) with the action of \( \Gamma_F \) trivial on \( N(T)/\pi N(T) \), and such that \( N(T)[1/\pi] = D^+(T)[1/\pi] \).

In addition, we have \( \varphi(\pi^b N(T)) \subset \pi^b N(T) \) and \( \pi^b N(T)/\varphi^*(\pi^b N(T)) \) is killed by \( q^{b-a} \).

The functor \( T \mapsto N(T) \) is an equivalence of categories between the category of lattices in crystalline representations and the category of modules satisfying all the above conditions.

We shall now show that \( D(V)^\psi=1 \) is not very far from being included in \( N(V) \). Indeed:

**Theorem A.3.** — If \( V \) is a crystalline representation of \( G_F \), whose Hodge-Tate weights are in \([a;b] \), then \( D(V)^\psi=1 \subset \pi^{a-1} N(V) \).

If in addition \( V \) has no quotient isomorphic to \( Q_p(a) \), then \( D(V)^\psi=1 \subset \pi^a N(V) \).
Before we prove the above statement, we will need a few results concerning the action of $\psi$ on $D(T)$. In lemmas A.4 through A.7, we will assume that the Hodge-Tate weights of $V$ are $\geq 0$. In particular, $N(T) \subset \varphi^*N(T)$ so that $\psi(N(T)) \subset N(T)$.

**Lemma A.4.** — If $m \geq 1$, then there exists a polynomial $Q_m(X) \in \mathbb{Z}_p[X]$ such that $\psi(\pi^{-m}) = \pi^{-m}(p^{m-1} + \pi Q_m(\pi))$.

**Proof.** — By the definition of $\psi$, it is enough to show that if $m \geq 1$, there exists a polynomial $Q_m(X) \in \mathbb{Z}[X]$ such that

$$\frac{1}{p} \sum_{\eta \in (\mathbb{Z}/p\mathbb{Z})^*} \frac{1}{(1 + X) - 1} = \frac{p^{m-1} + ((1 + X)^p - 1)Q_m((1 + X)^p - 1)}{(1 + X)^p - 1}.$$  

which is left to the reader.

**Lemma A.5.** — If $k \geq 1$, then $\psi(pD(T) + \pi^{-(k+1)}N(T)) \subset pD(T) + \pi^{-k}N(T)$. In addition, $\psi(pD(T) + \pi^{-1}N(T)) \subset pD(T) + \pi^{-1}N(T)$.

**Proof.** — If $x \in N(T)$, then one can write $x = \sum \lambda_i \varphi(x_i)$ with $\lambda_i \in A_F^+$ and $x_i \in N(T)$, so that $\psi(\pi^{-(k+1)}x) = \sum \psi(\pi^{-(k+1)}\lambda_i)x_i$. By the preceding lemma, $\psi(\pi^{-(k+1)}\lambda_i) \in pA_F + \pi^{-k}A_F^+$ whenever $k \geq 1$. The lemma follows easily, and the second claim is proved in the same way.

**Lemma A.6.** — If $k \geq 1$ and $x \in D(T)$ has the property that $\psi(x) - x \in pD(T) + \pi^{-k}N(T)$, then $x \in pD(T) + \pi^{-k}N(T)$.

**Proof.** — Let $\ell$ be the smallest integer $\geq 0$ such that $x \in pD(T) + \pi^{-\ell}N(T)$. If $\ell \leq k$, then we are done and otherwise lemma A.5 shows that $\psi(x) \in pD(T) + \pi^{-(\ell-1)}N(T)$, so that $\psi(x) - x$ would be in $pD(T) + \pi^{-\ell}N(T)$ but not $pD(T) + \pi^{-(\ell-1)}N(T)$, a contradiction if $\ell > k$.

**Lemma A.7.** — We have $D(T)^{\psi = 1} \subset \pi^{-1}N(T)$.

**Proof.** — We shall prove by induction that $D(T)^{\psi = 1} \subset p^kD(T) + \pi^{-1}N(T)$ for $k \geq 1$. Let us start with the case $k = 1$. If $x \in D(T)^{\psi = 1}$, then there exists some $j \geq 1$ such that $x \in pD(T) + \pi^{-j}N(T)$. If $j = 1$ we are done and otherwise the fact that $\psi(x) = x$ combined with lemma A.5 shows that $j$ can be decreased by 1. This proves our claim for $k = 1$.

We will now assume our claim to be true for $k$ and prove it for $k + 1$. If $x \in D(T)^{\psi = 1}$, we can therefore write $x = p^ky + n$ where $y \in D(T)$ and $n \in \pi^{-1}N(T)$. Since $\psi(x) = x$, we have $\psi(n) - n = p^k(\psi(y) - y)$ so that $\psi(y) - y \in \pi^{-1}N(T)$ (this is because $p^kD(T) \cap N(T) = D(T)^{\psi = 1}$).
\[ p^k N(T) \]. By lemma A.3, this implies that \( y \in pD(T) + \pi^{-1}N(T) \), so that we can write \( x = p^k(y' + n') + n = p^{k+1}y' + (p^kn' + n) \), and this proves our claim.

Finally, it is clear that our claim implies the lemma: if one can write \( x = p^ky_k + n_k \), then the \( n_k \) will converge for the \( p \)-adic topology to a \( n \in \pi^{-1}N \) such that \( x = n \). \hfill \Box

Proof of theorem A.3 — Clearly, it is enough to show that if \( T \) is a \( G_F \)-stable lattice of \( V \), then \( D(T)^{\psi=1} \subset \pi^{d-1}N(T) \). It is also clear that we can twist \( V \) as we wish, and we shall now assume that the Hodge-Tate weights of \( V \) are in \([0; h]\). In this case, the theorem says that \( D(T)^{\psi=1} \subset \pi^{-1}N(T) \), which is the content of lemma A.7 above.

Let us now prove that if a positive \( V \) has no quotient isomorphic to \( Q_p \), then actually \( D(T)^{\psi=1} \subset N(T) \). Recall that \( N(T) \subset \varphi^d(N(T)) \), since the Hodge-Tate weights of \( V \) are \( \geq 0 \), so that if \( e_1, \ldots, e_d \) is a basis of \( N(T) \), then there exists \( q_{ij} \in \mathbb{A}^+_F \) such that \( e_i = \sum_{j=1}^d q_{ij} \varphi(e_j) \). If \( \psi(\sum_{i=1}^d \alpha_i e_i) = \sum_{i=1}^d \alpha_i e_i \), with \( \alpha_i \in \pi^{-1}A^+_F \), then this translates into \( \psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j \) for \( 1 \leq j \leq d \).

Let \( \alpha_{i,n} \) be the coefficient of \( \pi^n \) in \( \alpha_i \), and likewise for \( q_{ij,n} \). Since \( \psi(1/\pi) = 1/\pi \), the equations \( \psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j \) then tell us that for \( 1 \leq j \leq d \):

\[
\sum_{i=1}^d \alpha_{i,-1} q_{ij,0} = \varphi(\alpha_{j,-1}).
\]

Since \( N(V)/\pi N(V) \simeq D_{\text{cris}}(V) \) as filtered \( \varphi \)-modules (cf the appendix to \[3\]), the above equations say that \( 1 \) is an eigenvalue of \( \varphi \) on \( D_{\text{cris}}(V) \). It is easy to see that if a representation has \( \geq 0 \) weights and \( D_{\text{cris}}(V)^{\psi=1} \neq 0 \), then \( V \) has a quotient isomorphic to \( Q_p \). \hfill \Box

Remark A.8 — It is proved in \[3\] that \( D_{\text{cris}}(V) = (B_{rig,F}^+ \otimes_F N(V))^{GF} \) and that if \( \text{Fil}^{-h} D_{\text{cris}}(V) = D_{\text{cris}}(V) \), then \( (t/\pi)^h B_{rig,F}^+ \otimes_F D_{\text{cris}}(V) \subset B_{rig,F}^+ \otimes_F N(V) \).

In all the above constructions, one could therefore replace \( D_{\text{rig}}^+(V) \) by \( B_{rig,F}^+ \otimes_F \pi^h N(V) \). For example, the image of \( \Omega_{V,h} \) is included in \( (B_{rig,F}^+ \otimes_F \pi^h N(V))^{\psi=1} \).

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