CURVES WITH LOW HARBOURNE CONSTANTS ON KUMMER AND ABELIAN SURFACES

XAVIER ROLLEAU

Abstract. We construct and study curves with low H-constants on abelian and K3 surfaces. Using the Kummer \((16_6)\)-configurations on Jacobian surfaces and some \((16_{10})\)-configurations of curves on \((1,3)\)-polarized Abelian surfaces, we obtain examples of configurations of curves of genus > 1 on a generic Jacobian K3 surface with H-constants < −4.

1. Introduction

The Bounded Negativity Conjecture predicts that for a smooth complex projective surface \(X\) there exists a bound \(b_X\) such that for any reduced curve \(C\) on \(X\) one has
\[ C^2 \geq b_X. \]
That conjecture holds in some cases, for instance if \(X\) is an abelian surface, but we do not know whether it remains true if one considers a blow-up of \(X\). With that question in mind, the H-constants have been introduced in [1].

For a reduced (but not necessarily irreducible) curve \(C\) on a surface \(X\) and \(\mathcal{P} \subset X\) a finite non empty set of points, let \(\pi : \tilde{X} \to X\) be the blowing-up of \(X\) at \(\mathcal{P}\) and let \(\tilde{C}\) denotes the strict transform of \(C\) on \(\tilde{X}\). Let us define the number
\[ H(C, \mathcal{P}) = \frac{\tilde{C}^2}{|\mathcal{P}|}, \]
where \(|\mathcal{P}|\) is the order of \(\mathcal{P}\). We define the Harbourne constant of \(C\) (for short the H-constant) by the formula
\[ H(C) = \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R}, \]
where \(\mathcal{P} \subset X\) varies among all finite non-empty subsets of \(X\) (note that there is a slight difference with the definition of Hadean constant of a curve given in [1, Remark 2.4], which definition exists only for singular curves; see Remark 4 for the details). Singular curves tend to have low H-constants. It is in general difficult to construct curves having low H-constants, especially if one requires the curve to be irreducible. The (global) Harbourne constant of the surface \(X\) is defined by
\[ H_X = \inf_C H(C) \in \mathbb{R} \cup \{-\infty\} \]

2000 Mathematics Subject Classification. Primary: 14J28.
where the infimum is taken among all reduced curves $C \subset X$. Harbourne constants and their variants are intensively studied (see e.g. [1], [11], [12], [14]); note that the finiteness of $H_X$ implies the BNC conjecture. Using some elliptic curve configurations in the plane [15], it is known that $H_{P^2} \leq -4$, and for any surface $X$ one has $H_X \leq H_{P^2} \leq -4$ (see [14]). However, the curves $(C_n)_{n \in \mathbb{N}}$ on $X \neq P^2$ with H-constant tending to $-4$ used to prove that $H_X \leq -4$ are not very explicit and they all satisfy $H(C_n) > -4$.

The H-constant is an invariant of the isogeny class of an abelian surface. Using the classical $(16_6)$ configuration $R_1$ of 16 genus 2 curves and 16 2-torsion points in a principally polarized abelian surface and a $(16_{10})$ configuration of 16 smooth genus 4 curves and 16 2-torsion points on a $(1,3)$-polarized abelian surface, plus the dynamic of the multiplication by $n \in \mathbb{Z}$ map, we construct explicitly some curves with low H-constants on abelian surfaces:

**Theorem 1.** Let $A$ be a simple abelian surface. There exists a sequence of curves $(R_n)_{n \in \mathbb{N}}$ in $A$ such that $R_n^2 \to \infty$ and $H(R_n) = -4$. If $A$ is the Jacobian of a smooth genus 2 curve, the curve $R_n$ can be chosen either as the union of 16 smooth curves or as an irreducible singular curve.

It is known that on two particular abelian surfaces with CM there exists a configuration $C$ of elliptic curves with $H(C) = -4$. Moreover for any elliptic curve configuration $C$ in an abelian surface $A$, one always has

$$H(C) \geq -4,$$

with equality if and only if the complement of the singularities of $C$ is an open ball quotient surface (for these previous results see [14]). Thus elliptic curve configurations with $H(C) = -4$ are rather special, in particular these configurations are rigid. Indeed to an algebraic family $(A_t, C_t)_t$ of such surfaces $A_t$, each containing a configuration $C_t$ of elliptic curves with H-constant equals to $-4$, such that $C_t$ varies algebraically with $A_t$, one can associate a family of ball quotient surfaces. Since ball quotient surfaces are rigid, the family $(A_t, C_t)_t$ is trivial and the pairs $(A_t, C_t)$ are isomorphic.

We observe that for our new examples of curves with $H(C) = -4$ there is no such links with ball quotient surfaces. Indeed the pairs $(A, C)$ we give such that $H(C) = -4$ have deformations.

We then consider the images of the curves $R_n$ in the associated Kummer surface $X$ and we obtain:

**Theorem 2.** Let $X$ be a Jacobian Kummer surface. For any $n > 1$, there are configurations $C_n$ of curves of genus $>1$ such that $H(C_n) = -4 \frac{a^4}{n^4} < -4$.

The H-constants of curves (and some related variants such as the $s$-tuple Harbourne constants) on K3 surfaces have been previously studied, by example in [8] and [12]. In [8], R. Laface and P. Pokora study transversal arrangements $C$ of rational curves on K3 surfaces and they give examples of
configurations $C$ with a low Harbourne constant. In their examples, one has $H(C) \geq -3.777$, with the exception of two examples on the Schur quartic and the Fermat quartic surfaces, both reaching

$$H(C) = -8.$$ 

In the last section, we then turn our attention to irreducible curves with low H-constants in abelian and Kummer surfaces, which are more difficult to obtain, some of which have been recently constructed in [16].

Acknowledgments. The author thanks T. Szemberg for sharing his observation that the H-constant of the Kummer configuration is $-4$ and P. Pokora for a very useful correspondence and his many criticisms. The author moreover thanks the referee for pointing out an error in a previous version of this paper and numerous comments which allowed to improve the exposition of this note.

2. Smooth hyperelliptic curves in abelian surfaces and H-constants

2.1. Preliminaries, Notations. By [3], an abelian surface $A$ contains a smooth hyperelliptic curve $C_0$ of genus $g$ if and only if it is a generic $(1, g-1)$-polarized abelian surface and $g \in \{2, 3, 4, 5\}$.

In this section, we study the configurations of curves obtained by translation of these hyperelliptic curves $C_0$ (of genus 2, 3, 4 or 5) by 2-torsion points and by taking pull-backs by endomorphisms of $A$. In the present sub-section, we recall some facts on the computation of the H-constants and some notations.

Let $C_1, \ldots, C_t$ be smooth curves in a smooth surface $X$ such that the singularities of $C = \sum_j C_j$ are ordinary (i.e. resolved after one blow-up). Let $\text{Sing}(C)$ be the singularity set of $C$; we suppose that it is non-empty. Let $f : \bar{X} \to X$ be the blow-up of $X$ at $\text{Sing}(C)$. For each $p$ in $\text{Sing}(C)$, let $m_p$ be the multiplicity of $C$ (we say that such a singularity $p$ is a $m_p$-point) and let $E_p$ be the exceptional divisor in $\bar{X}$ above $p$. Let us recall the following notation:

$$H(C, \mathcal{P}) = \frac{\bar{C}^2}{|\mathcal{P}|},$$

where $\bar{C}$ is the strict transform of a curve $C$ in the blowing-up surface at $\mathcal{P} \neq \emptyset$. The following formula is well known:

**Lemma 3.** Let $s$ be the cardinal of $\text{Sing}(C)$. One has

$$H(C, \text{Sing}(C)) = \frac{\bar{C}^2 - \sum_{p \in \mathcal{P}} m_p^2}{s} = \frac{\sum_{j=1}^{t} C_j^2 - \sum_{p \in \mathcal{P}} m_p}{s}.$$ 

**Proof.** One can compute $\bar{C}^2$ in two ways, indeed

$$\bar{C} = f^*C - \sum m_pE_p,$$
thus $\bar{C}^2 = C^2 - \sum_{p \in \mathcal{P}} m_p^2$ (that formula is valid for any configurations).

But $\bar{C} = \sum_{i=1}^t \bar{C}_i = \sum_{i=1}^t (f^* C_i - \sum_{p \in C_i} E_p)$, and since the singularities are ordinary, the curves $\bar{C}_i$ are disjoint, thus

$$\bar{C}^2 = \sum_{i=1}^t \bar{C}_i^2 = \sum_{i=1}^t C_i^2 - \sum_{p \in \text{Sing}(C)} m_p,$$

where we just use the fact that $\sum_{i=1}^t \sum_{p \in C_i} 1 = \sum_{p \in \text{Sing}(C)} m_p$. \qed

Recall that we define the H-constant of a curve $C$ by the formula

$$H(C) := \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R},$$

where $\mathcal{P} \subset X$ varies among all finite non-empty subsets of $X$.

**Remark 4.**

a) If $C$ is smooth one has $H(C) = \min(-1, C^2 - 1)$.

b) In [1, Remark 2.4] the Hadean constant of a singular curve $C$ on a surface $X$ is defined by the formula

$$H_{ad}(C) := \min_{\mathcal{P} \subset \text{Sing}(C), \mathcal{P} \neq \emptyset} H(C, \mathcal{P}).$$

Let $C$ be an arrangement of $n \geq 2$ smooth curves intersecting transversally (with at least one intersection point). In [7] is defined and studied the quantity $H(X, C) := H(C, \text{Sing}(C))$. An advantage of our definition of H-constant is that it is defined for any curves. Moreover with our definition, it is immediate that the global H-constant of the surface $X$ satisfies $H(X) = \inf H(C)$, where the infimum is taken over reduced curves $C$ in $X$.

Let $m \in \mathbb{N}^*$ and let $C \hookrightarrow X$ be a singular curve having singularities of multiplicity $m$ only (this will be the case for most of the curves in this paper).

Let $s$ be the order of $\text{Sing}(C)$.

**Lemma 5.** One has $H(C, \text{Sing}(C)) = \frac{C^2}{s} - m^2$.

The H-constant of $C$ is

$$H(C) = \min(-1, C^2 - m^2, H(C, \text{Sing}(C))).$$

**Proof.** For integers $0 \leq a \leq s$, $b \geq 0$, $c \geq 0$ such that $a + b + c > 0$, let $\mathcal{P}_{a,b,c}$ be a set of $a$ $m$-points, $b$ smooth points of $C$ and $c$ points in $X \setminus C$. Let

$$H_{a,b,c} = H(C, \mathcal{P}_{a,b,c}) = \frac{C^2 - am^2 - b}{a + b + c}.$$  

The border cases are $H_{1,0,0} = C^2 - m^2$, $H_{0,1,0} = C^2 - 1$ and $H_{0,0,1} = C^2$. If $a < \frac{c + C^2}{m^2 - 1}$ (case which occurs when $c$ is large) the function $b \to H_{a,b,c}$ is decreasing and converging to $-1$ when $b \to \infty$.

If $a \geq \frac{c + C^2}{m^2 - 1}$, the function $b \to H_{a,b,c}$ is increasing, thus if $a \neq 0$, one has

$$\inf_{b \geq 0} H_{a,b,c} = H_{a,0,c} = \frac{C^2 - am^2}{a + c}.$$
(note that even if \( a < \frac{C^2}{m^2} \), one still has \( \frac{C^2-am^2}{a} \geq -1 \)). If \( C^2-am^2 > 0 \), \( H_{a,0,c} \) is a decreasing function of \( c \), with limit 0, otherwise this is an increasing function and the infimum is attained for \( c = 0 \), which gives \( \frac{C^2-am^2}{a} \) (if \( a = 0 \), one gets \( C^2 \)). Then taking the minimum over \( a \), one obtains the result. \( \square \)

Let us recall (see [6]) that for \( a, b, n, m \in \mathbb{N}^* \), a \((a_n, b_m)\)-configuration is the data of two sets \( A, B \) of order \( a \) and \( b \), respectively, and a relation \( R \subset A \times B \), such that \( \forall \alpha \in A, \# \{(\alpha, x) \in R \} = n \) and \( \forall \beta \in B, \# \{(y, \beta) \in R \} = m \).

One has \( an = bm = \#R \). If \( a = b \) and \( n = m \), it is called a \((a_n)\)-configuration. If for \( \alpha \neq \alpha' \) in \( A \) the cardinality \( \lambda \) of \( \{(\alpha, x) \in R \} \cap \{(\alpha', x) \in R \} \) does not depend on \( \alpha \neq \alpha' \), this is called a \((a_n, b_m)\)-design and \( mn(n-1) = \lambda(a-1) \); \( \lambda \) is called the type of the design.

2.2. Construction of configuration from Genus 2 curves. Let \( A \) be a principally polarized abelian surface such that the principal polarization \( C_0 \) is a smooth genus 2 curve. One can choose an immersion such that \( 0 \in A \) is a Weierstrass point of \( C_0 \). The configuration of the 16 translates

\[ C_t = t + C_0, \quad t \in A[2] \]

of \( C_0 \) by the 2 torsion points of \( A \) is the famous \((16_0)\) Kummer configuration: there are 6 curves through each point in \( A[2] \), and each curve contains 6 points in \( A[2] \) (since \( C_tC_{t'} = 2 \) for \( t \neq t' \) in \( A[2] \), it is even a \((16_0)\)-design of type 2).

Let now \( n > 0 \) be an integer and let \([n] : A \to A\) be the multiplication by \( n \) map on \( A \). For \( t \in A[2] \), let us define \( D_t = [n]*C_t \), in other words

\[ D_t = \{x \mid nx \in C_t\} = \{x \mid nx + t \in C_0\}. \]

Since \([n]\) is étale, the curve \( D_t \) is a smooth curve, thus it is irreducible since its components are the pull back of an ample divisor. By [9] Proposition 2.3.5, since \( C_t \) is symmetric (i.e. \( [-1]*C_t = C_t \)), one has \( D_t \sim n^2C_t \) (in particular \( D_t^2 = 2n^4 \)). The curve

\[ W_n = [n]* \sum_{t \in A[2]} C_t = \sum_{t \in A[2]} D_t \]

has 16 irreducible components and \( 16n^4 \) ordinary singularities of multiplicity 6 (6-points), which are the torsion points \( A[2n] := \text{Ker} [2n] \). Each curve \( D_t \) contains \( 6n^4 \) 6-points; the configuration of curves \( D_t \) and singular points of \( W_n \) is a \((16_{6n^4}, 16n^4)\)-configuration. Using Lemma [3] we get:

**Lemma 6.** One has \( H(W_n, \text{Sing}(W_n)) = -4 \).

The Harbourne constant \( H_A \) of a surface \( A \) is an invariant of the isogeny class of \( A \) (see [14]). Thus if \( A \) is generic, it is isogeneous to the Jacobian of a smooth genus 2 curve, and we thus obtain the following:

**Proposition 7.** On a generic abelian surface \( A \), one has:

\[ H_A \leq -4. \]
Note that when \( A \) is isogeneous to the product of 2 elliptic curves \( E, E' \) (thus non generic in our situation), the H-constant of \( A \) verifies that \( H_A \leq -2 \), and \( H_A \leq -3 \) if \( E \) and \( E' \) are isogeneous (see [14]). Moreover, there are two examples of surfaces with CM for which \( H_A \leq -4 \).

**Remark 8.**

i) Suppose that \( n \) is odd, then
\[
D_t = \{ x \mid n(x + t) \in C_0 \} = D_0 + t.
\]
Moreover, if \( u \) is a 2-torsion point one has \( 2u = 0 \iff 2nu = 0 \), thus \( D_0 \) and each curve \( D_t \) contains 6 points of 2-torsion.

ii) Suppose that \( n \) is even. Let \( u \in A[2] \) be a 2-torsion point. One has \( u \in D_t \iff nu + t \in C_0 \iff t \in C_0 \). Therefore the 6 curves \( D_t \) with \( t \in A[2] \cap C_0 \) contain \( A[2] \), and the remaining curves do not contain any points from \( A[2] \).

2.3. **Genus 3 curves.** Let \( A \) be an abelian surface containing a hyperelliptic genus 3 curve \( C_0 \) such that 0 is a Weierstrass point. Then the 8 Weierstrass points of \( C_0 \) are contained in the set of 2-torsion points of \( A \). Let \( \mathcal{O} \) be the orbit of \( C_0 \) under the action of \( A[2] \) by translation and let \( a \) be the cardinal of \( \mathcal{O} \). The stabilizer \( S_t \) of \( C_0 \) acts as a fix-point free automorphism group of \( C_0 \). Thus considering the possibilities for the genus of \( C_0/S_t \) it is either trivial or an involution, therefore \( a = 16 \) or 8. By [5, Remark 1], the curve \( C_0 \) is stable by translation by a 2-torsion point, therefore \( a = 8 \). Let \( m \) be the number of curves in \( \mathcal{O} \) through one point in \( A[2] \) (this is well defined because \( A[2] \) acts transitively on itself). The sets of 8 genus 3 curves and \( A[2] \) form a \((8,16m)\)-configuration, thus \( m = 4 \). Moreover, since they are translates, two curves \( C, C' \in \mathcal{O} \) satisfy \( CC' = C^2 = 4 \), thus
\[
C \sum_{C' \in \mathcal{O}, C' \neq C} C' = 7 \cdot 4.
\]
If the singularities of the union of the curves in \( \mathcal{O} \) were only at the points in \( A[2] \) and ordinary, one would have
\[
C \sum_{C' \in \mathcal{O}, C' \neq C} C' = 8 \cdot 3.
\]
The configuration \( \mathcal{C} = \sum_{C \in \mathcal{O}} C \) contains therefore other singularities than the points in \( A[2] \) or the singularities are non ordinary. It seems less interesting from the point of view of H-constants. Observe that if the singularities at \( A[2] \) are ordinary, one has \( H(\mathcal{C}, A[2]) = -2 \). If there are other singularities, since the configuration is stable by translations by \( A[2] \), there are at least 16 more singularities.

2.4. **Construction of configurations from Genus 4 curves.** Traynard in [17], almost one century later Barth, Nieto in [3], and Naruki in [10] constructed \((16_{10})\) configurations of lines lying on a 3-dimensional family of quartic K3 surfaces \( X \) in \( \mathbb{P}^3 \); there exist two sets \( \mathcal{C}, \mathcal{C}' \) of 16 disjoint lines in \( X \) such that each line in \( \mathcal{C} \) meets exactly 10 ten lines in \( \mathcal{C}' \), and vice versa.
By the famous results of Nikulin characterizing Kummer surfaces, there exists a double cover $\pi : \tilde{A} \to X$ branched over $C$. That cover contains 16 $(-1)$-curves over $\pi^{-1}C$. The contraction $\mu : \tilde{A} \to A$ of these 16 exceptional divisors is an abelian surface and the image of these 16 curves is the set $A[2]$ of two torsion points of $A$.

We denote by $C_1, \ldots, C_{16}$ the 16 smooth curves images by $\mu_*\pi^*$ of the 16 disjoint lines in $C'$. By [3, Section 6], the 16 curves $C_1, \ldots, C_{16}$ are translates of each other by the action by the group $A[2]$ of 2-torsion points; the argument is that if $C'_i$ is a translate of $C_i$ by a 2-torsion point, then $\pi_*\mu^*C'_i$ is a line in the quartic $X$, but a such a generic quartic has exactly 32 lines.

**Proposition 9.** The curves $C_1, \ldots, C_{16}$ in $A$ are smooth of genus 4. The 16 2-torsion points $A[2]$ and these 16 curves form a $(16_{10})$-design of type 6: 10 curves though one point in $A[2]$, a curve contains 10 points in $A[2]$ and two curves meet at 6 points in $A[2]$. The H-constant of that configuration $\sum C_i$ is $H = -4$.

**Proof.** The 10 intersection points between the lines in $C$ and $C'$ are transverse, therefore by the Riemann-Hurwitz Theorem, the genus of the 16 irreducible components of $\pi^*C'$ is 4. The intersections of the 16 components in $\mu_*\pi^*C'$ are transverse (since $\pi^*C'$ is a union of disjoint curves) and that intersection holds over points in $A[2]$ (which is the image of the exceptional divisors of $\tilde{A}$).

Since the curves in $C$ and $C'$ form a $(16_{10})$ configuration, the 16 curves $C_1, \ldots, C_{16}$ and the 2 torsion points in $A$ have the described $(16_{10})$ configuration.

Since the strict transform in $\tilde{A}$ of the curves $C_i \neq C_j$ are two disjoint curves, the 6 intersection points of $C_i \neq C_j$ are 2-torsion points, the configuration is therefore a $(16_{10})$-design of type 6.

It is then immediate to compute the H-constant of $C = C_1 + \cdots + C_{16}$.

**Remark 10.** Since the 16 curves are the orbit of a curve by the group $A[2]$ of torsion points, one can change the notations and define $C_t = C_0 + t$ for $t \in A[2]$, for a chosen curve $C_0$ containing 0. As in sub-section 2.2 let us define $D_t = [n]^*C_t$; this is a smooth curve. It is then immediate to check that the $H$-constant of the curve $W_n = \sum D_t$ equals $-4$. We will use these configurations of curves in Section 3.

2.5. **Genus 5 curves.** By [4], a generic $(1, 4)$-polarized abelian surface contains a smooth genus 5 curve $C$ which is hyperelliptic, the set of Weierstrass points in $C$ is 12 2-torsion points, and $C$ is stable by a sub-group of $A[2]$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Thus the orbit of $C$ by the translations by elements of $A[2]$ is the union of 4 genus 5 curves.

The intersection of two of these curves equals $C^2 = 2g - 2 = 8$. Since each of these two curves contains 12 points in $A[2]$, the intersections are transverse and are on 8 points in $A[2]$. The 4 curves and the 16 2-torsion
points form a \((4_{12}, 16_3)\) configuration. The \(H\)-constant of that configuration is \(H = \frac{4 \cdot 8 - 16 \cdot 3}{16} = -1\).

### 3. Configurations of curves with low \(H\)-constant in Kummer surfaces

In this Section, we study the images in the Kummer surface \(Km(A)\) of the various curve configurations studied in Section 2 in abelian surfaces \(A\).

#### 3.1. The genus 2 case

We keep the notations and hypothesis of subsection 2.2. In particular, \(A\) is the Jacobian of a genus 2 curve. Let \(\mu : \hat{A} \to A\) be the blow-up of \(A\) at the 16 2-torsion points. We denote by \(D\) the strict transform in \(\hat{A}\) of a curve \(D \hookrightarrow A\). Let \(\pi : \hat{A} \to X\) be the quotient map by the automorphism \([-1]\). Since on \(A\) one has \([-1]^{*}C_t = [-1]^{*}(t + C_0) = C_t\), one obtains

\[ [-1]^{*}D_t = D_t \]

and the map \(D_t \to D_t' = \pi(D_t)\) has degree 2, thus \(D_{t'}^2 = \frac{1}{2}(D_t)^2\).

**Proposition 11.** Let be \(n > 1\). The configuration \(D\) of the 16 curves \(D_t'\) with \(t \in A[2]\) in the Kummer surface \(X\) has Harbourne constant

\[ H\left( \sum_{t \in A[2]} D_t' \right) = -4\frac{n^4}{n^4 - 1}. \]

**Proof.** If \(n\) is even, a curve \(D_t\) contains 16 or 0 points of 2 torsion depending if \(t \in C_0\) or not (thus there are 10 curves without points of 2 torsion, and 6 with). If \(n\) is odd, each curve \(D_t\) contains 6 points of 2-torsion and then one has:

\[ D_{t'}^2 = \frac{1}{2}(2n^4 - 6) = n^4 - 3. \]

If \(n\) is even, one has:

\[ D_{t'}^2 = \frac{1}{2}(2n^4 - 16) = n^4 - 8 \text{ or } D_{t'}^2 = n^4, \]

according if \(t \in A[2]\) is in \(C_0\) or not. The configuration \(D\) contains

\[ \frac{1}{2}(16n^4 - 16) = 8(n^4 - 1) \]

6-points and no other singularities. If \(n\) is even, then the configuration has 10 curves with self-intersection \(n^4\) and 6 curves with self-intersection \(n^4 - 8\). Thus if \(n\) is even one has

\[ H(D) = \frac{10n^4 + 6(n^4 - 8) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4\frac{n^4}{n^4 - 1} \sim -4, \]

which for \(n = 2\) gives \(H = -64/15 \simeq -4.26\).

If \(n\) is odd, one has 16 curves with self-intersection \(n^4 - 3\), and we get the same formula:

\[ H(D) = \frac{\sum D_{t'}^2 - 8(n^4 - 1)6}{8(n^4 - 1)} = \frac{16(n^4 - 3) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4\frac{n^4}{n^4 - 1}. \]
Remark 12. a) The H-constants of the various configurations are $<-4$.
b) For $n=1$, the H-constant is $-2$.

3.2. The genus 4 case. Let us consider the configuration $16_{10}$ considered in sub-section 2.4 of 16 genus 4 curves $C_t$, $t \in A[2]$ in a generic $(1,3)$-polarized abelian surface $A$. Let $X = \text{Km}(A)$ be the Kummer surface associated to $A$. Let $\mu : \tilde{A} \to A$ the blow-up at the points in $A[2]$, and $\pi : \tilde{A} \to X$ be the quotient map. Let us consider as in Remark 10 the 16 curves $D_t = [n]*C_t$, $t \in A[2]$ in $A$. Let be $D_t'$ the strict transform in $\tilde{A}$ of $D_t$ and $D_t' = \pi(D_t)$.

Proposition 13. For $n > 1$, the configuration $C = \sum_{t \in A[2]} D_t'$ in the Kummer surface $X$ has Harbourne constant

$$H(C) = -4 \frac{n^4}{n^4 - 1}.$$ 

Proof. The involution $[-1] : A \to A$ fixes the set $A[2]$ and stabilizes the configuration $C = \sum_{t \in A[2]} C_t$, since a curve $C_t$ in $C$ is determined by the 2-torsion points it contains, $[-1]$ stabilizes each curve $C_t$, $t \in A[2]$, and thus also one has $[-1]^* D_t = D_t$. Thus the restriction $\tilde{D}_t \to D_t'$ of $\pi$ has degree 2. Numerically, one has $D_t = n^2C_t$ and $C_t^2 = 6$. Since $D_t$ as a set is $\{x \in A \mid nx + t \in C_0\}$, a point $t' \in A[2]$ is in $D_t$ if and only if $nt' + t \in C_0$. Thus if $n$ is even, the curve $D_t$ contains 16 or 0 points of 2 torsion depending if $t \in C_0$ or not (thus there are 6 curves without points of 2 torsion, and 10 with), moreover one has:

$$D_t^2 = \frac{1}{2}(\tilde{D}_t)^2 = \frac{1}{2}(6n^4 - 16) = 3n^4 - 8 \text{ or } D_t'^2 = 3n^4,$$

according if $t \in A[2]$ is in $C_0$ or not. If $n$ is odd, each curve $D_t$ contains 10 points of 2-torsion and

$$D_t'^2 = \frac{1}{2}(6n^4 - 10) = 3n^4 - 5.$$ 

The configuration $C = \sum D_t'$ has $\frac{1}{2}(16n^4 - 16) = 8(n^4 - 1)$ 10-points and no other singularities. If $n$ is even, then the configuration contains 6 curves with self-intersection $3n^4$ and 10 curves with self-intersection $3n^4 - 8$, thus

$$H(C) = \frac{6 \cdot 3n^4 + 10(3 \cdot n^4 - 8 - 8(n^4 - 1)10}{8(n^4 - 1)} = -4 \frac{n^4}{n^4 - 1} \sim -4.$$ 

If $n$ is odd, one has 16 curves with self-intersection $3n^4 - 5$, and one gets the same formula. □

Remark 14. The multiplication by $n$ map $[n]$ on $A$ induces a rational map $[n] : X \dasharrow X$. The configurations $\sum D_t$ we are describing are the pull back by $[n]$ of a configuration in $X = \text{Km}(A)$ of 16 disjoint rational curves.
4. IRREDUCIBLE CURVES WITH LOW H-CONSTANT IN ABELIAN AND KUMMER SURFACES.

Obtaining irreducible curves with low Harbourne constant is in general a difficult problem. Let \( k > 0 \) be an integer. In [10], we prove that in a generic abelian surface polarized by \( M \) with \( M^2 = k(k+1) \) there exists a hyperelliptic curve \( \Gamma_k \) numerically equivalent to \( 4M \) such that \( \Gamma_k \) has a unique singularity of multiplicity \( 4k + 2 \). Thus:

**Proposition 15.** The H-constant of \( \Gamma_k \) is

\[
H(\Gamma_k) = \Gamma_k^2 - (4k + 2)^2 = -4.
\]

Let us study the case \( k = 1 \) and define \( T_1 = \Gamma_1 \). This is a curve of geometric genus 2 in an abelian surface \( A \) with one 6-point singularity, which we can suppose in 0. In [13] such a curve \( T_1 \) is constructed: \( A \) is the Jacobian of a genus 2 curve \( C_0 \) and \( T_1 \) is the image by the multiplication by 2 map of \( C_0 \) in \( A = J(C_0) \). The self-intersection of \( T_1 \) is \( T_1^2 = 32 \) and the singularity of \( T_1 \) is a 6-point. Let \( n \in \mathbb{N} \) be odd. The curve \( T_n = [n]^*T_1 \sim n^2T_1 \) has 6-points singularities at each points of \( A[n] \), the set of \( n \)-torsion points of \( A \). Let \([n] \) be the multiplication by \( n \) map. The following diagram of curve configurations in \( A \) is commutative

\[
\begin{array}{ccc}
\sum_{t \in A[2]} D_t \downarrow [n] & \xrightarrow{[2]} & T_n \\
\downarrow [n] & & \downarrow [n] \\
\sum_{t \in A[2]} C_t & \xrightarrow{[2]} & T_1
\end{array}
\]

Since the multiplication by 2 map [2] has degree 16 and the curves \( D_t \) are permuted by translations by elements of \( A[2] \), the map \( D_t \xrightarrow{[2]} T_n \) is birational, thus \( T_n \) is irreducible. Its singularities are 6-points over each \( n \)-torsion points.

**Theorem 16.** Let \( n \in \mathbb{N} \) be odd. The Harbourne constant of the irreducible curve \( T_n \to A \) is :

\[
H(T_n) = \frac{32n^4 - 36n^4}{n^4} = -4.
\]

Let \( \tilde{T}_n \) be the strict transform of \( T_n \) under the blowing-up map \( \tilde{A} \to A \) at the points from \( A[2] \). Since \( n \) is odd, among points in \( A[2] \), \( T_n \) contains only 0, thus \( T_n^2 = 32n^4 - 36 \). Moreover \([-1]^*T_n = T_n \) (it can be seen using the map \( C^{(2)} \to A \) that \([-1]^*T_1 = T_1 \), and therefore \([-1]^*T_n = T_n \)). The image of \( \tilde{T}_n \) on the Kummer surface \( X = \tilde{A}/[-1] \) is an irreducible curve \( W_n \) with \( \frac{1}{2}(n^4 - 1) \) 6-points if \( n \) is odd. The map

\[
\tilde{T}_n \to W_n
\]

has degree 2 and

\[
W_n^2 = 16n^4 - 18,
\]

thus
Proposition 17. Let $n \in \mathbb{N}$ be odd. The $H$-constant of the irreducible curve $W_n$ in the Kummer surface $X$ is

$$H(W_n) = \frac{-4n^4}{n^4 - 1}.$$  

In particular, for $n = 3$ one has $H(W_3) = -\frac{81}{20}$.

5. Some remarks on $H$-constants of abelian surfaces

Let $\phi : C \hookrightarrow A$ be an irreducible curve of geometric genus $g$ in an abelian surface $A$. Let $m_p = m_p(C)$ be the multiplicity of $C$ at a point $p$. One has

$$C^2 = 2g - 2 + 2\gamma,$$

where

$$\gamma \geq \sum_p m_p(m_p - 1),$$

with equality if all singularities are ordinary. Thus

$$H(C, \text{Sing}(C)) = \frac{1}{\#\text{Sing}(C)}(C^2 - \sum_{p \in \text{Sing}(C)} m_p^2) \geq \frac{1}{\#\text{Sing}(C)}(2g - 2 - \sum_{p \in \text{Sing}(C)} m_p)$$

with equality if all singularities are ordinary. From the previous construction in Section 2, one can ask the following

Problem 18. Does there exists an abelian surface containing a curve $C$ of geometric genus $g$ such that

$$-4 > \frac{1}{\#\text{Sing}(C)}(2g - 2 - \sum_{p \in \text{Sing}(C)} m_p)?$$

We were not able to find any example of such a curve. If the answer is no, it would imply that the bounded negativity conjecture holds true for surfaces which are blow-ups of abelian surfaces.

References

[1] Bauer T., Di Rocco S., Harbourne B., Huizenga J., Lundman A., Pokora P., Szemberg T., Bounded negativity and arrangement of lines, Int. Math. Res. Notices. (2015) 2015 (19): 9456–9471

[2] Bauer T., Harbourne B., Knutsen A.L.; Küronya A., Müller-Stach S., Roulleau X., Szemberg T., Negative curves on algebraic surfaces, Duke Math. J. 162 (2013), 1877–1894.

[3] Barth W., Nieto I., abelian surfaces of type $(1,3)$ and quartic surfaces with 16 skew lines, J. Algebraic Geom. 3 (1994), no. 2, 173–222.

[4] Borówka P., Ortega A., Hyperelliptic curves on $(1,4)$ polarised abelian surfaces, arXiv:1708.01270

[5] Borówka P., Sankaran G., Hyperelliptic genus 4 curves on abelian surfaces, Proc. Amer. Math. Soc. 145 (2017), 5023–5034

[6] Dolgachev I., Abstract configurations in Algebraic Geometry, The Fano Conference, 423–462, Univ. Torino, Turin, 2004.

[7] Laface R., Pokora P., Local negativity of surfaces with non-negative Kodaira dimension and transversal configurations of curves, arXiv:1602.05418
[8] Laface R., Pokora P., On the local negativity of surfaces with numerically trivial canonical class, to appear in Rend. Lin. Mat. e App.
[9] Birkenhake C., Lange H., Complex abelian varieties, Second edition. Grund. der Math. Wiss., 302. Springer-Verlag, Berlin, 2004. xii+635 pp.
[10] Naruki I., On smooth quartic embedding of Kummer surfaces, Proc. Japan Acad., 67, Ser. A (1991), pp. 223–225
[11] Pokora P., Tutaj-Gasińska H., Harbourne constants and conic configurations on the projective plane, Math. Nachr. 289 (2016), no. 7, 888–894.
[12] Pokora P., Harbourne constants and arrangements of lines on smooth hypersurfaces in $\mathbb{P}^3$, Taiwanese J. Math. 20 (2016), no. 1, 25–31
[13] Polizzi F., Rito C., Roulleau X., A pair of rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not the bidisk, preprint
[14] Roulleau X., Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations, Int. Math. Res. Notices (2017) 2017 (8): 2480–2496
[15] Roulleau X., Urzúa G., Chern slopes of simply connected complex surfaces of general type are dense in $[2,3]$, Annals of Math. 182 (2015), 287–306
[16] Roulleau X., Sarti A., Construction of Nikulin configurations on some Kummer surfaces and applications, arXiv 1711.05968
[17] Traynard E., Sur les fonctions thêta de deux variables et les surfaces Hyperelliptiques, Ann. Sci. École Norm. Sup. (3) 24 (1907), 77–177.

Xavier Roulleau,
Aix-Marseille Université, CNRS, Centrale Marseille,
I2M UMR 7373,
13453 Marseille, France
Xavier.Roulleau@univ-amu.fr

URL: https://old.i2m.univ-amu.fr/~roulleau.x/