A PROBABILITY APPROXIMATION FRAMEWORK: MARKOV PROCESS APPROACH

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ABSTRACT. We view the classical Lindeberg principle in a Markov process setting to establish a probability approximation framework by the associated Itô’s formula and Markov operator. As applications, we study the error bounds of the following three approximations: approximating a family of online stochastic gradient descents (SGDs) by a stochastic differential equation (SDE) driven by multiplicative Brownian motion, Euler-Maruyama (EM) discretization for multi-dimensional Ornstein-Uhlenbeck stable process, and multivariate normal approximation. All these error bounds are in Wasserstein-1 distance.

Keywords: Probability approximation, Markov process, Itô’s formula, Online stochastic gradient descent, Stochastic differential equation, Euler-Maruyama (EM) discretization, stable process, Normal approximation, Wasserstein-1 distance.

1. INTRODUCTION

Lindeberg principle provides an elegant proof for the classical central limit theorem of the sum of independent random variables [Lin22], it has been extensively applied to many research problems, see [Cha06, KM11, TV11, CCK14, BMP15, MP16, CCK17, CSZ17, KY17, WAP17, BCP18, CX19] and the references therein. In this paper, we shall view the classical Lindeberg principle in a Markov process setting, and use the well developed tools in stochastic analysis, such as Itô’s formula and infinitesimal generator, to establish a probability approximation framework.

In order to interpret our method, we first briefly recall the classical Lindeberg principle by the following example. Let \((\xi_i)_{i \geq 1}\) be a sequence of independent and identically distributed (i.i.d.) \(\mathbb{R}\)-valued random variables with \(\mathbb{E}[\xi_i] = 0, \mathbb{E}[\xi_i^2] = 1\) and \(\mathbb{E}|\xi_i|^3 < \infty\). Let \((\zeta_i)_{i \geq 1}\) be a sequence of independent standard normal distributed random variables and it is well known that \(\zeta_1 + \cdots + \zeta_n \sim \mathcal{N}(0, n)\) for any \(n \in \mathbb{N}\). Write \(\xi_{n,i} = \xi_i \sqrt{n}\) and \(\zeta_{n,i} = \frac{\zeta_i}{\sqrt{n}}\), and denote

\[X_n = \zeta_{n,1} + \cdots + \zeta_{n,n}, \quad Y_n = \xi_{n,1} + \cdots + \xi_{n,n}\]

Further denote \(Z_0 = X_n\) and \(Z_i = Z_{i-1} - \zeta_{n,i} + \xi_{n,i}\) for \(i \geq 1\), we easily see that \(Z_i\) is obtained by swapping \(\zeta_{n,i}\) in \(Z_{i-1}\) with \(\xi_{n,i}\). For any bounded 3rd order differentiable function \(h\), we have

\[
|\mathbb{E}[h(X_n)] - \mathbb{E}[h(Y_n)]| \leq \sum_{i=1}^{n} |\mathbb{E}[h(Z_{i-1})] - \mathbb{E}[h(Z_i)]| \leq Cn^{-3/2}\|h''\|\sum_{i=1}^{n} [\mathbb{E}|\xi_i|^3 + \mathbb{E}|\zeta_i|^3] \leq Cn^{-1/2}\|h''\|
\]

where \(\|\cdot\|\) is the uniform norm of continuous function and the second inequality is obtained by a 3rd order Taylor expansion.

Let us now explain the Lindeberg’s proof from a perspective of Markov process and view the above swap trick as a comparison of two Markov processes. Denote \(X_0 = 0, X_i = \zeta_{n,1} + \cdots + \zeta_{n,i}\) and \(Y_0 = 0, Y_i = \xi_{n,1} + \cdots + \xi_{n,i}\) for \(i \geq 1\), it is clear that \((X_i)_{0 \leq i \leq n}\) and \((Y_i)_{0 \leq i \leq n}\) are
both Markov processes. Formally, let \( X_j(i, x) \) the random variable \( X_j \) given \( X_i = x \in \mathbb{R} \), i.e., \( X_j(i, x) = x + \zeta_{n,i+1} + \ldots + \zeta_{n,j} \), it is obvious \( X_j(i, X_i) = X_j \) for \( j \geq i \). Similarly, we define \( Y_j(i, y) \) for \( j \geq i \). It is easy to see that \( Z_j = X_n(j, Y_j) = X_n(j, Y_j(j - 1, Y_{j-1})) \) and \( Z_{j-1} = X_n(j - 1, Y_{j-1}) = X_n(j, Y_j(j - 1, Y_{j-1})) \) for each \( 1 \leq j \leq n \), thus

\[
\begin{align*}
|\mathbb{E}[h(X_n)] - \mathbb{E}[h(Y_n)]| &\leq \sum_{j=1}^{n} |\mathbb{E}[h(Z_{j-1})] - \mathbb{E}[h(Z_j)]| \\
&= \sum_{j=1}^{n} |\mathbb{E}[h(X_n(j, Y_j(j - 1, Y_{j-1})))] - \mathbb{E}[h(X_n(j, Y_j(j - 1, Y_{j-1})))]|.
\end{align*}
\]

A rigorous proof will be given in Section 2 below with the help of the Chapman-Kolmogorov equation and the time homogeneity. Notice that these two new functions rather than directly compute \( E[h(Z_{j-1})] - E[h(Z_j)] \) in Lindeberg principle. Because \((X_j)_{0 \leq i \leq n}\) can be embedded into a Brownian motion \((B_t)_{0 \leq t \leq 1}\) which has a smoothing effect, we expect that Itô’s formula and the semigroup theory of Brownian motion will make \( E[h(X_n(j, Y_j(j - 1, Y_{j-1})))] \) and \( E[h(X_n(j, Y_j(j - 1, Y_{j-1})))] \) have better regularity than \( h \), see more details in Subsection 4.3. Since the above procedure only depends on Markov property, this perspective of viewing Lindeberg principle can be extended to other Markov processes.

The novelty of this paper is the following two aspects. (1) We view the procedure of the classical Lindeberg principle as a special Markov process and extend this point of view to general Markov process setting, using Itô’s formula of Markov process and Markov operator (see, e.g., [Gar85, BK02, EK09, KP13, Oks13]), we establish a probability approximation framework. Chatterjee [Cha06] extended Lindeberg principle to a family of dependent random variables, and established a general approximation error bound from which he identified the limiting spectral distribution of Wigner matrices with exchangeable entries. It is obvious to see from the SGD approximation below that our approximation framework also works for dependent random variables. (2) We apply our framework to three applications in the classical Wasserstein-1 distance: approximating online stochastic gradient descent (SGD) in machine learning by a stochastic differential equation (SDE), bounding the error between a SDE with \( \alpha \)-stable noise and its Euler-Maruyama (EM) discretization, and normal approximation.

For the first application, there have been many results on approximating SGD by a SDE, see for instance [TTV16, LTW17, AN19, FGL+19, HLLL19, LTW19, BS20, FDBD20] and the references therein. To the best of our knowledge, most of the known approximation results are about the error bounds over a family of test functions with bounded high order derivatives, from which it is not easy to obtain an approximation error bound in a probability metric. By restricting an SGD in a neighborhood of a local minimum and solving a Kolmogorov backward equation, Feng et al. [FGL+19] studied locally approximating the SGD before it jumps to another minimum. When their test functions have bounded \( k \)-th order derivatives with \( k = 0, ..., 4 \), the error bound is of order \( \eta \) (\( \eta \) is the learning rate), whereas the bound is improved to be of order \( \eta^2 \) as the test functions additionally have bounded 5-th and 6-th order derivatives. In [LTW19], Li et al. proposed stochastic modified equation, a SDE with multiplicative noise, to approximate SGD, their approximation error is defined through a family of test functions which has high order derivatives and a certain growth condition. If the test function is Lipschitz function family, it seems to us that their result cannot provide a convergence rate, see [LTW19, Definition 1 and Theorem 3]. In contrast, we will use our framework to get an explicit error bound between SGD and the associated SDE in the classical Wasserstein-1 distance, where our test function family is Lipschitz.
In the second application, we consider the EM discretization of \(\alpha\)-stable Ornstein-Uhlenbeck (OU) process with a constant step size \(\eta\), which leads to a heavy tailed AR(1) time series without second moment, [Cli83, DKL92, Res97, CLW12]. Using our framework, we establish an error bound of EM discretization and obtain a rate \(\eta^{2-\alpha}\) for \(\alpha \in (1, 2)\). It seems that there are not many results about EM discretization for SDE with \(\alpha\)-stable noise, see [JMW96, WY07, TA18, Liu19], most of them are about bounding strong approximations in a finite time interval. As the time tends to \(\infty\), these bounds blow up. The bound that we obtained is uniform with respect to the time, this means that our bound still holds true even the time tends to \(\infty\). Note that the discrete AR(1) time series are not independent.

The third application is normal approximations, which have recently been intensively studied by Stein’s method, see for instance, [CM07, RR09, CGS10, VV10, FSX19, Son20] and the references therein. In particular, Chatterjee and Meckes used an exchangeable pair method to obtain a bound for multivariate normal approximation in [CM08], because the test functions in [CM08] have bounded first and second derivatives, their bound cannot derive a convergence in Wasserstein-1 distance. By a direct calculation under our framework, we get a \(\frac{1}{\sqrt{n}}\) convergence rate up to \(\log n\) correction for multivariate normal approximation, which was established in [VV10, FSX19]. For Stein’s method, we refer the reader to [Xu19, CNX20, CNX+21, CNXY19, BU20] for stable approximation, [Che75, BH84, Bar88, AGG90, BHJ92] for Poisson approximation and [Bar90, BB92, BCL92, Loh92, NP09, Gur14, GM15, BD17, BDF17, Kas17, AH19] for other approximations.

Besides the applications to online SGD, EM discretization and normal approximation addressed in this paper, we hope that our new method can also be applied to many other probability approximations, e.g., diffusion approximation with constant step size and so on. We will study these research problems in the future paper.

In this paper, we focus on the approximation problems in Wasserstein-1 distance. However, it is clear to see from Theorem 2.1 that our approximation method also works for other metrics. For instance, if we consider bound measurable function \(h\), then the approximation will be in total variation metric.

The organization of this paper is as follows. We shall introduce our probability approximation framework and main theorem in next section. In Section 3, we will give the results about the three applications to SGD, EM discretization and normal approximation, where proofs are given in Subsections 4.1-4.3, respectively. Appendixes A and B are devoted to proving some auxiliary lemmas about the first application, while Appendixes C and D provide the proofs of auxiliary lemmas about the second and third applications, respectively.

Notations. We end this section by introducing some notations, which will be frequently used in sequel. The inner product of \(x, y \in \mathbb{R}^d\) is denoted by \(\langle x, y \rangle\) and the Euclidean metric is denoted by \(|x|\).

Let \(\mu\) and \(\nu\) be two probability distributions on \(\mathbb{R}^d\), their Wasserstein-1 distance is defined as

\[
d_W(\mu, \nu) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,
\]

where \(\text{Lip}(1) := \{h : \mathbb{R}^d \to \mathbb{R} : |h(x) - h(y)| \leq |x - y|\}\), \(X\) and \(Y\) are two random variables with distributions \(\mu\) and \(\nu\), respectively.

For a random variable \(X\), we denote by \(\mathcal{L}(X)\) its probability law. In addition, for any \(d\)-dimensional random vectors \(\xi_1, \xi_2\), we call \(\xi_1 \overset{d}{=} \xi_2\) if for any \(A \in \mathcal{B}(E)\), the Borel set of \(E\), we have

\[
\mathbb{P}(\xi_1 \in A) = \mathbb{P}(\xi_2 \in A).
\]
Let $C(\mathbb{R}^d, \mathbb{R})$ denote the collection of all continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $C^k(\mathbb{R}^d, \mathbb{R})$, $k \geq 1$, denote the collection of all $k$-th order continuously differentiable functions. For $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and $v_1, v_2, v_3, x \in \mathbb{R}^d$, the directional derivative $\nabla_v f(x)$, $\nabla_v \nabla_v f(x)$ and $\nabla_v \nabla_v \nabla_v f(x)$ are defined by

$$\nabla_v f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon},$$

$$\nabla_v \nabla_v f(x) = \lim_{\varepsilon \to 0} \frac{\nabla_v f(x + \varepsilon v_2) - \nabla_v f(x)}{\varepsilon},$$

$$\nabla_v \nabla_v \nabla_v f(x) = \lim_{\varepsilon \to 0} \frac{\nabla_v \nabla_v f(x + \varepsilon v_3) - \nabla_v \nabla_v f(x)}{\varepsilon},$$

respectively. Let $\nabla f(x) \in \mathbb{R}^d$ and $\nabla^2 f(x) \in \mathbb{R}^{d \times d}$ denote the gradient and the Hessian of $f$, respectively. It is known that $\nabla v f(x) = \langle \nabla f(x), v \rangle$ and $\nabla v_2 \nabla v_1 f(x) = \langle \nabla^2 f(x), v_1 v_2^T \rangle_{\text{HS}}$, where $T$ is the transpose operator and $\langle A, B \rangle_{\text{HS}} := \sum_{i,j=1}^{d} A_{ij} B_{ij}$ for $A, B \in \mathbb{R}^{d \times d}$. We define the operator norm of $\nabla^2 f(x)$ by

$$\|\nabla^2 f(x)\|_{\text{op}} = \sup_{|v_1|,|v_2|=1} |\nabla_v \nabla_v f(x)|,$$

$$\|\nabla^2 f(x)\|_{\text{op,}\infty} = \sup_{x \in \mathbb{R}^d} \|\nabla^2 f(x)\|_{\text{op}}.$$ We often drop the subscript "op" in the definitions above and simply write $|\nabla^2 f(x)| = \|\nabla^2 f(x)\|_{\text{op}}$ and $\|\nabla^2 f(x)\|_{\infty} = \|\nabla^2 f(x)\|_{\text{op,}\infty}$ if there is no confusion. Similarly we define

$$\|\nabla^3 f(x)\|_{\text{op}} = \sup_{|v_1|=1,|v_2|=1,2,3} |\nabla_v \nabla_v \nabla_v f(x)|$$

and $\|\nabla^3 f\|_{\text{op,}\infty}$ and the short-hand notations $\|\nabla^2 f(x)\|$ and $\|\nabla^3 f\|_{\infty}$.

Given a matrix $A \in \mathbb{R}^{d \times d}$, its Hilbert-Schmidt norm is $\|A\|_{\text{HS}} = \sqrt{\sum_{i,j=1}^{d} A_{ij}^2} = \sqrt{\text{Tr}(A^T A)}$ and its operator norm is $\|A\|_{\text{op}} = \sup_{|v|=1} |Av|$. We have the following relations:

$$(1.2) \quad \|A\|_{\text{op}} = \sup_{|v_1|,|v_2|=1} |\langle A, v_1 v_2^T \rangle_{\text{HS}}|, \quad \|A\|_{\text{op}} \leq \|A\|_{\text{HS}} \leq \sqrt{d} \|A\|_{\text{op}}.$$ 

Moreover, $C_0(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ with $d_1, d_2 \in \mathbb{N}$ denotes the set of all bounded continuous functions from $\mathbb{R}^{d_1}$ to $\mathbb{R}^{d_2}$ with the supremum norm defined by

$$\|f\| = \sup_{x \in \mathbb{R}^{d_1}} |f(x)|.$$ 

Denote by $C_{p_1, \ldots, p_k}$ some positive number depending on $k$ parameters, $p_1, \ldots, p_k$, whose exact values can vary from line to line.

2. THE FRAMEWORK AND MAIN THEOREM

Let $E$ be a Polish space. Let $(X_t)_{t \geq 0}$ be a continuous time homogeneous $E$-valued Markov process, and let $(Y_k)_{k \in \mathbb{Z}_+}$ be a discrete time homogeneous $E$-valued Markov process (note $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$). If $X_0 = x \in \mathbb{R}^d$, we denote the Markov process $(X_t)_{t \geq 0}$ by $(X_t^x)_{t \geq 0}$ to stress it starts from $x \in E$. Similarly for the notation $(Y^y_k)_{k \in \mathbb{Z}_+}$ for $y \in E$.

Notice that the process $(X_t)_{t \geq 0}$ is a time homogeneous $E$-valued Markov process, for any $0 \leq s \leq t < \infty, x \in E$ and $B \in B(E)$, the Borel sets of $E$, denote the transition probability function of $(X_t)_{t \geq 0}$ by $P_{s,t}(x,B)$, that is,

$$P_{s,t}(x,B) = \mathbb{P}(X_t \in B | X_s = x).$$

Then $P_{s,t}(x,B)$ satisfies the following properties (see, e.g., [Sat99, Section 10 of Chapter 2]):

1. it is a probability measure as a mapping of $B$ for any fixed $x$;
2. it is measurable in $x$ for any fixed $B$;
According to the time homogeneity, we denote
\[ Q(x) \] (2.3) corresponding transition probability function by
\[ Q(y, B) = \mathbb{P}(Y_j \in B | Y_i = x), \]
then the Chapman-Kolmogorov equation (4) can be written as
\[ P_t(x, B) = P_{s+t}(x, B), \quad s \leq 0, \]
then the Chapman-Kolmogorov equation (4) can be written as
\[ (2.2) \]
For the process \((Y_k)_{k \in \mathbb{Z}^+}, \) for any \(y \in E, B \in \mathcal{B}(E), i, j \in \mathbb{Z}^+ \) and \(i \leq j, \) we denote the corresponding transition probability function by \(Q_{i,j}(y, B),\) that is,
\[ (2.4) \]
According to the time homogeneity, we denote
\[ Q_j(y, B) = Q_{i,i+j}(y, B) \quad \text{for} \quad i \in \mathbb{Z}^+. \]
For \((X_t)_{t \geq 0},\) its infinitesimal generator is defined as
\[ A^X f(x) := \lim_{t \to 0} \frac{E[f(X_t^x) - f(x)]}{t}, \quad f \in \mathcal{D}(A^X), \]
where \(\mathcal{D}(A^X)\) is the domain of the operator \(A^X,\) whose exact form varies according to the concrete applications at hand. For \((Y_k)_{k \in \mathbb{Z}^+},\) its infinitesimal generator is defined as
\[ A^Y f(y) := E[f(Y_t^y) - f(y)], \quad f \in \mathcal{D}(A^Y), \]
where \(\mathcal{D}(A^Y)\) is the domain of the operator \(A^X,\) whose exact form varies according to the concrete applications at hand.

In order to avoid entering the semigroup theory in which we need to figure out function spaces and operator domains to justify our method, we use the concept of full generator family, which is usually easy to be verified by Itô’s formula in practice. More precisely, for a function \(f,\) we call \((f, A^X f)\) belongs to the full generator family of \(X_t\) if
\[ (2.5) \]
is a martingale with \(E[M_t] = 0\) for all \(t \geq 0,\) see [EK09, Chapter 4] for more details. In practice, it is easy for us to verify that a function belongs to a full generator family by Itô’s formula. For a more thorough discussion on the infinitesimal generators and Itô’s formula, we refer the reader to [EK09, Chapter 1 and Chapter 4], [Yos88, Chapter IX], [Oks13, Chapter 4] and the references therein.

Our first main result is a framework of comparing the distributions of \((X_t)_{t \geq 0}\) and \((Y_k)_{k \in \mathbb{Z}^+},\) which can be fitted into many probability approximations arising in concrete applications. The key ingredients of the proof are Markov semigroup, \((2.5)\) and infinitesimal generator in stochastic analysis.

**Theorem 2.1.** Let \(N \geq 2\) be a natural number and let \(h : E \to \mathbb{R}\) be a measurable function such that: (1). \(E|h(X_t^x)| < \infty \) and \(E|h(Y_k^y)| < \infty \) for all \(x \in E, y \in E, t \leq N \) and \(k \leq N;\) (2). the function \(u_k(x) := E[h(X_t^x) \mid X_0 = x]\) for \(k \geq 1\) satisfies \(E[A^X u_k(Y_j)] < \infty \) and
\[ \mathbb{E}|A^X u_k(X_t^j)| < \infty \text{ for all } 1 \leq j, k \leq N \text{ and } 0 \leq t \leq 1; \ (3). \ (u_k, A^X u_k) \text{ belongs to the full generator family of } X_t \text{ for } 1 \leq k \leq N. \] Then, for any \( x \in E, \)

\[ (2.6) \quad \mathbb{E}h(X_N^x) - \mathbb{E}h(Y_N^x) = \sum_{j=1}^{N} \left[ \mathbb{E}u_{N-j}(X_1^{Y_{j-1}^x}) - \mathbb{E}u_{N-j}(Y_1^{Y_{j-1}^x}) \right], \]

where, for simplicity, we write \( Y_{j-1} = Y_{j-1}^x \) for all \( j. \) In practice, we often rewrite (2.6) in the following form and then estimate each term on its right hand:

\[ (2.7) \quad \mathbb{E}h(X_N^x) - \mathbb{E}h(Y_N^x) = \mathcal{I}_h + \mathcal{II}_h, \]

where

\[ \mathcal{I}_h = \sum_{j=1}^{N-1} \int_0^1 \left[ \mathbb{E}A^X u_{N-j}(X_s^{Y_{j-1}^x}) - \mathbb{E}A^Y u_{N-j}(Y_{j-1}) \right] ds, \]

\[ \mathcal{II}_h = \mathbb{E} \left[ h(X_1^{Y_{N-1}^x}) - h(Y_{N-1}) \right] + \mathbb{E} \left[ h(Y_1) - h(Y_{N-1}) \right]. \]

In particular,

\[ (2.8) \quad d_W(\mathcal{L}(X_N), \mathcal{L}(Y_N)) \leq \sup_{h \in \text{Lip}(1)} (|\mathcal{I}_h| + |\mathcal{II}_h|). \]

**Remark 2.2.** In the above theorem, for \( 1 \leq j \leq N, \) the expectation \( \mathbb{E} \left[ u_{N-j} \left( X_{Y_{j-1}^x} \right) \right] \) actually means the following:

\[ \mathbb{E} \left[ u_{N-j} \left( X_{Y_{j-1}^x} \right) \right] = \int_E Q_{j-1}(x, dy) \int_E P_1(y, dz) \int_E h(u) P_{N-j}(z, du). \]

**Proof.** In the proof, we will often use Chapman-Kolmogorov equation and the following relation: for all \( x \in E \) and \( j \geq i, \)

\[ (2.9) \quad u_{j-i}(x) = \int_E h(y) P_{i,j}(x, dy), \]

where we have used the definition of \( u_k(.) \) and the time homogeneous property.

For \( N \geq 2, \) by (2.1) and (2.2), one can write

\[ \mathbb{E}h(X_N^x) = \int_E h(y) P_{0,N}(x, dy) \]

\[ = \int_E P_{0,1}(x, dz_1) \int_E h(y) P_{1,N}(z_1, dy) \]

\[ = \int_E u_{N-1}(z_1) P_{0,1}(x, dz_1), \]

where the last equality is by (2.9), therefore,

\[ \mathbb{E}h(X_N^x) = \int_E u_{N-1}(z_1) P_{0,1}(x, dz_1) - \int_E u_{N-1}(z_1) Q_{0,1}(x, dz_1) + \int_E u_{N-1}(z_1) Q_{0,1}(x, dz_1) \]

\[ = \mathbb{E} \left[ u_{N-1}(X_1^x) \right] - \mathbb{E} \left[ u_{N-1}(Y_1^x) \right] + \int_E u_{N-1}(z_1) Q_{0,1}(x, dz_1). \]

(2.10) By (2.9) and Chapman-Kolmogorov equation, we further have

\[ \int_E u_{N-1}(z_1) Q_{0,1}(x, dz_1) = \int_E Q_{0,1}(x, dz_1) \int_E h(y) P_{1,N}(z_1, dy) \]

\[ = \int_E Q_{0,1}(x, dz_1) \int_E P_{1,2}(z_1, dz_2) \int_E h(y) P_{2,N}(z_2, dy) \]
where the last equality is by Chapman-Kolmogrov equation and the following observations:

By a similar argument with (2.2), the time homogeneity and (2.1), we have

\[ (2.11) \]

where

\[ (2.12) \]

where the last equality is by Chapman-Kolmogrov equation and the following observations:

\[ \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) P_{1,2}(z_1, dz_2) = \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) P_{1,2}(z_1, dz_2) \]

\[ = \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) P_{1,2}(z_1, dz_2) - \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) Q_{1,2}(z_1, dz_2) \]

\[ + \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) Q_{1,2}(z_1, dz_2) \]

\[ = \mathbb{E}\left[u_{N-2}(X_1^{Y_i})\right] - \mathbb{E}\left[u_{N-2}(Y_1^{Y_i})\right] + \int_E u_{N-2}(z_2) Q_{0,2}(x, dz_2) \]

where the last equality is by Chapman-Kolmogrov equation and the following observations:

\[ \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) P_{1,2}(z_1, dz_2) = \int_E \mathbb{E}\left[u_{N-2}(X_1^{Y_i})\right] Q_{0,1}(x, dz_1) \]

\[ = \mathbb{E}\left[u_{N-2}(X_1^{Y_i})\right], \]

\[ \int_E Q_{0,1}(x, dz_1) \int_E u_{N-2}(z_2) Q_{1,2}(z_1, dz_2) = \int_E \mathbb{E}\left[u_{N-2}(Y_1^{Y_i})\right] Q_{0,1}(x, dz_1) \]

\[ = \mathbb{E}\left[u_{N-2}(Y_1^{Y_i})\right]. \]

Hence, we have

\[ \int_E u_{N-1}(z_1) Q_{0,1}(x, dz_1) = \mathbb{E}\left[u_{N-2}(X_1^{Y_i})\right] - \mathbb{E}\left[u_{N-2}(Y_1^{Y_i})\right] + \int_E u_{N-2}(z_2) Q_{0,2}(x, dz_2), \]

where \( Y_1 = Y_1^x \).

By the same argument, we can show that for all \( i = 1, 2, \ldots, N - 1 \),

\[ (2.11) \]

\[ \int_E u_{N-i}(z_i) Q_{0,i}(x, dz_i) = \mathbb{E}\left[u_{N-i-1}(X_1^{Y_i})\right] - \mathbb{E}\left[u_{N-i-1}(Y_1^{Y_i})\right] + \int_E u_{N-i-1}(z_{i+1}) Q_{0,i+1}(x, dz_{i+1}), \]

where \( Y_i = Y_i^x \). Combining these relations with (2.10) and noticing \( Y_0^x = x \), we obtain

\[ (2.12) \]

\[ \mathbb{E}h(X_N^x) = \sum_{i=1}^{N} \left( \mathbb{E}\left[u_{N-i}(X_1^{Y_i-1})\right] - \mathbb{E}\left[u_{N-i}(Y_1^{Y_i-1})\right]\right) + \int_E u_0(z_N) Q_{0,N}(x, dz_N). \]

Noticing \( u_0 = h \), this immediately implies (2.6).

Let us now calculate each term in the sum on the right hand side. For \( 1 \leq j \leq N \), we have

\[ \mathbb{E}u_{N-j}(X_1^{Y_j-1}) - \mathbb{E}u_{N-j}(Y_1^{Y_j-1}) \]

\[ = \mathbb{E}\left[u_{N-j}(X_1^{Y_j-1}) - u_{N-j}(Y_1^{Y_j-1})\right] - \mathbb{E}\left[u_{N-j}(Y_1^{Y_j-1}) - u_{N-j}(Y_{j-1})\right]. \]

When \( 1 \leq j \leq N - 1 \), by the condition (3), we have

\[ u_{N-j}(X_1^{Y_j-1}) - u_{N-j}(Y_{j-1}) = \int_0^1 \mathcal{A}^X u_{N-j}(X_s^{Y_j-1}) ds + \mathcal{M}_1, \]

where \((\mathcal{M}_t)_{0 \leq t \leq 1}\) is a martingale with mean 0, and thus

\[ \mathbb{E}\left[u_{N-j}(X_1^{Y_j-1}) - u_{N-j}(Y_{j-1})\right] = \mathbb{E}\left[\int_0^1 \mathcal{A}^X u_{N-j}(X_s^{Y_j-1}) ds\right]. \]
On the other hand, by conditional probability, we obtain
\[ \mathbb{E} \left[ u_{N-j}(Y_j^{Y_j-1}) - u_{N-j}(Y_{j-1}) \right] = \mathbb{E} \left[ \mathcal{A}^Y u_{N-j}(Y_{j-1}) \right]. \]
Hence, for \( 1 \leq j \leq N - 1, \)
\[ \mathbb{E} u_{N-j}(X_j(j - 1, Y_{j-1})) - \mathbb{E} u_{N-j}(Y_j) = \mathbb{E} \int_0^1 \left[ \mathcal{A}^X u_{N-j}(X_s^{Y_j-1}) - \mathcal{A}^Y u_{N-j}(Y_{j-1}) \right] \, ds. \]
Combining all the relations above, we immediately obtain the equalities (2.6) and (2.7) in the theorem, as desired. Moreover, (2.8) is an immediate corollary from (2.7) by the definition of Wasserstein-1 distance. The proof is complete. \( \square \)

3. Three Applications

We only consider in this section three applications: SDE’s approximation to online SGD, EM discretization for SDE driven by \( \alpha \)-stable process with \( \alpha \in (1, 2) \), and normal approximation. As we mentioned early, we focus on Wasserstein-1 distance, though Theorem 2.1 can be applied to approximation problems in other metrics, for instance, if we replace the \( \text{Lip}(1) \) function family by bounded measurable function family, the approximation turns to be in total variation metric. The other applications will be studied in the forthcoming paper.

3.1. Application 1: Online SGD and SDEs ([CLTZ16, LTW19]). For the first application, we concentrate on approximating a family of online SGDs by a SDE driven by multiplicative Brownian motion. Using our framework, we will obtain an explicit error bound in the classical Wasserstein-1 distance. We shall give two examples for Theorem 3.5 below, which are applications.

For the first application, see instance [TTV16, LTW17, AN19, FGL+19, LL19, LTW19, BS20, FDBD20] and the references therein.

Now, first we introduce the online SGD. Estimation of model parameters by minimizing an objective function is a fundamental idea in statistics. Let \( w^* \in \mathbb{R}^d \) be the true \( d \)-dimensional parameter vector. In common models, \( w^* \) is the minimizer of a convex objective \( P(w) : \mathbb{R}^d \to \mathbb{R} \), i.e.,
\[ w^* = \text{argmin} \left( P(w) := \mathbb{E}_{\zeta \sim \Pi} \psi(w, \zeta) = \int \psi(w, \zeta) \, d\Pi(\zeta) \right), \]
where \( \zeta \) denotes the random sample from a probability distribution \( \Pi \) and \( \psi(w, \zeta) \) is the loss function. The online SGD is a widely used optimization method for minimizing \( P(w) \).

The online SGD is an iterative algorithm, let \( w_0 = x \) and the \( k \)-th iterate \( w_k \) takes the following form,
\[ w_k = w_{k-1} - \eta \nabla \psi(w_{k-1}, \zeta_k), \quad k \geq 1, \]
where \( \eta \) is a small positive step-size known as the learning rate, \( \zeta_k \) is the \( k \)-th sample randomly drawn from the distribution \( \Pi \), and \( \nabla \psi(w_{k-1}, \zeta_k) \) denotes the gradient of \( \psi(w_{k-1}, \zeta_k) \) with respect to \( w \) at \( w = w_{k-1} \).

It is easily seen that online SGD (3.1) can be rewritten as
\[ w_k = w_{k-1} - \eta \nabla P(w_{k-1}) + \sqrt{\eta} V_\eta(w_{k-1}, \zeta_k), \]
where \( V_\eta(w_{k-1}, \zeta_k) = \sqrt{\eta} (\nabla P(w_{k-1}) - \nabla \psi(w_{k-1}, \zeta_k)). \) It is straightforward to check that
\[ \mathbb{E} \left[ V_\eta(w_{k-1}, \zeta_k) | w_{k-1} \right] = 0, \quad \text{Cov} \left[ V_\eta(w_{k-1}, \zeta_k), V_\eta(w_{k-1}, \zeta_k) | w_{k-1} \right] = \eta \Sigma(w_{k-1}), \]
where \( \Sigma(w_{k-1}) = \mathbb{E} \left[ (\nabla P(w_{k-1}) - \nabla \psi(w_{k-1}, \zeta_k)) (\nabla P(w_{k-1}) - \nabla \psi(w_{k-1}, \zeta_k))^T | w_{k-1} \right] \)
Now, we can consider the stochastic differential equation (SDE) as follows to approximate the above online SGD:

\[
(3.2) \quad d\hat{X}_t = -\nabla P(\hat{X}_t)dt + \left(\eta \Sigma(\hat{X}_t)\right)^{1/2} dB_t, \quad \hat{X}_0 = x,
\]

where \(B_t\) is a \(d\)-dimensional Brownian motion. For the research of above SDE with the noise term depending on a small parameter (the learning rate), we refer the reader to [Xi01, CF14, LTW17] and the references therein.

For further use, we shall assume:

**Assumption A1**

(i) There exist \(\theta_0 > 0\) and \(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 > 0\) such that for any \(v_1, v_2, v_3, x \in \mathbb{R}^d\), \(\nabla P(x)\) satisfies

\[
(3.3) \quad \langle v_1, \nabla v_1 \nabla P(x) \rangle \geq \theta_0 |v_1|^2, \quad |\nabla v_1 \nabla v_2 \nabla P(x)| \leq \theta_1 |v_1||v_2|,
\]

(ii) There exist \(\delta > 0\) such that for any \(x \in \mathbb{R}^d\) and non-zero vector \(\xi \in \mathbb{R}^d\), \(\Sigma(x)^{1/2}\) satisfies

\[
(3.4) \quad |\nabla v_1 \nabla v_2 \nabla v_3 \nabla P(x)| \leq \theta_2 |v_1||v_2|;
\]

\[
(3.5) \quad \|\nabla v_1 \Sigma(x)^{1/2}\|_{HS} \leq \theta_3 |v_1|, \quad \|\nabla v_2 \Sigma(x)^{1/2}\|_{HS} \leq \theta_4 |v_1||v_2|,
\]

\[
(3.6) \quad \|\nabla v_3 \nabla v_1 \Sigma(x)^{1/2}\|_{HS} \leq \theta_5 |v_1||v_2||v_3|.
\]

**Remark 3.1.** By integration, \(3.3)\) implies

\[
(3.8) \quad \langle x - y, \nabla P(x) - \nabla P(y) \rangle \geq \theta_0 |x - y|^2, \quad \forall x, y \in \mathbb{R}^d.
\]

In addition, from now on, we simply write a number \(C_{\theta_0, \ldots, \theta_5}\), depending on \(\theta_0, \ldots, \theta_5\), by \(C_0\) in shorthand.

Moreover, in order to ensure \(\mathbb{E}|w_k|^4 < \infty\), we further assume:

**Assumption A2**

There exists a constant \(\kappa > 0\) such that

\[
(3.9) \quad \mathbb{E}|\nabla \psi(x, \zeta) - \nabla \psi(y, \zeta)|^4 \leq \kappa^4 |x - y|^4, \quad \forall x, y \in \mathbb{R}^d.
\]

**Remark 3.2.** By \(3.9)\) and convexity inequality, for \(j = 1, 2, 3, 4\), it is easily seen that

\[
(3.10) \quad \mathbb{E}|\nabla \psi(x, \zeta)|^j \leq 2^{j-1}\mathbb{E}|\nabla \psi(x, \zeta) - \nabla \psi(0, \zeta)|^j + 2^{j-1}\mathbb{E}|\nabla \psi(0, \zeta)|^j
\]

(3.10) with \(\ell_0^j := \mathbb{E}|\nabla \psi(0, \zeta)|^j\).

In addition, since \(\Sigma(x) = \mathbb{E}\left[(\nabla P(x) - \nabla \psi(x, \zeta))(\nabla P(x) - \nabla \psi(x, \zeta))^T\right]\), by \(3.10)\), \(3.9)\) and the Cauchy-Schwarz inequality, we have

\[
(3.11) \quad \|\Sigma(x)^{1/2}\|_{HS} = \text{Tr}(\Sigma(x)) = \mathbb{E}|\nabla P(x) - \nabla \psi(x, \zeta)|^2
\]

\[
(3.12) \quad \|\Sigma(x)\|_{HS} \leq \mathbb{E}|\nabla P(x) - \nabla \psi(x, \zeta)|^2 \leq 2\ell_0^j + 2\kappa^2 |x|^2.
\]

Since \(\nabla P(x) = \mathbb{E}\nabla \psi(x, \zeta)\), by \(3.10)\), \(3.9)\) and the Cauchy-Schwarz inequality, we have

\[
(3.13) \quad |\nabla P(x) - \nabla P(y)| \leq \mathbb{E}|\psi(x, \zeta) - \psi(y, \zeta)| \leq \kappa|x - y|, \quad \forall x, y \in \mathbb{R}^d,
\]

which further implies

\[
(3.14) \quad |\nabla P(x)| \leq |\nabla P(x) - \nabla P(0)| + |\nabla P(0)| \leq \kappa|x| + |\nabla P(0)|.
\]
To illustrate the online SGD recursion in (3.1) and the Assumption A1 and Assumption A2, we consider the following two motivating examples. In appendix A, we will verify that the following two examples satisfy Assumption A1 and Assumption A2.

**Example 3.3.** (Model in [LTW19, Section 5]) Let $H \in \mathbb{R}^d$ be a symmetric, positive definite matrix. Define the sample objective

$$\psi(x, \zeta) = \frac{1}{2} (x - \zeta)^T H (x - \zeta) - \frac{1}{2} \text{Tr}(H),$$

where $\zeta \sim N(0, I_d)$. Then, the online SGD iterates in (3.1) become,

$$w_k = w_{k-1} - \eta H(w_{k-1} - \zeta_k),$$

which implies

$$\nabla \psi(x, \zeta) = H(x - \zeta), \quad \nabla P(x) = H x, \quad \Sigma(x) = H^2.$$

**Example 3.4.** (Variation of alternate Model in [LTW19, Section 5.1]) Let $H \in \mathbb{R}^{d \times d}$ be a symmetric, positive definite matrix, we diagonalize it in the form $H = Q D Q^T$, where $Q$ is an orthogonal matrix and $D$ is a diagonal matrix of eigenvalues. Let $\alpha \sim N(0, I_d)$, $\beta \sim N(0, I_d)$ and $\alpha$ is independent of $\beta$. Denote $\zeta = (\alpha, \beta)$ and define the loss function

$$\psi(x, \zeta) := \frac{1}{2} (Q^T x)^T [D + \text{diag}(\alpha)] (Q^T x) + \frac{\gamma}{2} (x - \beta)^T (x - \beta),$$

where $\text{diag}(\alpha)$ is a diagonal matrix, whose diagonal elements are each component of the vector $\alpha$ and $\gamma > 0$ is a tuning parameter. Therefore, the online SGD iterates in (3.1) become,

$$w_k = w_{k-1} - \eta \left[ Q [D + \text{diag}(\alpha_k)] Q^T w_{k-1} + \gamma (w_{k-1} - \beta_k) \right],$$

which implies

$$\nabla \psi(x, \zeta) = Q [D + \text{diag}(\alpha)] (Q^T x) + \gamma (x - \beta), \quad \nabla P(x) = H x + \gamma x$$

and

$$\Sigma(x) = \mathbb{E} \left[ (Q [D + \text{diag}(\alpha)] (Q^T x) - H x) (Q [D + \text{diag}(\alpha)] (Q^T x) - H x)^T \right]$$

$$+ \mathbb{E} \left[ (\gamma (x - \beta) - \gamma x) (\gamma (x - \beta) - \gamma x)^T \right]$$

$$= Q \text{diag}(Q x)^2 Q^T + \gamma^2 I_d = Q \left[ \text{diag}(Q x)^2 + \gamma^2 I_d \right] Q^T.$$

Now, we are at the position to state our theorem of the first application.

**Theorem 3.5 (Online SGD v.s. SDE).** Keep the same notations as above. Let $N \geq 2$ be a natural number. Suppose that Assumption A1 and Assumption A2 hold. Then, as $0 < \eta \leq \min \{1, \frac{\eta_0}{\mathbb{E}[10 + 7k + 4 + \gamma_0]} \}$, we have

$$d_W(\mathcal{L}(\tilde{X}_{\eta N}), \mathcal{L}(w_N)) \leq C_{\theta, \kappa, \epsilon_0} (1 + |x|^3) \left( 1 + \frac{d}{\delta^2} \right) (1 + |\ln \eta|) \eta,$$

where $(\tilde{X}_t)_{t \geq 0}$ is the diffusion process, which is defined by SDE (3.2), $(w_k)_{k \in \mathbb{Z}^+}$ is the online SGD iteration process, which is defined by (3.1).
3.2. Application 2: EM discretization for SDEs driven by \( \alpha \)-stable process with \( \alpha \in (1, 2) \) ([JMW96, TA18]). In recent years, the EM discretization for SDEs driven by \( \alpha \)-stable process has been studied by [JMW96, WY07, TA18, Liu19] in the finite time interval. In this subsection, we will consider a particular Ornstein–Uhlenbeck process driven by \( \alpha \)-stable process with \( \alpha \in (1, 2) \) and obtain a uniform convergence rate with respect to the time.

Let \( (Z_t)_{t \geq 0} \) be the \( d \)-dimensional rotationally symmetric \( \alpha \)-stable process, i.e., \( \mathbb{E}[e^{i(Z_t, \lambda)}] = e^{-t|\lambda|^\alpha} \), then we have \( Z_t \overset{d}{=} t^{\frac{1}{\alpha}} Z_1 \) (see, e.g., [Sat99, Theorem 14.3]) and the corresponding generator is

\[
\Delta_x f(x) = d \int_{\mathbb{R}^d} \frac{f(x + y) - f(x)}{|y|^{\alpha + d}} dy, \quad f \in C_b^2(\mathbb{R}^d),
\]

where

\[
d = \left( \int_0^{\infty} \frac{1 - \cos y}{y^\alpha + 1} dy \int_{S^{d-1}} |\langle e, \theta \rangle|^\alpha d\theta \right)^{-1}
\]

and \( e \) is an unit vector. Moreover, it is well known that \( d = \frac{\alpha \Gamma((d + \alpha)/2)}{\Gamma(d/2) \Gamma((2 - \alpha)/2)} \) (see, e.g., [BG68]).

Now, we consider the following SDEs:

\[
d\tilde{X}_t = -\frac{1}{\alpha} \tilde{X}_t dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d.
\]

Let \( \tilde{Z}_1, \tilde{Z}_2, \cdots \) be a sequence of random vectors independently drawn from the Pareto random variable \( \tilde{Z} \), which has the probability density function

\[
p(z) = \frac{\alpha}{V(\mathbb{S}^{d-1})|z|^{\alpha + d}} 1_{(1, \infty)}(|z|),
\]

where \( V(\mathbb{S}^{d-1}) \) is the surface area of \( \mathbb{S}^{d-1} \) and it is well known that \( V(\mathbb{S}^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \). The reason we choose Pareto random variable is that it has an explicit density function and thus can be easily sampled on computer.

We consider the following discrete Markov process with step size \( \eta \in (0, 1) \) to approximate the above SDEs: let initial value \( \tilde{Y}_0 = x \) and

\[
\tilde{Y}_{k+1} = \tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k + \frac{\eta^2}{\sigma} \tilde{Z}_{k+1}, \quad k \geq 0,
\]

where \( \sigma = \left( \frac{\alpha}{V(\mathbb{S}^{d-1})}\right)^\frac{1}{\alpha} \).

Then, we have the following theorem:

**Theorem 3.6** (EM discretization for SDEs driven by \( \alpha \)-stable process). *Keep the same notations as above. Let \( N \geq 2 \) be a natural number. Then, as \( \eta \in (0, 1) \), we have

\[
d_W(\mathcal{L}(\tilde{X}_{\eta N}), \mathcal{L}(\tilde{Y}_N)) \leq C_{c, \eta}(1 + |x|)\eta^\frac{2-\alpha}{\alpha},
\]

where \( (\tilde{X}_t)_{t \geq 0} \) and \( (\tilde{Y}_k)_{k \in \mathbb{Z}^+} \) are defined by (3.15) and (3.17), respectively.*

3.3. Application 3: Multivariate Normal CLT ([Ste72, CGS10]). Finally, we apply Theorem 2.1 to the multivariate normal approximation, and recover the results in [VY10, FSX19].

In this application, we denote the \( d \)-dimensional Brownian motion by \( (B_t)_{t \geq 0} \) and denote the \( d \)-dimensional standard normal distribution by \( N(0, I_d) \), that is, if \( B \sim N(0, I_d) \), then \( \mathbb{E}[e^{i(B, \lambda)}] = e^{-|\lambda|^2} \) for any \( \lambda \in \mathbb{R}^d \). Moreover, it is well known that \( B \overset{d}{=} B_1 \).
Theorem 3.7 (Multivariate normal CLT). Let $B \sim N(0, I_d)$ and $S_n = \sum_{i=1}^{n} \frac{x_i}{\sqrt{n}}$ with i.i.d. random vectors $(\xi_i)_{i \in \mathbb{N}}$ satisfying $E \xi_i = 0$, $E \xi_i \xi_i^T = I_d$ and $\sup_i E|\xi_i|^3 < \infty$. Then, we have
\[ d_W(\mathcal{L}(B), \mathcal{L}(S_n)) \leq \left[ \left( \frac{2}{3}d + 1 \right) E|B| + \frac{1}{3} E|\xi_1|^3 + E|\xi_1| \right] n^{-\frac{1}{2}} (1 + \ln n). \]

4. Proofs of Theorems 3.5, 3.6 and 3.7

In this section, with the help of Theorem 2.1, we focus on proving the Theorem 3.5, Theorem 3.6 and Theorem 3.7.

4.1. Proof of Theorem 3.5. We first give the following upper bounds of the processes $w_k$ and $\hat{X}_t$, which will be proved in Appendix A.

Lemma 4.1. Let $w_k$ be defined by (3.1) with $w_0 = x \in \mathbb{R}^d$. Then, as $\eta \leq \min\{1, \frac{\theta_0}{2(10 + 7c^3 + 7c_0^3)}\}$, for any $k \geq 1$, we have
\[ E|w_k|^4 \leq |w_0|^4 + C_{\theta, c, \ell_0^3}. \]

Lemma 4.2. Let $\hat{X}_t$ be the solution to the equation (3.2). Then, as $\eta \leq \min\{1, \frac{\theta_0}{4c^3}\}$, for any $t > 0$, we have
\[ E|\hat{X}_t|^2 < |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0^2}{\theta_0^2}, \]
\[ E|\hat{X}_t - x|^2 \leq C_{\theta, c, \ell_0^2}(1 + |x|^2)(t + \eta)t. \]

With the help of Malliavin calculus and Bismut’s formula, we can obtain the following estimates, which will be proved in Appendix B.

Lemma 4.3. Let $\hat{X}_t$ be the solution to the equation (3.2) and denote $P_t h(x) = E[h(\hat{X}_t^x)]$ for $h \in \text{Lip}(1)$. Then, as $\eta \leq \min\{1, \frac{\theta_0}{4c^3}\}$, for any $x, v, v_1, v_2, v_3 \in \mathbb{R}^d$ and $t > 0$, we have
\[ |\nabla_v(P_t h)(x)| \leq e^{-\frac{\theta_0}{4c^3}}|v|, \]
\[ |\nabla_v \nabla_{v_1} P_t h(x)| \leq C_{\theta} \left( 1 + \frac{\sqrt{d}}{\delta} \right) (1 + \frac{1}{\sqrt{\eta t}}) e^{-\frac{\theta_0}{4c^3}}|v_1||v_2|, \]
\[ |\nabla_{v_2} \nabla_{v_1} \nabla_{v_3} P_t h(x)| \leq C_{\theta} \left( 1 + \frac{d}{\delta^2} \right) (1 + \frac{1}{\eta t} + \frac{1}{t^2}) e^{-\frac{\theta_0}{4c^3}}|v_1||v_2||v_3|. \]

In order to use Theorem 2.1 to solve the problem, we also need the following lemma, which will be proved in Appendix A.

Lemma 4.4. Let $X_t = X_{nt}$ and $Y_k = w_k$. Denote $u_k(x) = E h(X_k^x)$ for $1 \leq k \leq N$. Then, as $\eta \leq \min\{1, \frac{\theta_0}{2(10 + 7c^3 + 7c_0^3)}\}$, we have
\[ \left| E \int_0^1 \left[ A^X u_k(X_k^x) - A^Y u_k(x) \right] ds \right| \leq C_{\theta, c, \ell_0^3}(1 + |x|^3)(1 + \frac{d}{\delta^2})(1 + \frac{1}{\eta k} + \frac{\eta}{(\eta k)^2}) \eta^2 e^{-\frac{\theta_0}{4c^3} \eta k}. \]

With the above results, we can give the proof of Theorem 3.5.

Proof of Theorem 3.5. In order to apply Theorem 2.1, we need to identify the $X_t$ and $Y_k$ therein in our setting and compute the corresponding $A^X$ and $A^Y$. Let $X_t = \hat{X}_{nt}$, $Y_k = w_k$, $X_0 = Y_0 = x \in \mathbb{R}^d$ and $N \geq 2$. Then, $u_k(x) = E h(X_k^x) = E h(\hat{X}_{nt}^x)$ for $k \geq 1$. Notice that $h \in \text{Lip}(1)$, the Cauchy-Schwarz inequality and (4.2) imply $E|h(X_k^x)| \leq |h(0)| + \sqrt{E|\dot{X}_{nt}|^2} < \infty$; Similarly, we can derive $E|h(Y_k^x)| < \infty$, that is, the condition (1) holds. In addition,
Lemma 4.5 implies that the function $u_k$ has bounded 1st, 2nd and 3rd order derivatives. Hence, by the Itô’s formula for SDE (see, e.g., [Oks13, Theorem 4.2.1]), we can see that the conditions (2) and (3) in Theorem 2.1 is satisfied.

Now we apply Theorem 2.1 to prove the theorem, it suffices to bound the two terms $I_h$ and $II_h$ in (2.7). For the term $I_h$, by Lemma 4.4 and (4.1), we have

$$|I_h| \leq C_{θ, κ, ℓ_0} \sum_{j=1}^{N-1} (1 + E[Y_j - 1]^3)(1 + \frac{d}{δ^2})(1 + \frac{1}{η(N - j)} + \frac{η}{[η(N - j)]^\frac{3}{2}}) e^{-\frac{η}{2} η(N - j) η^2} \leq C_{θ, κ, ℓ_0} (1 + |x|^3)(1 + \frac{d}{δ^2})η \left[\eta + \eta^\frac{1}{3} + η \int_1^N (1 + \frac{1}{ηv} + \frac{η}{[ηv]^\frac{3}{2}}) e^{-\frac{η}{2} ηv dv} \right] \leq C_{θ, κ, ℓ_0} (1 + |x|^3)(1 + \frac{d}{δ^2})(1 + |ln η|)η.$$

Then, since $η \leq 1$, it is straightforward to calculate

$$|I_h| \leq C_{θ, κ, ℓ_0} (1 + |x|^3)(1 + \frac{d}{δ^2})η \left[\eta + η^\frac{1}{3} + η \int_1^N (1 + \frac{1}{ηv} + \frac{η}{[ηv]^\frac{3}{2}}) e^{-\frac{η}{2} ηv dv} \right] \leq C_{θ, κ, ℓ_0} (1 + |x|^3)(1 + \frac{d}{δ^2})(1 + |ln η|)η.$$

For the term $II_h$, by the Cauchy-Schwarz inequality, (4.3) and (4.1), we have

$$E|h(X_{N-1}^N) − h(Y_{N-1})| \leq E|X_{N-1}^w − w_{N-1}| \leq C_{θ, κ, ℓ_0} (1 + |x|)η.$$

Recall (3.1), by Cauchy-Schwarz inequality, (3.10) and (4.1), we have

$$E|h(Y_N) − h(Y_{N-1})| \leq E|w_N − w_{N-1}| \leq η E|\nabla Ψ(w_{N-1}, ζ_N)| \leq C_{θ, κ, ℓ_0} (1 + |x|)η.$$

These imply

$$|II_h| \leq C_{θ, κ, ℓ_0} (1 + |x|)η.$$

Combining all of above, we have

$$d_W(\mathcal{L}(X_t), \mathcal{L}(w_t)) \leq C_{θ, κ, ℓ_0} (1 + |x|^3)(1 + \frac{d}{δ^2})(1 + |ln η|)η.$$

### 4.2. Proof of Theorem 3.6
We first give the following upper bounds of the processes $\tilde{Y}_k$ and $\tilde{X}_t$, which will be proved in Appendix C.

**Lemma 4.5.** Let $\tilde{Y}_k$ be defined by (3.17) with $\tilde{Y}_0 = x ∈ \mathbb{R}^d$. Then, as $η ∈ (0, 1]$, we have

$$E|\tilde{Y}_k| \leq C_{α, d}(1 + |x|).$$

**Lemma 4.6.** Let $\tilde{X}_t$ be the solution to the equation (3.15). Then, for any $x ∈ \mathbb{R}^d$ and $t > 0$, we have

(4.7) $E|\tilde{X}_t| \leq C_{α, d}(1 + |x|)$,

(4.8) $E|\tilde{X}_t - x| \leq C_{α, d}(1 + |x|)(t + t^\frac{*}{2})$

Moreover, by Itô’s formula, we have the following lemma, which will be proved in Appendix C.
Lemma 4.7. Let \((\tilde{X}_t)_{t \geq 0}\) and \((\tilde{Y}_k)_{k \geq 0}\) be defined by (3.15) and (3.17), respectively. Then, for any \(x \in \mathbb{R}^d\), \(\eta \in (0, 1]\), and \(f : \mathbb{R}^d \to \mathbb{R}\) satisfying \(\|\nabla f\| < \infty\) and \(\|\nabla^2 f\|_{HS} < \infty\), we have
\[
|E[f(\tilde{X}_t^x) - f(\tilde{Y}_1)]| \leq C_{\alpha,d}(1 + |x|)(\|\nabla f\| + \|\nabla^2 f\|_{HS})\eta^2.
\]

With the help of the heat kernel estimates of the \(\alpha\)-stable process, we can obtain the following estimates, which will be proved in Appendix C.

Lemma 4.8. Let \(\tilde{X}_t\) be the solution to the equation (3.15), and denote \(Q_t h(x) = E[h(\tilde{X}_t^x)]\) for \(h \in \text{Lip}(1)\). Then, for any \(x \in \mathbb{R}^d\) and \(t > 0\), we have
\[
|\nabla(Q_t h)(x)| \leq \|\nabla h\|e^{-\frac{t}{\alpha}}, \quad \|\nabla^2(Q_t h)(x)\|_{HS} \leq C_{\alpha,d}\|\nabla h\|t^{-\frac{1}{\alpha}}e^{-\frac{t}{\alpha}}.
\]

With the above results, we can give the proof of Theorem 3.6.

Proof of Theorem 3.6. In order to apply Theorem 2.1, we need to identify the \(X_t\) and \(Y_t\).

Let \(X_t = X_{\eta t}, Y_k = Y_k, X_0 = Y_0 = x \in \mathbb{R}^d\) and \(N \geq 2\). Now we apply (2.6) with \(u_k(x) = E[h(X_{\eta t})] = E[h(X_{\eta t}^x)]\) for \(k \geq 1\) to prove the theorem. Since \(h \in \text{Lip}(1)\), the (4.7) and Lemma 4.5 imply \(E[h(X_{\eta t}^x)] \leq \|h(0)\| + E|X_{\eta t}| < \infty\) and \(E[h(Y_k^x)] \leq \|h(0)\| + E|\tilde{Y}_k| < \infty\), respectively, the condition (1) is proved. By Lemma 4.8, we can further imply that the function \(u_k\) has bounded 1st and 2nd derivatives. Hence, by the Itô’s formula for SDE, we can see that the conditions (2) and (3) in Theorem 2.1 is satisfied. When \(1 \leq j \leq N - 1\), by Lemmas 4.7, 4.8 and 4.5, we have
\[
|E[u_{N-j}(X_{\eta t}^{j-1}) - u_{N-j}(Y_{\eta t}^{j-1})]| \leq C_{\alpha,d}(1 + |Y_{\eta t-1}|)(1 + [\eta(N - j)]^{-\frac{1}{\alpha}})E[-\frac{\eta(N - j)}{\alpha} \eta^2] \leq C_{\alpha,d}(1 + |x|)(1 + [\eta(N - j)]^{-\frac{1}{\alpha}})E[-\frac{\eta(N - j)}{\alpha} \eta^2],
\]
which implies
\[
\sum_{j=1}^{N-1} E[u_{N-j}(X_{\eta t}^{j-1}) - u_{N-j}(Y_{\eta t}^{j-1})] \leq C_{\alpha,d}(1 + |x|) \sum_{k=1}^{N-1} (1 + [\eta(N - j)]^{-\frac{1}{\alpha}})E[-\frac{\eta(N - j)}{\alpha} \eta^2] \leq C_{\alpha,d}(1 + |x|)\eta^\frac{2-\alpha}{\alpha}.
\]

When \(j = N\), by (4.8) and Lemma 4.5, we have
\[
|E[h(\tilde{X}_{\eta N}^x) - h(\tilde{Y}_{N-1})]| \leq E|\tilde{X}_{\eta N}^x - \tilde{Y}_{N-1}| \leq C_{\alpha,d}(1 + |Y_{\eta N-1}|)\eta^\frac{1}{\alpha} \leq C_{\alpha,d}(1 + |x|)\eta^\frac{1}{\alpha},
\]
and recall (3.17), Lemma 4.5 implies
\[
|E[h(\tilde{Y}_{N}) - h(\tilde{Y}_{N-1})]| \leq E|\tilde{Y}_{N} - \tilde{Y}_{N-1}| \leq C_{\alpha,d}(1 + |\tilde{Y}_{N-1}|)\eta^\frac{1}{\alpha} \leq C_{\alpha,d}(1 + |x|)\eta^\frac{1}{\alpha}.
\]

These imply
\[
|E[h(\tilde{X}_{\eta N}^x) - h(\tilde{Y}_{N}^x)]| \leq C_{\alpha,d}(1 + |x|)\eta^\frac{2-\alpha}{\alpha}.
\]

Combining all of above, we have
\[
|E[h(\tilde{X}_{\eta N}^x) - E[h(\tilde{Y}_{N})]| \leq C_{\alpha,d}(1 + |x|)\eta^\frac{2-\alpha}{\alpha}.
\]

□
4.3. **Proof of Theorem 3.7.** In order to use Theorem 2.1, we need the following properties for the semigroup of Brownian motion, which will be proved in Appendix D.

**Lemma 4.9.** Let \( h \in \text{Lip}(1) \) and denote \( P_t h(x) = \mathbb{E} h(B_t^x) \), then for any \( x, v, v_1, v_2 \in \mathbb{R}^d \) and \( t > 0 \), we have

\[
|\langle \nabla^2 (P_t h)(x + v) - \nabla^2 (P_t h)(x), v_1 v_2^T \rangle| \leq \frac{2 |v_1||v_2||v|}{t}, \tag{4.9}
\]

\[
|\Delta (P_t h)(x + v) - \Delta (P_t h)(x)| \leq \frac{2d}{t} |v|. \tag{4.10}
\]

With the above results, we can give the proof of Theorem 3.7.

**Proof of Theorem 3.7.** In order to apply Theorem 2.1, we first need to identify the \( X_t \) and \( Y_k \) therein in our setting and compute the corresponding \( A^X \) and \( A^Y \). Let \( X_t = B^1_t \), \( Y_k = S_k = \sum_{i=1}^k \frac{\xi_i}{\sqrt{n}} \), where \( \{\xi_i\}_{i \in \mathbb{Z}} \) is a sequence of i.i.d. random vectors satisfying \( \mathbb{E} \xi_i = 0 \), \( \mathbb{E} \xi_i \xi_i^T = I_d \) and \( \mathbb{E} |\xi_i|^3 < \infty \), \( X_0 = Y_0 = 0 \) and \( N = n \). Then, \( u_k(x) = \mathbb{E} h(X^k_t) = \mathbb{E} h(B^x_t) \) for \( k \geq 1 \), \( (D.1) \) and \( (D.2) \) below imply that \( u_k \) has bounded first and second order derivatives, whereby it is easy to obtain that the conditions (2) and (3) in Theorem 2.1 is satisfied by Itô’s formula of Brownian motion. In addition, the condition (1) can be derived easily from the fact \( h \in \text{Lip}(1) \), \( \mathbb{E} |B_t| < \infty \) and \( \mathbb{E} |\xi_i| < \infty \) for any \( t \geq 0 \), \( i = 1, 2, \cdots, n \). Then, it is straightforward to check that

\[
A^X u_k(x) = \lim_{t \to 0} \frac{\mathbb{E} u_k(X^x_t) - u_k(x)}{t} = \frac{1}{n} \lim_{t \to 0} \frac{\mathbb{E} u_k(B^x_t) - u_k(x)}{t} = \frac{1}{2n} \Delta u_k(x)
\]

and

\[
A^Y u_k(x) = \mathbb{E}[u_k(Y^x_t) - u_k(x)] = \mathbb{E} [u_k(Y^x_t) - u_k(x) - \langle \nabla u_k(x), \frac{\xi_1}{\sqrt{n}} \rangle] \tag{4.11}
\]

Hence,

\[
|A^X u_k(x) - A^Y u_k(x)| = \left| \frac{1}{2n} \Delta u_k(x) - \frac{1}{n} \mathbb{E} \left[ \int_0^1 \int_0^r \langle \nabla^2 u_k(x + s \frac{\xi_1}{\sqrt{n}}), \xi_1 \xi_1^T \rangle_{HS} ds dr \right] \right|.
\]

Now we apply Theorem 2.1 to prove the theorem, it suffices to bound the three terms \( I_h, II_h, III_h \) in (2.7). For the term \( I_h \), we rewrite it as

\[
I_h = \sum_{j=1}^{N-1} \mathbb{E} \left[ A^X u_{N-j}(Y_{j-1}) - A^Y u_{N-j}(Y_{j-1}) \right] + \sum_{j=1}^{N-1} \mathbb{E} \int_0^1 \left[ A^X u_{N-j}(X^Y_{s-1}) - A^X u_{N-j}(Y_{j-1}) \right] ds := I_{h,1} + I_{h,2}.
\]

For the first term, noticing that \( \mathbb{E} \xi_1 \xi_1^T = I_d \) and \( \langle \nabla^2 f(x), I_d \rangle_{HS} = \Delta f(x) \), by (4.11), we have

\[
\left| A^X f(x) - A^Y f(x) \right| = \left| \frac{1}{2n} \Delta f(x) - \frac{1}{n} \mathbb{E} \left[ \int_0^1 \int_0^r \langle \nabla^2 f(x + s \frac{\xi_1}{\sqrt{n}}), \xi_1 \xi_1^T \rangle_{HS} ds dr \right] \right| \leq \frac{1}{n} \int_0^1 \int_0^r \mathbb{E} \left| \langle \nabla^2 f(x + s \frac{\xi_1}{\sqrt{n}}) - \nabla^2 f(x), \xi_1 \xi_1^T \rangle_{HS} \right| ds dr.
\]
Lemma A.1. In Example 3.3, denote the smallest eigenvalue of the matrix \( A \) and \( \theta \).

Then, the desired result follows from the fact that

\[
\theta = \frac{1}{3\sqrt{n}} \mathbb{E} |\xi_1|^3 \sum_{j=1}^{n-1} \frac{1}{j} \leq \frac{1}{3\sqrt{n}} \mathbb{E} |\xi_1|^3(1 + \int_1^n \frac{1}{y} dy) = \frac{1}{3\sqrt{n}} \mathbb{E} |\xi_1|^3(1 + \ln n).
\]

For the second term, by (4.10) and the scaling property of \( B \), i.e., \( B_t = t^{1/2} B_1 \), we have

\[
| T_{h,2} | \leq \frac{1}{2n} \sum_{j=1}^{n-1} \int_{t}^{1} \mathbb{E} \left| \Delta u_{n-j}(Y_{j-1}) - \Delta u_{n-j}(Y_{j-1}) \right| ds
\]

\[
\leq d \int_{0}^{1} \mathbb{E} |B_n| ds \sum_{j=1}^{n-1} \frac{1}{n-j} = \frac{d}{\sqrt{n}} \mathbb{E} |B_t| \int_{0}^{1} s^{1/2} ds \sum_{j=1}^{n-1} \frac{1}{j} \leq \frac{2d}{3\sqrt{n}} \mathbb{E} |B_t|(1 + \ln n).
\]

It remains to estimate \( T \). By the scaling property of \( B_t \), it is easily seen that

\[
\mathbb{E} |h(X_1^{Y_{n-1}}) - h(Y_{n-1})| \leq \mathbb{E} |X_1^{Y_{n-1}} - Y_{n-1}| = \mathbb{E} |B_1| = \frac{1}{\sqrt{n}} \mathbb{E} |B_t|,
\]

\[
\mathbb{E} |h(Y_n) - h(Y_{n-1})| \leq \mathbb{E} |Y_n - Y_{n-1}| = E \frac{\xi_n}{\sqrt{n}}.
\]

These imply

\[
| T_{h,1} | \leq \left( \mathbb{E} |B_t| + \mathbb{E} |\xi_1| \right) \frac{1}{\sqrt{n}}.
\]

Collecting the estimates of \( T_{h,1} \) and \( T_{h,2} \), which hold true for all \( h \in \text{Lip}(1) \), we immediately obtain

\[
d_W (\mathcal{L}(B_t), \mathcal{L}(S_n)) \leq \left[ \frac{2}{3} d + 1 \right] \mathbb{E} |B_t| + \frac{1}{3} \mathbb{E} |\xi_1|^3 + E |\xi_1| n^{-1/2}(1 + \ln n).
\]

Then, the desired result follows from the fact that \( B \stackrel{d}{=} B_1 \).

**Appendix A. Proofs of Lemmas in Subsections 3.1 and 4.1**

A.1 **Verifying assumptions for two examples.** In this subsection, we verify Assumption A1 and Assumption A2 for Examples 3.3 and 3.4.

**Lemma A.1.** In Example 3.3, denote the smallest eigenvalue of the matrix \( H \) by \( \lambda_{\min}(H) \). Then, Assumption A1 and Assumption A2 hold for \( \theta_0 = \delta = \lambda_{\min}(H) \), \( \kappa = \| H \|_{HS} \) and \( \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = 0 \).

**Proof.** Recall

\[
\nabla P(x) = H x, \quad \Sigma(x) = H, \quad \nabla \psi(x, \zeta) = H(x - \zeta).
\]

Then, for any \( v, x, y \in \mathbb{R}^d \) and non-zero vector \( \xi \in \mathbb{R}^d \), it is easy to see that

\[
\langle v, \nabla_x \nabla P(x) \rangle = \langle v, H v \rangle \geq \lambda_{\min}(H) |v|^2,
\]

\[
\mathbb{E} |H(x - \zeta) - H(y - \zeta)|^4 \leq \| H \|^4_{HS} |x - y|^4
\]
and
\[ \xi^T H \xi \geq \lambda_{\min}(H) \| \xi \|^2. \]
Moreover, other results are clearly available from (A.1). \( \square \)

**Lemma A.2.** In Example 3.4, denote the smallest eigenvalue of the matrix \( H \) by \( \lambda_{\min}(H) \). Then, Assumption A1 and Assumption A2 hold for \( \theta_0 = \lambda_{\min}(H) + \gamma, \kappa^4 = 27(\|H\|_\text{HS}^4 + 3d^6 + \gamma^4), \delta = \gamma, \theta_1 = \theta_2 = 0, \theta_3 = \sqrt{d}\|Q\|_{\text{HS}}, \theta_4 = \sqrt{d}\|Q\|_{\text{HS}}^4(1 + \gamma^{-1} + \gamma^{-3}) \) and \( \theta_5 = 3\sqrt{d}\|Q\|_{\text{HS}}^5(2 + \gamma^{-3} + \gamma^{-5}) \).

**Proof.** Recall
\[ (A.2) \quad \nabla \psi(x, \zeta) = Q[D + \text{diag}(\alpha)](Q^T x) + \gamma(x - \beta), \quad \nabla P(x) = Hx + \gamma x, \]
\[ \Sigma(x)^{1/2} = Q\left[\text{diag}(Qx)^2 + \gamma^2 I_d\right]^{1/2} Q^T. \]
Then, for any \( v, x, y \in \mathbb{R}^d \) and non-zero vector \( \xi \in \mathbb{R}^d \), it is easy to see that
\begin{align*}
\mathbb{E}\left[Q[D + \text{diag}(\alpha)](Q^T x) + \gamma(x - \beta) - Q[D + \text{diag}(\alpha)](Q^T y) - \gamma(y - \beta)\right]^4 &\leq 27(\|H\|_{\text{HS}}^4 + \|Q\|_{\text{HS}}^4\|\alpha\|^4 + \gamma^4)|x - y|^4 \leq 27(\|H\|_{\text{HS}}^4 + 3d^6 + \gamma^4)|x - y|^4, \\
\xi^T \Sigma(x)^{1/2} \xi &\geq \|\xi\|^2.
\end{align*}
Moreover, by (A.2), it is easily seen that \( \theta_1 = \theta_2 = 0 \). For any \( v_1, v_2, v_3, x \in \mathbb{R}^d \), notice that \( \Sigma(x)^{1/2} \) is a diagonal matrix, by the chain rule and product rule, it is straightforward to calculate
\begin{align*}
\|\nabla_{v_1} \Sigma(x)^{1/2}\|_{\text{HS}} &= \frac{1}{2} \left\|\left(\Sigma(x)^{1/2}\right)^{-1} \nabla_{v_1} \Sigma(x)\right\|_{\text{HS}} \leq \sqrt{d}\|Q\|_{\text{HS}}^2|v_1|, \\
\|\nabla_{v_2} \nabla_{v_1} \Sigma(x)^{1/2}\|_{\text{HS}} &\leq \sqrt{d}\|Q\|_{\text{HS}}^4(1 + \gamma^{-1} + \gamma^{-3})|v_1||v_2|, \\
\|\nabla_{v_3} \nabla_{v_2} \nabla_{v_1} \Sigma(x)^{1/2}\|_{\text{HS}} &\leq 3\sqrt{d}\|Q\|_{\text{HS}}^5(2 + \gamma^{-3} + \gamma^{-5})|v_1||v_2||v_3|. \end{align*}
\( \square \)

**A.2. Proof of Lemma 4.1.** Recall (3.1), it is easily seen that
\begin{align*}
\mathbb{E}|w_k|^4 &= \mathbb{E}|w_{k-1}|^4 - 4\eta\mathbb{E}\left[|w_{k-1}|^2\langle \nabla \psi(w_{k-1}, \zeta_k), w_{k-1} \rangle\right] + 2\eta^2\mathbb{E}\left[|w_{k-1}|^2|\nabla \psi(w_{k-1}, \zeta_k)|^2\right] \\
&\quad - 4\eta^3\mathbb{E}\left[|\nabla \psi(w_{k-1}, \zeta_k)|^2\langle \nabla \psi(w_{k-1}, \zeta_k), w_{k-1} \rangle\right] + 4\eta^2\mathbb{E}\left[|\nabla \psi(w_{k-1}, \zeta_k), w_{k-1} |^2\right] \\
&\quad + \eta^4\mathbb{E}|\nabla \psi(w_{k-1}, \zeta_k)|^4.
\end{align*}
Since \( \zeta_k \) is independent of \( w_{k-1} \) for any \( k \geq 1 \), (3.8) yields
\begin{align*}
\mathbb{E}\left[|w_{k-1}|^2\langle \nabla \psi(w_{k-1}, \zeta_k), w_{k-1} \rangle\right] &= \mathbb{E}\left[|w_{k-1}|^2\langle \nabla (w_{k-1}, \zeta_k), w_{k-1} \rangle\right]|w_{k-1}| \\
= &\mathbb{E}\left[|w_{k-1}|^2\langle \nabla P(w_{k-1}) - \nabla P(0), w_{k-1} \rangle\right] + \mathbb{E}[|w_{k-1}|^2\langle \nabla P(0), w_{k-1} \rangle] \\
\geq &\theta_0 \mathbb{E}|w_{k-1}|^4 + \mathbb{E}[|w_{k-1}|^2\langle \nabla P(0), w_{k-1} \rangle],
\end{align*}
which implies
\begin{align*}
-4\eta\mathbb{E}\left[|w_{k-1}|^2\langle w_{k-1}, \nabla \psi(w_{k-1}, \zeta_k) \rangle\right] &\leq -4\theta_0 \eta \mathbb{E}|w_{k-1}|^4 + 4\eta\mathbb{E}\left[|w_{k-1}|^2|\nabla P(0)|^2\right] \\
&\leq -3\theta_0 \eta \mathbb{E}|w_{k-1}|^4 + \frac{27\|\nabla P(0)\|^4}{\theta_0^2} \eta,
\end{align*}
where the last inequality comes from Young’s inequality. In addition, by the Cauchy-Schwarz inequality and (3.10), we have
\[
\mathbb{E}[|w_{k-1}|^2 | \nabla \psi(w_{k-1}, \zeta_k)|^2] = \mathbb{E}[|w_{k-1}|^2 \mathbb{E}[|\nabla (w_{k-1}, \zeta_k)|^2 | w_{k-1}]]
\leq \mathbb{E}[2\kappa^2 |w_{k-1}|^4 + 2\ell_0^2 |w_{k-1}|^2] \leq 2(\kappa^2 + \ell_0^2) \mathbb{E}[|w_{k-1}|^4 + 2\ell_0^2],
\]
\[
\mathbb{E}[|\nabla \psi(w_{k-1}, \zeta_k)|^2 (\nabla \psi(w_{k-1}, \zeta_k), w_{k-1})] \leq \mathbb{E}[|\nabla \psi(w_{k-1}, \zeta_k)|^3 | w_{k-1}] 
\leq 4(\kappa^3 + \ell_0^3)(\mathbb{E}[|w_{k-1}|^4 + 1],
\]
\[
\mathbb{E}[|\nabla \psi(w_{k-1}, \zeta_k), w_{k-1}|]^2 \leq 2(\kappa^2 + \ell_0^2) \mathbb{E}[|w_{k-1}|^4 + 2\ell_0^2],
\]
\[
\mathbb{E}[|\nabla \psi(w_{k-1}, \zeta_k)|^4] = \mathbb{E}[|\nabla \psi(w_{k-1}, \zeta_k)|^4 | w_{k-1}] \leq 8\kappa^4 \mathbb{E}[|w_{k-1}|^4 + 8\ell_0^4].
\]
These imply
\[
\mathbb{E}[|w_k|^4 \leq \left[1 - 3\theta_0 \eta + 12(\kappa^2 + \ell_0^2) \eta^2 + 16(\kappa^3 + \ell_0^3) \eta^3 + 8\kappa^4 \eta^4\right] \mathbb{E}[|w_{k-1}|^4
\]
\[+ \frac{27}{\theta_0^2} \mathbb{E}[|\nabla P(0)|^4 \eta] + 12\ell_0^2 \eta^2 + 16(\kappa^3 + \ell_0^3) \eta^3 + 8\kappa^4 \eta^4\]
\[\leq (1 - \theta_0 \eta) \mathbb{E}[|w_{k-1}|^4 + C_{\theta, \kappa, \ell_0^2} \eta],
\]
where the second inequality is by the fact \(\eta \leq \min\{1, \frac{\theta_0}{2(10 + \kappa^4 + 7\ell_0^2)}\}\). Therefore,
\[
\mathbb{E}[|w_k|^4 \leq \left[1 - \theta_0 \eta\right]^k |w_0|^4 + C_{\theta, \kappa, \ell_0^2} \eta \sum_{j=0}^{k-1} (1 - \theta_0 \eta)^j \leq |w_0|^4 + C_{\theta, \kappa, \ell_0^2}.
\]

A.3. Proof of Lemma 4.2. Recall (3.2), by Itô’s formula, (3.11), (3.8) and the Young inequality, we have
\[
\frac{d}{ds} \mathbb{E}[\dot{X}_s^x]^2 = 2\mathbb{E} \left[\langle \dot{X}_s^x, -\nabla P(\dot{X}_s^x) \rangle \right] + \eta \mathbb{E} \left[\|\Sigma(\hat{X}_s^x)\|_{HS}^2\right]
\leq -2\mathbb{E}[\dot{X}_s^x, P(\dot{X}_s^x) - P(0)] + 2\mathbb{E}[|\dot{X}_s^x|^2 | P(0)] + 2\eta \mathbb{E} \left[\kappa^2 |\dot{X}_s^x|^2 + \ell_0^2\right]
\leq -\frac{3}{2} \theta_0 \mathbb{E}[|\dot{X}_s^x|^2] + \frac{2|\nabla P(0)|^2}{\theta_0} + 2\kappa^2 \eta \mathbb{E} \left[|\dot{X}_s^x|^2\right] + 2\eta \ell_0^2
\leq -\theta_0 \mathbb{E}[|\dot{X}_s^x|^2] + \frac{2|\nabla P(0)|^2}{\theta_0} + 2\ell_0^2,
\]
where the last inequality is by the fact \(\eta \leq \min\{1, \frac{\theta_0}{5\kappa^2}\}\). This inequality, together with \(\dot{X}_0^x = x\), implies
\[
\mathbb{E}[\dot{X}_t^x] \leq e^{-\theta_0 t} |x|^2 + \left(\frac{2|\nabla P(0)|^2}{\theta_0} + 2\ell_0^2\right) \int_0^t e^{-\theta_0 (t-s)} ds \leq |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \ell_0^2}{\theta_0^2},
\]
(4.2) is proved.
By the Cauchy-Schwarz inequality, the Itô isometry, (3.14) and (3.11), it is easy to verify
\[
\mathbb{E} |\dot{X}_t^x - x|^2 \leq 2\mathbb{E} \left[\left|\int_0^t -\nabla P(\dot{X}_r^x) dr\right|^2\right] + 2\mathbb{E} \left[\left|\eta \Sigma(\dot{X}_r)\right|^2 dB_r\right]^2
\leq 2t \mathbb{E}[|\nabla P(\dot{X}_t^x)|^2] + 2\eta \int_0^t \mathbb{E} \left[\left|\Sigma(\dot{X}_r)\right|^2\right]_{HS} ds dr
\leq C_{\kappa, \ell_0^2} (t + \eta) \int_0^t \left(\mathbb{E}[|\dot{X}_r^x|^2 + 1]\right) dr,
\]
which, together with (4.2), implies (4.3).

A.4. Proof of Lemma 4.4. Recall SDE (3.2), for any \( f \in C_b^2(\mathbb{R}^d) \), we have
\[
\mathcal{A}^X f(x) = \lim_{t \to 0} \frac{E f(X^x_t) - f(x)}{t} = -\eta \langle \nabla f(x), \nabla P(x) \rangle + \frac{1}{2} \eta^2 \langle \nabla^2 f(x), \Sigma(x) \rangle_{HS},
\]
Then, for any \( u_k(x) = \mathbb{E} h(X^x_{\eta_k}) \) with \( k \geq 1 \), we have
\[
\mathbb{E} \int_0^1 \mathcal{A}^X u_k(X^x_t) ds = -\eta \mathbb{E} \int_0^1 \langle \nabla u_k(X^x_s), \nabla P(X^x_s) \rangle ds + \frac{1}{2} \eta^2 \mathbb{E} \int_0^1 \langle \nabla^2 u_k(X^x_s), \Sigma(X^x_s) \rangle_{HS} ds
\]
Recall (3.1), by Taylor expansion, we have
\[
\mathcal{A}^Y u_k(x) = \mathbb{E}[u_k(Y^x_1) - u_k(x)] = \mathbb{E}[u_k(x_{1}) - u_k(x)]
\]
\[
= \mathbb{E}[\langle \nabla u_k(x), -\eta \nabla \psi(x, \zeta) \rangle + \frac{1}{2} \eta^2 \mathbb{E} \langle \nabla^2 u_k(x), \nabla \psi(x, \zeta) (\nabla \psi(x, \zeta))^T \rangle_{HS} + E[R^{u_k}(x)]
\]
\[
= \langle \nabla u_k(x), -\eta \nabla P(x) \rangle + \frac{1}{2} \eta^2 \langle \nabla^2 u_k(x), \Sigma(x) + \nabla P(x) (\nabla P(x))^T \rangle_{HS} + E[R^{u_k}(x)],
\]
where
\[
R^{u_k}(x) = \eta^2 \int_0^1 \int_0^t \langle \nabla^2 u_k(x - s\eta \nabla \psi(x, \zeta)) - \nabla^2 u_k(x), (\nabla \psi(x, \zeta))(\nabla \psi(x, \zeta))^T \rangle ds dr.
\]
Therefore, we have
\[
\mathbb{E} \int_0^1 [\mathcal{A}^Y u_k(X^x_t) - \mathcal{A}^Y u_k(x)] ds \leq \mathcal{J}_1 + \mathcal{J}_2 + \mathbb{E}|R^{u_k}(x)|,
\]
where
\[
\mathcal{J}_1 := \mathbb{E} \int_0^\eta \langle \nabla u_k(\hat{X}^x_s), \nabla P(\hat{X}^x_s) \rangle ds - \eta \langle \nabla u_k(x), \nabla P(x) \rangle + \frac{1}{2} \eta^2 \langle \nabla^2 u_k(x), \nabla P(x) (\nabla P(x))^T \rangle_{HS},
\]
\[
\mathcal{J}_2 := \frac{\eta}{2} \mathbb{E} \int_0^\eta \langle \nabla^2 u_k(\hat{X}^x_s), \Sigma(\hat{X}^x_s) \rangle_{HS} ds - \eta \langle \nabla^2 u_k(x), \Sigma(x) \rangle_{HS}.
\]
For \( \mathcal{J}_1 \), we have
\[
\mathcal{J}_1 \leq \mathbb{E} \int_0^\eta \langle \nabla u_k(\hat{X}^x_s), \nabla P(\hat{X}^x_s) - \nabla P(x) \rangle ds
\]
\[
+ \mathbb{E} \int_0^\eta \langle \nabla u_k(\hat{X}^x_s) - \nabla u_k(x), \nabla P(x) \rangle ds + \frac{1}{2} \eta^2 \langle \nabla^2 u_k(x), \nabla P(x) (\nabla P(x))^T \rangle_{HS}
\]
\[
:= \mathcal{J}_{11} + \mathcal{J}_{12}.
\]
By (4.4), (3.13), the Cauchy-Schwarz inequality and (4.3), one has
\[
\mathcal{J}_{11} \leq \kappa e^{-\frac{\theta^2 \eta k}{2}} \int_0^\eta E|\hat{X}^x_s - x| ds
\]
\[
\leq C_{\theta, \kappa} \kappa^2 (1 + |x|^2) e^{-\frac{\theta^2 \eta k}{2}} \int_0^\eta (s^2 + \eta s)^{\frac{1}{2}} ds \leq C_{\theta, \kappa} \kappa^2 (1 + |x|^2) \eta^2 e^{-\frac{\theta^2 \eta k}{2}}.
\]
Notice that
\[
\mathbb{E}(\nabla u_k(\hat{X}_x^e) - \nabla u_k(x), \nabla P(x))
\]
\[
= -\int_0^s \mathbb{E}(\nabla^2 u_k(x), \nabla P(\hat{X}_e^s)(\nabla P(x))^T_{\text{HS}}dv
\]
\[
+ \int_0^1 \mathbb{E}(\nabla^2 u_k(x + r(\hat{X}_x^e - x)) - \nabla^2 u_k(x), (\hat{X}_x^e - x)(\nabla P(x))^T_{\text{HS}}dr.
\]
By (4.5), (3.13), (3.14) and (4.6), we have
\[
J_{12} \leq \left| \int_0^n \int_0^s \mathbb{E}(\nabla^2 u_k(x), (\nabla P(\hat{X}_e^s) - \nabla P(x))(\nabla P(x))^T_{\text{HS}}dvds \right|
\]
\[
+ \left| \int_0^n \int_0^1 \mathbb{E}(\nabla^2 u_k(x + r(\hat{X}_x^e - x)) - \nabla^2 u_k(x), (\hat{X}_x^e - x)(\nabla P(x))^T_{\text{HS}}drds \right|
\]
\[
\leq C_{\theta, r, \epsilon_0} \alpha_k (1 + |x|)(1 + \frac{\sqrt{d}}{\delta})(1 + \frac{1}{\sqrt{\eta^2 k}}) e^{-\frac{\eta k}{2}} \int_0^n \int_0^s \mathbb{E}|\hat{X}_e^s - x|dvds
\]
\[
+ C_{\theta, r, \epsilon_0} \alpha_k (1 + |x|)(1 + \frac{d}{\delta^2})(1 + \frac{1}{\eta^2 k} + \frac{1}{(\eta k)^\frac{1}{x}}) e^{-\frac{\eta k}{2}} \int_0^n \int_0^1 r\mathbb{E}|\hat{X}_x^e - x|^2drds.
\]
Then, by the Cauchy-Schwarz inequality and (4.3), we can obtain
\[
J_{12} \leq C_{\theta, r, \epsilon_0} \alpha_k (1 + |x|^3)(1 + \frac{d}{\delta^2})(1 + \frac{1}{\eta^2 k} + \frac{1}{(\eta k)^\frac{1}{x}}) \eta^3 e^{-\frac{\eta k}{2}}.
\]
Hence, the fact \( \eta \leq 1 \) implies
\[
(A.3) \quad J_1 \leq C_{\theta, r, \epsilon_0} \alpha_k (1 + |x|^3)(1 + \frac{d}{\delta^2})(1 + \frac{1}{\eta^2 k} + \frac{\eta}{(\eta k)^\frac{1}{x}}) \eta^2 e^{-\frac{\eta k}{2}}.
\]
For \( J_2 \), recall \( \eta \Sigma(x) = \mathbb{E}[V_\eta(x, I)V_\eta(x, \xi)^T] \) with \( V_\eta(x, \xi) = \sqrt{\eta}(\nabla P(x) - \nabla \psi(x, \xi)) \), we have
\[
\eta \Sigma(x) - \eta \Sigma(y) = \mathbb{E}[V_\eta(x, \xi)(V_\eta(x, \xi) - V_\eta(y, \xi))^T] + \mathbb{E}[V_\eta(x, \xi) - V_\eta(y, \xi)]V_\eta(y, \xi)^T.
\]
By (3.10), the Cauchy-Schwarz inequality, (3.9) and (3.13), we further have
\[
(A.4) \quad \mathbb{E}|V_\eta(x, \xi)|^2 = \eta (\mathbb{E} |\nabla \psi(x, \xi)|^2 - |\nabla P(x)|^2) \leq \eta \mathbb{E} |\nabla \psi(x, \xi)|^2 \leq 2(\kappa^2|x|^2 + \ell_0^2)\eta,
\]
\[
(A.5) \quad \mathbb{E}[|V_\eta(x, \xi)| |V_\eta(x, \xi) - V_\eta(y, \xi)|] \leq C_{\alpha, \epsilon_0}(1 + |x|)|x - y|\eta.
\]
Then, (A.4), (4.6), (A.5) and (4.5) imply
\[
J_2 \leq \frac{1}{2} \mathbb{E} \left| \int_0^n (\nabla^2 u_k(\hat{X}_x^e) - \nabla^2 u_k(x), \eta \Sigma(x)_{\text{HS}}dvds \right|
\]
\[
+ \frac{1}{2} \mathbb{E} \left| \int_0^n (\nabla^2 u_k(X_x^e), \eta \Sigma(\hat{X}_x^e) - \eta \Sigma(x))_{\text{HS}}dvds \right|
\]
\[
\leq C_{\theta, r, \epsilon_0} \alpha_k (1 + |x|^3)(1 + \frac{d}{\delta^2})(1 + \frac{1}{\eta^2 k} + \frac{1}{(\eta k)^\frac{1}{x}}) e^{-\frac{\eta k}{2}} \eta \int_0^n \mathbb{E}|\hat{X}_x^e - x|dvds
\]
Following the Cauchy-Schwarz inequality, (4.2) and (4.3), by the same argument as the proof of (A.3), one has
\[ J_2 \leq C_{\theta, \kappa, \ell} (1 + |x|^3) (1 + \frac{d}{\delta^2}) (1 + \frac{1}{\eta k} + \frac{\eta}{(\eta k)^{\frac{3}{2}}}) \eta^2 e^{-\frac{a_0}{8} \eta k}. \]

In addition, by (4.6), Hölder’s inequality and (3.10), we have
\[ \mathbb{E} |R^u_k(x)| \leq C_{\theta, \kappa, \ell} (1 + \frac{d}{\delta^2}) (1 + \frac{1}{\eta^2 k} + \frac{1}{(\eta k)^{\frac{1}{2}}}) e^{-\frac{a_0}{8} \eta k} \eta^3 \int_0^1 \int_0^r (1 + |x|^3) ds dr \]
\[ \leq C_{\theta, \kappa, \ell} (1 + |x|^3) (1 + \frac{d}{\delta^2}) (1 + \frac{1}{\eta k} + \frac{\eta}{(\eta k)^{\frac{3}{2}}}) \eta^2 e^{-\frac{a_0}{8} \eta k}. \]

Combining all of above, we have
\[ |\mathbb{E} \int_0^1 [A^X u_k(X^x_s) - A^V u_k(x)] ds | \leq C_{\theta, \kappa, \ell} (1 + |x|^3) (1 + \frac{d}{\delta^2}) (1 + \frac{1}{\eta k} + \frac{\eta}{(\eta k)^{\frac{3}{2}}}) \eta^2 e^{-\frac{a_0}{8} \eta k}. \]

\[ \square \]

**APPENDIX B. PROOF OF LEMMA 4.3**

Under Assumption A1, we recall some preliminary of Malliavin calculus and derive standard estimates related to Malliavin calculus and SDE, which will be applied to prove the Lemma 4.3 in Subsection 4.1.

**B.1. Malliavin calculus of SDE (3.2).** For simplicity, denote \( B(x) := -\nabla P(x), \sigma(x) := (\Sigma(x))^{\frac{1}{2}} \). Then SDE (3.2) can be written as the following form:
\[ d\hat{X}_t = B(\hat{X}_t) dt + \sqrt{\eta} \sigma(\hat{X}_t) dB_t, \quad \hat{X}_0 = x. \]

Moreover, Assumption A1 in Subsection 3.1 can be rewritten as the following form:

**Assumption A1**
(i) There exist \( \theta_0 > 0 \) and \( \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \geq 0 \) such that for any \( v_1, v_2, v_3, x \in \mathbb{R}^d \), \( \nabla P(x) \) satisfies
\[ \langle v_1, \nabla v_1 B(x) \rangle \leq -\theta_0 |v_1|^2, \quad |\nabla v_1 \nabla v_2 B(x)| \leq \theta_1 |v_1||v_2|, \]
\[ |\nabla v_1 \nabla v_2 \nabla v_3 B(x)| \leq \theta_2 |v_1||v_2|; \]
and that any \( x, y \in \mathbb{R}^d \), \( \sigma(x) \) satisfies
\[ \| \nabla v_1 \sigma(x) \|_{HS} \leq \theta_3 |v_1|, \quad \| \nabla v_2 \nabla v_3 \sigma(x) \|_{HS} \leq \theta_4 |v_1||v_2|, \]
\[ \| \nabla v_3 \nabla v_2 \nabla v_1 \sigma(x) \|_{HS} \leq \theta_5 |v_1||v_2||v_3|. \]

(ii) There exists \( \delta > 0 \) such that for any \( x \in \mathbb{R}^d \) and non-zero vector \( \xi \in \mathbb{R}^d \), \( \sigma(x) \) satisfies
\[ \xi^T \sigma(x) \xi \geq \delta |\xi|^2. \]

Under the Assumption A1, there exists a unique solution to the SDE (B.1) and the SDE (B.1) has a unique non-degenerate invariant measure (see, e.g., [BR95, Cer96, DPG01, Eva14]).

Next, we briefly recall Bismut’s approach to Malliavin calculus, which is crucial to prove Lemma 4.3. Let \( \nu \in \mathbb{R}^d \) and \( \nabla \nu \hat{X}_t^x \) is defined by
\[ \nabla \nu \hat{X}_t^x = \lim_{\epsilon \to 0} \frac{\hat{X}_{t+\epsilon}^x - \hat{X}_t^x}{\epsilon}, \quad t \geq 0. \]
The above limit exists and satisfies
\begin{equation}
\frac{d}{dt} \nabla_x \dot{X}_t^x = \nabla B(\dot{X}_t^x) \nabla_x \dot{X}_t^x + \sqrt{\eta} \nabla \sigma(\dot{X}_t^x) \nabla_v \dot{X}_t^x dB_t, \quad \nabla_v \dot{X}_t^x = v.
\end{equation}

Then, we use the notations \(J^x_{s,t}\) with \(0 \leq s \leq t < \infty\) for the stochastic flow between time \(s\) and \(t\), that is,
\[\nabla_v \dot{X}_t^x = J^x_{0,t} v.\]

Note that we have the important cocycle property \(J^x_{0,s} J^x_{s,t} = J^x_{0,t}\) for all \(0 \leq s \leq t < \infty\). For a more thorough discussion on stochastic flow, we refer the reader to [Kun84, Kun97, Bah99, HM06] and the references therein.

For \(v_1, v_2 \in \mathbb{R}^d\), we can define \(\nabla_{v_1} v_1 \dot{X}_t^x\), which satisfies
\begin{align}
\frac{d}{dt} \nabla_{v_1} v_1 \dot{X}_t^x &= \nabla B(\dot{X}_t^x) \nabla_{v_1} v_1 \dot{X}_t^x + \nabla^2 B(\dot{X}_t^x) v_2 \dot{X}_t^x \nabla_{v_1} \dot{X}_t^x dt \\
&\quad + \sqrt{\eta} \nabla \sigma(\dot{X}_t^x) \nabla_{v_1} v_1 \dot{X}_t^x dB_t + \sqrt{\eta} \nabla^2 \sigma(\dot{X}_t^x) \nabla_{v_2} \dot{X}_t^x \nabla_{v_1} \dot{X}_t^x dB_t,
\end{align}

with \(\nabla_{v_2} \nabla_{v_1} \dot{X}_t^x = 0\). Moreover, for \(v_1, v_2, v_3 \in \mathbb{R}^d\), we can similarly define \(\nabla_{v_3} \nabla_{v_2} v_1 \dot{X}_t^x\) from above equation. Then, we have the following estimates:

**Lemma B.1.** For all \(x, v, v_1, v_2, v_3 \in \mathbb{R}^d\), as \(\eta \leq \min\{1, \frac{\theta_0}{4\theta_1}\}\), we have
\begin{align}
\mathbb{E}[|\nabla_v \dot{X}_t^x|^8] &\leq e^{-\theta_0 t} |v|^8,
\end{align}
\begin{align}
\mathbb{E}[|\nabla_{v_2} \nabla_{v_1} \dot{X}_t^x|^4] &\leq C_\eta e^{-\frac{\theta_0}{\theta_1} t} |v_1|^4 |v_2|^4,
\end{align}
and
\begin{align}
\mathbb{E}[|\nabla_{v_3} \nabla_{v_2} \nabla_{v_1} \dot{X}_t^x|^2] &\leq C_\eta e^{-\frac{\theta_0}{\theta_1} t} |v_1|^2 |v_2|^2 |v_3|^2.
\end{align}

**Proof.** Recalling (B.7), by Itô’s formula, (B.2) and (B.4), we have
\begin{align}
\frac{d}{ds} \mathbb{E}[|\nabla_v \dot{X}_s^x|^8] &= 8 \mathbb{E}[|\nabla_v \dot{X}_s^x|^6 (\nabla B(\dot{X}_s^x) \nabla_x \dot{X}_s^x, \nabla_v \dot{X}_s^x)] + 4\eta \mathbb{E}[|\nabla_v \dot{X}_s^x|^6 |\nabla \sigma(\dot{X}_s^x) \nabla_v \dot{X}_s^x|^2_{HS}] \\
&\quad + 24\eta \mathbb{E}[|\nabla_v \dot{X}_s^x|^4 |\nabla \sigma(\dot{X}_s^x) \nabla_x \dot{X}_s^x \nabla_v \dot{X}_s^x|^2] \\
\leq &\quad -4(2\theta_0 - 7\theta_1^3 \eta) \mathbb{E}[|\nabla_v \dot{X}_s^x|^8] \leq \theta_0 \mathbb{E}[|\nabla_v \dot{X}_s^x|^8],
\end{align}
where the last inequality is by the fact \(\eta \leq \frac{\theta_0}{4\theta_1}\). This inequality, together with \(\nabla_v \dot{X}_0^x = v\), implies
\[\mathbb{E}[|\nabla_v \dot{X}_t^x|^8] \leq e^{-\theta_0 t} |v|^8.\]

Using Itô’s formula to \(\zeta(t) = \nabla_{v_2} \nabla_{v_1} \dot{X}_t^x\), by (B.8), the Cauchy-Schwarz inequality, and \textbf{Assumption A1}, we have
\begin{align}
\frac{d}{ds} \mathbb{E}[|\zeta(s)|^4] &= 4 \mathbb{E}[|\zeta(s)|^2 (\nabla B(\dot{X}_s^x) \zeta(s) + \nabla^2 B(\dot{X}_s^x) \nabla_{v_2} \dot{X}_s^x \nabla_{v_1} \dot{X}_s^x, \zeta(s))] \\
&\quad + 2\eta \mathbb{E}[|\zeta(s)|^2 |\nabla \sigma(\dot{X}_s^x) \zeta(s) + \nabla^2 \sigma(\dot{X}_s^x) \nabla_{v_2} \dot{X}_s^x \nabla_{v_1} \dot{X}_s^x|^2_{HS}] \\
&\quad + 4\eta \mathbb{E}[|\nabla \sigma(\dot{X}_s^x) \zeta(s) + \nabla^2 \sigma(\dot{X}_s^x) \nabla_{v_2} \dot{X}_s^x \nabla_{v_1} \dot{X}_s^x|^2 |\zeta(s)|^2] \\
&\quad \leq -4\theta_0 \mathbb{E}[|\zeta(s)|^4] + 4 \theta_1 \mathbb{E}[|\nabla_{v_2} \dot{X}_s^x||\nabla_{v_1} \dot{X}_s^x||\zeta(s)|^3] \\
&\quad + 12\eta \mathbb{E}[|\zeta(s)|^2 (|\nabla \sigma(\dot{X}_s^x) \zeta(s)|^2_{HS} + |\nabla^2 \sigma(\dot{X}_s^x) \nabla_{v_2} \dot{X}_s^x \nabla_{v_1} \dot{X}_s^x|^2_{HS})] \\
&\quad \leq -4(\theta_0 - 3\theta_1^3 \eta) \mathbb{E}[|\zeta(s)|^4] + 4 \theta_1 \mathbb{E}[|\nabla_{v_2} \dot{X}_s^x||\nabla_{v_1} \dot{X}_s^x||\zeta(s)|^3] \\
&\quad + 12\eta \mathbb{E}[|\zeta(s)|^2 |\nabla_{v_2} \dot{X}_s^x|^2 |\nabla_{v_1} \dot{X}_s^x|^2].
By Young’s inequality, Cauchy-Schwarz inequality and (B.9), the fact $\eta \leq \min\{1, \frac{\theta_0}{2\tilde{\theta}_2}\}$ implies
\[
\frac{d}{ds} \mathbb{E}|\varsigma(s)|^4 \leq -4\left(\frac{7}{8}\theta_0 - 3\theta_2^2\eta\right)\mathbb{E}|\varsigma(s)|^4 + \frac{123\theta_3^4}{\theta_0^3} \mathbb{E}[[\nabla v_2 \dot{X}_s^x | \nabla v_1 \dot{X}_s^x] \\
+ \frac{12\theta_4^4}{\theta_0} \mathbb{E}[[\nabla v_2 \dot{X}_s^x | \nabla v_1 \dot{X}_s^x] \\
\leq -\frac{\theta_0}{2} \mathbb{E}|\varsigma(s)|^4 + C_9 e^{-\theta_0 t}|v_1|^4|v_2|^4.
\]

This inequality, together with $\phi(0) = 0$, implies
\[
\mathbb{E}|\varsigma(t)|^4 \leq C_9 \int_0^t e^{-\theta_0 s}|v_1|^4|v_2|^4 e^{-\frac{\theta_0}{2}(t-s)}ds \leq C_9 e^{-\frac{\theta_0}{2}|t|}|v_1|^4|v_2|^4.
\]

Furthermore, according to (B.2)-(B.6), a similar calculation implies (B.11).

Next, we use Bismut’s approach to Malliavin calculus for SDE (3.15)([Nor86]). Let $u \in L_{loc}^2([0, \infty) \times (\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$, i.e., $\mathbb{E}\int_0^t |u(s)|^2ds < \infty$ for all $t > 0$. Further assume that $u$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$; i.e., $u(t)$ is $\mathcal{F}_t$ measurable for $t \geq 0$. Define
\[
U_t = \int_0^t u(s)ds, \quad t \geq 0.
\]

For a $t > 0$, let $F_t : C([0, t], \mathbb{R}^d) \to \mathbb{R}$ be a $\mathcal{F}_t$ measurable map. If the following limit exists
\[
D_U F_t(B) = \lim_{\epsilon \to 0} \frac{F_t(B + \epsilon U) - F_t(B)}{\epsilon}
\]
in $L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R})$, then $F_t(B)$ is said to be Malliavin differentiable and $D_U F_t(B)$ is called the Malliavin derivative of $F_t(B)$ in the direction $U$.

Let $F_t(B)$ and $G_t(B)$ both be Malliavin differentiable, then the following product rule holds:
\[
D_U (F_t(B)G_t(B)) = F_t(B)D_U G_t(B) + G_t(B)D_U F_t(B).
\]

When
\[
F_t(B) = \int_0^t \langle a(s), dB(s) \rangle,
\]
where $a(s) = (a_1(s), \cdots, a_d(s))$ is a stochastic process adapted to the filtration $\mathcal{F}_s$ such that $\mathbb{E}\int_0^t |a(s)|^2ds < \infty$ for all $t > 0$, it is easy to check that
\[
D_U F_t(B) = \int_0^t \langle a(s), u(s) \rangle ds + \int_0^t \langle D_U a(s), dB_s \rangle.
\]

Then, we consider the following integration by parts formula, which is called Bismut’s formula. For Malliavin differentiable $F_t(B)$ such that $F_t(B)$, $D_U F_t(B) \in L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R})$, we have
\[
\mathbb{E}[D_U F_t(B)] = \mathbb{E}[F_t(B) \int_0^t \langle u(s), dB_s \rangle].
\]

Let $\phi \in \text{Lip}(1)$ and let $F_t(B) = (F_{t}^{1}(B), \cdots, F_{t}^{d}(B))$ be a $d$-dimensional Malliavin differentiable functional. The following chain rule holds:
\[
D_U \phi(F_t(B)) = \langle \nabla \phi(F_t(B)), D_U F_t(B) \rangle = \sum_{i=1}^{d} \partial_i \phi(F_t(B)) D_U F_{t}^{i}(B).
\]
Now, we come back to the SDE (3.2). Fixing $t \geq 0$ and $x \in \mathbb{R}^d$, the solution $X^x_t$ is a $d$-dimensional functional of Brownian motion $(B_s)_{0 \leq s \leq t}$.

The following Malliavin derivative of $X^x_t$ along the direction $U$ exists in $L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$ and is defined by

$$D_U \hat{X}^x_t(B) = \lim_{\epsilon \to 0} \frac{\hat{X}^x_t(B + \epsilon U) - \hat{X}^x_t(B)}{\epsilon}.$$  

We drop the $B$ in $D_U \hat{X}^x_t(B)$ and write $D_U \hat{X}^x_t = D_U \hat{X}^x_t(B)$ for simplicity. It satisfies the equation

$$dD_U \hat{X}^x_t = \nabla B(\hat{X}^x_t) D_U \hat{X}^x_t \, dt + \eta \frac{1}{2} \nabla \sigma(\hat{X}^x_t) D_U \hat{X}^x_t \, dB_t + \eta^2 \sigma(\hat{X}^x_t) u(t) \, dt,$$

and the equation has a unique solution:

$$D_U \hat{X}^x_t = \int_0^t J^x_{r,t} \eta \frac{1}{2} \sigma(\hat{X}^x_r) u(r) \, dr.$$

Noticing that $\nabla u \hat{X}^x_t = J^x_{0,t} v$, if we take

$$(B.16) \quad u(s) = \frac{1}{t} \eta^{-\frac{1}{2}} \sigma(\hat{X}^x_s)^{-1} \nabla \hat{X}^x_s, \quad 0 \leq s \leq t,$$

then (B.6) and (B.9) imply $u \in L^2_{\text{loc}}([0, \infty) \times (\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$. Since $\nabla u \hat{X}^x_r = J^x_{0,r} v$ and $J^x_{0,r} J^x_{r,t} = J^x_{0,t}$, for all $0 \leq r < t$, we have

$$(B.17) \quad D_U \hat{X}^x_t = \nabla u \hat{X}^x_t$$

and

$$(B.18) \quad D_U \hat{X}^x_s = \frac{s}{t} \nabla u \hat{X}^x_s, \quad 0 \leq s \leq t.$$  

Let $v_1, v_2 \in \mathbb{R}^d$, and define $u_i$ and $U_i$ as (B.16) and (B.12), respectively, for $i = 1, 2$. We can similarly define $D_{U_2} \nabla v_1 \hat{X}^x_s$, which satisfies the following equation: for $s \in [0, t]$,

$$dD_{U_2} \nabla v_1 \hat{X}^x_s = \left[\nabla B(\hat{X}^x_s) D_{U_2} \nabla v_1 \hat{X}^x_s + \nabla^2 B(\hat{X}^x_s) D_{U_2} \hat{X}^x_s \nabla v_1 \hat{X}^x_s + \eta \frac{1}{2} \nabla \sigma(\hat{X}^x_s) \nabla v_1 \hat{X}^x_s u_2(s)\right] \, ds + \eta \frac{1}{2} \left[\nabla \sigma(\hat{X}^x_s) D_{U_2} \nabla v_1 \hat{X}^x_s + \nabla^2 \sigma(\hat{X}^x_s) D_{U_2} \hat{X}^x_s \nabla v_1 \hat{X}^x_s\right] \, dB_s$$

$$= \left[\nabla B(\hat{X}^x_s) D_{U_2} \nabla v_1 \hat{X}^x_s + \frac{s}{t} \nabla^2 B(\hat{X}^x_s) \nabla v_2 \hat{X}^x_s \nabla v_1 \hat{X}^x_s \right] \, dB_s + \frac{1}{t} \nabla \sigma(\hat{X}^x_s) \nabla v_1 \hat{X}^x_s \sigma(\hat{X}^x_s)^{-1} \nabla v_2 \hat{X}^x_s \, ds$$

$$(B.19) \quad + \eta \frac{1}{2} \left[\nabla \sigma(\hat{X}^x_s) D_{U_2} \nabla v_1 \hat{X}^x_s + \frac{s}{t} \nabla^2 \sigma(\hat{X}^x_s) \nabla v_2 \hat{X}^x_s \nabla v_1 \hat{X}^x_s\right] \, dB_s,$$

with $D_{U_2} \nabla v_1 \hat{X}^x_0 = 0$, where the second equality is by (B.16) and (B.18).

For further use, we define

$$I^x_{v_1}(t) := \frac{1}{t} \int_0^t \langle \eta^{-\frac{1}{2}} \sigma(\hat{X}^x_s)^{-1} \nabla v_1 \hat{X}^x_s, dB_s \rangle,$$

$$R^x_{v_1,v_2}(t) := \nabla v_2 \nabla v_1 \hat{X}^x_t - D_{U_2} \nabla v_1 \hat{X}^x_t.$$

Then, we have the following upper bounds on Malliavin derivatives.
Lemma B.2. Let \( v_1, v_2 \in \mathbb{R}^d \) and 

\[
U_{i,s} = \int_0^s u_i(r)dr, \quad 0 \leq s \leq t, 
\]

where \( u_i(r) = \frac{1}{2} \eta^{-\frac{3}{2}} \sigma(\hat{X}_r^x)^{-1} \nabla_{v_i} \hat{X}_r^x \) for \( 0 \leq r \leq t \) and \( i = 1, 2 \). Then, as \( \eta \leq \min\{1, \frac{\theta_0}{489t^2}\} \), we have

\[
\mathbb{E}|D u_2 \nabla_{v_1} \hat{X}_s^x|^2 \leq C_\theta(1 + \frac{d^2}{\delta^2 t}) e^{-\frac{\theta_0}{2t^2}} |v_1|^2 |v_2|^2 
\]

and

\[
\mathbb{E}|D u_2 \nabla_{v_1} \hat{X}_s^x|^4 \leq C_\theta(1 + \frac{d^2}{\delta^2 t}) e^{-\frac{\theta_0}{2t^2}} |v_1|^4 |v_2|^4. 
\]

Proof. We only give the proof of (B.20) and the (B.21) can be proved in the same way.

Writing \( \zeta(s) = D u_2 \nabla_{v_1} \hat{X}_s^x \), by Itô’s formula, (3.3), Cauchy-Schwarz inequality, (B.2) and (B.4), we have

\[
\frac{d}{dr} \mathbb{E} \left| \zeta(r) \right|^2 = 2 \mathbb{E} \left[ \left( \nabla B(\hat{X}_r^x) \zeta(r) + \frac{r}{t} \nabla^2 B(\hat{X}_r^x) \nabla v_2 \hat{X}_r^x \nabla v_1 \hat{X}_r^x, \zeta(r) \right) \right] 
+ 2 \mathbb{E} \left[ \frac{1}{l} \nabla \sigma(\hat{X}_r^x) \nabla v_1 \hat{X}_r^x \sigma(\hat{X}_r^x)^{-1} \nabla v_2 \hat{X}_r^x, \zeta(r) \right] 
+ \eta \mathbb{E} \left[ \left| \nabla \sigma(\hat{X}_r^x) \zeta(r) + \frac{r}{t} \nabla^2 \sigma(\hat{X}_r^x) \nabla v_2 \hat{X}_r^x \nabla v_1 \hat{X}_r^x \right|_{HS}^2 \right] 
\leq -2 \theta_0 \mathbb{E} \left| \zeta(r) \right|^2 + 2 \eta \mathbb{E} \left[ \left| \nabla v_1 \hat{X}_r^x \right| \left| \nabla v_2 \hat{X}_r^x \right| \left| \nabla \sigma(\hat{X}_r^x)^{-1} \right|_{HS} \right] 
+ 2 \eta \mathbb{E} \left[ \theta_{1,3} \left| \zeta(r) \right|^2 + \theta_{1,4} \frac{r^2}{t^2} \left| \nabla v_1 \hat{X}_r^x \right|^2 \left| \nabla v_2 \hat{X}_r^x \right|^2 \right].
\]

By Young’s inequality, we further have

\[
\frac{d}{dr} \mathbb{E} \left| \zeta(r) \right|^2 \leq -2 \theta_0 \mathbb{E} \left| \zeta(r) \right|^2 + 2 \eta \mathbb{E} \left[ \frac{\theta_0}{32 \theta_1} \left| \zeta(r) \right|^2 + \frac{8 \theta_1}{3 \theta_0} \frac{r^2}{t^2} \left| \nabla v_1 \hat{X}_r^x \right|^2 \left| \nabla v_2 \hat{X}_r^x \right|^2 \right] 
+ 2 \eta \mathbb{E} \left[ \theta_{1,3} \left| \zeta(r) \right|^2 + \frac{\theta_{1,4}}{16 \theta_0} \left| \nabla v_1 \hat{X}_r^x \right|^2 \left| \nabla v_2 \hat{X}_r^x \right|^2 \right].
\]

Noticing that \( \eta \leq \frac{10}{16 \theta_0^2} \), by (B.6), Cauchy’s inequality and (B.9), we can get

\[
\frac{d}{dr} \mathbb{E} \left| \zeta(r) \right|^2 \leq -\left( \frac{13 \theta_0}{8} - 2 \theta_0^2 \eta \right) \mathbb{E} \left| \zeta(r) \right|^2 + \left( \frac{16 \theta_1^2}{15 \theta_0} + \frac{16 \theta_1^2}{3 \theta_0} \frac{1}{d^2 t^2} + 2 \theta_1^2 \eta \right) e^{-\frac{\theta_0}{4t^2}} |v_1|^2 |v_2|^2 
\leq -\frac{\theta_0}{4} \mathbb{E} \left| \zeta(r) \right|^2 + C_\theta \left( 1 + \frac{d^2}{\delta^2 t^2} \right) e^{-\frac{\theta_0}{4t^2}} |v_1|^2 |v_2|^2.
\]

This inequality, together with \( \zeta(0) = 0 \), implies

\[
\mathbb{E} \left| \zeta(s) \right|^2 \leq C_\theta \left( 1 + \frac{d^2}{\delta^2 t} \right) e^{-\frac{\theta_0}{2t^2}} |v_1|^2 |v_2|^2.
\]

Based on the results above, we have the following two lemmas:
Lemma B.3. Let unit vectors $v_1, v_2 \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then, for all $\eta \leq \min\{1, \frac{\delta_0}{4s\eta^2}\}$, we have

\begin{equation}
\mathbb{E}|\mathcal{I}_{v_1}^x(t)|^4 \leq \frac{10}{t^2} \frac{d^2}{\delta^4 \eta^2} |v_1|^4, \tag{B.22}
\end{equation}

\begin{equation}
\mathbb{E}|
abla_{v_2} \mathcal{I}_{v_1}^x(t)|^2 \leq C_\eta \frac{1}{t^2} \left(1 + \frac{d^2}{\delta^4}\right) \eta^{-1} |v_1|^2 |v_2|^2, \tag{B.23}
\end{equation}

\begin{equation}
\mathbb{E}|D_{v_2} \mathcal{I}_{v_1}^x(t)|^2 \leq C_\eta \frac{1}{t^2} \left(1 + \frac{d^2}{\delta^4} + \frac{d^2}{\delta^4} \right) |v_1|^2 |v_2|^2. \tag{B.24}
\end{equation}

Proof. By Burkholder’s inequality [Ren08, Theorem 2], (B.6) and (B.9), we have

$$\mathbb{E}|\mathcal{I}_{v_1}^x(t)|^4 = \mathbb{E}\left[\frac{1}{t} \int_0^t \langle \eta^{-\frac{1}{2}} \sigma(\dot{X}_s)^{-1} \nabla_{v_1} \dot{X}_s^x, dB_s \rangle \right]^4 \leq \frac{4\sqrt{2}}{t^4} \eta^{-2} \mathbb{E}\left( \int_0^t |\sigma(\dot{X}_s)^{-1} \nabla_{v_1} \dot{X}_s^x|^2 ds \right)^2 \leq \frac{10}{t^4} \frac{d^2}{\delta^4 \eta^2} \mathbb{E}\left( \int_0^t |\nabla_{v_1} \dot{X}_s^x|^2 ds \right)^2 \leq \frac{10}{t^3} \frac{d^2}{\delta^4 \eta^2} \int_0^t \mathbb{E}|\nabla_{v_1} \dot{X}_s^x|^4 ds \leq \frac{10}{t^2} \frac{d^2}{\delta^4 \eta^2} |v_1|^4.$$

For (B.23), the definition of $\mathcal{I}_{v_1}^x(t)$ yields

\[ \nabla_{v_2} \mathcal{I}_{v_1}^x(t) = \frac{1}{t} \eta^{-\frac{1}{2}} \int_0^t \left\langle -\sigma(X_s)^{-1} \nabla_{v_2} \sigma(X_s)^{-1} \nabla_{v_1} X_s^x + \sigma(X_s)^{-1} \nabla_{v_2} \nabla_{v_1} X_s^x, dB_s \right\rangle. \]

Then, by Itô isometry, (B.4) and (B.6), we have

$$\mathbb{E}|\nabla_{v_2} \mathcal{I}_{v_1}^x(t)|^2 \leq \frac{2}{t^2} \eta^{-1} \int_0^t \mathbb{E}[\theta_2^2 |\sigma(\dot{X}_s)^{-1}|^2 |\nabla_{v_1} \dot{X}_s^x|^2 |\nabla_{v_1} \dot{X}_s^x|^2 + |\sigma(\dot{X}_s)^{-1}|^2 |\nabla_{v_2} \nabla_{v_1} \dot{X}_s^x|^2] ds \leq \frac{2}{t^2} \eta^{-1} \int_0^t \mathbb{E}[\theta_2^2 \frac{d^2}{\delta^4} |\nabla_{v_1} \dot{X}_s^x|^2 |\nabla_{v_1} \dot{X}_s^x|^2 + \frac{d^2}{\delta^4} |\nabla_{v_2} \nabla_{v_1} \dot{X}_s^x|^2] ds.$$

By Cauchy-Schwarz inequality, (B.9) and (B.10), we further have

\[ \mathbb{E}|\nabla_{v_2} \mathcal{I}_{v_1}^x(t)|^2 \leq C_\eta \frac{1}{t^2} \eta^{-1} \int_0^t \left(1 + \frac{d^2}{\delta^4}\right) ds |v_1|^2 |v_2|^2 \leq C_\eta \frac{1}{t^2} \left(1 + \frac{d^2}{\delta^4}\right) \eta^{-1} |v_1|^2 |v_2|^2. \]

Recall (B.16) and (B.18), it is easy to see that $D_{v_2} \mathcal{I}_{v_1}^x(t)$ can be computed by (B.14) as

\[ D_{v_2} \mathcal{I}_{v_1}^x(t) = \frac{\eta^{-1}}{t^2} \int_0^t \langle \sigma(\dot{X}_s)^{-1} \nabla_{v_1} \dot{X}_s^x, \sigma(\dot{X}_s)^{-1} \nabla_{v_2} \dot{X}_s^x \rangle ds + \frac{\eta^{-1}}{t^2} \int_0^t \langle \sigma(\dot{X}_s)^{-1} D_{v_2} \nabla_{v_1} \dot{X}_s^x, dB_s \rangle ds \]

\[ - \frac{1}{t^2} \eta^{-\frac{1}{2}} \int_0^t \langle \sigma(\dot{X}_s)^{-1} \nabla \sigma(\dot{X}_s) \sigma(\dot{X}_s)^{-1} \frac{X}{t} \nabla_{v_2} \dot{X}_s^x \nabla_{v_1} \dot{X}_s^x, dB_s \rangle. \]

Then, by the Cauchy-Schwarz inequality, Itô isometry, (B.4), (B.6), (B.9) and (B.20), we have

\[ \mathbb{E}|D_{v_2} \mathcal{I}_{v_1}^x(t)|^2 \leq \frac{3}{t^2} \int_0^t \mathbb{E}[\sigma(\dot{X}_s)^{-1} |\nabla_{v_1} \dot{X}_s^x|^2 |\nabla_{v_2} \dot{X}_s^x|^2] ds + \frac{3}{t^2} \int_0^t \mathbb{E}[\sigma(\dot{X}_s)^{-1} |\nabla_{v_2} \dot{X}_s^x|^2 |\nabla_{v_1} \dot{X}_s^x|^2] ds \]

\[ + \theta_2^2 \frac{3}{t^2} \eta^{-1} \int_0^t \mathbb{E}[\sigma(\dot{X}_s)^{-1} |\nabla_{v_2} \dot{X}_s^x|^2 |\nabla_{v_1} \dot{X}_s^x|^2] ds. \]
\begin{equation}
\leq \frac{3}{t^2\eta} \left( \frac{1}{t\eta} + \theta^2 \right) \int_0^t \frac{d^2}{d\tau^2} \mathbb{E}[|\nabla_{v_1} \hat{X}_s|^2 | \nabla_{v_2} \hat{X}_s|^2] \, ds + \frac{3}{t^2\eta} \int_0^t \frac{d}{d\tau} \mathbb{E}[|D_{U_2} \nabla_{v_1} \hat{X}_s|^2] \, ds
\leq C_\eta \frac{1}{t\eta} \left( 1 + \frac{d^2}{\delta^2} + \frac{d^2}{\delta^2} \frac{1}{t\eta} \right) |v_1|^2 |v_2|^2. \tag{B.27}
\end{equation}

Furthermore, let $v_1, v_2, v_3 \in \mathbb{R}^d$, and define $u_i$ and $U_i$ as (B.16) and (B.12), respectively, for $i = 1, 2, 3$. From (B.19), we can similarly define $\nabla_v D_{U_2} \nabla_{v_1} \hat{X}_s$, which satisfies the following equation: for $s \in [0, t]$,
\begin{align*}
&d \nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s \\
&= [\nabla^2 B(\hat{X}_s) \nabla_{v_3} \hat{X}_s D_{U_2} \nabla_{v_1} \hat{X}_s + \nabla B(\hat{X}_s) \nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s \\
&+ \frac{s}{t} \nabla^3 B(\hat{X}_s) \nabla_{v_3} \hat{X}_s \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s + \frac{s}{t} \nabla^2 B(\hat{X}_s) \nabla_{v_3} \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s \\
&+ \frac{s}{t} \nabla^2 B(\hat{X}_s) \nabla_{v_3} \hat{X}_s \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s + \frac{1}{t} \nabla^3 \sigma(\hat{X}_s) \nabla_{v_3} \hat{X}_s \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s \sigma(\hat{X}_s)^{-1} \nabla_{v_2} \hat{X}_s] \\
&+ \eta^2 \left[ \nabla^2 \sigma(\hat{X}_s) \nabla_{v_3} \hat{X}_s D_{U_2} \nabla_{v_1} \hat{X}_s + \nabla \sigma(\hat{X}_s) \nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s \\
&+ \frac{s}{t} \nabla^3 \sigma(\hat{X}_s) \nabla_{v_3} \hat{X}_s \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s + \frac{s}{t} \nabla^2 \sigma(\hat{X}_s) \nabla_{v_3} \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s + \frac{s}{t} \nabla^2 \sigma(\hat{X}_s) \nabla_{v_3} \hat{X}_s \nabla_{v_2} \hat{X}_s \nabla_{v_1} \hat{X}_s \right] ds \\
&+ \eta \left[ \nabla^2 \sigma(\hat{X}_s) \nabla_{v_3} \hat{X}_s D_{U_2} \nabla_{v_1} \hat{X}_s + \nabla \sigma(\hat{X}_s) \nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s \right] ds \\
&= 0.
\end{align*}

Moreover, from (B.8) and (B.19), we can similarly define $D_{U_3} \nabla_{v_2} \nabla_{v_1} \hat{X}_s$ and $D_{U_3} D_{U_2} \nabla_{v_1} \hat{X}_s$, respectively.

Then, we have the following upper bounds on Malliavin derivatives.

\textbf{Lemma B.4. Let $v_i \in \mathbb{R}^d$ for $i = 1, 2, 3$, and let}
\begin{equation}
U_{i,s} = \int_0^s u_i(r) \, dr, \quad 0 \leq s \leq t,
\end{equation}
\text{where $u_i(r) = \frac{1}{t} \eta^{-\frac{1}{2}} \sigma(\hat{X}_r)^{-1} \nabla_{v_i} \hat{X}_r$ for $0 \leq r \leq t$. Then, for all $\eta \leq \min\{1, \frac{176}{\log q}\}$, we have}

\begin{align}
\mathbb{E}[|\nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s|^2] &\leq C_\theta \left( 1 + \frac{d^2}{\delta^4} \right) \left( 1 + \frac{1}{t} \right) e^{\frac{\eta^4}{4}} |v_1|^2 |v_2|^2 |v_3|^2, \tag{B.25} \\
\mathbb{E}[|D_{U_3} \nabla_{v_2} \nabla_{v_1} \hat{X}_s|^2] &\leq C_\theta \left( 1 + \frac{d^2}{\delta^4} \right) \left( 1 + \frac{1}{t} \right) e^{\frac{\eta^4}{4}} |v_1|^2 |v_2|^2 |v_3|^2, \tag{B.26} \\
\mathbb{E}[|D_{U_3} D_{U_2} \nabla_{v_1} \hat{X}_s|^2] &\leq C_\theta \left( 1 + \frac{d^2}{\delta^4} \right) \left( 1 + \frac{1}{t} \right) e^{\frac{\eta^4}{4}} |v_1|^2 |v_2|^2 |v_3|^2. \tag{B.27}
\end{align}

\textbf{Proof. Writing $\tau_1(s) = \nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_s$, by Itô’s formula, we have}
\begin{equation}
\frac{d}{dr} \mathbb{E}[|\tau_1(r)|^2] = 2 \mathbb{E}[\nabla^2 B(\hat{X}_r) \nabla_{v_3} \hat{X}_r D_{U_2} \nabla_{v_1} \hat{X}_r + \nabla B(\hat{X}_r) \tau_1(r) \\
+ \frac{1}{t} \nabla^3 B(\hat{X}_r) \nabla_{v_3} \hat{X}_r \nabla_{v_2} \hat{X}_r \nabla_{v_1} \hat{X}_r + \frac{1}{t} \nabla^2 B(\hat{X}_r) \nabla_{v_3} \nabla_{v_2} \hat{X}_r \nabla_{v_1} \hat{X}_r]
\end{equation}
\[ \begin{align*}
+ \frac{\tau}{t} \nabla^2 B(\hat{X}_r) \nabla_{v_2} \hat{X}_r \nabla_{v_3} \nabla_{v_4} \hat{X}_r^x + \frac{1}{t} \nabla^2 \sigma(\hat{X}_r^x) \nabla_{v_3} \hat{X}_r^x \nabla_{v_4} \hat{X}_r^x \sigma(\hat{X}_r^x)^{-1} \nabla_{v_2} \hat{X}_r^x \\
+ \frac{1}{t} \nabla \sigma(\hat{X}_r^x) \nabla_{v_3} \hat{X}_r^x \sigma(\hat{X}_r^x)^{-1} \nabla_{v_2} \hat{X}_r^x + \nabla_{v_4} \hat{X}_r^x \sigma(\hat{X}_r^x)^{-1} \nabla_{v_3} \nabla_{v_2} \hat{X}_r^x \\
- \frac{1}{t} \nabla \sigma(\hat{X}_r^x) \nabla_{v_3} \hat{X}_r^x \sigma(\hat{X}_r^x)^{-1} \nabla_{v_2} \hat{X}_r^x, \tau_1(r) \rangle \\
+ \eta \mathbb{E} \| \nabla^2 \sigma(\hat{X}_r^x) \nabla_{v_3} \hat{X}_r^x \nabla_{v_4} \hat{X}_r^x \nabla_{v_2} \hat{X}_r^x + \nabla \sigma(\hat{X}_r^x) \nabla_{v_4} \hat{X}_r^x \nabla_{v_3} \nabla_{v_1} \hat{X}_r^x \\
+ \frac{\tau}{t} \nabla^3 \sigma(\hat{X}_r^x) \nabla_{v_3} \hat{X}_r^x \nabla_{v_4} \hat{X}_r^x \nabla_{v_1} \hat{X}_r^x + \frac{\tau}{t} \nabla^2 \sigma(\hat{X}_r^x) \nabla_{v_3} \nabla_{v_2} \hat{X}_r^x \nabla_{v_1} \hat{X}_r^x \|^2_{\text{HS}}.
\end{align*} \]

It follows from Assumption A1 and the Cauchy-Schwarz inequality that

\[
\frac{d}{dr} \mathbb{E} |\tau_1(r)|^2 \\
\leq -2 \theta_0 \mathbb{E} |\tau_1(r)|^2 + 2 \theta_1 \mathbb{E} \left[ |\nabla_{v_3} \hat{X}_r^x| |D_{U_2} \nabla_{v_4} \hat{X}_r^x| |\tau_1(r)| \right] \\
+ 2 \mathbb{E} \left[ \left( \frac{\theta_2}{2} |\nabla_{v_3} \hat{X}_r^x| |D_{U_2} \nabla_{v_4} \hat{X}_r^x| |\tau_1(r)| + \theta_1 |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| |\nabla_{v_4} \hat{X}_r^x| |\tau_1(r)| \right) \right] \\
+ 2 \mathbb{E} \left[ \left( \theta_3 |\nabla_{v_3} \nabla_{v_4} \hat{X}_r^x| + \frac{\theta_1}{t} |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| |\sigma(\hat{X}_r^x)^{-1}|_{\text{HS}} \right) |\nabla_{v_2} \hat{X}_r^x| |\tau_1(r)| \right] \\
+ 2 \frac{\theta_3}{4} \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \nabla_{v_1} \hat{X}_r^x| + |\nabla_{v_1} \hat{X}_r^x| |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| \right) |\sigma(\hat{X}_r^x)^{-1}|_{\text{HS}} |\tau_1(r)| \right] \\
+ 2 \frac{\theta_3}{4} \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \hat{X}_r^x| |\nabla_{v_4} \hat{X}_r^x| |\sigma(\hat{X}_r^x)^{-1}|_{\text{HS}} \right) |\nabla_{v_3} \hat{X}_r^x| |\nabla_{v_2} \hat{X}_r^x| |\tau_1(r)| \right] \\
+ 5 \theta_3^2 \eta \mathbb{E} \left[ |\nabla_{v_3} \hat{X}_r^x| |D_{U_2} \nabla_{v_1} \hat{X}_r^x| |\tau_1(r)|^2 \right] \\
+ 5 \theta_3^2 \eta \mathbb{E} \left[ |\nabla_{v_3} \hat{X}_r^x|^2 |D_{U_2} \nabla_{v_1} \hat{X}_r^x|^2 \right] \\
+ 5 \eta \mathbb{E} \left[ \theta_3^2 |\tau_1(r)|^2 + \theta_2^2 |\nabla_{v_3} \hat{X}_r^x|^2 |\nabla_{v_2} \hat{X}_r^x|^2 |\nabla_{v_1} \hat{X}_r^x|^2 \right].
\]

Moreover, by (B.6) and Young’s inequality, we have

\[
\frac{d}{dr} \mathbb{E} |\tau_1(r)|^2 \\
\leq - (2 \theta_0 - 5 \theta_3^2 \eta) \mathbb{E} |\tau_1(r)|^2 + \mathbb{E} \left[ \theta_0 \left( \theta_3 \frac{\theta_2}{2} |\nabla_{v_3} \hat{X}_r^x| |D_{U_2} \nabla_{v_4} \hat{X}_r^x| + \theta_1 |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| |\nabla_{v_4} \hat{X}_r^x| \right) \right] \\
+ \mathbb{E} \left[ \left( \theta_3 \frac{\theta_1}{t} |\nabla_{v_3} \nabla_{v_4} \hat{X}_r^x| + \theta_1 \sigma(\hat{X}_r^x)^{-1} |\nabla_{v_2} \hat{X}_r^x| \right) |\nabla_{v_2} \hat{X}_r^x| \right] \\
+ \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \nabla_{v_1} \hat{X}_r^x| + |\nabla_{v_1} \hat{X}_r^x| |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| \right) \right] \\
+ \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \hat{X}_r^x| |\nabla_{v_4} \hat{X}_r^x| \right) \right] \\
+ 5 \theta_3^2 \eta \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \nabla_{v_1} \hat{X}_r^x| + |\nabla_{v_1} \hat{X}_r^x| |\nabla_{v_3} \nabla_{v_2} \hat{X}_r^x| \right) \right] \\
+ 5 \theta_3^2 \eta \mathbb{E} \left[ \left( \frac{\theta_1}{2} |\nabla_{v_3} \hat{X}_r^x| |\nabla_{v_4} \hat{X}_r^x| \right) \right],
\]

By the fact \( \eta \leq \min \{1, \frac{17 \theta_0}{60 \theta_3^2} \} \), the Cauchy-Schwarz inequality, (B.9), (B.21) and (B.10), we further have

\[
\frac{d}{dr} \mathbb{E} |\tau_1(r)|^2 \leq - \frac{\theta_0}{4} \mathbb{E} |\tau_1(r)|^2 + C_0 \left( 1 + \frac{d^2}{\delta^4} \right) (1 + \frac{1}{t^2}) e^{\frac{-\theta_0}{4} |v_1|^2 |v_2|^2 |v_3|^2}.
\]
This inequality, together with $\tau_1(0) = 0$, implies
\[
\mathbb{E}|\tau_1(s)|^2 \leq C_\theta (1 + \frac{d^2}{\delta^2}) (1 + \frac{1}{l^2}) |v_1|^2 |v_2|^2 |v_3|^2 \int_0^s e^{-\frac{t_0}{s}} e^{-\frac{\theta_1(x-s)}{s}} dr
\leq C_\theta (1 + \frac{d^2}{\delta^2}) (1 + \frac{1}{l^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2 |v_3|^2.
\]

With the above results, we have the following estimates:

**Lemma B.5.** Let $v_1, v_2, v_3 \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then, as $\eta < \min \{1, \frac{\theta_1}{4\delta^2} \}$, we have

\begin{align}
|{\mathcal R}_{\nu_1, \nu_2}^x(t)|^2 &\leq C_\theta (1 + \frac{d^2}{\delta^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2, \\
|\nabla_{v_3} {\mathcal R}_{\nu_1, \nu_2}^x(t)|^2 &\leq C_\theta (1 + \frac{d^2}{\delta^2}) (1 + \frac{1}{l^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2 |v_3|^2.
\end{align}

**Proof.** By the Cauchy-Schwarz inequality, (B.10) and (B.20), we have

\[
\mathbb{E}|{\mathcal R}_{\nu_1, \nu_2}^x(t)|^2 \leq 2\mathbb{E}|\nabla_{v_3} \nabla_{v_1} \hat{X}_t^x|^2 + 2\mathbb{E}|D_{U_2} \nabla_{v_1} \hat{X}_t^x|^2 \leq C_\theta (1 + \frac{d}{\delta^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2.
\]

By (B.11) and (B.25), we have

\[
\mathbb{E}|\nabla_{v_3} R_{\nu_1, \nu_2}^x(t)|^2 \leq 2\mathbb{E}|\nabla_{v_3} \nabla_{v_1} \hat{X}_t^x|^2 + 2\mathbb{E}|\nabla_{v_3} D_{U_2} \nabla_{v_1} \hat{X}_t^x|^2 \leq C_\theta (1 + \frac{d^2}{\delta^2}) (1 + \frac{1}{l^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2 |v_3|^2.
\]

By (B.26) and (B.27), we have

\[
\mathbb{E}|D_{U_3} R_{\nu_1, \nu_2}^x(t)|^2 \leq 2\mathbb{E}|D_{U_3} \nabla_{v_2} \nabla_{v_1} \hat{X}_t^x|^2 + 2\mathbb{E}|D_{U_3} D_{U_2} \nabla_{v_1} \hat{X}_t^x|^2 \leq C_\theta (1 + \frac{d^2}{\delta^2}) (1 + \frac{1}{l^2}) e^{-\frac{\theta_1}{s}} |v_1|^2 |v_2|^2 |v_3|^2.
\]

\[\square\]

**B.2. Proof of Lemma 4.3.** Recall $P_t h(x) = \mathbb{E}[h(\hat{X}^x_t)]$ for $h \in Lip(1)$, by Lebesgue’s dominated convergence theorem, the Cauchy-Schwarz inequality and (B.9), we have

\[
|\nabla_x \mathbb{E}[h(\hat{X}^x_t)]| = |\mathbb{E}[\nabla (h(\hat{X}^x_t)) \nabla_x \hat{X}^x_t]| \leq ||\nabla h|| \mathbb{E}|\nabla_x \hat{X}^x_t| \leq e^{-\frac{\theta_1}{s}} |v|,
\]

(4.4) is proved.

Denote

\[
h_\epsilon(x) = \int_{\mathbb{R}^d} f_\epsilon(y) h(x-y) dy,
\]

with $\epsilon > 0$ and $f_\epsilon$ is the density of the normal distribution $N(0, \epsilon^2 I_d)$. It is easy to see that $h_\epsilon$ is smooth, $\lim_{\epsilon \to 0} h_\epsilon(x) = h(x)$, $\lim_{\epsilon \to 0} \nabla h_\epsilon(x) = \nabla h(x)$ and $|h_\epsilon(x)| \leq C(1 + |x|)$ for all $x \in \mathbb{R}^d$ and some $C > 0$. Moreover, $||\nabla h_\epsilon|| \leq ||\nabla h|| \leq 1$. Then, by Lebesgue’s dominated convergence theorem, we have

\[
\nabla_{v_2} \nabla_{v_1} \mathbb{E}[h_\epsilon(\hat{X}^x_t)] = \mathbb{E}[\nabla^2 h_\epsilon(\hat{X}^x_t) \nabla_{v_2} \hat{X}^x_t \nabla_{v_1} \hat{X}^x_t] + \mathbb{E}[\nabla h_\epsilon(\hat{X}^x_t) \nabla_{v_2} \nabla_{v_1} \hat{X}^x_t],
\]

by (B.16) and (B.17), we further have

\[
\mathbb{E}[\nabla^2 h_\epsilon(\hat{X}^x_t) \nabla_{v_2} \hat{X}^x_t \nabla_{v_1} \hat{X}^x_t] = \mathbb{E}[\nabla^2 h_\epsilon(\hat{X}^x_t) D_{U_2} \hat{X}^x_t \nabla_{v_1} \hat{X}^x_t]
\]
where the last equality is by Bismut’s formula (B.15). These imply

\begin{equation}
(\text{B.22}) - (\text{B.24}) \text{ and } (\text{B.28}) - (\text{B.30}).
\end{equation}

Therefore, by Lebesgue’s dominated convergence theorem, the Cauchy-Schwarz inequality, (B.9), (B.22) and (B.28), we have

\begin{equation}
|\nabla \nu_2 \nabla \nu_1 \mathbb{E}[h(\hat{\mathcal{F}}^x)]| = |\lim_{\varepsilon \rightarrow 0} \nabla \nu_2 \nabla \nu_1 \mathbb{E}[h_\varepsilon(\hat{\mathcal{F}}^x)]| \leq \mathbb{E}[|\nabla \nu_1 \hat{\mathcal{F}}^x_2(t)|] + \mathbb{E}[|\mathcal{R}^x_{\nu_1,\nu_2}(t)|] \\
\leq C_\theta (1 + \sqrt{\frac{\delta}{\delta}})(1 + \frac{1}{\sqrt{\eta}}) e^{-\frac{\eta t}{\delta}} |\nu_1||\nu_2|,
\end{equation}

(4.5) is proved.

By (B.32) and Lebesgue’s dominated convergence theorem, we have

\begin{align*}
\nabla \nu_2 \nabla \nu_1 \mathbb{E}[h(\hat{\mathcal{F}}^x)] & = \nabla \nu_2 \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) h_\varepsilon(\hat{\mathcal{F}}^x)] + \nabla \nu_1 \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \mathcal{R}^x_{\nu_1,\nu_2}(t)] \\
& = \mathbb{E}[\nabla^2 h_\varepsilon(\hat{\mathcal{F}}^x) \nabla \nu_2 \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] + \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \mathcal{R}^x_{\nu_1,\nu_2}(t)] \\
& \quad + \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \mathcal{R}^x_{\nu_1,\nu_2}(t)] + \mathbb{E}[\nabla^2 h_\varepsilon(\hat{\mathcal{F}}^x) \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] \\
& \quad + \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \mathcal{R}^x_{\nu_1,\nu_2}(t)].
\end{align*}

by (B.16), (B.17) and (B.15), we further have

\begin{align*}
\mathbb{E}[\nabla^2 h_\varepsilon(\hat{\mathcal{F}}^x) \nabla \nu_2 \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] & = \mathbb{E}[\nabla^2 h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \hat{\mathcal{F}}^x_2(t)] \\
& = \mathbb{E}[D \nu_3(\nabla h_\varepsilon(\hat{\mathcal{F}}^x)) \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] \\
& = \mathbb{E}[D \nu_3(\nabla h_\varepsilon(\hat{\mathcal{F}}^x)) \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] - \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \hat{\mathcal{F}}^x_2(t)] \\
& \quad - \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \hat{\mathcal{F}}^x_2(t)] \\
& = \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] - \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \hat{\mathcal{F}}^x_2(t)] \\
& \quad - \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \hat{\mathcal{F}}^x_2(t)]
\end{align*}

and similarly

\begin{equation}
\mathbb{E}[\nabla^2 h_\varepsilon(\hat{\mathcal{F}}^x) \nabla \nu_2 \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] = \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) \mathcal{R}^x_{\nu_1,\nu_2}(t) \mathcal{I}^x_3(t)] - \mathbb{E}[\nabla h_\varepsilon(\hat{\mathcal{F}}^x) D \nu_3 \mathcal{R}^x_{\nu_1,\nu_2}(t)].
\end{equation}

Hence, by Lebesgue’s dominated convergence theorem, we have

\begin{equation}
|\nabla \nu_2 \nabla \nu_1 \nabla \nu_1 \mathbb{E}[h(\hat{\mathcal{F}}^x)]| = |\lim_{\varepsilon \rightarrow 0} \nabla \nu_2 \nabla \nu_1 \nabla \nu_1 \mathbb{E}[h_\varepsilon(\hat{\mathcal{F}}^x)]| \\
\leq \mathbb{E}[|\nabla \nu_1 \hat{\mathcal{F}}^x_2(t) \mathcal{I}^x_3(t)|] + \mathbb{E}[D \nu_3 \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] \\
+ \mathbb{E}[\nabla \nu_2 \hat{\mathcal{F}}^x_2 D \nu_3 \mathcal{I}^x_3(t)] + \mathbb{E}[\nabla \nu_1 \nabla \nu_1 \hat{\mathcal{F}}^x_2(t)] \\
+ \mathbb{E}[\nabla \nu_1 \hat{\mathcal{F}}^x_2 \nabla \nu_1 \mathcal{I}^x_3(t)] + \mathbb{E}[\mathcal{R}^x_{\nu_1,\nu_2}(t) \mathcal{I}^x_3(t)] \\
+ \mathbb{E}[D \nu_3 \mathcal{R}^x_{\nu_1,\nu_2}(t)] + \mathbb{E}[\nabla \nu_1 \mathcal{R}^x_{\nu_1,\nu_2}(t)].
\end{equation}

Then, the desired result follows from the Cauchy-Schwarz inequality, (B.9), (B.10), (B.20), (B.22)-(B.24) and (B.28)-(B.30).
APPENDIX C. PROOFS OF LEMMAS IN SUBSECTION 4.2

C.1. Proof of Lemma 4.5. Define $V(y):=(1+|y|^2)^{1/2}$, then
\[
\nabla V(y) = \frac{x}{V(y)}, \quad \nabla^2 V(y) = \frac{V(y)^2 I_d - yy^T}{V(x)^3},
\]
where $I_d$ denotes the $d \times d$ identity matrix. Hence, for any $y \in \mathbb{R}^d$
\[
|\nabla V(y)| \leq 1, \quad \|\nabla^2 V(y)\|_{HS} \leq 2\sqrt{d}.
\]
Notice that $|y| \leq V(y)$ and by (3.16), for any $k \geq 0$, we have
\[
V(\tilde{Y}_{k+1}) = V\left(\tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k + \frac{\eta^2}{\sigma} \tilde{Z}_{k+1}\right)
\]
\[
= V\left(\tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k\right) + V\left(\tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k + \frac{\eta^2}{\sigma} \tilde{Z}_{k+1}\right) - V\left(\tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k\right)
\]
\[
= V\left(\tilde{Y}_k\right) - \int_0^{\frac{\eta}{\alpha}} \langle \nabla V(\tilde{Y}_k - r\tilde{Y}_k), \tilde{Y}_k \rangle dr
\]
\[
+ \int_0^{\frac{\eta}{\alpha}} \langle \nabla V\left(\tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k + r\tilde{Z}_{k+1}\right), \tilde{Z}_{k+1}\rangle dr.
\]
Recall $\nabla V(x) = \frac{x}{V(x)}$, we have
\[
- \int_0^{\frac{\eta}{\alpha}} \langle \nabla V(\tilde{Y}_k - r\tilde{Y}_k), \tilde{Y}_k \rangle dr = - \int_0^{\frac{\eta}{\alpha}} \frac{\langle \tilde{Y}_k - r\tilde{Y}_k, \tilde{Y}_k \rangle}{\nabla V(\tilde{Y}_k - r\tilde{Y}_k)} dr
\]
\[
\leq - \int_0^{\frac{\eta}{\alpha}} \frac{(1-r)|\tilde{Y}_k|^2}{(|\tilde{Y}_k|^2 + 1)^2} dr
\]
\[
= - \int_0^{\frac{\eta}{\alpha}} (1-r) V\left(\tilde{Y}_k\right) dr + \int_0^{\frac{\eta}{\alpha}} \frac{1-r}{(|\tilde{Y}_k|^2 + 1)^2} dr
\]
\[
\leq - \left(\frac{\eta}{\alpha} - \frac{\eta^2}{2\alpha^2}\right) V\left(\tilde{Y}_k\right) + \frac{\eta}{\alpha} \leq - \frac{\eta}{2\alpha} V\left(\tilde{Y}_k\right) + \frac{\eta}{\alpha},
\]
where the last inequality is by the fact $\eta \leq 1 < \alpha$. In addition, notice that $\tilde{Z}_{k+1}$ is independent of $\tilde{Y}_k$, conditioned on $\tilde{Y}_k = y$, we have
\[
\mathbb{E}\left[\langle \nabla V\left((1 - \frac{\eta}{\alpha}) y + r\tilde{Z}_{k+1}\right), \tilde{Z}_{k+1}\rangle\right]
\]
\[
= \frac{\alpha}{V(S^{d-1})} \int_{|z| > 1} \frac{\langle \nabla V\left((1 - \frac{\eta}{\alpha}) y + rz\right), z\rangle - \langle \nabla V\left((1 - \frac{\eta}{\alpha}) y\right), z\rangle 1_{(0,\eta^{-\frac{1}{d}})}(|z|)}{|z|^{a+d}} dz
\]
\[
= \frac{\alpha}{V(S^{d-1})} \int_{1 < |z| < \eta^{-\frac{1}{d}}} \int_0^r \frac{\langle \nabla^2 V\left((1 - \frac{\eta}{\alpha}) y + sz\right), zz^T\rangle_{HS}}{|z|^{a+d}} ds dz
\]
\[
+ \frac{\alpha}{V(S^{d-1})} \int_{|z| = \eta^{-\frac{1}{d}}} \frac{\langle \nabla V\left((1 - \frac{\eta}{\alpha}) y + rz\right), z\rangle}{|z|^{a+d}} dz,
\]
which implies
\[
\mathbb{E}\left[\langle \nabla V\left((1 - \frac{\eta}{\alpha}) y + r\tilde{Z}_{k+1}\right), \tilde{Z}_{k+1}\rangle\right]
\]
which immediately implies

\[ \text{Hence, we have} \]

\[ \left| \mathbb{E} \left[ \int_{|z|>\frac{\eta}{2}} \frac{2\sqrt{d}|z|^2}{|z|^\alpha d^2} + \int_{|z|>\frac{\eta}{2}} \frac{|z|}{|z|^\alpha d^2} \right] \right| \leq \frac{2\alpha \sqrt{d}}{2-\alpha} r^\frac{\alpha - 2}{\alpha} + \frac{\alpha}{\alpha - 1} \eta^\frac{\alpha - 1}{\alpha}. \]

Hence, we have

\[ \left| \mathbb{E} \left[ \int_{0}^{\eta} \langle \nabla V \left( \tilde{Y}_k - \frac{\eta}{\alpha} \tilde{Y}_k + r \tilde{Z}_{k+1} \right), \tilde{Z}_{k+1} \rangle \right] \right| \leq \int_{0}^{\eta} \left( \frac{2\alpha \sqrt{d}}{2-\alpha} r^\frac{\alpha - 2}{\alpha} + \frac{\alpha}{\alpha - 1} \eta^\frac{\alpha - 1}{\alpha} \right) \right| dr = \left( \frac{\alpha \sqrt{d}}{(2-\alpha)^2} + \frac{\alpha}{(\alpha - 1)\sigma^2} \right) \eta. \]

Therefore, we have

\[ \mathbb{E}[V(\tilde{Y}_{k+1})] \leq \left( 1 - \frac{\eta}{2\alpha} \right)^{k+1} |x| + \left( \frac{\alpha \sqrt{d}}{(2-\alpha)^2} + \frac{\alpha}{(\alpha - 1)\sigma^2} + 1 \right) \eta \sum_{j=0}^{k} \left( 1 - \frac{\eta}{2\alpha} \right)^j \]

\[ \leq |x| + \frac{2\alpha^2 \sqrt{d}}{(2-\alpha)^2} + \frac{2\alpha^2}{(\alpha - 1)\sigma^2} + 2. \]

Hence, we have

\[ \mathbb{E}[|\tilde{Y}_k^x|] \leq \mathbb{E}[V(\tilde{Y}_k)] \leq C_{\alpha,d}(1 + |x|). \]

The proof is complete. \( \square \)

C.2. Proof of Lemma 4.6. Let \( \mathcal{L}^\alpha \) be the generator that corresponds to the process \( \tilde{X}_t \). Then, it is easy to check that for any \( f \in C^2_b(\mathbb{R}^d, \mathbb{R}^d) \),

\[ \mathcal{L}^\alpha f(x) = -\frac{1}{\alpha} \langle x, \nabla f(x) \rangle + \Delta^{\alpha/2} f(x). \]

Following [Wan13, Section 2], we know that the extended domain of the operator \( \mathcal{L}^\alpha \) is given as follows:

\[ \mathcal{D}(\mathcal{L}^\alpha) : \{ f \in C^2(\mathbb{R}^d, \mathbb{R}^d) : \int_{|z| \geq 1} \frac{f(x + z) - f(x)}{|z|^\alpha + d} dz < \infty \}. \]

Taking a function \( V \in C^2(\mathbb{R}^d, \mathbb{R}^d) \) such that \( V \geq 1 \), for \( |x| \leq 1 \), \( V(x) \) is bounded and for \( |x| > 1 \), \( V(x) = |x| \). It is easy to check that \( V \in \mathcal{D}(\mathcal{L}^\alpha) \), and \( \mathcal{L}^\alpha V \) is a well defined locally measurable function. Then, there exist \( c_2, c_3 > 0 \) and a compact set \( A \) such that for all \( x \in \mathbb{R}^d \),

\[ \mathcal{L}^\alpha V(x) \leq -c_2 V(x) + c_3 1_A(x). \]

This along with [MT93, Theorem 6.1] yields that the process \( (\tilde{X}_t^x)_{t \geq 0} \) is exponential ergodic, i.e., there exists a unique invariant probability \( \mu \) such that for all \( x \in \mathbb{R}^d \) and \( t > 0 \),

\[ \sup_{|f| \leq V + 1} \left| \mathbb{E}[f(X_t^x)] - \mu(f) \right| \leq C_{\alpha,d}(1 + V(x)) e^{-c_4 t} \]

for some constant \( c_4 > 0 \) and \( \mu(V) < \infty \) (here \( \mu(V) = \mathbb{E}[V(Z_1)] \), see, e.g., [CNXY19]). These further imply

\[ \mathbb{E}[|\tilde{X}_t^x|] \leq C_{\alpha,d}(1 + |x|). \]
Recall $\tilde{X}_t^x = x - \frac{1}{\alpha} \int_0^t \tilde{X}_r^x dr + Z_t$, by (4.7), we further have
\[
\mathbb{E}|X_t^x - x| \leq \frac{1}{\alpha} \int_0^t \mathbb{E}|\tilde{X}_r^x| dr + \mathbb{E}|Z_t| \leq C_{\alpha,d}(1 + |x|)(t + t^\frac{1}{2}).
\]

C.3. **Proof of Lemma 4.7.** Recall (3.15) and (3.17), we have
\[
\mathbb{E}[f(\tilde{X}_t^x) - f(\tilde{Y}_t)] = \mathbb{E}[f(x - \frac{1}{\alpha} \int_0^t \tilde{X}_r^x dr + Z_t) - f(x - \frac{\eta}{\alpha} x + \frac{\eta^2}{\sigma} \tilde{Z})] := \mathcal{J}_1 + \mathcal{J}_2,
\]
where
\[
\mathcal{J}_1 := \mathbb{E}[f(x - \frac{1}{\alpha} \int_0^t \tilde{X}_r^x dr + Z_t) - f(x - \frac{\eta}{\alpha} x + Z_t)],
\]
\[
\mathcal{J}_2 := \mathbb{E}[f(x - \frac{\eta}{\alpha} x + Z_t) - f(x - \frac{\eta}{\alpha} x)] - \mathbb{E}[f(x - \frac{\eta}{\alpha} x + \frac{\eta^2}{\sigma} \tilde{Z}) - f(x - \frac{\eta}{\alpha} x)].
\]

For $\mathcal{J}_1$, by Lemma 4.6, we have
\[
|\mathcal{J}_1| \leq \frac{\|\nabla f\|}{\alpha} \int_0^t \mathbb{E}|\tilde{X}_r^x - x| dr \leq C_{\alpha,d}(1 + |x|)\|\nabla f\| \int_0^t r^\frac{1}{2} dr \leq C_{\alpha,d}(1 + |x|)\|\nabla f\|\eta^\frac{1}{2}.
\]

For $\mathcal{J}_2$, by Itô’s formula, we have
\[
\mathbb{E}[f(x + \eta b(x) + Z_t) - f(x + \eta b(x))] = \int_0^t \mathbb{E}[\Delta^{\alpha/2} f(x + \eta b(x) + Z_r)] dr,
\]
and noting that $d_\alpha = \frac{\alpha}{(\alpha - 1) \sigma^\alpha}$, by Taylor expansion, we have
\[
\mathbb{E}[f(x - \frac{\eta}{\alpha} x + \frac{\eta^2}{\sigma} \tilde{Z}) - f(x - \frac{\eta}{\alpha} x)] = \frac{\eta^\frac{1}{2}}{\sigma} \mathbb{E}\left[\int_0^1 \langle \nabla f(x - \frac{\eta}{\alpha} x + \frac{\eta^2}{\sigma} t \tilde{Z}), \tilde{Z} \rangle dt\right]
\]
\[
= \frac{\eta^2}{\sigma} \int_{|z| > 1} \int_0^1 \frac{\alpha \langle \nabla f(x - \frac{\eta}{\alpha} x + \frac{\eta^2}{\sigma} t \tilde{Z}), z \rangle}{V(S^{d-1}) |z|^{\alpha + d}} dt dz
\]
\[
= \frac{\alpha \eta}{V(S^{d-1}) \sigma^\alpha} \int_{|z| > \frac{1}{\sqrt{\alpha}}} \int_0^1 \frac{\langle \nabla f(x - \frac{\eta}{\alpha} x + t \tilde{Z}), z \rangle}{|z|^{\alpha + d}} dt dz = \eta \Delta^{\alpha/2} f(x - \frac{\eta}{\alpha} x) - \mathcal{R},
\]
with
\[
\mathcal{R} = \eta d_\alpha \int_{|z| < \frac{1}{\sqrt{\alpha}}} \int_0^1 \frac{\langle \nabla f(x - \frac{\eta}{\alpha} x + t \tilde{Z}), z \rangle}{|z|^{\alpha + d}} dt dz.
\]
These imply
\[
|\mathcal{J}_2| \leq |\mathcal{R}| + \left| \int_0^t \mathbb{E}[\Delta^{\alpha/2} f(x - \frac{\eta}{\alpha} x + Z_r)] dr - \eta \Delta^{\alpha/2} f(x - \frac{\eta}{\alpha} x) \right|.
\]

It is easy to check that
\[
|\mathcal{R}| = \eta d_\alpha \int_{|z| < \frac{1}{\sqrt{\alpha}}} \int_0^1 \frac{\langle \nabla f(x - \frac{\eta}{\alpha} x + t \tilde{Z}), z \rangle}{|z|^{\alpha + d}} dt dz \leq \eta d_\alpha \int_{|z| < \frac{1}{\sqrt{\alpha}}} \int_0^1 \frac{|\nabla f(x - \frac{\eta}{\alpha} x + t \tilde{Z}) - \nabla f(x - \frac{\eta}{\alpha} x)|}{|z|^{\alpha + d - 1}} dt dz.
\]
\[ \leq C_{\alpha,d} \eta \|
abla^2 f\|_{HS} \int_{|z|<\frac{1}{n}} \frac{1}{|z|^{\alpha+d-2}} dz \leq C_{\alpha,d} \|
abla^2 f\|_{HS} \eta^{\frac{2}{d}}. \]

By [CNXY19, (2.23)], that is,
\[ |(\Delta^\frac{a}{2} f)(x) - (\Delta^\frac{a}{2} f)(y)| \leq \frac{2d_{\alpha}V(S_{d-1})\|
abla^2 f\|_{HS}}{\alpha(2-\alpha)(\alpha-1)} \|x-y\|^{2-\alpha}, \]
we further have
\[ \left| \int_0^{\eta} \mathbb{E}[\Delta^\alpha f(x - \frac{\eta}{\alpha} x + Z_r)] dr - \eta \Delta^\alpha f(x - \frac{\eta}{\alpha} x) \right| \]
\[ \leq \int_0^{\eta} \mathbb{E}[\Delta^\alpha f(x - \frac{\eta}{\alpha} x + Z_r)] - \eta \Delta^\alpha f(x - \frac{\eta}{\alpha} x) |dr\]
\[ \leq C_{\alpha,d} \|
abla^2 f\|_{HS} \int_0^{\eta} \mathbb{E}|Z_r|^{2-\alpha} dr = C_{\alpha,d} \|
abla^2 f\|_{HS} \int_0^{\eta} \mathbb{E}|Z_1|^{2-\alpha} r^{\frac{2-\alpha}{\alpha}} dr \leq C_{\alpha,d} \|
abla^2 f\|_{HS} \eta^{\frac{2}{d}}. \]

Therefore, we have
\[ |\mathbb{E}[f(\hat{X}_\eta) - f(\hat{Y}_1)| \leq C_{\alpha,d}(1 + |x|)(\|f\| + \|\nabla^2 f\|_{HS}) \eta^{\frac{2}{d}}. \]

\[ \square \]

**C.4. Proof of Lemma 4.8.** Let \( p(t, x) \) be the transition probability density of rotationally symmetric \( \alpha \)-stable process \((Z_t)_{t\geq0}\) and the following heat kernel estimates is well known (see, e.g., [BJ07, Lemma 5]), that is,
\[ |\nabla \log p(t, x)| \leq C_{\alpha,d} t^{\frac{1}{\alpha} - \frac{\alpha}{2}} \frac{t}{(t^\frac{1}{\alpha} + |x|)^{\alpha+d}}. \]

Recall the SDE (3.15),
\[ d\hat{X}_t = -\frac{1}{\alpha}\hat{X}_t dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d. \]

Such an equation can be solved explicitly
\[ \hat{X}_t = x e^{-\frac{t}{\alpha}} + \int_0^t e^{-\frac{s}{\alpha}} dZ_s, \]
see [Sat99, p.105]. It follows from (C.2) that the density of \((Q_t)_{t\geq0}\) is given by
\[ q(t, x, y) = p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x), \]
which further implies that for any \( h \in \text{Lip}(1) \) and \( x \in \mathbb{R}^d \), we have
\[ Q_t h(x) = \int_{\mathbb{R}^d} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) h(y) dy. \]

Now, we are at the position to prove the Lemma 4.8.

**Proof of Lemma 4.8.** For any \( x \in \mathbb{R}^d \) and \( t > 0 \), by integration by parts, we have
\[ \nabla (Q_t h)(x) = \int_{\mathbb{R}^d} \nabla_x p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) h(y) dy \]
\[ = -e^{-\frac{t}{\alpha}} \int_{\mathbb{R}^d} \nabla_y p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) h(y) dy \]
\[ = e^{-\frac{t}{\alpha}} \int_{\mathbb{R}^d} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) \nabla h(y) dy, \]
where \( \nabla_x \) means that the gradient operator acts on \( x \). Then, we have
\[ |\nabla (Q_t h)(x)| \leq \|\nabla h\| e^{-\frac{t}{\alpha}} \int_{\mathbb{R}^d} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) dy = \|\nabla h\| e^{-\frac{t}{\alpha}}. \]
Furthermore, by (C.1), we have

\[ \| \nabla (Q_t h)^2 (x) \|_{\text{HS}} \leq \int_{\mathbb{R}^d} e^{-\frac{2t}{\alpha}} \| \nabla y p(1 - e^{-t}, y - e^{-\frac{t}{\alpha} x}) \| \nabla h(y) \| dy \]

\[ \leq C_{\alpha,d} \| \nabla h \| (1 - e^{-t})^{-\frac{1}{\alpha}} e^{-\frac{2t}{\alpha}} \int_{\mathbb{R}^d} \frac{1 - e^{-t}}{(1 - e^{-t}) + |y - e^{-\frac{t}{\alpha} x}|^{\alpha+d}} dy \]

\[ \leq C_{\alpha,d} \| \nabla h \| (1 - e^{-t})^{-\frac{1}{\alpha}} e^{-\frac{2t}{\alpha}} \]

\[ = C_{\alpha,d} \| \nabla h \| (e^t - 1)^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} \leq C_{\alpha,d} \| \nabla h \| e^{-\frac{t}{\alpha}}. \]

\[ \square \]

APPENDIX D. PROOF OF LEMMA 4.9

In this section, we use the semigroup of \((B_t^{\pi})_{t \geq 0}\) and the formula of integration by parts to prove Lemma 4.9.

Proof of Lemma 4.9. Recall

\[ P_t h(x) = \mathbb{E} h(B_t^{\pi}) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} h(y) dy. \]

For any \(v, x_1, x_2 \in \mathbb{R}^d\) and \(f \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})\), denote the directional derivative of \(f(x_1, x_2)\) with respect to \(x_i\) by \(\nabla_{x_i} f(x_1, x_2), i = 1, 2\), respectively. Then, we have

(D.1)

\[ \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla_{v_1} x_1 e^{-\frac{|y-x|^2}{2t}} h(y) dy = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_1 \rangle h(y) dy, \]

which implies

\[ \nabla_{v_2} \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla_{v_2} \nabla_{v_1} x_1 e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_1 \rangle h(y) dy \]

\[ = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_2 \rangle \langle \frac{y-x}{t}, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle h(y) dy. \]

(D.2)

Hence, by integration by parts, we have

\[ \nabla_{v_3} \nabla_{v_2} \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla_{v_3} \nabla_{v_2} \nabla_{v_1} x_1 e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_2 \rangle \langle \frac{y-x}{t}, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle h(y) dy \]

\[ = -\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla_{v_3} \nabla_{v_2} \nabla_{v_1} x_1 e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_2 \rangle \langle \frac{y-x}{t}, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle h(y) dy \]

\[ = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} \langle \frac{y-x}{t}, v_2 \rangle \langle \frac{y-x}{t}, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \nabla_{v_3} h(y) dy \]

\[ = \mathbb{E} \left[ \langle \frac{B_t^{\pi}}{t}, v_2 \rangle \langle \frac{B_t^{\pi}}{t}, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \langle \nabla h(B_t^{\pi}), v_3 \rangle \right], \]

then, by Cauchy-Schwarz inequality, we have

\[ |\nabla_{v_3} \nabla_{v_2} \nabla_{v_1} P_t h(x)| \leq \| \nabla h \| v_3 \left[ \frac{1}{t} \mathbb{E} \langle B_t^{\pi}, v_2 \rangle \langle B_t^{\pi}, v_1 \rangle + \frac{1}{t} \langle v_2, v_1 \rangle \right] \]

\[ \leq |v_3| \left[ \frac{1}{t} |v_1||v_2| + \frac{1}{t} |v_2||v_1| \right] = \frac{2}{t} |v_1||v_2||v_3|, \]

(4.9) is proved.

Next, noticing that

\[ \Delta P_t h(x) = \langle \nabla^2 P_t h(x), I_d \rangle \]
and $I_d = \mathbb{E}[WW^T]$ with $W \sim N(0, I_d)$, it follows from (4.9) that
\[
\left| \Delta(P_th)(x+v) - \Delta(P_th)(x) \right| = \left| \langle \nabla^2(P_th)(x+v) - \nabla^2(P_th)(x), I_d \rangle_{HS} \right| \leq \frac{2d}{t} |v|.
\]

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