INTERACTIONS OF SEMILINEAR PROGRESSING WAVES IN TWO OR MORE SPACE DIMENSIONS

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Abstract. We analyze the behavior of the singularities of solutions of semilinear wave equations after the interaction of three transversal conormal waves. Our results hold for space dimensions two and higher, and for arbitrary $C^\infty$ nonlinearity. The case of two space dimensions in which the nonlinearity is a polynomial was studied by the author and Yiran Wang. We also indicate possible applications to inverse problems.

1. Introduction. We study the propagation of conormal singularities of solutions to semilinear wave equations which are in $H^s_{\text{loc}}(\mathbb{R}^n)$, $s > \frac{n}{4}$. There is no analogue of Hörmander’s Theorem for the propagation of $C^\infty$ singularities for such solutions of nonlinear wave equations (see Theorem 26.1.1 of [21]). However, Bony [7] showed that there is an analogue of Hörmander’s Theorem for $H^{s'}$ singularities (see Theorem 26.1.4 of [21]), provided $s' < 2s - \frac{n}{4}$. In general, for $s' > 2s - \frac{n}{4}$, singularities of solutions of nonlinear wave equations $H^s_{\text{loc}}(\mathbb{R}^n)$ are governed only by fine speed of propagation, see the examples constructed by Beals [1]. When the singularities of the solution to a semilinear wave equation are conormal at a given time, their future behavior can be more tractable in certain situations. We study this phenomenon when three transversal conormal waves interact in dimensions greater than or equal to three. This problem has a relatively long history beginning in the late 1970s with the work of Bony [6] and followed by many others, including Beals [4], Bony [10, 11], Chemin [12], Melrose and Ritter [30], Rauch and Reed [35], Sá Barreto [38, 39]; it has received more attention recently due to applications to inverse problems discovered by Kurylev, Lassas and Uhlmann [23] and followed by many others including [13, 25, 44].

We consider solutions to $P(y, D)u = \mathcal{Y}(y) f(y, u)$, $y \in \mathbb{R}^n$, $n \geq 3$, where $P(y, D)$ is a second order strictly hyperbolic operator, $\mathcal{Y} \in C^\infty_0$, and $f \in C^\infty$. We assume that $\mathcal{Y} = 0$ for negative times where the solution $u$ is the superposition of three elliptic conormal waves which intersect transversally at a codimension three submanifold $\Gamma$ which intersects the support of the nonlinearity. Bony [8] showed that as long as the incoming waves do not have caustics, no new singularities are formed before the triple interaction – transversal interactions of two waves do not produce new singularities. Bony [10, 11] and Melrose and Ritter [30] have shown that, in three dimensions (two space dimensions), after a triple interaction occurs on the support

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of the nonlinearity, the only possible additional singularities of the solution $u$ lie on $Q$ and $u$ will be conormal to $Q$ away from $\Gamma$ and the incoming hypersurfaces, see Fig.1. Other results of Bony [8] and Melrose and Ritter [31] show that in dimension three this persists until $Q$ or one of the incoming waves develops caustics. However, the results of Bony [10, 11] and Melrose and Ritter [30] do not guarantee that singularities on $Q$ will in fact exist. Examples of formation of singularities after the triple interaction for a linearized problem were first given by Rauch and Reed [35]. We compute the strength of singularity on the hypersurface emanating from the submanifold where the three waves interact and show that, unless some degeneracy occurs, it will be roughly of order $3s$. The case $n = 3$ with polynomial nonlinearity was proved in [42].

Caustics are often unavoidable, for instance if $n = 3$ and $P(y, D) = D^2_t - \Delta_g$, where $\Delta_g$ is the Laplacian with respect to a Riemannian metric $g$, and if two geodesics of the metric $g$ emanating from one point (the tip of the cone) intersect at another point, this causes a swallowtail singularity on the cone $Q$, see Fig.2.

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**Figure 1.** The interaction of three conormal plane waves in two space dimensions. The only possible singularities created by the triple interaction appear on the surface of the light cone. Fig.3, Fig.5 and Fig.4 below illustrate the higher dimensional cases.

**Figure 2.** A swallowtail singularity formed on the light cone emanating from a point in two space dimensions. This can be due to the existence of conjugate points of the geodesic flow in the case $P = D^2_t - \Delta_g$, $g$ a Riemannian metric in $\mathbb{R}^2$. The solution to (2.4) would remain conormal to $Q$ away from the caustic, but other singularities could be generated by the caustic. This figure resembles one after equation 5.1.24 in Duistermaat’s book [15].
Let us say that \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) are the incoming hypersurfaces which intersect transversally at \( \Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \). In three dimensions \( \Gamma \) is a point, \( \Omega \) is the forward light cone with tip at \( \Gamma \) and figure Fig.1 shows how the cone and the hypersurfaces behave after the triple interaction. A similar behavior would occur in higher dimensions if \( \Gamma \) were contained on a level surface of the time function, see equation (3.1) and Fig.3. However, this is not the case in general and we give examples below which illustrate this situation, see Fig.4 and Fig.5 below. In higher dimensions, in general \( \Omega \) and \( \Gamma \) do not separate after the triple interaction, see Fig.4. So the more appropriate version of the result of Melrose and Ritter and Bony to be applied in the higher dimensional case is that if the solution \( u \) to \( Pu = f(y, u) \) is conormal (in a suitable sense) to \( \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega \), for \( t < 0 \), then \( u \) is also conormal to \( \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega \) for \( t > 0 \). The methods of [10,11,30] apply in higher dimensions, as pointed out in [27].

The paper is divided in several sections plus three appendices. In Section 2 we explain the geometric framework of the problem and in Section 3 we give examples showing how the hypersurfaces and \( \Omega \) may look like in \( \mathbb{R}^4 \). In Section 4 we state the main theorem, and in Section 6 we outline the main ideas of the proof. In Section 7 we recall some properties of conormal distributions, and in Section 8 we recall the results of Melrose and Ritter [30], Bony [10,11] and Sá Barreto [38,39] about the interactions of conormal waves. In Section 9 we analyze the local geometry of the triple interaction and establish some normal forms for the surfaces and the operator in preparation for the definition of the spaces in Section 10. We prove Theorem 4.1 in Section 11. In Appendix A, we discuss one result about the propagation of singularities from the linear wave equation. Proposition 2.1, which in \( n = 3 \) is contained in [30], is proved in Appendix B and in Appendix C we prove Proposition 10.2, which for \( n = 3 \) is partly contained in [4]. In Section 5 we establish connections with inverse problems.

2. The Framework of the problem. Let \( \Omega \subset \mathbb{R}^n, n \geq 3, \) be an open subset, and let \( P(y, D) \) be a second order strictly hyperbolic operator with \( C^\infty \) coefficients in \( \Omega \). Let \( t \) be a time function for \( P(y, D) \) in \( \Omega \). This means that there exists an open set \( U \) such that \( \Omega \subset U \subset \mathbb{R} \times \mathbb{R}^{n-1} \) such that

\[
P(t, x, D) = \alpha^2(t, x)\partial_t^2 - \sum_{j,k} h_{jk}(t, x)\partial_{x_j}\partial_{x_k}, \quad |\alpha| > 0, \text{ in } U,
\]

and \( h_{jk}(t, x) \) is positive definite. We shall often use \( y = (t, x) \). We assume that \( \Omega \subset \mathbb{R}^n \) is bicharacteristically convex with respect to the level surfaces of the time function \( t \). More specifically, every maximally extended null bicharacteristic of \( P(y, D) \) over \( \Omega \) meets \( \Omega \cap \{ t < T \} \) for every \( T \).

Let \( \Sigma_j \subset \Omega, j = 1, 2, 3, \) be \( C^\infty \) hypersurfaces which are closed, connected and characteristic for \( P(y, D) \). Moreover, we assume that the surfaces intersect transversally in the sense that their normals are linearly independent over the submanifolds

\[
\Gamma_{jk} = \Sigma_j \cap \Sigma_k, \quad j \neq k, \text{ and } \Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3.
\]

Let \( v(y) = v_1(y) + v_2(y) + v_3(y) \), where \( v_j(y) \) is a conormal distribution of appropriate order with respect to \( \Sigma_j, j = 1, 2, 3, \) and assume that

\[
P(y, D)v_j(y) = 0, \quad y \in \Omega, \quad j = 1, 2, 3.
\]
We will analyze the propagation of singularities of solutions \( u(y) \in H^s_{\text{loc}}(\Omega) \), \( s > \frac{n}{2} \), of semilinear wave equations of the form

\[
P(y, D)u(y) = \gamma(y)f(y, u(y)), \ y \in \Omega, \quad u(y) = v(y), \ \text{for } t < -1,
\]

where \( t \) is a time function of \( P(y, D) \) in \( \Omega \), \( f \in C^\infty(\Omega \times \mathbb{R}) \), \( \gamma \in C^\infty_0(\Omega) \), \( \gamma = 0 \) when \( t < -1 \).

Our results also apply, with minor changes of the proof, to the forcing problem

\[
P(y, D)u(y) = f(y, u(y)) + g(y), \ y \in \Omega, \quad g(y) = u(y) = 0, \ \text{and } f(y, \bullet) = 0 \ \text{in } t < -1,
\]

where \( g(y) = g_1(y) + g_2(y) + g_3(y), \ g_j \) conformal to \( \Sigma_j \), \( j = 1, 2, 3 \). This is the form of the equation used in the applications to the nonlinear inverse problems as in [13, 23, 25, 44].

Let \( \Gamma \) be defined in (2.2) and let \( N^*\Gamma \setminus 0 \) denote its conormal bundle minus its zero section, and for each \( q \in \Gamma \), let \( N^*_q\Gamma \setminus 0 \) denote its fiber over \( q \). Let \( p(y, \eta) \) denote the principal symbol of \( P(y, D) \) and let \( H_p \) be its Hamilton vector field. For each \( (q, \eta) \in (N^*_q\Gamma \setminus 0) \cap p^{-1}(0) \), let \( \gamma^+(q, \eta) \) denote the forward null bicharacteristic for \( p \) passing through \( (q, \eta) \) and let

\[
\Lambda_q = \bigcup_{s > 0} \exp(sH_p \gamma) \left( (N^*_q\Gamma \setminus 0) \cap p^{-1}(0) \right) = \bigcup_{(q, \eta) \in (N^*_q\Gamma \setminus 0) \cap p^{-1}(0)} \gamma^+(q, \eta),
\]

denote the flow-out of \( (N^*_q\Gamma \setminus 0) \cap p^{-1}(0) \) by \( H_p \). Since \( \Gamma \) has dimension \( n - 3 \), the dimension of the fiber of \( N^*_q\Gamma \setminus 0 \) is equal to three, and it is well known, see for example [20] that \( \Lambda_q \) is a three dimensional \( C^\infty \) submanifold of \( T^*\Omega \setminus 0 \) which is isotropic, that is the canonical symplectic form vanishes on its tangent space. The manifold

\[
\Lambda = \bigcup_{q \in \Gamma} \Lambda_q
\]

is a \( C^\infty \) conic Lagrangian submanifold of \( T^*\Omega \setminus 0 \) with boundary \( \Lambda \cap (N^*\Gamma \setminus 0) \), since we are only considering the future oriented bicharacteristics. When \( n = 3 \), \( \Gamma = \{q\} \) and \( \Lambda_q = \Lambda \).

We will need to analyze the projection of the bicharacteristics \( \gamma^+(q, \eta) \) and the Lagrangian \( \Lambda_q \) from \( T^*\Omega \setminus 0 \) to \( \Omega \). As usual, we let

\[
\Pi : T^*\Omega \setminus 0 \rightarrow \Omega
\]

denote the canonical projection, and for a bicharacteristic \( \gamma^+(q, \eta) \), let

\[
\sigma(q, \eta) = \Pi(\gamma^+(q, \eta)), \quad (q, \eta) \in (N^*_q\Gamma \setminus 0) \cap p^{-1}(0), \quad \Omega_q = \Pi(\Lambda_q) = \bigcup_{(q, \eta) \in (N^*_q\Gamma \setminus 0) \cap p^{-1}(0)} \sigma(q, \eta),
\]

\[
\Omega = \bigcup_{q \in \Gamma} \Omega_q.
\]

In dimension \( n = 3 \), \( \Gamma \) is a point and \( \Omega_q = \Omega \) is the forward light cone for the operator \( P \) over \( \Gamma \), and \( \sigma(q, \eta) \) is a characteristic ray on the cone. It is well known
that $\Omega \setminus \Gamma$ is a $C^\infty$ manifold in a sufficiently small neighborhood of $\Gamma$. In other words, the map

$\Pi_A : \Lambda \setminus 0 \to \Omega$

with $\zeta \mapsto \Pi(\zeta)$,

is a local diffeomorphism near $\Gamma$, but not at $\Gamma$. This implies that locally in $\Omega \setminus \Gamma$, but near $\Gamma$, $\Lambda = \{(x, d\varphi(x))\}$, $\varphi \in C^\infty$, so $\Lambda$ is the graph of the differential of a $C^\infty$ function, see for example Theorem 5.3 of [18]. However $\Omega$ may develop caustics later, i.e. the map $\Pi_A$ is no longer a local diffeomorphism.

One also needs to consider the geometry of the three transversally intersecting hypersurfaces and the outgoing manifold $\Omega$, and we begin by establishing a local model for $\Omega$ and one of the hypersurfaces. We will also need this result in the proof of Proposition 8.3 below.

Our model for the operator and the hypersurfaces in $\mathbb{R}^n$ is

$P_0(y,D) = \partial_{y_1} \partial_{y_2} + \partial_{y_1} \partial_{y_2} + \partial_{\eta_1} \partial_{\eta_2}, \text{ if } n = 3,$

and

$P_0(y,D) = \partial_{y_1} \partial_{y_2} + \partial_{y_1} \partial_{y_2} + \partial_{y_1} \partial_{y_2} - \sum_{j,k=4}^n b_{jk}(y^{''}) \partial_{y_j} \partial_{y_k}, \text{ if } n \geq 4,$

where we denote $y = (y_1, y_2, y_3, y^{''}) \in \mathbb{R}^n$, $b_{jk} \in C^\infty$ and the corresponding quadratic form is positive definite. The function $t = y_1 + y_2 + y_3$ is a time function for $P(y,D)$. The case $n = 3$ was the model used by Beals [4]. In this case the principal symbol of $P(y,D)$ is given by

$p(y,\eta) = \eta_1 \eta_2 + \eta_1 \eta_3 + \eta_2 \eta_3 + \sum_{j,k=4}^n b_{jk}(y^{''}) \eta_j \eta_k,$

and we have that

$N^* \setminus 0 = \{y_1 = y_2 = y_3 = 0, \eta^{''} = 0\},$

To compute the flow-out of this submanifold under $H_p$, we need to solve the following set of equations:

$\dot{y}_1 = \eta_2 + \eta_3, \quad y_1(0) = 0, \quad \dot{y}_2 = \eta_1 + \eta_3, \quad y_2(0) = 0,$

$\dot{y}_3 = \eta_1 + \eta_2, \quad y_3(0) = 0, \quad \dot{y}_j = \partial_{y_0} p(y,\eta), \quad y_j(0) = y_{j0}, \quad j = 4, \ldots, n,$

$\dot{\eta}_1 = 0, \quad \eta_1(0) = \eta_{10}, \quad \dot{\eta}_2 = 0, \quad \eta_2(0) = \eta_{20}, \quad \dot{\eta}_3 = 0, \quad \eta_3(0) = \eta_{30},$

$\dot{\eta}_m = - \partial_{y_m} \sum_{j,k=4}^n b_{jk}(y^{''}) \eta_j \eta_k, \quad \eta_m(0) = 0, \quad j = 4, \ldots, n,$

where $(\eta_{10}, \eta_{20}, \eta_{30})$ satisfy (2.11). So we find that $\eta^{''} = 0$, $y^{''} = y_0^{''}$, and

$y_1 = s(\eta_{20} + \eta_{30}), \quad y_2 = s(\eta_{10} + \eta_{30}), \quad y_3 = s(\eta_{10} + \eta_{20}),$

and so

$\eta_{10} = \frac{1}{2s}(-y_1 + y_2 + y_3), \quad \eta_{20} = \frac{1}{2s}(y_1 - y_2 + y_3), \quad \eta_{30} = \frac{1}{2s}(y_1 + y_2 - y_3).$

In view of (2.11),

$\Omega_0 = \{\eta_0(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3 = 0\}.$
In fact this serves as a local model in the general case:

**Proposition 2.1.** Let \( \Sigma_j, j = 1, 2, 3 \) be \( C^\infty \) closed characteristic surfaces for \( P(y, D) \) that satisfy (2.2), then for every point \( q \in \Gamma \), there exist a neighborhood \( U_q \subset \Omega \), a neighborhood \( U_0 \) of \( 0 \in \mathbb{R}^n \) and a \( C^\infty \) diffeomorphism \( \Psi : U_0 \to U_q \) such that

\[
\Psi^* \Gamma = \{y_1 = y_2 = y_3 = 0\}, \quad \Psi^* \Sigma_1 = \{y_1 = 0\} \quad \text{and} \quad \Psi^* \Omega = \{y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3 = 0\}.
\]

For \( n = 3 \), this result is contained in Proposition 4.9 of [30]. We prove the general case in Appendix B for the convenience of the reader. One can put \( \Omega \) and two hypersurfaces into normal form, see for example [2] for \( n = 3 \). We will not pursue this since we will not need it here, but it is unlikely one can put \( \Omega \) and the three hypersurfaces simultaneously into normal form.

**Proposition 2.1** shows that near a point \( q \in \Gamma \), \( \Omega \) and \( \Sigma_1 \) are simply tangent along the \( C^\infty \) submanifold where they intersect. This implies that near \( \Gamma \), \( \Omega \) and the three hypersurfaces \( \Sigma_j, j = 1, 2, 3 \), intersect in this way. One can also see that from the fact that for each \( q \in \Gamma \), \( \Omega_q \) is contained in the light cone with vertex at \( q \), since \( \Sigma_j \), is characteristic and \( q \in \Sigma_j \), it is tangent to the light cone.

Also, since \( \Sigma_j \) is characteristic, \( \partial_t \) is transversal to \( \Sigma_j \) and vanishes nowhere, and this gives the future orientation across \( \Sigma_j \), which is the direction in which time increases. The rays on \( \Omega \) start on \( \Gamma \), are future oriented, and near \( \Gamma \), are on the side of \( \Sigma_j \) where \( \partial_t \) points to, and so they do not cross \( \Sigma_j \). This implies that \( \Omega \) will not eventually intersect \( \Sigma_j, j = 1, 2, 3 \), transversally, and in particular \( \Omega \) will not intersect \( \Gamma_{jk}, j, k = 1, 2, 3 \), transversally.

One also needs to inquire whether \( \Omega \) and \( \Sigma_j \) or \( \Gamma_{jk}, j = 1, 2, 3 \), can intersect tangentially at a place other than at the submanifold described above. In this case, we would have a ray \( r \) on \( \Omega \) which would intersect \( \Sigma_j \) tangentially. Suppose that \( r \) is the projection of the bicharacteristic \( \gamma_+ \) on \( \Lambda \). This would imply that \( \gamma_+ \) would intersect \( N^* \Sigma_j \). But since \( \Sigma_j \) is characteristic, the Hamilton vector field \( H_p \) is tangent to \( N^* \Sigma_j \), and this would imply that \( \gamma_+ \subset N^* \Sigma_j \setminus \{0\} \). So the entire ray would be contained in \( \Sigma_j \cap \Omega \).

So we conclude that \( \Sigma_j \) and \( \Omega \) are tangent along the \( C^\infty \) submanifold where they intersect, at least as long as \( \Sigma_j \) and \( \Omega \) remain \( C^\infty \), or for as long as no caustics form. This means that the higher dimensional picture is still somewhat similar to the three dimensional case, illustrated in Fig.1, where \( \Omega \) is tangent to the three hypersurfaces and is separated from the double intersections away from \( \Gamma \), but not from \( \Gamma \).

3. **Examples.** We give examples of the interaction of three waves in Minkowski space \( \mathbb{R}^4 \) which illustrate how \( \Gamma \) and \( \Omega \) can globally look like. In these examples \( \Omega \) does not develop caustics and there are no intersections of \( \Omega \) and \( \Gamma_{jk} \). The operator is the Minkowski wave operator in \( \mathbb{R}^4 \) with coordinates \((t, x), x = (x_1, x_2, x_3) : \)

\[
P(t, x) = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2.
\]

In the first example we take three plane waves:

\[
(3.1) \quad \Sigma_1 = \{t = x_1\}, \quad \Sigma_2 = \{t = x_2\} \quad \text{and} \quad \Sigma_3 = \{t = \frac{1}{\sqrt{2}}(x_1 + x_2)\},
\]

which will intersect transversally at
\[
\Gamma = \{t = 0, x_1 = 0, x_2 = 0\}.
\]
The conormal bundle to $\Gamma$ is

$$N^*\Gamma \setminus 0 = \{ t = 0, x_1 = 0, x_2 = 0, \xi_3 = 0 \},$$

and the flow out of $(N^*\Gamma \setminus 0) \cap \{ \tau^2 = |\xi|^2 \}$ is given by

$$\Lambda = \{ \tau = \tau_0, \xi = \xi_0, t = 2\tau s, x_1 = -2\xi_1 s, x_2 = -2\xi_2 s, x_3 = x_{03}, \tau = |\xi|, \xi_3 = 0 \}.$$

The projection of $\Lambda$ to $\mathbb{R}^4$ is given by

$$\Omega = \{(t, x) \in \mathbb{R}^4 : t = (x_1^2 + x_2^2)^{1/2} \}.$$

This can be viewed as a fiber bundle over $\Gamma$ where the fiber over a point $(0, 0, 0, x_{03}) \in \Gamma$ is the circle $t = (x_1^2 + x_2^2)^{1/2}$. An observer sitting at $(0, 0, x_3)$ will see a circular wave expanding with speed one, see Fig. 3. In this example $\Gamma \subset \{ t = 0 \}$ and the analysis of this interaction is similar to the three dimensional case.

![Figure 3](image-url)

**Figure 3.** The dotted line represents an expanding cylindrical wave, generated by the interaction of three plane waves given by (3.1) in $\mathbb{R}^4$, viewed by an observer in $\mathbb{R}^3$ as time increases. The speed in which the radius of the wave expands is equal to one.

In the second example, we again pick three plane waves, but this time they intersect at a submanifold $\Gamma$ which is not contained on a level surface of the time function $t$. As observed above, in this situation the newly formed wave $\Omega$ will not separate from $\Gamma$ as in the previous example. We just pick the surfaces

$$\Sigma_j = \{ t = x_j \}, \quad j = 1, 2, 3,$$

which meet at $\Gamma = \{ t = x_1 = x_2 = x_3 \}$. The conormal bundle to $\Gamma$ is given by

$$N^*\Gamma = \{(t, x, \tau, \xi) : t = x_1 = x_2 = x_3, \tau = \tau_0, \xi = \xi_0, \tau + \xi_1 + \xi_2 + \xi_3 = 0 \},$$

and so the Lagrangian $\Lambda$ obtained by the flow-out of $(N^*\Gamma \setminus 0) \cap \{ \tau^2 = |\xi|^2 \}$ is given by

$$\Lambda = \{ t = a + 2\tau s, x_j = a - 2\xi_j s, \xi_j = \xi_{0j}, \tau = \tau_0, \tau + \xi_1 + \xi_2 + \xi_3 = 0, \tau = |\xi|, a, s \in \mathbb{R} \}.$$

Its projection to $\mathbb{R}^4$ is given by

$$\Omega = \{(3t - x_1 - x_2 - x_3)^2 = (x_1 - x_2 - x_3 + t)^2 + (x_2 - x_1 - x_3 + t)^2 + (x_3 - x_2 - x_1 + t)^2 \}.$$

One can check that $t - x_1 - x_2 - x_3 = -2a$ on $\Omega$, where the parameter $a$ gives the position of a point on $\Gamma$. So to consider the behavior of $\Omega$ for a fixed time $t$, one should restrict the variable $a \in [t, A_1]$. The forward part of $\Omega$ for fixed $t$ and viewed by an observer in $\mathbb{R}^3$ is part of a cone with axis of symmetry $L = \{ x_1 = x_2 = x_3 \}$ and vertex at $(t, t, t)$, bounded by the planes $3t \leq x_1 + x_2 + x_3 \leq t + 2A_1$, see figure Fig. 4.
In the third example, which was used by Sá Barreto, Uhlmann and Wang \[41\] to study nonlinear inverse scattering, we pick three spherical waves centered at \(p_1 = (0, 0, 0, 0)\), \(p_2 = (0, 2a, 0, 0)\), and \(p_3 = (0, 0, 2b, 0)\) respectively. These are represented by three forward light cones:

\[
\begin{align*}
\tau &= \left( x_1^2 + x_2^2 + x_3^2 \right)^{\frac{1}{2}}, \\
\tau &= \left( x_1 - 2a \right)^2 + x_2^2 + x_3^2^{\frac{1}{2}}, \\
\tau &= \left( x_1^2 + (x_2 - 2b)^2 + x_3^2 \right)^{\frac{1}{2}}.
\end{align*}
\]

These waves will intersect transversally at the hyperbola

\[
\Gamma = \Gamma_{a,b} = \{ (t, x) : x_1 = a, x_2 = b, t = (x_3^2 + a^2 + b^2)^{\frac{1}{2}}, x_3\tau + t\xi_3 = 0 \}.
\]

whose conormal bundle is given by

\[
N^*\Gamma = \{ (t, x, \tau, \xi) : x_1 = a, x_2 = b, t = (x_3^2 + a^2 + b^2)^{\frac{1}{2}}, x_3\tau + t\xi_3 = 0 \}.
\]

If \(\Lambda\) is Lagrangian submanifold obtained by the forward flow-out of \((N^*\Gamma \setminus \emptyset) \cap \{ \tau^2 = |\xi|^2 \}\), then points \((t, x, \tau, \xi) \in \Lambda\) and \((t_0, x_0, \tau_0, \xi_0) \in (N^*\Gamma \setminus \emptyset) \cap \{ \tau^2 = |\xi|^2 \}\) are related by the following:

- for \(t_0 = |x_0|\), \(x_0 = (a, b, x_{03})\), \(\tau_0 = |\xi_0|\), \(\xi_0 = (\xi_{01}, \xi_{02}, \xi_{03})\),
- \(\xi_1 = \xi_{01}, \xi_2 = \xi_{02}, \xi_3 = \xi_{03}, \tau = \tau_0, x_{03}\tau_0 + t_0\xi_{03} = 0,\)
- \(x_1 = a - 2\xi_{01}s, \quad x_2 = b - 2\xi_{02}s, \quad x_3 = x_{03} - 2\xi_{03}s, \quad t = t_0 + 2\tau_0s, \quad s \in \mathbb{R}.
\]

The projection of \(\Lambda\) to \(\mathbb{R}^4\) is denoted by \(Q\) and it is given by

\[
Q = \{ (t, x) : (t - t_0)^2 = (x_3 - x_{03})^2 + (x_2 - b)^2 + (x_1 - a)^2; x_3 = \frac{x_{03}}{t_0}t \},
\]

which is again a fibered bundle over \(\Gamma\) whose fibers over the point \((t_0, x_0) = (t_0, a, b, x_{03}) \in \Gamma\) given by these equations. Notice that \(a\) and \(b\) are fixed, but \(x_{03}\) is not. So \(Q\) is described by five parameters, \((t, x_1, x_2, x_3)\) and \(x_{03}\), and two equations. One can check that the differential of the functions that define \(Q\) are linearly independent everywhere on \(\Omega\), except at \(\Gamma\), and so \(\Omega \setminus \Gamma\) is \(C^\infty\).
Interactions of semilinear waves

\[ x_1 = (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

\[ x_3 = (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

\[ x_1 = (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

\[ x_3 = (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

\[ x_3 = - (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

\[ x_3 = (t^2 - a^2 - b^2)^{1/2}, \text{ radius } R = 0. \]

Figure 5. The dotted line shows the surface (3.4) as \((x_1, x_2, x_3)\) vary for \(t\) fixed. Unlike the wave formed by the interaction of three plane waves considered above, which is an infinite cylinder, three spherical waves intersect along a bounded curve for fixed time. The level sets of this surface for \(\{x_3 = c\}\) are circles centered on the line \(\{x_1 = a, x_2 = b\}\).

One can also think about \(Q\) from the point of view of an observer in \(\mathbb{R}^3\). One has three spherical waves, centered at \(p_1, p_2\) and \(p_3\) and expanding with speed one. They first meet at the point \((a, b, 0)\) which is equidistant from their centers \(p_1, p_2\) and \(p_3\), at a time \(t_0 = (a^2 + b^2)^{1/2}\), which is equal to the distance between any of the centers to the point of interaction. After that, a wave centered at the line \(L = \{x_1 = a, x_2 = b\}\) will form for times \(t \geq (a^2 + b^2)^{1/2}\). It is convenient to express \(Q\) as

\[(3.4) \quad Q = \{(t, x) : (x_1 - a)^2 + (x_2 - b)^2 = \left[ (t^2 - x_3^2)^{1/2} - (a^2 + b^2)^{1/2} \right]^2, \quad x_3 = \frac{x_03}{t_0} t \},\]

For a fixed time \(t \geq t_0 = (a^2 + b^2 + x_03^2)^{1/2}, t > |x_03|\), and since \(x_3 = \frac{x_03}{t_0} t, t > |x_3|\). This is consistent with the fact that for fixed \(t\), the three spherical waves intersect along a bounded segment of the hyperbola and the surface of the newly formed wave is bounded for bounded times, see Fig. 5.

Now, with \(x_3\) fixed, and \(t > x_3\) increasing, this is an expanding circular wave centered at \((a, b)\) with radius

\[R = (t^2 - x_3^2)^{1/2} - (a^2 + b^2)^{1/2},\]
and therefore \( \frac{dR}{dt} = \frac{t}{(t^2 - x_j^2)^{\frac{3}{2}}} \geq 1 \), which will give the appearance that the circular wave is moving faster speed than the speed of light.

4. The Statement of the main result. Recall that \( \Omega \subset \mathbb{R}^n \) is an open subset and \( P(y,D) \), \( t, y(y), f(y,u) \), are as described above and that \( \Sigma_j \subset \Omega, 1 \leq j \leq 3 \), are \( C^\infty \) closed connected hypersurfaces satisfying (2.2), which defines \( \Gamma \).

In Section 7 we will introduce the class of conormal distributions of order \( m \) to a submanifold \( M \) of \( \Omega \), which we denote by \( I^m(\Omega, M) \). We will also discuss products of conormal distributions conormal to transversal hypersurfaces and we will prove that if \( v_j \in I^m(\Omega, \Sigma_j) \), are conormal distributions to \( \Sigma_j \), \( j = 1, 2, 3 \), then (4.1)

\[
v_1 v_2 v_3 = v_T + \sum_{j,k=1}^3 v_{jk} + \sum_{j=1}^3 w_j, \quad \text{and we define } V \text{ to be the principal part of } v_T,
\]

where \( v_T \) and \( V \) are product-type conormal distributions with respect to the submanifold \( \Gamma \), \( v_{jk} \) is a product-type conormal distribution associated to \( \Gamma_{jk} \) respectively, and \( w_j \) are conormal distributions to \( \Sigma_j \). By product-type conormal distributions, we mean their symbols are product-type symbols which are defined below.

We define \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma \) to be the restriction of \( (\partial_u^{\alpha} (yf))(y, u(y)) \) to \( \Gamma \). In local coordinates \( y = (y', y'^n) \), \( y' = (y_1, y_2, y_3) \), where \( \Sigma_j = \{ y_j = 0 \} \), and \( \Gamma = \{ (0, 0, 0, y'^n) \} \), \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma = (\partial_u^{\alpha} (yf))(0, 0, 0, y'^n) \). Since we are dealing with functions in \( H^s(\Omega) \), \( s > \frac{3}{2} \), they are continuous functions, and there is no problem defining this operation. This is not the restriction in the sense of distributions, which is not defined. When \( n = 3 \), \( \Gamma \) is a point and \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma \) is a constant. When \( n > 3 \) we will show in Proposition 8.3 below that under the hypotheses of Theorem 4.1 below, \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma \in C^\infty(\Gamma) \).

We will also need to make sense of the product \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma V \). When \( n = 3 \), this is obvious. We will show in Proposition 10.12 that and modulo smoother terms one can define \( (\partial_u^{\alpha} (yf))(y, u(y))|_\Gamma V \equiv \mathcal{S} V \), for any \( \mathcal{S} \in C^\infty(\Omega) \) such that \( \mathcal{S}(y) = (\partial_u^{\alpha} (yf))(y, u(y)) \), for all \( y \in \Gamma \).

As we discussed above, we need to make assumptions that guarantee that we are only dealing with singularities on \( \Sigma_j \), \( j = 1, 2, 3 \) and \( \Omega \) so we can use the work of Bony [10,11] and Melrose and Ritter [30]. We will make the following assumptions:

TR1. Assume that \( y \) is supported on a bicharacteristically convex relatively compact open subset \( U \subset \Omega \cap \{ t > -1 \} \).

TR2. Let \( \Omega \) be an open subset of \( U \) containing the intersection of singular support of \( v_j \), \( j = 1, 2, 3 \), and \( U \) (of course one may have \( U = \emptyset \)). Let \( \Lambda_0 \) denote the part of \( \Lambda \) emanating from \( \mathcal{N}(\Gamma \cap \emptyset) \setminus 0 \) and let \( \mathcal{Q}_0 \) be the corresponding part of \( \mathcal{Q} \). We shall assume that no caustics form on \( \mathcal{Q}_0 \) for a certain time on \( \mathcal{Q} \). More precisely, there exists \( T > -1 \) such that the map \( \Pi_\Lambda \) defined in (2.9) restricted to \( \Lambda_0 \) is a local diffeomorphism on \( \mathcal{Q} \) for \( t < T \).

It is important to realize that the equation is linear outside \( U \) and \( v_j \in C^\infty(\Omega \setminus \emptyset) \), \( j = 1, 2, 3 \), and no singularities of the solution \( u \) will form on the part of \( \mathcal{Q} \) coming from outside \( \mathcal{Q} \). After they leave \( U \), singularities of \( u \) will governed by propagation of singularities results for linear equations. As discussed in Proposition 2.1 Assumption TR.2 is always true if \( T + 1 \) is small enough. In applications to inverse problems [13,23,25,44], the singular support of \( v_j \) is contained in a small neighborhood of a ray \( \gamma_j \subset \Sigma_j, j = 1, 2, 3 \), and \( \gamma_1 \cap \gamma_2 \cap \gamma_3 = \{ q \} \subset \Gamma \).
Under these assumptions, eventual new singularities of the solution $u$ of (2.4) with initial data that $v$ which form on $Q$ are only due to the triple interaction, and then the results of Melrose and Ritter [30, 31], Bony [10, 11] and Sá Barreto [38, 39] guarantee if the solution $u$ of (2.4) is conormal (in a suitable sense) $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup Q$ in $\{ t < -1 \}$, then $u$ remains conormal to $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup Q$ in $\Omega \cap \{ t < T \}$. As mentioned above, one needs to include $Q$ in the past, since $Q$ will keep interacting with the three waves at $\Gamma$ after the first singularity is formed.

Our main theorem roughly says that, under the assumptions that no caustics form of $Q$, the principal part of the singularities of $u$ on $\Lambda$ and away from the hypersurfaces $\Sigma_j$, $j = 1, 2, 3$, and $\Gamma$ if given by $E_+(SV)$, where $V$ is defined in (4.1) and $\mathcal{G} \in C^\infty(\Omega)$, $\mathcal{G} = (\partial^\nu_y(yf))(y,u(y))|_{\Gamma}$. We discuss the regularity of $E_+(SV)$ in Appendix A.

Before we state it, we recall that if $E_+$ is the forward fundamental solution of $P(y, D)$, then the solution to (2.4) is given by

$$u = v + E_+(F(y, u)) = v + E_+([F(y, v + E_+ F(y, u))]), \text{ where } F(y, u) = \mathcal{G}(y)f(y, u).$$

More precisely, we have the following:

**Theorem 4.1.** Let $p(y, \eta)$ denote the principal symbol of $P(y, D)$, let $V$ be defined in (4.1) and suppose that assumptions TR.1 and TR.2 are satisfied. Let $u$ be the solution to (2.4) in $\Omega$ with initial data $v = v_1 + v_2 + v_3$, with $v_j \in I^{m-\frac{7}{4}+\hat{\nu}}(\Omega, \Sigma_j)$, $m < -\frac{1}{2}(n + 7)$, satisfying $P(y, D)v_j = 0$, $j = 1, 2, 3$, in $\Omega$. Moreover,

**H1.** let $(y_0, \eta_0) \in (N^\ast \Gamma \setminus 0) \cap \{ p = 0 \}$, and $(y_0, \eta_0) \notin N^\ast \Sigma_j$, $j = 1, 2, 3$,.

**H2.** and let $V_1 \subset V$ be closed conic neighborhoods of $(y_0, \eta_0)$, such that $V_1$ is contained in the interior of $V$ and that $V \cap (N^\ast \Sigma_j \setminus 0) = \emptyset$, $j = 1, 2, 3$.

**H3.** Let $A \in \Psi^0(\Omega)$ be a pseudodifferential operator of order zero, which is elliptic in $V_1$ and is of order $-\infty$ outside $V$.

**H4.** Let $\gamma_+$ be the forward bicharacteristic which starts at $(y_0, \eta_0)$ and let $C_1(\gamma_+) \subset C(\gamma_+)$ denote closed conic neighborhoods of $\gamma_+$, such that $C_1(\gamma_+)$ is contained in the interior of $C(\gamma_+)$ and that $C(\gamma_+) \cap (N^\ast \Sigma_j \setminus 0) = \emptyset$, $j = 1, 2, 3$.

**H5.** Let $B \in \Psi^0(\Omega)$ be a pseudodifferential operator of order zero, which is elliptic in $C_1(\gamma_+)$ and is of order $-\infty$ outside $C(\gamma_+)$.

We denote $F(y, u) = \mathcal{G}(y)f(y, u)$ and we have the following:

**C1.** $(\partial^\nu_y F)(y, u(y))|_{\Gamma} \in C^\infty(\Gamma)$, and

**C2.** if $E_+$ is the forward fundamental solution of $P(y, D)$, and if $\mathcal{G} \in C^\infty(\Omega)$ is such that $\mathcal{G} = (\partial^\nu_y F)(y, u(y)), y \in \Gamma$,

$$u, \quad E_+(A\mathcal{G}V) \in I^{3m-\frac{7}{4}}(\Omega \cap \{ t < T \}, \Omega) \quad \text{and microlocally away from } N^\ast \Gamma,$$

$$B[u(y) - E_+(A\mathcal{G}V)] \in I^{3m-\frac{7}{4}-1}(\Omega \cap \{ t < T \}, \Omega).$$

Some remarks about Theorem 4.1:

**R1.** This result explains the mechanism of the formation of the top singularities of $u$ on $Q : SV$ acts as a source on $\Gamma$.

**R2.** Result C1 is a consequence of the results of [10, 11, 30, 31].

**R3.** We know from [10, 11, 30, 31] that away from $\Sigma_j$ and $\Gamma$, $j = 1, 2, 3$, $u$ is conormal to $Q$ in $\Omega \cap \{ t < T \}$. Equation (4.2) gives the principal part of $u$ near $Q$ and away from $\Sigma_j$ and $\Gamma$. The principal symbol of $u$ in this region is given by equation (A.2) in Appendix A with $f$ replaced by $A(SV)$. Notice that for $A = Id$ on $V_1$, the initial condition of (A.2) captures $\mathcal{G}|_{\Gamma}$. 

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R4. We are also using the absence of caustics and the results of [10, 11, 30, 31] to guarantee that \( AE_+((I - B)F(y, u)) \) does not create new singularities on \( \Omega \).

R5. Sá Barreto and Wang [42] proved Theorem 4.1 in the case where \( n = 3 \) and \( f(y, u) \) is a polynomial in \( u \) with \( C^\infty \) coefficients.

R6. M. Beals [4] proved a local version of this result for \( n = 3 \) and the operator \( F_0(y, D) \) and \( \Sigma_j \) as in (2.10) above, \( v_j = y_j^{\alpha_j} \), \( m \in \mathbb{N}_0 \), and \( f(y, u) = a(y)u^3 \).

R7. The proof of C2 relies on a modification of the spaces introduced in [4].

R8. It is much harder to track lower order singularities of \( u \). In particular, if \( S \subset \Gamma \) is a relatively open subset and \( (\partial^3_\gamma(yf))(y, u)|_S = 0 \), we cannot say anything about the leading order singularities of \( u \) on the part of \( \Omega \) corresponding to the flow-out of \( S \).

We need to mention some known results about the propagation of singularities for semilinear wave equations when caustics occur. The case of a cusp was studied by Melrose [28, 29] and Beals [5], and they do not produce any additional singularities of the solution. In the case of a swallowtail caustic, Joshi and Sá Barreto [22] gave an example for a linearized equation, in the spirit of the example of Rauch and Reed for the transversal triple interaction, which showed that it produces singularities of the solution \( u \) to (2.4) of order 6 on the surface emanating from the swallowtail tip, just like \( \Omega \) emanates from \( \Gamma \). Delort [14], Lebeau [24] and Sá Barreto [40] had shown that in three dimensions these are the only possible additional singularity of the solution caused by a swallowtail caustic. Nothing is known about the propagation of singularities of solution of semilinear equations when more complex caustics appear.

5. Connections to inverse problems. We will not prove any theorems, we will only indicate how Theorem 4.1 can be used to determine \( (\partial^3_\gamma(yf))(y, u(y)) \) for every \( y \in \Gamma \), where \( u(y) \) is a solution to (2.4), from the top singularity of the solution \( u(y) \) on \( \Lambda \). We assume that the hypotheses the theorem hold throughout the discussion and \( t < T \). In the work mentioned above [13, 23, 25, 44], \( v_j \) is supported near a ray.

If one can vary \( \Sigma_j, j = 1, 2, 3 \), in such a way that the corresponding family of \( \Gamma \) sweeps an open subset \( \Omega_1 \subset \text{supp}(f(y, u)) \subset \Omega \), then one can determine on \( (\partial^3_\gamma(yf))(y, u(y)) \) for every \( y \in \Omega_1 \) by measuring the principal symbol of the solutions on \( \Omega \setminus \Omega_1 \).

As already mentioned above, [13, 23, 25, 44] use propagation of conormal singularities to treat nonlinear inverse problems. We briefly describe part of their methods and relate them to what is done here. These papers deal with the so-called source-to-solution map and the construction of suitable \( \Sigma_j \), and \( v_j \), \( j = 1, 2, 3 \), and how they can be used to solve the problem is a significant part of those papers. We are not concerned about this here; we assume that \( \Sigma_j, j = 1, 2, 3 \), are \( C^\infty \), intersect transversally, and we can somehow measure the top singularities of \( u \) on \( \Lambda \).

We should also say that [23, 25, 44] use singularities produced by the interaction of four waves. While this works well with the method we are about to describe, very little is known about the propagation of singularities of solutions of semilinear wave equation in this case. When four waves intersect transversally (in this case \( n \geq 4 \)) they produce infinitely triple and multiple interactions. Even though in principle the singularities produced after each interaction are weaker than those that generate them, this has not been proved, and a quadruple interaction can produce an open set of \( C^\infty \) singularities of the solution to (2.4); an example due to Beals can be found in [3]. One should say that the interaction of two tangent waves (like \( \Omega \) and each of the incoming hypersurfaces \( \Sigma_j \)) with a third wave that is transversal to
the tangent pair do produce new singularities of the solution to the nonlinear wave equation. An example of that, which is similar in nature to the example of Rauch and Reed [35] for the transversal tripe interaction, was given by Holt [19].

Let us denote \( \mathcal{Y} = F \), and let us consider a linearization of the nonlinear forcing problem

\[
P(y, D)u + F(y, u) = g(y) = g_0 + \varepsilon_1 g_1 + \varepsilon_2 g_2 + \varepsilon_3 g_3,
\]

\[
 u = g = 0, \quad t < 0,
\]

with \( g_0 \in C^\infty(\Omega) \), and \( g_j \in L^m(\Omega, \Sigma_j) \). We do not quite study the forcing problem in this paper, but as mentioned above, our result holds with little change to its proof.

One needs to assume that \(|g_0|_{H^s} < \delta\), and \(|g_j|_{H^s} \leq 1\), with \( s \) large enough, \( \delta \) and \( \varepsilon_j \), \( j = 1, 2, 3 \), small enough so we know the solution \( u(y) \) to (5.1) exists in \( \Omega \). One should say that in [13, 23, 25, 44], \( g_0 = 0 \).

The goal is to show that, for any \( k \), the solution \( u \) has a unique expansion of the form

\[
u = u_0 + \sum_{|\alpha| \leq k} \varepsilon^\alpha u_\alpha + \mathcal{E}_k, \quad \text{where} \quad \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \quad \text{and} \quad |\mathcal{E}_k|_{H^s} = O(|\varepsilon|^{k+1}).
\]

To do that, one writes

\[
F(y, u) = F(y, u_0 + (u - u_0)) = F(y, u_0) + \sum_{j=1}^N \frac{1}{j!} (\partial_u^j F)(y, u_0)(u - u_0)^j + \frac{1}{N!} (u - u_0)^{N+1} \int_0^1 (\partial_u^{N+1} F)(y, u_0 + t(u - u_0))(1 - t)^N \, dt,
\]

replaces (5.2) into (5.3), matches powers of \( \varepsilon \) in (5.1) and obtains a series of equations for \( u_\alpha \). One then shows that the remainder is of order \(|\varepsilon|^{k+1}\). The term of order zero satisfies

\[
P(y, D)u_0 + F(y, u_0(y)) = g_0,
\]

\[
u_0 = g_0 = 0, \quad t < 0.
\]

Since \( g_0 \in C^\infty(\Omega) \) and has small enough \( H^s \) norm, one can show that the solution \( u_0 \) exists in \( \Omega \) and \( u_0 \in C^\infty(\Omega) \). The terms \( u_\alpha, |\alpha| = 1 \) satisfy

\[
(P(y, D)u_0 + (\partial_u F)(y, u_0(y)))u_\alpha = g_\alpha,
\]

\[
u_\alpha = g_\alpha = 0, \quad t < 0,
\]

where \( g_1 = g_{1,0,0} \), \( g_2 = g_{0,1,0} \), \( g_3 = g_{0,0,1} \). The terms \( u_\alpha, |\alpha| > 1 \) satisfy a series of linear differential equations with forcing terms which are linear combinations, with \( C^\infty \) coefficients depending on derivatives of \( F(y, u_0(y)) \), of products of the previous solutions \( u_\beta \) with \(|\beta| < |\alpha| \). For example, the term \( u_{1,1,1} \) satisfies

\[
(P(y, D) + \partial_u F)(y, u_0(y))u_{1,1,1} + (\partial_u^2 F)(y, u_0(y))(u_{1,0,0}u_{1,1,1} + u_{0,1,0}u_{1,0,0}) = (\partial_u^3 F)(y, u_0(y))u_{1,0,0}u_{1,0,0}u_{0,0,1} = 0,
\]

\[
u_{1,1,1} = 0, \quad t < 0.
\]

It is shown in [13, 25, 44], using the calculus of paired Lagrangians of Greenleaf and Uhlmann [17], and Melrose and Uhlmann [26], that if \( E_+ \) is the forward fundamental
solution of $P(y, D)$, then microlocally near $\Lambda$ and away from $N^*\Sigma_j$, $j = 1, 2, 3$,
\begin{equation}
  u_{1,1,1} = -E_+((\partial_u^3 F)(y, u_0(y)))|_{\Gamma}u_{0,0,0}^0u_{0,1,0}^0u_{0,0,1}^0 + \text{smoother terms},
\end{equation}
where $u_{0}^0$ is the principal part of $u_{\alpha}$, $|\alpha| = 1$.

Notice this describes singularities of $u_{1,1,1}$, and not necessarily singularities of $u$. It is important to realize that equation (5.6) is linear, and its solution $u_{1,1,1}$ is a Lagrangian distribution with respect to $\Lambda$, away from $\Sigma_j$ and $\Gamma$, and this is not affected by appearance of caustics on $Q$. By assumption, $\Sigma_j$ do not have caustics.

We find that, away from $N^*\Sigma_j$, $j = 1, 2, 3$, $\sigma(u_{1,1,1})$, which is the principal symbol of $u_{1,1,1}$, satisfies (A.2), with $f$ replaced by $(\partial_u^3 F)(y, u_0(y))|_{\Gamma}u_{0,0,0}^0u_{0,1,0}^0u_{0,0,1}^0$.

Equation (A.2) implies that, modulo an integrating factor which takes into account half densities and the subprincipal symbol of $p$, $\sigma(u_{1,1,1})$ is constant along the integral curves of $H_p$, and since one knows the principal part of $u_{1,0,1}$, $u_{0,1,0}$ and $u_{0,1,1}$ (notice that the principal part of $u_{\alpha}$, $|\alpha| = 1$, which is given by (5.5) does not depend on $(\partial_u F)(y, u_0(y)))$ one can determine $(\partial_u^3 F)(y, u_0(y))|_{\Gamma}$, where $u_0(y)$ is the solution to (5.4), by measuring from $\sigma(u_{1,1,1})$ along any null-bicharacteristic of $p$ starting over the point $y \in \Gamma$.

This holds for any $g_0 \in C^\infty(\Omega)$, but since (5.1) is well posed for $||g||_{H^s}$ small, this determines $(\partial_u^3 F)(y, u(y))|_{\Gamma}$, where $u$ is the corresponding solution of (5.1), for any $g \in H^s(\Omega)$ with small enough norm.

In the case of Theorem 4.1 one cannot work with Lagrangian distributions, or paired Lagrangians distributions, to filter singularities of the solution $u$ to (2.4), since these spaces will not be closed under composition with $C^\infty$ functions. We do not work with a linearization of the equation, and we use the spaces introduced in Section 10 to show similarly to (5.7), microlocally near $\Lambda$ and away from $N^*\Sigma_j$, $j = 1, 2, 3$, the solution of (2.4) satisfies
\begin{equation}
  u = E_+ \left( (\partial_u^3 F)(y, u(y))|_{\Gamma} V \right) + \text{smoother terms}, \text{ where } V \text{ is defined in (4.1)}.
\end{equation}

The principal symbol of $u$ on $\Lambda$ is given by that of $E_+ \left( (\partial_u^3 F)(y, u(y))|_{\Gamma} V \right)$ on $\Lambda$ and away from $N^*\Sigma_j$, $j = 1, 2, 3$, satisfies (A.2), with $E_+ \left( (\partial_u^3 F)(y, u(y))|_{\Gamma} V \right)$ in place of $f$. Therefore, if one measures $\sigma(w)$ along any bicharacteristic which lies on $\Lambda$ and is away from the incoming surfaces $N^*\Sigma_j \setminus 0$, one obtains the quantity given by (A.2). Since one knows $v_1$, $v_2$ and $v_3$, one can determine $(\partial_u^3 F)(y, u(y))|_{\Gamma}$, where $u$ is the solution to (5.1), from $\sigma(u)$ on $\Lambda$.

A second application is similar to what is done in [13], which was carried out using the linearization method described above. However, we should say that [13] treats the case of a system, and it is not quite clear if the methods used here apply to that situation. Suppose one a priori knows the nonlinearity, and it is of the form $F(y, u) = a(y)u^3$. Suppose one knows the principal part of $P(y, D)$, but now one wants to determine the subprincipal symbol of $P(y, D)$ from the singularities of the solution $u$. One can measure the principal symbol of $w$ on $\Lambda$ away from $N^*\Sigma_j \setminus 0$, $j = 1, 2, 3$. In this case, $(\partial_u^3 F)(y, u) = 6a(y)$ is known, and the information about the subprincipal symbol of $P(y, D)$ is contained on the integrating factor used to solve (A.2). This determines the integral of the subprincipal symbol of $P(y, D)$ along bicharacteristics from the point on $(N^*\Gamma \setminus 0) \cap p^{-1}(0)$ where they start to the point where measurements are made. If one can repeat the argument for perturbations of $\Sigma_j$, $j = 1, 2, 3$, one finds the integral of the principal symbol over bicharacteristic starting over a family of $\Gamma$. In particular, one can arrange so that by differentiating
the integral along one bicharacteristic with respect to the starting point, this will determine the subprincipal symbol at \( \Gamma \).

6. **An outline of the proof of theorem 4.1.** To prove a theorem on propagation of singularities for solutions of semilinear wave equation one usually needs to work on suitable spaces, let’s say \( J_\alpha(\Omega) \), where \( \alpha \) is an element of some index family, which are defined according to the geometric features of the problem, and have two basic properties:

**J.1.** \( J_\alpha(\Omega) \) is a \( C^\infty \) algebra. Namely, if \( u_1, \ldots, u_N \in J_\alpha(\Omega) \) and \( f \in C^\infty(\Omega \times \mathbb{R}^N) \), then \( f(y, u_1, \ldots, u_N) \in J_\alpha(\Omega) \).

**J.2.** If \( P(y, D)u = F \in J_\alpha(\Omega) \), \( u = F = 0 \) for \( t < -1 \), then \( u, Du \in J_\alpha(\Omega) \).

Once these are defined, a standard bootstrapping argument, which will be explained below, gives a propagation theorem for semilinear equations \( P(y, D)u = f(y, u) \) for those spaces. For equations of the type \( P(y, D)u = f(y, u, Du) \) one would need to supplement both properties with estimates, but we are not pursuing this here.

We will use two such families of spaces. The ones associated with the propagation of conormality as in \([10, 11, 30, 38]\), and another which are the spaces discussed in Section 10. We mostly use the latter spaces to filter singularities in our proofs.

We explain the main ideas of the proof in a simplified version of the theorem. Here we shall suppose that \( \Omega \) is small enough such that there exist local coordinates

\[
y = (y', y^3), \quad y' = (y_1, y_2, y_3)
\]

such that \( \Sigma_j = \{ y_j = 0 \}, \; j = 1, 2, 3 \), valid in \( \Omega \), but the existence of such coordinates is not required in the proof of Theorem 4.1. We denote \( F(y, u) = y(y)f(y, u) \) and analyze the singularities of the solution of

\[
(6.1) \quad P(y, D)u = F(y, u),
\]

\[
u = v_1 + v_2 + v_3 \text{ for } t < -1.
\]

If \( s > \frac{n}{2} \), it is well known that \( H^s_{\text{loc}}(\Omega) \) is a \( C^\infty \) algebra – it is closed under composition with \( C^\infty \) functions. If \( v \in H^s_{\text{loc}}(\Omega) \), equation (6.1) can be solved by using a contraction mapping argument to show that, for small enough \( ||v||_s \), there exists a unique \( u \in H^s_{\text{loc}}(\Omega) \) that satisfies

\[
(6.2) \quad u = v + E_+(F(y, u)),
\]

where \( E_+ \) is the forward fundamental solution of \( P \) and

\[
E_+ : H^s_{\text{loc}}(\Omega) \longrightarrow H^{s+1}_{\text{loc}}(\Omega).
\]

Now to analyze the propagation of singularities, we assume that the solution exists and proceed as in \([4]\) and \([42]\). We take advantage of the fact that \( E_+(F(y, u)) \) is smoother than \( u \), we iterate this formula and obtain

\[
(6.3) \quad u = v + E_+[F(y, v + E_+(F(y, u)))].
\]

We shall appeal to the work of Rauch and Reed \([37]\) and Pirion \([34]\) which show that if \( v_j \in I^{m-\frac{3}{2}+\frac{1}{2}}(\Omega, \Sigma_j) \) with \( m < -1 \), and \( \Sigma_j = \{ y_j = 0 \} \), then \( v_j = v_j + \mathcal{E}_j \), where \( v_j = y^j_3 w_j \), \( k = k(m) \) is the non-negative integer such that \( -m - 2 \leq k(m) < -m - 1 \), \( \mathcal{E}_j \in C^\infty \) and \( w_j \in I^{m-\frac{3}{2}+\frac{1}{2}+k(m)}(\Omega, \Sigma_j) \). We then write

\[
v = v_1 + v_2 + v_3 = \nu + \mathcal{E}, \quad \nu = v_1 + v_2 + v_3, \quad \mathcal{E} \in C^\infty,
\]

and define \( \mathcal{W} = u - \nu = \mathcal{E} + E_+(F(y, u)) \).
Since $\nu = 0$ at $\Gamma$, $W = u$ at $\Gamma$. We then expand $F(y, \nu(y) + W)$ in Taylor series in $\nu$ centered at $W$:

$$
F(y, \nu + W) = F(y, W) + (\partial_u F)(y, W)\nu + \frac{1}{2}(\partial^2_u F)(y, W)\nu^2 + \frac{1}{6}(\partial^3_u F)(y, W)\nu^3 + \frac{\nu^4}{6} \int_0^1 (\partial^4_u F)(y, tW + (1-t)\nu)(1-t)^3 dt.
$$

We introduce a family of spaces in Section 10, see Definition 10.1, which we will use to filter singularities.

As stated in Theorem 4.1, we analyze the regularity of $u$ microlocally near $\Lambda$ and away from $N^*\Sigma_j$, $j = 1, 2, 3$. Notice that in these coordinates

$$
N^*\Sigma_1 = \{y_1 = 0, \eta_2 = \eta_3 = 0, \eta'' = 0\}, \quad N^*\Sigma_2 = \{y_2 = 0, \eta_1 = \eta_3 = 0, \eta'' = 0\}, \quad N^*\Sigma_3 = \{y_3 = 0, \eta_1 = \eta_2 = 0, \eta'' = 0\}, \quad N^*\Gamma = \{y_1 = y_2 = y_3 = 0, \eta'' = 0\},
$$

where $\eta''$ is the dual to $y''$. So, to stay away from $N^*\Sigma_j$, $j = 1, 2, 3$, we need to work microlocally in the region where $\eta_j$ is elliptic, $j = 1, 2, 3$, and by that we mean $\langle \eta_j \rangle \gtrsim \langle \eta \rangle$, $j = 1, 2, 3$.

We will prove the following claims:

**Claim 1** (proved in Proposition 11.1 below): The term

$$
\Re(y) = F(y, W) + (\partial_u F)(y, W)\nu + \frac{1}{2}(\partial^2_u F)(y, W)\nu^2 + \frac{\nu^4}{6} \int_0^1 (\partial^4_u F)(y, tW + (1-t)\nu)(1-t)^3 dt \text{ is smoother than } \frac{1}{6}(\partial^3_u F)(y, W)\nu^3.
$$

**Claim 2** (also proved in Proposition 11.1 below): The term

$$
\frac{1}{6}(\partial^3_u F)(y, W)\nu^3 = (\partial^3_u F)(y, W)(\nu_1\nu_2\nu_3) + \text{smoother terms.}
$$

However, since we do not know much about the singularities of $W$, we need to look closer at $(\partial^3_u F)(y, W)(\nu_1\nu_2\nu_3)$. We will show that microlocally where $\eta_j$ is elliptic, $j = 1, 2, 3$,

$$
\nu_1\nu_2\nu_3 = V + \text{smoother terms},
$$

where $V$ is a conormal distribution to $\Gamma$ and

**Claim 3** (proved in Proposition 11.3 below):

$$
(\partial^3_u F)(y, W)(\nu_1\nu_2\nu_3) = (\partial^3_u F)(0, 0, 0, y'', W(0, 0, 0, y''))V + \text{smoother terms.}
$$

But according to (6.4), $W(0, 0, 0, y'') = u(0, 0, 0, y'')$. So we find that near $\Omega$ and away from $\Sigma_j$, $j = 1, 2, 3$, which is the flow-out of the region of $(N^*\Gamma \setminus 0) \cap p^{-1}(0)$ where $\eta_j$ is elliptic, $j = 1, 2, 3$, equations (6.3), (6.5), (6.6) and (6.7) show that

$$
u = E_+ ((\partial^3_u F)(0, 0, 0, y'', u(0, 0, 0, y''))V) + \text{smoother terms.}
$$

When $n > 3$, we need to say more about $(\partial^3_u F)(0, 0, 0, y'', u(0, 0, 0, y''))V$. We use the results of Bony [10, 11] and Melrose and Ritter [30] to show that

$$
(\partial^3_u F)(0, 0, 0, y'', u(0, 0, 0, y'')) \in C^\infty(\Gamma)
$$

and since we have a fixed coordinate system, we may think of

$$
(\partial^3_u F)(0, 0, 0, y'', u(0, 0, 0, y'')) \in C^\infty(\Omega)
$$

which does not depend on $y_1, y_2, y_3$, and therefore $(\partial^3_u F)(0, 0, 0, y'', u(0, 0, 0, y''))V$ makes sense. In general, when we do not have fixed local coordinates, we will show
that for any two $\mathcal{G}, \mathcal{G}_1 \in C^\infty(\Omega)$ such that $\mathcal{G} - \mathcal{G}_1 = 0$ at $\Gamma$, $(\mathcal{G} - \mathcal{G}_1)V$ is smoother than $\mathcal{G}V$ or $\mathcal{G}_1V$ and we can say that

$$u = E_+(\mathcal{G}V) + \text{ smoother terms, if } \mathcal{G}(0, 0, 0, y'') = (\partial_\alpha^2 F)(0, 0, 0, y'', u(0, 0, 0, y''))$$

and its principal symbol is given by (A.2).

7. **Conormal distributions.** We first recall the definition of the class of conormal distributions to a $C^\infty$ closed submanifold $M \subset \Omega$ of codimension $k$. Let $V_M$ denote the Lie algebra of $C^\infty$ vector fields tangent to $M$. We will use $M$ to denote $C^\infty$ submanifolds of $\Omega$ and $\Sigma$ to denote $C^\infty$ hypersurfaces.

As in Hörmander [20], we say that $u \in I^m(\Omega, M)$, $m \in \mathbb{R}$, and $u$ is a conormal distribution to $M$ of order $m$, if for any $N \in \mathbb{N}_0$,

$$V_1 V_2 \ldots V_N u \in \infty H^loc_{-\frac{k}{2}}(\Omega), \ V_j \in V_M.$$  

The definition of the Besov spaces $\infty H^loc_{-\frac{k}{2}}(\Omega)$ can be found in Appendix B of [20].

It follows from the definition that

$$I^m(\Omega, M) \subset I^{m'}(\Omega, M), \ m \leq m'.$$

In general, if $W$ is a Lie algebra and $C^\infty$ module of $C^\infty$ vector fields, the space of conormal distributions with respect to $W$ is defined to be

$$IH^loc_{-\frac{k}{2}}(\Omega, W) = \{ u \in H^loc_{-\frac{k}{2}}(\Omega) : V_1 V_2 \ldots V_N u \in H^loc_{-\frac{k}{2}}(\Omega), \ s \in \mathbb{R}, V_j \in W, \ N \in \mathbb{N}_0 \}.$$  

This can also be defined in terms of Besov spaces, instead of Sobolev ones. We will consider spaces of conormal distributions related to the interaction of waves.

Let $\Sigma_j \subset \Omega, \ j = 1, 2, 3$, be closed $C^\infty$ hypersurfaces which intersect transversally at $\Sigma_i \cap \Sigma_j = \Gamma_{ij} i \neq j$, and at $\Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$. Let $\mathcal{Q}$ be defined as above. The following Lie algebras and $C^\infty$ modules of $C^\infty$ vector fields will play an important role in this paper:

$$W_j \text{ denotes the } C^\infty \text{ vector fields tangent to } \Sigma_j, \ j = 1, 2, 3,$$

$$W_{jk} \text{ denotes the } C^\infty \text{ vector fields tangent to } \Sigma_j \cup \Sigma_k, \ j = 1, 2, 3, \ j \neq k,$$

$$W_{123} \text{ denotes the } C^\infty \text{ vector fields tangent to } \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,$$

$$W_{123,0} \text{ denotes the } C^\infty \text{ vector fields tangent to } \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \mathcal{Q}.$$  

These Lie algebras are locally finitely generated. In local coordinates $y = (y', y'')$, $y' = (y_1, y_2, y_3)$, where $\Sigma_j = \{ y_j = 0 \}, \ j = 1, 2, 3$, we have

$$W_j = C^\infty - \text{ span of } \{ y_j \partial_{y_j}, \partial_{y_k}, \ k \neq j \},$$

$$W_{jk} = C^\infty(\Omega) - \text{ span of } \{ y_j \partial_{y_j}, y_k \partial_{y_k}, \partial_{y_m}, \ m \neq j, k \},$$

$$W_{123} = C^\infty(\Omega) - \text{ span of } \{ y_1 \partial_{y_1}, y_2 \partial_{y_2}, y_3 \partial_{y_3}, \partial_{y_m}, \ m \geq 4 \}.$$  

$W_{123,0} \text{ is also finitely generated, see [30].}$

The class of symbols $S^r(\mathbb{R}^n \times \mathbb{R}^k)$ is defined as the space of $C^\infty(\mathbb{R}^n \times \mathbb{R}^k)$ functions that satisfy

$$|D^\alpha_y D^\beta_{y''} b(y, y'')| \leq C_{\alpha, \beta}(1 + |y''|)^{r-|\beta|}, \ \alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^k.$$  

These spaces satisfy

$$S^r(\mathbb{R}^{n-k} \times \mathbb{R}^k) \subset S^{r'}(\mathbb{R}^{n-k} \times \mathbb{R}^k), \ r \leq r'.$$
Theorem 18.2.8 of [20] says that \( u \in I^m(\Omega, M) \) if and only if \( u \in C^\infty(\mathbb{R}^n \setminus M) \) and in a neighborhood of any point \( p \in M \), in local coordinates where

\[
(7.8) \quad y = (y', y'') , \quad y' = (y_1, y_2, \ldots, y_k), \quad \text{such that } M = \{ y_1 = y_2 = \ldots = y_k = 0 \},
\]

\( u \) is given by

\[
(7.9) \quad u(y) = \int_{\mathbb{R}^k} e^{iy' \cdot \eta'} a(y, \eta') \, d\eta', \quad a \in S^{m+\frac{n-2k}{n-k}}(\mathbb{R}_y^n \times \mathbb{R}^k_\eta).
\]

If one multiplies a conormal distribution \( u \in I^m(\Omega, M) \) by a \( C^\infty \) function \( f \) which vanishes on \( M \), \( fu \in I^m(\Omega, M) \), with \( m' < m \). This is made precise in the following

**Proposition 7.1.** (Proposition 18.2.3 of [20]) Let \( M \subset \Omega \) be a \( C^\infty \) submanifold of codimension \( k \). Let \( u \in I^m(\Omega, M) \) and let \( y = (y', y'') \), be local coordinates as in (7.8). If \( \alpha \in \mathbb{N}_0^k \), then \( y'^a u \in I^{m-|\alpha|}(\Omega, M) \).

Therefore, if \( u \in I^m(\Omega, M) \) satisfies (7.9), and \( y = (y', y'') \), satisfy (7.8), the Taylor expansion of its symbol about \( y' = 0 \) satisfies

\[
\frac{1}{\alpha!} y^{\alpha} \partial_{y''}^\alpha a(0, y'', \eta') = O(|y'|^{|\alpha|+1}),
\]

and therefore, by Borel summation formula,

\[
(7.10) \quad u(y) = \int_{\mathbb{R}^k} e^{iy' \cdot \eta'} a(0, y'', \eta') \, d\eta'' + \int_{\mathbb{R}^k} e^{iy' \cdot \eta'} b(0, y'', \eta') \, d\eta'' + \mathcal{E}, \quad \text{where } \mathcal{E} \in C^\infty, \text{ and}
\]

\[
b(0, y'', \eta') \sim \sum_{|\alpha| \geq 1} \frac{|\alpha|}{\alpha!} \partial_{y''}^\alpha a(0, y'', \eta') \in S^{m-1-\frac{2k}{n-k}}(\mathbb{R}_y^n \times \mathbb{R}^k_\eta).
\]

The principal symbol of \( u \) is defined to be

\[
[a(0, y'', \eta')] \in S^{m+\frac{n-2k}{n-k}}(\mathbb{R}_y^n \times \mathbb{R}^k)/S^{m-\frac{n-2k}{n-k}}(\mathbb{R}_y^n \times \mathbb{R}^k),
\]

which is the equivalence class of \( a(0, y'', \eta') \) in this quotient.

However, this definition is not coordinate invariant. Following [20], this issue is resolved if one thinks of conormal distributions as distributions acting on half-densities \( \Gamma^\frac{1}{2}_\Omega \) and their principal symbol as an element of the half-density bundle \( \Gamma^\frac{1}{2}_{N^*M} \) on the conormal bundle \( N^*(M) \). In local coordinates \((y', y'', \eta', \eta'')\) this is given by

\[
(7.11) \quad a(0, y'', \eta') |dy''|^{\frac{1}{2}} |d\eta''|^{\frac{1}{2}} \in S^{m+\frac{n-2k}{2}}(N^*(M), \Gamma^\frac{1}{2}_{N^*M}).
\]

One needs to realize that \( |d\eta''|^{\frac{1}{2}} \) is homogeneous of degree \( \frac{k}{2} \), and so this is a symbol of the order stated above.

We conclude that the principal symbol map is the isomorphism

\[
S^{m+\frac{n-2k}{2}}(N^*(M), \Gamma^\frac{1}{2}_{N^*M})/S^{m+\frac{n-2k}{2}-1}(N^*(M), \Gamma^\frac{1}{2}_{N^*M}) \longrightarrow I^m(\Omega, M; \Gamma^\frac{1}{2}_\Omega)/I^{m-1}(\Omega, M; \Gamma^\frac{1}{2}_\Omega) \to [a] \mapsto [a].
\]
7.1. Further properties of conormal distributions. First, we recall a result due to Rauch and Reed, Proposition 2.1 of [37] which is very important in the study of nonlinear equations:

**Proposition 7.2.** Let \( \Sigma \subset \Omega \) be a closed \( C^\infty \) hypersurface and \( u_j \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma) \), with \( m < -1, j = 1, 2, \ldots, N \). If \( f(y, x_1, \ldots, x_N) \in C^\infty \), then \( f(y, u_1, \ldots, u_N) \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma) \).

Next we recall properties of conormal distributions established by Rauch and Reed [37] and Piriou [34]. Let \( u \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), m < -1, \) where \( \Sigma \) is a \( C^\infty \) closed hypersurface on \( \Omega \) and in local coordinates (7.8), \( \Sigma = \{y_1 = 0\} \). Then

\[
\eta(y) = \frac{1}{2\pi} \int_\mathbb{R} e^{iy \cdot \eta} a(y, \eta) d\eta + \mathcal{E}, \quad a \in \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R}), \quad \mathcal{E} \in C^\infty.
\]

One can show that if \( m < -1, \) and \( k(m) \) is the non-negative integer such that \( -m - 2 \leq k(m) < -m - 1, \) we have

\[
\eta(y) = y^k v_k(y) + \mathcal{E}, \quad \text{such that} \quad \mathcal{E} \in C^\infty, \quad \text{and}
\]

\[
v_k(y) = \int_\mathbb{R} e^{iy \cdot \eta} b_k(y, \eta) d\eta, \quad b_k \in \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R}), \quad 0 \leq k \leq k(m),
\]

see for example [34], or [42] details. We then define, as in [34,37],

**Definition 7.3.** Let \( \Sigma \subset \Omega \) be a \( C^\infty \) closed hypersurface. For \( m < -1, \) the space \( I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma) \) consisting of elements \( u \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma) \) which vanish to order \( k(m)-1 \) at \( \Sigma, \) where \( k(m) \) is the only positive integer in the interval \( [-m-2, -m-1]. \)

In local coordinates where \( \Sigma = \{y_1 = 0\}, \) this means that \( u \) can be written as \( u = y_1^k v_k, \) with \( v_k \in I^{m-\frac{n}{2}+\frac{j}{4}+k}(\Omega, \Sigma), \) \( k \leq k(m). \)

And one can prove, see [34,37,42]

**Proposition 7.4.** Let \( \Sigma \subset \Omega \) be a \( C^\infty \) closed hypersurface and \( u \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \) with \( m < -1, \) then

\[
u(y) = v(y) + \mathcal{E}, \quad v \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \quad \mathcal{E} \in C^\infty.
\]

As a consequence of the definition of \( I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \) Proposition 7.1 and Proposition 7.2 we have:

**Proposition 7.5.** Let \( \Sigma \subset \Omega \) be a closed \( C^\infty \) hypersurface let \( u_j \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \) \( 1 \leq j \leq N, \) and \( m < -1, \) then \( u_1 u_2 \ldots u_N \in I^{m-\frac{n}{2}+\frac{j}{4}-(N-1)k(m)}(\Omega, \Sigma). \) In particular, if \( u \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \) then \( u^N \in I^{m-(N-1)k(m)-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma). \)

**Proof.** Suppose that \( \Sigma = \{y_1 = 0\}. \) If \( u_j \in I^{m-\frac{n}{2}+\frac{j}{4}}(\Omega, \Sigma), \) then by definition, there exists \( v_j \in I^{m-\frac{n}{2}+\frac{j}{4}+k(m)}(\Omega, \Sigma) \) such that \( u_j = y_1^{k(m)} v_j. \) So, \( u_1 \ldots u_N = y_1^{Nk(m)} v_1 \ldots v_N. \) Since \( m+k(m) < -1, \) it follows from Proposition 7.2 that

\[ v_1 \ldots v_N \in I^{m-\frac{n}{2}+\frac{j}{4}+k(m)}(\Omega, \Sigma). \]

Then Proposition 7.1 gives that \( u_1 \ldots u_N \in I^{m-\frac{n}{2}+\frac{j}{4}+k(m)-Nk(m)}(\Omega, \Sigma), \) and this proves the Proposition.
8. Propagation of singularities for the double and triple interactions.

We briefly recall the results of Bony \[9,10\] and Melrose and Ritter \[30\] about the evolution of one wave and the double and triple transversal interactions and we analyze the regularity of \(u|_\Gamma\).

Bony \[8,9\] proved the following result regarding the propagation of conormal regularity with respect to one hypersurface and two transversal hypersurfaces, see also \[30\]:

**Theorem 8.1.** Let \(u \in H^s_{\text{loc}}(\Omega)\), \(s > \frac{3}{2}\), satisfy \(P(y,D)u = f(y,u)\), \(f \in C^\infty\). Let \(\Sigma_1\) and \(\Sigma_2\) be closed \(C^\infty\) hypersurfaces in \(\Omega\) intersecting transversally, and let \(W_j\) and \(W_{jk}\) be the Lie algebras of \(C^\infty\) vector fields defined in (7.4):

1. If \(u \in IH^s_{\text{loc}}(\Omega,W_J)\) in \(t < 0\), then \(u \in IH^s_{\text{loc}}(\Omega,W_J)\).
2. If \(u \in IH^s_{\text{loc}}(\Omega,W_{jk})\) in \(t < 0\), then \(u \in IH^s_{\text{loc}}(\Omega,W_{jk})\).

In these cases the spaces \(J_*(\Omega)\) referred to in Section 6 are \(J_*(\Omega) = IH^s_{\text{loc}}(\Omega,W_J)\) or \(J_*(\Omega) = IH^s_{\text{loc}}(\Omega,W_{jk})\). The finite regularity spaces, where one only allows \(N\) derivatives with respect to the algebras \(W_j\) or \(W_{jk}\), also satisfy both properties, but we will not use this here.

Let \(W_{123}\) denote the Lie algebra of \(C^\infty\) vector fields tangent to \(\Sigma_j\), \(j = 1,2,3\). The purpose of this paper is to show that in general

if \(u \in IH^s_{\text{loc}}(\Omega,W_{123})\) in \(t < 0\), then \(u \notin IH^s_{\text{loc}}(\Omega,W_{123})\),

since singularities will form on \(\Omega\). One may ask whether

if \(u \in IH^s_{\text{loc}}(\Omega,W_{1230})\) in \(t < 0\) implies that \(u \in IH^s_{\text{loc}}(\Omega,W_{1230})\),

where \(W_{1230}\) denotes the Lie algebra of \(C^\infty\) vector fields tangent to \(\Sigma_j\), \(j = 1,2,3\), and \(\Omega\). The answer is not known to the author, because the Lie algebra \(W_{1230}\) is too degenerate at \(\Gamma\). However, one can construct a smaller space of distributions \(J(\Omega)\) associated with \(\Omega \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3\) which satisfies properties J.1 and J.2, defined in Section 6, and the following

1. \(IH^s_{\text{loc}}(\Omega,W_{123}) \subset J(\Omega) \subset IH^s_{\text{loc}}(\Omega,W_{1230})\),
2. If \(Pu = f(y,u)\) and \(u \in IH^s_{\text{loc}}(\Omega \backslash \Gamma,W_{1230})\), then \(u \in J(\Omega \backslash \Gamma)\).

The space \(J(\Omega)\) is defined by first blowing up \(\Gamma\) and then blowing up the intersections of the lift of \(\Omega\) and \(\Sigma_j\), \(j = 1,2,3\), and defining \(J(\Omega)\) by the action of vector fields tangent to the lift of \(\Sigma_j\), \(j = 1,2,3\), \(\Omega\) and the boundary of the blown-up manifold.

The blow-up of \(\Gamma\) is of course independent of choice of coordinates, but it is useful to express it in local coordinates \(y = (y_1,y_2,y_3,y''')\), where \(\Gamma = \{y_1 = y_2 = y_3 = 0\}\), by just taking polar coordinates with respect to the variables \(\{y_1,y_2,y_3\}\), \(\rho = (y_1^2 + y_2^2 + y_3^2)^{\frac{1}{2}}, \omega = \frac{1}{\rho}(y_1,y_2,y_3)\). The map \(\beta_1 : X_1 \overset{\text{def}}{=} S^2 \times [0,\infty) \times \mathbb{R}^{n-3} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^{n-3}, \omega, \rho, y''\mapsto (\rho \omega, y''\).

\(X_1\) is a manifold with boundary and \(\beta_1^* \Sigma_j\), \(j = 1,2,3\), and \(\beta_1^* \Omega\) intersect its boundary transversally. However \(\beta_1^* \Sigma_j\) and \(\beta_1^* \Omega\) still intersect tangentially to first order at \(L_j \in \beta_1^* (\Sigma_j \cap \Omega)\), and this needs to be resolved. The best way of doing this is to use non-homogeneous blow-ups. In local coordinates where \(\Sigma_j = \{x = 0\}\) and \(\Omega = \{x = z^2\}\) one defines \(R = (x^2 + z^4)^{\frac{1}{2}}\) and \((x/R^2, z/R^4)\); and so \(\omega_1^2 + \omega_2^4 = 1\). One now has a manifold with corners \(X_2\) and a blow-down map \(\beta_2 : X_2 \longrightarrow X_1\). The composition \(\beta = \beta_1 \circ \beta_2\) is defined as the blow-down map. The lifts \(\beta^* \Sigma_j\),
$j = 1, 2, 3$ intersect each other and the boundary of $X_2$ transversally and do not intersect $\beta^*\Omega$, while $\beta^*\Omega$ intersects the boundary of $X_2$ transversally. We define the space $J(\Omega)$ just like $IH^s(\Omega, W_j)$, but instead of $\Omega$ we now have $X_2$, and instead of $W_j$ we have $\tilde W$, which denotes the Lie algebra of $C^\infty$ vector fields on $X_2$ which are tangent to $\beta^*\Sigma_j$, $j = 1, 2, 3$ and $\beta^*\Omega$ and the boundary of $X_2$, and a suitable replacement for the Sobolev space $H^s$. We refer the reader to [30, 38, 39] for a more thorough discussion. One can also think of $J(\Omega)$ as being defined by the action of singular vector fields in $\Omega$, as in [4].

It is relatively simple to prove that a subspace of $H^s$, $s > \frac{n}{2}$, which consists of functions which are also stable under the application of vector fields, even in blown-up manifolds, are $C^\infty$ algebras. It is a lot harder to prove that the spaces propagate, i.e. they satisfy Property J.2 in Section 6. One obtains the following, see [38]:

**Theorem 8.2.** Let $u \in H^s_{\text{loc}}(\Omega)$, $s > \frac{n}{2}$, satisfy $P(y, D)u = f(y, u)$, $f \in C^\infty$. Let $\Sigma_j$, $j = 1, 2, 3$ be closed $C^\infty$ characteristic hypersurfaces intersecting transversally at $\Gamma$ and let $\Omega$ be as defined above. Suppose that assumptions T.1 to T.2 of Theorem 4.1 are satisfied. If $u \in J(\Omega \cap \{t < -1\})$ then $u \in J(\Omega \cap \{t < T\})$.

As mentioned in the introduction, we need the following result for $n > 3$:

**Proposition 8.3.** Under the conditions of Theorem 8.2, $u|_{\Gamma} \in C^\infty(\Gamma \cap \{t < T\})$.

**Proof.** It follows from the definition of $J(\Omega)$ that if $\mathbb{U}_j$, $j = 1, 2, 3$, are Lie algebras of vector fields tangent to $\Sigma_j \cup \Omega \cup \Gamma$, then

$$J(\Omega) \subset \sum_{j=1}^{3} IH^s_{\text{loc}}(\Omega, \mathbb{U}_j), \quad s > \frac{n}{2}.$$  

The space on the right of is not good enough to prove a propagation theorem; $J(\Omega)$ is much smaller, but it serves well for this purpose.

As discussed above, the space $J(\Omega)$ is defined by first blowing up $\Gamma$ and then blowing up lines of intersection of the lift of $\Omega$ and $\Sigma_j$, $j = 1, 2, 3$, and the lifts of $\Omega$ and $\Sigma_j$ are disjoint $C^\infty$ hypersurfaces on a manifold with corners $X_2$, and therefore one can use a partition of unity on $X_2$ to show that $J(\Omega)$ is a sum of three spaces corresponding to vector fields tangent the lifts of $\Omega$, one of the of the $\Sigma_j$, $j = 1, 2, 3$, and the boundary of the blown-up manifold.

It is tedious, but one can use the local model given by Proposition 2.1 and the definition of the blow-ups in terms of local coordinates given above to check that the vector fields in $\mathbb{U}_j$ lift to vector fields tangent to the lifts of $\Omega$, $\Sigma_j$ and the boundary of $X_2$, and this proves (8.1).

We also need to appeal to Proposition 2.1 to see that for every point $q \in \Gamma$, there exist a neighborhood $U_q \subset \Omega$, a neighborhood $U_0$ of $0 \in \mathbb{R}^n$ and a $C^\infty$ diffeomorphism $\Psi : U_0 \leftrightarrow U_q$ such that $\partial_{y_k} \in \Psi^* (\mathbb{U}_1)$, $k = 4, \ldots, n$, and so if $\varphi \in IH^s_{\text{loc}}(\Omega, U_1)$, then

$$\partial_{y_4}^k \ldots \partial_{y_n}^k \Psi^* \varphi(y) \in H^s_{\text{loc}}(\mathbb{R}^n), \quad k_j \in \mathbb{N}_0, \quad j = 4, \ldots, n, \quad s > \frac{n}{2},$$

and hence

$$\partial_{y_4}^k \ldots \partial_{y_n}^k (\Psi^* \varphi)(0, 0, 0, y''') \in H^{s-\frac{2}{3}}_{\text{loc}}(\mathbb{R}^{n-3}), \quad k_j \in \mathbb{N}_0, \quad j = 4, \ldots, n, \quad s > \frac{n}{2}.$$
and in particular $\Psi^\ast(\varphi)(0,0,0,y'') \in C^\infty(U_0 \cap \{(0,0,0,y'')\})$. The charts $(U_q \cap \Gamma, \Psi_q|\Gamma)$ give an atlas of $\Gamma$, and since the transition functions fix $\Gamma$, and we conclude

that $\varphi \in C^\infty(\Gamma)$.

Of course similar thing can be done with $\Sigma_2$ and $\Sigma_3$ and this gives the result. $\square$

We will use this result in Proposition 10.12 to make sense of $(\partial^1_n(yf))(y,u(y))|_V$.

9. Normal forms of the operator and the surfaces. Before we can define the generalization of Beals spaces, we need to show that the operator $P(y, D)$ and surfaces can be put into local normal form.

**Theorem 9.1.** Let $\Omega \subset \mathbb{R}^n$ be an open subset, $n \geq 3$, let $P(y, D)$ be a second order strictly hyperbolic operator in $\Omega$. Let $\Sigma_j \subset \Omega$, $j = 1, 2, 3$, be closed $C^\infty$ hypersurfaces that are characteristic for $P(y, D)$ and intersect transversally at $\Gamma_{jk} = \Sigma_j \cap \Sigma_k$ and at $\Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$. We have the following normal forms for $\Sigma_j$ and $P(y, D)$, where $\mathcal{L}$ denotes a differential operator of order one with $C^\infty$ coefficients.

**NF.1** If $q \in \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_3)$, there exist local coordinates $y = (y_1, y'')$ near $q$ such that

$$\Sigma_1 = \{y_1 = 0\} \quad \text{and} \quad P(y, D) = \sum_{j=2}^n b_{1j}(y)\partial_{y_1}\partial_{y_j} + \sum_{j,k=2}^n b_{jk}(y)\partial_{y_j}\partial_{y_k} + \mathcal{L},$$

where $b_{jk} \in C^\infty$. Similar formulas hold near $\Sigma_j$, $j = 2, 3$.

**NF.2** For $q \in (\Sigma_1 \cap \Sigma_2) \setminus \Sigma_3$, there exist local coordinates $y = (y_1, y_2, y'')$ near $q$ such that

$$\Sigma_j = \{y_j = 0\}, \quad j = 1, 2, \quad \text{and} \quad P(y, D) = b_{12}(y)\partial_{y_1}\partial_{y_2} + \sum_{j=3}^n b_{1j}(y)\partial_{y_1}\partial_{y_j} + \sum_{j=3}^n b_{2j}(y)\partial_{y_2}\partial_{y_j} + \sum_{j,k=3}^n b_{jk}(y)\partial_{y_j}\partial_{y_k} + \mathcal{L},$$

where $b_{jk} \in C^\infty$, $b_{12} \neq 0$ near $\Gamma_{12}$. Similar formulas hold near $(\Sigma_1 \cap \Sigma_3) \setminus \Sigma_2$ and near $(\Sigma_2 \cap \Sigma_3) \setminus \Sigma_1$.

**NF.3** For $q \in \Gamma$ there exist coordinates $y = (y_1, y_2, y_3, y'')$ valid in a neighborhood $U$ of $q$ such that

$$\Sigma_j = \{y_j = 0\}, \quad j = 1, 2, 3, \quad P(y, D) = b_{12}(y)\partial_{y_1}\partial_{y_2} + b_{13}(y)\partial_{y_1}\partial_{y_3} + b_{23}(y)\partial_{y_2}\partial_{y_3} + \mathcal{L} \quad \text{if } n = 3 \quad \text{and} \quad P(y, D) = b_{12}(y)\partial_{y_1}\partial_{y_2} + b_{13}(y)\partial_{y_1}\partial_{y_3} + b_{23}(y)\partial_{y_2}\partial_{y_3} + \sum_{j=4}^n b_{jk}(y)\partial_{y_j}\partial_{y_k} + \mathcal{L}, \quad \text{if } n > 3,$$

where $b_{jk} \in C^\infty$. Since $P(y, D)$ is strictly hyperbolic, $b_{jk} \neq 0$, near $\Gamma$ provided $j, k = 1, 2, 3$.

**Proof.** The proof of the last case contains the proofs of the other two cases, and we will concentrate on it. We start by choosing coordinates $Y = (Y_1, Y_2, Y_3, Y'')$ in a neighborhood $U_q$ of $q \in \Gamma$ such that $\Sigma_j = \{Y_j = 0\}$, $j = 1, 2, 3$, which can be done.
because the surfaces intersect transversally. Since $\Sigma_j$, $j = 1, 2, 3$, are characteristic for $P(Y, D)$, we must have

\[
P(Y, D) = a_{11}(Y)Y_1 \partial_{Y_1}^2 + a_{12}(Y)\partial_{Y_1}Y_2 + a_{13}(Y)\partial_{Y_1}Y_3 + a_{22}(Y)Y_2 \partial_{Y_2}^2 + a_{23}(Y)\partial_{Y_2}Y_3 + a_{33}(Y)Y_3 \partial_{Y_3}^2 + \sum_{j=1,k=4}^n a_{jk}(Y)\partial_{Y_j}Y_k + \mathcal{L}.
\]

(9.4)

**Lemma 9.2.** Because $P(Y, D)$ is strictly hyperbolic, $|a_{12}(Y)| > 0$, $|a_{13}(Y)| > 0$ and $|a_{23}(Y)| > 0$ for $Y \in U_q$, if $U_q$ is small enough.

**Proof.** Let us suppose that $a_{12}(0) = 0$. Recall $P(Y, D)$ is strictly hyperbolic with respect to a time function $t$. This means that if $p(Y, \eta)$ is the principal symbol of $P(Y, D)$, then for every $Y \in U_q$, $p(Y, \nabla_Y t(Y)) \neq 0$ and for every $\xi \in \mathbb{R}^n \setminus \mathbb{R} \nabla_Y t(Y)$, the polynomial

\[
\lambda(s) = p(Y, \xi + s \nabla_Y t(Y)) = 0
\]

has two distinct real roots.

Pick $\xi = (\xi_1, \xi_2, 0, \ldots, 0)$. Then, because $a_{12}(0) = 0$, $p(0, \xi) = 0$ and we find that

\[
p(0, \xi + s \nabla_Y t(0)) = s^2 p(0, \nabla_Y t(0)) + s \left(a_{13}(0)\partial_{Y_1}Y_2 t(0) + \sum_{k=4}^n a_{1k}(0)\partial_{Y_k}Y_2 t(0)\right) \xi_1 + s \left(a_{23}(0)\partial_{Y_2}Y_3 t(0) + \sum_{k=4}^n a_{2k}(0)\partial_{Y_k}Y_3 t(0)\right) \xi_2.
\]

One can just pick $\xi_1$ and $\xi_2$ such that the terms in $s$ add up to 0, and then the polynomial has a single double root $s = 0$. \hfill \Box

We want to find a $C^\infty$ change of variables $y = \Psi(Y)$ in $O_q \subset U_q$, by such that the hypersurfaces $\Sigma_j = \{y_j = 0\}$, $j = 1, 2, 3$, and such that the form of the operator (9.3) holds. In fact we can find $O_q$ and $X_j, W_j \in C^\infty(\partial_q)$ such that for $Y = (Y', Y'')$, $Y' = (Y_1, Y_2, Y_3)$,

$y_j = Y_jX_j(Y')$, $j = 1, 2, 3$, $|X_j(Y')| > 0$ and $y_j = Y_j$, $4 \leq j \leq n$, provided $n > 3$.

In this case we have

\[
\begin{align*}
\partial_{Y_1} &= (X_1 + Y_1 \partial_Y X_1)\partial_{y_1} + Y_2 \partial_{Y_2}X_2 \partial_{y_2} + Y_3 \partial_{Y_3}X_3 \partial_{y_3}, \\
\partial_{Y_2} &= Y_1 \partial_{y_2}X_1 \partial_{y_1} + (X_2 + Y_2 \partial_Y X_2)\partial_{y_2} + Y_3 \partial_{Y_3}X_3 \partial_{y_3}, \\
\partial_{Y_3} &= Y_1 \partial_{y_3}X_1 \partial_{y_1} + Y_2 \partial_{y_3}X_2 \partial_{y_2} + (X_3 + Y_3 \partial_Y X_3)\partial_{y_3}, \\
\partial_{y_k} &= \partial_{y_k}, \quad 4 \leq k \leq n.
\end{align*}
\]

(9.5)

Therefore (9.3) transforms into

\[
P(y, D) = \frac{Z_1}{X_1} y_1 \partial_{y_1}^2 + \frac{Z_2}{X_2} y_2 \partial_{y_2}^2 + \frac{Z_3}{X_3} y_3 \partial_{y_3}^2 + A_{12} \partial_{y_1} \partial_{y_2} + A_{13} \partial_{y_1} \partial_{y_3} + A_{23} \partial_{y_2} \partial_{y_3} + \sum_{j=1,k=4}^n A_{jk}(Y)\partial_{y_j} \partial_{y_k} + \mathcal{L}(y, \partial_y),
\]

Therefore, $P(y, D)$ satisfies (9.3) if and only if $X_1$, $X_2$ and $X_3$ are such that

\[
Z_1(Y, X_1, \nabla_Y X_1) = Z_2(Y, X_2, \nabla_Y X_2) = Z_3(Y, X_3, \nabla_Y X_3) = 0.
\]
If we denote \( \Theta_j = X_j + Y_j \partial_{Y_j} X_j \), the terms \( Z_j, j = 1, 2, 3 \), satisfy

\[
Z_j(Y, X_j, \nabla Y X_j) = a_{jj} \Theta_j^2 + \sum_{k=1, k \neq j}^3 a_{jk} \Theta_j \partial_{Y_k} X_j + \sum_{k=1, k \neq j}^3 a_{kk} Y_k Y_j (\partial_{Y_k} X_j)^2 + a_{kl} Y_l (\partial_{Y_l} X_j) (\partial_{Y_l} X_j),
\]

where in the last term \( a_{kl}, k, l \neq j, k \neq l \). In particular, \( Z_1 \) satisfies

\[
Z_1(Y, X_1, \nabla Y X_1) = a_{11} \Theta_1^2 + \sum_{k=2,3} a_{1k} \Theta_1 \partial_{Y_k} X_1 + \sum_{k=2,3} a_{kk} Y_k Y_1 (\partial_{Y_k} X_1)^2 + a_{23} Y_1 (\partial_{Y_2} X_1) (\partial_{Y_3} X_1).
\]

Notice that the system is not coupled, and each equation can be solved independently, and therefore this case includes the other two cases of the Proposition.

Since \( a_{12}(Y) \neq 0 \), \( a_{13}(Y) \neq 0 \) and \( a_{23}(Y) \neq 0 \), for \( |Y_j| \) small enough, \( j = 1, 2, 3 \), the first order PDE for \( Z_1 \) is non-characteristic with respect to \( \Sigma_2 \) or \( \Sigma_3 \). Therefore, fixed an initial data \( X_1(Y_1, 0, Y_3) \) or \( (X_1(Y_1, Y_2, 0)) \), there exists a unique \( X_1(Y) \) which is \( C^\infty \) in a neighborhood of \( q \), satisfying \( Z_1(Y, X_1, \nabla Y X_1) = 0 \).

Similarly, the differential equation \( Z_2 \) is non-characteristic with respect to \( \Sigma_1 \) or \( \Sigma_3 \) and the one for \( Z_3 \), is non-characteristic with respect to \( \Sigma_1 \) or \( \Sigma_2 \). Therefore, once suitable initial data is chosen, they have unique solutions near \( q \).

## 10. A generalization of M. Beals spaces.

We extend the definition of the spaces introduced by Beals [4] to \( \mathbb{R}^n \) and general operators \( (y, D) \) and hypersurfaces \( \Sigma_j, j = 1, 2, 3 \). According to [4], \( u \in H_{loc}^{s, k_1, k_2, k_3}(\mathbb{R}^3) \) if for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \),

\[
\langle D_y \rangle^{k_1} \langle D_{y_2} \rangle^{k_2} \langle D_{y_3} \rangle^{k_3} (D) \varphi u \in L^2(\mathbb{R}^3),
\]

or equivalently that

\[
\langle \eta_1 \rangle^{k_1} \langle \eta_2 \rangle^{k_2} \langle \eta_3 \rangle^{k_3} (\varphi u)(\eta) \in L^2(\mathbb{R}^3).
\]

In the discussion in [4], coordinates are fixed so that \( \Sigma_j = \{ y_j = 0 \}, j = 1, 2, 3 \). These spaces are not coordinate invariant, even if the change preserves \( \Sigma_j \) for \( y_j = 0 \). As already mentioned, in [4] the operator used in [4] is \( P_0(y, D) \) given by (2.10).

### 10.1. The local definition.

#### Definition 10.1.

Let \( \Omega \subset \mathbb{R}^n \) be an open subset and let \( \Sigma_j \subset \Omega, j = 1, 2, 3 \), be \( C^\infty \) closed hypersurfaces intersecting transversally at \( \Gamma_{jk} = \Sigma_j \cap \Sigma_k \) and at \( \Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \). Given a point \( q \in \Omega \), fix a neighborhood \( U \) of \( q \) and fix local coordinates \( y \) in \( U \). We say that \( u \in H_{loc}^{s, k_1, k_2, k_3}(U, \{ y \}), k_1, k_2, k_3 \in \mathbb{R}_+ \) and \( s \in \mathbb{R} \), if \( u \) satisfies the following conditions:

1. \( u \in H_{loc}^{s+k_1+k_2+k_3}(U), \) provided \( U \subset \Omega \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \).
2. If \( q \in \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_3) \), and \( y = (y_1, y''), y'' = (y_2, \ldots, y_n) \), are local coordinates such that \( \Sigma_1 = \{ y_1 = 0 \} \), and \( P(y, D) \) satisfies (9.1), \( u \in H_{loc}^{s, k_1, k_2, k_3}(U, \{ y \}) \) if it satisfies

\[
\langle D_{y''} \rangle^{k_2+k_3} \varphi u \in H_{loc}^{s+k_1}(U), \quad \varphi \in C_0^\infty(U),
\]

and similarly for \( q \in \Sigma_2 \) or \( \Sigma_3 \).

3. If \( q \in (\Sigma_1 \cap \Sigma_2) \setminus \Sigma_3 \), and \( y = (y_1, y_2, y'''), y''' = (y_3, \ldots, y_n) \) are local coordinates such that \( \Sigma_j = \{ y_j = 0 \}, j = 1, 2 \), and \( P(y, D) \) satisfies (9.2), \( u \in H_{loc}^{s, k_1, k_2, k_3}(U, \{ y \}) \) if it satisfies

\[
\langle D_{y'''} \rangle^{k_2+k_3} \varphi u \in H_{loc}^{s+k_1}(U), \quad \varphi \in C_0^\infty(U),
\]

\]}
and similarly for \( q \in \Sigma_1 \cap \Sigma_3 \) or \( q \in \Sigma_2 \cap \Sigma_3 \).

C4. If \( q \in \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \) and in local coordinates \( y = (y_1, y_2, y_3, y') \), \( y'' = (y_4, \ldots, y_n) \), if \( n \geq 4 \), such that \( \Sigma_j = \{ y_j = 0 \} \), \( j = 1, 2, 3 \), and \( P(y, D) \) satisfies (9.3), \( u \in H^{s,k_1,k_2,k_3}_{loc}(U, \{ y \}) \) if it satisfies

\[
\langle D_{y_1}, D_{y'}^{\gamma_1}(D_{y_2}, D_{y''}^{\gamma_2}(D_{y_3}, D_{y''}^{\gamma_3}u \varphi H^{s}_{loc}(U), \varphi \in C^\infty(U).
\]

The main difficulty with working with these spaces is that they depend on the choice of the coordinates for which their elements satisfy (10.2), (10.3) or (10.4), and this is a problem if one wants to use them to study the triple interaction. Before we can address this issue we need to establish properties of the spaces \( H^{s,k_1,k_2,k_3}_{loc}(U, \{ y \}) \):

**Proposition 10.2.** Let \( \Sigma_j \subset \Omega, j = 1, 2, 3 \), be \( C^\infty \) closed hypersurfaces intersecting transversally at \( \Gamma_{jk} = \Sigma_j \cap \Sigma_k \) and at \( \Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \). Given a point \( q \in \Omega \) and a neighborhood \( U \) of \( q \), and coordinates \( y \) valid in \( U \) which satisfy one of the assumptions of Definition 10.1, the corresponding spaces \( H^{s,k_1,k_2,k_3}_{loc}(U, \{ y \}) \) satisfy the following properties:

\begin{enumerate}
  \item[\textbf{P.1}] \( H^{s,k_1,k_2,k_3}_{loc}(U, \{ y \}) \subset H^{s',k'_1,k'_2,k'_3}_{loc}(U, \{ y \}) \), provided \( s \geq s', k_j \geq k'_j, j = 1, 2, 3 \).
  \item[\textbf{P.2}] If \( a_j > 0 \) and \( a_1 + a_2 + a_3 = 1 \), then
    \begin{equation}
    H^{s+1,k_1,k_2,k_3}_{loc}(U, \{ y \}) \subset H^{s,k_1+a_1,k_2+a_2,k_3+a_3}_{loc}(U, \{ y \}).
    \end{equation}
  \item[\textbf{P.3}] If \( s \geq 0 \) and \( k_j > \frac{a_j}{a_3} \), then
    \begin{equation}
    H^{s-k_1-k_2-k_3}_{loc}(U, \{ y \}) \subset L^\infty_{loc}(U).
    \end{equation}
  \item[\textbf{P.4}] If \( k_j > \frac{a_j}{a_3} \), then \( H^{0-,k_1,k_2,k_3}_{loc}(U, \{ y \}) \) is closed under multiplication, and for \( \delta > 0 \) small enough
    \begin{equation}
    |||\varphi u \varphi v|||_{-\delta,k_1,k_2,k_3} \leq C |||\varphi u|||_{-\delta,k_1,k_2,k_3} |||\varphi v|||_{-\delta,k_1,k_2,k_3}.
    \end{equation}
  \item[\textbf{P.5}] If \( k_j > \frac{a_j}{a_3} + 1, j = 1, 2, 3, s \in \mathbb{N}_0 \), and \( f(y, z) \in C^\infty(U \times \mathbb{R}) \), then \( u \in H^{s-,k_1,k_2,k_3}_{loc}(U, \{ y \}) \), then \( f(y, u) \in H^{s-,k_1,k_2,k_3}_{loc}(U, \{ y \}) \).
\end{enumerate}

If \( u \) is supported in a compact subset \( K \subset U \), and if there exist \( C_\delta \) such that for \( \delta > 0 \) small,

\[
||u||_{s-,k_1,k_2,k_3} \leq C_\delta,
\]

and \( \varphi \in C^\infty_0(U) \), then there exists a constant \( \hat{C} \), depending on \( C_\delta \), \( f \) and \( \varphi \) such that

\[
||\varphi f(y, u)||_{s-,k_1,k_2,k_3} \leq \hat{C}.
\]

We will prove this Proposition in Appendix C at the end of the paper. The proofs of items \textbf{P.1} and \textbf{P.2} easily follow from the definition. The proof of \textbf{P.3} and \textbf{P.4} when \( n = 3 \), are left as exercise for the reader in [4]. The proof for \( n > 3 \) is more technical and we include in the appendix for completeness. Property \textbf{P.5} is
not stated in [4], even in the three dimensional case. Property P.5 is enough for our purposes, but it is not sharp: \( s \geq 0 \) and \( k_j > \frac{n}{2} \), \( j = 1, 2, 3 \), should be enough.

Next we establish mapping properties for the fundamental solution of \( P(y, D) \) acting on the spaces \( H^{s,k_1,k_2,k_3}_{\text{loc}}(U, \{y\}) \). The case \( n = 3 \) was proved in [42].

**Proposition 10.3.** Let \( \Sigma_j \subset \Omega \), \( j = 1, 2, 3 \), be \( C^\infty \) closed hypersurfaces intersecting transversally at \( \Gamma_{jk} = \Sigma_j \cap \Sigma_k \) and at \( \Gamma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \). Let \( q \in \Omega \), let \( U \) be a neighborhood of \( q \) and let \( y \) be coordinates in \( U \) which satisfy the assumptions of Definition 10.1. for \( s \in \mathbb{R} \) and \( k_j \in \mathbb{R}_+ \), let \( H^{s,k_1,k_2,k_3}_{\text{loc}}(U, \{y\}) \) be the corresponding spaces defined in either (10.2), (10.3) or (10.4). Suppose that \( U \) is bicharacteristically convex with respect to \( P(y, D) \). If \( E_+ \) denotes the forward fundamental solution to \( P(y, D) \) and \( \varphi, \psi \in C^\infty_0(U) \), then

\[
(10.10) \quad \psi E_+ \varphi : H^{s,k_1,k_2,k_3}_{\text{loc}}(U, \{y\}) \to H^{s+1,k_1,k_2,k_3}_{\text{loc}}(U, \{y\}),
\]

is a bounded linear operator in terms of the norms given by (10.2) – (10.4).

**Proof.** We analyze the case where \( q \in \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \). The proofs of the other cases are very similar. We start by proving this result for \( k_j \in \mathbb{N}_0, j = 1, 2, 3 \). In this case we need to show that if \( P(y, D)u = \psi f \in H^{s,k_1,k_2,k_3}_{\text{loc}}(U, \{y\}) \), then for \( \alpha_j \in \mathbb{N}_0^{n-2} \), \( |\alpha_j| \leq k_j, j = 1, 2, 3 \),

\[
(10.11) \quad \| (\partial_{y_1}, \partial_{y_2})^{\alpha_1} (\partial_{y_2}, \partial_{y_3})^{\alpha_2} (\partial_{y_3}, \partial_{y'})^{\alpha_3} \varphi u \|_{H^{s+1}} \leq C \sum_{|\beta_j| \leq |\alpha_j|} \| (\partial_{y_1} \beta_1, \partial_{y_2} \beta_2, \partial_{y_3} \beta_3) \psi f \|_{H^s}, \quad \beta_j \in \mathbb{N}_0^{n-2}.
\]

This is based on a commutator method due to Bony [8, 9], see also Melrose and Ritter [30] and for that we need the operator in the normal form above, otherwise this would not work. In this case, the operator \( P(y, D) \) is given by (9.3) and then

\[
[P(y, D), \partial_{y_1}] = (\partial_{y_1} b_{12}) \partial_{y_2} \partial_{y_3} + (\partial_{y_1} b_{13}) \partial_{y_1} \partial_{y_3} + (\partial_{y_1} b_{23}) \partial_{y_2} \partial_{y_3} + \sum_{j=1,k=4}^{n} (\partial_{y_j} b_{jk}(y)) \partial_{y_j} \partial_{y_k} + \mathcal{L},
\]

where here and below \( \mathcal{L} \) denotes differential operator of order one with \( C^\infty \) coefficients. As discussed in Lemma 9.2 above, since \( P(y, D) \) is strictly hyperbolic, \( b_{23} \neq 0 \), and using the formula for \( P(y, D) \) given by (9.3) we can write

\[
(\partial_{y_1} b_{23}) \partial_{y_2} \partial_{y_3} = \frac{\partial_{y_1} b_{23}}{b_{23}} (P - b_{12} \partial_{y_1} \partial_{y_2} - b_{13} \partial_{y_1} \partial_{y_3} - \sum_{j=1,k=4}^{n} b_{jk}(y) \partial_{y_j} \partial_{y_k} - \mathcal{L}),
\]

and therefore we obtain

\[
(10.12) \quad [P(y, D), \partial_{y_1}] = a_1(y) P(y, D) + \mathcal{L}_{11}(y, D) \partial_{y_1} + \sum_{k=4}^{n} \mathcal{L}_{1k} \partial_{y_k} + \mathcal{L}.
\]

We can argue in the same way to obtain

\[
(10.13) \quad [P(y, D), \partial_{y_m}] = a_m(y) P(y, D) + \mathcal{L}_{m1}(y, D) \partial_{y_1} + \sum_{k=4}^{n} \mathcal{L}_{mk} \partial_{y_k} + \mathcal{L}, \quad m = 4, \ldots, n.
\]

Therefore, if \( P(y, D)u = f \),

\[
\mathcal{U}_T^T = \{ \partial_{y_1} u, \partial_{y_2} u, \ldots, \partial_{y_n} u \} \quad \text{and} \quad \mathcal{G}_T^T = \{ f, \partial_{y_1} f, \partial_{y_2} f, \ldots, \partial_{y_n} f \},
\]
one gets a system
\[ \mathcal{P}_1 \mathcal{U}_1 = \mathcal{M}_1 \mathcal{F}_1 \in H^s_{loc}(\Omega), \]
where \( \mathcal{P}_1 \) is a matrix of operators with diagonal principal part \( P(y, D) \text{Id} \) and \( \mathcal{M}_1 \) is a matrix of \( C^\infty \) functions. Since \( \mathcal{P} \) is a strictly hyperbolic system, we conclude that \( \mathcal{U}_1 \in H^s_{loc}(\Omega) \) and (10.11) holds for \( |\alpha| \leq 1 \).

Using the fact that for any three operators (as long as the compositions are well defined),
\[ (10.14) \quad [A, BC] = B[A, C] + [A, B]C, \]
we deduce from (10.12) and (10.13) that, for \( m \geq 4 \),
\[
[P(y, D), \partial_{y_1} \partial_{y_m}] = A_{1m}(y)P + B_{1m}(y)\partial_{y_1}P + C_{1m}(y)\partial_{y_m}P + \mathcal{L}_1(y, D)\partial_{y_1} + \\
\mathcal{L}_m(y, D)\partial_{y_m} + \mathcal{L}_{11}(y, D)\partial_{y_1}^2 + \mathcal{L}_{1m}(y, D)\partial_{y_1} \partial_{y_m} + \sum_{k=4}^n \mathcal{L}_{1k}\partial_{y_1} \partial_{y_k} + \\
\sum_{k=4}^n \mathcal{L}_{1k}\partial_{y_m} \partial_{y_k} + \mathcal{L}.
\]

We can rewrite this as
\[
[P(y, D), \partial_{y_1} \partial_{y_m}] = \sum_{|\beta|=0}^{1} a_{\beta}(y)(\partial_{y_1}, \partial_{y_m})^\beta P(y, D) + \sum_{|\beta|=0}^{2} \mathcal{L}_\beta(y, D)(\partial_{y_1}, \partial_{y'})^\beta,
\]
where \( a_{\beta} \in C^\infty \) and \( \mathcal{L}_\beta \) are differential operators with \( C^\infty \) coefficients of order one.

Using (10.14) and induction we arrive at
\[
[P(y, D), (\partial_{y_1}, \partial_{y'})^{\alpha}] = \sum_{|\beta|=0}^{|\alpha|-1} a_{\beta}(y)(\partial_{y_1}, \partial_{y'})^\beta P(y, D) + \sum_{|\beta|=0}^{|\alpha|} \mathcal{L}_\beta(\partial_{y_1}, \partial_{y'})^\beta.
\]

Using the same argument, we have
\[
[P(y, D), (\partial_{y_1}, \partial_{y'})^{\alpha_1}(\partial_{y_2}, \partial_{y'})^{\alpha_2}(\partial_{y_3}, \partial_{y'})^{\alpha_3}] = \\
\sum_{j=1}^{3} \sum_{|\beta_j|=0}^{|[\alpha_j]|-1} a_{\beta_1, \beta_2, \beta_3}(y)(\partial_{y_1}, \partial_{y'})^{\beta_1}(\partial_{y_2}, \partial_{y'})^{\beta_2}(\partial_{y_3}, \partial_{y'})^{\beta_3} P(y, D) + \\
\sum_{j=1}^{3} \sum_{|\beta_j|=0}^{|[\alpha_j]|} \mathcal{L}_{\beta_1, \beta_2, \beta_3}(y, D)(\partial_{y_1}, \partial_{y'})^{\beta_1}(\partial_{y_2}, \partial_{y'})^{\beta_2}(\partial_{y_3}, \partial_{y'})^{\beta_3}.
\]

If
\[
\mathcal{U}^T = (u, (\partial_{y_1}, \partial_{y'})^{\gamma_1}(\partial_{y_2}, \partial_{y'})^{\gamma_2}(\partial_{y_3}, \partial_{y'})^{\gamma_3} u), \quad \text{and}
\]
\[
\mathcal{F}^T = (f, (\partial_{y_1}, \partial_{y'})^{\gamma_1}(\partial_{y_2}, \partial_{y'})^{\gamma_2}(\partial_{y_3}, \partial_{y'})^{\gamma_3} f), \quad |\gamma_j| \leq |\beta_j| \leq |\alpha_j|, \quad j = 1, 2, 3,
\]
we obtain a system
\[
\mathcal{P}\mathcal{U} = \mathcal{F},
\]
where \( \mathcal{P} \) is a square matrix of differential operators of order two, and its principal part is \( P(y, D) \text{Id} \). So the system is strictly hyperbolic, and we conclude that \( \mathcal{U} \in H^s_{loc}(\Omega) \).

This proves the result for \( k_j \in \mathbb{N}_0, \ j = 1, 2, 3 \). The case \( k_j \in \mathbb{R}_+ \) follows from the Stein-Weiss interpolation Theorem [43], and we refer the reader to the proof of Proposition 2.13 of [42] for more details.
10.2. Coordinate invariance. As mentioned above, the spaces $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ depend on the choice of the coordinates that satisfy (10.2), (10.3) or (10.4), and if we want to use them to study properties of solutions of (2.4) we need to circumvent this problem. The idea is to show that, although we do not know if $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ is invariant under $C^{\infty}$ coordinate changes, this is true for elements of $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ which satisfy a semilinear wave equation with initial data $v = v_1 + v_2 + v_3$, $v_j \in I^m(\Omega,\Sigma_j)$, $j = 1, 2, 3$, and $m << 0$.

One might think that it would be better to define the spaces $H^{s,k_1,k_2,k_3}_{loc}(\Omega)$ as the family of $u \in H^{s}_{loc}(\Omega)$, $s > \frac{3}{2}$, such that

$$(10.15) \quad W^{s}_{23}W^{s}_{12}W^{s}_{13} \varphi u \in H^{s}(\Omega), \varphi \in C^{\infty}(\Omega), \quad k_j \in \mathbb{N}_0,$$

where $W_{jk}$ are the Lie algebras of vector fields defined in (7.4). In local coordinates where condition C.4 from Definition 10.1 holds, according to (7.5), this is equivalent to

$$((\partial_{y_{1}}, y_{2}\partial_{y_{2}}, y_{3}\partial_{y_{3}}, \partial_{y^\prime})^{k_1}(\partial_{y_{2}}, y_{1}\partial_{y_{1}}, y_{3}\partial_{y_{3}}, \partial_{y^\prime})^{k_2}(\partial_{y_{3}}, y_{1}\partial_{y_{1}}, y_{2}\partial_{y_{2}}, \partial_{y^\prime})^{k_3}) \varphi u \in H^{s}_{loc}(\Omega).$$

These spaces are smaller than $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ defined above, for $k_j \in \mathbb{N}_0$. They are also coordinate invariant, have properties P.1 to P.5 of Proposition 10.2 and satisfy Proposition 10.3. The problem with using them is that to prove Theorem 4.1 one does need to work with spaces for $k_j \in \mathbb{R}_+$, which also have to be $C^{\infty}$ algebras, and there is no obvious way of extending the definition (10.15) to $k_j \in \mathbb{R}_+$, and have the spaces remain $C^{\infty}$ algebras. Still, one could work with this invariant formulation and $k_j \in \mathbb{N}_0$, but one would have to work with $H^s$-based conormal distributions (instead of (7.1) which uses Besov spaces) and would only be able to prove Theorem 4.1 for symbols in $S^{m-}$ instead of $S^{m}$, and one would also need to assume that $-m - \frac{1}{2} \in \mathbb{N}_0$, which would be less than desirable.

We show that distributions $u \in I^{m-\frac{1}{2}+\frac{1}{2}}(\Omega,\Sigma_j) \cap H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ do not depend of the choice of $\{y\}$. We will use that to show that the family of elements of $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ which satisfy (2.4), with $v$ conormal to $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is independent of the choice of $\{y\}$ and this observation will allow us to overcome the coordinate dependence issue for solutions of (2.4) which are conormal to $\Sigma_j$, $j = 1, 2, 3$ for $t < -1$.

Now we consider the invariance of $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ under change of variables. If $U \cap \Sigma_j = \emptyset$, $j = 1, 2, 3$, we know that $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\}) = H^{s+k_1+k_2+k_3}_{loc}(U)$ is independent of the choice of coordinates. The other cases are more subtle and we introduce the following spaces:

**Definition 10.4.** If $U \subset \Omega$ is an open subset, we say that $u \in \mathcal{H}^{s,k_1,k_2,k_3}_{loc}(U)$ if

**Inv1.** For any $q \in U$, there exist local coordinates $\{y\}$ in a neighborhood $U_q$ of $q$ satisfying one of the conditions of Definition 10.1 so that $u \in H^{s,k_1,k_2,k_3}_{loc}(U_q,\{y\})$.

**Inv2.** Given any two sets of local coordinates $\{y\}$ and $\{Y\}$ which are defined in $U_q$ and satisfy the same one of the conditions of Definition 10.1, and a $C^{\infty}$ diffeomorphism $\Psi : U_q \rightarrow U_q$ such that if $Y = \Psi^{-1}(y)$, $u \in H^{s,k_1,k_2,k_3}_{loc}(U_q,\{y\})$ if and only if $\Psi^*u \in H^{s,k_1,k_2,k_3}_{loc}(U_q,\{Y\})$.

Next we consider elements of $H^{s,k_1,k_2,k_3}_{loc}(U,\{y\})$ which are also conormal distributions.
Proposition 10.5. Let $\Sigma_1 \subset \Omega$ be as above. Then for any $q \in \Omega$ and any neighborhood $U \subset \Omega$ of $q$ which has local coordinates valid in $U$ such that the condition C.2 of Definition 10.1 holds, $I^u(U, \Sigma_1) \cap H^{s,k_1,k_2,k_3}_{loc}(U, \{y\}) \subset \mathcal{H}^{s,k_1,k_2,k_3}_{loc}(U)$. In other words, the space $u \in I^u(U, \Sigma_1) \cap H^{s,k_1,k_2,k_3}_{loc}(U, \{y\})$ is invariant under change of variables that fix $\Sigma_1$.

Proof. Let $\{y\}$ be local coordinates in $U$ such that $\Sigma_1 = \{y_1 = 0\}$. A change of coordinates $Y = \Psi(y)$ in $U$ that fixes $\Sigma_1$, must satisfy
\begin{equation}
Y_1 = y_1 X_1(y), \quad |X_1(y)| > 0, \quad Y'' = Y''(y), \quad y_1 = Y_1 Z_1(Y), \quad |Z_1(Y)| > 0, \quad y'' = W''(Y), \quad j = 2, 3, \ldots, n,
\end{equation}
and hence
\begin{equation}
\partial_{y_1} = (X_1 + y_1 \partial_{y_1} X_1) \partial_{Y_1} + \sum_{k=2}^n \partial_{y_k} Y_k \partial_{Y_k},
\end{equation}
\begin{equation}
\partial_{y_j} = \left(\frac{1}{X_1} \partial_{y_j} X_1\right) Y_1 \partial_{Y_1} + \sum_{k=2}^n \partial_{y_k} Y_k \partial_{Y_k}, \quad j = 2, 3, \ldots, n.
\end{equation}

According to (10.2), if $k_j \in \mathbb{N}_0$, $j = 1, 2, 3$, $u \in H^{s,k_1,k_2,k_3}_{loc}(\Omega)$, if
\begin{equation}
(D_{y_1}^{k_1} u \in H^{s}_{loc}(U)) \text{ or } a(y'', \eta_1)(1 + |\eta_1|^{k_1}) \in L^2(\mathbb{R}^n).
\end{equation}

But of course, the pull-back of $\varphi u$ by $\Psi$ is a conormal distribution to $\Sigma_1$ and using (10.16) it would also be given by an oscillatory integral
\begin{equation}
\varphi u(y) = \int_{\mathbb{R}} e^{iy_1 \eta_1} a(y'', \eta_1) d\eta_1, \quad a \in S^m + \frac{1}{2} - \frac{1}{2}(\mathbb{R}^{n-1} \times \mathbb{R}),
\end{equation}
with $a(y'', \eta_1)$ compactly supported in $y''$, but then (10.18) is equivalent to
\begin{equation}
(D_{y_1}^{k_1} u \in H^{s}_{loc}(U)) \text{ or } a(y'', \eta_1)(1 + |\eta_1|^{k_1}) \in L^2(\mathbb{R}^n).
\end{equation}

We will need the following result about the inclusion of distributions conormal to a hypersurface into $\mathcal{H}^{s,k_1,k_2,k_3}_{loc}(\Omega)$:

We need the following result about the inclusion of distributions conormal to a hypersurface into $\mathcal{H}^{s,k_1,k_2,k_3}_{loc}(\Omega)$:
Proposition 10.6. Let \( \Omega \subset \mathbb{R}^n \) be an open neighborhood of the origin and \( y = (y_1, y_2, y_3, y_4) \), be coordinates in \( \Omega \) such that \( \Sigma_j = \{y_j = 0\} \), \( j = 1, 2, 3 \). If \( v_j \in H^{s,\infty}_0(\Omega, \Sigma_j) \), \( j = 1, 2, 3 \) and \( s \geq 0 \), then

\[
v_j \in H^{s}_0(\Omega), \quad s < -m - \frac{1}{2}, \quad j = 1, 2, 3,
\]

(10.19) \( v_1 \in \mathcal{H}^{r,\infty}_0(\Omega), \quad v_2 \in \mathcal{H}^{r,\infty}_0(\Omega) \) and \( v_3 \in \mathcal{H}^{r,\infty}_0(\Omega) \), provided \( \kappa \geq 0 \), \( r \geq 0 \) and \( \kappa + r < -m - \frac{1}{2} \).

The proof is a very simple application of the Fourier transform and details (for \( n = 3 \)) can be found in the proof of Proposition 2.15 of [42].

Next we consider solutions of semilinear wave equations in \( H^{s-k_1,k_2,k_3}_0(U, \{y\}) \). We recall that the spaces \( H^{s-k_1,k_2,k_3}_0(U, \{y\}) \) were defined by equation (10.5), Property \( \text{P.1} \), Proposition 10.2.

Proposition 10.7. Let \( q \in \Omega \cap \{t = c\} \), and let \( q \in U \subset U' \subset \Omega \) be relatively compact open subsets. Suppose that \( U \) is bicharacteristically convex with respect to \( P(y, D) \). Moreover, suppose that there exist local coordinates \( y \) in \( U' \) such that one of the normal forms required in Definition 10.1 holds in \( U' \) and let \( H^{s-k_1,k_2,k_3}_0(U, \{y\}) \) denote the restriction of functions \( u \in H^{s-k_1,k_2,k_3}_0(U', \{y\}) \) to \( U \).

Suppose that \( u \in H^{s}_0(\Omega), s > \frac{3}{2} \), satisfies (2.4) and \( u \in H^{s-k_1,k_2,k_3}_0(U \cap \{t < c\}, \{y\}) \), with \( s \geq 0 \), and \( k_j > \frac{3}{2} + 3, j = 1, 2, 3 \), then \( u \in H^{s-k_1,k_2,k_3}_0(U, \{y\}) \).

Proof. Let \( \chi \in C^\infty(\mathbb{R}) \), be such that \( \chi(t) = 1, t > c - \delta \) and \( \chi = 0 \) for \( t < c - 2\delta \), then

\[
P(y, D)(\chi u) = \chi f(y, u) + [P(y, D), \chi]u.
\]

Notice that \( [P(y, D), \chi]u \in H^{s-1-k_1,k_2,k_3}_0(U, \{y\}) \) is supported in \( t \in [c - 2\delta, c - \delta] \). If \( P(y, D)v = [P(y, D), \chi]u \),

\[
v = 0, \quad t < c - 2\delta,
\]

it follows from Proposition 10.3 that \( v \in H^{s-k_1,k_2,k_3}_0(U) \). If one writes \( u = \chi u + (1 - \chi)u \) and \( \chi u = v + w \), then

\[
P(y, D)w = \chi f(y, w + \vartheta), \quad \vartheta = v + (1 - \chi)u \in H^{s-k_1,k_2,k_3}_0(U, \{y\}),
\]

\[
w = 0, \quad t < c - 2\delta.
\]

Suppose that \( k_1 \leq \min\{k_2, k_3\} \). Since at every point, \( \gamma \in T^*\Omega \setminus 0 \), at least one of \( \eta_j \) is elliptic, it follows that \( u \in H^{s+1-k_1}_0(U \cap \{t < c\}) \) and we claim that this implies that \( u \in H^{s+1-k_1}_0(U) \). To see that, recall that \( v, \vartheta \in H^{s+1-k_1}_0(U) \) and we also know that \( u \in H^{s}_0(\Omega) \), with \( s > \frac{3}{2} \) and this implies that \( w \in H^{s}_0(\Omega) \). But in this case \( H^{s}_0(\Omega) \) is a \( C^\infty \) algebra and therefore \( \chi f(y, w + \vartheta) \) is in \( H^{s+1}_0(U) \) and

\[
w = E_+((\chi f)(y, w + \vartheta)) \in H^{s+1}_0(U).
\]

If \( k_1 = 0 \), this stops here. If not, then \( w + \vartheta, f(y, w + \vartheta) \in H^{s+1}_0(U) \), and hence \( w \in H^{s+2}_0(U) \). This process stops when we find that \( w \in H^{s+k_1+1}_0(U) \) and therefore \( \chi u = v + w \in H^{s+k_1}_0(U) \), but we already know that \( (1 - \chi)u = v + w \in H^{s+k_1}_0(U) \).

But in view of Property \( \text{P.2} \) in Proposition 10.2,

\[
w \in H^{s+k_1}_0(U) \subset H^{s+1-rac{3}{2}-rac{3}{2}}_0(U, \{y\}),
\]

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We remark this is where we need to work with these spaces for non-integers $k_j$, $j = 1, 2, 3$.

But $k_1 > \frac{n}{2} + 1$, so in view of Property 5 in Proposition 10.2, $\chi f(y, w + \vartheta) \in H^{s_{\loc}}_{\rm loc}(U, \{y\})$ and then by Proposition 10.3 and Property P.2

$$w \in H^{s_{\loc}}_{\rm loc}(U, \{y\} \subset H^{s_{\loc}}_{\rm loc}(U, \{y\}).$$

We repeat this argument and conclude that $w \in H^{s_{\loc}}_{\rm loc}(U, \{y\})$, and after finitely many steps, we conclude that $w \in H^{s_{\loc}}_{\rm loc}(U, \{y\})$. This proves the Proposition.

The following is a consequence of Proposition 10.7:

**Proposition 10.8.** Let $q \in \{t = c\} \cap \Omega$ and let $U \subset \Omega$ be a relatively compact bicharacteristically convex neighborhood of $q$. Suppose $U$ is equipped with local coordinates $\{y\}$ which satisfy one of the conditions of Definition 10.1. If $u \in H^{s}(U)$, $s > \frac{n}{2}$, satisfies (2.4) and $u \in \mathcal{H}^{s_{\loc}}_{\rm loc}(U \cap \{t < c\})$, $s \geq 0$ and $k_j > \frac{n}{2} + 3$, then $u \in \mathcal{H}^{s_{\loc}}_{\rm loc}(U, \{y\})$.

**Proof.** Suppose $\{Y\}$ is another set of local coordinates in $U$ satisfying the same conditions of Definition 10.1 as $\{y\}$, and let $Y = \Psi^{-1}(y)$. Since $u \in H^{s_{\loc}}_{\rm loc}(U, \{y\})$, then in particular $u \in H^{s_{\loc}}_{\rm loc}(U \cap \{t < c\}, \{y\})$. But by assumption, $\Psi^* u \in H^{s_{\loc}}_{\rm loc}(U \cap \{t < c\}, \{\Psi t < c\}, \{Y\})$, and so Proposition 10.7 implies that $\Psi^* u \in H^{s_{\loc}}_{\rm loc}(U, \{Y\})$. The same argument shows that if $\Psi^* u \in H^{s_{\loc}}_{\rm loc}(U, \{Y\})$, then $u \in H^{s_{\loc}}_{\rm loc}(U, \{y\})$.

As a consequence of Proposition 10.8 and Proposition 10.5 we have that

**Proposition 10.9.** Let $u \in H^{s}(\Omega)$, $s > \frac{n}{2}$, be a solution to (2.4). Assume that $\Omega$ is bicharacteristically convex with respect to $P(y, D)$. Let $\mu \geq 0$ and $k_j > \frac{n}{2} + 3$, and suppose that $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(\Omega \cap \{t < -1\})$, then $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(\Omega \cap \{t < -1\})$.

**Proof.** Let $U$ be a neighborhood of $q \in \{t = -1\}$ which is bicharacteristically convex with respect to $\{t = -1\}$ which is equipped with local coordinates $\{y\}$ that satisfy one of the conditions of Definition 10.1. By assumption, $u \in \mathcal{H}^{\mu, k_1, k_2, k_3}_{\rm loc}(U \cap \{t < -1\})$. Then Proposition 10.8 shows that $u \in \mathcal{H}^{\mu, k_1, k_2, k_3}_{\rm loc}(U)$.

Suppose $U$ and $\bar{U}$ are neighborhoods of a point $q \in \{t = -1\}$, respectively equipped with local coordinates $\{y\}$ and $\{\bar{y}\}$ which satisfy the same one of the conditions of Definition 10.1. Furthermore, suppose that $U$ and $\bar{U}$ are bicharacteristically convex with respect to $P(y, D)$, then every null bicharacteristic for $P(y, D)$ over $U$ (respectively $\bar{U}$) meets $U \cap \{t < -1\}$, (respectively $\bar{U} \cap \{t < -1\}$), and so $U \cap \bar{U}$ is also bicharacterically convex.

By assumption, $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(\Omega \cap \{t < -1\})$, and so $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(U \cap \bar{U} \cap \{t < -1\})$. Proposition 10.8 guarantees that $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(U \cap \bar{U})$. Of course here we are using that (2.4) has a unique solution in $U \cap \bar{U}$. This means that if $\{y\}$ and $\{\bar{y}\}$ are local coordinates in $U \cap \bar{U}$ satisfying the same one of the conditions of Definition 10.1, if $\Psi(\bar{y}) = y$, and $u \in H^{\mu_{\loc}}_{\rm loc}(U \cap \bar{U}, \{y\})$ then $\Psi^* u \in H^{\mu_{\loc}}_{\rm loc}(U \cap \bar{U}, \{\bar{y}\})$.

Since the support of $\Psi(\bar{y})$ is compact, after repeating this argument finitely many times we find that there exists $\varepsilon > 0$ such that $u \in \mathcal{H}^{\mu_{\loc}}_{\rm loc}(\Omega \cap \{t < -1 + \varepsilon\})$. 
We then repeat the argument for \( \{ t = -1 + \varepsilon \} \) instead of \( \{ t = -1 \} \), and find that there exists \( \varepsilon_1 > 0 \) such that \( u \in \mathcal{H}^{m-k_1,k_2,k_3}_\text{loc}(\Omega \cap \{ t < -1 + \varepsilon + \varepsilon_1 \}) \). Again, due to the compactness of the support of \( \mathcal{Y} \), we prove the Proposition by repeating the argument finitely many times.

The following result will be important in the proof of Theorem 4.1:

**Proposition 10.10.** Let \( u \in H^s_\text{loc}(\Omega) \), \( s > \frac{4}{7} \), be a solution to (2.4) and suppose that the initial data \( v = v_1 + v_2 + v_3 \), \( v_j \in \mathcal{H}^{m-\frac{4}{7}+\frac{1}{7}}_\text{loc}(\Omega, \Sigma_j) \), \( m < -\frac{1}{3}(n+7) \). Then \( u \in \mathcal{H}^{0,-m-\frac{4}{7},-m-\frac{4}{7},-m-\frac{4}{7}}_\text{loc}(\Omega) \).

**Proof.** Since \( v_j \in \mathcal{H}^{m-\frac{4}{7}+\frac{1}{7}}_\text{loc}(\Omega, \Sigma_j) \), it follows from Proposition 10.5 that,

\[
v_1 \in \mathcal{H}^{0,-m-\frac{4}{7},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega), \quad v_2 \in \mathcal{H}^{0,-\infty,-m-\frac{4}{7},-\infty}_\text{loc}(\Omega) \quad \text{and} \quad v_3 \in \mathcal{H}^{0,-\infty,-m-\frac{4}{7},-\infty}_\text{loc}(\Omega),
\]

and therefore for \( t < -1 \), \( u = v_1 + v_2 + v_3 \), \( \in \mathcal{H}^{0,-m-\frac{4}{7},-m-\frac{4}{7},-\frac{4}{7}}_\text{loc}(\Omega \cap \{ t < -1 \}) \).

Since \( m < -\frac{1}{3}(n+7) \), \( m-\frac{1}{3} > \frac{4}{7}+3 \), and therefore the result follows from Proposition 10.9.

We recall some results from [42] about products of conormal distributions.

**Proposition 10.11.** Let \( \Omega \subset \mathbb{R}^n \) be an open neighborhood of the origin and \( y = (y_1, y_2, y_3) \) be coordinates in \( \Omega \) such that \( \Sigma_j = \{ y_j = 0 \}, \ j = 1, 2, 3 \), and let \( v_j \in \mathcal{H}^{m-\frac{4}{7}+\frac{1}{7}}(\Omega, \Sigma_j) \), \( j = 1, 2, 3 \), \( m < -1 \). If

\[
a_1(y_1, y_2, y_3, y'), a_2(y_2, y_1, y_3, y''), a_3(y_3, y_1, y_2, y''') \in \mathcal{S}^m(\mathbb{R} \times \mathbb{R}^{n-1}),
\]

are respectively the principal symbols of \( v_1, v_2 \) and \( v_3 \) then

\[
v_1 v_2 = w_{12} + \mathcal{E}_{12}, \quad v_1 v_3 = w_{13} + \mathcal{E}_{13}, \quad \text{and} \quad v_2 v_3 = w_{23} + \mathcal{E}_{23},
\]

where \( w_{12} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega) \), \( w_{13} \in \mathcal{H}^{0,-\infty,-m-\frac{1}{3},-\infty}_\text{loc}(\Omega) \), \( w_{23} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega) \) and

\[
w_{12}(y) = \int_{\mathbb{R}} e^{i(y_1 y_1 + y_2 y_2 + y_3 y_3)} a_1(y_1, 0, y_3, y') a_2(y_2, 0, y_3, y'') dy_1 dy_2,
\]

\[
\mathcal{E}_{12} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-\infty}_\text{loc}(\Omega) + \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega),
\]

\[
w_{13}(y) = \int_{\mathbb{R}} e^{i(y_1 y_1 + y_2 y_2 + y_3 y_3)} a_1(y_1, y_2, 0, y'', y') a_3(y_3, 0, y_2, y''') dy_1 dy_3,
\]

\[
\mathcal{E}_{13} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-\infty}_\text{loc}(\Omega) + \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega),
\]

\[
w_{23}(y) = \int_{\mathbb{R}} e^{i(y_1 y_1 + y_2 y_2 + y_3 y_3)} a_2(y_2, y_1, 0, y''', y') a_3(y_3, y_1, 0, y''') dy_2 dy_3,
\]

\[
\mathcal{E}_{23} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-\infty}_\text{loc}(\Omega) + \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{4}{7},-\infty}_\text{loc}(\Omega).
\]

The product \( v_1 v_2 v_3 = V + \mathcal{E} \), with \( V \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-m-\frac{1}{3}}(\Omega) \),

\[
V(y) = \int_{\mathbb{R}^3} e^{i(y_1 y_1 + y_2 y_2 + y_3 y_3)} a_1(y_1, 0, y', y'') a_2(y_2, 0, y', y''') a_3(y_3, 0, y', y''') dy_1 dy_2 dy_3,
\]

\[
\mathcal{E} \in \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-m-\frac{1}{3}}(\Omega) + \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-m-\frac{1}{3}}(\Omega) + \mathcal{H}^{0,-m-\frac{1}{3},-m-\frac{1}{3},-m-\frac{1}{3}}(\Omega)
\]
One can interpret this result in terms of distributions which are conormal to $\Gamma$, but with product-type symbols. If $v_j$, $j = 1, 2, 3$ is conormal to $\Sigma_j$, with principal symbol $a_j \in S^m(N^*\Sigma_j \setminus 0)$. Suppose that in local coordinates in which $\Sigma_j = \{y_j = 0\}$, $a_j$ is of the form $a_j(y_j, \eta_j)|dy_j|^\frac{1}{2}|d\eta_j|^\frac{1}{2}$, $y_j' = (y_2, \ldots, y_n)$, $y_j'' = (y_1, y_3, \ldots, y_n)$ and $y_j''' = (y_1, y_2, y_4, \ldots, y_n)$. We rephrase (10.22) as

$$v_1v_2v_3 = V + \mathcal{E},$$

where $V$ has a product-type principal symbol $\sigma(V)$ which in local coordinates $y = (y_1, y_2, y_3, y''')$, $y''' = (y_4, \ldots, y_n)$, has the form

$$\sigma(V)(y'', \eta_1, \eta_2, \eta_3) = a_1(0, 0, 0, y'', \eta_1)a_2(0, 0, 0, y'', \eta_2)a_3(0, 0, 0, y'', \eta_3)|dy''|^\frac{1}{2}|d\eta_1|^\frac{1}{2}|d\eta_2|^\frac{1}{2}|d\eta_3|^\frac{1}{2}.$$

Notice that any transformation that fixes the hypersurfaces $\Sigma_j = \{y_j = 0\}$, $j = 1, 2, 3$, is of the form

$$\tilde{\gamma} = \Psi(y_1, y_2, y_3, y''') = (y_1F_1(y), y_2F_2(y), y_3F_3(y), F_4(y), \ldots, F_n(y)),$$

and therefore, $d\tilde{\gamma}_1 = (F_1(y) + y_1\partial_{y_1}F_1(y))dy_1$, and

$$\partial\tilde{\gamma}_1 = (F_1(y) + y_1\partial_{y_1}F_1(y))\partial_{y_1} + y_2\partial_{y_2}F_2(y)\partial_{y_2} + y_3\partial_{y_3}F_3(y)\partial_{y_3}.$$

Therefore,

$$\Psi^*V = a_1(0, 0, 0, \tilde{y}''', \tilde{\eta}_1)a_2(0, 0, 0, \tilde{y}''', \tilde{\eta}_2)a_3(0, 0, 0, \tilde{y}''', \tilde{\eta}_3)|d\tilde{y}'''|^\frac{1}{2}|d\tilde{\eta}_1|^\frac{1}{2}|d\tilde{\eta}_2|^\frac{1}{2}|d\tilde{\eta}_3|^\frac{1}{2},$$

and we conclude that $V$ is a well defined conormal distribution to $\Gamma$ with a product type symbol.

Now we want to make sense of the operation $(\partial^2_y(yf))(y, u)|_V$. There are several different ways in which one can do that, and this may be the simplest. First notice that if $\mathcal{G}, \mathcal{G} \in C^\infty(\Omega)$ and $\mathcal{G}(y) = \mathcal{G}(y)$ for $y \in \Gamma$, then by just considering the Taylor series expansion of $\mathcal{G} - \mathcal{G}$ at $\Gamma$, we have that

$$(\mathcal{G} - \mathcal{G})V \in \mathcal{X}^{0, m}(\Omega),$$

$$\mathcal{X}^{0, m}(\Omega) = \mathcal{X}^{0, m}_{loc}(\Omega) + \mathcal{X}^{0, m}_{loc}(\Omega) + \mathcal{X}^{0, m}_{loc}(\Omega).$$

In particular, if $(y, \eta) \in V_1 \subset V$, which are closed conic neighborhoods of $(y, \eta)$ such that $V_1$ is contained in the interior of $V$ and $V \cap N^*\Sigma_j = \emptyset$, $j = 1, 2, 3$, and if $A$ is a pseudodifferential operator of order zero which is elliptic in $V_1$ and it is of order $-\infty$ outside $V$, then provided $\mathcal{G} = \mathcal{G}$ at $\Gamma$,

$$A(\mathcal{G}V, A(\mathcal{G}V) \in H^{3m-\frac{1}{2}}_{loc}(\Omega),$$

and $A([\mathcal{G} - \mathcal{G}]V \in H^{3m-\frac{1}{2}}_{loc}(\Omega)$.

So modulo terms in $\mathcal{X}^{0, m}(\Omega)$, we define

$$(\partial^2_y(yf))(y, u(y))|_V \equiv \mathcal{G}V, \text{ if } \mathcal{G} \in C^\infty(\Omega), \mathcal{G}(y) = (\partial^2_y(yf))(y, u(y)), \ y \in \Gamma,$$



**Proposition 10.12.** Let $(y, \eta) \in N^*\Gamma \setminus 0 \cup (N^*\Sigma_j \setminus 0)$ and let $(y, \eta) \in V_1 \subset V$, which are closed conic neighborhoods of $(y, \eta)$ such that $V_1$ is contained in the interior of $\Omega$ and $V \cap N^*\Sigma_j = \emptyset$, $j = 1, 2, 3$. If the hypotheses of Theorem 8.2 are satisfied and $A$ is a pseudodifferential operator of order zero which is elliptic in $V_1$ and it is of order $-\infty$ outside $V$, then if $\mathcal{G} \in C^\infty(\Omega)$, $A(\mathcal{G}V) \in H^{3m-\frac{1}{2}}(\Omega, \Gamma) \subset H^{3m-\frac{1}{2}}(\Omega)$, and modulo $H^{3m-\frac{1}{2}}(\Omega)$,

$$A((\partial^2_y(yf))(y, u(y))|_V \equiv A(\mathcal{G}V),$$

provided $\mathcal{G} = (\partial^2_y(yf))(y, u(y))$ on $\Gamma$. 

**Inverse Problems and Imaging**

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11. The proof of Theorem 4.1. We follow the strategy of the proof of Theorem 4.1 of [42], which is in part based on the strategy of the proof of Theorem 4.1 of [4]. We also allow arbitrary $C^\infty$ nonlinearities $f(y,u)$, while in [42] the nonlinear term $f(y,u)$ is a polynomial in $u$ with $C^\infty$ coefficients. As we will see below, the key to the proofs of both theorems is (10.24).

First we discuss the global behavior of the solution $u$ of (2.4) and later we will analyze the microlocal behavior of $u$ near $\Gamma$ in the directions where $\langle \eta_1 \rangle > \langle \eta \rangle$, $j = 1, 2, 3$, and compute the principal part of the new singularities on $\Omega$.

Recall that the initial data of (2.4) is of the form $v = v_1 + v_2 + v_3$, with $v_j \in \mathcal{I}^{m-\frac{n}{2}+\frac{1}{j}}(\Omega, \Sigma_j)$ and $m < -\frac{n}{2}(n+7)$. So we conclude from Proposition 10.10 that $u \in \mathcal{H}_0^{0,\ldots, -m-\frac{1}{j}, -m-\frac{1}{j}}(\Omega)$.

We recall the formula for the expansion of the triple product $v_1v_2v_3$ given by (10.22), and motivated by this we already defined the spaces $\mathcal{K}^{0,m}(\Omega)$ in (10.23), and we now define

$$\mathcal{K}^{1,m}(\Omega) = \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega) + \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega) + \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega) + \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega).$$

To simplify the notation, we set,

$$(11.1) \quad F(y, u(y)) = \mathcal{Y}(y)f(y, u(y)),$$

where $f(y, u)$ and $\mathcal{Y}(y)$ are as in Theorem 4.1, so $F(y, u(y))$ is compactly supported and its support is contained in $\Omega \cap \{t \geq -1\}$. We use the forward fundamental solution to $P(y, D)$, which we have denoted by $E_+$, and the fact that $P(y, D)v = 0$, to write

$$(11.2) \quad u = v + E_+(F(y, u)),$$

and we use Proposition 7.4 to write

$$v_j = \nu_j + \mathcal{E}_j, \quad \nu_j \in \mathcal{I}^{m-\frac{n}{2}+\frac{1}{j}}(\Omega, \Sigma_j), \quad \mathcal{E}_j \in C^\infty, \quad \text{and hence } v = \nu + \mathcal{E},$$

where $\mathcal{E} \in C^\infty$ and $v = \nu_1 + \nu_2 + \nu_3$, $\nu_j \in \mathcal{I}^{m-\frac{n}{2}+\frac{1}{j}}(\Omega, \Sigma_j)$. Recall that $\nu_j$ vanishes to order $k(m)$ on $\Sigma_j$. To simplify the notation, we use (11.3) to write

$$u = v + E_+(F(y, u(y)) = v + \mathcal{W}, \quad \text{where } \mathcal{W} = \mathcal{E} + E_+(F(y, u, \nu + \mathcal{W})), \quad \text{and hence}$$

$$(11.4) \quad u = v + E_+(F(y, \nu + \mathcal{W})) + \mathcal{E}.$$  

Later we will need the fact that since $u = \mathcal{W} + \nu$ and since $\nu = 0$ on $\Gamma$,

$$(11.5) \quad \mathcal{W} = u \text{ on } \Gamma.$$

Since $m < -\frac{n}{2}(n+7)$, it follows that $-m + \frac{1}{j} > \frac{n}{2} + 3$, and we deduce from Property P.5 of Proposition 10.2 that $F(y, u) \in \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega)$. Then, it follows from Proposition 10.3 that

$$(11.6) \quad E_+(F(y, u)) \in \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega),$$

and so we conclude that

$$(11.7) \quad \mathcal{W} = \mathcal{E} + E_+(F(y, u)) \in \mathcal{H}^{0,\ldots, -m+\frac{1}{2}, -m-\frac{1}{2}}_{\text{loc}}(\Omega).$$

Our first result separates the terms with higher order of regularity of $F(y, \nu + \mathcal{W})$, and was stated in Claim 1 and Claim 2 in Section 6:
Proposition 11.1. Let \( u, F(y, u) \) be as above and let \( V \) be defined in (10.22). Then
\[
F(y, u + W) - (\partial_u^3 F)(y, W)V \in \mathcal{X}^{0,m}(\Omega),
\]
where \( \mathcal{X}^{0,m}(\Omega) \) was defined in (10.23).

Proof. We begin by taking the Taylor expansion of order three in \( u \) of \( F(y, u) \) centered at \( W \):
\[
F(y, W + \nu) = \sum_{j=0}^3 \frac{1}{j!} (\partial_u^j F)(y, W)\nu^j + \frac{1}{3!} \nu^3 \int_0^1 (\partial_u^3 F)(y, W + t\nu)(1 - t)^3 dt.
\]

First we consider the third order Taylor polynomial
\[
T_3(y) = \sum_{j=0}^3 \frac{1}{j!} (\partial_u^j F)(y, W)\nu^j.
\]

We know that
\[
\nu^2 = \nu_1^2 + \nu_2^2 + \nu_3^2 + 2\nu_1 \nu_2 + 2\nu_1 \nu_3 + 2\nu_2 \nu_3,
\]
\[
\nu^3 = \nu_1^3 + \nu_2^3 + \nu_3^3 + 3\nu_1^2 \nu_2 + 3\nu_1 \nu_2^2 + \nu_1^2 \nu_3 + 3\nu_2 \nu_3^2 + 3\nu_1 \nu_2 \nu_3 + 3\nu_2^2 \nu_3 + 6\nu_1 \nu_2 \nu_3.
\]
Therefore we write
\[
T_3(y) = F(y, W) + (\partial_u \theta)(y, W)(\nu_1 + \nu_2 + \nu_3) + \Theta(y) + (\partial_u^2 \theta)(y, W)\nu_1 \nu_2 \nu_3,
\]
where
\[
\Theta(y) = \frac{1}{2}(\partial_u^2 \theta)(y, W)(\nu_1^2 + \nu_2^2 + \nu_3^2 + 2(\nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3)) + \frac{1}{3!} (\partial_u^3 \theta)(y, W)(\nu_1^3 + \nu_2^3 + \nu_3^3 + 3\nu_1^2 \nu_2 + 3\nu_1 \nu_2^2 + \nu_1^2 \nu_3 + 3\nu_2 \nu_3^2 + 3\nu_1 \nu_2 \nu_3 + 3\nu_2^2 \nu_3 + 6\nu_1 \nu_2 \nu_3).
\]

First we consider the terms \( \nu_j^m, \nu_j^m \) or \( \nu_j \nu_k \) or \( \nu_j^m \). It follows from Proposition 10.2 that for \( \alpha_j \geq 1, \alpha_j \in \mathbb{N}_0 \),
\[
\nu_1^{\alpha_1} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega), \quad \nu_2^{\alpha_2} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega),
\]
\[
\nu_3^{\alpha_3} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega), \quad \nu_1^{\alpha_1} \nu_2^{\alpha_2} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega),
\]
\[
\nu_1^{\alpha_1} \nu_2^{\alpha_2} \nu_3^{\alpha_3} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega), \quad \nu_2^{\alpha_2} \nu_3^{\alpha_3} \in \mathcal{H}^{0, -m - \frac{1}{2}, \infty, \infty}(\Omega).
\]

In fact, in view of Proposition 7.5, the terms with \( \alpha_j > 1 \) are smoother, but we do not need this now. We know from (11.7) that \( W \in \mathcal{H}^{1, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}}(\Omega) \), and since \( m < -\frac{1}{2} \), it follows from Property P.5 of Proposition 10.2 that for any \( j \in \mathbb{N}_0 \),
\[
(\partial_u^j F)(y, W) \in \mathcal{H}^{1, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}}(\Omega),
\]
and therefore from Properties P.2 and P.4 of Proposition 10.2 we conclude that for any \( j \), and \( \alpha_j \geq 1 \),
\[
\nu_1^{\alpha_1} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m + \frac{1}{2}}(\Omega),
\]
\[
\nu_2^{\alpha_2} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m + \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}}(\Omega),
\]
\[
\nu_3^{\alpha_3} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}}(\Omega),
\]
\[
\nu_2^{\alpha_2} \nu_3^{\alpha_3} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}}(\Omega),
\]
\[
\nu_1^{\alpha_1} \nu_3^{\alpha_3} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m - \frac{1}{2}, -m + \frac{1}{2}, -m - \frac{1}{2}}(\Omega),
\]
\[
\nu_1^{\alpha_1} \nu_2^{\alpha_2} (\partial_u^j F)(y, W) \in \mathcal{H}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m + \frac{1}{2}}(\Omega).
\]
So we conclude that
\[ T_3(y) - (\partial^3_y F)(y, W)\nu_1 \nu_2 \nu_3 \in \mathcal{K}^{0,m}(\Omega). \]

But, we know from Proposition 10.11 that if \( V \) is given by (10.22), then \( \nu_1 \nu_2 \nu_3 - V \in \mathcal{K}^{0,m}(\Omega) \), and again from (11.10) and Property \( \textbf{P.4} \) of Proposition 10.2 we conclude that
\[ (11.12) \quad T_3(y) - (\partial^3_y F)(y, u)V \in \mathcal{K}^{0,m}(\Omega). \]

Next we consider the fourth order remainder in the Taylor’s expansion:

**Lemma 11.2.** Let \( W \) and \( \nu \) be as above. Then for any \( t \in [0,1] \), we have
\[
\begin{align*}
\nu^4(\partial^4_y F)(y, W + tv) &= \mathcal{B}_1(t) + \mathcal{B}_2(t) + \mathcal{B}_3(t), \\
\mathcal{B}_1(t) &\in \mathcal{H}_{\text{loc}}^{0, -m + \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}}(\Omega), \quad \mathcal{B}_2(t) \in \mathcal{H}_{\text{loc}}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}}(\Omega), \\
\text{and } \mathcal{B}_3(t) &\in \mathcal{H}_{\text{loc}}^{0, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}}(\Omega)
\end{align*}
\]

and moreover for every \( \delta > 0 \) and \( \varphi \in C^\infty_0(\Omega) \), there exists a constant \( C > 0 \) such that for all \( t \in [0,1] \),
\[
\begin{align*}
||\varphi \mathcal{B}_1(t)||_{-\delta, -m + \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}} &\leq C \quad ||\varphi \mathcal{B}_2(t)||_{-\delta, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}} \leq C, \\
||\varphi \mathcal{B}_3(t)||_{-\delta, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{4}} &\leq C.
\end{align*}
\]

**Proof.** We begin by expanding the term \( \nu^4 = (\nu_1 + \nu_2 + \nu_3)^4 \):
\[
\begin{align*}
\nu^4 &= \nu_1^4 + \nu_2^4 + \nu_3^4 + 4\nu_1^3\nu_2 + 4\nu_1^3\nu_3 + 4\nu_2^3\nu_1 + 4\nu_2^3\nu_3 + 4\nu_3^3\nu_1 + 4\nu_3^3\nu_2 + 12\nu_1^2\nu_2\nu_3 + 12\nu_1^2\nu_1\nu_3 + 12\nu_2^2\nu_1\nu_3 + 6\nu_1^2\nu_2^2 + 6\nu_1^2\nu_3^2 + 6\nu_2^2\nu_2^2.
\end{align*}
\]

We see that
\[
\begin{align*}
\partial_y \left[ \nu_1^4(\partial^4_y F)(y, W + tv) \right] &= (\partial_y \nu_1^4)(\partial^4_y F)(y, W + tv) + \nu_1^4(\partial_{yy} \partial^4_y F)(y, W + tv)(t\partial_{y, t} + \partial_{y, t} W) = \\
(\partial_y \nu_1^4)(\partial^4_y F)(y, W + tv) + \frac{t}{4}(\partial^4_y F)(y, W + tv)\partial_y \nu_1^5 &\quad (11.14)
\end{align*}
\]

Now we appeal to Proposition 7.5 and Proposition 10.6 to conclude that
\[ (11.15) \quad \nu_1^4 \in s^{m-(j-1)k(m)-\frac{3}{2}+\frac{1}{2}}(\Omega, \Sigma_1) \subset \mathcal{H}_{\text{loc}}^{0, -m + (j-1)k(m) + \frac{1}{2}, -\infty, \infty}(\Omega). \]

Since \( m < -\frac{3}{2} - \frac{7}{2}, k(m) \geq 1 \) and hence
\[
\begin{align*}
\partial_y \nu_1^4 &\in \mathcal{H}_{\text{loc}}^{0, -m + 3k(m) - \frac{1}{2}, -\infty, \infty}(\Omega) \subset \mathcal{H}_{\text{loc}}^{0, -m - \frac{1}{2}, -\infty, \infty}(\Omega), \\
\partial_y \nu_1^3 &\in \mathcal{H}_{\text{loc}}^{0, -m + 2k(m) - \frac{1}{2}, -\infty, \infty}(\Omega) \subset \mathcal{H}_{\text{loc}}^{0, -m + \frac{1}{2}, -\infty, \infty}(\Omega), \\
\partial_y \nu_2 &\in \mathcal{H}_{\text{loc}}^{0, -m + k(m) - \frac{1}{2}, -\infty, \infty}(\Omega) \subset \mathcal{H}_{\text{loc}}^{0, -m - \frac{1}{2}, -\infty, \infty}(\Omega).
\end{align*}
\]

We also know that
\[ (11.16) \quad \partial_y \nu_2 \in \mathcal{H}_{\text{loc}}^{0, -\infty, -m - \frac{1}{2}, -\infty}(\Omega) \text{ and } \partial_y \nu_3 \in \mathcal{H}_{\text{loc}}^{0, -\infty, -m + \frac{1}{2}, -\infty}(\Omega), \]
and
\[
\nu \in \mathcal{H}_{\text{loc}}^{0, -m - \frac{1}{2}, -\infty, \infty}(\Omega), \quad W \in \mathcal{H}_{\text{loc}}^{1, -m - \frac{1}{2}, -m - \frac{1}{2}, -\frac{1}{4}}(\Omega).
\]
and since \( m < -\frac{n}{2} - \frac{7}{2} \), we deduce from Property \( P.5 \) that
\[
(\partial_{y_i} \partial^a u)(y, W + t\nu), \quad (\partial^a \partial^b u)(y, W + t\nu), \quad (\partial^a F)(y, W + t\nu)
\] and
\[
(\partial^a F)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega)
\]
We also know that \( \partial_{y_i} W \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega) \) and so we conclude that
\[
\partial_{y_i} \left[ \nu_1(\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
and hence \( \nu_1(\partial^a u)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega) \).

The same argument used with respect to \( y_2 \) and \( y_3 \) respectively shows that
\[
\nu_2(\partial^a F)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega)
\]
and
\[
\nu_3(\partial^a F)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega).
\]

Now we consider the terms with \( \nu_1 \). We again write
\[
\partial_{y_i} \left[ \nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) \right] = (\partial_{y_i} \nu_1^2)(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) + 
\nu_1^2(\partial_{y_i}(y_2 + \nu_3))(\partial^a u)(y, W + t\nu) + 
\nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) + 
\nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu)(t\partial_{y_i} \nu + \partial_{y_i} W) = 
(\partial_{y_i} \nu_1^2)(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) + \frac{1}{4}(y_2 + \nu_3)(\partial^a u)(y, W + t\nu)(\partial_{y_i} \nu_1^2) + 
\nu_1^2(\partial_{y_i}(y_2 + \nu_3))(\partial^a u)(y, W + t\nu) + 
\nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) + 
\nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu)(t\partial_{y_i} (y_2 + \nu_3) + \partial_{y_i} W).
\]

Using (11.16) we conclude that
\[
\partial_{y_i} \left[ \nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega)
\]
and therefore
\[
\nu_1^2(y_2 + \nu_3)(\partial^a u)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega).
\]

Following this argument with respect to \( y_2 \) and \( y_3 \) we also find that
\[
\nu_2^2(y_1 + \nu_3)(\partial^a u)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega)
\]
and
\[
\nu_3^2(y_1 + \nu_3)(\partial^a u)(y, W + t\nu) \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega).
\]

The terms in \( \nu_j^2 \nu_k, j \neq k \), and \( \nu_j^2 \nu_k \nu_m, j \neq k, j \neq m \) and \( k \neq m \) can be handled in the same way and we obtain
\[
\partial_{y_i} \left[ \nu_1^2 \nu_2^2(\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_1^2 \nu_3^2(\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_2^2 \nu_3^2(\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_1^2 \nu_2 \nu_3 (\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_1^2 \nu_2 \nu_3 (\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_2^2 \nu_1 \nu_3 (\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega),
\]
\[
\partial_{y_i} \left[ \nu_3^2 \nu_1 \nu_2 (\partial^a u)(y, W + t\nu) \right] \in \mathcal{H}^{0,-m-\frac{1}{2},-m-\frac{7}{2}}(\Omega).
\]

The estimates in (11.13) follow by applying (10.9) and (10.8) at each step of the proof. This ends the proof of Lemma 11.2.
To finish the proof of the proof of Proposition 11.1 one needs to show that
\[ \int_0^1 \nu^A (\partial^a u) F(y, W + t\nu)(1 - t)^3 dt \in \mathcal{K}^{m, 0}(\Omega). \]
The integral is a well defined Riemann integral, as all functions here are continuous, the only issue is the boundedness of the integral in these spaces, but this follows from the estimates in (11.13) for each \( B_j(t), j = 1, 2, 3. \)

Next we consider the regularity of the solution \( u \) near \( \Gamma \). Let \( \chi_j \in C_0^\infty(U_j) \), \( U_j \) a small enough neighborhood of \( q_j \in \Gamma \), \( 1 \leq j \leq N \), and suppose that \( \sum_{j=1}^N \chi_j = 1 \) in a neighborhood of \( \Gamma \) in the support of \( Y(y) \). It follows from Proposition 10.11 that
\[ (1 - \chi)V \in \mathcal{K}^{0, m}(\Omega), \]
We already know that \( W \in \mathcal{H}^{1, -m - \frac{1}{2}, -m - \frac{1}{2}}_{\text{loc}}(\Omega) \), and since \( m < -\frac{1}{2}(n + 7) \), \( (\partial^A u) y, W \in \mathcal{H}^{1, -m - \frac{1}{2}, -m - \frac{1}{2}}_{\text{loc}}(\Omega) \). Since \( \mathcal{H}^{1, m}(\Omega) \) is contained in each factor of \( \mathcal{K}^{1, m}(\Omega) \),
\[
(1 - \sum_{j=1}^N \chi_j)V(\partial^A u)(y, W) \in \mathcal{K}^{0, m}(\Omega),
\]
and therefore, putting this together with (11.8), we find that
\[
F(y, W + \nu) - \sum_{j=1}^N \chi_j(\partial^A u F)(y, W)V \in \mathcal{K}^{0, m}(\Omega).
\]
So we need to discuss the regularity of the terms \( \chi_j V(\partial^A u F)(y, W) \) in the region near \( N^\ast \Gamma \setminus 0 = \{ y_1 = y_2 = y_3 = 0, \eta'' = 0 \} \),
and away from \( N^\ast \Sigma_1 = \{ y_1 = 0, \eta_2 = \eta_3 = 0, \eta'' = 0 \}, N^\ast \Sigma_2 = \{ y_2 = 0, \eta_1 = \eta_3 = 0, \eta'' = 0 \}, N^\ast \Sigma_3 = \{ y_3 = 0, \eta_1 = \eta_2 = 0, \eta'' = 0 \} \).
This region is characterized by
\[
\bigcup_{\varepsilon > 0} \{ (y, \eta) : \langle \eta_j \rangle \geq \varepsilon \langle \eta \rangle, j = 1, 2, 3 \}.
\]
So we define a pseudodifferential operator of order zero, \( A(D) \in \Psi^0(\Omega) \), with symbol
\[
\sigma(A)(\eta) = \psi(\eta_1) \psi(\eta_2) \psi(\eta_3),
\]
where \( \psi(s) = C^\infty(\mathbb{R}) \), \( \psi(s) = 0 \) for \( |s| \leq \frac{1}{2} \) and \( \psi(s) = 1 \) for \( |s| \geq 1 \).

We prove the following, which was stated as Claim 3 in Section 6:

**Proposition 11.3.** Let \( \Sigma_j, j = 1, 2, 3 \) and \( \Omega \) satisfy the hypotheses of Theorem 4.1. Let \( q \in \Gamma \), let \( U \) be a neighborhood of \( q \) and let \( y = (y', \eta'), y = (y_1, y_2, y_3) \) be local coordinates in \( U \) such that \( \Sigma_j = \{ y_j = 0 \}, j = 1, 2, 3 \) and \( P(y, D) \) is given by (9.3). Let \( \chi \in C_0^\infty(U) \). Suppose that \( v = v_1 + v_2 + v_3, v_j \in \mathcal{H}^{m - \frac{1}{2}, \frac{1}{2}, m - \frac{1}{2}}(\Omega, \Sigma_j) \), \( m < -\frac{1}{2}(n + 7) \), \( j = 1, 2, 3 \). Let
\[
a_1(y_2, y_3, \eta_1, \eta_3), a_2(y_1, y_3, \eta_2, \eta_3), a_3(y_1, y_2, \eta', \eta_3) \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}),
\]
be the principal symbols of \( v_1, v_2 \) and \( v_3 \) respectively, and assume they are elliptic. Let \( V \) be the principal part of the product \( v_1 v_2 v_3 \) given by (10.22). Let \( u \) be the
solution to (2.4) and let \( W \) be defined in (11.7). If \( A \in \Psi^0(\Omega) \) is defined in (11.24), then
\[
A_X \left[ (\partial^2_a F)(y, W)V - (\partial^2_a F)(0, 0, y', u(0, 0, 0, y''))V \right] \in H^{-\frac{3m-\frac{2}{r}}{2}}(\Omega),
\]
(11.25)
provided \( r \in (0, 1 - \frac{2}{m - \frac{2}{r}}) \), while
\[
A_X \left[ (\partial^2_u F)(0, 0, y', u(0, 0, 0, y'''))V \right] \in H^{-\frac{3m-\frac{2}{r}}{2}}(\Omega).
\]

Proof. This is a consequence of the following Lemma, which was proved in Proposition 4.3 of [42]:

**Lemma 11.4.** Let \( \alpha(\eta) = \alpha_1(\eta_1)\alpha_2(\eta_2)\alpha_3(\eta_3) \), with \( \alpha_j(\eta_j) \in S^m(\mathbb{R}) \) and \( m < -\frac{2}{3} \).

Let \( b(\eta_1, \eta_2, \eta_3) \) be such that for all \( \delta > 0 \),
\[
(11.26) \quad \langle \eta_1 \rangle^{-m - \frac{2}{3}} \langle \eta_2 \rangle^{-m - \frac{2}{3}} \langle \eta_3 \rangle^{-m - \frac{2}{3}} \langle (\eta_1, \eta_2, \eta_3) \rangle^{1 - \delta} b(\eta_1, \eta_2, \eta_3) \in L^2(\mathbb{R}^3).
\]

Then in the conic regions
\[
\Upsilon_{\mu_0, \mu_1} = \Upsilon_{\mu_0, \mu_1}(\eta_1) \cup \Upsilon_{\mu_0, \mu_1}(\eta_2) \cup \Upsilon_{\mu_0, \mu_1}(\eta_3),
\]
\[
\Upsilon_{\mu_0, \mu_1}(\eta_1) = \{ \mu_0 < \eta_1^2, \mu_1 \}, \quad \mu_0 < \eta_1^2, \mu_1 \},
\]
\[
\Upsilon_{\mu_0, \mu_1}(\eta_2) = \{ \mu_0 < \eta_1^2, \mu_1 \}, \quad \mu_0 < \eta_1^2, \mu_1 \},
\]
\[
\Upsilon_{\mu_0, \mu_1}(\eta_3) = \{ \mu_0 < \eta_1^2, \mu_1 \}, \quad \mu_0 < \eta_1^2, \mu_1 \}.
\]

the convolution \( \alpha \ast b \) satisfies, for \( r + \frac{2}{m - \frac{2}{r}} < 1 \),
\[
(11.27) \quad \alpha \ast b(\eta_1, \eta_2, \eta_3) = \alpha(\eta_1, \eta_2, \eta_3) \int_{\mathbb{R}^3} b(\eta) d\eta + \mathcal{E}(\eta_1, \eta_2, \eta_3), \quad \text{where}
\]
\[
\int_{\Upsilon_{\mu_0, \mu_1}} \left| \langle (\eta_1, \eta_2, \eta_3) \rangle^{-3m - \frac{2}{r} + \frac{3}{2}} \mathcal{E}(\eta_1, \eta_2, \eta_3) \right|^2 d\eta_1 d\eta_2 d\eta_3 < \infty.
\]

Now we can finish the proof of Proposition 11.3. We define \( b(\eta_1, \eta_2, \eta_3, y'') \) to be the partial Fourier transform in \( y' = (y_1, y_2, y_3) \) of \( \chi(y)(\partial^2_u F)(y, W) \):
\[
b(\eta_1, \eta_2, \eta_3, y'') = \mathcal{F}_{y'}(\chi(y)(\partial^2_u F)(y, W))(\eta_1, \eta_2, \eta_3, y'') = \int_{\mathbb{R}^3} e^{i(\eta_1 y_1 + \eta_2 y_2 + \eta_3 y_3)} \chi(y')(\partial^2_u F)(y', y'', W(y_1, y_2, y_3, y'')) dy'.
\]

We know from (10.22) that
\[
\mathcal{F}_{y'}(V) = a(\eta_1, \eta_2, \eta_3, y'') = a(\eta_1, \eta_2, \eta_3, y'') a_2(\eta_2, 0, 0, y'') a_3(\eta_3, 0, 0, y''),
\]
and we also know that \( \chi(y)(\partial^2_u F)(y, W) \in \mathcal{F}^{1-m-\frac{2}{r}, -m-\frac{2}{r}, -m-\frac{2}{r}}(U) \) and in particular,
\[
\langle \eta_1 \rangle^{-m - \frac{2}{3}} \langle \eta_2 \rangle^{-m - \frac{2}{3}} \langle \eta_3 \rangle^{-m - \frac{2}{3}} \langle (\eta_1, \eta_2, \eta_3) \rangle^{1 - \varepsilon} b(\eta_1, \eta_2, \eta_3, y'') \in L^2(\mathbb{R}^3, \eta_1, \eta_2, \eta_3 \times \mathbb{R}^3),
\]
and so we deduce from Lemma 11.4 that
\[
\mathcal{F}_{y'} \left[ V(y) \chi(y)(\partial^2_u F)(y, W) \right](\eta_1, \eta_2, \eta_3, y'') = \frac{1}{(2\pi)^3} a \ast b(y', y'') = a(\eta_1, \eta_2, \eta_3, y'') \int_{\mathbb{R}^3} b(\zeta_1, \zeta_2, \zeta_3, y'') d\zeta_1 d\zeta_2 d\zeta_3 + \mathcal{E}(\eta_1, \eta_2, \eta_3, y'') = a(\eta_1, \eta_2, \eta_3, y'') (\partial^2_u F)(0, 0, 0, y'', W(0, 0, 0, y'')) + \mathcal{E}(\eta_1, \eta_2, \eta_3, y'').
\]
where \( \mathcal{E}(\eta_1, \eta_2, \eta_3, y'') \) satisfies
\[
(11.28) \quad \int_{\mathbb{R}^{n-2}} \left| \left( (\eta_1, \eta_2, \eta_3) \right)^{-3m-\frac{2}{3}+r/2} \mathcal{E}(\eta_1, \eta_2, \eta_3, y'') \right|^2 d\eta_1 d\eta_2 d\eta_3 dy'' < \infty.
\]

Let \( A(D) \) be the pseudodifferential operator with symbol \( \sigma(A) (\eta) \) given by (11.24). We conclude from (11.28) that, provided \( r < 1 - \frac{2}{m+\frac{2}{3}} \),
\[
A(D) \mathcal{E}(y) = \mathcal{F}^{-1} (\sigma(A)(\eta) \mathcal{F} y'' (\mathcal{E}(\eta_1, \eta_2, \eta_3, y'')) \eta) \in H^{-3m-\frac{2}{3}+r/2}_{loc}(\Omega),
\]
where \( \mathcal{F} y'' \) is the partial Fourier transform in \( y'' \) and \( \mathcal{F} \) is the Fourier transform in \( y = (y', y'') \). Therefore, we conclude from (11.28) that
\[
A(D) \left[ \chi(y) (\partial^2_u F)(y, W)V - \chi(y) (\partial^2_u F)(0, 0, 0, y'', W(0, 0, 0, y''))V(y) \right] = (11.29) \quad A(D) \mathcal{E}(y) \in H^{-3m-\frac{2}{3}+\frac{r}{2}}_{loc}(\Omega), \quad \text{if } r < 1 - \frac{2}{m+\frac{2}{3}}.
\]
This concludes the proof of Proposition 11.3 \( \Box \)

Now we can finish the proof of Theorem 4.1.

**Proof.** We know from (11.23) that if \( A(D) \) is a pseudodifferential operator with symbol given by (11.24), it follows from the definition of \( \mathcal{K}_0, m \), that
\[
A(D) \left( F(y, W + \nu) - \sum_{j=1}^{N} \chi_j(y) (\partial^2_u F)(y, W)V \right) \in H^{3m-\frac{4}{3}}_{loc}(\Omega).
\]
But then in view of (11.29), as long as \( r < 1 - \frac{2}{m+\frac{2}{3}} \),
\[
A(D)(F(y, W + \nu) - \sum_{j=1}^{N} \chi_j(y) (\partial^2_u F)(0, 0, 0, y'', W(0, 0, 0, y''))V(y)) \in H^{-3m-\frac{2}{3}+\frac{r}{2}}_{loc}(\Omega).
\]
But recall that \( u = \nu + W \), and that \( \nu = 0 \) on \( \Gamma \) and so \( W = u \) on \( \Gamma \). Let \( \gamma_+ \) be a bicharacteristic on \( \Lambda \setminus \bigcup_{j=1}^{N*} \Sigma_j \) and let \( \gamma_+ \subset C_1(\gamma_+) \subset C(\gamma_+) \) be closed conic neighborhoods of \( \gamma_+ \) such that \( C_1(\gamma_+) \) is contained in the interior of \( C(\gamma_+) \) and \( C(\gamma_+) \) does not intersect \( N^* \Sigma_j, j = 1, 2, 3 \) or \( N^* \Gamma \setminus 0 \). Let \( B \) be a pseudodifferential operator of order zero which is elliptic on \( C_1(\gamma_+) \) and of order \(-\infty\) outside \( C(\gamma_+) \), then
\[
B [u - E_+ (A F(y, u(y))) |V] \in H^{3m-\frac{4}{3}+r/2}_{loc}(\Omega).
\]
If \( n = 3, \Gamma = \{ q \}, A (F(q, u(q)))V \in I^{3m+\frac{4}{3}}(\Omega, \Gamma) \) and \( E_+ (A (F(y, u(q)))V) \) has principal symbol given by (A.2).

If \( n > 3, \) we know from Proposition 10.24 that
\[
A ((\partial^2_y F)(y, u)|V) \in I^{3m-\frac{4}{3}+\frac{r}{2}}(\Omega, \Gamma) + H^{3m-\frac{4}{3}+\frac{r}{2}}_{loc}(\Omega)
\]
and again the principal symbol of \( E_+ (A ((\partial^2_y F)(y, u)|V)) \) is given by (A.2). We also know that away from \( N^* \Sigma_j, j = 1, 2, 3 \) and \( \Gamma u \) is conormal to \( \Omega \), so this is its principal symbol. \( \Box \)
Appendix A. A result on propagation of singularities for linear equations.

We recall one particular result about the propagation of singularities for solutions of the linear wave equation. This shows how one can compute the principal symbol of a conormal distribution which solves a linear wave equation. Let \( \Omega, \Sigma_j, j = 1, 2, 3 \), \( \Gamma \) and \( P(y, D) \) be as above. Let \( v(y) \) satisfy
\[
P(y, D)v = f \in \mathcal{I}^\mu(\Omega, \Gamma).
\]
\[
v = f = 0, \quad t < -1.
\]

The space \( \mathcal{I}^m(\Omega, M) \), where \( M \subset \Omega \) is a \( C^\infty \) submanifold, was defined in Section 7. The results of Melrose and Uhlmann [26] imply that microlocally near \( \Lambda \) and away from \( N^*\Gamma \), \( v \in \mathcal{I}^{m-\frac{1}{2}}(\Omega, Q) \) as long as there are no caustics on \( Q \).

Moreover, if \( \sigma(v) \) is the principal symbol of \( v \), it satisfies the following equation on \( \Lambda \):
\[
(i\mathcal{L}_{H_p} + c)v = 0 \quad \text{on} \: \Lambda,
\]

with initial condition \( \sigma(u) = \frac{e^{\pm i\pi}}{(2\pi)^{\frac{3}{2}}} R(p^{-1}\sigma(f)) \) on \( \Lambda \cap \Lambda_0 \),

\( \mathcal{L}_{H_p} \) denotes the Lie derivative with respect to \( H_p \), and \( \sigma(f) \) is the principal symbol of \( f \) and \( R \) is the map defined in Definition 4.11 of [26].

In fact, Melrose and Uhlmann give a much more precise description of \( v \) uniformly up to \( \Lambda \cap \Lambda_0 \), and globally independently of existence of caustics, in terms of paired Lagrangian distributions. Proposition 6.6 of [26] states that \( v \) is a paired Lagrangian distribution \( v \in \mathcal{I}^{m-\frac{1}{2}}(\Omega; \Lambda_0, \Lambda) \), where \( \Lambda_0 = N^*\Gamma \setminus 0 \) and \( \Lambda \) is the manifold defined above by the forward flow-out of \( \Lambda_0 \cap \{p = 0\} \).

Appendix B. The proof of proposition 2.1. Recall that our goal is to put \( Q \) and \( \Sigma_1 \) into a normal form while keeping \( \Gamma \) fixed. For \( n = 3 \) this is done in Proposition 4.9 of [30]. We have shown in Theorem 9.1 that for any \( q \in \Gamma \) there exists a neighborhood \( U_q \subset \Omega \) of \( q \) and a neighborhood \( U_0 \) of 0 and a map \( \Psi_0 : U_0 \longrightarrow U_q \) such that \( \Psi_0 \Gamma = \{y_1 = y_2 = y_3 = 0\} \) and \( \Psi^*P(y, D) \) is given by (9.3). We will show that, by shrinking \( U_0 \) if necessary, we can construct a map \( \Psi_1 : U_0 \longrightarrow U_0 \) which is equal to the identity on
\[
\Gamma = \{y_1 = y_2 = y_3 = 0\} \quad \text{and} \quad (\Psi_1 \circ \Psi_0)^*Q = \{y_1^2 + y_2^2 + y_3^3 - 2y_1y_2 - 2y_2y_3 - 2y_3y_1 = 0\}.
\]

We then construct another transformation \( \Psi_2 : U_0 \longrightarrow U_0 \) that keeps \( \Gamma \) and \( Q \) fixed and is such that \( (\Psi_2 \circ \Psi_1 \circ \Psi_0)^*\Sigma_1 = \{y_1 = 0\} \). The construction of \( \Psi_1 \) is done in three steps:

Step 1: We make a change of variables such that \( P(y, D) \) satisfies
\[
\alpha^{-1}(y'')P(y, D) = \partial_{y_1}\partial_{y_2} + \partial_{y_1}\partial_{y_2} + \partial_{y_1}\partial_{y_2} + \sum_{j,k=1}^{n} b_{jk}(y'')\partial_{y_j}\partial_{y_k} + \sum_{m=1}^{3} \sum_{j,k=1}^{n} y_m b_{jk,m}(y)\partial_{y_j}\partial_{y_k} + \mathcal{L},
\]

where \( \alpha(y'') \in C^\infty \) and \( \alpha > 0 \).

Step 2: Since \( H_{op} = \alpha H_p + pH_p \), and we are working in the region where \( p = 0 \), the factor \( \alpha \) does not really matter for the equation of \( \Omega \) since it can be viewed as
a reparametrization along the integral curves of \( H_p \). So we can view \( P(y, D) \) as a perturbation of \( P_0(y, D) \) as in (2.10). As a consequence of that we will show that (B.2)

\[
Q = \{ q(y) = 0 \}, \quad q(y) = q_0(y_1, y_2, y_3) + q'(y):
\]

\[
q'(y) = \sum_{|\alpha| = 3} (y_1, y_2, y_3)^\alpha q_\alpha(y).
\]

Step 3: We will show that there exists a family of diffeomorphisms \( \Psi_s, s \in [0, 1] \) such that \( \Psi_s \) fixes \( \Gamma \) and

\[
\Psi_s^*(q_0 + sq') = q_0.
\]

We begin by proving Step 3, assuming that Step 2 has been proven. Recall that in coordinates \( \{ y \} \) such that \( \Gamma \) is given by (B.1) the vector fields tangent to \( \Gamma \) are spanned over \( C^\infty \) by \( \{ y_j \partial_{y_k}, \partial_{y_m}, \partial_x, j, k = 1, 2, 3, m = 4, \ldots, n \} \). We will construct a vector field \( V_s \) of the form

\[
V_s = \partial_s + W_s, \quad W_s = \sum_{j,k=1}^3 a_{jk}(s, y)y_j \partial_{y_k}, \quad \text{such that } V_s(q_0 + sq') = 0.
\]

We then define \( \Psi_s \) as the diffeomorphism which satisfies,

\[
\Psi_0 = \text{Id}, \quad \frac{d}{ds} \Psi_s^*(f) = \Psi_s^*(V_s f), \quad f \in C^\infty, \quad s \in [0, 1].
\]

Then, in particular if \( f = q_0 + sq' \)

\[
\frac{d}{ds} \Psi_s^*(q_0 + sq') = \Psi_s^*(W_s(q_0 + sq') + q') = 0,
\]

and since \( \Psi_0 = \text{Id} \), and \( \Psi_s^*(q_0 + sq') \) is does not depend on \( s \), \( \Psi_1 \) is the desired map and we are done. Moreover, since \( W_s \) vanishes on \( \Gamma \), \( \Psi_s \) is equal to the identity on \( \Gamma \). This well known method of constructing diffeomorphisms goes back to Moser [33] and has been used in similar context in [32, 40].

Notice that since \( V_s \) is tangent to \( \Gamma \), \( \Psi_s \) fixes \( \Gamma \). So we just have to construct \( W_s \). Notice that

\[
\begin{align*}
y_1 \partial_{y_1} q_0 &= 2y_1(y_1 - y_2 - y_3), \quad y_2 \partial_{y_1} q_0 = 2y_2(y_1 - y_2 - y_3), \\
y_2 \partial_{y_2} q_0 &= 2y_3(y_1 - y_2 - y_3), \quad y_1 \partial_{y_2} q_0 = 2y_1(y_2 - y_1 - y_3), \\
y_2 \partial_{y_3} q_0 &= 2y_2(y_2 - y_1 - y_3), \quad y_3 \partial_{y_2} q_0 = 2y_3(y_2 - y_1 - y_3), \\
y_1 \partial_{y_3} q_0 &= 2y_1(y_3 - y_1 - y_2), \quad y_2 \partial_{y_3} q_0 = 2y_2(y_3 - y_1 - y_2), \\
y_3 \partial_{y_3} q_0 &= 2y_3(y_3 - y_1 - y_2).
\end{align*}
\]

It is not hard to check that \( y_j \partial_{y_k} q_0 \) span the homogeneous polynomials \( \{ y^\alpha, |\alpha| = 2 \} \) over \( \mathbb{R} \). So, since \( q \) satisfies (B.2), it follows that

\[
q'(y) = \sum_{jk} A_{jk}(y) y_j \partial_{y_k} q_0, \quad A_{jk}(0, y'') = 0,
\]

\[
yi \partial_{y_{m}}(sq') = s \sum_{\alpha,\beta=1}^3 B_{t_{\alpha\beta}}(y) y_\alpha \partial_{y_\beta} q_0, \quad B(0, y'') = 0.
\]

If we think of \( \Theta_0 = (y \partial_{y_\alpha} q_0)_{l,m=1,2,3} \), and \( \Theta' = (y_\alpha \partial_{y_\beta} q')_{(\alpha,\beta=1,2,3)} \) as column vectors, this says that \( q'(y) = A(y) \cdot \Theta_0(y) \) and there exists a matrix \( B(y) = (B_{t_{\alpha\beta}}(y)) \) such that

\[
\Theta' = B \Theta_0, \quad B(0, y'') = 0.
\]
We want to find a row vector \( W(s, y) = (v_{jk}(s, y)) \) such that
\[
W(s, y) \cdot (I + sB(y))\Theta_0 = A(y) \cdot \Theta_0.
\]
This is equivalent to requiring that \(((I + sB(y))^*W(s, y) - A(y)) \cdot \Theta_0 = 0\), where * indicates the adjoint of the matrix. This is satisfied if \((I + sB(y))^*W(s, y) = A(y)\). Since \(B(0, y'') = 0\), one can find \(W(s, y)\) for small \(y\). This proves Step 3.

Now we prove Step 1. We have shown that there exist local coordinates \(y = (y_1, y_2, y_3, y'')\) near every point of \(\Gamma\) such that (9.3) holds. Using the notation in that equation, we know that there exist \(C^\infty\) functions \(b_{jkm}(y)\) such that
\[
b_{jk}(y) = A_{jk}(y'') + \sum_{m=1}^{3} y_{m} b_{jkm}(y), \quad A_{jk}(y'') = b_{jk}(0, 0, 0, y'').
\]
and hence the operator \(P(y, D)\) can be written as
\[
P(y, D) = A_{12}(y'')\partial_{y_1}\partial_{y_2} + A_{13}(y'')\partial_{y_1}\partial_{y_3} + A_{23}(y'')\partial_{y_2}\partial_{y_3} + \sum_{j,k=4}^{n} A_{jk}(y'')\partial_{y_j}\partial_{y_k} + \sum_{m=1}^{3} (y_{m} b_{m12}(y)\partial_{y_1}\partial_{y_2} + y_{m} b_{m13}(y)\partial_{y_1}\partial_{y_3} + y_{m} b_{m23}(y)\partial_{y_2}\partial_{y_3}) + \sum_{m=1}^{3} \sum_{j,k=4}^{n} y_{m} A_{jk}(y'')\partial_{y_j}\partial_{y_k} + \mathcal{L}.
\]
We know that \(A_{jk} \neq 0, j, k = 1, 2, 3\). If we now make a change of variables
\[
(y_1, y_2, y_3, y'') \mapsto (\lambda_1(y'')y_1, \lambda_2(y'')y_2, \lambda_2(y'')y_3, y''),
\]
with \(\lambda_3 = \frac{\lambda_2 A_{12} A_{13}}{A_{12}}\) and \(\lambda_1 = \frac{\lambda_2 A_{23}}{A_{12}}\), and we denote \(\alpha(y'') = \lambda_1\lambda_2 A_{12} = \lambda_2\lambda_3 A_{13} = \lambda_2\lambda_3 A_{23}\), the operator satisfies
\[
\alpha^{-1}(y'')P(y, D) = \partial_{y_1}\partial_{y_2} + \partial_{y_1}\partial_{y_3} + \partial_{y_2}\partial_{y_3} + \sum_{j,k=4}^{n} \tilde{A}_{jk}(y'')\partial_{y_j}\partial_{y_k} + \sum_{m=1}^{3} (y_{m} \tilde{b}_{m12}(y)\partial_{y_1}\partial_{y_2} + y_{m} \tilde{b}_{m13}(y)\partial_{y_1}\partial_{y_3} + y_{m} \tilde{b}_{m23}(y)\partial_{y_2}\partial_{y_3}) + \sum_{m=1}^{3} \sum_{j,k=4}^{n} y_{m} \tilde{A}_{jk}(y'')\partial_{y_j}\partial_{y_k} + \mathcal{L}.
\]
and this proves Step 1.

To prove Step 2, one just needs to compare the model case with this perturbation. In (2.12) we solved the system of odes in the model case and we found that the map
\[
\exp(sH_{p_0}) : (N^\ast \Gamma \setminus \emptyset) \cap \{ p = 0 \} \rightarrow \Omega
\]
\[
(0, 0, 0, y'', \eta_{10}, \eta_{20}, \eta_{30}, 0) \rightarrow (y_1, y_2, y_3, y''),
\]
is homogeneous of degree one. In the perturbed case, we find that \((y_1, y_2, y_3) = \Theta_0(\eta_{10}, \eta_{20}, \eta_{30}) + O(||(\eta_{10}, \eta_{20}, \eta_{30})||^2)\), and we can check that by examining the system of odes, which implies (B.2).

So far we have constructed the map \(\Psi_1 \circ \Psi_0\) mentioned above. Since \(P(y, D)\) is a small perturbation of \(P_0(y, D)\), the \(\Omega\) and \(\Sigma_1\) should be small perturbations of the model. In particular, \(\Omega\) and \(\Sigma_1\) are tangent to first order along their intersection. In the model case, \(\Omega\) and \(\Sigma_1\) are simply tangent along the manifold \(\{y_1 = 0, y_2 = y_3\}\).
In the general case, they must be simply tangent along the manifold given by the flow of
\[ \{ y_1 = y_2 = y_3 = 0, \eta_2 = \ldots \eta_n = 0 \} \]
under \( H_n \), with \( P(y, D) \) given by (B.4). By examining the equations of the flow, we conclude that \( \Sigma_{(1)} \) and \( \Sigma_{(2)} \) are tangent to first order at \( \{ y_1 = 0, y_2 = y_3 \} \), and so \( \Sigma_{(1)} \) must be given by \( \Sigma_{(1)} = \{ y_1 = \varphi(y)(y_2 - y_3)^2 \}, \varphi \in C^\infty \).

As in the proof of Proposition 4.9 of [30], one just needs to verify that the change of coordinates
\[
Y_1 = y_1 - \varphi(y)(y_2 - y_3)^2, \quad Y_2 = y_2 + y_3 - \frac{1}{2} \varphi(y)(y_2 - y_3)^2, \quad Y_3 = (y_2 - y_3)[1 - \varphi(y)(2(y_2 + y_3) - y_1)]^{\frac{1}{2}},
\]
preserves \( \Gamma \), satisfies
\[
Y_1^2 + Y_2^2 + Y_3^2 - 2Y_1Y_2 - 2Y_1Y_3 - 2Y_2Y_3 = Y_1(Y_1 - 2(Y_2 + Y_3)) + (Y_2 - Y_3)^2 = y_1(y_1 - 2(y_2 + y_3)) + (y_2 - y_3)^2 = y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3,
\]
and \( \Sigma_{(1)} = \{ Y_1 = 0 \} \). This ends the construction of \( \Psi \).

Appendix C. The proof of proposition 10.2.

Proof. First we prove Property P.3. Let \( \varphi \in C^{\infty}_0(U) \) and \( \varphi u \in H^{0-k_1,k_2,k_3}_\text{loc}(U) \) with \( k_j > \frac{3}{2} \), we want to show that \( \varphi u \in L^\infty(U) \) and we just need to show that \( |\mathcal{F}(\varphi u)| \in L^1(\mathbb{R}^n) \). We define
\begin{equation}
W_\varphi(\eta) = \langle \eta_1, \eta'' \rangle^{k_1} \langle \eta_2, \eta'' \rangle^{k_2} \langle \eta_3, \eta'' \rangle^{k_3} \langle \eta \rangle^{-\delta},
\end{equation}

and so
\[
|||\mathcal{F}(\varphi u)|||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathcal{F}(\varphi u)(\eta)| d\eta = \int_{\mathbb{R}^n} [W_\varphi(\eta)]^{-1} |\mathcal{F}(\varphi u)(\eta)| d\eta \leq \left[ \int_{\mathbb{R}^n} [W_\varphi(\eta)]^{-2} d\eta \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}^n} [W_\varphi(\eta)]^2 |\mathcal{F}(\varphi u)(\eta)|^2 d\eta \right]^{\frac{1}{2}} = |||\varphi u|||_{H^{-\delta,k_1,k_2,k_3}} \left[ \int_{\mathbb{R}^n} [W_\varphi(\eta)]^{-2} d\eta \right]^{\frac{1}{2}}
\]

Notice that, if one sets \( t = z(1 + \rho^2)^{\frac{3}{2}}, \) then, provided \( 1 - 2k < 0 \),
\[
\int_{\mathbb{R}} (1 + \rho^2 + t^2)^{-k} dt = (1 + \rho^2)^{-k} + \frac{\rho}{2} \int_{\mathbb{R}} (1 + z^2)^{-k} dz = C(1 + \rho^2)^{-k} + \frac{\rho}{2}.
\]

So, by setting \( \rho = |\eta''| \), we obtain
\[
\int_{\mathbb{R}^n} [W_\varphi(\eta)]^{-2} d\eta = (C.2) \quad C \int_{\mathbb{R}^3} (1 + \eta^2 + \rho^2)^{-k_1} (1 + \eta^2 + \rho^2)^{-k_2} (1 + \eta^2 + \rho^2)^{-k_3} \rho^{n-4} d\rho d\eta_1 d\eta_2 d\eta_3 \leq C \int_{\mathbb{R}} (1 + \rho^2)^{-(k_1 + k_2 + k_3) + \frac{3}{2} - \frac{3}{2}} d\rho,
\]
which converges, since \( n - 2(k_1 + k_2 + k_3) < 0 \). So, \( |||\mathcal{F}(\varphi u)|||_{L^1(\mathbb{R}^n)} < \infty \) and hence \( \varphi u \in L^\infty(U) \).

Now we prove Property P.4.

Proof. The main ingredient in the arguments used below is the following Lemma:
Lemma C.1. (Rauch and Reed [36]) Suppose $K(\xi, \eta) = \sum_{j=1}^{k} K_j(\xi, \eta)$ and
\begin{equation}
\sup_{\xi} \int |K_j(\xi, \eta)|^2 d\eta < \infty \text{ or } \sup_{\eta} \int |K_j(\xi, \eta)|^2 d\xi < \infty.
\end{equation}
If $f, g \in L^2(\mathbb{R}^n)$ and $h(\xi) = \int K(\xi, \eta)f(\xi - \eta)g(\eta) d\eta$, it follows that $h \in L^2(\mathbb{R}^n)$ and
\begin{equation}
||h||_{L^2} \leq C||f||_{L^2}||g||_{L^2}.
\end{equation}

Let $\xi = (\xi_1, \xi_2, \xi_3, \xi''')$ and $\eta = (\eta_1, \eta_2, \eta_3, \eta''')$. For $\kappa = (k_1, k_2, k_3)$ and $\delta > 0$, let $W_\kappa(\eta)$ be defined in (C.1) and let
\begin{equation}
K_\kappa(\xi, \eta) = \frac{W_\kappa(\xi)}{W_\kappa(\xi - \eta)W_\kappa(\eta)}.
\end{equation}

Then,
\begin{equation}
\mathcal{F}(W_\kappa(D)uv)(\xi) = \int_{\mathbb{R}^4} K_\kappa(\xi, \eta) (W_\kappa(\xi - \eta)\hat{u}(\xi - \eta)) (W_\kappa(\eta)\hat{v}(\eta)) d\eta.
\end{equation}

So, according to Lemma C.1 we need to prove that $K_\kappa(\xi, \eta)$ can be decomposed as a sum $K_\kappa(\xi, \eta) = \sum_{j=1}^{M} K_{\kappa,j}(\xi, \eta)$, with $K_{\kappa,j}$ satisfying (3.3).

For $j = 1, 2, 3$, we shall denote,
\begin{equation}
E_j = \{(\xi_j, \xi'''_j, \eta_j, \eta''') : ||(\xi_j, \xi''') - (\eta_j, \eta''')|| \leq \frac{1}{2}||\xi_j, \xi'''||\},
\end{equation}
and
\begin{equation}
F_j = \{(\xi_j, \eta_j) : ||(\xi_j - \eta_j, \xi''' - \eta''')|| > \frac{1}{2}||\xi_j, \xi'''||\}.
\end{equation}

Notice that
\begin{equation}
\frac{1}{2}||\xi_j, \xi'''|| \leq ||(\eta_j, \eta''')|| \leq \frac{3}{2}||\xi_j, \xi'''|| \text{ on } E_j.
\end{equation}

Let $\chi_{E_j}$ and $\chi_{F_j}$ denote the characteristic functions of $E_j$ and $F_j$ respectively. For $J = (j_1, j_2, j_3)$, $M = (m_1, m_2, m_2)$, with $m_r, j_r = 0, 1$. Let
\begin{equation}
\chi^J_{E} = \chi^{j_1}_{E_1} \chi^{j_2}_{E_2} \chi^{j_3}_{E_3}, \quad \chi^M_{F} = \chi^{m_1}_{F_1} \chi^{m_2}_{F_2} \chi^{m_3}_{F_3},
\end{equation}
and write
\begin{equation}
K_\kappa(\xi, \eta) = \sum_{j_j+m_j=1} \chi^J_{E} \chi^M_{F} K_\kappa(\xi, \eta).
\end{equation}

In the case where $J = (1, 1, 1)$ and $M = (0, 0, 0)$, in virtue of (C.6), we have
\begin{equation}
\chi^J_{E} K_\kappa(\xi, \eta) \leq \frac{C}{W_\kappa(\xi - \eta)},
\end{equation}
and hence from (C.2),
\begin{equation}
\int_{\mathbb{R}^4} \chi^J_{E} \chi^M_{F} K_\kappa(\xi, \eta)^2 d\eta \leq \int_{\mathbb{R}^4} \frac{C}{W_\kappa^2(\xi - \eta)} d\eta \leq \int_{\mathbb{R}^4} \frac{C}{W_\kappa^2(\eta)} d\eta < \infty,
\end{equation}
provided $k_j > \frac{\delta}{\xi}$ and $\delta$ is small enough.

Next we consider the case $J = (1, 0, 1)$ and $M = (0, 1, 0)$. First, in virtue of the first inequality in (C.5), and then because of the definition of $F_j$, we have
\begin{equation}
K_\kappa(\xi, \eta) = \sum_{j_j+m_j=1} \chi^J_{E} \chi^M_{F} K_\kappa(\xi, \eta)^2 d\eta \leq \int_{\mathbb{R}^4} \frac{C}{W_\kappa^2(\xi - \eta)} d\eta < \infty,
\end{equation}
provided $k_j > \frac{\delta}{\xi}$ and $\delta$ is small enough.
Therefore, using (C.5) for \( j = 1, 3, \)
\[
\chi_E^M \chi_F^M K_M (\xi, \eta) \leq C \frac{\langle \xi_2, \xi'' \rangle^k_2}{W_\alpha (\xi - \eta) (\eta_2, \eta'' \rangle^k_2} \leq 
\]
\[
\frac{\langle \xi_1 - \eta_1, \xi'' - \eta'' \rangle^k_1 (\xi_3 - \eta_3, \xi'' - \eta'' \rangle^k_3 (\eta_2, \eta'' \rangle^k_2}
\]
Again, making a change of variables, one finds that for \( J = (1, 0, 1) \) and \( M = (0, 1, 0) \), as in (C.2),
\[
\int_{\mathbb{R}^4} \chi_E^M \chi_F^M K_M (\xi, \eta) d\eta < \infty,
\]
provided \( \alpha_{jk} > \frac{\alpha}{6} \) and \( \delta \) is small enough. The other terms are controlled in the same way, and the details are left to the reader. \( \square \)

Next we prove Property P.5:

**Proof.** By replacing \( f(y, u) \) with \( f(y, u) - f(y, 0) \), we may assume that \( f(y, 0) = 0 \) and since \( u \) is compactly supported we may assume that \( f(y, u(y)) \) is compactly supported in \( y \).

First we need to prove a particular case:

**Lemma C.2.** If \( u \in H^{0, k_1, k_2, k_3} (U, \{ y \}) \cap L^\infty (\mathbb{R}^n), k_j \in \mathbb{N}, j = 1, 2, 3, \) and \( f(y, s) \in C^\infty \), then \( f(y, u) \in H^{0, k_1, k_2, k_3} (U, \{ y \}) \).

**Proof.** The proof depends on three ingredients:

1. A special case of the Gagliardo-Nirenberg inequality, see for example [16]: For \( |\alpha| \leq m, \)
\[
\| D_y^\alpha u \|_{L^\infty} \leq C \| u \|_{L^\infty}^{1 - \frac{|\alpha|}{m}} \sum_{|\beta| \leq m} \| D_y^\beta u \|_{L^2} \tag{C.7}
\]

2. The following version of Hölder’s inequality:
\[
\| f_1 f_2 \ldots f_N \|_{L^2} \leq \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}} \ldots \| f_N \|_{L^{p_N}}, \text{ if } \sum_{j=1}^N \frac{1}{p_j} = 1 - \frac{1}{2} \tag{C.8}
\]

3. The following formula, which can be proven by induction:
\[
(D_{y_1}^\gamma D_{y'}^\gamma)^\gamma f(y, u) = \sum_{\beta_1, \ldots, \beta_k} C_{\beta_1, \ldots, \beta_k} (y, u) ((D_{y_1}^\beta D_{y'}^\beta)^\beta_1 u) \cdots ((D_{y_1}^\beta D_{y'}^\beta)^\beta_k u), \tag{C.9}
\]

provided \( |\beta_1| + |\beta_2| + \ldots |\beta_k| \leq |\gamma|, \ k = |\gamma| - 1. \)

Since \( u \in L^\infty \) and \( f \in C^\infty \), it follows from (C.9) and (C.8) that
\[
\| (D_{y_1}^\gamma D_{y'}^\gamma)^\gamma f(y, u) \|_{L^2} \leq C(\gamma, \| u \|_{L^\infty}) \sum_{\beta_1, \ldots, \beta_k} \| (D_{y_1}^\beta D_{y'}^\beta)^\beta_1 u) \cdots ((D_{y_1}^\beta D_{y'}^\beta)^\beta_k u) \|_{L^2} \leq 
\]
\[
C \sum_{\beta_1, \ldots, \beta_k} \| (D_{y_1}^\beta D_{y'}^\beta)^\beta_1 u) \|_{L^{p_1}} \cdots \| ((D_{y_1}^\beta D_{y'}^\beta)^\beta_k u) \|_{L^{p_N}},
\]
where \( p_j = \frac{2|\gamma|}{|\beta_j|}, \ j = 1, 2, \ldots, N. \)
Now using (C.7) we find that

\[ \left\| (D_{y_1}, D_{y''})^\gamma f(y, u) \right\|_{L^2} \leq C(\gamma, \|u\|_{L^\infty}) \left( \sum_{|\beta| \leq |\gamma|} \left\| (D_{y_1}, D_{y''})^\beta u \right\|_{L^2} \right). \]

We control \[ \left\| (D_{y_1}, D_{y''})^{\gamma_1} (D_{y_2}, D_{y''})^{\gamma_2} (D_{y_3}, D_{y''})^{\gamma_3} f(y, u) \right\|_{L^2} \] using the same argument. This ends the proof of the Lemma. \( \square \)

Next we prove that Property P.5 of Proposition 10.2 holds for \( s = 0 \):

\[ \text{Proof.} \text{ Since } \partial_{y_j} \text{ is elliptic, for at least one value of } j \in \{1, 2, 3, \ldots, n\}, \text{ if } u \in H_{loc}^{0, k_1, k_2, k_3}(U, \{y\}), \text{ and } k_j > \frac{p}{6} + 1, j = 1, 2, 3, \text{ one can pick } m_j \in \mathbb{N}_0 \text{ such that } \frac{p}{6} < m_j < k_j, \text{ and therefore } u \in H^{0, m_1, m_2, m_3}(U, \{y\}). \text{ Since } k_j > \frac{p}{6}, \text{ it follows from Prop. P.3 that } u \in L^\infty_{loc}(\mathbb{R}^n). \text{ It follows from Lemma C.2 that } f(y, u) \in H^{0, m_1, m_2, m_3}(U, \{y\}) \text{ with } 1 \leq m_j < k_j \text{ and for all } \|\varphi u\|_{-\delta, k_1, k_2, k_3} \leq C_\delta, \text{ there exists } K \text{ depending on } f, C_\delta \text{ such that } \|\varphi f(y, u)\|_{0, m_1, m_2, m_3} \leq K. \text{ But we know that } \]

\[ \begin{align*}
\partial_{y_j} f(y, u) &= (\partial_{y_j} f)(y, u) + (\partial_u f)(y, u) \partial_{y_j} u, \\
\partial_{y_j} f(y, u) &= (\partial_{y_j} f)(y, u) + (\partial_u f)(y, u) \partial_{y_j} u, \quad j \geq 4,
\end{align*} \]

Since \( (\partial_{y_j} f)(y, u) \in H_{loc}^{0, m_1, m_2, m_3}(U, \{y\}) \), \( (\partial_u f)(y, u) \in H_{loc}^{0, m_1, m_2, m_3}(U, \{y\}) \) and \( \partial_{y_j} u \in H_{loc}^{0, k_1 - 1, k_2, k_3}(Y, \{y\}) \), and since \( k_j > m_j > \frac{p}{6} \), and \( k_j - 1 > \frac{p}{6} \), it follows from Proposition 10.2 that

\[ \begin{align*}
\partial_{y_j} f(y, u) &\in H_{loc}^{0, r_1, m_2, m_3}(U, \{y\}), \quad r_1 = \min\{m_1, k_1 - 1\}, \\
\partial_{y_j} f(y, u) &\in H_{loc}^{0, r_1, m_2, m_3}(U, \{y\}), \quad j \geq 4, \quad r_1 = \min\{m_1, k_1 - 1\}.
\end{align*} \]

This implies that

\[ f(y, u) \in H_{loc}^{0, r_1 + 1, m_2, m_3}(U, \{y\}), \quad r_1 = \min\{m_1, k_1 - 1\}. \]

We have two possibilities: either \( r_1 + 1 = k_1 \) or \( r_1 + 1 = m_1 + 1 \). If \( r_1 + 1 = m_1 + 1 \in \mathbb{N}_0 \), in this case we repeat the argument for \( m_1 \) replaced by \( m_1 + 1 \). So after finitely many steps, we will find that \( r_1 + 1 = k_1 \) and so we conclude that \( f(y, u) \in H_{loc}^{0, k_1, m_2, m_3}(U, \{y\}) \) for an arbitrary function \( f \in C^\infty \). Now we repeat (C.10) for \( (\partial_{y_j}, \partial_{y''}) \), and we conclude that

\[ \begin{align*}
\partial_{y_j} f(y, u) &\in H_{loc}^{0, k_1, r_2, m_3}(\mathbb{R}^n), \quad r_2 = \min\{m_2, k_2 - 1\}, \\
\partial_{y_j} f(y, u) &\in H_{loc}^{0, k_1, r_2, m_3}(\mathbb{R}^n), \quad j \geq 4, \quad r_2 = \min\{m_2, k_2 - 1\}.
\end{align*} \]

We apply the same argument and conclude that

\[ f(y, u) \in H_{loc}^{0, k_1, k_2, m_3}(U, \{y\}). \]

We do this again with respect to \( (\partial_{y_3}, \partial_{y''}) \), and we are done. \( \square \)

Now we can prove that Property P.5 holds for \( s \in \mathbb{N}_0 \). We prove this induction. Suppose \( u \in H^{1, k_1, k_2, k_3}(U) \). Since

\[ \partial_{y_j} f(y, u) = (\partial_{y_j} f)(y, u) + (\partial_u f)(y, u) \partial_{y_j} u, \]

and the result holds for \( s = 0 \), so \( (\partial_{y_j} f)(y, u), (\partial_u f)(y, u) \in H_{loc}^{0, k_1, k_2, k_3}(U, \{y\}) \) and Property P.4, gives that that \( \partial_{y_j} f(y, u) \in H_{loc}^{0, -1, k_1, k_2, k_3}(U, \{y\}) \), and therefore \( f(y, u) \in H_{loc}^{1, k_1, k_2, k_3}(U, \{y\}) \). The same argument shows that if \( H_{loc}^{s, -1, k_1, k_2, k_3}(U) \) is a \( C^\infty \) algebra, so is \( H_{loc}^{0, k_1, k_2, k_3}(U, \{y\}) \). The bound on the norm also follows from the proof. \( \square \)
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