Adaptive Density-Matrix Renormalization-Group study of the disordered antiferromagnetic spin-1/2 Heisenberg chain

Alexander H. O. Wada$^{1}$ and José A. Hoyos$^{1,2}$

1Instituto de Física de São Carlos, Universidade de São Paulo, C. P. 369, São Carlos, São Paulo 13560-970, Brazil
2Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany

Using an adaptive strategy which enables the study of quenched disordered system via the density-matrix renormalization-group method, we compute the various ground-state spin-spin correlation measures of the spin-1/2 antiferromagnetic Heisenberg chain with random coupling constants, namely, the mean values of the bulk and of the end-to-end correlations, the typical value of the bulk correlations, and the distribution of the bulk correlations. Our results are in agreement with the predictions of the strong-disorder renormalization-group method. We do not find any hint of logarithmic corrections either in the bulk average correlations, which were recently reported by Shu et al. [Phys. Rev. B. 94, 174442, (2016)], or in the end-to-end average correlations. We report the existence of a logarithmic correction on the end-to-end correlations of the clean chain. Finally, we have determined that the distribution of the bulk correlations, when properly rescaled by an associated Lyapunov exponent, is a narrow and universal (disorder-independent) probability function.

Published in Phys. Rev. B 105, 104205 (2022); DOI: 10.1103/PhysRevB.105.104205

I. INTRODUCTION

One-dimensional random quantum systems display a rich plethora of phenomena and are important theoretical laboratories for strongly correlated quantum phenomena. A prominent phenomenon is the infinite-randomness criticality (IRC) [1]. Initially, it was thought to be exclusive to one-dimensional systems. In recent decades, however, it was found in many different and seemingly unrelated model systems. To name a few, IRC governs the paramagnet–ferromagnet transition of the random transverse-field Ising model (in any dimension) [2–5] and of the quenched disordered Hertz-Millis antiferromagnet [6, 7] and the metal–superconductor transition of rough thin films and nanowires [8, 9]. IRC is also found in out-of-equilibrium situations such as Floquet systems [10] and reaction-diffusion classical systems [11] (for a review, see, e.g., Refs. 12–15). Despite this plethora of theoretical situations, experimental checks of IRC are still rare. Early hints come from quasi-one-dimensional tetracyanoquinodimethan (TCNQ) compounds [16–19] (modeled by the Hamiltonian in Eq. (1)), which initiated this field of research. However, more accurate experiments and clearer signatures are still desirable. In this context, precise knowledge of the ground-state spin-spin correlation function (the main quantity studied in this work) has great relevance as it dictates the behavior of the structure factor at low temperatures [20, 21], which is experimentally accessible via neutron-scattering experiments. Finally, it is worth noting that, more recently, strong experimental evidence of infinite-randomness criticality was reported in itinerant magnets [22–24] (for a review, see Ref. 14) and in thin superconducting films [25].

A paradigmatic model exhibiting IRC is the random antiferromagnetic (AF) spin-1/2 XXZ chain

$$H = \sum_i J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z),$$  

(1)

where $S_i$ are the usual spin-1/2 operators associated with site $i$, the antiferromagnetic coupling constants $J_i > 0$ are independent and identically distributed random variables drawn from a distribution $P_D(J)$ [with $D$ parametrizing the disorder strength; see, for definiteness, Eq. (6)], and $\Delta$ is the anisotropy parameter. It is now well accepted that, for $-\frac{1}{2} < \Delta \leq 1$, the chain is critical and governed by an infinite-randomness fixed point where the arithmetic and geometric means (henceforth referred to as mean and typical values, respectively) of the spin-spin correlation function $[C^\alpha(r) = \langle S_i^\alpha S_{i+r}^\alpha \rangle]$, with $\langle \cdot \rangle$ denoting the ground-state average, behave quite differently. In the thermodynamic limit and for spins sufficiently far from each other, the mean value is

$$\langle C^\alpha(r) \rangle = \frac{(-1)^r}{12\pi^D} \left\{ \begin{array}{ll} c_{\alpha,o}, & \text{if } r \text{ is odd}, \\ c_{\alpha,e}, & \text{otherwise}, \end{array} \right. \quad (2)$$

with $\equiv$ denoting the arithmetic average over the disorder configurations. The exponent $\eta = 2$ is universal [i.e., does not depend on the details of $P_D(J)$], isotropic (i.e., $\alpha$ independent), and $\Delta$ independent [26] due to an enhancement in the ground-state symmetry from $SO(N) \to SU(N)$ (here, $N = 2$), a generic feature of $SO(N)$-symmetric AF random spin chains [27, 28]. The numerical prefactors $c_{\alpha,o,e}$, on the other hand, are nonuniversal (i.e., disorder dependent), anisotropic (i.e., $\alpha$ dependent), and $\Delta$ dependent. Surprisingly, it was conjectured that $c_{\alpha,o} - c_{\alpha,e} = 1$ is universal if $\alpha$ is a symmetry axis, i.e., for $\alpha = x, y$, and for any $\alpha$ when $\Delta = 1$ [21].

The typical value of the spin-spin correlation function,

$$C^\alpha_{\text{typ}}(r) = \exp \ln \langle C^\alpha_{\text{typ}}(r) \rangle \approx c_{\alpha,D} \exp \left[ -A_\alpha \times \langle r \gamma_D \rangle \right], \quad (3)$$

behaves quite differently. It decays stretched exponentially with universal and isotropic tunneling exponent $\gamma = \frac{1}{2}$ [26]. The numerical prefactor $A_\alpha$ is universal and anisotropic, and the Lyapunov exponent $\gamma_D$ is nonuniversal, isotropic, and $\Delta$ dependent. For the free-fermionic case ($\Delta = 0$), a single-parameter theory [29, 30] predicts that $\gamma_D = 8\pi^2 \text{var}(\ln J)$ [where $\text{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$ is the variance]. For the generic case ($-\frac{1}{2} < \Delta \leq 1$), however, $\gamma_D \propto \text{var}(\ln J)^{\frac{1}{1-2\Delta}}$ with $2K = [1 - \pi^{-1} \arccos(\Delta)]^{-1}$.

1 According to standard field-theory methods [31, 32], the Lyapunov expo-
Results (2) and (3) stem from the fact that the ground state is a random singlet which is captured by the strong-disorder renormalization-group (SDRG) method and, supposedly, are asymptotic exact [26]. It is worth mentioning that, at the free-fermion point $\Delta = 0$ [21, 30, 34, 35] and at the isotropic Heisenberg point $\Delta = 1$ [33, 36–38], these results (among other SDRG predictions) have been confirmed with increasing numerical precision over the years (see Refs. 12, 15 and references therein). Interestingly, however, a recent ground-state quantum Monte Carlo study found a logarithmic factor in the mean correlation function [39] at the isotropic point $\Delta = 1$. Namely, result (2) is corrected to

$$\mathcal{C}(r) \sim \ln^{\alpha} r / r^2,$$

with $0.3 \leq \sigma_2 \leq 0.7$. It is certainly desirable to understand the origin of this logarithmic correction, which is not predicted by the SDRG method.\(^2\) For the homogeneous (clean) system at the Heisenberg point $\Delta = 1$ [31, 41–47] and for the dirty system at $\Delta = -\frac{1}{2}$ (where disorder is perturbatively irrelevant) [48], logarithmic factors due to marginally irrelevant operators have been reported. Which marginal operator, if any, endows the logarithmic factor to $\mathcal{C}(r)$? Does the typical value also acquire a similar correction? Unfortunately, conventional perturbative field-theoretical methods cannot be applied at $\Delta = 1$ due to runaway flow of the disorder strength.

Furthermore, it is interesting to ponder the consequences of a possible logarithmic factor to the correlation function. Assuming that the resulting random singlet ground state is localized, i.e., the typical correlation is stretched exponentially small (regardless of logarithmic corrections), it is then possible to use the methods of Ref. 21 to relate $\mathcal{C}(r)$ to the von Neumann entanglement entropy

$$S_l = -\text{Tr} \rho_A \ln \rho_A,$$

where $\rho_A$ is the reduced density matrix of subsystem $A$ (of length $l$) obtained by tracing the degrees of freedom of the complementary subsystem $B$. To leading order in $l$, they are related via $S_l = -\ln 2 \sum_{\sigma=\pm1} \mathcal{C}(r) \sim (\ln l)^{1+\sigma}$. Thus, this would be an interesting violation of the area law if $\sigma_2 \neq 0$.

It is thus desirable to confirm the existence of the logarithmic factor found in Ref. 39. Therefore, we study the spin-spin correlation function of the random AF spin-1/2 Heisenberg chain [Eq. (1) with $\Delta = 1$] using the adaptive density-matrix renormalization-group (aDMRG) method, which is a recently introduced unbiased method for strongly disordered systems.

The remainder of this paper is organized as follows. In Sec. (II) we define the coupling constant distribution $P_D(J)$ and review the employed aDMRG method. In Sec. (III) we apply the DMRG method to the clean chain, and we show that both the bulk and the end-to-end correlation exhibit logarithmic factors. We then apply the aDMRG method to the disordered case and study the effects of disorder on the bulk mean and typical values of the correlation, the end-to-end correlation, and the distribution of the correlations. In all cases, our data are compatible with the absence of logarithmic factors.

Finally, we summarize and discuss our results in Sec. (IV).

\section{Disorder Description and Method}

We study the ground-state spin-spin correlation function of the random AF spin-1/2 Heisenberg chain. The model Hamiltonian is given by Eq. (1) with $\Delta = 1$. The coupling constants $0 < J_i < 1$ are uncorrelated random variables drawn from the probability distribution

$$P_D(J) = D^{-1} J^{1/D - 1},$$

where the disorder strength is parametrized by $D$. The disorder configurations of coupling constants $\{J_i\}$.

How the efficiency of the DMRG method diminishes when dealing with systems governed by infinite-randomness physics is notorious [35, 49–51]. The reason is due to a disorder-induced rough energy landscape with nearly degenerate local minima. The standard DMRG method then gets stuck in an excited/metastable state. As a result, the method fails to capture the rare spin pairs (or clusters) that are largely separated but highly entangled. Although rare, they are responsible for the leading contribution to the mean value of the spin-spin correlations.

In order to circumvent this problem, we employ the recently introduced adaptive aDMRG method [52] to obtain the ground-state spin-spin correlation function. The idea is to apply the standard DMRG method to a clean or nearly clean system (where it works efficiently well) in order to obtain a good representation of the ground state $|\psi_D\rangle$ and then modify it adiabatically by increasing the disorder strength $D$ in small steps $D \to D + \delta D$. Precisely, (i) we start with a disorder configuration $\{J_i\}$ drawn from (6) with $D = D_0 \ll 1$. The standard DMRG method is then applied, and $|\psi_D\rangle$ is obtained (after convergence). The next step is to (ii.a) increase the disorder strength to $D + \delta D$ while keeping the disorder configuration fixed; that is, we simply make the transformation $J_i \to J_i^{1+\delta D}$. (ii.b) Using the previously found ground state $|\psi_D\rangle$ as an input, the standard DMRG method is applied again, from which $|\psi_{D+\delta D}\rangle$ is obtained. (iii) Step (ii) is iterated until the desired disorder strength $D$ is reached.

We have used $D_0 = D = 1/16$ (the nearly clean system). Our DMRG code is implemented using the ITensor Library [53] on chains of $L$ spins with open boundary conditions. In each DMRG application we kept up to $N = 400$ states, which is enough to keep the truncation error below $T_{\text{err}} \sim 10^{-10}$. To ensure convergence, we used 20 sweeps for the initial state $|\psi_D\rangle$ and $\delta_{\text{sweeps}} = 4$ sweeps when increasing the disorder strength, i.e., when going from $|\psi_D\rangle \to |\psi_{D+\delta D}\rangle$.\(^2\) Recently, the subleading corrections to (2) in the SDRG framework were obtained, and no hint of logarithmic corrections was found [40].\(^2\)
III. NUMERICAL RESULTS

In this section, we report our numerical results using the aDMRG method on the various spin-spin correlation functions studied: the mean and typical values for the bulk, the mean end-to-end correlations, and the distribution of the bulk correlations. They are studied for the cases of homogeneous and randomly disordered chains. Finally, we have studied only chains with open boundary conditions.

III.1. The homogeneous AF Heisenberg chain

Due to the open boundary conditions, the system is not translation invariant, and therefore, we average over the various spin pairs of the same size; that is, the bulk correlation function is defined as

$$\langle \mathbf{S}_i \cdot \mathbf{S}_{i+r} \rangle$$

where in order to reduce the finite-size effects, we have excluded the L/4 spins closest to the open boundaries.

We plot in Fig. 1(a) $C(r)$ as a function of the spin separation $r$ for a chain of $L = 500$ spins (black squares). In Fig. 1(b) we plot the same data multiplied by $r$ in order to highlight the logarithmic correction.

The leading terms of the correlation function $\langle \mathbf{S}_i \cdot \mathbf{S}_{i+r} \rangle$ in the regime $1 \ll r \ll L$ are known to be $[45, 46]$

$$\langle \mathbf{S}_i \cdot \mathbf{S}_{i+r} \rangle = \frac{A(r)}{r} + b \frac{(-1)^r}{r^2},$$

with $a = 1$ and $b$ being an unknown constant (both of which we take as fitting parameters for our numerical data) and the function

$$A(r) = \frac{3}{\sqrt{8\pi^3}g} \left(1 - \frac{3}{16}g^2 + \frac{156\zeta(3) - 73}{384}g^3 + O(g^4)\right),$$

where $\zeta(s)$ is the Riemann zeta function and $g \equiv g(r)$ is obtained from

$$g^{-1} + \frac{1}{2} \ln g = \ln \left(2\sqrt{2\pi e^{\gamma+1}r}\right),$$

where $\gamma$ is the Euler constant. In both panels of Fig. 1 we fit our numerical data to the analytical expectation (8) and find that $a = 0.983(1)$ and $b = -0.452(9)$ (red solid line). Therefore, we confirm that, to leading order, $\langle \mathbf{S}_i \cdot \mathbf{S}_{i+r} \rangle \sim \sqrt{\ln r/r}$.

We now study the end-to-end correlation function. For large system sizes ($L \gg 1$), we expect that

$$C_{1L} = \langle \mathbf{S}_1 \cdot \mathbf{S}_L \rangle = c \frac{[\ln(L/L_0)]^\theta}{L^\eta},$$

where the surface correlation function exponent $\eta_s = 2\chi_v = 2$ $[54, 55]$. For the same reason as in the bulk correlations, we expect a logarithmic factor. To the best of our knowledge, however, the exponent $\theta$ is unknown.

Figure 2(a) shows $C_{1L}$ for system sizes ranging from $L = 40$ up to 500. Clearly, $C_{1L}$ does not decay as a simple power law $\sim L^{-\theta}$. In Fig. 2(b) we plot $C_{1L}L^\theta$, from which we fit Eq. (11) to our data taking $\theta, c,$ and $L_0$ as fitting parameters. We obtain $L_0 = 0.7(2), c = 1.5(3)$, and $\theta = 1.5(2)$. Evidently, the value of the exponent $\theta \approx 1.5$ should be interpreted only as an effective exponent since we are not performing a thorough finite-size study.

III.2. The disordered AF Heisenberg chain

In this section, we report our main results on the ground-state correlation function of the AF disordered Heisenberg chain $[\Delta = 1$ in the Hamiltonian (1)] using the aDMRG method.

As a benchmark, we start by computing the correlation function $C$ [as defined in Eq. (7)] for a single disorder realization of coupling constants $\{J_i\}$ using the exact diagonalization, the standard DMRG, and the aDMRG methods for a chain of $L = 10$ spins. As shown in Fig. (3), the standard DMRG method fails to reproduce the exact values, while the aDMRG method reproduces the exact ones within a relative error smaller than $10^{-3}$. We have repeated this benchmark for

3 If instead of (2) one defines $C(r) = \langle \mathbf{S}_{i-1/2} \cdot \mathbf{S}_{i+1/2} \rangle$ for even, and $2C(r) = \langle \mathbf{S}_{i-1/2} \cdot \mathbf{S}_{i-1/2} \rangle + \langle \mathbf{S}_{i+1/2} \cdot \mathbf{S}_{i+1/2} \rangle$ for odd, only the last digit of the fitting parameters to $a$ and $b$ changes.

4 The error in the last digit of the fitting parameters is obtained by removing the first tree data points (smallest $L$) from the fit.
numbers of other disorder realizations and have obtained the same result.\footnote{For further comparison between these methods, we refer the reader to Ref. 52.}

In order to illustrate the convergence of the adaptive strategy, we plot in Fig. 4 an additional analysis with respect to the (a) number of DMRG sweeps $\delta_{\text{sweeps}}$ necessary for convergence when increasing the disorder strength from $D \rightarrow D + \delta D$, (b) the disorder strength increment $\delta D$, and (c) the number of states needed to keep the DMRG error truncation $T_{\text{err}}$ below a certain threshold. We then compute the entanglement entropy $S_l$ as a function of the subsystem size $l$. Here, we show only a typical disorder realization of a chain $L = 50$ sites long with final disorder strength $D = 3.0625$. The subsystem $A$ consists of all spins from sites 1 to $l$. In (a), $S_l$ is shown for different numbers of DMRG sweeps $\delta_{\text{sweeps}}$ used in the adaptive strategy while the disorder increment $\delta D$ is kept fix. In (b), $S_l$ is shown for different values of $\delta D$ and fix $\delta_{\text{sweeps}}$. (c) The total number of states $N$ needed to ensure that the DMRG truncation error is not larger than $T_{\text{err}} = 10^{-10}$ as the disorder strength is increased.
plot in Fig. 4(c) the total number of states $N$ required to keep the DMRG truncation error below $T_{\text{err}} = 10^{-10}$ as the disorder strength $D$ is increased along the adaptive strategy. Clearly, the disorder gets larger, fewer states are necessary. We report that the entanglement entropy $S_1$ converges less rapidly than the correlation function $C$; that is, if the parameters used are enough to ensure the convergence of $S_1$, then $C$ is also converged.

Now we turn to the main results of this work. The first one is on the mean value of the bulk correlations $\overline{C}(r)$ [as defined in (7)] for chains of $L = 100$ spins and various disorder strengths $D$, shown in Fig. 5. In Fig. 5(a) we can see that $\overline{C}$ crosses over from the clean behavior $C_{\text{clean}} \sim (\sqrt{n}r)/r$ (black solid line) to the disordered one $\overline{C} \sim r^{-2}$ (red dashed line) with increasing disorder strength $D$, as expected.

In order to obtain a data collapse, we now follow the reasoning of Ref. 30. The first step is to relate the clean-dirty crossover length $\xi_D$ to the multiplicative prefactor $c_D$ of the correlation function $\overline{C} = c_D r^{-2}$. This is accomplished by assuming a sharp crossover at $r = \xi_D$, i.e., $C_{\text{clean}} = A(\xi_D)/\xi_D = \overline{C} = c_D / \xi_D^2$, and thus, $c_D \sim \xi_D A(\xi_D)$, with $A(r)$ defined in (9). The second step is to rescale the spin-spin separation $r$ in terms of $\xi_D$ (i.e., $r \rightarrow r/\xi_D$) and to rescale $\overline{C}$ accordingly. Thus, $\overline{C}_{\xi_D}/A(\xi_D) \sim (r/\xi_D)^{-2}$. The third step is to relate $\overline{C}$ to the disorder strength $D$. As explained in the Introduction, the associated Lyapunov exponent is

$$\gamma_D \equiv [\text{var}(\ln J)]^{-1/2} \nu.$$

with $2K = [1 - \pi^{-1} \arccos(\Delta)]^{-1}$ [32], and we ignore any possible multiplicative prefactor. As $\xi_D \approx D^{-1}$, then, $\xi_D \approx D^{-1}$ for $\Delta = 1$. In order to proceed, we need to take a final step: we assume that $A(\xi_D) \approx 1$ for $D > 1$. This is justified after we verify that $\xi_D$ is of order unity in the data in Fig. 5(a). While $A(\xi_D)$ cannot be dropped for $D < 1$, for our purposes we need the correct scaling function only in the strong-disorder regime. Thus, we plot $|\overline{C}|/D$ vs $rD$ in Fig. 5(b). The data collapse reasonably well apart from the deviations due to finite-size effects and $D$-dependent corrections to $A(\xi_D)$ (or $c_D$) in the $D < 1$ regime. Finally, in Fig. 5(c) we plot $\overline{C} D r^2$ as a function of $rD$, which should be compared to the clean case in Fig. 1(b). The increasing of the plateau for the largest values of $D$ suggests the nonexistence of the logarithmic factor. Notice that this is reached only for the largest values of $D$. For $D \approx 2$, the plateau seems like a shoulder and is strongly affected by the finite-size corrections. As reported in Ref. 30, this can mimic logarithmic factors. We remark that the numerical observation of this plateau is not a simple task to accomplish even in the free-fermion case $\Delta = 0$ with periodic boundary conditions [30].

We now extract the value of the exponent $\eta$ and the difference $c_0 - c_e$ between the numerical prefactors. We then analyze the data in Fig. 5(b) excluding the points which, due to strong finite-size effects, are out of the data collapse. The resulting data points are replotted Fig. 6(a) from which we fit Eq. (2) to each data set using $\eta$, $c_0$, and $c_e$ as fitting parameters. The corresponding values are plotted in Fig. 6(b) as a function of the disorder strength $D$. For small values of $D$, $\eta$ is simply an effective exponent due to the large associated crossover length. As $D$ increases, the crossover length shortens, and the effective exponent $\eta$ approaches the expected value $\eta = 2$. We observe analogous behavior for the difference $c_0 - c_e$. The fitted values for $D = 3.0625$ are

---

6 Reference 30 showed that these corrections, although smaller, exist even in the XX chain, where there are no logarithmic corrections to the clean correlation function.
the corresponding value of $\eta_r \approx 1.1$ is already far from the

clean value 2. This indicates that the end-to-end correlation is less affected by the clean-dirty crossover when compared to the bulk correlations. More interestingly, the logarithmic factor (if any) is strongly affected by disorder indicating its absence.

We now turn our attention to the typical value of the correlation function,

$$C_{\text{typ}} = \exp \left( \frac{\sum_{i=L/4}^{3L/4} \ln |S_i \cdot S_{i+r}|}{L/2-r} \right),$$

(13)

which is defined analogously to Eq. (7). This quantity is plotted in Fig. 8 as a function of the spin-spin separation $r$ for $L = 100$ and various values of $D$. Figure 8(a) shows $\ln C_{\text{typ}}$ very good data collapse based on Eq. (3). This is done by fitting Eq. (3) to the data in Fig. 8(a) in a region which, presumably, is only weakly affected by finite size (see magenta lines). The resulting collapsed data are shown in Fig. 8(b). The fitting pa-

$$\eta = 1.99(2) \quad \text{and} \quad c_o - c_e = 1.01(5).$$

For completeness, we report that these values were obtained by fitting the data within the range $x_{\text{min}} \leq rD \leq x_{\text{max}}$ where $(x_{\text{min}}, x_{\text{max}}) \approx (4, 10), (10, 20), (15, 30), (20, 40), (25, 50)$, and $(25, 60)$, for $D = 0.5625, 1.0625, \ldots, 3.0625$, respectively.

We now turn our attention to the typical value of the correlation function,

$$C_{\text{typ}} = \exp \left( \frac{\sum_{i=L/4}^{3L/4} \ln |S_i \cdot S_{i+r}|}{L/2-r} \right),$$

(13)

which is defined analogously to Eq. (7). This quantity is plotted in Fig. 8 as a function of the spin-spin separation $r$ for $L = 100$ and various values of $D$. Figure 8(a) shows $\ln C_{\text{typ}}$ very good data collapse based on Eq. (3). This is done by fitting Eq. (3) to the data in Fig. 8(a) in a region which, presumably, is only weakly affected by finite size (see magenta lines). The resulting collapsed data are shown in Fig. 8(b). The fitting pa-

In order to highlight a possible logarithmic factor, we plot $C_{1L} \approx \langle S_1 \cdot S_L \rangle$ for even $s_D$ and diminishes for larger $D$. A similar behavior was also found in the free-fermion case $\Delta = 0$ for the longitudinal correlation $C_{1L}^{\text{L}}$ [30].

In order to highlight a possible logarithmic factor, we plot in Fig. 7(b) the data from Fig. 7(a) with $C_{1L}$ multiplied by $L^{\eta_s(D)}$, with $\eta_s(D) = 1 + \delta_{D,0}$. In the clean case, $C_{1L}$ clearly has a logarithmic multiplicative factor [as already reported in Fig. 2(b)]. In the disordered case, our data are compatible with its absence. Even for the smallest value of disorder $D = 1/16$, the corresponding value of $\eta_r \approx 1.1$ is already far from the clean value 2.

7 The number in parentheses is an estimate of the error. It accounts for the statistical uncertainty of the fitted data and to how much the fitted value changes if we increase, shrink, or shift the fitting region by a few lattice spaces. In all cases, we verify that the reduced weighted error sum $\chi_r^2 \leq 2.$
In addition, we conclude that $\ln |C_{i,j}|/\sqrt{\gamma D} |i−j|$ converges to a nontrivial, narrow, and universal (disorder-independent) distribution for $\gamma D |i−j| \gg 1$. The same observation was reported in the free-fermion case $\Delta = 0$ [30].

Finally, we now study the distribution of spin-spin correlations. We restrain ourselves to the quantity $C = \langle S_{L/4} \cdot S_{M/4} \rangle$ for chains of size $L = 100$ and many values of $D$. In Fig. 9(a) we plot the distribution of $−\ln (C/c_D) / (A\sqrt{rD})$, with $r = L/2$ and $A$ and $c_D$ being the fitted values in Fig. 8(c). The data collapse for the largest values of $D$ confirms the conjecture of Ref. 26 which states that the distribution of $−\ln |C_{i,j}|/\sqrt{|i−j|}$ converges to a nontrivial distribution for large spin-spin separation $|i−j| \gg 1$. Here, parameters $A$ and $c_D$ are shown in Fig. 8(c) as a function of the disorder parameter $D$. In agreement with Eq. (3), $A$ is disorder independent for large $D$. We attribute the weak $D$ dependence to the large crossover length in the weak disorder limit. Similar behavior was found in the free-fermion case $\Delta = 0$ [30].

In Ref. 30, the distributions of the transverse ($C^t$) and longitudinal ($C^l$) correlations for the XX model $[\Delta = 0$ in Eq. (1)] were shown to be well fitted by

$$P(w) = B \exp \left\{ −\frac{w−w_1}{\delta_1} |i−j| − \left( \frac{\delta_2}{w−w_2} \right)^2 \right\}. \quad (14)$$

The first term in the exponential dictates the weak-correlation behavior $w \gg 1$, which, naively, is expected to be a Gaussian; that is, $\gamma_1$ is expected to be 2. Thus, $w_1$ and $\delta_1$ are, respectively, the associated mean and width. The second term in the exponential dictates the strong-correlation regime. A sharp cutoff, represented by $\delta_2$, is expected since the correlations $C_{i,j}$ cannot be arbitrarily large in absolute value. Thus,
w > w2. The parameters δ2 and γ2 are the associated width and exponent, respectively. The parameter B is just the normalization. The fitted values in that work for the transverse correlations are δ1 = 1.66, δ2 = 79, w1 = −1.45, w2 = −2.51, γ1 = 1.71, and γ2 = 0.41, which is plotted as a black solid line in Fig. 9(b). Surprisingly, it fits our data quite satisfactorily. It is thus tempting to conjecture that the distribution of the transverse correlation in the model Hamiltonian (1) is Δ independent in the infinite-randomness regime −1/2 < Δ ≤ 1. For comparison, we also plot the distribution of the longitudinal correlations C’ (blue dashed line) of the XX model obtained in Ref. 30.

Thus, we conclude that the distribution of ln[C’xx(i,j)/√γD|i−j|] in the long-distance regime γD|i−j| ≫ 1 converges to a nontrivial distribution which is narrow, universal (disorder independent), and possibly Δ independent. The same conclusions apply for the distribution of longitudinal correlations, except that it is Δ dependent.

IV. CONCLUSIONS AND DISCUSSION

In this work, we have studied various measures of the ground-state spin-spin correlations C_{i,j} = ⟨S_i S_j⟩ of the AF spin-1/2 Heisenberg chain [Δ = 1 in Eq. (1)] with random coupling constants. We applied the recently developed [52] adaptive strategy, which enabled us to study strongly (quenched) disordered systems using the unbiased DMRG method.

Our data are entirely compatible with the SDRG analytical predictions [21, 26]. Specifically, regarding the bulk correlations (2) in the regime 1 ≪ γD|i−j| ≪ L, we verified that the exponent η = 2 and prefactor difference c0 = c2 = 1 are universal (disorder independent). Our data confirm that the typical value of the correlations [Eq. (3)] decay stretched exponentially with the spin separation with universal exponent ψ = 1/2. Furthermore, we have confirmed the observation of Ref. 33 that the relevant length scale is the inverse Lyapunov exponent γ0 in Eq. (12), which plays the role of the clean system crossover length. This observation was made precise in the XX chain case (Δ = 0). In that case, this length scale is the inverse of the Lyapunov exponent of a single-parameter theory of the associated free-fermion system with particle-hole symmetry [29]. We have also studied the distribution of the spin-spin correlations for a fixed distance and confirmed the conjecture of Ref. 26 that ln[C_{i,j}]/√|i−j| converges to a nontrivial, narrow, and universal distribution for γD|i−j| ≫ 1. We have also found that, within our statistical precision, it is equal to that of the transverse correlations of the XX chain reported in Ref. [30]. It is thus tempting to conjecture that, besides being nontrivial, narrow, and universal, the distribution of ln[C_{i,j}]/√γD|i−j| does not depend (or weakly) on Δ.

One reported result that we have not confirmed is the logarithmic factor on the mean correlations of the disordered chain [39]. As we have shown, our data are compatible with its absence in both the mean (see Figs. 5 and 6) and typical (see Fig. 8) values of the bulk correlations, as well as in the mean value of the end-to-end correlation (see Fig. 7). Evidently, we cannot exclude (although it very implausible) a logarithmic factor appearing for system sizes larger than the ones studied here. If that is the case, we recall that the adaptive DMRG method employed here starts with the near-clean wave function, which does have a logarithmic factor in its two-point correlation. The fact that we do not detect it when the disorder strength is increased strongly suggests that the origin of the logarithmic factor, if one exists, is unrelated to that of the clean system.

Currently, it is unclear why the zero-temperature quantum Monte Carlo study of Ref. 39 predicts a logarithmic correction. The only suggestion that comes to us is finite-size effects. As shown in Fig. 5(c), the finite-size corrections are still strong for D ≃ 2 even for system sizes L ≃ 100. More importantly, the finite-size correction promotes a slow increase in the correlations which can be interpreted as a logarithmic correction [30]. Interestingly, D = 2 is the strongest disorder parameter value studied in Ref. 39. However, those authors considered periodic boundary conditions where finite-size effects are presumably smaller. In addition, they were able to study chains with sizes larger than ours.

Finally, we would like to point out that logarithmic factors are predicted by the SDRG method. They appear in the susceptibility and specific heat of infinite-randomness critical chains (but not in the correlations) [26] and in certain quantities of critical chains at a Kosterlitz-Thouless like transition [57–59]. Interestingly, logarithmic factors appear in the correlations of the clean Heisenberg chain (Δ = 1) and in the weakly disordered XXZ chain at the point Δ = −1/2. In both cases, the associated renormalization-group flow is of the Kosterlitz-Thouless type [48].

In conclusion, our numerical results are in agreement with those predicted by the SDRG method. They appear in the DMRG method without much more coding effort, and therefore, it adds to the toolbox of unbiased theoretical methods for disordered systems.
ACKNOWLEDGMENTS

We thank F. Alcaraz, R. Pereira, A. Sandvik, R. Juhasz, and N. Laflorencie for useful discussions. We acknowledge the financial support of the Brazilian agencies FAPESP and CNPq.

[1] D. S. Fisher, Phys. Rev. Lett. 69, 534 (1992).
[2] O. Motrunich, S.-C. Mau, D. A. Huse, and D. S. Fisher, Phys. Rev. B 61, 1160 (2000).
[3] C. Fich, A. P. Young, H. Rieger, and N. Kawashima, Phys. Rev. Lett. 81, 5916 (1998).
[4] I. A. Kovacs and F. Igloi, Phys. Rev. B 83, 174207 (2011).
[5] T. Voja and J. A. Hoyos, Phys. Rev. Lett. 112, 075702 (2014).
[6] J. A. Hoyos, C. Kotabage, and T. Voja, Phys. Rev. Lett. 99, 230601 (2007).
[7] T. Voja, C. Kotabage, and J. A. Hoyos, Phys. Rev. B 79, 024404 (2009).
[8] A. Del Maestro, B. Rosenow, M. Muller, and S. Sachdev, Phys. Rev. Lett. 101, 035701 (2008).
[9] A. Del Maestro, B. Rosenow, J. A. Hoyos, and T. Voja, Phys. Rev. Lett. 105, 145702 (2010).
[10] W. Berdanier, M. Kolodrubetz, S. A. Parameswaran, and R. Vasseur, Proceedings of the National Academy of Sciences 115, 9491 (2018).
[11] J. Hooyberghs, F. Igloi, and C. Vanderzande, Phys. Rev. Lett. 90, 100601 (2003).
[12] F. Igloi and C. Monthus, Physics Reports 412, 277 (2005).
[13] T. Voja, Journal of Physics A: Mathematical and General 39, R43 (2006).
[14] T. Voja, Journal of Low Temperature Physics 161, 299 (2010).
[15] F. Igloi and C. Monthus, The European Physical Journal B 91, 290 (2018).
[16] N. L. Bulavskii, A. V. Zvarykina, Y. S. Karimov, L. B. Lyuboviskii, and I. F. Shchegolev, Sov. Phys. JETP 35, 384 (1972).
[17] I. F. Shchegolev, physica status solidi (a) 12, 9 (1972).
[18] L. J. Azevedo and W. G. Clark, Phys. Rev. B 16, 3252 (1977).
[19] L. C. Tippie and W. G. Clark, Phys. Rev. B 23, 5846 (1981).
[20] K. Damle, O. Motrunich, and D. A. Huse, Phys. Rev. Lett. 84, 3434 (2000).
[21] J. A. Hoyos, A. P. Vieira, N. Laflorencie, and E. Miranda, Phys. Rev. B 76, 174425 (2007).
[22] S. Guo, D. P. Young, R. T. Macaluso, D. A. Browne, N. L. Henderson, J. Y. Chan, L. L. Henry, and J. F. DiTusa, Phys. Rev. Lett. 100, 017209 (2008).
[23] T. Westerkamp, M. Deppe, R. Kuchler, M. Brando, C. Geibel, P. Gegenwart, A. P. Pikul, and F. Steglich, Phys. Rev. Lett. 102, 206404 (2009).
[24] S. Ubaid-Kassit, T. Voja, and A. Schroeder, Phys. Rev. Lett. 104, 066402 (2010).
[25] Y. Xing, H.-M. Zhang, H.-L. Fu, H. Liu, Y. Sun, J.-P. Peng, F. Wang, X. Lin, X.-C. Ma, Q.-K. Xue, J. Wang, and X. C. Xie, Science 350, 542 (2015).
[26] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[27] V. L. Quito, F. L. S. Lopes, J. A. Hoyos, and E. Miranda, Eur. Phys. J. B 93, 17 (2020).
[28] V. L. Quito, F. L. S. Lopes, J. A. Hoyos, and E. Miranda, Phys. Rev. B 100, 224407 (2019).
[29] H. J. Mard, J. A. Hoyos, E. Miranda, and V. Dobrosavljevic, Phys. Rev. B 90, 125141 (2014).
[30] J. C. Getelina and J. A. Hoyos, The European Physical Journal B 93, 2 (2020).
[31] T. Giamarchi and H. J. Schulz, Phys. Rev. B 39, 4620 (1989).
[32] C. A. Doty and D. S. Fisher, Phys. Rev. B 45, 2167 (1992).
[33] N. Laflorencie, H. Rieger, A. W. Sandvik, and P. Hennelius, Phys. Rev. B 70, 054430 (2004).
[34] P. Hennelius and S. M. Girvin, Phys. Rev. B 57, 11457 (1998).
[35] N. Laflorencie and H. Rieger, Phys. Rev. Lett. 91, 229701 (2003).
[36] H. Tran and N. E. Boneskeel, Phys. Rev. B 84, 144420 (2011).
[37] A. M. Goldsborough and R. A. Romer, Phys. Rev. B 89, 214203 (2014).
[38] A. M. Goldsborough and G. Evenbly, Phys. Rev. B 95, 155136 (2017).
[39] Y.-R. Shu, D.-X. Yao, C.-W. Ke, Y.-C. Lin, and A. W. Sandvik, Phys. Rev. B 94, 174442 (2016).
[40] R. Juhasz, Phys. Rev. B 104, 054209 (2021).
[41] I. Affleck, D. Gepner, H. J. Schulz, and T. Ziman, Journal of Physics A: Mathematical and General 22, 511 (1989).
[42] R. P. Singh, M. E. Fisher, and R. Shankar, Phys. Rev. B 39, 2562 (1989).
[43] K. A. Hallberg, P. Horsch, and G. Martinez, Phys. Rev. B 52, R719 (1995).
[44] I. Affleck, Journal of Physics A: Mathematical and General 31, 4573 (1998).
[45] S. Lukyanov, Nuclear Physics B 522, 533 (1998).
[46] T. Hikihara and A. Furusaki, Phys. Rev. B 58, R583 (1998).
[47] A. W. Sandvik, AIP Conference Proceedings 1297, 135 (2010).
[48] Z. Ristivojevic, A. Petkovic, and T. Giamarchi, Nuclear Physics B 864, 317 (2012).
[49] A. Juozapavicius, S. Caprara, and A. Rosengren, Phys. Rev. B 56, 11097 (1997).
[50] K. Hamacher, J. Stolze, and W. Wenzel, Phys. Rev. Lett. 89, 127202 (2002).
[51] P. Ruggiero, V. Alba, and P. Calabrese, Phys. Rev. B 94, 035152 (2016).
[52] J. C. Xavier, J. A. Hoyos, and E. Miranda, Phys. Rev. B 98, 195115 (2018).
[53] M. Fishman, S. R. White, and E. M. Stoudenmire, “The ITensor software library for tensor network calculations,” (2020), arXiv:2007.14822.
[54] J. L. Cardy, Nuclear Physics B 240, 514 (1984).
[55] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, Journal of Physics A: Mathematical and General 20, 6397 (1987).
[56] D. S. Fisher and A. P. Young, Phys. Rev. B 58, 9131 (1998).
[57] E. Altan, Y. Kafri, A. Polkovnikov, and G. Refael, Phys. Rev. Lett. 93, 130402 (2004).
[58] T. Voja, J. A. Hoyos, P. Mohan, and R. Narayanan, J. Phys.: Condens. Matter 23, 094206 (2011).
[59] R. Juhasz, I. A. Kovacs, and F. Igloi, EPL (Europhysics Letters) 107, 47008 (2014).