Noncommutative 1-cocycle in the Seiberg-Witten map

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Abstract

We show that the Seiberg-Witten map for a noncommutative gauge theory involves a noncommutative 1-cocycle. The cocycle condition enforces a consistency requirement, which has been previously derived.

1 Introduction

The chiral anomaly \cite{1} and its consistency condition \cite{2} have been given a cohomological formulation in terms of infinitesimal gauge transformations by Stora and Zumino \cite{3}. Alternatively, Faddeev, Shatashvili, and Mickelsson \cite{4} used finite transformations to construct on the gauge group cocycles, which in infinitesimal form reproduce the anomaly, anomalous commutators \cite{5} and the consistency condition.

Apparently a similar story can be told for the Seiberg-Witten map in a noncommutative gauge theory \cite{6}. A consistency condition has been identified by Jurčo et al. \cite{7}, and a cohomological approach, in terms of infinitesimal quantities, has been constructed by Brace et al. \cite{8}. 
Here we continue this parallelism with anomaly theory by considering the Seiberg-Witten map in terms of finite transformations, and thereby construct a 1-cocycle, which is noncommuting.

In Section 2 we recall the definition and properties of a 1-cocycle, which is then extracted from the Seiberg-Witten map in Section 3.

## 2 1-cocycle (review)

Consider a group of transformations \{g\}, \((g_1 g_2 = g_{12})\) on some coordinates \(\xi: \xi \rightarrow \xi^g\). Let these transformations be implemented on functions of \(\xi, \Psi(\xi)\), by some operation \(U(g)\). The simplest action of \(U\) on \(\Psi\) would be \(U(g)\Psi(\xi) = \Psi(\xi^g)\). But this can be generalized by allowing a factor to appear:

\[
U(g)\Psi(\xi) = C(\xi, g)\Psi(\xi^g)
\]

\(C(\xi, I) = 1\).

If the \(U\)'s obey the group composition law

\[
U(g_1)U(g_2) = U(g_{12})
\]

then \(C\) must satisfy a condition, which follows by effecting a second transformation on (2.1):

\[
C(\xi, g_{12}) = C(\xi, g_1)C(\xi^{g_1}, g_2).
\]

A quantity that depends on one group element \((g)\) and possibly on the coordinates \((\xi)\) is called a 1-cochain. If it also satisfies (2.3) it is a 1-cocycle. When a 1-cocycle can be written as

\[
C(\xi, g) = C_0^{-1}(\xi)C_0(\xi^g)
\]

(2.4)

it is trivial: (2.4) certainly satisfies (2.3), but \(C\) can be removed from (2.1) by replacing \(\Psi\) by \(C_0\Psi\) and \(U\) by \(C_0UC_0^{-1}\). A trivial cocycle is called a coboundary. When a 1-cochain is written in exponential form

\[
C(\xi, g) = \exp(-i\gamma(\xi, g))
\]

\[
\gamma(\xi, I) = 0 \mod 2\pi\text{(integer)}
\]

(2.5)

the 1-cocycle condition (2.3) may be presented as

\[
\gamma(\xi^{g_1}, g_2) - \gamma(\xi, g_{12}) + \gamma(\xi, g_1) = 0 \mod 2\pi\text{(integer)}
\]

(2.6)
and the 1-cocycle is a trivial coboundary when
\[ \gamma(\xi, g) = \gamma_0(\xi^g) - \gamma_0(\xi) . \]

[Generalizations of the above include 2-cocycles (when the composition law for \( U \) acquires a modification) and 3-cocycles (when the composition law for \( U \) is nonassociative) [9].]

In the application to anomalies, \( \xi \) is the vector potential and \( g \) is a gauge transformation. Then the anomaly is the infinitesimal portion of \( \gamma \) and the consistency condition is the infinitesimal version of the 1-cocycle condition (2.6).

## 3 1-cocycle (Seiberg-Witten map)

The Seiberg-Witten map arises from the requirement that a noncommutative gauge potential \( \hat{A} \), viewed as a function of the commutative gauge potential \( A \), be stable against gauge transformations, in the sense that [6]
\[ \hat{A}(A) + D\hat{\Lambda}(A, \alpha) = \hat{A}(A + D\alpha) . \]

Here \( \hat{\Lambda} \) and \( \alpha \) are infinitesimal parameters of a noncommutative and commutative gauge transformation, respectively:
\[ D\hat{\Lambda} = d\hat{\Lambda} - i[\hat{A}, \hat{\Lambda}]_\star \]
\[ D\alpha = d\alpha - i[A, \alpha] . \]

As usual, the star product, involving the noncommutativity parameter \( \theta^{ij} \), forms the star commutator \( [\hat{A}, \hat{\Lambda}]_\star = \hat{A} \star \hat{\Lambda} - \hat{\Lambda} \star \hat{A} \). \( \hat{\Lambda} \) depends on \( A \) and \( \alpha \) with
\[ \hat{\Lambda}(A, 0) = 0 . \]

(\( \hat{A} \) and \( \hat{\Lambda} \) also depend on \( \theta \), but this will not be indicated explicitly.)

When the Seiberg-Witten map is extended to additional fields, transforming with the fundamental representation of the gauge group, a consistency condition has been derived. Define
\[ \hat{\Lambda}(A, \alpha) = \Lambda_{\alpha}(A) + \Lambda^{(2)}_{\alpha}(A) + \cdots . \]
\( \Lambda_{\alpha}(A) \) is the portion of \( \hat{\Lambda}(A, \alpha) \) that is linear in \( \alpha \); \( \Lambda^{(2)}_{\alpha}(A) \) is the quadratic part; in view of (3.3) there is no \( \alpha \)-independent contribution. The consistency condition then reads [7]
\[ \delta_{\alpha}\Lambda_{\beta} - \delta_{\beta}\Lambda_{\alpha} - i[\Lambda_{\alpha}, \Lambda_{\beta}]_\star + i\Lambda_{[\alpha, \beta]} = 0 . \]
Here
\[ \delta \Lambda = \Lambda (A + D \alpha) - \Lambda . \] (3.6)

[When \( \Lambda, \Lambda^{(2)} \) are written without an argument, the missing argument is understood to be \( A \); other arguments are indicated explicitly as in (3.6).]

Rather than considering the response to infinitesimal transformations as in (3.1), we use finite gauge transformations and posit the finite version of (3.1):
\[ \hat{A}^G(A) = \hat{A}(A^g) . \] (3.7)

Here \( A^g \) is the commutative gauge transformation of the commutative potential \( A \):
\[ A^g = g^{-1}A + g^{-1}i dg . \] (3.8)

Similarly \( \hat{A}^G \) is the noncommutative gauge transformation of the noncommutative potential \( \hat{A} \):
\[ \hat{A}^G = G^{-1} \star A \star G + G^{-1} \star i dG . \] (3.9)

\( G \) depends on \( A \) and \( g \).

Consider now \( \hat{A}(A^{g_1g_2}) \). We may view \( A^{g_1} \) as a new gauge potential \( A' \) and \( A^{g_1g_2} \) as the \( g_2 \)-transformation of \( A' \). Then (3.7) implies
\[ \hat{A}(A^{g_1g_2}) = \hat{A}(A^{g_2}) = G^{-1}(A', g_2) \star (\hat{A}(A') \star i d \star G(A', g_2)) \]
\[ = G^{-1}(A^{g_1}, g_2) \star (G^{-1}(A, g_1) \star (\hat{A}(A) \star i d \star G(A, g_1)) \star i d \star G(A^{g_1}, g_2)) . \] (3.10a)

Alternatively \( A^{g_1g_2} \) is also the \( g_12 \) transform of \( A \). Then (3.7) gives
\[ \hat{A}(A^{g_1g_2}) = \hat{A}(A^{g_{12}}) = G^{-1}(A, g_{12}) \star (\hat{A}(A) \star i d \star G(A, g_{12})) . \] (3.10b)

Comparing the two results in the equation
\[ G(A, g_{12}) = G(A, g_1) \star G(A^{g_1}, g_2) . \] (3.11)

This is the same as the 1-cocycle condition (2.4), except that star multiplication has replaced ordinary multiplication, namely, the 1-cocycle \( G \) is noncommutative.

The 1-cocycle would be trivial if it were given, analogously to (2.4), by
\[ G(A, g) = G_0^{-1}(A) \star G_0(A^g) \] (3.12)
which certainly satisfies (3.11). Moreover, using this trivial cocycle in (3.7)–(3.9) implies

$$G_0(A) \star (\hat{A}(A) \star + i \mathbf{d}) G_0^{-1}(A) = G_0(A^g) \star (\hat{A}(A^g) \star + i \mathbf{d}) G_0^{-1}(A^g).$$

(3.13)

This states that the transform of $\hat{A}(A)$ with the noncommuting gauge transformation $G_0^{-1}(A)$ results in a quantity that is invariant against commuting gauge transformations of $A$. Presumably this can only be true if $\hat{A}$ is a pure gauge: $\hat{A} = G_0^{-1} \star i \mathbf{d}G_0$, or a gauge transformation (by $G_0$) of an $A$-independent noncommuting potential $\hat{A}_0$:

$$\hat{A}(A) = G_0^{-1}(A) \star \hat{A}_0 \star G_0(A) + G_0^{-1}(A) \star i \mathbf{d}G_0(A).$$

(3.14)

$G_0$ also parameterizes an ambiguity in solutions to (3.11): If $G(A, g)$ solves (3.11), so does $G_0^{-1}(A) \star G(A, g) \star G_0(A^g)$. When this form for the cocycle/gauge transformation is used in (3.7) and terms are rearranged, we are left with

$$G_0(A^g) \star \hat{A}(A^g) \star G_0^{-1}(A) - i \mathbf{d}G_0(A^g) \star G_0^{-1}(A^g)$$

$$= G^{-1}(A, g) \star \left(G_0(A) \star \hat{A}(A) \star G_0^{-1}(A) - i \mathbf{d}G_0(A) \star G_0^{-1}(A) \star G(A, g)\right)$$

$$+ G^{-1}(A, g) \star i \mathbf{d}G(A, g).$$

(3.15)

This equation demonstrates the gauge covariance of the formalism.

In analogy to (2.5) and consistent with (3.1), (3.7), (3.9), an exponential form for $G$ may still be used:

$$G = e^{-i\hat{A}}.$$  

(3.16)

But a formula analogous to (2.6) cannot be established because for noncommuting quantities products of exponentials are not simply exponentials of summed exponents. Nevertheless, the expression (3.16) may be used to derive the consistency condition (3.5) from (3.11). We set $g_1 = e^{-i\alpha}$, $g_2 = e^{-i\beta}$, and label $\hat{A}$ by generator $\alpha$ or $\beta$. Thus

$$G(A, g_1) = I - i\hat{A}(A, \alpha) - \frac{1}{2}\hat{A}(A, \alpha) \star \hat{A}(A, \alpha) + \cdots.$$  

(3.17a)

It will be necessary to work to quadratic order, so according to (3.4) we have

$$G(A, g_1) = I - i\Lambda_{\alpha} - \frac{1}{2}\Lambda_{\alpha} \star \Lambda_{\alpha} - i\Lambda_{\alpha}^{(2)}.$$  

(3.17b)

Consequently

$$G(A^{g_1}, g_2) = I - i\Lambda_{\alpha}(A + D\alpha) - \frac{1}{2}\Lambda_{\beta} \star \Lambda_{\beta} - i\Lambda_{\beta}^{(2)}$$  

(3.17c)

$$G(A, g_{12}) = I - i\Lambda_{\alpha+\beta - \frac{1}{2}[\alpha, \beta]} - \frac{1}{2}\Lambda_{\alpha+\beta} \star \Lambda_{\alpha+\beta} - i\Lambda_{\alpha+\beta}^{(2)}$$

$$= I - i\Lambda_{\alpha} - i\Lambda_{\beta} - \frac{1}{2}\Lambda_{[\alpha, \beta]} - \frac{1}{2}(\Lambda_{\alpha} + \Lambda_{\beta}) \star (\Lambda_{\alpha} + \Lambda_{\beta}) - i\Lambda_{\alpha+\beta}^{(2)}.$$  

(3.17d)
The second equality in (3.17d) follows from the first by linearity: \( \Lambda_{\alpha+\beta} = \Lambda_{\alpha} + \Lambda_{\beta} \)
etc. After rearrangements, it follows from (3.11) that
\[
\delta_{\alpha}\Lambda_{\beta} \equiv \Lambda_{\beta}(A + D\alpha) - \Lambda_{\beta} = -\frac{i}{2}\Lambda_{[\alpha,\beta]} + \frac{i}{2}[\Lambda_{\alpha},\Lambda_{\beta}] - \Lambda_{\beta}^{(2)} - \Lambda_{\alpha}^{(2)} + \Lambda_{\alpha+\beta}^{(2)}. \tag{3.18}
\]
Taking the portion of (3.18) that is antisymmetric in \( \alpha \leftrightarrow \beta \)
leaves
\[
\delta_{\alpha}\Lambda_{\beta} - \delta_{\beta}\Lambda_{\alpha} = -i\Lambda_{[\alpha,\beta]} + i[\Lambda_{\alpha},\Lambda_{\beta}]^* \tag{3.19}
\]
This reproduces (3.3). Note that it was not necessary to introduce auxiliary fields in the fundamental representation to arrive at the consistency condition.

*Added Note:* We have learned that similar results are contained in a preprint by B. Jurčo, P. Schupp, and J. Wess, LMU-TPW 01-06, [hep-th/0106110](http://arxiv.org/abs/hep-th/0106110).

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[9] For a summary, see R. Jackiw in S. Treiman et al., Ref. [3].