On perpetual American options in a multidimensional Black–Scholes model

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ABSTRACT
We consider the problem of pricing perpetual American options written on dividend-paying assets whose price dynamics follow a multidimensional Black and Scholes model. For convex Lipschitz continuous reward functions, we give a probabilistic characterization of the fair price in terms of a reflected BSDE, and an analytical one in terms of an obstacle problem. We also provide the early exercise premium formula.

1. Introduction
In this paper, we consider the problem of pricing perpetual American options written on dividend-paying assets whose price dynamics follow the classical multidimensional Black and Scholes model. In this model, under the risk-neutral measure \( P \), the asset prices \( X^{s,x_1}, \ldots, X^{s,x_d} \) on \([s, \infty)\) evolve according to the stochastic differential equation

\[
X^{s,x,i}_t = x_i + \int_s^t (r - \delta_i) X^{s,x,i}_\theta \, d\theta + \sum_{j=1}^n \int_s^t \sigma_{ij} X^{s,x,i}_\theta \, dW^j_\theta, \quad t \geq s. \tag{1}
\]

In (1), \( W \) is a standard \( d \)-dimensional Wiener process, \( x_i > 0, i = 1, \ldots, d \), are the initial prices at time \( s \), \( r > 0 \) is the risk-free interest rate, \( \delta_i \geq 0, i = 1, \ldots, d \), are dividend rates and \( \sigma = \{\sigma_{ij}\}_{i,j=1,\ldots,d} \) is the volatility matrix. We assume that \( a = \sigma \cdot \sigma^* \), where \( \sigma^* \) is the transpose of \( \sigma \), is strictly positive definite.

Let \( T > 0 \) and \( \psi : \mathbb{R}^d \to \mathbb{R} \) be a nonnegative continuous function satisfying the linear growth condition. Under the measure \( P \), the value at time \( s \) of the American option with payoff function \( \psi \) and expiration time \( T > 0 \) is given by

\[
V_T(s, x) = \sup_{s \leq \tau \leq T} E e^{-r(\tau-s)} \psi(X^{s,x}_\tau), \tag{2}
\]
and the value of the perpetual option with payoff function $\psi$ is

$$V(s, x) = \sup_{\tau \geq s} E e^{-r(\tau - s)} \psi(X^s_x)$$

(3)

(see [13,14,25]). In (2), the supremum is taken over the set of all stopping times with values in $[s, T]$, and in (3), over the set of stopping times in $[s, \infty]$. In the event that $\tau = \infty$, we interpret $e^{-r(\tau - s)} \psi(X^s_x, x_\tau) = \lim_{t \to \infty} e^{-r(t - s)} \psi(X_t)$.

Nowadays, properties of $V_T$ are quite well investigated. It is known (see [7–9]) that $V_T$ can be represented by a solution of a reflected backward stochastic differential equation (RBSDE). A detailed study of the structure of this RBSDE, which in particular leads to the early exercise premium formula, is given in [17] (also see Section 3.1). The value $V_T$ can also be characterized analytically as a solution of some obstacle problem (or, in different terminology, variational inequality) (see [7–9,17] and Section 3.2). It is worth noting here that the analytical characterization relies heavily on the characterization via solutions of RBSDEs.

In the case of perpetual options less in known, except for put and call options in case $d = 1$, which were thoroughly investigated as early as in [21,22]. For a nice presentation of these results as well as some newer results and historical comments we refer the reader to the books [14,25]. Presumably, the main reason that less attention has been paid to $V$ than to $V_T$ is that perpetual options are not traded. On the other hand, in our opinion, perpetual American options are interesting for historical reasons and from a purely theoretical point of view. This motivated us to ask whether in the multidimensional case one can represent $V$ in terms of BSDEs or solutions of obstacle problems. The answer is ‘yes’ and the desired representations of $V$ can be derived in a quite elegant way from those of $V_T$. The main idea is as follows. Intuitively, $V$ is the limit of $V_T$ as $T \to \infty$. This suggests that properties of $V$ we are interested in can be derived by studying the behaviour, as $T \to \infty$, of the solution of the RBSDE with terminal condition at time $T$, which is used to represent $V_T$. By modifying some results from the recent paper [19], we show that the idea sketched above is realizable. As a result we show that for convex and Lipschitz continuous $\psi$ the value function $V$ is represented by a solution of some RBSDE with terminal condition 0 at infinity and we get the exercise premium formula. We also show that $V$ is a unique solution of some obstacle problem. Finally, we estimate that rate of convergence of $V_T$ to $V$. It seems that some of our results (the representation in terms of RBSDEs, rate of convergence) are new even in the case of the classical call/put option and $d = 1$.

2. Preliminaries

Let $\Omega = C([0, T]; \mathbb{R}^d)$ and $X$ be the canonical process on $\Omega$. For $(s, x) \in [0, T] \times \mathbb{R}^d$ let $P_{s,x}$ denote the law of the process $X^{s,x} = (X^{s,x,1}, \ldots, X^{s,x,d})$ defined by (1) and $\mathcal{F}^s_{t}$ denote the completion of $\sigma(X_\theta; \theta \in [s, t])$ with respect to the family $\{P_{s,\mu}; \mu \text{ a finite measure on } \mathcal{B}({\mathbb{R}^n})\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$. Let $a = \sigma \cdot \sigma^*$. Using Itô’s formula and Lévy’s characterization of the Wiener process one can check (see [17, Section 2] for details) that

$$X^i_t = x_i + \int_s^t (r - \delta_i) X^i_\theta \, d\theta + \sum_{j=1}^d \int_s^t \sigma^i_j X^j_\theta \, dB^j_{s,\theta}, \quad t \geq s, \quad P_{s,x} - \text{a.s.},$$

(4)
where, under the measure $P_{s,x}$, $(B_{s,t}, t \geq s)$ is a standard $d$-dimensional $(\mathcal{F}_{s}^{t})$-Wiener process on $[s, \infty)$. It is well known that the unique solution of (4) is of the form

$$X^i_t = x^i_t \exp \left( (r - \delta_i - a_{ii}(t-s)) + \sum_{j=1}^{d} \sigma_{ij}B_{s,t}^j \right), \quad t \geq s, \quad P_{s,x} - \text{a.s.} \quad (5)$$

Since $\tilde{B}^i_t := \sum_{j=1}^{d} \sigma_{ij}B_{s,t}^j$ is a continuous martingale with the quadratic variation $(\tilde{B}^i_{s,t})_t = a_{ii}(t-s), t \geq s$, the process $X^i_t$ can be written as

$$X^i_t = x^i_t e^{(r-\delta_i)(t-s)} N^i_{s,t}, \quad t \geq s, \quad (6)$$

where

$$N^i_{s,t} = \exp(-(t-s)a_{ii}/2 + \tilde{B}^i_{s,t}) = \exp(-(\tilde{B}^i_{s,t})/2 + \tilde{B}^i_{s,t}) \quad t \geq s,$$

is an $(\mathcal{F}_{s}^{t})$-martingale under $P_{s,x}$. Let $D = \{ x = (x_1, \ldots, x_d) : x_i > 0, i = 1, \ldots, d \}$. From (5) it follows that if $x \in D$, then $P_{s,x}(X_t \in D, t \geq s) = 1$.

Below we recall some known results on the pricing of American options with finite expiration time $T > 0$. They will be needed in the next section.

In this paper, we assume that the payoff function satisfies the following condition:

(A1) $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a nonnegative convex function which is Lipschitz continuous, i.e. there is $L > 0$ such that $|\psi(x) - \psi(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^d$.

In particular,

$$\psi(x) \leq C(1 + |x|), \quad x \in \mathbb{R}^d, \quad (7)$$

with $C = \max\{L, \psi(0)\}$. Furthermore, since $\psi$ is convex, for a.e. $x \in \mathbb{R}^d$ there exist the usual partial derivatives $\nabla_1 \psi(x), \ldots, \nabla_d \psi(x)$ of $\psi$ at $x$. Furthermore, by Alexandrov’s theorem (see, e.g. [1, Theorem 7.10]), there is a set $N \subset \mathbb{R}^d$ of Lebesgue measure zero such that $\psi$ has second-order derivatives at $x$ for every $x \in \mathbb{R}^d \setminus N$. We denote them by $\nabla^2 \psi(x)$.

Let $T_{s,T}$ denote the set of all $(\mathcal{F}_{s}^{t})$-stopping times with values in $[s, T]$. The fair price (or value) $V_T(s,x)$ of the American option with expiration time $T$ and payoff function $\psi$ is given by

$$V_T(s,x) = \sup_{\tau \in T_{s,T}} E_{s,x} e^{-r(T-s)} \psi(X_\tau). \quad (8)$$

Let $L = \psi(X)$. Note that $E_{s,x}|N^i_{s,T}|^2 = e^{a_{ii}(T-s)}$, so by (6) and Doob’s inequality, $E_{s,x} \sup_{s \leq t \leq T} |X^i_t|^2 < \infty$, $i = 1, \ldots, d$. By this and (7), $E_{s,x} \sup_{s \leq t \leq T} |L_t|^2 < \infty$. Therefore, by [7, Theorem 5.2], for every $(s,x) \in [0, T] \times \mathbb{R}^d$ there exists a unique solution $(Y_{T,s,x}, K_{T,s,x}, Z_{T,s,x})$, on the space $(\Omega, \mathcal{F}_{T}^{s}, P_{s,x})$, of the RBSDE with coefficient $f(y) = -ry$,
\( y \in \mathbb{R} \), terminal condition \( \psi(X_T) \) and barrier \( L \), that is linear RBSDE of the form

\[
\begin{cases}
Y^{T,s,x}_t = \psi(X_T) - \int_t^T r Y^{T,s,x}_t \, d\theta + \int_t^T \mathbb{K}^T_{s,x} \, dB_{s,\theta}, & t \in [s, T], \\
Y^{T,s,x}_t \geq \psi(X_t), & t \in [s, T], \\
K^T_{s,x} = 0, & \text{for every } t \in [s, T].
\end{cases}
\]

(9)

For a precise definition of a solution, we refer the reader to [7]. Here let us only note that \( E_{s,x} \int_s^T \left| Z^{T,s,x}_{0,\theta} \right|^2 \, d\theta < \infty \), so the process

\[ M^{T,s,x}_t = \int_s^t Z^{T,s,x}_{0,\theta} \, dB_{s,\theta}, \quad t \in [s, T], \]

is a martingale under \( P_{s,x} \). Let \( L_{BS} \) denote the Black–Scholes operator defined by

\[
L_{BS} = \sum_{i=1}^d (r - \delta_i) x_i \partial x_i + \frac{1}{2} \sum_{i,j=1}^d a_{ij} x_i x_j \partial_{x_i x_j},
\]

where \( \partial x_i, \partial^2_{x_i x_j} \) denote the partial derivatives in the distribution sense. In [7, Theorem 8.5] it is also proved that for every \( (s, x) \in [0, T] \times \mathbb{R}^d \),

\[ Y^{T,s,x}_t = u_T(t, X_t), \quad t \in [s, T], \quad P_{s,x}-\text{a.s.}, \]

(10)

where \( u_T \) is the unique viscosity solution to the obstacle problem

\[
\begin{align*}
\min \{ u_T - \psi, -\partial x u_T - L_{BS} u_T + ru_T \} &= 0 \quad \text{in } [0, T] \times \mathbb{R}^d, \\
u_T(T, \cdot) &= \psi \quad \text{on } x \in \mathbb{R}^d.
\end{align*}
\]

(11)

The process \( \bar{Y}^{T,s,x}_t \) defined as \( \bar{Y}^{T,s,x}_t = e^{-r(t-s)} Y^{T,s,x}_t \), \( t \in [s, T] \), is the first component of the solution of RBSDE with coefficient \( f = 0 \), terminal condition \( e^{-rT} \psi(X_T) \) and barrier \( e^{-rT} \psi(X_t), t \in [s, T] \). Therefore from (10) with \( t = s \) and [7, Proposition 2.3] (or [8, Proposition 3.3]) it follows that \( V_T = u_T \). Let

\[
L_{BS} = \sum_{i=1}^d (r - \delta_i) x_i \nabla x_i + \frac{1}{2} \sum_{i,j=1}^d a_{ij} x_i x_j \nabla_{ij}.
\]

In [17, Theorem 2] it is proved that under (A1), for every \( (s, x) \in [0, T] \times D \),

\[ K^{T,s,x}_t = \int_s^t \Phi(X_\theta, u_T(\theta, X_\theta)) \, d\theta, \quad t \in [s, T], \quad P_{s,x}-\text{a.s.}, \]

(12)

where

\[
\Phi(x, y) = \Psi^-(x) \mathbf{1}_{(\infty, \psi(x)]}(y), \quad \Psi^- = \max\{-\Psi, 0\}
\]

(13)

and

\[
\Psi(x) = -r \psi(x) + L_{BS} \psi(x) \text{ if } x \in D \setminus N, \quad \Psi(x) = 0 \text{ if } x \in N.
\]

(14)

Note that from (5) it follows that if \( (s, x) \in [0, \infty) \times D \) and \( t \in (s, T) \), then under the measure \( P_{s,x} \) the random variable \( X_t \) has density with respect to the Lebesgue measure.
Therefore $K^{T,s,x}$ is independent of $N$ and the way we define $\Psi$ on $N$. Note also that
\[
\Phi(x, 0) = \Psi^-(x), \quad \Phi(x, u_T(s, x)) = \Psi^-(x)1_{\{u_T(s, x) = \psi(x)\}}, \quad (s, x) \in [0, T] \times D,
\]
since $u_T(s, x) \geq \psi(x) \geq 0$.

3. Perpetual options

To shorten notation, in this section we set $V(x) = V(0, x)$, $\mathcal{F}_t = \mathcal{F}_t^0$, $P_x = P_{0,x}$, and we denote by $E_x$ the expectation with respect to $P_x$. With this notation (3) takes the form
\[
V(x) = \sup_{\tau \in \mathcal{T}} E_x e^{-r\tau} \psi(X_\tau),
\]
where $\mathcal{T}$ is the set of all $(\mathcal{F}_t)$-stopping times. In the event that $\tau = \infty$, we interpret $e^{-r(\tau-s)}\psi(X_\tau) = \lim_{t \to \infty} e^{-r(t-s)}\psi(X_t)$.

3.1. Stochastic representation of the value function

Assume (A1) and let
\[
Y^T_t = u_T(t, X_t), \quad K^T_t = \int_0^t \Phi(X_s, u_T(s, X_s)) \, ds, \quad t \in [0, T].
\]
By (10) and (12), $Y^T$ and $K^T$ are independent of $x$ versions of $Y^{T,0,x}$ and $K^{T,0,x}$, respectively. Since $V_T = u_T$, we have
\[
V_T(t, X_t) = Y^T_t = u_T(t, X_t), \quad t \in [0, T], \quad P_x-\text{a.s.}
\]
By the first equation in (9) we have
\[
M^{T,0,x}_t = Y^{T,0,x}_t - Y^{T,0,x}_0 - \int_0^t rY^{T,0,x}_s \, ds + K^{T,0,x}_t, \quad t \geq 0,
\]
so $M^{T,0,x}$ also has a version independent of $x$, which we denote by $M^T$. Set
\[
\tilde{Y}^T_t = e^{-rt} Y^T_t, \quad \tilde{K}^T_t = \int_0^t e^{-rs} dK^T_s, \quad \tilde{M}^T_t = \int_0^t e^{-rs} dM^T_s, \quad t \in [0, T].
\]
Since
\[
Y^T_t = \psi(X_T) - \int_t^T rY^T_s \, ds + \int_t^T dK^T_s - \int_t^T dM^T_s, \quad t \in [s, T],
\]
integrating by parts we obtain
\[
\tilde{Y}^T_t = e^{-rt} \psi(X_T) + \int_t^T d\tilde{K}^T_s - \int_t^T d\tilde{M}^T_s, \quad t \in [0, T].
\]
We will also need the following condition.
For every \( x \in D \),

\[
(\text{a}) \quad \lim_{t \to \infty} E_x e^{-rt} \psi(X_t) = 0, \quad (\text{b}) \quad E_x \int_0^\infty e^{-rt} \Psi^-(X_t) \, dt < \infty. \tag{18}
\]

**Remark 3.1:** (i) Condition (18) can be equivalently stated as

\[
(\text{a}) \quad \lim_{t \to \infty} e^{-rt} P_t \psi(x) = 0, \quad (\text{b}) \quad R_r \Psi^-(x) < \infty,
\]

where \((P_t)_{t \geq 0}\) (resp. \((R_r)_{r > 0}\)) is the semigroup (resp. resolvent) associated with \( X \).

(ii) Assume that \( r > 0 \). Clearly (18)(a) is satisfied for all \( x \in D \) if \( \psi \) is bounded. By (6),

\[
E_x X^t_i = x_i e^{(r - \delta_i) t}, \quad t \geq 0.
\]

Therefore (18)(a) is satisfied, for all \( x \in D \), for general Lipschitz continuous \( \psi \) if \( \delta_i > 0 \), \( i = 1, \ldots, d \). Similarly, (18)(b) is satisfied, for all \( x \in D \), if \( \Psi^- \) is bounded or \( \delta_i > 0 \), \( i = 1, \ldots, d \), and there is \( c > 0 \) such that

\[
\Psi^-(x) \leq c(1 + |x|), \quad x \in \mathbb{R}^d. \tag{19}
\]

We are going to show that if (18) is satisfied for some \( x \in D \), then \( \bar{Y}^T \) converges as \( T \to \infty \) to a process \( \bar{Y}^x \) being the first component of the solution \((\bar{Y}^x, \bar{K}^x, \bar{M}^x)\) of the reflected BSDE which informally can be written as

\[
\bar{Y}^x_t = \int_t^\infty d\bar{K}^x_s - \int_t^\infty d\bar{M}^x_s, \quad t \geq 0. \tag{20}
\]

We will also show that \( \bar{K}^x \) has the representation

\[
\bar{K}^x_t = \int_0^t e^{-rs} \Phi(X_s, e^{s \bar{Y}^x_s}) \, ds, \quad t \geq 0, \tag{21}
\]

so in fact \((\bar{Y}^x, \bar{M}^x)\) is a solution of the usual BSDE

\[
\bar{Y}^x_t = \int_t^\infty e^{-rs} \Phi(X_s, e^{s \bar{Y}^x_s}) \, ds - \int_t^\infty d\bar{M}^x_s, \quad t \geq 0. \tag{22}
\]

Equation (20) is a very special case of nonlinear reflected BSDEs treated in [11]. For existence and uniqueness results for general infinite horizon BSDEs with \( L^2 \)-data we refer the reader to [4] (equations in \( \mathbb{R}^d \)) and [10] (equations in Hilbert spaces). Roughly speaking, in [11] it is proved that if the coefficient \( f \) of the equation satisfies a generalized Lipschitz condition, its terminal condition is square-integrable and its barrier \( \bar{L} \) is continuous and satisfies the condition \( E_x \sup_{t \geq 0} |\bar{L}_t|^2 < \infty \), then the equation has a unique square-integrable solution. In (20), the coefficient \( f \) is equal to zero, terminal condition is equal to zero and the barrier has the form \( \bar{L}_t = e^{-rt} \psi(X_t) \), \( t \geq 0 \). In general, under (A1) and (A2), this barrier does not satisfy the aforementioned assumption from [11], so the results of [11] are not directly applicable to our situation. Assumption (A2)(a) says that \( \bar{L}_t = e^{-rt} \psi(X_t) \), \( t \geq 0 \), has the property that \( \lim_{t \to \infty} E_x \bar{L}_t = 0, x \in D \). We shall see that this condition on \( \bar{L} \) together with (A2)(b) guarantee the existence of a unique solution of (20) such that its first component \( \bar{Y}^x \) is of Doob’s class (D), i.e. it has (in general) weaker integrability properties than the solution considered in [11].
Before giving the definition of solutions of (20) and (22) let us recall that a continuous $(\mathcal{F}_t)$-adapted process $Y$ is said to be of class (D) under the measure $P_x$ if the collection $\{Y_\tau : \tau \in T, \tau \text{ finite} \}$ is uniformly integrable under $P_x$. Let $\mathcal{L}^1(P_x)$ denote the space of continuous processes with finite norm $\|Y\|_{x,1} = \sup\{E_x|Y_\tau| : \tau \in T, \tau \text{ finite} \}$. It is known that $\mathcal{L}^1(P_x)$ is complete (see [5, Theorem VI.22]). Moreover, if $Y^n$ are of class (D) and $Y^n \to Y$ in $\mathcal{L}^1(P_x)$, then $Y$ is of class (D) (see [19, Section 3]).

**Definition 3.1:** (i) We say that a triple $(\bar{Y}^x, \bar{K}^x, \bar{M}^x)$ of adapted continuous processes is a solution of the reflected BSDE (20) with lower barrier $\bar{L}_t = e^{-rt}\psi(X_t)$ if $\bar{Y}^x$ is of class (D), $\bar{M}^x$ is a local martingale with $\bar{M}_0^x = 0$, $\bar{K}^x$ is an increasing process with $\bar{K}_0 = 0$, and for every $T > 0$,

$$
\begin{align*}
\bar{Y}_t^x &= \bar{Y}_T^x + \int_t^T \bar{K}_s^x - \int_t^T d\bar{M}_s^x, \quad t \geq 0, \\
\bar{Y}_t^x &\geq \bar{L}_t, \quad t \in [0, T], \\
\int_0^T (\bar{Y}_t^x - \bar{L}_t) d\bar{K}_t^x &= 0, \\
\bar{Y}_T^x &\to 0 \text{ } P_x\text{-a.s. as } T \to \infty.
\end{align*}
$$

(ii) We say that a pair $(\bar{Y}^x, \bar{M}^x)$ of adapted continuous processes is a solution of the BSDE (22) if $\bar{Y}^x$ is of class (D), $\bar{M}^x$ is a local martingale with $\bar{M}_0^x = 0$, for every $T > 0$, $\int_0^T e^{-rt}\Phi(X_t, e^{rt}\bar{Y}_t^x) dt < \infty$ $P_x$-a.s. and moreover,

$$
\begin{align*}
\bar{Y}_t^x &= \bar{Y}_T^x + \int_t^T e^{-rs}\Phi(X_s, e^{rs}\bar{Y}_s^x) ds - \int_t^T d\bar{M}_s^x, \quad t \geq 0, \\
\bar{Y}_T^x &\to 0 \text{ } P_x\text{-a.s. as } T \to \infty.
\end{align*}
$$

**Remark 3.2:** Assume that for some $x \in D$ there exists a solution $(\bar{Y}^x, \bar{M}^x, \bar{K}^x)$ of (23). Then

(i) $e^{-rt}\psi(X_t) = 0$ $P_x$-a.s. on $\{\tau = \infty\}$ because by our convention, on the set $\{\tau = \infty\}$ we have $e^{-r(\tau-s)}\psi(X_t) = \lim_{t \to \infty} e^{-r(t-s)}\psi(X_t) \leq \lim_{t \to \infty} \bar{Y}_t^x = 0$ $P_x$-a.s.

(ii) For every $\tau \in T$,

$$
E_x\bar{Y}_0^x \geq E_x\bar{Y}_\tau^x \geq E_xe^{-r\tau}\psi(X_\tau).
$$

To see this, consider a localizing sequence $\{\tau_n\}$ for $\bar{M}^x$. Since

$$
\bar{Y}_t^x = \bar{Y}_0^x - \int_0^t d\bar{K}_s^x + \int_0^t d\bar{M}_s^x, \quad t \geq 0,
$$

we have $E_x\bar{Y}_0^x \geq \liminf_{n \to \infty} E_x\bar{Y}_{t \wedge \tau_n}^x$. Applying Fatou's lemma yields the desired inequalities.

We start with uniqueness results for BSDE (22) and RBSDE (23).

**Proposition 3.1:** Assume that $\psi$ satisfies (A1) and (18) for some $x \in D$. Then there is at most one solution of (23). Similarly, there is at most one solution of (22).
Proof: Suppose that \((\bar{Y}^i, \bar{K}^i, \bar{M}^i), i = 1, 2\), are solutions of (23). Write \(\bar{Y} = \bar{Y}^1 - \bar{Y}^2\), \(\bar{K} = \bar{K}^1 - \bar{K}^2\), \(\bar{M} = \bar{M}^1 - \bar{M}^2\). Then, by Remark 3.2,

\[
\bar{Y}_t = \bar{Y}_0 - \int_0^t d\bar{K}_s + \int_0^t d\bar{M}_s, \quad t \geq 0.
\]

By the Meyer–Tanaka formula (see, e.g. [23, Theorem IV.68]),

\[
\bar{Y}^+_t \leq \bar{Y}^+_T + \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} d\bar{K}_s - \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} d\bar{M}_s.
\]

(25)

Since \(\bar{L}_t \leq \bar{Y}^1_t \wedge \bar{Y}^2_t \leq \bar{Y}_t\), we have

\[
\int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} d\bar{K}_s = \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} (\bar{Y}^1_s - \bar{Y}^2_s)^{-1}(\bar{Y}^1_s - \bar{Y}^2_s \wedge \bar{Y}_s) d\bar{K}_s \leq 0.
\]

By the above inequality and (25), \(E_x \bar{Y}^+_T \leq E_x \bar{Y}^+_T\). Since \(E_x \bar{Y}^+_T \to 0\) as \(T \to \infty\), we see that \(\bar{Y}^+_T = 0\), \(t \geq 0\), \(P_x\)-a.s. In the same way we show that \((-\bar{Y})^+_T = 0\), \(t \geq 0\), \(P_x\)-a.s. Thus \(\bar{Y}^1 = \bar{Y}^2\). That \(\bar{M}^1 = \bar{M}^2\) and \(\bar{K}^1 = \bar{K}^2\) now follows from uniqueness of the Doob–Meyer decomposition of \(\bar{Y}\).

The proof of the second assertion is similar. Suppose that \((\bar{Y}^1, \bar{M}^1), (\bar{Y}^2, \bar{M}^2)\) are solutions of (22). Let \(\bar{Y} = \bar{Y}^1 - \bar{Y}^2\), \(\bar{M} = \bar{M}^1 - \bar{M}^2\). Applying the Meyer–Tanaka formula yields

\[
\bar{Y}^+_T \leq \bar{Y}^+_T + \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} e^{-rs}(\Psi(X_s, e^{rs} \bar{Y}^1_s) - \Psi(X_s, e^{rs} \bar{Y}^2_s)) ds - \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} d\bar{M}_s.
\]

But

\[
(\Phi(x, y_1) - \Phi(x, y_2))(y_1 - y_2)
\]

\[
= \Psi^{-1}(x)(\mathbf{1}_{(-\infty, \psi(x))}(y_1) - \mathbf{1}_{(-\infty, \psi(x))}(y_2))(y_1 - y_2) \leq 0,
\]

so \(\bar{Y}^+_T \leq \bar{Y}^+_T - \int_t^T \mathbf{1}_{\{\bar{Y}^1_s > \bar{Y}^2_s\}} d\bar{M}_s\). To prove that \(\bar{Y}^1 = \bar{Y}^2\) and \(\bar{M}^1 = \bar{M}^2\), it suffices now to repeat the argument from the proof of the first assertion.

We show next the existence of a solution of (23). To this end, we need some notation. By (17) with \(T = n\),

\[
\bar{Y}^n_t = e^{-rn} \psi(X_n) + \int_t^n e^{-rs} \Phi(X_s, e^{rs} \bar{Y}^n_s) ds - \int_t^n d\bar{M}_t^n, \quad t \in [0, n].
\]

(27)

We put

\[
\bar{Y}^n_t = \bar{Y}_t^n, \quad \bar{M}_t^n = \bar{M}_t^n, \quad t < n, \quad \bar{Y}_t^n = 0, \quad \bar{M}_t^n = \bar{M}_t^n, \quad t \geq n.
\]

The proof of the following theorem is a modification of the proof of [19, Propositions 4.1, 4.2].
Theorem 3.2: Assume that $\psi$ satisfies (A1) and (18) for some $x \in D$. Then there exists a unique solution $(\bar{Y}^x, \bar{M}^x)$ of (22) on $(\Omega, \mathcal{F}, P_x)$. Moreover,

$$E_x \int_0^\infty e^{-rt} \Phi(X_t, e^{rt} \bar{Y}^x_t) \, dt \leq 2E_x \int_0^\infty e^{-rt} \Psi^-(X_t) \, dt,$$  

$$\lim_{n \to \infty} \|\bar{Y}^n - \bar{Y}^x\|_{x,1} = 0$$

and for every $q \in (0, 1)$,

$$\lim_{n \to \infty} E_x \sup_{t \geq 0} |\bar{Y}^n_t - \bar{Y}^x_t|^q = 0. \tag{30}$$

Proof: Uniqueness follows from Proposition 3.1. The proof of the existence and (28)–(30) is divided into two steps.

Step 1. We shall prove some a priori estimates for the process $\bar{Y}^n$ and the difference $\delta \bar{Y} := \bar{Y}^m - \bar{Y}^n$. Specifically, we shall prove that

$$\|\delta \bar{Y}\|_{x,1} \leq E_x \left( e^{-rm} \psi(X_m) + e^{-rn} \psi(X_n) + \int_0^m e^{-rt} \Psi^-(X_t) \, dt \right),$$  

$$E_x \sup_{t \geq 0} |\delta \bar{Y}_t|^q \leq \frac{1}{1-q} E_x \left( e^{-rm} \psi(X_m) + e^{-rn} \psi(X_n) + \int_0^m e^{-rt} \Psi^-(X_t) \, dt \right)^q$$

for every $q \in (0, 1)$, and for every $t \geq 0$,

$$E_x \int_0^t e^{-rs} \Phi(X_s, e^{rs} \bar{Y}^n_s) \, ds \leq E_x \left( \bar{Y}^n_t + 2 \int_0^t e^{-rs} \Psi^-(X_s) \, ds \right). \tag{33}$$

By (27),

$$\bar{Y}^n_t = \bar{Y}^n_0 - \int_0^t 1_{[0,n]}(s) e^{-rs} \Phi(X_s, e^{rs} \bar{Y}^n_s) \, ds + \int_0^t 1_{[0,n]}(s) \, d\bar{M}^n_s, \quad t \in [0, n]. \tag{34}$$

Moreover,

$$\bar{Y}^n_t = \bar{Y}^n_0 - \int_0^t 1_{[0,n]}(s) e^{-rs} \Phi(X_s, e^{rs} \bar{Y}^n_s) \, ds + \int_0^t dV^m_s + \int_0^t 1_{[0,n]}(s) \, d\bar{M}^m_s, \quad t \geq 0,$$

where

$$V^m_t = 0, \quad t < n, \quad V^m_t = -\bar{Y}^n_t, \quad t \geq n.$$

Hence

$$\delta \bar{Y}_t = \delta \bar{Y}_0 + R_t + \int_0^t (1_{[0,n]}(s) \, d\bar{M}^m_s - 1_{[0,n]}(s) \, d\bar{M}^n_s), \quad t \geq 0,$$

with

$$R_t = -\int_0^t 1_{[0,n]}(s) e^{-rs} (\Phi(X_s, e^{rs} \bar{Y}^m_s) - \Phi(X_s, e^{rs} \bar{Y}^n_s)) \, ds$$

$$- \int_0^t 1_{[n,n]}(s) e^{-rs} \Phi(X_s, e^{rs} \bar{Y}^m_s) \, ds + \int_0^t d(V^m - V^n_s).$$
By the Meyer–Tanaka formula, for $t < m$ we have

$$|\delta \tilde{Y}_m| - |\delta \tilde{Y}_t| \geq \int_t^m \text{sign}(\delta \tilde{Y}_{s-}) \, d(\delta \tilde{Y})_s,$$

where $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x \leq 0$. Therefore, for $t < m$,

$$|\delta \tilde{Y}_t| = E_x(|\delta \tilde{Y}_t| |\mathcal{F}_t) \leq E_x\left(|\delta \tilde{Y}_m| - \int_t^m \text{sign}(\delta \tilde{Y}_{s-}) \, dR_s |\mathcal{F}_t) .$$

From this it follows that for $t \in [0, m]$,

$$|\delta \tilde{Y}_t| \leq E_x\left(|\delta \tilde{Y}_m| + \int_t^m 1_{[0,n]}(s) e^{-rs} \text{sign}(\delta \tilde{Y}_s)(\Phi(X_s, e^{r \tilde{Y}_s^m}) - \Phi(X_s, e^{r \tilde{Y}_s^n})) \, ds

+ \int_t^m 1_{(n,m]}(s) e^{-rs} \text{sign}(\delta \tilde{Y}_s) \Phi(X_s, e^{r \tilde{Y}_s^m}) \, ds + |V_m^m| + |V_n^n| |\mathcal{F}_t) .

By (26),

$$\int_t^m 1_{[0,n]}(s) e^{-rs} \text{sign}(\delta \tilde{Y}_s)(\Phi(X_s, e^{r \tilde{Y}_s^m}) - \Phi(X_s, e^{r \tilde{Y}_s^n})) \, ds \leq 0.$$

Since $\tilde{Y}_t^n = 0$ for $t \geq n$, it follows from (26) that

$$\int_t^m 1_{(n,m]}(s) e^{-rs} \text{sign}(\delta \tilde{Y}_s) \Phi(X_s, e^{r \tilde{Y}_s^m}) \, ds \leq \int_t^m e^{-rs} \Psi^-(X_s) \, ds.$$

Furthermore, $\delta \tilde{Y}_m = 0$ and

$$|V_m^m| + |V_n^n| = |\tilde{Y}_m^m| + |\tilde{Y}_n^n| = e^{-r m} \psi(X_m) + e^{-r n} \psi(X_n).$$

Therefore, for $t \in [0, m]$ we have

$$|\delta \tilde{Y}_t| \leq E_x\left(e^{-r m} \psi(X_m) + e^{-r n} \psi(X_n) + \int_t^m e^{-rs} \Psi^-(X_s) \, ds |\mathcal{F}_t) =: N_t,$$

from which (31) follows. By the above inequality and [2, Lemma 6.1],

$$E_x \sup_{0 \leq t \leq m} |\delta \tilde{Y}_t|^q \leq (1 - q)^{-1} (E_x N_m)^q,$$

which shows (32). To prove (33), we first observe that by the Meyer–Tanaka formula,

$$E_x |\tilde{Y}_t^n| - E_x |\tilde{Y}_0^n| \geq E_x \int_t^m \text{sign}(\tilde{Y}_{s-}) \, d\tilde{Y}_s^n.$$

By the above inequality and (34), for $t < n$ we have

$$E_x |\tilde{Y}_t^n| - E_x |\tilde{Y}_0^n| \geq -E_x \int_0^t 1_{[0,n]}(s) \text{sign}(\tilde{Y}_s^n) e^{-rs} \Phi(X_s, e^{r \tilde{Y}_s^n}) \, ds. \quad (35)$$
By Doob’s inequality and (29),

\[ \frac{\bar{Y}_m}{\Phi_1} \]

Hence it follows that

Step 2. We will prove the existence of a solution of (22) and (29), (30). From (18) and (31) which when combined with (35) proves (33).

On the other hand, for every \( t \geq 0 \),

\[
\int_0^t e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) \, ds \leq \int_0^t e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) - \Phi(X_s, 0) \, ds + \int_0^t e^{-rs} \Phi(X_s, 0) \, ds
\]

\[
= - \int_0^t \operatorname{sign}(\bar{Y}_s^n) e^{-rs} (\Phi(X_s, e^{rs} \bar{Y}_s^n) - \Phi(X_s, 0)) \, ds
\]

\[
+ \int_0^t e^{-rs} \Phi(X_s, 0) \, ds
\]

\[
\leq - \int_0^t \operatorname{sign}(\bar{Y}_s^n) e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) \, ds + 2 \int_0^t e^{-rs} \Psi^-(X_s) \, ds,
\]

which when combined with (35) proves (33).

Step 2. We will prove the existence of a solution of (22) and (29), (30). From (18) and (31) it follows that \( \| \bar{Y}_n - \bar{Y}_m \|_{x_1} \to 0 \) as \( n, m \to \infty \). Therefore, there exists a process \( Y^x \in \mathcal{L}^1(P_x) \) of class (D) such that (29) is satisfied. By (18) and (31), \( \lim_{n,m \to \infty} \mathbb{E}_x \sup_{t \geq 0} |\bar{Y}_n - \bar{Y}_m| \to 0 \). Since the space \( \mathcal{D}^q(P_x) \) is complete, the last convergence and (29) imply that \( Y^x \in \mathcal{D}^q(P_x) \) and (30) is satisfied. By (8) and (16), \( \bar{Y}_t^n \leq \bar{Y}_t^{n+1} \), \( t \geq 0 \), \( P_x \)-a.s. By this and (30),

\[
\lim_{n \to \infty} 1_{\{e^{rt} \bar{Y}_t^n \leq \Psi(X_t)\}} = 1_{\{e^{rt} \bar{Y}_t \leq \Psi(X_t)\}}, \quad t \geq 0, \quad P_x \text{-a.s.}
\]

Hence

\[
\lim_{n \to \infty} \mathbb{E}_x \Phi(X_t, e^{rt} \bar{Y}_t^n) = \mathbb{E}_x \Phi(X_t, e^{rt} \bar{Y}_t), \quad t \geq 0, \quad P_x \text{-a.s.,}
\]

(36)

so applying Fatou’s lemma we conclude from (33) that for every \( T > 0 \),

\[
\mathbb{E}_x \int_0^T e^{-rt} \Phi(X_t, e^{rt} \bar{Y}_t^n) \, dt \leq \mathbb{E}_x \left( \bar{Y}_T^n + 2 \int_0^T e^{-rt} \Psi^-(X_t) \, dt \right).
\]

(37)

From (30) it follows that \( \bar{Y}_T^n \to 0 \) in probability \( P_x \) as \( T \to \infty \). As a consequence, since \( \bar{Y}_x \) is of class (D), \( \mathbb{E}_x \bar{Y}_T^n \to 0 \). Letting \( T \to \infty \) in (37), we therefore get (28). By (27),

\[
\bar{Y}_t^n = \bar{Y}_T^n + \int_t^T e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) \, ds - \int_t^T d\bar{M}_s^n, \quad t < T \leq n.
\]

Since \( \bar{M}_n^n \) is a martingale, it follows that

\[
\bar{Y}_t^n = \mathbb{E}_x \left( \bar{Y}_T^n + \int_t^T e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) \, ds | \mathcal{F}_t \right), \quad t < T \leq n.
\]

(38)

By Doob’s inequality and (29),

\[
\lim_{n \to \infty} P_x (\sup_{0 \leq t \leq T} |\mathbb{E}_x(\bar{Y}_T^n - \bar{Y}_T^n | \mathcal{F}_t)| > \epsilon) \leq \epsilon^{-1} \lim_{n \to \infty} \mathbb{E}_x|\bar{Y}_T^n - \bar{Y}_T^n| = 0.
\]

(39)

By (18), (36) and the dominated convergence theorem,

\[
\lim_{n \to \infty} \mathbb{E}_x \int_0^T e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s^n) - \Phi(X_s, 0) \, ds = 0.
\]

(40)
From (38)–(40) we deduce that
\[\bar{Y}_t^x = E_x \left( \bar{Y}_T^x + \int_t^T e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s) \, ds \mid \mathcal{F}_t \right) .\]

Letting \( T \to \infty \) and using (28) and the fact that \( \lim_{T \to \infty} E_x \bar{Y}_T = 0 \) yields
\[\bar{Y}_t^x = E_x \left( \int_0^\infty e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s) \, ds \mid \mathcal{F}_t \right) .\]

Let \( \tilde{M}^x \) be a càdlàg version of the martingale
\[t \mapsto E_x \left( \int_0^\infty e^{-rs} \Phi(X_s, e^{rs} \bar{Y}_s) \, ds \mid \mathcal{F}_t \right) - \bar{Y}_0.\] (41)

One can check that \((\bar{Y}^x, \tilde{M}^x)\) is a solution of (22). \(\square\)

**Remark 3.3:** Since \( \tilde{M}^x \) is a version of (41), it follows from (28) and (A2)(b) that it is a closed martingale. Hence (see, e.g. [23, Theorem I.12]), \( \tilde{M}_\infty^x = \lim_{t \to \infty} \tilde{M}_t^x \) exists \(P_x\)-a.s. and \( \tilde{M}^x \) is a martingale on \([0, \infty]\). Therefore (20) is satisfied \(P_x\)-a.s. and \( E_x \tilde{M}_\infty^x = E_x \tilde{M}_0^x = 0 \). As a result,
\[E_x \bar{Y}_0^x = E_x \int_0^\infty e^{-rt} \Phi(X_t, e^{rt} \bar{Y}_s^x) \, dt.\] (42)

**Corollary 3.3:** Let the assumption of Theorem 3.2 hold.

(i) If \((\bar{Y}^x, \tilde{M}^x)\) is a solution of (22), then \((\bar{Y}^x, \tilde{K}^x, \tilde{M}^x)\) with \(\tilde{K}^x\) defined by (21) is a solution of (20).

(ii) Conversely, if \((\bar{Y}^x, \tilde{K}^x, \tilde{M}^x)\) is a solution of (20), then \(\tilde{K}^x\) admits the representation (21).

**Proof:** To prove (i), we only have to show that \(\bar{Y}^x, \tilde{K}^x\) have the properties formulated in the second line of (23). By (30), \(\bar{Y}^x_t \geq \bar{L}_t, t \geq 0\), since by construction we have \(\bar{Y}^x_n \geq \bar{L}_t, t \in [0, n]\), for every \(n \geq 1\). Clearly \(\tilde{K}^0 = 0\) and \(\tilde{K}^x\) is continuous and increasing. Since we know that \(\bar{Y}^x_t \geq \bar{L}_t, t \geq 0\), from (13) and (21) it follows that
\[\int_0^T (\bar{Y}^x_t - \bar{L}_t) \, d\tilde{K}^x_t = \int_0^T (\bar{Y}^x_t - e^{-rt} \psi(X_t)) e^{-rt} \Psi^-(X_t) \mathbf{1}_{(-\infty, \psi(X_t))}(e^{rt} \bar{Y}^x_t) \, dt = 0,
so \(\tilde{K}^x\) satisfies the minimality condition. Part (ii) follows from (i) and the first part of Proposition 3.1. \(\square\)

**Corollary 3.4:** Assume that (A1), (A2) are satisfied. Then

(i) \(V(x) = E_x \bar{Y}_0^x, x \in D.\) Moreover, \(e^{rt} \bar{Y}_t^x = V(X_t), t \geq 0, P_x\)-a.s. for every \(x \in D.\)

(ii) \(\lim_{T \to \infty} V_T(t, x) = V(x)\) for all \(t \geq 0\) and \(x \in D.\) Moreover, for every \(x \in D,
\[V(x) - V_T(0, x) \leq e^{e^{-rT} \psi(X_T) + \int_T^\infty e^{-rt} \Psi^-(X_t) \, dt}, \quad T > 0.\] (43)
Indeed, by Corollary 3.4(i), the process \( V(t, x) = \max(0, x - K_t) \) satisfies
\[
\mathbb{E}_x \int_0^\infty e^{-rt} \psi(X_t) \, dt < \infty, 
\]
have $|V(x) - V(y)| \leq L|x - y|$ for $x, y \in D$. (ii) Since $\psi(x) \leq C(1 + |x|)$, $x \in D$, for all $T > 0$ and $t \in [0, T]$ we have $V_T(t, x) \leq V_T(0, x) \leq C + C \sup_{t \in T_0, T} E x e^{-rt} |X_T|$. Since $\delta_i \geq 0$, $i = 1, \ldots, d$, for any $\tau \in T_0, T$ we also have $|X_\tau| \leq \sum_{i=1}^d X_0 e^{rt} N_{0, \tau}^i$. Since $E x N_{0, \tau}^i = 1$, $i = 1, \ldots, d$, this proves (ii). 

\[ \begin{align*}
L_{BS} v &= rv - \Phi(\cdot, v), \quad v \geq \psi \quad \text{in } D, \\
\lim_{t \to \infty} e^{-rt} P_t v(x) &= 0, \quad x \in D.
\end{align*} \]

Before formulating a precise definition of a solution of (45), we first give some remarks on the connection between (45) and the obstacle problem. Roughly speaking, since $\Phi$ is given by (13), the first line of (45) means that

\[ v \geq \psi \quad \text{and} \quad L_{BS} v = rv + \begin{cases} -\Psi^- & \text{on } \{v = \psi\}, \\
0 & \text{on } \{v > \psi\}. \end{cases} \]

Note also that the measure $\nu$ on $D$ defined as

\[ \nu(dx) = \Psi^{-}(x) 1_{\{v(x) = \psi(x)\}} \, dx \]

has the property that

\[ \int_D (v - \psi)(x) \, v(dx) = 0. \]

This means that the pair $(v, \nu)$ is a solution of the so-called complementarity system associated with the obstacle problem

\[ \min\{v - \psi, -L_{BS} v + rv\} = 0 \quad \text{on } D. \]

The ‘minimality condition’ (47) says that $\nu$ (sometimes called the obstacle reaction measure associated with the solution $v$ of (48)) acts only when $v$ touches the obstacle $\psi$. The fact that $(v, \nu)$ satisfies (46), (47) may be viewed as an analytic counterpart to the first two lines of (23). For more information about this kind of correspondence between reflected BSDE and solutions to complementarity systems associated with obstacle problems see [17,18] (parabolic case) and [15,24] (elliptic case).

Of course, to give a rigorous definition of (45) (or (46)) we have to specify in what sense the equation in the first line of (45) is satisfied. We are interested in solutions of (45) in some Sobolev space. Since we require that $v \geq \psi$ and $\psi$ satisfies (7), it is natural to work with Sobolev space with some weight $\rho$ such that $\int_{R^d} (1 + |x|^2)^2 \rho^2(x) \, dx < \infty$. As in [16,17], our choice of the weight is $\rho(x) = (1 + |x|^2)^{-\gamma}$ with some $\gamma > (2 + d)/4$. Then, by an elementary calculation, $\int_{R^d} \rho^2(x) \, dx < \infty$ and $\int_{R^d} |x|^2 \rho^2(x) \, dx < \infty$. In particular, if $\psi
satisfies (A1) and $\Psi$ satisfies (19), then
\[
\int_{\mathbb{R}^d} |\psi(x)|^2 \varrho^2(x) \, dx < \infty, \quad \int_{\mathbb{R}^d} |\Psi^{-}(x)|^2 \varrho^2(x) \, dx < \infty. \tag{49}
\]

Define
\[
L^2_{\varrho}(D) = L^2(D; \varrho^2 \, dx), \quad H^1_{\varrho}(D) = \{ u \in L^2_{\varrho}(D) : \sum_{j=1}^d \sigma_{ij} x_i u_{x_j} \in L^2_{\varrho}, \ i = 1, \ldots, d \},
\]
and for $\phi, \varphi \in C^\infty_0(D)$ set
\[
B^{\text{BS}}_{\varrho}(\phi, \varphi) = \sum_{i=1}^d \int_D (r - \delta_i) x_i \partial_{x_i} \phi(x) \varphi(x) \varrho^2(x) \, dx
- \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij} \partial_{x_i} \phi(x) \partial_{x_j} (x_i x_j \varphi(x) \varrho^2(x)) \, dx.
\]

One can check that there is $c > 0$ such that
\[
B^{\text{BS}}_{\varrho}(\phi, \varphi) \leq c \| \phi \|_{H^1_{\varrho}(D)} \| \varphi \|_{H^1_{\varrho}(D)}.
\]

Therefore the form $B^{\text{BS}}_{\varrho}$ can be extended to a bilinear form on $H^1_{\varrho}(D) \times H^1_{\varrho}(D)$, which we still denote by $B^{\text{BS}}_{\varrho}$. For an open set $U \subset \mathbb{R}^d$, we define the spaces $H^1(U), H^2(U)$ in the usual way.

**Definition 3.2:** We say that $v \in H^1_{\varrho}(D)$ is a variational solution of the semilinear problem
\[
L^{\text{BS}}_\varrho v = rv - \Phi(\cdot, v), \quad v \geq \psi \tag{50}
\]
if $v(x) \geq \psi(x)$ for $x \in D$, $\Phi(\cdot, v) \in L^2_{\varrho}(D)$ and the equation in (50) is satisfied in the weak sense, i.e. for every $\varphi \in H^1_{\varrho}(D)$,
\[
B^{\text{BS}}_{\varrho}(v, \varphi) = (rv - \Phi(\cdot, v), \varphi)_{L^2_{\varrho}(D)}. \tag{51}
\]

Recall that by Alexandrov’s theorem (see, e.g. [1, Theorem 7.10]), $\psi$ has second-order derivatives at $x$ for a.e. $x \in \mathbb{R}^d$. Consequently, $L^{\text{BS}}_\varrho \psi$ appearing in the definition of $\Phi$ (see (14)) is well defined for a.e. $x \in \mathbb{R}^d$. Of course, in general, $\psi$ does not have second-order derivatives in the distribution sense given by locally integrable functions.

Below we show that under (A1), (A2) and (19) a weak solution of (50) really exists, and in fact has the second-order derivatives in distribution sense given by locally integrable functions. Note that by Remark 3.1, if $\delta_i > 0, i = 1, \ldots, d$, then (A1) and (3.5) imply (A2).

**Proposition 3.7:** Assume that (A1), (A2) and (19) are satisfied. If $v$ is a variational solution of (50) then $v \in H^2_{\text{loc}}(D)$. In particular,
\[
L^{\text{BS}}_\varrho v(x) = rv(x) - \Phi(x, v(x)) \quad \text{for a.e. } x \in D. \tag{52}
\]
Proof: Fix a bounded open set \( U \) such that \( U \subset \bar{U} \subset D \). Let \( \xi \in C^\infty_0(U) \) and \( \varphi = \xi/\varrho^2 \). Then \( \varphi \in H^1_\varrho \), so from (51) it follows that
\[
B^{BS}(\nu, \xi) = (\nu r - \Phi(\cdot, \nu), \xi)_{L^2(\mathbb{R}^d, dx)},
\]
where \( B^{BS} \) is defined as \( B^B_\varrho \) but with \( \varrho = 1 \). Therefore \( \nu \) is a weak solution, in the space \( H^1(U) \), of the problem \( L_{BS} \nu = rv - \Phi(\cdot, \nu) \) in \( U \). To show that \( \nu \in H^2_{loc}(D) \) we make a well-known change of variables, which reduces the study of (52) to the study of an equation with uniformly elliptic operator \( \tilde{L} \) defined as
\[
\tilde{L} = \sum_{i=1}^d (r - \delta_i - a_{ii}/2) \partial_{x_i} + \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{x_i} \partial_{x_j}.
\]
More precisely, write \( e^x = (e^{x_1}, \ldots, e^{x_d}) \) for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), and then define \( \tilde{v}(x) = \nu(e^x), \tilde{\Phi}(x) = \Phi(e^x, \tilde{v}(x)) \) and \( \tilde{U} = \{ x \in \mathbb{R}^d : e^x \in U \} \). An elementary computation shows that \( \tilde{v} \in H^1(\tilde{U}) \) and \( \tilde{L} \tilde{v} = \tilde{r} \tilde{v} - \tilde{\Phi} \) in \( \tilde{U} \). By [6, Theorem 1, Section 6.3], \( \tilde{v} \in H^2(\tilde{U}) \), from which it follows that \( \nu \in H^2(U) \). Because of arbitrariness of \( U, \nu \in H^2_{loc}(D) \). The equality (52) now follows by a standard argument (see Remark (ii) following [6, Section 6.3, Theorem 1]).

Theorem 3.8: Assume that (A1), (A2) and (19) are satisfied. Then \( V \) is a variational solution of (50).

Proof: Let \( W_\varrho = \{ u \in L^2(0, T; H^1_\varrho) : u_t \in L^2(0, T; H^{-1}_\varrho) \} \). In [17] it is proved that for every \( T > 0, V_T \in W_\varrho \) and \( V_T \) is a variational solution of the Cauchy problem
\[
\partial_t V_T + L_{BS} V_T = r V_T - \Phi(\cdot, V_T), \quad V_T(T, \cdot) = \psi,
\]
i.e. \( V_T \geq \psi \) and (53) is satisfied in the weak sense. In particular, for any test function \( \eta \in C^\infty_0((0, T) \times D) \) we have
\[
\int_0^T \langle \partial_t V_T(t), \eta(t) \rangle \, dt + \int_0^T B^{BS}_\varrho (V_T(t), \eta(t)) \, dt = \int_0^T (r V_T(t) - \Phi(\cdot, V_T(t)), \eta(t))_{L^2_\varrho} \, dt,
\]
where \( V_T(t) = V_T(t, \cdot), \eta(t) = \eta(t, \cdot) \) and \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( L^2(0, T; H^1_\varrho) \) and \( L^2(0, T; H^{-1}_\varrho) \). From this one can deduce that for every \( \varphi \in C^\infty_0(D) \),
\[
\int_0^1 \int_D \partial_t V_T(t, x) \varphi \varrho^2(x) \, dx \, dt + \sum_{i=1}^d \int_0^1 \int_D (r - \delta_i) x_i \partial_{x_i} V_T(t, x) \varphi(x) \varrho^2(x) \, dx \, dt
\]
\[
- \frac{1}{2} \sum_{i,j=1}^d \int_0^1 \int_D a_{ij} \partial_{x_i} V_T(t, x) \partial_{x_j} (x_i x_j \varphi(x) \varrho^2(x)) \, dx \, dt
\]
\[
= \int_0^1 \int_D (r V_T(t, x) - \Phi(x, V_T(t, x)) \varphi(x) \varrho^2(t, x) \, dx \, dt,
\]
that is
\[ \int_0^1 \langle \partial_t V_T(t), \varphi \rangle \, dt + \int_0^1 B_{\varrho}^{BS}(V_T(t), \varphi) \, dt = \int_0^1 (rV_T(t) - \Phi(\cdot, V_T(t)), \varphi)_{L_0^2} \, dt. \] (54)

By Corollary 3.4(ii), for every \( x \in D \), \( V_T(0, x) \to V(x) \) and \( V_T(1, x) \to V(x) \). Furthermore, by Lemma 3.6(ii), \(|V_T(1, \cdot) - V(0, \cdot)| \varphi \|^2 \) is bounded by the function \( x \mapsto 2C(1 + |x|)|\varphi(x)| \varrho^2(x) \), which is integrable on \( D \) since \( \varphi \in C_0^\infty(D) \). Therefore applying the dominated convergence theorem we get
\[ \lim_{T \to \infty} \int_0^T (\partial_t V_T(t), \varphi)_{L_0^2} \, dt = \lim_{T \to \infty} \int_D (V_T(1, x) - V_T(0, x))\varphi(x)\varrho^2(x) \, dx = 0. \]

Suppose that \( \text{supp}[\varphi] \subset U \) for some relatively compact open set \( U \subset D \). By Lemma 3.6, \(|\partial_{x_i} V_T| \leq L \) a.e. for all \( i = 1, \ldots, d \) and \( T > 0 \), and \( V_T \) are bounded on \((0, 1) \times U \) uniformly in \( T > 0 \). By this and Corollary 3.4(ii), \( V_T \to V \) weakly in \( L^2(0, 1; H^1(U)) \). Therefore, for \( i = 1, \ldots, d \), we have
\[ \lim_{T \to \infty} \int_0^1 \int_D (r - \delta_i)x_i \partial_{x_i} V_T(t, x) \varphi(x) \varrho^2(x) \, dx \, dt \]
\[ = \int_D (r - \delta_i)x_i \partial_{x_i} V(x) \varphi(x) \varrho^2(x) \, dx \]
and
\[ \lim_{T \to \infty} \sum_{j=1}^d \int_0^1 \int_D a_{ij}x_i \partial_{x_i} V_T(t, x) \partial_{x_i}(x_i x_j \varphi(x) \varrho^2(x)) \, dx \, dt \]
\[ = \sum_{j=1}^d \int_D a_{ij}x_i \partial_{x_i} V(x) \partial_{x_i}(x_i x_j \varphi(x) \varrho^2(x)) \, dx. \]

Hence
\[ \lim_{T \to \infty} \int_0^1 B_{\varrho}^{BS}(V_T(t), \varphi) \, dt = B_{\varrho}^{BS}(V, \varphi). \] (55)

Since \( V_T \leq V_{T'} \) if \( T \leq T' \), in fact \( V_T \not\to V \) as \( T \to \infty \). Therefore \( \Phi(\cdot, V_T) \to \Phi(\cdot, V) \) pointwise. Furthermore, by Lemma 3.6(ii), \( V_T(t, x) \leq C(1 + |x|) \), and by the definition of \( \Psi \) we have \( \Phi(x, V_T(t, x)) \leq \Psi^-(x), (t, x) \in (0, 1) \times D \). Hence \(|(rV_T - \Phi(\cdot, V_T))\varphi \|^2 \) is bounded by the function \( x \mapsto (rC(1 + |x|) + \Psi^-(x))|\varphi(x)| \varrho^2(x) \), which is integrable on \((0, 1) \times D \) by (49) and the fact that \( \varphi \in C_0^\infty(D) \). Therefore applying the dominated convergence theorem, we get
\[ \lim_{T \to \infty} \int_0^1 \int_D (rV_T(t, x) - \Phi(x, V_T(t, x)))\varphi(x) \varrho^2(x) \, dx \, dt \]
\[ = \int_D (rV(x) - \Phi(x, V(x)))\varphi(x) \varrho^2(x) \, dx, \]
Proposition 3.9: Under the assumptions of Theorem prove uniqueness in the class of functions having the same property. (45)

Let \( \bar{v} \). Let \( \bar{v} \to \infty \) the extension of Itô’s formula proved by Krylov (see [20, Chapter II, §10, Theorem 1]) we have

\[
Z \bar{v} = \bar{v}(Z_0) + \sum_{i,j=1}^{d} \int_0^{t \wedge \tau_n} \partial_{x_i} \bar{v}(Z_s) \sigma_{ij} dB_{0,s} + \int_0^{t \wedge \tau_n} L \bar{v}(Z_s) \, ds, \quad t \geq 0.
\]

Define \( Y_t = v(X_t), t \geq 0 \). Since \( v(X) = \bar{v}(Z) \), it follows that

\[
Y_{t \wedge \tau_n} = Y_0 + \sum_{i,j=1}^{d} \int_0^{t \wedge \tau_n} L_{BS} v(X_s) \, ds + R_{t \wedge \tau_n}, \quad t \geq 0,
\]

where \( R_t = \sum_{i,j=1}^{d} \int_0^t \sigma_{ij} \partial_{x_i} v(X_s) \, dB_{0,s} \). Since \( P_x(X_t \in D, t \geq 0) = 1 \), \( \tau_n \to \infty \) \( P_x \)-a.s. as \( n \to \infty \). Therefore letting \( n \to \infty \) in (57) shows that it holds true with \( t \wedge \tau_n \) replaced by \( t \). Let \( \bar{Y}_t = e^{-rt} Y_t \). Integrating by parts we obtain

\[
\bar{Y}_t = \bar{Y}_0 + \int_0^t (-re^{-rs} Y_s) \, ds + \int_0^t e^{-rs} \, dY_s
\]

\[
= \bar{Y}_0 + \int_0^t e^{-rs} (-rv + L_{BS}v)(X_s) \, ds + \int_0^t e^{-rs} \, dR_s
\]

\[
= \bar{Y}_0 - \int_0^t e^{-rs}(\Phi(X_s, v^1(X_s)) - \Phi(X_s, v^2(X_s))) \, ds + \int_0^t e^{-rs} \, dR_s.
\]

Repeating now the argument from the proof of Proposition 3.1 we show that \( E_x \bar{Y}_0^+ \leq E_x \bar{Y}_{t}^+, t \geq 0 \). In much the same way we show that \( E_x \bar{Y}_0^- \leq E_x \bar{Y}_{t}^-, t \geq 0 \). Hence \( E_x|\bar{Y}_0| \leq E_x|\bar{Y}_t| = e^{-rt} E_x|v(X_t)| = e^{-rt} P_t|v|(x) \), which converges to zero as \( t \to \infty \). Thus \( |v(x)| = E_x \bar{Y}_0 = 0 \). 

\[\Box\]
In the case of American call and American put on single asset explicit formulas for the solution of (45) are known (see e.g. [12,14,21,25]). Assume that \( d = 1 \) and \( \psi = (K - x)^+, x \in \mathbb{R} \). Then from (13) and (14) it follows that

\[
\Phi(x, \psi(x)) = \begin{cases} 
(rK - \delta x)^+ & \text{if } \psi(x) \leq \psi(x), \\
0 & \text{if } \psi(x) > \psi(x).
\end{cases}
\]

Let \( \psi \) be a variational solution of (45). Then \( \psi \geq \psi \) and \( \psi \) satisfies the equation

\[
L_{BS} \psi = r \psi + \begin{cases} 
-(rK - \delta x)^+ & \text{on } \{ \psi = \psi \}, \\
0 & \text{on } \{ \psi > \psi \},
\end{cases}
\]

it the weak sense (see the definition preceding Proposition 3.7). Furthermore, by Proposition 3.7, \( \psi \in H^2_{loc}(D) \) and (58) is satisfied for a.e. \( x \in (0, \infty) \). In fact much more can be said. McKean [21] (see also [12] and [14, Chapter 2, Theorem 7.2]) showed that \( \psi \) has the form

\[
\psi(x) = \begin{cases} 
K - x, & 0 \leq x \leq b, \\
(K - b)(x/b)\gamma, & x > b,
\end{cases}
\]

where \( \gamma = -(1/\sigma)(v + \sqrt{v^2 + 2r}) \), \( v = -(1/2)\sigma + (r - \delta)/\sigma \) and \( b = \gamma K/(\gamma - 1) \). In particular, we see that \( \psi \) is \( C^1((0, \infty) \cap C^2((0, \infty) \setminus \{ b \}) \) and \( \{ \psi > \psi \} = \{ b, \infty \}, \{ \psi = \psi \} = [0, b] \). Furthermore, from (59) it follows that (58) is satisfied for every \( x \in (0, b) \cup (b, \infty) \). For the corresponding formulas for \( \psi \) in the case of American call, we refer the reader to [14, Theorem 6.7]. Let \( C \) denote the continuation region for the stopping problem (2) with \( s = 0 \), that is \( C = \{(t, x) \in [0, \infty)^2 : V_t(0, x) > \psi(x) = (K - x)^+ \} \), and let \( C_t \) be the \( t \) section of \( C \), i.e. \( C_t = \{ x \geq 0 : (t, x) \in C \} \). In [12] (see also [14, Section 2.7]) it is proved that \( C_t = (b(t), \infty), t > 0 \) for some continuous function \( b(t) \) called the free boundary for the parabolic obstacle problem (11)). Furthermore, the constant \( b \) of (59) is the limit, as \( t \to \infty \), of \( b(t) \).

4. Examples

Below we give examples of payoff functions satisfying (A1), (A2) and (19). In all the examples \( \Psi^- \) is computed in the subset \( D \cap \{ \psi > 0 \} \) (see Remark 3.4(ii)).

Example 4.1: Let \( d = 1 \).

\[
\psi(x) = (x - K)^+, \quad \Psi^-(x) = (\delta x - rK)^+ \quad \text{(call)}
\]

\[
\psi(x) = (K - x)^+, \quad \Psi^-(x) = (rK - \delta x)^+ \quad \text{(put)}
\]

The assumptions (A1) and (A2) are satisfied if \( r > 0 \) in case of put option, and if \( r > 0, \delta > 0 \) in case of call option. By (43), for put option we have

\[
V(x) - V_T(0, x) \leq e \left( Ke^{-rT} + rK \int_T^\infty e^{-rt} \, dt \right) = 2eKe^{-rT}, \quad x > K.
\]

For call option, \( V(x) - V_T(0, x) \leq 2eKe^{-\delta T}, T > 0, x \in (0, K) \).
Example 4.2: In the examples below $d \geq 2$. In all the cases where $\psi$ is bounded, (A1) and (A2) are satisfied if $r > 0$. In the other cases they are satisfied if $r > 0$ and $\delta_i > 0$, $i = 1, \ldots, d$.

(i) Index options and spread options.

\[
\psi(x) = \left( \sum_{i=1}^{d} w_i x_i - K \right)^+, \quad \Psi^-(x) = \left( \sum_{i=1}^{d} w_i \delta_i x_i - rK \right)^+ \quad \text{(call)}
\]

\[
\psi(x) = \left( K - \sum_{i=1}^{d} w_i x_i \right)^+, \quad \Psi^-(x) = \left( rK - \sum_{i=1}^{d} w_i \delta_i x_i \right)^+ \quad \text{(put)}
\]

(ii) Call on max option.

\[
\psi(x) = (\max\{x_1, \ldots, x_d\} - K)^+, \quad \Psi^-(x) = \left( \sum_{i=1}^{d} \delta_i 1_{B_i}(x_i) x_i - rK \right)^+, \quad \text{where } B_i = \{ x \in \mathbb{R}^d : x_i > x_j, j \neq i \}.
\]

(iii) Put on min option.

\[
\psi(x) = (K - \min\{x_1, \ldots, x_d\})^+, \quad \Psi^-(x) = \left( rK - \sum_{i=1}^{d} \delta_i 1_{C_i}(x_i) x_i \right)^+, \quad \text{where } C_i = \{ x \in \mathbb{R}^d : x_i < x_j, j \neq i \}.
\]

(iv) Multiple strike options.

\[
\psi(x) = (\max\{x_1 - K_1, \ldots, x_d - K_d\})^+, \\
\Psi^-(x) = \left( \sum_{i=1}^{d} 1_{B_i}(x - K)(\delta_i x_i - rK_i) \right)^+ \quad \text{with } K = (K_1, \ldots, K_d).
\]

Example 4.3: No explicit solution of (50) seem possible in the multidimensional cases considered in Example 4.2. Note however, that (44) gives an integral formula for $V$. For instance, in case $d = 2$ and $\psi(x) = (\max\{x_1, x_2\} - K)^+$ (see Example 4.2(ii)), we have

\[
V(x) = E_x \int_0^\infty e^{-rt} \left( (\delta_1 X_t^1 - rK)^+ 1_{[X_t^1 < X_t^2]} 1_{[V(X_t) = X_t^1 - K > 0]} + (\delta_2 X_t^2 - rK)^+ 1_{[X_t^2 < X_t^1]} 1_{[V(X_t) = X_t^2 - K > 0]} \right) dt.
\]  

As in the case of options with finite exercise time (see [3, Proposition 2.7] and the remarks following it), formula (60) (and similar formulas for other options considered in Example 4.2) has the potential to be used in a numerical valuation procedure. However, as remarked in [3], its implementation may be a challenge.
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