Adiabatic Gravitational Perturbation During Reheating

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We study the possibilities of parametric amplification of the gravitational perturbation during reheating in single-field inflation models. Our result shows that there is no additional growth of the super-horizon modes beyond the usual predictions.

I. INTRODUCTION

As was realized in [1] that parametric resonance instability occurs during reheating period when the inflaton field φ oscillates. Since gravitational perturbation is coupled to the inflaton by the Einstein equation, it may also experience parametric resonance amplification during this stage. This issue has been studied in Refs [2] and [3], and recently re-examined by Finelli and Brandenberger [4].

The gravitational potential Φ can be calculated by solving the linearized Einstein equation, however, in the case of the adiabatic perturbation with scales far outside the Hubble radius (the wavenumber $k \to 0$), it is convenient to work with Bardeen parameter,

$$\zeta = \frac{2 \Phi + H^{-1} \dot{\Phi}}{3(1+w)} + \Phi.$$  \hspace{1cm} (1)

In Eq.(1) the dot denotes the derivative with respective to time, $H$ is the Hubble expansion rate and $w = p/\rho$ is the ratio of the pressure to the density of the background. In the limit of $k \to 0$, the Bardeen parameter satisfies \[3,5,6\]

$$\frac{3}{2}(1+w)H\dot{\zeta} = 0.$$  \hspace{1cm} (2)

During the stage of reheating, Eq. (2) becomes \[6\]:

$$\ddot{\zeta} = 0.$$  \hspace{1cm} (3)

Recently Finelli and Brandenberger \[4\] pointed out that when the inflaton field oscillates, $\dot{\phi} = 0$ occurs periodically, so it is possible to have $\dot{\zeta} \neq 0$. If it happens, the cosmological perturbation will undergo parametric amplification without violating causality. Specifically, they have considered a inflaton model with potential $V(\phi) = m^2 \phi^2/2$ and solved it numerically. They found that $\zeta$ is constant in time \[4\].

In this paper, we extend their work and consider a general single-field inflaton potential. We examine the evolution of $\zeta$ during the reheating stage, and find no changes of $\zeta$ in time. To begin with, we derive the equation of motion of the Bardeen parameter $\zeta$ in perturbation theory, then we will take two specific models by numerical calculations to illustrate our general analytical result.

II. PERTURBATION THEORY AND ANALYTIC ARGUMENT

Working with the longitudinal gauge, the perturbed metric can be expressed in terms of the gravitational potential $\Phi$

$$ds^2 = (1 + 2\Phi)dt^2 - a(t)(1 - 2\Phi)dx_i dx^i,$$  \hspace{1cm} (4)

where $a(t)$ is the scale factor. The perturbed Einstein equation gives \[4\].
\[ \ddot{\Phi} + 3H \dot{\Phi} + \left[ \frac{k^2}{a^2} + 2(\dot{H} + H^2) \right] \Phi = \kappa^2 (\dot{\phi} + H \dot{\phi}) \delta \phi, \quad (5) \]

\[ \delta \phi + 3H \dot{\delta \phi} + \left( \frac{k^2}{a^2} + V'' \right) \delta \phi = 4 \Phi \dot{\phi} - 2V \Phi, \quad (6) \]

\[ \dot{\Phi} + H \Phi = \frac{1}{2} \kappa^2 \dot{\phi} \delta \phi, \quad (7) \]

where \( \kappa^2 = 8\pi G \), \( V \) is inflaton potential, \( \delta \phi \) is the perturbation to the field \( \phi \), and a prime denotes the derivative with respect to \( \phi \).

Inserting (7) into (5) one can obtain the equation of motion for \( \Phi \),

\[ \ddot{\Phi} + \left( H - 2 \frac{\ddot{\phi}}{\dot{\phi}} \right) \dot{\Phi} + \left[ \frac{k^2}{a^2} + 2 \dot{H} - 2H \frac{\ddot{\phi}}{\dot{\phi}} \right] \Phi = 0. \quad (8) \]

To eliminate the singularities in the equation above when the inflaton field \( \phi \) oscillates, one can make use of Sasaki-Mukhanov variable

\[ Q \equiv \delta \phi + \frac{\dot{\phi}}{H} \Phi, \quad (9) \]

then Eq.(8) can be re-written as

\[ \ddot{Q} + 3H \dot{Q} + \left[ V'' + \frac{k^2}{a^2} + 2(\dot{H} + H^2) \right] Q = 0. \quad (10) \]

The Bardeen parameter \( \zeta \) is related to \( Q \) by

\[ \zeta = \frac{H}{\dot{\phi}} Q. \quad (11) \]

In the expanding universe, the inflaton field \( \phi \) satisfies the equation of motion,

\[ \ddot{\phi} + 3H \dot{\phi} + V' = 0. \quad (12) \]

Differentiating (12) with respect to \( \alpha \equiv \ln a \) (note that \( \dot{\alpha} = H \)), we get

\[ H^{-1}(\dot{\phi})'' + 3\ddot{\phi} + (V'' + 3\dot{H})H^{-1} \dot{\phi} = 0, \quad (13) \]

where \( (\cdot)'' \equiv \frac{d^2}{d\tau^2} \).

Since \( \ddot{H} = -\kappa^2 \dot{\phi}^2/2 \) and \( \dddot{H} = -\kappa^2 \dot{\phi} \), we have the following relation

\[ 2H^{-2} \dddot{H} \dot{\phi} - H^{-2} \dddot{\phi} \dot{H} = 0. \quad (14) \]

Subtracting (14) from (13), and simplifying it, we can obtain

\[ \left( \frac{\dot{\phi}}{H} \right)'' + 3H \left( \frac{\dot{\phi}}{H} \right)' + \left[ V'' + 2 \left( \frac{\dot{H}}{H} + 3H \right) \right] \frac{\dot{\phi}}{H} = 0. \quad (15) \]

Differentiating \( Q = \left( \frac{\dot{\phi}}{H} \right) \zeta \) with respect to time \( t \), we have

\[ \dot{Q} = \left( \frac{\dot{\phi}}{H} \right) \zeta + \left( \frac{\dot{\phi}}{H} \right) \zeta, \quad (16) \]

\[ \ddot{Q} = \left( \frac{\dot{\phi}}{H} \right)'' + 2 \left( \frac{\dot{\phi}}{H} \right)' \zeta + \left( \frac{\dot{\phi}}{H} \right) \zeta. \quad (17) \]

Plugging \( Q, \dot{Q} \) and \( \ddot{Q} \) above into Eq.(11), and making use of Eq.(13), finally we obtain an equation of motion for \( \zeta \)

\[ \frac{\dot{\phi}}{H} \zeta + \left( 2 \frac{\dot{\phi}}{H} - 2 \frac{\dot{H}}{H^2} \dot{\phi} + 3 \dot{\phi} \right) \zeta + \frac{k^2}{a^2} \left( \frac{\dot{\phi}}{H} \right) \zeta = 0. \quad (18) \]

Clearly, the solution of Eq.(18) is that \( \zeta = 0 \) when \( \dot{\phi} = 0 \) (note that \( \ddot{\phi} \neq 0 \) at the time when \( \dot{\phi} = 0 \)). On the other hand, for \( \dot{\phi} = 0 \), Eq.(6) gives rise to \( \zeta = 0 \). We conclude that \( \zeta \) keeps unchanged during reheating. We should point out that the potential \( V \) in our analytical proof above is not specified, and our result applies for general single-field inflaton models.
III. NUMERICAL EXAMPLES

To illustrate the analytical result in the last section, we take two specific models as examples by directly solving the perturbed Einstein equation with numerical calculations. The first model is $V(\phi) = \lambda \phi^4 / 4$, the second one is a massive inflaton with self-coupled interaction, $V(\phi) = m^2 \phi^2 / 2 + \lambda \phi^4 / 4$. In the latter model two parameters are introduced. One is the inflaton mass $m$, another is the self-interaction coupling constant $\lambda$. In our numerical calculations, for this model we take $m^{-1}$ to be the units of the time and leave $\lambda / m^2$ as a free parameter, at the same time, we only choose $\lambda / m^2 = 1 \times 10^{-3} m_{pl}^{-2}, 1 m_{pl}^{-2}, 1 \times 10^3 m_{pl}^{-2}$ (where $m_{pl}$ is the Planck mass) as illustrations.

Fig.1 and Fig.2 present the evolution of $Q$ and $\zeta$ for these two models, from which we can see that $Q$ does not change over a period of time and $\zeta$ does not change during the zero crossing of $\dot{\phi}$ in the reheating stage (Note that Eqs. (2) and (3) hold only for the adiabatic perturbation with the wavenumber $k \to 0$, and we take $k = 0$ for simplicity in the numerical calculations.).

IV. DISCUSSION

In this paper we have studied the evolution of perturbations in the inflationary cosmology and found no additional growth of gravitational fluctuations due to the oscillating inflaton field during reheating. Our result is valid for any single-field inflaton potential. For the multiple-field models Bassett et al. [7] recently pointed out that there are possibilities of amplification of long wavelength perturbation. This important open issue deserves further study.

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\footnote{The model considered by Finelli and Brandenberger \cite{finelli} corresponds to the limit of $\lambda \to 0$.}
Figure Captions

Fig.1 Evolution of $Q$ and $\zeta$ as a function of time in the model $V(\phi) = \lambda \phi^4 / 4$. The initial condition is chosen as $Q = -1$ when $\phi = 0.2m_{\text{pl}}$. The solid and dashed lines represent the evolution of $Q$ and $\zeta$ respectively. Time is expressed in units of $(m_{\text{pl}} \sqrt{\lambda})^{-1}$.

Fig.2 Evolution of $Q$ and $\zeta$ as a function of time for a massive inflaton $V(\phi) = m^2 \phi^2 / 2 + \lambda \phi^4 / 4$. (a), (b), and (c) correspond to $\lambda/m^2 = 1 \times 10^{-3}m_{\text{pl}}^{-2}$, $1m_{\text{pl}}^{-2}$, $1 \times 10^3m_{\text{pl}}^{-2}$ respectively. Time is in units of $m^{-1}$. 
Fig. 1
Fig. 2(a)
