Quantum Non-abelian Toda Field Theories

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We derive an explicit, exactly conformally invariant form for the action for the non-abelian Toda field theory. We demonstrate that the conformal invariance conditions, expressed in terms of the \( \beta \)-functions of the theory, are satisfied to all orders, and we use our results to obtain a value for the central charge agreeing with previous calculations.
1. Introduction

Toda field theories provide examples of conformal field theories with a rich and interesting structure, and consequently have been investigated from various points of view over a considerable period \[1\] \[2\] \[3\]. The standard Toda field theories are each associated with a Lie algebra; to be more precise, with the canonical grading of a Lie algebra, in which the Cartan subalgebra furnishes the zero-grade part of the algebra, and the positive and negative grade components are generated by the step operators corresponding to positive and negative roots respectively. More generally, one can associate a generalised Toda field theory with an arbitrary integral grading of a Lie algebra, one for which the zero-grade component contains some of the step operators in addition to the Cartan subalgebra generators. Such theories were first considered in Ref. \[4\], where however the authors were principally interested in integral gradings corresponding to integral embeddings of \(sl(2)\) into the Lie algebra. More general integral gradings have been considered in Refs. \[5\] \[6\] \[7\], and it is these which we shall describe. (Half-integral gradings are discussed in Ref. \[8\].) Such generalised Toda theories are termed “non-abelian” since the zero-grade component is now non-abelian. The Lagrangian for the standard, or abelian Toda field theory is characterised by a set of fields with standard kinetic terms and simple exponential interaction terms; the non-abelian Toda theories, on the other hand, have a non-linear \(\sigma\)-model type of kinetic term (in fact, the action for a Wess-Zumino-Witten model \[9\]) together with more complex interactions featuring polynomials multiplying exponentials. Non-abelian Toda theories were first introduced some time ago, but have recently been receiving rather more attention \[10\] \[11\] \[12\] \[3\]. In particular it has been shown how they can be derived by Hamiltonian reduction of the Wess-Zumino-Witten (WZW) model \[5\]. (This reduction can also be implemented by gauging the WZW model, as discussed in detail in the abelian context in Ref. \[13\].) By virtue of this relation with the WZW model, the non-abelian Toda theories should be exactly conformally invariant at the quantum level, just as for the ordinary Toda theory, and this can be shown by construction of the energy-momentum tensor for the non-abelian Toda theory from that for the corresponding WZW model \[14\] \[5\]. Our goal in this paper is to show explicitly how to construct an exactly conformally invariant action for the non-abelian Toda theory. The classical action for the non-abelian Toda theory, obtained from the WZW action by the reduction process, is of the non-linear \(\sigma\)-model type \[5\], and the conditions for such an action to be conformally invariant have a well-known formulation \[15\] \[14\] in terms of the renormalisation-group \(\beta\)-functions for the
theory [17]. We shall show how these conditions may be satisfied to all orders by adjusting
the couplings in the classical action and also adding to the action a dilaton field, coupling
to the two-dimensional scalar curvature. We corroborate our results by showing that we
reproduce the known result for the central charge [5] for the non-abelian Toda theory.

The essence of our method is to identify the conformal invariance condition for the
scalar potential in the non-abelian Toda action, expressed in terms of the \( \beta \)-function,
with the Virasoro condition for the scalar potential to be a primary field (or rather, to
be precise, with the zeroth order term in the Laurent expansion of this condition). The
Virasoro condition can be given explicitly in terms of the Casimir operator for the Lie
algebra associated with this non-abelian Toda theory; on the other hand, the conformal
invariance condition can be determined explicitly in terms of the metric and dilaton fields
for the \( \sigma \) model, exploiting the fact that the kinetic terms are those of a WZW model, up to
multiplication by constants. Thus we can identify these constants and the required dilaton
field. This procedure has elements in common with ideas used in discussing quasi-exactly-
soluble quantum mechanical systems [18] and the energy-momentum tensor of conformal
field theories based on the generalised Sugawara construction [19]. Recently it has been
used very successfully to derive exact metric and dilaton fields for string black holes [20]
[21], yielding results which, in the case of Witten’s original string black hole solution [22],
have been checked to fourth order in perturbation theory [23]. The technique has already
been used to obtain the central charge and dilaton for the ordinary abelian Toda theory
[24], and we shall refer back to this paper for a fuller explanation of some of the results we
shall use. In the abelian case it is possible to derive the results more quickly by assuming
the general form of the Toda action and then simply solving the conformal invariance
conditions, and this was in fact done some time ago [25]; however, this would be more
difficult in the non-abelian case without the guidance provided by the knowledge that the
scalar potential satisfies the Virasoro condition.

The plan of the paper is as follows: in Section 2 we define non-abelian Toda field
theories, focussing on their derivation by reduction of a WZW model. In Section 3 we
discuss the conformal invariance conditions for a field theory in terms of the renormalisation
group \( \beta \)-functions for a non-linear \( \sigma \)-model, while in Section 4 we obtain a differential
equation for the potential term in the non-abelian Toda theory from the Virasoro condition.
In Section 5 we compare the \( \beta \)-function condition for the potential with this differential
equation, which enables us to read off the exact quantum form of the action for the non-
abelian Toda theory. As a check we derive the central charge for the theory. In Section 6 we
exemplify the above considerations with reference to the particular case of a non-abelian Toda theory based on the Lie algebra $B_2$. Finally in Appendix A we define our notation and conventions for Lie algebras, and prove two theorems in group theory which we need in the main text.

2. Non-abelian Toda theories

In this section we define non-abelian Toda theories and fix our notation. Non-abelian Toda theories were first introduced by generalising the Lax pair representation for a conventional Toda theory (which uses the canonical grading for the associated Lie algebra) to the case of a Lie algebra with arbitrary grading. However, we shall find it more convenient here to obtain the non-abelian Toda theory by Hamiltonian reduction (or equivalently gauging) of a Wess-Zumino-Witten model. We start by discussing the notion of grading a Lie algebra $G$. We suppose that the Lie algebra $G$ has a set of simple roots $\alpha \in \Delta$ and a corresponding set of positive roots $\Phi^+$. We define the dual simple roots (or fundamental weights) $\alpha'$, $\alpha \in \Delta$ to satisfy

$$2 \frac{\beta', \alpha}{\alpha, \alpha} = \delta_{\alpha \beta}$$

so that $\alpha'$ is given explicitly by

$$\alpha' = \sum_{\beta \in \Delta} A^{\alpha \beta} \beta,$$  \hspace{1cm} \hspace{1cm} (2.1)

where $A^{\alpha \beta}$ is the inverse of the Cartan matrix $A_{\alpha \beta}$. We can introduce an integral grading of the Lie algebra as follows: suppose we select a subset of the simple roots $\tilde{\Delta} \subset \Delta$. It will also be convenient to define $\tilde{\Phi}^+$ to be the set of positive roots which are sums of $\tilde{\alpha}$, $\tilde{\alpha} \in \tilde{\Delta}$, and $\tilde{\Phi}^-$ to be the corresponding set of negative roots. We now define

$$\delta = 2 \sum_{\tilde{\alpha} \in \Delta \setminus \tilde{\Delta}} \frac{\tilde{\alpha}'}{\tilde{\alpha}, \tilde{\alpha}}. \hspace{1cm} \hspace{1cm} (2.2)$$

The corresponding element $\delta.H = \delta_i H_i$ of the Cartan subalgebra now acts as a grading operator; $\delta$ has the properties

$$\delta \tilde{\alpha} = 0, \hspace{1cm} \tilde{\alpha} \in \tilde{\Delta}$$

$$\delta \alpha = 1, \hspace{1cm} \alpha \in \Delta \setminus \tilde{\Delta}$$

from which follows

$$[\delta.H, E_{\tilde{\alpha}}] = 0, \hspace{1cm} \tilde{\alpha} \in \tilde{\Delta}, \hspace{1cm} \hspace{1cm} (2.4)$$

$$[\delta.H, E_{\alpha}] = E_{\tilde{\alpha}}, \hspace{1cm} \alpha \in \Delta \setminus \tilde{\Delta}. \hspace{1cm} \hspace{1cm} (2.5)$$
We may now decompose $G$ into $G_\pm, G_0$ where $X \in G$ is assigned to $G_+, G_-$ or $G_0$ according as $[\delta H, X] = nX$ where the integer $n$ is positive, negative or zero respectively. Hence $G_0$ is generated by $E_{\pm \tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}^+$ and by the Cartan subalgebra generators. This is not the most general integral grading possible, since with our definition the simple roots are always assigned a grade of one or zero; this is necessary to ensure the existence of the Drinfeld-Sokolov gauge and hence to guarantee a polynomial realisation of the Kac-Moody algebra (see later). It is convenient now to select a basis for the Cartan subalgebra which is in accord with the decomposition. We start with the Cartan subalgebra generators $\tilde{\alpha} H$, $\tilde{\alpha} \in \tilde{\Delta}$. These span some $r^\tilde{G}$-dimensional subspace of the Cartan subalgebra, where $r^\tilde{G}$ is the number of simple roots in $\tilde{\Delta}$. We select an orthonormal basis $H_i$, $i = 1, 2, \ldots, r^\tilde{G}$ for this subspace. We then extend this basis to an orthonormal basis $H_i$, $i = 1, 2, \ldots, r^G$ (where $r^G$ is the dimension of $G$) for the whole Cartan subalgebra. We then have

$$\tilde{\alpha}_i = 0, \quad [H_i, E_{\tilde{\alpha}}] = \tilde{\alpha}_i E_{\tilde{\alpha}} = 0 \quad (i = r^\tilde{G} + 1, \ldots, r^G).\tag{2.6}$$

Then we see from (2.6) that $G_0$ itself splits into a direct sum of $\tilde{G}$, the Lie algebra whose generators are $E_{\pm \tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}^+$ and $H_i$, $i = 1, 2, \ldots, r^\tilde{G}$, together with $r^G - r^\tilde{G}$ factors of $R$. $G_+$ is generated by $E_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}^+ \setminus \tilde{\Phi}^+$, while $G_-$ is generated by $E_{-\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}^+ \setminus \tilde{\Phi}^+$. This decomposition of $G$ is called a grading of the Lie algebra. From now on we shall assume the index $i$ runs from 1 to $r^\tilde{G}$, while the index $\bar{i}$ will run from $r^\tilde{G} + 1$ to $r^G$. The index $i$ will be assumed to take values from 1 to $r^G$. We shall raise and lower indices, and perform contractions, using the ordinary Kronecker delta. As a special case of the grading of the Lie algebra we may take $\tilde{\Delta} = \emptyset$, in which case $G_0$ is simply the Cartan subalgebra and $G_\pm$ consist of the algebras generated by $E_\alpha$, $\alpha \in \Phi^+$ and by $E_{-\alpha}$, $\alpha \in \Phi^+$. This is called the canonical grading, and will correspond to the usual Toda theory (termed abelian since $G_0$ is abelian in this case). The general case of a non-abelian $G_0$ will correspond to a non-abelian Toda theory.

We assume henceforth that the Lie group $M_G$ whose Lie algebra is $G$ is maximally non-compact. We can then make a generalised Gauss decomposition of a group element $g \in M_G$ according to the grading, writing

$$g = g_- g_0 g_+ \tag{2.7}$$

where

$$g_- = \exp\left( \sum_{\tilde{\alpha} \in \tilde{\Phi}^+ \setminus \tilde{\Phi}^+} \phi_{-\tilde{\alpha}}^\tilde{\alpha} E_{-\tilde{\alpha}} \right), \quad g_+ = \exp\left( \sum_{\tilde{\alpha} \in \tilde{\Phi}^+ \setminus \tilde{\Phi}^+} \phi_{\tilde{\alpha}}^\tilde{\alpha} E_{\tilde{\alpha}} \right). \tag{2.8}$$
and where
\[ g_0 = \tilde{g} \exp\left( \sum_{i=r^g+1}^{r^g} r^i H_i \right), \quad \tilde{g} = \tilde{g}_0 \tilde{g}_+ \]
with
\[ \tilde{g}_- = \exp\left( \sum_{\bar{\alpha} \in \Phi^+} \phi_{\bar{\alpha}}^H E_{-\bar{\alpha}} \right), \quad \tilde{g}_+ = \exp\left( \sum_{\bar{\alpha} \in \Phi^+} \phi_{\bar{\alpha}}^+ E_{\bar{\alpha}} \right), \]
(2.10)
\[ \tilde{g}_0 = \exp(\sum_{i=1}^{r^g} r^i H_i). \]
The parameters \( \{ \phi_{\bar{\alpha}}, \phi_{\bar{\alpha}}^+ (\bar{\alpha} \in \Phi \setminus \tilde{\Phi}^+), \phi_\alpha, \phi_\alpha^+ (\alpha \in \tilde{\Phi}^+), r^i (i = 1, \ldots, r^G) \} \) may be regarded as co-ordinates on the group manifold \( M_G \).

We now write down the action for the Wess-Zumino-Witten (WZW) model \([9]\) defined on a group manifold \( M_G \):
\[ kS_{WZW}(g) = -\frac{k}{8\pi} \int_S d^2 \text{tr}(g^{-1} \partial_\mu gg^{-1} \partial^\mu g) + \frac{ik}{12\pi} \int_B d^3 x \varepsilon^{\mu\nu\rho} \text{tr}(g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g) \]
(2.11)
where \( g \in M_G \), and where \( B \) is a 3-dimensional ball whose surface is the two-dimensional worldsheet \( S \). We assume that the group generators are normalised as in Eq. (A.5). We are using here the conventions of Ref. \([27]\). The level \( x \) is defined in terms of \( k \) by \( x = \frac{2k}{\psi^2} \), where \( \psi \) is the highest root in \( \mathcal{G} \). It is conventional to normalise so that \( \psi^2 = 2 \), in which case \( x = k \), but we cannot in general normalise both \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) in this fashion simultaneously, and so we prefer to leave \( \psi^2 \) arbitrary. For a compact group \( M_G \) with Lie algebra \( \mathcal{G} \), \( x \) is restricted to be an integer, but there is no such constraint in the non-compact case considered here. This action is invariant under the transformations
\[ g(z, \bar{z}) \rightarrow \Omega_L(z) g(z, \bar{z}) \Omega_R(\bar{z}), \]
(2.12)
where we have introduced holomorphic and antiholomorphic co-ordinates \( z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1 \). The generators of the transformations (2.12) are the currents
\[ J(z) = k \partial gg^{-1}, \quad \bar{J}(\bar{z}) = kg^{-1} \partial g, \]
(2.13)
which generate two commuting copies of the Kac-Moody algebra. With a view to defining the non-abelian Toda theory, we first pick elements \( M_\pm \) in \( \mathcal{G}_\pm \) such that
\[ [\delta H, M_\pm] = \pm M_\pm. \]
(2.14)
In general, any choice of $M_{\pm}$ satisfying Eq. (2.14) will suffice for our purposes. However, in order that the non-abelian Toda theory may provide a polynomial realisation of the Kac-Moody algebra, one needs to be able to use the Drinfeld-Sokolov gauge\cite{26}. This is ensured if $M_{-}$ satisfies\cite{7}

$$\text{Ker}(\text{ad}M_{-}) \cap \mathcal{G}_{+} = \{0\}$$

(2.15)

and if, as we have arranged, the simple roots all have grade one or zero. It appears\cite{7} that there is one and only one possible such $M_{-}$ up to conjugation by the little group of $\delta.H$. Moreover, if Eq. (2.15) is satisfied, then one can find an $M_{+}$ such that $\{\delta.H, M_{\pm}\}$ generate an $sl(2)$ embedding\cite{7}, i.e. in addition to Eq. (2.14) we also have

$$[M_{+}, M_{-}] = 2\delta.H.$$  

(2.16)

However there is no need to use this particular $M_{+}$. (Conversely, given an $sl(2)$ embedding into $\mathcal{G}$ one can ask when one can find $\delta$ and $M_{\pm}$ satisfying Eqs. (2.14) and (2.15). This question is answered in Ref. \cite{8}.) The non-abelian Toda theory is now obtained from the WZW model in Eq. (2.11) by imposing the following constraints on the Kac-Moody currents in Eq. (2.13)\cite{5}:

$$J_{+} = kM_{+}, \quad \bar{J}_{-} = kM_{-}, \quad (2.17)$$

where $J_{+}$, $\bar{J}_{-}$ represent the projections of $J$, $\bar{J}$ onto $\mathcal{G}_{+}$, $\mathcal{G}_{-}$ respectively. The action $kS_{WZW}(g)$ in Eq. (2.12) now reduces classically to

$$S_{\text{NAT}}(g_{0}) = kS_{WZW}(g_{0}) - k \int d^{2}x V(\phi_{-}^{\alpha}, \phi_{+}^{\alpha}, r^{i})$$

(2.18)

where

$$V(\phi_{-}^{\alpha}, \phi_{+}^{\alpha}, r^{i}) = \frac{1}{8\pi} \text{tr}[M_{+}g_{0}M_{-}^{-1}g_{0}^{-1}]$$

(2.19)

and with $g_{0}$ as given by Eqs. (2.9), (2.10). Eq. (2.18) represents the action for the non-abelian Toda theory. The kinetic part of the action consists of a standard WZW action for the group $M_{\mathcal{G}_{0}}$ whose Lie algebra is $\mathcal{G}_{0}$. The remaining term in Eq. (2.18) yields a scalar potential term, typically consisting of a sum of terms of the form $f(\phi_{\pm}^{\alpha})e^{l(r^{i})}$ where $f$, $l$ are polynomials ($l$ is linear). Corresponding to the decomposition of $\mathcal{G}_{0}$ into a direct sum of $\tilde{\mathcal{G}}$ together with factors of $R$, the WZW action $S_{WZW}(g_{0})$ may be written

$$S_{WZW}(g_{0}) = S_{WZW}(\tilde{g}) - \sum_{i=r_{\tilde{g}}+1}^{r_{\tilde{g}}} S_{0}(r^{i})$$

(2.20)
where $S_0(r)$ is the action for a free massless scalar field,

$$S_0(r) = \int d^2 x \partial_\mu r \partial^\mu r$$

and where $\tilde{g}$ is defined in Eqs. (2.9), (2.10).

In the following Sections we shall determine the correct quantum form of Eq. (2.18) to ensure exact conformal invariance at the quantum level. This will entail modifying the couplings and adding a dilaton field coupling to the two-dimensional curvature.

### 3. Conformal Invariance Conditions

The original WZW model in Eq. (2.11) is exactly conformally invariant at the quantum level. In the standard language of conformal field theory, this means that the energy-momentum tensor is traceless and its independent components $T(z)$, $\tilde{T}(\tilde{z})$ generate the Virasoro algebra. Moreover, for the WZW model $T(z)$ and $\tilde{T}(\tilde{z})$ can be written in terms of the Kac-Moody currents in (2.13) according to the Sugawara construction [28] (for a review see also [27]) as

$$T(z) = \frac{1}{2k + c_G} \text{tr}(J^2), \quad \tilde{T}(\tilde{z}) = \frac{1}{2k + c_G} \text{tr}(\tilde{J}^2),$$

where $c_G$ is the value of the quadratic Casimir in the adjoint representation of $G$, which is related to the dual Coxeter number $h^G$ by $h^G = \frac{c_G}{\psi^2}$. The central charge is given by

$$c = \frac{k \dim G}{k + \frac{1}{2} c_G}.$$  

We wish to maintain exact conformal invariance for the non-abelian Toda theory. The energy-momentum tensor components of the reduced theory are given by [14] [15]

$$T(z) = \frac{1}{2k + c_G} \text{tr}(J^2) - \text{tr}(\delta.H\partial J)$$

(with a similar expression for $\tilde{T}(\tilde{z})$), where the additional term (with $\delta$ given by Eq. (2.3)) is required to ensure that the energy-momentum tensor commutes with the constraints Eq. (2.17).

Our main purpose is to discuss the implications of the requirement of exact conformal invariance for the precise form of the action for the non-abelian Toda theory, which was given classically by Eq. (2.18). To this end it is convenient to discuss conformal invariance
in the non-linear $\sigma$-model formulation. The action for a general non-linear $\sigma$-model may be written

$$S(\phi) = \frac{\lambda}{8\pi} \int d^2 x \{ G_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + e^{\mu \nu} B_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j + \frac{1}{\lambda} D(\phi) R^{(2)} + V(\phi) \} \quad (3.4)$$

where $e^{\mu \nu}$ is the two-dimensional alternating symbol, $\{ \phi^j \}$ represent co-ordinates on some target manifold with metric $G_{ij}$ and antisymmetric tensor field $B_{ij}$ defined on it, $D$ is the dilaton field coupling to the two-dimensional scalar curvature $R^{(2)}$, and $V$ is the tachyon field. (The terminology derives from string theory; our conventions are equivalent to taking $\alpha' = \frac{2}{\lambda}$ in Ref. [16], where $\alpha'$ is the string coupling.) The conformal invariance conditions for the $\sigma$-model Eq. (3.4) may be written\[15]\[16] \[19\]

$$B_G^{ij} \equiv \beta_G^{ij} + \frac{2}{\lambda} \nabla_i \partial_j D + 2 \partial(i W_{ij}) = 0 \quad (3.5a)$$

$$B_B^{ij} \equiv \beta_B^{ij} + \frac{2}{\lambda} H^k_{ij} \partial_k D + 2 H^k_{ij} W_k = 0 \quad (3.5b)$$

$$B_V \equiv \beta_V - 2 V + \frac{1}{\lambda} \partial^k D \partial_k V + W^i \partial_i V = 0 \quad (3.5c)$$

where $\beta_G^{ij}$, $\beta_B^{ij}$ and $\beta_V$ are the standard renormalisation group $\beta$-functions for $G_{ij}$, $B_{ij}$ and $V$, and $H_{ijk}$ is the torsion, defined by $H_{ijk} = 3 \nabla_{[i} B_{jk]}$. $W_i$ is a vector field which can be determined perturbatively within a given renormalisation scheme. The results which we shall be using for the $\beta$ functions imply a renormalisation scheme in which $W_i$ vanishes.

When $B_G^{ij}$ and $B_B^{ij}$ both vanish, the quantity $B^D$ given by

$$B^D \equiv \beta^D + \frac{1}{\lambda} \partial^k D \partial_k D + W^k \partial_k D, \quad (3.6)$$

where $\beta^D$ is the dilaton $\beta$-function, becomes constant\[29]\[16] and is then related to the central charge $c$ for the conformal field theory by

$$c = 3 B^D. \quad (3.7)$$

In the case of the WZW model Eq. (2.11), the target manifold is the group $M_G$. There is no tachyon field $V$, and the metric $G_{ij}$ and $B_{ij}$ may be read off by comparing Eqs. (2.11) and (3.4). We have

$$G_{ij} = e_{ai} e_{aj}, \quad \lambda = k, \quad (3.8)$$

where the vielbein $e_{ai}$ is defined by

$$ig^{-1} \partial_i g = e_{ia} T_a, \quad (3.9)$$
with $T_a$ denoting a generic element of the Lie algebra, i.e. either $E_{\pm\alpha}$ or $H_i$. In fact $B_{ij}$ can only be defined locally, but we have

$$H_{ijk} = \frac{1}{2} f_{abc} e_{ai} e_{bj} e_{ck}$$

(3.10)

These values for $G_{ij}$ and $H_{ijk}$ satisfy Eq. (3.5) with vanishing $D$ and $W_i$, in virtue of the fact that the generalised curvature given by

$$\nabla_{jkl} = R_{jkl}^i + 2\partial[|H_i]j + 2H^m_{j[l}H^i_{k]m}$$

(3.11)

vanishes, which is sufficient to imply the vanishing of $\beta_{ij}^G$ and $\beta_{ij}^B$ [30]. The central charge given by Eq. (3.7) can be shown to reproduce the exact value of Eq. (3.2) at least to $O(k^{-2})$ in an expansion in powers of $\frac{1}{k}$ [31].

The question which now arises is the following: how can we perform the reduction of the WZW action to the non-abelian Toda action in such a way as to preserve the exact conformal invariance? The main problem is that the non-abelian Toda action in Eq. (2.18) contains a scalar potential and hence $V$ in Eq. (3.4) is no longer zero, as was the case for the WZW model, but rather is given by Eq. (2.19). Hence we have to satisfy the additional conformal invariance condition Eq. (3.5c) in addition to Eqs. (3.5a,b). It is certainly not obvious at first sight that Eq. (3.5c) will be satisfied by the potential term which appears in Eq. (2.19). The key is to identify Eq. (3.5c) with the condition for $V$ to be a primary field of conformal dimensions $h = \bar{h} = 1$ (in string theory terms, the condition for $V$ to be a physical tachyon field). As such it should automatically be satisfied, and we shall see later that indeed it is. We can explicitly write down the corresponding Virasoro constraint on $V$ and by comparing with Eq. (3.5c) we will be able to deduce the required modifications of the coupling constants in Eq. (2.18). We will also find that a non-zero dilaton field is now required.

4. The Virasoro Constraint

This section will be devoted to the discussion of the explicit form of the Virasoro constraint on the potential $V$ in the non-abelian Toda theory. The zero modes in Laurent expansions of the Kac-Moody currents $J, \bar{J}$ act as differential operators $\mathcal{J}^L, \mathcal{J}^R$ on functions of the co-ordinates of $M_G$, namely $\phi^\pm_\alpha, \alpha \in \Phi^+, r^{\bar{\alpha}}_{i}, \bar{\alpha}_i \in \bar{\Delta}, r^i, i = r^G + 1, \ldots, r^G$. The components of these operators, defined by

$$\mathcal{J}^L_a = \text{tr}(T_a \mathcal{J}^L), \quad \mathcal{J}^R_a = \text{tr}(T_a \mathcal{J}^R),$$

(4.1)
are in fact the generators of left and right multiplication by Lie algebra elements, \( \text{v}_\text{iz.} \)

\[
\mathcal{J}_a^L g = T_a g, \quad \mathcal{J}_a^R g = g T_a.
\]  

(4.2)

Mathematically, \( \mathcal{J}^L, \mathcal{J}^R \) may be regarded as left- and right-invariant vector fields respectively on the group manifold. Correspondingly, the Virasoro generators corresponding to the energy-momentum operator in (3.3) may be written as differential operators acting on functions of the co-ordinates on the group manifold \( M_G \). In particular, the Virasoro operator \( L_0 \) can be expressed as a differential operator by replacing the zero modes of the currents in \( T(z) \) in Eq. (3.3) by \( \mathcal{J}_a^L \), and similarly \( \bar{L}_0 \) may also be expressed as an operator by replacing the zero modes of the currents in \( \bar{T}(\bar{z}) \) by \( \mathcal{J}_a^R \). The results are

\[
L_0 = \frac{1}{2k + c^G} \text{tr}[(\mathcal{J}^L)^2] - \text{tr}(\delta H \mathcal{J}^L),
\]

\[
\bar{L}_0 = \frac{1}{2k + c^G} \text{tr}[(\mathcal{J}^R)^2] - \text{tr}(\delta H \mathcal{J}^R).
\]

(4.3)

The expressions \( \text{tr}[(\mathcal{J}^L)^2] \) and \( \text{tr}[(\mathcal{J}^R)^2] \) are in fact both equal to the Casimir operator \( C^G \) for the group \( G \). Moreover, since \( \delta H \) is in the Cartan subalgebra, \( \text{tr}(\delta H \mathcal{J}^L) = \text{tr}(\delta H \mathcal{J}^R) \). Hence \( L_0 \) and \( \bar{L}_0 \) coincide as operators. The Virasoro conditions for the potential \( V(\phi^-_\alpha, \phi^+_\alpha, r^i) \) in Eq. (2.19) to be a primary field of conformal dimensions \( h = \bar{h} = 1 \) take the form

\[
(L_0 + \bar{L}_0 - 2)V(\phi^-_\alpha, \phi^+_\alpha, r^i) = 0, \quad (L_0 - \bar{L}_0)V(\phi^-_\alpha, \phi^+_\alpha, r^i) = 0.
\]

(4.4)

The second of Eqs. (4.4) is automatically satisfied; the first becomes, using Eq. (4.3),

\[
\left( \frac{C^G}{k + \frac{1}{2} c^G} - 2\text{tr}(\delta H \mathcal{J}^R) - 2 \right)V(\phi^-_\alpha, \phi^+_\alpha, r^i) = 0.
\]

(4.5)

It thus becomes important to have some knowledge of the form of the Casimir operator for co-ordinates corresponding to the generalised Gauss decomposition of Eqs. (2.7)–(2.10). This problem has been addressed in Ref. [32] and the results are summarised in a recent book by Leznov and Saveliev [33]. These results were restated and amplified in Ref. [24], to which we refer the reader for proofs of the identities we shall use here. It is convenient to begin by introducing operators \( X_{\pm \alpha}^L, X_{\pm \alpha}^R, \alpha \in \Phi^+ \), defined by

\[
X_{-\alpha}^L(g-\bar{g} -) = E_{-\alpha}(g-\bar{g} -), \quad X_{-\alpha}^R(g-\bar{g} -) = (g-\bar{g} -)E_{-\alpha},
\]

\[
X_{+\alpha}^L(\bar{g}+g +) = E_{\alpha}(\bar{g}+g +), \quad X_{+\alpha}^R(\bar{g}+g +) = (\bar{g}+g +)E_{\alpha}.
\]

(4.6)
The operators $X^{L,R}_{-\alpha}, \alpha \in \Phi^+$ act only on $g_-\bar{g}_-$ and contain only the variables $\phi^{\alpha}_-$, and similarly the operators $X^{L,R}_{+\alpha}$ act only on $\bar{g}_+g_+$ and contain only the variables $\phi^{\alpha}_+$. It can be shown after some algebra that the Casimir operator $C^g$ is given by

$$C^g = \sum_{\beta \in \Phi^+} 2e^{-r^i\beta_i}X^R_{-\beta}X^L_{+\beta} + 2\rho^i \frac{\partial}{\partial r^i}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{G}} \alpha. \quad (4.8)$$

It is clear in fact that $X^R_{-\alpha}$ acts only on $\bar{g}_-$, and contains only $\phi^\alpha_-$, and $X^L_{+\alpha}$ acts only on $\bar{g}^+$, and contains only $\phi^\alpha_+$. They can be given explicitly by analogous expressions to those obtained in Ref. [24] for the canonical decomposition; for instance,

$$X^L_{+\bar{\alpha}} = \sum_{n=0}^{\infty} \frac{b_n}{n!} N_{\alpha\bar{\beta}_1...\bar{\beta}_n} \phi^{\tilde{\beta}_1}...\phi^{\tilde{\beta}_n} \partial^{(\alpha+\bar{\beta}_1+...+\bar{\beta}_n)}$$

where $N_{\alpha\bar{\beta}_1...\bar{\beta}_n}$ is the coefficient of $E^{\alpha+\bar{\beta}_1+...+\bar{\beta}_n}$ in $[E_{\bar{\beta}_n},...[E_{\bar{\beta}_2},[E_{\bar{\beta}_1},E_{\bar{\alpha}}]]...]$, $b_n$ are the Bernoulli numbers, and $\partial_{\pm\alpha} = \frac{\partial}{\partial \phi^{\pm}_\alpha}$. $X^R_{-\alpha}$ may be obtained from Eq. (4.9) by replacing the subscript + by − and inserting a factor $(-1)^n$ in the summation. The most important property to notice is that $X^L_{+\alpha}$ contains only derivatives with respect to $\phi^\beta_+$ for $\beta \geq \bar{\alpha}$, and $X^R_{-\alpha}$ contains only derivatives with respect to $\phi^\beta_-$ for $\beta \geq \bar{\alpha}$. In particular, for the highest root $\tilde{\beta} \in \tilde{\Phi}^+$,

$$X^L_{+\tilde{\beta}} = \frac{\partial}{\partial \phi^\tilde{\beta}_+}, \quad X^R_{-\tilde{\beta}} = \frac{\partial}{\partial \phi^\tilde{\beta}_-}. \quad (4.10)$$

We will not give expressions for $X^{L,R}_{\pm\alpha}$ here, as they are even more complicated and will not be required in any case. It suffices to note that they have the property just mentioned, namely that $X^L_{+\alpha}$ contains only derivatives with respect to $\phi^\tilde{\beta}_+$ for $\tilde{\beta} \geq \bar{\alpha}$, and similarly for $X^R_{-\alpha}$. Finally it is straightforward to show using the methods of Ref. [24] that

$$J^R_i = \frac{\partial}{\partial r^i} - \sum_{\bar{\alpha} \in \bar{\Phi}^+} \phi^\alpha_{+}\bar{\alpha}_i \partial_{+\bar{\alpha}} + \ldots$$

$$J^R_i = \frac{\partial}{\partial r^i} + \ldots \quad (4.11)$$

where, in accord with the notation of Eq. (4.2), $J^R_i$ is the operator inducing multiplication of $g$ on the right by $H_i$. We have omitted terms involving derivatives with respect to
φ̅_+,  α ∈ Φ^+ \setminus ̃Φ^+, since these do not contribute when acting on V(φ̅_+, φ̅_+, r^i). There are similar expressions for J_i^L, J_i^L'. Using the fact that δ. ̃α = 0 for ̃α ∈ ̃Φ^+, which follows from (2.3), we have

\[ \text{tr}(δ. H J^R) = \text{tr}(δ. H J^L) = δ_1 \frac{∂}{∂r^i} = δ_1 \frac{∂}{∂r^i}. \]  

(4.12)

Using Eqs. (4.7), (4.12), we can now write the Virasoro condition Eq. (4.3) explicitly. Moreover, since V is independent of φ̅_+, the operators X^L,R_± ̃α can be omitted from the expression for C^G in Eq. (4.7) for our purposes.

It is possible to check explicitly that the Virasoro condition Eq. (4.5) is satisfied, with V(φ̅_+, φ̅_+, r^i) as given by Eq. (2.19). In fact, we have the stronger results

\[ C^G V(φ̅_+, φ̅_+, r^i) = 0, \]  

(4.13a)

\[ (\text{tr}(δ. H J_R) + 1)V(φ̅_+, φ̅_+, r^i) = 0. \]  

(4.13b)

These identities provide a concise statement of the properties of the potential V which guarantee conformal invariance at the quantum level. The proof of Eq. (4.13a) is straightforward but quite lengthy, and we relegate it to the Appendix; Eq. (4.13b) follows straightforwardly from Eqs. (2.8), (2.14), (2.19) and (4.12).

5. The tachyon β-function

In this Section we obtain a more explicit form for the tachyon β-function with a view to comparing it with the explicit form of the Virasoro constraint for V obtained in the previous Section. This will enable us to determine the modified form of the non-abelian Toda action in Eq. (2.18) which will ensure full conformal invariance. We postulate that the correct action for the non-abelian Toda theory is given by

\[ S_{NAT}(g_0) = k' S_{WZW}(g) - k'' \sum_{i=1}^{\ell^G} S_0(r^i) + \mu \int d^2 x V(φ̅_+, φ̅_+, r^i) + \int d^2 x D(r^i, r^i) R^{(2)}. \]  

(5.1)

where V(φ̅_+, φ̅_+, r^i) is as defined in Eq. (2.19), and μ is a constant (which we will not determine). We wish to discover the form of D and to obtain k', k'' in terms of the coupling k of the original WZW model in Eq. (2.11). Before proceeding further, we need a more explicit form for the tachyon β-function which appears in Eq. (3.5a). For a general
\(\sigma\)-model given by Eq. (3.4), the tachyon \(\beta\)-function can be calculated perturbatively, and takes the form \[ \beta^V = \left[ -\frac{1}{\lambda} \nabla^2 + 2 \frac{1}{\lambda^2} H^i_{\ kl} H^{jkl} \nabla_i \nabla_j + \ldots \right] V \] \hfill (5.2)

However, to attain our goal of exact conformal invariance we need an exact expression for the tachyon \(\beta\)-function. It can be seen by power counting that the tachyon \(\beta\)-function must take the form \[ \beta^V = \Omega V \] \hfill (5.3)

with \(\Omega\) depending only on the metric \(G_{ij}\) and the antisymmetric tensor field \(B_{ij}\) in Eq. (3.4) (i.e. not on \(D\) or \(V\)). The exact form of \(\Omega\) is not known in the general case, but it will be sufficient to have the exact result for the particular instance of the WZW model. This will enable us to compute the exact form for \(\Omega\) for the action in Eq. (5.1), since the value of \(\Omega\) for a sum of kinetic terms such as appears in Eq. (5.1) is the sum of the values of \(\Omega\) for each individual kinetic term, and the value of \(\Omega\) for the free field kinetic action is trivial (simply \(-\partial^2\)). For the WZW model of Eq. (2.11), the calculation of \(\Omega\) is essentially the same as the calculation of the anomalous dimension of \(g\) \cite{34}, and hence the exact result for \(\Omega\) can be deduced as \cite{36}

\[ \Omega = -\frac{1}{k + \frac{1}{2} c^G} \nabla^2. \] \hfill (5.4)

The perturbative result given in Eq. (5.2) is readily seen to agree with Eq. (5.4) upon use of Eqs. (3.8), (3.10), with \(c^G \delta_{\ ab} = f_{acd} f_{bcd}\). It will be convenient from our point of view to write \(\nabla^2\) in the equivalent form

\[ \nabla^2 = \frac{1}{\sqrt{G}} \nabla_i \sqrt{G} G^{ij} \nabla_j, \] \hfill (5.5)

where \(G = |\text{det}(G_{ij})|\). Finally we can identify the conformal invariance condition for the tachyon, Eq. (5.5), with the Virasoro condition on \(V\), Eq. (4.3). Denoting the coordinates of \(\tilde{G}\) generically by \(x^{\tilde{a}}\), and the corresponding metric for \(S_{WZW}(\tilde{g})\) by \(\tilde{G}_{\tilde{a}\tilde{b}}\), we have

\[ -\frac{1}{k'} + \frac{1}{2} c^{\tilde{G}} \frac{1}{\sqrt{\tilde{G}}} \partial_{\tilde{a}} \sqrt{\tilde{G}} \tilde{G}^{\tilde{a} \tilde{b}} \partial_{\tilde{b}} + \frac{1}{k''} \sum_{\tilde{i} = \gamma + 1}^{\tilde{G}} \frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{k'} \partial_{\tilde{a}} D \tilde{G}^{\tilde{a} \tilde{b}} \partial_{\tilde{b}} - \frac{1}{k''} \sum_{\tilde{i} = \gamma + 1}^{\tilde{G}} \partial_{\tilde{a}} D \partial_{\tilde{a}} \tilde{G}^{\tilde{a} \tilde{b}} \partial_{\tilde{b}} \tilde{D} - \frac{c^{\tilde{G}}}{k + \frac{1}{2} c^{\tilde{G}}} - 2(\delta, H, J_R) \] \hfill (5.6)
where \( \partial_i \equiv \frac{\partial}{\partial r^i} \), etc, and where

\[
C^G = \sum_{\beta \in \tilde{\Phi}^+} 2e^{-r^i \tilde{\beta}_i} X^R_{-\beta} X^L_{+\beta} + 2\rho_i \frac{\partial}{\partial r^i}
+ \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^i}
\]

as in Eq. (4.7), but recalling that the operators \( X^L_R \pm \tilde{\alpha} \) can be omitted from \( C^G \) when acting on \( V(\phi^-_{\tilde{\alpha}}, \phi^+_{\tilde{\alpha}}, r^i) \). The metric \( \tilde{G}_{\tilde{a} \tilde{b}} \), the dilaton and the couplings \( k', k'' \) can now be read off by comparing the LHS and RHS of Eq. (5.6). Firstly, we manifestly must have

\[
k'' = \kappa
\]

where for convenience we define

\[
\kappa = k + \frac{1}{2} \rho^G.
\]

Moreover, the metric \( \tilde{G}_{\tilde{a} \tilde{b}} \) can be read off by comparing the double derivatives with respect to \( \phi^\pm_{\tilde{\alpha}}, r^i \) on the LHS and RHS of Eq. (5.6). These terms in the Casimir Eq. (5.7) are exactly the same as they would appear in the Casimir for \( \tilde{G} \), and so the metric \( \tilde{G}_{\tilde{a} \tilde{b}} \) is proportional to the standard metric for the WZW model corresponding to \( G \). The normalisation is fixed by requiring that classically \( k' = k \), which implies

\[
k' = \kappa - \frac{1}{2} \rho^G.
\]

In particular, we now have

\[
G^{\tilde{a} \tilde{b}} = -\delta^{\tilde{a} \tilde{b}}.
\]

Using the property of \( X^L_{+\tilde{\alpha}} \) and \( X^R_{-\tilde{\alpha}} \) mentioned in Section 4, namely that they contain only derivatives with respect to \( \phi^\beta_{\pm} \) respectively for \( \beta > \tilde{\alpha} \), we can show using row and column operations that

\[
G = \det G = \exp(2r^i \tilde{\rho}_i),
\]

where

\[
\tilde{\rho} = \frac{1}{2} \sum_{\alpha \in \tilde{\Phi}^+} \tilde{\alpha}.
\]

The form of \( X^R_{+\tilde{\alpha}} \) given in Eq. (4.9) (with the similar form for \( X^R_{-\tilde{\alpha}} \)) also implies a form for the metric \( \tilde{G}^{\tilde{a} \tilde{b}} \) which guarantees that

\[
\frac{1}{\sqrt{\tilde{G}}} \partial_{\tilde{a}} \sqrt{\tilde{G}} \tilde{G}^{\tilde{a} \tilde{b}} \partial_{\tilde{b}}
\]

contains no contributions linear in

14
derivatives $\frac{\partial}{\partial \phi_{\pm}}$. Hence the only terms linear in derivatives on the LHS of Eq. (5.6) come from $\frac{1}{k' + \frac{1}{2} c' \rho'} \sqrt{G} \partial_i \sqrt{G} \partial_j \partial_i \tilde{\alpha}$. Using Eqs. (5.11), (5.10) and (5.12), we find

$$\frac{1}{k' + \frac{1}{2} c' \rho'} \sqrt{G} \partial_i \sqrt{G} \partial_j \partial_i \tilde{\alpha} = \frac{1}{k} \partial_i \partial_i + 2 \frac{1}{k'} \tilde{\rho} \partial_i \partial_i. \quad (5.14)$$

Comparing the terms first order in derivatives on the LHS and RHS of Eq. (5.6), we find using Eqs. (2.2) and (2.3)

$$\frac{2}{k}(\rho \tilde{\rho} + \rho \tilde{\rho}) + \frac{1}{2} \delta_i \partial_i = -\frac{1}{k'} \partial_i D \partial_i - \frac{1}{k'} \partial_i D \partial_i, \quad (5.15)$$

where

$$\rho' = \rho - \tilde{\rho}, \quad (5.16)$$

with $\rho, \tilde{\rho}$ as given in Eqs. (1.8), (5.13). We show in Appendix A that

$$\rho' \tilde{\alpha} = 0, \quad \tilde{\alpha} \in \tilde{\Phi}^+, \quad (5.17)$$

from which it follows, using Eq. (2.6) that

$$\rho'_i = 0. \quad (5.18)$$

Similarly, it follows that

$$\delta_i = 0. \quad (5.19)$$

We finally deduce

$$D = -2 \rho_i r^i - \frac{1}{2} \kappa \delta_i r^i, \quad (5.20)$$

which, using Eqs. (5.10), (5.18), (5.19) and the fact that $\tilde{\rho_i} = 0$, we may write in the form

$$D = -2 \rho'_i r^i - \kappa \delta_i r^i. \quad (5.21)$$

It is in fact essential that $D$ should not depend on $r^i$ (or indeed any of the variables associated with $\tilde{G}$): The metric, anti-symmetric tensor and dilaton should satisfy all the conformal invariance conditions Eq. (3.5). The metric and antisymmetric tensor field $G_{\tilde{a}\tilde{b}}$ and $B_{\tilde{a}\tilde{b}}$ are exactly those corresponding to the WZW model for $\tilde{G}$. They satisfy Eq. (3.5a) without the need for a dilaton field, in other words $\beta^G_{\tilde{a}\tilde{b}}$ and $\beta^B_{\tilde{a}\tilde{b}}$ vanish. The Christoffel symbols $\Gamma_{\tilde{a}\tilde{b}}^\tilde{c}$ are in general non-zero, and so if $D$ had any dependence on $r^i$ or $\phi_{\pm}$ then the RHS of Eq. (3.5a) could not vanish. (We could not remedy this by invoking a non-zero
$W_i$ since $D$ and $W_i$ appear in each of Eqs. (3.5) in the combination $\partial_i D + 2W_i$, and hence it is really this quantity which we identify in Eq. (5.20). We choose however to assume a renormalisation prescription for which $W_i$ vanishes.) It is thus a good check on our method that $D$ only depends on $r^i$. As a second, even more stringent check on our results we shall now compute the central charge, obtaining agreement with previous results. The central charge is given by Eqs. (3.6), (3.7). The contributions to $c$ from $\beta^D$ in Eq. (3.6) consist of the sum of the central charge for the $(r^G - r^{\tilde{G}})$ scalars $r^i$, and the central charge for the WZW model corresponding to $\tilde{G}$. So we have, using Eq. (5.21),

$$
c = r^\mathcal{G} - r^{\mathcal{G}} + \frac{k'dim\tilde{G}}{k' + \frac{1}{2}c^{\mathcal{G}}} - \frac{12}{\kappa}(\rho' + \kappa\delta)^2,
$$

(5.22)

which becomes, using Eqs. (2.4), (5.10), (5.16) and (5.17),

$$
c = r^\mathcal{G} - r^{\mathcal{G}} + \frac{1}{\kappa}dim\tilde{G}(\kappa - \frac{1}{2}c^{\mathcal{G}})
+ \frac{12}{\kappa}\rho^2
- \frac{12}{\kappa}(\rho + \kappa\delta)^2.
$$

(5.23)

Using the Freudenthal-deVries strange formula this expression simplifies to

$$
c = dim\mathcal{G}_0 + \frac{12}{\kappa}(\rho + \kappa\delta)^2,
$$

(5.24)

where

$$
dim\mathcal{G}_0 = r^\mathcal{G} - r^{\mathcal{G}} + dim\tilde{G}
$$

(5.25)

is the dimension of $\mathcal{G}_0$. Eq. (5.24) is the formula for the central charge obtained previously by other methods[3], generalising the result for the abelian Toda field theory computed in Ref. [2].

6. An explicit example

Our discussion so far has been somewhat abstract, and so it seems appropriate to present a simple example for which we can give explicit expressions for all the quantities involved. We take the case of the algebra $B_2$, which is also the case discussed in Ref.[11]. The Cartan matrix is given by

$$
A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.
$$

(6.1)
Denoting the simple roots by two-vectors $\alpha_1, \alpha_2$, the positive roots are $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = \alpha_1 + 2\alpha_2\}$. The corresponding generators can be written in a $4 \times 4$ matrix representation as

\[
E_{\alpha_1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad E_{\alpha_2} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
E_{\alpha_3} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad E_{\alpha_4} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(6.2)

with the generators corresponding to the negative roots, which we shall denote as $E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3},$ and $E_{\alpha_4}$, given by the transposes of $E_{\alpha_1} - E_{\alpha_4}$. We pick the Cartan subalgebra generators to be

\[
H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(6.3)

The generators have been normalised according to Eq. (A.5). The commutation relations take the form

\[
[H, E_{\alpha_i}] = \alpha_i E_{\alpha_i},
\]

(6.4)

where

\[
\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
2
\end{pmatrix}, \quad \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
-1
\end{pmatrix}, \quad \alpha_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
1
\end{pmatrix}, \quad \alpha_4 = \frac{1}{\sqrt{2}} \begin{pmatrix}
2 \\
0
\end{pmatrix}.
\]

(6.5)

so that we have

\[
\alpha_1^2 = \alpha_2^2 = 2, \quad \alpha_3^2 = \alpha_4^2 = 1.
\]

(6.6)

We now pick $\tilde{\Delta}$ to consist of the single simple root $\alpha_2$. We then find from Eqs. (2.2), (2.3),

\[
\delta = \alpha_1 + \alpha_2 = \alpha_3
\]

(6.7)

so that

\[
\delta H = \frac{1}{\sqrt{2}} (H_1 + H_2).
\]

(6.8)

It is then easy to check explicitly that

\[
\delta \alpha_1 = 1, \quad \delta \alpha_2 = 0, \quad [\delta, H, E_{\alpha_1}] = E_{\alpha_1}, \quad [\delta, H, E_{\alpha_2}] = 0,
\]

(6.9)
as in Eqs. (2.4), (2.5). We now parametrise a group element $g$ as in Eqs. (2.7)–(2.10), with

\[
\begin{align*}
g &= g_0 g_+, \quad g_0 = \tilde{g} e^{\frac{i}{\lambda} H}, \quad \tilde{g} = \tilde{g}_0 \tilde{g}_+, \quad \tilde{g}_0 = e^{\frac{i}{\lambda} H}, \\
g_- &= e^{(\phi_1^- E_{-\alpha_1} + \phi_3^- E_{-\alpha_3} + \phi_4^- E_{-\alpha_4})}, \quad \tilde{g}_- = e^{\phi_2^- E_{-\alpha_2}}, \\
g_+ &= e^{(\phi_1^+ E_{\alpha_1} + \phi_3^+ E_{\alpha_3} + \phi_4^+ E_{\alpha_4})}, \quad \tilde{g}_+ = e^{\phi_2^+ E_{\alpha_2}},
\end{align*}
\]

(6.10)

where \(\{\alpha_2, H, \delta H\}\) form an orthogonal basis for the Cartan subalgebra constructed as specified in Section 2.

We finally need to pick the elements $M_{\pm}$. From the considerations presented in the Appendix, the simplest choice would be $M_{\pm} = E_{\pm \alpha_1}$. However, this leads to a fairly trivial potential depending only on $r$ and $s$ in Eq. (6.10). This is an example of a general phenomenon: if we take $M_{\pm} = E_{\pm \bar{\gamma}_L}$, where $\bar{\gamma}_L$ is a simple root in $\Delta \setminus \bar{\Delta}$, then the potential $V(\phi^\Delta, \phi_{1+}^\Delta, r^i)$ depends only on the variables $r^i$, and is in fact simply the potential for the corresponding ordinary (i.e. abelian) Toda theory. This is because

\[
\tilde{g}_-^{-1} E_{\bar{\gamma}_L} \tilde{g}_- = E_{\bar{\gamma}_L}, \quad \tilde{g}_+ E_{-\bar{\gamma}_L} \tilde{g}_+^{-1} = E_{-\bar{\gamma}_L}
\]

(6.11)

for $\bar{\gamma}_L \in \Delta \setminus \bar{\Delta}$. Hence to make things more interesting, we pick

\[
M_{\pm} = E_{\pm \alpha_1},
\]

(6.12)

which, since $\alpha_3 = \alpha_1 + \alpha_2$, in view of Eq. (6.9) also have the properties Eq. (2.14). This $M_-$ also satisfies Eq. (2.15), guaranteeing the existence of the Drinfeld-Sokolov gauge. Hence, in this case $\{M_{\pm}, \delta H\} = \{E_{\pm \alpha_1}, \alpha_2, H\}$ generate a non-canonical embedding of $A_1$ in $B_2$. We can now calculate the classical action as given by Eq. (2.18), (2.20). The easiest way to do this is to use the Polyakov-Wiegmann identity

\[
S_{WZW}(\tilde{g}_0) = S_{WZW}(\tilde{g}_-) + S_{WZW}(\tilde{g}_0) + S_{WZW}(\tilde{g}_+)
- \frac{1}{4\pi} \int d^2x [\partial g \partial \bar{g} \bar{g} - \partial \bar{g} \bar{g} \partial g] + \frac{1}{4\pi} \int d^2x [\partial \bar{g} \bar{g} \partial g - \partial g \partial \bar{g} \bar{g}] + \frac{1}{4\pi} \int d^2x [\partial \bar{g} \bar{g} \partial g - \partial g \partial \bar{g} \bar{g}]
\]

(6.13)

with $g_0, \tilde{g}_\pm$ and $\tilde{g}_0$ as given by Eq. (6.10). Using Eq. (A.5), we obtain

\[
S_{NAT}(g_0) = - \frac{k}{8\pi} \int d^2x [\partial r \partial \bar{r} + \partial s \partial \bar{s} + 2e^{-2r} \partial \phi_2^+ \partial \phi_2^- + V(\phi_2^+, r, s)],
\]

(6.14)

where

\[
V(\phi_2^+, r, s) = e^{-s} + \phi_2^+ \phi_2^- e^{r-s}.
\]

(6.15)
This action is of the form (2.20) with \( s \) as the free scalar field. We may write it in the form Eq. (3.4) with a metric \( G_{ij} \) and an antisymmetric tensor field \( B_{ij} \) given by

\[
G_{ij} = \frac{s}{b} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_{\tilde{a}\tilde{b}} \end{pmatrix},
\]

\[
\tilde{G}_{rr} = -1, \quad \tilde{G}_{r+} = \tilde{G}_{+r} = -e^r, \quad \tilde{G}_{r-} = \tilde{G}_{-r} = -e^r,
\]

\[
B_{+\pm} = \tilde{B}_{+\pm} = -B_{-\pm} = \tilde{B}_{-\pm} = ie^r,
\]

all other components being zero (with \( \pm \) denoting the components corresponding to \( \phi^\pm_2 \)). \( \tilde{G}_{ij} \) and \( \tilde{B}_{ij} \) are then the metric and antisymmetric tensor field for \( S_{WZW}(\tilde{g}) \). The Christoffel symbols and torsion are then given by

\[
\Gamma^r_{+-} = -\frac{1}{2} e^r, \quad \Gamma^+_r = \Gamma^-_r = \frac{1}{2}, \quad H_{+-r} = \frac{i}{2} e^r,
\]

with all other components not related by symmetry vanishing. We then readily check that the generalised curvature given by Eq. (3.11) vanishes, which is sufficient to ensure that the \( \beta \)-functions \( \beta^G_{ij} \) and \( \beta^B_{ij} \) vanish. This is of course a consequence of the fact that the kinetic part of the action Eq. (3.14) is simply the that for the WZW model for \( A_1 \) coupled to a free scalar. Hence Eqs. (3.5a,b) are satisfied without the dilaton \( D \). We now wish to determine \( D \) and the modified couplings \( k' \) and \( k'' \) in Eq. (5.1) using Eq. (5.6), and we shall see that the form for \( D \) which emerges will be such as not to contribute to Eqs. (3.5a,b). Moreover, once we have made the identification Eq. (5.6), the third conformal invariance condition Eq. (3.5d) is guaranteed to be satisfied; the LHS of Eq. (5.6) is the operator which acts on the potential \( V \) in Eq. (3.5d), and the RHS annihilates the potential \( V(\phi^\pm_2, r, s) \) in Eq. (6.14), as we shall now demonstrate explicitly. We first of all need an explicit form for the Casimir in Eq. (4.7). We find for the operators \( X^L_{+\alpha} \) defined in Eq. (4.6):
the operators $X^R_{\alpha_i}$ being given by the same expressions but with $+ \rightarrow -$. (These expressions were calculated using the algebraic manipulation package REDUCE.) The Casimir $C^G$ is now given by Eq. (4.7), with Eqs. (4.8), (6.5), as

$$C^G = 2(e^{r-s}X^R_{\alpha_1}X^L_{\alpha_2} + e^{-r}X^R_{\alpha_2}X^L_{\alpha_3} + e^{-s}X^R_{\alpha_3}X^L_{\alpha_4} + e^{-r-s}X^R_{\alpha_4}X^L_{\alpha_2}) + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial r} + 3 \frac{\partial}{\partial s}. \quad (6.19)$$

It is then easy to check that

$$C^G V(\phi^\pm_2, r, s) = \left(2e^{-r} \frac{\partial}{\partial \phi^+_2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial r} + 3 \frac{\partial}{\partial s}\right) V(\phi^\pm_2, r, s) = 0, \quad (6.20)$$

with $V(\phi^\pm_2, r, s)$ as given by Eq. (6.15). We also have, from Eq. (4.12) and (6.7),

$$\text{tr}(\delta H J^R) = \frac{\partial}{\partial r} \quad (6.21)$$

from which it follows immediately that

$$(\text{tr}(\delta H J^R) + 1)V(\phi^\pm_2, r, s) = 0, \quad (6.22)$$

and hence from Eqs. (6.20) and (6.22), we see that indeed the RHS of Eq. (5.6) annihilates $V(\phi^\pm_2, r, s)$ in this example. We must now identify the quantities on the LHS of Eq. (5.6). With $\tilde{G}_{\alpha\beta}$ given by Eq. (6.16), we have

$$\sqrt{\tilde{G}} = e^{r}, \quad (6.23)$$

and so the LHS of Eq. (5.6) becomes

$$\frac{1}{k' + 1} (2e^{-r} \frac{\partial}{\partial \phi^+_2} \frac{\partial}{\partial \phi^-_2} + e^{-r} \frac{\partial}{\partial r} e^r \frac{\partial}{\partial r} + \frac{1}{k''} \frac{\partial^2}{\partial s^2}) + \left(\frac{1}{k'} \frac{\partial}{\partial r} D \frac{\partial}{\partial r} + \frac{1}{k''} \frac{\partial}{\partial s} D \frac{\partial}{\partial s}\right), \quad (6.24)$$

incorporating the obvious fact that $D$ cannot depend on $\phi^\pm_2$, since no single derivatives with respect to $\phi^\pm_2$ appear. The RHS of Eq. (5.6) as given explicitly by Eqs. (6.20) and (6.21) takes the form

$$\frac{1}{\kappa} (2e^{-r} \frac{\partial}{\partial \phi^+_2} \frac{\partial}{\partial \phi^-_2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial r} + 3 \frac{\partial}{\partial s}) + \frac{\partial}{\partial s} + 1, \quad (6.25)$$

where, since $c^G = 3$ for $B_2$, $\kappa = k + \frac{3}{2}$. Comparing (6.24) and (6.25), we must take

$$k' = \kappa - 1, \quad k'' = \kappa, \quad D = (3 + \kappa)s, \quad (6.26)$$

20
which agrees with the general result Eq. (5.21), since, from Eqs. (5.16), (6.3) and (6.7), we have
\[
\rho' = \frac{3}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\delta.
\] (6.27)

emphasised in discussing the general case, the expression for \( D \) in Eq. (6.26) has the vital property that it does not depend on the variables associated with \( \tilde{G} \). For instance, had the dilaton depended on \( r \), then from Eqs. (3.5a, b), (6.17), there would have been contributions from the dilaton to \( B_{ij}^G \), \( B_{ij}^B \), which would therefore not vanish (since \( \beta_{ij}^G \), \( \beta_{ij}^B \) are zero).

7. Conclusions

We have demonstrated how the process of hamiltonian reduction of the WZW model may be implemented at the quantum level, to furnish an action for the non-abelian Toda theory which is exactly conformally invariant. The quantum action differs from the classical version by the adjustment of the coupling constants and by the addition of a dilaton field. The scalar potential \( V(\phi^\alpha, \phi^\beta, r^i) \) satisfies the simple identities (4.13) which guarantee that it satisfies the Virasoro constraint (at zeroth order in the Laurent expansion in \( z \)).

These identities are proved in the Appendix; the first of them follows from a simple identity Eq. (A.16) for the ordinary Lie algebra Casimir (i.e. not the operator form of the Casimir), for the Lie subalgebra \( G_0 \) which is the grade-zero part of \( \mathcal{G} \).

The exact results for string black hole solutions, which were first obtained using methods similar to those used here, have recently been rederived using an approach based on the exact quantum effective action for the gauged WZW model [37]. (See also Ref. [38]. This method presents conceptual advantages over the present technique and moreover can also yield exact results for the antisymmetric tensor field, if present. The antisymmetric tensor field cannot be treated by the methods used here since it does not contribute to \( L_0 \), the Virasoro operator. It would be desirable to extend the quantum effective action methods to the present Toda field theory case.

Our initial interest in non-abelian Toda field theories was awakened by the fact [11] that the kinetic sector of the non-abelian Toda theory based on \( B_2 \) was equivalent to Witten’s string black hole [22]. However, this has only been shown at the classical level at present, and indeed the proof proceeds by eliminating one of the fields using its equation of motion. It is therefore not clear how one would demonstrate this equivalence at the quantum level.
At the most naive level, the equations of motion derived from the quantum action we derived in Section 5 do not appear to produce the equations of motion derived from the exact quantum action for the string black hole\[20][37] even if we again allow ourselves to eliminate one of the fields using its equation of motion. This point deserves more careful investigation, though.

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Appendix A.

In this Appendix we summarise our conventions for Lie algebras, and also prove some mathematical results, in particular Eqs. (4.13) and (5.17). In general a Lie algebra $\mathcal{G}$ can be specified by a Cartan subalgebra $\mathcal{H}$ of mutually commuting generators with a basis $\{H_i, \ i = 1, \ldots, r^G\}$, where $r^G$ is the rank of $\mathcal{G}$, together with a set of step operators $E_\alpha$ corresponding to the roots $\alpha$, satisfying

\begin{equation}
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha(H_i)E_\alpha, \quad (A.1)
\end{equation}

regarding the roots as elements of $\mathcal{H}^*$, i.e. as maps from $\mathcal{H}$ to $R$. In terms of some ordered basis $\gamma_1, \ldots, \gamma_{r^G}$ for the roots, a root $\beta$ can be defined as positive if the first coefficient in the expansion of $\beta$ in terms of the $\gamma_i$ is positive. A simple root is a positive root which cannot be written as the sum of two other positive roots. The Cartan-Killing form is defined for two generators $X, Y$ of $\mathcal{G}$ by

\begin{equation}
(X, Y) = \text{tr}(\text{ad}X \text{ad}Y), \quad (A.2)
\end{equation}

where

\begin{equation}
(\text{ad}X)Z = [X, Z]. \quad (A.3)
\end{equation}

We define

\begin{equation}
\gamma_{ij} = (H_i, H_j), \quad \gamma_{\alpha\beta} = (E_\alpha, E_\beta). \quad (A.4)
\end{equation}

All this is perfectly general; however, it is convenient to consider the case of orthonormal generators, normalised so that

\begin{equation}
\text{tr}(H_i H_j) = \delta_{ij}, \quad \text{tr}(E_{-\alpha} E_\beta) = \delta_{\alpha\beta}. \quad (A.5)
\end{equation}
In these circumstances we have
\[
\gamma_{ij} = c^G \delta_{ij}, \quad \gamma_{-\alpha\beta} = c^G \delta_{\alpha\beta},
\]  \hspace{1cm} (A.6)
where \( c^G \) is the eigenvalue of the quadratic Casimir in the adjoint representation. We now write \( \alpha(H_i) = \alpha_i \), so that \( \alpha_i \) are orthonormal co-ordinates for the root vectors, and we perform contractions using the ordinary Dirac delta function, i.e. \( \alpha.\beta = \alpha_i \beta_i \). We also have
\[
[E_{\alpha}, E_{-\alpha}] = \alpha.H = \alpha_i H_i.
\]  \hspace{1cm} (A.7)
Defining \( \psi \) to be the highest root in \( \mathcal{G} \), we have
\[
h^G = \frac{c^G \psi^2}{\psi^2}
\]  \hspace{1cm} (A.8)
and also the Freudenthal-De Vries strange formula [39],
\[
24 \rho^2 = h^G \psi^2 \dim \mathcal{G}
\]  \hspace{1cm} (A.9)
where \( \rho \) is defined in Eq. (4.8). We should explain at this point the relation between our present conventions and notation, and those used in Ref. 24. In Ref. 24 we found it useful not to restrict ourselves to the case of an orthonormal basis for \( \mathcal{H} \). We then defined the inner product between roots as follows: first we defined \( H_\alpha \) by requiring \( (H_\alpha, H) = \alpha(H) \) for all \( H \in \mathcal{H} \). We then defined \( \langle \alpha, \beta \rangle = (H_\alpha, H_\beta) \). With an orthonormal basis, these quantities are related to those used here by \( \alpha.H = c^G H_\alpha \) and \( \alpha.\beta = c^G <\alpha, \beta> \). As a consequence of Eq. (A.8), we then have \( \langle \psi, \psi \rangle = \frac{1}{h^G} \).

We now prove Eq. (5.17). Given a root \( \alpha \), the Weyl reflection \( w(\alpha) \) corresponding to \( \alpha \) acts on roots \( \beta \) by
\[
\beta \rightarrow \beta^{w(\alpha)} = \beta - 2 \frac{\beta.\alpha}{\alpha.\alpha} \alpha.
\]  \hspace{1cm} (A.10)
If \( \beta \) is a root, then \( \beta^{w(\alpha)} \) is a root also. Moreover, if \( \alpha \) is a simple root and \( \beta \) is a positive root (other than \( \alpha \) itself), then \( \beta^{w(\alpha)} \) is also a positive root; for \( w(\alpha) \) only changes the coefficient of \( \alpha \) in an expansion of \( \beta \) in terms of simple roots, and the coefficients of the simple roots in the expansion of \( \beta^{w(\alpha)} \) must all be of the same sign. In other words if \( \alpha \) is simple, then \( w(\alpha) \) permutes the positive roots except \( \alpha \) amongst themselves. We thus have, for a simple root \( \alpha \in \Delta \),
\[
\rho \rightarrow \rho^{w(\alpha)} = \rho - \alpha,
\]  \hspace{1cm} (A.11)
where \( \rho \) is defined in Eq. (1.8). Comparing Eqs. (A.10), (A.11), we find
\[
2 \frac{\rho \alpha}{\alpha \alpha} = 1, \quad \alpha \in \Delta, \quad (A.12)
\]
from which it follows that
\[
\rho = \sum_{\alpha \in \Delta} \alpha', \quad (A.13)
\]
with \( \alpha' \) as defined in Eq. (2.2).

Now consider \( \tilde{\rho} \) as defined in Eq. (5.16). Weyl reflections in a simple root of \( \tilde{\alpha} \in \tilde{\Delta} \) permute the positive roots (except \( \tilde{\alpha} \)) in \( \Phi^+ \) amongst themselves and also permute the positive roots (except \( \tilde{\alpha} \)) in \( \tilde{\Phi}^+ \) amongst themselves (since \( G_0 \) is a subalgebra). Hence they permute the roots in \( \Phi^+ \setminus \tilde{\Phi}^+ \) amongst themselves (since \( \rho' \) does not contain \( \tilde{\alpha} \)). So we have
\[
\rho^{w(\tilde{\alpha})} = \rho', \quad \tilde{\alpha} \in \tilde{\Delta}. \quad (A.14)
\]
Comparing with Eq. (A.10), we see that
\[
\rho' \cdot \tilde{\alpha} = 0, \quad \tilde{\alpha} \in \tilde{\Delta}, \quad (A.15)
\]
which is what we needed to prove.

We now turn to the proof of Eq. (4.13.) The Casimir \( C^G = \text{tr}[(J^L)^2] \) may be written
\[
C^G = \sum_{\alpha \in \Phi^+} (J^L \alpha J^L \alpha + J^L \alpha J^L \alpha) \quad + \text{cyclic terms} \quad (A.16)
\]
where \( J^L \alpha \) is the operator which induces multiplication of \( g \) on the left by \( E_\alpha \) and \( J^L_i \) is the operator inducing multiplication on the left by \( H_i \). It is easy to see from the definitions (4.6) that
\[
J^L_{-\alpha} = X^L_{-\alpha}, \quad (A.17)
\]
however there is no such simple expression for \( J^L_\alpha \). It is now convenient, using
\[
[J^L_\alpha, J^L_{-\alpha}] = \alpha_i J^L_i \quad (A.18)
\]
to rewrite Eq. (A.16) in the form
\[
C^G = \sum_{\alpha \in \Phi^+} (2J^L_\alpha J^L_{-\alpha} - \alpha_i J^L_i) + J^L_i J^L_i. \quad (A.19)
\]
The reason for writing $C^C$ in this form is that we can now replace the summation over $\alpha$ in the first term on the RHS of Eq. (A.19) by a summation over $\tilde{\alpha} \in \tilde{\Phi}^+$ when $C^G$ acts on $V(\phi^\tilde{\alpha}, \phi_{\tilde{\alpha}}^+, r^i)$. This is because, from Eq. (A.17) and the fact, mentioned in Section 3, that $X_{\tilde{\alpha}}^L$ contains only derivatives with respect to $\phi^\beta$ for $\beta \geq \tilde{\alpha}$, it follows that every term in $J^L_{\tilde{\alpha}} J^{-L}_{\tilde{\alpha}}$ contains at least one derivative with respect to $\phi^\beta$, for some $\beta \in \tilde{\Phi}^+ \setminus \tilde{\Phi}^+$. (This is not the case for $J^L_{\tilde{\alpha}} J^{-L}_{\tilde{\alpha}}$, which can in fact yield terms containing only a single derivative with respect to $r^i$ for some $i$. This is clear from Eqs. (A.18) and (4.11).) We now have

$$C^G V(\phi^\tilde{\alpha}, \phi_{\tilde{\alpha}}^+, r^i) = (C^G_0 - 2 \rho' J^L_i) V(\phi^\tilde{\alpha}, \phi_{\tilde{\alpha}}^+, r^i)$$  \hspace{1cm} (A.20)

where $C^G_0$ is the Casimir operator for $G_0$, and $\rho'$ is as defined in Eq. (5.16). It can readily be shown, using the definitions of the operators $J$ as generators of multiplication by Lie algebra elements, that

$$(C^G_0 + 2 \rho' J^L_i) V(\phi^\tilde{\alpha}, \phi_{\tilde{\alpha}}^+, r^i) = \text{tr}[(O M_+) g_0 M_- g_0^{-1}]$$  \hspace{1cm} (A.21)

where

$$O M_+ = C^G_0 M_+ - \rho' [H_i, M_+]$$  \hspace{1cm} (A.22)

with $C^G_0$ the ordinary adjoint Casimir for $G_0$, i.e.

$$C^G_0 M_+ = \sum_{\tilde{\alpha} \in \tilde{\Phi}^+} ([E_{\tilde{\alpha}}, [E_{-\tilde{\alpha}}, M_+] + [E_{-\tilde{\alpha}}, [E_{\tilde{\alpha}}, M_+]] + [H_i, [H_i, M_+]].$$  \hspace{1cm} (A.23)

It is clear that any $M_+$ with the property (2.14) must be a sum of $E_{\gamma}$ with $< \delta, \gamma >= 0$. From Eq. (2.4), any such $\gamma$ must be of the form

$$\gamma = \tilde{\alpha} + \bar{\gamma}_L,$$  \hspace{1cm} (A.24)

where $\tilde{\alpha}$ is some root in $\tilde{\Phi}^+$, and $\bar{\gamma}_L$ is a simple root in $\Delta \setminus \tilde{\Delta}$, subject only to the restriction that $\gamma$ is indeed a root. Hence, from Eq. (A.22), it will be sufficient to prove

$$C^G_0 E_{\gamma} = \rho' \gamma > E_{\gamma}$$  \hspace{1cm} (A.25)

for a root of the form Eq. (A.24). We can write

$$E_{\gamma} = N[E_{\tilde{\alpha}}, E_{\bar{\gamma}_L}]$$  \hspace{1cm} (A.26)
for some constant $N$. The action of the adjoint Casimir $C^G_0$ commutes with the action of any elements of $G_0$, and hence in particular with $E_{\tilde{\alpha}}$. Hence we have

$$C^G_0 E_\gamma = N[E_{\tilde{\alpha}}, C^G_0 E_{\tilde{\gamma}_L}]. \quad (A.27)$$

We can write

$$C^G_0 E_{\tilde{\gamma}_L} = 2 \sum_{\tilde{\alpha} \in \Phi^+} ([E_{\tilde{\alpha}}, [E_{-\tilde{\alpha}}, E_{\tilde{\gamma}_L}]] + [H_i, [H_i, E_{\tilde{\gamma}_L}]] - 2\tilde{\rho}_i[H_i, E_{\tilde{\gamma}_L}], \quad (A.28)$$

where $\tilde{\rho}$ is as defined in Eq. (5.16). Since $\tilde{\gamma}_L$ is a simple root, the first term on the RHS vanishes, and we have

$$C^G_0 E_{\tilde{\gamma}_L} = (\tilde{\gamma}_L - 2\tilde{\rho}) \cdot \tilde{\gamma}_LE_{\tilde{\gamma}_L}. \quad (A.29)$$

Using Eqs. (A.13) and (2.2), we can write

$$\tilde{\gamma}_L^2 = 2\rho \cdot \tilde{\gamma}_L, \quad (A.30)$$

and hence we have, from Eqs. (A.29), (A.30) and (5.16),

$$C^G_0 E_{\tilde{\gamma}_L} = \rho' \cdot \tilde{\gamma}_LE_{\tilde{\gamma}_L}. \quad (A.31)$$

Finally, combining Eqs. (A.13), (A.24), (A.26), (A.27) and (A.31), we obtain Eq. (A.25).
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