Area Quantization in Quasi-Extreme Black Holes

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Abstract

We consider quasi-extreme Kerr and quasi-extreme Schwarzschild-de Sitter black holes. From the known analytical expressions obtained for their quasi-normal modes frequencies, we suggest an area quantization prescription for those objects.

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The question of the quantization of the black hole horizon area is well posed and has been considered long ago by Bekenstein [1], being a major issue since then [2], [3], [4], [5], [6], [7], [8], [9]. The microscopic origin of the black hole entropy [10], [11] is also an unanswered question. There are attempts to partially understand these questions using string theory [12], [13] as well as the canonical approach of quantum gravity [14], [15], [16], [17]. Recently, the quantization of the black hole area has been considered [5], [6] as a result of the absorption of a quasi-normal mode excitation. Bekenstein’s idea for quantizing a black hole is based on the fact that its horizon area, in the nonextreme case, behaves as a classical adiabatic invariant [1], [4]. It is worthwhile studying how quasi-extreme holes would be quantized. It is specially interesting to investigate this case since we analytically know the quasi-normal mode spectrum of some black holes of that kind, namely the quasi-extreme Kerr [18] and quasi-extreme Schwarzschild-de Sitter (i.e., near-Nariai) [19] solutions. The quasi-normal modes of black holes are the characteristic, ringing frequencies which result from their perturbations [20] and provide a unique signature of these objects [21], possible to be observed in gravitational waves. Besides, quasi-normal modes have been used to obtain further information of the space-time structure, as for example in [22], [23], [24], [25], and [26].

Furthermore, gravity in such extreme configurations are an excellent laboratory for the understanding of quantum gravity, and information about the quantum structure of space-time can be derived in such contexts by means of general setups [27].

The first case of interest to us where the black hole quasi-normal mode spectrum is analytically known is the quasi-extreme Kerr black hole. In this case, the specific angular momentum of the hole, \(a\), is very nearly its mass \(M\) (\(a \approx M\)). Detweiler [28] was able to show that in such a case there is an infinity of quasi-normal modes given by [18]

\[
\omega_n M \approx \frac{m}{2} - \frac{1}{4m} \exp\left[\frac{\xi - 2n\pi}{2\delta} + i\eta\right],
\]

where \(n = 0, 1, \ldots\) labels the solution, \(m\) is an integer labeling the axial mode of the perturbation, while \(\xi, \delta, \) and \(\eta\) are constants. We note that (1) is valid for \(\ell = m\), where \(\ell\) is the multipole index of the perturbation. For details we refer the reader to [18].
In Boyer-Lindquist coordinates the Kerr solution reads

$$ds^2 = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Delta}{\Sigma}dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + 2Ma^2r\sin^2\theta \right)\sin^2\theta d\phi^2,$$

where

$$\Delta = r^2 - 2Mr + a^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta.$$  

$M$ and $0 \leq a \leq M$ are the black hole mass and specific angular momentum ($a = J/M$), respectively. The horizons are at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

In units $G = c = 1$, the black hole horizon area and its surface gravity (temperature) are given, respectively, by

$$A = 4\pi (r_+^2 + a^2).$$

$$\kappa = \frac{1}{4A} (r_+ - r_-).$$

Based on Bohr’s correspondence principle (“for large quantum numbers, transition frequencies should equal classical frequencies”), Hod [5] has considered the asymptotic limit $n \to \infty$ for the quasi-normal mode frequencies $\omega_n$ of a Schwarzschild black hole in order to determine the spacing of its equally spaced quantum area spectrum. That asymptotic quasi-normal mode spectrum was obtained numerically by Nollert [29]. Recently, Motl [8] has computed analytically $\Re(\omega_n)$ as $n \to \infty$, finding agreement with the numerical value of Nollert. In this large $n$ limit, Hod [5] then assumed that a Schwarzschild black hole mass should increase by $\delta M = \hbar \Re(\omega_n)$ when it absorbs a quantum of energy $\hbar \Re(\omega_n)$.

As in Ref. [5], we expect that the real part of the quasi-normal mode frequency for large $n$ corresponds to an addition of energy equal to $\hbar \Re(\omega_n)$ to the quasi-extreme Kerr black hole mass as it falls into its event horizon. Then, taking the limit $n \to \infty$ for $\omega_n$ in (1) we simply have

$$\omega_n \approx \frac{m}{2M}, \quad n \to \infty.$$  

Contrary to the Schwarzschild case, where the limit $n \to \infty$ gives highly damped modes, for the present case, it gives virtually undamped modes with frequencies close to the upper limit of the superradiance interval [30], $0 < \omega < \infty$. 

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\(m \Omega\), where \(\Omega = 4 \pi a/A\) is angular velocity of the horizon. The quasi-normal mode spectrum [1] of near-extreme Kerr black holes leads to interesting consequences, as recently analysed by Glampedakis and Anderson [18].

Furthermore, here the angular momentum \(\bar{h} m\) adds to the angular momentum \(J = M a\) associated with the Kerr solution. We then have a pair of variations for black hole parameters given by

\[
\delta M = \frac{\bar{h} m}{2 M} ; \quad \delta J = \bar{h} m \quad \Rightarrow \quad \delta a = \bar{h} m \left(\frac{1}{M} - \frac{a}{2 M^2}\right).
\]

In what follows we will consider \(\bar{h} = 1\) and for the sake of brevity \(m = 1\).

The variation of the horizon area is related to the first law of black hole thermodynamics,

\[
\delta M = \kappa \delta A + \Omega \delta J.
\]

Making use of relations (8), we can obtain from (5) that, for a near-extreme hole \((a \approx M)\), the area variation is given by

\[
\delta A = 8 \pi \left(1 + \sqrt{\frac{M - a}{2 M}}\right),
\]

up to first order in \((M - a)^{1/2}\).

Therefore, for strictly extreme holes, we simply have \(\delta A = 8 \pi\).

For \(a \approx M\), we can express \(\kappa\) and \(\Omega\), respectively, as

\[
\kappa \approx \frac{1}{16 \pi} \frac{\sqrt{M^2 - a^2}}{M^2} \left[1 - \frac{\sqrt{M^2 - a^2}}{M}\right]\]

and

\[
\Omega \approx \frac{a}{2 M^2} \left(1 - \frac{\sqrt{M^2 - a^2}}{M} + \frac{M^2 - a^2}{M^2}\right).
\]

Finally, from (8), (10), (11), and (12), to order \((M - a)^{3/2}\), we obtain

\[
\kappa \delta A + \Omega \delta J \approx \frac{1}{2 M},
\]

in agreement with the first law of black hole thermodynamics [2]. Thus we can prescribe the quantization of a quasi-extreme Kerr black hole area as

\[
A_n = n \delta A \ell_p^2 \simeq 8 \pi \ell_p^2 n.
\]
where \( n = 1, 2, ... \) and \( \ell_P \) is the Planck length.

A second case where we can obtain information about the black hole parameters involved in its quantization is the near-extreme Schwarzschild-de Sitter (S-dS) black hole. This is the case when the mass of the black hole is increased as to arrive near the limit \( M_N = R/3\sqrt{3} \), where the constant \( R \) is related to the cosmological constant \( \Lambda \) by \( R^2 = 3/\Lambda \). This is the Nariai limit \[31\], for which the black hole and cosmological horizons coincide. The S-dS metric is \[32\]

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

where

\[
f(r) = 1 - \frac{2M}{r} - \frac{r^2}{R^2},
\]

and \( 0 \leq M \leq M_N \) is the black hole mass. The roots of \( f(r) \) are \( r_b, r_c \) and \( r_0 = -(r_b + r_c) \), where \( r_b \) and \( r_c \) are the black hole and cosmological horizon radii, respectively. To each horizon there is a surface gravity, given by \( \kappa_{b,c} = \frac{1}{2} \frac{df}{dr} |_{r=r_{b,c}} \). For \( \kappa_b \) we have the expression

\[
\kappa_b = \frac{(r_c - r_b)(r_b - r_0)}{2R^2r_b}.
\]

As in the Kerr case, we will perform the variation of \( M \). It is useful to write \( M \) and \( R \) in terms of \( r_b \) and \( r_c \) as

\[
2MR^2 = r_br_c(r_b + r_c),
\]

\[
R^2 = r_b^2 + r_br_c + r_c^2.
\]

The analytical quasi-normal mode spectrum for the quasi-extreme S-dS black hole has been recently derived by Cardoso and Lemos \[19\] and reads

\[
\omega_n = \kappa_b \left[ \sqrt{\frac{V_0}{\kappa_b^2} - \frac{1}{4} - i(n + \frac{1}{2})} \right],
\]

where \( n = 0, 1, ... \), and \( V_0 = \kappa_b^2 \ell (\ell + 1) \), for scalar and electromagnetic perturbations, and \( V_0 = \kappa_b^2 (\ell + 2)(\ell - 1) \) for gravitational perturbations.

Since we are considering the near extreme limit of the S-dS solution, for which \( (r_c - r_b)/r_b << 1 \), it is suitable for our purposes to write the black hole mass as

\[
M = M_N + \mu = \frac{R}{3\sqrt{3}} + \mu.
\]
Therefore, since \( R = \sqrt{3/\Lambda} \) is fixed, the use of (18) and (19) leads us to

\[
\delta M = \delta \mu = \frac{r_b \Delta r \delta r_b}{2R^2},
\]  

(22)

where \( \Delta r = r_c - r_b \).

Similarly as we did for the Kerr case, here we can consider \( \delta M = \hbar \Re(\omega_n) \) and in view of (20) and (17) write (\( \hbar = 1 \))

\[
\delta M = \frac{\Delta r}{2r_b^2} \sqrt{(\ell + 2)(\ell - 1) - \frac{1}{4}},
\]  

(23)

where we have used \( R^2 \sim 3r_b^2 \) and considered \( V_0 \) for gravitational perturbations.

The variation of the black hole horizon area,

\[
\delta A_b = 8\pi r_b \delta r_b,
\]  

(24)

then gives us

\[
\delta A_b = 24\pi \sqrt{(\ell + 2)(\ell - 1) - \frac{1}{4}},
\]  

(25)

for gravitational quasi-normal modes and \( \delta A_b = 24\pi \sqrt{\ell(\ell + 1) - \frac{1}{4}} \) otherwise.

Thus we can prescribe the quantum area spectrum for a quasi-Nariai black hole as

\[
A_{b_n} = n\delta A_b \ell_P^2 \simeq 12\pi \sqrt{15}\ell_P^2 n,
\]  

(26)

where \( n = 1, 2, \ldots \), or, in the case of scalar or electromagnetic perturbations, \( 12\pi \sqrt{7}\ell_P^2 n \).

In summary, with the knowledge of the analytical quasi-normal mode spectrum of near extreme Kerr and near extreme S-dS black holes, as given in [18] and [19], we have prescribed how their horizon area would be quantized. This was done by simply assuming they have a uniformly spaced area spectrum given by \( A_n = \delta A \ell_P^2 n \), where \( \delta A \) is the area variation caused by absorption of a quasi-normal mode. This was done in analogy with the Schwarzschild case, where the spacing of its area spectrum was determined by means of the knowledge of its asymptotic ("large \( n \)") quasi-normal mode frequencies [5]. In the cases regarded here, the results for the spacing of the area spectrum differ from that for Schwarzschild, as well as for non-extreme Kerr [33] black holes, in which cases, the spacing is predicted to be given by
4ln3. This factor comes from the real part of the asymptotic quasi-normal mode frequencies of those black holes [5], [33]. Such a difference may be justified due to the quite different nature of the asymptotic quasi-normal mode spectrum of the near extreme black holes we considered. Furthermore, it should be no a priori reason for expecting the same behaviour for the asymptotic quasi-normal mode frequencies of near extreme and non-extreme black holes.

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