On the spacetime structure of infrared divergencies in QED

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Abstract

We investigate analytic properties of string-integrated correlation functions and propagators with emphasis on their infrared behaviour. These are relevant in various models of quantum field theory with massless fields, including QED.

1 Motivation

A conceptually new approach to QED is presented in [12] (see also [11,16]). It is designed to better understand the long-distance behaviour of QED, including the uncountable superselection structure of charged states due to their “asymptotic photon clouds”, and the infraparticle nature of the electron. The latter is manifest in a sharp lower end of the mass spectrum with a singular set-on of the continuum due to the attached soft photons. The new approach properly addresses and solves the problem with the quantum Gauss Law on the physical Hilbert space: if the charged field were a local quantum field, the integrated electric flux at spacelike infinity would commute with it and cannot act as the generator of the U(1) symmetry. The tight relations among these physical features have been known since long [3,4,7,8], while a way to incorporate them into a model was so far lacking – apart from a simple model in 1+1 spacetime dimensions [15].

The most prominent role in the new approach is played by an auxiliary quantum field formally defined as

$$ \phi(x,e) = \int_{\Gamma_{x,e}} A^K_{\mu}(y)dy^\mu = \int_0^\infty ds A^K_{\mu}(x+se)e^\mu $$

(introduced in more detail below; $\Gamma_{x,e}$ is a straight curve from $x$ along a direction $e$ to infinity, and the superscript $K$ stands for “Krein space” emphasizing the indefinite metric of the usual Feynman gauge Fock space). It is the main purpose of these notes to investigate details of its infrared behaviour in position space.

The infrared superselection sectors of QED arise by exponentiating the field $\phi(x,e)$, smeared with suitable functions $c(e)$. Because Eq. (1.1) is infrared divergent as it stands, a suitable infrared cut-off function $\nu(k)$ is needed that allows to extend the Fourier transform of the two-point function to $k = 0$ (as a distribution). With an appropriate regularization by a mass $m \to 0$, the correlation functions of exponentiated fields (“vertex operators”)

$$ N_\nu(c) \cdot :e^{i\phi(x,e)}:e $$

(1.2)

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contain an overall factor $e^{-d_{m,v}(C,C)}$, where $d_{m,v}(C,C)$ is an integral diverging to $+\infty$ as $m \to 0$, unless $C(e) \equiv \sum_i q_i c_i(e) = 0$. In the limit, the factor converges to zero unless $C(e) = 0$:

$$e^{-d_{m,v}(C,C)} \to \delta_{C,0}. \quad (1.3)$$

This factor entails that states with different “charge functions” $C$ are mutually orthogonal, and produces an uncountable number of superselection rules. The physical meaning of the charge functions is that of “photon clouds” attached to charged particles [12]. States created by the exponential field acting on the vacuum can formally be regarded as coherent photon states lying outside the vacuum Fock space, and these coherent states belong to inequivalent representations of the Maxwell field whenever their photon clouds (i.e., their smearing functions $c(e)$) differ.

In [12], a “dressed Dirac field”

$$\psi_{q,c}(x) = e^{iq\phi(x,c)} \cdot \psi_0(x) \quad (1.4)$$

is introduced, where $q$ is the unit of electric charge. This field arises by subjecting the free Dirac field to the “trivial” interaction density

$$q \cdot \partial_{\mu} \phi(x,c) j_{\mu}(x). \quad (1.5)$$

Being a total derivative, Eq. (1.5) does not contribute to the total action, and gives rise to a trivial scattering matrix. The non-perturbative construction of the dressed Dirac field is meant as a first step towards the full perturbative QED, by splitting into two parts a QED interaction density that can be defined on a positive-definite subspace of the indefinite Fock space (i.e., a Krein space). Although the interaction density Eq. (1.5) is “trivial”, it drastically changes the algebraic structure of the charged field. The dressed Dirac field is string-localized (see Sect. 2) and falls outside the regime of, say, Wightman quantum field theory. Unlike the free Dirac field, it creates states enjoying infrared features of QED that cannot be attained in the usual local approach to QED (including the quantum Gauss Law, the photon cloud superselection structure, and the associated breakdown of Lorentz invariance). But the dressed Dirac field does not interpolate between different scattering states. A nontrivial S-matrix is only produced when also the “true” interaction $q \cdot A_{\mu}^K j_{\mu}(x)$ is turned on.

In some disguise, the exponentiated escort field has previously made its appearance as a “dressing factor” in the Faddeev-Kulish prescription [9] to prevent the formal vanishing of the LSZ limit, and hence of the scattering matrix of QED [5, 18]. But (apart from further differences in the detailed formula), the dressing factor in [9] is not part of the charged field but rather of the states in which the S-matrix has to be evaluated.

Scattering theory in each of its formulations exploits the asymptotic large-time behaviour of correlation functions. Thus, a future modification of scattering theory adapted to theories with infraparticles will need detailed information about correlation functions of the infrafield. In the case of the dressed Dirac field, these involve correlation functions of the vertex operators. This is one of our motivations to study the latter.

The new approach to QED itself is not the topic of this paper, except for the short remarks in Sect. 2. For more, we refer to [12]. Our topic are correlation functions of vertex operators and their analytic properties. In particular, the coefficients $q$ in Eq. (1.2) may be regarded as free parameters, unrelated to the electric charge. The results also bear on technical aspects of the very setup of string-localized quantum field theory [13], e.g., how much smearing of the string directions is necessary.
2 Preliminaries

The basic idea of the new approach to QED [11, 13, 15] is to use “string-localized potentials” $A_\mu(x, e)$ for the free Maxwell field:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x, e) - \partial_\nu A_\mu(x, e).$$  (2.1)

They depend on a spacelike four-direction $e$, and enjoy the axiality property

$$e^\mu A_\mu(x, e) = 0.$$  (2.2)

It is suggestive to think of them as “axial gauge” potentials; but the choice of $e$ is not a gauge-fixing condition; rather the potentials $A_\mu(x, e)$ for all $e$ coexist on the same Hilbert space where $F_{\mu\nu}$ is defined. More precisely, $A_\mu(x, e)$ is a distribution also in the variable $e$ and requires a smearing with suitable functions $c(e)$. The latter are required to have total weight

$$\int d\sigma(e) c(e) = 1,$$

so that $A_\mu(x, c)$ are still potentials for $F_{\mu\nu}$.

The terminology “string-localized” refers to the fact that they can be defined as integrals (well defined in the distributional sense) over the field along a “string” of direction $e$:

$$A_\mu(x, e) := \int_0^\infty ds F_{\mu\nu}(x + se)e^\nu.$$  (2.3)

String-localization is an algebraic property: $A(x, e)$ commutes with $F(y)$ provided $y$ is spacelike separated from the string $x + \mathbb{R}_+ e$. The definition Eq. (2.3) ensures both Eq. (2.1) and Eq. (2.2). Thus, “axiality” is a consequence of localization along a string.

The Hilbert space for the free Maxwell field can be directly obtained from the massless unitary Wigner representations of helicities $\pm 1$ [18] without the detour through a potential, and can thus be seen as a primary entity. But in perturbative QED one usually starts from a local potential $A^K_\mu(x)$, say in the Feynman gauge, that is defined on an indefinite Krein space, and then defines

$$F^K_{\mu\nu}(x) = \partial_\mu A^K_\nu(x) - \partial_\nu A^K_\mu(x).$$  (2.4)

When the Krein space is reduced to the physical Hilbert space by the Gupta-Bleuler (or BRST) prescription, the Maxwell field becomes equivalent to the one on the Fock space over the Wigner representation; while the potential $A^K_\mu$ ceases to exist. In contrast, the string-localized potentials $A_\mu(x, e)$ (being functionals of the Maxwell field) exist on both the Krein space and on the physical Hilbert space.

Because on the Krein space both Eq. (2.1) and Eq. (2.4) hold, the two potentials differ by an operator-valued gauge transformation

$$A^K_\mu(x, e) = A^K_\mu(x) + \partial_\mu \phi(x, e).$$  (2.5)

The quantity $\phi(x, e)$, baptized “escort field” in [16], turns out to be given by Eq. (1.1). Thus, it is string-localized and – as we shall see – infrared divergent. But its derivative is well-defined as a string-integral over $\partial A^K$ that decays fast enough as $s \to \infty$.

There are now several new options [12] to construct QED without the need to work in Krein spaces. The first is to replace the usual interaction density $q A^K j$, defined on a Krein space, by $q A(e)j$, defined on the Wigner Hilbert space (where the string-dependence is a total derivative and should not affect the resulting theory [16]). This option requires string-localized propagators of the potentials $A(e)$, that are the topic of Sect. 4.

The “hybrid” option indicated in the introduction is to split $q A(e)j$ into $q \partial \phi(e)j$ (which is a total derivative) and $q A^K j$, and study the theory with the “trivial interaction density” $q \partial \phi(e)j$...
first. This model can actually be constructed non-perturbatively, leading to a rigorous and IR-finite definition of the vertex operators Eq. (1.2) and the dressed Dirac field Eq. (1.4). The analytic properties of vertex operators, in particular, the space-time structure of their correlation functions, is the topic of the main part of this paper, Sect. 2 and Sect. 3.

The second step, the perturbation of the dressed Dirac field with the QED interaction density $qA^K\cdot j$ (and the reasons why this does not re-introduce Hilbert space non-positivity) requires a rather big effort and is addressed in [12].

Let us begin with an inventory of the basic quantities that are needed in the various approaches. The two-point function $W_0$ and the Feynman propagator $G_0^F$ of the massless scalar Klein-Gordon field coincide (up to a factor of $i$) as a function of $x = x_1 - x_2$ outside the singular support, which is the null-cone $(x_1 - x_2)^2 = 0$. As distributions, they are given in position space as boundary values of analytic functions

$$W_0(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \frac{1}{\left(x^0 - i\varepsilon\right)^2 - \vec{x}^2} = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \frac{1}{\left(x^0 - i\varepsilon\right)^2 - \left(x^0\right)^2}, \quad (2.6)$$

$$G_0^F(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \frac{1}{\left(x^0 - i\varepsilon\right)^2 - \vec{x}^2}, \quad (2.7)$$

respectively. The commutator function (Pauli-Jordan function) can be written as

$$C_0(x) := i(W_0(x) - W_0(-x)) = \frac{1}{2\pi} \text{sign}(x^0) \delta(x^2). \quad (2.8)$$

For the indefinite Feynman gauge vector potential, one has

$$\langle A^K_\mu(x)A^K_\nu(x') \rangle = -\eta_{\mu\nu}W_0(x - x'), \quad (2.9)$$

$$i\langle TA^K_\mu(x)A^K_\nu(x') \rangle = -\eta_{\mu\nu}G_0^F(x - x'). \quad (2.10)$$

By Eq. (1.1), the ensuing two-point function and Feynman propagator of the escort field are given as double integrals

$$-(ee')\int_0^\infty ds'\int_0^\infty ds F(x + se - s'e') \equiv -(ee')(I_{-e'}I_eF)(x), \quad (F = W_0 \text{ resp. } G_0^F). \quad (2.11)$$

The notation $I_e$ stands for the string-integration as in Eq. (1.1) or Eq. (2.3). The operations $I_e$ commute among each other and with derivatives as long as all integrals exist as distributions, and it holds

$$e^\mu\partial_\mu(I_e f)(x) = -f(x). \quad (2.12)$$

If $f$ is a function or a distribution, $I_e f$ can only be defined if $f$ has sufficiently rapid decay. Because $W_0$ and $G_0^F$ fall off in configuration space like $1/x^2$, their first string-integrations are finite, whereas the second string-integration diverges logarithmically and has to be regularized.

The well-definedness as a distribution is a more subtle issue than the convergence of an integral. In Fourier space, the string-integration is a multiplication with another distribution:

$$I_e e^{-ikx} = \int_0^\infty ds \, e^{-ik(x + se)} = \lim_{\varepsilon \downarrow 0} \frac{-i}{(ke) - i\varepsilon} \cdot e^{-ikx}. \quad (2.13)$$

The existence of a product of distributions has to be analyzed by microlocal methods, such as Hörmander’s criterion for the wave front sets. But the latter is only a sufficient condition, and by cancellations of singularities the product may be better behaved than the wave front sets may tell. Since in this work we are interested mainly in the behaviour in position space, we
Because \(1\) diverges logarithmically at \(k\). We just notice here that the Fourier transforms of \(W_0\) and \(G_0^F\) scale like \(k^{-2}\). The IR divergence arises because with two additional denominators as in Eq. (2.11), the Fourier integrals would diverge logarithmically at \(k = 0\).

Because \(\frac{1}{(y - s)^2} (\text{where } y = (ke))\) is well-defined, \((I_e I_e f)(x)\) is well-defined as a distribution in \(x\), provided the decay of \(f\) is fast enough. But because \(\frac{1}{(y - s)^2(y + s)}\) is ill-defined, \((I_e I_e f)(x)\) is always ill-defined. In position space, this can be easily understood because the integrand in Eq. (2.11) depends only on \(s - s'\). In particular, \(\langle A(x, c)A(x', c') \rangle\) is ill-defined at \(e = e'\). Not least for this reason, one should consider the string-integrals also as distributions in \(e\) and \(c'\).

We shall see in Sect. 3.2 that the singularity at \(e = e'\) is integrable in \(e\) and \(e'\) w.r.t. the invariant measure of \(S^2\), so that smeared expressions like \(\langle A(x, c)A(x', c') \rangle\) are well-defined even when the supports of \(c(e)\) and \(c'(e')\) overlap.

Because \(W_0\) and \(G_0^F\) are homogeneous distributions in \(x\) of degree \(-2\), \(I_e W_0\) and \(I_e G_0^F\) are homogeneous both in \(e\) and in \(x\) of degree \(-1\). We shall (most of the time) restrict \(e\) to the open set of spacelike vectors, because these are the directions needed in the intended applications [12]. Because of homogeneity in \(e\), we may as well restrict \(e\) to the unit spacelike hyperboloid \(H_1 = \{e \in \mathbb{R}^4 : e^2 = -1\}\) [10]; but it will be advantageous to display factors “\(e^{2\nu}\)” explicitly, so as to maintain homogeneity in \(e\). Yet, smearing in \(e\) is always understood with normalized \(e\).

The quantities of interest in various applications [12] are:

(i) Two-point functions of \(A_\mu(x, e) = (I_e F_{\mu\nu})(x)e^\nu\) or \(A^K_\mu(x, e) = A^K_\mu(x) + \partial_\mu \phi(x, e)\). (These distributions are identical in the Wigner space version and the Krein space version.)

(ii) Mixed two-point functions between \(A^K_\mu(x)\) and \(\phi(x, e)\).

(iii) IR-regularized two-point functions of \(\phi(x, e)\) and their exponentials.

(iv) Propagators (= time-ordered two-point functions) of \(A_\mu(x, e)\).

Propagators of \(\phi\) with itself would require time-ordering and IR regularization, which would be very delicate to implement simultaneously. Fortunately, such objects do not occur [12].

(i) define a string-localized quantum field theory. Apart from a local contribution, they involve contributions with one or two string-integrations over derivatives of Eq. (2.6). Thanks to the derivatives, these string-integrations are IR-finite and do not need a regularization. They are far simpler than (iii) without derivatives.

(ii) and (iii) arise in the “hybrid” approach of [12], where the escort field without derivative appears in the exponent of a regularized normal-ordered Weyl operator (vertex operator). Their computation and analysis is our first main topic.

(iv) are needed in perturbation theory when a current is coupled to a string-localized potential. They involve one or two string-integrations over derivatives of Eq. (2.7). Again, thanks to the derivatives, an IR regularization is not needed. Their computation and analysis is the second main topic of this work.

Thus, we shall study the IR-finite expressions

\[
(I_e W_0)(x), \quad (I_e I_e I_{e_1} \partial W_0)(x), \quad (I_e I_e I_{e_1} \partial G_0^F)(x)
\] (2.14)

from which the other IR-finite quantities of interest arise via

\[
(I_e \partial W_0)(x) = \partial(I_e W_0)(x), \quad (I_e I_{e_1} \partial W_0)(x) = \partial(I_e I_{e_1} \partial W_0)(x); \quad \text{and the IR-regularized expression}
\]

\[
(I_e I_{e_1} W_0)_c(x).
\] (2.16)
3 Two-point function

3.1 One string-integration

The distribution $W_0(x)$ is defined as the boundary value of the analytic function $-\frac{1}{(2\pi)^\frac{1}{2}}\frac{1}{z}$ in the complex forward tube $\text{Im} z^0 < 0$. We may thus write $z = x - i\varepsilon u$ where $x$ is real and $u$ a forward timelike (unit) vector. By Lorentz invariance, the distributional limit $\varepsilon \downarrow 0$ is independent of $u$. So, because $e$ is spacelike, one may choose $u$ perpendicular to $e$. Let in this section $F(x) = \frac{1}{(x - i\varepsilon u)^2}$, where the distributional limit $\varepsilon \downarrow 0$ is always understood.

The string-integral over $F(x)$ can be written as

$$f(x,e) := -(I_e F)(x) \equiv -\int_0^\infty \frac{ds}{(x + se - i\varepsilon u)^2} = -\int_0^\infty \frac{ds}{(x - i\varepsilon u)^2 + 2s(xe) + s^2 e^2}. \quad (3.1)$$

The point is that the complex denominator cannot vanish for real $s$, and $i\varepsilon$ appears only in the parameter $(x - i\varepsilon u)^2$. The elementary integration gives

**Lemma 3.1.**

$$(2\pi)^2(I_e W_0)(x) = f(x,e) = \frac{\frac{1}{2} \log \frac{-(xe)+i\sqrt{\det_{x,e}}}{-(xe)-i\sqrt{\det_{x,e}}}}{i\sqrt{\det_{x,e}}} = \frac{\frac{1}{2} \log \frac{-(xe)+\sqrt{-\det_{x,e}}}{-(xe)-\sqrt{-\det_{x,e}}}}{\sqrt{-\det_{x,e}}}, \quad (3.2)$$

where

$$\det_{x,e} := (x - i\varepsilon u)^2 e^2 - (xe)^2 \quad (3.3)$$

is the Gram determinant, with the imaginary shift of $x$ being implicitly understood.

The logarithm of the quotient is understood as the difference of two logarithms with their branch cut along $\mathbb{R}_+$. In particular, Eq. (3.2) does not depend on the choice of the branch of the square root (because numerator and denominator would simultaneously switch sign). As a function on $\mathbb{R}^4 \times H_1 \times H_1$ (i.e., putting $\varepsilon = 0$), Eq. (3.2) is ill-defined only when $x^2 = 0$ or $\det_{x,e} = 0$. The $i\varepsilon$-prescription in $\det_{x,e}$ defines Eq. (3.2) as a distribution.

When $x$ and $e$ lie in a common spacelike plane, one may without loss of generality assume $x^0 = 0$ and $e^0 = 0$. Then

$$\sqrt{x^2 e^2} \cdot f(x,e) = \frac{\alpha}{\sin \alpha}, \quad (3.4)$$

where $\alpha = \angle(\vec{x}, \vec{e}) \in [0, \pi)$. It has a singularity at $\alpha = \pi$ reflecting the fact that the string $x + R_+ e$ passes through the origin. Yet, upon smearing in $\vec{e}$, this singularity is integrable w.r.t. the invariant measure of $S^2$ and does not need an $i\varepsilon$ prescription to define it as a distribution in $\vec{e}$.

More generally, the defining integral Eq. (3.1) may be singular whenever $(x + se)^2$ can become zero for $s \geq 0$, i.e., geometrically, when the string $x + R_+ e$ hits the null-cone. This happens necessarily if $x$ is timelike or lightlike (the string starts inside or on the null-cone). If $x$ is spacelike, the string may touch the null-cone or pierce it twice. However, inspection of Eq. (3.2) shows that with the $i\varepsilon$-prescription, the singular support of $f(x,e)$ is at $x^2 = 0$ (the string starts on the null-cone) and at $\det_{x,e} = 0$, $(xe) \geq 0$ (the string touches the null-cone).

Somewhat unexpected from its unsymmetric definition $f(x,e) = -(I_e F)(x)$, this function (where it is regular) is symmetric in $x \leftrightarrow e$. The symmetry can be understood by a change of integration variables $s \to \frac{1}{s}$ in Eq. (3.1).

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1In particular, negative factors must not be cancelled under the logarithm: this would produce errors of $2\pi i$!
3.2 Two string-integrations

The two-point function of the escort field is the twofold string-integral over $W_0(x)$, multiplied by the factor $-(c e')$. The presence of this factor jeopardizes the positivity of the inner product defined by the two-point function, which is essential on the way to the superselection structure via Eq. (3.29), outlined in the introduction. To secure positivity, one has to impose that $e$ and $e'$ are smeared within a spacelike surface [12] perpendicular to a timelike unit vector $u$. Without loss of generality, we may pick

$$u = u_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow e = \begin{pmatrix} 0 \\ e \end{pmatrix}, \quad e' = \begin{pmatrix} 0 \\ e' \end{pmatrix},$$

(3.5)

Thus, smearing functions $c(e) = c(\bar{e})$ are elements of $C^\infty(S^2)$. Vectors $y$ with $y^0 = 0$ will be called “purely spatial”.

Because the two-point function $W_0(x)$ is homogeneous of degree $-2$, $(I_e W_0)(x)$ is homogeneous of degree $-1$, and the second string-integration would diverge logarithmically. In momentum space,

$$(I_{-e'} I_e W_0)(x) \triangleq \lim_{\varepsilon, \varepsilon' \downarrow 0} \int \frac{e^{-ikx}}{((ke) - i\varepsilon)((ke') + i\varepsilon')} \, dk \, d\mu_k$$

(3.6)
diverges at $k = 0$. We therefore have to regularize it in the infrared. The regularization extends the momentum space distribution to $k = 0$. This is done by replacing $e^{-ikx}$ by $e^{-ikx} - v(k)$ where $v(k)$ is any smooth test function with $v(0) = 1$. Thus, we define

$$(I_{-e'} I_e W_0)_{v}(x) := \lim_{\varepsilon, \varepsilon' \downarrow 0} \int \frac{e^{-ikx} - v(k)}{((ke) - i\varepsilon)((ke') + i\varepsilon')} \, dk \, d\mu_k.$$  

(3.7)

Because of the symmetry $e \leftrightarrow -e'$, we shall in the sequel write $e = e_1$ and $-e' = e_2$, so that $(I_{e_2} I_{e_1} W_0)_{v}(x)$ is symmetric in $e_1 \leftrightarrow e_2$.

We want to gain insight into the distribution $(I_{e_2} I_{e_1} W_0)_{v}(x)$ in position space, leaving the regulator function $v$ unspecified. It is clearly not possible to compute the integral Eq. (3.7) when $v(k)$ is not specified. The strategy is therefore to compute instead the cut-off integral

$$(I_{e_2}^{a} I_{e_1} W_0)(x) \equiv \int_{0}^{a} ds_2 \, (I_{e_1} W_0)(x + s_2 e_2) = \frac{1}{(2\pi)^2} \int_{0}^{a} ds \, f(x + s e_2, e_1)$$

(3.8) – which can be done analytically – and use that

$$\partial_\mu (I_{e_2} I_{e_1} W_0)_{v}(x) = (I_{e_2} I_{e_1} \partial_\mu W_0)(x) = \lim_{a \to \infty} (I_{e_2}^{a} I_{e_1} \partial_\mu W_0)(x) = \lim_{a \to \infty} \partial_\mu (I_{e_2}^{a} I_{e_1} W_0)(x),$$

(3.9)

where the first equality follows from the definition Eq. (3.7), the second holds because $\partial_\mu W_0$ decays sufficiently fast to make the integral converge in $a$, and the last is obvious from the definition of the integral operations. Thus, the difference is independent of $x$, and the result for $(I_{e_2} I_{e_1} W_0)_{v}(x)$ is obtained by replacing the cut-off dependent but $x$-independent term by another (unknown) $x$-independent term. Specifically, we will show in the remainder of this section

$$(I_{e_2}^{a} I_{e_1} W_0)(x) = \frac{1}{(2\pi)^2} \left[ \frac{1}{2} f(e_1, e_2) \cdot \log \left( \frac{A(e_2)^2}{(x - i\varepsilon u)^2} \right) + \frac{H(x; e_1, e_2)}{(e_1 e_2)} \right] + O(1/a),$$

(3.10)

where $f(e_1, e_2) = f(e_2, e_1)$ is the same distribution as in Eq. (3.1), and the distribution $H$ is symmetric in $e_1 \leftrightarrow e_2$ and homogeneous of degree 0 separately in $x, e_1,$ and $e_2$. We conclude

$$(I_{e_2} I_{e_1} W_0)_{v}(x) = \frac{1}{(2\pi)^2} \left[ - \frac{1}{2} f(e_1, e_2) \cdot \log(-\mu_0^2 \cdot (x - i\varepsilon u)^2) + \frac{H(x; e_1, e_2)}{(e_1 e_2)} \right],$$

(3.11)

where $\mu_0$ carries the dependence on the regulator function $v$ and may depend on $e_1$ and $e_2$. 
To prepare the computation of \( H(x; e_1, e_2) \) in Eq. (3.10) and Eq. (3.11), we need some definitions. We shall denote by
\[
\det_{y_1,y_2,y_3} := y_1^2 y_2^2 y_3^2 - y_1^2 (y_2 y_3)^2 - y_2^2 (y_1 y_3)^2 - y_3^2 (y_1 y_2)^2 + 2(y_1 y_2)(y_1 y_3)(y_2 y_3)
\] (3.12)
the Gram determinant of vectors \( y_1, y_2, y_3 \). For \( i, j, k \in \{1, 2, 3\} \) pairwise distinct, the cofactors of \( y_i^2 \) are the \( 2 \times 2 \) Gram determinants \( \det_{y_j,y_k} \), and we shall denote the cofactors of \( (y_i y_j) \) by
\[
\Lambda_k = (y_i y_k)(y_j y_k) - y_k^2 (y_i y_j).
\] (3.13)
We shall need a few trivial facts, proven by elementary computation.

**Lemma 3.2.** It holds
\[
\partial_{y_i} \det_{y_1,y_2,y_3} = 2 \begin{pmatrix} \Lambda_3 & \Lambda_2 & \Lambda_1 \\ \Lambda_3 & \det_{y_1,y_3} & \det_{y_1,y_2} \\ \det_{y_1,y_3} & \det_{y_1,y_2} & \Lambda_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.
\] (3.14)
If \( G \) is the Gram matrix and \( L \) the matrix in Eq. (3.14), then \( GL = LG = \det_{y_1,y_2,y_3} \), i.e., if \( \det_{y_1,y_2,y_3} \neq 0 \), then \( \det^{-1}_{y_1,y_2,y_3} L = G^{-1} \).

**Lemma 3.3.** For \( i, j, k \in \{1, 2, 3\} \) pairwise distinct, it holds
\[
y_i^2 \det_{y_i,y_j,y_k} = \det_{y_i,y_k} \det_{y_i,y_j} - \Lambda_i^2.
\] (3.15)
Notice that with Lorentzian metric the vanishing of a Gram determinant does not require the linear dependence of the vectors, see, however Lemma 4.2 and Lemma 4.3.

The inverse of the Gram determinant \( \det_{x,e_1,e_2} \) will play a major role. It is understood as the distributional boundary value from the forward tube \( x - i \varepsilon u \). Because \((e_i u) = 0 \), this simply means that \( x^2 \) is understood as \((x - i \varepsilon u)^2 \) while all other scalar products are real. For properties of Gram determinants in Lorentzian metric, see Sect. 4.

It is convenient to define \( \gamma = \angle(e_1, e_2) \), so that \( \sqrt{e_1^2 e_2^2} \cos \gamma = -(e_1 e_2) \) and \( \det_{e_1,e_2} = e_1^2 e_2^2 \sin^2 \gamma \). One trivially has

**Lemma 3.4.** The distribution \( f(x,e_2) \) in Lemma 3.1 with \( x \) substituted by \( e_1 \) equals
\[
\sqrt{e_1^2 e_2^2} \cdot f(e_1,e_2) = \frac{\gamma}{\sin \gamma}.
\] (3.16)
The singularity at \( \gamma = \pi \) is integrable w.r.t. the invariant measure on \( S^2 \times S^2 \).

**Remark 3.5.** The singularity is not integrable along one-dimensional submanifolds of \( S^2 \). Thus, strings must not be further restricted than \( e^2 = -1 \) (which is trivial by homogeneity) and \( e^0 = 0 \).

Next, by using Lemma 3.3 with \( y_1 = e_1, y_2 = e_2, y_3 = x, \) and \( i = 1 \) and \( i = 2 \), respectively, one can define the homogeneous functions \( \zeta_1(x,e_1,e_2) \) and \( \zeta_2(x,e_1,e_2) \) by
\[
\pm e^{\pm \zeta_1} = \frac{\Lambda_1 \pm \sqrt{\det_{e_1,e_2} \det_{x,e_1}}}{\sqrt{e_1^2 \det_{x,e_1,e_2}}}, \quad \pm e^{\pm \zeta_2} = \frac{\Lambda_2 \pm \sqrt{\det_{e_1,e_2} \det_{x,e_2}}}{\sqrt{e_2^2 \det_{x,e_1,e_2}}}.
\] (3.17)
When \( e_1, e_2 \) and \( x \) are purely spatial, the geometry is Euclidean. Then all diagonal cofactors \( (2 \times 2 \) Gram determinants) are \( \geq 0 \) and \( e_1^2 \det_{x,e_1,e_2} \geq 0 \). In this case, \( \zeta_1 \) and \( \zeta_2 \) are real. The same is true when \( x^\perp \), the component of \( x \) perpendicular to the plane spanned by \( e_1 \) and \( e_2 \), is spacelike, because such \( x \) can be boosted to become purely spatial while \( e_1 \) and \( e_2 \) are preserved.

We can now state the result.
**Proposition 3.6.** Let $e_i$ ($i = 1, 2$) be purely spatial and linearly independent ($\det e_1, e_2 \neq 0$). Denote by $D = \frac{x(y-x^1 e_1 e_2)}{x^1 e_1 y^1 e_2}$ the normalized Gram determinant. Then the distribution $H$ in Eq. (3.10) and Eq. (3.11) is

$$H(x; e_1, e_2) = -\frac{\cos \gamma}{2 \sin \gamma} \left[ \gamma \log \left( \frac{\sin^4 \gamma}{D} \right) + \pi(\zeta_1 + \zeta_2) - \frac{i}{2} \left\{ \text{Li}_2(e^{i\gamma} e^{\zeta_1} e^{\zeta_2}) + (e^{\zeta_1} \leftrightarrow -e^{-\zeta_1}) \right\} \right]. \quad (3.18)$$

The limit $\gamma \to 0$ ($e_1$ and $e_2$ parallel) is regular, while the limit $\gamma \to \pi$ ($e_1$ and $e_2$ antiparallel) is singular, but integrable w.r.t. the invariant measure on $S^2 \times S^2$.

Via Eq. (3.11), this formula determines the regularized double string-integrated distribution $(I_{e_2} I_{e_1} W_0)_{e_1, e_2}(x)$ on $\mathbb{R}^4 \times S^2 \times S^2$, up to the unknown additive dependence on the regulator function $v$ via $\mu_v(e_1, e_2)$.

Its relevance for the intended applications to QED will be discussed at the end of Sect. 3.3.

**Sketch of proof:** For Eq. (3.10) we need to compute the integral over $f(x + s e_2, e_1)$ in Eq. (3.8). We begin with $x^1$ (the component of $x$ perpendicular to $e_1$ and $e_2$) spacelike. Then there is a boost preserving $e_1$ and $e_2$, such that $(uA\nu) = 0$. Thus, we may without loss of generality assume that also $x$ is purely spatial, and $\zeta_1$ and $\zeta_2$ are real.

The clue to compute Eq. (3.8) analytically is the change of integration variable

$$C(s) = e^{\arcsinh(s \Gamma_1 + \Gamma_2)} = s \Gamma_1 + \Gamma_2 + \sqrt{(s \Gamma_1 + \Gamma_2)^2 + 1}, \quad (3.19)$$

where

$$\Gamma_1 := \frac{\det e_1, e_2}{\sqrt{e_1^1} \det e_1 e_2}, \quad \Gamma_2 := -\frac{\Lambda_1}{\sqrt{e_1^1} \det e_1 e_2} \quad (3.20)$$

with $\Lambda_1$ defined as in Eq. (3.13) for $y_1 = e_1, y_2 = e_2, y_3 = x$. Then $C(0) = e^{-\zeta_1}$ and $C(a) = \log(2\Lambda_1) + O(\frac{1}{\Lambda_1})$ for large $a$, and one finds

$$f(x + s e_2, e_1) = \frac{\Gamma_1}{\sqrt{\det e_1, e_2}} \frac{\log \frac{e^{\frac{\zeta_2}{2\Lambda_1}}(1-e^{-\gamma} e^{\frac{\zeta_2}{2}} C(s))(1+e^{-\gamma} e^{\frac{\zeta_2}{2}} C(s)^{-1})}{e^{\gamma} e^{\zeta_2} (1-e^{-\gamma} e^{\zeta_2} C(s))(1+e^{\gamma} e^{\zeta_2} C(s)^{-1})}}{i(C(s) + C(s)^{-1})}. \quad (3.21)$$

Because $\zeta_i$ are real and $\Gamma_1$ is positive, one can cancel the factor $\frac{e^{\frac{\zeta_2}{2\Lambda_1}}}{\sqrt{\det e_1, e_2}}$ in the difference of logarithms without risk of changing the branches, see footnote [1].

The integral is solved by

$$2\Gamma_1 \log \frac{(1-e^{-\gamma} e^{\zeta_2} C(s))(1+e^{-\gamma} e^{\zeta_2} C(s)^{-1})}{C(s) + C(s)^{-1}} = \frac{d}{ds} \left[ \text{Li}_2(-e^{-i\gamma} e^{\zeta_2} C(s)) - \text{Li}_2(e^{i\gamma} e^{\zeta_2} C(s)) \right], \quad (3.22)$$

and likewise for $\gamma \to -\gamma$. By working out the values at $s = 0$ and $s = a$, and writing the result symmetrically in $e_1 \leftrightarrow e_2$, one arrives at Eq. (3.18).

The behaviour at the branch cuts of the dilogarithm function, determined by the imaginary part of $x^0$, can be worked out explicitly in both limits $\gamma \to 0$ and $\gamma \to \pi$. The singularity at $\gamma = \pi$ is like $\frac{O(\log(\pi-\gamma))}{\pi-\gamma}$ and hence integrable.

When $x$ is not spacelike, the variables and functions are defined by analytic continuation in $x - i\varepsilon u$ in the forward tube, and the distribution as the boundary value $\varepsilon \downarrow 0$. The claims follow by analytic continuation in $x^0 - i\varepsilon u$ in the forward tube.

The regular behaviour at $\gamma = 0$ ($e_1 = e_2$) is expected because $(I_e F)(x) = \int_0^\infty s \, ds \, F(x + se)$ is well-defined when regularized as in Eq. (3.7). The singular behaviour at $\gamma = \pi$ ($e_1 = -e_2$) is expected because $I_{-e} I_e$ is never defined.
Corollary 3.7. Let \( \tilde{f}(e,e') := f(e,-e') \) and \( \tilde{H}(x-x';e,e') := H(x-x';e,-e') \), which are symmetric under \( e \leftrightarrow -e' \). The regularized two-point function of the escort field

\[
(2\pi)^2 \langle \phi(x,c)\phi(x',c') \rangle_v = \frac{(e'e')}{2} \tilde{f}(e,e') \log \left( -\mu^2_v \cdot (x-x'-i\varepsilon u)^2 \right) + \tilde{H}(x-x';e,e')
\]  

(3.23)

is a distribution on \( \mathbb{R}^4 \times S^2 \times S^2 \setminus \{(x,e,e') : e = e'\} \).

When smeared with the constant function \( c_0(e) = \frac{1}{4\pi} \), Eq. (3.23) simplifies drastically. The averaging with \( c_0 \) can be done already in the momentum space representation Eq. (3.7):

Lemma 3.8. (see [12]) For purely spatial \( e \), \( u^0 \) the standard timelike unit vector as above, and \( k^2 = 0 \) one has

\[
\int_{S^2} d\sigma(\vec{c}) \ c_0(\vec{c}) \frac{e}{(ek)_\pm} = \frac{u_0}{(u_0 k)^2} - \frac{k}{(u_0 k)^2}.
\]  

(3.24)

Because \( (\frac{u_0}{(u_0 k)^2} - \frac{k}{(u_0 k)^2})^2 = -\frac{1}{(u_0 k)^2} \), which is the denominator for two string-integrations in the direction \( u_0 \), one obtains the desired result by computing \( (I_{u_0}I_{u_0}W_0)_v \) along the same lines as \( (I_{v_1}I_{v_1}W_0)_v \) before. The result is

Lemma 3.9. For \( c_0(\vec{c}) = \frac{1}{4\pi} \), one has

\[
(2\pi)^2 \langle \phi(x,c_0)\phi(x',c_0) \rangle_v = -\frac{1}{2} \log \left( -\mu^2_v \cdot ((x-x')^2)_- \right) + \tilde{H}(x-x';c_0,0),
\]  

(3.25)

with \( \tilde{H}(x;c_0,0) = \frac{x^0}{2r} \log \left( \frac{x^0 - i\varepsilon + r}{(x^0 - i\varepsilon) - r} \right) \) (\( r = |\vec{x}| \)).

Sketch of proof: By a direct computation of the cut-off integral Eq. (3.10), which for two parallel strings becomes an elementary integral. The claim for the regularized integral follows by the argument preceding Eq. (3.10), which leaves the constant \( \tilde{\mu}_v \) unspecified.

3.3 Vertex operator correlations and commutation relations

We briefly sketch the definition of operators \( :e^{i\phi(g\otimes c)}:_v \) (smeared in both \( x \) and \( e \)) through a massless limit [12]:

\[
:e^{i\phi(g\otimes c)}:_v := \lim_{m \to 0} e^{-\frac{\tilde{\theta}(0)^2}{2}d_{m,v}(e,c)} \cdot :e^{i\phi_m(g\otimes c)}: \equiv \frac{e^{i\phi(g\otimes c)}:}{e^{-w_v(g\otimes c,g\otimes c)}},
\]  

(3.27)

where \( \phi_m \) is the IR-regular massive escort field whose two-point function diverges in the limit \( m \to 0 \), and

\[
d_{m,v}(e,e') = -(ee') \int \frac{d\mu_m(k)v(k)}{((ke') - i\varepsilon)((ke') + i\varepsilon)}.
\]  

(3.28)

smeared over \( e \) and \( e' \) is the divergent part. The second writing in Eq. (3.27) refers to a normal-ordering prescription w.r.t. the non-positive two-point function \( w_v \) given by Eq. (3.23). However, correlations of Eq. (3.27) evaluated in the limit of massive vacuum states define a positive functional. The correlations can be worked out by the Weyl formula. The crucial feature is that they contain the IR divergent parts in the combination

\[
e^{-\frac{1}{2}d_{m,v}(C,C)}
\]  

(3.29)
where \( C(e) = \sum_i \tilde{g}_i(0)c_i(e) \). Because \( d_{\omega',\omega}(C, C) \) diverges to \(+\infty\) unless \( C = 0 \), one obtains in the limit a Kronecker delta \( \delta_{C,0} \). This defines an uncountable superselection rule: the GNS Hilbert space splits into an uncountable direct sum of subspaces \( \mathcal{H}_C \) which carry inequivalent representations of the Weyl subalgebra generated by \( e^{i\phi(g \otimes c)} \) with \( \tilde{g}(0) = 0 \).

Therefore, correlation functions of \( e^{i\phi(g \otimes c)} \) are given by choosing \( g \) (see [12]). The correlation functions of vertex operators are given by

\[
\langle e^{i\phi(g \otimes c_1)} \cdot \cdots \cdot e^{i\phi(g_n \otimes c_n)} \rangle = \delta_{C,0} \cdot \prod_{i<j} e^{-w_v(g_i \otimes c_i, g_j \otimes c_j)}. \tag{3.30}
\]

When \( w_v \) given by Eq. (3.23) is inserted, one may collect the contributions from \( \log \mu_v^2(e, -e') \) in the exponent as

\[
\sum_{i<j} \frac{\tilde{g}_i(0)\tilde{g}_j(0)}{8\pi^2} \lambda_v(c_i, c_j) \tag{3.31}
\]

with the symmetric bilinear form

\[
\lambda_v(c, c') := \int d\sigma(c) c(c') \int d\sigma'(c') c'(c') \frac{\pi - \theta(c', c')}{\tan \theta(c, c')} \log \mu_v(e, -e'), \tag{3.32}
\]

where \( \theta(c', c') = \angle(c, c') \). Because \( \sum_i \tilde{g}_i(0)c_i = 0 \), the exponential of the sum factorizes:

\[
\exp \left( \sum_{i<j} \frac{\tilde{g}_i(0)\tilde{g}_j(0)}{8\pi^2} \lambda_v(c_i, c_j) \right) = \prod_i e^{-\frac{\tilde{g}_i(0)^2}{16\pi^2} \lambda_v(c_i, c_i)} \tag{3.33}
\]

and one may conveniently absorb the factors as \( c \)-dependent normalizations of the fields.

Vertex operators are defined by choosing \( g(y) = q\delta_y(x) \), hence \( \tilde{g}(0) = q \):

\[
V_{q\epsilon}(x) := N_{v\epsilon} \cdot :e^{i\phi(x,c)}:_v. \tag{3.34}
\]

**Corollary 3.10.** (see [13]) The correlation functions of vertex operators are

\[
\langle V_{q_1c_1}(x_1) \cdots V_{q_nc_n}(x_n) \rangle = \delta_{\sum_q c_i,0} \cdot \prod_{i<j} \left( \frac{-1}{(x_i - x_j)^2} \right)^{\frac{q_i q_j}{8\pi^2} (c_i, c_j)} e^{-\frac{q_i q_j}{16\pi^2} H(x_i - x_j; c_i, c_j)}, \tag{3.35}
\]

where

\[
\langle c, c' \rangle := \int d\sigma(c) c(c') \int d\sigma'(c') c'(c') \frac{\pi - \theta(c', c')}{\tan \theta(c, c')}. \tag{3.36}
\]

**Proof:** The statement follows by combining the definition of Eq. (3.34) with Eq. (3.30), where \( w_v \) is specified by Eq. (3.23) and Prop. 3.6. The formula Eq. (3.36) uses Eq. (3.16).

We conclude this section with some miscellaneous results about correlations of vertex operators, with only indications of proofs.

### 3.3.1 Commutation relations

**Proposition 3.11.** The vertex operators satisfy anyonic commutation relations

\[
V_{q\epsilon}(x)V_{q'\epsilon'}(x') = e^{iq\lambda'(x-x';c,c')} \cdot V_{q'\epsilon}(x')V_{q\epsilon}(x), \tag{3.37}
\]

where \( \beta(x - x'; c, c') \) arises by smearing with \( c(e) \) and \( c'(e') \) the escort commutator function

\[
\beta(x - x'; e, e') = i[\phi(x, e), \phi(x', e')] = -(ee')(I_{-e'}I_{0}) (x - x'). \tag{3.38}
\]

**Proof:** The claim follows from the fact that vertex operators are defined as limits of multiples of Weyl operators, Eq. (3.27) and Eq. (3.34). The escort commutator function does not suffer from the IR divergence because the Fourier transform of \( C_0 \) vanishes at \( k = 0 \).
For \( x^0 = x'^0 \), Eq. (3.38) vanishes because the equal-time commutator vanishes and \( e, e' \) are purely spatial. Otherwise it is a rather simple geometric quantity in terms of the intersection of the null-cone with the planar wedge \( x - x' + \mathbb{R}e - \mathbb{R}e' \), which is a subset of a circle. It can be written in a symmetric form, by writing \( e_1 = e \) and \( e_2 = -e' \) as before, so that \( \gamma = \angle(e_1, e_2) \):

**Lemma 3.12.** (see [13]) Denote by \( A \) the total arc-length of the intersection of the null-cone with the planar wedge \( x + \mathbb{R}e_1 + \mathbb{R}e_2 \) of opening angle \( \gamma \). Then

\[
\beta(x; e_1, e_2) = -\text{sign}(x^0) \frac{A}{4\pi \tan \gamma}.
\] (3.39)

**Proof:** Perform the double string-integral in Eq. (3.38) in Euclidean polar coordinates \((r, \varphi)\) of the plane \( x + \mathbb{R}e_1 + \mathbb{R}e_2 \). The change of coordinates contributes the Jacobi determinant \( \frac{1}{\sin \gamma} \). Using Eq. (3.19), the relevant integral can be written as

\[
\int r\, dr\, d\varphi\, \chi_W(r, \varphi)\delta(r^2 - x^{12}) = \frac{1}{2} A,
\] (3.40)

where \( \chi_W \) is the characteristic function of the wedge, and \( x^1 \) is the component of \( x \) perpendicular to the plane. Finally, \((e_1, e_2) = -\cos \gamma \). Collecting all factors, yields Eq. (3.39). \( \square \)

The intersection of the wedge with the null-cone is empty if the two strings are spacelike separated, in particular when \( x^{12} < 0 \). In this case \( A = 0 \) and the vertex operators commute.

### 3.3.2 Spectrum

One would like to know the Fourier transform of the two-point function of vertex operators because its \( c \)-dependent energy-momentum distribution supported in the interior of \( V^+ \) reveals the energy-momentum spectrum of the state created by the vertex operator [12]. It has to be added to the mass-shell energy-momentum of the free Dirac particle. The sum is the spectrum of the infraparticle state generated by the dressed Dirac field Eq. (1.4).

We do not know how to compute this Fourier transform for general \( c \), other than by working out an exponential series of convolution products of Eq. (3.23) (which is impractical). However, the case of the constant smearing function \( c(e) = c_0 = \frac{1}{2\pi} \), for which the two-point function drastically simplifies, see Lemma 3.9 allows to quantify the ensuing dissolution of the mass-shell in a special case.

By Eq. (3.25), the two-point function of the vertex operator \( V_{qc_0} \) is

\[
\langle V_{qc_0}^*(x_1)V_{qc_0}(x_2) \rangle = \left[ \frac{x^0}{x^{02} + i\varepsilon x^0} \right]^{\frac{1}{2\pi}} \alpha \left[ \frac{\alpha}{(x^2 - i\varepsilon x^0)} \right]^{\frac{1}{2\pi}}, \quad (x = x_1 - x_2, r = |\vec{x}|).
\] (3.41)

Here, \( \alpha := \frac{\gamma}{2\pi} \), which in the application to QED [12] is the fine structure constant.

Eq. (3.41) equals \((ix^0 + \varepsilon)^{-\frac{1}{2}} \) multiplied with a power series in \( \frac{\alpha^2}{4\pi} \). This structure allows to extract quantitative details of the Fourier transform, and hence of the rotationally invariant energy-momentum distribution \( \rho(\omega, \vec{k}) \) in the state created by \( V_{qc_0} \) [14]. By putting \( r = 0 \), one concludes that the distribution \( \rho(\omega) = \int d^3 k \rho(\omega, \vec{k}) \) of energies decays like \( \omega^{\frac{3}{2} - 1} \). By applying powers of the Laplacian before putting \( r = 0 \), one can compute averages of powers of \(|\vec{k}|^2\) at fixed energy \( \omega \). E.g., the average of the invariant masses \( \omega^2 - |\vec{k}|^2 \) at given energy \( \omega \) is found to be \( \frac{2}{\pi} \cdot \omega^2 + O(\alpha^2) \) with variance \( \frac{4}{9} \frac{2}{\pi} \cdot \omega^4 + O(\alpha^2) \). These data are roughly compatible with an inverse-power-law distribution \( \sim (\omega^2 - |\vec{k}|^2)^{1-\frac{3}{2}} \) near the mass-shell \(|\vec{k}| = \omega^2 \).
3.3.3 Scattering

Vertex operator correlation functions Eq. (3.35) involve exponentials of the distribution \( \tilde{H} \) (smearred in the strings), and the same is true for correlations of the dressed Dirac field Eq. (1.4). Asymptotic properties of \( \tilde{H} \) at large times will become relevant in a future scattering theory. At this moment, a scattering theory for infrafields like the dressed Dirac field has not yet been formulated. The LSZ method fails because of the absence of a sharp mass-shell, and for the same reason, “asymptotic creation operators” needed in the Haag-Ruelle theory have not been found. Apart from the lack of the mass-shell, a major obstacle is the product structure of vertex operator correlations that is very different from that of free correlations. It probably entails that “two-infraparticle states” are not tensor products as in the Fock space.

A toy model without a mass gap would be a scattering theory for vertex operators without the Dirac field. A candidate would be a version of Buchholz’ scattering theory for massless waves that is successful in the two-dimensional vertex operator model (no \( \tilde{H} \)-terms) and yields an S-matrix that is a complex phase. If the same method is naively applied in the case at hand, the \( \tilde{H} \)-terms in the correlations Eq. (3.35) can produce a result of modulus > 1, which jeopardizes the direct interpretation of the relevant asymptotic limit as a scattering amplitude. Nevertheless, the computation illustrates how features of \( \tilde{H} \) will play a role in scattering theory.

The would-be scattering amplitude is a limit of four-point correlations relative to two-point correlations, when the positions go to infinity like \( x \pm t \ell, x' \pm t \ell' \) in future and past lightlike directions. Let \( \ell, \ell' \) be two non-parallel future directed lightlike vectors. Consider the quantity

\[
\lim_{t \to \infty} \frac{\langle V_c(f_{\ell t})^* V_c(f_{\ell' t})^* V_c(f_{\ell-t} V_c(f_{\ell-t} t) \rangle}{\langle V_c(f_{\ell t})^* V_c(f_{\ell' t}) \rangle \langle V_c(f_{\ell' t})^* V_c(f_{\ell-t}) \rangle} \tag{3.42}
\]

with smearing functions \( f_{\ell t}(x) = f(x-\ell t) \) etc. shifted in lightlike directions. To simplify matters, we absorb the charge factor \( q \) into the string smearing functions \( c \) which therefore have arbitrary total weight \( q \).

**Lemma 3.13.** Let \( \ell, \ell' \) be future directed lightlike vectors, and \( c, c' \in C^\infty(S^2) \). Then in the limit \( t \to \infty \), Eq. (3.42) converges to

\[
e^{-i\frac{\langle c, c' \rangle}{8\pi^2}} \cdot e^{\frac{1}{4\pi^2} \left( \tilde{H}(\ell + \ell'; c, c') + \tilde{H}(\ell + \ell'; c, c') - \tilde{H}(\ell - \ell'; c, c') - \tilde{H}(\ell - \ell'; c, c') \right)}, \tag{3.43}
\]

The second factor is, in general, not a complex phase. Replacing \( c \) by \( -c \), if necessary, its modulus may be > 1.

**Sketch of proof:** By the structure Eq. (3.35) of vertex operator correlations, the four-point function is a product of six factors. One can convince oneself that in the limit the smearing functions in \( x \) can be neglected, so that two of the six factors in the numerator cancel against the denominator. This essentially leaves the four factors

\[
e^{-\frac{1}{4\pi^2} \tilde{H}(x + t\ell - x' - t\ell', c, c')}, \quad e^{-\frac{1}{4\pi^2} \tilde{H}(x' - t\ell' - x + t\ell', c, c')}
\]

\[
\frac{1}{e^{\frac{1}{4\pi^2} \tilde{H}(x + t\ell - x' + t\ell', c, c')}} \cdot \frac{1}{e^{\frac{1}{4\pi^2} \tilde{H}(x' + t\ell' - x + t\ell', c, c')}}
\]

\[
(\langle x + t\ell - x' - t\ell' - i\varepsilon u \rangle^2)^{\frac{\langle c, c' \rangle}{8\pi^2}}, \quad (\langle x' + t\ell' - x + t\ell - i\varepsilon u \rangle^2)^{\frac{\langle c, c' \rangle}{8\pi^2}}, \tag{3.44}
\]

Because \( \ell \pm \ell' \) are timelike resp. spacelike, the denominators in Eq. (3.44) are dominated by powers of \( -t^2((\ell \pm \ell') \cdot u) = 2t^2 \mp \varepsilon ((\ell \pm \ell') \cdot u) \). Since \((\ell \ell') > 0\), the spacelike cases in the first line give \([2t^2(\ell \ell')(1 + \varepsilon)]^{\frac{\langle c, c' \rangle}{8\pi^2}}\) each. The timelike cases in the second line give \([2t^2(\ell \ell')(1 + \varepsilon)]^{\frac{\langle c, c' \rangle}{8\pi^2}}\)
each. Together, they yield the first factor in Eq. (3.43). Because $H$ is a homogeneous function, the numerators in Eq. (3.44) yield the second factor in Eq. (3.43). Now, one may regard $L_\pm = \ell \pm \ell'$ as a pair of orthogonal vectors, one timelike and one spacelike, and otherwise independent. For $L_-$ purely spacelike, $H(L_-; c', c) + H(L_-; c, c')$ is a real function with a nontrivial dependence on $L_-$. It cannot be cancelled by the real part of $\tilde{H}(L_+; c, c') + \tilde{H}(L_+; c', c)$.

The conclusion is only avoided if all strings are orthogonal to both $\ell$ and $\ell'$, in which case $H(\ell \pm \ell'; e_1, e_2)$ are independent of $\ell, \ell'$. Namely, in this case $\zeta(\ell \pm \ell'; e_1, e_2)$ are independent of $\ell, \ell'$. But by Rem. 3.5 this would require a smearing in the intersection of the sphere $S^2$ with a plane (i.e., a circle $S^1$), which would in turn render, e.g., the exponents Eq. (3.36) ill-defined.

### 3.4 Derivative formula

The derivative of Eq. (3.11) w.r.t. $x$ is surprisingly simple to compute.

**Lemma 3.14.** Let $F(x) = \frac{1}{x^2}$. For $x^2, e_1^2, det_{x,e_1}, det_{x,e_2}$ and $det_{x,e_1,e_2}$ all non-zero, it holds

$$P := \langle I_{e_1}, I_{e_2}, \partial F \rangle(x) = \frac{1}{2} \left[ f(e_1, e_2) \partial_x + f(x, e_2) \partial_{e_1} + f(x, e_1) \partial_{e_2} \right] \log \det_{x,e_1,e_2}. \quad (3.45)$$

**Proof:** By definition as a convergent integral over the derivative of $\frac{1}{(x + \sum_{i=0}^{2} e_i)^2}$, $P$ is a linear combination $b_1 e_1 + b_2 e_2 + b_3 x$. From Eq. (2.12) we know that $(e_i P) = f(x, e_j)$ for $i,j \in \{1, 2\}$ pairwise distinct. From Eq. (3.11) we know that $(x P) = f(e_1, e_2)$. Then

$$\begin{pmatrix} (e_1 P) \\ (e_2 P) \\ (x P) \end{pmatrix} = \begin{pmatrix} f(x, e_2) \\ f(x, e_1) \\ f(e_1, e_2) \end{pmatrix} = G \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (3.46)$$

where $G$ is the Gram matrix, is solved for the coefficient functions $b_i$ by using Lemma 3.2. □

A remarkable feature of Eq. (3.45) is its formal symmetry in the three vector variables, despite the unsymmetric definition. The symmetry can be understood by a change of variables $(s_1, s_2) \rightarrow (\frac{1}{s_1}, \frac{s_2}{s_1})$ in the defining double string-integral.

Given Lemma 3.14, to specify $\langle I_{e_2} I_{e_1} \partial W_0 \rangle(x)$ for $e_1, e_2 \in H_1$ purely spatial, it suffices to specify the $i\varepsilon$-prescriptions at the possibly singular configurations. For $e_1, e_2 \in H_1$ purely spatial and $x$ replaced by $x - i\varepsilon u$, the distributions $f(x, e_i)$ and $f(e_1, e_2)$ are well-defined. It remains to consider the reciprocal of $det_{x,e_1,e_2}$ in Eq. (3.45) when $det_{x,e_1,e_2} = 0$. Because $i\varepsilon u$ appears only in the term $(x - i\varepsilon u)^2 det_{x,e_1,e_2}$, the distribution is well-defined unless $det_{x,e_1,e_2} = 0$, i.e., when $e_1 \neq \pm e_2$. The case $e_1 = e_2$ is regular by Prop. 3.6. Thus we conclude

**Proposition 3.15.**

$$-(2\pi)^2 \langle I_{e_2} I_{e_1} \partial W_0 \rangle(x) = \frac{1}{2} \left[ f(e_1, e_2) \partial_x + f(x, e_2) \partial_{e_1} + f(x, e_1) \partial_{e_2} \right] \log \det_{x,e_1,e_2}. \quad (3.47)$$

with the distributions $f(x, e_1)$ as specified in Sect. 3.1 and $f(e_1, e_2)$ given by Eq. (3.16), is well-defined as a distribution on $\mathbb{R}^3 \times \mathbb{R}^2 \setminus \{(x, e_1, e_2) : e_1 = -e_2\}$. The singularity at $e_1 = -e_2$ is integrable w.r.t. the invariant measure on $\mathbb{R}^3 \times \mathbb{R}^2$.

The result can also be obtained with a cumbersome computation of the derivative of $\langle I_{e_2} I_{e_1} \partial W_0 \rangle(x)$ given by Eq. (3.11) and Eq. (3.15). This may be taken as a nontrivial check of Eq. (3.15).
Propagator

While the IR-regularized \textit{two-point function} of the escort field is needed for the non-perturbative construction of the dressed Dirac field in [12], \textit{propagators} are needed in perturbation theory. Defining QED as a perturbative expansion with interaction density \( q A_\mu(e) j^\mu \) in the Wigner space, where the two-point function of \( A_\mu(e) \) is

\[
\langle A_\mu(x, e) A_\nu(x', e') \rangle = -\left[ \eta_{\mu\nu} + e_\nu \Gamma_\mu^\alpha \partial_\alpha + e'_\mu \Gamma_\nu^\beta \partial_\beta + (ee') \Gamma_\mu^\alpha \Gamma_\nu^\beta \partial_\alpha \partial_\beta \right] W_0(x - x'),
\]

(4.1)
one needs only string-integrations over \textit{derivatives} of the massless two-point function and propagator. These are IR-regular. We therefore focus on the analog of Eq. (3.47) for the propagator.

At first sight, the case of the propagator should be very parallel to that of the two-point function except for a different \( i\varepsilon \)-prescription: \( x^2 - i\varepsilon \) rather than \( (x - i\varepsilon u)^2 \). Instead of an “analytic continuation through the forward tube”, one needs an analytic continuation in the variable \( x^2 \).

For the single string-integral of the propagator, it suffices to define the derivative of \( f(x,e) \) by simply substituting \( (x - i\varepsilon u)^2 \) in Eq. (3.42) and Eq. (3.43) by \( x^2 - i\varepsilon \).

For the double string-integral over the derivative, however, this prescription is not sufficient because the reciprocal of \( \det_{x,e_1,e_2} \) becomes

\[
\frac{1}{\det_{x,e_1,e_2} - i\varepsilon \det_{x,e_1,e_2}}. \quad (4.2)
\]

In the Wigner space approach to QED, the strings are not restricted to be purely spatial [12]. But in Lorentzian metric, if \( e_i \in H_1 \) are not purely spatial, \( \det_{x,e_1,e_2} \) may vanish in a larger submanifold of configurations, see Lemma 4.3. For such \( e_1, e_2 \), Eq. (4.2) does not define a distribution in \( x \) at \( \det_{x,e_1,e_2} = 0 \). The following Lemma 4.1, which is a corollary to Lemma 3.3 allows to reduce the set of configurations where Eq. (4.2) is not defined.

Lemma 4.1. Suppose that \( \det_{y_1,y_2,y_3} = 0 \). Then, for \( i, j, k \in \{1, 2, 3\} \) pairwise distinct, hold:
(i) All “diagonal” cofactors \( \det_{y_i,y_j} \) have the same sign, or vanish.
(ii) If \( \det_{y_i,y_j} = 0 \), then also the cofactors \( \Lambda_i \) and \( \Lambda_j \) vanish.

Proof: (i) By Lemma 3.3 if \( \det_{y_1,y_2,y_3} = 0 \), then \( \det_{y_i,y_j} \det_{y_j,y_k} = \Lambda_i^2 \geq 0 \). Thus \( \det_{y_i,y_j} \) and \( \det_{y_j,y_k} \) cannot be non-zero with opposite sign.
(ii) is obvious from Lemma 3.3

Because by Lemma 4.1(i) the three two-variable Gram determinants cannot have opposite signs when \( \det_{x,e_1,e_2} = 0 \), one may define the reciprocal of \( \det_{x,e_1,e_2} \) as a pullback of a boundary value of an analytic function in the variables \( x^2 - i\varepsilon, e_1^2 - i\varepsilon, \) and \( e_2^2 - i\varepsilon \), i.e.,

\[
\frac{1}{\det_{x,e_1,e_2} - i\varepsilon (\det_{x,e_1,e_2} + \det_{x,e_1} + \det_{x,e_2})}. \quad (4.3)
\]

This prescription, applied to Eq. (3.45), defines the double string-integral over the derivative of the propagator as a distribution everywhere on \( \mathbb{R} \times H_1 \times H_1 \), except on configurations where all four Gram determinants vanish simultaneously. When this happens, then by Lemma 3.3 (or Lemma 4.1(ii)), also the non-diagonal cofactors \( \Lambda_i \) of the Gram determinant vanish.

The following Lemmas allow to characterize configurations with vanishing Gram determinants in Lorentzian metric, and in particular the configurations where Eq. (4.2) is not defined.

Lemma 4.2. Suppose that \( \det_{y_1,y_2} \neq 0 \). Then \( \det_{y_1,y_2,y_3} = 0 \) if and only if there exist \( \alpha, \beta \in \mathbb{R} \) and \( \ell \) lightlike \((\ell^2 = 0)\) such that

\[
y_3 = \alpha y_1 + \beta y_2 + \ell \quad \text{and} \quad (\ell y_1) = (\ell y_2) = 0. \quad (4.4)
\]
At this moment, we do not know how to naturally extend the propagator to these sets. See, for some \( \alpha \) if and only if either \( \ell := y_3 - \alpha y_1 - \beta y_2 \), an elementary computation gives
\[
\det_{y_1,y_2,y_3} = \det_{y_1,y_2,\ell} = \ell^2 \det_{y_1,y_2}.
\] (4.6)
Hence \( \det_{y_1,y_2,y_3} = 0 \) if and only if \( \ell^2 = 0 \).

Analogous statements with analogous proofs hold for \( n \times n \) Gram determinants (which are trivially zero in 4 dimensions for \( n > 4 \)). The version for \( n = 2 \) is

**Lemma 4.3.** Suppose that \( y_1^2 \neq 0 \). Then \( \det_{y_1,y_2} = 0 \) if and only if there exists \( \alpha \in \mathbb{R} \) and \( \ell \) lightlike such that
\[
y_2 = \alpha y_1 + \ell \quad \text{and} \quad (\ell y_1) = 0.
\] (4.7)

**Lemma 4.4.** Let \( y_1^2 \neq 0, \ y_2^2 \neq 0 \). If \( \det_{y_1,y_2,y_3} = \det_{y_1,y_2} = 0 \) and either \( \det_{y_1,y_3} = 0 \) or \( \det_{y_2,y_3} = 0 \), then the set \( \{y_1,y_2,y_3\} \) is linearly dependent.

**Proof:** Assume \( \det_{y_1,y_3} = 0 \). Define
\[
v := y_2 - \frac{(y_1 y_2)}{y_1^2} y_1, \quad w := y_3 - \frac{(y_2 y_3)}{y_2^2} y_2.
\]
One computes that \( v^2 = w^2 = 0 \) because \( \det_{y_1,y_2} = \det_{y_1,y_3} = 0 \), and \( (vw) = 0 \) because \( \Lambda_2 = 0 \) by Lemma 4.1(ii). Because two lightlike vectors that are orthogonal must be linearly dependent, it follows that \( v \) and \( w \), and consequently \( y_1, y_2, y_3 \) are linearly dependent. The case with \( \det_{y_2,y_3} = 0 \) is similar, replacing \( y_1 \leftrightarrow y_2 \).

The conclusion of Lemma 4.4 is not true if \( \det_{y_1,y_2} \neq 0 \) but the other two \( 2 \times 2 \) determinants are zero. It is easy to find counter examples.

We can now classify the configurations where the inverse Gram determinant is not defined by the prescription Eq. (4.3).

**Lemma 4.5.** For \( (x,e_1,e_2) \in \mathbb{R}^4 \times H_1 \times H_1 \) it holds \( \det_{x,e_1,e_2} = \det_{x,e_2} = \det_{e_1,e_2} = 0 \) if and only if either
\[
e_1 = \pm e_2, \quad x = \alpha e_1 + \ell
\] (4.8)
for some \( \alpha \in \mathbb{R} \) and lightlike \( \ell \) with \((e_1 \ell) = 0\), or
\[
e_2 = a' e_1 + \ell', \quad x = \alpha_1 e_1 + \alpha_2 e_2
\] (4.9)
for some \( \alpha', \alpha_1, \alpha_2 \in \mathbb{R} \) and lightlike \( \ell' \) with \((e_1 \ell') = (e_2 \ell') = 0\). In either case, the set \( \{x,e_1,e_2\} \) is linearly dependent.

**Proof:** The “If” statements are trivial by linear dependency. Conversely, \( \det_{e_1,e_2} = 0 \) implies \( e_2 = a e_1 + \ell' \) with \( \ell'^2 = 0 \) by Lemma 4.3. It then trivially follows that also \( (e_2 \ell') = 0 \). If \( \ell = 0 \), then \( a = \pm 1 \), and Lemma 4.3 applied to \( \det_{x,e_1} = 0 \) entails Eq. (4.8). If \( \ell' \neq 0 \), then \( e_1 \) and \( e_2 \) are not linearly dependent. Because the three vectors \( x, e_1, e_2 \) are linearly dependent by Lemma 4.4, \( x \) must be a linear combination of \( e_1 \) and \( e_2 \). This is Eq. (4.9).

The configurations of type Eq. (4.8) and Eq. (4.9) both have codimension 3 in \( \mathbb{R}^4 \times H_1 \times H_1 \). At this moment, we do not know how to naturally extend the propagator to these sets. See,
however, [10] where it was proven with microlocal methods that the string-localized propagator of $A_\mu(x,e)$, when smeared in the strings, can be defined on all of $\mathbb{R}^4$. This suggests that one can find coordinates in $\mathbb{R}^4 \times H_1 \times H_1$

with respect to which the singularities of the inverse determinant at configurations as in Lemma 4.5 are integrable, similarly to the integrability of $\frac{\alpha}{\sin \alpha}$ observed earlier. In [10], it is also shown that products of smeared string-localized propagators have natural extensions as distributions in $x$ in $\mathbb{R}^4 \setminus \{0\}$, exactly as in point-localized QFT.

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