ABELIAN ÉTALE COVERINGS OF MODULAR CURVES OVER LOCAL FIELDS

Abstract. We relate a part of the abelian étale fundamental group of curves over local fields to the component group of the Néron model of the jacobian. We apply the result to the modular curve $X_0(p)/\mathbb{Q}_p$ to show that the unramified abelian covering $X_1(p) \rightarrow X_0(p)$ (Shimura covering) uses up all the possible ramification over the special fiber of $X_0(p)$.

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1. Introduction

The geometric class field theory tries to classify the abelian étale coverings of a proper smooth variety $X$, and there is an arithmetic interest when the base field is a number field or a local field. There is a well-formulated class field theory for these curves over global/local fields (e.g. Kato-Saito [7], Saito [16]) but there are very few examples where the abelian étale coverings are explicitly classified. The most remarkable known example is the case of modular curve $X_0(p)$ over $\mathbb{Q}$ (Mazur [9], Introduction, Theorem (2)), which makes use of the deep theory of integral Hecke algebras. We give the corresponding result for $X_0(p)$ over $\mathbb{Q}_p$, by proving general results for the curves over local fields.

For a proper variety $X$ over a field $K$, let $\pi_1^{ab}(X)$ be the abelian étale fundamental group of $X$. There is a natural surjection $\pi_1^{ab}(X) \rightarrow G_K^{ab}$ where $G_K^{ab} = \text{Gal}(K_{ab}/K)$ is
the Galois group of the maximal abelian extension of $K$, and we denote the kernel by $\pi_1^{ab}(X)^{\text{geo}}$:  

$$0 \longrightarrow \pi_1^{ab}(X)^{\text{geo}} \longrightarrow \pi_1^{ab}(X) \longrightarrow G_K^{ab} \longrightarrow 0$$

For a proper smooth geometrically irreducible variety $X$ over a local field $K$, i.e. a complete discrete valuation field $K$ with finite residue field $F$ with $\text{char} F = p$, we showed in our previous paper [17] that $\pi_1^{ab}(X)^{\text{geo}}$ is an extension of $\mathbb{Z}^r$ by a finite torsion group $\pi_1^{ab}(X)^{\text{geo}}_{\text{tor}}$. Here the rank $r$ is the $F$-rank (the dimension of the maximal $F$-split subtorus) of the special fiber of the Néron model of the Albanese variety of $X$.

In this paper, we confine ourselves to the case of curves, and investigate the most mysterious part of $\pi_1^{ab}(X)^{\text{geo}}$, namely $\pi_1^{ab}(X)^{\text{geo}}_{\text{ram}}$ which classifies the abelian étale coverings of the generic fiber which are “completely ramified over the special fiber”, defined as follows. Let $X_F$ be the special fiber of the minimal regular model $\mathcal{X}$ over the integer ring $O_K$ (Abyhankar [1]). Then we have a natural map $\pi_1^{ab}(X)^{\text{geo}} \longrightarrow \pi_1^{ab}(X_F)^{\text{geo}}$ which is surjective if $X$ has a $K$-rational point, and denote the kernel by $\pi_1^{ab}(X)^{\text{geo}}_{\text{ram}}$:

$$0 \longrightarrow \pi_1^{ab}(X)^{\text{geo}}_{\text{ram}} \longrightarrow \pi_1^{ab}(X)^{\text{geo}} \longrightarrow \pi_1^{ab}(X_F)^{\text{geo}} \longrightarrow 0$$

Then our main result is:

**Theorem 1.1.** Assume that $X$ admits a $K$-rational point.

(i) (Theorem 5.2) The dual of the prime-to-$p$ part of the finite abelian group $\pi_1^{ab}(X)^{\text{geo}}_{\text{ram}}$ injects to the component group $\Phi$ of the Néron model of the jacobian variety of $X$.

(ii) (Theorem 4.1) Assume moreover that the absolute ramification index $e$ of $K$ is less than $p - 1$, $\text{char} K = 0$, and $X$ has semistable reduction over $O_K$. Then the $p$-primary part of $\pi_1^{ab}(X)^{\text{geo}}_{\text{ram}}$ vanishes.

Note that if $X_F$ is smooth, i.e. $X$ has good reduction, $\pi_1^{ab}(X)^{\text{geo}} \to \pi_1^{ab}(X_F)^{\text{geo}}$ is isomorphic on the prime-to-$p$ part by the proper smooth base change theorem on $H^1_{et}$. As $\pi_1^{ab}(X_F)^{\text{geo}}$ can be calculated in principle from the special fiber $X_F$, we have control on the whole of $\pi_1^{ab}(X)$ in the case $e < p - 1$, $\text{char} K = 0$ and $X$ has semistable reduction, in terms of the special fiber of the Néron model of the jacobian of $X$.

In the latter part of this paper, we give an application of these results to the modular curve $X_0(p)$, where $X_0(p)$ is the usual modular curve classifying the elliptic curves with $\Gamma_0(p)$-structures ($p$ is a prime). For the curve $X_0(p)/\mathbb{Q}_p$, above theorem enables us to compute the group $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}$ completely:

**Theorem 1.2.** $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}$ has the following structure:

$$0 \longrightarrow \Phi(J_0(p)) \longrightarrow \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}} \longrightarrow \mathbb{Z}^r \longrightarrow 0$$

where $\Phi(J_0(p)) \cong \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}} = \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{tor}}$ is a cyclic group of order equal to the numerator of $\frac{p-1}{24}$, and $r = \frac{g+1}{2}$ where $g$ is the genus of $X_0(p)$ and $h$ is the number of the supersingular points defined over $\mathbb{F}_p$. 
Acknowledgements. This paper was mostly written in 2001, as a part of the author’s master thesis at the University of Tokyo. The author would like to express his sincere gratitude to his thesis adviser K. Kato for suggesting the problem and his constant encouragement. He would like to thank K. Ban for helpful discussions, and B. Conrad for many comments on the first draft.

Notations. Throughout this paper, a local field $K$ means a complete discrete valuation field with finite residue field $F$, with char $F = p$. $O_K$ is the integer ring of $K$. For any field $K$, $\overline{K}$ is a separable closure of $K$, and $G_K = \text{Gal}(\overline{K}/K)$ is the absolute Galois group of $K$. For a variety $X$ over any field $K$, $\overline{X} = X \times_K \text{Spec}(\overline{K})$.

For an abelian group $X$, its Pontrjagin dual is denoted by $X^\vee = \text{Hom}(X, \mathbb{Q}/\mathbb{Z})$. $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$.

For a scheme $X$ over $O_K$, we denote the generic fiber and the special fiber of $X$ respectively by $X_K, X_F$. For any group scheme $X$, $X[m]$ denotes the kernel of the multiplication-by-$m$ map, which is also a group scheme. We use the notation $\mu_N = \mathbb{G}_m[N]$ for the group scheme of $N$-th roots of unity over an arbitrary base, without specifying the base scheme.

2. Preliminaries on abelian étale fundamental groups

Here we review some generalities on abelian étale fundamental groups, and fix the notations.

2.1. Geometric abelian étale fundamental groups. For any noetherian connected scheme $X$, the étale fundamental group $\pi_1(X)$ is a profinite group classifying finite étale coverings of $X$ ([SGA1]), and we denote the maximal abelian quotient by $\pi_1^{ab}(X)$. When $X$ is a proper variety over a field $K$, there is an exact sequence (ibid., Exposé IX, Th. 6.1):

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to G_K \to 1$$

where $\overline{K}$ is a separable closure of $K$, $\overline{X} = X \times_K \text{Spec}(\overline{K})$, and $G_K = \text{Gal}(\overline{K}/K)$ is the absolute Galois group of $K$. Therefore we have surjection $\pi_1^{ab}(X) \to G_K^{ab}$ where $G_K^{ab} = \text{Gal}(K_{ab}/K)$ is the Galois group of the maximal abelian extension of $K$, and we denote the kernel by $\pi_1^{ab(X)}_{\text{geo}}$:

$$0 \to \pi_1^{ab}(X)_{\text{geo}} \to \pi_1^{ab}(X) \to G_K^{ab} \to 0$$

Remark 2.1. When $X$ has a $K$-rational point $x$, (2.1) and consequently also (2.2) has a splitting, and $\pi_1^{ab}(\overline{X})_{G_K} \cong \pi_1^{ab}(X)_{\text{geo}}$ where the former group is the coinvariant with respect to the $G_K$-action by inner automorphisms. In this case, $\pi_1^{ab}(X)_{\text{geo}}$ has a geometric interpretation as the group classifying the abelian finite étale coverings of $X$ in which $x$ splits completely ([8]).
Now assume that \( K \) is a local field, i.e. a complete discrete valuation fields with a finite residue field \( F \) with \( \text{char} F = p \), and let \( \mathcal{X} \) be a proper flat model of \( X \) over the integer ring \( O_K \), i.e. a proper flat scheme \( \mathcal{X} \rightarrow \text{Spec} O_K \) with \( \mathcal{X} \times_O K \text{Spec} K \cong X \), and let \( X_F = \mathcal{X} \times_O K \text{Spec} F \) be the special fiber. Then we have a canonical surjection:

\[
\pi_1^{ab}(X) \rightarrow \pi_1^{ab}(\mathcal{X}) \cong \pi_1^{ab}(X_F)
\]

Here the latter isomorphism follows by [SGA1] Exposé X, Th. 2.1 (a part of proper base change theorem for \( H^1_{et} \)). We have the corresponding map on the geometric part by the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_1^{ab}(X)^{geo} & \rightarrow & \pi_1^{ab}(X) & \rightarrow & G^q_K & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \pi_1^{ab}(X_F)^{geo} & \rightarrow & \pi_1^{ab}(X_F) & \rightarrow & G_F & \rightarrow & 0
\end{array}
\]

**Definition 2.2.** Denote the kernel of left and middle vertical maps of (2.3) respectively by \( \pi_1^{ab}(X)^{geo}_{\mathcal{X}-\text{ram}} \), \( \pi_1^{ab}(X)^{geo}_{\mathcal{X}-\text{ram}} \). (These groups classify the abelian étale coverings of \( X \) which become completely ramified over a part of \( X_F \) when extended to the coverings over \( X_F \).)

By definition, we have an exact sequence:

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_1^{ab}(X)^{geo}_{\mathcal{X}-\text{ram}} & \rightarrow & \pi_1^{ab}(X)^{geo} & \rightarrow & \pi_1^{ab}(X_F)^{geo} & \rightarrow & 0
\end{array}
\]

As \( \pi_1^{ab}(X) \) is the Pontrjagin dual of the étale cohomology group \( H^1_{et}(X, \mathbb{Q}/\mathbb{Z}) \), we can pass to the dual and translate the above definition into the language of \( H^1_{et} \). By the Hochschild-Serre spectral sequence, we have a short exact sequence:

\[
0 \rightarrow H^1(G_K, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1_{et}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})^{G_K} \rightarrow 0
\]

where \((-)^{G_K}\) is the Galois invariant. (Here we used \( H^2(G_K, \mathbb{Q}/\mathbb{Z}) = 0 \) for local fields \( K \), and the same holds when we replace \( K \) by the finite field \( F \).) We can view this exact sequence as the Pontrjagin dual of (2.2), and therefore we have a canonical isomorphism:

\[
(\pi_1^{ab}(X)^{geo})^\vee \cong H^1_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})^{G_K}
\]

Hence by taking the Pontrjagin dual of the exact sequence (2.4), we have:

**Lemma 2.3.** Let \( X \) be a proper geometrically irreducible variety over a local field \( K \), and let \( \mathcal{X} \) be its proper flat model over \( O_K \), and denote the special fiber by \( X_F \). Then there is a canonical exact sequence of abelian groups:

\[
\begin{array}{cccccc}
H^1_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})^{G_K} & \rightarrow & H^1_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}) & \rightarrow & (\pi_1^{ab}(X)^{geo}_{\mathcal{X}-\text{ram}})^\vee & \rightarrow & 0
\end{array}
\]

**Remark 2.4.** Note that when \( X \) has a \( K \)-rational point, it gives a splitting of the exact sequence (2.2), and consequently all the vertical arrows of (2.3) are surjective, and exact sequences (2.4) and (2.6) turn out to be the short exact sequences.
2.2. Étale cohomology groups. Here we review some of the properties of the group $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^G_K$ for a proper geometrically irreducible curve $X$ over a field $K$.

First recall that, for prime-to-$p$ part, we have:

**Lemma 2.5** \((\text{SGA}4, \text{Exposé IX, Corollaire 4.7})\). Let $\ell$ be a prime different from $\text{char } K$. For a proper geometrically connected curve $X$ over a field $K$, $H^1_{et}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \text{Pic}^0(X)[\ell^\infty](-1)$ as $G_K$-modules.

More generally, considering the long exact sequence induced from the short exact sequence:

$$0 \longrightarrow \frac{1}{N}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we have $H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}) = H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})[N]$. As $\mathbb{Q}/\mathbb{Z} = \lim 
_{\longrightarrow} \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ and the étale cohomology commutes with the directed inductive limits \((\text{SGA}4 \text{ Exposé VII, Théorème 5.7})\), we deduce that:

$$H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z}) = \lim 
_{\longrightarrow} H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}) = \bigcup 
_N H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})$$

(In particular, $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})$ is a torsion module.) Therefore we have:

\begin{equation}
H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^G_K = \bigcup 
_N H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})[N]^G_K = \bigcup 
_N H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})^G_K
\end{equation}

and can reduce the calculation of $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^G_K$ to that of $H^1_{et}(\overline{X}, \mathbb{Z}/NZ)^G_K$, for which we can make use of the following generalization of Lemma 2.5:

**Lemma 2.6** (Milne [11], Theorem 3.9, Proposition 4.16). For any proper geometrically irreducible curve over a field $K$ and integer $N \geq 1$, we have a canonical isomorphism of $G_K$-modules:

$$H^1_{et}(\overline{X}, \mathbb{Z}/NZ) \cong \text{Hom}_{\text{gp}}(\mu_N, \text{Pic}^0(X)[N])$$

where $\text{Hom}_{\text{gp}}$ denotes the $G_K$-module consisting of homomorphisms as group schemes. In particular, we have:

$$H^1_{et}(\overline{X}, \mathbb{Z}/NZ)^G_K \cong \text{Hom}_{K-\text{gp}}(\mu_N, \text{Pic}^0(X)[N])$$

where $\text{Hom}_{K-\text{gp}}$ denotes the abelian group consisting of homomorphisms defined over $K$.

**Remark 2.7.** This interpretation shows that, $(\pi^b_1(X)^{\text{geo}})^\vee \cong H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^G_K$ is the maximal $\mu$-type subgroup of $\text{Pic}^0(X)$ in the terminology of Mazur [9], I-3. This re-interprets the Mazur’s theory of the $\mu$-type subgroup of the jacobian of the modular curves as the theory of $\pi^b_1(X)^{\text{geo}}$ of the modular curves.
3. THE PRIME-TO-$p$ PART OF $\pi_1^{ab}(X)_{\text{geo}}^{\text{ram}}$

From this section, we assume that $X$ is a proper smooth geometrically irreducible curve over a local field $K$. In this case, the proper flat regular model $\mathcal{X}$ always exists by Abyhankar [1], and denote its special fiber by $X_F$.

Let $J = \text{Pic}^0(X)$ be the jacobian variety of $X$, and denote the Néron model and its special fiber respectively by $J$ and $J_F$. The quotient of $J_F$ by its connected component of the identity $J^0_F$ is a finite étale group $\Phi(J)$ over $F$, which we call the group of components of $J$ (or of $J$, by an abuse of language).

Now we assume that $X$ admits a $K$-rational point (see Remark 2.4), and our starting point of the investigation is the following:

**Lemma 3.1** (Raynaud [12], [SGA7] Exposé IX, (12.1.12)). Assume that $X$ admits a $K$-rational point. Then we have $\text{Pic}^0(X)_F \cong J^0_F$ as smooth group schemes over $F$.

This lemma, in view of Lemma 2.3 and 2.6, assures that $\pi_1^{ab}(X)_{\mathcal{X}}^{\text{geo}}^{\text{ram}}$ is independent of the model $\mathcal{X}$. As we assume the existence of $K$-rational point in the rest of this paper, we will suppress the notation $\mathcal{X}$ from $\pi_1^{ab}(X)_{\mathcal{X}}^{\text{geo}}^{\text{ram}}$.

In this section we will treat the prime-to-$p$ part of $\pi_1^{ab}(X)_{\text{geo}}^{\text{ram}}$, and the main result is stated as follows:

**Theorem 3.2.** Assume that $X$ admits a $K$-rational point. Then there is an injection of finite abelian groups $(\pi_1^{ab}(X)_{\mathcal{X}}^{\text{geo}}^{\text{ram}})_{\text{not } p} \hookrightarrow \Phi(J)$, where $(-)_{\text{not } p}$ denotes the prime-to-$p$ part ($p = \text{char } F$).

We will start by fixing some notations on Galois modules, and give the proof of the above theorem. As the theorem can be proved by establishing the injectivity on $\ell$-primary parts for each prime number $\ell \neq p = \text{char } F$, we fix an $\ell \neq p$ in the remainder of this section.

### 3.1. Galois modules

For any smooth group scheme $X$ of finite type over $K$, define pro-$\ell$ (resp. ind-$\ell$) group scheme $T_\ell(X)$ (resp. $X[\ell^{\infty}]$) by:

$$T_\ell(X) = \lim_{\leftarrow} X[\ell^n], \quad X[\ell^{\infty}] = \lim_{\rightarrow} X[\ell^n]$$

where $X[m]$ denotes the kernel of the multiplication-by-$m$ map which is an étale group scheme over $K$, and inductive (resp. projective) limit is taken with respect to the canonical inclusions (resp. multiplication-by-$\ell$ maps). We use the same notations for group schemes over $O_K$ or $F$. We often identify $T_\ell(X), X[\ell^{\infty}]$ with the associated Galois modules, which are $\mathbb{Z}_\ell$-modules and satisfies:

$$X[\ell^{\infty}] \cong T_\ell(X) \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell), \quad T_\ell(X) = \text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X[\ell^{\infty}])$$

Now we fix the notations concerning Galois modules. Let $G = G_K$ or $G_F$ be the absolute Galois group, and let $M$ be an arbitrary $\mathbb{Z}_\ell$-module with a continuous action
of $G$. As usual, we write $\mathbb{Z}_\ell(1) = T_\ell(G_m)$, and define the Tate twist $M(r)$ of $M$ for $\forall r \in \mathbb{Z}$ by:

$$M(r) = \begin{cases} 
M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(r) & (r > 0) \\
M & (r = 0) \\
\text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell(-r), M) & (r < 0)
\end{cases}$$

as Galois modules. $M \mapsto M(r)$ gives exact functors of Galois modules for $\forall r \in \mathbb{Z}$, and we have canonical isomorphisms $M(r)(s) \cong M(r+s)$ for $\forall r, s \in \mathbb{Z}$. The Galois action on $\mathbb{Z}_\ell(1)$ is via the cyclotomic character $\chi: G \to \mathbb{Z}_\ell^\times$. The Galois invariant and coinvariant is denoted respectively by $M_G, M_G^\vee$.

**Definition 3.3.** The $\chi$-part $M^\chi$ of $M$ is defined by $M^\chi = (M(-1)^G)(1)$. It is canonically identified with the maximal subgroup of $M$ on which $G$ acts via the cyclotomic character $\chi$.

Note that the functors $M \mapsto M^G, M \mapsto M^\chi$ are both left exact.

### 3.2. Tate modules of the jacobians of curves

Now we return to the curve $X$ over a local field $K$. Applying Lemma 2.5 and Lemma 3.1, we can restate the Lemma 2.3 in the present case, in view of Remark 2.4, as follows:

**Lemma 3.4.** For a proper smooth geometrically irreducible curve $X$ over a local field $K$ which admits a $K$-rational point, there is a canonical short exact sequence of abelian groups:

$$0 \rightarrow J^0_F[\ell^\infty](-1)^G_K \rightarrow J[\ell^\infty](-1)^G_K \rightarrow (\pi^{ab}_1(X)_{\text{geo}})_\ell^\vee \rightarrow 0$$

where $(\pi^{ab}_1(X)_{\text{ram}})^\vee$ denotes the $\ell$-primary part of $(\pi^{ab}_1(X)_{\text{ram}})^\vee$.

Using the canonical isomorphism $M(-1)^G \cong M^\chi(-1)$ of $G_F$-modules, we can express this lemma by the short exact sequence of $G_F$-modules:

$$0 \rightarrow J^0_F[\ell^\infty]^\chi \rightarrow J[\ell^\infty]^\chi \rightarrow (\pi^{ab}_1(X)_{\text{geo}})_\ell^\vee(1) \rightarrow 0$$

Now we relate this group with the component group $\Phi(J)$. For this we need the description of $\Phi(J)$ by the Galois module associated to $J$, stated as follows:

**Lemma 3.5 (SGA7 Exposé IX, Proposition 11.2).** For an abelian variety $A$ over a local field $K$, denote the Néron model and its special fiber respectively by $A, A_F$. Let $A^0_F$ be the connected component of the identity, and denote the component group by $\Phi(A) = A_F/A^0_F$. For any prime $\ell \neq \text{char } F$, there is a canonical isomorphism of $G_F$-modules:

$$\Phi(A)_\ell \cong \text{Coker } (T_\ell(A)^I \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow (T_\ell(A) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell))^I)$$

where $\Phi(A)_\ell$ is the $\ell$-primary part of $\Phi(A)$, and $(-)^I$ denotes the invariant by the action of the inertia group $I \subset G_K$. 

By the canonical isomorphism of $G_F$-modules $T_\ell(A)^I \cong T_\ell(A^0_F)$ (SGA7 Exposé IX, Proposition 2.2.5, (2.2.3.3)), we can restate the above lemma by the short exact sequence of $G_F$-modules:

\[
0 \longrightarrow A^0_F[\ell^\infty] \longrightarrow A[\ell^\infty]^I \longrightarrow \Phi(A)_\ell \longrightarrow 0
\]

**Proof of Theorem 3.2.** Applying the left exact functor $M \mapsto M^\chi$ to the short exact sequence for $A = J$, we have the exact sequence:

\[
0 \longrightarrow J^0_F[\ell^\infty]^\chi \longrightarrow J[\ell^\infty]^\chi \longrightarrow \Phi(J)_\ell^\chi
\]

Comparing this with the short exact sequence (3.1), we have a canonical injection $(\pi_1^b(X)^{\text{geo}}_{\text{ram}})_\ell^\vee \longrightarrow \Phi(J)_\ell^\chi(-1)$. As $\Phi(J)_\ell^\chi(-1) \cong \Phi(J)_\ell(-1)^G$ injects to $\Phi(J)_\ell$ as finite abelian groups and $\ell \neq p$ was arbitrary, we have proven the theorem. \hfill \Box

### 3.3. Example: elliptic curves

Let $X = E$ be an elliptic curve over the local field $K$ (see Silverman [15], Chap. IV). For $A = E$, we have the exact sequence:

\[
0 \longrightarrow E^0_F[\ell^\infty]^\chi \longrightarrow E[\ell^\infty]^\chi \longrightarrow (\pi_1^b(E)^{\text{geo}}_{\text{ram}})_\ell^\vee(1) \longrightarrow 0
\]

When $E$ has good reduction over $O_K$, trivially $(\pi_1^b(X)^{\text{geo}}_{\text{ram}})_{\text{not} \cdot p} = 0$. If $E$ has additive reduction, $E^0_F$ is a unipotent group, therefore $E^0_F[\ell^\infty] = 0$. Hence we have canonical isomorphisms:

\[
E[\ell^\infty]^\chi \cong (\pi_1^b(E)^{\text{geo}}_{\text{ram}})_\ell^\vee(1), \quad E[\ell^\infty]^I \cong \Phi(E)_\ell
\]

which gives $\Phi(E)^\chi \cong (\pi_1^b(E)^{\text{geo}}_{\text{ram}})_\ell^\vee(1)$. Moreover, in this case we know that the $G = G_F$ acts trivially on $\Phi(E)$, hence we have:

**Proposition 3.6.** If $E$ has good or additive reduction over $O_K$, the injection in Th. 3.2 is a bijection $(\pi_1^b(E)^{\text{geo}}_{\text{ram}})_{\text{not} \cdot p} \cong \Phi(E)(-1)^G \cong \Phi(E)[q - 1]$, where $q = |F|$.

In the multiplicative reduction case, this is not necessarily true.

### 4. The $p$-primary part for low absolute ramification case

As in the previous section, we consider a proper smooth geometrically irreducible curve $X$ over a local field $K$ which admits a $K$-rational point. Let $e \in \mathbb{N}$ denote the absolute ramification index of $K$, i.e. the normalized valuation of $p = \text{char } F$ in $K$. In this section, we prove the following:

**Theorem 4.1.** Assume $e < p - 1$ and $\text{char } K = 0$. If $X$ has semistable reduction over $O_K$, then $(\pi_1^b(X)^{\text{geo}}_{\text{ram}})_p = 0$, where $(-)_p$ denotes the $p$-primary part ($p = \text{char } F$).

**Remark 4.2.** This theorem is a generalization of the Prop. 7 of Kato-Saito [7], where it is proved in the good reduction case. Here we employ a completely different method of proof.
Our task here is to analyze the $H^1_{et}(X, \mathbb{Z}/p^n\mathbb{Z})^{G_K} \cong \text{Hom}_{K_{et}}(\mu_{p^n}, \text{Pic}^0(X)[p^n])$ (Lemma 2.6). For this purpose, we recall the structure of the $p^n$-torsion points $J[p^n]$ of the Jacobian $J = \text{Pic}^0(X)$, following [SGA7], Exposé IX. In the rest of this section, we assume that $X$ has semi-stable reduction over $O_K$. In this case, $J$ has semi-stable reduction, i.e., the connected component of the identity of $J$, we know that $J_0[p^n]$ is a quasi-finite group scheme over $O_K$ ([SGA7] Exposé IX, Lemme 2.2.1).

Now recall that any quasi-finite group scheme $\mathcal{G}$ over $O_K$ has the canonical and functorial decomposition $\mathcal{G} = \mathcal{G}^f \amalg \mathcal{G}'$ where $\mathcal{G}^f$ (the fixed part of $\mathcal{G}$) is finite flat over $O_K$ and $\mathcal{G}'$ has empty special fiber ([EGA] II,(6.2.6), [SGA7] Exposé IX, (2.2.3.1)).

Coming back to our case, $J_0[p^n]$ has the decomposition:

$$J_0[p^n] = J_0[p^n]^f \amalg J_0[p^n]'$$

where $J_0[p^n]^f$ is a finite flat group scheme. We denote the generic fiber of $J_0[p^n]^f$ by $J_0[p^n]^f$, which is a subgroup of $J[p^n]$, the generic fiber of $J_0[p^n]$ (recall $(J_0)^K = J$). Note that $J_0[p^n], J_0[p^n]^f$ have a common special fiber $J_0[p^n]$, which is canonically isomorphic to $\text{Pic}^0(X_F)[p^n]$ by Lemma 3.1 (recall the running hypothesis that $X$ has a $K$-rational point).

Moreover, we have additional information by the semi-stability hypothesis ([SGA7] Exposé IX, (5.5.8)):

$$J[p^n]/J_0[p^n]^f \cong M_K \otimes (\mathbb{Z}/p^n\mathbb{Z}) = (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))/K$$

where $M$ denotes the character group of the toric part of $J_F$, which is an unramified Galois module, free of finite rank over $\mathbb{Z}$. We consider $M$ as an étale group scheme over $O_K$, therefore justifying the notation $M_K$. (Note that in general we have to take the corresponding object for the dual abelian variety, but we have the autoduality of the Jacobian in the present case.)

Now we apply the result of Raynaud [13] to obtain the following lemma:

**Lemma 4.3.** Under the hypothesis $e < p - 1$ and $\text{char } K = 0$, the following natural homomorphism is an isomorphism:

$$H^1_{et}(X_F, \mathbb{Z}/p^n\mathbb{Z})^{G_F} \cong H^1_{et}(X, \mathbb{Z}/p^n\mathbb{Z})^{G_K}$$

**Proof.** By Lemma 2.6, we have:

$$H^1_{et}(X_F, \mathbb{Z}/p^n\mathbb{Z})^{G_F} \cong \text{Hom}_{F_{et}}(\mu_{p^n}, J_F[p^n])$$

$$H^1_{et}(X, \mathbb{Z}/p^n\mathbb{Z})^{G_K} \cong \text{Hom}_{K_{et}}(\mu_{p^n}, J[p^n])$$
By (1.1), we have an exact sequence:

$$0 \longrightarrow \text{Hom}_{K^\text{-gp}}(\mu_{p^n}, J^0[p^n]^f) \longrightarrow \text{Hom}_{K^\text{-gp}}(\mu_{p^n}, J[p^n]) \longrightarrow \text{Hom}_{K^\text{-gp}}(\mu_{p^n}, (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))_K)$$

But by Corollaire 3.3.6 of Raynaud [13], we have:

$$\text{Hom}_{K^\text{-gp}}(\mu_{p^n}, (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))_K) \cong \text{Hom}_{O_K^\text{-gp}}(\mu_{p^n}, M \otimes (\mathbb{Z}/p^n\mathbb{Z})) = 0$$

which vanishes simply because $\mu_{p^n}$ is connected and $M \otimes (\mathbb{Z}/p^n\mathbb{Z})$ is étale. Therefore we have, using [13], Corollaire 3.3.6 again,

$$\text{Hom}_{K^\text{-gp}}(\mu_{p^n}, J[p^n]) \cong \text{Hom}_{K^\text{-gp}}(\mu_{p^n}, J^0[p^n]^f) \cong \text{Hom}_{O_K^\text{-gp}}(\mu_{p^n}, J_F^0[p^n]_F)$$

Therefore it suffices to see that $\text{Hom}_{O_K^\text{-gp}}(\mu_{p^n}, J_F^0[p^n]_F) \cong \text{Hom}_{F^\text{-gp}}(\mu_{p^n}, J_F^0[p^n])$, which is equivalent by Cartier duality ([SGA3] Exposé VIIA, (3.3.1)) to:

$$\text{Hom}_{O_K^\text{-gp}}(J_F^0[p^n]_F^\ast, \mathbb{Z}/p^n\mathbb{Z}) \cong \text{Hom}_{F^\text{-gp}}(J_F^0[p^n]^\ast, \mathbb{Z}/p^n\mathbb{Z})$$

where $\ast$ denotes the Cartier dual $\mathcal{G}^\ast = \text{Hom}(\mathcal{G}, \mathbb{G}_m)$. But as $\mathbb{Z}/p^n\mathbb{Z}$ is étale, this is clear by the Hensel’s lemma (e.g. [EGA] IV, (18.5.12)).

**Proof of Theorem 4.1.** By Lemma 2.3, it is enough to show that the canonical homomorphism $H^1_{\text{et}}(X_F, \mathbb{Q}_p/\mathbb{Z}_p)^{G_F} \longrightarrow H^1_{\text{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p)^{G_K}$ is an isomorphism. But by exactly the same argument as in (2.7) we have:

$$H^1_{\text{et}}(X_F, \mathbb{Q}_p/\mathbb{Z}_p)^{G_F} = \bigcup_n H^1_{\text{et}}(X_F, \mathbb{Q}_p/\mathbb{Z}_p)[p^n]^{G_F} = \bigcup_n H^1_{\text{et}}(X_F, \frac{1}{p^n}\mathbb{Z}/\mathbb{Z})^{G_F}$$

$$H^1_{\text{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p)^{G_K} = \bigcup_n H^1_{\text{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p)[p^n]^{G_K} = \bigcup_n H^1_{\text{et}}(X, \frac{1}{p^n}\mathbb{Z}/\mathbb{Z})^{G_K}$$

Therefore the desired isomorphism is deduced from the corresponding result in the each $p$-power torsion level, namely the Lemma 13.

Combining with Th. 3.2 we have:

**Corollary 4.4.** Assume $e < p - 1$ and char $K = 0$. If $X$ admits a $K$-rational point and has semistable reduction over $O_K$, then there is an injection of finite abelian groups $(\pi_1^{ab}(X)_{\text{ram}})^\vee \longrightarrow \Phi(J)$.

**Remark 4.5.** We have the canonical perfect duality:

$$\Phi \otimes_{\mathbb{Z}} \Phi \longrightarrow \mathbb{Q}/\mathbb{Z}$$

deduced by the autoduality of the jacobian and [SGA7] Exposé IX, (11.4.1) or Conjecture 1.3 (proven in this case). This gives the canonical identification $\Phi \cong \Phi^\vee$, which enables us to state Cor. 4.4 as the surjection $\Phi(J) \longrightarrow \pi_1^{ab}(X)_{\text{ram}}$. 


5. The case of the modular curves

In this section, we apply the results of preceding section to the modular curve $X_0(p)/\mathbb{Q}_p$ for a prime $p$, which is a proper smooth geometrically irreducible curve over $\mathbb{Q}_p$. For the definition and basic properties of the modular curve $X_0(p)$, we refer to Mazur [9]. The result of this section is summarized as follows:

**Theorem 5.1.** $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}$ has the following structure:

$$
0 \longrightarrow \Phi(J_0(p)) \longrightarrow \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}} \longrightarrow \hat{\mathbb{Z}}_p \longrightarrow 0
$$

where $\Phi(J_0(p)) \cong \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}} = \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{tor}}$ is a cyclic group of order equal to the numerator of $\frac{p-1}{12}$, and $r = \frac{g+h-1}{2}$ where $g$ is the genus of $X_0(p)$ and $h$ is the number of the supersingular points defined over $\mathbb{F}_p$.

**Remark 5.2.** This result should be interpreted as the local analogue of the theorem of Mazur which asserts that $\pi_1^{ab}(X_0(p)/\mathbb{Q})^{\text{geo}} \cong \Phi(J_0(p))$ ([9, Introduction, Theorem (2)]). The fact that $\pi_1^{ab}(X_0(p)/\mathbb{Q})^{\text{geo}}$ is equal to $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}} = (\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{tor}}$ shows that the maximal abelian étale covering of $X_0(p)/\mathbb{Q}$ “uses up” all the ramification allowed at the special fiber at $p$. The origin of this phenomenon remains to be clarified.

### 5.1. The part $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}}$

First, we apply the results of the preceding sections to determine the part $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}}$. As $e = 1$ for $\mathbb{Q}_p$ and $X_0(p)$ admits a $\mathbb{Q}_p$-rational point (e.g. the $\infty$-cusp), and moreover $X_0(p)$ has semistable reduction over $\mathbb{Z}_p$ (Deligne-Rapoport [6, V-6, or Mazur [9], II-1]), we know by Cor. 4.4 that $(\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}})^{\vee}$ injects to $\Phi(J_0(p))$, where $J_0(p)$ is the jacobian of $X_0(p)$. By Mazur-Rapoport [10], we know that:

**Proposition 5.3** ([10, Theorem (A.1), b)]. $\Phi(J_0(p))$ is a cyclic group of order equal to the numerator of $\frac{p-1}{12}$.

Because the Shimura covering ([9, Cor. (2.3)]) gives a cyclic étale covering of $X_0(p)/\mathbb{Q}_p$ of order equal to the numerator of $\frac{p-1}{12}$, which is by definition completely ramified over one of the component of the special fiber, we have:

**Proposition 5.4.** $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{\text{geo}}_{\text{ram}}$ is isomorphic to $\Phi(J_0(p))$.

**Remark 5.5.** Similar results hold for the modular curves $X_H(p)$ between $X_1(p)$ and $X_0(p)$ corresponding to any subgroup $H \subset (\mathbb{Z}/p\mathbb{Z})^\times / \{\pm 1\}$ at least for the prime-to-$p$ part, by the results in Conrad-Edixhoven-Stein [3].

### 5.2. The part $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{\text{geo}}$

We denote by $X_0(p)/\mathbb{Z}_p$ the minimal regular model of $X_0(p)/\mathbb{Q}_p$, following [11], and denote the special fiber by $X_0(p)/\mathbb{F}_p$. Here we determine the part $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{\text{geo}}$, thereby completing the proof of Th. 5.1. The result is:

**Proposition 5.6.** $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{\text{geo}}$ is a free module of over $\hat{\mathbb{Z}}$ with the rank equal to $\frac{g+h-1}{2}$, where $g$ is the genus of $X_0(p)$ and $h$ is the number of the supersingular points defined over $\mathbb{F}_p$. 
Proof. As each of the component of the special fiber $X_0(p)_{\mathbb{F}_p}$ of $X_0(p)$ is (geometrically) isomorphic to $\mathbb{P}^1$, we know that $\text{Pic}^0(X_0(p)_{\mathbb{F}_p})$ is a torus with the character group canonically isomorphic (as the $\mathbb{G}_{\mathbb{F}_p}$-module) to the first homology group $H_1(\Gamma, \mathbb{Z})$ of the graph $\Gamma$ of $X_0(p)_{\mathbb{F}_p}$ ([SGA7] Exposé IX, 12.3). This graph is described in Mazur-Rapoport [10], §3. In particular, if we denote the genus of $X_0(p)$ by $g$, the total number of supersingular points are $g + 1$, and $H_1(\Gamma, \mathbb{Z})$ is a free module over $\mathbb{Z}$ of rank $g$.

Now by (2.5), (2.7), and Lemma 2.6, we have:

$$\pi_{1,ab}^{ab}(X_0(p)/\mathbb{F}_p)_{\text{geo}} \cong \left( \bigcup_N \text{Hom}_{\mathbb{F}_p}(\mu_N, \text{Pic}^0(X_0(p)_{\mathbb{F}_p})([N])) \right)^\vee$$

$$\cong \left( \bigcup_N \text{Hom}_{\mathbb{G}_{\mathbb{F}_p}}(H_1(\Gamma, \mathbb{Z}/N\mathbb{Z}), \mathbb{Z}/N\mathbb{Z}) \right)^\vee$$

$$\cong \lim_N \left( \text{Hom}(H_1(\Gamma, \mathbb{Z}/N\mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}}, \mathbb{Q}/\mathbb{Z}) \right)^\vee$$

$$\cong \lim_N (H_1(\Gamma, \mathbb{Z}/N\mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}} \otimes \mathbb{Z}/N\mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}} \otimes \hat{\mathbb{Z}}$$

where $H_1(\Gamma, \mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}}$ is the $\mathbb{G}_{\mathbb{F}_p}$-coinvariant of $H_1(\Gamma, \mathbb{Z})$. The last isomorphism follows from the fact that $H_1(\Gamma, \mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}}$ is clearly a finitely generated $\mathbb{Z}$-module. Now it only remains to determine the group $H_1(\Gamma, \mathbb{Z})_{\mathbb{G}_{\mathbb{F}_p}}$ explicitly.

As we know that the vertices of $\Gamma$ is fixed by $\mathbb{G}_{\mathbb{F}_p}$ as the irreducible components are defined over $\mathbb{F}_p$, and the edges of $\Gamma$ which correspond to the supersingular points of $X_0(p)$ are fixed or interchanged by pairs by the Frobenius automorphism of $\mathbb{G}_{\mathbb{F}_p}$, according to whether the field of definition of the corresponding supersingular point is $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ (these possibilities can be visibly read off from the Table 6 of [2]).

Therefore the Frobenius acts on the basis of the homology group formed by the pair of the edges corresponding to the pair of supersingular points defined over $\mathbb{F}_{p^2}$ by multiplication by $-1$, which means that when we take the $\mathbb{G}_{\mathbb{F}_p}$-coinvariant, exactly these bases vanish. Therefore if we denote the number of the supersingular points defined over $\mathbb{F}_p$ and the pairs of supersingular points defined over $\mathbb{F}_{p^2}$ respectively by $h$ and $j$ (therefore $g + 1 = h + 2j$), the $\mathbb{G}_{\mathbb{F}_p}$-coinvariant of $H_1(\Gamma, \mathbb{Z})$ is a free module of rank $g - j = \frac{a + h - 1}{2}$. (This rank is equal to the rank defined in the general setting in Saito [10], II-Def. 2.5.)

□

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