ANALYTIC TORSION AND R-TORSION OF WITT REPRESENTATIONS ON MANIFOLDS WITH CUSPS

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Abstract. We establish a Cheeger-Müller theorem for orthogonal representations satisfying a Witt condition on a noncompact manifold with cusps. This class of spaces includes all non-compact hyperbolic spaces of finite volume, but we do not assume that the metric has constant curvature nor that the link of the cusp is a torus. We use renormalized traces in the sense of Melrose to define the analytic torsion and we relate it to the intersection R-torsion of Dar of the natural compactification to a stratified space. Our proof relies on our recent work on the behavior of the Hodge Laplacian spectrum on a closed manifold undergoing degeneration to a manifold with fibered cusps.

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Introduction

The celebrated theorem of Cheeger and Müller establishes the equality of Reidemeister and analytic torsion on an odd-dimensional closed manifold equipped with a flat Euclidean bundle. This was originally conjectured by Ray-Singer [RS71], proven by Cheeger and Müller [Che79, Mül78] and subsequently extended by Müller [Müll93] and Bismut-Zhang [BZ92]. The importance and usefulness of the theorem stems from the fact that the Reidemeister torsion,
or R-torsion, is a combinatorial invariant of simplicial complexes while the analytic torsion is a smooth invariant defined via the spectrum of the Hodge Laplacian. This connection is behind many applications in topology, number theory, and mathematical physics.

One particularly interesting aspect of this theorem is that it allows us to use analysis to study the size of the torsion in homology. For example, in [Che79, Example 1.3] Cheeger points out that if

\[
\begin{array}{c}
F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \ldots \xrightarrow{d^n} F^n \xrightarrow{d^n} 0
\end{array}
\]

is a complex of free Abelian groups and \( K^i = F^i \otimes \mathbb{R} \) then the Reidemeister torsion (after some canonical choices) is given by

\[
R\text{-torsion} = \prod \frac{|H^{2k+1}(F^\bullet)_{\text{torsion}}|}{|H^{2k}(F^\bullet)_{\text{torsion}}|}.
\]

A recent paper of Bergeron-Venkatesh [BV13] exploits this relationship to study the growth of torsion in group homology by studying the analytic torsion of locally symmetric spaces. This paper has initiated a lively discussion [BV13, Mül12, MM13, MP13c, MP13b, MP14, BMZ11, BSV14, MFP14], which has recently expanded to non-compact hyperbolic spaces. Analytic torsion in this context was first studied by Park [Par09] who proved that a relation discovered by Fried [Fri86] between analytic torsion and dynamical zeta functions continues to hold on noncompact hyperbolic spaces. Recently there has been an impressive sequence of papers by Müller and Pfaff [MP12, Pfa14b, MP13a, Pfa12, Pfa14a, Pfa13] in which the Selberg trace formula is used to great effect in analyzing analytic torsion.

The methods in the papers cited above are closely tied to the algebraic structure of locally symmetric spaces. In [ARS14] and the present paper we use the geometric microlocal analysis methods of Melrose to extend the study of analytic torsion to a larger class of spaces devoid of this algebraic structure. Indeed, in [ARS14] we have established a Cheeger-Müller theorem on non-compact manifolds with fibered cusp ends, a class of manifolds including many locally symmetric spaces of rank one.

Specifically, let \( N \) be the interior of a manifold with boundary \( \overline{N} \) and assume that the boundary participates in a fiber bundle of closed manifolds

\[
Z \rightarrow \partial N \xrightarrow{\phi} Y.
\]

Let \( x \) be a ‘boundary defining function’ for \( \partial N \), that is, a smooth function on \( \overline{N} \) that vanishes precisely at \( \partial N \) and with non-vanishing differential there. A metric \( g_d \) on \( N \) is a fibered cusp metric, or \( d \)-metric, if it is asymptotically of the form

\[
g_d \sim \frac{dx^2}{x^2} + x^2 g_Z + \phi^* g_Y
\]

where \( g_Z + \phi^* g_Y \) is a submersion metric on \( \partial M \), and an ‘even’ \( d \)-metric if \( g_Z \) and \( g_y \) are functions of \( x^2 \), see [Vai01, ARS14, Definition 7.3] for more details. Let \( F \rightarrow \overline{N} \) be vector bundle with flat connection \( \nabla^F \) and \( g_F \) a compatible bundle metric smooth all the way down to \( \partial N \). In fact we assume that \( g_F \) is even, meaning that it extends smoothly to the double of \( N \) across \( \partial N \).

The analytic torsion of \((N, g_d, F, g_F)\) is defined in [ARS14, §9] following [Mel93] by means of the renormalized trace of the heat kernel, since the usual operator trace is infinite. We prove that this analytic torsion can be expressed in terms of the Reidemeister torsion of \( \overline{N} \).
relative to \( \partial N \) and in terms of the Reidemeister torsion of Dar [Dar87], \( I\tau^m \) of the singular space \( \hat{N} \) obtained from \( N \) by ‘coning-off’ the fibers of the fiber bundle \( \phi \).

**Theorem 1 (ARS14, Corollary 11.2).** Let \((N,g_d)\) be an odd-dimensional Riemannian manifold with fibered cusp ends and an even \(d\)-metric. Let \( \alpha : \pi_1(N) \to \text{GL}(k, \mathbb{R}) \) be a unimodular representation meaning that \( |\det \alpha| = 1 \). Endow the associated flat vector bundle \( F \to M \) with an even bundle metric \( g_F \) and assume that it is strongly acyclic in that

\[
\mathcal{H}^\ast(H/Y; F) = 0, \quad \mathcal{H}^\ast_{L^2}(M; F) = 0.
\]

The logarithm of analytic torsion satisfies

\[
\text{LAT}(N,g_d,F) = \log \frac{I\tau^m(\hat{N},\alpha)\tau(\partial N,\alpha)^\frac{1}{2}}{I\tau^m(C_\phi \partial N,\alpha)} = \log \tau(N,\partial N,\alpha) + \frac{1}{2} \log \tau(\partial N,\alpha).
\]

where \( C_\phi \partial N \) is the mapping cylinder of \( \partial N \to Y \).

In this paper we specialize from fibered cusps to non-fibered cusps (i.e., we assume that \( Y \) is a point), and we replace the strong acyclicity condition on \( \alpha \) with a much weaker ‘Witt condition’.

We can describe exactly how we will manage this extension by recalling the proof of Theorem 1 in the case \( Y = \text{pt} \). Let \( M \) be a smooth closed manifold obtained by doubling \( N \) across \( \partial N = Z \), and \( F \) a flat bundle over \( M \). We consider a family of metrics \( \varepsilon \mapsto g_{\varepsilon,hc} \) that, in a tubular neighborhood of \( Z \), has the form

\[
g_{\varepsilon,hc} = \frac{dx^2}{x^2 + \varepsilon^2} + (x^2 + \varepsilon^2)g_Z.
\]

This can be visualized as stretching the manifold \( M \) in the direction normal to the hypersurface \( Z \) until it has two infinite cusp ends in place of the hypersurface.

In fact, while for \( \varepsilon > 0 \) the metrics \( g_{\varepsilon,hc} \) are smooth Riemannian metrics on \( M \), as \( \varepsilon \to 0 \) the metric degenerates along \( Z \), and has two non-singular limits: a hyperbolic cusp metric on \( M \setminus Z \) and a complete metric with cylindrical ends on the normal bundle of \( Z, N_MZ \cong Z \times \mathbb{R} \). The de Rham operator \( \delta_{\text{DR}} = d + \delta \) associated to \( g_{\varepsilon,hc} \) has a model operator corresponding to each of these limiting spaces. On \( M \setminus Z \), we obtain \( \delta_{\text{DR},hc} \), the de Rham operator of the limiting fibered hyperbolic cusp metric. The other model operator is actually on \( \mathbb{R} \). It is the de Rham operator \( \delta_{\text{DR},b} \) of a metric with cylindrical ends, but twisted by the ‘vertical cohomology bundle’,

\[
H^\ast(Z; F) \to \mathbb{R}.
\]

In [ARS14] we carried out a careful analysis of the spectrum of \( \delta_{\text{DR},hc} \) as \( \varepsilon \to 0 \) by describing the precise asymptotics of the Schwartz kernels of the resolvent and heat kernel. In particular we proved that there are finitely many eigenvalues of \( \delta_{\text{DR},hc} \) that converge to zero as \( \varepsilon \to 0 \). We call these the ‘small eigenvalues’ and denote the product of the non-zero small eigenvalues by \( \det(\delta_{\text{DR}})_{B_{sm}} \).

**Theorem 2 (ARS14, Theorem 10.2).** If \( \log \det(\delta_{\text{DR}})_{B_{sm}} \) is polyhomogeneous in \( \varepsilon \), the metric \( g_{\varepsilon,hc} \) is of ‘product-type’ and the flat bundle is Witt in that

\[
H^{\dim Z/2}(Z; F) = 0.
\]
then the determinant of the Laplacian satisfies

\[
\begin{align*}
\lim_{\epsilon \to 0} \log \det \partial_{\text{DR}, \epsilon}^2 &= \log \det \partial_{\text{DR}, \epsilon}^2 + \log \det \partial_{\text{DR}, b}^2 - \lim_{\epsilon \to 0} \log \det (\partial_{\text{DR}}^2)_{\mathbb{B}_{sm}}.
\end{align*}
\]

The acyclicity condition we imposed in the fibered cusp setting implies that the last two terms in (1) do not contribute to the analytic torsion. To remove the acyclicity condition and compute the limit of analytic torsion as \(\epsilon \to 0\), we will establish that \(\log \det (\partial_{\text{DR}}^2)_{\mathbb{B}_{sm}}\) is polyhomogeneous in \(\epsilon\) and compute its finite part as \(\epsilon \to 0\) (Corollary 3.4) and we will compute the determinants of the model operators \(\partial_{\text{DR}, b}^2\) (2.2 especially (2.15)). Another consequence of removing the acyclicity condition is that the analytic torsion should be thought of not as a number, but as a function that assigns a number to each basis of the cohomology \(H^*(M; F)\), so we will also compute the behavior of a basis of harmonic forms as \(\epsilon \to 0\) ((3.30)). These pieces together determine the limit of analytic torsion as \(\epsilon \to 0\).

In Theorem 5.4 we determine how the Reidemeister torsion of \(M\) is related to the Reidemeister torsion of \(N\). All together we establish the following theorem in Corollary 6.2.

**Theorem 3** (A Cheeger-Müller theorem for manifolds with cusps). Let \(F \rightarrow N\) be a flat Euclidian bundle satisfying the Witt condition. For compatible choices of orthonormal bases of cohomology, \(\mu_N\) of \(\text{IH}^*_{m}(N; F) \cong H^*_{L^2}(N; F)\) and \(\mu_{CZ}\) of \(\text{IH}^*_{m}(CZ)\)

\[
\text{LAT}(N, \mu_N, F) = \log \left( \frac{\text{I}_m^\tau(N, \mu_N, E)}{\text{I}_m^\tau(CZ, \mu_{CZ}, F)} \right) - \frac{\chi(Z; F)}{8} \log 2
\]

\[
- \frac{1}{2} \sum_{q=0}^{m-1} (-1)^q \dim H^q(Z; F) [(m - 1 - 2q) \log(m - 1 - 2q)]
\]

where \(CZ\) is the cone over \(Z\).

In the literature cited above, a theorem close to ours is an interesting Cheeger-Müller theorem due to Pfaff [Pfa13]. This theorem applies to noncompact hyperbolic manifolds with cusps \(N\) of odd dimension \(m\) and flat vector bundles \(F\) induced by the irreducible representations of \(SO^0(m, 1)\) or \(\text{Spin}(m, 1)\) that are not invariant with respect to the Cartan involution. Pfaff uses constructions of Harder [Har75] to define a canonical Reidemeister torsion \(\tau_{Eis}(N; F)\) (a similar construction is used by Calegari and Venkatesh [CV12] in three dimensions). Let \(C\) be a neighborhood of the cusps Pfaff uses the renormalized trace of Melrose to define analytic torsion and is then able to compute the difference

\[
\log \tau_{Eis}(N; F) - \log \left( \frac{\text{AT}(N; F)}{\text{AT}(C, \partial C; F)} \right)
\]

in terms of the rank of \(F\), the Betti numbers and volume of \(\partial C\) and some weights associated to the holonomy representation of \(F\). Notice that in this setting, the Witt condition is never satisfied.

A very interesting preprint, using different methods, has recently been posted by Vertman [Ver14] that includes a Cheeger-Müller theorem for flat unitary bundles over three dimensional manifolds with product-type cusps satisfying the Witt condition.

For hyperbolic surfaces cusp formation corresponds to converging to the boundary of Teichmüller space and so has been the subject of much study. For example Seeley and Singer [SS88] studied the \(\partial\) operator as a cusp is formed. In Propositions 7.1 and 7.2 we recover results of Wolpert and Burger [Wol87, Wol90, Wol10, Bur88] on the asymptotics of
small eigenvalues and the blow-up of the determinant.

This paper is organized as follows: Section 1 recalls our conventions for cusp metrics and analytic torsion. After this section we work in the context of a closed manifold $M$ with hypersurface $Z$ along which the metric is degenerating to form cusp ends. In §2 we analyze the model operator $D_b$ on $\mathbb{R}$ and compute its contribution to the asymptotics of analytic torsion. Then section 3 is devoted to the study of the small eigenvalues including their polyhomogeneity in $\varepsilon$ and culminating in the computation of the corresponding determinant. This section also includes an analysis of the asymptotics of an appropriately chosen basis of harmonic forms. These results are collected in §4 and yield the asymptotics of analytic torsion along degeneration to a manifold with cusp ends.

Section 5 contains our study of Reidemeister torsion, particularly of how the R-torsion of the closed manifold $M$ relates to the R-torsion of $N$, the manifold with cusp ends. Then in §6 we combine this study with our analysis of analytic torsion to obtain our Cheeger-Müller theorem. In the final section, §7 we specialize to dimension two and explain the relevance of our results to families of hyperbolic metrics approaching the boundary of Teichmüller space.

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1. Cusp metrics and analytic torsion

In this section we recall the definition of cusp metrics and a very useful replacement for the tangent bundle that is adapted to the geometry. We also recall the definition of analytic torsion on closed manifolds and manifolds with asymptotic cusps.

1.1. Analytic torsion. On a closed Riemannian manifold $(M, g)$ of dimension $m$, the heat kernel of any Laplace-type operator satisfies

$$\text{Tr}(e^{-t\Delta}) \sim t^{-m/2} \sum_{k \geq 0} a_k t^k \text{ as } t \to 0, \quad \text{Tr}(e^{-t\Delta}) - \dim \ker \Delta = O(e^{-t\lambda_1}) \text{ as } t \to \infty$$

with $\lambda_1 > 0$. Hence its zeta function

$$\zeta(s) = \zeta(s; \Delta) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \text{Tr}(e^{-t\Delta} - P_{\ker \Delta}) \frac{dt}{t}$$

extends from a holomorphic function on $\text{Re } s > m/2$ to a meromorphic function on all of $\mathbb{C}$ which has at worst simple poles and is regular at the origin. If $F \to M$ is a flat vector bundle with flat metric, and $\Delta_q$ is the Hodge Laplacian on $F$-valued differential forms of degree $q$, then

$$\text{LAT}(M, g; F) = \frac{1}{2} \sum_q (-1)^q q \zeta'(0; \Delta_q)$$

is the logarithm of the analytic torsion of $(M, g; F)$. 
If $F$ is acyclic, that is if $H^\ast(M; F) = 0$, then the analytic torsion is independent of the choice of Riemannian metric. Otherwise, we choose a basis \{$\mu^q_j$\} of each $H^q(M; F)$ and let $\omega^q$ be an orthonormal basis of harmonic representatives with respect to the metric $g_{e, hc}$; then we define

$$\text{LAT}(M, \{\mu^q_j\}, F) = \text{LAT}(M, g_{e, d}; F) - \log \left(\prod_{q=0}^n [\mu^q | \omega^q]^2 \right),$$

where $[\mu^q | \omega^q] = | \det W^q |$ with $W^q$ the matrix such that $\mu^q_i = \sum_j W^q_{ij} \omega^q_j$.

It is this quantity that is independent of the choice of metric.

1.2. Cusp metrics. Let $L$ be a smooth manifold with boundary $Z$. Let $x$ be a smooth, non-negative function on $L$ that vanishes precisely on $Z$ and such that $dx$ does not vanish anywhere on $Z$. We call such a function a ‘boundary defining function’ for $Z$, or ‘bdf’ for short. We fix a choice of bdf, and our constructions will depend (mildly) on this choice.

Let us single out a subset of the vector fields on $L$,

$$\mathcal{V}_\phi(L) = \{ V \in \mathcal{C}^\infty(L; TL) : V \text{ is tangent to } Z, \text{ and } Vx \in \mathcal{O}(x^2) \}$$

and point out that there is a vector bundle over $L$ whose space of sections is $\mathcal{V}_\phi(L)$. We denote this bundle

$$\phi TL \rightarrow L$$

and refer to it as the ‘$\phi$-tangent bundle’ of $L$. (The $\phi$ more generally denotes a fibration on the boundary of $L$; in our present context the fibration is $Z \rightarrow Z \rightarrow \text{pt}$.) The $\phi$-tangent bundle is isomorphic to the usual tangent bundle of $L$. This isomorphism is canonical over $L^\circ$, but not over all of $L$. The dual bundle

$$\phi T^*L \rightarrow L$$

is called the ‘$\phi$-cotangent bundle’ of $L$. Note that $\frac{dx}{x}$ is a section of $\phi T^*L$ that is non-degenerate at $Z = \{x = 0\}$.

We can use $x$ to rescale the $\phi$-tangent bundle at $Z$ (see [Mel93, Chapter 8]), and we refer to the bundle

$$\text{hc} TL = \frac{1}{x} \phi TL$$

as the hc-tangent bundle or ‘hyperbolic cusp tangent bundle’. Its dual bundle

$$\text{hc} T^*L \rightarrow L$$

is the hc-cotangent bundle of $L$, and we point out that the one form $\frac{dx}{x}$, as a section of $\text{hc} T^*L$, is non-degenerate at $Z$. Similarly if $z$ is a local coordinate on $Z$ then $xdz$, as a local section of $\text{hc} T^*L$, is non-vanishing at $x = 0$.

An hc-metric is a bundle metric on the hc-tangent bundle. The simplest hc-metrics are those that in some collar neighborhood of $Z$ of the form $[0, 1]_x \times Z$ take the form

$$g_{hc, pt} = \frac{dx^2}{x^2} + x^2 g_Z$$

with $g_Z$ a metric on $Z$ independent of $x$. We refer to such metrics as product-type hc-metrics. An hc-metric $g_{hc}$ is product-type to order $\ell$ if there is a product-type metric $g_{hc, pt}$ such that

$$g_{hc} - g_{hc, pt} \in x^\ell \mathcal{C}^\infty(L; S^2(\text{hc} T^*L))$$
where $S^2(\text{hc} T^* L)$ denotes the bundle of symmetric bilinear forms on $\text{hc} T^* L$. In this paper our results will hold for hc-metrics that are product-type to order 2.

The heat kernel of a Laplace-type operator associated to an hc-metric is not as well-behaved as the corresponding object on a closed manifold ([Vai01], [ARS14, §7]). First, the heat kernel is possibly not trace class. Fortunately it is well-behaved enough that we can make sense of its renormalized trace

$$R \text{Tr} \left( e^{-t\Delta} \right) = \text{FP} \text{Tr}(x^2 e^{-t\Delta}).$$

Moreover, from [ARS14, §7] and the appendix of [AR13], the asymptotics of the renormalized trace of the heat kernel are more complicated as $t \to 0$:

$$R \text{Tr} \left( e^{-t\Delta} \right) \sim t^{-m/2} \sum_{k \geq 0} a_{k/2} t^{k/2} + t^{-1/2} \sum_{k \geq 0} b_{k/2} t^{k/2} \log t.$$

And finally, one does not always have exponential converge of $R \text{Tr} \left( e^{-t\Delta} \right)$ to dim ker $\Delta$ as $t \to \infty$. We will deal with these differences by adding appropriate additional assumptions.

Let us say that a flat bundle $F$ is Witt if, upon restricting to $Z$, we have

$$H^{v/2}(Z; F) = 0$$

where $v = \dim Z = m - 1$. If $\Delta$ is a Hodge Laplacian associated to a Witt bundle, then we know from [ARS14] that

$$R \text{Tr} \left( e^{-t\Delta} \right) - \dim \ker \Delta = O(e^{-t\lambda_1}) \text{ as } t \to \infty \text{ for some } \lambda_1 > 0.$$

If $g_{\text{hc}}$ is product-type to order two and $m$ is odd, then $a_{m/2} = b_{1/2} = 0$. Again, if $\Delta$ is a Hodge Laplacian associated to a Witt bundle, the zeta function

$$\zeta(s; \Delta) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^s R \text{Tr} \left( e^{-t\Delta} - P_{\ker \Delta} \right) \frac{dt}{t}$$

is a holomorphic function on $\text{Re } s > m/2$ that extends to a meromorphic function on $\mathbb{C}$, with at worst double poles but regular at the origin. Thus for flat Witt bundles we may define analytic torsion for a cusp manifold just as for a closed manifold.

1.3. Cusp degeneration. We say that a closed Riemannian manifold $(M, g)$ with a two-sided hypersurface $Z$ is undergoing cusp degeneration if the metric is degenerating from a smooth metric to a cusp metric on $M \setminus Z$. We will carry out these degenerations in a controlled fashion by studying ‘cusp surgery metrics’.

Let us start by performing the ‘radial blow-up’ of $Z \times \{0\}$ in $M \times [0,1]_\varepsilon$. Recall that this is a smooth manifold with corners,

$$X_s = [M \times [0,1]_\varepsilon, Z \times \{0\}],$$

obtained by replacing $Z \times \{0\}$ with its inward pointing spherical normal bundle (see [Mel93]). Figure II represents the space $X_s$. There is a natural map, known as the blown-down map,

$$\beta : X_s \longrightarrow M \times [0,1]_\varepsilon$$

obtained by collapsing the new boundary hypersurface of $X_s$ back to $Z \times \{0\}$. 
The manifold $X_s$ has three boundary hypersurfaces. One, $\beta^{-1}(\{\varepsilon = 1\})$, will not be relevant to our studies and will be cheerfully ignored. The other two are $\beta^{-1}(Z \times \{0\})$, known as the surgery boundary and denoted $\mathcal{B}_{sb}$, and

$$\mathcal{B}_{sm} = \beta^{-1}(M \times \{0\} \setminus Z \times \{0\}),$$

where the $m$ in the subscript recalls that this is where most of $M \times \{0\}$ ended up. Given any blow-down map, the ‘interior lift of a set’ is equal to the closure of the lift of that set minus the set being blown-up; thus $\mathcal{B}_{sm}$ is the interior lift of $M \times \{0\}$, which we denote

$$\mathcal{B}_{sm} = \beta^\sharp(M \times \{0\}).$$

There is a natural choice of boundary defining function for $\mathcal{B}_{sb}$, which we fix once and for all:

$$\rho_{sb} = \sqrt{x^2 + \varepsilon^2}.$$

When there is no possibility of confusion, we will denote this simply as $\rho$.

The interior of $\mathcal{B}_{sb}$ can be identified with the normal bundle to $Z$ in $M$; $\mathcal{B}_{sb}$ corresponds to its fiberwise compactification. The normal bundle to $Z$ is trivial by assumption, and so we have

$$\mathcal{B}_{sb} \cong Z \times [-\pi/2, \pi/2].$$

(Of course any closed interval would serve, but our usual choice of coordinates will correspond to $[-\pi/2, \pi/2]$, so we use this interval throughout.) We endow $\mathcal{B}_{sb}$ with a trivial fibration

$$Z \to \mathcal{B}_{sb} \xrightarrow{\phi_+} [-\pi/2, \pi/2].$$

Analogously to the hc-tangent bundle, we will define a ‘cusp surgery tangent bundle’ or $\varepsilon, \text{hc}$-bundle. First let $\pi_\varepsilon : X_s \to [0, 1]_\varepsilon$ be the composition of $\beta$ with the obvious projection and define

$$\varepsilon T X_s = \ker \pi_\varepsilon \subseteq TX_s.$$

Next let

$$\mathcal{V}_{\varepsilon, \phi} = \{ V \in C^\infty(X_s; \varepsilon TX_s) : V|_{\mathcal{B}_{sb}} \text{ tangent to fibers of } \phi_+ \text{ and } V\rho \in O(\rho^2) \}$$

and define $\varepsilon, \phi TX_s$ so that $\mathcal{V}_{\varepsilon, \phi}$ is its space of sections. Finally, let

$$\varepsilon, \text{hc} TX_s = \frac{1}{\rho} \varepsilon, \phi TX_s,$$
and let \( \varepsilon, \text{hc} T^*X_s \) denote the dual bundle. The one-forms
\[
\frac{dx}{\rho}, \quad \rho \, dz,
\]
where \( z \) denotes a coordinate along \( Z \), lift from the interior of \( X_s \) to a spanning set of sections of \( \varepsilon, \text{hc} T^*X_s \). Note that, as sections of \( \varepsilon, \text{hc} T^*X_s \), these do not degenerate at \( Z \).

A cusp surgery metric is a bundle metric on \( \varepsilon, \text{hc} TX_s \). We say that an \( \varepsilon, \text{hc} \)-metric is of \textbf{product type} if there is a tubular neighborhood \( \text{Tub}(Z) \cong [-1, 1]_x \times Z \subseteq M \) around \( Z \) in which the metric takes the form
\[
g_{\varepsilon, \text{hc}, \text{pt}} = \frac{dx^2}{x^2 + \varepsilon^2} + (x^2 + \varepsilon^2)g_Z
\]
where \( g_Z \) is a metric on \( Z \) that is independent of both \( x \) and \( \varepsilon \). We say that an \( \varepsilon, \text{hc} \)-metric \( g_{\varepsilon, \text{hc}} \) is of \textbf{product type to order} \( \ell \) if
\[
g_{\varepsilon, \text{hc}} - g_{\varepsilon, \text{hc}, \text{pt}} \in \rho^\ell C^\infty(X_s; S^2(\varepsilon, \text{hc} T^*X_s))
\]
for some product type metric \( g_{\varepsilon, \text{hc}, \text{pt}} \), where \( S^2(\varepsilon, \text{hc} T^*X_s) \) denotes the bundle of symmetric two-tensors on \( \varepsilon, \text{hc} T^*X_s \).

Let \( F \rightarrow X_s \) be a flat Euclidean vector bundle and let
\[
\bar{\partial}_{\text{DR}} = d + \delta
\]
be the corresponding de Rham operator. We will consider this as an operator on the bundle
\[
E = \Lambda^* \varepsilon, \text{hc} T^*X_s \otimes F.
\]
One of the advantages to using the \( \varepsilon, \text{hc} \)-cotangent bundle, as opposed to the usual cotangent bundle of \( X_s \), is that the leading order behavior of \( \bar{\partial}_{\text{DR}} \) will be described by tractable model operators, discussed below. We are interested in the action of \( \bar{\partial}_{\text{DR}} \) as an unbounded operator on \( L_{\varepsilon, \text{hc}}(M; E) \), the natural \( L^2 \) space associated to an \( \varepsilon, \text{hc} \)-metric \( g_{\varepsilon, \text{hc}} \) and the bundle metric on \( F \). However, for some constructions it will be easier to work with
\[
L_{\varepsilon, b}^2(M; E) = \rho^{v/2} L_{\varepsilon, \text{hc}}^2(M; E),
\]
where \( v = \dim Z = m - 1 \). Thus our main object of interest is the operator
\[
D_{\text{DR}} = \rho^{v/2} \bar{\partial}_{\text{DR}} \rho^{-v/2}
\]
acting as an unbounded operator on \( L_{\varepsilon, b}^2(M; E) \).

If \( g_{\varepsilon, \text{hc}} \) is of product-type to order two, then we have simple expressions for the model operators of \( D_{\text{DR}} \). First let us write
\[
\Lambda^\ell \left( \varepsilon, \text{hc} T^* \text{Tub}(Z) \right) \cong \rho^\ell T^*Z \oplus \frac{dx}{\rho} \wedge \left( \rho^{\ell - 1} T^{\ell - 1} Z \right);
\]
this splitting distinguishes between forms with a \( dx \) and forms without a \( dx \). With respect to this splitting, a direct computation tells us that \( D_{\text{DR}} \) is given by
\[
D_{\text{DR}} = \left( \begin{array}{cc}
\rho \partial_x + \left( N_Z - \frac{1}{2} v \right) \frac{z}{\rho} & -\rho \partial_x + (N_Z - \frac{1}{2} v) \frac{z}{\rho} \\
\frac{1}{\rho} \delta^Z_{\text{DR}} & -\frac{1}{\rho} \delta^Z_{\text{DR}}
\end{array} \right),
\]
near \( \mathfrak{V}_{ab} \) up to higher order terms in \( \rho \) as \( \varepsilon, \text{hc} \) differential operators. Here \( N_Z \) is the number operator on \( Z \) that multiplies a differential form by its degree.
The first model operator, known as the **vertical operator**, is

\[(1.3) \quad D_v = \rho D_{\text{dR}}|_{\mathcal{B}_{sb}} = \begin{pmatrix} \partial_{\text{dR}}^Z & 0 \\ 0 & -\partial_{\text{dR}}^Z \end{pmatrix}.\]

Its null space forms a vector bundle over \(\mathcal{B}_{sb}\) which is just the space of scaled harmonic forms on \(Z\), thought of as a trivial vector bundle over \([-\pi/2, \pi/2]\) and then pulled-back along \(\phi_+\). We will denote this bundle by

\[\rho^N\mathcal{H}^*(Z; F) \to \mathcal{B}_{sb}.\]

The second model operator, known as the **horizontal operator**, is defined by

\[(1.4) \quad D_h u = \Pi_h (0 - \langle X \rangle \partial_X + (N_Z - \frac{1}{2}v) \frac{X}{\langle X \rangle}) \tilde{u},\]

where \(\Pi_h\) denotes the projection onto \(Z\)-harmonic forms, \(u\) is a section of ker \(D_v\) and \(\tilde{u}\) is any choice of extension off \(\mathcal{B}_{sb}\). In terms of (1.2), the operator \(D_h\) is given by

\[D_h = \begin{pmatrix} 0 & -\langle X \rangle \partial_X + (N_Z - \frac{1}{2}v) \frac{X}{\langle X \rangle} \\ \langle X \rangle \partial_X + (N_Z - \frac{1}{2}v) \frac{X}{\langle X \rangle} & 0 \end{pmatrix}, \quad \langle X \rangle = \sqrt{1 + X^2}\]

acting on \(C^\infty(\mathbb{R}^X; \rho^N\mathcal{H}^*(Z; F) \oplus \frac{dX}{\langle X \rangle} \wedge \rho^N\mathcal{H}^*(Z; F))\), where the restriction of \(\rho^N\mathcal{H}^k(Z; F)\) to \(\mathcal{B}_{sb}\) is well defined as a section of \(\Lambda^k(\varepsilon, \text{hc}T^*X_s)\). Thus, \(D_h\) is a \(b\)-operator in the sense of Melrose [Mel93]. If \(F\) is a Witt bundle, then \(D_h\) is Fredholm [ARS14, Lemma 2.1].

Finally, \(D_{\text{dR}}\) induces an operator on \(\mathcal{B}_{sm}\). This face is the manifold with boundary \(M_0 = [M; Z]\) and

\[D_d = D_{\text{dR}}|_{\mathcal{B}_{sm}}\]

is the twisted de Rham operator corresponding to the hc-metric \(g_0 = g_{\varepsilon, \text{hc}}|_{\mathcal{B}_{sm}}\) and the flat bundle \(F|_{\mathcal{B}_{sm}}\).

## 2. Analysis of the Model Operator

Let \(M\) be a closed manifold of odd dimension \(m\), \(Z\) a two-sided hypersurface with fixed boundary defining function \(x\) and \(g_{\varepsilon, \text{hc}}\) a cusp surgery metric of product-type to second order. If \(F \to X_s\) is a flat vector bundle of Witt type, then we have seen that there is a model \(b\)-operator

\[D_b = \begin{pmatrix} 0 & -\langle X \rangle \partial_X + (N_Z - \frac{1}{2}v) \frac{X}{\langle X \rangle} \\ \langle X \rangle \partial_X + (N_Z - \frac{1}{2}v) \frac{X}{\langle X \rangle} & 0 \end{pmatrix}, \quad \langle X \rangle = \sqrt{1 + X^2}\]

acting on \(C^\infty(\mathbb{R}^X; \rho^N\mathcal{H}^*(Z; F) \oplus \frac{dX}{\langle X \rangle} \wedge \rho^N\mathcal{H}^*(Z; F))\). In this section we study this operator and its contribution to the asymptotics of analytic torsion.
2.1. Null space of the horizontal operator. First let us compute its null space, and that of its square. Note that if
\[
\begin{pmatrix}
0 & -(X)\partial_X + (N_Z - \frac{1}{2}v)\frac{X}{(X)} \\
-(X)\partial_X + (N_Z - \frac{1}{2}v)\frac{X}{(X)} & 0
\end{pmatrix}
\begin{pmatrix}
f(X) \\
h(X)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
then the projections of \( f \) and \( h \) onto the spaces of forms of fixed vertical degree \( k \) (that is, having degree \( k \) in \( Z \)), which we denote \( f_k \) and \( h_k \) respectively, are also in the null space of \( D_b \). Now since
\[
P(a) = \langle X \rangle \partial_X + a\frac{X}{(X)} = \langle X \rangle^{-a}(\langle X \rangle \partial_X)\langle X \rangle^a
\]
we see that
\[
\langle X \rangle^{-(k-v/2)}(\langle X \rangle \partial_X (\langle X \rangle^{(k-v/2)} f_k(X))) = 0 \implies f_k(X) = C\langle X \rangle^{v/2-k}
\]
and that
\[
-\langle X \rangle^{(k-v/2)}(\langle X \rangle \partial_X (\langle X \rangle^{-(k-v/2)} h_k(X))) = 0 \implies h_k(X) = C\langle X \rangle^{k-v/2}.
\]
Thus we have found
\[
\ker D_b = \text{span}\left\{ \begin{pmatrix} u(X)^{v/2-k} \\ v(X)^{k-v/2} \end{pmatrix} : u, v \in \rho^k\mathcal{H}^k(Z;F), k \in \mathbb{N}_0 \right\}.
\]
We are interested in \( D_b \) as an unbounded operator on \( L^2_b \), i.e., with the measure \( \frac{dX}{(X)} \) on \( \mathbb{R} \). With respect to this measure, \( \langle X \rangle^a \) is in \( L^2 \) iff \( a < 0 \), and hence the \( L^2 \) kernel of \( D_b \) is
\[
(2.1) \quad \ker_{L^2} D_b = \text{span}\left\{ \begin{pmatrix} u(X)^{v/2-k} \\ v(X)^{k-v/2} \end{pmatrix} : u \in \rho^k\mathcal{H}^k(Z;F), k > v/2, \quad v \in \rho^k\mathcal{H}^k(Z;F), k < v/2 \right\}.
\]
Next consider
\[
D_b^2 = \begin{pmatrix} -P(\frac{1}{2}v - N_Z)P(N_Z - \frac{1}{2}v) & 0 \\
0 & -P(N_Z - \frac{1}{2}v)P(\frac{1}{2}v - N_Z) \end{pmatrix}
\]
and note that, for \( j \in C^\infty(\mathbb{R}) \),
\[
P(-a)P(a)j = 0 \implies P(a)j = C\langle X \rangle^a \implies j = C\ell_a(X) + C'\langle X \rangle^{-a},
\]
with \( \ell_a(X) = \langle X \rangle^{-a}\int_0^X \langle s \rangle^{2a-1} ds \).

Notice that as \( X \to \pm\infty \), \( |\ell_a(X)| \) is of order \( |X|^{a} \) for \( a \neq 0 \), while for \( a = 0 \), \( \ell_0(X) = \sinh^{-1}(X) \), so \( \ell_a \) is never in \( L^2 \) with respect to the density \( \frac{dX}{(X)} \). Hence the null space of \( D_b^2 \) acting on smooth sections of \( \langle X \rangle^N\mathcal{H}^k(Z) \) is
\[
(2.2) \quad \ker_{L^2} D_b^2 = \text{span}\left\{ \begin{pmatrix} u(X)^{v/2-k} + u'\ell_{k-v/2}(X) \\ v(X)^{k-v/2} + v'\ell_{v/2-k}(X) \end{pmatrix} : u, u', v, v' \in \langle X \rangle^k\mathcal{H}^k(Z;F), k \in \mathbb{N}_0 \right\}.
\]
Since the function \( \ell_a(X) \) is never in \( L^2_b \),
\[
\ker_{L^2} D_b^2 = \ker_{L^2} D_b,
\]
which could also have been deduced from the formal self-adjointness of $D_b$. Let us emphasize in particular that $D_b^2$ has no $L^2$-kernel on forms of total degree (i.e., degree in $\frac{dX}{(X)}$ plus degree in $Z$) equal to zero or $m$.

2.2. Analytic torsion contribution of the horizontal operator. Let

$$(D_b^2)_{j,k} = D_b^2|_{\Lambda^j R_\chi \backslash \langle X \rangle ^k \mathcal{H}^k(Z;F)}, \quad j \in \{0, 1\}, \ k \in \{0, \ldots, v\}$$

where $v = \dim Z = m - 1$. Each of these is a Laplace-type operator on $\mathbb{R}_\chi$ and we denote the corresponding zeta function by $\zeta_{j,k}(s)$. From [ARS14, Theorem 8.1], we know that the contribution of the horizontal operator $D_b$ to the asymptotics of analytic torsion is through

$$\frac{1}{2} \sum (-1)^{j+k}(j+k)\zeta'_{j,k}(0).$$

In this subsection we will compute this contribution.

From the previous subsection we see that the heat kernel of $D_b$ satisfies

$$e^{-tD_b^2}|_{\rho^j \mathcal{H}^q(Z;F) \oplus \frac{dX}{(X)} \wedge \rho^{q-1} \mathcal{H}^{q-1}(Z;F)} = \begin{pmatrix} e^{tP(\frac{v-2q}{2})P(\frac{2q-v}{2})} & 0 \\ 0 & e^{tP(\frac{2(q-1)-v}{2})P(\frac{v-2(q-1)}{2})} \end{pmatrix}.$$  

We introduce the abbreviation

$$F_a^R = R \text{Tr}(e^{tP(a)}P(-a))$$

and note that

$$R \text{Tr}(e^{-tD_b^2})|_{\rho^j \mathcal{H}^q(Z;F) \oplus \frac{dX}{(X)} \wedge \rho^{q-1} \mathcal{H}^{q-1}(Z;F)} = b_q F_{(v-2q)/2}^R + b_{q-1} F_{(2(q-1)-v)/2}^R,$$

where $b_q = \dim \mathcal{H}^q(Z; F)$. So we can write

$$\sum (-1)^{j+k}(j+k)R \text{Tr}(e^{-t(D_b^2)_{j,k}}) = \sum q (-1)^q b_q F_{(v-2q)/2}^R + b_{q-1} F_{(2(q-1)-v)/2}^R.$$  

Let us use Poincaré duality on $Z$ and the corresponding identification of $\mathcal{H}^q(Z; F)$ with $\mathcal{H}^{v-q}(Z; F)$ to group together all of the terms acting on vertical harmonic forms of degree $q$ and $v-q$,

$$(-1)^q b_q \left[ qF_{(v-2q)/2}^R - (q+1)F_{(2q-v)/2}^R - (-1)^v(v-q)F_{(v-2(v-q))/2}^R - (-1)^v(v-q+1)F_{(2(v-q)-v)/2}^R \right]$$

$$= (-1)^q b_q \left[ (q - (-1)^v(v-q+1))F_{(v-2q)/2}^R + (-q-1+(-1)^v(v-q))F_{(2q-v)/2}^R \right].$$

Summing the expression (2.5) over $q < \frac{v}{2}$ yields (2.4), so overall, (2.3) is equal to

$$\frac{1}{2} \sum_{q=0}^v (-1)^q b_q \left[ (q - (-1)^v(v-q+1)) (-\log \det -P(\frac{v-2q}{2})P(\frac{2q-v}{2})) \right]$$

$$+((-q-1+(-1)^v(v-q)) (-\log \det -P(\frac{2q-v}{2})P(\frac{v-2q}{2})).$$

It thus suffices to compute the determinant of $-P(-a)P(a)$ on $\mathbb{R}$ (endowed with the metric $dX^2/\langle X \rangle^2$ and bdf $\rho = \langle X \rangle^{-1}$),

$$\det (-P(-a)P(a)) = e^{-C_\rho P(-a)P(a)(0)}.$$
Our strategy will be to compute the variation in $a$ of the renormalized trace (see (2.10)) and use this to compute the determinant (see (2.13)). Once we have computed these one-dimensional determinants, we return to (2.6) in (2.14).

Let us start with the two cases we can compute directly.

**Lemma 2.1.** When $a = 0$, we have

\[ e^{tP(0)^2}(X, X') = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|\sinh^{-1}(X) - \sinh^{-1}(X')|^2}{4t} \right), \]

\[ R \text{Tr} \left( e^{tP(0)^2} \right) = \frac{\log 2}{\sqrt{4\pi t}}, \quad \zeta_{-P(0)^2}(s) = 0, \quad \log \det -P(0)^2 = 0. \]

When $a = -1$, we have

\[ e^{tP(1)P(-1)}(X, X') = e^{-t}e^{tP(0)^2}(X, X'), \quad R \text{Tr} \left( e^{tP(1)P(-1)} \right) = \frac{e^{-t}\log 2}{\sqrt{4\pi t}}, \]

\[ \int_0^\infty t^s R \text{Tr} \left( e^{tP(1)P(-1)} \right) \frac{dt}{t} = \frac{\Gamma(s - 1/2) \log 2}{\sqrt{\pi}}, \quad \log \det (-P(1)P(-1)) = 2\log 2. \]

**Proof.** In the coordinate $u = \sinh^{-1}(X)$,

\[ P(a) = \partial_u + a \tanh u, \quad -P(-a)P(a) = -\partial_u^2 + a^2 - (a^2 + a) \sech^2 u. \]

Hence $-P(0)^2$ is the Euclidean Laplacian and $-P(1)P(-1)$ is the Euclidean Laplacian plus one. In the former case the restriction to the diagonal is $(4\pi t)^{-1/2}$ and in the latter $e^{-t}(4\pi t)^{-1/2}$; as these are independent of $u$, the renormalized trace is computed by multiplying these expressions by the renormalized volume. The renormalized volume is, for our choice of measure $dX/(X)$ and of boundary defining function $\varrho = \langle X \rangle^{-1}$,

\[ \int_R dX/\langle X \rangle = 2 \int_{R^+} dX/\langle X \rangle = 2 \int_0^1 \frac{d\varrho}{\varrho \sqrt{1 - \varrho^2}} = 2 \text{FP} \int_0^1 \frac{d\varrho}{\varrho \sqrt{1 - \varrho^2}}, \]

\[ = 2 \text{FP}(\log(\sqrt{1 - \varepsilon^2} + 1) - \log \varepsilon) = 2 \text{FP}(\log 2 - \log \varepsilon + O(\varepsilon^2)) = 2 \log 2. \]

This proves that $R \text{Tr} \left( e^{tP(0)^2} \right) = \frac{\log 2}{\sqrt{4\pi t}}$. It is easy to see that the renormalized Mellin transform over $\mathbb{R}_t^+$ of a power of $t$ is equal to zero; indeed, let us define, for any function $f(t)$ with an asymptotic expansion in $t$ as $t \to 0$ and $t^{-1}$ as $t \to \infty$:

\[ (2.7) \quad \mathcal{M}_0(f, s) = \int_0^1 t^s f(t) \frac{dt}{t}, \quad \mathcal{M}_\infty(f, s) = \int_1^\infty t^s f(t) \frac{dt}{t}. \]

Each of these extends to a meromorphic function on $\mathbb{C}$, which we denote by the same symbol, and the renormalized Mellin transform of $f$ is

\[ \mathcal{M}(f, s) = \mathcal{M}_0(f, s) + \mathcal{M}_\infty(f, s). \]

If $f(t) = t^\nu$, then $\mathcal{M}_0(f, s) = \frac{1}{s+\nu}$ and $\mathcal{M}_\infty(f, s) = -\frac{1}{s+\nu}$, and so $\mathcal{M}(f, s) = 0$. Hence we see that the zeta function of $-P(0)^2$ is identically zero.

For $a = -1$ we have, for any $s > \frac{1}{2}$,

\[ \int_0^\infty t^s \text{Tr}(e^{tP(1)P(-1)}) \frac{dt}{t} = \frac{\log 2}{\sqrt{\pi}} \int_0^\infty t^{s-1/2} e^{-t} \frac{dt}{t} = \frac{\Gamma(s - 1/2) \log 2}{\sqrt{\pi}} \sim -2 \log 2 + O(s); \]
hence
\[
\log \det -P(1)P(-1) = -\frac{\partial}{\partial s} \bigg|_{s=0} \zeta(s) = -\frac{\partial}{\partial s} \bigg|_{s=0} \left( \frac{\Gamma(s - 1/2) \log 2}{\Gamma(s) \sqrt{\pi}} \right) = 2 \log 2.
\]

Next let us compute the variation of the renormalized trace for arbitrary $a$. First note that
\[
\partial_a P(a) = \partial_a \left( \langle X \rangle^{-a} \langle X \rangle \partial X \langle X \rangle^a \right) = -\log \langle X \rangle P(a) + P(a) \log \langle X \rangle = [P(a), \alpha]
\]
where $\alpha$ denotes $\log \langle X \rangle$. Similarly
\[
\partial_a P(-a) = [\alpha, P(-a)], \quad \partial_a (-P(-a)P(a)) = -\alpha P(-a)P(a) + 2P(-a)\alpha P(a) - P(-a)P(a)\alpha.
\]
Next, by Duhamel’s formula, we have
\[
\frac{\partial}{\partial a}^R \text{Tr}(e^{tP(-a)P(a)}) = -R \text{Tr} \left( \int_0^t e^{\tau P(-a)P(a)} \partial_\alpha (-P(-a)P(a)) e^{(t-\tau)P(-a)P(a)} \right) d\tau = T_1 + R_1
\]
where
\[
T_1 = -t^R \text{Tr} (e^{tP(-a)P(a)} \partial_\alpha (-P(-a)P(a))), \quad R_1 = - \int_0^t R \text{Tr} \left[ e^{\tau P(-a)P(a)} \partial_\alpha (-P(-a)P(a)), e^{(t-\tau)P(-a)P(a)} \right] d\tau.
\]
Note that since the renormalized trace does not vanish on commutators, $R_1$ is not automatically zero.

Let us focus first on $T_1$. We can rewrite it as
\[
T_1 = -t^R \text{Tr} (e^{tP(-a)P(a)}(-\alpha P(-a)P(a) + 2P(-a)\alpha P(a) - P(-a)P(a)\alpha))
\]
\[
= -t^R \text{Tr} \left( -P(-a)P(a)e^{tP(-a)P(a)}\alpha + 2P(a)e^{tP(-a)P(a)}P(-a)\alpha - e^{tP(-a)P(a)}P(-a)P(a)\alpha \right.
\]
\[
\left. + \left[ P(-a)P(a), e^{tP(-a)P(a)}\alpha \right] - 2 \left[ P(a), e^{tP(-a)P(a)}P(-a)\alpha \right] \right).
\]
In turn let us write this as $T_2 + R_2$ where $R_2$ consists of the summands involving commutators. From the uniqueness of the solution to the heat equation, we note that $P(-a)e^{P(a)P(-a)} = e^{P(-a)P(a)}P(-a)$, and so we can write $T_2$ as
\[
T_2 = -2t^R \text{Tr}(P(a)P(-a)e^{tP(a)P(-a)}\alpha) + 2t^R \text{Tr}(P(-a)P(a)e^{tP(-a)P(a)}\alpha)
\]
\[
= 2t^R \partial_t \left( \text{Tr}(e^{tP(a)P(-a)}\alpha) - \text{Tr}(e^{tP(-a)P(a)}\alpha) \right)
\]
which we can rewrite as $2t^R \partial_t^R \text{Str}(e^{tP_2(a)}\alpha)$, where
\[
\hat{P}(a) = \begin{pmatrix} 0 & P(-a) \\ P(a) & 0 \end{pmatrix} \quad \text{and} \quad \text{Str} \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \text{Tr}(A) - \text{Tr}(D).
\]

Thus we have
\[
(2.8) \quad \frac{\partial}{\partial a}^R \text{Tr}(e^{tP(a)P(-a)}) = 2t^R \partial_t^R \text{Str}(e^{t\hat{P}_2(a)}\alpha) + R_1 + R_2
\]
where $R_1$ and $R_2$ involve renormalized traces of commutators.
To address these terms we will make use of appropriate trace defect formulæ. We will use the same conventions as in [Mel93] regarding Mellin transform and indicial operators, namely:

$$M(u)(\lambda) = \int_0^\infty x^{-i\lambda} u(x) \frac{dx}{x}, \quad (M^{-1} v)(x) = \frac{1}{2\pi} \int_{\text{Im} \lambda = \eta} x^{i\lambda} v(\lambda) \, d\lambda$$

$$I(A, \lambda) = [x^{-i\lambda} A x^{i\lambda}]_\partial.$$

The latter means that $I(A, \lambda)$ acts on a section $u$ over the boundary by choosing an extension $\tilde{u}$ off of the boundary, applying $x^{-i\lambda} A x^{i\lambda}$ to $\tilde{u}$, and then restricting back to the boundary; the result is independent of the choice of extension.

**Lemma 2.2.** On any manifold with boundary $M$ with a fixed choice of bdf $\varrho$,

a) [Mel93, Lemma 5.10] If $A$ is a $b$-pseudodifferential operator and $B$ is a smoothing $b$-pseudodifferential operator, then

$$R \text{Tr}([A, B]) = \frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_\varrho (\partial_\lambda I(A, \lambda) I(B, \lambda)) \, d\lambda,$$

where the trace in the integrand is the trace $\text{Tr}_\varrho : \Psi^{-\infty} (\partial M) \rightarrow \mathbb{C}$.

b) With $A$ and $B$ as above,

$$R \text{Tr}([A, B \log \varrho]) = -\frac{1}{4\pi} \int_{\mathbb{R}} \text{Tr}_\varrho (\partial_\lambda^2 I(A, \lambda) I(B, \lambda)) \, d\lambda.$$

**Proof.** Following [MN96] (cf. [MR04, Alb09]), let us use Riesz renormalization to define the renormalized trace,

$$R \text{Tr}(B) = \text{FP} \text{Tr}(\varrho^z B)$$

for any operator $B$ such that $\varrho^z B$ is trace-class for large enough $\text{Re}(z)$. We have

$$R \text{Tr}([A, B]) = \text{FP} \text{Tr}(\varrho^z [A, B]) = \text{FP} \text{Tr}([\varrho^z, A] B) = \text{FP} \text{Tr}(z \varrho^z \tilde{A}(z) B) = \text{Res}_{z=0} \text{Tr}(\varrho^z \tilde{A}(z) B)$$

where $\tilde{A}(z) = \frac{A - \varrho^{-z} A \varrho^z}{z}$. Note that this is a holomorphic function of $z$ and has indicial operator

$$I(\tilde{A}(z), \lambda) = \frac{I(A, \lambda) - I(A, \lambda + \frac{1}{z})}{z}$$

with Taylor expansion at $z = 0$ given by

$$-\sum_{k \geq 1} \frac{z^{k-1}}{k!} \partial_\lambda^k I(A, \lambda).$$

In particular, we point out that

(2.9) $I(\tilde{A}(0), \lambda) = i \partial_\lambda I(A, \lambda) = -I([A, \log \varrho], \lambda)$.

To compute the residue $\text{Res}_{z=0} \text{Tr}(\varrho^z \tilde{A}(z) B)$ note that

$$\int_0^\delta \frac{\varrho^k}{\varrho} \, d\varrho = \frac{\delta^{z+k}}{z+k}$$
so only the term with \( k = 0 \) contributes to the residue at \( z = 0 \). Hence \( \text{Tr}(\varphi^z \tilde{A}(z) B) \) extends to a meromorphic function with a simple pole at \( z = 0 \) and residue equal to
\[
\left. \text{Tr}_\partial (\tilde{A}(0) B) \right|_{\varphi = 0} = \frac{1}{2\pi} \int \text{Tr}_\partial (\tilde{I}(\tilde{A}(0) B, \lambda)) \, d\lambda = \frac{i}{2\pi} \int \text{Tr}_\partial (\partial_\lambda I(A, \lambda) I(B, \lambda)) \, d\lambda
\]
as required.

Now replace \( B \) with \( B \log \varphi \). As before, since
\[
\int_0^\delta \varphi^z (\varphi^k \log \varphi) \frac{d\varphi}{\varphi} = \frac{\delta^{z+k} \log \delta}{z+k} - \frac{\delta^{z+k}}{(z+k)^2},
\]
only the terms with \( k = 0 \) contribute to the residue. Furthermore, for small \( z \) we have \( \delta^z = 1 + z \log \delta + \mathcal{O}(z^2) \), and so
\[
\frac{\delta^z \log \delta}{z} - \frac{\delta^z}{z^2} \sim \log \delta - \frac{1 + z \log \delta}{z^2} + \mathcal{O}(1) = -\frac{1}{z^2} + \mathcal{O}(1).
\]
We see that to compute the residue we will only need (minus one times) the \( \mathcal{O}(z) \) term in \( \text{Tr}(\tilde{A}(z) B) \). Since the \( \mathcal{O}(z) \) term in the expansion of \( I(\tilde{A}(z), \lambda) \) is \( \tilde{z} \partial_\lambda^2 I(A, \lambda) \), we find
\[
\text{Res}_{z=0} \text{Tr}(\varphi^z \tilde{A}(z) B \log \varphi) = -\frac{1}{4\pi} \int \text{Tr}_\partial (\partial_\lambda^2 I(A, \lambda) I(B, \lambda)) \, d\lambda.
\]

To compute the indicial operator of \( P(a) \) let us recall that our bdf is \( \varphi = |X|^{-1} \), so that
\[
X = \text{sign}(X) \sqrt{\varphi^{-2} - 1}, \quad \langle X \rangle \partial_X = -\text{sign}(X) \sqrt{1 - \varphi^2 \varphi \partial_\varphi}
\]
and hence
\[
P(a) = \varphi^a \left( -\text{sign}(X) \sqrt{1 - \varphi^2 \varphi \partial_\varphi} \right) \varphi^{-a}, = -\text{sign}(X) \sqrt{1 - \varphi^2 (\varphi \partial_\varphi - a)}
\]
\[
I(P(a), \lambda) = \begin{cases} -i\lambda + a & \text{at } X \to +\infty, \\ i\lambda - a & \text{at } X \to -\infty. \end{cases}
\]
It follows that \( I(-P(-a) P(a), \lambda) = \lambda^2 + a^2 \) and \( I(e^{tP(-a)P(a)}, \lambda) = e^{-t(\lambda^2 + a^2)} \) at both ends of \( \mathbb{R} \).

We can use this observation and Lemma 2.2 to compute \( R_1 \) and \( R_2 \). Indeed, from (2.9) and the fact that \( \alpha = -\log \varphi \) that
\[
I(\partial_a (-P(-a) P(a)), \lambda) = I(-[P(-a), \log \varphi] P(a) + P(-a) [P(a), \log \varphi]), \lambda)
\]
\[
= -(\frac{1}{2} \partial_\lambda I(P(-a), \lambda)) I(P(a), \lambda) + P(-a) (\frac{1}{2} \partial_\lambda I(P(a), \lambda)) = 2a
\]
(the same at both ends of \( \mathbb{R} \)). Hence
\[
R_1 = -\int_0^t R \text{Tr} \left[ e^{\tau P(-a) P(a)} \partial_a (-P(-a) P(a)), e^{(t-\tau) P(-a) P(a)} \right] d\tau
\]
\[
= -2 \int_0^t \frac{i}{2\pi} \int_{\mathbb{R}} \partial_\lambda \left( e^{-\tau(a^2 + \lambda^2)2a} \right) e^{-(t-\tau)(a^2 + \lambda^2)} \, d\lambda d\tau
\]
where we multiply by two in applying the trace defect formula since we have the same contribution from each end of \( \mathbb{R} \). Since the integrand is odd in \( \lambda \), we see that \( R_1 = 0 \).
Next for $R_2$ we have (again multiplying by two to take into account both ends of $\mathbb{R}$)

$$R_2 = -t \, R\text{Tr} \left[ \left[ P(-a)P(a), e^{tP(-a)P(a)} \right] \right] - 2 \left[ P(a), e^{tP(-a)P(a)} P(-a) \right]$$

$$= \frac{t}{2\pi} \int_{\mathbb{R}} \left( -\left( \partial^2 \right)_a I(P(-a)P(a), \lambda) e^{-t(a^2 + \lambda^2)} + 2(\partial^2 \right)_a I(P(a), \lambda)) e^{-t(a^2 + \lambda^2)} I(P(-a), \lambda) \right) d\lambda$$

$$= \frac{t}{\pi} \int_{\mathbb{R}} e^{-t(a^2 + \lambda^2)} d\lambda = \sqrt{\frac{t}{\pi}} e^{-ta^2},$$

and so altogether

$$(2.10) \quad \frac{\partial}{\partial a} R\text{Tr}(e^{tP(-a)P(a)}) = \sqrt{\frac{t}{\pi}} e^{-ta^2} - 2t \frac{\partial}{\partial t} R\text{Str}(e^{tP_2(\log a)}).$$

**Lemma 2.3.** When $a = 1$, we have

$$\log \det (-P(-1)P(1)) = 0.$$

**Proof.** Note that since $R\text{Str} \left( e^{tP_2(\log a)} \right)$ is an odd function of $a$ for each fixed $t$, as are its derivatives in $t$, we have

$$\frac{\partial}{\partial a} \left( R\text{Tr}(e^{tP(-a)P(a)}) - R\text{Tr}(e^{tP(a)P(-a)}) \right) = 2 \sqrt{\frac{t}{\pi}} e^{-ta^2}$$

Since $R\text{Str}(e^{tP_2(0)}) = 0$, integrating from 0 to $a$ yields

$$(2.11) \quad R\text{Tr}(e^{tP(-a)P(a)}) - R\text{Tr}(e^{tP(a)P(-a)}) = 2 \int_0^a \sqrt{\frac{t}{\pi}} e^{-tb^2} db.$$

Hence from Lemma 2.1 we have

$$R\text{Tr} \left( e^{tP(-1)P(1)} \right) = R\text{Tr} \left( e^{tP(1)P(-1)} \right) + 2 \int_0^1 \sqrt{\frac{t}{\pi}} e^{-tb^2} db = \frac{e^{-t} \log 2}{\sqrt{\pi t}} + \Gamma(s + 1/2) \frac{e^{-tb^2}}{db}. $$

Consider this integral as a function of $t$. Writing it alternately as

$$2 \int_0^1 \sqrt{\frac{t}{\pi}} e^{-tb^2} db = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv$$

we see that it is $O(t^{1/2})$ as $t \to 0$ and $1 + O(t^{-1/2}e^{-t})$ as $t \to \infty$. It follows that the integral

$$2 \int_0^\infty t^{s} \int_0^1 \sqrt{\frac{t}{\pi}} e^{-tb^2} db \frac{dt}{t}$$

exists for all $s$ with real part in $(-1/2, 0)$. For these $s$ we can use Fubini’s theorem to find

$$2 \int_0^\infty t^{s} \int_0^1 \sqrt{\frac{t}{\pi}} e^{-tb^2} db \frac{dt}{t} = \frac{\Gamma(s + 1/2) \log 2}{\sqrt{\pi}} - \frac{\Gamma(s + 1/2)}{\sqrt{\pi} \Gamma(s)}.$$

We also know that $\mathcal{M} (R\text{Tr}(e^{tP(1)P(-1)}), s) = \frac{\Gamma(s + 1/2) \log 2}{\sqrt{\pi} \Gamma(s)}$, so altogether it follows that the zeta function of $-P(-1)P(1)$ is equal to

$$\zeta_1(s) = \frac{1}{\Gamma(s)} \mathcal{M} (R\text{Tr}(e^{tP(-1)P(1)}), s) = \frac{\Gamma(s - 1/2) \log 2}{\sqrt{\pi} \Gamma(s)} - \frac{\Gamma(s + 1/2)}{\sqrt{\pi} \Gamma(s)} \sim -1 + O(s^2).$$
and hence
\[
\log \det -P(-1)P(1) = -\partial_s|_{s=0} \zeta_1(s) = 0. \]

Next, using the notation of [2,7], we point out that
\[
\frac{\partial}{\partial a} \mathcal{M}_0 \left( R \text{Tr} \left( e^{tP(-a)P(a)} \right), s \right) = \mathcal{M}_0 \left( \frac{\partial}{\partial a} R \text{Tr} \left( e^{tP(-a)P(a)} \right), s \right),
\]
\[
\frac{\partial}{\partial a} \mathcal{M}_\infty \left( R \text{Tr} \left( e^{tP(-a)P(a)} \right), s \right) = \mathcal{M}_\infty \left( \frac{\partial}{\partial a} R \text{Tr} \left( e^{tP(-a)P(a)} \right), s \right),
\]
since this interchange is justified in the particular regions of \( s \in \mathbb{C} \) where these functions are holomorphic. Hence, for \( a \neq 0 \), writing \( \zeta_a(s) = \frac{1}{\Gamma(s)} \mathcal{M} \left( R \text{Tr} \left( e^{tP(-a)P(a)} \right), s \right) \), we have
\[
\frac{\partial}{\partial a} \zeta_a(s) = \frac{1}{\Gamma(s)} \mathcal{M} \left( \sqrt{\frac{t}{\pi}} e^{-ta^2} - 2t \partial_t R \text{Str} \left( e^{tP^2(a)} \log \varrho \right) \right) =: \tilde{\zeta}_a(s) + \hat{\zeta}_a(s).
\]
We now examine \( \tilde{\zeta}_a(s) \) and \( \hat{\zeta}_a(s) \) in a neighborhood of \( s = 0 \).

For \( \tilde{\zeta}_a(s) \), note that
\[
\int_0^\infty t^{s+1/2} e^{-ta^2} \frac{dt}{t} = a^{-2s-1} \int_0^\infty y^{s+1/2} e^{-y} \frac{dy}{y} = a^{-2s-1} \Gamma(s + 1/2)
\]
and hence,
\[
\tilde{\zeta}_a(s) = \frac{a^{-2s-1} \Gamma(s + 1/2)}{\sqrt{\pi} \Gamma(s)} \sim \frac{1}{a} s + \mathcal{O}(s^2) \text{ as } s \to 0.
\]

For \( \hat{\zeta}_a(s) \), we start by integrating by parts to find
\[
\hat{\zeta}_a(s) = \frac{2s}{\Gamma(s)} \mathcal{M} \left( R \text{Str} \left( e^{tP^2(a)} \log \varrho \right), s \right).
\]
Indeed, the integration by parts is justified for each of \( \mathcal{M}_0 \) and \( \mathcal{M}_\infty \) in the region where it is holomorphic, and the resulting boundary terms cancel out when we add together the meromorphically continued \( \mathcal{M}_0 \) and \( \mathcal{M}_\infty \). As above, \( \mathcal{M}_0 \left( R \text{Str} \left( e^{tP^2(a)} \log \varrho \right), s \right) \) extends meromorphically from \( \text{Re } s > 1/2 \) to the complex plane with simple poles at a subset of \( \left\{ \frac{1}{2} - N \right\} \). For \( a \neq 0 \), \( \mathcal{M}_\infty \left( R \text{Str} \left( e^{tP^2(a)} \log \varrho \right), s \right) \) extends meromorphically from \( \text{Re } s < 0 \) to the complex plane with a single, simple pole at \( s = 0 \) and residue
\[
\hat{R}_a = - \left( \text{Tr}(\Pi_{\ker L^2} P(a) \log \varrho) - \text{Tr}(\Pi_{\ker L^2} P(-a) \log \varrho) \right),
\]
where \( \Pi_{\ker L^2} P(b) \) is the orthogonal projection onto the \( L^2 \)-null space of \( P(b) \).

We will need a more explicit formula for this residue. We have shown that
\[
\ker L^2 P(a) = \begin{cases} \text{span}\{\langle X \rangle^{-a}\} & \text{if } a > 0 \\ \{0\} & \text{if } a \leq 0. \end{cases}
\]
Let us write

\[ c_k = \| (X)^{-k} \|_{L^2}^2 = \int_X (X)^{-2k} dX = 2 \int_0^1 \frac{d\rho}{\rho \sqrt{1 - \rho^2}} \left( r^{k-1}(1-r)^{1/2-1} dr = B(k,1/2) = \frac{\Gamma(k)\Gamma(1/2)}{\Gamma(k+1/2)}. \right. \]

So, for \( b > 0 \), the Schwartz kernel of the projection onto \( \text{ker} P(b) \) is given by

\[ K_\Pi(X,X') = \frac{1}{c_b} \langle X \rangle^{-b} \langle X' \rangle^{-b} \frac{dX'}{dX}. \]

and hence

\[ \text{Tr}(\Pi_{\text{ker} P(b)} \log \varrho) = \frac{1}{c_b} \int_X (X)^{-2b} \log \left( \frac{1}{\langle X \rangle} \right) dX = \frac{1}{2c_b} \partial_b (c_b). \]

Thus, for \( a \neq 0 \), the residue equals

\[ \hat{R}_a = - \left( \text{Tr}(\Pi_{\text{ker} P(a)} \log \varrho) - \text{Tr}(\Pi_{\text{ker} P(-a)} \log \varrho) \right) = \begin{cases} \frac{1}{2ca} \partial_a c_a & \text{if } a > 0 \\ \frac{1}{2c|a|} \partial_a c_{|a|} & \text{if } a < 0, \end{cases} \]

which determines the behavior of \( \hat{\zeta}_a(s) \) near \( s = 0 \). Indeed, since \( \frac{s}{\Pi(s)} = s^2 + \mathcal{O}(s^3) \) and \( \mathcal{M}(R\text{Str} \left( e^{tP^2(a)} \log \varrho \right), s) = \frac{1}{s} \hat{R}_a + \mathcal{O}(1) \), we have

\[ \hat{\zeta}_a(s) = 2\hat{R}_a s + \mathcal{O}(s^2). \]

Thus altogether we have

\[ \frac{\partial}{\partial a} \zeta_a(s) = \tilde{\zeta}_a(s) + \hat{\zeta}_a(s) \sim \left( \frac{1}{a} + 2\hat{R}_a \right) s + \mathcal{O}(s^2), \]

and, interchanging \( \partial_a \) and \( -\partial_s \big|_{s=0} \), this shows that

\[ \frac{\partial}{\partial a} \log \det (-P(-a)) = -\frac{1}{a} - 2\hat{R}_a = \begin{cases} \frac{\partial}{\partial a} (- \log a + \log c_a) & \text{if } a > 0 \\ \frac{\partial}{\partial a} (\log |a| + \log c_{|a|}) & \text{if } a < 0. \end{cases} \]

Hence there exist constants \( C_{\pm} \) such that

\[ \log \det (-P(-a)) = \begin{cases} C_+ - \log a + \log c_a & \text{if } a > 0 \\ C_- + \log |a| + \log c_{|a|} & \text{if } a < 0. \end{cases} \]

Note that \( c_1 = 2 \), so from Lemmas 2.1 and 2.3 we have

\[ \log \det (-P(1)P(-1)) = 2 \log 2 = C_+ + \log 2 \]

\[ \log \det (-P(-1)P(1)) = 0 = C_+ + \log 2, \]

which shows that \( C_- = \log 2 \) and \( C_+ = -\log 2 \). Thus altogether we have shown that

\[ \log \det (-P(-a)P(a)) = \begin{cases} \log(2|a|c_{|a|}) & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ \log \left( \frac{c_a}{2t} \right) & \text{if } a > 0 \end{cases} \]
Finally, let us compute the contribution to the analytic torsion we have been looking for. From (2.6) and (2.13), we get

$$
\frac{1}{2} \sum_{q < \frac{1}{2}} (-1)^q b_q \left[ (q - (-1)^v (v - q + 1)) \left( -\log \det \left( -P(\frac{v-2q}{2})P(\frac{2q-v}{2}) \right) \right) \right.
$$

\[ + ((-q - 1 + (-1)^v (v - q))) \left( \log \det \left( -P(\frac{2v-q}{2})P(\frac{v-2q}{2}) \right) \right) \left] - \frac{1}{2} \sum_{q < \frac{1}{2}} (-1)^q b_q \left[ (q - (-1)^v (v - q + 1)) \log((v - 2q)c_{v/2-q}) \right.
$$

\[ + ((-q - 1 + (-1)^v (v - q))) \log(\frac{1}{v-2q} c_{v/2-q}) \left],
\]

so that

$$
(2.14) \quad \text{LAT}(\{-\pi/2, \pi/2\}, D_b, H^s(Z; F)) = \frac{1}{2} \sum_{q < \frac{1}{2}} (-1)^q b_q \left[ (1 + (-1)^v) \log c_{v/2-q} + ((-1)^v(2v - 2q + 1) - (2q + 1)) \log(v - 2q) \right].
$$

In particular, when $v$ is even, we obtain

$$
(2.15) \quad \text{LAT}(\{-\pi/2, \pi/2\}, D_b, H^s(Z; F)) = \sum_{q < \frac{1}{2}} (-1)^q b_q \left[ \log c_{v/2-q} + (v - 2q) \log(v - 2q) \right].
$$

3. CUSP DEGENERATION AND SMALL EIGENVALUES

Let $g_{\varepsilon, hc}$ be an $\varepsilon$-hc metric of product-type to order two and $F \rightarrow X_s$ a flat Euclidean vector bundle. In [ARS14, §§4-5] we showed that, as long as $F|_Z$ is Witt, there exists a $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$
\text{Spec}(D_{dr}) \cap S_\delta(0) = \emptyset \quad \text{for all } \varepsilon < \varepsilon_0,
$$

and such that every eigenvalue in $\mathbb{B}_\delta(0)$ converges to zero as $\varepsilon \rightarrow 0$. We call the eigenvalues of $D_{dr}$ in $\mathbb{B}_\delta(0)$ the small eigenvalues and the sum of their eigenspaces the small eigenforms, $\Omega^k_{\text{small}}(M; \varepsilon)$; note that the space of harmonic forms $\ker D^2_{dr} = \ker D_{dr}$ is a subspace of $\Omega^k_{\text{small}}(M; \varepsilon)$. Write $d_\varepsilon = \rho^{v/2} d \rho^{-v/2}$, $\partial_\varepsilon = \rho^{v/2} \partial \rho^{-v/2}$, so that $d_{\text{dr}} = d + d_\varepsilon$ and $D_{\text{dr}} = d_\varepsilon + \partial_\varepsilon$; the $\varepsilon$ subscripts are used to remind us that all of the operators except $d$ depend on $\varepsilon$. Note that the small eigenvalues of $D_{\text{dr}}$ and of $\partial_{\text{dr}}$ are the same, and the eigenforms differ by a factor of $\rho^{v/2}$, so we may usually speak without ambiguity. As we will see in Corollary 3.4 below, the product of the positive eigenvalues in a given degree are polyhomogeneous in $\varepsilon$. The quantity

$$
\log \tau_{\text{small}}(\Delta_q) = \text{FP log} \prod_{\lambda \in \text{Spec}_{\text{small}}(\Delta_q) \setminus \{0\}} \lambda \quad \text{repeated with multiplicity}
$$

is thus well-defined, where $\Delta_q$ denotes the action of $D^2_{\text{dr}}$ on forms of degree $q$. In this section we will compute the contribution to the analytic torsion from these small eigenvalues, which by [ARS14, Theorem 10.2] is given by

$$
(3.1) \quad -\frac{1}{2} \sum_{q=0}^{m} (-1)^q \log \tau_{\text{small}}(\Delta_q).
$$
3.1. **Surgery long exact sequence.** From [ARS14] §5, the dimension of the space of small eigenforms is equal to
\[
\dim \Omega^*_\text{small} = \dim \ker D_d + \dim \ker D_b.
\]
We can express these dimensions directly in terms of the topology of \(M, Z\), and \(M_0 = \mathcal{B}_\text{sm} = [M; Z]\). Let us write \(\widehat{M}_0\) for the singular space obtained from \(M_0\) by coning off \(\partial M_0\), which is two copies of \(Z\); so we have \(\widehat{M}_0 = M_0 \cup CZ \cup CZ\). It follows from work of Hausel, Hunsicker, Mazzeo [HHM04] and Lemma 8.19 of [ARS14] that
\[
\begin{aligned}
H^k_{(2)}(M_0; F) &\cong \text{IH}^k_{\text{sm}}(\widehat{M}_0; F) \\
&\cong \begin{cases} 
H^k(M_0; F) & k \leq \frac{m-1}{2} \\
\text{Im} \left( H^k(M_0, \partial M_0; F) \to H^k(M_0; F) \right) & k = \frac{m}{2} \\
H^k(\partial M_0; F) & k > \frac{m-1}{2}
\end{cases}
\end{aligned}
\]
where \(H^k_{(2)}(M_0; F)\) denotes the \(k\)th \(L^2\)-cohomology group of \((M_0, g_0)\) with coefficients in \(F\). On the other hand, from the computation of the \(L^2\)-null space of the horizontal model operator in §2.1 we see that for \(k \leq \frac{m-1}{2}\),
\[
\dim \Omega^k_{\text{small}} = \dim H^k(M_0; F) + \dim H^{k-1}(Z; F).
\]
For positive \(\varepsilon\), a subspace of dimension \(\dim \ker \Delta_{\varepsilon, \text{hc}} = \dim \ker D_{\text{dR}}\) of these eigenforms will correspond to the eigenvalue zero, and the rest will correspond to positive small eigenvalues.

This suggests that, to understand the small eigenforms corresponding to non-zero small eigenvalues, we look for a long exact sequence linking \(H^k(M_0; F), H^k(M; F)\) and \(H^{k-1}(Z; F)\). Observe first that \(M\) is homeomorphic to the union of \(M_0\) with \(Z \times (-1, 1)\), with an overlap region homotopic to \(Z \sqcup Z\). The associated Mayer-Vietoris sequence is
\[
\ldots \to H^{k-1}(Z; F) \oplus H^{k-1}(Z; F) \xrightarrow{\partial_{k-1}} H^k(M; F) \to H^k(M_0; F) \oplus H^k(Z; F) \xrightarrow{\tilde{j}_k} H^k(Z; F) \oplus H^k(Z; F) \to \ldots,
\]
where the map \(\tilde{j}_k\) is given by \(\tilde{j}_k(\mu, \lambda) = (\iota^*_+ \mu - \iota^*_- \lambda, \iota^*_- \mu - \lambda)\) and \(\iota^+_\pm : \partial_\pm M_0 \hookrightarrow M_0\) is the natural inclusion. Note that \(\tilde{j}_k\) is injective when restricted to \(H^k(Z; F)\). By using the identification
\[
(H^k(Z, F) \oplus H^k(Z, F)) \xrightarrow{\tilde{j}_k} H^k(Z, F)
\]
we obtain from (3.2) a new long exact sequence
\[
\ldots \to H^{k-1}(Z; F) \xrightarrow{\partial_{k-1}} H^k(M; F) \xrightarrow{i_k} H^k(M_0; F) \xrightarrow{j_k} H^k(Z; F) \to \ldots
\]
with \(j_k(\mu) = \iota^*_+ \mu - \iota^*_- \mu\).

Moreover, for \(k \leq \frac{m-1}{2}\), we can replace \(H^k(M_0; F)\) with \(H^k_{(2)}(M_0; F)\). Replacing singular cohomology with Hodge cohomology (notationally by replacing \(H\) with \(\mathcal{H}\)) endows these spaces with inner products, so for \(k \leq \frac{m-1}{2}\), we have short exact sequences
\[
0 \to (\ker \partial_{k-1})^\perp \to \mathcal{H}^k(M; F) \to \ker j_k \to 0,
\]
and hence the dimension of the space of eigenforms corresponding to positive small eigenvalues is equal to
\[ \dim \Omega_{\text{small}}^k(M; F) - \dim \mathcal{H}^k(M; F) = \dim \ker \partial_{k-1} + \dim(\ker j_k)^\perp. \]
This in turn suggests that we relate the vector spaces
\[ \mathcal{H}_+^k(Z; F) := \ker \partial_k, \quad \mathcal{H}_+^k(M; F) := (\ker j_k)^\perp \]
to the space of small eigenforms with positive eigenvalue, and that we relate their respective orthocomplements, which we denote \( \mathcal{H}_+^k(Z; F) \) and \( \mathcal{H}_+^k(M; F) \), to the space of harmonic forms on \( M \). Indeed, as the notation suggests, we will see in the next subsection that these vector spaces are restrictions of the subspaces of small eigenforms of \( D_{\text{dR}}^2 \) with positive eigenvalue, or with zero eigenvalue, to the two boundary faces of \( X_s \).

Finally, from (3.3), we deduce the decomposition
\[ H^k(M; F) = L^2\mathcal{H}_+^k(M; F) \oplus L^2\mathcal{H}_+^{k-1}(Z; F) \quad \text{for} \quad k \leq \frac{m-1}{2}. \]

For \( k > \frac{m-1}{2} \), we need to consider the dual of the long exact sequence (3.3) under Poincaré duality. Notice that the long exact sequence (3.3) could have alternatively been obtained by looking at the long exact sequence associated to the pair \( (M, M_0) \) and by using the Thom isomorphism \( H_{\text{dR}}^{k+1}(Z \times (-1, 1); F) \cong H^k(Z; F) \) along with the identification \( H_{(2)}^k(M; F) \cong H^k(M_0; F) \) for \( k \leq \frac{m-1}{2} \). This means that the long exact sequence dual to (3.3) can be obtained by looking at the long exact sequence associated to the pair \( (M, Z) \):
\[ \cdots \rightarrow H^k_c(M_0; F) \rightarrow H^k(M_0; F) \rightarrow H^k(Z; F) \rightarrow H^k_c(M_0; F) \rightarrow \cdots. \]
Indeed, it is obtained from (3.5) by using the identification \( H^k_c(M_0; F) \cong H_{(2)}^k(M_0; F) \) for \( k > \frac{m-1}{2} \). We get
\[ \cdots \rightarrow H^k_{(2)}(M_0; F) \rightarrow \hat{i}_k \rightarrow H^k(M_0; F) \rightarrow \hat{j}_k \rightarrow H^k_{(2)}(M_0; F) \rightarrow \cdots \]
where \( \hat{i}_k, \hat{j}_k \) and \( \hat{\partial}_k \) are Poincaré duals of the maps \( j_m - k, j_m - k \) and \( \partial_{m-1-k} \) in (3.3). Using the Hodge \( * \)-operators of \( g_2 \) and \( g_0 \), we can define for \( k > \frac{m}{2} \) the space of harmonic forms
\[ \mathcal{H}_+^k(Z; F) := *_Z \mathcal{H}_+^{m-1-k}(Z; F), \quad \mathcal{H}_+^k(Z; F) := *_Z \mathcal{H}_+^{m-1-k}(Z; F), \]
\[ L^2\mathcal{H}_+^k(M_0; F) := *_{g_0} L^2\mathcal{H}_+^{m-k}(M_0; F), \quad L^2\mathcal{H}_+^k(M_0; F) := *_{g_0} L^2\mathcal{H}_+^{m-k}(M_0; F), \]
so that for \( k > \frac{m}{2} \) we have the decompositions
\[ H^k(Z; F) = \mathcal{H}_+^k(Z; F) \oplus \mathcal{H}_-^k(Z; F), \]
\[ H^k_{(2)}(M_0; F) = L^2\mathcal{H}_+^k(M_0; F) \oplus L^2\mathcal{H}_-^k(M_0; F), \]
\[ H^k(M; F) = L^2\mathcal{H}_+^k(M_0; F) \oplus \mathcal{H}_-^k(Z; F). \]
This last decomposition is of course related to the long exact sequence (3.6) via the natural identifications
\[ \mathcal{H}_+^k(Z; F) \cong \ker \hat{j}_k \quad \text{and} \quad L^2\mathcal{H}_+^k(M_0; F) \cong \ker \hat{i}_k \quad \text{for} \quad k > \frac{m}{2}. \]
Lemma 3.2. Suppose $L^2\mathcal{H}^\frac{m}{2}_H(M_0; F) := L^2\mathcal{H}^\frac{m}{2}_H(M_0; F) \cong \mathcal{H}^\frac{m}{2}_{(2)}(M_0; F)$, \( H^\frac{m}{2}(M; F) = L^2\mathcal{H}^\frac{m}{2}_H(M_0; F) \oplus \mathcal{H}^\frac{m}{2}_{(2)}(Z; F) \oplus \mathcal{H}^\frac{m}{2}_H(Z; F). \)

Indeed, using the natural identification $H^\frac{m}{2}_{(2)}(M_0; F) \cong \text{Im} \left[ H^\frac{m}{2}_H(M_0; F) \rightarrow H^\frac{m}{2}_H(M; F) \right]$, this can be checked directly using the commutative diagram

\[
\begin{array}{cccccc}
H^\frac{m}{2}_{(2)}(Z; F) & \longrightarrow & H^\frac{m}{2}_H(M_0; F) & \longrightarrow & H^\frac{m}{2}_H(M; F) & \longrightarrow & H^\frac{m}{2}_H(Z; F) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^\frac{m}{2}_{(2)}(Z; F) & \overset{\partial^\frac{m}{2}}{\longrightarrow} & H^\frac{m}{2}_H(M; F) & \longrightarrow & H^\frac{m}{2}_H(M_0; F) & \longrightarrow & H^\frac{m}{2}_H(Z; F),
\end{array}
\]

where the top and bottom rows come from the long exact sequences \((3.5)\) and \((3.3)\). Alternatively, we could proceed more analytically and deduce the decompositions \((3.10)\) from Theorem \(3.3\) below.

### 3.2. Positive small eigenvalues.

As discussed in the previous subsection, the surgery long exact sequences \((3.3)\) and \((3.5)\) suggest that the vector spaces $\mathcal{H}^k(Z; F)$ and $\mathcal{H}^k_+(M_0; F)$ correspond to restrictions to the boundary faces of $X_s$ of the eigenforms corresponding to positive small eigenvalues. In this subsection, we will both see that this is indeed the case and compute the rate at which these eigenvalues approach zero as $\varepsilon \to 0$. We first make the following simple observation:

**Lemma 3.1.** If $\lambda_\varepsilon$ is a positive small eigenvalue with eigenform $u_\varepsilon \neq 0$, $D^2_{\text{dR}} u_\varepsilon = \lambda_\varepsilon u_\varepsilon$, then $\tilde{d}_\varepsilon u_\varepsilon$ and $\tilde{\delta}_\varepsilon u_\varepsilon$ are also eigenforms with the same small eigenvalue $\lambda_\varepsilon$. Furthermore, at least one of these two eigenforms is non-zero.

**Proof.** The first statement is immediate since both $\tilde{d}_\varepsilon$ and $\tilde{\delta}_\varepsilon$ commute with $D_{\text{dR}}$. To see that $du_\varepsilon$ and $\tilde{\delta}_\varepsilon u_\varepsilon$ cannot be both zero, it suffices to notice that for $\varepsilon > 0$,

$$0 \neq \lambda_\varepsilon u_\varepsilon = \tilde{d}_\varepsilon(\tilde{\delta}_\varepsilon u_\varepsilon) + \tilde{\delta}_\varepsilon(\tilde{d}_\varepsilon u_\varepsilon).$$

The basis for the computation of the decay rates of the small eigenvalues is the following lemma.

**Lemma 3.2.** Suppose $u_\varepsilon$ is a section of $\Lambda^k(\varepsilon, h c^* T^* X_s)$, with $k \leq \frac{m-1}{2}$, such that

- $\Pi_{\text{small}} u_\varepsilon = u_\varepsilon$ and $\|u_\varepsilon\|_{L^2_b} = 1$;
- $u_\varepsilon$ is polyhomogeneous on $X_s$;
- $u_\varepsilon|_{\mathcal{B}_{sb}} = 0$ and $u_0 := u_\varepsilon|_{\mathcal{B}_{sm}}$ is such that $\|u_0\|_{L^2_b} = 1$;
- $\tilde{\delta}_\varepsilon u_\varepsilon = 0$;
- $j_k([u_0]) \neq 0$, where $[u_0] \in H^k_{(2)}(M_0; F)$ is the cohomology class associated to $u_0$.

Then

$$\|\tilde{d}_\varepsilon u_\varepsilon\|_{L^2_b}^2 = \frac{1}{4c(m-1)/2-k} \||j_k([u_0])||_{L^2 \varepsilon^{m-1-2k}}^2 + o(\varepsilon^{m-1-2k}),$$

where $c(k) = \frac{1}{2k-1}$.
where \( c_k = B(k, 1/2) \) is the constant from [2.12] and \( j_k \) is understood as a map into the harmonic forms on \( Z \) (identified with the cohomology by the de Rham isomorphism). Moreover, the \((k+1)\)-form \( v_\varepsilon = \frac{d_x u_\varepsilon}{\|d_x u_\varepsilon\|_{L^2_\varepsilon}} \) is well-defined and

- \( \Pi_{\text{small}} v_\varepsilon = v_\varepsilon \) and \( \|v_\varepsilon\|_{L^2_\varepsilon} = 1 \);
- \( v_\varepsilon \) is polyhomogeneous on \( X_\varepsilon \);
- \( v_\varepsilon|_{\mathcal{B}_{sm}} = 0 \) and \( v_\varepsilon|_{\mathcal{B}_{sb}} \) is such that \( \|v_\varepsilon\|_{L^2_\varepsilon} = 1 \);
- \( \tilde{d}_\varepsilon v_\varepsilon = 0 \);
- \( v_\varepsilon \in \ker \partial_k \subset \mathcal{H}^k(Z; F) \cong \ker_{L^2} D_{b}^{k+1} \) is a non-zero multiple of \( j_k([u_0]) \in \mathcal{H}^k(Z; F) \).

In particular, if \( u_\varepsilon \) is an eigenform associated to a small eigenvalue \( \lambda_\varepsilon \), then

\[
\lambda_\varepsilon = \|\tilde{d}_\varepsilon u_\varepsilon\|^2_{L^2_\varepsilon} = \frac{1}{c_{(m-1)/2-k}} \|j_k([u_0])\|^2_{L^2_\varepsilon} \varepsilon^{m-1-2k} + o(\varepsilon^{m-1-2k}),
\]

and \( v_\varepsilon \) is also an eigenform for the small eigenvalue \( \lambda_\varepsilon \).

**Proof.** Since \( j_{(m-1)/2}([u_0]) \) is always 0 by the Witt condition, the theorem statement is empty for \( k = (m-1)/2 \), so we may assume that \( k < (m-1)/2 \). With this understood, let \( u_0 \) be the restriction of \( u_\varepsilon \) to \( \mathcal{B}_{sm} \). Since \( \Pi_{\text{small}} u_\varepsilon = u_\varepsilon \), we know from [ARS14] that \( u_0 \) is in the \( L^2 \)-kernel of \( D_{dR}|_{\mathcal{B}_{sm}} \). This is helpful in describing the expansion of \( u_0 \) near \( \mathcal{B}_{sm} \cap \mathcal{B}_{sb} \), which has two components, which we write as \( \partial_+ \) and \( \partial_- \). Near \( \partial_\pm \), using the splitting into tangential and normal parts, we see from (1.2) that

\[
0 = \left( \begin{array}{cc} |x|^{\ell} u_\pm \varepsilon & 0 \\ 0 & |x|^{\ell'} v_\pm \end{array} \right) + \left( \begin{array}{c} o(|x|^{\ell}) \\ o(|x|^{\ell'}) \end{array} \right), \quad \text{for some } u_\pm, v_\pm \in \mathcal{H}^k(Z; F) \text{ and } \ell, \ell' \in \mathbb{R}.
\]

Indeed, clearly, the coefficients \( u_\pm \) and \( v_\pm \) must be in \( \mathcal{H}^k(Z; F) \) for \( u_0 \) to be in the kernel of \( D_{dR}|_{\mathcal{B}_{sm}} \). Then the powers \( \ell \) and \( \ell' \) are obtained by solving the equation

\[
\left( \begin{array}{cc} 0 & -|x| \partial_x |x| + (k - \frac{1}{2})v \\ |x| \partial_x |x| + (k - \frac{1}{2})v & 0 \end{array} \right) \left( \begin{array}{c} o(|x|^{\ell}) \\ o(|x|^{\ell'}) \end{array} \right) = 0,
\]

which gives \( \ell = \frac{m-1}{2} - k \), \( \ell' = k - \frac{m-1}{2} \). Since \( \ell' < 0 \) and \( u_0 \) is in \( L^2_\varepsilon(M_0; E) = \|x\|^m L^2_{\varepsilon c}(M_0; E) \), this means \( v_\pm = 0 \), so that in fact we have that

\[
(3.11) \quad u_0 = |x|^{-\frac{m-1}{2} + k} \left( \begin{array}{c} u_\pm \\ 0 \end{array} \right) + o(|x|^{-\frac{m-1}{2} - k}), \quad \text{for some } u_\pm \in \mathcal{H}^k(Z; F).
\]

Now, since \( j_k[u_0] = u_+ - u_- \) and \( j_k[u_0] \neq 0 \), we must have \( u_+ \neq u_- \), so at least one must be nonzero. Therefore the expansion of \( u_\varepsilon \) at \( \mathcal{B}_{sb} \) must have a nonzero term of order \( \varepsilon^{\frac{m-1}{2} - k} \). Moreover, since \( u_+ \neq u_- \), the coefficient of this term cannot be of the form (3.12) and hence cannot be in the \( L^2 \)-kernel of \( d_b = P(k - \frac{m-1}{2}) \).

A priori, \( u_\varepsilon \) could have lower order terms at \( \mathcal{B}_{sb} \). Notice however that \( u_\varepsilon \) cannot have terms of order less than \( \varepsilon^{\frac{m-1}{2} - k} \) at \( \mathcal{B}_{sb} \) that are in the kernel of \( d_b \). Indeed, if there were such a term of order \( \varepsilon^\alpha (\log \varepsilon)^q \) with \( \alpha < \frac{m-1}{2} - k \), or with \( \alpha = \frac{m-1}{2} - k \) and \( q > 0 \), the coefficient \( u_{\alpha, q} \) of this term would be in the kernel of \( d_b \), so of the form

\[
(3.12) \quad u_{\alpha, q} = \omega(1 + X^2)^{\frac{m-1}{2} - \frac{k}{2} - \frac{1}{2}} \quad \text{for some } \omega \in \mathcal{H}^k(Z; F).
\]

Let \( \rho_{sm} = \frac{\varepsilon}{|p|} \); it is a boundary defining function for \( \mathcal{B}_{sm} \). Since \( \sqrt{1 + X^2} = \frac{\varepsilon}{|p|} \); near \( \mathcal{B}_{sm} \), \( \varepsilon^\alpha (\log \varepsilon)^q u_{\alpha, q} \) would be of order \( \rho_{sm}^{\alpha - \frac{m-1}{2} + k} (\log \rho_{sm})^q \). Since \( u_\varepsilon \) is bounded, this forces \( u_{\alpha, q} = 0 \).
Notice then that exterior differential $d$ does not depend on $\varepsilon$, and that $\tilde{d}_s = \rho^{m-1 \over 2} d \rho^{-m-1 \over 2}$. Therefore, to understand $u_\varepsilon$, we must examine the first term in the expansion of $u_\varepsilon$ at $\mathfrak{B}_s$ (resp. $\mathfrak{B}_{sb}$) which is not in the kernel of $d_0$ (resp. $d_b$); the discussion above indicates that such a term exists. Let us denote it by $u_{\text{lead}}$ and suppose for contradiction that $u_{\text{lead}}$ occurs at order $\varepsilon^\alpha (\log \varepsilon)^q$ with either $\alpha < m-1 \over 2 - k$ or $\alpha = m-1 \over 2 - k$ and $q > 0$, and with $(\alpha, q)$ minimal. Then consider $\tilde{d}_s u_\varepsilon$; it is polyhomogeneous on $X_s$ with a nontrivial term of order either $\varepsilon^{\alpha-1} (\log \varepsilon)^q$ or $\varepsilon^\alpha (\log \varepsilon)^q$ (the loss of an order happens if and only if the term is at $\mathfrak{B}_{sb}$ and is not a section of $\ker D_v$). Let $w_\varepsilon$ be $\tilde{d}_s u_\varepsilon$ divided by its leading-order coefficient in $\varepsilon$, so that $w_\varepsilon$ has expansions at $\mathfrak{B}_{sb}$ and $\mathfrak{B}_s$, with restriction $w_0$ to $\mathfrak{B}_s$ and/or restriction $w_b$ to $\mathfrak{B}_{sb}$ being nontrivial.

Since $\tilde{d}_s \Pi_{\text{small}} = \Pi_{\text{small}} \tilde{d}_s$, $w_\varepsilon$ is also in the range of $\Pi_{\text{small}}$, and so $w_0$ and $w_b$ are in $|x|^{(m-1)/2} \mathcal{H}^\varepsilon_{L^2}(M_0; F)$ and $\ker L^2 D_b$ respectively. In particular, we immediately see that $w_b$ must be a section of $\ker D_v$. Suppose that $u_{\text{lead}}$ is at $\mathfrak{B}_{sb}$ but is not a section of the $\ker D_v = \rho^N \mathcal{H}^* (Z; F) \longrightarrow \mathfrak{B}_{sb}$; then the Hodge decomposition on $Z$ would imply that $w_b$ is not a section of $\ker D_v$, which is a contradiction. So $u_{\text{lead}}$ must either be at $\mathfrak{B}_s$ or a section of $\ker D_v$ at $\mathfrak{B}_{sb}$.

Next suppose it is at $\mathfrak{B}_s$ at order $(\alpha, q)$; then by the Hodge decomposition on $\mathfrak{B}_s = M_0$, the coefficient of $u_{\text{lead}}$ at $\mathfrak{B}_s$ cannot be in $L^2_0 (M_0; E)$, so must have a term of order zero or smaller at $\mathfrak{B}_{sb} \cap \mathfrak{B}_s$, which would contradict our assumption that $u_\varepsilon$ is bounded with $u_\varepsilon |_{\mathfrak{B}_{sb}} = 0$. Thus $w_0 = 0$, $w_b \neq 0$, and $u_{\text{lead}}$ occurs at $\mathfrak{B}_{sb}$ and is a section of $\ker D_v$. Furthermore, the same logic as in the previous paragraph tells us that no term at $\mathfrak{B}_{sb}$ within one order of $u_{\text{lead}}$, inclusive, is not a section of $\ker D_v$; if it were, $w_b$ would have a contribution from $\tilde{d}_s$ applied to that term and would not be a section of $\ker D_v$ either. We conclude that $u_{\text{lead}}$ occurs at $\mathfrak{B}_{sb}$, is equal to $u_{\alpha, q} \varepsilon^\alpha (\log \varepsilon)^q$, and that $d_b u_{\alpha, q} = w_b$.

Since $w_b \in \ker L^2 (d_b + \delta_b)$ and $d_b^2 = 0$, we must have $u_{\alpha, q} \in \ker (\delta_b d_b)$ but is not contained in $\ker d_b$. By examining (2.2), we see that there must be a nontrivial term of order $\langle X \rangle^{m-1 \over 2 - k}$ in the expansion of the first component of $u_{\alpha, q}$ at the boundary of $\mathfrak{B}_{sb}$. Since $\langle X \rangle = \frac{2}{\varepsilon}$, this means that $u_{\alpha, q} \varepsilon^\alpha (\log \varepsilon)^q$ is of order $\rho^{m-1 \over 2 - k} \varepsilon^{-(m-1 \over 2 - k)} (\log \varepsilon)^q$ at $\mathfrak{B}_{sb} \cap \mathfrak{B}_s$. However, unless $\alpha \geq (m-1)/2 - k$ and $q = 0$ if $\alpha = m-1 \over 2 - k$, this would contradict the assumption that $u_\varepsilon$ is bounded at $\mathfrak{B}_s$. And since there is in fact a nontrivial term of order $(m-1)/2 - k$ at $\mathfrak{B}_{sb}$, we must in fact have $\alpha = (m-1)/2 - k$ and $q = 0$.

Thus let $u_b = u_{(m-1)/2 - k, 0}$ be the coefficient of the term of order $m-1 \over 2 - k$ of $u_\varepsilon$ at $\mathfrak{B}_{sb}$. By the argument above, $u_b$ must be an element of the form (2.2). To be consistent with (3.11), this means that

$$(3.13) \quad u_b = \left( {u_+ - u_- \over 0} \right) \left( {X^2 + 1 \over 2} \right)^{m-1 \over 2 - k} f_k (X) + \left( {u_+ + u_- \over 0} \right) \left( {X^2 + 1 \over 2} \right)^{m-1 \over 2 - k} + \nu$$

with $\nu \in \ker L^2_b D_b$ and where

$$f_k (X) = \frac{2}{C^{m-1 \over 2 - k}} \int_0^X \langle s \rangle^{2k-m} ds \quad \text{is such that } \lim_{X \to \pm \infty} f(X) = \pm 1.$$
Now a simple computation using the explicit form of $d_b$ shows that the leading order term $w_b$ is precisely the vector with zero in the first factor and

$$\frac{1}{c(m-1)/2-k}(u_+ - u_-)(X^2 + 1)^{\frac{k^2 - m-1}{2}}$$

in the second factor. Squaring and then integrating with respect to $b$-surgery densities, this finally gives

$$\|d_\varepsilon u_\varepsilon\|^2_{L^2_b} = \left( \frac{1}{c(m-1)/2-k} \|u_+ - u_-\|^2_{L^2} \right) \varepsilon^{m-1-2k} + o(\varepsilon^{m-1-2k}).$$

From there, the properties of $v_\varepsilon$ are easily obtained. The only slightly tricky part is to show that $v_\varepsilon$ is polyhomogeneous. However, we may write

$$v_\varepsilon = \varepsilon^{-(m-1)/2+k} \tilde{d}_\varepsilon u_\varepsilon / \|\varepsilon^{-(m-1)/2+k} \tilde{d}_\varepsilon u_\varepsilon\|_{L^2_b}.$$ 

Since $\|\varepsilon^{-(m-1)/2+k} \tilde{d}_\varepsilon u_\varepsilon\|^2_{L^2_b}$ is bounded and has limit a positive constant as $\varepsilon$ approaches zero, it follows from a direct series expansion construction that its inverse is also polyhomogeneous in $\varepsilon$ with limit a positive constant as $\varepsilon$ approaches zero. Since the product of a polyhomogeneous function of $\varepsilon$ and a polyhomogeneous function on $X_s$ is polyhomogeneous on $X_s$, the result follows. 

We can now consider the projections. In the following theorem shows that their leading-order behavior at $\mathfrak{B}_{sm}$ and $\mathfrak{B}_{sb}$ is what we expect. In the statement of the result, we will tacitly make the following natural identification following from (2.1),

$$\ker L^2 D^2_b \Lambda^{k(\varepsilon, k\varepsilon^* X_s)}_{\mathfrak{B}_{sb}} \cong \begin{cases} \mathcal{H}^{k-1}(Z; F), & k < \frac{m-1}{2}, \\ \mathcal{H}^{k}(Z; F), & k \geq \frac{m-1}{2}. \end{cases}$$

**Theorem 3.3.** Recall from [ARS14] that for some index family $K \geq 0$, $\Pi_{\text{small}} \in \Psi^{-\infty, K}_{b,s}(X_s; E)$ is the projection onto eigenforms of $D_{\text{dR}}$ corresponding to small eigenvalues, where $E = \Lambda^*(\varepsilon, d^* X_s) \otimes F$ and where $\Psi^{-\infty, K}_{b,s}$ is the space of operators whose kernels are smooth and polyhomogeneous on $X_s$ with index family $K$. Then let $\Pi_{\text{small}}^k$ be the subspace of $\Pi_{\text{small}}$ consisting of forms of pure degree $k$. We have:

(i) $\Pi_{\text{small}}^k = \Pi_H^k + \Pi^k_+$, where $\Pi_H^k, \Pi^k_+ \in \Psi^{-\infty, K}_{b,s}(X_s; E)$ are projections onto the harmonic forms and onto the eigenforms associated to positive small eigenvalues respectively.

(ii) Each of the projections $\Pi_H^k$ and $\Pi^k_+$ can itself be written as a sum of two projections in $\Psi^{-\infty, K}_{b,s}(X_s; E)$,

$$\Pi^k = \Pi_H^k \mathfrak{B}_{sm} + \Pi_H^k \mathfrak{B}_{sb}, \quad \Pi^k_+ = \Pi^k_+ \mathfrak{B}_{sm} + \Pi^k_+ \mathfrak{B}_{sb},$$

where $\Pi_{H, sm}^k$ restricts to the projection onto $L^2 \mathcal{H}^k_H(M_0; F)$ on $\mathfrak{B}_{sm}$ and restricts to zero on $\mathfrak{B}_{sb}$, while $\Pi_{H, sb}^k$ vanishes on $\mathfrak{B}_{sm}$ and on $\mathfrak{B}_{sb}$ restricts to the projection onto $\mathcal{H}^k_H(Z; F)$ for $k \leq \frac{m-1}{2}$ and onto $\mathcal{H}^k_H(Z; F)$ for $k \geq \frac{m+1}{2}$. Similarly, $\Pi^k_+ \mathfrak{B}_{sm}$ restricts to the projection onto $L^2 \mathcal{H}^k_+(M_0; F)$ on $\mathfrak{B}_{sm}$ and vanishes on $\mathfrak{B}_{sb}$, while $\Pi^k_+ \mathfrak{B}_{sb}$ vanishes on $\mathfrak{B}_{sm}$ and on $\mathfrak{B}_{sb}$ restricts to the projection onto $\mathcal{H}^k_+(Z; F)$ for $k \leq \frac{m-1}{2}$ and onto $\mathcal{H}^k_+(Z; F)$ for $k \geq \frac{m+1}{2}$. Furthermore, the image of each of these projections admits a basis of polyhomogeneous forms on $X_s$. 
Lemma 3.2, we see that for \( X \) involved in this proof, as all are polyhomogeneous on homogeneity of the projection to polyhomogeneity of a basis works for any of the projections to each element yields a polyhomogeneous basis for \( \varepsilon \) homogeneous basis of \( \text{Im } \Pi \).

Suppose it is true for degree \( j \), then by Lemma 3.2, we have

\[
\Pi^k \ = \ \Pi^k \ |_{\ker \Pi^k} \oplus \Pi^k \ |_{\text{range } \Pi^k} = \Pi^k \ |_{\ker \Pi^k} \oplus \Pi^k \ |_{\text{range } \Pi^k} = \Pi^k \ |_{\ker \Pi^k} \oplus \Pi^k \ |_{\text{range } \Pi^k}.
\]

Thus, \( \Pi^k \ |_{\ker \Pi^k} \) is a subspace of \( \ker \Pi^k \) and \( \Pi^k \ |_{\text{range } \Pi^k} \) is its range.

Proof. In degree \( k = 0 \), \( \Pi^0 \) is just the projection onto \( \ker D^2_{\text{dr},k=0} \), which is \( \rho = \Gamma_{\text{flat}}(M; F) \), where \( \Gamma_{\text{flat}}(M; F) \) is the space of flat sections of \( F \).

Since \( \Gamma_{\text{flat}}(M; F) \) does not depend on \( \varepsilon \), we clearly see that \( \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \) and that its range admits a basis of polyhomogeneous sections of \( F \) on \( X \) is trivial in degree zero, \( \Pi^0 = \Pi^0 = \Pi^0 \), and \( \Pi^0 = \Pi^0 \ |_{\ker \Pi^0} \). Thus, we can take \( \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \) and \( \Pi^0 = \Psi_{b,s}^\infty(X; E) \) and \( \Pi^0 = \Psi_{b,s}^\infty(X; E) \).

Indeed, the argument is standard and proceeds as follows. Take a basis of harmonic forms corresponding to \( \ker \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \) and \( \ker \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \). By Lemma 3.2, we see that for \( u_\varepsilon \) and \( v_\varepsilon \) in the range of \( \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \),

\[
\langle v_\varepsilon, D^2_{\text{dr},u_\varepsilon} \rangle_L^2 = \langle \tilde{d}_\varepsilon v_\varepsilon, \tilde{d}_\varepsilon u_\varepsilon \rangle_L^2 = \O(\varepsilon^{-m}).
\]

Since \( D_{\text{dr}} \) and \( \Pi_{\text{small}} \) commute, this means that \( D_{\text{dr}} \Pi^0 \subseteq \Psi_{b,s}^\infty(X; E) \).

From here, we now proceed inductively to prove the rest of the theorem for \( k \leq \frac{m-1}{2} \).

First, we take \( \Pi^k \ |_{\ker \Pi^k} \) to be the projection onto the range of \( \varepsilon^k \ |_{\ker \Pi^k} \). We claim that it has all the required properties. Indeed, it is clear that \( \tilde{d}_\varepsilon \Pi^k \ |_{\ker \Pi^k} \).
Next we construct $\Pi^{k}_{H,b}$. If $(\ker \partial_{k})^\perp = 0$, then $\Pi^{k}_{H,b} = 0$. If not, let $\omega \in (\ker \partial_{k})^\perp$ be a non-zero element and let
\[
 u_b = (1 + X^2)^{\frac{k}{2} - \frac{m-1}{2}} \begin{pmatrix} 0 \\ \omega \end{pmatrix}
\]
be the corresponding element in $\ker L^2 D_b$. Let $\chi \in C^\infty_c(\mathbb{R})$ be a cutoff function taking values between 0 and 1, with $\chi(t) = 1$ for $|t| < \delta$ and $\chi(t) = 0$ for $|t| > 2\delta$, where $\delta > 0$ is chosen small enough, and consider $u_{\varepsilon} := \chi(x) u_b(x/\varepsilon)$. By direct computation, $\tilde{d}_{\varepsilon} u_{\varepsilon} = 0$, $u_{\varepsilon}|_{\mathcal{B}_s^{\infty}} = 0$, and $\tilde{d}_{\varepsilon} u_{\varepsilon}$ is polyhomogeneous and bounded on $X_s$, vanishing at both $\mathcal{B}_s^{\infty}$ and $\mathcal{B}_s^b$.

First set
\[
 v_{\varepsilon} := u_{\varepsilon} - \Pi^{k}_{+\delta} u_{\varepsilon}.
\]
Since $\tilde{d}_{\varepsilon} \Pi^{k}_{+\delta} = 0$, we still have that $\tilde{d}_{\varepsilon} v_{\varepsilon} = 0$. By the properties of $\Pi^{k}_{+\delta}$, we also have that $v_{\varepsilon}|_{\mathcal{B}_s^{\infty}} = u_{\varepsilon}|_{\mathcal{B}_s^{\infty}} = u_b$ and $v_{\varepsilon}|_{\mathcal{B}_s^{\infty}} = 0$. By construction, $\Pi^{k}_{+\delta} v_{\varepsilon} = 0$, which implies by the inductive step that $\Pi^{k-1}_{+\delta} \tilde{d}_{\varepsilon} v_{\varepsilon} = 0$. However, by the Hodge decomposition $\tilde{d}_{\varepsilon} v_{\varepsilon}$ is orthogonal to the harmonic forms, and since $\tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} v_{\varepsilon} = 0$, the inductive hypothesis implies that $\tilde{d}_{\varepsilon} v_{\varepsilon}$ is also orthogonal to the image of $\Pi^{k-1}_{+\delta}$. Therefore $\Pi^{k-1}_{+\delta} \tilde{d}_{\varepsilon} v_{\varepsilon} = 0$. Finally, note that $\Pi^{k}_{+\delta} u_{\varepsilon}$ is also polyhomogeneous and bounded on $X_s$, vanishing to positive order at both $\mathcal{B}_s^{\infty}$ and $\mathcal{B}_s^b$. Therefore $\tilde{d}_{\varepsilon} \Pi^{k}_{+\delta} u_{\varepsilon}$ is polyhomogeneous on $X_s$, and (3.16) implies that it is in $\varepsilon^{-\frac{1}{2} - k + 1} L^2$ and thus bounded, and in fact vanishing at both $\mathcal{B}_s^{\infty}$ and $\mathcal{B}_s^b$. The same is therefore true for $\tilde{d}_{\varepsilon} v_{\varepsilon}$.

Now set
\[
 \mu_{\varepsilon} = - (\tilde{d}_{\varepsilon} + \tilde{d}_{\varepsilon} - \Pi_{\mathcal{B}_s^{\infty}})^{-1} (\tilde{d}_{\varepsilon} v_{\varepsilon}).
\]
By the Hodge decomposition, $\mu_{\varepsilon}$ is a pure form of degree $k$ with $\tilde{d}_{\varepsilon} \mu_{\varepsilon} = 0$ and $\Pi_{\mathcal{B}_s^{\infty}} \mu_{\varepsilon} = 0$. By the resolvent construction and mapping properties of surgery operators from [ARS14], $\mu_{\varepsilon}$ is polyhomogeneous on $X_s$, bounded, and vanishes to positive order at both $\mathcal{B}_s^{\infty}$ and $\mathcal{B}_s^b$.

Finally, set $w_{\varepsilon} = v_{\varepsilon} + \mu_{\varepsilon}$; we have that
\[
 D^2_{dR} w_{\varepsilon} = \tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} v_{\varepsilon} + \tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} \mu_{\varepsilon} = \tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} v_{\varepsilon} + \tilde{d}_{\varepsilon} (\tilde{d}_{\varepsilon} + \tilde{d}_{\varepsilon} - \Pi_{\mathcal{B}_s^{\infty}}) \mu_{\varepsilon} = \tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} v_{\varepsilon} - \tilde{d}_{\varepsilon} \tilde{d}_{\varepsilon} v_{\varepsilon} = 0.
\]
So $w_{\varepsilon}$ is harmonic. And since $w_{\varepsilon} = u_{\varepsilon} - \Pi^{k}_{+\delta} u_{\varepsilon} + \mu_{\varepsilon}$, $w_{\varepsilon}$ has the same restrictions at $\mathcal{B}_s^{\infty}$ and $\mathcal{B}_s^b$ as $u_{\varepsilon}$, namely 0 and $u_b$.

Thus, taking a basis $\omega_1, \ldots, \omega_p$ of $(\ker \partial_{k-1})^\perp$, we can find harmonic forms $w_1, \ldots, w_p$ on $X_s$, polyhomogeneous and bounded on $X_s$, such that
\[
 w_i|_{\mathcal{B}_s^{\infty}} = 0, \quad w_i|_{\mathcal{B}_s^b} = (1 + X^2)^{\frac{k}{2} - \frac{m-1}{2}} \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}
\]
and with $[w_i|_{x=\varepsilon}] \in H^k(M; F)$ a positive multiple of $\partial_{k-1} [\omega_i]$ for $c > 0$. We can thus define $\Pi^{k}_{H,b}$ to be the projection on the span of $w_1, \ldots, w_p$.

To construct $\Pi^{k}_{H,\mathcal{B}_s^{\infty}}$ we can proceed in a similar fashion. If ker $j_k = \{0\}$, then we can just pick $\Pi^{k}_{H,\mathcal{B}_s^{\infty}} = 0$. Otherwise, take a class $\mu \in \ker j_k$ and choose a class $\tau \in H^k(M; F)$ such that $i_k(\tau) = \mu$. Represent $\tau$ by a smooth form $v$ of degree $k$ on $M$; without loss of generality, we can assume that in a tubular neighborhood $Z \subset M$, $v$ is of the form
\[
 v|_{Z \times (-1,1)_x} = \omega
\]
with $\omega \in \mathcal{H}^k(Z; F)$ independent of $x$. In particular, if $k = \frac{m-1}{2}$, this means by the Witt condition that $\omega = 0$ so that $v|_{Z \times (-1,1)_x} = 0$. If instead $k < \frac{m-1}{2}$, then the norm of $v \in$
\[L^2\Lambda^k(M_\varepsilon; F)\] is uniformly bounded as \(\varepsilon\) approaches zero. Thus, for any \(k \leq \frac{m-1}{2}\), we have that the form \(v_\varepsilon := \rho^{\frac{m-1}{2}} v\) on \(X_s\) is in \(L^2_b\). Since \(dv = 0\), it is such that \(\delta_\varepsilon v_\varepsilon = 0\). Moreover, we have that \(v_\varepsilon|_{\mathfrak{B}_{sb}} = 0\), while \(v_\varepsilon|_{\mathfrak{B}_{sm}}\) represents the class \(\mu \in \ker j_k \subset H^2_0(M_0; F)\). Consider then

\[u_\varepsilon := v_\varepsilon - (\Pi^k_{H,b} + \Pi^k_+) v_\varepsilon.\]

Then we still have that \(\bar{\delta}_\varepsilon u_\varepsilon = 0\), \(u_\varepsilon|_{\mathfrak{B}_{sb}} = 0\) and \(u_\varepsilon|_{\mathfrak{B}_{sm}}\) represents the class \(\mu\). We also have that \(\Pi^{k-1}_{\text{small}} \delta_\varepsilon u_\varepsilon = 0\), so the form

\[\mu_\varepsilon = - (\bar{\delta}_\varepsilon + \delta_\varepsilon - \Pi_{\text{small}})^{-1} \bar{\delta}_\varepsilon u_\varepsilon\]

is a well-defined \(k\)-form with \(\Pi^{k}_{\text{small}} \delta_\varepsilon = 0\). As in the construction of \(\Pi^k_{H,b}\), the form \(w_\varepsilon = u_\varepsilon + \mu_\varepsilon\) is harmonic with \(w_\varepsilon|_{\mathfrak{B}_{sb}} = 0\) and with \(w_\varepsilon|_{\mathfrak{B}_{sm}}\) the harmonic representative of the class \(\mu\).

Thus, starting with with a basis \(\mu_1, \ldots, \mu_\ell\) of \(\ker j_k\), we can construct harmonic forms \(w_1, \ldots, w_\ell\) such that \(w_i|_{\mathfrak{B}_{sb}} = 0\), \(w_i|_{\mathfrak{B}_{sm}}\) represents \(\mu_i\) and \(\Pi^k_{H,b} w_i = 0\). Then we define \(\Pi^k_{H,\mathfrak{B}_{sm}}\) to be the projection on the range of \(w_1, \ldots, w_\ell\).

Finally, we set

\[\Pi^k_{+,\mathfrak{B}_{sm}} = (\Pi^k_{+,b} + \Pi^k_{H,b} + \Pi^k_{H,\mathfrak{B}_{sm}})^\perp \subset \Pi^{k}_{\text{small}};\]

as before, it is polyhomogeneous with a basis which is polyhomogeneous on \(X_s\); since we understand the restrictions of every space on the right-hand side, we conclude that it has the appropriate restrictions at \(\mathfrak{B}_{sm}\) and \(\mathfrak{B}_{sb}\). Let \(u_\varepsilon \in \Pi^k_{+,\mathfrak{B}_{sm}}\); then consider \(\varepsilon^{-\frac{m-1}{2}} \bar{\delta}_\varepsilon u_\varepsilon\), which we claim is equal to zero. Suppose not. Then it is certainly in the image of \(\Pi^{k-1}_{\text{small}}\), and as it is in the image of \(\delta_\varepsilon\), it must be in the image of one of the \(+\) projections. By duality and the fact that \(d_\varepsilon \Pi^{k-1} = 0\), it is orthogonal to everything in the image of \(\Pi^{k-1}_{+,b}\), so it must be in the image of \(\Pi^{k-1}_{+,\mathfrak{B}_{sm}}\). However, \(\Pi^k_{+,b} u_\varepsilon = 0\); therefore, by the inductive hypothesis, \(\Pi^{k-1}_{+,\mathfrak{B}_{sm}} \delta_\varepsilon u_\varepsilon = 0\). Hence \(\delta_\varepsilon \Pi^k_{+,\mathfrak{B}_{sm}} = 0\) as required. Then Lemma 3.1 immediately gives the required estimates for \(D_{\text{dr}}^2 \Pi^k_{+,\mathfrak{B}_{sm}}\). This completes the inductive step, and the proof for \(k \leq \frac{m-1}{2}\).

For \(k \geq \frac{m+1}{2}\), we obtain the corresponding results by applying Poincaré duality.

Finally, if \(m\) is even, then, applying Lemma 3.2 as in the case \(k = 0\) to \(\Pi^{\frac{m-1}{2}}_{+,\mathfrak{B}_{sm}}\), we obtain the part of \(\Pi^{\frac{m}{2}}_{+,b}\) projecting on \(\varepsilon^{\frac{m-1}{2}} d_\varepsilon \text{Im} \Pi^{\frac{m}{2}}_{+,\mathfrak{B}_{sm}}\). Using the Hodge \(*\)-operator we get the other part of \(\Pi^{\frac{m}{2}}_{+,b}\). Since each eigenspace of positive small eigenvalues is even dimensional and formed of pairs of eigenfunctions by Lemma 3.1, we see from the statement of the theorem when \(k \neq \frac{m}{2}\) that we get all of \(\Pi^{\frac{m}{2}}_{+,\mathfrak{B}_{sm}}\), so \(\Pi^{\frac{m}{2}}_{+,\mathfrak{B}_{sm}} = 0\). \(\square\)

**Corollary 3.4.** In every degree, the product of positive small eigenvalues is polyhomogeneous in \(\varepsilon > 0\). Furthermore, for \(k \leq (m-1)/2\), the product of all positive small eigenvalues of \(D_{\text{dr}}^2\) in degree \(k\) is asymptotic to

\[\left(\frac{1}{c_{k/2}}\right)^{e^{m-1-2k}} \det((j_k)_{-})^2 \cdot \left(\frac{1}{c_{(k-1)}}\right)^{e^{m-1-2(k-1)}} \det((j_{k-1})_{-})^2\]
as \( \varepsilon \searrow 0 \). For \( k \geq (m+1)/2 \), the product is the same as the product for \( m-k \). If \( m \) is even and \( k = \frac{m}{2} \), then the product is asymptotic to

\[
\left( \frac{1}{c_{1/2}} \varepsilon^{\dim \mathbb{H}_+^{m-1}(Z;F)} |\det((j_{\frac{m}{2}})_{\perp})|^2 \right)^2
\]
as \( \varepsilon \searrow 0 \).

**Proof.** By Theorem 3.3, the product of positive small eigenvalues in degree \( k \) is polyhomogeneous in \( \varepsilon \) since it is given by \( \det(\Pi^k_{+} D_\text{dr}^{-1} \Pi^k_{+}) \). In the asymptotic behavior of this product for \( k \leq \frac{m-1}{2} \), the first term comes from the eigenvalues corresponding to \( \Pi_{+,\mathfrak{B}_{sm}}(v) \) of Theorem 3.3, the positive small eigenvalues in degree \( m \) immediately by Poincaré duality. For the last statement, it suffices to notice that from part (vi) of Theorem 3.3, the positive small eigenvalues in degree \( \frac{m}{2} \) are the same as those corresponding to \( \Pi_{+,\mathfrak{B}_{sm}}(v) \) of degree \( m \). Indeed, if \( u_0 \) and \( v_0 \) are orthogonal harmonic representatives of classes in \( \mathbb{H}_+^k(M_0;F) \) and \( u_\varepsilon \) and \( v_\varepsilon \) are extensions in the range of \( \Pi_{\text{small}} \), then similarly to (3.15), we have that

\[
o(\varepsilon^{m-1-2k}) = \langle v_\varepsilon, \tilde{\Delta}_\varepsilon u_\varepsilon \rangle_{L^2} = \langle \tilde{d}_\varepsilon v_\varepsilon, \tilde{u}_\varepsilon \rangle_{L^2}
\]
from which we see that \( \tilde{j}_k(u_0) = u_+ - u_- \) and \( \tilde{j}_k(v_0) = v_+ - v_- \) are orthogonal as claimed. The second term comes from the small eigenvalues corresponding to \( \Pi_{+,\mathfrak{B}}^{-k} \), which by Lemma 3.1 are the same as those corresponding to \( \Pi_{+,\mathfrak{B}_{sm}}^{-k} \). The second statement in the theorem follows immediately by Poincaré duality. For the last statement, it suffices to notice that from part (v) of Theorem 3.3, the positive small eigenvalues in degree \( \frac{m}{2} \) are the same as the small eigenvalues coming from \( \Pi_{\mathfrak{B}_{sm}}^{m-1} \).

We can now compute (3.18). First note that, for \( q \leq (m-1)/2 \), with \( \Delta_q = \tilde{\partial}_{\text{dr}}^2 \) in degree \( q \), taking the finite part in \( \varepsilon \) gives

\[
\log \tau_{\text{small}}(\Delta_q) = -\dim \mathbb{H}_+^q(Z;F) \log(c_{(m-1)/2-q}) + 2 \log |\det(j_{q})_{\perp}| - \dim \mathbb{H}_+^{q-1}(Z;F) \log(c_{(m-1)/2-(q-1)}) + 2 \log |\det(j_{q-1})_{\perp}| =: a_q + a_{q-1}
\]
with the convention that \( a_{m-1} = 0 \) since \( \dim \mathbb{H}_+^{(m-1)/2}(Z;F) = 0 \) and \( (j_{(m-1)/2})_{\perp} = 0 \). Next, since \( \log \tau_{\text{small}}(\Delta_q) = \log \tau_{\text{small}}(\Delta_{m-q}) \) and \( a_{(m-1)/2} = 0 \), we have that for \( m \) odd,

\[
(3.18) \quad \frac{1}{2} \sum_{q=0}^{m} (-1)^q q \log \tau_{\text{small}}(\Delta_q) = \frac{1}{2} \sum_{q=0}^{(m-1)/2} \left[ (-1)^q q + (-1)^{m-q}(m-q) \right] (\log \tau_{\text{small}}(\Delta_q))
\]

\[
= -\frac{1}{2} \sum_{q=0}^{(m-1)/2} (-1)^q(2q - m)(\log \tau_{\text{small}}(\Delta_q)) = -\frac{1}{2} \sum_{q=0}^{(m-1)/2} (-1)^q(2q - m)(a_q + a_{q-1})
\]

\[
= \sum_{q=0}^{(m-1)/2} (2q - m)(a_q + a_{q-1})
\]

\[
= \sum_{q=0}^{(m-1)/2} (-1)^q(a_q + a_{q-1}) = \sum_{q=0}^{(m-1)/2} (-1)^q(- \dim \mathbb{H}_+^q(Z;F) \log(c_{(m-1)/2-q}) + 2 \log |\det(j_{q})_{\perp}|).\]
When $m$ is even, then using the fact that
\[
\log \tau_{\text{small}}(\Delta_Z) = -2 \dim H_{+}^{m-1}(Z; F) \log(c_1) + 4 \log |\det((j_m)_{\perp})|,
\]
a similar computation shows that
\[
(3.19) \quad -\frac{1}{2} \sum_{q=0}^{m} (-1)^q q \log \tau_{\text{small}}(\Delta_q) = 0.
\]

3.3. Harmonic bases. The second consequence of the analysis in [ARS14] that we will use is about the asymptotics of harmonic forms as $\varepsilon \to 0$. Recall from 1.1 that the analytically defined metric invariant quantity is
\[
\Lambda(M, \{\mu^q\}, F) = \Lambda(M, g_{\varepsilon, d}; F) - \log \left( \prod_{q=0}^{n} |\mu^q|\omega_{\varepsilon}^{(-1)q} \right)
\]
where $\mu = \{\mu^q\}$ is a fixed basis of $H^*(M; F)$, $\omega_{\varepsilon}$ is an orthonormal basis of harmonic representatives with respect to the metric $g_{\varepsilon, hc}$ and where $[\mu^q|\omega_{\varepsilon}^q] = |\det W^q|$ with $W^q$ the matrix such that
\[
\mu^q_i = \sum_j W^q_{ij} \omega_{\varepsilon}^j.
\]
In this section we compute the asymptotic expansion of $\log \left( \prod_{q=0}^{n} |\mu^q|\omega_{\varepsilon}^{(-1)q} \right)$. We are interested in the coefficient of $\varepsilon^0$, as terms dependent on $\varepsilon$ will cancel out with those in the expansion of $\Lambda(M, g_{\varepsilon, d}; F)$.

To compute this contribution, we will make a specific choice for the basis $\mu$. Namely, we let $\mu^k_{M_0}$ and $\mu^k_Z$ be bases of orthonormal harmonic representatives for $M_0$ and for $Z$ with respect to the metrics $g_0$ and $g_Z$ respectively; by an orthogonal transformation we can also assume without loss of generality that they are compatible with the decompositions
\[
(3.20) \quad H^k_{m}(M_0; F) = L^2H^k_{H}(M_0; F) \oplus L^2H^k_{+}(M_0; F),
\]
\[
(3.21) \quad H^k(Z; F) = H^k_{H}(Z; F) \oplus H^k_{+}(Z; F).
\]
Then take $\mu^k$ to be a subset of $(\mu^k_{M_0}, \mu^k_{Z})$ which is a basis compatible with the canonical decomposition
\[
(3.22) \quad H^k(M; F) = (\ker j_k) \oplus (\ker \delta_{k-1})^\perp =: H^k_{H}(M_0; F) \oplus H^k_{H}(Z; F) \quad \text{for } k \leq \frac{m-1}{2}.
\]
Similarly, for $k > \frac{m-1}{2}$, we take $\mu^k$ to be a subset of $(\mu^k_{M_0}, \mu^k_{Z})$ compatible with the canonical decomposition
\[
(3.23) \quad H^k(M) = H^k_{H}(M_0; F) \oplus H^k_{H}(Z; F), \quad k > \frac{m-1}{2}.
\]
With these choices, the constant term in the asymptotic expansion of $[\mu^k|\omega_{\varepsilon}^k]$ is coming from $\mu^k_{Z}$ if $k \leq \frac{m-1}{2}$ and from $\mu^k_Z$ otherwise. Precisely, let $\alpha$ be a harmonic form on $Z$ with $[\alpha] \in H^k_{H}(Z; F)$ and $\|\alpha\|_{L^2} = 1$ and let $\beta \in H^k(Z; F^*)$ be a harmonic form Poincaré dual to $\alpha$, so that
\[
(3.24) \quad \int_Z \alpha \wedge \beta = 1.
\]
If $k \leq \frac{m-1}{2}$, the element $[\nu] \in H^k(M; F)$ corresponding to the form $\alpha$ in the decomposition (3.22) can be represented by a form $\nu$ with support in a tubular neighborhood $(-1, 1) \times Z$ of $Z$ in $M$ such that

\begin{equation}
\int_{(-1,1)\times Z} \nu \land \beta = 1.
\end{equation}

On the other hand, by Theorem 3.3, the harmonic form $\omega_\varepsilon$ with respect to $g_{\varepsilon, \text{hc}}$ of $L^2$-norm equal to 1 representing a positive multiple of the class $[\nu]$ in $H^k(M; F)$ is asymptotically of the form

\[ \omega_\varepsilon \sim \frac{1}{\sqrt{c_{m-1}-(k-1)}} \langle X \rangle^{k-1 - \frac{m-1}{2}} \left( \rho^{-\frac{m-1}{2}} dx \land \rho^{k-1} \alpha \right) = \varepsilon^{k-1 - \frac{m-1}{2}} \langle X \rangle^{2k-2-(m-1)} \left( \frac{dX}{\langle X \rangle} \right) \land \alpha \]

when $\varepsilon$ approaches 0. Thus, from (3.25), we see that asymptotically as $\varepsilon$ tends to zero, $[\nu] \sim \gamma_\varepsilon [\omega_\varepsilon]$ with

\[ \gamma_\varepsilon^{-1} = \int_{-\infty}^{\infty} \varepsilon^{k-1 - \frac{m-1}{2}} \langle X \rangle^{2k-2-(m-1)} \frac{dX}{\langle X \rangle} = \varepsilon^{k-1 - \frac{m-1}{2}} \sqrt{c_{m-1}-(k-1)}. \]

This implies that for $k \leq \frac{m-1}{2}$,

\begin{equation}
\log[\mu^k | \omega^k] = - \dim H_{H}^{k-1}(Z; F) \left( \frac{1}{2} \log c_{m-1} - (k - 1 - \frac{m-1}{2}) \log \varepsilon \right) + o(1)
\end{equation}

as $\varepsilon$ tends to zero. When $k > \frac{m-1}{2}$, the form $\nu$ representing the cohomology class $[\nu]$ corresponding to the form $\alpha$ under the decomposition (3.23) is such that its restriction to $Z$ is $[\alpha]$, in other words,

\begin{equation}
\int_{Z} \nu \land \beta = 1.
\end{equation}

Another application of Theorem 3.3 shows that the harmonic form $\omega_\varepsilon$ with respect $g_{\varepsilon}$ of $L^2$-norm equal to 1 representing a positive multiple of the class $[\nu]$ in $H^k(M; F)$ is asymptotically of the form

\begin{equation}
\omega_\varepsilon \sim \frac{1}{\sqrt{c_{k-\frac{m-1}{2}}}} \langle X \rangle^{\frac{m-1}{2} - k} \left( \rho^{-\frac{m-1}{2}} \rho^k \alpha \right) = \varepsilon^{k-\frac{m-1}{2}} \langle X \rangle^{2k-2-(m-1)} \left( \frac{dX}{\langle X \rangle} \right) \land \alpha
\end{equation}

as $\varepsilon$ tends to zero. From (3.27), we thus see that

\[ [\nu] \sim \sqrt{c_{k-\frac{m-1}{2}}} [\omega_\varepsilon] \]

as $\varepsilon$ tends to zero. Taking the logarithm, we obtain that for $k > \frac{m-1}{2}$,

\begin{equation}
\log[\mu^k | \omega^k] = \dim H_{H}^{k}(Z; F) \left( \frac{1}{2} \log c_{k-\frac{m-1}{2}} + \left( \frac{m-1}{2} - k \right) \log \varepsilon \right) + o(1)
\end{equation}

Using Poincaré duality on $Z$ and the fact $m$ is odd, we see from (3.26) and (3.29) that the constant term in the expansion of $- \log[\Pi_{q=0}^{m} [\mu^q | \omega^q]^{(-1)^q}]$ is given by

\begin{equation}
- \sum_{q=1}^{m-1} (-1)^{q+1} \dim H_{H}^{q-1}(Z; F) \log c_{\frac{m-1}{2}-(q-1)} = - \sum_{q=0}^{m-1} (-1)^q \dim H_{H}^{q}(Z; F) \log c_{\frac{m-1}{2}-q}.
\end{equation}
4. CUSP DEGENERATION AND ANALYTIC TORSION

Let $M$ be a closed manifold with a two-sided hypersurface $Z$. We endow $X_s$ with a flat Euclidean bundle $F \to X_s$ and an $\varepsilon$, hc metric $g_{\varepsilon, hc}$ and in this section we determine the limit as $\varepsilon \to 0$ of analytic torsion.

In [ARS14, Theorem 8.1] we have computed the constant term in the expansion as $\varepsilon \to 0$ of the logarithm of analytic torsion:

\[
\begin{align*}
(4.1) \quad \text{FP LAT}(M, g_{\varepsilon, hc}; F) &= \text{LAT}([M; Z], g_0; F) \\
&\quad + \text{LAT}([-\pi/2, \pi/2], D_b, \mathcal{H}^*(Z; F)) - \frac{1}{2} \sum (-1)^q \log \tau_{\text{small}}(\Delta_q).
\end{align*}
\]

We have computed the last two terms, which leads to the following theorem.

**Theorem 4.1.** Let $M$ be a closed manifold, $Z$ a two-sided hypersurface in $M$, and $F \to X_s$ a flat Euclidean vector bundle. Let $g_{\varepsilon, hc}$ be an $\varepsilon$, hc metric, product-type to order two, and let $\mu^k_M$ and $\mu^k_Z$ be orthonormal bases of harmonic representatives with respect to the metrics $g_0$ and $g_Z$; then let $\mu$ be a choice of basis for $\mathcal{H}^*(M; F)$ compatible with the decompositions (3.22) and (3.23). If $m$ is odd, then

\[
(4.2) \quad \text{LAT}(M, \mu; F) = \text{LAT}([M; Z], \mu_0; F)
\]

\[
\quad + \sum_{q=0}^{m-1} (-1)^q \left[ 2 \log |\det(j_q)\| + \dim \mathcal{H}^q(Z; F) ((m - 1 - 2q) \log(m - 1 - 2q)) \right].
\]

If instead $m$ is even, then $\text{LAT}(M, \mu; F) = 0$ and

\[
\text{LAT}([M; Z], \mu_0; F) = m \sum_{q=\frac{m-1}{2}}^{\frac{m+1}{2}} (-1)^q \dim \mathcal{H}^q(Z; F) \log(m - 1 - 2q).
\]

**Proof.** When $m$ is odd, the term $\text{LAT}([-\pi/2, \pi/2], D_b, \mathcal{H}^*(Z; F))$ is computed in (2.15) and equals

\[
\sum_{q=0}^{\frac{v-1}{2}} (-1)^q \dim \mathcal{H}^q(Z; F) \left[ \log c_{v/2-q} + (v - 2q) \log(v - 2q) \right].
\]

The term $-\frac{1}{2} \sum (-1)^q \log \tau_{\text{small}}(\Delta_q)$ is computed in (3.18) and equals

\[
\sum_{q=0}^{(m-1)/2} (-1)^q (- \dim \mathcal{H}^q_+(Z; F) \log(c_{v/2-q}) + 2 \log |\det(j_q)\|).
\]

Finally, recall from \[3.3\] that

\[
\text{LAT}(M, \mu, F) = \text{LAT}(M, g_{\varepsilon, hc}; F) - \log (\prod_{q=0}^{\infty} |\mu^q|\omega^q_{\varepsilon}^{(-1)^q})
\]

and from \[3.30\] that the constant term in the expansion of $- \log (\prod_{q=0}^{\infty} |\mu^q|\omega^q_{\varepsilon}^{(-1)^q})$ as $\varepsilon \to 0$ is given by

\[
\sum_{q=0}^{m-1} (-1)^q (\text{dim } \mathcal{H}^q_+(Z; F) \log c_{v/2-q}).
\]
The formula when $m$ is odd then follows immediately from the formulas above and the fact that
\[
\dim \mathcal{H}^q(Z; F) = \dim \mathcal{H}^q_1(Z; F) + \dim \mathcal{H}^q_1(Z; F),
\]
which yields cancellation of all the terms involving the constants $c_k$. When $m$ is even, then $\text{LAT}(M, \mu, F) = 0$ by the argument of [2.14]. We also computed that
\[
-\frac{1}{2} \sum (-1)^q \log \tau_{\text{small}}(\Delta_q) = 0,
\]
so the result follows from (4.1) and (2.14) with $v$ odd.

Applying this result to the case where $M_0 = N_0 \sqcup N_0$ is the disjoint union of two copies of $(N_0, g_0)$, an even-dimensional manifold with cusp with link $Z$, and flat Euclidean vector bundle $F$ satisfying the Witt condition, we obtain the following.

**Corollary 4.2.** The analytic torsion of an $m$-dimensional manifold ($m$ even) with cusp $(N_0, g_0)$ with link $Z$, equipped with a flat vector bundle $F$ satisfying the Witt condition, is given by
\[
\text{LAT}([M; Z], \mu_0; F) = \frac{m}{2} \sum_{q \leq \frac{m-1}{2}} (-1)^q \dim \mathcal{H}^q(Z; F) \log (m - 1 - 2q),
\]
where $\mu_0$ is a basis of $L^2$-harmonic forms on $N_0$ with values in $F$.

## 5. Cusp degeneration and Reidemeister torsion

We assume as in the previous section that $F \to X_s$ is a flat Euclidean vector bundle such that $H^{m-1}(Z; F) = \{0\}$ (the Witt condition). Moreover, we will now assume that $m$ is odd. To study the change of the $R$-torsion under a pinching surgery, we will make use of the long exact sequence (3.3). As a complex, we will denote this long exact sequence by $\mathcal{H}_1$.

Let
\[
\overline{M} = \mathfrak{B}_{sm} \bigcup (CZ \sqcup CZ), \quad \mathfrak{B}_{sm} \bigcap (CZ \sqcup CZ) = Z \sqcup Z,
\]
be the singular space associated to $\mathfrak{B}_{sm}$, where $CZ$ is the disjoint union of the cones of each connected component of $Z$. To relate the $R$-torsion of $M$ with an appropriate intersection $R$-torsion on $\overline{M}$, we will need another exact sequence, namely the Mayer-Vietoris sequence obtained by writing $\overline{M}$ as the union of $\mathfrak{B}_{sm}$ with $CZ \sqcup CZ$.

The pseudomanifold $\overline{M}$ has a natural stratification of depth one, with singular stratum given by a disjoint union of points. Let $T$ be a choice of triangulation on $\overline{M}$ compatible with this stratification and the decomposition (5.1). Recall from [ARS14] that we can then use $T$ and its first barycentric subdivision $T'$ to define the complex of cochains $R^*_m(\overline{M}, \alpha)$ where $\alpha$ is the orthogonal representation induced by the holonomy of $F$. This complex has natural restrictions to $CZ \sqcup CZ$ and $Z \sqcup Z$, so there is an induced Mayer-Vietoris short exact sequence of finite dimensional complexes
\[
0 \longrightarrow R^*_m(\overline{M}; F) \longrightarrow C^*_T(M_0; F) \oplus R^*_m(CZ \sqcup CZ; F) \longrightarrow C^*_T(Z \sqcup Z; F) \longrightarrow 0.
\]

Here, $C^*_T(M_0; F) = C^*_T(\mathfrak{B}_{sm}) \otimes_{\pi_1(\mathfrak{B}_{sm})} \mathbb{R}^q$, where $\mathbb{R}^q$ is seen as a $\mathbb{Z}\pi_1(\mathfrak{B}_{sm})$-module via the representation $\alpha : \pi_1(M) \to O(q)$ given by the holonomy of $F$, $\tilde{T}'$ is the lift of $T'$ to
the universal cover \( \widetilde{\mathcal{B}}_{sm} \) of \( \mathcal{B}_{sm} \), and \( C^*_{\widetilde{T}}(\widetilde{\mathcal{B}}_{sm}) \) is the group of cochains associated to the triangulation \( \widetilde{T} \). Similarly, we have that

\[
C_{\tau}(Z \sqcup Z; F) = \bigoplus_i \left[ C^*_{\widetilde{T}}(Z_i) \otimes_{\mathbb{Z}_1} \mathbb{R}^q \oplus C^*_{\widetilde{T}}(Z_i) \otimes_{\mathbb{Z}_1} \mathbb{R}^q \right],
\]

where \( i \) labels the connected components of \( Z \). By [ARS14, Proposition 8.14], the short exact sequence \( (5.2) \) induces a Mayer-Vietoris long exact sequence involving intersection cohomology \( (5.3) \)

\[
\cdots \rightarrow \mathbb{H}_m^k(M; F) \xrightarrow{\partial_k} \mathbb{H}_m^k(M_0; F) \oplus \mathbb{H}_m^k(CZ \sqcup CZ; F) \xrightarrow{\partial_k} \mathbb{H}_m^k(Z \sqcup Z; F) \rightarrow \cdots,
\]

where \( \mathbb{H}_m^k(M; F) = \mathbb{H}_m^k(M, \alpha) \) and \( \mathbb{H}_m^k(CZ \sqcup CZ; F) = \mathbb{H}_m^k(CZ \cup CZ, \alpha) \). We will denote the long exact sequence \( (5.3) \) by \( \mathcal{H}_2 \).

**Theorem 5.1.** Given any bases for \( \mathbb{H}^k(M; F), \mathbb{H}_m^k(M; F), \mathbb{H}_m^k(CZ; F), \mathbb{H}^k(Z; F), \) and \( \mathbb{H}^k(M_0; F) \), we define corresponding bases for \( \mathbb{H}_m^k(CZ \cup CZ; F) = \mathbb{H}_m^k(CZ; F) \oplus \mathbb{H}_m^k(CZ; F) \), and \( \mathbb{H}^k(Z \sqcup Z; F) = \mathbb{H}^k(Z; F) \oplus \mathbb{H}^k(Z; F) \) via the direct sum. Using these bases to define the \( R \)-torsions of \( M, \mathcal{M}, \mathcal{B}_{sm}, CZ, CZ \sqcup CZ, Z, Z \sqcup Z, \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we have

\[
\tau(M; F) = \frac{I \tau^m(M; F) \tau(Z; F) \tau(\mathcal{H}_2)}{I \tau^m(CZ; F)^2} \tau(\mathcal{H}_1).
\]

**Proof.** By the formula of Milnor [Mil66], we have that

\[
\tau(M_0; F) = \tau(M; F) \tau(Z; F) \tau(\mathcal{H}_1)
\]

and

\[
\tau(M_0; F) I \tau^m(CZ; F)^2 = I \tau^m(M; F) \tau(Z; F)^2 \tau(\mathcal{H}_2).
\]

Combining these two relations gives the result. Note that the direct sum assumption is used to write, for example, \( \tau(Z \sqcup Z; F) = (\tau(Z; F))^2 \). \( \square \)

We now make a particular choice of bases for these spaces that allows a direct comparison with \( (4.12) \) and also makes some of the terms in \( (5.4) \) more explicit, in particular \( \tau(\mathcal{H}_2) \tau(\mathcal{H}_1)^{-1} \). Recall from section 3.3 the decompositions:

(5.5) \( \mathbb{H}_m^k(M_0; F) = L^2 \mathbb{H}_H^k(M_0; F) \oplus L^2 \mathbb{H}_H^{k-1}(M_0; F) \),

(5.6) \( \mathbb{H}^k(Z; F) = \mathbb{H}_H^k(Z; F) \oplus \mathbb{H}_H^k(Z; F) \),

(5.7) \( \mathbb{H}^k(M; F) = (\ker j_k) \oplus (\ker \partial_{k-1})^\perp =: \mathbb{H}_H^k(M_0; F) \oplus \mathbb{H}_H^{k-1}(Z; F) \) for \( k \leq \frac{m-1}{2} \);

(5.8) \( \mathbb{H}^k(M) = \mathbb{H}_H^k(M_0; F) \oplus \mathbb{H}_H^{k+1}(Z; F), \quad k > \frac{m-1}{2} \).

We use the same bases as in section 3.3 for \( \mathbb{H}_m^k(M; F) \), \( \mathbb{H}^k(Z; F) \), and \( \mathbb{H}^k(M; F) \); namely, orthonormal bases \( \mu^k_{M_0}, \mu^k_{Z}, \) and \( \mu^k \) compatible with the decompositions \( (5.5), (5.7) \) and \( (5.8) \).

For \( k \leq \frac{m-1}{2} \), we use the canonical identification \( \mathbb{H}^k(M_0; F) = \mathbb{H}_m^k(M; F) \) to get a corresponding basis for \( \mathbb{H}^k(M_0; F) \). Similarly, for \( k \geq \frac{m+1}{2} \), the canonical identification \( \mathbb{H}_c^k(M_0; F) = \mathbb{H}_m^k(M; F) \) gives a corresponding basis for \( \mathbb{H}_c^k(M_0; F) := \mathbb{H}^k(\mathcal{B}_{sm}, \partial \mathcal{B}_{sm}; F) \).
We also need to make a choice of basis of $\text{IH}_m^k(CZ; F)$; we will take the one induced by our choice of basis for $H^k(Z; F)$ and the canonical identification

$$\text{IH}_m^k(CZ; F) = \begin{cases} H^k_H(Z; F), & k \leq \frac{m-1}{2}, \\ \{0\}, & k > \frac{m-1}{2}. \end{cases}$$

Note immediately that since $Z$ is even-dimensional, we have $\tau(Z; F) = 1$ by [Che79, Proposition 1.19]. It remains to make a choice of basis for $H^k(M_0; F)$ when $k > \frac{m-1}{2}$, but at the moment we can at least make a partial computation of $\tau(H_2)\tau(H_1)^{-1}$.

**Lemma 5.2.** With the choice of bases made above, the contribution to $\tau(H_2)\tau(H_1)^{-1}$ coming from cohomology classes of degree $k \leq \frac{m-1}{2}$ is given by

$$\prod_{k < \frac{m-1}{2}} \left( |\det(j_k) : H^k(M_0; F) \to H^k(Z; F))| \right)^{(-1)^k}.$$

**Proof.** First we compute the contribution to $\tau(H_2)$. To do this, it suffices to notice that the restriction of $j^*_k$ to the second factor induces the identity map $j^*_k : \text{IH}_m^k(CZ \sqcup CZ; F) \to H^k(Z \sqcup Z; F)$ with respect to our choice of bases, while $i^*_k$ composed with the projection on the first factor gives the canonical identification $\text{IH}_m^k(M; F) = H^k(M_0; F)$ (for $k \leq \frac{m-1}{2}$) used to choose our basis for $H^k(M_0; F)$.

As for $\tau(H_1)$, the decompositions (5.5) and (5.7) are such that in the long exact sequence $H_1$,

$$\text{im}(i_k) = H^k_H(M_0; F), \quad \text{im} j_k = H^k(Z; F), \quad \text{im} \partial_k = H^k_H(Z; F) \subset H^{k+1}(M; F).$$

With our choice of bases, this means that the contribution to $\tau(H_1)$ coming from cohomology classes of degree $k \leq \frac{m-1}{2}$ is given by

$$\prod_{k < \frac{m-1}{2}} \left( \frac{|\det(i_k)\perp||\det(\partial_k)\perp|}{|\det(j_k)\perp|} \right)^{(-1)^k},$$

where the subscript $\perp$ for a map $d$ denotes the restriction $d\perp : (\ker d)\perp \to \text{im} d$ of $d$ to the orthogonal complement of its kernel with respect to the $L^2$-inner product. Since $(i_k)\perp : H^k_H(M_0; F) \to H^k_H(M_0; F)$ and $(\partial_k)\perp : H^k_H(Z; F) \to H^k_H(Z; F)$ are the identity maps on $H^k_H(M_0; F)$ and $H^k_H(Z; F)$ for our choices of bases, the result follows. \hfill $\square$

In degree $k > \frac{m-1}{2}$, we will take advantage of some cancellations occurring between $\tau(H_1)$ and $\tau(H_2)$ to compute $\tau(H_2)\tau(H_1)^{-1}$ directly. First, notice that for $k > \frac{m-1}{2}$, the long exact sequence $H_2$ corresponds to the relative long exact sequence associated to the pair $(\mathcal{B}_{sm}, \partial\mathcal{B}_{sm})$,

$$\cdots \to H^k_c(M_0; F) \overset{i^*_k}{\longrightarrow} H^k(M_0; F) \overset{j^*_k}{\longrightarrow} H^k(Z \sqcup Z; F) \overset{\partial^*_k}{\longrightarrow} \cdots, \quad k \geq \frac{m+1}{2},$$

$$\cdots \to H^k_c(M_0; F) \overset{i^*_k}{\longrightarrow} H^k(M_0; F) \overset{j^*_k}{\longrightarrow} H^k(Z \sqcup Z; F) \overset{\partial^*_k}{\longrightarrow} \cdots, \quad k \geq \frac{m+1}{2},$$
under the canonical identification $H^k_c(M_0; F) = \text{IH}^k_{\text{m}}(M; F)$. This leads to the following commutative diagram between the long exact sequences $\mathcal{H}_1$ and $\mathcal{H}_2$ when $k \geq \frac{m+1}{2}$:

\[(5.11)\]

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & H^k_c(M_0; F) & \overset{i'_k}{\longrightarrow} & H^k(M_0; F) & \overset{j'_k}{\longrightarrow} & H^k(Z \sqcup Z; F) & \overset{\partial'_k}{\longrightarrow} & H^k_{c+1}(M_0; F) & \longrightarrow \cdots \\
\downarrow \alpha_k & & \downarrow \text{id} & & \downarrow \beta_k & & \downarrow \alpha_{k+1} & \\
\cdots & \longrightarrow & H^k(M; F) & \overset{i_k}{\longrightarrow} & H^k(M_0; F) & \overset{j_k}{\longrightarrow} & H^k(Z; F) & \overset{\partial_k}{\longrightarrow} & H^{k+1}(M; F) & \longrightarrow \cdots ,
\end{array}
\]

where $\alpha_k : H^k_c(M_0; F) \to H^k(M; F)$ is the standard push-forward map and the map $\beta_k$ is given by

$$\beta_k : H^k(Z; F) \oplus H^k(Z; F) \to H^k(Z; F) \quad (\mu_+, \mu_-) \mapsto \mu_+ - \mu_-,$$

and the canonical identification $H^k(Z \sqcup Z; F) = H^k(Z; F) \oplus H^k(Z; F)$. This definition suggests that we take a different orthonormal basis of harmonic forms on $H^k(Z; F) \oplus H^k(Z; F)$. Namely, if $\nu_1, \ldots, \nu_k$ is our chosen basis for $H^k(Z; F)$, then we take the basis

$$\left(\frac{\nu_1}{\sqrt{2}}, \frac{\nu_2}{\sqrt{2}}, \ldots, \frac{\nu_k}{\sqrt{2}}\right), \left(\frac{\nu_1}{\sqrt{2}}, -\frac{\nu_2}{\sqrt{2}}, \ldots, -\frac{\nu_k}{\sqrt{2}}\right),$$

for $H^k(Z \sqcup Z; F) = H^k(Z; F) \oplus H^k(Z; F)$. Since this change of basis is orthogonal, it has no effect on the torsion of $Z \sqcup Z$, and in particular (5.4) still holds if we compute the torsion of $Z \sqcup Z$ with respect to this new basis. We can now make the following simple observations.

**Lemma 5.3.** Recall the map $\hat{i}_k$ defined in (3.6). For $k \geq \frac{m+1}{2}$ the following assertions hold:

1. $\ker \alpha_k = \ker \hat{i}_k = \mathcal{H}^k_c(M_0; F) \subset \text{IH}^k_{\text{m}}(M; F) = H^k_c(M_0; F);$  
2. $\text{im} \alpha_k = \text{im} \hat{i}_k \cong \mathcal{H}^k_c(M_0; F) \subset H^k(M; F);$  
3. If $\omega_1, \ldots, \omega_k$ is an orthonormal basis of $\mathcal{H}^k_c(Z; F)$, then

\[
\partial'_k \left(\frac{\omega_1}{\sqrt{2}}, \frac{\omega_2}{\sqrt{2}}, \ldots, \frac{\omega_k}{\sqrt{2}}\right) = 0;
\]

4. $\beta_k \circ j'_k \circ i_k(\mathcal{H}^k_c(Z; F)) = 0;$  
5. The composition

\[
\mathcal{H}^k_c(Z; F) \overset{i_k}{\longrightarrow} H^k(M_0; F) \overset{j'_k}{\longrightarrow} H^k(Z; F) \oplus H^k(Z; F) \overset{\text{pr}_d}{\longrightarrow} H^k(Z; F) \longrightarrow \mathcal{H}^k_c(Z; F)
\]

is the identity map, where $\text{pr}_d$ is the projection

$$\text{pr}_d : H^k(Z; F) \oplus H^k(Z; F) \to H^k(Z; F) \quad (\mu_1, \mu_2) \mapsto \frac{\mu_1 + \mu_2}{2}.$$

In particular, the map $i_k$ is injective when restricted to $\mathcal{H}^k_c(Z; F) \subset H^k(M; F)$.

**Proof.** The proof of (1) and (2) follows from the identification of $H^k_c(M_0; F) \cong \text{IH}^k_{(2)}(M_0; F)$ when $k \geq \frac{m+1}{2}$. For (3), it follows by noticing that, still under the identification $H^k_c(M_0; F) \cong H^k_{(2)}(M_0; F)$, we have that $\partial'_k \circ i_k = \hat{j}_k$ where

$$i_k : H^k(Z; F) \to H^k(Z; F) \oplus H^k(Z; F) \quad \mu \mapsto (\mu, \mu)$$

and

$$j_k : H^k(Z; F) \to H^k(Z; F) \oplus H^k(Z; F) \quad \omega \mapsto (\omega, -\omega).$$
is the diagonal inclusion. For (4), this is by exactness of the bottom sequence in (5.11), since
\[ \beta_k \circ j_k^* \circ i_k = j_k \circ i_k = 0. \]
Finally, (5) follows from (3.9) and the definition of the map \( i_k \).

Therefore, removing the span of \( (\sqrt{\frac{1}{2}} \omega_k, \sqrt{\frac{1}{2}} \omega_k), \ldots (\sqrt{\frac{1}{2}} \omega_k, \sqrt{\frac{1}{2}} \omega_k) \) in \( H^k(Z \sqcup Z; F) \), \( H^k(H_1; F) \) in \( H^k(M_0; F) \), \( H^k_1(Z; F) \) in \( H^k(M; F) \) and \( i_k(H^k_1(Z; F)) \) in \( H^k(M_0; F) \), we obtain from (5.11) the following commutative diagram of long exact sequences:

\[
\begin{array}{c}
\cdots \longrightarrow H^k(M_0; F) \longrightarrow H^k(M_0; F)/i_k(H^k_1(Z; F)) \longrightarrow H^k(Z; F) \longrightarrow \cdots \\
\downarrow \text{Id} & \downarrow \text{Id} & \downarrow \text{Id} \\
\cdots \longrightarrow H^k_1(M_0; F) \longrightarrow H^k_1(M_0; F)/i_k(H^k_1(Z; F)) \longrightarrow H^k(Z; F) \longrightarrow \cdots.
\end{array}
\]

In this diagram, the contribution of the top row to \( \tau(H_1)^{-1} \tau(H_2) \) is cancelled by the contribution from the bottom row. Therefore, in degree \( k \geq \frac{m+1}{2} \), the only contributions to \( \tau(H_1)^{-1} \tau(H_2) \) come from

- \( \partial^i_k \) when restricted to the span of \( (\sqrt{\frac{1}{2}} \omega_k, \sqrt{\frac{1}{2}} \omega_k), \ldots (\sqrt{\frac{1}{2}} \omega_k, \sqrt{\frac{1}{2}} \omega_k) \) in \( H^k(Z \sqcup Z; F) \);
- \( i_k : H^k_1(Z; F) \rightarrow i_k(H^k_1(Z; F)) \); and
- \( j^*_k : i_k(H^k_1(Z; F)) \rightarrow j^*_k \circ i_k(H^k_1(Z; F)) \cong H^k_1(Z; F) \).

To reach this conclusion, we have tacitly assumed that we have chosen a basis of \( H^k(M_0; F) \) for \( k \geq \frac{m+1}{2} \) which includes a basis of \( i_k(H^k_1(Z; F)) \). We can go one step further and choose this basis so that the corresponding basis of \( i_k(H^k_1(Z; F)) \) is the image under \( i_k \) of the chosen basis on \( H^k_1(Z; F) \). With this choice, \( i_k \) does not contribute to \( \tau(H_1)^{-1} \tau(H_2) \) and we are left with the contributions of \( \partial^i_k \) and \( j^*_k \).

We are now ready to state the refinement of Theorem 5.1.

**Theorem 5.4.** Let \( \mu^k_{M_0} \) and \( \mu^k_Z \) be bases of \( IH^k_m(M; F) \) and \( H^k(Z; F) \), orthonormal with respect to the metrics \( g_0 \) and \( g_Z \) respectively, and compatible with the decompositions (5.5). Let \( \mu^k \) be a basis of \( H^k(M; F) \) compatible with (5.7) and (5.8) and choose the basis for \( IH^k_m(CZ; F) \) induced by (5.9). Using these bases to define the corresponding R-torsions, we have the relation

\[
\log \tau(M; F) = \log \frac{I_{\tau^m(M; F)}}{I_{\tau^m(CZ; F)}} + 2 \sum_{k=\frac{m+1}{2}} (-1)^k \log | \det((j_k)_\perp) | - \frac{\chi(Z; F)}{4} \log 2.
\]

**Proof.** By [Che79 Proposition 1.19], we know that \( \tau(Z; F) = 1 \). By Theorem 5.1, it remains therefore to compute \( \tau(H_1)^{-1} \tau(H_2) \). By Lemma 5.2 and the discussion above, it remains to compute the contributions coming from \( j^*_k \) and \( \partial^i_k \) in degree \( k \geq \frac{m+1}{2} \).

First, notice that with the choice of basis we have made for \( H^k(M_0; F) \), \( j^*_k \) is almost an isometry; more precisely, \( j^*_k \sqrt{\frac{1}{2}} \) is an isometry. The contribution of \( j^*_k \) to \( \tau(H_1)^{-1} \tau(H_2) \) is therefore given by

\[
\prod_{k=\frac{m+1}{2}} | \det(j^*_k : i_k(H^k_1(Z; F)) \rightarrow j^*_k(i_k(H^k_1(Z; F)))) |^{-(-1)^k} = \prod_{k=\frac{m+1}{2}} 2^{-(-1)^k \dim H^k_1(Z; F)}.
\]
On the other hand, identifying the span of \( \left( \frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_2}{\sqrt{2}} \right), \ldots \left( \frac{\alpha_{m-1}}{\sqrt{2}}, \frac{\alpha_{m-1}}{\sqrt{2}} \right) \) isometrically with \( \mathcal{H}^k_\mu(Z; F) \), we see that the map \( \sqrt{2} \partial'_k : \mathcal{H}^k_\mu(Z; F) \to L^2 \mathcal{H}^{m-k+1}_\mu(M_0; F) \) corresponds to \( j_k \), hence to the adjoint of \( j_{m-k-1} : L^2 \mathcal{H}^{m-k-1}_\mu(M_0; F) \to \mathcal{H}^{m-k-1}_\mu(Z; F) \). Thus, the contribution of \( \partial'_k : \mathcal{H}^k_\mu(Z; F) \to L^2 \mathcal{H}^{k+1}_\mu(M_0; F) \) to \( \tau(\mathcal{H}_1)^{-1} \tau(\mathcal{H}_2) \) is given by
\[
(5.15) \quad \prod_{k < \frac{m-1}{2}} (\sqrt{2}^{-\dim \mathcal{H}^k_\mu(Z; F)}) | \det((j_k)_\perp)| \kappa^{-k}.
\]
Combining (5.14) and (5.15) with Lemma 5.2 and using the fact that \( \dim \mathcal{H}^k(Z; F) = \dim \mathcal{H}^k_\mu(Z; F) + \dim \mathcal{H}^k_\mu(Z; F) \), along with Poincaré duality, gives the result. \( \square \)

**Remark 1.** Even though we have made a specific choice of basis for \( \mathcal{H}^k(M_0; F) \) to prove Theorem 5.4, the final result is independent of such a choice.

6. A Cheeger-Müller theorem for Witt representations on manifolds with cusps

Let \( (M, g) \) be a closed Riemannian manifold with a flat bundle \( F \to M \) corresponding to an orthogonal representation of the fundamental group of \( M \). The classical Cheeger-Müller theorem gives an equality of analytic torsion and R-torsion for every choice of basis \( \mu \) of the homology groups:
\[
\text{LAT}(M, \mu, F) = \log \tau(M, \mu, F).
\]

We have analyzed the behavior of both sides of this equation under analytic cusp surgery - the left-hand side in Theorem 4.1 and the right-hand side in Theorem 5.4. In this section we use these results to conclude a Cheeger-Müller theorem for Witt representations on manifolds with cusps.

**Theorem 6.1.** Let \( M \) be an odd-dimensional manifold, \( Z \subseteq M \) a two sided hypersurface, \( g_{c, hc} \) a cusp surgery metric which is product-type to order two, and \( F \to X \) a flat Euclidean vector bundle satisfying the Witt condition. Let \( \mu^k_{M_0} \) and \( \mu^k_Z \) be bases of \( \mathcal{H}^k(M_0; F) \cong L^2 g_0 \mathcal{H}^k(M_0; F) \) and \( \mathcal{H}^k(Z; F) \), consisting of harmonic forms which are orthonormal with respect to the metrics \( g_0 \) and \( g_2 \). Choose the basis \( \mu_{CZ} \) for \( \mathcal{H}^k_{CZ}(CZ; F) \) induced by (5.9). Using these bases to define the corresponding R-torsions and analytic torsion, we have the following formula:
\[
(6.1) \quad \text{LAT}([M; H], \mu_{M_0}, F) = \log \left( \frac{I_{\tau^m(M_0, \mu_{M_0}, F)}}{I_{\tau^m(CZ, \mu_{CZ}, F)^2}} \right) - \frac{\chi(Z; F)}{4} \log 2 - \sum_{q=0}^{m-1} (-1)^q \dim \mathcal{H}^q(Z; F) [(m - 1 - 2q) \log(m - 1 - 2q)].
\]

**Proof.** Assume (by an orthogonal change of basis, if necessary) that \( \mu_{M_0} \) and \( \mu_{CZ} \) respect the decompositions (3.20), then pick the basis \( \mu \) for \( \mathcal{H}^k(M; F) \) induced by (3.22) and (3.23); the theorem then follows immediately from Theorems 4.1 and 5.4 together with the Cheeger-Müller theorem on \( M \). \( \square \)

In particular, applying this result to the case where \( M_0 = N_0 \sqcup N_0 \) is the disjoint union of two copies of \( (N_0, g_0) \), a manifold with cusp with link \( Z \), where \( g_0 \) is product-type to
order two and $E$ is a flat vector bundle on $N_0$ satisfying the Witt condition, we obtain the following:

**Corollary 6.2.** Let $\mu_{N_0}$ and $\mu_Z$ be bases of $IH^k_m(\overline{N}_0; E) \cong L^2_{g_0} H^k(N_0; E)$ and $H^k(Z; E)$ respectively, consisting of harmonic forms orthonormal with respect to $g_0$ and $g_Z$. The canonical identification \([5,9]\) gives a basis $\mu_{CZ}$ for $IH^k_m(CZ)$. Using these bases to define the corresponding $R$-torsions and analytic torsion, we have the following formula:

\[
\text{LAT}(N_0, \mu_{N_0}, E) = \log \left( \frac{I_r^m(\overline{N}_0, \mu_{N_0}, E)}{I_r^m(CZ, \mu_{CZ}, F)} \right) - \frac{\chi(Z; E)}{8} \log 2 \\
- \frac{1}{2} \sum_{q=0}^{m-1} (-1)^q \dim H^q(Z; E) [(m - 1 - 2q) \log(m - 1 - 2q)].
\]

Note that if $E$ is a flat vector bundle on $N_0$, it extends to a flat vector bundle on the double $M_0$ which can then be pulled back to obtain a flat vector bundle on $X_s$, allowing us to apply the main theorem.

7. Cusp degeneration and the boundary of Teichmüller space

In addition to analyzing the analytic torsion, our results can also be used to analyze the behavior of families of hyperbolic metrics on surfaces which approach the boundary of Teichmüller space, giving a new perspective on results of Wolpert and of Burger \([Wol87, Wol90, Wol10, Bur88]\). In particular, we analyze the so-called ‘plumbing construction’. First we describe this construction: let $R_0$ be a hyperbolic surface with nodes $p_1, \ldots, p_m$; each node represents a pair of cusps at punctures $a_i$ and $b_i$ of $R_0 \setminus \{p_1, \ldots, p_m\}$. Let $U^j_i, j = 1, 2$, be neighborhoods of $a_i$ and $b_i$ respectively. Suppose without loss of generality that each $U^j_i$ is a disk of radius $\gamma_0$ for some fixed $\gamma_0 > 0$ and that we have local coordinates $z_i : U^1_i \to U$ and $w_i : U^2_i \to U$ with $z_i(a_i) = 0$, $w_i(b_i) = 0$ and such that the hyperbolic metric $g_0$ takes the forms

\[
\left( \frac{|dz_i|}{|z_i| \log |z_i|} \right)^2 \quad \text{and} \quad \left( \frac{|dw_i|}{|w_i| \log |w_i|} \right)^2
\]

in terms of these coordinates. Then, for a sufficiently small choice of $\gamma_0$, there exists an open set $V$ disjoint from each $U^j_i$ and a set of Beltrami differentials $\nu_i, i = 1, \ldots, 3g - 3 - m$, which are supported in $V$ and which span the tangent space of the boundary of Teichmüller space at $R_0$. For $s$ sufficiently close to the origin in $\mathbb{C}^{3g - 3 - m}$, we can solve the Beltrami equation for $\nu(s) = \sum_{i=1}^m s_i \nu_i$ and obtain a family of hyperbolic surfaces (with cusps) which we denote $R_{\nu(s)}$. The hyperbolic metrics $g_s$ on these surfaces are smooth in $s$ and are conformal to the hyperbolic metric $g_0$ on $R_0$ in each $U^j_i$ \([Wol90, Wol10]\).

We now introduce a degeneration parameter $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{C}^m$ describing the opening of the nodes. For each $\sigma$ sufficiently close to the origin, and each $i \in [1, m]$, we remove a pair of disks $D^1_{\sigma_i}$ of radius $|\sigma_i|$ around the $i$th node and identify the annuli $U^1_i \setminus D^1_{\sigma_i}$ and $U^2_i \setminus D^2_{\sigma_i}$, in complex coordinates $(z, w)$, by $zw = \sigma_i$. This produces new Riemann surface $R_{\sigma,s}$ spanning a neighborhood of $R_0$ in Teichmüller space \([Wol87]\). Each surface $R_{\sigma,s}$ can be equipped with a unique hyperbolic metric $g_{\sigma,s}$ in the conformal class specified by the complex structure.
To describe the behavior of $g_{\sigma,s}$ as $\sigma \to 0$, notice that the local model describing the opening of the node, the degeneration fixture

$$\mathcal{P} = \{(z,w,\tau) : zw = \tau, |z|, |w|, |	au| < 1\},$$

is a complex manifold fibering over the disk $D = \{|	au| < 1\}$ with fiber above $\tau$ naturally identified with the annulus $|\tau| < |z| < 1$ for $\tau \neq 0$. On each fiber, the unique complete hyperbolic metric in the conformal class specified by the complex structure is given by

$$g_{\mathcal{P},\tau} = \left(\frac{\pi}{\log|	au|} \csc \left(\frac{\pi \log|z|}{\log|	au|}\right) \frac{|dz|}{z}\right)^2 = \left(\frac{\pi}{\log|	au|} \csc \left(\frac{\pi \log|w|}{\log|	au|}\right) \frac{|dw|}{w}\right)^2.$$

At $\tau = 0$ this model degenerate to give two cusps as in (7.1). In fact, making the change of variables $x = \cot \left(\frac{\pi \log|z|}{\log|	au|}\right)$, $\theta = \arg z$, we obtain

$$g_{\mathcal{P},\tau} = \frac{dx^2}{x^2 + \epsilon^2} + (x^2 + \epsilon^2)d\theta^2, \quad \text{with} \quad \epsilon = \frac{-\pi}{\log|	au|},$$

which is precisely the degeneration model considered in the present paper.

For each sufficiently small $\sigma$, we can use this model and construct an approximate hyperbolic metric $h_{\sigma,s}$ by gluing. More precisely, let $g_{\mathcal{P},\sigma}$ be the metric which on each $U^j_i \setminus D^i_{\sigma_i}$ is given by the metric $g_{\mathcal{P},\sigma_i}$. Then let $\eta$ be a cutoff function on $R_0$ which is zero within a distance $\gamma_0/2$ of each node and identically 1 outside a distance $2\gamma_0$ of each node, and whose gradient has support in a union of annuli with inner radius $\gamma_0/2$ and outer radius $2\gamma_0$ about each node. Finally, as in [Wol10], define a new metric $h_{\sigma,s}$ on $R_0$ by

$$h_{\sigma,s} = g_0^\eta g_{\mathcal{P},\sigma}^{1-\eta}.$$

This family of metrics is smooth in $(\sigma,s)$ away from the nodes [Wol10]. The metrics $h_{\sigma,s}$ are not necessarily exactly hyperbolic; their curvature may not be identically $-1$ on the annuli where $\nabla \eta$ may be nonzero. However, their curvature may be computed directly; call it $K_{\sigma,s}$. Then we must have $g_{\sigma,s} = e^{2\varphi_{\sigma,s}} h_{\sigma,s}$, where $\varphi_{\sigma,s}$ is the solution of the prescribed-curvature equation

$$-\Delta h_{\sigma,s} \varphi_{\sigma,s} - K_{\sigma,s} = e^{2\varphi_{\sigma,s}},$$

where $\Delta h_{\sigma,s}$ is the Laplacian with non-negative spectrum corresponding to the metric $h_{\sigma,s}$. As shown in [Wol87, p.293], the metric $h_{\sigma,s}$ is a good approximation of $g_{\sigma,s}$ in the sense that

$$\lim_{(\sigma,s) \to 0} g_{\sigma,s}/h_{\sigma,s} = 1,$$

see also [Wol90] for an expansion at $(\sigma,s) = 0$.

In [Wol87], Wolpert also obtains estimates for the small eigenvalues and for the determinants of the Laplacian of the metrics $g_{\sigma,s}$, see also [Bur88] for a sharpening of the eigenvalue asymptotics. Since the metrics $h_{\sigma,s}$ are exactly of the form (7.2) in a fixed neighborhood of the nodes, our work may be used to recover and extend these results in many cases. Indeed, we may apply directly our results to $h_{e^{-\pi/\epsilon_{\tau,s}}}$, so using (7.4) together with the prescribed curvature equation (7.3) and its solution $\varphi_{e^{-\pi/\epsilon_{\tau,s}}}$, we can derive corresponding results for $g_{e^{-\pi/\epsilon_{\tau,s}}}$. 

\[\]
7.1. Small eigenvalues. First we analyze the behavior of the small eigenvalues of $\Delta_{g_{\varepsilon,s}}$. The curvature $K_{e^{-\pi/s_T,s}}$ and the corresponding solution of the prescribed curvature equation are analyzed carefully in [Wol90]. From [Wol90, Section 3], we conclude that $||K_{e^{-\pi/s_T,s}}+1||_{C^0} \to 0$ as $\varepsilon \to 0$, uniformly in $s$. (In fact, $||K_{e^{-\pi/s_T,s}}+1||_{C^0} = C\varepsilon^2 + O(\varepsilon^4)$). As a consequence, it is proved in [Wol90, Section 4] that for any $\delta$ there exists an $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$, then for all sufficiently small $\varepsilon$ and all points on the surface,

$$(1-\delta)h_{e^{-\pi/s_T,s}} \leq g_{e^{-\pi/s_T,s}} \leq (1+\delta)h_{e^{-\pi/s_T,s}}.$$  

By the well-known result of Dodziuk [Dod82, Prop. 3.3], the quotients of the nonzero eigenvales of the Laplacians for $(R, g_{e^{-\pi/s_T,s}})$ and $(R, h_{e^{-\pi/s_T,s}})$ therefore approach 1 as $\varepsilon \to 0$, which allows us to apply our analysis of the small eigenvalues in section 3 to conclude:

Proposition 7.1. As $\varepsilon$ goes to zero, the positive small eigenvalues $\lambda_{\varepsilon,s}$ of $\Delta_{g_{e^{-\pi/s_T,s}}}$ satisfy $\lambda_{\varepsilon,s} \sim c\varepsilon + o(\varepsilon)$, where $c$ can be computed explicitly for each small eigenvalue using the methods of section 3.

As a particular case, consider the situation with $i = 1$. From the long exact sequence (3.3), there is one small eigenvalue if the manifold $R_\varepsilon$ becomes disconnected in the limit and zero if it does not. If it becomes disconnected and the volumes of the two connected components of the limit are $V_1$ and $V_2$, we conclude from Lemma 3.2 that the leading asymptotic of the single small eigenvalue is $\frac{\pi V_1}{\pi V_2} \varepsilon$. This agrees with the result of Burger [Bur88], who computed these eigenvalue asymptotics, to the same accuracy and with specific values of $c$, using methods involving a comparison with the graph Laplacian.

7.2. Determinant. We can also analyze the determinant of the Laplacian $\Delta_{g_{e^{-\pi/s_T,s}}}$. Observe that it is easy to show, by applying the maximum principle to the prescribed curvature equation, that whenever $\varepsilon$ is small enough so that $||K_{e^{-\pi/s_T,s}}+1||_{C^0} \leq 1/2$, then there is a constant $C$ such that $||\varphi_{e^{-\pi/s_T,s}}||_{C^0} \leq C$. By substituting these bounds into the prescribed curvature equation, we also get a $C^0$ bound for $\Delta \varphi_{e^{-\pi/s_T,s}}$. Therefore, for sufficiently small $\varepsilon$, there is a universal constant $C$ such that

$$||\varphi_{e^{-\pi/s_T,s}}||_{C^0} + ||\Delta \varphi_{e^{-\pi/s_T,s}}||_{C^0} \leq C.$$  

We may now apply the Polyakov formula in the form from [OPS98]:

$$\log \det \Delta_{g_{e^{-\pi/s_T,s}}} = \log \det \Delta_{h_{e^{-\pi/s_T,s}}} - \frac{1}{12\pi} \int_R (|\nabla \varphi_{e^{-\pi/s_T,s}}|^2 + 2K_{e^{-\pi/s_T,s}})dh_{e^{-\pi/s_T,s}}.$$  

Using integration by parts and the bounds we have just proven, it is straightforward to show that for sufficiently small $\varepsilon$, there is a universal constant $C$ (possibly different from the one above) such that

$$(7.5) \quad |\log \det \Delta_{g_{e^{-\pi/s_T,s}}} - \log \det \Delta_{h_{e^{-\pi/s_T,s}}}| \leq C.$$  

We now claim:

Proposition 7.2. As $\varepsilon \to 0$, $\log \det \Delta_{h_{e^{-\pi/s_T,s}}} \to -\infty$.

Proof. Since $h_{e^{-\pi/s_T,s}}$ is a family of cusp surgery metrics, the log determinant in question has a polyhomogeneous expansion as $\varepsilon \to 0$. We need to investigate all the divergent terms in this expansion, so we need to be specific about the leading orders.
Recall from [ARS14, Sec. 7] that the polyhomogeneous expansion of the log determinant is obtained by a pair of renormalized pushforwards: first an integration in the spatial variables to get the renormalized trace, then an integration in time to get the log determinant. From [ARS14, Eq. (7.18)], the first integral shows that the renormalized trace is an element of 
\[ A^{-2,(-2+2N_0)\mathbb{C}(-1+2N_0),0\mathbb{C}0(\mathcal{E},\mathcal{F})}; \]
that is, it has order \(-2\) at \(B_{tf}\), \(-2\) at \(B_{tff}\), and order \(0\) at \(B_{af}\). At \(B_{tff}\), the term of order \(-2\) comes from \(B_{\varepsilon,\tau}\) and the term of order \(-1\) comes from \(B_{tff} \subset \Delta_{HX}\). The second pushforward sends both \(B_{tff}\) and \(B_{af}\) to \(\varepsilon = 0\), so the log determinant has index set equal to \(-2\mathbb{C}(0\mathbb{C}0)\), which would normally yield a term of order \(\varepsilon^{-2}\) coming from \(B_{tff}\) as the leading term. However, the term of order \(\varepsilon^{-2}\) is actually zero. To see why, note that it comes from the leading-order term in the expansion of the diagonal heat kernel at \(B_{\varepsilon,\tau}\), which is a constant multiple of \(\tau^{-2}\). Performing two renormalized pushforwards, first in \(x\) and then in \(\tau\), on such a term yields zero. Therefore the true leading term is of order \(\varepsilon^{-1}\) and comes from \(B_{tff} \subset \Delta_{HX}\). We conclude that there exist constants \(C_1, C_2, C_3,\) and \(C_4\) such that, as \(\varepsilon \to 0\),
\[(7.6) \ \log \det \Delta_{g_{e-\pi/\varepsilon,s}} \sim C_1 \varepsilon^{-1} + C_2 \log \log \varepsilon + C_3 \log \varepsilon + C_4.\]
If \(C_1 < 0\), or if \(C_1 = 0\) and \(C_2 > 0\), et cetera, we are done.

However, from the definition of the determinant and the renormalized pushforward theorem, we know that
\[ C_1 = -\frac{1}{2} \int_0^\infty A_{tff} d\sigma, \]
where \(A_{tff}\) is the coefficient of the \(\varepsilon^{-1}\) term in the expansion of the renormalized trace at \(B_{tff}\). From [ARS14, Sec. 7],
\[ A_{tff} = N_{tff}(A) = e^{-\sigma^2 \Delta_{S_1}} \frac{1}{4\pi \sigma} \exp \left( -\frac{|\cdot|^2_{g,NB_{tff}}}{2\sigma^2} \right) \mu_{e,\phi,NB_{sh} \times Y \mathcal{H}/H}. \]
Therefore, restricting to the diagonal and integrating yields
\[ C_1 = -\frac{1}{8\pi} \int_0^\infty \text{Tr}(e^{-\sigma^2 \Delta_{S_1}}) \sigma^{-2} d\sigma = -\frac{1}{16\pi} \int_0^\infty \text{Tr}(e^{-t\Delta_{S_1}}) t^{-3/2} dt. \]
This integral may be evaluated and gives \(C_1 = -\frac{1}{16\pi} \Gamma(-1/2)\zeta_{Riem}(-1/2)\), which is negative. This completes the calculation. \(\Box\)

Combining this lemma with (7.5), we obtain
\[ \log \det \Delta_{g_{e-\pi/\varepsilon,s}} \to -\infty \text{ as } \varepsilon \to 0, \]
which agrees with the result of Wolpert [Wol87, Theorem 5.3] when we use the description of the determinant in terms of the Selberg Zeta function.
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