Solving Directed Laplacian Systems in Nearly-Linear Time through Sparse LU Factorizations

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Abstract

In this paper, we show how to solve directed Laplacian systems in nearly-linear time. Given a linear system in an $n \times n$ Eulerian directed Laplacian with $m$ nonzero entries, we show how to compute an $\epsilon$-approximate solution in time $O(m \log^{O(1)}(n) \log(1/\epsilon))$. Through reductions from [Cohen et al. FOCS’16], this gives the first nearly-linear time algorithms for computing $\epsilon$-approximate solutions to row or column diagonally dominant linear systems (including arbitrary directed Laplacians) and computing $\epsilon$-approximations to various properties of random walks on directed graphs, including stationary distributions, personalized PageRank vectors, hitting times, and escape probabilities. These bounds improve upon the recent almost-linear algorithms of [Cohen et al. STOC’17], which gave an algorithm to solve Eulerian Laplacian systems in time $O((m + n^2 \log n \log \log n) \log^{O(1)}(n \epsilon^{-1}))$.

To achieve our results, we provide a structural result that we believe is of independent interest. We show that Eulerian Laplacians (and therefore the Laplacians of all strongly connected directed graphs) have sparse approximate LU-factorizations. That is, for every such directed Laplacian $L$, there is a lower triangular matrix $L_L$ and an upper triangular matrix $L_U$, each with at most $O(n)$ nonzero entries, such that their product $L_L L_U$ spectrally approximates $L$ in an appropriate norm. This claim can be viewed as an analogue of recent work on sparse Cholesky factorizations of Laplacians of undirected graphs. We show how to construct such factorizations in nearly-linear time and prove that, once constructed, they yield nearly-linear time algorithms for solving directed Laplacian systems.

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1 Introduction

A matrix $M$ is \textit{(column) diagonally dominant} if $M_{ii} \geq \sum_{j \neq i} |M_{ji}|$ for all $i$. Such matrices, which notably include Laplacians of graphs, are ubiquitous in computer science, with applications spanning scientific computing, graph theory, machine learning, and the analysis of random processes, among others. For \textit{symmetric} diagonally dominant matrices, which include the Laplacians of \textit{undirected} graphs, Spielman and Teng gave an algorithm in 2004 \cite{33} to solve the corresponding linear systems in nearly-linear time. Since then, the ability solve such linear systems has emerged as a powerful algorithmic primitive, serving as a crucial subroutine in the design of faster algorithms for a long list of problems (e.g., \cite{2,6,7,10,12,13,22,24–26,29,31,32}).

Moreover, the pursuit of faster, simpler, and more parallelizable Laplacian system solvers has driven numerous algorithmic advances, comprising both improvements in the underlying linear algebraic machinery \cite{5,14–16,18,19,21,30,34} and applications of their ideas and techniques to problems in other domains (e.g., \cite{1,8,11,21,23}).

However, while this approach has been incredibly successful for symmetric linear systems and undirected graph optimization problems, comparable results for their asymmetric or directed counterparts have proven quite elusive. In particular, the techniques for solving Laplacian systems seemed to rely intrinsically on multiple properties of undirected graphs, and, until recently, the best algorithms in the directed case simply treated the Laplacians as unstructured matrices and applied general linear algebraic routines, leading to super-quadratic running times.

Two recent papers \cite{3,4} suggested that it may be possible to close this gap, potentially laying the foundation for a new class of nearly-linear time algorithms for directed graphs and asymmetric linear systems. The first paper \cite{3} showed that linear systems involving several natural classes of asymmetric matrices, including Laplacians of directed graphs, general square column diagonally dominant matrices, and their transposes (called \textit{row diagonally dominant} matrices), could be reduced with only polylogarithmic overhead to solving linear systems in the Laplacians of Eulerian graphs. They further showed how to use these solvers, with only polylogarithmic overhead, to compute a wide range of fundamental quantities associated with random walks on directed graphs, including the stationary distribution, personalized PageRank vectors, hitting times, and escape probabilities. The paper combined these reductions with an algorithm to solve Eulerian Laplacian systems in time $O(m^{3/4}n + mn^{2/3})$ to achieve faster (but still significantly super-linear) algorithms for all of these problems.\footnote{Following the notation and terminology of the previous papers, we use $O$ notation to suppress terms that are polylogarithmic in $n$, the natural condition number of the problem $\kappa$, and the desired accuracy $\epsilon$. We use the term “nearly-linear” for algorithms that run in time $O(m) = O(m)\log^{O(1)}(m\kappa\epsilon)$, and “almost linear” for algorithms that run in time $O(m(n\kappa^{-1})^{o(1)})$.}

The second paper \cite{4} gave an improved solver for Eulerian systems that runs in almost-linear time $O(m + m\log n \log \log n)$, providing almost-linear time algorithms for all of the problems reduced to such a solver in \cite{3}.

In this paper, we close the algorithmic gap between the directed and undirected cases (up to polylogarithmic factors) by providing an algorithm to solve Eulerian Laplacian systems in time $\tilde{O}(m)$. Combining this with the reductions from \cite{3} yields the first nearly-linear time algorithms for all of the problems listed above.

To achieve our results, we prove a structural result that we believe to be of independent interest. We show that Laplacians of strongly connected directed graphs have sparse approximate $LU$-factorizations. More precisely, we show that for every such directed Laplacian $L \in \mathbb{R}^{n \times n}$, there is a lower triangular matrix $L' \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $U \in \mathbb{R}^{n \times n}$ such that both
matrices have at most $\tilde{O}(n)$ nonzero entries, and $\mathbf{LU}$ spectrally approximates $\mathbf{L}$ in an appropriate norm. We show how to construct such factorizations in nearly-linear time for the special case of Laplacians of strongly connected directed graphs where weighted in-degree equals weighted out-degree for every vertex. Such Laplacians are known as Eulerian Laplacians. We prove that once constructed, these factorizations yield nearly-linear time algorithms for solving directed Laplacian systems, and also yield sparse LU-factorizations of all Laplacians of strongly connected directed graphs.

This claim is analogous to the result first obtained by Lee, Peng, and Spielman in [20], which showed that undirected Laplacians have sparse Cholesky factorizations, and our algorithm builds on a combination of the techniques in [19, 20] as well as the sparsification machinery developed in [4]. Unfortunately, as we will discuss below (Subsection 1.2), there were several aspects of the approach in [19] that relied strongly on properties of symmetric Laplacians that are not present in the asymmetric setting, and obtaining a similar result for directed Laplacians required the development of new algorithmic and analytical techniques.

### 1.1 Our Results

The main technical result in this paper is the following theorem, which asserts that we can compute sparse approximate LU factorizations of an Eulerian Laplacian in nearly-linear time. Eulerian Laplacians have a special property that a standard symmetrization of these ($\mathbf{L} + \mathbf{L}^T$) are PSD matrices and define useful norms. Our main algorithmic result for Eulerian Laplacians is:

**Theorem 1.1 (Sparse LU).** Given an Eulerian Laplacian $\mathbf{L} \in \mathbb{R}^{n \times n}$ with $m$ nonzero entries and any $\epsilon \in (0, 1/2)$, and $\delta < 1/n$, in $O(m + n\epsilon^{-8}\log^O(1)(1/\delta))$ time, with probability at least $1 - O(\delta)$ the algorithm $\text{EULERIANLU}(\mathbf{L}, \delta, \epsilon)$, produces lower and upper triangular matrices $\mathbf{L} \in \mathbb{R}^{n \times n}$ and $\mathbf{U} \in \mathbb{R}^{n \times n}$ such that for some symmetric PSD matrix $\mathbf{F} \approx \text{poly}(n) \cdot (\mathbf{L} + \mathbf{L}^T)/2$, $\langle \mathbf{LU} \rangle^\top \mathbf{F}^\top \langle \mathbf{LU} \rangle \geq 1/O(\log^2 n) \cdot \mathbf{F}$, $\|\mathbf{F}^\top/2 (\mathbf{L} - \mathbf{LU}) \mathbf{F}^\top/2\|_2 \leq \epsilon$ and $\max\{\text{nnz}(\mathbf{L}), \text{nnz}(\mathbf{U})\} \leq n \log^O(1)(1/\delta) \cdot \epsilon^{-6}$.

For simplicity, we assume all real number computations are exact throughout this paper. However, we believe a crude numerical stability analysis for constant length inputs in the fixed point precision model similar to the one sketched in Appendix A of [3] is possible.

The conditions of this theorem can easily be shown to yield that the pseudoinverse of $\mathbf{LU}$ is a good preconditioner for solving $\mathbf{Lx} = \mathbf{b}$ for Eulerian $\mathbf{L}$. Consequently, we show in Section 2 as a corollary of this main theorem that we can solve Eulerian Laplacian systems in nearly-linear time.

**Corollary 1.2 (Nearly-Linear Time Solver for Eulerian Laplacians).** Given an Eulerian Laplacian $\mathbf{L} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \perp \mathbf{1}$, and $\epsilon \in (0, 1/2)$, in $O(m \log^O(1)n \log(1/\epsilon))$ time we can w.h.p. compute an $\epsilon$-approximate solution $\hat{\mathbf{x}}$ to $\mathbf{Lx} = \mathbf{b}$ in the sense that $\|\hat{\mathbf{x}} - \mathbf{L}^\top \mathbf{b}\|_{\mathbf{UL}} \leq \epsilon \|\mathbf{L}^\top \mathbf{b}\|_{\mathbf{UL}}$ where $\mathbf{UL} = (\mathbf{L} + \mathbf{L}^\top)/2$.

Combining this result with the reductions from general directed Laplacians to Eulerian Laplacians from [3] then leads to nearly-linear time algorithms for computing quantities related to directed random walks when the mixing time is $\text{poly}(n)$. Such a reduction is identical to the incorporation of the previous almost-linear time Eulerian solver outlined in Appendix D of [4]. It results in running times that’s $O(m \log^O(1)n \log^O(1) t_{mix} \log(1/\epsilon))$ linear up to polylogarithmic factors in $n$, $m$, and a natural condition number-like quantity related to the mixing time for the random walks, $t_{mix}$. The problems for which we readily obtain such a running time include:
• solving row diagonally dominant (or column diagonally dominant) linear systems including arbitrary (non-Eulerian) directed Laplacian systems
• computing the stationary distribution of a Markov chain
• personalized PageRank
• obtaining polynomially good estimates of the mixing time of a Markov chain
• computing the hitting time from one vertex to another
• computing escape probabilities for any triple of vertices
• approximately computing all-pairs commute times

1.2 Overview of Approach

Here we present the broad algorithmic and analytical approach we take to provably solving Eulerian Laplacian systems in nearly-linear time. Our algorithm is broadly inspired by recent nearly-linear time (undirected) Laplacian system solvers based on performing repeated vertex elimination and Schur complement sparsification to compute a sparse approximate Cholesky factorization [18, 19].

Initially, one might hope to obtain our results by simply adapting the algorithm in [19], which is arguably the simplest known for provably solving undirected Laplacian systems in nearly-linear time. However, there are multiple substantial barriers to adapting any undirected Laplacian system solver based on Cholesky factorization to solve arbitrary Eulerian Laplacians. We obtain our results by providing new algorithmic and analytical tools to systematically overcome these issues.

In the remainder of this subsection we outline the high-level ideas that underlie these tools and connect them to the later sections that present them rigorously. For the details of the mathematical notation we use see Subsection 1.4.

Vertex Elimination and Approximate LU-Factorization (Section 2) To solve a system of equations in an Eulerian Laplacian \( L \in \mathbb{R}^{V \times V} \), we leverage the well-known fact that it suffices to compute a matrix \( Z \) that is a good preconditioner or approximate pseudoinverse (see Definition 2.4).

As was shown in [4], if \( Z \) can be applied in nearly-linear time and \( ZL \approx I_{\text{ini}}(Z) \) in a suitable norm, then this suffices to solve Laplacian systems in \( L \) in nearly-linear time through an iterative method known as preconditioned Richardson iteration (see Lemma 2.5).

Whereas [4] constructed \( Z \) through repeated sparsification and squaring, here we take an approach inspired by [18, 19] based on vertex elimination and sparsification of Schur complements. For any square matrix \( A \in \mathbb{R}^{n \times n} \) where \( F, C \) partition \([n]\) and \( A_{FF} \) is invertible, it holds that

\[
A = \begin{bmatrix}
A_{FF} & A_{FC} \\
A_{CF} & A_{CC}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
A_{CF}A_{FF}^{-1} & I
\end{bmatrix} \begin{bmatrix}
A_{FF} & 0 \\
0 & A_{CC} - A_{CF}A_{FF}^{-1}A_{FC}
\end{bmatrix} \begin{bmatrix}
I & A_{FF}^{-1}A_{FC} \\
0 & I
\end{bmatrix}.
\]

We can easily invert the upper and lower triangular matrices in this factorization:

\[
\begin{bmatrix}
I & 0 \\
A_{CF}A_{FF}^{-1} & I
\end{bmatrix}^{-1} = \begin{bmatrix}
I & 0 \\
-A_{CF}A_{FF}^{-1} & I
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
I & A_{FF}^{-1}A_{FC} \\
0 & I
\end{bmatrix}^{-1} = \begin{bmatrix}
I & -A_{FF}^{-1}A_{FC} \\
0 & I
\end{bmatrix}.
\]

\(^2\)If one wishes to compute commute times for a number of pairs greater than the number of edges in the graph, the runtime will be nearly-linear in the output size instead of the number of the number of edges.
This reduces solving linear systems in $A$ to solving smaller linear systems in $A_{FF}$ and the Schur complement $A_{CC} - A_{CF}A_{FF}^{-1}A_{FC}$.

The recent work of $[18, 19]$ leverages the fact that, when $A$ is a undirected Laplacian, its Schur complement is as well. By cleverly choosing the block $F$ of vertices to eliminate, sparsifying the Schur complement, and recursing, these papers compute efficient preconditioners. In the work of $[19]$, each $F$ is simply a single coordinate or vertex of the associated graph. In this case eliminating this vertex induces a Schur complement with the vertex removed and an appropriately weighted clique added to its neighbors. Repeated elimination and recursion then yields a $LU$ factorization of the original matrix. In the setting of $[19]$ this matrix is symmetric and therefore the lower and upper triangular matrices are transposes of each other and the factorization is known as a Cholesky factorization. By picking the vertex to eliminate randomly and sparsifying the cliques directly $[19]$ showed that a sparse approximate Cholesky factorization of the Laplacian can be obtained in nearly-linear time.

It is easy to see that for Eulerian Laplacians, it is also the case that its Schur complements are Eulerian Laplacians. Eliminating a single vertex corresponds to deleting that vertex and adding a weighted complete bipartite graph, or biclique, from the vertices that yield incoming edges to the eliminated vertex to the vertices that yield outgoing edges from the eliminated vertex. Consequently, $[19]$ suggests a natural approach for solving directed Laplacian systems and producing sparse $LU$-factorizations of Eulerian Laplacians: repeatedly eliminate random vertices and directly sparsify the bicliques that eliminating these vertices induces.

Unfortunately, there are multiple issues with this approach. First, sparsifying directed Eulerian Laplacians is more delicate than sparsifying undirected Laplacians and even showing that sparsifiers exist was a major contribution of $[4]$. Second, analyzing the error induced by sparsification is much more difficult for directed Laplacians and reasoning about this error was the major contributor to the almost linear rather than nearly-linear running time of $[4]$. Third, the reasoning of $[19]$ required rather tight bounds on the sparsity induced by sparsification, and showing that their algorithm works without this seems impossible.

Nevertheless, we show how to overcome these issues by proving a biclique sparsification technique with favorable properties for our analysis, providing new results on the error induced by Schur complement sparsification as it relates to the quality of a preconditioner, and modifying the $[19]$ algorithm to only eliminate carefully chosen sets of vertices and alternate with full sparsification of the resulting graph. We discuss each of these further below.

Unbiased Degree Preserving Vertex Elimination (Section 3): The first immediate issue in leveraging the insights from $[19]$ to develop a a single vertex elimination algorithm capable of producing sparse $LU$ factorizations of an Eulerian Laplacians is to determine how to sparsify the bicliques that elimination creates. On the one hand, the analysis of $[19]$ crucially leverages that the sparsification procedure is unbiased, i.e. it produces a Laplacian that is in expectation the sparsified graph, with low variance so that matrix martingale arguments can be used to bound the error induced by the entire procedure. On the other the known sparsification results for Eulerian graphs $[4]$ require that the sparsified graph is Eulerian and typically work by exactly preserving the in-degrees and out-degrees of vertices.

Unfortunately, these two constraints on a sparsification procedure seem at odds with each other as independent sampling to create an unbiased estimator likely does not preserves degrees. Moreover, it is difficult to see how to completely drop either constraint. However, a different algorithmic primitive suggests optimism. Consider the edges of the vertex we eliminate. If we treat these edges
as flows coming into and out of the vertex, then running a standard flow path decomposition on the these flows will result in a collection of flow paths – and if we treat these flow paths as edges that bypass the middle vertex, we get a graph on the neighbors of the eliminated vertex. This graph will be sparse and have the same in- and out-degree for every neighbor as the biclique created by elimination. We show that it is possible to randomize this flow decomposition in such a way that the sparse graph becomes an unbiased estimator of the dense biclique.

Formally, in Section 3 we remedy this issue by providing an efficient biclique sparsification procedure that is unbiased, with low variance, that exactly preserves the in and out degrees of vertices. This procedure works by careful sampling edges from the biclique so that no vertex has an excess of in-degree or out-degree and then carefully sampling from the remaining graph so that the conditional expectation is preserved. Crucially this procedure is not independent sampling from the entire graph. Moreover, we show that this procedure has low variance as desired. We hope the ideas of Section 3 could be useful for sparsification more broadly.

**Schur Complement Blow Up and Bounding Error Increase (Section 4 and Section 6):**
While our biclique sparsification procedure of Section 3 provides hope of developing a single vertex elimination algorithm, it leaves significant issues in bounding the error induced by sparsification.

First, a key fact used in the previous analysis [19] is that, when recursing, the Schur complement matrices that result are always spectrally dominated by the original matrix. This is used to show that the overall error of the algorithm is small as long as the error is small in every recursive phase. It is not even obvious how to define the right notion of “spectral domination,” in our context since the typical notion only applies to positive semidefinite matrices. Moreover, once it is appropriately defined (using notions from [4]), this spectral domination property fails even for simple graphs like the directed cycle, which can increase by a factor that is linear in the number of vertices.

Nevertheless, in Section 4 we show that it is possible to efficiently find a large fraction of the vertices such that eliminating them only causes a constant multiplicative growth in the relevant norms. We show that (highly) row-column diagonally dominant sets are well-behaved in this sense and thus we can simply perform random vertex elimination restricted to these large sets. We note that though an analogous notion for undirected graphs was used in [17, 20], it played a different role there. They show that Jacobi iterations for the Laplacian minor defined by a diagonally dominant set converges quickly and also, that a sparse approximation of Schur complement can be constructed efficiently. On the other hand, we use the row-column diagonal dominance to show that an appropriately defined spectral norm of the Schur complement does not blow up.

Unfortunately, if the multiplicative errors from eliminating the highly row-column diagonally dominant sets compounded, the overall error would still be too large for our elimination procedure to yield a good preconditioner. The previous solvers [3, 4] essentially measured the quality of a preconditioner by bounding error in the norm induced by the undirected Laplacian obtained from an Eulerian Laplacian by removing the direction on every edge. However the compounding error from repeated Schur complement sparsification seems prohibitively large in this norm. In some sense, this was the critical issue which prevented [4] from achieving a nearly-linear running time.

To circumvent this we develop a deeper understanding of the norms we can use to prove convergence of iterative solvers for directed Laplacian systems. We show that there is good deal of flexibility in the family of norms that can be used to prove convergence of iterative solvers for directed Laplacian systems (See Section 6). Formally, we prove that based on our vertex elimination sequence, we can construct a new matrix (Equation (2)) that is both sufficient for proving the con-
vergence of preconditioned Richardson iteration and for which the errors induced by repeated Schur complement sparsification is small. This matrix is a careful combination of the undirected graph norms induced by the sequence of Schur complement matrices we encountered. We show this matrix spectrally dominates the “large” matrices created by vertex elimination and therefore allows us to convert the local errors induced by one round of sparsification into a global bound on the quality of a preconditioner (Lemma 2.3). We ultimately prove that our solver converges quickly in this new norm, before converting the final error guarantee back to the norms we care about in Corollary 1.2 for application purposes. We believe that these tools for analyzing the error induced by sparsifying Schur complements could be useful for reasoning about asymmetric matrices more broadly.

Sparsification, Algorithm Design, and Analysis (Section 2 and Section 5) The above gives the key building blocks for developing a nearly-linear time algorithm for solving Eulerian Laplacian systems. It suggests that picking a block of row-column diagonally dominant vertices, eliminating them with our unbiased degree-preserving vertex elimination procedure, and then analyzing this in a carefully chosen norm induced by the recursion could work. However, there is still one more issue that arises—the sparsity of the resulting Schur complements this procedure induces grows much faster then in [19]. To overcome this, we simply leverage black box previous work from [4] that Eulerian Laplacians have sparsifiers which can be efficiently computed. Consequently, by occasionally sparsifying the resulting Schur complement we can fix this final issue.

Putting together these pieces yields our algorithm. Algorithm 1 gives a rough sketch of this algorithm, which alternates between vertex elimination on carefully chosen subsets and sparsification.

\begin{algorithm}
\caption{Overall Sketch of the Algorithm}
\begin{algorithmic}
\State \For {$i \leftarrow 1$ \KwTo $O(\log n)$} \Do
\State Choose a highly row-column diagonally dominant set $F \subset V$ of size proportional to $|V|$.
\EndFor
\For {$v \in F$} \Do
\State \textbf{Vertex Elimination $v$}: Use the routine \texttt{SINGLEVERTEXELIM} to eliminate vertex $v$ and to add a sparse approximation of its Schur complement.
\EndFor
\State \textbf{Sparsify Graph}: If the number of edges in the resulting graph is above a threshold, then use the routine \texttt{SPARSIFYEULERIAN} to sparsify the whole graph.
\State $V \leftarrow V \setminus F$
\Endalgorithmic
\end{algorithm}

Although this algorithm only describes how to eliminate vertices, as in Gaussian elimination, this also gives an LU factor decomposition. For a more detailed description of the algorithm, see Algorithm 3 and routines it calls. Details of line 4 of Algorithm 1 can be found in Algorithm 5. See Algorithm 4 for a description of \texttt{SINGLEVERTEXELIM}.

We provide the analysis of this procedure (assuming the pieces of the rest of the paper) in Section 2 and in Section 5 we provide the careful martingale analysis of elimination and sparsification within a single highly row column diagonally dominant block. Though it has several pieces, ultimately, we believe this framework for solving Eulerian solvers is simpler than that in [4] and we hope that these pieces may find further use and possibly lead to even simpler algorithms.

1.3 Paper Outline

The presentation of these results is split into several pieces. In Section 2 we give the main algorithm that computes an approximate LU-factorization of an Eulerian Laplacian and prove its correctness
as well as Theorem 1.1 and Corollary 1.2, assuming the analysis of later parts of the paper. In Section 3, we analyze our single vertex elimination or biclique sparsification procedure. In Section 4 we analyze the error induced by sparsifying Schur complements of highly diagonally dominant sets. In Section 5 we perform the matrix martingale analysis of the error incurred by single vertex elimination and graph sparsification. In Section 6 we provide proofs of facts regarding the norms we use to analyze the overall error of our approximate LU factorization procedure. In Appendix A we show how to find RCDD subsets and in Appendix B we provide matrix facts we use throughout.

1.4 Preliminaries

Matrices: For a square matrix \( \mathbf{M} \), we denote its symmetric part by \( \mathbf{U}_\mathbf{M} \overset{\text{def}}{=} (1/2)(\mathbf{M} + \mathbf{M}^\top) \). Typically, \( \mathbf{U}_\mathbf{M} \succeq 0 \) and the kernel of \( \mathbf{M} \) and \( \mathbf{M}^\top \) are same, i.e. \( \ker(\mathbf{M}) = \ker(\mathbf{M}^\top) \).

For matrix \( \mathbf{A} \in \mathbb{R}^{n \times n} \) and subsets \( F, C \subseteq [n] \) we let \( \mathbf{A}_{FC} \in \mathbb{R}^{F \times C} \) denote the sub-matrix of \( \mathbf{A} \) corresponding to the \( F \) and \( C \) entries. Furthermore, for \( v \in \mathbb{R}^N \) and \( F \subseteq [n] \) we let \( v_F \) denote the restriction of \( v \) to the coordinates of \( v \). Consequently, if \( F, C \subseteq [n] \) partition \([n]\) then we have that
\[
[\mathbf{A}v]_F = \mathbf{A}_{FF}v_F + \mathbf{A}_{FC}v_C \quad \text{and} \quad v^\top \mathbf{A}v = v_F^\top \mathbf{A}_{FF}v_F + v_F^\top \mathbf{A}_{FC}v_C + v_C^\top \mathbf{A}_{CF}v_F + v_C^\top \mathbf{A}_{CC}v_C.
\]

Such a partition naturally leads to the notion of Schur complements. We will use \( \text{Sc}(\mathbf{A}, C) \) to denote \( n \)-by-\( n \) matrix where the only non-zeros are on \( C \), and these variables corresponding to the result of eliminating all variables in \( F \). Formally, we let \( \mathbf{0}_{FC} \in \mathbb{R}^{F \times C} \) be the all zero matrix, from which we can write this Schur complement as:
\[
\text{Sc}(\mathbf{A}, F) \overset{\text{def}}{=} \begin{bmatrix}
\mathbf{0}_{FF} & \mathbf{0}_{FC} \\
\mathbf{0}_{CF} & \mathbf{A}_{CC} - \mathbf{A}_{CF} \mathbf{A}_{FF}^{-1} \mathbf{A}_{FC}
\end{bmatrix}.
\]

Norms: Given PSD \( \mathbf{H} \in \mathbb{R}^{n \times n} \), we define a semi-norm on vector \( ||\cdot||_\mathbf{H} \) by \( ||x||_\mathbf{H} = \sqrt{x^\top \mathbf{H} x} \). For any norm \( ||\cdot|| \) defined on \( \mathbb{R}^n \) we define the seminorm it induces on \( \mathbb{R}^{n \times n} \) for all \( \mathbf{A} \in \mathbb{R}^{n \times n} \) by \( ||\mathbf{A}|| = \max_{x \neq 0} ||\mathbf{A}x||/||x|| \). When we wish to make clear that we are considering this ratio we use the \( \to \) symbol; e.g., \( ||\mathbf{A}||_{\mathbf{H} \to \mathbf{H}} = \max_{x \neq 0} ||\mathbf{A}x||_\mathbf{H}/||x||_\mathbf{H} \) but we may also simply write \( ||\mathbf{A}||_{\mathbf{H}} \overset{\text{def}}{=} ||\mathbf{A}||_{\mathbf{H} \to \mathbf{H}} \).

Pseudoinverse and Square Roots: For symmetric PSD matrix \( \mathbf{A} \) we use \( \mathbf{A}^\dagger \) to denote its Moore-Penrose pseudoinverse of \( \mathbf{A} \), we use \( \mathbf{A}^{1/2} \) to denote its square root, i.e. the unique PSD matrix such that \( [\mathbf{A}^{1/2}]^2 = \mathbf{A} \), and we use \( \mathbf{A}^{1/2} \) to denote \( [\mathbf{A}^\dagger]^{1/2} = [\mathbf{A}^{1/2}]^\dagger \).

Upper and Lower Triangular Matrices: We say a square matrix \( \mathbf{U} \) is upper triangular if it has non-zero entries \( \mathbf{U}(i, j) \neq 0 \) only for \( i \leq j \) (i.e. above the diagonal). Similarly, we say a square matrix \( \mathbf{L} \) is lower triangular if it has non-zero entries \( \mathbf{L}(i, j) \neq 0 \) only for \( i \geq j \) (i.e. below the diagonal). Often, we work with matrices that are not upper or lower triangular, but for which we know a permutation matrix \( \mathbf{P} \) s.t. \( \mathbf{P} \mathbf{U} \mathbf{P}^\top \) is upper (respectively lower) triangular. For computational purposes, this is essentially equivalent to having a upper or lower triangular matrix, and we refer to such matrices as upper (or lower) triangular. The algorithms we develop for factorization will always compute the necessary permutation.
Asymmetric Matrix Approximation: We use the asymmetric matrix approximation definition of [4] and say that matrix $B$ is said to be an $\epsilon$-approximation of matrix $A$ if and only if $U_A$ is symmetric PSD with $\ker(U_A) \subseteq \ker(A - B) \cap \ker((A - B)^\top)$ and $\|U_A^{1/2}(A - B)U_A^{1/2}\|_2 \leq \epsilon$. In this paper we only use this definition in the restricted setting where $\ker(A) = \ker(A^\top)$.

Row Column Diagonal Dominant (RCDD): For square matrix $A$ we say that a subset of the coordinates $F$ is $\alpha$-RCDD if $\sum_{j \in F, j \neq i} |L_{ij}| \leq \frac{1}{1 - \alpha} |L_{ii}|$ and $\sum_{j \in F, j \neq i} |L_{ji}| \leq \frac{1}{1 - \alpha} |L_{ii}|$.

Directed Laplacians: We follow the conventions of [3] and say matrix $L \in \mathbb{R}^{n \times n}$ is a directed Laplacian if its off diagonal entries are non-positive, i.e. $L_{i,j} \leq 0$ for all $i \neq j$, and $1^\top L = 0$, i.e. $L_{ii} = -\sum_{j \neq i} L_{ji}$ for all $i$. For every directed Laplacian $L \in \mathbb{R}^{n \times n}$ we associate a graph $G_L = (V,E,w)$ with vertices $V = [n]$, and edges $(i,j)$ of weight $w_{ij} = -L_{ji}$, for all $i \neq j \in [n]$ with $L_{ji} \neq 0$. Occasionally we write $L = D - A^\top$ to denote that we decompose $L$ into the diagonal matrix $D$ (where $D_{ii} = L_{ii}$ is the out degree of vertex $i$ in $G_L$) and non-negative matrix $A$ (which is weighted adjacency matrix of $G_L$, with $A_{ij} = w_{ij}$ if $(i,j) \in E$, and $A_{ij} = 0$ otherwise).

Letting $\chi_v \in \mathbb{R}^n$ be the vector whose $v$'th coordinate is set to one and all others to be zero and $b_{(u,v)} = \chi_v - \chi_u$, a directed Laplacian can be written as $L = \sum_{(v,u) \in E} w_{(v,u)} b_{(u,v)} \chi_v^\top$.

Finally, we call a matrix $L$ is an Eulerian Laplacian if it is a directed Laplacian with $LL = 0$. Note that $L$ is an Eulerian Laplacian if and only if its associated graph is Eulerian.

2 Main Algorithm

In this section, we give an overview of the key components of our algorithm for LU factorization and show how to use an LU factorization it generates to solve Eulerian Laplacians. We also give proofs of our two main statements on Eulerian Laplacians (Theorem 1.1 and Corollary 1.2) assuming auxiliary statements proven in this section and later in the chapter.

Standard LU Factorization. The core of our algorithm is a sparsified version of LU factorization. The starting point for standard LU factorization is the formula for eliminating a single variable. Letting $C = V \setminus \{v\}$, the formula for eliminating a single variable in a matrix $L$ is

$$L = \begin{bmatrix} L_{vv} & L_{vC} \\ L_{Cv} & L_{CC} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{L_{vv}}L_{Cv} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & L_{CC} - \frac{1}{L_{vv}}L_{Cv}L_{vC} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{L_{vv}}L_{vC} \\ 0 & I \end{bmatrix}$$

We have decomposed the matrix into a lower triangular matrix, a block diagonal matrix, and an upper triangular matrix. Recursively applying the same elimination procedure to the matrix $L_{CC} - \frac{1}{L_{vv}}L_{Cv}L_{vC}$ on the block diagonal, we can then get a decomposition into a product of a sequence of lower triangular matrices times a sequence of upper triangular matrices. Since the product of lower triangular matrices is lower triangular, and similarly the product of upper triangular matrices is upper triangular, we can then collect these into one lower triangular factor $L$ and an upper triangular factor $U$ and write $L = LU$. However, one can check that in fact the factors $L$ and $U$ have a simple structure: If $d_i$ is the diagonal entry corresponding to the variable being eliminated in round $i$, and $c_i$ is the corresponding column scaled by $d_i^{-1/2}$, and $r_i$ the row scaled scaled by
$d_i^{-1/2}$ (e.g. so that in the first round $d_1 = L_{vv}$, $c_1 = d_1^{-1/2}L_{vV}$, and $r_1 = d_1^{-1/2}L_{Vv}$), then

$$L = \begin{bmatrix} c_1 & 0 & 0 & \cdots \\ c_2 & 0 & 0 & \cdots \\ c_3 & 0 & 0 & \cdots \\ \vdots & & & \end{bmatrix}$$

and

$$U = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots \\ 0 & r_2 & r_3 & \cdots \\ 0 & 0 & r_3 & \cdots \\ \vdots & & & \end{bmatrix}.$$

For us, it will be more convenient to consider all rows and columns to be of length $n$, and abusing notation by redefining $c_2 \leftarrow \begin{pmatrix} 0 \\ c_2 \end{pmatrix}$ and so on, we can then write

$$L = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$$

and

$$U = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \end{bmatrix}$$

and

$$L = LU = \sum_i c_i r_i.$$

The fact that $C = V \setminus \{v\}$ allows us to write the top-left block as the scalar entry $L_{vv}$, giving

$$SC(L, C) = \begin{bmatrix} 0 \\ 0 \\ L_{CC} - \frac{1}{L_{vv}}L_{Vv}L_{vC} \end{bmatrix}.$$

and after simple algebraic manipulations,

$$L = \frac{1}{L_{vv}}L_{Vv}L_{vV} + SC(L, C) = c_1 r_1 + SC(L, C) \quad (1)$$

**LU Factorization of Eulerian Laplacians.** Let us restrict our attention to LU factorization of a matrix $L$ which is a strongly connected Eulerian Laplacian. In general, LU factorization may run into trouble if one tries to eliminate a variable with a zero diagonal entry. But, one can show that this will never happen if the input matrix is an Eulerian Laplacian – except that the final diagonal entry will always be zero.

We introduce notation for the directed Laplacian corresponding to the star graph of edges coming into and going out of a vertex $v$ of in the graph corresponding to $L$, which we write as

$$St[L]_v = \sum_{(v,u) \in E} w_{(v,u)} b_{(u,v)} \chi^T_v + \sum_{(u,v) \in E} w_{(u,v)} b_{(v,u)} \chi^T_u,$$

Rearranging Equation (1), we have

$$SC(L, V \setminus \{v\}) = L - \frac{1}{L_{vv}}L_{Vv}L_{vV} = L - ST[L]_v + ST[L]_v - \frac{1}{L_{vv}}L_{Vv}L_{vV}.$$

Importantly, when $L$ is a strongly connected Eulerian Laplacian, then so is $SC(L, V \setminus \{v\})$, in fact Eulerian Laplacians are closed under taking Schur complements, and strong connectivity is preserved. Furthermore, $L - ST[L]_v$ is simply the directed Laplacian corresponding to the graph of $L$ with $v$ and the edges incident on it removed, and $ST[L]_v - \frac{1}{L_{vv}}L_{Vv}L_{Vv}$, is a directed Laplacian of a weighted biclique on the neighbors of $v$. In general, neither of these parts of the Schur complement are individually Eulerian, but together they are.
Our algorithm. As stated in Theorem 1.1, we assume that the input matrix, \( L \), is a strongly connected Eulerian Laplacian. Our algorithm is based on standard LU factorization as outlined above. Since each elimination creates a biclique on the neighbors of the vertex being eliminated, the graph corresponding to the remaining Schur complement matrix can quickly grow dense. To remedy this, we develop a method for computing sparse approximations of the Schur complement, without ever having to write down the dense biclique. This throws up several difficulties: The sparse approximation must accumulate very little error over many iterations and it must preserve Eulerianness. See Subsection 1.2 for an overview of the challenges and the techniques we use to resolve these difficulties.

The main algorithm for LU factorization, Algorithm 3, has \( p_{\text{max}} \) phases or iterations. The routine for a single phase, given in Algorithm 2, iteratively eliminates vertices belonging to a random set selected by Algorithm 5. We need to perform the eliminations in phases because eliminating from the sets selected by Algorithm 5 helps us constrain the growth in the norm of certain matrices, which is necessary for controlling overall error accumulation.

In every iteration within any phase of the algorithm, a vertex, say \( v \), is eliminated. We can afford to store the row and column that this creates in our LU factorization, but we cannot afford to compute the dense biclique that is created in the graph corresponding to the Schur complement.

Since the biclique directed Laplacian \( \tilde{S}(L) = \frac{1}{\alpha} \tilde{L}(v,v) \tilde{L}(v,v) \) is dense in general, we use the routine \textsc{SingleVertexElim}(\cdot), given in Section 3, to sparsify it during every elimination, while ensuring we always output an Eulerian Laplacian as our overall sparse approximation of \( \text{Sc}(L, V \setminus \{v\}) \). To increase the accuracy of our approximation, we average the result of \( \tilde{O}(1) \) calls to \textsc{SingleVertexElim}(\cdot) at every elimination step, and this results in a slow increase in the the total number of edges at every iteration. We will use \textsc{SparsifyEulerian}(\cdot) from [4] to sparsify the current graph every once in a while to keep the total edge count low enough. This is a routine to sparsify any Eulerian graph, and can be found in [4]. Recalling the notion of asymmetric matrix approximation from Subsection 1.4, the guarantees stated in Theorem 3.16 of [4] can be stated as follows:

\textbf{Theorem 2.1} (Eulerian Spectral Sparsification - Theorem 3.16 of [4] Rephrased). For Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \) and \( \epsilon, \delta, \in (0,1) \) with probability at least \( 1 - \delta \) the \textsc{SparsifyEulerian}(\( L, \delta, \epsilon \)) computes in \( \tilde{O}(\text{mnz}(L) + n \epsilon^{-2} \log(1/\delta)) \) time an Eulerian Laplacian \( \tilde{L} \in \mathbb{R}^{n \times n} \) such that

1. \( \tilde{L} \) is an \( \epsilon \)-asymmetric spectral approximation of \( L \).
2. \( \tilde{L} \) has \( \tilde{O}(n \epsilon^{-2} \log(1/\delta)) \) non-zeros.
3. the weighted in and out degrees of the graphs associated with \( L \) and \( \tilde{L} \) are identical.

Let the vertices be labeled in the order in which we eliminate them so that in phase \( p \) of the algorithm, we eliminate vertices \( (i_p + 1) \ldots i_{p+1} \). The phases are numbered from 0 to \( p_{\text{max}} \) – 1, while vertices are numbered from 1 to \( n \). Note, this implies that at the start of phase \( p \), we have a graph on \( n - i_p \) vertices, starting initially with \( i_0 = 0 \), before any eliminations have taken place. We then index the elimination steps using the superscript \( (i) \) to denote the state of the algorithm just after we make the \( i^{\text{th}} \) elimination step.

We denote the intermediate matrices produced by our approximate elimination steps using \( S^{(i)} \). Each \( S^{(i)} \) as an \( n \times n \) matrix that’s non-zero only on entries that correspond to pairs of un-eliminated variables, specifically in the block \([i+1,n] \times [i+1,n] \). It is a sparse approximation of the Schur complement of \( L \) onto the remaining variables, with errors coming from two sources:
1. The randomized single vertex elimination procedure, \textsc{SingleVertexElim}(\cdot).

2. The global sparsification procedure \textsc{SparsifyEulerian}(\cdot).

We then use $L^{(i)}$ to denote the original matrix perturbed by the perturbations introduced by our elimination steps:

$$L^{(i+1)} \overset{\text{def}}{=} L^{(i)} + \left(S^{(i+1)} - \text{Sc} \left(L^{(i)}, [i+1,n]\right) \right).$$

The quantity $S^{(i)}$ is formally defined in Algorithm 3, along with a set of indices $i_j$ (sometimes denoted $i_p$) for $j = 0 \ldots p_{\max} - 1$, where $p_{\max}$ is the total number of phases. Recall the index $i_j$ is used to denote the state after the $i_j$th elimination step, and the index of the initial state is $i_0 = 0$.

Using a matrix martingale concentration inequality, we prove the following statement about the distortion bounded in each phase. (See Section 5 for the proof.)

**Theorem 2.2.** Given an $n \times n$ Eulerian matrix $L$ with $m$ non-zeros and a $0.1$-RCDD subset $J$, and an error parameter $\epsilon \leq 1/2$, and probability bound $\delta \leq 1/n$, \textsc{SinglePhase}(L, $\delta$, $\epsilon$) (Algorithm 2) creates with probability $1 - O(\delta)$ matrices $\tilde{S}$, $\tilde{L}$, $\tilde{U}$, where $\tilde{S}$ is an Eulerian Laplacian, and $\tilde{L}$, $\tilde{U}$ are upper and lower triangular respectively. The algorithm also finds a subset $\tilde{J}$ such that $\tilde{S} = \text{Sc} \left(\tilde{L}, V \setminus \tilde{J}\right)$ for the matrix $\tilde{L} = \tilde{S} + \tilde{L}' \tilde{U}'$ such that $\tilde{L}$ is an $\epsilon$-approximation of $L$ and $|\tilde{J}| \geq \frac{1}{2} |J|$. Furthermore, the number of non-zeros in $\tilde{S}$, $\tilde{L}'$, and $\tilde{U}'$ is at most $\tilde{O}(n\epsilon^{-6}\log^5(1/\delta))$ and the runtime is at most $\tilde{O}(m + n\epsilon^{-8}\log^7(1/\delta) + \epsilon^{-10}\log^2(1/\delta))$ where the $O$ notation additionally hides $O(\log\log 1/\delta)$ factors.

This means that with high probability we can bound the distortion within each phase by:

$$\left\|U_{\tilde{S}(i_p)}^{1/2} \left(L^{(i_p)} - L^{(i_{p+1})}\right) U_{\tilde{S}(i_p)}^{1/2}\right\| \leq \theta_p \epsilon,$$

while we also have $(n - i_{p+1}) \leq 0.99(n - i_p)$

We now define a PSD matrix which we use for measuring the accumulation of errors:

$$F^{(p)} \overset{\text{def}}{=} \sum_{p' \leq p} \theta_{p'} U_{\tilde{S}(i_{p'})}.$$

For convenience, we denote $F^{(p_{\max})}$ by $F$:

$$F \overset{\text{def}}{=} F^{(p_{\max})} = \sum_{0 \leq p < p_{\max}} \theta_p U_{\tilde{S}(i_p)}. \quad (2)$$

We will set $\theta_p = \frac{1}{p_{\max}}$ so that $\sum_{p=0}^{p_{\max}-1} \theta_p = 1$. We will bound the final error in terms of $F$ as stated in the following lemma, which we state for general matrices.

**Lemma 2.3.** Consider a sequence of $n$-by-$n$ matrices $S^{(0)}, S^{(1)}, \ldots, S^{(n)}$ such that

1. $S^{(0)}$ has non-zero entries only on the indices $[i+1,n]$,

2. The left/right kernels of $S^{(i)}$ are equal, and the after restricting $S^{(i)}$ to the indices $[i+1,n]$, the kernel of the resulting matrix equals the coordinate restriction of the vectors in the kernel of $S$. Formally, $\ker(S^{(i)}_{[i+1,n],[i+1,n]}) = \{b_{[i+1,n]} : b \in \ker(S^{(0)})\}$. 

12
Algorithm 2: SINGLEPhase($\mathbf{L}, \delta, \epsilon$)

**Input:** an Eulerian Laplacian $\mathbf{L}$ on vertex set $V$, and an error parameter $\epsilon$

**Output:** Set of vertices eliminated, $F$, $\mathbf{S}^{[F]}$ an Eulerian Laplacian on $V \setminus F$, matrices $\mathbf{U}^{[F]}$, $\mathbf{L}^{[F]}$ with non-zeros only in the rows/columns corresponding to $F$ respectively that are upper/lower triangle upon rearranging the vertices in $F$, and $\mathbf{L} \approx \mathbf{S}^{[F]} + \mathbf{L}^{[F]} \mathbf{U}^{[F]}$

1. $P \leftarrow \Theta(\log^2(1/\delta)/\epsilon^2)$
2. $\epsilon' \leftarrow \Theta(\epsilon/P)$
3. Compute a sparsifier of $\mathbf{L}$, $\mathbf{S}^{(0)} \leftarrow$ SPARSIYEUERIAL($\mathbf{L}, \delta/P, \epsilon'$)
4. $T \leftarrow \tilde{O}(n\epsilon^{-6}\log^5(1/\delta))$ (the upper bound on $\text{nnz}(\mathbf{S}^{(0)})$ from Theorem 2.1)
5. Pick a $0.1$-RCDD subset (See Algorithm 5 and Appendix A) of vertices $F^{(0)}$ from $\mathbf{S}^{(0)}$
6. Initialize $\mathbf{U}^{(0)}$, $\mathbf{L}^{(0)} \leftarrow 0$
7. Set $k_{\text{max}} \leftarrow \lfloor |F^{(0)}|/2 \rfloor$
8. for $k = 1 \ldots k_{\text{max}}$ do
   9. Among vertices in $F^{(k-1)}$ with degree at most twice the average, pick a random $v_k$.
10. $F^{(k)} \leftarrow F^{(k-1)} \setminus \{v_k\}$
11. Set $d^{(k)} \leftarrow \mathbf{S}^{(k-1)}(v_k, v_k)$
12. Update the factorization: set $\mathbf{U}^{(k)}$ to $\mathbf{U}^{(k-1)}$ with row $v_k$ replaced by $\frac{1}{d^{(k)}} \mathbf{S}^{(k-1)}(v_k, :)$ and $\mathbf{L}^{(k)}$ to $\mathbf{L}^{(k-1)}$ with column $v_k$ replaced by $\mathbf{S}^{(k-1)}(v_k, :)$.
13. Set $l^{(k)}, r^{(k)}$ to length $n$ vectors containing the the off-diagonal non-zeros in the column and row of $v_k$ in $\mathbf{S}^{(k-1)}$ respectively.
14. Initialize the first matrix of the inner loop to be the exact Schur complement of pivoting out $v_k$ from $\mathbf{S}^{(k-1)}$: $\mathbf{S}^{(k, 0)} \leftarrow \mathbf{S}^{(k-1)} - \frac{1}{d^{(k)}} l^{(k)} r^{(k)\top}$
15. for $t = 1 \ldots P$ do
16.   $\mathbf{S}^{(k,t)} \leftarrow \mathbf{S}^{(k,t-1)} - \frac{1}{P} \left( \text{SINGLEVERTEXELIM} \left( d^{(k)}, l^{(k)}, r^{(k)} \right) - \frac{1}{d^{(k)}} l^{(k)} r^{(k)\top} \right)$, (see Algorithm 4)
17.   if $\text{nnz}(\mathbf{S}^{(k,P)}) \geq 2T$ then
18.     $\mathbf{S}^{(k)} \leftarrow \text{SPARSIYEUERIAL}(\mathbf{S}^{(k,P)}, \delta/P, \epsilon')$
19.   else
20.     $\mathbf{S}^{(k)} \leftarrow \mathbf{S}^{(k,P)}$
21. Return $\mathbf{S}^{(k_{\text{max}})}$, $\mathbf{U}^{(k_{\text{max}})}$, $\mathbf{L}^{(k_{\text{max}})}$, and $k_{\text{max}}$.
Algorithm 3: EulerianLU($L, \delta, \epsilon$)

**Input:** an Eulerian Laplacian $L$ and error parameter $0 < \epsilon < 1/2$

**Output:** lower and upper triangular matrices $\mathcal{L}, \mathcal{U}$ whose product approximates $L$

1. $L \leftarrow \text{SparsifyEulerian}(L, \delta/2, O(\epsilon/\log n))$
2. $S^{(0)} \leftarrow L$ and set $\mathcal{L}, \mathcal{U}$ to be empty matrices.
3. $j \leftarrow 0$, $i_j \leftarrow 0$
4. while $i_j < n$ (i.e., for $p_{\text{max}} = O(\log n)$ iterations) do
5.   $(T, \mathcal{L}', \mathcal{U}', k_{\text{max}}) \leftarrow \text{SinglePhase}(S(i_j), O(\delta n), O(\epsilon/\log n))$
6.   $j \leftarrow j + 1$
7.   $i_j \leftarrow i_{j-1} + k_{\text{max}}$, $S(i_j) \leftarrow T$
8.   Insert the nonzero vectors from the partial LU factorization $\mathcal{L}', \mathcal{U}'$ into their corresponding locations $\mathcal{L}$ and $\mathcal{U}$, respectively.
9. return $\mathcal{L}, \mathcal{U}$

3. The undirectification of each $S(i), U_S(i) = \frac{1}{2}(S(i) + (S(i))^\top)$ is positive semi-definite.

Let $M = M^{(0)} = S^{(0)}$, and define matrices $M^{(1)}, M^{(2)}, \ldots, M^{(n)}$ iteratively by

$$M^{(i+1)} \overset{\text{def}}{=} M^{(i)} + \left( S^{(i+1)} - \text{Sc} \left( M^{(i)}, [i+1, n] \right) \right) \quad \forall 0 \leq i < n.$$

If for a subsequence of indices $1 = i_0 < i_1 < i_2 < \ldots < i_{p_{\text{max}}} = n$ associated scaling parameters $0 < \theta_0, \theta_1, \ldots, \theta_{p_{\text{max}}-1} < 1/2$, and some global error $0 < \epsilon < 1/2$, we have for every $0 \leq p < p_{\text{max}}$:

$$\left\| U^{1/2}_{i_p} \left( M^{(i_p)} - M^{(i_{p+1})} \right) U^{1/2}_{i_p} \right\| \leq \theta_p \epsilon,$$

then for a matrix-norm defined from the symmetrization of the $S(i_p)$ matrices and the scaling parameters:

$$F = \sum_{0 \leq p < p_{\text{max}}} \theta_p U_{S(i_p)},$$

we have:

1. for each $0 \leq i \leq n$,

$$\left\| F^{1/2} \left( M - M^{(i)} \right) F^{1/2} \right\|_2 \leq \epsilon,$$

2. The final matrix $M^{(n)}$ satisfies

$$M^{(n)}^\top F^{1/2} M^{(n)} \geq \frac{1}{10p^2} \cdot F.$$

We will prove Lemma 2.3 in Section 6, after first bounding the errors within each phase. The bounds on the new norm from Part 2 of Lemma 2.3 and the ability to solve linear systems in $L^{(n)}$ enable us to solve linear systems in $L$. To formalize this, we need to draw upon the definition of approximate pseudoinverses from [4].
and, we get Equation (5) \[ \text{Definition 2.4 (Approximate Pseudoinverse). Matrix } Z \text{ is an } \epsilon \text{-approximate pseudoinverse of matrix } M \text{ with respect to a symmetric positive semidefinite matrix } F, \text{ if } \ker(F) \subseteq \ker(M) = \ker(M^T) = \ker(Z) = \ker(Z^T), \text{ and } \|I_{im(M)} - ZM\|_{\mathbb{F} \rightarrow \mathbb{F}} \leq \epsilon. \]

The reason why approximate pseudoinverses are useful is that if one precondition with a solver for an approximate pseudoinverse, one can quickly solve the original system.

**Lemma 2.5** (Preconditioned Richardson, [4] Lemma 4.2, pg. 30). Let \( b \in \mathbb{R}^n \) and \( M, Z, F \in \mathbb{R}^{n \times n} \) such that \( F \) is symmetric positive semidefinite, \( \ker(F) \subseteq \ker(M) = \ker(M^T) = \ker(Z) = \ker(Z^T), \) and \( b \in \text{im}(M). \) Then if one performs \( t \geq 0 \) iterative refinement steps with step size \( \eta > 0, \) one obtains a vector \( x_t = \text{PreconRichardson}(M, Z, b, \eta, t) \) such that

\[
\|x_t - M^Tb\|_F \leq \|I_{im(M)} - \eta ZM\|_{\mathbb{F} \rightarrow \mathbb{F}} \|M^Tb\|_F.
\]

Furthermore, preconditioned Richardson implements a linear operator, in the sense that \( x_t = Z_t b, \) for some matrix \( Z_t \) only depending on \( Z, M, \eta \) and \( t. \)

We now argue that the properties of the approximate LU factorization produced by our algorithm imply that a solver for systems in it is an approximate pseudoinverse of the original Laplacian.

**Lemma 2.6.** Suppose we are given matrices \( L, \bar{L} \) and a positive semi-definite matrix \( F \) such that \( \ker(F) \subseteq \ker(L) = \ker(L^T) = \ker(\bar{L}) = \ker(\bar{L}^T) \) and

1. \( \|F^{1/2}(L - \bar{L})F^{1/2}\|_2 \leq \epsilon, \)

2. \( \bar{L}^T F \bar{L} \geq \gamma F. \)

Then \( \bar{L} \) is an \( \sqrt{\epsilon^2 \gamma^{-1}} \)-approximate pseudoinverse for \( L \) w.r.t. the norm \( F. \)

**Proof.** With a slight abuse of notation for notational convenience we let \( I \) denote \( I_{im(L)} = \Pi \) throughout this proof. The condition we need to show \( \|I - \bar{L}^T L\|_{\mathbb{F} \rightarrow \mathbb{F}} \leq \sqrt{\epsilon^2 \gamma^{-1}} \) is equivalent to

\[
\left(I - \bar{L}^T L\right)^T F \left(I - \bar{L}^T L\right) \leq \epsilon^2 \gamma^{-1} F.
\]

By rearranging a factor of \( \bar{L} \) on the LHS, we get

\[
\left(I - \bar{L}^T L\right)^T F \left(I - \bar{L}^T L\right) = \left(L - \bar{L}\right)^T \bar{L}^T \bar{L} F \bar{L}^T \left(L - \bar{L}\right)
\]
\[
\leq \gamma^{-1} \left(L^{(n)} - L\right)^T F \left(L^{(n)} - L\right), \tag{3}
\]

where in the last inequality we used \( \bar{L}^T \bar{L} F \bar{L}^T \leq \gamma^{-1} F. \) Condition 1, i.e., \( \|F^{1/2} \left(L^{(n)} - L\right) F^{1/2}\|_2 \leq \epsilon \) is equivalent to

\[
\left(L^{(n)} - L\right)^T F \left(L^{(n)} - L\right) \leq \epsilon^2 F. \tag{5}
\]

Combining Equation (4) and Equation (5), we get \( \left(I - \bar{L}^T L\right)^T F \left(I - \bar{L}^T L\right) \leq \epsilon^2 \gamma^{-1} F. \) \( \Box \)

\(^{3}\)Note that the ordering of \( Z \) and \( M \) is crucial: this definition is not equivalent to \( \|I_{im(M)} - MZ\|_{\mathbb{F} \rightarrow \mathbb{F}} \) being small.
We would like for the error guarantees of our solver to be in terms of $U_L$. In order to provide such guarantees, we need to relate this matrix to $F$.

**Lemma 2.7.** $U_L/O(\log(n)) \leq F \leq O(n^2 \log^5 n) \cdot U_L$

**Proof.** Since $F = \sum_{p' \leq p} \theta_{p'} U_{s(p)}$ and $\theta_{p'} = \frac{1}{O(\log n)}$, we have $F \geq \frac{1}{O(\log n)} U_L$. We have by Lemma 2.3, Part 2 that with probability $1 - O(\delta)$,

$$F \geq O(\log^2 n) \cdot \tilde{L}^T \tilde{F}^T \tilde{L} \leq O(\log^4 n) \cdot L^T F^T L \leq O(\log^5 n) \cdot L^T U_L L \leq O(n^2 \log^5 n) \cdot U_L,$$

where we used Lemma B.3 and Lemma 2.3 Part 1 for the second step and Lemma 13 from [3] pg. 19 for the last step. □

We have now stated the key theorems and lemmas needed to analyze correctness. With these tools, we can obtain the main theorem statement about finding sparse LU factorizations (Theorem 1.1) as follows.

**Proof of Theorem 1.1.** It is clear from the statement of the algorithm and the guarantees of Theorem 2.2 that EULERIANLU (Algorithm 3) outputs an LU factorization with the sparsity claimed and with the claimed bound on running time and error probability. The remaining correctness guarantees were proven as Lemma 2.7, and both parts of Lemma 2.3, respectively. □

We now have all the tools we need to obtain a fast solver for strongly connected Eulerian Laplacian systems.

**Proof of Corollary 1.2.** Suppose we have an Eulerian Laplacian $L$ and find a $1/O(\log^2(n))$-approximate LU factorization in nearly-linear time using Theorem 1.1 in the sense that $\|F^{1/2}(L - 2U)L^{1/2}\|_2 \leq 1/O(\log^2 n)$. Because it is an LU factorization, we can solve systems in it in linear time. By Lemma 2.6, such a solver is an $O(1)$-approximate pseudoinverse of $L$ with respect to $F$, provided we pick an appropriately small constant in the error guarantee we invoke our LU factorization algorithm EULERIANLU (Algorithm 3) with. By Lemma 2.5, if we precondition the original system with this solver, we can find a solution $x$ to the original system with $\epsilon/poly(n)$ error in the sense that $\|x - L^T b\|_F \leq \epsilon \cdot poly(n) \cdot \|L^T b\|_F$ in nearly-linear time. Since $F \approx poly(n) U_L$, this implies $\|x - L^T b\|_U_L \leq \epsilon \cdot \|L^T b\|_U_L$. □

### 3 Unbiased Degree Preserving Vertex Elimination

In this section we provide and analyze SINGLEVERTEXELIM, see Algorithm 4, which produces a sparse approximation of the clique created by Gaussian Elimination on an Eulerian directed Laplacian. It can be implemented to run in $O(\deg(v) \log \deg(v))$ time where $\deg$ is the combinatorial degree of the vertex $v$ being eliminated, i.e. the number of vertices incident to it.

The algorithm has three key features. When including self-loops, it preserves the weighted in and out degree of each vertex. This ensures the graph created by replacing the clique with the sparse approximation is still Eulerian. Note that we may get self-loops which will cancel out and change the degree of vertices, but it won’t change the fact that each vertex still has in-degree equal to out-degree. Secondly, it produces a sparse approximation of the biclique created by elimination. Thirdly, it achieves these guarantees while being an unbiased estimate of the biclique.
Algorithm 4: SingleVertexElim($l, r$)

Input: $l, r \in \mathbb{R}_{\geq 0}^n$ such that $1^\top l = 1^\top r$.
Output: $N \in \mathbb{R}^{n \times n}$

1. $s \leftarrow 1^\top l$
2. if $s = 0$ then
   3. return 0
4. else if $\min(l) \leq \min(r)$ then
   5. $i \leftarrow \arg\min(l)$
   6. Pick index $j \leftarrow k$ with probability $r(k)/\min(l)$
   7. return $\ell(i)\chi_i\chi_j^\top + \text{SingleVertexElim}(l - \ell(i)\chi_i, r - \ell(i)\chi_j)$
8. else
   9. $i \leftarrow \arg\min(r)$
10. Pick index $j \leftarrow k$ with probability $l(k)/\min(r)$
11. return $r(i)\chi_j\chi_i^\top + \text{SingleVertexElim}(l - r(i)\chi_j, r - r(i)\chi_i)$

Lemma 3.1. The matrix $A$ returned by SingleVertexElim($l, r$) has $\text{nnz}(A) \leq \text{nnz}(l) + \text{nnz}(r)$. Furthermore, the algorithm makes at most $\text{nnz}(l) + \text{nnz}(r)$ recursive calls to itself each of which is made to a vectors $l$ and $r$ with non-negative entries satisfying $1^\top l = 1^\top r$.

Proof. We prove this by induction on $\text{nnz}(l) + \text{nnz}(r)$. Base case: $\text{nnz}(l) + \text{nnz}(r) = 0$, then $l, r = 0$, so $A = 0$, and $\text{nnz}(A) = 0$. This proves the base case. For the inductive step, we suppose $\text{nnz}(l) + \text{nnz}(r) = k + 1$ and that the lemma holds whenever $\text{nnz}(l) + \text{nnz}(r) \leq k$. Without loss of generality consider the case of $\min(l) \leq \min(r)$. Note that $l - l(i)\chi_i, r - l(i)\chi_j \geq 0$, and that

\[
\text{nnz}(l - l(i)\chi_i) + \text{nnz}(r - l(i)\chi_j) \leq k
\]

so by the induction hypothesis with $A' = \text{SingleVertexElim}(l - l(i)\chi_i, r - l(i)\chi_j)$ we have $\text{nnz}(A') \leq k$, and so $\text{nnz}(A) \leq k + 1$. This proves the lemma by induction. The number of recursive calls can be bounded in the same way. \hfill \Box

Lemma 3.2. The matrix $A$ returned by SingleVertexElim($l, r$) has only non-negative entries and satisfies $A1 = l$, and $1^\top A = r^\top$.

Proof. We prove the lemma by induction on $\text{nnz}(l) + \text{nnz}(r)$. It is true in the base case $\text{nnz}(l) + \text{nnz}(r) = 0$, where $l, r = 0$, so $A = 0$, and $A1 = 0 = l$, and $1^\top A = 0^\top = r^\top$.

For the inductive step, we suppose $\text{nnz}(l) + \text{nnz}(r) = k + 1$ and that the lemma holds whenever $\text{nnz}(l) + \text{nnz}(r) \leq k$. W.l.o.g. consider the case of $\min(l) \leq \min(r)$.

Let $A' = \text{SingleVertexElim}(l - l(i)\chi_i, r - l(i)\chi_j)$. By the induction hypothesis,

\[
A1 = l(i)\chi_i\chi_j^\top 1 + A'1 = l(i)\chi_i + l - l(i)\chi_i = l
\]

and similarly

\[
1^\top A = 1^\top l(i)\chi_i\chi_j^\top + 1^\top A' = l(i)\chi_j^\top + r^\top - l(i)\chi_j^\top = r
\]

\hfill \Box
Lemma 3.3. Given \( l, r \in \mathbb{R}^n \) s.t. both have non-negative entries and \( 1^\top l = 1^\top r = s \), let \( A = \text{SingleVertexElim}(l, r) \). Then \( \mathbb{E}[A] = lr^\top / s \).

Proof. We prove the lemma by induction on \( \text{nnz}(l) + \text{nnz}(r) \). It is true in the base case \( \text{nnz}(l) + \text{nnz}(r) = 0 \), where \( l, r = \vec{0} \), so \( A = 0 \), so \( \mathbb{E}[A] = 0 \). For the inductive step, we suppose \( \text{nnz}(l) + \text{nnz}(r) = k + 1 \) and that the lemma holds whenever \( \text{nnz}(l) + \text{nnz}(r) \leq k \). Without loss of generality consider the case of \( \min(l) \leq \min(r) \). In this case we have

\[
\mathbb{E}[A] = \sum_j \frac{r(j)}{s} \left( l(i) \chi_i \chi_j^\top + \frac{1}{s - l(i)}(l - l(i) \chi_i)(r - l(i) \chi_j)^\top \right)
\]

\[
= \sum_j \frac{l(i)}{s} \chi_i r(j) \chi_j^\top + \frac{1}{s - l(i)}(l - l(i) \chi_i) r^\top - l(i) \frac{1}{s - l(i)}(l - l(i) \chi_i) r^\top
\]

\[
= l(i) \chi_i r^\top \left( \frac{1}{s - l(i)} + \frac{l(i)}{s - l(i)} \right) + lr^\top \frac{1}{s - l(i)} \left( 1 - \frac{l(i)}{s} \right)
\]

\[
= lr^\top / s.
\]

A crucial matrix used in analyzing the elimination of a single vertex is the Schur complement of the star incident on the eliminated vertex in the undirectification of the whole matrix.

Definition 3.4. Given an Eulerian Laplacian \( L \) and a vertex \( v \), let

\[
U_{\text{local}} = \text{ST}[U_L]_v = \frac{1}{U_L(v,v)} U_L(:,v) U_L(v,:).
\]

Note that

\[
U_{\text{local}} \preceq \text{ST}[U_L]_v.
\]

And hence for a random choice of vertex \( v \) in a graph with \( n \) remaining vertices

\[
\mathbb{E}_v[U_{\text{local}}] \preceq \frac{2}{n} U_L
\]

(6)

Lemma 3.5 (Single Vertex Elimination Routine). There is a routine \( \text{SingleVertexElim} \) that takes the in and out adjacency list vectors \( l \) and \( r \) of a vertex \( u \) in an Eulerian Laplacian with \( d \) non-zeros, and produces a matrix \( A \) with at most \( d \) non-zeros such that the error matrix

\[
X = \frac{1}{r^\top} lr^\top - A
\]

satisfies

1. \( X1 = 0, \ X^\top 1 = 0 \), and
2. \( \mathbb{E}[X] = 0 \), and
3. For the local undirectification, \( U_{\text{local}} \) as given in Definition 3.4, we have \( \|U_{\text{local}}^{1/2} X U_{\text{local}}^{1/2}\|_2 \leq 4 \).
Proof. We already have Parts 1 and 2 from Lemma 3.2 and Lemma 3.3.

For Part 3, we note that by $A$ (by Lemma 3.2) has the sum of the absolute value of its entries in the $i$th row and $i$th column of at most $l(i) + r(i)$. This similarly applies to the the expectation $l(i) \cdot r^\top$. Thus, the sums for the error matrix $X$ are at most double this: $2(l(i) + r(i))$.

Now, we define a diagonal matrix $D_{\text{local}}$, whose $i$th diagonal entry is $\frac{l(i) + r(i)}{2}$. Because the sums of the absolute values of the $i$th row and column are at most $4(D_{\text{local}})_{ii}$, we have

$$\|D_{\text{local}}^{1/2}XD_{\text{local}}^{1/2}\| \leq 4.$$

Now, $\|U_{\text{local}}^{11/2}D_{\text{local}}^{1/2}\|_2 \leq 1$ by Lemma B.5 (proved in Appendix B, $U_{\text{local}}^{\dagger} \preceq D_{\text{local}}^{-1}$) and consequently

$$\|U_{\text{local}}^{11/2}XU_{\text{local}}^{1/2}\|_2 = \|(U_{\text{local}}^{11/2}D_{\text{local}}^{1/2})(D_{\text{local}}^{1/2}XD_{\text{local}}^{1/2})(U_{\text{local}}^{11/2}D_{\text{local}}^{1/2})^\top\|_2 \leq 4.$$ 

\[\square\]

4 Robustly Bounded Schur Complement Sets

As we have discussed, one of the key difficulties in applying repeated vertex elimination to solve Eulerian Laplacian systems is that unlike with symmetric Laplacian systems the Schur complement of an Eulerian Laplacian may be much larger than that of the original Laplacian. Consequently, if we simply eliminate an arbitrary set of vertices the error we incur may too large to ensure we compute an effective preconditioner.

To circumvent this we eliminate vertices in phases where in each phase we only eliminate vertices that do not "blow up" the Schur complement, i.e. induce Schur complements that are not spectrally dominated by a small multiple of the current Laplacian. Formally, we only eliminate vertices from what we call robustly bounded schur complement sets defined below. These are sets, where even under a small amount of spectral error, the Schur complement is not too much larger than the original graph.

Definition 4.1 (Robustly Bounded Schur Complement Set). Given an Eulerian Laplacian $L$, a robustly bounded Schur complement set vertex set $J$, is a subset of the vertices of $L$, such that for any $\tilde{J} \subseteq J$ and any $\tilde{L}$ that $1/2$-approximates $L$ we have $U_{\text{Sc}(\tilde{L},\tilde{J})} \preceq O(1)U_L$.

In this section, we formally show that $\alpha$-RCDD subsets of the vertices are robustly bounded Schur complement sets and therefore we can easily find such sets. The main result of this section is the following formalization of this claim.

Lemma 4.2 ($\alpha$-RCDD Sets are Robustly Bounded). Given an Eulerian Laplacian $L$, for any fixed constant $\alpha$, an $\alpha$-RCDD subset $J$ of the vertices of $L$ is robustly bounded.

We prove this lemma in several pieces. First in Subsection 4.1 we provide a general lemma about how much a Schur complement of an arbitrary asymmetric matrix can increase. Then in Subsection 4.2 we show how to apply this to Eulerian Laplacians to bound show how much a Schur complement of an RCDD subset of an Eulerian Laplacian can increase. Finally, in Subsection 4.3 we show that this analysis is robust to asymmetric approximation and with this prove Lemma 4.2.
4.1 Bounding General Schur Complements

Here we provide a general lemma that upper bounds the symmetrization of the Schur complement of a general matrix spectrally via its symmetrization. This lemma is the main tool we use to reason about the Schur complement of subsets of a $\alpha$-RCDD subset of an Eulerian Laplacian.

**Lemma 4.3.** If $N \in \mathbb{R}^{n \times n}$ satisfies $U \overset{\text{def}}{=} U_N \succeq 0$ and $F, C \subseteq [n]$ is a partition of $[n]$ where $U_{FF} > 0$ then $U_{\text{Sc}(N,F)} \preceq (1 + \alpha)U$ provided the following condition holds

$$M \overset{\text{def}}{=} \begin{bmatrix} [N_{FF}^{-1}]^T U_{FF} N_{FF}^{-1} & 0_{FC} \\ 0_{CF} & 0_{CC} \end{bmatrix} \preceq \alpha [N]^T UN^T.$$

**Proof.** Let $z \in \mathbb{R}^n$ be arbitrary and let $x, y \in \mathbb{R}^n$ be defined so that $x_C = y_C = z_C$, $x_F \overset{\text{def}}{=} -N_{FF}^{-1} N_{FC} z_C$, and $y_F = -U_{FF}^{-1} U_{FC} z_C$. (Note $N_{FF}$ is invertible by Lemma B.1 as $U_{FF} > 0$.) Now, this definition was chosen so that

$$\mathbf{N}x = \begin{bmatrix} N_{FF} & N_{FC} \\ N_{CF} & N_{CC} \end{bmatrix} \begin{bmatrix} -N_{FF}^{-1} N_{FC} z_C \\ z_C \end{bmatrix} = \begin{bmatrix} \tilde{0}_F \\ \text{Sc}(N,F) z_C \end{bmatrix},$$

and therefore $x^T \mathbf{N}x = z^T \text{Sc}(N,F) z$. Furthermore, note that

$$z^T Uz = z_F^T U_{FF} z_F + 2z_C^T U_{CF} z_F + z_C^T U_{CC} z_C$$

and since $U_{FF} > 0$ we have that $z^T Uz$ is minimized over $z_F$ when $z_F = -U_{FF}^{-1} U_{FC} z_C$ and thus $y^T U y \leq z^T Uz$. Furthermore, $y_F - x_F = N_{FF}^{-1} [Ny]_F$, $x_C = y_C$, and Lemma B.2 yields

$$\|x - y\|^2_U = (x_F - y_F) U_{FF} (x_F - y_F) = \|[Ny]_F\|^2_{N_{FF}^{-1} U_{FF} N_{FF}^{-1}}.$$

Further, by the $U$-orthogonality of $y$ and $x - y$ (as $x - y$ is supported on $F$ and $Uy$ is 0 on $F$),

$$\|x\|^2_U = \|y\|^2_U + \|x - y\|_U \leq (1 + \alpha) \|y\|^2_U.$$

As $\|x\|^2_U = z^T \text{Sc}(N,F) z = z^T U_{\text{Sc}(N,F)} z$ and $\|y\|^2_U \leq \|z\|^2_U$ the result follows. \qed

4.2 Schur Complements of Eulerian Laplacians

Here we show how to apply the Schur complement bounds of the previous subsection to bound the increase in Schur complements for Eulerian Laplacians. In particular we bound the blowup of the Schur complements as we pivot away $\alpha$-RCDD subsets of vertices. The main result we prove is the following.

**Lemma 4.4.** Suppose that $L = D - A^\top \in \mathbb{R}^{n \times n}$ is an Eulerian Laplacian, and $F \subseteq [n]$ is an $\alpha$-RCDD subset. Then $U_{\text{Sc}(L,F)} \preceq (3 + \frac{2}{\alpha})U_L$.

To prove this lemma, first we provide the following lemma, which is a self contained fact about Eulerian Laplacians that will allow us to leverage Lemma 4.3.

**Lemma 4.5.** Suppose that $L = D - A^\top \in \mathbb{R}^{n \times n}$ is an Eulerian Laplacian associated with directed graph $G = (V,E,w)$. Then $L^\top D^{-1} L \preceq 2U_L$. 

20
Proof. Let \( x \in \mathbb{R}^n \) be arbitrary and recall that

\[
[Lx]_i = \sum_{(i,j) \in E} w_{ij}(x(i) - x(j)) \quad \text{and} \quad D_i = \sum_{(i,j) \in E} w_{ij}.
\]

Furthermore, by Cauchy-Schwarz we have that

\[
[Lx]^2 = \left( \sum_{(i,j) \in E} w_{ij}(x(i) - x(j)) \right)^2 \leq \left( \sum_{(i,j) \in E} w_{ij} \right) \cdot \left( \sum_{(i,j) \in E} w_{ij}(x(i) - x(j))^2 \right).
\]

Consequently,

\[
x^\top L^\top D^{-1} Lx = \sum_{i \in [n]} \frac{|Lx|^2_i}{D_i} \leq \sum_{i \in [n]} \sum_{(i,j) \in E} w_{ij}(x(i) - x(j))^2 = 2 \cdot x^\top U_L x.
\]

Using Lemma 4.3 and Lemma 4.5 we can prove the following, a key bound on the increase of Schur complements of Eulerian Laplacians.

**Lemma 4.6.** Suppose that \( L = D - A^\top \in \mathbb{R}^{n \times n} \) is an Eulerian Laplacian, let \( U \overset{\text{def}}{=} U_L \) and let \( F, C \subseteq [n] \) be a partition of \([n]\) such that \( U_{FF} \succeq \frac{1}{\alpha} D_{FF} \) then \( U_{Sc(L,F)} \succeq (1 + 2\alpha) U \).

**Proof of Lemma 4.6.** First note that \( L_{FF} \) and \( U_{FF} \) must be RCDD as \( L \) is Eulerian, and since \( U_{FF} \succeq \frac{1}{\alpha} D_{FF} \) it is the case that \( U_{FF} \) is invertible and so is \( L_{FF} \). Consequently, as \( U_{FF} \succeq L_{FF}^{-1} U_{FF} L_{FF} \) from our general bounds on harmonic symmetrizations we have that \( [L_{FF}^{-1}]^\top U_{FF} [L_{FF}^{-1}] \succeq U_{FF}^{-1} \). Furthermore, as \( U_{FF} \succeq \frac{1}{\alpha} D_{FF} \) we have that \( U_{FF}^{-1} \succeq \alpha D_{FF}^{-1} \) and therefore that

\[
\begin{bmatrix}
[L^{-1}_{FF}]^\top U_{FF} L^{-1}_{FF} & 0_{FC} \\
0_{CF} & 0_{CC}
\end{bmatrix} \succeq \begin{bmatrix}
\alpha D^{-1}_{FF} & 0_{FC} \\
0_{CF} & 0_{CC}
\end{bmatrix} \succeq \alpha D^{-1} \succeq \alpha [L^\top]^\top D^{-1} [L^\top] L.
\]

As \( [L^\top]^\top D^{-1} [L^\top] \preceq 2 \cdot U \) by Lemma 4.5 the result then follows from Lemma 4.3.

Using Lemma 4.6 we can now prove Lemma 4.4.

**Proof of Lemma 4.4.** Since \( F \) is an \( \alpha \)-RCDD subset \( \frac{1}{1+\alpha} D_{FF} - (U_A)_{FF} \) is SDD and PSD. Consequently, \( U_{FF} = \frac{\alpha}{1+\alpha} D_{FF} + \frac{1}{1+\alpha} D_{FF} - (U_A)_{FF} \succeq \frac{\alpha}{1+\alpha} D \) and the claim follows from Lemma 4.6.

### 4.3 Schur Complement Stability

Here, we show that Schur complements are robust to small changes to the original matrix. Using this stability result Lemma 4.7, and the results of the previous subsection we prove the main claim of this section, Lemma 4.2.

**Lemma 4.7.** Suppose \( N \in \mathbb{R}^{n \times n} \) satisfies \( U \overset{\text{def}}{=} U_N \succeq 0 \) and \( \ker(N) = \ker(N^\top) \) and suppose \( F, C \subseteq [n] \) is a partition of \([n]\) where \( U_{FF} \succeq 0 \). If \( \tilde{N} \) \( \epsilon \)-approximates \( N \) then \( U_{Sc(N,F)} \succeq (\frac{1+\epsilon}{1-\epsilon})^2 U_{Sc(N,F)} \).

21
Proof. As in the proof of Lemma 4.3, given any vector \( z \) we define \( x \) and \( \tilde{x} \) such that \( x_C = \tilde{x}_C = z_C \), \( x_F \overset{\text{def}}{=} -N_{FF}^{-1}N_{FC}z_C \), \( \tilde{x}_F \overset{\text{def}}{=} -\tilde{N}_{FF}^{-1}\tilde{N}_{FC}z_C \). (Note \( N_{FF} \) is invertible by Lemma B.1 as \( U_{FF} \succ 0 \).) As in the proof of Lemma 4.3 this yields \( z^T U_{\text{Sparsify}(N,F)} z = x^T U x \) and \( z^T U_{\text{Sc}(\tilde{N},F)} z = \tilde{x}^T \tilde{U} \tilde{x} \). Furthermore, direct calculation reveals that

\[
\tilde{x}_F - x_F = -\tilde{N}_{FF}^{-1}[\tilde{N}x]_F = -\tilde{N}_{FF}^{-1}[(\tilde{N} - N)x]_F.
\]

By the assumption of \( \tilde{N} \), we have

\[
\| (\tilde{N} - N)x \|_{\tilde{U}^\dagger} \leq \frac{1}{1 - \epsilon} \| (\tilde{N} - N)x \|_{U^\dagger} \leq \frac{\epsilon}{1 - \epsilon} \| x \|_U.
\]

By Lemma B.2 and the fact that \( [(\tilde{N} - N)x]_C = 0 \) this gives

\[
\| \tilde{x}_F - x_F \|_{\tilde{U}_{FF}} = \left\| \tilde{N}_{FF}^{-1}[\tilde{N} - N)x]_F \right\|_{\tilde{U}_{FF}} \leq \left\| (\tilde{N} - N)x \right\|_{U^\dagger} \leq \frac{\epsilon}{1 - \epsilon} \| x \|_U.
\]

Therefore we can write

\[
\| z \|_{U_{\text{Sc}(\tilde{N},F)}}^2 = \| \tilde{x} \|_{\tilde{U}}^2 \leq (\| z \|_{\tilde{U}} + \| \tilde{x} - x \|_{\tilde{U}})^2 \leq \left( 1 + \frac{\epsilon}{1 - \epsilon} \right) \| x \|_{U}^2 \leq \left( 1 + \frac{\epsilon}{1 - \epsilon} \right)^2 \| z \|_{U_{\text{Sc}(\tilde{N},F)}}^2.
\]

\[\square\]

Proof of Lemma 4.2. By Lemma 4.4 and the fact that subsets of \( \alpha \)-RCD subsets are \( \alpha \)-RCD we have that \( U_{\text{Sc}(L,\tilde{J})} \preceq O(1)U_L \). and Lemma 4.7. The result then follows by applying Lemma 4.7 since for Eulerian \( L \) we have \( \ker(L) = \ker(L)^\top \). \[\square\]

5 Single Phase Analysis

In this section we prove Theorem 2.2 which gives the guarantees of SinglePhase (Algorithm 2), our main subroutine for eliminating blocks of vertices.

We start by analyzing the running time of SinglePhase in Algorithm 2. The following Lemma 5.1 bounds both the number of times the entire graph is sparsified, i.e. SparsifyEulerian, as well as the total running time of the algorithm. To prove this lemma, we bound how much the sparsity of the graph increases as we eliminate vertices, i.e. call SingleVertexElim (Algorithm 4).

**Lemma 5.1 (SinglePhase (Algorithm 2) Running Time).** Suppose \( L \) is an Eulerian Laplacian on \( n \) vertices with \( m \) non-zeros. In the for-loop of SinglePhase (Algorithm 2), SparsifyEulerian is called \( O(P) = O(\epsilon^{-2} \log^2(1/\delta)) \) times. Consequently, the total running time for SinglePhase is

\[
\tilde{O}(m + P(T + TP/n)) = \tilde{O}(m + n \epsilon^{-8} \log^7(1/\delta) + \epsilon^{-10} \log^9(1/\delta))
\]

where here the \( \tilde{O} \) notation additionally hides \( O(\log \log 1/\delta) \) factors.
Proof. Note that for all \( k \in [0, k_{\text{max}}] \) the Eulerian Laplacian \( S^{(k)} \) is non-zero on at most \( n - k \geq n - k_{\text{max}} \geq n/2 \) vertices. Since in each iteration of the algorithm a vertex \( v \) of at most twice the average degree and then adds \( O(\deg(v)) = O(\text{nnz}(S^{(k)}/n)) \) edges are added to the graph from each of the \( P \) calls to \textsc{SingleVertexElim} and then sparsification only decreases sparsity we have that

\[
\text{nnz}(S^{(k+1)}) \leq \left(1 + O\left(\frac{P}{n}\right)\right) \text{nnz}(S^{(k)}) \leq \text{nnz}(S^{(k)}) \exp(O(P/n)).
\]

Consequently, for all \( t > 0 \) we have that \( \text{nnz}(S^{(k+t)}) \leq \exp(O(Pt/n)) \text{nnz}(S^{(k)}) \). Consequently, it takes at least \( t = \Omega(n/P) \) iterations for the sparsity of \( \text{nnz}(S^{(k)}) \) to double and we have that \textsc{SparsifyEulerian} is called at most \( O(k_{\text{max}}/(n/P)) = O(P) \) times.

With the exception of the first one, each invocations of \textsc{SparsifyEulerian} is on a graph with sparsity is at most \((1 + O(P/n))T\). Consequently, the running time for all the sparsification calls combined (ignoring \( O(\log \log \delta) \) factors) is

\[
\widetilde{O}(m + P \cdot (T + TP/n)) = \widetilde{O}(m + n \epsilon^{-8} \log^7(1/\delta) + \epsilon^{-10} \log^9(1/\delta)).
\]

This also upper bounds the running time required for vertex eliminations, as \textsc{SingleVertexElim} (Algorithm 4) can be implemented to run in time \( O(d \log d) \), where \( d \) is the combinatorial degree of the vertex being eliminated. With probability \( 1 - O(\delta) \), this also upper bounds the running time required for performing the random vertex selections of low degree vertices, which can be implemented using a simple rejection sampling approach. \( \square \)

To prove Algorithm 2 it only remains to show that in \textsc{SinglePhase} it is the case that

\[
\Pr \left[ \left\| U^{1/2}_L (L^{(k_{\text{max}})} - L) U^{1/2}_L \right\|_2 > \epsilon \right] \leq O(\delta) \tag{7}
\]

where \( L \) is the input to \textsc{SinglePhase} and \( L^{(k_{\text{max}})} \) is the output. To do this, we set up a matrix martingale as follows. For the inner loops (inside \( k \), inside the \( t \) loops) of \textsc{SinglePhase}. We define the change in each step to be:

\[
X^{(k,t)} \overset{\text{def}}{=} S^{(k,t)} - S^{(k,t-1)}.
\]

Each \( X^{(k,t)} \) has zero expectation, hence we can define the following zero-mean martingale sequence

\[
M^{(k,t)} = \sum_{k=1}^k \sum_{t=1}^P \text{ if } \hat{k} < k \quad M^{(k,t)} = \sum_{\tilde{k}, \tilde{t} \leq (k,t)} X^{(\hat{k}, \tilde{t})}.
\]

Here and for the remainder of this section we overload the \( \leq \) and \(< \) notation to handle pair of variables in the lexicographical sense, e.g. \( (\hat{k}, \hat{t}) \leq (k, t) \) if and only if \( \hat{k} < k \) or \( \hat{k} = k \) and \( \hat{t} \leq t \).

Note that this martingale does not include the changes introduced by calls to \textsc{SparsifyEulerian} and therefore it may be the case that \( L^{(k)} \neq L + M^{(k,P)} \). To track the changes caused by \textsc{SparsifyEulerian}, we further define changes from the sparsification steps. We let

\[
Z^{(0)} \overset{\text{def}}{=} S^{(0)} - S \text{ and } Z^{(k)} \overset{\text{def}}{=} S^{(k)} - S^{(k,P)} \text{ for all } k > 0
\]

and with this notation define the output matrix \( L^{(k)} \) and intermediate matrices \( L^{(k,t)} \) by

\[
L^{(k)} = L + M^{(k,P)} + \sum_{\hat{k}=0}^k Z^{(\hat{k})} \quad \text{and} \quad L^{(k,t)} = L + M^{(k,t)} + \sum_{\hat{k}=0}^{k-1} Z^{(\hat{k})}.
\]

23
Now we wish to analyze this martingale conditioned on certain high probability events holding which make the martingale safe or stable for analysis. We defined martingale safety as follows.

**Definition 5.2 (Martingale Safety).** We let \( \text{SAFE}^{(k,t)} \) denote the event that the martingale is safe until \((k, t)\), for \( t \in \{1, \ldots, P + 1\} \) which we define as the following two conditions holding:

1. All calls to SPARSIFYEULERIAN strictly before the \( k \)th elimination have been successful, e.g. 
   \[ \| U_{S(k,p)}^{1/2} Z^{(k)} U_{S(k,p)}^{1/2} \|_2 \leq \epsilon' \text{ for all } \hat{k} < k. \]

2. For all indices \((\hat{k}, \hat{t}) < (k, t)\) we had 
   \[ \| U_{L}^{1/2} M^{(\hat{k},\hat{t})} U_{L}^{1/2} \|_2 \leq \epsilon/2. \]

With this notation of Martingale safety established we define the following truncated Martingale as one where the steps incur no additional error once it fails. Formally, we let

\[
X^{(k,t)} \overset{\text{def}}{=} \begin{cases} X^{(k,t)} & \text{if } \text{SAFE}^{(k,t)} \\ 0 & \text{otherwise.} \end{cases}
\] (8)

Note that the following sequence of sums of \( X^{(k,t)} \) is another zero mean martingale:

\[
M^{(k,t)} = \sum_{(\hat{k},\hat{t}) \leq (k,t)} X^{(\hat{k},\hat{t})}.
\]

To analyze this martingale we first establish the following consequences of \( \text{SAFE}^{(k,t)} \).

**Lemma 5.3.** If \( \text{SAFE}^{(k,t)} \) holds,

- then for all \((\hat{k}, \hat{t}) < (k, t)\),
  \[ \| U_{L}^{1/2} (L^{(\hat{k},\hat{t})} - L) U_{L}^{1/2} \|_2 \leq \epsilon \text{ and } \| U_{L}^{1/2} (L^{(k)} - L) U_{L}^{1/2} \|_2 \leq \epsilon \] (9)

- and we have for all \( \hat{k} < k \)
  \[ U_{S(\hat{k})} \preceq O(1) \cdot U_{L} \] (10)

**Proof of Lemma 5.3.** We consider the ordering \((k, 1) < (k, 2) < \ldots < (k, P) < (k, P + 1) < (k + 1, 1) < \ldots \) and prove by induction on this ordering, that Equation (10) holds as well as

\[ \sum_{k<k} \| U_{L}^{1/2} Z^{(k)} U_{L}^{1/2} \|_2 \leq \frac{C \epsilon}{P} \cdot N^{(k,t)} \] (11)

Where \( N^{(k,t)} \) is defined as the number of calls SPARSIFYEULERIAN before \((k,t)\) and \( C \) is defined as a constant such that guarantee of Lemma 5.1 that \( N^{(k,t)} = O(P) \) ensures \( C \epsilon N^{(k,t)} / P \leq \epsilon/2 \). Since \( L^{(k,t)} - L = M^{(k,t)} + \sum_{\hat{k}<k} Z^{(\hat{k})} \), by triangle inequality and Part 2 of Definition 5.2 this suffices to prove the result.

As our base case, consider the index \((1,1)\). We call SPARSIFYEULERIAN\((L, \delta/P, \epsilon')\) to compute \( S^{(0)} \), and \( \text{SAFE}^{(1,1)} \) guarantees this call succeeded. So Theorem 2.1 immediately tells us that \( \| U_{L}^{1/2} Z^{(0)} U_{L}^{1/2} \|_2 \leq \epsilon' \leq \frac{C \epsilon}{P} \), establishing Equation (11) for this index. Equation (9) follows
from a triangle inequality combined with Part 2 of Definition 5.2 and Equation (10) follows from Lemma 4.2.

Next, we consider proving the inductive statements when $\text{SAFE}^{(k,t+1)}$ holds, assuming the induction hypothesis holds for $\text{SAFE}^{(k,t)}$. In this case, the condition in Equation (10) remains unchanged, so it follows immediately from the induction hypothesis for $\text{SAFE}^{(k,t)}$. The sum $\sum_{k<k} \|U_L^{1/2} Z(\hat{k}) U_L^{1/2}\|_2$ and upper bound we want for it also remain unchanged, so again we get Equation (11). This gives Equation (9), from a triangle inequality combined with Part 2 of Definition 5.2. Now we consider proving the inductive statements when $\text{SAFE}^{(k+1,1)}$ holds, assuming the induction hypothesis holds for $\text{SAFE}^{(k,P+1)}$. From the induction hypothesis for $\text{SAFE}^{(k,P+1)}$, we have that $\|U_L^{1/2} (L(k,P) - L) U_L^{1/2}\|_2 \leq \epsilon$ and from Lemma 4.2, we then get $U_{S(k,P)} \leq O(1) \cdot U_L$. This ensures that if a call to SPARSIFYEULERIAN was made at the end of $k$th elimination, then since $\text{SAFE}^{(k+1,1)}$ guarantees the call succeeded, we have

$$\left\| U_L^{1/2} Z(\hat{k}) U_L^{1/2} \right\|_2 \leq O(1) \left( \left\| U_{S(k,P)}^{1/2} Z(\hat{k}) U_{S(k,P)}^{1/2} \right\|_2 + O(1) \epsilon' \right) \leq \frac{C\epsilon}{P}.$$ 

This then proves Equation (11) for $\text{SAFE}^{(k+1,1)}$, which gives Equation (9) from a triangle inequality combined with Part 2 of Definition 5.2. Finally, by Lemma 4.2, we then get $U_{S(k)} \leq O(1) \cdot U_L$, which proves Equation (10) for $\text{SAFE}^{(k+1,1)}$.

Note that this lemma shows that if we can prove $\Pr[-\text{SAFE}^{(n+1,1)}] \leq O(\delta)$, it will imply Equation (7) and prove Theorem 2.2. To prove this, note that $\|U_L^{1/2} M(k,t) U_L^{1/2}\|_2 > \epsilon/2$ implies $\|U_L^{1/2} \tilde{M}(k,t) U_L^{1/2}\|_2 > \epsilon/2$. Hence, when upper bounding the probability of $\Pr[-\text{SAFE}^{(k,t)}]$, we can instead consider the higher probability event

$$-\left( \left( \forall \hat{k} \leq k \left\| U_{S(k,P)}^{1/2} Z(\hat{k}) U_{S(k,P)}^{1/2} \right\|_2 \leq \epsilon' \right) \land \left( \forall (\hat{k},i) \leq (k,P) \left\| U_L^{1/2} \tilde{M}(\hat{k},i) U_L^{1/2} \right\|_2 \leq \epsilon/2 \right) \right)$$

To bound this we use the following rectangular matrix martingale result from [35].

**Lemma 5.4** (Matrix Freedman (from Cor 1.3. of [35])). Let $E^{(1)} \ldots E^{(N)}$ be a sequence of matrices and let $E_{j-1} \left[ E^{(j)} \right]$ denote the expectation of $E^{(j)}$ conditioned on $E^{(j-1)}, E^{(j-2)}, \ldots, E^{(1)}$. If $E_i \left[ E^{(i)} \right] = 0$ and $\|E^{(i)}\|_2 \leq \rho$ with probability 1 for all $i$ then for any error $t$ we have:

$$\Pr \left[ \exists k \geq 0 \text{ s.t.} \left\| \sum_{j \leq k} E^{(j)} \right\|_2 \geq t \text{ AND } \left\| \sum_{j \leq k} E_{j-1} \left[ E^{(j)} E^{(j)\top} + E^{(j)\top} E^{(j)} \right] \right\|_2 \leq \sigma^2 \right] \leq n \cdot \exp \left( \frac{-t^2}{100 (\sigma^2 + t\rho)} \right).$$

As our bounds normalize by $U_L$ for simplicity we can define rescaled quantities:

$$\tilde{M}^{(k,t)} \overset{\text{def}}{=} U_L^{1/2} M^{(k,t)} U_L^{1/2} \text{ and } \tilde{X}^{(k,t)} \overset{\text{def}}{=} U_L^{1/2} X^{(k,t)} U_L^{1/2}$$
Together with a union bound, this allows us to bound $\Pr[-\text{SAFE}^{(k_{\text{max}}+1,1)}]$ by:

\[
\Pr[-\text{SAFE}^{(k_{\text{max}}+1,1)}]
\leq \sum_k \Pr \left[ \left\| U_{S(k,P)}^{1/2} Z^{(k)} S(k,P) U_{S(k,P)}^{1/2} \right\|_2 > \epsilon \right]
\]

(12)

\[
+ \Pr \left[ \exists (k, t) \text{ s.t. } \left\| \tilde{M}^{(k,t)} \right\|_2 \geq s \text{ AND } \sum_{(k, t) \leq (k, t)} \mathbb{E}_{(k, t) \leq (k, t)} \left[ \tilde{X}^{(k,t)} \tilde{X}^{(k,t)^\top} + \tilde{X}^{(k,t)^\top} \tilde{X}^{(k,t)} \right] \right] \leq \sigma^2
\]

(13)

\[
+ \Pr \left[ \exists (k, t) \text{ s.t. } \left\| \sum_{(k, t) \leq (k, t)} \mathbb{E} \left[ \tilde{X}^{(k,t)} \tilde{X}^{(k,t)^\top} + \tilde{X}^{(k,t)^\top} \tilde{X}^{(k,t)} \right] \right\|_2 > \sigma^2 \right].
\]

(14)

Each call to SPARSIFYEULERIAN is made with error probability parameter $\delta/P$ and by Lemma 5.1, we call the routine at most $O(P)$ times, so by a union bound and Theorem 2.1, the probability at some call fails is at most $O(\delta)$, which bounds the term (12).

The other two terms rely on properties of the truncated Martingales, which in turn rely on the condition $\text{SAFE}^{(k,t)}$. To bound these first we provide the following simple lemma which uses Lemma 5.3 to bound the error of $\tilde{X}^{(k,t)}$.

Lemma 5.5. We have $\left\| \tilde{X}^{(k,t)} \right\|_2 \leq O(1/P)$ over the entire support of $\tilde{X}^{(k,t)}$ unconditionally.

Proof. Note that if $\text{SAFE}^{(k,t)}$ no longer holds, the truncation process sets $\tilde{X}(k, t) = 0$. Otherwise, since by Lemma 5.3 we have $U_{\text{local}}^{(k)} \leq U_{S(k-1)} \leq O(1) \cdot U_L$, and therefore

\[
\left\| \tilde{X}^{(k,t)} \right\|_2 = \left\| U_{\text{local}}^{1/2} X^{(k,t)} U_{\text{local}}^{1/2} \right\|_2 \leq O(1) \left\| U_{\text{local}}^{1/2} X^{(k,t)} U_{\text{local}}^{1/2} \right\|_2 \leq O\left( \frac{1}{P} \right).
\]

Here the last condition follows from the rescaling factor of $1/P$, and the bounds on the error of the single vertex elimination algorithm given by Lemma 3.5 Part 3.

Lemma 5.5 implies the steps of $\tilde{M}^{(k,t)}$ have norm bounded by $O(1/P)$. Consequently, Lemma 5.4 yields that for $\sigma^2 = \Theta(\log(1/\delta))$ and $s = \epsilon^2$, with $P = \Theta(\epsilon^{-2} \log^2(1/\delta))$ and $\log(1/\delta) \geq \Omega(\log n)$, the probability Equation (13) is upper bounded by

\[
n \exp \left( \frac{-\epsilon^2}{100(\sigma^2 + \epsilon^2)} \right) \leq O(\delta).
\]

(15)

Consequently, all that remains is to bound (14) which we do with the following lemma.

Lemma 5.6. For $\sigma^2 = \Theta(\log(1/\delta))$ we have

\[
\mathbb{P} \left[ \exists (k, t) \text{ s.t. } \left\| \sum_{(k, t) \leq (k, t)} \mathbb{E} \left[ \tilde{X}^{(k,t)} \tilde{X}^{(k,t)^\top} + \tilde{X}^{(k,t)^\top} \tilde{X}^{(k,t)} \right] \right\|_2 > \sigma^2 \right] \leq O(\delta).
\]
Proof.

\[
\mathbb{P}
\left[
\exists (k,t) \text{ s.t.}
\left\lfloor \sum_{(k,w) \leq (k,t)} \mathbb{E}_{< (k,t)} \left[ \hat{X}^{\top}(k,w) \hat{X}^{\top} + \hat{X}^{\top}(k,w) \hat{X}^{\top} \right] \right\rfloor > \sigma^2
\right]
\leq \mathbb{P}
\left[
\exists (k,t) \text{ s.t.}
\left\lfloor \sum_{(k,w) \leq (k,t)} \mathbb{E}_{< (k,t)} \left[ \hat{X}^{\top}(k,w) \hat{X}^{\top} \right] \right\rfloor > \frac{\sigma^2}{2}
\right]
\leq \mathbb{P}
\left[
\exists (k,t) \text{ s.t.}
\left\lfloor \sum_{(k,w) \leq (k,t)} \mathbb{E}_{< (k,t)} \left[ \hat{X}^{\top}(k,w) \hat{X}^{\top} \right] \right\rfloor > \frac{\sigma^2}{2}
\right]
\]  

(16)

We will bound each of these two terms by $O(\delta)$, giving the desired result. Since the proofs for bounding each of the terms, (16) and (17) are essentially identical, we only bound (16).

When $\text{SAFE}^{(k,t)}$ does not hold $\hat{X}^{(k,t)} = 0$. However, when $\text{SAFE}^{(k,t)}$ holds, then by Lemma 5.3,

\[
U_{\text{local}(k)} \preceq U_{S(k-1)} \preceq O(1) U_L,
\]

in which case

\[
X^{(k,t)^\top} U_L^\dagger X^{(k,t)^\top} \preceq O(1) X^{(k,t)^\top} U_{\text{local}(k)}^\dagger X^{(k,t)^\top}.
\]

This implies that

\[
\hat{X}^{(k,t)^\top} \hat{X}^{(k,t)} = U_L^{1/2} X^{(k,t)^\top} U_L^{1/2} \hat{X}^{(k,t)} U_L^{1/2} X^{(k,t)^\top} U_L^{1/2} \preceq O(1) U_{\text{local}}^{1/2} U_{\text{local}}^{1/2} U_{\text{local}}^{1/2} U_{\text{local}}^{1/2}
\]

\[
\preceq \frac{O(1)}{P^2} U_L^{1/2} U_{\text{local}}^{1/2}
\]

where in the last line we used that $\|U_{\text{local}}^{1/2} X^{(k,t)^\top} U_{\text{local}}^{1/2}\|_2 = O(1/P)$ by Lemma 3.5. This bound holds unconditionally, so clearly we also have

\[
\mathbb{E}_{< (k,t)} \left[ \hat{X}^{(k,t)^\top} \hat{X}^{(k,t)} \right] \preceq \frac{O(1)}{P^2} U_L^{1/2} U_{\text{local}}^{1/2}
\]

Summing for a fixed $k$ over the $P$ samples made in one round of elimination, we get

\[
\sum_t \mathbb{E}_{< (k,t)} \left[ \hat{X}^{(k,t)^\top} \hat{X}^{(k,t)} \right] \leq \frac{O(1)}{P} U_L^{1/2} U_{\text{local}}^{1/2}
\]

We define $W_0 \overset{\text{def}}{=} 0$ and $W_k \overset{\text{def}}{=} \sum_{k \leq l} \sum_t \mathbb{E}_{< (k,t)} \left[ \hat{X}^{(k,t)^\top} \hat{X}^{(k,t)} \right]$. This gives $0 \preceq W_k - W_{k-1}$.

Starting from Lemma 5.5, direct algebraic manipulations then imply that if $\text{SAFE}^{(k,t)}$ holds, then

\[
\left\| \hat{X}^{(k,t)^\top} \hat{X}^{(k,t)} \right\|_2 \leq O\left(1/P^2\right),
\]

and when $\text{SAFE}^{(k,t)}$ does not hold, we get this bound trivially from $\hat{X}^{(k,t)} = 0$. Triangle inequality then gives

\[
\|W_k - W_{k-1}\| \leq O(P/P^2) = O(1/P)
\]

(18)
and if we let $\mathbb{E}_{<k}[\cdot]$ denote expectation of $W_k$ over the random choice of vertex to eliminate, conditional on all the random choices of the algorithm until the $k$th elimination, then by Equation (6),

$$
\mathbb{E}_{<k}[W_k - W_{k-1}] \preceq \frac{O(1)}{P(n-k)} \Pi.
$$

where lapid is the projection matrix orthogonal to the all ones vector. Summing over all $k \leq k$, and using $k \leq n/2$ gives

$$
\sum_{k \leq k} \mathbb{E}_{<k}[W_k - W_{k-1}] \preceq O\left(\frac{1}{P}\right) \Pi. \tag{19}
$$

We now construct a zero mean martingale which we will use to bound the probability in term (16), by an application of Lemma 5.4. Let

$$
V_k \overset{\text{def}}{=} W_k - W_{k-1} - \mathbb{E}_{<k}[W_k - W_{k-1}] = W_k - \mathbb{E}_{<k}[W_k].
$$

$V_k$ is zero-mean conditional on the random choices up to step $k$ and so $R_k = \sum_{j=1}^k V_j$ is a zero-mean martingale.

$$
R_k = \sum_{j=1}^k W_j - W_{j-1} - \mathbb{E}_{<j}[W_j - W_{j-1}] = W_k - \sum_{j=1}^k \mathbb{E}_{<j}[W_j - W_{j-1}].
$$

Let $H_k = \sum_{j\leq k} \mathbb{E}_{<j} V_j V_j^\top + V_j^\top V_j = \sum_{j\leq k} \mathbb{E}_{<j} 2V_j^2$. Note that unconditionally

$$
H_k = 2 \sum_{j\leq k} \mathbb{E}_{<j} V_j^2 = 2 \sum_{j\leq k} \mathbb{E}_{<j} \left(W_j - W_{j-1} - \mathbb{E}_{<j}[W_j - W_{j-1}]\right)^2 \preceq 2 \sum_{j\leq k} \mathbb{E}_{<j} (W_j - W_{j-1})^2
$$

$$
\preceq 2 \sum_{j\leq k} \mathbb{E}_{<j} \|W_j - W_{j-1}\| \preceq \frac{O(1)}{P^2} \Pi. \tag{20}
$$

Note, in the line above, we are using that $W_j - W_{j-1} \succeq 0$, i.e. this difference is PSD, and we used that for any symmetric matrix $A$, $\mathbb{E}(A - \mathbb{E}A)^2 = \mathbb{E}A^2 - (\mathbb{E}A)^2$, and when $A$ is PSD, we hence get $\mathbb{E}(A - \mathbb{E}A)^2 \preceq \mathbb{E}A^2$.

Let $\omega^2 = \frac{C}{P}$, for some absolute constant $C$ chosen such that $\Pr[\exists i: \lambda_{\max}(H_i) > \omega^2] = 0$ by Equation (20). Now, by Equation (19) we get

$$
\Pr[\exists k: \lambda_{\max}(W_k) > \sigma^2] = \Pr \left[ \exists k: \lambda_{\max} \left( R_k + \sum_{j=1}^k \mathbb{E}_{<j}[W_j - W_{j-1}] \right) > \sigma^2 \right]
$$

$$
\leq \Pr \left[ \exists k: \lambda_{\max}(R_k) > \sigma^2 - \frac{O(1)}{P} \right]
$$

$$
= \Pr \left[ \exists k: \lambda_{\max}(R_k) \geq \sigma^2 - \frac{O(1)}{P} \text{ and } \lambda_{\max}(H_k) \leq \omega^2 \right].
$$

28
We now want to apply Lemma 5.4. By Equation (18)

\[ \|V_k\| = \left\| W_k - W_{k-1} - \mathbb{E}_{<k} [W_k - W_{k-1}] \right\| \]

\[ \leq \max \left\{ \|W_k - W_{k-1}\|, \|\mathbb{E}_{<k} [W_k - W_{k-1}]\| \right\} \] (since both terms are PSD)

\[ \leq O(1) \frac{1}{P}, \]

which gives us a value for the norm control parameter \( R \). Thus by Lemma 5.4, and using \( \sigma^2 = \Theta(\log(1/\delta)) \), \( \log(1/\delta) \geq \Omega(\log n) \), and \( \omega^2 = O(1) \frac{1}{P^2} \), we get for an appropriate choice of constants that

\[ \Pr \left[ \exists i : \lambda_{\text{max}}(R_i) \geq \sigma^2 - O(1) \frac{1}{P} \text{ and } \lambda_{\text{max}}(H_i) \leq \omega^2 \right] \leq n \exp \left( -\frac{(\sigma^2 - O(1) \frac{1}{P})^2}{\sigma^2 + O(1) \frac{1}{P^2}} \right) \leq \delta. \]

This completes the bound on the probability term (16), and similarly, we can show the term (17) is bounded by \( \delta \).

We now how everything we need to prove Theorem 2.2.

Proof of Theorem 2.2. The running time guarantees we need were established in Lemma 5.1. Based on Lemma 5.3, we observed earlier that \( \Pr[-\text{SAFE}^{(n+1,1)}] \leq O(\delta) \) implies Equation (7). Our bounds on each of the terms (12), (13) (see Equation (15)) and (14) (see Lemma 5.6) establish this, hence proving the theorem.

\[ \square \]

6 Bounding Error Accumulations

In this section we study of the overall accumulation of errors resulting from the single phases, and prove Lemma 2.3. In particular, we’re interesting in the operator \( \mathbf{F} \) obtained by summing the undirectifications of the Eulerian Laplacians obtained at the end of each phase.

We first show the overall error accumulation from Part 1. For this we need to invoke Lemma B.2 of [4], which we state below for completeness.

Lemma 6.1. (Lemma B.2 of [4]) For all \( \mathbf{A} \in \mathbb{R}^{n \times n} \) and symmetric PSD \( \mathbf{M}, \mathbf{N} \in \mathbb{R}^{n \times n} \) such that \( \ker(\mathbf{M}) \subseteq \ker(\mathbf{A}^\top) \) and \( \ker(\mathbf{N}) \subseteq \ker(\mathbf{A}) \) we have

\[ \left\| \mathbf{M}^{-1/2} \mathbf{A} \mathbf{N}^{-1/2} \right\|_2 = \max_{x,y \neq 0} \frac{x^\top \mathbf{A} y}{\sqrt{(x^\top \mathbf{M} x)(y^\top \mathbf{N} y)}} = 2 \cdot \max_{x,y \neq 0} \frac{x^\top \mathbf{A} y}{x^\top \mathbf{M} x + y^\top \mathbf{N} y}. \]

where in each of the maximization problems we define 0/0 to be 0.

Proof of Lemma 2.3 Part 1. Recall that the assumption fo error per phase gives:

\[ \left\| \mathbf{U}^{\top 1/2}_{p} \left( \mathbf{M}_{(ip)} - \mathbf{M}_{(ip+1)} \right) \mathbf{U}^{\top 1/2}_{p} \right\|_2 \leq \theta_p, \]

for any \( 0 \leq p < p_{\text{max}} \).
By Lemma 6.1, this implies for every $p$
\[
2x \left( M^{(i_p)} - M^{(i_{p+1})} \right) y \leq \theta_p \epsilon \left( x^\top U_{S^{(i_p)}} x + y^\top U_{S^{(i_p)}} y \right).
\]
So summing over these gives
\[
2x \left( M^{(0)} - M^{(i)} \right) y \leq \epsilon \left( x^\top F x + y^\top F y \right).
\]
Again by Lemma 6.1, this gives
\[
\left\| F^{1/2} \left( M^{(i)} - M^{(0)} \right) F^{1/2} \right\| \leq \epsilon.
\]

We now turn our attention to the additional condition on $F$ outlined in Part 2. For this proof, we define a new matrix $\hat{F}$, which is made up of the various Schur complements of the final matrix $M^{(n)}$ onto the corresponding intermediate spaces.

Let $I_p = \{i_p \ldots n\}$ be the set of vertices remaining after first $p$ phases. Let $\hat{F} \overset{\text{def}}{=} \sum_{0 \leq p < \text{pmax}} \theta_p \cdot U_{Sc(M^{(n)},\{i_p \ldots n\})}$
\[\text{(21)}\]
where $\sum_p \theta_p = 1$ and $\theta_p \geq 1/O(\text{pmax}) = 1/O(\log n)$.

Rewriting $\hat{F}$ as differences between consecutive steps shows that it is in fact close to $F$.

**Lemma 6.2.** The matrices $F$ and $\hat{F}$ as defined in Equations 2 and 21 respectively satisfy:
\[
\hat{F} \approx_{O(\text{pmax})} F,
\]
where $\text{pmax} = O(\log n)$ is the number of invocations of SINGLEPHASE (Algorithm 2) by EULERIANLU (Algorithm 3).

**Proof.** The key observation is that because the Schur complement steps after step $i_p$ are completely contained among the vertices $\{i_p, \ldots n\}$, the difference between $F$ and $\hat{F}$ can be bounded using the discrepancies at the steps.

Formally, the choice of pivots means we have
\[
U_{Sc(M^{(n)},\{i_p \ldots n\})} = U_{S^{(i_p)}} + \sum_{p' \geq p} \left( U_{M^{(i_{p'})}} - U_{M^{(i_{p'}+1)}} \right),
\]
which when substituted into the formula for $\hat{F}$ gives:
\[
\hat{F} = \sum_{0 \leq p < \text{pmax}} \theta_p U_{S^{(i_p)}} + \sum_{0 \leq p < \text{pmax}} \sum_{p' \geq p} \theta_p \left( U_{M^{(i_{p'})}} - U_{M^{(i_{p'}+1)}} \right).
\]
Collecting the terms related to $F$, and reversing the summation on the $p'$s turns this into:
\[
\hat{F} = F + \sum_{p' \geq p} \left( \sum_{p \leq p'} \theta_p \right) \left( U_{M^{(i_{p'})}} - U_{M^{(n)}} \right).
\]

By triangle inequality we then get:

\[ \left\| F^{1/2} (\hat{F} - F) F^{1/2} \right\|_2 \leq \sum_{p' \leq p} \left( \sum_{p \leq p'} \theta_p \right) \left\| F^{1/2} \left( U_{M^{(p')}} - U_{M^{(n)}} \right) F^{1/2} \right\|_2 \]

Since \( \sum \theta_p = 1 \), the above is at most \( \epsilon \max \) provided the maximum error over any consecutive sequences of phases is \( \epsilon \), which happens \( 1 - O(\delta) \) by Part 1 of Lemma 2.3 shown above. Since the \( \epsilon \) argument to EULERIANLU (Algorithm 3) is required to be \( \leq 1 \), the desired result follows. \( \square \)

**Lemma 6.3.** Let \( M = M^{(0)} \) be a (possibly asymmetric) matrix, \( M^{(1)}, M^{(2)}, \ldots, M^{(n)} \) be the intermediate elimination states with errors defined in Lemma 2.3 using \( I_0 = V, \ldots, I_{p_{\max} - 1} \) that are nested subsets of indices, i.e., \( I_0 \subseteq I_{p+1} \subseteq \ldots \subseteq I_{p_{\max} - 1} \) and \( c_0, c_1, \ldots, c_p \) be constants. Then the matrix \( \hat{F} \) as defined above in Equation (21) and \( \hat{M} = M^{(n)} \) satisfy

\( \hat{F} \preceq \hat{M}^T \hat{F} \hat{M} \).

Throughout this section, we will frequently use the following definition, which allows us to extend the view of Schur complements as inverses of coordinate restrictions of the inverse of a matrix \( (SC (M, C) = (M^{-1})_{CC}) \) to the setting pseudo-inverses. Such characterization requires this definition of restricting pseudo-inverses to a subset of coordinates,

**Definition 6.4 (Projected coordinate restriction).** Consider any \( M \in \mathbb{R}^{n \times [n]} \), and \( C, F \) a partition of \([n]\), where \( M_{FF} \) is invertible. Let \( S = SC (M, C) \). We define the projected coordinate restriction of \( M^I \) to \( C \) as

\[ M^I [C] \overset{\text{def}}{=} P_S (M^I)_{CC} P_{S^T}. \]

Now, let \( Z \) be any matrix with \( Z + Z^T \succeq 0 \) and define

\[ C_Z \overset{\text{def}}{=} Z^T \left( \frac{Z + Z^T}{2} \right)^\dagger Z = \left( \frac{Z^\dagger + Z^{\dagger T}}{2} \right)^\dagger. \]

**Lemma 6.5.** If \( M \in \mathbb{R}^{n \times n} \) such that \( \ker (M) = \ker (M^\dagger) \), and \( I \) is a subset of indices, \( \bar{I} = [n] \setminus I \), such that \( M_{II} \), the principal minor of \( M \) on the indices outside of \( I \), is invertible, then

(i) \( C_{SC(M,I)} = SC (C_M, I) \),

(ii) \( U_{SC(M,I)} \preceq C_{SC(M,I)} \preceq C_M \).

**Proof.** To prove (i), we invoke the characterization of Schur complements as minors of inverses (which we formalize in Appendix C, specifically Lemma C.1) to write the Schur complement as \( SC (M, I) = (M^I [I])^\dagger \). It follows that

\[ C_{SC(M,I)} = \left( \frac{SC (M, I)^\dagger + SC (M, I)^{\dagger T}}{2} \right)^\dagger = \left( \frac{M^I [I] + M^{\dagger T} [I]}{2} \right)^\dagger = \left( C_{M[I]}^\dagger \right)^\dagger = SC (C_M, I), \]

where to see \( M^I [I] + M^{\dagger T} [I] = C_{M[I]}^\dagger \), we need the fact that left and right kernels of the involved matrices agree, from which it also follows that \( (C_M)_{II} \) is invertible (otherwise some nonzero vector
in \( \ker((C_M)_{I^T}) \) would also lie in \( \ker(M_{I^T}) \), and the latter matrix is invertible). This ensures \( C_M^t[I] \) is well-defined.

The first inequality in \( (ii) \) is an immediate consequence of Lemma B.2. For the second inequality in \( (ii) \), we note that the monotonicity property of the Schur complement stated in Lemma B.4 implies that \( Sc(C_M, I) \subseteq C_M \). Combining this with \( (i) \) yields the asserted bound. \( \square \)

**Lemma 6.6.** Let \( M \in \mathbb{R}^{n \times n} \) be a matrix such that \( \ker(M) = \ker(M^T) \), with subset of indices \( I \subseteq J \subseteq [n] \) such that the principal minors of \( M \) on \([n] \setminus I\) and \([n] \setminus J\) are both invertible. Then

\[
\left( M^t U_{Sc(M, I)} M^{t\top} \right) [J] \preceq U_{Sc(M, J)}^{t\top},
\]

and

\[
\left( M^t U_{Sc(M, J)} M^{t\top} \right) [I] \preceq U_{Sc(M, I)}^t.
\]

**Proof.** By Lemma C.2, \( Sc(Sc(M, J), I) = Sc(M, I) \). By Lemma 6.5 Part (ii) and the assumption that \( I \subseteq J \), \( U_{Sc(M, I)} \preceq C_{Sc(M, J)} \), so

\[
\left( M^t U_{Sc(M, I)} M^{t\top} \right) [J] \preceq \left( M^t C_{Sc(M, J)} M^{t\top} \right) [J]
\]

Since the kernels agree, and \( C_{Sc(M, J)} \) is supported on the submatrix with row and column indices in \( J \), we can replace \( M^t \) and \( M^{t\top} \) with their respective projected coordinate restrictions \( M^t[I] = Sc(M, J)^t \) and \( M^{t\top}[J] = Sc(M, J)^{t\top} \). We thus have

\[
\left( M^t U_{Sc(M, I)} M^{t\top} \right) [J] \preceq Sc(M, J)^t C_{Sc(M, J)} Sc(M, J)^{t\top}
\]

\[
= Sc(M, J)^t \left( Sc(M, J) U_{Sc(M, J)} Sc(M, J) \right) Sc(M, J)^{t\top}
\]

\[
= U_{Sc(M, J)}^t,
\]

which proves Equation (22).

To prove Equation (23), we can restrict \( M^t \) and \( M^{t\top} \) to indices in \( J \) as above to rewrite its left-hand side as

\[
\left( M^t U_{Sc(M, J)} M^{t\top} \right) [I] = \left( Sc(M, J)^t U_{Sc(M, J)} Sc(M, J)^{t\top} \right) [I] = C_{Sc(M, J)}^t[I] = C_{Sc(M, I)}^t.
\]

We have \( C_{Sc(M, I)} \geq U_{Sc(M, I)} \) by Lemma 6.5, so \( C_{Sc(M, I)}^t \preceq U_{Sc(M, I)}^t \), which yields Equation (23). \( \square \)

**Proof of Lemma 6.3.** Because \( \tilde{F} \) is a convex combination of the \( Sc(\tilde{S}, I_p) \) over the \( p \)s (see Equation (21)), it suffices to show

\[
U_{Sc(\tilde{S}, I_p)} \preceq \tilde{S}^t \tilde{F} \tilde{S}
\]

for each \( p \). Also, because of the (operator) monotonicity of Schur complements given in Lemma B.4, we can instead show the stronger condition:

\[
U_{Sc(\tilde{S}, I_p)} \preceq Sc(\tilde{S}^t \tilde{F} \tilde{S}, I_p) = \left( \left( \tilde{S}^t \tilde{F} \tilde{S} \right)^+ [I_p] \right)^t.
\]

32
Inverting both sides then reduces it to $U^\dagger_{Sc(\bar{S},I_p)} \succeq \left(\bar{S}^\dagger \hat{F} \bar{S}^\dagger\right)[I_p]$. Using the definition of $\hat{F}$ gives

$$\left(\bar{S}^\dagger \hat{F} \bar{S}^\dagger\right)[I_p] = \sum_{0 \leq p' < p_{\text{max}}} \theta_{p'} \left(\bar{S}^\dagger U_{Sc(\bar{S},I_{p'})} \bar{S}^\dagger\right)[I_p].$$

We now consider the terms separately and note that, by Lemma 6.6, $\left(\bar{S}^\dagger U_{Sc(\bar{S},I_{p'})} \bar{S}^\dagger\right)[I_p] \preceq U^\dagger_{Sc(\bar{S},I_p)}$ for every $p'$. Taking a convex combination of these inequalities thus completes the proof.

We can now conclude things formally.

**Proof of Lemma 2.3 Part 2.** By Lemma 6.3, $\hat{F} \preceq (L^{(n)})^\top \hat{F}^\dagger L^{(n)}$. By Lemma 6.2, we have with probability $1 - O(\delta)$ that $\hat{F} \approx O(\log(n)) F$. Thus, we have $1/O(\log^2 n) \cdot F \preceq (L^{(n)})^\top F^\dagger L^{(n)}$. □

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A Finding an $\alpha$-RCDD Block

Here provide a basic result showing we can find large $\alpha$-RCDD blocks of vertices efficiently. This results is a natural analog of a result in [20]. The main result of this section is the following theorem analyzing Algorithm 5.

**Theorem A.1.** Given a directed graph $G$, the function $\text{FindRCDDBlock}(G, \alpha)$ (Algorithm 5) outputs an $\alpha$-RCDD set of vertices of size at least $n/16(1+\alpha)$ in time $O(m \log(1/\delta))$ with probability at least $1 - O(\delta)$.

**Proof.** The proof of this theorem follows immediately from Lemma A.3 which in turn uses Lemma A.2 which are proved in the remainder of this section. \hfill \Box

**Algorithm 5: FindRCDDBlock($G, \alpha$)**

- **Input:** a directed graph $G$ and a parameter $\alpha$
- **Output:** an $\alpha$-RCDD set of vertices $F$ of size at least $n/16$

1. $F \leftarrow \emptyset$
2. while $|F| < \frac{n}{16(1+\alpha)}$ do
3. Let $F$ be $k = \frac{n}{8(1+\alpha)}$ vertices sampled uniformly at random.
4. Remove from $F$ each vertex that is not $\alpha$-RCDD with respect to $F$.

**Lemma A.2.** Let $L \in \mathbb{R}^{n \times n}$ be an Eulerian Laplacian and let $F \subseteq V$ be a random subset of size $k$. Then the expected number of $i \in F$ such that $\sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha}|L_{ii}|$ is at most $k^2(1+\alpha)/n$.

**Proof.** We have that for all $j \neq i$

$$
\Pr[j \notin F | i \in F] = \prod_{i \in [k-1]} \left(1 - \frac{1}{n-i}\right) = \prod_{i \in [k-1]} \left(\frac{n-i-1}{n-i}\right) = \frac{n-k}{n-1}
$$
and therefore $\Pr \left[ j \notin F \mid i \in F \right] = \frac{k-1}{n}$. Since $\sum_{j \in F, j \neq i} |L_{ij}| = L_{ii}$ we have that $E \left[ \sum_{j \in F, j \neq i} |L_{ij}| \right] = \left( \frac{k-1}{n-1} \right) L_{ii}$ and that by Markov’s inequality

$$\Pr \left[ \sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha} L_{ii} \mid i \in F \right] \leq \left( \frac{k-1}{n-1} \right) (1+\alpha)$$

and since

$$\Pr \left[ i \notin F \right] = \prod_{i \in [k]} \left( 1 - \frac{1}{n+1-i} \right) = \prod_{i \in [k]} \frac{n-i}{n+1-i} = \frac{n-k}{n} = 1 - \frac{k}{n}$$

we have that $\Pr \left[ i \notin F \right] = \frac{k}{n}$ and the expected number $i \in F$ such that $\sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha}$ is at most

$$\sum_{i \in [n]} \Pr \left[ i \in F, \sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha} \right] = \sum_{i \in [n]} \Pr \left[ i \in F \right] \Pr \left[ \sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha} L_{ii} \mid i \in F \right] \leq k \left( \frac{k-1}{n-1} \right) (1+\alpha)$$

Since $k \leq n$ we have $(k-1)/(n-1) \leq k/n$ and the result follows. □

**Lemma A.3.** Let $L \in \mathbb{R}^{n \times n}$ be an Eulerian Laplacian, let $F \subseteq V$ be a random subset of size $k$ and let $F' \subseteq V$ be the elements $i \in V$ for which $\sum_{j \in F, j \neq i} |L_{ij}| \leq \frac{1}{1+\alpha} |L_{ii}|$ and $\sum_{j \in F, j \neq i} |L_{ji}| \leq \frac{1}{1+\alpha} |L_{ii}|$ then with probability at least $1/2$ we have

$$|F'| \geq k \left[ 1 - \frac{4k}{(1+\alpha)n} \right]$$

and therefore for $k = \frac{n}{8(1+\alpha)}$, $L_{F',F'}$ is $\alpha$-RCDD with $|F'| \geq \frac{n}{16(1+\alpha)}$ with probability at least $1/2$.

**Proof.** Applying Lemma A.2 to $L$ and $L^\top$ we see that the expected number of elements $i \in F$ for which $\sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha} |L_{ii}|$ is at most $k^2 (1+\alpha)/n$ and the expected number of elements $i \in F$ for which $\sum_{j \in F, j \neq i} |L_{ij}| \geq \frac{1}{1+\alpha} |L_{ii}|$ is at most $k^2 (1+\alpha)/n$ consequently an expected $2k^2 (1+\alpha)/n$ are removed from $F$ to get $F'$. Consequently, by Markov’s inequality with probability at least $1/2$ at most $2k^2 (1+\alpha)/n$ are removed from $F$ to get $F'$ □

### B Matrix Facts

Here we provide some general matrix facts we use throughout the paper.

**Lemma B.1.** If $N$ is a square matrix with $U_N > 0$ then $N$ is invertible.

**Proof.** If $d \neq 0$ with $Nd = 0$ then $0 = d^\top Nd = d^\top U_N d$ contradicting $U_N > 0$. □

**Lemma B.2.** If $N$ is a matrix with $\ker(N) = \ker(N^\top) = \ker(U_N)$ and $U_N \succeq 0$ then $U_N \preceq N^\top U_N^\top N$.

37
Proof. This was previously shown with slightly different hypotheses in Equation 2.1 and Theorem 2.2 in [27] and Lemma 13 in [3], and in the current form in Lemma B.9 in [4]. For completeness, we include the proof from [4], which we reproduce here almost verbatim.

Let \( N = U + V \), where

\[
N^\top U^\dagger N = (U + V)^\top U^\dagger U + (U + V)^\top U^\dagger V + V^\top U^\dagger U + V^\top U^\dagger V.
\]

Our kernel assumptions imply that \( U^\top U^\dagger V = V \) and \( V^\top U^\dagger U = V^\top \), so we obtain

\[
N^\top U^\dagger N = U^\top U^\dagger U + V + V^\top + V^\top U^\dagger V = \ub + V^\top U^\dagger V \succeq \ub,
\]

where the final inequality used the assumption that \( \ub \succeq 0 \) to guarantee that \( V^\top U^\dagger V \succeq 0 \).

Lemma B.3. Let \( \tilde{L}, L, F \) be arbitrary matrices with \( \ker(\tilde{L}) = \ker(L^\top) = \ker(L) = \ker(L^\top) = \ker(F) = \ker(F^\top) \). If \( \|F^{+}/(L - \tilde{L})F^{+}/2\| \leq \epsilon \) and \( \gamma F \leq \tilde{L}^\top F^{+} \tilde{L} \), then \( L^\top F^{+} \approx O(\epsilon / \gamma + \epsilon^2 / \gamma) \cdot \tilde{L}^\top F^{+} \tilde{L} \).

Proof. We have

\[
\|F^{+}/(L - \tilde{L})F^{+}/2\| \leq \epsilon
\]

\[
\|Lx\|_{F^+} - \|\tilde{L}x\|_{F^+} \leq \epsilon \sqrt{\gamma} \cdot \|Lx\|_{F^+} \forall x
\]

\[
\|x^\top L^\top F^{+} Lx - x^\top \tilde{L}^\top F^{+} \tilde{L}x\| \leq O \left( \frac{\epsilon}{\sqrt{\gamma}} + \epsilon^2 \right) \cdot x^\top L^\top F^{+} \tilde{L}x \forall x,
\]

which is one definition of the desired condition.

Lemma B.4. For any positive semi-definite matrix \( P \succeq 0 \) and any subset of variables \( I \), we have

\[
\text{Sc}(P, I) \preceq P.
\]

Proof. This follows from the optimization definition of Schur complements:

\[
x^\top \text{Sc}(P, I) x = \min_{\tilde{x} : \tilde{x}_I = x_I} \tilde{x}^\top P \tilde{x}.
\]

A formal proof of this fact can be found in Lemma B.2 (proven in Appendix C) of [28].

Lemma B.5. Suppose \( a \) is a vector with positive entries and \( D \) is the diagonal matrix with \( a \) on the diagonal, and \( d = \mathbf{1}^\top a \), then \( U = D - \frac{1}{d} D a a^\top \) is satisfies

\[
U^\dagger \succeq D^{-1}.
\]

38
To prove this lemma, we will use a standard fact about pseudo-inverses:

**Fact B.6.** Suppose $A$ is a symmetric matrix and $X$ is a non-singular matrix, and that $P$ is the projection onto the image of $X^\top AX$. Then,

$$(X^\top AX)^\dagger = PX^{-1}A^\dagger(X^{-1})^\top P$$

**Proof of Lemma B.5.** Note that one can check that $U$ is in fact the undirected Laplacian of a weighted complete graph and so kernel $U$ is exactly the span of $1$, and $U$ is PSD. Let $P = I - 11^\top$ denote the projection onto the image of $U$. Let $v = D^{-1/2}1_{d+2}$, and $V = I - vv^\top$. Note that $U = D^{1/2}VD^{1/2}$ and $V^\dagger = V$. Hence

$$U^\dagger = (D^{1/2}VD^{1/2})^\dagger = PD^{-1/2}V^\dagger D^{-1/2}P = PD^{-1/2}VD^{-1/2}P \preceq PD^{-1}P \preceq D^{-1}.$$  

\[\square\]

### C Pseudo-Inverses and Schur Complements

In this appendix, we formally justify our view of Schur complements as taking inverses of minors of inverses in the setting of pseudo-inverses. Such characterization requires Definition 6.4 which defines the appropriate way to restrict pseudo-inverses to a subset of coordinates. Our main equivalence statement is:

**Lemma C.1.** Consider any $M \in \mathbb{R}^{[n] \times [n]}$, and $F, C$ a partition of $[n]$, where $M_{FF}$ is invertible. Then we have

$$\text{Sc}(M, C) = \left(M^\dagger [C]\right)^\dagger.$$  

Using this characterization, we can also show the next lemma, which tells us that a Schur complement onto a set can be computed by blockwise elimination, or by first eliminating a some variables and then eliminating more. This is a well-known result for invertible matrices, or symmetric matrices, but we extend it to the case of singular, asymmetric matrices.

**Lemma C.2.** Consider any $M \in \mathbb{R}^{[n] \times [n]}$, and $F, C$ a partition of $[n]$, where $M_{FF}$ is invertible. Let $F_1, F_2$ be a partition of $F$, and let $C_1 = [n] \setminus F_1$. Suppose $M_{F_1F_1}$ is invertible. Then we have

$$\text{Sc}(\text{Sc}(M, C_1), C) = \text{Sc}(M, C)$$

and

$$\left(M^\dagger[C_1]\right)[C] = \left(M^\dagger[C_1]\right).$$

### C.1 Pseudo-Inverse of a Product

Given a real matrix $M \in \mathbb{R}^{m \times n}$ with kernel $\ker(M)$, we let $P_M$ denote the orthogonal projection onto its columns pace $\ker(M)^\perp$, and $Q_M = I_{n \times n} - P_M$ denote the orthogonal projection onto $\ker(M)$. Recall that such orthogonal projections are symmetric matrices. Note also that $MP_M = M$, and so $MQ_M = 0$. Similarly, we can show $Q_M^\top M = 0$.

In this subsection, we prove the following helpful lemma that characterizes the pseudo-inverse of a product.
Lemma C.3. Consider real matrices $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times n}$, and $C \in \mathbb{R}^{n \times n}$, where $A$ and $C$ are invertible. Let $M = ABC$. Then $M^\dagger = P_M C^{-1} B^\dagger A^{-1} P_M^\top$.

Before proving this lemma we recall a standard fact about pseudo-inverses (e.g. see [9], 2nd ed. p. 453).

Fact C.4. The pseudo-inverse of $M$, denoted by $M^\dagger$, is the unique operator satisfying

1. $M = MM^\dagger M$.
2. $M^\dagger = M^\dagger MM^\dagger$.
3. $(M^\dagger M)^\top = M^\dagger M = P_M$.
4. $(MM^\dagger)^\top = MM^\dagger = P_M^\top$.

To prove Lemma C.3, we also need a simple observation about the projection operations related to $M$ and $B$.

Claim C.5.

1. $P_M C^{-1} Q_B = 0$.
2. $Q_B^\top A^{-1} P_M^\top = 0$.

Proof.

$P_M C^{-1} Q_B = M^\dagger M C^{-1} Q_B = M^\dagger A B C^{-1} Q_B = M^\dagger A B Q_B = M^\dagger A 0 = 0$.

and

$Q_B^\top A^{-1} P_M^\top = Q_B^\top A^{-1} A B C M^\dagger = Q_B^\top B C M^\dagger = 0 C M^\dagger = 0$.

Proof of Lemma C.3. Let $N = P_M C^{-1} B^\dagger A^{-1} P_M^\top$. We want to show $N = M^\dagger$. We prove this by verifying the four conditions of Fact C.4. First we verify Condition 1:

$N M N = M P_M C^{-1} B^\dagger A^{-1} P_M^\top M = M C^{-1} B^\dagger A^{-1} M = A B C^{-1} B^\dagger A^{-1} A B C = A B B^\dagger B C = M$.

Second, we similarly verify Condition 2:

$N M N = M P_M C^{-1} B^\dagger A^{-1} P_M^\top M = P_M C^{-1} B^\dagger A^{-1} P_M^\top = P_M C^{-1} B^\dagger A^{-1} P_M^\top = N$.

Third, we verify Condition 3. First, observe that

$N M = N M P_M = P_M C^{-1} P_B C P_M = P_M C^{-1} (P_B + Q_B) C P_M = P_M$.

where to obtain the last equality we used Claim C.5, Part 1. Hence $(N M)^\top = P_M = P_M = N M$, which establishes the condition. Condition 4 can be verified similarly.
C.2 Pseudo-Inverses and Schur Complements

We now utilize the above characterization of projections to prove the full characterization of Schur complements as pseudoinverses as stated in Lemma C.1. Throughout the rest of this section, we will use $C$ and $F$ to denote the partition of variables:

$$M = \begin{bmatrix} M_{FF} & M_{FC} \\ M_{CF} & M_{CC} \end{bmatrix}.$$  

and furthermore assume $M_{FF}$ is invertible. Note that this assumption allows us to invoke $M_{FF}^{-1}$, and can write the Schur complement as:

$$\text{Sc}(M, C) = M_{CC} - M_{CF}M_{FF}^{-1}M_{FC}.$$  

Lemma C.6. Under the assumptions at the start of this subsection,

$$\ker(M) = \left\{ \left( -M_{FF}^{-1}M_{FC}b \right) \mid \text{Sc}(M, C) b = 0 \right\}$$  

Proof. Consider a vector $[a; b]$ in the null space of $M$ where $a$ is on the $F$ coordinates and $b$ is on the $C$ coordinates. Invoking the block-wise characterization of $M$ gives:

$$M_{FF}a + M_{FC}b = 0,$$
$$M_{CF}a + M_{CC}b = 0.$$  

Since $M_{FF}$ is invertible, the first condition is equivalent to

$$a = -M_{FF}^{-1}M_{FC}b$$  

and substituting this, as well as the characterization of Schur complement, into the second condition gives

$$\text{Sc}(M, C) b = ( -M_{CF}M_{FF}^{-1}M_{FC} + M_{CC} ) b = M_{CF}a + M_{CC}b = 0.$$  

Lemma C.7. Consider the assumptions from the start of this subsection. For convenience of notation, let $S = \text{Sc}(M, C)$, and $0_{FF}$, $0_{FC}$ and $0_{CF}$ denote the $|F| \times |F|$, $|F| \times |C|$, and $|C| \times |F|$ all-zeros matrices respectively. Then

$$P_M \begin{bmatrix} 0_{FF} & 0_{FC} \\ 0_{CF} & P_S \end{bmatrix} = \begin{bmatrix} 0_{FF} & 0_{FC} \\ 0_{CF} & P_S \end{bmatrix}$$  

Proof. Recall that the column space of $M$ can also be characterized as the vectors $y$ orthogonal to the null space of $M$, or formally

$$v^\top y = 0 \quad \text{for all } v \text{ s.t. } Mv = 0.$$  

By Lemma C.6, such vectors $v$ can be written as

$$v = \left( -M_{FF}^{-1}M_{FC}b \right)$$  

41
where \( \text{Sc}(M, C)b = 0 \).

Now consider a vector \( x \) whose partition into coordinates in \( F \) and \( C \) we denote as \([x_F; x_C]\). We have
\[
PM \begin{bmatrix}
0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{bmatrix} \begin{bmatrix} x_F \\
x_C
\end{bmatrix} = PM \begin{bmatrix} 0 \\
P_S x_C
\end{bmatrix},
\]
while by definition of \( P_S \),
\[
b^\top P_S x_C = 0
\]
for any \( b \) in the null space of \( \text{Sc}(M, C) \). So for any \( v \) s.t. \( Mv = 0 \), we have
\[
v^\top \begin{bmatrix} 0 \\
P_S x_C
\end{bmatrix} = b^\top P_S x_C = 0.
\]

Thus \( PM \begin{bmatrix} 0 \\
P_S x_C
\end{bmatrix} = \begin{bmatrix} 0 \\
P_S x_C
\end{bmatrix} \).

This means for any \( x \),
\[
PM \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{bmatrix} x = PM \begin{bmatrix} 0 \\
P_S x_C
\end{bmatrix} = \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{bmatrix} x,
\]
and the claim follows.

\(\square\)

**Proof.** (of Lemma C.1) Let \( S = \text{Sc}(M, C) \). Recall the standard factorization
\[
M = \begin{bmatrix} I & 0 \\
M_{CF}M^{-1}_{FF} & I
\end{bmatrix}\begin{bmatrix} M_{FF} & 0 \\
0 & S
\end{bmatrix}\begin{bmatrix} I & M^{-1}_{FF}M_{FC} \\
0 & I
\end{bmatrix}.
\]

By Lemma C.3
\[
M^\dagger = PM \begin{bmatrix} I & M^{-1}_{FF}M_{FC} \\
0 & I
\end{bmatrix}^{-1}\begin{bmatrix} M^{-1}_{FF} & 0 \\
0 & S^\top
\end{bmatrix}\begin{bmatrix} I & 0 \\
M_{CF}M^{-1}_{FF} & I
\end{bmatrix}^{-1}P_M^\top.
\]

One can show (e.g. simple multiplication or by applying the formula for blockwise inversion) that
\[
\begin{bmatrix} I & 0 \\
B & I
\end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\
-B & I
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & B \\
0 & I
\end{bmatrix}^{-1} = \begin{bmatrix} I & -B \\
0 & I
\end{bmatrix}
\]

So
\[
M^\dagger = PM \begin{bmatrix} I & -M^{-1}_{FF}M_{FC} \\
0 & I
\end{bmatrix}\begin{bmatrix} M^{-1}_{FF} & 0 \\
0 & S^\top
\end{bmatrix}\begin{bmatrix} I \\
-M_{CF}M^{-1}_{FF} & I
\end{bmatrix}P_M^\top.
\]

Now, by applying Lemma C.7 to get rid of \( P_M \) and \( P_M^\top \), we get
\[
\begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{bmatrix} M^\dagger \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S^\top
\end{bmatrix}
= \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{bmatrix} \begin{bmatrix} I & -M^{-1}_{FF}M_{FC} \\
0 & I
\end{bmatrix}\begin{bmatrix} M^{-1}_{FF} & 0 \\
0 & S^\top
\end{bmatrix}\begin{bmatrix} I \\
-M_{CF}M^{-1}_{FF} & I
\end{bmatrix} \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S^\top
\end{bmatrix}
= \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & P_S S^\top P_S^\top
\end{bmatrix}
= \begin{bmatrix} 0_{FF} & 0_{FC} \\
0_{CF} & S^\top
\end{bmatrix}.
\]
where in the last equality, we used the fact that \( S^\dagger \) has the same kernel as \( S^T \), and \((S^\dagger)^T \) has the same kernel as \( S \) (which are in turn consequences of Fact C.4). But, directly computing the matrix product also tells us that

\[
\begin{pmatrix}
0_{FF} & 0_{FC} \\
0_{CF} & P_S
\end{pmatrix}
M^\dagger
\begin{pmatrix}
0_{FF} & 0_{FC} \\
0_{CF} & P_S^T
\end{pmatrix}
= \begin{pmatrix}
0_{FC} & 0_{FC} \\
0_{CF} & P_S(M^\dagger)CCP_S^T
\end{pmatrix}.
\] (26)

By comparing Equations (25) and (26), we arrive at the desired conclusion, after noting that by Definition 6.4, we have \( M^\dagger[C] = P_S(M^\dagger)CCP_S^T \), and \( A = B \) if and only if \( A^\dagger = B^\dagger \).

**Claim C.8.** Consider any \( M \in \mathbb{R}^{[n] \times [n]} \), and \( F, C \) a partition of \([n]\), where \( M_{FF} \) is invertible. Let \( F_1, F_2 \) be a partition of \( F \), and let \( C_1 = [n] \setminus F_1 \). Suppose \( M_{F_1F_1} \) is invertible. Then \( (\text{Sc} (M, C_1))_{F_2F_2} \) is invertible.

**Proof.** The key observation we need is that

\[ (\text{Sc} (M, C_1))_{F_2F_2} = \text{Sc} (M_{FF}, F_2). \]

This holds because

\[
(\text{Sc} (M, C_1))_{F_2F_2} = (M_{C_1C_1} - M_{C_1F_1}M_{F_1F_1}^{-1}M_{F_1C_1})_{F_2F_2} = M_{F_2F_2} - M_{F_2F_1}M_{F_1F_1}^{-1}M_{F_1F_2} = \text{Sc} (M_{FF}, F_2).
\]

By a standard factorization, we have

\[
M_{FF} = \begin{pmatrix}
I & 0 \\
M_{F_2F_1}M_{F_1F_1}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
M_{F_1F_1} & 0 \\
0 & \text{Sc}(M_{FF}, F_2)
\end{pmatrix}
\begin{pmatrix}
I & M_{F_1F_1}^{-1}M_{F_1F_2} \\
0 & I
\end{pmatrix},
\]

from which we conclude that unless both \( M_{F_2F_1} \) and \( \text{Sc}(M_{FF}, F_2) \) are full rank, \( M_{FF} \) is not full rank. As \( M_{FF} \) is invertible and hence full rank, we conclude \( \text{Sc}(M_{FF}, F_2) \) is full rank and hence invertible.

**Proof of Lemma C.2.** Note that by Lemma C.1,

\[ \text{Sc} (\text{Sc} (M, C_1), C) = \text{Sc} (M, C) \]

is equivalent to

\[ \left(M^\dagger[C_1]\right)[C] = \left(M^\dagger[C]\right), \]

so it suffices to show the latter. We assume that \( M_{FF} \) and \( M_{F_1F_1} \) are both invertible. Hence by Claim C.8, \( (\text{Sc} (M, C_1))_{F_2F_2} \) is invertible, which ensure the Schur complement \( \text{Sc} (\text{Sc} (M, C_1), C) \) is well-defined. Let \( S = \text{Sc} (M, C) \) and \( S_1 = \text{Sc} (M, C_1) \), and \( T = \text{Sc} (\text{Sc} (M, C_1), C) = \text{Sc} (S_1, C) \).

Next we observe that

\[
(M^\dagger[C_1])[C] = S_1^\dagger[C] = P_T(S_1^\dagger)CCP_T^T = P_T(PS_1(M^\dagger)_{C_1C_1}P_S^\dagger)CCP_T^T
= \begin{pmatrix}
0_{F_2F_2} & 0_{F_2C} \\
0_{CF_2} & P_T
\end{pmatrix}
PS_1(M^\dagger)_{C_1C_1}P_S^\dagger
\begin{pmatrix}
0_{F_2F_2} & 0_{F_2C} \\
0_{CF_2} & P_T^T
\end{pmatrix}
\]
But, by Lemma C.7,

\[ \mathbf{P}_{\mathbf{S}} \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \mathbf{P}_{\mathbf{T}} = \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \left( \mathbf{P}_{\mathbf{S}} \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \mathbf{P}_{\mathbf{T}} = \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \right) \]

So

\[(\mathbf{M}^{\dagger}[C_1])[C] = \left( \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \right) \left( \mathbf{M}^{\dagger} \right)_{C_1} \left( \begin{pmatrix} 0_{F_2 F_2} & 0_{F_2 C} \\ 0_{CF_2} & \mathbf{P}_{T}^{\top} \end{pmatrix} \right)_{CC} = \mathbf{P}_{T} \left( \mathbf{M}^{\dagger} \right)_{CC} \mathbf{P}_{T}^{\top} \]

We can show by Lemma C.6, that \( \mathbf{P}_{\mathbf{T}} = \mathbf{P}_{\mathbf{S}} \), because \( \ker(\mathbf{T}) = \ker(\mathbf{S}) \) as both kernel arise as coordinate restrictions of \( \ker(\mathbf{M}) \). Formally \( \ker(\mathbf{T}) = \{ \mathbf{b}_C : \mathbf{b} \in \ker(\mathbf{S}_1) \} \), and \( \ker(\mathbf{S}_1) = \{ \mathbf{b}_{C_1} : \mathbf{b} \in \ker(\mathbf{M}) \} \), so \( \ker(\mathbf{T}) = \{ \mathbf{b}_C : \mathbf{b} \in \ker(\mathbf{M}) \} = \ker(\mathbf{S}) \). Similarly, we get \( \mathbf{P}_{\mathbf{T}}^{\top} = \mathbf{P}_{\mathbf{S}}^{\top} \).

Thus \( (\mathbf{M}^{\dagger}[C_1])[C] = \mathbf{P}_{\mathbf{S}} (\mathbf{M}^{\dagger})_{CC} \mathbf{P}_{\mathbf{S}}^{\top} = \mathbf{M}^{\dagger}[C] \).

\(\square\)