EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR BERTRAND AND COURNOT MEAN FIELD GAMES

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Abstract. We study a system of partial differential equations used to describe Bertrand and Cournot competition among a continuum of producers of an exhaustible resource. By deriving new a priori estimates, we prove the existence of classical solutions under general assumptions on the data. Moreover, under an additional hypothesis we prove uniqueness.

Keywords: mean field games, Hamilton-Jacobi, Fokker-Planck, coupled systems, optimal control, nonlinear partial differential equations

MSC: 35K61

1. Introduction

Our purpose is to study the following system of partial differential equations:

\[
\begin{align*}
(i) & \quad u_t + \frac{1}{2} \sigma^2 u_{xx} - ru + H(t, u_x, [mu_x]) = 0, \quad 0 < t < T, \ 0 < x < L \\
(ii) & \quad m_t - \frac{1}{2} \sigma^2 m_{xx} - (G(t, u_x, [mu_x])m)_x = 0, \quad 0 < t < T, \ 0 < x < L \\
(iii) & \quad m(0, x) = m_0(x), \ u(T, x) = u_T(x), \quad 0 \leq x \leq L \\
(iv) & \quad u(t, 0) = m(t, 0) = 0, \ u_x(t, L) = 0, \quad 0 \leq t \leq T \\
v) & \quad \frac{1}{2} \sigma^2 m_x(t, L) + G(t, u_x(t, L), [mu_x])m(t, L) = 0, \quad 0 \leq t \leq T
\end{align*}
\]

where \( T > 0 \) and \( L > 0 \) are given constants, \( m_0 \) and \( u_T \) are known smooth functions, and \( H \) and \( G \) are defined below in Section 1.1. We mention for now that \( H \) and \( G \) depend on the variable \( mu_x \) in a nonlocal way, in particular they are functions of

\[
\int_0^L u_x(t, x)m(t, x)dx.
\]

System (1.1) was introduced by Chan and Sircar in [9] to represent a mean field game in which producers compete to sell an exhaustible resource. Here we view the producers as a continuum of rational agents whose “density” is given by the function \( m(t, x) \) governed by a Fokker-Planck equation. Each of them must solve an optimal control problem corresponding to the Hamilton-Jacobi-Bellman equation (1.1)(i). Further details will be given below in Section 1.1. Mean field games were introduced in [21][19] to describe differential games with large numbers of players represented by a continuum. Most recent results deal with models of the form

\[
\begin{align*}
u_t + \frac{1}{2} \sigma^2 u_{xx} - ru + H(t, x, u_x) = V[m] \\
m_t - \frac{1}{2} \sigma^2 m_{xx} - (G(t, x, u_x)m)_x = 0
\end{align*}
\]

where \( V[m] \) is a monotone function. There have been a number of existence and uniqueness theorems proved when \( V[m] \) depends on \( m \) both locally [23, 5, 4, 8, 7, 14, 15] and non-locally [6].

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More recently some progress has been made toward similar results for cases where the Hamiltonian depends on \(m\) in a nonlinear way \([13, 16, 17]\). However, none of these results address systems where the coupling happens in the nonlocal part of the Fokker-Planck equation.

Applications of mean field games to economics have attracted much recent interest; see \([1, 2, 12]\) for surveys of the topic. The model we study here, which comes from \([9]\), describes Bertrand or Cournot economic competition in the mean field limit (i.e. for a continuum of producers/consumers). It resembles the model proposed by Guéant, Lasry, and Lions to model oil production \([18]\), and it appears in a more complex and highly nonlinear form in \([10]\) to describe the response of traditional oil producers to new technological developments such as renewable energy and “fracking”. In \([10]\) the authors point out, “There does not exist anything like general existence and uniqueness theorems for PDE systems of this kind.” This statement has inspired the present work, in which we prove existence and uniqueness for (1.1).

We would like to mention in this context a recent result of Burger, et al. \([3]\) which provides an existence and uniqueness theorem for a mean field games model of knowledge growth introduced by Lucas and Moll \([22]\). Their model also involves a coupled Boltzmann/Hamilton-Jacobi system of equations in which the coupling occurs through an integral over the space variable. This structure is natural for applications to economics, since aggregate quantities such as market price or total production are expressed mathematically as averages with respect to the density of agents. For this reason it is desirable to develop techniques to analyze PDE systems of this type.

In this article we prove that under general conditions there exists a classical solution to (1.1), which under a certain restriction is also unique. By “classical solution” we mean that the equations in (1.1) hold pointwise. We consider only the case where \(\sigma > 0\) so that the equations are of parabolic type. Existence is obtained by applying the Leray-Schauder fixed point theorem; accordingly, the main effort of this paper is to provide \textit{a priori estimates} of solutions. A new feature of our analysis is the estimation of the nonlocal term \(\int_0^L u_x(t, x) m(t, x)\). Although traditional methods provide estimates of \(u\) in \(L^\infty\), it is not immediately clear how to obtain similar estimates for the gradient \(u_x\). In Section 2.3 we exploit the structure of (1.1) by directly computing the time derivative of the nonlocal term, careful analysis of which allows us to derive higher order regularity.

The remainder of this paper is organized as follows. In the rest of the introduction we give definitions of the functions \(H\) and \(G\) from (1.1), introduce some notation and give our main assumptions on the data. Section 2 is devoted to a priori estimates and constitutes the core of this paper. In Section 3 we prove the existence of solutions. Finally, in Section 4 we prove uniqueness under an additional hypothesis.

1.1. \textbf{Specification and explanation of the model.} We summarize the interpretation of (1.1) as follows. Let \(t\) be time and \(x\) be the producer’s capacity. We assume there is a large set of producers and represent it as a continuum. We say \(m(t, x)\) is the “density” of producers at time \(t\), so that

\[
\eta(t) := \int_0^L m(t, x) dx, \quad 0 \leq \eta(t) \leq 1
\]

represents the total mass of producers remaining with positive stock. Note that \(\eta(t)\) is a decreasing function in time.

The first equation in (1.1) is the Hamilton-Jacobi-Bellman (HJB) equation for the maximization of profit. Each producer’s capacity is driven by a stochastic differential equation

\[
dX(s) = -q(s, X(s)) ds + \sigma 1_{X(s)>0} dW(s),
\]
where \( q \) is determined by the price \( p \) through a linear demand schedule
\begin{equation}
q = D^{(\eta)}(p, \bar{p}) = a(\eta) - p + c(\eta)\bar{p}, \ \eta > 0.
\end{equation}

In \( (1.4) \) \( \bar{p} \) represents the market price, that is, the average price offered by all producers. This is given by
\begin{equation}
\bar{p}(t) = \frac{1}{\eta(t)} \int_0^L p^*(t, x)m(t, x)dx,
\end{equation}
where \( p^*(t, x) \) is the Nash equilibrium price. The coefficients \( a(\eta) \) and \( c(\eta) = 1 - a(\eta) \) are defined by
\begin{equation}
a(\eta) := \frac{1}{1 + \epsilon \eta}, \ c(\eta) := \frac{\epsilon \eta}{1 + \epsilon \eta}
\end{equation}
for a given fixed competition parameter \( \epsilon \geq 0 \). The case \( \epsilon = 0 \) corresponds to monopoly, while perfect competition is given by \( \epsilon = +\infty \). Thus each producer competes with all the others by responding to the market price.

We define the value function
\begin{equation}
\mu(t, x) := \sup_{p} E \left\{ \int_t^T e^{-r(s-t)} p(s) q(s) ds + u_T(X(T)) \mid X(t) = x \right\}
\end{equation}
where \( q(s) \) is given in terms of \( p(s) \) by \( (1.4) \). The optimization problem \( (1.7) \) has the corresponding Hamilton-Jacobi-Bellman equation
\begin{equation}
\frac{\partial \mu}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \mu}{\partial x^2} - ru + \max_{p} \left[ (a(\eta(t)) - p + c(\eta(t))\bar{p}(t))(p - u_x(t, x)) \right] = 0.
\end{equation}

The optimal \( p^*(t, x) \) satisfies the first order condition
\begin{equation}
p^*(t, x) = \frac{1}{2} (a(\eta(t)) + c(\eta(t))\bar{p}(t) + u_x(t, x))
\end{equation}
and we take \( q^*(t, x) \) to be the corresponding demand
\begin{equation}
q^*(t, x) = \frac{1}{2} (a(\eta(t)) + c(\eta(t))\bar{p}(t) - u_x(t, x)).
\end{equation}

This leads to Equation \( (1.11) \) where
\begin{equation}
H(t, u_x, [mu_x]) := q^*(t, x)^2 = \frac{1}{4} (a(\eta(t)) + c(\eta(t))\bar{p}(t) - u_x)^2
\end{equation}
On the other hand, the density of producers is driven by the Fokker-Planck equation \( (1.11) \) where
\begin{equation}
G(t, u_x, [mu_x]) := q^*(t, x) = \frac{1}{2} (a(\eta(t)) + c(\eta(t))\bar{p}(t) - u_x)
\end{equation}
The coupling takes place through the average price function, which, thanks to \( (1.5) \) and \( (1.9) \), is given by
\begin{equation}
\bar{p}(t) = \frac{1}{2\eta(t)} \left( a(\eta(t)) + \frac{1}{\gamma(t)} \int_0^L u_x(t, x)m(t, x)dx \right)
\end{equation}
We have taken Dirichlet boundary conditions at \( x = 0 \) as in \( [9] \). On the other hand, rather than taking \( L = +\infty \) and working on an unbounded domain, we have taken Neumann boundary conditions at \( x = L \), which represents a diffusion which is reflected at this boundary point. We can think of \( L \) as an upper limit on the capacity of any given producer.
1.2. Notation and assumptions. Throughout this article we define \( Q_T := (0, T) \times (0, L) \) to be the domain, \( S_T := ([0, T] \times \{0, L\}) \cup \{\{T\} \times [0, L]\} \) to be the parabolic boundary, and at times \( \Gamma_T := ([0, T] \times \{0\}) \cup \{\{T\} \times [0, L]\} \) to be the parabolic half-boundary. For any domain \( X \) in \( \mathbb{R} \) or \( \mathbb{R}^2 \) we define \( L^p(X), p \in [1, +\infty] \) to be the Lebesgue space of \( p \)-integrable functions on \( X \); \( C^0(X) \) to be the space of all continuous functions on \( X \); \( C^\alpha(X), 0 < \alpha < 1 \) to be the space of all Hölder functions with exponent \( \alpha \) on \( X \); and \( C^{n+\alpha}(X) \) to be the set of all functions whose \( n \) derivatives are all in \( C^\alpha(X) \). For a subset \( X \subseteq \overline{Q_T} \) we also define \( C^{1,2}(X) \) to be the space of all functions on \( X \) which are locally continuously differentiable in \( t \) and twice locally continuously differentiable in \( x \). By \( C^{\alpha/2,\alpha}(X) \) we denote the set of all functions which are locally Hölder continuous in time with exponent \( \alpha/2 \) and in space with exponent \( \alpha \).

We will denote by \( C \) a generic constant, which depends only on the data (namely \( u_T, m_0, L, T, \sigma, r \) and \( \epsilon \)). Its precise value may change from line to line.

Throughout we take the following assumptions on the data:

1. \( u_T(x) \) and \( m_0(x) \) are functions in \( C^{2+\gamma}([0, L]) \) for some \( \gamma > 0 \).
2. \( u_T \) and \( m_0 \) satisfy compatible boundary conditions: \( u_T(0) = u_T'(L) = 0 \) and \( m_0(0) = m_0(L) = 0 \).
3. \( m_0 \geq 0 \) and \( \int_0^L m_0(x)dx = 1 \), i.e. \( m_0 \) is a probability density.
4. \( u_T \geq 0 \) and \( u_T' \geq 0 \), i.e. \( u_T \) is non-negative and non-decreasing.

Remark 1.1. Of all the assumptions, the stipulation that \( u_T \) be non-negative and non-decreasing seems the least essential; it is not necessary for most estimates. However, it appears to be needed to prove the a priori bounds of Section 2.3.

2. A priori estimates

The goal of this section is to estimate various norms of solutions to \((1.1)\) using constants depending only on the data. In Section 2.1 we prove some standard results, including the usual “energy” type estimate on the quantity \( \int_0^T \int_0^L u_T^2 m \, dx \, dt \). Then in Section 2.2 we prove a priori bounds on the solution to the Hamilton-Jacobi equation using classical techniques for parabolic equations. Section 2.3 is our most original contribution; there we show that the term \( \int_0^L u_T(t, x)m(t, x)dx \) is a priori bounded uniformly in \( t \). Finally, in Section 2.4 we use the previous estimates to prove higher regularity.

2.1. Basic a priori estimates.

Proposition 2.1 (Main a priori estimates). Suppose \((u, m)\) is a pair of smooth functions satisfying \((1.1)\). Then

\[
(u(t, x) \geq 0, m(t, x) \geq 0, \|m(t)\|_{L^1(0, L)} \leq \|m_0\|_{L^1(0, L)} \forall t \in [0, T], \forall x \in [0, L],
\]

Moreover, for some \( C > 0 \) depending on the data, we have

\[
\int_0^T \int_0^L mu_x^2 \, dx \, dt \leq C, \tag{2.2}
\]

and

\[
\int_0^T \int_0^L m|G(t, u_x, [mu_x])|^2 \, dx \, dt = \int_0^T \int_0^L mH(t, u_x, [mu_x]) \, dx \, dt \leq C. \tag{2.3}
\]
Proof. We start by proving (2.1). Let \( \phi \) be any function in \( C^2(\mathbb{R}) \) with \( \phi'(0) = 0 \). Multiply (1.1)(ii) by \( \phi'(m) \) and integrate by parts to get
\[
(2.4) \quad \int_0^L \phi(m)(t,x)dx = -\frac{1}{2}\sigma^2 \int_0^L \int_0^L \phi''(m)m_x^2dxds + \int_0^t \int_0^L \phi''(m)m_xG_m dxds.
\]
Take \( \phi(s) = (s_-)^{2+\delta} \) where \( s_- := (|s| - s)/2 \) and let \( \delta \to 0 \) to deduce
\[
(2.5) \quad \int_0^L m_-(t,x)^2dx \leq \frac{1}{2}\sigma^2 \int_0^L \int_0^L \phi''(m)^2dxds + 2\int_0^t \int_0^L (m_-)_xG_m dxds
\]
where we note that \((m_0)_- = 0\). By Gronwall’s inequality we obtain \( m_-(t,x) \equiv 0 \), which proves positivity. On the other hand, if we take \( \phi(s) = s^{1+\delta} \) and let \( \delta \to 0 \), then we deduce \( \|m(t)\|_{L^1(0,L)} \leq \|m_0\|_{L^1(0,L)} \).

Also, since we have
\[
(2.6) \quad -u_t - \frac{\sigma^2}{2}u_{xx} + ru \geq 0,
\]
we can deduce using similar arguments that \( u \geq e^{-rT} \min_x u(T,x) \). Hence, in particular, we have \( u \geq 0 \) by the assumption \( u_T \geq 0 \). Thus we have proved (2.1).

Next we prove (2.2), from which follows (2.3). Multiply (1.1)(i) by \( m \) and (1.1)(ii) by \( u \) and integrate by parts to get
\[
(2.7) \quad \int_0^L u_T(x)m(T,x)dx - \int_0^L u(0,x)m_0(x)dx = r \int_0^T \int_0^L um dxdt - \int_0^T \int_0^L mH(t,u_x, [mu_x]) dxdt - \int_0^L \int_0^L mu_x G(t,u_x, [mu_x]) dxdt.
\]
Since \( u \geq 0 \) and \( m \geq 0 \), we get
\[
(2.8) \quad \int_0^L u_T(x)m(T,x)dx + \int_0^T \int_0^L m|mu_x G(t,u_x, [mu_x]) + H(t,u_x, [mu_x])| dxdt \geq 0,
\]
and then using the fact that \( \|m(t)\|_{L^1} \leq 1 \) we can rewrite this as
\[
(2.9) \quad \int_0^T \int_0^L mu_x^2 dxdt \leq \int_0^T \int_0^L m(a + c\tilde{p})^2 dxdt + 4\|u_T\|_{L^1} = \int_0^T \eta(t)(a + c\tilde{p})^2 dt + 4\|u_T\|_{L^1},
\]
since \( a + c\tilde{p} \) does not depend on \( x \).

To analyze the right-hand side, we observe that
\[
a(\eta(t)) + c(\eta(t))\tilde{p}(t) = \frac{2}{2 + \epsilon\eta(t)} + \frac{\epsilon}{2 + \epsilon\eta(t)} \int_0^L u_x(t,x)m(t,x)dx.
\]
By Cauchy-Schwartz we see that
\[
(2.10) \quad |a(\eta(t)) + c(\eta(t))\tilde{p}(t)| \leq 1 + \frac{\epsilon\eta^{1/2}(t)}{2 + \epsilon\eta(t)} \left( \int_0^L u_x^2(t,x)m(t,x)dx \right)^{1/2}.
\]
which implies, using the fact that $0 \leq \eta(t) \leq 1$,
\[
\eta(t)(a(\eta(t)) + c(\eta(t))\bar{p}(t))^2 \leq (1 + 1/\delta)\eta(t) + (1 + \delta)\frac{e^2\eta^2(t)}{(2 + e\eta(t))^2} \int_0^L u_x^2(t, x)m(t, x)dx \\
\leq 1 + 1/\delta + \frac{(1 + \delta)e}{2 + \epsilon} \int_0^L u_x^2(t, x)m(t, x)dx
\]
for an arbitrary $\delta > 0$. By choosing $\delta = 1/\epsilon$, then (2.9) becomes
\[
\int_0^T \int_0^L m^2_x \, dx dt \leq (2 + \epsilon)(1 + \epsilon)T + 4(2 + \epsilon)\|u_T\|_{\infty}
\]
which yields (2.2). As for (2.3), we combine (2.2) with (2.10) and the definition of $G$ and $H$. □

We may now deduce certain a priori bounds on the Fokker-Planck equation, which will be useful later on.

**Lemma 2.2** (Regularity of $m$). Suppose $(u, m)$ is a pair of smooth functions satisfying (1.1).
Then there exists a constant $C > 0$ depending on the data such that
\[(2.11) \quad \int_0^T \int_0^L \frac{m_x^2}{m + 1} \, dx dt \leq C.\]

**Proof.** Multiply (1.1)(ii) by $\ln(m + 1)$ and integrate by parts to get
\[(2.12) \quad \frac{d}{dt} \int_0^L \phi(m(t)) \, dx = -\frac{\sigma^2}{2} \int_0^L \frac{m_x^2}{1 + m} \, dx - \int_0^L Gm m_x \, dx \\
\leq -\frac{\sigma^2}{4} \int_0^L \frac{m_x^2}{1 + m} \, dx + \frac{1}{\sigma^2} \int_0^L G^2 m \, dx,
\]
where $\phi(m) = (1 + m)\ln(1 + m) - m$. By Equation (2.3) in Proposition 2.1 and using the fact that $m_0$ is bounded and $\phi(m) \geq 0$, we get
\[(2.13) \quad \frac{\sigma^2}{4} \int_0^T \int_0^L \frac{m_x^2}{1 + m} \, dx \leq \int_0^L \phi(m_0) \, dx + \frac{1}{\sigma^2} \int_0^T \int_0^L G^2 m \, dx dt \leq C.\]

### 2.2. A priori bounds for the Hamilton-Jacobi equation.
Let $f(t) := a(\eta(t)) + c(\eta(t))\bar{p}(t)$.
Then (1.1)(i) reads as
\[(2.14) \quad u_t + \frac{\sigma^2}{2} u_{xx} - ru + \frac{1}{4}(f(t) - u_x)^2 = 0,
\]
from which we can estimate
\[(2.15) \quad -u_t - \frac{\sigma^2}{2} u_{xx} + ru \leq \frac{1}{2} f(t)^2 + \frac{1}{2} u_x^2.
\]
From Proposition 2.1 we know that $f \in L^2(0, T)$ with an a priori bound on its norm. Using classical arguments, this is enough to infer an $L^\infty$ estimate on $u$ as well as an $L^2$ estimate on $u_x$, as the following proposition makes clear.
Lemma 2.3. Suppose $u$ is a smooth function on $[0, T] \times [0, L]$ satisfying

\begin{equation}
- u_t - ku_{xx} \leq g(t) + ju_x^2, \ u(T, x) = u_T
\end{equation}

where $j, k$ are positive constants, $g(t) \geq 0$ is an integrable function on $[0, T]$, and $u_T = u_T(x)$ is a smooth function on $[0, L]$. Assume that $u$ is bounded below, that $u(t, 0) = 0$, and $u_x(t, L) = 0$. Then there exists a constant $C = C(j, k, u_T, \|g\|_{L^1(0, T)})$ such that

\begin{equation}
\|u\|_\infty + \int_0^T \int_0^L u_x^2 \ dx \ dt \leq C.
\end{equation}

Proof. Let $w(t, x) = \exp \left\{ \lambda \left( u(t, x) + \int_0^t g(s) \ ds \right) \right\} - 1$ for $\lambda = j/k$. Then

\begin{equation}
- w_t - kw_{xx} \leq (j - k\lambda)\lambda u_x^2 = 0.
\end{equation}

In particular $w$ satisfies the maximum principle, i.e.

\[
\max_{[0, T] \times [0, L]} w = \max_{\Gamma_T} w
\]

where $\Gamma_T = ([0, T] \times \{0\}) \cup (\{T\} \times [0, L])$. To see this, it suffices to take $\mu = \max_{\Gamma_T} w$, then multiply \eqref{218} by $(w - \mu)_+$ and integrate by parts (note that $w_x(t, L) = 0$) to get

\[
\int_0^L (w(t, x) - \mu)_+^2 \ dx = -k \int_0^T \int_0^L (w - \mu)_+^2 \ dx \ dt + \int_0^L (w(T, x) - \mu)_+^2 \ dx \leq 0.
\]

It follows that $w \leq \mu$ everywhere, from which we deduce that $u \leq \frac{1}{\lambda} \ln(\mu)$. On the other hand, by the definition of $w$ we can directly compute

\[
\mu = \max_{x \in [0, L]} \exp \left\{ \lambda \left( u_T(x) + \int_0^T g(t) \ dt \right) \right\} - 1,
\]

which is a constant depending only on $\|u_T\|_\infty$ and $\|g\|_{L^1(0, T)}$.

Using the same computation with $w$ instead of $(w - \mu)_+$ we get

\[
\int_0^T \int_0^L w_x^2 \ dx \ dt \leq \frac{1}{k} \int_0^L w(T, x)_+^2 \ dx \leq \frac{u_T^2 L}{k}.
\]

Since $w_x(t, x) = \lambda \exp \left\{ \lambda \left( u(t, x) + \int_0^t g(s) \ ds \right) \right\} u_x(t, x)$ and $u$ is bounded below, we deduce

\[
\int_0^T \int_0^L u_x^2 \ dx \ dt \leq \frac{\mu^2 L}{\lambda^2 k} e^{2\lambda \|u\|_\infty}.
\]

$\square$

2.3. Analysis of nonlocal term. In order to obtain higher regularity on $u$, we need to analyze the nonlocal coupling term $\int_0^L u_x(t, x)m(t, x) \ dx$. In particular, we will show it is bounded.

In the case when $\sigma = 0$, we have a fortuitous identity which follows from integration by parts. Differentiate \eqref{111} to get

\[
u_{xt} - ru_x - \frac{1}{2} (a + c \bar{p} - u_x) u_{xx} = 0,
\]

noting that $\sigma = 0$. Then multiply by $m$ and \eqref{112} to get

\[
\frac{d}{dt} \int_0^L u_x(t, x)m(t, x) \ dx = r \int_0^L u_x(t, x)m(t, x) \ dx,
\]
which means
\[ \int_0^L u_x(t,x)m(t,x)dx = e^{r(T-t)} \int_0^L u_x(T,x)m(T,x)dx. \]

Thus, as long as \( u_T \) is smooth, we know that the term \( \int_0^L u_x(t,x)m(t,x)dx \) is bounded uniformly in \( t \). This in turn implies that \( \bar{p}(t) \) is bounded, which allows us to analyze the regularity of \( u \) by classical methods for parabolic equations.

Unfortunately, when \( \sigma > 0 \) this formal calculation fails; we get instead
\[
(2.19) \quad e^{rt} \frac{d}{dt} e^{-rt} \int_0^L u_x(t,x)m(t,x)dx = -\frac{\sigma^2}{2}u_x(t,0)m_x(t,0) - \frac{\sigma^2}{2}u_{xx}(t,L)m(t,L)
\]

with no estimates on the boundary terms. On the other hand, thanks to the following lemma, we note that each of the terms \( u_x(t,0)m_x(t,0) \) and \( u_{xx}(t,L)m(t,L) \) has a definite sign. This will allow us to prove that each of these terms is integrable in time with an a priori bound on its \( L^1(0,T) \) norm.

**Lemma 2.4.** Suppose \((u,m)\) is a pair of smooth functions satisfying (1.1). Then \( u_x(t,x) \geq 0 \), \( m_x(t,0) \geq 0 \), and \( u_{xx}(t,L) \leq 0 \) for all \((t,x) \in [0,T] \times [0,L]\).

**Proof.** Notice that, by Proposition 2.1, \( u \) and \( m \) are both non-negative, hence their minimum is attained at \( u(t,0) = 0 \) and \( m(t,0) = 0 \), respectively. It follows that \( u_x(t,0) \) and \( m_x(t,0) \) are non-negative for all \( t \in [0,T] \).

Differentiate (1.1) (i) to get
\[
(2.20) \quad u_{xt} + \frac{\sigma^2}{2}u_{xxx} - ru_x - \frac{1}{2}(a + c\bar{p} - u_x)u_{xx} = 0.
\]

Note that \( u_x(t,0) \) and \( 0 = u_x(t,L) \) are both non-negative. As \( u_T' \) is also non-negative, it follows that \( u_x \geq 0 \) everywhere by a classical maximum principle argument. Thus \( u_x(t,L) = 0 \) is a minimum of \( u_x \), so \( u_{xx}(t,L) \leq 0 \). \( \square \)

Returning to (2.19), we note that the terms on the right-hand side have definite but opposite signs. To show that each of them is integrable in time, we localize away from each boundary point. Once this is accomplished, we can see that the term \( \int_0^L u_x(t,x)m(t,x)dx \) is bounded. We prove this in the following:

**Proposition 2.5.** Suppose \((u,m)\) is a smooth solution of (1.1).

(i) For any \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that
\[
(2.21) \quad \left| \int_0^{L-\delta} u_x(t,x)m(t,x)dx \right| \leq C_\delta \quad \forall t \in [0,T].
\]

(ii) For any \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that
\[
(2.22) \quad \left| \int_\delta^L u_x(t,x)m(t,x)dx \right| \leq C_\delta \quad \forall t \in [0,T].
\]

(iii) By (i) and (ii), there exists a constant \( C \) depending only on the data such that
\[
(2.23) \quad \left| \int_0^L u_x(t,x)m(t,x)dx \right| \leq C \quad \forall t \in [0,T].
\]
Proof. Take a smooth, non-negative function $\zeta = \zeta(x)$ on $[0, L]$ to be further specified later. Multiply (2.20) by $\zeta m$ and integrate by parts using (1.1)(ii) to get

\begin{equation}
(2.24) \quad e^{rt} \frac{d}{dt} \left( e^{-rt} \int_0^L u_x(t)m(t)\zeta \, dx \right) = -\frac{\sigma^2}{2} \zeta(L)u_{xx}(t, L)m(t, L) - \frac{\sigma^2}{2} \zeta(0)u_x(t, 0)m_x(t, 0) - \frac{\sigma^2}{2} \int_0^L \zeta_xu_x \, dx - \frac{\sigma^2}{2} \int_0^L \zeta_{xx}u_{xx} \, dx - \frac{1}{2} (a + c\bar{p}) \int_0^L \zeta_xu_x \, dx + \frac{1}{2} \int_0^L \zeta_xu_x^2 \, dx.
\end{equation}

Let us estimate the time integral of each of the terms in the second line of (2.24). First we have

\begin{equation}
(2.25) \quad \int_0^T \sigma^2 \int_0^L \zeta_xu_x \, dx \, dt \leq \sigma^2 \|\zeta_x\|_\infty \int_0^T \int_0^L \left\{ u_x^2(m + 1) + \frac{m^2}{m + 1} \right\} dx dt \leq C \|\zeta_x\|_\infty.
\end{equation}

Using Proposition 2.1 and Lemma 2.3, Likewise,

\begin{equation}
(2.26) \quad \int_0^T \frac{\sigma^2}{2} \int_0^L \zeta_{xx}u_{xx} \, dx \, dx \, dt \leq \frac{\sigma^2}{2} \|\zeta_{xx}\|_\infty \int_0^T \int_0^L \left\{ (u_x^2 + 1)m \right\} dx dt \leq C \|\zeta_{xx}\|_\infty,
\end{equation}

and finally

\begin{equation}
(2.27) \quad \int_0^T \int_0^L \zeta_xu_x^2 \, dx \, dt \leq \|\zeta_x\|_\infty \int_0^T \int_0^L \left\{ (a + c\bar{p})^2 + \int_0^L u_x^2 \, dx \right\} dt \leq C \|\zeta_x\|_\infty.
\end{equation}

To summarize, we may write (2.24) as

\begin{equation}
(2.29) \quad e^{rt} \frac{d}{dt} \left( e^{-rt} \int_0^L u_x(t)m(t)\zeta \, dx \right) = -\frac{\sigma^2}{2} \zeta(L)u_{xx}(t, L)m(t, L) - \frac{\sigma^2}{2} \zeta(0)u_x(t, 0)m_x(t, 0) + I_\zeta(t),
\end{equation}

where

\begin{equation}
\int_0^T |I_\zeta(t)| \, dt \leq C(\zeta)
\end{equation}

such that $C(\zeta)$ depends only on $\|\zeta_x\|_\infty, \|\zeta_{xx}\|_\infty$, and previous estimates.

Now let us prove (i). We specify that $\zeta(x) = 0$ for $L - \delta/2 \leq x \leq L$, $\zeta(x) = 1$ for $0 \leq x \leq L - \delta$, and $0 \leq \zeta \leq 1$. Then we can assume $C(\zeta) \leq C_\delta$, where $C_\delta$ is some constant proportional to $1/\delta^2$ for $\delta > 0$ small. Integrating (2.29) over $[0, T]$ we get

\begin{equation}
(2.30) \quad \frac{\sigma^2}{2} \int_0^T e^{-rt}u_x(t, 0)m_x(t, 0) \, dt
\end{equation}

\begin{equation}
= \int_0^L u_x(0, x) \, m_0(x) \zeta \, dx - e^{-rT} \int_0^L u'_T(x)m(T, x)\zeta \, dx + \int_0^T e^{-rt}I_\zeta(t) \, dt.
\end{equation}

Now on the one hand, using the fact that $u_x \geq 0$ and that $u$ is bounded (Lemma 2.3), we have

\begin{equation}
(2.31) \quad \int_0^L |u_x(0, x)m_0(x)\zeta| \, dx = \int_0^L u_x(0, x)m_0(x)\zeta \, dx \leq \|m_0\|_\infty \int_0^L u_x(0, x) \, dx = \|m_0\|_\infty u(0, L) \leq C.
\end{equation}

On the other hand, since $\int_0^L m(t, x) \, dx \leq 1$ for all $t$, we have

\begin{equation}
(2.32) \quad \int_0^L |u'_T(x)m(T, x)\zeta| \, dx \leq \|u'_T\|_\infty.
\end{equation}
Using the fact that \( u_x \geq 0 \) and \( m_x(t, 0) \geq 0 \) from Lemma 2.4, we deduce that

\[
\int_0^T |u_x(t, 0)m_x(t, 0)|dt \leq e^{rT} \int_0^T e^{rt}u_x(t, 0)m_x(t, 0)dt \leq C_\delta.
\]

Finally, integrating \( 2.29 \) this time over \([t, T]\) we get

\[
\int_0^L u_x(t, x)m(t, x)\zeta \, dx = \frac{\sigma^2}{2} \int_t^T e^{-rt}u_x(t, 0)m_x(t, 0)dt + e^{-rt} \int_0^L u'_T(x)m(T, x)\zeta \, dx - \int_t^Te^{-rt}I_\zeta(t)dt
\]

from which we obtain \( 2.21 \).

In a similar way we can prove (ii). This time we specify that \( \zeta(x) = 0 \) for \( 0 \leq x \leq \delta/2, \zeta(x) = 1 \) for \( \delta \leq x \leq L \), and \( 0 \leq \zeta \leq 1 \). Again we can assume \( C(\zeta) \leq C_\delta \), where \( C_\delta \) is some constant proportional to \( 1/\delta^2 \) for \( \delta > 0 \) small. Integrating \( 2.29 \) over \([0, T]\) we get

\[
\int_0^T e^{-rt}u_{xx}(t, L)m(t, L)dt = -\int_0^L u_x(0, x)m_0(x)\zeta \, dx + e^{-rt} \int_0^L u'_T(x)m(T, x)\zeta \, dx - \int_t^Te^{-rt}I_\zeta(t)dt.
\]

Now since \( u_{xx}(t, L) \leq 0 \) from Lemma 2.4 and \( m \geq 0 \), we can deduce

\[
\int_0^T |u_{xx}(t, L)m(t, L)|dt \leq C_\delta
\]

using the same estimates as in the proof of part (i). Now integrating \( 2.29 \) over \([t, T]\) we get

\[
\int_0^L u_x(t, x)m(t, x)\zeta \, dx = \frac{\sigma^2}{2} \int_t^T e^{-rt}u_x(t, 0)m_x(t, 0)dt + e^{-rt} \int_0^L u'_T(x)m(T, x)\zeta \, dx - \int_t^Te^{-rt}I_\zeta(t)dt
\]

from which we infer \( 2.22 \).

Finally, \( 2.23 \) follows from \( 2.21 \) and \( 2.22 \) by fixing any \( \delta < L/2 \).

**Corollary 2.6.** Suppose \((u, m)\) is a smooth solution of \( 1.1 \). Then for some constant \( C \) depending only on the data,

\[
|a(\eta(t)) + c(\eta(t))\bar{p}(t)| \leq C \quad \forall t \in [0, T].
\]

**Proof.** By the definition of \( a, c, \) and \( \bar{p} \), it is enough to have \( |\int_0^T u_x(t, x)m(t, x)dx| \leq C \) for all \( t \in [0, T] \), which we get from Proposition 2.5. \( \square \)

### 2.4. Full regularity of \( u \)

Let us return to \( 2.13 \):

\[
u_t + \frac{\sigma^2}{2}u_{xx} - ru + \frac{1}{4}(f(t) - u_x)^2 = 0,
\]

where we recall \( f(t) := a(\eta(t)) + c(\eta(t))\bar{p}(t) \). We can now write

\[
-u_t - \frac{\sigma^2}{2}u_{xx} + ru \leq C_1 + \frac{1}{2}u_x^2,
\]
where $C_1$ is the constant coming from Corollary 2.6. It is now possible to obtain global estimates on $|u_x|$, which we will then be able to “bootstrap” to gain higher regularity on $u$.

**Lemma 2.7.** For $(u, m)$ a smooth solution of (1.1), we have
\begin{equation}
|u_x(t, x)| \leq C \quad \forall (t, x) \in [0, T] \times [0, L].
\end{equation}
for some constant $C$ depending only on the data.

**Proof.** By Lemma 2.4 we already know $u_x \geq 0$. It suffices to prove that $u_x \leq C$. We claim
\begin{equation}
|u_x(t, 0)| \leq C.
\end{equation}
To see this, set $v = e^{u/\sigma^2} - 1$ and use (2.39) to get
\begin{equation}
-v_t - \frac{\sigma^2}{2} v_{xx} \leq C_1 \frac{1}{\sigma^2} e^{\|u\|_{\infty}/\sigma^2} =: \tilde{C}_1.
\end{equation}
Note that $v(t, 0) = v_x(t, L) = 0$. Set $\tilde{v} = v + Me^{-x}$ where $M$ is large enough that
\begin{equation}
v_x(T, x) = \frac{1}{\sigma^2} e^{uT(x)/\sigma^2} u_T(x) \leq Me^{-L}
\end{equation}
and
\begin{equation}
\tilde{C}_1 \leq \frac{\sigma^2}{2} Me^{-L}.
\end{equation}
Then, on the one hand, we have $-\tilde{v}_t - \frac{\sigma^2}{2} \tilde{v}_{xx} \leq 0$, and since $\tilde{v}_x(t, L) = -Me^{-L} \leq 0$ we get as before the maximum principle
\begin{equation}
\max_{[0,T] \times [0,L]} \tilde{v} \leq \max \tilde{v}.
\end{equation}
On the other hand, we have $\tilde{v}_x(T, x) \leq 0$ and so $\tilde{v}(T, x) \leq \tilde{v}(T, 0) = M$ for all $x \in [0, L]$. It follows that $\tilde{v}(t, 0) = M$ is the global maximum of $\tilde{v}$, hence $\tilde{v}_x(t, 0) \leq 0$ at each $t \in [0, T]$. Recalling the definition of $\tilde{v}$ we get
\begin{equation}
\frac{1}{\sigma^2} e^{u/\sigma^2} u_x(t, 0) \leq M,
\end{equation}
and since $u \geq 0$ we get $u_x(t, 0) \leq M\sigma^2$, which is the desired estimate.

Taking into account the assumption that $u_T$ is smooth, we have thus shown
\begin{equation}
\max_{\Gamma_T} |u_x| \leq C, \quad \Gamma_T := ([0, T] \times [0, L]) \cup ([T] \times [0, L])
\end{equation}
where $C$ depends only on the data.

Now differentiate (2.14) to get
\begin{equation}
- u_{xt} - \frac{\sigma^2}{2} u_{xxx} + ru_x + \frac{1}{2}(f - u_x)u_{xx} = 0.
\end{equation}
Then $w(t, x) = u_x(t, x)e^{-rt}$ satisfies
\begin{equation}
-w_t - \frac{\sigma^2}{2} w_{xx} + \frac{1}{2}(f - u_x)w_x = 0.
\end{equation}
By classical arguments, $w$ satisfies the maximum principle, i.e.
\begin{equation}
|w(t, x)| \leq \max_{\Gamma_T} |w| \leq \max_{\Gamma_T} |u_x|,
\end{equation}
from which it follows that
\begin{equation}
|u_x(t, x)| \leq e^{rT} \max_{\Gamma_T} |u_x| \leq C.
\end{equation}
Corollary 2.7 permits us to obtain higher order estimates for $u$.

**Proposition 2.8.** There exists a constant $C$ depending only on the data such that if $(u, m)$ is a smooth solution of (1.1), then for some $\alpha > 0$

$$
\|u\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} + \|m\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \leq C
$$

where $C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ is the parabolic Hölder space defined on $\overline{Q_T} = [0, T] \times [0, L]$.

**Proof.** Observe that $u$ and $u_x$ each satisfy a linear parabolic equation with coefficients which are bounded by constants depending on the data. By [20, Theorems IV.5.2 and IV.5.3] now gives the conclusion. Applying [20, Theorem IV.9.1], we have that $u$ is bounded in $L^p(0, T; W^{2,p}(0, L)) \cap W^{1,p}(\overline{Q_T})$ for any $p > 1$ and thus $u, u_x$ and $u_{xx}$ are all bounded in a Hölder space $C^\beta(\overline{Q_T})$ for some $\beta > 0$. Then by [20, Theorem III.7.1] $m$ has an a priori bound in $L^\infty(\overline{Q_T})$. Further, we observe that (1.1) (ii) can be written

$$
m_t - \frac{\sigma^2}{2} m_{xx} - \frac{1}{2} (a + c\varphi - u_x) m_x + \frac{1}{2} u_{xx} m = 0,
$$

which also has coefficients bounded by the data. Using the same technique as in Lemma 2.7 we obtain an a priori estimate on $m_x(0, x)$. Then (2.19) can be used to directly estimate the term \( \int_0^L u_x(t, x) m(t, x) dx \) in $C^1([0, T])$. We can now see that (1.1) (i), (ii) are both parabolic equations with coefficients estimated in Hölder spaces by constants depending only on the data. Applying [20, Theorems IV.5.2 and IV.5.3] now gives the conclusion.\; \square

### 3. Existence

We now prove the main result of this paper.

**Theorem 3.1.** There exists a classical solution of (1.1).

**Proof.** We use the Leray-Schauder fixed point theorem. Consider the map $(u, m) \mapsto (v, f) = T(u, m; \tau)$ given by solving the following parametrized set of PDE systems:

$$
\begin{align*}
(i) & \quad v_t + \frac{1}{\alpha} \sigma^2 v_{xx} - rv + \tau H(t, u_x, [mu_x]) = 0, & 0 < t < T, 0 < x < L \\
(ii) & \quad f_t - \frac{1}{\alpha} \sigma^2 f_{xx} - \tau (G(t, v_x, [mv_x])f) = 0, & 0 < t < T, 0 < x < L \\
(iii) & \quad f(0, x) = \tau m_0(x), v(T, x) = \tau u_T(x), & 0 \leq x \leq L \\
(iv) & \quad v(t, 0) = f(t, 0) = 0, v_x(t, L) = 0, & 0 \leq t \leq T \\
(v) & \quad \frac{1}{\alpha} \sigma^2 f_x(t, L) + \tau G(t, v_x(t, L), [mv_x])f(t, L) = 0, & 0 \leq t \leq T
\end{align*}
$$

for $\tau \in [0, 1]$. Let $X$ be the space of all $(u, m)$ such that $u$ and $u_x$ are both Hölder continuous, say in $C^\beta(\overline{Q_T})$, and $m$ is in $W^{1,\infty}(0, T; L^1(0, L))$. Then by inspecting the definitions of $G$ (1.12) and $H$ (1.11) we find that $H(t, u_x, [mu_x])$ and $G(t, u_x, [mu_x])$ are both Hölder continuous as well. By [20, Theorem IV.5.2] there is a solution $v$ of (3.1) (i) satisfying (iii) and (iv) such that $v \in C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ for some $\alpha > 0$ and

$$
\|v\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \leq C \|(u, m)\|_X = C(\|u\|_{C^{\alpha/2,\alpha}(\overline{Q_T})} + \|u_x\|_{C^{\alpha/2,\alpha}(\overline{Q_T})} + \|m\|_{W^{1,\infty}(0, T; L^1(0, L))})
$$

for some generic constant $C$. Next, we write (3.1) (ii) as

$$
f_t - \frac{1}{2} \sigma^2 f_{xx} - \tau G(t, v_x, [mv_x])f + \frac{\tau}{2} v_{xx} f = 0,
$$

and we note that the coefficients are Hölder continuous. Furthermore, because $m \in W^{1,\infty}(0, T; L^1(0, L))$ we can see that

$$
G(t, v_x(t, L), [mv_x]) = \frac{2}{2 + e\eta(t)} + \frac{e}{2 + e\eta(t)} \int_0^L v_x(t, y)m(t, y)dy
$$
is independent of $x$ and has a bounded time derivative. Then we can apply [20] Theorem IV.5.3 to get a solution $f \in C^{1+\alpha/2,2+\alpha}(Q_T)$ satisfying (3.1) (iii) and (v) such that

$$
\|f\|_{C^{1+\alpha/2,2+\alpha}(Q_T)} \leq C\|\|v\|_{C^{1+\alpha/2,2+\alpha}(Q_T)} + \|m\|_{W^{1,\infty}(0,T;L^1(0,L))} \leq C\|(u,m)\|_X.
$$

It follows that $T : X \times [0,1] \to X$ is a well-defined and compact mapping, since $C^{1+\alpha/2,2+\alpha}(Q_T) \times C^{1+\alpha/2,2+\alpha}(Q_T)$ is compact in $X$ by the Arzelà-Ascoli Theorem.

Now for $\tau = 0$ we have $T(u,m;0) = 0$ for all $(u,m)$ by standard theory for the linear heat equation. On the other hand, if $(u,m) \in X, \tau \in [0,1]$ is such that $(u,m) = T(u,m;\tau)$ then $(u,m) \in C^{1+\alpha/2,2+\alpha}(Q_T) \times C^{1+\alpha/2,2+\alpha}(Q_T)$ is a solution of (1.1) with $m_0,u_T, G$ and $H$ replaced by $\tau m_0, \tau u_T, \tau G$ and $\tau H$, respectively. Then the a priori estimates of Section 2 carry through uniformly in $\tau \in [0,1]$, and so by Proposition 2.8 we obtain a constant $C_0$ depending only on the data such that

$$
\|(u,m)\|_X \leq C_0.
$$

By the Leray-Schauder fixed point theorem (see e.g. [11] Theorem 10.6), there exists $(u,m)$ such that $(u,m) = T((u,m);1)$, which is a classical solution to (1.1).

4. Uniqueness

The structure of (1.1) makes uniqueness a nontrivial issue. Unlike traditional mean field games in which uniqueness is verified by a straightforward use of the “energy identity” (thanks to the fact that the coupling is monotone [21]), System (1.1) does not allow such an argument. Our uniqueness result will rely heavily on the fact that solutions are smooth with a priori bounds, and it will hold only when $\epsilon$ is small.

We now proceed to state and prove the main uniqueness result.

**Theorem 4.1.** There exists $\epsilon_0 > 0$ sufficiently small such that for any $\epsilon \leq \epsilon_0$, (1.1) has at most one classical solution.

**Proof.** Let $(u_1,m_1)$ and $(u_2,m_2)$ be two solutions. Define, for $i = 1,2$,

$$
H_i = \frac{1}{4}(a(\eta_i(t)) + c(\eta_i(t))\bar{p}_i(t) - \partial_x u_i(t,x))^2, \quad G_i = \frac{1}{2}(a(\eta_i(t)) + c(\eta_i(t))\bar{p}_i(t) - \partial_x u_i(t,x))
$$

where $\eta_i(t)$ and $\bar{p}_i(t)$ are defined according to the definitions in (1.2) and (1.3), with $u$ and $m$ replaced by $u_i$ and $m_i$. Then, in particular, $u = u_1 - u_2$ satisfies

$$
u_t + \frac{\sigma^2}{2}u_{xx} - ru + H_1 - H_2 = 0, \quad u(T,\cdot) \equiv 0
$$

while $m = m_1 - m_2$ satisfies

$$
m_t - \frac{\sigma^2}{2}m_{xx} - (G_1m_1 - G_2m_2)_x = 0, \quad m(0,\cdot) \equiv 0.
$$

Let us introduce some notation. Observe that

$$
a(\eta_i(t)) + c(\eta_i(t))\bar{p}_i(t) = \frac{2}{2 + \epsilon\eta_i(t)} + \frac{\epsilon}{2 + \epsilon\eta_i(t)} \int_0^L \partial_x u_i(t,x)m_i(t,x)dx.
$$

With this in mind, we define

$$
A_i(t) = \frac{2}{2 + \epsilon\eta_i(t)}, \quad B_i(t) = \frac{\epsilon}{2 + \epsilon\eta_i(t)} \int_0^L \partial_x u_i(t,x)m_i(t,x)dx.
$$

Notice that $2G_i = A_i + B_i - \partial_x u_i$ and $H_i = G_i^2$. 

\[13\]
Now multiply (4.1) by $m_1 - m_2$ and (4.2) by $u_1 - u_2$, integrate by parts and add to get the typical energy identity for mean field games:

\[(4.3) \quad \int_0^T \int_0^L e^{-rt}[(H_1 - H_2)(m_1 - m_2) + (G_1m_1 - G_2m_2)(\partial_x u_1 - \partial_x u_2)]dxdt = 0,\]

which can be rearranged to get

\[(4.4) \quad \int_0^T \int_0^L e^{-rt}m_1[H_1 - H_2 + (\partial_x u_1 - \partial_x u_2)G_1]dxdt + \int_0^T \int_0^L e^{-rt}m_2[H_2 - H_1 + (\partial_x u_2 - \partial_x u_1)G_2]dxdt = 0.\]

This, in turn, can be rearranged to give

\[(4.5) \quad - \int_0^T e^{-rt}(A_1(t)^2 - A_2(t)^2)(\eta_1(t) - \eta_2(t))dt + \int_0^T \int_0^L e^{-rt}(m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2dxdt = I_1 + I_2\]

where

\[(4.6) \quad I_1 := \int_0^T e^{-rt}(\eta_1(t) - \eta_2(t))\{B_1(t)^2 - B_2(t)^2 + 2A_1(t)B_1(t) - 2A_2(t)B_2(t)\}dt\]

and

\[(4.7) \quad I_2 := 2\int_0^T \int_0^L e^{-rt}(m_2\partial_x u_1 - m_1\partial_x u_2)(A_1(t) + B_1(t) - A_2(t) - B_2(t))dxdt.\]

By a simple computation we have

\[(4.8) \quad - \int_0^T e^{-rt}(A_1(t)^2 - A_2(t)^2)(\eta_1(t) - \eta_2(t))dt \geq \frac{8\epsilon}{(1 + \epsilon)^3} \int_0^T e^{-rt}(\eta_1(t) - \eta_2(t))^2dt\]

so we can write

\[(4.9) \quad 8\epsilon \int_0^T e^{-rt}(\eta_1(t) - \eta_2(t))^2dt + \int_0^T \int_0^L e^{-rt}(m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2dxdt \leq I_1 + I_2.\]

Our main task is to estimate $I_1$ and $I_2$. For this we will use the a priori estimates from Section 2 which say in particular that $|\partial_x u_i| \leq C_0$ for some $C_0$ depending only on the data and on $\epsilon_0$, where $\epsilon \leq \epsilon_0$. Since we are going to make $\epsilon$ small, we need to keep in mind that the constant $C_0$ does not change for decreasing values of $\epsilon$.

First, we estimate $|B_1(t) - B_2(t)|$. We have

\[(4.10) \quad B_1(t) - B_2(t) = \frac{\epsilon}{2 + \epsilon\eta_1(t)} \int_0^L \partial_x u_1(t, x)m_1(t, x)dx - \frac{\epsilon}{2 + \epsilon\eta_2(t)} \int_0^L \partial_x u_2(t, x)m_2(t, x)dx \]

\[= \left(\frac{\epsilon}{2 + \epsilon\eta_1(t)} - \frac{\epsilon}{2 + \epsilon\eta_2(t)}\right) \int_0^L \partial_x u_1(t, x)m_1(t, x)dx + \frac{\epsilon}{2(2 + \epsilon\eta_2(t))} \int_0^L \partial_x(u_1(t, x) - u_2(t, x))(m_1(t, x) + m_2(t, x))dx \]

\[+ \frac{\epsilon}{2(2 + \epsilon\eta_2(t))} \int_0^L (\partial_x u_1(t, x) + \partial_x u_2(t, x))(m_1(t, x) - m_2(t, x))dx.\]
Observe that

\[
\frac{1}{2 + \epsilon \eta_1(t)} - \frac{1}{2 + \epsilon \eta_2(t)} \leq \frac{\epsilon |\eta_1(t) - \eta_2(t)|}{4},
\]

\[
\int_0^L |m_1(t, x) - m_2(t, x)|dx \leq L^{1/2} \left( \int_0^L |m_1(t, x) - m_2(t, x)|^2dx \right)^{1/2}, \quad \text{and}
\]

\[
\int_0^L (\partial_x u_1 + \partial_x u_2)(m_1 - m_2)dx, \leq \sqrt{2} \left( \int_0^L (\partial_x u_1 + \partial_x u_2)^2(m_1 - m_2)dx \right)^{1/2}
\]

by the Cauchy-Schwartz inequality. So using the uniform pointwise bounds on \(u_x(t, x)\), we obtain

\[
(4.11) \quad |B_1(t) - B_2(t)| \leq \frac{\epsilon}{2\sqrt{2}} \left( \int_0^L (\partial_x u_1 - \partial_x u_2)^2(m_1 + m_2) \, dx \right)^{1/2}
+ \frac{\epsilon^2}{4} |\eta_1(t) - \eta_2(t)| + \frac{\epsilon}{4} C_0 L^{1/2} \left( \int_0^L |m_1 - m_2|^2dx \right)^{1/2}.
\]

On the other hand, we have \(|B_i(t)| \leq C_0 \epsilon/2 \) for \(i = 1, 2\). We deduce from (4.11) that

\[
(4.12) \quad |B_1(t)^2 - B_2(t)^2| \leq \frac{\epsilon^2}{2\sqrt{2}} C_0 \left( \int_0^L (\partial_x u_1 - \partial_x u_2)^2(m_1 + m_2) \, dx \right)^{1/2}
+ \frac{\epsilon}{4} C_0 |\eta_1(t) - \eta_2(t)| + \frac{\epsilon^2}{4} C_0^2 L^{1/2} \left( \int_0^L |m_1 - m_2|^2dx \right)^{1/2}.
\]

We also have

\[
(4.13) \quad |A_1(t) - A_2(t)| \leq \frac{\epsilon}{2} |\eta_2(t) - \eta_1(t)|.
\]

as well as \(|A_i(t)| \leq 1 \) for \(i = 1, 2\). Combine this with (4.11) to get

\[
(4.14) \quad |A_1(t)B_1(t) - A_2(t)B_2(t)| \leq \frac{\epsilon}{2\sqrt{2}} \left( \int_0^L (\partial_x u_1 - \partial_x u_2)^2(m_1 + m_2) \, dx \right)^{1/2}
+ \frac{\epsilon^2}{4} (1 + C_0) |\eta_1(t) - \eta_2(t)| + \frac{\epsilon}{4} C_0 L^{1/2} \left( \int_0^L |m_1 - m_2|^2dx \right)^{1/2}.
\]

From (4.12) and (4.14) it follows that

\[
(4.15) \quad I_1 \leq \frac{\epsilon^2 C_0 + \epsilon}{2\sqrt{2}} \int_0^T e^{-rt} (\eta_1(t) - \eta_2(t)) \left( \int_0^L (\partial_x u_1 - \partial_x u_2)^2(m_1 + m_2) \, dx \right)^{1/2} dt
+ \frac{\epsilon^3 C_0 + \epsilon^2 (1 + C_0)}{4} \int_0^T e^{-rt}(\eta_1(t) - \eta_2(t))^2 dt +
\]

\[
\frac{\epsilon^2 C_0^2 + \epsilon C_0}{4} L^{1/2} \int_0^T e^{-rt}(\eta_1(t) - \eta_2(t)) \left( \int_0^L |m_1 - m_2|^2dx \right)^{1/2} dt
\]

\[
\leq P_1(\epsilon) \int_0^T \int_0^L e^{-rt}(m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2 \, dx dt + P_2(\epsilon) \int_0^T \int_0^L e^{-rt}(m_2 - m_1)^2 dx dt.
\]
where $P_1(\epsilon), P_2(\epsilon) \to 0$ as $\epsilon \to 0$. As for $I_2$, setting $D(t) = A_1(t) + B_1(t) - A_2(t) - B_2(t)$, we write

\begin{equation}
I_2 = \int_0^T \int_0^L e^{-rt} D(t)(m_2 - m_1)(\partial_x u_1 + \partial_x u_2)dxdt
\end{equation}

\[ + \int_0^T \int_0^L e^{-rt} D(t)(m_2 + m_1)(\partial_x u_1 - \partial_x u_2)dxdt \]

\[ \leq 2C_0 L^{1/2} \int_0^T e^{-rt} |D(t)| \left( \int_0^L (m_2 - m_1)^2 dx \right)^{1/2} dt \]

\[ + \sqrt{2} \int_0^T e^{-rt} |D(t)| \left( \int_0^L (m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2 \right)^{1/2} dxdt. \]

By (4.11) and (4.13) we have

\begin{equation}
I_2 \leq P_3(\epsilon) \int_0^T \int_0^L e^{-rt}(m_2 - m_1)^2 dxdt + P_4(\epsilon) \int_0^T \int_0^L e^{-rt}(m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2 dxdt
\end{equation}

where $P_3(\epsilon), P_4(\epsilon) \to 0$ as $\epsilon \to 0$. By (4.15) and (4.17), Equation (4.9) becomes

\begin{equation}
\int_0^T \int_0^L e^{-rt}(m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2 dxdt \leq \frac{P_2(\epsilon) + P_3(\epsilon)}{1 - P_1(\epsilon) - P_4(\epsilon)} \int_0^T \int_0^L e^{-rt}(m_2 - m_1)^2 dxdt.
\end{equation}

Now we consider the Fokker-Planck equation (4.2). Recall that $m_1 = m_1 - m_2$. Multiply by $m$ and integrate by parts to get

\begin{equation}
\frac{1}{2} \int_0^L m^2(t, x) dx = \frac{\sigma^2}{2} \int_t^T \int_0^L m_x dxdt - \int_0^T \int_0^L m_x(G_1 m_1 - G_2 m_2) dxdt
\end{equation}

\[ \leq -\frac{\sigma^2}{4} \int_0^T \int_0^L m_x^2 dxdt + \frac{1}{\sigma^2} \int_0^T \int_0^L (G_1 m_1 - G_2 m_2)^2 dxdt \]

\[ \leq \frac{2}{\sigma^2} \int_0^T \int_0^L G_1^2 (m_1 - m_2)^2 dxdt + \frac{2}{\sigma^2} \int_0^T \int_0^L |G_1 - G_2|^2 m_2^2 dxdt. \]

Recall that $G_i = \frac{1}{2}(A_i + B_i - \partial_x u_i)$. Then, using (4.11), (4.13), and the fact that $G_1$ and $m_2$ are bounded by some constant depending on the data, we obtain

\begin{equation}
\int_0^L (m_1(t, x) - m_2(t, x))^2 dx \leq C \int_0^T \int_0^L (m_1 - m_2)^2 dxdt + P_5(\epsilon) \int_0^T \int_0^L (\partial_x u_1 - \partial_x u_2)^2 m_2 dxdt
\end{equation}

where $P_5(\epsilon) \to 0$ as $\epsilon \to 0$. By Gronwall’s Lemma, we have

\begin{equation}
\sup_{t \in [0, T]} \int_0^L (m_1(t, x) - m_2(t, x))^2 dx \leq e^{CT} P_5(\epsilon) \int_0^T \int_0^L (\partial_x u_1 - \partial_x u_2)^2 m_2 dxdt,
\end{equation}

and then by appealing to (4.18) we have

\begin{equation}
\sup_{t \in [0, T]} \int_0^L (m_1(t, x) - m_2(t, x))^2 dx \leq \frac{e^{CT} P_5(\epsilon)(P_2(\epsilon) + P_3(\epsilon))}{1 - P_1(\epsilon) - P_4(\epsilon)} \int_0^T \int_0^L e^{-rt}(m_2 - m_1)^2 dxdt
\end{equation}

\[ \leq \frac{Te^{CT} P_5(\epsilon)(P_2(\epsilon) + P_3(\epsilon))}{1 - P_1(\epsilon) - P_4(\epsilon)} \sup_{t \in [0, T]} \int_0^L (m_1(t, x) - m_2(t, x))^2 dx. \]

Fix $\epsilon$ small enough; then (4.22) implies

\begin{equation}
\sup_{t \in [0, T]} \int_0^L (m_1(t, x) - m_2(t, x))^2 dx = 0,
\end{equation}

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i.e. $m_1 = m_2$. Returning to (4.18) we also have

$$
(4.24) \quad \int_0^T \int_0^L (m_1 + m_2)(\partial_x u_1 - \partial_x u_2)^2 \, dx \, dt = 0,
$$

and so appealing to (4.11) and (4.13) we have $A_1 = A_2$ and $B_1 = B_2$. Then it is straightforward to show that $u_1 = u_2$ by multiplying (4.11) by $u = u_1 - u_2$ and integrating by parts. Noting that $u_x$ has an a priori bound, we have

$$
(4.25) \quad \frac{1}{2} \int_0^L u^2(t, x) \, dx + \frac{\sigma^2}{4} \int_t^T \int_0^L u_x^2 \, dx \, dt \leq C \int_t^T \int_0^L u^2 \, dx \, dt,
$$

and we conclude that $u = 0$ by Gronwall’s Lemma. \hfill \Box



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