Tracking shocked dust: state estimation for a complex plasma during a shock wave

Neil P. Oxtoby, Jason F. Ralph, Céline Durniak and Dmitry Samsonov

Department of Electrical Engineering and Electronics, University of Liverpool, Liverpool, L69 3GJ, United Kingdom

(Dated: 16 December 2011)

We consider a two-dimensional complex (dusty) plasma crystal excited by an electrostatically-induced shock wave. Dust particle kinematics in such a system are usually determined using particle tracking velocimetry. In this work we present a particle tracking algorithm which determines the dust particle kinematics with significantly higher accuracy than particle tracking velocimetry. The algorithm uses multiple extended Kalman filters to estimate the particle states and an interacting multiple model to assign probabilities to the different filters. This enables the determination of relevant physical properties of the dust, such as kinetic energy and kinetic temperature, with high precision. We use a Hugoniot shock-jump relation to calculate a pressure-volume diagram from the shocked dust kinematics. Calculation of the full pressure-volume diagram was possible with our tracking algorithm, but not with particle tracking velocimetry.

Keywords: complex plasma, dusty plasma, shock, Hugoniot, state estimation, tracking, extended Kalman filter, interacting multiple model

I. INTRODUCTION

A complex plasma consists of mesoscopic particles of 'dust' suspended within an ionized gas — a low-density plasma consisting of neutral atoms, ions and electrons.\(^{1,2}\) The dust particles repel each other due to acquiring a net negative charge from collisions with electrons in the plasma (and less frequently, ions). Confined dust can form ordered, crystal-like structures.\(^{3,4}\) In dusty plasma monolayer experiments, single particles of dust are resolvable using a digital camera and illumination by a two-dimensional (2D) laser sheet, making dusty plasma monolayers a useful macromolecular-like system for exploring the microscopic kinematics of condensed-matter systems. The setup shown in Fig. 5 in Sec. IV can be used in experiments to generate and observe a range of phenomena including Mach cones,\(^{5}\) shock waves,\(^{6}\) tsunamis (steepening effect) and solitons.\(^{7}\)

Monitoring the dust behavior is a dynamic-state estimation problem. Recursive estimation of a dynamic state (position/velocity/acceleration) based on remote measurements is often referred to as 'tracking', with the tracked object called the 'target'.\(^{8,9}\) The most widely-used technique for tracking individual dusty plasma particles based on measurements of their positions is known as particle tracking velocimetry (PTV), as defined in the experiments of Refs. 10–13, for example. In PTV, the average velocity of a target over one sample period is determined by differencing consecutive position measurements. The instantaneous velocity of a target can be estimated by including a Bayesian inference step in the recursion. This prediction step is based on a priori knowledge of the target dynamics. The Kalman filter\(^{8,9,14}\) is an example of a recursive estimator which combines remote measurements with a predictive step suitable for linear dynamics. Nonlinear dynamics can be handled by the extended Kalman filter (EKF), which uses a truncated series expansion of the nonlinear dynamical equation (usually first- or second-order). The Kalman filter is an optimal estimator for linear systems and Gaussian precision, in the sense that the minimum mean-square error is obtained. No such result holds for nonlinear systems, but the EKF is widely used because it treats the nonlinearity explicitly and typically performs very well when the initial errors and noises are not too large.\(^{8}\) A linear Kalman filter has been used to track a single particle in a simulated dust crystal in Ref. 15. In that work, the x- and y-dimensions were filtered independently, which limits the accuracy of the prediction.

In this work we present and implement an EKF-based algorithm to track myriad (thousands of) dusty plasma particles during the shock wave experiment of Ref. 16, and during a computer simulation of a dusty plasma shock wave. Central to our algorithm is an interacting multiple model\(^{17,18}\) (IMM) tracker which, using a predetermined set of EKFs, automatically switches to the EKF most appropriate to the changeable dust kinematics. Synthetic data from the computer simulation (described in Section II) was used to characterize the algorithm performance in Section III, showing significant improvement over PTV. The algorithm was applied to experimental data in Section IV, where a pressure-volume diagram was generated using a Hugoniot shock-jump relation. The dusty plasma community is largely unfamiliar with modern target tracking algorithms, that originate in aerospace engineering,\(^{8,9}\) so we have included extensive technical details in an appendix.

II. DUST DYNAMICS

Each charged dust particle in a dusty plasma creates a Coulomb-like potential that is screened by the combination of other particles and the ionized gas. The dust can be treated as point-sources of charge when the radii \(R\) are much smaller than the Debye screening length \(\lambda_D\), which in turn is smaller than the inter-particle separation \(r\). The effective potential experienced by particle \(j\) due
to particle $k$ is a screened, repulsive Coulomb interaction of the Debye-Hückel/Yukawa form:

$$
\phi_{j,k}(\vec{r}_{j,k}) = k_0 Q_d^2 e^{-r/\lambda_D},
$$

where $Q_d$ is the particle charge, $k_0 = 1/(4\pi\varepsilon_0)$ is Coulomb’s constant, the plasma screening distance is the Debye length $\lambda_D$, and the particles are separated by $\vec{r}_{j,k} = r \hat{r}_{j,k}$. A hat denotes a unit vector. Taking the negative of the gradient of the two-particle potential yields the effective interaction force between pairs of charged dust particles:

$$
\vec{F}_{j,k}(\vec{r}_{j,k}) = \frac{k_0 Q_d^2}{\lambda_D^2} e^{-\hat{r}} (\hat{r}^{-2} + \hat{r}^{-1}) \hat{r}_{j,k},
$$

where the dimensionless inter-particle separation is defined by $\hat{r} \equiv r/\lambda_D$. In monolayer experiments, the dust is confined by an external potential which typically has a shallow parabolic profile (or similar in-plane circular geometry). The dust particles align themselves in an imperfect hexagonal lattice. Such a dust crystal was the initial condition for the work in this paper: for both experiment and simulation. The particles were subjected to an external force of short duration that induced a shock wave, as in the experiment of Ref. 16 and in the simulation Run 7 of Ref. 7. The dust was imaged at 1000 frames per second in both the experiment and the computer simulation (where synthetic images were generated).

The total force on the $j$th particle, $\vec{F}_j$, is the sum of all two-particle interaction forces $\vec{F}_j = \sum_k \vec{F}_{j,k}$, plus a damping/drag force due to collisions with the plasma, plus the force due to the global confinement potential, and the short-lived external force (see Ref. 7 for details).

### III. Algorithm Performance

The performance of any tracking algorithm is characterized by the errors in the tracks (estimated states). A computer simulation is necessary for this because the true behaviour of the targets, the ‘ground truth’, is unknown in an experiment. In this section we present errors in the dust particle tracks (position and velocity) using the ground truth from a molecular dynamics simulation of shocked dust. PTV errors were also calculated for comparison. Section III A contains an overview of our tracking algorithm, with extensive details included as an appendix.

The dynamics of the simulated particles were determined using a fifth-order Runge-Kutta numerical integration of the Newtonian equations of motion for position $\vec{q}(t)$ and velocity $\vec{v}(t)$: $d\vec{q}(t)/dt = \vec{v}(t)$, $d\vec{v}(t)/dt = \vec{F}/m$, where $m$ is the uniform particle mass (see Ref. 7 for parameter values used in the simulation).

We simulated $N = 3000$ particles in a weakly-confined 2D dust crystal monolayer, subject to an impulsive external force that induced a shock wave, as illustrated in Fig. 1. The shock wave traversed the captured field of view (the ‘scene’) in one second, with 1000 simulated images generated at millisecond intervals ($\Delta t = 10^{-3}$s). These images were designed to be faithful replicas of those from experiments so that the tracking algorithm was tested with realistic data.

#### A. Tracking algorithm

The general procedure for tracking dust (details in Appendix A) starts with particle detection followed by track initialization. Each dust particle typically illuminates a few pixels in an image. The pixel-intensity-weighted centroid of each illuminated region constitutes a particle detection, or measurement. For each subsequent image, track management (also known as track maintenance) associates particle measurements to existing particle tracks, or optionally initiates new tracks from unmatched measurements. Occasionally a particle track will not have a corresponding measurement. Such a missed detection can occur when a dust particle moves out of the 2D laser illumination sheet, or when two particles are close enough to illuminate a single region of pixels. After a number of consecutive missed detections, a particle track is terminated. The final stage of the tracking procedure is our focus in this paper: state estimation. State estimation uses Bayesian inference to predict the target behavior based on a physical model describing expected target dynamics. This prediction is combined with measurement in a
weighted sum with weightings assigned so as to minimize the expected error in the estimate. For this purpose we used a discrete-time extended Kalman filter (EKF)\(^8,9\) due to the nonlinear-time extended Kalman filter and measurement matrices. The weights are determined by the uncertainties in the prediction and measurement (stored in covariance matrices). These uncertainties are directly influenced by two filter-design parameters known as the process noise and the measurement noise, respectively. Tuning these design parameters allows one to controlably modify the weighted average between measured and predicted values.

When the dust is excited by an electrostatic force, the dynamical model used for prediction is augmented by an additional force. In target-tracking nomenclature, this temporary deviation from the original dynamical model is known as a maneuver. Multiple models can be combined in an efficient manner using the interacting multiple model (IMM) approach\(^8,17,18\). Just as an EKF produces a state estimate by combining a predicted value with a measured value in a weighted sum, the IMM algorithm can be thought of as combining multiple state estimates (from multiple EKFs, for example) in a weighted sum. The weights for each model in the IMM depend on the probabilities of each dynamical model being correct, as determined from the measurement-based likelihoods. Just as the weights in an EKF are directly affected by filter-design parameters (the process noise and measurement noise), the IMM weights are directly affected by selectable parameters known as mode-switching probabilities. As the name suggests, mode-switching probabilities affect how quickly the IMM switches between modes at the onset and at the conclusion of a maneuver.

We used an IMM with three prediction models (EKFs) for the dust dynamics. The first EKF predicted dust behaviour based on the dust interactions. This is known as the base model, or mode, of the IMM. The second EKF model added an external force of fixed magnitude in the positive-X direction, which accounted for the electrostatic excitation force that generated the shock. The third EKF model extended the base model by adding an external force in the negative-X direction, which improved the tracking of particles reflected off the dust cloud bulk. The second and third EKFs are known as maneuvering modes. Technical details of the three EKFs and the IMM are contained in the appendix in sections A.3 and A.4, respectively.

We partitioned the myriad targets into subsets, which were tracked independently across multiple computers using our EKF-based IMM trackers. This partitioning was necessitated by the exponential dependence of computational load on the number of targets per tracker, \(N_T\): the multi-target state vector in two dimensions scales as \(2N_T\), with the corresponding covariance matrix scaling as the square of this. As discussed in the appendix, this would consume (at double precision) in excess of two gigabytes of memory per time step to track thousands of particles. The subset state estimates were combined offline into a single myriad target state, while ensuring that particles were counted only once each – there was no unchecked overlap of tracks where a single particle was accidentally tracked by multiple trackers. Such track overlap occur due to particle association errors caused by missed detections or intersecting particle trajectories. Tracker sizes substantially less than 100 are manageable on a desktop computer. Since \(N_T\) is a design parameter in myriad target tracking, we considered \(N_T = 1, 6\) and 12 particle(s) per tracker for comparison. With the goal of minimizing track errors, within limitations imposed by the heavy computational resource requirements of multi-target tracking, we found that target loss was slightly reduced for larger \(N_T\), due primarily to the reduced likelihood of track overlap. Average RMS error levels did not vary significantly with \(N_T\).

### B. IMM performance

The point of the IMM is to detect dust particle maneuvers. Therefore, we measured the IMM performance by comparing the IMM tracker errors with those from a single-mode tracker that did not handle maneuvers: an EKF tracker based on the Yukawa interactions alone.

Figure 2 shows a typical result for all tracker configurations. The bottom plot shows that the IMM trackers were a factor of four better than the EKF trackers (1\% down to 0.25\%) at recovering particles lost during the confusion created by the shock wave formation between \(t = 0.2s\) and \(0.4s\) (the maneuver). The top and middle plots show that this comes at the cost of increased root-mean-square (RMS) errors for estimated position (≈ 4\% larger error) and velocity (≈ 20\% larger error). This target loss in the simulation was due to the aforementioned track overlap during maneuvers.

### C. Bulk quantity: kinetic energy map

In Figure 2 we have shown that IMM trackers produce significantly higher-accuracy estimates of the dust particle kinematics than using PTV, particularly for estimating velocity. In this section we use this higher-accuracy data to estimate a bulk physical quantity: the average kinetic energy

\[
KE_{av} = \frac{1}{2}mv^2.
\]

Here \(m\) is the particle mass, \(v\) is the particle speed, and the overbar indicates the average over multiple particles within each bin in a \(20 \times 20\) grid across the scene (field of view). The bins are shown in Fig. 1, which contains the
FIG. 2. RMS errors in position (top) and velocity (center), and percentage target loss (bottom) as functions of time. Tracker configuration: $N = 1500$ total targets, $N_T = 12$ PPT. Compare the IMM tracker (solid red line) with the base mode EKF (dotted blue line) and PTV (dashed green line).

simulated image at $t = 0.499s$, enhanced for presentation purposes (larger dots and increased brightness).

A snapshot of the kinetic energy map at $t = 0.499s$, estimated using $N = 3000$ with $N_T = 1$, is shown in Fig. 3(b), with the corresponding true map in Fig. 3(a) showing excellent agreement. Bins containing fewer than five particles are ignored and are crossed out in the figure. Similar results were achieved with $N_T = 6$ and 12.

The time-dependence of the accuracy of the estimated kinetic energy map can be visualized by combining selected snapshots of a grid cross-section, or 'slice', taken in the direction of shock-wave propagation. We consider

the central slice highlighted in Fig. 1. Figure 4 contains a three-dimensional representation of snapshots of this slice taken every 50ms, with the height showing the kinetic energy. Figure 4(a) is the ground truth calculated from the simulation, which compares favorably with the average kinetic energy estimate for in Figure 4(b). The peak of nearly-constant height in the foreground is the shock-front. The highest peak (for small $X$) shows dust particles initially excited by the external force, then later reflected (at lower speed) off the dust cloud. The kinetic energy valley between this excitation/reflect and the shock front is where the dust cloud re-crystallizes in the wake of the shock wave.

FIG. 3. Snapshots of the kinetic energy map at $t = 0.499s$: (a) truth; (b) IMM-tracker estimate using $N = 3000$ with $N_T = 1$. The outer three rows/columns of empty bins have been omitted. Particle locations are overlaid for reference. Nonempty bins containing fewer than five particles were ignored and are crossed out.

FIG. 4. Kinetic energy map cross-section slice (50ms-spaced snapshots): (a) Truth (from the simulation); (b) IMM-tracker estimate using $N = 3000$ with $N_T = 1$.

IV. EXPERIMENT

In the experiment reported in Ref. 16, a shock wave was generated by a short sequence of negative voltage pulses along a wire adjacent to the left side of the dusty plasma, and slightly below the plane of the dust cloud. The vertical component of this excitation caused many dust particles to move out of the laser illumination sheet, and hence to disappear, for up to 10 consecutive frames. These missed detections were the primary obstacle to tracking dust particles in the left of the scene. We tracked a total of $N = 1860$ particles — those that were present in all of the first few images.

Figure 6 shows a snapshot of the experimental results at $t = 432ms$. Shown are the estimated positions and average particle number density, velocity, and kinetic temperature in the direction of the shock wave propagation ($X$). Averages were calculated in each of 50 vertical bins across the scene, with the standard deviation for each bin shown as error bars (PTV) and a shaded region (IMM tracker). The kinetic temperature of particles in each vertical bin was calculated as half the particle mass multiplied by the mean-squared velocity deviation within the bin. Missed detections affected the PTV results severely, with many particle velocities (and consequently kinetic temperature values) missing — note the large gaps in the PTV plots in Fig. 6. Contrast this with the IMM tracker.
where the problem did not occur. The significant target loss which occurred in the shock formation zone (left of the scene) due to missed detections was not a problem because we were more interested in the shock itself. For a recent discussion of problems and limitations of PTV using high-speed cameras, we refer the reader to Ref. 20.

The shock-jump (Hugoniot) relations\cite{21} use conservation laws to relate some physical properties of a fluid upstream (behind) and downstream (ahead) of an ideal shock front. Specifically, the pressure, density and velocity are related by three equations that also include the shock speed. For known initial conditions downstream, and known shock speed, only two of the upstream properties need to be determined to find the third unknown behind the shock. We consider conservation of momentum across the shock front, which gives the second shock relation\cite{21}

\[ P_2 - P_1 = mn_{1.2}(U_s - v_{1.2})(v_2 - v_1). \] (4)

Here \( P_{1,2}, n_{1,2} \) and \( v_{1,2} \) are the pressure, number density and particle velocity downstream (1) and upstream (2) from the shock. Our IMM tracker results were used to determine average bin values for \( v_{1.2} \) and \( n_1 \). The shock speed \( U_s = dX_s(t)/dt \) was determined from the shock front position \( X_s(t) \), which was fitted to the tracker results for peak number density \( n_X(t) \). An ideal shock would travel with constant speed \( c_s \). Real shocks slow down due to damping, which here is a drag force primarily due to ion/neutral collisions with the dust particles. To first order, \( U_s(t) = c_s - \gamma t \). We determined \( c_s,expt = 42.7 \pm 1.8\text{mm/s}, \gamma,expt = 11.1 \pm 1.9\text{mm/s}^2 \) in the experiment, and \( c_s,sim = 53.8 \pm 1.5\text{mm/s}, \gamma,sim = 1.7 \pm 1.9\text{mm/s}^2 \) in the simulation. Higher dissipation in the experiment \( \gamma,expt > \gamma,sim \) suggests either underestimated dissipation in the simulation, or additional unmodelled sources of dissipation. Substituting \( m, n_1, v_1, v_2 \) and \( U_s \) into equation (4) yields the shock-induced pressure jump \( P_2 - P_1 \).

This is plotted in Fig. 7(a) as a function of time, and in Fig. 7(b) as a function of \( n_1(t_s)/n_2(t) \) — the inverse of the shock-induced compression. Here we were restricted to times where the shock was present (after formation at \( t_s \) and before leaving the scene): \( t_s \leq t \leq 600\text{ms} \). Figure 7(b) is essentially a pressure-volume, or P-V, diagram. Experiment (crosses, broken line) and simulation (triangles, solid line) displayed the same qualitative behavior: shock-induced pressure decreased over time and with increasing volume. Pressure decreased over time due to the shock wave losing energy as it traversed the dust.
cloud. Pressure was expected to decrease with increasing volume, based on basic thermodynamic arguments (e.g., Boyle’s law for an ideal gas). We speculate that the quantitative differences between experiment and simulation were due in part to dust particle uniformity in the simulation versus non-uniformity in the experiment. For example, we would expect a non-uniform distribution of values for particle mass, charge, etc. in the experiment. The large gaps in the PTV velocities shown in Fig. 6 meant that a P-V diagram could not be calculated using PTV.

![Graph of shock-induced pressure jump](image)

FIG. 7. Shock-induced pressure jump. Experiment (crosses, dotted line) and simulation (triangles, solid line) showing a steady decrease in shock-induced pressure jump as a function of (a) time, and (b) inverse compression.

V. CONCLUSION

We have considered a monolayer dusty plasma crystal disturbed by an electrostatically-induced shock wave, presenting results from a computer simulation and the experiment of Ref. 16. The dust particle kinematics were estimated recursively using an algorithm that combines measurements of position (from images) with predicted behavior based on multiple models for the particle dynamics. The algorithm provided accurate estimates for the motional states of the dust particles (position, velocity and acceleration) and also selected which of the motion models most accurately reflected the measured data. Estimation of dynamic states based on remote measurements is often referred to as target tracking, which here was a nonlinear dynamic estimation problem involving myriad maneuvering objects (up to 3000). Extensive analysis of the results from computer simulations allowed us to quantify the algorithm performance, which was significantly more accurate than using particle tracking velocimetry (a factor of three reduction in the average error). When applied to the experimental data of Ref. 16, the target tracking algorithm demonstrated further superiority to PTV — the tracking algorithm was robust to missed detections. Missed detections prevented calculation of particle velocities for PTV. Using Hugoniot shock-jump relations and tracker-estimated kinematics of the dust, the shock-induced pressure jump and compression was calculated for the dust cloud. As expected for a non-ideal shock, pressure decreased as the shock wave lost energy over time. Pressure also decreased as a function of increasing volume (inverse compression). Qualitative agreement was observed between experiment and simulation.

In this paper we have shown that the improvement in precision and reliability of estimating dust kinematics using a target tracking algorithm is important for determining the physics of dusty plasmas. Our approach can be applied in future experiments to reliably determine quantities such as kinetic energy, kinetic temperature, diffusion coefficients, and dynamic viscosity.

ACKNOWLEDGMENTS

This work was supported by the Engineering and Physical Sciences Research Council of the United Kingdom (Grant EP/G007918). N.P.O. acknowledges use of high-throughput computational resources provided by the eScience team at the University of Liverpool.

Appendix A: State Estimation and Tracking

As overviewed in Section III A, estimation of a dynamic state consists of combining observations (measurements) with predictions. In this appendix we present details of the general procedure used in this paper for tracking multiple targets. It consists of the following steps. Remote measurements of the targets are made, usually at regular intervals. Following track initialization, track maintenance occurs after each measurement. Measurements of the multiple target states (often only positions) are either associated to existing tracks, or used to initiate new tracks. Tracks lacking a measurement (known as a missed detection) are terminated after multiple consecutive missed detections have occurred. State estimation is then implemented and the procedure is repeated for each subsequent measurement.
1. Measurement: image processing

Dusty plasma observations consist of a time sequence of images taken at (typically) regular intervals separated by \( \Delta t \). A snapshot at \( t = 0.353 \) s from the experiment of Ref. 16 is shown in Figure 8(a). The image has been enhanced for display purposes (increased particle size and overall image brightness). Similar enhancement was used on the synthetic image in Figure 8(b).

Our synthetic images emulate the experimental images in Refs. 6, 16, 24, and 25: we also use grayscale, 8-bit, tagged image file format (TIFF) images with a 1024-pixel square field of view. The field of view, or scene, is 27.3 mm wide for the experiment, and 80 mm wide for the simulation, corresponding to respective spatial resolutions of approximately 26.7 \( \mu \)m and 78.1 \( \mu \)m per pixel (37.5 and 12.8 pixels per mm). The wider field of view in the simulation enabled the entire dust cloud to be viewed.

After thresholding the image (setting pixel intensities below some threshold equal to zero), a particle is identified as a contiguous group of bright pixels. Using the moment method, the intensity-weighted centre-of-mass of this group of pixels gives the measured/detected position of a particle:\(^{24,26}\)

\[
\vec{z}_p = \sum_{j=1}^{N_p} \frac{I_j}{I_\Sigma} \vec{x}_j, \tag{A1}
\]

where the intensity of each bright pixel is \( I_j \), and \( \vec{x}_j \) is the 2 \( \times \) 1 vector containing the \( x \)- and \( y \)-pixel number for the \( j \)-th pixel in the group containing \( N_p \) contiguous bright pixels. Here the total intensity of the group of bright pixels is \( I_\Sigma = \sum_{j=1}^{N_p} I_j \). For particles illuminating more than a few pixels, the moment method yields a typical measurement precision of around one tenth of a pixel in each direction. More advanced techniques can improve this precision, such as those considered in Refs. 24 and 26, at the cost of heavier computational load. However, the potential improvement is limited if measurements are not the dominant source of inaccuracies.

2. Track maintenance

Maintaining continuous tracks involves a decision process known as ‘measurement-to-track association’. Each measurement is associated to a track, dismissed as a false alarm, or saved as a possible new track. Missed detections must also be handled, since object detection can never be perfect. For example, dust particles may move out of the two-dimensional laser sheet and no longer be illuminated. We deleted tracks after nine consecutive missed detections. Measurement-to-track association is performed online as part of the tracking algorithm.

Track maintenance is complicated when partitioning myriad targets. Unless using centralized, online data fusion for all trackers (i.e., a parallel tracking algorithm), track ‘overlap’ may occasionally occur. This is where a single particle is tracked by multiple trackers/computers. Track overlap often results when particle trajectories intersect. Here we delete such duplicate tracks as part of our offline data fusion process.

3. The extended Kalman filter

We process the measurements and estimate the dust particle states (two-dimensional position \( q \), velocity \( v \) and
The Kalman gain is a matrix version of the ratio of measurement to measurement uncertainty in the Kalman filter:

$$
\hat{x}(k+1) = \hat{x}(k) + K(k) \cdot [z(k) - z(k|k-1)].
$$

(A2)

Here the innovation [\(z(k) - z(k|k-1)\)] is weighted by the Kalman gain \(K(k) = S_z(k) \cdot H^T \cdot [H \cdot S_z(k) \cdot H^T + R(k)]^{-1}\). The innovation is the difference between the actual measurement of position \(z(k)\) and the expected measurement \(z(k|k-1) = H \cdot \hat{x}(k) = (q_a(k), q_v(k))^T\). Here the measurement matrix is

$$
H = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

(A3)

The Kalman gain is a matrix version of the ratio of track uncertainty to total uncertainty (track plus measurement uncertainty), with the measurement uncertainty given by \(R(k)\), the expected covariance matrix for the measurement. Note that matrix multiplication applies throughout — the dots are to guide the eye.

The updated state \(\hat{x}(k+)\) is now predicted forward to the next time step using a discretization of the governing dynamical process equations given in Sec. II:

$$
\hat{x}(k+1) = f(\hat{x}(k+))
$$

\[
\begin{pmatrix}
q_a(k+) + (\Delta t)v_x(k+) + (\Delta t)^2 F^2_x(\hat{x}(k+)) / 2m \\
v_x(k+) + (\Delta t) F^1_x(\hat{x}(k+)) / m \\
F^1_x(\hat{x}(k+)) / m \\
q_v(k+) + (\Delta t)v_y(k+) + (\Delta t)^2 F^2_y(\hat{x}(k+)) / 2m \\
v_y(k+) + (\Delta t) F^1_y(\hat{x}(k+)) / m \\
F^1_y(\hat{x}(k+)) / m
\end{pmatrix}
\]

(A4)

where \(\Delta t\) is the time step. Here we have ignored higher-order terms from the Taylor expansion of the full dynamical model equation, which includes the (Gaussian random) process noise \(\nu(k): \hat{x}(k+1) = f(\hat{x}(k+)) + \nu(k)\).

The process noise assigns a selectable Gaussian uncertainty to the predicted state \(f(\hat{x})\), which increases the state covariance by \(Q(k)\), the process noise covariance (matrix). For the first-order EKF the covariance in the predicted state becomes

$$
S_\hat{x}(k+1) = f'(\hat{x}) \cdot S_\hat{x}(k+) \cdot f'(\hat{x})^T + Q(k),
$$

(A5)

where the first-order term from the aforementioned Taylor expansion involves the Jacobian matrix

$$
f'(\hat{x}) = \frac{\partial f}{\partial \hat{x}} |_{\hat{x}=\hat{x}(k+)}.
$$

In the standard Kalman filter, the Jacobian is replaced by the prediction matrix that represents the linear dynamics. As there is no truncated series expansion in the standard Kalman filter, the covariance is a true representation of the prediction error for linear dynamics with Gaussian uncertainty. In the EKF, the covariance in the prediction is an approximate representation of the prediction error. The first-order EKF is a piecewise linearization of the nonlinear particle dynamics. This linear approximation should be accurate while the errors in the state estimates are small compared to the non-linear nature of the underlying dynamical function. For our purposes, when the dust particles are in a crystalline state, this approximation is excellent. In more dynamic moments (during the shock wave, for example), the approximation begins to break down and so the tracker performance is expected to suffer, but not to fail. This is because the process noise in \(Q(k)\) affords us the ability to assign a weight to our confidence in the accuracy of the linearized prediction. Similarly, the measurement noise in \(R(k)\) reflects our confidence in the measurement accuracy. These values can be controllably varied to improve the filter accuracy (see Ref. 30 for an example of such filter ‘tuning’).

Larger linearization errors can be dealt with by assigning a lower weight to the prediction, relative to the measurement. Alternatively, more advanced techniques can be used. These include unscented filters and particle filters.

When tracking multiple targets with a single tracker, the single-particle state vectors are stacked to form a large vector and the filter matrices are correspondingly resized in a block-diagonal fashion. For 3000 particles, this would be an 18000-element vector, with corresponding matrices containing up to 18000×18000 elements. At double precision, this amounts to approximately 2.5 gigabytes of storage per time step, as well as many floating-point operations. For this reason, we partitioned the myriad targets into subsets that were manageable as individual multi-target trackers.

### 4. Interacting multiple model estimator

The prediction step inherent in any Kalman-filter-based technique for dynamic state estimation is based on a physical model for the underlying dynamics of the target, as discussed for an EKF in the Section A 3. If the underlying dynamics change, then the filter performance (accuracy) will decrease due to the reduced ability of the model to predict the changed/changing dynamics of the target. Such dynamical changes, called maneuvers, can
often be described by adding an input \( u(k) \) to the prediction step of the filter. Approaches to handling an unknown input fall into two broad categories\(^8\): modeling the unknown input as a random process; and estimating the unknown input in real time. Input estimation can be computationally costly as it involves either maintaining a history (sliding window) of estimates that is used to detect a maneuver, or it involves increasing the state dimension by augmenting the state vector with the input to be estimated. In the case where the underlying system dynamics is known to switch between a finite number of modes, it can be significantly less costly to run a filter for each dynamical model than to perform input estimation. Such multiple model approaches to state estimation use a Bayesian framework where the prior probabilities for each model being correct are updated to the posterior probability at each point in time. In an interacting multiple model\(^{17,18}\) state estimator, the mode probabilities \( \mu_j \) are used in conjunction with tunable Markovian mode-switching probabilities \( p_{i,j} \) to calculate a mixed (weighted) initial condition at each time, which is filtered using each possible current model. All of this comes at a fraction of the cost of input estimation, with significantly improved performance during maneuvers\(^8\) and the added benefit of being decision-free (no maneuver detection is necessary).

The IMM algorithm proceeds as follows. Mixing probabilities \( \mu_{ij}(k|1) \) are calculated from the probability \( \mu_i(k−1) \) that mode \( i \) was in effect at time step \( k−1 \), given that mode \( j \) is in effect at time step \( k \), conditioned on the measurements up to \( k−1 \):

\[
\mu_{ij}(k|1|k−1) = \frac{p_{ij} \mu_i(k−1)}{\bar{c}_j} \quad i, j = 1 \ldots s, \quad (A7)
\]

where \( \bar{c}_j \) ensures normalization and we use \( s = 3 \) modes (see below). The mode probabilities \( \mu_j(k) \) are stored along with the state vectors \( \hat{x}^j(k) \). The mixing probabilities are used to calculate the mixed initial condition

\[
\hat{x}^{Dj}(k−1|k−1) = \sum_{i=1}^s \hat{x}^i(k−1|k−1) \mu_{ij}(k|1|k−1), \quad (A8)
\]

which is input to the \( j \)th filter, along with the corresponding covariance

\[
P^{0j} = \sum_{i=1}^s \mu_{ij} \left\{ P^i + [\hat{x}^i - \hat{x}^{0j}] \cdot [\hat{x}^i - \hat{x}^{0j}]^T \right\}, \quad (A9)
\]

where we have dropped the \((k−1|k−1)\) arguments for presentation purposes. Each filter has a likelihood function associated with it, \( \Lambda_j(k) \), which is a Gaussian probability reflecting the likelihood of obtaining the new measurement \( z(k) \), given the predicted measurement using the mixed initial condition, \( \hat{x}^j(k|k−1) = H \cdot \hat{x}^j(k−1|k−1) \). After running the filters and calculating the likelihood functions, the mode probabilities are updated to

\[
\mu_j(k) = \frac{\Lambda_j(k) \bar{c}_j}{c}, \quad (A10)
\]

where \( c \) is a normalization constant.

We use a three-mode IMM, with a model each for before, during and after the shock wave. The first model is an EKF based on our work in Ref. 30, designed for a dust crystal. We refer to this as the base mode. The second and third modes are referred to as the maneuvering modes, where the target dynamics deviate from the base mode (usually for short periods of time). The second mode includes an input \( u(k) \) to account for an external force in the positive-X direction, extending the prediction equation (A4) to

\[
\dot{x}(k + 1) = f(\hat{x}(k+)) + G_v u(k)
\]

\[
= f(\hat{x}(k+)) + \begin{pmatrix} (\Delta t)^2 / 2 \\ \Delta t \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} F_{\text{shock}}(k), \quad (A11)
\]

where \( G_v \) is the input gain matrix for the unknown effective force of tunable magnitude \( u(k) = F_{\text{shock}}(k) \). The third mode is designed to account for particles reflected in the opposite direction, and so the prediction equation has the same form as (A11), but with \( u(k) = F_{\text{after}}(k) \) in the negative-X direction (towards the left of the scene). The remainder of the EKF equations are unchanged by the inclusion of these inputs. For ephemeral maneuvers, the base mode should be the best description of the dust dynamics for most of the time. Thus, we choose the following mode-switching probabilities

\[
p_{i,j} = \begin{pmatrix} 0.80 & 0.10 & 0.10 \\ 0.30 & 0.60 & 0.10 \\ 0.40 & 0.10 & 0.50 \end{pmatrix}, \quad (A12)
\]

which sum to unity along each row, as required. Larger values along the diagonal of \( p_{i,j} \) lead to slower switching between modes. This can also be tuned in the opposite direction if faster mode switching is desired. Faster mode switching tends to yield lower peak RMS error during a maneuver, at the expense of greater RMS error during quiescent periods\(^8\). Initially, the mode probabilities are equal to one other: \( \mu_1 = \mu_2 = \mu_3 = 1/3 \). IMM mode probabilities can be used to visualize the maneuvers, as done for a shocked dusty plasma in Ref. 31.

5. Range of influence

The Yukawa force in Equation (2) decreases both exponentially and polynomially with increasing particle separation, thereby reducing the range of influence that each dust particle has on other particles. Beyond this range, two dust particles exert negligible forces on each other. Here we define the range of influence \( r_{\text{max}} \) relative to the nearest neighbor separation \( r_{\text{NN}} \) as the separation at which the two-particle Yukawa force magnitude drops to 1% of the Yukawa force due to the nearest
REFERENCES

1. R.L. Merlino and J.A. Goree. Dusty plasmas in the laboratory, industry, and space. *Physics Today*, 57:32, 2004.
2. P.K. Shukla and B. Eliasson. Colloquium: Fundamentals of dust-plasma interactions. *Reviews of Modern Physics*, 81:25, 2009.
3. H. Thomas, G.E. Morfill, V. Demmel, J. Goree, B. Feuerbacher, and D. Möhlmann. Plasma crystal: Coulomb crystallization in a dusty plasma. *Phys. Rev. Lett.*, 73:652, 1994.
4. A. Melzer, T. Trottenberg, and A. Piel. Experimental determination of the charge on dust particles forming coulomb lattices. *Physics Letters A*, 191:301, 1994.
5. D. Samsonov, J. Goree, Z.W. Ma, A. Bhattacharjee, H.M. Thomas, and G.E. Morfill. Mach cones in a coulomb lattice and a dusty plasma. *Phys. Rev. Lett.*, 83:3649, 1999.
6. D. Samsonov, S.K. Zhdanov, R.A. Quinn, S.I. Popel, and G.E. Morfill. Shock melting of a two-dimensional complex (dusty) plasma. *Phys. Rev. Lett.*, 92:255004, 2004.
7. C. Durniak, D. Samsonov, N.P. Oxtoby, J.F. Ralph, and S. Zhdanov. Molecular-dynamics simulations of dynamic phenomena in complex plasmas. *Plasma Science, IEEE Transactions on*, 38:2412, 2010.
8. Y. Bar-Shalom, X.R. Li, and T. Kirubarajan. *Estimation with Applications to Tracking and Navigation*. Wiley, New York, 2001.
9. J.F. Ralph. *Target Tracking*, volume 5: Dynamics and Control, chapter 251. Wiley, 2010.
10. D. Samsonov, J. Goree, H.M. Thomas, and G.E. Morfill. Mach cone shocks in a two-dimensional yukawa solid using a complex plasma. *Phys. Rev. E*, 61:5557, 2000.
11. C. Boesse, M. Henry, T. Hyde, and L. Matthews. Digital imaging and analysis of dusty plasmas. *Advances in Space Research*, 34:2374, 2004.
12. B. Liu, J. Goree, V.E. Fortov, A.M. Lipaev, V.I. Molotkov, O.F. Petrov, G.E. Morfill, H.M. Thomas, and A.V. Ilev. Dusty plasma diagnostics methods for charge, electron temperature, and ion density. *Physics of Plasmas*, 17:053701, 2010.
13. Y. Feng, J. Goree, and B. Liu. Viscoelasticity of 2d liquids quantified in a dusty plasma experiment. *Phys. Rev. Lett.*, 105:025002, 2010.
14. R.E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82:35, 1960.
15. V. Hadziavdic, F. Melandsø, and A. Hanssen. Particle tracking from image sequences of complex plasma crystals. *Physics of Plasmas*, 13:055304, 2006.
16. D. Samsonov and G.E. Morfill. High-speed imaging of a shock in a complex plasma. *Plasma Science, IEEE Transactions on*, 36:1020, 2008.
17. H.A.P. Blom. An efficient filter for abruptly changing systems. In *Proceedings of the 23rd Conference on Decision and Control*. IEEE, 1984.
18. H.A.P. Blom and Y. Bar-Shalom. The interacting multiple model algorithm for systems with markovian switching coefficients. *IEEE Transactions on Automatic Control*, 33:780, 1988.
19. U. Konopka, G.E. Morfill, and L. Ratke. Measurement of the interaction potential of microspheres in the sheath of a rf discharge. *Phys. Rev. Lett.*, 84:891, 2000.
20. Y. Feng, J. Goree, and B. Liu. Errors in particle tracking velocimetry with high-speed cameras. *Review of Scientific Instruments*, 82:053707, 2011.
21. J.W. Bond, Jr., K.M. Watson, and J.A. Welch, Jr. *Atomic Theory of Gas Dynamics*. Addison-Wesley, Reading, 1965.
22. B.J. Alder and T.E. Wainwright. Decay of the velocity autocorrelation function. *Phys. Rev. A*, 1:18, 1970.
23. A. Gavrikov, I. Shakhova, A. Ivanov, O. Petrov, N. Vorona, and V. Fortov. Experimental study of laminar flow in dusty plasma liquid. *Physics Letters A*, 336:378, 2005.
24. Y. Feng, J. Goree, and B. Liu. Accurate particle position measurement from images. *Review of Scientific Instruments*, 78:053704, 2007.
25. J.F. Ralph, D. Samsonov, C. Durniak, and G.E. Morfill. Tracking dust: tracking and state estimation for dusty plasmas. In *Proceedings of SPIE*, 7336, 73361H, 2009.
26. Y. Ivanov and A. Melzer. Particle positioning techniques for dusty plasma experiments. *Review of Scientific Instruments*, 78:033506, 2007.
27. S.J. Julier and J.K. Uhlmann. Unscented filtering and nonlinear estimation. *Proceedings of the IEEE*, 92:401, 2004.
28. R.P.S. Mahler. Multitarget bayes filtering via first-order multitarget moments. *Aerospace and Electronic Systems, IEEE Transactions on*, 39:1152, 2003.
29. M.S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp. A tutorial on particle filters for online nonlinear/non-gaussian bayesian tracking. *Signal Processing, IEEE Transactions on*, 50:174, 2002.
30. N.P. Oxtoby, J.F. Ralph, D. Samsonov, and C. Durniak. Tracking interacting dust: comparison of tracking and state estimation techniques for dusty plasmas. In *Proceedings of SPIE*, 7698, 76980C, 2010; arXiv:1112.3924 [physics.plasm-ph]
31. N.P. Oxtoby, J.F. Ralph, C. Durniak, and D. Samsonov. Visualizing a Dusty Plasma Shock Wave via Interacting Multiple-Model Mode Probabilities. *Plasma Science, IEEE Transactions on*, 39:2772, 2011; arXiv:1112.3947 [physics.plasm-ph]