LP MAXIMAL BOUND AND SOBOLEV REGULARITY OF TWO-PARAMETER AVERAGES OVER TORI

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ABSTRACT. We investigate $L^p$ boundedness of the maximal function defined by the averaging operator $f \rightarrow A^s_t f$ over the two-parameter family of tori $T^s_t := \{(t + s \cos \theta) \cos \phi, (t + s \cos \theta) \sin \phi, s \sin \theta) : \theta, \phi \in [0, 2\pi)\}$ with $c_0 t > s > 0$ for some $c_0 \in (0, 1)$. We prove that the associated (two-parameter) maximal function is bounded on $L^p$ if and only if $p > 2$. We also obtain $L^p$--$L^q$ estimates for the local maximal operator on a sharp range of $p, q$. Furthermore, the sharp smoothing estimates are proved including the sharp local smoothing estimates for the operators $f \rightarrow A^s_t f$ and $f \rightarrow A^{t_0 t}_s f$. For the purpose, we make use of Bourgain–Demeter’s decoupling inequality for the cone and Guth–Wang–Zhang’s local smoothing estimates for the 2 dimensional wave operator.

1. INTRODUCTION

The maximal functions generated by (one-parameter) dilations of a given hypersurface have been extensively studied (for example, [30, Ch. 11], [24, 16, 17, 10, 7], and references therein) since Stein’s seminal work on the spherical maximal function [31]. Most of investigations were restricted to the one-parameter maximal functions. Meanwhile, the maximal operators involved with more than one-parameter family of dilations were considered by some authors (see [28] for multiparameter lacunary maximal functions and [9, 25] for related results). For example, Cho [8] and Heo [14] obtained such results built on the $L^2$ method which requires sufficient decay of the Fourier transform of the associated surface measures. However, in those results, boundedness on sharp range is generally unknown. Two-parameter maximal functions associated to homogeneous surfaces were studied by Marletta–Ricci [21], and Marletta–Ricci–Zienkiewicz [22], who obtained their boundedness on the sharp range. In those works, homogeneity makes it possible to deduce $L^p$ boundedness from that of a one-parameter maximal operator. Not much is so far known about the maximal functions which are genuinely of multiparameter.

In this paper we are concerned with a maximal function which is generated by averages over a natural two-parameter family of tori in $\mathbb{R}^3$. Let us set

$$\Phi^s_t(\theta, \phi) = \{(t + s \cos \theta) \cos \phi, (t + s \cos \theta) \sin \phi, s \sin \theta).$$

For $0 < s < t$, we denote $T^s_t = \{\Phi^s_t(\theta, \phi) : \theta, \phi \in [0, 2\pi)\}$, which is a parametrized torus in $\mathbb{R}^3$. We consider a measure on $T^s_t$, which is given by

$$\langle f, \sigma^s_t \rangle = \int_{[0, 2\pi)^2} f(\Phi^s_t(\theta, \phi)) \, d\theta d\phi.$$
Convolution with the measure $\sigma_t^c$ gives rise to a 2-parameter averaging operator $A_t^c f := f * \sigma_t^c$. Let $0 < c_0 < 1$ be a fixed constant. We begin our discussion with the maximal operator

$$f \rightarrow \sup_{0 < t} |A_t^c f|,$$

which is generated by the averages over (isotropic) dilations of the torus $T_1^{c_0}$. It is not difficult to see that $f \rightarrow \sup_{0 < t} |A_t^c f|$ is bounded on $L^p$ if and only if $p > 2$. Indeed, writing $f * \sigma_t^c = \int f * \mu_t^c d\phi$, where $\mu_t^c$ is the measure on the circle $\{ t\Phi_1^c(\phi, \theta) : \theta \in [0, 2\pi) \}$. Since these circles are subsets of 2-planes containing the origin, $L^p$ boundedness of $f \rightarrow \sup_{t>0} |f * \mu_t^c|$ for $p > 2$ can be obtained using the circular maximal theorem [4]. In fact, we need $L^p$ boundedness of the maximal function given by the convolution averages in $\mathbb{R}^2$ over the circles $C((t/c_0)e_1, t)$, which are not centered at the origin. Here, $C(y, r)$ denotes the circle $\{ x \in \mathbb{R}^2 : \| x - y \| = r \}$. However, such a maximal estimate can be obtained by making use of the local smoothing estimate for the wave operator (see, for example, [23]). Failure of $L^p$ boundedness of $f \rightarrow \sup_{0 < c} |A_t^c f|$ for $p \leq 2$ follows if one takes $f(x) = \tilde{\chi}(x)|x_3|^{-1/2}| \log |x_3||^{-1/2-\epsilon}$ for a small $\epsilon > 0$, where $\tilde{\chi}$ is a smooth positive function supported in a neighborhood of the origin.

In the study of the averaging operator defined by hypersurface, nonvanishing curvature of the underlying surface plays a crucial role. However, the torus $T_1^{c_0}$ has vanishing curvature. More precisely, the Gaussian curvature $K(\theta, \phi)$ of $T_1^{c_0}$ at the point $\Phi_1^{c_0}(\theta, \phi)$ is given by

$$K(\theta, \phi) = \frac{\cos \theta}{c_0(1 + c_0 \cos \theta)}.$$ 

Notice that $K$ vanishes on the circles $\Phi_1^{c_0}(\pm \pi/2, \phi), \phi \in [0, 2\pi)$. Decomposing $T_1^{c_0}$ into the parts which are away from and near those circles, we can show, in an alternative way, $L^p$ boundedness of $f \rightarrow \sup_{0 < c} |A_t^c f|$ for $p > 2$. The part away from the circles has nonvanishing curvature. Thus, the associated maximal function is bounded on $L^p$ for $p > 3/2$ ([31]). Meanwhile, the other parts near the circles can be handled by the result in [17].

2-parameter maximal function. We now consider a two-parameter maximal function

$$Mf(x) = \sup_{0 < s < c_0 t} |A_t^c f(x)|.$$ 

Here, the supremum is taken over on the set $\{(t, s) : 0 < s < c_0 t\}$ so that $T_s^t$ remains to be a torus. Unlike the one-parameter maximal function, (nontrivial) $L^p$ on $M$ can not be obtained by the same argument as above which relies $L^p$ boundedness of a related circular maximal function in $\mathbb{R}^2$. In fact, to carry out the same argument, one needs $L^p$ boundedness of the maximal function given by the (convolution) averages over the circles $C(se_1, t)$ while supremum is taken over $0 < s < c_0 t$. However, Talagrand’s construction [32] (also see [13 Corollary A.2]) shows that this (two-parameter) maximal function can not be bounded on any $L^p, p \neq \infty$.

The following is our first result, which is somewhat surprising in that the two-parameter maximal function $M$ has the same $L^p$ boundedness as the one-parameter maximal function $f \rightarrow \sup_{0 < t} |A_t^c f|$.

**Theorem 1.1.** The maximal operator $M$ is bounded on $L^p$ if and only if $p > 2$. 


Localized maximal function. The localized spherical and circular maximal functions which are defined by taking supremum over radii contained in a compact interval included in \((0, \infty)\) have \(L^p\)-improving property, that is to say, the maximal operators are bounded from \(L^p\) to \(L^q\) for some \(p < q\). Schlag \cite{26} and Schlag–Sogge \cite{27} characterized the almost complete typeset of \(p,q\) except the endpoint cases. One of the authors \cite{20} obtained most of the remaining endpoint cases. There are also results in which dilation parameter sets were generalized to sets of fractal dimensions (for example, see \cite{1, 29}).

In analogue to those results concerning the localized spherical and circular maximal operators, it is natural to investigate \(L^p\)-improving property of \(\mathcal{M}_c\) which is defined by
\[
\mathcal{M}_c f(x) = \sup_{(t,s) \in J} \left| A^s_t f(x) \right|.
\]
Here \(J\) is a compact subset of \(J_* := \{(t,s) \in \mathbb{R}^2 : 0 < s < t\}\). The next theorem gives \(L^p-L^q\) bound on \(\mathcal{M}_c\) on a sharp large of \(p,q\).

**Theorem 1.2.** Set \(P_1 = (5/11, 2/11)\) and \(P_2 = (3/7, 1/7)\). Let \(Q\) be the open quadrangle with vertices \((0,0)\), \((1/2,1/2)\), \(P_1\), and \(P_2\) which includes the half open line segment \([0,0), (1/2,1/2)\). (See Figure 1) Then, the estimate
\[
(1.2) \quad \| \mathcal{M}_c f \|_{L^q} \lesssim \| f \|_{L^p}
\]
holds if \((1/p, 1/q) \in Q\).

Conversely, if \((1/p, 1/q) \notin \overline{Q} \setminus \{(1/2,1/2)\}\), then the estimate (1.2) fails.

**Smoothing estimates for \(A^s_t\).** Smoothing estimates for averaging operators have a close connection to the associated maximal functions. Especially, the local smoothing estimate for the wave operator was used by Mockenhaupt–Seeger–Sogge \cite{24} to provide an alternative proof of the circular maximal theorem. Recent progress \cite{18,2,19} on the maximal functions associated with the curves in higher dimensions were also achieved by relying on local smoothing estimates (also see \cite{25}). Analogously, our proofs of Theorem \ref{thm:1.1} and \ref{thm:1.2} are also based on 2-parameter local

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The typeset of \(\mathcal{M}_c\)}
\end{figure}
smoothing estimates for the averaging operator $A^s_t$, which are of independent interest. In the following, we obtain the sharp two-parameter local smoothing estimate for $A^s_t$.

**Theorem 1.3.** Let $p \geq 2$ and $\psi$ be a smooth function with its support contained in $J_*$. Set $\tilde{A}^s_t f(x) = \psi(t,s) A^s_t f(x)$. Then, the estimate

$$\|\tilde{A}^s_t f\|_{L^p_\alpha(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}$$

holds if $\alpha < \min\{1/2, 4/p\}$.

The result in Theorem 1.3 is sharp in that $\tilde{A}^s_t$ can not be bounded from $L^p$ to $L^p_\alpha$ if $\alpha > \min\{1/2, 4/p\}$ (see Section 5 below). Using the estimate (1.3), one can deduce results concerning the dimension of a union of the torus $x + \Gamma^s_t$, $(x,t,s) \in E \subset \mathbb{R}^3 \times J_*$. See [12].

We also obtain the sharp local smoothing estimate for the 1-parameter operator $f \rightarrow A^{c_0 t}_t f$.

**Theorem 1.4.** Let $\chi_0 \in C_\infty_c(0,\infty)$. Let $p \geq 2$ and $0 < c_0 < 1$. Then, for $\alpha < \min\{1/2, 3/p\}$, we have

$$\|\chi_0(t) A^{c_0 t}_t f\|_{L^p_\alpha(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$  

The estimate above is sharp since $f \rightarrow \chi_0(t) A^{c_0 t}_t f$ fails to be bounded from $L^p_\alpha$ to $L^p_\alpha(\mathbb{R}^4)$ if $\alpha > \min\{1/2, 3/p\}$ (Section 5). The next theorem gives the sharp regularity estimate for $A^s_t$ when $s,t$ fixed.

**Theorem 1.5.** Let $0 < s < t$. If $\alpha < \min\{1/2, 2/p\}$, then we have

$$\|A^s_t f\|_{L^p_\alpha(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$  

If $\alpha > \min\{1/2, 2/p\}$, $\tilde{A}^s_t$ is not bounded from $L^p(\mathbb{R}^3)$ to $L^p_\alpha(\mathbb{R}^3)$ (Section 5). One can compare the local smoothing estimates in Theorem 1.3 and 1.4 with the regularity estimate in Theorem 1.5. The 2-parameter and 1-parameter local smoothing estimates have extra smoothing of order up to $2/p$ and $1/p$, respectively, when $p > 8$ (see Figure 2).

For $p < 2$, it is easy to show that there is no additional smoothing (local smoothing) for the operators $A^s_t$ and $\chi_0(t) A^{c_0 t}_t$ when compared with the estimates with fixed $s,t$ (Theorem 1.5). That is to say, $\tilde{A}^s_t$ fails to be bounded from $L^p(\mathbb{R}^3)$ to
Theorem 2.1. \(\text{Zhang [11]}\) (also see \([27]\)). The following is a consequence of the sharp local smoothing due to Guth–Wang–Wang from (2.2). Indeed, since

\[ q \]

\[ I \]

\[ p \]

local smoothing estimates for \(\hat{g}\). In addition to \(\hat{g}\) and \(\hat{g}^{-1}\) as well as \(\hat{g}^\circ\) for \(\tau \in (0, 1]\), we write \(\hat{g} \subset A_\lambda\) if there is a constant \(C > 0\) such that \(B \leq CA\).

2. Local smoothing Estimates for \(A_{s,t}\)

In this section we are mainly concerned with estimates for the averaging operator under frequency localization of the input function. We obtain the estimates by making use of the decoupling inequality and the local smoothing estimate for the wave operator.

We denote

\[ A_\lambda = \{ \eta \in \mathbb{R}^2 : 2^{-\lambda} \leq |\eta| \leq 2\lambda\}, \quad A^\circ_\lambda = \{ \eta \in \mathbb{R}^2 : |\eta| \leq 2\lambda\}, \]

respectively. Similarly, we set \(\mathbb{I} = [1, 2]\) and \(\mathbb{I}^\circ = [0, 2]\), and we denote \(\mathbb{I}_\tau = \tau \mathbb{I}\) and \(\mathbb{I}^\circ_\tau = \tau \mathbb{I}^\circ\) for \(\tau \in (0, 1]\).

We now consider the 2-d wave operator

\[ W_{\pm}(y, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\iota(y \cdot \eta \pm \iota t/|\eta|)} \hat{g}(\eta) d\eta. \]

The following is a consequence of the sharp local smoothing due to Guth–Wang–Zhang \([11]\) (also see \([27]\)).

**Theorem 2.1.** Let \(2 \leq p \leq q, 1/p + 3/q \leq 1, \) and \(\lambda \geq 1\). Then, the estimate

\[ \| W_{\pm} g \|_{L^q(\mathbb{R}^2 \times \mathbb{I}^\circ)} \leq C \lambda^{\frac{3}{2} + \frac{1}{p} - \frac{1}{q} - \epsilon} \| g \|_{L^p} \]

holds for any \(\epsilon > 0\) whenever \(\supp \hat{g} \subset A_\lambda\).

**Proof.** It is sufficient to show the estimate for \(W_+\) since that for \(W_-\) follows by conjugation and reflection. When the interval \(\mathbb{I}^\circ\) is replaced by \(\mathbb{I}\), the desired conjugate follows from the known estimates and interpolation. Indeed, for \(1 \leq p \leq q \leq \infty\) and \(1/p + 3/q \leq 1\), we have

\[ \| W_+ g \|_{L^s(\mathbb{R}^2 \times \mathbb{I})} \leq C \lambda^{\frac{3}{2} + \frac{1}{p} - \frac{1}{q} - \epsilon} \| g \|_{L^p} \]

whenever \(\supp \hat{g} \subset A_\lambda\). This is a consequence of interpolation between the sharp \(L^p\) local smoothing estimates for \(p = q \geq 4\) \([11]\) and the estimate \(\| W_+ g \|_{L^\infty(\mathbb{R}^2 \times \mathbb{I})} \leq C \lambda^{\frac{3}{2}} \| g \|_{L^1}\) (e.g., see \([27]\)).

By dyadically decomposing \(\mathbb{I}^\circ\) away from 0 and scaling, one can deduce (2.1) from (2.2). Indeed, since

\[ W_+ g(x, \tau t) = W_+ g(\tau \cdot)(x/\tau, t), \]

and so does \(\chi_0(t) A_{\alpha} \| f \|_{L^p(\mathbb{R}^3)} \) to \(L^p_\alpha(\mathbb{R}^4)\) if \(\alpha > \min(2/p', 1/2)\) and \(1 \leq p \leq 2\).

Finally, we remark that our result for two parameter 2-dimensional tori can be extended to multiparameter tori in higher dimensions. We will address this issue elsewhere.
Proposition 2.3. Let \( R \) is closely related to the averaging operator \( \text{supp} \) whenever \( \text{check the estimate (2.4)} \) for \( (\omega g W) \) have
conical region with a small angle. later to obtain estimate for functions whose Fourier supports are included in a 
boundedly overlapping rectangles of dimension \( h \)
rescaling gives the estimate
\[
\| W_+ g \|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C \tau^{\frac{1}{2} - \frac{1}{p} \lambda^\frac{3}{2} - \frac{1}{q} - \frac{1}{p} + \epsilon} \| g \|_{L^p}
\]
for any \( \epsilon > 0 \) if \( \text{supp} \tilde{g} \subset A_{\lambda} \) if \( \tau \lambda \geq 1 \). When \( \tau \sim \lambda^{-1} \), by scaling and an easy estimate we also have \( \| W_+ g \|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq \lambda^{2/p - 3/q} \| g \|_{L^p} \). Now, since \( p \geq 2 \), decomposing \( \mathbb{R}^n = (\bigcup_{\tau > (2\lambda)^{-1}} \Pi_{\tau}) \subset \Pi_{\lambda^{-1}} \), and taking sum over the intervals, we get
\[
\| W_+ g \|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C \max \{ \lambda^{\frac{1}{2} - \frac{1}{p} + \epsilon}, \lambda^{\frac{3}{2} - \frac{1}{q} - \frac{1}{p} + \epsilon} \} \| g \|_{L^p} \leq \lambda^{\frac{1}{2} - \frac{1}{p} + \epsilon} \| g \|_{L^p}
\]
for any \( \epsilon > 0 \).

As a consequence of Theorem 2.2 we also have the next lemma, which we use later to obtain estimate for functions whose Fourier supports are included in a conical region with a small angle.

Lemma 2.2. Let \( 2 \leq p \leq q \leq \infty \), \( 1/p + 3/q \leq 1 \), and \( \lambda \geq 1 \). Suppose that \( \lambda \leq h \leq \lambda^2 \). Then, for any \( \epsilon > 0 \) there is a constant \( C \) such that
\[
\| W_\pm g \|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C \lambda^{\frac{1}{2} - \frac{1}{p} + \frac{1}{q} - \frac{1}{p} + \epsilon} \| g \|_{L^p}
\]
whenever \( \text{supp} \tilde{g} \subset I_h \times \Pi_{\lambda^\epsilon} \).

Proof. As before, it is sufficient to consider \( W_+ \). By interpolation we only need to check the estimate (2.4) for \( (p, q) = (4, 4), (2, 6), (2, \infty) \), and \((\infty, \infty)\). Since \( \lambda \leq h \), \( \text{supp} \tilde{g} \subset \{ \eta : \| \eta \| \sim h \} \). So, the estimate (2.4) for \( (p, q) = (4, 4), (2, 6) \) and \((\infty, \infty)\) is clear from (2.1). Thus, it suffices to verify (2.4) for \( p = q = \infty \), that is to say,
\[
\| W_\pm g \|_{L^\infty(\mathbb{R}^3 \times \Pi^p)} \leq C \lambda h^{-1/2} \| g \|_{L^\infty}
\]
whenever \( \text{supp} \tilde{g} \subset I_h \times \Pi_{\lambda^\epsilon} \). To show this, we cover \( I_h \times \Pi_{\lambda^\epsilon} \) by as many as \( C \lambda h^{-1/2} \)
boundedly overlapping rectangles of dimension \( h \times h^{1/2} \) whose principal axis contains the origin, and consider a partition of unity \( \{ \tilde{\omega}_\nu \} \) subordinated to those rectangles such that \( (\alpha, \beta) \)-th derivatives of \( \tilde{\omega}_\nu \) in the directions of the principal and its normal directions is bounded by \( C h^{-\alpha} h^{-\beta/2} \). (In fact, one can also use \( \omega_\nu(\eta) \) in the proof of Proposition 2.3 below replacing \( \lambda \) by \( h \).) Consequently, we have \( W_\pm g = \sum_\nu W_\pm \chi_\nu(D) g \). It is easy to see that the kernel of the operator \( g \to W_\pm \chi_\nu(D) g \) has a uniformly bounded \( L^1 \) norm for \( t \in \Pi^p, \nu \). Therefore, we get the desired estimate.

\[\square\]

2.1. Two-parameter propagator. We define an operator \( \mathcal{U} \) by
\[
\mathcal{U}f(x, t, s) = \int e^{i(x \xi + t(\xi + s) + s) \xi} d\xi,
\]
is closely related to the averaging operator \( A^\tau_\lambda \) and the wave operator \( W_\pm \). In fact, we obtain the estimates for \( \mathcal{U} \) making use of those for \( W_\pm \).

Let \( \mathbb{J}_0 = \{(t, s) : 0 < s < c_0 t \} \) and \( \mathbb{J}_\tau = (I \times \Pi) \cap \mathbb{J}_0 \). To obtain the required estimates for our purpose, we consider the estimates over \( \mathbb{R}^3 \times \mathbb{J}_\tau \) for small \( \tau \).

Proposition 2.3. Let \( 2 \leq p \leq q \leq \infty \) satisfy \( 1/p + 3/q \leq 1 \), and let \( 0 < \tau \leq 1 \) and \( \lambda \geq \tau^{-1} \). (a) If \( \lambda \leq h \leq \tau \lambda^2 \), then for any \( \epsilon > 0 \) the estimate
\[
\| \mathcal{U}f \|_{L^p(\mathbb{R}^3 \times \mathbb{J}_\tau)} \leq \tau^{(\frac{1}{2} - \frac{1}{p})} \lambda^{\frac{3}{2} - \frac{1}{q} - \frac{1}{p} + \frac{1}{p} + \epsilon} \| g \|_{L^p}
\]
holds whenever $\supp \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{I}_h$. Moreover, (b) if $\supp \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{I}_h^\circ$, then we have the estimate (2.5) with $h = \lambda$. (c) If $h \gtrsim \tau \lambda^2$, then we have
\begin{equation}
\|Uf\|_{L^q_t(L^r_x)} \lesssim \tau^{1/2} \lambda^{1/2 + 1/q - 2/r + \epsilon} h^{1/2 - 1/r} \|f\|_{L^p_t(L^r_x)}
\end{equation}
whenever $\supp \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{I}_h$.

For a bounded measurable function $m$, we denote by $m(D)$ the multiplier operator defined by $\mathcal{F}(m(D)f)(\xi) = m(\xi) \hat{f}(\xi)$. In what follows, we occasionally use the following lemma.

**Lemma 2.4.** Let $\xi = (\xi', \xi'') \in \mathbb{R}^k \times \mathbb{R}^{d-k}$. Let $\chi$ be an integrable function on $\mathbb{R}^k$ such that $\hat{\chi}$ is also integrable. Suppose $\|m(D)f\|_q \leq B\|f\|_p$ for a constant $B > 0$, then we have $\|m(D)\chi(D')f\|_q \leq B\|\hat{\chi}\|_1\|f\|_p$.

This lemma follows from the identity
\begin{equation}
m(D)\chi(D')f(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} \hat{\chi}(y)(m(D)f)(x + y, x'')dy,
\end{equation}
which is a simple consequence of the Fourier inversion. The desired inequality is immediate from Minkowski’s inequality and translation invariance of $L^p$ norm.

**Proof of Proposition 2.3.** We make use of the decoupling inequality for the cone [5] and the sharp local smoothing estimate (Lemma 2.2) for $\mathcal{W}_+$. We first show the case (a) where $\lambda \lesssim h \lesssim \tau \lambda^2$. To this end, we prove the estimate (2.6) under the additional assumption that $q \geq 6$. We subsequently extend the range by interpolation between the consequent estimates and (2.5) for $(p, q) = (4, 4)$, which we prove later.

Fixing $x_3$ and $s$, we define an operator $T_{x_3}^s$ by setting
\begin{equation}
T_{x_3}^s \mathcal{F}(\xi) = \int e^{i(x_3 \xi_3 + s|\xi|)} \hat{F}(\xi, \xi_3)d\xi_3, \quad \xi = (\xi', \xi_3).
\end{equation}
Then, we observe that
\begin{equation}
Uf(x, t, s) = \mathcal{W}(T_{x_3}^s f)(\bar{x}, t).
\end{equation}

Let $\mathfrak{G}_\lambda \subset \mathbb{S}^1$ be a collection of $\sim \lambda^{-1/2}$-separated points. By $\{w_\nu\}_{\nu \in \mathfrak{G}_\lambda}$, we denote a partition of unity on the unit circle $\mathbb{S}^1$ such that $w_\nu$ is supported in an arc centered at $\nu$ of length about $\lambda^{-1/2}$ and $|\partial^\nu/\partial^k w_\nu| \lesssim \lambda^{k/2}$. For each $\nu \in \mathfrak{G}_\lambda$, we set $\omega_\nu(\xi) = w_\nu(\xi/|\xi|)$ and
\begin{equation}
\mathcal{W}_\nu g(\bar{x}, t) = \int e^{i(x+\bar{t}/|\xi|)(\xi)} \omega_\nu(\xi) \hat{g}(\xi) d\xi.
\end{equation}

Let $\hat{\chi} \in \mathcal{S}(\mathbb{R})$ such that $\hat{\chi} \geq 1$ on $\mathbb{I}$ and $\supp \mathcal{F}(\hat{\chi}) \subset [-1/2, 1/2]$. Note that the Fourier transform of $\hat{\chi}(t)\mathcal{W}_\nu g(\bar{x}, t)$ is supported in the set $\{\xi, \tau : \tau - |\xi| \leq 1, \xi/|\xi| \in \supp \omega_\nu, |\xi| \sim \lambda\}$ if $\supp \hat{g} \subset \mathcal{A}_\lambda$. Thus, by Bourgain–Demeter’s $l^2$ decoupling inequality [3] followed by Hölder’s inequality, we have
\begin{equation}
\left(\sum_{\nu \in \mathfrak{G}_\lambda} \mathcal{W}_\nu g\right)^2 \lesssim \lambda^{3/2} \frac{1}{q - 2/r + \epsilon} \left(\sum_{\nu \in \mathfrak{G}_\lambda} \|\hat{\chi}(t)\mathcal{W}_\nu g\|_{L^2_t(L^r_x)}^p\right)^{1/p}
\end{equation}
for any $\epsilon > 0$, $q \geq 6$, and $p \geq 2$, provided that $\supp \hat{g} \subset \mathcal{A}_\lambda$. Note that $Uf(x, t, s) = \sum_{\nu} \mathcal{W}_\nu(T_{x_3}^s f)(\bar{x}, t)$ and $\mathcal{W}_\nu(T_{x_3}^s f)(\bar{x}, t) = U\omega_\nu(D)f(x, t, s)$.
freezing $s, x_3$, we apply the inequality (2.7), followed by Minkowski’s inequality, to get

\[
\|\mathcal{U}f\|_{L^q_t(L^q \times I_t)} \lesssim \lambda^{\frac{3}{2} - \frac{1}{p}} - \frac{1}{2} - \frac{1}{p} \left( \sum_{\nu \in \mathbb{N}_+^3} \|\tilde{\chi}(t)(\mathcal{U}f_{\nu})\|_{L^p_{x,t}}^{\frac{1}{p}} \right)^{1/p}
\]

for $q \geq 6$ where $f_{\nu} = \omega_{\nu}(\tilde{D})f$. We now claim that

\[
\|\tilde{\chi}(t)(\mathcal{U}f_{\nu})\|_{L^q_t(L^q \times I_t)} \lesssim \tau^{\frac{1}{2} - \frac{1}{q}} \lambda^{1 - \frac{1}{p} + \frac{1}{2}} - \frac{1}{2} - \frac{1}{p} \|f_{\nu}\|_{L^p}
\]

holds for $1/p + 3/q \leq 1$. Note that $(\sum_{\nu} \|f_{\nu}\|_{L^p}^{p})^{1/p}$ for $1 \leq p \leq \infty$. Thus, from (2.8) and (2.9) the estimate (2.5) follows for $q \geq 6$.

To obtain (2.9), we begin by showing

\[
\|\tilde{\chi}(t)(\mathcal{U}f_{\nu}(\cdot, s))\|_{L^q_t(L^q)}^{\frac{1}{q}} \leq C\|e^{is|D|}f_{\nu}\|_{L^q_t(L^q)}^{\frac{1}{q}}
\]

To do this, we apply the argument used to show Lemma 2.4. Let us set

\[
\tilde{\chi}_{\nu}(t, \xi) = e^{it(|\xi| - \xi \cdot \nu)}\tilde{\omega}_{\nu}(\xi)\varphi(\xi / \lambda)
\]

so that $\tilde{\chi}_{\nu}(t, \xi)\tilde{f}_{\nu}(\xi) = e^{it(|\xi| - \xi \cdot \nu)}\tilde{f}_{\nu}(\xi)$. Here $\tilde{\omega}_{\nu}(\xi)$ is a angular cutoff function given in the same manner as $\omega_{\nu}(\xi)$ such that $\tilde{\omega}_{\nu}\omega_{\nu} = \omega_{\nu}$. Then, a computation shows that

\[
|\nu \cdot \nabla_{\xi}^k(\nu_\perp \cdot \nabla_{\xi}^l)\tilde{\chi}_{\nu}(t, \xi)| \lesssim (1 + |t|)^{k+l} \lambda^{\frac{1}{2} - \frac{1}{q}}(1 + \lambda^{1/2})^{-N}(1 + \lambda^{1/2})^{-N}
\]

for any $N$ where $\nu_\perp$ denotes a unit vector orthogonal to $\nu$. Indeed, this can be easily seen via rotation and scaling (i.e., setting $\nu = e_1$ and scaling $\xi_1 \rightarrow \lambda \xi_1$ and $\xi_2 \rightarrow \lambda^{1/2} \xi_2$). Thus, using the above inequality for $0 \leq k, l \leq 2$ and integration by parts, we see $(|\tilde{\chi}_{\nu}(t, \xi)|^2)_{|t|} \leq C(1 + |t|)^4$ for a constant $C > 0$. Since $\mathcal{U}f_{\nu}(x, t, s) = F^{-1}(e^{it\nu \cdot \xi + \frac{|s|}{\lambda}}\tilde{\chi}_{\nu}(t, \xi)\tilde{f}_{\nu}(\xi))$, we have

\[
\mathcal{U}f_{\nu}(x, t, s) = \int (\tilde{\chi}_{\nu}(t, \cdot))^{\vee}(\eta) e^{is|D|f_{\nu}(\tilde{x} - \eta + tv, x_3))}d\eta.
\]

By Minkowski’s inequality and changing variables $\tilde{x} \rightarrow x - \eta - tv$ we see that the left hand side of (2.10) is bounded by $C\|\tilde{\chi}(t)(1 + |t|)^{4}\|_{L^q_t(L^q)}\|e^{is|D|f_{\nu}}\|_{L^q_t(L^q)}^{\frac{1}{q}}$. Therefore, we get the desired inequality (2.10).

Let us set

\[
\chi_{\nu}(\xi) = e^{is(|\xi| - \xi \cdot \nu)}\tilde{\omega}_{\nu}(\xi)\varphi(\xi / \lambda)\varphi(\xi_3 / h),
\]

where $\xi \cdot \nu, \xi_3$. Since $\lambda \lesssim h$, similarly as before, one can easily see $\|\chi_{\nu}\|_{1} \leq C$ for a constant. Thus, by Lemma 2.4 we have $\|e^{is|D|f_{\nu}}\|_{L^q_t(L^q)}^{\frac{1}{q}} \lesssim \|e^{is|D|f_{\nu}}\|_{L^q_t(L^q)}^{\frac{1}{q}}$. Combining this and (2.10) yields

\[
\|\mathcal{U}f_{\nu}\|_{L^q_t(L^q \times I_t)} \lesssim \|e^{is|D|f_{\nu}}\|_{L^q_t(L^q \times I_t)} \lesssim \lambda^{\frac{3}{2} - \frac{1}{p} - \frac{1}{2} - \frac{1}{p} \|\|e^{is|D|f_{\nu}}\|_{L^q_t(L^q \times I_t)}^{\frac{1}{q}}(L^q_t(L^q \times x_3, (R^2 \times I_t)))},
\]

where $\tilde{x}_{\nu} = \nu \cdot \tilde{x}$ and $\tilde{x}_{\nu}' = \nu_\perp \cdot \tilde{x}$. For the second inequality we use Bernstein’s inequality (see, for example, [23, Ch.5]) and Minkowski’s inequality together with the fact that the projection of supp $\tilde{f}$ to span{e_\nu} is contained in an interval of length $\lesssim \lambda^{1/2}$.

Note that the projection supp $\tilde{f}$ to span{e_\nu} is contained in the rectangle $I_x \times I_t$. By rotation the matter is reduced to obtaining estimate for the 2-d wave operator. That is to say, the inequality (2.9) follows for $q \geq 6$ if we show

\[
\|\mathcal{W}_t g\|_{L^q(R^2 \times I_t)} \lesssim \tau^{\frac{1}{2} - \frac{1}{q}} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{q} \|g\|_{L^p}
\]
for $1/p + 3/q \leq 1$ whenever $\text{supp } \hat{g} \subset \mathbb{I}_h \times \mathbb{I}_\nu^\lambda$. This inequality is an immediate consequence of (2.4) and scaling. Indeed, as before, after scaling (i.e., (2.3)) we apply Lemma 2.2 with $\text{supp } \mathcal{F}(g(\tau \cdot)) \subset \mathbb{I}_{\tau h} \times \mathbb{I}^\lambda_{\nu, x}$. To this end, we use the condition $h \leq \tau \lambda^2$, equivalently, $\tau h \leq (\tau \lambda)^2$.

We now have the estimate (2.5) for $6 \leq q$, $2 \leq p$, and $1/p + 3/q \leq 1$. In order to prove it in the full range, by interpolation we only have to show (2.5) for $p = q = 4$.

Let us define $f_\pm$ by setting $f_\pm(\xi) = \chi(0, \infty)(\pm \xi_2)\hat{f}(\xi)$ where $\chi_E$ denotes the character function of a set $E$. Then, changing variables $\xi_2 \to \pm \sqrt{\rho^2 - \xi_1^2}$, we write

$$
\mathcal{U}f(x, t, s) = \sum_{\pm} \int e^{i(x_\perp x_\perp t + y \cdot s)} e^{i(2 \pm \xi_2)} \mathcal{F}(\mathcal{S}_{\pm} \hat{f}_\pm)(\rho, \xi_3) d\rho d\xi_3,
$$

where

$$
\mathcal{F}(\mathcal{S}_{\pm} \hat{f}_\pm)(\rho, \xi_3) = \pm \int e^{i(x_\perp x_\perp t + y \cdot s)} e^{i(2 \pm \xi_2)} \hat{f}_\pm(\xi_1, \pm \sqrt{\rho^2 - \xi_1^2}, \xi_3) \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} d\xi_1.
$$

We observe the following, which is a consequence of the estimate (2.2) with $p = q = 4$ and the finite speed of propagation of the wave operator:

$$
\|\mathcal{W}_+ g\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim \tau \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \|\mathcal{W}_+ g\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} + h^{-N} \|t^{-N} \mathcal{W}_+ g\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)}
$$

for any $N$ whenever $\text{supp } \hat{g} \subset \{\xi : |\xi| \sim h\}$. Indeed, to show this we decompose $g = g_1 + g_2 := g\chi_1(y_2) + g\chi_1(y_2)$. By finite speed of propagation (in fact, by straightforward kernel estimate) we have $\|\mathcal{W}_+ g_2\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim h^{-N} \|t^{-N} \mathcal{W}_+ g\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)}$. Meanwhile, by scaling and (2.2) with $p = q = 4$, we have $\|\mathcal{W}_+ g_1\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim \tau \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \|g\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)}$. Combining those two estimates, we obtain (2.11).

We now note that $\mathcal{U}f(x, t, s) = \sum_{\pm} \mathcal{W}_+ (\mathcal{S}_{\pm} \hat{f}_\pm)(x_\perp, t, s) + \text{supp } \mathcal{F}(\mathcal{S}_{\pm} \hat{f}_\pm) \subset \{\xi : |\xi| \sim h\}$ since $\lambda \leq h$. Here, we regard $(x_\perp, t)$ and $s$ as the spatial and temporal variables, respectively. Applying (2.11) to $\mathcal{W}_+(\mathcal{S}_{\pm} \hat{f}_\pm)$ with $g = \mathcal{S}_{\pm} \hat{f}_\pm$, we obtain

$$
\|\mathcal{U}f\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim \sum_{\pm} \left(\tau \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \|\mathcal{S}_{\pm} \hat{f}_\pm\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)} + h^{-N} \|t^{-N} \mathcal{S}_{\pm} \hat{f}_\pm\|_{L^4_{x,t,\nu}(\mathbb{R}^3 \times \mathbb{R}^+)}\right).
$$

Reversing the change of variables $\xi_2 \to \pm \sqrt{\rho^2 - \xi_1^2}$, we note that $\mathcal{S}_{\pm} \hat{f}_\pm(x_\perp, t) = \mathcal{W}_+ f_\pm(\cdot, x_\perp)(x_\perp, t)$. Recalling $\text{supp } \mathcal{F} \subset A_\lambda \times \mathbb{I}_h$, we see that the second term in the right hand side is bounded by a constant times $h^{-N/2} \|f\|_{L^4}$. Since $\text{supp } \mathcal{F}(\cdot, x_\perp) \subset A_\lambda$ for all $x_\perp$, using Lemma 2.2 for $p = q = 4$, we obtain (2.5) for $p = q = 4$. This completes the proof of (a).

The case (b) in which $\text{supp } \hat{f} \subset A_\lambda \times \mathbb{I}^\lambda_{\nu, x}$ can be handled without change. We only need to note that the Fourier support of $f_\nu$ is included in $\{\xi : |(\xi \cdot \nu, \xi_3)| \sim \lambda\}$, instead of $\{\xi : |(\xi \cdot \nu, \xi_3)| \sim h\}$, if $f_\nu \neq 0$.

We now consider the case (c) where $\text{supp } \hat{f} \subset A_\lambda \times \mathbb{I}_h$ with $\tau \lambda^2 \leq h$. The estimate (2.6) is easier to show. We note that the Fourier transform of

$$
eq e^{is(|\xi| - |\xi_3|)} \varphi(\xi/h) \varphi(\xi_3/h)
$$

has uniformly bounded $L^1$ norm. One can easily verify this using $\partial_j s(|(\lambda \xi_3, h \xi_3)| - |h \xi_3|) = O(1)$ on $A_\lambda \times \mathbb{I}_h$ if $\tau \lambda^2 \leq h$. Thus, by Lemma 2.4 we have $\|\mathcal{U}f(\cdot, t, s)\|_{L^4} \lesssim \|e^{itD}f\|_{L^4}$ uniformly in $s$. So, taking integration in $t$, $s$, we get

$$
\|\mathcal{U}f\|_{L^4(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim \tau \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \|e^{itD}f\|_{L^4(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim \tau \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}} \|e^{itD}f\|_{L^4(\mathbb{R}^3 \times \mathbb{R}^+)}.
$$
2.3. When \( \supp \hat{f} \subseteq \mathcal{A}_{\lambda}^\circ \times \mathbb{R}, \lambda \leq 1/\tau. \) If \( \supp \hat{f} \subseteq \mathcal{A}_{1/\tau}^\circ \times \mathbb{R}^{\tau}, \) the sharp estimates are easy to obtain.

**Lemma 2.5.** Let \( 1 \leq p \leq q \leq \infty \) and \( \tau \in (0, 1]. \) Suppose \( \supp \hat{f} \subseteq B(0, 1/\tau) := \{x : |x| < 1/\tau\}. \) Then, for a constant \( C > 0 \) we have
\[
\|\mathcal{A}_t^s f\|_{L^q_{\tau,s}(\mathbb{R}^3 \times J_\tau)} \leq C\tau^{-\frac{3}{\tau}}\|f\|_{L^p}.
\]

**Proof.** Since \( \mathcal{A}_t^s \) is a convolution operator and \( \supp \hat{f} \subseteq B(0, 1/\tau), \) Bernstein’s inequality gives \( \|\mathcal{A}_t^s f\|_{L^q_{\tau,s}} \lesssim \tau^{-\frac{3}{\tau}}\|\mathcal{A}_t^s f\|_{L^p} \) for any \( s, t \in \mathbb{R}. \) Thus, we have
\[
\|\mathcal{A}_t^s f\|_{L^q_{\tau,s}} \lesssim \tau^{-\frac{3}{\tau}}\|f\|_{L^p}, \quad \forall s, t \in \mathbb{R}.
\]
The inequality (2.15) follows by integration in \( t, s \) over \( J_\tau. \)

**Proposition 2.6.** Let \( 1 \leq p \leq q \leq \infty, \) \( \tau \lesssim 1, \) and \( h \gtrsim 1/\tau. \) Suppose \( \supp \hat{f} \subseteq \mathcal{A}_{1/\tau}^\circ \times 1_h. \) Then, we have
\[
\|\mathcal{A}_t^s f\|_{L^q_{\tau,s}(\mathbb{R}^3 \times J_\tau)} \lesssim \tau^{1/q}(\tau h)^{-\frac{1}{2}}h^{-\frac{3}{\tau}}\|f\|_{L^p}.
\]

**Remark 1.** Following the argument in the proof of Proposition 2.3 and using Theorem 2.1 and Lemma 2.2, one can see without difficulty that \( f \to U f(x, -t, s) \) satisfies the same estimates in Proposition 2.3 in place of \( U. \) Then, by conjugation and reflection it follows that the estimates also hold for \( f \to U f(x, \pm t, -s). \)

2.2. Estimates for the averaging operator \( \mathcal{A}^\circ. \) Making use of the estimates for \( U \) in Section 2.1 (Proposition 2.3), we obtain estimates for the averaging operator \( \mathcal{A}_t^\circ \) while assuming the input function is localized in the Fourier side. These estimates are to play crucial roles in proving Theorem 1.1, 1.2, and 1.3.

We relate \( \mathcal{A}^\circ \) to \( U \) via asymptotic expansion of the Fourier transform of \( d\sigma_t^\circ. \)

Note that
\[
\widehat{d\sigma_t^\circ}(\xi) = \int_0^{2\pi} e^{-is\sin \theta \xi_3} \widehat{d\mu}(t + s \cos \theta \xi) d\theta,
\]
where \( d\mu \) denotes the normalized arc length measure on the unit circle. We recall the well known asymptotic expansion of the Bessel function (for example, see [30]):
\[
\widehat{d\mu}(\xi) = \sum_{\pm, 0 \leq j \leq N} C_j^\pm |\xi|^{-\frac{j}{2}} e^{\pm i\theta} + E_N(|\xi|), \quad |\xi| \gtrsim 1
\]
for some constants \( C_j^\pm \) where \( E_N \) is a smooth function satisfying
\[
|\langle d\sigma_t^\circ \rangle E_N(r) | \leq C r^{-l-(N+1)/4}, \quad 0 \leq l \leq N,
\]
for \( r \gtrsim 1 \) and a constant \( C > 0 \) where \( N' = [(N+1)/4]. \) We use (2.13) by taking \( N \) large enough.

Combining (2.12) and (2.13) gives an asymptotic expansion for \( \mathcal{F}(d\sigma_t^\circ), \) which we utilize by decomposing \( f \) in the Fourier side. We consider the cases \( \supp \hat{f} \subset \{\xi : |\xi| > 1/\tau\} \) and \( \supp \hat{f} \subset \{\xi : |\xi| \leq 1/\tau\}, \) separately.
Proof. To prove (2.17) it is sufficient to show, for a positive constant $C$,
\[
\|A^*_f\|_{L^q_t} \leq C(\tau h)^{-\frac{1}{2}}h^\frac{1}{2} \|f\|_{L^p_t}, \quad \forall (t,s) \in \mathbb{J}_r.
\]
In fact, integration over $\mathbb{J}_r$ yields (2.17).

For simplicity, we denote $v_\phi = (\cos \phi, \sin \phi)$, and we note that
\[
A^*_f(x) = (2\pi)^{-3} \int \int e^{i((x-\xi\phi) \cdot \xi + x_3 \xi_3 - (v_\phi \cdot \xi) t)} \tilde{f}(\xi) d\phi d\theta d\xi.
\]
Since $\text{supp} \, \tilde{f} \subset A^\alpha_\theta \times \mathbb{I}_h$, we may disregard the factor $e^{-itv_\phi \cdot \xi}$ using Lemma 2.4.

Indeed, let $\rho \in C_c(A^\alpha_\theta)$ such that $\rho = 1$ on $A_1$. Setting $\rho_1(\xi) = \rho(\xi)e^{itv_\phi \cdot \xi}$, we see $\|\mathcal{F}(\rho_1^\alpha)\|_1 \leq C$ for a constant $C > 0$ and $|t| \leq 1$. Thus, by Minkowski's inequality and Lemma 2.4 we have
\[
\|A^*_f\|_{L^q_t} \leq \sup_{\phi} \|\int \int e^{i(x-\xi\phi) \cdot \xi} \tilde{f}(\xi) d\theta d\xi\|_{L^q_t}
\]
for $|t| \leq 1$. We denote $\xi_\phi = (v_\phi \cdot \xi_3)$, and notice that $|s\xi_\phi| \geq 1$ since $h \tau \geq 1$. So, using (2.13), we have
\[
\int e^{-itv_\phi \cdot \phi} d\theta = \sum_{\pm, 0 \leq j \leq N} C^\pm_j |s\xi_\phi|^{-\frac{1}{2}-j} e^{\pm i\theta} \Xi_j + E_N(s|\xi_\phi|).
\]

To show (2.15), we obtain only the estimates for the operators $m^{\pm}_s(D), E_N(s|D_\phi|)$ whose multipliers are given by
\[
m^{\pm}_s(\xi) := |s\xi_\phi|^{-1/2} e^{\pm i\theta} \Xi_j, \quad E_N(s|\xi_\phi|).
\]

Contributions from the multiplier operators associated with the other terms can be handled similarly but those are easier. Since $|\xi| < 2$ and $|\xi_3| \sim h \geq 1/\tau$, we use the Mikhlin multiplier theorem and Lemma 2.4 to see
\[
\|m^{\pm}_s(D)f\|_{L^q_t} \lesssim (\tau h)^{-\frac{1}{2}} \|\int e^{i(x-\xi\phi) \cdot \xi} \tilde{f}(\xi) d\xi\|_{L^q_t} \leq (\tau h)^{-\frac{1}{2}} \|f\|_{L^p_t}.
\]
Since $\text{supp} \, \tilde{f} \subset A^\alpha_\theta \times \mathbb{I}_h$, by Bernstein’s lemma we have $\|f\|_{L^q_t} \lesssim h^{\frac{1}{2}} \|f\|_{L^p_t}$. This gives the desired estimates for $m^{\pm}_s(D)$. For the multiplier operator $E_N(s|D_\phi|)$, note from (2.14) that $\partial_\xi^\alpha \Lambda^*|s\xi_\phi|^{-N'} E_N(|s\xi_\phi|) \leq C(|s\xi_\phi|^{-|\alpha|})$ for $|\alpha| \leq N'$ and a constant $C > 0$. Using the Mikhlin multiplier theorem again, we have
\[
\|E_N(s|D_\phi|)f\|_{L^q_t} \lesssim \|\int e^{i(x-\xi\phi) \cdot \xi} \tilde{f}(\xi) d\xi d\theta\|_{L^q_t}.
\]
Since $\text{supp} \, \tilde{f} \subset A^\alpha_\theta \times \mathbb{I}_h$, we see, as before, that the right hand side is bounded by $C(h)^{-N} h^{1/p-1/q} \|f\|_{L^p_t}$. Thus, the desired estimate for $E_N(s|D_\phi|)$ follows. \hfill \Box

When $\lambda \geq 1$, to handle the case supp $\tilde{f} \subset A^\lambda_\theta \times \mathbb{I}_h$ we need more than the estimates with fixed $t,s$. We need the smoothing estimates obtained in Section 2.1.

**Proposition 2.7.** Let $2 \leq p \leq q \leq \infty$, $1/p + 1/q \leq 1$, and $1 \leq \lambda \leq 1/\tau \lesssim h$.

Suppose supp $\tilde{f} \subset A^\lambda_\theta \times \mathbb{I}_h$. Then, for any $e > 0$ we have the following:
\[
(2.19) \quad \|A^*_f\|_{L^q_t(\mathbb{R}^3)} \leq r^\frac{1}{q}(\tau h)^{-\frac{1}{2}} h^\frac{1}{2} \lambda^\frac{1}{2} \|f\|_{L^p_t}, \quad 1/p + 3/q \leq 1,
\]
\[
(2.20) \quad \|A^*_f\|_{L^q_t(\mathbb{R}^3)} \leq r^\frac{1}{q}(\tau h)^{-\frac{1}{2}} h^\frac{1}{2} \lambda^\frac{1}{2} \|f\|_{L^p_t}, \quad 1/p + 3/q > 1.
\]
To show Proposition 2.7 as mentioned above, we use the asymptotic expansion of the Fourier transform of $d\sigma_t^\pm$. Let us set
\[ m_t^\pm(\xi, t, s) = \int e^{-i(\xi \cdot s)} e^{\lambda t |\xi|} a_1(\theta, t, s) d\theta, \]
where $a_1(\theta, t, s) = (t + s \cos \theta)^{-2(t+1)/2}$. Putting (2.12) and (2.13) together, we have
\[ \overline{d\sigma_t^\pm}(\xi) = \sum_{\pm,0 \leq l \leq N} M_t^\pm(\xi, t, s) + \mathcal{E}(\xi, t, s) \]
for $|\xi| \gtrsim 1$ where
\[ M_t^\pm(\xi, t, s) = C_1|\xi|^{-l-\frac{3}{2}} e^{\pm i|\xi|} m_t^\pm(\xi, t, s), \quad \xi = 0, \ldots, N, \]
\[ \mathcal{E}(\xi, t, s) = \int e^{-i\xi \cdot s} E_N((t + s \cos \theta)|\xi|) d\theta. \]

Proof. We first show (2.13). From (2.21) we need to obtain estimates for the operators associated to the multipliers $M_t^\pm$ and $\mathcal{E}$. The major contributions are from $M_t^\pm(D, t, s)$. We claim that
\[ \|M_t^\pm(D, t, s)f\|_{L^p_t(L^q_x (\mathbb{R}^3 \times \mathcal{S}))} \lesssim \tau^\frac{1}{h} (\tau h)^{-\frac{1}{2} h} \lambda^\frac{1}{h} \frac{1}{\tau} - \frac{1}{l} + \epsilon \|f\|_{L^p} \]
holds for $p \leq q$ and $1/p + 3/q \leq 1$. To show this, we consider the operator $e^{\pm i(t\xi)} m_t^\pm(D, t, s)$. Note that $m_t^\pm(\xi, t, s) = \int e^{-i\xi \cdot s} m_t^\pm(\xi, t, s) d\theta$. By the stationary phase method, we have
\[ m_t^\pm(\xi, t, s) = \sum_{\pm,0 \leq j \leq N} B_j^\pm |\xi|^{-\frac{1}{2} - j} e^{\pm i|\xi|} + \tilde{E}_N^\pm(s|\xi|), \quad (t, s) \in \mathcal{J}_t \]
for $|\xi| \gtrsim 1$. Here, $B_j^\pm$ and $\tilde{E}_N^\pm$ depend on $t, s$. However, $(\partial / \partial \theta)^k a_1$ is uniformly bounded since $s < \epsilon_0 t$, i.e., $(t, s) \in \mathcal{J}_0$, so $B_j^\pm$ are uniformly bounded and $\tilde{E}_N^\pm$ satisfies (2.14) in place of $E_N$ as long as $(t, s) \in \mathcal{J}_t$.

For the error term $\tilde{E}_N^\pm(s|\xi|)$, we can replace it, similarly as before, by $|\xi|^{-N'}$ using the Mikhlin multiplier theorem. Thus, using (2.14) and Bernstein’s inequality in $x_3$ (see, for example, [34] Ch.5), we obtain
\[ \|\chi_{|x_3|}(t, s) e^{\pm i(t\xi)} \tilde{E}_N^\pm(s|\xi|) f\|_{L^p_t(L^q_x (\mathbb{R}^3 \times x_1))} \lesssim (\tau h)^{-N'} (\tau h)^{3} \lambda^{3/2} \tau^{1/2} + \frac{1}{l} + \epsilon \|f\|_{L^p} \]
for $p, q$ satisfying $1/p + 3/q \leq 1$ since supp $\tilde{f} \subset \mathcal{A}_\lambda \times \mathcal{I}_h$, $s \in \mathcal{I}_t$, and $\tau h \gtrsim 1$. Recalling (2.25), we consider the multiplier operator given by
\[ a_{l,t,s}^\pm(\xi) = \sum_{\pm,0 \leq j \leq N} B_j^\pm |\xi|^{-\frac{1}{2} - j}. \]

Since $\lambda \lesssim 1/\tau \lesssim h$, using the same argument as before (e.g., Lemma 2.4), we may replace $e^{\pm i|\xi|}$ with $e^{\pm i|\xi|}$, and $a_{l,t,s}^\pm$ by the Mikhlin multiplier theorem, we have
\[ \|\chi_{|x_3|}(t, s) e^{\pm i(t\xi)} a_{l,t,s}^\pm(D) f\|_{L^p_t(L^q_x (\mathbb{R}^3 \times x_1))} \lesssim (\tau h)^{-\frac{1}{2} h} \lambda^{\frac{3}{2}} \tau^{1/2} + \frac{1}{l} + \epsilon \|f\|_{L^p} \]
for $1/p + 3/q \leq 1$. Combining this and (2.26), we obtain
\[ \|\chi_{|x_3|}(t, s) M_t^\pm(D, t, s) f\|_{L^p_t(L^q_x (\mathbb{R}^3 \times x_1))} \lesssim (\tau h)^{-\frac{1}{2} h} \lambda^{\frac{3}{2}} \tau^{1/2} + \frac{1}{l} + \epsilon \|f\|_{L^p}. \]
Thus, taking integration in $s$ gives (2.24).
We now consider the contribution of the error term $E$ in (2.21), whose contribution is less significant. It can be handled by using the estimates for fixed $(t,s) \in J_{\tau}$. Recalling (2.21), we set
\[ E_N^2(\theta) := E_N^2(\theta, t, s, \xi) = |\xi|^N E_N((t + s \cos \theta)|\xi|). \]
We have $|\partial_\theta E_N^2(\theta)| \lesssim 1$ uniformly in $n, \theta$ for $(t,s) \in J_{\tau}$ since $(t + s \cos \theta) \gtrsim 1 - c_0$ for $(t,s) \in J_{\tau}$. By the stationary phase method [15, Theorem 7.7.5] one can obtain a similar expansion as before:
\[ \int e^{-is_3 \sin \theta} E_N^2(\theta) d\theta = \sum_{\pm, 0 \leq u \leq M} D_w^\pm |s\xi_3|^{-\frac{1}{2} - u} e^{\pm is_3} + E_M^\prime(|s\xi_3|) \]
for $(t,s) \in J_{\tau}$. Here, $E_M^\prime$ satisfies the same bounds as $E_N$ (i.e., (2.14)) and $M \leq N/4$. $D_w^\pm$ and $E_M^\prime$ depend on $t, s$, but they are harmless as can be seen by the Mikhlin multiplier theorem. The contribution from $E_M^\prime$ can be directly controlled by the Mikhlin multiplier theorem. Since supp $f \subset \mathbb{A}_\lambda \times I_h$, Bernstein’s inequality gives
\[ \left\| \int e^{-is_3 \sin \theta} E_N((t + s \cos \theta)|D|) d\theta f \right\|_{L^q} \lesssim (\tau h)^{-\frac{1}{2}} \lambda^{-\frac{1}{4}} \|f\|_2. \]
for $(t,s) \in J_{\tau}$. Note that the implicit constant here does not depend on $t, s$. Thus, integration in $s, t$ gives
\[ \|E(D, t, s)f\|_{L^q(\mathbb{R}^3 \times J_{\tau})} \leq C_{\tau} \lambda^{-\frac{1}{4}} \lambda^{-\frac{1}{4}} \|f\|_p \]
for $1 \leq p \leq q \leq \infty$. So, the contribution of $E(D, t, s)f$ is acceptable. Therefore, from (2.21) and (2.24), we obtain (2.19).

Putting (2.21), (2.22), (2.23), and (2.25) together, by Plancherel’s theorem one can easily see $\|A^\prime f\|_{L^2} \lesssim (\tau h)^{-\frac{1}{2}} \lambda^{-\frac{1}{4}} \|f\|_2$. Thus, integration in $s, t$ gives
\[ \|A^\prime f\|_{L^2(\mathbb{R}^3 \times J_{\tau})} \lesssim h^{-\frac{1}{2}} \lambda^{-\frac{1}{4}} \|f\|_2, \]
which is (2.20) for $p = q = 2$. Interpolation between this and the estimate (2.19) for $p, q$ satisfying $1/p + 3/q = 1$ gives (2.20) for $1/p + 3/q > 1$.

2.4. When $\text{supp} \hat{f} \subset \mathbb{A}_\lambda \times \mathbb{R}$ and $\lambda \gtrsim 1/\tau$. We have the following estimate.

**Proposition 2.8.** Let $2 \leq p \leq q \leq \infty$ satisfy $1/p + 1/q \leq 1$. (a) If $1/\tau \lesssim \lambda \lesssim h \lesssim \tau \lambda^2$, then for any $\epsilon > 0$ we have the estimates
\[ \|A^\prime f\|_{L^p(\mathbb{R}^3 \times J_{\tau})} \lesssim \tau^{-\frac{1}{2}} h^{-\frac{1}{2}} h^{-\frac{1}{2} + \epsilon} \lambda^{-\frac{1}{2}} \|f\|_{L^p} \]
for $1/p + 3/q > 1$, and
\[ \|A^\prime f\|_{L^q(\mathbb{R}^3 \times J_{\tau})} \lesssim \tau^{-\frac{1}{2}} h^{-1 + \frac{1}{2} + \epsilon} \lambda^{-\frac{1}{2} - \frac{1}{4}} \|f\|_{L^q} \]
for $1/p + 3/q \leq 1$ whenever $\text{supp} \hat{f} \subset \mathbb{A}_\lambda \times I_h$. (b) If $\text{supp} \hat{f} \subset \mathbb{A}_\lambda \times \mathbb{R}$, the estimates (2.30) and (2.31) hold with $h = \lambda$. (c) Suppose $1/\tau \lesssim \lambda \lesssim h \gtrsim \lambda^2 \tau$, then the estimates (2.19) and (2.20) hold whenever $\text{supp} \hat{f} \subset \mathbb{A}_\lambda \times I_h$.

We can prove Proposition 2.8 in the same manner as Proposition 2.7 using the expansions (2.21) and (2.25). By (2.28) we may disregard the contribution from $E$. Thus, we only need to handle $M^\prime$. Moreover, one can easily see the contribution from the multiplier operator $E_N^\prime(s|D|)$ is acceptable. In fact, we have the following.
Lemma 2.9. Let \( 2 \leq p \leq q \leq \infty \) and \( 1/p + 1/q \leq 1 \). If \( \text{supp} \hat{f} \subset A_\lambda \times I_h \) and \( h \geq \lambda \), then we have the estimates

\[
\left\| (D_e^{\pm iD}) E_N^{\mp}(s | D) |f| \right\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{p} h^{\lambda - \frac{1}{2} + \frac{1}{q}} \frac{\lambda}{p} \frac{\lambda}{q} + \frac{1}{q} \|f\|_{L^p}
\]

for \( 1/p + 3/q \leq 1 \), and

\[
\left\| (D_e^{\pm iD}) E_N^{\mp}(s | D) |f| \right\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{p} h^{\lambda - \frac{1}{2} + \frac{1}{q}} \frac{\lambda}{p} \frac{\lambda}{q} + \frac{1}{q} \|f\|_{L^p}
\]

for \( 1/p + 3/q > 1 \). If \( \text{supp} \hat{f} \subset A_\lambda \times I_h^3 \), (2.32) and (2.33) hold with \( h = \lambda \).

Proof. We first consider the case \( \text{supp} \hat{f} \subset A_\lambda \times I_h \) and \( h \geq \lambda \). The estimate (2.32) is easy to show by using (2.1) and Bernstein’s inequality (for example, see 1.1). Note that (2.33) with \( p = q = 2 \) follows by Plancherel’s theorem. Thus, interpolation between this estimate and (2.32) for \( 1/p + 3/q = 1 \) gives (2.33) for \( 1/p + 3/q > 1 \). If \( \text{supp} \hat{f} \subset A_\lambda \times I_h^3 \), the estimates (2.32) and (2.33) with \( h = \lambda \) follow in the same manner. We omit the detail. \( \square \)

Proof of Proposition 2.8. Recalling (2.25) and comparing the estimates (2.32) and (2.33), we notice that it is sufficient to consider the estimates for the multiplier operators defined by \( B^\pm_\lambda \xi \in [-\frac{1}{2}, \frac{1}{2}]^\lambda | \xi | \). Therefore, the matter is reduced to obtaining, instead of \( A_\lambda^s \), the estimates for the operators

\[
C_\pm^s f(x, t, s) := |D_e^{\pm iD} E_N^{\mp}(s | D)| - \frac{1}{2} \mathcal{U} f(x, \kappa t, \pm s), \quad \kappa = \pm,
\]

which constitute the major part. We first consider the case (a): \( 1/\tau \leq \lambda \leq h \leq \tau \lambda^2 \) and \( \text{supp} \hat{f} \subset A_\lambda \times I_h \). Note that \( \|C_\pm^s f(\cdot, s, t)\|_{L^r(R^3)} \lesssim (\tau \lambda h)^{-1/2} \|\mathcal{U} f(\cdot, \kappa t, \pm s)\|_{L^r(R^3)} \) for \( \kappa = \pm \). Thus, by (2.24) and Remark 2.20, we get

\[
\|C_\pm^s f\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{-1/2} h^{1/2 + \epsilon} \lambda^{1 - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^p}, \quad \kappa = \pm
\]

for \( 1/p + 3/q \leq 1 \). Therefore, we obtain (2.31). So (2.30) follows from interpolation with (2.24).

If \( \text{supp} \hat{f} \subset A_\lambda \times I_h^3 \), by the estimate (2.25) with \( \lambda = h \) ((b) in Lemma 2.3) we get the desired estimates (2.31) and (2.30) with \( h = \lambda \). This proves (b).

If \( 1/\tau \leq \lambda \), \( h \geq \lambda^2 \tau \), and \( \text{supp} \hat{f} \subset A_\lambda \times I_h \), the estimate (2.19) follows by (2.6). As a result, we get (2.29) by interpolation between (2.29) and (2.19). \( \square \)

Since the main contribution to the estimate for \( A_\lambda^s f \) is from \( C_\lambda^s f \), by the same argument in the proof of Proposition 2.8 one can easily obtain the next.

Corollary 2.10. Let \( \alpha, \beta \in \mathbb{N}_0 \). (a) If \( 1/\tau \leq \lambda \leq h \leq \tau \lambda^2 \), then for any \( \epsilon > 0 \)

\[
\|\partial_x^\alpha \partial_t^\beta A_\lambda^s f\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{2} h^{\beta + \alpha + \lambda - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^p}, \quad 1/p + 3/q > 1,
\]

(2.34)

\[
\|\partial_x^\alpha \partial_t^\beta A_\lambda^s f\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{2} h^{\beta + \alpha + \lambda - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^p}, \quad 1/p + 3/q > 1,
\]

hold whenever \( \text{supp} \hat{f} \subset A_\lambda \times I_h \). (b) If \( \text{supp} \hat{f} \subset A_\lambda \times I_h^3 \), we obtain the above two estimates with \( h = \lambda \). (c) When \( 1/\tau \leq \lambda \) and \( h \geq \lambda^2 \tau \), for any \( \epsilon > 0 \) we have

\[
\|\partial_x^\alpha \partial_t^\beta A_\lambda^s f\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{2} h^{\beta + \alpha + \lambda - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^p}, \quad 1/p + 3/q \leq 1,
\]

(2.35)

\[
\|\partial_x^\alpha \partial_t^\beta A_\lambda^s f\|_{L^r(R^3 \times I_s, L^p)} \lesssim \tau^{1/2}(\tau h) - \frac{1}{2} h^{\beta + \alpha + \lambda - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^p}, \quad 1/p + 3/q \leq 1,
\]

whenever \( \text{supp} \hat{f} \subset A_\lambda \times I_h \).
Remark 2. By (2.21) and (2.25) it follows that
\[ |\hat{\sigma}_j(\xi)| \lesssim (1 + |\xi_3|)^{-1/2}(1 + |\xi|)^{-1/2}. \]
Furthermore, if $|\xi| \lesssim 1$, we have $|\hat{\sigma}_j(\xi)| \sim |\xi|^{-1/2}$ for $|\xi|$ large enough. Therefore, by Plancherel’s theorem one can see that the $L^2$ to $L^2_{1/2}$ estimate for $A^*_t$ is optimal. One can also see that the part of the surface $T^*_t$ near the sets $\{\Phi^*_s(\pm \pi/2, \phi) : \phi \in [0, 2\pi)\}$ is responsible for the worst decay while the Fourier transform of the part (of the surface) away from the sets enjoys better decay.

3. Two-parameter maximal and smoothing estimates

In this section we prove Theorem 1.1, 1.2, and 1.3. First, we recall an elementary lemma, which enables us to relate the local smoothing estimate to the estimate for the maximal function.

Lemma 3.1. Let $1 \leq p \leq \infty$, and let $I$ and $J$ be closed intervals of length 1 and $\ell$, respectively. Suppose $G$ be a smooth function on the rectangle $R = I \times J$. Then, for any $\lambda, h > 0$, we have
\[
\sup_{(t,s) \in I \times J} |G(t, s)| \lesssim (1 + \lambda^1/p)(\ell^{-1/p} + h^{-1/p})\|G\|_{L^p(R)} + (\ell^{-1/p} + h^{-1/p})\lambda^{-1/p'}\|\partial_t G\|_{L^p(R)} + (1 + \lambda^1/p)h^{-1/p'}\|\partial_s G\|_{L^p(R)} + \lambda^{-1/p'}h^{-1/p'}\|\partial_s \partial_t G\|_{L^p(R)},
\]

Proof. We first recall the inequality
\[
\sup_{t \in I'} |F(t)| \lesssim |I'|^{-1/p}\|F\|_{L^p(I')} + \|F\|_{L^p(I')} \|\partial_t F\|_{L^p(I')},
\]
which holds whenever $F$ is a smooth function defined on an interval $I'$ (for example, see [20]). By Young’s inequality we have
\[
\sup_{t \in I'} |F(t)| \lesssim |I'|^{-1/p}\|F\|_{L^p(I')} + \lambda^{-1/p}\|F\|_{L^p(I')} + \lambda^{-1/p'}\|\partial_t F\|_{L^p(I')}.
\]
for any $\lambda > 0$. We use this inequality with $F = G(\cdot, s)$ and $I' = I$ to get
\[
\sup_{(t,s) \in I \times J} |G(t, s)| \lesssim (1 + \lambda^2)|\sup_{s \in J}|G(t, s)|\|G(t, s)\|_{L^p(I)} + \lambda^{-1/p'}\|\sup_{s \in J}|\partial_t G(t, s)|\|_{L^p(I)}.\]

Then, we apply the above inequality again to $G(t, \cdot)$ and $\partial_t G(t, s)$ with $I' = J$ taking $\lambda = h$. \qed

In what follows, we frequently use the Littlewood-Paley decomposition. Let $\varphi \in C^\infty_c((1 - 2^{-13}, 2 + 2^{-13}))$ such that $\sum_{j=\infty}^{\infty} \varphi(s/2^j) = 1$ for $s > 0$. We set $\varphi_j(s) = \varphi(s/2^j)$, $\varphi_{<j}(s) = \sum_{k<j} \varphi_k(s)$, and $\varphi_{>j}(s) = \sum_{k>j} \varphi_k(s)$. For a given $f$ we define $f^k_j$ and $f^k_{<j}$ by
\[
\mathcal{F}(f^k_j) = \varphi_j(|\xi|)\varphi_k(|\xi_3|)\hat{f}(\xi), \quad \mathcal{F}(f^k_{<j}) = \varphi_{<j}(|\xi|)\varphi_{<k}(|\xi_3|)\hat{f}(\xi),
\]
and $f^k_{>j}$, $f^k_{\leq j}$, $f^k_{\geq j}$, $f_{<j}$, and $f_{\geq j}$, etc are similarly defined. In particular, we have $f = \sum_{j,k} f^k_j$. 

3.1. Proof of Theorem 1.1

By a standard argument using scaling, it is sufficient to show $L^p$ boundedness of a localized maximal operator

$$\mathcal{M}f(x) = \sup_{0 < s < c_0} |A^s f(x)|.$$  

Furthermore, we only need to show that $\mathcal{M}$ is bounded on $L^p$ for $2 < p \leq 4$ since the other estimates follow by interpolation with the trivial $L^\infty$ bound. To this end, we consider

$$\mathcal{M}_n f(x) = \sup_{(t,s) \in J_{2-n}} |A^s f(x)|, \quad n \geq 0.$$  

In order to obtain estimates for $\mathcal{M}_n$, we consider $\mathcal{M}_n f_j^k$ for each $j, k$. The correct bounds in terms of $n$, not to mention $j, k$, are also important for our purpose.

**Lemma 3.2.** Let $k, j \geq n$. \(\overset{(a)}{\text{(a)}}\) If $j \leq k < 2j - n$, we have

$$\|\mathcal{M}_n f_j^k\|_{L^q} \lesssim \begin{cases} 2^{n(\frac{4}{p} - \frac{1}{q})} & 0 < \frac{4}{p} - \frac{1}{q} < 1, \frac{1}{p} + \frac{3}{q} \geq 1, \\ 2^{n(\frac{4}{p} - \frac{1}{q})} & \sqrt{q} \geq 1. \end{cases}$$

\(\overset{(b)}{\text{(b)}}\) For $\mathcal{M}_n f_j^{< k}$, the same bounds hold with $k = j$. \(\overset{(c)}{\text{(c)}}\) If $2j - n \leq k$, then we have

$$\|\mathcal{M}_n f_j^k\|_{L^q} \lesssim \begin{cases} 2^{n(\frac{4}{p} - \frac{1}{q})} & 0 < \frac{4}{p} - \frac{1}{q} < 1, \frac{1}{p} + \frac{3}{q} \geq 1, \\ 2^{n(\frac{4}{p} - \frac{1}{q})} & \sqrt{q} \geq 1. \end{cases}$$

**Proof.** Let $n_0$ be the smallest integer such $2^{-n_0+1} \leq c_0$. If $n \geq n_0$, then $J_{2-n} = \mathbb{I} \times J_{2-n}$. Since $n \leq k, j$, using Lemma 3.1 one can obtain \((a), (b), (c)\) from \((a), (b), (c)\) in Corollary 2.10 respectively. For $n < n_0$, we can not directly apply Lemma 3.1. However, this can be easily overcome by a simple modification. Indeed, we cover $\bigcup_{n=0}^{n_0-1} J_{2-n}$ with essentially disjoint closed dyadic cubes $Q$ of side length $L \in (2^{-\gamma}(1 - c_0), 2^{-\gamma}(1 - c_0)]$ so that $\bigcup Q \subset J_0 := \{(t, s) : 2^{1-n_0} s < 2^{-1}(1 + t_0) t, 1 \leq t \leq 2\}$. Thus, we note

$$\|\sup_{(t,s) \in J_{2-n}} |A^s f|\|_{L^q} \lesssim \sum Q \|\sup_{(t,s) \in Q} |A^s f|\|_{L^q},$$

for $n < n_0$. We may now apply Lemma 3.1 to $A^s f$ and $Q$. Since $\bigcup Q \subset Q_0$, we clearly have the same maximal bounds up to a constant multiple for $n < n_0$. \(\square\)

We denote $Q^m_0 = J_0 \cap (J_{2-1} \times J_{2-n})$ for simplicity. Then, it follows that

$$\mathcal{M} f(x) = \sup_{m \geq 0} \sup_{(t,s) \in Q^m_0} |A^s f|.$$  

Decomposing $f = \sum_{j,k} f_j^k$, we have

$$\mathcal{M} f(x) \leq \mathcal{M}^1 f + \mathcal{M}^2 f + \mathcal{M}^3 f + \mathcal{M}^4 f,$$

where

$$\mathcal{M}^1 f = \sup_{m \geq 0} \sup_{(t,s) \in Q^m_0} |A^s f|,$$

$$\mathcal{M}^2 f = \sup_{m \geq 0} \sup_{(t,s) \in Q^m_0} |A^s f|,$$

$$\mathcal{M}^3 f = \sup_{m \geq 0} \sup_{(t,s) \in Q^m_0} |A^s f|,$$

$$\mathcal{M}^4 f = \sup_{m \geq 0} \sup_{(t,s) \in Q^m_0} |A^s f|.$$  

The maximal operators $\mathcal{M}^1, \mathcal{M}^2$ and $\mathcal{M}^3$ can be handled by using the $L^p$ bounds on the Hardy-Littlewood maximal and the circular maximal functions.
We first handle $\mathfrak{N}_1 f$. We set $K = F^{-1}(\varphi_{\leq 1}(\xi))$ and $K_3 = F^{-1}(\varphi_{\leq 1}(\xi_3))$. Since $F(f_{\leq 1}^m)(\xi) = \varphi_{\leq 1}(\xi)\varphi_{=m}(\xi)\hat{f}(\xi)$ and $\varphi_{=m}(t) = \varphi_{\leq 1}(2^{-m}t)$, we have

$$f_{\leq 1}^m(x) = 2^{2l+m} \int f(x-y)\hat{K}(2^l\hat{y})K_3(2^m y_3)dy.$$

Hence, it follows that

$$A_1^* f_{\leq 1}^m(x) = 2^{2l+m} \int_{T^2} \int f(x-y)\hat{K}(2^l(\hat{y} - \hat{z}))K_3(2^m (y_3 - z_3))dy d\sigma_1^m(z).$$

If $(t, s) \in Q_l^m$, $|\hat{K}(2^l(\hat{y} - \hat{z}))K_3(2^m (y_3 - z_3))| \leq C(1 + 2^l|\hat{y}|)^{-M}(1 + 2^m|y_3|)^{-M}$ for any $M$. By a standard argument using dyadic decomposition, we see

$$\mathfrak{N}_1^1 f(x) \lesssim HH_3 f(x),$$

where $\tilde{H}$ and $H_3$ denote the 2-d and 1-d Hardy-Littlewood maximal operators acting on $\hat{z}$ and $x_3$, respectively. The right hand side is bounded by the strong maximal function. Thus, $\mathfrak{N}_1^1$ is bounded on $L^p$ whenever $p > 1$.

Next, we consider $\mathfrak{N}_2$. Since $f_{\leq 1}^m(x) = 2^{2l} \int f_{\geq m}(\cdot, x_3)\cdot \hat{K}(2^l(\hat{y} - t + s \cos \theta)\varphi_0)\cdot d\theta d\phi dy$.

Note that $s < \cot \leq 2^{-l}$, so we have $|\hat{K}(2^l(\hat{y} - t + s \cos \theta)\varphi_0)| \lesssim C(1 + 2^l|\hat{y}|)^{-M}$ for any $M$. Similarly as above, this gives

$$|A_1^* f_{\geq m}^m(x)| \lesssim \int_0^{2\pi} \tilde{H} f_{\geq m}(\tilde{x}, x_3 - s \sin \theta) d\theta \lesssim \int_0^{2\pi} \tilde{H} H_3 f(\tilde{x}, x_3 - s \sin \theta) d\theta$$

For the second inequality, we use $f_{\geq m} = f - f_{\leq m}$ and $|f|, |f_{\leq m}| \leq H_3 f$. As a result, we have

$$\mathfrak{N}_2^2 f(x) \lesssim \sup_{s > 0} \int_0^{2\pi} \tilde{H} H_3 f(\tilde{x}, x_3 - s \sin \theta) d\theta.$$

To handle the consequent maximal operator, we use the following simple lemma.

**Lemma 3.3.** For $p > 2$, we have the estimate

$$\left\| \sup_{0 < s < 1} \int g(x_3 - s \sin \theta) d\theta \right\|_{L^p_{x_3}} \lesssim \left\| g \right\|_{L^p}.$$

**Proof.** Let us define $\tilde{g}$ on $\mathbb{R}^2$ by setting $\tilde{g}(z, x_3) = g(x_3)$ for $x_3 \in \mathbb{R}$ and $-10 \leq z \leq 10$, and $\tilde{g}(z, x_3) = 0$ if $|z| > 10$. Note that $\int g(x_3 - s \cos \theta) d\theta = \int \tilde{g}(z - s \cos \theta, x_3 - s \sin \theta) d\theta$ for $|z| \leq 1, 0 < s < 1$. So, $\sup_{0 < s < 1} \int \int g(x_3 - s \sin \theta) d\theta \lesssim M_{cr} \tilde{g}(z, x_3)$ for $|z| \leq 1$, where $M_{cr}$ denotes the circular maximal operator. By the circular maximal theorem [4], $\left\| \sup_{0 < s < 1} \int g(x_3 - s \sin \theta) d\theta \right\|_{L^p_{x_3}}$ is bounded above by a constant times $\left\| \tilde{g} \right\|_{L^{p, 1}_{L^p_{x_3}}} = 20^{1/p} \left\| g \right\|_{L^p_{x_3}}$ for $p > 2$. \hfill \square

Therefore, by Lemma 3.3 and $L^p$ boundedness of $\tilde{H}$ and $H_3$ we see that $\mathfrak{N}_2^2$ is bounded on $L^p$ for $p > 2$.

$\mathfrak{N}_3$ can be handled similarly. Since $f_{\geq m}^m = 2^m \int f_{\geq l}(\tilde{x}, \cdot) \cdot K_3(2^m \cdot)(x_3)$, we get

$$A_1^* f_{\geq m}^m(x) = 2^m \int f_{\geq l}(\tilde{x} - (t + s \cos \theta)\varphi_0, x_3 - y_3)K_3(2^m (y_3 - s \sin \theta))d\theta d\phi dy_3.$$
Thus, \( \mathcal{N}_f(x) \leq M_{cr}[(H_3 H \bar{f})(\cdot, x_3)](\bar{x}) \). Using the circular maximal theorem, we see that \( \mathcal{N}_f \) is bounded on \( L^p \) for \( p > 2 \).

Finally, we consider \( \mathcal{N}^1 \). For simplicity, we set

\[
\mathcal{A}_{m,k}^l = \sup_{(t,s) \in \mathbb{Q}^m} |A^l_{t,j} f^k|.
\]

Decomposing \( \sum_{j \geq 1, k \geq m} = \sum_{m \leq k \leq j} + \sum_{j < k \leq j-m} + \sum_{k \leq j \leq m, m \leq (2j-m)} < k \), we have

\[
\mathcal{N}^1 f \leq \sup_{m \geq 0} \mathcal{S}^1_m f + \sup_{m \geq 0} \mathcal{S}^2_m f + \sup_{m \geq 0} \mathcal{S}^3_m f,
\]

where

\[
\mathcal{S}^1_m f = \sum_{m \leq k \leq j} \mathcal{A}_{m,k}^l, \quad \mathcal{S}^2_m f = \sum_{j \leq k \leq j-m} \mathcal{A}_{m,k}^l, \quad \mathcal{S}^3_m f = \sum_{j \leq j, m \leq (2j-m)} < k \mathcal{A}_{m,k}^l.
\]

Here, \( \max(a, b) \) denotes \( \max(a, b) \). Thus, the matter is reduced to showing, for \( \kappa = 1, 2, 3 \),

\[
\| \sup_{m \geq 0} \mathcal{S}^\kappa_m f \|_{L^p} \lesssim C \| f \|_p, \quad p \in (2, 4).
\]

We consider \( \mathcal{S}^\kappa_m f \) first. Recalling (3.1), by scaling we have

\[
\mathcal{A}_{m,k}^l f(x) = \mathcal{M}_{m-l}(f^{j}_j(2^{-l} \cdot))(2^l x) = \mathcal{M}_{m-l}[f(2^{-l} \cdot)]^{j-l}_j(2^l x).
\]

So, reindexing \( k \rightarrow k + l \) and \( j \rightarrow j + l \) gives

\[
\mathcal{S}^\kappa_m f(x) \leq \sum_{m-l \leq k \leq j} \mathcal{M}_{m-l}[f(2^{-l} \cdot)]^{j-l}_j(2^l x).
\]

Thus, the imbedding \( \ell^p \subset \ell^\infty \) and Minkowski’s inequality yield

\[
\| \sup_{m \geq 0} \mathcal{S}^\kappa_m f \|_{L^p} \leq \left( \sum_{m \geq 0} \| \mathcal{M}_{m-l}[f(2^{-l} \cdot)]^{j-l}_j \|_{L^p} \right)^p.
\]

We now use (\( \tilde{b} \)) in Lemma 3.2 (with \( n = m - l \)) for \( \mathcal{M}_{m-l}[f(2^{-l} \cdot)]^{j-l}_j(2^l \cdot) \). Thus, by the first estimate in (3.2) with \( k = j \), we have

\[
\| \sup_{m \geq 0} \mathcal{S}^\kappa_m f \|_{L^p} \leq \sum_{m \geq 0} 2^{m-l} 2^{-\epsilon j} \left( \sum_{m \leq l} \sum_{l \leq m} 2^{-a(m-l)} 2^{-bj} \| f_{j+l} \|_{L^p} \right)^p
\]

for any \( \epsilon > 0 \) for \( 2 < p \leq 4 \). Taking \( \epsilon > 0 \) small enough, we have

\[
\| \sup_{m \geq 0} \mathcal{S}^\kappa_m f \|_{L^p} \leq \sum_{m \geq 0} \sum_{m \geq 0} 2^{-a(m-l)} 2^{-bj} \| f_{j+l} \|_{L^p}^p
\]

for some positive numbers \( a, b \) for \( 2 < p \leq 4 \). Changing the order of summation, we see the right hand side is bounded above by \( C \sum_{j \geq 0} 2^{-bj} \sum_{l \geq 0} \| f_{j+l} \|_{L^p}^p \), which is bounded by \( C \| f \|_p^p \), as can be seen, for example, using the Littlewood-Paley inequality. Consequently, we obtain (3.3) for \( \kappa = 1 \).

We now consider \( \mathcal{S}^\kappa_m f \). As before, by the imbedding \( \ell^p \subset \ell^\infty \), Minkowski’s inequality, (3.2), and reindexing \( k \rightarrow k + l \) and \( j \rightarrow j + l \), we get

\[
\| \sup_{m \geq 0} \mathcal{S}^\kappa_m f \|_{L^p} \leq \sum_{m \geq 0} \left( \sum_{j \leq k \leq 2^j(m-l)} \| \mathcal{M}_{m-l}[f(2^{-l} \cdot)]^{j-l}_j \|_{L^p} \right)^p.
\]
The first inequality in (3.2) with \( n = m - l \) gives
\[
\| \sup_{m \geq l} \mathcal{S}_2^{m,l}f \|_{L^p}^p \leq \sum_{m \geq l} \left( \sum_{j < k \leq 2j - (m-l)} 2^{-j} \sum_{j \leq k < 2j - (m-l)} 2^k \| f_{j+l} \|_{L^p} \right)^p
\]
for any \( \epsilon > 0 \) for \( 2 < p \leq 4 \). Note that \( m - l < j \) for the inner sum, which is bounded by a constant times \( \sum_{m-l \leq j} 2^{-2j(1/2 - 1/p)} 2^{j} \| f_{j+l} \|_{L^p} \) by taking sum over \( k \) with an \( \epsilon > 0 \) small enough. Since \( p > 2 \), similarly, we have
\[
\| \sup_{m \geq l} \mathcal{S}_2^{m,l}f \|_{L^p}^p \lesssim \sum_{m \geq l} \sum_{m-l \leq j} 2^{-a(m-l)} 2^{-bj} \| f_{j+l} \|_{L^p}^p
\]
for some \( a, b > 0 \) and \( 2 < p \leq 4 \). Thus, the right hand is bounded above by \( C\| f \|_{L^p}^p \). This proves (3.3) for \( \kappa = 2 \).

Finally, we consider \( \mathcal{S}_3^{m,l}f \), which we can handle in the same manner as before. Via the imbedding \( \ell^p \subset \ell^\infty \), (3.3), and reindexing after applying Minkowski’s inequality we have
\[
\| \sup_{m \geq l} \mathcal{S}_2^{m,l}f \|_{L^p}^p \lesssim \sum_{m \geq l} \sum_{m-l \leq j} \| \mathcal{M}_n[f(2^{-l} \cdot)](2^j \cdot) \|_{L^p}^p,
\]
where \( n := m - l \). Breaking \( \sum_{0 \leq j, n \leq (2j - n) - k} = \sum_{0 \leq j \leq n \leq k} + \sum_{n < j, (2j - n) - k} \), we apply the first estimate in (3.3) to get
\[
\| \sup_{m \geq l} \mathcal{S}_2^{m,l}f \|_{L^p}^p \lesssim \sum_{m \geq l} 2^{np(\frac{1}{2} - \frac{1}{p'})} (S_1^p + S_2^p)
\]
for any \( \epsilon > 0 \) and \( 2 < p \leq 4 \), where
\[
S_1 := \sum_{0 \leq j \leq n \leq k} 2^{(j+k)(\frac{1}{2} - \frac{1}{p'})} \| f_{j+l} \|_{L^p}, \quad S_2 := \sum_{n < j, (2j - n) - k} 2^{(j+k)(\frac{1}{2} - \frac{1}{p'})} \| f_{j+l} \|_{L^p}.
\]
For the second sum \( S_2 \), we note that \( k > j > n \). Thus, taking \( \epsilon > 0 \) small enough, we get
\[
\sum_{m \geq l} 2^{np(\frac{1}{2} - \frac{1}{p'})} S_2^p \lesssim \sum_{m \geq l} \sum_{m-l \leq j} 2^{-a(m-l)} 2^{-bj} \| f_{j+l} \|_{L^p}^p
\]
for some \( a, b > 0 \) since \( p > 2 \). Thus, the right hand side is bounded by \( C\| f \|_{L^p}^p \). To handle \( S_1 \), note that \( \sum_{0 \leq j \leq n \leq k} 2^{(j+k)(\frac{1}{2} - \frac{1}{p'})} \| f_{j+l} \|_{L^p}^p \lesssim 2^{n(p'-1)(\frac{1}{2} - \frac{1}{p'})} \). Thus, by Hölder’s inequality we have
\[
S_1^p \lesssim 2^{n(p'-1)(\frac{1}{2} - \frac{1}{p'})} \sum_{0 \leq j \leq n \leq k} 2^{(j+k)(\frac{1}{2} - \frac{1}{p'})} \| f_{j+l} \|_{L^p}^p.
\]
Hence, changing the order of summation, we get
\[
\sum_{m \geq l} 2^{np(\frac{1}{2} - \frac{1}{p'})} S_1^p \lesssim \sum_{0 \leq j} 2^{j(\frac{1}{2} - \frac{1}{p'})} S_{1,j}^p,
\]
where
\[
S_{1,j} := \sum_{m \geq l \geq 0} \sum_{m-l \leq k} 2^{(m-l)(\frac{1}{2} - \frac{1}{p'})} 2^{k(-\frac{1}{2} + \frac{1}{p'})} \| f_{j+l} \|_{L^p}^p.
\]
Therefore, since \( 2 < p \leq 4 \), taking a sufficiently small \( \epsilon > 0 \), we obtain the desired inequality \( \sum_{m \geq l \geq 0} 2^{np(\frac{1}{2} - \frac{1}{p'})} S_1^p \lesssim \| f \|_{L^p}^p \) if we show that \( S_{1,j} \lesssim \| f \|_{L^p}^p \) for \( 0 \leq j \). To this end, rearranging the sums, we observe
\[
S_{1,j} = \sum_{0 \leq k} \sum_{0 \leq l \leq m \leq l+k} 2^{(m-l)(\frac{1}{2} - \frac{1}{p'})} 2^{k(-\frac{1}{2} + \frac{1}{p'})} \| f_{j+l} \|_{L^p}^p \lesssim \sum_{0 \leq k} \sum_{0 \leq l} \| f_{j+l} \|_{L^p}^p.
\]
Since \( \sum_{0 \leq k} \| f_{j,k} \|^p_{L^p} \lesssim \| f_j \|^p_{L^p} \), by the same argument as above it follows that \( S^p_{1,j} \leq C \| f \|^p_{L^p} \). Consequently, we obtain \( 3.3.1 \) for \( \kappa = 3 \).

3.2. Proof of Theorem 1.2. Since \( J \) is a compact subset of \( J_x \), there are constants \( c_0 \in (0,1) \), and \( m_1, m_2 > 0 \) such that
\[
J \subset \{(t,s) : m_1 \leq s \leq m_2, s < c_0 t\}.
\]

Therefore, via finite decomposition and scaling it is sufficient to show that the maximal operator
\[
\mathcal{M}_c f(x) := \sup_{(t,s) \in J_0} |A_t^s f(x)|
\]
is bounded from \( L^p \) to \( L^q \) for \( (1/p, 1/q) \in \text{int } Q \). To do this, we decompose \( f = f_{\geq 0} + f_{< 0} \), hence
\[
\mathcal{M}_c f \lesssim \mathcal{M}_c f_{\geq 0} + \mathcal{M}_c f_{< 0} + \mathcal{M}_c f_{< 0}.
\]
The last two operators are easy to deal with. As before, we have \( \mathcal{M}_c f_{< 0}(x) \lesssim (1 + |x|^{-4}) |f_j(x)| \), hence \( \mathcal{M}_c f_{< 0} \| L^q \lesssim \| f \|_{L^p} \), for \( 1 \leq p \leq q < \infty \). Concerning \( \mathcal{M}_c f_{< 0} \), we use Lemma 3.4 and (2.17) to get
\[
\| \mathcal{M}_c f_{< 0} \|_{L^q} \lesssim 2^{k(-\frac{1}{4} + \frac{3}{2})} \| f \|_{L^p}, \quad 1 \leq p \leq q < \infty,
\]
for \( k \geq 0 \). So, it follows that \( \| \mathcal{M}_c f_{< 0} \|_{L^q} \lesssim \| f \|_{L^p} \) for \( 2 < p < q \). Thus, we only need to show that \( \mathcal{M}_c f_{\geq 0} \) is bounded from \( L^p \) to \( L^q \) for \( (1/p, 1/q) \in \text{int } Q \).

Decomposing \( f_{\geq 0} = \sum_{j \geq 0} (f_{j+}^c + \sum_{j \leq k \leq 2j} f_j^k + \sum_{k > 2j} f_j^k) \), we have
\[
\mathcal{M}_c f_{\geq 0} \leq \sum_{j \geq 0} (\mathcal{G}_j f + \mathcal{G}_j^2 f),
\]
where
\[
\mathcal{G}_j f = \mathcal{M}_c f_{j+}^c + \sum_{j \leq k \leq 2j} \mathcal{M}_c f_j^k, \quad \mathcal{G}_j^2 f = \sum_{k > 2j} \mathcal{M}_c f_j^k.
\]

We first show \( L^p - L^q \) bound on \( \mathcal{M}_c f_{\geq 0} \) for \( (1/p, 1/q) \) contained in the interior of the triangle \( T \) with vertices \((1/4, 1/4), P_1, \) and \((1/2, 1/2)\) (see Figure 1). The first estimate in (3.3) with \( 2^n \sim 1 \) gives
\[
\| \mathcal{M}_c f_j^k \|_{L^q} \lesssim 2^{k(-\frac{1}{4} + \frac{3}{2}) + 2^{j(-1 + \frac{3}{4})}} \| f \|_{L^p}, \quad 1/p + 3/q \geq 1,
\]
for \( 0 \leq j \leq k \leq 2j \). \( \mathcal{G}_j^2 f \) satisfies the same bound with \( k = j \). Note that \( -3/2 + 7/(2p) - 1/(2q) < 0, -1 + 2/p < 0, \) and \( 1/p + 3/q > 1 \) if \( (1/p, 1/q) \in \text{int } T \) (Figure 1). Thus, using those estimates, we get
\[
\sum_{j \geq 0} \| \mathcal{G}_j f \|_{L^q} \lesssim \sum_{j \geq 0} (2^{k(-1 + \frac{3}{4})} + 2^{j(-1 + \frac{3}{4})}) \| f \|_{L^p} \lesssim \| f \|_{L^p}
\]
for \( (1/p, 1/q) \in \text{int } T \). We now consider \( \sum_{j \geq 0} \mathcal{G}_j^2 f \). By the first estimate in (3.3) with \( 2^n \sim 1 \), we have
\[
\sum_{j \geq 0} \| \mathcal{G}_j^2 f \|_{L^q} \lesssim \sum_{0 \leq j \leq k \leq 2j} 2^{j(-1 + \frac{3}{4})} \| f \|_{L^p} \lesssim \| f \|_{L^p}
\]
for \( (1/p, 1/q) \in \text{int } T \). Thus, \( \mathcal{M}_c f_{\geq 0} \) is bounded from \( L^p \) to \( L^q \) for \( (1/p, 1/q) \in \text{int } T \).

Next, we show \( L^p - L^q \) bound on \( \mathcal{M}_c f_{\geq 0} \) for \( (1/p, 1/q) \in \text{int } T \)'s where \( T' \) is the quadrangle with vertices \((1/4, 1/4), (0,0), P_3, \) and \( P_2 \) (see Figure 1). Note that \( 1/p + 3/q < 1 \) if \( (p,q) \in \text{int } T' \). By the second estimate of (3.3) with \( 2^n \sim 1 \), we have
\[
\| \mathcal{M}_c f_j^k \|_{L^q} \lesssim 2^{j(-1 + \frac{3}{4})} 2^{k(-1 + \frac{3}{4})} \| f \|_{L^p}, \quad 1/p + 3/q < 1
\]
for $0 \leq j \leq k \leq 2j$. $M_j f_j^{\leq j}$ satisfies the same bound with $k = j$. Thus,
\[
\sum_{j \geq 0} \| G_j f_j \|_{L^p} \lesssim \sum_{j \geq 0} (2^{j(\frac{1}{p} - \frac{1}{4} + \varepsilon)} + 2^{j(\frac{1}{p} - \frac{1}{2} - 1 + 2\varepsilon)}) \| f \|_{L^p} \lesssim \| f \|_{L^p}
\]
for $(1/p, 1/q) \in \text{int } Q'$ since $1/p - 3/4 < 0$ and $3/p - 2/q < 1$ for $(1/p, 1/q) \in \text{int } Q'$. Similarly, the second estimate of (3.3) with $2^n \sim 1$ gives
\[
\sum_{j \geq 0} \| G_j f_j \|_{L^p} \lesssim \sum_{k > 2j \geq 0} 2^{j(\frac{1}{p} - \frac{1}{4} + \varepsilon)} 2^{k(\frac{1}{p} - \frac{1}{4} + \varepsilon)} \| f \|_{L^p} \lesssim \sum_{j \geq 0} 2^{j(-1 + \frac{1}{p} - \frac{1}{4} + \varepsilon)} \| f \|_{L^p}
\]
for $(1/p, 1/q) \in \text{int } Q'$. Note that $-1 + 3/p - 2/q < 0$ for $(1/p, 1/q) \in \text{int } Q'$, so it follows that $\sum_{j \geq 0} \| G_j f_j \|_{L^p} \lesssim \| f \|_{L^p}$ for $(1/p, 1/q) \in \text{int } Q'$. Thus, $f \to M_{c} f_{j \geq 0}$ is bounded from $L^p$ to $L^q$ for $(1/p, 1/q) \in \text{int } Q'$. Consequently, $f \to M_{c} f_{j \geq 0}$ is bounded from $L^p$ to $L^q$ for $(1/p, 1/q) \in \text{int } Q' \cup \text{int } Q''$. Thus, via interpolation $f \to M_{c} f_{j \geq 0}$ is bounded from $L^p$ to $L^q$ for $(1/p, 1/q) \in \text{int } Q$. This complete the proof of Theorem 1.3.

3.3. Proof of Theorem 1.3 We set $\mathbb{D}_r = \mathbb{R}^3 \times \mathbb{J}_r$. By $L^p_{\alpha, x}$ we denote the $L^p$ Sobolev space of order $\alpha$ in $x$, and set $L^p_{\alpha, x}(\mathbb{D}_r) = L^p_{\alpha, x}(\mathbb{J}_r; L^p_{\alpha, x}(\mathbb{R}^3))$. We prove Theorem 1.3 making use of the next lemma.

Proposition 3.4. Let $\tau \in (0, 1]$ and $8 \leq p < \infty$. If $\alpha < 4/p$, then we have
\[
\| \tilde{A}_\xi f \|_{L^p_{\tau}(\mathbb{D}_r)} \lesssim \tau^{-\frac{3}{2p}} \| f \|_{L^p}.
\]

It is not difficult to see that the bound $\tau^{-3/p}$ is sharp up to a constant by using a frequency localized smooth function. Assuming Proposition 3.4 for the moment, we prove Theorem 1.3.

Proof of Theorem 1.3. Since $\psi \in C^\infty_c(\mathbb{J}_r)$, as before, there are constants $c_0 \in (0, 1)$, and $m_1, m_2 > 0$ such that $\sup \psi \subset \{(t, s) : m_1 \leq s \leq m_2, s < c_0\}$. By finite decomposition and scaling, we may assume $\sup \psi \subset \{(t, s) : 1 \leq s \leq 2, s < c_0\}$.

We now consider the Fourier transform of the function $(x, t, s) \to \tilde{A}_\xi f(t, x, s)$:
\[
F(\zeta) = S(\zeta) \tilde{f}(\xi) := \int \int \int e^{-i(t \tau + s \sigma + \Phi_\zeta(\Theta, \Phi; \tau, \sigma))} \psi(t, s) d\theta d\phi ds dt \tilde{f}(\xi),
\]
where $\zeta = (\xi, \tau, \sigma)$. Let us set $m^\zeta(\xi) = (1 + |\zeta|^2)^{\alpha/2}, \varphi_0 = \varphi_{<0}(|1|)$, and $\varphi_0 = 1 - \varphi_0$. To prove Theorem 1.3 we need to show $\| F^{-1}(m^\zeta F) \|_{L^p} \lesssim \| f \|_{L^p}$. Since $\| F^{-1}(\varphi_0 m^\zeta F) \|_{L^p} \lesssim \| f \|_{L^p}$, we only have to show
\[
\| F^{-1}(\varphi_0 m^\zeta F) \|_{L^p} \lesssim \| f \|_{L^p}.
\]

For a large positive constant $C$, we set $\varphi_\ast(\zeta) = \varphi_{<0}(|\tau|/C|\xi|)$ and $\varphi_\ast(\zeta) = \varphi_{<0}(|\sigma|/C|\xi|)$. We also set $\varphi_\ast = 1 - \varphi_\ast$ and $\varphi_\ast = 1 - \varphi_\ast$. Thus, we have
\[
\varphi_\ast \varphi_\ast + \varphi_\ast \varphi_\ast + \varphi_\ast \varphi_\ast + \varphi_\ast \varphi_\ast = 1.
\]
If $|\tau| \geq C|\xi|$, integration by parts in $t$ gives $|S(\zeta)| \lesssim (1 + |\tau|)^{-N}$ for any $N$. Since $|\tau| \geq C|\xi|$ and $|\sigma| \leq C|\xi|$ on the support of $\varphi_\ast \varphi_\ast$, one can easily see $\| F^{-1}(\varphi_\ast \varphi_\ast \varphi_\ast m^\zeta F) \|_{L^p} \lesssim \| f \|_{L^p}$ for any $\alpha$. The same argument also shows that $\| F^{-1}(\varphi_\ast \varphi_\ast \varphi_\ast m^\zeta F) \|_{L^p}, \| F^{-1}(\varphi_\ast \varphi_\ast \varphi_\ast \varphi_\ast m^\zeta F) \|_{L^p} \lesssim \| f \|_{L^p}$ for any $\alpha$. Now, we note that $|\tau| \leq C|\xi|$ and $|\sigma| \leq C|\xi|$ on the support of $\varphi_\ast \varphi_\ast$. Thus, by the Mikhlin multiplier theorem
\[
\| F^{-1}(\varphi_\ast \varphi_\ast \varphi_\ast \varphi_\ast m^\zeta F) \|_{L^p} \lesssim \| F^{-1}(m^\zeta F) \|_{L^p},
\]
where \( \tilde{m}^\alpha(\zeta) = (1 + |\zeta|^2)^{\alpha/2} \). Since \( \text{supp} \, \psi \subset \{(t, s) : 1 \leq s \leq 2, s < c_0 t\} \), the right hand side is bounded above by \( \|A_t^* f\|_{L^p(D_1)} \). Therefore, using Proposition 3.4, we get \( \|F^{-1}(\varphi_+ \varphi_0^* \tilde{m}^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p} \). \( \square \)

In what follows, we prove Proposition 3.4 using the estimates obtained in Section 2.2

**Proof of Proposition 3.4.** Let \( n \) be an integer such that \( 2^n \leq 1/\tau < 2^{n+1} \). Then, we decompose

\[
A_t^* f = A_t^* f_{\leq n} + \sum_{k \geq n} A_t^* f_k^{<0} + \sum_{0 \leq j < n \leq k} A_t^* f_j^k + I_t^* f + \Pi_t^* f,
\]

where

\[
I_t^* f = \sum_{j \geq n, k > 2j - n} A_t^* f_k^j, \quad \Pi_t^* f = \sum_{n \leq j \leq k \leq 2j - n} A_t^* f_j^k + \sum_{n \leq j} A_t^* f_j^{<n}.
\]

Note that \( \|A_t^* f_{\leq n}\|_{L^p} \lesssim \tau^{-\alpha} \|A_t^* f\|_{L^p} \). Also, \( \|A_t^* f_{\leq n}\|_{L^p} \lesssim \tau^{-\alpha+1/p} \|f\|_{L^p} \lesssim \tau^{-1/p} \|f\|_{L^p} \) since \( \alpha < 4/p \). Similarly, using (2.17), we have \( \|A_t^* f_{\leq n}\|_{L^p} \lesssim \tau^{1/p - 1/2} \|f\|_{L^p} \) for \( k \geq n \). Taking sum over \( k \) gives

\[
\|\sum_{k \geq n} A_t^* f_k^{<0}\|_{L^p} \lesssim \sum_{k \geq n} \tau^{2(\alpha - \frac{1}{2})k} \tau^{-\frac{1}{2}j} \|f\|_{L^p} \lesssim \tau^{-1/p} \|f\|_{L^p}
\]

for \( \sigma < 4/p \). Let \( \sigma < 4/p \) be fixed. Then, for \( 0 \leq j < n \leq k \), by (2.19) it follows that \( \|A_t^* f_j^k\|_{L^p} \lesssim \tau^{\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + \frac{1}{2} k(\alpha - \frac{1}{2})} \|f\|_{L^p} \) for \( p \geq 4 \). Thus, we see that

\[
\|\sum_{0 \leq j < n \leq k} A_t^* f_j^k\|_{L^p} \lesssim \tau^{\frac{1}{p} - \alpha} \|f\|_{L^p} \lesssim \tau^{-\frac{1}{p}} \|f\|_{L^p}.
\]

Therefore, it remains to show the estimates for the operators \( I_t^* \) and \( \Pi_t^* \). Using (c) and (a) in Proposition 2.8 we obtain, respectively,

\[
\|A_t^* f_j^k\|_{L^p} \lesssim \tau^{\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + k(\alpha - \frac{1}{2})} \|f\|_{L^p}, \quad j \geq n, k > 2j - n
\]

and

\[
\|A_t^* f_j^k\|_{L^p} \lesssim \tau^{-\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + k(\alpha - \frac{1}{2})} \|f\|_{L^p}, \quad n \leq j \leq k \leq 2j - n
\]

for any \( \sigma > 0 \) and \( p \geq 4 \). Besides, (b) in Proposition 2.8 (2.31) with \( h = \lambda \) gives

\[
\|A_t^* f_j^k\|_{L^p} \lesssim \tau^{-\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + k(\alpha - \frac{1}{2})} \|f\|_{L^p}, \quad n \leq j \leq k \leq 2j - n
\]

for any \( \sigma > 0 \) and \( p \geq 4 \). Therefore, recalling \( p > 8 \) and \( \alpha < 4/p \), we get

\[
\|I_t^* f\|_{L^p} \lesssim \tau^{\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + \sum_{j \geq n, k > 2j - n} \frac{1}{2} k(\alpha - \frac{1}{2})} \|f\|_{L^p} \lesssim \tau^{-\frac{1}{p}} \|f\|_{L^p}
\]

and

\[
\|\Pi_t^* f\|_{L^p} \lesssim \tau^{-\frac{1}{p} \frac{1}{2} j - (\frac{1}{2} + \sigma) + \sum_{n \leq j \leq k \leq 2j - n} \frac{1}{2} k(\alpha - \frac{1}{2})} \|f\|_{L^p} \lesssim \tau^{-\frac{1}{p}} \|f\|_{L^p}.
\]

This completes the proof. \( \square \)

4. **ONE-PARAMETER LOCAL SMOOTHING AND ESTIMATE WITH FIXED \( t, s \)**

In this section we prove Theorems 1.4 and 1.5.

4.1. **One-parameter propagator.** In order to prove Theorem 1.4 we make use of local smoothing estimate for the operator \( f \rightarrow U f(x, t, c_0 t) \). For the two-parameter propagator \( U \), we can handle the associated operators \( e^{itD} \) and \( e^{isD} \) separately so that the sharp smoothing estimates are obtained by utilizing the decoupling and local smoothing inequalities for the cone in \( \mathbb{R}^{2+1} \). However, for the sharp estimate for \( f \rightarrow U f(x, t, c_0 t) \) a similar approach does not work. Instead, we make use of the decoupling inequality for the conic surface \( (\xi, |\xi| + c_0|\xi|) \) in \( \mathbb{R}^{3+1} \). (See [5] and Theorem 2.1 of [3]).
Proposition 4.1. Set $\tilde{U}_\pm f(x,t) = Uf(x,t,\pm c_0 t)$. Let $1 \leq \lambda \leq h \leq \lambda^2$. Then, if $6 \leq p \leq \infty$, for any $\epsilon > 0$ we have
\begin{equation}
\|\tilde{U}_\pm f\|_{L^p_{\epsilon, r}(\mathbb{R}^3 \times [1, 2])} \lesssim \lambda^{\frac{3}{2} - \frac{\epsilon}{2}} h^{\frac{3}{2} + \frac{\epsilon}{2}} \|f\|_{L^p}
\end{equation}
whenever $\text{supp} \tilde{f} \subset \Lambda_2 \times I_2$. Also, the same bound with $h = \lambda$ holds for $4 \leq p \leq \infty$ whenever $\text{supp} \tilde{f} \subset \Lambda_2 \times I_1^2$.

Proof. When $p = \infty$, the estimate (4.1) is already shown in the previous section (see (2.5)). Thus, we focus on the estimates (4.1) for $p = 4, 6$, and the other estimates follow by interpolation.

We first consider the case $\text{supp} \tilde{f} \subset \Lambda_2 \times I_2$, for which (4.1) hold on a larger range $4 \leq p \leq \infty$. To show (4.1), we make use of the decoupling inequality associated to the conic surfaces
\[ \Gamma_\pm = \{ (\xi, P_\pm(\xi)) : \xi \in A_1 \times I_1^2 \} \]
where $P_\pm(\xi) := |\xi| \pm c_0 |\xi|$. In fact, we use the $l^p$ decoupling inequality for the conic surfaces [5, 3]. To this end, we first check that the Hessian matrix of $P_\pm$ is of rank 2. Indeed, a computation shows that
\[ \text{Hess } P_\pm(\xi) = \frac{1}{|\xi|^3} \begin{pmatrix} \xi_2^2 & -\xi_1 \xi_2 & 0 \\ -\xi_1 \xi_2 & \xi_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pm \frac{c_0}{|\xi|^3} \begin{pmatrix} \xi_2^2 + \xi_3^2 & -\xi_1 \xi_2 & -\xi_1 \xi_3 \\ -\xi_1 \xi_2 & \xi_1^2 + \xi_3^2 & -\xi_1 \xi_3 \\ -\xi_1 \xi_3 & -\xi_1 \xi_3 & \xi_1^2 + \xi_3^2 \end{pmatrix}. \]

Note that $\text{Hess } P_\pm(\xi)\xi = 0$, so $\Gamma$ has a vanishing principal curvature in the direction of $\xi$. By rotational symmetry in $\xi$, to compute the eigenvalues of $\text{Hess } P_\pm(\xi)$ it is sufficient to consider the case $\xi_1 = 0$ and $\xi_2 = |\xi| \neq 0$. Consequently, one can easily see that the matrix $\text{Hess } P_\pm(\xi)$ has two nonzero eigenvalues
\[ |\xi|^{-1} \pm c_0 |\xi|^{-1}, \pm c_0 |\xi|^{-1}. \]

Let us denote by $\mathcal{Q}_\lambda$ a collection of points which are maximally $\sim \lambda^{-1/2}$ separated in the set $S^2 \cap \{ \xi : |\xi| \geq 2^{-3} \xi_3 \}$. Let $\{ W_\mu \}_{\mu \in \mathcal{Q}_\lambda}^{}$ denote a partition of unity subordinated to a collection of finitely overlapping spherical caps centered at $\mu$ of diameter $\sim \lambda^{-1/2}$ which cover $S^2 \cap \{ \xi : |\xi| \geq 2^{-3} \xi_3 \}$ such that $|\varphi^H W_\mu| \lesssim \lambda^{a(\mu)}$. Denote $\Omega_\mu(\xi) = W_\mu(\xi/|\xi|)$. Since $\text{supp} \tilde{f} \subset \Lambda_2 \times I_2^2$, we have $f = \sum_{\mu \in \mathcal{Q}_\lambda} f_\mu$ where $f_\mu = \mathcal{F}^{-\lambda}(\Omega_\mu \tilde{f})$. So, we can write
\[ \tilde{U}_\pm f(x,t) = \sum_{\mu \in \mathcal{Q}_\lambda} \tilde{U}_\pm f_\mu(x,t) = \sum_{\mu \in \mathcal{Q}_\lambda} \int \tilde{e}^{i(x \cdot \xi + \xi \cdot P_\pm(\xi))} \tilde{f}_\mu(\xi) d\xi. \]

Since $\Gamma_\pm$ are conic surfaces with two nonvanishing curvatures in $\mathbb{R}^4$, we have the following $l^p$-decoupling inequality:
\begin{equation}
\| \tilde{\chi}(\xi) \tilde{U}_\pm f \|_{L^4_{\epsilon, r}} \lesssim \lambda^{1 - \frac{\epsilon}{2}} \left( \sum_{\mu \in \mathcal{Q}_\lambda} \| \tilde{\chi}(\xi) \tilde{U}_\pm f_\mu \|_{L^p_{\epsilon, r}} \right)^{1/p}
\end{equation}
for $p \geq 4$. (See [5 and 3, Theorem 1.4].) Here $\tilde{\chi} \in \mathcal{S}(\mathbb{R})$ such that $\tilde{\chi} \geq 1$ on $I$ and $\text{supp} \mathcal{F}(\tilde{\chi}) \subset [-1/2, 1/2]$. Using Lemma 2.1 as before, we see $\| \tilde{\chi}(\xi) \tilde{U}_\pm f_\mu \|_{L^4_{\epsilon, r}} \lesssim \| \tilde{\chi}(\xi) e^{i(|\xi| / 2) \mu} \sum_{\mu \in \mathcal{Q}_\lambda} f_\mu \|_{L^4_{\epsilon, r}}$, where $\mu = (\mu_1, \mu_2)$. Thus, a change of variables gives $\| \tilde{\chi}(\xi) \tilde{U}_\pm f_\mu \|_{L^4_{\epsilon, r}} \lesssim \| f_\mu \|_{L^p}$ for $1 \leq p \leq \infty$. Since $(\sum_{\mu} \| f_\mu \|_p^p) \lesssim \| f \|_p$ for $p \geq 2$, combining the estimates and (4.2) with $p = 4$, we obtain
\[ \| \tilde{U}_\pm f \|_{L^4_{\epsilon, r}} \lesssim \lambda^{1 - \frac{\epsilon}{2}} \| f \|_{L^4}. \]
Interpolation with the easy $L^\infty$ estimate \((2.4)\) with \(p = q = \infty\) gives \((3.1)\) with \(h = \lambda\) for \(4 \leq p \leq \infty\).

Now, we consider the case \(\text{supp } \hat{f} \subset A_\lambda \times I_h\) with \(\lambda \leq h \leq \lambda^2\). Recall the partition of unity \(\{w_\nu\}_{\nu \in \mathcal{G}_h}\) on the unit circle \(\mathbb{S}^1\) and \(f_\nu = \omega_\nu(D)f\). Note that \(\hat{U}_\pm f_\nu(\cdot, x, t)\), \(\nu \in \mathcal{G}_h\) have Fourier supports contained in finitely overlapping rectangles of dimension \(\lambda \times \lambda^{1/2}\). So, we have

\[
\|\sum_{\nu \in \mathcal{G}_h} \hat{U}_\pm f_\nu(\cdot, x, t)\|_p \lesssim \lambda^{1/2-\frac{1}{p}}\left(\sum_{\nu \in \mathcal{G}_h} \|\hat{U}_\pm f_\nu(\cdot, x, t)\|_p^p\right)^{1/p}
\]

for \(2 \leq p \leq \infty\), which is a simple consequence of the Plancherel theorem and interpolation (for example, see Lemma 6.1 in [33]). Integration in \(x_3\) and \(t\) gives

\[
(4.3) \quad \|\hat{U}_\pm f\|_{L^p_{x,t}(\mathbb{R}^2 \times \mathbb{R})} \lesssim \lambda^{\frac{1}{2}} \left(\sum_{\nu \in \mathcal{G}_h} \|\hat{U}_\pm f_\nu\|_{L^p_{x,t}(\mathbb{R}^2 \times \mathbb{R})}^p\right)^{1/p}, \quad 2 \leq p \leq \infty.
\]

We proceed to obtain estimates for \(\|\hat{U}_\pm f_\nu\|_{L^p_{x,t}(\mathbb{R}^3)}\). Using Lemma \((2.4)\) and changing variables \(x \to x - (\nu, 0)t\), we see \(\|\hat{U}_\pm f_\nu\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim \|e^{\pm itc_\lambda D}f_\nu\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})}\). Similarly, we also have \(\|e^{\pm itc_\lambda D}f_\nu\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim \|\hat{U}_\pm f_\nu\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})}\), where

\[
\hat{U}_\pm h(x, t) = \int e^{i\left(x - (\nu, 0)t\right)(\xi_1^2 + \xi_2^2 + \xi_3^2)}\hat{h}(\xi)d\xi.
\]

Therefore, from \((4.3)\) it follows that

\[
(4.4) \quad \|\hat{U}_\pm f\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim \lambda^{\frac{1}{2}} \left(\sum_{\nu \in \mathcal{G}_h} \|\hat{U}_\pm f_\nu\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})}^p\right)^{1/p}, \quad 2 \leq p \leq \infty.
\]

Note that Fourier transform of \(f\) is contained in \(\{\xi : |\xi| \sim \lambda\}\) because \(\lambda \leq h\). To estimate \(\hat{U}_\pm f_\nu\), freezing \(\nu^\perp \cdot \hat{x}\), we use the \(\ell^2\) decoupling inequality \([5]\) (i.e., \((2.7)\)) with \(p = 2, q = 6,\) and \(\lambda = h\) with respect to \(\nu \cdot \hat{x}\), \(x_3\) variables. Thus, by the decoupling inequality followed by Minkowski’s inequality, we get

\[
\|\hat{U}_\pm f_\nu\|_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim h^\epsilon \left(\sum_{\nu \in \mathcal{G}_h} \|\hat{\chi}(t)\hat{U}_\pm f_\nu\|_{L^2_{x,t}}^2\right)^{1/2},
\]

where \(\mathcal{F}(f_\nu)(\xi) = \omega_\nu(\nu \cdot \xi, \xi_3)f_\nu(\xi)\). Since \#\{\nu : f_\nu \neq 0\} \lesssim \lambda h^{-1/2}\), by Hölder’s inequality it follows that

\[
\|\hat{U}_\pm f_\nu\|_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim h^\epsilon (\lambda h^{-1/2}) \left(\sum_{\nu \in \mathcal{G}_h} \|\hat{\chi}(t)\hat{U}_\pm f_\nu\|_{L^6_{x,t}}^6\right)^{1/6}.
\]

Lemma \((2.4)\) and a similar argument as before yield \(\|\hat{\chi}(t)\hat{U}_\pm f_\nu\|_{L^6_{x,t}} \lesssim \|f_\nu\|_6\). Hence, \(\|\hat{U}_\pm f_\nu\|_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim \lambda^2 h^{-1+6\epsilon} \sum_{\nu \in \mathcal{G}_h} \|f_\nu\|_6^6 \lesssim \lambda^2 h^{-1+6\epsilon}\|f_\nu\|_{L^6_{x,t}}^6\). Therefore, combining this and \((4.4)\) with \(p = 6\), we obtain \((4.1)\) for \(p = 6\).

\[\square\]

4.2. Proof of Theorem \((1.4)\) We denote \(L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R}) = L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})\). By an argument similar to the proof of Theorem \((1.3)\) it is sufficient to show that

\[
\|\hat{A}^{\alpha f}_t f\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R}^3)}, \quad \alpha < 3/p
\]

for a constant \(c_0 \in (0, 1)\). We use the decomposition \((3.6)\) with \(s = c_0 t\) and \(n = 0\) to have

\[
\hat{A}^{\alpha f}_t f = \hat{A}^{\alpha f}_{t < 0} f + \sum_{k \geq 0} \hat{A}^{\alpha f}_{t \leq 0} f^k + \Gamma^{\alpha f}_t f + \Pi^{\alpha f}_t f.
\]
The estimates for \(A_t^{\alpha t}f\) and \(\sum_{k\geq 0} A_t^{\alpha t}f_k\) follow from (2.10) and (2.18) for fixed \(t, s\). Indeed, we have \(\|A_t^{\alpha t}f\|_{L^{p,\beta}(\mathbb{R}^3)} \lesssim \|f\|\) and
\[
\sum_{k\geq 0} \|A_t^{\alpha t}f_k\|_{L^{p,\beta}(\mathbb{R}^3)} \lesssim \sum_{k\geq 0} 2^{(3/p-1/2)k} \|f\|_p \lesssim \|f\|_p
\]
for \(p > 6\).

We obtain the estimates for \(I_t^{\alpha t}\) and \(\Pi_t^{\alpha t}\) using the next proposition.

**Proposition 4.2.** (a) If \(1 \leq \lambda < h \leq \lambda^2\), then for any \(\epsilon > 0\) we have
\[
\|A_t^{\alpha t}f\|_{L^{p,\beta}_{\alpha,\epsilon}(\mathbb{R}^3)} \lesssim \lambda^{1-\frac{1}{p}} h^{-1+\frac{1}{p}+\epsilon\alpha} \|f\|_{L^p}
\]
for \(6 \leq p \leq \infty\) whenever supp \(\hat{f} \subset A_{\lambda} \times I_h\). (b) If supp \(\hat{f} \subset A_{\lambda} \times \mathbb{I}_{\lambda}^h\), the estimate (4.5) holds with \(h = \lambda\) for \(4 \leq p \leq \infty\). (c) If \(1 \leq \lambda < h \leq \lambda^2\), we have
\[
\|A_t^{\alpha t}f\|_{L^{p,\beta}_{\alpha,\epsilon}(\mathbb{R}^3)} \lesssim \lambda^{-\frac{1}{p}} h^{-\frac{1}{p}+\epsilon}\|f\|_{L^p}
\]
for \(4 \leq p \leq \infty\) whenever supp \(\hat{f} \subset A_{\lambda} \times I_h\).

Assuming this for the moment, we finish the proof of Theorem 1.3. By (a) and (b) in Proposition 4.2, we have
\[
\|\Pi_t^{\alpha t}f\|_{L^{p,\beta}_{\alpha,\epsilon}(\mathbb{R}^3)} \lesssim \sum_{j\geq 0} 2^{(1-\frac{1}{p})j} \sum_{k\leq 2j} 2^{k(\frac{1}{p}+\epsilon+\alpha)} \|f\|_{L^p}.
\]
Since \(p > 6\) and \(\alpha < 3/p\), taking \(\epsilon > 0\) small enough, we have the right hand side bounded above by \(C\|f\|_{L^p}\). Finally, using (c) in Proposition 4.2 we obtain
\[
\|\Pi_t^{\alpha t}f\|_{L^{p,\beta}_{\alpha,\epsilon}(\mathbb{R}^3)} \lesssim \sum_{j\geq 0} \sum_{k\geq 2j} 2^{j(\frac{1}{p}+\epsilon+\alpha)+k(\frac{1}{p}-\epsilon)} \|f\|_{L^p} \lesssim \|f\|_{L^p}
\]
for \(p > 6\) and \(\alpha < 3/p\).

To complete the proof, it remains to prove Proposition 4.2. For the purpose we closely follow the proof of Proposition 2.8.

**Proof of Proposition 4.2.** We recall (2.21), (2.22), and (2.23). As seen in the proof of Proposition 2.8 using the Mikhlin multiplier theorem, we can handle \(E(t, s)\) as if it is \(|\xi|^{-N} |\xi|^{-1}\) (see (2.27)). Likewise, we can replace \(E_N(c, t, |\xi|)\) by \((c, t, |\xi|)^{-N}\). Thus, the matter is reduced to obtaining estimates for the operators
\[
\tilde{C}_x f(x, t) := |D|^{-\frac{1}{2}} |D|^{-\frac{1}{2}} e^{i(\xi|D|)\pm t} f(x), \quad \kappa = \pm
\]
(cf. (2.34)). Thus, it is sufficient to show that the desired bounds on \(A_t^{\alpha t}\) also hold on \(\tilde{C}_x\).

We first consider the case (a). Note \(\|\tilde{C}_x f\|_{L^p_m(\mathbb{R}^3)} \lesssim (\lambda h)^{-1/2} e^{i(\xi|D|)\pm t} \|f\|_{L^p_m(\mathbb{R}^3)}\) since supp \(\hat{f} \subset A_{\lambda} \times I_h\). By Proposition 1.1 we get
\[
\|\tilde{C}_x f\|_{L^p_m(\mathbb{R}^3)} \lesssim \lambda^{1-\frac{1}{p}} h^{-\frac{1}{p}+\epsilon}\|f\|_{L^p}, \quad \kappa = \pm
\]
for \(6 \leq p \leq \infty\) as desired. In fact, the estimates for \(e^{i(\pm t(D)|\pm c_0t(D)|)} f\) follow by conjugation and reflection as before (cf. Remark 1.1). Also, note that \(\|\tilde{C}_x f\|_{L^p_{\alpha,\epsilon}} \lesssim \lambda^{-2} e^{i\alpha t|D|\pm c_0t|D|} \|f\|_{L^p_{\alpha,\epsilon}}\) when supp \(\hat{f} \subset A_{\lambda} \times I_h\). Thus, we get the estimate in the case (b) in the same manner.

Finally, we consider the case (c). Since supp \(\hat{f} \subset A_{\lambda} \times I_h\) and \(\lambda^2 \leq h\), applying Mikhlin’s multiplier theorem and Lemma 2.4 successively, we see \(\|\tilde{C}_x f\|_{L^p_{\alpha,\epsilon}} \lesssim \)
(\lambda h)^{-1/2} \| e^{i(\kappa t |D| \pm c a |D^1|)} f \|_{L^p_2} \lesssim (\lambda h)^{-1/2} \| e^{i(\kappa t |D| \pm c a D^1)} f \|_{L^p_2}. \) Thus, by a change of variables we have
\[
\| \tilde{C}^s f \|_{L^p_{s^3}(\mathbb{R}^3 \times I)} \lesssim (\lambda h)^{-1/2} \| e^{i\kappa t |D| |f|} \|_{L^p_{s^3}(\mathbb{R}^3 \times I)}
\]
for \( 1 \leq p \leq \infty \) and \( \kappa = \pm \). Therefore, for \( 4 \leq p \leq \infty \), the desired estimate follows from (2.1).

4.3. Estimates with fixed \( s,t \). In this subsection we prove Theorem 4.4. We consider estimates for \( A^s_t \) with fixed \( 0 < s < t \).

**Lemma 4.3.** Let \( 0 < s < t \). Let \( 1 \leq p \leq q \leq \infty \), \( 1/p + 1/q \leq 1 \), and \( h \geq \lambda \sim 1 \). Suppose \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \). Then, we have
\[
\| A^s_t f \|_{L^p_2} \lesssim h^{1/2} \| f \|_{L^p_2}.
\]

**Proof.** Recalling (2.21), (2.22), and (2.23), we see that the main contribution comes from \( C^s_\pm \) (see (2.33)). Applying Mikhlin’s theorem and Lemma 2.4, we see that
\[
\| C^s_\pm f(\cdot, s) \|_{L^p_2} \lesssim h^{-1/2} \| e^{i\kappa t |D|} f \|_{L^p_2} \lesssim h^{-1/2} \| e^{i\kappa t |D|} f \|_{L^p_2}.
\]
Thus, Bernstein’s inequality gives the desired estimate \( \| C^s_\pm f \|_{L^p_2} \lesssim h^{1/2} \| f \|_{L^p_2}\) since \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \) and \( \lambda \sim 1 \).

**Lemma 4.4.** Let \( 0 < s < t \) and \( p \geq 2 \). (a) If \( 1 \leq \lambda \leq h \leq \lambda^2 \), then for any \( \epsilon > 0 \)
\[
(4.6) \quad \| A^s_t f \|_{L^p_2} \lesssim \lambda^{-1/2} \| f \|_{L^p_2}
\]
holds whenever \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \). (b) If \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \), we have the estimate (4.6) with \( h = \lambda \). (c) If \( 1 \leq \lambda \) and \( \lambda^2 \leq h \), then for any \( \epsilon > 0 \)
\[
\| A^s_t f \|_{L^p_2} \lesssim \lambda^{-1/2} \| f \|_{L^p_2}
\]
holds whenever \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \).

**Proof.** As before, it is sufficient to show that \( C^s_\pm \) (2.34) satisfies the above estimates in place of \( A^s_t \). Note that
\[
\| C^s_\pm f \|_{L^p_2} \lesssim (\lambda h)^{-1/2} \| \mathcal{U} f(\cdot, \kappa t, \pm \xi) \|_{L^p_2}.
\]
For all the cases (a), (b), and (c), the desired estimates for \( p = 2 \) follows by Plancherel’s theorem. Thus, we only need to show the estimates for \( p = \infty \). For the cases (a) and (b) the estimates for \( p = \infty \) follow from (2.5) of the corresponding cases (a) and (b) with \( p = q = \infty \) (Remark 1). Since \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \) and \( 1 \leq \lambda \) and \( \lambda^2 \leq h \), by Lemma 2.4 we note that
\[
\| \mathcal{U} f(\cdot, \kappa t, \pm \xi) \|_{L^p_\infty} \lesssim \| e^{i(\kappa t |D| \pm c a |D^1|)} f \|_{L^p_\infty} \lesssim \sum_{\pm} \| e^{i|D| \pm c a D^1} f \|_{L^p_\infty},
\]
where \( \hat{f}_\pm(\xi) = \chi(0,\infty)(\pm &xi) \hat{f}(\xi) \). Since \( \text{supp} \, \hat{f} \subset \mathcal{A}_\lambda \times \mathbb{R}_h \), the estimate for \( p = \infty \) in the case (c) follows from (2.1).

**Proof of Theorem 4.3.** Since \( A^s_t f \) is bounded from \( L^2 \) to \( L^2_{1/2} \), it is sufficient to show \( A^s_t f \) is bounded from \( L^p \) to \( L^p_2 \) for \( p > 4 \) and \( \alpha > 2/p \).

We use the decomposition (3.6) with \( n \sim 1 \). Note that \( \| A^s_t f \|_{L^p_2} \lesssim \| A^s_t f \|_{L^p_2} \lesssim \| A^s_t f \|_{L^p_2} \). By Lemma 4.3 we have
\[
\| A^s_t f \|_{L^p_2} \lesssim \sum_{\kappa \geq 0} \| A^s_t f \|_{L^p_2} \lesssim \sum_{\kappa \geq 0} 2^{\kappa(\alpha - 1/2)} \| f \|_{L^p_2} \lesssim \| f \|_{L^p_2}
\]
for \( \alpha < 2/p \) and \( p > 4 \). Since \( \alpha < 2/p \), using (a) and (b) in Lemma 4.3 with an \( \epsilon \) small enough, we have
\[
\| f \|_{L^p_2} \lesssim \sum_{0 \leq j \leq k \leq 2j} 2^{(1 - \frac{1}{2})j} 2^{k(\alpha - 1/2 + 1/4 + \epsilon)} \| f \|_{L^p_2} \lesssim \| f \|_{L^p_2}
\]
for $p \geq 2$. Similarly, using (c) in Lemma 1.3 we obtain

$$\| \hat{I}_t f \|_{L^p_{\alpha, x}} \lesssim \sum_{j \geq 0} \sum_{k \geq 2j} 2^{(\alpha - \frac{1}{p} + c)k} 2^{-\frac{j}{p}} \| f \|_{L^p} \lesssim \| f \|_{L^p}$$

for $p > 4$ and $\alpha < 2/p$. \hfill \Box

5. Sharpness of the results

In this section, considering specific examples, we show sharpness of the estimates in Theorem 1.2, 1.3, 1.4, and 1.5 except for some endpoint cases.

5.1. Necessary conditions on $(p, q)$ for (1.2) to hold. We show that if (1.2) holds, then the following hold true:

(a) $p \leq q$,  \hspace{1cm} (b) $3 + 1/q \geq 7/p$,  \hspace{1cm} (c) $1 + 2/q \geq 3/p$, \hspace{1cm} (d) $3/q \geq 1/p$.

This shows that (1.2) fails unless $(1/p, 1/q)$ is contained in the closure of $Q$.

To show (a)–(d), it is sufficient to consider $M_0$ (see (3.3)) instead of $M_c$ with $J_1 = \{(t, s) \in [1, 2]^2 : s < c t \}$. The condition (a) is clear since $A^*_t$ is an translation invariant operator, which can not be bounded from $L^p$ to $L^4$ if $p > q$. It can also be seen by a simple example. Indeed, let $f_R$ be the characteristic function of a ball of radius $R \gg 1$ which is centered at the origin. Then, \( M_0 f_R(x) \sim 1 \) for $|x| \leq R/2$, so we have \( \| M_0 f_R \|_{L^p} / \| f_R \|_{L^p} \gtrsim R^{3/q - 3/p} \). Thus, $M_0$ can be bounded from $L^p$ to $L^q$ only if $p \leq q$.

To show (b), let $f_r$ denote the characteristic function of the set

\[
\{(x_1, x_2, x_3) : |x_1| < r^2, |x_2| < r, |x_3| < r^4\}
\]

for a small $r > 0$. One can easily see that $M_0 f_r(x) \sim r^3$ if $x_1 \sim 1$, $|x_2| \lesssim r$, and $x_3 \sim 1$. This gives

\[\| M_0 f_r \|_{L^p} / \| f_r \|_{L^p} \gtrsim r^{3+\frac{4}{p} - \frac{3}{q}}.\]

Therefore, letting $r \to 0$ shows that the maximal operator is bounded from $L^p$ to $L^q$ only if (b) holds. Now, for (c) we consider the characteristic function of

\[\{(\bar{x}, x_3) : |\bar{x}| - 1 < r, |x_3| < r^2\},\]

which we denote by $\tilde{f}_r$. Note that $M_0 \tilde{f}_r \sim r$ if $|\bar{x}| \lesssim r$ and $x_3 \sim 1$. So, we have

\[\| M_0 \tilde{f}_r \|_{L^p} / \| \tilde{f}_r \|_{L^p} \gtrsim r^{1+\frac{4}{p} - \frac{3}{q}},\]

which gives (c) by taking $r \to 0$. Finally, to show (d), let $\tilde{f}_r$ be the characteristic function of the $r$-neighborhood of $T^0_c$. Then, $|M_0 \tilde{f}_r(x)| \approx 1$ if $|x| \lesssim r$. Thus, it follows that $\| M_0 \tilde{f}_r \|_{L^p} / \| \tilde{f}_r \|_{L^p} \gtrsim r^{\frac{4}{p} - \frac{3}{q}}$. So, letting $r \to 0$, we obtain (d).

5.2. Sharpness of smoothing estimates. Let $c_0 \in (0, 8/9)$, and let $\psi$ be a smooth function supported in $[1/2, 2] \times [(1 - 2^{-4})c_0, (1 + 2^{-3})c_0]$ such that $\psi = 1$ if $(t, s) \in [3/4, 7/4] \times [(1 - 2^{-5})c_0, (1 + 2^{-5})c_0]$. Then, we consider

\[\tilde{A}^*_t f(x) = \psi(t, s) A^*_t f(x).\]

We first claim that the estimates (1.3), (1.4), and (1.5) imply $\alpha \leq 4/p$, $\alpha \leq 3/p$, and $\alpha \leq 2/p$, respectively.

Let $\zeta_0$ be a function such that $\text{supp} \zeta_0 \subset [-10^{-2}, 10^{-2}]$ and $\zeta_0(s) > 1$ if $|s| < c_1$ for a small constant $0 < c_1 \ll c_0$. Let $\zeta_\ast \in C_c([-2, 2])$ such that $\zeta_\ast = 1$ on $[-1, 1]$. Two-parameter averages over tori 27
Note that \( \tilde{T}_{x_1}^0 := \mathbb{T}_{x_1}^0 \cap \{ x : ||x|-1| < 10c_1, x_3 > 0 \} \) can be parametrized by a smooth radial function \( \phi \). That is to say,

\[
\tilde{T}_{x_1}^0 = \{ (\bar{x}, \phi(\bar{x})) : ||\bar{x}| - 1| < 10c_1 \}.
\]

For a large \( R \gg 1 \), we consider

\[
f_R(x) = e^{iR(x_3 + \phi(\bar{x}))} \zeta_0(R(x_3 + \phi(\bar{x}))) \zeta_s(||\bar{x}| - 1|/c_1).
\]

Then, we claim that

\[
|A_t^s f_R(x)| \gtrsim 1, \quad (x, t, s) \in S_R,
\]

where \( S_R = \{ (x, t, s) : |x| \leq 1/(CR), |t - 1| \leq 1/(CR), |s - c_0| \leq 1/(CR) \} \) for a large constant \( C > 0 \). Indeed, note that

\[
A_t^s f(x) = \int_{\tilde{T}_{x_1}^0} e^{iR(x_3 + \phi(\bar{x}) - y_3)} \zeta_0(R(x_3 + \phi(\bar{y} - \bar{x}) - y_3)) \zeta_s(||\bar{x} - \bar{y}|| - 1|/c_1) d\sigma_t^s(\bar{y}).
\]

If \( |x| \leq 1/(CR) \) and \( |\bar{y} - 1| \leq 2c_1 \), we have \( |\phi(\bar{y} - \bar{x})| \leq 1/(CR) \) and \( |x_3 + \phi(\bar{y} - \bar{x}) - y_3| \lesssim 1/(CR) \) when \( y_3 = \phi(\bar{y}) \), i.e., \( y \in \tilde{T}_{x_1}^0 \). Thus, \( |A_t^s f(x)| \approx 1 \) if \( |x| \leq 1/(CR) \). Furthermore, if \( |t - 1| \leq 1/(CR) \) and \( |s - c_0| \leq 1/(CR) \), the integration is actually taken over a surface which is \( O(1/(CR)) \) perturbation of the surface \( \tilde{T}_{x_1}^0 \). Thus, taking \( C \) large enough, we see that (5.1) holds.

By Mikhlin’s theorem it follows that \( \| \tilde{A}_t^s g \|_{L^p(\mathbb{R}^3)} \gtrsim \| (1 + |D_A|)^{a/2} \tilde{A}_t^s g \|_{L^p(\mathbb{R}^3)} \). Note that \( \tilde{f}_R(\xi) = 0 \) if \( \xi \notin [(1 - 10^{-2})R, (1 + 10^{-2})R] \). Since \( \mathcal{F}(A_t^s f)(\xi) = \hat{f}(\xi) \mathcal{F}(\phi_t^s)(\xi) \), we see

\[
\| \tilde{A}_t^s f \|_{L^p(\mathbb{R}^3)} \gtrsim \| R^a \mathcal{F}(\phi_t^s) \|_{L^p(\mathbb{R}^3)} \gtrsim \| R^a \|_{L^p(S_R)} \gtrsim R^{a-5/p}.
\]

For the last inequality we use (5.1). Since \( \| f_R \|_{L^p} \approx R^{-1/p} \), (5.3) implies that \( \alpha \leq 4/p \). Fixing \( t = 1 \) and \( s = c_0 \), by (5.1) we similarly have \( \| A_t^s f \|_{L^p(\mathbb{R}^3)} \gtrsim R^{a-5/p} \). Thus, (5.3) holds only if \( \alpha \leq 2/p \). Concerning \( A_t^s g \), by (5.1) it follows that \( |A_t^s g_R(x)| \gtrsim 1 \) if \( |t - 1| \leq 1/(CR) \) and \( |x| \leq 1/(CR) \) for \( C \) large enough. Thus, \( \| A_t^s g \|_{L^p(\mathbb{R}^3)} \gtrsim \| A_t^s g_R \|_{L^p(\mathbb{R}^3)} \gtrsim R^{a-4/p} \). Therefore, (1.4) implies \( \alpha \leq 3/p \). This proves the claim.

Therefore, to show sharpness of the estimates (1.3)–(1.5). We need only to show that each of the estimates (1.6), (1.7), and (1.8) holds only if \( \alpha \leq 1/2 \). To do this, we consider

\[
g_R(x) = e^{iR(x_3 + c_0)} \zeta_0(R(x_3 + c_0)) \zeta(|x|).
\]

Then, we have

\[
g_R(x) = \int_{T_{x_1}} e^{iR(x_3 + c_0) - y_3} \zeta_0(R(x_3 + c_0) - y_3) \zeta_s(|x - y|) d\sigma_t^s(\bar{y}).
\]

Recalling (1.1), we see that the integral is nonzero only if \( |R(x_3 + c_0) - s \sin \theta| \lesssim 2/C \). Since \( |x_3 + c_0 - s| \lesssim 1/C \), the integral is taken over the set \( \mathbb{T} := \Phi_t^s(\theta, \phi) \cap |1 - \sin \theta| \lesssim 1/R \). Note that the surface area of \( \mathbb{T} \) is about \( R^{-1/2} \), thus (5.2) follows.

Since \( g_R(\xi) = 0 \) if \( \xi \notin [(1 - 10^{-2})R, (1 + 10^{-2})R] \), following the same argument as above, from (5.2) we obtain \( \| A_t^s g \|_{L^p_{t,s}} \gtrsim R^{a - 1/2 - 1/p} \). Hence, (1.3) implies that \( \alpha \leq 1/2 \).
Regarding (1.4), we consider \( \tilde{S}_R := \{(x, t, s) : |x|, |t - 1| \leq 1/C, |x_3 + c_0 - c_0t| \leq 1/C R \} \) for a large constant \( C \gg c_0 \). Then, we have \( |A_{x,t}^{c_0 t} g_R(x)| \gtrsim R^{-1/2} \) for \( (x, t) \in \tilde{S}_R \), thus we see (1.4) implies \( \alpha \leq 1/2 \).

Finally, for (1.5), fixing \( t = 1 \) and \( s = c_0 \), we consider \( \tilde{S}_R := \{x : |x| \leq 1/C, |x_3| \leq 1/C R \} \) for a constant \( C > 0 \). Then, it is easy to see \( |A_{x,t}^{c_0} g_R(x)| \gtrsim R^{-1/2} \) for \( x \in \tilde{S}_R \) if we take \( C \) large enough. Similarly as before, we have \( \|A_{x,t}^{c_0} g_R\|_{L^p_{\alpha,x}} \gtrsim R^\alpha R^{-1/2-1/p} \). Therefore, (1.5) implies \( \alpha \leq 1/2 \) because \( \|g_R\|_{L^p} \sim R^{-1/p} \).

Acknowledgements

This work was supported by the National Research Foundation (Republic of Korea) grant no. 2022R1A4A1018904.

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30 JUYOUNG LEE AND SANGHYUK LEE

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