Flexible Density Tempering Approaches for State
Space Models with an Application to Factor
Stochastic Volatility Models

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Abstract

Duan and Fulop (2015) proposed a tempering or annealing approach to
Bayesian inference for time series state space models. In such models the likely-
lihood is often analytically and computationally intractable. Their approach
generalizes the annealed importance sampling (AIS) approach of Neal (2001)
and Del Moral et al. (2006) when the likelihood can be computed analytically.
Annealing is a sequential Monte Carlo approach that moves a collection of pa-
rameters and latent state variables through a number of levels, with each level
having its own target density, in such a way that it is easy to generate both the
parameters and latent state variables at the initial level while the target den-
sity at the final level is the posterior density of interest. A critical component
of the annealing or density tempering method is the Markov move component
that is implemented at every stage of the annealing process. The Markov move
component effectively runs a small number of Markov chain Monte Carlo iter-
ations for each combination of parameters and latent variables so that they are
better approximations to that level of the tempered target density. Duan and
Fulop (2015) used a pseudo marginal Metropolis-Hastings (PMMH) approach
with the likelihood estimated unbiasedly in the Markov move component. One
of the drawbacks of this approach, however, is that it is difficult to obtain good
proposals when the parameter space is high dimensional, such as for a high di-
mensional factor stochastic volatility models. We propose using instead more
flexible Markov move steps that are based on particle Gibbs and Hamiltonian
Monte Carlo and demonstrate the proposed methods using a high dimensional stochastic volatility factor model. An estimate of the marginal likelihood is obtained as a byproduct of the estimation procedure.

Keywords: Factor stochastic volatility model; Hamiltonian Monte Carlo; Particle Gibbs; Particle Markov chain Monte Carlo.

1 Introduction

We consider the problem of Bayesian inference over the parameters and the latent states in non-Gaussian non-linear state space models, and in particular a factor stochastic volatility model, which is a particular example of a high dimensional state space model. However, we believe that most of the lessons learnt are likely to apply more generally. Joint inference over both the parameters and latent states in such models can be challenging because the likelihood is an integral over the latent state variables. This integral is often analytically intractable and can also be computationally challenging to evaluate numerically when the dimensions of the latent states is high. In an important contribution in the literature, Andrieu et al. (2010) proposed two particle Markov chain Monte Carlo (PMCMC) methods for state space models. The first is the particle marginal Metropolis-Hastings (PMMH) method, where the parameters are generated with the unobserved latent states integrated out. The second is the particle Gibbs (PG) method, which generates the parameters conditional on the states. Both methods run a particle filter algorithm within an MCMC at each iteration. They show that for any finite number of particles, the target density used by these two algorithms has the joint posterior density of the parameters and states as a marginal density. Their work has been extended by Lindsten and Schön (2012), Lindsten et al. (2014), Olsson and Ryden (2011), Mendes et al. (2018), and Gunawan et al. (2018).

An alternative approach to MCMC is the annealed importance sampling (AIS) approach discussed by Neal (2001), which is a sequential importance sampling method where samples are first drawn from an easily-generated distribution and then moved towards the target distribution through Markov kernels. It is a useful method for estimating expectations with respect to the posterior of the parameters when it is possible to evaluate the likelihood pointwise. In general, the AIS method has some advantages compared to MCMC, and more generally, PMCMC approaches: (a) the Markov chain generated by an MCMC sampler can often be trapped in local modes and it is difficult to assess whether the chain has mixed adequately and converged to its invariant density, whereas AIS explores the parameter space more efficiently.
when the target distribution is multimodal, and assessing convergence to the posterior is much less of an issue. (b) it is often difficult to use MCMC to estimate the marginal likelihood, which is of interest because it is often used to choose between models (Kass and Raftery, 1995; Chib and Jeliazkov, 2001). Using AIS to estimate the likelihood is far more straightforward. (c) MCMC algorithms are not parallelizable in general, whereas it is straightforward to parallelize the AIS algorithm as discussed in Neal (2001), Del Moral et al. (2006), and Duan and Fulop (2015).

Duan and Fulop (2015) extends the annealing/tempering method of Neal (2001), and Del Moral et al. (2006) to time series state space models having intractable likelihoods and call their approach the Density Tempered Sequential Monte Carlo (SMC) algorithm. Their SMC approach includes three steps: reweighting, resampling, and Markov moves which are discussed in Section 2. A critical component of the annealing or density tempering method is the Markov move component that is implemented at every stage of the annealing process. The Markov move component effectively runs a small Markov chain Monte Carlo for each combination of the parameters and latent variables to help diversify the collection of parameters and latent variables so that they are better approximations to the tempered target density at that level or temperature. Duan and Fulop (2015) use a pseudo marginal Metropolis-Hastings (PMMH) approach with the likelihood estimated unbiasedly by the particle filter in the Markov move component. There are two issues with the Markov move based on PMMH. First, it is computationally costly to follow the guidelines of Pitt et al. (2012) for high dimensional state space models and set the optimal number of particles in the particle filter such that the variance of the log of the estimated likelihood is around 1; we show this in our empirical example in Section 5. It is also possible to implement the correlated PMMH of Deligiannidis et al. (2017) which is more efficient than standard PMMH for low dimensional state space models. However, given that we consider the high dimensional state space model in Section 5, even the correlated PMMH would get stuck unless a very large number of particles is used to ensure the variance of the log of the estimated likelihood is around 1. Second, it is hard to implement the PMMH Markov move efficiently when the dimension of the parameter space in the model is large because it is difficult to obtain good proposals for the parameters. This is because the first and second derivatives with respect to the parameters can only be estimated, while a random walk proposal is easy to implement but is very inefficient in high dimensions.

Our contribution is to develop and study two flexible annealing approaches that use Markov move steps that are more efficient than the density tempered SMC approach of Duan and Fulop (2015) and can handle a high dimensional parameter space. The first is based on the PG algorithm of Andrieu et al. (2010) and we
call it Annealed Importance Sampling for Intractable Likelihood with particle Gibbs (AISIL-PG). The second is based on Hamiltonian Monte Carlo (HMC) algorithm discussed by Neal (2011) and we call it Annealed Importance Sampling for Intractable Likelihood with Hamiltonian Monte Carlo (AISIL-HMC).

The AISIL-PG and AISIL-HMC approaches refine the density-tempered SMC methodology proposed by Duan and Fulop (2015) to allow for more flexible Markov moves and apply to any state space model. In particular, they allow for particle Metropolis within Gibbs moves and Hamiltonian Monte Carlo moves. Such a generalisation is important for two reasons. First, it permits applications to much higher dimensional parameter spaces and much better proposals than random walks; in particular, in some cases we are able to sample from the full conditional distribution using Gibbs steps. Second, using particle Metropolis within Gibbs (PMwG) Markov moves can substantially reduce the number of particles required because it is then unnecessary for the variance of the log of the likelihood estimate to be around 1 as we show in Section 5. The AISIL methods also provide an estimate of the marginal likelihood as a by product. We illustrate the proposed AISIL methods empirically using a sample of daily US stock returns to estimate a univariate stochastic volatility (SV) model and a multivariate high dimensional factor SV model.

The rest of the article is organised as follows. Section 2 outlines the basic state space model and the AISIL approach to such a model. Section 3 discusses in detail the application of the AISIL-PG and AISIL-HMC methods to the univariate SV model. Section 4 introduces the factor SV model, and discusses in detail the application of the AISIL-PG and AISIL-HMC methods to this model. Section 5 presents empirical results for both the univariate SV and for the factor SV models. The article also has an online supplement that contains some further technical empirical results. We use the following notation in both the main paper and the online supplement. Eq. (1), Sec. 1, Alg. 1 and Sampling Scheme 1, etc. refer to the main article, while Eq. (S1), Sec. S1, Alg. S1, and Sampling Scheme S1, etc. refer to the supplement.

## 2 Flexible Density Tempering

This section first introduces the basic state space model and then discusses the two flexible AISIL approaches.

### 2.1 The Basic State Space Model

We consider a state space model where the latent states $X_t$ determine the evolution of the system. The initial density of the latent state $X_1 = x_1$ is $f_1(x_1)$ and the
transition density of $X_t = x_t$ given $X_{1:t-1} = x_{1:t-1}$ is $f_t^\theta(x_t|x_{t-1})$ for $t \geq 2$. The observations $Y_t = y_t$ for $t = 1, \ldots, T$ are linked to the latent states through the observation equation $g_t^\theta(y_t|x_t)$. Denote a $d$ dimensional Euclidean space by $\mathbb{R}^d$. We assume that: (i) the state vector $x_t \in X$ and $y_t \in Y$ where $X \subset \mathbb{R}^{d_x}$ and $Y \subset \mathbb{R}^{d_y}$; (ii) the parameter vector $\theta \in \Theta$, where $\Theta$ is a subset of $\mathbb{R}^{d_\theta}$.

Our aim is to perform Bayesian inference over the latent states and the parameters conditional on the observations $y_{1:T}$. By using Bayes rule, the joint posterior of the latent states and the parameters is

$$p(\theta, x_{1:T}|y_{1:T}) = \frac{p(x_{1:T}, y_{1:T}|\theta)p(\theta)}{p(y_{1:T})},$$

where

$$p(x_{1:T}, y_{1:T}|\theta) = f_1^\theta(x_1) \prod_{t=2}^T f_t^\theta(x_t|x_{t-1}) \prod_{t=1}^T g_t^\theta(y_t|x_t);$$

$p(\theta)$ is the prior density of $\theta$ and $p(y_{1:T})$ is the marginal likelihood of $y_{1:T}$. For notational simplicity we will often write $x = x_{1:T}$ and $y = y_{1:T}$.

### 2.2 The Annealing Approach for State Space Models

This section discusses our proposed AISIL estimation method. The main idea of the proposed method is to begin with an easy to sample distribution and propagate a particle cloud $\{\theta^{(p)}_{1:M}, x^{(p)}_{1:M}, W^{(p)}_{1:M}\}$ through a sequence of tempered target densities $\xi_a(\theta, x)$, for $p = 0, \ldots, P$, to the posterior density of interest which is much harder to sample from directly. The tempered densities are defined as

$$\xi_a(\theta, x) = \eta_a(\theta, x)/Z_a,$$

where

$$Z_a = \int \eta_a(\theta, x) d\theta dx$$

and

$$\eta_a(\theta, x) = (\pi_0(\theta, x))^{1-a_p} (p(y|\theta, x)p(x|\theta)p(\theta))^{a_p},$$

The tempering sequence $a_{0:P}$ is such that $a_0 = 0 < a_1 < \ldots < a_P = 1$. If it is both easy to generate from the densities $p(\theta)$ and $p(x|\theta)$, and evaluate them, then we take $\pi_0(x, \theta) = p(\theta)p(x|\theta)$, and then

$$\eta_a(\theta, x) = p(y|\theta, x)^{a_p} p(x|\theta)p(\theta).$$

We now discuss the general AISIL approach that is summarized by Alg. 1. The initial particle cloud $\{\theta^{(0)}_{1:M}, x^{(0)}_{1:M}, W^{(0)}_{1:M}\}$ is obtained by generating the $\{\theta^{(0)}_{1:M}, x^{(0)}_{1:M}\}$ from $\pi_0(x, \theta)$, and giving the particles equal weight, i.e., $W^{(0)}_{1:M} = 1/M$. The weighted particles (particle cloud) $\{\theta^{(p-1)}_{1:M}, x^{(p-1)}_{1:M}, W^{(p-1)}_{1:M}\}$ at the $(p-1)$st level (or stage) of the annealing is an estimate of $\xi_{a_{p-1}}(\theta, x)$. Based on this estimate of $\xi_{a_{p-1}}(\theta, x)$, the
AISIL algorithm goes through the following steps to obtain an estimate of $\xi_{a_p}(\theta, x)$. The move from $\xi_{a_{p-1}}(\theta, x)$ to $\xi_{a_p}(\theta, x)$ is implemented by reweighting the particles by the ratio of the two unnormalized densities $\eta_{a_p}/\eta_{a_{p-1}}$, yielding the following weights,

$$W_{1:M}^{(p)} = \frac{w_{1:M}^{1:M}}{\sum_{j=1}^{M} w_j},$$

where $w_i = W_{i}^{(p-1)} \eta_{a_p}(\theta_i, x_i)/\eta_{a_{p-1}}(\theta_i, x_i) = W_{i}^{(p-1)} p(y|\theta_i, x_{a_{p-1}}^{a_p-a_{p-1}}).$

We follow Del Moral et al. (2012) and choose the tempering sequence adaptively to ensure a sufficient amount of particle diversity by selecting the next value of $a_p$ automatically such that the effective sample size (ESS) stays close to some target value $\text{ESS}_T$ chosen by the user. The effective sample size ESS is used to measure the variability in the $W_{i}^{(p)}$, and is defined as $\text{ESS} = \left( \sum_{i=1}^{M} \left(W_{i}^{(p)}\right)^2 \right)^{-1}$. The ESS varies between 1 and $M$, with a low value of ESS indicating that the weights are concentrated only on a few particles and a value of $M$ means that the particles are equally weighted. We achieve an ESS close to the target value by evaluating the ESS over a grid $a_{1,G,p}$ of potential $a_p$ values and select as $a_p$ that value of the value of $a_{j,p}$ whose ESS is the closest to $\text{ESS}_T$. We then resample the particles $\{\theta_{1:M}^{(p)}, x_{1:M}^{(p)}\}$ proportionally to their weights $W_{1:M}^{(p)}$, which means that the resampled particles then have equal weight $W_{1:M}^{(p)} = 1/M$. This has the effect of eliminating particles with negligible weights and replicating the particles with large weights, so the ESS is now $M$.

Repeatedly reweighting and resampling can seriously reduce the diversity of the particles, leading to particle depletion. The particle cloud $\{\theta_{1:M}^{(p)}, x_{1:M}^{(p)}, W_{1:M}^{(p)}\}$ may then not be a good representation of $\xi_{a_p}(\theta, x)$. To improve the approximation of the particle cloud to $\xi_{a_p}(\theta, x)$, we carry out $R$ Markov move steps for each particle using a Markov kernel $K_{a_p}$ which retains $\xi_{a_p}(\theta, x)$ as its invariant density. Thus, at each annealing schedule, we run a short MCMC scheme for each of the $M$ particles.

Step 1 and Steps 2a-2d of Alg. 1 are standard and apply to any model with slight modification so it is only necessary to discuss the Markov move step in detail for each model. We propose two Markov move algorithms that leave the target density $\xi_{a_p}(\theta, x)$ invariant. The first algorithm is based on particle Gibbs (Andrieu et al., 2010) which samples the latent states $x_{1:T}$ and is denoted as the AISIL-PG algorithm. The second algorithm is based on the Hamiltonian Monte Carlo proposal (Neal, 2011) to sample the latent states $x_{1:T}$ and is denoted as the AISIL-HMC algorithm. By using the results from Del Moral et al. (2006), the AISIL algorithm provides consistent inference for the target density $p(\theta, x_{1:T}|y_{1:T})$ as $M$ goes to infinity. See also Beskos et al. (2016) for recent consistency results for the adaptive
Algorithm 1 Generic AISIL Algorithm

1. Set \( p = 0 \) and initialise the particle cloud \( \{ \theta^{(0)}_1, \theta^{(0)}_M, x^{(0)}_1, x^{(0)}_M, W^{(0)}_1, W^{(0)}_M \} \), by generating the \( \{ \theta^{(0)}_1, \theta^{(0)}_M \} \) from \( \pi_0 (x, \theta) \), and giving them equal weight, i.e., \( W^{(0)}_i = 1/M \), for \( i = 1, \ldots, M \).

2. While the tempering sequence \( a_p < 1 \) do
   a. Set \( p \leftarrow p + 1 \)
   b. Find \( a_p \) adaptively by searching across a grid of \( a_p \) to maintain effective sample size (ESS) around some constant \( \text{ESS}_T \).
   c. Compute new weights, 
      \[
      W^{(p)}_1 = \frac{w^{(1:M)}_1}{\sum_{j=1}^{M} w_j} \quad \text{where} \quad w_i = \frac{W^{(p-1)}_i \eta_{a_p}(\theta_i, x_i)}{\eta_{a_{p-1}}(\theta_i, x_i)}.
      \] (1)
   d. Resample \( \{ \theta^{(p)}_1, x^{(p)}_1 \} \) using the weights \( W^{(p)}_1 \) to obtain \( \{ \theta^{(p)}_1, x^{(p)}_1, W^{(p)}_1 = 1/M \} \).
   e. Markov moves
      i. Let \( K_{a_p}(\theta, x, \cdot) \) be a Markov kernel having invariant density \( \xi_{a_p}(\theta, x) \). For \( i = 1, \ldots, M \), move each \( \theta^{(p)}_i, x^{(p)}_i \) \( R \) times using the Markov kernel \( K_{a_p} \) to obtain \( \{ \tilde{\theta}_i, \tilde{x}_i \} \).
      ii. Set \( \{ \tilde{\theta}^{(p)}_1, \tilde{x}^{(p)}_1 \} \leftarrow \{ \tilde{\theta}_{1:M}, \tilde{x}_{1:M} \} \).

3 Applying the AISIL method to the Univariate Stochastic Volatility Model

This section discusses the application of the AISIL methods to the simple univariate SV model. We introduce this univariate model first as it forms the basis of the factor SV model studied later.

\[
y_t = \exp (x_t/2) \epsilon_t, \quad x_{t+1} = \mu + \phi (x_t - \mu) + \tau \eta_t, \quad x_1 \sim N \left( \mu, \frac{\tau^2}{1 - \phi^2} \right),
\] (2)

with \( \epsilon_t \sim N(0,1) \) and \( \eta_t \sim N(0,1) \). We call \( x_{1:T} \) the latent volatility process. The unknown parameters \( \theta \) are \( \mu, \phi, \) and \( \tau^2 \). We place the following prior distributions
on the parameters, and assume that the parameters are independent apriori: (a) \( p(\mu) \propto I(-10 < \mu < 10) \). (b) We follow Jensen and Maheu (2010) and choose the prior for \( \tau^2 \) as inverse Gamma \( IG(v_0/2, s_0/2) \) with \( v_0 = 10 \) and \( s_0 = 0.5 \). (c) We restrict the persistence parameter \(-1 < \phi < 1\) to ensure stationarity, and follow Kim et al. (1998) and choose the prior for \( \phi \) as \((\phi + 1)/2 \sim Beta(a_0, b_0)\), with \( a_0 = 100 \) and \( b_0 = 1.5 \).

3.1 AISIL-HMC Markov Moves

This section discusses the application of the AISIL-HMC Markov moves for the univariate stochastic volatility (SV) model in Eq. (2). We first define the appropriate sequence of intermediate target density for \( p = 1, ..., P \),

\[
\eta_{ap}(\theta, x) = p(y|\theta, x)^{ap} p(x|\theta) p(\theta),
\]

where \( p(y|\theta, x)^{ap} = \prod_{t=1}^{T} g_\theta(y_t|x_t)^{ap}, \ p(x|\theta) = f_\theta(x_1) \prod_{t=2}^{T} f_\theta(x_t|x_{t-1}) \),

and \( p(\theta) \) is the prior of \( \theta \). Alg. 2 proposes the following Markov move steps, which are discussed in more detail in Section S1.

**Algorithm 2** A single Markov move step based on Hamiltonian Monte Carlo for univariate SV models

For each of the \( i = 1, ..., M \), particles.

(i) Sample \( x_{i,1:T}|y_{1:T}, \theta_i \) using Hamiltonian Monte Carlo.

(ii) Sample \( \mu_i|x_{i,1:T}, y_{1:T}, \theta_i, -\mu \)

(iii) Sample \( \phi_i|x_{i,1:T}, y_{1:T}, \theta_i, -\phi \)

(iv) Sample \( \tau_{i,2}^2|x_{i,1:T}, y_{1:T}, \theta_i, -\tau^2 \)

3.2 AISIL-PG Markov Moves

This section discusses the application of AISIL-PG Markov moves to a univariate stochastic volatility (SV) model. The main idea of Markov moves based on particle Gibbs is to define a sequence of tempered target densities on an augmented space that includes all of the parameters and all the particles generated by particle filter algorithm. The augmented target density has the \( \xi_{ap}(\theta, x) \) as a marginal density for \( p = 1, ..., P \). The particle filter is discussed in Section 3.2.1. The augmented target density is discussed in Section 3.2.2.
3.2.1 Sequential Monte Carlo/Particle Filtering

We first describe the particle filter methods used to obtain sequential approximations to the densities $p(x_{1:t}|y_{1:t}, \theta)$ for $t = 1, \ldots, T$. The particle filter algorithm consists of recursively producing a set of weighted particles $\{x^{(j)}_{1:t}, \hat{W}^{(j)}_{t}\}_{j=1}^{N}$ such that the intermediate densities $p(x_{1:t}|y_{1:t}, \theta)$, defined on the sequence of spaces $\{\chi^t; t = 1, \ldots, T\}$, are approximated by

$$
\hat{p}(x_{1:t}|y_{1:t}, \theta) = \sum_{j=1}^{N} \hat{W}^{(j)}_{t} \delta_{x^{(j)}_{1:t}}(dx_{1:t}),
$$

where $\delta_{a}(dx)$ is the Dirac delta distribution located at $a$. In more detail, given $N$ samples $\{x^{(j)}_{t-1}, \hat{W}^{(j)}_{t-1}\}_{j=1}^{N}$ (representing the filtering density $p(x_{t}|y_{1:t-1}, \theta)$ at time $t-1$, we use

$$
p(x_{t}|y_{1:t}, \theta) \propto \int p(y_{t}|x_{t}, \theta) p(x_{t}|x_{t-1}, \theta) p(x_{t-1}|y_{1:t-1}, \theta) dx_{t-1},
$$

to obtain the particles $\{x^{(j)}_{t}, \hat{W}^{(j)}_{t}\}_{j=1}^{N}$ (representing $p(x_{t}|y_{1:T}, \theta)$), by first drawing from a known and easily sampled proposal density function $m(x_{t}|x_{t-1}, y_{t}, \theta)$ and then computing the unnormalised weights $\tilde{w}^{(j)}_{t}$ to account for the difference between the target posterior density and the proposal, where

$$
\tilde{w}^{(j)}_{t} = \hat{W}^{(j)}_{t-1} \frac{p(y_{t}|x^{(j)}_{t}, \theta) p(x^{(j)}_{t}|x^{(j)}_{t-1}, \theta)}{m_{t}(x^{(j)}_{t}|x^{(j)}_{t-1}, y_{t}, \theta)}, \quad j = 1, \ldots, N;
$$

and then normalize $\hat{W}^{(j)}_{t} = \tilde{w}^{(j)}_{t} / \sum_{r=1}^{N} \tilde{w}^{(r)}_{t}$.

Chopin (2004) shows that as the number of time periods $t$ increases, the normalised weights of the particle system become concentrated on only few particles and eventually the normalised weight of a single particle converges to one. This is known as the ‘weight degeneracy’ problem. One way to reduce the impact of weight degeneracy is to include resampling steps in a particle filter algorithm. A resampling scheme is defined as $\mathcal{M}(\tilde{a}^{1:N}_{t} \hat{W}^{1:N}_{t-1})$, where $\tilde{a}^{j}_{t}$ indexes a particle in $\{x^{(j)}_{1:t}, \hat{W}^{(j)}_{t}\}_{j=1}^{N}$ and is chosen with probability $\hat{W}^{(j)}_{t}$. Some assumptions on the proposal density $m(x_{t}|x_{t-1}, y_{t}, \theta)$ and resampling scheme are given in Section S4. In our empirical application in Section 5, we use the bootstrap filter with $p(x_{t}|x_{t-1}, \theta)$ as a proposal density and systematic resampling.
The likelihood estimate is a by-product of the particle filter

\[
\hat{p}(y_{1:T}|\theta) = \hat{p}(y_1|\theta) \prod_{t=2}^{T} \hat{p}(y_t|y_{t-1}, \theta) = \prod_{t=2}^{T} \left\{ \frac{1}{N} \sum_{j=1}^{N} \tilde{w}^{(j)}_t \right\}.
\]

and is unbiased, i.e. \( E(\hat{p}(y_{1:T}|\theta)) = p(y_{1:T}|\theta) \) (see Proposition 7.4.1 of Del Moral (2004)). The likelihood estimate is an essential component of the PMMH Markov move of the density tempered SMC algorithm of Duan and Fulop (2015).

### 3.2.2 AISIL-PG Augmented Target Density

This section discusses the appropriate sequence of intermediate augmented target densities \( \xi_{ap}(\theta, x_{1:T}) \), for \( p = 1, ..., P \), for the state space model described in Section 2.1. It includes all the random variables which are produced by the particle filter method. Let \( U_{1:T}^{1:N} = (x_{1:T}^{1:N}, \tilde{a}_{1:T-1}^{1:N}) \) denote the vector of particles. The joint distribution of the particles given the parameters \( \theta \) is

\[
\psi(U_{1:T}|\theta) := \prod_{j=1}^{N} m_1(x_1^j|\theta, y_1) \prod_{t=2}^{T} \left\{ \mathcal{M}\left(\tilde{a}_{t-1}^{1:N} | \tilde{W}_{t-1}^{1:N}\right) \prod_{j=1}^{N} m_t(x_t^j | x_{t-1}^j, y_t, \theta) \right\}.
\]

The backward simulation algorithm introduced by Godsill et al. (2004) is used to obtain a sample from the particle approximation of \( p(x_{1:T}|y_{1:T}, \theta) \). by sampling the indices \( J_T, J_{T-1}, ..., J_1 \) sequentially. We denote the selected particles and trajectory by \( x_{1:T}^{j,T} = (x_{1:t}^{j}, ..., x_{T}^{j}) \) and \( j_{1:T} \), respectively. Further, let us denote \( U_{1:T}^{-j_{1:T}} \) as the collection of all random variables except the chosen particle trajectory. The AISIL constructs a sequence of tempered densities \( \xi_{ap}(\theta, x_{1:T}) \), \( p = 0, ..., P \), based on the following augmented target density

\[
\tilde{\xi}_{ap}(x_{1:T}^{j_{1:T}}, j_{1:T}, U_{1:T}^{-j_{1:T}}, \theta) = \frac{\xi_{ap}(\theta, d x_{1:T}^{j_{1:T}})}{m_1(d x_1^1|\theta, y_1) \prod_{t=2}^{T} \tilde{W}_{t-1}^{1:N} m_t(x_t^j | x_{t-1}^j, \theta) \prod_{t=2}^{T} \tilde{w}_t^{(j)} \sum_{l=1}^{N} w_{t-1}^{(j)} f_\theta(x_t^j | x_{t-1}^j, \theta)} \]

\[
\times \psi(U_{1:T}|\theta) \prod_{t=2}^{T} \tilde{W}_{t-1}^{1:N} m_t(x_t^j | x_{t-1}^j, \theta) \prod_{t=2}^{T} \tilde{w}_t^{(j)} \sum_{l=1}^{N} w_{t-1}^{(j)} f_\theta(x_t^j | x_{t-1}^j, \theta)
\]

(3)

The next lemma gives the important properties of the target density. Its proof is in Section S3.

**Lemma 1.** (i) The target distribution in Eq. (3) has the marginal distribution

\[
\tilde{\xi}_{ap}(x_{1:T}^{j_{1:T}}, j_{1:T}, \theta) = N^{-T} \xi_{ap}(\theta, x_{1:T}^{j_{1:T}}).
\]
(ii) The conditional distribution \( \tilde{\xi}_{ap} (j_{1:T} | U_{1:T}, \theta) \) is

\[
\tilde{\xi}_{ap} (j_{1:T} | U_{1:T}, \theta) = \frac{\prod_{t=2}^{T} w_{t-1}^{j_{t-1}} f_{t}^{\theta} (x_{t}^{j_{t}} | x_{t-1}^{j_{t-1}})}{\sum_{l=1}^{N} w_{t-1}^{l} f_{t}^{\theta} (x_{t}^{l} | x_{t-1}^{l})}
\]

Let \( \theta := (\theta_1, ..., \theta_H) \) be a partition of the parameter vector into \( H \) components, where each component may be a vector. Alg. 3 gives the Markov move based on particle Gibbs algorithm.

**Algorithm 3** Markov move based on the Particle Gibbs algorithm

For \( i = 1, ..., M \), we propose the following Markov steps

1. For \( h = 1, ..., H \), sample \( \theta_{ih}^{*} \) from the proposal density
   \[
   q_h \left( \cdot | j_{1:T}, x_{i,1:T}^{j_{1:T}}, \theta_{-ih}, \theta_{ih} \right).
   \]

2. Accept the proposed values \( \theta_{ih}^{*} \) with probability
   \[
   \min \left( 1, \frac{\xi_{ap} (\theta_{ih}^{*} | x_{i,1:T}^{j_{1:T}}, j_{i,1:T}, \theta_{-ih}) q_h \left( \theta_{ih} | \theta_{-ih}, \theta_{ih}^{*}, x_{i,1:T}^{j_{1:T}}, j_{i,1:T} \right)}{\xi_{ap} (\theta_{ih} | x_{i,1:T}^{j_{1:T}}, j_{i,1:T}, \theta_{-ih}) q_h \left( \theta_{ih}^{*} | \theta_{-ih}, \theta_{ih}, x_{i,1:T}^{j_{1:T}}, j_{i,1:T} \right)} \right).
   \]

3. Sample \( U_{i,1:T}^{(j_{i,1:T}^{*})} \sim \tilde{\xi}_{ap} (\cdot | j_{1:T}, x_{i,1:T}^{j_{1:T}}, \theta) \) using the conditional sequential Monte Carlo (CSMC) algorithm given in Alg. S4.

4. Sample \( J_{i,1:T} = j_{i,1:T} \sim \tilde{\xi}_{ap} (j_{i,1:T} | U_{1:T}, \theta) \) using the backward simulation algorithm given in Alg. S5.

Note that the expression

\[
\psi (U_{1:T} | \theta) m_1 (d x_{1}^{j_{1}} | \theta, y_1) \prod_{t=2}^{T} \tilde{W}_{t-1}^{j_{t-1}} m_t \left( x_{t}^{j_{t}} | x_{t-1}^{j_{t-1}}, \theta \right),
\]

appearing in \( \tilde{\xi}_{ap} \) given in Eq. (3) is the density under \( \tilde{\xi}_{ap} \) of all the variables that are generated by the particle filter algorithm conditional on a pre-specified path. This is the conditional sequential Monte Carlo algorithm of Andrieu et al. (2010). It is a sequential Monte Carlo algorithm in which a particle \( x_{1:T}^{j_{1:T}} = (x_{1}^{j_{1}}, ..., x_{T}^{j_{T}}) \), and the associated sequence of ancestral indices are kept unchanged with all other particles and indices resampled and updated. Alg. S4 in Section S6 describes the conditional sequential Monte Carlo algorithm and is the key part of the Markov move steps. Alg. 4 gives the Markov move step based on particle Gibbs for the univariate SV model.
Algorithm 4 Markov moves based on the Particle Gibbs algorithm for univariate SV models

1. Sample $\mu_i | j_{i,1:T}, x_{i,1:T}^j, y_{1:T}, \theta_{i,-\mu}$

2. Sample $\phi_i | j_{i,1:T}, x_{i,1:T}^j, y_{1:T}, \theta_{i,-\phi}$

3. Sample $\tau_i^2 | j_{i,1:T}, x_{i,1:T}^j, y_{1:T}, \theta_{i,-\tau^2}$

4. Sample $U_{i,1:T}^{(-j_{i,1:T})} \sim \tilde{\xi}_a (\cdot | j_{i,1:T}, x_{i,1:T}^j, \theta_i)$ using the conditional sequential Monte Carlo (CSMC) algorithm given in Alg. S4.

5. Sample $J_{i,1:T} = j_{i,1:T} \sim \tilde{\xi}_a (j_{i,1:T} | U_{i,1:T}, \theta_i)$ using the backward simulation algorithm given in Alg. S5.

Steps 1 to 3 use the proposal in Section 3.1

3.3 Estimating the Marginal Likelihood

The marginal likelihood $p(y_{1:T})$ is often used in the Bayesian literature to compare models (Chib and Jeliazkov, 2001). An advantage of the AISIL method is that it offers a natural way to estimate the marginal likelihood. We note that $p(y_{1:T}) = Z_{a_{p-1}}$, $Z_{a_0} = 1$, so that

$$p(y_{1:T}) = \prod_{p=1}^P \frac{Z_{a_p}}{Z_{a_{p-1}}} \text{ with } \frac{Z_{a_p}}{Z_{a_{p-1}}} = \int \left( \frac{\eta_{a_p}(\theta, x)}{\eta_{a_{p-1}}(\theta, x)} \right) \xi_{a_{p-1}}(\theta, x) \, d\theta d\mathbf{x}.$$ 

Because the particle cloud $\{\theta_{1:M}^{(p-1)}, x_{1:M}^{(p-1)}, W_{i,1:M}^{(p-1)}\}$ obtained after iteration $p - 1$ approximates $\tilde{\xi}_{a_{p-1}}(\theta, x)$, the ratio above is estimated by

$$\frac{Z_{a_p}}{Z_{a_{p-1}}} = \sum_{i=1}^M W_i^{(p-1)} \frac{\eta_{a_p}(\theta_i^{(p-1)}, x_i^{(p-1)})}{\eta_{a_{p-1}}(\theta_i^{(p-1)}, x_i^{(p-1)})},$$

giving the marginal likelihood estimate

$$\hat{p}(y_{1:T}) = \prod_{p=1}^P \frac{Z_{a_p}}{Z_{a_{p-1}}}.$$
4 The Multivariate Factor Stochastic Volatility Model

4.1 Model

The factor SV model is a parsimonious multivariate stochastic volatility model that is often used to model a vector of stock returns; see, for example, Chib et al. (2006) and Kastner et al. (2017). Suppose that $P_t$ is a $S \times 1$ vector of daily stock prices and define $y_t := \log P_t - \log P_{t-1}$ as the log-return of the stocks. We model $y_t$ as a factor SV model

$$y_t = \beta f_t + V_t^{1/2} \epsilon_t, \ (t = 1, ..., T) \tag{5}$$

where $f_t$ is a $K \times 1$ vector of latent factors (with $K \ll S$), and $\beta$ is a $S \times K$ factor loading matrix of unknown parameters. We model the latent factors as $f_t \sim N(0, D_t)$ and $\epsilon_t \sim N(0, I)$. The time varying variance matrices $V_t$ and $D_t$ depend on unobserved random variables $h_t = (h_{1t}, ..., h_{St})$ and $\lambda_t = (\lambda_{1t}, ..., \lambda_{Kt})$ such that

$$V_t := \text{diag} \{ \exp(h_{1t}), ..., \exp(h_{St}) \}, \ D_t := \text{diag} \{ \exp(\lambda_{1t}), ..., \exp(\lambda_{Kt}) \}.$$

Each of the log stochastic volatilities $\lambda_{kt}$ and $h_{st}$ are assumed to follow independent first order autoregressive processes, with

$$h_{st} - \mu_{es} = \phi_{es} (h_{st-1} - \mu_{es}) + \eta_{est}, \eta_{est} \sim N \left(0, \tau_{es}^2 \right), s = 1, ..., S \tag{6}$$

and

$$\lambda_{kt} = \phi_{fk} \lambda_{kt-1} + \eta_{fkt}, \eta_{fkt} \sim N \left(0, \tau_{fk}^2 \right), k = 1, ..., K. \tag{7}$$

Conditional Independence in the factor SV model

The key to making the estimation of the factor SV model tractable is that given the values of $(y_{1:T}, f_{1:T}, \beta)$, the factor model in Eq. (5) separates into $S + K$ independent components consisting of $K$ univariate SV models for the latent factors with $f_{kt}$ the $tt$ ‘observation’ of the $kth$ factor univariate SV model and $S$ univariate SV models for the idiosyncratic errors with $\epsilon_{st}$ the $tt$ ‘observation’ on the $st$th error SV model. Section 4.2 discusses the AISIL-HMC method for the factor SV model and Section 4.3 discusses the AISIL-PG method.
4.2 Application of the AISIL-HMC method to the Multivariate Factor Stochastic Volatility Model

This section discusses the application of the AISIL-HMC method to the multivariate factor SV model described in Section 4.1. We first define the appropriate sequence of intermediate target densities,

\[ \xi_{ap}(\theta, h_{1:T}, \lambda_{1:T}, f_{1:T}) = p(y_{1:T}|f_{1:T}, h_{1:T}, \lambda_{1:T}, \theta)^{ap} p(f_{1:T}|h_{1:T}, \lambda_{1:T}, \theta) \times p(\lambda_{1:T}, h_{1:T}|\theta) p(\theta), \]

where \( \theta = \begin{cases} \theta_{es} = \{\mu_{es}, \phi_{es}, \tau_{es}^2\}_{s=1}^{S}, \theta_{fk} = \{\phi_{fk}, \tau_{fk}^2\}_{k=1}^{K}, \beta \end{cases} \),

\[
p(y_{1:T}|f_{1:T}, h_{1:T}, \lambda_{1:T}, \theta)^{ap} = \prod_{t=1}^{T} \prod_{s=1}^{S} p(y_{st}|\beta_{s}, f_{t}, h_{st})^{ap},
\]

\[
p(f_{1:T}|\lambda_{1:T}) = \prod_{t=1}^{T} \prod_{k=1}^{K} p(f_{kt}|\lambda_{kt}),
\]

\[
p(\lambda_{1:T}|\theta_f) = \prod_{k=1}^{K} \prod_{t=2}^{T} p(\lambda_{kt}|\lambda_{kt-1}, \theta_{fk}) p(\lambda_{k1}|\theta_{fk}),
\]

\[
p(h_{1:T}|\theta_e) = \prod_{s=1}^{S} \prod_{t=2}^{T} p(h_{st}|h_{st-1}, \theta_{es}) p(h_{s1}|\theta_{es}) .
\]

(8)

We propose this following Markov move steps, with details in Section S2.1.
Algorithm 5 Markov moves based on Hamiltonian Monte Carlo for multivariate factor SV model

for each $i = 1, ..., M$,

1. for $s = 1 : S$
   
   (a) Sample $\mu_{i,es} | h_{is,t}^i, \theta_{i, -\mu_{i,es}}$
   
   (b) Sample $\phi_{i,es} | h_{is,t}^i, \theta_{i, -\phi_{i,es}}$
   
   (c) Sample $\tau_{i,es}^2 | h_{is,t}^i, \theta_{i, -\tau_{i,es}^2}$

2. For $k = 1, ..., K$
   
   (a) Sample $\phi_{i,fk} | \lambda_{ik,t}, \theta_{i, -\phi_{i,fk}}$
   
   (b) Sample $\tau_{i,fk}^2 | \lambda_{ik,t}, \theta_{i, -\tau_{i,fk}^2}$

3. Sample $\beta_i | \theta_{i, -\beta}, f_{i,1:T}, h_{i,1:T}, \lambda_{k,1:T}, y_{1:T}$

4. Sample $f_{i,t} | \theta, h_{i,1:T}, \lambda_{k,1:T}, y_{1:T}$

5. For $s = 1, ..., S$, sample $h_{is,t}^i | \theta, f_{1:T}, y_{1:T}$ using Hamiltonian Monte Carlo proposal.

6. For $k = 1, ..., K$, sample $\lambda_{ik,t} | \theta, f_{1:T}, y_{1:T}$ using Hamiltonian Monte Carlo proposal.

4.3 Application of the AISIL-PG method to the Multivariate Factor Stochastic Volatility Model

This section discusses the application of the AISIL-PG method to a multivariate factor SV model. Similarly to Section 4.2, Eq. (8) gives the appropriate sequence of intermediate target densities.

Augmented Intermediate Target Density for the Factor SV model

This section provides an appropriate augmented tempered target density for the factor SV model. This augmented tempered target density includes all the random variables produced by $S + K$ univariate particle filter methods that generate the factor log-volatilities $\lambda_{k,1:T}$ for $k = 1, ..., K$ and the idiosyncratic log-volatilities $h_{s,1:T}$ for $s = 1, ..., S$, as well as the latent factors $f_{1:T}$ and the parameters $\theta$. We use Eq. (6) to specify the particle filters for idiosyncratic SV log-volatilities $h_{s,1:T}$ for $s = 1, ..., S$, and equation 7 to specify the univariate particle filters that generate the factor log-volatilities $\lambda_{k,1:T}$ for $k = 1, ..., K$. We denote the $N$ weighted samples at time $t$ for the factor log-volatilities by $(\lambda_{kt}^{1:N}, \hat{W}_{ft}^{1:N})$ and $(h_{st}^{1:N}, \hat{W}_{est}^{1:N})$ for the idiosyn-
cratic error log-volatilities. The corresponding proposal densities are \( m_{fk1} (\lambda_{k1}|\theta_{fk}) \), \( m_{fkt} (\lambda_{kt}|\lambda_{kt-1}, \theta_{fk}) \), \( m_{es1} (h_{s1}|\theta_{es}) \), and \( m_{est} (h_{st}|h_{st-1}, \theta_{es}) \) for \( t = 2, ..., T \). We denote the resampling schemes by \( \mathcal{M}\left( \tilde{a}_{est-1}^{1:N}\mid W_{est-1}^{1:N} \right) \), where each \( \tilde{a}_{est-1} = k \) indexed a particle in \( (h_{st}^{1:N}, W_{est}^{1:N}) \) and is chosen with probability \( W_{est}^{k} \). \( \mathcal{M}\left( \tilde{a}_{fkt}^{1:N}\mid W_{fkt}^{1:N} \right) \) is defined similarly. We then denote the vectors of particles by

\[
\mathbf{U}_{es,1:T} := (h_{s,1}^{1:N}, ..., h_{s,T}^{1:N}, \tilde{a}_{es,1}^{1:N}, ..., \tilde{a}_{es,T-1}^{1:N})
\]

and

\[
\mathbf{U}_{fk,1:T} := (\lambda_{k,1}^{1:N}, ..., \lambda_{k,T}^{1:N}, \tilde{a}_{fkt,1}^{1:N}, ..., \tilde{a}_{fkt,T-1}^{1:N}).
\]

The joint distribution of the particles given the parameters is

\[
\psi_{es}\left( U_{es,1:T}^{1:N}\mid \theta_{es} \right) := \prod_{j=1}^{N} m_{es1} (h_{s,j}^{1}|y_{1}, \theta_{es}) \prod_{t=2}^{T} \left\{ \mathcal{M}\left( \tilde{a}_{est-1}^{1:N}\mid W_{est-1}^{1:N} \right) \prod_{j=1}^{N} m_{est} (h_{st,j}^{1}\mid \tilde{a}_{est-1}^{j}, \theta_{es}) \right\},
\]

for \( s = 1, ..., S \) and

\[
\psi_{fk}\left( U_{fk,1:T}^{1:N}\mid \theta_{fk} \right) := \prod_{j=1}^{N} m_{fk1} (\lambda_{k,j}^{1}\mid y_{1}, \theta_{fk}) \prod_{t=2}^{T} \left\{ \mathcal{M}\left( \tilde{a}_{fkt-1}^{1:N}\mid W_{fkt-1}^{1:N} \right) \prod_{j=1}^{N} m_{fkt} (\lambda_{k,t}^{j}\mid \tilde{a}_{fkt-1}^{j}, \theta_{fk}) \right\},
\]

for \( k = 1, ..., K \). Next, we define indices \( J_{es,1:T} \) for \( s = 1, ..., S \), the selected particle trajectory \( h_{s,1:T}^{j_{es}} = (h_{s,1}^{j_{es}}, ..., h_{s,T}^{j_{es}}) \), indices \( J_{fk,1:T} \) for \( k = 1, ..., K \) and the selected particle trajectory \( \lambda_{k,1:T}^{j_{fk}} = (\lambda_{k,1}^{j_{fk}}, ..., \lambda_{k,T}^{j_{fk}}) \).

The augmented intermediate target density in this case consists of all of the
The next lemma gives the important properties of the target density. Its proof is in Section S3.

Lemma 2. The target distribution in Equation Eq. (9) has the following marginal distribution

\[
\tilde{\xi}_{ap} \left( h_{1:T}^{J_{es,1:T}}, J_{es,1:T}, \theta^{J_{fk,1:T}}, J_{fk,1:T}, \theta, f_{1:T} \right) = \frac{\xi_{ap} \left( d\theta, dJ_{es,1:T}^{h_{1:T}}, d\lambda_{1:T}^{J_{fk,1:T}}, df_{1:T} \right)}{N^T(S+K)}
\]

Alg. 6 describes the AISIL-PG Markov moves.

\[
\tilde{\xi}_{ap} \left( h_{1:T}^{J_{es,1:T}}, J_{es,1:T}, \theta^{J_{fk,1:T}}, J_{fk,1:T}, \theta, f_{1:T} \right) := \frac{\xi_{ap} \left( d\theta, dJ_{es,1:T}^{h_{1:T}}, d\lambda_{1:T}^{J_{fk,1:T}}, df_{1:T} \right)}{N^T(S+K)} \times \\
\prod_{s=1}^{S} \psi_{es} \left( U_{es,1:T} | \theta_{es} \right) \prod_{t=2}^{T} \tilde{w}_{est-1} \psi \left( h_{est-1}^{J_{es,1:T}}, \lambda_{est-1}^{J_{es,1:T}}, \theta_{es} \right) \times \prod_{t=2}^{T} \sum_{l=1}^{N} \tilde{w}_{l,est-1} \psi \left( h_{est}^{J_{es,1:T}}, \lambda_{est}^{J_{es,1:T}}, \theta_{es} \right) \\
\prod_{k=1}^{K} \psi_{fk} \left( U_{fk,1:T} | \theta_{fk} \right) \prod_{t=2}^{T} \tilde{w}_{fkt-1} \psi \left( \lambda_{fkt-1}^{J_{fk,1:T}}, \lambda_{fkt-1}, \theta_{fk} \right) \times \prod_{t=2}^{T} \sum_{l=1}^{N} \tilde{w}_{l,fkt-1} \psi \left( \lambda_{fkt}^{J_{fk,1:T}}, \lambda_{fkt}, \theta_{fk} \right).
\]
Algorithm 6 Markov moves based on particle Gibbs for multivariate factor SV model

for $i = 1, ..., M$,

1. For $s = 1, ..., S$,
   
   (a) Sample $\mu_{is} | h_{is,1:T}, j_{is,1:T}, \theta_i, -\mu_{is}, f_{1:T}, y_{s,1:T}$
   
   (b) Sample $\phi_{is} | h_{is,1:T}, j_{is,1:T}, \theta_i, -\phi_{is}, f_{1:T}, y_{s,1:T}$
   
   (c) Sample $\tau_{is}^2 | h_{is,1:T}, j_{is,1:T}, \theta_i, -\tau_{is}^2, f_{1:T}, y_{s,1:T}$

2. For $l = 1, ..., L$,
   
   (a) Sample $\phi_{ikl} | j_{ikl,1:T}, j_{ikl,1:T}, \theta_i, -\phi_{ikl}, f_{1:T}, y_{1:T}$
   
   (b) Sample $\tau_{ikl}^2 | j_{ikl,1:T}, j_{ikl,1:T}, \theta_i, -\tau_{ikl}^2, f_{1:T}$

3. Sample $\beta_i | h_{is,1:T}, j_{is,1:T}, \lambda_{ikl,1:T}, j_{ikl,1:T}, \theta_i, f_{1:T}, y_{1:T}$

4. Sample $f_{1:T} | h_{is,1:T}, j_{is,1:T}, \lambda_{ikl,1:T}, j_{ikl,1:T}, \theta_i, y_{1:T}$ for $t = 1, ..., T$.

5. For $s = 1, ..., S$, sample

   $$U_{is,1:T}^{(-j_{is,1:T})} \sim \tilde{\xi}_a \left( U_{is,1:T}^{(-j_{is,1:T})} | h_{is,1:T}, j_{is,1:T}, \theta_i, f_{1:T}, y_{1:T} \right)$$

   using conditional sequential Monte Carlo algorithm.

6. For $s = 1, ..., S$, sample $j_{is,1:T} \sim \tilde{\xi}_a \left( j_{is,1:T} | U_{is,1:T}, \theta_i \right)$

   using backward simulation algorithm

7. For $l = 1, ..., L$, sample

   $$U_{ikl,1:T}^{(-j_{ikl,1:T})} \sim \tilde{\xi}_a \left( U_{ikl,1:T}^{(-j_{ikl,1:T})} | \lambda_{ikl,1:T}, j_{ikl,1:T}, \theta_i, f_{1:T} \right)$$

8. For $l = 1, ..., L$, sample $j_{ikl,1:T} \sim \tilde{\xi}_a \left( j_{ikl,1:T} | U_{ikl,1:T}, \theta_i \right)$. using the backward simulation algorithm.

Step 1 and 2 are discussed in Section 4.2. Step 3 and 4 are discussed in Section S5.

5 Examples

5.1 Examples: Univariate Stochastic Volatility Model

This section illustrates the AISIL-PG and AISIL-HMC methods by applying them to the univariate stochastic volatility (SV) model discussed in Section 3 and compare
their performance to the Efficient PMMH+PG sampler of Gunawan et al. (2018). We apply the methods to a sample of daily US food industry stock returns data obtained from the Kenneth French website, using a sample from December 11th, 2001 to the 11th November 2013, a total of 3001 observations.

Table 1 summarizes the estimation results for the univariate SV model estimated using the Efficient PMMH+PG, AISIL-HMC, and AISIL-PG methods. The AISIL estimates are obtained using 10 independent runs, each with $M = 560$ samples, to generate a total of 5600 samples for each algorithm. We set the constant $\text{ESS}_T = 0.8M$, and use the estimates of Efficient PMMH+PG sampler as the “gold standard” to assess the accuracy of the two annealing approaches. The Efficient PMMH+PG MCMC chain consists of 50000 iterates with another 5000 iterates used as burn-in.

Table 1 shows that the AISIL-PG estimates are very close to the Efficient PMMH+PG estimates for all parameters even with as few as 150 particles and $R = 10$ Markov move steps. Fig. 2 shows the kernel density estimates of the marginal posterior of the univariate stochastic volatility parameters as well as the estimates of the log-volatility estimated using the Efficient PMMH+PG sampler and the AISIL-PG method with different numbers of particles and Markov move steps. The density estimates from the AISIL-PG method are very close to the density estimates of the Efficient PMMH+PG sampler for all the parameters for all setup. The posterior mean estimates of the log-volatilities $h_{1:T}$ are also indistinguishable.

The estimates of $\tau^2$ and $\phi$ using the AISIL-HMC method with the number of leapfrog steps set at $L = 50$ and with $M = 10$ is a little different to the estimates of the efficient PMMH+PG samplers, but it gets closer when the number of leapfrog increases to $L = 100$ and $R = 20$. The AISIL-HMC method requires more tuning than the AISIL-PG method. Fig. 1 shows the kernel density estimates of the marginal posteriors of the univariate SV parameters and the estimates of the log-volatility estimated using the Efficient PMMH+PG sampler and the AISIL-HMC method with different numbers of leapfrog and Markov move steps. The figure confirms that the density estimates of $\tau^2$ and $\phi$ with $L = 50$ and $M = 10$ are different to the estimates from the Efficient PMMH+PG sampler, but get closer when the number of leapfrog and Markov move steps increases to $L = 100$ and $M = 20$. The posterior mean estimates of the log-volatilities $h_{1:T}$ are very close to the estimates from the Efficient PMMH+PG sampler. Table 1 also shows the estimates of the log of the marginal likelihood estimated using AISIL-HMC and AISIL-PG methods and shows that the estimated standard errors of the estimates of the log of the marginal likelihood estimated using AISIL-HMC is bigger than for the AISIL-PG.

The density tempered sequential Monte Carlo samplers of Duan and Fulop (2015) use pseudo marginal Metropolis-Hastings (PMMH) Markov move steps and follow
the guideline of Pitt et al. (2012) to set optimal number of particles in the particle filter such that the variance of log of estimated likelihood is around 1 and this requires around 5000 particles. We do not compare with their approach because it is clear that our AISIL-PG approach is more efficient and it only needs 250 particles.

Table 1: Results for the univariate SV model estimated using Efficient PMMH+PG, AISIL-HMC, and AISIL-PG samplers for US food stock returns data with $T = 3001$. The results for the AISIL methods are obtained using 10 independent runs of each algorithm. Time is the time in minutes for one run of the algorithm. The estimated standard errors of the estimates are in brackets.

| Method          | N   | L  | R  | $\mu$      | $\phi$    | $\tau^2$ | $\log \hat{p}(y)$ | Time |
|-----------------|-----|----|----|------------|-----------|----------|-------------------|------|
| Efficient PMMH + PG | 100 | -  | -  | -0.4886    | 0.9853    | 0.0226   | -                 | -    |
| HMC             | -   | 50 | 10 | -0.5004    | 0.9823    | 0.0283   | -3668.58         | 19   |
| HMC             | -   | 100| 10 | -0.4934    | 0.9839    | 0.0252   | -3665.29         | 20   |
| HMC             | -   | 100| 20 | -0.4888    | 0.9849    | 0.0234   | -3665.88         | 38   |
| PF              | 150 | -  | 10 | -0.4996    | 0.9850    | 0.0231   | -3669.39         | 42   |
| PF              | 250 | -  | 10 | -0.4986    | 0.9851    | 0.0229   | -3666.14         | 57   |
| PF              | 250 | -  | 20 | -0.4797    | 0.9850    | 0.0229   | -3666.22         | 112  |
Figure 1: The kernel density estimates of the marginal posterior densities of the univariate SV parameters and log volatilities for the Efficient PMMH+PG (MCMC) and AISIL-HMC samplers. The bottom right panel plots the estimated posterior means of the log volatilities for the two samplers.
Figure 2: The kernel density estimates of the marginal posterior densities of the univariate SV parameters and log volatilities for the Efficient PMMH+PG (MCMC) and AISIL-PG samplers. The bottom right panel plots the estimated posterior means of the log volatilities for the two samplers.
5.2 Examples: Multivariate Factor Stochastic Volatility Model

This section compares the performance of the AISIL-PG, AISIL-HMC and the Efficient PMMH +PG methods by applying them to the multivariate factor SV model discussed in Section 4.1 using one factor. We use a sample of daily returns for \( S = 26 \) value weighted industry portfolios, from December 11th, 2001 to 29th November 2005, a total of 1000 observations. The data was obtained from the Kenneth French website and the industry portfolios we use are listed Section S7. We regard the Efficient PMMH + PG sampler as the “gold standard” and ran it for 50000 iterates with another 5000 iterates used as burn in.

The density tempered SMC of Duan and Fulop (2015) uses the PMMH Markov steps and follows the Pitt et al. (2012) guidelines to set the optimal number of particles in the particle filter such that the variance of the log of the estimated likelihood to be around 1. The PMMH Markov step generates the parameters with latent factors, factor and idiosyncratic log-volatilities integrated out as this results in a \((S + K)\) dimensional state space models. The tempered measurement density at the \( p \)th stage then becomes

\[
N \left( y_t; 0, \beta D_t \beta' + V_t \right)^{a_p},
\]

Eq. (6) gives the state transition densities for the idiosyncratic log-volatilities \((s = 1, \ldots, S)\) and Eq. (7) gives the state transition equations for the factor log-volatilities \((k = 1, \ldots, K)\).

Table 3 shows the variance of log of the estimated likelihood for different number of particles evaluated at posterior means of the parameters obtained using the Efficient PMMH+PG sampler of Gunawan et al. (2018) with \( a_p = 1 \). It shows that even with 5000 particles, the PMMH Markov step would get stuck. Deligiannidis et al. (2017) proposed the correlated PMMH method and it is possible to implement it in the Markov move step instead of the standard PMMH method of Andrieu et al. (2010). The correlated PMMH correlates the random numbers used in constructing the estimators of the likelihood at current and proposed values of the parameters and sets the correlation very close to 1 to reduce the variance of the difference in the logs of estimated likelihoods at the current and proposed values of the parameters appearing in the Metropolis-Hastings (MH) acceptance ratio. Deligiannidis et al. (2017) show that the correlated PMMH can be much more efficient and significantly reduce the number of particles required than the standard PMMH approach for small dimension of latent states. In this example, we consider a high (27 dimensional) dimensional
factor SV model of latent states in the one factor-Factor SV model. Mendes et al. (2018) found that it is very challenging to preserve the correlation between the logs of the estimated likelihoods for such a high dimensional state space model. The Markov move based on the correlated PMMH approach would also get stuck unless enough particles are used to ensure the variance of the log of estimated likelihood is around 1. A second issue with the PMMH Markov move step as in Duan and Fulop (2015) is that the dimension of the parameter space in the factor SV model is large making it very hard to implement the PMMH Markov step efficiently. The reason is that it is very difficult to obtain good proposals for the parameters, because the first and second derivatives with respect to the parameters are not available analytically and can only be estimated. As noted by Sherlock et al. (2015), it is in general even more difficult to obtain accurate estimate of gradient of the log posterior than it is to obtain accurate estimates of the log-posterior. The random walk proposal is easy to implement but it is very inefficient in high dimensions.

The AISIL estimates were obtained with 10 independent runs with $M = 280$ samples to generate a total of 2800 samples for each algorithm. For the AISIL methods, we set the constant $\text{ESS}_T = 0.8M$. We set $L = 100$, $R = 20$, and $R = 50$ for the AISIL-HMC method, and we set $R = 10$, $N = 250$ for the AISIL-PG method. Table 2 shows some of the estimates of parameters of the factor SV model estimated using the Efficient PMMH+PG, AISIL-HMC, and the AISIL-PG methods. We found that AISIL-PG estimates are very close to the Efficient PMMH+PG estimates for all the parameters. The estimates of the AISIL-HMC method with $R = 20$ are a little different to the estimates of the efficient PMMH+PG sampler, but they get closer when $R$ is increased to 50. For example, the estimates of $\tau_{19}^2$ for AISIL-PG, AISIL-HMC-R20, AISIL-HMC-R50, efficient PMMH+PG are 0.0441, 0.0393, 0.0434, and 0.0435, respectively. Figures 4 and 5 show the kernel density estimates of the marginal posteriors of four of the $\phi_{es}$ and $\tau_{es}^2$ parameters estimated using AISIL-PG, AISIL-HMC with $R = 20$ and $R = 50$ Markov steps. These two figures confirm that (i) the density estimates from the AISIL-PG method are close to the the density estimates from the Efficient PMMH+PG method, (ii) the density estimates from AISIL-HMC with $R = 20$ is a little different to the density estimates of the efficient PMMH+PG method, but they get closer when $R$ is increased to 50. Figure 3 shows the posterior mean estimates of the log-volatilities for $t = 1, \ldots, T$ for some of the stock returns. The posterior mean estimates of the log-volatilities estimated using all the methods are close to the estimates from the Efficient PMMH+PG sampler. Note that both the AISIL-PG and AISIL-HMC estimates give very similar results and both give independent draws from the posterior distribution of interest so that it is unnecessary to deal with the convergence and autocorrelation issues of PMCMC.
and MCMC samplers in general. Mendes et al. (2016) and Gunawan et al. (2018) found that it is necessary to use a combination of PMMH+PG sampler to reduce the autocorrelation in the Markov chain in the factor SV model.

Table 2 also shows the estimates of the log of the marginal likelihood for the one factor model. We can clearly see that the estimates of the log of the marginal likelihood estimated using AISIL-HMC is bigger than AISIL-PG. The estimates of the log of the marginal likelihood can be used to select the best model. We estimated 1 to 4 factor models using the AISIL-PF method. The logs of the marginal likelihoods for the four models, with standard deviations in brackets, obtained with 10 independent runs for each factor model are $-26157.25 (22.13), -25574.41 (46.12), -25432.81 (73.11)$, and $-25441.72 (76.11)$, respectively. The three factor models is the best model in terms of maximizing the marginal likelihood.
Table 2: Factor SV model estimated using Efficient PMMH+PG , AISIL-HMC, and AISIL-PG samplers for the US stock returns data with $T = 1000$, $S = 26$, and $K = 1$. Time is in Minutes. The results are obtained with 10 independent runs of each algorithm. The table gives: (i) the posterior mean estimates of some of the parameters with standard errors in brackets. (ii) the estimates of the log of the marginal likelihood $\hat{\log p(y_{1:T})}$ based on an average of the 10 runs as well as the standard error of the estimate in brackets. (iii) the average value of the number of annealing steps $P$ averaged over the 10 runs.

| Param.       | Efficient PMMH+PG | AISIL-HMC-R20 | AISIL-HMC-R50 | AISIL-PG |
|--------------|------------------|---------------|---------------|----------|
| $\phi_{c3}$ | 0.9454 (0.0180)  | 0.9490 (0.0161) | 0.9452 (0.0178) | 0.9451 (0.0178) |
| $\phi_{c5}$ | 0.9483 (0.0178)  | 0.9477 (0.0180) | 0.9488 (0.0173) | 0.9468 (0.0183) |
| $\phi_{c19}$ | 0.9738 (0.0102)  | 0.9758 (0.0094) | 0.9735 (0.0102) | 0.9731 (0.0099) |
| $\phi_{c20}$ | 0.9831 (0.0077)  | 0.9863 (0.0067) | 0.9842 (0.0073) | 0.9836 (0.0077) |
| $\tau_{c3}^2$ | 0.0538 (0.0176)  | 0.0479 (0.0135) | 0.0538 (0.0172) | 0.0527 (0.0170) |
| $\tau_{c5}^2$ | 0.0377 (0.0118)  | 0.0373 (0.0116) | 0.0370 (0.0112) | 0.0379 (0.0120) |
| $\tau_{c19}^2$ | 0.0435 (0.0114)  | 0.0393 (0.0102) | 0.0434 (0.0115) | 0.0441 (0.0116) |
| $\tau_{c20}^2$ | 0.0405 (0.0125)  | 0.0337 (0.0102) | 0.0394 (0.0121) | 0.0403 (0.0123) |
| $\mu_{c3}$  | -1.7294 (0.1593) | -1.7251 (0.1562) | -1.7296 (0.1556) | -1.7355 (0.1545) |
| $\mu_{c5}$  | -0.5392 (0.1436) | -0.5349 (0.1450) | -0.5387 (0.1499) | -0.5436 (0.1441) |
| $\mu_{c19}$ | -1.3112 (0.3157) | -1.3075 (0.3427) | -1.3132 (0.3066) | -1.3058 (0.2945) |
| $\mu_{c20}$ | -1.0516 (0.5057) | -1.0702 (0.5825) | -1.0468 (0.5210) | -1.0697 (0.5071) |
| $\beta_{31}$  | 0.9118 (0.0759)  | 0.9090 (0.0715) | 0.9054 (0.0694) | 0.9051 (0.0666) |
| $\beta_{51}$  | 1.0894 (0.0925)  | 1.0863 (0.0858) | 1.0816 (0.0846) | 1.0811 (0.0824) |
| $\beta_{191}$  | 0.5897 (0.0514)  | 0.5873 (0.0482) | 0.5852 (0.0472) | 0.5856 (0.0460) |
| $\beta_{201}$  | 1.0156 (0.0858)  | 1.0121 (0.0807) | 1.0074 (0.0779) | 1.0077 (0.0754) |
| $\log \hat{p}(y_{1:T})$ | -26222.16 (84.16) | -26237.87 (102.91) | -26157.25 (22.13) |
| $P$             | 270.20 (102.26)  | 266.30 (99.87)  | 288.50 (20.07)  |
| Time per AISIL iter. | 2.85 (1.92) | 7.29 (3.87) | 5.45 (2.00) |
| Total Time      | 770.07 (102.26) | 1941.33 (22.13) | 1572.33 (20.07) |
Figure 3: Plot of log-volatilities $h_{1:T}$ for US stock returns data estimated using the PMCMC, AISIL-HMC, and AISIL-PG methods.

Figure 4: The kernel density estimates of the marginal posterior densities of $\phi_{\epsilon_3}$ for the US stock returns data for four representative $\phi_{\epsilon_3}$. The density estimates are for the Efficient PMMH +PG, AISIL-HMC, and AISIL-PG methods.
Figure 5: The kernel density estimates of the marginal posterior densities of $\tau^2_{\epsilon_8}$ for the US stock returns data for four representative $\tau^2_{\epsilon_8}$. The density estimates are for the Efficient PMMH +PG, AISIL-HMC, and AISIL-PG methods.

Table 3: The Variance of the log-likelihood for the PMMH step for different numbers of particles for the US stock returns dataset with $T = 1000$, $S = 26$, and $K = 1$ with the tempered sequence $a_\alpha$ set to 1, evaluated at the posterior means of the parameters obtained from the Efficient PMMH+PG sampler of Gunawan et al. (2018), Time denotes the time in seconds to compute one log of the estimated likelihood.

| Number of Particles | Variance of log-likelihood | Time  |
|---------------------|----------------------------|-------|
| 250                 | 2198.72                    | 4.86  |
| 500                 | 1164.51                    | 9.88  |
| 1000                | 813.53                     | 20.13 |
| 2500                | 439.05                     | 50.43 |
| 5000                | 345.85                     | 99.24 |

6 Conclusions

Duan and Fulop (2015) introduce the density tempering sequential Monte Carlo (SMC) sampler for state space models. A critical component of the annealing or density tempering method is the Markov move component that is implemented at every stage of the annealing process. Duan and Fulop (2015) used a pseudo marginal
(PMMH) approach with the likelihood estimated unbiasedly using the particle filter in the Markov move component. There are two issues in basing the Markov move on PMMH sampling, (1) As our empirical example in Section 5 shows, for a high dimensional state space model it is computationally costly to follow the guidelines of Pitt et al. (2012) to set the optimal number of particles in the particle filter such that the variance of log of estimated likelihood is around 1; (2) it is difficult to implement the PMMH Markov move step efficiently when the dimension of the parameter space in the model is large.

Our contribution is to develop flexible annealing approach with a Markov move step that is more efficient than the density tempered SMC of Duan and Fulop (2015) and that can handle a high dimensional parameter. We propose two Markov move algorithms, one that is based on particle Gibbs (Andrieu et al., 2010) and another based on Hamiltonian Monte Carlo (Neal, 2011). We call the first Annealing Importance Sampling for Intractable Likelihood with particle Gibbs (AISIL-PG) and the second AISIL-HMC. We found that the AISIL-HMC method requires substantially more tuning parameters, such as the step size $\epsilon$, the number of leapfrog $L$, and the mass matrix $M$, but the method itself is relatively easy to implement. Conversely, the AISIL-PG method seems to be more robust and has less tuning parameters, but it has been our experience that understanding and using the Markov move based on PG is much harder than the corresponding method using HMC. There are two main reasons for this, (i) The target distributions of PG are non-standard, (ii) The PG method has steps such as conditional sequential Monte Carlo that include generating the particles conditional on the reference particle trajectory. These complications are particularly difficult for practitioners. We demonstrate the usefulness of the proposed methods when estimating a factor SV model. An estimate of the marginal likelihood can be used to choose the best model.
Online Supplementary material for ‘Flexible Density Tempering Approaches for State Space Models with an application to Factor Stochastic Volatility Models’

We use the following notation in the supplement. Eq. (1), Alg. 1, and Sampling Scheme 1, etc, refer to the main paper, while Eq. (S1), Alg. S1, and Sampling Scheme S1, etc, refer to the supplement.

S1 Markov move steps for the univariate SV model

This section discusses the Markov move steps in Alg. 2.

S1.1 Step (i) of Alg. 2: Sampling the latent volatilities using Hamiltonian Monte Carlo

This section discusses the Hamiltonian Monte Carlo (HMC) proposal to sample the high dimensional latent state vector \( x_{1:T} \). Suppose we want to sample from a \( T \)-dimensional distribution with pdf proportional to \( \exp(\mathcal{L}(x)) \), where \( \mathcal{L}(x) = \log p(x|\theta, y) \). In Hamiltonian Monte Carlo (Neal, 2011), we augment an auxiliary momentum vector \( r \) having the same dimension as the latent state vector \( x \) with the density \( p(r) = N(r|0,M) \), where \( M \) is a mass matrix. We define the joint conditional density of \( (x, r) \) as

\[
p(x, r|\text{rest}) \propto \exp(-H(x, r)),
\]

where \( H(x, r) = -\mathcal{L}(x) + \frac{1}{2} r^T M^{-1} r \) is called the Hamiltonian. In an idealised Hamiltonian step, the state vectors \( x \) and the momentum variables \( r \) move continuously according to the differential equations

\[
\frac{dx}{dt} = \frac{\partial H}{\partial r} = M^{-1} r, \quad \frac{dr}{dt} = -\frac{\partial H}{\partial x} = \nabla_x \mathcal{L}(x),
\]

where \( \nabla_x \) denotes the gradient with respect to \( x \). In practice, this continuous time Hamiltonian dynamics needs to be approximated by discretizing time, using a step size \( \epsilon \). We can then simulate the evolution over time of \( (x, r) \) via the “leapfrog” integrator using Alg. S1.
Algorithm S1 The Hamiltonian Monte Carlo part of a single Markov move step

Given initial values of $\mathbf{x}$, $\epsilon$, $L$, where $L$ is the number of leapfrog updates

Sample $\mathbf{r} \sim N(0, M)$

For $i = 1$ to $L$

Set $(\mathbf{x}^*, \mathbf{r}^*) \leftarrow \text{Leapfrog} (\mathbf{x}, \mathbf{r}, \epsilon)$ (See Alg. S2)

end for

With probability

$$
\alpha = \min \left( 1, \frac{\exp \left( \mathcal{L}(\mathbf{x}^*) - \frac{1}{2} \mathbf{r}^T M^{-1} \mathbf{r}^* \right)}{\exp \left( \mathcal{L}(\mathbf{x}) - \frac{1}{2} \mathbf{r}^T M^{-1} \mathbf{r} \right)} \right).
$$

set $\mathbf{x} = \mathbf{x}^*$, $\mathbf{r}^* = -\mathbf{r}$, else retain current $\mathbf{x}$.

Algorithm S2 One step of the leapfrog algorithm

Input $i$, $L$, $\epsilon$.

if $i = 1$, \quad $\mathbf{r} = \mathbf{r} + \epsilon \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}) / 2$,

$\mathbf{x} = \mathbf{x} + \epsilon M^{-1} \mathbf{r}$,

if $i = L$ \quad $\mathbf{r} = \mathbf{r} + \epsilon \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}) / 2$, else \quad $\mathbf{r} = \mathbf{r} + \epsilon \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x})$

For the univariate SV model Eq. (2), we need the gradient of the log-likelihood $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x})$ with respect to each of the latent volatilities. The required gradient for $t = 1$ is

$$
\nabla_{x_1} \mathcal{L}(\mathbf{x}) = a_p \left( -0.5 + 0.5 y_1^2 \exp (x_1) \right) - \frac{(1 - \phi^2)}{\tau^2} (x_1 - \mu) + \frac{\phi}{\tau^2} (x_2 - \mu - \phi (x_1 - \mu)) ;
$$

the gradient for $1 < t < T$ is

$$
\nabla_{x_t} \mathcal{L}(\mathbf{x}) = a_p \left( -0.5 + 0.5 y_t^2 \exp (-x_t) \right) + \frac{\phi}{\tau^2} (x_{t+1} - \mu - \phi (x_t - \mu)) - \frac{1}{\tau^2} (x_t - \mu - \phi (x_{t-1} - \mu)) ;
$$

and, for $t = T$, the gradient is

$$
\nabla_{x_T} \mathcal{L}(\mathbf{x}) = a_p \left( -0.5 + 0.5 y_T^2 \exp (-x_T) \right) - \frac{1}{\tau^2} (x_T - \mu - \phi (x_{T-1} - \mu)) .
$$

The performance of HMC depends strongly on choosing suitable values for the mass matrix $M$, the step size $\epsilon$, and the number of leapfrog steps $L$. The step size $\epsilon$ determines how well the leapfrog integration approximates the Hamiltonian dynamics. If we set it too large, then it may yield a low acceptance rate, but if we set it too small, then it becomes computationally expensive to obtain distant proposals.
Similarly, if we set $L$ too small, the proposal will be close to the current value of the latent state vectors, resulting in undesirable random walk behaviour. If we set $L$ too large, HMC will generate proposals that retrace their steps. Our article adopts an adaptive method based on Garthwaite et al. (2015) to select $\epsilon$ that yields a specified average acceptance probability across all $M$ particles. In our application, we set $L$ by experimentation as in Section 5.

The precision matrix of the AR(1) process of the latent states is a sparse tridiagonal matrix whose diagonal elements are equal to $0.5a_p + \frac{(1+\phi^2)}{\tau^2}$ with the exception of the first and last diagonal elements which are equal to $0.5a_p + \frac{1}{\tau^2}$; the super and sub diagonal elements are equal to $-\phi \tau^2$. To speed up the computations in our applications, we take $M$ as the diagonal of this precision matrix.

**S1.2 Steps (ii)-(iv) of Alg. 2**

For $i = 1, \ldots, M$, we sample $\mu_i | \mathbf{x}_{1:T}, \mathbf{y}_{1:T}, \theta_{-\mu}$ from $N(\mu_{\mu}, \sigma_{\mu}^2)$ truncated within $(-10, 10)$, where

$$\sigma_{\mu}^2 = \frac{\tau_i^2}{1 - \phi_i^2 + (T - 1) (1 - \phi_i)^2}$$

and

$$\mu_{\mu} = \sigma_{\mu}^2 \sum_{t=2}^{T} x_{it} (1 - \phi_i^2) + \sum_{t=2}^{T} x_{it} - \phi_i x_{it} + \phi_i^2 x_{it-1} - \phi_i x_{it-1}.$$

We sample the persistence parameter $\phi_i$ by drawing a proposed value $\phi_i^*$ from $N(\mu_{\phi}, \sigma_{\phi}^2)$ truncated within $(-1, 1)$, where

$$\sigma_{\phi}^2 = \frac{\tau_i^2}{\sum_{t=2}^{T} (x_{it-1} - \mu_i)^2 - (x_{i1} - \mu_i)^2}$$

and

$$\mu_{\phi} = \sigma_{\phi}^2 \sum_{t=2}^{T} (x_{it} - \mu_i) (x_{it-1} - \mu_i),$$

and accept with probability

$$\min \left( 1, \frac{p(\phi_i^*) \sqrt{1 - \phi_i^{2*}}}{p(\phi_i) \sqrt{1 - \phi_i^2}} \right).$$

We sample $\tau_i^2$ from IG $(v_1/2, s_1/2)$, where $v_1 = v_0 + T$ and $s_1 = s_0 + (1 - \phi_i^2) (x_{i1} - \mu_i)^2 + \sum_{t=2}^{T} (x_{it} - \mu_i - \phi_i (x_{it-1} - \mu_i))^2$. 

S3
S2  Markov moves for Factor SV model

S2.1  Markov move steps for HMC

For \( s = 1, \ldots, S \) and \( k = 1, \ldots, K \) we choose the priors for the persistence parameters \( \phi_{es} \) and \( \phi_{fk} \), the priors \( \tau_{2 es}^2, \tau_{2 fk}^2 \), and \( \mu_{es} \) as in Section 3.1. For every unrestricted element of the factor loadings matrix \( \beta \), we follow Kastner et al. (2017) and choose independent Gaussian distributions \( N(0, 1) \). These prior densities cover most possible values in practice.

Section S5 discusses parameterization and identification issues regarding the factor loading matrix \( \beta \) and the latent factors \( f \).

For \( s = 1, \ldots, S \), the gradient of the log-likelihood \( \nabla_{h_s} \mathcal{L}(h_s) \) with respect to each of the latent volatilities \( h_{s1:T} \) is required. The required gradient is, for \( t = 1 \),

\[
\nabla_{h_{s1}} \mathcal{L}(h_s) = a_p \left(-0.5 + 0.5 (y_{s1} - \beta_s f_1)^2 \exp(h_{s1})\right) - \frac{(1 - \phi_{es}^2)}{\tau_{2 es}^2} (h_{s1} - \mu_{es})
\]

\[
+ \frac{\phi_{es}}{\tau_{2 es}^2} (h_{s2} - \mu_{es} - \phi_{es} (h_{s1} - \mu_{es})),
\]

for \( 1 < t < T \), the required gradient is

\[
\nabla_{h_{st}} \mathcal{L}(h_s) = a_p \left(-0.5 + 0.5 (y_{st} - \beta_s f_t)^2 \exp(-h_{st})\right) + \frac{\phi_{es}}{\tau_{2 es}^2} (h_{st+1} - \mu_{es} - \phi_{es} (h_{st} - \mu_{es})) - \frac{1}{\tau_{2 es}^2} (h_{st} - \mu_{es} - \phi_{es} (h_{st-1} - \mu_{es})).
\]

and, for \( t = T \), the required gradient is

\[
\nabla_{h_{sT}} \mathcal{L}(h_s) = a_p \left(-0.5 + 0.5 (y_{sT} - \beta_s f_T)^2 \exp(-h_{sT})\right) - \frac{1}{\tau_{2 es}^2} (h_{sT} - \mu_{es} - \phi_{es} (h_{sT-1} - \mu_{es})).
\]

For \( k = 1, \ldots, K \), the gradient of the log-likelihood \( \nabla_{\lambda_{k1:T}} \mathcal{L}(\lambda_k) \) with respect to each of the latent volatilities \( \lambda_{k1:T} \) is required. The required gradient is, for \( t = 1 \),

\[
\nabla_{\lambda_{k1}} \mathcal{L}(\lambda_k) = a_p \left(-0.5 + 0.5 f_{k1}^2 \exp(\lambda_{k1})\right) - \frac{(1 - \phi_{fk}^2)}{\tau_{2 fk}^2} \lambda_{k1} + \frac{\phi_{fk}}{\tau_{2 fk}^2} (\lambda_{k2} - \phi_{fk} \lambda_{k1}),
\]

for \( 1 < t < T \), the required gradient is

\[
\nabla_{\lambda_{kt}} \mathcal{L}(\lambda_k) = a_p \left(-0.5 + 0.5 f_{kt}^2 \exp(-\lambda_{kt})\right) + \frac{\phi_{fk}}{\tau_{2 fk}^2} (\lambda_{kt+1} - \phi_{fk} \lambda_{kt}) - \frac{1}{\tau_{2 fk}^2} (\lambda_{kt} - \phi_{fk} \lambda_{kt-1}).
\]
and, for $t = T$, the required gradient is 
\[
\nabla_{\lambda_k T} \mathcal{L} (\lambda_k) = a_p (-0.5 + 0.5f^2_{kT} \exp (-\lambda_{kT})) - \frac{1}{\tau_{fT}} (\lambda_{kT} - \phi_{fT} \lambda_{kT-1}).
\]

For $s = 1, \ldots, S$, sample $\mu_{s}$ from $N (\mu_{s}, \sigma^2_{\mu_{s}})$ truncated within $(-10, 10)$, where 
\[
\sigma^2_{\mu_{s}} = \frac{\tau^2_{\mu_{s}}}{1 - \phi^2_{\mu_{s}} + (T - 1) (1 - \phi_{\mu_{s}})^2}
\]
and 
\[
\mu_{\mu_{s}} = \frac{\sigma^2_{\mu_{s}} h_{is1} (1 - \phi^2_{\mu_{s}}) + \sum_{t=2}^{T} h_{ist} - \phi_{\mu_{s}} h_{ist} + \phi^2_{\mu_{s}} h_{ist-1} - \phi_{\mu_{s}} h_{ist-1}}{\tau^2_{\mu_{s}}}
\]

For $s = 1, \ldots, S$, we sample $\phi_{i,s}$ by drawing a proposed value $\phi^*_s$ from $N (\mu_{\phi_{s}}, \sigma^2_{\phi_{s}})$ truncated within $(-1, 1)$, where 
\[
\sigma^2_{\phi_{s}} = \frac{\tau^2_{\phi_{s}}}{\sum_{t=2}^{T} (h_{ist-1} - \mu_{s})^2 - (h_{is1} - \mu_{s})^2}
\]
and 
\[
\mu_{\phi_{s}} = \sigma^2_{\phi_{s}} \frac{\sum_{t=2}^{T} (h_{ist} - \mu_{s}) (h_{ist-1} - \mu_{s})}{\tau^2_{\phi_{s}}}
\]

The candidate is accepted with probability 
\[
\min \left( 1, \frac{p (\phi^*_s) \sqrt{1 - \phi^2_{s}}}{p (\phi_{s}) \sqrt{1 - \phi^2_{s}}} \right).
\]

For $s = 1, \ldots, S$, we sample $\tau^2_{i,s}$ from IG $(v_{i,s}/2, s_{i,s}/2)$, where $v_{i,s} = v_{0,s} + T$ and $s_{i,s} = s_{0,s} + (1 - \phi^2_{i,s}) (h_{is1} - \mu_{s})^2 + \sum_{t=2}^{T} (h_{ist} - \mu_{s} - \phi_{i,s} (h_{ist-1} - \mu_{s}))^2$. For $k = 1, \ldots, K$, we sample $\phi_{i,fk}$ by drawing a proposed value $\phi^*_{i,fk}$ from $N (\mu_{\phi_{f,k}, \sigma^2_{\phi_{f,k}}})$ truncated within $(-1, 1)$, where 
\[
\sigma^2_{\phi_{f,k}} = \frac{\tau^2_{\phi_{f,k}}}{\sum_{t=2}^{T-1} \lambda^2_{ikt}}, \quad \text{and} \quad \mu_{\phi_{f,k}} = \frac{\sum_{t=2}^{T} \lambda_{ikt} \lambda_{ikt-1}}{\sum_{t=2}^{T-1} \lambda^2_{ikt}}
\]

For $k = 1, \ldots, K$, we sample $\tau^2_{i,fk}$ from IG $(v_{1,fk}/2, s_{1,fk}/2)$, where $v_{1,fk} = v_{0,fk} + T$ and $s_{1,fk} = s_{0,fk} + (1 - \phi^2_{i,fk}) \lambda^2_{ikt1} + \sum_{t=2}^{T} (\lambda_{ikt} - \phi_{i,fk} \lambda_{ikt-1})^2$. 

S5
S3 Proofs

Proof of Lemma 1. We prove the lemma by carrying out the marginalisation. The marginal distribution \( \tilde{\xi}_{ap} \left( x_{1:T}^{j_{1:T}}, j_{1:T}, \theta \right) \) is obtained by integrating \( \tilde{\xi}_{ap} \left( x_{1:T}^{j_{1:T}}, j_{1:T}, U_{1:T}^{-j_{1:T}}, \theta \right) \) over \( \left( x_{1:T}^{(-j_{1:T}), \tilde{a}_{1:T-1}} \right) \). We begin by integrating over \( \left( x_{T}^{(-j_{1:T}), \tilde{a}_{T-1}} \right) \) to obtain

\[
\tilde{\xi}_{ap} \left( x_{1:T}^{j_{1:T}}, j_{1:T}, x_{1:T-1}^{(-j_{1:T}), \tilde{a}_{1:T-2}}, \tilde{a}_{1:T-2}, \theta \right) = \frac{\xi_{ap} \left( d\theta, dx_{1:T}^{j_{1:T}} \right)}{N^T} \tag{S1}
\]

\[
\psi \left( x_{1:T-1}^{N}, \tilde{a}_{1:T-2} | \theta \right) m_{1} \left( dx_{1}^{j_{1}}, \theta_{1} \right) \prod_{t=2}^{T-1} \tilde{W}_{t-1}^{j_{t}} m_{t} \left( x_{t}^{j_{t}} | x_{t-1}^{j_{t}}, \theta \right) \prod_{t=2}^{T-1} \tilde{a}_{t-1}^{j_{t}} f_{t} \left( x_{t}^{j_{t}} | x_{t-1}^{j_{t}} \right).
\]

We similarly repeat this for \( t = T - 1, ..., 1 \) to obtain

\[
\tilde{\xi}_{ap} \left( d\theta, dx_{1:T}^{j_{1:T}}, j_{1:T} \right) = \frac{\xi_{ap} \left( d\theta, dx_{1:T}^{j_{1:T}} \right)}{N^T}. \tag{S2}
\]

\[\square\]

Proof of Lemma 2. Similarly to the proof of Lemma 1, we can show that the marginal distribution

\[
\tilde{\xi}_{ap} \left( h_{1:T}^{J_{s:1:T}}, J_{s:1:T}, \lambda_{1:T}^{J_{fk:1:T}}, J_{fk:1:T}, \theta, f_{1:T} \right) = \frac{\xi_{ap} \left( d\theta, dh_{1:T}^{J_{s:1:T}}, d\lambda_{1:T}^{J_{fk:1:T}}, df_{1:T} \right)}{N^T(S+K)},
\]

is obtained by integrating \( \tilde{\xi}_{ap} \left( U_{s:1:T}, U_{fk:1:T}, \theta, f_{1:T} \right) \) over \( \left( h_{s:1:T}^{-j_{s:1:T}}, \tilde{a}_{s:1:T-1} \right) \) for \( s = 1, ..., S \), and \( \left( \lambda_{fk:1:T}^{(-j_{fk:1:T})}, \tilde{a}_{fk:1:T-1} \right) \) for \( k = 1, ..., K \). \[\square\]

S4 Assumptions

In Section 3.2.1, we use the particle filter to approximate the joint filtering densities \( \{ p \left( x_{t} | y_{1:t} \right) : t = 1, ..., T \} \) sequentially using \( N \) particles, \( \left\{ x_{t}^{N}, \tilde{W}_{t}^{N} \right\} \), drawn from some proposal densities \( m_{1} \left( x_{1} | y_{1}, \theta \right) \) and \( m_{t} \left( x_{t} | x_{t-1}, y_{1:t}, \theta \right) \) for \( t \geq 2 \). For \( t \geq 1 \), we follow (Andrieu et al., 2010) and define,

\[
S_{t}^{\theta} := \{ x_{1:t} \in \chi : \pi \left( x_{1:t} | \theta \right) > 0 \} \quad \text{and} \quad Q_{t}^{\theta} := \{ x_{1:t} \in \chi : \pi \left( x_{1:t-1} | \theta \right) m_{t} \left( x_{t} | \theta, x_{1:t-1}, y_{1:t} \right) > 0 \}.
\]

Assumption S1. (Andrieu et al., 2010) We assume that \( S_{t}^{\theta} \subseteq Q_{t}^{\theta} \) for any \( \theta \in \Theta \) and \( t = 1, ..., T \)
Assumption S1 is always satisfied in our implementation because we use the boot-
strap filter with \( p(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{\theta}) \) as a proposal density, and Assumption S1 is satisfied
if \( p(\mathbf{y}_t|\mathbf{x}_t, \mathbf{\theta}) > 0 \) for all \( \mathbf{\theta} \).

**Assumption S2.** For any \( j = 1, ..., N \) and \( t = 1, ..., T \), the resampling scheme
\( \mathcal{M} \left( \tilde{a}_{t-1}^{1:N} | \tilde{W}_{t-1}^{1:N} \right) \) satisfies
\( \Pr \left( \tilde{A}_{t-1}^{k} = j | \tilde{W}_{t-1}^{1:N} \right) = \tilde{W}_{t-1}^j \). (Chopin and Singh, 2015; Andrieu et al., 2010).

Assumption 2 is satisfied by the popular resampling schemes, such as multinomial,
systematic, residual resampling.

**S5 Sampling the factor loading matrix \( \mathbf{\beta} \) and the
latent factors \( f_{1:T} \)**

In this section, we discuss the parameterization of the factor loading matrix and the
latent factors, and how they are sampled from their full conditional distribution.

To identify the parameters of the model for the factor loading matrix \( \mathbf{\beta} \), it is
necessary to impose some further constraints. Usually, the factor loading matrix \( \mathbf{\beta} \)
is assumed lower triangular in the sense that \( \beta_{sk} = 0 \) for \( k > s \) and furthermore, one
of two constraints are used. i) The first is that the \( \mathbf{f}_{kt} \) have unit variance (Geweke
and Zhou, 1996); ii) or, alternatively, assume that \( \beta_{ss} = 1 \), for \( s = 1, ..., S \), and
the variance of \( \mathbf{f}_t \) is diagonal but unconstrained. The main drawback of the lower
triangular assumption on \( \mathbf{\beta} \) is that the resulting inference can depend on the order in
which the components of \( y_t \) are chosen (Chan et al., 2017). Therefore, in our empirical
work, we reorder the elements of \( \mathbf{y} \) so that the lower triangular constraints are not
in conflict with the given dataset. Further, as noted by Kastner et al. (2017), the
second set of constraints impose that the first \( K \) variables are leading the factors, and
making the variable ordering dependence stronger. We follow Kastner et al. (2017)
and leave the diagonal elements \( \beta_{ss} \) unrestricted and set the level \( \mu_{2k} = 0 \) of factor
log-volatilities \( \lambda_{kt} \) to zero for \( k = 1, ..., K \). Let \( z_s \) denote the number of unrestricted
elements in row \( s \) and then, we define

\[
F_{is} = \begin{bmatrix}
\begin{array}{cccc}
    f_{i1} & \cdots & f_{iz_1} \\
    \vdots & & \vdots \\
    f_{i1T} & \cdots & f_{iz_sT}
\end{array}
\end{bmatrix},
\]

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and
\[ V_{is} = \begin{bmatrix} \exp (h_{is1}) & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \exp (h_{isT}) \end{bmatrix}, \]
then sampling the factor loadings \( \beta'_{is, s} = (B_{is1}, ..., B_{isz_s}) \) for \( s = 1, ..., S \), conditionally on \( f_{i1:T} \) can be done independently for each \( s \), by performing a Gibbs-update from
\[ \beta'_{is, s} | f_{i1:T}, y_{s1:T} \sim N_{k_s} (a_{sT}, b_{sT}), \quad (S3) \]
where
\[ b_{sT} = \left( a_p \left( F_{is} V_{is}^{-1} F_{is} \right) + B_0^{-1} I_z \right)^{-1}, \]
and
\[ a_{sT} = b_{sT} F_{is}' \left( a_p V_{is}^{-1} y_{s1:T} \right). \]

Sampling of \( \{ f_{it} \} | y_t, \{ h_{it1} \}, \{ h_{it2} \}, \beta \). After completing some algebra, we can show that \( \{ f_{it} \} \) can be sampled from
\[ \{ f_{it} \} | y_t, \{ h_{it} \}, \{ \lambda_{it} \}, \beta \sim N (a_t, b_t), \quad (S4) \]
where
\[ b_t = \left( a_p \left( B' V_t^{-1} B \right) + D_t^{-1} \right)^{-1}, \]
and
\[ a_t = b_t B' \left( a_p V_t^{-1} y_t \right). \]
S6 Particle Filter, Conditional Particle Filter, and backward simulation Algorithm

Algorithm S3 Generic Particle Filter Algorithm

Inputs: $\mathbf{y}_{1:T}$, $N$, $\theta$

Outputs: $\mathbf{x}_{1:T}^{1:N}$, $\mathbf{a}_{1:T-1}^{1:N}$, $\tilde{w}_{1:T}^{1:N}$

1. For $t = 1$
   
   (a) Sample $\mathbf{x}_t^j$ from $m_1(\mathbf{x}_1|\mathbf{y}_1, \theta)$, for $j = 1, \ldots, N$

   (b) Calculate the importance weights

   $$\tilde{w}_1^j = \frac{p(\mathbf{y}_1|\mathbf{x}_1^j, \theta) p(\mathbf{x}_1^j|\theta)}{m_1(\mathbf{x}_1^j|\mathbf{y}_1, \theta)}, \ j = 1, \ldots, N.$$  

   and normalise them to obtain $\tilde{W}_1^{1:N}$.

2. For $t > 1$

   (a) Sample the ancestral indices $\tilde{a}_{t-1}^{1:N} \sim \mathcal{M}(\tilde{a}_{t-1}^{1:N}|\tilde{W}_{t-1}^{1:N})$.

   (b) Sample $\mathbf{x}_t^j$ from $m_t\left(\mathbf{x}_t|\mathbf{x}_{t-1}^{\tilde{a}_{t-1}^j}, \theta\right)$, $j = 1, \ldots, N$.

   (c) Calculate the importance weights

   $$\tilde{w}_t^j = \frac{p(\mathbf{y}_t|\mathbf{x}_t^j, \theta) p(\mathbf{x}_t^j|\mathbf{x}_{t-1}^{\tilde{a}_{t-1}^j}, \theta)}{m_t\left(\mathbf{x}_t^j|\mathbf{x}_{t-1}^{\tilde{a}_{t-1}^j}, \theta\right)}, \ j = 1, \ldots, N.$$  

   and normalised to obtain $\tilde{W}_t^{1:N}$.
**Algorithm S4** Conditional Sequential Monte Carlo algorithm

**Inputs:** $N$, $\theta$, $y_{1:T}$, $x_{1:T}^{j_1}$, and $j_1:T$

**Outputs:** $x_{1:T}^{1:N}$, $a_{1:T-1}^{1:N}$, $\tilde{w}_{1:T}^{1:N}$

1. For $t = 1$
   
   (a) Sample $x_t^j$ from $m_1(x_t^j | y_1, \theta)$, for $j \in \{1, ..., N\} \setminus \{j_1\}$.
   
   (b) Calculate the weights
   
   $$ \tilde{w}_t^j = \frac{p(y_t^j | x_t^j, \theta)^{a_p} p(x_t^j | \theta)}{m_1(x_t^j | y_t, \theta)}, j = 1, ..., N. $$
   
   and normalised to obtain $\tilde{W}_{1:T}^{1:N}$.

2. For $t > 1$
   
   (a) Sample the ancestral indices $\tilde{a}_{t-1}^{(j_t)} \sim M(\tilde{a}_{t-1}^{(j_t)} | \tilde{W}_{1:T}^{1:N})$.
   
   (b) Sample $x_t^j$ from $m_t(x_t^j | x_{t-1}^{\tilde{a}_{t-1}^j}, \theta)$, $j = 1, ..., N \setminus \{j_t\}$.
   
   (c) Calculate the weights
   
   $$ \tilde{w}_t^j = \frac{p(y_t^j | x_t^j, \theta)^{a_p} p(x_t^j | x_{t-1}^{\tilde{a}_{t-1}^j}, \theta)}{m_t(x_t^j | x_{t-1}^{\tilde{a}_{t-1}^j}, \theta)}, j = 1, ..., N. $$
   
   and normalised to obtain $\tilde{W}_{1:T}^{1:N}$.

---

**Algorithm S5** The Backward simulation algorithm

1. Sample $J_T = j_T$ conditional on $(U_{1:T}, \theta)$, with probability proportional to $w_T^{j_T}$, and choose $x_T^{j_T}$.

2. For $t = T - 1, ..., 1$, sample $J_t = j_t$ conditional on $(u_{1:t}, j_{t+1:T}, x_{t+1}^{j_{t+1}}, ..., x_T^{j_T})$, and with probability proportional to $w_t^{j_t} f_{\theta}(x_{t+1}^{j_{t+1}} | x_t^{j_t})$, and choose $x_t^{j_t}$.
Table S1: The list of industry portfolios

| Stocks                               |
|--------------------------------------|
| 1 Coal                               |
| 2 Health Care and Equipment          |
| 3 Retail                             |
| 4 Tobacco                            |
| 5 Steel Works                        |
| 6 Food Products                      |
| 7 Recreation                         |
| 8 Printing and Publishing            |
| 9 Consumer Goods                     |
| 10 Apparel                           |
| 11 Chemicals                         |
| 12 Textiles                          |
| 13 Fabricated Products               |
| 14 Electrical Equipment              |
| 15 Automobiles and Trucks            |
| 16 Aircraft, ships, and Railroad Equipment |
| 17 Industrial Mining                 |
| 18 Petroleum and Natural Gas         |
| 19 Utilities                          |
| 20 Telecommunication                 |
| 21 Personal and Business Services    |
| 22 Business Equipment                 |
| 23 Transportation                    |
| 24 Wholesale                          |
| 25 Restaurants, Hotels, and Motels   |
| 26 Banking, Insurance, Real Estate    |
S8 Additional Empirical Results for Factor SV model

Figure S1: The kernel density estimates of the marginal posterior densities of \( \beta \) for the US stock returns data for four representative \( \beta \). The density estimates are for the PMCMC, AISIL-HMC, and AISIL-PG methods.
Figure S2: The Kernel Density Estimates of marginal posterior densities of $\mu$ for the US stock returns data for four representative $\mu$. The density estimates are for the PMCMC, AISIL-HMC, and AISIL-PG methods.

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