Electroweak symmetry breaking and mass spectra in six-dimensional gauge-Higgs grand unification

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Abstract

The mass spectra of the standard model particles are reproduced in the SO(11) gauge-Higgs grand unification in the six-dimensional warped space without introducing exotic light fermions. Light neutrino masses are explained by the gauge-Higgs seesaw mechanism. We evaluate the effective potential of the 4d Higgs boson appearing as a fluctuation mode of the Aharonov-Bohm phase \(\theta_H\) in the extra-dimensional space, and show that the dynamical electroweak symmetry breaking takes place with the Higgs boson mass \(m_H \sim 125\) GeV and \(\theta_H \sim 0.1\). The Kaluza-Klein mass scale in the fifth dimension is approximately given by \(m_{KK} \sim 1.230\) TeV/\(\sin \theta_H\).
1 Introduction

The discovery of the last piece of the standard model (SM) particle, the Higgs boson, seems to imply that the non-vanishing vacuum expectation value (VEV) of the Higgs field spontaneously breaks the electroweak (EW) gauge symmetry $SU(2)_L \times U(1)_Y$ to the electromagnetic gauge symmetry $U(1)_{EM}$. Almost all observational data at low-energy experiments are consistent with the SM. The SM is a good low-energy effective theory. However, it is not clear whether or not the Higgs boson is a genuine fundamental scalar field, which usually suffers from the so-called gauge hierarchy problem.

There are several proposals to overcome the problem by making use of symmetries. One of them is the gauge-Higgs unification. In this theory, the Higgs boson is identified with a part of the extra dimensional component of gauge fields in higher dimensional spacetime $[1, 5]$. It is described as a four-dimensional (4D) fluctuation mode of the Aharonov-Bohm (AB) phase $\theta_H$ along the extra-dimensional space.

The $SU(2)_L \times U(1)$ EW unification of the SM is formulated as the $SO(5) \times U(1)$ gauge-Higgs unification in the five-dimensional (5D) Randall-Sundrum (RS) warped space $[6, 16]$. According to Refs. $[11, 14]$, its phenomenology at low energies under the mass scale of the first Kaluza-Klein (KK) modes is almost the same as in the SM for the AB phase $\theta_H \lesssim 0.1$. $Z'$ bosons, which are the first KK modes of $\gamma$, $Z$, and $Z_R$, are predicted around $7 \sim 9$ TeV range for $\theta_H = 0.1 \sim 0.07$. $Z'$ bosons can be produced at 14 TeV LHC. $[15]$ At electron-positron linear colliders at the energies of $250$ GeV $\sim 1$ TeV, the interference effects among $\gamma$, $Z$ and $Z'$ bosons give distinct signals of the gauge-Higgs unification. $[16]$

To incorporate the $SU(3)_C$ gauge symmetry in higher dimensional gauge theories and gauge-Higgs unification scenario, grand unified theories (GUTs) based on a GUT gauge group $G_{GUT} \supset G_{SM} := SU(3)_C \times SU(2)_L \times U(1)$ have been discussed in Refs. $[17, 37]$. The $SO(11)$ gauge-Higgs grand unified theory (GHGUT) is proposed in the 5D Randall-Sundrum (RS) warped space in Ref. $[32]$. The EW Higgs boson is identified with a part of the 5th dimensional component of the $SO(11)$ gauge bosons. The $SO(11)$ gauge symmetry is reduced first by two different orbifold boundary conditions (BCs) on the UV and IR branes in the RS space. It is reduced to $SO(10)$ by the BC on the UV brane, and to $SO(4) \times SO(7)$ by the BC on the IR brane, the resultant symmetry being $SO(4) \times SO(6)$. Secondly the brane scalar on the UV brane, which is an $SO(10)$ spinor $[16]$, spontaneously breaks $SO(10)$ to $SU(5)$ by the Higgs mechanism on the UV brane. As a net result the
SO(11) gauge symmetry is reduced to the SM gauge symmetry $G_{\text{SM}}$, which is dynamically broken to $SU(3)_C \times U(1)_{\text{EM}}$ by the Hosotani mechanism.

Zero modes of 5D SO(11) 32 fermions in the bulk are identified with each generation of quarks and leptons. In Ref. [35] it is found that the observed mass spectra of the quarks and leptons can be reproduced, while there appear additional exotic particles having unacceptably small masses.

Recently, to avoid these light exotic particles, SO(11) GHGUT in six-dimensional (6D) hybrid warped space has been proposed in Ref. [38]. For neutrinos a new seesaw mechanism in 6D hybrid warped space has been formulated by using a 5D symplectic Majorana fermion [39], which generalizes the well-known 4D seesaw mechanism [40].

This paper is organized as follows. In Sec. 2, 6D SO(11) GHGUT is introduced. The matter content is specified and the action is given that contains both 6D bulk and 5D brane terms. In Sec. 3, a summary is given for the mass spectrum of the 4D gauge and scalar bosons originating from the 6D SO(11) gauge bosons. In Sec. 4, we derive the mass spectrum of 4D SM fermions. By using these mass spectra, we evaluate the effective potential $V_{\text{eff}}(\theta_H)$ in Sec. 5 to show that the dynamical EW symmetry breaking takes place and the 4D Higgs boson mass $m_H = 125.1$ GeV is obtained. Section 6 is devoted to a summary and discussions. In the Appendix basics for the KK expansion in 6D warped space are explained.

2 6D SO(11) GHGUT

We construct an SO(11) gauge-Higgs grand unified model on the six dimensional hybrid warped space introduced in Ref. [38]. The metric of generalized Randall-Sundrum (RS) space [38,41] is given by

\[ ds^2 = e^{-2\sigma(y)} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dv^2 \right) + dy^2, \]  

(2.1)

where $e^{-2\sigma(y)}$ is a warped factor and $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$. $\sigma(y)$ satisfies $\sigma(y) = \sigma(-y) = \sigma(y+2L_5)$ and $\sigma(y) = k|y|$ for $|y| \leq L_5$. The fifth dimension with the coordinate $y$ behaves as the EW dimension, whereas the sixth dimension with the coordinate $v$ is $S^1$ ($v \sim v + 2\pi R_6$) and behaves as the GUT dimension. Two spacetime points $(x^\mu, y, v)$ and $(x^\mu, -y, -v)$ are identified by the $\mathbb{Z}_2$ transformation. As a result the spacetime has the same topology as the orbifold $M^4 \times T^2/\mathbb{Z}_2$. The spacetime (2.1) solves the Einstein
equations with brane tensions at $y = 0$ and $y = L_5$. The bulk region is the anti-de-Sitter space with a negative cosmological constant $\Lambda = -10k^2$. The five-dimensional branes at $y = 0$ and $y = L_5$ have the same topology as $M^4 \times S^1$.

The extra dimensional space has four fixed points under $\mathbb{Z}_2$: $(y_0, v_0) = (0, 0)$, $(y_1, v_1) = (L_5, 0)$, $(y_2, v_2) = (0, \pi R_6)$, and $(y_3, v_3) = (L_5, \pi R_6)$. In terms of the conformal coordinate $z = e^{k y} (1 \leq z \leq z_L = e^{k L_5})$ in the region $0 \leq y \leq L_5$, the metric becomes

$$ds^2 = \frac{1}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dv^2 + \frac{dz^2}{k^2} \right).$$

The fifth and sixth dimensional Kaluza-Klein (KK) mass scales are given by $m_{KK_5} = \pi k / (e^{k L_5} - 1)$ and $m_{KK_6} = R_6^{-1}$. The warp factor is supposed to be large; $z_L = e^{k L_5} \gg 1$. $m_{KK_6}$ is expected to be a GUT scale while $m_{KK_5}$ is $O(10)$ TeV so that $m_{KK_6} \gg m_{KK_5}$.

Parity transformations $P_j$ ($j = 0, 1, 2, 3$) around the four fixed points are defined as $(x^\mu, y_j + y, v_j + v) \rightarrow (x^\mu, y_j - y, v_j - v)$. Only three of the four parity transformations $P_j$ are independent. They satisfy the relation $P_3 = P_2 P_0 P_1 = P_1 P_0 P_2$.

We adopt orbifold boundary conditions (BCs) such that $P_0 = P_1$ and $P_2 = P_3$, which enables us to avoid the problem of unwanted light exotic fermions mentioned in the Introduction. More specifically, we choose BCs such that $P_0 = P_1$ and $P_2 = P_3$ break $SO(11)$ to $SO(4) \times SO(7)$ and $SO(10)$, respectively. The orbifold BCs reduce $SO(11)$ symmetry to the Pati-Salam symmetry $SO(4) \times SO(6) \simeq SU(2)_L \times SU(2)_R \times SU(4)_C =: G_{PS}$. We note that the 5th and 6th dimensional loop translations $U_5 : (x^\mu, y, v) \rightarrow (x^\mu, y + 2 L_5, v)$ and $U_6 : (x^\mu, y, v) \rightarrow (x^\mu, y, v + 2 \pi R_6)$ are related to $P_j$'s by $U_5 = P_1 P_0 = P_3 P_2$ and $U_6 = P_2 P_0 = P_3 P_1$.

### 2.1 6D bulk and 5D brane fields and orbifold boundary conditions

The matter content in the $SO(11)$ GHGUT consists of 6D $SO(11)$ gauge bosons $A_M$, 6D $SO(11)$ 32 Weyl fermions $\Psi^{\alpha}_{32}(x, y, v)$ ($\alpha = 1, 2, 3, 4$), 6D $SO(11)$ 11 Dirac fermions $\Psi^\beta_{11}(x, y, v)$ and $\Psi^{\beta}_{11}(x, y, v)$ ($\beta = 1, 2, 3$) in the six-dimensional bulk space, and a 5D $SO(11)$ 32 brane scalar boson $F_{32}(x, v)$ and 5D $SO(11)$ 1 brane symplectic Majorana fermions $\chi^\beta_1(x, v)$ ($\beta = 1, 2, 3$) on the UV brane $y = 0$ [38]. Their orbifold BCs are listed below.
• For the 6D $SO(11)$ gauge boson $A_M$, the orbifold BCs are given by
\[
\begin{pmatrix}
A_\mu \\
A_\nu \\
A_\bar{v}
\end{pmatrix}
(x, y_j - y, v_j - v) = P_j
\begin{pmatrix}
A_\mu \\
-A_y \\
-A_v
\end{pmatrix}
(x, y_j + y, v_j + v)P_j^{-1},
\]
where in the $SO(11)$ vector representation, we take the orbifold BCs $P_0 = P_1$ and $P_2 = P_3$ as
\[
P_0^{\text{vec}} = P_1^{\text{vec}} = \text{diag}(I_4,-I_7), \quad P_2^{\text{vec}} = P_3^{\text{vec}} = \text{diag}(I_{10},-I_1).
\]
By using $(P_0 = P_1, P_2 = P_3)$, the parity assignment of $A_\mu$, $A_y$, and $A_v$ are summarized in Table 1.

• For the four 6D $SO(11)$ spinor 32 bulk Weyl fermions $\Psi^{\alpha}_{32}(x, y, v) \ (\alpha = 1, 2, 3, 4)$, the orbifold BCs are given by
\[
\Psi^{\alpha}_{32}(x, y_j - y, v_j - v) = \eta_j^\alpha \gamma P_j^{\text{sp}} \Psi^{\alpha}_{32}(x, y_j + y, v_j + v),
\]
where $\eta_j^\alpha = \pm 1$. 6D Dirac matrices $\gamma^a \ (a = 1, 2, \cdots, 6)$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ ($\eta^{ab} = \text{diag}(-I_1, I_5)$), and $\gamma := -i\gamma^5\gamma^6 = \gamma_{6D}\gamma_{5D} = \gamma_{4D}\gamma_{6D}$, $\gamma_{6D} = I_2 \otimes \sigma^3 \otimes I_2$, $\gamma_{6D} = I_4 \otimes \sigma^3$. $P_j^{\text{sp}}$'s are
\[
P_0^{\text{sp}} = P_1^{\text{sp}} = I_2 \otimes \sigma^3 \otimes I_8, \quad P_2^{\text{sp}} = P_3^{\text{sp}} = I_{16} \otimes \sigma^3.
\]
To ensure the 6D $SO(11)$ chiral anomaly cancellation, we assign $\gamma_{6D}^7 = +1$ for $\alpha = 1, 2$; $\gamma_{6D}^7 = -1$ for $\alpha = 3, 4$. We take $\eta_j^{1,2} = -1; \eta_j^3 = 1$, and $\eta_{0,2}^4 = -\eta_{1,3}^4 = 1$. The parity assignment
\[
\begin{pmatrix}
\eta_2^\alpha \gamma P_2^{\text{sp}} \\
\eta_3^\alpha \gamma P_3^{\text{sp}} \\
\eta_0^\alpha \gamma P_0^{\text{sp}} \\
\eta_1^\alpha \gamma P_1^{\text{sp}}
\end{pmatrix}
\]
of 4D left- and right-handed component of $\Psi^{\alpha}_{32}$ are summarized in Table 2. We find that $\Psi^{\alpha}_{32} \ (\alpha = 1, 2, 3)$ has zero modes, corresponding to one generation of quarks and leptons for each $\alpha$. The corresponding names adopted in Ref. [35] are also listed in Table 2 for $\alpha = 1, 2, 3$.

• For the three 6D $SO(11)$ vector 11 bulk Dirac fermions $\Psi^{\beta}_{11}(x, y, v)$ and $\Psi^{(\beta)}_{11}(x, y, v)$ ($\beta = 1, 2, 3$), the orbifold BCs are given by
\[
\Psi^{(\beta)}_{11}(x, y_j - y, v_j - v) = \eta_j^{(\beta)} \gamma P_j^{\text{vec}} \Psi^{(\beta)}_{11}(x, y_j + y, v_j + v),
\]
where $\eta_{0,1}^\beta = -\eta_{2,3}^\beta = -1$ for $\Psi^{\beta}_{11}$; $\eta_{0,1}^\beta = \eta_{2,3}^\beta = -1$ for $\Psi^{(\beta)}_{11}$. The parity assignment ($\eta_0^{3} \gamma P_0^{\text{vec}} = \eta_1^{3} \gamma P_1^{\text{vec}}, \eta_2^{3} \gamma P_2^{\text{vec}} = \eta_3^{3} \gamma P_3^{\text{vec}}$) of 4D left- and right-handed component of $\Psi^{(\beta)}_{11}$ are summarized in Table 3.
• For the 5D SO(11) spinor 32 brane scalar field on the UV brane at $y = 0$, $\Phi_{32}(x, v)$, the orbifold BCs are given by

$$\Phi_{32}(x, v_j - v) = \eta_j P_{sp}^j \Phi_{32}(x, v_j + v),$$ 

(2.9)

where $j = 0, 2$, $\eta_0 = -\eta_2 = -1$. The components of the $G_{PS}$ (1, 2, 4) have zero modes, one of which corresponds to the $SU(5)$ 1 of $SO(10)$ 16. It is responsible for reducing $SO(11)$ to $SU(5)$ at the UV brane $y = 0$. The BCs of $\Phi_{32}$ are summarized in Table 4.

• For the 5D SO(11) singlet fermions $\chi^\beta(x, v) = \chi^\beta_1(x, v)$ ($\beta = 1, 2, 3$) at $y = 0$, their orbifold BCs are given by

$$\chi^\beta(x, v_j - v) = \overline{\gamma} \chi^\beta(x, v_j + v) \quad (j = 0, 2).$$ 

(2.10)

These brane fields also satisfy the 5D symplectic Majorana condition $\chi^C = \overline{\chi}$, $\overline{\chi} := i \Gamma \chi$, where $\Gamma := \gamma^{54D} \gamma^6$:

$$\chi = \begin{pmatrix} \xi_+ \\ \eta_+ \\ \xi_- \\ \eta_- \end{pmatrix}, \quad \chi^C = \begin{pmatrix} +\eta^C_+ \\ -\xi^C_+ \\ -\eta^C_- \\ +\xi^C_- \end{pmatrix} = e^{i\delta C} \begin{pmatrix} -\sigma^2 \eta^*_+ \\ -\sigma^2 \xi^*_+ \\ +\sigma^2 \eta^*_- \\ +\sigma^2 \xi^*_- \end{pmatrix} = \begin{pmatrix} +\xi_- \\ -\eta_- \\ -\xi_- \\ +\eta_- \end{pmatrix} = \overline{\chi}. \quad \text{(2.11)}$$

Here the generation index $\beta$ has been suppressed.

| $G_{PS}$ | $A_{\mu}$ | $A_y$ | $A_v$ |
|---------|-------|-----|-----|
| (3,1,1) | (+,+) | (−,−) | (−,−) |
| (1,3,1) | (+,+) | (−,−) | (−,−) |
| (1,1,15) | (+,+) | (−,−) | (−,−) |
| (2,2,6) | (+,+) | (+,+) | (+,+) |
| (2,2,1) | (+,−) | (+,+) | (+,+) |
| (1,1,6) | (+,−) | (−,+)) | (−,+) |

Table 1: Parity assignment $(P_0, P_2) = (P_1, P_3)$ of $A_{\mu}$, $A_y$, and $A_v$ in $G_{PS} = SU(2)_L \times SU(2)_R \times SU(4)_C$.

### 2.2 Action

The action consists of the 6D bulk and 5D brane terms.
\[
\begin{array}{cccc}
G_{32} & \text{Left} & \text{Right} & \text{name} \\
(2, 1, 4) & (-, +) & (+, -) & \nu \ u_j \\
(1, 2, 4) & (+, +) & (-, -) & \bar{e} \ d_j \\
(2, 1, 4) & (+, -) & (+, +) & \bar{e}' \ d'_j \\
(1, 2, 4) & (-, -) & (+, +) & \bar{u}' \ u'_j \\
\end{array}
\]

Table 2: Parity assignment \([P_2 \ P_3 \ P_1] \) in \((2.7)\) of \(\Psi_{32}^\alpha\) \((\alpha = 1, 2, 3, 4)\) in \(G_{32}\).

\[
\begin{array}{cccc}
G_{32} & \text{Left} & \text{Right} & \text{name} \\
(2, 1, 4) & (+, -) & (-, +) & \nu' \ u_j \\
(1, 2, 4) & (-, -) & (+, +) & \bar{e}' \ d'_j \\
(2, 1, 4) & (-, +) & (+, -) & \bar{u}' \ u'_j \\
(1, 2, 4) & (+, +) & (-, -) & \bar{e}' \ d'_j \\
\end{array}
\]

Table 3: Parity assignment \((P_0, P_2) = (P_1, P_3)\) for \(\Psi_{11}^{(\beta)}\) \((\beta = 1, 2, 3)\) in \(G_{32}\).

2.2.1 Bulk terms

The bulk part of the action is given by

\[
S_{\text{bulk}} = S_{\text{bulk}}^{\text{gauge}} + S_{\text{bulk}}^{\text{fermion}},
\]

where \(S_{\text{bulk}}^{\text{gauge}}\) and \(S_{\text{bulk}}^{\text{fermion}}\) are bulk actions of gauge and fermion fields, respectively. The action of \(SO(11)\) gauge field \(55 \ A_M(x, y, v)\) is

\[
S_{\text{bulk}}^{\text{gauge}} = \int d^6x \sqrt{-\det G} \left[ - \text{tr} \left( \frac{1}{4} F^{MN} F_{MN} + \frac{1}{2 \xi} (f_{g})^2 + \mathcal{L}_{gh} \right) \right],
\]

where \(f_{g}\) is the fermion field and \(\mathcal{L}_{gh}\) is the fermion part of the action.
Table 4: Parity assignment \((P_0, P_2)\) for \(\Phi_{32}\) in \(G_{PS}\).

\[
\begin{array}{ccc}
\Phi_{32} & G_{PS} & \text{BCs} \\
(2, 1, 4) & (-, +) \\
(1, 2, 4) & (+, +) \\
(2, 1, 4) & (-, -) \\
(1, 2, 4) & (+, -) \\
\end{array}
\]

where \(\sqrt{-\det G} = 1/kz^6\), \(z = e^{ky}\), \(M, N = 0, 1, 2, 3, 5, 6\), and the field strength \(F_{MN}\) is defined by

\[
F_{MN} := \partial_M A_N - \partial_N A_M - ig[A_M, A_N].
\]  

(2.14)

We take the following gauge fixing and ghost terms

\[
f_{gf} = z^2 \left\{ \eta^{\mu\nu} D_\mu^c A^e_\nu + D_6^c A^e_6 + \xi k^2 z^2 D_z^e \left( \frac{A^e_6}{z^2} \right) \right\},
\]

\[
\mathcal{L}_{gh} = \bar{c} \left\{ \eta^{\mu\nu} D_\mu^c D_\nu^{c+q} + D_6^c D_6^{c+q} + \xi k^2 z^2 D_z^c \frac{1}{z^2} D_z^{c+q} \right\} c.
\]  

(2.15)

where \(\mu, \nu = 0, 1, 2, 3\), \(\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)\), and \(A_M = A_M^c + A_M^e\). \(D_M^c B = \partial_M B - ig[A_M, B]\) and \(D_M^{c+q} B = \partial_M B - ig[A_M, B]\) where \(B = A^e_\mu, A^e_6/z^2, A^e_6\) and \(c\).

The action of fermions in \(SO(11)\) spinor and vector representations \(32\) and \(11\), \(\Psi^\alpha_{32}(x, y, v)\), \(\Psi^\beta_{11}(x, y, v)\) and \(\Psi'^\beta_{11}(x, y, v)\), is given by

\[
S_{\text{fermion bulk}} = \int d^6 x \sqrt{-\det G} \left\{ \sum_{\alpha=1}^{4} \Psi^\alpha_{32} \mathcal{D}(c \psi^\alpha_{32}) \Psi^\alpha_{32} + \sum_{\beta=1}^{3} \Psi^\beta_{11} \mathcal{D}(c \psi^\beta_{11}) \Psi^\beta_{11} + \sum_{\beta=1}^{3} \Psi'^\beta_{11} \mathcal{D}(c \psi'^\beta_{11}) \Psi'^\beta_{11} \right\}.
\]  

(2.16)

where \(\overline{\Psi^\alpha_{32}} := i \Psi^\dagger_{32} \gamma^0\). \(c_{\psi^\alpha_{32}}, c_{\psi^\beta_{11}}\) and \(c_{\psi'^\beta_{11}}\) are bulk mass parameters, and \(\mathcal{D}(c)\) is a covariant derivative given by

\[
\mathcal{D}(c) = z \left\{ \gamma^\mu D_\mu + \gamma^6 D_6 - \frac{5}{2} \sigma'^\gamma D_z + \sigma'^\gamma + \gamma^5 D_z + i c \sigma'^\gamma \right\},
\]  

(2.17)

where \(\sigma'(y) := d\sigma(y)/dy\) and \(\sigma'(y) = k\) for \(0 < y < L_5\).

Here, we introduce \(\tilde{\Psi}\) defined by

\[
\tilde{\Psi} := \frac{1}{z^5/2} \Psi, \quad \left( D_z - \frac{5}{2} \right) \tilde{\Psi} = z^{5/2} D_z \tilde{\Psi},
\]  

(2.18)
which turns out convenient to discuss mass spectra for fermions in Sec. The bulk part of the bulk fermion action becomes

\[
S = \int d^4x \int_0^{2\pi R_6} dv \int_{z_L}^{z_U} \frac{dz}{k} \left[ \bar{\Psi}^{32}_{\alpha} \left( \gamma^\mu D_\mu + \gamma^6 D_v + \sigma'\gamma^5 D_z + i\epsilon c_{\Psi^{32}} \frac{\sigma'}{z} \gamma^6 \right) \right] \Psi^{32}_\alpha \\
+ \bar{\Psi}^{11}_\beta \left( \gamma^\mu D_\mu + \gamma^6 D_v + \sigma'\gamma^5 D_z + i\epsilon c_{\Psi^{11}} \frac{\sigma'}{z} \gamma^6 \right) \Psi^{11}_\beta \\
+ \bar{\Psi}^{11}_\beta \left( \gamma^\mu D_\mu + \gamma^6 D_v + \sigma'\gamma^5 D_z + i\epsilon c_{\Psi^{11}} \frac{\sigma'}{z} \gamma^6 \right) \Psi^{11}_\beta].
\] (2.19)

2.2.2 Action for the brane scalar \( \Phi^{32} \) and the Higgs mechanism

We consider the 5D brane terms for a brane scalar \( \Phi^{32}(x,v) \):

\[
S^{\text{brane}}_{5D \text{ brane}} = \int d^6x \sqrt{-\det G} \delta(y) L_{\Phi^{32}},
\]

\[
L_{\Phi^{32}} := -(D_\mu \Phi^{32})^\dagger D^\mu \Phi^{32} - (D_v \Phi^{32})^\dagger D^v \Phi^{32} - \lambda_{\Phi^{32}} (\Phi^{32}_\dagger \Phi^{32} - |w|^2)^2,
\] (2.20)

where \( \lambda_{\Phi^{32}} \) and \( w \) are constants, and

\[
D_\mu \Phi^{32} = \left( \partial_\mu - igA_{\mu}^{SO(11)} \right) \Phi^{32} = \left\{ \partial_\mu - \frac{ig}{\sqrt{2}} \sum_{j<k}^{11} A_{\mu}^{(jk)} T_{jk}^{sp} \right\} \Phi^{32},
\]

\[
D_v \Phi^{32} = \left( \partial_v - igA_{v}^{SO(11)} \right) \Phi^{32} = \left\{ \partial_v - \frac{ig}{\sqrt{2}} \sum_{j<k}^{11} A_{v}^{(jk)} T_{jk}^{sp} \right\} \Phi^{32}.
\] (2.21)

We denote the 5D brane scalar field \( \Phi^{32} \) as

\[
\Phi^{32} := \begin{pmatrix} \Phi^{16} \\ \Phi^{16} \end{pmatrix}
\] (2.22)

and assume that the component \((1,2,4)\) of \( G_{PS}(= SU(2)_L \times SU(2)_R \times SU(4)_C) \) for \( \Phi^{32} \) develops the nonvanishing VEV:

\[
\langle \Phi^{16} \rangle = \begin{pmatrix} 0_4 \\ 0_4 \\ v_4 \\ 0_4 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w \end{pmatrix}, \quad \langle \Phi^{16} \rangle = 0,
\] (2.23)

where one component of \((1,2,4)\) of \( G_{PS}(\subset SO(10)) \) corresponds to \( 1_{+5} \) of \( SU(5) \times U(1)_Z(\subset SO(10)) \). The VEV breaks \( SO(11) \) to \( SU(5) \). Note that branching rules of \( SO(11) \ 32 \) and \( 55 \) representations for \( SO(11) \supset SO(10) \supset SU(5) \) are given by \( 32 = \ldots \)
\((16) \oplus (1\overline{6}) = (10 \oplus \overline{5} \oplus 1) \oplus (1\overline{0} \oplus 5 \oplus 1)\) and \(55 = (45) \oplus (10) = (24 \oplus 10 \oplus 1\overline{0} \oplus 1) \oplus (5 \oplus \overline{5})\), respectively. (For further information, see e.g., Refs. \([42–44]\).)

We also use the 5D brane scalar field \(\tilde{\Phi}_{32}\) related to \(\Phi_{32}\) by

\[
\tilde{\Phi}_{32} = \left(\begin{array}{c}
\frac{-\hat{R} \Phi^*_{16}}{\hat{R} \Phi^*_{16}} \\
\frac{-\hat{R} \Phi^*_{16}}{\hat{R} \Phi^*_{16}}
\end{array}\right), \quad \hat{R} = \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^3. \tag{2.24}
\]

For \(\langle \Phi_{32} \rangle \neq 0\),

\[
\hat{R} \langle \Phi^*_{16} \rangle = \left(\begin{array}{c}
0_4 \\
0_4 \\
0_4 \\
\tilde{v}_4
\end{array}\right), \quad \tilde{v}_4 = \left(\begin{array}{c}
0 \\
0 \\
w^* \\
0
\end{array}\right). \tag{2.25}
\]

The combination of the nonvanishing VEV \(\langle \Phi_{32} \rangle\) on the UV brane (at \(y = 0\)) and the orbifold BCs \(P_j(j = 0, 1, 2, 3)\) reduces \(SO(11)\) to the SM gauge group \(G_{SM}(= SU(3)_C \times SU(2)_L \times U(1)_Y)\).

### 2.2.3 Action for the singlet brane fermion \(\chi_1\)

The action for \(\chi_1^\beta(x, v)\), which satisfies the symplectic Majorana condition \((2.11)\), is

\[
S_{\text{fermion}}^{\text{brane}} = \int d^6x \sqrt{-\text{det}G \delta(y)} L_{\chi_1},
\]

\[
L_{\chi_1} := \frac{1}{2} \chi_1^\beta (\gamma^\mu \partial_\mu + \gamma^6 \partial_v) \chi_1^\beta - \frac{1}{2} M_{\beta \beta'} \chi_1^\beta \chi_1^{\beta'}, \tag{2.26}
\]

where \(M_{\beta \beta'}\) is a constant matrix.

### 2.2.4 Brane masses and interactions

On the UV brane there can be \(SO(11)\)-invariant brane interactions among the bulk fermions, the singlet brane fermion, and the brane scalar. We consider

\[
S_{\text{fermion}}^{\text{brane}} = \int d^6x \sqrt{-\text{det}G \delta(y)} \left( L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 \right),
\]
\[
\mathcal{L}_1 := -\kappa^{\alpha\beta} \overline{\Psi}_{32} \Gamma^a \Phi_{32} \cdot (\Psi_1^\beta)_a - \kappa^{\alpha\beta*} \overline{\Psi}_{11} \Phi_{32}^\dagger \Gamma^a (\Psi_2^\alpha)_a,
\]
\[
\mathcal{L}_2 := -\kappa^{\alpha\beta} \overline{\Psi}_{32} \Gamma^a \Phi_{32} \cdot (\Psi_1^\beta)_a - \kappa^{\alpha\beta*} \overline{\Psi}_{11} \Phi_{32}^\dagger \Gamma^a (\Psi_2^\alpha)_a,
\]
\[
\mathcal{L}_3 := -2\mu_{11}^{\beta\gamma} \Psi_{11}^\beta \Psi_{11}^\gamma,
\]
\[
\mathcal{L}_4 := -\kappa^{\alpha\beta} \overline{\Psi}_{32} \Gamma^a \Phi_{32} \cdot (\Psi_1^\beta)_a - \kappa^{\alpha\beta*} \overline{\Psi}_{11} \Phi_{32}^\dagger \Gamma^a (\Psi_2^\alpha)_a,
\]
\[
\mathcal{L}_5 := -\kappa^{\alpha\beta} \overline{\Psi}_{32} \Gamma^a \Phi_{32} \cdot (\Psi_1^\beta)_a - \kappa^{\alpha\beta*} \overline{\Psi}_{11} \Phi_{32}^\dagger \Gamma^a (\Psi_2^\alpha)_a,
\]
\[
\mathcal{L}_6 := -2\mu_{11}^{\beta\gamma} \Psi_{11}^\beta \Psi_{11}^\gamma,
\]
\[
\mathcal{L}_7 := -\kappa^{\alpha\beta} \overline{\lambda}_1 \Phi_{32}^\dagger \overline{\Psi}_{32}^\alpha - \kappa^{\alpha\beta*} \overline{\lambda}_1 \Phi_{32}^\dagger \overline{\Psi}_{32}^\alpha \lambda_1^\beta,
\]
\[
\mathcal{L}_8 := -\kappa^{\alpha\beta} \overline{\lambda}_1 \Phi_{32}^\dagger \overline{\Psi}_{32}^\alpha - \kappa^{\alpha\beta*} \overline{\lambda}_1 \Phi_{32}^\dagger \overline{\Psi}_{32}^\alpha \lambda_1^\beta,
\]
(2.27)

where \(\kappa's\) and \(\mu's\) are coupling constants. Note that \(\overline{\Psi}_{11}^\beta \Psi_{11}^\alpha\) and \(\overline{\Psi}_{11}^{\beta*} \Psi_{11}^\alpha\) are forbidden by the parity assignment of \(\overline{\Psi}_{11}^\alpha\) and \(\Psi_{11}^\beta\) shown in Table 3.

### 2.2.5 Mass terms for fermions on the UV brane

From the action in Eqs. (2.26) and (2.27), the quadratic mass terms on the UV brane for the bulk and brane fermions with \(\langle \Phi_{32} \rangle \neq 0\) are given by

\[
S_{\text{fermion}}^{\text{brane, mass}} = \int d^4x \int_0^{2\pi R_6} dv \left( \sum_{j=1}^8 \mathcal{L}_j^m + \mathcal{L}_{\chi_1}^m \right),
\]
(2.28)

where

\[
\mathcal{L}_1^m = -\mu_1^{\alpha\beta} \left\{ -i2\overline{\epsilon}^\alpha \tilde{E}^\beta + \nu^\alpha \tilde{N}^\beta \right\} - 2\overline{d}^\alpha \tilde{D}^\beta + \sqrt{2}\overline{\nu} \tilde{S}^\beta \right\} + \text{h.c.},
\]
\[
\mathcal{L}_2^m = -\mu_1^{\alpha\beta} \left\{ i2\overline{\epsilon}^\alpha \tilde{E}^\beta + \nu^\alpha \tilde{N}^\beta \right\} - 2\overline{d}^\alpha \tilde{D}^\beta + \sqrt{2}\overline{\nu} \tilde{S}^\beta \right\} + \text{h.c.},
\]
\[
\mathcal{L}_3^m = -2\mu_{11}^{\beta\gamma} \left\{ \overline{D}^\beta \tilde{D}^\gamma + \overline{D}^\gamma \tilde{D}^\beta + \overline{E}^\beta \tilde{E}^\gamma + \overline{E}^\gamma \tilde{E}^\beta \right\}
+ \tilde{N}^\beta \tilde{N}^\gamma + \tilde{N}^\gamma \tilde{N}^\beta + \tilde{S}^\beta \tilde{S}^\gamma \right\},
\]
\[
\mathcal{L}_4^m = -\mu_2^{\alpha\beta} \left\{ -i2\overline{\epsilon}^{\alpha*} \tilde{E}^{\beta*} + \nu^{\alpha*} \tilde{N}^{\beta*} \right\} - 2\overline{d}^{\alpha*} \tilde{D}^{\beta*} + \sqrt{2}\overline{\nu} \tilde{S}^{\beta*} \right\} + \text{h.c.},
\]
\[
\mathcal{L}_5^m = -\mu_2^{\alpha\beta} \left\{ i2\overline{\epsilon}^{\alpha*} \tilde{E}^{\beta*} + \nu^{\alpha*} \tilde{N}^{\beta*} \right\} - 2\overline{d}^{\alpha*} \tilde{D}^{\beta*} + \sqrt{2}\overline{\nu} \tilde{S}^{\beta*} \right\} + \text{h.c.},
\]
\[
\mathcal{L}_6^m = -2\mu_{11}^{\beta\gamma} \left\{ \overline{D}^{\beta*} \tilde{D}^{\gamma*} + \overline{D}^{\gamma*} \tilde{D}^{\beta*} + \overline{E}^{\beta*} \tilde{E}^{\gamma*} + \overline{E}^{\gamma*} \tilde{E}^{\beta*} \right\}
+ \tilde{N}^{\beta*} \tilde{N}^{\gamma*} + \tilde{N}^{\gamma*} \tilde{N}^{\beta*} + \tilde{S}^{\beta*} \tilde{S}^{\gamma*} \right\},
\]
\[
\mathcal{L}_7^m = -\overline{m}_B^{\alpha\beta} \left( \chi^\beta \overline{\nu}^{\alpha} + \nu^{\alpha} \chi^\beta \right),
\]

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from the action (2.20), we find the mass terms for the 6th dimensional component of the

\[ \mathcal{L}_m^m = -\frac{\hat{m}_B}{\sqrt{k}} (\chi^\beta \dot{\nu}^\alpha + \nu^\alpha \chi^\beta) , \]

\[ \mathcal{L}_{\chi_1}^m = -\frac{1}{2} M^{\beta \beta'} \chi^\alpha \chi^{\beta'} . \]  

(2.29)

Here \( 2\mu_1^{\alpha \beta} := \sqrt{2} \kappa_1^{\alpha \beta} w, 2\mu_2^{\alpha \beta} := \sqrt{2} \kappa_2^{\alpha \beta} w, 2\mu_3^{\alpha \beta} := \sqrt{2} \kappa_3^{\alpha \beta} w, m_B^2 / \sqrt{k} := \tilde{\kappa}_1^{\alpha \beta} w \) and \( \tilde{m}^2_B / \sqrt{k} := \kappa_1^{\alpha \beta} w \). For the sake of simplicity \( \chi^\beta_1 \) has been denoted as \( \chi^\beta \). All \( \mu \) parameters are dimensionless quantities, whereas \( m_B^2, \tilde{m}^2_B \) and \( M^{\beta \beta'} \) have the dimension of mass.

2.2.6 Mass terms for gauge bosons on the UV brane

By replacing \( \Phi_{32} \) to its VEV \( \langle \Phi_{32} \rangle \) in Eq. (2.20), the mass terms for the 4D components of the \( SO(11) \) gauge fields \( A_\mu \) can be read off as

\[ \mathcal{L}^{4D \ \text{gauge mass}} = -\frac{g^2 w^2}{8} \left\{ (A_{15}^{15} - A_{16}^{16})^2 + (A_{16}^{15} + A_{25}^{25})^2 \right. \]

\[ + (A_{17}^{17} - A_{28}^{28})^2 + (A_{18}^{18} + A_{27}^{27})^2 + (A_{19}^{29} - A_{10}^{20})^2 + (A_{11}^{11} + A_{29}^{29})^2 \]

\[ + (A_{35}^{35} + A_{46}^{46})^2 + (A_{36}^{36} - A_{45}^{45})^2 + (A_{47}^{47} + A_{48}^{48})^2 + (A_{38}^{38} - A_{47}^{47})^2 \]

\[ + (A_{39}^{39} + A_{410}^{410})^2 + (A_{310}^{310} - A_{49}^{49})^2 + (A_{57}^{57} - A_{68}^{68})^2 + (A_{58}^{58} + A_{67}^{67})^2 \]

\[ + (A_{59}^{59} - A_{610}^{610})^2 + (A_{510}^{510} + A_{69}^{69})^2 + (A_{79}^{79} - A_{810}^{810})^2 + (A_{710}^{710} + A_{89}^{89})^2 \]

\[ + (A_{23}^{23} - A_{114}^{114})^2 + (A_{31}^{31} - A_{24}^{24})^2 + (A_{12}^{12} - A_{234}^{234})^2 + (A_{12}^{12} + A_{56}^{56} + A_{78}^{78} + A_{9,10}^{9,10})^2 \]

\[ + (A_{11}^{11})^2 + (A_{211}^{211})^2 + (A_{311}^{311})^2 + (A_{411}^{411})^2 + (A_{511}^{511})^2 \]

\[ \left. + (A_{611}^{611})^2 + (A_{711}^{711})^2 + (A_{811}^{811})^2 + (A_{911}^{911})^2 + (A_{1011}^{1011})^2 \right\} , \]  

(2.30)

where the notation is the same as one in Ref. [35]. We omit them here. All the 31 components of \( SO(11)/SU(5) \) obtain large brane masses via \( \langle \Phi_{32} \rangle \neq 0 \). Each mass term changes the 5th dimensional BCs on the UV brane for the corresponding fields. Similarly, from the action (2.20), we find the mass terms for the 6th dimensional component of the
\[ \mathcal{L}_{\text{brane mass}}^{6\text{th dim gauge}} = - \frac{g^2 u^2}{8} \left\{ \left( A_{v}^{15} - A_{v}^{26} \right)^2 + \left( A_{v}^{16} + A_{v}^{25} \right)^2 + \left( A_{v}^{17} - A_{v}^{28} \right)^2 + \left( A_{v}^{18} + A_{v}^{27} \right)^2 + \left( A_{v}^{19} - A_{v}^{210} \right)^2 + \left( A_{v}^{110} + A_{v}^{29} \right)^2 + \left( A_{v}^{35} + A_{v}^{46} \right)^2 + \left( A_{v}^{36} - A_{v}^{45} \right)^2 + \left( A_{v}^{37} + A_{v}^{48} \right)^2 + \left( A_{v}^{38} - A_{v}^{47} \right)^2 + \left( A_{v}^{49} + A_{v}^{1010} \right)^2 + \left( A_{v}^{410} - A_{v}^{69} \right)^2 + \left( A_{v}^{57} - A_{v}^{68} \right)^2 + \left( A_{v}^{58} + A_{v}^{67} \right)^2 \right\} \]

(2.31)

These mass terms give masses for the corresponding fields by changing the BCs. We shall see more details in Section 3.

### 2.3 EW Higgs boson and twisted gauge

The orbifold BCs break \( SO(11) \) to \( G_{PS} \), and further the nonvanishing VEV of the 5D brane scalar \( \Phi_{32} \) reduces \( G_{PS} \) to \( G_{SM} \). The bilinear terms of the action of gauge fields in (2.13) are written down as

\[ (S_{\text{bulk}}^{\text{gauge}})_{\text{quadratic}} \]

\[ = \int d^4 x dv \frac{dz}{k z^2} \sum_{j<k} \left[ \frac{1}{2} A_{j \lambda}^{(jk)} \left\{ \hat{\eta}^{\lambda \rho} \left( \Box + k^2 \mathcal{P}_5 + \partial^2 \right) \right. - \left( 1 - \frac{1}{\xi} \right) \partial^\lambda \partial^\rho \right\} A_{\rho}^{(jk)} \]

\[ + \frac{1}{2} k^2 A_{j \lambda}^{(jk)} \left( \Box + \xi k^2 \mathcal{P}_5 + \partial^2 \right) A_{\lambda}^{(jk)} + \bar{c}^{(jk)} \left( \Box + \xi k^2 \mathcal{P}_5 + \partial^2 \right) c^{(jk)} \]

\[ \hat{\eta}_{\lambda \rho} = \hat{\eta}^{\lambda \rho} = \text{diag}(-1, +1, +1, +1, +1) \]

\[ \Box = \eta^{\mu \nu} \partial_\mu \partial_\nu, \quad \partial^\lambda = \hat{\eta}^{\lambda \rho} \partial_\rho, \quad \mathcal{P}_5 = z^2 \frac{1}{\partial z} \frac{\partial}{\partial z^2} \partial z, \quad \mathcal{P}_z = \frac{1}{\partial z} \frac{\partial}{\partial z} \frac{1}{\partial z} \frac{\partial}{\partial z}. \]

(2.32)

Here \( \lambda, \rho \) run over 0, 1, 2, 3, 6 so that \( A_\lambda \hat{\eta}^{\lambda \rho} B_\rho = A_\mu \eta^{\mu \nu} B_\nu + A_\nu B_\nu \). The 4D and 6th dimensional components \( A_\mu \) and \( A_v \) have additional bilinear terms that come from the 5D brane scalar term \( \mathcal{L}_{\Phi_{32}} \) in (2.20) with \( \langle \Phi_{32} \rangle \neq 0 \). The parity assignments of \( A_\mu \), \( A_z \), and \( A_v \) are summarized in Table 1. The components (2, 2, 1) of \( A_z \) and \( A_v \) have parity (+, +). The components (2, 2, 1) of \( A_v \) obtain masses through the brane mass terms.
in Eq. (2.31), so only the components \((2, 2, 1)\) of \(A_z\), which is identified with the SM Higgs scalar fields, have zero modes. Three of them are absorbed by \(W\) and \(Z\) bosons. The remaining neutral component, the observed Higgs boson, also becomes massive by radiative corrections through the Hosotani mechanism as shown below. We note that the components \((2, 2, 1)\) of \(A_v\) get additional masses of order \(gR_6^{-1}\) by radiative corrections at one loop. The components \((3, 1, 1), (1, 3, 1),\) and \((1, 1, 15)\) of \(A_\mu\) have parity \((+, +)\).

\(G_{PS}\) is spontaneously broken to \(G_{SM}\) by the nonvanishing VEV of the component \((1, 2, 4)\) of \(\Phi_{32}\).

The zero modes of \(A_z\) are physical degrees of freedom which cannot be gauged away. By using the KK mode expansion discussed in Appendix A.2, the 6th dimensional \(n = 0\) modes (\(v\)-independent modes) of \(A_z^{a=11}\) are expanded, in terms of mode functions \(\{h_n^{(++)}(z)\}\) for the parity \((+, +)\) boundary condition, as

\[
A_z^{a=11} = \frac{1}{\sqrt{2\pi R_6}} \left\{ \phi_H^{a(0)}(x) u_H(z) + \sum_{n=1}^{\infty} \phi_H^{a(n)}(x) h_n^{(++)}(z) \right\} \quad (a = 1, 2, 3, 4),
\]

where the zero mode function in the fifth dimension is given by

\[
u_H(z) = \left[ \frac{3}{k(z_L^2 - 1)} \right]^{1/2} z^2.
\]

The four-component real field \(\phi_H^{a(0)}(x)\) is identified with the EW Higgs doublet field in the SM. We will show later that the \(\phi_H := \phi_H^{4(0)}\) dynamically obtain the nonvanishing VEV

\[
\langle \phi_H \rangle \neq 0.
\]

The VEV breaks \(G_{SM}\) to \(SU(3)_C \times U(1)_{EM}\). \(\phi_H^{1,2,3(0)}\) are absorbed by \(W\) and \(Z\) bosons.

Under a general gauge transformation \(A'_M = \Omega A_M \Omega^{-1} + (i/g) \Omega \partial_M \Omega^{-1}\), new gauge potentials satisfy the following new BCs:

\[
P_j' = \Omega (x, y_j - y, v_j - v) P_j (x, y_j + y, v_j + v)\bar{P}_j (x, y_j - y, v_j - v)\Omega^{-1} (x, y_j + y, v_j + v)^{-1}.
\]

It is convenient to work in the twisted gauge discussed, e.g., in Ref. [35]. With the following large gauge transformation \(\Omega(y; \alpha)\), \(\hat{\theta}_H\) is transformed to \(\hat{\theta}_H = 0:\)

\[
\Omega(y) = \exp \left\{ \frac{i}{\sqrt{4\pi R_6}} \int_y^L dy \bar{u}_H(y) T_{4,11} \right\} = \exp \{i\hat{\theta}(z) T_{4,11} \}.
\]
\[ \theta(z) = \theta_H \frac{z^3_L - z^3}{z^3_L - 1} \text{ for } 1 \leq z \leq z_L. \] 

The new BC matrices \( P_j' \) are given by

\[ \tilde{P}_j = \Omega(0)^2 P_j = e^{2i \theta_H T_{4,11}} P_j \quad (j = 0, 2), \]
\[ \tilde{P}_k = P_k \quad (k = 1, 3). \] (2.36)

For fields in the \( SO(11) \) vector and spinor representations, the BC matrices are given by

\[ \tilde{P}_{0vec} = \begin{cases} \begin{pmatrix} \cos \theta_H & -\sin \theta_H \\ -\sin \theta_H & -\cos \theta_H \end{pmatrix} & \text{in the } 4 - 11 \text{ subspace,} \\ I_3 & \text{in the } 1, 2, 3 \text{ subspace,} \\ -I_6 & \text{in the } 5, \cdots, 10 \text{ subspace,} \end{cases} \]

\[ \tilde{P}_{2vec} = \begin{cases} \begin{pmatrix} \cos \theta_H & -\sin \theta_H \\ -\sin \theta_H & -\cos \theta_H \end{pmatrix} & \text{in the } 4 - 11 \text{ subspace,} \\ I_9 & \text{otherwise,} \end{cases} \]

\[ \tilde{P}_{0sp} = \begin{cases} \left( \pm \cos \frac{\theta_H}{2} - i \sin \frac{\theta_H}{2}, -i \sin \frac{\theta_H}{2} \mp \cos \frac{\theta_H}{2} \right) & \text{for } \begin{cases} \psi \\ \hat{\psi} \end{cases}, \\ \end{cases} \]

\[ \tilde{P}_{2sp} = \begin{cases} \pm i \cos \frac{\theta_H}{2} - \cos \frac{\theta_H}{2} & \text{for } \begin{cases} \psi \\ \hat{\psi} \end{cases}, \\ \end{cases} \] (2.37)

where

\[ \psi = \begin{pmatrix} \nu \\ \nu' \end{pmatrix}, \begin{pmatrix} e \\ e' \end{pmatrix}, \begin{pmatrix} u_j \\ u'_j \end{pmatrix}, \begin{pmatrix} d_j \\ d'_j \end{pmatrix}, \]
\[ \hat{\psi} = \begin{pmatrix} \hat{\nu} \\ \hat{\nu}' \end{pmatrix}, \begin{pmatrix} \hat{e} \\ \hat{e}' \end{pmatrix}, \begin{pmatrix} \hat{u}_j \\ \hat{u}'_j \end{pmatrix}, \begin{pmatrix} \hat{d}_j \\ \hat{d}'_j \end{pmatrix}. \] (2.38)

### 3 Mass spectrum of bosons

In this section we examine the spectrum of gauge fields, particularly for 6th-dimensional \( n = 0 \) KK modes (\( \nu \)-independent modes) because we are interested in the mass spectrum of the SM particles and the effective potential \( V_{\text{eff}}(\theta_H) \). In the following we omit the modes which are odd under the 6th dimensional loop translation \( U_6 = P_2 P_0 = P_3 P_1 = -1 \). Those modes have masses \( \geq \frac{1}{2}m_{\text{KK}} \).
| No. of generators | \( A_\mu \) | \( A_2 \) | \( A_\nu \) |
|-------------------|------------|------------|------------|
| (1) \( G_{SM} \)  | 12         | \((N\ N)\) | \((D\ D)\) | \((D\ D)\) |
| (2) \( SU(5)/G_{SM} \) | 12 | \((N\ N)\) | \((D\ D)\) | \((D\ D)\) |
| (3) \( G_{PS}/G_{SM} \) | 9 | \((D_{eff}\ N)\) | \((D\ D)\) | \((D\ D)\) |
| (4) \( SO(10)/(SU(5) \cup G_{PS}) \) | 12 | \((D_{eff}\ N)\) | \((D\ D)\) | \((D\ D)\) |
| (5) \( SO(5)/SO(4) \) | 4 | \((D\ D)\) | \((N\ N)\) | \((D_{eff}\ N)\) |
| (6) \( SO(7)/SO(6) \) | 6 | \((D\ D)\) | \((N\ N)\) | \((D_{eff}\ N)\) |

Table 5: The Venn diagram for symmetry breaking pattern and the BCs \( \left( \begin{array}{cc} P_2 & P_3 \\ P_0 & P_1 \end{array} \right) \) of the \( SO(11) \) gauge field are summarized. \( N \) and \( D \) stand for Neumann and Dirichlet conditions, respectively. \( D_{eff} \) represents the effective Dirichlet condition given by (3.3).
The BCs for gauge fields in the absence of the brane terms are given by

\[
\begin{align*}
N : & \quad \frac{\partial}{\partial z} A_\mu = 0 \quad \text{for parity} = +, \\
D : & \quad A_\mu = 0 \quad \text{for parity} = -,
\end{align*}
\]

\[
\begin{align*}
N : & \quad \frac{\partial}{\partial z} \left( \frac{1}{z} A_z \right) = 0 \quad \text{for parity} = +, \\
D : & \quad A_z = 0 \quad \text{for parity} = -,
\end{align*}
\]

\[
\begin{align*}
N : & \quad \frac{\partial}{\partial z} A_v = 0 \quad \text{for parity} = +, \\
D : & \quad A_v = 0 \quad \text{for parity} = -,
\end{align*}
\]

\[ (3.1) \]

at \( z = 1 \) (\( y = 0 \)) and \( z = z_L \) (\( y = L \)). In the presence of the brane mass terms \( |g A_\mu (\Phi_{32})|^2 \) on the UV brane, the Neumann BC \( N \) is modified to an effective Dirichlet BC \( D_{\text{eff}} \) for the 5th dimensional zero mode of the \( SO(11)/SU(5) \) components of \( A_\mu \) and \( A_v \). \[35\]

The equation of motion for \( A^a_\mu \) in the \( y \)-coordinate is given in the form

\[
\left( \Box + e^{\sigma(y)} \frac{\partial}{\partial y} e^{-3\sigma(y)} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial v^2} - \frac{g^2 w^2}{2} \delta(y) \right) A^a_\mu = \left( 1 - \frac{1}{\xi} \right) \partial_\mu (\partial^\nu A^a_\nu + \partial_v A^a_v) - \cdots \quad (3.2)
\]

where the right-hand side involves interaction terms. Suppose that \( A^a_\mu \) is parity even, namely \( A^a_\mu(x, -y, -v) = +A^a_\mu(x, y, v) \). Note that \( A^a_v \) is parity odd so that \( \partial_v A^a_v \) is parity even. By integrating the equation \( \int_{-\epsilon}^\epsilon dy \cdots \) and taking the limit \( \epsilon \to 0 \), one finds \( \partial A^a_\mu / \partial y \big|_{y=\epsilon} = (g^2 w^2 / 4) A^a_\mu \big|_{y=0} \). In the \( z \) coordinate the Neumann condition is changed to the effective Dirichlet condition

\[
D_{\text{eff}}(\omega) : \left( \frac{\partial}{\partial z} - \omega \right) A^a_\mu = 0, \quad \omega = \frac{g^2 w^2}{4k}, \quad \text{at} \quad z = 1^+.
\]

(3.3)

Note that in the current six-dimensional model the mass dimensions of \( g \) and \( w \) are \([g] = M^{-1}, [w] = M^{3/2}\) so that \( \omega \) is dimensionless. When \( \omega \neq 0 \), \( A^a_\mu \) develops a cusp at \( y = 0 \). The lowest mode of \( A^a_\mu \) becomes massive, whose mass is \( O(m_{\text{KK}5}) \). The value of \( \omega \) depends on the brane mass terms. The BCs for the gauge fields are summarized in Table 5.

In the twisted gauge \( \tilde{A}_M = \Omega(z) A_M \Omega(z)^{-1} + i g \Omega(z) \partial_M \Omega(z)^{-1} \) where \( \Omega(z) = e^{i\theta(z) T_{4,11}} \). \( \theta(z) \) is defined in \((2.35)\). The \((k, 4), (k, 11), \) and \((4, 11) \) components of the \( SO(11) \) gauge fields are changed as

\[
A^k_4 = \cos \theta(z) \tilde{A}^k_4 - \sin \theta(z) \tilde{A}^{k,11}_M \quad (k \neq 4, 11),
\]

\[
A^k_{11} = \sin \theta(z) \tilde{A}^k_4 + \cos \theta(z) \tilde{A}^{k,11}_M,
\]

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while the other components remain unchanged. As in Ref. [35], the BCs at \( z = z_L \) determine wave functions for \( \tilde{A}_\mu, \tilde{A}_z, \) and \( \tilde{A}_v \):

\[
\begin{align*}
\tilde{A}_\mu &\quad N: C(z; \lambda), \quad D: S(z; \lambda), \\
\tilde{A}_z &\quad N: S'(z; \lambda), \quad D: C'(z; \lambda), \\
\tilde{A}_v &\quad N: S(z; \lambda), \quad D: C'(z; \lambda),
\end{align*}
\]

(3.5)

where \( C(z; \lambda) \) and \( S(z; \lambda) \) are defined in (A.2) and (A.3).

From the BCs for the gauge fields summarized in Table 5, the components (2) \( SU(5)/G_{SM} \), (4) \( SO(10)/(SU(5)\cup G_{PS}) \), and (6) \( SO(7)/SO(6) \) are parity odd under the 6th dimensional loop translation \( U_6 \), while the components (1) \( G_{SM} \), (3) \( G_{PS}/G_{SM} \), and (5) \( SO(5)/SO(4) \) are parity even under \( U_6 \). Thus, we find that the low-lying modes of the components (2), (4), (6) of \( A_M \) have masses of \( O(m_{KK_6}) \), while the low-lying modes of the components (1), (3), and (5) of \( A_M \) have masses less than or around \( O(m_{KK_5}) \).

We summarize the formulas determining the mass spectrum of 6th dimensional \( n = 0 \) modes of \( A_M \), for which the arguments in Ref. [35] remain intact. We use the same notation as in Ref. [35]. \( A^{a_L}_M \) and \( A^{a_R}_M \) \( (a_L, a_R = 1, 2, 3) \) stand for \( SU(2)_L \) and \( SU(2)_R \) components of \( A_M \), respectively.

- \( A_\mu \) components.

(i) \( (\tilde{A}^{a_L}_\mu, \tilde{A}^{a_R}_\mu, \tilde{A}^{a, 11}_\mu) \) \( (a = 1, 2) \): For \( W \) and \( W_R \) towers, there are 6th dimensional \( n = 0 \) modes, and there are also 5th dimensional zero modes in the absence of brane terms. In the presence of the brane terms in (2.30), their mass spectra of the 5th dimensional modes are given by

\[
2C'(SC' + \lambda \sin^2 \theta_H) - \omega C(2SC' + \lambda \sin^2 \theta_H) = 0,
\]

(3.6)

where \( \omega := g^2 w^2/4k \). Let us denote the average magnitude of \( C(1; \lambda) \) and \( C'(1; \lambda) \) by \( \overline{C} \) and \( \overline{C}' \), respectively. When \( \omega \overline{C} \gg \overline{C}' \), the spectrum is determined by

\[
\begin{align*}
W \text{ tower: } 2S(1; \lambda)C'(1; \lambda) + \lambda \sin^2 \theta_H &= 0, \\
W_R \text{ tower: } C(1; \lambda) &= 0.
\end{align*}
\]

(3.7)
Indeed, in typical cases examined below, $\bar{C}/\bar{C}' \sim 10^6$ so that the condition $\omega \bar{C} \gg \bar{C}'$ is well satisfied for $w \gg (m_{\text{KK}})^{3/2}$. The mass of $W$ boson $m_W = m_{W(0)}$ is given by

$$m_W \simeq \sqrt{\frac{3}{2}} k z_L^{-3/2} \sin \theta_H = \frac{\sqrt{3} \sin \theta_H}{\sqrt{2 \pi} \sqrt{z_L}} m_{\text{KK}},$$

(3.8)

(ii) $\tilde{A}_\mu^3, \tilde{C}_\mu, \tilde{B}_\mu^Y, \tilde{A}_\mu^{3,11}$: For $\gamma$, $Z$ and $Z_R$ towers, there are 6th dimensional $n = 0$ modes, and there are also 5th dimensional zero modes in the absence of brane terms. Here $C_\mu = \sqrt{2/5} A_\mu^{3 R} + \sqrt{3/5} A_\mu^{0 C}$ and $B_\mu^Y = \sqrt{3/5} A_\mu^{3 R} - \sqrt{2/5} A_\mu^{0 C}$, where $A_\mu^{0 C} = (A_\mu^{56} + A_\mu^{78} + A_\mu^{10})/\sqrt{3}$. In the presence of the brane terms in (2.30), the mass spectra of the 5th dimensional modes are given by

$$C' \{ 2C'(SC' + \lambda \sin^2 \theta_H) - \omega C(5SC' + 4\lambda \sin^2 \theta_H) \} = 0.$$  

(3.9)

For $\omega \bar{C} \gg \bar{C}'$

$$\gamma \text{ tower: } C'(1; \lambda) = 0,$$
$$Z \text{ tower: } 5S(1; \lambda) C'(1; \lambda) + 4\lambda \sin^2 \theta_H = 0,$$
$$Z_R \text{ tower: } C(1; \lambda) = 0.$$  

(3.10)

The mass of $Z$ boson $m_Z = m_{Z(0)}$ is given by

$$m_Z \simeq \sqrt{\frac{12}{5}} k z_L^{-3/2} \sin \theta_H = \frac{m_W}{\cos \theta_W}, \quad \sin^2 \theta_W = \frac{3}{8}.$$  

(3.11)

The relation for the $Z$ tower in (3.10) can be written, by using $\cos^2 \theta_W = \frac{5}{8}$, as

$$Z \text{ tower: } 2S(1; \lambda) C'(1; \lambda) + \frac{\lambda}{\cos^2 \theta_W} \sin^2 \theta_H = 0.$$  

(3.12)

The value $\sin^2 \theta_W = \frac{3}{8}$ is valid at the GUT scale. The gauge coupling constants evolve as the energy scale, following the renormalization group equation (RGE).

It is not clear whether $\sin^2 \theta_W$ evolves to the observed value at low energies as there are KK modes which do not respect $SU(5)$ below the GUT scale. We leave solving the RGE for future investigation. When we evaluate the effective potential $V_{\text{eff}}(\theta_H)$ at the EW scale in Section 5, we adopt the formula (3.12) where the observed value, $\sin^2 \theta_W \simeq 0.2312$, at the EW scale is inserted.
(iii) $\tilde{A}^{4,11}_\mu$: For $\tilde{A}^4$ tower, there is 6th dimensional $n = 0$ modes, but there is no 5th dimensional zero modes. The mass spectra of the 5th dimensional modes is given by

$$\tilde{A}^4 \text{ tower: } S(1; \lambda) = 0.$$  \hfill (3.13)

(iv) For $SU(3)_C$ gluons, there are 6th dimensional $n = 0$ modes, and there are also 5th dimensional zero modes which correspond to 4d gluons. The mass spectrum of the 5th dimensional massive modes is given by

$$\text{gluon tower: } C'(1; \lambda) = 0.$$  \hfill (3.14)

(v) For $X$-gluons, there are 6th dimensional $n = 0$ modes, and there are also 5th dimensional zero modes in the absence of brane terms. In the presence of the brane terms in $\text{(2.30)}$, the mass spectrum of the 5th dimensional modes is given by

$$X\text{-gluon tower: } C''(1; \lambda) - \omega C(1; \lambda) = 0.$$  \hfill (3.15)

For $\omega \bar{C} \gg \bar{C}^\prime$

$$X\text{-gluon tower: } C(1; \lambda) \simeq 0.$$  \hfill (3.16)

(vi) For $X$-bosons, there are no 6th dimensional $n = 0$ modes.

(vii) For $X'$-bosons, there are no 6th dimensional $n = 0$ modes.

(viii) For $Y$, $Y'$-bosons, there are no 6th dimensional $n = 0$ modes.

• $A_z$ components.

(i) $A^{ab}_z$ ($1 \leq a < b \leq 3$) and $A^{jk}_z$ ($5 \leq j < k \leq 10$): There are 6th dimensional $n = 0$ modes, but there are no 5th dimensional zero modes. Their mass spectrum of the 5th dimensional modes is given by

$$C'(1; \lambda) = 0.$$  \hfill (3.17)

(ii) $A^{a4}_z, A^{a11}_z$ ($a = 1, 2, 3$): There are 6th dimensional $n = 0$ modes for $A^{a4}_z, A^{a11}_z$ ($a = 1, 2, 3$). There are no 5th dimensional zero modes for $A^{a4}_z$ while there are 5th dimensional zero modes for $A^{a11}_z$. Their mass spectra of the 5th dimensional modes are given by

$$S(1; \lambda)C'(1; \lambda) + \lambda \sin^2 \theta_H = 0.$$  \hfill (3.18)
(iii) $A_{4}^{11}$: Higgs tower. There are 6th dimensional $n = 0$ modes, and there is also a 5th dimensional zero mode. The mass spectrum of the 5th dimensional modes is given by

\[
Higgs \text{ tower: } S(1; \lambda) = 0. 
\] (3.19)

(iv) $A_{z}^{ak}(a = 1, 2, 3, k = 5, ..., 10)$: There are no 6th dimensional $n = 0$ modes.

(v) $A_{z}^{k4}, A_{z}^{k11}$ $(k = 5, ..., 10)$: There are no 6th dimensional $n = 0$ modes.

- $A_{v}$ components.

(i) $A_{v}^{ab}$ $(1 \leq a < b \leq 3)$ and $A_{v}^{jk}$ $(5 \leq j < k \leq 10)$: There are 6th dimensional $n = 0$ modes, but there are no 5th dimensional zero modes. Their mass spectra of the 5th dimensional modes are given by

\[
C'(1; \lambda) = 0. 
\] (3.20)

(ii) $A_{v}^{a4}, A_{v}^{a11}(a = 1, 2, 3)$: There are 6th dimensional $n = 0$ modes for $A_{v}^{a4}$ and $A_{v}^{a11}(a = 1, 2, 3)$. There are no 5th dimensional zero modes for $A_{v}^{a4}$ while there are 5th dimensional zero modes for $A_{v}^{a11}(a = 1, 2, 3)$ in the absence of brane terms. In the presence of the brane terms in (2.31), one find that at $z = 1$,

\[
\begin{pmatrix}
\cos \theta_{H}C' & -\sin \theta_{H}S' \\
\sin \theta_{H}(C - \omega C') & \cos \theta_{H}(S - \omega S')
\end{pmatrix}
\begin{pmatrix}
\beta_{a4}^{v} \\
\beta_{a11}^{v}
\end{pmatrix}
= 0. 
\] (3.21)

Thus, their mass spectra of the 5th dimensional modes are given by

\[
C'(1; \lambda) \{S(1; \lambda) - \omega S'(1; \lambda)\} + \lambda \sin^{2} \theta_{H} = 0. 
\] (3.22)

The average magnitude of $S(1; \lambda)$ ($= \overline{S}$) is much larger than that of $S'(1; \lambda)$ ($= \overline{S'}$). In typical cases $\overline{S}/\overline{S'} \sim 10^{4}$. Hence the spectrum is determined by

\[
C'(1; \lambda)S(1; \lambda) + \lambda \sin^{2} \theta_{H} = 0. 
\] (3.23)

(iii) $A_{v}^{4,11}$: $A_{v}$-Higgs tower. There are 6th dimensional $n = 0$ modes, and there are also 5th dimensional zero modes in the absence of brane terms. In the presence of the brane terms in (2.31), their mass spectra of the 5th dimensional modes are given by

\[
A_{v}\text{-Higgs tower: } S(1; \lambda) - \omega S'(1; \lambda) = 0, 
\] (3.24)
which is well approximated by

\[ A_v \text{-Higgs tower: } S(1; \lambda) = 0. \] (3.25)

The components \( A_{v}^{a11} (a = 1 \sim 4) \) acquire large 1 loop corrections of order \( g_w M_{KK6} \) to their masses so that their contributions to the \( \theta_H \)-dependent part of \( V_{\text{eff}}(\theta_H) \) become negligible and may be dropped.

(iv) \( A_{v}^{a k} (a = 1, 2, 3, k = 5, \ldots, 10) \): There are no 6th dimensional \( n = 0 \) modes.

(v) \( A_{v}^{k4}, A_{v}^{k11} (k = 5, \ldots, 10) \): There are no 6th dimensional \( n = 0 \) modes.

4 Mass spectrum of fermions

In this section we determine the mass spectrum of quarks and leptons. For up-type quarks, there are no brane interactions on the UV brane \((y = 0)\) as in the 5D \( SO(11) \) GHGUT. For down-type quarks, charged leptons, and neutrinos, both 6D bulk mass terms and brane mass terms on the UV brane are important to reproduce the observed mass spectra.

In the following, we will calculate mass spectra for one set of a 6D \( SO(11) \) \( 32 \) Weyl fermion \( \Psi_{32}^{\alpha} (\alpha = 1, 2 \text{ or } 3) \), 6D \( SO(11) \) \( 11 \) Dirac fermions \( \Psi_{11}^\beta \) and \( \Psi_{11}^{\beta} (\beta = \alpha) \), and 5D \( SO(11) \) singlet \( \chi_{1}^{\alpha} \), which is identified with one generation of the SM quarks and leptons. We will also discuss mass spectra for the 6D \( SO(11) \) \( 32 \) Weyl fermion \( \Psi_{32}^{\alpha=4} \) denoted by \( \Psi_{32}' \). We will omit the superscripts \( \alpha \) and \( \beta \). We are interested in EW scale physics and we keep only 6th dimensional \( n = 0 \) KK modes because the masses of 6th dimensional \( n \neq 0 \) modes are much larger than \( O(m_{KK5}) \) as \( m_{KK6} \gg m_{KK5} \). That is, we will discuss mass spectra for the \( G_{PS} (2, 1, 4) \) and \( (1, 2, 4) \) components of \( \Psi_{32} \), the \( G_{PS} (1, 1, 6) \) components of \( \Psi_{11} \), the \( G_{PS} (2, 2, 1) \) and \( (1, 1, 1) \) components of \( \Psi_{11}' \), and the \( G_{PS} (2, 1, 4) \) and \( (1, 2, 4) \) components of \( \Psi_{32}' \). We will denote \( c_{\Psi_{32}}^{\alpha \neq 4}, c_{\Psi_{11}}^{\beta}, c_{\Psi_{11}'}^{\beta}, \) and \( c_{\Psi_{32}'}^{\alpha=4} \) as \( c_0, c_1, c_2, \) and \( c'_0 \), respectively. In this paper we assume that \( c_0, c_1, c'_0 \geq 0 \) while \( c_2 \) can be negative. The calculation method employed in the following is the same as the one employed in the 5D \( SO(11) \) GHGUT discussed in Ref. [35].
4.1 Up-type quark

(i) $Q_{EM} = +2/3$: $u_j, u'_j$ ($\Psi_{32}$)

From the action (2.19), we find the equations of motion for up-type quarks with $\gamma^7_{6D} = +1$:

$$-i\delta \left( \begin{array}{c} u^+_L \\ u^+_R \end{array} \right): \left( -k \hat{D}_-(c_0) + i\partial_v \right) \left( \begin{array}{c} \tilde{u}^-_R \\ \tilde{u}^-_L \end{array} \right) + \sigma^\mu \partial_\mu \left( \begin{array}{c} \tilde{u}^+_L \\ \tilde{u}^+_R \end{array} \right) = 0,$$

$$i\delta \left( \begin{array}{c} u^+_R \\ u^+_L \end{array} \right): \sigma^\mu \partial_\mu \left( \begin{array}{c} \tilde{u}^+_R \\ \tilde{u}^+_L \end{array} \right) + \left( -k \hat{D}_+(c_0) + i\partial_v \right) \left( \begin{array}{c} \tilde{u}^+_L \\ \tilde{u}^+_R \end{array} \right) = 0. \quad (4.1)$$

Here $\sigma^\mu = (I_2, \tilde{\sigma})$, $\sigma^\mu = (-I_2, \tilde{\sigma})$, and

$$\hat{D}_\pm(c) = \pm \left( \frac{\partial}{\partial z} + i\theta'(z) T_{4,11} \right) + \frac{c}{z},$$

$$T_{4,11} = \begin{cases} \frac{1}{2} \tau_1 & \text{for } \psi, \\
-\frac{1}{2} \tau_1 & \text{for } \hat{\psi}, \end{cases} \quad (4.2)$$

where $\psi, \hat{\psi}$ are the pairs defined in (2.38). For up-type quark with $\gamma^7_{6D} = -1$, namely for the top quark, the equations become

$$\left( k \hat{D}_+(c_0) - i\partial_v \right) \left( \begin{array}{c} \tilde{u}^-_R \\ \tilde{u}^-_L \end{array} \right) + \sigma^\mu \partial_\mu \left( \begin{array}{c} \tilde{u}^+_L \\ \tilde{u}^+_R \end{array} \right) = 0,$$

$$\overline{\sigma}^\mu \partial_\mu \left( \begin{array}{c} \tilde{u}^-_R \\ \tilde{u}^-_L \end{array} \right) + \left( k \hat{D}_-(c_0) - i\partial_v \right) \left( \begin{array}{c} \tilde{u}^+_L \\ \tilde{u}^+_R \end{array} \right) = 0. \quad (4.3)$$

In the twisted gauge, the equations of motion become

$$\left\{ \begin{array}{l} \sigma^\mu \partial_\mu \tilde{u}^+_R = (kD_+(c_0) - i\partial_v) \tilde{u}^+_L, \\
\sigma^\mu \partial_\mu \tilde{u}^-_L = (kD_-(-c_0) - i\partial_v) \tilde{u}^-_R, \\
\sigma^\mu \partial_\mu \tilde{u}^-_R = (-kD_-(c_0) + i\partial_v) \tilde{u}^-_L, \\
\sigma^\mu \partial_\mu \tilde{u}^+_L = (-kD_+(c_0) + i\partial_v) \tilde{u}^+_R, \end{array} \right\} \qquad (4.4)$$

where

$$D_\pm(c) = \pm \frac{\partial}{\partial z} + \frac{c}{z} \quad (4.5)$$

and the relation between the twisted gauge and original one is given by

$$\left( \begin{array}{c} u \\ u' \end{array} \right) = \left( \begin{array}{cc} \cos \frac{1}{2} \theta(z) & -i \sin \frac{1}{2} \theta(z) \\ -i \sin \frac{1}{2} \theta(z) & \cos \frac{1}{2} \theta(z) \end{array} \right) \left( \begin{array}{c} \tilde{u} \\ \tilde{u}' \end{array} \right) =: \Omega(z)^{-1} \left( \begin{array}{c} \tilde{u} \\ \tilde{u}' \end{array} \right). \quad (4.6)$$

Let us check KK mode expansions for $u_L$ and $u_R$. Since $u_L$ has the parity assignment

$$\left( \begin{array}{cc} P_2 & P_3 \\ P_0 & P_1 \end{array} \right) = \left( \begin{array}{cc} + & + \\ + & + \end{array} \right),$$

we can expand it by using parity even combinations:

$$u_L(x, y, v) = \sum_{n=0}^{\infty} u^C_{nL}(x, y) f^C_n(v) + \sum_{n=1}^{\infty} u^S_{nL}(x, y) f^S_n(v)$$
They have the following BCs at \((y_0, y_1) = (0, L_5)\):

\[
\begin{align*}
\begin{cases}
    u_{0L}(x, y_j - y) &= +u_{0L}(x, y_j + y) \\
    u_{0R}(x, y_j - y) &= -u_{0R}(x, y_j + y) \\
    u'_{0L}(x, y_j - y) &= -u'_{0L}(x, y_j + y) \\
    u'_{0R}(x, y_j - y) &= +u'_{0R}(x, y_j + y)
\end{cases}
\end{align*}
\]
where \( j = 0, 1 \). The equations of motion in the twisted gauge \([4.6]\) for the \( \gamma_{6D} = +1 \) fields become

\[
\begin{align*}
\{ \bar{\sigma}^\mu \partial_\mu \tilde{u}_{+0R} & = +kD_+(c_0)\tilde{u}_{+0L} , \\
\sigma^\mu \partial_\mu \tilde{u}_{+0L} & = +kD_-(c_0)\tilde{u}_{+0R} , \\
\{ \bar{\sigma}^\mu \partial_\mu \tilde{u}_{+0L}' & = +kD_+(c_0)\tilde{u}_{+0R}' , \\
\sigma^\mu \partial_\mu \tilde{u}_{+0R}' & = +kD_-(c_0)\tilde{u}_{+0L}' . \\
\end{align*}
\]

(4.13)

It follows that \( k^2D_+D_-\tilde{u}_{+0R} = \Box \tilde{u}_{+0R} = m^2\tilde{u}_{+0R} \) etc. The BCs at \( z = z_L \) in the twisted gauge are the same as those in the original gauge:

\[
\tilde{u}_{+0R} = 0, \quad D_+\tilde{u}_{+0L} = 0, \quad \tilde{u}_{+0L}' = 0, \quad D_-\tilde{u}_{+0R}' = 0 .
\]

(4.14)

In terms of the basis functions \( C_{R/L}(z; \lambda, c) \), \( S_{R/L}(z; \lambda, c) \) given in Appendix B of Ref. [35], which satisfy

\[
\begin{align*}
D_+(c) \begin{pmatrix} C_L \\ S_L \end{pmatrix} & = \lambda \begin{pmatrix} S_R \\ C_R \end{pmatrix} , \\
D_-(c) \begin{pmatrix} C_R \\ S_R \end{pmatrix} & = \lambda \begin{pmatrix} S_L \\ C_L \end{pmatrix} , \\
C_R & = C_L = 1 , \quad S_R = S_L = 0 \quad \text{at} \quad z = z_L ;
\end{align*}
\]

(4.15)

one can write the mode functions for \( \gamma_{6D} = +1 \) as

\[
\begin{align*}
\begin{pmatrix} \tilde{u}_{+0R} \\ \tilde{u}_{+0R}' \end{pmatrix} & = \begin{pmatrix} \alpha_R^a S_R(z; \lambda, c_0) \\ \alpha_R^a C_R(z; \lambda, c_0) \end{pmatrix} f_R(x) , \quad \bar{\sigma}^\mu \partial_\mu f_R(x) = mf_L(x) , \\
\begin{pmatrix} \tilde{u}_{+0L} \\ \tilde{u}_{+0L}' \end{pmatrix} & = \begin{pmatrix} \alpha_L^a C_L(z; \lambda, c_0) \\ \alpha_L^a S_L(z; \lambda, c_0) \end{pmatrix} f_L(x) , \quad \sigma^\mu \partial_\mu f_L(x) = mf_R(x) .
\end{align*}
\]

(4.16)

The equations of motion \([4.13]\) in the bulk \( 0 < y < L \) \( (1 < z < z_L) \) yield

\[
\begin{pmatrix} -k\lambda & m \\ m & -k\lambda \end{pmatrix} \begin{pmatrix} \alpha_R^a \\ \alpha_L^a \end{pmatrix} = 0 , \quad \text{etc.}
\]

(4.17)

so that one finds

\[
m = k\lambda .
\]

(4.18)

The BCs at \( z = 1 \) in the twisted gauge follow from those in the original gauge:

\[
\begin{align*}
u_{+0R} = 0 & \Rightarrow \tilde{u}_{+0R} \cos \frac{\theta_H}{2} - i\tilde{u}_{+0R}' \sin \frac{\theta_H}{2} = 0 , \\
D_-u_{+0R}' = 0 & \Rightarrow -iD_-\tilde{u}_{+0R} \sin \frac{\theta_H}{2} + D_-\tilde{u}_{+0R}' \cos \frac{\theta_H}{2} = 0 .
\end{align*}
\]

(4.19)

(4.20)
We write the above equations in the matrix form:

\[
M_u \begin{pmatrix} \alpha_R^u \\ \alpha_R^u \end{pmatrix} := \begin{pmatrix} \cos \frac{\theta_H}{2} S_R^0 & -i \sin \frac{\theta_H}{2} C_R^0 \\ -i \sin \frac{\theta_H}{2} \lambda C_L^0 & \cos \frac{\theta_H}{2} S_L^0 \end{pmatrix} \begin{pmatrix} \alpha_R^u \\ \alpha_R^u \end{pmatrix} = 0 ,
\]

where we have used short-hand notation such as \( S_R^0 \) standing for \( S_L^0(1; \lambda, c_0) \). In the following we will use the same notation. From \( \det M_u = 0 \), we find the mass formula of the up-type quarks

\[
S_L^0 S_R^0 + \sin^2 \frac{\theta_H}{2} = 0 .
\]  

The same formula holds for \( \Psi_{32} \) with \( \gamma_{6D} = -1 \). The formula (4.22) is the same as in the 5D \( \text{SO}(11) \) GHGUT in Ref. [35]. For \( \lambda z_L \ll 1 \), the up-type quark mass spectrum is given by

\[
m_u \simeq \begin{cases} 
  k z_{L}^{-1} \sqrt{1 - 4c_0^2} \sin \frac{1}{2} \theta_H & \text{for } c_0 < \frac{1}{2} \\
  k z_{L}^{-1/2-c_0} \sqrt{4c_0^2 - 1} \sin \frac{1}{2} \theta_H & \text{for } c_0 > \frac{1}{2} 
\end{cases}.
\]

4.2 Down-type quark

(ii) \( Q_{EM} = -\frac{1}{3} \): \( d_j', d_j, D_j, D_{j-} \) (\( \Psi_{32}, \Psi_{11} \))

Consider \( \Psi_{32} \) with \( \gamma_{6D} = +1 \). Parity even modes at \( y = 0 \) with \( (P_0, P_2) = (+, +) \) are \( d_{+L}, d_{+R}, D_{+R} \) and \( D_{-L} \). From the action (2.19) and the \( \mathcal{L}_1^m \) and \( \mathcal{L}_3^m \) terms in (2.28), one finds the equations of motion for down-type quarks:

\[
-i \delta \begin{pmatrix} \frac{d_{+L}}{d_{+L}} \\ \frac{d_{+R}}{d_{+R}} \end{pmatrix} : \left( -k \hat{D}_{+L}(c_0) + i \partial_v \right) \left( \frac{d_{+R}}{d_{+R}} \right) + \sigma^\mu \partial_\mu \left( \frac{d_{+L}}{d_{+L}} \right) = 0 ,
\]

\[
i \delta \begin{pmatrix} \frac{d_{+L}}{d_{+L}} \\ \frac{d_{+R}}{d_{+R}} \end{pmatrix} : \sigma^\mu \partial_\mu \left( \frac{d_{+R}}{d_{+R}} \right) + \left( -k \hat{D}_{+R}(c_0) + i \partial_v \right) \left( \frac{d_{+L}}{d_{+L}} \right) = 2 \mu_1 \delta(y) \left( 0 \right) \hat{D}_{-L} ,
\]

\[
-i \delta \hat{D}_{+L} : \left( -k \hat{D}_{+L}(c_1) + i \partial_v \right) \hat{D}_{+R} + \sigma^\mu \partial_\mu \hat{D}_{+L} = 0 ,
\]

\[
i \delta \hat{D}_{+R} : \sigma^\mu \partial_\mu \hat{D}_{+R} + \left( -k \hat{D}_{+R}(c_1) + i \partial_v \right) \hat{D}_{+L} = 2 \mu_{11} \delta(y) \hat{D}_{-L} ,
\]

\[
-i \delta \hat{D}_{-L} : \left( k \hat{D}_{-L}(c_1) - i \partial_v \right) \hat{D}_{-R} + \sigma^\mu \partial_\mu \hat{D}_{-L} = \delta(y) \left\{ 2 \mu_{11} \hat{D}_{-R} + 2 \mu_1 \hat{d}_{+R} \right\} ,
\]

\[
i \delta \hat{D}_{-R} : \sigma^\mu \partial_\mu \hat{D}_{-R} + \left( k \hat{D}_{-R}(c_1) - i \partial_v \right) \hat{D}_{-L} = 0 .
\]  

(4.24)
For 6th dimensional \( n = 0 \) KK modes, the equations of motion reduce to

\[
\begin{align*}
(a) & \quad - k \ddot{D}_-(c_0) \begin{pmatrix} \dddot{d}_{+R} \\ \dddot{d}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \dddot{d}_{+L} \\ \dddot{d}'_{+L} \end{pmatrix} = 0, \\
(b) & \quad \sigma^\mu \partial_\mu \begin{pmatrix} \dddot{d}_{+R} \\ \dddot{d}'_{+R} \end{pmatrix} = 0, \\
(c) & \quad \sigma^\mu \partial_\mu \begin{pmatrix} \dddot{d}_{+R} \\ \dddot{d}'_{+R} \end{pmatrix} - k \dddot{D}_+(c_0) \begin{pmatrix} \dddot{d}_{+L} \\ \dddot{d}'_{+L} \end{pmatrix} = 2\mu_1 \delta(y) \begin{pmatrix} 0 \\ \dddot{D}_{-L} \end{pmatrix}, \\
(d) & \quad \sigma^\mu \partial_\mu \begin{pmatrix} \dddot{d}_{+R} \\ \dddot{d}'_{+R} \end{pmatrix} - k \dddot{D}_+(c_1) \dddot{d}_{+L} = 2\mu_{11} \delta(y) \dddot{D}_{-L}, \\
(e) & \quad - k \dddot{D}_-(c_1) \dddot{D}_{+R} + \sigma^\mu \partial_\mu \dddot{D}_{+L} = 0, \\
(f) & \quad \sigma^\mu \partial_\mu \dddot{D}_{-R} + k \dddot{D}_-(c_1) \dddot{D}_{-L} = 2\mu_{11} \delta(y) \dddot{D}_{+R} + 2\mu_1 \delta(y) \dddot{d}'_{+R}, \\
(g) & \quad k \dddot{D}_+(c_1) \dddot{D}_{-R} + \sigma^\mu \partial_\mu \dddot{D}_{-L} = 2\mu_{11} \delta(y) \dddot{D}_{+R} + 2\mu_1 \delta(y) \dddot{d}'_{+R}, \\
(h) & \quad \sigma^\mu \partial_\mu \dddot{D}_{-R} + k \dddot{D}_-(c_1) \dddot{D}_{-L} = 0. \quad (4.25)
\end{align*}
\]

To obtain BCs at \( y = 0 \), we integrate the above \((a), (d), (f), (g)\) in the vicinity of \( y = 0 \) for parity-odd fields:

\[
\begin{align*}
(a) & \quad \Rightarrow 2\dddot{d}_{+R}(x, \epsilon) = 0, \\
(d) & \quad \Rightarrow -2 \dddot{d}'_{+L}(x, \epsilon) = 2\mu_1 \dddot{D}_{-L}(x, 0), \\
(f) & \quad \Rightarrow -2 \dddot{D}_{+L}(x, \epsilon) = 2\mu_{11} \dddot{D}_{-L}(x, 0), \\
(g) & \quad \Rightarrow 2 \dddot{D}_{-R}(x, \epsilon) = 2\mu_{11} \dddot{D}_{+R}(x, 0) + 2\mu_1 \dddot{d}'_{+R}(x, 0). \quad (4.26)
\end{align*}
\]

For parity-even fields, we evaluate the equations of motion at \( y = +\epsilon \) by using the above conditions:

\[
\begin{align*}
(c) & \quad \Rightarrow \dddot{D}_{+} \dddot{d}_{+L}(x, \epsilon) = 0, \\
(b) & \quad \Rightarrow -k \dddot{D}_{-} \dddot{d}'_{+R} + \mu_1 k \dddot{D}_{+} \dddot{D}_{-R} = 0, \\
(e) & \quad \Rightarrow -k \dddot{D}_{-} \dddot{D}_{+R}(x, \epsilon) + \mu_{11} k \dddot{D}_{+} \dddot{D}_{-R} = 0, \\
(h) & \quad \Rightarrow k \dddot{D}_{-} \dddot{D}_{-R} + \mu_{11} k \dddot{D}_{+} \dddot{D}_{+L} + \mu_1 k \dddot{D}_{+} \dddot{d}'_{+L} = 0, \quad (4.27)
\end{align*}
\]

where we used the equations of motion \((d)\) and \((g)\) at \( y = +\epsilon \).

We recall that in the twisted gauge all fields obey free-field equations in the bulk. Their eigenmodes are determined by the BCs on the IR brane. The mass spectra can be fixed by the BCs at the UV brane. In the twisted gauge the mode functions are given by

\[
\begin{pmatrix}
\dddot{d}_{+R} \\
\dddot{d}'_{+R} \\
\dddot{D}_{+R} \\
\dddot{D}_{-R}
\end{pmatrix} = \begin{pmatrix}
\alpha_{R}^{dR} S_{R}(z; \lambda, c_0) \\
\alpha_{R}^{dR} C_{R}(z; \lambda, c_0) \\
\alpha_{R}^{D+} C_{R}(z; \lambda, c_1) \\
\alpha_{R}^{D-} S_{L}(z; \lambda, c_1)
\end{pmatrix} f_{R}(x),
\]

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where $f_R(x), f_L(x)$ satisfy the relations in (4.16) and $m = k\lambda$. The BCs at $z = 1^+$ in the twisted gauge are converted to

\[
K \begin{pmatrix}
\alpha^d_R \\
\alpha^d_R' \\
\alpha^D_R \\
\alpha^D_R'
\end{pmatrix} = 0 ,
\]

\[
K = \begin{pmatrix}
\cos \frac{\theta_H}{2} S^0_R & -i \sin \frac{\theta_H}{2} C^0_R & 0 & 0 \\
-i \sin \frac{\theta_H}{2} C^0_L & \cos \frac{\theta_H}{2} S^0_L & 0 & -\mu_1 \lambda C^1_R \\
0 & 0 & \lambda S^1_L & -\mu_{11} C^1_R \\
i\mu_1 \sin \frac{\theta_H}{2} S^0_R & -\mu_1 \cos \frac{\theta_H}{2} C^0_R & -\mu_{11} C^1_R & S^1_L
\end{pmatrix} .
\] (4.29)

From $\det K = 0$, we find the mass spectrum formula for the down-type quarks:

\[
S^0_L S^0_R + \sin^2 \frac{\theta_H}{2} = -\frac{\mu_{11}^2 S^0_R C^0_L S^1_L C^1_R}{\mu_{11}^2 (C^1_R)^2 - (S^1_L)^2} .
\] (4.30)

The same formula is obtained for $\Psi_{32}$ with $\gamma^7_{6D} = -1$. For $\lambda z_L \ll 1$, the down-type quark mass spectrum is given by

\[
m_d \simeq \begin{cases}
k z^c_0 - c_1 - 1 \frac{\mu_{11}}{\mu_1} \sqrt{(1 - 2c_0)(1 + 2c_1)} \sin \frac{\theta_H}{2} & \text{for } c_0 < \frac{1}{2} , \\
k z^{c_1 - 1/2} - c_1 - 1 \frac{\mu_{11}}{\mu_1} \sqrt{(2c_0 - 1)(2c_1 + 1)} \sin \frac{\theta_H}{2} & \text{for } c_0 > \frac{1}{2} .
\end{cases}
\] (4.31)

Combining (4.23) and (4.31), one finds

\[
\frac{m_d}{m_u} = z^c_0 - c_1 \frac{\mu_{11}}{\mu_1} \sqrt{rac{1 + 2c_1}{1 + 2c_0}} .
\] (4.32)

### 4.3 Charged lepton

(iii) $Q_{EM} = -1$: $e, e', E'_+, E'_-$ ($\Psi_{32}, \Psi'_{11}$)

Parity even modes at $y = 0$ with $(P_0, P_2) = (+, +)$ are $e_{+L}, e'_{+R}, E'_{+L}$ and $E'_{-R}$. From the action (2.19) and the $\mathcal{L}_5^a$ and $\mathcal{L}_6^a$ terms in (2.28), one finds the equations of motion for
charged leptons:

\[-i\delta \left( \frac{\tilde{e}^+_L}{\tilde{e}^+_R} \right): (- k \hat{D}_-(c_0) + i\partial_x) \left( \tilde{e}^+_R \right) + \sigma^\nu \partial_\mu \left( \tilde{e}^+_L \right) = 2i\bar{\mu}_2 \delta(y) \begin{pmatrix} \tilde{E}'_R \\ 0 \end{pmatrix},\]

\[i\delta \left( \frac{e^+_R}{e^+_R} \right): \sigma^\nu \partial_\mu \left( \tilde{e}^+_R \right) + (- k \hat{D}_+(c_0) + i\partial_x) \left( \tilde{e}^+_L \right) = 0,\]

\[-i\delta E'^{+}_{R} : (- k \hat{D}_-(c_2) + i\partial_x) \tilde{E}'_R + \sigma^\nu \partial_\mu \tilde{E}'_L = 2\mu'^{11}_1 \delta(y) \tilde{E}'_R,\]

\[i\delta E'^{+}_{R} : \sigma^\nu \partial_\mu \tilde{E}'_R + (- k \hat{D}_+(c_2) + i\partial_x) \tilde{E}'_L = 0,\]

\[-i\delta E'^{-}_{L} : (k \hat{D}_+(c_2) - i\partial_x) \tilde{E}'^-_R + \sigma^\nu \partial_\mu \tilde{E}'^-_L = 0,\]

\[i\delta E'^{+}_{L} : \sigma^\nu \partial_\mu \tilde{E}'^-_R + (k \hat{D}_-(c_2) - i\partial_x) \tilde{E}'^-_L = \delta(y) \{ -2i\bar{\mu}_2 \tilde{e}^+_{L} + 2\mu'^{11}_1 \tilde{E}'_L \} \ . \quad (4.33)\]

For 6th dimensional $n = 0$ KK modes, the equations of motion for charged leptons reduce to

\[\begin{align}
(a) & : - k \hat{D}_-(c_0) \left( \tilde{e}^+_R \right) + \sigma^\nu \partial_\mu \left( \tilde{e}^+_L \right) = 2i\bar{\mu}_2 \delta(y) \begin{pmatrix} \tilde{E}'_R \\ 0 \end{pmatrix}, \\
(b) & : \sigma^\nu \partial_\mu \left( \tilde{e}^+_R \right) - k \hat{D}_+(c_0) \left( \tilde{e}^+_L \right) = 0, \\
(c) & : - k \hat{D}_-(c_2) \tilde{E}'_R + \sigma^\nu \partial_\mu \tilde{E}'_L = 2\mu'^{11}_1 \delta(y) \tilde{E}'_R, \\
(d) & : \sigma^\nu \partial_\mu \tilde{E}'_R - k \hat{D}_+(c_2) \tilde{E}'_L = 0, \\
(e) & : k \hat{D}_+(c_2) \tilde{E}'^-_R + \sigma^\nu \partial_\mu \tilde{E}'^-_L = 0, \\
(f) & : \sigma^\nu \partial_\mu \tilde{E}'^-_R + k \hat{D}_-(c_2) \tilde{E}'^-_L = \delta(y) \{ -2i\bar{\mu}_2 \tilde{e}^+_{L} + 2\mu'^{11}_1 \tilde{E}'_L \} \ . \quad (4.34)\]

To obtain boundary conditions at $y = 0$, we integrate the above $(a), (d), (e), (h)$ in the vicinity of $y = 0$ for parity-odd fields:

\[(a) \Rightarrow 2\tilde{e}^+_R(x, \epsilon) = 2i\bar{\mu}_2 \tilde{E}'^-_L(x, 0),\]

\[(d) \Rightarrow -2\tilde{e}^+_{L}(x, \epsilon) = 0,\]

\[(e) \Rightarrow 2\tilde{E}'^+_R(x, \epsilon) = 2\mu'^{11}_1 \tilde{E}'^-_R(x, 0),\]

\[(h) \Rightarrow -2\tilde{E}'^-_L(x, \epsilon) = -i2\bar{\mu}_2 \tilde{e}^+_{L}(x, 0) + 2\mu'^{11}_1 \tilde{E}'_L(x, 0). \quad (4.35)\]

For parity-even fields, we calculate the equations of motion at $y = +\epsilon$ by using the above conditions:

\[(b) \Rightarrow \hat{D}_- \tilde{e}'^-_R(x, \epsilon) = 0,\]

\[(c) \Rightarrow \hat{D}_+ \tilde{e}'^-_L + i\bar{\mu}_2 D_\tilde{E}'^-_L = 0,\]

\[(f) \Rightarrow D_+ \tilde{E}'^-_L(x, \epsilon) + \mu'^{11}_1 D^- \tilde{E}'^-_L = 0,\]

\[(g) \Rightarrow D_+ \tilde{E}'^-_R + i\bar{\mu}_2 \hat{D}_- \tilde{e}'^-_R - \mu'^{11}_1 D^- \tilde{E}'^-_R = 0, \quad (4.36)\]
where we used the equations of motion (d) and (h) at $y = +\epsilon$.

By using the BCs on the IR brane, the mode functions of charged leptons in the twisted gauge are given by

$$
\left( \begin{array}{c}
\tilde{e}_{+R} \\
\tilde{e}^\prime_{+R} \\
\tilde{E}^\prime_{+R} \\
\tilde{E}_{-R} \\
\end{array} \right) = \left( \begin{array}{c}
\alpha_R^e S^0_R(z; \lambda, c_0) \\
\alpha_R^e C_R(z; \lambda, c_0) \\
\alpha_R^{E^\prime} S_R(z; \lambda, c_2) \\
\alpha_R^{E^\prime} C_L(z; \lambda, c_2) \\
\end{array} \right) f_R(x),
$$

$$
\left( \begin{array}{c}
\tilde{e}_{+L} \\
\tilde{e}^\prime_{+L} \\
\tilde{E}^\prime_{+L} \\
\tilde{E}_{-L} \\
\end{array} \right) = \left( \begin{array}{c}
\alpha_L^e S_L(z; \lambda, c_0) \\
\alpha_L^e C_L(z; \lambda, c_0) \\
\alpha_L^{E^\prime} S_R(z; \lambda, c_2) \\
\alpha_L^{E^\prime} S_L(z; \lambda, c_2) \\
\end{array} \right) f_L(x). 
\tag{4.37}
$$

As in the case of down-type quarks, the BCs at $z = 1^+$ in the twisted gauge are converted to

$$
K \left( \begin{array}{c}
\alpha_R^e \\
\alpha_R^e \\
\alpha_R^{E^\prime} \\
\alpha_R^{E^\prime} \\
\end{array} \right) = 0,
$$

$$
K = \begin{pmatrix}
\cos \frac{\theta_H}{2} S^0_R & -i \sin \frac{\theta_H}{2} C^0_R & 0 & -i \tilde{\mu}_2 C^0_L \\
-i \sin \frac{\theta_H}{2} \lambda C^0_L & \cos \frac{\theta_H}{2} \lambda S^0_L & 0 & 0 \\
0 & 0 & S^2_R & -\mu^2_{11} C^2_L \\
i \tilde{\mu}_2 \cos \frac{\theta_H}{2} \lambda C^0_L & \tilde{\mu}_2 \sin \frac{\theta_H}{2} \lambda S^0_L & -\mu^2_{11} \lambda S^0_L & \lambda S^2_R \\
\end{pmatrix}. \tag{4.38}
$$

From $\text{det}K = 0$, we find the mass spectrum formula for the charged leptons:

$$
S^0_L S^0_R + \sin^2 \frac{\theta_H}{2} = -\frac{\tilde{\mu}^2_2 S^0_L C^0_R S^0_R C^2_L}{\mu^2_{11} (C^2_L)^2 - (S^2_R)^2}. \tag{4.39}
$$

The mass spectrum of charged leptons, for which $\lambda z_L \ll 1$, is given by

$$
m_e \simeq \begin{cases} 
km_{11}^{z_L^{1+c_2-c_0}} \frac{\mu_{11}^2}{\mu^2} \sqrt{(2c_0 + 1)(1 - 2c_2)} \sin \frac{\theta_H}{2} & \text{for } c_2 < \frac{1}{2}, \\
km_{11}^{-1+c_2-c_0} \frac{\mu_{11}^2}{\mu^2} \sqrt{(2c_0 + 1)(2c_2 - 1)} \sin \frac{\theta_H}{2} & \text{for } c_2 > \frac{1}{2},
\end{cases} \tag{4.40}
$$

Here we have made use of $(m_t/m_\tau)^2, (m_c/m_\mu)^2, (m_u/m_e)^2 \gg 1$ which assures $|S^0_L S^0_R| \ll \sin^2 \frac{1}{2} \theta_H$, and have assumed that $\lambda^2 \ll \mu^2_{11}$ for $c_2 > \frac{1}{2}$ and $\lambda^2 \ll (\mu^2_{11} z^2_{2c_2-1})^2$ for $c_2 < \frac{1}{2}$.
so that $\mu_{11}^2 (C_L^2)^2 \gg (S_R^2)^2$. Combining (4.23) and (4.40), one finds

$$
\frac{m_e}{m_\mu} = \begin{cases}
\frac{\mu_{11}'}{\mu_2} \frac{1 - 2c_2}{2c_0 - 1} & \text{for } c_0 < \frac{1}{2}, \ c_2 > \frac{1}{2}, \\
\frac{1}{2} - c_0 \frac{\mu_{11}'}{\mu_2} \frac{2c_2 - 1}{2c_0 - 1} & \text{for } c_0 < \frac{1}{2}, \ c_2 > \frac{1}{2}.
\end{cases}
$$

(4.41)

It will be seen below that in a typical example for the third generation $c_2 = -0.7$ and $\mu_{11}' C_L^L / S_R^R \sim 2.6$.

One comment is in order about the masses of exotic charged leptons $\hat{e}, \hat{e}', \hat{E}_\pm'$ with charge $Q_{EM} = +1$. No zero modes exist for $\hat{e}$ and $\hat{e}'$, and all modes have masses larger than $\frac{1}{2} m_{KK6}$. On the other hand $\hat{E}_\pm'$ have zero modes. They acquire masses through the Hosotani mechanism and brane interactions. We suppose that $c_2 < \frac{1}{2}$ and/or $\mu_{11}'$ is sufficiently large so that their lightest masses are $O(m_{KK6})$.

4.4 Neutrino

(iv) $Q_{EM} = 0$: $\nu, \nu', N_{\pm}'', \tilde{N}_{\pm}', S_{\pm}'$, $\eta_-$ ($\Psi_{32}, \Psi_{11}', \chi_1$)

6D $SO(11)$ spinor and vector bulk fermion $\Psi_{32}$ and $\Psi_{11}'$ and a 5D $SO(11)$ singlet symplectic Majorana brane fermion $\chi$ appear in this sector. From the action in Eqs. (2.19) and $L_{\mu}^m, L_{\nu}^m, L_{\chi}^m$ in (2.28), the equations of motion for the bulk fermions become

$$
\begin{align*}
-i \theta \begin{pmatrix} \nu_{+L}^\dagger \\ \nu_{+R} \end{pmatrix} & : (- k \hat{D}_- (c_0) + i \partial_\nu) \begin{pmatrix} \nu_{+R} \\ \nu_{+L} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \nu_{+L} \\ \nu_{+R} \end{pmatrix} = \delta(y) \begin{pmatrix} 2 \tilde{\mu}_2 \tilde{N}_{-R}' \\ (m_B / \sqrt{\kappa}) \xi_- \end{pmatrix}, \\
-i \theta \begin{pmatrix} \tilde{N}_{-L}' \\ N_{-L}' \end{pmatrix} & : (- k \hat{D}_- (c_2) + i \partial_\nu) \begin{pmatrix} \tilde{N}_{-R}' \\ S_{-L}' \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \tilde{N}_{-L}' \\ S_{-L}' \end{pmatrix} = \delta(y) \begin{pmatrix} 2 \mu_{11}' \tilde{N}_{-R}' \\ 2 \mu_{11}' \tilde{S}_{-L}' \end{pmatrix}, \\
-i \theta \begin{pmatrix} \tilde{N}_{+L}' \\ N_{+L}' \end{pmatrix} & : (- k \hat{D}_+ (c_0) + i \partial_\nu) \begin{pmatrix} \tilde{N}_{+R}' \\ S_{+L}' \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \tilde{N}_{+L}' \\ S_{+L}' \end{pmatrix} = \delta(y) \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
-i \theta \begin{pmatrix} \tilde{N}_{+R}' \\ N_{+R}' \end{pmatrix} & : (- k \hat{D}_+ (c_2) + i \partial_\nu) \begin{pmatrix} \tilde{N}_{+L}' \\ S_{+L}' \end{pmatrix} = \delta(y) \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}
$$

31
\[ -i\delta \left( \begin{array}{c} \hat{N}_{-L}^t \\ \hat{N}_{-R}^t \\ \hat{S}_{-R}^t \\ \end{array} \right) : \left( k\hat{D}_+(c_2) - i\partial_v \right) \left( \begin{array}{c} \hat{N}_{-R}^t \\ \hat{N}_{-R}^t \\ \hat{S}_{-R}^t \\ \end{array} \right) + \sigma^\mu \partial_\mu \left( \begin{array}{c} \hat{N}_{-L}^t \\ \hat{N}_{-L}^t \\ \hat{S}_{-L}^t \\ \end{array} \right) = \delta(y) \left( \begin{array}{c} 0 \\ \sqrt{2}\mu_2 \nu_{+R} + 2\mu'_{11} \hat{S}_{+R}^t \\ 0 \end{array} \right) , \]

\[ i\delta \left( \begin{array}{c} \hat{N}_{+L}^t \\ \hat{N}_{+R}^t \\ \hat{S}_{+R}^t \\ \end{array} \right) : \sigma^\mu \partial_\mu \left( \begin{array}{c} \hat{N}_{+R}^t \\ \hat{N}_{+R}^t \\ \hat{S}_{+R}^t \\ \end{array} \right) + \left( k\hat{D}_-(c_2) - i\partial_v \right) \left( \begin{array}{c} \hat{N}_{+L}^t \\ \hat{N}_{+L}^t \\ \hat{S}_{+L}^t \\ \end{array} \right) = \delta(y) \left( \begin{array}{c} 2\mu'_{11} \hat{N}_{+L}^t \\ -2i\mu_2 \nu_{+L} + 2\mu'_{11} \hat{N}_{+L}^t \\ 0 \end{array} \right) . \]

(4.42)

The equations for the 5D brane symplectic Majorana fermion \( \chi_1 \) are

\[-i\delta_{\eta_-}^\dagger : \sigma^\mu \partial_\mu \eta_+ - i\partial_v \xi_- - \frac{m_B}{\sqrt{k}} \nu_{-R} - M\eta_-^C = 0 , \]

\[ i\delta_{\xi_-}^\dagger : \sigma^\mu \partial_\mu \xi_- - i\partial_v \eta_- - \frac{m_B}{\sqrt{k}} \nu_{-L} - M\xi_-^C = 0 . \]

(4.43)

Note that \( \eta_- \) has a zero mode but \( \xi_- \) has no zero mode in the 6th dimensional direction.

For 6th dimensional \( n = 0 \) KK modes, the equations of motion become

\[ \begin{align*}
  & (a) : - k\hat{D}_-(c_0) \left( \tilde{\nu}_{+R} \right) + \sigma^\mu \partial_\mu \left( \tilde{\nu}_{+L} \right) = \delta(y) \left( \begin{array}{c} 2i\mu_2 \hat{N}_{-R}^t \\ 0 \end{array} \right) , \\
  & (b) : - k\hat{D}_-(c_0) \left( \tilde{\nu}_{+R} \right) - k\hat{D}_+(c_0) \left( \tilde{\nu}_{+L} \right) = \delta(y) \left( \begin{array}{c} \sqrt{2}\nu_{-R} \\ 0 \end{array} \right) , \\
  & (c) : \sigma^\mu \partial_\mu \left( \tilde{\nu}_{+R} \right) - k\hat{D}_+(c_0) \left( \tilde{\nu}_{+L} \right) = \delta(y) \left( \begin{array}{c} 2\mu'_{11} \hat{N}_{+R}^t \\ 2\mu'_{11} \hat{N}_{+R}^t \end{array} \right) , \\
  & (d) : k\hat{D}_+(c_2) \left( \tilde{N}_{+R}^t \right) + \sigma^\mu \partial_\mu \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (e) : k\hat{D}_+(c_2) \left( \tilde{N}_{+R}^t \right) - k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (f) : - k\hat{D}_-(c_2) \left( \tilde{N}_{-R}^t \right) + \sigma^\mu \partial_\mu \left( \tilde{N}_{-L}^t \right) = \delta(y) \left( \begin{array}{c} 2\mu'_{11} \hat{N}_{-R}^t \\ 2\mu'_{11} \hat{N}_{-R}^t \end{array} \right) , \\
  & (g) : - k\hat{D}_-(c_2) \left( \tilde{N}_{-R}^t \right) - k\hat{D}_-(c_2) \left( \tilde{N}_{-L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (h) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+R}^t \right) - k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (i) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+R}^t \right) + k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (j) : \sigma^\mu \partial_\mu \left( \tilde{N}_{-R}^t \right) - k\hat{D}_-(c_2) \left( \tilde{N}_{-L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (k) : \sigma^\mu \partial_\mu \left( \tilde{N}_{-R}^t \right) + k\hat{D}_-(c_2) \left( \tilde{N}_{-L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (l) : \sigma^\mu \partial_\mu \left( \tilde{N}_{-R}^t \right) + k\hat{D}_-(c_2) \left( \tilde{N}_{-L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (m) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+L}^t \right) + k\hat{D}_+(c_2) \left( \tilde{N}_{+R}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (n) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+R}^t \right) + k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (o) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+R}^t \right) + k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) , \\
  & (p) : \sigma^\mu \partial_\mu \left( \tilde{N}_{+R}^t \right) + k\hat{D}_+(c_2) \left( \tilde{N}_{+L}^t \right) = \delta(y) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) .
\end{align*}\]
\begin{align}
(q) : \left\{ \sigma^\mu \partial_\nu \eta - \frac{m_B}{\sqrt{k}} \nu'_{+R} - M \eta_C \right\} \delta(y) &= 0 .
\end{align}

Here \( \hat{D}_\pm(c) \) is given by (4.2). To find the form of \( \hat{D}_\pm(c_2) \) for \( (\hat{N}', N', S') \) in \( \Psi'_{11} \), let us recall that the original and twisted gauges are related by \( \Psi'_{11} = \Omega(z) \tilde{\Psi}'_{11} \) where \( \Omega(z) = e^{i \theta(z) T_{4,11}} \). Hence

\begin{align}
\psi_3' &= \tilde{\psi}_3' , \quad \begin{pmatrix} \psi_4' \\ \psi_{11}' \end{pmatrix} = \begin{pmatrix} \cos \theta(z) & \sin \theta(z) \\ -\sin \theta(z) & \cos \theta(z) \end{pmatrix} \begin{pmatrix} \tilde{\psi}_4' \\ \tilde{\psi}_{11}' \end{pmatrix} .
\end{align}

As

\begin{align}
\psi_4' &= \frac{1}{\sqrt{2}} (\hat{N}' - N') , \quad \psi_3' = i \frac{1}{\sqrt{2}} (\hat{N}' + N') , \quad \psi_{11}' = S' ,
\end{align}

one finds

\begin{align}
\begin{pmatrix} \tilde{\hat{N}}' \\ \tilde{\hat{N}}' \\ \tilde{\hat{S}}' \end{pmatrix} &= \tilde{\Omega}(z) \begin{pmatrix} \hat{N}' \\ \hat{N}' \\ \hat{S}' \end{pmatrix} ,
\end{align}

\begin{align}
\tilde{\Omega}^{-1}(z) &= \frac{1 + \cos \theta(z)}{2} \begin{pmatrix} 1 & -\cos \theta(z) \\ 1 & \cos \theta(z) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sin \theta(z) \\ \sin \theta(z) \end{pmatrix} .
\end{align}

It follows that

\begin{align}
\hat{D}_-(c_2) \begin{pmatrix} \tilde{\hat{N}}'_{+R} \\ \tilde{\hat{N}}'_{+R} \\ \tilde{\hat{S}}'_{+R} \end{pmatrix} &= \tilde{\Omega}^{-1}(z) D_-(c_2) \begin{pmatrix} \tilde{\hat{N}}'_{+R} \\ \tilde{\hat{N}}'_{+R} \\ \tilde{\hat{S}}'_{+R} \end{pmatrix}
\end{align}

and so on.

To obtain boundary conditions at \( y = 0 \), we integrate the above \((a), (d), (e), (f), (j), (m), (n), (o)\) \( \left( \frac{1}{2} \int_{-\epsilon}^{+\epsilon} dy \cdots \right) \) in the vicinity of \( y = 0 \) for parity-odd \((x^5 = y)\) fields \( \nu'_{+R}, \nu'_{+L}, \hat{N}'_{+R}, \hat{N}'_{+L}, S'_{+R}, S'_{+L}, \hat{N}'_{-L}, N'_{-L} \):

| Condition | Result |
|-----------|--------|
| \((a)\) | \( + \tilde{\nu}_{+R}(x, \epsilon) = i \tilde{\mu}_2 \hat{N}'_{-R}(x, 0) \) |
| \((d)\) | \( - \tilde{\nu}_{+L}'(x, \epsilon) = - \frac{\tilde{\mu}_2}{\sqrt{2}} \hat{S}'_{-L}(x, 0) + \frac{m_B}{2 \sqrt{k}} \eta_-(x) \) |
| \((e)\) | \( + \tilde{\hat{N}}'_{+R}(x, \epsilon) = \mu_{11}' \hat{N}'_{-R}(x, 0) \) |
(f) \Rightarrow + \tilde{N}_{-R}'(x,\epsilon) = \mu_1^{11}\tilde{N}_{-R}'(x,0),
(j) \Rightarrow - \tilde{S}_{+L}'(x,\epsilon) = \mu_1^{11}\tilde{S}_{+L}'(x,0),
(m) \Rightarrow + \tilde{S}_{-L}'(x,\epsilon) = -\frac{\mu_2}{\sqrt{2}}\tilde{\nu}'_{+R}(x,0) + \mu_1^{11}\tilde{S}_{-L}'(x,0),
(n) \Rightarrow - \tilde{\eta}_{-L}'(x,\epsilon) = \mu_1^{11}\tilde{\eta}_{-L}'(x,0),
(o) \Rightarrow - \tilde{\eta}_{-L}'(x,\epsilon) = -i\tilde{\nu}_{+L}(x,0) + \mu_1^{11}\tilde{\eta}_{+L}(x,0).
\tag{4.49}

For parity-even \( (x = y) \) fields \( \nu_{+L}, \nu_{+R}, \tilde{N}_{+L}', \tilde{N}_{+L}', S_{+R}', S_{-L}', \tilde{N}_{-R}', \tilde{N}_{-R}' \), we evaluate the equations of motion at \( y = +\epsilon \) by making use of the above conditions to find

\[ (b) \Rightarrow - \hat{D}_-(c_0)\tilde{\nu}_{+R}' - \frac{\tilde{\mu}_2}{\sqrt{2}}\hat{D}_+\tilde{S}_{-R}' - \frac{m_B^2}{2k^2}\tilde{\nu}'_{+R} - \frac{m_B M}{2k^{3/2}}\eta_C = 0, \]
\[ (c) \Rightarrow \hat{D}_+(c_0)\tilde{\nu}_{+L}' + i\tilde{\nu}_2\hat{D}_-\tilde{N}_{-L}' = 0, \]
\[ (g) \Rightarrow - \hat{D}_-(c_0)\tilde{S}_{+R}' + \mu_1^{11}\hat{D}_+\tilde{S}_{-R}' = 0, \]
\[ (h) \Rightarrow \hat{D}_+(c_0)\tilde{\eta}_{+L}' + \mu_1^{11}\hat{D}_-\tilde{\eta}_{-R}' = 0, \]
\[ (i) \Rightarrow \hat{D}_+(c_0)\tilde{\eta}_{+L}' + \mu_1^{11}\hat{D}_-\tilde{\eta}_{-R}' = 0, \]
\[ (k) \Rightarrow \hat{D}_+(c_2)\tilde{\eta}_{-R}' - \mu_1^{11}\hat{D}_-\tilde{\eta}_{+R}' = 0, \]
\[ (l) \Rightarrow \hat{D}_+(c_2)\tilde{\eta}_{-R}' + i\tilde{\nu}_2\hat{D}_-\tilde{\nu}_{+R}' - \mu_1^{11}\hat{D}_-\tilde{\eta}_{+R}' = 0, \]
\[ (p) \Rightarrow \hat{D}_-(c_2)\tilde{S}_{-L}' - \frac{\tilde{\mu}_2}{\sqrt{2}}\hat{D}_+\tilde{\nu}_{+L}' + \mu_1^{11}\hat{D}_-\tilde{S}_{+L}' = 0, \tag{4.50} \]
at \( y = +\epsilon \).

There are nine fields in the neutral fermion sector which intertwine with each others. To simplify the discussions, we consider the case in which \( m_B^2/k^2, m_B M/k^2, \mu_1^{11} \gg \tilde{\mu}_2 \). In the \( \tilde{\mu}_2 = 0 \) limit, the equations of motion and boundary conditions in the neutral sector, \( 4.44 \), \( 4.49 \), and \( 4.50 \), split into two parts. The first sector, the neutrino sector-1, contains \( \tilde{\nu}_{+R}' \), \( \tilde{\nu}_{+L}' \) in \( \Psi_{32} \) and \( \eta_- \) in \( \chi_1 \), while the second sector, the neutrino sector-2, contains other components in \( \Psi_{32} \) and \( \Psi_{11} \). \( \tilde{\mu}_2 \neq 0 \) mixes the neutrino sector-1 and sector-2.

In the twisted gauge, the mode functions are determined by the BCs on the IR brane. The mass spectra can be fixed by the BCs on the UV brane. For the neutrino sector-1,
by using the boundary conditions at \( z = z_L \), their mode functions can be written as

\[
\begin{pmatrix}
\tilde{\nu}_R \\
\tilde{\nu}_\nu \\
\eta_C
\end{pmatrix} = \begin{pmatrix}
\alpha_\nu S_R(z; \lambda, c_0) \\
\alpha_\nu' C_R(z; \lambda, c_0) \\
-i \alpha_\eta' \sqrt{k}
\end{pmatrix} f_R(x), \quad \sigma^\mu\partial_\mu f_R(x) = k\lambda f_L(x),
\]

\[
\begin{pmatrix}
\tilde{\nu}_L \\
\tilde{\nu}_\nu \\
\eta_L
\end{pmatrix} = \begin{pmatrix}
\alpha_\nu C_L(z; \lambda, c_0) \\
\alpha_\nu' S_L(z; \lambda, c_0) \\
-i \alpha_\eta \sqrt{k}
\end{pmatrix} f_L(x), \quad \sigma^\mu\partial_\mu f_L(x) = k\lambda f_R(x).
\] (4.51)

Here \( f_{R/L}(x) \) are chosen in the neutrino sector-1 such that

\[
\sigma^\mu\partial_\mu f_L(x) = k\lambda f_R(x),
\]

\[
\eta_C = -e^{i\delta_C} \sigma^2 f_L(x) \quad \Rightarrow \quad \eta_C = f_R(x).
\] (4.52)

Setting \( \tilde{\mu}_2 = 0 \), one finds that the BCs at \( z = 1^+ \) in the twisted gauge can be written as

\[
\hat{D}_-(c_0) \tilde{\nu}_+^{(R)} + \frac{\mu_2}{\sqrt{2}} \hat{D}_- \hat{S}^{(R)} - \frac{m_B}{2\sqrt{k}} \eta_C = 0.
\] (4.53)

Setting \( \mu_2 = 0 \), one finds that the BCs at \( z = 1^+ \) in the twisted gauge can be written as

\[
K_{\nu_1} \begin{pmatrix}
\alpha_\nu \\
\alpha_\nu' \\
\alpha_\eta
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta_H}{2} S^0_R & \sin \frac{\theta_H}{2} C^0_R & 0 \\
-\sin \frac{\theta_H}{2} C^0_L & \cos \frac{\theta_H}{2} S^0_L & \frac{m_B}{2k} \\
-\sin \frac{\theta_H}{2} S^0_R & \cos \frac{\theta_H}{2} C^0_R & -k \lambda + M \frac{m_B}{m_B}
\end{pmatrix} \begin{pmatrix}
\alpha_\nu \\
\alpha_\nu' \\
\alpha_\eta
\end{pmatrix} = 0.
\] (4.54)

From \( \det K_{\nu_1} = 0 \), we find the mass spectrum formula for the neutrino sector-1:

\[
\det K_{\nu_1} = -k \lambda + M \frac{m_B}{m_B} \left( S^0_L S^0_R + \sin^2 \frac{\theta_H}{2} \right) = \frac{m_B}{2k} S^0_R C^0_R = 0.
\] (4.55)

This gives the gauge-Higgs seesaw mechanism for the small neutrino masses. For \( \lambda z_L \ll 1 \) and \( k \lambda \ll |M| \), the neutrino mass is given by

\[
m_\nu \simeq -2 m_u M z_L^{2c_0+1} \div \frac{1}{(2c_0 + 1) m_B^2}.
\] (4.56)

where \( m_u \) is given by (4.23). As shown in Ref. 38, the gauge-Higgs seesaw mechanism is characterized by a \( 3 \times 3 \) mass matrix

\[
\begin{pmatrix}
m_D & m_D & m_D \\
m_D & \tilde{m}_B & \tilde{m}_B \\
m_D & \tilde{m}_B & M
\end{pmatrix}
\] (4.57)

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in the 4d effective theory in which $m_D = m_u$ and $\tilde{m}_B \sim m_B z_L^{c_0 - 1/2}$. The Majorana mass $M$ may take a moderate value.

Next, we consider the neutrino sector-2. The BCs at the IR brane determine the mode functions in the twisted gauge to be

$$\begin{pmatrix}
\tilde{\tilde{N}}_{+R} \\
\tilde{\tilde{N}}_{+R} \\
\tilde{\tilde{S}}_{+R} \\
\tilde{\tilde{N}}_{-R} \\
\tilde{\tilde{N}}_{-R} \\
\tilde{\tilde{S}}_{-R}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{\tilde{N}} + S_R(z; \lambda, c_2) \\
\alpha_{\tilde{N}} + S_R(z; \lambda, c_2) \\
\alpha_{\tilde{S}}^+ C_R(z; \lambda, c_2) \\
\alpha_{\tilde{N}} + C_L(z; \lambda, c_2) \\
\alpha_{\tilde{N}} + C_L(z; \lambda, c_2) \\
\alpha_{\tilde{S}}^- S_L(z; \lambda, c_2)
\end{pmatrix} f_R(x) ,$$

$$\begin{pmatrix}
\tilde{\tilde{N}}_{+L} \\
\tilde{\tilde{N}}_{+L} \\
\tilde{\tilde{S}}_{+L} \\
\tilde{\tilde{N}}_{-L} \\
\tilde{\tilde{N}}_{-L} \\
\tilde{\tilde{S}}_{-L}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{\tilde{N}} + C_L(z; \lambda, c_2) \\
\alpha_{\tilde{N}} + C_L(z; \lambda, c_2) \\
\alpha_{\tilde{S}}^+ S_L(z; \lambda, c_2) \\
-\alpha_{\tilde{N}} - S_R(z; \lambda, c_2) \\
-\alpha_{\tilde{N}} - S_R(z; \lambda, c_2) \\
-\alpha_{\tilde{S}}^- C_L(z; \lambda, c_2)
\end{pmatrix} f_L(x) . \quad (4.58)
$$

By making use of \[\[4.47] and \[\[4.48], the BCs at the UV brane are expressed as

$$K_{\nu_2} \begin{pmatrix}
\alpha_{\tilde{N}}' + & \alpha_{\tilde{N}}' + & \alpha_{\tilde{S}}' + & \alpha_{\tilde{N}}' - & \alpha_{\tilde{S}}' - 
\end{pmatrix}^T = 0 , \quad (4.59)
$$

where

$$K_{\nu_2} = \begin{pmatrix} A & -\mu_{11} B \\ -\mu_{11} B & A \end{pmatrix} ,$$

$$A = \begin{pmatrix}
\frac{1 + \cos \theta_H}{2} S_R^2 & \frac{1 - \cos \theta_H}{2} S_R^2 & \frac{\sin \theta_H}{\sqrt{2}} C_R^2 \\
\frac{1 - \cos \theta_H}{2} S_R^2 & \frac{1 + \cos \theta_H}{2} S_R^2 & -\frac{\sin \theta_H}{\sqrt{2}} C_R^2 \\
-\frac{\sin \theta_H}{\sqrt{2}} C_L^2 & \frac{\sin \theta_H}{\sqrt{2}} C_L^2 & \cos \theta_H S_L^2
\end{pmatrix} ,$$

$$B = \begin{pmatrix}
\frac{1 + \cos \theta_H}{2} C_L^2 & \frac{1 - \cos \theta_H}{2} C_L^2 & \frac{\sin \theta_H}{\sqrt{2}} S_L^2 \\
\frac{1 - \cos \theta_H}{2} C_L^2 & \frac{1 + \cos \theta_H}{2} C_L^2 & -\frac{\sin \theta_H}{\sqrt{2}} S_L^2 \\
-\frac{\sin \theta_H}{\sqrt{2}} S_R^2 & \frac{\sin \theta_H}{\sqrt{2}} S_R^2 & \cos \theta_H C_R^2
\end{pmatrix} . \quad (4.60)$$
Note that
\[
K'_{\nu_2} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} K_{\nu_2} \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} = \begin{pmatrix} A' & -\mu'_{11} B' \\ -\mu'_{11} B' & A' \end{pmatrix},
\]
\[
V = V^{-1} = \begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2} & -2^{-1/2} \\ 0 & 0 \end{pmatrix},
\]
\[
A' = \begin{pmatrix} S^2_R & 0 & 0 \\ 0 & \cos \theta_H S^2_R & \sin \theta_H C^2_R \\ 0 & -\sin \theta_H C^2_R & \cos \theta_H S^2_R \end{pmatrix},
\]
\[
B' = \begin{pmatrix} C^2_L & 0 & 0 \\ 0 & \cos \theta_H C^2_L & \sin \theta_H S^2_L \\ 0 & -\sin \theta_H S^2_L & \cos \theta_H C^2_R \end{pmatrix}.
\]

From \(\det K_{\nu_2} = \det K'_{\nu_2} = 0\) the mass spectra for the neutrino sector-2 are found to be
\[
\det K_{\nu_2} = (S^2_R S^2_R - \mu^2_{11} C^2_L C^2_L) \left\{ (S^2_L S^2_R + \sin^2 \theta_H)^2 \\
+ \mu^2_{11} \left( 2 \sin^2 \theta_H \cos^2 \theta_H - C^2_R C^2_R S^2_R - C^2_L C^2_L S^2_L \right) \\
+ \mu^4_{11} \left( S^2_L S^2_R + \cos^2 \theta_H \right)^2 \right\} = 0.
\]

In the \(\mu'_{11} = 0\) limit, or in the absence of the brane interactions, (4.62) becomes
\[
(S^2_R)^2 (S^2_L S^2_R + \sin^2 \theta_H)^2 = 0.
\]

For large \(\mu'_{11}\), it becomes
\[
(C^2_L)^2 (S^2_L S^2_R + \cos^2 \theta_H)^2 \approx 0.
\]

For \(\tilde{\mu}_2 \neq 0\) the neutrino sector-1 and sector-2 mix through the boundary conditions. One needs to solve
\[
K \begin{pmatrix} \alpha_{\nu} & \alpha_{\nu'} & \alpha_\eta & \alpha_{\tilde{N}^+_1} & \alpha_{N^+_1} & \alpha_{S^+_1} & \alpha_{\tilde{N}^-_1} & \alpha_{N^-_1} & \alpha_{S^-_1} \end{pmatrix}^T = 0,
\]
\[
K = \begin{pmatrix} K_{\nu_1} & 0 & -i\tilde{\mu}_2 D \\ 0 & A & -\mu'_{11} B \\ i\tilde{\mu}_2 C & -\mu'_{11} B & A \end{pmatrix},
\]
\[
C = \begin{pmatrix} 0 & 0 & 0 \\ \cos \frac{1}{2} \theta_H C^0_L & \sin \frac{1}{2} \theta_H S^0_L & 0 \\ \sin \frac{1}{2} \theta_H C^0_L & \cos \frac{1}{2} \theta_H S^0_L & 0 \\ -\sqrt{2} S^0_R & \sqrt{2} C^0_R & 0 \end{pmatrix},
\]
\[
\text{37}
\]
\[
D = \begin{pmatrix}
\frac{1 - \cos \theta_H}{2} C_L^2 & \frac{1 + \cos \theta_H}{2} C_L^2 & -\frac{\sin \theta_H}{\sqrt{2}} S_L^2 \\
\frac{-\sin \theta_H}{2} S_R^2 & \frac{\sin \theta_H}{2} S_R^2 & \frac{\cos \theta_H}{\sqrt{2}} C_R^2 \\
0 & 0 & 0
\end{pmatrix}
\] (4.65)

det \( K \) is evaluated to be

\[
det K = F_0 + F_2 (\tilde{\mu}_2)^2 + F_4 (\tilde{\mu}_2)^4,
\]

\[
F_0 = \det K_{\nu_1} \cdot \det K_{\nu_2},
\]

\[
F_2 = \left\{ \frac{m_B}{4k} (S_L^2 S_R^0 + \cos^2 \frac{1}{2} \theta_H) + \frac{k\lambda + M}{2m_B} C_L^0 S_L^0 \right\} 
\times \left\{ C_L^2 S_R^2 (2X + \sin^2 \theta_H) (X + \sin^2 \theta_H) 
- \mu_{11}'^2 \left[ (C_L^3)^2 S_L^2 (2X + \sin^2 \theta_H) + C_L^2 (S_R^2)^3 (2X + 1 + \cos^2 \theta_H) 
+ C_L^2 S_R^2 (X^2 - 2) \sin^2 \theta_H \cos^2 \theta_H \right] + \mu_{11}'^4 C_L^2 S_R^2 (X + \cos^2 \theta_H) (2X + 1 + \cos^2 \theta_H) \right\}
+ \frac{k\lambda + M}{2m_B} \left\{ (S_R^2)^2 - \mu_{11}'^2 (C_L^2)^2 \right\} \left\{ \left[ (1 + \mu_{11}'^2) X + \sin^2 \theta_H + \mu_{11}'^2 \cos^2 \theta_H \right] \cos \theta_H \sin^2 \theta_H 
+ C_R^0 S_R^0 \left[ C_R^2 S_R^2 (X + \sin^2 \theta_H) - \mu_{11}'^2 C_L^2 S_R^2 (X + \cos^2 \theta_H) \right] \right\},
\]

\[
F_4 = -\frac{k\lambda + M}{32m_B} (S_L^2 S_R^2 + \cos^2 \frac{1}{2} \theta_H) 
\times \left\{ (S_R^2)^2 \left[ 16 (S_L^2 S_R^2)^2 + (20 - 4 \cos 2\theta_H) S_L^2 S_R^2 + 5 - 4 \cos 2\theta_H - \cos 4\theta_H \right] 
- \mu_{11}'^2 (C_L^2)^2 \left[ 16 (S_L^2 S_R^2)^2 + (12 + 4 \cos 2\theta_H) S_L^2 S_R^2 + 5 + 4 \cos 2\theta_H - \cos 4\theta_H \right] \right\},
\]

where \( X = S_L^2 S_R^2 \). In evaluating the effective potential \( V_{\text{eff}}(\theta_H) \), we shall use the approximate formula \( \det K \sim \det K_{\nu_1} \det K_{\nu_2} \) for small \( \tilde{\mu}_2 \).

### 4.5 Dark fermion

(v) \( \Psi_{32}' = \Psi_{32}'^{3=4} \)

The 6D \( SO(11) \) \( 32 \) Weyl fermion \( \Psi_{32}' \), which may be called a dark fermion multiplet, has 6th dimensional \( n = 0 \) KK modes. For \( \Psi_{32}' \) one needs not introduce brane interactions. We consider the action (2.19) for the dark fermion sector \( \Psi_{32}' \). From the parity assignment
shown in Table 2, the mass formula for the dark fermions is found to be

\[ S^0_L S^0_R + \cos^2 \frac{\theta_H}{2} = 0. \] (4.67)

For \( \lambda z_L \ll 1 \), the dark fermion mass is given by

\[
m_{\text{Dark}} \simeq \begin{cases} 
    k z_L^{-1} \sqrt{1 - 4 c_0^2} \cos \frac{\theta_H}{2} & \text{for } c_0' < 1/2, \\
    k z_L^{-1/2} c_0' \sqrt{4 c_0^2 - 1} \cos \frac{\theta_H}{2} & \text{for } c_0' > 1/2.
\end{cases}
\] (4.68)

As in the case of the 5D SO(11) GHGUT discussed in Ref. [35], the bulk mass parameter of the dark fermions, \( c_0' \), must be relatively small not to become light exotic particles.

5 Effective potential

We evaluate the Higgs effective potential \( V_{\text{eff}}(\theta_H) \) by using the mass spectrum formulas of the 6th-dimensional \( n = 0 \) KK modes of the SO(11) gauge bosons and fermions. The evaluation is done in the same manner as in Ref. [35].

One-loop effective potential from each KK tower is given by [9, 45, 46]

\[
V_{\text{eff}}(\theta_H) = \pm \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \sum_n \ln \left( p^2 + m_n(\theta_H)^2 \right)
\] (5.1)

where \( m_n(\theta_H) \) is the mass spectrum of the KK tower and we take + and − sign for bosons and fermions, respectively. When the mass spectrum \( \{m_n = k \lambda_n\} \) is determined by \( 1 + \tilde{Q}(\lambda_n) f(\theta_H) = 0 \), (5.1) can be written as

\[
V_{\text{eff}}(\theta_H) = \pm I \left[ Q(q); f(\theta_H) \right]
\]

\[
I \left[ Q(q); f(\theta_H) \right] = \frac{(k z_L^{-1})^4}{(4\pi)^2} \int_0^\infty dq q^3 \ln[1 + Q(q) f(\theta_H)],
\] (5.2)

where \( Q(q) = \tilde{Q}(iq z_L^{-1}) \). We utilize the mass spectra \( m_n(\theta_H) \) of the SO(11) bulk gauge and fermions obtained in Sec. 3 and 4. Note that only \( \theta_H \)-dependent mass spectra contribute to the \( \theta_H \)-dependent part of \( V_{\text{eff}}(\theta_H) \). It contains the SO(11) bulk gauge field, SO(11) spinor and vector fermion fields.

We summarize those mass spectra. The SO(11) gauge boson contribution is almost the same as in the 5D SO(11) GHGUT. The difference from the 5D case is that the Y boson contributions can be neglected in the current scheme as they have masses of
\( O(m_{\text{KK}}) \). The relevant mass spectra of \( SO(11) \) gauge fields \( A_\mu \) and \( A_z \) are given, from \((3.7), (3.12) \) and \((3.18) \), by

\[
W^\pm \text{ tower} : 1 + \frac{\lambda}{2C'(1; \lambda)S(1; \lambda)} \sin^2 \theta_H = 0 ,
\]

\[
Z \text{ tower} : 1 + \frac{\lambda}{2 \cos^2 \theta_W C'(1; \lambda)S(1; \lambda)} \sin^2 \theta_H = 0 ,
\]

\[
A^a_4, A^a_{11} (a = 1, 2, 3) : 1 + \frac{\lambda}{C'(1; \lambda)S(1; \lambda)} \sin^2 \theta_H = 0 .
\]

The mass spectra of up- and down-type quarks, charged leptons, neutrinos, and dark fermions are given, from \((4.22), (4.30), (4.39), (4.55), (4.62) \) and \((4.67) \), by

(i) : \( Q_{EM} = +\frac{2}{3} : 1 + \frac{\sin^2 \frac{1}{2} \theta_H}{S_L^0 S_R^0} = 0 , \)

(ii) : \( Q_{EM} = -\frac{1}{3} : 1 + \frac{\sin^2 \frac{1}{2} \theta_H}{S_L^0 S_R^0 + \frac{\mu^2 S_L^0 C_R^0 S_L^1 C_R^1}{\mu^2 (C_R^1)^2 - (S_L^0)^2}} = 0 , \)

(iii) : \( Q_{EM} = -1 : 1 + \frac{\sin^2 \frac{1}{2} \theta_H}{S_L^0 C_R^0 + \frac{\mu^2 S_L^0 C_R^0 S_L^2 C_L^2}{\mu^2 (C_L^2)^2 - (S_L^0)^2}} = 0 , \)

(iv-1) : \( Q_{EM} = 0 : 1 + \frac{\sin^2 \frac{1}{2} \theta_H}{S_L^0 S_R^0 + \frac{m_B^2/k}{2(k\lambda + M)} S_L^0 C_R^0} = 0 , \)

(iv-2) : \( Q_{EM} = 0 : 1 + \frac{f_1(\theta_H)}{f_2} = 0 , \)

\[
f_1(\theta_H) = (1 - \mu'_{11}) \left\{ 2(1 + \mu'_{11}) S_L^2 S_R^2 + 2 \mu'_{11} + (1 - \mu'_{11}) \sin^2 \theta_H \right\} \sin^2 \theta_H ,
\]

\[
f_2 = \mu'^4_{11} + 2 \mu'^4_{11} S_L^2 S_R^2 + (1 + \mu'^4_{11}) (S_L^2 S_R^2)^2 - \mu'^2_{11} \left\{ (C_R^2 S_R^2)^2 + (C_L^2 S_L^2)^2 \right\} ,
\]

(v) : \( Q_{EM} = +\frac{2}{3}, -\frac{2}{3}, -1, 0: 1 + \frac{\cos^2 \frac{1}{2} \theta_H}{S_L^0 S_R^0} = 0 , \)

where \( m_B^2/k^2, m_B M/k^2, \mu'_{11} \gg \tilde{\mu}_2 \) has been assumed for (iv-1) and (iv-2). (v) is the dark fermion mass spectrum. \( c'_o \) stands for the bulk mass of the 6D \( SO(11) \) spinor bulk Weyl fermion \( \Psi_{32}^{\alpha=4} \).

We evaluate the effective potential \( V^\text{eff}(\theta_H) = V^\text{gauge}_{\text{eff}}(\theta_H) + V^\text{fermion}_{\text{eff}}(\theta_H) \). In the \( R_\xi \) gauge \( V^\text{gauge}_{\text{eff}}(\theta_H) \) is decomposed into

\[
V^\text{gauge}_{\text{eff}}(\theta_H) = V^W_{\text{eff}} + V^Z_{\text{eff}} + V^{A_4^a, A_{11}^a}_{\text{eff}} ,
\]
\[ V_{\text{eff}}^{W^\pm}(\theta_H) = 2(3 - \xi^2)I \left[ \frac{1}{2} Q_0(q,1); \sin^2 \theta_H \right], \]
\[ V_{\text{eff}}^{Z}(\theta_H) = (3 - \xi^2)I \left[ \frac{Q_0(q,1)}{2 \cos^2 \theta_W}; \sin^2 \theta_H \right], \]
\[ V_{\text{eff}}^{A_{\text{eff}}^{10},A_{\text{eff}}^{11}}(\theta_H) = 3\xi^2I \left[ Q_0(q,1); \sin^2 \theta_H \right]. \]  

(5.5)

Here
\[
Q_0(q,c) = \frac{z_L}{q^2} \hat{F}_{c}^{++}(q) \hat{F}_{c}^{--}(q),
\]
\[
\hat{F}_{c}^{\pm \pm}(q) = \hat{F}_{c}^{1} \pm \frac{1}{2} \hat{F}_{c}^{2}(q z_L^{-1},q),
\]
\[
\hat{F}_{\alpha,\beta}(u,v) = I_{\alpha}(u)K_{\beta}(v) - e^{-i(\alpha-\beta)\pi}K_{\alpha}(u)I_{\beta}(v).
\]  

(5.6)

where \( I_{\alpha}(u) \) and \( K_{\beta}(u) \) are modified Bessel functions.

The fermion part \( V_{\text{eff}}^{\text{fermion}}(\theta_H) \) is evaluated in a similar manner. Following the decomposition in (5.4) and taking into account four degrees of freedom for each Dirac fermion and the color factor 3 for quarks, one can write as
\[
V_{\text{eff}}^{\text{fermion}}(\theta_H) = V_{\text{eff}}^{(i)} + V_{\text{eff}}^{(ii)} + V_{\text{eff}}^{(iii)} + V_{\text{eff}}^{(iv-1)} + V_{\text{eff}}^{(iv-2)} + V_{\text{eff}}^{(v)},
\]
\[
V_{\text{eff}}^{(i)}(\theta_H) = -12I \left[ Q_0(q,c_0); \sin^2 \frac{1}{2} \theta_H \right],
\]
\[
V_{\text{eff}}^{(ii)}(\theta_H) = -12I \left[ Q_{(ii)}(q,c_0,c_1,\mu_1,\mu_{11}); \sin^2 \frac{1}{2} \theta_H \right],
\]
\[
V_{\text{eff}}^{(iii)}(\theta_H) = -4I \left[ Q_{(iii)}(q,c_0,c_2,\bar{\mu}_2,\mu'_{11}); \sin^2 \frac{1}{2} \theta_H \right],
\]
\[
V_{\text{eff}}^{(iv-1)}(\theta_H) = -4 \cdot \frac{1}{2} I \left[ \tilde{Q}_{(iv-1)}(q,c_0,m_B,M;\theta_H); 1 \right],
\]
\[
V_{\text{eff}}^{(iv-2)}(\theta_H) = -4I \left[ \tilde{Q}_{(iv-2)}(q,c_2,\mu'_{11};\theta_H); 1 \right],
\]
\[
V_{\text{eff}}^{(v)}(\theta_H) = -32I \left[ Q_0(q,c_0'); \cos^2 \frac{1}{2} \theta_H \right],
\]  

(5.7)

where
\[
Q_{(ii)}(q) = \frac{z_L}{q^2} \left\{ \hat{F}_{c_0}^{++} \hat{F}_{c_0}^{--} + \frac{\mu_1^2 \hat{F}_{c_0}^{+-} \hat{F}_{c_0}^{-+} \hat{F}_{c_1}^{+-} \hat{F}_{c_1}^{-+}}{\mu_{11}^2 (\hat{F}_{c_1}^{+-})^2 + (\hat{F}_{c_1}^{++})^2} \right\}^{-1},
\]
\[
Q_{(iii)}(q) = \frac{z_L}{q^2} \left\{ \hat{F}_{c_0}^{++} \hat{F}_{c_0}^{--} + \frac{\bar{\mu}_2^2 \hat{F}_{c_0}^{+-} \hat{F}_{c_0}^{-+} \hat{F}_{c_2}^{+-} \hat{F}_{c_2}^{-+}}{\mu_{11}^2 (\hat{F}_{c_1}^{+-})^2 + (\hat{F}_{c_1}^{--})^2} \right\}^{-1},
\]
\[
Q_{(iv-1)}(q) = \frac{z_L}{q^2} \left\{ \hat{F}_{c_0}^{++} \hat{F}_{c_0}^{--} - \frac{i m_B^2/k^2}{2(i q z_L^{-1} + M/k)} \hat{F}_{c_0}^{+-} \hat{F}_{c_0}^{-+} \right\}^{-1},
\]  

(5.8)
\[ \tilde{Q}_{(iv-1)}(q, \theta_H) = (Q_{(iv-1)}(q) + Q_{(iv-1)}(q)^*) \sin^2 \frac{1}{2} \theta_H \]
\[ + (Q_{(iv-1)}(q)Q_{(iv-1)}(q)^*) \sin^4 \frac{1}{2} \theta_H , \]
\[ \tilde{Q}_{(iv-2)}(q, \theta_H) = \frac{Z_L^2}{Q^4} \left\{ \frac{h_1}{h_2} \right\} , \]
\[ h_1 = (1 - \mu_{11}^2) \left\{ \frac{2q^2}{z_L} (1 + \mu_{11}^2) \hat{F}_{c_2}^{++} \hat{F}_{c_2}^{--} + 2\mu_{11}^2 + (1 - \mu_{11}^2) \sin^2 \theta_H \right\} \sin^2 \theta_H , \]
\[ h_2 = \frac{Z_L^2}{Q^4} \mu_{11}^2 + 2\mu_{11}^2 \frac{Z_L}{Q^2} \hat{F}_{c_2}^{++} \hat{F}_{c_2}^{--} + (1 + \mu_{11}^2) (\hat{F}_{c_2}^{++} \hat{F}_{c_2}^{--})^2 \]
\[ + \mu_{11}^2 \left\{ (\hat{F}_{c_2}^{++} \hat{F}_{c_2}^{--})^2 + (\hat{F}_{c_2}^{++} \hat{F}_{c_2}^{--})^2 \right\} . \] (5.8)

As \( Q_{(iv-1)}(q) \) is not a real function, \( V_{eff}^{(iv-1)}(\theta_H) \) has been converted to the integral involving \( \tilde{Q}_{(iv-1)}(q; \theta_H) \).

The Higgs mass \( m_H \) is determined, at the minimum \( \theta_H = \theta_H^{\text{min}} \) of \( V_{eff}(\theta_H) \), by
\[ m_H^2 = \frac{1}{f_H^2} \frac{d^2 V_{eff}(\theta_H)}{d\theta_H^2} \bigg|_{\theta_H=\theta_H^{\text{min}}} , \]
\[ f_H = \frac{\sqrt{6}}{g} \sqrt{\frac{2\pi R_6 k}{z_L^2 - 1}} = \frac{\sqrt{6}}{g_w} \sqrt{\frac{k}{(1 - z_L^2)(z_L^2 - 1)}}, \]
\[ g_w = \frac{e}{\sin \theta_W} = g \sqrt{\frac{k}{2\pi R_6(1 - z_L^2)}}. \] (5.9)

Note that the 4D neutral Higgs field \( H(x) \) is related to \( \phi_H(x) \) in (2.33) by
\[ \phi_H(x) = \theta_H f_H + H(x) . \] (5.10)

We give results for the effective potential and the Higgs mass for typical sample values of the parameters. In the following calculation, we take into account the 6th dimensional \( n = 0 \) modes of 6D bulk gauge bosons \( A_M \), the third generation \( SO(11) \) spinor and vector fermions \( \Psi^3_{32}, \Psi^{3}_{11}, \Psi^\beta_{11}, \Psi^\beta_{11} \), and dark fermions \( \Psi^3_{32} \). Contributions coming from the first and second generations of quarks and leptons are numerically negligible. Some of the parameters in the fermion sector are fixed by the observed masses of quarks and leptons in their approximate forms given in Eqs. (4.23), (4.31), (4.40), and (4.56).

The calculation algorithm is almost the same as in 5D \( SO(11) \) GHGUT given in Sec. 5 of Ref. [35]. We are interested in the effective potential \( V_{eff}(\theta_H) \) at the electroweak scale.

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Some of the relations derived in this paper are those at the GUT scale. The value of \( \sin^2 \theta_W \) evolves from \( \frac{3}{8} \) at the GUT scale to the observed value at the electroweak scale by the renormalization group equation (RGE). The RGE analysis of the coupling constants is beyond the scope of the current paper. In this paper we content ourselves to insert the observed value of \( \sin^2 \theta_W \) into the formulas of \( m_Z \) and \( V_{\text{eff}}(\theta_H) \). As input parameters we take \( \alpha_{\text{EM}}^{-1} = 127.916 \), \( \sin^2 \theta_W = 0.2312 \), \( m_Z = 91.1876 \) GeV, \( m_t = 171.17 \) GeV, \( m_b = 4.18 \) GeV, \( m_r = 1.776 \) GeV and \( m_{\nu_e} = 0.1 \) eV. The evaluation is carried out in the following steps.

(1) We first pick the values for \( z_L \) and \( \theta_H \). So far consistent sets of parameters have been found only for \( 30 \lesssim z_L \lesssim 40 \).

(2) From \( m_Z \), (3.12), \( k \) and \( m_{\text{KK}_5} \) are determined.

(3) \( c_0 \) is fixed from \( m_t \) by (4.23).

(4) Some of the parameters in the brane interactions on the UV brane remain free. We take \( c_1 = 0, c_2 = -0.7, \) and \( M = -10^7 \) GeV. These three parameters are kept fixed in the evaluation below. We temporarily assign a value for \( \mu_{11} = \mu_{11}' \). Then, \( \mu_1, \tilde{\mu}_2, \) and \( m_B \) are determined by (4.32), (4.41) and (4.56), or by

\[
\mu_1 \simeq \sqrt{\frac{1 + 2c_1}{1 + 2c_0} z_L^{c_0 - c_1} \frac{m_t}{m_b} \mu_{11}'},
\]

\[
\tilde{\mu}_2 \simeq \sqrt{\frac{1 - 2c_2}{1 - 2c_0} z_L^{c_2 - c_0} \frac{m_t}{m_\tau} \mu_{11}'},
\]

\[
m_B = \sqrt{- \frac{2M m_t^2}{m_{\nu_e}} z_L^{2c_0 + 1} \frac{z_L^{2c_0 + 1}}{1 + 2c_0}}.
\]

(5) Given the value of the bulk mass parameter \( c'_0 \) of the dark fermion multiplet, the effective potential \( V_{\text{eff}}(\theta_H) \) can be evaluated. We adjust the value of \( c'_0 \) such that the minimum of \( V_{\text{eff}}(\theta_H) \) is located at the initial value of \( \theta_H \). In this manner a consistent parameter set has been obtained.

(6) The Higgs boson mass \( m_H \) is determined from (5.9), which can be viewed as a function of \( z_L, \theta_H \) and \( \mu_{11}' \), namely \( m_H(\theta_H, z_L, \mu_{11}') \).

(7) By demanding that the resulting \( m_H \) is the observed value, namely \( m_H(\theta_H, z_L, \mu_{11}') = 125.09 \pm 0.24 \) GeV, \( \mu_{11}' \) is determined with the given \( \theta_H \) and \( z_L \).

(8) Take a different \( \theta_H \). By repeating the above procedure one finds a new value for \( \mu_{11}' \).

The physical value of \( \theta_H \) need to be determined, for instance, from the observation of the \( Z' \) bosons (the 1st KK bosons \( \gamma^{(1)}, Z^{(1)}, Z_R^{(1)} \)) and their masses. In the \( SO(5) \times U(1) \)
gauge-Higgs electroweak unification many of the physical quantities depend on the value of $\theta_H$, but not on the details of the parameters in the theory. It has been shown that $m_{Z(1)} \gtrsim 7 \text{ TeV}$ ($m_{\text{KK}5} \gtrsim 8.5 \text{ TeV}$) to be consistent with the current LHC data. \cite{15,16}

The results for the effective potential $V_{\text{eff}}(\theta_H)$ in the $R_{\xi=0}$ gauge are depicted in Figure 1 for which $z_L = 35$, $\theta_H = 0.15$, and $m_H = 125.1 \text{ GeV}$. The bulk and brane mass parameters are chosen as $z_L = 35$, $\theta_H = 0.15$, $m_H = 125.12 \text{ GeV}$, $c_0 = 0.3325$, $c_1 = 0$, $c_2 = -0.7$, $c'_0 = 0.5224$, $\mu_1 \simeq 11.18$, $\tilde{\mu}_2 \simeq 0.7091$, $\mu_{11} = \mu'_{11} = 0.108$, $m_B = 1.145 \times 10^{12} \text{ GeV}$, $M = -10^7 \text{ GeV}$. \(5.12\)

The resultant $m_{\text{KK}5} = \pi k z_L^{-1}$, $k$, and $f_H$ are given by

$$m_{\text{KK}5} \simeq 8.236 \text{ TeV}, \quad k \simeq 8.913 \times 10^4 \text{ GeV}, \quad f_H \simeq 1.642 \times 10^3 \text{ GeV}. \quad (5.13)$$

In Table 6 we have tabulated various sets of the parameters which yield $m_H = 125.1 \pm 0.1 \text{ GeV}$. As $\theta_H$ increases, $m_{\text{KK}}$ decreases. One sees $\theta_H \lesssim 0.15$ in order for $m_{\text{KK}} > 8 \text{ TeV}$ to be consistent with the current LHC data. We have observed that $\theta_H$ cannot be too small in order to reproduce $m_H \sim 125 \text{ GeV}$.

In Figure 2, $m_{\text{KK}5}$ is plotted as a function of $\theta_H$. Data points in Table 6 are fitted by

$$m_{\text{KK}5} \sim \frac{\alpha}{(\sin \theta_H)^\beta}, \quad \alpha = 1.230 \text{ TeV}, \quad \beta = 1.000 \quad (5.14)$$

in the range $0.05 < \theta_H < 0.30$. A similar relation has been found in the $SO(5) \times U(1)$ gauge-Higgs EW unification where $\alpha = 1.352 \text{ TeV}$ and $\beta = 0.786$. \cite{11,12} The difference between the two is expected to originate from the different gauge group structure of the models and the different matter content. So far we have found consistent parameter sets only for $c_1 = 0$ and $c_2 \leq 0$.

6 Summary and discussions

In this paper, we discussed 6D $SO(11)$ GHGUT in the 6D hybrid warped space. The GUT gauge symmetry $SO(11)$ is reduced to the Pati-Salam symmetry $G_{\text{PS}}(= SU(2)_L \times SU(2)_R \times SU(4)_C)$ by the orbifold BCs, and the symmetry $G_{\text{PS}}$ is spontaneously broken
Figure 1: The effective potential $V_{\text{eff}}(\theta_H)$ in the $R_{\xi=0}$ gauge in which $z_L = 35$ and the minimum of $V_{\text{eff}}(\theta_H)$ is located at $\theta_H = 0.15$. For the top two figures, the blue solid line shows the effective potential with both gauge and fermion effects. The red and green dashed lines represent the contributions of fermions and gauge fields, respectively. For the bottom left figure, the green solid line represents the total contributions of gauge fields. The red and purple dashed lines represent $Z$ and $W$ boson contributions. In the bottom right figure the red solid line represents the total contribution of all fermions. The green dashed lines represent contributions of top quark. The blue dashed line represents that of the neutrino-2 sector. The black dashed line represents the contribution of dark fermions. Contributions coming from other fermions are negligible.

to the SM gauge symmetry $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$ by the nonvanishing VEV of the 5D brane scalar field $\Phi_{32}$ on the UV brane. Finally, the EW gauge symmetry $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{\text{EM}}$ by the Hosotani mechanism. The SM Higgs boson appears as a zero mode of the 5th dimensional components of the $SO(11)$ gauge field.

The SM fermions are contained in $\Psi_{32}$, $\Psi_{11}$, $\Psi'_{11}$, $\chi_1$. The SM fermions as well as the $W$ and $Z$ bosons acquire masses via the Hosotani mechanism. In Sec. 4, we showed that the quark and lepton mass spectra can be reproduced by choosing the parameters of the bulk masses and the brane interactions. In the 5D GHGUT in Ref. [35], there necessarily have arisen light exotic vectorlike fermions accompanied with SM fermions,
\[\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\theta_H & \mu_{11} = \mu_{11}^L & m_{KK_5} [\text{TeV}] & c_0 & c_0' & m_H [\text{GeV}] & \tilde{\mu}_2 \\
\hline
0.05 & 0.1730 & 24.63 & 0.3289 & 0.5960 & 125.17 & 1.138 \\
0.06 & 0.1590 & 20.53 & 0.3292 & 0.5828 & 125.07 & 1.046 \\
0.07 & 0.1480 & 17.60 & 0.3294 & 0.5714 & 125.13 & 0.9736 \\
0.08 & 0.1400 & 15.40 & 0.3297 & 0.5625 & 125.03 & 0.9208 \\
0.09 & 0.1330 & 13.70 & 0.3300 & 0.5544 & 125.08 & 0.8746 \\
0.10 & 0.1270 & 12.33 & 0.3303 & 0.5471 & 125.17 & 0.8350 \\
0.11 & 0.1225 & 11.21 & 0.3307 & 0.5414 & 125.05 & 0.8052 \\
0.12 & 0.1180 & 10.28 & 0.3311 & 0.5356 & 125.14 & 0.7755 \\
0.13 & 0.1145 & 9.495 & 0.3315 & 0.5311 & 125.06 & 0.7523 \\
0.14 & 0.1110 & 8.821 & 0.3320 & 0.5264 & 125.12 & 0.7291 \\
0.15 & 0.1080 & 8.236 & 0.3325 & 0.5224 & 125.12 & 0.7091 \\
0.16 & 0.1055 & 7.725 & 0.3331 & 0.5192 & 125.03 & 0.6925 \\
0.17 & 0.1030 & 7.275 & 0.3337 & 0.5158 & 125.02 & 0.6759 \\
0.18 & 0.1005 & 6.875 & 0.3343 & 0.5125 & 125.11 & 0.6593 \\
0.19 & 0.0985 & 6.517 & 0.3349 & 0.5099 & 125.05 & 0.6459 \\
0.20 & 0.0965 & 6.195 & 0.3356 & 0.5073 & 125.05 & 0.6326 \\
0.21 & 0.0945 & 5.903 & 0.3363 & 0.5047 & 125.21 & 0.6192 \\
0.22 & 0.0930 & 5.639 & 0.3371 & 0.5030 & 125.00 & 0.6092 \\
0.23 & 0.0910 & 5.398 & 0.3379 & 0.5003 & 125.17 & 0.5959 \\
0.24 & 0.0895 & 5.177 & 0.3387 & 0.4986 & 125.14 & 0.5858 \\
0.25 & 0.0880 & 4.974 & 0.3395 & 0.4969 & 125.16 & 0.5758 \\
0.26 & 0.0867 & 4.786 & 0.3404 & 0.4956 & 125.11 & 0.5671 \\
0.27 & 0.0855 & 4.613 & 0.3413 & 0.4945 & 125.04 & 0.5590 \\
0.28 & 0.0840 & 4.452 & 0.3423 & 0.4928 & 125.16 & 0.5490 \\
0.29 & 0.0830 & 4.302 & 0.3432 & 0.4921 & 125.05 & 0.5423 \\
0.30 & 0.0818 & 4.163 & 0.3442 & 0.4911 & 125.07 & 0.5342 \\
\hline
\end{array}\]

Table 6: Parameter sets which give dynamical EW symmetry breaking with \(m_H = 125.1 \pm 0.1\) GeV. Here \(z_L = 35, c_1 = 0.0, c_2 = -0.7,\) and \(M = -10^7\) GeV. \(c_2\) can be varied without affecting \(m_{KK_5}\) and \(c_0\). For \(\theta_H = 0.05\), for instance, a set of \(c_2 = -0.9\) and \(\mu_{11} = \mu_{11}^L = 0.1174\) yields \(m_H = 125.05\) GeV, \(c_0' = 0.6008\), and \(\tilde{\mu}_2 = 0.4084\).

which is experimentally unacceptable. In quite contrast to the case of the 5D GHGUT, the non-SM fermions have masses of either \(O(m_{KK_5})\) or \(O(m_{KK_5})\) in the current 6D model. Thus the problem of light exotic fermions has been solved.

From the analysis of the effective potential \(V_{\text{eff}}(\theta_H)\) in Sec. 5 we have seen that the EW symmetry is dynamically broken by the AB phase \(\theta_H\). The Higgs boson mass \(m_H = 125.1\) GeV can be obtained for \(0.05 \lesssim \theta_H \lesssim 0.30\), namely for \(25\) TeV \(\gtrsim m_{KK_5} \gtrsim 4.1\) TeV.

In the present paper we have treated each generation of quarks and leptons independently. It is necessary to incorporate the flavor mixing in the gauge-Higgs grand unification. The mixing arises from the brane interactions. For quarks and charged leptons the mixing would result from the matrix structure of various \(\kappa\)'s and \(\mu\)'s in the brane interactions [2.27] of \(\Psi_{32}, \Psi_{11}\) and \(\Phi_{32}\). On the other hand the dominant mixing in the
neutrino sector would arise from the matrix structure in the mass $M^{\beta\beta'}$ associated with the symplectic Majorana fields $\chi_1$ in (2.26). The large mixing angles observed in the neutrino sector might be related to this special circumstance.

We would like to add a comment on the effective potential $V_{\text{eff}}^6(\theta_6^H)$ of the AB phase $\theta_6^H$ along the 6th dimension. From Table 5, not only the $SO(5)/SO(4)$ components of the 5th dimensional gauge field $A_y$ but also those of the 6th dimensional gauge field $A_v$ have zero modes in the absence of brane interactions. As in the effective potential $V_{\text{eff}}(\theta_H)$ for the AB phase $\theta_H$ along the 5th dimension, the $SO(11)$ bulk gauge bosons $A_M$, the $SO(11)$ bulk fermions $\Psi_{32}^\alpha$, $\Psi_{11}^\beta$, and $\Psi_{11}'^\beta$, contribute to $V_{\text{eff}}^6(\theta_6^H)$. If there were only gauge fields, it can be shown that the minimum of $V_{\text{eff}}^6(\theta_6^H)$ would be located at $\theta_6^H = \frac{1}{2} \pi$ (mod. $\pi$). In other words, the EW symmetry would be broken by the 6th dimensional AB phase $\theta_6$ and the $W$ and $Z$ bosons would acquire masses of $O(m_{KK_6})$. The situation is similar to that in the effective potential $V_{\text{eff}}(\theta_H)$ in the 5D $SO(11)$ GHGUT. [32, 35] Further the contribution from the $SO(11)$ bulk spinor fermions $\Psi_{32}^\alpha$ to $V_{\text{eff}}^6(\theta_6^H)$ has little $\theta_6^H$ dependence as is inferred from Table 2. The $\theta_6^H$-dependent contributions coming from a set of $SO(11)$ bulk vector fermions $\Psi_{11}^\beta$ and $\Psi_{11}'^\beta$ almost cancel each other as can be deduced from Table 3. Hence, with the minimal fermion content presented in this paper the minimum of $V_{\text{eff}}^6(\theta_6^H)$ would be located at around $\theta_6^H = \pi/2$ (mod.$\pi$). This undesirable result can be avoided by introducing additional $SO(11)$ bulk vector fermions $\Psi_{11}^\beta$ with appropriate boundary conditions.
The phase $\theta_H$ plays a crucial role in the $SO(11)$ gauge-Higgs grand unification. It is found that the 5D KK mass scale $m_{KK_5}$ is determined by $\theta_H$ in a very good approximation by the formula (5.14). Similar relations are expected for the 4D Higgs cubic and quartic couplings as in the $SO(5) \times U(1)$ gauge-Higgs EW unification. For $m_{KK_5}(\theta_H)$ we have recognized the difference between the gauge-Higgs EW and grand unification. We need experimental data to find which one is preferred.

We will come back to these issues in the near future.

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A Basics for 6D Kaluza-Klein expansion

We present KK mode expansions in 6D hybrid warped space with the metric given in (2.1).

A.1 6D gauge fields

The equations of motion for free gauge fields in the 6D hybrid warped space are given by

\[
\left\{ \eta^{\lambda \rho} (\Box + k^2 P_5 + \partial_v^2) - \left( 1 - \frac{1}{\xi} \right) \partial^{\lambda} \partial^{\rho} \right\} A_\rho = 0 ,
\]

\[
(\Box + \xi k^2 P_z + \partial_v^2) A_z = 0 ,
\]

(A.1)

where $P_5$ and $P_z$ are defined in Eq. (2.32). These $P_5$ and $P_z$ differ from $P_4$ and $P_z$ in the 5d case. Basis mode functions in the $z$-coordinate in the 6d case are given, except for zero modes, by

\[
C(z; \lambda) = \frac{\pi}{2} \lambda z^{3/2} z_L^{1/2} F_{\frac{1}{2}, \frac{3}{2}} \left( \lambda z, \lambda z_L \right),
\]

\[
S(z; \lambda) = -\frac{\pi}{2} \lambda z^{3/2} z_L^{1/2} F_{\frac{1}{2}, \frac{3}{2}} \left( \lambda z, \lambda z_L \right),
\]

\[
C'(z; \lambda) = \frac{\pi}{2} \lambda^2 z^{3/2} z_L^{1/2} F_{\frac{1}{2}, \frac{1}{2}} \left( \lambda z, \lambda z_L \right),
\]

\[
S'(z; \lambda) = -\frac{\pi}{2} \lambda^2 z^{3/2} z_L^{1/2} F_{\frac{1}{2}, \frac{1}{2}} \left( \lambda z, \lambda z_L \right),
\]

(A.2)
where \( F_{\alpha,\beta}(u, v) = J_{\alpha}(u)Y_{\beta}(v) - Y_{\alpha}(u)J_{\beta}(v) \). They can be expressed as

\[
C(z; \lambda) = z \cos \lambda(z - z_L) - \frac{1}{\lambda} \sin \lambda(z - z_L),
\]

\[
S(z; \lambda) = \left( z + \frac{1}{\lambda^2 z_L} \right) \sin \lambda(z - z_L) + \left( \frac{1}{\lambda} - \frac{z}{\lambda z_L} \right) \cos \lambda(z - z_L),
\]

\[
C'(z; \lambda) = -\lambda z \sin \lambda(z - z_L),
\]

\[
S'(z; \lambda) = \lambda z \cos \lambda(z - z_L) + \frac{z}{z_L} \sin \lambda(z - z_L),
\]

(A.3)

and satisfy

\[
\mathcal{P}_5 \left( \frac{C}{S} \right) = z^2 \frac{d}{dz} \frac{1}{z^2} \frac{d}{dz} \left( \frac{C}{S} \right) = -\lambda^2 \left( \frac{C}{S} \right),
\]

\[
\mathcal{P}_z \left( \frac{C'}{S'} \right) = \frac{d}{dz} z^2 \frac{d}{dz} \left( \frac{C'}{S'} \right) = -\lambda^2 \left( \frac{C'}{S'} \right),
\]

\[
C(z_L; \lambda) = z_L, \quad C'(z_L; \lambda) = 0, \quad S(z_L; \lambda) = 0, \quad S'(z_L; \lambda) = \lambda z_L.
\]

(A.4)

These basis functions have been employed in the text. We note that the basis functions for fermions, \( C_{L/R}(z; \lambda, c) \) and \( S_{L/R}(z; \lambda, c) \), remain the same as in the 5d case defined in Ref. [35].

### A.2 6D scalar fields

The action of a massless 6D scalar field \( \Phi(x, y, v) \) is given by

\[
S_{\text{scalar}}^{\text{bulk}} = \int d^6x \sqrt{-\det G} G^{MN} \partial_M \Phi(x, y, v) \partial_N \Phi(x, y, v),
\]

(A.5)

where \( M, N = 0, 1, 2, 3, 5, 6 \). The equation of motion for the scalar field \( \Phi(x, z, v) \) is given by

\[
z^2 \left\{ \Box_4 + \partial_v^2 + k^2 z^4 \frac{1}{z^4} \frac{\partial}{\partial z} \right\} \Phi(x, z, v) = 0.
\]

(A.6)

It follows that \( \phi(x, z, v) = z^{-\nu} \Phi(x, z, v) \) (\( \nu = \frac{5}{2} \)) satisfies

\[
\left\{ \Box_4 + \partial_v^2 + k^2 \left( \frac{\partial^2}{\partial z^2} + \frac{1}{z} \frac{\partial}{\partial z} - \frac{\nu^2}{z^2} \right) \right\} \phi(x, z, v) = 0.
\]

(A.7)

The Bessel equation is given by

\[
\left( \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + 1 - \frac{\nu^2}{z^2} \right) u(z) = 0,
\]

(A.8)
whose solutions are given by the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$. Hence a mode function can be written as $\phi(x, z, v) = [\alpha J_\nu(\lambda z) + \beta Y_\nu(\lambda z)]\phi_\lambda(x, v)$ where $\phi_\lambda(x, v)$ satisfies

$$\{\Box_4 + \partial_v^2 - k^2 \lambda^2\} \phi_\lambda(x, v) = 0 . \tag{A.8}$$

The orbifold boundary condition for the scalar field $\phi$ is given by

$$\Phi(x, y_j - y, v_j - v) = P_j \Phi(x, y_j + y, v_j + v) . \tag{A.9}$$

In the $y$ coordinate $\phi(x, y, v) = e^{-\nu \sigma(y)}\Phi(x, y, v)$ satisfies the same boundary condition as (A.9) and Eq. (A.6) becomes

$$\{\Box_4 + \partial_v^2 + e^{\frac{1}{2}\sigma} (\partial_v^2 + \nu \sigma'' - \nu^2 \sigma'^2)\} \phi = 0 . \tag{A.10}$$

Due caution must be taken as $\sigma''(y) = 2k\{\delta_{2L_5}(y) - \delta_{2L_5}(y - L_5)\}$. If $\phi$ is parity even in the $y$ coordinate, the Neumann ($N$) condition becomes, in the $z$ coordinate ($1 \leq z \leq z_L$),

$$N : \left(\frac{\partial}{\partial z} + \frac{\nu}{z}\right) \phi = 0 \quad \text{at} \quad z = 1^+ \text{ or } z_L^- , \tag{A.11}$$

as can be confirmed by integrating (A.10) over $y$ from $-\varepsilon$ to $+\varepsilon$ (or from $L_5 - \varepsilon$ to $L_5 + \varepsilon$).

If $\phi$ is parity odd in the $y$ coordinate, it satisfies the Dirichlet ($D$) condition $\phi = 0$ at $z = 1$ or $z_L$.

(i) $(P_0 = P_1, P_2 = P_3) = (+, +)$

The 6D scalar field $\Phi(x, y, v)$ is expanded as

$$\Phi(x, y, v) = e^{-\nu \sigma(y)}\left\{\sum_{n=0}^{\infty} \widetilde{\phi}_n^C(x, y)f_n^C(v) + \sum_{n=1}^{\infty} \widetilde{\phi}_n^S(x, y)f_n^S(v)\right\} , \tag{A.12}$$

where $\widetilde{\phi}_n^C(x, y)$ and $\widetilde{\phi}_n^S(x, y)$ satisfy

$$\begin{cases} 
\widetilde{\phi}_n^C(x, -y) = \widetilde{\phi}_n^C(x, y) \\
\widetilde{\phi}_n^C(x, L_5 - y) = \widetilde{\phi}_n^C(x, L_5 + y)
\end{cases} , \tag{A.13}$$

and Fourier modes $f_n^C(v)$ and $f_n^S(v)$ are given by

$$f_n^C(v) = \begin{cases} 
\frac{1}{\sqrt{2\pi R_6}} & (n = 0) \\
\frac{1}{\sqrt{\pi R_6}} \cos \frac{nv}{R_6} & (n \geq 1)
\end{cases} ,$$

$$f_n^S(v) = \frac{1}{\sqrt{\pi R_6}} \sin \frac{nv}{R_6} ,$$

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\[ f_n^S(v) = \frac{1}{\sqrt{\pi R_6}} \sin \frac{n\nu}{R_6} \quad (n \geq 1) . \tag{A.14} \]

\(\tilde{\phi}_n^C\) and \(\tilde{\phi}_n^S\) satisfy the Neumann and Dirichlet conditions at \(z = 1, z_L\), respectively. There is a zero mode for \(\tilde{\phi}_n^C\):

\[ \lambda_0^C = 0 , \quad \tilde{\phi}_n^C(x, z) = \phi_{n,0}^C(x) z^{-\nu} , \tag{A.15} \]

which corresponds to a mode constant in the 5th dimension. Other modes can be written as

\[ \tilde{\phi}_n^{C,S}(x, z) = Z_\nu(\lambda z) \phi_{n,\lambda}^{C,S}(x) \, , \]

\[ Z_\nu(\lambda z) = \alpha_{n,\lambda}^{C,S} J_\nu(\lambda z) + \beta_{n,\lambda}^{C,S} Y_\nu(\lambda z) . \tag{A.16} \]

Eigenvalues \(\lambda > 0\) and the relative coefficient \(\alpha/\beta\) are determined by the boundary conditions. Noting a recursion relation

\[ \left( \frac{d}{dz} + \frac{\nu}{z} \right) Z_\nu(\lambda z) = \lambda Z_{\nu-1}(\lambda z) , \]

one finds eigenvalues \(\{\lambda_{\ell}^{C,S}\} (\ell \geq 1)\) from

\[ \tilde{\phi}_n^C : \quad Z_{\nu-1}(\lambda_{\ell}^C) = Z_{\nu-1}(\lambda_{\ell}^C z_L) = 0 \, , \]

\[ \tilde{\phi}_n^S : \quad Z_{\nu}(\lambda_{\ell}^S) = Z_{\nu}(\lambda_{\ell}^S z_L) = 0 , \tag{A.17} \]

Thus \(\Phi\) is expanded as

\[ \Phi(x, z, v) = e^{\nu \sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \phi_{n,\ell}^C(x) Z_\nu(\lambda_{\ell}^C z) f_n^C(v) \right. \]

\[ + \left. \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \phi_{n,\ell}^S(x) Z_\nu(\lambda_{\ell}^S z) f_n^S(v) \right\} , \tag{A.18} \]

where \(\tilde{\phi}_{n,\ell}^{C,S}(x)\) satisfies

\[ \left\{ \Box_4 - (m_{n,\ell}^{C,S})^2 \right\} \phi_{n,\ell}^{C,S}(x) = 0 \, , \]

\[ (m_{n,\ell}^{C,S})^2 = \frac{n^2}{R_6^2} + (k\lambda_{\ell}^{C,S})^2 . \tag{A.19} \]

There is one massless mode \(\phi_{0,0}^C(x)\) with \(m_{0,0}^C = 0\).

(ii) \((P_0 = P_1, P_2 = P_3) = (-,-)\)
The expansion is given by

\[ \Phi(x, z, v) = e^{\nu \sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \phi^S_{n,\ell}(x) Z_\nu(\lambda^S_\ell z) f_n^C(v) \right. \\
\left. + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} \phi^C_{n,\ell}(x) Z_\nu(\lambda^C_\ell z) f_n^S(v) \right\}, \]  

(A.20)

There is no massless mode.

(iii) \((P_0 = P_1, P_2 = P_3) = (+, -)\)

\[ \Phi(x, z, v) = e^{\nu \sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \phi^C_{n,\ell}(x) Z_\nu(\lambda^C_\ell z) g^C_{n+\frac{1}{2}}(v) \right. \\
\left. + \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \phi^S_{n,\ell}(x) Z_\nu(\lambda^S_\ell z) g^S_{n+\frac{1}{2}}(v) \right\}, \]  

(A.21)

where

\[ g^C_{n+\frac{1}{2}}(v) = \frac{1}{\sqrt{\pi R_6}} \cos \left( \frac{n + \frac{1}{2}}{v} \right), \quad g^S_{n+\frac{1}{2}}(v) = \frac{1}{\sqrt{\pi R_6}} \sin \left( \frac{n + \frac{1}{2}}{v} \right). \]  

(A.22)

Note that the mass spectrum is given by

\[ (m_{n,\ell}^{C,S})^2 = \left( \frac{n + \frac{1}{2}}{R_6^2} \right)^2 + \left( k \lambda_{C,S}^{n,\ell} \right)^2 \geq \left( \frac{1}{2 R_6} \right)^2. \]  

(A.23)

(iv) \((P_0 = P_1, P_2 = P_3) = (-, +)\)

\[ \Phi(x, z, v) = e^{\nu \sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \phi^S_{n,\ell}(x) Z_\nu(\lambda^S_\ell z) g^C_{n+\frac{1}{2}}(v) \right. \\
\left. + \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \phi^C_{n,\ell}(x) Z_\nu(\lambda^C_\ell z) g^S_{n+\frac{1}{2}}(v) \right\}. \]  

(A.24)

### A.3 6D fermion field

Let us consider a free 6D Weyl fermion \(\Psi(x, y, v)\) with \(\gamma^7_{6D} = +1\):

\[ \Psi(x, y, v) = e^{\frac{\gamma^7}{2} \sigma(y)} \xi(x, z, v), \quad \bar{\Psi}(x, z, v) = \begin{bmatrix} \xi(x, z, v) \\ \eta(x, z, v) \\ 0 \\ 0 \end{bmatrix}. \]
\[
\gamma^7_{6D} = \begin{pmatrix} I_4 & -I_4 \end{pmatrix} = \gamma^5_{4D} \cdot (-i \gamma^5 \gamma^6), \quad \gamma^5_{4D} = \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}.
\] (A.25)

Its action is given by
\[
I = \int d^6x \sqrt{-\det G} \Psi \left\{ \gamma^A E_A^M D_M + i c k \gamma^6 \right\} \Psi \\
= \int d^4x \int_{2\pi R_6}^{2\pi R_6} dv \int_{z_L}^{z_U} dz \left[ -k D_-(c) + i \partial_v \sigma^\mu \partial_\mu - k D_+(c) + i \partial_v \right] \left[ \xi \eta \right] \tag{A.26}
\]
where \( D_\pm(c) \) is defined in (4.5). As an example we consider the case in which the BCs are given by
\[
\Psi(x, y_j - y, v_j - v) = -i \gamma^5 \gamma^6 P_j \Psi(x, y_j + y, v_j + v) = \gamma^7_{6D} \gamma^5_{4D} P_j \Psi(x, y_j + y, v_j + v). \tag{A.27}
\]

Equations of motion are
\[
(-k D_-(c) + i \partial_v) \xi + \sigma^\mu \partial_\mu \eta = 0, \\
\sigma^\mu \partial_\mu \xi + (-k D_+(c) + i \partial_v) \eta = 0. \tag{A.28}
\]

To find eigenmodes, we separate 4D and 5, 6th dimensional parts as
\[
\xi(x, z, v) = \tilde{\xi}(z, v) f_R(x), \quad \sigma^\mu \partial_\mu f_R(x) = m f_L(x), \tag{A.29}
\]
\[
\eta(x, z, v) = \tilde{\eta}(z, v) f_L(x), \quad \sigma^\mu \partial_\mu f_L(x) = m f_R(x).
\]

It follows that
\[
(-k D_-(c) + i \partial_v) \tilde{\xi} + m \tilde{\eta} = 0, \\
m \tilde{\xi} + (-k D_+(c) + i \partial_v) \tilde{\eta} = 0. \tag{A.30}
\]

In the \( y \) coordinate \( \xi \) and \( \eta \) satisfy the BCs
\[
\begin{pmatrix} \xi \\ \eta \end{pmatrix} (x, y_j - y, v_j - v) = P_j \begin{pmatrix} \xi \\ -\eta \end{pmatrix} (x, y_j + y, v_j + v). \tag{A.31}
\]

(i) \( (P_0 = P_1, P_2 = P_3) = (+, +) \)

In this case
\[
\xi(x, y_j - y, v_j - v) = +\xi(x, y_j + y, v_j + v), \\
\eta(x, y_j - y, v_j - v) = -\eta(x, y_j + y, v_j + v). \tag{A.32}
\]
Mode functions can be written as

$$\xi(x, y, v) = \left\{ \sum_{n=0}^{\infty} \tilde{\xi}^C_n(y) f^C_n(v) + \sum_{n=1}^{\infty} \tilde{\xi}^S_n(y) f^S_n(v) \right\} f_R(x) ,$$

$$\eta(x, y, v) = \left\{ \sum_{n=0}^{\infty} \tilde{\eta}^S_n(y) f^C_n(v) + \sum_{n=1}^{\infty} \tilde{\eta}^C_n(y) f^S_n(v) \right\} f_L(x) ,$$

(A.33)

where the \( f^C_n(v), f^S_n(v) \) are defined in (A.14). \( \tilde{\xi}^C_n \) and \( \tilde{\eta}^C_n \) satisfy the BCs

$$\tilde{\xi}^C_n(y_j - y) = +\tilde{\xi}^C_n(y_j + y) , \quad \tilde{\xi}^S_n(y_j - y) = -\tilde{\xi}^S_n(y_j + y) ,$$

$$\tilde{\eta}^C_n(y_j - y) = +\tilde{\eta}^C_n(y_j + y) , \quad \tilde{\eta}^S_n(y_j - y) = -\tilde{\eta}^S_n(y_j + y) .$$

(A.34)

By inserting (A.33) into (A.28), one finds equations for \( \tilde{\xi}^C_n, \tilde{\eta}^C_n \) in the \( z \) coordinate

\( 1 \leq z \leq z_L \):

$$\begin{pmatrix}
-kD_{-} & m & 0 \\
0 & -kD_{+} & \frac{i n}{R_6} \\
0 & -\frac{i n}{R_6} & -kD_{-} \\
\end{pmatrix}
\begin{pmatrix}
\tilde{\xi}^C_0 \\
\tilde{\eta}^S_0 \\
\tilde{\xi}^C_n \\
\end{pmatrix} = 0 \quad (n \geq 1).$$

(A.35)

It follows that

$$ (k^2 D_{+} D_{-} - m^2) \tilde{\xi}^C_0 = 0 ,$$

$$ (k^2 D_{-} D_{+} - m^2) \tilde{\eta}^S_0 = 0 ,$$

$$ (k^2 D_{+} D_{-} + \frac{2 k c n}{R_6} \frac{1}{z} + \frac{n^2}{R_6^2} - m^2) \left( \tilde{\xi}^C_n \pm i \tilde{\xi}^S_n \right) = 0 ,$$

$$ (k^2 D_{-} D_{+} + \frac{2 k c n}{R_6} \frac{1}{z} + \frac{n^2}{R_6^2} - m^2) \left( \tilde{\eta}^S_n \pm i \tilde{\eta}^C_n \right) = 0 \quad (n \geq 1).$$

(A.36)

The equations for the zero \( n = 0 \) modes in the 6th dimension, \( \tilde{\xi}^C_0 \) and \( \tilde{\eta}^S_0 \), are reduced to the Bessel equation. In terms of the basis functions \( C_{R/L}(z; \lambda, c) \), \( S_{R/L}(z; \lambda, c) \) given in

Appendix B of Ref. 35, solutions satisfying the BCs are given by

$$\tilde{\xi}^C_{0,0} = a_0 z^c , \quad \tilde{\eta}^S_{0,0} = 0 , \quad m_0 = 0 ,$$

$$\tilde{\xi}^C_{0,\ell} = a_{\ell} C_R(z; \lambda, c) , \quad \tilde{\eta}^S_{0,\ell} = a_{\ell} S_L(z; \lambda, c) ,$$

$$S_L(1; \lambda, c) = 0 , \quad m_{\ell} = k\lambda_{\ell} > 0 \quad (A.37)$$

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where \( a_\ell \) is a normalization constant. There is one massless mode for the right-handed component. For the \( n \neq 0 \) modes the solutions are more involved.

(ii) \((P_0 = P_1, P_2 = P_3) = (-, -)\)

In this case mode functions of \( \xi \) and \( \eta \) can be written as

\[
\xi(x, y, v) = \left\{ \sum_{n=0}^{\infty} \xi_n(y) f_n^C(v) + \sum_{n=1}^{\infty} \xi_n(y) f_n^S(v) \right\} f_R(x),
\]
\[
\eta(x, y, v) = \left\{ \sum_{n=0}^{\infty} \eta_n(y) f_n^C(v) + \sum_{n=1}^{\infty} \eta_n(y) f_n^S(v) \right\} f_L(x),
\]

(A.38)

The equations for the zero \((n = 0)\) modes in the 6th dimension, \( \tilde{\xi}_0^S \) and \( \tilde{\eta}_0^C \), are

\[
\begin{pmatrix}
-kD_- \\
m
-kD_+
\end{pmatrix}
\begin{pmatrix}
\tilde{\xi}_0^S \\
\tilde{\eta}_0^C
\end{pmatrix} = 0,
\]

(A.39)

and solutions satisfying the BCs are given by

\[
\tilde{\xi}_{0,0}^S = 0, \quad \tilde{\eta}_{0,0}^C = a_0 z^{-c}, \quad m_0 = 0,
\]
\[
\tilde{\xi}_{0,\ell}^S = a_\ell S_R(z; \lambda_\ell, c), \quad \tilde{\eta}_{0,\ell}^C = a_\ell C_L(z; \lambda_\ell, c),
\]
\[
S_R(1; \lambda_\ell, c) = 0, \quad m_\ell = k\lambda_\ell > 0
\]

(A.40)

where \( a_\ell \) is a normalization constant. There is one massless mode for the left-handed component.

(iii) \((P_0 = P_1, P_2 = P_3) = (+, -)\)

In this case mode functions can be written as

\[
\xi(x, y, v) = \left\{ \sum_{n=0}^{\infty} \xi_n(y) g_{n+\frac{1}{2}}^C(v) + \sum_{n=0}^{\infty} \xi_n(y) g_{n+\frac{1}{2}}^S(v) \right\} f_R(x),
\]
\[
\eta(x, y, v) = \left\{ \sum_{n=0}^{\infty} \eta_n(y) g_{n+\frac{1}{2}}^C(v) + \sum_{n=1}^{\infty} \eta_n(y) g_{n+\frac{1}{2}}^S(v) \right\} f_L(x),
\]

(A.41)

where \( g_{n+\frac{1}{2}}^{C,S}(v) \) are defined in (A.22). There are no zero modes in the 6th dimension.

(iv) \((P_0 = P_1, P_2 = P_3) = (-, +)\)

In this case mode functions can be written as

\[
\xi(x, y, v) = \left\{ \sum_{n=0}^{\infty} \xi_n(y) g_{n+\frac{1}{2}}^C(v) + \sum_{n=0}^{\infty} \xi_n(y) g_{n+\frac{1}{2}}^S(v) \right\} f_R(x),
\]
\[
\eta(x, y, v) = \left\{ \sum_{n=0}^{\infty} \eta_n(y) g_{n+\frac{1}{2}}^C(v) + \sum_{n=1}^{\infty} \eta_n(y) g_{n+\frac{1}{2}}^S(v) \right\} f_L(x).
\]

(A.42)

There are no zero modes in the 6th dimension.
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