Abstract

In 1985, Edward Nelson, who formulated the theory of stochastic mechanics, made an interesting remark on Bell’s theorem. Nelson analysed the latter in the light of classical fields that behave randomly. He found that if a stochastic hidden variable theory fulfils certain conditions, the inequality of Bell can be violated. Moreover, Nelson was able to prove that this may happen without any instantaneous communication between the two spatially separated measurement stations. Since Nelson’s article got almost overlooked by physicists, we try to review his comments on Bell’s theorem. We argue that a modification of stochastic mechanics published recently by Fritsche and Haugk can be extended to a theory which fulfils the requirements of Nelson’s analysis. The article proceeds to derive the quantum mechanical formalism of spinning particles and the Pauli equation from this version of stochastic mechanics. Then, we investigate Bohm’s version of the EPR experiment. Additionally, other setups, like entanglement swapping or time and position correlations, are shortly explained from the viewpoint of our local hidden-variable model. Finally, we mention that this theory could perhaps be relativistically extended.

Contents

1 Introduction 2

2 Nelson’s analysis of Bell’s theorem 3
  2.1 The setup of the EPR experiment in theory . . . . . . . . . . . . . . . . . . . . . . . . . 3
  2.2 Active locality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.3 Passive locality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  2.4 Nelson’s theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

3 What the failure of passive locality implies – a deeper analysis 9

4 Locality in the hidden variable theory of Fritsche and Haugk 11
  4.1 Derivation of the single-particle Schrödinger equation from Brownian motion . . . 11
  4.2 Quantum states in superpositions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
  4.3 Locality in the theory of Fritsche and Haugk with single-particle systems . . . . . 17
  4.4 Derivation of the many-particle Schrödinger equation . . . . . . . . . . . . . . . . . . . 18
  4.5 Locality in the theory of Fritsche and Haugk with many-particle systems . . . . . . 20
  4.6 Entangled states in the model of Fritsche and Haugk . . . . . . . . . . . . . . . . . . . . 20

5 Spin in a Fritsche-Haugk like hidden variable theory 22
  5.1 Preliminary considerations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
  5.2 Stern-Gerlach effect . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
  5.3 Pauli equation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
  5.4 Spin operator . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 26

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1 Introduction

In 1935, Einstein, Podolsky and Rosen (EPR) wrote an article in which they denied that quantum theory would be a complete theory of nature [1]. Around 1951, Bohm gave a more testable outline of the so-called “EPR paradox” [2, 3]. He described a thought experiment with one source that ejects particles having opposite spin to two spatially separate Stern-Gerlach magnets of variable orientations, see Fig. 1. Then, in 1964, Bell published a theorem about this paradox in the form of an inequality. It made clear that hidden variable theories fulfilling certain conditions would contradict quantum mechanics. Bell called these conditions “locally causal” and explained them in detail in [4].

In his first publication on that topic, Bell wrote: “If hidden parameters would be added to quantum mechanics, there must be a mechanism, whereby the setting of one measuring device can influence another spatially separated device” and the signal involved has to “propagate instantaneously” [5]. However, Bell’s first contribution underwent several modifications. Over the years, more and more instructive proofs of his inequality were constructed by him and others. A collection of all of Bell’s fundamental articles can be found in [6]. Finally in 1969, Clauser, Holt, Shimony and Horne (CHSH) brought the inequality of Bell into a form suitable for experimental investigation [7].

In 1985, Nelson tried to analyse Bell’s theorem with full mathematical rigour. The result of his analysis was that Bell’s definition of “locally causal”, which is the starting point to derive Bell’s inequality, could be divided into two separate conditions. Both were necessary to derive the inequality, but only one of them has to hold if a hidden parameter theory does not include instantaneous signals. Nelson published his result twice [8, 9]. (However, a small correction was added later [10], in order to make Nelson’s theorem compatible with Mermin’s presentation [11].) Furthermore, there are studies from other mathematicians, for example [12], which give further insightful analysis of Nelson’s work. Unfortunately, Nelson’s articles on Bell’s inequality got almost overlooked by physicists. The reason for this might be that Nelson found his own interpretation of quantum mechanics [13, 14] to be at variance with the requirements of his theorem for a model without instantaneous signalling effects.

This article is organised as follows: In section 2, we review Nelson’s contribution towards a mathematically rigorous understanding of Bell’s inequality. The physical implications of that contribution are discussed in section 3. A recent modification of Nelson’s stochastic mechanics [15] is the topic of section 4. It is shown that this modification fulfils Nelson’s requirements for a local hidden variable model which is compatible with quantum mechanics. In section 5, we extend this version of stochastic mechanics in order to include spinning particles described by the Pauli equation. Entangled spin states are discussed in section 6, where it is shown in detail how their correlations emerge without any instantaneous signalling effects. In section 7, we review some arguments from the literature against Nelson’s thoughts on Bell’s theorem. We come to the conclusion that these arguments are unfounded. The article closes with section 8, where we mention the possible use of our theory as a reasonable foundation of quantum mechanics in curved space-times.

The sections 2 and 3 will need some understanding of mathematical probability theory. For a general introduction to this theory, see [16–19].
Stern-Gerlach magnets, we write their outcomes on the state space using the axis dependent notation \( \Omega^j \). The results at the detectors could depend on the preparation of the particles by the source. We set up two families of stochastic processes \( \phi_{j1t} : \Omega_j \rightarrow \Omega'_j \) and \( \phi_{j2t} : \Omega_j \rightarrow \Omega'_j \). They are called “spin trajectories” and denote the time dependent spin values of the \( j \)-th particles that are sent to detector 1 and 2. The outcomes in the state spaces of \( \phi_{j1t} \) and \( \phi_{j2t} \) are elements of sets \( \Omega'_j = \Omega'_j \equiv \Omega'_j \equiv \{\uparrow, \downarrow\} \). The sets \( \Omega_j \) and \( \Omega_j \) belong to sample spaces of two probability spaces whose definitions will be given later.

As parameter of the spin trajectories, the vector \( \vec{z} \) corresponds to the axis around which the described particle rotates. If it rotates in the left direction with respect to \( \vec{z} \), we have \( \phi_{1j1} = \downarrow \), and a rotation in the right direction is defined by \( \phi_{1j1} = \uparrow \). The value \( t \) is the time during which the particle flies. Both particles start at time \( t = 0 \) at the source and arrive at a time \( t = T \), where \( T > 0 \), in a Stern-Gerlach magnet.

When the particles arrive in the Stern-Gerlach magnets at time \( t = T \), the magnetic field of these devices could change the particle’s properties, including the spin. In an EPR experiment, the parts of the detectors that might influence the particles are the axes \( \vec{\mu} \) and \( \vec{\nu} \). Those axes can be chosen by the experimenter freely at will. Due to the Stern-Gerlach magnets, we have to describe our measurement results by device dependent random variables. For the two detectors 1 and 2, we define two axis dependent families of random variables \( D_{j1\vec{\mu}} : \Omega_j \rightarrow \Omega'_j \) and \( D_{j2\vec{\nu}} : \Omega_j \rightarrow \Omega'_j \). The outcomes of these random variables are \( \Omega'_j = \Omega'_j = \Omega'_j = \{\uparrow, \downarrow\} \), and for simplicity, we will assume that \( \Omega_j = \Omega_j \) and \( \Omega_j = \Omega_j \). Since the random variables \( D_{j1\vec{\mu}} \) and \( D_{j2\vec{\nu}} \) depend on the setting of the Stern-Gerlach magnets, we write their outcomes on the state space using the axis dependent notation \( \sigma_{j1\vec{\mu}} \) and \( \sigma_{j2\vec{\nu}} \). However, the way, how the events generated by \( D_{j1\vec{\mu}} \) and \( D_{j2\vec{\nu}} \) are connected to the events generated by \( \phi_{j1t} \) and \( \phi_{j2t} \), will be postponed at the moment.

The results at the detectors could depend on the preparation of the particles by the source. We say that the corresponding events happen at preparation stage. They are generated by the family of product random variables

\[
S_{j1j2} \equiv \phi_{j1t=0} \otimes \phi_{j2t=0} : (\Omega_j \times \Omega_j) \rightarrow (\Omega'_j \times \Omega'_j).
\]
The axis dependent family of product random variables

\[ X_{j\bar{\mu}2\bar{\sigma}} \equiv D_{j\bar{\mu}} \otimes S_{j12} \otimes D_{j2\bar{\sigma}} : (\Omega_{1D} \times \Omega_{j1} \times \Omega_{j2} \times \Omega_{2D}) \mapsto (\Omega'_{j1D} \times \Omega'_{j1} \times \Omega'_{j2} \times \Omega'_{2D}) \]  

(2)
generates events on an enlarged probability space. We define the sigma algebra \( \mathcal{F} \) on that space by the power set \( \mathcal{F} = \mathcal{P} (\Omega_{1D} \times \Omega_{j1} \times \Omega_{j2} \times \Omega_{2D}) \). The probability measure of the enlarged probability space will be denoted by \( \mathbb{P} \).

Now, we introduce a new family of random variables

\[ \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}} : (\Omega_{1D} \times \Omega_{j1} \times \Omega_{j2} \times \Omega_{2D}) \mapsto (\Omega'_{j1D} \times \Omega'_{j2D}) \]  

(3)
whose definition is

\[ \{ X_{j1\bar{\mu}2\bar{\sigma}} \in (\tilde{\sigma}_{j1\bar{\mu}} \times \Omega'_{j1} \times \Omega'_{j2} \times \tilde{\sigma}_{j2\bar{\sigma}}) \} \equiv \{ \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}} \in (\tilde{\sigma}_{j1\bar{\mu}} \times \tilde{\sigma}_{j2\bar{\sigma}}) \} . \]  

(4)

It provides information about outcomes which happen simultaneously at the two detectors. The probabilities of the axis dependent events in Eq. (4) will be abbreviated by

\[ \mathbb{P} (\{ \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}} \in (\tilde{\sigma}_{j1\bar{\mu}} \times \tilde{\sigma}_{j2\bar{\sigma}}) \}) = \mathbb{P}_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}}} (\tilde{\sigma}_{j1\bar{\mu}} \times \tilde{\sigma}_{j2\bar{\sigma}}) . \]  

(5)
The expression \( \mathbb{P}_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}}} \) defines an axis dependent family of probability measures for events of the form \( \tilde{\sigma}_{j1\bar{\mu}} \times \tilde{\sigma}_{j2\bar{\sigma}} \). We say that these events are generated by the family of random variables \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}} \).

There may be situations where we are only interested in the outcomes at a single detector. For this reason, we define the notations \( \sigma_{j1\bar{\mu}} \equiv \tilde{\sigma}_{j1\bar{\mu}} \times \Omega'_{2D} \) and \( \sigma_{j2\bar{\sigma}} \equiv \Omega'_{1D} \times \tilde{\sigma}_{j2\bar{\sigma}} \). The events \( \sigma_{j1\bar{\mu}} = \uparrow \equiv (\tilde{\sigma}_{j1\bar{\mu}} = \uparrow) \times \Omega'_{2D} \) and \( \sigma_{j1\bar{\mu}} = \downarrow \equiv (\tilde{\sigma}_{j1\bar{\mu}} = \downarrow) \times \Omega'_{2D} \) generate a sigma algebra which we denote by \( \mathcal{F}_{j1\bar{\mu}} \). Similarly, the events \( \sigma_{j2\bar{\sigma}} = \uparrow \equiv \Omega'_{1D} \times (\tilde{\sigma}_{j2\bar{\sigma}} = \uparrow) \) and \( \sigma_{j2\bar{\sigma}} = \downarrow \equiv \Omega'_{1D} \times (\tilde{\sigma}_{j2\bar{\sigma}} = \downarrow) \) generate the sigma algebra \( \mathcal{F}_{j2\bar{\sigma}} \). From these definitions, it follows that \( \mathcal{F}_{j1\bar{\mu}} \) (or \( \mathcal{F}_{j2\bar{\sigma}} \)) contains the information about the outcomes at detector 1 (or 2) only.

Since the family of random variables \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\sigma}} \) is defined on the enlarged probability space, it will give us the possibility to investigate the relationship between the outcomes at the detectors and the ones at preparation stage. We let a third sigma algebra \( \mathcal{F}_{JS} \) contain the information about the outcomes at the source. Accordingly, \( \mathcal{F}_{JS} \) is generated by the events \( S_1 \equiv \{ X_{j1\bar{\mu}2\bar{\sigma}} \in (\Omega'_{j1D} \times \uparrow \times \downarrow \times \Omega'_{2D}) \} \) and \( S_2 \equiv \{ X_{j1\bar{\mu}2\bar{\sigma}} \in (\Omega'_{j1D} \times \downarrow \times \uparrow \times \Omega'_{2D}) \} \).

We can condition the events in \( \mathcal{F} \) with respect to the sigma algebra \( \mathcal{F}_{JS} \). The sigma algebra \( \mathcal{F}_{JS} \) is generated by two disjoint events \( S_1 \) and \( S_2 \). The probability of each of these events to happen is \( >0 \) and their union is the sure event of the enlarged probability space. In this special case, the conditional probability of an event \( A \in \mathcal{F} \) with respect to \( \mathcal{F}_{JS} \) is given by (see [16] at section 33, or [19] at page 126):

\[ \mathbb{P} (A | \mathcal{F}_{JS}) \equiv \sum_{i=1}^{2} \frac{1}{\mathbb{P} (S_i)} \int_{S_i} 1_A d\mathbb{P} \, dS_i . \]  

(6)
Eq. (6) is the definition of a random variable for outcomes in the enlarged probability space. It can be interpreted as the revised probability of an event \( A \) to happen with respect to the extra information about which events in \( \mathcal{F}_{JS} \) occur for a given outcome. Obviously, \( \mathbb{P} (A | \mathcal{F}_{JS}) \) fulfills the following properties:

- For all events \( A \in \mathcal{F} \), the random variable \( \mathbb{P} (A | \mathcal{F}_{JS}) \) is measurable with respect to \( \mathcal{F}_{JS} \).
- We have \( 0 \leq \mathbb{P} (A | \mathcal{F}_{JS}) \leq 1 \) for all events \( A \in \mathcal{F} \) and outcomes on the enlarged probability space. If \( A \) is the sure event \( \Omega_{1D} \times \Omega_{j1} \times \Omega_{j2} \times \Omega_{2D} \), the conditional probability \( \mathbb{P} (A | \mathcal{F}_{JS}) \) is unity for all outcomes on the enlarged probability space. In case \( A \) is the impossible event \( \emptyset \), the random variable \( \mathbb{P} (A | \mathcal{F}_{JS}) \) is zero for all outcomes.
For every sequence of pairwise disjoint events \( (A_n)_{n \in \mathbb{N}} \in \mathcal{F} \), the equality \( \sum_{n=1}^{\infty} P (A_n | \mathcal{F}_S) = \sum_{n=1}^{\infty} P (A_n | \mathcal{F}_S) \) holds.

Since the union of \( S_1 \) and \( S_2 \) is the sure event, the unconditional probability of any event \( A \in \mathcal{F} \) can be computed with the expectation value \( \text{EX} \left[ P (A | \mathcal{F}_S) \right] = \int P (A | \mathcal{F}_S) \, dP = P (A) \).

For any event \( A \in \mathcal{F} \) and another event \( x_{jZ} \in \mathcal{F}_S \), where \( x_{jZ} \neq \emptyset \), it follows from Eq. (6) that \( P (A \cap x_{jZ} | \mathcal{F}_S) = P (A | \mathcal{F}_S) \) for outcomes in \( x_{jZ} \), and \( P (A \cap x_{jZ} | \mathcal{F}_S) = 0 \) for outcomes not in \( x_{jZ} \).

In case that the events on the enlarged probability space are generated by \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}} \), we will use the following notation:

\[
P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\bar{\sigma}_{j1\bar{\mu}} \times \bar{\sigma}_{j2\bar{\nu}} | \mathcal{F}_S) \equiv P \left\{ \left( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}} \in (\bar{\sigma}_{j1\bar{\mu}} \times \bar{\sigma}_{j2\bar{nu}}) \right) | \mathcal{F}_S \right\} . \tag{7}
\]

Now, all the mathematical structures needed to analyse the EPR experiment have been defined and we can proceed with the necessary locality conditions.

### 2.2 Active locality

At first, Nelson defined two different forms of locality: Active locality and passive locality. The meaning of active locality is:

Whatever axes the experimenter selects at one measurement device, e.g. at 2, it does not change the probabilities of an event happening at 1, as long as 1 does not lie in the future cone of 2. Hence, active locality corresponds to a “no-signalling” condition. If it is violated, one can send instantaneous signals between two spatially separated locations. We can mathematically define active locality as follows:

We let the area of measurement station 1 be disjoint from the future cone of station 2. With the axis \( \bar{\mu} \) of 1 being left constant, different axes \( \bar{\nu} \neq \bar{\nu}' \) of Stern-Gerlach magnet 2 are chosen. The two pairs of axes influence the measurement results at the two devices through the random variables \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}} \) and \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{nu}} \). We define a theory to be actively local if \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}} \) and \( \phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}} \) are equivalent for probability of an event \( \sigma_{j1\bar{\mu}} \equiv \bar{\sigma}_{j1\bar{\mu}} \times \bar{\sigma}_{j2\bar{\nu}} \), or

\[
P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j1\bar{\mu}}) = P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j1\bar{\mu}}). \tag{8}
\]

A similar condition should hold for the events \( \sigma_{j2\bar{\nu}} \equiv \bar{\sigma}_{j2\bar{\nu}} \times \bar{\sigma}_{j2\bar{\nu}} \) at Stern-Gerlach magnet 2, provided that the spatial region of 2 is disjoint from the future cone of magnet 1:

\[
P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j2\bar{\nu}}) = P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j2\bar{\nu}}). \tag{9}
\]

In Eq. (8), we have \( \bar{\nu}' \neq \bar{\mu} \) for the axes of detector 1, with the axis \( \bar{\mu} \) of magnet 2 being left constant.

By Eq. (7), we can condition the probabilities of \( \sigma_{j1\bar{\mu}} \in \mathcal{F}_{j1\bar{\mu}} \) and \( \sigma_{j2\bar{\nu}} \in \mathcal{F}_{j2\bar{\nu}} \) with respect to \( \mathcal{F}_S \). The events in \( \mathcal{F}_S \) contain no information about the results at the detectors. Instead, they only define, what has happened at the source. The latter is located disjoint from the future cones of the two measurement devices. Therefore, in an actively local theory, the probabilities of the events in \( \mathcal{F}_S \) cannot be changed by the detector’s axes. Accordingly, it follows from Eqs. (8) and (9) that \( P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j1\bar{\mu}} | \mathcal{F}_S) \) and \( P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j2\bar{\nu}} | \mathcal{F}_S) \) will have similar properties as in the equations (8) and (9):

\[
P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j1\bar{\mu}} | \mathcal{F}_S) = P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j1\bar{\mu}} | \mathcal{F}_S), \tag{10}
\]

\[
P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j2\bar{\nu}} | \mathcal{F}_S) = P_{\phi_{j1\bar{\mu}} \otimes \phi_{j2\bar{\nu}}} (\sigma_{j2\bar{\nu}} | \mathcal{F}_S). \tag{11}
\]
2.3 Passive locality

In this subsection, we consider the joint probability of \( \sigma_{j1}\mu \in F_{j1}\mu \) and \( \sigma_{j2}\nu \in F_{j2}\nu \). It gives information about events which happen simultaneously at the spatially separated locations 1 and 2. We say that passive locality holds if

\[
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow \right) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow \right) = 1, \tag{13}
\]

for every pair of axes \( \mu \) and \( \nu \).

It is possible to have active locality without passive locality. Furthermore, a theory that violates passive locality does not have to incorporate any signalling mechanism between the spatially separated measurement stations 1 and 2. However, the true physical implications of the passive locality condition will be postponed at the moment.

2.4 Nelson’s theorem

With the definitions above, quantum mechanics fulfills the following properties:

\[
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow \right) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow \right) = 1, \tag{13}
\]

\[
|E(\mu, \nu) + E(\mu', \nu') + E(\mu', \nu') - E(\mu, \nu')| > 2. \tag{14}
\]

Equation (13) implies that if the axes of the Stern-Gerlach magnets are the same, the spin values measured at 1 and 2 are always opposite. Using \( P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \uparrow) = 1 \) and \( P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j2}\nu = \downarrow) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j2}\nu = \uparrow) = 1 \), we can rewrite Eq. (13) in the following form:

\[
\begin{align*}
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow) & = P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow), \\
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \uparrow) & = P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} (\sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow). \tag{15}
\end{align*}
\]

This makes clear that the events at the detectors are equivalent if the same axes at the separated measurement devices were chosen.

The function \( E(\mu, \nu) \) in Eq. (14) is called correlation coefficient. It is defined through:

\[
E(\mu, \nu) \equiv P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow \right) + P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow \right) - P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \downarrow \right) - P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu = \downarrow \cap \sigma_{j2}\nu = \uparrow \right). \tag{16}
\]

It was shown by Nelson that Eq. (14) is in conflict with theories where Eq. (13) and active as well as passive locality hold. Nelson’s proof [8, 9] (with corrections in [10]) proceeds as follows:

We are interested in the properties of the conditional probabilities of the events \( \sigma_{j1}\mu \in F_{j1}\mu \) and \( \sigma_{j2}\nu \in F_{j2}\nu \) given the sigma algebra \( F_{j}\mathcal{S} \). At first, we will investigate, what the equivalence property of Eq. (17) combined with the assumption of passive locality implies for these conditional probabilities.

If the axes at the two detectors are the same, i.e. \( \mu = \nu \), we have, due to passive locality:

\[
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu \cap \sigma_{j2}\nu \right) = P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu \mid F_{j}\mathcal{S} \right) P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j2}\nu \mid F_{j}\mathcal{S} \right). \tag{17}
\]

Since \( 0 \leq P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j2}\nu \mid F_{j}\mathcal{S} \right) \leq 1 \), we get

\[
P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu \cap \sigma_{j2}\nu \mid F_{j}\mathcal{S} \right) \leq P_{\phi_{j1}\mu \otimes \phi_{j2}\nu} \left( \sigma_{j1}\mu \mid F_{j}\mathcal{S} \right). \tag{18}
\]
With the definitions of $\sigma_{1\mu}$ and $\sigma_{2\mu}$, it follows from Eq. (5), the properties of the conditional probabilities stated in section 2.1, and Eq. (7) that

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu}) = P (\{\phi_{1\mu} \otimes \phi_{2\mu} \in \sigma_{1\mu}\})$$

$$= \text{EX} [P ( \{\phi_{1\mu} \otimes \phi_{2\mu} \in \sigma_{1\mu}\} | F_j S)]$$

$$= \text{EX} [P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} | F_j S)]$$

(19)

and, similarly,

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} \cap \sigma_{2\mu}) = \text{EX} [P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} \cap \sigma_{2\mu} | F_j S)].$$

(20)

Using Eq. (15), we observe that an event $\sigma_{1\mu}$ at detector 1 implies an equivalent event $\sigma_{2\mu}$ at 2, i.e.

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu}) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} \cap \sigma_{2\mu}) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu}).$$

(21)

Hence, the expectation values in Eqs. (19) and (20) are equal. Accordingly, we get with Eq. (18):

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} \cap \sigma_{2\mu} | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} | F_j S),$$

(22)

and

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} \cap \sigma_{2\mu} | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} | F_j S).$$

(23)

Now, we will look at the consequences of active and passive locality for the conditional probabilities. Passive locality demands for events $\sigma_{1\mu} = \uparrow$ at 1 and $\sigma_{2\mu} = \downarrow$ at 2 that:

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow \cap \sigma_{2\mu} = \downarrow | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow | F_j S) \text{ \times } P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S).$$

(24)

Active locality implies that the actions happening at 2 can only affect anything in the future cone of 2. Since 1 does not lie in that cone, nothing what happens in 2 can affect the events measured at 1. Therefore, the random variables $\phi_{1\mu} \otimes \phi_{2\mu}$ and $\phi_{1\mu}N \otimes \phi_{2\mu}$ are equivalent for the probability of $\sigma_{1\mu} = \uparrow$:

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow | F_j S) = P_{\uparrow}.$$

(25)

An analogous expression is true for the events at 2:

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S).$$

(26)

Without loss of generality, we can select an axis $\mu = \tilde{\nu}$ with $\tilde{\nu}$ from Eq. (26) in the Eqs. (15), (22) and (24). It then follows from Eq. (15) that the event at detector 2 in the right hand side of Eq. (20) implies an equivalent event of $\sigma_{1\mu} = \uparrow$ at 1. Using the Eqs. (22) and (23) with an axis $\mu = \tilde{\nu}$, we can conclude that

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S) = P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S)$$

$$= P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow | F_j S) \equiv P_{\uparrow}. \text{ \quad (27)}$$

Plugging Eq. (25) and Eq. (27) back to Eq. (24), we arrive at

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{1\mu} = \uparrow \cap \sigma_{2\mu} = \downarrow | F_j S) = P_{\uparrow} P_{\uparrow}.$$

(28)

The events $\sigma_{2\mu} = \downarrow$ and $\sigma_{2\mu} = \uparrow$ are disjoint and their union is the sure event. According to section 2.1, the sum of the conditional probabilities with respect to a sigma algebra is equal to unity for such events. Therefore, we get with Eq. (27):

$$P_{\phi_{1\mu}\otimes\phi_{2\mu}} (\sigma_{2\mu} = \downarrow | F_j S) = 1 - P_{\uparrow}. \text{ \quad (29)}$$

7
Since active locality holds, a similar result can be written for the event \( \sigma_{j1\mu} = \downarrow \) at 1:

\[
\begin{align*}
P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \downarrow | F_j S) &= P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \downarrow | F_j S) \\
&= 1 - P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \downarrow | F_j S) = 1 - P_{\mu}.
\end{align*}
\]  

(30)

Due to passive locality and the Eqs. (29) and (30), we have for the events \( \sigma_{j1\mu} = \downarrow \) and \( \sigma_{j2\sigma} = \uparrow \):

\[
\begin{align*}
P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \downarrow \cap \sigma_{j2\sigma} = \uparrow | F_j S) &= (1 - P_{\mu}) (1 - P_{\sigma}).
\end{align*}
\]  

(31)

In the same way, we can derive the relations

\[
\begin{align*}
P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \uparrow \cap \sigma_{j2\sigma} = \downarrow | F_j S) &= P_{\mu} (1 - P_{\sigma})
\end{align*}
\]  

(32)

and

\[
\begin{align*}
P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} = \uparrow \cap \sigma_{j2\sigma} = \uparrow | F_j S) &= (1 - P_{\mu}) P_{\sigma}.
\end{align*}
\]  

(33)

Using the Eqs. (32), (33), (28) and (31), we may define the function

\[
E (\mu, \nu | F_j S) = P_{\mu} (1 - P_{\sigma}) + (1 - P_{\mu}) P_{\sigma} - P_{\mu} P_{\sigma} - (1 - P_{\mu}) (1 - P_{\sigma}).
\]  

(34)

The conditional probabilities in Eq. (34) are all in the range \( 0 \leq P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} \cap \sigma_{j2\sigma} | F_j S) \leq 1 \). Therefore, the following inequality can be computed with four arbitrary axes \( \mu, \nu, \mu' \) and \( \nu', \nu' \) at the two stations 1 and 2:

\[
|E (\mu, \nu | F_j S) + E (\mu' \nu | F_j S) + E (\mu' \nu' | F_j S) - E (\mu, \nu' | F_j S)| \leq 2.
\]  

(35)

If we replace the conditional probabilities in Eq. (35) by their corresponding unconditional probability measures, we arrive at the correlation coefficient from Eq. (14). The unconditional probabilities are also in the range \( 0 \leq P_{\phi_{j1\mu}\otimes\phi_{j2\sigma}} (\sigma_{j1\mu} \cap \sigma_{j2\sigma}) \leq 1 \), and they are given by the expectation values of the conditional probabilities. Hence, an inequality analogous to Eq. (35) must be true for them:

\[
|E (\mu, \nu) + E (\mu', \nu') + E (\mu' , \nu) - E (\mu, \nu' )| \leq 2.
\]  

(36)

Equation (36) is called Clauser-Holt-Shimony-Horne inequality [7]. It is a version of the statement in Bell’s first article [5]

\[
|E (\mu, \nu) - E (\mu', \nu') - E (\nu', \nu')| \leq 1,
\]  

(37)

which can be similarly derived. Both Eq. (36) and Eq. (37) are violated in quantum mechanics and this violation was confirmed experimentally in 1982 [20].

The analysis of Nelson leads to two possibilities: Since Bell’s inequality can only be derived assuming that **active and passive locality** holds, in a theory that reproduces quantum mechanics, either **active locality** or **passive locality** has to be violated. A violation of active locality would mean that action at a distance is possible. The physical implications of a violation of passive locality will be explored in the next section.

It should be remarked that Jarret arrived at a similar conclusion [21], just one year before Nelson published his articles. Although Jarret was not able to identify the correct probability space given by quantum mechanics for the EPR experiment, one can find a preliminary definition of active and passive locality in his publication.
3 What the failure of passive locality implies – a deeper analysis

Stochastic processes can be constructed such that they act on each particle actively local, but still are violating passive locality. This was shown by Nelson with two examples [8, 9]. The first example was a rather ad-hoc construction of a stochastic process fulfilling the Klein-Gordon equation. The second example, mentioned briefly at p. 537 in [9] and, with some more detail, at p. 450 in [8], was a field theoretic variant of stochastic mechanics. It was originally developed by Guerra and Ruggiero [22], and it allowed to interpret various bosonic field theories, including the Maxwell field, the vector meson field and the linearised gravitational field, as the result of a stochastic process [23–25]. Unfortunately, this theory had no tempting particle interpretation as Nelson’s original theory had, where the particles move on stochastic paths. Therefore, the program of Guerra and Ruggiero was never developed extensively.

In 2003, Fritsche and Haugk published another modification of stochastic mechanics [15]. Unlike the model of Guerra and Ruggiero, the work in [15] constructs a non-relativistic hidden variable theory for fermions. The present article aims to show that the theory of Fritsche and Haugk also violates the model of Guerra and Ruggiero, the work in [15] constructs a non-relativistic hidden variable theory without any communication between the two separated measurement stations of the EPR experiment. However, in order to reach our goal, we need to evaluate the implications of Nelson’s theorem further.

We begin by analysing some deeper consequences of passive locality. Faris showed in [12] that if passive locality is combined with relation (13) from quantum mechanics, it immediately leads to another condition, which he calls “deterministic passive locality”. Suppose two spatially separated events happen or not happen always together with probability 1. Then, deterministic passive locality states that these events are determined by some random event at a prior preparation stage. Deterministic passive locality is defined mathematically as follows:

Let the events \( \sigma_{j1\hat{\mu}} \in \mathcal{F}_{j1\hat{\mu}} \) at 1 and \( \sigma_{j2\hat{\mu}} \in \mathcal{F}_{j2\hat{\mu}} \) at 2 be equivalent with respect to \( P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \) and passive locality hold. Then, there must be an event \( x_{jZ} \in \mathcal{F}_{jS} \) at preparation stage, which is equivalent to both \( \sigma_{j1\hat{\mu}} \) and \( \sigma_{j2\hat{\mu}} \).

In his contribution, Faris states that a similar result is presented by Redhead in [26] at pp. 101-102. Redhead claims, it would have been discovered at first by Suppes and Zanotti [27]. Recently, another derivation was given by Conway and Kochen [28] (who called their result “free will theorem” to emphasise its implication of the particle’s random behaviour). They relate the statement given below to the famous Kochen-Specker theorem [29]. The articles of Conway, Kochen and Specker are using geometric arguments. In contrast, the proof of Faris relies on probability theory, and explicitly uses the property of passive locality. The two different methods to prove the same lemma demonstrate an interesting connection between Bell’s theorem and the one of Kochen-Specker. That the inequalities of Bell and the theorem of Kochen-Specker are in some way related to each other was also pointed out by Mermin [30] on the grounds of a different argument. The derivation below will follow closely the lines of Faris:

As shown in section 2.4, it results from the equivalence property of Eq. (15) and the passive locality condition of Eq. (12) that

\[
P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \cap \sigma_{j2\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) = P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) \tag{38}
\]

and, similarly,

\[
P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \cap \sigma_{j2\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) = P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j2\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right). \tag{39}
\]

Using the assumption of passive locality again, we get with Eq. (38) and Eq. (39):

\[
P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) = P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) \tag{38}
\]

\[
= P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j2\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right)
\]

\[
= \left( P_{\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\mu}}} \left( \sigma_{j1\hat{\mu}} \bigm\vert \mathcal{F}_{jS} \right) \right)^2. \tag{40}
\]
Eq. (10) implies, that the random variable \( P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) \) can only have the values 1 and 0. The union event of the outcomes in the enlarged probability space for which \( P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) \) has the value 1 will be denoted by \( x_{j\bar{z}} \). From this definition and Eq. (10), it follows that \( P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) \) is equal to the indicator function of \( x_{j\bar{z}} \):

\[
P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) = 1_{x_{j\bar{z}}}. \tag{41}
\]

According to section 2, the value 1 will be denoted by \( x \).

Due to Eq. (47) and the definition of \( x \), we have \( x_{j\bar{z}} \in F_{jS} \).

The Eqs. (46), (42), (41) and (44), we can conclude that

\[
P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}) = P (x_{j\bar{z}}). \tag{45}
\]

In section 2.1, we have learnt that in case of \( x_{j\bar{z}} \neq \emptyset \), the conditional probability of the intersection between the two events \( \{\phi_{1}\hat{J} \otimes \phi_{2}2 \in \sigma_{j1\hat{J}}\} \) and \( x_{j\bar{z}} \in F_{jS} \) is equal to

\[
P \left( \{\phi_{1}\hat{J} \otimes \phi_{2}2 \in \sigma_{j1\hat{J}}\} \cap x_{j\bar{z}} \mid F_{jS} \right) = \begin{cases} P (\{\phi_{1}\hat{J} \otimes \phi_{2}2 \in \sigma_{j1\hat{J}}\} \mid F_{jS}) & \text{for outcomes in } x_{j\bar{z}}; \\ 0 & \text{for outcomes not in } x_{j\bar{z}}. \end{cases} \tag{46}
\]

On the other hand, we get with \( x_{j\bar{z}} = \emptyset \):

\[
P \left( \{\phi_{1}\hat{J} \otimes \phi_{2}2 \in \sigma_{j1\hat{J}}\} \cap x_{j\bar{z}} \mid F_{jS} \right) = P (\emptyset \mid F_{jS}) = 0 \text{ for all outcomes}. \tag{47}
\]

The Eqs. (46), (42) and (41) lead to the relation

\[
P \left( \{\phi_{1}\hat{J} \otimes \phi_{2}2 \in \sigma_{j1\hat{J}}\} \cap x_{j\bar{z}} \mid F_{jS} \right) = P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) = 1_{x_{j\bar{z}}}. \tag{48}
\]

Due to Eq. (47) and the definition of \( x_{j\bar{z}} \), Eq. (48) also holds in case of \( x_{j\bar{z}} = \emptyset \). Computing the expectation value from both sides of Eq. (48) yields

\[
P \left( \{\phi_{1}\hat{J} \otimes \phi_{2}2 \in (\sigma_{j1\hat{J}} \times \Omega_{j2D})\} \cap x_{j\bar{z}} \right) = P (x_{j\bar{z}}). \tag{49}
\]

Eqs. (15) and (49) imply that the event \( \sigma_{j1\hat{J}} \in F_{j1\hat{J}} \) is equivalent to an event \( x_{j\bar{z}} \in F_{jS} \). It follows from Eq. (38), Eq. (49) and the second line of Eq. (10) that

\[
P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j1\hat{J}}|F_{jS}) = (P_{\phi_{1}\hat{J} \otimes \phi_{2}2} (\sigma_{j2\hat{J}}|F_{jS}))^2. \tag{50}
\]
Using Eq. (41) together with Eq. (50), we arrive at
\[
\mathbb{P}_{\phi_1 \otimes \phi_2} (\sigma_{j1\mu} | F_j S) = \mathbb{P}_{\phi_1 \otimes \phi_2} (\sigma_{j2\mu} | F_j S) = 1_{x_j z}.
\]

A similar calculation as in the Eqs. (43)-(49) but for \(\sigma_{j2\mu}\) then shows that \(\sigma_{j2\mu} \in F_{j1\mu}\) is equivalent to the same event \(x_j z \in F_j S\) which \(\sigma_{j1\mu} \in F_{j2\mu}\) is equivalent to.

Deterministic passive locality states that the events at the detectors must be predetermined by another event at preparation stage. The failure of this condition implies for a hidden variable description of the EPR experiment that during the time of its flight from the source to the measurement apparatus, each entangled particle has to undergo a stochastic process which constantly changes its spin orientation from up to down and vice-versa. This must happen such that the particle’s spin value at the detector is not determined by the spin value the particle has had at the source.

Due to Eq. (13), the two spin values in each entangled particle pair must be opposite at every time until the particles arrive at the detectors. It may be that the experimenters had chosen the same axes of the Stern-Gerlach magnets at the two separated measurement stations. In this case, the spin values of the entangled particles must remain to be opposite after the particles have passed these magnets. Finally, for a theory compatible with relativity theory, all procedures involved must preserve active locality.

If these conditions can be fulfilled, Bell’s inequality will be violated without instantaneous signalling between the two measurement stations 1 and 2. As shall be shown in the next sections of this article, there is no reason to believe that hidden variable theories are unable to do so.

4 Locality in the hidden variable theory of Fritsche and Haugk

4.1 Derivation of the single-particle Schrödinger equation from Brownian motion

In 2003, Fritsche and Haugk published a modification of Nelson’s stochastic mechanics [15]. In contrast to Nelson’s original contribution [13, 14], the model of Fritsche and Haugk is an actively local theory with a non-markovian stochastic process. Moreover, it can, with a slight modification, be made to violate passive locality. Here, we will not discuss in detail how the (many-particle) Schrödinger equation (and the corresponding quantum mechanical operators) can be derived from the stochastic process involved because this is fully described in the original article of Fritsche and Haugk. Instead, the focus of this section is on superposed states, and we show the theory to be actively local even if many-particle systems are considered. In turn, we will have enough material to explain how the correlations in entangled states emerge.

The starting point to derive the modified stochastic mechanics of [15] is to interpret the energy time uncertainty relation \(\Delta E \Delta t \approx \hbar/2\) as an indication that in quantum mechanics energy is only conserved strictly in physical interactions, but not otherwise. Particles are assumed to be under statistical energy fluctuations. In the theory of Fritsche and Haugk, there occurs a deviation \(\Delta E\) of the particle’s energy \(E\) during a time interval \(\Delta t\), after which the initial energy \(E\) has to be restored on the average. In other words: Kinetic energy is thought to be conserved on the average only. The second assumption is in spirit of Ballentine’s famous statistical ensemble interpretation [31]. In this interpretation, the quantum state vector \(|\psi\rangle\) does not represent an individual particle, but a statistical ensemble of infinitely many of them.

For this reason, the probability to find the electron of a hydrogen atom in a volume \(\Omega\) at time \(t\), given by
\[
p(\Omega, t) \equiv \int_{\Omega} \psi^* (\vec{r}, t) \psi (\vec{r}, t) \, d^3 r,
\]
where \(\psi (\vec{r}, t)\) is the electron’s wave-function, describes the following situation: In order to get all the information of \(p(\Omega, t)\) experimentally, one has to prepare infinitely many identical hydrogen atoms in
the same state, including the same orientation in space. Then, one has to measure at each atom if one finds the electron in Ω at time t after the preparation of the state. Finally, one must compute the electron’s probability of being there with all the results from the infinitely large ensemble.

Accordingly, one has to deal with an infinitely large number of sample trajectories in order to describe the probabilistic time evolution of the statistical ensemble of electrons described by a quantum mechanical system. The idea of energy conservation on the average for the individual particles of this ensemble is implemented in [15] as follows: Frütsche and Haugk divide all the sample trajectories into two sub-ensembles with equal average velocities, and therefore the same average kinetic energy. In the sub-ensemble \( \{A\} \), each particle undergoes a Brownian motion with a negative friction coefficient during a time-interval \( \Delta t \). The particles in the sub-ensemble \( \{B\} \) undergo the same process, but with a positive friction coefficient. After each time step, some of the particles are interchanged between \( \{A\} \) and \( \{B\} \), until the kinetic energy of these sub-ensembles agrees again. At this point, the entire procedure is repeated.

For a single particle of rest mass \( m_0 \), we start our discussion with two Langevin equations for \( j \)-th trajectories:

\[
\begin{align*}
  m_0 \ddot{x}_j^A(t) + \frac{m_0}{\tau} \dot{x}_j^A(t) &= F_{j}^{\text{ext}}(t) + F_{j}^{\text{Brown}}(t), \\
  m_0 \ddot{x}_j^B(t) - \frac{m_0}{\tau} \dot{x}_j^B(t) &= -F_{j}^{\text{ext}}(t) + F_{j}^{\text{Brown}}(t).
\end{align*}
\]

The three dimensional vector \( x_j^{A(B)}(t) = (x_{j1}^{A(B)}(t), x_{j2}^{A(B)}(t), x_{j3}^{A(B)}(t)) \) denotes the hidden position variable for the \( j \)-th sample trajectory of a particle in the \( \{A\} (\{B\}) \) ensemble at time \( t \), where \( j \in \mathbb{N} \) and \( 1 \leq j \leq S \in \mathbb{N} \). The time dependent trajectories are defined on a coarse grained timescale \( \tau \) at which the Brownian motion process takes place. The vector \( F_{j}^{\text{ext}}(t) = (F_{j1}^{\text{ext}}(t), F_{j2}^{\text{ext}}(t), F_{j3}^{\text{ext}}(t)) \) is an arbitrary external force field and \( F_{j}^{\text{Brown}}(t) = (F_{j1}^{\text{Brown}}(t), F_{j2}^{\text{Brown}}(t), F_{j3}^{\text{Brown}}(t)) \) denotes the random force the \( j \)-th particle is subjected to. This random force is assumed to have a Gaussian distribution in each coordinate \( k \). The probability of an outcome \( F_{jk}^{\text{Brown}}(t) \) is given by

\[
P(F_{jk}^{\text{Brown}}(t)) = \frac{1}{\sqrt{2\pi} \sqrt{m_0 k_B T}} e^{-\frac{(F_{jk}^{\text{Brown}}(t))^2}{2m_0 k_B T}}.
\]

In Eq. (54), \( \tau_{\text{coll}} \) is a mean time of momentum transfer, \( k_B \) is Boltzmann’s constant and \( T \) is the temperature of the heat-bath that surrounds the \( j \)-th particle. The Gaussian distribution defined by Eq. (54) should be independent for every coordinate \( 1 \leq k \leq 3 \) and trajectory \( j \) of the random force field.

One can separate the sample trajectories \( x_j^{A(B)} \) into a sum caused by the random and the external force:

\[
x_j^{A(B)}(t) = \tilde{x}_j^{A(B)}(t) + \bar{x}_j^{A(B)}(t).
\]

Since \( F_{jk}^{\text{Brown}}(t) \) is independent in each coordinate, there does not exist any dependence between different components of the velocities \( \tilde{x}_j^{A(B)}(t), \bar{x}_j^{A(B)}(t) \) and \( \bar{x}_j^{A(B)}(t) \), as well as there is no dependence between \( \tilde{x}_j^{A(B)}(t) \) and \( \bar{x}_j^{A(B)}(t) \). With this property in mind, Frütsche and Haugk average in \( \{A\} \) and \( \{B\} \) about all \( j \)-th sample trajectories \( x_j^{A(B)}(t) \) after the two sub-ensembles were made infinitely large (\( S \to \infty \)). Finally, they arrive at two differential equations:

\[
\begin{align*}
  \frac{\partial}{\partial t} (\bar{v}^B - \bar{u}^B) + (\bar{v}^B + \bar{u}^B) \cdot \nabla (\bar{v}^B - \bar{u}^B) - \nabla \Delta (\bar{v}^B - \bar{u}^B) &= \frac{1}{m_0} F_{\text{ext}}, \\
  \frac{\partial}{\partial t} (\bar{v}^A + \bar{u}^A) + (\bar{v}^A - \bar{u}^A) \cdot \nabla (\bar{v}^A + \bar{u}^A) + \nabla \Delta (\bar{v}^A + \bar{u}^A) &= \frac{1}{m_0} F_{\text{ext}}.
\end{align*}
\]
In Eq. (56), $\vec{F}^{\text{ext}} \equiv (F_{1}^{\text{ext}}(\vec{r}, t), F_{2}^{\text{ext}}(\vec{r}, t), F_{3}^{\text{ext}}(\vec{r}, t))$ denotes the average external force at point $\vec{r} \equiv (r_1, r_2, r_3)$ and time $t$. The vector
\[
\vec{v}^{A(B)} = \sum_{j=1}^{\infty} \frac{\vec{e}_j(\vec{r}, t) \vec{x}_j^{A(B)}(t)}{\sum_{j=1}^{\infty} \vec{e}_j(\vec{r}, t)} = \left( v_{1}^{A(B)}(\vec{r}, t), v_{2}^{A(B)}(\vec{r}, t), v_{3}^{A(B)}(\vec{r}, t) \right)
\] (57)
is the average velocity of the ensemble, where
\[
\vec{e}_j(\vec{r}, t) \equiv \begin{cases} 
1 & \text{if } x_j^{A(B)}(t) \text{ at time } t \text{ is in a small volume } d^3r \text{ around } \vec{r}, \\
0 & \text{otherwise}.
\end{cases}
\] (58)

Additionally, the osmotic average velocity $\vec{u}^{A(B)}$ appears in Eq. (56). It is connected to the probability density $\rho^{A(B)}(\vec{r}, t) d^3r$ of finding a particle at point $(\vec{r}, t)$ in the $\{A\} (\{B\})$ ensemble via
\[
\vec{u}^{A(B)} \equiv -\vec{V} \ln \left( \frac{\rho^{A(B)}(\vec{r}, t)/\rho_0}{\rho_0} \right) = \left( u_{1}^{A(B)}(\vec{r}, t), u_{2}^{A(B)}(\vec{r}, t), u_{3}^{A(B)}(\vec{r}, t) \right).
\] (59)

In Eq. (59), $\rho_0$ is a constant reference density without physical importance, and $\nu \equiv \frac{k_B T}{m_0}$ is the friction coefficient of the Brownian motion, which depends on the temperature of the heat-bath and Boltzmann’s constant. Without loss of generality, $\nu$ can be written as
\[
\nu \equiv \frac{\hbar}{2m_0}.
\] (60)

Since we derive a non-relativistic system, it is possible that for some times, the random and the external forces are near infinity. Under these circumstances, a particle may be driven with an infinitely large velocity to $\vec{r} = \infty$ at a certain time-step $t$. However, in physical situations important for the non-locality issues discussed later in this article, relativistic effects certainly will occur, and particles are not allowed to move faster than the speed of light. Then, an ensemble of quantum particles which is emitted at a point $(\vec{r}_0, t_0)$, for example by a decaying atomic nucleus, can only reach points $(\vec{r}, t)$ within the future cone of $(\vec{r}_0, t_0)$. Accordingly, we restrict the functions $\vec{u}, \vec{v}, \rho(\vec{r}, t)$ such that they are defined only for these $(\vec{r}, t)$ in the future cone.

Before the two ensembles underwent a Brownian motion, it was ensured that their average velocity was equal:
\[
\vec{v}^{B} = \vec{v}^{A}.
\] (61)
It is shown in [15] that this constraint implies $\vec{u}^{B} = \vec{u}^{A}$, too. After the first time step, Eq. (61) does not hold anymore. Since the $\{A\}$ particle system gains kinetic energy compared to the $\{B\}$ ensemble, we will get
\[
|\vec{v}^{B}(\vec{r}, t + \Delta t)| < |\vec{v}^{A}(\vec{r}, t + \Delta t)|.
\] (62)
We can compute the arithmetic mean
\[
\vec{v}(\vec{r}, t + \Delta t) = \frac{\vec{v}^{A}(\vec{r}, t + \Delta t) + \vec{v}^{B}(\vec{r}, t + \Delta t)}{2}
\] (63)
of all particles in the two ensembles. Fortunately, it is always possible to exchange some trajectories from the $\{A\}$ ensemble whose velocity is larger than $|\vec{v}(\vec{r}, t + \Delta t)|$ with as many trajectories from the $\{B\}$ system whose velocity is smaller than $|\vec{v}(\vec{r}, t + \Delta t)|$, until
\[
\vec{v}^{B}(\vec{r}, t + \Delta t) = \vec{v}^{A}(\vec{r}, t + \Delta t) \equiv \vec{v}(\vec{r}, t + \Delta t)
\] (64)
The trajectories \{A\} and \{B\} are ordered by their velocity and their index \(j\). We may begin the exchange procedure by looking at the subset of those trajectories that have the highest velocity in the \{A\} ensemble, and the subset of those trajectories that have the lowest velocity in the \{B\} system. In each of these two subsets, we identify the trajectory which has the highest index \(j\). Then, we change the ensembles of the two trajectories that we have found. After this has been done, we check whether Eq. (64) is fulfilled. In case Eq. (64) does not hold, we restart the exchange procedure again. The trajectory exchange is completed when we arrive at Eq. (64). Then, the diffusion process will proceed, but for this time with our new division of the trajectories into the two sub-ensembles.

Every time after the trajectory exchange procedure is completed, we may compute the arithmetic mean of the two diffusion equations in (56). Since Eq. (64) then holds, we arrive at

\[
\frac{\partial}{\partial t} \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} - \vec{u} \cdot \vec{\nabla} \vec{u} + \nu \Delta \vec{u} = \frac{1}{m_0} \vec{F}_{\text{ext}}.
\]

(65)

If the Brownian forces were absent, we would have \(\vec{u} = 0\), with Eq. (65) becoming the equation of motion of a classical particle.

For the description of quantum systems, we will assume the Brownian forces to be present. We may simplify Eq. (65) with the definition of a function

\[
\psi(\vec{r},t) \equiv \pm \sqrt{\rho(\vec{r},t)} e^{i\varphi(\vec{r},t)}.
\]

(66)

This function is connected to the ensemble average velocity \(\vec{v}\) via \(\vec{v} \equiv \frac{\hbar}{m_0} \vec{\nabla} \varphi(\vec{r},t)\). (We note that it is only possible to formulate the latter equation because one can show \(\vec{v}\) to be curl free.) After some computations, Fritsche and Haugk are able to cast Eq. (65) directly in form of the time dependent Schrödinger equation:

\[
i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \hat{H}(\vec{r},t)\psi(\vec{r},t).
\]

(67)

In Eq. (67), \(\hat{H}(\vec{r},t)\) denotes the Hamiltonian operator

\[
\hat{H}(\vec{r},t) \equiv -\frac{\hbar^2 \Delta}{2m_0} + \hat{V}_{\text{ext}}(\vec{r},t),
\]

(68)

with \(\hat{V}_{\text{ext}}(\vec{r},t)\) as the potential of the external force \(\vec{F}_{\text{ext}} \equiv -\vec{\nabla} \hat{V}_{\text{ext}}(\vec{r},t)\). The stochastic process for the infinitely large ensemble of single particles leads to a differential equation encoding the ensemble average velocities \(\vec{v}\) and \(\vec{u}\) in a wave-function. This implies for the interpretation of a quantum mechanical state \(|\psi\rangle\) that all the information \(|\psi\rangle\) carries is about the evolution of average velocities from a statistical ensemble. It does reveal nothing about individual particles. One should mention that the quantum mechanical operators, their eigenvalues and uncertainty relations can be derived from the given formalism, as is fully documented in [15].

4.2 Quantum states in superpositions

Before we can discuss the locality properties of the Fritsche-Haugk theory and the physics behind entanglement in many-particle systems, we turn our attention to superpositions of two states in a single-particle system. The Schrödinger equation is linear. Therefore, the sum of two solutions \(\psi_1(\vec{r},t)\) and \(\psi_1(\vec{r},t)\), i.e.

\[
\Psi(\vec{r},t) \equiv a\psi_1(\vec{r},t) + b\psi_1(\vec{r},t)
\]

(69)

with coefficients \(a, b \in \mathbb{C}\), is also a solution of Eq. (67). With the state in Eq. (69), one would get mixing terms in \(\rho(\vec{r},t)\):

\[
\rho(\vec{r},t) = a^2\psi_1^2(\vec{r},t) + b^2\psi_1^2(\vec{r},t) + a^*b\psi_1^*(\vec{r},t)\psi_1(\vec{r},t) + b^*a\psi_1^*(\vec{r},t)\psi_1(\vec{r},t).
\]

(70)
In turn, the Eqs. (69) would become quite complicated. Accordingly, we need to make a basis change in order to get simpler equations. Since both states $\psi_\uparrow(\vec{r},t)$ and $\psi_\downarrow(\vec{r},t)$ are solutions of the same associated Schrödinger equation, we can write the latter without loss of its physical information in the form given below:

$$
\int \frac{\partial \tilde{\Psi}(\vec{r},t)}{\partial t} = \tilde{H}(\vec{r},t)\tilde{\Psi}(\vec{r},t)
$$

$$
= i\hbar \frac{\partial}{\partial t} (b\psi_1(\vec{r},t) + a\psi_1(\vec{r},t)) = \tilde{H}(\vec{r},t) (a\psi_1(\vec{r},t) + b\psi_1(\vec{r},t))
$$

$$
\int \frac{\partial \psi_1(\vec{r},t)}{\partial t} = \tilde{H}(\vec{r},t) (a\psi_1(\vec{r},t)) + \frac{\partial}{\partial t} (b\psi_1(\vec{r},t)) = \tilde{H}(\vec{r},t) (b\psi_1(\vec{r},t)).
$$

(72)

Using basis vectors in Dirac notation defined through $|\psi_\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |+\rangle$ and $|\psi_\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |--\rangle$, we get the following coefficients in coordinate representation:

$$
c_1(\vec{r},t) \equiv \langle \psi_1|\Psi(\vec{r},t) \rangle = a\psi_1(\vec{r},t) \quad \text{and} \quad c_1(\vec{r},t) \equiv \langle \psi_1|\Psi(\vec{r},t) \rangle = b\psi_1(\vec{r},t).
$$

Hence, the overall state vector $|\Psi(\vec{r},t)\rangle$ in Eq. (69) becomes

$$
|\Psi(\vec{r},t)\rangle = c_1 |\psi_\uparrow\rangle + c_1 |\psi_\downarrow\rangle,
$$

(74)

and the probability density can be written without mixing terms:

$$
\rho(\vec{r},t) = \langle \Psi(\vec{r},t)|\Psi(\vec{r},t) \rangle
= a^2\psi_\uparrow^*(\vec{r},t)\psi_\uparrow(\vec{r},t) + b^2\psi_\downarrow^*(\vec{r},t)\psi_\downarrow(\vec{r},t)
= a^2\rho_\uparrow(\vec{r},t) + b^2\rho_\downarrow(\vec{r},t).
$$

(75)

For $a^2$ and $b^2$, it follows from $\int \rho_\uparrow(\vec{r},t)d^3r = 1$ and $\int \rho(\vec{r},t)d^3r = 1$ that $a^2 + b^2 = 1$. For a given particle in the entire ensemble which is represented by the state in Eq. (74), we have a probability of $a^2$ that the said particle belongs to the sub-ensemble described by $\psi_\uparrow(\vec{r},t)$. Similarly, the probability to find that particle in the sub-ensemble represented by $\psi_\downarrow(\vec{r},t)$ is $b^2$.

Using the methods from section 4.1, we can derive the two Schrödinger equations for the $\uparrow$ and $\downarrow$ states in Eq. (72). We start with four Langevin equations

$$
\begin{align*}
\frac{m_0\dot{\omega}_j^A(t)}{\tau} + \frac{m_0\dot{\omega}_j^B(t)}{\tau} &= \vec{F}_{\text{ext}}(t) + \vec{F}_{\text{Brown}}(t), \\
\frac{m_0\dot{\omega}_j^A(t)}{\tau} - \frac{m_0\dot{\omega}_j^B(t)}{\tau} &= \vec{F}_{\text{ext}}(t) + \vec{F}_{\text{Brown}}(t), \\
\frac{m_0\dot{\omega}_j^B(t)}{\tau} + \frac{m_0\dot{\omega}_j^A(t)}{\tau} &= \vec{F}_{\text{ext}}(t) + \vec{F}_{\text{Brown}}(t), \\
\frac{m_0\dot{\omega}_j^B(t)}{\tau} - \frac{m_0\dot{\omega}_j^A(t)}{\tau} &= \vec{F}_{\text{ext}}(t) + \vec{F}_{\text{Brown}}(t),
\end{align*}
$$

(76)

for the hidden trajectory variables that correspond to the single particles in the sub-ensembles $\{A \uparrow\}, \{B \uparrow\}, \{A \downarrow\}$ and $\{B \downarrow\}$. After the same steps that we have already done in section 4.1, we are led to the following differential equations:

$$
\begin{align*}
\frac{\partial}{\partial t} (\vec{v}_1^B - \vec{u}_1^B) + (\vec{v}_1^B + \vec{u}_1^B) \cdot \vec{\nabla} (\vec{v}_1^B - \vec{u}_1^B) - \nu \Delta (\vec{v}_1^B - \vec{u}_1^B) &= \frac{1}{m_0} \vec{F}_{\text{ext}}, \\
\frac{\partial}{\partial t} (\vec{v}_1^A + \vec{u}_1^A) + (\vec{v}_1^A - \vec{u}_1^A) \cdot \vec{\nabla} (\vec{v}_1^A + \vec{u}_1^A) + \nu \Delta (\vec{v}_1^A + \vec{u}_1^A) &= \frac{1}{m_0} \vec{F}_{\text{ext}}, \\
\frac{\partial}{\partial t} (\vec{v}_1^B - \vec{u}_1^B) + (\vec{v}_1^B + \vec{u}_1^B) \cdot \vec{\nabla} (\vec{v}_1^B - \vec{u}_1^B) - \nu \Delta (\vec{v}_1^B - \vec{u}_1^B) &= \frac{1}{m_0} \vec{F}_{\text{ext}}, \\
\frac{\partial}{\partial t} (\vec{v}_1^A + \vec{u}_1^A) + (\vec{v}_1^A - \vec{u}_1^A) \cdot \vec{\nabla} (\vec{v}_1^A + \vec{u}_1^A) + \nu \Delta (\vec{v}_1^A + \vec{u}_1^A) &= \frac{1}{m_0} \vec{F}_{\text{ext}}.
\end{align*}
$$

(77)
The vectors $\vec{v}_1$, $\vec{u}_1$, $\vec{v}_2$, $\vec{u}_2$ are the ensemble average velocities of the $\uparrow$ and $\downarrow$ states. In this notation, the four equations in (77) would define two separate ensembles. Every particle would belong to either the $\{\uparrow\}$ or the $\{\downarrow\}$ state for all the time. Nevertheless, for a given single particle of a superposed state, there has to be a probability greater than 0 that this particle can be found in each one of these states.

Fortunately, the two ensembles are no more separate if we demand that some of the fast particles in the $\{A \uparrow\}$ $\{A \downarrow\}$ ensemble are exchanged with the same number of slow particles from the $\{B \uparrow\}$ $\{B \downarrow\}$ system. This has to be done with as many particles until

$$\vec{v}_1^B(\vec{r}, t) = \vec{v}_1^A(\vec{r}, t), \quad \vec{u}_1^B(\vec{r}, t) = \vec{u}_1^A(\vec{r}, t), \quad \vec{v}_2^B(\vec{r}, t) = \vec{v}_2^A(\vec{r}, t), \quad \vec{u}_2^B(\vec{r}, t) = \vec{u}_2^A(\vec{r}, t) \quad (78)$$

holds. If we add the $\{A \uparrow\}$ and $\{B \downarrow\}$, as well as the $\{A \downarrow\}$ and the $\{B \uparrow\}$ ensembles, we arrive at two equations:

$$\begin{align*}
\frac{\partial}{\partial t} (\vec{v}_1^A + \vec{v}_1^B) + \left( \vec{v}_1^A \cdot \vec{\nabla} \vec{v}_1^B - \vec{u}_1^B \cdot \vec{\nabla} \vec{u}_1^B \right) &+ \left( \vec{v}_1^A \cdot \vec{\nabla} \vec{v}_1^A - \vec{u}_1^A \cdot \vec{\nabla} \vec{u}_1^A \right) + \nu \Delta (\vec{u}_1^B + \vec{u}_1^A) = \frac{1}{m_0} \left( \vec{F}_{\text{ext}}^A + \vec{F}_{\text{ext}}^B \right), \\
\frac{\partial}{\partial t} (\vec{v}_1^A + \vec{v}_2^A) + \left( \vec{v}_2^A \cdot \vec{\nabla} \vec{v}_1^B - \vec{u}_1^A \cdot \vec{\nabla} \vec{u}_1^B \right) &+ \left( \vec{v}_2^A \cdot \vec{\nabla} \vec{v}_2^A - \vec{u}_2^A \cdot \vec{\nabla} \vec{u}_2^A \right) + \nu \Delta (\vec{u}_2^B + \vec{u}_2^A) = \frac{1}{m_0} \left( \vec{F}_{\text{ext}}^A + \vec{F}_{\text{ext}}^B \right). \quad (79)
\end{align*}$$

They can be summed together, yielding:

$$\frac{\partial}{\partial t} (\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4) + \left( \vec{v}_3 \cdot \vec{\nabla} \vec{v}_4 - \vec{u}_4 \cdot \vec{\nabla} \vec{u}_4 \right) + \left( \vec{v}_4 \cdot \vec{\nabla} \vec{v}_3 - \vec{u}_3 \cdot \vec{\nabla} \vec{u}_3 \right) + \nu \Delta (\vec{u}_3 + \vec{u}_4) = \frac{2}{m_0} \left( \vec{F}_{\text{ext}}^A + \vec{F}_{\text{ext}}^B \right). \quad (80)$$

Both equations in (79) contain $\{A\}$ and $\{B\}$, as well as $\{\uparrow\}$ and $\{\downarrow\}$ particle systems. These equations describe the behaviour of equations of motion under the same random and external forces on the average. Hence, we have:

$$\vec{v}_1 = \vec{v}_1^A \equiv \vec{v}_1, \quad \vec{v}_2 = \vec{v}_2^A \equiv \vec{v}_2, \quad \vec{u}_1 = \vec{u}_1^A \equiv \vec{u}_1, \quad \vec{u}_2 = \vec{u}_2^A \equiv \vec{u}_2. \quad (81)$$

Using Eq. (81), we get $2\vec{v}_1 = \vec{v}_1^B + \vec{v}_1^A$ and $2\vec{v}_2 = \vec{v}_2^B + \vec{v}_2^A$, as well as $2\vec{u}_1 = \vec{u}_1^B + \vec{u}_1^A$ and $2\vec{u}_2 = \vec{u}_2^B + \vec{u}_2^A$. Now, Eq. (80) becomes:

$$\frac{\partial}{\partial t} (\vec{v}_1 + \vec{v}_2) + \left( \vec{v}_1 \cdot \vec{\nabla} \vec{v}_2 - \vec{u}_2 \cdot \vec{\nabla} \vec{u}_2 \right) + \nu \Delta (\vec{u}_2 + \vec{u}_1) = \frac{1}{m_0} \left( \vec{F}_{\text{ext}}^A + \vec{F}_{\text{ext}}^B \right). \quad (82)$$

This is a sum of two separate differential equations for the ensemble averages of the $\uparrow$ and the $\downarrow$ state. Its physical information may be written as

$$\begin{align*}
\frac{d}{dt} \vec{v}_1 - \vec{u}_1 \cdot \vec{\nabla} \vec{u}_1 + \nu \Delta \vec{u}_1 &= \frac{1}{m_0} \vec{F}_{\text{ext}}, \\
\frac{d}{dt} \vec{v}_1 - \vec{u}_1 \cdot \vec{\nabla} \vec{u}_1 + \nu \Delta \vec{u}_1 &= \frac{1}{m_0} \vec{F}_{\text{ext}}. \quad (83)
\end{align*}$$

from which the two Schrödinger equations (72) can be derived. The two Schrödinger equations in (72) can then be added together and one finally arrives at Eq. (71). The two conditions of Eq. (78)
and Eq. (31) imply that the average velocities \( \bar{v}_i = \bar{v}_j = \bar{v} \) and \( \bar{u}_i = \bar{u}_j = \bar{u} \) are equal. Hence, in the interpretation of stochastic mechanics, Eq. (31) describes two oscillating ensembles with identical dynamics on the average.

The procedure above also works with an arbitrary number of states \(|\psi_1(\vec{r}, t)\rangle, |\psi_2(\vec{r}, t)\rangle, \ldots, |\psi_N(\vec{r}, t)\rangle\) that are summed up to a general superposition with weighting factors \( a_i \in \mathbb{C} \):

\[
|\Psi(\vec{r}, t)\rangle \equiv \sum_{i=1}^{N} a_i |\psi_i(\vec{r}, t)\rangle.
\] (84)

The weighting factors \( a_i \) define the probability of a given particle to be found in the \( i \)-th state \( |\psi_i(\vec{r}, t)\rangle \) of the superposition. Therefore, the ensembles in the corresponding \( N \) diffusion equations must contain different numbers of particles. More precisely, each state \( |\psi_i(\vec{r}, t)\rangle \) has to be computed with an ensemble of \( Sa_i^2 \) trajectories \( j_1, j_2, j_2, \ldots, j_{Sa_i^2} \). This ensures that there is a chance of \( a_i^2 \) to find a particle in the system associated with \( |\psi_i(\vec{r}, t)\rangle \).

Furthermore, the exchange procedure must be defined differently. Two trajectories with indices \( j \) and \( j' \) are now exchanged between two ensembles denoted by \( \{Bi\} \) and \( \{Ak\} \), where \( i, k \in \mathbb{N} \) are numbers that range from 1 to \( N \). The indices \( i \) and \( k \) define the ensembles corresponding to the states \( |\psi_i(\vec{r}, t)\rangle \) and \( |\psi_k(\vec{r}, t)\rangle \) between which the trajectories are interchanged. We note that \( i \) and \( k \) have to be chosen randomly, but with equal probability for each state. The different probabilities of finding a given particle in the separate states \( |\psi_i(\vec{r}, t)\rangle \) and \( |\psi_k(\vec{r}, t)\rangle \) only arise due to the unequal size of their corresponding ensembles.

After the necessary interchanges took place, one may add the \( N \) differential equations in one step and split them up into separate equations for the 1, 2, 3, \ldots, \( N \) systems, analogous to Eq. (33). We remark that because of Eq. (30), nothing about the weighting factors \( a_i^2 \) in the probability density appears in the ensemble averages \( \bar{v} \) and \( \bar{u} \). Hence, one has to re-introduce this information after one gets from the separated system in Eq. (72) to the superposed Eq. (71).

### 4.3 Locality in the theory of Fritsche and Haugk with single-particle systems

In this section, we turn to the locality properties of the stochastic process described above. We observe that in the Langevin equations (53), there is no dynamical coupling between different trajectories \( \vec{x}^{A(B)}_{j(1)}(t) \) and \( \vec{x}^{A(B)}_{j'(1)}(t) \), where \( j \neq j' \). The force field \( \vec{F}^{\text{ext}}(t) \) only acts on the \( j \)-trajectory, and \( \vec{F}^{\text{Brown}}_j(t) \) is required to be independent for each \( j \). Nevertheless, the time evolution of different trajectories will be correlated because of the trajectory exchange mechanism.

We will now assume that something happens to a large set of trajectories at a point \((\vec{r}', t')\). For example, the particle speed may be altered there by external forces controlled by the experimenter. In case \((\vec{r}', t')\) lies in the past cone of another point \((\vec{r}, t)\), some of the changed trajectories may arrive at \((\vec{r}, t)\). We will denote the changed trajectories that reach \((\vec{r}, t)\) by \( \vec{x}^{A(B)}_{j(1)}(t) \).

Let \( \vec{x}^{A(B)}_{j(1)}(t) \) be another trajectory at \((\vec{r}, t)\). The exchange procedure decides whether \( \vec{x}^{A(B)}_{j(1)}(t) \) will change its ensemble. This decision depends on the magnitude of the particle’s own velocity \( |\vec{x}^{A(B)}_{j(1)}(t)| \) and the velocity distribution of all trajectories from the \{A\} and \{B\} ensembles that have arrived at \((\vec{r}, t)\), including the trajectories \( \vec{x}^{A(B)}_{j'(1)}(t) \) which were changed at \((\vec{r}', t')\).

For the next time step, dependent on the trajectory exchanges required for Eq. (34) to hold, \( \vec{x}^{A(B)}_{j'(1)}(t) \) might be placed in the \{B\} \{A\} ensemble. In case \( \vec{x}^{A(B)}_{j(1)}(t) \) changes its ensemble, this trajectory will, at least for the next time-step, be subject to the differential equation of the new ensemble, and we will have to rename it as \( \vec{x}^{B(A)}_{j'(1)}(t) \).

Since the interchange procedure explicitly depends on the modified trajectories \( \vec{x}^{A(B)}_{j(1)}(t) \), the particle trajectory \( \vec{x}^{A(B)}_{j(1)}(t) \) may be influenced at \((\vec{r}, t)\) by what has happened at \((\vec{r}', t')\). Moreover, for
that kind of influence it is not required that $\tilde{x}_{j_1(1)}^{A(B)}(t)$ ever went through the point $(\vec{r}', t')$. This effect is important in double slit experiments, where a particle’s trajectory, after the passage of one slit, depends on whether the second slit is open or not, no matter if the particle actually went through the second slit.

However, it is important to note that the trajectories $\tilde{x}_{j_1(1)}^{A(B)}(t)$ can only be influenced at $(\vec{r}, t)$ through the exchange mechanism if $(\vec{r}', t')$, where some intervention was made, lies in the past cone of $(\vec{r}, t)$. In turn, there does not exist any violation of active locality.

By writing this, one should be aware that we are still discussing a non-relativistic system. The external and Brownian forces were given in a non covariant form. Hence, these forces are still able to modify the velocity of the particles instantaneously, even though we have restricted the particle’s maximum speed to lightspeed in section 4.1. This last problem could only be resolved with a covariant formulation of the Fritsche-Haugk theory. Fortunately, as we will see in section 4.3, the discussion of entanglement does not depend on that problem. The trajectories corresponding to different particle states of a many-particle system will turn out to be completely uncoupled, at least if inter-particle forces are not present.

Finally, we emphasise that the exchange procedure should not be seen as a dynamical coupling mechanism between different trajectories. Instead, the comparison of a particle’s velocity with the velocity distributions of the $\{A\}$ and $\{B\}$ ensembles is merely a model to account for the idea of kinetic energy conservation on the average in a stochastic theory. That there are in fact no dynamical couplings between the individual trajectories of the ensemble can be seen experimentally in self-interferometry experiments. The typical example is neutron interferometry [32], where “usually, at one time, there is only one neutron in the interferometer, if at all because at that time, the next neutron has not yet been born and is still contained in the uranium nuclei of the reactor fuel” [33].

### 4.4 Derivation of the many-particle Schrödinger equation

In this section, we want to investigate whether active locality persists when we consider many-particle systems. The N-particle Schrödinger equation can be derived if the three dimensional coordinate space of the single-particle problem is expanded into a $3N$ dimensional space. We start with the Langevin systems. The N-particle Schrödinger equation can be derived if the three dimensional coordinate space between the individual trajectories of the ensemble can be seen experimentally in self-interferometry.

4.4 Derivation of the many-particle Schrödinger equation

In this section, we want to investigate whether active locality persists when we consider many-particle systems. The N-particle Schrödinger equation can be derived if the three dimensional coordinate space of the single-particle problem is expanded into a $3N$ dimensional space. We start with the Langevin equations of the $j$-th particle trajectory in the ensemble of the $n$-th particle state, where $n \in \mathbb{N}$ and $1 \leq n \leq N$:

\[
\begin{align*}
    m_0 \ddot{x}_{jn}^A(t) + \frac{m_0}{\tau} \dot{x}_{jn}^A(t) &= \vec{F}_{\text{ext}}^A(t) + \vec{F}_{\text{Brown}}^A(t) + \sum_{n' \neq n}^{N} \sum_{n=1}^{N} \vec{F}_{\text{ext}}^A(x_{jn}^A(t), x_{j'n}^A(t), t), \\
    m_0 \ddot{x}_{jn}^B(t) - \frac{m_0}{\tau} \dot{x}_{jn}^B(t) &= \vec{F}_{\text{ext}}^B(t) + \vec{F}_{\text{Brown}}^B(t) + \sum_{n' \neq n}^{N} \sum_{n=1}^{N} \vec{F}_{\text{ext}}^B(x_{jn}^B(t), x_{j'n}^B(t), t) .
\end{align*}
\]

(85)

In Eq. (85), there is a sum of additional force terms $\vec{F}_{\text{ext}}^A(x_{jn}^{A(B)}(t), x_{j'n}^{A(B)}(t), t)$ corresponding to inter-particle forces between the particle states $n$ and $n' \neq n$. The inter-particle forces are defined to be equal for the $\{A\}$ and $\{B\}$ ensembles:

\[
\vec{F}_{\text{ext}}^A(x_{jn}^A(t), x_{j'n}^A(t), t) = \vec{F}_{\text{ext}}^B(x_{jn}^B(t), x_{j'n}^B(t), t) .
\]

(86)

From the two Langevin equations (85), Fritsche and Haugk derive the following differential equations for the $\{A\}$ and $\{B\}$ ensembles, completely analogous to the single-particle case:

\[
\begin{align*}
    \frac{\partial}{\partial t} (\vec{v}_{\text{NB}}^N - \vec{u}_{\text{NB}}^N) + (\vec{v}_{\text{NB}}^N + \vec{u}_{\text{NB}}^N) \cdot \vec{\nabla}^N (\vec{v}_{\text{NB}}^N - \vec{u}_{\text{NB}}^N) - \nu \Delta^N (\vec{v}_{\text{NB}}^N - \vec{u}_{\text{NB}}^N) &= \vec{F}_{\text{N,ext}}, \\
    \frac{\partial}{\partial t} (\vec{v}_{\text{NA}}^N + \vec{u}_{\text{NA}}^N) + (\vec{v}_{\text{NA}}^N - \vec{u}_{\text{NA}}^N) \cdot \vec{\nabla}^N (\vec{v}_{\text{NA}}^N + \vec{u}_{\text{NA}}^N) + \nu \Delta^N (\vec{v}_{\text{NA}}^N + \vec{u}_{\text{NA}}^N) &= \vec{F}_{\text{N,ext}}.
\end{align*}
\]

(87)
In contrast to the single-particle case, $\vec{F}_{\text{N,ext}}$ is a 3N dimensional vector of the form

$$\vec{F}_{\text{N,ext}} \equiv \left( \frac{\vec{F}^{\text{ext}}(\vec{r}_{n},t)}{m_{0,n=1}} + \sum_{n'}^{N} \frac{\vec{F}^{\text{ext}}(\vec{r}_{n},\vec{r}_{n'},t)}{m_{0,n'=1}}, \ldots, \frac{\vec{F}^{\text{ext}}(\vec{r}_{n,N},t)}{m_{0,n=N}} + \sum_{n'}^{N} \frac{\vec{F}^{\text{ext}}(\vec{r}_{n,N},\vec{r}_{n',t})}{m_{0,n=N}} \right).$$

(88)

Its $n$-th component $\vec{F}_{n,\text{ext}}$ consists of the average single-particle force $\vec{F}^{\text{ext}}_{n}(\vec{r}_{n},t)$ and a sum of average inter-particle forces $\vec{F}^{\text{ext}}_{n}(\vec{r}_{n},\vec{r}_{n'},t)$. Both forces $\vec{F}^{\text{ext}}_{n}(\vec{r}_{n},t)$ and $\vec{F}^{\text{ext}}_{n}(\vec{r}_{n},\vec{r}_{n'},t)$ act on the $n$-th particle, which has rest mass of $m_{0n}$.

The three dimensional ensemble average velocities $\vec{u}$ and $\vec{v}$ of the single-particle case are similarly replaced by 3N dimensional vectors. They depend on the 3N coordinates $\vec{r}_{n}$ corresponding to slow particles from the ensemble $\{n\}$ only one state $n$ present. This would cause a general impossibility to define single-particle operators, which act on ensembles corresponding to different particle states, even if no inter-particle forces are present. Therefore, the exchange procedure can well be confined within each single-particle state only.

In order to derive the many-particle Schrödinger equation, an interchange between some trajectories corresponding to slow particles from the ensemble $\{n\}$ and the same number of fast particles from the ensemble $\{n'\}$ is necessary after each time-step, until

$$\vec{v}_{\text{NA}(B)} \equiv \left( \vec{v}_{1}^{A(B)}(\vec{r}_{1},t), \ldots, \vec{v}_{N}^{A(B)}(\vec{r}_{N},t) \right), \quad \vec{u}_{\text{NA}(B)} \equiv \left( \vec{u}_{1}^{A(B)}(\vec{r}_{1},t), \ldots, \vec{u}_{N}^{A(B)}(\vec{r}_{N},t) \right).$$

(89)

The $n$-th component of $\vec{u}_{\text{NA}(B)}$ and $\vec{v}_{\text{NA}(B)}$ is computed only with the trajectories $\vec{x}_{jn}^{A(B)}(t)$ of the $n$-th particle state. This gives rise to the coordinate dependence shown in Eq. (87).

In order to derive the many-particle Schrödinger equation, an interchange between some trajectories corresponding to slow particles from the ensemble $\{n\}$ and the same number of fast particles from the ensemble $\{n'\}$ is necessary after each time-step, until

$$\vec{v}_{\text{NA}}(\vec{r}_{n},t) = \vec{v}_{\text{NB}}(\vec{r}_{n},t) \equiv \vec{v}_{n} = (\vec{v}_{1}(\vec{r}_{1},t), \vec{v}_{2}(\vec{r}_{2},t), \ldots, \vec{v}_{N}(\vec{r}_{N},t))$$

(90)

holds. Here, we have to note an additional subtlety that does only arise in many-particle systems. It has to be ensured that exchanges between two trajectories $\vec{x}_{jn}^{A(B)}(t)$ and $\vec{x}_{jn'}^{B(A)}(t)$ are confined within the same particle state $n = n'$. An interchange between trajectories of different states $n$ and $n' \neq n$ would couple ensembles corresponding to different particle states, even if no inter-particle forces are present. This would cause a general impossibility to define single-particle operators, which act only on one state $n$ of a many-particle system. Fortunately, due to the specific coordinate dependence of the ensemble average velocities $\vec{v}_{\text{NA}(B)}$ and $\vec{v}_{\text{NA}(B)}$ in Eq. (87), the components of Eq. (90) must agree for every state $n$ separately. Therefore, the exchange procedure can well be confined within each single-particle state only.

Every time after the trajectory exchange is completed, one can compute the arithmetic mean of the two differential equations in (87). Analogous to the single-particle case, we get the diffusion equation

$$\frac{d}{dt} \vec{v}_{n} - \vec{u}_{n} \cdot \vec{v}_{n} + \nu \Delta \vec{v}_{n} = \vec{F}_{\text{N,ext}}.$$  

(91)

From Eq. (91), the many-particle Schrödinger equation

$$i\hbar \frac{\partial \Psi (\vec{r}_{n},t)}{\partial t} = \hat{H} (\vec{r}_{n},t) \Psi (\vec{r}_{n},t)$$

(92)

can be derived. In Eq. (92), the Hamiltonian operator is given by

$$\hat{H} (\vec{r}_{n},t) = \sum_{n=1}^{N} \frac{-\hbar^{2} \Delta_{n}}{2m_{0n}} + \hat{V}_{\text{ext}} (\vec{r}_{n},t).$$

(93)

Its external potential

$$\hat{V}_{\text{ext}} (\vec{r}_{n},t) = \sum_{n=1}^{N} \hat{V}_{\text{ext}} (\vec{r}_{n},t) + \frac{1}{2} \sum_{n=1}^{N} \sum_{n'=1}^{N} \hat{V}_{\text{ext}} (\vec{r}_{n},\vec{r}_{n'},t)$$

(94)
is connected to the \( n \)-th component of \( \tilde{F}_{N,ext} \) via

\[
\tilde{F}_{N,ext}^n = -\frac{\nabla N}{m_{0n}} V_{ext} \left( \tilde{r}^N, t \right).
\]  

### 4.5 Locality in the theory of Frötsche and Haugk with many-particle systems

In this section, we assume all inter-particle terms to be absent. We observe that with the absence of inter-particle forces like \( \tilde{F}_{ext} \) \( x^{A(B)}(t), x^{A(B)}(t') \), there are no terms in Eq. (85) which interconnect different trajectories \( x^{A(B)}(t) \) and \( x^{A(B)}(t') \) from separate particle states \( n' \neq n \). A similar statement is also true for different components of the ensemble average velocities \( \bar{v}^N_n \) and \( \bar{w}^N_n \), because they are calculated with the \( x^{A(B)}(t) \) trajectories of each single-particle state \( n \) separately.

The decision whether a trajectory \( x_{jn}^{A(B)}(t) \) changes to \( x_{jn}^{B(A)}(t) \) during the interchange procedure only relies on the magnitude of the \((j, n)\)-th particle’s velocity \( |x_{jn}^{A(B)}(t)| \) and the velocity distribution of all particles that belong to the state \( n \) and are in the past cone of \( x_{jn}^{A(B)}(t) \). Therefore, the trajectories \( x_{jn}^{A(B)}(t) \) are never coupled in any way to the trajectories \( x_{jn'}^{A(B)}(t) \) of other particle states \( n' \neq n \). As a result, we conclude that active locality holds in the hidden variable theory of Frötsche and Haugk.

### 4.6 Entangled states in the model of Frötsche and Haugk

That there does not exist any dynamical coupling between the different particle states which a many-particle system is composed of doesn’t imply that correlations between these states are forbidden. In section 6, we will introduce further conditions on the Gaussian distributed Brownian forces \( \tilde{F}_{Brown} \) in a completely local framework. Those conditions will be such that under certain circumstances, the trajectories \( x_{jn}^{A(B)}(t) \) change their ensemble from \( \{A\} \) \( \{B\} \) simultaneously with the trajectories \( x_{jn'}^{A(B)}(t) \) from another state \( n' \neq n \). Then, correlations between ensembles corresponding to different particle states will emerge. The resulting many-particle wave-functions are of the form

\[
|\Psi(\tilde{r}_1, \tilde{r}_2, t)\rangle = \frac{1}{\sqrt{2}} |\psi_1(\tilde{r}_1, t)\rangle \otimes |\psi_2(\tilde{r}_2, t)\rangle + \frac{-1}{\sqrt{2}} |\psi_2(\tilde{r}_1, t)\rangle \otimes |\psi_1(\tilde{r}_2, t)\rangle,
\]  

and are commonly called entangled. They were observed for the first time in massive fermionic systems with a pair of entangled protons in 1976 [34].

The state vector \( |\Psi(\tilde{r}_1, \tilde{r}_2, t)\rangle \) in Eq. (96) is an element of the product Hilbert space \( \mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \). It is a superposition of the two product states

\[
|\chi_1(\tilde{r}_1, \tilde{r}_2, t)\rangle \equiv |\psi_1(\tilde{r}_1, t)\rangle \otimes |\psi_2(\tilde{r}_2, t)\rangle \quad \text{and} \quad |\chi_2(\tilde{r}_1, \tilde{r}_2, t)\rangle \equiv |\psi_2(\tilde{r}_1, t)\rangle \otimes |\psi_1(\tilde{r}_2, t)\rangle,
\]

where \( |\psi_1(\tilde{r}_1, t)\rangle, |\psi_2(\tilde{r}_1, t)\rangle \in \mathcal{H}_1 \) and \( |\psi_1(\tilde{r}_2, t)\rangle, |\psi_2(\tilde{r}_2, t)\rangle \in \mathcal{H}_2 \).

The entangled state solves a many-particle Schrödinger equation with an Hamiltonian operator

\[
\hat{H}(\tilde{r}_1, \tilde{r}_2) = \sum_{n=1}^{2} -\frac{\hbar^2 \Delta N}{2m_{0n}} + \sum_{n=1}^{2} \tilde{V}_{ext}(\tilde{r}_n, t),
\]

which has no inter-particle forces and whose external potentials agree at each time \( t \):

\[
\tilde{V}_{ext}(\tilde{r}_1, t) = \tilde{V}_{ext}(\tilde{r}_2, t).
\]
The Schrödinger equation associated with $|\Psi(\vec{r}_1, \vec{r}_2, t)\rangle$ can be written, analogous to Eq. (72), as:

$$ih \frac{\partial}{\partial t} |\Psi(\vec{r}_1, \vec{r}_2, t)\rangle = \hat{H}(\vec{r}_1, \vec{r}_2, t) |\Psi(\vec{r}_1, \vec{r}_2, t)\rangle$$

(100)

$$= \left\{ \begin{array}{l}
   ih \frac{\partial}{\partial t} \left( \frac{1}{\sqrt{2}} |\chi_1(\vec{r}_1, \vec{r}_2, t)\rangle \right) \\
   + \frac{1}{\sqrt{2}} |\chi_2(\vec{r}_1, \vec{r}_2, t)\rangle \\
   + \hat{H}(\vec{r}_1, \vec{r}_2, t) \left( \frac{1}{\sqrt{2}} |\chi_1(\vec{r}_1, \vec{r}_2, t)\rangle \right) \\
   + \hat{H}(\vec{r}_1, \vec{r}_2, t) \left( \frac{1}{\sqrt{2}} |\chi_2(\vec{r}_1, \vec{r}_2, t)\rangle \right)
\end{array} \right.$$

(101)

This system consists of two particle ensembles $\vec{r}_1$ and $\vec{r}_2$. The particles from $\vec{r}_1$ can be in different states $|\psi_1(\vec{r}_1)\rangle$ or $|\psi_2(\vec{r}_1)\rangle$, whereas the particles from $\vec{r}_2$ can be in the states $|\psi_1(\vec{r}_2)\rangle$ or $|\psi_2(\vec{r}_2)\rangle$. In order to describe the dynamics of the $\vec{r}_1$ particles, the trajectories in the $\{A1\}$ ($\{B1\}$) ensemble of $|\psi_1(\vec{r}_1)\rangle$ must be exchanged with the trajectories of the $\{B2\}$ ($\{A2\}$) ensemble of $|\psi_2(\vec{r}_1)\rangle$, and a similar procedure has to be done with the $\vec{r}_2$ particles. In the meantime, one has to ensure that each $j$-th pair of particle trajectories in the $\vec{r}_1$ and $\vec{r}_2$ systems changes its state simultaneously. We will show in the context with Bohm’s EPR experiment how this can be achieved although the trajectories of the two different ensembles are disconnected from each other. For now, we only remark that without inter-particle-forces, it is in fact a requirement for the particle trajectories $x_j^{A(\vec{B})}(t)$ and $x_j^{A(\vec{B})}(t)$ not to be interconnected directly in any way. This may also be seen with the following example by Nelson [8]:

If one induces an external measurement process on the $n$-th particle state of a multi-particle system, one may get information about the hidden variable $x_j^{A(\vec{B})}(t)$. The measurement device may change $x_j^{A(\vec{B})}(t)$ in a way that correlations with all the other particle trajectories $x_j^{A(\vec{B})}(t)$, where $n' \neq n$ are lost. Since the different particle systems are uncoupled, an interaction happening on the ensemble of the $n$-th particle state can in no way influence the properties of the particles in the other states $n'$.

Let $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$ be the product Hilbert space above and define an Hamiltonian operator

$$\hat{H} \equiv \hat{H}_1 \otimes 1_2 + 1_1 \otimes \hat{H}_2,$$

(102)

where $\hat{H}_1 \in \mathcal{H}_1$ and $\hat{H}_2 \in \mathcal{H}_2$. The time evolution for any observable $\hat{A}_1 \in \mathcal{H}_1$ in the Heisenberg picture

$$\hat{A}_H(t) = e^{i\hat{H}t} \hat{A}_1 e^{-i\hat{H}t} = e^{i\hat{H}t} \hat{A}_1 e^{-i\hat{H}t} \otimes 1_2$$

(103)

is completely independent of $\mathcal{H}_2$, even if $|\Psi(\vec{r}_1, \vec{r}_2, t)\rangle \in \mathcal{H}$ is described by the entangled state of Eq. (96). As Nelson put it:

“No matter how systems may be correlated, there is no way of telling, what [external] forces the second system is being subjected to, as long as they are not dynamically coupled.”

In this context, we want to mention that Nelson’s original theory could not reproduce the example above [8] because he identified $\vec{v}$ and $\vec{u}$ not as average velocities of an ensemble but assumed them to be the trajectories of individual particles. With this assertion, neither the splitting of the many-particle Schrödinger equation (101), nor any sort of exchange procedure between the particles in the $\{A1\}$ ($\{B1\}$) and $\{B2\}$ ($\{A2\}$) system was possible. Accordingly, active locality gets lost in Nelson’s original formulation of stochastic mechanics, similarly as it gets lost in Bohmian mechanics. There, (see [62] for a general introduction) it is well known that the derivation of the many-particle Schrödinger equation is only possible provided that a highly non-trivial dependency between the Bohmian “trajectories” of different particle states is introduced, which is at odds with the theory of relativity and without any physical foundation. Interestingly, Nelson himself has shown recently [35] that $\vec{v}$ and $\vec{u}$ cannot correspond to properties of individual particles since this would lead to measurable differences between his original theory and ordinary quantum mechanics. A similar observation was made for Bohmian “trajectories” earlier [36].
5 Spin in a Fritsche-Haugk like hidden variable theory

5.1 Preliminary considerations

Nelson’s derivation of the Schrödinger equation was quickly extended to include particle spin, but without proper physical interpretation [37]. Over the years, more physically motivated derivations of the Pauli equation were developed. One by Faris [38, 39], and another one by Fritsche and Haugk [40]. Unfortunately, these frameworks did either violate active locality or they did not contain any mechanism that would change a particle’s spin state after it is emitted. Thereby, none of these theories led to a violation of deterministic passive locality only. In this section, we propose a theory of spinning particles providing such a mechanism.

Beginning with an example of [40], we try to interpret the spin of a particle as rotation around a chosen axis \( \vec{z} \) (see Fig. 3). We let the particle rotate around this axis under energy fluctuations \( \Delta E \Delta t \approx \hbar / 2 \). For the rotational energy, we have the expression \( E = \Theta \omega^2 / 2 \) from classical mechanics. In this equation, \( \Theta = m_0 r^2 \) denotes the inertial tensor with \( m_0 \) as the particle’s mass and \( r \) as the distance of the particle from the chosen axis \( \vec{z} \). The particle’s angular velocity is given by \( \omega = 2 \pi T \), where \( T \) is the time needed for the particle to perform a complete rotation.

If we identify \( \Delta t \) with \( T \) and \( \Delta E \) with \( E \), we may write
\[
\Delta E \Delta t = 1 / 2 m_0 r \cdot 2 \pi \frac{r}{T} T = m_0 vr \pi = \pi s \approx \hbar / 2. \tag{104}
\]
In eq. (104), \( v = 2 \pi \frac{r}{T} \) is the magnitude of the particle’s velocity and
\[
s = m_0 vr \approx \frac{\hbar}{2 \pi} \tag{105}
\]
defines the magnitude of its angular momentum. Apparently, \( s \) is independent of the particle’s mass \( m_0 \) and the radius \( r \) of the rotation. The arguments above were only approximations. For later, it will be useful if we set the magnitude of the spin angular momentum to \( s = \hbar / 2 \) on the average.

From the time dependent Schrödinger equation, one can derive the theorem of Ehrenfest:
\[
\frac{d}{dt} \langle \hat{\vec{v}} \rangle = \langle \hat{\vec{F}}^{\text{ext}} \rangle. \tag{106}
\]
In Eq. (106), \( \langle \hat{\vec{v}} \rangle \) is the expectation value of the velocity operator and \( \langle \hat{\vec{F}}^{\text{ext}} \rangle \) is the expectation value of the external force. Ehrenfest’s theorem implies that the angular momentum operator \( \hat{\vec{L}} = \hat{\vec{r}} \times m_0 \hat{\vec{v}} \) may not vanish in the case of \( \hbar \to 0 \). On the other hand, the quantum mechanical spin should always go to zero in the classical limit by Eq. (105). Hence, the particle’s spin must be completely independent of \( \hat{\vec{L}} \) and we must expand our description.

5.2 Stern-Gerlach effect

If particles of spin \( 1/2 \) are not specially prepared, they occur in a superposition of \( \uparrow \) and \( \downarrow \) states. This superposition can be derived from a similar computation as in section 4.2. Here, however, the
trajectories $\vec{x}^{(A/B)}(t)$ of Eq. (76) describe particles in rotational motion. Their rotation can be in the left or in the right direction with respect to the chosen axis $\vec{z}$. Particles rotating in the left direction are denoted by $\uparrow$, whereas particles rotating in the right direction are identified by $\downarrow$. After some calculations, we eventually arrive at the four differential equations of Eq. (77).

As explained in section 4.2, a given particle in an ensemble described by a superposed quantum state must have a non-vanishing probability to be found in each one of the component states. Accordingly, we have to exchange trajectories between the $\{A \uparrow\}$ ($\{A \downarrow\}$) and $\{B \downarrow\}$ ($\{B \uparrow\}$) ensembles before we get to the differential equations (79). A nice drawing of this process is given by R. Penrose [41] for Dirac particles. It is this procedure, whereby the possibility for a violation of deterministic passive locality does arise.

At the end of the calculation in section 4.2, we were able to describe the behaviour of the superposed system by a probability density

$$\rho(\vec{r}, t) = a^2 \rho_\uparrow(\vec{r}, t) + b^2 \rho_\downarrow(\vec{r}, t),$$

and a quantum state

$$|\Psi(\vec{r}, t)\rangle \equiv \left( \begin{array}{c} a \psi_\uparrow(\vec{r}, t) \\ b \psi_\downarrow(\vec{r}, t) \end{array} \right) = a\psi_\uparrow(\vec{r}, t) |+\rangle + b\psi_\downarrow(\vec{r}, t) |--\rangle.$$  \hspace{1cm} (108)

Both are in form of a sum of entities from the two rotating particle ensembles.

If an external magnetic field $B_z(\vec{r}, t)$ is applied in the direction of $\vec{z}$, the energy of $\psi_\uparrow(\vec{r}, t)$ increases with a magnitude of $\mu_B |a|^2$, whereas the energy of $\psi_\downarrow(\vec{r}, t)$ decreases by $\mu_B |b|^2$. The constant $\mu_B$ is called Bohr magneton

$$\mu_B \equiv \frac{e\hbar}{2m_0}. \hspace{1cm} (109)$$

It emerges since the relativistic Dirac equation is, in contrast to the Pauli equation to be derived here, linear in all its derivatives. The necessity of this linearity, which is absent in the non-relativistic theory, changes the interaction of the angular momentum with magnetic fields. If one does linearise the Schrödinger equation in a similar manner as it is typically done for Dirac’s equation, one gets the same value for $\mu_B$ [42].

The changes of the quantum mechanical energy densities, which are caused by the interaction of the spin rotation with the magnetic field $B_z(\vec{r}, t)$, are given by the expressions

$$\mu_\uparrow(\vec{r}, t) \equiv \mu_B B_z(\vec{r}, t) |a|^2 \psi_\uparrow^*(\vec{r}, t)\psi_\uparrow(\vec{r}, t)$$

and

$$\mu_\downarrow(\vec{r}, t) \equiv -\mu_B B_z(\vec{r}, t) |b|^2 \psi_\downarrow^*(\vec{r}, t)\psi_\downarrow(\vec{r}, t).$$

With their sum

$$\mu(\vec{r}, t) \equiv \mu_\uparrow(\vec{r}, t) + \mu_\downarrow(\vec{r}, t),$$

the total change $\mu(\vec{r}, t)$ for both spinors can be cast as

$$\mu(\vec{r}, t) = \Psi^+ (\vec{r}, t) \mu_B B_z \Psi(\vec{r}, t),$$

where we have used the definition

$$B_z \equiv B_z(\vec{r}, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (114)

The different coupling of the $\psi_{\uparrow(\downarrow)}(\vec{r}, t)$ states to magnetic fields gives rise to the Stern-Gerlach effect [43]. If atoms from an incident beam with equally distributed magnetic energy density enter a strong
magnet, the two spin systems are subjected to different forces $\vec{F}_{\text{ext}}^j(t) \neq \vec{F}_{\text{ext}}^i(t)$. Accordingly, they get deflected in opposite directions and we are left with disconnected trajectory ensembles corresponding to the different spin states. Due to active locality, the exchange procedure can only happen between trajectories $x_{j(i)}^{A(B)}(t)$ that are located at the same point $(\vec{r}, t)$ in space-time. Hence, after the separation by the Stern-Gerlach magnet, the exchange procedure can only take place between trajectories of the same spin ensemble. This is observed in double Stern-Gerlach experiments [44]. If one places a second Stern-Gerlach magnet behind the first and let it have the same directional orientation, the parts of the beam that were previously separated, do not split up again. In this way, particle ensembles in a beam can be prepared to contain one spin state only.

The repeated Stern-Gerlach experiment demonstrates both the locality and the non-markovian property of the stochastic process involved. The stochastic differential equation that a particle is subject to depends, due to the coupling of different trajectories in the exchange procedure, on what has happened to the entire ensemble before. Therefore, the stochastic process that acts on a single-particle has a kind of memory about the past interactions between the magnets and the ensemble in which the individual particle is currently placed in. Nevertheless, the stochastic process is a local one. After the two spin systems are spatially separated, they do not influence each other anymore.

### 5.3 Pauli equation

From here on, we can closely follow [40] to derive the remaining parts of quantum mechanics for a particle with spin. In [15], the Schrödinger equation under a magnetic field with its Hamiltonian

$$\hat{H}(\vec{r}, t) \equiv \frac{(-i\hbar \vec{\nabla} - e\vec{A}(\vec{r}, t))^2}{2m_0} + e\Phi(\vec{r}, t) + \hat{V}_{\text{ext}}(\vec{r}, t)$$

(115)

was derived from a Brownian motion process. If the two rotating systems undergo this process, their overall energy density may be written as superposition

$$w(\vec{r}, t) = \psi_1^*(\vec{r}, t)\hat{H}(\vec{r}, t)\psi_1(\vec{r}, t) + \psi_2^*(\vec{r}, t)\hat{H}(\vec{r}, t)\psi_2(\vec{r}, t) = \Psi^\dagger(\vec{r}, t)\hat{H}\Psi(\vec{r}, t),$$

(116)

where we have used the definition

$$\hat{H} \equiv \hat{H}(\vec{r}, t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

(117)

The Hamiltonian should include the energy from the interaction of the spin rotation with $B_z(\vec{r}, t)$.

Hence, $\mu(\vec{r}, t)$ must be added to $w(\vec{r}, t)$, and we get $\hat{H} + \mu B_z$ for the Hamiltonian operator. We observe that the effect of $\mu B_z$ on each spin state is that of a potential. Since the time dependent Schrödinger equation was derived under an arbitrary external potential, we can write a first equation for a spinning particle:

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = (\hat{H} + \mu B_z)\Psi(\vec{r}, t).$$

(118)

If we want to measure the particle’s spin rotation around an axis $\vec{z}'$ that is different from the axis defined by the $z$ component $B_z(\vec{r}, t)$ of the magnetic field, we must change the coordinate system of $\vec{B}(\vec{r}, t)$. This can be done via linear transformations $\vec{B}'(\vec{r}, t) \equiv \vec{Q}\vec{B}(\vec{r}, t)$, where $\vec{Q}$ defines a $3 \times 3$ rotation matrix which transforms $(B_x(\vec{r}, t), B_y(\vec{r}, t), B_z(\vec{r}, t))$ into $\vec{B}'(\vec{r}, t) = (B_x(\vec{r}, t), B_y(\vec{r}, t), B_z(\vec{r}, t))$. It is known from classical mechanics (see e.g. [45]) that by using Euler angles $\psi$, $\phi$ and $\theta$ as shown in Fig. 4, any $3 \times 3$ rotation matrix can be converted to the following $2 \times 2$ form:

$$\vec{Q} \equiv \begin{pmatrix} e^{\frac{i}{2}(\psi + \phi) \cos (\theta/2)} & i e^{\frac{i}{2}(\psi - \phi) \sin (\theta/2)} \\ i e^{-\frac{i}{2}(\psi + \phi) \sin (\theta/2)} & e^{-\frac{i}{2}(\psi - \phi) \cos (\theta/2)} \end{pmatrix}.$$  

(119)
In order to apply the $2 \times 2$ matrix from Eq. (119) on $\vec{B}(\vec{r}, t)$, we must rewrite the magnetic field:

$$
\begin{pmatrix}
B_z(\vec{r}, t) & B_x(\vec{r}, t) - iB_y(\vec{r}, t) \\
B_x(\vec{r}, t) + iB_y(\vec{r}, t) & -B_z(\vec{r}, t)
\end{pmatrix}.
$$

(120)

Then, one can compute $\vec{B}'$ via

$$
\vec{B}' = \mathbf{Q}\vec{B} \mathbf{Q}^+.
$$

(121)

We note that a magnetic field $\vec{B}(\vec{r}, t) = (0, 0, B_z(\vec{r}, t))$ has a matrix form which is equal to $\mathbf{B}_z$ in Eq. (114). If we use the Pauli matrices $\sigma_\mathbf{x}$, $\sigma_\mathbf{y}$ and $\sigma_\mathbf{z}$, we can simplify equation (120) considerably. At first, we observe that

$$
\mathbf{B} = B_z(\vec{r}, t)\sigma_\mathbf{z} + B_y(\vec{r}, t)\sigma_\mathbf{y} + B_x(\vec{r}, t)\sigma_\mathbf{x}.
$$

(122)

Writing the Pauli matrices in form of a vector $\vec{\sigma} = (\sigma_\mathbf{x}, \sigma_\mathbf{y}, \sigma_\mathbf{z})$, we arrive at

$$
\vec{B} = \vec{B}(\vec{r}, t)\vec{\sigma}.
$$

(123)

It has to be derived, how $\Psi(\vec{r}, t)$ will transform under a rotation of the coordinate system. At each point $(\vec{r}, t)$, the change of the particle’s energy density, which is caused by the interaction of the spin with the magnetic field, should be invariant under a rotation:

$$
\mu(\vec{r}, t) = \mu'(\vec{r}, t).
$$

(124)

Making the assumption that

$$
\Psi'(\vec{r}, t) \equiv \mathbf{Q}\Psi(\vec{r}, t),
$$

(125)

we get the following expression for $\mu'(\vec{r}, t)$:

$$
\mu'(\vec{r}, t) = \mu B \Psi^+(\vec{r}, t) \mathbf{Q}^+ \mathbf{B}' \Psi(\vec{r}, t).
$$

(126)

Since $\mathbf{Q}$ in Eq. (119) is unitary, the following identity holds, provided we use Eq. (121):

$$
\mathbf{Q}^+ \mathbf{B}' \mathbf{Q} = \mathbf{Q}^+ \mathbf{Q} \mathbf{B} \mathbf{Q} \mathbf{Q}^+ = \mathbf{B}.
$$

(127)

The equations (127) and (126) imply that the condition of Eq. (124) is fulfilled. This justifies our assumption from Eq. (125).

Using Eq. (126), Eq. (127) and equation (123), we can write the overall energy density of a particle ensemble under a general magnetic field:

$$
w(\vec{r}, t) = \Psi^+(\vec{r}, t) \left( \mathbf{H} + \mu B \vec{B}(\vec{r}, t) \vec{\sigma} \right) \Psi(\vec{r}, t).
$$

(128)
Again, the energy from the interaction of the spin with the magnetic field has the form of a potential. Hence, we finally arrive at the time dependent Pauli equation for an arbitrary magnetic field:

$$\hbar i \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left( \hat{H} + \mu_B \vec{B}(\vec{r}, t) \vec{\sigma} \right) \Psi(\vec{r}, t).$$

(129)

In order to describe the direction of the spin rotation around the axis $\vec{z}'$, only the Euler angles $\theta$ and $\phi$ are necessary, and one can select $\psi = -\pi/2$ without loss of generality. The projection of the rotated axis $\vec{z}'$ in the $\vec{z}/\vec{y}$ plane of the non-rotated coordinate system encloses an angle $\varphi$ with the $\vec{x}$ axis. This angle is related to the Euler angle $\phi$ by $\phi \equiv \varphi + \pi/2$ (see Fig. 4). Inserting $\varphi + \pi/2$ and the choice of $\psi = -\pi/2$ in the matrix of Eq. (119) usually simplifies the calculations.

A common situation is that the magnetic field $\vec{B}(\vec{r}, t)$ of a Stern-Gerlach apparatus lies in a coordinate system which is rotated with respect to the orientation of an incoming spinor $\Psi'(\vec{r}, t)$. Especially in EPR experiments, one has to compute $\Psi(\vec{r}, t) = Q^+ \Psi'(\vec{r}, t)$. Applying the adjoint of $Q$ with the aforementioned simplifications on a $\Psi'(\vec{r}, t) = \psi'_\uparrow(\vec{r}, t) |+\rangle$ spinor yields

$$\Psi(\vec{r}, t) = \psi'_\uparrow(\vec{r}, t) \left( e^{-i\varphi/2} \cos(\theta/2) |+\rangle + e^{i\varphi/2} \sin(\theta/2) |-\rangle \right),$$

(130)

and we get a similar result for a $\Psi'(\vec{r}, t) = \psi'_\downarrow(\vec{r}, t) |-\rangle$ state:

$$\Psi(\vec{r}, t) = \psi'_\downarrow(\vec{r}, t) \left( -e^{-i\varphi/2} \sin(\theta/2) |+\rangle + e^{i\varphi/2} \cos(\theta/2) |-\rangle \right).$$

(131)

5.4 Spin operator

Still following the lines of [40], we will derive the so-called spin-operator. Both rotating particle ensembles have a spin angular momentum of $\pm \hbar/2$ on the average. Therefore, the overall expectation value of the spin in the $\vec{z}$ direction can be written as weighted sum

$$\langle S_{\vec{z}} \rangle \equiv \frac{\hbar}{2} \int \left( |a|^2 \psi^*_\uparrow(\vec{r}, t) \psi_\uparrow(\vec{r}, t) - |b|^2 \psi^*_\downarrow(\vec{r}, t) \psi_\downarrow(\vec{r}, t) \right) d^3 r.$$

(132)

We can simplify this expression with the definition of a matrix

$$\hat{S}_{\vec{z}} \equiv \frac{\hbar}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = \frac{\hbar}{2} \hat{\sigma}_z,$$

(133)

and obtain

$$\langle S_{\vec{z}} \rangle = \int \psi^+(\vec{r}, t) \hat{S}_{\vec{z}} \psi(\vec{r}, t) d^3 r.$$

(134)

For the spin with respect to the direction of an unit vector $\vec{z}' = \hat{\mu}$ from an arbitrary rotated coordinate system, we get, analogous to Eq. (123), the relation

$$\hat{S}_{\vec{z}'} = \frac{\hbar}{2} \hat{\mu} \hat{\sigma}.$$

(135)

Equation (135) is the so-called spin operator for the $\vec{z}'$ direction. It is convenient to write the axis $\vec{z}'$ in spherical coordinates as $\vec{z}' \equiv (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$. We observe that $\hat{\mu}$ depends on the same angles that are used for the spinors of Eq. (130) and Eq. (131). Finally, the average value $\langle S_{\vec{z}'} \rangle$ becomes

$$\langle S_{\vec{z}'} \rangle = \int \psi^+(\vec{r}, t) \hat{S}_{\vec{z}'} \psi(\vec{r}, t) d^3 r = \hat{\mu} \langle \hat{S}_{\vec{z}} \rangle,$$

(136)
Having derived the Pauli equation, the correct transformation properties of the spinors in rotated magnetic fields, and finally the Spin operator for arbitrary magnetic fields, we should look again at the Stern-Gerlach effect. Usually, it is said in textbooks that a Stern-Gerlach magnet would simply be a measurement device for eigenvalues of the Spin-Operator. However, a closer look shows that this is an oversimplification, that would only work for an infinitesimal small Stern-Gerlach magnet. In the real world, the magnetic field of a Stern-Gerlach magnet is intrinsically distributed over a broad part of the space. In order to take this into account, we have to solve the Pauli equation with a rather complicated magnetic field. One then finds that the results may differ from the naive eigenvalue computations. Such effects are especially important in experimental setups where the particle beam successively goes through two Stern-Gerlach magnets. The reader might consult the articles [46] and [47] for further information on this issue.

6 Entangled systems with spin

6.1 Transformation of the entangled state into equations of motion

As in the case of the many-particle Schrödinger equation, we can extend the coordinate space to derive the N-particle Pauli equation. From now on, we will assume that there are no forces which interconnect the different particle states.

We start our discussion with a state
\[
\Psi(\vec{r}_1, \vec{r}_2, t) \equiv \frac{1}{\sqrt{2}} (\psi_1(\vec{r}_1, t)|-\rangle \otimes \psi_2(\vec{r}_2, t)|+\rangle - \psi_2(\vec{r}_1, t)|+\rangle \otimes \psi_1(\vec{r}_2, t)|-\rangle)
\]
\[
= \frac{1}{\sqrt{2}} \chi_1(\vec{r}_1, \vec{r}_2, t) + \frac{-1}{\sqrt{2}} \chi_2(\vec{r}_1, \vec{r}_2, t).
\]
(138)

It shall describe two ensembles of entangled particles with the same rest mass \(m_0\) and opposite spin. The one-particle spin states in the first line of Eq. (138) are all eigenstates with respect to a chosen axis \(\vec{z}\) and in the second line, we have used the functions
\[
\chi_1(\vec{r}_1, \vec{r}_2, t) \equiv \psi_1(\vec{r}_1, t)|-\rangle \otimes \psi_2(\vec{r}_2, t)|+\rangle \quad \text{and} \quad \chi_2(\vec{r}_1, \vec{r}_2, t) \equiv \psi_2(\vec{r}_1, t)|+\rangle \otimes \psi_1(\vec{r}_2, t)|-\rangle.
\]
(139)

The state in Eq. (138) is the solution of a two-particle Pauli equation without external electromagnetic fields. This leads to a Hamiltonian operator
\[
\hat{H}(\vec{r}_1, \vec{r}_2, t) \equiv \left( \sum_{n=1}^{2} -\frac{\hbar^2 \Delta_n}{2m_0} + \sum_{n=1}^{2} \hat{V}_{\text{ext}}(\vec{r}_n, t) \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]
(140)

whose external potentials for the two particle systems agree at each time \(t\):
\[
\hat{V}_{\text{ext}}(\vec{r}_1, t) = \hat{V}_{\text{ext}}(\vec{r}_2, t).
\]
(141)

Similar as in Eq. (101), the Pauli equation for the entangled state in Eq. (138) can be written as:
\[
\frac{i \hbar}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, t) = \hat{H}(\vec{r}_1, \vec{r}_2, t) \Psi(\vec{r}_1, \vec{r}_2, t)
\]
\[
= \left\{ \begin{array}{l}
\frac{i \hbar}{\partial t} \left( \frac{1}{\sqrt{2}} \chi_1(\vec{r}_1, \vec{r}_2, t) \right) = \hat{H}(\vec{r}_1, \vec{r}_2, t) \left( \frac{1}{\sqrt{2}} \chi_1(\vec{r}_1, \vec{r}_2, t) \right) \\
+ \\
\frac{i \hbar}{\partial t} \left( \frac{-1}{\sqrt{2}} \chi_2(\vec{r}_1, \vec{r}_2, t) \right) = \hat{H}(\vec{r}_1, \vec{r}_2, t) \left( \frac{-1}{\sqrt{2}} \chi_2(\vec{r}_1, \vec{r}_2, t) \right)
\end{array} \right.
\]
(142)
The osmotic velocity is connected to the probability density of many-particle states via

\[ \rho(\vec{r}_1, \vec{r}_2, t) = \frac{1}{2} \rho_1(\vec{r}_1, \vec{r}_2, t) + \frac{1}{2} \rho_2(\vec{r}_1, \vec{r}_2, t), \]

where

\[ \rho_1(\vec{r}_1, \vec{r}_2, t) = \chi_1^+(\vec{r}_1, \vec{r}_2, t) \chi_1(\vec{r}_1, \vec{r}_2, t) = \psi_1^*(\vec{r}_1, t) \psi_1(\vec{r}_1, t) \langle - | - \rangle \otimes \psi_2^*(\vec{r}_2, t) \psi_2(\vec{r}_2, t) \langle + | + \rangle \equiv \rho_{11}(\vec{r}_1, t) \otimes \rho_{21}(\vec{r}_2, t) \] (145)

and

\[ \rho_2(\vec{r}_1, \vec{r}_2, t) = \chi_2^+(\vec{r}_1, \vec{r}_2, t) \chi_2(\vec{r}_1, \vec{r}_2, t) = \psi_2^*(\vec{r}_1, t) \psi_2(\vec{r}_2, t) \langle + | + \rangle \otimes \psi_1^*(\vec{r}_2, t) \psi_1(\vec{r}_2, t) \langle - | - \rangle \equiv \rho_{21}(\vec{r}_1, t) \otimes \rho_{11}(\vec{r}_2, t). \] (146)

The osmotic velocity is connected to the probability density of many-particle states via

\[ \bar{u}^N(\vec{r}_N, t) = -\frac{\hbar}{2m_0} \bar{\nabla}^N \ln \left( \rho(\vec{r}_N, t) / \rho_0 \right). \] (147)

Hence, after the tensor product in \( \rho_1(\vec{r}_1, \vec{r}_2, t) \) is converted by the logarithm laws into a sum, we get for \( \bar{u}^N_{\chi_1}(\vec{r}_1, \vec{r}_2, t) \):

\[ \bar{u}^N_{\chi_1}(\vec{r}_1, \vec{r}_2, t) = -\frac{\hbar}{2m_0} \bar{\nabla}^N \ln \left( \rho_{11}(\vec{r}_1, t) / \rho_0 \right) + \ln \left( \rho_{21}(\vec{r}_2, t) / \rho_0 \right) \]

\[ = \begin{pmatrix} -\frac{\hbar}{2m_0} \bar{\nabla} \ln \left( \rho_{11}(\vec{r}_1, t) / \rho_0 \right) \\ -\frac{\hbar}{2m_0} \bar{\nabla} \ln \left( \rho_{21}(\vec{r}_2, t) / \rho_0 \right) \end{pmatrix} = \begin{pmatrix} \bar{u}_{\chi_1}(\vec{r}_1, t) \\ \bar{u}_{\chi_1}(\vec{r}_2, t) \end{pmatrix}. \] (148)

Similarly, \( \bar{u}^N_{\chi_2} \) in the Pauli equation for \( \chi_2(\vec{r}_1, \vec{r}_2, t) \) is of the form \( \bar{u}^N_{\chi_2}(\vec{r}_N, t) \equiv \begin{pmatrix} \bar{u}_{\chi_2}(\vec{r}_1, t) \\ \bar{u}_{\chi_2}(\vec{r}_2, t) \end{pmatrix} \).

The definition of the average velocity \( \bar{v}^N(\vec{r}_N, t) \) for a many-particle ensemble is

\[ \bar{v}^N(\vec{r}_N, t) = \frac{\hbar}{m_0} \bar{\nabla}^N \varphi(\vec{r}_N, t), \] (149)

where \( \varphi(\vec{r}_N, t) \) is the phase of the many-particle wave-function

\[ \Psi(\vec{r}_N, t) \equiv \pm \sqrt{\rho(\vec{r}_N, t)} e^{i\varphi(\vec{r}_N, t)}. \] (150)

Writing \( \chi_1(\vec{r}_1, \vec{r}_2, t) \) in the form of Eq. (150), i.e.

\[ \chi_1(\vec{r}_1, \vec{r}_2, t) = \sqrt{\rho_{11}(\vec{r}_1, t)} e^{i\varphi_{11}(\vec{r}_1, t)} \otimes \sqrt{\rho_{21}(\vec{r}_2, t)} e^{i\varphi_{21}(\vec{r}_2, t)}, \] (151)

we can identify the overall phase of \( \chi_1(\vec{r}_1, \vec{r}_2, t) \) as

\[ \varphi_{11}(\vec{r}_1, \vec{r}_2, t) = \varphi_{11}(\vec{r}_1, t) + \varphi_{21}(\vec{r}_2, t). \] (152)

Using the Eqs. (149) and (152), we arrive at

\[ \bar{v}^N_{\chi_1}(\vec{r}_1, \vec{r}_2, t) = \frac{\hbar}{m_0} \bar{\nabla}^N \varphi_1(\vec{r}_1, \vec{r}_2, t) = \begin{pmatrix} \bar{v}_1(\vec{r}_1, t) \\ \bar{v}_1(\vec{r}_2, t) \end{pmatrix}, \] (153)

28
and we get \( \tilde{v}^N_{21}(\vec{r}_1, \vec{r}_2, t) \equiv \left( \tilde{v}_1(\vec{r}_1, t) \begin{array}{c} \tilde{u}_1(\vec{r}_1, t) \\ \tilde{u}_1(\vec{r}_2, t) \end{array} \right) \) in completely the same way.

The expressions for \( \tilde{u}^N_{11}, \tilde{u}^N_{21}, \tilde{v}^N_{11}, \tilde{v}^N_{21} \) imply that the Pauli equation \( \text{(143)} \) with the entangled state in Eq. \( \text{(158)} \) can be decomposed into 2 two component differential equations, where \( \tilde{F}^{\text{ext}}_2(\vec{r}_2, t) = \tilde{F}^{\text{ext}}_1(\vec{r}_1, t) \):

\[
\begin{cases}
\frac{d}{dt} \tilde{v}_1(\vec{r}_1, t) - \tilde{u}_1(\vec{r}_1, t) \cdot \tilde{v}_1(\vec{r}_1, t) + \nu \Delta \tilde{u}_1(\vec{r}_1, t) = \frac{1}{m_0} \tilde{F}^{\text{ext}}_1(\vec{r}_1, t), \\
\frac{d}{dt} \tilde{v}_1(\vec{r}_2, t) - \tilde{u}_1(\vec{r}_2, t) \cdot \tilde{v}_1(\vec{r}_2, t) + \nu \Delta \tilde{u}_1(\vec{r}_2, t) = \frac{1}{m_0} \tilde{F}^{\text{ext}}_2(\vec{r}_2, t), 
\end{cases}
\]

(154)

The physical information of Eq. \( \text{(154)} \) is equivalent to what is described by Eq. \( \text{(143)} \). However, Eq. \( \text{(154)} \) contains the ensemble averages \( \tilde{u} \) and \( \tilde{v} \) of each particle system directly. All equations in \( \text{(154)} \) can be similarly derived as the two equations in \( \text{(143)} \), but the derivation must be done for the \( \vec{r}_1 \) and \( \vec{r}_2 \) particle ensembles separately.

### 6.2 How the correlations of entangled states emerge in the local hidden variable theory

Having converted the entangled state into equations of motion that can be derived from the previously discussed hidden variable model, we will investigate in detail how the correlations of entangled systems emerge from the viewpoint of these hidden variables.

The source of the EPR experiment produces particles of a defined spin direction with respect to the axis \( \vec{z} \). However, the source cannot prepare the spin states in separate beams like a Stern-Gerlach magnet because it does not divide the two spin ensembles spatially. Hence, the spin states of the particles emerging from the source are in a superposition of an \( \uparrow \) and a \( \downarrow \) state. According to section 5.2, the individual particles in such a system have to undergo an exchange procedure between the \{A \( \uparrow \)\} and \{B \( \downarrow \)\} or the \{A \( \downarrow \)\} and \{B \( \uparrow \)\} ensembles. The trajectories \( \tilde{x}^{A(B)}_{j1(1)j1}(t) \) of the \( \vec{r}_1 \) particles have to switch between their \( \tilde{v}_1^{A}(\vec{r}_1, t) \) and \( \tilde{v}_1^{B}(\vec{r}_1, t) \) system, as well as between their \( \tilde{v}_1^{A}(\vec{r}_1, t) \) and \( \tilde{v}_1^{B}(\vec{r}_1, t) \) ensemble in Eq. \( \text{(154)} \) during their flight. An analogous procedure has to happen with the trajectories \( \tilde{x}^{A(B)}_{j1(1)j2}(t) \) of the \( \vec{r}_2 \) ensembles.

We observe that it is possible, to threat the \( \vec{r}_1 \) and \( \vec{r}_2 \) particle systems completely disconnected because they are uncoupled in Eq. \( \text{(154)} \). These ensembles may even be spatially separated, in a way that there is no contact between them. A local measurement device that acts on one particle ensemble, say \( \vec{r}_1 \), might change the potential \( \tilde{V}^{\text{ext}}_{1}(\vec{r}_1, t) \) in Eq. \( \text{(140)} \). Along with this goes a change of the external force \( \tilde{F}^{\text{ext}}_{1}(\vec{r}_1, t) \). If the action of the device was such that \( \tilde{F}^{\text{ext}}_{1}(\vec{r}_1, t) \neq \tilde{F}^{\text{ext}}_{2}(\vec{r}_2, t) \) at a certain time \( t \), the equations governing the \( \vec{r}_1 \) particle system would be different from the equations of the \( \vec{r}_2 \) system. As a result, correlations in Eq. \( \text{(154)} \) between both ensembles would be lost. Obviously, the dynamics of the \( \vec{r}_2 \) ensemble is not changed by something that happens with the \( \vec{r}_1 \) ensemble.

For the discussion of the EPR experiment, it is important to note that there will still be correlations between the two particle systems if the experimenters at both measurement stations 1 and 2 choose the settings of their devices in a way that \( \tilde{F}^{\text{ext}}_{1}(\vec{r}_1, t) = \tilde{F}^{\text{ext}}_{2}(\vec{r}_2, t) \) for all times \( t \). This is exactly the situation when the two separated Stern-Gerlach magnets are rotated in the same axis \( \vec{z} \).

In order to connect the present discussion with section 2, we recall the definition of the spin trajectories as stochastic processes \( \phi_{j1t} : \tilde{F}^{\text{Brown}}(t) \rightarrow \{\uparrow, \downarrow\} \) and \( \phi_{j2t} : \tilde{F}^{\text{Brown}}(t) \rightarrow \{\uparrow, \downarrow\} \) for times \( 0 \leq t < T \). We write \( \phi_{j1t} = \uparrow \) if the \( j \)-th trajectory of the \( \vec{r}_1(\vec{r}_2) \) system is in the \( \uparrow \) ensemble at time \( t \). Similarly, we have \( \phi_{j1t} = \downarrow \) in case that the \( j \)-th trajectory of the \( \vec{r}_1(\vec{r}_2) \) system is in the \( \downarrow \) ensemble at time \( t \).
When the particles arrive in the space-like separated detectors at time \( T \), they are affected by the forces of the magnetic fields of the Stern-Gerlach magnets. These fields depend on the axes chosen by the experimenters. After the particles have left the Stern-Gerlach magnets, their spin does not change anymore. Hence, we can describe the final result of the \( j \)-th experiment with two axis dependent random variables \( D_{j1\mu} : F_{j1}^{\text{Brown}}(T) \mapsto \{\uparrow, \downarrow\} \) and \( D_{j2\nu} : F_{j2}^{\text{Brown}}(T) \mapsto \{\uparrow, \downarrow\} \), where the external forces \( F_{j2}^{\text{ext}}(T) \) point in the same direction as the axes \( \vec{\mu} (\vec{\nu}) \).

One gets exact anti-correlations between the spins of the \( j \)-th particle pair during its flight if one can ensure for times \( 0 \leq t < T \) that

\[
\phi_{j1t\epsilon} \neq \phi_{j2t\epsilon}.
\]  

Eq. (155) gives rise to the entangled state in Eq. (158). In the viewpoint of our hidden variable theory, the wave-function in Eq. (158) describes two separate ensembles, whose particles have spin states that are oscillating in an opposite way.

When the entangled particles have passed the Stern-Gerlach magnets at times \( t \geq T \), their spin values should still be anticorrelated if the axes of the devices where chosen to be the same. Accordingly, we have to write

\[
D_{j1\mu} \neq D_{j2\nu}.
\]  

If a trajectory changes its ensemble from \( \{A\} \) \( \{B\} \) to \( \{B\} \) \( \{A\} \), the spin state of that trajectory has to change to the opposite value at the same time. Similarly, whenever a particle changes its spin, its \( \{A\} \) \( \{B\} \) ensemble must change also. Transitions like \( \{B\} \mapsto \{A\} \), \( \{B\} \mapsto \{A\} \), \( \{A\} \mapsto \{A\} \) or \( \{B\} \mapsto \{B\} \) are not allowed. Otherwise, the trajectories affected would not be described any more by a superposition of an \( \uparrow \) and a \( \downarrow \) state in which energy conservation holds on the average.

The source of an EPR experiment repeatedly produces pairs of entangled particles. In each of these pairs, one particle belongs to the \( \vec{r}_1 \) system while the other particle is associated with \( \vec{r}_2 \). Therefore, the same index \( j \) can be used for the two trajectories of an entangled particle pair without loss of generality. The entangled particles start at the source with opposite spin and the same initial speed, due to momentum conservation. Accordingly, the trajectories that have the same index \( j \) in the \( \vec{r}_1 \) and \( \vec{r}_2 \) system will be placed in the same \( \{A\} \) \( \{B\} \) ensemble at time \( t = 0 \), but in the opposite spin system. Now assume that, because of some reason, the velocities of these paired particles coincide at times \( 0 \leq t < T \):

\[
\vec{z}^{A(B)}_{\vec{r}_1(\vec{r}_2)}(t) = \vec{z}^{A(B)}_{\vec{r}_1(\vec{r}_2)}(t).
\]  

Then, the ensemble average velocities of the \( \vec{r}_1 \) and \( \vec{r}_2 \) systems will also agree at times \( 0 \leq t < T \). The decision whether a trajectory \( \vec{z}^{A(B)}_{\vec{r}_1(\vec{r}_2)}(t) \) from the \( \vec{r}_1 \) \( \vec{r}_2 \) particle system changes its ensemble from \( \{A\} \) \( \{B\} \) to \( \{B\} \) \( \{A\} \) depends on the magnitude of its velocity \( \vec{z}^{A(B)}_{\vec{r}_1(\vec{r}_2)}(t) \) and on the velocity distribution of all particles from \( \vec{r}_1 \) \( \vec{r}_2 \) that are in the past cone of \( \vec{z}^{A(B)}_{\vec{r}_1(\vec{r}_2)}(t) \). The exchange mechanism, as it was defined in section 4.1, works in the same way for all trajectories, no matter if they are in \( \vec{r}_1 \) or \( \vec{r}_2 \). In case Eq. (157) holds, it follows that the trajectories which have the same index \( j \) in the \( \vec{r}_1 \) \( \vec{r}_2 \) ensembles will switch their \( \{A\} \) \( \{B\} \) particle system in the same way during their flight. Each placement in another \( \{A\} \) \( \{B\} \) ensemble changes a particle's spin state to the opposite. Since the spin ensemble of the entangled particles was opposite from the beginning at \( t = 0 \), their spin trajectories will have opposite values \( \phi_{j1t\epsilon} \neq \phi_{j2t\epsilon} \) for times \( 0 \leq t < T \).

The velocities of all particles depend on the random forces \( F_{j1}^{\text{Brown}}(t) \), the external forces \( F_{j2}^{\text{ext}}(t) \) and the trajectory exchange procedure. The exchange mechanism works in the same way for the particles in \( \vec{r}_1 \) and \( \vec{r}_2 \). Furthermore, we must have

\[
F_{j1}^{\text{ext}}(t) = F_{j2}^{\text{ext}}(t)
\]  

(158)
for all times $0 \leq t \leq T$ if we are to observe exact correlations. Hence, any remaining differences in the velocity of two entangled particles can be caused by the random forces $\vec{F}^\text{Brown}_{j1(2)}$ only. Provided that we impose the condition

$$\vec{F}^\text{Brown}_{j1}(t) = \vec{F}^\text{Brown}_{j2}(t)$$

for all $j$ and all times $t$, Eq. (157) and thus Eq. (155) hold at least for times $0 \leq t < T$. If the experimenters had chosen the same axes $\vec{\mu} = \vec{\nu}$ at the Stern-Gerlach magnets, the external forces induced by these devices are equal. The particles arrive in the magnets at time $T$. Thereby, Eq. (155) also holds for time $T$ and all $j$. After the particles have passed the Stern-Gerlach magnets, their spin values do not change anymore and we can replace Eq. (155) by the random variables in Eq. (156).

We conclude that exact correlations between the spins measurement results of particles with the same index $j$ in $\vec{r}_1$ and $\vec{r}_2$ can be observed.

On the other hand if the experimenters have chosen different axes at the Stern-Gerlach magnets, the external forces for the $j$-th particles are different at time $T$, i.e.

$$\vec{F}^\text{ext}_{j1}(T) \neq \vec{F}^\text{ext}_{j2}(T).$$

In that case, the two particles which have the same index $j$ in the $\vec{r}_1$ and $\vec{r}_2$ ensembles are subject to different equations of motion. It then follows from the discussion above that spin measurement results will not be exactly correlated for each $j$-th particle pair.”

In order to fulfill the requirement of Eq. (159), a theory must not necessarily incorporate a signalling mechanism between the unchpled particle systems $\vec{r}_1$ and $\vec{r}_2$. This can be easily seen by an explicit construction of a local model in which Eq. (159) holds. The only constraint on the random forces was that they should have a Gaussian distribution. Now consider the following mechanism: A real random number generator sets a value $\lambda_j \in \mathbb{R}$ for each $j$-th particle pair at preparation stage. This value $\lambda_j$ is similar to the hidden parameter in Bell’s original work [5], but we define it to be independent for each entangled particle pair generated by the source. In our model, $\lambda_j$ serves as the starting value for two pseudo random number generators of the same type. Pseudo random number generators are deterministic procedures which always produce the same series provided that they are initialised with the same starting value $\lambda_j$. After they were initialised at $t = 0$, we let the two generators produce the Gaussian distributed and time dependent series $\vec{F}^\text{Brown}_{j1}(t, \lambda_j)$ and $\vec{F}^\text{Brown}_{j2}(t, \lambda_j)$. These series are the random forces of the $j$-th trajectories for the $\vec{r}_1$ and $\vec{r}_2$ particles. Since they only depend on their common starting value $\lambda_j \in \mathbb{R}$, Eq. (159) holds at each time. There is no signalling mechanism between the two disconnected forces because they simply depend on their common past.

At first sight, one may think that our procedure would set up a deterministic passively local theory. Indeed, for $\vec{F}^\text{Brown}_{j1}(t, \lambda_j)$ and $\vec{F}^\text{Brown}_{j2}(t, \lambda_j)$ there is a parameter $\lambda_j$ at preparation stage which determines all later outcomes of these forces. However, one does not have access to the outcomes of the Gaussian distributed random forces in the probability space, but only to the two spin values in the state space of $\phi_{j1\hat{\mu}} \otimes \phi_{j2\hat{\nu}}$.

The proof in section 3 was with regards to elements of the sigma algebras $\mathcal{F}_{jS}$, $\mathcal{F}_{j1\hat{\mu}}$ and $\mathcal{F}_{j2\hat{\nu}}$. These sigma algebras contain those events on the probability space that determine the accessible spin results on the state space of the EPR experiment. The set of outcomes in the probability space for the Gaussian distributed force fields $\vec{F}^\text{Brown}_{j1(2)}(t, \lambda_j)$ is considerably larger than the set of outcomes in the state space because the latter consists of only two spin values. Hence, every element in $\mathcal{F}_{jS}$, $\mathcal{F}_{j1\hat{\mu}}$ and $\mathcal{F}_{j2\hat{\nu}}$ must contain many outcomes of the force fields which all lead to the same spin result in the state space. This implies that the condition of deterministic passive locality corresponds to the following situation:

For every entangled particle that flies to a detector, an event on the probability space of the random force field has to happen at the first time step. All outcomes in this event must lead to the same particular spin state of the particle at that time. By deterministic passive locality, the spin result of a given particle at preparation stage has to determine the particle’s spin value at the detector. Accordingly, the event for the Brownian force field at the first time step has to determine another event.
The latter consists of outcomes of the random force field at a time when the particle has arrived in the Stern-Gerlach magnet. Most notably, all outcomes in this predetermined event must be such that they lead to the same particular spin state of the particle in the detector. We will illustrate below that deterministic passive locality can be easily violated, even though the outcomes of the time dependent random force may be predetermined by a hidden parameter for each entangled particle pair.

We let the spin trajectory \( \phi_{j=1,1,t} \) of a particle, which is in a pair \( j = 1 \) and will fly to detector 1, start in the \( \{ A \uparrow \} \) ensemble. The magnitude of the velocity of this particle at time \( t = 0 \) can be denoted by \( |\hat{x}^A_{j=1,1}(t=0)| \), and the particle’s spin trajectory at that time is \( \phi_{j=1,1,t=0} = \uparrow \). In the following example, we will assume that the external forces are zero until the particle arrives in a detector. For a given random starting value \( \lambda_{j=1} \in \mathbb{R} \) of the pseudo random number generators, we denote the outcome of the Brownian force at time \( t = 1 \) by

\[
\hat{F}^\text{Brown}_{j=1,1}(t=1, \lambda_{j=1}) \equiv \xi_1.
\]

Since \( |\hat{x}^A_{j=1,1}(t)| \) is a function of \( \hat{F}^\text{Brown}_{j=1,1}(t, \lambda_{j-1}) \), it may be that \( \xi_1 \) leads to a particle velocity where \( |\hat{x}^A_{j=1,1}(t=1)| \leq |\hat{v}(\hat{r}_1, t=1)| \). According to the definition of the trajectory exchange procedure from section 4.1, we are only allowed to replace a particle from \( \{ A \} \) to \( \{ B \} \) if the magnitude of its velocity is at least higher than \( |\hat{v}(\hat{r}_1, t)| \). Hence, no change of the spin trajectory will take place at \( t = 1 \) and we can write:

\[
\phi_{j=1,1,t=0} = \phi_{j=1,1,t=1} = \uparrow.
\]

Similarly, at time \( t = 2 \), we denote the outcome of the random force by

\[
\hat{F}^\text{Brown}_{j=1,1}(t=2, \lambda_{j=1}) \equiv \xi_2,
\]

where \( \xi_2 \) may be that \( |\hat{x}^A_{j=1,1}(t=2)| \leq |\hat{v}(\hat{r}_1, t=2)| \). In this case, no change of the spin trajectory is allowed and we can conclude that

\[
\phi_{j=1,1,t=0} = \phi_{j=1,1,t=1} = \phi_{j=1,1,t=2} = \uparrow.
\]

There could be another pair \( j = 2 \), for which the particle 1 that flies towards detector 1 starts in the \( \{ A \uparrow \} \) ensemble. The forces related to this particle may be driven by another random starting value \( \lambda_{j=2} \neq \lambda_{j=1} \). Without loss of generality, it is possible that we get

\[
\hat{F}^\text{Brown}_{j=2,1}(t=1, \lambda_{j=2}) \equiv \delta_1 \approx \xi_1
\]

for \( \lambda_{j=2} \) at \( t = 1 \), with \( \delta_1 \) such that \( |\hat{x}^A_{j=2,1}(t=1)| \leq |\hat{v}(\hat{r}_1, t=1)| \), and thereby

\[
\phi_{j=2,1,t=0} = \phi_{j=2,1,t=1} = \uparrow.
\]

Yet for \( t = 2 \), we may arrive at

\[
\hat{F}^\text{Brown}_{j=2,1}(t=2, \lambda_{j=2}) \equiv \delta_2 \gg \xi_2,
\]

where \( \delta_2 \) causes a significant change in the velocity and leads to \( |\hat{x}^A_{j=2,1}(t=2)| > |\hat{v}(\hat{r}_1, t=2)| \). As a consequence, the particle in the \( \{ A \uparrow \} \) ensemble may be placed in the \( \{ B \downarrow \} \) ensemble at time \( t = 2 \). Without loss of generality, we will assume that such a replacement happens for the trajectory \( j = 2 \). Then, \( \hat{x}^A_{j=2,1}(t=2) \) changes to \( \hat{x}^B_{j=2,1}(t=2) \) and the spin trajectory becomes

\[
\phi_{j=2,1,t=0} = \phi_{j=2,1,t=1} = \uparrow = \phi_{j=2,1,t=2}.
\]
That this behaviour corresponds to the violation of passive locality can be easily seen if we select an angle $\theta = 2\pi$ at the measurement device 1. By Eq. (150), the spin up state of Eq. (163) at $t = 2$ would lead with certainty to a spin up result in detector 1. Similarly, using Eq. (164), a spin down state can be measured with certainty at the same detector for the spin state of Eq. (167) at $t = 2$. These are opposite measurement results for two particles $j = 1$ and $j = 2$ that both started with spin up at the source and made their way to the same detector. Hence, there can be no event at preparation stage that determines a later event at the measurement devices. This corresponds to the failure of deterministic passive locality.

We have to remark that even if the model with the pseudo random number generators may look terribly artificial, it would nevertheless be difficult to test. The deterministic functions $F_{j1(2)}^{\text{Brown}}(t, \lambda_j)$ depend on $\lambda_j$, which is assumed to be a real random number. Therefore, the statistics of the outcomes measured at time $t = T$ at the detectors would equal the statistics of real random numbers, and this is actually observed in entanglement experiments.

At first sight, it seems that a natural objection against models with a common cause $\lambda_j$ would be so-called entanglement-swapping experiments [48]. In those setups, the entangled particle pairs are generated by two spatially separated sources. Fortunately, these experiments can be described with the model above, but one needs to make some minor changes: The area of the particle source in section 2.1 has to be replaced by a slice in which the two particle sources are. That slice must be disjoint from the future cones of the two Stern-Gerlach magnets and all proofs in section 2 and 3 work as before. In section 6.2, Eq. (159) has to hold for the Brownian forces at the two separated particle sources. Nevertheless, it is possible to find a spatial region where the past cones of the two sources intersect. At this place, a common cause $\lambda_j \in \mathbb{R}$ could well be established, leading to Eq. (159). Hence, even with entangled particles that are produced by two independent sources, there is no need for hidden variable theories to include an instantaneous connection between spatially separated regions.

It is well known that entanglement phenomena are not restricted to spin states. For example, according to quantum mechanics, exact correlations between positions of entangled particles should be observed when they are emitted by the source at almost the same time. A first outline of an experiment to investigate this behaviour was presented in 1989 [49], and the analogue of this setup for photons was realised in 2005 [50]. Recently, a proposal for a similar experiment with material particles has been made [51], which still waits for its realisation. Here, we only mention that the correlations predicted by quantum mechanics for these situations are naturally explained by our local model. Obviously, the discussion above is valid as long as the entangled particles in each pair are ejected within an interval smaller than the course grained timescale $\tau = \frac{\hbar}{2\pi|T|}$ of the Brownian motion process. In turn, we are lead to Eq. (157), which describes particles that behave exactly correlated in velocity and time. After an integration of the particle velocities, we get exact time and position correlations.

In order to violate Bell’s inequality with a local model of the EPR experiment, it does suffice to assume that the Gaussian distributed random forces $F_{j1}^{\text{Brown}}(t)$ and $F_{j2}^{\text{Brown}}(t)$ have a kind of “memory” of their past. Nevertheless, the author has to admit the somewhat incomplete nature of this article. The model given above is of course terribly artificial. It serves only as an example to show that a local hidden variable theory which violates Bell’s inequality can be built in principle. To give a more physical reason for Eq. (159) and to replace the rather arbitrary algorithm for the trajectory exchange procedure, one would need further studies. In fact, passive locality may be violated in many other local hidden variable theories, which are all sharing the property that the events at one measurement apparatus do not influence the events at another spatially separated device. Besides from Nelson, who gave two examples of such theories, a different model with similar locality features was published recently [52]. The authors of this article did not study Bell’s theorem in the language of advanced mathematical probability, as it was done by Nelson, but they seem to share his opinion about the meaning of passive locality. Here, we wish to leave it at that with a citation by Nelson from a popular talk. It shows that behaviour which is similar to entanglement is in fact commonly observed in stochastically evolving systems, and not at all confined to quantum mechanical objects:

“The snowflake problem is this: The inexhaustible variety of snowflakes makes it evident
that chance plays a major role in their development, yet they always preserve hexagonal symmetry – how does a portion of the snowflake growing at random on one side know to grow in precisely the same fashion as its partner all the way over on the other side? This is mysterious but not beyond understanding" [53].

“Similar phenomena are well known to occur in the study of random fields. [...] Perhaps the results of correlated spin experiments are ultimately no more and no less, mysterious than is the random growth of snowflakes with hexagonal symmetry” [9].

An explicit example of classical Brownian particles that exhibit entanglement-like phenomena is studied in [54].

7 Objections against Nelson’s theorem

7.1 The argument of Greenberger et al. – Bell’s theorem without inequalities

The essential argument in Nelson’s (and Jarret’s) reasoning is that a family of device dependent product random variables $\phi_{j,1}^{\vec{\mu}} \otimes \phi_{j,2}^{\vec{\nu}}$ may generate events at spatially separated locations for which passive locality fails, i.e.

$$P_{\phi_{j,1}^{\vec{\mu}} \otimes \phi_{j,2}^{\vec{\nu}} (\sigma_{j,1}^{\vec{\mu}} \cap \sigma_{j,2}^{\vec{\nu}} | F_{j,S})} \neq P_{\phi_{j,1}^{\vec{\mu}} \otimes \phi_{j,2}^{\vec{\nu}} (\sigma_{j,1}^{\vec{\mu}} | F_{j,S})} \cdot P_{\phi_{j,1}^{\vec{\mu}} \otimes \phi_{j,2}^{\vec{\nu}} (\sigma_{j,2}^{\vec{\nu}} | F_{j,S})}.$$ (168)

Nelson and Jarret explicitly wrote that this can happen without any signalling mechanism in the theory that produces the events in question. The latter statement, although mathematically correct, has raised criticisms from physicists since it was made. For example, in his famous introductory textbook, Ballentine [55] writes:

“There have been some debates as to whether the factorisation of the probability is justified by the locality condition alone, or whether it requires some additional stronger assumptions. If so, the contradiction may be blamed on those additional assumptions. The issue is subtle, but fortunately it is now irrelevant since the new proof in [56] does not make use of probability”.

The assertion that the work [56] by Greenberger et al. does not make use of probability is true. However, their arguments do not account for deterministic hidden variable theories of a sort where the events at the particle source are equivalent to the ones at the detectors. The authors are aware of this, noting at page 1138 in their article:

“The salvaging strategy could consist in [...] a stochastic local theory in contrast to a deterministic local theory which has been considered so far in this paper”.

They then go on, writing that these “stochastic local theories” would be impossible. They claim that this would have been thoroughly investigated by Bell [57], Clauser and Horne [58]. Furthermore, they mention so-called “equivalence theorems”, which were proven by Stapp [59] and Fine [60]. Those theorems are said to show that each stochastic local hidden variable theory could be duplicated by a deterministic passively local one. However, all articles cited by [56] only consider theories where passive locality is assumed to hold. (Bell [57] uses passive locality explicitly in his Eq. (5), Clauser and Horne [58] make this assumption in their Eqs. (2) and (2'), Stapp [59] does the same in his Eq. (7a) and also Fine [60] on pp. 1306 and 1308.) In fact, an equivalence to a deterministic passively local theory can only be proven in this case. We conclude that the proof in [56] does not call actively local hidden variable theories into question in which passive locality fails. That the arguments of Greenberger et al. do not apply for stochastic hidden variable theories has also been shown by Khrennikov [61] in a detailed analysis.
7.2 The argument of Dürr – Bell’s theorem with a single probability measure $P$

Furthermore, there are researchers who try to derive Bell’s theorem in the form of Eq. (37), but without the assumption of passive locality. They (e.g. [62] on p. 182) do not write the inequality with a whole family of axis dependent probability measures $P_{\phi_1\mu \otimes \phi_2\nu}$ for device dependent events $\sigma_{j1\mu} \in F_{j1\mu}$ and $\sigma_{j2\nu} \in F_{j2\nu}$. Instead, they are using one single detector independent measure $P$ for events that are thought to be produced by the source and, nevertheless, as dependent on the axes of the detectors. This is problematic because if we compute the quantum mechanical correlation coefficient

$$E(\mu, \nu) \equiv \langle \Psi | (\mu\sigma_1) \otimes (\nu\sigma_2) | \Psi \rangle$$

(169)

with the entangled state in Eq. (138), we are definitely led to Eq. (16) and to a family of axis dependent measures $P_{\phi_1\mu \otimes \phi_2\nu}$.

By using one probability measure for axis dependent events generated by the particle source, we would only be able to investigate theories where an intrinsic particle property is defined at preparation stage that determines the later results at the two detectors for each conceivable pair of axes. Yet it may be possible that the source generates particles whose properties are modified later by the Stern-Gerlach magnets. In this case, a hidden variable model of the EPR experiment has to use an enlarged probability space. The events at preparation stage take place in a smaller space and their probabilities do not depend on the detector’s axes, whereas the events at the detectors happen in the enlarged probability space. A natural assumption is that the events at the two separated measurement stations are generated by a family of device dependent product random variables $\phi_{j1\mu} \otimes \phi_{j2\nu}$. Accordingly, we arrive at a family of axis dependent probability measures $P_{\phi_{j1\mu} \otimes \phi_{j2\nu}}$ for the joint probabilities of the events at the two Stern-Gerlach magnets. Since the probability measures from the formalism of quantum mechanics are of the same structure, we believe that Nelson’s mathematical analysis of the EPR experiment is the correct one.

At this point, we want to remark that the impossibility to describe the EPR experiment with one single probability space, and the need for the inclusion of the detector settings in the corresponding random variables was also mentioned recently by Khrennikov [63]. Nevertheless, it is possible to confine oneself to a single and axis independent measure $P$, as long as active and passive locality are explicitly stated to hold. With that proposition and the proof in section 3, one can trace the events $\sigma_{j1\mu} \in F_{j1\mu}$ and $\sigma_{j2\nu} \in F_{j2\nu}$ for an axis $\mu$ at the detectors back to equivalent events in $F_{jS}$ at the common preparation stage. We can repeat this procedure with two other pairs of equivalent events for different axes $\nu$ and $\nu'$. Eventually, after some further calculations, we will arrive at an inequality which is analogue to Eq. (37) but defined only for events in $F_{jS}$. Due to active locality, the events at preparation stage can not be modified later by the settings of the detectors. Hence, with the assumptions of active and passive locality, we can use an axis independent probability measure $P$ to analyse the EPR experiment. A proof of Bell’s theorem that goes exactly this way can be found in [12].

7.3 The argument of Redhead – stability against disturbances

A different argument against stochastic hidden variable theories was raised by Redhead [26] (who presents in his book at p. 98 a proof of Bell’s inequality which is very similar to that of Nelson, although with some differences). Redhead argues that stochastic hidden parameter theories would be impossible and advances this opinion with the following observation: He asserts that under natural conditions, one can never exclude small disturbances, which might happen erroneously to a single-particle (maybe through a random disturbance by the gases of the air in the laboratory). Redhead then defines what he calls “the most general perturbation that is not a phase shift” as

$$u(\bar{n}, \vartheta) \equiv e^{i(\bar{n}\vartheta)} e^{\vartheta^2/2},$$

(170)
where $\vec{n}$ is an arbitrary unit vector and $\vartheta$ is an angle about $0 \leq \vartheta < 4\pi$. The perturbation should be confined to one part of the overall Hilbert space, e.g. $\mathcal{H}_2$ only. Accordingly, the entangled state in Eq. (138) will transform to

$$
\Psi'(\vec{r}_1, \vec{r}_2, t) \rightarrow \frac{1}{\sqrt{2}} (\psi_1(\vec{r}_1, t) \lvert - \rangle \otimes u(\vec{n}, \vartheta) \psi_2(\vec{r}_2, t) \lvert + \rangle ) + \frac{1}{\sqrt{2}} (\psi_2(\vec{r}_1, t) \lvert + \rangle \otimes u(\vec{n}, \vartheta) \psi_1(\vec{r}_2, t) \lvert - \rangle ).
$$

(171)

Redhead concludes that the exact correlations observed in EPR experiments will be lost with $\Psi'(\vec{r}_1, \vec{r}_2, t)$ for certain axes of the detectors. Nevertheless, the phenomenon of entanglement can be experimentally verified through distances over 100 kilometres in the air [64], where small perturbations acting on single particles cannot be avoided.

Yet, in spirit of Ballentine’s ensemble interpretation [31], a quantum mechanical state may not describe individual particles, but rather an infinitely large statistical ensemble of trajectories. In this case, a perturbation like Eq. (170) would correspond to a disturbance of an entire statistical ensemble of infinitely many particles for a sufficiently long time. Moreover, if defined as above, $u(\vec{n}, \vartheta)$ acts on every particle of that ensemble in exactly the same way. This has nothing to do with a small erroneous disturbance that may happen accidentally to individual particles in the laboratory.

In this context, we want to show that our theory is, in some sense, especially robust against small disturbances, at least if they act on individual particles. For a long distance entanglement experiment, it is a reasonable requirement to expect large uncorrelated external forces $\vec{F}_{\text{ext}}^2(\vec{r}_2, t)$ and $\vec{F}_{\text{ext}}^1(\vec{r}_1, t)$, which act on the particles during their flight, to be absent. It is well known that without external forces, the Langevin equations [52] lead to a Maxwell distribution of the particle’s velocity. Hence, most $\vec{r}_{1(2)}$ particles in the $\{A\}$ or $\{B\}$ ensembles have a velocity around one centre $\approx \lvert \vec{v}_c^A(\vec{r}_{1(2)}, t) \rvert$ or $\approx \lvert \vec{v}_c^B(\vec{r}_{1(2)}, t) \rvert$.

Without loss of generality, we will assume that a small disturbance $\vec{\delta}_{j1}(t)$ induces a slightly different velocity of the $j$-th $\vec{r}_1$ particle in the $\{A \uparrow\}$ ensemble at a point $(\vec{r}_1, t)$ in space-time:

$$
\vec{x}_{j1}(t) \equiv \vec{x}_{j1}^A(t) + \vec{\delta}_{j1}(t).
$$

(172)

We define $\vec{\delta}_{j1}(t)$ to be small if the half-width of the particle’s velocity distribution is by far larger than $\lvert \vec{\delta}_{j1}(t) \rvert$. Moreover, the disturbance should be small enough that the perturbation can’t change the statistics of the majority of particles through the trajectory exchange mechanism. The latter
requirement is equivalent to the condition
\[ \| \vec{v}(\vec{r}_1, t) \| - | \vec{v}_e^A(\vec{r}_1, t) | > | \delta_{j1}(t) | . \]  
(173)

Additionally, we expect the components of \( \delta_{j1}(t) \) to be fluctuating with zero mean.

A particle can only be replaced from the \{A \} to the \{B \} ensemble if the magnitude of its velocity \( | \vec{x}_{j1}^A(t) | \) is higher than \( | \vec{v}(\vec{r}_1, t) | \). The disturbance does not contribute to \( | \vec{v}(\vec{r}_1, t) | \) because the average of each component in \( \delta_{j1}(t) \) is assumed to be zero. Hence,
\[ \| \vec{v}(\vec{r}_1, t) \| - | \vec{x}_{j1}^A(t) | < | \delta_{j1}(t) | \]  
(174)

has to hold if the perturbation \( \delta_{j1}(t) \) is the primary reason for \( x_{j1}^A(t) \) to be placed in an ensemble. The particles fulfilling Eq. (174) are in a small velocity interval of width \( | \delta_{j1}(t) | \) around \( | \vec{v}(\vec{r}_1, t) | \) (see Fig. 4).

Since \( | \delta_{j1}(t) | \) is assumed to be much smaller than the half-width of the particle’s velocity distribution, the number of trajectories for which Eq. (174) holds will be low. Due to equation (173), the interval in which the disturbance is able to change the ensemble of a particle is far away from the centre \( | \vec{v}_e^A(\vec{r}_1, t) | \) where most trajectories are found. This further reduces the number of particles that can be affected by \( \delta_{j1}(t) \).

All in all, we can conclude that the number of particles whose ensemble is changed solely because of the disturbance \( \delta_{j1}(t) \) is very low. For this reason, the disturbance can only cause a vanishingly small reduction of the efficiency of the entanglement in an EPR experiment, but not its breakdown. Hence, the theory we have constructed is especially robust against small disturbances unavoidable in a laboratory, in contrast to the claims by [26].

8 Conclusions and outlook

8.1 What has been achieved

The most popular argument against hidden variable theories with particle trajectories is, as commonly taken from Bell’s words [that those theories] “require a mechanism, whereby the setting of one measuring device can influence the reading of another device” and “moreover, the signal involved must propagate instantaneously”. As shown by Nelson, whose analysis we reviewed here, this assertion is physically unfounded. Specially designed stochastic processes can violate Bell’s inequality at two separated locations without any contact at all. We have mathematically analysed, what such a behaviour really does imply for a stochastic theory, and we have constructed a corresponding one.

8.2 Possibility of relativistic extension

Unfortunately, our theory has, at least to this day, no relativistic extension. Stochastic mechanics in the original form in which Nelson wrote it down had severe conceptual difficulties in the relativistic domain. These problems arose due to the Markov property of the stochastic process one had to use. In fact, it was shown that the definition of a Markov process on Minkowski space would lead either to determinism or to the violation of causality [65–67]. Nevertheless, with some different assumptions for the stochastic process involved, the Klein-Gordon equation in four dimensions [68–73] as well as the Dirac equation in 1+1 dimensions [74–76] could be derived in various investigations. Furthermore, there also exist attempts to derive the Dirac equation in 3 + 1 dimensions from various versions of stochastic mechanics e. g. [77, 78]. Yet, the derivation of the Dirac equation from an actively local stochastic process still seems to be a completely open problem. Therefore, we will try to analyse if some of the methods used by Fritsche and Haugk could perhaps help to solve it.
At first, we want to emphasise that unlike Nelson’s original formulation, the theory of Fritsche and Haugk describes a non-markovian process and it thereby avoids the no-go theorems of [65–67]. However, in order to rewrite the Fritsche-Haugk model in a manifestly covariant form, one would need a fully formulated theory of special relativistic Brownian motion. Regrettably, although such a theory certainly must exist (since non-relativistic Brownian motion does), it seems that till this day, no one gave a suitable model based on a relativistic generalisation of the Langevin equations. (Although there are highly interesting attempts in this direction [79–82].)

Fortunately, there are other ways to describe a relativistic system that evolves stochastically. It is shown in [15] on p. 386 that after an approximation, the Eqs. in (56) can be transformed into the Navier-Stokes equations. We have in Eq. (56) for the \{B\} system:

\[
\frac{\partial}{\partial t} (\vec{v}^B - \vec{u}^B) + (\vec{v}^B + \vec{u}^B) \cdot \vec{\nabla} \left( \vec{v}^B - \vec{u}^B \right) - \nu \Delta \left( \vec{v}^B - \vec{u}^B \right) = \frac{1}{m_0} \vec{F}^{\text{ext}} \tag{175}
\]

If we set \( \vec{\nabla} \rho^B(\vec{r},t) = 0 \), we observe that

\[
\vec{u}^B \equiv -\nu \vec{\nabla} \ln \left( \rho^B(\vec{r},t)/\rho_0 \right) = -\frac{\nu}{\rho^B(\vec{r},t)} \vec{\nabla} \rho^B(\vec{r},t) = 0. \tag{176}
\]

With \( p(\vec{r},t) \) denoting the pressure and \( \rho_m \equiv m_0 \rho^B(\vec{r},t) \) as the mass density, the following relation holds:

\[
\rho_m \vec{F}^{\text{ext}} = -\vec{\nabla} p(\vec{r},t). \tag{177}
\]

Inserting the result of Eq. (176) and the relation (177) into Eq. (175), we arrive at the Navier-Stokes equation of a fluid

\[
\rho_m \frac{\partial}{\partial t} \vec{v}^B + \rho_m \left( \vec{v}^B \cdot \vec{\nabla} \vec{v}^B \right) - \nu \rho_m \Delta \vec{v}^B + \vec{\nabla} p(\vec{r},t) = 0. \tag{178}
\]

For Eq. (178), various manifestly covariant formulations have been proposed, see [83–86], and for a general review [87]. In order to derive the relativistic version of Eq. (178) from first principles, one typically starts with relativistic Boltzmann equations [88–95]. Due to the close similarity between Eq. (178) and Eq. (56), it seems at least plausible that it might be possible to formulate the theory of Fritsche and Haugk in a relativistic setting.

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