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Symmetry-protected topological phases in spin ladders with two-body interactions

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Spin-1/2 two-legged ladders respecting interleg exchange symmetry $\sigma$ and spin rotation symmetry $D_x$ have new symmetry-protected topological (SPT) phases which are different from the Haldane phase. Three of the new SPT phases are $t_x,t_y,t_z$, which all have symmetry-protected twofold degenerate edge states on each end of an open chain. However, the edge states in different phases have different responses to magnetic field. For example, the edge states in the $t_y$ phase will be split by the magnetic field along the $z$ direction, but not by the fields in the $x$ and $y$ directions. We give the Hamiltonian that realizes each SPT phase and demonstrate a proof-of-principle quantum simulation scheme for Hamiltonians of the $t_0$ and $t_z$ phases based on the coupled-QED-cavity ladder.

I. INTRODUCTION
Symmetry-protected topological (SPT) phases are formed by gapped short-range-entangled quantum states that do not break any symmetry.1 Contrary to the trivial case, quantum states in the nontrivial SPT phases cannot be transformed into direct product states via local unitary transformations which commute with the symmetry group. Meanwhile, two different states belong to the same SPT phase if and only if they can be transformed into each other by symmetric local unitary transformations.2 A nontrivial SPT phase is different from the trivial SPT phase because of the existence of nontrivial edge states on open boundaries. This nontrivial property is protected by symmetry, because once the symmetry is removed, the SPT phases can be smoothly connected to the trivial phase without phase transitions.3 The well known Haldane phase4 is an example of SPT phase in one dimension. Topological insulators5–10 are examples of SPT phases in higher dimensions.

The bosonic SPT phases are classified by projective representations (which describe the edge states at open boundaries or at the positions of impurities1–13) of the symmetry group in one dimension.14 and by group cohomology theory in higher dimensions.15 We also have a systematic understanding of free fermion SPT phases16 and some interacting fermion SPT phases17–20. Using those general results, fourteen new one-dimensional SPT phases protected by $D_2 \times \sigma$ spin rotation and time reversal symmetry are proposed in Ref. 21.

In this work, we will discuss two-legged spin-1/2 ladder models with two-body anisotropic Heisenberg interactions respecting $D_2 \times \sigma \times \sigma$ symmetry. Here $\sigma$ is the inter-chain exchanging symmetry and $D_2 = \{E,R_x,R_y,R_z\}$, where $E$ is the identity and $R_x$ ($R_y$, $R_z$) is a 180° rotation of the spin along $x$ ($y$, $z$) direction. The symmetry $D_2 \times \sigma$ protects seven nontrivial SPT phases. Four of them, $t_0,t_x,t_y,t_z$, can be realized in spin-1/2 ladder models. The $t_0$ phase is the Haldane phase, and the $t_x$, $t_y$, $t_z$ phases are new because of their different edge states. We provide a simple two-body Hamiltonian for each SPT phase, and study the phase transitions between these phases. We also discuss possible physical realizations of the SPT Hamiltonians and demonstrate a proof-of-principle implementation scheme based on a coupled quantum electrodynamics (QED) cavity ladder.

The paper is organized as follows. In Sec. II, we introduce the projective representations of the underlying symmetry group of the two-legged spin-1/2 ladder and discuss the possible SPT phases. We then numerically study the phase diagram and the phase transitions of the spin-ladder Hamiltonians in Sec. III. In Sec. IV, we discuss the physical realizations of the SPT Hamiltonians and demonstrate a proof-of-principle quantum simulation scheme based on a QED cavity ladder. Finally, we summarize in Sec. V.

II. PROJECTIVE REPRESENTATIONS OF THE SYMMETRY GROUP AND SPT PHASES

The symmetry group $D_2 \times \sigma$ for our two-legged spin-1/2 ladder is Abelian. All its eight representations are one-dimensional. The following two-spin states on a rung form four different one-dimensional representations of the symmetry group: $|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \downarrow \rangle - |\downarrow \uparrow \uparrow \rangle)$, $|1,x\rangle = \frac{1}{\sqrt{2}}(|\downarrow \uparrow \downarrow \rangle - |\uparrow \downarrow \downarrow \rangle)$, $|1,y\rangle = \frac{1}{\sqrt{2}}(|\downarrow \downarrow \uparrow \rangle + |\uparrow \downarrow \downarrow \rangle)$, and $|1,z\rangle = \frac{1}{\sqrt{2}}(|\downarrow \uparrow \downarrow \rangle + |\uparrow \downarrow \downarrow \rangle)$. The subscripts 1,2 label the different spins on the same rung. The group $D_2 \times \sigma$ has eight projective representations (see Table I), which describe eight different SPT phases22 of spin ladder models. One of the projective representations is one-dimensional and trivial. The other seven nontrivial ones are two-dimensional, which describe the seven kinds of twofold degenerate edge states of the seven nontrivial SPT phases.

The degeneracy at the edge in each nontrivial SPT phase is protected by the symmetry. To lift such a degeneracy, we need to add perturbations to break the $D_2 \times \sigma$ symmetry. The operators that split the edge degeneracy are called active operators.21 The active operators are different in different SPT phases, which provide experimental methods to probe different SPT phases.

We will mainly study five of the eight SPT phases which can be realized in two-legged spin-1/2 ladders: one trivial SPT...
The rungs of the two spins at a rung, and $S_z$ means $S_{1,i} \cdot S_{1,i+1} - S_{2,i} \cdot S_{2,i+1}$.

| $R_e$ | $R_i$ | $\sigma$ | Active operators | SPT phases |
|-------|-------|-------|-----------------|------------|
| $E_0$ | 1     | 1     | $I$             | Rung-singlet, $t_x \times t_x$, $t_y \times t_y$, $t_z \times t_z$ |
| $E_1$ | $i\sigma_x$ | $\sigma_y$ | $(S_{1,i}^x, S_{1,i}^y, S_{1,i}^z)$ | $t_x \times t_y$, $t_x \times t_z$, $t_y \times t_z$ |
| $E_2$ | $i\sigma_z$ | $I$ | $(S_{1,i}^z, S_{1,i}^y, S_{1,i}^z)$ | $t_x \times t_y$, $t_x \times t_z$, $t_y \times t_z$ |
| $E_3$ | $i\sigma_x$ | $I$ | $(S_{1,i}^x, S_{1,i}^y, S_{1,i}^z)$ | $t_x \times t_y$, $t_x \times t_z$, $t_y \times t_z$ |
| $E_4$ | $i\sigma_y$ | $I$ | $(S_{1,i}^y, S_{1,i}^y, S_{1,i}^z)$ | $t_x \times t_y$, $t_x \times t_z$, $t_y \times t_z$ |
| $E_5$ | $i\sigma_z$ | $i\sigma_x$ | $(S_{1,i}^x, S_{1,i}^z, S_{1,i}^z)$ | $t_x$ |
| $E_6$ | $i\sigma_x$ | $i\sigma_y$ | $(S_{1,i}^x, S_{1,i}^z, S_{1,i}^z)$ | $t_y$ |
| $E_7$ | $i\sigma_y$ | $i\sigma_z$ | $(S_{1,i}^y, S_{1,i}^z, S_{1,i}^z)$ | $t_z$ |

If the system has translational symmetry, there are four different rung-singlet phases: rung-$(0,0)$ phase, rung-$|1, x\rangle$ phase, rung-$|1, y\rangle$ phase, and rung-$|1, z\rangle$ phase. If the system does not have translational symmetry, then there is no difference between the four rung-singlet phases and there will be only one rung-singlet phase.

The other three SPT phases can be realized by stacking two of the above ones: the SPT phases corresponding to $E_1$, $E_2$, $E_4$ can be realized by stacking $t_x$ and $t_y$, $t_x$ and $t_z$, and $t_y$ and $t_z$, respectively. The $t_0$ phase can also be realized by stacking three phases $t_x$, $t_y$, $t_z$.

The nontrivial SPT phases $t_0, t_x, t_y, t_z$ can be realized in two-legged spin-1/2 ladders. The Hamiltonian that realizes these phases is simply given as $H = H_L + H_T$, where $H_L$ is the interaction along the leg (or longitudinal direction) and $H_T$ is the interaction along the rung (or transverse direction). We assume $H_T$ takes the following form:

$$H_L = \sum_{j} \left[ (\delta S_{1,i}^x S_{1,i+1}^x + S_{1,i}^y S_{1,i+1}^y + S_{1,i}^z S_{1,i+1}^z) + (\delta S_{2,i}^x S_{2,i+1}^x + S_{2,i}^y S_{2,i+1}^y + S_{2,i}^z S_{2,i+1}^z) \right].$$

where $J > 0$, $\delta \sim 1$, and $S_{m,i}^m (m = x, y, z)$ is the spin operator at the $j$th leg and $j$th rung.

By tuning the interaction $H_T$, we can obtain four SPT phases $t_0, t_x, t_y, t_z$. For example, the following model

$$H_0 = H_L + \lambda \sum_i S_{1,i} \cdot S_{2,i},$$

can realize the $t_0$ phase. When $\lambda > 0$, the rung-singlet state $(|0, 0\rangle)$ is lower in energy and the system falls into the rung-$|0, 0\rangle$ phase, which is a trivial SPT phase. When $\lambda < 0$, the rung-triplet ($|1, x\rangle, |1, y\rangle, |1, z\rangle$) are lower in energy, and effectively we obtain a spin-1 anisotropic Heisenberg model, which belongs to the Haldane phase $t_0$.

By partially flipping the sign of interactions along the rung, we obtain the Hamiltonian for the $t_x$ phase:

$$H_x = H_L + \lambda \sum_i (- S_{1,i}^x S_{1,i+1}^x + S_{2,i}^x S_{2,i+1}^x + S_{1,i}^y S_{1,i+1}^y + S_{2,i}^y S_{2,i+1}^y + S_{1,i}^z S_{1,i+1}^z + S_{2,i}^z S_{2,i+1}^z).$$

When $\lambda > 0$, it falls into a trivial SPT phase which corresponds to the rung-$|1, z\rangle$ product state when $\lambda < 0$, the low energy degrees of freedom in each rung are given by three states $|1, x\rangle, |1, y\rangle, |0, 0\rangle$, and the resultant model belongs to the $t_x$ phase. Note that the Hamiltonians (1) and (2) (with the respective ground states $t_0$ and $t_x$) can be transformed into each other by the unitary transformation: $U_j(\pi) = \exp(i\pi \sum_i S_{1,i}^x)$. However, this does not mean that $t_0$ and $t_x$ belong to the same state since $U_j(\pi)$ does not commute with the symmetry group $D_2 \times \sigma$. Applying similar arguments, we may have $H_{L} = H_{L} + \lambda \sum_i (S_{1,i}^x S_{1,i+1}^x - S_{1,i}^y S_{1,i+1}^y - S_{1,i}^z S_{1,i+1}^z)$ for the $t_x$ phase, and $H_T = H_T + \lambda \sum_i (- S_{1,i}^x S_{1,i+1}^x + S_{1,i}^y S_{1,i+1}^y + S_{1,i}^z S_{1,i+1}^z)$ for the $t_x$ phase. Table I shows that different SPT phases have different active operators. For example, in the $t_0$ phase, the edge degeneracy can be lifted by either $S_{1,i}^x$ or $S_{1,i}^y$. This means that the edge states can be polarized by a homogeneous magnetic field along any direction. In the $t_x$ phase, $S_{1,i}^x$ and $S_{1,i}^y$ are not active operators, indicating that a weak homogeneous magnetic field in the $x$-$y$ plane will not split the edge degeneracy. These properties can be verified by a finite-size exact diagonalization study of the Hamiltonian (1) and (2).21

III. PHASE TRANSITIONS AND THE PHASE DIAGRAM

To study the phase transitions, we consider the following model which contains both $t_0$ and $t_x$ phases:

$$H = H_L + \sum_i [\lambda_{xy}(S_{1,i}^x S_{2,i}^x + S_{1,i}^y S_{2,i}^y + \lambda S_{1,i}^z S_{2,i}^z)] + \lambda \sum_i (S_{1,i}^x S_{1,i+1}^x + S_{1,i}^y S_{1,i+1}^y + S_{1,i}^z S_{1,i+1}^z).$$

Since the $\lambda_{xy}$-$\lambda_z$ plane phase diagrams of the above model are quite different for $\delta \neq 0$ and $\delta = 1$, we will discuss them separately.

A. Case $\delta \neq 1$

1. Phase diagram

We only consider the case that anisotropy is weak, so we set $\delta = 0.9$ for simplicity. The phase diagram Fig. 1 (with $\delta = 0.9$) is obtained via infinite time-evolving block decimation (tEBD) algorithm.24 The phase diagram is symmetric along the $XY$-like phases. The origin $\lambda_{xy}, \lambda_z = (0, 0)$ is a multicritical point linking all the phases. On the upper half plane $\lambda_z > 0$, there are only three phases. The lower half plane is more interesting. The limit $\lambda_z \to -\infty$ corresponds to the Ising stripe-Néel phase, while $\lambda_z \to +\infty$ corresponds to the rung-$|0, 0\rangle$/rung-$|1, z\rangle$ phase.

In the intermediate region, we have two $XY$-like phases and two SPT phases: $XY$-stripe Néel phase and $XY$-Néel phase. For $\delta \neq 1$, the exchange interactions along the legs are anisotropic in the $x$-$y$ plane; consequently the two $XY$-like phases have a finite excitation gap and are ordered in $y$.  

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direction if \( \delta < 1 \) (or ordered in \( x \) direction if \( \delta > 1 \)). The \( t_0/t_z \) phase is located between the XY-stripe Néel/XY-Néel phase and the Ising stripe Néel phase. The phase diagram at \( \lambda_z = -0.5J \) is also shown in Fig. 2, where we illustrate the symmetry breaking orders and the entanglement spectrum (\( \Delta \rho = \rho_1 - \rho_2 \) is the difference between the two biggest Schmidt eigenvalues) in each phase. For \( \delta = 1 \), the two XY phases vanish, and the SPT phase \( t_0/t_z \) touches the trivial phase rung-(1,1)/rung-(0,0) directly (we will study this case in detail later).

2. Semiclassical explanation of the phase diagram

Notice that both of the SPT phases are sandwiched by two ordered phases. This suggests that they originate from quantum fluctuations caused by the competition between the different classical orders. To understand this point better, it will be interesting to compare the phase diagram Fig. 1 with that from the semiclassical approach in which the ground state is approximated by a direct product state. A semiclassical phase diagram (Fig. 3 with \( \delta = 0.9 \), \( \lambda_z = -0.5J \)) is obtained by minimizing the energy of the following trial wave function:

\[
|\psi\rangle_{sc} = \prod_i (a_1\phi_{1,2} + a_2\phi_{3,4} + a_3\phi_{1,3} + a_4\phi_{2,4})
\]

\[
\otimes (a_3\phi_{2,1} + a_1\phi_{2,4} + a_3\phi_{1,3} + a_4\phi_{4,2})
\]

where \( \phi_1 = -\frac{1}{\sqrt{2}}(|1,1\rangle + |1,0\rangle) \), \( \phi_2 = \frac{1}{\sqrt{2}}(|1,1\rangle - |1,0\rangle) \), \( \phi_3 = |1,2\rangle \), \( \phi_4 = |0,0\rangle \) are four bases of each rung, and \( a_1, a_2, a_3, a_4 \) are four trial parameters.

Comparing the semiclassical phase diagram with Fig. 2, we find that the two phase diagrams are similar except for the absence of two SPT phases in Fig. 3. Importantly, the locations of the quantum SPT phases are close to the phase boundaries between the Ising ordered phase (\( M_{\text{Ising}}^z \neq 0 \)) and the \( xy \)-planar ordered phases (\( M_{\text{xy}}^z \neq 0 \)) of the semiclassical phase diagram. Similar situations occur in spin-1 \( XXZ \) chain, where a Haldane phase exists between two ordered phases.\(^{1,25}\) This suggests that we can roughly obtain a quantum phase diagram with a semiclassical approach by “inserting” a SPT phase at the phase boundary of two different classically ordered phases.

Furthermore, the semiclassical picture even indicates some important information of the quantum SPT phases. For instance, it tells us why the \( t_0 \) phase is different from the \( t_z \) phase. The classical phase boundary between the two ordered phases is located at \( |\lambda_{xy}^c| = |\lambda_z^c| \). At the point \( \lambda_{xy}^c = \lambda_z^c < 0 \), the states have the same lowest energy if the spins are antiferromagnetically ordered along the leg and are parallel along the rung polarizing in \( y-z \) plane (namely, the classical ground states are highly degenerate). Quantum fluctuations (which are enhanced by the degeneracy) will drive the system into the higher energy states (e.g., some spins are pointing off the \( y-z \) plane) with a certain weight. As a consequence, the ground state is short-range correlated with a finite excitation gap. Furthermore, edge states exist at each boundary. The edge states can be considered as an effective spin whose magnetic moment is half of the total momentum of the two spins at a rung. Since the two spins in the same rung are parallel, the edge states carry free magnetic moment and respond to magnetic field along arbitrary directions. On the other hand, at the point \( \lambda_{xy}^c = -\lambda_z > 0 \), the lowest energy classical states...
are those that the two spins are parallel along the $z$ direction and antiparallel in the $x$-$y$ plane [see Fig. 4(b)]. Quantum fluctuations around the sphere drive this state to the $t_z$ phase. In the $t_z$ phase, the effective edge spins have no net magnetic moment in the $x$-$y$ plane, so they will not respond to the magnetic field in the $x$-$y$ plane. Similar things happen in the $t_x$ and $t_y$ phases. Above arguments are applicable to other systems with different symmetry groups, and provide a guidance to seek SPT phases and analyze their physical properties.

B. Case $\delta = 1$

1. Finite size effect in numerical method

When establishing the phase diagram Figs. 1 and 2, we have approximated the ground state by a matrix product state (MPS), which is obtained through the iTEBD method. The matrices in the MPS have a finite dimension $D$, which introduces a cutoff to the number of Schmidt eigenvalues in the entanglement Hilbert space. While the energy of the actual ground state can be estimated using a finite-$D$ MPS with high accuracy, the order parameters are usually overestimated due to the finite dimension of the matrix. Furthermore, in some cases the order parameters (which are finite if $D$ is finite) even vanish in the infinite $D$ limit. So a scaling of the order parameters with respect to the dimension $D$ is necessary in order to obtain the correct phase diagram. In the following we will illustrate, via finite size scaling, that the two $XY$ phases disappear if $\delta = 1$.

The phase diagram in Fig. 5(a) with $\delta = 1, \lambda_z = -0.5J$ is obtained by setting $D = 30$. To evaluate the order parameters of each phase in the infinite $D$ limit, we perform numerical calculations with different $D$'s, as shown in Fig. 6. Performing finite size scaling and extrapolating to infinite $D$, we find that the maximum value of $M^x_z$ in the $XY$-strip Néel phase (and the same for $M^{xy}_z$ in the $XY$-Néel phase) vanishes as a power law of $D^{-3.803}$, when shifted by a constant $M_0$ (see Fig. 7). The fact that $M_0 \neq 0$ shows that $M'^x_z$ in the $XY$-strip Néel phase (and the same for $M'^{xy}_z$ in the $XY$ Néel phase) is finite in the limit $D \to \infty$, and the phase diagrams at finite $D$ are qualitatively correct if $\delta \neq 1$.

2. Vanishing of the $XY$ phases and $U(1)$ symmetry-protected phase transitions

Physically, the disappearance of the two $XY$ phases at $\delta = 1$ can be explained by quantum fluctuations.

When $\delta = 1$, the Hamiltonian (3) has an enhanced symmetry $[U(1) \times Z_2] \times \sigma$, where the $U(1)$ subgroup means rotation of the spins along $z$ axis and $Z_2$ is generated by a rotation of the spin for $\pi$ around $x$ axis. Since the continuous symmetry $U(1)$ will never spontaneously break in one dimension, if the $XY$ phases exist, they will be quasi-long-ranged ordered in $x$-$y$ plane and gapless.

However, strong quantum fluctuations gap out these states and drive them into the SPT phases. From the semiclassical approach, the enhanced symmetry results in a larger degeneracy of the semiclassical ground states near $\lambda_{xy} \sim \lambda_z$. The enlarged degeneracy of the semiclassical ground states enhances the quantum fluctuations and gaps out all the states (except the states at the critical points). Consequently, the $XY$ phases disappear, and the nontrivial SPT phase $t_0/t$, touches the trivial
Magnetic dipole-dipole interaction and exchange interaction $H_T = \sum_i [\eta (\mathbf{S}_{1i} \cdot \mathbf{S}_{2i}) + \gamma (\mathbf{S}_{1i} \cdot \mathbf{S}_{2i} - \mathbf{S}_{1i}^2 \mathbf{S}_{2i}^2)]$ with $\lambda_z = \eta - 2\nu$ and $\lambda_{xy} = \eta + \nu$. However, in real materials, the magnetic dipole-dipole interaction is too weak to support the SPT phases. On the other hand, axial anisotropy interaction $D_z(S^z_1 + S^z_2)^2 = 2D_z(S^z_1 S^z_2 - 1)$ can yield the Hamiltonian (3) by $H_T = \sum_i [\eta (\mathbf{S}_{1i} \cdot \mathbf{S}_{2i}) + D_z(S^z_{1i} + S^z_{2i})^2]$ with $\lambda_z = 2D_z + \eta$, $\lambda_{xy} = \eta$. Note that a weak effective spin-1 axial anisotropy term exists in the material (C$_4$H$_{12}$N)$_2$CuBr$_4$.27 If this term is negative and is strong enough in some material, then the $t_1$ phase will be realized.

On the other hand, due to the nearest-neighbor-only interactions, it is tempting to consider the possibility of simulating these Hamiltonians in a non-condensed-matter setting. Here we provide a proof-of-principle implementation scheme for the Hamiltonians for the $t_0$ phase (1) and for the $t_1$ phase (2) based on the coupled-harmonic-oscillator array. Here we only consider the isotropic case with $\delta = 1$. Note that it is also possible to implement the more general anisotropic cases with additional Raman lasers along the ladder. Our scheme may be extended to systems including solid spins interacting with arrays of coupled transmission line resonators, or ions in a Coulomb crystal. As a concrete example, we illustrate the scheme using a coupled-QED-cavity-ladder [see Fig. 8(a)], where the quantized cavity fields couple to their nearest neighbors in the longitudinal $(L)$ and the transverse $(T)$ directions via photon hopping.25 The spin degrees of freedom on each site are encoded in the internal states of a single atom trapped inside the cavity.

In Fig. 8(b), we show the coupling scheme for a minimal model with three internal states, with $|\uparrow\downarrow_{j,k}\rangle$, $|\downarrow\uparrow_{j,k}\rangle$ corresponding to hyperfine states in the ground state manifold and $|e_{j,k}\rangle$ corresponding to an electronically excited state. Two far-detuned external laser fields with Rabi frequencies $\Omega_{j,k}^{\mu} = \Omega^\mu e^{i\phi_{\mu}}$ ($\mu = L, T$) couple $|\uparrow\downarrow_{j,k}\rangle$ and $|e_{j,k}\rangle$, while two far-detuned cavity modes with Rabi frequencies $G^\mu$ couple $|\downarrow\uparrow_{j,k}\rangle$ and $|e_{j,k}\rangle$, with all the detunings satisfying $|\Delta_{1L}^L - \Delta_{1L}^T|$, $|\Delta_{1T}^L - \Delta_{1T}^T|$, $|\Delta_{1T}^\mu - \Delta_{1L}^\mu| \gg G^\mu, \Omega^\mu, |\Delta_{1L}^L - \Delta_{1L}^T|$. Hence we have two independent Raman paths in the longitudinal and in the transverse directions, and a virtually populated excited state $|e_{j,k}\rangle$. Finally, the hyperfine states are coupled using a resonant radio-frequency (rf) dressing field with Rabi frequency $\Omega_{rf}^{rf} = \Omega^{rf} e^{i\phi_{rf}}$. One can show that all the cavity modes are virtually excited under the condition $|\Delta_{1L}^L - \Delta_{1L}^T| \sim \Omega^\mu \gg g^\mu / \sqrt{2N}$, where $g^\mu = G^\mu \Omega^\mu (\Delta_{1L}^L + \Delta_{1L}^T)/2\Delta_{1L}^T \Delta_{1L}^T$.26
vμ is the coupling rate between cavities along the direction μ, and N is the number of sites in the longitudinal direction.29

After adiabatic elimination of the excited states and the cavity modes, the resulting effective Hamiltonian gives rise to effective spin-spin interactions between the neighboring sites, with the pseudospins in the effective Hamiltonian given by the superpositions of the hyperfine states: \( |\uparrow j,k⟩ = (e^{iϕ_{j,k}} |\uparrow j,k⟩ + |\downarrow j,k⟩)/\sqrt{2} \) and \( |\downarrow j,k⟩ = (-|\uparrow j,k⟩ + e^{-iϕ_{j,k}} |\downarrow j,k⟩)/\sqrt{2} \).

Depending on the magnitude and the relative phases of the coupling fields, the effective Hamiltonian can support either the \( t_0 \) phase or the \( t_1 \) phase. In particular, with \( \Omega^{\mu} \sim 0.75|Δ_2^μ - Δ_1^μ| \) and \( |θ_{j,k+1}^L - θ_{j,k}^L| = π \), we get the Hamiltonian for the \( t_0 \) phase when \( θ_{j,k}^L = ϕ_{j,k} \) = 0 for arbitrary \( j,k \); and we get the Hamiltonian for the \( t_1 \) phase when \( |θ_{j,k}^L - θ_{j,k}^T| = |ϕ_{1,k} - ϕ_{2,k}| = π \) and \( ϕ_{j,k} = ϕ_{j,k+1} \) for arbitrary \( j,k \). These conditions can be achieved by modulating the relative phases of the external coupling fields so that they are either 0 or π between neighboring sites. The interaction rates \( J \) and \( λ \) in Hamiltonian (1),(2) are given as \( J \sim 1.14(g^2)^2v^\mu(Δ_1^μ)^2\lambda \) and \( λ = -1.14(g^2)^2v^\mu(Δ_1^μ)^2J^2 \). For typical experimental parameters,30 \( Ω^{\mu} \sim 100 \text{ MHz}, G^{\mu} \sim 100 \text{ MHz}, |Δ_1^μ| \sim 1 \text{ GHz} \quad (i = 1, 2), v^\mu \sim 10 \text{ MHz}, \) and \( Ω^{\mu} \sim 100 \text{ MHz}, we have \( J \sim 0.11 \text{ MHz}, with the magnitude of \( λ/J \) widely tunable by adjusting the ratio between \( Ω^L \) and \( Ω^T \).

The single-site addressability of the atoms in the cavity ladder allows much freedom in probing the properties of the system. For the SPT phases in which we are interested, we need to measure the response of the edge states to the magnetic field. As the pseudospins are related to the hyperfine states via a rotation, we may implement an effective magnetic field on the pseudospins by imposing the appropriately rotated operations on the edge states. For instance, to apply an effective magnetic field in the \( z \) direction, we need to apply a \( σ_z \) rotation on the hyperfine states (assuming \( ϕ_{j,k} = 0 \)), which can be achieved for example by adding a resonant rf field between the hyperfine states, with the effective Rabi frequency corresponding to the magnitude of the effective magnetic field. To probe the response of the edge states to the effective field, we may measure the polarization of the ground state of the system. Alternatively, as the effective magnetic field may induce an energy splitting between the spin states, we can measure the existence of energy splitting as a response of the SPT phases to the external field. As an example, for the \( t_1 \) phase under an effective magnetic field along \( z \), we need to implement the following steps on the two edge sites at one end of the open boundaries: (i) adiabatically turn on resonant rf fields between the hyperfine states so that the degeneracy of the edge states is lifted and an energy splitting appears between \( |\uparrow_{\text{edge}}⟩ \) and \( |\downarrow_{\text{edge}}⟩ \); (ii) in the presence of the effective magnetic field, apply an effective resonant coupling field between the pseudospins, which corresponds to two-photon detuned Raman fields between the hyperfine states with the Stark shifts equal to the Rabi frequency of the effective resonant coupling fields between the pseudospins; (iii) after some time of evolution, rapidly turn off all the coupling fields, then apply a π/2 pulse on the hyperfine states, so that pseudospin population is projected onto that of the hyperfine states; (iv) the population of the hyperfine state can be probed for example by a high-fidelity hyperfine state readout technique based on cavity-enhanced fluorescence.31 If the edge states are responsive to the effective magnetic field, one will observe Rabi oscillations in the measured fluorescence. Measuring the magnetic field response in all three spatial directions will allow us to establish the signature of the \( t_1 \) phase.

V. CONCLUSION

In summary, we have studied nontrivial quantum phases \( t_0 \) (the Haldane phase) and \( t_1 \) (a new SPT phase different from the Haldane phase) protected by \( D_2 \times σ \) symmetry in a spin-1/2 ladder model. The model has simple two-body interactions and a rich phase diagram. We then provided a semiclassical understanding of the physical properties of the SPT phases, and discussed the general principles for the search of these novel phases. Finally, we have proposed a proof-of-principle quantum simulation scheme of nontrivial SPT phases in cold atom systems.

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APPENDIX A: CONSTRUCTING THE HAMILTONIANS WITH SPT PHASES

In this Appendix, we demonstrate the method21 by which we obtain the Hamiltonian \( H_0, H_x, H_y, \) and \( H_z \) in the main text. The procedure contains three steps: (i) construct a matrix product state (MPS) wave function with given edge states which are described by the projective representations; (ii) construct the parent Hamiltonian for the MPS using projection operators; (iii) simplify the parent Hamiltonian by adiabatic deformations. In the first step, we need to know the projective
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and linear representations of the symmetry group, together with the Clebsch-Gordan (CG) coefficients. In the following, we will first present necessary information and then discuss the construction of the Hamiltonian step-by-step.

The four projective representations of \( D_2 \times \sigma \) corresponding to the \( t_0, t_1, t_2, t_3 \) phases are given in Table II (we only provide the representation matrices for the generators), and the eight linear representations are listed in Table III.

The Hilbert space of the direct product of two projective representations can be reduced to a direct sum of linear representations. The CG coefficients (we assume that all the CG coefficients are real numbers) are

\[
E_3 \otimes E_3 = A_g \oplus B_{2g} \oplus B_{3g}, \quad C_{A_g} = \sigma_z, \\
C_{B_{1g}} = \sigma_y, \quad C_{B_{2g}} = \sigma_y, \quad C_{B_{3g}} = I, \\
E_5 \otimes E_5 = B_{1g} \oplus B_{2g} \oplus A_u \oplus B_{3u}, \quad C_{A_u} = \sigma_x, \\
C_{B_{1g}} = I, \quad C_{B_{2g}} = i\sigma_y, \quad C_{B_{3u}} = \sigma_z, \\
E_6 \otimes E_6 = B_{2g} \oplus B_{3g} \oplus A_u \oplus B_{3u}, \quad C_{A_u} = i\sigma_y, \\
C_{B_{2g}} = \sigma_x, \quad C_{B_{3u}} = \sigma_z, \\
E_7 \otimes E_7 = A_g \oplus B_{2g} \oplus B_{1u} \oplus B_{3u}, \quad C_{A_g} = i\sigma_y, \\
C_{B_{2g}} = I, \quad C_{B_{1u}} = \sigma_x, \quad C_{B_{3u}} = \sigma_z,
\]

where \( |m\rangle = C_{\alpha_\beta}(|\alpha\rangle|\beta\rangle) \), \( |m\rangle \) is the basis of a linear representation, and \(|\alpha\rangle, |\beta\rangle\) are the bases of two projective representations.

In the following, we will illustrate the method to obtain the Hamiltonian of the \( t_0 \) phase as an example.

The first step is obtaining the MPS. From Table II, the edge states of the \( t_0 \) phase are described by the \( E_3 \) projective representation. In an ideal MPS, every rung is represented by a direct product of two \( E_3 \) projective representations, which can be reduced to four linear representations, \( E_3 \otimes E_3 = A_g \oplus B_{1g} \oplus B_{2g} \oplus B_{3g} \). From Table III, \( B_{1g}, B_{2g}, B_{3g} \) correspond to the bases \([1, z], [1, y], [1, x] \), respectively. The basis \( A_g \) (or \([0, 0]\)) is absent on every rung in the MPS state. Thus the support space for the ideal MPS is the Hilbert subspace \( \oplus([1, z] \oplus [1, y] \oplus [1, x]) \), where \( i \) is the index of rung. From the CG coefficients (A1), we can write such an ideal MPS which is invariant (up to a phase) under the symmetry group:

\[
|\psi\rangle = \sum_{\{m_1, \ldots, m_N\}} \text{Tr}(A_{m_1} \cdots A_{m_N})|m_1 \cdots m_N\rangle,
\]

with \( A_m = e^{i\theta_{BC_m}}B_m \). Here \( B \) is the CG coefficients of decomposing the product representations \( E_3 \otimes E_3 \) into a one-dimensional representation (here we choose \( B = C^{A_g} \)), and \( e^{i\theta} \) can be absorbed into the spin bases. Now we have

\[
A^{[1, x]} = \sigma_x, \quad A^{[1, y]} = \sigma_y, \quad A^{[1, z]} = \sigma_z.
\]

The second step is constructing the parent Hamiltonian, which is a sum of projectors. Each projector is a projection onto the ground state subspace of two neighboring rungs. Assuming the orthonormal bases for the MPS state of two neighboring rungs \( i, i + 1 \) are \( |\psi_i, \psi_{i+1}\rangle \), then the projector is \( P_{i,i+1} = -\langle\sum_{\alpha=1}^{N}\langle\psi_i|\psi_{i+1}\rangle|\psi_{i+1}\rangle \), and the resultant parent Hamiltonian \( H_{\text{ex}} = \sum_i P_{i,i+1} \) is given as

\[
H_{\text{ex}} = \sum_i \left[ \frac{5}{12} (S_{1,i} + S_{2,i}) \cdot (S_{1,i+1} + S_{2,i+1}) - \frac{2}{3} S_{1,i} \cdot S_{2,i} - \frac{2}{3} (S_{1,i} \cdot S_{2,i})(S_{1,i+1} \cdot S_{2,i+1}) + \frac{1}{3} (S_{1,i} \cdot S_{2,i+1})(S_{2,i} \cdot S_{1,i+1}) + \frac{1}{3} (S_{1,i} \cdot S_{2,i+1})(S_{2,i} \cdot S_{1,i+1}) \right].
\]

The final step is deforming the Hamiltonian. It can be shown that only the first two terms in (A6) are important. To see this, we introduce the parameter \( d \):

\[
H = J \sum_i \left[ \frac{5}{12} (S_{1,i} \cdot S_{1,i+1} + S_{2,i} \cdot S_{1,i+1}) - \frac{2}{3} S_{1,i} \cdot S_{2,i} \right] + d \sum_i \left[ \frac{5}{12} (S_{1,i} \cdot S_{2,i+1} + S_{2,i} \cdot S_{1,i+1}) - \frac{2}{3} (S_{1,i} \cdot S_{2,i})(S_{1,i+1} \cdot S_{2,i+1}) + \frac{1}{3} (S_{1,i} \cdot S_{2,i+1})(S_{2,i} \cdot S_{1,i+1}) + \frac{1}{3} (S_{1,i} \cdot S_{2,i+1})(S_{2,i} \cdot S_{1,i+1}) \right].
\]

Note that when \( d/J = 1 \), Eq. (A7) is the same as (A6). Now we study the ground state energy and entanglement spectrum (through a time-evolving block decimation method) to see if there is a phase transition when \( d \) is varied.
Figure 9 shows that the energy is a smooth function of parameter $d/J \in [0,1]$. Furthermore, the entanglement spectrum remains degenerate in $d/J \in [0,1]$. Thus we only need nearest-neighbor exchanges to realize the Haldane ($t_0$) phase, which leads to the Hamiltonian $H_0$ in the main text.

The active operators in Table II are obtained as the following. In the Hilbert space spanned by the twofold ground states are fourfold degenerate; (b) entanglement spectrum.

We find that $S^z_1$ and $\sigma_z$ belong to the same linear representation $B_{3g}$ under the symmetry operation. This means that in the low energy limit (i.e., in the ground state subspace), these two operators have similar behavior. So we can identify $S^z_1$ as an active operator. Similarly, the operators $S^+_1$ and $S^-_1$ are active operators corresponding to $\sigma_+ \pm \sigma_-$, respectively.

Similar to (A6), we can construct the exactly solvable Hamiltonian $H_{\text{ex}}$ of the $t_1$ phase. It is the same as (A6) except that every $\mathbf{A} \cdot \mathbf{B}$ term is replaced by $A_x B_x - A_y B_y$. This Hamiltonian can be simplified into the form of $H_{\text{ex}}$ [Eq. (3) of the main text] without any phase transition ($H_x$ and $H_y$ are obtained similarly).

The active operators in $t_1$ phase can be easily obtained: $S^+_1, S^z_1, S^-_1$. Notice that $S^+_1, S^-_1$ are not active operators, meaning that the edge states in the ground state will not respond to the uniform magnetic in $x$ and $y$ directions. To check this result, we perform a finite-size exact diagonalization of the solvable model $H_{\text{ex}}$. As shown in Fig. 10, only the magnetic field along $z$ direction can split the ground state degeneracy. These properties are valid in the whole $t_1$ phase in the thermodynamic limit. This verifies the conclusion that only $S^z_1$ is the active operator.

This interesting result indicates that we can distinguish $t_1$ from $t_0$ by the response to magnetic fields. In $t_0$ phase, arbitrarily small magnetic field can split the degeneracy of the ground states, showing that the edge states carry free

![Image](image_url)
magnetic moments. According to Curie’s law, the magnetic susceptibility will diverge at low temperature. But in the phase, the edge states only carry magnetic moment in the direction, so the magnetic susceptibility within the plane does not diverge at low temperature, but it does diverge if the magnetic field is along the direction. These results are also verified numerically; see Fig. 11. The behavior of low-temperature magnetic susceptibility is measurable, which allows us to distinguish different SPT phases experimentally.

APPENDIX B: IMPLEMENTING THE LADDER HAMILTONIAN

In this Appendix, we discuss in more detail the implementation scheme for the Hamiltonian with SPT phases. To keep our discussion general, we consider a two-dimensional (2D) coupled-harmonic-oscillator array. Later, we will relate this general scheme to the specific example of a cavity ladder as in the main text.

Consider a 2D array: on each site of the array, two independent harmonic oscillators exist and couple with those on the neighboring sites via energy tunneling. We label these independent harmonic oscillators as and . This can be achieved by requiring the frequency difference between different types of oscillators to be sufficiently large, and by setting up specific coupling schemes between neighboring sites. The Hamiltonian for this 2D coupled-harmonic-oscillator array can be written as \( H_1 = \sum_{j,k} \sum_{\mu=L,T} (v^L a^\dagger_{j,k} a^\mu_{j,k+1} + v^T a^\dagger_{j,k} a^\mu_{j+1,k} + \text{H.c.}) \), \( \text{(B1)} \)

where \( a^\mu_{j,k} \) (\( a^\dagger_{j,k} \)) is the annihilation operator for the harmonic oscillator labeled \( L \) (\( T \)) on the \( j \)th site transversally and the \( k \)th site longitudinally. The coupling strength along the longitudinal (transverse) direction is given by \( v_L \) (\( v_T \)).

The harmonic oscillators on each site interact with a two-level system \(| \uparrow_{j,k} \rangle, | \downarrow_{j,k} \rangle \), and the coupling rates are \( \gammatur{}_{j,k} = g^L e^{i \varphi_{j,k}} \) and \( \gamma^T_{j,k} = g^T e^{i \varphi_{j,k}} \), respectively. This is illustrated in Fig. 12. The interaction Hamiltonian is

\[
H_2 = \sum_{j,k} \sum_{\mu=L,T} \left( g^\mu_{j,k} e^{i \Delta^\mu} a^\mu_{j,k} S^z_j + \text{H.c.} \right), \tag{B2}
\]

where \( S^z_j = | \uparrow_j \rangle \langle \uparrow_j | + | \downarrow_j \rangle \langle \downarrow_j | \), and \( \Delta^L \) (\( \Delta^T \)) denotes the detuning of the corresponding harmonic oscillator mode (see Fig. 12).

Finally, the two-level system on each site is coupled by a resonant dressing field, with the Hamiltonian

\[
H_3 = \sum_{j,k} \sum_{\mu=L,T} (\Omega^\mu_{j,k} S^+_{j,k} + \text{H.c.}), \tag{B3}
\]

where \( \Omega^\mu_{j,k} = \Omega^\mu e^{i \varphi_{j,k}} \) is the Rabi frequency.

Starting from the full Hamiltonian \( H = H_1 + H_2 + H_3 \), we will eventually adiabatically eliminate the harmonic oscillator modes and derive an effective Hamiltonian for the dynamics of the coupled two-level systems throughout the array.

Before doing so, let us first introduce the following transformations:

\[
a^\mu_{j,k} = \frac{1}{\sqrt{MN}} \sum_{m,n} \exp \left[ -i \left( \frac{2\pi j m}{M} + \frac{2\pi k n}{N} \right) \right] a^\mu_{m,n}, \tag{B4}
\]

\[
| \uparrow'_{j,k} \rangle = \frac{1}{\sqrt{2}} (e^{-i \varphi_{j,k}} | \uparrow_{j,k} \rangle + | \downarrow_{j,k} \rangle), \tag{B5}
\]

\[
| \downarrow'_{j,k} \rangle = \frac{1}{\sqrt{2}} (| \uparrow_{j,k} \rangle + e^{-i \varphi_{j,k}} | \downarrow_{j,k} \rangle). \tag{B6}
\]
where $M$ and $N$ are the total number of sites in the longitudinal and transverse direction, respectively. While Eq. (B4) diagonalizes $H_1$, Eqs. (B5) and (B6) define the pseudospin basis $\{|\uparrow \downarrow \rangle_{j,k}\}$.

With these, the Hamiltonians become

$$H'_1 = \sum_{m,n} \sum_{\mu=L,T} \omega_{m,n}^{\mu} a_{m,n}^{\mu} a_{m,n}^{\dagger},$$

$$H_2 = \sum_{j,k} \sum_{\mu=L,T} \left[ \frac{g_{j,k}^\mu}{\sqrt{MN}} \sum_{m,n} \exp(-i\frac{2\pi m}{N} + \frac{2\pi n}{M}) \omega_{m,n}^{\mu} a_{m,n}^{\mu} \Delta^{\mu} \left( e^{-i\phi_{j,k}} \frac{1}{\sqrt{2}} S_{j,k}^z + e^{-i2(\phi_{j,k} - \Delta^{\mu}t)} \left( \frac{1}{2} S_{j,k}^{+} + S_{j,k}^{-} \right) + \text{H.c.} \right) \right],$$

$$H'_3 = \sum_{j,k} \sum_{\mu=L,T} \sqrt{2}\Omega_{l}^{\mu} S_{j,k}^\mu,$$

where $\omega_{m,n}^{L} = 2v^{L}\cos(\frac{2\pi n}{N})$ and $\omega_{m,n}^{T} = 2v^{T}\cos(\frac{2\pi n}{M})$. The pseudospin operators are defined through the pseudospin basis states: $S_{j,k}^{\uparrow} = |\uparrow \rangle_{j,k}(\downarrow \rangle_{j,k}$ and $S_{j,k}^{\downarrow} = |\downarrow \rangle_{j,k}(\uparrow \rangle_{j,k}$.

We now go to the rotating frame via the transformation $H' \rightarrow R - i R \frac{dR}{dt}$, and

$$H'' = \sum_{i} \sum_{j,k} \left[ \frac{g_{j,k}^\mu}{\sqrt{MN}} \sum_{m,n} \exp(-i\frac{2\pi m}{N} + \frac{2\pi n}{M}) \omega_{m,n}^{\mu} a_{m,n}^{\mu} \Delta^{\mu} \left( e^{-i\phi_{j,k}} \frac{1}{\sqrt{2}} S_{j,k}^z + e^{-i2(\phi_{j,k} - \Delta^{\mu}t)} \left( \frac{1}{2} S_{j,k}^{+} + S_{j,k}^{-} \right) + \text{H.c.} \right) \right].$$

Under the condition $|\Delta^{T} - \Delta^{L}| \sim |\Delta^{\mu}| \sim |\Delta^{\mu} \pm \sqrt{2}\Omega^{T}| \sim \sqrt{2}\Omega^{T} \gg \frac{g^{\mu}}{\sqrt{MN}}, \omega_{m,n}^{\mu}$, the oscillator modes are virtually populated. We may adiabatically eliminate $a_{m,n}^{\mu}$ and describe the dynamics of the system using the resultant effective Hamiltonian:  

$$H_{\text{eff}} = \sum_{j,k} \sum_{\mu=L,T} \sum_{m,n} \left\{ \left(\frac{g_{j,k}^\mu}{MN} \left[ \frac{1}{\Delta^{\mu} - \omega_{m,n}^{\mu}} (S_{j,k}^\mu)^2 + \frac{1}{4} \Delta^{\mu} - \omega_{m,n}^{\mu} + \sqrt{2}\Omega^{T} \right] S_{j,k}^{\mu} \right) \right\}.$$  

Adopting the formulas $\sum_{n} \cos(\frac{2\pi n}{N}) = 0$ and $\sum_{n} \cos^2(\frac{2\pi n}{N}) = \sum_{n} \cos(\frac{2\pi n}{N}) \frac{2n}{N} \equiv \frac{N}{2}$, and keeping only on-site and nearest-neighbor interactions, we can further simplify the effective Hamiltonian,

$$H_{\text{eff}} \approx \sum_{j,k} \sum_{\mu=L,T} \left\{ \left(\frac{g_{j,k}^\mu}{2(\Delta^{\mu})^2 - 2(\Omega^{T})^2} \right) \left( S_{j,k}^\mu \right)^2 \left( \frac{(g_{j,k}^\mu)^2(\Omega^{T})^2}{2(\Delta^{\mu})^2 - 2(\Omega^{T})^2} \right) \left( S_{j,k}^\mu \right) \right\}.$$  

While the first two terms on the first line in Eq. (B12) are constant and can be dropped, the third term is a Stark shift in the pseudospin basis, and can be canceled via local optical elimination. This corresponds to applying a rf or Raman field with appropriate magnitude and phase between the hyperfine states such that the effective Stark shift is canceled. Then, under the condition $3(\Delta^{\mu})^2 = 2(\Omega^{T})^2$, we have

$$H_{\text{eff}} \approx \left( -1 \right) \frac{3g_{j,k}^\mu v_{L}^2}{2(\Omega^{T})^2} \sum_{j,k} \left\{ \left( -1 \right) \frac{v_{L}}{(\Delta^{\mu})^2} S_{j,k}^{\uparrow} S_{j,k}^{\downarrow} \right\}.$$  

(B13)
Equation (B13) gives the most general form of the effective Hamiltonian using our setup. For the spin-ladder Hamiltonians we considered in the main text, we may take $M = 2$ so that $j = 1, 2$ in the summations. Then, depending on the magnitudes and the relative phases of the coupling fields, we have either the Hamiltonian for the $i_0$ phase ($i_0$ model) or the Hamiltonian for the $i_2$ phase ($i_2$ model). In particular, with $|\theta_{j,k+1}^T - \theta_{j,k}^T| = \pi$, $\theta_{j,k}^T = \varphi_{j,k} = 0$ for arbitrary $|j,k|$, the Hamiltonian reduces to the $i_0$ model

$$H_0 = J \sum_{j=1}^{2} \left( S_{j,k}^x S_{j,k+1}^x + S_{j,k}^y S_{j,k+1}^y + S_{j,k}^z S_{j,k+1}^z + S_{j,k}^{\pi/2} S_{j,k+1}^{\pi/2} \right) + \lambda \sum_k \left( S_{1,k}^x S_{2,k}^x + S_{1,k}^y S_{2,k}^y + S_{1,k}^z S_{2,k}^z \right),$$

where the interaction rate $J = \frac{3(\epsilon_{j,k}^x \epsilon_{j,k+1}^x + \epsilon_{j,k}^y \epsilon_{j,k+1}^y)}{2}$. On the other hand, when $|\theta_{j,k+1}^L - \theta_{j,k}^L| = |\theta_{j,k}^T - \theta_{j,k}^T| = \pi$, $|\varphi_{j,k} - \varphi_{j,k+1}| = \pi$ and $\varphi_{j,k} = \varphi_{j,k+1}$, the Hamiltonian reduces to the $i_2$ model,

$$H_2 = J \sum_{j=1}^{2} \left( S_{j,k}^x S_{j,k+1}^x + S_{j,k}^y S_{j,k+1}^y + S_{j,k}^z S_{j,k+1}^z + S_{j,k}^{\pi/2} S_{j,k+1}^{\pi/2} \right) + \lambda \sum_k \left( S_{1,k}^x S_{2,k}^x - S_{1,k}^y S_{2,k}^y - S_{1,k}^z S_{2,k}^z \right).$$

A straightforward example for the realization of the coupled-harmonic-oscillator (QED) cavity array, as shown in Fig. 5(a) in the main text. In such a system, atoms or solid spins interact with the quantized cavity fields, which couple to their neighboring ones across both the longitudinal and transverse directions via photon hopping. The two-level system in our general model can be replaced by a three-level structure, with two low-lying hyperfine states and an electronically excited state. Correspondingly, the coupling $g_{j,k}^x$ ($g_{j,k}^y$) in the general model is replaced by a Raman path in the longitudinal (transverse) direction, with an external laser field and a cavity mode each contributing a leg in the Raman coupling. Note that due to the large difference in the two-photon detuning of the Raman couplings, the two Raman paths are effectively independent. It is then straightforward to work out the correspondence: $g^x = g^y = g_{j,k}^x$ and $\Delta^x = |\Delta^x - \Delta^y|$, where $g^x$ is the Rabi frequency for the atom-cavity coupling and $\Delta^x$ is the detuning (cf. Fig. 5 in the main text). Importantly, one may realize Hamiltonians of different SPT phases ($i_0$ or $i_2$) by adjusting the phases of the Rabi frequencies $\Omega_{j,k}^x$ and $\Omega_{j,k}^y$. For typical experimental parameters, $\Omega^x \sim 100 \text{ MHz}$, $G^x \sim 100 \text{ MHz}$, $|\Delta^x| \sim 1 \text{ GHz}$ ($\pi = 1, 2$), $\omega^x \sim 10 \text{ MHz}$, and $\Omega^y \sim 100 \text{ MHz}$, we have $J \sim 0.15 \text{ MHz}$, with the magnitude of $\lambda/J$ widely tunable by adjusting the ratio between $\Omega^x$ and $\Omega^y$. 

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