Fluctuation Induced Critical Behavior at Nonzero Temperature and Chemical Potential

K. Splittorff\textsuperscript{(a)}, J. T. Lenaghan\textsuperscript{(b)}, and J. Wirstam\textsuperscript{(c)\textsuperscript{*}}

\textsuperscript{(a)} Department of Physics and Astronomy, SUNY, Stony Brook, New York 11794
\textsuperscript{(b)} Department of Physics, University of Virginia, 382 McCormick Rd., Charlottesville, VA 22904-4714
\textsuperscript{(c)} Swedish Defense Research Agency (FOI), S-172 90 Stockholm, Sweden

We discuss phase transitions in relativistic systems as a function of both chemical potential and temperature. The presence of a chemical potential explicitly breaks Lorentz invariance and may additionally break other internal symmetries. This introduces new subtleties in the determination of the critical properties. We discuss separately three characteristic effects of a nonzero chemical potential. Firstly, we consider only the explicit breaking of Lorentz invariance using a scalar field theory with a global $U(1)$ symmetry. Secondly, we study the explicit breaking of an internal symmetry in addition to Lorentz invariance using two–color QCD at nonzero baryonic chemical potential. Finally, we consider the spontaneous breaking of a symmetry using three-color QCD at nonzero baryonic and isospin chemical potential. For each case, we derive the appropriate three-dimensional effective theory at criticality and study the effect of the chemical potential on the fixed point structure of the $\beta$–functions. We find that the order of the phase transition is not affected by the explicit breaking of Lorentz invariance but is sensitive to the breaking of additional symmetries by the chemical potential.

I. INTRODUCTION

The concept of universality is a powerful tool in the description of second–order phase transitions. In many cases, a mean field treatment of the Landau theory for the order parameter provides a sufficient description of the universal properties of the phase transition. However, in some cases, a mean–field analysis is not reliable since higher–order corrections to the Landau theory may result in $\beta$–functions which do not have stable fixed points. The form of the higher–order corrections depends upon the detailed symmetries of the problem. In particular, higher–order corrections to the $\beta$–function may drive a phase transition which is second–order at the level of mean–field theory to be first order on account of quantum fluctuations. Such phase transitions are called fluctuation induced phase transitions. The mechanism by which this occurs at the level of the $\beta$–function is simple to understand. A second–order phase transition requires the existence of nontrivial stable fixed points for the $\beta$–function since such a transition is characterized by a diverging correlation length. Quantum corrections may drive this fixed points towards instability or even make it vanish \cite{1–5}.

In this article, we discuss the influence of a nonzero chemical potential on the fixed point structure of $\beta$–functions for systems which are relativistically invariant at zero temperature and zero chemical potential. Both the temperature and the chemical potential explicitly break the O(4) relativistic invariance to O(3).\textsuperscript{1} The presence of a chemical potential may also break additional symmetries both explicitly and spontaneously through, for example, Bose condensation or Cooper pairing. Hence, there are (at least) three characteristic effects of the chemical potential:

- the explicit breaking of Lorentz invariance,
- the explicit breaking of internal symmetries in addition to Lorentz invariance,
- the spontaneous breaking of symmetries.

We consider three different theories to illustrate the physics issues associated with each of the three above effects. First we consider a complex scalar field theory with a global $U(1)$ symmetry. We introduce a chemical potential for this $U(1)$ charge. This allows for additional kinetic terms in the Landau effective theory and these kinetic terms break Lorenz invariance explicitly. In the $U(1)$ model, no additional symmetries are explicitly broken and hence it is ideal

\textsuperscript{*}Former address: Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

\textsuperscript{1}Throughout this article, we will work in Euclidean space.
to illustrate the first point above. In particular, we use this example to illustrate dimensional reduction from four to three dimensions at nonzero temperature and chemical potential.

To exemplify the second and third points we consider the chiral symmetry restoring phase transition in massless QCD with two and three colors, respectively. The chemical potential may alter the pattern of chiral symmetry breaking which in turn changes the spectrum of Goldstone excitations. We examine how this change in the chiral symmetry breaking pattern affects the order of the temperature induced chiral phase transition as a function of chemical potential. The chiral order parameters discussed here are all color singlets. The possibility of superconducting phases has been studied using renormalization group techniques in [9]. In that case gauge fields can play an important role and are incorporated into a Landau-Ginzburg theory.

To be more specific, consider two–color QCD. In this case, a baryon chemical potential breaks the $U(2N_f)$ classical chiral symmetry explicitly to $U(N_f) \times U(N_f)$. In the hadronic low energy spectrum, this change in the chiral symmetry breaking pattern manifests itself in that the Goldstone bosons have different baryon charges. At low energy, the Goldstone modes are weakly coupled and hence correctly charged ones will form boson condensates when the baryon chemical potential is sufficiently high and the temperature sufficiently low. The formation of this Bose condensate breaks $U(N_f) \times U(N_f)$ spontaneously to $Sp(N_f) \times Sp(N_f) \times U(1)$. As the temperature increases the Bose condensate melts and chiral symmetry is restored. We study the order of this phase transition. See figure 1 for an illustration.

The case of three–color QCD is somewhat special in that the baryonic chemical potential couples only indirectly to the effective theory since the chiral order parameter is not charged under baryon number. As such, the presence of a baryonic chemical potential does not violate the usual chiral symmetry breaking pattern $SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$. This in turn implies that the $\beta$-functions are not affected by the presence of a baryonic chemical potential. This point was previously raised in [10], and here we discuss the region in the $(\mu, T)$-plane where such a simplification applies.

Finally we consider a chemical potential for the third component of isospin in three-color QCD with two light quark flavors. Here the situation is almost identical to the one described for two–color QCD at nonzero baryon chemical potential. An isospin chemical potential explicitly breaks chiral symmetry. The pions form an isospin triplet and with increasing isospin chemical potential a pion condensate forms [13]. This condensate breaks chiral symmetry at low temperature but as the temperature is raised chiral symmetry is restored. We study the order of this phase transition using the $c$-expansion.

The approach followed in this article begins with the four–dimensional Landau theory for the relevant order parameter $\Phi$. This effective theory is the most general renormalizable Lagrangian consistent with the relevant symmetries. Because of the explicit breaking of Lorentz invariance the kinetic term may take a nonstandard form ($B$ is the charge matrix defined below):

$$
\text{Tr} \left[ \overline{\partial_0 \Phi} \partial_0 \Phi \right] + v^2 (\mu, T) \text{Tr} \left[ \overline{\partial_i \Phi} \partial_i \Phi \right] + \mu q_1 (\mu, T) \text{Tr} \left[ B \Phi \overline{\partial_0} \Phi \right] + \mu q_2 (\mu, T) \text{Tr} \left[ B \Phi \partial_0 \Phi \right].
$$

The functions $q_1$ and $q_2$ are constrained by the symmetries of the underlying microscopic Lagrangian and determine the conserved current for the charge to which the chemical potential is associated. This is analogous to how the chemical potential enters in chiral perturbation theory [11–18], the difference being mainly that the Goldstone field has fewer components than the generalized order parameter field of the Landau theory. Before we start our analysis let us emphasize that the presence of fixed points only implies that the phase transition can be of second order, since higher dimensional operators may render the phase transition first order. Additionally, the possible existence of a nonperturbative fixed point may drive a phase transition to second order. Let us also point out that in addition to their intrinsic interest, our results should be useful for the interpretation of lattice gauge theory results. In general, lattice gauge theory simulations at nonzero chemical potential suffer from the notorious sign problem. Two–color QCD at nonzero baryonic and isospin chemical potential and three-color QCD at nonzero isospin chemical potential have positive measures and so are special cases of dense gauge theories which can be simulated by standard Monte–Carlo methods. Hence it is possible to conduct first principle numerical computations at nonzero isospin chemical potential using the standard methods [27]. Such simulations have exposed [28,29] a rich phase diagram in the $(\mu I, T)$-plane. For $T = 0$ the pion condensate sets in when $\mu I$ exceeds the pion mass. This transition into the pion phase is second order. For temperatures on the order of the pion mass and higher the phase transition changes from second order to first order. This scenario has been explained within the context of chiral perturbation theory [18]. Making a direct comparison to the predictions in the present paper is delicate since the lattice simulations necessarily work at a nonzero quark mass.

The paper is organized as follows. In the next section, we discuss the case where the introduction of a chemical potential only leads to a breaking of the Lorentz invariance (or, in Euclidean space, of the O(4) rotational symmetry). In sec. III we extend the analysis to $N_c = 2$ QCD, where, as mentioned above, additional global flavor symmetries are broken. Three-color QCD is discussed in sec. IV and sec. V, and we end with our conclusions in sec. VI.
II. WHEN $\mu$ BREAKS ONLY LORENTZ INVARIANCE

In this section, we consider a theory in which the chemical potential does not break any internal symmetries. The simplest possible example is the given by a complex scalar field theory with a $U(1)$ symmetry to which a chemical potential is coupled. This example extends the one given in [19] to nonzero chemical potential. The Lagrangian is

$$L_{4d} = \partial_t \Phi^* \partial_t \Phi + v^2 \partial_i \Phi^* \partial_i \Phi + \mu [\Phi \partial_\Phi \Phi^* - (\partial_t \Phi) \Phi^*] + (m^2 - \mu^2) \Phi^* \Phi + \lambda (\Phi^* \Phi)^2.$$  \hspace{1cm} (2.1)

The Lagrangian reveals the standard coupling of $\mu$ to the zeroth component of the conserved current. It is the lowest order coupling of the chemical potential to the order parameter field consistent with the $U(1)$ invariance.

The analysis of the critical behavior proceeds in four steps:

- 1.) Fourier decompose the fields and integrate over $x_0 \in [0, 1/T]$.
- 2.) Determine the propagators in the resulting three–dimensional theory.
- 3.) Integrate out the massive Matsubara modes to get the effective three–dimensional theory.
- 4.) Study the stability of the fixed points of the $\beta$–functions in this three–dimensional effective theory.

Writing out the Lagrangian in Eq. (2.1) in terms of the real components of the order parameter, $\Phi \equiv a + ib$, we find

$$L_{4d} = \partial_0 a \partial_0 a + v^2 \partial_i a \partial_i a + \partial_0 b \partial_0 b + v^2 \partial_i b \partial_i b - 2\mu (a \partial_t b - b \partial_t a) + (m^2 - \mu^2) (a^2 + b^2) + \lambda (a^2 + b^2)^2.$$ \hspace{1cm} (2.2)

Since the temporal direction is compact, we may Fourier decompose the fields as

$$a(x_0, \vec{x}) = T \sum_{n=-\infty}^{\infty} e^{i\omega_n x_0} a_n(\vec{x}) \quad \text{and} \quad b(x_0, \vec{x}) = T \sum_{n=-\infty}^{\infty} e^{i\omega_n x_0} b_n(\vec{x}).$$ \hspace{1cm} (2.3)

Dimensionally reducing this theory amounts to inserting these expressions into the action and integrating $x_0$ from 0 to $1/T$. Using the fact that $a_{-n} = a_n^*$ and $b_{-n} = b_n^*$, we get

$$L_{3d} = T \sum_{n=-\infty}^{\infty} [v^2 \partial_i a_n \partial_i a_n^* + v^2 \partial_i b_n \partial_i b_n^* + \{\omega_n^2 + m^2 - \mu^2\} (a_n a_n^* + b_n b_n^*) - 2\mu \omega_n (a_n b_n^* - b_n a_n^*)] + \lambda - \text{terms}.$$ \hspace{1cm} (2.4)

Next, we write the three-dimensional Lagrangian in terms of the real fields $a_n \equiv c_n + id_n$ and $b_n \equiv c_n + if_n$:

$$L_{3d} = T \sum_{n=-\infty}^{\infty} [\{v^2 ((\partial_c c_n)^2 + (\partial_d d_n)^2) + (\partial_f f_n)^2\} + \{\omega_n^2 + m^2 - \mu^2\} (c_n^2 + d_n^2) + f_n^2] \quad \text{and} \quad -4i\mu \omega_n (d_n c_n - f_n c_n) \lambda - \text{terms}.$$ \hspace{1cm} (2.5)

The propagator matrix in the $(c_n, d_n)$-sector and equivalently in the $(c_n, f_n)$-sector is

$$\left( \begin{array}{cc} v^2 p^2 + \omega_n^2 + m^2 - \mu^2 & -2i\mu \omega_n \\ -2i\mu \omega_n & v^2 p^2 + \omega_n^2 + m^2 - \mu^2 \end{array} \right).$$ \hspace{1cm} (2.6)

Given the propagator, we can derive the effective three–dimensional theory by integrating out the massive Matsubara modes which in the present case are all the nonzero modes. In doing so we extend Ginsparg’s analysis [19] to $\mu \neq 0$. We first choose the vacuum expectation value of the order parameter to be in the direction of $a_0$, i.e. $b$ is the Goldstone boson. The leading order contribution in $\lambda$ from the nonzero Matsubara modes in the effective three–dimensional theory of the massless zeroth Matsubara mode is then the one-loop correction to the mass of the $a_0$. Schematically in going from four dimensions to three dimensions we have

$$(m^2 - \mu^2)a_0^2 \rightarrow (m^2 - \mu^2)a_0^2 + \lambda T \left( 3 \sum_{n \neq 0} a_n a_{-n} + \sum_{n \neq 0} b_n b_{-n} \right) a_0^2.$$ \hspace{1cm} (2.7)

Using the propagator in Eq. (2.6), each of the summations over $n \neq 0$ leads to a correction term
\[ M^2(T, m^2, \mu^2) = \frac{1}{2} \lambda T \sum_{n \neq 0} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left( \frac{1}{v^2p^2 + m^2 - (\mu + i\omega_n)^2} + \frac{1}{v^2p^2 + m^2 - (\mu - i\omega_n)^2} \right). \] (2.8)

Inserting
\[ \frac{1}{v^2p^2 + m^2 - (\mu \pm i\omega_n)^2} = \int_0^\infty dt \, e^{-(v^2p^2 + m^2 - (\mu \pm i\omega_n)^2)t} \] (2.9)
and integrating first over \( p \) then over \( t \) we get
\[ M^2(T, m^2, \mu^2) = \frac{\lambda T}{2} \frac{\pi^{(d-1)/2}}{(2\pi v)^{d-1}} \Gamma \left( \frac{3}{2} - \frac{d}{2} \right) \sum_{n \neq 0} \left[ (m^2 - (\mu + i\omega_n)^2)^{(d-3)/2} + (m^2 - (\mu - i\omega_n)^2)^{(d-3)/2} \right]. \] (2.10)

Assuming that \( T \gg |m| \), we can expand the argument of the sum in powers of \( m^2/\omega_n^2 \). In the limit \( d \to 4 \) and ignoring the regularization of the next to leading order terms, we find
\[ M^2(T, m^2, \mu^2) = -\frac{\lambda T^2}{4\pi v^4} \sum_{n=1}^{\infty} \left[ \frac{2}{\pi} + \frac{1}{1 + (\mu/\omega_n)^2} \frac{m^2}{\omega_n^2} - \frac{1 - 3(\mu/\omega_n)^2}{4(1 + (\mu/\omega_n)^2)^2} \frac{m^4}{\omega_n^4} + \mathcal{O} \left( \frac{m^6}{\omega_n^6} \right) \right]. \] (2.11)

The leading order term, \( M^2(T, m^2, \mu^2) = \lambda T^2/(12v^3)(1 + \mathcal{O}(m^2/(2\pi T)^2)) \), is independent of both \( m \) and \( \mu \). Moreover, this holds independently of the ratio \( \mu^2 \) of \( \mu \) and \( T \). In other words, at leading order in \( m^2/(2\pi T)^2 \), the correction takes the same form as at \( \mu = 0 \) and we find
\[ L_{3d-eff} = \partial_i a_0 \partial_i a_0 + \partial_i b_0 \partial_i b_0 + \left( m^2 - \mu^2 + \frac{\lambda T^2}{3v^4} \right) a_0^2 + \lambda (a_0^2 + b_0^2)^2. \] (2.12)

Hence, if the \( U(1) \) symmetry is broken at zero temperature, we reproduce the standard result for the symmetry restoration temperature. Of course, for the theory to be self-consistent at \( T_c \), the condition \( T_c \gg |m| \) must be fulfilled. From Eq. (2.12), we see that \( T_c^2 = 3v^4(\mu^2 - m^2)/\lambda \). If \( m^2 < 0 \) and \( \lambda \ll 1 \), the consistency requirement is always fulfilled. If the coupling constant increases to \( \lambda \simeq 1 \) while \( m^2 < 0 \), we still have \( T_c \gg |m| \) if \( \mu^2 \gg -m^2 \). In this case, however, the one-loop approximation is no longer trustworthy for such large values of \( \lambda \), and the results not reliable. On the other hand, when \( m^2 > 0 \) the situation is different. Since \( T_c^2 > 0 \), a necessary condition is \( \mu \geq m \), and unless \( \lambda \) is extremely small, we must have \( \mu \gg m \) for consistency.

The point to be taken from this analysis is that the presence of the chemical potential in the effective three-dimensional theory is felt only through the direct term, \(-\mu^2 \bar{\Phi} \Phi\). At criticality the quadratic mass-like terms vanish, and the appropriate effective theory in three dimensions at criticality contains only spatial derivative terms and quartic couplings. Specifically, in the \( U(1) \)-model considered above, the effective Lagrangian is reduced to,
\[ L_{3d-eff}(T_c) = \partial_i a_0 \partial_i a_0 + \partial_i b_0 \partial_i b_0 + \lambda (a_0^2 + b_0^2)^2. \] (2.13)

The \( \beta \)-function for this theory is well studied [26]. It has stable fixed points and consequently the phase transition is second order. Again we want to emphasize that this holds true independently of the value of \( \mu \) as long as \( T_c \gg |m| \).

This ends our discussion of the \( U(1) \) model. In conclusion: There is no effect of the explicit breaking of Lorentz invariance on the order of the phase transition as long as \( T_c \gg |m| \).

### III. WHEN \( \mu \) BREAKS INTERNAL SYMMETRIES

We now consider the case in which the chemical potential explicitly breaks symmetries in addition to Lorentz invariance. A good example is provided by QCD with two colors and two flavors at nonzero baryonic chemical

---

2This is not the case if we consider an imaginary chemical potential. In that case, the nonzero Matsubara modes become massless when \( \mu = \omega_n \). Consequently, they must be included in the effective theory for the phase transition.

3See, for example, Refs. [20-22] where the relation to the effective potential is also explained in detail. The renormalization of the speed of light and the emergence of new critical behavior associated with dynamics near the critical point is discussed in Ref. [23].
potential. For simplicity, we first ignore the effects of the $U_A(1)$ axial anomaly. At the end of this section, we state the role of the axial anomaly on the order of the phase transition. A complete treatment of the structure of the $\beta$-functions is given in Ref. [24]. Here we explain in detail how the dimensional reduction assumed in [24] comes about.

QCD with two colors and two massless flavors enjoys a $U(2N_f)$ classical invariance. This invariance is explicitly broken to $U(N_f) \times U(N_f)$ when $\mu \neq 0$. This remaining symmetry is spontaneously broken down to $Sp(N_f) \times Sp(N_f) \times U(1)$ for any nonzero value of $\mu$ by the formation of a diquark condensate (see eg. [11,12]). For illustration see figure 1. This diquark condensate signals the onset of a superfluid phase. We assume that the superfluid order parameter field can be represented by a complex anti-symmetric matrix,

$$\Phi \equiv \begin{pmatrix} X_1 & X_2 & X_3 \\ -X_2^T & X_3 \\ -X_3^T & -X_1 \end{pmatrix} \quad (3.1)$$

where $X_i^T = -X_i$ and $X_3^T = -X_3$. In this basis, the baryonic charge matrix is

$$B \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.2)$$

The four–dimensional Landau theory for QCD with two colors which is appropriate for the renormalization group analysis of the temperature induced phase transition must be invariant under the same $U(2N_f)$ symmetry as the microscopic Lagrangian, and so is given by (cf. [11,12])

$$L_0 = \frac{1}{2} \text{Tr}[(\partial_\nu \Phi^\dagger)(\partial_\nu \Phi) - 2\mu \{\Phi, B\}\partial_0 \Phi^\dagger - 2\mu^2(\Phi \Phi^\dagger + \Phi B \Phi^\dagger B)] + V \quad , \quad (3.3)$$

where the potential is

$$V = \frac{m^2}{2} \text{Tr} [\Phi^\dagger \Phi] + \lambda_1 (\text{Tr} [\Phi^\dagger \Phi])^2 + \lambda_2 \text{Tr} [(\Phi^\dagger \Phi)^2] \quad . \quad (3.4)$$

In the four–dimensional theory, the quadratic part of the Lagrangian becomes, by using the representation in Eq. (3.1),

$$L_{\text{quad}} = \frac{1}{2} \text{Tr}[(\partial_\nu \Phi^\dagger)(\partial_\nu \Phi)] + \frac{m^2}{2} \text{Tr} [\Phi^\dagger \Phi] - \mu^2 \text{Tr} \left[\Phi \Phi^\dagger + \Phi B \Phi^\dagger B\right] - \mu \text{Tr} \left[\{\Phi, B\}\partial_0 \Phi^\dagger\right] \quad (3.5)$$

$$= \frac{1}{2} \text{Tr}[\partial_\nu X_1^\dagger \partial_\nu X_1] + \text{Tr}[\partial_\nu X_2^\dagger \partial_\nu X_2] + \frac{1}{2} \text{Tr}[\partial_\nu X_3^\dagger \partial_\nu X_3]$$

$$+ m^2 \text{Tr}[X_2^\dagger X_2] + \frac{1}{2}(m^2 - 4\mu^2)\text{Tr}[X_1^\dagger X_1 + X_2^\dagger X_2 + X_3^\dagger X_3] - 2\mu \text{Tr}[X_1 \partial_0 X_1^\dagger - X_3 \partial_0 X_3^\dagger] .$$
These are precisely the same dispersion relations as we found in the zeroth Matsubara modes from \( X \) we now derive their dispersion relations in the three-dimensional theory. After setting \( m \) exponents [25].

Potential and with the anomaly present, the phase transition remains second order, however, now with \( O(6) \) critical point [24]. Consequently, the phase transition is of second order in the presence of the axial anomaly. At zero chemical potential, the baryon chemical potential does not break the \( \beta \)-function develops a fixed point. Hence, the order of the phase transition is not determined at one-loop level.

If one takes the axial anomaly into account, the symmetry is reduced to \( O(2) \) and the \( \beta \)-function develops a fixed point [24]. Consequently, the phase transition is of second order in the presence of the axial anomaly. At zero chemical potential and with the anomaly present, the phase transition remains second order, however, now with \( O(6) \) critical exponents [25].

From this we conclude that the fixed point structure of the \( \beta \)-functions is affected by the chemical potential only through the explicit breaking of the flavor symmetries. This result holds as long as \( T_c \gg |m| \), and the consistency requirement is fulfilled for all values of \( \mu \) if \( m^2 < 0 \). On the other hand, when \( m^2 > 0 \), self-consistency requires that \( \mu \gg m \). The case \( m^2 < 0 \) mimics the phase structure for massless quarks while the case with \( m^2 > 0 \) resembles two–color QCD when the quarks have a common, nonzero mass. Some caution must be exercised when considering infinitesimally small values of \( \mu \), since in that case the masses of \( X_2 \)-modes in \( \Phi \) are not well separated from those of \( X_1 \) and \( X_3 \). The results given above only apply when the mass scales are well separated. This ends our discussion of two color QCD. We shall return to discuss the explicit breaking of internal symmetries in addition to Lorentz invariance in section V.

### IV. THREE–COLOR QCD AT NONZERO BARYONIC CHEMICAL POTENTIAL

We now consider the chiral phase transition in QCD with three colors and massless quarks at nonzero baryonic chemical potential. The baryon chemical potential does not break the \( SU(N_f) \times SU(N_f) \) chiral symmetry and if
\( \mu_B < M_{\text{Nucleon}}/3 \) no charged condensate is induced. Hence the order parameter remains unaltered, i.e. the symmetry breaking pattern is unaltered and so is the number of massless modes at the phase transition. For larger chemical potentials, the behavior of the order parameter is more complicated and we shall not discuss this case here. Following [7], we assume that this chiral symmetry breaking order parameter field can be parametrized by a complex \( N_f \times N_f \Phi \) for the values of \( \mu_B \) under consideration. Because the baryonic chemical potential explicitly breaks Lorentz invariance, the allowed form of the kinetic term is

\[
\text{Tr}[\partial_\mu \Phi^\dagger \partial_\nu \Phi] + v^2(\mu, T) \text{Tr}[\partial_\mu \Phi^\dagger \partial_\mu \Phi] + i \mu q(\mu, T) \left( \text{Tr}[\Phi^\dagger \partial_\mu \Phi] - \text{Tr}[\Phi \partial_\nu \Phi^\dagger] \right). \tag{4.1}
\]

Since all components of the order parameter field have zero baryonic charge, the current carries no charge. In chiral perturbation theory, the situation is analogous since the pions do not carry baryonic charge. Consequently, the baryonic chemical potential does not appear in that context. At the chiral phase transition, it is possible that the charge function \( q(\mu, T) \) is nonzero for \( \mu \neq 0 \) since such a term is not excluded by the global symmetries. As we have shown above, however, the influence of \( q(\mu, T) \) on the fixed point structure is negligible as long as \( T_c \gg |m| \).

In Ref. [10], Hsu and Schwetz considered the \( \beta \)-functions for massless QCD at nonzero baryonic chemical potential. They suggested that the linear derivative terms can be neglected along the entire phase transition in the \((\mu, T)\)-phase diagram. Consequently, they found that the order of the chiral phase transition does not change in the \((\mu, T)\)-plane. Based on our analysis here, we agree with their arguments as long as \( T_c \gg |m| \). For \( T_c \sim m \), the consistency of the approach breaks down and an alternative expansion scheme must be employed. Such a theory must interpolate between the critical behavior in the three–dimensional theory relevant for \( T = 0 \) and the full four–dimensional Landau theory relevant at \( T = 0 \). We are not aware of the existence of such a scheme. In addition let us also stress that for low temperature and high baryon chemical potential the symmetry breaking pattern is still under debate.

**V. THREE–COLOR QCD AT NONZERO ISOSPIN CHEMICAL POTENTIAL**

Finally, we discuss the influence on the fixed point structure of the \( \beta \)-functions if we allow for different chemical potentials for different flavors. A physically relevant case is given by \( N_c = 3 \) with two massless quarks at nonzero baryonic chemical potential, \( \mu_B = \mu_u + \mu_d \), and nonzero isospin chemical potential, \( \mu_I = \mu_u - \mu_d \). The isospin chemical potential breaks the flavor invariance explicitly and we are therefore in a similar situation as in Sec. III. Chiral perturbation theory at nonzero \( \mu_I \) has been discussed in [13–15,18]. Since the \( \beta \)-functions and the associated fixed-point structure have not been studied earlier in the literature, we will describe these issues and the results in some detail.

At \( \mu_I = 0 \), the flavor symmetry is \( SU(2) \times SU(2) \times U_A(1) \times U_V(1) \) and again following [7] we assume that the chiral symmetry breaking order parameter field can be parametrized by a complex \( N_f \times N_f \) matrix \( \Phi \) transforming as

\[
\Phi \rightarrow U \Phi V, \tag{5.1}
\]

where \( U, V \in U(N_f) \). At nonzero isospin chemical potential, the four–dimensional Landau theory is

\[
L_0 = \frac{1}{2} \text{Tr}[(\partial_\mu \Phi^\dagger)(\partial_\nu \Phi)] - \frac{\mu_I}{4} \text{Tr}[\tau_3, \Phi \partial_0 \Phi^\dagger] + \text{h.c.} + \frac{\mu_I^2}{4} \text{Tr}[\Phi \tau_3 \Phi^\dagger \tau_3 - \Phi \Phi^\dagger] + V, \tag{5.2}
\]

where the potential is given by

\[
V = \frac{m^2}{2} \text{Tr}[\Phi^\dagger \Phi] + \lambda_1 (\text{Tr}[\Phi^\dagger \Phi])^2 + \lambda_2 (\text{Tr}[\Phi^\dagger \Phi])^2 + c [\text{det}(\Phi) + \text{det}(\Phi^\dagger)]. \tag{5.3}
\]

To account for the possible presence of the axial anomaly at the chiral phase transition we have included the term with the proportionality constant \( c \). For \( c \neq 0 \) the \( U(1) \) axial invariance is explicitly broken. The chiral condensate breaks \( SU(2) \times SU(2) \) to \( SU(2) \). The remaining \( SU(2) \) is broken explicitly when the isospin chemical potential is nonzero. Furthermore if \( \mu_I > |m| \) then the \( U_V(1) \) is spontaneously broken as a pion condensate forms preferring a particular direction in isospin space, say

\[
\Phi_0 \equiv i \phi_0 \tau_2. \tag{5.4}
\]

As for two–color QCD at nonzero baryonic chemical potential, the isospin chemical potential in three–color QCD splits the masses of the charged and the uncharged modes. At criticality, only the zeroth charged Matsubara modes are massless. The effective three–dimensional theory at criticality is
\[ L_{3d-off}(T_c) = \frac{1}{2} \text{Tr}[(\partial_i \Phi^d)(\partial_i \Phi)] + \lambda_1 (\text{Tr}[\Phi^d \Phi])^2 + \lambda_2 \text{Tr}[(\Phi^d \Phi)^2] + c \left[ \det(\Phi) + \det(\Phi^d) \right] \]  

(5.5)

where \( \Phi \) now has four real components \( a, b, d, \) and \( f \)

\[ \Phi = \begin{pmatrix} 0 \\ d + i f \\ a + ib \\ 0 \end{pmatrix}. \]  

(5.6)

In order to determine the one loop \( \beta \)-functions, we expand the effective 3 dimensional Lagrangian about the vacuum \( \Phi \to \Phi_0 + \Phi \) and make use of the background field method at one-loop level. Ignoring the axial anomaly for the moment, the \( \beta \)-functions are

\[ \beta_1 = \kappa \partial \lambda_1 = -\epsilon \lambda_1 + \frac{1}{\pi^2} \left[ 6\lambda_1^2 + 4\lambda_1 \lambda_2 \right], \]  

(5.7)

\[ \beta_2 = \kappa \partial \lambda_2 = -\epsilon \lambda_2 + \frac{1}{\pi^2} \left[ 5\lambda_2^2 + 6\lambda_1 \lambda_2 \right]. \]  

(5.8)

where \( \kappa \) is the arbitrary mass scale. To order \( \epsilon \), the fixed points \((\lambda_1^*, \lambda_2^*)\) and the eigenvalues of the stability matrix \( S \) at the fixed points are

- \( \lambda_1^* = \lambda_2^* = 0 \). The eigenvalues of \( S \) at \((\lambda_1^*, \lambda_2^*)\) are \(-\epsilon \) and \(-\epsilon \) and hence this fixed point is not stable.
- \( \lambda_1^* = 0, \lambda_2^* = \epsilon \pi^2/5 \). The eigenvalues of \( S \) at \((\lambda_1^*, \lambda_2^*)\) are \(-\epsilon/5 \) and \(-\epsilon \) and hence this fixed point is not stable.
- \( \lambda_1^* = \epsilon \pi^2/6, \lambda_2^* = 0 \). The eigenvalues of \( S \) at \((\lambda_1^*, \lambda_2^*)\) are \(0\) and \(\epsilon \) and hence this fixed point is marginally stable.

There is one marginally stable fixed point which implies that the order of the phase transition is not determined at one-loop in the absence of the axial anomaly.

The effects of the axial anomaly can be understood by power-counting arguments. For \( N_f = 2 \), the anomaly term in (5.5) is a mass-like operator. The explicit breaking of the \( U_A(1) \) then splits the masses of the modes with the effect that only two Matsubara modes are massless at criticality. The effective theory of these massless modes at \( T_c \) is the familiar \( O(2) \) symmetric \( \phi^4 \) theory and the phase transition is of second order [26].

Let us compare this result to the one at \( \mu_I = 0 \) derived in [7]. For \( c = 0 \) and \( \mu_I = 0 \) the phase transition is first order induced by fluctuation. A nonzero \( \mu_I \) induces a marginally stable fixed point. For \( c \neq 0 \), the phase transition may be second order. If so it is characterized by \( O(4) \) critical exponents at \( \mu_I = 0 \) and by \( O(2) \) at \( \mu_I \neq 0 \).

Introducing the baryonic chemical potential in addition to a nonzero isospin chemical potential does not affect the order of the chiral phase transition. The baryonic chemical potential does not explicitly break any additional symmetries and in the range we consider it does not lead to additional condensates. In conclusion: Our treatment of 3 color QCD confirms that explicit and spontaneous breaking of symmetries induced by the chemical potential changes the fixed point structure of the \( \beta \)-functions and hence the predicted order of the phase transition can change. The combined predictions are summarized in figure 2.

VI. CONCLUSIONS

We have described the effect of a chemical potential on the order of the temperature induced phase transitions in relativistic systems. We have focused on the stability of the fixed points of the \( \beta \)-functions and have studied three examples. The examples studied illustrates three main effects of the chemical potential: 1) the explicit breaking of Lorentz invariance, 2) the explicit breaking of global symmetries and 3) spontaneous breaking of symmetries through Bose condensation. The discussion is in this way relevant for all Landau theories which are relativistically invariant at zero temperature and chemical potential.

Our examination of the \( U(1) \) model shows that the effect of Lorentz breaking in the Landau theory for the order parameter does not affect the renormalization group equations as long as \( T \gg |m| \). The \( \beta \)-functions are effected by a nonzero chemical potential only through the breaking of internal symmetries in addition to Lorentz invariance. We have illustrated this by examining QCD with two colors and two flavors. The existence of fluctuation induced phase transitions in two–color QCD is studied in further detail in [24].

As another example of how the chemical potential affects the stability of the \( \beta \)-functions, we considered the chiral phase transition in ordinary three color QCD at nonzero baryonic chemical potential. In this case the order parameter field is neutral with respect to the charge and the Lorentz breaking in the effective theory is suppressed. In the range
of temperatures and chemical potentials under consideration no additional symmetries are broken by the baryonic chemical potential. In agreement with [10] we conclude that the order of the phase transition in QCD does not change. However, we stress that this result is only self consistent for $T_c \gg |m|$. The situation is quite different when we consider a nonzero isospin chemical potential in three–color QCD. In that case part of the order parameter field does have a nonzero third component of isospin. It is only these components which give massless zeroth Matsubara modes in the effective three–dimensional theory at criticality. The modes with a zero third component of isospin remain massive at the phase transition when the isospin chemical potential is nonzero. Since the number of degrees of freedom at the phase transition changes the $\beta$-functions change. This leads to a new stability pattern of the fixed point and as such to a different prediction for the order of the phase transition.

Finally let us stress that the analysis as performed here does not address all caveats associated with fluctuation induced phase transitions. For example fixed points outside the reach of perturbation theory can change the conclusions.

ACKNOWLEDGMENTS

It is our pleasure to thank A.D. Jackson and R.D. Pisarski for useful conversations and comments on the manuscript. K.S. wishes to thank T. Schäfer and J.J.M. Verbaarschot for pointing out an error at a crucial point. The work of K.S. was supported by the Rosenfeld foundation. J.T.L. is supported by the U.S. DOE grant DE-FG02-97ER41027.
| $N_c = 2$, $\mu_B \neq 0$ | No $U_A(1)$ breaking | $U_A(1)$ breaking |
|-----------------------------|-----------------------|-------------------|
| $N_f = 2$                   | Inconclusive          | 2$_{\text{nd}}$-order (FI 1$_{\text{st}}$-order at $\mu_B = 0$) |
| $N_f = 4$                   | 2$_{\text{nd}}$-order (FI 1$_{\text{st}}$-order at $\mu_B = 0$) | FI 1$_{\text{st}}$-order (FI 1$_{\text{st}}$-order at $\mu_B = 0$) |
| $N_f \geq 6$                | 2$_{\text{nd}}$-order (FI 1$_{\text{st}}$-order at $\mu_B = 0$) | 2$_{\text{nd}}$-order (FI 1$_{\text{st}}$-order at $\mu_B = 0$) |

| $N_c = 3$, $\mu_I \neq 0$ | No $U_A(1)$ breaking | $U_A(1)$ breaking |
|-----------------------------|-----------------------|-------------------|
| $N_f = 2$                   | Inconclusive          | 2$_{\text{nd}}$-order (FI 1$_{\text{st}}$-order at $\mu_I = 0$) |

FIG. 2. The predicted order of the chiral phase transition in massless QCD with two and three colors at respectively nonzero baryon chemical potential and nonzero isospin chemical potential.

[1] P. Bak, S. Krinsky, and D. Mukamel, Phys. Rev. Lett. **36**, 52 (1976).
[2] S. A. Brazovsky and I.E. Dzyaloshinsky, JETP Lett. **21**, 164 (1975); S. A. Brazovsky, I.E. Dzyaloshinsky, and B.G. Kukharenko, Sov. Phys. JETP **43**, 1178 (1976).
[3] H. H. Jacobson and D. J. Amit, Ann. Phys. **133**, 57 (1981); D. J. Amit, *Field theory, the renormalization group and critical phenomena* (McGraw-Hill, New York, 1978).
[4] S. Coleman and E. Weinberg, Phys. Rev. **D7**, 1888 (1973).
[5] H. Yamagishi, Phys. Rev. **D23**, 1880 (1981).
[6] R.D. Pisarski and D.L. Stein, Phys. Rev. **B23**, 3549 (1981).
[7] R. D. Pisarski and F. Wilczek, Phys. Rev. **D29**, 338 (1984).
[8] P. Arnold and L. G. Yaffe, Phys. Rev. **D49**, 3003 (1994); *erratum ibid.* **D55**, 1114 (1997).
[9] D.M. Sedrakian, D. Blaschke, K.M. Shahabasyan, and D.N. Voskresensky, Astrofiz. **44**, 443, (2001); K. Iida and G. Baym, Phys.Rev. **D63**, 074018 (2001); Erratum-ibid. **D66**, 059903 (2002); Phys.Rev. **D65** 014022 (2002); Phys.Rev. **D66**, 014015 (2002); I. Giannakis, H-C. Ren, Phys.Rev. **D65**, 054017 (2002); E. Nakano, T. Suzuki, and H. Yabu, [hep-ph/0205095](https://arxiv.org/abs/hep-ph/0205095).
[10] S.D.H. Hsu and M. Schwetz, Phys. Lett. **B432**, 203 (1998).
[11] J.B. Kogut, M.A. Stephanov, and D. Toublan, Phys. Lett. **B464**, 183 (1999).
[12] J.B. Kogut, M.A. Stephanov, D. Toublan, J.J.M. Verbaarschot, and A. Zhitnitsky, Nucl. Phys. **B582**, 477 (2000).
[13] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001); Phys. Atom. Nucl. **64**, 834 (2001); Yad. Fiz. **64**, 899 (2001).
[14] K. Splittorff, D. T. Son, and M. A. Stephanov, Phys. Rev. **D64**, 016003 (2001).
[15] J. B. Kogut and D. Toublan, Phys. Rev. **D64**, 034007 (2001).
[16] J. T. Lenaghan, F. Sannino and K. Splittorff, Phys. Rev. **D65**, 054002 (2002).
[17] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. **B620**, 290 (2002).
[18] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. **B639**, 524 (2002).
[19] P. Ginsparg, Nucl. Phys. **B170**, 388 (1980).
[20] H. E. Habor and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981); H. E. Habor and H. A. Weldon, Phys. Rev. **D25**, 502 (1982).
[21] J. I. Kapusta, Phys. Rev. **D24**, 426 (1981).
[22] K. M. Benson, J. Bernstein, and S. Dodelson, Phys. Rev. **D44**, 2480 (1991).
[23] D. Boyanovsky and H. J. de Vega, Phys. Rev. **D65**, 085038 (2002).
[24] J. Wirstam, J. T. Lenaghan, and K. Splittorff, hep-ph/0210447, to appear in PRD.
[25] J. Wirstam, Phys. Rev. **D62**, 045012 (2000).
[26] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. **B21**, 3876 (1980).
[27] M. Alford, A. Kapustin, and F. Wilczek, Phys. Rev. **D59**, 054502 (1999).
[28] J.B. Kogut and D.K. Sinclair, Phys. Rev. **D66**, 014508 (2002).
[29] J.B. Kogut and D.K. Sinclair, Phys. Rev. **D66**, 034505 (2002).