Palindromic closures using multiple antimorphisms

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Abstract

Generalized pseudostandard word $u$, as introduced in 2006 by de Luca and De Luca, is given by a directive sequence of letters from an alphabet $\mathcal{A}$ and by a directive sequence of involutory antimorphisms acting on $\mathcal{A}^*$. Prefixes of $u$ with increasing length are constructed using pseudopalindromic closure operator.

We show that generalized Thue–Morse words $t_{b,m}$, with $b, m \in \mathbb{N}$ and $b, m \geq 2$, are generalized pseudostandard words if and only if $t_{b,m}$ is a periodic word or $b \leq m$. This extends the result of de Luca and De Luca obtained for the classical Thue–Morse words.

Keywords: palindromic closure, generalized Thue–Morse word, involutory antimorphism

1. Introduction

Palindromic closure of a finite word $w$ is the shortest palindrome having $w$ as a prefix. This concept was introduced in 1997 by Aldo de Luca for words over the binary alphabet. In \cite{10}, De Luca showed that any standard Sturmian word is a limit of a sequence of palindromes $(w_n)$, where $w_0$ equals the empty word and $w_{n+1}$ is the palindromic closure of $w_n\delta_{n+1}$ for some letter $\delta_{n+1}$ from the binary alphabet. And, vice versa, the limit of such a sequence is always a standard Sturmian word. This construction was extended to any finite alphabet $\mathcal{A}$ by Droubay, Justin and Pirillo in \cite{13}, and words arising by their construction are called standard episturmian words. The sequence $\delta_1\delta_2\delta_3\ldots$ of letters that are added successively at each step is referred to as the directive sequence of the standard episturmian word.

An important generalization of standard episturmian words appeared in \cite{11}, where the palindromic closure is replaced by $\vartheta$-palindromic closure with $\vartheta$ an

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arbitrary involutory antimorphism of the free monoid $\mathcal{A}^*$. The corresponding words are called $\vartheta$-standard words, or pseudostandard words.

A further generalization was provided in 2008 by Michelangelo Bucci, Aldo de Luca, Alessandro De Luca, Luca Q. Zamboni in [8], where the sequence $(w_n)$ is allowed to start with an arbitrary finite word $w_0$, and the limit word is a $\vartheta$-standard word with the seed $w_0$. Words obtained by these generalizations are in some sense quite similar to the standard episturmian words: by results of Bucci and De Luca [9], any $\vartheta$-standard word with a seed is a morphic image of a standard episturmian word.

The described constructions of $(w_n)$ guarantee that the language of a standard episturmian word contains infinitely many palindromes.

Droubay, Justin and Pirillo in [13] deduced that any finite word $w$ contains at most $|w| + 1$ distinct palindromes (where $|w|$ stands for the length of $w$). A word $w$ with exactly $|w| + 1$ palindromes is called rich (in [14]), or full (in [7]). An infinite word is rich if every finite factor of this word is rich. Examples of rich words include all episturmian words, see [13], two interval coding of rotations, see [3], words coding interval exchange transformation with symmetric permutation, see [2], etc.

The notion of rich word was generalized as well: the concept of $\vartheta$-rich word was introduced in [18]. A $\vartheta$-rich word is saturated by $\vartheta$-palindromes, fixed points of the involutory antimorphism $\vartheta$, up to the highest possible level. Another generalization of the concept of richness is a measure of how many palindromes are missing in a certain sense. This quantity is called the palindromic defect and it was first considered in [7]. Analogously to standard episturmian words, one can prove that every $\vartheta$-standard word is $\vartheta$-rich, and every standard word with seed has finite palindromic defect.

Again, both generalizations of rich words are not too far from the original notion. In particular, any uniformly recurrent $\vartheta$-rich word and any uniformly recurrent word with finite defect is just a morphic image of a rich word, see [16]. Nevertheless, the operation of palindromic closure enables us to construct a big class of rich words.

Up to this point, we were concerned with properties of words with respect to one fixed involutory antimorphism $\vartheta$ on $\mathcal{A}^*$.

In the last Section of the paper [11], De Luca and de Luca introduced an even more general concept - generalized pseudostandard words. They considered a set $\mathcal{I}$ of involutory antimorphisms over $\mathcal{A}^*$ and beside a directive sequence $\Delta = \delta_1 \delta_2 \delta_3 \ldots$ of letters from $\mathcal{A}$, also a directive sequence of antimorphisms $\Theta = \vartheta_1 \vartheta_2 \vartheta_3 \ldots$ from $\mathcal{I}$. The construction of $(w_n)$ starts with the empty word $w_0$ and recursively, $w_n$ is the $\vartheta_n$-palindromic closure of $w_{n-1} \delta_n$.

Then De Luca and de Luca focused on the prominent Thue–Morse word $u_{TM}$. They showed that $u_{TM}$ is a generalized pseudostandard word with the directive sequences $\Delta = 01111\ldots = 01^\omega$ and $\Theta = RERERE\ldots = (RE)^\omega$, where $R$ denotes the mirror image operator, and $E$ the antimorphism which exchanges
the letters $0 \leftrightarrow 1$.

The example of the Thue–Morse word illustrates that generalized pseudo-standard words substantially differ from the previous notions where only one antimorphism is used. This is due to the fact that the Thue–Morse word is not a morphic image of a standard episturmian word as all $\vartheta$-standard words are. This can be seen when comparing the factor complexities of a standard episturmian word, which is of the form $an + b$ except for finitely many integers $n$ (see [13]), and the factor complexity of the Thue–Morse word (see [6] or [12]).

The results of [5] confirm that the notion of the generalized pseudostandard word is very fruitful. In particular, Blondin-Massé, Paquin, Tremblay, and Vuillon showed that any standard Rote word is a generalized pseudostandard word. Since the Rote words are defined over the binary alphabet, their directive sequences $\Theta$ contain antimorphisms $R$ and $E$ only.

In this article we focus on the so-called generalized Thue–Morse words. Given two integers $b$ and $m$ such that $b > 1$ and $m > 1$, we denote the generalized Thue–Morse word by $t_{b,m}$. The alphabet of $t_{b,m}$ is $\mathcal{A} = \mathbb{Z}_m = \{0, \ldots, m - 1\}$. For a given integer base $b$, the number $s_b(n)$ denotes the digit sum of the expansion of number $n$ in the base $b$. The word $t_{b,m}$ is defined

$$t_{b,m} = (s_b(n) \mod m)_{n=0}^{\infty}.$$  

In this notation the classical Thue–Morse word equals $t_{2,2}$. As shown in [19], the language of $t_{b,m}$ is closed under a finite group containing $m$ involutory antimorphisms. This group is isomorphic to the dihedral group $I_2(m)$. Our aim in this paper is to prove the following theorem:

**Theorem 1.1.** The generalized Thue–Morse word $t_{b,m}$ is a generalized pseudostandard word if and only if $b \leq m$ or $b - 1 = 0 \pmod{m}$.

Unlike the case of the standard words with seed, very little is known about the properties of generalized pseudostandard words. In the last section, we propose several questions the answering of which would bring a better understanding of the structure of such words.

Our motivation for the study of generalized pseudostandard words stems from a desire to find $G$-rich words recently introduced in [15]. Words that are rich in the original sense are $G$-rich with respect to $G = \{R, Id\}$. In [19], the last author showed that the words $t_{b,m}$ are $I_2(m)$-rich. In particular, the classical Thue–Morse word is $H$-rich with $H = \{E, R, ER, Id\}$. Using the result of [3], one can also show that the Rote words are $H$-rich (for definition of the Rote words see [17]). These examples are almost all the examples of $G$-rich words we know for which the group $G$ is not isomorphic to $\{R, Id\}$. We believe that generalized pseudostandard words can provide many other new examples.
2. Preliminaries

By \( \mathcal{A} \) we denote a finite set of symbols usually called the alphabet. A finite word \( w \) over \( \mathcal{A} \) is a string \( w = w_0w_1 \cdots w_{n-1} \) with \( w_i \in \mathcal{A} \). For its length \( n \) we write \( |w| \). The set of all finite words over \( \mathcal{A} \), including the empty word \( \varepsilon \), together with the operation of concatenation of words, form the free monoid \( \mathcal{A}^* \).

A morphism of \( \mathcal{A}^* \) is a mapping \( \varphi : \mathcal{A}^* \rightarrow \mathcal{A}^* \) satisfying \( \varphi(wv) = \varphi(w)\varphi(v) \) for all finite words \( w, v \in \mathcal{A}^* \). A morphism is uniquely given by the images \( \varphi(a) \) of all letters \( a \in \mathcal{A} \). If, moreover, there exists a letter \( b \in \mathcal{A} \) and a non-empty word \( w \in \mathcal{A}^* \) such that \( \varphi(b) = bw \), then the morphism \( \varphi \) is called a substitution.

If a word \( w \in \mathcal{A}^* \) can be written as a concatenation \( w = uvz \), then \( v \) is called a factor of \( w \). If \( u \) is the empty word, then \( v \) is called a prefix of \( w \); if \( z \) is the empty word, then \( v \) is called a suffix of \( w \). If \( \varphi \) is a substitution, then for every \( n \in \mathbb{N} \) the word \( \varphi^n(b) \) is a prefix of \( \varphi^{n+1}(b) \).

An infinite word \( \mathbf{u} \) over \( \mathcal{A} \) is a sequence \( u_0u_1u_2 \cdots \in \mathcal{A}^\mathbb{N} \). The set of all factors of \( \mathbf{u} \) is denoted by \( \mathcal{L}(\mathbf{u}) \) and referred to as the language of \( \mathbf{u} \). The action of a morphism \( \varphi : \mathcal{A}^* \rightarrow \mathcal{A}^* \) can be naturally extended to \( \mathcal{A}^\mathbb{N} \) by \( \varphi(\mathbf{u}) = \varphi(u_0u_1u_2 \cdots) = \varphi(u_0)\varphi(u_1)\varphi(u_2) \cdots \). If \( \varphi(\mathbf{u}) = \mathbf{u} \) for some infinite word \( \mathbf{u} \), then the word \( \mathbf{u} \) is a fixed point of the morphism \( \varphi \). Every substitution \( \varphi \) has a fixed point, namely the infinite word which has prefix \( \varphi^n(b) \) for every \( n \); this infinite word is denoted \( \varphi^\infty(b) \).

**Example 2.1.** The Thue–Morse word \( \mathbf{u}_{TM} \) is a fixed point of the substitution \( \varphi_{TM} \) which maps \( 0 \mapsto \varphi_{TM}(0) = 01 \) and \( 1 \mapsto \varphi_{TM}(1) = 10 \). The substitution has two fixed points: the Thue–Morse word \( \mathbf{u}_{TM} = \varphi^\infty(0) \) and the word \( \varphi^\infty(1) \).

When manipulating a fixed point \( \mathbf{u} \) of a substitution \( \varphi \), we will need the notion of an ancestor: We say that a word \( w = w_0w_1 \cdots w_k \) is a \( \varphi \)-ancestor of a word \( v \in \mathcal{L}(\mathbf{u}) \) if the following three conditions are satisfied:

- \( v \) is a factor of \( \varphi(w_0w_1 \cdots w_k) \),
- \( v \) is not a factor of \( \varphi(w_1 \cdots w_k) \),
- \( v \) is not a factor of \( \varphi(w_0w_1 \cdots w_{k-1}) \).

**Example 2.2.** Consider the Thue–Morse word

\[
\mathbf{u}_{TM} = 01101001100101101011001 \cdots = \varphi_{TM}(0)\varphi_{TM}(1)\varphi_{TM}(1)\varphi_{TM}(0)\varphi_{TM}(1)\varphi_{TM}(0) \cdots
\]

The factor \( v = 010011 \) has an ancestor \( w = 1101 \), since \( v \) is a factor of \( \varphi_{TM}(1101) = 10100110 \), and \( w \) is neither a factor of \( \varphi_{TM}(101) = 100110 \) nor a factor of \( \varphi_{TM}(110) = 101001 \). In fact, \( w \) is the unique ancestor of \( 010011 \).

The factor \( v = 010 \) is a factor of \( \varphi(11) = 1010 \) and a factor of \( \varphi(00) = 0101 \). Thus \( v = 010 \) has two ancestors, namely 11 and 00.
Let us now define the key notion of this article. The mapping $\Psi : A^* \rightarrow A^*$ satisfying

$$\Psi(uv) = \Psi(v)\Psi(u) \quad \text{for any } u, v \in A^* \quad \text{and} \quad \Psi^2 \text{ equal to the identity}$$

is an involutory antimorphism. Any antimorphism $\Psi$ is determined by the images of the letters from $A$. The restriction of $\Psi$ to the alphabet $A$ is a permutation $\pi$ on $A$ with cycles of length 1 or 2 only.

A word $w \in A^*$ is a $\Psi$-palindrome if $\Psi(w) = w$. The notion pseudopalindrome is also used.

A $\Psi$-palindromic closure of a factor $w \in A^*$ is the shortest $\Psi$-palindrome having $w$ as a prefix. The $\Psi$-palindromic closure of $w$ is denoted by $w^\Psi$. If $w = uq$ such that $q$ is the longest $\Psi$-palindromic suffix of $w$, then

$$w^\Psi = uq\Psi(u).$$

**Example 2.3.** There are two distinct involutory antimorphisms on the alphabet $A = \{0, 1\}$: $R$, the mirror image, and $E$, the antimorphism exchanging letters, i.e., $E(0) = 1$ and $E(1) = 0$. Put $u = 0110110$. Then $u$ is an $R$-palindrome, and thus $u^R = u$. The word $u$ is not an $E$-palindrome, since $E(u) = E(0110110) = 1001001 \neq u$. The $E$-palindromic closure of $u$ is $u^E = 01101101001$ since the longest $E$-palindromic suffix of $u$ is 10.

In [11], de Luca and De Luca generalized the notion of standard word considering the set $I$ of all involutory antimorphisms on $A^*$ instead of just one fixed antimorphism. We will denote by $I^\mathbb{N}$ the set of all infinite sequences over $I$.

**Definition 2.4.** Let $\Theta = \vartheta_1 \vartheta_2 \vartheta_3 \ldots \in I^\mathbb{N}$ and $\Delta = \delta_1 \delta_2 \delta_3 \ldots \in A^\mathbb{N}$. Denote

$$w_0 = \varepsilon \quad \text{and} \quad w_n = \left(w_{n-1} \delta_n\right)^{\vartheta_n} \quad \text{for any } n \in \mathbb{N}, n \geq 1.$$

The word

$$u_\Theta(\Delta) = \lim_{n \to \infty} w_n$$

is called a generalized pseudostandard word with the directive sequence of letters $\Delta$ and the directive sequence of antimorphisms $\Theta$.

Let us stress that the definition of $u_\Theta(\Delta)$ is correct as $w_n$ is a prefix of $w_{n+1}$ for any $n$.

**Example 2.5.** Consider the directive sequence of letters $\Delta = 0(101)^\omega$ and the directive sequence of antimorphisms $\Theta = (RE)^\omega$. Then

$$w_0 = \varepsilon$$

$$w_1 = 0^R = 0$$

$$w_2 = (01)^E = 01$$
The authors of [11] proved that the famous Thue–Morse word $w_{TM}$ is a generalized pseudostandard word with directive sequences

\[ \Delta = 01^\omega \quad \text{and} \quad \Theta = (ER)^\omega. \]

2.1. The Generalized Thue–Morse words and their properties

In Introduction, we defined a generalized Thue–Morse word by \( (1) \), i.e., its \( n \)-th letter is the digit sum of the expansion of \( n \) in base \( b \) taken modulo \( m \). It can be shown that \( t_{b,m} \) is a fixed point of the substitution \( \varphi_{b,m} \) over the alphabet \( \mathbb{Z}_m \):

\[ \varphi(k) = \varphi_{b,m}(k) = k(k + 1)(k + 2)\ldots(k + b - 1) \quad \text{for every} \quad k \in \mathbb{Z}_m \quad (2) \]

where letters are expressed modulo \( m \). As already stated in [11], \( t_{b,m} \) is periodic if and only if \( b = 1 \pmod{m} \).

(Note about our subsequent notation: When dealing with letters from \( \mathbb{Z}_m \), we will consider all operations modulo \( m \). We will denote the relation \( x = y \pmod{m} \) by \( x \equiv y \pmod{m} \), to ease the notation.)

The language of \( t_{b,m} \) has many symmetries: denote by \( I_2(m) \) the group generated by antimorphisms \( \Psi_x \) defined for every \( x \in \mathbb{Z}_m \) by

\[ \Psi_x(k) = x - k \quad \text{for every} \quad k \in \mathbb{Z}_m. \quad (3) \]

This group - usually called the \textit{dihedral group of order} \( 2m \) - contains \( m \) morphisms and \( m \) antimorphisms. As shown in [19], if \( w \) is a factor of \( t_{b,m} \), then \( \nu(w) \) is a factor of \( t_{b,m} \) for every element \( \nu \) of the group \( I_2(m) \).

Let us list some properties of the generalized Thue–Morse word we will use later. They are not hard to observe. (See also [19].)

\textbf{Properties} of \( t_{b,m} \)

1. Let \( b \neq m \). If \( v = v_0v_1\cdots v_{k-1} \) is a factor of \( t_{b,m} \) of length \( k \geq 2b + 1 \), then there exists \( j \in \{0, 1, \ldots, k - 2\} \) such that \( v_j + 1 \neq m \) \( v_{j+1} \). Such index \( j \) will be called \textit{jump} in \( v \). It is important to note here that we always start indices from 0. Sometimes, when no confusion can occur, we will say that there is a jump between the letters \( v_j \) and \( v_{j+1} \).
2. If \( b \neq m \), then a factor \( v \) of length at least \( 2b + 1 \) has uniquely determined \( \varphi \)-ancestors.

3. If \( v = v_0v_1 \ldots v_k \) is a \( \Psi \)-palindrome and an index \( j \) is a jump in \( v \), then also the index \( k - j \) is a jump in \( v \).

4. \( \Psi_x \varphi = \varphi \Psi_x \varphi + 1 \) for every \( x \in \mathbb{Z} \).

5. For every \( \Psi \in I_2(m) \) there exists a unique \( \Psi' \in I_2(m) \) such that \( \Psi \varphi = \varphi \Psi' \).

6. If \( w \neq \varepsilon \) is a \( \Psi \)-palindrome for some \( \Psi \in I_2(m) \), then for every antimorphism \( \Psi' \in I_2(m) \) such that \( \Psi' \neq \Psi \) we have \( \Psi'(w) \neq w \).

7. If \( \Psi \neq \Psi' \), then for every letter \( a \in A \), \( \Psi(a) \neq \Psi'(a) \).

### 3. Proof of Theorem 1.1

Proof of Theorem 1.1 will be split into Propositions 3.1 and 3.9.

**Proposition 3.1.** Let \( m, b \in \mathbb{Z} \). Denote
\[
\Delta = 0^{(12 \ldots (b-1))} \in \mathbb{Z}_m^\omega \quad \text{and} \quad \Theta = \left( \Psi_0 \Psi_1 \ldots \Psi_{m-1} \right)^\omega \in I_2(m)^\omega. \tag{4}
\]

If \( b \leq m \) or \( b = 1 \pmod{m} \), then the generalized pseudostandard word \( u_\Theta(\Delta) \) with directive sequences \( \Delta \) and \( \Theta \) equals \( t_{b,m} \).

Axel Thue found the classical Thue–Morse word \( t_{2,2} \) when he searched for infinite words without overlapping factors, i.e., words without factors of the form \( v = ws = pw \) such that \( |w| > |s| \). The authors of [4] showed that the generalized Thue–Morse word \( t_{b,m} \) is overlap-free if and only if \( b \leq m \). It is worth to mention that the same condition appears in our characterization of non-periodic words \( t_{b,m} \) which are the generalized pseudostandard words.

For parameters \( b = m = 2 \), Proposition 3.1 was shown in [11]. The following example illustrates that the assumption \( b \leq m \) is crucial for validity of Proposition 3.1.

**Example 3.2.** Consider \( b = 4 \) and \( m = 2 \). On the alphabet \( A = \mathbb{Z}_2 \), we have \( \Psi_0(k) = 0 - k = k \) and \( \Psi_1(k) = 1 - k \) for any letter \( k \). In the notation of Example 2.3, it means \( \Psi_0 = R \) and \( \Psi_1 = E \). Therefore the sequences \( \Delta \) and \( \Theta \) from Proposition 3.1 coincide with sequences \( \Delta \) and \( \Theta \) from Example 2.5. The generalized Thue–Morse word \( t_{4,2} \) starts as
\[
t_{4,2} = 01011010010110101010010110100101010110100101\ldots
\]

Note that, using the notation from Definition 2.4, \( w_8 \) is not a prefix of \( t_{4,2} \) and thus the generalized pseudostandard word \( u_\Theta(\Delta) \) from Proposition 3.1 does not correspond to \( t_{4,2} \).

Propositions 3.1 and 3.9 rely on several technical lemmas. The first one settles the case for periodic Thue–Morse words.
Lemma 3.3. Let \( b = m \). 1. The word \( u_\Theta(\Delta) \) with the directive sequences \( \Delta \) and \( \Theta \) given in (4) equals \( t_{b,m} \).

Proof. Let \( n = \sum_{i=0}^{k} a_i b^i \) be the expansion of the number \( n \) in the base \( b \). The assumption \( b = m \) implies \( b^i = m \) for any \( i \in \mathbb{N} \). With respect to (1), we can write

\[
t_{b,m}(n) = s_b(n) = \sum_{i=0}^{k} a_i = m \sum_{i=0}^{k} a_i b^i = n .
\]

Since \( \Delta = 0 \left(1 \cdots (b-1)\right)^\omega \) equals \( (01 \cdots (m-1))^{\omega} \), we have showed that \( t_{b,m} = \Delta \). Moreover, the sequence of antimorphisms \( \Theta \) can be indexed by natural numbers as \( \Theta = \Psi_0 \Psi_1 \Psi_2 \Psi_3 \cdots \) where \( \Psi_n = \Psi_x \) for \( n = m \). Clearly by (3)

\[
\Psi_n(012 \ldots n) = 012 \ldots n = (012 \ldots n)^\Psi_n . \tag{5}
\]

Let the words \( w_n \) have the meaning as in Definition 2.4. We will show by induction that \( w_{n+1} = 0123 \cdots n \) for any \( n \in \mathbb{N} \). We have \( w_1 = (0)^{\Psi_0} = 0 = \Psi_0(0) \). Using definition of \( w_{n+1} \) and (5) we get

\[
w_{n+1} = (w_n \delta_{n+1})^{\Psi_n} = \left((012 \cdots (n-1))n\right)^\Psi_n = 012 \cdots (n-1)n .
\]

This means that \( t_{b,m} = \lim_{n \to \infty} w_n \), as desired. \( \square \)

We can now concentrate on the non-periodic Thue–Morse words, i.e., on the case \( b \neq m \), which will be treated using several lemmas.

Lemma 3.4. If \( \Psi \in I_2(m) \) is an antimorphism and \( p \in \mathcal{A}^* \) is a \( \Psi \)-palindrome such that \( \varphi(a_1 a_2) a_3 \) is a suffix of \( p \) for some letters \( a_1, a_2, a_3 \in \mathcal{A} \), then there exists a word \( w \) of length at least 2 and antimorphism \( \Psi' \in I_2(m) \) such that

\[
p = \Psi(a_3) \varphi(w) a_3 , \quad \Psi'(w) = w \quad \text{and} \quad \Psi \varphi = \varphi \Psi' .
\]

Proof. Let \( p = p_0 p_1 \ldots p_n \). Since \( p \) has a suffix \( \varphi(a_1 a_2) a_3 \) of length \( 2b + 1 \), according to Property \( \# \) \( p \) has a jump position. The jump position of \( p \) is either \( n - 1 \) or \( n - b - 1 \). As \( p \) is a \( \Psi \)-palindrome, the index 0 or \( b \) is a jump of \( p \). It implies that a prefix of \( p \) is of the form \( a'_3 \varphi(a'_2 a'_1) \) for some letters \( a'_1, a'_2, a'_3 \). Thus \( p = \Psi(a_3) \varphi(w) a_3 \) for some word \( w \) with \( |w| \geq 2 \). As \( p = \Psi(p) \), using Property \( \# \) we get \( \varphi(w) = \Psi(\varphi(w)) = \varphi \Psi'(w) \). Since \( \varphi \) is injective, we have \( w = \Psi'(w) \). \( \square \)

Lemma 3.5. Fix \( n \in \mathbb{N} \) and \( k \in \{2, \ldots, b-1\} \). Put \( \Psi = \Psi_{(b-1)n+k} \). The longest \( \Psi \)-palindromic suffix of the factor

\[
v = \varphi^n(0) \varphi^n(1) \varphi^n(2) \cdots \varphi^n(k-1) k
\]

is \( \Psi(k) \varphi^n(1) \varphi^n(2) \cdots \varphi^n(k-1) k \) and thus

\[
v^n = \varphi^n(0) \varphi^n(1) \varphi^n(2) \cdots \varphi^n(k-1) \varphi^n(k) . \tag{6}
\]
Proof. First we show that \( u = \Psi(k)\varphi^n(1)\varphi^n(2) \ldots \varphi^n(k-1) k \) is a \( \Psi \)-palindromic suffix of the factor \( v \). It is easy to check that the last letter of \( \varphi^n(0) \) is the letter \((b-1)n\). In our notation \( \Psi(k) = (b-1)n + k = (b-1)n \). Therefore \( \Psi(k) \) is the last letter of \( \varphi^n(0) \), and thus \( u \) is a suffix of \( v \). To show that \( u \) is a \( \Psi \)-palindromic suffix, we need to show

\[
\Psi\big(\varphi^n(i)\big) = \varphi^n(k-i) \quad \text{for any } i = 1, 2, \ldots, k-1.
\]

Using Property 4, we get

\[
\Psi\varphi^n = \Psi_{(b-1)n+k}\varphi^n = \varphi^n\Psi_k,
\]

and thus \( \Psi\big(\varphi^n(i)\big) = \varphi^n\big(\Psi_k(i)\big) = \varphi^n(k-i) \) for all \( i \), including \( i = 0 \) and \( i = k \).

Now we show by contradiction that \( u \) is the longest \( \Psi \)-palindromic suffix of \( v \). Consider the minimal \( n \) for which the statement is false, i.e., the longest \( \Psi \)-palindromic suffix of \( v \) - denote it by \( p \) - is longer than \( u \). Since \( 01 \ldots (k-1)k = (01 \ldots (k-1)k)^\Psi \), the minimal \( n \) is \( \geq 1 \). As \( |p| > |u| \), \( p \) has a suffix \( u \) and we can apply Lemma \( 3.4 \). Therefore \( p \) has the form \( p = \Psi(k)\varphi(w)k \), where \( w \) is a \( \Psi' \)-palindrome and \( \varphi\Psi' = \Psi\varphi \). According to Property 4, we have \( \Psi' = \Psi_{(b-1)(n-1)+k} \). In particular, \( \Psi'(k)wk \) is \( \Psi' \)-palindromic suffix of \( \varphi^{n-1}(0)\varphi^{n-1}(1)\varphi^{n-1}(2) \ldots \varphi^{n-1}(k-1)k \).

As \( |p| = 2 + |\varphi(w)| > |u| \), necessarily \( |w| > |\varphi^{n-1}(1)\varphi^{n-1}(2) \ldots \varphi^{n-1}(k-1)| \). It means that \( \varphi^{n-1}(0)\varphi^{n-1}(1)\varphi^{n-1}(2) \ldots \varphi^{n-1}(k-1)k \) has the longest \( \Psi' \)-palindromic suffix longer than \( \Psi'(k)\varphi^{n-1}(1)\varphi^{n-1}(2) \ldots \varphi^{n-1}(k-1)k \) - contradiction with the minimality of \( n \).

Lemma 3.6. Let \( v \) be a factor with the suffix \( \varphi((a-1)a)1 \) and let \( b \neq m \). Put \( \Psi = \Psi_{a+b} \). Under these assumptions, the longest \( \Psi \)-palindromic suffix \( p \) of the factor \( v \) is of length at least 2. Moreover, for \( |p| \) and the parameters \( a \) and \( b \), the following holds:

1. if \( |p| \geq b+1 \), then \( p = \Psi(1)\varphi(w)1 \), where \( w \) is a \( \Psi_{a+1} \)-palindrome of length at least 2;
2. if \( 3 \leq |p| \leq b \), then \( a + b = m \) and \( b > m \);
3. if \( |p| = 2 \), then either \( a + b \neq m \) or \( a + b = m \) with \( b \leq m \).

Proof. Since the last two letters of \( v \) are \( (a + b - 1)1 \), and \( \Psi(1) = a + b - 1 \), the word \( v \) has a palindromic suffix of length 2.

If \( p \) itself has the suffix \( \varphi((a-1)a)1 \), then the form of \( p \) is given by Lemma \( 3.4 \) as \( p = \Psi(1)\varphi(w)1 \). According to Properties 4 and 5 in Section 2.1, we have \( \Psi_{a+b} = \Psi_{a+1} \) and thus \( w \) is a \( \Psi_{a+1} \)-palindrome.

Let \( p \) be shorter than the suffix \( \varphi((a-1)a)1 \). It means that \( p \) is a suffix of the factor \( a(a+1) \ldots (a+b-2)a(a+1) \ldots (a+b-1)1 \). Since \( b-1 \neq m \), we
have a jump between letters $a + b - 2$ and $a$. Let us discuss the following two cases separately:

i) If $a + b \neq_m 1$, then the other jump is between the last two letters $a + b - 1$ and 1. In the $\Psi$-palindrome, jump positions must be symmetric with respect to the center, and thus the only two candidates for the palindromic suffix are $(a+b-2)a(a+1)\ldots(a+b-1)1$ and $(a+b-1)1$. Since $\Psi(1) = \Psi_{a+b}(1) \neq_m a+b-2$, only the latter possibility $p = \Psi(1)1$ occurs.

ii) If $a+b =_m 1$, then $a(a+1)\ldots(a+b-2)a(a+1)\ldots(a+b-1)1$ has only one jump, namely, as we mentioned above, between letters $a+b-2$ and $a$. Therefore the longest palindromic suffix $p$ does not contain any jumps. It implies, that $p$ is a suffix of $a(a+1)\ldots(a+b-1)1$. Let $k \in \{0,1,\ldots,m-1\}$ be a letter such that $p = (a+k)(a+k+1)\ldots(a+b-1)1$. Then $a+k = \Psi(1) = a+b-1 =_m 0$. Or equivalently, $k =_m b-1$.

If $b \leq m$, the equality $k =_m b-1$ has the only solution $k = b-1$, i.e., $p = (a+b-1)1 = \Psi(1)1$, as before.

If $b > m$, then the smallest $k \in \{0,1,\ldots,b-1\}$ solving $k =_m b-1$, satisfies $k \leq b-1 - m \leq b-3$, and as well $k > 0$ (since $k = b-1 \neq_m 0$). As $p = (a+k)(a+k+1)\ldots(a+b-1)1$ has the length $|p| = b-k+1$, we get $3 \leq |p| \leq b$.

The following claim addresses the question of the length of the longest $\Psi$-palindromic suffix of the factor $\varphi^n(0)1$.

**Claim 3.7.** Let $b \neq_m 1$. Put $q = \min \{i \in \mathbb{N} : i > 0$ and $i(b-1) =_m 0\}$ and for a fixed $n \in \mathbb{N}$ denote $\Psi = \Psi_{(b-1)n+1}$.

1. If $b \leq m$, then the longest $\Psi$-palindromic suffix of the factor $\varphi^n(0)1$ is of length 2.
2. If $b > m$ and $n < q$, then the longest $\Psi$-palindromic suffix of the factor $\varphi^n(0)1$ is of length 2.
3. If $b > m$ and $n = q$, then the longest $\Psi$-palindromic suffix of the factor $\varphi^n(0)1$ is of length greater than 2 and less than $b+1$.

**Proof.** As $b-1 \neq_m 0$, we have $q \geq 2$. First we show that Claim holds for $n = 0$ and $n = 1$.

Consider $n = 0$. The factor 01 is a $\Psi_1$-palindrome as $\Psi_1(0) = 1$.

Consider $n = 1$. The factor $\varphi(0)1 = 01\ldots(b-1)1$ has only one jump, namely between two last letters $b-1$ and 1 because of $b-1 \neq_m 0$. Thus $\varphi(0)1$ cannot have $\Psi$-palindromic suffix longer than 2. Since $\Psi(b-1) = \Psi_b(b-1) = 1$, the factor $\varphi(0)1$ has the longest $\Psi$-palindromic suffix of length 2.

Now suppose that there exists an index $n$ such that $\varphi^n(0)1$ has a $\Psi$-palindromic suffix of length at least 3. Consider the smallest such $n$. Obviously, $n \geq 2$. Denote by $p$ the longest $\Psi$-palindromic suffix of $\varphi^n(0)1$. We will apply Lemma 3.6 with $v = \varphi^n(0)1 = \varphi\left(\varphi^{n-1}(0)\right)1$. Since the last letter of $\varphi^{n-1}(0)$ equals to
(n - 1)(b - 1), we denote \(a = (n - 1)(b - 1)\). For this choice of \(a\), the antimorphism \(\Psi = \Psi_{(b - 1)n + 1} = \Psi_{a+b}\) is as required in Lemma 3.6. Since \(|p| \geq 3\), only Cases 1 and 2 from Lemma 3.6 apply:

Case 1: \(p = \Psi(1)\varphi(w)1 = (a + b - 1)\varphi(w)1\) for some factor \(w\) of length \(|w| \geq 2\) and \(w\) a \(\Psi_{a+1}\)-palindrome.

Let us realize that \(\varphi^n(0)1\) is a prefix of \(t_{b,m}\) for any \(n\) and thus \(\varphi^n(0)\varphi(1)\) is its prefix as well. Since \((a + b - 1)\) is the last letter of \(\varphi(a)\), we can deduce that \(\varphi(a)\varphi(w)\varphi(1)\) is a suffix of \(\varphi^n(0)\varphi(1)\) and thus \(aw1\) is a suffix of \(\varphi^{n-1}(0)1\). Moreover, \(aw1\) is a \(\Psi_{a+1}\)-palindrome of length \(\geq 4\). This means that \(\varphi^{n-1}(0)1\) has a \(\Psi\)-palindromic suffix of length greater than 2, where \(\Psi' = \Psi_{a+1} = \Psi_{(n-1)(b-1)+1}\). This is a contradiction with the minimality of \(n\).

Case 2: \(a + b = m 1, b > m\).
Since we denoted \(a = (n - 1)(b - 1)\), we have \(n(b - 1) = m 0\). The smallest \(n\) satisfying this equality was denoted by \(q\).

We can conclude: If \(b \leq m\) then for all \(n\), the longest \(\Psi\)-palindromic suffix of \(\varphi^n(0)1\) has length 2; if \(b > m\) then for all \(n < q\), the longest \(\Psi\)-palindromic suffix \(\varphi^n(0)1\) has length 2; if \(b > m\) and \(n = q\), then the longest \(\Psi\)-palindromic suffix \(\varphi^n(0)1\) has length among numbers 3, 4, \(\ldots\), \(b\).

**Lemma 3.8.** Let \(b \neq m 1\). Denote \(q = \min\{i \in \mathbb{N} : i > 0 \text{ and } i(b - 1) = m 0\}\). Fix \(n \in \mathbb{N}\) and put \(\Psi = \Psi_{(b - 1)n + 1}\).

1. If \(b \leq m\), then \(\left(\varphi^n(0)1\right)_{\Psi} = \varphi^n(0)\varphi^n(1)\).
2. If \(b > m\) and \(n < q\), then \(\left(\varphi^n(0)1\right)_{\Psi} = \varphi^n(0)\varphi^n(1)\).
3. If \(b > m\), then \(\left(\varphi^b(0)1\right)_{\Psi}\) is not a prefix of \(t_{b,m}\).

**Proof.** Let us denote by \(s\) the length of the longest \(\Psi\)-palindromic suffix of \(\varphi^n(0)1\). We will apply Claim 3.7.

If \(s = 2\), we clearly have \(\left(\varphi^n(0)1\right)_{\Psi} = \varphi^n(0)\Psi(\varphi^n(0))\). According to Property \(\Psi\varphi^n = \varphi^n\Psi_1\) and thus \(\Psi(\varphi^n(0)) = \varphi^n(\Psi_1(0)) = \varphi^n(1)\).

Consider now \(s \in \{3, 4, \ldots, b\}\). Then \(\left(\varphi^n(0)1\right)_{\Psi}\) has length \(2|\varphi^n(0)| + 2 - s\). From the form of the substitution \(\varphi\), and the fact that \(t_{b,m}\) is its fixed point, it follows that a jump in \(t_{b,m} = u_0u_1u_2 \cdots\) can occur only on indices \(i - 1 = b - 1\).

Since \(\varphi^n(0)\) is a prefix of \(t_{b,m}\), the prefix of the palindrome \(\left(\varphi^n(0)1\right)_{\Psi}\) with length \(|\varphi^n(0)|\), has jumps on positions \(i - 1 = b - 1\). Jumps in any palindrome occur symmetrically with respect to the center of the palindrome. The length
of the palindrome \( (\varphi^n(0)1)^\Psi \) is \((2 - s) \mod b\). As \(2 - s \neq b\), jumps in the left part of \( (\varphi^n(0)1)^\Psi \) are not compatible with the jump positions in \( t_{b,m} \) and thus \( (\varphi^n(0)1)^\Psi \) cannot be a prefix of \( t_{b,m} \).

Now we are ready to complete the proof of Proposition 3.1 for the non-periodic Thue–Morse words.

**Proof.** From Lemma 3.8, Part 1, we get the first identity in the following list; the others follow from Lemma 3.5:

1) \( (\varphi^n(0)1)^\Psi = \varphi^n(0)\varphi^n(1) \) if \( \Psi = \Psi_{(b-1)n+1} \) and \( b \leq m \);

2) \( (\varphi^n(0)\varphi^n(1)2)^\Psi = \varphi^n(0)\varphi^n(1)\varphi^n(2) \) if \( \Psi = \Psi_{(b-1)n+2} \);

\[ \vdots \]

\[ b-1 \) \( (\varphi^n(0)\varphi^n(1)\ldots\varphi^n(b-2)(b-1))^\Psi = \varphi^n(0)\varphi^n(1)\ldots\varphi^n(b-1) \) if \( \Psi = \Psi_{(b-1)n+b-1} \).

Since \( t_{b,m} = \lim_{n \to \infty} \varphi^n(0) \), this together with the simple fact

\[ \varphi^n(0)\varphi^n(1)\ldots\varphi^n(b-1) = \varphi^n(\varphi(0)) = \varphi^{n+1}(0) \]

finishes the proof of Proposition 3.1. \( \square \)

**Proposition 3.9.** Let \( m, b \in \mathbb{Z} \). If \( b > m \) and \( b \not\equiv 1 \pmod{m} \), then \( t_{b,m} \) is not a generalized pseudostandard word.

**Proof.** First, we show that a pseudopalindromic prefix of \( t_{b,m} \) which is longer than \( b \) is an image of a shorter pseudopalindromic prefix of \( t_{b,m} \).

Since the word \( t_{b,m} = 01 \cdots (b-1)1 \cdots \) has its first jump equal to \( b - 1 \), every its pseudopalindromic prefix \( p \) longer than \( b \) has a jump \(|p| - b\). This implies that \( p = \varphi(p') \) for some prefix \( p' \). Since \( \Psi(p) = p \) for some antimorphism \( \Psi \in I_2(m) \), according to Property 5, we have \( \Psi(\varphi(p')) = \varphi(\Psi'(p')) = p = \varphi(p') \) for some antimorphism \( \Psi' \in I_2(m) \). Since \( \varphi \) is injective, the last equality implies \( \Psi'(p') = p' \), and thus \( p' \) is a \( \Psi' \)-palindromic prefix.

One can see that for all \( n, \varphi^n(0) \) and \( \varphi^n(01) \) are pseudopalindromic prefixes. Next, we show that for each \( n \in \mathbb{N} \), the only palindromic prefix of \( t_{b,m} \) which is longer than \( |\varphi^n(0)| \) and shorter than \( 2|\varphi^n(0)| + 2 \), is the prefix \( \varphi^n(0)\varphi^n(1) \).

This part of the proof will proceed by contradiction: Suppose that \( n \) is the minimal integer for which the claim does not hold. Clearly \( n > 1 \), since the claim can be easily verified for \( n = 1 \). Using the fact that every pseudopalindromic
prefix of $t_{b,m}$ is a $\varphi$-image of a shorter one, we can immediately see that even for $n−1$ the statement does not hold, which is a contradiction with the minimality of $n$.

Since $b > 2$, there is no pseudopalindromic prefix of length $|\varphi^n(0)|−1$. For the lengths of the words $w_i$ from Definition 2.4 we have that $|w_{i+1}| ≤ 2|w_i| + 2$ for all $i$. Therefore, for each $n$, there exists an index $i$ such that $w_i = \varphi^n(0)$ and $w_{i+1} = \varphi^n(0)\varphi^n(1)$. Let $\Psi$ be the antimorphism which fixes $w_i$, i.e., $w_{i+1} = (w_i)\Psi$. The lengths of $w_i$ and $w_{i+1}$ imply that the longest $\Psi$-palindromic suffix of $w_i$ is of length 2.

Since the last letter of $\varphi^n(0)$ is the letter $n(b-1)$, the antimorphism $\Psi$ satisfies $\Psi(1) = n(b-1)$ and thus $\Psi = \Psi_{n(b-1)+1}$.

Set $n = q$ where $q$ is the order of $(b-1)$. It follows from Part 3 of Lemma 3.8 that the $\Psi_{q(b-1)+1}$-palindromic closure of $w_i$ is not a prefix of $t_{b,m}$.

4. Comments and open questions

1. As shown in Proposition 3.1, the word $t_{3,4}$ is a generalized pseudopalindromic word and its directive sequences are $\Delta = 0(12)^{\omega}$ and $\Theta = (\Psi_0\Psi_1\Psi_2\Psi_3)^{\omega}$.

One can easily check that the pairs

$$\Delta = 0(21)^{\omega}, \Theta = (\Psi_1\Psi_2\Psi_3\Psi_0)^{\omega} \quad \text{and} \quad \Delta = 01(12)^{\omega}, \Theta = \Psi_0\Psi_2\Psi_3(\Psi_0\Psi_1\Psi_2\Psi_3)^{\omega}$$

also correspond to the word $t_{3,4}$.

The authors of [5] study this phenomenon for the generalized pseudopalindromic word on the binary alphabet, where $\Delta \in \{0,1\}^\mathbb{N}$ and $\Theta \in \{R,E\}^\mathbb{N}$. They defined the notion of a normalized bisequence and showed (Theorem 27 in [5]) that every pseudostandard word is generated by a unique normalized bisequence. Moreover, for any generalized pseudopalindromic word $u_{\Theta}(\Delta)$, a simple algorithms which transforms the pair $\Delta, \Theta$ into the normalized bisequence is given.

**Question:** Is it possible to generalize the notion of a normalized bisequence for the case of a multi-literal alphabet?

2. It is well known the factor complexity of standard episturmian words is bounded by $(\#A−1)n + 1$. In particular, on binary alphabet these words which are not periodic are precisely standard Sturmian words and their factor complexity is $C(n) = n + 1$.

In [3], the authors conjectured that generalized pseudostandard words on binary alphabet have their factor complexity bounded by $4n + \text{const}$. The factor complexity of binary generalized Thue–Morse words can be found in [20]. The word $t_{2k+1,2}$ is periodic, and thus its factor complexity is bounded by a constant. The word $t_{2k,2}$ is aperiodic and its factor complexity is $\leq 4n$ for any parameter $k$. It means that even $t_{4,2}$ and $t_{6,2}$ (which
are not generalized pseudopalindromic words) have a small complexity. It, of course, does not contradict the conjecture.

The factor complexity of generalized Thue–Morse words on any alphabet is deduced in [19]. If the word $t_{b,m}$ is aperiodic, then

$$(qm - 1)n \leq C(n) \leq qmn,$$

where $q$ is the order of $b - 1$ in the additive group $\mathbb{Z}_m$, i.e. $q$ is the minimal positive integer such that $q(b - 1) = _m 0$.

The factor complexity of any infinite word can be derived from knowledge of its bispecial factors. Each aperiodic standard episturmian word $u$ has a nice structure of its bispecial factors. (A factor $w$ is bispecial if and only if $w$ is a palindromic prefix of $u$.)

**Question:** Is it possible to describe the structure of bispecial factors for a generalized pseudostandard word?

3. It is known [13] that classical standard palindromic words with a periodic directive sequence $\Delta = (\delta_1 \delta_2 \ldots \delta_k)^\omega$ are invariant under a substitution. For example, the Tribonacci word has the directive sequence $\Delta = (012)^\omega$ and simultaneously, it is a fixed point of the substitution $\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$.

Let us denote $s_{b,m}$ the generalized pseudostandard word with

$$\Delta = 0 \left( 12 \ldots (b - 1) \right)^\omega \quad \text{and} \quad \Theta = \left( \Psi_0 \Psi_1 \ldots \Psi_{m-1} \right)^\omega.$$

If $b \leq m$, then $s_{b,m} = t_{b,m}$ and obviously $s_{b,m}$ is invariant under the substitution described in (2).

**Question:** Is the word $s_{b,m}$ a fixed point of a substitution if $b > m$?

Acknowledgements

The second author acknowledges financial support from the Czech Science Foundation grant 13-03538S and the last author acknowledges financial support from the Czech Science Foundation grant 13-35273P.

References

[1] Jean-Paul Allouche and Jeffrey Shallit, *Sums of digits, overlaps, and palindromes*, Discrete Math. Theoret. Comput. Sci. **4** (2000), 1–10.

[2] Peter Baláži, Zuzana Masáková, and Edita Pelantová, *Factor versus palindromic complexity of uniformly recurrent infinite words*, Theoret. Comput. Sci. **380** (2007), no. 3, 266–275.

[3] A. Blondin Massé, S. Brlek, S. Labbé, and L. Vuillon, *Palindromic complexity of codings of rotations*, Theoret. Comput. Sci. **412** (2011), no. 46, 6455–6463.
[4] Alexandre Blondin Massé, Srecko Brlek, Amy Glen, and Sébastien Labbé, *On the critical exponent of generalized Thue-Morse words*, Discrete Math. Theoret. Comput. Sci. 9 (2007), no. 1.

[5] Alexandre Blondin-Massé, Geneviève Paquin, Hugo Tremblay, and Laurent Vuillon, *On generalized pseudostandard words over binary alphabets*, J. Integer Seq. 16 (2013), no. Article 13.2.11.

[6] Srecko Brlek, *Enumeration of factors in the Thue-Morse word*, Discrete Appl. Math. 24 (1989), no. 1-3, 83–96.

[7] Srecko Brlek, Sylvie Hamel, Maurice Nivat, and Christophe Reutenauer, *On the palindromic complexity of infinite words*, Int. J. Found. Comput. Sci. 15 (2004), no. 2, 293–306.

[8] Michelangelo Bucci, Aldo de Luca, Alessandro De Luca, and Luca Q. Zamboni, *On different generalizations of episturmian words*, Theoret. Comput. Sci. 393 (2008), no. 1-3, 23–36.

[9] Michelangelo Bucci and Alessandro De Luca, *On a family of morphic images of Arnoux-Rauzy words*, LATA ’09: Proceedings of the 3rd International Conference on Language and Automata Theory and Applications (Berlin, Heidelberg), Springer-Verlag, 2009, pp. 259–266.

[10] Aldo de Luca, *Sturmian words: structure, combinatorics, and their arithmetics*, Theoret. Comput. Sci. 183 (1997), no. 1, 45 – 82.

[11] Aldo de Luca and Alessandro De Luca, *Pseudopalindrome closure operators in free monoids*, Theoret. Comput. Sci. 362 (2006), no. 1-3, 282–300.

[12] Aldo de Luca and Stefano Varricchio, *Some combinatorial properties of the Thue-Morse sequence and a problem in semigroups*, Theoret. Comput. Sci. 63 (1989), no. 3, 333–348.

[13] Xavier Droubay, Jacques Justin, and Giuseppe Pirillo, *Episturmian words and some constructions of de Luca and Rauzy*, Theoret. Comput. Sci. 255 (2001), no. 1-2, 539–553.

[14] Amy Glen, Jacques Justin, Steve Widmer, and Luca Q. Zamboni, *Palindrome richness*, European J. Combin. 30 (2009), no. 2, 510–531.

[15] Edita Pelantová and Štěpán Starosta, *Languages invariant under more symmetries: overlapping factors versus palindromic richness*, to appear in Discrete Math., preprint available at http://arxiv.org/abs/1103.4051 (2011).

[16] , *Almost rich words as morphic images of rich words*, Int. J. Found. Comput. Sci. 23 (2012), no. 05, 1067–1083.
[17] Günter Rote, *Sequences with subword complexity 2n*, J. Number Th. 46 (1993), 196–213.

[18] Štěpán Starosta, *On theta-palindromic richness*, Theoret. Comput. Sci. 412 (2011), no. 12-14, 1111–1121.

[19] ———, *Generalized Thue-Morse words and palindromic richness*, Kybernetika 48 (2012), no. 3, 361–370.

[20] John Tromp and Jeffrey Shallit, *Subword complexity of a generalized Thue-Morse word*, Inf. Process. Lett. (1995), 313–316.