The Batalin-Vilkovisky formalism with the Virasoro symmetry

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Abstract

We introduce the Virasoro symmetry in the BV formalism and give an explicit construction of the anti-bracket, which is Virasoro invariant. It is shown that the master equation with this anti-bracket has an infinite number of solutions. The base space of the BV formalism is a fermionic version of the Virasoro manifold $Diff(S^1)/S^1$. We discuss also the Ricci tensor of this fermionic manifold.

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There is much interest in the BV formalism going beyond the original purpose to BRST quantize the gauge theory\textsuperscript{[1]}. Its extended viability has been proved by the recent applications to the non-critical string\textsuperscript{[2]} and the string field theory\textsuperscript{[3]}. The geometry of the formalism has been considerably clarified in refs 4 and 5.

Recently it has been discussed \textsuperscript{[6,7]} that the fermionic symplectic structure of the BV formalism can be given by a fermionic Kähler 2-form as a special case. The base space of the BV formalism with this symplectic structure is the so-called fermionic Kähler manifold, which is a fermionic version of the usual Kähler manifold. The fermionic Kähler 2-form has been determined by introducing an isometry\textsuperscript{[6]}. The anti-bracket defined with such 2-form is invariant by an isometry transformation. Some interesting solutions of the master equation have been discussed in this case.

The Virasoro symmetry is an interesting isometry to study in this regard. In this note we show that the BV formalism can incorporate the Virasoro symmetry as well, and construct the anti-bracket which is invariant by the Virasoro transformation. Since we deal with an infinite dimensional algebra, the machinery developed in ref. 6 is not applicable. Some years ago non-linear realization of the Virasoro \textsuperscript{[8]} algebra was studied by using the CCWZ formalism\textsuperscript{[9]}. They have shown that the quotient of the Virasoro group by its one parameter central group is indeed the Kähler manifold, called the Virasoro manifold. (It is commonly denoted by $Diff(S^1)/S^1$.) The idea is that we use the technique in ref. 8 to study the corresponding fermionic Kähler manifold. An explicit construction of this fermionic Virasoro manifold is given. It may be used as a base space of the BV formalism. Then the anti-bracket is Virasoro invariant. We study the master equation of the BV formalism in this case and discover an infinite number of Virasoro invariant solutions.

In ref. 8 they calculated the Ricci tensor of the bosonic Virasoro manifold:

$$R_{\alpha\beta} = -\frac{26}{12}(\alpha^3 - \frac{1}{13}\alpha)\delta_{\alpha+\beta,0}$$

(1)

with $\alpha, \beta = \pm 1, \pm 2, \cdots$. (The meaning of the indices will be clear in the text.) The curious coincidence between this Ricci tensor and the Virasoro anomaly raised vivid interest at that time. It was originally discovered by Bowick and Rajeev\textsuperscript{[10]}. 

Namely they calculated a curvature of the holomorphic vector bundle over the (bosonic) Kähler manifold $Diff(S^1)/S^1$, in which the fibre is either a string Fock space or simply the vacuum of the Fock space. In this note we examine the Ricci tensor (1) for the fermionic Virasoro manifold. We find that it is vanishing.

2. To start with, we shall recall the basic formulae of the symplectic geometry$^{[3,5]}$. Consider a 2D manifold parametrized by coordinates $y^i = (\phi^1, \phi^2, \ldots, \phi^D, \xi^1, \xi^2, \ldots, \xi^D)$ with $\phi$'s and $\xi$'s bosonic and fermionic respectively. Suppose that it has a symplectic structure given by a non-degenerate 2-form

$$\omega = dy^i \wedge dy^j \omega_{ij}, \quad (2)$$

which is fermionic and closed

$$d\omega = 0.$$  

These equations read in components

$$(-)^{ik}\partial_i \omega_{jk} + (-)^{ji}\partial_j \omega_{ki} + (-)^{kj}\partial_k \omega_{ij} = 0, \quad (3)$$

$$\omega_{ij} = -(-)^{ij}\omega_{ji}. \quad (4)$$

Here we have used the short-hand notation for the grassmannian parity of the coordinates $\varepsilon(y^i) = i$ in the sign factor. By this notation we have $\varepsilon(\omega_{ij}) = i + j + 1$. Define the inverse of $\omega_{ij}$ by

$$\omega_{ij} \omega^{jk} = \omega^{kj} \omega_{ji} = \delta_i^k. \quad (5)$$

Then eqs (3) and (4) may be written respectively as

$$(-)^{(i+1)(k+1)}\omega^{il} \partial_l \omega_{jk} + (-)^{(j+1)(i+1)}\omega^{jl} \partial_l \omega^{ki} + (-)^{(k+1)(j+1)}\omega^{kl} \partial_l \omega^{ij} = 0,$$

$$\omega^{ij} = -(-)^{(i+1)(j+1)}\omega^{ji}. \quad (6)$$

With this fermionic symplectic structure the anti-bracket of the BV formalism is given by

$$\{A, B\} = (-)^{i[\varepsilon(A)+1]} \partial_i A \omega^{ij} \partial_j B,$$
We also define a nilpotent second order differential operator by
\[ \Delta \equiv \frac{1}{\rho} (-)^i \partial_i (\rho \omega^{i j} \partial_j). \tag{7} \]
They are related with each other by
\[ \Delta (AB) = \Delta A \cdot B + (-)^{\varepsilon(A)} A \Delta B + (-)^{\varepsilon(A)} \{A, B\}. \tag{8} \]
The operator (7) is nilpotent if
\[ \Delta \left[ \frac{1}{\rho} (-)^i \partial_i (\rho \omega^{ij}) \right] = 0. \tag{9} \]

We may introduce an isometry in the manifold. It is realized by a set of Killing vectors \( V^{Ai}(y) \), \( A = 1, 2, \cdots, N \), which obey the Lie algebra of a group \( G \)
\[ V^{Ai} \partial_i V^{Bj} - V^{Bi} \partial_i V^{A j} = f^{ABC} V^{Cj}, \]
with structure constants \( f^{ABC} \). The grassmannian parities are assigned as \( \varepsilon(V^{Ai}) = i \). Then the fermionic symplectic structure \( \omega_{ij} \) satisfies the Killing condition
\[ \mathcal{L}_{V^A} \omega_{ij} \equiv V^{Ak} \partial_k \omega_{ij} + \partial_i V^{Ak} \omega_{kj} - (-)^{ij} \partial_j V^{Ak} \omega_{ki} = 0. \tag{10} \]
In terms of the inverse \( \omega^{ij} \) this condition becomes
\[ \mathcal{L}_{V^A} \omega^{ij} \equiv V^{Ak} \partial_k \omega^{ij} - \omega^{jk} \partial_k V^{A j} + (-)^{(i+1)(j+1)} \omega^{jk} \partial_k V^{Ai} = 0. \tag{11} \]

We may find an explicit form of \( \omega_{ij} \) as a simultaneous solution of eqs (3) and (10). Owing to the Killing condition (11), the anti-bracket (6) is invariant by the isometry transformations given by the Killing vectors
\[ \delta y^i = \varepsilon^A V^{Ai}, \]
in which \( \varepsilon^A \) are global parameters. In ref. 6 this program has been worked out by extending the isometry of the hermitian symmetric space

3. Assuming that it is infinite dimensional \( (D = \infty) \), we can introduce also the Virasoro symmetry in the manifold. The resulting manifold is a fermionic version
of the Virasoro manifold $\text{Diff}(S^1)/S^1$ discussed in ref. 8. It is convenient to summarize the CCWZ formalism for the bosonic Virasoro manifold, since we will construct the fermionic one on that basis.

Consider the Virasoro algebra

$$[L_a, L_b] = (a - b)L_{a+b}, \quad (12)$$

without the anomaly. The Virasoro manifold $\text{Diff}(S^1)/S^1$ is the quotient of the Virasoro group by its one parameter central group generated by $L_0$. A standard way to parametrize this coset space is to write a Virasoro group element

$$g = \exp(i \sum_{a, \alpha \neq 0} \phi^\alpha L_a \delta^\alpha_a), \quad (13)$$

where $\phi^\mu$ can be used as coordinates of the manifold $\text{Diff}(S^1)/S^1$. The generators $L_a$ satisfy the hermitian condition $L_a^\dagger = L_{-a}$, so that the manifold admits a complex structure as

$$(\phi^\alpha)^* = \phi^{-\alpha}. \quad (14)$$

The Cartan-Maurer 1-form is defined by

$$g^{-1}dg = \sum_a e^a L_a. \quad (15)$$

By exterior differentiation we get the Cartan-Maurer equation

$$de^a = -\frac{1}{2} \sum_{b,c} (b - c) \delta^a_{b+c} e^b e^c. \quad (16)$$

In components it reads

$$\frac{\partial}{\partial \phi^\alpha} e^a_\beta - \frac{\partial}{\partial \phi^\beta} e^a_\alpha = -\sum_{b,c} (b - c) \delta^a_{b+c} e^b_\alpha e^c_\beta. \quad (17)$$

When multiplied from the left by an element of the Virasoro group, the group element (13) transforms as

$$e^i \sum_a e^a L_a \cdot g = \exp(i \sum_{a, \alpha \neq 0} \Phi^\alpha(\phi) L_a \delta^\alpha_a) \cdot h. \quad (17)$$
Here $\epsilon^a$ are global parameters of the transformation and $h$ is the so-called compensator

$$h = e^{i\lambda(\phi)L_0},$$

with an appropriate function $\lambda(\phi)$. This defines non-linear transformation of the coordinates

$$\phi^\alpha \rightarrow \Phi^\alpha(\phi) = \phi^\alpha + \sum_a \epsilon^a R_a^\alpha(\phi) + O((\epsilon^a)^2),$$

in which $R_a$ are the Killing vectors of the manifold. Under this the coefficients $\epsilon^a, (a \neq 0)$, transform as

$$\epsilon^a \rightarrow e^{-i\lambda(\phi)a} \epsilon^a = (1 - i\lambda(\phi)a + O(\lambda^2))\epsilon^a,$$

(no sum over $a$).

The Virasoro manifold $Diff(S^1)/S^1$ has the (bosonic) symplectic structure

$$\Omega_{\alpha\beta} = \sum_a f(a)\epsilon^a\epsilon^{-a},$$

in which

$$f(a) = Aa^3 + Ba,$$

with arbitrary constants $A$ and $B$. Indeed we can check that it satisfies

$$\frac{\partial}{\partial \phi^\alpha}\Omega_{\beta\gamma} + \frac{\partial}{\partial \phi^\beta}\Omega_{\gamma\alpha} + \frac{\partial}{\partial \phi^\gamma}\Omega_{\alpha\beta} = 0,$$

by using the Cartan-Maurer equation (16). Having the complex structure given by eq. (14) the Virasoro manifold $Diff(S^1)/S^1$ is a Kähler manifold.

4. So far we have summarized the CCWZ formalism for the Virasoro manifold $Diff(S^1)/S^1$ [8]. We now consider a new manifold by introducing fermionic coordinates $\xi^\alpha$ corresponding to $\phi^\alpha$, with

$$(\xi^\alpha)^* = \xi^{-\alpha}.$$  

We associate the transformation law

$$\xi^\alpha \rightarrow \xi^\beta \frac{\partial}{\partial \phi^\beta}\Phi^\alpha(\phi).$$
The Killing vectors of the new manifold are given by

$$V_i^a = (R_{a}^{\alpha} \xi_{\beta} \frac{\partial}{\partial \phi^{\beta}} R_{a}^{\alpha})$$

(24)

Consider the following matrix

$$\omega_{ij} = \begin{pmatrix} \omega_{\phi \phi} & \omega_{\phi \xi} \\ \omega_{\xi \phi} & \omega_{\xi \xi} \end{pmatrix}$$

(25)

with eq. (20). First of all the symmetric property (4) is evident. Secondly this $$\omega_{ij}$$ satisfies eq. (3) by means of eq. (22). Finally it satisfies also the Killing condition (10). For instance the Killing condition for the block matrix $$\omega_{\phi \phi}$$ reads

$$V_i^b \partial_i \{ \sum_a f(a) e^a_{[\alpha} \xi_{\beta]} \frac{\partial}{\partial \phi^{\gamma}} R_{b}^{\delta} \cdot e^{a}_{\gamma]} \}$$

$$= -[\frac{\partial}{\partial \phi^{\alpha}} R_{b}^{\delta} \sum_a f(a) e^a_{[\alpha} \xi_{\beta]} \frac{\partial}{\partial \phi^{\gamma}} e^{a}_{\gamma]} + \frac{\partial}{\partial \phi^{\alpha}} \frac{\partial}{\partial \phi^{\gamma}} R_{b}^{\delta} \cdot \xi_{\gamma} R_{a}^{\delta} \cdot e^{a}_{\delta} + \xi_{\gamma} R_{a}^{\delta} \cdot e^{a}_{\delta} - i \lambda a e^a_{\alpha},$$

(26)

The other part of the Killing condition can be easily checked. Thus we have obtained the closed fermionic 2-form (2) with eqs (25). The manifold with the symplectic structure given by this 2-form is a fermionic version of the Virasoro manifold Diff($S^1$)/$S^1$ discussed in ref. 8, called the fermionic
Virasoro manifold. It is a Kähler manifold, since we have the complex structure
by eqs (14) and (23). The matrix (25) can be inverted as

\[
\omega^{ij} = \left( \begin{array}{cc} \omega^{\phi\phi} & \omega^{\phi\xi} \\ \omega^{\xi\phi} & \omega^{\xi\xi} \end{array} \right)
= \left( \begin{array}{cc} 0 & \Omega^{\alpha\beta} \\ \Omega^{\alpha\beta} & \xi^\gamma \frac{\partial}{\partial \phi^\gamma} \Omega^{\alpha\beta} \end{array} \right),
\]

(27)

by eq. (5). Here \(\Omega^{\alpha\beta}\) is the inverse of the block matrix \(\Omega_{\alpha\beta}\):

\[
\Omega_{\alpha\beta} \Omega^{\beta\gamma} = \Omega^{\gamma\beta} \Omega_{\beta\alpha} = \delta_\gamma^\alpha.
\]

(An explicit form of \(\Omega^{\alpha\beta}\) can be found by solving these equations perturbatively
around the origin of the coordinates.) Owing to the Killing condition (11) the
anti-bracket (6) with eq. (27) is invariant by the Virasoro transformations given
by the Killing vectors (24):

\[
\delta y^i = \sum_a \epsilon^a V_a^i.
\]

(28)

5. We may be interested in solving the master equation of the BV formalism

\[
\Delta \Psi = 0,
\]

with \(\Delta\) defined by eq. (7) where the symplectic structure \(\omega^{ij}\) is given by eq. (27).

It can be written as

\[
\Delta_0 \Psi + \{\log \rho, \Psi\} = 0,
\]

in which \(\Delta_0\) is the normal-ordered form of \(\Delta\)

\[
\Delta_0 = (-)^i \omega^{ij} \partial_j \partial_i,
\]

(29)

and the second piece is the anti-bracket defined by eq. (6). By eq. (8) we further
calculate the l.h.s. to find

\[
\frac{1}{\rho} \Delta_0 (\rho \Psi) - \frac{1}{\rho} (\Delta_0 \rho) \Psi = 0.
\]

(30)
The normal-ordered operator $\Delta_0$ with eq. (27) is nilpotent by using

$$\Omega^{\alpha\eta} \frac{\partial}{\partial \phi^\eta} \Omega^{\beta\gamma} + \Omega^{\beta\eta} \frac{\partial}{\partial \phi^\eta} \Omega^{\gamma\alpha} + \Omega^{\gamma\eta} \frac{\partial}{\partial \phi^\eta} \Omega^{\alpha\beta} = 0,$$

which follows from eq. (22). If we require that

$$\Delta_0 \rho = 0,$$

(31)

$\Delta$ is also nilpotent[5]. This is a stronger condition than eq. (9). Then eq. (30) becomes

$$\Delta_0 \Psi_\rho = 0,$$

(32)

with $\Psi_\rho = \rho \Psi$. We find that $\Psi_\rho$ and $\rho$ obey the same master equation with $\Delta_0$.

A special solution to eq. (31) or (32) is given by

$$S_0 = \sum_a \Omega^{\alpha\beta} \xi^\alpha \xi^\beta = \sum_a f(a) e^a e_a \xi^\alpha \xi^\beta,$$

which is invariant by the Virasoro transformations (28). This solution generates infinitely many other solutions such that

$$S = \sum_{\alpha=0}^\infty c_\alpha (S_0)^\alpha,$$

(33)

with arbitrary constants $c_\alpha$.

6. The rest of this note is dedicated to calculation of the Ricci tensor of the fermionic Virasoro manifold. It can be done quite similarly to the bosonic case[8]. First of all we have to make the technique elaborated in ref. 8 applicable for calculating the Ricci tensor of the fermionic Kähler manifold. The fermionic Kähler manifold is a complex supermanifold which can be parametrized by holomorphic supercoordinates

$$z^\mu = (z^\alpha, \zeta^\alpha), \quad \alpha = 1, 2, \cdots, D,$$

and their complex conjugates with $z$’s and $\zeta$’s bosonic and fermionic respectively. It has a symplectic structure given by the closed 2-form

$$\omega = idz^\mu \wedge dz^\nu \gamma_{\mu\nu}.$$
Here $\gamma_{\mu\nu}$ is the fermionic metric of the manifold, i.e., the grassmannian parity is 
$\varepsilon(\gamma_{\mu\nu}) = \varepsilon(\mu) + \varepsilon(\nu) + 1$. This 2-form is a special case of the general one given by 
eq (2). There exist a fermionic Kähler potential such that 
$$
\gamma_{\mu\nu} = \partial_{\mu} \partial_{\nu} K.
$$
The inverse metric $\gamma^{\mu\nu}$ is defined by 
$$
\gamma_{\mu\nu} \gamma^{\nu\eta} = \gamma^{\eta\mu},
$$
and satisfies 
$$
\gamma^{\mu\nu} = (-)^{(\mu+1)(\nu+1)} \gamma_{\mu\nu}.
$$
The affine connections are given by 
$$
\Gamma_{\mu\nu}^\eta = \partial_{\mu} \gamma_{\nu\rho} \cdot \gamma^{\eta\rho}, \quad \Gamma_{\mu\nu}^\eta = \partial_{\nu} \gamma_{\mu\rho} \cdot \gamma^{\rho\eta},
$$
and other components are vanishing. The covariant derivative for vectors is defined by 
$$
D_{\mu} A_{\nu} = \partial_{\mu} A_{\nu} - \Gamma_{\mu\nu}^\rho A_{\rho}, \quad D_{\mu} A_{\nu} = \partial_{\mu} A_{\nu} + A^\rho \Gamma_{\rho\mu}^\nu,
$$
c.c.c.
The curvature tensor is given by 
$$
R_{\nu\mu\sigma}^{\eta} = \partial_{\nu} \Gamma_{\mu\sigma}^\eta, \quad R_{\mu\nu\sigma}^{\eta} = -(-)^{\mu\nu} \partial_{\mu} \Gamma_{\nu\sigma}^\eta,
$$
c.c.c.
Other components are vanishing. We obtain the Ricci tensor as 
$$
R_{\mu\nu} = -(-)^{\mu\nu} R_{\nu\mu} = -(-)^{\mu\nu+\sigma} R_{\nu\mu\sigma}.
$$
The fermionic Kähler manifold can admit an isometry. It is realized by a set of holomorphic Killing vectors $R^{A\mu}(z)$ and their complex conjugates $\overline{R}^{A\mu}(\overline{z})$. They satisfy the Lie algebra 
$$
R^{A\mu} \partial_{\mu} R^{B\nu} - R^{B\mu} \partial_{\mu} R^{A\nu} = f^{ABC} R^{C\nu},
$$
and the Killing condition 
$$
\mathcal{L}^A_{\gamma_{\mu\nu}} \equiv [R^{A\rho} \partial_{\mu} + R^{A\rho} \partial_{\mu}] \gamma_{\mu\nu} + \partial_{\mu} R^{A\rho} \gamma_{\rho\nu} + (-)^{\mu\nu} \partial_{\mu} \overline{R}^{A\rho} \gamma_{\rho\mu} = 0,
$$
or equivalently
\[
\mathcal{L}^A \gamma^\mu\nu \equiv \left[ R^A\partial_\rho + R^A\partial_\rho \right] \gamma^\mu\nu - \gamma^\mu\rho \partial_\rho \overline{R}^A\nu
\]
\[
- (-)^{(\mu+1)(\nu+1)} \gamma^\mu\rho \partial_\rho \overline{R}^A\mu = 0,
\]
due to eq. (34). It is worth checking that the Ricci tensor given by eq. (36) is indeed covariant by the transformations
\[
\delta \mathbf{z}^\alpha = \delta^A R^A\alpha (\mathbf{z}), \quad \text{c.c.,}
\]
with global parameters $\epsilon^A$. Note that the sign factor $(-)^\sigma$ in eq. (36) does a right work for this. A little calculation shows that eq. (38) can be written as
\[
\partial_\mu (R^A\rho \gamma^\mu\nu) + (-)^{\mu\nu} \partial_\mu (\overline{R}^A\rho \gamma_{\mu\nu}) = 0 \quad \text{(Killing equation).}
\]
By multiplying by the Killing vectors, the curvature tensor takes the form
\[
(R^{AB})^\eta_\sigma = (R^{B\mu} \overline{R}^A\nu - R^A\mu \overline{R}^B\nu) R^A\mu\sigma
\]
\[
= (D^A D^B - f^{ABC} D^C)_{\sigma}^\eta,
\]
in which
\[
D^A \equiv R^A\mu D_\mu + \overline{R}^A\mu D^A_\mu.
\]
We consider the difference operator
\[
\varphi^A \equiv \mathcal{L}^A - D^A.
\]
It is important to note that it does not contain any derivative and operates as a matrix on tensors. For instance on a tensor $T_{\mu\nu}$ we have
\[
\varphi^A T_{\mu\nu} \equiv [(R^A\rho \partial_\rho + R^A\rho \partial_\rho) T_{\mu\nu} + \partial_\mu R^A\rho T_{\nu\mu} + (-)^{\mu(\nu+\rho)} \partial_\rho \overline{R}^A\rho T_{\mu\nu}]
\]
\[
- [(R^A\rho \partial_\rho + R^A\rho \partial_\rho) T_{\mu\nu} - R^A\rho T_{\nu\mu}] - (-)^{\mu(\nu+\sigma)} \overline{R}^A\rho \Gamma^{\rho\mu\sigma} T_{\mu\nu}
\]
\[
= (\varphi^A)_{\mu}^\rho T_{\rho\nu} + (-)^{\mu(\nu+\rho)} (\varphi^A)^\rho_\nu T_{\rho\mu},
\]
with
\[
(\varphi^A)^\rho_\mu = - D_\mu R^A\rho, \quad (\varphi^A)^\rho_\nu = - D_\nu \overline{R}^A\rho.
\]
Due to the Killing equation (40) the matrices are related by
\[
(\varphi^A)^\rho_\mu \overline{\gamma}_{\rho\mu} = - (-)^{\mu\nu} (\varphi^A)^\rho_\nu \overline{\gamma}_{\rho\mu},
\]
which will be useful later. The curvature tensor given by (41) can be expressed in terms of the difference operator
\[(R^{AB})_\sigma^\eta = (\varphi^{[A}_\sigma \varphi^{B]} - f^{ABC} \varphi^C)_\sigma^\eta,\] (44)
by using the formulae
\[[L^A, L^B] = f^{ABC} L^C, \quad [L^A, D^B] = f^{ABC} D^C.\]

7. We now come back to the fermionic Virasoro manifold discussed previously. It has been shown that it is a Kähler manifold. However the coordinates \((\phi^\alpha, \xi^\alpha)\) and \((\phi^{-\alpha}, \xi^{-\alpha}), \alpha > 0\), are mixed under the transformation (17), so that they can not be identified with the holomorphic coordinates \((z^\alpha, \zeta^\alpha)\) and \((\bar{z}^\alpha, \bar{\zeta}^\alpha)\) discussed just above. In order to get the holomorphic coordinates we further decompose the group element (13) as
\[g = \exp(i \sum_{a, \alpha \neq 0} \phi^\alpha L_a \delta_a^\alpha) = \exp(i \sum_{a > 0} z^\alpha L_a \delta_a^\alpha) \exp(i \sum_{a > 0} w^\alpha L_{-a} \delta_a^\alpha) \exp(u L_0).\] (45)
The product of the last two exponentials is an element of the subgroup generated by \(L_a, a \leq 0\). By requiring the two expressions of \(g\) to match, \(w^\alpha\) and \(u\) are found as functions of \(z^\alpha\) and \(\bar{z}^\alpha\). They are calculated as power series
\[w^\alpha = \bar{z}^\alpha + \cdots,\]
\[u = \sum_{\alpha > 0} \alpha |z^\alpha|^2 + \cdots.\]
We multiply eq. (45) by a group element as has been done in eq. (17). Then \(z^\alpha\) transforms holomorphically:
\[z^\alpha \longrightarrow \Phi^\alpha(z) = z^\alpha + \sum_a \epsilon^a R^\alpha_a(z) + O((\epsilon^A)^2).\]
The transformation law of \(\bar{z}^\alpha\) is obtained by taking complex conjugation of it. (For details we would like to refer to ref. 8.) Correspondingly to \(z^\alpha\) and \(\bar{z}^\alpha, \alpha > 0\) we introduce fermionic coordinates \(\zeta^\alpha\) and \(\bar{\zeta}^\alpha\) with the transformation law
\[\zeta^\alpha \longrightarrow \zeta^\beta \frac{\partial}{\partial \bar{z}^\beta} \Phi^\alpha(z).\]
The supercoordinates $z^\mu = (z^\alpha, \zeta^\alpha)$ and $\overline{z}^\mu = (\overline{z}^\alpha, \overline{\zeta}^\alpha)$ can be taken as the holomorphic coordinates of the fermionic Virasoro manifold.

In these new coordinates the Killing vectors (24) become holomorphic, i.e.,

$$R^\mu_a = (R'_{a\alpha}(z), \zeta^\beta \frac{\partial}{\partial z^\beta} R'_{a\alpha}(z)), \quad (46)$$

and their complex conjugates

$$(R_a^\mu(z))^* \equiv \overline{R}_{-a}^\mu(\overline{z}). \quad (47)$$

The Lie algebra (37) reads

$$R^\mu_a \partial_\mu R^\nu_b - R^\mu_b \partial_\mu R^\nu_a = -i(a-b)R^\nu_{a+b}. \quad (48)$$

The explicit form of (46) may be found in a power series of $z^\alpha$ and $\overline{z}^\alpha$:

$$R'_{a\alpha} = \delta_{a\alpha} + \frac{i}{2}(2a-\alpha)z^{\alpha-a} + \cdots, \quad (a > 0),$$

$$R'_{0\alpha} = -i\alpha z^\alpha + \cdots, \quad (48)$$

$$R'_{-a\alpha} = -i(2a+\alpha)z^{\alpha+a} + \cdots, \quad (a > 0),$$

where no sum is taken over $a$ and $z^\alpha = 0$ for $\alpha \leq 0$.\[8\]

For the fermionic Virasoro manifold the curvature tensor (44) reads

$$(R_{-a\beta})_{\rho}^\sigma = (\varphi_{-a}\varphi_{b} - \varphi_{b}\varphi_{-a} - i(a+b)\varphi_{-a+b})_{\rho}^\sigma, \quad (a, b, \sigma, \rho > 0). \quad (49)$$

We shall evaluate it at the origin, $z^\alpha = \zeta^\alpha = 0$. This is sufficient to calculate the Ricci tensor (1) for the fermionic Virasoro manifold, since it can be determined everywhere by the isometry of the manifold. (Hereafter all the calculations will be valid in the neighbourhood of the origin.) The difference operator $\varphi_{-a}, a > 0$, can be evaluated by means of (42) with (48):

$$(\varphi_{-a})^\nu_{\mu} = \begin{pmatrix} (\varphi_{a}^z)_{\alpha}^\beta (\varphi_{b}^\zeta)_{\alpha}^\beta \\ (\varphi_{a}^z)_{\alpha}^\beta (\varphi_{b}^\zeta)_{\alpha}^\beta \\ i\delta_{a}^{\alpha+b}(2a+\beta) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ i\delta_{a}^{\alpha+b}(2a+\beta) \end{pmatrix}, (50)$$

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while the difference operator \( \varphi_a, a > 0 \), can be computed by using eqs (43), (47) and the above result:

\[
(\varphi_a)_\mu^\nu = \begin{pmatrix}
if (\beta - a)\delta_\alpha^{\beta-a} \frac{1}{f(\beta)}(a + \beta) & 0 \\
0 & -if (\beta - a)\delta_\alpha^{\beta-a} \frac{1}{f(\beta)}(a + \beta)
\end{pmatrix}.
\]  

(51)

Here we have known the metric \( \gamma_{\mu\nu} \) at the origin as

\[
\gamma_{\mu\nu} = \begin{pmatrix}
0 & f(\alpha)\delta_\alpha^\beta \\
f(\alpha)\delta_\alpha^\beta & 0
\end{pmatrix}
\]

from eq. (25) with eqs (20) and (21). These difference operators are identical with those obtained for the bosonic Virasoro manifold except for the doubling due to the fermionic coordinates. Therefore the Ricci tensor corresponding to eq. (1) can be calculated closely following ref. 8. Namely with eqs (50) and (51) we calculate the r.h.s. of eq. (49). Owing to eqs (41), (46) and (48) the result is identified with the curvature tensor \( R^\sigma_\beta_\alpha_\rho \) where \( \alpha \) and \( \beta \) are bosonic indices. Then we take the trace over \( \rho \) and \( \sigma \). Remarkably the infinite sum converges:

\[
\sum_{\text{bosonic}} R^\sigma_\beta_\alpha_\sigma = \sum_{\text{fermionic}} R^\sigma_\beta_\alpha_\sigma = \frac{26}{12}(\alpha^3 - \frac{1}{13}\alpha)\delta_{\alpha+\beta,0},
\]

as has been shown in ref. 8. Note that it does not depend on the function \( f(\alpha) \) at all. Consequently we obtain the Ricci tensor

\[
R^\alpha_\beta = -R^\beta_\alpha = -\sum_{\text{all}}(-)^\sigma R^\sigma_\beta_\alpha_\sigma = 0
\]

with bosonic indices \( \beta \) and \( \alpha \).

8. In this note we have given an explicit construction of the anti-bracket of the BV formalism. The base space of the BV formalism is the fermionic Virasoro manifold. The Ricci tensor, whose form curiously coincided with the central charge of the Virasoro algebra in the bosonic \( Diff(S^1)/S^1 \), turned out to be vanishing in the fermionic one. We have studied the master equation of the BV formalism and
found an infinite number of Virasoro invariant solutions, eq. (33). The physical meaning of these solutions is that they could be physical states of the topological $\sigma$-model\cite{11} on the (bosonic) $\text{Diff}(S^1)/S^1$ \cite{12}.

Finally we would like to remark that the anti-bracket having the Kac-Moody symmetry can be similarly constructed along the arguments in this note. Non-linear realization of the Kac-Moody algebra is necessary to do this. It has been done in ref. 13.

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