Can nonlinear parametric oscillators solve random Ising models?

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We study large networks of parametric oscillators as heuristic solvers of random Ising models. In these networks, known as coherent Ising machines, the model to be solved is encoded in the dissipative coupling between the oscillators, and a solution is offered by the steady state of the network. This approach relies on the assumption that mode competition steers the network to the ground-state solution. By considering a broad family of frustrated Ising models, we instead show that the most efficient mode generically does not correspond to the ground state of the Ising model.

We infer that networks of parametric oscillators close to threshold are intrinsically not Ising solvers. Nevertheless, the network can find the correct solution if the oscillators are driven sufficiently above threshold, in a regime where nonlinearities play a predominant role. We find that for all probed instances of the model, the network converges to the ground state of the Ising model with a finite probability.

Introduction. Solving large-scale optimization problems has been a quest of uttermost importance during the last decades. In addition to physics, optimization problems are central in several fields of modern science such as finance [1], life science [2], biophysics and bioinformatics [3], and artificial intelligence [4]. Many of these problems belong to the non-deterministic polynomial (NP-hard) complexity class [5]: The computational time required to reach the optimal solution scales exponentially with the size of the problem, making the search for the exact solution often unfeasible using conventional computers, even for problems of realistic sizes.

A viable route to tackle some NP-hard problems is offered by the possibility of mapping them onto classical Ising models [6]. Solving the original NP-hard problem translates into finding the ground-state (GS) configuration of the corresponding Ising Hamiltonian, which is also a NP-hard task [7]. A number of heuristic algorithms have been developed aiming at efficiently finding at least approximate solutions for the Ising model. Notable examples include the Metropolis algorithm [8], simulated annealing [9], and quantum annealing [10–12]. In the last years, networks of coupled parametric oscillators (POs) have emerged as a novel promising heuristic Ising solver [13–18]. This platform, called parametric-oscillator coherent Ising machine (PO-CIM), simulates the dynamics of coupled artificial Ising spins to efficiently find the ground state of the corresponding Ising model, specified by the coupling matrix of the network.

In a PO-CIM, below the oscillation threshold, the POs are in a squeezed vacuum state [19–22]. Above threshold, the POs undergo a series of pitchfork bifurcations [23]: For a given coupling matrix defining the graph of the system, mode competition selects the most efficient mode of the network, i.e., the mode that minimizes the overall loss/gain ratio. In this mode, each oscillator has a binary phase [23, 24]. The assumption behind the working principle of PO-CIMs is that the most efficient mode, in terms of phases, is given by the ground-state configuration of the corresponding Ising model [13–18].

Here, we show that this naive explanation is insufficient to capture the true working principle of PO-CIMs. By considering a paradigmatic family of random graphs, we demonstrate that the most efficient mode of the network at threshold, in general, does not correspond to the Ising solution. This implies that parametric oscillators at threshold do not solve the desired Ising problem. Next, by means of a low-level numerical simulation that closely mimics temporal dynamics within a network of coupled parametric oscillators [25], we show that, if each oscillator is pumped individually, the system flows towards the correct Ising solution for a pump power sufficiently above the oscillation threshold.

Linear system. We open by discussing the behaviour of the parametric oscillator’s network in the linear regime, i.e., when the pump power is set close to the oscillation threshold and nonlinearities are negligible. We will show that a network of \( N \) POs coupled by a coupling matrix \( C \) does not work as an Ising solver close to threshold. To achieve this goal, we denote the steady-state configuration of the network by \( A = (\overline{A}^{(1)}, \ldots, \overline{A}^{(N)}) \), where the real number \( \overline{A}^{(j)} \) is the amplitude of the \( j \)-th oscillator (in the rotating frame of the pump). The overline denotes the steady-state value. We next identify each oscillator with an Ising variable \( \sigma = \text{sgn} \overline{A} \) and check whether this configuration minimizes the Ising energy

\[
E(\{\sigma_j\}) = -\frac{1}{2} \sum_{j,k} C_{jk} \sigma_j \sigma_k.
\]  

Before proceeding, it is important to discuss what are the possible behaviours of the system at threshold.

Recently, it was found that small POs networks at threshold can behave significantly differently from coupled Ising spins. Indeed, the system can show dynamics beyond the Ising picture such as persistent coherent beats [26, 27] or POs that remain in the zero-amplitude
state above threshold [25], where the Ising variable is not defined. These behaviours were ascribed to the fact that the POs network at threshold converges to the amplitude and phase configuration dictated by the eigenvectors of the coupling matrix that correspond to the eigenvalue with maximal real part (denoted respectively most-efficient eigenvectors and eigenvalue hereon).

Persistent coherent beats at threshold are generically observed when the most-efficient eigenvalue of the coupling matrix is degenerate (degenerate graphs). In this case, the steady-state configuration of the system is a linear combination of the most-efficient eigenvectors \( \{ v_{\text{max},m} \} \) of \( \mathbf{A} = \sum_m c_m v_{\text{max},m} \), for some coefficients \( \{ c_m \} \) determined by the initial conditions. If the coupling matrix includes an energy-preserving part, the network periodically explores all the possible phase configurations admitted by the linear combination, and beats are found [25–27]. For purely dissipative coupling, the network converges to a specific Ising configuration \( \sigma = \text{sgn}\mathbf{A} \) (i.e. purely dissipative coupling) that can represent a valid Ising graph. Specifically, we take \( C_{jk} = \pm 1 \) with \( \pm 1 \) randomly chosen with equal probability.

We first analyze the fraction of zero and degenerate graphs, respectively \( F_{\text{deg}} \) and \( F_{\text{zero}} \), as a function of the number of oscillators \( N \). The result is shown in Fig. 1. With our numerical resources, due to the exponentially increasing number of SK graphs as a function of \( N \) (there are \( 2^{N(N-1)/2} \) SK graphs for a given \( N \)), we can systematically analyze all graphs only up to \( N = 7 \). Thus, for \( N > 7 \), we randomly sample \( 10^7 \) graphs and assume this to be a fair sample of the total graph population. As evident from the figure, \( F_{\text{deg}} \) and \( F_{\text{zero}} \) quickly decay to zero, indicating that the probability of finding zero or degenerate graphs becomes vanishingly small as \( N \) is increased. The result in Fig. 1 therefore allows us to focus from now on only on non-degenerate SK graphs.

Next, we analyze if the steady state of the network at threshold is the correct Ising solution. To do so, we proceed as follows: For different values of \( N \), we randomly select a number \( W \) of non-degenerate SK graphs. For each graph, we compute the most-efficient eigenvector \( v_{\text{max}} \) and the corresponding Ising energy \( E(\{ \sigma \}) \) as in Eq. (1) with \( \sigma = \text{sgn}v_{\text{max}} \). Then, we compare the energy computed from \( v_{\text{max}} \) with the ground-state energy \( E_{\text{GS}} \) of the corresponding Ising model. If \( E(\{ \sigma \}) = E_{\text{GS}} \), we conclude that the network at threshold for that graph works as a PO-CIM.

We randomly select \( W = 300 \) non-degenerate SK graphs and compute the success fraction \( F_S(N) \) defined as the fraction of graphs for which the parametric oscillator network at threshold finds a correct Ising solution. To estimate the numerical precision of our algorithm, we repeat the same procedure five times and plot in Fig. 2 the average and standard deviation of the results obtained by this technique. We find that the network at threshold works as a PO-CIM for small network sizes \( N < 6 \), but for larger values of \( N \), the success fraction exponentially decays, as highlighted by the exponential fit in the inset. The data for \( N = 80, 100 \) are absent from the logarithmic

![FIG. 1. Finite-size scaling of degenerate graphs (left panel) and zero graphs (right panel). The statistics was performed by considering the totality of the graphs for \( N \leq 7 \), and by randomly sampling \( 10^7 \) SK graphs for \( N > 7 \). Within our numerical precision, we find that, for \( N > 16 \), the 100% of the SK graphs is non-degenerate.](image1)

![FIG. 2. Finite-size scaling of \( F_S \) obtained by randomly sampling 300 non-degenerate SK graphs. We find \( F_S = 100\% \) for \( N < 6 \), and it exponentially decreases for \( N \geq 6 \), as highlighted by the exponential fit (red solid line) in the inset.](image2)
plot since we find $F_S = 0$ within our numerical precision. The unavoidable conclusion is that the success probability of a POs network near threshold to find the Ising ground of a random large graph is exceedingly small.

**Nonlinear system.** We now extend the previous analysis to the nonlinear regime by computing the success fraction when the pump power is above the oscillation threshold. Our goal is to show that, even if the network fails to behave as PO-CIM at threshold, nonlinear effects can induce the POs network to find correct Ising solutions sufficiently above threshold. We show how this fact critically depends on the form of the nonlinearity.

To analyze the system above threshold, we simulate the dynamics of the POs network by means of the low-level numerical simulation described in [25]. The scheme of the simulation is shown in Fig. 3. At a given round trip $n \in \mathbb{N}$, the signal consists of a $N$-dimensional complex vector $A_n$. At the initial round trip $n = 0$, we assign random initial conditions $A_0$. The signal is injected together with a pump field of power $h$ into a parametric amplifier (PA), which amplifies the real parts of the fields $\text{Re}(A_n)$ and suppresses the imaginary parts $\text{Im}(A_n)$ following the appropriate nonlinear wave equation [28].

The amplified signal is then sent into the coupling device, which is connected to the cavity by two couplers (CC1 and CC2 in Fig. 3) that split the input field according to the transmission and reflection coefficients $T_{c,m}$ and $R_{c,m} = \sqrt{1 - T_{c,m}^2}$, with $m = 1, 2$ for CC1 and CC2, respectively. We take the coupling matrix as $C_{jk} = \pm \alpha$, where $\alpha > 0$ is the strength of the dissipative coupling and the signs are randomly chosen with equal probability. While the value of $\alpha$ is irrelevant for the linear analysis, it affects the behaviour of the system in the nonlinear regime, as shown later on. The coupled field is then re-injected into the cavity, and eventually the signal is sent to an output coupler (OC), with transmission and reflection coefficients $T_{\text{out}}$ and $R_{\text{out}}$, and then measured. The field at the $n+1$ round trip is therefore given by

$$A_{n+1}^{(j)} = R_{\text{out}} \sum_k Q_{jk} PA\{A_{n}^{(k)}\}. \tag{2}$$

Here we define $Q = a\mathbb{1} + b\mathbb{C}$, where $\mathbb{1}$ is the identity matrix, $a = R_{c,1}R_{c,2}$ and $b = T_{c,1}T_{c,2}$, and $PA\{A_{n}^{(j)}\}$ denotes the fields after the parametric amplification.

The dynamics of the POs network above threshold crucially depends on the nonlinearity, encoded in the $PA\{A_{n}^{(j)}\}$ in Eq. (2). In particular, we focus on nonlinearities due to pump depletion, which is the most relevant process in many experimental contexts. We consider two common cases of pump depletion: (i) The case when all POs are pumped by the same pump field:

$$PA\{A_{n}^{(j)}\} = A_{n}^{(j)} + \left[ gh - 2g^2 \sum_k (A_{n}^{(k)})^2 (A_{n}^{(j)})^* \right], \tag{3}$$

where the star denotes complex conjugation, and (ii) the case when each PO is independently driven by its own pump field:

$$PA\{A_{n}^{(j)}\} = A_{n}^{(j)} + \left[ gh - 2g^2 \sum_k (A_{n}^{(k)})^2 (A_{n}^{(j)})^* \right], \tag{4}$$

where $g$ in Eqs. (3) and (4) quantify the amount of pump depletion within the nonlinear medium.

The case of pump depletion described in Eq. (3) can be treated analytically. In the steady state, one has $A_{n+1}^{(j)} = A_{n}^{(j)}$, which from Eq. (2) yields $Q\mathbb{X} = [(1 + gh - 2g^2)|R_{\text{out}}|^{-1}]\mathbb{X}$. This implies that $\mathbb{X}$ is an eigenvector of the coupling matrix, for any pump power $h$. Since the steady state at threshold is given by $\mathbb{v}_\text{max}$ and $Q\mathbb{X}$ does not depend on $h$, the steady-state configuration of the POs network remains proportional to $\mathbb{v}_\text{max}$ for all $h$. The POs network with the nonlinearity in Eq. (3) is therefore not an Ising solver.

The POs network can operate as a heuristic Ising solver if the nonlinearity in Eq. (4) is considered. To analyze the behaviour of the network above threshold, we resort to the numerical simulation of the coherent network dynamics described above. We randomly select $W = 300$ SK graphs and we compute the success fraction $F_S$ for different values of $h$ above the threshold $h_{\text{th}}$. For each graph and $\Delta h$, we repeat the simulation $M = 100$ times assigning different random initial conditions $A_0$ at each repetition. At the end of each repetition $\mu = 1, \ldots, M$, we obtain the Ising energy $E_\mu(\{\sigma_j\})$ computed from Eq. (1) with $\sigma = \text{sgn}\mathbb{X}$, and compare it with the calculated ground-state energy $E_{\text{GS}}$ of the Ising model. Out of $M$ repetitions, the condition $E_\mu(\{\sigma_j\}) = E_{\text{GS}}$ is found $M_{\text{suc}}$ times: If $M_{\text{suc}} > 0$ the network works as a heuristic Ising solver, whereas if $M_{\text{suc}} = 0$ we conclude that within our numerical precision the network fails to behave as a PO-CIM.

The result of the simulation is shown in Fig. 4(a) for a network size $N = 16$. We focus on, $g = 0.09$, $T_{c,1} = T_{c,2} = 0.2$, $T_{\text{out}} = 0.1$, and show $F_S$ as a function of $\Delta h$ for different values of $\alpha$, as in the legends. We
see that the success fraction, starting from the threshold value $F_S \simeq 29\%$ (see also Fig. 2), increases monotonically and approaches $F_S = 100\%$ upon increasing $\Delta h$ [29]. Also, our data show that smaller values of $\alpha$ enhance the convergence of the network to the correct Ising solution, since the network reaches the nonlinear PO-CIM regime for smaller $\Delta h$ with smaller $\alpha$ [27].

In addition to the success fraction $F_S$, a deeper insight on the behaviour of the system is provided by the success probability, quantifying how often the Ising GS is found out of $M$ repetitions, for a given SK graph and system parameters: $P_{\text{succ}}(C) := M_{\text{succ}}/M$. To compute the success probabilities, we perform a statistical average over the scanned SK graphs as follows: We first compute for a given set of parameters the distribution of the success probabilities $H(P_{\text{succ}})$, which quantifies the fraction of SK graphs with success probability $P_{\text{succ}}$. Three prototype cases of $H(P_{\text{succ}})$ for three different values of $\Delta h$ are shown in Fig. 5. We checked that the observed behavior is common to all values of $\alpha$ [30]. At threshold, $H(P_{\text{succ}})$ is bimodal (the SK graphs have either $P_{\text{succ}} = 0$ or $P_{\text{succ}} = 1$) because the POs network deterministically converges to the configuration dictated by the most-efficient eigenvectors of the coupling matrix. Above threshold, $H(P_{\text{succ}})$ starts to be nonzero also for $0 < P_{\text{succ}} < 1$. This implies that, in the nonlinear regime, the POs network finds correct Ising solutions in a nondeterministic manner, highlighting the heuristic nature of PO-CIMs [14–18]. As $\Delta h$ is further increased above threshold, $H(P_{\text{succ}})$ becomes peaked around smaller and smaller values of $P_{\text{succ}}$.

From the distributions $H(P_{\text{succ}})$, we compute the average success probability $\overline{P}_{\text{succ}}$ [30], which is shown in Fig. 4(b). For $\Delta h = 0$, $\overline{P}_{\text{succ}} = F_S$ since $H(P_{\text{succ}})$ is bimodal at threshold. As $\Delta h$ is increased above threshold, $\overline{P}_{\text{succ}}$ increases until a maximum value is reached. By further increasing $\Delta h$, $\overline{P}_{\text{succ}}$ decreases towards zero, due to the fact that $H(P_{\text{succ}})$ becomes more and more peaked around $P_{\text{succ}} = 0$ for pump powers high above threshold. This non-monotonic behaviour of $\overline{P}_{\text{succ}}$ reveals that there exists an optimal value of pump power sufficiently above the oscillation threshold that maximizes the efficiency of the PO-CIM.

Conclusions. To conclude, we reported a detailed analysis of a network of $N$ coupled parametric oscillators (POs), analyzing the regimes of parameters where the network can be used as a coherent Ising machine (PO-CIM). Close to the oscillation threshold, the POs adjust their amplitudes and phases to converge to the minimal-loss configuration of the network, dictated by the eigenvectors of coupling matrix of the system corresponding to the maximal eigenvalue. We considered a paradigmatic family of Ising graphs (the SK model) and showed that the minimal-loss configuration often does not coincide with the ground-state solution of the corresponding Ising model. Instead, if each PO is driven individually, the network finds good Ising solutions with finite success probability in the nonlinear regime of operation, for pump powers sufficiently above the oscillation threshold.

On one hand, our findings show that parametric oscillators networks at threshold are intrinsically not Ising solvers, due to the linearity of the system at threshold. On the other hand, they highlight the role of nonlinear effects in realizing PO-CIMs. Our findings are an important step towards the realization and optimization of networks of coupled POs as heuristic Ising solver. We note that nonlinearities lead to an improvement of the success ratio, analogous to recently proposed feedback mechanisms that force all the POs to have the same amplitude [31, 32]. One intriguing possibility is that the key effect of nonlinearities is the reduction of the amplitude inhomogeneity with respect to the linear solution.

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FIG. 4. (a) Success fraction $F_S$ for $N = 16$ and different values of the coupling strength $\alpha$, as a function of the pump strength above threshold $\Delta h$ for the nonlinearity as in Eq. (4). (b) Average success probability $\overline{P}_{\text{succ}}$ as a function of $\Delta h$ computed from the distributions in Fig. 5.

FIG. 5. Distributions $H(P_{\text{succ}})$ of the success probabilities in three prototype cases, computed using 100 bins, for $\alpha = 0.02$ and for $\Delta h$ as in the legends. At threshold ($\Delta h = 0$) the distribution is bimodal, and it evolves towards a peaked distribution around $P_{\text{succ}} = 0$ by increasing $\Delta h$ [30].
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Supplemental Material for “Can nonlinear parametric oscillators solve random Ising models?”

I. NUMERICAL DATA OF THE SUCCESS PROBABILITY DISTRIBUTIONS

We here show the data of the distributions $H(P_{\text{succ}})$ used to compute the average success probability $\overline{P}_{\text{succ}}$ in Fig. 5 of the main text, for all the values of $\alpha$. As explained in the main text, we use the numerical parameters $N = 16$, $g = 0.09$, $T_{c,1} = T_{c,2} = 0.2$, and $T_{\text{out}} = 0.1$. The statistics is performed over $W = 300$ randomly chosen SK graphs. By repeating the experiment $M = 100$ times for each graph, and by determining the number of repetitions $M_{\text{succ}}$ for which the POs network finds correct Ising solutions, we define the success probability as $P_{\text{succ}} = M_{\text{succ}}/M$. The distribution $H(P_{\text{succ}})$ quantifies the fraction of SK graphs with success probability $P_{\text{succ}}$.

![Distributions $H(P_{\text{succ}})$ for different values of $\Delta h$](image)

FIG. S1. Distributions $H(P_{\text{succ}})$ for $\alpha = 0.02$ and different values of $\Delta h$ as in the legends.

The distributions $H(P_{\text{succ}})$ for $\alpha = 0.02, 0.05, 0.07$ are shown for different values of $\Delta h$ in Figs. S1, S2, and S3. From these data, the average success probability is computed as $\overline{P}_{\text{succ}} = \sum_{P_{\text{succ}}} H(P_{\text{succ}})P_{\text{succ}}$. As evident from the figures, the distributions $H(P_{\text{succ}})$ evolve as a function of $\Delta h$ in a qualitatively similar way for all the three values of $\alpha$: The distribution is bimodal at threshold, i.e., $H(P_{\text{succ}}) \neq 0$ only for $P_{\text{succ}} = 0$ or $P_{\text{succ}} = 1$, and becomes peaked around smaller and smaller values of $P_{\text{succ}}$ as $\Delta h$ increases.
FIG. S2. Distributions $H(P_{\text{succ}})$ for $\alpha = 0.05$ and different values of $\Delta h$ as in the legends.
FIG. S3. Distributions $H(P_{\text{succ}})$ for $\alpha = 0.07$ and different values of $\Delta h$ as in the legends.