Identifying codes of lexicographic product of graphs

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Abstract
Gravier et al. [6] investigated the identifying codes of Cartesian product of two graphs. In this paper we consider the identifying codes of lexicographic product $G[H]$ of a connected graph $G$ and an arbitrary graph $H$, and obtain the minimum cardinality of identifying codes of $G[H]$ in terms of some parameters of $G$ and $H$.

Key words: Identifying code; lexicographic product.

1 Introduction
In this paper, we only consider finite undirected simple graphs with at least two vertices. For a given graph $G$, we often write $V(G)$ for the vertex set of $G$ and $E(G)$ for the edge set of $G$. For any two vertices $u$ and $v$ of $G$, $d_G(u, v)$ denotes the distance between $u$ and $v$ in $G$. Given a vertex $v \in V(G)$, we define $B_G(v) = \{u | u \in V(G), d_G(u, v) \leq 1\}$. A code $C$ is a nonempty set of vertices. For a code $C$, we say that $C$ covers $v$ if $B_G(v) \cap C \neq \emptyset$; We say that $C$ separates two distinct vertices $x$ and $y$ if $B_G(x) \cap C \neq B_G(y) \cap C$. An identifying code of $G$ is a code which covers all the vertices of $G$ and separates any pair of distinct vertices of $G$. If $G$ admits at least one identifying code, we say $G$ is identifiable and denote the minimum cardinality of all identifying codes of $G$ by $I(G)$.

The concept of identifying codes was introduced by Karpovsky et al. [9] to model a fault-detection problem in multiprocessor systems. It was noted in [3, 4] that determining the identifying code with the minimum cardinality in a graph is an NP-complete problem. Many researchers have focused on the study of identifying codes in some restricted classes of graphs, for example, paths [1], cycles [1, 5, 12], and hypercubes [2, 8, 10, 11].

Gravier et al. [6] investigated the identifying codes of Cartesian product of two cliques. In this paper, we consider the identifying codes of lexicographic product $G[H]$ of a connected graph $G$ and an arbitrary graph $H$. In Section 2, we introduce two new families of codes which are closely related to identifying codes, and compute

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the minimum cardinalities of the two codes for paths and cycles, respectively. In Section 3, we give the sufficient and necessary condition when \( G[H] \) is identifiable, and obtain the minimum cardinality of identifying codes of \( G[H] \) in terms of some parameters of \( G \) and \( H \).

# Two new families of codes

For a graph \( H \), let \( C' \subseteq V(H) \) be a code which separates any pair of distinct vertices of \( H \), we use \( I'(H) \) to denote the minimum cardinality of all possible \( C' \); let \( C'' \subseteq V(H) \) be a code which separates any pair of distinct vertices of \( H \) and satisfies \( C'' \not\subseteq B_H(v) \) for every \( v \in V(H) \), we use \( I''(H) \) to denote the minimum cardinality of all possible \( C'' \).

The two parameters \( I'(H) \) and \( I''(H) \) are used to compute the minimum cardinality of identifying codes of \( G[H] \) of graphs \( G \) and \( H \) (see Theorem 3.4). In this section we shall compute the two parameters for paths and cycles, respectively.

Given an integer \( n \geq 3 \), let \( P_n \) be the path of order \( n \) and \( C_n \) be the cycle of order \( n \). Suppose

\[
V(P_n) = \{0, 1, \ldots, n-1\}, \quad E(P_n) = \{ij | j = i + 1, i = 0, \ldots, n-2\};
\]

\[
V(C_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \quad E(C_n) = \{ij | j = i + 1, i \in \mathbb{Z}_n\}.
\]

**Example 1** \( I'(P_3) = 2 \) and \( I''(P_3) \) is not well defined; \( I'(P_4) = 3 \) and \( I''(P_4) = 4 \); \( I'(P_5) = I''(P_5) = 3 \); \( I'(P_6) = 3 \) and \( I''(P_6) = 4 \).

For \( P_4 \), \( \{0, 1, 2\} \) is an identifying code, but \( \{0, 1, 2\} \subseteq B_{P_4}(1) \) and \( \{0, 1, 3\} \) cannot separate 0 and 1. For \( P_5 \), \( \{0, 2, 4\} \) separates any pair of distinct vertices. For \( P_6 \), \( \{1, 2, 3\} \) separates any pair of distinct vertices, but \( \{1, 2, 3\} \subseteq B_{P_6}(2) \).

**Example 2** \( I'(C_4) = 3 \) and \( I''(C_4) = 4 \); \( I'(C_5) = 3 \) and \( I''(C_5) = 4 \); \( I'(C_6) = I''(C_6) = 3 \); \( I'(C_7) = I''(C_7) = 4 \); \( I'(C_9) = I''(C_9) = 6 \); \( I'(C_{11}) = I''(C_{11}) = 6 \).

For \( C_4 \), \( \{0, 1, 2\} \) is an identifying code, but \( \{0, 1, 2\} \subseteq B_{C_4}(1) \). For \( C_5 \), \( \{0, 1, 2\} \) is an identifying code, but \( \{0, 1, 2\} \subseteq B_{C_5}(1) \) and \( \{0, 1, 3\} \) cannot separate 0 and 1. For \( C_6 \), both \( \{3, 4, 5\} \) and \( \{0, 2, 4\} \) separate any pair of distinct vertices. For \( C_7 \), \( \{3, 4, 5, 6\} \) separates any pair of distinct vertices. For \( C_9 \), both \( \{3, 4, 5, 6, 7, 8\} \) and \( \{0, 2, 4, 6, 7, 8\} \) separate any pair of distinct vertices. For \( C_{11} \), \( \{3, 4, 5, 8, 9, 10\} \) separates any pair of distinct vertices.

The minimum cardinality of identifying codes of a path or a cycle was computed in [1 5].

**Proposition 2.1** ([1 5]) (i) For \( n \geq 3 \), \( I(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1 \);

(ii) For \( n \geq 6 \), \( I(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ \frac{n+3}{2}, & n \text{ is odd}. \end{cases} \)

In order to compute the two parameters for paths and cycle, we need the following useful lemma.
Lemma 2.2 Let \( H \) be an identifiable graph.

(i) \( I(H) - 1 \leq I'(H) \leq I(H) \);
(ii) If \( \Delta(H) \leq |V(H)| - 2 \), then \( I(H) - 1 \leq I'(H) \leq I''(H) \leq I(H) + 1 \), where \( \Delta(H) \) is the maximum degree of \( H \).

**Proof.** Let \( C' \) be a code which separates any pair of distinct vertices of \( H \).

(i) Since there exists at most one vertex \( v \) not covered by \( C' \), \( C' \cup \{v\} \) is an identifying code of \( H \).

(ii) Note that there exists at most one vertex \( v \) such that \( C' \subseteq B_H(v) \). Since \( \Delta(H) \leq |V(H)| - 2 \), there exists \( v_0 \in V(H) \setminus B_H(v) \) such that \( C'' = C' \cup \{v_0\} \) is a code which separates any pair of distinct vertices of \( H \) and satisfies \( C'' \subseteq B_H(w) \) for every \( w \in V(H) \). It follows that \( I'(H) \leq I''(H) \leq I'(H) + 1 \). By (i), (ii) holds. \( \square \)

For two integers \( i \leq j \), let \( [i,j] = \{i,i+1,\ldots,j\} \).

**Proposition 2.3** For \( n \geq 7 \), \( I'(P_n) = I''(P_n) = \lfloor \frac{n}{2} \rfloor + 1 \).

**Proof.** By Lemma 2.2, \( I'(P_n) = I(P_n) \) or \( I(P_n) - 1 \). If \( I'(P_n) = I(P_n) - 1 \), then there exists a code \( W' \) of size \( I(P_n) - 1 \) such that \( W' \) separates any pair of distinct vertices of \( P_n \) and \( B_{P_n}(i_0) \cap W' = \emptyset \) for a unique \( i_0 \in [0,n-1] \).

Case 1. \( n \) is odd. Let \( W = ([0,i_0] \cap W') \cup \{i-1|i \in [i_0+1,n-1] \cap W'\} \subseteq [0,n-2] \). Since \( W \) covers all vertices of \( P_{n-1} \), \( W \) is an identifying code of \( P_{n-1} \). By Proposition 2.1

\[
\frac{n+1}{2} = I(P_{n-1}) \leq |W| = |W'| = I(P_n) - 1 = \frac{n-1}{2},
\]

a contradiction.

Case 2. \( n \) is even. By Proposition 2.1 \( |W'| = I(P_n) - 1 = \frac{n}{2} \).

Case 2.1. \( i_0 \neq 0 \) and \( i_0 \neq n-1 \). Then \( i_0 - 1,i_0,i_0 + 1 \notin W' \), and \( i_0 - 2,i_0 - 3,i_0 - 4,i_0 + 2,i_0 + 3,i_0 + 4 \in W' \), so \( 4 \leq i_0 \leq n-5 \). Let \( W = W' \cap [0,i_0-1] \) and \( \overline{W} = \{i - i_0 - 1|i \in W' \cap [i_0 + 1,n-1]\} \). Then \( W \) is an identifying code of \( P_{i_0} \) and \( \overline{W} \) is an identifying code of \( P_{n-i_0-1} \). By Proposition 2.1, we have

\[
\frac{n}{2} = |W'| = |W| + |\overline{W}| \geq I(P_{i_0}) + I(P_{n-i_0-1}) = \left[ \frac{i_0}{2} \right] + 1 + \left[ \frac{n - i_0 - 1}{2} \right] + 1 = \frac{n + 2}{2},
\]

a contradiction.

Case 2.2. \( i_0 = 0 \) or \( n-1 \). Without loss of generality, assume \( i_0 = n-1 \). Then \( n-1,n-2 \notin W' \), and \( n-3,n-4,n-5 \in W' \). We can observe the following results:

\[
|W' \cap [i,i+3]| \geq 2, i \in [0,n-4], \quad (1)
\]

\[
|W' \cap [0,2]| \geq 2, \quad (2)
\]

\[
|W' \cap [0,4]| \geq 3. \quad (3)
\]

Case 2.2.1. \( n = 4k \). By (1) and (2), \( 2k = |W'| \geq 2 \left[ \frac{n-5-3}{4} \right] + 3 + 2 = 2k + 1 \), a contradiction.

Case 2.2.2. \( n = 4k+2 \). By (1) and (3), \( 2k+1 = |W'| \geq 2 \left[ \frac{n-5-5}{4} \right] + 3 + 3 = 2k+2 \), a contradiction.

Therefore, \( I'(P_n) = I(P_n) \). Note that \( I''(P_n) = I'(P_n) \) when \( I'(P_n) \geq 4 \). By Proposition 2.1, the desired result follows. \( \square \)
Proposition 2.4 \( I'(C_n) = I''(C_n) = \begin{cases} \frac{n}{2}, & \text{n is even and } n \geq 8, \\ \frac{n+3}{2}, & \text{n is odd and } n \geq 13. \end{cases} \)

Proof. If \( I'(C_n) < \left\lceil \frac{n}{2} \right\rceil \), then there exists a code \( W' \) such that \(|W'| = I'(C_n) < \left\lceil \frac{n}{2} \right\rceil \) and \( W' \) separates any pair of distinct vertices of \( C_n \). It follows that there exists \( i_0 \) such that \( i_0, i_0 + 1 \notin W' \). Without loss of generality, assume \( n - 1, 0 \notin W' \). Since \( W' \) is also a subset of \( V(P_n) \) and \( B_{C_n}(j) \cap W' = B_{P_n}(j) \cap W' \) for any \( j \in [0, n - 1] \), \( W' \) separates any pair of distinct vertices of \( P_n \). By Proposition 2.3, \( \left\lceil \frac{n}{2} \right\rceil \leq I'(P_n) \leq |W'| < \left\lceil \frac{n}{2} \right\rceil \), a contradiction. Hence \( I'(C_n) \geq \left\lceil \frac{n}{2} \right\rceil \).

Case 1. \( n \) is even and \( n \geq 8 \). By Proposition 2.1 and Lemma 2.2, \( \frac{n}{2} \leq I'(C_n) \leq I(C_n) = \frac{n}{2} \). Hence \( I'(C_n) = \frac{n}{2} \).

Case 2. \( n \) is odd and \( n \geq 13 \). By Proposition 2.1 and Lemma 2.2, \( I'(C_n) = \frac{n+3}{2} \) or \( \frac{n+1}{2} \). If \( I'(C_n) = \frac{n+1}{2} \), then there exists a code \( W' \) of size \( \frac{n+1}{2} \) such that \( W' \) separates any pair of distinct vertices of \( C_n \) and \( B_{C_n}(i_0) \cap W' = \emptyset \) for a unique \( i_0 \in [0, n - 1] \). Without loss of generality, assume \( i_0 = 1 \). Then \( 0, 1, 2 \notin W' \) and \( 3, 4, 5, n - 3, n - 2, n - 1 \in W' \). We can observe the following results:

\[
\begin{align*}
|W' \cap [i, i + 3]| & \geq 2, i \in [6, n - 7], \\
|W' \cap [6, 11]| & \geq 3.
\end{align*}
\]

Case 2.2.1. \( n = 4k + 1 \). By (1), \( 2k + 1 = |W'| \geq 2\left\lceil \frac{n-9}{4} \right\rceil + 6 = 2k + 2 \), a contradiction.

Case 2.2.2. \( n = 4k + 3 \). By (1) and (5), \( 2k + 2 = |W'| \geq 2\left\lceil \frac{n-9}{4} \right\rceil + 6 + 3 = 2k + 3 \), a contradiction.

Therefore, \( I'(C_n) = \frac{n+3}{2} \).

Since \( I''(C_n) = I'(C_n) \) when \( I'(C_n) \geq 4 \), the desired result follows. \( \square \)

3 Main results

The lexicographic product \( G[H] \) of graphs \( G \) and \( H \) is the graph with the vertex set \( V(G) \times V(H) = \{ (u, v) | u \in V(G), v \in V(H) \} \), and the edge set \( \{ (u_1, v_1), (u_2, v_2) \} \) if \( d_G(u_1, u_2) = 1 \), or \( u_1 = u_2 \) and \( d_H(v_1, v_2) = 1 \). For any two distinct vertices \((u_1, v_1), (u_2, v_2)\) of \( G[H] \), we observe that

\[
d_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} 1, & \text{if } u_1 = u_2, d_H(v_1, v_2) = 1, \\
2, & \text{if } u_1 = u_2, d_H(v_1, v_2) \geq 2, \\
& \text{if } u_1 \neq u_2.
\end{cases}
\]

For \( u \in V(G) \), let \( N_G(u) = B_G(u) \setminus \{u\} \). For any \( u_1, u_2 \in V(G) \), define \( u_1 \equiv u_2 \) if and only if \( B_G(u_1) = B_G(u_2) \) or \( N_G(u_1) = N_G(u_2) \). Hernando et al. \[7\] proved that “\( \equiv \)” is an equivalent relation and the equivalence class of a vertex is of three types: a class of size 1, a clique of size at least 2, an independent set of size at least 2. Denote all equivalence classes by

\[
W_1, \ldots, W_p, U_1, \ldots, U_k, V_1, \ldots, V_l,
\]

where
(i) $|W_q| = 1$, $q = 1, \ldots, p$;
(ii) for any $u_1, u_2 \in U_i$, $i = 1, \ldots, k$, $B_G(u_1) = B_G(u_2)$;
(iii) for any $u_1, u_2 \in U_j$, $j = 1, \ldots, l$, $N_G(u_1) = N_G(u_2)$.

Denote $s(G) = |U_1| + \cdots + |U_k| - k$, $t(G) = |V_1| + \cdots + |V_l| - l$. We give an algorithm of computing $s(G)$ and $t(G)$ in Appendix.

For $u \in V(G)$ and $C \subseteq V(H)$, let $C^u = \{(u, v) | (u, v) \in V(GH), v \in C\}$. For $S \subseteq V(GH)$, let $S_u = \{v | v \in V(H), (u, v) \in S\}$. Note that $(S_u)^u = H^u \cap S$, where $H^u = (V(H))^u$. By (8), we have

$$B_{GH}(u, v)) = (B_H(v))^u \cup \bigcup_{w \in N_G(u)} H^w, \quad (8)$$

$$B_{GH}(u, v) \cap S = ((B_H(v)) \cap S_u)^u \cup \bigcup_{w \in N_G(u)} (S_w)^w. \quad (9)$$

In the rest of this section we always assume that $G$ is a connected graph and $H$ is an arbitrary graph.

**Theorem 3.1** The lexicographic product $G[H]$ of graphs $G$ and $H$ is identifiable if and only if

- (i) $H$ is identifiable and $\Delta(H) \leq |V(H)| - 2$, or
- (ii) both $G$ and $H$ are identifiable.

**Proof.** Suppose $G[H]$ is identifiable. If $H$ is not identifiable, then there exist two distinct vertices $v_1, v_2$ of $H$ with $B_H(v_1) = B_H(v_2)$. By (8), $B_{GH}(u, v_1) = B_{GH}(u, v_2)$ for $u \in V(G)$. This contradicts the condition that $G[H]$ is identifiable.

If $\Delta(H) = |V(H)| - 1$ and $G$ is not identifiable, then there exist $v \in V(H)$ and two distinct vertices $u_1, u_2$ of $G$ such that

$$B_H(v) = V(H) \text{ and } B_G(u_1) = B_G(u_2).$$

By (8), we have

$$B_{GH}(u_1, v) = H^{u_1} \cup \bigcup_{u \in N_G(u_1)} H^u = \bigcup_{u \in B_G(u_1)} H^u = \bigcup_{u \in B_G(u_2)} H^u = B_{GH}(u_2, v).$$

This contradicts the condition that $G[H]$ is identifiable.

Therefore, (i) or (ii) holds.

Conversely, suppose (i) or (ii) holds. Assume that $G[H]$ is not identifiable. Therefore, there exist two distinct vertices $(u_1, v_1), (u_2, v_2)$ such that $B_{GH}((u_1, v_1)) = B_{GH}((u_2, v_2))$. If $u_1 \neq u_2$, then $d_G(u_1, u_2) = 1$. It follows that $B_G(u_1) = B_G(u_2)$ and $B_H(v_1) = B_H(v_2) = V(H)$, contrary to (i) and (ii). If $u_1 = u_2$, then $v_1 \neq v_2$. By (8), $B_H(v_1) = B_H(v_2)$, contrary to the condition that $H$ is identifiable. \qed

**Remark.** Let $r$ be a positive integer and $\Gamma$ be a graph. Given a vertex $v \in V(\Gamma)$, define $B_G^{(r)}(v) = \{u | u \in V(\Gamma), d_{\Gamma}(u, v) \leq r\}$. An $r$-identifying code of $\Gamma$ is a code which $r$-covers all the vertices of $\Gamma$ and $r$-separates any pair of distinct vertices of $\Gamma$ (see [9] for details). Identifying codes in this paper are 1-identifying codes. If $r \geq 2$, then $G[H]$ does not admit any $r$-identifying code. Indeed, by (8), $B_{GH}^{(r)}((u, v_1)) = B_{GH}^{(r)}((u, v_2))$ for $r \geq 2$. 

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Lemma 3.2 If $S$ is an identifying code of $G[H]$, then for any vertex $u$ of $G$, $S_u$ separates any pair of distinct vertices of $H$. Moreover, with reference to (7),

(i) if $k \neq 0$, then there exists at most one vertex $u \in U_i$ satisfying $S_u \subseteq B_H(v)$ for a vertex $v$ of $H$, where $i = 1, \ldots, k$;

(ii) if $l \neq 0$, then there exists at most one vertex $u \in V_j$ satisfying $S_u \cap B_H(v) = \emptyset$ for a vertex $v$ of $H$, where $j = 1, \ldots, l$.

Proof. Assume that there exist $u_0 \in V(G)$ and two distinct vertices $v_1, v_2$ of $H$ such that $S_{u_0} \cap B_H(v_1) = S_{u_0} \cap B_H(v_2)$. By (7), $B_G((u_0, v_1)) \cap S = B_G((u_0, v_2)) \cap S$, contrary to the condition that $S$ is an identifying code of $G[H]$.

(i) Assume that there exist two distinct vertices $u_1, u_2 \in U_i$ such that $S_{u_1} \subseteq B_H(v_1)$ and $S_{u_2} \subseteq B_H(v_2)$. Since $B_G(u_1) = B_G(u_2)$, by (7) we have

$$B_G((u_1, v_1)) \cap S = (S_{u_1})^u \cup \bigcup_{u \in B_G(u_1)} (S_u)^u = \bigcup_{u \in B_G(u_2)} (S_u)^u = B_G((u_2, v_2)) \cap S.$$ 

Since $S$ is an identifying code of $G[H]$, $(u_1, v_1) = (u_2, v_2)$, a contradiction.

(ii) Assume that there exist two different vertices $u_1, u_2 \in V_j$ such that $S_{u_1} \cap B_H(v_1) = S_{u_2} \cap B_H(v_2) = \emptyset$. Since $N_G(u_1) = N_G(u_2)$, by (7) we have

$$B_G((u_1, v_1)) \cap S = \bigcup_{u \in N_G(u_1)} (S_u)^u = \bigcup_{u \in N_G(u_2)} (S_u)^u = B_G((u_2, v_2)) \cap S.$$ 

Since $S$ is an identifying code of $G[H]$, $(u_1, v_1) = (u_2, v_2)$, a contradiction. \qed

In equivalence classes (7) of $V(G)$, choose $\overline{u}_i \in U_i, i = 1, \ldots, k$, and $\overline{v}_j \in V_j, j = 1, \ldots, l$. Let $\overline{W}_0 = \bigcup_{q=1}^k \overline{W}_q \cup \{\overline{u}_1, \ldots, \overline{u}_k, \overline{v}_1, \ldots, \overline{v}_l\}$ and $\overline{U}_i = U_i \setminus \{\overline{u}_i\}, i = 1, \ldots, k$, $\overline{V}_j = V_j \setminus \{\overline{v}_j\}, j = 1, \ldots, l$. Therefore, we have a partition of $V(G)$:

$$\overline{W}_0, \overline{U}_1, \ldots, \overline{U}_k, \overline{V}_1, \ldots, \overline{V}_l. \quad (10)$$

Lemma 3.3 Let $C$ be an identifying code of graph $H$, and let $C', C''$ be two codes which separate any pair of distinct vertices of $H$ and $C'' \subset B_H(v)$ for every vertex $v$ of $H$. With reference to (10),

$$S = \bigcup_{u \in \overline{W}_0} (C')^u \cup \bigcup_{i=1}^k \bigcup_{u \in \overline{U}_i} (C'')^u \cup \bigcup_{i=1}^l \bigcup_{u \in \overline{V}_i} C^u$$

is an identifying code of $G[H]$.

Proof. For any $u \in V(G)$, we have

$$S_u = \begin{cases} C', & \text{if } u \in \overline{W}_0, \\ C'', & \text{if } u \in \bigcup_{i=1}^k \overline{U}_i, \\ C, & \text{if } u \in \bigcup_{j=1}^l \overline{V}_j. \end{cases}$$

Since $G$ is connected, there exists a vertex $w$ adjacent to $u$. By (6), $S$ covers all vertices of $G[H]$. For any two distinct vertices $(u_1, v_1), (u_2, v_2) \in V(G[H])$, we only need to show that

$$B_G((u_1, v_1)) \cap S \neq B_G((u_2, v_2)) \cap S. \quad (11)$$
To prove (11), it is sufficient to show that there exists \((u_0, v_0) \in S\) such that
\[
d_{G[H]}((u_0, v_0), (u_1, v_1)) \leq 1, \quad d_{G[H]}((u_0, v_0), (u_2, v_2)) \geq 2 \tag{12}
\]
or
\[
d_{G[H]}((u_0, v_0), (u_2, v_2)) \leq 1, \quad d_{G[H]}((u_0, v_0), (u_1, v_1)) \geq 2. \tag{13}
\]

**Case 1.** \(u_1 \neq u_2\). Then there exists \(u_0 \in V(G) \setminus \{u_1, u_2\}\) such that \(d_G(u_1, u_0) = 1\) and \(d_G(u_2, u_0) \geq 2\), or \(d_G(u_1, u_0) \geq 2\) and \(d_G(u_2, u_0) = 1\). Take \(v_0 \in S_{u_0}\). Then \((u_0, v_0) \in S\). By (9), (12) or (13) holds.

**Case 2.** \(u_1 \equiv u_2\).

**Case 2.1.** \(u_1 = u_2\). Since \(S_{u_1}\) separates \(v_1\) and \(v_2\), \(B_H(v_1) \cap S_{u_1} \neq B_H(v_2) \cap S_{u_1} = B_H(v_2) \cap S_{u_2}\). By (9), (11) holds.

**Case 2.2.** \(u_1 \neq u_2\) and \(B_G(u_1) = B_G(u_2)\). Then \(u_1\) and \(u_2\) are adjacent and fall into some \(U_i\). It follows that \(u_1 \in U_i\) or \(u_2 \in U_i\). Without loss of generality, suppose \(u_1 \in U_i\). Pick \(u_0 = u_1\). Since \(C'' \not\subseteq B_H(v_1)\), there exists \(v_0 \in C''\) such that \((u_0, v_0) \in S\) and \(d_H(v_0, v_1) \geq 2\). By (9), (13) holds.

**Case 2.3.** \(u_1 \neq u_2\) and \(N_G(u_1) = N_G(u_2)\). Then \(u_1\) and \(u_2\) are at distance 2 and fall into some \(V_j\). It follows that \(u_1 \in V_j\) or \(u_2 \in V_j\). Without loss of generality, suppose \(u_1 \in V_j\). Pick \(v_0 = u_1\). Since \(C\) covers \(v_1\), there exists \(v_0 \in C\) such that \((u_0, v_0) \in S\) and \(d_H(v_0, v_1) \leq 1\). By (9), (12) holds.

\[\square\]

**Theorem 3.4** Suppose (i) or (ii) holds in Theorem 3.1.

(i) If \(\Delta(H) \leq |V(H)| - 2\), then
\[
I(G[H]) = (|V(G)| - s(G) - t(G))I'(H) + s(G)I''(H) + t(G)I(H); \tag{14}
\]

(ii) If \(\Delta(H) = |V(H)| - 1\), then
\[
I(G[H]) = (|V(G)| - t(G))I'(H) + t(G)I(H). \tag{15}
\]

**Proof.** (i) By Theorem 3.1, \(I(H)\) and \(I'(H)\) are well defined. Since \(V(H)\) separates any pair of distinct vertices of \(H\) and \(V(H) \not\subseteq B_H(v)\) for every \(v \in V(H)\), \(I''(H)\) is well defined.

Let \(S\) be an identifying code of \(G[H]\) with the minimum cardinality, by Lemma 3.2
\[
I(G[H]) = |S| = \sum_{i=1}^{p} |S_u| + \sum_{i=1}^{k} |S_u| + \sum_{i=1}^{l} |S_v| \geq (p + k + l)I'(H) + (\sum_{i=1}^{k} |V_i| - l)I''(H) + (\sum_{i=1}^{l} |V_i| - l)I(H)
\]

Let \(C\) be an identifying code of \(H\) with the minimum cardinality. Let \(C'\) and \(C''\) be two codes with the minimum cardinality such that they separate any pair of distinct vertices of \(H\) and \(C'' \not\subseteq B_H(v)\) for every vertex \(v\) of \(H\). By Lemma 3.3
\[
I(G[H]) \leq |S| = (|V(G)| - s(G) - t(G))I'(H) + s(G)I''(H) + t(G)I(H).
\]

Therefore, (14) holds.

(ii) By Theorem 3.1 both \(G\) and \(H\) are identifiable. So \(I(H)\) and \(I'(H)\) are well defined. Owing to \(B_G(u_1) \neq B_G(u_2)\) for any two distinct vertices \(u_1, u_2\) of \(G\), we get \(k = 0\) in (7) and (10). Similar to the proof of (i), (15) holds.

Combining Propositions 2.1, 2.3, 2.4 and Theorem 3.4 we have
Corollary 3.5 Let $G$ be a connected graph of order $m$ ($m \geq 2$).

(i) For $n \geq 7$, $I(G[P_n]) = m\left\lfloor \frac{m}{7} \right\rfloor + 1$;

(ii) For $n \geq 12$, $I(G[C_n]) = \begin{cases} \frac{mn}{2}, & n \text{ is even}, \\ \frac{m(n+3)}{2}, & n \text{ is odd}. \end{cases}$

Appendix

| Algorithm |
|-----------|
| **Input** | Graph $G$ |
| **Output** | $W_1, \ldots, W_p, U_1, \ldots, U_k, V_1, \ldots, V_t$ //the equivalent classes of $V(G)$ |
| $s(G), t(G)$ |

**Step 1.** Preparation/Input the adjacent matrix $A$ of $G$ and $A + E$ ($E$ is an identity matrix).

1. $V(G) = \{1, \ldots, m\}; E(G) = \{ij|ij \text{ are adjacent in } G\}$
2. for $i = 1, \ldots, m$ do
3. for $j = 1, \ldots, m$ do
4. if $j = i$ then $a_{ij} := 0$ and $\pi_{ij} := 1$
5. else if $ij \in E$ then $a_{ij} := 1$ and $\pi_{ij} := 1$
6. else $a_{ij} := 0$ and $\pi_{ij} := 0$
7. end-if
8. end-if
9. end-for
10. end-for
11. for $i = 1, \ldots, m$ do
12. $A_i := (a_{i1}, \ldots, a_{im}); \overline{A}_i := (\overline{a}_{i1}, \ldots, \overline{a}_{im})$
13. end-for

**Step 2.** Output the equivalent classes of $V(G)$

14. $i := 1; p := 1; k := 1; l := 1; I := \emptyset$
15. while $i \leq m$ do
16. if $i \in I$ then $i := i + 1 //i \equiv i_0$ for some $i_0 < i$
17. else if $i \leq m - 1$ then $W_p = \{i\}; U_k = \{i\}; V_l = \{i\}$ and do
18. for $j = i + 1, \ldots, m$ do
19. if $\overline{A}_j = \overline{A}_i$, then $I := I \cup \{j\}$ and $U_k := U_k \cup \{j\} //B_G(j) = B_G(i)$
20. else if $A_j = A_i$ then $I := I \cup \{j\}$ and $V_l := V_l \cup \{j\} //N_G(j) = N_G(i)$
21. end-if
22. end-if
23. if $|U_k| > 1$ then output $U_k$ and $k := k + 1$
24. else if $|V_l| > 1$ then output $V_l$ and $l := l + 1$
25. else output $W_p$ and $p := p + 1 //i \not\equiv j$ for any $j \in V(G)$
26. end-if
27. end-if
28. $i := i + 1$
29. else $W_p := \{i\}, i := i + 1$ and output $W_p$
30. end-if
31. end-if
32. end-while

**Step 3.** Compute $s(G)$ and $t(G)$

33. $s = 0; t = 0$
34. if $k > 1$ then
35. for $i = 1, \ldots, k - 1$ do
36. $s := s + |U_i|$
37. end-for
38. if $k > 1$ then
39. for $i = 1, \ldots, k - 1$ do
40. $t := t + |V_i|$
41. end-for
42. output $s(G) = s$ and $t(G) = t$
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