DISTINGUISHING SETS OF STRONG RECURRENCE FROM VAN DER CORPUT SETS

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ABSTRACT

Sets of recurrence, which were introduced by Furstenberg, and van der Corput sets, which were introduced by Kamae and Mendés France, as well as variants thereof, are important classes of sets in Ergodic Theory. In this paper, we construct a set of strong recurrence which is not a van der Corput set. In particular, this shows that the class of enhanced van der Corput sets is a proper subclass of sets of strong recurrence. This answers some questions asked by Bergelson and Lesigne.

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1. Introduction

Throughout this paper, \( \mathbb{N} \) stands for the natural numbers excluding zero, while \( \mathbb{N}_0 \) stands for the natural numbers including zero. In addition, whenever we consider a measure on the torus \( \mathbb{T} \), it is implicit that the underlying \( \sigma \)-algebra is the Borel \( \sigma \)-algebra on \( \mathbb{T} \), so the measure is a Borel measure on \( \mathbb{T} \).

Inspired by the Poincaré Recurrence Theorem, Furstenberg gave the following natural definition of sets of recurrence:

**Definition 1.1.** A set \( D \subseteq \mathbb{N} \) is called a set of recurrence if for any measure preserving system \((X, \mathcal{A}, \mu, T)\) and any \( A \in \mathcal{A} \) with \( \mu(A) > 0 \), there exists \( n \in D \) such that \( \mu(A \cap T^{-n}A) > 0 \).

From the Poincaré Recurrence theorem one derives that \( \mathbb{N} \) is a set of recurrence. In fact, one can easily derive from it that for any \( k \in \mathbb{N} \), the set \( k\mathbb{N} = \{kn : n \in \mathbb{N}\} \) is a set of recurrence. More generally, if \((n_i)_{i \in \mathbb{N}}\) is a strictly increasing sequence of natural numbers, then it is known that the set \( S = \{n_j - n_i : i, j \in \mathbb{N}, i < j\} \) is a set of recurrence. A more sophisticated example is the following (cf. [BL08, Proposition 1.22]): if \( f \) is a (non-zero) polynomial with coefficients in \( \mathbb{N}_0 \) and zero constant term, then the sets \( D_1 = \{f(p - 1) : p \in \mathbb{P}\} \) and \( D_2 = \{f(p + 1) : p \in \mathbb{P}\} \) are sets of recurrence (\( \mathbb{P} \) denotes the set of prime numbers).

The notion of sets of recurrence has deep connections with combinatorics and number theory, as illustrated by Theorem 1.2. To state this theorem, we need the notion of upper density: For a set \( S \subseteq \mathbb{N} \), we define its upper density as \( \overline{d}(S) := \limsup_{n \to \infty} \frac{1}{n} |S \cap \{1, 2, ..., n\}| \).

**Theorem 1.2** (Cf. [BL08, pages 3-4 and Theorem 3.1]). Let \( D \subseteq \mathbb{N} \). Then the following are equivalent:

1. \( D \) is a set of recurrence.
2. \( D \) is intersective, i.e. for any \( S \subseteq \mathbb{N} \) with \( \overline{d}(S) > 0 \), there exist \( x, y \in S \) such that \( x - y \in D \), or equivalently, for any \( S \subseteq \mathbb{N} \) with \( \overline{d}(S) > 0 \), we have \( (S - S) \cap D \neq \emptyset \).
3. For any \( S \subseteq \mathbb{N} \) with \( \overline{d}(S) > 0 \), there exists \( n \in D \) such that \( \overline{d}(S \cap (S - n)) > 0 \).
4. For any sequence \((u_n)_{n \in \mathbb{N}}\) taking values in \( \{0, 1\} \) and satisfying

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+h}u_n = 0 \quad \text{for every } h \in D
\]

we have that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.
\]
Later, Kamae and Mendés France in [KMF78] introduced the notion of van der Corput (vdC) sets in connection with the theory of uniform distribution on $\mathbb{T}$. An equivalent definition of vdC sets, given by Ruzsa in [Ruz84], is the following:

**Definition 1.3.** A set $D \subseteq \mathbb{N}$ is a vdC set if the following holds: If $\sigma$ is a non-negative finite measure on $\mathbb{T}$ such that the Fourier transform of $\sigma$ vanishes on $D$ (i.e. $\hat{\sigma}(h) := \int_{\mathbb{T}} e^{-2\pi i h t} \, d\sigma(t) = 0$ for every $h \in D$), then $\sigma$ is continuous (i.e. $\sigma(\{t\}) = 0$ for every $t \in \mathbb{T}$).

Van der Corput sets also connect with number theory, as illustrated by the following theorem of Bergelson and Lesigne:

**Theorem 1.4** (Cf. [BL08, Definition 2 and Theorem 1.8]). A set $D \subseteq \mathbb{N}$ is a vdC set if and only if the following holds: Whenever a sequence $(u_n)_{n \in \mathbb{N}}$ of complex numbers with modulus 1 satisfies that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+h \overline{u}_n} = 0 \quad \text{for every } h \in D,$$

then we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.$$

Kamae and Mendés France derived from the classical van der Corput Lemma [Cor31] in uniform distribution that $\mathbb{N}$ is a van der Corput set (which is why they gave this name to this class of sets). In fact, in [KMF78] they proved that for any $k \in \mathbb{N}$, the set $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$ is a van der Corput set. In addition, in the same paper they show that if $(n_i)_{i \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the set $S = \{n_j - n_i : i, j \in \mathbb{N}, \ i < j\}$ is a van der Corput set. Moreover, the examples $D_1$ and $D_2$ described above are van der Corput sets (see [BL08, Proposition 1.22]).

One may observe that the examples of van der Corput sets given above are also examples of sets of recurrence. As Kamae and Mendés France showed in [KMF78, Theorem 2], every vdC set is a set of recurrence. In fact, when showing that a given set is a set of recurrence, one often implicitly shows that it also a vdC set. This raises the natural question of whether the opposite is true as well, i.e. is every set of recurrence also a vdC set? It turns out that the answer is negative, as was shown by Bourgain in the following theorem:

**Theorem 1.5** (Cf. [Bou87]). There is a set $R \subseteq \mathbb{N}$ which is a set of recurrence but not a vdC set.

Bourgain’s argument in the proof of Theorem 1.5 is constructive and finitary, and it strongly inspired the work presented in this article.
Following [BL08], a natural strengthening of the notion of sets of recurrence is the following definition of sets of strong recurrence:

**Definition 1.6** (Cf. [BL08, Definition 5]). An infinite set \( D \subseteq \mathbb{N} \) is a set of strong recurrence if, given any measure preserving system \((X, \mathcal{A}, \mu, T)\) and any set \( A \in \mathcal{A} \) with \( \mu(A) > 0 \), we have that

\[
\limsup_{m \to \infty} \mu(A \cap T^{-m}A) > 0.
\]

From the definitions, it is obvious that any set of strong recurrence is also a set of recurrence. Forrest in [For91] gave an example of a set of recurrence which is not a set of strong recurrence, thereby showing that sets of strong recurrence form a proper subclass of the class of sets of recurrence.

In the same way sets of strong recurrence provide a quantitative strengthening of sets of recurrence, following [BL08], we also have the following quantitative strengthening of van der Corput sets:

**Definition 1.7** (Cf. [BL08, Definition 4 and Theorem 2.1]). An infinite set \( D \subseteq \mathbb{N} \) is called an enhanced van der Corput set if it satisfies the following: Whenever \( \sigma \) is a non-negative finite measure on \( \mathbb{T} \) such that

\[
\lim_{d \to \infty} \tilde{\sigma}(d) = 0
\]

then \( \sigma \) is continuous (i.e. \( \sigma(\{t\}) = 0 \) for every \( t \in \mathbb{T} \)).

Again from the definitions, it is clear that every enhanced vdC set is also a vdC set. On the other hand, all the examples of vdC sets given before, serve also as examples of enhanced vdC sets. This gives rise to the interesting question of whether the classes of vdC sets and enhanced vdC sets coincide, or if there is a vdC set which is not an enhanced vdC set.

Parallel to the case of vdC sets and sets of recurrence, it is known that every enhanced vdC set is a set of strong recurrence (see [BL08, Proposition 3.5]). Hence we derive that the examples of vdC sets given before, which as remarked are also enhanced vdC sets, serve as examples of sets of strong recurrence as well. In [BL08], Bergelson and Lesigne asked the following questions:

**Question 1.8.** Is every set of strong recurrence an enhanced vdC set?

**Question 1.9.** Is there any inclusion between the collection of sets of strong recurrence and the collection of vdC sets?

Since Bourgain’s proof of Theorem 1.5 was rather involved, Question 1.8 was commented as “perhaps quite difficult” by Bergelson and Lesigne. In this paper, we establish the following theorem, which addresses these questions:
**Theorem 1.10.** There is a set $R \subseteq \mathbb{N}$ which is a set of strong recurrence but not a van der Corput set.

This theorem (whose proof will be presented in Section 3) provides a negative answer to Question 1.8 (the set $R$ is a set of strong recurrence, but not a vdC set, and thus it is not an enhanced vdC set either). In addition, it partially answers Question 1.9, as it shows that not every set of strong recurrence is a vdC set. It remains an interesting question whether every vdC set is a set of strong recurrence.

The paper is organized as follows. In Section 2, we collect some background from Fourier Analysis that is needed in this paper. Section 3 contains the proof of Theorem 1.10. The proof hinges on a finitistic result, Theorem 3.5, which is then proved in Section 4.

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### 2. Tools from Fourier Analysis

In this section, we collect notation, terminology and results from Fourier Analysis that are needed in later sections. Most of the material is well known and is presented for the convenience of the reader. Recall that $\lambda$ denotes the Lebesgue measure on $\mathbb{T}$. If $f$ is a function in $L^1(\mathbb{T}, \lambda)$, then we denote by $f \, d\lambda$ the measure whose Radon-Nikodym derivative (with respect to $\lambda$) is $f$.

For a function $f \in L^1(\mathbb{T}, \lambda)$, its Fourier transform is $\hat{f} : \mathbb{Z} \to \mathbb{C}$, $\hat{f}(k) = \int_{\mathbb{T}} f(t)e^{-2\piikt} \, d\lambda(t)$. We record the following classical property of the Fourier transform for later use: If $f, g \in L^2(\mathbb{T}, \lambda)$, then $fg \in L^1(\mathbb{T}, \lambda)$ and

$$\text{(1)} \quad \text{for every } k \in \mathbb{Z}, \quad \hat{fg}(k) = \sum_{m \in \mathbb{Z}} \hat{f}(m)\hat{g}(k-m).$$

If $\sigma$ is a bounded complex valued measure on the torus $\mathbb{T}$, then its Fourier transform is the function: $\hat{\sigma} : \mathbb{Z} \to \mathbb{C}$, $\hat{\sigma}(k) = \int_{\mathbb{T}} e^{-2\piikt} \, d\sigma(t)$. It is well known (see for example [Kat04, Section I.7]) that the Fourier transform $\hat{\sigma}$ uniquely determines the measure $\sigma$. 


Recall the notion of weak* convergence of probability measures: If \((\mu_n)_{n \in \mathbb{N}}\), \(\mu\) are probability measures on \(\mathbb{T}\), then \(\mu_n \to \mu\) in the weak* sense as \(n \to \infty\) if for every continuous function \(f\) on \(\mathbb{T}\), we have \(\int_{\mathbb{T}} f \, d\mu_n \to \int_{\mathbb{T}} f \, d\mu\), as \(n \to \infty\).

**Remark 2.1.** For \((\mu_n)_{n \in \mathbb{N}}\), \(\mu\) probability measures on \(\mathbb{T}\), it is known that \(\mu_n \to \mu\) in the weak* sense as \(n \to \infty\) is equivalent to \(\lim_{n \to \infty} \mu_n(k) = \hat{\mu}(k)\) for every \(k \in \mathbb{Z}\).

Let us now recall the definition of the convolution of two measures on \(\mathbb{T}\). If \(\mu, \nu\) are non-negative finite measures on \(\mathbb{T}\), their convolution \(\mu * \nu\) is the non-negative finite measure on \(\mathbb{T}\) characterised by the property that

\[
\int_{\mathbb{T}} f(t) \, d(\mu * \nu)(t) = \int_{\mathbb{T}^2} f(x + t) \, d\mu(x) \, d\nu(t) \quad \text{for every } f \in C(\mathbb{T}).
\]

Identifying a function \(f\) with the measure \(f \, d\lambda\), it makes sense to take the convolution of two functions and the convolution of a function with a measure. The operation of convolution is associative and commutative. Furthermore, for all \(k \in \mathbb{Z}\), the following identity holds

\[
(2) \quad \hat{\mu * \nu}(k) = \hat{\mu}(k) \hat{\nu}(k).
\]

The same identity holds for convolution of functions, as well as for convolution of a function with a measure.

For \(n \in \mathbb{N}\), the classical Dirichlet and Fejer kernels are the functions \(D_n, F_n : \mathbb{T} \to \mathbb{C}\), whose formulas are given by:

\[
D_n(t) = \sum_{k=-n}^{n} e^{2\pi ikt}, \quad F_n(t) = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) e^{2\pi ikt} = \frac{1}{n} \left(\frac{\sin(\pi nt)}{\sin(\pi t)}\right)^2.
\]

It is not difficult to verify that \(0 \leq F_n \leq n\) and \(F_n(t)F_n(nt) = F_{nm}(t)\). A property of the Fejer kernel that we will need is the following:

**Lemma 2.2.** Let \(R\) and \(L\) be positive integers and let \(f(t) = \sum_{|m| \leq RL} a_m e^{2\pi i nt}\) be a trigonometric polynomial. Assume that \(f(t) \geq 0\) for all \(t \in \mathbb{T}\). Then, \(f \leq 4R f * F_L\).

**Proof.** Consider the function \(K := 2F_{2RL} - F_{RL}\) on \(\mathbb{T}\), where \(F_{2RL}, F_{RL}\) denote the respective Fejer kernels. Calculating explicitly we have that \(\hat{K}(m) = 1\) for all \(|m| \leq RL\). Therefore, by equation (2) we have that \(\hat{f}(m) = \hat{f} * \hat{K}(m)\) for all \(m \in \mathbb{Z}\), which implies that \(f = f * K\). Observe that \(K \leq 2F_{2RL}\) and that \(\frac{F_{2RL}(t)}{2R} = \frac{F_{2R}(Lt)}{2R} \leq 1\). Using the previous and the non-negativity of \(f\) we obtain that \(f = f * K \leq f * 2F_{2RL} \leq 4R f * F_L\), as desired. \(\square\)

We will also need the following proposition:
Proposition 2.3. If a function \( f : \mathbb{N}_0 \to \mathbb{R} \) is non-negative, decreasing and convex \(^1\), and satisfies that \( f(\ell) = 0 \) for some \( \ell \in \mathbb{N}_0 \), then the real trigonometric polynomial \( s(t) = \sum_{m \in \mathbb{Z}} f(|m|)e^{2\pi i m t} \) is non-negative.

Proof. For \( \ell = 0 \), \( s(t) = 0 \) for all \( t \in \mathbb{T} \) and the conclusion holds trivially. Assume that the proposition holds for some \( \ell \in \mathbb{N}_0 \), and let \( f : \mathbb{N}_0 \to \mathbb{R} \) be non-negative, decreasing and convex with \( f(\ell + 1) = 0 \). Consider the function \( g : \mathbb{N}_0 \to \mathbb{R} \), \( g(m) = f(m) - f(\ell)(\ell + 1)\widehat{F}_{\ell+1}(m) \), where \( F_{\ell+1} \) denotes the respective Fejer kernel.

Then it is not difficult to show that \( g \) is also non-negative, decreasing, convex and \( g(\ell) = 0 \). Therefore, \( g \) satisfies the conditions of the induction hypothesis, and hence the real trigonometric polynomial \( r(t) = \sum_{m \in \mathbb{Z}} g(|m|)e^{2\pi i m t} \) is non-negative. Recall that \( \widehat{F}_{\ell+1}(t) \geq 0 \) for all \( t \in \mathbb{T} \), so the trigonometric polynomial \( r(t) + f(\ell)(\ell + 1)F_{\ell+1}(t) \) is real and non-negative. Calculating, one sees that \( s(t) = r(t) + f(\ell)(\ell + 1)F_{\ell+1} \), so \( s(t) \) is indeed non-negative and this concludes the proof of the Proposition. \( \square \)

If the Fourier transform of an integrable function \( s : \mathbb{T} \to \mathbb{C} \) satisfies \( \widehat{s}(m) = 0 \) for all \( |m| \geq N \), then \( s \) is a trigonometric polynomial of degree \(< N \) and

\[
\frac{1}{N} \sum_{j=0}^{N-1} s\left(\frac{j}{N}\right) = \widehat{s}(0).
\]

Remark 2.4. A measure \( \mu \) on the torus \( \mathbb{T} \) is said to be supported on the \( N \)-th roots of unity if \( \mu\left( \mathbb{T} \setminus \left\{ \frac{k}{N} : k = 0, 1, \ldots, N-1 \right\} \right) = 0 \). In that case, the Fourier transform \( \widehat{\mu} \) is periodic with period \( N \). Indeed for \( k \in \mathbb{Z} \) we have

\[
\widehat{\mu}(k+N) = \int_{\mathbb{T}} e^{-2\pi i (k+N)t} \, d\mu(t) = \sum_{j=0}^{N-1} e^{-2\pi i (k+N)\frac{j}{N}} \mu\left(\left\{ \frac{j}{N}\right\}\right) = \sum_{j=0}^{N-1} e^{-2\pi i k\frac{j}{N}} \mu\left(\left\{ \frac{j}{N}\right\}\right) = \int_{\mathbb{T}} e^{-2\pi i k t} \, d\mu(t) = \widehat{\mu}(k).
\]

3. A Set of Strong Recurrence that is not a van der Corput Set

In this section, we prove Theorem 1.10 by constructing a set \( R \) which is a set of strong recurrence, but not a vdC set. This construction hinges on a finitistic result, Theorem 3.5, which is proved in Section 4. The proof of Theorem 3.5 is the most technical part of this paper and it is heavily inspired by the work of Bourgain in [Bou87]. Before we can state Theorem 3.5 and prove Theorem 1.10, we need some definitions and lemmas.

\(^1\)The definition of convexity is that \( f(b) \leq \frac{b-a}{c-a} f(c) + \frac{c-b}{c-a} f(a) \), whenever \( a \leq b \leq c \).
From now on, given a positive integer \( n \), we denote the set \( \{0, 1, \ldots, n-1\} \) by \([n]\).

**Definition 3.1.** Let \( \epsilon > 0 \) and let \( R \subseteq \mathbb{N} \). We say that \( R \) is a set of \( \epsilon \)-recurrence if there exists some \( n \in \mathbb{N} \) such that for all sets \( E \subseteq [n] \) with \( |E| > \epsilon n \), there is some \( r \in R \) such that \( E \cap (E - r) \neq \emptyset \).

**Definition 3.2.** Let \( \epsilon > 0 \) and let \( R \subseteq \mathbb{N} \). We say that \( R \) is an \( \epsilon \)-vdC set if whenever a probability measure \( \mu \) on \( \mathbb{T} \) satisfies that \( \hat{\mu}(r) = 0 \) for every \( r \in R \), then \( \mu([0]) \leq \epsilon \).

**Example 3.3.** For \( 0 < \epsilon < 1 \), take an \( N \in \mathbb{N} \) with \( N > \frac{1}{\epsilon} \) and consider the set \( R := \{1, \ldots, N-1\} \). Then, it is not difficult to show that if \( E \) is a subset of \([N] = \{0, 1, \ldots, N-1\} \) with \( |E| > \epsilon N \), then there is some \( r \in R \) such that \( E \cap (E - r) \neq \emptyset \). Hence, \( R \) is a set of \( \epsilon \)-recurrence. On the other hand, if \( \mu \) is a probability measure on \( \mathbb{T} \) such that \( \hat{\mu}(r) = 0 \) for every \( r \in \{1, \ldots, N-1\} \), then using the fact that the Fejer kernel is non-negative, we obtain that

\[
N \mu([0]) = F_N(0) \mu([0]) \leq \int_{\mathbb{T}} F_N(t) \, d\mu(t) = \hat{\mu}(0) = 1,
\]

which implies that \( \mu([0]) \leq \frac{1}{N} < \epsilon \). Therefore, the set \( R \) is also an \( \epsilon \)-vdC set.

Example 3.3 shows that while there are no finite sets of recurrence or finite vdC sets, for every \( \epsilon > 0 \) there are finite sets of \( \epsilon \)-recurrence and finite \( \epsilon \)-vdC sets.

**Proposition 3.4.** If \( R \) is a set of \( \epsilon \)-recurrence, then there exist arbitrarily large \( n \in \mathbb{N} \) such that for all sets \( E \subseteq [n] \) with \( |E| > \epsilon n \), there is some \( r \in R \) such that \( E \cap (E - r) \neq \emptyset \).

**Proof.** \( R \) is a set of \( \epsilon \)-recurrence, and hence there exists an \( m \in \mathbb{N} \) such that whenever \( E \subseteq [m] \) satisfies that \( |E| > \epsilon m \), then there is some \( r \in R \) such that \( E \cap (E - r) \neq \emptyset \).

For \( k \in \mathbb{N} \) consider the natural number \( km \). Let \( E \subseteq [km] \) with \( |E| > \epsilon km \). For every \( j \in \{0, 1, \ldots, k-1\} \), define \( E_j := E \cap [jm, (j+1)m-1] \). Then, \( E = \bigcup_{j=0}^{k-1} E_j \) and the union is disjoint, which implies that \( |E| = \sum_{j=0}^{k-1} |E_j| \). Since \( |E| > \epsilon km \), there is some \( j \in \{0, 1, \ldots, k-1\} \) such that \( |E_j| > \epsilon m \). For this \( j \), let

\[
H = E_j - jm = E \cap [jm, (j+1)m-1] - jm.
\]

Then \( H \subseteq [m] \) and \( |H| = |E_j| > \epsilon m \). Therefore, by assumption on \( m \), there exists some \( r \in R \) such that \( H \cap (H - r) \neq \emptyset \). Take \( h \in H \cap (H - r) \) and note that \( h, h+r \in H \).

Now, let \( \ell = h + jm \). Then \( \ell, \ell + r \in E_j \), and therefore \( \ell \in E_j \cap (E_j - r) \). As a result, \( E \cap (E - r) \supseteq E_j \cap (E_j - r) \neq \emptyset \), and this concludes the proof of the proposition. \( \square \)

The following theorem is the finitistic analog of Theorem 1.10:
Theorem 3.5. For every \( j \in \mathbb{N} \) and for every \( \epsilon \in (0, \frac{1}{3}) \) there is a finite set \( R_j \subseteq \mathbb{N} \) which is a set of \( (\frac{1}{4}) \)-recurrence but not an \( \epsilon \)-vdC set.

Note that if \( \mu \) is a probability measure on \( \mathbb{T} \) and \( \mu(\{0\}) > \frac{1}{2} \), then for every \( n \in \mathbb{N} \) we have

\[
|\widehat{\mu}(n)| = \left| \int_{\mathbb{T}} e^{-2\pi int} d\mu(t) \right| \geq \mu(\{0\}) - \left| \int_{\mathbb{T}\backslash\{0\}} e^{-2\pi int} d\mu(t) \right| > 0.
\]

Therefore, for \( \epsilon \in [\frac{1}{2}, 1] \), every non-empty subset of \( \mathbb{N} \) is an \( \epsilon \)-vdC set. In this sense, Theorem 3.5 cannot be improved.

Theorem 3.5 will be proved in the next section. For now, using it, we will prove the following theorem.

Theorem 3.6. For every \( \epsilon \in (0, \frac{1}{3}) \), there is a set \( R \subseteq \mathbb{N} \) which is a set of strong recurrence but not an \( \epsilon \)-vdC set.

From the definitions, it is clear that since \( R \) is not an \( \epsilon \)-vdC set, then it is also not a vdC set. Therefore, Theorem 3.6 implies Theorem 1.10.

Proof of Theorem 3.6. Let \( \epsilon \in (0, \frac{1}{3}) \). Take an \( \epsilon' \in (0, \frac{1}{2}) \) such that \( \frac{\epsilon'}{1 - \epsilon'} > \epsilon \). For every \( i \in \mathbb{N} \), we use Theorem 3.5 to find a finite set \( Q_i \) which is a set of \( (\frac{1}{4}) \)-recurrence but not an \( \epsilon' \)-vdC set. Then there exists a probability measure \( \beta_i \) on \( \mathbb{T} \) such that \( \widehat{\beta_i}(q) = 0 \) for every \( q \in Q_i \) and \( \beta_i(\{0\}) > \epsilon' \).

Since \( Q_i \) is a set of \( (\frac{1}{4}) \)-recurrence, there is a \( \kappa_i \in \mathbb{N} \) satisfying that for all \( E \subseteq [\kappa_i] \) with \( |E| > \frac{\epsilon}{4} \), there is some \( q \in Q_i \) such that \( E \cap (E - q) \neq \emptyset \). \( Q_i \) is finite, so we may assume that \( Q_i \subseteq \{1, \ldots, \kappa_i\} \) (for otherwise, we can choose a larger \( \kappa_i \) using Proposition 3.4).

Let \( (R_j)_{j \in \mathbb{N}} \) be a sequence of subsets of \( \mathbb{N} \) having the following properties:

- For every \( j \in \mathbb{N} \) there exists an \( i(j) \in \mathbb{N} \) such that \( R_j = Q_{i(j)} \).
- For every \( i \in \mathbb{N} \) there exist infinitely many \( j \in \mathbb{N} \) such that \( Q_i = R_j \).

An example of a sequence with those properties is the one defined by \( R_1 = Q_1 \), \( R_2 = Q_1 \), \( R_3 = Q_2 \), \( R_4 = Q_1 \), \( R_5 = Q_2 \), \( R_6 = Q_3 \) and so on, i.e. the first term of \( (R_j)_{j \in \mathbb{N}} \) is \( Q_1 \), then the next two terms are \( Q_1, Q_2 \), then the next three terms are \( Q_1, Q_2, Q_3 \) and we continue in the same fashion.

For \( j \in \mathbb{N} \), and for \( i := i(j) \), let \( n_j = \kappa_i \) and consider the measures

\[
\mu_j = \beta_i \quad \text{and} \quad \nu_j = \mu_j - \epsilon' \delta_0,
\]

where \( \delta_0 \) denotes the Dirac mass at \( 0 \in \mathbb{T} \). Then \( (\mu_j)_{j \in \mathbb{N}} \) is a sequence of probability measures on \( \mathbb{T} \) satisfying that \( \widehat{\mu_j}(\ell) = 0 \) for every \( \ell \in R_j \). In addition, \( \mu_j(\{0\}) > \epsilon' \), and therefore \( (\nu_j)_{j \in \mathbb{N}} \) is a sequence of non-negative measures on \( \mathbb{T} \).
For every $j \in \mathbb{N}$, let $M_j \in \mathbb{N}$ with $M_j > \frac{1}{2} n_j (n_j + 1)$. We will construct a positive trigonometric polynomial $b_j$ having the following properties:

- $\hat{b}_j(m) = 0$, for $m \in \mathbb{Z}$ with $|m| > M_j$
- $\hat{b}_j(m) = \hat{v}_j(m) = \mu_j(m) - \epsilon'$, for $m \in \mathbb{Z}$ with $0 < |m| \leq n_j$
- $\hat{b}_j(0) = 1$.

To do this, first consider the real trigonometric polynomial whose Fourier transform is given by

$$
\hat{a}_j(m) = \begin{cases} 
0, & \text{if } |m| > n_j \\
\hat{\nu}_j(m)\frac{|m|}{M_j}, & \text{if } |m| \leq n_j.
\end{cases}
$$

Note that $\hat{a}_j(-m) = \overline{\hat{a}_j(m)}$, and hence $a_j$ is indeed real-valued. In addition for every $m \in \mathbb{Z}$, $|\hat{\nu}_j(m)| = \left| \int_T e^{-2\pi imt} \, d\nu_j(t) \right| \leq \nu_j(T) = 1 - \epsilon'$, and therefore

$$
||a_j||_{\infty} = \sup_{t \in \mathbb{T}} |a_j(t)| \leq \sum_{m=-n_j}^{n_j} |\hat{a}_j(m)| = \sum_{m=-n_j}^{n_j} |\hat{\nu}_j(m)| \frac{|m|}{M_j} < \sum_{m=-n_j}^{n_j} \frac{|m|}{M_j} < \epsilon'.
$$

This implies that $a_j + \epsilon'$ is a positive trigonometric polynomial. Note that also $F_{M_j} \ast \nu_j$ is a non-negative trigonometric polynomial, and hence the sum $b_j := a_j + \epsilon' + F_{M_j} \ast \nu_j$ is a positive trigonometric polynomial. Calculating, one sees that the Fourier transform of $b_j$ has the desired properties.

Now, inductively we will define two new sequences $(c_j)_{j \in \mathbb{N}}$, $(d_j)_{j \in \mathbb{N}}$ of trigonometric polynomials and a sequence $(N_j)_{j \in \mathbb{N}}$ of natural numbers. Let $c_1 := b_1$ and $N_1 := M_1$. For each $j \geq 2$ take $N_j \in \mathbb{N}$ such that $N_j > 2(M_{j-1} + 1)N_{j-1}$ and define

$$d_j, \ c_j : \mathbb{T} \to \mathbb{R}, \ d_j(t) = b_j(2N_j t) \text{ and } c_j(t) = c_{j-1}(t)d_j(t).$$

From the definition it is clear that for every $j \in \mathbb{N}$, $c_j$ and $d_j$ are positive trigonometric polynomials. Writing $b_j$ as a linear combination of characters we see that

$$
\hat{b}_j(m) = \hat{d}_j(2N_j m) \text{ for every } m \in \mathbb{Z} \text{ and } \hat{d}_j(m) = 0 \text{ when } m \text{ is not divisible by } 2N_j.
$$

Claim 1. For every $j \in \mathbb{N}$, the Fourier transform $\hat{c}_j$ satisfies the following:

- $\hat{c}_j(m) = 0$ for $m \in \mathbb{Z}$ with $|m| \geq N_{j+1}$
- $\hat{c}_j(m) = \hat{\nu}_{j+1}(m)$ for $m \in \mathbb{Z}$ with $|m| < N_{j+1}$
- $\hat{c}_j(0) = 1$ for every $j \in \mathbb{N}$.
- $\hat{c}_j(2N_j m) = -\epsilon'$ for every $m \in R_j$. 

The proof of Claim 1 will be presented after we finish the proof of the theorem.

Recall that \( \lambda \) denotes the Lebesgue measure on \( \mathbb{T} \). For every \( j \in \mathbb{N} \), \( c_j \) is a non-negative Borel measurable function on \( \mathbb{T} \) and \( \int_{\mathbb{T}} c_j(t) \, d\lambda(t) = \hat{c}_j(0) = 1 \). Consider the sequence of probability measures \( \sigma_j = c_j \, d\lambda, \) \( j \in \mathbb{N} \), on \( \mathbb{T} \). Since the space of probability measures on \( \mathbb{T} \) is weak* compact, we can consider a weak* limit point \( \sigma \) of \( (\sigma_j)_{j \in \mathbb{N}} \). Let \( \rho := \frac{1}{1+\epsilon} \sigma + \frac{\epsilon}{1+\epsilon} \delta_0 \). It is clear that \( \sigma, \rho \) are probability measures on \( \mathbb{T} \).

Consider the set \( R := \{ r \in \mathbb{N} : \hat{\rho}(r) = 0 \} \). Since \( \rho(\{0\}) \geq \frac{\epsilon'}{1+\epsilon'} > \epsilon \), \( R \) is not an \( \epsilon \)-vdC set. We claim that the set \( R \) contains the set \( 2N_j \) for all \( j \in \mathbb{N} \).

Since \( \sigma \) is a weak* limit point of \( (\sigma_\ell)_{\ell \in \mathbb{N}} \), we may assume that \( \sigma_\ell \to \sigma \) in the weak* sense as \( \ell \to \infty \) (for otherwise, we can pass to a subsequence). Fix a \( j \in \mathbb{N} \) and let \( m \in R_j \subseteq \{1, ..., n_j\} \). For every \( \ell > j \) we have \( 2N_j m \leq 2N_j n_j < N_\ell \) and therefore \( \hat{c}_\ell(2N_j m) = \hat{c}_{\ell-1}(2N_j m) = ... = \hat{c}_j(2N_j m) \).

Combining that with Remark 2.1, we obtain that

\[
\hat{\sigma}(2N_j m) = \lim_{\ell \to \infty} \hat{\sigma}_\ell(2N_j m) = \hat{c}_j(2N_j m) = -\epsilon'.
\]

As a result, \( \hat{\rho}(2N_j m) = \frac{1}{1+\epsilon'} \hat{\sigma}(2N_j m) + \frac{\epsilon'}{1+\epsilon'} 0 = 0 \), which in turn implies that \( 2N_j m \in R \).

We will now show that the union \( \bigcup_{j \in \mathbb{N}} 2N_j \) is a set of strong recurrence. Since \( R \) contains this union, it will follow that \( R \) is a set of strong recurrence as well. Let \( (X, A, \mu, S) \) be a measure preserving system and let \( A \subseteq A \) with \( \mu(A) > 0 \). Take an \( i \in \mathbb{N} \) such that \( i > \frac{2}{\mu(A)} \). From the construction of the sequence \( (R_j)_{j \in \mathbb{N}} \), there exist infinitely many \( j \)'s such that \( R_j = Q_i \) and \( n_j = \kappa_i \).

Fix such a \( j \) and consider the measure-preserving transformation \( T := S^{2N_j} \).

For every \( x \in X \) define the set \( E(x) := \{ i \in [\kappa_i] : T^n x \in A \} \) and consider the function

\[
f : X \to \mathbb{R}_{\geq 0}, \quad f(x) = \frac{1}{\kappa_i} |E(x)| = \frac{1}{\kappa_i} \sum_{n=0}^{\kappa_i - 1} \mathbb{1}_A \circ T^n (x).
\]

Since \( T \) preserves \( \mu \), we have \( \int_X f(x) \, d\mu(x) = \mu(A) > \frac{2}{i} \). Using this it is not difficult to derive that

\[
\mu(\{ x : |E(x)| \geq \frac{\kappa_i}{i} \}) = \mu(\{ x : f(x) \geq \frac{1}{i} \}) > \frac{1}{i}.
\]

Note that for every \( x \in X \), \( E(x) \) is a subset of \( \{0, 1, ..., \kappa_i - 1\} \), so there are \( 2^{\kappa_i} \) possibilities for \( E(x) \). Combining this with (6), we obtain a set \( E \subseteq \{0, 1, ..., \kappa_i - 1\} \) such that

\[
|E| \geq \frac{\kappa_i}{i} \quad \text{and} \quad \mu(\{ x : E(x) = E \}) > \frac{1}{i^{2\kappa_i}}.
\]

Let \( C := \{ x : E(x) = E \} \). Recall that \( Q_i \) is a set of \( (\frac{1}{i}) \)-recurrence, and since \( |E| \geq \frac{\kappa_i}{i} \), there exists some \( r \in Q_i (= R_j) \) such that \( E \cap (E - r) \neq \emptyset \). Take an \( n \in E \cap (E - r) \). Then \( n, n+r \in E \), and
therefore for every \( x \in C \) we have \( T^n x, T^{n+r} x \in A \), which implies that \( C \subseteq T^{-n}(A \cap T^{-r}A) \). As a result,
\[
\frac{1}{2^{n+1}} < \mu(C) \leq \mu(T^{-n}(A \cap T^{-r}A)) = \mu(A \cap T^{-r}A) = \mu(A \cap S^{-2N_j r} A),
\]
and \( r \in Q_i = R_j \), whence \( 2N_j r \in 2N_j R_j \subseteq R \).

So finally, we proved the following: For every \( j \in \mathbb{N} \) satisfying that \( R_j = Q_i \), there exists some \( r \in R_j \) such that
\[
\mu(A \cap S^{-2N_j r} A) > \frac{1}{2^{n+1}}.
\]
Since there are infinitely many \( j \)'s satisfying that \( R_j = Q_i \), and since \( 2N_j R_j \subseteq R \) and \( N_j \to \infty \) as \( j \to \infty \), we obtain that
\[
\limsup_{r \to +\infty} \mu(A \cap S^{-r} A) \geq \frac{1}{2^{n+1}} > 0.
\]

The measure preserving system \((X, \mathcal{A}, \mu, S)\) and the set \( A \in \mathcal{A} \) were arbitrarily chosen, and hence \( R \) is a set of strong recurrence. This concludes the proof of the theorem. \( \square \)

Now, we will present the proof of Claim 1:

**Proof of Claim 1.** First, by induction in \( j \), we will show that

\[
(7) \quad \text{if } |m| \geq N_{j+1}, \text{ then } \hat{c}_j(m) = 0.
\]

Let \( m \in \mathbb{Z} \) with \( |m| \geq N_2 \). Then, since \( c_1 = b_1 \) and \( N_2 > M_1 \), we have that \( \hat{c}_1(m) = \hat{b}_1(m) = 0 \). Assume that (7) holds for some \( j \) and let \( m \in \mathbb{Z} \) with \( |m| \geq N_{j+2} \). Then using (1) and the induction hypothesis we obtain that

\[
\hat{c}_{j+1}(m) = c_j d_{j+1}(m) = \sum_{|k| < N_{j+1}} \hat{c}_j(k) d_{j+1}(m - k).
\]

Observe that \( b_{j+1} \) is a trigonometric polynomial with ‘highest power term’ at most \( M_{j+1} \) (this means that \( \hat{b}_{j+1}(\ell) = 0 \) for every \( \ell \in \mathbb{Z} \) with \( |\ell| > M_{j+1} \)). Since \( d_{j+1}(t) = b_{j+1}(2N_{j+1} t) \), we have that \( d_{j+1} \) is a trigonometric polynomial with ‘highest power term’ at most \( 2N_{j+1} M_{j+1} \), i.e.
\[
\hat{d}_{j+1}(\ell) = 0 \quad \text{for every } \ell \in \mathbb{Z} \text{ with } |\ell| > 2N_{j+1} M_{j+1}.
\]
For \( k \in \mathbb{Z} \) with \( |k| < N_{j+1} \), we have \( |m - k| > N_{j+2} - N_{j+1} > 2N_{j+1} M_{j+1} \), and therefore \( \hat{d}_{j+1}(m - k) = 0 \). As a result, all the terms in the sum in (8) are 0, which in turn implies that \( \hat{c}_{j+1}(m) = 0 \). This concludes the induction.

Let \( m \in \mathbb{Z} \) with \( |m| < N_{j+1} \). Using (1) and (7), we obtain that

\[
\hat{c}_{j+1}(m) = c_j d_{j+1}(m) = \sum_{|k| < N_{j+1}} \hat{c}_j(k) \hat{d}_{j+1}(m - k)
\]
For \( k \in \mathbb{Z} \) with \(|k| < N_{j+1}\), if \( k \neq m \), then \( m - k \) is not a multiple of \( 2N_{j+1} \), and hence from (5) we get that \( \hat{d}_{j+1}(m - k) = 0 \). Therefore in (9) the only possible non-zero term in the sum is the one corresponding to \( k = m \), and thus

\[
\hat{c}_j(m) = \hat{c}_{j-1}(m)\hat{d}_j(0) = \hat{c}_{j-1}(m)\hat{b}_j(0) = \hat{c}_{j-1}(m).
\]

In particular, this also implies that \( \hat{c}_j(0) = \hat{c}_{j-1}(0) = \ldots = \hat{c}_1(0) = \hat{b}_1(0) = 1 \) for every \( j \in \mathbb{N} \).

Finally, let \( m \in R_j \). Then applying (7) to \( \hat{c}_{j-1} \), we obtain that

\[
\hat{c}_j(2N_jm) = \sum_{k \in \mathbb{Z}} \hat{c}_{j-1}(k)\hat{d}_j(2N_jm - k)
\]

Observe that the only \( k \in \mathbb{Z} \) with \(|k| < N_j\) satisfying that \( 2N_jm - k \) is a multiple of \( 2N_j \) is \( k = 0 \), and therefore, by (5), the only non-zero term in the sum in (11) is the one corresponding to \( k = 0 \). As a result, \( \hat{c}_j(2N_jm) = \hat{c}_{j-1}(0)\hat{d}_j(2N_jm) = \hat{b}_j(m) = \hat{\mu}_j(m) - \epsilon' = -\epsilon' \), where the last equality is due to the fact that \( \hat{\mu}_j = 0 \) on \( R_j \). This concludes the proof of Claim 1.

While Theorem 3.5 states that for every \( \epsilon \in (0, \frac{1}{2}) \) and every \( j \in \mathbb{N} \), there is a set \( R \) of \( \frac{1}{j} \)-recurrence which is not an \( \epsilon \)-vdC set, in Theorem 3.6 we managed to prove the existence of set \( R \) of strong recurrence which is not an \( \epsilon \)-vdC set only for \( \epsilon \in (0, \frac{1}{3}) \). An interesting question that arises naturally is whether Theorem 3.6 can be extended so that it covers the case \( \epsilon \in (0, \frac{1}{2}) \):

**Question 3.7.** Is it true that for every \( \epsilon \in (0, \frac{1}{2}) \) there is a set \( R \) which is a set of strong recurrence but not an \( \epsilon \)-vdC set?

### 4. A Finitistic Theorem

In this final section, we present the proof of Theorem 3.5. In order to give the proof, we first need to state and prove a series of lemmas.

#### 4.1. A Combinatorial Lemma.

**Lemma 4.1.** Let \( j, Q \in \mathbb{N} \), with \( Q \) even and sufficiently large depending on \( j \), and let \( P \in \mathbb{N} \) with \( P > Q \log 2j^2 \). Then, for any \( E \subseteq [Q^P] = \{0, 1, \ldots, Q^P - 1\} \) with \(|E| > \frac{Q^P}{2} \), there is a point \( y \in E - E \) such that, when written in base \( Q \), i.e., \( y = \sum_{i=0}^{P-1} y_i Q^i \), \( y_i \in \{0, 1, \ldots, Q - 1\} \), one of the digits, say \( y_s \), satisfies \( \frac{Q}{2} \leq y_s < \frac{Q}{2} + 8j \) and all the other digits satisfy that \( 1 \leq y_i < 8j \).
In order to prove Lemma 4.1 we need another two lemmas, which we will state and prove right now.

**Remark 4.2.** From now on, we will work several times with the group of integers mod $Q$, i.e. with $\mathbb{Z}_Q = \mathbb{Z}/Q\mathbb{Z} = \{0, \ldots, Q-1\}$. On $\mathbb{Z}_Q$ we put the metric $d$ defined by $d(m + Q\mathbb{Z}, n + Q\mathbb{Z}) = \min\{|m - n + kQ| : k \in \mathbb{Z}\}$.

**Lemma 4.3** (Cf. [Bou87, Lemma 3.5]). Let $\ell, Q \in \mathbb{N}$ with $Q$ even and let $P \in \mathbb{N}$ with $P > Q \log \ell$. Consider the space

$$
\mathbb{Z}_Q^P = \{(x_0, x_1, \ldots, x_{P-1}) : x_i \in \mathbb{Z}_Q \text{ for } i \in \{0, \ldots, P-1\}\}.
$$

Then, for any $B \subseteq \mathbb{Z}_Q^P$ with $|B| > \frac{Q^P}{\ell}$, there is a pair of points $x, x' \in B$ and some integer $s \in \{0, 1, \ldots, P-1\}$ such that

- $x_i = x'_i$ for $i = 0, 1, \ldots, s-1$
- $x_s = 0$ and $x'_s = \frac{Q}{2}$
- $d(x_i, x'_i) \leq 2$ for $i = s+1, \ldots, P-1$.

The proof of this lemma is adapted from Bourgain’s proof of Lemma 3.5 in [Bou87].

**Proof.** For a set $A \subseteq \{0, 1, \ldots, P-1\}$, we define the relation $\sim_A$ on $\mathbb{Z}_Q^P$ as follows: For $x, y \in \mathbb{Z}_Q^P$, $x \sim_A y : \iff x_i = y_i$ for all $i \in A$ and $d(x_i, y_i) \leq 1$ for all $i \notin A$. It is obvious that $\sim_A$ is reflexive ($x \sim_A x$ for all $x \in \mathbb{Z}_Q^P$) and symmetric ($x \sim_A y \iff y \sim_A x$). Now, let $B \subseteq \mathbb{Z}_Q^P$ with $|B| > \frac{Q^P}{\ell}$.

For every $s \in \{0, 1, \ldots, P\}$, we denote the set $\{0, 1, \ldots, s-1\}$ by $[s]$ (where $[0] = \emptyset$). Consider the set $B_s := \{y \in \mathbb{Z}_Q^P : y \sim_{[s]} x \text{ for some } x \in B\}$. Then, $B = B_P \subseteq B_{P-1} \subseteq \cdots \subseteq B_0 \subseteq \mathbb{Z}_Q^P$.

**Claim:** For some $s \in \{0, 1, \ldots, P-1\}$, there is some $y = (y_0, y_1, \ldots, y_P) \in B_{s+1}$ such that replacing the $s$ coordinate of $y$ with any $k \in \mathbb{Z}_Q$ we still get a point in $B_{s+1}$, i.e.

$$
y(k) := (y_0, \ldots, y_{s-1}, k, y_{s+1}, \ldots, y_{P-1}) \in B_{s+1} \text{ for all } k \in \mathbb{Z}_Q.
$$

**Proof of Claim.** We proceed by contradiction. Assume that the claim doesn’t hold. Then for every $s \in \{0, 1, \ldots, P-1\}$ and every $y \in B_{s+1}$ there is a $k \in \mathbb{Z}_Q$ such that $y(k) \notin B_{s+1}$. Since $y = y(y_s) \in B_{s+1}$ we conclude that there exists a $k \in \mathbb{Z}_Q$ such that $y(k) \notin B_{s+1}$ but $y(k-1) \in B_{s+1}$. Then, $y' = y(k) \in B_s \setminus B_{s+1}$ and $y'$ only disagrees with $y$ at the $s$ coordinate. So we have that for all $s \in \{0, 1, \ldots, P-1\}$ and all $y \in B_{s+1}$ there is some $y' \in B_s \setminus B_{s+1}$ such that $y'$ only disagrees with $y$ at the $s$ coordinate.
Fix \( s \in \{0, 1, ..., P - 1\} \). Then, according to the above, we can define a map \( \phi : B_{s+1} \to B_s \setminus B_{s+1} \) so that for every \( y \in B_{s+1} \), \( \phi(y) \) disagrees with \( y \) only at the \( s \) coordinate. Then

\[
|B_{s+1}| = |\phi^{-1}(B_s \setminus B_{s+1})| = \left| \bigcup_{z \in B_s \setminus B_{s+1}} \phi^{-1}\{\{z\}\} \right| = \sum_{z \in B_s \setminus B_{s+1}} |\phi^{-1}\{\{z\}\}|.
\]

Let \( z \in B_s \setminus B_{s+1} \). For each \( y \in \phi^{-1}\{\{z\}\} \), we have that \( y \) and \( z = \phi(y) \) disagree only at the \( s \) coordinate. Since there are exactly \( Q - 1 \) points in \( \mathbb{Z}_Q^P \) which disagree with \( z \) only at the \( s \) coordinate, we obtain that \( |\phi^{-1}\{\{z\}\}| \leq Q - 1 \). Using (12) we get that \( |B_{s+1}| \leq (Q - 1)|B_s \setminus B_{s+1}| \), or equivalently \( |B_s \setminus B_{s+1}| \geq \frac{|B_{s+1}|}{Q-1} \), and therefore \( |B_s| \geq \frac{Q}{Q-1}|B_{s+1}| \).

Then, inductively we obtain that

\[
|B_0| \geq \left( \frac{Q}{Q-1} \right)^P |B_1| \geq ... \geq \left( \frac{Q}{Q-1} \right)^P |B_P| = \left( \frac{Q}{Q-1} \right)^P |B| \geq \left( \frac{Q}{Q-1} \right)^P \frac{Q^P}{P}.
\]

Since \( P > Q \log \ell \), from (13) we obtain that

\[
|B_0| > \left( \frac{Q}{Q-1} \right)^{Q \log \ell} \frac{Q^P}{P} = Q^P \ell^{Q \log Q - \log(\log(1)) - 1} \geq Q^P,
\]

where the last inequality in (14) is due to \( \log Q - \log(\log(1)) \geq \frac{1}{Q} \). On the other hand, \( B_0 \subseteq \mathbb{Z}_Q^P \), and therefore \( |B_0| \leq Q^P \), which is a contradiction according to (14).

\( \triangle \)

Now we are ready to finish the proof of the lemma. We use the claim to find an \( s \in \{0, 1, ..., P - 1\} \) and a \( y = (y_0, y_1, ..., y_{P-1}) \) such that for every \( k \in \mathbb{Z}_Q \), \( (y_0, ..., y_{s-1}, k, y_{s+1}, ..., y_{P-1}) \in B_{s+1} \). Consider the elements

\[
z = (y_0, ..., y_{s-1}, 0, y_{s+1}, ..., y_{P-1}) \text{ and } z' = (y_0, ..., y_{s-1}, \frac{Q}{2}, y_{s+1}, ..., y_{P-1}),
\]

both of which lie in \( B_{s+1} \). Now, take \( x, x' \in B \) such that \( x \sim_{[s+1]} z \) and \( x' \sim_{[s+1]} z' \). Then, for \( i \in \{0, 1, ..., s - 1\} \) we have \( x_i = z_i = y_i = z'_i = x'_i \). In addition \( x_s = z_s = 0 \), \( x'_s = z'_s = \frac{Q}{2} \) and for \( i \in \{s + 1, ..., P - 1\} \) we have \( d(x_i, x'_i) \leq d(x_i, z_i) + d(z_i, z'_i) + d(z'_i, x'_i) \leq 1 + 0 + 1 = 2 \).

The second lemma we need is the following stronger form of the Poincaré Recurrence Theorem:

**Lemma 4.4.** Let \( (X, \mathcal{B}, \mu, T) \) be a m.p.s. and let \( E \in \mathcal{B} \) with \( \mu(E) > 0 \). Then for some \( n \leq \frac{2}{\mu(E)} \) we have \( \mu(E \cap T^{-n}E) \geq \frac{\mu(E)^2}{2} \).

For sake of completeness, we give the proof of this lemma.
Proof. For every $n \in \mathbb{N}$ consider the function $f_n = 1_{T^n E}$. Let $\alpha = \mu(E)$ and take $R \in \mathbb{N}$ with $R \geq \frac{2}{\alpha}$. Using the Cauchy-Schwarz inequality we obtain that

\[(R\alpha)^2 = \left( \int_X \sum_{n=1}^{R} f_n \, d\mu \right)^2 \leq \int_X \left( \sum_{n=1}^{R} f_n \right)^2 \, d\mu = \sum_{n=1}^{R} \int_X f_n^2 \, d\mu + 2 \sum_{n,m=1}^{R, n \neq m} \int_X f_n f_m \, d\mu = \]

\[= R\alpha + 2 \sum_{n,m=1}^{R, n \neq m} \mu(T^{-n} E \cap T^{-m} E) \leq R\alpha + (R^2 - R) \max_{i,j \in \{1,...,R\}} \mu(T^{-j} E \cap T^{-i} E) \]

As a result

\[(15) \max_{i,j \in \{1,...,R\}} \mu(T^{-j} E \cap T^{-i} E) \geq \frac{R\alpha^2 - \alpha}{R - 1} \geq \frac{\alpha^2}{2}. \]

Since $R$ was arbitrary, (15) holds for every $R \in \mathbb{N}$ with $R \geq \frac{2}{\alpha}$. Take $R_0 := \lceil \frac{2}{\alpha} \rceil$ (the ceiling of $\frac{2}{\alpha}$, i.e. the smallest integer which is greater than or equal $\frac{2}{\alpha}$). Then $R_0$ satisfies (15), and hence there exist $i, j \in \{1,...,R_0\}$ with $i < j$ such that $\mu(T^{-j} E \cap T^{-i} E) \geq \frac{\alpha^2}{2}$. Taking $n := j - i$ we have that $1 \leq n \leq \frac{2}{\alpha} = \frac{2}{\mu(E)}$ and $\mu(E \cap T^{-n} E) \geq \frac{\alpha^2}{2} = \frac{\mu(E)^2}{2}$. \qed

Now, we are ready to present the proof of Lemma 4.1:

Proof of Lemma 4.1: Let $j, Q \in \mathbb{N}$, with $Q$ even and sufficiently large depending on $j$, and let $P \in \mathbb{N}$ with $P > Q \log 2j^2$. In addition, let $E \subseteq [Q^P] = \{0,1,...,Q^P - 1\}$ with $|E| > \frac{Q^P}{2}$. We will identify $[Q^P]$ with the product $X = \mathbb{Z}_Q^P$ through the expansion in base $Q$ digits. Give $X$ the normalised counting probability measure $\mu$ and consider the measure-preserving transformation

\[T : X \to X, \ T(x_0, x_1, ..., x_{P-1}) = (x_0 + 4, x_1 + 4, ..., x_{P-1} + 4) \]

where the addition is mod $Q$. Then $\mu(E) = \frac{|E|}{Q^P} > \frac{1}{4}$, so by Lemma 4.4, there is some $n \in \mathbb{N}$ with $1 \leq n \leq \frac{2}{\mu(E)} < 2j$ such that $\mu(T^{-n} E \cap E) \geq \frac{\mu(E)^2}{2}$. For such an $n$, consider the set $B := T^{-n} E \cap E$. Then $|B| = \mu(B)Q^P > \frac{Q^P}{2j^2}$ and we can apply Lemma 4.3 for the set $B$ (and for $\ell = 2j^2$) to obtain a pair of points $x, x' \in B$ and some integer $s \in \{0,1,...,P - 1\}$ such that

- $x_i = x'_i$ for $i \in \{0,1,...,s - 1\}$
- $x_s = 0$, $x'_s = \frac{Q}{2}$
- $d(x_i, x'_i) \leq 2$ for $i \in \{s + 1,...,P - 1\}$

Since $x' \in B$, we have that $T^n x' \in E$. Also $x \in B \subseteq E$.

$T^n x' \neq x$: $T^n x' = (x'_0 + 4n, ..., x'_{P-1} + 4n)$ and $x = (x_0, ..., x_{P-1})$. Since $4n < 8j$, we have $1 \leq x'_s + 4n = \frac{Q}{2} + 4n < \frac{Q}{2} + 8j$. $Q$ is chosen sufficiently large depending on $j$, so we may assume that $\frac{Q}{2} + 8j < Q$. Therefore, we have that $1 \leq x'_s + 4n < Q$ and $x_s = 0$, which in particular gives
that $x_s \neq x'_s + 4n \mod Q$. As a result therefore $T^n x' \neq x$.

Now, we see $T^n x', x$ as elements of $\mathbb{N}$. Consider $y = \max\{T^n x' - x, x - T^n x'\}$ where now the subtraction is taken in $\mathbb{Z}$. Then $y \in E - E$ and $y > 0$. Let $y = \sum_{i=0}^{P-1} y_i Q^i$ be the base $Q$ expansion of $y$. Because of the possible borrow of digits when we perform the subtraction we have that $y_i \mod Q \in \{x'_i + 4n - x_i \mod Q, x_i - x'_i - 4n \mod Q, x_i - x'_i - 4n - 1 \mod Q\}$.

In every case, since $1 \leq n < 2j$ we have that $Q^2 \leq y_s < Q^2 + 8j$ and for $i \neq s$ we have $1 \leq y_i < 8j$. This concludes the proof of the lemma. □

4.2. Some measures on the $N$-th roots of unity. The following lemma gives rise to some measures that will be useful in the proof of the Theorem 3.5:

**Lemma 4.5** (Cf. [Bou87, Lemma 5.1]). Let $\ell, Q \in \mathbb{N}$, with $Q$ even and sufficiently large depending on $\ell$. Then there is an absolute positive constant $^2 C$ such that for every $k \in \mathbb{N}$, there is a non-negative finite measure $\sigma$ on the torus $\mathbb{T}$ satisfying the following:

- $\sigma$ is supported on $Q^{k+1}$-th roots of unity, so by Remark 2.4 its Fourier transform is periodic with period $Q^{k+1}$
- $\hat{\sigma}(0) \leq 1 + C \frac{\ell^3}{Q^2}$
- $\hat{\sigma}(m) = 1$ for $m \in \mathbb{Z}$ with $1 \leq m \frac{Q}{Q^k} \leq \ell$
- $\hat{\sigma}(m) = -1$ for $m \in \mathbb{Z}$ with $\frac{Q}{2} \leq m \frac{Q}{Q^k} \leq \frac{Q}{2} + \ell$.

**Proof.** $Q$ is sufficiently large depending on $\ell$, so we may assume that $Q > 4\ell$. Let $N := Q^{k+1}$ and consider the real trigonometric polynomials

$$p : \mathbb{T} \to \mathbb{C}, \quad p(x) = \sum_{m \in \mathbb{Z}} \left(1 - \cos \frac{2\pi \ell Q^k}{N} \frac{|m|}{N}\right) e^{2\pi i m x}$$

$$r : \mathbb{T} \to \mathbb{C}, \quad r(x) = 2 \cos \left(\ell Q^k 2\pi x\right) - \cos \left(\left(\frac{N}{2} - \ell Q^k\right) 2\pi x\right) - \cos \left(\left(\frac{N}{2} + \ell Q^k\right) 2\pi x\right).$$

In addition, let $F_{Q^k}$ denote the $Q^k$-th Fejer kernel and consider the real trigonometric polynomial

$$s(x) = 16\ell p \ast F_{Q^k}(x) + r(x)p(x).$$

We want to show that $s \geq 0$. Consider the function

$$f : \mathbb{N}_0 \to \mathbb{R}, \quad f(m) = \begin{cases} 1 - \cos \frac{2\pi \ell Q^k - m}{N}, & \text{if } 0 \leq m \leq \ell Q^k \\ 0, & \text{otherwise} \end{cases}$$

$^2$This constant $C$ is independent of $\ell$ and $Q$. From the proof one sees that we can choose for example $C = 320$. 

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We have assumed that $4\ell < Q$, and hence for every $x \in [0, \ell Q^k]$, we have $0 \leq \frac{\ell Q^k - x}{N} < \frac{\pi}{2}$. Then the function $x \mapsto 1 - \cos 2\pi \frac{\ell Q^k - x}{N}$ is decreasing and convex in the interval $[0, \ell Q^k]$, which implies that $f$ is decreasing and convex in $\mathbb{N}_0$. Obviously, $f$ is non-negative, and therefore $f$ satisfies the conditions in Proposition 2.3. As a result, we obtain that the trigonometric polynomial $p(x) = \sum_{m \in \mathbb{Z}} f(|m|) e^{2\pi i m x}$ satisfies $p \geq 0$. Using Lemma 2.2 we obtain that $p \leq (4\ell p) * F_{Q_k}$ and hence $4p \leq (16\ell p) * F_{Q_k}$. In addition, $|r(x)| \leq 4$, so we have that $-r(x)p(x) \leq |r(x)|p(x) \leq 4p(x) \leq (16\ell p) * F_{Q_k}$, and therefore $s(x) \geq 0$ for every $x \in \mathbb{T}$.

We can then consider the non-negative finite measure $\sigma$ on the torus $\mathbb{T}$, defined by

$$\sigma = \frac{1}{2} \left( \delta_{\frac{1}{N}} + \delta_{\frac{1}{-N}} \right) + \frac{1}{N} \sum_{n=0}^{N-1} s\left( \frac{n}{N} \right) \delta_{\frac{n}{N}}.$$ 

$\sigma$ is clearly supported on the $N$-th roots of unity. Now we need to check that the Fourier transform of $\sigma$ has the desired properties. To do that, we first compute the Fourier transforms of $p$ and $r$.

$$\hat{p}(m) = \begin{cases} 
1 - \cos \left( 2\pi \frac{\ell Q^k - |m|}{N} \right), & \text{if } |m| \leq \ell Q^k \\
0, & \text{otherwise}.
\end{cases}$$  

(16)

On the other hand, for the Fourier transform of $r$ we have

$$\hat{r}(m) = \begin{cases} 
1, & \text{if } |m| = \ell Q^k \\
-\frac{1}{2}, & \text{if } |m| = \frac{N}{2} - \ell Q^k \text{ or } |m| = \frac{N}{2} + \ell Q^k \\
0, & \text{otherwise}.
\end{cases}$$  

(17)

Next, we will compute the Fourier transform of $s$. For every $m \in \mathbb{Z}$, we have

$$\hat{s}(m) = 16\ell \hat{p}(m) \overline{F_{Q_k}(m)} + \sum_{t \in \mathbb{Z}} \hat{r}(t) \hat{p}(m - t).$$ 

(18)

We now split in cases depending on $m \in \mathbb{Z}$.

**Case 1.** If $|m| \geq N = Q^{k+1}$:

If $|m| > Q^k$, so $\overline{F_{Q_k}(m)} = 0$ and therefore using (18) and (17) we obtain

$$\hat{s}(m) = \hat{p}(m - \ell Q^k) + \hat{p}(m + \ell Q^k) - \frac{1}{2} \hat{p}(m + \frac{N}{2} - \ell Q^k) - \frac{1}{2} \hat{p}(m - \frac{N}{2} - \ell Q^k) - \frac{1}{2} \hat{p}(m - \frac{N}{2} + \ell Q^k).$$

For $Q$ sufficiently large depending on $\ell$, all the arguments in the above expression have absolute value greater than $\ell Q^k$, and hence from (16) we obtain that $\hat{s}(m) = 0$. So we proved that

$$\hat{s}(m) = 0 \text{ for every } m \in \mathbb{Z} \text{ with } |m| \geq N = Q^{k+1}.$$ 

(19)
Case 2. If \( m = 0 \):

\( Q \) is sufficiently large depending on \( \ell \), so we may assume that \( \ell Q^k < \frac{N}{2} - \ell Q^k \). Then using (18), (16) and (17) we obtain

\[
\tilde{s}(0) = 16\ell\hat{\gamma}(0)\tilde{F}_{Q^k}(0) = 16\ell \left(1 - \cos 2\pi \frac{\ell}{Q}\right) \leq 16\ell \times 20 \left(\frac{\ell}{Q}\right)^2 \leq C \frac{\ell^3}{Q^3},
\]

where for the first inequality we use the fact that for every \( x \in \mathbb{R} \), \( 1 - \cos(2\pi x) \leq 20x^2 \).

Case 3. If \( Q^k \leq |m| \leq \ell Q^k \):

\( |m| \geq Q^k \), so \( \tilde{F}_{Q^k}(m) = 0 \). \( Q \) is sufficiently large depending on \( \ell \), so we may assume that \( \frac{Q}{2} - 2\ell > \ell \). Then, for every \( t \in \mathbb{Z} \) with \( |t| \geq \frac{N}{2} - \ell Q^k \) we have that \( |m - t| \geq |t| - |m| \geq \left(\frac{Q}{2} - 2\ell\right)Q^k > \ell Q^k \), which implies that \( \hat{\gamma}(m - t) = 0 \). Using (18) we then get

\[
\tilde{s}(m) = \sum_{|t| < \frac{N}{2} - \ell Q^k} \hat{\gamma}(t) \hat{\gamma}(m - t).
\]

For the \( t \)'s in the above sum, we know that \( \hat{\gamma}(t) \) is non-zero exactly when \( t = \ell Q^k \) or \( t = -\ell Q^k \).

Subcase 3.1. If \( m > 0 \), then \( |m - \ell Q^k| = \ell Q^k - m < \ell \) and therefore

\[
\hat{\gamma}(m - \ell Q^k) = 1 - \cos \left(2\pi \frac{\ell Q^k - |m - \ell Q^k|}{N}\right) = 1 - \cos \left(2\pi \frac{m}{N}\right).
\]

On the other hand, \( |m + \ell Q^k| > \ell Q^k \) and hence \( \hat{\gamma}(m + \ell Q^k) = 0 \). Combining all those, we obtain that \( \tilde{s}(m) = \hat{\gamma}(m - \ell Q^k) = 1 - \cos \left(2\pi \frac{m}{N}\right) \).

Subcase 3.2. If \( m < 0 \), then similarly one shows that \( \tilde{s}(m) = \hat{\gamma}(m - \ell Q^k) = 1 - \cos \left(2\pi \frac{m}{N}\right) \).

So after all we have that

\[
\tilde{s}(m) = 1 - \cos \left(2\pi \frac{m}{N}\right) \text{ for every } m \in \mathbb{Z} \text{ with } Q^k \leq |m| \leq \ell Q^k.
\]

Case 4. If \( Q^k - N \leq m \leq \ell Q^k - N \):

Again, since \( Q \) is chosen sufficiently large depending on \( \ell \), we may assume that \( \ell Q^k - N < -Q^k \) and \( 2\ell Q^k - \frac{N}{2} < -\ell Q^k \). Then \( m < -Q^k \), and thus \( \tilde{F}_{Q^k}(m) = 0 \). Therefore, using (18) we get that \( \tilde{s}(m) = \sum_{t \in \mathbb{Z}} \hat{\gamma}(t) \hat{\gamma}(m - t) \). If \( t \in \mathbb{Z} \) is such that \( \hat{\gamma}(t) \neq 0 \), then \( -t \leq -\frac{N}{2} + \ell Q^k \), so \( m - t \leq 2\ell Q^k - \frac{N}{2} < -\ell Q^k \) and therefore \( \hat{\gamma}(m - t) = 0 \). As a result we obtain

\[
\tilde{s}(m) = 0 \text{ for every } m \in \mathbb{Z} \text{ with } Q^k - N \leq m \leq \ell Q^k - N.
\]

Case 5. If \( \frac{N}{2} \leq m \leq \frac{N}{2} + \ell Q^k \):

\( m \geq \frac{N}{2} > Q^k \), so \( \tilde{F}_{Q^k}(m) = 0 \) and using (18) we have that \( \tilde{s}(m) = \sum_{t \in \mathbb{Z}} \hat{\gamma}(t) \hat{\gamma}(m - t) \). The only
\( t \in \mathbb{Z} \) such that \( \hat{r}(t) \neq 0 \) and \( |m - t| \leq \ell Q^k \) is \( t = \frac{N}{2} + \ell Q^k \). Therefore using (16) and (17) we obtain that

\[
\hat{s}(m) = -\frac{1}{2} \hat{p} \left( m - \frac{N}{2} - \ell Q^k \right) = -\frac{1}{2} \left( 1 - \cos 2\pi \frac{\ell Q^k - |m - N/2 - \ell Q^k|}{N} \right).
\]

Observe that \( |m - N/2 - \ell Q^k| = N/2 + \ell Q^k - m \). Therefore we showed that

\[
(23) \quad \hat{s}(m) = -\frac{1}{2} \left( 1 - \cos 2\pi \frac{m - N/2}{N} \right) \quad \text{for every} \quad m \in \mathbb{Z} \quad \text{with} \quad \frac{N}{2} \leq m \leq \frac{N}{2} + \ell Q^k.
\]

**Case 6.** If \(-\frac{N}{2} \leq m \leq -\frac{N}{2} + \ell Q^k\):

\( Q \) is chosen sufficiently large depending on \( \ell \), so we may assume that \(-\frac{N}{2} + \ell Q^k < -Q^k \). Then \( m < -Q^k \), and thus \( \hat{F}_{Q^k}(m) = 0 \). Using (18) we then obtain \( \hat{s}(m) = \sum_{t \in \mathbb{Z}} \hat{r}(t) \hat{p}(m - t) \). The only \( t \in \mathbb{Z} \) such that \( \hat{r}(t) \neq 0 \) and \( |m - t| \leq Q^k \) is \( t = -\frac{N}{2} + \ell Q^k \). Therefore using (16) and (17) we obtain that

\[
\hat{s}(m) = -\frac{1}{2} \hat{p} \left( m + \frac{N}{2} - \ell Q^k \right) = -\frac{1}{2} \left( 1 - \cos 2\pi \frac{\ell Q^k - |m + N/2 - \ell Q^k|}{N} \right).
\]

Observe that \( |m + N/2 - \ell Q^k| = -N/2 + \ell Q^k - m \). Therefore we showed that

\[
(24) \quad \hat{s}(m) = -\frac{1}{2} \left( 1 - \cos 2\pi \frac{m + N/2}{N} \right) \quad \text{for every} \quad m \in \mathbb{Z} \quad \text{with} \quad -\frac{N}{2} \leq m \leq -\frac{N}{2} + \ell Q^k.
\]

Now we are ready to compute the Fourier transform of the measure \( \sigma \). From the definition of \( \sigma \), we get that for every \( m \in \mathbb{Z} \)

\[
(25) \quad \hat{\sigma}(m) = \frac{1}{2} \left( \hat{\sigma}_+(m) + \hat{\sigma}_-(m) \right) + \frac{1}{N} \sum_{n=0}^{N-1} s \left( \frac{n}{N} \right) \hat{\sigma}_-(m) = \cos \left( 2\pi \frac{m}{N} \right) + \frac{1}{N} \sum_{n=0}^{N-1} s \left( \frac{n}{N} \right) e^{-2\pi\imath m \frac{n}{N}}.
\]

For \( m = 0 \), using (3) and (20) we get

\[
\hat{\sigma}(0) = 1 + \frac{1}{N} \sum_{n=0}^{N-1} s \left( \frac{n}{N} \right) = 1 + \hat{s}(0) \leq 1 + C \ell^3 Q^2.
\]

Using (25) and the Fourier inversion formula on \( s \), we obtain that

\[
\hat{\sigma}(m) - \cos \left( 2\pi \frac{m}{N} \right) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{t \in \mathbb{Z}} \hat{s}(t) e^{2\pi\imath \frac{mt}{N}} e^{-2\pi\imath \frac{nt}{N}} = \sum_{t \in \mathbb{Z}} \hat{s}(t) \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi\imath (t-m) \frac{n}{N}} = \sum_{t \in \mathbb{Z}} \hat{s}(t) \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi\imath (t+m) \frac{n}{N}} = \sum_{t \in \mathbb{Z}} \hat{s}(t+m) \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi\imath \frac{nt}{N}} = \sum_{t \in \mathbb{Z}} \hat{s}(tN + m).
\]

From equation (19) we have that \( \hat{s}(t) = 0 \) for \( |t| \geq N \). Thus if \( 0 < m \leq N \), then

\[
(26) \quad \hat{\sigma}(m) = \cos \left( 2\pi \frac{m}{N} \right) + \hat{s}(m) + \hat{s}(m - N).
\]
For \( m \in \mathbb{Z} \) with \( 1 \leq m/Q^k \leq \ell \), we have that \( Q^k \leq m \leq \ell Q^k \) and \( Q^k - N \leq m - N \leq \ell Q^k - N \).

Therefore from equations (26), (21) and (22) we have

\[
\hat{\sigma}(m) = \cos \left( \frac{2\pi m}{N} \right) + \left( 1 - \cos \left( \frac{2\pi m}{N} \right) \right) + 0 = 1.
\]

On the other hand, for \( m \in \mathbb{Z} \) with \( Q/2 \leq m/Q^k \leq Q/2 + \ell \) we have \( N/2 \leq m \leq N/2 + \ell Q^k \) and \(-N/2 \leq m - N \leq \ell Q^k - N/2\). Therefore from equations (26), (23) and (24) we obtain

\[
\hat{\sigma}(m) = \cos \left( \frac{2\pi m}{N} \right) - \frac{1}{2} \left( 1 - \cos \frac{2\pi m - N/2}{N} \right) - \frac{1}{2} \left( 1 - \cos \frac{2\pi m - N/2}{N} \right) = \\
= \cos \left( \frac{2\pi m}{N} \right) + \cos \left( \frac{2\pi m - \pi}{N} \right) - 1 = -1.
\]

\[\square\]

4.3. The proof of Theorem 3.5. The following remark is also useful:

**Remark 4.6.** If \( R \) is a set of \( \epsilon \)-recurrence, then there exists \( n \in \mathbb{N} \) such that whenever \( E \subseteq [n] \) and \( |E| > \epsilon n \) we have \( (E - E) \cap R \neq \emptyset \). But we have \( (E - E) \cap R \subseteq [n] \), so in fact \( (E - E) \cap (R \cap [n]) \neq \emptyset \), which means that \( R \cap [n] \) is a set of \( \epsilon \)-recurrence.

Finally, we have assembled all the tools that we need in order to prove Theorem 3.5. For the convenience of the reader, we state Theorem 3.5 again, before proving it.

**Theorem 3.5.** For every \( j \in \mathbb{N} \) and for every \( \epsilon \in (0, \frac{1}{2}) \) there is a finite set \( R_j \subseteq \mathbb{N} \) which is a set of \((\frac{1}{j})\)-recurrence but not an \( \epsilon \)-vdC set.

**Proof.** Let \( j \in \mathbb{N} \) and \( \epsilon \in (0, \frac{1}{2}) \). Choose an even \( Q \in \mathbb{N} \) sufficiently large depending on \( j \), so that we can apply Lemma 4.5 (with \( \ell = 8j \)) and Lemma 4.1. Since \( \epsilon < \frac{1}{2} \) we may choose \( Q \) so large that

\[
\frac{1}{1 + \left( 1 + C \frac{(8j)^3}{Q^2} \right)^{[Q \log 2j^2] + 1}} > \epsilon.
\]

Let \( P := [Q \log 2j^2] + 1 \). For each \( k \in \{0, 1, ..., P - 1\} \), apply Lemma 4.5 with \( \ell = 8j \) to get a measure \( \sigma_k \). Consider the non-negative finite measure \( \sigma = \sigma_0 * \sigma_1 * ... * \sigma_{P-1} \) on \( T \). For every \( k \in \{0, 1, ..., P - 1\} \) we have \( \widehat{\sigma_k}(0) \leq 1 + C \frac{(8j)^3}{Q^2} \), and therefore

\[
0 \leq \sigma(T) = \widehat{\sigma}(0) = \widehat{\sigma_0}(0)\widehat{\sigma_1}(0)...\widehat{\sigma_{P-1}}(0) \leq \left( 1 + C \frac{(8j)^3}{Q^2} \right)^P.
\]
Now, consider the measure \( \mu = \frac{\sigma + \delta_\rho}{\sigma + 1} \). Then \( \mu(T) = \frac{\sigma(T) + \delta_\rho(T)}{\sigma + 1} = 1 \), i.e. \( \mu \) is a probability measure on \( T \). Consider the set \( S_j := \{ r \in \mathbb{N} : \hat{\mu}(r) = 0 \} \). Since

\[
\mu(\{0\}) = \frac{\sigma(\{0\}) + 1}{1 + \sigma(T)} \geq \frac{1}{1 + \sigma(T)} \geq \frac{1}{1 + \left(1 + C(8j)^3/Q^2\right)^P} > \epsilon,
\]

we have that \( S_j \) is not an \( \epsilon \)-vdC set.

**Claim:** \( S_j \) is a set of \((\frac{1}{j})\)-recurrence.

**Proof of Claim.** Let \( E \subseteq \{0, 1, \ldots, Q^P - 1\} \) with \(|E| > \frac{Q^P}{2} \). Since \( P > Q \log 2j^2 \), we can apply Lemma 4.1 to obtain a \( y \in E - E \) and an \( s \in \{0, 1, \ldots, P - 1\} \) such that when \( y \) is written in base \( Q \), i.e.

\[
y = \sum_{i=0}^{P-1} y_i Q^i, \quad y_i \in \{0, 1, \ldots, Q - 1\},
\]

we have that \( \frac{Q}{2} \leq y_s < \frac{Q}{2} + 8j \) and for \( i \neq s \), we have \( 1 \leq y_i < 8j \). We will show that \( y \in S_j \).

Recall that for each \( k \in \{0, 1, \ldots, P - 1\} \), \( \hat{\sigma}_k \) is periodic with period \( Q^{k+1} \), and therefore

\[
\hat{\sigma}(y) = \prod_{k=0}^{P-1} \hat{\sigma}_k(y) = \prod_{k=0}^{P-1} \hat{\sigma}_k\left(\sum_{i=0}^{P-1} y_i Q^i\right) = \prod_{k=0}^{P-1} \hat{\sigma}_k\left(\sum_{i=0}^{k} y_i Q^i\right).
\]

Recall that \( Q \) is sufficiently large depending on \( j \), so we may assume that \( \frac{s}{Q-1} \leq 1 \). Then from the hypothesis on \( y \) we have that

\[
\frac{Q}{2} \leq \frac{\sum_{i=0}^{s} y_i Q^i}{Q^s} \leq \frac{Q}{2} + 8j - 1 + \frac{(8j - 1)}{Q^s} \sum_{i=0}^{s-1} Q^i \leq \frac{Q}{2} + 8j.
\]

Then, using the properties of \( \hat{\sigma}_s \) (see Lemma 4.5), we obtain that \( \hat{\sigma}_s(\sum_{i=0}^{s} y_i Q^i) = -1 \). On the other hand, for \( k \neq s \) we have

\[
1 \leq \frac{\sum_{i=0}^{k} y_i Q^i}{Q^k} = y_k + \sum_{i=0}^{k-1} y_i Q^{i-k} \leq 8j,
\]

and using the properties of \( \hat{\sigma}_k \) (see Lemma 4.5), we obtain that \( \hat{\sigma}_k(\sum_{i=0}^{k} y_i Q^i) = 1 \). Then, using (27) we obtain \( \hat{\sigma}(y) = -1 \) and therefore \( \hat{\mu}(y) = \frac{\hat{\sigma}(y) + \delta_\rho(y)}{\sigma(T) + 1} = 0 \), which means that \( y \in S_j \). As a result, \( y \in (E - E) \cap S_j \), which is to say that \( E \cap (E - y) \neq \emptyset \). So, after all we proved that for any \( E \subseteq \{0, 1, \ldots, Q^P - 1\} \) with \(|E| > \frac{Q^P}{2} \), there is some \( y \in S_j \) such that \( E \cap (E - y) \neq \emptyset \), and therefore \( S_j \) is a set of \((\frac{1}{j})\)-recurrence. This concludes the proof of the claim. \( \square \)

Finally, from Remark 4.6, we know that we can find a finite set \( R_j \subseteq S_j \) such that \( R_j \) is again a set of \((\frac{1}{j})\)-recurrence. Of course, \( \hat{\mu}(y) = 0 \) for every \( y \in R_j \) (\( \subseteq S_j \)) and therefore \( R_j \) is not an \( \epsilon \)-vdC set. Hence, \( R_j \) is a set with the desired properties, and this concludes the proof of the theorem. \( \square \)
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