Mattig’s relation and dynamical distance indicators

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We discuss how the redshift (Mattig) method in Friedmann cosmology relates to dynamical distance indicators based on Newton’s gravity (Teerikorpi 2011). It belongs to the class of indicators where the relevant length inside the system is the distance itself (in this case the proper metric distance). As the Friedmann model has Newtonian analogy, its use to infer distances has instructive similarities to classical dynamical distance indicators. In view of the theoretical exact linear distance-velocity law, we emphasize that it is conceptually correct to derive the cosmological distance via the route: redshift (primarily observed) → space expansion velocity (not directly observed) → metric distance (physical length in "cm"). Important properties of the proper metric distance are summarized.

1 Introduction

Newtonian acceleration $\ddot{r}$ of a particle in the force field of a spherical mass $M$

$$\frac{1}{r^2} = -\frac{1}{G} \times \frac{\ddot{r}}{M}.$$ (1)

offers known ways to infer the distance $d$ of a gravitating system. Such dynamical distance indicators often belong to "physical" (Sandage et al. 2006) or "one-step" (Jackson 2007) methods, in contrast to relative distance indicators such as standard candles and rods. The latter give luminosity and angular size distances. The distance type from physical indicators depends on the method; essential is that the distance is directly obtained in cm, with no midway distance as a reference.

Note that $r$ in Eq. (1) can be either a length within a remote system whose distance is wanted or it can be the distance itself from the observer to the system.

1.1 When $r$ is within a distant system

When the acceleration can be related to an observable velocity quantity $V$, Teerikorpi (2011; T2011) grouped dynamical distance indicators according to how the $M/d$ degeneracy in the equation $M = c_0 \delta dV^2$ is overcome if one can express $M/d$ in another form containing known quantities and the unknown $d$ ($\theta$ is the angle the size $r$ is observed at). T2011 discussed cases $M \propto r^n$ where $n = 0, 2$ or 3, and the distance $d$ can be determined.

1.2 When $r$ is the desired distance $d$ from us

In some cases the length in Eq.(1) is the distance $d$ from us to an object, as in a binary system with the observer on one of the bodies. Idealized examples could be the Earth-Moon and Sun-Earth pairs. Assuming a circular orbit, the radial acceleration leads to $d = \frac{GM}{V^2}$, with the orbital speed from the period $T = \frac{2\pi d}{V}$. This leaves $M \propto d^3$ so one needs an auxiliary relation $M \propto d^n$, where $n \neq 3$. For the Earth-Moon pair one writes $GM_E = gr_E^2$ (so $n = 0$). For the Sun the gravitational redshift $z_g$ gives (T2011) $GM_{\text{sun}} = z_g c^2 \theta_{\text{sun}} d$ ($n = 1$).

Such cases involved circular orbits. In the Galaxy-M31 pair and in the flow of dwarf galaxies away from the Local Group the motion is (nearly) radial and one cannot directly infer the acceleration in Eq.(1). However, with data on the mass and "bang time" one can derive, by Tolman-Bondi-type calculations, the expected distance-velocity relation that gives the distance if the radial velocity is measured (e.g., Gromov et al. 2001).

A supreme distance indicator of this category with radial motion is the Friedmann universe itself. Its special features are discussed below.

2 The Friedmann universe

The spheres cut from this world model are known to expand in a Newtonian way (McCrea & Milne 1934). In terms of internal hyperspherical coordinates, with the scale factor $a$ chosen to be dimensionless and the comoving radial coordinate $r_c$ so that the metric distance is $d = ar_c = r_c$ at the present time ($a = 1$):

$$\frac{1}{(ar_c)^2} = -\frac{1}{G} \times \frac{\ddot{ar}_c}{M_{\text{eff}}},$$ (2)

where the effective mass $M_{\text{eff}} = \frac{4\pi}{3}(ar_c)^3(\rho_m - 2\rho_\Lambda)$ ($\rho_m$ is the density of the gravitating matter and $\rho_\Lambda$ is...
that of the \Lambda term). We note that Eq. (2) is exact in the Friedmann model, not an approximation valid, say, at short distances (Baryshev 2008; Baryshev & Teerikorpi 2012).

Here the dynamical equation is, of course, Einstein’s field equation. With the symmetries underlying the world model it leads to Friedmann’s equations (and Eq. (2)). The puzzling similarity of Eq. (2) to Eq. (1) has led to discussions on its meaning (e.g., Layzer 1954) and if it could throw light on conceptual problems in Friedmann cosmology (e.g., Baryshev 2008). Here we simply ask if the analogy extends to how one obtains the (well-known) distance indicator in the Friedmann world.

We cannot measure the acceleration of space expansion \( \ddot{a} \) between us and the remote object. However, we can replace \( \ddot{a} \) in Eq. (2) by \(-q\ddot{a}_c^2/a\), where \( q \) is the deceleration parameter, constant on all scales \( r_c \) at the epoch \( t \) (fixed \( a(t) \)), due to the system’s regularity. So

\[
(\ddot{a}_c)^2 = \frac{G}{q} \times \frac{M_{\text{eff}}}{a r_c},
\]

which resembles the expressions often appearing for classical dynamical distance indicators (T2011). Writing \( V = \dot{a} r_c \) and \( d = a r_c \), we see also here the \( M/d \) degeneracy. It can be overcome, as \( M_{\text{eff}} \propto d^3 \), so \( n = 3 \). Inserting the expression for \( M_{\text{eff}} \) into Eq. (3) one gets

\[
V = \dot{a} r_c = (\frac{2}{q})^{1/2} \times a r_c = H(t) d,
\]

the linear distance-velocity law in the Friedmann world. Here \( \gamma = \frac{4\pi G}{q}(2\rho_{\Lambda} - \rho_m) \) is constant for a fixed scale factor. It is this law that offers a route to the distance.

In a landmark study, Harrison (1993) underlined that the relation \( V = H d \) is exact. \( V \) is the rate at which the (proper) metric distance \( d \) between two objects at rest in space grows. The Hubble constant \( H_0 \) fixes this directly unobservable law in its current state.

The expansion rate \( V \) cannot be directly measured. In classical indicators if the radial velocity is needed it is inferred using the Doppler effect. The expansion rate between us and the object is also got from the spectral line shift but now using the Lemaître (1927) effect that gives the exact sense of the cosmological redshift.

### 3 From redshift to velocity to distance

In his classic paper, Mattig (1958) solved the task of deriving the formula for the luminosity distance for a given redshift. His aim was to express the apparent magnitude of a standard candle as a function of redshift, and the focus was not on the metric distance and the exact linear expansion law (which had not yet been adequately discussed at the time).\(^1\)

After Harrison’s (1993) work, it is natural to underline the exact link between the expansion velocity and the metric distance. It is also instructive and logical when considering the expansion law as a distance indicator, to derive first the present moment \( V(z) \) for the redshift \( z \) and then divide it by \( H_0 \). The mathematics is essentially as in Mattig’s original way.

In general, the light emitted from a galaxy having the redshift \( z \), has gone through a comoving radial coordinate interval \( r_c \) before having reached us. Then one can write for the present rate of increase of the metric distance between us and the galaxy, using \( adr_c = c dt \):

\[
V(z) = (\frac{dr_c}{dt})_0 = (\dot{a})_0 c \int_{a_0/(1+z)}^{\infty} \frac{da}{a \dot{a}}.
\]

One may express the derivative \( \dot{a} \) in terms of \( a \) using the first Friedmann equation. Changing the variable from \( a \) to \( z \), noting how \( \rho_m \) and \( \rho_{\Lambda} \) depend on \( a \), and dividing by \( H_0 \) gives the known formula

\[
d(z) = \frac{V(z)}{H_0} = \frac{c}{H_0} \int_{0}^{z} \frac{dx}{(\Omega_m(1 + x)^3 + \Omega_{\Lambda} + \omega(1 + x)^2)^{1/2}}
\]

where \( \omega = 1 - \Omega_m - \Omega_{\Lambda} \) (\( = 0 \) for flat space). Only for the flat pure \( \Lambda \) model the expansion velocity is the “classical Doppler” \( V(z) = cz \). Eq. (6) can be generalised to include energy transfer between the matter components (Teerikorpi et al 2003; Gromov et al 2004).

### 4 Concluding remarks

We showed the steps from redshift to metric distance in order to sum up the idiosyncracies of the redshift method where the system is the universe as described by the Friedmann model. This machinery as a dynamical distance indicator involves tightly packed all the ingredients of cosmological physics, such as the Cosmological Principle, general relativity, space expansion (instead of motion in static space), and the Lemaître redshift effect. At the same time, it is instructive to see that each step has analogies with classical dynamical distance indicators (Eqs. (2)-(5)).

Remarkably, when the density parameters of the Friedmann model \( \Omega_m \) and \( \Omega_{\Lambda} \) are fixed, a single observation of the dimensionless spectral redshift allows one to derive the current space expansion rate (in cm/s) between us and the object in question. Then, the exact

\(^1\) Wolfgang Mattig derived his result as student when he had to give a talk on cosmology for his graduate exam. A letter to the present authors reveals that during the short preparation time "I also studied Heckmann’s book on 'Theorien der Kosmologie' and I found it extremely insufficient that the relations \( z(m) \) and \( N(m) \) are given in series expansions. I tried to find a closed form for the simplest case, zero cosmological constant and flat space. I succeeded within the two weeks and got through the examination with a good result.” He also noted: “the reaction of the cosmological community was nearly zero, only Allan Sandage discussed my relations in ApJ 133 (1961). He rendered accessible my results [which] were published in German, in an East-German journal.” The letter is printed in Baryshev & Teerikorpi (2002). For interesting remarks, see Sandage (1995).
distance-velocity relation (with the Hubble constant $H_0$) leads to the proper metric distance, as expressed in units of cm. This is the correct, straightforward and simple way to speak about the derivation of the distance within the standard cosmology.

We find it useful to finish with some notes on the end result, the metric distance. McVittie (1974) wrote that "distance is a measure of remoteness". In his analysis of different distance types that appear in cosmology he did not prefer any one of them above the others. However, with the Friedmann model as the basis of cosmology, there are good reasons to view the "tape measure" metric distance $d$ as basic. It comes close to our ordinary physical conception of distance, even though it cannot be directly measured.

This distance appears in the exact expansion law. It is also additive, unlike, say, the luminosity distance. Also, for $z = \infty$ it gives the particle horizon (e.g., $2H_0$ in the Einstein-de Sitter model ($\Omega_\Lambda = \omega = 0$ in Eq.(6)).

The metric distance is the backbone distance related in known ways to the work horses of practical cosmology: luminosity and angular size distances. The link is especially simple in flat models, where $d_{\text{lin}} = (1+z)d$ and $d_{\text{ang}} = (1+z)^{-1}d$ and the metric distance $d$ is the geometric mean of those other distances.

Finally, the current metric distance is the best "measure of remoteness" characterizing the location of a distant object in the galaxy universe. So, if its metric distance is 800 Mpc, we at once know that it is 1000 times more distant than the Andromeda nebula. Therefore, the metric distance is also the choice, say, in science news about very distant galaxies, preferably with the familiar light-year as unit.

The metric distance in light-years may be so large that its numerical value exceeds the age of the universe in years. This may disquiet an attentive reader, wondering how the light had enough time to cross such a distance. Then the balloon analogy by Eddington (1930), illustrating the Hubble law and the changing metric distance, also offers an intelligible solution to this "age paradox". In fact, the "light travel distance" is obtained by weighing the integrand in Eq.(6) by the redshift factor $(1+x)^{-1}$.

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2 In non-flat models $d_{\text{lin}} = (1+z)d_{\text{ext}}$ and $d_{\text{ang}} = (1+z)^{-1}d_{\text{ext}}$, where $d_{\text{ext}}$ is the external metric distance (so termed by Baryshev & Teerikorpi 2012), which Mattig (1958) derived for zero-$\Lambda$ dust models as his Eq.(12). It coincides with the internal distance $d$ in flat models, being otherwise related to $d$ as $d_{\text{ext}}(z) = R_H(k\omega)^{-1/2}sn((k\omega)^{1/2}d_{\text{ext}})$, where $sn = \sin$ when $\omega > 0$ (and $k = 1$) and $sn = \sinh$ when $\omega < 0$ (and $k = -1$). $R_H$ is the Hubble distance $c/H_0$.

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**References**

Baryshev, Yu., 2008, in Practical Cosmology Vol. II., Yu.V. Baryshev, I.N. Taganov, P. Teerikorpi (eds.), Russian Geographical Society Saint-Petersburg 2008, p. 20 (arXiv: gr-qc/0810.0153)

Baryshev, Yu. & Teerikorpi, P., 2002, *Discovery of Cosmic Fractals* (World Scientific, Singapore)

Baryshev, Yu. & Teerikorpi, P., 2012, *Fundamental Questions of Practical Cosmology* (Springer, Berlin)

Eddington, A.S., 1930, MNRAS 90, 668

Gromov, A., Baryshev, Yu., Tuson, D., Teerikorpi, P., 2001, Gravitation & Cosmology 7, 140

Gromov, A., Baryshev, Yu., Teerikorpi, P., 2004, A&A 415, 813

Jackson, N., 2007, Living Rev. Relativity 10, (2007), 4 (http://www.livingreviews.org/lrr-2007-4)

Harrison, E.R., 1993, ApJ 403, 28

Layzer, D., 1954, AJ 59, 268

Lemaitre, G., 1927, Ann. Soc. Sci. Brux. 47, 49 (the English translation in MNRAS 91, 483 (1931))

Mattig, W., 1958, AN 284, 109

McCrea, W.H., & Milne, E.A., 1934, Quart. J. Math. 5, 73

McVittie, G.C., 1974, QJRAS 15, 246

Sandage, A., Tammann, G.A., Saha, A., Reindl, B., Macchetto, F.D., Panagia, N., 2006, ApJ 653, 843

Sandage, A.R., 1995, in: The Deep Universe, eds. B. Binggeli & R. Buser (Berlin, Springer), p.16

Teerikorpi, P., Gromov, A., Baryshev, Yu., 2003, A&A 407, L9

Teerikorpi, P., 2011, A&A 531, A10