Genuine Bell locality and its maximal violation in quantum networks

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Abstract
In $K$-locality networks, local hidden variables emitted from classical sources are distributed among limited observers. We explore genuine Bell locality in classical networks, where, regarding all local hidden variables as classical objects that can be perfectly cloned and spread throughout the networks, any observer can access all local hidden variables plus shared randomness. In the proposed linear and nonlinear Bell-type inequalities, there are more correlators to reveal genuine Bell locality than those in the $K$-locality inequalities, and their upper bounds can be specified using the probability normalization of the predetermined probability distribution. On the other hand, the no-cloning theorem limits the broadcast of quantum correlations in quantum networks. To explore genuine Bell nonlocality, the stabilizing operators play an important role in designing the segmented Bell operators and assigning the incompatible measurements for the spatially separated observers. We prove the maximal violations of the proposed Bell-type inequalities tailored for the given qubit distributions in quantum networks.

1. Introduction

As one of the central features in quantum foundations, Bell theorem states that quantum correlations cannot be reproduced or predicted by any classical local theory [1–4]. In ontological models and local hidden variables (LHV), the locality of spacelike events and the realism therein constrain the strength of classical correlations, which are upper-bounded by the Bell inequalities. Quantum theory inconsistent with local realism predicts stronger correlations that violate Bell inequalities. Thanks to quantum information science, two-particle and multiparticle quantum correlations have been extensively investigated. In addition to its immense influence on quantum foundations, Bell nonlocality as a quantum resource has also led to applications in quantum information processing such as quantum cryptography [5], reductions in communication complexity [6], private random number generation [7], and self-testing protocols [8].

To test the strength of Bell nonlocality, an associated Bell test is devised for a given Bell inequality. Therein, a source initially emits a state of two or more particles received by spatially separated observers, who each perform local measurements with a random measurement setting as the input and then obtain the measurement outcome as the output. The observers are to verify whether or not the strength of the input-output correlations violates the tested Bell inequality. Although there is still a decent chance of generating Bell-violating correlations using randomly chosen measurement bases [9, 10], the Bell inequality and local observables in the Bell test should be deliberately set so that the prepared entangled state can achieve the maximum violation. In particular, given the prepared entangled state as a stabilizer state, the stabilizing operators can be used to assign the local observables and design the associated Bell operators.

As an example, denote the four two-qubit Bell states by $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)$ and $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle)$. The operators $\sigma_x \otimes \sigma_z (\sigma_x \otimes \sigma_x)$ can stabilize a two-dimensional subspace spanned by two orthogonal two-qubit Bell states $|\Phi^{+}\rangle$ and $|\Phi^{-}\rangle (|\Phi^{+}\rangle$ and $|\Psi^{+}\rangle)$. The intersection of a two-dimensional subspace stabilized by $\sigma_z \otimes \sigma_z$ and $\sigma_x \otimes \sigma_x$ is the Bell state $|\Phi^{+}\rangle$. Regarding the Clauser–Horne–Shimony–Holt (CHSH) inequality as the Bell inequality tailored for $|\Phi^{+}\rangle$, two segmented Bell operators therein are devised proportional to these two stabilizing operators. In detail, let the local measurement pairs for separate Alice and
Bob in the CHSH test be \((A_0, A_1)\) and \((B_0, B_1)\), respectively, and the Bell-CHSH operator be 
\[ B_{\text{CHSH}} = \sum_{i,j=0}^{1} (-1)^i A_i \otimes B_j. \] 
The CHSH inequality reads \(\langle B_{\text{CHSH}} \rangle_{\text{LHV}} \leq 2\), where \(\langle \cdot \rangle\) denotes the expectation value of \(\cdot\). To achieve the maximum CHSH value, assign the observables \(\sigma_i = \frac{1}{\sqrt{2}} (A_0 + A_1)\) and \(\sigma_x = \frac{1}{\sqrt{2}} (A_0 - A_1)\) for the first qubit; and assign the observables \(\sigma_y = B_0, \sigma_z = B_1\) for the second qubit. Consequently, the Bell-CHSH operator comprises two segmented Bell operators: \((A_0 + A_1)B_0 = \sqrt{2} (\sigma_1 \otimes \sigma_z)\) and \((A_0 - A_1)B_1 = \sqrt{2} (\sigma_1 \otimes \sigma_x)\) and is proportional to the addition of two stabilizing operators. As a result, the maximum CHSH value \(2\sqrt{2}\) can be achieved using the Bell state \(|\Phi^+\rangle\).

Recently, Bell nonlocality in quantum networks has attracted much research attention. A quantum network involves multiple independent sources, and each of them initially emits an entangled state of two or more qubits distributed among specific observers. Since long-distance quantum networks involving large-scale multi-users are the essential goals of upcoming quantum communication, it is fundamental to study their nonlocality strength in the network [11–15]. Specifically, the bilocality and \(K\)-locality correlations in the two-source and later the \(K\)-sources cases, respectively, have been extensively studied [16–25]. In most of the literature, however, the notions of bilocality and \(K\)-locality implicitly refer to the restricted distributions of local hidden variables. For example, in the classical bilocal scenario, a source \(e_i\) sends the local hidden variable \(x_i\) to Alice and Bob, and the other independent separate source \(e_2\) sends the local hidden variable \(y_2\) to Charlie and Bob. In this case, only Bob can access both \(x_1\) and \(y_2\). Regarding local hidden variables as classical objects that can be perfectly cloned and then spread throughout the networks, hereafter genuine Bell locality refers to the correlation strength in the scenario in which all local hidden variables and shared randomness are accessible to each observer in the classical networks. In this case, genuine Bell nonlocality refers to the correlations that cannot be reproduced or predicted in the genuine Bell local scenario.

Here, we explore Bell nonlocality in the \(K\)-source quantum networks. Therein, each quantum source emits the two-qubit Bell state to spatially separated observers. An observer can receive two or more qubits from different quantum sources and then perform a joint measurement on these accessible qubits. The paper is organized as follows. In section 2 and 3, we explore the two-source networks (\(K = 2\)) and the \(K\)-source star-networks, respectively. We propose linear Bell inequalities that respect genuine Bell locality and, on the other hand, can be maximally violated using the prepared state \(|\Phi^+\rangle^{\otimes K}\) distributed in the quantum networks. In section 4, we propose nonlinear Bell inequalities tailored for the distributed product state \(|\Phi^+\rangle^{\otimes K}\). In section 5, we investigate the Bell nonlocality in \((N, K, m)\)-networks as an extension of the star-networks. Finally, we revisit the two-source quantum networks in section 6. There, a source allows for the emission of three-qubit Greenberger-Horne-Zeilinger (GHZ) states instead of the two-qubit state \(|\Phi^+\rangle\). Since two-qubit Bell states and GHZ states are stabilizer states, the stabilizing operators are demonstrated to play a substantial role in constructing the Bell-type inequalities and assigning local observables. In the following, the observables \(\sigma_x, \sigma_y, \text{ and } \sigma_z\) on the qubit \(i\) are denoted by \(X_i, Y_i, \text{ and } Z_i\), respectively.

### 2. Linear Bell inequalities in two-source networks

We review the bilocal model as follows. The source \(e_1\) \((e_2)\) initially sends two particles involving the local hidden variable \(\lambda_1\) \((\lambda_2)\) to Alice and Bob (Charlie and Bob). Alice, Bob, and Charlie perform measurements on their accessible particles, labeled by \(x_i, y_i, \text{ and } z_i\) and obtain outcomes denoted by \(a_{x_i, y_i, z_i}\) and \(c_{x_i, y_i, z_i}\), respectively. By bilocality, the tripartite distribution can be written in the factorized form

\[
P(a_x, b_y, c_z | x, y, z) = \sum_{x, y, z} P(\lambda_1, \lambda_2) P(a_x | x, \lambda_1) P(b_y | y, \lambda_2) P(c_z | z, \lambda_2).
\]

The response function for Alice depends only on the hidden state \(\lambda_1\), the one for Charlie only on \(\lambda_2\); while the one for Bob on \(\lambda_1\) and \(\lambda_2\). Furthermore, it is assumed that two sources \(e_1\) and \(e_2\) are independent and uncorrelated. We denote \(\rho(\lambda_i)\) the distribution of the local hidden variable \(\lambda_i\) and the joint distribution \(\rho(\lambda_1, \lambda_2)\) can be factorized as \(\rho(\lambda_1) \rho(\lambda_2)\) in the bilocal model. A typical nonlinear bilocal inequality can be characterized as [16–19]

\[
|J_{\text{Bil}}| + |J_{\text{Bil}}| \leq 1.
\]

Violating (2) just indicates that the correlation strength in the network is non-bilocal. It is also claimed that any LHV in the bilocal model respects the linear Bell inequality

\[
|J_{\text{LHV}}| + |J_{\text{LHV}}| \leq 1,
\]

which indicates that even classical correlations obeying (3) can admit bi-nonlocality that violates (2) [16–19].

A few remarks on bilocality and non-bilocality are in order. Firstly, to achieve the maximum quantum values of (2) and (3), the following prepare-and-measure scenario is usually proposed in most of the literature. The source \(e_1\) emits the two-qubit states \(|\psi^+\rangle\) of qubits 1 and 2 sent to Alice and Bob, respectively; the source
denotes the bound-entangled Smolin state
\[
\rho_{\text{Smolin}} = \frac{1}{4} \left( | \Phi^+ \rangle \langle \Phi^+ |_{34} + | \Phi^- \rangle \langle \Phi^- |_{34} + | \Psi^+ \rangle \langle \Psi^+ |_{34} + | \Psi^- \rangle \langle \Psi^- |_{34} \right),
\]
where the one-rank projector is denoted by \( \rho \). Furthermore, these methods of preparing states can be prepared with emitted states \( | e_1 \rangle_{23} = | e_2 \rangle_{14} \in \{ | \Phi^\pm \rangle, | \Psi^\pm \rangle \} \) for each emission. On the one hand, according to the second line of (4), the probability normalization plays an important role in constructing Bell inequalities throughout the paper.
On the other hand, since $I_{00}I_{01}I_{10} = I_{11}$, only three of these four operators $I_{00}, I_{10}, I_{11}$ and $I_{11}$ are independent and can stabilize the two-dimensional subspace spanned by $|\Phi^+\rangle_{12}|\Phi^-\rangle_{34}$ and $|\Psi^-\rangle_{12}|\Psi^-\rangle_{34}$. That is, the maximum value in (7) can be achieved using the mixed state $\rho_1 = q_1|\Phi^+\rangle_{12}|\Phi^-\rangle_{34} + (1 - q_1)|\Psi^-\rangle_{12}|\Psi^-\rangle_{34}$.

Here, we propose the second linear Bell inequality similar to (6). Denote the correlator

$$I'_{x'y'} = \frac{1}{2} \sum_{x,y} (-1)^{x+y} a'_x b'_y c'_x d'_y,$$

where $a'_x$ and $c'_x, b'_y, d'_y$ are the outcomes of the observables $A'_x, C'_y, B'_y$ and $D'_y$ measured by Alice, Charlie and Bob, respectively, and $x', y', z' \in \{0, 1\}$ and $a'_y, b'_y, c'_y, d'_y \in \{-1, 1\}$. Another joint probability in classical networks is defined as

$$P_{I'_{x'y'}} = \sum_{x'y'} P \left( \frac{(a'_y + (-1)^y a'_y) (c'_y + (-1)^y c'_y)}{2} = 1 | \lambda \right) P(\lambda)$$

$$= \sum_{x'y'} P \left( \frac{(a'_y + (-1)^y a'_y) b'_y c'_y + (-1)^y c'_y)}{2} = 1 | \lambda \right) P(\lambda),$$

(8)

and the probability normalization requires that $\sum_{x'y'} P_{I'_{x'y'}} = 1$. Note that $P_{I'_{x'y'}} = \langle |I'_{x'y'}|^2 \rangle_{LHV}$, and hence the second linear Bell-type inequality reads

$$\langle I_{00}\rangle_{LHV} + \langle I_{01}\rangle_{LHV} + \langle I_{10}\rangle_{LHV} + \langle I_{11}\rangle_{LHV}$$

$$\leq \langle I'_{00}\rangle_{LHV} + \langle I'_{01}\rangle_{LHV} + \langle I'_{10}\rangle_{LHV} + \langle I'_{11}\rangle_{LHV}$$

$$= \sum_{x'y'} P_{I'_{x'y'}} = 1,$$

(9)

In the quantum region, the key is to take advantage of the stabilizing operator $-Y_i \otimes Y_j = (Z_i \otimes Z_j)(X_i \otimes X_j)$ of $|\Phi^+\rangle_{ij}$. Denote the segmented Bell operators $I'_{x'y'} = \{A'_x + (-1)^y A'_x\} \{B'_y + (-1)^y C'_y\}$ and set the observables $Z_i \rightarrow -Y_i = \frac{1}{\sqrt{2}}(C'_x + C'_z), Z_j \rightarrow Y_j = \frac{1}{\sqrt{2}}(C'_x - C'_z), X_i \rightarrow Z_i = \frac{1}{\sqrt{2}}(C'_y + C'_z), X_j \rightarrow Z_j = \frac{1}{\sqrt{2}}(C'_y - C'_z), Y_4 \rightarrow \frac{1}{\sqrt{2}}(C'_x + C'_z)$ and $X_4 \rightarrow Z_4 = \frac{1}{\sqrt{2}}(C'_y + C'_z)$. As a result, $I'_{00} = I_{00}, I'_{01} = \frac{1}{2} Z_1 Z_2 Z_3 Y_4, I'_{10} = \frac{1}{2} Z_1 Z_2 Z_3 Y_4$ and $I'_{11} = 0$. Note that $I''_y$ can also be obtained by replacing the Pauli observable $X_k$ by $Y_k$. In this case, we have

$$\max \{ \langle I'_{00}\rangle_Q + \langle I'_{01}\rangle_Q - \langle I''_{01}\rangle_Q \}$$

$$= \max \{ \langle I'_{10}\rangle_Q + \langle I'_{11}\rangle_Q + \langle I''_{11}\rangle_Q \}$$

$$= 2,$$

(10)

Similarly, since $I'_{00} I'_{01} I'_{02} = I'_{11}$, only three of these four operators $I'_{00}, I'_{01}, I''_{01}$ and $I'_{11}$ are independent and can stabilize a two-dimensional subspace spanned by $|\Phi^+\rangle_{12}|\Phi^-\rangle_{34}$ and $|\Psi^-\rangle_{12}|\Psi^-\rangle_{34}$. That is, the maximum quantum value (10) can be achieved using the state $\rho_2 = q_2|\Phi^+\rangle_{12}|\Phi^-\rangle_{34} + (1 - q_2)|\Psi^-\rangle_{12}|\Psi^-\rangle_{34}$. Finally, to achieve (6) and (10) simultaneously, the relation $\rho_1 = \rho_2 = |\Phi^+\rangle|\Phi^-\rangle$ with $q_1 = q_2 = 1$ must hold.

A few remarks are in order. Denote the Bell states $|\psi^+\rangle = I_{2} \otimes H|\Phi^+\rangle, |\phi^-\rangle = I_{2} \otimes H|\Psi^-\rangle$, where

$$I_{2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, H' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$
3. Linear Bell inequalities for K-source star-networks

In the K-source star-networks, the source 1 emits the particles i and (K + 1) that are sent to Bob and Alice1, respectively, 1 ≤ i ≤ K. Similarly, we construct two linear Bell inequalities. Denote Bob’s K-bit input bit string by \( y = y_1y_2 \cdots y_K \) with the Hamming weight \( W(y) \) and the one-bit output by \( b_y; \) Alice1’s input bit by \( x_i \) and output by \( a_{i_i}^{(j)} \), where the input bits \( x_1, x_2, \ldots, x_K, y_1, y_2, \ldots, y_K \in \{0, 1\} \) and the measurement outcomes \( b_{x_j}, a_{i_1}^{(j)}, \ldots, a_{i_K}^{(j)} \in \{1, -1\} \). Regarding the genuine Bell locality, Alice1, Alice2, …, AliceK and Bob share randomness, and the local hidden variable \( \lambda = \lambda_1 \cup \lambda_2 \cdots \cup \lambda_K \), where \( \lambda_i \) is the local hidden variable emitted from the \( i \)-th classical source. Note that one and only one of the pre-assigned values among these \( 2^K \) variables is 1 and all the others are 0 since the bits \( b_x, a_{i_1}^{(j)}, \ldots, a_{i_K}^{(j)} \) are predetermined in classical causal models. Denote the correlator \( I_y = \frac{1}{2^K} \prod_{j=1}^{K} (a_{i_1}^{(j)} + (-1)^{i_1}a_{i_1}^{(j)}) b_y \) and the probability distribution

\[
p_y = \sum_{I_y} P\left( \frac{1}{2^K} \prod_{j=1}^{K} (a_{i_1}^{(j)} + (-1)^{i_1}a_{i_1}^{(j)}) b_y \right) = 1\lambda P(\lambda),
\]

and the probability normalization requires that \( \sum_y p_y = 1 \). Note that \( p_y = \langle I_y \rangle_{\text{LHV}} \), and the first linear Bell inequality reads

\[
\sum_y \langle I_y \rangle_{\text{LHV}} = \sum_y p_y = 1.
\]

In the quantum region, Alice and Bob measure the observables \( A_{i_1}^{(j)} \) and \( B_y = \prod_{j=1}^{K} B_{i_1}^{(j)} \), respectively. By assigning the local observable \( Z_i = \frac{1}{\sqrt{2}}(A_{i_1}^{(j)} + A_{i_1}^{(j)}) \), \( X_i = \frac{1}{\sqrt{2}}(A_{i_1}^{(j)} - A_{i_1}^{(j)}) \), and \( B_{i_1}^{(j)} = Z_i Y_i \), where \( Y_i = y_i + 1 \mod 2 \). In this case, the segmented Bell operator \( I_y = \frac{1}{2^K} \prod_{j=1}^{K} (A_{i_1}^{(j)} + (-1)^{i_1}A_{i_1}^{(j)}) B_y \) can be express as

\[
I_y = \frac{1}{2^K} \prod_{j=1}^{K} (Z_i Y_{i_1} Y_i) = \langle \Phi^+ \rangle_{\text{LHV}}.
\]

Since \( Z_i Z_{K+i} | \Phi^+ \rangle_{\text{LHV}} = X_i X_{K+i} \), it is easy to verify that \( \sqrt{2^K} I_y \) is a stabilizing operator of the quantum state \( \prod_{j=1}^{K} | \Phi^+ \rangle_{\text{LHV}} \). As a result, we have

\[
\max \left( \sum_y \langle I_y \rangle_{\text{Q}} \right) = \sqrt{2^K}.
\]

Although it is very complicated to find out the subspace stabilized by these \( 2^K \) operators \( \sqrt{2^K} I_{090, \ldots, 0} \), and \( \sqrt{2^K} I_{0 \ldots, 1} \), at least \( K \) stabilizing operators \( \sqrt{2^K} I_y \) with \( W(y) = 1 \) are linearly independent. Since there are \( 2^K \) independent stabilizing operators for the product states \( \prod_{j=1}^{K} | \Phi^+ \rangle_{\text{LHV}} \), we propose the second linear Bell inequality so that at least another \( K \) linearly-independent stabilizing operators are exploited as the second Bell operators.

Similarly, denote Bob’s K-bit input string by \( y' = y_1' y_2' \cdots y_K' \) and the outcome by \( b_y' \); Alice1’s input bit by \( x_i' \) and outcome by \( a_{i_1}^{(j)} \), where the input bits \( x_1', x_2', \ldots, x_K', y_1', y_2', \ldots, y_K' \in \{0, 1\} \) and output bits \( b_{x_j}, a_{i_1}^{(j)}, \ldots, a_{i_K}^{(j)} \in \{1, -1\} \). Denote the correlator \( I_{y'} = \frac{1}{2^K} \prod_{j=1}^{K} (a_{i_1}^{(j)} + (-1)^{i_1}a_{i_1}^{(j)}) b_{y'} \) and the probability distribution

\[
p_{y'} = \sum_{I_{y'}} P\left( \frac{1}{2^K} \prod_{j=1}^{K} (a_{i_1}^{(j)} + (-1)^{i_1}a_{i_1}^{(j)}) b_{y'} \right) = 1\lambda P(\lambda),
\]

According the probability normalization \( \sum_{y'} p_{y'} = 1 \) and the identity \( p_{y'} = \langle I_{y'} \rangle_{\text{LHV}} \), the second linear Bell inequality reads

\[
\sum_{y'} (-1)^W(y') \langle I_{y'} \rangle_{\text{LHV}} = \sum_{y'} p_{y'} = 1.
\]

In the quantum region, Alice and Bob measure the observables \( A_{i_1}^{(j)} \) and \( B_{y'} = \prod_{j=1}^{K} B_{i_1}^{(j)} \), respectively. By assigning the local observable \( Z_i = \frac{1}{\sqrt{2}}(A_{i_1}^{(j)} + A_{i_1}^{(j)}) \), \( Y_i = \frac{1}{\sqrt{2}}(A_{i_1}^{(j)} - A_{i_1}^{(j)}) \), and \( B_{i_1}^{(j)} = Z_i Y_i' \), the
segmented Bell operator denoted by $I_y' = \frac{1}{\pi} \prod_{i=1}^{K} (A_y^{(i)} + (-1)^y A_y^{(i)}) B_y'$ can be re-expressed as
\begin{equation}
I_y' = \frac{1}{\sqrt{2K}} \prod_{i=1}^{K} (Z_i Z_{K+i})^y (Y_i Y_{K+i})^y.
\end{equation}

It is easy to verify that $\sqrt{2K} (1)^{(y_y)} I_y'$ is a stabilizing operator of the quantum state $\prod_{i=1}^{K} |\Phi^+\rangle_{i\bar{K}+i}$, and we have
\begin{equation}
\max\{\sum_y (-1)^y (I_y') \} = \sqrt{2K}.
\end{equation}

Therein, $K$ operators $\sqrt{2K} (1)^{(y_y)} I_y'$, with $W(y') = 1$, are linearly independent. In the end, these two linear Bell inequalities (12) and (16) can be combined into one that reads
\begin{equation}
\sum_y \langle (I_y) + (-1)^{(y_y)} (I_y') \rangle \leq 2.
\end{equation}

In the quantum region, we have
\begin{equation}
\max\{\sum_y (I_y) + (-1)^y (I_y') \} = 2\sqrt{2K},
\end{equation}
which can be achieved only by the product state $\prod_{i=1}^{K} |\Phi^+\rangle_{i\bar{K}+i}$. The input bits and strings $x, x', z, z', y,$ and $y'$ in the above two Bell tests should be revised as $0x, 1x', 0z, 1z', 0y, \text{and} \, 1y'$, respectively.

At the end of the section, we prove (20) as follows. Without loss of generality, assign Alice's local observables as $Z_{K+i} \rightarrow \frac{1}{2 \cos \theta_i} (A_0^{(i)} + A_1^{(i)})$, $X_{K+i} \rightarrow \frac{1}{2 \sin \theta_i} (A_0^{(i)} - A_1^{(i)})$, where $0 < \theta_i < \frac{\pi}{2}$ and $i = 1, 2, ..., K$; Bob's local observables are assigned as $B_y = \prod_{i=1}^{K} Z_i X_i$, where the input $K$-bit string $y = y_1 y_2 \cdots y_K$. As a result, the segmented Bell operator reads
\begin{equation}
I_y = \prod_{i=1}^{K} (\cos \theta_i Z_i X_{K+i})^y (\sin \theta_i X_i)^y.
\end{equation}

Since the operators $Z_i Z_{K+i}$ and $X_i X_{K+i}$ each stabilize the state $|e_i\rangle_{i\bar{K}+i}$ given the product state $\prod_{i=1}^{K} |e_i\rangle_{i\bar{K}+i}$ distributed in the $K$-source quantum networks, we have
\begin{equation}
\langle (I_y) \rangle_Q = \prod_{i=1}^{K} (\cos \theta_i \sin \theta_i)^y,
\end{equation}
and then
\begin{align}
\max \sum_y \langle (I_y) \rangle_Q &= \max \prod_{i=1}^{K} (\cos \theta_i \sin \theta_i) \\
&= \max (\sqrt{2}) \prod_{i=1}^{K} \sin(\theta_i + \frac{\pi}{4}) \\
&= 2\sqrt{2}.
\end{align}

That is, the maximum of $\sum_y \langle (I_y) \rangle_Q$ can be achieved by setting $\theta_1 = \theta_2 = \cdots = \theta_K = \frac{\pi}{4}$. Similarly, assign the observables
\begin{align}
Z_{K+i} \rightarrow \frac{1}{2 \cos \theta_i} (A_0^{(i)} + A_1^{(i)}), \quad Y_{K+i} \rightarrow \frac{1}{2 \sin \theta_i} (A_0^{(i)} - A_1^{(i)}),
\end{align}
we have $\max \sum_y (-1)^y (I_y') \langle Q \rangle = 2\sqrt{2}$. Therefore, we prove the maximum violation (20). Finally, combining the result (7) and (10), we reach (20) in the case $K = 2$.

4. Non-linear Bell-type inequalities for $K$-source star-network

The nonlinear Bell inequalities in the $K$-source ($K \geq 2$) classical star networks reads
\begin{align}
\sum_y \langle (I_y)^{y_y} \rangle_{LHV} &\leq \sum_y \langle (I_y')^{y_y} \rangle_{LHV} \\
&= \sum_y p_y'^{y_y} \\
&\leq 2^{K-1}.
\end{align}

where the rational number $r = \frac{2r+1}{2r+1}, u, v \in N$, and it is required that $t = rK < 2$. The equality of the second inequality in (24) holds if $p_y = 2^{-K}$. Similarly, we have $\sum_y (-1)^{y_y} \langle (I_y')^{y_y} \rangle_{LHV} \leq 2^{K-1}$, and hence
\begin{align}
\sum_y \langle (I_y)^{y_y} \rangle_{LHV} + (-1)^{y_y} \langle (I_y')^{y_y} \rangle_{LHV} \leq 2^{K-1}.
\end{align}
To find the maximal violation of the nonlinear Bell inequalities, we introduce the function
\[ f(\theta) = \cos t \theta + \sin t \theta, \quad 0 < t < 2 \text{ and } 0 < \theta < \frac{\pi}{2}. \]
It is easy to verify that \( \frac{df}{d\theta}(\theta)\big|_{\theta=\pi/2} = 0 \) and \( \frac{df}{d\theta}(\theta)\big|_{\theta=\pi/2} = 0 \), which leads to
\[ \cos t \theta + \sin t \theta \leq \max_{\theta}(\cos t \theta + \sin t \theta) = 2^{1/2}. \tag{25} \]
Denote the K-bit string \( y = y_1 y_2 \cdots y_K \), where \( y_i = y_1 + 1 \mod 2. \) Let \( t = rK \) have
\[ (I_{y_1}^{(y)} + I_{y_1}^{(y)}) \]
\[ = \Pi_{i=1}^{K} (\cos^2 \theta_j \sin^2 \theta_j) + \Pi_{i=1}^{K} (\cos^2 \theta_j \sin^2 \theta_j) 
= \Pi_{i=1}^{K} (\cos^2 \theta_j \sin^2 \theta_j) + \Pi_{i=1}^{K} (\cos^2 \theta_j \sin^2 \theta_j) \]
\[ = \Pi_{i=1}^{K} \alpha_i^2 + \Pi_{i=1}^{K} \beta_i^2, \tag{26} \]
where the condition either \( \alpha_i = \cos \theta_j \) and \( \beta_i = \sin \theta_j \) or \( \alpha_i = \sin \theta_j \) and \( \beta_i = \cos \theta_j \) holds. Next, we exploit the following lemma.

**Lemma 1. (Mahler inequality)** Let \( \alpha_i \) and \( \beta_i \) be nonnegative real numbers; then,
\[ \Pi_{i=1}^{K} \alpha_i^2 + \Pi_{i=1}^{K} \beta_i^2 \leq \Pi_{i=1}^{K} (\alpha_i + \beta_i)^{1/2}. \tag{27} \]
where the equality holds if \( \alpha_i = \beta_i \) for any i. Combining (25), (26) and (27), we have
\[ \max_{y} (I_{y_1}^{(y)} + I_{y_1}^{(y)} \) \]
\[ \text{max}_{y} \sum_y (I_{y_1}^{(y)} + (-1)^{I_{y_2}^{(y)}}) = 2^{K - \sqrt{2}}. \tag{28} \]
where all the equalities of these inequalities hold if \( \theta_j = \frac{\pi}{2} \) for any y. Denote the K-bit string \( y \) with \( y_1 = 0 \) by \( y = 0y_2 \cdots y_K = 00 \), and we have
\[ \max_{y} \sum_y (I_{y_1}^{(y)} + (-1)^{I_{y_2}^{(y)}}) = 2^{K - \sqrt{2}}. \tag{29} \]
where the equality holds if \( \theta_j = \frac{\pi}{2} \) for any y. Denote the K-bit string \( y \) with \( y_1 = 0 \) by \( y = 0y_2 \cdots y_K = 00 \), and we have
\[ \max_{y} \sum_y (I_{y_1}^{(y)} + (-1)^{I_{y_2}^{(y)}}) = 2^{K - \sqrt{2}}. \tag{29} \]
We reach the maximal violation of the nonlinear Bell inequalities, which leads to
\[ \max_{y} \sum_y (I_{y_1}^{(y)} + (-1)^{I_{y_2}^{(y)}}) = 2^{K + 1 - \sqrt{2}}. \tag{30} \]
As a result, K plays a dominant role in revealing maximum quantum values in (28)–(30).

5. Bell inequalities for \((N, K, m)\)-networks

The star-networks can be extended as the \((N, K, m)\)-networks as follows. There are \(N\) independent sources \(e_1, e_2, \ldots, e_N\) and \(K + m\) observers/receivers Alice_1, Alice_2, ..., Alice_K, Bob_1, ..., Bob_m. The source \(e_i\) sends the particles \((2i - 1)\) to Alice_i and Bob_j, respectively, where \(i = 1, 2, \ldots, K\). The sources \(e_j\) sends two particles \(2i' - 1\) and \(2i'\) to some Bob_j and Bob_j', where \(K + 1 \leq i' \leq N, j', j' \in \{1, 2, \ldots, m\}\). Therein, Bob_j receives \(n_j\) (\(n_j \geq 2\)) particles with particle indices \(j_1, j_2, \ldots, j_{n_j}\) from different \(n_j\) sources. Without loss of generality, let \(j_1 < j_2 < \cdots < j_{n_j}\), and \(j_1 = 2j\) if \(1 \leq j \leq K\). Denote Alice_i’s input and output by \(x_i\) and \(a_{(i)}\), respectively, and Bob_j’s \(n_j\)-bit input string by \(y_{j'}^{(i)} = y_{j_1}^{(i)} y_{j_2}^{(i)} \cdots y_{j_{n_j}}^{(i)}\) and one-bit output by \(b_{j'}^{(i)}\), where the input bits \(y_{j_1}^{(i)}, y_{j_2}^{(i)}, \ldots, y_{j_{n_j}}^{(i)}\), \(x_i \in \{0, 1\}\), and the output \(b_{j'}^{(i)}, a_{(i)} \in \{1, -1\}\). Denote the correlator
\[ I_{Y}^{(N, K, m)} = \frac{1}{2} \prod_{i=1}^{N} (a_{(i)}^{(i)} + (-1)^{Y_i} a_{(i)}^{(i)}) \]
where \(Y\) denotes the K-bit string \(y_1 y_2 \cdots y_K\). \(y_j = y_{j}^{(i)}\) In the classical \((N, K, m)\)-networks, the local hidden variable \(\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_N\) and shared randomness are distributed throughout the network. Denote the probability
\[ p_Y = \prod_{i=1}^{N} (a_{(i)}^{(i)} + (-1)^{Y_i} a_{(i)}^{(i)}) \]
and the first linear Bell-type local inequality reads

\[ \]
In the Bell test of the second Bell inequality, denote Alice’s input and output by \( x_i \) and \( a_{s(i)} \), respectively, and Bob’s \( n_i \)-bit input string by \( y_{i(j)} \) and \( b_{r(j)} \) and one-bit output by \( b_{r(j)} \), where the input bits \( y_{i(j)} \), \( y_{i(j)}', \ldots, y_{i(j)}' \), \( x_i \in \{0, 1\} \) and the output bits \( b_{r(j)} \) are \( \in \{1, -1\} \). Denote the correlator
\[
I_Y^{(N,K,m)} = \frac{1}{2^n} \prod_{i=1}^{m} b_{r(i)} \prod_{j=1}^{K} (a_{s(i)} + (-1)^{y_{i(j)}} a_{s(i)}) \text{, where } Y \text{ denotes the K-bit string } y_{1} y_{2} y_{j} \ldots y_{K},
\]
\( y_{j} = y_{j}' \). In the classical region, denote the probability
\[
p_Y y' = \sum_{\alpha} P\left( \frac{1}{2^n} \prod_{i=1}^{m} (a_{s(i)} + (-1)^{y_{i(j)}} a_{s(i)}) \prod_{j=1}^{K} = 1(\lambda) P(\lambda), \text{ and the second linear Bell-type local inequality reads}
\]
\[
\begin{align*}
\sum_{\alpha} y'(1)^{(Y)} (I_Y^{(N,K,m)})_{LV} \\
\leq \sum_{\alpha} (I_Y^{(N,K,m)})_{LV}
\end{align*}
= 1.
\]

In the quantum region, let the observables \( A_{s(i)} = \frac{1}{\sqrt{2}} (Z_{i} + (-1)^{y_{i(j)}} X_{i}) \), \( A_{s(i)} = \frac{1}{\sqrt{2}} (Z_{i} + (-1)^{y_{i(j)}} Y_{i}) \), and
\[
B_{r(j)} = \prod_{i=1}^{m} Z_{i}^{y_{i(j)}} X_{i}^{y_{i(j)}} \text{ (} B_{r(j)} = \prod_{i=1}^{m} Z_{i}^{y_{i(j)}} Y_{i}^{y_{i(j)}} \text{) in the first (second) linear Bell inequality. Hence, the}
\]
segmented Bell operator for the first Bell inequality reads
\[
I_Y^{(N,K,m)} = \frac{1}{\sqrt{2^K}} \prod_{i=1}^{N} (Z_{i} - i Z_{i}) Y_{i}(Y_{i} - i Y_{i}) Y_{i},
\]  
(31)
and the one for the second Bell inequality reads
\[
I_Y^{(N,K,m)} = \frac{1}{\sqrt{2^K}} \prod_{i=1}^{N} (Z_{i} - i Z_{i}) Y_{i}(Y_{i} - i Y_{i}) Y_{i},
\]  
(32)
And we have
\[
\max\{\sum_{\alpha} (I_{Y}^{(N,K,m)})_{Q}\} = \sqrt{2^K}, \quad \alpha = Y, \ Y'.
\]  
(33)
Finally note that \( 2N \) operators \( \sqrt{2^K} I_{Y}^{(N,K,m)} \) - \( 1(\lambda) \sqrt{2^K} I_{Y}^{(N,K,m)} \) with \( W(Y) = 1 \) and \( W(Y') = 1 \) equal to one are linearly independent and each of them can stabilize the state \( \prod_{i=1}^{K} \Phi^{(X_{i} = i X_{i})} \). Hence only \( \prod_{i=1}^{K} \Phi^{(X_{i} = i X_{i})} \) can reach the maximum quantum values (31) and (32) simultaneously. We omit the construction of non-linear Bell inequalities and their maximal violation in the \((N, K, m)\)-networks that are very close to those in the \(K\)-source star-networks. In the end, it should be noted that \(K\)-source star-networks can be recovered from \((N, K, m)\)-networks by setting \(N = m = K\) and \(Bob_{1} = \cdots = Bob_{K} = Bob\). We omit the proofs of (31), (32), and (33) since they are almost the same as those of (14), (18), and (20). In addition, one can also construct the nonlinear Bell inequalities of the \((N, K, m)\)-networks, which are very similar to those in the \(K\)-source star-networks.

6. Two-Source scenario revisited

In the above discussion, all quantum sources each emit two-qubit Bell states in quantum networks. Here we revisit the two-source networks with a quantum source emitting three-qubit GHZ states. In details, source \( e_{1} \) emits a state of particles 1 and 2, which are sent to Alice and Bob; source \( e_{2} \) emits a state of particles 3, 4, and 5. We consider two different particle distributions as follows.

Case (a) Particles 3 and 4 are sent to Bob and particle 5 is sent to Charlie. The one-bit inputs for Alice and Charlie are \( x \) and \( z \), respectively, and the three-bit input string for Bob is \( y_{3} y_{4} y_{5} \). The output bits for Alice and Charlie are denoted by \( a_{x} \), \( b_{y_{3} y_{4}} \), and \( c_{z} \). Consider the correlators
\[
I_{y_{3} y_{4} y_{5}} = \frac{1}{4} (a_{x} + (-1)^{y_{3}} a_{x}) b_{y_{3} y_{4}} (c_{z} + (-1)^{y_{3} + y_{4} + c_{z}}), (y_{3} y_{4} y_{5}) \in \{001, 000, 100, 110\}
\]
and
\[
I_{y_{3} y_{4} y_{5}} = \frac{1}{4} (a_{x} + (-1)^{y_{3}} a_{x}) b_{y_{3} y_{4}} (c_{z} + (-1)^{y_{3} + y_{4} + c_{z}}), (y_{3} y_{4} y_{5}) \in \{010, 000, 100, 101\}.
\]
Similarly, denote the probability distribution in classical networks.
On the other hand, we have
\[
P_{\lambda} = \sum_{x} P\left( \frac{1}{4}(a_{0} + (-1)^{i}a_{i})(c_{0} + (-1)^{k}c_{k}) \right) = 1 |\lambda\rangle P(\lambda)
\]
\[
= \sum_{x} P\left( \frac{1}{4}(a_{0} + (-1)^{i}a_{i})b_{xj\gamma}^{j} (c_{0} + (-1)^{k}c_{k}) \right) = 1 |\lambda\rangle P(\lambda)
\]
\[
= \sum_{x} P\left( \frac{1}{4}(a_{0} + (-1)^{i}a_{i})b'_{xj\gamma}^{j} (c_{0} + (-1)^{k}c_{k}) \right) = 1 |\lambda\rangle P(\lambda)
\]
(34)
(35)

Note that \(\langle I_{yj\beta\gamma}^{j} \rangle_{LHV} = \langle I_{yj\beta\gamma}^{j} \rangle_{LHV} = p_{j(y_{j}+x_{j}+1)}\), and we have the first linear Bell inequality
\[
\sum_{j} \sum_{i=0}^{1} P_{j\lambda} \leq 1.
\]
(36)

In the quantum region, the source \(c_{1}\) prepares \(|\Phi^{+}\rangle_{12}\), and \(c_{2}\) prepares the three-qubit GHZ state \(|GHZ\rangle_{345}\) with the density matrix \([GHZ]\)_{345} = \(\frac{1}{2}(I_{x} + X_{x}Z_{z})Z_{x}(I_{y} + Z_{y}X_{y})(I_{z} + Z_{z}X_{z})\). Given the input \((z, z)\), Alice (Charlie) measures the observable \(A_{s} = \frac{1}{2}(Z_{z} + (-1)^{s}X_{z})\) \((C_{z} = \frac{1}{2}(Z_{z} + (-1)^{s}X_{z}))\); given the input string \(y_{1}y_{2}\), Bob measures the observable \(B_{xj\beta\gamma}^{j} = \prod_{i=2}^{n} Z_{i}^{y}X_{i}^{x}\) or \(B_{xj\beta\gamma}^{j} = \prod_{i=2}^{n} Z_{i}^{y}X_{i}^{x}\) if \((y_{1}y_{2}) \in s\) or \((y_{1}y_{2}) \in s'\), respectively. Four independent operators 2\(I_{yj\beta\gamma}^{j}= (Z_{y}(Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z})\) also stabilize the state \(|\Phi^{+}\rangle_{12} \rangle |GHZ\rangle_{345}. Combining (36) and (37), we derive the linear Bell inequality that reads
\[
\sum_{j} \sum_{i=0}^{1} P_{j\lambda} \leq 1.
\]
(37)

It is to verify that these four independent operators \(2I_{yj\beta\gamma}^{j}= (Z_{y}(Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z}), 2I_{yj\beta\gamma}^{j}= (Z_{y}Z_{x}Z_{z})\) also stabilize the state \(|\Phi^{+}\rangle_{12} \rangle |GHZ\rangle_{345}. Combining (36) and (37), we derive the linear Bell inequality that reads
\[
\sum_{j} \sum_{i=0}^{1} P_{j\lambda} + \sum_{j} \sum_{i=0}^{1} P_{j\lambda} \leq 2.
\]

On the other hand, we have
\[
\max \left\{ \sum_{j} \sum_{i=0}^{1} P_{j\lambda} + \sum_{j} \sum_{i=0}^{1} P_{j\lambda} \right\} = 4,
\]
which can be achieved using the state \(|\Phi^{+}\rangle_{12} \rangle |GHZ\rangle_{345}. Case (b) In this case, particles 3, 4, and 5 are sent to Bob, Charlie1, and Charlie2, respectively. Denote the one-bit inputs for Alice, Charlie1, and Charlie2 by \(y_{3}, z_{3}\), and \(z_{5}\); the one-bit outputs by \(a_{3}^{(1)}, c_{3}^{(s)}\) and \(c_{5}^{(s)}\), respectively. Denote the two-bit string input and one-bit output for Bob by \(y_{j5}\) and \(b_{j5}\), respectively. In classical networks, we introduce the expectation values of the correlators
\[
\langle I_{yj\gamma}^{j} \rangle = \frac{1}{8} \langle (a_{0}^{(1)} + (-1)^{i}a_{1}^{(1)})(c_{0}^{(4)} + (-1)^{k}c_{1}^{(4)})(c_{0}^{(5)} + (-1)^{l}c_{1}^{(5)}) \rangle
\]
and
\[
\langle I_{yj\gamma}'^{j} \rangle = \frac{1}{8} \langle (a_{0}^{(1)} + (-1)^{i}a_{1}^{(1)})(c_{0}^{(4)} + (-1)^{k}c_{1}^{(4)})(c_{0}^{(5)} + (-1)^{l}c_{1}^{(5)}) \rangle.
\]
Similarly, denote the probability distribution in classical networks
\[
P_{\lambda} = \sum_{x} P\left( \frac{1}{8}(a_{0}^{(1)} + (-1)i^{i}a_{1}^{(1)})(c_{0}^{(4)} + (-1)^{k}c_{1}^{(4)})(c_{0}^{(5)} + (-1)^{l}c_{1}^{(5)}) \right) = 1 |\lambda\rangle P(\lambda)
\]
\[
= \sum_{x} P\left( \frac{1}{8}(a_{0}^{(1)} + (-1)^{i}a_{1}^{(1)})(c_{0}^{(4)} + (-1)^{k}c_{1}^{(4)})(c_{0}^{(5)} + (-1)^{l}c_{1}^{(5)})b_{j5} \right) = 1 |\lambda\rangle P(\lambda),
\]
(38)
and we have \(\langle I_{yj\gamma}^{j} \rangle_{LHV} = p_{yj\gamma\beta\gamma}^{j}\) and \(\langle I_{yj\gamma}'^{j} \rangle_{LHV} = p_{yj\gamma\beta\gamma}^{j}. The Bell-type inequality reads
\[
\sum_{j} \sum_{i=0}^{1} P_{j\lambda} \leq 1.
\]
(39)

In the quantum region, given the input \((z_{3}, z_{5})\), Alice (Charlie1, Charlie2) measures the observable
\[
A_{x}^{(1)} = \frac{1}{2}(Z_{3} + (-1)^{s}X_{3})\ (C_{z}^{(s)} = \frac{1}{2}(Z_{z} + (-1)^{s}X_{z})), C_{z}^{(s)} = \frac{1}{2}(Z_{z} + (-1)^{s}X_{z})\); given the input string \(y_{j5}\), Bob measures the observable \(B_{xj\beta\gamma}^{j} = \prod_{i=2}^{n} Z_{i}^{y}X_{i}^{x}\). Denote the correlator operators
\[
I_{x_{23}} = \frac{1}{8} (A_0^{(4)} + (-1)^{x_0} A_1^{(1)}) B_{x_{23}} (C_0^{(4)} + (-1)^{x_1} C_1^{(1)}) (C_0^{(5)} + (-1)^{x_2} C_1^{(4)}) (C_0^{(5)} + (-1)^{x_3} C_1^{(5)})
\]
and
\[
I'_{x_{23}} = \frac{1}{8} (A_0^{(4)} + (-1)^{x_0} A_1^{(1)}) B_{x_{23}} (C_0^{(4)} + (-1)^{x_1} C_1^{(1)}) (C_0^{(5)} + (-1)^{x_2} C_1^{(4)}) (C_0^{(5)} + (-1)^{x_3} C_1^{(5)}).
\]
These eight operators 2√2 \( I_{x_{23}} \) and 2√2 \( I'_{x_{23}} \) in \([Z_1 Z_2, X_1 X_3] \otimes [Z_1 Z_2, X_1 X_3, X_2 X_3 X_4 Z_5 - X_1 X_3 X_5] \) stabilize the state |Φ^+\>_12 |GHZ\>_345. As a result,
\[
\max \sum_{x_{23}} \langle (-1)^{x_{23}} | I_{x_{23}} \rangle_Q + (-1)^{x_{23}} \langle I'_{x_{23}} \rangle_Q = 2\sqrt{2},
\]
which can be achieved using the state |Φ^+\>_12 |GHZ\>_345.

7. Conclusions

In conclusion, we investigate Bell nonlocality with variant particle distributions in the networks. As for genuine Bell locality, local hidden variables and randomness can be perfectly cloned and spread throughout the networks. In quantum networks, each quantum source emits the two-qubit Bell states. Instead of brutal Bell locality, local hidden variables and randomness can be perfectly cloned and spread throughout the networks. In the upcoming study, we will explore Bell nonlocality with stabilizer states such as graph states [27, 28] and codewords of stabilizer-based quantum error correction distributed in quantum networks [22]. The local observables therein can be made up of “cut-graft-mixing” stabilizing operators and logical operators [27]. In addition, we seek Bell inequalities tailored for the non-maximal entangled states distributed in the networks and the trade-off of randomness and nonlocality therein [29, 30]. In addition, we will further study the steering effect and the self-test in the networks [31, 32].

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Data availability statement

No new data were created or analysed in this study.

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References

[1] Einstein A, Podolsky B and Rosen N 1935 Can quantum mechanical description of physical reality be considered complete? Phys. Rev. 47 777
[2] Bell J S 1964 On the Einstein-Podolsky-Rosen paradox Physics 1 195
[3] Bae K and Son W 2018 Generalized nonlocality criteria under the correlation symmetry Phys. Rev. A 98 022116
[4] Son W, Lee J and Kim M S 2006 Generic Bell Inequalities for Multipartite Arbitrary Dimensional Systems Phys. Rev. Lett. 96 060406
[5] Acin A, Brunner N, Gisin N, Massar S, Pironio S and Scarani V 2007 Device-Independent Security of Quantum Cryptography Against Collective Attacks Phys. Rev. Lett. 98 230501
[6] Buhrman H, Cleve R, Massar S and de Wolf R 2010 Nonlocality and communication complexity Rev. Mod. Phys. 82 665
[7] Pironio S et al 2010 Random numbers certified by Bell’s theorem Nature 464 1021
[8] Reichhardt B W, Unger F and Vazirani U 2013 Classical command of quantum systems Nature 496 456
[9] Liang Y-C, Harrigan N, Bartlett S D and Rudolph T 2010 Nonclassical correlations from randomly chosen local measurements Phys. Rev. Lett. 104 050401
[10] Furkan Senel C, Lawson T, Kaplan M, Markham D and Diamanti E 2015 Demonstrating genuine multipartite entanglement and nonseparability without shared reference frames Phys. Rev. A 91 052118
[11] Wang N N, Pozas-Kerstjens A, Zhang C, Liu B-H, Huang Y-F, Li C-F, Guo G-C, Gisin N and Tavakoli A 2023 Certification of nonclassicality in all links of a photonic star network without assuming quantum mechanics Nat. Comm. 14 2153
[12] Mao Y-L, Li Z-D, Steffenlango A, Guo B, Liu B, Xu S, Gisin N, Tavakoli A and Fan J 2023 Recycling nonlocality in quantum star networks Phys. Rev. Research 5 013104
[13] Pozas-Kerstjens A, Gisin N and Renou M-O 2023 Proofs of network quantum nonlocality in continuous families of distributions Phys. Rev. Lett. 130 090201
[14] Pozas-Kerstjens A, Gisin N and Tavakoli A 2022 Full network nonlocality Phys. Rev. Lett. 128 010403

10
[15] Gisin N, Bancel J D, Cai Y, Remy P, Tavakoli A and Zambrini E 2020 Constraints on nonlocality in networks from no-signaling and independence Nat. Comm. 11 2378
[16] Branciard C, Gisin N and Pironio S 2010 Characterizing the nonlocal correlations created via entanglement swapping Phys. Rev. Lett. 104 170401
[17] Branciard C, Rosset D, Gisin N and Pironio S 2012 Bilocal versus nonbilocal correlations in entanglement-swapping experiments Phys. Rev. A 85 032119
[18] Carvacho G, Andreoli F, Santodonato L, Bentivegna M, Chaves R and Sciarrino F 2017 Experimental violation of local causality in a quantum network Nat. Commun. 8 14775
[19] Saunders D J, Bennett A J, Branciard C and Pryde G J 2017 Experimental demonstration of nonbilocal quantum correlations Sci. Adv. 3 e1602743
[20] Tavakoli A, Skrzypczyk P, Cavalcanti D and Acín A 2014 Nonlocal correlations in the star-network configuration Phys. Rev. A 90 062109
[21] Luo M-X 2018 Computationally efficient nonlinear bell inequalities for quantum networks Phys. Rev. Lett. 120 140402
[22] Hsu L-Y and Chen Exploring Bell C-H 2021 nonlocality of quantum networks with stabilizing and logical operators Phys. Rev. Res. 3 023139
[23] Sneha Munshi A K 2022 Pan, Characterizing nonlocal correlations through various n-locality inequalities in a quantum network Phys. Rev. A 105 032216
[24] Munshi S and Pan A K 2021 Generalized n-locality inequalities in star-network configuration and their optimal quantum violations Phys. Rev. A 104 042217
[25] Tavakoli A, Pozas-Kerstjens A, Luo M-X and Renou M-O 2022 Bell nonlocality in networks Rep. Prog. Phys. 85 056001
[26] Smolin J A 2001 Four-party unlockable bound entangled state Phys. Rev. A 63 032306
[27] Hsu L-Y 2006 Bell-type inequalities edembled in the subgraph of graph states Phys. Rev. A 73 042308
[28] Cabello A, Gühne O and Rodríguez D 2008 Mermin inequalities for perfect correlations Phys. Rev. A 77 062106
[29] Acín A, Massar S and Pironio S 2012 Randomness versus nonlocality and entanglement Phys. Rev. Lett. 108 100402
[30] Wooltorton L, Brown P and Colbeck R Tight analytic bound on the trade-off between device-independent randomness and nonlocality arXiv:2205.00124v1
[31] Jones B D-M, Šupić I, Uola R, Brunner N and Skrzypczyk P 2021 Network Quantum Steering Phys. Rev. Lett. 127 170405
[32] Šupić I, Bowles J, Renou M-O, Acín A and Hoban M J Quantum networks self-test all entangled states arXiv:2201.05032