Diffraction by a Dirichlet right angle on a discrete planar lattice

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Abstract

A problem of scattering by a Dirichlet right angle on a discrete square lattice is studied. The waves are governed by a discrete Helmholtz equation. The solution is looked for in the form of the Sommerfeld integral. The Sommerfeld transformant of the field is built as an algebraic function. The paper is a continuation of [1].
1 Introduction

This paper continues the research presented in [1]. A 2D discrete square lattice is under consideration. The lattice bears a discrete Helmholtz equation with a 5-point stencil. The
first quadrant of the lattice is blocked by setting the field equal to zero there. The problem of
diffraction of an incident plane wave by the blocked angle is studied. The motivation and the
literature review for such a problem can be found in [1].

A new formalism has been developed for this problem. Similarly to continuous problems
of diffraction in angular domains, a branching surface $S_3$ is introduced in the physical discrete
plane, and the diffraction problem is reformulated as a propagation problem on this surface
using the reflection principle. An analog of the Sommerfeld integral for field representation is
introduced. This integral is a contour integral on the dispersion diagram of the lattice, which is
a compact Riemann surface $R$ equipped with a structure of a complex manifold. Topologically,
$R$ is a torus. The integrand is a differential form that is multivalued on the dispersion diagram
and possesses prescribed poles corresponding to the incident wave and reflected waves. The
contour of integration depends on the position of the observation point, and “slides” along the
surface as the observation point moves. We introduce a 3-sheet covering of $R$, named $R_3$, to
take into account the multivaluedness of the integrand.

The integrand form contains an unknown function referred to as the Sommerfeld trans-
formant of the field. It obeys a certain functional problem. In [1] the authors found this
transformant in terms of elliptic functions. However, such a representation is not convenient.
Moreover, it can be proven that such a transformant should be an algebraic function, thus,
a representation through the elliptic functions is somewhat unnecessarily complicated. The
aim of the current paper is to build the Sommerfeld transformant of the field as an algebraic
function, and then to study the properties of the field.

We should note that Riemann surfaces and Abelian integrals have been used for solving
diffraction problem before. The context of the application of this theory was the Riemann–
Hilbert problem on Riemann surfaces [2, 3, 4, 5].

The paper is organized as follows.

In Section 2 some preliminary steps are made. The initial diffraction problem is formulated
and is reformulated as a propagation problem on a branched lattice $S_3$. The Riemann surface
$R$ related to the dispersion diagram of the lattice is introduced. The Riemann surface $R_3$
 describing all plane waves on $S_3$ is built. Both Riemann surfaces are equipped with a structure of
a complex manifold. The Sommerfeld integral for the field is written, and Functional problem 1
for the Sommerfeld transformant $A$ is formulated. The functional problem is constituted in
finding a function meromorphic on a given Riemann surface, having prescribed poles on it with
known residues.

In Section 3 we introduce some basic mathematical concepts that are necessary to solve the
functional problem. We introduce the transformations of $R_3$ and $R$ (the desk transformations
in terms of [6]). These transformations enable to convert Functional problem 1 into a set of
partial problems that are slightly simpler. The key idea for solving the functional problem is to
introduce for each Riemann surface a field (in the algebraic meaning of this word) of functions
meromorphic on this surface. The Sommerfeld transformant belongs to some of these fields, $K_3$,
which is an extension of a simpler field $K_1$. Then, one can construct a basis of this extension,
which is a set of three functions $[1, F_1, F_2]$, two of which are quite difficult to find. Once the
basis is built, one can find the function $A$ by constructing relatively simple coefficients of the
expansion.

In Section 4 we build the basis function \([1, F_1, F_2]\). This is a tricky part of the paper. The functions \(F_{1,2}\) are found as cubic roots of functions \(G_{1,2}\) specially tailored on the base of Abel’s theorem. The Abel’s theorem is then replaced by an algebraic condition.

In Section 5 we build the coefficients of expansion of the Sommerfeld transformant \(A\) using the basis \([1, F_1, F_2]\). This is done in an elementary way by studying the zeros and poles of \(F_1, F_2,\) and \(A\).

In Section 6 a detailed description of the numerical procedure is given, and some results are demonstrated.

The Appendix contains the proof of the validity of the Sommerfeld representation of the wave field, some statements completing the algebraization of Abel’s theorem, and the proof of the theorem on which the computation of the functions \(F_{1,2}\) is based.

The paper is written mainly in the Theorem / Proof / Remark style for simplicity of understanding.

There are two important issues, clearly related to the diffraction problem considered in the current paper, namely, they are the computation of the directivity of the far field and building a low-frequency approximation (the limiting case of the continuous medium). Both issues are interesting and not elementary. We do not include them into the paper due to the lack of space, and are planning to write a separate work on them.

2 The Sommerfeld integral for a discrete wedge diffraction problem

2.1 Problem formulation

Consider a planar square lattice whose nodes have integer indices \((m, n)\). Let the homogeneous discrete Helmholtz equation

\[
\begin{align*}
  u(m, n - 1) + u(m, n + 1) + u(m - 1, n) + u(m + 1, n) + (K^2 - 4)u(m, n) &= 0 \\
\end{align*}
\]

be valid in the domain

\[
m < 0 \quad \text{or} \quad n < 0
\]

(see Fig. 1). The wavenumber parameter \(K\) has a positive real part and a small positive imaginary part corresponding to an energy absorption.

The set of nodes with

\[
(m = 0 \text{ and } n \geq 0) \text{ or } (n = 0 \text{ and } m \geq 0)
\]

is the boundary of the domain. We assume that this boundary is of the Dirichlet type, so

\[
u(m, n) = 0
\]

on it.
In order to describe the incident wave, we introduce plane waves on such a lattice:

\[ w_{m,n} = w_{m,n}(x, y) = x^m y^n, \]  

provided that the pair of wavenumber parameters \((x, y)\) obey the dispersion equation

\[ \hat{D}(x, y) = 0, \]  

\[ \hat{D}(x, y) \equiv x + x^{-1} + y + y^{-1} + K^2 - 4. \]  

One can see that (4) guarantees fulfillment of the homogeneous Helmholtz equation (1) by \(w\).

The total wave for our problem is a sum of the incident wave and the scattered wave:

\[ u(m, n) = u_{\text{in}}(m, n) + u_{\text{sc}}(m, n), \]

where

\[ u_{\text{in}}(m, n) = w_{m,n}(x_{\text{in}}, y_{\text{in}}) = x_{\text{in}}^m y_{\text{in}}^n, \]  

is the incident plane wave, and \(x_{\text{in}}\) and \(y_{\text{in}}\) are wavenumber parameters. Indeed, they obey the dispersion equation:

\[ \hat{D}(x_{\text{in}}, y_{\text{in}}) = 0, \]  

For simplicity of the problem formulation, we assume that the wave travels into the direction of positive \(m\) and \(n\). Since the waves have some attenuation, this means that

\[ |x_{\text{in}}| < 1, \quad |y_{\text{in}}| < 1. \]
We introduce the angle of incidence by the relation

$$\phi_{\text{in}} \equiv \arctan\left(\frac{y_{\text{in}} - y_{\text{in}}^{-1}}{x_{\text{in}} - x_{\text{in}}^{-1}}\right).$$  

(8)

Such a definition of the angle may seem not obvious, but it is motivated by the saddle point argument in [1], see equation (47) there. We assume that angle $\phi_{\text{in}}$ is real, and

$$0 < \phi_{\text{in}} < \pi/2.$$  

(9)

The scattered wave $u_{\text{sc}}$ should obey the radiation condition, i.e. it should decay at infinity. The aim is to find $u_{\text{sc}}$.

Remark. For a $K$ with $\text{Im}[K^2] \neq 0$ and under the condition (9) the scattered field $u_{\text{sc}}$ belongs to the space $l_2(\mathbb{Z}^2)$. The solution is unique. The proof of uniqueness can be found in Appendix 4.

2.2 Reformulation of the diffraction problem on a branched discrete lattice

In the current paper we use the Sommerfeld integral technique for an angular domain of a discrete lattice. It is well-known that the Sommerfeld method starts with applying the principle of reflection, with the aim to get rid of the scatterers and to obtain a diffraction problem on a branched surface.

Introduce the angle

$$\tan \phi = n/m,$$

(10)

such that the domain shown in Fig. 1 is $\pi/2 \leq \phi \leq 2\pi$. Take the solution $u(m, n)$ in this domain. Reflect the domain with respect to the horizontal axis and define the function $-u(m, -n)$ on it. Connect the initial domain and the reflected domain by merging the nodes on the boundaries at $m \geq 0, n = 0$. As the result, get the discrete angular domain $\pi/2 \leq \phi \leq 7\pi/2$ with some function $u(m, n)$ defined on it. The following proposition is valid:

**Proposition 1** The Helmholtz equation (1) is valid on the merged nodes $m > 0, n = 0$.

To prove this proposition, one can directly check (1) at corresponding nodes. The value of $u(m, n)$ is zero at them due to the boundary condition, and $u(m, n)$ is odd with respect to $n$ by construction.

Thus, we applied the reflection principle once, deleted a horizontal part of the boundary, and obtained a wider discrete angular domain. By repeating this procedure two more times, now with respect to the vertical part of the boundary, obtain a function $u$ defined on a branched discrete surface $S_3$ shown in Fig. 2. The boundaries that should be merged with each other are shown by equal Roman numbers.

The surface $S_3$ has three points over each point $(m, n) \neq (0, 0)$; the origin is the branch point, thus it has three sheets over a discrete planar lattice, and any single-valued function on
$S_3$ is periodic with respect to $\phi$ with the period $6\pi$, i.e. the branch point has order three. The function $u(m,n)$ built on $S_3$ by the reflections obeys (1) at any point of $S_3$ except the origin. The equation connects $u$ at some node with its four neighbors in the mesh.

There are four incident plane waves on $S_3$ (shown in Fig. 2). One can summarize these waves in the following table:

| angle of incidence $\phi'$ | amplitude | formula |
|---------------------------|-----------|---------|
| $\phi_{in} + \pi$         | 1         | $x_{in}^m y_{in}^n$ |
| $3\pi - \phi_{in}$        | -1        | $-x_{in}^m y_{in}^{-n}$ |
| $\phi_{in} + 4\pi$        | 1         | $x_{in}^{-m} y_{in}^{-n}$ |
| $6\pi - \phi_{in}$        | -1        | $-x_{in}^{-m} y_{in}^{n}$ |

Each plane wave is visible in the domain $\phi' - \pi < \phi < \phi' + \pi$, where $\phi'$ is the angle of incidence of the corresponding wave.

As it is shown in [1], the Sommerfeld integral provides a solution of the diffraction problem on $S_3$, i.e. it yields a function $u(m,n)$ single valued on $S_3$, obeying (1) everywhere except the origin, and composed of the incident waves (each at its visibility domain), and the scattered field decaying as $\sqrt{m^2 + n^2} \to \infty$. The restriction of such a solution onto the angle $\pi/2 \leq \phi \leq 2\pi$ is a solution of the initial diffraction problem.

**Remark.** The structure of the problem on $S_3$ gives important clues to the functional problem for the Sommerfeld’s transformant. Namely, it makes clear why it is necessary to study a 3-sheet covering of $\mathbb{R}$ (see below), and why the transformant has four poles. For the first question, as it
is shown in [1], the number of sheets of the branched discrete lattice is equal to the multiplicity of the transformant over $\mathbf{R}$. For the second question, each pole corresponds to a plane incident wave on $\mathbf{S}_3$.

2.3 Riemann surfaces $\mathbf{R}$ and $\mathbf{R}_3$

The equation (4) can be solved with respect to $y$ for some fixed $x$:

$$y(x) = y_\pm(x) = -\frac{K^2 - 4 + x + x^{-1}}{2} \pm \sqrt{\left(\frac{K^2 - 4 + x + x^{-1}}{2}\right)^2 - 4}$$

(11)

The function $y(x)$ is a two-valued function. It is easy to check that its two values have the property $y_+(x)y_-(x) = 1$. The Riemann surface of $y(x)$ is denoted by $\mathbf{R}$ and is shown in Fig. 3.

![Fig. 3: Contours $\sigma_\alpha$ and $\sigma_\beta$ on $\mathbf{R}$](image)

Branch points of $\mathbf{R}$ are $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}$:

$$\eta_{1,1} = -\frac{d}{2} - \frac{i\sqrt{4 - d^2}}{2}, \quad \eta_{1,2} = -\frac{d}{2} + \frac{i\sqrt{4 - d^2}}{2}, \quad d = K^2 - 2,$$

(12)

$$\eta_{2,1} = -\frac{d}{2} + \frac{\sqrt{d^2 - 4}}{2}, \quad \eta_{2,2} = -\frac{d}{2} - \frac{\sqrt{d^2 - 4}}{2}, \quad d = K^2 - 6.$$  

(13)

All branch points are of the second order. One can see that

$$y(\eta_{2,1}) = y(\eta_{1,1}) = 1, \quad y(\eta_{2,2}) = y(\eta_{1,2}) = -1.$$

The branch points are connected by cuts on $\mathbf{R}$ (shown by bold lines in the figure). The cuts are conducted in such a way that $|y(x)| = 1$ on them. This corresponds to the conditions

$$\text{Im}[K^2 - 4 + x + x^{-1}] = 0, \quad -2 < \text{Re}[K^2 - 4 + x + x^{-1}] < 2.$$  

(14)

The sides of the cuts marked by the same Roman numbers are attached to each other.
We select the physical sheet (or sheet 1) of $R$ as the sheet on which $|y| \leq 1$. Let $\Xi(x)$ be the value of the function $y(x)$ on the physical sheet, i.e.

$$\Xi(x) = -\frac{K^2 - 4 + x + x^{-1}}{2} + \frac{\sqrt{(K^2 - 4 + x + x^{-1})^2 - 4}}{2},$$

(15)

where the square root is taken in its “arithmetical” sense (with a cut along the negative half-axis of the argument). Obviously,

$$\Xi^{-1}(x) = -\frac{K^2 - 4 + x + x^{-1}}{2} - \frac{\sqrt{(K^2 - 4 + x + x^{-1})^2 - 4}}{2},$$

(16)

and this is the value of $y(x)$ on sheet 2.

The Riemann surface $R$ is compactified, i.e. the infinite points of sheet 1 and of sheet 2 are added (this means that $x$ takes values on the Riemann sphere $\overline{\mathbb{C}}$). One can check that $\Xi(\infty) = 0$. It is important that the infinite points are not branch points of $R$, i.e. a bypass about an infinity does not change the sheet of the Riemann surface.

The authors made some efforts in [1] to demonstrate that $R$ is topologically a torus. In particular, Fig. 4 and Fig. 6 of [1] demonstrate coordinates $(\alpha, \beta)$ on the torus, both taking values on a circle $0 \leq \alpha, \beta < 2\pi$.

Introduce the function $\Upsilon(x)$ single-valued on $R$ and defined on the physical sheet (sheet 1) by

$$\Upsilon(x) = x(\Xi(x) - \Xi^{-1}(x)) = x\sqrt{(K^2 - 4 + x + x^{-1})^2 - 4}.$$  

(17)

The square root takes the arithmetical value. Indeed, on sheet 2 this function is equal to

$$\Upsilon(x) = -x\sqrt{(K^2 - 4 + x + x^{-1})^2 - 4}.$$  

The importance of this function is explained by its equivalent form:

$$\Upsilon(x) = \sqrt{(x - \eta_{1,1})(x - \eta_{1,2})(x - \eta_{2,1})(x - \eta_{2,2})}.$$  

(18)

One can see that on the physical sheet

$$y(x) = -\frac{K^2 - 4 + x + x^{-1}}{2} + \frac{\Upsilon(x)}{2x}.$$  

(19)

The function $\Upsilon(x)$ is the irrationality of $\Xi(x)$, thus the Riemann surface of $\Upsilon(x)$ is the same as of $y(x)$, i.e. this surface is $R$.

Similarly to [1], construct Riemann surface $R_3$ as follows. Take six copies of the compactified complex plane of $x$ (the sheets), make cuts in them (the same as for $R$, see by (14)), and assembly surface according to scheme shown in Fig. 4 by Roman numbers. Note that this surface is not introduced as a Riemann surface of a certain function, and our aim below is to find such functions.
Introduce notations for the points of the compactified complex plane $\mathbb{C}$ and for the Riemann surfaces $\mathbf{R}$, $\mathbf{R}_3$. The points of $\mathbf{R}_3$ will be indicated by the $\hat{}$ decoration, the points of $\mathbf{R}$ will be indicated by the $\tilde{}$ decoration, and the points of $\mathbb{C}$ will exist without decorations. For example,

$$\tilde{x} \in \mathbf{R}_3, \quad \hat{x} \in \mathbf{R}, \quad x \in \mathbb{C}.$$ 

Introduce the projections

$$\tilde{x} \xrightarrow{P_{3:1}} \hat{x} \xrightarrow{P_{1:0}} x.$$ 

Let both projections keep the **affix** (the value of $x$), so, the projections $P_{1:0}(\cdot)$ and $P_{1:0}(P_{3:1}(\cdot))$ take an affix of a point of a Riemann surface. Let the projection $P_{3:1}$ map the sheets 1, 3, 5 of $\mathbf{R}_3$ onto sheet 1 of $\mathbf{R}$, and the sheets 2, 4, 6 of $\mathbf{R}_3$ onto sheet 2 of $\mathbf{R}$. The points on the cuts are served by continuity and cause no problem.
We keep the following convention in the whole paper: everywhere \( \hat{x} \) is the projection of \( \hat{x} \), and \( x \) is a projection of \( \hat{x} \) and \( \bar{x} \). Indeed, this is valid for any letter instead of \( x \) (it may be, say, \( a \) or \( b \)). This can be written as
\[
\hat{\cdot} \equiv P_{3:1}(\cdot), \quad \cdot \equiv P_{1:0}(\hat{\cdot}) \equiv P_{1:0}(P_{3:1}(\cdot)),
\]
(20)
where \( \cdot \) stays for any letter, possibly with indexes, but without a decoration.

To specify the points \( \bar{x} \) on \( \mathbb{R}^3 \) and \( \hat{x} \) on \( \mathbb{R} \), we introduce notation \( \bar{X}(x,j) \) and \( \hat{X}(x,j) \). The notation \( \bar{x} = \bar{X}(x,j) \) denotes the point having the affix \( x \in \mathbb{C} \) and lying on the sheet number \( j \in \{1, 2, 3, 4, 5, 6\} \), as it is shown in Fig. 4. Similarly, for the notation \( \hat{x} = \hat{X}(x,j) \) the index \( j \) takes values in \( \{1, 2\} \), and the sheets are shown in Fig. 3. Indeed,
\[
P_{1:0}(\hat{X}(x,j)) = x, \\
P_{3:1}(\bar{X}(x,j)) = \hat{X}(x,j'), \quad j' = 1 \text{ for } j = 1, 3, 5, \quad j' = 2 \text{ for } j = 2, 4, 6.
\]

One can check directly that:

**Proposition 2** Projections \( P_{3:1} \) and \( P_{1:0} \) are continuous.

This fact is important. Namely, \( \mathbb{R}_3 \) is a 3-sheet covering of \( \mathbb{R} \) (see [7]). Note that each point \( \hat{x} \in \mathbb{R} \) (including the branch points) has exactly three preimages \( P_{3:1}^{-1}(\hat{x}) \). A covering of a torus without branch points is also a torus, thus \( \mathbb{R}_3 \) has topology of a torus. We explored this feature in [1] (see Fig. 13 there).

If function \( f(\hat{x}) \) is single valued on \( \mathbb{R} \), there is no difficulty to define it on \( \mathbb{R}_3 \) as a single-valued function
\[
f(\hat{x}) = f(P_{3:1}(\hat{x})) = P(x).
\]
Conversely, if \( f(\hat{x}) \) is single-valued on \( \mathbb{R}_3 \), the function \( f(\hat{x}) \) is generally three-valued on \( \mathbb{R} \), and \( f(x) \) is six-valued on \( \mathbb{C} \). Functions \( \Upsilon(\hat{x}) \) and \( y(\hat{x}) \) are single-valued; functions \( \Upsilon(x) \) are \( y(x) \) are two-valued.

Consider the oriented contours \( \sigma_\alpha \) and \( \sigma_\beta \) on \( \mathbb{R} \) (see Fig. 3) starting and ending at \( \eta_{2,1} \). Contours \( \sigma_\alpha \) and \( \sigma_\beta \) play an important role in [1] and here. The contour \( \sigma_\alpha \) is homotopic to the “real waves” contour on \( \mathbb{R} \), i.e. to the contour on which the propagation angles
\[
\phi = \arctan\left(\frac{y(x) - y^{-1}(x)}{x - x^{-1}}\right)
\]
are real, and which tends to an arc of the unit circle as \( \text{Im}[K] \to 0 \). The points on the “real waves” contour become saddle points when the far field is estimated [1]. The contour \( \sigma_\beta \) is homotopic to the unit circle on the physical sheet of \( \mathbb{R} \); this is the integral path for the Green’s function of a discrete plane (see [1]). The contours \( \sigma_\alpha \) and \( \sigma_\beta \) form a canonical dissection of \( \mathbb{R} \), i.e. \( \mathbb{R} \) cut along \( \sigma_\alpha \) and \( \sigma_\beta \) becomes simply connected.

The paths \( \sigma_\alpha \) and \( \sigma_\beta \) are generators of the fundamental group \( \pi_1 \) of \( \mathbb{R} \) (note that \( \pi_1 \) is commutative). Thus, many topological properties of \( \mathbb{R} \) are connected with \( \sigma_\alpha \) and \( \sigma_\beta \). In particular, comparing Fig. 3 with Fig. 4, we find that
Proposition 3  a) The preimage $P_{3:1}^{-1}(\sigma_3)$ is a set of three copies of $\sigma_3$ (on sheets 1,3,5). The preimage $P_{3:1}^{-1}(\sigma_5)$ is a connected three-sheet covering of $\sigma_5$.

b) Let $R'$ be any covering of $R$, and let $P : R' \to R$ be the corresponding projection. Let $P^{-1}(\sigma_3)$ be three copies of $\sigma_3$, and let $P^{-1}(\sigma_5)$ be a connected three-sheet covering of $\sigma_5$. Then $R'$ is equivalent to $R_3$.

2.4 A structure of a complex manifold on $R_3$ and contour integration

Introduce a structure of a complex manifold on $R$ and on $R_3$. By definition [8], a complex manifold is a union of possibly intersecting neighborhoods $U_s$ in each of which a local complex variable $\tau_s$ can be introduced, describing the neighborhood in a trivial way (there exists a continuous bijection between $U_s$ and some open circle in the $\tau_s$ plane). Transitions between the local variables in the intersections of the neighborhoods should be biholomorphic.

For both $R$ and $R_3$, a possible choice of local variables is as follows. In each small neighborhood not including the infinity or $\eta_{j,l}$ (i.e. almost everywhere), one can take $\tau_s = x$ as a local variable. In the neighborhoods of the points with $x = \eta_{j,l}$ one can choose $\tau_s = \sqrt{x - \eta_{j,l}}$. The presence of the square root provides a one-to-one correspondence between a neighborhood of zero in the domain of the local variable and the neighborhood of the branch point of $R$ or $R_3$. Near the infinities one can take $\tau_s = x^{-1}$.

Indeed, one can introduce the structure of a complex manifold on the whole $\mathbb{C}$ by taking the local variable $\tau_s = x^{-1}$ at the neighborhood of the infinity and $x$ in the finite part of $\mathbb{C}$.

Thus, $\mathbb{C}$, $R$, and $R_3$ become complex manifolds, and one can make an important note: all points of $\mathbb{C}$, $R$, or $R_3$ (including the infinities and the branch points) are regular from the point of view of the complex structure.

Define a single-valued function $f$ on $R$ or on $R_3$. At each neighborhood $U_s$ one can express it as a function of the local variable: $f = f(\tau_s)$. The function $f$ is analytic in $U_s$ if $f(\tau_s)$ is analytic. Thus, definition of analyticity becomes local. A function has a pole or zero at some point belonging to $U_s$ if $f(\tau_s)$ has a pole or zero at the corresponding point. The order of the pole/zero of a function is also defined with respect to the local variable.

The concept of a pole / zero at some point on the complex manifolds $R$ or on $R_3$ differs from that on $\mathbb{C}$. For example the function $\Upsilon(\hat{x})$ has simple zeros on $R$ at each of the points $\eta_{j,l}$, although it may seem surprising. Moreover, the function

$$\frac{1}{\Upsilon^2(\hat{x})} = \frac{1}{(x - \eta_{1,1})(x - \eta_{2,1})(x - \eta_{1,2})(x - \eta_{2,2})}$$

has double poles at each $\eta_{j,l}$ as a function on $R$, while the same function, but considered on $\mathbb{C}$, has simple poles at those points. Function $\Upsilon(\hat{x})$ has double poles at the infinities on $R$ and on $R_3$.

A meromorphic function on $R$ or on $R_3$ (or on any other compact Riemann surface with a structure of complex manifold) is a function single-valued on the Riemann surface and having a finite number of singularities, each of which is a pole of finite order.
A differential 1-form on a complex manifold is as a set of expressions \( f_s(\tau_s) \, d\tau_s \) (each defined in \( U_s \)), such that
\[
\frac{f_j}{f_s} = \frac{d\tau_s}{d\tau_j}
\]
in \( U_j \cap U_s \). The form is analytic in some domain (or everywhere) if \( f_j \) is analytic there (or if all \( f_j \) are analytic). It is important for our consideration that the form \( dx/\Upsilon(\tilde{x}) \) is analytic everywhere in \( \mathbb{R}_3 \).

Poles and zeros of forms and functions are defined also in the local variables. A residue of a 1-form is defined invariantly, i.e., the residue does not depend on the choice of the local variable. Namely, a 1-form has residue equal to \( a \) at some point \( \tau_j' \) if it can be locally represented using \( \tau_j \) as
\[
\left( \frac{a}{\tau_j - \tau_j'} + c(\tau_j) \right) d\tau_j,
\]
where \( c \) is some function regular near \( \tau_j' \). Note that the residue of a function is not invariant.

Below we use contour integration on the Riemann surface \( \mathbb{R}_3 \) based on the structure of a complex manifold on \( \mathbb{R}_3 \). This integration is introduced also locally. Let a 1-form \( f_j \, d\tau_j \) and an oriented contour \( \gamma \) on \( \mathbb{R}_3 \) be given. In each neighborhood \( U_j \) passed by \( \gamma \) one can define a term
\[
\int_{\gamma_j} f_j \, d\tau_j,
\]
where \( \gamma_j \) is some part of \( \gamma \cap U_j \). The total integral is a sum of all such parts provided that the concatenation of all \( \gamma_j \) is \( \gamma \). The condition (21) guarantees that integration is invariant with respect to the choice of the local variables. The Cauchy’s theorem is inherited from usual complex analysis in a trivial way (in each \( U_j \)). The theorem states that the contour of integration can be freely deformed in some domain on \( \mathbb{R}_3 \), provided the integrand (a 1-form) is analytic in this domain. The result of integration remains unchanged under this deformation.

Introduce a Riemann surface over a complex manifold. Namely, let \( \mathcal{R} \) be a complex manifold described above, \( \mathcal{R}' \) be a manifold of real dimension 2, and \( P : \mathcal{R}' \to \mathcal{R} \) be a continuous mapping. \( \mathcal{R}' \) is a Riemann surface if all \( P^{-1}(U_s), U_s \subset \mathcal{R} \) have structure of a Riemann surface. Using this definition, one can show that \( \mathbb{R}_3 \) is a 3-sheet Riemann surface over \( \mathbb{R} \). Besides, one can see that \( \mathbb{R}_3 \) over \( \mathbb{R} \) is a Riemann surface without branch points. To prove this, one can consider the neighborhoods of \( \eta_{j,l} \) in corresponding local variables.

2.5 The main result of [1] (the Sommerfeld integral)

Let \( \tilde{x}_1, \ldots, \tilde{x}_4 \) be the points of \( \mathbb{R}_3 \) defined by
\[
\tilde{x}_1 = \tilde{X}(x_{in}, 3), \quad \tilde{x}_2 = \tilde{X}(x_{in}, 4), \quad \tilde{x}_3 = \tilde{X}(x_{in}^{-1}, 6), \quad \tilde{x}_4 = \tilde{X}(x_{in}^{-1}, 1).
\]
One can see that
\[
y(\tilde{x}_1) = y(\tilde{x}_4) = y_{in}, \quad y(\tilde{x}_2) = y(\tilde{x}_3) = y_{in}^{-1}.
\]
It is shown in [1] that the point \( \hat{x}_1 \) corresponds to the incident wave, and three other points correspond to the reflected waves of \( S_3 \) (this can be seen from the “wavenumbers” \( x \) and \( y \)). Note that the choice of sheets for these points is made according to contours of integration in the Sommerfeld integral (see integration contours in Fig. 4 and Theorem 1 below).

Following [1], let us formulate the functional problem for the function \( A(\hat{x}) \), which is the Sommerfeld transformant of the total wave (see below):

**Functional problem 1.**

a) The function \( A(\hat{x}) \) should be meromorphic on \( \mathbb{R}^3 \).

b) Function \( A(\hat{x}) \) should have four poles on \( \mathbb{R}^3 \) at the points \( \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \) of order 1, and no other poles.

c) The residues of \( A(\hat{x}) \) at the poles are specified by setting the residues of the poles of the form \( A(\hat{x})dx/\Upsilon(\hat{x}) \). Namely, this form should have the residues \( -(2\pi i)^{-1} \) at the points \( \hat{x}_1 \) and \( \hat{x}_3 \), and the residues \( (2\pi i)^{-1} \) at the points \( \hat{x}_2 \) and \( \hat{x}_4 \).

Some comments should be made. According to the first condition, \( A(x) \) should be an analytic function, 6-valued on \( \mathbb{C} \). It should have branch points of order two at \( \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2} \). The sheets of the Riemann surface of \( A \) should be linked in the way shown in Fig. 4. Function \( A(\hat{x}) \) should be regular as \( |x| \to \infty \) on each of the six sheets of \( \mathbb{R}^3 \).

The residues of the function \( A(\hat{x}) \) can be specified, provided \( x \) is taken as a local variable near the poles. The residues at \( \hat{x}_j \), \( j = 1, \ldots, 4 \) are equal to

\[-(2\pi i)^{-1}Y_1, \quad (2\pi i)^{-1}Y_2, \quad -(2\pi i)^{-1}Y_3, \quad (2\pi i)^{-1}Y_4,\]

where

\[ Y_j = \Upsilon(\hat{x}_j). \]

One can see that \( Y_2 = -Y_1, Y_4 = -Y_3 \).

Obviously, Functional problem 1 defines \( A \) up to an arbitrary additive constant (at least). Moreover:

**Proposition 4** If \( A(\hat{x}) \) and \( A'(\hat{x}) \) are different solutions of Functional problem 1, then \( A(\hat{x}) - A'(\hat{x}) \) is a constant.

To prove this proposition, note that the function \( A(\hat{x}) - A'(\hat{x}) \) is regular everywhere on \( \mathbb{R}^3 \), thus it is constant (by Liouville’s theorem for Riemann surfaces, see [9] chapter 13).

Function \( A \), formally, has been found in [1] (see (70) there), and it has been expressed in elliptic functions, and this solution is hardly practical. However, the formulation of the problem is of algebraic nature, so one can expect a purely algebraic solution.

The main result of [1] related to the problem of diffraction in the domain shown in Fig. 1 can be formulated as a theorem:

**Theorem 1** Let the function \( A(\hat{x}) \) obey Functional problem 1. Introduce the contours

\[ \Gamma_2 = J_3 + J_2 + J'_1 + J'_4, \quad (22) \]
\[ \Gamma_3 = J_2 + J'_1 + J'_4 + J'_3, \]  
\[ \text{where the contours } J_3, J_2, J'_1, J'_4, J'_3 \text{ on } \mathbb{R}_3 \text{ are shown in Fig. 4.} \]

Define the functions \( u_j(m, n), m, n \in \mathbb{Z}, j = 2, 3, \) by the Sommerfeld integral:

\[ u_j(m, n) = \int_{\Gamma_j} w_{m,n}(\hat{x}, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})}. \]  
\[ \text{Then} \]
\[ a) \ u_2(m, n) = u_3(m, n) \text{ for } m, n \leq 0. \]
\[ b) \text{ The function} \]

\[ u(m, n) = \begin{cases} 
  u_2(m, n), & m \leq 0, \ n > 0 \\
  u_3(m, n), & n \leq 0, \ m > 0 \\
  u_2(m, n) = u_3(m, n), & m \leq 0, \ n \leq 0 
\end{cases} \]

\[ \text{defined in the domain shown in Fig. 1 obeys the diffraction problem formulated in Subsection 2.1.} \]

The proof can be found in [1]. An alternative proof is given in Appendix 1.

**Remark.** The notation for the contours \( \Gamma_2 \) and \( \Gamma_3 \) is kept similar to that of [1], where a family of Sommerfeld contours on a torus is introduced. Totally, one can define 12 contours \( \Gamma_1 \ldots \Gamma_{12} \), describing \( u(m, n) \) on the whole branched surface \( S_3 \). Contours \( J_2 \) and \( J'_1 \) encircle the point \( x = 0 \) on corresponding sheets. Contours \( J_3, J'_4, J'_3 \) encircle the infinities on corresponding sheets.

According to the formula (24), adding an arbitrary constant to \( A \) does not change the integral.

The main result of this section is as follows: to find the solution of the diffraction problem, one should first solve the Functional problem 1, and then substitute \( A(\tilde{x}) \) into the Sommerfeld integral (24).

### 3 Mathematical basics of solving the functional problem for \( A(x) \)

#### 3.1 Symmetries of the Riemann surface

Consider the cyclical substitution of sheets

\[ 1 \to 3 \to 5 \to 1, \quad 2 \to 4 \to 6 \to 2. \]  
\[ \text{(25)} \]

This substitution of sheets generates a symmetry (referred to as \( \Lambda \)) of \( \mathbb{R}_3 \), Namely,

\[ \Lambda(\tilde{X}(x, j)) = \tilde{X}(x, j'), \]  
\[ \text{(26)} \]
where \( j' \) is obtained from \( j \) by applying the mapping (25).

One can see that if a function \( f(\tilde{x}) \) is meromorphic on \( \mathbb{R}_3 \) then the same is valid for \( f(\Lambda(\tilde{x})) \). Moreover, since \( P_{3,1}(\Lambda(\tilde{x})) = P_{3,1}(\tilde{x}) \), \( \Lambda \) does not change the value of \( \Upsilon(\tilde{x}) \):

\[
\Upsilon(\Lambda(\tilde{x})) = \Upsilon(\tilde{x}).
\]

Another symmetry (referred to as \( \Pi \)) is defined on \( \mathbb{R} \) and on \( \mathbb{R}_3 \) as follows:

\[
\Pi(\tilde{X}(x,j)) = \tilde{X}(x,3-j), \quad \Pi(\tilde{X}(x,j)) = \tilde{X}(x,7-j).
\]

One can see that if a function \( f(\tilde{x}) \) is meromorphic on \( \mathbb{R}_3 \) then the same is valid for \( f(\Pi(\tilde{x})) \).

A direct check shows that

\[
\Upsilon(\Pi(\hat{x})) = -\Upsilon(\hat{x}), \quad \Upsilon(\Pi(\tilde{x})) = -\Upsilon(\tilde{x}).
\]

Obviously,

\[
\Pi(\Pi(\hat{x})) = \hat{x}, \quad \Pi(\Pi(\tilde{x})) = \tilde{x}, \quad \Lambda(\Lambda(\tilde{x})) = \tilde{x}.
\]

The mappings \( \Lambda \) and \( \Pi \) are desk transformations of \( \mathbb{R}_3 \) in terms of [6].

The symmetries \( \Pi \) and \( \Lambda \) can be used to simplify the formulation of the functional problem for \( A(\tilde{x}) \). These simplifications can be formulated as the following propositions.

**Proposition 5** If \( A \) is a solution of the functional problem formulated above, then

\[
A(\Pi(\tilde{x})) = A(\tilde{x}).
\]

To prove this proposition, note that the poles of \( A \) have the property \( \Pi(\tilde{x}_1) = \tilde{x}_2, \Pi(\tilde{x}_3) = \tilde{x}_4 \), moreover, the residues at corresponding poles are the same. Thus, \( A(\Pi(\tilde{x})) \) also obeys the functional problem. Taking into account Proposition 4, and the fact that at some branch points \( \Pi(\tilde{x}) = \tilde{x} \), obtain the result.

**Proposition 6** Let \( A \) be a solution of Functional problem 1. Then it can be represented as a sum of 3 components:

\[
A(\tilde{x}) = A_0(\tilde{x}) + A_1(\tilde{x}) + A_2(\tilde{x}),
\]

where \( A_1, A_2, A_3 \) are meromorphic on \( \mathbb{R}_3 \), and

\[
A_0(\Lambda(\tilde{x})) = A_0(\tilde{x}), \quad A_1(\Lambda(\tilde{x})) = \varpi A_1(\tilde{x}), \quad A_2(\Lambda(\tilde{x})) = \varpi^{-1} A_2(\tilde{x})
\]

for all \( \tilde{x} \in \mathbb{R}_3 \),

\[
\varpi \equiv e^{2\pi i/3}.
\]

The proof is given by explicit formulae:

\[
A_0(\tilde{x}) = \frac{1}{3}(A(\tilde{x}) + A(\Lambda(\tilde{x})) + A(\Lambda(\Lambda(\tilde{x})))).
\]
\[ A_1(\tilde{x}) = \frac{1}{3} \left( A(\tilde{x}) + \varpi^2 A(\Lambda(\tilde{x})) + \varpi A(\Lambda(\Lambda(\tilde{x}))) \right), \] (35)
\[ A_2(\tilde{x}) = \frac{1}{3} \left( A(\tilde{x}) + \varpi A(\Lambda(\tilde{x})) + \varpi^2 A(\Lambda(\Lambda(\tilde{x}))) \right). \] (36)

Indeed, (34), (35), (36) constitute a discrete Fourier transform on each set \( \{ \tilde{x}, \Lambda(\tilde{x}), \Lambda(\Lambda(\tilde{x})) \} \).

By construction, the functions \( A_0(\tilde{x}), A_1(\tilde{x}), A_2(\tilde{x}) \) obey the following functional problem:

**Functional problem 2.**

a) Functions \( A_0, A_1, A_2 \) should be meromorphic on \( \mathbb{R}_3 \).

b) Each of the functions \( A_0, A_1, A_2 \) is allowed to have 12 poles located at \( \tilde{x}_j, \Lambda(\tilde{x}_j), \Lambda(\Lambda(\tilde{x}_j)) \), \( j \in \{1, 2, 3, 4\} \). All poles should be of order 1.

c) All residues of the poles of the functions \( A_0, A_1, A_2 \) are prescribed. For each of these functions they are equal to \( -(6\pi i)^{-1}Y_1 \) at the points \( \tilde{x}_1, \tilde{x}_2 \), and to \( -(6\pi i)^{-1}Y_3 \) at the points \( \tilde{x}_3, \tilde{x}_4 \). The residues at the poles \( \Lambda(\tilde{x}_j) \) and \( \Lambda(\Lambda(\tilde{x}_j)) \) can be found from these values and the relations (32).

d) Functions \( A_0, A_1, A_2 \) should obey relations (32).

Similarly to Proposition 4, the solution of this functional problem is unique for \( A_1, A_2 \), and defined up to an additive constant for \( A_0 \).

### 3.2 Fields of functions meromorphic on Riemann surfaces

Above, we considered wave fields, i.e. solutions of a (discrete) Helmholtz equation. Here and below we consider algebraic fields that are sets of elements, on which the arithmetic operations are defined and possess usual properties. We hope that the usage of the term “field” should not cause a confusion. In more details, we are going to study functional fields, i.e. the elements of fields are some functions. A set of functions is a field if a sum, a product, a difference, or a ratio of two elements of the set belongs to this set. Indeed, in the last case the denominator should not be identically equal to zero.

It is obvious that the following proposition is correct (see [9], chapter 11):

**Proposition 7** Let \( \mathcal{R} \) be a compact Riemann surface with a structure of complex manifold defined on it. Then the set of all functions meromorphic on \( \mathcal{R} \) is a field.

We will use three Riemann surfaces as \( \mathcal{R} \), namely \( \mathcal{C}, \mathcal{R}, \) and \( \mathcal{R}_3 \). The fields of functions meromorphic on them will be denoted by \( K_0, K_1, K_3 \). We are particularly interested in the field \( K_3 \), since \( A(\tilde{x}) \in K_3 \).

Let \( \mathcal{R} \) be a Riemann surface over the Riemann surface \( \mathcal{R}' \) (both compact), and let \( P \) the corresponding projection \( P : \mathcal{R} \to \mathcal{R}' \). Let the fields of meromorphic functions on \( \mathcal{R} \) and \( \mathcal{R}' \) be denoted by \( K \) and \( K' \), respectively. We say that \( f \in K \) belongs also to \( K' \) if \( f \) is single-valued on \( K' \), i.e. if there exists a function \( f' \in K' \) such that \( f(x) = f'(P(z)) \), \( x \in \mathcal{R} \). In the same sense, any function \( f' \in K' \) belongs to \( K \), since \( f(x) = f'(P(z)) \) is a definition of an appropriate function \( f(x) \). The existence of continuous mappings \( P_{3:1} \) and \( P_{1:0} \) leads to the following statement:
Proposition 8 The following inclusions are valid:

\[ K_0 \subset K_1 \subset K_3. \]

In the usual terms (see, for example, [10]), \( K_1 \) is an extension of \( K_0 \), and \( K_3 \) is extension of \( K_1 \).

Remark. Mappings

\[ f(\hat{x}) \rightarrow f(\Pi(\hat{x})), \quad f(\hat{x}) \in K_1, \quad (37) \]

\[ f(\tilde{x}) \rightarrow f(\Lambda(\tilde{x})), \quad f(\tilde{x}) \in K_3 \]

are authomorphisms of \( K_1 \) and \( K_3 \), respectively. As it is common for the Galois theory [6], the following proposition links field authomorphisms and extensions:

Proposition 9 The elements of \( K_1 \) invariant with respect to (37) belong to \( K_0 \). The elements of \( K_3 \) invariant with respect to (38) belong to \( K_1 \).

Indeed, these facts are obvious for the surfaces under consideration.

The field \( K_0 \) of functions \( f(x) \) meromorphic on the Riemann sphere \( \bar{C} \) consists of all rational functions of \( x \) (a rational function is a ratio of two polynomials). For the field \( K_1 \), the following proposition is valid:

Proposition 10 Each element of \( z \in K_1 \) can be uniquely represented as

\[ z(\hat{x}) = q_0(x) + q_1(x)\Upsilon(\hat{x}), \quad q_0, q_1 \in K_0. \]

(39)

Proof. Let be \( x \in \bar{C} \). The functions

\[ \frac{z(\hat{x}) + z(\Pi(\hat{x}))}{2} \quad \text{and} \quad \frac{z(\hat{x}) - z(\Pi(\hat{x}))}{2\Upsilon(\hat{x})} \]

are meromorphic on \( \bar{C} \) since they are invariant with respect to (37). They are \( q_0 \) and \( q_1 \), respectively. The elements with \( q_1 \equiv 0 \) are the elements of \( K_1 \) that belong to \( K_0 \). □

One can see that \( K_1 \) is an algebraic extension of \( K_0 \), i.e. an irrationality \( \Upsilon(x) \) is added to the field \( K_0 \). The irrationality is a solution of the algebraic equation of order 2, whose coefficients belong to \( K_0 \):

\[ \Upsilon^2 - f = 0, \quad f(x) = (x - \eta_{1,1})(x - \eta_{1,2})(x - \eta_{2,1})(x - \eta_{2,2}). \]

(40)

One can see that (39) is an expansion of the form

\[ z(\hat{x}) = q_0(x)\omega_0(\hat{x}) + \cdots + q_{j-1}(x)\omega_{j-1}(\hat{x}), \quad (41) \]
where \( j = 2; \, q_0, q_1 \in K_0 \), and the set
\[
\Omega_{1:0} \equiv [\omega_0, \ldots, \omega_{j-1}] = [1, \Upsilon(\hat{x})]
\] (42)
is the basis of the extension. In [9], chapter 23 it is shown that such a basis always exists. The number \( j \) of elements of the basis is referred to as the degree of extension of \( K_1 \) over \( K_0 \). This degree is equal to 2, and it is the same as number of sheets of \( R \) over \( \mathbb{C} \).

Consider the field \( K_3 \), i.e. the field of functions meromorphic on \( R_3 \). The following statement is valid:

**Theorem 2** Let there exist non-zero functions \( F_1(\bar{x}), F_2(\bar{x}) \in K_3 \) having properties
\[
F_1(\Lambda(\bar{x})) = \wp F_1(\bar{x}),
\] (43)
\[
F_2(\Lambda(\bar{x})) = \wp^{-1} F_2(\bar{x}).
\] (44)
Then for any function \( z(\bar{x}) \in K_3 \) there exists a (unique) representation
\[
f(\bar{x}) = q_0(\bar{x}) + q_1(\bar{x}) F_1(\bar{x}) + q_2(\bar{x}) F_2(\bar{x}),
\] (45)
where \( q_0, q_1, q_2 \in K_1 \).

The functions \( F_1(\bar{x}) \) and \( F_2(\bar{x}) \) are solutions of cubic equations with coefficients belonging to \( K_1 \):
\[
F_1^3 - G_1 = 0, \quad F_2^3 - G_2 = 0, \quad G_{1,2} \in K_1.
\] (46)

Indeed, this statement means that the extension \( K_3 \) over \( K_1 \) has a basis
\[
\Omega_{3:1} = [1, F_1, F_2]
\] (47)
The functions \( F_{1,2} \) should be single-valued on \( R_3 \), three-valued on \( K_1 \), and six-valued on \( \mathbb{C} \).

**Proof.** a) Consider the combinations similar to (34), (35), (36):
\[
f_0(\bar{x}) = \frac{1}{3} (f(\bar{x}) + f(\Lambda(\bar{x})) + f(\Lambda(\Lambda(\bar{x})))),
\] (48)
\[
f_1(\bar{x}) = \frac{1}{3} (f(\bar{x}) + \wp^{-1} f(\Lambda(\bar{x})) + \wp f(\Lambda(\Lambda(\bar{x})))),
\] (49)
\[
f_2(\bar{x}) = \frac{1}{3} (f(\bar{x}) + \wp f(\Lambda(\bar{x})) + \wp^{-1} f(\Lambda(\Lambda(\bar{x})))).
\] (50)
Obviously, if \( f_0, f_1, f_2 \) are known, the function \( f \) can be reconstructed by
\[
f(\bar{x}) = f_0(\bar{x}) + f_1(\bar{x}) + f_2(\bar{x}).
\] (51)
Note that
\[ f_0(\Lambda(\tilde{x})) = f_0(\tilde{x}), \quad f_1(\Lambda(\tilde{x})) = \varpi f_1(\tilde{x}), \quad f_2(\Lambda(\tilde{x})) = \varpi^{-1} f_2(\tilde{x}). \quad (52) \]

According to Proposition 9, \( f_0 \in K_1 \), and one can take \( q_0 = f_0 \). The coefficients \( q_1 \) and \( q_2 \) are chosen as
\[ q_1 = f_1(\tilde{x})/F_1(\tilde{x}), \quad q_2 = f_2(\tilde{x})/F_2(\tilde{x}). \]

These functions are invariant with respect to (37), thus they belong to \( K_1 \). The uniqueness of the representation (45) (provided the functions \( F_1 \) and \( F_2 \) are fixed) follows from the construction of the coefficients.

b) Due to (43) and (44), the functions \( G_1 = (F_1)^3 \) and \( G_2 = (F_2)^3 \) are invariant with respect to (38), thus, \( F_{1,2} \in K_1 \). □

The functions \( F_1 \) and \( F_2 \) do exist, they will be built explicitly below. A corollary of Theorem 2 is that the Sommerfeld transformant \( A \) has a representation
\[ A(\tilde{x}) = q_0(\hat{x}) + q_1(\hat{x})F_1(\tilde{x}) + q_2(\hat{x})F_2(\tilde{x}), \quad (53) \]
where \( q_j(\hat{x}) \) belong to \( K_1 \) being rational functions of \( x \) and \( \Upsilon(\hat{x}) \). Comparing formulae (34)–(36) with (48)–(50) we conclude that
\[ A_0(\tilde{x}) = q_0(\hat{x}), \quad A_1(\tilde{x}) = q_1(\hat{x})F_1(\tilde{x}), \quad A_2(\tilde{x}) = q_2(\hat{x})F_2(\tilde{x}). \quad (54) \]

Finding the functions \( F_1(\tilde{x}) \) and \( F_2(\tilde{x}) \) is an unusual problem since no function whose Riemann surface is \( R_3 \) is given \textit{a priori}. Finding the coefficients \( q_j(x) \) is, conversely, an almost trivial task when the basis (47) is built. They are constructed by using the knowledge of poles and residues of the Sommerfeld transformant \( A \).

The choice of functions \( F_1 \) and \( F_2 \) is not unique. Each of them can be multiplied by any nonzero element of \( K_1 \), still keeping the properties (43), (44). Below we are trying to construct the functions having the simplest structure, i.e., having as small amount of poles / zeros on \( R_3 \) as possible.

\textbf{Remark.} Consider the expansion (53). The basis functions \( F_1 \) and \( F_2 \) are constructed below depending on \( K \) and not depending on \( x_{in} \). The coefficients \( q_0, q_1, q_2 \), conversely, do depend on \( x_{in} \). The structure of functions \( F_{1,2} \) guarantee the validity of conditions a) and d) of the Functional problem 2, while the choice of the coefficients \( q_{0,1,2} \) guarantee the conditions b) and c).

### 3.3 Abelian integral of the first kind on \( R \)

Introduce the Abelian integral of the first kind on \( R \). A detailed description of this subject can be found, e.g., in [9], chapter 12. Since \( R \) is a torus, a surface of genus 1, there is one Abelian integral analytic everywhere (indeed, defined up to a constant factor and a constant additive term). This Abelian integral is an integral of a differential 1-form, which is analytic everywhere.
on $\mathbb{R}$. As we have mentioned, $dx/Y$ is such a form, thus the Abelian integral of the first kind is

$$\chi(\hat{x}) = \int_{\eta_{2,1}}^{\hat{x}} \frac{dx'}{Y(\hat{x}')}. \tag{55}$$

The choice of the starting point $\eta_{2,1}$ is arbitrary. The integral is assumed to be taken along some oriented contour $\gamma$ on $\mathbb{R}$ connecting $\eta_{2,1}$ and $\hat{x}$: $\chi(\hat{x}) = \chi(\hat{x}, \gamma)$. Note that (55) is an Abelian integral on $\mathbb{R}$ as well.

The integrals (55) taken along the closed contours $\sigma_\alpha$ and $\sigma_\beta$ on $\mathbb{R}$ are the periods of $\chi(\hat{x})$ referred to as $T_\alpha$ and $T_\beta$:

$$T_\alpha = \int_{\sigma_\alpha} \frac{dx}{Y(\hat{x})}, \quad T_\beta = \int_{\sigma_\beta} \frac{dx}{Y(\hat{x})}. \tag{56}$$

The function $\chi$ is used below as a multiple-valued mapping between $\mathbb{R}$ and the complex plane of $\chi$. Introduce also an inverse mapping $\psi : \chi \to \hat{x}$. We will use the following properties of the mappings $\chi$ and $\psi$ that can be found in any textbook on elliptic functions, e.g. [9].

**Proposition 11**

a) Let $\gamma_0$ be some contour connecting $\eta_{2,1}$ with $\hat{x}$. Then all values of $\chi(\hat{x}, \gamma)$ are $\chi(\hat{x}, \gamma_0) + jT_\alpha + lT_\beta$, $j, l \in \mathbb{Z}$.

b) Mapping $\psi$ is a bijection of $\mathbb{R}$ and the elementary parallelogram with vertexes $(0, T_\alpha, T_\beta + T_\alpha, T_\beta)$ and with the opposite sides glued together.

As it follows from this proposition, mapping $\psi$ is defined correctly and is bi-periodic:

$$\psi(\chi + T_\alpha) = \psi(\chi + T_\beta) = \psi(\chi). \tag{57}$$

Being cut along the contours $\sigma_\alpha$ and $\sigma_\beta$, the surface $\mathbb{R}$ becomes (topologically) a parallelogram, as it is shown in Fig. 5.

![Fig. 5](image-url)  
**Fig. 5**: Elementary parallelogram in the $\chi$-plane and coordinates $(\alpha, \beta)$ on a torus
Proposition 11 can be used for introduction of coordinates $\alpha$ and $\beta$ on $R$, revealing the structure of $R$ as the structure of a torus. Namely, the coordinates $\alpha$ and $\beta$ can be introduced as linear combinations
\[
\alpha = c_{1,1}\text{Re}[\chi] + c_{1,2}\text{Im}[\chi], \quad \beta = c_{2,1}\text{Re}[\chi] + c_{2,2}\text{Im}[\chi],
\]
with the coefficients $c_{j,k}$ found from the following equations
\[
c_{1,1}\text{Re}[T_{\alpha}] + c_{1,2}\text{Im}[T_{\alpha}] = 2\pi, \quad c_{1,1}\text{Re}[T_{\beta}] + c_{1,2}\text{Im}[T_{\beta}] = 0, \quad c_{2,1}\text{Re}[T_{\alpha}] + c_{2,2}\text{Im}[T_{\alpha}] = 0, \quad c_{2,1}\text{Re}[T_{\beta}] + c_{2,2}\text{Im}[T_{\beta}] = 2\pi.
\]

The coordinates $(\alpha, \beta)$ on $R$ are shown in Fig. 5, right. The surface $R$ is displayed schematically as a torus, i.e. $R$ is deformed in an appropriate way. The resulting surface is compact, thus, the infinities are represented as two points on it. The coordinate lines of $\alpha$ and $\beta$ on the initial representation of $R$ are close\(^1\) to those shown in Fig. 4 of [1].

The torus $R$ corresponds to the parallelogram
\[
R : \quad 0 \leq \alpha < 2\pi, \quad 0 \leq \beta < 2\pi,
\]
while, according to Proposition 3, the torus $R_3$ corresponds to the parallelogram
\[
R_3 : \quad 0 \leq \alpha < 6\pi, \quad 0 \leq \beta < 2\pi.
\]
Each point $(\alpha, \beta) \in R$ has three preimages $P^{-1}_{3;1}(\alpha, \beta)$: $(\alpha, \beta), (\alpha + 2\pi, \beta), (\alpha + 4\pi, \beta)$ on $R_3$.

The symmetries $\Lambda$ and $\Pi$ have the following representations in the coordinates $(\alpha, \beta)$:
\[
\Lambda : \quad \alpha \to \alpha + 2\pi, \quad \beta \to \beta,
\]
\[
\Pi : \quad \alpha \to 4\pi - \alpha, \quad \beta \to -\beta.
\]

### 4 Finding the basis functions $F_1, F_2$

#### 4.1 Elementary meromorphic functions on $R$

Our aim is to build functions $F_{1,2}$. For this, we will use an auxiliary function
\[
M(\hat{x}) = M(\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2; \hat{x})
\]
($\hat{x} \in R$ is a variable, $\hat{b}_{1,2}, \hat{a}_{1,2} \in R$ are parameters) such that: $M \in K_1$, it has poles only at $\hat{b}_{1,2}$, the poles are simple if $\hat{b}_1 \neq \hat{b}_2$ or double if $\hat{b}_1 = \hat{b}_2$, and it has zeros only at $\hat{a}_{1,2}$ (simple zeros if $\hat{a}_1 \neq \hat{a}_2$ or a double zero if $\hat{a}_1 = \hat{a}_2$).

Indeed, each such function (if it exists) is defined up to a constant factor. This factor is not important for us and we suppress it in the Ansätze written below.

Such a function exists not for any set $(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2)$. A criterion of existence of such a function $M$ is known:

---

\(^1\)Coordinates $\alpha$ and $\beta$ are close to the coordinates $\alpha$ and $\beta$ defined in [1], but not exactly the same. Note that the requirement that $\beta = \pi$ on the “real waves” line is not fulfilled in the new formulation.
Theorem 3 Function $M(\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2; \hat{x})$ exists iff there exist oriented contours $\gamma_1$ and $\gamma_2$ on $R$, such that $\gamma_1$ goes from $\hat{a}_1$ to $\hat{b}_1$, $\gamma_2$ goes from $\hat{a}_2$ to $\hat{b}_2$, and

$$\int_{\gamma_1} \frac{dx}{\Upsilon(x)} + \int_{\gamma_2} \frac{dx}{\Upsilon(\hat{x})} = 0. \tag{61}$$

This is a particular case of the Abel's theorem (see [11], chapter 10). The criterion has a transcendent (non-algebraic) character. Surprisingly, it is possible to formulate an algebraic criterion of existence of such a function. Below we formulate a set of propositions describing different choices of $\hat{a}_{1,2}, \hat{b}_{1,2}$. We formulate two cases important for our consideration as Propositions 12 and 13 here, and, for completeness, formulate two more cases as Propositions 17 and 18 in Appendix 2. In all cases the proofs are quite elementary and are based on a detailed study of a form (39) for $M$.

Proposition 12 Let be $\hat{b}_1 = \hat{b}_2 = \eta_{l,l}$. The function $M(\eta_{l,l}, \eta_{l,l}, \hat{a}_1, \hat{a}_2; \hat{x})$ exists iff $a_1 = a_2$.

Proof. A general Ansatz of a function $M \in K_1$ having a double pole at $\eta_{l,l}$ on $R$ and no other poles is

$$M = \frac{1}{x - \eta_{l,l}} + c.$$  

(We remind that this expression corresponds to a double pole, since the local variable near $\eta_{l,l}$ is $\tau = \sqrt{x - \eta_{l,l}}$.) Then, $M$ has a zero at $\hat{a}_1$ if $c$ is chosen in such a way that

$$M(\hat{x}) = \frac{x - a_1}{x - \eta_{l,l}}. \tag{62}$$

Indeed, the other zero on $R$ should have the same affix $a_1$. □

Proposition 13 Let a double pole be located at the point $\hat{b}_1$, and a simple zero be located at $\hat{b}_2$, such that $b_1 = b_2 = b$ not equal to the infinity or any of the branch points. Let another simple zero be located at $\eta_{l,l}$. The function $M(\hat{b}_1, \hat{b}_2, \eta_{l,l}; x)$ exists iff

$$\frac{\Upsilon(\hat{b}_1)}{(\eta_{l,l} - b)^2} + \frac{\Upsilon(\hat{b}_1)}{\eta_{l,l} - b} + \frac{\Upsilon(\hat{b}_1)}{2} = 0. \tag{63}$$

Proof. A general Ansatz for a function with a double pole at $\hat{b}_1$ and regular at $\hat{b}_2$ is as follows:

$$M(\hat{x}) = \frac{\Upsilon(\hat{x})}{(x - b)^2} + \frac{\Upsilon(\hat{b}_1)}{(x - b)^2} + \frac{\Upsilon(\hat{b}_1)}{x - b} + c. \tag{64}$$

Function $M(\hat{x})$ has a zero at $\hat{b}_2$ if

$$c = \frac{\Upsilon(\hat{b}_1)}{2}, \tag{65}$$

where

$$\Upsilon(\hat{x}) \equiv d^2 \Upsilon(\hat{x}).$$

Since $\Upsilon(\eta_{l,l}) = 0$, the condition $M(\eta_{l,l}) = 0$ reads as (63). Function $M$ is given by (64), (65). □
4.2 Building the elements $F_1$ and $F_2$ of the basis $\Omega_{3.1}$

Here we describe the most tricky result of the paper, namely, we build functions $F_{1,2}$ (we remind that the choice of these functions is not unique).

The difficulty of building $F_{1,2}$ is as follows. According to Theorem 2, say, $F_1(\tilde{x})$ is a cubic radical of a function $G_1 \in K_1$. Since function $F_1$ is not allowed to have branch points on $R$, all poles and zeros of $G(\hat{x})$ on $R$ should have order $3\nu$, $\nu \in \mathbb{Z}$. At the same time, $G_1$ cannot be a cube of another function from $K_1$, otherwise $F_1(\Lambda(\tilde{x})) = F_1(\tilde{x})$ and the condition (43) cannot be fulfilled. So, it is necessary to construct a function having triple poles and zeros, but which is not a cube of a meromorphic function. Clearly, there are no such functions on $\mathbb{C}$, but, surprisingly, such functions exist on $R$:

**Theorem 4** Let $\hat{a}, \hat{b}, \hat{c}$ be three points on $R$. Let $\gamma_1, \gamma_2, \gamma_3$ be oriented contours connecting the points $\hat{a}$ and $\hat{b}$, $\hat{b}$ and $\hat{c}$, and $\hat{c}$ and $\hat{a}$, respectively. Let the concatenation $\gamma_1 + \gamma_2 + \gamma_3$ be homotopic to the contour $\sigma_3$ shown in Fig. 3. Let be

$$\int_{\gamma_1} \frac{dx}{\Upsilon(x)} = \int_{\gamma_2} \frac{dx}{\Upsilon(x)} = \int_{\gamma_3} \frac{dx}{\Upsilon(x)} = \frac{T_3}{3}.$$  \hspace{1cm} (66)

Then

a) Function

$$G_1 = \frac{\Lambda(\hat{a}, \hat{b}, \hat{c}; \hat{x})}{\Lambda(\hat{b}, \hat{c}, \hat{a}; \hat{x})} \in K_1$$ \hspace{1cm} (67)

has a triple pole at $\hat{a}$, a triple zero at $\hat{b}$, and no other zeros or poles.

b) Function

$$G_2 = \frac{\Lambda(\hat{a}, \hat{b}, \hat{c}; \hat{x})}{\Lambda(\hat{c}, \hat{a}, \hat{b}; \hat{x})} \in K_1$$ \hspace{1cm} (68)

has a triple pole at $\hat{a}$, a triple zero at $\hat{c}$, and no other zeros or poles.

c) Function $F_1 = (G_1)^{1/3}$ is meromorphic on $R_3$ and obeys (43).

d) Function $F_2 = (G_2)^{1/3}$ is meromorphic on $R_3$ and obeys (44).

The proof of the theorem is given in Appendix 3. According to this proof, one can take an arbitrary point on $R$ as $\hat{a}$, find $\hat{b}$ and $\hat{c}$ as functions of $\hat{a}$, and build corresponding functions $F_1$ and $F_2$. For convenience, fix the point

$$\hat{a} = \eta_{2.1}.$$ \hspace{1cm} (69)

By definition (66) of $\hat{b}, \hat{c}$, and by using the mapping $\psi$, one can write

$$\hat{b} = \psi(T_3/3), \quad \hat{c} = \psi(2T_3/3).$$ \hspace{1cm} (70)

Introduce the notations

$$\hat{b} = \hat{b}, \quad \hat{c} = \hat{c}$$

to stress that these $\hat{b}$ and $\hat{c}$ are some fixed values. For the selected $\hat{a}, \hat{b}, \hat{c}$, the functions $F_{1,2}$ are given by the following proposition.
Proposition 14 Let \( \hat{a}, \hat{b}, \hat{c} \) be defined by (69), (70). Then

**a)** Functions \( F_1(\tilde{x}) \) and \( F_2(\tilde{x}) \) have form

\[
F_1(\tilde{x}) = \left( \frac{\Upsilon(\hat{x})}{(x-b)^2} + \frac{\Upsilon(b)}{(x-b)^2} \right) - \frac{1}{3} \left( \frac{x-b}{x-\eta_{2,1}} \right) \]

**b)** The affix \( b \) of \( \hat{b} \) obeys a fourth-order algebraic equation

\[
h_0 + h_1 b + h_2 b^2 + h_3 b^3 + h_4 b^4 = 0,
\]

where

\[
h_0 = \eta_{2,1} + 3\eta_{1,2} - \eta_{2,1}^2 \eta_{1,2} + \eta_{2,1} \eta_{1,2}^2,
\]

\[
h_1 = -4(1 + 2\eta_{2,1} + \eta_{1,2}^2),
\]

\[
h_2 = 6(\eta_{2,1} + \eta_{1,2} + \eta_{2,1}^2 \eta_{1,2} + \eta_{2,1} \eta_{1,2}^2),
\]

\[
h_3 = -4\eta_{2,1}(\eta_{2,1} + 2\eta_{1,2} + \eta_{2,1} \eta_{1,2}),
\]

\[
h_4 = \eta_{2,1} - \eta_{1,2} + 3\eta_{2,1}^2 \eta_{1,2} + \eta_{2,1} \eta_{1,2}^2.
\]

**c)** The affix \( b \) can be found by solving the ordinary differential equation

\[
\frac{dx}{d\chi} = \Upsilon(x)
\]

for the function \( x(\chi) \) on the segment \( \chi \in [0, T_\beta/3] \). The initial data is \( x(0) = \eta_{2,1} \). The result is defined by \( b = x(T_\beta/3) \).

**Proof.** a) Construct the ratios (67), (68) for the selected points \( \hat{a}, \hat{b}, \hat{c} \). Note that according Proposition 12, the affixes of \( b \) and \( c \) coincide, and they are located on different sheets of \( \mathbb{R} \):

\[
b = c, \quad \hat{c} = \Pi(\hat{b}).
\]

Besides, this proposition yields

\[
M(\eta_{2,1}, \eta_{2,1}, \hat{b}, \hat{c}; \tilde{x}) = \frac{x-b}{x-\eta_{2,1}}.
\]

According to Proposition 13,

\[
M(\hat{b}, \hat{b}, \eta_{2,1}, \hat{c}; \tilde{x}) = \frac{\Upsilon(\tilde{x})}{(x-b)^2} + \frac{\Upsilon(b)}{(x-b)^2} + \frac{\Upsilon(b)}{x-b} + \frac{\Upsilon(b)}{2}.
\]
provided
\[
\frac{\Upsilon(b)}{(\eta_{2,1} - b)^2} + \frac{\dot{\Upsilon}(b)}{\eta_{2,1} - b} + \frac{\ddot{\Upsilon}(b)}{2} = 0 \tag{77}
\]
is valid. According to the same proposition,\[
M(\hat{c}, \hat{c}, \eta_{2,1}, \eta_{1,1}, \hat{b}; \hat{x}) = -\Upsilon(\hat{x}) = \frac{(x - b)^2}{(x - b)^2} + \dot{\Upsilon}(b) + \ddot{\Upsilon}(b) \tag{78}
\]
(we change the sign of (64) for convenience; it can be done since all \(M\)-functions are defined up to a constant factor). The condition of existence of \(M(\hat{c}, \hat{c}, \eta_{2,1}, \eta_{1,1}, \hat{b}; \hat{x})\) is also (77). Substituting (75), (76), and (78) into (67) and (68), obtain (71) and (72).

Let us prove point b). Consider the condition (77). Divide it by \(\Upsilon(\hat{b})\) and note that ratios \(\frac{\dot{\Upsilon}(\hat{b})}{\Upsilon(\hat{b})}\) and \(\frac{\ddot{\Upsilon}(\hat{b})}{\Upsilon(\hat{b})}\) are rational functions of \(b\). Thus, (77) is an algebraic equation for \(b\). After some algebra get (73).

The proof of c) is elementary: (74) is another form of (55). □

Several remarks should be made regarding Proposition 14.

**Remark 1.** To solve the ordinary differential equation (74) near the branch point \(\eta_{2,1}\) one can use the local variable on \(\mathbb{R}\), namely,
\[
\tau = \tau(x) = \sqrt{x - \eta_{2,1}},
\]
as the dependent variable for the ODE. One can rewrite (74) as
\[
\frac{d\tau}{d\chi} = \frac{1}{2} \sqrt{(r^2 + \eta_{2,1} - \eta_{1,1})(r^2 + \eta_{2,1} - \eta_{1,2})(r^2 + \eta_{2,1} - \eta_{2,2})}. \tag{79}
\]
Thus, one can solve (79) in some small neighborhood of \(\eta_{2,1}\), and then solve (74) on the remaining part of the segment \(\chi \in [0, T_{\beta}/3]\).

**Remark 2.** By construction, \(F_1\) has three simple poles at \(P^{-1}_{3,1}(\eta_{2,1})\) and three simple zeros at \(P^{-1}_{3,1}(\hat{b})\). There are no other poles or zeros. The properties of \(F_2\) are similar, but it has simple zeros at \(P^{-1}_{3,1}(\hat{c})\).

**Remark 3.** The algebraic condition (73) is slightly weaker than the transcendent condition (68). Namely, equation (73) is fulfilled if there exist any contours \(\gamma_1, \gamma_2, \gamma_3\) cyclically connecting the points \(\eta_{2,1}\) and two points on \(\mathbb{R}\) having affix \(b\), such that
\[
\int_{\gamma_1} \frac{dx}{\Upsilon(\hat{x})} = \int_{\gamma_2} \frac{dx}{\Upsilon(\hat{x})} = \int_{\gamma_3} \frac{dx}{\Upsilon(\hat{x})}. \tag{80}
\]
The concatenation of the contours \(\gamma_1 + \gamma_2 + \gamma_3\) is not necessarily homotopic to \(\sigma_\beta\), thus, each of the integrals is not necessarily equal to \(T_{\beta}/3\).
A detailed study shows that the equation (73) has four roots: $b_1, b_2, b_3, b_4$, such that

$$
\int_{\eta_{2,1}}^{b_1} \frac{dx}{\Upsilon(\hat{x})} = \pm \frac{T_\beta}{3} + \mu T_\beta + \nu T_\alpha, \tag{81}
$$

$$
\int_{\eta_{2,1}}^{b_2} \frac{dx}{\Upsilon(\hat{x})} = \pm \frac{T_\alpha}{3} + \mu T_\beta + \nu T_\alpha, \tag{82}
$$

$$
\int_{\eta_{2,1}}^{b_3} \frac{dx}{\Upsilon(\hat{x})} = \pm \frac{T_\alpha + T_\beta}{3} + \mu T_\beta + \nu T_\alpha, \tag{83}
$$

$$
\int_{\eta_{2,1}}^{b_4} \frac{dx}{\Upsilon(\hat{x})} = \pm \frac{T_\alpha - T_\beta}{3} + \mu T_\beta + \nu T_\alpha. \tag{84}
$$

The integrals are defined up to the sign and up to the integers $\mu, \nu$, which depend on the particular choice of the integration contour. One can see that only $b_1$ fits the condition (68), i.e. $b = b_1$.

**Remark 4.** The functions $F_{1,2}$ are defined by (71) ambiguously. This ambiguity follows from that of the cubic radical, i.e. the result can be multiplied by $\varpi$ or $\varpi^{-1}$. Let us remove this ambiguity. The symmetry $\Pi$ converts $F_1$ into $F_2$ (up to multiplication by some cubic root of 1):

$$
F_1(\Pi(\tilde{x})) = \delta F_2(\tilde{x}), \quad \delta \in \{1, \varpi, \varpi^{-1}\}. \tag{85}
$$

To prove this, use (29) and note that

$$
\frac{\Upsilon(\hat{x})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{x-b} + \frac{\Upsilon(\hat{b})}{2} \rightarrow - \frac{\Upsilon(\hat{x})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{x-b} + \frac{\Upsilon(\hat{b})}{2}
$$

To remove some of the ambiguity of determining $F_1$ and $F_2$, fix the value $\delta = 1$, thus fixing

$$
F_1(\Pi(\tilde{x})) = F_2(\tilde{x}). \tag{86}
$$

Thus, one can choose the cubic root defining $F_1$ arbitrarily, and then the choice for $F_2$ follows from (86).

**Remark 5.** The product of functions $F_1$ and $F_2$ is rational. Namely, it belongs to $K_1$ since

$$
F_1(\Lambda(\tilde{x}))F_2(\Lambda(\tilde{x})) = F_1(\tilde{x})F_2(\tilde{x}),
$$

and, then, it belongs to $K_0$ since

$$
F_1(\Pi(\tilde{x}))F_2(\Pi(\tilde{x})) = F_1(\tilde{x})F_2(\tilde{x}).
$$
One can easily see that \( F_1(\tilde{x})F_2(\tilde{x}) \) should have a simple pole at \( x = \eta_{2,1} \) and a simple zero at \( x = b \). Studying the function at infinity, one can find that

\[
F_1(\tilde{x})F_2(\tilde{x}) = \frac{1}{((\Upsilon(b))^2/4 - 1)^{1/3}} x - b.
\]

(87)

5 Constructing the Sommerfeld transformant \( A(x) \)

Here we assume that \( x_{in} \) and \( x_{in}^{-1} \) are not equal to \( b \) or to \( \eta_{j,l} \).

Let us build the Sommerfeld transformant \( A(\tilde{x}) \) in the form (31), i.e. let us find functions \( A_0, A_1, A_2 \) obeying Functional problem 2. These functions have form (54), where the coefficients \( q_j(\hat{x}) \) belong to \( K_1 \).

According to Proposition 10,

\[
q_j(\hat{x}) = q'_j(x) + q''_j(x)\Upsilon(\hat{x}), \quad j = 0, 1, 2,
\]

(88)

where \( q'_j(x) \) and \( q''_j(x) \) belong to \( K_0 \), i.e. they are rational functions of \( x \). The aim of this section is to find the functions \( q'_j(x) \) and \( q''_j(x) \).

Functions \( A_0, A_1, A_2 \) obey conditions a) and d) of Functional problem 2 by construction. Condition b) and c) are provided by Propositions 15 and 16, respectively.

**Proposition 15** Functions \( A_{0,1,2} \) defined by (54) obey condition b) of Functional problem 2 iff the coefficients \( q'_j(x) \), \( q''_j(x) \) have form

\[
q'_0(x) = \frac{s_1}{x - x_{in}} + \frac{s_2}{x - x_{in}^{-1}} + s_0,
\]

(89)

\[
q''_0(x) = \frac{s_3}{x - x_{in}} - \frac{s_3}{x - x_{in}^{-1}},
\]

(90)

\[
q'_1(x) = (x - \eta_{2,1}) \left( \frac{s_4}{x - x_{in}} + \frac{s_5}{x - x_{in}^{-1}} - \frac{(s_6 + s_7)\Upsilon(b)}{x - b} \right),
\]

(91)

\[
q''_1(x) = (b - \eta_{2,1}) \left( \frac{s_6}{x - x_{in}} + \frac{s_7}{x - x_{in}^{-1}} - \frac{s_6 + s_7}{x - b} \right),
\]

(92)

\[
q'_2(x) = (x - \eta_{2,1}) \left( \frac{s_8}{x - x_{in}} + \frac{s_9}{x - x_{in}^{-1}} + \frac{(s_{10} + s_{11})\Upsilon(b)}{x - b} \right),
\]

(93)

\[
q''_2(x) = (b - \eta_{2,1}) \left( \frac{s_{10}}{x - x_{in}} + \frac{s_{11}}{x - x_{in}^{-1}} - \frac{s_{10} + s_{11}}{x - b} \right).
\]

(94)

where \( s_0, \ldots, s_{11} \) are some arbitrary parameters.
Proof. To find the Ansatz for each of the coefficients \( q_j' \), \( q_j'' \) let us prove the following statements based on Functional problem 2.

1. Let \( x \) be not equal to \( \infty \), \( \eta_{i,t} \), \( b \), \( x_{in} \), or \( x_{in}^{-1} \). Then all functions \( q_j', q_j'' \), \( j = 0, 1, 2 \) are regular at \( x \).

   The proof is as follows. Consider some particular \( j \). Let \( \nu \) be the highest pole order of the functions \( q_j', q_j'' \) at \( x \). Note that \( F_{1,2}(x) \neq 0 \) and \( \Upsilon(x) \neq 0 \). Thus, the pole of order \( \nu \) will appear on the sheet 1 or 2 (corresponding residues cannot be compensated both). This contradicts to Functional problem 2.

   The statements 2, 3, and 4 are similar to statement 1, so we omit their proofs.

2. The functions \( q_k', q_k'' \), \( k = 0, 1, 2 \) are regular at the points \( \eta_{1,1}, \eta_{1,2}, \eta_{2,2} \).

3. The functions \( q_k', x^2 q_k'' \), \( k = 0, 1, 2 \) are regular at infinity.

4. The functions \( q_k', q_k'' \) have simple poles at \( x_{in} \) and \( x_{in}^{-1} \).

   Slightly more subtle consideration is needed for the values \( x \) equal to \( \eta_{2,1} \) and \( b \), since functions \( F_1 \) and \( F_2 \) have poles and zeros at these affixes. The following statements can be checked:

5. The functions \( q_0'(x), q_0''(x), (x - \eta_{2,1})^{-1} q_1'(x), q_1'(x), (x - \eta_{2,1})^{-1} q_2'(x), q_2'(x) \) are regular at \( x = \eta_{2,1} \).

6. Functions \( q_0' \) and \( q_0'' \) are regular at \( b \). \( q_j'(x), q_j''(x), j = 1, 2 \) can have simple poles at \( b \). The following identities should be valid:

\[
\lim_{x \to b}[q_1'(x) - \Upsilon(b)q_1''(x)] = 0, \quad \lim_{x \to b}[q_2'(x) + \Upsilon(b)q_2''(x)] = 0,
\]

One can see that (98)–(94) are the most general formulae for rational functions obeying statements 1–6. □

Proposition 16 Functions \( A_{0,1,2} \) defined by (54) obey condition c) of Functional problem 2 iff Proposition 15 is fulfilled, and

\[
s_1 = i Y_1/(6\pi), \quad s_2 = i Y_3/(6\pi), \quad s_3 = 0, \quad (95)
\]

\[
(x_{in} - \eta_{2,1})s_4 F_1(\bar{x}_1) + (b - \eta_{2,1})s_6 Y_1 F_1(\bar{x}_1) = i Y_1/(6\pi), \quad (96)
\]

\[
(x_{in} - \eta_{2,1})s_4 F_1(\bar{x}_2) - (b - \eta_{2,1})s_6 Y_1 F_1(\bar{x}_2) = i Y_1/(6\pi), \quad (97)
\]

\[
(x_{in} - \eta_{2,1})s_8 F_2(\bar{x}_1) + (b - \eta_{2,1})s_{10} Y_1 F_2(\bar{x}_1) = i Y_1/(6\pi), \quad (98)
\]

\[
(x_{in} - \eta_{2,1})s_8 F_2(\bar{x}_2) - (b - \eta_{2,1})s_{10} Y_1 F_2(\bar{x}_2) = i Y_1/(6\pi), \quad (99)
\]
The additive parameter $s_0$ can be chosen arbitrarily (e.g. $s_0 = 0$).

The proof is straightforward. One should substitute (89)–(94) into (88), (54) and check the validity of condition c) of Functional problem 2.

The equations (96)–(103) can be easily solved. The result can be written using (87) as follows:

$$s_4 = Z \Upsilon(\hat{x}_1) \frac{1}{x_{\text{in}} - b} (F_2(\hat{x}_1) + F_2(\hat{x}_2)), \quad (104)$$

$$s_6 = Z \left( \frac{1}{b - \eta_{2,1}} + \frac{1}{x_{\text{in}} - b} \right) (F_2(\hat{x}_1) - F_2(\hat{x}_2)), \quad (105)$$

$$s_8 = Z \Upsilon(\hat{x}_1) \frac{1}{x_{\text{in}} - b} (F_1(\hat{x}_1) + F_1(\hat{x}_2)), \quad (106)$$

$$s_{10} = Z \left( \frac{1}{b - \eta_{2,1}} + \frac{1}{x_{\text{in}} - b} \right) (F_1(\hat{x}_1) - F_1(\hat{x}_2)), \quad (107)$$

$$s_5 = Z \Upsilon(\hat{x}_3) \frac{1}{x_{\text{in}}^{-1} - b} (F_2(\hat{x}_3) + F_2(\hat{x}_4)), \quad (108)$$

$$s_7 = Z \left( \frac{1}{b - \eta_{2,1}} + \frac{1}{x_{\text{in}}^{-1} - b} \right) (F_2(\hat{x}_3) - F_2(\hat{x}_4)), \quad (109)$$

$$s_9 = Z \Upsilon(\hat{x}_3) \frac{1}{x_{\text{in}}^{-1} - b} (F_1(\hat{x}_3) + F_1(\hat{x}_4)), \quad (110)$$

$$s_{11} = Z \left( \frac{1}{b - \eta_{2,1}} + \frac{1}{x_{\text{in}}^{-1} - b} \right) (F_1(\hat{x}_3) - F_1(\hat{x}_4)), \quad (111)$$

where

$$Z = \frac{i((\hat{\Upsilon}(\hat{b}))^2/4 - 1)^{1/3}}{12\pi}. \quad (112)$$

Finally, the Sommerfeld transformant is found. The formulae that should be used for computations are (104)–(111), (89)–(94), (53), (71), (72). The value $b$ is defined from (73) or from the ODE (74). The transformant is substituted into the Sommerfeld integral (24).
6 Numerical examples

In this section we are demonstrating the ideas of the paper using some numerical examples. We take real values of $K$, having in mind the limit $\text{Im}[K] \rightarrow +0$.

6.1 Computation of $\hat{b}$

For practical computations, we propose two following algorithms for computing the value $b$ with a high accuracy.

Algorithm 1:

1. Compute $T_\beta$ by numerical integration.

2. Find $b$ approximately by solving numerically the ordinary differential equation (74) on the segment $\chi \in [0, T_\beta/3]$. Take $x(0) = \eta_{2,1}$. The value $x(T_\beta/3)$ is the approximation for $b$. Denote it by $b'$.

3. Using $b'$ as a starting approximation, solve (73) by Newton’s method. As a result, after several iterations, get a refined value of $b$ with a machine accuracy.

Since the first two steps are necessary only to obtain the starting approximation for Newton’s method used on the third step, very coarse meshes can be used for numerical integration and for solving the ordinary differential equation. The Newton’s method is very cheap, and, several iterations provide the value of $b$ having the machine accuracy.

Another algorithm can be developed, taking the algebraic equation (73) as the starting point. The algorithm is as follows.

Algorithm 2:

1. Solve equation (73) and find four values: $b_1, b_2, b_3, b_4$. This can be done even explicitly.

2. For each affix $b_j$ take points $\hat{b}_j$ and $\check{b}_j$ on two different sheets of $\mathbb{R}$. Totally there will be eight candidates for $\hat{b}$.

3. For each value $\hat{b}_j$ or $\check{b}_j$ construct the function

$$G_1(\tilde{x}) = \left( \frac{\Upsilon(\hat{x})}{(x - b)^2} + \frac{\Upsilon'(\hat{b})}{(x - b)^2} + \frac{\Upsilon''(\hat{b})}{2} \right)^{-1} \frac{x - b}{x - \eta_{2,1}}, \quad (113)$$

and check the variation of $\text{Arg}[G_1]$ along the contour $\sigma_\beta$. and $\sigma_\alpha$. There should exist only one value of $\hat{b}$ (among the eight candidates), for which the conditions (130) and (132) are valid. This value of $\hat{b}$ is what we are looking for.
Indeed, Algorithm 1 and Algorithm 2 should yield the same result. We use Algorithm 1 below.

Let be $K = 0.5$. First, find the period $T_\beta$ (see (56)). For the numerical integration, use contour $\sigma_\beta$ shown in Fig. 6, left. The positions of the branch points $\eta_{j,l}$ are shown by stars.

Find the correct values of $\Upsilon(\hat{x})$ on this contour. The contour passes through the point $x = 1$ on sheet 1, thus, one can fix $\Upsilon(1)$, and then utilize the continuity. One can see that $\Upsilon(1) = \pm 0.9682i$, and one should choose the correct sign. Take $K$ close to 0.5, but having a small positive imaginary part, say $K = 0.5 + 0.1i$. The values of $\Upsilon(1)$ for this $K$ are $\pm(-0.1810 + 0.9722i)$, and they correspond to the values $y_1 = 0.7895 + 0.4361$ and $y_2 = 0.9705 - 0.5361$, respectively. One can see that $|y_1| < 1$ and $|y_2| > 1$. Thus, one should choose $\Upsilon(1) = -0.1810 + 0.9722i$ for $K = 0.5 + 0.1i$. By continuity, $\Upsilon(1) = 0.9682i$ for $K = 0.5$. This reasoning yields also that $\text{Im}[\Upsilon(1)] > 0$ on the physical sheet for all real $0 < K < 2$.

The integral for $T_\beta$ can be easily computed for $K = 0.5$:

$$T_\beta = -1.6219 + 2.4884i.$$  

For demonstration purposes, solve the equation (74) (or its equivalent form (79)) on the segment $\chi \in [0, T_\beta]$ taking the initial value $x(0) = \eta_{2,1}$. The result is the trajectory going from $\eta_{2,1}$ to $\eta_{2,2}$ along one of the sheets of $\mathbb{R}$ and returning back along another sheet. The trajectory ends almost exactly at $\eta_{2,1}$. The projection of this trajectory onto the $x$-plane is shown in Fig. 6, right.

According to Algorithm 1, solve equation (74) for $\chi \in [0, T_\beta/3]$, with $x(0) = \eta_{2,1}$. As the result, get the position of the point $\hat{b}$, i.e. the affix $b$ and the value $\Upsilon(\hat{b})$. We use this value as a starting approximation for $b$ and refer to it as $b'$. If the ODE is solved by the simplest Euler’s scheme on a mesh of 100 nodes,

$$b' = 0.2917 + 0.1858i, \quad \Upsilon(\hat{b}') = -0.2437 + 0.7958i.$$

The position of $b'$ is shown in Fig. 6, right, by a circle. The value of $\Upsilon(\hat{b}')$ is needed to conclude that $b'$ belongs to sheet 2 of $\mathbb{R}$. 

Fig. 6: Contour for finding $T_\beta$ (left), solution of equation (74), right
The value of $b'$ obtained so far can be considered as a rough approximation for this parameter. According to Algorithm 1, one can solve (73) by the Newton’s method to refine the value. The process stabilizes after 4 steps, and the result is:

$$b = 0.295390040273516 + 0.186354378894278i.$$ 

One can see that the starting approximation $b'$ happens to be quite close to the exact root of (73). This is a clear demonstration of consistency of our approach.

Taking $K$ belonging to a dense grid covering the segment $[10^{-3}, 1]$ and repeating the procedure described above, one can obtain the values $\hat{b}(K)$. They are presented graphically in Fig. 7. The affix $b$ is shown by its real and imaginary part. The value $\Upsilon(\hat{b})$ is necessary only to select a correct sheet of $\mathbb{R}$, so we displayed the imaginary part of it.

![Graph of $\mathrm{Re}[b]$, $\mathrm{Im}[b]$, and $\mathrm{Im}[\Upsilon(\hat{b})]$ as functions of $K$.](image)

**Fig. 7:** The values of $\mathrm{Re}[b]$, $\mathrm{Im}[b]$, and $\mathrm{Im}[\Upsilon(\hat{b})]$ as functions of $K$

An important conclusion that can be made from Fig. 7 (and of course one can prove this analytically) is that

$$b \to 1 \quad \text{as} \quad K \to 0,$$

and $\hat{b}$ is located on sheet 2.

### 6.2 Examination of $G_1 = (F_1)^3$

Fix $K = 0.5$ and use the value of $\hat{b}$ found in the previous subsection. Construct the function $G_1(\hat{x})$ by the formula (113). Check numerically the validity of the conditions (130), (132).
To check this, we build hodographs of $G_1$ on $\sigma_\alpha$ and on $\sigma_\beta$, i.e. we plot the values of $G_1(\hat{x})$ for $\hat{x}$ running along the contours $\sigma_\alpha$ and $\sigma_\beta$. As the result, we get oriented contours in the complex plane of $G_1$.

The contour homotopic to $\sigma_\beta$ has been already built (see Fig. 6, left). The contour homotopic to $\sigma_\alpha$ and convenient for numerical computations is shown in Fig. 8. The contour passes the value $x = 1$ on sheet 1 on the way down.

![Fig. 8: A contour homotopic to $\sigma_\alpha$](image)

The hodographs of $G_1(\hat{x})$ on $\sigma_\alpha$ and $\sigma_\beta$ are shown in Fig. 9, left and right, respectively. The origin is marked by letter O in both graphs. One can see that the hodograph for $\sigma_\alpha$ encircles the origin for a single time in the positive direction, and the hodograph for $\sigma_\beta$ does not encircle the origin at all. Thus, the conditions for $G_1$ are valid.

Indeed, a similar check can be performed for $G_2 = (F_2)^3$. 

![Fig. 9: Hodographs of $G$ on $\sigma_\alpha$ (left) and on $\sigma_\beta$ (right)](image)
6.3 Building the wave $u(m, n)$

Take the value of $K$ equal to 0.5. For simplicity, take angle of incidence $\phi_{in} = \pi/4$. By symmetry, $x_{in} = y_{in}$ and we are looking for the solution of the equation $\hat{D}(x_{in}, x_{in}) = 0$ corresponding to the wave traveling in the positive direction with respect to $m$ and $n$:

$$x_{in} = y_{in} = \frac{4 - K^2 + iK\sqrt{8 - K^2}}{4} = 0.9375 + 0.3480i.$$

For the total wave, we use an alternative form of the Sommerfeld integral (24), namely (117), (118) with the transformant $A$ represented by (53). We take 50000 nodes on each contour for integration in the $x$-plane.

The real part of the total wave is shown in Fig. 10.

![Fig. 10: The real part of $u(m, n)$](image)

The field pattern corresponds to what can be expected. The field is zero at the boundary, and there are visible zones of the reflected waves.

In Fig. 11 we plot the scattered wave $u_{sc}$ only. The real part is in the left, while the imaginary part is in the right. One can see cylindrical wave scattered by the angle vertex.
7 Conclusion

Let us summarize the process of solving the problem of diffraction by a Dirichlet right angle on a discrete plane. Note that the problem is characterized by two parameters: by the wavenumber parameter $K$ of the Helmholtz equation (1) and by the incident angle $\phi_m$ defined by (8). The procedure is as follows:

1. The consideration is based on the structure of the Riemann surface $R$. This surface is described by four branch points $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}$. These branch points depend only on $K$, and they are found from (12), (13).

2. One should find the period $T_{\beta}$. This can be done by computing the corresponding integral in (56). The contour is shown in Fig. 3, but it is practical to perform the integration along the unit circle in the negative direction. Function $\Upsilon(\hat{x})$ is given by the last expression of (17). The branch of $\Upsilon(\hat{x})$ is chosen in such a way that $y(\hat{x})$ defined by (19) has property $|y(\hat{x})| < 1$.

3. The parameter $\hat{b}$ should be found. This is the point on $R$, thus it is characterized by the affix $b$ and the branch of $\Upsilon(b)$.

Algorithm 1 described in Subsection 6.1 can be used for this. According to this algorithm, first, the differential equation (74) is solved numerically on the segment $\chi \in [0, T_{\beta}/3]$, and an approximation $b'$ of the parameter $b$ becomes obtained. Besides, the sheet of $R$ on which $b$ is located becomes determined. Second, an algebraic equation (73) is solved iteratively using $b'$ as the starting approximation.

4. The functions $F_1(\hat{x})$ and $F_2(\hat{x})$ are constructed by (71) and (72). Note that these functions depend on $K$ as a parameter.

5. The Sommerfeld transformant of the total wave $A(\hat{x})$ is built using (104)–(111), (89)–(94), (53), (71), (72).

6. The function $u(m, n)$ is built using the Sommerfeld integral (24).
The whole consideration is held in the framework of the Sommerfeld integral. The structure of the integral may seem slightly unusual, however, as we demonstrate in [1], it is a natural generalization of the Sommerfeld integral for angular domains known for the continuous case.

We build the functional field $K_3$ to which the Sommerfeld transformant belongs. This field is common for all incident angles. The functional field $K_3$ is represented as the basis $\Omega_{3,1}$ composed of three functions, two of which, $F_1$ and $F_2$ should be built. The construction of the basis is a non-trivial procedure. Then, for a particular angle of incidence $\phi_{\text{in}}$ we find the Sommerfeld transformant $A(\tilde{x})$. This task is tedious, but quite simple. The coefficients $q'_j(x), g_j(x)''$ are rational functions, and one should find these functions obeying some known restrictions and having some known poles. This structure of solution seems to be deeply linked with the embedding procedure [12, 13].

8 Acknoledgements

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Appendix 1. Proof of Theorem 1

Consistency of the Sommerfeld integral

To prove statement a) of the theorem, it is sufficient to show that

\[
\int_{\gamma_3} w_{m,n}(x, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})} = \int_{\gamma_3} w_{m,n}(x, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})} = 0 \tag{114}
\]

for \( m \leq 0 \) and \( n \leq 0 \). One can see that both \( x \) and \( y(\hat{x}) \) tend to \( \infty \) at the infinities of the sheets 2 and 4 of \( \mathbb{R}^3 \). Thus, the function \( w_{m,n}(x, y(\hat{x})) \) (see (3)) does not grow as \( |x| \to \infty \). According to the conditions imposed on \( A \), the integrals are equal to zero.

Validity of the discrete Helmholtz equation

We are starting to prove validity of statement b) of the theorem. Substitute the representation (24) with (22) for \( m < 0 \), or with (23) for \( n < 0 \) into the equation (1). Note that the discrete Laplace operator acts only on \( w \). A direct check shows that (1) is valid.

Radiation condition

Let us demonstrate that \( u(m, n) \) obeys the radiation condition formulated in the form of the limiting absorption principle. For this, deform the contours \( \Gamma_2 \) and \( \Gamma_3 \) homotopically as follows:

\[
\Gamma_2 = \lambda_1 + \lambda_2 + \lambda_3, \quad \Gamma_3 = \lambda_3 + \lambda_4 + \lambda_5, \tag{115}
\]

where contours \( \lambda_1, \ldots, \lambda_5 \) are shown in Fig. 12. Sheets 5 and 6 are not shown. Contours \( \lambda_1 \) and \( \lambda_2 \) are drawn around corresponding cuts (we remind that the cuts are conducted along the sets of \( x \) for which \( |y(\hat{x})| = 1 \)). Contours \( \lambda_4 \) and \( \lambda_5 \) are unit circles. Contour \( \lambda_3 \) encircles \( x_{\text{in}} \).

Note that \( |x_{\text{in}}| < 1 \).

One can see that

\[
\int_{\lambda_3} w_{m,n}(\hat{x}, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})} = u_{\text{in}}. \tag{116}
\]
As the result, the following representations of the field are obtained:

\[ u(m, n) = u_{\text{in}}(m, n) + \int_{\lambda_1 + \lambda_2} w_{m,n}(\hat{x}, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})} \quad \text{for } m \leq 0, \]  

\[ u(m, n) = u_{\text{in}}(m, n) + \int_{\lambda_4 + \lambda_5} w_{m,n}(\hat{x}, y(\hat{x})) A(\hat{x}) \frac{dx}{\Upsilon(\hat{x})} \quad \text{for } n \leq 0. \]  

Consider the exponential factor \( w_{m,n} = x^m y^n \) of the representation (117). For each point of the representation contours, \(|y| = 1\) and \(|x| > 1\). Since \(m \leq 0\), the result should decay for large negative \(m\). Besides, the field should decay for constant negative \(m\) and growing positive \(n\) due to the oscillatory nature of factor \(w\) on the contours \(\lambda_1\) and \(\lambda_2\).

Similarly, for the representation (118), \(|x| = 1\) and \(|y| > 1\) in the exponential factor, thus the field should decay for large negative \(n\).

Thus, we obtain that the total field is a sum of the incident field and a decaying field.

**Boundary conditions**

Let us check the boundary condition \(u = 0\) on the side \(m \geq 0, n = 0\). For this, use the representation (24), (23).

On the boundary \(m \geq 0, n = 0\), the contour of the Sommerfeld integral can be deformed
into two unit circles drawn in sheet 3 and 4 (see Fig. 13). Namely,

$$u(m, 0) = \int_{\lambda_4 + \lambda_6} x^m A(\tilde{x}) \frac{dx}{Y(\tilde{x})}. \quad (119)$$

Due to the symmetry (30), this integral is zero.

The situation is slightly more subtle with the boundary \( m = 0, n \geq 0 \). Instead of \( \Pi \), we need another symmetry \( \Pi' \) of the Riemann surface \( \mathbb{R}_3 \). The symmetry \( \Pi' \) is defined as

$$\Pi'(\tilde{X}(x, 2)) = \tilde{X}(x^{-1}, 2) \quad (120)$$

in the neighborhood of the point \( x = 1 \) on sheet 2, and then continued analytically onto the whole \( \mathbb{R}_3 \). The mapping \( \Pi' \) transforms \( \mathbb{R}_3 \) into \( \mathbb{R}_3 \) keeping its complex structure, but it is not a desk transformation, since it does not maintain the affix. Note that \( y(\tilde{x}) = y(\Pi'(\tilde{x})) \).

One can show that the function \( A(\Pi'(\tilde{x})) \) obeys the same functional problem as \( A(\tilde{x}) \). Besides, there exists a stationary point of \( \Pi' \),

$$A(\Pi'(\tilde{x})) = A(\tilde{x}) \quad (121)$$

The contour of integration \( \Gamma_2 \) can be transformed into the contour shown in Fig. 14 (it is composed of \( \lambda_7, \lambda_8 \), and two polar terms). The resulting contour is symmetrical with respect to the mapping \( \Pi' \). The wave factor is also symmetrical:

$$w_{0,n}(\tilde{x}, y(\tilde{x})) = w_{0,n}(\Pi'(\tilde{x}), y(\Pi'(\tilde{x}))).$$

Thus, finally, the Sommerfeld integral on the boundary yields zero due to the symmetry.
Appendix 2. Two propositions for algebraization of the Abel’s theorem

Here we put two more cases of the criterion of building the function $M$.

**Proposition 17** Let $a_1, a_2, b_1, b_2$ are all distinct values not equal to infinity or to $\eta_{j,l}$. A function $M(\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2; \hat{x})$ exists iff

$$
\frac{\Upsilon(\hat{a}_2)}{(a_2 - b_1)(a_2 - b_2)} + \frac{\Upsilon(\hat{b}_1)}{(a_2 - b_1)(b_1 - b_2)} + \frac{\Upsilon(\hat{b}_2)}{(a_2 - b_2)(b_2 - b_1)} = \\
\frac{\Upsilon(\hat{a}_1)}{(a_1 - b_1)(a_1 - b_2)} + \frac{\Upsilon(\hat{b}_1)}{(a_1 - b_1)(b_1 - b_2)} + \frac{\Upsilon(\hat{b}_2)}{(a_1 - b_2)(b_2 - b_1)}. \tag{122}
$$

**Proof.** Let us try to construct function $M(\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2; \hat{x})$ explicitly. First, construct function $M \in K_1$ (possibly depending on some parameters) having poles only at $\hat{b}_1$ and $\hat{b}_2$. An obvious Ansatz, up to a common constant factor, is as follows:

$$
M(\hat{x}) = \frac{\Upsilon(\hat{x})}{(x - b_1)(x - b_2)} + \frac{g_1}{x - b_1} + \frac{g_2}{x - b_2} + c. \tag{123}
$$

for arbitrary complex values $g_1, g_2, c$. As above, $x, a_1, a_2, b_1, b_2$ are the affixes of the points $\hat{x}, \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$, respectively.

For arbitrary $g_1, g_2$, this function has poles at four points of $\mathbb{R}$: at $\hat{b}_1, \hat{b}_2, \Pi(\hat{b}_1), \Pi(\hat{b}_2)$. Choose the values of $g_1, g_2$ such that they suppress the poles at $\Pi(\hat{b}_1), \Pi(\hat{b}_2)$. One can see that
the appropriate function is as follows:

\[ M(\hat{x}) = \frac{\Upsilon(\hat{x})}{(x-b_1)(x-b_2)} + \frac{\Upsilon(\hat{b}_1)}{(x-b_1)(b_1-b_2)} + \frac{\Upsilon(\hat{b}_2)}{(x-b_2)(b_2-b_1)} + c. \] (124)

Now let us fix the zeros. Choose parameter \( c \) in such a way that \( M(\hat{a}_1) = 0 \):

\[ c = -\left[ \frac{\Upsilon(\hat{a}_1)}{(a_1-b_1)(a_1-b_2)} + \frac{\Upsilon(\hat{b}_1)}{(a_1-b_1)(b_1-b_2)} + \frac{\Upsilon(\hat{b}_2)}{(a_1-b_2)(b_2-b_1)} \right]. \] (125)

Finally, the condition guaranteeing that \( \hat{a}_2 \) is also a zero is the equation (122). □

We claim that (122) is an algebraic analog of the analytic equation (61) in the case of distinct \( a_1, a_2, b_1, b_2 \). The function \( M(\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2; \hat{x}) \) is given by (124), (125).

**Proposition 18** Let be \( \hat{b}_1 = \hat{b}_2 (= \hat{b}) \), \( a_{1,2} \neq b \), \( a_1 \neq a_2 \), and neither of the points \( a_{1,2}, b \) is equal to \( \eta_{j,l} \) or infinity. A function \( M(\hat{b}, \hat{b}, \hat{a}_1, \hat{a}_2; \hat{x}) \) exists iff

\[ \frac{\Upsilon(\hat{a}_1)}{(a_1-b_1)(a_1-b_2)} + \frac{\Upsilon(\hat{b})}{(a_1-b)(a_2-b)} = \frac{\Upsilon(\hat{a}_2)}{(a_2-b)(a_2-b)} + \frac{\Upsilon(\hat{b})}{(a_2-b)} + \frac{\Upsilon(\hat{b})}{a_2-b}. \] (126)

**Proof.** The most general Ansatz for function \( M \in K_1 \) having a pole of order 2 at \( \hat{b} \) is as follows (this Ansatz does not include a common constant factor):

\[ M(\hat{x}) = \frac{\Upsilon(\hat{x})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{(x-b)^2} + \frac{\Upsilon(\hat{b})}{x-b} + c, \] (127)

where

\[ \dot{\Upsilon}(\hat{x}) \equiv \frac{d\Upsilon(\hat{x})}{dx}. \] (128)

The constant \( c \) is chosen in such a way that \( M(\hat{a}_1) = 0 \):

\[ c = -\left[ \frac{\Upsilon(\hat{a}_1)}{(a_1-b)^2} + \frac{\Upsilon(\hat{b})}{(a_1-b)^2} + \frac{\Upsilon(\hat{b})}{a_1-b} \right]. \] (129)

Finally, this function is zero at \( \hat{a}_2 \) if (126) is valid. The function \( M \) is given by (127), (129). □

**Appendix 3. Proof of Theorem 4**

All three functions in the numerators and denominators of (67), (68) do exist according to Theorem 3 and the condition (66). The order of poles and zeros can be checked directly. Thus, the statements a) and b) of the theorem are valid by construction.

Let us prove c) (statement d) is similar). Note that \( F_1 = G^{1/3}_1(\hat{x}) \) is a three-valued function of \( \hat{x} \) having no branch points over \( \mathbb{R} \), since the pole and the zero have order 3. Thus, \( F_1 \) has some
3-sheet Riemann surface $R'$ without branching over $R$. Consider the projection $P : R' \to R$. According to Proposition 3 b), it is sufficient to show that $P^{-1}(\sigma_{\beta})$ is three disjoint copies of $\sigma_{\beta}$ and that $P^{-1}(\sigma_{\alpha})$ is a connected 3-sheet covering of $\sigma_{\alpha}$, to prove that $R'$ is $R_3$, and thus that $F_1$ is meromorphic on $R_3$.

To establish the validity Proposition 3 b) one should study the variation of $\text{Arg}[G_1]$ along the contours $\sigma_{\alpha}$ and $\sigma_{\beta}$. Denote these variations by the Var symbol. Note that such a variation is $2\pi j$, $j \in \mathbb{Z}$, since $G_1$ is meromorphic on $R$ and the contours are closed. One should prove that

$$\frac{\text{Var}_{\sigma_{\beta}} \text{Arg}[G_1]}{2\pi} \equiv 0 \pmod{3}$$

and

$$\frac{\text{Var}_{\sigma_{\alpha}} \text{Arg}[G_1]}{2\pi} \neq 0 \pmod{3}$$

to establish the validity of Proposition 3 b). Moreover, the condition

$$\frac{\text{Var}_{\sigma_{\alpha}} \text{Arg}[G_1]}{2\pi} \equiv 1 \pmod{3},$$

guarantees (43), since

$$\text{Arg}[F_1]^{\Lambda(\tilde{\xi})} = \text{Var}_{\sigma_{\alpha}} \text{Arg}[F_1] = \frac{1}{3} \text{Var}_{\sigma_{\alpha}} \text{Arg}[G_1],$$

(note that $P_{3:1}^{-1}(\sigma_{\alpha})$ connects $\eta_{2,1}$ with $\Lambda(\eta_{2,1})$). For $G_2$, (132) should be replaced with

$$\frac{\text{Var}_{\sigma_{\alpha}} \text{Arg}[G_2]}{2\pi} \equiv -1 \pmod{3}. \quad (133)$$

Let us prove (130), (132). Consider the points $\hat{a}$, $\hat{b}$, $\hat{c}$ obeying the condition of Theorem 4. Note that $b$ and $c$ are functions of $\hat{a}$:

$$\begin{align*}
\hat{b} &= b(\hat{a}) = \psi(\chi(\hat{a}) + T_{\beta}/3), \\
\hat{c} &= c(\hat{a}) = \psi(\chi(\hat{a}) + 2T_{\beta}/3).
\end{align*} \quad (134)$$

The image $\chi(\sigma_{\beta})$ is a continuous path connecting the points 0 and $T_{\beta}$. Deform $\sigma_{\beta}$ into $\sigma'_{\beta}$ such that $\chi(\sigma'_{\beta})$ is a straight segment connecting these points. During this deformation, the variation $\text{Var}_{\sigma_{\beta}} \text{Arg}[G_1]$ can change only by $6\pi j$, since the contour can cross the poles or zeros of order 3. Deform $\sigma_{\alpha}$ the same way, i.e. such that $\chi(\sigma'_{\alpha})$ is a straight line connecting $\chi(\hat{a})$ with $\chi(\hat{a}) + T_{\beta}$. Note that $\chi(\gamma_j)$ become parallel $\chi(\sigma_{\beta})$, thus generally $\gamma_j$ do not cross $\sigma'_{\beta}$ (see Fig. 15).

Define a function

$$M'(\hat{a}; \hat{x}) = \frac{M(\hat{a}, \hat{a}, \hat{b}(\hat{a}), \hat{c}(\hat{a}); \hat{x})}{M(\hat{a}, \hat{a}, \hat{b}(\hat{a}), \hat{c}(\hat{a}); \infty)}. \quad (135)$$

One can see that the function is normalized for definiteness; the infinity point is taken on sheet 1. According to (67), one can see that (up to a constant factor)

$$G_1 = M'(\hat{a}; \hat{x})/M'(\hat{b}; \hat{x}). \quad (136)$$
By construction (see the explicit formulae from Proposition 18), function $M'(\hat{a}; \hat{x})$ depends analytically on $\hat{a}$. Let the point $\hat{a}'$ move from $\hat{a}$ to $\hat{b}$ along $\gamma_1$. Note that $\gamma_1$ does not cross $\sigma_\beta$, thus, as $\hat{a}'$ travels along $\gamma_1$, the argument variation of $M'(\hat{a}'; \hat{x})$ along $\sigma'_\beta$ does not change. Thus,

$$\text{Var}_{\sigma'_\beta}[\text{Arg}[G_1]] = 0. \quad (137)$$

Now let us prove (132). As $\hat{a}'$ travels along $\gamma_1$ from $\hat{a}$ to $\hat{b}$, $\sigma'_\alpha$ is crossed either by a double pole or by a simple zero. A careful study of the orientation of the contours states that in the first case $\text{Var}_{\sigma'_\alpha}[G_1]$ changes by $4\pi$, and in the second case by $-2\pi$. In both cases (132) is valid. □

### Appendix 4. Uniqueness of the solution of the diffraction problem

Let there be two different scattered waves, $u_{sc}(m, n)$ and $u'_{sc}(m, n)$, both obeying the conditions of the diffraction problem. Consider the difference

$$v(m, n) = u_{sc}(m, n) - u'_{sc}(m, n).$$

Let us show that it is equal to zero. Function $v$ obeys equation (1), radiation condition and the homogeneous Dirichlet boundary conditions. Moreover, it belongs to $l_2(\mathbb{Z}^2)$.

Introduce a single index $\mu$ numbering all non-boundary nodes of the domain shown in Fig. 1 in a reasonable order. Consider the function

$$v_\mu = v(m(\mu), n(\mu)).$$

The equation (1) can be rewritten in the form

$$\sum_{\mu'} C_{\mu, \mu'} v_{\mu'} + K^2 v_\mu = 0, \quad (138)$$

where

$$C_{\mu, \mu'} = \begin{cases} 
1 & \text{if } \mu \text{ and } \mu' \text{ are neighboring nodes}, \\
-4 & \mu = \mu', \\
0 & \text{otherwise} 
\end{cases}$$
Note that $C_{\mu,\mu'} = C_{\mu',\mu}$, and $C_{\mu,\mu'}$ is a real (infinite) matrix. Take the complex conjugation of (138):

$$\sum_{\mu'} C_{\mu,\mu'} \bar{v}_{\mu'} + \bar{K}^2 \bar{v}_{\mu} = 0,$$

(139)

Multiply (138) by $\bar{v}_{\mu}$. Perform summation over all $\mu$. Multiply (139) by $v_{\mu}$ and also perform a summation. Subtract the second sum from the first one. The result is

$$(K^2 - \bar{K}^2) \sum_{\mu} \bar{v}_{\mu} v_{\mu} = 0.$$

Indeed, this yields $v_{\mu} \equiv 0$. 
