A few more comments on secularly growing loop corrections in strong electric fields

E. T. Akhmedov\textsuperscript{1,3) and F.K.Popov\textsuperscript{2,3)}

1) International Laboratory of Representation Theory and Mathematical Physics, National Research University Higher School of Economics, Russian Federation

2) Institutskii per, 9, Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Russia

3) B. Cheremushkinskaya, 25, Institute for Theoretical and Experimental Physics, 117218, Moscow, Russia

Abstract

We extend the observations of our previous paper JHEP 1409, 071 (2014) [arXiv:1405.5285]. In particular, we show that the secular growth of the loop corrections to the two–point correlation functions is gauge independent: we observe the same growth in the case of the static gauge for the constant background electric field. Furthermore we solve the kinetic equation describing photon production from the background fields, which was derived in our previous paper and allows one to sum up leading secularly growing corrections from all loops. Finally, we show that in the constant electric field background the one–loop correction to the current of the produced pairs is not zero: it also grows with time and violates time translational and reversal invariance of QED on the constant electric field background.

I. INTRODUCTION

Schwinger’s pair creation\textsuperscript{1] is a well studied phenomenon. However, in our recent paper [2] we show that in QED on strong electric field backgrounds there are loop corrections which grow with time\textsuperscript{1}. We use Schwinger–Keldysh diagrammatic technique and consider a constant electric field, $E_z = const$, and electric pulse, $E_z(t) \propto \frac{1}{\cosh^2(t/T)}$. We show that after a long enough evolution in a constant field background (or as $T \to \infty$ for the case of the pulse), loop corrections become of the order of the tree–level contribution. This effect is overlooked in the standard literature on the subject (see e.g. [4] – [31]). At the same time it is also a quite general phenomenon for other strong background fields: See [32], [33], [34], [35], [36], [37], [38], [39] for the same kind of effects in de

\textsuperscript{1} The presence and non-cancelation of infrared divergences in QED on strong backgrounds was first noticed in [3] in a different context.
Sitter space and for a review. Furthermore, such a situation is quite a well known phenomenon in non–stationary condensed matter theory, if one does not take into account the time variation of the level populations.

The growth of the loop corrections to the two–point correlation functions has bright physical consequences. In particular in [2], we observe that particle number density, \( \langle \alpha^+_\mu \alpha^\mu \rangle \), which is an element of the photon’s Keldysh propagator, grows with time even if at the initial state it was zero. Thus, there is photon production together with charged particles from the background electric field. (If interactions between quantum charged and gauge fields are turned on, photons are produced by the background field together with the charged particles rather than by accelerating products of the pair creation, i.e. photons are produced even if the density of the charged pairs is zero.) This is true for the both types of electric backgrounds under consideration — constant and pulse.

Furthermore, in [2] we show that the photon’s retarded and advanced propagators and all propagators of the charged particles do not receive such secularly growing corrections at the first loop. Also vertexes do not receive corrections that grow with time. These observations allowed us to simplify the system of the Dyson–Schwinger equations to take into account leading corrections. Along these lines we derive in [2] the kinetic equation for the photon production in the strong field backgrounds. The solution of this equation allows one to sum up leading secularly growing corrections from all loops.

All these observations in [2] have been made in the \( A_\mu = (0, 0, 0, A_3 = Et) \) gauge in the case of the constant field background. One of the goals of the present paper is to show that our result is gauge independent, i.e. we would like to repeat the calculation in the \( A_\mu = (A_0 = -Ez, 0, 0, 0) \) gauge. The point is that the observations of [2] are based on the fact that there is no energy conservation in time dependent backgrounds. Hence, it may seem unclear what the reason for the same phenomenon in the static gauge under consideration is, once there is energy conservation for the single particle problem. In this note we clarify this point.

Another goal of this note is to solve the aforementioned kinetic equation for photons. And finally we would like to see the impact of these effects on the current of the produced charged particles. The point is that at tree–level this current is zero in the constant electric field background, because of the invariance of QED under the time translational and reversal invariance on the eternally and everywhere constant field background. We show that at the loop order these symmetries are broken and the current receives non–zero contributions that grow with time.

II. SETUP OF THE PROBLEM

We consider, here, a massive scalar field coupled to an electromagnetic field in (3+1) dimensions:

\[
S = \int d^4x \left[ |D_\mu \phi|^2 - m^2 \phi^2 - \frac{1}{4} F_{\mu \nu}^2 - j_{\mu} A_{\mu} \right],
\]

where \( D_\mu = \partial_\mu - ieA_\mu \). We divide the full gauge potential into two pieces \( A_\mu = A_{\mu}^{cl} + a_\mu \) — classical, \( A_{\mu}^{cl} \), and quantum, \( a_\mu \), parts. Throughout this paper, we denote the external gauge-potential \( A_{\mu}^{cl} \)
as $A_\mu$. If not otherwise stated, in this note we study the constant field background in static gauge, where $A_0(z) = -Ez$ and $\vec{A} = 0$.

The quantization of the gauge field is straightforward. One just has to choose a convenient gauge for $a_\mu$. Below we choose Feynman gauge. For the charged scalars the situation is not so transparent because we use exact harmonics in the background field rather than plane waves. So we give here a few comments on how to quantize the theory in such a situation.

Introducing the following notations $\vec{k} = (k_0, k_1, k_2)$, $\vec{k}_\perp = (k_1, k_2)$ and $d^3 \vec{k} = dk_0d^2\vec{k}_\perp$, we expand the charged scalar fields in harmonics as follows:

$$\phi(x, t) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \left\{ a_\vec{k} f_{k_\perp} \left( z - \frac{k_0}{eE} \right) e^{-ik_0t+i\vec{k}_\perp \cdot \vec{x}_\perp} + b_\vec{k}^* f_{k^*_\perp} \left( -z - \frac{k_0}{eE} \right) e^{ik_0t-i\vec{k}_\perp \cdot \vec{x}_\perp} \right\}. \quad (1)$$

The function $f_{k_\perp} \left( z - \frac{k_0}{eE} \right)$ satisfies the following differential equation:

$$- \left[ \left( \partial_t + ieEz \right)^2 - \partial_z^2 - \partial_\perp^2 + m^2 \right] f_{k_\perp} \left( z - \frac{k_0}{eE} \right) e^{-ik_0t+i\vec{k}_\perp \cdot \vec{x}_\perp} =$$

$$= \left[ \partial_z^2 + \left( k_3 - eEt \right)^2 - k_\perp^2 + m^2 \right] f_{k_\perp} \left( z - \frac{k_0}{eE} \right) e^{-ik_0t+i\vec{k}_\perp \cdot \vec{x}_\perp} = 0. \quad (2)$$

Solutions of $f_{k_\perp}(\vec{x}_\perp)$ are related via a Fourier transformation, which we give below, to those of:

$$\left[ \partial_t^2 + \left( k_3 + eEt \right)^2 + k_\perp^2 + m^2 \right] f_{k_\perp} \left( t + \frac{k_3}{eE} \right) = 0. \quad (3)$$

This equation defines harmonic functions in the temporal, $A_3 = Et$, gauge (see e.g. [2]). The Fourier relation in question can be seen after the change of variables $k_0 - eEz = -eEZ$ and $eET = k_3 + eEt$. Then the solutions of $f_{k_\perp}$ and $f_{k^*_\perp}$ are related as follows:

$$\int_{-\infty}^{+\infty} dT f_{k_\perp} (T) e^{-ieETZ} = f_{k_\perp} (Z) \quad (4)$$

We use this Fourier relation throughout the paper and we give the explicit form of $f_{k_\perp}$ below.

From the commutation relations $[a_\vec{k}, a_\vec{k}'^+] = [b_\vec{k}, b_\vec{k}'^+] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$ the commutation relations between $\phi$ and its conjugate momentum $\pi = (\partial_t - ieEz) \phi^*$ takes the standard form:

$$[\phi (t, \vec{x}_1), \pi (t, \vec{x}_2)] = i \int \frac{d^3 \vec{k}}{(2\pi)^3} \left( k_0 - eEz_2 \right) e^{i\vec{k}_\perp \cdot (\vec{x}_1 - \vec{x}_2)} \times$$

$$\times \left[ f_{k_\perp} \left( z_1 - \frac{k_0}{eE} \right) f_{k^*_\perp} \left( z_2 - \frac{k_0}{eE} \right) - f_{k_\perp} \left( -z_1 + \frac{k_0}{eE} \right) f_{k^*_\perp} \left( -z_2 + \frac{k_0}{eE} \right) \right] = i\delta^{(3)} (\vec{x}_1 - \vec{x}_2)$$

The last equality follows from the Fourier transformation (4). Also one has to use the conservation of the Wronskian for the solutions of (3).

The free Hamiltonian for the charged scalars is diagonal:
we find that the photon’s Keldysh propagator has the following form:

$$n \kappa \text{h.c.}$$

Because of the latter fact we will see that various particle creation processes will be allowed when the interaction with the quantum gauge field, $a_\mu$, will be turned on.

### III. ONE–LOOP CORRECTION

Because the free Hamiltonian, $H_0$, is not bounded from below, the field theory under consideration is in the non-stationary situation. Hence, to calculate correlation functions one has to apply the Keldysh-Schwinger (KS) diagrammatic technique instead of the Feynman one [41, 42]. In such a formalism every particle is described by the matrix propagator, whose entries are the Keldysh propagator $G^K_{\mu\nu} = \frac{1}{2} \langle \{ a_\mu(x), a_\nu(y) \} \rangle$, and the retarded and advanced propagators $G^{A,R}_{\mu\nu} = \mp \theta(\mp \Delta t) \langle [ a_\mu(x), a_\nu(y) ] \rangle$ (and the same for the scalar fields, with $a_\mu \rightarrow \phi$).

For our discussion it is instructive to see how the Keldysh propagators behave if the quantum average is done with the use of an arbitrary state $|\psi\rangle$. Performing the harmonic expansion of the quantum part, $a_\mu(x)$, of the photon field

$$a_\mu(x) = \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-iq|t+i\vec{q}\cdot\vec{x}}}{\sqrt{2|q|}} + h.c.,$$

we find that the photon’s Keldysh propagator has the following form:

$$G^K_{\mu\nu}(x_1, x_2) = \frac{1}{2} \int \frac{d^3 \vec{q} d^3 \vec{q}’}{(2\pi)^6} \left\{ n_{\mu\nu}(\vec{q}, \vec{q}’) \frac{e^{iq_1 x_1-iq’_2 x_2}}{\sqrt{|q| |q’|}} + \kappa_{\mu\nu}(\vec{q}, \vec{q}’) \frac{e^{-iq x_1-iq’ x_2}}{\sqrt{|q| |q’|}} + h.c. \right\}. \quad (6)$$

Here $n_{\mu\nu}(\vec{q}, \vec{q}’) = \langle \psi | \alpha_{\vec{q}\mu}^{\dagger} \alpha_{\vec{q}’\nu} \psi \rangle$, $\kappa_{\mu\nu}(\vec{q}, \vec{q}’) = \langle \psi | \alpha_{\vec{q}\mu}^{\dagger} \alpha_{\vec{q}’\nu} \psi \rangle$ and $q \cdot x = |q|t - \vec{q} \cdot \vec{x}$. Furthermore, $h.c.$ stands for the quantities containing $\langle \psi | \alpha_{\vec{q}\mu}^{\dagger} \alpha_{\vec{q}’\nu} \psi \rangle = n_{\mu\nu}(\vec{q}, \vec{q}’) - g_{\mu\nu} \delta^{(3)}(\vec{q} - \vec{q}’)$ and $\kappa_{\mu\nu}^*(\vec{q}, \vec{q}’) = \langle \psi | \alpha_{\vec{q}\mu}^{\dagger} \alpha_{\vec{q}’\nu}^{\dagger} \psi \rangle$.

Furthermore from [11] we find that scalar field’s Keldysh propagator is as follows:
The propagator in question has the form of (6), where
\[ n_G a \] by all annihilation operators under consideration (\( t \to \infty \)).

Here \( n, k \) containing \( \delta \), the reason why we present (6) and (7) here is that loop corrections contribute to the retarded and advanced propagators and also to the vertexes. This is a quite generic phenomenon:

Performing the same calculation as in [2] one can see that the one–loop correction to the photon’s Keldysh propagator in the limit \( \frac{t_1 + t_2}{2} = t \to \infty \), when \( t_1 - t_2 = \text{const} \). The initial state that we consider here is the one that is annihilated by all annihilation operators under consideration (\( a, b, a' \)). i.e. the tree–level Keldysh propagators \( G^K \) and \( D^K \) look as (3) and (7) with all \( n \) and \( \kappa \) equal to zero.

At the same time the form of the retarded and advanced propagators does not depend on the state \( |\psi\rangle \). In [2] it was shown that there are no large (growing with time) loop corrections to the retarded and advanced propagators and also to the vertexes. This is a quite generic phenomenon: see e.g. [42] for the similar situations in different theories. It is straightforward to show that the same is true in the static gauge. Hence, we continue with the discussion of the Keldysh propagators. The reason why we present (6) and (7) here is that loop corrections contribute to \( n \) and \( \kappa \) in the Keldysh propagators of both fields.

### A. Correction to the photon’s Keldysh propagator

We start with the one–loop correction to the photon’s Keldysh propagator in the limit \( \frac{t_1 + t_2}{2} = t \to \infty \), when \( t_1 - t_2 = \text{const} \). The initial state that we consider here is the one that is annihilated by all annihilation operators under consideration (\( a, b, a' \)). i.e. the tree–level Keldysh propagators \( G^K \) and \( D^K \) look as (3) and (7) with all \( n \) and \( \kappa \) equal to zero.

Performing the same calculation as in [2] one can see that the one–loop correction to the propagator in question has the form of (3), where
\[ n_{\mu \nu}(\bar{q}, \bar{q}', t) = \delta^{(2)}(\bar{q}_\perp - \bar{q}'_\perp) \ n_{\mu \nu}(q_3, q'_3, \bar{q}_\perp, t) \] and
\[ \kappa_{\mu \nu}(\bar{q}, \bar{q}', t) = \delta^{(2)}(\bar{q}_\perp - \bar{q}'_\perp) \ \kappa_{\mu \nu}(q_3, q'_3, \bar{q}_\perp, t) \].

The latter quantities are as follows:

\[ D^K(x_1, x_2) = \frac{1}{2} \left\langle \left\{ \phi(x_1), \bar{\phi}(x_2) \right\} \right\rangle = \frac{1}{2} \int \frac{d^3\bar{k}d^3\bar{k}'}{(2\pi)^6} \times \]
\[ \times \left\{ n^+(\bar{k}, \bar{k}') e^{ik_0t_1 - i\bar{k}_\perp \cdot \bar{x}_{1\perp}} e^{-ik'_0t_2 + i\bar{k}'_\perp \cdot \bar{x}_{2\perp}} f^*_{\bar{k}_\perp} \left( z_1 - \frac{k_0}{eE} \right) f_{\bar{k}'_\perp} \left( z_2 - \frac{k'_0}{eE} \right) + \right. \]
\[ \left. + \ k^+(\bar{k}, \bar{k}') e^{-ik_0t_1 + i\bar{k}_\perp \cdot \bar{x}_{1\perp}} e^{-ik'_0t_2 + i\bar{k}'_\perp \cdot \bar{x}_{2\perp}} f_{\bar{k}_\perp} \left( z_1 - \frac{k_0}{eE} \right) f^*_{\bar{k}'_\perp} \left( -z_2 - \frac{k'_0}{eE} \right) + h.c. \right\} . \]
\[ n_{\mu \nu} (q_3, q_3', q_{\perp}, t) \approx e^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{dk_0}{2\pi} \int_t^{t+4} dt_3 dt_4 \frac{e^{-i(k_0+k')_0(t_3-t_4)}e^{-iq[t_3+iq'|t_4]}}{2\sqrt{|q||q'|}} \times \\
\times \int dz_3 e^{iqz_3} \left[ f_{k_\perp} \left( z_3 - \frac{k_0}{eE} \right) \tilde{D}_{\mu} f_{|\vec{q}_{\perp}+\vec{k}|} \left( -z_3 - \frac{k_0}{eE} \right) \right] \times \\
\times \int dz_4 e^{-iqz_4} \left[ f^*_{k_\perp} \left( z_4 - \frac{k_0}{eE} \right) \tilde{D}_{\mu} f^*_{|\vec{q}_{\perp}+\vec{k}|} \left( -z_4 - \frac{k_0}{eE} \right) \right], \quad (8) \]

where \( D_{\mu} f_{p_\perp} \left( \pm z - p_0/eE \right) = -(ip_0 \pm ieEZ, i\vec{p}_\perp, \partial_z) f_{p_\perp} \left( \pm z - p_0/eE \right) \) and \( f_1 \tilde{D}_{\mu} f_2 = (D_{\mu} f_1) f_2 - f_1 (D_{\mu}^* f_2) \); \( t_0 \) is the moment of time after which we adiabatically turn on interactions between charged scalars, \( \phi \), and quantum gauge fields, \( a_{\mu} \). In these expressions we neglect the difference between \( t_{1,2} \) and \( t \) in the limit under consideration. This is mathematically rigorous if \( n_{\mu \nu} \) and \( \kappa_{\mu \nu} \) have a divergence as \( t \to +\infty \) and if we would like to single out only the leading contributions. Otherwise we do such an approximation just to estimate the quantities under consideration. The physical meaning of such loop corrections is discussed in [2].

Let us consider \( n_{\mu \nu} \) in (9). In order to estimate the expression in (8) we make the change of integration variables to: \( t' = \frac{t_3 + t_4}{2}, \tau = t_3 - t_4 \). Then, we obtain the \( \tau \)-integral in the range \([t_0 - t, t - t_0] \), but its integrand is rapidly oscillating for large \( \tau \), as \( t \to +\infty \) and \( t_0 \to -\infty \). Hence, we can extend the upper and lower limits of the \( \tau \)-integration to plus and minus infinity, respectively. Then, the integral over \( \tau \) leads to the \( \delta \)-function in the following expression:

\[ n_{\mu \nu} (q_3, q_3', q_{\perp}, t) \approx e^2 \int_0^t dt' \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{dk_0}{2\pi} \delta \left( \frac{|q| + |q'|}{2} + k_0' + k_0 \right) \frac{e^{-i(|q|+|q'|)t'}}{2\sqrt{|q||q'|}} \times \\
\times \int dz_3 e^{iqz_3} \left[ f_{k_\perp} \left( z_3 - \frac{k_0}{eE} \right) \tilde{D}_{\mu} f_{|\vec{q}_{\perp}+\vec{k}|} \left( -z_3 - \frac{k_0}{eE} \right) \right] \times \\
\times \int dz_4 e^{-iqz_4} \left[ f^*_{k_\perp} \left( z_4 - \frac{k_0}{eE} \right) \tilde{D}_{\mu} f^*_{|\vec{q}_{\perp}+\vec{k}|} \left( -z_4 - \frac{k_0}{eE} \right) \right], \quad (9) \]

We further make the following change of integration variables \( Z = \frac{z_3 + z_4}{2} \) and \( z = z_3 - z_4 \). Also we change \( k_0 \to k_0 - eEZ \) and \( k_0' \to k_0' + eEZ \). This change of integration variables allows us to simplify the integral over \( Z \), which leads to a \( \delta \)-function establishing that \( q_3 = q_3' \). As a result, \( n_{\mu \nu} (q_3, q_3', q_{\perp}, t) = \delta (q_3 - q_3') n_{\mu \nu} (q', t) \), where
\[ n_{\mu\nu}(\vec{q}, t) \approx e^2 (t - t_0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|q|} \times \]
\[ \times \int \! dz e^{-izq_z} \left[ f_{k_\perp} \left( z - \frac{k_0}{eE} \right) \overrightarrow{D}_\mu f^*_{|\vec{q}_\perp + \vec{k}_\perp|} \left( -z + \frac{k_0 + |q|}{eE} \right) \right] \times \]
\[ \times \left[ f_{k'_\perp} \left( -z - \frac{k_0}{eE} \right) \overrightarrow{D}_\nu f^*_{|\vec{q}_\perp + \vec{k}_\perp|} \left( z + \frac{k_0 + |q|}{eE} \right) \right]. \quad (10) \]

To obtain this expression from (9) we have used that \( |q| = |q'| \) due to the presence of \( \delta^{(2)}(\vec{q}_\perp - \vec{q}'_\perp) \delta (q_3 - q'_3) \) in \( n_{\mu\nu}(\vec{q}, \vec{q}', t) \). Also we evaluate the integral over \( t' \) in (9).

Finally, making the Fourier transformation (4), one can straightforwardly see that (10) coincides with the expression for \( n_{\mu\nu} \) obtained in [2]. Thus, \( n_{\mu\nu} \) is divergent as \((t - t_0) \to \infty\). This divergence signals the presence of the photon production which starts right after the moment \( t_0 \), when the interactions are turned on. It brakes the time reversal and translational invariance of QED on the constant field background. We discuss the physical meaning of all these observations in [2] in greater detail.

Let us continue now with the consideration of \( \kappa_{\mu\nu} \). In [2] we show that it does not receive growing contributions. (This, in particular, shows that the initial state for the photons is the appropriate vacuum state.) Now we are going to show, that in the static gauge, \( \kappa_{\mu\nu} \) also does not grow with time. Similarly to the case of \( n_{\mu\nu} \) here we also get that \( \kappa_{\mu\nu}(q_3, q'_3, \vec{q}_\perp) \propto \delta(q_3 + q'_3) \). Then, we take the limit \( t \to \infty \) and \( t_0 \to -\infty \) in (5). This way we find that \( \kappa_{\mu\nu}(q_3, \vec{q}_\perp) \propto \delta(k_0 + k'_0 + |q|)\delta(k_0 + k'_0 - |q|) \).

Hence, one can integrate out \( k'_0 \) to find that:

\[ \kappa_{\mu\nu} (\vec{q}, t \to +\infty) \approx -2e^2 \frac{1}{|q|} \delta (2|q|) \int \frac{d^3k}{(2\pi)^3} \int dz e^{\ii 2q_z z} \times \]
\[ \times \left[ f_{k_\perp} \left( z - \frac{k_0}{eE} \right) \overrightarrow{D}_\mu f^*_{|\vec{q}_\perp + \vec{k}_\perp|} \left( -z + \frac{k_0 + |q|}{eE} \right) \right] \left[ f_{k'_\perp} \left( z - \frac{k_0}{eE} \right) \overrightarrow{D}_\nu f^*_{|\vec{q}_\perp + \vec{k}_\perp|} \left( -z + \frac{k_0 + |q|}{eE} \right) \right]. \]

The obtained expression contains only convergent integrals and, hence, is finite, if \( q \neq 0 \).

### B. Correction to the Keldysh propagator of the charged particles

The one-loop correction to the scalar Keldysh propagator, in the limit \( t = (t_1 + t_2)/2 \to \infty \) and \( t_1 - t_2 = \text{const} \), can also be expressed as (7) where \( n^\pm (\vec{k}, \vec{k}', t) = \delta^{(2)} (\vec{k}_\perp - \vec{k}'_\perp) \) \( n^\pm (k_0, k'_0, \vec{k}_\perp, t) \) and similarly for the case of \( \kappa^\pm \). In this case, for example,
\[ n^+ \left( k_0, k_0', \vec{k}_\perp, t \right) = e^2 \int \frac{d^3 \vec{q}}{(2\pi)^3} \int \frac{dk_0''}{2\pi} \int_{t_0}^t dt_3 dt_4 \frac{e^{-i(q + k_0'')(t_3 - t_4)} e^{-ik_0 t_4 + ik_0' t_4}}{2|q|} \times \]

\[ \times \int dz_3 e^{iqz_3} \left[ f_{k_{\perp}} \left( z_3 - \frac{k_0}{eE} \right) \vec{D}_\mu f_{\vec{q}_{\perp} + \vec{k}_\perp} \left( -z_3 - \frac{k_0''}{eE} \right) \right] \times \]

\[ \times \int dz_4 e^{-iqz_4} \left[ f_{k_{\perp}} \left( z_4 - \frac{k_0}{eE} \right) \vec{D}_\mu f_{\vec{q}_{\perp} + \vec{k}_\perp} \left( -z_4 - \frac{k_0''}{eE} \right) \right] \]

and \[ \kappa^+ \left( k_0, k_0', \vec{k}_\perp, t \right) = -2e^2 \int \frac{d^3 \vec{q}}{(2\pi)^3} \int \frac{dk_0''}{2\pi} \int_{t_0}^{t_3} dt_3 dt_4 \frac{e^{-i(q + k_0'')(t_3 - t_4)} e^{-ik_0 t_4 - ik_0' t_4}}{2|q|} \times \]

\[ \times \int dz_3 e^{iqz_3} \left[ f_{k_{\perp}} \left( z_3 - \frac{k_0}{eE} \right) \vec{D}_\mu f_{\vec{q}_{\perp} + \vec{k}_\perp} \left( -z_3 - \frac{k_0''}{eE} \right) \right] \times \]

\[ \times \int dz_4 e^{-iqz_4} \left[ f_{k_{\perp}} \left( z_4 - \frac{k_0}{eE} \right) \vec{D}_\mu f_{\vec{q}_{\perp} + \vec{k}_\perp} \left( -z_4 - \frac{k_0''}{eE} \right) \right] . \] (11)

There are similar expressions for \( n^- \) and \( \kappa^\pm \).

In [2] we show that none of the \( n^\pm \) and \( \kappa^\pm \) receive corrections that grow with time. To make the same conclusion here we perform the same trick as at the end of the previous subsection. For example, let us consider \( n^+ \) and take \( t \to +\infty \) and \( t_0 \to -\infty \). Then, performing the same transformations as at the end of the previous subsection, we find:

\[ n^+ \left( k_0, k_0', \vec{k}_\perp, t \to +\infty \right) \approx e^2 \delta(k_0 - k_0') \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{2|q|} \times \]

\[ \times \int dz_3 e^{iqz_3} \left[ f_{k_{\perp}} \left( z_3 - \frac{k_0}{eE} \right) \vec{D}_\mu f_{\vec{q}_{\perp} + \vec{k}_\perp} \left( -z_3 + \frac{k_0 + |q|}{eE} \right) \right] \right|^2 . \] (12)

This expression contains only convergent integrals. Hence, \( n^+ \) cannot contain contributions that grow with time. Using the same line of arguments one can draw the same conclusion for the case of \( n^- \) and \( \kappa^\pm \).

**IV. DISCUSSION**

We would like to present here some additional physical consequences of the observations made above and in our previous paper.

**A. Remarks on the loop correction to the current of the created particles**

Since we have shown that the result of [2] is gauge independent, we prefer to use the temporal gauge, i.e. \( A_\mu = (0, 0, 0, -Et) \), because then the situation is easier to generalize to more physically natural situations such as the pulse background.

The fact that \( n^\pm \) do not grow with time does not necessarily mean that there is no charge particle production generated by loops. First, it is worth stressing here that the correct particle
number in the temporal gauge is \( n^\pm (\vec{k}, t) \left| f_{k\perp} \left( \pm t + \frac{k_3}{eE} \right) \right|^2 \) rather than \( n^\pm \) itself. Second, although \( n^\pm (\vec{k}, t \to -\infty) = 0 \), \( \kappa^\pm (\vec{k}, t \to -\infty) \) is the case that \( n^\pm (\vec{k}, t \to +\infty) = n^\pm \neq 0 \), \( \kappa^\pm (\vec{k}, t \to +\infty) = \kappa^\pm \neq 0 \). This kind of behavior of \( n^\pm \) and \( \kappa^\pm \) is clearly another sign of the breaking of the time translational and reversal invariance of the theory, which is respected at tree–level.

What physical consequences should all this have? In e.g. [43, 44] it was shown that the tree–level current of the produced pairs,

\[
\langle : J_3 : \rangle_{\text{tree}} = 2e \int \frac{dp_3 d^2 \vec{p}_\perp}{(2\pi)^3} (p_3 + eEt) \left| f_{p\perp} \left( t + \frac{p_3}{eE} \right) \right|^2 - \frac{1}{2 \omega_{p\perp} (p_3 + eEt)} ,
\]

is vanishing. Here \( \omega_{p\perp} (p_3 + eEt) = \sqrt{m^2 + \vec{p}_\perp^2 + (p_3 + eEt)^2} \) and the last term under the integral cancels UV divergent contribution to the current, if it is present (see e.g. [44], [43]). To see the vanishing of (13) one has to convert the integration variables \( p_3 \to p_{ph} = p_3 + eEt \) and to note that \( |f_{p\perp} (p_{ph})|^2 \) is an even function of \( p_{ph} \). Thus, the current vanishes just as a consequence of the time translation and time reversal invariance of the theory in the constant electric field.

At loop order, time translational and reversal invariance is broken. Hence, we can expect that the one–loop correction to the current will be non–vanishing. In fact, the correction is given by

\[
\langle : J_3 : \rangle_{\text{loop}} = 4e \int \frac{dp_{ph} d^2 \vec{p}_\perp}{(2\pi)^3} \left\{ n^+_p (p_{ph}) |f_{p\perp} (p_{ph})|^2 + \Re \left[ \kappa^+_p (p_{ph}) f^2_{p\perp} (p_{ph}) \right] \right\}_{p_{ph}},
\]

where we denote \( n^+_p (p_3 + eEt) = n^+_p (p_{ph}) \) and similarly for \( \kappa^+_p \). Here \( n^\pm \) and \( \kappa^\pm \) are indeed functions of \( p_{ph} = p_3 + eEt \):

\[
n^+_p (p_{ph}) \approx \frac{e}{E} \int_{-\infty}^{p_{ph}} d\tau \int_{-\infty}^{\infty} d\sigma \; \frac{e^{-2i|\sigma|\tau}}{2|q|} \times \left[ f^*_{p\perp} \left( \frac{\tau + \frac{k_{ph}}{eE}}{eE} \right) \frac{D^*_\mu f [\vec{p}_\perp - \vec{q}_\perp]}{\vec{p}_\perp \vec{q}_\perp} \left( \frac{\tau + \frac{k_{ph} - q_3}{eE}}{eE} \right) \right] ,
\]

and

\[
\kappa^+_p (p_{ph}) \approx -\frac{2e}{E} \int_{-\infty}^{p_{ph}} d\tau \int_{-\infty}^{\infty} d\sigma \; \frac{e^{-2i|\sigma|\tau}}{2|q|} \times \left[ f^*_{p\perp} \left( \frac{\tau + \frac{k_{ph}}{eE}}{eE} \right) \frac{D^*_\mu f [\vec{p}_\perp - \vec{q}_\perp]}{\vec{p}_\perp \vec{q}_\perp} \left( \frac{\tau + \frac{k_{ph} - q_3}{eE}}{eE} \right) \right] .
\]

where \( D^*_\mu f_{p\perp} (t + \frac{p_3}{eE}) \equiv (\partial_t, ip_1, ip_2, ip_3 + ieE) f_{p\perp} (t + \frac{p_3}{eE}) \). Furthermore, to derive (14) we use that \( \kappa^- \) is just the complex conjugate of \( \kappa^+ \) and \( n^- (p_{ph}) = n^+ (-p_{ph}) \), which is straightforward to show.

It is not hard to see that (14) is not zero. The point is that \( n^+ \) and \( \kappa^+ \) are not even functions of \( p_{ph} \). For any choice of the harmonic functions, \( f_{k\perp} \), these quantities do vanish as \( p_{ph} \to -\infty \) and approach finite non-zero constants as \( p_{ph} \to +\infty \).
In order to estimate (14), we note that in–harmonics behave as:

\[ f_{p_{\perp}}(p_{\text{ph}}) \propto \left( \frac{p_{\text{ph}}}{m} \right)^{\frac{p_{\perp}^2 + m^2}{2eE}} \exp \left[ i \frac{p_{\text{ph}}^2}{2eE} \right] \frac{1}{\sqrt{2 \left( m^2 + p_{\perp}^2 + p_{\text{ph}}^2 \right)}} \]

when \( p_{\text{ph}} \to -\infty \) and

\[ f_{p_{\perp}}(p_{\text{ph}}) \approx \alpha_{p_{\perp}} \cdot \left( \frac{p_{\text{ph}}}{m} \right)^{\frac{p_{\perp}^2 + m^2}{2eE}} \exp \left[ i \frac{p_{\text{ph}}^2}{2eE} \right] \frac{\beta_{p_{\perp}}}{\sqrt{2 \left( m^2 + p_{\perp}^2 + p_{\text{ph}}^2 \right)}} \]

when \( p_{\text{ph}} \to +\infty \). Here \( \alpha_{p_{\perp}} \) and \( \beta_{p_{\perp}} \) are functions of \( p_{\perp} \), obeying the condition \( |\alpha_{p_{\perp}}|^2 - |\beta_{p_{\perp}}|^2 = 1 \). Then, defining \( n_{p_{\perp}}^+ (p_{\text{ph}} = +\infty) = n_{p_{\perp}}^+ \), \( \kappa_{p_{\perp}}^+ (p_{\text{ph}} = +\infty) = \kappa_{p_{\perp}}^+ \) and using the same approximations as in [32], we obtain

\[ \langle : J_3 : \rangle_{\text{loop}} \propto e^2 (t - t_0) \int d^2 \mathbf{p}_{\perp} \left\{ n_{p_{\perp}}^+ |\beta_{p_{\perp}}|^2 + \text{Re} \left[ \kappa_{p_{\perp}}^+, \alpha_{p_{\perp}} \beta_{p_{\perp}} \right] \right\}, \quad (16) \]

This expression is similar to the one obtained in [13], [43], [32], [44], [43] in the pulse background. The crucial difference with the tree–level result for the pulse background, however, comes from the fact that \( n^+ \sim e^2 \) and \( \kappa^+ \sim e^2 \) are the results of the one–loop contribution.

B. Remarks on the solution of the kinetic equation and summation of the leading loop corrections

In [2] we show that \( n_{\mu\nu} \) for the photons is equal to \( n_{\mu\nu}(q, t) = \pi_{\mu\nu} n_q(t) \), where \( \pi_{\mu\nu} \) is time independent, symmetric, transversal, \( q^\mu \pi_{\mu\nu} = 0, q^2 = 0 \), tensor. Then from the system of Dyson-Schwinger equations we derive a kinetic equation for \( n_q(t) \):

\[ \frac{\partial n_q(t)}{\partial t} = \Gamma_1(q) [1 + n_q(t)] - \Gamma_2(q) n_q(t) \quad \text{,} \quad (17) \]

where

\[ \Gamma_1(q) \approx e^2 \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} d\tau \frac{e^{-2|q|\tau}}{|q|} \left[ f_{k_{\perp}} \left( \tau + \frac{k_3}{eE} \right) \overleftarrow{D}_{\mu} f_{\overleftarrow{k}_{\perp} - q_{\perp}} \left( \tau + \frac{k_3 - q_3}{eE} \right) \right] \times \]

\[ \times \left[ f_{k_{\perp}}^* \left( \tau - \frac{k_3}{eE} \right) \overrightarrow{D}_{\mu} f_{\overrightarrow{k}_{\perp} - q_{\perp}} \left( \tau - \frac{k_3 - q_3}{eE} \right) \right] \]

and

\[ \Gamma_2(q) \approx e^2 \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} d\tau \frac{e^{-2|q|\tau}}{|q|} \left[ f_{k_{\perp}}^* \left( \tau + \frac{k_3}{eE} \right) \overrightarrow{D}_{\mu} f_{\overrightarrow{k}_{\perp} - q_{\perp}} \left( \tau + \frac{k_3 - q_3}{eE} \right) \right] \times \]

\[ \times \left[ f_{k_{\perp}} \left( \tau - \frac{k_3}{eE} \right) \overleftarrow{D}_{\mu} f_{\overleftarrow{k}_{\perp} - q_{\perp}} \left( \tau - \frac{k_3 - q_3}{eE} \right) \right] \quad \text{(18)} \]
The physical meaning of (17) is transparent. The first term on the right hand side describes the photon production by the background field, while the second term accounts for the decay of the produced photons into charged pairs. These processes are allowed in the presence of the background field. The absence of other terms describing other processes is explained by their suppression by higher powers of $e^2$ [2]. The solution of (17) sums up leading corrections, i.e. unsuppressed powers of $e^2(t - t_0)$, from all loops. Here we would like to find/compare $\Gamma_1$ and $\Gamma_2$ and, hence, to solve this kinetic equation.

To find the relation between $\Gamma_1$ and $\Gamma_2$, note that generic harmonic functions look like (see e.g. [43], [44]):

$$f_{k\perp}(t + \frac{k_3}{eE}) = AD^{-\frac{1}{2} + i\frac{m^2 + k^2}{2eE}} \left[ e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{eE}} \left(k_3 + eEt\right) \right] + BD^{-\frac{1}{2} - i\frac{m^2 + k^2}{2eE}} \left[ -e^{i\frac{\pi}{4}} \sqrt{\frac{2}{eE}} \left(k_3 + eEt\right) \right].$$

(19)

Where $A$ and $B$ some constants. For example, for the in–harmonics $B = 0$. Then, one can see that $f_{k\perp}^*(t + \frac{k_3}{eE})$ is equal to $f_{k\perp}(t + \frac{k_3}{eE})$ under the exchange of $eE \rightarrow -eE$ and $\vec{k} \rightarrow -\vec{k}$. Using this relation and the change of $\vec{k} \rightarrow \vec{q} - \vec{k}$ under the integrals in (18), one can show that $\Gamma_1 = \Gamma_2$. The same is also true for the case of out–harmonics. As a result, for such a choice of the harmonic functions, the leading one–loop correction to $n_{\mu\nu}(\vec{q}, t)$ is exact and we have the linear growth in all loops. This means that the time translational and reversal invariance cannot be restored after summation of all loops.

V. ACKNOWLEDGEMENTS

Our work was partially supported by the grant for the support of the leading scientific schools SSch–1500.2014.2 and by our grants from the Dynasty foundation. The work of FKP is done under the partial support of the RFBR grant 14-02-31446-mol-a. The work of ETA was done under the financial support from the Government of the Russian Federation within the framework of the implementation of the 5-100 Programme Roadmap of the National Research University Higher School of Economics.

[1] J. S. Schwinger, Phys. Rev. 82, 664 (1951).
[2] E. T. Akhmedov, N. Astrakhantsev and F. K. Popov, JHEP 1409, 071 (2014) [arXiv:1405.5285 [hep-th]].
[3] E. T. Akhmedov and E. T. Musaev, “Comments on QED with background electric fields,” New J. Phys. 11, 103048 (2009) [arXiv:0901.0424 [hep-ph]].
[4] E. S. Fradkin and D. M. Gitman, Fortsch. Phys. 29, 381 (1981).
[5] D. M. Gitman, E. S. Fradkin and S. M. Shvartsman, Fortsch. Phys. 36, 643 (1988).
[6] S. P. Gavrilov, D. M. Gitman and S. M. Shvartsman, Sov. Phys. J. 23, 257 (1980).
[7] N. B. Narozhnyi and A. I. Nikishov, Teor. Mat. Fiz. 26, 16 (1976).
[8] A. I. Nikishov, Teor. Mat. Fiz. 20, 48 (1974).
