Relation between the solitons of Yang-Mills-Higgs fields in 2+1 dimensional Minkowski space-time and anti-de Sitter space-time

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Abstract

The Yang-Mills-Higgs-Bogomolny equations in both 2+1 dimensional Minkowski space-time and 2+1 dimensional anti-de Sitter space-time are known to be integrable and their soliton solutions have already been obtained. In this paper we show that there is a natural relation between the Lax pairs and soliton solutions in these two space-times when the curvature changes from 0 to $-1$. The change of the asymptotic behaviors of the solitons are also discussed.

1 Introduction

The Yang-Mills-Higgs-Bogomolny equations in both 2+1 dimensional Minkowski space-time and 2+1 dimensional anti-de Sitter space-time are known to be integrable [1, 2, 3, 4]. There are several ways to solve them explicitly. Darboux transformation method is one of them, which gives an easy way to obtain explicit soliton solutions [5, 6]. Since the Lax pairs in both Minkowski and anti-de Sitter cases are known, the Darboux transformations can be constructed separately in these two cases.

The standard anti-de Sitter space-time has curvature $-1$. Naturally we can consider the anti-de Sitter space-time with constant curvature $-1/\rho^2$ ($\rho > 0$). When $\rho \to +\infty$, the space-time tends to the Minkowski space-time. In this paper, we consider the following problem: When $\rho$ changes from 1 to $+\infty$, whether the solitons in the anti-de Sitter space-time change to solitons in the Minkowski space-time?

In Section 2, the Yang-Mills-Higgs-Bogomolny equations and their Lax pairs for general $\rho$ are considered. When $\rho = 1$ and $\rho \to +\infty$, they become the known equations and their Lax pairs for Minkowski and anti-de Sitter cases. In Section 3, the Darboux transformation is discussed. Using the Darboux transformation, we construct solitons in $SU(2)$ case in Section 4 and give some examples. When $\rho$ changes from 1 to $+\infty$, the shape of solitons changes a lot. However, when the coordinates of the space-time depend on $\rho$ suitably, the position of the solitons keeps in a finite region and the solitons in part of the anti-de Sitter space-time change to the solitons in the Minkowski space-time.
2 Yang-Mills-Higgs-Bogomolny equations and their Lax pairs

Let $M$ be a three dimensional Lorentz manifold with metric $g = (g_{\mu \nu})$. \{$A_\mu$ | $\mu = 1, 2, 3$\} is a gauge potential and $\Phi$ is a (scalar) Higgs field, both of which are valued in the Lie algebra of an $N \times N$ matrix Lie group $G$.

The Yang-Mills-Higgs-Bogomolny equation \[1, 7\] is

$$D\Phi = *F,$$  

or, written in terms of the components,

$$D_\mu \Phi = \frac{1}{2\sqrt{|g|}}g_{\mu \nu}e^{\nu}F_{\alpha \beta},$$  

where the action of the covariant derivative $D_\mu = \partial_\mu + A_\mu$ on $\Phi$ is $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$, $\partial_\mu = \partial/\partial x^\mu$. \{$F_{\mu \nu}$\} is the curvature corresponding to \{$A_\mu$\}, hence $F_{\mu \nu} = [D_\mu, D_\nu]$.

The 2+1 dimensional anti-de Sitter space-time of constant curvature $-1/\rho^2$ ($\rho > 0$) is the hyperboloid $U^2 + V^2 - X^2 - Y^2 = \rho^2$ in $\mathbb{R}^{2,2}$ with the metric

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2.$$  

By defining

$$r = \frac{\rho}{U + X} - \rho + 1, \quad x = \frac{Y}{U + X}, \quad t = -\frac{V}{U + X},$$

a part of the 2+1 dimensional anti-de Sitter space-time with $U + X > 0$ is represented by the Poincaré coordinates $(r, x, t)$ with $r > -\rho + 1$ and the metric is

$$ds^2 = \frac{\rho^2}{(r + \rho - 1)^2}(-dt^2 + dv^2 + dx^2) = \frac{\rho^2}{(r + \rho - 1)^2}(dr^2 + du dv)$$

where $u = x + t$, $v = x - t$. Clearly, when $\rho \to +\infty$, the metric on this part of the 2+1 dimensional anti-de Sitter space-time tends to the flat Minkowski metric on the whole $\mathbb{R}^{2,1}$. In order to consider the change of the solitons with respect to $\rho$, we only need to consider the solutions in this part.

With the metric (3) and the orientation $(v, u, r)$, (2) becomes

$$D_u \Phi = \frac{r + \rho - 1}{\rho}F_{ur}, \quad D_v \Phi = -\frac{r + \rho - 1}{\rho}F_{vr}, \quad D_r \Phi = -\frac{2(r + \rho - 1)}{\rho}F_{uv}.$$  

When $\rho = 1$, it is reduced to

$$D_u \Phi = rF_{ur}, \quad D_v \Phi = -rF_{vr}, \quad D_r \Phi = -2rF_{uv}.$$  

[4] showed that it had a Lax pair

$$(rD_r + \Phi - 2(\zeta - u)D_u)\psi = 0, \quad \left(2D_v + \frac{\zeta - u}{r}D_r - \frac{\zeta - u}{r^2}\Phi\right)\psi = 0$$
where $D_\mu \psi = \partial_\mu \psi + A_\mu \psi$ and $\zeta$ is a complex spectral parameter. That is, (8) is the integrability condition of the over-determined system (3).

When $\rho > 0$, the Yang-Mills-Higgs-Bogomolny equation (3) can be derived from (7) by substituting $r \rightarrow r + \rho - 1$ and $\Phi \rightarrow \rho \Phi$. Moreover, since $\zeta$ is a constant in (8), we can replace $\zeta$ by $\rho \zeta$. After the substitution $r \rightarrow r + \rho - 1$, $\Phi \rightarrow \rho \Phi$, $\zeta \rightarrow \rho \zeta$, (9) leads to the Lax pair of (3):

$$
\begin{align*}
((r + \rho - 1)D_r + \rho \Phi - 2(\rho \zeta - u)D_u)\psi &= 0, \\
\left(2D_v + \frac{\rho \zeta - u}{r + \rho - 1}D_r - \frac{\rho(\rho \zeta - u)}{(r + \rho - 1)^2} \Phi\right)\psi &= 0.
\end{align*}
$$

(10)

It is easy to check directly that the integrability condition of (10) is the Yang-Mills-Higgs-Bogomolny equation (3).

When $\rho \rightarrow +\infty$, the metric (5) becomes the standard Minkowski metric

$$
\begin{align*}
ds^2 &= -dt^2 + dr^2 + dx^2 = dr^2 + du \, dv,
\end{align*}
$$

(11)

the Yang-Mills-Higgs-Bogomolny equation (3) becomes

$$
D_u \Phi = F_{ur}, \quad D_v \Phi = -F_{vr}, \quad D_r \Phi = -2F_{uv},
$$

(12)

and the Lax pair (10) becomes

$$
\begin{align*}
(D_r + \Phi - 2\zeta D_u)\psi &= 0, \\
(2D_v + \zeta D_r - \zeta \Phi)\psi &= 0.
\end{align*}
$$

(13)

Remark 1 If we substitute

$$
r \rightarrow x, \quad \zeta \rightarrow \frac{1}{\lambda}, \quad u \rightarrow y + t, \quad v \rightarrow y - t,
$$

(14)

then (13) is changed to

$$
\begin{align*}
(\lambda D_x - D_t - D_y + \lambda \Phi)\psi &= 0, \\
(\lambda D_t - \lambda D_y - D_x + \Phi)\psi &= 0,
\end{align*}
$$

(15)

which is just the Lax pair given by [4].

3 Darboux transformations

For $\rho \rightarrow +\infty$ and $\rho = 1$, [3] and [4] gave the construction of the Darboux matrix separately based on a general method [5]. Here we show that these are the two special cases for general $\rho$. 

3
For $\rho = 1$, the Darboux transformation is given as follows [3]. Let $Z = \text{diag}(\zeta_1, \cdots, \zeta_N)$ be a diagonal matrix and satisfies

$$
\partial_r Z - \frac{2(Z - u)}{r} (\partial_u Z) = 0, \quad \partial_v Z + \frac{Z - u}{2r} (\partial_r Z) = 0,
$$

(16)

$H = (h_1, \cdots, h_N)$ where $h_j$ is a solution of (3) with $\zeta = \zeta_j$, then $G = \zeta - HZH^{-1}$ is a Darboux matrix for (3). That is, for any solution $\psi$ of the Lax pair (3), $\tilde{\psi} = G\psi$ satisfies

$$(r \tilde{D}_r + \tilde{\Phi} - 2(\zeta - u) \tilde{D}_u)\tilde{\psi} = 0, \quad \left(2\tilde{D}_v + \frac{\zeta - u}{r} \tilde{D}_r - \frac{\zeta - u}{r^2} \tilde{\Phi}\right)\tilde{\psi} = 0$$

(17)

where $\tilde{D}_\mu = \partial_\mu + \tilde{A}_\mu$ and $\tilde{\Phi}, \tilde{A}_\mu$ are other functions in the Lie algebra of $G$.

When $\rho > 1$, similar conclusion is obtained by the substitution (3) and $Z \to \rho Z$. Hence the Darboux matrix is given by

$$G(r, u, v, \zeta) = \zeta - \frac{u}{\rho} - S(r, u, v), \quad S(r, u, v) = H \left(Z - \frac{u}{\rho}\right) H^{-1}$$

(18)

where $Z = \text{diag}(\zeta_1, \cdots, \zeta_N)$ satisfies

$$
\partial_r Z - \frac{2(\rho Z - u)}{r + \rho - 1} \partial_u Z = 0, \quad \partial_v Z + \frac{\rho Z - u}{2(r + \rho - 1)} \partial_r Z = 0,
$$

(19)

$H = (h_1, \cdots, h_N)$ and $h_j$ is a solution of (14) with $\zeta = \zeta_j$. It can be checked that $S$ satisfies

$$(r + \rho - 1)(\partial_r S + [A_r, S]) - 2\rho(\partial_u S + [A_u, S])S + \rho[\Phi, S] - 2S = 0,$$

$$2(\partial_v S + [A_v, S]) + \frac{\rho}{r + \rho - 1}(\partial_r S + [A_r, S])S - \frac{\rho^2}{(r + \rho - 1)^2}[\Phi, S]S = 0.$$  

(20)

By direct computation, we know that for any solution $\psi$ of (10), $\tilde{\psi} = G\psi$ satisfies

$$((r + \rho - 1)\tilde{D}_r + \rho \tilde{\Phi} - 2(\rho \zeta - u) \tilde{D}_u)\tilde{\psi} = 0, \quad \left(2\tilde{D}_v + \frac{\rho \zeta - u}{r + \rho - 1} \tilde{D}_r - \frac{\rho(\rho \zeta - u)}{(r + \rho - 1)^2} \tilde{\Phi}\right)\tilde{\psi} = 0$$

(21)

with $\tilde{D}_\mu = \partial_\mu + \tilde{A}_\mu$ ($\mu = u, v, r$),

$$\tilde{A}_u = A_u,$$

$$\tilde{A}_v = A_v + \frac{\rho}{2(r + \rho - 1)}(\partial_r S + [A_r, S]) - \frac{\rho^2}{2(r + \rho - 1)^2}[\Phi, S],$$

$$\tilde{A}_r = A_r - \frac{1 + \rho(\partial_u S + [A_u, S])}{r + \rho - 1},$$

$$\tilde{\Phi} = \Phi - \frac{1 + \rho(\partial_u S + [A_u, S])}{\rho}.$$  

(22)

Hence $G$ is really a Darboux matrix for (14).
According to (19), each \( \zeta_j \) \((j = 1, \cdots, N)\) is a constant or a non-constant solution of
\[
\partial_r \zeta - \frac{2(\rho \zeta - u)}{r + \rho - 1} \partial_u \zeta = 0, \quad \partial_u \zeta + \frac{\rho \zeta - u}{2(r + \rho - 1)} \partial_r \zeta = 0. \tag{23}
\]

The general non-constant solution is given implicitly by
\[
v - \frac{(r + \rho - 1)^2}{\rho \zeta - u} = C_1(\zeta, \rho) \tag{24}
\]
where \( C_1 \) is an arbitrary function, which is meromorphic to \( \zeta \) and smooth to \( \rho \in (0, +\infty) \).

In order to consider the limit for \( \rho \to +\infty \), we rewrite (24) as
\[
v - \frac{(r + \rho - 1)^2}{\rho \zeta - u} + \frac{\rho - 1}{\zeta} = C(\zeta, \rho). \tag{25}
\]
Here \( C(\zeta, \rho) \) is also an arbitrary function, which is holomorphic to \( \zeta \) and smooth to \( \rho \).
Moreover, suppose that \( \lim_{\rho \to +\infty} C(\zeta, \rho) \) exists.

When \( \rho = 1 \), (25) becomes
\[
v - \frac{r^2}{\zeta - u} = C(\zeta, 1) \tag{26}
\]
which is given by [3]. When \( \rho \to +\infty \), (23) becomes
\[
v - \frac{u}{\zeta^2} - \frac{2r}{\zeta} = C(\zeta, +\infty) - \frac{1}{\zeta}. \tag{27}
\]

When the group \( G = U(N) \), there should be more constraints on \( \zeta_j \)'s and \( h_j \)'s in the construction of Darboux matrix. They are:
\[
\zeta_j = \zeta_0 \text{ or } \bar{\zeta}_0 \text{ for certain fixed } \zeta_0, \tag{28}
\]
\[
h_j^* h_k = 0 \text{ if } \zeta_j \neq \zeta_k,
\]
as mentioned in [3, 4]. If so, after the Darboux transformation, \( \tilde{\Phi} \in u(N), \tilde{A}_\mu \in u(N) \) provided that \( \Phi \in u(N), A_\mu \in u(N) \).

4 Soliton solutions in SU(2) case

Single soliton solutions are given by Darboux transformations from the trivial seed solution \( \Phi = 0, A_u = A_v = A_r = 0 \). In the construction of \( S = H(Z - u/\rho)H^{-1}, Z = \text{diag}(\zeta_1, \cdots, \zeta_N) \) where \( \zeta_j \) is a constant or a non-constant solution of (23), \( h_j \) is a column solution of (10) with \( \zeta_j \).

With the action of the Darboux matrix \( G = \zeta - u/\rho - S \), (22) gives
\[
\tilde{A}_u = 0, \quad \tilde{A}_v = \frac{\rho \partial_r S}{2(r + \rho - 1)}, \quad \tilde{A}_r = -\frac{1 + \rho \partial_u S}{r + \rho - 1}, \quad \tilde{\Phi} = -\frac{1 + \rho \partial_u S}{\rho}. \tag{29}
\]
Here we only consider the case where all $\zeta_j$’s are constants. When $\zeta_j$’s are non-constant solutions of (23), we can obtain solutions in similar ways. However, in the latter case, solutions may only defined when $t$ is large then some constant \[6\]. Now $h_j$ satisfies
\[
\partial_r h_j - \frac{2(r \zeta_j - u)}{r + \rho - 1} \partial_u h_j = 0, \quad \partial_v h_j + \frac{\rho \zeta_j - u}{2(r + \rho - 1)} \partial_r h_j = 0.
\]
(30)
The general solution is
\[
h_j = \omega(\zeta_j)
\]
(31)
where
\[
\omega(\zeta) = v - \frac{(r + \rho - 1)^2}{\rho \zeta - u} + \frac{\rho - 1}{\zeta}.
\]
(32)
When $\rho = 1$,
\[
\omega(\zeta) = v - \frac{r^2}{\zeta - u},
\]
(33)
which is the same as the result in \[6\]. When $\rho \to +\infty$,
\[
\omega(\zeta) \to v - \frac{u}{\zeta^2} - \frac{2r}{\zeta} + \frac{1}{\zeta}.
\]
(34)
With the substitution (14),
\[
\omega(\lambda^{-1}) \to (1 - \lambda^2)y - (1 + \lambda^2)t - 2\lambda x + \lambda.
\]
(35)
This coincides with \[3\].

When $G = SU(2)$, the conditions (28) should be satisfied. Hence we want $\zeta_1 = \zeta_0, \zeta_2 = \bar{\zeta}_0$ for some $\zeta_0 \in \mathbb{C}$ and
\[
H = \begin{pmatrix}
\alpha(\tau) & -\beta(\tau) \\
\beta(\tau) & \alpha(\tau)
\end{pmatrix}
\]
(36)
where $\alpha$, $\beta$ are two holomorphic functions of $\tau = \omega(\zeta_0)$. Let $\sigma(\tau) = \beta(\tau)/\alpha(\tau)$, then
\[
S = \frac{\zeta_0 - \bar{\zeta}_0}{1 + |\sigma|^2} \begin{pmatrix} 1 & \bar{\sigma} \\ \sigma & |\sigma|^2 \end{pmatrix} + \bar{\zeta}_0 - \frac{u}{\rho}.
\]
(37)

\[
\bar{\Phi} = -\partial_u S - \frac{1}{\rho} = \frac{\zeta_0 - \bar{\zeta}_0}{1 + |\sigma|^2} \begin{pmatrix} |\sigma|^2 u & \sigma^2 \sigma_u - \bar{\sigma}_u \\ \sigma^2 \bar{\sigma}_u - \sigma_u & -(|\sigma|^2)_u \end{pmatrix}
\]
(38)
and
\[
- \text{tr} \bar{\Phi}^2 = \frac{8(\text{Im} \zeta_0)^2}{(1 + |\sigma|^2)^2} |\partial_u \sigma|^2.
\]
(39)

When $\sigma(z)$ is a given meromorphic function of $z$ which is independent of $\rho$, then by (34) and (33),
\[
\sigma(\tau)|_{\rho \to +\infty} = \sigma \left( v - \frac{u}{\zeta_0^2} - \frac{2r}{\zeta_0} + \frac{1}{\zeta_0} \right), \quad \sigma(\tau)|_{\rho = 1} = \sigma \left( v - \frac{r^2}{\zeta_0 - u} \right).
\]
(40)
Hence when $\rho \to +\infty$ and $\rho = 1$, the solutions tend to the soliton solutions in the Minkowski and anti-de Sitter space-time respectively.

These are single soliton solutions. Each solution depends on a complex constant $\zeta_0$ and a meromorphic function $\sigma$. Multi-soliton solutions can be constructed by successive Darboux transformations [5,6]. For simplicity, here we only consider the change of single soliton solutions with respect to $\rho$.

**Example 1:** $\sigma(\tau)$ is a polynomial of $\tau$ without multiple zero

In this case, the solutions are always localized. When $\rho = 1$, the behavior of the asymptotic solution as $t \to \infty$ varies according to the roots of $\sigma(\tau)$ [3]. Suppose $\tau_0$ is a root of $\sigma(\tau)$, then (1) if $|\text{Im}\, \tau_0| << 1$, it corresponds to a ridge in the graph of $-\text{tr}\, \tilde{\Phi}^2$; (2) if $\text{Im}\, \tau_0 >> 1$, it corresponds to a peak; (3) if $\text{Im}\, \tau_0 << -1$, it corresponds to nothing. However, when $\rho \to +\infty$, each root of $\sigma(\tau)$ corresponds to a peak [3]. The following figures 1–5 show the change of the solution with respect to $\rho$ for fixed $t = 10$, where $\zeta_0 = 2i$,

$$\sigma(\tau) = (\tau - 2)(\tau - 6)(\tau + 6)(\tau - 2i)(\tau - 6i)(\tau + 6i).$$  \hspace{1cm} (41)

In these figures the vertical axis is $(-\text{tr}\, \tilde{\Phi}^2)^{1/16}$. 

Fig. 1. $\rho = 1$

Fig. 2. $\rho = 2$
Example 2: $\sigma(\tau) = \sin(\tau/20)$

For both finite and infinite $\rho$, the solution is always non-localized. For finite $\rho$, it behaves very complicated. However, for infinite $\rho$, (40) shows that the solution is invariant if $(x, r)$ is changed to $(x', r')$ with $\text{Re}[(1 - \zeta_0^{-2})(x' - x) - 2\zeta_0^{-1}(r' - r)] = 40\pi k$ ($k$ is an arbitrary integer). Hence the solution is periodic in one direction. Figures 6–8 show this solution for $\rho = 1, 30, +\infty$ with $t = 10, \zeta_0 = 2i$. In these figures the vertical axis is $(-\text{tr} \Phi^2)^{1/8}$. 

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Figure 1: $\rho = 1$

Figure 2: $\rho = 2$

Figure 3: $\rho = 5$

Figure 4: $\rho = 20$

Figure 5: $\rho = +\infty$

Figure 6: $\rho = 1$

Figure 7: $\rho = 30$

Figure 8: $\rho = +\infty$