On the Møller Energy-Momentum Complex of the Melvin Magnetic Universe

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Abstract
We use the Møller energy-momentum complex to calculate the energy of the Melvin magnetic universe. The energy distribution depends on the magnetic field.
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Introduction
The subject of energy-momentum localization in general relativity continues to be an open one because there is no given yet a generally accepted expression for the energy-momentum density. Even they are coordinate dependent, various energy-momentum complexes give the same energy distribution for a given space-time.

Aguirregabiria, Chamorro and Virbhadra [1] obtained that the energy-momentum complexes of Einstein [2], Landau and Lifshitz [3], Papapetrou [4], and Weinberg [5] give the same result for the energy distribution for any...
Kerr–Schild metric. Also, recently, Virbhadra investigated [6] if these definitions lead to the same result for the most general nonstatic spherically symmetric metric and found they disagree. Only the energy-momentum complex of Einstein gives the same expression for the energy when the calculations are performed in the Kerr–Schild Cartesian and Schwarzschild Cartesian coordinates. The Møller energy-momentum complex allows to compute the energy in any coordinate system.

Some results recently obtained [7]-[10] sustain that the Møller energy-momentum complex is a good tool for obtaining the energy distribution in a given space-time. Also, in his recent paper, Lessner [11] gave his opinion that the Møller definition is a powerful concept of energy and momentum in general relativity. Very important is the Cooperstock [12] hypothesis which states that the energy and momentum are confined to the regions of non-vanishing energy-momentum tensor of the matter and all non-gravitational fields. Also, Chang, Nester and Chen [13] showed that the energy-momentum complexes are actually quasilocal and legitimate expression for the energy-momentum.

In this paper we calculate the energy distribution of the Melvin magnetic universe in the Møller prescription. We use geometrized units ($G = 1, c = 1$) and follow the convention that Latin indices run from 0 to 3.

Energy in the Møller Prescription

The Melvin magnetic universe [14], [15] is described by the electrovac solution to the Einstein–Maxwell equations and consists in a collection of parallel magnetic lines of forces in equilibrium under their mutual gravitational attraction. The Einstein–Maxwell equations are

\[
R^k_i - \frac{1}{2} g^k_i R = 8 \pi T^k_i, \tag{1}
\]

\[
\frac{1}{\sqrt{-g}} \left( \sqrt{-g} F^{ik} \right)_k = 4 \pi J^i, \tag{2}
\]

\[
F_{ij,k} + F_{jk,i} + F_{ki,j} = 0. \tag{3}
\]

The energy-momentum tensor of the electromagnetic field is given by

\[
T^k_i = \frac{1}{4 \pi} \left[ -F_{im} F^{km} + \frac{1}{4} g^k_i F_{mn} F^{mn} \right]. \tag{4}
\]
The electrovac solution corresponds to \( J^i = 0 \) and is given by the metric

\[
ds^2 = L^2 (dt^2 - dr^2 - r^2 d\theta^2) - L^{-2} r^2 \sin^2 \theta \, d\varphi^2,
\]

where

\[
L = 1 + \frac{1}{4} B_0^2 r^2 \sin^2 \theta.
\]

The Cartan components of the magnetic field are

\[
H_r = L^{-2} B_0 \cos \theta,
H_\theta = -L^{-2} B_0 \sin \theta.
\]

\( B_0 \) is the magnetic field parameter and is a constant in the solution given by (5) and (6).

The energy-momentum tensor has the non-vanishing components

\[
T^{1\,1} = -T^{2\,2} = \frac{B_0^2 (1 - 2 \sin^2 \theta)}{8 \pi L^4},
T^{0\,0} = -T^{3\,3} = \frac{B_0^2 L^4}{8 \pi},
T^{1\,2} = -T^{2\,1} = \frac{2 B_0^2 \sin \theta \cos \theta}{8 \pi L^4}.
\]

The Møller energy-momentum complex \( \mathcal{M}^k_i \) [16] is given by

\[
\mathcal{M}^k_i = \frac{1}{8 \pi} \chi^{kl}_i \mathcal{J}^l,
\]

where

\[
\chi^{kl}_i = -\chi^{lk}_i = \sqrt{-g} \left( \frac{\partial g_{mn}}{\partial x^l} - \frac{\partial g_{im}}{\partial x^m} \right) g^{km} g^{ni}.
\]

Also, \( \mathcal{M}^k_i \) satisfies the local conservations laws

\[
\frac{\partial \mathcal{M}^k_i}{\partial x^k} = 0.
\]

\( \mathcal{M}^0_0 \) is the energy density and \( \mathcal{M}^\alpha_0 \) are the momentum density components.

The energy and momentum are given by
\[ E = \iiint M_0^0 dx^1 dx^2 dx^3 = \frac{1}{8\pi} \iiint \frac{\partial X_0^0}{\partial x^1} dx^1 dx^2 dx^3. \quad (12) \]

For the Melvin magnetic universe we obtain

\[ \chi_0^{01} = \frac{B_0^2 r^3 \sin^3 \theta}{(1 + 1/4 B_0^2 r^2 \sin^2 \theta)} \]
\[ \chi_0^{02} = \frac{B_0^2 r^2 \cos \theta \sin^2 \theta}{(1 + 1/4 B_0^2 r^2 \sin^2 \theta)}. \quad (13) \]

After some calculations, applying the Gauss theorem and plugging (13) into (12) we obtain the energy distribution

\[ E(r) = \frac{1}{3} B_0^2 r^3 - \frac{1}{15} B_0^4 r^5 + \frac{1}{70} B_0^6 r^7. \quad (14) \]

Put the \( G \) and \( c \) at their places we get

\[ E(r) = \frac{1}{3} B_0^2 r^3 - \frac{1}{15} \frac{G}{c^4} B_0^4 r^5 + \frac{1}{70} \frac{G^2}{c^8} B_0^6 r^7. \quad (15) \]

The first term represents twice of the classical value of energy [17] obtained in the Landau and Lifshitz and Papapetrou prescriptions. The other terms are due to the relativistic correction.

**Discussion**

Many results recently obtained sustain the viewpoint of Bondi [18]. He gave his opinion that a nonlocalizable form of energy is not admissible in relativity.

We obtain the energy distribution of the Melvin magnetic universe using the energy-momentum complex of Møller. The energy depends on the magnetic field. The result is different as that obtained by Xulu [17] using the energy-momentum complexes of Landau and Lifshitz and those of Papapetrou. The first term represents twice of the classical value of energy [17] obtained in the Landau and Lifshitz and Papapetrou prescriptions. The third term, that is due to the relativistic correction, is twice of value of energy obtained in [17]. Also, the Møller energy-momentum complex does not need to carry out calculations in any particular coordinates.
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