ON A CRITERION FOR LOG-CONVEX DECAY IN NON-SELFADJOINT DYNAMICS

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ABSTRACT. The short-time and global behaviour are studied for autonomous linear evolution equations defined by generators of uniformly bounded holomorphic semigroups in a Hilbert space. A general criterion for log-convexity in time of the norm of the solution is treated. Strict decrease and differentiability at the initial time results, with a derivative controlled by the lower bound of the negative generator, which is proved strictly accretive with equal numerical and spectral abscissas.

1. INTRODUCTION

The subjects here are the global and the short-time behaviour of the solutions to the Cauchy problem of an autonomous linear evolution equation, throughout with data $u_0 \neq 0$,

$$\partial_t u + Au = 0 \quad \text{for } t > 0, \quad u(0) = u_0 \quad \text{in } H.$$ (1)

In case the generator $-A$ is non-selfadjoint, this is particularly interesting. "Non-self-adjoint operators is an old, sophisticated and highly developed subject" to quote the recent treatise of Sjögren [Sjö19]; also the exposition of Helffer [Hel13, Ch. 13] on their pseudo-spectral theory could be mentioned; or [TE05].

Logarithmically convex decay of the solutions was seemingly first studied in the author’s paper [Joh18]. This is given a more concise exposition here, with additional examples.

The main purpose below, however, is to improve the results in [Joh18] by adding in Section 2 a much sharper necessary condition on $A$ for the log-convex decay, leading to the improved Theorem 7 below.

It is assumed that $A$ is an accretive operator with domain $D(A)$ in a complex Hilbert space $H$, with norm $| \cdot |$ and inner product $(\cdot | \cdot)$, and that $-A$ generates a uniformly bounded, holomorphic $C_0$-semigroup $e^{-tA}$ for $t$ in an open sector having the form $\Sigma_\delta = \{ z \in \mathbb{C} | -\delta < \arg z < \delta \}$. Focus is here on the “height” function

$$h(t) = |e^{-tA}u_0|. \quad (2)$$

This was shown in [Joh18] to be a log-convex function, that is, for $0 \leq r \leq s \leq t < \infty$

$$|e^{-rA}u_0| \leq |e^{-sA}u_0|^{1-s/r} |e^{-tA}u_0|^{s/r}, \quad (3)$$

if and only if the possibly non-normal generator $-A$ has the special property that for every $x \in D(A^2)$,

$$2(\text{Re}(Ax | x))^2 \leq \text{Re}(A^2x | x)|x|^2 + |Ax|^2 |x|^2. \quad (4)$$

The present paper and [Joh18] grew out of the author’s joint work [CJ18b, CJ18a] on the inverse heat equation and its well-posedness under the Dirichlet condition. But the main parts also apply to solutions of the similar Neumann problem studied in [Joh20, Joh19a, Joh19b].

To elucidate the importance of (3), hence of (4), two remarks are made.

1° The log-convexity in (3) implies that the solutions $u$ of (1) have important global properties in common with those of the heat equation (the case $A = -\Delta$ in $H = L_2(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^n$). Namely, the height function $h(t) = |e^{tA}u_0|$ of (1) is

(i) strictly positive ($h > 0$),
(ii) strictly decreasing ($h' < 0$),
(iii) strictly convex ($h'' > 0$).

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Here the strict decrease and strict convexity combine to a noteworthy and precise dynamical property. For example, even if $A$ has eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, they do not give rise to oscillations in the size of the solution $e^{-tA}u_0$—this is ruled out by strict convexity, which thus can be seen as a stiffness in the decay of $h(t)$.

In addition, (1) also shares the short-time behaviour with the heat equation, for in terms of the numerical range $\nu(A) = \{ \langle Ax | x \rangle \mid x \in D(A), \| x \| = 1 \}$ and its lower bound $m(A) = \inf \Re \nu(A)$, the onset of decay of $h \in C^\omega([0,\infty)) \cap C([0,\infty))$ is constrained by the properties:

(i) $h(t)$ is right differentiable at $t = 0$, with

(ii) $h'(0) \leq -m(A) < 0$ for $|u_0| = 1$, though

(ii) $h'(0) = -\Re(Au_0 | u_0)$ whenever $u_0 \in D(A), \|u_0\| = 1$.

For the considered $A$, (iv)–(vi) follow from log-convexity; cf. the below Theorem 4.4.

More generally, one could try to work with the $A$ that merely have strictly convex height functions, but this class is not easy to characterise. One may therefore view (3) as a very large class of (possibly non-normal) generators having the described dynamical properties in common with the selfadjoint cases.

Secondly, the operators satisfying (4) may be seen to comprise the $A$ that are selfadjoint, $A^* = A$, or normal, $A^*A = AA^*$. But as observed in [TI18a], one only needs the following two half-way houses,

$$D(A) \subset D(A^*), \quad |Ax| \geq |A^*x| \text{ for every } x \in D(A). \quad (5)$$

This property is hyponormality for unbounded operators, as studied by Janas [Jan94]. Clearly $A$ is normal if and only if both $A$, $A^*$ are hyponormal, so this operator class is quite general. As symmetric operators have a full inclusion $A \subset A^*$, they are also encompassed by the hyponormal class. But there is more:

**Example 1.1.** Truly hyponormal operators are easily exemplified: for the advection-diffusion operators $A^\pm u = -u'' \pm u'$ in $L^2(\alpha, \beta)$, for $\alpha < \beta$ in $\mathbb{R}$, it is classical that the minimal realisation $A^\pm_{min}$ has the domain $D(A^\pm_{min}) = H^2_0(\alpha, \beta)$ because of the ellipticity (cf. [Gru09] Thm. 6.24). The maximal realisation has domain $D(A^\pm_{max}) = H^2(\alpha, \beta)$, for when $f = -u'' \pm u'$ holds for $u, f \in L^2$, then $-u'' \pm u \in L^2$ as primitives of $f$, so $u'' \in L^2$; hence $u'' \in L^2$ and its lower bound $m(A^\pm_{min})$.

Partial integration for $u \in H^2(\alpha, \beta)$ yields $\| -u'' \pm u'\|^2 = \|u''\|^2 + \|u'\|^2 = \|u'\|^2 - |u'(\beta)|^2 - |u'(\alpha)|^2$, where the last two terms vanish for $u \in D(A^\pm_{min})$, so that $|A^\pm_{min}u| = \|A^\pm_{min}u\|$. Hence the $A^\pm_{min}$ are nonnormal, but nonetheless hyponormal.

That every hyponormal operator $A$ in $H$ necessarily satisfies the log-convexity condition (4) is recalled from [Joh18] for the reader’s convenience: the inclusion $D(A) \subset D(A^*)$ gives at once for $x \in D(A^2)$ that

$$2(\Re(AX | x))^2 \leq \frac{1}{2}|(A + A^*)x|^2 |x|^2 \leq \langle |Ax|^2 + \Re(A^2x | x) \rangle |x|^2, \quad (7)$$

for in the last step the norm inequality in (5) gives, because $D(A^2) \subset D(A) \subset D(A^*)$, that

$$|(A + A^*)x|^2 = |Ax|^2 + |A^*x|^2 + 2 \Re(AX | A^*x) \leq 2|Ax|^2 + 2 \Re(A^2x | x). \quad (8)$$

It is noteworthy, though, that whilst hyponormality expresses a certain interrelationship between $A$ and its adjoint, criterion (4) instead involves $A$ and its square $A^2$. In addition it was exemplified in [Joh18] that (4) is unfulfilled for certain explicitly given $A \in B(H)$, even for some symmetric $n \times n$-matrices, $n \geq 2$.

Moreover, the mixed Dirichlet–Neumann and Dirichlet–Robin realisations $A^\pm_{DN}$ and $A^\pm_{DR}$, respectively, are variational and elliptic, so they generate holomorphic semigroups in $L^2(\alpha, \beta)$. But none of them are hyponormal, cf. Example 5.5 below. This delicate situation around the $A^\pm$ should motivate the present analysis of the generators that have log-convex decay. It is envisaged that (4) can give interesting examples when $A$ is a suitable realisation of a partial differential operator.

In the above discussion of log-convexity of $h(t)$, its importance for the dynamics of (11) was explained in (i)–(vi) via the more general strict convexity. So it is natural to pose the question: does log-convexity have advantages in itself? At least it gives rise to the (perhaps new) proof technique used in the next section.
2. A NEW NECESSARY CONDITION FOR LOG-CONVEX DECAY

The reader is assumed familiar with semigroup theory, for which [EN00, Paz83] could be references; the simpler Hilbert space case is exposed e.g. in [Gri99] Ch. 14.

It is recalled that there is a bijection between the $C_0$-semigroups $e^{-tA}$ in $\mathbb{B}(H)$ that are uniformly bounded, i.e. $\|e^{-tA}\| \leq M$ for $t \geq 0$, and holomorphic in $\Sigma_\delta \subset \mathbb{C}$ for $\delta \in [0, \frac{1}{2}]$, and the densely defined, closed operators $A$ in $H$ satisfying a resolvent estimate $|\lambda|\|(A + \lambda I)^{-1}\| \leq C$ for all $\lambda \in \{0\} \cup \Sigma_{\delta+\pi/2}$.

It is classical that, since $\sigma(A) \subset \{z \in \mathbb{C} | \Re z \geq \varepsilon\}$ for some $\varepsilon > 0$, there is a bound $\|e^{-tA}\| \leq M_{\eta} e^{-\eta t}$ for $t \geq 0$, $0 < \eta < \varepsilon$. This yields the crude decay estimate

$$h(t) \leq M_{\eta} e^{-\eta t}|u_0|. \tag{9}$$

In general the possible $\eta$ are restricted by $0 \leq \eta < \sigma(A)$ in terms of the spectral abscissa of $A$,

$$\sigma(A) = \inf \Re \sigma(A). \tag{10}$$

The below analyses all rely on the recent result that such semigroups consist of injections, which, mentioned for precision, holds without the uniform boundedness:

**Lemma 2.1** ([Joh20, Joh19]). If $-A$ generates a holomorphic semigroup $e^{-tA}$ in $\mathbb{B}(X)$ for some complex Banach space $X$, and $e^{-tA}$ is holomorphic in the open sector $\Sigma_\delta \subset \mathbb{C}$ given by $|\arg z| < \delta$ for some $\delta > 0$, then $e^{-tA}$ is injective on $X$ for each such $z$.

The injectivity is clearly equivalent to the geometric property that two solutions $e^{-tA}u$ and $e^{-tA}w$ to the differential equation $u' + Au = 0$ cannot have any points of confluence in $X$ for $t > 0$ when $v \neq w$. One obvious consequence of this is its backward uniqueness: $u(T) = 0$ implies $u(t) = 0$ for $0 \leq t \leq T$.

Lemma 2.1 is also important because it allows a calculation of $h'(t)$, $h''(t)$, using differential calculus in Banach spaces as exposed e.g. by Hörmander [Hor83] Ch. 1 or Lang [Lan72]. This uses that $u(t) = e^{-tA}u_0 \neq 0$ for all $t > 0$ when $u_0 \neq 0$, cf. Lemma 2.1 whence $h(t) > 0$.

As the inner product on $H$, despite its sesquilinearity, is differentiable on the induced real vector space $H_\mathbb{R}$ with derivative $(\cdot | \cdot) + (\cdot | \cdot)$ at $(x, y) \in H_\mathbb{R} \oplus H_\mathbb{R}$, which applies to the composite map between open sets $\mathbb{R}_+ \to (H_\mathbb{R} \setminus \{0\}) \oplus (H_\mathbb{R} \setminus \{0\}) \to \mathbb{R}_+ \to \mathbb{R}_+$ given by $t \mapsto \sqrt{(u(t) | u(t))}$, the Chain Rule for real Banach spaces gives

$$h'(t) = \frac{(u' | u) + (u | u')}{2\sqrt{(u | u)}} = -\frac{\Re(Au | u)}{|u|}; \tag{11}$$

$$h''(t) = \frac{(A^2u | u) + 2(Au | Au) + (u | A^2u)}{2|u|^3} - \frac{(\Re(Au | u))^2}{|u|^3}. \tag{12}$$

The second line follows from the first, since $u'' = (e^{-tA}u_0)'' = A^2 e^{-tA}u_0 = A^2 u$.

When $A$ satisfies (4), the short-time behaviour at $t = 0$ is via the information on $h'(0)$ in (iv)–(vi) specifically controlled by $\sigma(A)$, and not by its spectrum $\sigma(A)$. Moreover, the proofs in [Joh18] also gave that $h'(0) = \inf h' < 0$, which when combined with (vi) shows that $A$ is a bit better than accretive ($m(A) \geq 0$) in the sense that its numerical range is contained in the open right half-plane, $\sigma(A) \subset \{z \in \mathbb{C} | \Re z > 0\}$. It seems useful to call $A$ a positively accretive operator, when it has this property (milder than strict accretivity [Kat93]), and it was shown in [Joh18] that (4) implies this.

But there is a significantly sharper necessary condition, which is given already now because of the novelty. Its proof exploits the log-convexity directly:

**Proposition 2.2.** If the generator $A$ has log-convex height functions $h(t)$ on $[0, \infty]$ for every $u_0 \neq 0$ and the one-sided derivative $h'(0)$ exists and fulfills $h'(0) = -\Re(Au_0 | u_0)$ when $u_0 \in D(A)$ with $|u_0| = 1$, then $A$ is strictly accretive and

$$m(A) = \sigma(A) > 0. \tag{13}$$

**Proof.** The log-convexity means that the continuous function $\log(h(t))$ is convex on $[0, \infty]$, so its graph lies entirely above each of its half-tangents. Applying this at $t = 0$ for $u_0 \in D(A)$, $|u_0| = 1$, and invoking (5), one finds that

$$\log h(0) + \frac{1}{h(0)} h'(0) \leq \log h(t) \leq \log M_{\eta} - t\eta \quad \text{for } t > 0. \tag{14}$$
Indeed, \( h(t) \) extends to \( t < 0 \) in a \( C^1 \)-function along its (half-)tangent at \( t = 0 \), after which the Chain Rule applies to \( \log h(t) \). (Differentiability of \( h(t) \) holds for \( t > 0 \) by \( \text{(11)} \), for \( t \leq 0 \) by construction.)

Now, the above inequalities being valid for all \( t > 0 \), the graphs of the two first order polynomials cannot intersect, so their slopes fulfill \( h'(0) \leq -\eta \) (as \( h(0) = |u_0| = 1 \)). Hence \( -h'(0) \geq \sigma(A) \), as the spectral abscissa is the supremum of the possible \( \eta \); cf. \( \text{(9)} \) ff. The assumption on \( h'(0) \) in the statement now gives that for any \( u_0 \in D(A) \) having \( |u_0| = 1 \),

\[
\Re(Au_0 | u_0) \geq \sigma(A),
\]

This entails the inequality \( m(A) \geq \sigma(A) \), hence strict accretivity since \( \sigma(A) > 0 \).

However, the strict inequality \( m(A) > \sigma(A) \) is impossible, for it would imply that \( \mathcal{V}(A) \) is contained in the closed half-plane \( \Pi_{m(A)} = \{ z \mid \Re z \geq m(A) \} \) and that \( \mathbb{C} \setminus \Pi_{m(A)} = \{ z \mid \Re z < m(A) \} \) contains some \( \lambda \in \sigma(A) \) as well as \( \mathbb{R} \) in the resolvent set \( \rho(A) \); but then \( \sigma(A) \) and \( \rho(A) \) intersect the same connectedness component of \( \mathbb{C} \setminus \mathcal{V}(A) \), contradicting \( \text{[Paz83 Thm. 1.3.9]} \). Hence \( m(A) = \sigma(A) \) as claimed. \( \square \)

3. MAIN RESULTS

For the reader’s sake, some basics are recalled here: a positive function \( f : \mathbb{R} \to [0, \infty] \) is log-convex if \( \log f(t) \) is convex, or more precisely, for all \( r \leq t \) in \( \mathbb{R} \) and \( 0 < \theta < 1 \),

\[
f((1 - \theta)t + \theta r) \leq f(r)^{1-\theta} f(t)^{\theta}.
\]

(16)

Note, though, that \( t^{\theta} \) and \( t^{1-\theta} \) do not require their continuous extensions to \( t = 0 \) when we take \( f = h \) below, for since \( e^{-it} \) is holomorphic, \( h(t) > 0 \) or equivalently \( e^{-it}u_0 \neq 0 \) holds for \( t \geq 0 \) by Lemma 2.1.

For the intermediate point \( s = (1 - \theta)t + \theta r \) an exercise yields \( \theta = (s - r)/(t - r) \), so log-convexity therefore means that, for \( 0 \leq r < s < t \),

\[
f(s) \leq f(r)^{1-\theta} f(t)^{\theta},
\]

(17)

This leads to \( \text{(3)} \) for the semigroup. There \( A \) is just a positive scalar if \( \operatorname{dim} H = 1 \), so \( \text{(3)} \) is then an identity. For \( \operatorname{dim} H > 1 \), the possible validity of \( \text{(3)} \) is by no means obvious to discuss for the operator function \( e^{-it} \) in \( \mathbb{B}(H) \).

In general log-convexity is stronger than strict convexity for non-constant functions:

**Lemma 3.1.** If \( f : I \to [0, \infty] \) is log-convex on an interval or halfline \( I \subset \mathbb{R} \), then \( f \) is convex—and if \( f \) is not constant in any subinterval, then \( f \) is strictly convex on \( I \).

**Proof.** Convexity on \( I \) follows from Young’s inequality for the dual exponents \( 1/\theta \) and \( 1/(1 - \theta) \):

\[
f((1 - \theta)t + \theta r) \leq f(r)^{1-\theta} f(t)^{\theta} \leq (1 - \theta)f(r) + \theta f(t).
\]

(18)

In case \( f(r) \neq f(t) \), the last inequality will be strict, as equality holds in Young’s inequality if and only if the numerators are identical (cf. \( \text{[NAP06 p. 141]} \)). This yields strict convexity in this case.

If there is a common value \( C = f(r) = f(t) \) for some \( r < t \) in \( I \), there is by assumption a \( u \in ]r,t[ \) so that \( f(u) \neq f(r) \), and because of the convexity of \( f \) this entails that \( f(u) < f(r) = f(t) \); when \( r < s \leq u \) one may write \( s = (1 - \theta)t + \theta u \) and \( s = (1 - \omega)t + \omega t \) for suitable \( \theta, \omega \in ]0,1[ \), so clearly

\[
f(s) \leq (1 - \theta)f(r) + \theta f(u) \quad < (1 - \theta)f(r) + \theta f(t) = C = (1 - \omega)f(r) + \omega f(t);
\]

(19)

similarly for \( u \leq s < t \); so \( f \) is strictly convex. \( \square \)

As examples it is noted that whilst \( e^t \) is log-convex, \( f(t) = e^t - 1 \) is not log-convex as \( (\log f)' < 0 \). However, when \( f : I \to ]0, \infty[ \) is log-convex, so is the stretched function defined for \( a < b \) in \( I \) as

\[
f_{a,b}(t) = \begin{cases} f(t) & \text{for } t < a, \\ f(a) & \text{for } a \leq t < b, \\ f(b - t) & \text{for } b \leq t. \end{cases}
\]

(20)

This follows from the geometrically obvious fact that the convexity of \( \log f \) survives the stretching. Since \( f_{a,b} \), clearly is not strictly convex, the last assumption of Lemma 3.1 is necessary. Moreover, a small exercise yields, cf. [Joh18].
Lemma 3.2. If \( f : [0, \infty] \to \mathbb{R}_+ \) is convex and \( f(t) \to 0 \) for \( t \to \infty \), then \( f \) is strictly monotone decreasing.

By now it is obvious that if a height function \( h(t) \) is log-convex on \([0, \infty]\) for some \( u_0 \neq 0 \), it fulfills the first assumption in Lemma 3.2 by the convexity statement in Lemma 3.1, and the second assumption holds because of (9). Therefore such \( h(t) \) is necessarily strictly decreasing on \([0, \infty]\) —hence non-constant in any subinterval, and by Lemma 3.4, therefore strictly convex.

That \( h(t) > 0 \) allows an analysis of its log-convexity using a characterisation of the log-convex \( C^2 \)-functions as the solutions to a differential inequality:

Lemma 3.3. If \( f \in C([0, \infty], \mathbb{R}_+) \) is \( C^2 \) for \( t > 0 \), the following are equivalent:

1. \( f''(t)^2 \leq f(t)f'''(t) \) holds whenever \( 0 < t < \infty \).
2. \( f(t) \) is log-convex on the open halfline \([0, \infty)\), cf. (17).

In the affirmative case \( f(t) \) is log-convex also on the closed halfline \([0, \infty]\).

Proof. By the assumptions \( F(t) = \log f(t) \) is defined for \( t > 0 \) and \( C^2 \) for \( t > 0 \) and

\[
F''(t) = \left( \frac{f'(t)}{f(t)} \right)' = \frac{f''(t)f(t) - f'(t)^2}{f(t)^2}.
\] (21)

Hence (I) is equivalent to \( F''(t) \geq 0 \) for \( t > 0 \), which is the criterion for the \( C^2 \)-function \( F \) to be convex for \( t > 0 \); which is a paraphrase of the condition (II) for log-convexity of the positive function \( f(t) \) for \( t > 0 \).

Letting \( r \to 0^+ \) for fixed \( s < t \), the continuity of \( f(r) \) and of, say \( \exp(-r\log f(r)) \), yields that (17) is valid for \( 0 < r < s < t \). So \( f \) is log-convex on \([0, \infty]\).

The formulation of the lemma was inspired by the discussion of convexity notions in [NP06]. Whilst \( f \in C^2 \) is convex if and only if \( f'' \geq 0 \), this positivity is clearly fulfilled if \( f \) satisfies (I), as \( f(t) \) is assumed—but the positivity then holds in a qualified way, equivalent to log-convexity, since (I) \( \iff \) (II).

The differential inequality in (I) of Lemma 3.3 is straightforwardly seen to amount to the following for \( h(t) \), cf. (11)–(12):

\[
2(\text{Re}(Au|u)^2) \leq (\text{Re}(A^2u|u) + |Au|^2)|u|^2.
\] (22)

Obviously this is fulfilled for every \( t > 0 \) when \( A \) satisfies (4) above, for \( u(t) = e^{-tA}u_0 \) belongs to the subspace \( D(A^n) \subset D(A^2) \) for every \( n \geq 2 \), and all \( u_0 \in H \), when the semigroup is holomorphic. Moreover, the continuity of \( h(t) \) and of its derivatives \( h', h'' \) given above show that \( h \in C^2 \) for \( t > 0 \). So according to Lemma 3.3 condition (4) implies that \( h(t) = |e^{-tA}u_0| \) is log-convex on the closed half-line \([0, \infty]\).

Conversely, when the height function \( h(t) \) is log-convex for each \( u_0 \neq 0 \), then the generator \(-A\) fulfills (4). Indeed, \( h \) then fulfills (I) above by the log-convexity, hence (22) holds. Especially it is seen by insertion of an arbitrary \( u_0 \in D(A^2) \) in (22) and commutation of \( A \) and \( A^2 \) with the semigroup that

\[
2(\text{Re}(e^{-tA}u_0|e^{-tA}u_0)^2) \leq (\text{Re}(e^{-tA}A^2u_0|e^{-tA}u_0) + |e^{-tA}Au_0|^2)|e^{-tA}u_0|^2.
\] (23)

By passing to the limit for \( t \to 0^+ \) it follows by continuity that (4) holds for \( x = u_0 \).

Consequently (4) characterises the generators \(-A\) of uniformly bounded, analytic semigroups having log-convex height functions for all non-trivial initial data.

The above discussion now allows the following sharpening of [Joh18] Thm. 2.5:

Theorem 3.4. When \(-A\) denotes a generator of a uniformly bounded, holomorphic \( C_0\)-semigroup \( e^{-tA} \) in a complex Hilbert space \( H \), then the following properties are equivalent:

1. \( 2(\text{Re}(Ax|x)^2) \leq \text{Re}(A^2x|x)^2 + |Ax|^2|x|^2 \) for every \( x \in D(A^2) \).
2. \( h(t) = |e^{-tA}u_0| \) is log-convex for every \( u_0 \neq 0 \); that is, whenever \( 0 \leq r < s < t \),

\[
|e^{-rA}u_0| \leq |e^{-tA}u_0| \implies |e^{-rA}u_0| \leq |e^{-sA}u_0| \leq |e^{-tA}u_0|.
\] (24)

In the affirmative case, \( h(t) \) is for \( u_0 \neq 0 \) strictly positive, strictly decreasing and strictly convex on the closed halfline \([0, \infty]\) and moreover differentiable from the right at \( t = 0 \), with a derivative in \([-\infty, 0]\), which for \( |u_0| = 1 \) satisfies

\[
h'(0) = \inf_{t>0} h'(t) \leq -m(A) < 0;
\] (25)
and if $u_0 \in D(A)$ with $|u_0| = 1$, then $h \in C^1([0, \infty[, [\mathbb{R}]) \cap C^\infty([\mathbb{R}_+, \mathbb{R}])$ and

$$h'(0) = -\Re(Au_0 | u_0).$$

(26)

Furthermore $\sigma(A) = m(A) > 0$ holds, in particular such $A$ are strictly accretive.

**Proof.** That (I) $\iff$ (II) was seen in the considerations after Lemma 3.3. The strict positivity was derived after Lemma 3.2: strict decrease and strict convexity after Lemma 3.2.

Convexity of $h$ entails $h''(t) \geq 0$ for $t > 0$, so $h'(t)$ is increasing on $\mathbb{R}_+$ and $\lim_{t \to 0^+} h'(t) = \inf_{t > 0} h'$ exists in $[-\infty, 0]$, as $h' < 0$. By the Mean Value Theorem, some $t' \in [0, t]$ fulfills

$$(h(t) - h(0))/t = h'(t') < 0.$$  

(27)

Therefore $h(t)$ is (extended) differentiable from the right at $t = 0$, with $h'(0) = \inf h'$. Since the strong continuity and strict decrease of $h$ gives $|e^{-tA}u_0| \to 1$ for $t \to 0^+$, an application of (11) yields

$$h'(0) = \inf h' \leq \limsup_{t \to 0^+} h'(t) \leq \limsup_{t \to 0^+} (-m(A)|e^{-tA}u_0|) \leq -m(A).$$

(28)

In case $u_0 \in D(A)$ and $|u_0| = 1$, one can exploit $h'(0) = \lim_{t \to 0^+} h'(t)$ by commuting $A$ with $e^{-tA}$ in (11), which in the limit gives, because of the strong continuity at $t = 0$ and the continuity of inner products,

$$h'(0) = \lim_{t \to 0^+} -\Re(e^{-tA}Au_0 | e^{-tA}u_0) = -\Re(Au_0 | u_0).$$

(29)

Finally, the last line of the statement results from Proposition 2.2. \qed

The conclusions of the theorem apply in particular to every hyponormal generator $-A$, cf. the account in (7) that such $A$ always satisfy the criterion (4).

It is instructive to review condition (4) in case the generator $A$ is variational. That is, for some Hilbert space $V \subset H$ algebraically, topologically and densely and some sesquilinear form $a: V \times V \to \mathbb{C}$, which is $V$-bounded and $V$-elliptic in the sense that (with $\| \cdot \|$ denoting the norm in $V$) for some $C_0 > 0$

$$\Re(a(u, u)) \geq C_0 \|u\|^2 \quad \text{for all } u \in V,$n

(30)

it holds for $A$ that $(Au | v) = a(u, v)$ for all $u \in D(A)$ and $v \in V$. Lax–Milgram’s lemma on the properties of $A$ is exposed in [Gru09, Ch. 12] and [Hel13, Ch. 3]. It is classical that $-A$ generates a holomorphic semigroup $e^{-tA}$ in $\mathbb{B}(H)$; an explicit proof is e.g. given in [CJ18a, Lem. 4].

For such $A$, the log-convexity criterion (4) can be stated for $V$-elliptical variational $A$ as a comparison of sesquilinear forms, cf. [John19],

$$(\Re(a(u, u)))^2 \leq \Re(a_{\Re}(Au, u))(u | u) \quad \text{for } u \in D(A^2).$$

(31)

**Example 3.5.** To see that variational operators need not be hyponormal, one may take $H = L^2(\mathbb{R}, |)\beta)$, with norm $\|f\|_0 = (\int_\mathbb{R} f^2(\beta) dx)^{1/2}$, for reals $\alpha < \beta$ and let $V = \{v \in H^1(\alpha, \beta) \mid u(\alpha) = 0\}$ be a subspace of the first Sobolev space with norm given by $\|f\|_1 = \int_\alpha^\beta (|f(\beta)|^2 + |f'(\beta)|^2) dx$ and the sequeilinear forms

$$a(u, v) = \int_\alpha^\beta u'(x)v'(x) + u(x)v(x) dx.$$  

(32)

This is clearly $V$-bounded, and also $V$-elliptic: partial integration gives $\Re(a(u, u)) = \|u''\|_0^2 + \frac{1}{\|u\|^2} \|u(\beta)\|^2$, and $\Re(a(u, u)) \geq C_0\|u\|^2_1$ follows for all $u \in V$ and e.g. $C_0 = \min\left(\frac{1}{\|u\|^2}, (\beta - \alpha)^{-2}\right)$ by ignoring the last term and using Poincaré’s inequality (its standard proof, e.g. [Gru09, Thm. 4.29], applies to $V$).

The induced $A^+_{DN}$ acts in the distribution space $\mathcal{D}'(\alpha, \beta)$ of Schwartz [Sch66] as $A^+_{DN}u = -u'' + u'$, which is the advection-diffusion operator with a mixed Dirichlet and Neumann condition,

$$D(A^+_{DN}) = \{u \in H^2(\alpha, \beta) \mid u(\alpha) = 0, u'(\beta) = 0\}.$$  

(33)

(The Dirichlet realisation of $u'' - u''$ has been studied at length; cf. [TE05, Ch. 12].)

As $(A^+_{DN})^*$ is induced by $a(v, u)$, one finds similarly $(A^+_{DN})^*u = -u'' - u' = -A_{DR}u$ with the domain characterised by a mixed Dirichlet and Robin condition,

$$D((A^+_{DN})^*) = D(A_{DR}) = \{u \in H^2(\alpha, \beta) \mid u(\alpha) = 0, u'(\beta) + u(\beta) = 0\}. $$

(34)
As both $D(A_{DN}^+)\text{ and } D((A_{DN}^+)^*)$ contain functions outside their intersection, (5) shows that neither $A_{DN}^+$ nor $(A_{DN}^+)^* = A_{DR}^+$ is hyponormal. This is part of the motivation for the study of condition (4).

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