Relative coherent modules

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Abstract. Several authors have introduced various type of coherent-like rings and proved analogous results on these rings. It appears that all these relative coherent rings and all the used techniques can be unified. In [2], several coherent-like rings are unified. In this manuscript we continue this work and we introduce coherent-like module which also emphasizes our point of view by unifying the existed relative coherent concepts. Several classical results are generalized and some new results are given.

Key Words. n-\mathcal{X}-coherent modules, n-\mathcal{X}-coherent rings

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1 Introduction

Throughout this paper $R$ will be an associative (non necessarily commutative) ring with identity, and all modules will be unital left $R$-modules (unless specified otherwise). In this section, first some fundamental concepts and notations are stated. Let $n$ be a non-negative integer and $M$ an $R$-module. Then $M$ is said to be $n$-presented if there is an exact sequence of $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow$, where each $F_i$ is a finitely generated free module. In particular, 0-presented and 1-presented modules are finitely generated and finitely presented modules, respectively. $M$ is said to be infinitely presented if it is $n$-presented for every positive integer $n$. A ring $R$ is called (left) coherent, if every finitely generated (left) ideal is finitely presented, equivalently every finitely presented $R$-module is 2-presented and so infinitely presented. The coherent rings were first appear in Chase’s paper [5] without being mentioned by name. The term coherent was first used by Bourbaki in [3]. Since then, coherent rings have became a vigorously active area of research. For background on coherence for commutative rings, we refer the reader to [9]. A ring $R$ is called (left) $n$-coherent ring if every $(n-1)$-presented (left) ideal is $n$-presented, equivalently every $n$-presented $R$-module is $(n+1)$-presented. Also, it is clear that 0-coherent (resp, 1-coherent) rings are just Noetherian (resp; coherent) rings. The $n$-coherent rings by Costa in [6] introduced, for more details see [2, 8, 11, 16, 17]. In [8], Kabbaj et al. introduced the concept of $n$-coherent modules, and $M$ is called $n$-coherent module if it is $(n-1)$-presented and every $(n-1)$-presented submodule of $M$ is $n$-presented, the 1-coherent modules are just the coherent modules, see [3].

In this paper, we introduce the $n$-$\mathcal{X}$-coherent modules. Let $n$ be an integer, $M$ be an $R$-module and $\mathcal{X}$ be a class of submodules of $M$. Then, $M$ is said to be $n$-$\mathcal{X}$-coherent if $\mathcal{X}_{n-1}$ is non empty and every submodule of $\mathcal{X}_{n-1}$ is in $\mathcal{X}_n$, where $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ are two classes of $(n-1)$-presented modules and $n$-presented modules in $\mathcal{X}$, respectively. In particular, if $\mathcal{X}$ is a class of $R$-modules and $M = R$, then $R$ is said to be an $n$-$\mathcal{X}$-coherent ring if every $R$-module of $\mathcal{X}_n$ is in $\mathcal{X}_{n+1}$ (see [2]). Our main aim is to show that the well-known Glaz, Smaili, Dobbes, Mahdou, Kabbaj, Chase, Greenberg and Scrivanti characterization of coherent modules and coherent rings hold true for any $n$-$\mathcal{X}$-coherent module and any $n$-$\mathcal{X}$-coherent ring. So, in Section 2, first we study some results of $n$-$\mathcal{X}$-coherent modules on short exact sequences, factor modules, homomorphism of $R$-modules and direct sum of $R$-modules. Also in this section, several results on transfer of $n$-$\mathcal{X}$-coherence are developed and then in end, another characterizations of $n$-$\mathcal{X}$-coherence using the notion of
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thickness are given (see Theorems 2.3, 2.7, 2.9, 2.12, 2.14, 2.16, 2.20 and Proposition 2.23). Finally, in Section 3, with considering pullback diagram, some characterizations of $n$-$\mathcal{X}$-coherent rings are studied (see Theorems 3.3 and 3.5).

2 Relative coherent modules

Among the many generalizations of the notion of a coherent ring, we recall the following one: $R$ is said to be (left) $J$-coherent, if every finitely generated (left) ideal of $R$ contained in $\text{Rad}(R)$, the radical of $R$, is finitely presented [7]. Also, $R$ is said to be (left) $\text{Nil}_*$-coherent, if every finitely generated (left) ideal of $R$ contained in $\text{Nil}(R)$, the nilradical of $R$, is finitely presented [13]. Here, we introduce the following definition of coherence which generalizes all the definitions above.

Definition 2.1. Let $n$ be an integer, $M$ be an $R$-module and $\mathcal{X}$ be a class of submodules of $M$. Let $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ be two classes of $(n-1)$-presented modules and $n$-presented modules in $\mathcal{X}$, respectively. We say that $M$ is (left) $n$-$\mathcal{X}$-coherent, if $\mathcal{X}_{n-1}$ is non empty and every module of $\mathcal{X}_{n-1}$ is in $\mathcal{X}_n$.

Examples 2.2. (1) If $\mathcal{X}$ is the class of all submodules of $M$ and $n = 1$ then $M$ is $n$-$\mathcal{X}$-coherent if and only if it is pseudo coherent. If, in addition, $M$ is finitely generated then $M$ is $n$-$\mathcal{X}$-coherent if and only if it is coherent (see [12]).

(2) If $\mathcal{X}$ is the class of all submodules of $M$ contained in $\text{Nil}(R)M$ and $n = 1$ then $M$ is $n$-$\mathcal{X}$-coherent if and only if it is $\text{Nil}_*$-coherent.

(3) Let $R$ be a semisimple ring and let $\mathcal{X}$ be any non empty class of submodules of an $R$-module $M$. Then $M$ is $n$-$\mathcal{X}$-coherent for every integer $n$.

(4) Let $M$ be an $R$-module and let $\mathcal{X}$ be a class of all finitely generated projective submodules of $M$. Then $M$ is $n$-$\mathcal{X}$-coherent.

(5) Let $K$ be a field and $E$ be a $k$-vector space with infinite rank. Consider $R = K \times E$ the trivial extension of $K$ by $E$. If $\mathcal{X}$ is the class of all 2-presented $R$-submodules of $M$. Then $M$ is $n$-$\mathcal{X}$-coherent, since every 2-presented $R$-submodules of $M$ is projective. But, if $\mathcal{X}^1$ is the class of all 1-presented $R$-submodules of any desirable $R$-module $M_1$, then there is an
Let \( R \)-module \( M_i \) such that \( M_i \) is not \( 2, \mathcal{X}^1 \)-coherent, since if any \( M_i \) is \( 2, \mathcal{X}^1 \)-coherent, then \( R \) is regular, a contradiction (see [15]).

(6) Let \( R_{n+1} = R_n \times M_n \) be the trivial extension, where \( R_i \) is a non-noetherian commutative ring for any \( i \geq 0 \). Consider \( M_0 = \frac{R_0}{I} \) for a finitely generated ideal \( I \) of \( R_0 \). If \( \mathcal{X} \) is the class of all \( R \)-submodules of \( M_{n+1} = \frac{R_{n+1}}{M_n} \). Then \( M_{n+1} \) is not \((n + 2), \mathcal{X}\)-coherent for every \( n \geq 0 \) (see [17]).

For a morphism \( \varphi : A \to B \) and a class \( \mathcal{X} \) of submodules of \( A \), we denote by \( \varphi(\mathcal{X}) \) the class of submodules of \( B \) of the form \( \varphi(N) \) with \( N \) in \( \mathcal{X} \).

The following theorem is a generalization of [9, Theorem 2.2.1].

**Theorem 2.3.** Let \( 0 \to M_1 \xrightarrow{h} M_2 \xrightarrow{s} M_3 \to 0 \) be an exact sequence of \( R \)-modules and \( \mathcal{X} \) and \( \mathcal{Y} \) two classes of submodules of \( M_1 \) and \( M_2 \), respectively. Then:

1. \( M_2 \) is \( n, \mathcal{Y} \)-coherent, if \( M_3 \) is \( n, \mathcal{Y} \)-coherent and \( M_1 \) is \( n \)-coherent.

2. \( M_1 \) is \( n, \mathcal{X} \)-coherent if \( M_2 \) is \( n, \mathcal{X} \)-coherent.

3. \( M_3 \) is \( n, \mathcal{Y} \)-coherent if \( M_2 \) is \( n, \mathcal{Y} \)-coherent and \( h(M_1) \) is \((n - 1)\)-presented in \( \mathcal{Y} \).

**Proof.** (1) Let \( N_2 \) be an \((n - 1)\)-presented submodule in \( \mathcal{Y} \). Our aim is to prove that \( N_2 \) is \( n \)-presented. For that, consider the following commutative diagram with exact rows and columns:

\[
\begin{array}{c}
0 \to K \to N_2 \to s(N_2) \to 0 \\
0 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & M_1 & M_2 & M_3 & 0 \\
\end{array}
\]

In view of the exactness of the first row, it suffices to show that both \( K \) and \( s(N_2) \) are \( n \)-presented. Since \( N_2 \) is an \((n - 1)\)-presented module in \( \mathcal{Y} \), \( s(N_2) \) is an \((n - 1)\)-presented module in \( s(\mathcal{Y}) \), so \( s(N_2) \) is \( n \)-presented. By the exactness of the top row, \( K \) is an \((n - 1)\)-presented submodule of \( M_1 \) which is \( n \)-coherent. Then \( K \) is \( n \)-presented.

(2) Let \( N_1 \) be an \((n - 1)\)-presented submodule in \( \mathcal{X} \). Since \( h \) is injective, \( h(N_1) \cong N_1 \) which is \((n - 1)\)-presented, so \( h(N_1) \) is an \((n - 1)\)-presented module of \( h(\mathcal{X}) \), and then by hypothesis, \( h(N_1) \) is \( n \)-presented, so is \( N_1 \).
(3) Let $N_3$ be an $(n-1)$-presented submodule in $s(\mathcal{Y})$. Notice that $\text{Im}(h) = \ker(s) \subseteq s^{-1}(N_3)$. $M_1$ is $n$-presented, since $M_2$ is $n$-$\mathcal{Y}$-coherent. So, we get using the horseshoe lemma to the following diagram is commutative with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K_1 & \rightarrow & K_2 & \rightarrow & K_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_{n-1} & \rightarrow & F_{n-1} \oplus F'_{n-1} & \rightarrow & F'_{n-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_0 & \rightarrow & F_0 \oplus F'_0 & \rightarrow & F'_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M_1 & \rightarrow & s^{-1}(N_3) & \rightarrow & N_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 0 & 0 & 0 & & & & & \\
\end{array}
\]

Where $F_i$ and $F'_i$ are finitely generated free modules for every $i \in \{0, \ldots, n-1\}$. Due to the exactness of the right vertical sequence, it suffices to prove that $K_3$ is finitely generated. For that, it is sufficient to prove that $K_2$ is finitely generated. Since the middle vertical sequence is exact, it suffices to show that $s^{-1}(N_3)$ is $n$-presented. We have that $N_3$ is in $s(\mathcal{Y})$, then $s^{-1}(N_3)$ is in $s^{-1}(s(\mathcal{Y}))$ and so it is in $\mathcal{Y}$. And since $N_3$ is $(n-1)$-presented, so is $s^{-1}(N_3)$ and consequently, it is $n$-presented. This implies that $K_2$ is finitely generated as desired. 

Consider a short exact sequence as in Theorem 2.3 if, for some class $\mathcal{Y}$ of submodules of $M_2$, we have that $s(\mathcal{Y}) = 0$, then $M_2$ is $n$-$\mathcal{Y}$-coherent if $M_1$ is $n$-coherent. For example, for $I = \text{ann}(M_3)$ and $\mathcal{Y}$ the class of submodules of $IM_2$, it is evident that $M_2$ is $n$-$\mathcal{Y}$-coherent if and only if $IM_2$ is $n$-coherent. This can be seen just by the definition of $n$-coherence and also if we take in Theorem 2.3 the short exact sequence $0 \rightarrow IM_2 \xrightarrow{h} M_2 \xrightarrow{s} \frac{M_2}{IM_2} \rightarrow 0$. 


In what follows, for a submodule $N$ of an $R$-module $M$ and a class $\mathcal{X}$ of submodules of $M$, we will denote by $\frac{M}{N}$ the class of quotient modules $\frac{L}{N}$ where $L \in \mathcal{X}$ and contains $N$. The following corollary generalizes [1, Corollary 2.3].

**Corollary 2.4.** Let $M$ be an $R$-module, $N$ a submodule of $M$ and $\mathcal{X}$ a class of submodules of $M$. The following assertions hold:

1. If $M$ is $n$-$\mathcal{X}$-coherent, $N$ is $(n - 1)$-presented and each module in $\mathcal{X}$ contains $N$, then $\frac{M}{N}$ is $n$-$\frac{\mathcal{X}}{N}$-coherent.

2. Assume that $\frac{\mathcal{X}}{N}$ is non empty. Then $M$ is $n$-$\mathcal{X}$-coherent if $\frac{M}{N}$ is $n$-$\frac{\mathcal{X}}{N}$-coherent and $N$ is $n$-coherent.

**Proof.** Let $\pi : M \to \frac{M}{N}$ be the canonical surjection. It is evident that, if $\mathcal{X}$ is the class of submodules $K$ of $M$ containing $N$ then $\frac{\mathcal{X}}{N}$ is the class of quotient modules $\frac{K}{N}$ with $K$ is in $\mathcal{X}$. Applying Theorem 2.3 to the following exact sequence: $0 \to N \to M \to \frac{M}{N} \to 0$, we get the result.

**Lemma 2.5.** Let $\mathcal{X}$ and $\mathcal{Y}$ two classes of submodules of $M$ with $\mathcal{X} \subseteq \mathcal{Y}$. If $M$ is $n$-$\mathcal{Y}$-coherent, then $M$ is $n$-$\mathcal{X}$-coherent.

For some submodule $K$ of an $R$-module $M$ and a class $\mathcal{Y}$ of submodules of $M$, we denote by $tr_K(\mathcal{Y})$ the class of submodules of $K$ of the form $K \cap Y$ with $Y \in \mathcal{Y}$. Also, we denote by $f(\mathcal{Y})$ the class of submodules of the form $f(Y)$ with $Y \in \mathcal{Y}$. The following proposition is a generalization of [9, Corollary 2.2.2].

**Proposition 2.6.** Let $f : M \to N$ be a homomorphism of $R$-modules and $\mathcal{X}$ and $\mathcal{Y}$ two classes of submodules of $M$ and $N$, respectively. The following assertions hold:

1. If $M$ is $n$-$\mathcal{X}$-coherent, then $\ker(f)$ is $n$-$tr_{\ker(f)}(\mathcal{X})$-coherent.

2. If $N$ is $n$-$\mathcal{Y}$-coherent, then $\operatorname{Im}(f)$ is $n$-$tr_{\operatorname{Im}(f)}(\mathcal{Y})$-coherent.

3. If $M$ is $n$-$\mathcal{X}$-coherent and $\ker(f)$ is an $(n - 1)$-presented module in $\mathcal{X}$, then $\operatorname{Im}(f)$ is $n$-$f(\mathcal{X})$-coherent.

4. If $N$ is $n$-$\mathcal{Y}$-coherent and $\operatorname{Im}(f)$ is an $(n - 1)$-presented module in $\mathcal{Y}$, then $\operatorname{Coker}(f)$ is $n$-$\frac{\mathcal{X}}{\operatorname{Im}(f)}$-coherent.
Proof. The two first assertions follow by applying (2) of Theorem 2.3 and Lemma 2.5 to the following exact sequences:

\[ 0 \to \text{Ker}(f) \to M \to \frac{M}{\text{Ker}(f)} \to 0 \]

and

\[ 0 \to \text{Im}(f) \xrightarrow{i} N \to \text{Coker}(i) \to 0. \]

The two last assertions follow by applying (3) of Theorem 2.3 to the following exact sequences:

\[ 0 \to \text{Ker}(f) \to M \to \text{Im}(f) \to 0 \]

and

\[ 0 \to \text{Im}(f) \to N \to \text{Coker}(f) \to 0. \]

Now, we set the result concerning the coherence of the direct sum of modules. It generalizes [9, corollary 2.2.3]. Let \((M_i)_{i \in I}\) be a family of \(R\)-modules and \(\mathcal{X}^i\) a class of submodules of \(M_i\), for each \(i \in I\). We will denote by \(\bigoplus_{i \in I} \mathcal{X}^i\) the class of modules of the form \(\bigoplus_{i \in I} N_i\) with each \(N_i\) is in \(\mathcal{X}^i\).

**Theorem 2.7.** Let \((M_i)_{i \in \{1, \ldots, m\}}\) be a finite family of \(R\)-modules and \(\mathcal{X}^1\) a class of submodules of \(M_i\) for every \(i = 1, \ldots, m\). Then, \(\bigoplus_{i=1}^m M_i\) is an \(n\)-\((\bigoplus_{i=1}^m \mathcal{X}^i)\)-coherent \(R\)-module if and only if \(M_i\) is \(n\)-\(\mathcal{X}^i\)-coherent for all \(i = 1, \ldots, m\).

**Proof.** The “only if” part follows easily using Lemma 2.5 (2) of Theorem 2.3 and the following exact sequence: \(0 \to M_i \to \bigoplus_{j=1}^m M_j \to \frac{\bigoplus_{i=1}^m M_i}{M_i} \to 0\).

For the “if” part, consider an \((n-1)\)-presented submodule \(N\) of \(\bigoplus_{i=1}^m M_i\) and the canonical projection \(\pi_i : \bigoplus_{i=1}^m M_i \to M_i\). Thus, \(\pi_i(N) \in \mathcal{X}^i_{n-1}\) for all \(i = 1, \ldots, m\). Then, \(\pi_i(N)\) is \(n\)-presented for all \(i = 1, \ldots, m\). We have the following exact sequence:

\[ 0 \to \bigoplus_{j=1, i \neq j}^m \pi_j(N) \to N \to \pi_i(N) \to 0. \]

Consequently by [17, Theorem 1], \(N\) is \(n\)-presented, which completes the proof. ■

The following result is a generalization of [9, Corollary 2.2.5].
Corollary 2.8. Let $R$ be a commutative ring, $M$ be a finitely generated $R$-module and $N$ be an $n$-$\mathcal{X}$-coherent module for some class $\mathcal{X}$ of submodules of $N$. Then $\text{Hom}_R(M, N)$ is $n$-$\mathcal{Y}$-coherent where $\mathcal{Y}$ is the class of submodules of $\text{Hom}_R(M, N)$ which are isomorphic to a module in $\text{tr}_A(\sum_{i=1}^k \mathcal{X})$.

Proof. Since $M$ is finitely generated, there is an exact sequence of $R$-modules $0 \to K \to R^k \to M \to 0$ for some non-negative integer $k$. As $\text{Hom}_R(\cdot, N)$ is a left-exact functor, $\text{Hom}_R(M, N) \cong A$, where $A$ is a submodule of $\text{Hom}_R(R^k, N) \cong N^k$. By Theorem 2.7, $N^k$ is $n$-$\bigoplus_{i=1}^k \mathcal{X}$-coherent, and so by Proposition 2.6, $A$ is $n$-$\text{tr}_A(\sum_{i=1}^k \mathcal{X})$-coherent, which completes the proof.

We finish this section with some transfer results. First, we present a generalization of [8, Theorem 2.5].

Theorem 2.9. Let $I$ be an $(n-1)$-presented two-sided ideal of a ring $R$ and $M$ be an $R_I$-module and $\mathcal{X}$ a class of submodules of $M$. Then $M$ is $n$-$\mathcal{X}$-coherent as an $R$-module if and only if $M$ is $n$-$\mathcal{X}$-coherent as an $R_I$-module.

For the proof, we need the following lemma.

Lemma 2.10 ([8], lemmas 2.6 and 2.7). Let $R \to S$ be a ring homomorphism and $m$ a non-negative integer. Then the following assertions hold:

1. If $S$ is $m$-presented as an $R$-module, then every $m$-presented $S$-module is an $m$-presented $R$-module.

2. If $S$ is $(m-1)$-presented as an $R$-module and $M$ a $S$-module, then $M$ is $m$-presented $S$-module if $M$ is an $m$-presented $R$-module.

Proof of Theorem 2.9. Let $R \to \frac{R}{I}$ be the canonical homomorphism and $N$ be a submodule of $M$. Using Lemma 2.10, we get the following equivalences: $N$ is an $(n-1)$-presented $\frac{R}{I}$-submodule of $\mathcal{X}$ if and only if it is $(n-1)$-presented $R$-submodule of $\mathcal{X}$. Consequently, $M$ is an $n$-$\mathcal{X}$-coherent as an $\frac{R}{I}$-module if and only if it is $n$-$\mathcal{X}$-coherent as an $R$-module.

Assume that $S \geq R$ is a unitary ring extension. Then, the ring $S$ is called right $R$-projective in case, for any right $S$-module $M_S$ with an $S$-module $N_S, N_R | M_R$ implies $N_S | M_S$, where
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$N \mid M$ means that $N$ is a direct summand of $M$. The ring extension $S \geq R$ is called a finite normalizing extension in case there is a finite subset \( \{ s_1, \ldots, s_n \} \subseteq S \) such that $S = \sum_{i=1}^n s_i R$ and $s_i R = Rs_i$ for $i = 1, \ldots, n$. A finite normalizing extension $S \geq R$ is called an almost excellent extension in case $S$ is flat, $S R$ is projective, and the ring $S$ is right $R$-projective (see [12]).

In the following, we mainly consider the properties of $n$-$\mathcal{X}$-coherent modules and $n$-$\mathcal{X}$-coherent rings under an almost excellent extension of commutative rings.

**Theorem 2.11.** Let $S \geq R$ be an almost excellent extension, $M$ a $S$-module and $\mathcal{Y}$ a class of submodules of $M$. Then, the following statements are equivalent:

1. $M$ is $n$-$\mathcal{X}$-coherent as an $R$-module;
2. $\text{Hom}_{R}(S, M)$ is $n$-$\mathcal{Y}$-coherent where $\mathcal{Y}$ is the class of submodules of $\text{Hom}_{R}(S, M)$;
3. $M$ is $n$-$\mathcal{X}$-coherent as an $S$-module.

**Proof.** (1) $\implies$ (2) $S$ is a finitely generated $R$-module. So (2) follows from Corollary 2.8.

(2) $\implies$ (3) Assume that $N$ is an $(n-1)$-presented submodule of $M$ in $\mathcal{X}$. We show that $N$ is $n$-presented. By [18, Lemma 1.1], $M \cong K$, where $K$ is a direct summand of $\text{Hom}_{R}(S, M)$. Therefore by hypothesis and Corollary 2.6, $K$ is $n$-$\text{tr}_{K}(\mathcal{Y})$-coherent. So, we deduce that $N$ is $n$-presented.

(3) $\implies$ (1) By [17, Theorem 5], $S$ is an $n$-presented $R$-module. So, if $N$ is an $(n-1)$-presented submodule of $R$-module $M$ in $\mathcal{X}$, then by Lemma 2.10 and (3), $N$ is an $n$-presented submodule of $R$-module $M$, and hence $M$ is $n$-$\mathcal{X}$-coherent as an $R$-module. 

In what follows, we will denote by $\mathcal{X}_k$, for some class of $R$-modules $\mathcal{X}$ and an integer $k$, the subclass of $k$-presented submodules of $\mathcal{X}$ (which we assume they exist). For an $R$-module $M$ and a class $\mathcal{X}$ of submodules of $M$, we will denote by $S \otimes \mathcal{X}$ the class $\{ S \otimes N, \text{where } N \text{ is a module of } \mathcal{X} \}$.

**Theorem 2.12.** Let $S \geq R$ be an almost excellent extension, $M$ a $S$-module and $\mathcal{X}$ a class of submodules of $M$. Then, the following statements are equivalent:

1. $M$ is $n$-$\mathcal{X}$-coherent as an $R$-module;
2. $S \otimes_{R} M$ is $n$-$S \otimes \mathcal{X}$-coherent;
(3) $M$ is $n\mathcal{X}$-coherent as an $S$-module.

**Proof.** (1) $\implies$ (2) Assume that $N$ is an $(n-1)$-presented submodule of $S \otimes_R M$ in $S\otimes\mathcal{X}$. So, there is a submodule $I$ in $\mathcal{X}$ such that $N = S \otimes_R I$. By [17] Lemma 4, $I \in \mathcal{X}_{n-1}$ as an $R$-module, and so by (1), $I \in \mathcal{X}_n$. Hence by Lemma 2.10, we deduce that $N$ is $n$-presented.

(2) $\implies$ (1) Assume that $N$ is an $(n-1)$-presented submodule of $M$ in $\mathcal{X}$. Then by [17] Lemma 4 and (2), $N \in \mathcal{X}_n$.

(1) $\implies$ (3) and (3) $\implies$ (1) are trivial.

Corollary 2.13. Let $S \geq R$ be an almost excellent extension and $\mathcal{X}$ a class of $R$-modules. Then, $R$ is $n\mathcal{X}$-coherent if and only if $S$ is $nS\otimes\mathcal{X}$-coherent.

**Proof.** It is particulary of Theorem 2.12.

The next result generalizes [8] Theorem 2.11 and [9] Corollary 2.2.5.

**Theorem 2.14.** Let $R \rightarrow S$ be a ring homomorphism making $S$ a faithfully flat right $R$-module, $M$ be an $R$-module and $\mathcal{X}$ a class of submodules of $M$. Then

(1) $M$ is $n\mathcal{X}$-coherent, if $S \otimes_R M$ is an $n\otimes\mathcal{X}$-coherent $S$-module.

(2) $M$ is $n\mathcal{X}$-coherent if and only if $S \otimes_R M$ is an $n\otimes\mathcal{X}_{n-1}$-coherent $S$-module.

**Proof.** (1) Let $N$ be an $(n-1)$-presented module of $\mathcal{X}$, then $S \otimes N$ is an $(n-1)$-presented module of $S\otimes\mathcal{X}$ (since $S$ is flat). Then, $S \otimes N$ is $n$-presented, so is $N$ since $S$ is faithfully flat.

(2) $\iff$ This is a direct consequence of (1).

(\iff) Assume that $K$ is an $(n-1)$-presented submodule of $S \otimes_R M$ in $S\otimes\mathcal{X}_{n-1}$. So, there is an $(n-1)$-presented submodule $N$ in $\mathcal{X}_{n-1}$ such that $K = S \otimes_R N$. By hypothesis, $N$ is in $\mathcal{X}_n$, and so by [9] Theorem 2.1.9, $K$ is $n$-presented.

**Corollary 2.15.** Let $R \rightarrow S$ be a ring homomorphism making $S$ a faithfully flat right $R$-module and $\mathcal{X}$ a class of ideals of $R$. Then $R$ is an $n\mathcal{X}$-coherent ring, if $S$ is an $nS\otimes\mathcal{X}$-coherent ring.

**Proof.** It is enough to take $M = R$. 

\[ \square \]
Question: Let $R \to S$ be a ring homomorphism making $S$ a faithfully flat right $R$-module and $\mathcal{X}$ a class of ideals of $R$. If $R$ is $n$-$\mathcal{X}$-coherent, then what conditions on the fibers $R \to S$ are required in order that $S$ is $n$-$S \otimes \mathcal{X}$-coherent?

Now, we give a generalization of the classical result due to Chase in [5] stating that $R$ is coherent if and only if the annihilator of any element $a$ of $R$ is finitely generated and the intersection of any two finitely generated ideals in $R$ is also finitely generated.

We say that a class of modules $\mathcal{X}$ is said to be closed under finite sums if, for every finite family of modules $\{M_i\}_{i \in I}$ in $\mathcal{X}$, $\sum_{i \in I} M_i$ is also in $\mathcal{X}$. A class $\mathcal{X}$ is said to be closed under cyclic submodules if, whenever $N$ is a cyclic submodule of a module in $\mathcal{X}$, it is also in $\mathcal{X}$.

The following theorem generalizes [5, Theorem 2.2].

**Theorem 2.16.** Let $M$ be an $R$-module and let $\mathcal{X}$ be a class of submodules of $M$ such that $\mathcal{X}_0$ is closed under finite sum and closed under cyclic submodules. Then $M$ is left $1$-$\mathcal{X}$-coherent if and only if $(0 :_R a)$ is a finitely generated of $M$ for any $a \in M$ such that $Ra$ is in $\mathcal{X}_0$ and the intersection of any two submodules of $M$ in $\mathcal{X}_0$ is finitely generated.

**Proof.** Suppose that $M$ is 1-$\mathcal{X}$-coherent and let $a$ be in $M$ such that $Ra$ is in an element $N$ of $\mathcal{X}_0$, then $Ra \in \mathcal{X}_0$. Then, $Ra$ is in $\mathcal{X}_1$. Considering the exact sequence: $0 \to (0 :_R a) \to R \to Ra \to 0$, we get that $(0 :_R a)$ is a finitely generated ideal of $R$.

Now, let $N$ and $L$ be in $\mathcal{X}_0$, then $N + L \in \mathcal{X}_0$. Then, by hypothesis, $N + L$ is in $\mathcal{X}_1$ and $N \oplus L$ is finitely generated as an $R$-module. Via the exact sequence $0 \to N \cap L \to N \oplus L \to N + L \to 0$, we get that $N \cap L$ is a finitely generated submodule of $M$.

Conversely, let $N \in \mathcal{X}_0$, then there exist $a_1, \ldots, a_p \in M$ such that $N = \sum_{i=1}^{p} Ra_i$. We prove by induction on $p$ that $N$ is 1-presented. If $p = 1$, $(0 :_R a_1)$ is finitely generated submodule of $M$.

Hence, $N$ is 1-presented by the exactness of the sequence $0 \to (0 :_R a_1) \to R \to N \to 0$. For the induction step (with $p > 1$), consider the following exact sequence $0 \to (\sum_{i=1}^{p-1} Ra_i) \cap Ra_p \to (\sum_{i=1}^{p-1} Ra_i) \oplus Ra_p \to N \to 0$. By hypothesis on $\mathcal{X}_0$, we have $Ra_p$ and $\sum_{i=1}^{p-1} Ra_i$ are in $\mathcal{X}_0$, then they are in $\mathcal{X}_1$, thus $(\sum_{i=1}^{p-1} Ra_i) \oplus Ra_p$ is 1-presented. Then, $(\sum_{i=1}^{p-1} Ra_i) \cap Ra_p$ is a finitely generated ideal of $M$. Therefore, $N$ is 1-presented.

Let $I$ be an ideal of $R$ and $\mathcal{X}$ be the class of ideals $J$ of $R$ contained in $I$. $R$ is 1-$\mathcal{X}$-coherent if
Corollary 2.17. Let $\mathcal{X}$ be a class of ideals of $R$ such that $\mathcal{X}_0$ is closed under finite sum and closed under cyclic submodules. Then $R$ is left $1$-$\mathcal{X}$-coherent if and only if $(0 :_R a)$ is a finitely generated of $R$ for any $a \in R$ such that $Ra$ is in $\mathcal{X}_0$ and the intersection of any two ideals in $\mathcal{X}_0$ is finitely generated.

Corollary 2.18. Let $I$ be an ideal of $R$. Then $I$ is quasi-coherent if and only if $(0 :_R a)$ is a finitely generated ideal of $R$ for any $a \in I$ and the intersection of any two left (resp., right) ideals contained in $I$ is finitely generated.

As an application of the previous results established in this section, we get the following result on the coherence of the amalgamated algebra along an ideal which is proved differently in [1].

Corollary 2.19. Let $R_1$ and $R_2$ be two unitary associative rings and let $f : R_1 \rightarrow R_2$ be a ring homomorphism. Let $J$ be a finitely generated ideal of $\text{Nil}(R_2)$ such that $f^{-1}(J)$ is a finitely generated ideal of $R_1$.

Then $R_1 \bowtie f J$ is $1$-$\text{Nil}$-coherent $R_1$-module if and only if $R_1$ and $f(R_1) + J$ are $1$-$\text{Nil}$-coherent.

Proof. The direct implication is proved directly using corollary 2.6 and the fact that $p_{R_2}(A \bowtie f J) = f(A) + J$ and $p_{R_1}(A \bowtie f J) = A$ for any ideal $A$ of $R_1$, where $p_{R_1}$ and $p_{R_2}$ are respectively the projection of $R_1 \bowtie f J$ on $R_1$ and $R_2$.

For the inverse, in light of theorem, it is sufficient to prove that $(0 : (a, f(a) + j))$ is a finitely generated of $R_1 \bowtie f J$ for any $a \in R_1$ and $j \in J$ such that $R(a, f(a) + j)$ is in the nil-radical of $R_1 \bowtie f J$ and the intersection of any two submodules of $R_1 \bowtie f J$ in the nilradical is finitely generated. For that, it is easy to prove that $(0 :_{R_1} (a, f(a) + j)) = (0 :_{R_1} a) \cap (0 :_{R_1} f(a) + j)$ and for any two ideals of $R_1$, we have that $(N \bowtie f J) \cap (I \bowtie f J) = (N \cap I) \bowtie f J$.

Now, we give some transfer results. We start with a generalization of [8, Theorem 2.13] and [9, Theorem 2.4.3]. Let $(M_i, i \in \{1, \ldots, p\})$ be a family of modules and $\mathcal{X}^i$ a class of submodules of $M_i$ for $i \in \{1, \ldots, p\}$. We will denote by $\prod_{i=1}^p \mathcal{X}^i$ the class of the submodules $\prod_{i=1}^p N_i$ with each $N_i$ is in $\mathcal{X}^i$.

Theorem 2.20. Let $(R_i, 1 \leq i \leq p)$ be a family of rings. Let $(M_i, 1 \leq i \leq p)$ be a family of $R_i$-modules, $p \geq 1$ an integer, $\mathcal{X}^i$ a class of submodules of $M_i$ for any integer $i \in \{1, \ldots, p\}$ and
\( \mathcal{X} = \prod_{i=1}^{p} \mathcal{X}^i. \) Then \( M = \prod_{i=1}^{p} M_i \) is \( n \)-\( \mathcal{X} \)-coherent \( R \)-module if only if \( M_i \) is \( n \)-\( \mathcal{X}^1 \)-coherent \( R_i \)-module for each \( i = 1, ..., p \), where \( R = \prod_{i=1}^{p} R_i. \)

**Proof.** \((\Longrightarrow)\) Let \( p = 2 \). Consider the short exact sequence \( 0 \to R_2 \to R_1 \times R_2 \to R_1 \to 0 \), where \( R_1 \) is an \( n \)-presented \( R_1 \times R_2 \)-module, since by [8] Lemma 2.14, \( R_1 \) is an infinitely presented \( R_1 \times R_2 \)-module. So, if \( N_1 \) is a submodule of \( M_1 \) in \( \mathcal{X}^1_{n-1} \), then by Lemma 2.10(1), \( N_1 \) is in \( \mathcal{X}^1_n \). Therefore by Lemma 2.10(2), \( N_1 \) is in \( \mathcal{X}^1_1 \). Similary, if \( N_2 \) is a submodule of \( M_2 \) in \( \mathcal{X}^2_{n-1} \), then \( N_2 \) is in \( \mathcal{X}^2_2 \).

\((\Longleftarrow)\) Suppose that, for every \( i \in \{1, ..., p\} \), \( M_i \) is \( n \)-\( \mathcal{X}^i \)-coherent. Let \( N \) be a module of \( \mathcal{X}^i_{n-1} \). Then there exist \( N_1, ..., N_p \) in \( \prod_{i=1}^{p} \mathcal{X}^i \) such that \( N = \prod_{i=1}^{p} N_i \). For each \( i \in \{1, ..., p\} \), \( N_i \) is in \( \mathcal{X}^i_{n-1} \). And so, \( N_i \) is in \( \mathcal{X}^i_{n-1} \). Hence, it is in \( \mathcal{X}^i_{1} \) by \( n \)-\( \mathcal{X}^i \)-coherence of \( M_i \). Consequently, \( N \) is also in \( \mathcal{X}^i_{n} \), and so we deduce that \( M \) is an \( n \)-\( \mathcal{X} \)-coherent \( R \)-module. \( \blacksquare \)

**Examples 2.21.** Let \( (R_i, 1 \leq i \leq n) \) be a family of rings in Example 2.2(6), and also let \( (M_i, 1 \leq i \leq n) \) be a family of \( R_i \)-modules in Example 2.2(6). Consider \( R = \prod_{i=1}^{n} R_i \) and \( M = \prod_{i=1}^{n} M_i. \) If \( \mathcal{X}^i \) is a class of submodules of \( M_i \) for any integer \( i \in \{1, ..., n\} \) and \( \mathcal{X} = \prod_{i=1}^{n} \mathcal{X}^i. \) Then by Example 2.2(6) and Theorem 2.20 \( M \) is not \( (n + 1) \)-\( \mathcal{X} \)-coherent \( R \)-module.

**Corollary 2.22.** Let \( (R_i, 1 \leq i \leq p) \) be a family of rings, \( p \geq 1 \) an integer, \( \mathcal{X}^i \) be a class of ideals of \( R_i \) for any integer \( i \in \{1, ..., p\} \) and \( \mathcal{X} = \prod_{i=1}^{p} \mathcal{X}^i. \) Then \( R = \prod_{i=1}^{p} R_i \) is \( n \)-\( \mathcal{X} \)-coherent if only if \( R_i \) is \( n \)-\( \mathcal{X}^1 \)-coherent for each \( i = 1, ..., p. \)

We end this section by establishing another characterization of \( n \)-\( \mathcal{X} \)-coherence using the notion of thickness. Recall that a class of modules \( \mathcal{Y} \) is said to be thick if it is closed under direct summands and whenever we are given a short exact sequence \( 0 \to A \to B \to C \to 0 \) with two out of the three terms \( A, B, C \) in \( \mathcal{Y} \), so is the third module \( B \). \([9\) Theorem 2.5.1] and \([4\) Theorem 2.4], it is proved that when \( R \) is coherent, the class of \( n \)-presented \( R \)-modules is thick. Here, we set the following generalization.

**Proposition 2.23.** Let \( n \) be a non negative integer and \( \mathcal{X} \) a class of \( R \)-modules which is closed under direct summand and kernels of epimorphisms. The following assertions are equivalent:

1. \( R \) is \( n \)-\( \mathcal{X} \)-coherent;
2. The class $\mathcal{X}_{n-1}$ is thick;

3. $\mathcal{X}_{n-1} = \mathcal{X}_\infty$.

**Proof.** (3) $\implies$ (2). It suffices to show that $\mathcal{X}_\infty$ is thick which is easily deduced using [9, Theorem 2.1.2], since $\mathcal{X}_\infty = \bigcap_{k \geq 0} \mathcal{X}_k$.

(2) $\implies$ (1). Let $I \in \mathcal{X}_{n-1}$, then there is an exact sequence of $R$-modules $0 \to K \to F_0 \to I \to 0$, where $K \in \mathcal{X}_{n-2}$ and $F_0$ is finitely generated and free. Since $\mathcal{X}_{n-1}$ is thick and both $I$ and $F_0$ are in $\mathcal{X}_{n-1}$, we get that $K \in \mathcal{X}_{n-1}$ and so $I \in \mathcal{X}_n$.

(1) $\implies$ (3). Let $I \in \mathcal{X}_{n-1}$. By the coherence of $R$, $I \in \mathcal{X}_n$. Using the same argument as in (2) $\implies$ (1), we can obtain an $(n+1)$-presentation of $I$. Iterating this procedure yields a finite $m$-presentation of $I$ for all $m \geq n$. Hence $I \in \bigcap_{m \geq 0} \mathcal{X}_m = \mathcal{X}_\infty$. \hfill \Box

3 On the coherence of pullbacks

By a ring, we mean a commutative ring with identity. Considering a commutative square of rings and ring homomorphisms of the following form:

$$
\begin{array}{ccc}
R & \xrightarrow{i_1} & R_1 \\
\downarrow{i_2} & & \downarrow{j_1} \\
R_2 & \xrightarrow{j_2} & R_0
\end{array}
$$

(1)

Recall that (1) is called a pullback diagram if $R = \ker(j_1 \circ p_1 - j_2 \circ p_2)$, where $p_1$ be the projection of $R$ on $R_1$ and $p_2$ be the projection of $R$ on $R_2$.

In the following, we say that a class $\mathcal{X}$ of modules satisfies the property $(\ast)$ proper if for every module $M \in \mathcal{X}_1$, there exists an exact sequence $0 \to K \to R^k \to M \to 0$ with $K \in \mathcal{X}$. We, also, consider a pullback diagram (1) with $i_1$ is surjective.

The following lemma can be found in [14, Proposition 1(c)].

**Lemma 3.1.** An $R$-module $M$ is finitely generated if and only if $R_i \otimes_R M$ is a finitely generated $R_i$-module, for $i = 1, 2$.

The following proposition generalizes [10, Proposition 2.1] and [14, Proposition 2].
Proposition 3.2. Let $M$ be an $R$-module and let $\mathcal{X}$ be a class of $R$-modules satisfying the property $(\ast)$ and let $\mathcal{Y}^i$ be a class of $R_i$-modules, for $i = 1, 2$, such that $R_i \otimes \mathcal{X}_k$ is a subclass of $\mathcal{Y}_k^i$ for every integer $k$, for $i = 1, 2$.

Suppose that $\text{Tor}_j^R(R_i, M)$ is in $\mathcal{Y}_{n-j}^i$ for $1 \leq j \leq n$ and $i = 1, 2$, then $M$ is in $\mathcal{X}_n$ if and only if $R_i \otimes_R M$ is in $\mathcal{Y}_n^i$ for $i = 1, 2$.

Proof. The proof is by induction on $n$.

The case $n = 0$ follows easily from 3.1.

Now, assume that $\text{Tor}_j^R(R_i, M)$ is in $\mathcal{Y}_{n-j}^i$ for $1 \leq j \leq n$ and $i = 1, 2$.

Let $M$ be in $\mathcal{X}_n$. Then, we have an exact sequence of $R$-modules

$$0 \rightarrow K \rightarrow R^k \rightarrow M \rightarrow 0 \quad (2)$$

We have that $\text{Tor}_j^R(R_i, K) \cong \text{Tor}_{j+1}^R(R_i, M)$, $j \geq 1$ then $R_i \otimes K$ is in $\mathcal{Y}_{n-1}^i$.

Now, we tensor the short exact sequence (2) with $R_i$ over $R$ and we obtain the following two exact sequences:

$$0 \rightarrow K_i \rightarrow R_i^k \rightarrow R_i \otimes_R M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Tor}_1^R(R_i, M) \rightarrow R_i \otimes_R K \rightarrow R_i \otimes_R K_i \rightarrow 0$$

with $K_i = \text{ker}(1_{R_i} \otimes f)$.

From the previous two exact sequences, we can deduce that $K_i$ is in $\mathcal{Y}_{n-1}^i$ which implies that $R_i \otimes_R M$ is in $\mathcal{Y}_n^i$, $i = 1, 2$.

For the converse, suppose that $R_i \otimes_R M$ is in $\mathcal{Y}_n^i$, then $K_i$ is in $\mathcal{Y}_{n-1}^i$. And so $R_i \otimes K$ is in $\mathcal{Y}_{n-1}^i$.

Hence $M$ is in $\mathcal{X}_n$. \qed

The following theorem generalizes [14, Theorem 4].

Theorem 3.3. Let $M$ be an $R$-module and let $\mathcal{X}$ be a class of $R$-modules satisfying the property $(\ast)$ and let $\mathcal{Y}^i$ be a class of $R_i$-modules, for $i = 1, 2$, such that $R_i \otimes \mathcal{X}_k$ is a subclass of $\mathcal{Y}_k^i$ for every integer $k$, for $i = 1, 2$.

Suppose that, for each $M \in \mathcal{X}_n$, we have that $\text{Tor}_j^R(R_i, M)$ is in $\mathcal{Y}_{n+1-j}^i$ for $1 \leq j \leq n + 1$ and $i = 1, 2$.

Then $R$ is $n$-$\mathcal{X}$-coherent if $R_i$ is $n$-$\mathcal{Y}^i$-coherent, $i = 1, 2$.

Proof. Let $M$ be an $R$-module in $\mathcal{X}_n$. By Proposition 3.2 $R_i \otimes_R M$ is in $\mathcal{Y}_n^i$. Then, by the coherence of $R_i$, $R_i \otimes_R M$ is in $\mathcal{Y}_{n+1}^i$. Again, by Proposition 3.2 $M$ is in $\mathcal{X}_{n+1}$. And so $R$ is
Corollary 3.4. Let $M$ be an $R$-module and let $\mathcal{X}$ be a class of $R$-modules satisfying the property $(\ast)$ and let $\mathcal{Y}^i$ be a class of $R_i$-modules, for $i = 1, 2$, such that $R_i \otimes \mathcal{X}_k$ is a subclass of $\mathcal{Y}^i_k$ for every integer $k$, for $i = 1, 2$.

Suppose that $R_i$ is $n\mathcal{Y}^i$-coherent, $i = 1, 2$.

Then $R$ is $n\mathcal{X}$-coherent if and only if for all $I \in \mathcal{X}_n$, we have that $\text{Tor}^R_i(R_i, \frac{R}{I})$ is in $\mathcal{Y}^i_{n+1-j}$ for $1 \leq j \leq n + 1$ and $i = 1, 2$.

Proof. The only if assertion follows from Theorem 3.3, we will prove the converse.

Let $I \in \mathcal{X}_n$, then we have an exact sequence of the form

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\pi} \frac{R}{I} \longrightarrow 0. \tag{3}$$

Tensoring the sequence (3) with $R_i$ ($i = 1, 2$) over $R$ and put $H_i = \ker(1 \otimes \pi)$, we obtain two exact sequences

$$0 \longrightarrow H_i \longrightarrow R_i \xrightarrow{1 \otimes \pi} R_i \otimes_R \frac{R}{I} \longrightarrow 0. \tag{4}$$

and

$$0 \longrightarrow \text{Tor}^R_i(R_i, \frac{R}{I}) \longrightarrow R_i \otimes_R I \xrightarrow{\pi} H_i \longrightarrow 0. \tag{5}$$

Using the coherence of $R$ and the exactness of the sequences (3), (4) and (5), we get that $\text{Tor}^R_i(R_i, \frac{R}{I})$ is in $\mathcal{Y}^i_n$.

Now, since $I$ is in $\mathcal{X}_n$, we have an exact sequence

$$0 \longrightarrow P \longrightarrow R^i \longrightarrow I \longrightarrow 0. \tag{6}$$

Using a similar argument, we get that $\text{Tor}^R_i(R_i, I)$ is in $\mathcal{Y}^i_{n-1}$, and hence $\text{Tor}^R_i(R_i, \frac{R}{P})$ is also in $\mathcal{Y}^i_{n-1}$. Again, since $P$ is in $\mathcal{X}_{n-1}$, we have an exact sequence

$$0 \longrightarrow P_0 \longrightarrow R^i \longrightarrow P \longrightarrow 0. \tag{7}$$

Using a similar argument, we get that $\text{Tor}^R_i(R_i, P)$ is in $\mathcal{Y}^i_{n-2}$, and hence $\text{Tor}^R_i(R_i, I)$ is also in $\mathcal{Y}^i_{n-2}$. Iterating with the same argument, we get that for each $I \in \mathcal{X}_n$, we have that $\text{Tor}^R_j(R_i, \frac{R}{I})$ is in $\mathcal{Y}^i_{n+1-j}$ for $1 \leq j \leq n + 1$ and $i = 1, 2$. 

$\blacksquare$
Now, we establish necessary and sufficient conditions for the coherence of the pullback diagram.

**Theorem 3.5.** Let $\mathcal{X}$ be a class of $R$-modules satisfying the property $(\ast)$ and let $\mathcal{Y}^i$ be a class of $R_i$-modules, for $i = 1, 2$.

Suppose that, for each $M \in \mathcal{X}_n$, we have that $\text{Tor}_j^R(R_i, M)$ is in $\mathcal{Y}^i_{n+1-j}$ for $1 \leq j \leq n + 1$ and $i = 1, 2$. Suppose, moreover, that for any module in $Y_i \in \mathcal{Y}^i_n$, there exists a module $X_i \in \mathcal{X}_n$ such that $R_i \otimes_R X_i \simeq Y_i$, for $i = 1, 2$.

Then $R$ is $n$-$\mathcal{X}$-coherent if and only if $R_i$ is $n$-$\mathcal{Y}^i$-coherent, $i = 1, 2$.

**Proof.** The direct sense of the equivalence is proved in Theorem 3.3. For the converse, let $N_i$ be a $R_i$-module of $\mathcal{Y}^i_n$, ($i = 1, 2$). By hypothesis, $N_i = R_i \otimes_R N_i'$, where $N_i'$ is in $\mathcal{X}_n$. Then, by the coherence of $R$, $N_i'$ is in $\mathcal{X}_{n+1}$. Hence, $N_i$ is in $\mathcal{Y}^i_{n+1}$, $i = 1, 2$. $\blacksquare$

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