On Relations for Zeros of $f$-Polynomials and $f^+$-Polynomials

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Received: 10 July 2016 / Revised: 13 October 2016 / Accepted: 1 November 2016 /
Published online: 19 January 2017
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Abstract Let $\Phi$ be an irreducible (possibly noncrystallographic) root system of rank $l$ of type $P$. For the corresponding cluster complex $\Delta(P)$, which is known as pure $(l-1)$-dimensional simplicial complex, we define the generating function of the number of faces of $\Delta(P)$ with dimension $i-1$, which is called $f$-polynomial. We show that the $f$-polynomial has exactly $l$ simple real zeros on the interval $(0,1)$ and the smallest root for the infinite series of type $A_l$, $B_l$, and $D_l$ monotone decreasingly converges to zero as the rank $l$ tends to infinity. We also consider the generating function (called the $f^+$-polynomial) of the number of faces of the positive part $\Delta^+(P)$ of the complex $\Delta(P)$ with dimension $i-1$, whose zeros are real and simple and are located in the interval $(0,1)$, including a simple root at $t=1$. We show that the roots in decreasing order of $f$-polynomial alternate with the roots in decreasing order of $f^+$-polynomial.

Keywords Growth function · The Jacobi polynomials

Mathematics Subject Classification (2010) 12D10

1 Introduction

Let $\Phi$ be an irreducible (possibly noncrystallographic) root system of rank $l$ of type $P \in \{A_l \ (l \geq 1), \ B_l \ (l \geq 2), \ D_l \ (l \geq 4), \ E_l \ (l = 6, 7, 8), \ F_4, \ G_2, \ H_3, \ H_4, \ I_2(p) \ (p \geq 3)\}$. Let $\Phi^+$ be a positive system for $\Phi$ with corresponding simple system $\Pi$. The cluster complex
\(\Delta(P)\) introduced by Fomin-Zelevinsky [8] is a pure \((l - 1)\)-dimensional simplicial complex whose ground set is the set \(\Phi_{\geq -1} := \Phi^+ \sqcup (-\Pi)\) of almost-positive roots and its geometric realization is homeomorphic to a sphere. Let \(\Delta_+(P)\) denote the induced subcomplex of \(\Delta(P)\) on the vertex set \(\Phi^+\). The complex \(\Delta_+(P)\) is referred to as the positive part of \(\Delta(P)\). Let \(f_i\) (resp. \(f^+_i\)) denote the number of faces of \(\Delta(P)\) (resp. \(\Delta_+(P)\)) with dimension \(i - 1\). We call the sequence \((f_0, f_1, \ldots, f_{l-1})\) (resp. \((f^+_0, f^+_1, \ldots, f^+_{l-1})\)) the \(f\)-vector of the complex \(\Delta(P)\) (resp. \(\Delta_+(P)\)). We define the \(f\)-polynomial and \(f^+\)-polynomial of type \(P\) in the formal variable \(t\) by

\[
f_P(t) := \sum_{i=0}^{l} f_{i-1}(P)(-t)^i, \quad f^+_P(t) := \sum_{i=0}^{l} f^+_{i-1}(P)(-t)^i,
\]

where \(f_{-1}(P) := 1\) and \(f^+_{-1}(P) := 1\). In this article, we will study the zero loci of the \(f\)-polynomial. In a similar manner to [12], we will show that the \(f\)-polynomial has an unexpected strong connection with orthogonal polynomials. By making the best use of these properties, we will show that the \(f\)-polynomial has exactly \(l\) simple real zeros on the interval \((0, 1)\) and the smallest root for the infinite series of type \(A_l, B_l, D_l\) monotonically decreases to zero as the rank \(l\) tends to infinity. Furthermore, we obtain inequalities for zeros of \(f\)-polynomials and \(f^+\)-polynomials.

In [12], the authors studied the zero loci of the skew-growth function (see [17]) of a dual Artin monoid of finite type (see [2]). They noted that the skew-growth function of type \(P\) coincides with the \(f^+\)-polynomial of the same type.\(^3\) Due to this coincidence, in this article we mainly study the \(f^+\)-polynomial instead of the skew-growth function. Suggested by some numerical experiments, they conjectured the following 1, 2, and 3 (see [12]).\(^4\)

1. \(f^+_P(t)/(1-t)\) is an irreducible polynomial over \(\mathbb{Z}\), up to the trivial factor \(1 - 2t\) for the types \(A_l, B_l, D_4\).
2. \(f^+_P(t)\) has \(l = \text{rank}(P) (= \deg(N_{\text{dual}^+}^P))\) simple real roots on the interval \((0, 1)\), including a simple root at \(t = 1\).
3. The smallest root of \(f^+_P(t)\) monotonously decreasingly converges to 0 as the rank \(l\) tends to infinity for the infinite series of type \(A_l, B_l, D_l\).

By analogy with 1, 2, and 3, and also inspired by some numerical experiments (see Appendix II for the figures of the zero loci of the functions of types \(A_{20}, B_{20}, D_{20}\), and

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1 In [8], the notion of cluster complex \(\Delta(P)\) was originally defined for the case where \(\Phi\) is crystallographic. As was noted in [7], the constructions in [8] extend verbatim to the noncrystallographic types.
2 The study of the skew-growth function of a monoid and its zero loci is motivated by the study of the partition functions associated with the monoid, since the partition functions are given by certain residue formula at the zero loci of the skew-growth function (see [18, Section 11, Theorem 6 and Section 12]).
3 This fact is proved in [1, Corollary 2.5]. In addition to this fact, it is known that the skew-growth function is identified with the generating function of Möbius invariants (called the characteristic polynomial) of the lattice of non-crossing partitions (see [2, 3, 13]).
4 In [16], the author also gave three conjectures on the skew-growth function for an Artin monoid of finite type.
E8), we conjecture the following. We note that the fact that \( f_P(t) \) has \( l = \text{rank}(P) \) simple real roots on the interval \((0, 1)\) has been proved essentially in [4].

**Conjecture 1** \( f_P(t) \) is an irreducible polynomial over \( \mathbb{Z} \), up to the trivial factor \( 1 - 2t \) for the types \( A_l (l : \text{odd}), B_l (l : \text{odd}), D_l (l : \text{odd}), E_7, \) and \( H_3 \).

**Conjecture 2** The smallest root of \( f_P(t) \) monotonously decreasingly converges to 0 as the rank \( l \) tends to infinity for the infinite series of type \( A_l, B_l, \) and \( D_l \).

**Remark 1.1** Conjecture 1 is approved for types \( A_l (1 \leq l \leq 30), B_l (2 \leq l \leq 30), D_l (4 \leq l \leq 30), E_6, E_7, E_8, F_4, G_2, H_3, H_4, \) and \( I_2(p) \) (\( p \geq 3 \)) by using the software package Mathematica on the Tables A and B in Appendix I. In Section 2, we will show that they are expressed in higher (logarithmic) derivatives of the polynomials of \( f_P(t) \) in decreasing order (i.e., \( 1 > t_{p,1} > t_{p,2} > \cdots > t_{p,l} > 0 \)) and let \( t_{p,v}^+, v = 1, 2, \ldots, l = \text{rank}(P) \), be the zeros of \( f_P(t) \) in decreasing order (i.e., \( 1 = t_{p,1} > t_{p,2} > \cdots > t_{p,l} > 0 \)).

**Conjecture 3** The system \( \{t_{p,v}^+, v = 1, 2, \ldots, l = \text{rank}(P)\} \) alternates with the system \( \{t_{p,v}\}_{v=1}^l \). That is, \( t_{p,v} > t_{p,v+1} > t_{p,v+1}, \) \( (v = 1, \ldots, l-1) \).

The aim of the present paper is to give affirmative answers to Conjectures 2 and 3, and give another proof of Brändén’s results. In Section 2, we first prepare some useful functional (2.1), (2.2), and (2.3) for later use. In the functional equation for type \( A_l \) (resp. \( B_l \)), the \( f \)-polynomial of type \( A_l \) (resp. \( B_l \)) has a simple relation with the \( f^+ \)-polynomial of the same type. Although for type \( D_l \) the \( f \)-polynomial does not have a simple relation with the \( f^+ \)-polynomial of the same type, the \( f \)-polynomial of type \( D_l \) has a rather complicated relation with the \( f \)-polynomial of type \( B_l \). Hence, from the Eq. 2.2 and the equation (6.4) in [12], we may say that the \( f \)-polynomial of type \( D_l \) has a certain relation with \( f^+ \)-polynomial of type \( D_l \). Secondly, for the three infinite series \( A_l, B_l, \) and \( D_l \) of \( f \)-polynomials, we show that they are expressed in higher (logarithmic) derivatives of the polynomials of the form \( t^p(1-t)^q \). In analogy with the classical Rodrigues’s formula in the theory of orthogonal polynomials [19], we call the formula Rodrigues type formula. As a consequence of the formulae, the polynomials are expressed by using orthogonal polynomials. In

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5In [4], the author showed that the \( h \)-polynomial \( h_P(t) \) of type \( P \) has \( l \) simple real roots on the interval \((−∞, 0)\). The \( h \)-polynomial \( h_P(t) \) is combined with \( f_P(t) \) as

\[
 f_P(t) = (1-t)^l h_P \left( \frac{t}{l-1} \right). \tag{1.1}
\]

From Eq. 1.1, this implies that the \( f \)-polynomial of type \( P \) has \( l \) simple real roots on the interval \((0, 1)\).
Section 3, we will show that the series of the \( f \)-polynomials of type \( A_l \) and \( B_l \) satisfies 3-term recurrence relation, respectively, and the series of the \( f \)-polynomials of type \( D_l \) satisfies 4-term recurrence relation. In Section 4, we will give another proof of Brändén’s results except for types \( D_l \). Namely, we will show that the \( f \)-polynomial of type \( P \) has \( l \) simple real roots on the interval \((0, 1)\). For type \( A_l \) (resp. \( B_l \)), the \( f \)-polynomial, up to a constant factor, coincides with a certain orthogonal polynomial. By these properties, Brändén’s results for types \( A_l \) and \( B_l \) are approved. For exceptional types \( E_6, E_7, E_8, F_4 \), and \( G_2 \) and non-crystallographic types \( H_3, H_4 \), and \( I_2(p) \), we will construct Sturm’s sequence on the interval \((0, 1)\). Due to the Sturm Theorem (see for instance \([20, \text{Theorem 3.3}])\), Brändén’s results for them are approved. In Section 5, we will give another proof of Brändén’s results for types \( D_l \), by applying the intermediate value theorem to the Eq. 2.3. In Section 6, we will prove Conjecture 2 affirmatively. For series \( A_l \), due to the Eq. 2.1, the smallest zero locus of \( f_{A_l}(t) \) coincides with the smallest zero locus of \( f_{A_{l+1}}^+(t) \). From Theorem 6.1 in \([12]\), we show that Conjecture 2 for series \( A_l \) is true. For series \( B_l \), from the Eq. 2.8, we show that the smallest zero locus of \( f_{B_l}(t) \) coincides with the smallest zero locus of the shifted Legendre polynomial \( \tilde{P}_l^{(0,0)}(t) := P_l^{(0,0)}(2t - 1) \) (the shifting of Legendre polynomial \( P_l^{(0,0)}(t) \)) of degree \( l \). From \([19, \text{Theorem 6.21.3}])\), we have that the smallest zero locus of Legendre polynomial \( P_l^{(0,0)}(t) \) monotonously decreasingly converges to zero as the rank \( l \) tends to infinity. Therefore, we show that Conjecture 2 for series \( B_l \) is true. The proof for the series \( D_l \) uses again the Eq. 2.3, where the polynomials of type \( D_l \) are expressed by those of type \( B_l \) so that the roots of type \( D_l \) are sandwiched by the roots of type \( B_l \). Hence, we show that Conjecture 2 for types \( D_l \) is true. In Section 7, we will prove Conjecture 3 except for types \( D_l \) affirmatively. For series \( A_l \), due to the Eq. 2.1, we have that Conjecture 3 for series \( A_l \) is true. Since the \( f \)-polynomial of type \( B_l \), up to a constant factor, coincides with the shifted Legendre polynomial \( \tilde{P}_l^{(0,0)}(t) \), due to Proposition 6.6 in \([12]\), we have that Conjecture 3 for series \( B_l \) is true. In Section 8, we will prove Conjecture 3 for types \( D_l \) affirmatively. The proof is divided into two parts and is more complicated. In Part I, we discuss location of the following roots \( t_{D_l,l+1,v}^+ \) and \( t_{D_l,l+1,v}^- \) \((v = 1, \ldots, \lfloor l/2 \rfloor)\). In Part II, we discuss location of the following roots \( t_{D_l,l/2+1,v}^- \) and \( t_{D_l,l/2+1,v}^+ \) \((v = 1, \ldots, \lfloor l/2 \rfloor)\).

2 Rodrigues Type Formulae and Orthogonal Polynomials

In this section, we first prepare some useful propositions for later use. Next, for the three infinite series \( A_l, B_l \), and \( D_l \) of \( f \)-polynomials, we show that they are expressed in higher (logarithmic) derivatives of the polynomials of the form \( t^p(1 - t)^q \). In analogy with the classical Rodrigues’s formula \([19]\), we call the formula Rodrigues type formula. As a consequence of the formulae, the polynomials are expressed by a use of orthogonal polynomials.

**Proposition 2.1**

1. For type \( A_l \), the following identity holds for \( l = 1, 2, \ldots: \)

\[
(1 - t) f_{A_l}(t) = f_{A_{l+1}}^+ (t). \tag{2.1}
\]

2. For type \( B_l \), the following identity holds for \( l = 2, 3, \ldots: \)

\[
f_{B_l}(t) + f_{B_{l-1}}(t) = 2 f_{B_l}^+(t). \tag{2.2}
\]
3. The following identity holds for \( l = 4, 5, \ldots \):

\[
f_{D_l}(t) = \frac{l-2}{2(l-1)} f_{B_l}(t) + \frac{l}{2(l-1)} (1-2t) f_{B_{l-1}}(t).
\]

(2.3)

**Proof** 1. From Table A, the coefficient of \((-t)^k\) in \((1-t) f_{A_l}(t)\) is computed as

\[
\frac{1}{l+2} \binom{l}{k} (l+k+2) + \frac{1}{l+2} \binom{l}{k-1} (l+k+1) = \frac{1}{l+1} \binom{l+1}{k} (l+k+1).
\]

This coincides with the coefficient of \((-t)^k\) of \(N_{G_{A_{l+1}}}^+(t)\) in Table A in [12].

2. From Table A, the coefficient of \((-t)^k\) on the LHS of Eq. 2.2 is

\[
\binom{l}{k} (l+k) + \binom{l}{k-1} (l+k-1) = 2 \binom{l}{k} (l+k-1).
\]

This coincides with the coefficient of \((-t)^k\) of \(2 f_{B_l}^+(t) (= 2N_{G_{B_l}^+}(t))\) in Table A in [12].

3. From Table A, the coefficient of \((-t)^k\) on the RHS of Eq. 2.3 is

\[
\frac{l-2}{2(l-1)} \frac{(l+k)!}{(l-k)!k!} + \frac{l}{2(l-1)} \frac{(l+k-1)!}{(l-1-k)!k!} + \frac{l(l+k-2)!}{(l-1)!}\frac{1}{(l+k)!} = \frac{(l+k-2)!}{(l-k)!k!} (l(k+1)+k(k-1)) .
\]

This coincides with the coefficient of \((-t)^k\) on the left hand side in Table A.

**Theorem 2.2** (Rodrigues type formula) For types \( A_l \) \((l \geq 1)\), \( B_l \) \((l \geq 2)\), and \( D_l \) \((l \geq 4)\), we have the formulae:

\[
t(1-t) f_{A_l}(t) = \frac{1}{(l+1)!} \frac{d^l}{dt^l} \left[ t^{l+1}(1-t)^{l+1} \right],
\]

(2.4)

\[
f_{B_l}(t) = \frac{1}{l!} \frac{d^l}{dt^l} \left[ t^l (1-t)^l \right],
\]

(2.5)

\[
f_{D_l}(t) = \frac{1}{(l-1)!} \frac{d^{l-1}}{dt^{l-1}} \left[ t^{l-1}(1-t)^l \right] + \frac{1}{(l-2)!} \frac{d^{l-2}}{dt^{l-2}} \left[ t^l (1-t)^{l-2} \right] = \frac{1}{(l-1)!} \frac{d^{l-2}}{dt^{l-2}} \left[ t^{l-2}(1-t)^{l-2} \right] \left\{ (l-1) - (3l-2)t + (3l-2)t^2 \right\}.
\]

(2.6)

**Proof** Type \( A_l \): Due to Theorem 2.1 in [12], the right hand side of Eq. 2.4 coincides with \(t f_{A_{l+1}}^+(t)\). Thanks to Eq. 2.1, this coincides with the left hand side of Eq. 2.4.

Type \( B_l \): The right hand side of Eq. 2.5 is calculated as

\[
\frac{1}{l!} \frac{d^l}{dt^l} \left[ t^l (1-t)^l \right] = \frac{1}{l!} \frac{d^l}{dt^l} \left[ \sum_{k=0}^{l} (-1)^k \binom{l}{k} t^l + k \right] = \sum_{k=0}^{l} (-1)^k \frac{(l+k)!}{(l-k)!k!} k.
\]
This gives RHS of the expression of \( f_{Bl}(t) \) in Table A.

**Type \( D_l \):** We compute the right hand side of Eq. 2.6.

\[
\frac{1}{(l-1)!} \frac{d^{l-1}}{dt^{l-1}} \left[ \sum_{k=0}^{l} (-1)^k \binom{l}{k} t^l + k \right] + \frac{1}{(l-2)!} \frac{d^{l-2}}{dt^{l-2}} \left[ \sum_{k=0}^{l-2} (-1)^k \binom{l-2}{k} t^l + k \right]
\]

\[
= \sum_{k=0}^{l} (-1)^k \binom{l}{k} \frac{l(l+k-1)!}{(l-k)!k!} t^k + \sum_{k=0}^{l-2} (-1)^k \binom{l-2}{k} \frac{(l+k)!}{(l-2-k)!k!(k+2)!} t^{k+2}
\]

\[
= \sum_{k=0}^{l} (-1)^k \binom{l}{k} \frac{l(l+k-1)!}{(l-k)!k!} t^k + \sum_{k=2}^{l} (-1)^k \binom{l+k-2}{k} \frac{(l+k-2)!}{(l-k)!k!(k-2)!} t^k
\]

\[
= \sum_{k=0}^{l} (-1)^k \binom{l}{k} \left( \binom{l+k-1}{k} + \binom{l-2}{k} \binom{l-2+k}{k} \right) t^k.
\]

This gives RHS of the expression of \( f_{Dl}(t) \) in Table A. 

For \( l \in \mathbb{Z}_{\geq 0} \) and \( \alpha, \beta \in \mathbb{R}_{\geq -1} \), let \( P_l^{(\alpha, \beta)}(x) \) be the Jacobi polynomial (c.f. [19, 2.4]). Let us introduce the shifted Jacobi polynomial of degree \( l \) by setting

\[
\tilde{P}_l^{(\alpha, \beta)}(t) := P_l^{(\alpha, \beta)}(2t - 1).
\]

**Fact 2.3 ([19, (4.3.1)])** The shifted Jacobi polynomial satisfies the following equality

\[
(t - 1)^\alpha t^\beta \tilde{P}_l^{(\alpha, \beta)}(t) = \frac{1}{l!} \frac{d^l}{dt^l} \left[ (t - 1)^{l+\alpha} t^{l+\beta} \right].
\]

Comparing two formulae in Theorem 2.2 and Fact 2.3, we obtain expression of the \( f \)-polynomials for types \( A_l, B_l, \) and \( D_l \) by shifted Jacobi polynomials.

**Corollary 2.4**

\[
f_{A_l}(t) = \frac{(-1)^l}{l+1} \tilde{P}_l^{(1,1)}(t), \quad \text{(2.7)}
\]

\[
f_{B_l}(t) = (-1)^l \tilde{P}_l^{(0,0)}(t), \quad \text{(2.8)}
\]

\[
f_{D_l}(t) = (-1)^{l-1} (1-t) \tilde{P}_{l-1}^{(1,0)}(t) + (-1)^l t^2 \tilde{P}_{l-2}^{(0,2)}(t). \quad \text{(2.9)}
\]

**Remark 2.5** From Eq. 2.3, we obtain the following expression of the \( f \)-polynomial for type \( D_l \) by shifted Legendre polynomials.

\[
(-1)^l f_{D_l}(t) = \frac{l-2}{2(l-1)} \tilde{P}_l^{(0,0)}(t) - \frac{l}{2(l-1)} (1-2t) \tilde{P}_{l-1}^{(0,0)}(t). \quad \text{(2.10)}
\]

We obtain the following functional equation.

**Corollary 2.6** For each type \( P \in \{A_l \ (l \geq 1), B_l \ (l \geq 2), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}, \) the \( f \)-polynomial satisfies the following equation

\[
f_P(t) = (-1)^l f_P(1-t),
\]

where \( l \) is the rank of \( P \).
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Proof Case I: types $A_l$, $B_l$, and $D_l$.

First, we recall an elementary fact: If a polynomial $g(t)$ satisfies $g(t) = g(1-t)$, then the function $f(t) := \frac{d}{dt} g(t)$ satisfies $f(t) = (-1)^l f(1-t)$. Applying this fact to the Rodrigues type formulae (2.4), (2.5), and (2.6), we obtain the equation $f_{Pl}(t) = (-1)^l f_{Pl}(1-t)$.

Case II: exceptional types $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$ and non-crystallographic types $H_3$, $H_4$, and $I_2(p)$.

By using the software package Mathematica on the Table B in Appendix I, we obtain the results. 

3 Recurrence Relations for Types $A_l$ ($l \geq 1$), $B_l$ ($l \geq 2$), and $D_l$ ($l \geq 4$)

As an application of the Rodrigues type formulae, we show that the the series of $f$-polynomials for types $A_l$ ($l \geq 1$), $B_l$ ($l \geq 2$), and $D_l$ satisfy either 3-term or 4-term recurrence relations (Theorem 3.1).

Theorem 3.1 For type $A_l$ and $B_l$, the following 3-term recurrence relations hold.

\[
(l+4)f_{A_{l+2}}(t) = (2l+5)(1-2t)f_{A_{l+1}}(t) - (l+1)f_{A_l}(t),
\]
\[
(l+2)f_{B_{l+2}}(t) = (2l+3)(1-2t)f_{B_{l+1}}(t) - (l+1)f_{B_l}(t).
\]

(3.1)

For type $D_l$, the following 4-term recurrence relation holds.

\[
f_{D_{l+3}}(t) = (a_l + b_l t)f_{D_{l+2}}(t) + (c_l + d_l t + e_l t^2)f_{D_{l+1}}(t) + (f_l + g_l t)f_{D_l}(t).
\]

(3.2)

Here, $a_l$, $b_l$, $c_l$, $d_l$, $e_l$, $f_l$, and $g_l$ are the following rational functions:

\[
a_l = \frac{(l+1)(5l^2 + 4l - 21)}{(l-1)(l+3)(5l + 4)},
\]
\[
b_l = \frac{2(l+1)(5l^2 + 4l - 21)}{(l-1)(l+3)(5l + 4)},
\]
\[
c_l = \frac{l(5l^2 + 14l + 5)}{(l-1)(l+3)(5l + 4)},
\]
\[
d_l = \frac{4l(2l + 1)(5l + 9)}{(l-1)(l+3)(5l + 4)},
\]
\[
e_l = \frac{4l(2l + 1)(5l + 9)}{(l-1)(l+3)(5l + 4)},
\]
\[
f_l = \frac{(l+1)(5l + 9)}{(l+3)(5l + 4)},
\]
\[
g_l = \frac{2(l+1)(5l + 9)}{(l+3)(5l + 4)}.
\]

Proof Let us consider the $k$th coefficient of $f_{D_l}(t)$ up to the sign $(-1)^k$:

\[
C(l, k) := \binom{l}{k} \left( \frac{l+k-1}{k} \right) + \binom{l-2}{k-2} \left( \frac{l+k-2}{k} \right)
\]
\[
= \frac{(l+k-2)!}{(l-k)!(k)!^2} \left( l(l+k-1) + k(k-1) \right).
\]
We compute the coefficient of the term \((-t)^k\) on the right hand side of Eq. 3.2.

\[
a_l \cdot C(l + 2, k) - b_l \cdot C(l + 2, k - 1) + c_l \cdot C(l + 1, k) - d_l \cdot C(l + 1, k - 1)
+ e_l \cdot C(l + 1, k - 2) + f_l \cdot C(l, k) - g_l \cdot C(l, k - 1)
\]

\[
= \frac{(l + 3 - k)!}{(l + 3 - k)! (k!)^2 (l - 1) (l + 3) (5l + 4)} \times \left[ (l + 1) (5l^2 + 4l - 21) (l + k) (l + k - 1) (l + k - 2) \right.
\]

\[
\times (l - k + 3) [(l + 2) (l + k + 1) + k (k - 1)]
+ 2 (l + 1) (5l^2 + 4l - 21) k^2 (l + k - 1) (l + k - 2)
\times (l + 2) (l + k) + (k - 1) (k - 2)]
+ l (5l^2 + 14l + 5) (l - k + 2) (l + k - 1) (l + k - 2)
\times (l + 1) (l + k + k (k - 1))
+ 4l (2l + 1) (5l + 9) k^2 (l + k - 2) (l - k + 3)
\times (l + 1) (l + k - 1) + (k - 1) (k - 2)
\]

\[
= \frac{(l + k + 1)!}{(l + 3 - k)! (k!)^2} (6 + 2k + k^2 + 5l + kl + l^2)
\]

\[
= C(l, k).
\]

As an application of the recurrence relation, we observe the following.

**Corollary 3.2** For each types \(P_l = A_l (l \geq 1), B_l (l \geq 2)\), the \(f\)-polynomial \(f_{Pl}(t)\) is divisible by \(2t - 1\) if and only if \(l\) is odd.

Therefore, due to Eq. 2.3, we obtain the following.

**Remark 3.3** The \(f\)-polynomial \(f_{D_l}(t)\) \((l \geq 4)\) is divisible by \(2t - 1\) if and only if \(l\) is odd.

Although for each types \(P_l = A_l (l \geq 1), B_l (l \geq 2)\) the \(f\)-polynomial \(f_{Pl}(t)\) is a solution of the Gauss hypergeometric differential equation, the \(f\)-polynomial \(f_{D_l}(t)\) is a solution of the following Fuchsian equation of third-order.

**Remark 3.4** The \(f\)-polynomial \(f_{D_l}(t)\) satisfies the following Fuchsian ordinary differential equation of third-order. The proof is left to the reader.

\[
t(t - 1)(2t - 1) \frac{d^3 y}{dt^3} + [(l + 6) (t^2 - t) + 2] \frac{d^2 y}{dt^2} - l(l - 1)(2t - 1) \frac{dy}{dt} - l(l - 1)(l + 2) y = 0.
\]
4 Another Proof of Brändén’s Results Except for Types $D_l$

In the present section, we prove, except for types $D_l$, the following theorem, which approves Brändén’s results in another manner. The proof for types $D_l$ will be given in the next Section 5.

**Theorem 4.1** The $f$-polynomial $f_P(t)$ for any finite type $P$ has rank$(P)$ simple roots on the interval $(0, 1)$.

**Proof** Case I: type $A_l$ ($l \in \mathbb{Z}_{\geq 1}$) and $B_l$ ($l \in \mathbb{Z}_{\geq 1}$).

This is an immediate consequence of the formulae (2.7) and (2.8), since the Jacobi polynomials $\tilde{P}_l^{(1,1)}$ and $\tilde{P}_l^{(0,0)}$ are well known to have $l$ simple roots on the interval $(0, 1)$ (see [19, Theorem 3.3.1]).

Case II: Exceptional types and non-crystallographic types

We apply the Euclid division algorithm for the pair of polynomials $f_0 := f_P$ and $f_1 := f_P'$. So, we obtain, a sequence $f_0, f_1, f_2, \ldots$ of polynomials in $t$ such that $f_k = f_{k-1} q_{k-1} + f_{k+1}$ for $k = 1, 2, \ldots$ (where $q_{k-1}$ is the quotient and $f_{k+1}$ is the remainder).

Then, we prove the following fact by direct calculations case by case.

**Fact 4.2** (i) The degrees of the sequence $f_0, f_1, f_2, \ldots$ of polynomials descend one by one, and $f_l$ is a non-zero constant.

(ii) The sequence $f_0(0), f_1(0), -f_2(0), \ldots, (-1)^{l-1} f_l(0)$ has constant sign, and the sequence $f_0(1), f_1(1), -f_2(1), \ldots, (-1)^{l-1} f_l(1)$ has alternating sign.

Applying the Sturm theorem (see for instance [20, Theorem 3.1]), we observe that $f_0$ has $l$ distinct roots on the interval (0, 1).

This completes a proof of Theorem 4.1.

5 Another Proof of Brändén’s Results for Types $D_l$ ($l \geq 4$)

In this section, we prove the following theorem, which approves Brändén’s results for types $D_l$ ($l \geq 4$) in another manner.

**Theorem 5.1** The polynomial $f_{D_l}(t)$ has $l$ simple roots on the interval $(0, 1)$.

**Proof** First, let $t_{B_l,v}, v = 1, 2, \ldots, l$, be the zeros of $f_{B_l}(t)$ in decreasing order (i.e., $1 > t_{B_l,1} > t_{B_l,2} > \cdots > t_{B_l,l} > 0$). Then, the following fact is known (see [19, Theorem 3.3.2]).

**Fact 5.2** The system $\{t_{B_l,v}\}$ alternates with the system $\{t_{B_l,v} + 1\}$, that is,

$$t_{B_l,v} + 1 > t_{B_l,v} > t_{B_l,v} + 1, \quad (v = 1, \ldots, l).$$

**Case I:** $l = 2k$

We consider $2k$ open intervals $(t_{B_{2k,1}} - 1, v, t_{B_{2k-1,2k-v}}) (v = 1, 2, \ldots, k)$ and $(t_{B_{2k-1,2k-v}, t_{B_{2k,k+1-v}}}) (v = 1, 2, \ldots, k)$. We note that $t_{B_{2k-1,k}} = 1/2$. On the intervals $(t_{B_{2k,2k+1-v}, t_{B_{2k-1,2k-v}}}) (v = 1, 2, \ldots, k)$, the polynomials $f_{B_{2k}}(t)$ and
Corollary 5.3

1. For the case where $l = 2k$, we obtain the following properties for $v = 1, 2, \ldots, k$:

   $t_{D_{2k}}, 2k+1-v \in (t_{B_{2k}}, 2k+1-v, t_{B_{2k-1}}, 2k-v),$
   $t_{D_{2k}}, k+1-v \in (t_{B_{2k-1}}, k+1-v, t_{B_{2k}}, k+1-v).$

2. For the case where $l = 2k + 1$, we obtain the following properties for $v = 1, 2, \ldots, k$:

   $t_{D_{2k+1}}, 2k+2-v \in (t_{B_{2k+1}}, 2k+2-v, t_{B_{2k}}, 2k+1-v), t_{D_{2k+1}}, k+1 = 1/2,$
   $t_{D_{2k+1}}, k+1-v \in (t_{B_{2k}}, k+1-v, t_{B_{2k+1}}, k+1-v).$

6 Proof of Conjecture 2

In this section, we prove the following theorem, which approve Conjecture 2. Let us fix notation: for $P \in \{ A_l \ (l \geq 1), B_l \ (l \geq 1), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, f_2(p) \ (p \geq 3)\}$, let $t_{P,v}, v = 1, 2, \ldots, l = \text{rank}(P)$ be the zeros of $f_P(t)$ in decreasing order (i.e., $1 > t_{P,1} > t_{P,2} > \cdots > t_{P,l} > 0$).

Theorem 6.1 For each series of types $P_l = A_l, B_l, D_l$, the smallest zero locus of $f_P(t)$ monotonously decreasingly converges to zero as the rank $l$ tends to infinity.

Proof Type $A_l$: Due to Theorem 6.1 in [12], the smallest zero locus of $f_{A_{l+1}}^+(t)$ monotonously decreasingly converges to zero as the rank $l$ tends to infinity. Hence, from Eq. 2.1, we show that the smallest zero locus of $f_{A_l}(t)$ monotonously decreasingly converges to zero. Type $B_l$: Recall a fact on the distribution of the zeros of $\tilde{P}_l^{(0,0)}(t)$ ($= (-1)^l f_{B_l}(t)$) ([19, Theorem 6.21.3]).

Fact 6.2 (Bruns [5]) Let $\bar{x}_v = \bar{x}_{l,v}, v = 1, 2, \ldots, l$, be the zeros of $\tilde{P}_l^{(0,0)}(t)$ in decreasing order. Let $\theta_v = \theta_{l,v} \in (0, \pi), v = 1, 2, \ldots, l$, be the real number defined by

$$\cos \theta_v = 2\bar{x}_v - 1.$$
Then, the inequalities hold as follows: \( \frac{v-1}{l+\frac{1}{2}} \pi < \theta_v < \frac{v}{l+\frac{1}{2}} \pi \) \((v = 1, 2, \ldots, l)\).

Therefore, we have that the smallest zero locus of \( f_{Bl}(t) \) monotonously decreasingly converges to zero.

Type \( D_l \): Due to Corollary 5.3, we show that the smallest zero locus \( t_{D_l,l} \) of \( f_{D_l}(t) \) is an element of the interval \((t_{Bl,l}, t_{Bl,l-1})\). Therefore, we conclude that the smallest zero locus \( t_{D_l,l} \) monotonously decreasingly converges to zero.

### 7 Proof of Conjecture 3 Except for Types \( D_l \)

In this section, we prove, except for types \( D_l \), the following theorem, which approves Conjecture 3. The proof for types \( D_l \) will be given in the next Section 8. Let us fix notation: for \( P \in \{ A_l \ (l \geq 1), B_l \ (l \geq 1), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\} \), let \( t_{P,v} = 1, 2, \ldots, l = \text{rank}(P) \), be the zeros of \( f_P(t) \) in decreasing order (i.e., \( 1 = t_{P,1} > t_{P,2} > \cdots > t_{P,l} > 0 \)).

**Theorem 7.1** The following inequalities hold for any type \( P \)

\[
 t_{P,v} > t_{P,v+1} > t_{P,v+1}, \quad (v = 1, \ldots, l-1).
\]

**Proof** I. Case for types \( A_l \) and \( B_l \)

First, for type \( A_l \), we recall the following fact (see [19, Theorem 3.3.2]).

**Fact 7.2** The system \( \{ t_{A_l,v}^+, v = 2 \} \) alternates with the system \( \{ t_{A_l+1,v}^+, v = 1 \} \), that is,

\[
 t_{A_l,v}^+ > t_{A_l,v}^+ > t_{A_l+1,v+1}^+, \quad (v = 2, \ldots, l).
\]

Hence, from Eq. 2.1, we have the following inequalities

\[
 t_{A_l,v} > t_{A_l,v+1} > t_{A_l,v+1}, \quad (v = 1, \ldots, l-1).
\]

Next, we recall \( f_{Bl}(t) = (-1)^{l} P_l^{(0,0)}(t) \). Due to Proposition 6.6 in [12], we have the following inequalities:

\[
 t_{Bl,v} > t_{Bl,v+1} > t_{Bl,v+1}, \quad (v = 1, \ldots, l-1).
\]

II. Exceptional types and non-crystallographic types.

In Section 4, for \( P \in \{ E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\} \), we constructed a Strum sequence \( f_0(t), f_1(t), -f_2(t), \ldots, (-1)^{l-1} f_l(t) \) on \([0, 1]\). In [12, Section 4], for the pair of polynomials \( f_0^+ := f_P(t)/(1-t) \) and \( f_1^+ := (f_0)' \), we constructed a Strum sequence \( f_0^+(t), f_1^+(t), -f_2^+(t), \ldots, (-1)^{l-1} f_l^+(t) \) on \([0, 1]\). For \( t_0 \in [0, 1] \), let \( V(f_P, t_0) \) (resp. \( V(f_P^+, t_0) \)) be the number of sign changes in the sequence \( f_0(t), f_1(t), -f_2(t), \ldots, (-1)^{l-1} f_l(t) \) (resp. \( f_0^+(t), f_1^+(t), -f_2^+(t), \ldots, (-1)^{l-2} f_l^+(t) \)). For \( t_0 \in [0, 1] \), we put

\[
 V(f_P, t_0) := V(f_P, t_0) - V(f_P, t_0), \quad V(f_P^+, t_0) := V(f_P^+, t_0) - V(f_P^+, t_0).
\]

Then, we prove the following fact case by case.
Fact 7.3 There exist sequences \( \{ \alpha_{P,v} \}_{v=1}^{l-1} \) and \( \{ \alpha_{P,v}^+ \}_{v=1}^{l-1} \) of real numbers which satisfy inequalities
\[ 0 < \alpha_{P,j} < \alpha_{P,j}^+ < \cdots < \alpha_{P,1} < \alpha_{P,1}^+ < 1 \]
\[ \overline{V}(f_P, \alpha_{P,i}) = l - i, \quad \overline{V}(f_P, \alpha_{P,i}^+) = l - i, \]
\[ \overline{V}(f_P, \alpha_{i,P}) = l - i - 1, \quad \overline{V}(f_P, \alpha_{i,P}^+) = l - i, \quad (i = 1, \ldots, l - 1). \]

Therefore, due to the Sturm theorem (see for instance [20, Theorem 3.1]), we have that
\[ t_{P,v} \in (\alpha_{P,v}, \alpha_{P,v}^-) \quad \text{and} \quad t_{P,v}^+ \in (\alpha_{P,v}^-, \alpha_{P,v}^-) \quad (v = 2, \ldots, l), \]
where we put \( \alpha_{P,j}^- := 0. \) Hence, we have the following inequalities:
\[ t_{P,v} > t_{P,v+1}^+ > t_{P,v+1}, \quad (v = 1, \ldots, l - 1). \]

Example For each \( P \in \{ E_l \mid (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) (p \geq 3) \}, \) we give an example of two kinds of sequences \( \{ \alpha_{P,v} \}_{v=1}^{l-1} \) and \( \{ \alpha_{P,v}^+ \}_{v=1}^{l-1} \).

\[ E_6: \]
\[ \alpha_{E_6,5} = 7/200, \alpha_{E_6,5}^+ = 1/10, \alpha_{E_6,4} = 21/100, \alpha_{E_6,4}^+ = 1/4, \alpha_{E_6,3} = 2/5, \]
\[ \alpha_{E_6,3}^+ = 3/5, \alpha_{E_6,2} = 13/20, \alpha_{E_6,2}^+ = 7/10, \alpha_{E_6,1} = 17/20, \alpha_{E_6,1}^+ = 9/10. \]

\[ E_7: \]
\[ \alpha_{E_7,6} = 1/50, \alpha_{E_7,6}^+ = 1/10, \alpha_{E_7,5} = 3/20, \alpha_{E_7,5}^+ = 1/5, \alpha_{E_7,4} = 8/25, \]
\[ \alpha_{E_7,4}^+ = 2/5, \alpha_{E_7,3} = 13/25, \alpha_{E_7,3}^+ = 3/5, \alpha_{E_7,2}^+ = 7/10, \alpha_{E_7,2} = 4/5, \]
\[ \alpha_{E_7,1}^+ = 87/100, \alpha_{E_7,1} = 19/20. \]

\[ E_8: \]
\[ \alpha_{E_8,7} = 19/2000, \alpha_{E_8,7}^+ = 1/100, \alpha_{E_8,6} = 11/100, \alpha_{E_8,6}^+ = 1/5, \alpha_{E_8,5} = 1/4, \]
\[ \alpha_{E_8,5}^+ = 3/10, \alpha_{E_8,4}^+ = 21/50, \alpha_{E_8,4} = 49/100, \alpha_{E_8,3}^+ = 3/5, \alpha_{E_8,3} = 7/10, \]
\[ \alpha_{E_8,2} = 77/100, \alpha_{E_8,2}^+ = 4/5, \alpha_{E_8,1} = 9/10, \alpha_{E_8,1}^+ = 19/20. \]

\[ F_3: \]
\[ \alpha_{F_3,3} = 1/20, \alpha_{F_3,3}^+ = 1/10, \alpha_{F_3,2} = 7/20, \alpha_{F_3,2}^+ = 2/5, \alpha_{F_3,1} = 7/10, \alpha_{F_3,1}^+ = 4/5. \]

\[ G_2: \alpha_{G_2,1} = 1/6, \alpha_{G_2,1}^+ = 1/2. \]

\[ H_3: \alpha_{H_3,2} = 7/100, \alpha_{H_3,2}^+ = 1/10, \alpha_{H_3,1} = 11/20, \alpha_{H_3,1}^+ = 3/5. \]

\[ H_4: \alpha_{H_4,2} = 9/500, \alpha_{H_4,2} = 1/5, \alpha_{H_4,2}^+ = 31/100, \alpha_{H_4,1} = 2/5, \alpha_{H_4,1}^+ = 7/10, \]
\[ \alpha_{H_4,1}^+ = 4/5. \]

\[ I_2(p \geq 3): \alpha_{I_2(p),1} = 1/p, \alpha_{I_2(p),1}^+ = 1/2. \]

8 Proof of Conjecture 3 for Types \( D_l \) \( (l \geq 4) \)

In the case where \( l = 4 \), we approve Conjecture 3 by hand calculation. Throughout this section, we assume \( l \geq 5. \)

Theorem 8.1 The following inequalities hold for \( v = 1, \ldots, l - 1 \)
\[ t_{D_l,v} > t_{D_l,v+1}^+ > t_{D_l,v+1}. \]

Proof First, we recall an equation from [12, Section 5]
\[ f_{D_l}(t) = (-1)^{l-3} (l - 2)! \left( \left( 1 - t \right) H_l^{(l-3)}(t) - (l - 3) H_l^{(l-4)}(t) \right). \]
First, we define two kinds of functions

1. By definition, we have

\[ H^{(i)}_l(t) := \frac{d^i}{dt^i} H_l(t) \quad \text{for } 0 \leq i \leq l - 3. \]

From Lemma 5.2 in [12], let \( u_1 > u_2 > \cdots > u_{l-1} > u_l = 0 \) be all roots of the polynomial \( H^{(l-4)}_l(t) = 0 \). Let \( 1 > v_1 > v_2 > \cdots > v_{l-1} > 0 \) be the \( l - 1 \) roots of \( H^{(l-3)}_l(t) = 0 \) so that one has the inequalities:

\[ u_1 > v_1 > u_2 > v_2 > \cdots > u_{l-1} > v_{l-1} > u_l. \]

Moreover, in the proof of Lemma 5.2 in [12], we have shown the following fact. \( \square \)

**Fact 8.2** \( t_{D_l,l+1-v}^+ \in (u_{l+1-v}, v_{l-v}) \), \( (v = 1, \ldots, l - 1) \) and \( t_{D_l,l}^+ = 1 \).

Next, the proof for Theorem 8.1 is divided into two parts.

**Part I** We discuss location of the following roots

\[ t_{D_l,l+1-v}^+ \quad \text{and} \quad t_{D_l,l+1-v}^+ (v = 1, \ldots, \lfloor l/2 \rfloor). \]

First, we define two kinds of functions

\[
\tilde{f}_{D_l}(t) := \frac{(-1)^{l-3}}{(l-2)!} \left( (1 - 2t)H^{(l-3)}_l(t) - 2(l - 3)H^{(l-4)}_l(t) \right),
\]

\[
K_l(t) := f_{D_l}^+(t) - \tilde{f}_{D_l}(t) = \frac{(-1)^{l-3}}{(l-2)!} \left( tH^{(l-3)}_l(t) + (l - 3)H^{(l-4)}_l(t) \right).
\]

For the polynomial \( \tilde{f}_{D_l}(t) \), we have the following proposition.

**Proposition 8.3** The following identities hold for \( l = 5, 6, \ldots: \)

\[
\tilde{f}_{D_l}(t) = \frac{1}{(l-2)!} \frac{d^{l-3}}{dt^{l-3}} \left[ t^{l-3}(1-t)^{l-3}(1-2t) \left( (l-2)-(3l-4)t+(3l-4)t^2 \right) \right], \tag*{(8.1)}
\]

\[
\tilde{f}_{D_l}(t) = \frac{l+2}{l} f_{D_l}(t) - \frac{2}{l} f_{B_l}(t). \tag*{(8.2)}
\]

\[
\tilde{f}_{D_l}(t) = (-1)^l \tilde{f}_{D_l}(1-t). \tag*{(8.3)}
\]

**Proof.** 1. By definition, we have

\[ 2f_{D_l}^+(t) = \tilde{f}_{D_l}(t) + \frac{(-1)^{l-3}}{(l-2)!} H^{(l-3)}_l(t). \]

We recall the Rodrigues type formula for \( f_{D_l}^+(t) \) from Theorem 5.1 in [12]

\[
f_{D_l}^+(t) = \frac{1}{(l-2)!} \frac{d^{l-3}}{dt^{l-3}} \left[ t^{l-3}(1-t)^{l-2} \left( (l-2)-(3l-4)t+(3l-4)t^2 \right) \right].
\]

By calculating \( 2f_{D_l}^+(t) - \frac{(-1)^{l-3}}{(l-2)!} H^{(l-3)}_l(t) \), we obtain the result.

2. From Eqs. 2.6 and 8.1, we have

\[ \tilde{f}_{D_l}(t) - f_{D_l}(t) = \frac{2}{(l-1)!} \frac{d^{l-2}}{dt^{l-2}} \left[ t^{l-1}(1-t)^{l-1} \right] \]

\[ = \frac{2}{l} f_{D_l}(t) - \frac{2}{l} f_{B_l}(t). \]

This completes the proof.
3. Since \( f_{B_l}(t) = (-1)^l f_{B_l}(1-t) \) and \( f_{D_l}(t) = (-1)^l f_{D_l}(1-t) \), we obtain the result.

\[ \square \]

Remark 8.4 As a consequence of Eq. 8.2, the polynomial \( \tilde{f}_{D_l}(t) \) is divisible by \( 2t - 1 \) if and only if \( l \) is odd.

We recall a formula from [12, Section 5].

**Formula B.** Set \( h^{(i)}_k(t) := \left( \frac{d}{dt} \right)^i (t(t-1))^k \) for \( 0 \leq i \leq k \). Then, we have

\[
\begin{align*}
    h^{(2i-1)}_k \left( \frac{1}{2} \right) &= 0 \quad (i = 1, \ldots, k), \\
    h^{(2i)}_k \left( \frac{1}{2} \right) &= \left( -\frac{1}{4} \right)^{k-i} \frac{k!(2i)!}{(k-i)!i!} \quad (i = 1, \ldots, k).
\end{align*}
\]

**Proof** of Formula B. By induction on \( i \), we obtain the following equations:

\[
\begin{align*}
    h^{(2i-1)}_k(t) &= \sum_{j=1}^{i} \frac{k!(2i-1)!}{(i-j)!(2j-1)!(k-i-j+1)!}(t^2-t)^{k-i-j+1}(2t-1)^{2j-1} \quad (8.4) \\
    h^{(2i)}_k(t) &= \sum_{j=0}^{i} \frac{k!(2i)!}{(i-j)!(2j)!(k-i-j)!}(t^2-t)^{k-i-j}(2t-1)^{2j}. \quad (8.5)
\end{align*}
\]

These equations hold for \( 0 \leq i \leq k \).

From Eqs. 8.4 and 8.5, we obtain the following formula.

**Formula C.** We have

\[
\begin{align*}
    h^{(l-3)}_{l-3}(0) &= (-1)^{l-1}(l-3)! \text{,} \quad h^{(l-3)}_{l-3}(0) = 0.
\end{align*}
\]

**Proposition 8.5** The polynomial \( \tilde{f}_{D_l}(t) \) has \( l \) simple roots on the interval \((0, 1)\).

**Proof** For the case where \( l = 2k \), we consider \( 2k \) open intervals \((t_{D_{2k,2k+1-v}}, t_{D_{2k,2k-v}}) (v = 1, 2, \ldots, k-1)\), \((t_{D_{2k,k+1}}, 1/2)\) and \((1/2, t_{D_{2k,k}})\), \((t_{D_{2k,k+1-v}}, t_{D_{2k,k-1-v}}) (v = 1, 2, \ldots, k-1)\). For a non-zero real number \( \theta \), we put

\[
\text{sgn}(\theta) := \theta/|\theta|.
\]

From the discussion in the proof of Theorem 5.1, we have shown the following properties

\[
\text{sgn}(f_{B_{2k}}(t_{D_{2k,2k+1-v}})) = (-1)^v \quad (v = 1, 2, \ldots, k).
\]

Moreover, due to **Formula B**, we have

\[
\begin{align*}
    f_{B_{2k}}(1/2) &= \left( -\frac{1}{4} \right)^{k} \frac{(2k)!}{k!k!} \quad \text{and} \quad f_{D_{2k}}(1/2) = \left( -\frac{1}{4} \right)^{k} \frac{(2k-2)(2k-2)!}{k!(k-1)!}.
\end{align*}
\]

Hence, we have

\[
\text{sgn}(\tilde{f}_{D_{2k}}(1/2)) = (-1)^k.
\]

\[\footnote{Although, in [12, Section 5], the index \( i \) for \( h^{(2i-1)}_k(t) \) (resp. \( h^{(2i)}_k(t) \)) ranges from 1 to \( \lfloor (k+1)/2 \rfloor \) (resp. \( \lfloor k/2 \rfloor \)), the range of the index \( i \) can be extended to \( 0 \leq i \leq k \).} \]
Due to the identity (8.2), we can show \( \tilde{f}_{D_{2k}}(t_{D_{2k},2k+1-v})f_{D_{2k}}(t_{D_{2k},2k-v}) < 0 \), \( (v = 1, \ldots, k − 1) \) and \( \tilde{f}_{D_{2k}}(t_{D_{2k},k+1})f_{D_{2k}}(1/2) < 0 \). Thanks to intermediate value theorem, for each interval \( (t_{D_{2k},2k+1-v}, t_{D_{2k},2k-v})(v = 1, 2, \ldots, k − 1) \) and \( (t_{D_{2k},2k+1}, 1/2) \), there exists at least one root of \( \tilde{f}_{D_{2k}}(t) \). From Eq. (8.3), the set of roots of \( \tilde{f}_{D_{2k}}(t) \) are symmetric with respect to the reflection centered at \( t = 1/2 \). Therefore, we have that for each interval \( (t_{D_{2k},2k+1-v}, t_{D_{2k},2k-v})(v = 1, 2, \ldots, k − 1) \) and \((1/2, t_{D_{2k},k})\), there exists at least one root of \( \tilde{f}_{D_{2k}}(t) \). Since the polynomial \( \tilde{f}_{D_{2k}}(t) \) is of precise degree \( 2k \), we conclude that, in each intervals \( (t_{B_{2k},2k+1-v}, t_{B_{2k},2k-v})(v = 1, 2, \ldots, k − 1) \), \((1/2, t_{D_{2k},k})\), and \((t_{B_{2k-k-1,2k+1-v}}, t_{B_{2k-k-1,2k+1-v}})(v = 1, 2, \ldots, k − 1) \), there is one and only one root of the polynomial \( \tilde{f}_{D_{2k}}(t) \). For the case where \( l = 2k + 1 \), in a similar manner to case where \( l = 2k \), we conclude that the polynomial \( \tilde{f}_{D_{l}}(t) \) has \( l \) simple roots on the interval \( (0, 1) \).

Let \( \tilde{t}_{D_{l},v} = 1, 2, \ldots, l \), be the zeros of \( \tilde{f}_{D_{l}}(t) \) in decreasing order (i.e., \( 1 > \tilde{t}_{D_{l},1} > \tilde{t}_{D_{l},2} > \cdots > \tilde{t}_{D_{l},l} > 0 \)).

**Proposition 8.6** 1. For the case where \( l = 2k \), we obtain the following properties for \( v = 1, 2, \ldots, k − 1 \):

\[
\begin{align*}
\tilde{t}_{D_{2k},2k+1-v} &\in (t_{D_{2k},2k+1-v}, t_{D_{2k},2k-v}), \\
\tilde{t}_{D_{2k},k+1} &\in (t_{D_{2k},k+1}, 1/2), \\
\tilde{t}_{D_{2k},k} &\in (1/2, t_{D_{2k},k}), \\
\tilde{t}_{D_{2k},k+1-v} &\in (t_{D_{2k},k+1-v}, t_{D_{2k},k-v}).
\end{align*}
\]

2. For the case where \( l = 2k + 1 \), we obtain the following properties for \( v = 1, 2, \ldots, k − 1 \):

\[
\begin{align*}
\tilde{t}_{D_{2k+1},2k+2-v} &\in (t_{D_{2k+1},2k+2-v}, t_{D_{2k+1},2k+1-v}), \\
\tilde{t}_{D_{2k+1},k+2} &\in (t_{D_{2k+1},k+2}, 1/2), \\
\tilde{t}_{D_{2k+1},k+1} &\in (1/2, t_{D_{2k+1},k+1}), \\
\tilde{t}_{D_{2k+1},k+1-v} &\in (t_{D_{2k+1},k+1-v}, t_{D_{2k+1},k-v}).
\end{align*}
\]

**Proposition 8.7** 1. For the case where \( l = 2k \), we obtain the following properties for \( v = 1, 2, \ldots, k \):

\[
\begin{align*}
\tilde{t}_{D_{2k},2k+1-v} &\in (u_{2k+1-v}, v_{2k-v}).
\end{align*}
\]

2. For the case where \( l = 2k + 1 \), we obtain the following properties for \( v = 1, 2, \ldots, k \):

\[
\begin{align*}
\tilde{t}_{D_{2k+1},2k+2-v} &\in (u_{2k+2-v}, v_{2k+1-v}), \\
\tilde{t}_{D_{2k+1},k+1} &\in (1/2).
\end{align*}
\]

**Proof** By applying intermediate value theorem to the polynomial

\[
\tilde{f}_{D_{l}}(t) = \left(\frac{-1}{(l-2)!}\right) \left((1-2t)H^{(l-3)}_{l}(t) - 2(l-3)H^{(l-4)}_{l}(t)\right).
\]

we obtain the results. We omit details.

**Proposition 8.8** 1. The polynomial \( K_{l}(t) \) has \( l \) simple roots on the interval \( (0, 1) \).

2. Let \( t_{K_{l},v}, v = 1, 2, \ldots, l \), be the zeros of \( K_{l}(t) \) in decreasing order (i.e., \( 1 > t_{K_{l},1} > t_{K_{l},2} > \cdots > t_{K_{l},l} = 0 \)). For \( v = 1, 2, \ldots, l - 1 \), we have

\[
t_{K_{l},l} = 0, t_{K_{l},l-v} \in (v_{l-v}, u_{l-v}).
\]

3. \( \frac{d}{dt}K_{l}(t)|_{t=0} = l - 2 \).
Proof. 1.–2. By applying intermediate value theorem to the polynomial $K_t(t)$, we obtain the results. We omit details.

3. By definition, we have $K_t'(0) = \frac{(-1)^{l-3}}{(l-3)!} H_t^{(l-3)}(0)$.

Due to Formula C, we have

$$H_t^{(l-3)}(0) = (l - 2)h_t^{(l-3)}(0) + (3l - 4)h_t^{(l-3)}(0) = (-1)^{l+1}(l - 2)!.$$  

Hence, we obtain $\frac{d}{dt} K_t(t) \bigg|_{t=0} = l - 2$. \qed

**Proposition 8.9** 1. For the case where $l = 2k$, we obtain the following properties for $v = 1, 2, \ldots, k - 1$:

$$t_{D_{2k},2k+1-v}^+ \in (\bar{t}_{D_{2k},2k+1-v}, \bar{t}_{D_{2k},2k-v}), t_{D_{2k},k+1}^+ \in (\bar{t}_{D_{2k},k+1}, 1/2).$$

2. For the case where $l = 2k + 1$, we obtain the following properties for $v = 1, 2, \ldots, k - 1$:

$$t_{D_{2k+1},2k+2-v}^+ \in (\bar{t}_{D_{2k+1},2k+2-v}, \bar{t}_{D_{2k+1},2k+1-v}), t_{D_{2k+1},k+2}^+ \in (\bar{t}_{D_{2k+1},k+2}, 1/2).$$

**Proof** For the case where $l = 2k$, we consider $k$ open intervals $(\bar{t}_{D_{2k},2k+1-v}, \bar{t}_{D_{2k},2k-v})$ ($v = 1, 2, \ldots, k - 1$), $(\bar{t}_{D_{2k},k+1}, 1/2)$. From Proposition 8.8, we have shown the following properties:

$$\text{sgn}(K_{2k}(\bar{t}_{D_{2k},2k+1-v})) = (-1)^{v+1}, \ (v = 1, \ldots, k).$$

Since $f_{D_{2k}}^+(t) = \tilde{f}_{D_{2k}}(t) + K_{2k}(t)$, we have

$$\text{sgn}(f_{D_{2k}}^+(\bar{t}_{D_{2k},2k+1-v})) = (-1)^{v+1}, \ (v = 1, \ldots, k).$$

Moreover, due to Formula B, we have

$$f_{D_{2k}}^+(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k - 4)(2k - 3)!}{k!(k - 2)!}.$$  

From [12, Section 5], we have

$$\text{sgn}(f_{D_{2k}}^+(v_{2k-v})) = (-1)^v, \ (v = 1, 2, \ldots, k).$$

From Fact 8.2, we obtain the following results for $v = 1, 2, \ldots, k - 1$:

$$t_{D_{2k},2k+1-v}^+ \in (\bar{t}_{D_{2k},2k+1-v}, v_{2k-v}), t_{D_{2k},k+1}^+ \in (\bar{t}_{D_{2k},k+1}, 1/2).$$

From Proposition 8.7, we also have for $v = 1, 2, \ldots, k - 1$:

$$t_{D_{2k+1},2k+2-v}^+ \in (\bar{t}_{D_{2k+1},2k+2-v}, v_{2k+2-v}), t_{D_{2k+1},k+2}^+ \in (\bar{t}_{D_{2k+1},k+2}, 1/2).$$

For the case where $l = 2k + 1$, we consider $k$ open intervals $(\bar{t}_{D_{2k+1},2k+2-v}, t_{D_{2k+1},k+2-v})(v = 1, 2, \ldots, k - 1)$, $(\bar{t}_{D_{2k+1},k+2}, 1/2)$. From Proposition 8.8, we have shown the following properties:

$$\text{sgn}(K_{2k+1}(\bar{t}_{D_{2k+1},2k+2-v})) = (-1)^{v+1}, \ (v = 1, \ldots, k).$$

Since $f_{D_{2k+1}}^+(t) = \tilde{f}_{D_{2k+1}}(t) + K_{2k+1}(t)$, we have

$$\text{sgn}(f_{D_{2k+1}}^+(\bar{t}_{D_{2k+1},2k+2-v})) = (-1)^{v+1}, \ (v = 1, \ldots, k).$$

Moreover, due to Formula B, we have

$$f_{D_{2k+1}}^+(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k - 1)!}{k!(k - 1)!}.$$  

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From [12, Section 5], we have
\[ \text{sgn}(f^+_{2k+1}(v_{2k+1-v})) = (-1)^v, \ (v = 1, 2, \ldots, k). \]

From Fact 8.2, we obtain the following results for \( v = 1, 2, \ldots, k - 1: \)
\[ t^+_{D2k+1,2k+2-v} \in (\tilde{t}_D_{2k+1,2k+2-v}, v_{2k+1-v}), \ t^+_{D2k+1,k+2} \in (\tilde{t}_D_{2k+1,k+2}, 1/2). \]

From Proposition 8.7, we also have for \( v = 1, 2, \ldots, k - 1: \)
\[ t^+_{D2k+1,2k+2-v} \in (\tilde{t}_D_{2k+1,2k+2-v}, \tilde{t}_D_{2k+1,2k+1-v}), \ t^+_{D2k+1,k+2} \in (\tilde{t}_D_{2k+1,k+2}, 1/2). \]

**Lemma 8.10**
\[ f^+_{D_l}(t) = \frac{1}{2(l - 1)}(lt - 2)f_{B_l}(t) + \frac{1}{2(l - 1)}\{2(2l - 1)t^2 - 5lt + 2l\}f_{B_l-1}(t). \quad (8.6) \]

**Proof** First, we recall an equality from (6.4) in [12]
\[ f^+_{D_l}(t) = \frac{l - 2}{2l - 1}f_{B_l}(t) + \left(\frac{l + 1}{2l - 1} - t\right)f^+_{B_l-1}(t). \]
Second, from Eq. 2.2, we have
\[ \frac{l - 2}{4l - 2}\left(f_{B_l}(t) + f_{B_l-1}(t)\right) + \frac{1}{2}\left(\frac{l + 1}{2l - 1} - t\right)\left(f_{B_l-1}(t) + f_{B_l-2}(t)\right). \]
Moreover, \( f_{B_l}(t) = (-1)^l \tilde{P}^{(0,0)}_l(t) \) satisfies the 3-term recurrence relation
\[ l f_{B_l}(t) = (2l - 1)(1 - 2t)f_{B_l-1}(t) - (l - 1)f_{B_l-2}(t). \]
Hence, we obtain
\[ \frac{l - 2}{4l - 2}\left(f_{B_l}(t) + f_{B_l-1}(t)\right) + \frac{1}{2}\left(\frac{l + 1}{2l - 1} - t\right)\left(f_{B_l-1}(t) + f_{B_l-2}(t)\right) \]
\[ = \frac{1}{2(l - 1)}(lt - 2)f_{B_l}(t) + \frac{1}{2(l - 1)}\{2(2l - 1)t^2 - 5lt + 2l\}f_{B_l-1}(t). \]
This coincides with the right hand side of Eq. 8.6.

As a corollary of Lemma 8.10, we have.

**Corollary 8.11** \( t^+_{D_l,l+1-v} \in (t_{B_l,l+1-v}, t_{B_l,l-v}), \ (v = 1, \ldots, l - 1), \ t^+_{D_l,1} = 1. \)

**Proof** We note that the coefficient \( \frac{l - 2}{4l - 2}(2l - 1)^2 - 5lt + 2l \) in Eq. 8.6 takes positive values on the interval \((0, 1)\). Due to Lemma 8.10, the sequence \( f^+_{D_l}(t_{B_l,l}), f^+_{D_l}(t_{B_l,l-1}), \ldots, f^+_{D_l}(t_{B_l,1}) \) has alternating sign. By applying intermediate value theorem to the polynomial \( f^+_{D_l}(t) \), we obtain the results.

Lastly, we prove the following proposition.

**Proposition 8.12** We obtain the following inequalities for \( v = 1, 2, \ldots, \lfloor l/2 \rfloor - 1: \)
\[ t^+_{D_l,l+1-v} < t^+_{D_l,l-v}. \]

**Proof** By combining Corollary 5.3 and Corollary 8.11, we obtain the results.
Remark 8.13 1. For $l = 2k$, we have $1/2 < t_{D_{2k}}^+$. 
2. For $l = 2k + 1$, we have $1/2 < t_{D_{2k+1}}^+$. 

Proof 1. This is an immediate consequence of Corollary 5.3. 
2. Since $t_{B_{2k+1}} = 1/2$, from Corollary 8.11, we obtain the result.

Part II 
We discuss location of the following roots:

$t_{D_i}, \lceil l/2 \rceil + 1 - \nu$ and $t_{D_i}, \lceil l/2 \rceil + 1 - \nu$ ($\nu = 1, \ldots, \lceil l/2 \rceil$).

We recall the assumption $l \geq 5$.

Theorem 8.14 The following inequalities hold for $\nu = 1, \ldots, \lceil l/2 \rceil - 1$

$t_{D_i}, \lfloor l/2 \rfloor - \nu > t_{D_i}, \lceil l/2 \rceil + \nu > t_{D_i}, \lceil l/2 \rceil - \nu$.

Proof First, we prepare a proposition.

Proposition 8.15 We have the following properties:

$t_{D_i}, \lceil l/2 \rceil + 1 - \nu \in (t_{B_i}, \lceil l/2 \rceil + 1 - \nu, t_{B_{i-1}}, \lceil l/2 \rceil - \nu)$, ($\nu = 1, \ldots, \lceil l/2 \rceil - 1$).

Proof First, from Corollary 8.11, we obtain

$t_{D_i}, \lceil l/2 \rceil + 1 - \nu \in (t_{B_i}, \lceil l/2 \rceil + 1 - \nu, t_{B_{i-1}}, \lceil l/2 \rceil - \nu)$, ($\nu = 1, \ldots, \lceil l/2 \rceil - 1$).

Next, we note that the coefficients $lt - 2$ and $\frac{1}{l(l-1)} (2(2l - 1)t^2 - 5lt + 2l)$ in Eq. 8.6 take positive values on the interval $(2/l, 1)$. Hence, by applying intermediate value theorem to the polynomial $f_{D_i}^+(t)$, we obtain the results.

Next, from Corollary 5.3, we have

$t_{D_i}, \lfloor l/2 \rfloor + 1 - \nu \in (t_{B_{i-1}}, \lfloor l/2 \rfloor + 1 - \nu, t_{B_i}, \lfloor l/2 \rfloor - \nu)$, ($\nu = 1, \ldots, \lceil l/2 \rceil$).

Therefore, we have the following inequalities:

$t_{D_i}, \lfloor l/2 \rfloor < t_{D_i}, \lceil l/2 \rceil < \cdots < t_{D_i}, 2 < t_{D_i}, 1 < 1$.

Acknowledgements The author was very glad to participate in the symposium “the 3rd Franco-Japanese-Vietnamese Symposium on Singularities” Hanoi, Vietnam, November 30–December 4, 2015. The author
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thanks all the organizers. The author is grateful to Kyoji Saito for enlightening discussions and his great encouragement. The author is grateful to Mutsuo Oka for his warm encouragement. This research was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

Appendix I

From [6, Section 8], we make a list of $f$-polynomials of types $A_l, B_l, \text{ and } D_l$. Table A contains three infinite series $A_l \ (l \geq 1), \ B_l \ (l \geq 2)$ and $D_l \ (l \geq 4)$. Table B contains the remaining exceptional types $E_6, E_7, E_8, F_4, \text{ and } G_2$ and non-crystallographic types $H_3, H_4, \text{ and } I_2(p)$. We note that in [8, Proposition 3.7], the $f$-polynomials of crystallographic types are essentially determined.

Table A

$$f_{A_l}(t) = \sum_{k=0}^{l} (-1)^k \frac{1}{l+2} \binom{l}{k} \binom{l+k+2}{k} t^k,$$

$$f_{B_l}(t) = \sum_{k=0}^{l} (-1)^k \frac{l}{k} \binom{l+k}{k} t^k,$$

$$f_{D_l}(t) = \sum_{k=0}^{l} (-1)^k \left( \binom{l}{k} \binom{l+k-1}{k} + \binom{l-2}{k} \binom{l+k-2}{k} \right) t^k.$$

Table B

$$f_{E_6}(t) = 1 - 42t + 399t^2 - 1547t^3 + 2856t^4 - 2499t^5 + 833t^6,$$

$$f_{E_7}(t) = 170t + 945t^2 - 5180t^3 + 14105t^4 - 20202t^5 + 14560t^6 - 4160t^7,$$

$$f_{E_8}(t) = 1 - 128t + 2408t^2 - 17936t^3 + 67488t^4 - 140448t^5 + 163856t^6 - 100320t^7 + 25080t^8,$$

$$f_{F_4}(t) = 1 - 28t + 133t^2 - 210t^3 + 105t^4,$$

$$f_{G_2}(t) = 1 - 8t + 8t^2,$$

$$f_{H_3}(t) = 1 - 18t + 48t^2 - 32t^3,$$

$$f_{H_4}(t) = 1 - 64t + 344t^2 - 560t^3 + 280t^4,$$

$$f_{I_2(p)}(t) = 1 - (p + 2)t + (p + 2)t^2.$$

Appendix II

The zero loci in the complex plane of the $f$-polynomial $f_P(t)$ for types $A_{20}, B_{20}, D_{20}, \text{ and } E_8$ are exhibited in the following figures, where the zeros are indicated by +.
In Remark 4.4 in [12], for an Artin monoid $G^+_P$ and a dual Artin monoid $G^\text{dual+}_P$ of type $P$, the authors observed that the derivative at $t = 1$ of the skew-growth function $N_{G^+_P}(t)$ for the Artin monoid $G^+_P$ coincides with that of the dual Artin monoid $G^\text{dual+}_P$. In addition to this observation, we have found the following equalities:

$$N'_{G^+_P}(1) = N'_{G^\text{dual+}_P}(1) = \frac{(-1)^l}{|W|} (lh) \prod_{i=2}^l (e_i - 1),$$

where $h$ is the Coxeter number and $e_1, e_2, \ldots, e_l$ are the exponents of the corresponding finite reflection group $W$ of type $P$.

In [12], we have studied the polynomial $\hat{N}_{G^+_P}(t) = f_{D_l}^+(t)/(1 - t)$. We note that the polynomial $\hat{N}_{G^+_P}(t)$ satisfies the following Fuchsian ordinary differential equation of third-order. The proof is left to the reader.

$$t(t - 1)[2(l - 1)t - l] \frac{d^3y}{dt^3} + [(l + 8)(l - 1)t^2 - (l^2 + 6l - 2)t + 2l] \frac{d^2y}{dt^2} - [(l - 1)(2l^2 - 5l - 2)t - (l^3 - 2l^2 - l - 2)] \frac{dy}{dt} - (l - 1)^3(l + 2)y = 0.$$

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