Muons decay in a laser field

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We investigate the change in the decay rate of a muon caused by embedding it in the field of a laser. A previous paper found that the change could be large, as much as an order of magnitude. We find the more intuitive result that the change is small and give analytic expressions for the small corrections.

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1. INTRODUCTION

There is interest and work on the properties of elementary systems when they are placed in strong electromagnetic fields\textsuperscript{[1, 2, 3]}. Recently, in an attempt to extend this work to unstable systems, Liu, Li, and Berakdar\textsuperscript{4} (LLB) calculated the effect of a strong laser field on the decay rate of muons. They found the lifetime could be changed from its normal value of $2 \times 10^{-6}$ seconds to $5 \times 10^{-7}$ seconds or even less. This conclusion was challenged by Narozhny and Fedotov, who offer an abbreviated calculation to support their criticism\textsuperscript{[5, 6]}. If LLB were correct, this would be a very interesting result. We have done our own calculation and, unfortunately, also reach a very different conclusion. We find the effects of the laser to be very small and give explicit expressions for these small effects.

Although the idea of the problem is straightforward, the actual calculation is somewhat complicated and LLB did the complicated part numerically. Because of this difference, and because we get such a different result, we will present our calculation in some detail and only after our results are apparent will we compare with LLB. Also because our calculation is analytic, we don’t need to make definite choices about the properties of the laser. We will assume only that the energies of the photons are about $0.1 – 1.0$ eV and that the magnitude of the laser field amplitude is $10^6 – 10^7$ V/cm (as used in LLB). The next section gives our work, the following section compares with, and discusses, LLB, including the fact that we have somewhat different starting points; we use the Volkov wavefunction\textsuperscript{[7, 8, 9, 10]} for circular polarization while they use an approximation to the wavefunction for linear polarization. The last section repeats our conclusions.

2. FORMALISM

The process is muon decay,

\begin{equation}
\mu^- (P) \rightarrow e^- (p) + \bar{\nu}_e (q_1) + \nu_\mu (q_2),
\end{equation}

where the arguments are our labels for the momentum. We will assume the photons from the laser are along the $z$-axis with momentum $k^\mu = (\omega, 0, 0, \omega)$ and circular polarization,

\begin{align}
A^\mu (x) &= a n_1^\mu \cos k \cdot x + a n_2^\mu \sin k \cdot x, \\
n_1^\mu &= (0, 1, 0, 0), \\
n_2^\mu &= (0, 0, 1, 0).
\end{align}
The electron wave function is then
\[ \psi(x) = e^{-\frac{iea}{p \cdot k}p \cdot x} e^{-\frac{iea^2}{2p \cdot k}k \cdot x} e^{-ip \cdot x} \left( 1 + \frac{e^2 A}{2p \cdot k} \right) u(p) \] (5)

where we have taken the electron to be in the \(xz\) plane and thereby avoided a factor \(-i \frac{ea}{p \cdot k} p_y \cos k \cdot x\) in the exponential. From the second exponential factor the electron has an effective momentum and mass
\[ q^\mu = p^\mu + \frac{e^2 a^2}{2p \cdot k} k^\mu, \] (6)
\[ m^2 = m_0^2 + e^2 a^2, \] (7)

where \(m_0 = 0.511\) MeV. Note that \(q \cdot k = p \cdot k\) and \(q_x = p_x\). Following the standard procedure we use the generating function for Bessel functions \(^{14}\),
\[ e^{\frac{1}{2}z(t-1/t)} = \sum_{\ell=-\infty}^{\infty} t^\ell J_\ell(z) \] (8)

to rewrite the first factor in (5) as
\[ e^{-\frac{iea}{p \cdot k}p \cdot x} = \sum_{\ell=-\infty}^{\infty} J_\ell(D)e^{-it \cdot k \cdot x} \] (9)

with \(D = \frac{eap \cdot x}{p \cdot k}\). Momentum conservation is then
\[ P^\mu + \ell k^\mu = q^\mu + q_1^\mu + q_2^\mu \] (10)

and the matrix element, for a given value of \(\ell\), is
\[ R_\ell = \frac{G}{\sqrt{2}} \bar{u}(q_2) \gamma^5 (1 - \gamma_5) u(P) \bar{u}(p) \left[ \Delta_0 + \Delta_1 \gamma_1 \frac{k}{k} + \Delta_2 \gamma_2 \frac{k}{k} \right] \gamma_\alpha (1 - \gamma_5) \gamma_\beta(q_1) \] (11)

where
\[ \Delta_0 = J_\ell(D) \] (12)
\[ \Delta_1 = \frac{1}{2} \frac{ea}{2p \cdot k} (J_{\ell+1}(D) + J_{\ell-1}(D)) \] (13)
\[ \Delta_2 = -i \frac{ea}{2} \frac{1}{2p \cdot k} (J_{\ell+1}(D) - J_{\ell-1}(D)). \] (14)

Note that the argument of the electron spinor is still \(p\).

We square the matrix element in the usual way, using FORM \(^{11}\), and integrate out the momentum of the neutrinos in the usual way using
\[ \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^4(Q - q_1 - q_2) q_1^\alpha q_2^\beta = \frac{\pi}{24} (Q^2 g^{\alpha\beta} + 2Q^\alpha Q^\beta) \Theta(Q^2). \] (15)

The total width is then
\[ \Gamma = \sum_{\ell=-\infty}^{\infty} \Gamma_\ell \] (16)

with the width for each \(\ell\) given as an integral over the electron energy and angle
\[ \Gamma_\ell = \frac{1}{3072\pi^3 M} \int dE |q| \int dz \Theta(Q^2) |T_\ell|^2 \] (17)

where \(E\) is \(q^0\) (the Jacobian connecting \(p^\mu\) and \(q^\mu\) is unity) and \(M\) is the muon mass.
The square of the matrix element, \( |T_\ell|^2 \), summed over spin and integrated over the neutrino momentum, is

\[
|T_\ell|^2 = 128G^2 \left\{ J_\ell^2(D) [3P \cdot q(M^2 + m^2) - 4(P \cdot q)^2 - 2M^2m^2] \right. \\
+ J_\ell^2(D) \frac{e^2a^2}{2q \cdot k} [2q \cdot k(M^2 - P \cdot q) + P \cdot k(4P \cdot q - 3M^2 - m^2)] \\
+ \ell J_\ell^2(D) [q \cdot k(2M^2 - 4P \cdot q) - P \cdot k(3M^2 - 8P \cdot q + 3m^2)] \\
+ 2\ell J_\ell^2(D) \frac{e^2a^2}{q \cdot k} P \cdot k(2q \cdot k - P \cdot k) \\
+ \frac{e^2a^2}{4q \cdot k} [J_{\ell+1}^2(D) + J_{\ell-1}^2(D)][-4P \cdot qP \cdot k + 3M^2P \cdot k + m^2P \cdot k + 2q \cdot k(P \cdot q - M^2)] \\
+ \frac{e^2a^2}{q \cdot k} \ell [J_{\ell+1}^2(D) + J_{\ell-1}^2(D)]P \cdot k(P \cdot q - k) \\
+ \left. + ie(P, q, k, n_2) \frac{ea}{2q \cdot k} J_\ell(D) [J_{\ell+1}(D) - J_{\ell-1}(D)](3M^2 + m^2 - 4P \cdot q) \\
+ ie(P, q, k, n_2) \frac{ea}{q \cdot k} \ell J_\ell(D) [J_{\ell+1}(D) - J_{\ell-1}(D)](2P \cdot q - k) \\
+ ie(P, q, n_1, n_2) \frac{e^2a^2}{4q \cdot k} [J_{\ell+1}^2(D) - J_{\ell-1}^2(D)](3M^2 + m^2 - 4P \cdot q) \\
+ ie(P, q, n_1, n_2) \frac{e^2a^2}{2q \cdot k} \ell [J_{\ell+1}^2(D) - J_{\ell-1}^2(D)](2P \cdot k - 2q \cdot k) \\
+ ie(q, q, n_1, n_2) \frac{e^2a^2}{2q \cdot k} [J_{\ell+1}^2(D) - J_{\ell-1}^2(D)](M^2 - P \cdot q) \right\} 
\] 

(18)

where \( e^{0123} = -i \).

If we set \( a = 0 \) then \( D = 0 \) so \( J_\ell = 0 \) for \( \ell \neq 0 \) and \( J_0(0) = 1 \) and we get the usual expression for muon decay from the first line of (18)

\[
\Gamma^0 = \frac{G^2M^5}{192\pi^3} 
\]

(19)

where terms proportional to the electron mass have been neglected. Note that \( \Gamma^0 \) is not the same as \( \Gamma_{\ell=0} \).

The crucial thing is the limits on the integrals in Eq. (17). These are determined by the \( \Theta \) function,

\[
\int dE \int dz \Theta(Q^2) = \int_m^{\frac{\ell}{2}} dE \int_{-1}^1 dz + \int_{\frac{\ell}{2}}^{\frac{\ell}{2} + \ell\omega} dE \int_{z_L(E)}^1 dz, \quad (\ell > 0) 
\]

(20)

\[
\int dE \int dz \Theta(Q^2) = \int_m^{\frac{\ell}{2}} dE \int_{-1}^1 dz + \int_{\frac{\ell}{2} + \ell\omega}^{M + \ell\omega} dE \int_{-1}^{z_L(E)} dz, \quad (\ell < 0) 
\]

(21)

where

\[
z_L(E) = -\frac{M^2 + 2M\ell\omega - 2E(M + \ell\omega)}{2\ell\omega E}. 
\]

(22)

It is important to notice that \( z_L(M + \ell\omega) = -1 \) and \( z_L(M + \ell\omega) = +1 \). We can rewrite these limits, for both signs of \( \ell \), as

\[
\int dE \int dz \Theta(Q^2) = \int_m^{\frac{M}{2}} dE \int_{-1}^1 dz + \int_{\frac{M}{2}}^{M + \ell\omega} dE \int_{z_L(E)}^1 dz. 
\]

(23)

We will treat these two terms separately.
Consider the first term in (23). Because the limits of integration are not functions of $\ell$, we can immediately do the sum over $\ell$ in Eq. (16). The only things we need to do the sums are the relations

$$\sum_{\ell=-\infty}^{\infty} J_{\ell}(z) J_{\ell+n}(z) = J_n(0),$$

(24)

which follows from 9.1.75 in Abramowitz and Stegun [14], the recursion relation for Bessel functions, $\ell J_{\ell}(z) = \frac{1}{z}(J_{\ell+1}(z) + J_{\ell-1}(z))$, and $J_{-\ell}(z) = (-1)^{\ell} J_{\ell}(z)$. The most important sum is

$$\sum_{\ell=-\infty}^{\infty} J_{\ell}^2(D) = 1$$

(25)

because that replaces the $J_{\ell}^2(D)$ in the first line of (18) by unity. After doing all the sums Eq. (18) becomes

$$\sum_{\ell=-\infty}^{\infty} |T_{\ell}|^2 = 128 G^2 \left\{ 3P \cdot q(M^2 + m^2) - 4(P \cdot q)^2 - 2M^2 m^2 \right\}$$

$$+ 2 \frac{e^2 a^2}{(q \cdot k)^2} q \cdot k (q \cdot k - P \cdot k)$$

$$- \frac{e^2 a^2}{q \cdot k} \epsilon(P, k, n_1, n_2)(2P \cdot k - 2q \cdot k)\}$$

(26)

where there has been a lot of cancellation. This can be easily integrated by hand

$$\Gamma = \Gamma^0 \left(1 + \frac{8}{3} \frac{e^2 a^2}{M^2} \left(\frac{5}{3} - \ln \frac{M}{m} - \frac{2\omega}{M} \ln \frac{M}{m} + \frac{5}{2} \frac{\omega}{M} \right)\right)$$

(27)

where again the first line of (26) gives $\Gamma^0$. Since $ea$ is much less than 1 MeV the additional laser dependent terms are very small, smaller than the electron mass terms which were neglected in (18). The terms which are even further suppressed by the factor of $\omega/M$ come from the $\epsilon$ term in (26).

Now consider the correction from the second term in (23). For $ea \sim 10^{-4}$ MeV and $\omega \sim 1$ eV, the argument of the Bessel function, $D$, varies with $E$ and $z$ from zero to $\sim 1.5 \times 10^4$. Bessel functions become very small once the index, $\ell$, becomes greater than the argument so $\ell$ is limited by $\ell \leq D$. In other words $\omega \ell$ will always be much less than 1 MeV and this correction will be small because the range of the energy integration is small. So define a function of the energy as

$$F(E, \omega, \ell) = \int dEE\beta \int_{z_L}^{1} dz |\overline{T_{\ell}}|^2$$

(28)

where $|\overline{T_{\ell}}|^2$ is given by Eq. (18) without the prefactor of $128 G^2$. Then the correction to the width is given by

$$\Gamma_C \frac{\Gamma^C}{\Gamma^0} = \frac{8}{M^6} \sum_{\ell=-\infty}^{\infty} \left[ F(E = \frac{M}{2} + \ell \omega, \omega, \ell) - F(E = \frac{M}{2}, \omega, \ell) \right].$$

(29)

Now if we expand the first term in a Taylor series

$$F(E = \frac{M}{2} + \ell \omega, \omega, \ell) = F(E = \frac{M}{2}, \omega, \ell) + \ell \omega \frac{dF}{dE} |_{E=\frac{M}{2}} + \frac{\ell^2 \omega^2}{2} \frac{d^2F}{dE^2} |_{E=\frac{M}{2}} + \cdots$$

(30)

then, since $z_L \left(\frac{M}{2}\right) = -1$, we can again do the sum over $\ell$ before we do the integral. We will approximate (29) by keeping only the first nonzero term; since

$$\frac{dF}{dE} |_{E=\frac{M}{2}} = \frac{M}{2} \int_{-1}^{1} dz |\overline{T_{\ell}}|^2$$

(31)
the correction, Eq. (29), is
\[ \frac{\Gamma C}{\Gamma^0} = \frac{4\omega}{M^2} \int_{-1}^{1} dz \sum_{\ell=-\infty}^{\infty} \ell |T_{\ell}|^2. \] (32)

Now in Eq. (18) the only nonzero sums come from lines 3, 4, 7, 9, 10, and 12 and after a bit of work we get
\[ \frac{\Gamma C}{\Gamma^0} = 8e^{2a^2/2} \left( 1 + 3 \frac{\omega}{M} + 4 \frac{\omega^2}{M^2} \right) \ln \frac{M}{m} - 1 - 4 \frac{\omega}{M} - 2 \frac{\omega^2}{M^2} \right]. \] (33)

These terms are small but not necessarily smaller than those in (27). In fact the largest term cancels and the total effect of the laser, the sum of (27) and (33), is to change the width from \( \Gamma^0 \) to
\[ \Gamma = \Gamma^0 \left\{ 1 + 8e^{2a^2/2} \left[ \frac{\omega}{M} + 4 \frac{\omega^2}{M^2} \right] \ln \frac{M}{m} + \frac{2}{3} - 3 \frac{\omega}{2M} - 2 \frac{\omega^2}{M^2} \right\}. \] (34)

Let’s be clear about what has been done. Eq. (27) is an exact solution for the part of the phase space where the integral over the electron energy is between \( m \) and \( M^2 \). Eq. (33) is an approximation for the part of the phase space where the integral is between \( M^2 \) and \( M^2 + \ell \omega \), in that it is the first nonzero term in an expansion in powers of \( \ell \omega \). As a consistency check we have calculated the contribution to the second nonzero term in the expansion (the second derivative term in (30)) from the first line of (18) (which should be the largest contribution) and found
\[ \frac{\Gamma C}{\Gamma^0} \sim 32e^{2a^2m^2/2} \left[ -1 + \ln \frac{M}{m} \right] \] (35)
which is the same as zero since we did not keep terms \( \sim \frac{M}{m} \) in (27) or (33).

3. Comparison with LLB

We have assumed the laser radiation is circularly polarized because, as is well known [2], that case is easier. For linear polarization the wavefunction in Eq. (5) is multiplied by an extra factor
\[ e^{i \frac{2a^2}{\pi} \sin(2k \cdot x)} \] (36)
and the second term in the definition of \( q^\mu \), Eq. (6), has an extra \( \frac{1}{2} \). Because of this term the generating function for Bessel functions must be used twice and the square of the matrix element involves two sums over Bessel functions. LLB use linear polarization but approximate the wavefunction by omitting this term thus avoiding the double sum. Dropping this term is not the same as doing circular polarization because the vector potential, \( A^\mu \), is different. The effect of this approximation on the muon lifetime is unknown but it seems unlikely this could explain the huge difference in our results. Given their definition of the wavefunction we agree with their expression for the S matrix. We do not, however, agree with their expression for the partial width. We believe their \( W_\ell = \frac{G^2}{96\pi^2} \int dE \cdots \) should be \( W_\ell = \frac{G^2}{48\pi^3 M} \int dE \cdots \). (\( W_\ell \) is what we call \( \Gamma_\ell \).) Otherwise their \( \Gamma^0 \) would be too big by a factor of \( \frac{5}{3} \). This is probably a misprint. If not, it would partially explain their small lifetimes.

LLB find a shorter lifetime which means they find a larger width. On the other hand they don’t get much contribution from what we call negative \( \ell \). (We have defined \( \ell \) differently – what we call negative \( \ell \) LLB call positive.) If we consider only positive \( \ell \) then, for example, Eq. (25) is replaced by
\[ \sum_{\ell=0}^{\infty} J^2_\ell(z) = \frac{1}{2} \left( 1 + J^2_0(z) \right) \] (37)
and \( \Gamma^0 \), the contribution from the first line of (18), is replaced by approximately \( \frac{1}{4} \Gamma^0 \). But this is a change in the wrong direction. LLB get a larger width despite having little contribution from negative \( \ell \). As we have seen
the other terms in (18) go as $e^2 a^2/M^2$. Furthermore a *longer* lifetime might be easier to understand as a kind of Zeno effect; the interactions with the laser photons make the decay keep starting over. But that is not what LLB find.

We disagree with LLB on the lower limit of the electron energy. We get the lower limit to be the (effective) electron mass, $m$, they get $m + \ell \omega$. Again it is hard to believe that makes much difference.

So about the only place left to look for a difference is the numerical integrations. To do these requires definite values for $ea$ and $\omega$. We used $ea = 1.69 \times 10^{-4}$ MeV and $\omega = 1.17$ eV (the Nd:YAG laser of LLB) and integrated (18) using the limits (20) or (21). A few results are shown in the table

| $\ell$ | $\Gamma_{\ell}/\Gamma_0$ |
|-------|-------------------------|
| 0     | 3.463 $10^{-3}$         |
| 1     | 3.455 $10^{-3}$         |
| 10    | 3.427 $10^{-3}$         |
| 100   | 1.921 $10^{-3}$         |
| 200   | 6.940 $10^{-4}$         |
| 500   | 7.391 $10^{-5}$         |
| 1000  | 1.013 $10^{-5}$         |
| 5000  | 7.866 $10^{-8}$         |

We used routines for the Bessel functions from Numerical Recipes\cite{12} and the integration routine VEGAS\cite{13}. The partial widths for negative $\ell$ were indistinguishable from those of positive $\ell$ for a given $|\ell|$. The integration was sufficiently fast that we could do each of the $\Gamma_{\ell}$ for $\ell$ up to 500 with the result

$$\frac{1}{\Gamma_0} \sum_{\ell=-500}^{500} \Gamma_{\ell} = 0.96$$

(38)

where we estimate the error from the numerical integration to be less than 0.01. If we assume the $\ell$ dependence of $\Gamma_{\ell}$ is linear from $\ell = 500$ to 1000 we get another contribution to (35) of 0.04; if we make the same assumption for $\ell$ from 1000 to 5000 we get another 0.04. These are surely overestimates because the $\ell$ dependence must fall faster than linear. We could do a better job for $|\ell| > 500$ but these are sufficient to show the magnitude of the contribution to the total width that could be expected from higher $\ell$. Thus the conclusion is that the total width cannot be very different than $\Gamma_0$, in agreement with our more precise arguments above.

4. CONCLUSIONS

We have considered muon decay in the electromagnetic field of a laser. Our discussion is entirely analytic with the only approximation a Taylor series expansion of the squared matrix element for the region of electron energy between $M^2$ and $M^2 + \ell \omega$, a distance of less than $10^{-2}$ MeV. We find the effect of a laser on muon decay is very small, of order $e^2 a^2/M^2$, $e^2 a^2 \omega/M^3$, or $e^2 a^2 \omega^2/M^4$, where $ea \sim 2 \times 10^{-4}$ MeV, $\omega \sim 1$ eV and $M$ is the muon mass, 105.66 MeV. Given the dimensions of the Fermi coupling constant the decay width must have five powers of energy. Once we find the coefficient of the $M^5$ term is the same as in the absence of the laser we know the effect of the laser is very slight because the only other energies in the problem are $ea$, $\omega$, and $m$. We dropped the electron mass where possible so there are surely corrections of order $e^2 a^2 m^2/M^4$ which are numerically larger than some of the terms we included. But all of these corrections are tiny, smaller than the known corrections of order $m^2/M^2$. Our result is given by Eq. (34).
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[1] S. Chelkowski, T. Zuo, and A. D. Bandrauk, Phys. Rev. A 52, 2977 (1995); C. Muller, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. D 74, 074017 (2006); N. B. Narozhny, Phys. Rev. D 20, 1313 (1979); B. A. Remington, D. Arnet, R. P. Drake, and H. Takabe, Science 284, 1488 (1999); G. A. Mourou, T. Tajima, and S. V. Bulanov, Rev. Mod. Phys. 78, 309 (2006); M. Markland and P. K. Shukla, Rev. Mod. Phys. 78, 591 (2006); K. W. D. Ledingham, P. McKenna, and R. P. Shinghal, Science 300, 1107 (2003); D. Umstadter, Nature(London) 404, 239 (2000); S. Chelkowski, A. D. Bandrauk, and P. B. Corkum, Phys. Rev. Lett. 93, 083602 (2004).

[2] T. M. Tinsley, Phys. Rev. D 71, 073010 (2005).

[3] D. A. Dicus, W. W. Repko, and T. M. Tinsley, Phys. Rev. D 76, 025005 (2007).

[4] A.-H. Liu, S.-M. Li, and J. Berakdar, Phys. Rev. Letters 98, 251803 (2007).

[5] N. B. Narozhny and A. M. Fedotov, Phys. Rev. Letters 100, 219101 (2008).

[6] A.-H. Liu, S.-M. Li, and J. Berakdar, Phys. Rev. Letters 100, 219102 (2008).

[7] D. M. Volkov, Z. Phys. 94, 250 (1935).

[8] D. M. Volkov, Zh. Eksp. Teor. Fiz. 7, 1286 (1937).

[9] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Quantum Electrodynamics (Butterworth-Heinemann, Linacre House, Jordan Hill, Oxford OX2 8DP, 1982), 2nd ed.

[10] C. Szymanowski, V. Veïard, R. Taïeb, and A. Maquet, Phys. Rev. A 56, 3846 (1997).

[11] J. A. M. Vermaseren, Symbolic Manipulations with FORM CAN Experite Centre, 1991; J. A. M. Vermaseren, “New Features of FORM” math-ph/0010025.

[12] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes in Fortran 77, 2nd edition (Cambridge University Press, 1992), pg 223.

[13] G. P. Lepage, J. Comp. Phys. 27, 192 (1978); G. P. Lepage, “VEGAS: An Adaptive Multidimensional Integration Program,” Publication CLNS-80/447, Cornell University.

[14] M. Abramowitz and I. S. Stegun, Handbook of Mathematical Functions (Dover Publications, Inc., New York, 1968).

[15] The width including the electron mass terms is known to be $\Gamma = \Gamma_0(1 - 8r^2 + 8r^6 - r^8 - 12r^4 \ln r^2)$, $r = \frac{m}{M}$, and comes from the first line of (18) if $J_2(D)$ is set to unity.