Quasiperiodic Modulated-Spring Model

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Abstract

We study the classical vibration problem of a chain with spring constants which are modulated in a quasiperiodic manner, \textit{i.e.}, a model in which the elastic energy is $\sum_j k_j (u_{j-1} - u_j)^2$, where $k_j = 1 + \Delta \cos[2\pi \sigma (j - 1/2) + \theta]$ and $\sigma$ is an irrational number. For $\Delta < 1$, it is shown analytically that the spectrum is absolutely continuous, \textit{i.e.}, all the eigen modes are extended. For $\Delta = 1$, numerical scaling analysis shows that the spectrum is purely singular continuous, \textit{i.e.}, all the modes are critical.
I. INTRODUCTION

For some years, the electronic properties of quasiperiodic systems have been extensively studied. In particular, a number of studies are devoted to a one-dimensional quasiperiodic tight-binding model \[1\],

\[-\psi_{n+1} - \psi_{n-1} + \lambda V(n\sigma)\psi_n = E\psi_n,\]

(1.1)

where \(V(x)\) is a periodic function with period 1 (\(V(x + 1) = V(x)\)) and \(\sigma\) is an irrational number.

The properties of the energy spectra and the eigen states of (1.1) are generically as follows \[2,3\]: When the strength of the potential \(\lambda\) is small, all the states are extended, \(i.e.,\) the energy spectrum is purely absolutely continuous. As \(\lambda\) is increased, localized states appear, which are separated from extended states by mobility edges in the spectrum. That is, the spectrum has absolutely continuous parts and pure point parts, and they are separated by mobility edges. When \(\lambda\) is large enough, all the states are localized, \(i.e.,\) the spectrum is pure dense point.

The well-known exceptions are the Harper model \[4\] and the Fibonacci model \[5,6\]. In the Harper model \((V(x) = \cos(2\pi x))\), all the states are extended for \(\lambda < 2\), while all the states are localized for \(\lambda > 2\). At \(\lambda = 2\), all the states are critical and the energy spectrum is purely singular continuous. These properties are caused by the existence of the duality \[4\]. It is self-dual at the critical point \(\lambda = 2\).

In the Fibonacci model \(V\)'s take two values 1 and \(-1\) arranged by the Fibonacci sequence. So \(V(x)\) is piecewise constant. The spectrum is purely singular continuous and all the states are critical irrespective of \(\lambda\).

In this paper, we study the dynamics of a one-dimensional array of equal mass particles connected by springs modulated in a quasiperiodic manner. The model called the modulated spring (MS) model is introduced in Sec. 2. A quasiperiodic case was first studied by De Lange and Janssen \[7\]. Their numerical results show the intricate Cantor-set like structures of the
spectra. Janssen and Kohmoto [8] applied the multifractal method [9, 10] and conjectured that the model has a pure singular continuous spectrum at $\Delta = 1$ and has a mixed one for $\Delta < 1$. However, their system sizes are not large enough. In fact, their claim for $\Delta < 1$ is not correct as shown below.

Incidentally the MS model is related to a two-dimensional tight-binding model in a magnetic field with next-nearest-neighbor hoppings [11, 13, 14] which is introduced in Sec. 3. Using the result of Thouless [11] for the Lyapunov exponent it is shown that all the eigen modes are extended for $\Delta < 1$ in Sec. 4. For $\Delta = 1$ numerical scaling analyses are required to determine the types of the spectrum. The results are presented in Sec. 5 and it is concluded conclude that all the eigen modes are critical for $\Delta = 1$. This is presumably due to the fact that the spring constants can be infinitesimally close to zero.

II. MODULATED SPRING (MS) MODEL

The equations of motion of the one-dimensional MS model are

$$\omega^2 u_j = k_j(u_j - u_{j-1}) + k_{j+1}(u_j - u_{j+1}) \quad (2.1)$$

where $u$’s are displacements of the atoms and $k$’s are the spring constants:

$$k_j = 1 + \Delta \cos[2\pi\sigma(j - 1/2) + \theta]. \quad (2.2)$$

Note that $\Delta$ cannot be greater than one since the spring constants $k$’s must be positive. Thus $0 < \Delta \leq 1$. The MS model can be regarded as a one-dimensional electronic tight-binding model by rewriting (2.1) with (2.2) as

$$-k_j u_{j-1} - k_{j+1} u_{j+1} + V_j u_j = Eu_j, \quad (2.3)$$

with

$$V_j = k_j + k_{j+1} - 2 = 2\Delta \cos(\pi\sigma)\cos(2\pi\sigma j + \theta), \quad (2.4)$$

and

$$E = \omega^2 - 2. \quad (2.5)$$
Consider the tight-binding model on the square lattice with nearest-neighbor hoppings $t_a$ and $t_b$, next-nearest-neighbor hopping $t_c$, and magnetic flux per plaquette $\phi$ (see Fig. 1). In the Landau gauge ($\vec{A} = (0, By)$) the tight-binding equation is

$$
-t_a(\psi_{m-1,n} + \psi_{m+1,n}) - t_b(e^{i2\phi\psi_m}\psi_{m,n-1}e^{-i2\pi\phi_m}\psi_{m,n+1})
-t_c(e^{-i2\pi\phi(m-1/2)}\psi_{m+1,n+1} + e^{i2\pi\phi(m+1/2)}\psi_{m-1,n-1})
-e^{i2\pi\phi(m+1/2)}\psi_{m+1,n-1} + e^{-i2\pi\phi(m-1/2)}\psi_{m+1,n+1}) = E\psi_{m,n}. \quad (3.1)
$$

Since this equation is translation invariant in the y-direction, the Bloch theorem is applied and one may write

$$
\psi_{m,n} = e^{-ik_n}\phi_m. \quad (3.2)
$$

By substituting (3.2), (3.1) reduces to a one-dimensional tight-binding equation

$$
-(t_a + 2t_c\cos[k + 2\pi\phi(m - 1/2)]\phi_{m-1} - (t_a + 2t_c\cos[(k + 2\pi\phi(m + 1/2)])\phi_{m+1}
-2t_b\cos(k + 2\pi\phi m)\phi_m = E\phi_m. \quad (3.3)
$$

Since the original 2d problem has a symmetry with respect to an interchange of $x$ and $y$ (3.3) has a duality. Substitute

$$
\phi_m = \sum_j f_j e^{i[Km+(2\pi\phi m-k)]j}, \quad (3.4)
$$

to (3.3), then one gets

$$
-(t_b + 2t_c\cos[K + 2\pi\phi(j - 1/2)]f_{j-1} - (t_b + 2t_c\cos[(K + 2\pi\phi(j + 1/2)])f_{j+1}
-2t_a\cos(K + 2\pi\phi j)f_j = Ef_j \quad (3.5)
$$

which has the same form as (3.3).
IV. LYAPUNOV EXPONENT

Using the duality in the last Sec.3 Thouless [11] obtained the Lyapunov exponent $\gamma$ (inverse of the localization length) for (3.3) as

$$
\gamma = \begin{cases} 
\ln(t_b + \sqrt{t_b^2 - 4t_c^2}) - \ln(t_a + \sqrt{t_a^2 - 4t_c^2}) & \text{for } 2t_c < t_a < t_b, \\
\ln(t_b + \sqrt{t_b^2 - 4t_c^2}) - \ln(2t_c) & \text{for } t_a < 2t_c < t_b, \\
0 & \text{for } t_a < t_b < 2t_c.
\end{cases}
$$

(4.1)

From (4.1) the Lyapunov exponent is strictly positive for $2t_c < t_b$ and $t_a < t_b$. Thus all the eigen states of (3.3) are exponentially localized and the spectrum is pure point. The duality (3.4) which maps (3.3) to (3.5) by exchanging $t_a$ and $t_b$ is essentially a Fourier transformation. So a localized state is mapped to an extended state. Therefore all the states are extended for if

$$2t_c < t_b \text{ and } t_a < t_b
$$

(4.2)

Note that the MS model (2.3) with (2.4) and (3.3) are the same if

$$t_a = 1, \ t_c = \Delta/2, \ t_b = \Delta \cos(\pi \sigma) \text{and} \ k = \theta + \pi.
$$

(4.3)

For $\Delta < 1$, (4.2) is satisfied. This means that all the eigen modes of the MS model (2.1) is extended for $\Delta < 1$.

For $\Delta = 1$ we have $2t_c = t_a$ and $t_b < t_a$. In the dual situation ($2t_c = t_b, t_a < t_b$) one has $\gamma = 0$ from (4.1) and the states are either critical or extended. Thus in the original situation the eigen modes are either critical or localized. In order to determine whether the eigen modes are critical or localized for $\Delta = 1$ numerical analysis is required. It is presented in the next section.

V. NUMERICAL SCALING ANALYSIS

We take $\sigma$ to be $(\sqrt{5} - 1)/2$ (the inverse of the golden mean) and $\theta$ to be 0. This irrational number $\sigma$ is approached by rational numbers $F_{n-1}/F_n$ ($F_{n-1}/F_n \to \sigma$ as $n \to \infty$), where
$F_n$ is the $n$th Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = F_1 = 1$. Thus in the numerical calculations $\sigma$ is approximated by $F_{n-1}/F_n$ which makes the system periodic with period $F_n$. The spectrum consists of $F_n$ bands. As $n$ is increased, i.e., as $F_{n-1}/F_n$ approaches to $\sigma$, each band splits into sub-bands and the width of each band approaches to zero. We analyze scaling behaviors of the band widths for large $n$’s. For the band width, we adopt the width of $\omega^2$ or $E$ in (2.3) and denote it as $\delta(\omega^2)$.

For localized modes the band widths are expected to decrease exponentially with $F_n$, while for extended modes they should behave as $1/F_n$ [15]. Thus, if we define a scaling exponent $\alpha$ by $1/F_n \sim \delta(\omega^2)^\alpha$, eigen modes are localized if $\alpha$ is 0 and are extended if $\alpha$ is 1. In addition, extended modes with $\alpha = 1/2$ may exist. These correspond to the remnant of van Hove singularities at the edges of band clusters. This behavior was observed in the Harper model in the extended region [12]. For critical modes, which correspond to a singular continuous spectrum, $\alpha$ takes a continuous range of values. As $n$ is increased by 2 or 3, each band splits into three subbands. Therefore, each point in the spectrum of the quasiperiodic limit ($n \to \infty$) is specified by an infinite series of 1, 0 and -1, which represents the upper, middle and lower sub-bands, respectively. In Fig.1 $F_n\delta(\omega^2)$ is plotted against $n$ for several points in the spectrum for (a) $\Delta = 0.95$ and (b) $\Delta = 1.0$ [10]. For $\Delta = 0.95$, the widths scale as $\delta(\omega^2) \sim 1/F_n$ except for the points identified by $\{C_1C_2\cdots -1 -1 -1\cdots\}$ or $\{C_1C_2\cdots 1 1 1 \cdots\}$ where they scale as $\delta(\omega^2) \sim (1/F_n)^2$. The latter scaling with $\alpha = 1/2$ are remnants of the van Hove singularities. The spectrum is absolutely continuous, i.e., the modes are extended at least for those in Fig. 2(a).

For $\Delta = 1$, $\delta(\omega^2)$ scales with various powers of $1/F_n$ as shown in Fig. 2(b). It is an interesting result that the lowest edge band $\{-1 -1 -1 -1 \cdots\}$ seems to scale exactly as $\delta(\omega^2) \sim (1/F_n)^3$. The scaling $\alpha = 1/3$ is different from the edge scaling of absolutely continuous spectra $\alpha = 1/2$ which represents a remnant of a van Hove singularity. Thus it seems that usual extended modes do not exist in the vicinity of $\omega = 0$.

In order to clarify the global properties of the spectrum, we perform the multifractal analysis [9,10]. We calculate $f(\alpha)$ of the whole spectrum which is defined by $\Omega(\alpha)d\alpha \sim F_n^{f(\alpha)}$,
where \(\Omega(\alpha)d\alpha\) is the number of bands whose scaling exponents \(\alpha\)’s lie in an interval \([\alpha, \alpha + d\alpha]\). When we calculate \(f(\alpha)\) using the statistical mechanics-like formalism \([\text{[10]}]\) a smooth \(f(\alpha)\) on \([\alpha_{\text{min}}, \alpha_{\text{max}}]\) is always obtained for a finite system and it is not possible to distinguish spectral types. Therefore it is crucial to take an extrapolation \(n \to \infty\).

In Fig. 3, (a) \(\alpha_{\text{max}}^{(n)}\) and (b) \(f_{\text{max}}^{(n)}\) (calculated with \(\sigma_n = F_{n-1}/F_n\)) for \(\Delta = 0.95\) are plotted. It is clearly seen that \(\alpha_{\text{max}} = 1\) and \(f(\alpha_{\text{max}}) = 1\) in the \(n \to \infty\) limit. Similar results are obtained for other values of \(\Delta\) which are less than one. These numerical results are consistent with the analytical result in Sec. 4 that the spectrum is absolutely continuous for \(\Delta < 1\).

In Fig. 4, \(f(\alpha)\) which is obtained by extrapolation \(n \to \infty\) for \(\Delta = 1\) is shown. This gives the convincing evidence that the spectrum is purely singular continuous rather than point like. Thus we conclude that all the eigen modes of the MS spring model \((2.1)\) is critical at \(\Delta = 1\).

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[16] Note that, when ∆ = 1, k’s for certain j’s become exactly zero if n=3m+1 (m: integer).

Since the infinitely long chain is divided into finite chains in this case all the modes are
trivially localized and δ’s are zero. This is caused by the choice θ = 0 in (2.2) and is not a generic feature. Thus we disregard the data for n = 3m + 1 when Δ = 1.0.
FIGURES

1 The square lattice with nearest-neighbor and next-nearest-neighbor hoppings in the magnetic field $\vec{B}$ whose flux quanta per plaquette is $\phi$.

2 $F_n \delta(\omega)$ against $n$ for several modes with (a) $\Delta = 0.95$ and (b) $\Delta = 1.0$.

3 (a) $\alpha_{max}(n)$ and (b) $f_{max}(n)$ against $1/n$ for $\Delta = 0.95$.

4 $f(\alpha)$ for spectrum with $\Delta = 1$. 