THE LINE WIDTH DIFFERENCE OF NEUTRALS AND IONS INDUCED BY MHD TURBULENCE

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ABSTRACT

We address the problem of the different line widths of coexistent neutrals and ions observed in molecular clouds and explore whether this difference can arise from the effects of magnetohydrodynamic (MHD) turbulence acting on partially ionized gas. Among the three fundamental modes of MHD turbulence, we find that fast and slow modes do not contribute to line width differences. We focus on the Alfvénic component, and consider the damping of Alfvén modes by taking into account both neutral-ion collisions and neutral viscosity. We confirm that the line width difference can be explained by the differential damping of the Alfvénic turbulence in ions and the hydrodynamic turbulence in neutrals, and find it strongly depends on the properties of MHD turbulence. We consider various regimes of turbulence corresponding to different media magnetizations and turbulent drivings. In the case of super-Alfvénic turbulence, when the damping scale of Alfvénic turbulence is below the Alfvénic scale $l_A$, the line width difference does not depend on magnetic field strength. In other turbulence regimes, however, the dependence is present and evaluation of magnetic field from the observed line width difference is possible.

Key words: ISM: clouds – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

The interstellar medium (ISM) is turbulent and magnetized (see Armstrong et al. 1995; Chepurnov & Lazarian 2010). As the densest part of the ISM, molecular clouds are typical environments where many observational phenomena can only be correctly understood in the framework of MHD turbulence (see McKee & Ostriker 2007 and references therein).

The turbulence in molecular clouds takes place in partially ionized gas. This makes turbulence more complicated and induces new effects related to the relative motion of neutrals and ions. The difference between neutral and ion line widths has been detected by a number of observations of turbulent molecular clouds (Houde et al. 2000a, 2000b; Lai et al. 2003). The narrower line profiles of ions have been explained by modeling bulk motions of neutral flows and their frictions with ions, which are trapped along magnetic field lines (Houde et al. 2004). This process was interpreted as ambipolar diffusion (AD) (see Shu 1992). Moreover, Houde et al. (2002) argued that the ion-to-neutral line width ratio is related to the orientation of the magnetic field, which would open a new way to study magnetic fields.

An important study that explored the line width difference was by Li & Houde (2008; henceforth LH08) who, for the first time, related the difference to the different turbulence truncation for neutrals and ions. In fact, they attributed the difference between the turbulent velocity dispersion spectra of coexistent neutrals and ions to their different turbulent energy dissipation scales, relating it to the AD concept. Their approach provided the first plausible explanation of the line width difference based on the concept of ubiquitous interstellar turbulence. Referring to some of the available studies of turbulence in partially ionized gas, LH08 proposed a technique to determine the AD scale, as well as the strength of the plane-of-the-sky component of magnetic field. Their work has been followed in attempts to measure magnetic field strength in molecular clouds (see Hezareh et al. 2010, 2014).

To provide a quantitative interpretation of the neutral-ion line width difference as arising from the differential damping of the turbulence cascade, a theoretically justified and numerically tested treatment of MHD turbulence is required. MHD turbulence has been a focus of intensive investigations in the past decade, which have changed the subject considerably (see reviews by Lazarian et al. 2012; Brandenburg & Lazarian 2014 and references therein). For instance, the anisotropy of Alfvénic turbulence is an essential part of the MHD turbulence cascade (Goldreich & Sridhar 1995; hereafter GS95) and this has been corroborated by numerical studies of mode decomposition (Cho & Lazarian 2002, 2003; Kowal et al. 2009). Ignoring this is erroneous for many applications of turbulence, such as the acceleration and propagation of cosmic rays (e.g., Yan & Lazarian 2004; Yan 2015), propagation of heat (e.g., Lazarian 2006), turbulent magnetic reconnection (e.g., Lazarian & Vishniac 1999; Eyink et al. 2013), and other astrophysical problems.

Studies that treat MHD turbulence in the partially ionized gas include Lithwick & Goldreich (2001) and Lazarian et al. (2004). Those studies focused on only one damping mechanism, such as neutral-ion collisions in Lithwick & Goldreich (2001) and viscosity in neutrals in Lazarian et al. (2004). Lithwick & Goldreich (2001) dealt with a high-$\beta$ and ion-dominated medium. Because the neutral fraction under their consideration is sufficiently small, they argued that the cascade of Alfvén modes can survive the neutral-ion collisional damping and is truncated at the transverse length scale of the proton gyroradius. And slow modes are damped at the proton diffusion scale, at which protons can diffuse across an eddy during its turnover time. Lazarian et al. (2004) analyzed turbulence damping in view of the magnetic reconnection, and discussed the effects of neutral viscosity in high Prandtl number turbulence (i.e., the turbulence in the media with viscosity much larger than resistivity; see numerical simulations in Cho et al. 2002a). To achieve a comprehensive picture...
of the damping process, both damping effects should be taken into account.

Our main goal for the present work is to explain the neutral-ion line width difference through the damping process of MHD turbulence in a partially ionized medium. The study is based on the physically motivated and numerically tested picture of MHD turbulence as a composition of the cascades of Alfvén, fast, and slow modes. We consider mostly Alfvénic modes in this paper, and provide a detailed treatment of their damping as well as decoupling of neutral fluid from the Alfvénic motions. We consider the effects of scale-dependent anisotropy associated with the cascade, and find that it is very important for understanding the physics of neutral-ion interactions at sufficiently small scales. To gain a general solution, we will study super- and sub-Alfvénic turbulent plasmas separately. To study the damping process in partially ionized plasma, we treat ion–electron and neutral fluids separately. This two-fluid approach is fully described in Zaqarashvili et al. (2011) and studied numerically by Tilley & Balsara (2010). We use this understanding of turbulence to obtain expressions for the difference between neutral and ion squared velocity dispersions in various turbulence regimes. We will also compare our results with those in earlier studies.

We organize this paper as follows. We brieﬂy present the scaling laws of the MHD turbulence cascade in Section 2. Section 3 contains our investigation of the damping process in partially ionized plasma and explicit damping scales in different turbulence regimes of Alfvénic turbulence. Section 4 brieﬂy discusses the damping of fast and slow modes. Following that, in Section 5 we illustrate the effect of distinctive damping scales of ions and neutrals on their different velocity dispersion spectra in various situations. Some important results are extracted and summarized in Section 6. Section 7 introduces the application to molecular clouds. Discussions are given in Section 8. Finally, Section 9 summarizes our results.

2. PROPERTIES OF MHD CASCADE

It is known that small scale MHD perturbations can be decomposed into Alfvén, fast, and slow modes (see Dobrowolny et al. 1980). However, there exists an opinion that such a decomposition is not meaningful within the strong compressible MHD turbulence due to the high coupling of the modes (see Stone et al. 1998). Numerical simulations show that the cascade of Alfvén modes can be treated independently due to the weak back-reaction from slow and fast modes (Cho & Lazarian 2003). This also agrees with the theoretical arguments in the pioneering GS95 study (see also Lithwick & Goldreich 2001). The decomposition was usually discussed in the literature for the case of a strong background magnetic field with inﬁnitesimal ﬂuctuations. Cho & Lazarian (2002, 2003) dealt with perturbations of substantial amplitude and clearly showed the statistical nature of the procedure. A potentially more accurate decomposition was suggested by Kowal & Lazarian (2010). In addition to Fourier transformations, they introduced wavelet transformations that follow the local magnetic ﬁeld direction. Their study conﬁrmed the results in Cho & Lazarian (2003).

We ﬁrst discuss the Alfvénic cascade, which is expected to carry most of the MHD turbulence energy (see Cho & Lazarian 2005).

MHD turbulence can be subdivided into super- and sub-Alfvénic regimes, determined by the initial turbulent energy relative to the magnetic energy (see Brandenburg & Lazarian 2013 for more details). When we are dealing with super-Alfvénic turbulence (i.e., Alfvénic Mach number $M_A = V_L/V_A > 1$, where $V_L$ is turbulent velocity at the injection scale of turbulence $L$, and $V_A = \frac{B}{\sqrt{4\pi \rho}}$ is Alfvén velocity), we have (see Lazarian 2006)

$$k_l \sim k_L$$  \hspace{1cm} (1)  

$$v_l \sim V_L \left( \frac{L}{L} \right)^{1/3}$$  \hspace{1cm} (2)  

for $L_A < 1/k < L$, and

$$k_l \sim k_L^{-1} (k_L L)^{2/3}$$  \hspace{1cm} (3)  

$$v_l \sim V_A \left( \frac{L}{L} \right)^{1/3} = V_L \left( \frac{L}{L} \right)^{1/3}$$  \hspace{1cm} (4)  

for $1/k < L_A$. Here $L_A = L M_A^{3}$ is the Alfvénic scale, where the magnetic field becomes dynamically important and turbulence anisotropy develops (Lazarian 2006).

The cascading rate is given by

$$\tau_{\text{cas}}^{-1} = k^{2/3} L^{-1/3} V_L, \quad L_A < 1/k < L,$$  \hspace{1cm} (5a)  

$$\tau_{\text{cas}}^{-1} = k_{\perp}^{2/3} L^{-1/3} V_L, \quad 1/k < L_A.$$  \hspace{1cm} (5b)  

where the rate given by Equation (5a) is a usual Kolmogorov cascading rate for hydroodynamic turbulence, whereas Equation (5b) corresponds to the GS95 cascading of a strong balanced cascade of Alfvénic turbulence. We then turn to the sub-Alfvénic case ($M_A < 1$). Weak turbulence exists in a range $l_u < 1/k < L$, where $l_u = L M_u^{3}$ is defined as the transition scale from weak to strong turbulence (Lazarian & Vishniac 1999). In this paper, we spare the discussion on weak turbulence because of its very limited spatial range.

When we arrive at the strong turbulence region with scales smaller than $l_u$, scalings become

$$k_l \sim L^{-1} (k_L L)^{2/3} M_A^{4/3},$$  \hspace{1cm} (6)  

and

$$v_l \sim V_d \left( \frac{l_u}{l} \right)^{1/3} = V_L \left( \frac{l_u}{L} \right)^{1/3} M_A^{4/3}.$$  \hspace{1cm} (7)  

The corresponding cascading rate is

$$\tau_{\text{cas}}^{-1} = v_l / l_u = k_{\perp}^{2} L^{-1} V_L / M_A^{4/3}.$$  \hspace{1cm} (8)  

We see that the cascade proceeds faster at smaller scales.

The other two basic modes in MHD turbulence are fast and slow modes, which are compressible. The cascade of slow modes evolves passively and follows the same GS95 scaling as described above (GS95, Cho & Lazarian 2003; Lithwick &
Goldreich 2003). Fast modes have weak coupling with Alfvén modes and show isotropic distribution. The cascade of fast modes is radial in the Fourier space and fast modes have scaling relations compatible with acoustic turbulence (Cho & Lazarian 2002). In GS95-type turbulence, the energy cascade of fast modes takes longer than the eddy-turnover time (Cho & Lazarian 2003; Yan & Lazarian 2004). The cascade rate is

\[ \tau^{-1}_{\text{cas}} = \left( \frac{k}{L} \right)^2 \frac{V_A^2}{V_f}, \quad M_A > 1, \quad (9a) \]

\[ \tau^{-1}_{\text{cas}} = \left( \frac{k}{L} \right)^2 \frac{V_L^2}{V_f}, \quad M_A < 1. \quad (9b) \]

The damping analysis in the rest of the paper is based on these properties of MHD turbulence.

3. DAMPING OF ALFVÉNIC CASCADE IN PARTIALLY IONIZED PLASMA

Our study of the damping process of turbulence is based on the linear analysis of MHD perturbations. In this section we discuss the decoupling scales, and then present the damping scales in different turbulence regimes.

3.1. Decoupling Scale

Decoupling can happen when neutrals decouple from ions or vice versa. In a mostly neutral medium, neutrals decouple at a larger scale compared to ions. The neutral-ion decoupling scale \( k_{\text{dec,ni}} \) is determined by the condition that the frequency of Alfvén waves is equal to neutral-ion collisional frequency, namely,

\[ k_{\text{dec,ni}} V_A = \nu_{ni}. \quad (10) \]

Here \( \nu_{ni} = \gamma_d \rho_i \) is the neutral-ion collisional frequency. \( \gamma_d \) is the drag coefficient defined in Shu (1992) and \( \nu_{ni} \) is related to the ion-neutral collisional frequency \( \nu_{in} \) by \( \nu_{ni} \rho_n = \nu_{in} \rho_i \).

**Super-Alfvénic turbulence.** In the case of super-Alfvénic turbulence, turbulence performs an isotropic Kolmogorov cascade until reaching \( l_A \). Then turbulent eddies get more and more elongated along magnetic field lines. For the decoupling scale in the GS95 turbulence regime, in combination with Equation (3), Equation (10) yields

\[ k_{\text{dec,ni}} = \nu_{ni}^3 \frac{V_{L}^{3/2} V_{L}^{3/2}}{V_f} \sqrt{1 + \frac{V_A}{\nu_{ni} l_A}}, \quad 1/k_{\text{dec,ni}} < l_A. \quad (11) \]

If \( k_{\text{dec,ni}}^{-1} \ll l_A \), \( \frac{V_A}{\nu_{ni} l_A} \) becomes much smaller than 1. Then we can approximately get,

\[ k_{\text{dec,ni}} \sim \nu_{ni}^{3/2} L^{1/2} V_L^{3/2}. \quad (12) \]

Notice that the component of \( k_{\text{dec,ni}} \) perpendicular to magnetic field is

\[ k_{\text{dec,ni},\perp} = \nu_{ni}^{3/2} L^{1/2} V_L^{3/2}. \quad (13) \]

It shows \( k_{\text{dec,ni},\perp} \sim k_{\text{dec,ni}} \) when \( k_{\text{dec,ni}}^{-1} \ll l_A \) due to an increasing anisotropy with decreasing scales.

**Sub-Alfvénic turbulence.** For the sub-Alfvénic case, anisotropy applies to all scales below the injection scale and is prominent in the strong turbulence regime. By inserting Equation (6) in Equation (10) in strong turbulence regime, we can obtain

\[ k_{\text{dec,ni}} = \nu_{ni}^{3/2} L^{1/2} V_L^{3/2} V_A^{1/2} \sqrt{1 + \frac{V_A M_A^2}{\nu_{ni} L}}, \quad 1/k_{\text{dec,ni}} < l_A. \quad (14) \]

Similarly, for a small decoupling scale, the second term in the square root can be neglected. Then \( k_{\text{dec,ni}} \) can be approximated by its perpendicular component. That is,

\[ k_{\text{dec,ni}} \sim k_{\text{dec,ni},\perp} = \nu_{ni}^{3/2} L^{1/2} V_L^{3/2} V_A^{1/2} \]

\[ = \nu_{ni}^{3/2} L^{1/2} V_L^{3/2} M_A^{-1/2}. \quad (15) \]

It is different from \( k_{\text{dec,ni},\perp} \) in a super-Alfvénic case (Equation (13)) by \( M_A^{-1/2} \).

Below the neutral-ion decoupling scale, Alfvén waves cannot be sustained in neutrals. Neutral fluid starts to evolve along the hydrodynamic cascade, while ions still experience frequent collisions with neutrals and the MHD cascade proceeds in ions. Reaching the scale where ions also decouple from neutrals, the two species are essentially decoupled from each other. In that case, the Alfvén speed in ions becomes \( V_A = \frac{B}{\sqrt{4 \pi \rho_i}} \), where \( \rho_i \) is ion density. The ion-neutral decoupling scale \( k_{\text{dec,ini}}^{-1} \) is determined by

\[ k_{\text{dec,ini}} V_A = \nu_{ini}. \quad (16) \]

The expressions of \( k_{\text{dec,ini}} \) are

\[ k_{\text{dec,ini}} = \frac{\nu_{ini}}{V_A} \sqrt{1 + \frac{V_A}{\nu_{ini} l_A}}, \quad 1/k_{\text{dec,ini}} < l_A \quad (17) \]

for super-Alfvénic turbulence, and

\[ k_{\text{dec,ini}} = \frac{\nu_{ini}}{V_A} \sqrt{1 + L M_A^{-4} \frac{\nu_{ini}}{V_A}}, \quad 1/k_{\text{dec,ini}} < l_A \quad (18) \]

for sub-Alfvénic turbulence.

3.2. Damping Scale of Alfvénic Turbulence

In the molecular cloud context, the general dispersion relation of Alfvén waves incorporating damping through both neutral-ion collisions and neutral viscosity takes the form

\[ \omega^3 + i \left( \tau^{-1} + (1 + \chi) \nu_{in} \right) \omega^2 - \left( k^2 \cos^2 \theta V_{Ai}^2 + \chi \tau^{-1} \nu_{in} \right) \omega - i \left( \tau^{-1} + \nu_{in} \right) k^2 \cos^2 \theta V_{Ai}^2 = 0, \quad (19) \]

where \( \tau^{-1} = k^2 \nu_{in} \), representing collision frequency of neutrals (Lazarian et al. 2004). Here \( \nu_{in} \) is the kinematic viscosity in neutrals,

\[ \nu_{in} = \frac{1}{n_{in} c_{in}^2} c_{in}^2 = \frac{1}{n_{in} n_{in} c_{in}^2} c_{in} = \frac{P_{in}}{\rho_i}. \quad (20) \]

\( \chi \) is defined as \( \rho_n / \rho_i \). If we set \( \tau^{-1} = 0 \), the above equation recovers the classic dispersion relation of Alfvén waves found by Piddington (1956).

The complex wave frequency is expressed as \( \omega = \omega_R + i \omega_I \). By assuming weak damping (i.e., \( |\omega| \ll |\omega_R| \)), we obtain the
approximate analytic solution
\[
\omega_R^2 = \frac{F_1(\tau^{-1}_v, \nu_{ni})}{F_2(\tau^{-1}_v, \nu_{ni})},
\]

\[
\omega_1 = -\left[\tau^{-1}_v(1 + \chi) \nu_{ni} + k^2 \cos^2 \theta V_{Ai}^2 \right] \nu_{ni},
\]

\[
2\left[ k^2 \cos^2 \theta V_{Ai}^2 + \chi \tau^{-1}_v \nu_{ni} + \left( \tau^{-1}_v + (1 + \chi) \nu_{ni} \right)^2 \right],
\]

where
\[
F_1(\tau^{-1}_v, \nu_{ni}) = \left( k^2 \cos^2 \theta V_{Ai}^2 + \chi \tau^{-1}_v \nu_{ni} \right)^2 + \left( \tau^{-1}_v + (1 + \chi) \nu_{ni} \right) k^2 \cos^2 \theta V_{Ai}^2,
\]

\[
F_2(\tau^{-1}_v, \nu_{ni}) = \chi \tau^{-1}_v \nu_{ni} + k^2 \cos^2 \theta V_{Ai}^2 + \left( \tau^{-1}_v + (1 + \chi) \nu_{ni} \right)^2.
\]

The absolute value of the imaginary component \(|\omega_1|\) is the rate of damping. By equating the damping rate and the cascading rate of turbulence, we can determine the scale where the cascade of Alfvénic turbulence is truncated, namely, the damping scale \(k^{-1}_\text{dam}\). In what follows, we consider “damping scales” to be the damping scales of Alfvénic turbulence. We will specifically discuss the damping scale of the hydrodynamic turbulence in neutrals in Section 3.6.

Derivation of \(k_\text{dam}\) directly from Equation (21b) is not easy and the resulting expression may be too complicated to illuminate the physical meaning. We found that at high wave frequency limit, \(\nu_{ni}, \nu_{in} \ll \omega\), the wave frequency can be reduced to

\[
\omega_R^2 = k^2 \cos^2 \theta V_{Ai}^2,
\]

\[
\omega_1 = -\frac{\nu_{in}}{2}.
\]

It signifies that neutral viscosity can only influence the behavior of the Alfvén waves over the scales, where ions are coupled with neutrals. That is reasonable because when ions are decoupled from neutrals, the viscosity in neutrals does not affect the MHD turbulence carried by ions. Hence, the damping scales incorporating two damping effects should be calculated in the coupling regime. And, since ions remain coupled with neutrals until the relatively small ion-neutral decoupling scale, the assumption of strong coupling for Alfvén waves is valid over extended scales down to \(k^{-1}_\text{dec.in}\). Therefore, we are able to employ \(\omega_1\), which is analytically derived at the limit of strong coupling, to calculate \(k_\text{dam}\) of Alfvénic turbulence.

We first rewrite Equation (19) as

\[
\omega^3 - \omega^2 \omega + \nu_{ni} \left[ i(1 + \chi) \omega^2 - \chi \tau^{-1}_v \omega - i \omega_k \right] + \tau^{-1}_v \left( \omega^2 - \omega_k \right) = 0,
\]

where \(\omega_k = k \cos \theta V_{Ai}\). At low wave frequencies, namely, \(\omega \ll \nu_{in}\), after some simplifications, the above equation becomes

\[
(1 + \chi) \nu_{ni} \omega^2 + i \left( \chi \tau^{-1}_v \nu_{ni} + \omega_k^2 \right) \omega - \nu_{ni} \omega_k = 0.
\]

The approximate solutions under the weak-damping assumption are then

\[
\omega_R^2 = k^2 \cos^2 \theta V_{Ai}^2,
\]

\[
\omega_1 = -\frac{\epsilon_n}{2} \left( \tau^{-1}_v + \omega_k^2 \right).
\]

We see that the real part of the wave frequency (Equations (26a) and (23a)) corresponds to the classic Alfvén waves. Given this simplified expression of \(\omega_1\), we are able to obtain \(k_\text{dam}\) analytically.

1) **Super-Alfvénic turbulence:** In Kolmogorov turbulence regime, the equality between \(|\omega_1|\) (Equation (26b)) and \(\tau^{-1}_v\) (Equation (5a)) yields

\[
k_\text{dam} = 2 \pi \epsilon_n^{-3/4} L^{-1} \sqrt{2} \left( 2 \nu_{ni} + \frac{V_{Ai}^2}{\nu_{in}} \right)^{-3/4}, \quad l_A < 1/k_\text{dam} < L.
\]

Here \(\cos \theta = \omega_k\) is just \(k_j/k\). We adopt the scaling relation given by Equation (1).

In the MHD turbulence regime, according to the critical balance condition given by GS95, \(k_j V_A = k_j \nu_{in}\), namely, \(k_j V_A = \tau^{-1}_v\). Due to \(|\omega_1| = \epsilon_j V_A\) (Equation (26a)), the damping condition \(\tau^{-1}_v = |\omega_1|\) is equivalent to

\[
|\omega_1| = |\omega_0|.
\]

By taking advantage of Equation (3), the above equation and Equation (26) give the corresponding damping scale

\[
k_{\text{dam}.i} = \left[ \left( \nu_{ni} + \frac{V_{Ai}^2}{\nu_{in}} \right) + \frac{8 \nu_{ni} l_A}{\epsilon_n} \right],
\]

\[
k_{\text{dam}} = k_{\text{dam}.i} \sqrt{1 + l_A k_{\text{dam}.i}}, \quad 1/k_\text{dam} < l_A.
\]

The damping of Alfvénic turbulence depends on the angle between \(k\) and \(B\). Here we assume that the parallel and perpendicular components of \(k_{\text{dam}}\) with respect to the local magnetic field are related by GS95 scaling relation (Equation (3) for super-Alfvénic turbulence), which describes the scale-dependent anisotropy of turbulent eddies, and has been proved by numerical simulations (Cho & Lazarian 2002, 2003). The accuracy of this approximation is discussed in Appendix A by providing more detailed calculations. We find the scaling relation to be sufficiently accurate and therefore use it for the rest of the paper.

2) **Sub-Alfvénic turbulence:** In a strong MHD turbulence regime, by using Equations (26) and (6), the condition \(|\omega_1| = |\omega_0|\) yields

\[
k_{\text{dam}.i} = \left[ \left( \nu_{ni} + \frac{V_{Ai}^2}{\nu_{in}} \right) + \frac{8 \nu_{ni} l_A}{\epsilon_n} \right],
\]

\[
k_{\text{dam}} = k_{\text{dam}.i} \sqrt{1 + L M_A^{-4} k_{\text{dam}.i}}, \quad 1/k_\text{dam} < l_A.
\]

It is very similar to Equation (29). In fact, we find that Equation (30) can be directly obtained from Equation (29) by replacing \(l_A\) with \(L M_A^{-4}\). This difference comes from the
different scaling relations of super- and sub-Alfvénic turbulence (see Equations (3) and (6)).

The damping scales presented here can be used in a general situation when neutral-ion collisions and neutral viscosity play comparable roles in turbulence damping. Next, we will discuss the relative importance of the two damping effects in an effectively coupled regime, where neutral viscosity can play a significant role (Section 3.3). Then we study the simplified dispersion relations in limit cases dealing with only one dominant damping effect (Sections 3.4 and 3.5). The damping scale introduced in each case has a simpler form and applies to different situations.

### 3.3. Relative Importance of Neutral-ion Collisional and Neutral Viscous Damping

When it comes to astrophysical applications, it is important to evaluate the relative importance of the two damping effects. The starting point is the damping rate derived from the simplified general dispersion relation at low wave frequencies (Equation (25)). The two terms in Equation (26) represent the contributions from the two damping effects. Their ratio,

\[ r = \frac{\pi^{-1} \nu_{ni}}{\omega^2_k}, \quad (31) \]

reflects the relative role of neutral viscosity, as compared to neutral-ion collisional damping. This expression can be further evaluated by taking into account different turbulence regimes. For super-Alfvénic turbulence, it becomes

\[ r \approx 0.8 \left( \frac{T}{10^3 \, K} \right)^{1/2} \left( \frac{B}{10 \, \mu G} \right)^{-2} \frac{\rho}{10^{-20} \, \text{g cm}^{-3}} \xi, \]

\[ l_A < 1/k < L, \quad (32a) \]

\[ r \approx 0.4 \left( \frac{T}{10^3 \, K} \right)^{1/2} \left( \frac{B}{10 \, \mu G} \right)^{-2} \frac{\rho}{10^{-20} \, \text{g cm}^{-3}} \xi \left( \frac{l}{l_A} \right)^{2/3}, \]

\[ 1/k < l_A, \quad (32b) \]

And for sub-Alfvénic turbulence, when \( 1/k < l_n \), we have

\[ r \approx 0.4 \left( \frac{T}{10^3 \, K} \right)^{1/2} \left( \frac{B}{10 \, \mu G} \right)^{-2} \frac{\rho}{10^{-20} \, \text{g cm}^{-3}} \xi \left( \frac{l}{l_A} \right)^{2/3} M_A^{-2/3}. \]

\[ \xi \]

Here \( \cos \theta \) in \( \omega_k \) is derived from the scalings presented in Section 2. Specifically, \( r > 1 \) indicates that neutral viscosity is the dominant damping effect. Conversely, it can be safely neglected. Notice that \( r \) increases with decreasing length scales. It can always exceed 1 at a sufficiently small scale. But recall that neutral viscosity has no effect on Alfvén waves at high wave frequencies when the two fluids are decoupled. Therefore, the criteria is applicable to determine the relative importance of the two damping mechanisms at a certain scale in the coupling regime. By comparing Equations (32b) and (33), we find with the same \( T, B, \rho, \xi, \) and \( l \), that sub-Alfvénic turbulence is more likely to have \( r > 1 \) than super-Alfvénic turbulence. We will explore the applicability of the criteria for models of molecular clouds in Section 3.7.

The damping scales for a joint damping effect can be further simplified when only one damping effect is dominant. We again perform the analysis in super- and sub-Alfvénic turbulence separately.

1. **Super-Alfvénic turbulence**: \( k_{\text{dam}} \) in Kolmogorov turbulence regime \( [L, l_A] \) (Equation (27)) can be reduced to

\[ k_{\text{dam}} = \frac{2 \pi^{1/2} \nu_{ni}^{3/2}}{\xi^{3/2} L^{1/2} V_A^{3/2}}, \quad r < 1, \]

\[ k_{\text{dam}} = \frac{2 \pi^{3/2}}{\xi^{3/2} \nu_{ni}^{3/2} \xi^{1/2} L^{1/2} V_A^{3/2}}, \quad r > 1. \]

In GS95 type turbulence regime \( (< l_A) \), when \( r < 1 \), \( k_{\text{dam}} \) (Equation (29a)) becomes

\[ k_{\text{dam}} = \frac{2 \nu_{ni}}{V_A}. \]

The resulting \( k_{\text{dam}} \) is

\[ k_{\text{dam}} = (2 \nu_{ni})^{3/2} L^{1/2} V_A^{1/2} \frac{1 + \frac{V_A}{2 \nu_{ni} L}}{2 \nu_{ni} L}. \]

If we take into account \( k_l \ll k \) and \( k \sim k_l \) at scales much smaller than \( l_A \), \( k_{\text{dam}} \) can be approximated by its perpendicular component

\[ k_{\text{dam}} \sim k_{\text{dam}, \perp} = (2 \nu_{ni})^{3/2} L^{1/2} V_A^{1/2}. \]

Notice that different from the total \( k_{\text{dam}} \), its perpendicular component \( k_{\text{dam}, \perp} \) does not have a dependence on \( V_A \) or \( B \). In addition, going back to the approximate expression of \( k_{\text{dec,n}} \) at \( k_{\text{dec,n}}^{-1} \ll l_A \) (Equation (12)), we find

\[ k_{\text{dam}} \approx 2 k_{\text{nic}}. \]

In the opposite situation, when \( r > 1 \) the damping scale in Equation (29) has the form

\[ k_{\text{dam, \parallel}} = \frac{2 V_A}{\xi \nu_{ni} l_A^2}, \]

\[ k_{\text{dam}} = \left( \frac{2}{\xi \nu_{ni} l_A^2} \right)^{3/2} L^{1/2} V_A^{3/2} \left[ 1 + \left( \frac{\xi \nu_{ni} l_A^{1/2}}{2 l_A V_A} \right)^2 \right]. \]

With \( l_n \ll l_A \) taken into account, \( k_{\text{dam}} \) is approximated by

\[ k_{\text{dam}} \approx \frac{3 \pi^{3/2}}{\xi \nu_{ni}^{3/2} L^{1/2} V_A^{3/2} \xi}, \]

which is the same as that given by Equation (34b). It shows that \( k_{\text{dam}} \) due to neutrals’ viscous damping is uniform over all scales in super-Alfvénic turbulence.

2. **Sub-Alfvénic turbulence**: In the strong turbulence regime, at \( r < 1 \), Equation (30) takes the form

\[ k_{\text{dam, \parallel}} = \frac{2 \nu_{ni}}{V_A}, \]

\[ k_{\text{dam}} = (2 \nu_{ni})^{3/2} L^{1/2} V_A^{1/2} \left[ 1 + \frac{V_A M_A^3}{2 \nu_{ni} L} \right]. \]
\( k_{\text{dam}} \) can be simplified to
\[
k_{\text{dam}} \sim k_{\text{dam}, A} = (2\nu_i) \frac{3}{2} \frac{k^3}{L^2} V_a^2 V_A^2
\]
\[
= (2\nu_i) \frac{3}{2} \frac{k^3}{L^2} V_a^2 \frac{3}{2} M_A^{-\frac{1}{2}},
\]
due to strong anisotropy at small scales. Compared with \( k_{\text{dam}, A} \) in super-Alfvénic case (Equation (37)), the only difference is \( M_A^{-1/2} \) in the above expression. It can also be related to \( k_{\text{dec}, n} \) in Equation (15) by
\[
k_{\text{dam}} \approx 2\pi k_{\text{dec}, n}.
\]

Also, we find that \( k_{\text{dam}, A} \) remains the same as that in the super-Alfvénic case (Equation (35)). The different turbulence properties between super- and sub-Alfvénic turbulence can only affect \( k_{\text{dam}, A} \) in this case. At \( r > 1 \), Equation (30) is simplified to,
\[
k_{\text{dam}, A} = \frac{|k|}{\xi_n \nu_i L M_A^2},
\]
\[
k_{\text{dam}} = \left( \frac{2}{\xi_n \nu_i} \right)^{\frac{3}{2}} L^2 \frac{1}{V_A} V_A \frac{3}{2} \frac{1}{1 + \left( \xi_n \nu_i M_A^2 \right)^{\frac{1}{2}}}.
\]

Under the consideration of \( \xi_n \ll L \), \( k_{\text{dam}} \) can be approximated by
\[
k_{\text{dam}} \approx \frac{3}{2} \frac{\xi_n}{\nu_i} \frac{3}{2} L^2 \frac{1}{V_A} \frac{3}{2} M_A^{-\frac{1}{2}}.
\]

The expressions of the damping scales presented above are derived by solving the general dispersion relation that includes both damping effects at the strong coupling limit (Equation (25)). In most cases, only one damping mechanism plays the dominant role. Knowing the relative importance of the neutral-ion collisional and viscous damping, we only need to deal with a simplified dispersion relation considering one damping effect. In this spirit, we will perform the analysis only focusing on the dominant damping process in the following subsections.

### 3.4. Damping of Alfvénic Cascade Due to Neutral-ion Collisions

In the case where damping due to neutral viscosity is negligible (i.e., \( r < 1 \)), by setting \( \nu_i^A = 0 \) in Equation (19), we can obtain the well-known dispersion relation considering only neutral-ion collisions (see, e.g., Piddington 1956; Kulsrud & Pearce 1969; Soler et al. 2013b),
\[
\omega^3 + i(1 + \chi) \nu_m \omega^2 - k^2 \nu_m^2 \omega^2 V_A^2 = 0.
\]

We obtain the damping rate \( |\omega_1| \) by approximately solving the above equation under the weak-damping assumption. The approximate solutions are
\[
\omega_R^2 = \frac{k^2 \nu_m^2 \omega^2 V_A^2 (1 + \chi) \nu_m^2 + k^2 \nu_m^2 \omega^2 V_A^2}{(1 + \chi) \nu_m^2 + k^2 \nu_m^2 \omega^2 V_A^2},
\]
\[
\omega_1 = -\frac{\nu_m \chi k^2 \nu_m^2 \omega^2 V_A^2}{2((1 + \chi) \nu_m^2 + k^2 \nu_m^2 \omega^2 V_A^2)}.
\]

The solution can be further simplified at the strong coupling limit,
\[
\omega_R^2 = k^2 \cos^2 \theta V_A^2,
\]
\[
\omega_1 = -\frac{\xi_n \omega_R^2}{2\nu_i}.
\]

At the weak coupling limit, the solution becomes the same as Equation (23). By comparing with \( \tau_{\text{cas}}^{-1} \), we find that the ratio \( |\omega_1|/\tau_{\text{cas}}^{-1} \) depends on both the coupling degree of the two fluids and turbulence properties,

1) **Super-Alfvénic turbulence**: We first consider super-Alfvénic turbulence. We find (Equations (48b), (1), (3) and (5))
\[
\frac{|\omega_1|}{\tau_{\text{cas}}} \sim k^{4/3}, \quad l_A < k_{\text{dec}, n}^{-1} < L,
\]
\[
\frac{|\omega_1|}{\tau_{\text{cas}}} \sim k^{2/3}, \quad k_{\text{dec}, n}^{-1} < l_A,
\]

at scales above \( k_{\text{dec}, n}^{-1} \), and (Equations (23b), (5))
\[
|\omega_1| \sim k^{-2/3}
\]
below \( k_{\text{dec}, n}^{-1} \). This means the damping is likely to occur in the coupled regime. Therefore, same as discussed in Section 3.2, we can use \( \omega_1 \) at the strong coupling limit (Equation (48b)) to calculate the damping scale.

By equalizing \( |\omega_1| \) (Equation (48b)) and \( \tau_{\text{cas}}^{-1} \) (Equation (5)), the damping scale is given by
\[
k_{\text{dam}} = 2\pi \frac{3}{2} \frac{3}{2} \frac{L^2}{V_A^2} \frac{3}{2} \frac{1}{N L \nu_i^2},
\]
\[
l_A < 1/k_{\text{dam}} < L,
\]
\[
k_{\text{dam}} = \left( \frac{2}{\xi_n} \right)^{\frac{3}{2}} L^2 V_A \frac{3}{2} \frac{1}{1 + \left( \xi_n \nu_i M_A^2 \right)^{\frac{1}{2}}} \frac{1}{2\nu_i l_A},
\]
\[
1/k_{\text{dam}} < l_A,
\]

where \( \xi_n = \rho_n/\rho \). The scaling relations between \( k_{\text{dam}} \) and \( k_{\text{dec}, n} \) are taken from Equation (1) for (51a) and from Equation (3) for (51b). The perpendicular component of \( k_{\text{dam}} \) in Equation (51b) is
\[
k_{\text{dam}, A} = \left( \frac{2}{\xi_n} \right)^{\frac{3}{2}} L^2 V_A \frac{3}{2},
\]
which is independent of magnetic field strength.

2) **Sub-Alfvénic turbulence**: We then move to sub-Alfvénic turbulence. When \( k_{\text{dec}, n} \) is in the strong turbulence regime, \( |\omega_1|/\tau_{\text{cas}}^{-1} \) becomes (Equations (48b), (23b), (8), (6))
\[
\frac{|\omega_1|}{\tau_{\text{cas}}} \sim k^{2/3}, \quad k_{\text{dec}, n}^{-1} < k^{-1} < l_A,
\]
\[
\frac{|\omega_1|}{\tau_{\text{cas}}} \sim k^{-2/3}, \quad k^{-1} < k_{\text{dec}, n}^{-1}.
\]

Similar to the super-Alfvénic case, it also shows that \( |\omega_1|/\tau_{\text{cas}}^{-1} = 1 \) can be reached in the coupled regime. Analogously, we equate \( |\omega_1| \) (Equation (48b)) and \( \tau_{\text{cas}}^{-1} \) (Equation (8)),

\[
|\omega_1| \sim k^{-2/3},
\]

\[
|\omega_1| \sim k^{-2/3}.
\]
in combination with Equation (6), and derive

\[ k_{\text{dam}} = \left( \frac{2\nu_{\text{n}}}{\xi_n} \right)^{\frac{3}{2}} L^2 \frac{V_L^{-2}}{V_A^2} \frac{1}{\eta_{\text{n}} M_{\text{A}}^2} \left[ 1 + \frac{\xi_n V_A M_{\text{A}}^2}{2\nu_{\text{n}} L} \right]. \]  

(54)

At small scales, it can be approximated by \( k_{\text{dam,1}} \),

\[ k_{\text{dam,1}} = \left( \frac{2\nu_{\text{n}}}{\xi_n} \right)^{\frac{3}{2}} L^2 V_L^{-2} V_A^{-1} \left( \frac{2\nu_{\text{n}}}{\xi_n} \right)^{\frac{3}{2}} L^2 V_L^{-2} M_{\text{A}}^{-1/2}. \]  

(55)

Another important issue is that when solving the dispersion relation (Equation (46)), we find that the properties of the solutions depend on the value of \( \chi \). Soler et al. (2013b) pointed out that when \( \chi < 8 \), we can always get a complex wave frequency from the dispersion relation, whereas when \( \chi > 8 \), there is an interval of parallel wavenumbers \( [k^+, k^-] \) where only purely imaginary solutions exist. This “cutoff” region was identified earlier by Kulsrud & Pearce (1969), corresponding to the range of no propagation of Alfvén waves. The physical reason has been discussed in Soler et al. (2013b) as the friction force dominates over the magnetic tension force and suppresses the oscillatory modes within the “cutoff” (see also Moschovias 1987; Kamaya & Nishi 1998). Here it is necessary to reexamine the cutoff region by taking the scale-dependent anisotropy into account. By using the scaling relations given in Section 2, we derive the full expressions of \( k^+ \) and \( k^- \) for both super- and sub-Alfvénic turbulence (see Appendix B).

Soler et al. (2013b) identified the “cutoff” by studying the polynomial discriminant of the dispersion relation. In fact, there is another convenient method to approximately locate the “cutoff” region. We find that the ratio \( |\omega_1|/|\omega_R| \) (Equation (48)) increases with \( k \) at large scales, but decreases with \( k \) at small scales (Equation (23)). The damping is relatively weak at both large and small scales, except for the intermediate scales, where \( |\omega_1| \) becomes comparable to and even exceeds \( |\omega_R| \). This heavily damped region corresponds to the “cutoff.” Its boundary wavenumbers can be calculated by setting

\[ |\omega_R| = |\omega_1|. \]  

(56)

In the coupled regime, the expressions in Equations (48) and (56) give

\[ k_{\text{c,1}} = \frac{2\nu_{\text{n}} V_A}{\xi_n}. \]  

(57)

At high wave frequencies, the solutions in Equation (47) are reduced to Equation (23). Using Equations (23) and (56) yields

\[ k_{\text{c,2}} = \frac{\nu_{\text{n}}}{2V_A}. \]  

(58)

The minimum \( \chi \) required for the existence of “cutoff” wavelengths can also be roughly estimated by setting \( k_{\text{c,1}} = k_{\text{c,2}} \).

Then we turn to the expressions of \( k_{\parallel}^{\pm} \) given by Soler et al. (2013b),

\[ k_{\parallel}^{\pm} = \frac{\nu_{\text{n}}}{V_A} \left[ \frac{\chi^2 + 20\chi - 8}{8(1 + \chi)^3} \pm \frac{\chi^{1/2}(\chi - 8)^{1/2}}{8(1 + \chi)^3} \right]^{-1/2}. \]  

(59)

They stand when \( \chi > 8 \). At the limit of large \( \chi \), the above expressions approximate to

\[ k_{\parallel}^{+} \approx \frac{2\nu_{\text{n}} V_{\text{n}}}{V_A}, \]  

(60a)

\[ k_{\parallel}^{-} \approx 0.6\frac{\nu_{\text{n}}}{V_A}, \]  

(60b)

which are very close to Equations (57) and (58). It confirms that the “cutoff” set by \( |\omega_R| = |\omega_1| \) provides a good approximation of the nonpropagating region with \( \omega_R = 0 \).

Meanwhile, as described in Section 3.2, \( |\omega_R| = |\omega_1| \) in a strong coupling and a strong MHD turbulence regime is equivalent to the damping condition and provides the damping scale. Therefore, \( k_{\text{dam}} \) corresponds to the lower-limit wave-number \( k_{\parallel}^{+} \) of the cutoff. That explains how \( k_{\parallel}^{+} \) in Equation (57) is the same as \( k_{\text{dam},1} \) in Equation (35) at \( \xi_n \sim 1 \).

Moreover, the consistency between the expressions of \( k^+ \) (Appendix B) and \( k_{\text{dam}} \) indicates that the calculation of the cutoff region provides an alternative approach of determining damping scales in the case of neutral-ion collisional damping. We will numerically compare \( k^+ \) and \( k_{\text{dam}} \) in Section 3.7. However, it is worthwhile to mention that, same as with the critical balance, this approach only applies when \( k_{\text{dam}} \) is in strong MHD turbulence. Another limitation is that it requires a low ionization degree (i.e., \( \chi > 8 \)), which is usually true in molecular clouds.

The cutoff arises due to the linear interaction between the two fluids. The main difference of turbulence damping from wave cutoff is the involvement of nonlinear turbulence cascade. But indeed, we see the correlation between the boundary of the cutoff \( k_{\parallel}^{+} \) and \( k_{\text{dam}} \). The physical reason is that Alvénic turbulence has an Alvénic rate \( (k_{\parallel} V_A) \) equal to the eddy-turnover rate \( (k_{\text{c,1}} V_A) \). The critical balance between the wave-like motions parallel to magnetic field and mixing motions of magnetic field lines in the perpendicular direction bridges the linear waves and nonlinear turbulence cascade (GS95; Cho et al. 2002b; Cho & Lazarian 2003). To better seek the physical connections between \( k_{\parallel}^{+} \) and \( k_{\text{dam}} \), hereafter we use the condition \( |\omega_R| = |\omega_1| \) to confine the cutoff region \( [k_{\parallel}^{+}, k_{\parallel}^{-}] \), which is also shown by numerical results to have a marginal difference from the non-propagation region \( [k^+, k^-] \) with \( \omega_R = 0 \). We list the expressions of \( k_{\parallel}^{\pm} \) for different turbulence regimes in Appendix B.

### 3.5. Damping of Alvénic Cascade due to the Viscosity of Neutrals

When neutral viscosity is the dominant damping effect (i.e., \( r > 1 \)), the general dispersion relation (Equation (19)) can be simplified to

\[ \omega^3 + i\tau_{\nu}^{-1}\omega^2 - k^2 \cos^2 \theta_{\text{Ai}}^2 \omega - i\tau_{\nu}^{-1}k^2 \cos^2 \theta_{\text{Ai}}^2 = 0. \]  

(61)

The approximation \( \nu_{\text{n}} = 0 \) used here removes the effect of neutral-ion collisions, as well as their coupling. Recall that neutral viscosity can only affect Alvén waves when neutrals and ions are coupled (Section 3.2). But in terms of damping, it is applicable over scales where neutral-ion collisional damping is negligible, and so that a simplified damping rate can be
Table 1

| Processes                              | Neutral-ion Collisions | Neutral Viscosity |
|----------------------------------------|------------------------|------------------|
| Turbulence                             | Super      | Sub    | Super      | Sub    |
| Scales                                 | $k_{\text{dam}}$     | $k_{\text{dam}}$ |
|                                        | $(51a)$    | $(51b)$     | $(54)$     | $(63a)$    | $(63b)$    | $(65)$    |
| or $(164a)$ or $(165a)$                |                     |                 |            |            |            |

Table 1 summarizes the damping scales derived in Sections 3.4 and 3.5. When comparing the results in Section 3.3 for the corresponding limit situations, we find that in an almost neutral plasma (i.e., $\xi \sim 1$) the two approaches come to the same results. The expressions in Table 1 provide a convenient way to evaluate $k_{\text{dam}}$ in different turbulent regimes. We numerically test the accuracy of these analytical $k_{\text{dam}}$ from Table 1 in Section 3.7.

3.6. Damping Scale of the Hydrodynamic Turbulence in Neutrals

After neutrals decouple from ions (i.e., $k > k_{\text{dec,ni}}$) hydrodynamic turbulence starts to evolve in neutral fluid, with a cascading rate

$$\tau_{\text{cas}}^{-1} = k^{2/3}L^{-1/3}V_L.$$  (67)

The dissipation mechanisms of turbulence in neutrals also include both neutral-ion collisions and viscosity in neutrals. Their relative importance can be determined by the ratio of their damping rates,

$$r_n = \frac{\tau_{\text{ni}}^{-1}}{\tau_{\text{ni}}} = \frac{\nu_n}{\nu_{\text{ni}}}.$$  (68)

On the neutral-ion decoupling scale, assuming we are in GS95-type turbulence regime, $\tau_{\text{ni}} = (V_A k_{\text{dec,ni}})$ is equal to $\tau_{\text{cas}}^{-1}$, owing to the critical balance. And $r_n$ at $k_{\text{dec,ni}}$ can be written as

$$r_n = \frac{c_n}{V_A} \frac{k_{\text{dec,ni}}}{\cos\theta},$$  (69)

which is smaller than 1 as a result of $I_n \ll k_{\text{dec,ni}}^{-1}$. In this case, neutral-ion collisions dominate damping. But since $\tau_{\text{cas}}^{-1}/\tau_{\text{ni}} \propto k^{2/3}$, the hydrodynamic cascade in neutrals at scales below $k_{\text{dec,ni}}^{-1}$ remains unaffected by their collisions with ions. On the other hand, $r_n$ increases with $k$, so viscosity will become the dominant damping effect at small scales when $r_n > 1$, and eventually truncate the hydrodynamic cascade. By equating the viscous damping rate and turbulence cascading rate, namely $\tau_{\text{ni}}^{-1} = \tau_{\text{cas}}^{-1}$, we get the corresponding viscous scale,

$$k_v = \nu_n^{-1} L^{-1/3} V_L^{1/3}.$$  (70)

$k_{\text{ni}}^{-1}$ is the damping scale where the hydrodynamic cascade terminates. In a typical molecular cloud, the damping scale of neutrals is much smaller than that of the Alfvénic cascade.

It is worthwhile to clarify that in a particular situation, damping of Alfvénic turbulence can happen before neutrals decouple from ions. In this case the above analysis cannot apply because no turbulence exists in neutral fluid at $k > k_{\text{dam}}$. But the turbulence cascade in ions may reemerge below the damping scale because magnetic field perturbations are not suppressed and drive velocity fluctuations in the damping regime (Lazarian et al. 2004). We will not provide a detailed discussion about this situation in this work; we refer the reader to Lazarian et al. (2004) for more extensive information.

3.7. Numerical Tests in Models of Typical Molecular Clouds

We list the parameters used for typical super- and sub-Alfvénic molecular clouds in Table 2. $L$ and $V_L$ are chosen to have typical values for the ISM. We set different $V_L$ values for the two models of sub-Alfvénic molecular clouds. The $\gamma$ value is taken from the calculations in Draine et al. (1983). Other parameters are taken from Lazarian et al. (2004). We adopt the mean molecular mass of ions and neutrals to be $m_i = 29m_H$ and $m_n = 2.3m_H$ (see Shu 1992). We define $\beta = \frac{2\beta}{V_L^2}$ for the whole plasma, and $\beta = \frac{2\beta}{V_L^2}$ for the ion–electron fluid. We will designate them as Model 1, 2, and 3 in the following discussions.
We then apply the analytical expressions of damping scales to molecular clouds. With the parameters used, we numerically solve the general dispersion relation (Equation (19)), and compare the numerically derived damping scales with the analytical ones listed in Table 1.

Figure 1 illustrates the normalized damping rate as a function of the normalized wave number for Model 1. Open circles are the analytical result for neutral-ion collisional damping (Equation (47b)). Its simplification in the coupling regime is shown by filled circles (Equation (48b)) and provides a good approximation over a wide range of wavenumbers. Triangles represent the general analytical solution, including two damping processes (Equation (21b)). It is consistent with the numerical damping rate (solid line). No effect of neutral viscosity can be seen in this case. Purely imaginary solutions are omitted in the numerical result. The cutoff (shade region) is confined by \( k^+ \) and \( k^- \) (Equation (164)), and \( k^- \) overlaps with \( k^-_c \) (Equation (167)). Clearly, \( k^+ \) coincides with the \( k_{\text{dam}} \) calculated using our analytical expression (Equation (51b)). They both also coincide with the wavenumber where the turbulence cascade (dash–dotted line) is truncated by damping. Other wavenumbers, \( l^{-1}_A \), \( k_{\text{dec,ni}} \) (Equation (11)), and \( k_v \) (Equation (70)) are also denoted by vertical dashed lines.

Figures 2(a) and (b) present the results for Model 2 and 3. Notice the damping both happens in strong MHD turbulence regime. Damping in Model 2 exhibits a similar behavior to Model 1. Our analytical \( k_{\text{dam}} \) (Equation (54)) agrees well with \( k^+ \) (Equation (165a)) and the numerical result.

However, Model 3 shows distinctive features. First, neutral viscosity dominates damping. The analytical damping rate including both damping effects (Equation (21b), triangles) agrees well with the numerical result (solid line, Equation (19)), which is larger than that of neutral-ion collisional damping obtained by numerically solving Equation (46; dashed line). The inclusion of neutral viscosity leads to a larger damping scale and a wider cutoff region than the predictions given by neutral-ion collisional damping alone. The actual boundaries of the cutoff region are \( k_{\text{dam}} \) (i.e., \( k^+_c \), Equation (66)) and \( k^-_c \) (Equation (168)). \( k^-_c \) and \( k^+ \) (Equation (165b)) overlap.

Furthermore, Model 3 falls into the situation we discussed at the end of Section 3.6, where damping happens before neutrals decouple from ions (i.e., \( k_{\text{dam}} < k_{\text{dec,ni}} \)). The dotted line shows the damping rate given by Equation (62). It does not exactly align with the actual solution, but can still provide a good approximation (Equation (66)) of \( k_{\text{dam}} \), where the damping rate intersects the cascading rate.

On the other hand, the difference between Model 2 and 3 offers a good opportunity to examine the criteria, \( r \), which we developed in Section 3.3 (Equation (31)). We replot Figures 2(a) and (b) in Figures 2(c) and (d). Solid and dashed lines still represent the numerical damping rates with and without neutral viscosity. The squares show the joint contribution from both neutral-ion collisional and neutral viscous damping, given by Equation (26b). It is a good approximation of the total damping rate at large scales. The dash–dotted line shows \( r \) as a function of scales. Neutral viscosity overwhelms neutral-ion collisional effect at the scales where \( r \) exceeds 1. In coupling regime, \( r \) is smaller than 1 over all scales in Model 2. But \( r \) exceeds 1, together with the appearance of neutral viscosity becoming important in Model 3. Therefore, we are convinced that \( r \) is capable of benchmarking the relative importance of neutral viscosity in turbulence damping.

4. Damping of Compressible Modes in Partially Ionized Plasma

This paper is mostly devoted to the damping of Alfvén modes of MHD turbulence. A comprehensive study on the damping of compressible modes in different ISM conditions will be presented in a later paper. For completeness, here we include a brief discussion on damping of fast and slow modes.

4.1. Decoupling Scale

Fast modes are isotropic and have the neutral-ion decoupling wavenumber

\[
\kappa_{\text{dec,ni}} = v_{\text{ni}}/V_A. \tag{71}
\]
Different from Alfvén and fast modes, in low-\(\beta\) medium, compression from slow modes can induce additional acoustic waves in neutrals after they decouple from ions. The neutral-ion decoupling scale of slow modes is then determined by the acoustic waves developed in neutrals. The expression of \(k_{\text{dec,ni}}\) in slow modes is

\[
k_{\text{dec,ni}} = k_{V_i} c_{s} \frac{\nu_i}{\nu_{ni}}
\]

**4.2. Damping Scale of MHD Turbulence**

We proceed as before for Alfvén modes. We first focus on the derivation of the dispersion relation for magnetoacoustic modes in a partially ionized two-fluid plasma given by Soler et al. (2013a; also see Zaqarashvili et al. 2011). We again assume weak damping \(|\omega_1| \ll |\omega_R|\) and approximately attain the solutions at low and high wave frequencies. At the limit of low wave frequency, \(\omega \ll \nu_{ni}\), we find

\[
\omega_R^2 = \frac{1}{2} \left[ (c_s^2 + V_A^2) \pm \sqrt{(c_s^2 + V_A^2)^2 - 4c_s^2 V_A^2 \cos^2 \Theta} \right] k^2.
\]

\[\text{(73a)}\]

\[
\omega_1 = -\frac{k^2 \left[ \xi_i V_A^2 (c_s^2 k^2 - \omega_R^2) + \xi_i c_s^2 \omega_R^2 \right]}{2\nu_{ni} \left[ k^2 (c_s^2 + V_A^2) - 2\omega_R^2 \right]}.
\]

\[\text{(73b)}\]

The sound speed \(c_s\) used here is defined as

\[
c_s = \sqrt{c_s^2 \xi_i + c_m^2 \xi_m} = \sqrt{\frac{\gamma kT (2n_i + n_m)}{\rho}}.
\]

\[\text{(74)}\]
The classic magnetosonic waves are regained in the real part, with the sign \( \pm \) corresponding to fast and slow waves, respectively. In a low-\( \beta \) (\( \beta \lesssim 1 \)) environment, as commonly seen in molecular clouds, Equation (73) reduces to

\[
\omega_R^2 = \frac{V_{A}^2 k^2}{2}, \quad \omega_I = -\frac{c_sn V_{A}^2 k^2}{2\nu_{ni}},
\]

for fast modes, and

\[
\omega_R^2 = c_s^2 k^2 \cos^2 \theta, \quad \omega_I = -\frac{c_sc_c^2 k^2}{2\nu_{ni}},
\]

for slow modes.

At the converse limit \( \omega \gg \nu_{ni} \), we get the approximate analytical solutions

\[
\omega_R^2 = \frac{1}{2} \left( \epsilon_s^2 + V_{A}^2 \right) \pm \sqrt{\left( \epsilon_s^2 + V_{A}^2 \right)^2 - 4c_{si}^2 V_{A}^2 \cos^2 \theta} k^2,
\]

\[
\omega_I = -\frac{\nu_{ni}}{2}.
\]

Equations (73) and (77) are consistent with the earlier results in Ferriere et al. (1988). In a low-\( \beta \) condition, \( |\omega_R| \) in the above solution can be simplified to \( V_{A}k \) for fast modes and \( c_s k \cos \theta \) for slow modes.

In compressible turbulence, neither fast nor slow modes can provide as efficient shearing motions as Alfvén modes. Thus the neutral viscosity induced by the shear is generally less important than the neutral-ion collisions in damping compressible modes. We can further examine their relative importance by comparing their damping rates in a strongly coupled regime. In a low-\( \beta \) medium, the ratio of viscous to neutral-ion collisional damping rates is (Equations (20), (75b) and (71))

\[
r_I = \frac{c_{si} l_n k_{\text{dec,ini}}}{\xi_n V_{A}} = \frac{2}{\xi_n} \frac{c_{si} l_n k_{\text{dec,ini}}}{V_{A}},
\]

for fast modes, and (Equations (20), (76b) and (72))

\[
r_s = \frac{c_{si} l_n k_{\text{dec,ini}}}{\xi_n} = \frac{2}{\xi_n} l_n k_{\text{dec,ini}},
\]

for slow modes, where we assume \( k \approx k_{\text{c}} \) due to prominent turbulence anisotropy and \( c_s \approx c_{si} \) due to low ionization. Because \( l_n \) is usually much smaller than the decoupling scale, and \( c_{si} < V_{A} \) in a low-\( \beta \) condition, the above ratios are both much smaller than 1, indicating that neutral viscous damping can be safely neglected in comparison with neutral-ion collisional damping. Next we present the damping scales of fast and slow modes due to neutral-ion collisions.

(1) **Damping of fast modes:** The comparison between \( |\omega_I| \) (Equations (75b) and (77b)) and \( \tau_{\text{c,ini}}^{-1} \) (Equation (9)) shows

\[
\frac{|\omega_I|}{\tau_{\text{c,ini}}} \sim k^{1/2}, \quad k \ll k_{\text{dec,ini}},
\]

for super- and sub-Alfvénic turbulence, respectively. It takes the form

\[
k_{\text{dam}} = \begin{cases} \frac{2\nu_{ni}}{\xi_n} V_{A}^{2/3} l_n^{-1/3} V_{A}^{2/3}, & M_A > 1, \\ \frac{2\nu_{ni}}{\xi_n} V_{L}^{2/3} L_{\text{c}}^{-1/3} V_{L}^{-2}, & M_A < 1, \end{cases}
\]

when \( \beta \) is small.

Using the set of parameters of Model 1, Figure 3(a) illustrates the damping of the cascade of fast modes. The same symbols are used as in Figure 1. The solid line is the numerical damping rate by solving the dispersion relation for compressible modes, Equation (51) in Soler et al. (2013a). Open circles are the damping rate from Equations (75b) and (77b). The dash-dotted line is the cascading rate from Equation (9a). Its intersection with the damping rate corresponds to the damping scale given by Equation (82a), which is indicated by the vertical dashed line.

(2) **Damping of slow modes:** Slow modes have the same \( \tau_{\text{c,ini}} \) as Alfvén modes (see Section 2). Before tackling the damping of the turbulence cascade, we first deal with the cutoff region of slow modes. From Equations (76), (3), and (6), we deduce

\[
k_{\text{c,||}}^- = \frac{2\nu_{ni}}{\xi_n c_s L_{\text{c}}}, \quad 1/k_{\text{c,||}}^- < l_n,
\]

for super-Alfvénic turbulence, and

\[
k_{\text{c,||}}^- = \frac{2\nu_{ni}}{\xi_n c_s L_{\text{c}} M_A}, \quad 1/k_{\text{c,||}}^- < l_n,
\]

for sub-Alfvénic turbulence. Using Equation (77) at a low-\( \beta \) limit, \( |\omega_R| = |\omega_I| \) gives

\[
k_{\text{c,||}} = \frac{\nu_{ni}}{2c_{si}}.
\]

The expressions of \( k_{\text{c,||}}^- \) are given in Appendix B.

The inspection of the damping rate in Figure 3(b) shows the cutoff starts earlier before \( |\omega_I| \) (solid line) intersects with \( \tau_{\text{c,ini}}^{-1} \) (dash-dotted line), which results in \( k_{\text{dam}} = k_{\text{c,||}}^- \). The analytical damping rate (open circles) are from Equations (76b) and (77b). The solid and dashed lines represent the usual slow modes and the slow modes in neutrals, respectively. We see there are two nonpropagating intervals in the usual slow modes. Their outer boundaries correspond to \( k_{\text{c}}^- \) (Equation (169)),
Regarding the new sort of slow modes in neutrals, it is actually acoustic waves sustained by neutrals that are induced by the compression from the usual slow modes (Zaqarashvili et al. 2011; Soler et al. 2013b). It has wave frequency as

\[ \omega_1^2 = c_{\text{sn}}^2 k^2, \]

\[ \omega_1 = -\frac{\nu_m}{2} \sin \theta, \quad k_{\text{dec,ni}} < k < k_{\text{dec,ni}}. \] (87a)

\[ \omega_1 = -\frac{\nu_m}{2}, \quad k > k_{\text{dec,ni}}. \] (87b)

We further examine \( \omega_1 \) (Equation (76b)) and \( \tau_{\text{cas}}^{-1} \) (Equations (5b), (8)) at \( k_c^+ \), using Equations (3) and (6), and find

\[ \left| \omega_1(k_c^+) \right| = \left| \omega_R(k_c^+) \right| = c_c k_c^+, \] (88a)

\[ \tau_{\text{cas}}^{-1}(k_c^+) = V_A k_c^+. \] (88b)

Equation (88b) stands for both super- and sub-Alfvénic turbulence. It shows \( \left| \omega_1 \right|/\tau_{\text{cas}}^{-1} = c_c/V_A < 1 \) at the scale \( k_c^{+} \) in a low-\( \beta \) plasma. That is, the equality \( \left| \omega_1 \right| = \tau_{\text{cas}}^{-1} \) happens at a smaller scale than \( 1/k_c^{+} \). Consequently, the \( k_{\text{dam}} \) of slow modes is represented by \( k_c^{+} \).

Another constraint on the \( k_{\text{dam}} \) of slow modes comes from Alfvénic cascade. Because slow modes are slaved to Alfvén modes and do not cancel themselves (GS95; Cho & Lazarian 2002), if Alfvén modes are damped first, the cascade of slow modes subsequently terminates. Hence more exactly, we have

\[ k_{\text{dam}} = \min(k_c^+, k_{\text{dam, Alfvén}}). \] (89)

as the damping scale of slow modes.

Figure 3. Damping rates of compressible modes in the Model 1 cloud. The same symbols are used as in Figure 1. Solid lines are numerical solutions to the dispersion relation (Equation (57) in Zaqarashvili et al. 2011). In Figure (a), the analytical damping rate (open circles) is from Equations (75b) and (77b). In Figure (b), purely imaginary solutions are omitted in the numerical result. Solid and dashed lines are “ion” and “neutral” slow modes, respectively. Open circles correspond to Equations (76b) and (77b). The outer boundaries of the two cutoff regions \( k_c^{+} \) are given by Equation (169).

5. DIFFERENCE BETWEEN THE VELOCITY DISPERSION SPECTRA OF NEUTRALS AND IONS

We start with the Alfvén modes. Because the energy spectra of turbulence vary in different \( M_\Lambda \) domains, it is necessary to perform the analysis in super- and sub-Alfvénic turbulence separately.

5.1. Super-Alfvénic Turbulence

The 3D energy spectrum of the GS95-type turbulence is given by Cho et al. (2002b),

\[ E(k_\perp, k_i) = \frac{1}{3 \pi} \frac{\nu_m^2}{V_A} l^{-1/3} k^{-10/3} \exp \left( -\frac{l_\perp^{1/3}}{k_c^{1/3} k_i^{2/3}} \right). \] (90)

Here we replace the injection scale of strong MHD turbulence in the original equation with \( l_\Lambda \). By integrating the above expression over \( k_c^{1/3} \), the turbulent energy spectrum density for super-Alfvénic turbulence follows

\[ E(k_\perp) = \frac{2}{3} L^{-2/3} V_A^2 k^{-5/3}. \] (91)

at \( k^{-1} < l_\Lambda \). The Kolmogorov turbulence at \( [L, l_\Lambda] \) has

\[ E(k) = \frac{2}{3} L^{-2/3} V_A^2 k^{-5/3}. \] (92)

Equations (91) and (92) apply to both neutrals and ions above \( k_{\text{dec,ni}}^{-1} \). However, in the GS95 turbulence regime \( (k^{-1} < l_\Lambda) \), for scales smaller than \( k_{\text{dec,ni}}^{-1} \), Equation (91) only applies to ions. Neutrals begin to carry a hydrodynamic cascade independently, with \( k \) being isotropic and \( k = k_\perp \) at \( k_{\text{dec,ni}}^{-1} \) having an energy spectrum

\[ E_n(k) = \frac{2}{3} L^{-2/3} V_A^2 k^{-5/3}. \] (93)
At the scale \( k_{\text{dam}} \), the Alfvénic turbulence cascade terminates in ion–electron fluid, but the hydrodynamic cascade proceeds in neutrals until reaching \( k_\nu \).

Because the energy spectra differ in different regimes, we discuss the following cases.

1. \( 1/k_{\text{dec,ni}} > 1/k_{\text{dam}} > l_\lambda \): In this case, ions and neutrals have the same turbulent energy spectra (i.e., \( E(k) = E_\nu(k) \)). Since the square of velocity dispersion is proportional to the integration of the energy spectrum in \( k \) space (LH08), the squared velocity dispersion at \( k \) is

\[
\sigma_\nu^2(k) \sim \int_{k}^{k_{\text{dam}}} E(k)dk = L^{-2/3}V_L^2 k^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}
\]  

(94)

for ions and

\[
\sigma_n^2(k) \sim \int_{k}^{k_{\text{dam}}} E_n(k)dk = L^{-2/3}V_L^2 k^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}
\]  

(95)

for neutrals. It is worth noting that in observations \( k^{-1} \) is the subcloud scale at which the corresponding velocity dispersion is measured. The difference between the squared velocity dispersions of neutrals and ions is

\[
\Delta \sigma^2 = \sigma_n^2(k) - \sigma_\nu^2(k) = L^{-2/3}V_L^2 (k_{\text{dam}}^{-2/3} - k_\nu^{-2/3}).
\]  

(96)

It results from the different integration domains due to the different damping scales of the turbulence cascade in neutrals and ions. With a small viscous scale of neutrals, the above equation becomes

\[
\Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}.
\]  

(97)

Thus the damping scale of ions can be determined from the measurement of \( \Delta \sigma^2 \).

2. \( l_\lambda > 1/k_{\text{dec,ni}} > 1/k_{\text{dam}} \): The Alfvénic turbulence is anisotropic in this regime. Since neutrals and ions carry the same Alfvénic turbulence before they decouple, we only need to focus on the velocity dispersions at scales smaller than \( 1/k_{\text{dec,ni}} \). From \( k_{\text{dec,ni}} \), isotropic turbulence arises in neutrals, thus the turbulence in the two fluids follow different cascades. For ions, the squared velocity dispersion at \( k > k_{\text{dec,ni}} \) is given by

\[
\sigma_\nu^2(k) \sim \int_{k_{\text{dec,ni}}}^{k} E_\nu(k)dk = L^{-2/3}V_L^2 k_{\text{dec,ni}}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3},
\]  

(98)

whereas for neutrals, it is

\[
\sigma_n^2(k) \sim \int_{k_{\text{dec,ni}}}^{k} E_n(k)dk = L^{-2/3}V_L^2 k_{\text{dec,ni}}^{-2/3} - L^{-2/3}V_L^2 k_\nu^{-2/3}.
\]  

(99)

To obtain the exact expression of \( \Delta \sigma^2 \), we start the integration from \( k_{\text{dec,ni}} \). Considering that at \( k_{\text{dec,ni}} \), neutrals and ions still share the same energy spectrum, we get

\[
\sigma_\nu^2(k_{\text{dec,ni}}) \sim \int_{k_{\text{dec,ni}}}^{k_{\text{dam}}} E_\nu(k)dk = L^{-2/3}V_L^2 k_{\text{dec,ni}}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}.
\]  

(100)

When the integration applies to neutrals, we have

\[
\sigma_n^2(k_{\text{dec,ni}}) \sim \int_{k_{\text{dec,ni}}}^{k_\nu} E_n(k)dk = L^{-2/3}V_L^2 k_{\text{dec,ni}}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}.
\]  

(101)

Thus the difference of the squared velocity dispersions of neutrals and ions can be obtained,

\[
\Delta \sigma^2 = \sigma_n^2(k_{\text{dec,ni}}) - \sigma_\nu^2(k_{\text{dec,ni}}) = L^{-2/3}V_L^2 (k_{\text{dec,ni}}^{-2/3} - k_\nu^{-2/3}).
\]  

(102)

As \( 1/k_\nu \) is negligibly small, \( \Delta \sigma^2 \) can be written as

\[
\Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}.
\]  

(103)

Given the turbulence driving, it only depends on the perpendicular component of the damping scale of ions.

3. \( 1/k_{\text{dec,ni}} > l_\lambda > 1/k_{\text{dam}} \): This case is similar to Case (2), but the energy spectra of ions and neutrals begin to diverge only from \( l_\lambda \). Thus we just need to replace \( k_{\text{dec,ni}} \) with \( l_\lambda \) in Equations (100) and (101), and get the same \( \Delta \sigma^2 \) as expressed in Equation (102).

Figure 4(a) displays \( E(k) \) as a function of \( k \) using the parameters of Model 1, corresponding to Case (2). The shaded area illustrates \( \Delta \sigma^2 \). Figure 4(b) shows \( E(k) \) of sub-Alfvénic turbulence as a comparison, which we will discuss in the next subsection.

Figure 5 simulates the observed \( \sigma^2_n(k) \) and \( \sigma^2_\nu(k) \) as a function of length scale (i.e., \( k^{-1} \)) using the parameters of Model 1. It shows that neutrals have larger velocity dispersion compared to that of ions, due to the smaller turbulence damping scale in neutrals. This results in a wider line width of neutrals than ions from an observational point of view. Furthermore, because the viscous scale \( k_\nu^{-1} \) is relatively small with the parameters used for a typical molecular cloud, the curve for neutrals can be considered as passing (0, 0) point. It corroborates that we can safely neglect the term with \( k_\nu \) in the expressions of \( \Delta \sigma^2 \). Figure 5(a) displays the results down to the scale \( k_\nu^{-1} \). Figure 5(b) zooms in on \([k_\nu^{-1}, k_{\text{dec,ni}}^-] \). They are derived using Equations (98) (solid line) and (99) (dash–dotted line). The dashed line in Figure 5(b) is calculated by taking the approximation,

\[
\sigma_\nu^2(k) \sim L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}}^{-1}.
\]  

(104)

It replaces \( k_{\text{dec,ni}} \) in Equation (98) by \( k \). The dashed line almost overlaps the solid one. It shows at small scales, this change does not make a significant difference for the velocity dispersion spectrum. Therefore, we can use Equations (104) and (99) to compare with the observed velocity dispersion spectra and get the same expression of \( \Delta \sigma^2 \) as Equation (103).

5.2. Sub-Alfvénic Turbulence

In strong turbulence regime (i.e., \( k^{-1} < l_\nu \)), the 3D energy spectrum is (Cho et al. 2002b)

\[
E(k_\perp, k_\parallel) = \frac{1}{3\pi} V_L^2 l_\nu^{-1/3} k_\perp^{-10/3} \exp \left( -L^{1/3} \frac{k_\parallel}{M_A^{1/3} k_\perp^{2/3}} \right).
\]  

(105)
where \( l_t \) is the injection scale of strong MHD turbulence. The corresponding 1D energy spectrum can be obtained by integrating over \( k \),

\[
\frac{E(k)}{L^2 \nu^3} \sim \frac{1}{2} L^{-2/3} V_L^2 M_N^{2/3} k^{-5/3}, \quad 1/k < l_t.
\]  

(106)

We consider the case where \( l_t > 1/k_{\text{dec,ni}} < 1/k_{\text{dam}} \). At scales larger than the neutral-ion decoupling scale, the Alfvénic turbulence cascade proceeds in the strongly coupled two fluids. At \( k_{\text{dec,ni}} \), the squared velocity dispersion of ions is given by

\[
\sigma_i^2(k_{\text{dec,ni}}) \sim \int_{k_{\text{dec,ni}}}^{k_{\text{dam}}} E(k) \, dk \\
= L^{-2/3} V_L^2 M_N^{2/3} k_{\text{dec,ni}}^{-2/3} - L^{-2/3} V_L^2 M_N^{2/3} k_{\text{dam}}^{-2/3}.
\]  

(107)

which is different from Equation (100) by a \( M_N^{2/3} \) factor.

The energy spectrum of neutrals with their hydrodynamic cascade starting in the strong sub-Alfvénic turbulence regime takes the form,

\[
E_n(k) = \frac{2}{3} L^{-2/3} V_L^2 M_N^{2/3} k^{-5/3}. 
\]  

(108)

The squared velocity dispersion for neutrals is then

\[
\sigma_n^2(k_{\text{dec,ni}}) \sim \int_{k_{\text{dec,ni}}}^{k_{\text{dam}}} E_n(k) \, dk \\
= L^{-2/3} V_L^2 M_N^{2/3} k_{\text{dec,ni}}^{-2/3} - L^{-2/3} V_L^2 M_N^{2/3} k_{\text{dam}}^{-2/3}. 
\]  

(109)
Similar to super-Alfvénic turbulence case, in practice, we can use

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 M_A^{2/3} k^{-2/3} - L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3} \]  

(100)

to approximate

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 M_A^{2/3} k^{-2/3} - L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3} \]  

(101)

at small scales due to high anisotropy when comparing with observations. The difference of the squared velocity dispersions of ions and neutrals is

\[ \Delta \sigma^2 = L^{-2/3}V_L^2 M_A^{2/3} (k_{\text{dam}, \perp}^{-2/3} - k_{\nu}^{-2/3}). \]  

(112)

It can also be written as

\[ \Delta \sigma^2 \sim L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3} \]  

(113)

when \( k_{\nu}^{-1} \) is much smaller than \( k_{\text{dam}, \perp}^{-1} \).

Figure 4(b) shows the energy spectrum corresponding to Model 2. The shade region in Figure 4(b) shows \( \Delta \sigma^2 \) expressed by Equation (112). In contrast to super-Alfvénic turbulence, anisotropy applies over all scales in both weak and strong turbulence regimes of sub-Alfvénic turbulence.

Finally, we discuss a particular case where \( k_{\text{dam}, \perp}^{-1} > k_{\nu}^{-1} \). We take the situation of Model 3 as an example. For Model 3, the turbulence is damped when neutrals and ions are still strongly coupled and behave as one fluid. Thus the Alfvénic turbulence is truncated in both neutrals and ions at \( k_{\text{dam}, \perp} \), but the turbulence cascade in ions may resume at small scales (see Lazarian et al. 2004) as mentioned earlier. This will lead to a larger velocity dispersion and wider line width of ions than neutrals. Although this model contradicts the existing observational facts, it still deserves special attention because this particular turbulence regime may not be covered by current limited observational data.

We found among all the cases discussed for both super- and sub-Alfvénic turbulence, only in the Kolmogorov turbulence regime of super-Alfvénic turbulence, \( \Delta \sigma^2 \) has a dependence on total \( k_{\text{dam}, \perp} \). For all the other cases, \( \Delta \sigma^2 \) is only related to the perpendicular component of the damping scale \( k_{\text{dam}, \perp} \). We show in Section 7 that weather \( \Delta \sigma^2 \) depends on \( k_{\text{dam}, \perp} \) or \( k_{\text{dam}, \perp} \) plays a crucial role in determining magnetic field strength.

In a typical molecular cloud condition, compressible modes are more severely damped compared with Alfvén modes, having the turbulence truncation in a strongly coupled regime (i.e., \( k_{\text{dam}} < k_{\text{dec}, \nu} \)). The turbulent energy spectra of compressible modes dissipate at the same scale for the coupled neutrals and ions, so damping of fast and slow modes does not contribute to the difference between the squared velocity dispersions of neutrals and ions.

6. A SUMMARY OF IMPORTANT RESULTS

Because of the multitude of turbulence regimes and damping effects, we provided the expressions of \( k_{\text{dam}} \) and \( \Delta \sigma^2 \) in a wide variety of situations. From an observational point of view, a recapitulation of the results in most typical situations might be useful. They are summarized as follows.

Super-Alfvénic; Kolmogorov; neutral-ion collisions—In the case of Kolmogorov turbulence and damping dominated by neutral-ion collisions, the damping scale is (Equation (51a))

\[ k_{\text{dam}} = \frac{3}{2} \pi n_i \xi_n \frac{3}{2} L \tau V_L^3 V_A^{-2} \]  

(114)

The squared velocity dispersions are (Case 1 in Section 5.1)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 k_{\nu}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3} \]  

(115a)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 k_{\nu}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3} \]  

(115b)

Their difference depends on the total \( k_{\text{dam}, \perp} \).

\[ \Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3}. \]  

(116)

Super-Alfvénic, GS95-type, neutral-ion collisions—The damping scale perpendicular to the magnetic field is (Equation (52))

\[ k_{\text{dam}, \perp} = \frac{1}{\xi_n} \frac{3}{2} L \tau V_L^3 V_A^{-2} \]  

(117)

which is independent of \( B \). The squared velocity dispersions are (Case 2 in Section 5.1)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 k_{\nu}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3} \]  

(118a)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 k_{\nu}^{-2/3} - L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3} \]  

(118b)

Their difference depends on \( k_{\text{dam}, \perp} \).

\[ \Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam}, \perp}^{-2/3}. \]  

(119)

Sub-Alfvénic, strong, neutral-ion collisions—In the strong turbulence regime, the perpendicular damping scale can be expressed as a function of \( B \) or \( V_A \) (Equation (55))

\[ k_{\text{dam}, \perp} = \frac{1}{\xi_n} \frac{3}{2} L \tau V_L^3 V_A^{-2} \]  

(120)

The squared velocity dispersions and their difference are (Section 5.2)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 M_A^{2/3} k_{\nu}^{-2/3} - L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3} \]  

(121a)

\[ \sigma_1^2(k) \sim L^{-2/3}V_L^2 M_A^{2/3} k_{\nu}^{-2/3} - L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3} \]  

(121b)

\[ \Delta \sigma^2 \sim L^{-2/3}V_L^2 M_A^{2/3} k_{\text{dam}, \perp}^{-2/3}. \]  

(121c)

In fact, the first situation with Kolmogorov turbulence is not common in molecular clouds. The dependence of \( \Delta \sigma^2 \) on \( k_{\text{dam}, \perp} \) instead of \( k_{\text{dam}, \perp} \) is a direct consequence of turbulence anisotropy, which is critical in determining the magnetic field from the observed line width differences.

7. APPLICATION TO MOLECULAR CLOUDS

We next apply the theory of turbulence damping developed above to molecular clouds.

7.1. Velocity Dispersion Spectra

We start with the observed difference in spectral line widths between neutrals and ions. In Section 5, we present the analytical expressions of velocity dispersion spectra for neutrals and ions. In observations, if we can attain data for velocity dispersions on different length scales, by fitting to the observational data, we expect to see

\[ \sigma_1^2(k) = pk^m, \]  

(122a)

\[ \sigma_1^2(k) = pk^m - q, \]  

(122b)

where \( p, q, \) and \( m \) are all fit parameters. The analytical formulae of \( \sigma_1^2(k) \) and \( \sigma_1^2(k) \) in Section 6 suggest that \( p \) is a coefficient
determined by the injection condition of turbulence, and $m$ corresponds to the power index of the velocity dispersion spectrum, which is related to the power index of the energy spectrum of turbulence. $q$ represents the difference between the squared velocity dispersions of neutrals and ions, $\Delta\sigma^2$.

In fact, obtaining the actual velocity distribution from observational data is an interesting but difficult issue, with many observational techniques (e.g., centroids) showing an inability to get spectra in realistic compressible turbulence (Esquivel & Lazarian 2005). By measuring the velocity dispersions of coexistent neutrals and ions at different length scales of a molecular cloud, LH08 found that the lower envelopes of the neutral and ion velocity dispersions can be fitted by power laws, as shown in Equation (122). However, there are several caveats before one attempts to apply the theory developed in this paper to similar observations.

1. The $k_l$ and $k_s$ in all our expressions should be treated as inverse of scales measured parallel and perpendicular to the local mean magnetic field, instead of ordinary Fourier components of wave vectors. However, in observations we can only measure variables integrated along the line of sight (LOS), where only the global system of reference applies (Cho & Lazarian 2003; Esquivel & Lazarian 2005, see Section 8.2).

2. The lower envelope of the LOS velocity dispersions may not exactly follow the actual velocity dispersion spectrum. The limitations related to recovering the 3D dispersion from the observed 2D values were identified in Falceta-Gonçalves et al. (2010) by carrying out a number of MHD simulations with different sonic and Alfvénic Mach numbers. They showed that the discrepancy can be significant in some particular cases (see Figure 2 in their work), and therefore the accuracy of a technique that uses these dispersions is also limited.

Furthermore, the theoretically motivated relation between the observed spectral indices and that of the underlying spectrum of Alfvénic turbulence (see Lazarian & Pogosyan 2004, 2006, 2008) cannot account for the aforementioned relation between the observed 2D and true 3D statistics. Instead, the empirical relation between them is only approximate, as explained in Falceta-Gonçalves et al. (2010).

3. We only present analytical velocity dispersion spectra for Alfvén modes due to their sole contribution in linewidth difference. In reality, however, compressible modes are also involved in turbulent motions. For example, fast modes alone with an energy spectral index $-3/2$ (Cho & Lazarian 2002) or $-2$ (Kowal & Lazarian 2010), can lead to a shallower ($m = -1/2$) or steeper ($m = -1$) velocity dispersion spectrum. It suggests that the actual spectral index also depends on the components of turbulence and energy distribution among different modes.

4. The measurement of the velocity dispersion spectra is also affected by length range. The spectral index near the damping scale can distort the value measured in the inertial range. Besides, further distortions can also come from the weak turbulence range in sub-Alfvénic turbulence.

5. The scaling relations and scale-dependent anisotropy of turbulence with an arbitrary spectrum index are presented in the appendix in Lazarian & Vishniac (1999), while considering magnetic reconnection in GS99 type turbulence. In a situation where other types of turbulence are present, the above analysis should be validated by adjusting the scaling relations shown in Section 2 accordingly.

6. Because the expressions of velocity dispersion spectra differ in different turbulence regimes, the specific turbulence regime should be primarily determined for the quantitative interpretation of the difference in velocity dispersions, provided that the actual spectra are able to be extracted from observations.

7.2. Evaluation of Magnetic Field Strength

In earlier studies attention was paid exclusively to obtaining magnetic field in molecular clouds. Determination of magnetic field strength is of fundamental importance in understanding the dynamics and processes (e.g., star formation), arising in molecular clouds.

In the preceding sections, we provided analytical expressions for damping scales of Alfvénic turbulence in different turbulence regimes. They are not only essential in explaining different line widths of neutrals and ions, but also serve as a possible tool for measuring magnetic field strength through their explicit dependence on magnetic field.

The relative importance of neutral viscosity can be determined by $r$ (Section 3.3) with environment parameters provided. As discussed earlier, neutral viscous damping can lead to wider lines of ions than neutrals. In accordance with the existing observations on the relative narrowing of ion lines, here we only discuss the cases with negligible neutral viscosity.

1. Dependence of $B$ on $k_{\text{dam}}$

In super-Alfvénic turbulence, we rewrite Equation (51) to get

$$B = \frac{4\pi\rho}{\xi_n} \left( \frac{2\nu_{\text{ni}}}{\xi_n} \right)^2 - \frac{L}{k_{\text{dam}}^2}, \quad 1/k_{\text{dam}} > l_\Lambda, \quad (123a)$$

$$B = \frac{4\pi\rho}{\xi_n^3} \left( \frac{2\nu_{\text{ni}}}{\xi_n} \right)^2 - \frac{L}{k_{\text{dam}}^2} L^{-3}, \quad 1/k_{\text{dam}} < l_\Lambda. \quad (123b)$$

When $k_{\text{dam}} > l_\Lambda$, $B$ depends on total $k_{\text{dam}}$. For $k_{\text{dam}} > l_\Lambda$, we also express $B$ in terms of the total $k_{\text{dam}}$ in Equation (123b) by rewriting Equation (51a). But we actually see from Equation (52) that the perpendicular component of $k_{\text{dam}}$ is independent of $B$ and hence $B$ is only related with $k_{\text{dam},||}$. As is known, anisotropy starts at $l_\Lambda$ and increases rapidly with decreasing scales. This leads to a quite weak dependence of $B$ on $k_{\text{dam}}$. Furthermore, if only $k_{\text{dam},\perp}$ can be determined observationally, there is no way to evaluate $B$ when $k_{\text{dam}} > l_\Lambda$.

For $k_{\text{dam}} < l_\Lambda$ in sub-Alfvénic turbulence, we can write $B$ in terms of $k_{\text{dam}}$ from Equation (54). However, given the simple expression of $k_{\text{dam},\perp}$ by Equation (55), we can reach a convenient form of $B$ as

$$B = \sqrt{4\pi\rho} \left( \frac{2\nu_{\text{ni}}}{\xi_n} \right)^{-3} L^{-1} k_{\text{dam},\perp}^{4}, \quad 1/k_{\text{dam}} > l_\text{tr}, \quad (124)$$

Here we show the dependence of $B$ on the damping scale. In observations, in order to evaluate magnetic field strength, $k_{\text{dam}}$ should be first estimated from the measurements on $\Delta\sigma^2$. By combining $\Delta\sigma^2$ expressed by $k_{\text{dam}}$ and expressions of $B$ presented here, $B$ can be rewritten in terms of observational parameters.

2. Method 1: evaluation of $B$ from $\Delta\sigma^2$
We again start from super-Alfvénic turbulence. Our analysis showed that the difference between the squared velocity dispersions of neutrals and ions is (Section 6)

\[
\Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam}}^{-2/3}, \quad 1/k_{\text{dam}} > l_A, \tag{125a}
\]

\[
\Delta \sigma^2 \sim L^{-2/3}V_L^2 k_{\text{dam},\perp}^{-2/3}, \quad 1/k_{\text{dam}} < l_A, \tag{125b}
\]

where the term containing \(k_{\text{dam}}^{-2/3}\) is neglected due to its relatively small value. The above relations show that damping scales can be easily obtained, namely

\[
\Delta \sigma^2 L^{-3}V_L^3 = k_{\text{dam}} \quad 1/k_{\text{dam}} > l_A, \tag{126a}
\]

\[
\Delta \sigma^2 L^{-3}V_L^3 = k_{\text{dam},\perp} \quad 1/k_{\text{dam}} < l_A. \tag{126b}
\]

As discussed earlier, \(k_{\text{dam},\perp}\) in a GS95-type turbulence regime is independent of \(B\) (Equation (120)). Therefore, \(B\) can only be expressed in terms of \(\Delta \sigma^2\) for \(k_{\text{dam}}^{-1} > l_A\). Combining Equations (123a) and (125a), we get

\[
B = 4(\pi v_{\text{th}})^{1/2} \rho^{1/2} \xi_n^{-1/2} L^{1/2} V_L^{-3/2} \Delta \sigma^2. \tag{127}
\]

We see that, to determine \(B\), parameters \(\Delta \sigma^2\), \(V_L\), and \(L\) are needed. \(V_L\) can be taken as the global turbulent velocity measured at the cloud size \(\sim L\).

For sub-Alfvénic turbulence, we consider \(k_{\text{dam}}\) in the strong turbulence regime. Similar to super-Alfvénic case, with \(V_L\), \(\Delta \sigma^2\), and \(L\) measured using the aforementioned method, Equation (121c) and (122) together yield

\[
B = \sqrt{4\pi \rho} \left(\frac{2 v_{\text{th}}}{\xi_n}\right)^{-1} L^{-1} V_L^2 \left(\Delta \sigma^2\right)^{-1} \tag{128}
\]

for \(k_{\text{dam}}^{-1} < l_A\). Once the magnetic field value is determined, damping scale can also be deduced from either Equation (121c) or Equation (124).

(3) Method 2: evaluation of \(B\) from velocity dispersion spectra

An alternative method for estimating \(B\) is by using velocity dispersion spectra. For our limited purpose of exemplifying possibilities of this method, we disregard the limitations discussed in Section 7.1 and consider a model cloud where only Alfvénic turbulence exists. Accordingly, \(m = -2/3\) stands, corresponding to the theoretical expectation of strong Alfvénic cascade (GS95) and numerical studies (e.g., Cho & Lazarian 2003; Kowal & Lazarian 2010).

In super-Alfvénic turbulence, by comparing Equations (115) and (118) with Equation (122), we find

\[
p = L^{-2/3} V_L^2, \tag{129}
\]

and

\[
\left(\frac{p}{q}\right)^{3} = \begin{cases} k_{\text{dam}} \quad l_A < 1/k_{\text{dam}} < L, \\ k_{\text{dam},\perp} \quad 1/k_{\text{dam}} < l_A. \end{cases} \tag{130}
\]

Similar to Method 1, \(B\) value can only be obtained when \(k_{\text{dam}}\) is in the isotropic turbulence regime \([L, l_A]\). From Equation (123a) and above relations, we have

\[
B = 4(\pi v_{\text{th}})^{1/2} \rho^{1/2} \xi_n^{-1/2} p^{-1/2} q. \tag{131}
\]

In the sub-Alfvénic turbulent cloud, comparing Equations (121) and (122) yields

\[
p = L^{-2/3} V_L^2 M_{\text{ref}}^2, \tag{132}
\]

and

\[
k_{\text{dam},\perp} = \left(\frac{p}{q}\right)^{1/2}. \tag{133}
\]

Given the estimated \(k_{\text{dam},\perp}\) from the above equation, \(B\) can be obtained from Equation (124),

\[
B = \sqrt{4\pi \rho} \left(\frac{2 v_{\text{th}}}{\xi_n}\right)^{-3/2} L^{-1} V_L^2 \left(\frac{p}{q}\right)^{3/2}. \tag{134}
\]

Or more conveniently, we can directly deduce \(B\) from Equation (132),

\[
B = \sqrt{4\pi \rho} \ p^{-3/2} L^{-1} V_L^2. \tag{135}
\]

(4) Effect of angle between the magnetic field and LOS

When we consider Alfvénic turbulence, the turbulent motions that account for the observed velocity dispersions are in a direction perpendicular to the magnetic field. In the above equations for measuring \(B\), we assume that the LOS direction is perpendicular to the mean magnetic field. But in practice, an observer can only measure the projected velocity dispersion along LOS, namely, \(\sigma_{\text{LOS}}\). We next briefly discuss the effect of this observational limitation on the evaluation of \(B\).

Consider first sub-Alfvénic turbulence. All eddies of different sizes basically align with the global mean magnetic field \(B_0\). Then the LOS component of \(\sigma\) is

\[
\sigma_{\text{LOS}} = \sigma \sin \alpha, \tag{136}
\]

where \(\alpha\) is the angle between \(B_0\) and LOS. Accordingly,

\[
\Delta \sigma_{\text{LOS}}^2 = \Delta \sigma^2 \sin^2 \alpha. \tag{137}
\]

Therefore, the measured \(B\) can be biased by an angle-dependent factor.

In a super-Alfvénic turbulent cloud, if a telescope’s beam size is larger than \(l_A\), most of the contribution in the observed \(\sigma\) is from isotropic hydro-like motions, so the LOS direction is irrelevant. But if the beam size is smaller than \(l_A\), within each \(l_A\)-size eddy, all smaller eddies can be considered aligning along the local mean magnetic field of the \(l_A\)-size eddy, \(B(l_A)\). Similar to the case in sub-Alfvénic turbulence, the turbulent velocity at scale \(l (l < l_A)\) has a projection on LOS direction as \(v_{\text{LOS}} = v_t \sin \alpha\). Here \(\alpha\) is the angle between \(B(l_A)\) and the LOS. Since each LOS crosses regions with random \(B(l_A)\) orientations, the observed \(\sigma_{\text{LOS}}^2\) is an average over all orientations,

\[
\left\langle \sigma_{\text{LOS}}^2 \right\rangle = \sigma^2 \left\langle \sin^2 \alpha \right\rangle = \sigma^2 \int_{-\alpha_2}^{\alpha_1} \sin^2 \alpha d\alpha, \tag{138}
\]

where \((\alpha_1, \alpha_2)\) is the range of angles between the \(l_A\)-size eddies and LOS. Accordingly, \(\Delta \sigma_{\text{LOS}}^2 = \Delta \sigma^2 \left\langle \sin^2 \alpha \right\rangle\).6

6 The randomly oriented anisotropies are in accordance with our analysis, which is carried out in local reference system, whereas the observed \(\Delta \sigma_{\text{LOS}}^2\) averaging over a distribution of \(\alpha\) is obtained in a global reference system. See Section 8.2 for discussions of the two reference systems.
The orientation of the magnetic field relative to the LOS introduces additional uncertainties for determining the magnetic field from line width differences.

5. Importance of determining turbulence regimes
We introduced two observational methods for evaluating magnetic field strength from the observed line width difference. Method 1 makes use of the constant $\Delta \sigma^2$ measured on a single scale. It can also avoid the uncertainties from the fitted velocity dispersion spectra using multiple-scale measurements in Method 2 as well as from the involvement of compressible modes, thus Method 1 is more promising to obtain reliable estimates of $B$. In addition to magnetic field strength and damping scale, the advantage of Method 2 is that it may provide extra information on the turbulent energy spectrum. Since the results reported in this paper reveal that damping scales, as well as magnetic field, vary with different turbulence regimes, both methods require knowledge of the turbulence regime prior to the estimation of $B$. Only then can one apply the appropriate expression of $B$ in terms of available observational parameters in the corresponding situation.

To demonstrate the importance of identifying the turbulence regime in a magnetic field measurement, we apply Equation (123a) for isotropic super-Alfvénic turbulence, as the measured magnetic field $B'$ in different turbulence regimes. We first carry out a numerical test in super-Alfvénic turbulence using the parameters of Model 1, but keep $M_A$ as a free parameter by adjusting $V_L$. Figure 6(a) exhibits the error of the measured magnetic field $B'$ using Equation (123a) as a function of $k_{\text{dam}}l_A$. We found that at scales smaller than $l_A$, $B'$ underestimates the real $B$, which shows in a sense how much Equation (123a) deviates from the real magnetic field strength when it is applied in anisotropic super-Alfvénic turbulence regime. It suggests that in a super-Alfvénic cloud, when $k_{\text{dam}}^{-1}$ is slightly smaller than $l_A$, the effect of turbulence anisotropy can be included in the error bars. However, when $k_{\text{dam}}^{-1}$ is sufficiently smaller than $l_A$, turbulence local anisotropy develops to such an extent that the magnetic field strength given by Equation (123a) will be far away from the actual value. Take the Model 1 cloud for example, $k_{\text{dam}}l_A \sim 344$, the corresponding error is as large as $\sim 80\%$.

We then perform another test in sub-Alfvénic turbulence using the parameters from Model 2, but keep $L$ as a free parameter. Figure 6(b) presents the error of the measured $B'$ by employing Equation (123a) in comparison with the real $B$, as a function of $k_{\text{dam}}l_tr$. It shows that in the strong turbulence regime in a sub-Alfvénic cloud, Equation (123a) can underestimate the magnetic field strength considerably, even when $1/k_{\text{dam}}$ is comparable to $l_tr$. At a smaller scale, for instance, $k_{\text{dam}}l_tr \sim 1.5 \times 10^6$ in the Model 2 cloud, the error reaches the maximum $\sim 100\%$.

The above tests indicate that the essential base of magnetic field determination using these methods is to identify the turbulence properties.

8. DISCUSSIONS

8.1. AD Scale and Damping Scale
The concepts of the AD Reynolds number $R_{\text{AD}}$ and the associated AD scale are often used in the literature (e.g., LH08, Li et al. 2008; McKee et al. 2010). It is necessary to clarify the relation between the AD scale and the damping scale discussed in this paper.

$R_{\text{AD}}$ is defined as (Myers & Khersonsky 1995; Zweibel & Brandenburg 1997; Zweibel 2002; Li et al. 2006)

$$R_{\text{AD}} = \frac{\nu_l}{\nu_M},$$

(139)

where $\nu_l$ is the characteristic velocity of a fluid element across field lines, $l$ is the size of the fluid element over which the magnetic field has significant variation, and $\nu_M$ is magnetic viscosity, which is also called the “effective magnetic diffusivity” in AD context (Zweibel 2002; Biskamp 2003).

$$\nu_M = \frac{B^2}{4\pi \rho_n \nu_{ni}} = \frac{V_{\text{damp}}^2}{\nu_{ni}},$$

(140)
In a medium with a low ionization fraction, the drift velocity between the neutrals and ions can be solved from the balance between the Lorentz force and drag force acting on ions (Shu 1992),

$$v_{AD} \approx \frac{V_{An}^2}{v_{ni} l} = \frac{v_M}{l}. \quad (141)$$

Thus $R_{AD} = 1$ is equivalent to

$$v_t = v_{AD}, \quad (142)$$

which signifies the equality between the characteristic flow velocity and the drift velocity. The corresponding length is the AD scale,

$$l_{AD} = \frac{v_M}{v_t} = \frac{V_{An}^2}{v_{ni} v_t} \quad (143)$$

In terms of timescales, $R_{AD} = 1$ can also be written as

$$\frac{l}{v_t} = \frac{1}{v_{AD}}. \quad (144)$$

In MHD turbulence, the left side in the above equation is comparable to the turbulence eddy-turnover time of the Alfvén modes $l_{\perp}/v_t$, the inverse of which is the turbulence cascading rate $\tau_{cas}^{-1}$. The AD time on the right side is

$$\frac{l}{v_{AD}} = \frac{l^2 v_{ni}}{V_{An}^2} = \frac{v_{ni}}{V_{An}^2 k^2}, \quad (145)$$

which has a similar form of the inverse damping rate of Alfvén waves due to neutral-ion collisions in a strongly coupled regime (see Equation (48b)). Thus, qualitatively, $R_{AD} = 1$ can be used to characterize the relative importance between the flow motions induced by magnetic perturbations and AD. However, it is challenging to use the value of $R_{AD}$ to determine the scale where the turbulent motions are damped out by the AD effect. In comparison with the damping condition proposed in this work $\tau_{cas}^{-1} = |\omega_1|$, Equation (144), which originates from $R_{AD} = 1$, is incapable of capturing (1) properties of MHD turbulence including turbulence anisotropy and scalings of different modes, for example, the cascading time of fast modes is longer than $1/v_t$ by a factor of $V_{ph}/v_t$, where $V_{ph}$ is the phase speed of fast modes; (2) variance in propagation and dissipation behavior of different MHD waves. Moreover, a missing process in Equation (144) is wave cutoff, which requires the involvement of the propagating waves’ period. In the case of slow modes in a low-$\beta$ medium, wave motions take more time than eddy-like motions. Turbulence vanishes as the waves become nonpropagating due to neutral-ion collisional damping. Accordingly, the damping scale is given by the cutoff scale of waves.

$R_{AD}$ is related with neutral-ion collisional damping; whereas the other damping effect discussed in this work, neutral viscous damping, can be seen in the Reynolds number $Re$. $Re$ is defined as (Lang 1980; Reynolds 1883)

$$Re = \frac{l v_t}{v_n}, \quad (146)$$

where $v_n$ is the kinematic viscosity in neutrals (Equation (20)). The condition $Re = 1$ can be written as

$$\frac{l}{v_t} = \frac{l^2}{v_n}. \quad (147)$$

The inverse of the right side is the collision frequency of neutrals $\tau_n^{-1}$. Regarding the left side, it can be treated as the eddy-turnover time of the hydrodynamic turbulence in the neutral fluid. Then $Re = 1$ corresponds to the damping condition and yields the viscous scale of neutrals. Otherwise, if we apply the eddy-turnover time of the Alfvén modes in MHD turbulence to the left side, $Re = 1$ is equivalent to the damping scale of Alfvén modes due to neutral viscosity in the strongly coupled ions and neutrals.

In brief, $R_{AD}$ and $Re$ are related to neutral-ion collisional damping and neutral viscous damping, respectively. In nonmagnetic turbulence, $Re = 1$ is sufficient for deriving the viscous damping scale. Provided the Kolmogorov scalings are used, it can lead to the same expression in Equation (70), which is obtained by equalling the cascading rate and damping rate in neutrals after they decouple from ions. But in magnetic turbulence, the presence of the magnetic field substantially affects the dynamics of turbulence and develops turbulence anisotropy and wave motions. Consequently, $R_{AD} = 1$ is too crude to properly determine the damping scale, as well as its dependence on magnetic field strength. Even for the Alfvén modes in a particular regime, $[L, l_A]$ in super-Alfvénic turbulence, where magnetic energy is dominated by the kinematic energy of the fluid, and the eddy-turnover time can be expressed as $l/v_t$ for isotropic turbulence, the expression of AD time is still problematic because the wave propagation direction is disregarded.

### 8.2. Local and Global Reference Systems

The scaling relations employed in this work are only valid in the local reference system, with $k_\parallel$ and $k_\perp$ being the wavenumbers measured relative to the local magnetic field direction and turbulence anisotropy being scale dependent (Lazarian & Vishniac 1999; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002b). However, from the observational point of view, only the scales projected on the plane of sky and quantities averaged along the LOS can be measured in the global frame of reference with regard to the mean magnetic field, which is the only reference frame available for observations integrated along the LOS (Esquivel & Lazarian 2005; Lazarian & Pogosyan 2012). Accordingly, the anisotropy attained in the global reference system is scale independent (see detailed discussions in, e.g., Cho & Vishniac 2000).\(^7\)

Concerning the current problem of the observed line width difference, it is related to the velocity dispersion spectrum and turbulent energy spectrum. The spectral index of turbulent energy, as well as velocity dispersion, remains the same in

\(^7\) It is necessary to point out that the scale-dependent global anisotropy seen from the numerical simulations by Vestuto et al. (2003) is an artifact of their isotropic energy injection and insufficient driving scale (peak driving at $k_{peak} L/2\pi = 8$). That creates a transient scale-dependent anisotropy, which vanishes as the inertial range increases.

For studies seeking for the correspondence between the GS95 model and observational data (e.g., Heyer et al. 2008; Heyer & Brunt 2012), the primary concern is in the global frame of reference, which is only accessible to an observer; GS95 scalings are distorted and unobservable.
global reference system as the perpendicular spectrum in the local reference system, because turbulent energy is concentrated (1) in the direction perpendicular to the local magnetic field in anisotropic turbulence; (2) on large scales with a rapid decay of energy toward smaller scales. Point (1) suggests \( E(k) \sim E(k_f) \), and the scaling of turbulence in the global reference frame is determined by perpendicular fluctuations, thus \( E(k_f) \) cannot manifest itself. Point (2) suggests that the magnetic perturbation on a small scale \( \delta B_l/B_0 \), which determines the degree of local anisotropy, is dominated by that on the injection scale \( \delta B_l/B_0 \), which defines the degree of global anisotropy. Therefore, the variety in the shapes of the individual eddies is lost, but only the largest eddy as an average over all smaller eddies is left in the energy spectrum measured in a global reference system. Thus, the spectral index of turbulent energy is uniform, regardless of reference systems, orientation of coordinate axis, and LOS direction (as numerically shown in, e.g., Beresnyak 2014, 2015).

On the one hand, the parameters on the outer scale of turbulence (i.e., \( L \) and \( V_L \)) used in our formulae ensure their applicability in the global reference system that pertains to observations. On the other hand, \( k_{\text{dam},1} \) instead of \( k_{\text{dam}} \) determines the actual value of the damping scale in anisotropic turbulence where turbulent motions cease, and the expression of \( k_{\text{dam},1} \) is obtained by using the scaling relations in the local reference system. Globally isotropic turbulence can have very prominent anisotropy on small scales. Understanding and distinguishing both reference systems is essential for connecting turbulence theories and observations.

In addition, the equilibrium state in the linear analysis is assumed to be a homogeneous partially ionized plasma. But in a real molecular cloud, when the inhomogeneity in density is significant, the local \( M_A \) may differ considerably from the global value at large scales. In particular, substantial density fluctuations expected in high-\( M_A \) turbulence can appreciably change the local value of \( V_A \), affecting the locally measured \( M_A \). In this situation, turbulence that is globally sub-Alfvénic can have super-Alfvénic dense clumps (Burkhart et al. 2009).

Considering this complexity, for such high-density subvolumes our analysis should be modified to use the local parameters of turbulence rather than those for the entire cloud.

8.3. Turbulence in Partially Ionized Gas: Comparison with Selected Earlier Work

Waves in partially ionized gas are well studied (e.g., Braginskii 1965; Kumar & Roberts 2003; Zaqarashvili et al. 2011; Soler et al. 2013a, 2013b). However, MHD turbulence is different from linear waves. Difference in the properties of compressible and incompressible motions, and anisotropy of MHD turbulence must be taken into account. Although it is frequently assumed that MHD turbulence is isotropic, for some astrophysical applications (e.g., cosmic ray propagation and acceleration), it has already been shown that this improper assumption results in many wrong conclusions (see Yan & Lazarian 2002, 2004, 2008). Numerical studies in Cho & Lazarian (2002, 2003) quantified the coupling between the fast, slow, and Alfvén modes, and provided the basis for our present study. In particular, these studies show that it is legitimate to consider the Alfvénic cascade independently from the cascades of other modes (see also GS95; Lithwick & Goldreich 2001; Cho et al. 2002b).

In this paper, we focused on the Alfvénic cascade because in the molecular cloud environment, the other fundamental components, fast modes, and slow modes are damped out in a strongly coupled regime, and do not have differential damping in neutrals and ions. Thus the line width difference solely depends on the properties of the Alfvén modes, while the velocity dispersion measured on a large scale depends on the amplitudes of all three turbulent modes. This potentially opens a possibility of studying the ratio between the intensities of compressible components to that of the Alfvénic component.

Our analysis did not take into account the possible differences in masses of different ionized and neutral species, or their effect on differential damping. This is based on the assumption that the frequencies of the magnetic perturbations we study are much smaller than the resonance frequencies of ions with different masses. This approximation is well motivated for the problem of line width differences that we address.

We consider MHD turbulence in partially ionized gas. This type of turbulence has been studied both numerically and analytically (see Lithwick & Goldreich 2001; Mac Low 2004; Lazarian et al. 2004; Tilley & Balsara 2010). In particular, Lithwick & Goldreich (2001; henceforth LG01) studied super-Alfvénic compressible MHD turbulence in an ion-dominated medium with \( \beta \gtrsim 1 \). In that work neutral–neutral collisions were neglected, and only neutral-ion collisional damping was considered. Through a different approach by including the force that neutrals exert on ions in the momentum equation in the derivation of the dispersion relation, LG01 obtained the damping rates in asymptotic limits (Equation (39) in LG01),

\[
\omega_1 = -\frac{1}{2} \frac{m_i n_i}{m_n n_n} \frac{1}{V_{\text{fr}}} \left\{ \frac{1}{2} (k L_N)^2, \quad k L_N \ll 1 \right. \tag{148a}
\]

\[
\omega_1 = -\frac{1}{2} \frac{m_i n_i}{m_n n_n} \frac{1}{V_{\text{fr}}} \left\{ 1, \quad k L_N \gg 1 \right. \tag{148b}
\]

Here \( m_n \) and \( m_i \) are neutral and ion mass. \( L_N = \frac{c_i}{V_{\text{fr}}} \left( \frac{m_n}{m_i} \right)^{1/2} \) is the neutral mean free path (Equation (37) in LG01), but has different definition from our \( L_N \). In comparison, the damping rates presented in this paper are general. Specifically, in decoupling regime, Equation (23b) in this paper agrees with the above expression at \( k L_N \gg 1 \). In the other limit, when the viscosity of neutrals dominates damping (\( k L_N \ll 1 \)), Equation (62) in this paper recovers Equation (148a) in an ion-dominated environment, where the neutral mean free path is determined by their interactions with ions. An important point is that instead of the approximation \( k \approx k_a \) adopted throughout LG01, we explicitly distinguish the expressions of \( k_{\text{dam},1} \) and \( k_{\text{dam}} \), which turns out to be crucial in interpreting the observed difference between the velocity dispersion spectra of neutrals and ions.

A different treatment of MHD turbulence in partially ionized gas was given in Lazarian et al. (2004). The authors predicted interesting new effects that are the resurrection of the MHD cascade in ions. But their paper only dealt with the effect of neutral drag, which corresponds to \( r > 1 \) in the present study. Here we consider both neutral viscosity and the neutral-ion collisional damping. However, our analysis did not cover all the cases of media introduced in Lazarian et al. (2004).

The difference between the line widths of coexistent neutrals and ions is a known observational fact (Houde...
et al. 2000a, 2000b; Lai et al. 2003 and references therein). Houde et al. (2004) related this difference to AD. The later work LH08 first provided the explanation in light of the differential damping of turbulence in the fluids of neutrals and ions. They derived a single AD scale from $R_{AD} = 1$ (see Section 8.1), which also allowed them to deduce the plane-of-the-sky component of the magnetic field strength in astrophysical applications. Different from their approach, we apply the damping condition $\tau_{\text{dam}}^{-1} = |\omega|$, which incorporates both the turbulence properties and wave behavior. The resulting damping scale is shown to have varying expressions in different turbulence regimes, and also be affected by wave cutoff.

Burkhart et al. (2015) investigated the behavior of Alfvén modes in neutrals and ions by performing two-fluid MHD simulations. They confirmed that neutrals and ions form hydro and MHD cascades separately below the neutral-ion decoupling scale. They found that Alfvén modes in super-Alfvénic turbulence damp below the neutral-ion decoupling scale. This is in accordance with our findings on the relation between the neutral-ion decoupling and damping scales (see Equation (38)). A detailed quantitative comparison with the simulations about the dependence of damping on turbulence parameters will be a good test of our theoretical work.

8.4. Magnetic Field Measurements

Our study reveals different regimes that entail various dependence of $\Delta \sigma^2$ on $B$. In the case where super-Alfvénic turbulence damps out in GS95 turbulence regime (i.e., $k_{\text{dam}} < k_0$), $\Delta \sigma^2$ is independent of the $B$ value. When damping occurs in other turbulence regimes, $B$ has distinctive expressions in terms of not only $\Delta \sigma^2$, but also turbulence parameters (Equations (127) and (128)), which stem from the respective turbulence properties. There is not a single universal expression relating the magnetic field strength with the observed line width difference. Disregarding the effect of various turbulence regimes can result in severely biased estimates of $B$. This suggests a phased approach on evaluating magnetic field strength, where one first identifies the regime of turbulence with other techniques at hand, then obtains the magnetic field value with higher precision from the measured line width difference, and finally checks the correspondence between the magnetization result and the turbulence regime.

To preliminarily distinguish the regimes of turbulence, we need additional data. The initial magnetic field value may only be approximate based on measurements using other techniques, such as the Chandrasekhar–Fermi technique (see Chandrasekhar & Fermi 1953; Falceta-Gonçalves et al. 2008; Houde et al. 2013) or Zeeman observations (Crutcher et al. 2010). There are also new techniques that have been developed for measuring turbulence anisotropy (Lazarian & Pogosyan 2000; Lazarian et al. 2001; Esquivel & Lazarian 2005, 2010, 2011; Tofflemire et al. 2011), which can assist in distinguishing super- and sub-Alfvénic turbulence. With this information, one can then use the corresponding expression to obtain the actual value of magnetic field strength and test the consistency. This combination can make the evaluation of magnetic field strength more reliable.

It is well-known that magnetic fields are notoriously difficult to measure in astrophysics. For instance, one of the popular ways of measuring a magnetic field is based on the Chandrasekhar–Fermi technique, which is known to have limited accuracy related both to the technique (see Chandrasekhar & Fermi 1953; Falceta-Gonçalves et al. 2008; Houde et al. 2013 and references therein) and to the assumption that grains are equally aligned at different depths in the cloud. The variations of grain alignment degree that follow from the modern theory of grain alignment (see Lazarian 2007 for a review) introduce an additional complication for the quantitative study of magnetic fields with the Chandrasekhar–Fermi technique. We restrict the discussion on evaluating the magnetic field to serve only our limited purpose of providing theoretical guidelines that can be useful in the development of the future techniques.

9. SUMMARY

Motivated by the observed line width differences between molecular neutral and ion species, we advanced the theory of turbulence damping in partially ionized gas. This is the first so detailed analytical study that we are aware of on the differential dissipation of neutrals and ions participating within Alfvén, fast, and slow mode motions associated with magnetic turbulence.

Following the up-to-date understanding of MHD turbulence, we acquired different expressions of damping scales in various turbulence regimes. With the aim to fully capture the variety of astrophysical situations, we considered both super- and sub-Alfvénic turbulence. In terms of damping, we considered both neutral-ion collisions and neutral viscosity, and identified the conditions where only one damping effect is dominant.

Compressible modes are found to be irrelevant in interpreting line width differences because they are damped out when neutrals and ions are still strongly coupled in the molecular cloud environment, whereas Alfvén modes exhibit different turbulent damping scales of neutral and ion fluids, and thus can account for the observed line width differences. The line width differences depend on turbulence properties. We provided the expressions of the difference between the squared velocity dispersions of neutrals and ions in various regimes of turbulence.

On the basis of our study, we proposed new methods of studying the magnetic field for both super- and sub-Alfvénic turbulence, which require additional procedures for identifying the turbulence regime from more observational inputs to ensure reliable magnetic field estimates. The methods are intended to synergistically augment the existing ways of studying magnetic fields in turbulent molecular clouds and interstellar media.

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APPENDIX A
TURBULENT ENERGY DISSIPATION

The anisotropy of turbulence damping was studied earlier in Yan & Lazarian (2004). However, their study was focused on the fast modes for which the effect of anisotropic damping was found to be extremely important. The cascading of fast modes is radically different from that of Alfvén modes, which we mainly deal with in this paper. Therefore the approach in Yan & Lazarian (2004) should be modified to take into account the efficient redistribution of energies over different directions in the k space as the Alfvén modes cascade.

In what follows, we use the explicit form of the Alfvén modes’ energy tensor and calculate the damping for different directions of the wave vector at a particular $k_i$. Then, because the directions of wave vectors are randomized in the process of Alfvénic cascading, it is reasonable to consider the total energy dissipation integrated over all angles between $k$ and $B$.

Here we only discuss the case where the energy is dissipated through neutral-ion collisions.

(1) Super-Alfvénic turbulence: In Kolmogorov turbulence regime (i.e., $l_A < k^{-1} < L$) the turbulent energy spectrum is isotropic. The energy transfer rate along the cascade is

$$\eta_\epsilon(k) = E(k)\tau_{\text{cas}}^{-1} = \frac{2}{3} V_L^2 L^{-2} k^{-1},$$

where $\tau_{\text{cas}}^{-1}$ is the cascading rate from Equation (5a) and $E(k)$ is given by Equation (92). The energy loss due to neutral-ion collisions during one eddy-turnover time, namely, the energy dissipation rate is

$$\eta_\eta(k) = E(k)|\omega_1| = \frac{\xi_n}{\nu_{\text{fin}}} V_L^2 V_A^4 L^{-2/3} k^{1/3} \left\langle \cos^2 \theta \right\rangle,$$

$$= \frac{\xi_n}{\nu_{\text{fin}}} V_L^2 V_A^4 L^{-2/3} k^{1/3}$$

(150)

where $\omega_1$ is taken from Equation (48b), and $\left\langle \cos^2 \theta \right\rangle$ is the statistical average over all angles. The scale on which neutral-ion collisions are efficient enough to cut off the Alfvénic turbulence is the damping scale. If we set

$$\eta_{d}(k) = \eta_\eta(k),$$

we arrive at

$$k_{\text{dam}} = \frac{3}{2\pi} \frac{3}{n_i} \frac{3}{\xi_n} L^{-1/2} V_L^3 V_A^{-3/2},$$

(152)

which is the same as Equation (51a).

In GS95-type turbulence regime ($k^{-1} < l_A$), turbulent energy has anisotropic distribution in the k space. Following the above method, the energy cascading rate is

$$\eta_\epsilon(k_i, k) = E(k_i, k)\tau_{\text{cas}}^{-1} = \frac{1}{3\pi} V_A^3 L^{-2/3} k^{-2/3} \left\langle \cos^2 \theta \right\rangle \exp \left( -l_A^{-1/3} k_i l_\perp^{-1/3} \right),$$

(153)

where $E(k_i, k)$ is the 3D energy density given by Equation (90). We can see that $\eta_\epsilon(k_i, k)$ has the largest value when $k_i \sim 0$. This means that the energy cascade is highly anisotropic and mainly acts in the direction perpendicular to the magnetic field. At a given $k_i$, instead of using the scaling indicated by Equation (3), we integrate the above equation over all directions, and get

$$\eta_k(k_i) = \frac{1}{3\pi} V_A^3 L^{-2/3} k^{-2/3} \int_0^{k_{\text{max}}} \exp \left( -l_A^{-1/3} k_i l_\perp^{-1/3} \right) dk_{\parallel} \approx \frac{1}{3\pi} V_L^3 L^{-4/3} k_i^{-2/3}.$$

Here $k_{\parallel}^{-1}$ extends to a sufficiently small scale, so that we can neglect the second term in the integral. The 3D energy dissipation rate is

$$\eta_{d}(k_i, k) = E(k_i, k)|\omega_1| = \frac{\xi_n}{\nu_{\text{fin}}} V_A^4 L^{-2/3} k_i^{-1/3} \int_0^{k_{\text{max}}} \exp \left( -l_A^{-1/3} k_i l_\perp^{-1/3} \right) dk_{\parallel} \approx \frac{\xi_n}{3\nu_{\text{fin}}} V_A^4 L^{-4/3} k_i^{-4/3}.$$

(156)

By equaling $\eta_{d}(k_i)$ and $\eta_k(k_i)$, we get the perpendicular damping scale

$$k_{\text{dam}, \perp} = \left( \frac{\nu_{\text{fin}}}{\xi_n} \right)^{3/2} L^{1/2} V_A^{-3/2},$$

(157)

which is only different from Equation (52) by a constant.

We note that the energy dissipation rate is limited by the energy cascading rate. By using the parameters of Model 1, Figure 7(a) shows $\eta_\epsilon(k_i, k)$ (dashed line) and $\eta_{d}(k_i, k)$ (solid line) as a function of $\theta$ at $k_i l_A = 50$ as an example. There are a few directions in which $\eta_\epsilon(k_i, k)$ is larger than $\eta_{d}(k_i, k)$, which cannot happen in reality. By taking this effect into account, we set $\eta_{d}(k_i, k)$ as the actual $\eta_{d}(k_i, k)$ when $\eta_{d}(k_i, k)$ exceeds $\eta_\epsilon(k_i, k)$ in a certain propagation direction. Then we numerically integrate the modified $\eta_{d}(k_i, k)$ over all directions. Figure 7(b) displays the resulting $\eta_{d}(k_i)$ (dashed line) and $\eta_\epsilon(k_i)$ (solid line) as a function of $k_i$ in a GS95-type turbulence regime. Unlike the original $\eta_\epsilon(k_i)$ given by Equation (156) (open circles), the modified $\eta_{d}(k_i)$ is always lower than or equal to $\eta_\epsilon(k_i)$. The scale where $\eta_{d}(k_i)$ becomes comparable to $\eta_\epsilon(k_i)$ is the damping scale, which can be represented by $k_{\text{dam}, \perp}$ given by Equation (157), as indicated by the vertical dashed line. It also corresponds to the intersection between the original $\eta_{d}(k_i)$ and $\eta_\epsilon(k_i)$.

If we increase $B$ by 10 times, the results (thinner lines) overlap the ones with smaller $B$. It confirms our earlier conclusion that $k_{\text{dam}, \perp}$ is independent of $B$ in GS95-type turbulence for super-Alfvénic case.

(2) Sub-Alfvénic turbulence: In strong turbulence at $k^{-1} < l_A$, the 3D energy density is expressed in Equation (105). Similar to the super-Alfvénic case, we have the energy
cascading rate
\[
\eta_c(k_\perp, k_i) = E(k_\perp, k_i) \tau_{\text{cas}}^{-1} \\
= \frac{1}{3\pi} L^2 L^{-2/3} M_A^{-1/3} k_\perp^{-8/3} \\
\times \exp\left(-L^{1/3} \frac{k_\parallel}{M_A^{1/3} k_\perp^{2/3}}\right). \tag{158}
\]

It becomes
\[
\eta_c(k_\perp) = \frac{1}{3\pi} V_L^2 L^{-2/3} M_A^{-1/3} k_\perp^{-8/3} \\
\times \int_0^{k_{\text{max}}} \exp\left(-L^{1/3} \frac{k_\parallel}{M_A^{1/3} k_\perp^{2/3}}\right) dk_\parallel \\
\approx \frac{1}{3\pi} V_L^2 L^{-1} M_A k_\perp^{-2}, \tag{159}
\]
at a certain \(k_\perp\). The energy dissipation rate is
\[
\eta_d(k_\perp, k_i) = E(k_\perp, k_i) |\omega_1| \\
= \frac{\xi_n}{6\pi\nu_\parallel} V_L^2 L^{-2/3} M_A^{-1/3} V_A^{-2/3} k_\perp^{10/3} k_i^2 \\
\times \exp\left(-L^{1/3} \frac{k_\parallel}{M_A^{1/3} k_\perp^{2/3}}\right). \tag{160}
\]

By integrating over all directions, we get
\[
\eta_d(k_\perp) = \frac{\xi_n}{6\pi\nu_\parallel} V_L^4 L^{-1/3} V_A^{-10/3} k_\perp^{-10/3} \int_0^{k_{\text{max}}} k_i^2 \\
\times \exp\left(-L^{1/3} \frac{k_\parallel}{M_A^{1/3} k_\perp^{2/3}}\right) dk_\parallel \\
\approx \frac{\xi_n}{3\pi\nu_\parallel} V_L^{16/3} L^{-4/3} V_A^{-4/3} k_\perp^{-4/3}. \tag{161}
\]

The equality between \(\eta_c(k_\perp)\) and \(\eta_d(k_\perp)\) leads to
\[
k_{\text{dam},\perp} = \left(\frac{\nu_\parallel}{\nu_\parallel}\right)^{3/2} L^{1/2} V_L^{-2} V_A^{1/2}, \tag{162}
\]

which is only different from Equation (55) by a constant.

We again modify \(\eta_c(k_\perp, k_i)\) to not be larger than \(\eta_c(k_\perp, k_i)\) in any direction. Figure 8 shows the integrals \(\eta_c(k_\perp)\) and \(\eta_d(k_\perp)\) by using parameters from Model 2. The same symbols as Figure 7(b) are used here. Different from the super-Alfvénic case, when we increase \(B\) by 10 times to 0.87 mG, both \(\eta_c(k_\perp)\) and \(\eta_d(k_\perp)\) (thicker lines) shift, together with \(k_{\text{dam},\perp}\) given by Equation (162). It shows in strong sub-Alfvénic turbulence, \(k_{\text{dam},\perp}\) indeed depends on \(B\). And the dependence is well described by Equation (162) (or Equation (55)).

Assuming that the randomization of wave vectors in the cascading process is so fast that we get an upper estimate of the
dissipation that provides an order of unity correspondence with the results in the main text of the paper. One can argue that the dissipation of energy at angles close to 90° (see Figure 7(a)) may happen faster than the replenishment of energy during the cascade. However, if this happens, we expect that the Alfvénic cascade with \( k_L \gg k_0 \) stops. The consistency of the estimates in the appendix and main text supports this idea. Naturally, numerical studies of the cascade in ion–electron and neutral fluids can test the accuracy of our assumption.

**APPENDIX B**

**THE CUTOFF REGION OF MHD WAVES IN DIFFERENT TURBULENCE REGIMES**

The cutoff region is introduced in a GS95-type turbulence regime in super-Alfvénic turbulence and the strong turbulence regime in sub-Alfvénic turbulence.

1. **Alfvén modes:** By setting the discriminant of Equation (46) to be zero, the lower and upper limit wavenumbers of the non-propagation region are given by (Soler et al. 2013b),

\[
k^\pm = \frac{\nu_{\text{ni}}}{V_{\text{Ai}}} \cos \theta \left[ \chi^2 + 20\chi - 8 \pm \chi^{1/2}(\chi - 8)^{3/2} \right]^{1/2} / 8(1 + \chi)^3.
\]

(163)

Notice that \( \cos \theta \) is also a function of \( k \). Under the consideration of scale-dependent anisotropy, we arrive at new expressions for the limit wavenumbers of the cutoff region. By applying Equation (3), the above expression becomes

\[
k^+ = (f(\chi)) \frac{3}{2} \left[ \epsilon_{\text{f}} c_s^2 L V_L^{-3} \right]^{1/2} \times \sqrt{1 + (f(\chi)) \frac{3}{2} \epsilon_{\text{f}} c_s^2 L^{-1} V_L^{-2} V_{\text{Ai}}^2},
\]

(164a)

\[
k^- = (g(\chi)) \frac{3}{2} \left[ \epsilon_{\text{f}} c_s^2 L V_L^{-3} \right]^{1/2} \times \sqrt{1 + (g(\chi)) \frac{3}{2} \epsilon_{\text{f}} c_s^2 L^{-1} V_L^{-2} V_{\text{Ai}}^2},
\]

(164b)

for super-Alfvénic turbulence. And in combination with Equation (6), (163) becomes

\[
k^+ = (f(\chi)) \frac{3}{2} \left[ \epsilon_{\text{f}} c_s^2 L V_L^{-2} V_{\text{Ai}}^{-1} \right]^{1/2} \times \sqrt{1 + (f(\chi)) \frac{3}{2} \epsilon_{\text{f}} c_s^2 L^{-1} V_{\text{Ai}}^4 V_L^{-3}},
\]

(165a)

\[
k^- = (g(\chi)) \frac{3}{2} \left[ \epsilon_{\text{f}} c_s^2 L V_L^{-2} V_{\text{Ai}}^{-1} \right]^{1/2} \times \sqrt{1 + (g(\chi)) \frac{3}{2} \epsilon_{\text{f}} c_s^2 L^{-1} V_{\text{Ai}}^4 V_L^{-3}},
\]

(165b)

for sub-Alfvénic turbulence, where

\[
f(\chi) = \frac{\chi^2 + 20\chi - 8}{8(1 + \chi)^3} + \frac{\chi^{1/2}(\chi - 8)^{3/2}}{8(1 + \chi)^3},
\]

\[
g(\chi) = \frac{\chi^2 + 20\chi - 8}{8(1 + \chi)^3} - \frac{\chi^{1/2}(\chi - 8)^{3/2}}{8(1 + \chi)^3},
\]

(166)

and \( \epsilon_{\text{f}} = \rho_f/\rho_e \). We can clearly see that the expressions of \( k_c^+ \) given by Equations (164a) and (165a) are very similar to the damping scales given by Equations (51b) and (54).

In the case of Alfvén modes, as long as the cutoff region \((k_c^+, k_c^-)\) resides in the GS95-type turbulence regime, we always have \( k_c^- = k_{\text{dam}} \); its simplified forms in different situations can be found in Table 1.

For \( k_c^- \) in super-Alfvénic turbulence, Equations (58) and (3) together give

\[
k_c^- = \frac{\nu_{\text{in}}}{2V_{\text{Ai}}} \sqrt{1 + \frac{L M_{\text{A}}^{-4}}{c_s^4 \xi_{\text{A}}}}.
\]

(167)

\( k_c^- \) in sub-Alfvénic turbulence is then (Equations (58) and (6))

\[
k_c^- = \frac{\nu_{\text{in}}}{2V_{\text{Ai}}} \sqrt{1 + \frac{L M_{\text{A}}^{-4} \nu_{\text{in}}^4}{2 c_s^4 \xi_{\text{A}}}},
\]

(168)

(2) **Slow modes:** The limit wavenumbers of the cutoff are

\[
k_c^+ = \frac{2 \nu_{\text{in}}}{c_s^4 \xi_{\text{A}} L M_{\text{A}}^{-4} \left( 1 + \frac{2 \nu_{\text{in}} L M_{\text{A}}^{-4}}{c_s^4 \xi_{\text{A}}} \right)^{1/2}},
\]

(169)

for super-Alfvénic turbulence, and

\[
k_c^+ = \frac{2 \nu_{\text{in}}}{c_s^4 \xi_{\text{A}} L M_{\text{A}}^{-4} \left( 1 + \frac{2 \nu_{\text{in}} L M_{\text{A}}^{-4}}{c_s^4 \xi_{\text{A}}} \right)^{1/2}},
\]

(170)

for sub-Alfvénic turbulence.

**APPENDIX C**

A SUMMARY OF THE NOTATIONS USED IN THIS PAPER

| Notation | Description |
|----------|-------------|
| \( k \) | Wave number |
| \( k_L \) | \( k \) component parallel to the local magnetic field |
| \( k_{\perp} \) | \( k \) component perpendicular to the local magnetic field |
| \( L \) | Injection scale of turbulence |
| \( L_{\text{S}} \) | Injection scale of strong turbulence |
| \( l_{\text{dec}} \) | Transition scale from weak to strong turbulence |
| \( V_{\text{t}} \) | Turbulent velocity at \( L \) |
| \( V_{\text{tl}} \) | Turbulent velocity at a scale \( l \) |
| \( V_{\text{ni}} \) | Turbulent velocity at \( l_{\text{ni}} \) |
| \( \mathcal{B} \) | Magnetic field |
| \( V_{\text{A}} \) | Alfvén speed |
| \( V_{\text{Ai}} \) | Alfvén speed of ion–electron gas |
| \( M_{\text{A}} \) | Alfvén Mach number |
| \( \tau_{\text{cfs}}^{-1} \) | Cascading rate |
| \( \nu_{\text{ni}} \) | Ion-neutral collision frequency |
| \( \nu_{\text{in}} \) | Ion-neutral collision frequency |
| \( \nu_{\text{in}} \) | Ion density |
| \( \nu_{\text{ni}} \) | Ion density |
| \( \rho_{\text{f}} \) | Total density |
| \( \rho_{\text{f}} \) | Neutral-ion decoupling scale |
| \( \rho_{\text{f}} \) | Ion-neutral decoupling scale |
| \( \kappa_{\text{dam}}^{-1} \) | Damping scale |
| \( \kappa_{\text{dam}}^{-1} \) | Viscous scale |
| \( \omega_{\text{R}} \) | Wave frequency |
| \( \omega_{\text{s}} \) | Real part of wave frequency |
| \( \omega_{\text{p}} \) | Imaginary part of wave frequency |
| \( \theta \) | Wave propagation angle with regard to magnetic field |
| \( \xi_{\text{i}} \) | Ion fraction |
