A MAP/PH(1),PH(2)/2 Production Inventory System with Multiple Servers and Varying Service Rates

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Abstract. The paper comprises an (s,S) production inventory facility in which multiple vacations and varying service rates are considered for the multiple servers. The arrival of customers constitutes a Markovian Arrival Process (MAP) and service completion times follow the phase type distributions with representations \((\alpha, S)\) and \((\beta, T)\) respectively. Manufacturing needs to begin at the moment when the stock position falls to s. Service is offered at a lower rate if the stock level ranges from 0 to s and the service time distribution has the representations \((\alpha, \eta_1 S)\) and \((\beta, \eta_2 T)\), \(0 < \eta_1, \eta_2 < 1\) respectively. If the stock level reaches the maximum S, production is ceased to operate. 1-limited service policy, Bernoulli service schedule, and Exhaustive service disciplines are considered for the servers. A suitable cost function is outlined as per performance assessment. The impact of negative correlated inter arrival times on cost function is illustrated. Also, a relative study of expected cost function on different service modes is presented.

Keywords: Markovian arrival process, multiple vacations, varying service rate, cost analysis.

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1. Introduction

The servers in the vacation model may become inaccessible or completely stop service for a random period to perform some secondary jobs, lack of work, unavailability of stocks, the emptiness of the system, or server failure. The methodology of queueing systems with one or more vacations has been studied by Levy and Yechiali [1]. The analysis of experience on models of vacation is found to be developing quickly. Such literature consists of the reports of Teghem [2] and Doshi [3]. Krishnamoorthy et.al [4] discussed a production facility with server vacation and execution time positive. Some performance measures such as expected duration of the manufacturing process, the time duration for the manufacturing process on the active mode, etc. are discussed. In certain situations, the servers work at a reduced rate is better than completely stopping service and revert to the original rate when the stock position reaches a predetermined level. Jose and Salini [5] studied an (s, S) production inventory model with varying execution rates for service and the system allows retrials. The arrival of the customer is as stated by the Markovian Arrival Process (MAP) and in the steady state, they obtained some relevant performance measures. A single server production inventory system with a dual rate of service
and exponentially distributed processing time was analyzed by Bradley [6]. Krishnakumar and Madheswari [7] detailed a queue with N parallel and identical servers, in which all servers avail Bernoulli server vacations. They discussed different services such as 1-limited service and exhaustive service as a particular case. Shan Gao and Zaiming Liu [8] selected an M/G/1 queue with single and operating condition vacations and vacation interruptions. At the instant of departure, they developed the blueprint for the distribution of static queue length by considering the Bernoulli’s schedule for vacation interruption.

A single server G-queueing system with retrials, variant multiple vacations, vacation disturbances, and different service rates were analyzed by Rajadurai [9]. The author discussed some performance and reliability measures. Krishnakumar et al. [10] analyzed a queueing system with more than one server and severs avail Bernoulli vacation scheduling service. A production inventory model with a further trial of customers and rate of service follows the phase type distribution was analyzed by Salini and Jose [11]. They studied the system with varying production rates by considering Poisson distribution for the arrival process. Krishnamoorthy and Dhanya Shajin [12] investigated a single server retrial inventory system in which consumers report as per the MAP and time took for the execution of service follows the phase type distribution. In the model, they discussed a search in the orbit, reneging of consumers, and long run system probability. Keilson and Servi [13] studied a system, where the service discipline for the vacation period was Bernoulli. They discussed the random walk model for the G1/G/1 vacation system. The present paper is scheduled as furnished below. Section 2 comprises the analysis of the model. Stability, cost analysis, and concluding remarks are presented in sections 3, 4 and 5 respectively.

| Reference | Production single multiple varying vacation schedule | Inventory | server | rates | Bernoulli | 1-Limited | Exhaustive |
|-----------|-----------------------------------------------------|-----------|-------|-------|-----------|-----------|------------|
| S.Gao and Z.Liu(2013) | Yes | Yes | | | |
| K.P.Jose and S.S.Nair(2017) | | | | | |
| S.S.Nair and K.P.Jose(2017) | | | | | |
| J.R.Bradley(2005) | | | | | |
| Krishnamoorthy and V.C.Narayanan(2011) | | | | | |
| Krishnamoorthy and D. Shajin(2017) | Yes | | | | |
| B.K.Kumar et al.(2009) | | Yes | Yes | Yes | Yes |
| B.K.Kumar et al.(2008) | | Yes | Yes | | |
| Y.Levy and U.Yechiali(1976) | | | Yes | | |
| P.Rajadurai(2018) | Yes | Yes | | | |
| J.Keilson and L.Servi(1986) | Yes | Yes | Yes | Yes | Yes |
| This model | Yes | Yes | Yes | Yes | Yes |

The comparison table 1 shows some research works dealing with the inventory system with different vacation scheduling service. Among all the reported papers production inventory with multiple servers under different vacation scheduling service is a new area. There are no related papers reported in this research area.

2. Analysis of the Model

A production inventory system is selected in which the process of consumer arrival into the system is determined by MAP with representation $(D_0, D_1)$ of order $p$ and the times required for the service of server 1 and server 2 follow the phase type distributions with representations
\((\alpha, S)_q\) and \((\beta, T)_r\), respectively. We assume that when the stock level decelerates to \(s\), the manufacturing procedure started and at the same instant the service rates are reduced. The time required for producing an item is exponentially distributed with rate \(\gamma\). The vacation times of servers 1 and 2 are distributed exponentially with parameters \(\theta_1\) and \(\theta_2\). The production process gets started when the stock level reaches to \(s\), and then service rates are reduced and service time distributions are \((\alpha, \eta_1 S)_q\) and \((\beta, \eta_2 T)_r\), where \(0 < \eta_1, \eta_2 < 1\) and \(\eta_1\) and \(\eta_2\) are the reduction parameters. Servers come back to normal rate when the stock level crosses \(s\) and the production process is discontinued when the stock level enhances to its maximum \(S\).

Having finalized the facility, the servers start serving the next customer with probability \(q_1\) and \(q_2\), if there are customers in the waiting area and the stock level is positive; otherwise they start vacations with their complementary probabilities \(0 < p_1, p_2 < 1\). If there are no buyers or stock in the system or both it again chooses vacation. The servers continue with multiple vacations until they discover at least one member in the system with stock level positive.

\(\kappa_1 = p(S - s) + pS, \kappa_2 = pq(S - s) + pq(S - 1), \kappa_3 = pqr(S - s) + pqr(S - 1), \kappa_4 = pq(S - s) + pqr(S - 2)\).

\(e_{\kappa_1+\kappa_2+\kappa_3}(\kappa_1)\) is a \((\kappa_1 + \kappa_2 + \kappa_3) \times 1\) column vector with first \(\kappa_1\) element is 1 and all other entries are zero.

\(N(t), I(t), C(t), F(t)\) respectively indicate the number of buyers in the system, level of stock, the status of servers 1 and 2, and production status at time \(t \geq 0\). Also, \(N_0(t), N_1(t),\) and \(N_2(t)\) denote the phase of the process of arrival, the process of service of server 1, and 2 at time \(t\) respectively. \(e\) indicates column vector of ones of proper dimension.

\[
C(t) = \begin{cases} 0, & \text{if the two servers in the system are on vacation} \\ 1, & \text{if server 1 is functioning and server 2 is on vacation} \\ 2, & \text{if server 1 is on vacation and the server 2 is functioning} \\ 3, & \text{if both servers are functioning} \end{cases}
\]

\[
F(t) = \begin{cases} 0, & \text{if the production procedure gets discontinued} \\ 1, & \text{if the production procedure starts functioning} \end{cases}
\]

The CTMC \(X(t) = \{(N(t), C(t), F(t), I(t), N_0(t), N_1(t), N_2(t)), t \geq 0\}\) has the state space \((1 \leq n_0 \leq p, 1 \leq n_1 \leq q, 1 \leq n_2 \leq r)\)

\[
\Omega = \Omega^1 \cup \Omega^2 \cup \Omega^3 \cup \Omega^4
\]

\[
\Omega^1 = (i, 0, 0, k, n_0)|s + 1 \leq k \leq S \cup (i, 0, 1, k, n_0)|0 \leq k \leq S - 1; i \geq 0
\]

\[
\Omega^2 = (i, 1, 0, k, n_0, n_1)|s + 1 \leq k \leq S \cup (i, 1, 1, k, n_0, n_1)|1 \leq k \leq S - 1; i \geq 1
\]

\[
\Omega^3 = (i, 2, 0, k, n_0, n_2)|s + 1 \leq k \leq S \cup (i, 2, 1, k, n_0, n_2)|1 \leq k \leq S - 1; i \geq 1
\]

\[
\Omega^4 = (i, 3, 0, k, n_0, n_1, n_2)|s + 1 \leq k \leq S \cup (i, 3, 1, k, n_0, n_1, n_2)|2 \leq k \leq S - 1; i \geq 2
\]

The following are the details regarding conversions in the Markov chain.

a) Conversions caused by the arrival of buyers \((1 \leq n_0 \leq p, 1 \leq n_1 \leq q, 1 \leq n_2 \leq r)\)

\[
(i, 0, 0, k, n_0) \xrightarrow{D_1} (i + 1, 0, 0, k, n_0); i \geq 0, s + 1 \leq k \leq S
\]

\[
(i, 0, 1, k, n_0) \xrightarrow{D_1} (i + 1, 0, 1, k, n_0); i \geq 0, 0 \leq k \leq S - 1,
\]

\[
(i, 1, 0, k, n_0, n_1) \xrightarrow{D_1 \otimes I} (i + 1, 1, 0, k, n_0, n_1); i \geq 1, s + 1 \leq k \leq S,
\]
(i, 1, 1, k, n₀, n₁) \xrightarrow{D₁ \otimes Iᵢ} (i + 1, 1, 1, k, n₀, n₁); i ≥ 1, 1 ≤ k ≤ S − 1,

(i, 2, 0, k, n₀, n₂) \xrightarrow{D₂ \otimes Iᵢ} (i + 1, 2, 0, k, n₀, n₂); i ≥ 1, s + 1 ≤ k ≤ S,

(i, 2, 1, k, n₀, n₂) \xrightarrow{D₁ \otimes Iᵢ} (i + 1, 2, 1, k, n₀, n₂); i ≥ 1, 1 ≤ k ≤ S − 1,

(i, 3, 0, k, n₀, n₁, n₂) \xrightarrow{D₁ \otimes Iₚ \otimes Iᵢ} (i + 1, 3, 0, k, n₀, n₁, n₂); i ≥ 2, s + 1 ≤ k ≤ S

(i, 3, 1, k, n₀, n₂) \xrightarrow{D₁ \otimes Iₚ \otimes Iᵢ} (i + 1, 2, 1, k, n₀, n₂); i ≥ 2, 1 ≤ k ≤ S − 1

b) conversions due to the completion of service (1 ≤ n₀ ≤ p, 1 ≤ n₁ ≤ q, 1 ≤ n₂ ≤ r)

and for i=1, we take p₁ = p₂ = 1

(i, 1, 0, s + 1, n₀, n₁) \xrightarrow{Iₚ \otimes p₁S₀} (i − 1, 0, 1, s, n₀); i ≥ 1

(i, 1, 0, k, n₀, n₁) \xrightarrow{Iₚ \otimes p₁S₀} (i − 1, 0, 0, k − 1, n₀); i ≥ 1, s + 2 ≤ k ≤ S

(i, 1, 0, k, n₀, n₁) \xrightarrow{Iₚ \otimes q₁S₀α} (i − 1, 1, 0, k − 1, n₀); i ≥ 1, 1 ≤ k ≤ s

(i, 1, 1, k, n₀, n₁) \xrightarrow{Iₚ \otimes p₁S₀} (i − 1, 0, 1, k − 1, n₀); i ≥ 1, s + 1 ≤ k ≤ S − 1

(i, 1, 1, k, n₀, n₁) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 1, s, n₀); i ≥ 1

(i, 2, 0, s + 1, n₀, n₂) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 1, s, n₀); i ≥ 1, 1 ≤ k ≤ s

(i, 2, 0, k, n₀, n₂) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 0, k − 1, n₀); i ≥ 1, s + 2 ≤ k ≤ S

(i, 2, 0, k, n₀, n₂) \xrightarrow{Iₚ \otimes q₂T₀β} (i − 1, 2, 0, k − 1, n₀); i ≥ 1, s + 2 ≤ k ≤ S

(i, 2, 1, k, n₀, n₂) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 1, k − 1, n₀); i ≥ 1, s + 1 ≤ k ≤ S

(i, 2, 1, k, n₀, n₂) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 1, k − 1, n₀); i ≥ 1, 1 ≤ k ≤ s

(i, 2, 1, k, n₀, n₂) \xrightarrow{Iₚ \otimes p₂T₀} (i − 1, 0, 1, k − 1, n₀); i ≥ 1, s + 1 ≤ k ≤ S − 1

(i, 2, 1, k, n₀, n₂) \xrightarrow{Iₚ \otimes q₂T₀β} (i − 1, 2, 1, k − 1, n₀); i ≥ 1, 1 ≤ k ≤ s

(i, 2, 1, k, n₀, n₂) \xrightarrow{Iₚ \otimes q₂T₀β} (i − 1, 2, 1, k − 1, n₀); i ≥ 1, s + 1 ≤ k ≤ S − 1
For i=2, choose $p_1 = p_2 = 1$ and for $i \geq 2$

\[
(i, 3, 0, s + 1, n_0, n_1, n_2) \xrightarrow{I_p \otimes I_q \otimes p_2 T^0} (i - 1, 1, 1, s, n_0, n_1); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 0, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes I_q \otimes p_2 T^0} (i - 1, 1, 0, k - 1, n_0, n_1); s + 2 \leq k \leq S
\]

\[
(i, 3, 0, s + 1, n_0, n_1, n_2) \xrightarrow{I_p \otimes p_1 S^0 \otimes I_r} (i - 1, 1, s, n_0, n_2)
\]

\[
(i, 3, 0, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes p_1 S^0 \otimes I_r} (i - 1, 2, 0, k - 1, n_0, n_2); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 0, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_1 S^0 \otimes I_r} (i - 1, 1, k - 1, n_0, n_1); s + 2 \leq k \leq S
\]

\[
(i, 3, 0, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_2 p_2 T^0} (i - 1, 1, 1, k - 1, n_0, n_1); 2 \leq k \leq s
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_1 S^0 \otimes I_r} (i - 1, 1, 1, k - 1, n_0, n_1); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_2 S^0 \otimes I_r} (i - 1, 2, 1, k - 1, n_0, n_2); 2 \leq k \leq s
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_1 S^0 \otimes I_r} (i - 1, 2, 1, k - 1, n_0, n_2); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes \eta_2 S^0 \otimes I_r} (i - 1, 3, 1, k - 1, n_0, n_1, n_2); 2 \leq k \leq s
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes [\eta_1 q_1 S^0 + \eta_2 q_2 T^0 \beta]} (i - 1, 3, 0, k - 1, n_0, n_1, n_2); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes [\eta_1 q_1 S^0 + \eta_2 q_2 T^0 \beta]} (i - 1, 1, 0, k - 1, n_0, n_1); 2 \leq k \leq s
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes [\eta_1 q_1 S^0 + \eta_2 q_2 T^0 \beta]} (i - 1, 1, 1, k - 1, n_0, n_1, n_2); s + 1 \leq k \leq S - 1
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes [\eta_1 q_1 S^0 + \eta_2 q_2 T^0 \beta]} (i - 1, 1, 2, 1, k - 1, n_0, n_2); 2 \leq k \leq s
\]

\[
(i, 3, 1, k, n_0, n_1, n_2) \xrightarrow{I_p \otimes [\eta_1 q_1 S^0 + \eta_2 q_2 T^0 \beta]} (i - 1, 1, 3, 1, k - 1, n_0, n_1, n_2); s + 1 \leq k \leq S - 1
\]

c) conversions caused by the manufacturing of an item ($1 \leq n_0 \leq p, 1 \leq n_1 \leq q, 1 \leq n_2 \leq r$)

\[
(i, 0, 1, k, n_0) \xrightarrow{\gamma I_p} (i, 0, 1, k + 1, n_0); i \geq 0, 0 \leq k \leq S - 2
\]

\[
(i, 0, 1, k, n_0) \xrightarrow{\gamma I_p} (i, 0, 1, k, n_0); i \geq 0
\]

\[
(i, 1, 1, k, n_0, n_1) \xrightarrow{\gamma I_{pq}} (i, 1, 1, k + 1, n_0, n_1); i \geq 1, 1 \leq k \leq S - 2
\]

\[
(i, 1, 1, k, n_0, n_1) \xrightarrow{\gamma I_{pq}} (i, 1, 0, k, n_0, n_1); i \geq 1
\]

\[
(i, 2, 1, k, n_0, n_2) \xrightarrow{\gamma I_{pr}} (i, 2, 1, k + 1, n_0, n_2); i \geq 1, 1 \leq k \leq S - 2
\]

\[
(i, 2, 1, k, n_0, n_2) \xrightarrow{\gamma I_{pr}} (i, 2, 0, k, n_0, n_2); i \geq 1
\]

\[
(i, 3, 1, k, n_0, n_2) \xrightarrow{\gamma I_{par}} (i, 3, 1, k + 1, n_0, n_1, n_2); i \geq 2, 2 \leq k \leq S - 2
\]

\[
(i, 3, 1, k, n_0, n_2) \xrightarrow{\gamma I_{par}} (i, 3, 0, k, n_0, n_1, n_2); i \geq 2
\]

d) conversions due to completion of vacation of servers ($1 \leq n_0 \leq p, 1 \leq n_1 \leq q, 1 \leq n_2 \leq r$)

\[
(i, 0, 0, k, n_0) \xrightarrow{I_p \otimes \theta_1 \alpha} (i, 0, 1, k, n_0, n_1); i \geq 1, s + 1 \leq k \leq S
\]
For $i=0,1$ choose $\theta_1 = \theta_2 = 0$, and $p_1 = p_2 = 1$

\[(i, 0, 0, k, n_0) \quad \rightarrow \quad (i, 1, 0, k, n_0); \quad i \geq 0, o, k \leq S \]

\[(i, 0, 0, k, n_0) \quad \rightarrow \quad (i, 2, 0, k, n_0); \quad i \geq 1, s + 1 \leq k \leq S \]

\[(i, 0, 1, k, n_0) \quad \rightarrow \quad (i, 2, 1, k, n_0); \quad i \geq 1, k \leq S - 1 \]

\[(i, 1, 0, k, n_0) \quad \rightarrow \quad (i, 3, 0, k, n_0); \quad i \geq 2, s + 1 \leq k \leq S \]

\[(i, 1, 0, k, n_0) \quad \rightarrow \quad (i, 3, 1, k, n_0); \quad i \geq 2, 2 \leq k \leq S - 1 \]

\[(i, 1, 1, k, n_0) \quad \rightarrow \quad (i, 3, 1, k, n_0); \quad i \geq 2, 2 \leq k \leq S - 1 \]

\[(i, 2, 0, k, n_0) \quad \rightarrow \quad (i, 3, 0, k, n_0); \quad i \geq 2, s + 1 \leq k \leq S \]

\[(i, 2, 1, k, n_0) \quad \rightarrow \quad (i, 3, 1, k, n_0); \quad i \geq 2, 2 \leq k \leq S - 1 \]

e) conversions that makes no change in the first four coordinates of $\Omega$

\[(1 \leq n_0 \leq p_1 \leq 1 \leq n_1 \leq q, 1 \leq n_2 \leq r)\]

For $i=0,1$ choose $\theta_1 = \theta_2 = 0$, and $p_1 = p_2 = 1$

\[(i, 0, 0, k, n_0) \quad \rightarrow \quad (i, 0, 0, k, n_0); \quad i \geq 0, o, k \leq S \]

\[(i, 0, 1, k, n_0) \quad \rightarrow \quad (i, 0, 1, k, n_0); \quad i \geq 0 \]

\[(i, 1, 0, k, n_0) \quad \rightarrow \quad (i, 1, 0, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 1, 1, k, n_0) \quad \rightarrow \quad (i, 1, 1, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 2, 0, k, n_0) \quad \rightarrow \quad (i, 2, 0, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 2, 1, k, n_0) \quad \rightarrow \quad (i, 2, 1, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 3, 0, k, n_0) \quad \rightarrow \quad (i, 3, 0, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 3, 1, k, n_0) \quad \rightarrow \quad (i, 3, 1, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 3, 2, k, n_0) \quad \rightarrow \quad (i, 3, 2, k, n_0); \quad i \geq 0, s \leq k \leq S \]

\[(i, 3, 3, k, n_0) \quad \rightarrow \quad (i, 3, 3, k, n_0); \quad i \geq 0, s \leq k \leq S \]
\[
(i, 3, 1, k, n_0, n_1, n_2) \rightarrow (i, 3, 1, k, n_0, n_1, n_2); \quad s + 1 \leq k \leq S - 1
\]

The generator matrix of the CTMC is \( Q = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \ldots \\ 0 & K_{00} & K_{01} & 0 & 0 & \ldots \\ 1 & K_{10} & K_{11} & K_{12} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \)

### 3. Stability Condition and Steady State Analysis

We define a matrix \( \bar{K} = K_0 + K_1 + K_2 \) of order \((\pi_1 + \pi_2 + \pi_3 + \pi_4)\) which is irreducible so there exist a transition probability vector \( \Pi = (\pi[c], c = 0, 1, 2, 3) \) of order 1 \( \times (\pi_1 + \pi_2 + \pi_3 + \pi_4) \) satisfying \( \Pi \bar{K} = 0 \) and \( \Pi e = 1 \)

The process is continued until the consecutive difference in the value of \( R \) is less than a specified tolerance criterion.

From the standard drift condition derived by Nuets[14], \( \Pi K_0 e < \Pi K_2 e \) is necessary and sufficient condition for the stability of the QBD process. Since the system is stable, there exist a steady state probability vector \( X = (X_0, X_1, \ldots) \). The sub vectors of \( X \) can be calculated from the equations \( XQ = 0 \), \( Xe = 1 \) and \( X_i = X_{i-1} * R, i = 3, 4, 5, \ldots \) where each \( X_0 = (y_0, 0, 0, s+1, y_0, 0, 0, s, \ldots y_0, 0, 1, s-1) \)

\[
X_1 = \begin{cases} 
(y_1, 0, 0, s+1, \ldots y_1, 0, 0, s, \ldots y_1, 0, 1, s-1, y_1, 1, 0, s-1 \ldots y_1, 1, 1, s) \\
(y_1, 1, 1, 1, \ldots y_1, 1, 1, s-1 \ldots y_1, 2, 0, s+1 \ldots y_1, 2, 0, s, y_1, 2, 1, s-1 \ldots y_1, 2, 1, s-1) 
\end{cases}
\]

and for \( i \geq 2 \), \( X_i = \begin{cases} 
(y_i, 0, 0, s+1, \ldots y_i, 0, 0, s, y_i, 0, 1, s-1, y_i, 1, 0, s-1 \ldots y_i, 1, 1, s-1 \ldots y_i, 2, 0, s+1 \ldots y_i, 2, 0, s, y_i, 2, 1, s-1 \ldots y_i, 2, 1, s-1) \\
(y_i, 3, 0, s+1, \ldots y_i, 3, 0, s, y_i, 3, 1, s-1, y_i, 3, 1, 1, s-1 \ldots y_i, 3, 1, 1, s-1) 
\end{cases}
\]

where \( y_{i,0,j,k} = \begin{cases} 
(y_i, 0, 0, k, 1, \ldots y_i, 0, 0, k, p), s + 1 \leq k \leq S, i \geq 0, \\
(y_i, 0, 1, k, 1, \ldots y_i, 0, 1, k, p), 0 \leq k \leq S - 1, i \geq 0, 
\end{cases} \)

\[
y_{i,1,j,k} = \begin{cases} 
(y_i, 1, 0, k, 1, \ldots y_i, 1, 0, k, p, q), s + 1 \leq k \leq S, i \geq 1, \\
y_{i,1,j,k} = \begin{cases} 
(y_i, 1, 1, k, 1, \ldots y_i, 1, 1, k, p, q), 1 \leq k \leq S - 1, i \geq 1, 
\end{cases} \)
\]

\[
y_{i,2,j,k} = \begin{cases} 
(y_i, 2, 0, k, 1, \ldots y_i, 2, 0, k, p, r), s + 1 \leq k \leq S, i \geq 1, \\
y_{i,2,j,k} = \begin{cases} 
(y_i, 2, 1, k, 1, \ldots y_i, 2, 1, k, p, r), 1 \leq k \leq S - 1, i \geq 1, 
\end{cases} \)
\]

\[
y_{i,3,j,k} = \begin{cases} 
(y_i, 3, 0, k, 1, \ldots y_i, 3, 0, k, p, q, r), s + 1 \leq k \leq S, i \geq 2, \\
y_{i,3,j,k} = \begin{cases} 
(y_i, 3, 1, k, 1, \ldots y_i, 3, 1, k, p, q, r), 2 \leq k \leq S - 1, i \geq 2, 
\end{cases} \)
\]

Among the nonnegative solutions of the equation \( R^2 K_2 + RK_1 + K_0 = 0 \), the minimum is \( R \) with its spectral radius is below one. \( R \) is computed from \( R = -K_0(K_1)^{-1} - R^2 K_2(K_1)^{-1} \) and it is approximated by successive substitution method \( R_0 = 0, R_{n+1} = -K_0(K_1)^{-1} - R_n^2 K_2(K_1)^{-1}, n = 0, 1, 2, \ldots \) The process is continued until the consecutive difference in the value of \( R \) is less than a specified tolerance criterion.
3.1. Performance Assessment

(i) Expected number of buyers in the system:

\[ \varphi_{EC} = X_1 e^{(\lambda_1 + \lambda_2 + \lambda_3)} + X_2 [2(I - R)^{-1} + R(I - R)^{-2}] e^{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \]

(ii) The odds against having buyers in the system when all of the servers are on vacation:

\[ \varphi_{ECV} = X_1 e^{(\lambda_1 + \lambda_2 + \lambda_3)}(\lambda_1) + X_2 [2(I - R)^{-1} + R(I - R)^{-2}] e^{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}(\lambda_1) \]

(iii) Expected switching rate:

\[ \varphi_{SWR} = \sum_{i=1}^{\infty} y_{i,1,0,s+1} [I_l \otimes S^0] e + \sum_{i=1}^{\infty} y_{i,2,0,s+1} [I_l \otimes T^0] e \]

\[ + \sum_{i=2}^{\infty} y_{i,3,0,s+1} [I_l \otimes (S^0 \otimes T^0)] e \]

(iv) Expected inventory level:

\[ \varphi_{EI} = \sum_{i=0}^{\infty} \sum_{k=s+1}^{S} \sum_{n=1}^{p} k y_{i,0,0,k,n} + \sum_{i=0}^{\infty} \sum_{k=1}^{S-1} \sum_{n=1}^{p} k y_{i,0,1,k,n} \]

\[ + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} \sum_{n=1}^{p} \sum_{q=1}^{q} k y_{i,1,0,k,n,q} + \sum_{i=1}^{\infty} \sum_{k=1}^{S-1} \sum_{n=1}^{p} \sum_{q=1}^{q} k y_{i,1,1,k,n,q} \]

\[ + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} \sum_{n=1}^{p} \sum_{r=1}^{r} k y_{i,2,0,k,n,r} + \sum_{i=1}^{\infty} \sum_{k=1}^{S-1} \sum_{n=1}^{p} \sum_{r=1}^{r} k y_{i,2,1,k,n,r} \]

\[ + \sum_{i=2}^{\infty} \sum_{k=s+1}^{S} \sum_{n=1}^{p} \sum_{q=1}^{q} \sum_{r=1}^{r} k y_{i,3,0,k,n,q,r} + \sum_{i=2}^{\infty} \sum_{k=2}^{S-1} \sum_{n=1}^{p} \sum_{q=1}^{q} \sum_{r=1}^{r} k y_{i,3,1,k,n,q,r} \]

(v) Expected inventory level when all the servers are active:

\[ \varphi_{EIA} = \sum_{i=2}^{\infty} \sum_{k=s+1}^{S} \sum_{n=1}^{p} \sum_{q=1}^{q} \sum_{r=1}^{r} k y_{i,3,0,k,n,q,r} \]

\[ + \sum_{i=2}^{\infty} \sum_{k=2}^{S-1} \sum_{n=1}^{p} \sum_{q=1}^{q} \sum_{r=1}^{r} k y_{i,3,1,k,n,q,r} \]

(vi) The mean number of buyers going away after getting service:

\[ \varphi_{EDS} = \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,1,0,k} (I_p \otimes S^0) e + \sum_{i=1}^{\infty} \sum_{k=1}^{S} y_{i,1,1,k} (I_p \otimes \eta_1 S^0) e \]
The expected total cost of the system per unit per unit time is given by $D$:

$$
D = \sum_{i=1}^{\infty} \sum_{k=s+1}^{S-1} y_{i,1,k} (I_p \otimes S^0) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,2,k} (I_p \otimes T^0) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,2,k} (I_p \otimes \eta_2 T^0) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,3,1,k} (I_p \otimes (S^0 \oplus T^0)) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,3,1,k} (I_p \otimes (\eta_1 S^0 \oplus \eta_2 T^0)) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,3,1,k} (I_p \otimes (S^0 \oplus T^0)) + \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,3,1,k} (I_p \otimes (S^0 \oplus T^0))$

4. Cost Analysis and Correlation

Two sets of different values for $D_0$ and $D_1$ are given and we compared the total cost in 1-limited service policy, Bernoulli service schedule, and exhaustive service policy. Choose the values of parameters $\alpha = [0.3 \ 0.5 \ 0.2]$, $\beta = [0.3 \ 0.4 \ 0.3]$, $S = \begin{bmatrix} -6 & 3 & 0 \\ 1 & -4 & 2 \\ 2 & 0 & -5 \end{bmatrix}$, $T = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -5 & 2 \\ 1 & 0 & -3 \end{bmatrix}$, $S^0 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$, $T^0 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

Map with negative correlation(MAP(-)):

$$
D_0 = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -450.5 & 445.995 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.02 & 0 & 1.98 \\ 445.995 & 4.505 & 0 \end{bmatrix}
$$

The expected total cost of the system per unit per unit time is given by $T_{cost} = (C + (S - s) c_1 \psi_{SR} + c_2 \psi_{ET} + c_3 \psi_{EC} + c_4 \psi_{EDS}$ where $C, c_1, c_2, c_4, c_3$ represents the fixed cost, procurement cost, cost due to service, carrying cost of inventory, and customers per unit per unit time. Table 2 represents the impact of the values of $\eta_1$ and $\eta_2$ on $T_{cost}$ for three different vacation scheduling services, in MAP(-), keeping values of other parameters fixed. The optimum values of $T_{cost}$ are given in Table 3 for three different vacation scheduling services in negative correlation.

| $\eta_1$ | $T_{cost}(B)$ | $T_{cost}(E)$ | $T_{cost}(1-L)$ | $\eta_2$ | $T_{cost}(B)$ | $T_{cost}(E)$ | $T_{cost}(1-L)$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 46.5779 | 46.6269 | 64.5783 | 0.2 | 275.4639 | 275.4645 | 319.1891 |
| 0.6 | 47.2248 | 47.3762 | 63.2040 | 0.3 | 276.5423 | 276.5491 | 312.9121 |
| 0.7 | 48.0809 | 48.3643 | 61.9414 | 0.4 | 278.5554 | 278.5697 | 306.9276 |
| 0.8 | 49.1979 | 49.6523 | 60.7846 | 0.5 | 281.6433 | 281.6663 | 300.3990 |
| 0.9 | 50.6401 | 51.3225 | 59.7266 | 0.6 | 285.9822 | 286.0158 | 294.2849 |
Table 3: Optimum values of $T_{\text{cost}}$ for $\eta_1(S=10, s=4, C=100, c_1=56, c_2=1, c_3=10, c_4=6, \eta_2=0.7, \gamma=3.7, \theta_1=5, \theta_2=10, p_1=0.01, p_2=0.05)$ and for $\eta_2(S=10, s=4, C=150, c_1=3, c_2=1, c_3=10, c_4=250, \theta_1=5, \theta_2=10, \gamma=3.7, p_1=0.001, p_2=0.0001)$

| $\eta_1$ | $T_{\text{cost}}(B)$ | $T_{\text{cost}}(E)$ | $\eta_2$ | $T_{\text{cost}}(B)$ | $T_{\text{cost}}(E)$ |
|----------|------------------------|------------------------|----------|------------------------|------------------------|
| 0.4      | 45.8368                | 45.5089                | 0.2      | 247.2532               | 242.3833               |
| 0.5      | 45.60928               | 45.3034                | 0.3      | 246.3530               | 241.7443               |
| 0.6      | 45.4678                | 45.1862                | 0.4      | 245.8386               | **241.5076**           |
| **0.7**  | **45.4229**            | **45.1683**            | **0.5**  | **245.7691**           | 241.7356               |
| 0.8      | 45.4852                | 45.2613                | 0.6      | 246.2076               | 242.4970               |
| 0.9      | 45.6672                | 45.4787                | 0.7      | 247.2238               | 243.8710               |

5. Concluding Remarks

In this paper, we analyzed a production inventory system with more than one server and different service rates. Further, a comparison among $1$-limited service policy, Bernoulli service schedule, and Exhaustive service policy are also undertaken using Matrix Analytic Method. The main difficulty for this work is to find a closed form solution for the expected cost function. One can use the algorithmic method to find the optimal value of the expected cost. It is possible to extend the model in varied dimensions by assuming more than two servers and any arbitrary distribution for lead time.

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