The holomorphic convexity of reductive Kähler covering surfaces in non-archimedean case

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Abstract

In this article, we consider the holomorphic convexity of a weakly 1-complete Kähler surface without two ends, where the continuous plurisubharmonic exhaustion function is subanalytic with the possible degeneracy set that is holomorphically foliated. This situation arises naturally when we consider the harmonic map associated with a reductive linear representation into a locally compact Euclidean building. The role of subanalyticity is essential here to get a proper holomorphic map onto a Riemann surface with connected fibers. With this result, we prove the holomorphic convexity of any Galois covering over a compact Kähler surface which does not have two ends, and admits a discrete reductive representation defined over a non-archimedean locally finite local field of the Galois group. Furthermore, we generalize the result of Katzarkov and Ramachandran from projective surfaces to Kähler surfaces.

1 Introduction.

It is well known that on a Stein manifold, we can construct a continuous strictly plurisubharmonic exhaustion function, which can be further shown to be real analytic. The converse is the famous Levi problem solved by H. Grauert: a complex manifold that admits a continuous strictly plurisubharmonic exhaustion function is Stein. In particular, it is holomorphically convex.

It is natural to ask what will happen if the exhaustion function on the manifold is only weakly plurisubharmonic with a nonempty degeneracy set of its Levi form. In this situation, we say that the complete complex manifold is
weakly 1-complete. While the holomorphic convexity is not true in general, we can seek additional conditions under which it is still valid. One interesting situation is provided by the archimedean case in the work of Katzarkov and Ramachandran [KR98], i.e. when we consider the harmonic map associated with a reductive linear representation into a Riemannian locally symmetric space of noncompact type. The result of Napier and Ramachandran (cf. Theorem 4.8, [NR95]) gives the essential tool to handle this situation.

Here, we consider the non-archimedean case when the harmonic map associated with a reductive linear representation maps into a locally compact Euclidean building. We want to get an analogous result as in the archimedean case, and there is some modification needed. Foremost, the plurisubharmonic exhaustion function is only continuous subanalytic, not real analytic. The main purpose of this article is to provide a uniform way to handle this situation without using the factorization theorem (cf. Theorem 1.3, [KR98]). Since the factorization theorem relies heavily on the structure of the algebraic groups and may not be available for more general Kähler groups, the advantage of this process is to generalize the main result of [KR98] from projective surfaces to Kähler surfaces. Besides, the method of using factorization theorem need to deform the original representation, which could change its kernel drastically (cf. example on page 529, [KR98]). Thus it proves the holomorphic convexity of the universal covering only, not for the given intermediate Galois covering surface with a discrete reductive Galois group and without two ends. In this sense, the method provided here fills a gap in the non-archimedean case.

We need the following definition to state our results.

**Definition 1.1.** We say a finitely generated group $\Gamma$ is reductive if it admits a faithful representation into a complex Lie group $G$, and its monodromy group (i.e. the Zariski closure of its image) is a reductive subgroup of $G$.

Briefly, we will call a connected Galois covering $X'$ of a compact Kähler surface $X$ with a linear reductive infinite Galois group $\Gamma$ simply as a reductive covering surface $X'$ (over $X$).

Then the main theorem to handle this non-archimedean case is:

**Theorem 1.2 (Main Theorem).** Let $X'$ be a reductive covering surface that does not have two ends and $\mathcal{F}$ be a holomorphic foliation of codimension 1 defined by a (closed) holomorphic 1-form $\theta$. If there exists a subanalytic plurisubharmonic exhaustion function $f$ which is constant on the leaves of
\( F \), then \( X' \) admits a proper holomorphic map onto a Riemann surface with connected fibers. In particular, \( X' \) is holomorphically convex.

In section 2, we give the strategy of using harmonic maps in both archimedean and non-archimedean cases. In particular, we show the assumptions in the main theorem are fulfilled in the non-archimedean case. In section 3, we give the proof of the main theorem, with an emphasis on how to find a compact leaf using the resolution of subanalytic functions. Especially, we get a Sard-type theorem for a subanalytic function with its level set real-analytically foliated almost everywhere. Eventually, in section 4, we first provide the non-archimedean version on holomorphic convexity, then use it to show the holomorphic convexity of the reductive covering surface which does not have two ends following the work by Katzarkov and Ramachandran [KR98].

2 The strategy of using harmonic maps.

Let \( X' \) be a reductive covering surface over \( X \) with the Galois group \( \Gamma \) and assume that the representation \( \rho : \Gamma \to G \), where \( G \) is a complex Lie group, is discrete.

If \( G \) is defined over the archimedean field (i.e. \( \mathbb{R} \) or \( \mathbb{C} \)), the intrinsically associated non-positively curved space \( N(\rho) = G/K \) (here \( K \) is the unique maximal compact subgroup of \( G \)) is a Riemannian locally symmetric space of non-compact type and \( G \) acts isometrically on \( G/K \) by left action. If \( G \) is defined over a non-archimedean locally finite local field (i.e. a finite extension \( E_p \) of \( \mathbb{Q}_p \) where \( p \) is a non-archimedean place), the intrinsically associated non-positively curved space \( N(\rho) \) is a locally compact Euclidean building and \( G \) acts on it by isometries.

**Definition 2.1.** A map \( u : X' \to N(\rho) \) is called \( \Gamma \)-equivariant if for any \( \gamma \in \Gamma \) and \( x \in X' \), we have

\[
u(\gamma \cdot x) = \rho(\gamma) \cdot u(x),\]

where \( \Gamma \) acts on \( X' \) by the deck transformation and \( \rho(\Gamma) \subseteq G \) acts on \( N(\rho) \) as isometries.

First, let us recall the strategy of using harmonic maps for the archimedean case. Next, we adjust it for the non-archimedean case. The results in this section are well known and we include them for completeness.
2.1 Archimedean case.

Since the target \(G/K\) is topologically contractible, we can construct a non-
constant continuous \(\Gamma\)-equivariant map \(v\), as shown by Lasell and Ramachandran (cf. proof of Lemma 2.1, [LR96]). Since \(X = X'/\Gamma\) is compact, the map \(v\) is of finite energy.

Now we need the following special case of the theorem by Corlette (cf. Corollary 3.5, [Cor88]).

**Theorem 2.2.** Let \(X'\) be a reductive covering surface and \(G\) be a semisimple Lie group without compact factors. Let \(K\) be the maximal compact subgroup of \(G\) so that \(G/K\) carries the structure of a Riemannian locally symmetric space of noncompact type. Then there exists a nonconstant \(\Gamma\)-equivariant harmonic map \(u : X' \rightarrow G/K\) with finite energy.

If \(G\) is defined over \(\mathbb{C}\), by Weil’s restriction of scales, we can see \(G(\mathbb{C})\) as the set of \(\mathbb{R}\)-points of \(\text{Res}_{\mathbb{C}\mid \mathbb{R}}(G)\). Now we may assume that \(G\) is a real reductive Lie group, and the target is a real Riemannian locally symmetric space \(G/K\) of noncompact type and has strongly negative curvature in the complexified sense. From the Bochner-Siu-Sampson formula, we have \(u\) is pluriharmonic in the sense that its restriction to any complex analytic curve on \(X'\) is harmonic (cf. Theorem 2, [Mok92]).

We can define a holomorphic structure on the smooth complex vector bundle \(u^*(T^\mathbb{C}_N)\), with respect to which the one-form \(\theta := \partial u\) is a holomorphic section of \(T^{(1,0)}_{X'} \otimes u^*(T^\mathbb{C}_N)\) on \(X'\). The distribution given by the kernel of \(\theta\) defines a holomorphic foliation \(\mathcal{F}\) on the complement of a complex analytic subvariety \(V\) of codimension \(\geq 2\) (i.e. the set of indeterminacy of \(\theta\)). In our situation, \(V\) is a set of discrete points. The above construction can be found in the work by Mok [Mok92] or Siu [Siu87].

On each leaf of \(\mathcal{F}\), we have

\[
\overline{\partial} u = \overline{\partial} u = 0,
\]

because \(u\) is real-valued. Thus \(u\) is constant on each leaf of \(\mathcal{F}\).

Define

\[
f(x) = d^2(u(x), y_0)
\]

where \(y_0\) is a fixed point on the target and \(d\) is the distance function induced by the Riemannian structure there. Then \(d^2\) is strictly convex and real analytic. Because \(\Gamma\) is discrete, the \(f(x)\) is a real analytic plurisubharmonic...
exhaustion function on $X'$. Also, it is constant on the leaves of the holomorphic foliation $\mathcal{F}$ defined by the holomorphic 1-form $\theta = \partial u$.

We summarize the above argument as:

**Lemma 2.3.** Let $X'$ be a reductive covering surface with group $\Gamma$ and the representation $\rho : \Gamma \to G$ be discrete. Assume $G$ is a real Lie group defined over the archimedean field, then there exists a real analytic plurisubharmonic exhaustion function $f$ on $X'$ which is constant on the leaves of a holomorphic foliation $\mathcal{F}$ defined by a (closed) holomorphic 1-form $\theta = \partial u$, where $u$ is the associated $\Gamma$-equivariant harmonic map from $X'$ to $N(\rho)$.

Now we are in the place to apply the theorem 4.6 of [NR95], and we rewrite it for the Kähler surfaces.

**Theorem 2.4.** Let $X'$ be a Galois covering of a compact Kähler surface with an infinite covering group $\Gamma$ which does not have two ends and $f$ be a continuous plurisubharmonic exhaustion function which is constant on the leaves of a given holomorphic foliation $\mathcal{F}$ of complex codimension 1. If there exists one level set of $f$ which is real analytic and the action of $\Gamma$ is properly discontinuous (discrete), then $X'$ admits a proper holomorphic map onto a Riemann surface with connected fibers. In particular, it is holomorphically convex.

We include the proof here for the convenience of readers.

**Proof.** If $X'$ has at least three ends, then it admits a proper holomorphic map onto a Riemann surface with connected fibers by a classical result of Napier and Ramachandran (cf. Theorem 3.4 [NR95]). Now we consider the case when $X'$ has exactly one end. The real analytic level set $X'_r := \{ x \in X' : f(x) = r \}$ is Levi-flat in the sense that $\sqrt{-1} \partial \bar{\partial} f = 0$ on the maximal complex 1-dimension complex analytic tangent subspace contained in $X'_r$, since this 1-dimension complex manifold is given by the holomorphic leaves of $\mathcal{F}$ and $f$ is constant there. Choose a relatively compact neighborhood $U$ of the given level set $X'_r$ where $f$ is real analytic, and consider $g(x) = e^{-c \cdot f(x)}$, where $c$ is a sufficiently large constant to be determined. Write $v = v_1 + v_2$, where $v_1 \in T_x^{(1,0)}(X'_{f(x)})$ and $v_2$ is in its orthogonal complement. Denote $\mathcal{L}(g)$ to be the Levi-form of function $g$, then

$$\mathcal{L}(g)(v, v) = e^{-c \cdot f} \left( c^2 \cdot |\partial g(v)|^2 - c \cdot \mathcal{L}(f)(v, v) \right)$$

$$= c \cdot e^{-c \cdot f} \left( c^2 |\partial f(v_2)|^2 - \mathcal{L}(f)(v_2, v_2) \right)$$

5
Since $U$ is relatively compact, we can assume that $|\partial f(v_2)|^2 \geq \mu|v_2|^2$ and $\mathcal{L}(f)(v_2, v_2) \leq \nu|v_2|^2$ for some $\mu, \nu > 0$. Then $\mathcal{L}(g)(v, v) \geq c \cdot e^{-c g(c \cdot \mu - \nu)} > 0$ if we choose $c > \nu/\mu$.

This shows that $g(x) = e^{-c f(x)}$ is a strictly plurisubharmonic function on a neighborhood $U$ of $X'_r$. Also, $r$ is a regular value of $f$. Denote $\Omega_1$ to be the union of relatively compact sublevel set $\{x \in X' : f(x) < r\}$ with all compact components of its complement and $X' - \Omega_1$ to be the unique noncompact component (the end). Since $\Gamma$ is properly discontinuous, we can find $\gamma \in \Gamma$ such that $\Omega_2 = \gamma(\Omega_1)$ satisfies $\overline{\Omega_2} \cap \overline{\Omega_1} = \emptyset$. Then the connected component $W := X' - (\Omega_1 \cup \Omega_2)$ is a weakly 1-complete Kähler surface with three ends.

Thus we have a proper holomorphic map $\tau$ from $W$ onto a Riemann surface $R$ with connected fibers. By analytic continuation, we can extend $\tau$ to $X'$. Precisely, choose $U$ to be an open set in the regular values of $\tau$ in $R$ and two distinct points $p$ and $q$ there. Then the connected Kähler manifold $V := X' - \tau^{-1}(\{p, q\})$ is weakly 1-complete with three ends. Thus there exists a connected Riemann surface $S$ and a proper holomorphic map $\sigma : V \to S$ with connected fibers. The maps $\tau$ and $\sigma$ together give a holomorphic map from $X'$ to some Riemann surface $T$ by identifying each point $x$ in $U - \{p, q\}$ with $\sigma(\tau^{-1}(x))$ in $S$.

\[\square\]

Combine the results of lemma 2.3 and theorem 2.4, we get the following well-known result in the archimedean case:

**Theorem 2.5.** Let $X'$ be a reductive covering surface with the representation being discrete and defined over the archimedean field, then $X'$ is holomorphically convex.
2.2 Non-archimedean case.

Now the target $Y = N(\rho)$ intrinsically associated is a locally compact Euclidean building. It is still contractible and we can similarly construct a nonconstant $\Gamma$-equivariant map $v$ with finite energy into $Y$, for $X$ is compact.

The existence of a nonconstant $\Gamma$-equivariant harmonic map of finite energy now is guaranteed by the theory of Gromov and Schoen (cf. Theorem 7.1, [GS92] for locally compact F-connected complexes). For our special situation, we have:

**Theorem 2.6.** Let $X'$ be a reductive covering surface and $G$ be a reductive complex Lie group defined over a non-archimedean locally finite local field. Denote $Y = N(\rho)$ to be the locally compact Euclidean building associated. If there exists a nonconstant $\Gamma$-equivariant map $v : X' \to Y$ with finite energy, then there exists a $\Gamma$-equivariant harmonic map $u$ of least energy, in the sense that the restriction of $u$ to a small ball about any point is energy minimizing.

Let us recall the definition of singular and regular set for harmonic maps into locally compact F-connected complexes.

**Definition 2.7.** Let $u$ be a harmonic map from a Riemannian manifold $X$ to a locally compact $F$-connected complex $Y$. A point $x \in X$ is called a regular point if there exists $\sigma > 0$ such that $u(B_\sigma(x))$ is contained in one $F$-flat (in the condition of buildings, we use the terminology “apartment” instead). Otherwise, it is called a singular point. Denote the set of regular points and that of singular points as $\mathcal{R}(u)$ and $\mathcal{S}(u)$ respectively.

Also, the definition of a pluriharmonic map is stated as:

**Definition 2.8.** We say that a harmonic map $u$ from a Kähler manifold to a locally compact $F$-connected complex is pluriharmonic if it is pluriharmonic in the usual sense on its regular set $\mathcal{R}(u)$.

Although we can only do the smooth differential geometry calculation on $\mathcal{R}(u)$, the set $\mathcal{S}(u)$ has Hausdorff codimension at least 2 and the Bochner-Siu-Sampson formula is still valid (cf. Theorem 7.3, [GS92]). The result is:

**Theorem 2.9.** A finite energy equivariant harmonic map from a Kähler manifold into a locally compact $F$-connected complex is pluriharmonic.
By the work of Eyssidieux (cf. section 3.3.2 of [Eys04]), there exists a closed real semipositive \((1,1)\)-form \(\omega_\rho\) (i.e. a semi-Kähler \((1,1)\)-form) on the spectral covering \(s: \hat{X}' \to X'\). It can be written as
\[
\omega_\rho = s^* \sqrt{-1} \sum_{i=1}^{N} \lambda_i \wedge \bar{\lambda}_i
\]
where \(\lambda_i\)'s are holomorphic 1-forms well-defined on \(X'\). Notice that these \(\lambda_i\)'s differ from each other by an isometric action, thus their kernels coincide and define the holomorphic foliation \(\mathcal{F}\) on \(\mathcal{R}(u)\). Furthermore, the singular set \(\mathcal{S}(u)\) is complex analytic of codimension 1 and \(\lambda_i\) is bounded for \(u\) is Lipschitz in the interior, we have that \(\mathcal{F}\) can be extended across \(\mathcal{S}(u)\) and is well defined by holomorphic 1-form \(\theta\) on the spectral covering \(\hat{X}'\).

Now we need the following property on the distance function of a locally compact Euclidean building.

**Lemma 2.10.** Let \(y_0\) be a fixed point on a locally compact Euclidean building \(Y\) of dimension \(k\), then the square of a distance function defined as
\[
d^2(y) := d^2(y, y_0)
\]
is subanalytic.

**Proof.** Assume the strata containing \(y_0\) with the lowest dimension is \(Z_0\) and write the union of all apartments containing \(Z_0\) as \(\bigcup_{i=1}^{l} Y_i\). On each \(Y_i\), it is a Euclidean space of dimension \(k\) and the distance function \(d_i^2(y) := d^2(y, y_0)|_{Y_i}\) is obviously real analytic. Also, being real analytic or subanalytic is a local property and we only need to consider the intersection with a small neighborhood of \(y_0\).

Now denote the chamber whose closure contains \(Z_0\) as \(C_0\) (i.e. \(C_0\) has \(Z_0\) as frontier or \(C_0 = Z_0\)). Then the graph of function \(d^2(y)\) is the projection of the direct product \(d_1^2(y) \times \ldots \times d_l^2(y)\) identified at \(C_0\). By definition, \(d^2(y)\) is a continuous real subanalytic function.

In addition, \(d^2\) is strictly convex because the target is non-positively curved (in the sense of [GS92]). Composing with the pluriharmonic map \(u\), we get that
\[
f(x) = d^2(u(x), y_0)
\]
is a continuous subanalytic plurisubharmonic exhaustion function on $X'$, for the action of $\Gamma$ is discrete. Furthermore, the function $f$ is constant on the leaves lying in $\mathcal{R}(u)$ as in the archimedean case. This is still true on $\mathcal{S}(u)$ because we can approach the leaves inside it by leaves in the regular set $\mathcal{R}(u)$, where $f$ is a constant. The above argument is summarized as the following lemma.

**Lemma 2.11.** Let $X'$ be a reductive covering surface with group $\Gamma$ and let $\rho : \Gamma \rightarrow G$ be a discrete representation. If $G$ is defined over a non-archimedean locally finite local field, then there exists a holomorphic foliation $\mathcal{F}$ given by a holomorphic 1-form $\theta$ being well defined on the spectral covering $\hat{X}'$ of $X'$. Also, there exists a continuous subanalytic plurisubharmonic exhaustion function $f$ on $X'$ and it is constant on the leaves of $\mathcal{F}$.

This shows that all the requirements in the main theorem naturally appear in the non-archimedean case.

### 3 The proof of Main Theorem

In this section, we set $M = X'$. The indeterminacy set $\mathcal{I}$ given by the holomorphic 1-form $\theta$ defining $\mathcal{F}$ is of complex codimension 2, which is a set of discrete points when $M$ is a complex surface. Then $f(\mathcal{I})$ is also a set of discrete values in $\mathbb{R}$, because $f$ is proper. For the following choices of level sets of $f$, we always avoid this set.

Now we divide the proof of the main theorem into the following two cases:

#### 3.1 Case I: On the existence of one real analytic level set of $f$

In this case, we have at least one real analytic level set of $f$. Then we can imitate the proof of theorem 2.4 to get a proper holomorphic map onto a Riemann surface with connected fibers.

#### 3.2 Case II: All level sets of $f$ are not real analytic.

This case can be seen as the non-archimedean version of Theorem 2.4. First, we define critical/regular values and regular/singular sets for subanalytic functions.
Definition 3.1. Let $N$ be a subanalytic subset of a real analytic manifold $M$, and $f$ be a real-valued continuous subanalytic function on $N$, we say that $r \in f(N)$ is a critical value of $f$ if $f^{-1}(r)$ is not a real analytic submanifold of $M$. Otherwise, it is called a regular value. Denote the set of critical values as $C(f)$.

We say that $x \in N$ is a regular point if $f$ is real analytic at $x$; otherwise, we say that $x$ is a singular point. The set of regular and singular points are called regular set and singular set of $f$ respectively.

Remark. This definition of regular/singular set for subanalytic functions is different from that for harmonic maps in Definition 2.7.

The existence of one compact leaf is the key step to getting a proper holomorphic map onto a Riemann surface with connected fibers. Usually, we would try to show that the leaf coincides with a level set of one holomorphic function or a level set of two linearly independent real-valued functions which are naturally closed, by the comparison of dimensions. While here, we have only one real-valued function $f$, and the dimension of its level set is 1 higher than that of the leaves. In this situation, we need to use the fact that $f$ is subanalytic to analyze its singular set lying in a fixed level set (which is also closed as the intersection of two closed subsets), which is of dimension lower by 1 and has a chance to coincide with the leaves inside it. The subanalytic structure is essential here.

This most difficult part is stated as:

Theorem 3.2. Let $M$ be a Galois covering surface of a compact Kähler surface and $f$ be a continuous subanalytic plurisubharmonic exhaustion function which is constant on the leaves of a given codimension 1 holomorphic foliation $\mathcal{F}$. If every level set of $f$ is not real analytic, then for almost every $r \in f(M)$, the set of singular points lying in the level set $f^{-1}(r)$ is given by a discrete union of compact leaves of $\mathcal{F}$.

The proof of this lemma is divided into following five sections a)-e).

a) Propositions of subanalytic functions/sets.

We list here all the properties of subanalytic functions/sets needed (cf. the work by Bierstone and Milman [BM88] for proof).

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1Here we use the convention that the number not lying in the image set (i.e. $\forall r \notin f(N)$) is a regular value.
Proposition 3.3 (Theorem 0.1, [BM88]: Uniformization theorem). Suppose \( M \) is a real-analytic manifold and \( X \) is a closed subanalytic subset of \( M \). Then there is a real analytic manifold \( N \) of the same dimension as \( X \) and a proper real analytic map \( \phi : N \to M \) such that \( \phi(N) = X \).

Proposition 3.4 (Proposition 5.3, [BM88]: Resolution of subanalytic functions). Suppose \( M \) is a real analytic manifold and \( f \) is a continuous subanalytic function on \( M \). Then there exists a real analytic manifold \( N \), of the same dimension as \( M \), and a proper surjective real analytic map \( \phi : N \to M \) such that \( f \circ \phi \) is real analytic on \( N \).

Proposition 3.5 (Theorem 6.1, [BM88]: low dimensional subanalytic sets). Let \( M \) be a real analytic manifold and let \( X \) be a subanalytic subset of \( M \). If \( \dim(M) \leq 2 \) or \( \dim(X) \leq 1 \), \( X \) is semianalytic.

Proposition 3.6 (Theorem 3.10, [BM88]: theorem of the complement). Let \( M \) be a real analytic manifold and let \( X \) be a subanalytic subset of \( M \). Then \( M - X \) is subanalytic.

Proposition 3.7 (Theorem 7.5, [BM88]: subanalyticity of the regular set). Let \( N \) be a real analytic manifold and let \( f : N \to \mathbb{R} \) be a continuous subanalytic function. Then the regular set of \( f \) is an open subanalytic subset of \( N \).

Proposition 3.8 (Theorem 3.14, [BM88]: Bound on the number of fibers for subanalytic map). Let \( M \) and \( N \) be real analytic manifolds and \( X \) be a relatively compact subset of \( M \). Let \( \phi : X \to N \) be a subanalytic map. Then the number of connected components of a fiber \( \phi^{-1}(y) \) is bounded locally on \( N \).

Also, the subanalytic sets are “nicely” stratified in the sense that any point admits a locally compact neighborhood which can be written as the union of smooth real analytic manifolds of different dimensions, called strata. The lower dimension strata are either disjoint from the higher dimensional strata, or are contained in them as the frontier (cf. section 16, [Loj64] for details).
b) Global picture.
Due to Proposition 3.4, there exists the following resolution of $f$:

$$
\begin{array}{c}
N \xrightarrow{\phi} M \\
\downarrow g \quad \downarrow f \\
\mathbb{R}
\end{array}
$$

where $N$ is a 4-dimensional real analytic manifold, $\phi$ is a real analytic map of generically rank 4 onto $M$ and $g = f \circ \phi$ is a real analytic function on $N$. Denote $E \subseteq M$ to be the singular set of $f$. By Proposition 3.7, $M - E$ is open and subanalytic. Then $E$ is a closed subanalytic subset of $M$ by Proposition 3.6. Let $D$ be the set on $N$ where $\phi$ has a lower rank, then $D$ is of real dimension at most 3. On the complement of $\phi(D)$, $f = g \circ \phi^{-1}$ is real analytic. Thus we show that $E \subseteq \phi(D)$ is of real dimension at most 3.

c) Local picture.
Locally, every $x \in M$ admits a relatively compact open subset $V$ of $M$ which is compatible with the foliation $\mathcal{F}$. Write the coordinate on $V \cong \Delta \times \Delta$ as $\{z_1, z_2\}$, where $\Delta$ denotes the unit disc on complex plane. The transversal direction to the leaves is given by $z_1$ and the leaf direction is given by $z_2$. Since $f$ is constant on the leaves, locally $f = f(z_1)$ descends to a function $\tilde{f}(z_1)$ on $\Delta$. In conclusion, we have the following diagram:

$$
\begin{array}{c}
\Delta \times \Delta \xrightarrow{\pi_1} \Delta \\
\downarrow f \\
\mathbb{R} \\
\downarrow \tilde{f}
\end{array}
$$

where $\pi_1$ denotes the projection onto the first factor.
Denote $\tilde{E} = \{z_1 \in \Delta : \tilde{f} \text{ is not real analytic at } z_1\}$ to be the singular set of $\tilde{f}$, then locally

$$E \cap V = \tilde{E} \times \Delta.$$

Since $\tilde{E}$ is a 1-dimensional subanalytic subset of $\Delta$, it is semianalytic by Proposition 3.5. Then $E$ can be seen as a disc bundle over the semianalytic set $\tilde{E}$. This is a generalization of the disc bundle over manifolds (cf. work of Diederich and Ohsawa [DO85]) and we sought a similar result about the holomorphic convexity of the total space.
In conclusion, we have the following commutative diagram:

\[
\begin{array}{ccc}
N \overset{\phi}{\longrightarrow} M & \xleftarrow{\text{inclusion}} & V \cong \Delta \times \Delta \\
\downarrow g & & \downarrow \pi_1 \\
\mathbb{R} & \overset{(f)}{\longrightarrow} & \Delta
\end{array}
\]

d) Stratification.

We now stratify \( \tilde{E} \) into 1-dimensional and 0-dimensional real analytic manifolds: i.e.

\[
\tilde{E} = (\bigcup_i \tilde{E}_i^0) \cup (\bigcup_j \tilde{E}_j^1)
\]

where \( \tilde{E}_i^0 \)'s are the 0-strata (points) and \( \tilde{E}_j^1 \)'s are the 1-strata. Here and following, we use an upper index to show the dimension of strata.

This local stratification gives the corresponding part of \( E \) as

\[
E = (\bigcup_i E_i^2) \cup (\bigcup_j E_j^3)
\]

where the 2-strata \( E_i^2 \) is the leaf going through \( \tilde{E}_i^0 \) and the 3-strata \( E_j^3 \) is locally \( \tilde{E}_j^1 \times \Delta \) and globally the union of all leaves going through points in \( \tilde{E}_j^1 \). We can summarize this by saying that \( E \) is foliated by \( \mathcal{F} \).

e) Sard-type theorem for subanalytic function \( \hat{f} \).

For the classical Sard’s theorem, we need some kind of smoothness for the functions. While here, the function is only assumed to be continuous and subanalytic. We need to use the stratification of subanalytic sets to provide an analogous version.

From now on, we focus on the restriction of \( f \) to its singular set \( E \), denoted as

\[
\hat{f} = f\big|_E.
\]

Then \( \hat{f} \) is a continuous subanalytic function defined on a 3-dimensional closed subanalytic set \( E \).

Due to Proposition 3.3, we have a 3-dimension real analytic manifold \( F \) and a
proper real analytic map \( \psi_1 : F \to E \) such that \( \psi_1(F) = E \). Then \( k := \hat{f} \circ \psi_1 \) is a subanalytic function on a real analytic manifold \( F \). By applying Proposition 3.4, the resolution of subanalytic functions, to \( k \), we get the following diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\psi_2} & F \\
\downarrow{g} & & \downarrow{k} \\
& & E \\
\end{array}
\]

Here \( G \) is a 3-dimensional real analytic manifold, \( g \) is a real analytic function and \( \psi_2 \) is a proper real analytic map onto \( F \). Then \( \psi := \psi_1 \circ \psi_2 \) is a proper real analytic map from a 3-dimensional analytic manifold \( G \) onto \( E \).

Denote the set of critical values of the real analytic function \( g \) in the usual sense as \( C(g) \). For any \( r \in \mathbb{R} - C(g) \), \( g^{-1}(r) \) is a 2-dimensional real analytic manifold. Then \( \dim \psi(g^{-1}(r)) \leq 2 \). Since \( \hat{f} \) is constant on leaves, its level set \( \hat{f}^{-1}(r) \) is of real dimension at least 2. Also, \( \hat{f}^{-1}(r) = \psi(g^{-1}(r)) \) as an easy consequence of the following lemma.

**Lemma.** If \( h = g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z \) and \( f \) is surjective, then \( g^{-1}(z) = f(h^{-1}(z)) \), for any \( z \in Z \).

**Proof.** One direction is obvious: \( f(h^{-1}(z)) \subseteq g^{-1}(z) \). Conversely, for any \( y \in g^{-1}(z) \), there exists \( x \in X \) with \( f(x) = y \), then \( z = g(y) = g(f(x)) = h(x) \). So \( x \in h^{-1}(z) \). This shows that \( g^{-1}(z) \subseteq f(h^{-1}(z)) \) and thus we have the equality. \( \square \)

This forces \( \hat{f}^{-1}(r) \) to be of real dimension 2 and to coincide with one leaf of \( \mathcal{F} \) (This in fact shows that \( \hat{f}^{-1}(r) \) is not only real analytic but also complex analytic). Then \( r \) is a regular value for \( \hat{f} \). This shows that \( C(\hat{f}) \subseteq C(g) \) is of measure zero thanks to the classical Sard’s theorem for the real analytic function \( g \).

Then the Sard-type theorem for the subanalytic function \( \hat{f} \) is proved. We can rewrite for a slightly more general situation without any difficulty.

**Theorem 3.9** (Sard-type theorem for special subanalytic functions). Let \( N \) be a closed subanalytic subset of a real analytic manifold \( M \) and \( f \) be a real subanalytic function defined on \( N \). Assume that there exists a real analytic foliation \( \mathcal{F} \) of real codimension 1 such that the level set of \( f \) is foliated by \( \mathcal{F} \).
with the exception at most on a subset of measure zero in $f(N) \subseteq \mathbb{R}$, then the set of critical values of $f$ is of measure zero in $\mathbb{R}$.

**Remark.** The Sard-type theorem is true for general subanalytic functions, as subanalytic sets are nicely stratified and locally (path-)connected (see the work by Bolte, Daniilidis and Lewis [BDL06] for details). For our special situation here, it is much simpler as the level sets are foliated by holomorphic leaves almost everywhere.

Now we finish the proof of Theorem 3.2. For any $r \notin \mathcal{C}(\hat{f}) \cup f(I)$ (a set of measure zero), we have that

$$\hat{f}^{-1}(r) = f^{-1}(r) \cap E \neq \emptyset,$$

for no level set of $f$ is real analytic. Furthermore, this singular set of $f$ lying in $f^{-1}(r)$ (which is closed) is of real codimension 2 and thus coincides with the discrete union of some holomorphic leaves of $F$ contained in $f^{-1}(r)$. The closed leaf is compact because it is contained in the level set of an exhaustion function $f$. This shows the existence of at least one compact leaf in generic level sets of $f$.

QED.

The remaining step to prove the main theorem is classical. As in the proof of Theorem 2.4, we can assume that $M$ has exactly one end. Given a compact subset $K$, choose $a > \max_{x \in K} f(x)$ to be a regular value of $f$ (cf. Definition 3.1), which is possible by the Sard-type Theorem 3.9. Let $\Omega_a$ be the connected component of $\{x \in M : f(x) < a\}$ containing $K$. Denote $U_i$ to be the relatively compact components of $\Omega_a - K$, where $1 \leq i \leq N$ is finitely many.

Take a regular value

$$b > \max \{a, \max_{1 \leq i \leq N} \max_{x \in U_i} f(x)\},$$

then $M - \overline{\Omega_b}$ is connected, where $\Omega_b = \{x \in X' : f(x) < b\}$. If $\partial \Omega_b$ is not connected, say $\partial \Omega_b = A \cup B$ where $A$ and $B$ are disjoint nonempty compact subsets, then $M - \overline{\Omega_b}$ is a connected weakly 1-complete Kähler surface with three ends and thus admits a proper holomorphic map onto a Riemann surface with connected fibers. By the analytic continuation
argument as in the proof of Theorem 2.4, we get a proper holomorphic map from $X'$ onto a Riemann surface with connected fibers.

Now assume $\partial \Omega_c$ is connected and is not real analytic for every value $c \geq b$. By Theorem 3.2, almost each $\partial \Omega_c$ contains a compact leaf of $\mathcal{F}$. Now let $Q$ be the set of points in $M$ which lie in some compact leaves of $\mathcal{F}$. Then $Q$ is nonempty. We will use the connectedness argument below.

Firstly, we show that $Q$ is open following the work by Napier and Ramachandran (cf. proof of Proposition 2.3, [NR97]). Let $x_0 \in Q$ and $L_0$ be the compact leaf it lies in. It can be globally defined as $\theta = dF$ for some holomorphic function $F : U \to \mathbb{C}$ in a relatively compact neighborhood $U$ of $L_0$ in $M - \left( F^{-1}(F(x_0)) - L_0 \right)$. Since $\partial U$ is compact, we can find a neighborhood $V \subseteq \mathbb{C}$ of $F(x_0)$ such that $F^{-1}(V) \cap \partial U = \emptyset$. Hence the leaf given by $F^{-1}(w)$ for any $w \in V$ is also compact. Thus $x_0 \in F^{-1}(V) \subseteq Q$ and $Q$ is open.

Next, we show that $Q$ is closed by considering the volume of these compact leaves. The volume of closed subvarieties of dimension $k$ is a homotopy invariant, for its volume form $d\text{Vol} = \omega^k$ is closed where $\omega$ is the Kähler form on $M$. Assume that $x_n \to x_0$ when $n \to \infty$ and $L_n$ is the compact leaf containing $x_n$. Denote $L_0$ to be the leaf going through $x_0$. The volumes of $L_n$'s are bounded by $V$ and $L_n$'s are of bounded recurrence $N$ (cf. Proposition 3.8) in the sense of Definition 3 of [Mok92]. Then $L_0$ is also of finite volume, which is bounded by $N \cdot V$. Because $M$, as a Galois covering of a compact Kähler surface, is of bounded geometry, we must have $L_0$ is also compact. This shows that $x_0 \in Q$ and that $Q$ is closed.

Now, since $Q$ is nonempty, open, and closed and $M$ is connected, we must have $Q = M$. Denote the holomorphic equivalent relation $\mathcal{R}$ given by these compact leaves (i.e. $x \sim y$ if and only if they lie in the same compact leaf), we get a Riemann surface $R = M/\mathcal{R}$ and the natural holomorphic map $\alpha : M \to R$ which is proper and has connected fibers by construction. In particular, $M$ is holomorphically convex. This finished the proof of the main theorem.

### 4 Applications.

With the main theorem working as a replacement of Theorem 4.8 of [NR95], we get from Lemma 2.11 the following “non-archimedean version” theorem on holomorphic convexity of any Galois covering over a compact Kähler surface which does not have two ends and admits a discrete reductive representation.
defined over a non-archimedean locally finite local field of the Galois group.

**Theorem 4.1.** Let $X'$ be a reductive covering surface over $X$ with group $\Gamma$ which does not have two ends and $G$ be a complex Lie group defined over a non-archimedean locally finite local field. If the representation $\rho : \Gamma \to G$ is discrete, then $X'$ admits a proper holomorphic map onto a Riemann surface with connected fibers. In particular, it is holomorphically convex.

Now we can provide a generalization of Theorem 1.2 of [KR98]. The proof there uses the factorization theorem (cf. Theorem 1.3 [KR98]) to deal with the non-archimedean case, which is unavailable for general Kähler groups. With the above theorem at hand, we can generalize their result from projective (algebraic) surfaces to Kähler surfaces.

**Theorem 4.2.** If $X' \to X$ is a reductive covering surface which does not have two ends, then $X'$ is holomorphically convex.

Even though the essential adjustment is only in the non-archimedean case, we include all the details from [KR98] here for the convenience of readers and point out explicitly where we apply the theorem 4.1 instead of the factorization theorem.

Firstly, we can deform the reductive representation $\rho : \Gamma \to G$ to a properly discontinuous representation $\rho'$ as follows. By taking a finite covering of $X'$, we may assume that group $G$ is torsion-free. Write $G = S \times C$, where $C$ is the maximal compact abelian factor contained in $G$ and $S$ is the semisimple part of $G$.

Denote $\rho_C : \Gamma \to G \to C = G/S$. By Lemma 2.1 of [KR98], every abelian representation $\rho_C$ can be replaced with another representation $\rho'_C : \Gamma \to C$ which is defined over $\mathbb{Q}$ and whose kernel is commensurable to $\ker(\rho_C)$. Similarly, we consider the semisimple representation $\rho_S : \Gamma \to G \to S = G/C$. By Lemma 2.2 of [KR98], we have a neighborhood $W$ of $\rho_S$ where any representation has a Zariski dense image. In particular, there exists a Zariski dense representation $\rho'_S$ defined over $\mathbb{Q}$ in $W$. Now consider the construction of adelic type for $\rho'_S$. Since $\Gamma$ is finitely generated, $\rho'_S$ is defined over a finite extension $E$ of $\mathbb{Q}$ and we consider the completion with distinct valuations. When the valuation place is non-archimedean, say $p$, we have the completion $E_p$ with action on a locally compact Euclidean building $B_p$; when the valuation place is archimedean, say $\nu$, we have the completion $E_{\nu}$ with action on a Riemannian locally symmetric space $\text{Symm}_{\nu}$ of noncompact type.
Consider the direct product $Y_1 = \prod_{\nu} \text{Symm}_\nu \times \prod_p B_p$ with the action of $\Gamma$ on the diagonal, where only finitely many terms can be nontrivial. Since $E_p$ is discrete in $\prod_{\nu} E_\nu \times \prod_p E_p$, the image of $\rho_S$ acts discretely on $Y_1$.

After doing the deformation on $\rho_C$ and $\rho_S$ separately, we get a discrete representation $\rho'$ of $\Gamma$ over $\overline{\mathbb{Q}}$, acting on $Y = Y_1 \times \mathbb{C}^k$, where $Y_1$ comes from the deformation of $\rho_S$ and $\mathbb{C}^k$ comes from the deformation of $\rho_C$. Denote $Y = N(\rho')$, then $N(\rho')$ is a non-positively curved space and we can construct a $\Gamma$-equivariant pluriharmonic map $u : X' \to N(\rho')$, which comes from harmonic maps into each factors as shown in section 2.

Now recall the definition of the rank of a pluriharmonic map.

**Definition 4.3.** The rank of a pluriharmonic map is the codimension of the holomorphic foliation it induces.

In addition, since any open Riemann surface is Stein, we can rewrite Lemma 3.1 of [KR98] as:

**Lemma 4.4.** Let $X'$ be a connected noncompact complete complex surface admitting a continuous plurisubharmonic exhaustion function $g$ which is generically (i.e. away from a proper complex analytic subvariety) strictly plurisubharmonic. Then $X'$ is holomorphically convex.

Now we are ready to prove Theorem 4.2. As $X'$ is of complex dimension 2, the rank of a pluriharmonic map $u$ is either 1 or 2.

**Proof of Theorem 4.2.** We consider the following two cases.

*Case 1.* There exists a representation $\rho'$ defined over $\overline{\mathbb{Q}}$ for which the associated pluriharmonic map $u$ has rank 2.

This means the dimension of the leaves of the foliation $\mathcal{F}$ given by $u$ is generically zero. Let $X''$ be the Galois covering that corresponds to the image of $\rho' : \Gamma \to G \leq GL_N(\overline{\mathbb{Q}})$, i.e.
Take \( u : X'' \rightarrow N(\rho') \) to be the associated \( \Gamma \)-equivariant pluriharmonic map and define \( g(x) = d^2(u(x), y_0) \), where \( d \) is the distance function of \( N(\rho') \) and \( y_0 \) is a fixed point there. The degeneracy locus of \( \sqrt{-1} \partial \bar{\partial} g \) given by the leaves of this foliation is a set \( B \) of discrete points and the function \( g \) is strictly plurisubharmonic away from this subset. In addition, it is an exhaustion function for \( \rho' \) is discrete. By Lemma 4.4, \( X'' \) is holomorphically convex. Then we may consider its Cartan-Remmert reduction \( S(X'') \), which is a complex surface. By the result of Napier [Nap90], the only obstruction of the holomorphic convexity for a complex surface is the existence of an infinite chain (i.e. a connected noncompact analytic set all of whose irreducible components are compact). Assume \( Z'' \) is a finite chain of compact irreducible components in \( X'' \) whose lift to \( X' \) is an infinite chain \( Z' \), then \( \ker \rho' \) has to be infinite.

Now we are in the place to apply the result by Lasell and Ramachandran (cf. Theorem 1.1, [LR96]):

**Lemma 4.5.** Let \( X' \) be a Galois covering of a compact Kähler surface \( X \) with group \( \Gamma \) and a reductive representation \( \rho : \Gamma \rightarrow G \) into a reductive complex Lie group. Let \( Z' = \bigcup_{i=1}^{\infty} D_i \) be a connected analytic subspace of \( X' \), all the irreducible components \( D_i \)'s of which are compact. If \( \rho(\pi_1(D_i)) \) is trivial for any \( i \), we have \( \rho \) restricted to \( \pi_1(Z') \) factors through a finite group \( \Delta_n \), the order of which depends only on the rank \( n \) of \( G \).

Because the image of \( \rho : \pi_1(D_i) \rightarrow G \) is finite for each \( i \), by the above Lemma, we have \( \rho(\pi_1(Z')) \) is also finite. Since \( \rho \) is faithful, this means \( \pi_1(Z') \) is finite, which contradicts the fact that \( Z' \) is an infinite chain.

We summarize the above result as the following

**Lemma 4.6.** Let \( X_2 \rightarrow X_1 \rightarrow X \) be a tower of infinite Galois coverings over a compact Kähler surface \( X \). Assume that the Galois group \( \Gamma_2 = \text{Galois}(X_2/X) \) admits a(an) (almost) faithful linear reductive representation into a reductive complex Lie group and the intermediate covering \( X_1 \) is holomorphically convex, then \( X_2 \) is also holomorphically convex.

**Case 2:** there exists a neighborhood \( W \) of \( \rho \) such that for every representation \( \rho' \in W \) defined over \( \overline{\mathbb{Q}} \), the rank of the associated pluriharmonic map \( u \) is 1.

As before, we may assume \( \rho' \) is a discrete representation with Zariski dense image in a reductive complex Lie group \( G \) over \( \overline{\mathbb{Q}} \). Since any group with two
ends contains an infinite cyclic group $\mathbb{Z}$ of finite index, we know the image of $\rho'$ cannot have two ends.

If $N(\rho')$ contains no Euclidean factor, we apply Theorem 2.5; if there exists at least one non-trivial Euclidean building factor in $N(\rho')$ for any $\rho' \in W$, we use Theorem 4.1. In both cases, we get a holomorphic map from some intermediate covering surface $X''$ onto a Riemann surface with connected fibers, which shows $X''$ is holomorphically convex. Then by Lemma 4.6, we know $X'$ is holomorphically convex.

\[ \Box \]

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