COSMOLOGICAL MODELS IN TWO
SPACETIME DIMENSIONS

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Abstract

Various physical properties of cosmological models in (1 + 1) dimensions are investigated. We demonstrate how a hot big bang and a hot big crunch can arise in some models. In particular, we examine why particle horizons do not occur in matter and radiation models. We also discuss under what circumstances exponential inflation and matter/radiation decoupling can happen. Finally, without assuming any particular equation of state, we show that physical singularities can occur in both untilted and tilted universe models if certain assumptions are satisfied, similar to the (3 + 1)-dimensional cases.
1 Introduction

Gravitational theory in two spacetime dimensions continues to provide theorists with an interesting “laboratory” for exploring the foundations of classical and quantum gravity. The reduction in the number of degrees of freedom markedly reduces the complexity of a system of (3 + 1)-dimensional gravity/matter field equations. The resultant simplicity ought to provide an important understanding of the issues associated with short-distance problems, black holes, information loss, topology change, singularities and the cosmological constant problem.

It is well-known that the price one pays for such simplicity is that one is forced to adopt a new system of field equations in the gravitational sector, since the \( (N+1) \)-dimensional Einstein equations are trivial when \( N = 1 \). A number of approaches [1] have been adopted to address this problem including the incorporation of extended geometrical structures, non-locality, constant curvature theories, non-linear curvature terms, or dilaton couplings.

Perhaps the simplest approach is to set the only measure of curvature in two dimensions, the Ricci scalar, equal to the trace of the conserved stress-energy tensor [2]. Specifically,

\[
R = 8\pi G T^a_a, \quad \nabla_a T^a_b = 0
\]  
(1)

where \( R \) is the curvature scalar, \( G \) is the \( (1 + 1) \)-dimensional gravitational constant and \( T \) is the trace of the energy-momentum tensor. A number of features single out such a theory for consideration. The gravity/matter interaction is qualitatively the same as that in general relativity: stress-energy acts upon spacetime, telling it how to curve, and spacetime in turn acts upon stress-energy, telling it how to move. Viewed as a particular type of “dilaton gravity”, it is the only such theory in which the dilaton classically decouples from the gravity/matter system [3]. Furthermore, the system (1) can viewed as the \( D \rightarrow 2 \) limit of the \( D \)-dimensional Einstein equations \( (D = N + 1) \) provided the \( D \)-dimensional gravitational constant \( G_D \) is appropriately rescaled by \( (D - 2) \) [4]. Finally, despite its simplicity, this theory has a number of remarkable classical and semi-classical features, including a well-defined Newtonian limit, black holes, a post-Newtonian expansion, FRW cosmologies, gravitational collapse, and black hole radiation [2,3,5]. These features suggest that the theory is potentially a very useful tool in studying quantum gravitational and cosmological effects. Since its classical features are so similar to those of \( (3 + 1) \)-dimensional general relativity, one might hope that its quantization would bear a similar resemblance to \( (3 + 1) \)-dimensional quantum gravity.

To this end, we investigate here the cosmological properties of a \((1 + 1)\)-dimensional universe based on the gravitational field equation (1). We begin in section 2 by outlining the basic mathematical and physical components of the \((1 + 1)\)-dimensional theory based on (1) as pertains to the study of cosmology. The Raychaudhuri equation and some of its solutions, and the Stefan-Boltzmann law (i.e. the temperature of a radiation universe) are also obtained. In section 3 we show that a hot big bang and a hot big crunch arise in an universe with a certain kind of matter-radiation mixture and a particular equation of state. In addition, as a consequence of the lower-dimensionality of the spacetime, particle horizons do not exist in a \((1 + 1)\)-dimensional Friedmann-Robertson-Walker (FRW) universe in which matter and radiation are decoupled. In section 4 we investigate decoupling and inflation, showing under what circumstances such processes arise. The final section concerns physical singularities of both FRW and tilted universe models for general equations of state. We show that infinite density singularities can occur in both models if certain energy conditions are satisfied. We also give an explicit model in which a “whimper” singularity can arise. Conclusions and an appendix concerning a specific model which can realize inflation round out our work.

Throughout this paper we take speed of light to be unity but retain Newton’s constant \( G \) explicitly. We use the metric signature \((-+\)).

2 FRW Cosmology

In this section we set up the general formalism suitable for analyzing \((1 + 1)\)-dimensional cosmologies.

We shall assume that there exists locally a well-defined normalized velocity two vector \( U^a \) \((U^a U_a = -1)\) describing the motion of the matter in the universe, and a uniquely defined symmetric one metric

\[
h_{ab} = g_{ab} + U_a U_b,
\]  
(2)
which is a local projection tensor \((h^a_b h^b_c = h^a_c)\) projecting into the one-dimensional instantaneous rest space of observers moving with \(U^a\). This space is orthogonal to \(U^a\), where \(h^a_a = 1, h^a_b U^b = 0\). Thus the spacetime metric \(g_{ab}\) induces a spatial metric \(h_{ab}\) on the rest space of \(U^a\). As with the \((3 + 1)\)-dimensional case \([6]\), one can write the first covariant derivative \(\nabla_b U^a\) of \(U^a\) in \(1 + 1\) dimensions as

\[
\nabla_b U^a = h^a_c h^b_d \nabla_d U^c - U^b U^d \nabla_d U^a = V_{ab} - \frac{dU_a}{dt} U_b
\]

or

\[
\nabla_b U^a = \theta h_{ab} - \frac{dU_a}{dt} U_b,
\]

where \(V_{ab} = h^a_c h^b_d \nabla_d U^c = \theta h_{ab}\) and \(\theta \equiv \theta^a_a\) is the length expansion. The expansion tensor \(\theta_{ab}\) is a symmetric tensor satisfying \(\theta^a_a U^b = 0\); it has only one degree of freedom which may be expressed by its trace, \(\theta\). The usual vorticity and shear tensors are missing in (4) because there are no rotations and shears in one spatial dimension.

A spatially homogeneous and isotropic (these two concepts are equivalent in \(1 + 1\) dimensions, and will be explained in section 5) model of the universe is a perfect fluid spacetime with the properties:

\[
T_{ab} = p h_{ab} + \rho U^a U_b
\]

and

\[
\frac{dU^a}{dt} = 0.
\]

Thus the only non-zero kinematic quantity is \(\theta\). (6) implies \(U_a = -t_a\), where \(t\) measures proper time along all world lines \([6]\). Now, the rest spaces defined at each point by \(h_{ab}\) mesh together to form a family of one-dimensional space-like lines \(L_t\) orthogonal to \(U^a\). Using the comoving coordinate \(x\) and \(t\), we can write the \((1 + 1)\)-dimensional Friedmann-Robertson-Walker (FRW) metric as

\[
ds^2 = -dt^2 + \frac{a^2(t)}{1 - kx^2} dx^2,
\]

where the usual angular coordinates have been suppressed. A change of variable \(dx^2/(1 - kx^2) \rightarrow dx^2\) implies

\[
ds^2 = -dt^2 + a^2(t)dx^2,
\]

where \(a(t)\) is the cosmic scale factor. Thus in \(1 + 1\) dimensions the three different cosmological models corresponding to values \(k = 0, \pm 1\) in (7) do not affect the time evolution of \(a(t)\), unlike \((3 + 1)\)-dimensional FRW universe models. \(k = 0(-1)\) still describes a spatially open flat (hyperbolic) universe. \(k = 1\) describes a closed universe. In addition, in any FRW model, \(a(t)\) is related to the length expansion \(\theta\) through

\[
\frac{da}{dt} = \frac{\theta}{a}.
\]

For the sections 3 and 4, we shall proceed under the assumption that the universe is homogeneous and isotropic.

The Raychaudhuri equation (the time evolution equation of the expansion \(\theta\)) in \(1 + 1\) dimensions can be easily derived. We start with the Ricci identity applied to the velocity two vector \(U^a\)

\[
2\nabla_b [\nabla_d] U^a = R_{abcd} U^b.
\]

Multiplying (10) by \(U^d\) and then projecting on \(a\) and \(c\) and using (3), we get

\[
(h^a_c h^b_d) \frac{dV_{cd}}{dt} - (\frac{dU_a}{dt})(\frac{dU_b}{dt}) - (h^a_c h^b_d) \nabla_d (\frac{dU_c}{dt}) + V_{ad} V^d_b + R_{abcd} U^c U^d = 0.
\]
By contracting $a$ and $b$ in (11), one finds
\[ \frac{d\theta}{dt} + \theta^2 - \nabla_a \left( \frac{dU^a}{dt} \right) + R_{ab} U^a U^b = 0 \] (12)
on using $V_{ab} = \theta h_{ab}$. Using $R_{ab} U^a U^b = 4\pi G(\rho - p)$, which follows from $R_{ab} = (1/2)g_{ab} R$ in $1+1$ dimensions \cite{7} and (1), for a perfect fluid and conditions of homogeneity and isotropy ($dU^a/dt = 0$), the Raychaudhuri equation (12) becomes
\[ \frac{d\theta}{dt} + \theta^2 + 4\pi G(\rho - p) = 0. \] (13)
Thus the Raychaudhuri equation in $1+1$ dimensions is quite similar to the $(3+1)$-dimensional one (see e.g. \cite{6}) except for the sign of the pressure term. This difference prohibits inflation from happening in $1+1$ dimensions under the influence of scalar fields (as we shall see in section 4). Furthermore, the term $\rho - p$ plays the role as the active gravitational mass density and thus the strong energy condition, SEC, in $1+1$ dimension becomes $\rho - p > 0$. In addition, in section 5, we shall show that (13) plays an important role on physical singularities in $(1+1)$-dimensional universe models.

For a perfect fluid, we obtain the field equation
\[ \frac{d^2a}{dt^2} = 4\pi G(p - \rho)a \] (14)
on using (9) and (13). With the comoving velocity two-vector $U^a = (1,0)$, the conservation law
\[ \nabla^a T_{ab} = 0 \] (15)
becomes
\[ \frac{dp}{da} = \frac{d[a(p + \rho)]}{dt}, \quad \text{or} \quad \frac{d(\rho a)}{da} = -p. \] (16)
This equation describes the exchange of energy between matter-radiation and gravitational energy. By considering barotropic perfect fluids (those where $p = p(\rho)$ being at least $C^1$, so that there is a well-defined speed of sound $v_s = (dp/d\rho)^{1/2}$), we can integrate (16).

Consider the equation of state $p = (\gamma - 1)\rho$ with the condition $2 \geq \gamma \geq 1$ as an example. The upper limit corresponds to a pure radiation universe ($\rho_m = 0$), the lower limit to a pure matter one ($\rho_r = 0$), We obtain
\[ p_m = 0, \quad \rho_m \propto a^{-1} \] (17)
for a pressure-free matter, and for radiation
\[ p_r = \rho_r, \quad \rho_r \propto a^{-2} \] (18)
on using (16). Both matter and radiation stress-energy tensors satisfy (16).

In the case of interactions, one may propose a more general situation
\[ \rho_r = \rho_{ro} a^\beta, \quad \rho_m = \rho_{mo} a^\alpha, \] (19)
where $\rho_{ro}$ and $\rho_{mo}$ are non-zero constants in general. For simplicity, we may assume that the fluids do not interact significantly, thereby assuming that both matter and radiation have almost the same velocity two-vector. In other words, the total energy-momentum tensor of the fluids is still in a perfect fluid form with $\rho = \rho_m + \rho_r$ and $p = \rho_r$ (pressure-free matter). Now, substituting (19) into (16), one obtains
\[ a^{\alpha - \beta} = -\frac{\rho_{ro}}{\rho_{mo}} \left( \frac{\beta + 2}{\alpha + 1} \right). \] (20)
Note that this equation and the weak energy condition (WEC) \((\rho_m \geq 0, \rho_r \geq 0)\) impose restrictions on the values of \(\alpha\) and \(\beta\). If \(a(t)\) is not a constant, we must have \(\alpha = -1\) and \(\beta = -2\) (a non-interacting mixture of matter and radiation) or \(\alpha = \beta\) with \(-2 < \alpha < -1\).

Now the field equation (14) becomes

\[
\frac{d^2 a}{dt^2} = -4\pi G \rho_{mo} a^{\alpha+1}.
\] (21)

The general solution is

\[
t = \frac{1}{\sqrt{K_1}} \int \frac{da}{\sqrt{K_2 - a^{\alpha+2}}} = \frac{a}{\sqrt{K_1 K_2}} \beta F_1 \left(\frac{1}{2}, \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2}, \frac{a^{\alpha+2}}{K_2}\right)
\] (22)

where \(K_1 = 8\pi G \rho_{mo}/(\alpha + 2)\), \(K_2\) is an integration constant and \(\beta F_1\) is the hypergeometric function. This equation seems to imply that evolution of \(a(t)\) depends only on \(\alpha\) (or \(\rho_m\)). However closer inspection reveals that this is not the case; unless \(\beta = -2\) (the non-interacting case), the value of \(\beta\) restricts \(\alpha\) via (20) and the WEC. In this manner the radiation density indirectly affects the evolution of \(a(t)\).

Henceforth, unless otherwise stated, we consider the non-interacting case. It is clear that the dynamical equations for the \((1+1)\)-dimensional FRW cosmology are (14), (16) and the equation of state \(p = p(\rho)\). For a mixture of non-interacting matter and radiation universe, we have

\[
\frac{d^2 a}{dt^2} = -4\pi G \rho_{mo} \cdot
\] (23)

Thus, such a universe is a dynamical one which began with a “big bang” in its evolutionary history. For a pure radiation universe \((i.e. \ \rho_m = 0)\), \(d^2 a/dt^2 = 0\), and so the scale factor varies linearly with \(t\) or it is a constant. The spacetime is globally equivalent to flat space since the associated energy-momentum tensor has vanishing trace and consequently the \((1 + 1)\)-dimensional curvature tensor vanishes everywhere.

To close this section we obtain the Stefan-Boltzmann law in \((1+1)\) dimensions; this will yield a relation between \(a(t)\) and \(T\) (temperature) in a radiation universe. Consider, for simplicity, a one dimensional spatial cavity with length \(l\) and whose walls (just two end-points) are maintained at a temperature \(T\). Analogous to the derivation in three spatial dimensions \((e.g. [8])\), one can show that the number of states of photon in a frequency interval \([f, f + df]\) is

\[
g(f)df = ldf.
\] (24)

The density states \(g(f)\) is therefore constant. From this and the Bose-Einstein distribution function, we find that the amount of energy, \(dE_f\), carried by the \(dn_f\) photons with frequencies between \(f\) and \(f + df\) is

\[
dE_f = hf dn_f = \frac{lnhf \cdot df}{\exp[\frac{kT}{hf}] - 1}.
\] (25)

Now, it is easy to see that the Stefan-Boltzmann law in one spatial dimension is given by

\[
E = \int dE_f = lh \int_0^\infty \frac{f df}{\exp[\frac{kT}{hf}] - 1}
\] (26)

where \(E\) is the total energy in the cavity. Rescaling \((hf/kT) = x\), we see that

\[
\frac{E}{l} = \rho_r \propto T^2.
\] (27)

Condition (18) then implies

\[
a \propto T^{-1}.
\] (28)
Thus, in a (1 + 1) or (3 + 1)-dimensional FRW universe, the scale factor is inversely proportional to the temperature of radiation. As $a(t)$ approaches zero, the temperature $T$ becomes infinite in both cases.

3 The Big Bang and Particle Horizons

For a non-interacting mixture of radiation and matter, one possible solution to (23) is

$$a(t) = -2\pi G \rho_{mo} t^2 + At, \quad (29a)$$

where $A = (da/dt)$ at $t = 0$. Consider a universe containing such a (non-interacting) mixture of matter and radiation. As $a(t)$ approaches 0, $\rho_r$ has no effect on the spacetime and the time evolution of $a(t)$, even though $\rho_r$ could be much greater than $\rho_m$. At $t = 0$ or $A/2\pi G \rho_{mo}$, $a(t) \to 0$ and correspondingly, $\rho$ diverges as does the curvature scalar $R = -8\pi G \rho_m$ and the temperature $T \propto a^{-1}$. Thus we have a hot big bang and a hot big crunch in a universe with a non-interacting mixture of matter and radiation.

Note also that for a pure matter ($\rho_r = 0$) universe, one has [5]

$$a(t) = a_o[1 - 2\pi G \rho_o(t - \frac{1}{\sqrt{2\pi G \rho_o}})^2], \quad (29b)$$

where $\rho_o$ and $a_o$ are the matter density and cosmic scale factor at maximum expansion. Obviously, as $t \to 0$ and $2/\sqrt{2\pi G \rho_o}$, $a(t) \to 0$; consequently, $T, \rho_m$, and $R$ all diverge as expected, since radiation has no effect on spacetime in the former case.

For weakly interacting mixtures of matter and radiation, recall that these obey (19), (20) and the WEC. Hence they must satisfy $-2 < \alpha < -1$ in (22). Since \(_2F_1(\frac{1}{2}, \frac{1}{\alpha + 2}; \frac{\alpha + 3}{\alpha + 2}; 0)\) vanishes, it is clear that this kind of weakly interacting mixture yields a big bang. For example, when $\alpha = -3/2$, (22) yields

$$t^2 = (\tau - \frac{8(\sqrt{K_2})^3}{3\sqrt{K_1}})^2 = \frac{16}{9K_1}(2K_2 + \sqrt{a})^2(2K_2 - \sqrt{a}), \quad (30)$$

where $\tau$ is the translated proper time. One can see that the scale factor vanishes at $\tau = 0$ and at $\tau = 16(\sqrt{K_2})^3/3\sqrt{K_1}$ (see fig.(1)), yielding respectively a big bang and a big crunch. In addition, the temperature $T \propto a^{-3/4}$ and $\rho \propto a^{-3/2}$ diverge. One can further study the dynamics of the scale factor $a(t)$ if the assumption of “weak interactions” is relaxed; that is, one assumes the fluids interact significantly and thus the total energy-momentum tensor will not become the perfect fluid form.

In the case of the FRW models particle horizons will occur if the integral

$$I = a(t_1) \int_{t_o}^{t_1} \frac{dt}{a(t)} \quad (31)$$

converges as $t_o \to 0$ (big bang) or $-\infty$ (big bang free), where $t_o$ and $t_1$ are the time of emission of light and time of reception respectively. $I$ is the proper distance of the source of emission at $t_1$. For a (1 + 1)-dimensional universe with non-interacting mixture of radiation and matter, we have (29a). It is not difficult to see that the integral

$$\int \frac{dt}{a(t)} = \frac{1}{A} \ln \left(\frac{4\pi G \rho_{mol} t}{2A - 4\pi G \rho_{mol} t}\right) + \text{constants} \quad (32)$$

diverges as one approaches the big bang singularity, $t_o \to 0$. In a pure matter ($\rho_r = 0$) universe with the solution (29b), we obtain a similar integral as in (32). Thus particle horizons do not exist in these two cases. A pure radiation universe yields a cosmic scale factor $a \propto t$ (or constant). The absence of a big bang in such a universe implies that the integral $[\int dt/a(t)] = \log t + \text{constant}$ is undefined as $t \to -\infty$ and thus no particle horizons exist. This effect is expected since a pure radiation universe is globally flat. Therefore a (1 + 1)-dimensional FRW universe with a non-interacting matter/radiation mixture or pure radiation yields no particle horizons.
This former effect is easily seen to be a consequence of the lower dimensionality of the spacetime. Consider, for example, the matter-dominated scenario (\( \rho_c = 0 \)). For \( a(t) \propto t^n \) only those models which have \( n < 1 \) possess a particle horizon, since (31) implies \( I = t_1/(1 - n) \) [9]. The time-time component of the (3 + 1)-dimensional Einstein field equation gives \( (d^2a/dt^2) = -(4\pi G/3)\rho a \) with \( p = 0 \). The dimensionality of our spacetime (four) implies \( \rho \propto a^{-3} \) and so \( a^2(d^2a/dt^2) = -4\pi G/3 \). It is easy to see that in this case \( a \propto t^{2/3} \) thus \( n < 1 \). That is, particle horizons exist. However in \( 1 + 1 \) dimensions, the dimensionality of the spacetime (two) implies \( a \propto \rho^{-1} \) implying \( n > 1 \), \textit{i.e.} particle horizons do not occur.

4 Decoupling and Inflation

In this section, we first discuss an interesting process called decoupling. Consider a \((1+1)\)-dimensional universe that contains only pure radiation (\( \rho_m = 0 \)), and has a constant cosmic scale factor. Consequently the temperature and density are also constant. There is no “dynamics” at all. That is, it is a “steady-state” universe without a beginning and an end. The stress-energy tensor for all matter is traceless.

Suppose that there is some physical mechanism in which the stress-energy tensor from some form(s) of matter develops a non-zero trace (\textit{i.e.} which ‘decouples’ some matter from the radiation). For simplicity, assume that the change is instantaneous (\( \rho_c \) abruptly decreases). Conservation of energy implies

\[
\rho_c = \rho_r' + \rho_m'
\]

where the prime denotes quantities after decoupling. We also assume \( a(t) \) and \( da/dt \) are continuous at the moment of decoupling, which we take to be \( t_o \). From the viewpoint of a \((1 + 1)\)-dimensional observer, this time marks the birth of the matter and radiation filled universe. After a time \( t_d - t_o \) this universe will end in a big crunch (see fig.2), where \( t_d = A/(2\pi G\rho_m'K_2^3) \), where \( K_2 \) and \( A \) are defined respectively in (22) and (29a).

However, one might argue that the pure radiation universe is “absolutely static”: \( \rho_r \) is always constant and spacetime does not evolve. How could some of the radiation spontaneously convert into matter and what mechanisms provide the conversion? These mechanisms cannot come from the outside universe since by definition there is no physical system outside the universe; there is no way it can be “perturbed”. Decoupling, then, can only occur if there are some kinds of macroscopic \((1 + 1)\)-dimensional quantum fluctuation processes to provide the mechanisms, the conformal anomaly being the most obvious candidate. However one is then left with the puzzle of the choice of \( t_o \), since the mechanism of the conformal anomaly should be operative at all times. It would be interesting to see if a two-dimensional universe could in fact be born via such a mechanism.

We turn next to a consideration of inflationary processes. The original form of the inflationary universe model in \( 3 + 1 \) dimensions [10] was developed to address the flatness and horizon problems, (and subsequently the monopole problem). Although these two cosmological problems are not present in \( 1 + 1 \) dimensions, it is instructive to investigate under what circumstances inflation can occur.

For simplicity, we assume that there exists a classical scalar field \( \phi \) with an effective potential \( V(\phi) \) such that the false vacuum of \( \phi \) has a constant energy density \( \rho_f = V(\phi_{false}) \), similar to the \((3 + 1)\)-dimensional case (see e.g. [10]). \( T_{ab} \) then has the perfect fluid form [11]

\[
T_{ab} = -g_{ab}V(\phi_{false}) = -g_{ab}\rho_f
\]

which shows that \( p = -\rho_f \). The pressure of the false vacuum is negative and constant.

Now one can assume that a \((1 + 1)\)-dimensional homogeneous and isotropic matter and radiation filled universe begins from a hot big bang and is in a symmetric phase and then gradually approaches the false vacuum state (the broken symmetry) as the temperature drops. The total energy-momentum tensor of the scalar field becomes equal to (33). To see what happens next, it is easiest to use (14) and (33) to get

\[
\frac{d^2a}{dt^2} = -K^2a
\]
where \( K^2 = 8\pi G\rho_f \) is constant. This equation has the solution

\[
a \propto \cos(K(t - t_0))
\]

which is not inflationary at all. The reason is that the field equation (1) in 1 + 1 dimensions implies that a negative pressure tends to slow down the expansion rate of the universe \([d^2a/dt^2] < 0\). In 3 + 1 dimensions, however, a negative pressure tends to provide an acceleration \([d^2a/dt^2] > 0\). Therefore we conclude that there is no inflation in 1 + 1 dimensions analogous to a (3+1)-dimensional FRW universe under the influence of scalar fields with positive false vacuum energy density.

The only mechanisms under which exponential inflation can occur involve adopting unconventional assumptions. Essentially one needs a mechanism under which the constant \( K^2 \) on the right hand side of (34) becomes negative. For example one could consider taking the false vacuum energy density to be negative. The possibility of the existence of such a negative energy density depends on particle theories in 1 + 1 dimensions which are beyond the scope of this paper. Another possibility is that the matter density violates the WEC, that is, \( \rho_{mo} < 0 \) but \( \rho_{ro} > 0 \). On using (20), we see that \( \alpha < -2 \) or \( \alpha > -1 \) (recall that \( \beta = \alpha \) for a non-constant \( a(t) \)). Now, if \( \alpha = 0 \) (\( \rho_m = -|\rho_{mo}| = -2\rho_{ro} \)), the solution to (21) is

\[
t = \int \frac{da}{\sqrt{K_1a}}, \quad K_1 = 4\pi G|\rho_{mo}|
\]

where \( K_2 \) (the integration constant) is set to be zero. This integral obviously yields

\[
a(t) = a_i \exp[\sqrt{K_1}(t - t_i)], \quad a_i \equiv a(t_i).
\]

Thus the universe can exponentially inflate under the influence of exotic matter. During inflation, the energy density of the pressure-free matter is negative and constant. Further mechanisms involve a time-dependent constant of gravity which permits \( G \) to become negative or taking \( \gamma > 2 \) in the equation of state (implying that the speed of sound becomes greater than the speed of light).

None of the above mechanisms is particularly attractive, and furthermore, none is necessary. The field equation (1) simultaneously implies that particle horizons in some cases (e.g. pure matter or a non-interacting mixture) and inflation (under the influence of a positive false vacuum density scalar field) do not occur in a (1 + 1)-dimensional universe. It would appear that the lack of structure in the lower-dimensional universe banishes both the standard problems of (3 + 1)-dimensional cosmology and the mechanisms which could solve them.

### 5 Singularities in FRW and Tilted Models

In 3+1 dimensions, spatially homogeneous cosmological models in general have spacetime singularities [12]. Specifically, when one considers the spatially homogeneous tilted model, that is, the fluid velocity four-vector is no longer everywhere and everytime orthogonal to the homogeneous surfaces, one has either an infinite density singularity in which \( \rho \) diverges or a finite density singularity in which the tilted angle \( \beta \) is unbounded [13]. In section 3 we have shown that infinite density big bang singularities indeed occur in certain kinds of FRW universes with the equation of state \( p = (\gamma - 1)\rho \).

In this section, without assuming any particular equation of state, we shall show that infinite density singularities can occur in all FRW perfect fluid universes (except the pure radiation model) if certain energy conditions are satisfied. We extend these results to (1 + 1)-dimensional spatially homogeneous tilted models, showing that both finite and infinite density singularities can also happen in this context. We first present the basic equations and then show that the aforementioned physical singularities may occur in both untilted and tilted cases.

As with the (3 + 1)-dimensional tilted model [14], we may define the quantities \( \beta, n_a, \bar{e}_a \) and \( c_a \) by the following relations in 1 + 1 dimensions using \( \tilde{h}_{ab} = g_{ab} + n_an_b \),

\[
cosh\beta \equiv -u^an_a, \quad \tilde{h}^a_bu^b = (\sinh\beta)\bar{e}^a, \quad \tilde{h}^a_bn_b = -(\sinh\beta)c^a,
\]
where \( \tilde{c}_a n^a = c_a u^a = 0 \), \( c^a c_a = \tilde{c}^a \tilde{c}_a = 1 \) and \( \beta(t) \) is the hyperbolic angle of tilt (\( t \) is the proper time of the fluid). As before, \( h_{ab} = g_{ab} + u_a u_b, n^a (n^a n_a = -1) \) is a geodesic vector field and normal to \( L_t \), the lines of homogeneity. \( c^a \) is the direction of \( n^a \) normal to \( u^a \). \( \tilde{c}^a \) is the direction of the projection of \( u^a \) in the lines \( L_t \). Since \( n_a \) is the normal geodesic vector field, we can write

\[
  n_a = -\tilde{\tau}_a, \tag{39}
\]

and consequently,

\[
  \nabla_b n_a = \tilde{\theta}_{ab}, \quad \tilde{\theta}_{ab} n^b = 0. \tag{40}
\]

In addition, one can decompose \( u^a \) and \( n^a \) as

\[
  u^a = (\sinh \beta) c^a + (\cosh \beta) n^a, \quad n^a = -(\sinh \beta) c^a + (\cosh \beta) u^a. \tag{41}
\]

By virtue of (41), one can write [14]

\[
  dt = (\cosh \beta) dt. \tag{42}
\]

The first covariant derivative of \( u^a \) is given by

\[
  \nabla_b u_a = -\left( \frac{d\beta}{dt} \right) c_a n_b + (\cosh \beta) \tilde{\theta}_{ab} + (\sinh \beta) \nabla_b \tilde{c}_a, \tag{43}
\]

on using (41). Consequently, the acceleration and expansion of the fluid \( \theta \equiv \nabla_a u^a \) are respectively given by

\[
  \frac{du_a}{dt} = \frac{d(\sinh \beta)}{dt} c_a + (\sinh \beta \cosh \beta) \tilde{\theta}_{ab} c^b + \sinh \beta \frac{d\tilde{c}_a}{dt}, \tag{44}
\]

and

\[
  \theta = \frac{d(\cosh \beta)}{dt} + (\cosh \beta) \tilde{\theta} + (\sinh \beta) \nabla_u \tilde{c}^a. \tag{45}
\]

In 1 + 1 dimensions (45) can be further simplified as follows: if one constructs normalized co-ordinates \( \tilde{t}, \tilde{x} \), comoving with the geodesic normals, then in these co-ordinate \( n^a = \delta^a_0, \tilde{h}_{a0} = 0, \tilde{h}_{11} = g_{11} \). Furthermore,

\[
  ds^2 = -dt^2 + g_{11}(\tilde{t}) d\tilde{x}^2. \tag{46}
\]

(In 3 + 1 dimensions, in general, \( ds^2 = -d\tilde{t}^2 + g_{\mu\nu}(\tilde{t}) d\tilde{x}^\mu d\tilde{x}^\nu, \mu, \nu = 1, 2, 3 \), see e.g. [15]). Using (46) it is easy to see that \( \nabla_0 \tilde{c}^a = \nabla_1 \tilde{c}^a = 0 \) since \( \tilde{c}^a = (0, \tilde{c}) \) and \( \tilde{c} = \tilde{c}(\tilde{t}) \). This form of \( \tilde{c}^a \) with respect to the metric (46) is obvious since \( n^a \tilde{c}_a = 0 \) and \( \tilde{c}^a \) which spans \( L_t \), is invariant under the group of isometry (one-dimensional spatial translations) on \( L_t \) and therefore it depends only on the “tilde time”. Consequently we have in 1 + 1 dimensions

\[
  \nabla_a \tilde{c}^a = 0, \tag{47}
\]

which simplifies (45).

For a perfect fluid one obtains from the components \( u_a \nabla_b T^{ab} = 0 \) and \( h^{ca} \nabla_b T^{ab} = 0 \) of the conservation equation (15)

\[
  (\cosh \beta) \frac{d\ln w}{dt} + \theta = 0, \quad (\sinh \beta) \frac{d\ln r}{dt} + e^a \frac{du^a}{dt} = 0, \tag{48}
\]

where \( w(t) \equiv \exp[\int \frac{dp}{r + p}] \) and \( r(t) \equiv \exp[\int \frac{dp}{(r + p)}] \). Another expression for acceleration may be obtained from (48):

\[
  \frac{du^a}{dt} = (\tanh \beta) \frac{dp}{dr} \tilde{c}^a. \tag{49}
\]
This equation shows that \( du^a / dt \) is parallel to \( e^a \) and vanishes when pressure is constant (or zero). Finally, combining (44), the second equation of (48) and the equation \( \tilde{c}^a \tilde{d}c_a / dt = 0 \), one has

\[
\frac{d\ln(r \sinh \beta)}{dt} + \tilde{g}_{ab} \tilde{c}^a \tilde{c}^b = 0. \tag{50}
\]

This equation shows that \( \beta \) is either zero or non-zero for all \( t \).

In 3 + 1 dimensions, \( \sigma_{ab} = 0 \Rightarrow \beta = 0 \) [14]. Consequently, the only shear-free spatially homogeneous perfect fluid universe models are the FRW models. A detailed proof involving \((\mu, \nu)\) and \((0, \nu)\) field equations and the Jacobi identities can be found in ref. [14]. In 1 + 1 dimensions, however, both spatial shear and rotation vanish. We must have an FRW universe, which is not tilted if \( du_a / dt = 0 \). In this case, if \( du_a / dt \neq 0 \) then \( \beta \neq 0 \) from (49) and (50) implies that a tilted model stays tilted. We shall exclude the possibility that a universe can have \( du_a / dt \neq 0 \) and \( \beta = 0 \) at the same time (for a spatially inhomogeneous universe, this situation is possible).

Conversely, since \( (du_a / dt) = 0 \) implies from (49) that \( \beta = 0 \) (FRW universes cannot be tilted), the following theorem is obvious.

**Theorem 1**: \( \beta = 0 \Leftrightarrow du_a / dt = 0 \) for all spatially homogeneous universe models in 1 + 1 dimensions.

In other words, the necessary and sufficient condition for a 1 + 1 dimensional universe to be a FRW model is that \( \beta = 0 \). Theorem 1 is clearly not true in 3 + 1 dimensions. Note also that in 3 + 1 dimensions, in a strict mathematical sense, spatially isotropic universe models imply spatially homogeneous untilted models (the converse is not true). It is obvious that in 1 + 1 dimensions, the only untilted models are FRW models. Hence one gets the following simple theorem:

**Theorem 2**: For an untilted (1 + 1)-dimensional universe, it is spatially homogeneous if and only if it is spatially isotropic.

Now, rather than assuming any particular equation of state, we show that physical singularities may happen in both FRW and tilted models under certain assumptions. Our approach is quite similar to that of Collins and Ellis [13]. The spacetime we consider will satisfy the following criteria.

**Assumption 1**: \((M, g_{ab})\) is a connected (1 + 1)-dimensional \( C^\infty \) Hausdorff manifold \( M \) with at least \( C^2 \) Lorentz metric \( g \) and it is inextendible and hole-free.

In a strong intuitive sense, this assumption avoids the situation in which spacetimes which are otherwise non-singular but simply have points “artificially removed” would be considered singular. In addition, we have assumed that \((M, g_{ab})\) has been extended as far as possible.

Similar to equation (2.2) of Ellis and King [16], we have the following assumption.

**Assumption 2**: A \( C^1 \) equation of state \( p = p(\rho) \) for the fluid is given, and is such that the following conditions

\[
1 \geq dp / d\rho \geq 0, \quad \rho \geq 0, \quad p \geq 0, \quad \rho \geq p
\]

hold at all times.

The first inequality expresses the fact that the speed of sound cannot exceed unity (i.e. the speed of light) in 1 + 1 dimensions, and that the initial value problem is well posed [17]. The second and fourth ones are the WEC and SEC (strong energy condition) respectively. The third one is imposed since we are not going to involve a false vacuum (or equivalently, a cosmological constant) in our discussion. Assumption 2 implies that there is a time \( t_o \) such that \( \rho_o + p_o > 0 \), where \( \rho_o = \rho(t_o) \) and \( p_o = p(t_o) \).

The average length scale, \( R(t) \), is defined as \((dR / dt)(1/R) = \theta \) (\( R(t) = a(t) \) in FRW models). We assume that the universe is expanding at the time \( t_o \).

**Assumption 3**: On each world line there is a time \( t_o \) such that \((dR / dt)o(1/R)_o \equiv H_o > 0 \).
We first consider FRW models. Recall that in such models, (14) and (16) are the dynamical equations. In addition, because of (16), one can integrate (14) to obtain the Friedmann equation in $1+1$ dimensions,

$$
\frac{dR}{dt}^2 = -8\pi G R^2 \rho + C,
$$

where $C$ is a constant. Now (16) and the first inequality show that

$$
0 < R \leq R_o \Rightarrow (\rho_o + p_o)\left(\frac{R_o}{R}\right)^2 \geq (\rho + p) \geq (\rho_o + p_o)\left(\frac{R_o}{R}\right) \geq \rho_o + p_o > 0.
$$

The mean value theorem of a $C^1$ curve with the first inequality of assumption 2 implies that $1 \geq (p_2 - p_1)/(\rho_2 - \rho_1) \geq 0$ for the $C^1$ curve $p = (\rho)$. However we have just shown that $(\rho + p) \geq (\rho_o + p_o)$. So, it must be true that

$$
p \geq p_o, \quad \rho \geq \rho_o.
$$

Conditions (52), (53) and the mean value theorem yield

$$
(\rho_o + p_o)(\frac{R_o^2}{R^2} - 1) + \rho_o - p_o \geq \rho - p \geq 0,
$$

and

$$
(\rho_o + p_o)(\frac{R_o^2}{R^2} - 1) + \rho_o \geq \rho \geq \frac{1}{2}(\rho_o + p_o)(\frac{R_o}{R} - 1) + \rho_o.
$$

Note that the term $\rho - p$ does not have the similar expression for its lower bound as $\rho + p$ and $\rho$. In fact, we have used the SEC rather than (52) and (53) (a point to which we shall later return). From (52), (54), (55), (14) and (51), it is clear that:

For each value of $R$ with $0 < R < R_o$, $\frac{1}{R} \frac{d^2 R}{dt^2}$ and $\frac{1}{R} \frac{dR}{dt}$ are bounded above and below by finite bounds, as are $\rho$, $\rho + p$ and $\rho - p$.

In addition, (14) and the SEC show that

$$
0 < R \leq R_o \Rightarrow \left(\frac{d^2 R}{dt^2}\right) < 0.
$$

Let the universe be regular and well-defined for $T_1 < t \leq t_o$. Now condition (57) and assumption 3 show that

$$
T_1 < t \leq t_o \Rightarrow 0 < R \leq R_o \Rightarrow \frac{dR}{dt} \geq \left(\frac{dR}{dt}\right)_o = H_o R_o > 0.
$$

Thus

$$
0 < R \leq R_o (1 - H_o(t_o - t)),
$$

for $T_1 < t \leq t_o$. Obviously, $R(t)$ is a monotonically decreasing function if one goes backward in time from $t_o$. The geometry of the universe is regular when $R > 0$ due to condition (56), and thus the initial data, $[\rho, R, dR/dt]$, for the (14) and (16) on a spacelike line (constant $t$) is well-behaved. In addition, recall that we have assumed that initial value problem is well posed in $1+1$ dimensions (assumption 2). Consequently, we do not expect some other singularity to intervene at a time $T_1$ or earlier times when $R$ is still bounded away from zero. Assumption 1 states that if it is possible to extend the universe, it is so extended. Hence, (59) implies that $R \to 0$ a finite time ago. Without loss of generality, we take $R \to 0$ as $t \to 0^+$. Now incorporating conditions (52) and (55), the following conclusion can be drawn:

All $(1+1)$-dimensional FRW universe models satisfying assumptions 1-3, and (14) and (16) must have physical singularities, that is, as $t \to 0^+$, $R \to 0$, $\rho \to \infty$, $\rho + p \to \infty$ and $R_{ab}R^{ab} \propto (p - \rho)^2 \to \infty$. 

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Note that both $p$ and $\rho$ diverge as $R \to 0$, and thus it is possible that $p - \rho$ (along with $R_{ab}R^{ab}$) is finite. Unless $p = \rho$, we exclude any equations of state in which the divergence in $p$ “cancels” the divergence in $\rho$ and makes $R_{ab}R^{ab}$ finite.

It is clear that this conclusion does not hold for a universe of pure radiation since it does not satisfy assumption 3. In addition, recall that zero was taken to be the lower bound for the term $(\rho - p)$ in (54). If it is possible to find a lower bound for $\rho - p$ similar to the forms of (52) and (55), then $\rho_r - p_r$ will diverge as $R \to 0$. However, in the special case of a non-interacting mixture with equation of state $p = (\gamma - 1)\rho$, this obviously cannot be true since one always has $\rho_r - p_r = 0$. It is due to the fact that for this equation of state and field equation (1), radiation has no effect on the structure of $(1 + 1)$-dimensional spacetimes.

Finally, by considering the family of geodesic curves normal to $L_\theta$, the family of lines of homogeneity, we show that infinite density or “whimper” singularities may happen in $(1 + 1)$-dimensional tilted models. The reason we do not consider the family of fluid flow lines is that $du_a/dt \neq 0$ and we do not know directly how $\nabla_a (du^a/dt)$ varies with $R(t)$. On the other hand, for the geodesic normals the acceleration vanishes so that we can argue as before. We define the average length scale for the normal geodesic as $(dR/dt)(1/R) = \theta$. Relative to the vector field $n^a$, the Ricci tensor takes the form

$$R_{ab} = \frac{1}{2}g_{ab}R = 4\pi Gg_{ab}(\bar{\rho} - \bar{p})$$

where $T = p - \rho = \bar{p} - \bar{p}$. Here $\bar{\rho} = \rho(cosh^2\beta) + p(sinh^2\beta)$ and $\bar{p} = p + (\rho + p)(sinh^2\beta)$ are the energy density and pressure respectively measured in the normal geodesic frame (see (1.33b) of [14]). Thus the Raychaudhuri equation for $\theta$ takes exactly the same form as (13) or (14) if $R(t)$ is being used instead. Because of the similarity to the previous equations, much of the argument is parallel to the previous case. Then the analogues of (57), (58) and (59) show that $\bar{R} \to 0$ a finite time ago (as before, we shall take $\tilde{t} \to 0^+$). Note that relative to $n^a$, the energy-momentum tensor is not that of a perfect fluid and therefore (15) will not reduce to (16).

In order to get the analogue of condition (55), we use the quantity $w(t)$, the fluid enthalpy defined in (48). This definition and assumption 2 yield

$$\rho \geq \rho_o \Rightarrow \frac{\rho}{\rho_o} \geq \frac{w}{w_o} \geq \left(\frac{\rho}{\rho_o}\right)^{1/2}.$$  \hspace{1cm} (61)

Now (45), (47) and the first equation of (48) give

$$\frac{d(ln(\bar{R}w\cosh\beta))}{dt} = 0.$$  \hspace{1cm} (62)

According to this equation, one has

$$w(cosh\beta) \propto \frac{1}{R},$$

which is a generalization of condition (55). Since $\tilde{t} \to 0^+$, $\bar{R} \to 0$ we conclude that $\lim_{t \to 0^+} \rho(cosh\beta) \to \infty$. It is easy to see that one either has

$$\tilde{t} \to 0^+ \Rightarrow \rho \to \infty, \quad g^{ab}R_{ab} \to \infty,$$  \hspace{1cm} (64)

or

$$\tilde{t} \to 0^+ \Rightarrow \beta \to \infty$$

with $\rho$ bounded. From the normal geodesic viewpoint, (64) corresponds to an infinite density singularity. On the other hand, (65) does not give a diverging Ricci scalar. An observer moving on these geodesics will experience a tidal force determined by the geodesic deviation equation in $1 + 1$ dimension,

$$R_{abcd}X^c n^b n^d = (1/2)g^{ef}R_{ef}(-g_{ac} - g_{ad}n^d g_{be}n^b)X^c = 4\pi G(\rho - p)X_a.$$  \hspace{1cm} (66)
where $X^a$ is the normal deviation vector from a geodesic to an infinitesimally nearby geodesic. In some chosen frames if $g_{ab}$ blows up as $\tilde{t} \to 0^+$, the tidal force will blow up as well. Looked at from the fluid viewpoint, similar events happen. One also has a infinite density singularity in (64). For the situation in (65), one can choose $[n^a, u^a]$ as a non-orthogonal basis for the “tilted spacetime”, we have, in this basis, $g_{nu} = n^a u_a = -\cosh \beta$ which blows up when $\tilde{t} \to 0^+$. As (42) implies $\tilde{dt} > dt$, there is a finite value $T_1$ such that when $\tilde{t} \to 0^+$ on these fluid flow lines, $t \to T_1^+$. Of course, the bad behaviour of the metric component may be a co-ordinate singularity rather than a physical one. Hence, as $t \to T_1^+$ along the fluid flow lines, either the flow lines may run into a finite density singularity or it is possible to extend the flow lines beyond $t = T_1$. In this fashion a “frame-dependent whimper” singularity can occur in (65).

Conclusions
Our exploration of $(1 + 1)$ dimensional cosmologies using (1) has shown that such cosmologies for the most part bear a strong qualitative resemblance to their $(3 + 1)$ dimensional counterparts. The most notable departure from this is, perhaps, the absence of both particle horizons and the possibility of realizing an inflationary scenario without violating the WEC or imposing some other unconventional condition. The most significant resemblance is the persistent occurrence of singularities for virtually all models except the radiation dominated scenario. Whether or not quantum gravitational and quantum cosmological effects will be able to appropriately treat such singularities is a subject for further study.

Acknowledgements
This work was supported by the Natural Sciences and Engineering Research Council of Canada. We would like to thank G. Ellis for discussions.

Appendix
In this appendix we give a toy model for inflationary scenario in two spacetime dimensions.
Suppose the life history of a $(1 + 1)$-dimensional universe is divided into three epochs according to one of the two following scenarios:

1. Big Bang (non-interacting matter + radiation) $\to$ Inflation (exotic matter and radiation) $\to$
   (non-interacting matter + radiation)

2. Big Bang (non-interacting matter + radiation) $\to$ Inflation (exotic matter and radiation) $\to$
   radiation only.

The second sequence is trivial since the universe suddenly becomes static. Note that just before inflation, all matter spontaneously and instantly converts into exotic matter and consequently this exotic matter “accelerates” the expansion. As discussed in section 4 some physical mechanism operative at macroscopic distance scales in $1 + 1$ dimensions is required to carry out such a conversion; a scalar field with negative energy density false vacuum would work, for example.

Assuming that such a mechanism is possible, we continue to study sequence 1. At each transition between the two epochs, both $a(t)$ and $da/dt$ are assumed to be continuous.

For epoch 1 (the matter and radiation era, $0 \leq t \leq t_i$), we choose $t = 0$ as the time when $a(t) = 0$ (origin of the universe). Then

$$a(t) = -Bt^2 + At, \quad B = 2\pi G \rho_{mo},$$

$$\frac{da}{dt} = -2Bt + A. \quad (A2)$$

For epoch 2 (the inflationary era, $t_i \leq t \leq t_f$), all the matter instantly becomes exotic and the negative energy density is constant during inflation:

$$a = a_i \exp[\sqrt{K_1}(t - t_i)], \quad a_i \equiv a(t_i) \text{ of } (A1),$$

$$\frac{da}{dt} = a_i \sqrt{K_1} \exp[\sqrt{K_1}(t - t_i)]. \quad (A4)$$
Note that $a_i K_1 = 2B$ which expresses the fact that all ordinary matter becomes exotic. The continuity of $a(t)$ and $da/dt$ at $t_i$ implies

$$ t_i^\pm = (1/2 \pm \sqrt{3}/6)(A/B). \quad (A5) $$

There are two solutions for $t_i$. However, $t_i^+ > t = A/2B$, the time when the universe starts to contract; inflation can only start at $t_i^-$. Knowledge of $A$ and $B$ (or $\rho_{mo}$), permits one to determine $t_i^-$. In other words, the “initial conditions” $A$ and $B$ exactly determine the time when the ordinary matter will becomes exotic.

For epoch $3$ all the exotic matter instantly convert back to ordinary matter. For simplicity we assume $a(0) = 0$. For the interval $t_f \leq t \leq t_o$:

$$ a(t) = -B't^2 + A't, \quad B' = 2\pi G\rho_{mo}. \quad (A6) $$

Note that, $\rho_{mo} = (a_i/a_f)\rho_{mo}'$. Thus $a_f K_1 = 2B'$. Similarly, the continuity of $a(t)$ and $da/dt$ at $t_f$ implies

$$ t_f^- = (1/2 - \sqrt{3}/6)(A'/B'). \quad (A7) $$

$t_f^+$ is omitted for the same reason as above. Since $B' = B(a_f/a_i)$ and $t_f^- > t_i^-$, we must have $A' > A(a_f/a_i)$. Knowledge of $A'$ and $\rho_{mo}'$ permits determination of $t_f^-$, that is, the time when all the exotic matter become ordinary again.

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