Identification of Time-varying Frequencies and Model Parameters for Large Flexible On-orbit Satellites Using a Recursive Algorithm

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Today, the primary identification methods in time domain for spacecraft are based on singular value decomposition (SVD), such as the eigensystem realization algorithm (ERA) or stochastic subspace identification (SSI), which requires significant computation time. However, some control problems, such as self-adaptive control, need the latest modal parameters to update controller parameters online. To improve computational efficiency, the fast approximated power iteration (FAPI) recursive algorithm, which avoids SVD, is applied as an alternative method to identify the time-varying frequencies of large flexible satellites. Moreover, an improved recursive form is also proposed to obtain the time-varying input matrix in a state-space model by rewriting the relation of the input and output data. In numerical simulations, the time-varying model of the Engineering Test Satellite-VIII (ETS-VIII) is established. The results illustrate that this recursive algorithm can implement time-varying parameter online identification and it has a better computational efficiency than the SVD-based methods.

Key Words: On-orbit Identification, Time-varying Modal Parameters, Large Flexible Satellite, Recursive Algorithm, State-space Model

Nomenclature

| Symbol | Description |
|--------|-------------|
| $k$   | time instant |
| $t$   | time |
| $\Delta t$ | sampling interval |
| $u$   | control torque input |
| $y$   | system output |
| $A$   | system matrix |
| $B$   | input matrix |
| $C$   | output matrix |
| $x$   | state vector in state-space equation |
| $w_n$ | system process noise |
| $v_n$ | measurement noise |
| $r$   | number of inputs in state-space equation |
| $m$   | number of outputs in state-space equation |
| $n$   | system order in state-space equation |
| $U/Y$ | input/output Hankel matrix |
| $M$   | number of Hankel matrix blocks |
| $u_M/Y_M$ | input/output data sequences |
| $\Gamma$ | system observability matrix |
| $M$   | mass matrix in dynamic equation |
| $E$   | damping matrix in dynamic equation |
| $K$   | stiffness matrix in dynamic equation |
| $I$   | input influence matrix |
| $\delta$ | state vector in dynamic equation |
| $I$   | identity matrix |
| $\psi$ | attitude angle of satellite |
| $\theta$ | rotation angle of satellite solar panel |
| $J$   | rotational inertia of satellite |

$F$: flexibility influence coefficient  
$r$: position vector in coordinate system  
$F$: skew-symmetric matrix of vector $r$  
$T$: coordinate transformation matrix  
$\eta$: modal coordinate of flexible appendage  
$\Omega$: modal stiffness matrix of appendage  
$\zeta$: damping ratio of appendage  
$\omega$: pseudo-frequency of satellite  
$\varsigma$: damping ratio of satellite  
$\lambda$: conjugate complex eigenvalue  
$\Lambda$: diagonal eigenvalue matrix  
$\Sigma$: eigenvector matrix  
$k$: number of selected vibration modes for each appendage  
$\Phi$: modal matrix of appendage  
$s$: vibration displacement of appendage  
$\beta$: forgetting factor in FAPI algorithm  
$R$: covariance matrix  
$\xi$: input vector in FAPI algorithm  
$E$: expectation  

Subscripts

$r$: central rigid body of satellite  
s1/s2: solar panel 1/2 of satellite  
a1/a2: antenna reflector 1/2 of satellite

Superscripts

$\dagger$: Moore-Penrose inverse  
$\wedge$: system identification value  
$T$: matrix transposition

1. Introduction

Identification experiments that determine the modal parameters for on-orbit spacecraft have been performed in some
For instance, in the 1980s and 90s, Juang et al. identified the modal parameters and corresponding state-space model parameters for the Galileo spacecraft and Hubble Space Telescope using the eigensystem realization algorithm (ERA).\textsuperscript{5,7} However, these identification experiments are mostly based on the linear time-invariant (LTI) system. From the perspective of actual operations, the configuration properties of these rigid-flexible coupling spacecraft may require alteration in orbit for many reasons, such as reflector deployment,\textsuperscript{8} antenna rotation,\textsuperscript{9} docking with a satellite\textsuperscript{10} or capturing another moving body. Therefore, it is necessary to accurately identify the system parameters of the on-orbit spacecraft when the system is linear time-varying (LTV).

In existing on-orbit identification methods for spacecraft, ERA has been successfully applied to the identification of the LTI modal parameters and corresponding state-space model for various spacecraft.\textsuperscript{3–7,11,12} The ERA is a typical system realization method. This approach constructs the Hankel matrix using complete input and output (I-O) data sequences over a period of time and utilizes singular value decomposition (SVD) at each time step to extract the observability matrices. Then, the system model matrices \{A, B, C\} and corresponding modal parameters are determined. The ERA is based on an LTI system; if the system of interest is LTV, repeated experimental methods are often used to identify the time-varying system model such as pseudo-modal subspace method and time-varying eigensystem realization algorithm (TV-ERA).\textsuperscript{13,14} The repeated experimental methods assume that multiple experiments are performed during system identification and that the system undergoes the same time-varying change by a different input in each experiment. Then, the LTV state-space model and modal parameters are determined. However, because these identification methods for an LTV state-space model are still based on SVD (e.g., pseudo-modal subspace method, TV-ERA, or other repeated experimental methods\textsuperscript{13}), and the procedures of SVD require significant computation costs in system identification, a faster identification method is required.

Some control problems, such as self-adaptive control, often need to obtain the latest dynamic parameters for updating the controller parameters in real-time. In addition, the identified system parameters can help to track and monitor the on-orbit working conditions of spacecraft. Consequently looking for a fast identification method for on-orbit spacecraft is useful. A recursive subspace method called fast approximated power iteration (FAPI) was developed by Badeau et al.\textsuperscript{15} to estimate the system observability matrix. Compared to the existing repeated experimental methods based on SVD, FAPI is derived from signal subspace projection tracking technology, and the LTV signal subspace matrix (i.e., observability matrix) is tracked by updating I-O data in real-time. Therefore, the FAPI method does not need to perform SVD for a Hankel matrix at each time step and consequently computes quickly.

Recursive subspace algorithms have been used in recent years for LTI/LTV system modal parameter identification in the mechanical engineering field, but few studies have investigated using recursive methods to identify the spacecraft LTV model. In this paper, we apply a recursive algorithm to the on-orbit identification of a large flexible satellite. By designing the I-O signals, the time-varying frequencies are obtained. Furthermore, in the existing recursive algorithm, after the matrices \{A(k), C(k)\} are determined recursively, the least squares method can usually be used to estimate the matrix \(B(k)\). Different from the least squares method, an improved recursive method is developed by rewriting the I-O relation to extract the input matrix from the new signal subspace matrix. With the employment of this improved recursive form, the state-space model \([A(k), B(k), C(k)]\) of an LTV system is determined.

The primary purpose of this paper is to discuss the identification of satellite time-varying parameters using recursive algorithm. Frequently-used identification algorithms, which are based on SVD, significantly increase computational complexity and are not suitable for online controller parameter updates. Consequently, the recursive algorithm will be applied in this paper to improve computational efficiency and implement online identification. In addition, an improved recursive form, which is based on the FAPI algorithm, for identifying matrix \(B(k)\) is also presented. In numerical simulations, the model of the Engineering Test Satellite-VIII (ETS-VIII) is established by considering the rotation of solar panels. The numerical results demonstrate that the recursive algorithm is effective in the identification of time-varying frequencies and state-space model parameters. The comparisons of computation time also show that the recursive algorithm has faster computation speeds than the frequently-used SVD method.

This paper is organized as follows. The equation description of a rigid-flexible coupling satellite model is reviewed in Section 2. Section 3 simply introduces the procedure of the FAPI recursive subspace algorithm. In Section 4, identification of the satellite time-varying modal parameters is presented using the FAPI method, and an improved recursive form based on the FAPI algorithm is also developed to identify the time-varying input matrix \(B(k)\) of the state-space model. In Section 5, the simulation results of frequency parameters and corresponding state-space model of the ETS-VIII spacecraft are presented. Some conclusions are presented in Section 6.

2. Equation Description of Rigid-flexible Coupling Satellites

Rigid-flexible coupling satellite structures can be constructed in the form of a central rigid body with \(N\) large flexible appendages such as solar panels or antennas. It is worth noting that this paper does not consider a satellite’s translational motion but is only concerned with rotational motion. Furthermore, it is assumed that the angular velocity \(\hat{\psi}\) of the satellite is very small, namely, \(\hat{\psi} \approx 0\), and the solar panels are rotating with a uniform rotational speed of \(\hat{\theta}\). When a \(3 \times 1\) dimensional control torque \(u(t)\) is applied to the satellite body, the rigid-flexible coupling equation of the satellite

\[
\eqref{eq:1}
\]
with \( N \) appendages is described by (detailed modeling procedure can be found in other studies\(^{16-19} \)):

\[
J(t) \ddot{x} + \sum_{i} F_i(t) \dot{y}_i = u(t)
\]  
(1)

\[
F_i(t) \ddot{y}_i + \ddot{y}_i + 2\xi_i \Omega_i \dot{y}_i + \Omega_i^2 y_i = 0, \quad i = \{1, 2, ..., N\}
\]  
(2)

Select a state vector \( \delta \) as:

\[
\delta = \begin{bmatrix} \psi^T \quad \eta_1^T \quad \eta_2^T \cdot \cdot \cdot \eta_N^T \end{bmatrix}^T, \quad i = \{1, 2, ..., N\}
\]  
(3)

and Eqs. (1) and (2) can be expressed as:

\[
M(t) \ddot{\delta} + E \dot{\delta} + K \delta = Lu(t)
\]  
(4)

Then, the state equation can be established as:

\[
\dot{x}(t) = A(t) x(t) + B(t) u(t)
\]  
(5)

where the \( x(t) \) is state vector, and the system matrix \( A(t) \) and input matrix \( B(t) \) are as follows:

\[
A(t) = \begin{bmatrix} 0 & I \\ -M^{-1}(t) K & -M^{-1}(t) E \end{bmatrix}
\]

\[
B(t) = \begin{bmatrix} 0 \\ M^{-1}(t) K \end{bmatrix}
\]

The satellite output signal \( y(t) \) is selected as:

\[
y(t) = \begin{bmatrix} \psi^T \\ s^T \\ \psi^T \\ s^T \end{bmatrix}^T
\]

Then, the measurement equation of the satellite can be written as:

\[
y(t) = C x(t) = \begin{bmatrix} H \\ H \end{bmatrix} \begin{bmatrix} \delta \\ \delta \end{bmatrix}
\]  
(6)

where the matrix \( H \) is

\[
H = \begin{bmatrix} I \\ \Phi_i \end{bmatrix}, \quad i = \{1, 2, ..., N\}
\]

In numerical simulations, the modal matrices are obtained by finite element modeling (FEM). In actual on-orbit identification, the system output signals are measured using the gyro on the central rigid body and the sensors on the appendages.

3. FAPI Recursive Subspace Tracking Algorithm

The discrete form of Eqs. (5) and (6) with process and measurement noises can be described as:

\[
x(k+1) = A(k)x(k) + B(k)u(k) + w_n(t)
\]  
(7)

\[
y(k) = C(k)x(k) + v_n(t)
\]  
(8)

The I-O data sequences at time step \( k \) are described by:

\[
u_M(k) = \begin{bmatrix} u^T(k) \\ u^T(k+1) \\ \cdot \cdot \cdot \\ u^T(k+M-1) \end{bmatrix}
\]  
(9)

\[
y_M(k) = \begin{bmatrix} y^T(k) \\ y^T(k+1) \\ \cdot \cdot \cdot \\ y^T(k+M-1) \end{bmatrix}
\]  
(10)

Therefore, Eqs. (7) and (8) can be written as:

\[
y_M(k) = \Gamma(k)x(k) + \Delta(k)u_M(k)
\]  
(11)

where

\[
\Gamma(k) = \begin{bmatrix} C(k) \\ C(k+1)A(k) \\ \vdots \\ C(k+M-1)A(k+M-2) \ldots A(k) \end{bmatrix}
\]

\[
\Delta(k) = \begin{bmatrix} 0 \\ C(k+1)B(k) \\ \vdots \\ C(k+M-1)A(k+M-2) \ldots B(k) \end{bmatrix}
\]

A random vector \( \xi \) is defined, where \( \xi \) denotes a sample vector of a time series in the time domain spectral analysis.According to the signal subspace projection theory, for the vector \( \xi \) and its signal subspace (observability matrix) \( \Gamma \), we solve the following optimization problem:

\[
J(\Gamma(k)) = \mathbb{E} \left[ \xi^T(k) - \Gamma(k) \Gamma^T(k) \xi(k) \right]^2
\]  
(12)

and find a matrix \( \Gamma(k) \) to let \( J(\Gamma(k)) \) be minimal. Clearly, \( J(\Gamma(k)) \) will be minimal if \( \Gamma(k) \Gamma^T(k) \) projects \( \xi(k) \) into the signal subspace. In this case, the matrix \( \Gamma(k) \) contains \( n \) dominant eigenvectors spanning the signal subspace. Assuming that the matrix \( \Gamma(k) \) is full rank, the covariance matrix \( R_{\xi \xi}(k) \) of the data vector \( \xi(k) \) is thus

\[
R_{\xi \xi}(k) = \mathbb{E} \left[ \xi(k) \xi(k)^T \right]
\]  
(13)

Furthermore, if the expectation in Eq. (12) is rewritten by the form of exponential weighting sum as:

\[
J(\Gamma(k)) = \sum_{i=1}^{k} \beta^{i-k} \left[ \xi(i) - \Gamma(k) \Gamma^T(k) \xi(i) \right]^2
\]  
(14)

Equation (14) is identical to Eq. (12) except for the exponential weighting sum instead of the expectation. In Eq. (14), \( \beta^{i-k} \) denotes the different weighting coefficient for each time instant \( i = 1, 2, \ldots, k \). The old I-O data are forgotten and the new I-O data are added by introducing the forgetting factor \( \beta \) (0 < \( \beta \) < 1). Data updating in the recursive algorithm is implemented. Therefore, Eq. (13) can also be replaced using the exponential weighting sum as:

\[
R_{\xi \xi}(k) = \sum_{i=1}^{k} \beta^{i-k} \xi(i) \xi^T(i)
\]  
(15)

In Eq. (14), the primary goal of the FAPI method is to construct the projection of \( \xi(k) \) onto the column vectors of \( \Gamma(k) \) as:

\[
h(k) = \Gamma^T(k-1) \xi(k)
\]  
(16)

To approximate \( \Gamma^T(k-1) \xi(i) \) (\( i = 1, 2, \ldots, k \)) in Eq. (14), Eq. (16) is applied by subspace projection approximation
and a modified cost function is as follows:

$$J'(\Gamma(k)) = \sum_{i=1}^{k} \beta^{i-1} \|\xi(i) - \Gamma(k)h(i)\|^2$$  \hspace{1cm} (17)$$

which is quadratic in the elements of $\Gamma(k)$. Equation (17) is minimized if:

$$\Gamma(k) = R_{\xi(k)}R_{\eta(k)}^{-1}$$ \hspace{1cm} (18)

By defining an auxiliary matrix $Z(k-1)$ and vector $v(k)$ to satisfy the following relations:

$$v(k) = Z(k-1)h(k)$$ \hspace{1cm} (19)

where the initial value $Z(0)$ of matrix $Z(k-1)$ is selected as a positive definite matrix, then the error $e(k)$ in Eq. (16) can be computed as follows:

$$e(k) = \xi(k) - \Gamma(k-1)h(k)$$ \hspace{1cm} (20)

A modified orthonormal matrix $\Theta(k)$ is defined as

$$\Theta(k) = \mathbf{I} - \tau(k)g(k)g^T(k)$$ \hspace{1cm} (21)

where

$$g(k) = v(k)(\beta + h^T(k)v(k))^{-1}$$

$$\tau(k) = \|e(k)\|^2 \left(1 + \|e(k)\|^2\|g(k)\|^2 + \sqrt{1 + \|e(k)\|^2\|g(k)\|^2}\right)$$

For the specific computation procedures about vectors $g(k)$ and $\tau(k)$, please see Badeau et al. In the FAPI algorithm, the new projection approximation and orthonormal matrix $\Theta(k)$ are applied to establish approximate relations between matrices $\Gamma(k-1)$ and $\Gamma(k)$ as:

$$\Gamma(k) = \Gamma(k-1)\Theta(k)$$ \hspace{1cm} (22)

In Badeau et al., Proposition 4.3 and the corresponding proof proved that the practical recursive form of $\Gamma(k)$ satisfies:

$$\Gamma(k) = (\Gamma(k-1) + e(k)\bar{g}^T(k))\Theta(k)$$ \hspace{1cm} (23)

Finally, substituting Eq. (21) in Eq. (23) yields:

$$\Gamma(k) = \Gamma(k-1) + e'(k)\bar{g}^T(k)$$ \hspace{1cm} (24)

where $e'(k) = (1 - \tau(k))\|g(k)\|^2 e(k) - \tau(k)\Gamma(k-1)g(k)$.

The theory description for the FAPI algorithm is introduced in Badeau et al., so we will not go into detail on this method here. Table 1 shows the procedures of the FAPI method. Its computational cost is $4mnMn + O(n^2)$ flops per iteration, whereas the complexities of the identification method commonly used, which is based on SVD, is at least $O(n^3)$. Therefore, the computational cost of FAPI is lower than the SVD method. Particularly, when the order of spacecraft system is high, the advantage of computational efficiency for the recursive method is obvious.

In the FAPI algorithm, the I-O data $u(k)$ and $y(k)$ must be applied to construct the vector $\xi(k)$ of the FAPI algorithm. For this purpose, Appendix A introduces a data-preprocessing method to construct and update vector $\xi(k)$.

### 4. Identification of Satellite Time-varying Parameters

In this section, the time-varying state-space model $[A(k), B(k), C(k)]$ will be identified using an improved FAPI algorithm by reconstructing the I-O data sequences. Once the model matrices are obtained, the LTV modal parameters can be determined. For future applications, the state-space model matrices identified can also be used to verify the ground numerical model of the satellite.

#### 4.1. Identification of time-varying modal parameters

For another set of identified model parameters $[\hat{A}(k), \hat{B}(k), \hat{C}(k)]$, Eqs. (7) and (8), without the process and measurement noises, can be described by:

$$\dot{x}(k+1) = \hat{A}(k)x(k) + \hat{B}(k)u(k)$$ \hspace{1cm} (25)

$$y(k) = \hat{C}(k)x(k)$$ \hspace{1cm} (26)

Based on the FAPI recursive algorithm, the system observability matrix $\hat{\Gamma}(k)$ identified can be determined. Then, the system matrix $\hat{A}(k)$ with $\hat{\Gamma}(k)$ identified can be constructed as:

$$\hat{A}(k) = [\hat{\Gamma}_1(k+1) \hat{\Gamma}_2(k)]$$ \hspace{1cm} (27)

where $\hat{\Gamma}_1(k+1)$ and $\hat{\Gamma}_2(k)$ are the first $m \times (M-1)$ rows of $\hat{\Gamma}(k+1)$ and the last $n \times (M-1)$ rows of $\hat{\Gamma}(k)$, respectively. After the matrix $\hat{\Gamma}(k)$ has been obtained, the first $m$ rows of $\hat{\Gamma}(k)$ can be extracted to determine the output matrix $\hat{C}(k)$ identified.

If system matrix $\hat{A}(k)$ has been derived, the identification of the time-varying modal parameters (i.e., pseudo-modal parameters) of the satellite can be implemented. The eigenvalue decomposition of system matrix $\hat{A}(k)$ at time $k$ is:

$$\hat{\Lambda}(k) = \Sigma(k)\Lambda(k)\Sigma^{-1}(k)$$ \hspace{1cm} (28)

where $\Lambda(k)$ is the diagonal eigenvalue matrix, and $\Sigma(k)$ is the corresponding time-varying matrix of eigenvectors. $\Lambda(k) = \text{diag}(\lambda_1(k), \lambda_2(k), \ldots, \lambda_n(k))$, in which $\lambda_i(k) (i = 1, 2, \ldots, n)$ contains time-varying conjugate complex eigenvalues. The $\text{th}$ pseudo-eigenvalue is $\lambda_i(k) = \exp(\Delta t)\zeta_i(k)\pm j\omega_i(k)\Delta t$, in which $-\zeta_i(k)$ and $\omega_i(k)$ are referred to as
the ith satellite pseudo-damping ratio and pseudo-damped natural frequency, respectively; \( \Delta t \) is the sampled time, and \( j = \sqrt{-1} \).

In this part, the LTV matrices \( \{ \hat{A}(k), \hat{C}(k) \} \) and modal parameters of the satellite are obtained. How to utilize this set \( \{ \hat{A}(k), \hat{C}(k) \} \) to confirm the input matrix \( \hat{B}(k) \) recursively is discussed in the subsequent sections.

4.2. An improved recursive form for identifying the input matrix \( \hat{B}(k) \)

Generally, multiple sets of experimental data and the least squares method can be applied to determine the time-varying input matrix \( \hat{B}(k) \) when using the repeated experiments identification method.\(^{13,14} \) For the recursive method, after the matrices \( \{ \hat{A}(k), \hat{C}(k) \} \) are obtained, the least squares method is used to estimate input matrix \( \hat{B}(k) \). In this part, an improved recursive form based on the FAPI method is proposed to confirm the time-varying input matrix.

In Eq. (26), the following relation at time \( k+1 \) is obviously:

\[
\hat{x}(k+1) = \hat{C}^\dagger(k+1)y(k+1)
\]

(29)

Substituting Eq. (29) into (25) yields:

\[
\hat{C}^\dagger(k+1)y(k+1) = \hat{B}(k)u(k) + \hat{A}(k)\hat{C}^\dagger(k)y(k)
\]

(30)

namely:

\[
y(k+1) = \hat{C}(k+1)\hat{B}(k)u(k) + \hat{C}(k+1)\hat{A}(k)\hat{C}^\dagger(k)y(k)
\]

(31)

As a result, the recursive form of each time instant can be written as:

\[
y(k+2) = \hat{C}(k+2)\hat{A}(k+1)+\hat{C}(k+2)\hat{B}(k+1)u(k+1) + \hat{C}(k+2)\hat{A}(k+1)\hat{C}^\dagger(k)y(k)
\]

\[\vdots\]

\[
y(k+M) = \hat{C}(k+M)\hat{A}(k+M-1)\dot{\hat{A}}(k)\hat{C}^\dagger(k) + \hat{C}(k+M)\hat{B}(k+1)u(k+1) + \cdots + \hat{C}(k+M)\hat{B}(k+M-1)\hat{A}(k+M-1)\dot{\hat{A}}(k)\hat{C}^\dagger(k)\]

(32)

Then, the generalized form of Eqs. (31) and (32) is:

\[
\hat{y}_m(k) = \hat{O}(k)u(k) + \hat{E}(k)\hat{u}_m(k)
\]

(33)

where

\[
\hat{y}_m(k) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+M) \end{bmatrix}, \quad \hat{u}_m(k) = \begin{bmatrix} u(k+1) \\ \vdots \\ u(k+M-1) \end{bmatrix}, \quad \hat{O}(k) = \begin{bmatrix} \hat{C}(k+1)\hat{B}(k) \\ \hat{C}(k+2)\hat{A}(k+1)\hat{B}(k) \\ \vdots \\ \hat{C}(k+M)\hat{A}(k+M-1)\hat{B}(k) \end{bmatrix}
\]

\[
\hat{E}(k) = \begin{bmatrix} 0 & 0 & 0 \\ \hat{C}(k+2)\hat{B}(k+1) & 0 & 0 \\ \vdots & \ddots & 0 \\ \hat{C}(k+M)\hat{B}(k+M-1) \\ \hat{C}(k+M)\hat{A}(k+M-1)\hat{B}(k+1) \end{bmatrix}
\]

(34)

Considering \( \hat{u}_m(k) \) and \( \hat{y}_m(k) \) as the new I-O data sequences and applying the FAPI method again, the new signal subspace matrix \( \hat{O}(k) \) can be obtained recursively as:

\[
\hat{O}(k) = \begin{bmatrix} \hat{C}(k+1) \\ \hat{C}(k+2)\hat{A}(k+1) \\ \vdots \\ \hat{C}(k+M)\hat{A}(k+M-1)\hat{B}(k) \end{bmatrix}
\]

(34)

\[
\hat{E}(k) = \begin{bmatrix} 0 & 0 & 0 \\ \hat{C}(k+2)\hat{B}(k+1) & 0 & 0 \\ \vdots & \ddots & 0 \\ \hat{C}(k+M)\hat{B}(k+M-1) \\ \hat{C}(k+M)\hat{A}(k+M-1)\hat{B}(k+1) \end{bmatrix}
\]

Then, the algorithm presented can be also applied here for online identification.

4.3. Summary of identification procedures for time-varying frequency and input matrix

Step 1: Construct the input vector \( \xi(k) \) based on the description in Appendix A. Then, feed the vector \( \xi(k) \) to the FAPI algorithm. The observability matrix \( \hat{F}(k) \) is determined using the FAPI algorithm.

Step 2: Estimate matrices \( \hat{A}(k) \) and \( \hat{C}(k) \) from matrix \( \hat{F}(k) \). The time-varying modal parameters of the satellite sys-
tem can be derived from the eigenvalue decomposition of matrix $\hat{A}(k)$.

**Step 3:** After $\hat{A}(k)$ and $\hat{C}(k)$ are derived from Step 2, re-write the I-O data relation of the state-space equation as Eq. (33). The new I-O sequences $\tilde{u}_m(k)$ and $\tilde{y}_m(k)$ are then derived.

**Step 4:** Repeat Step 1 to construct the new input vector of the FAPM algorithm using new I-O sequences $\tilde{u}_m(k)$ and $\tilde{y}_m(k)$. The new subspace matrix $\hat{O}(k)$ is obtained recursively, and then matrix $\hat{B}(k)$ is determined by matrices $\hat{A}(k)$, $\hat{C}(k)$ and $\hat{O}(k)$.

5. Numerical Simulations

In the following, a numerical model for the Engineering Test Satellite-VIII (ETS-VIII) is established. Identification of the time-varying frequencies and corresponding state-space model $\{\hat{A}(k), \hat{B}(k), \hat{C}(k)\}$ of the satellite are considered.

### 5.1 Parameters of the ETS-VIII model

The ETS-VIII was launched in 2006 by Japan to provide digital communications for mobile telephones and other mobile devices. It is a geosynchronous satellite at an altitude of approximately 35,786 km. The satellite has four large flexible appendages including a pair of deployable antenna reflectors and a pair of solar array panels.22) The solar panels rotate around the pitch axis at a rate of 360 deg/day so that they continually face the Sun; the solar panel rotation of the satellite can cause a maximum 25% change in the system parameter.23) Consequently, the satellite can be studied as an LTV system.

We simplify the satellite model based on the following conditions: the antenna reflectors are considered to be a plane truss structure, and the central rigid body of the satellite is considered to be a solid cuboid. The panels and reflectors are hinged to the central rigid body by a linkage. The four appendages are composed of a homogeneous material. The gravitational gradient torque is omitted here because ETS-VIII is geosynchronous. It is assumed that the entire satellite’s center of mass is the coordinate origin, and the origin of each appendage coordinate is established at the hinge joint of the central rigid body and appendage. The configuration of the simplified ETS-VIII is shown in Fig. 1, where the notations $s_1, s_2, a_1, a_2$ are used to denote the north/south solar panels and the A/B antenna reflectors, respectively.

In simulations, the size of the satellite model is shown in Fig. 2. The frequencies of panels $s_1/s_2$ and antennas $a_1/a_2$ are given by FEM in Table 2. The damping ratio is $\xi_{s1} = \xi_{s2} = \xi_{a1} = \xi_{a2} = 0.01$. The values of other parameters are given in Appendix B. In these simulations, we select the first four frequencies for each appendage. Therefore, the order of the satellite system is $n = (3 + 4 \times 4) \times 2 = 38$. The rotation speed $\dot{\theta}$ of the solar panels is given as $\dot{\theta} = 0.6$ deg/s in the numerical simulation experiments. The parameters of the recursive method are given as follows: system sampling interval $\Delta t = 0.01$ s, and the Hankel matrix parameter is $M = 20$ to make sure the rank of Hankel matrix is larger than the system order. From past experience, the forgetting factor is usually selected as $0.95 \leq \beta < 1$ in recursive subspace method, so the forgetting factor is $\beta = 0.98$ in this simulation.

### 5.2 Design of satellite input and output signals

To simulate the control torque signal produced by the satellite reaction wheel, the input signal designed for the simulations is shown in Fig. 3. In addition, the situation of sensors placement for collecting the output signal of the appendages is also considered: the out-of-plane vibration response signals of some nodes on each appendage are selected as the output signals (see Fig. 4). Consequently, output matrix $C$ of Eq. (6) is constructed by extracting the corresponding elements from the modal matrix $\Phi_i$ ($i = \{s_1, s_2, a_1, a_2\}$). The measurement noise is a stationary zero-mean Gaussian random noise, and the signal-noise-ratio (SNR) is selected as 50 dB. We only discuss system identification in the open-loop system for simplicity. The closed-
5.3. Simulation results

Applying the recursive algorithm, the frequencies of ETS-VIII can be obtained. The original frequencies for the satellite model established are shown in Fig. 5, where the first three frequencies are zero because these correspond to three rigid modes of the satellite. In addition to these three rigid frequencies, the first five vibration frequencies, from 4th to 8th, are selected.

Figure 6 compares the results of the time-invariant 4th and 6th to 7th frequencies between the original and identified values as computed by the recursive subspace method. Figure 7 shows the time-varying 5th and 8th frequency values computed by the recursive method. The results of the identified frequencies computed by the pseudo-modal method based on SVD are shown in Fig. 8.

From Figs. 6 to 8, it can be seen that the improved recursive method can adequately track the satellite modal parameters. The average relative error of each frequency is shown in Table 3. (A total of 30 experiments implemented for each algorithm.) The error results show that both the recursive method and pseudo-modal method can identify the time-varying frequencies. Although the relative error for the recursive method is higher than the SVD method, the maximum error remained below 3%.

For different SNRs, Tables 4 and 5 illustrate that the identification results of frequency using the FAPI and classic SVD-based methods, respectively. From Tables 4 and 5, we can see that when SNR $\geq 20$ in this simulation, the two methods have similar identification accuracy. However, when SNR $< 20$, a lower SNR significantly influences the results identified for the recursive algorithm. The simulation results also prove that the SVD technology has a certain degree of noise immunity. To improve the identification accuracy of the recursive algorithm at a lower SNR, some noise reduction processing for the I/O data before identification can be considered, such as the wavelet filtering technique.

The computational efficiency of these two methods is now considered. Table 6 shows the computation time of the simulations in MATLAB for different row numbers $M$ in the Hankel matrix; the column number in the Hankel matrix is set to 20 for the two methods. The simulation results also illustrate the conclusion of analysis for computation complexity in Section 3: because SVD is required in the pseudo-modal method, the computation time is found to be much larger than that of the recursive method. Particularly, when the dimensions of the Hankel matrix increase, the difference in computational time between these two methods is obvious.

Identification of the state-space model can also be implemented using the improved recursive form in Section 4. Because the different system state-space realizations satisfy the same I-O relation, to verify the model identified, the same test inputs are applied to the original model $\{A(k), B(k), C(k)\}$ and the model identified $\{\hat{A}(k), \hat{B}(k), \hat{C}(k)\}$ with the initial state conditions; the response values are then compared. However, because the initial state vector $\hat{x}(0) = T^{-1}(0)x(0)$ for the model identified is found to be different from the initial $x(0)$ of the original system before the test inputs are given to compute the responses, the initial conditions
To compute the initial value

Then, the initial state vector \( \mathbf{x}(0) \) for the original system is determined. Estimation of the initial state vector \( \hat{\mathbf{x}}(0) \) for the identified system can be implemented using Eq. (38). In addition to the proposed recursive method, the pseudo-modal method, which is based on SVD, is also applied to identify the ETS-VIII state-space model in this simulation.\(^{13,24}\) To verify the state-space model identified and used for controller design, we design a simple output feedback controller using the linear quadratic regulator (LQR) method to control the attitude angle and attitude angular velocity of the satellite. Using a known LQR controller, closed-loop responses without noise for the same test inputs are compared in Figs. 9 and 10. The angle and angular velocity responses of the state-space model estimated are shown to be generally consistent with the original LTV system.

In simulation, the SNR = 50 dB is high. However, the SNR in practice usually cannot attain 50 dB. Therefore, to verify the identification ability of the proposed method at a lower SNR, when the SNR is selected as 40, 30 and 20 dB, the response results of attitude angle \( \psi \) for the approach proposed and SVD-based method are shown in Figs. 11 and 12, respectively. The results in Fig. 11 illustrate that the method proposed can obtain the state-space model parameters when the SNR is lower in practical implementation.

Finally, if the rotation speed \( \dot{\theta} \) of the solar panels is increased to \( \dot{\theta} = 6 \text{ deg/s} \), and other simulation conditions are the same as mentioned above, then the results of attitude angle \( \psi \) are shown in Fig. 13(a). In another simulated example, the comparison results of satellite attitude angles \( \dot{\psi} \) are shown in Fig. 13(b) when the sampling interval \( \Delta t \) is se-

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**Table 4.** Error comparison of frequencies of the original and identified models for different SNRs using the FAPI method.

| Frequency | SNR = 40 (%) | SNR = 30 (%) | SNR = 20 (%) | SNR = 10 (%) |
|-----------|-------------|-------------|-------------|-------------|
| \( \omega_1 \) | 2.3356      | 2.8754      | 7.5679      | 10.7804     |
| \( \omega_2 \) | 2.0860      | 2.7774      | 6.5678      | 7.9703      |
| \( \omega_3 \) | 1.8567      | 2.4956      | 4.4567      | 7.5687      |
| \( \omega_4 \) | 2.1426      | 2.6574      | 5.8467      | 8.5674      |
| \( \omega_5 \) | 2.5689      | 3.1854      | 8.2580      | 11.5686     |

**Table 5.** Error comparison of frequencies of the original and identified models for different SNRs using the SVD method.

| Frequency | SNR = 40 (%) | SNR = 30 (%) | SNR = 20 (%) | SNR = 10 (%) |
|-----------|-------------|-------------|-------------|-------------|
| \( \omega_1 \) | 2.4467      | 2.7802      | 6.8606      | 8.8576      |
| \( \omega_2 \) | 2.2457      | 2.8043      | 6.4367      | 7.2346      |
| \( \omega_3 \) | 1.8471      | 2.2694      | 4.4258      | 6.3487      |
| \( \omega_4 \) | 2.0643      | 2.7825      | 5.9854      | 7.1889      |
| \( \omega_5 \) | 2.6940      | 2.9063      | 7.9856      | 9.1571      |

**Table 6.** Comparison of the average computation time with \( M \) block rows of the Hankel matrix in MATLAB (unit: s).

| Identification method | \( M = 10 \) | \( M = 15 \) | \( M = 20 \) |
|----------------------|-------------|-------------|-------------|
| Recursive algorithm  | 16.67       | 27.45       | 44.63       |
| SVD method           | 34.56       | 48.53       | 76.51       |

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lected as $\Delta t = 0.1$ s. From Figs. 13(a) and (b), it is seen that, when the panels rotate faster or the sampling interval increases, the state-space model parameter can still be determined accurately.

6. Conclusions

In this paper, the authors attempt to provide an alternative method that differs from commonly used approaches that employ SVD to identify the time-varying modal parameters and corresponding state-space model of large rigid-flexible coupling satellites. Consequently, the time-varying frequencies of a satellite are obtained using the FAPI recursive method. Moreover, an improved recursive form based on the FAPI algorithm for identifying the input matrix is developed by rewriting the relation of system I-O data sequences.

The results show that the recursive algorithm can identify the time-varying frequencies of a satellite (Figs. 6–8) using the I-O data designed (Figs. 3 and 4). The results of Tables 4 and 5 illustrate that when the SNR $> 20$ dB, the recursive algorithm is capable of identifying the time-varying frequencies. However, when the SNR $= 10$ dB in simulation, because SVD is avoided in the FAPI method, noises significantly influence the results. To improve the accuracy of this method at a lower SNR, some noise reduction processing of the I/O data before identification, such as a wavelet filtering technique, can be considered.

The comparison results of computation time in Table 6 illustrate that the FAPI method has better computational efficiency than SVD-based methods. In addition, the results of test responses show that the improved recursive form can identify the input matrices in Figs. 9–13 under different SNRs and simulation conditions.
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Appendix

A. I-O data pre-processing to construct the vector $\xi$ for the FAPI algorithm

To establish input vector $\xi$ in the FAPI algorithm, there are several methods. Here, we introduce a way to construct vector $\xi$ using URV matrix decomposition. The detailed procedures of this method are described in Tasker et al. (25)

Firstly, the Hankel matrix of input $u(k)$ is written as:

$$ U(k) = \begin{bmatrix} u(1) & u(2) & \cdots & u(k) \\ u(2) & u(3) & \cdots & u(k+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(M) & u(M+1) & \cdots & u(k+M-1) \end{bmatrix} $$

(Hankel matrix $Y(k)$ for output $y(k)$ is similarly formulated. Parameter $M$ is selected as the proper number to ensure that the rank of the Hankel matrix is larger than system order $n$.

The recursive update for $U(k)$ can be expressed as:

$$ (U(k)U^T(k))^{-1} \equiv (U(k-1)U^T(k-1) + u_M(k)u_M^T(k))^{-1} = (U(k-1)U^T(k-1))^{-1} - \frac{y(k)u_M^T(k)(U(k-1)U^T(k-1))^{-1}}{1 + \alpha(k)} $$

(A.2)

where parameters $y(k)$ and $\alpha(k)$ are defined as:

$$ y(k) = (U(k-1)U^T(k-1))^{-1}u_M(k) $$

$$ \alpha(k) = u_M^T(k)y(k) $$

and the matrix that is normal to $U(k)$ is also defined as:

$$ U^\perp(k) = I - U^T(k)[U(k)U^T(k)]^{-1}U(k) $$

(A.3)

Then:
\[ \mathbf{U}^{T}(k)\mathbf{U}(k)\mathbf{U}^{T}(k)^{-1} = \mathbf{I} - \mathbf{U}^{\perp}(k) \]

\[
= \begin{bmatrix}
I - \mathbf{U}^{\perp}(k - 1) & 0 \\
0 & 0
\end{bmatrix}
- \begin{bmatrix}
\mathbf{U}^{T}(k)\mathbf{y}(k)\mathbf{y}^{T}(k)\mathbf{U}(k - 1) \\
\mathbf{y}^{T}(k)\mathbf{U}(k - 1)
\end{bmatrix} \frac{1 + \alpha(k)}{1 + \alpha(k)^2}
\]

leading to:

\[
\mathbf{U}^{\perp}(k) = \begin{bmatrix}
\mathbf{U}^{\perp}(k - 1) & 0 \\
0 & 0
\end{bmatrix}
- \frac{1}{\sqrt{1 + \alpha(k)^2}} \begin{bmatrix}
\mathbf{U}^{T}(k - 1)\mathbf{y}(k) \\
\mathbf{y}^{T}(k)\mathbf{U}(k - 1)
\end{bmatrix} - 1 \left( \frac{1}{\sqrt{1 + \alpha(k)^2}} \right)
\]

where the notation \( \lceil \cdot \rceil \) denotes double floor. Therefore:

\[
\mathbf{Y}(k)\mathbf{U}^{\perp}(k) = \mathbf{Y}(k - 1) - \frac{1}{\sqrt{1 + \alpha(k)^2}} \begin{bmatrix}
\mathbf{U}^{T}(k - 1)\mathbf{y}(k) \\
\mathbf{y}^{T}(k)\mathbf{U}(k - 1)
\end{bmatrix} - 1 \left( \frac{1}{\sqrt{1 + \alpha(k)^2}} \right)
\]

\[
\mathbf{w}^{T}(k) = \mathbf{y}^{T}(k)\mathbf{U}^{T}(k - 1) = \mathbf{u}^{T}_{M}(k)(\mathbf{U}(k - 1)\mathbf{U}^{T}(k - 1))^{-1} \mathbf{U}^{T}(k - 1) / \sqrt{1 + \alpha(k)^2}
\]

where \( \mathbf{z}(k) \) is what we need for the FAPI algorithm as vector \( \mathbf{\xi}(k) \) using the I-O data, and the updating of \( \mathbf{U}(k)\mathbf{U}^{T}(k) \) and \( \mathbf{Y}(k)\mathbf{U}^{T}(k) \) in Eqs. (A.7) and (A.8) can be expressed as follows:

\[
(U(k)U^{T}(k))^{-1} = (U(k - 1)U^{T}(k - 1))^{-1} - \frac{y(k)y^{T}(k)}{\sqrt{1 + \alpha(k)}}
\]

\[
Y(k)U^{T}(k) = Y(k - 1)U^{T}(k - 1) - z(k)y^{T}(k)
\]

B. Some simulation parameters for the ETS-VIII spacecraft

The derivation of \( \mathbf{J}(t) \) is described by:

\[
\mathbf{J}(t) = \mathbf{J}_{r} - m_{s}\mathbf{\tilde{r}}_{r} + \sum_{i} T_{i}^{T}(t)\mathbf{T}_{i}(t) - \sum_{i} m_{s}\mathbf{\tilde{r}}_{i}
\]

\[
i = \{s_{1}, s_{2}, a_{1}, a_{2}\}
\]

and some other simulation parameters are shown as follows:

\[ m_{s} = 2400 \text{ kg}, \quad m_{s_{1}} = m_{s_{2}} = 115.2 \text{ kg}, \]

\[ m_{a_{1}} = m_{a_{2}} = 165 \text{ kg} \]

\[
\mathbf{J}_{s_{1}} = \begin{bmatrix}
3611.7 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{J}_{s_{2}} = \begin{bmatrix}
0 & 18.2 & 0
\end{bmatrix}
\]

\[
\mathbf{J}_{a_{1}} = \begin{bmatrix}
1683.4 & 0 & 853.8
\end{bmatrix}
\]

\[
\mathbf{J}_{a_{2}} = \begin{bmatrix}
0 & 3226.8 & 0
\end{bmatrix}
\]

\[
\mathbf{J} = \begin{bmatrix}
8858.5 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{J}_{r} = \begin{bmatrix}
0 & 7762.5 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}} = \begin{bmatrix}
0 & -0.4827 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{s_{1}} = \begin{bmatrix}
0.4827 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{a_{1}} = \begin{bmatrix}
0 & -0.4827 & -5
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{s_{2}} = \begin{bmatrix}
0.4827 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{a_{2}} = \begin{bmatrix}
0 & -0.4827 & 5
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{s_{1}} = \begin{bmatrix}
0.4827 & 0 & -1.175
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{a_{1}} = \begin{bmatrix}
0 & 1.175 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{s_{2}} = \begin{bmatrix}
0 & 1.175 & 0
\end{bmatrix}
\]

\[
\mathbf{\tilde{r}}_{a_{2}} = \begin{bmatrix}
0 & -1.175 & 0
\end{bmatrix}
\]

\[
\mathbf{T}_{s_{1}} = \begin{bmatrix}
-\cos(\dot{\theta}) & 0 & \sin(\dot{\theta})
0 & -1 & 0
\sin(\dot{\theta}) & 0 & \cos(\dot{\theta})
\end{bmatrix}
\]

\[
\mathbf{T}_{s_{2}} = \begin{bmatrix}
\cos(\dot{\theta}) & 0 & -\sin(\dot{\theta})
0 & 1 & 0
\sin(\dot{\theta}) & 0 & \cos(\dot{\theta})
\end{bmatrix}
\]

\[
\mathbf{T}_{a_{1}} = \mathbf{T}_{a_{2}} = \begin{bmatrix}
1 & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{bmatrix}
\]

\[ y. Miyazaki\]

Associate Editor