ON ASYMPTOTIC BEHAVIOUR OF DIRICHLET INVERSE

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Abstract. Let \( f(n) \) be an arithmetic function with \( f(1) \neq 0 \) and let \( f^{-1}(n) \) be its reciprocal with respect to the Dirichlet convolution. We study the asymptotic behaviour of \( |f^{-1}(n)| \) with regard to the asymptotic behaviour of \( |f(n)| \) assuming that the latter one grows or decays with at most polynomial or exponential speed. As a by-product, we obtain simple but constructive upper bounds for the number of ordered factorizations of \( n \) into \( k \) factors.

1. Introduction

Let \( f : \mathbb{N} \to \mathbb{R} \) be an arithmetic function. The set of those \( f(n) \) with \( f(1) \neq 0 \) endowed with the Dirichlet convolution defined as

\[
(f * g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d), \quad n \in \mathbb{N},
\]

forms an abelian group. The identity element \( \varepsilon(n) \) is given by \( \varepsilon(1) = 1, \varepsilon(n) = 0 \) for all \( n \geq 2 \), and we denote by \( f^{-1}(n) \) the corresponding inversion of \( f(n) \), i.e.,

\[
(f * f^{-1})(n) = (f^{-1} * f)(n) = \varepsilon(n), \quad n \in \mathbb{N}. \tag{1.1}
\]

We call \( f^{-1}(n) \) the Dirichlet inverse of \( f(n) \) and note that \( f^{-1}(n) \) can be determined recursively via (1.1) as

\[
f^{-1}(1) = \frac{1}{f(1)} \quad \text{and} \quad f^{-1}(n) = -\frac{1}{f(1)} \sum_{d|n, d<n} f\left(\frac{n}{d}\right) f^{-1}(d), \quad n \geq 2. \tag{1.2}
\]

Alternatively, \( f^{-1}(n) \) can be found in the following nonrecurrent way:

\[
f^{-1}(n) = \sum_{k=1}^{\Omega(n)} \frac{(-1)^k}{f(1)^{k+1}} \sum_{d_1 \cdots d_k = n, d_1, \cdots, d_k \geq 2} f(d_1) \cdots f(d_k), \quad n \geq 2, \tag{1.3}
\]

where \( \Omega(n) \) is the number of prime factors of \( n \) counted with multiplicities. The formula (1.3) can be obtained from [6, Theorem 2.2] using the evident identity

\[
(af)^{-1}(n) = \frac{1}{a} f^{-1}(n), \quad n \geq 1, \quad a \in \mathbb{R} \setminus \{0\}. \tag{1.4}
\]

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As a consequence of (1.4), we will always claim that \( f(1) = 1 \), unless otherwise stated. In particular, as was noticed by Hille [9], taking \( f(n) = -1 \) for all \( n \geq 2 \), one gets \( f^{-1}(n) = H(n) \), where

\[
H(n) = \sum_{k=1}^{\Omega(n)} H_k(n) = \sum_{k=1}^{\Omega(n)} \sum_{d_1 \cdots d_k = n \atop d_1, \ldots, d_k \geq 2} 1, \quad n \geq 2,
\]

is the number of ordered factorizations of \( n \) and \( H_k(n) \) is the number of ordered factorizations of \( n \) into \( k \) factors where each factor is greater than or equal to 2.

In the analysis of various problems there appears a necessity to control the growth or decay rate of both \( f(n) \) and \( f^{-1}(n) \), simultaneously. For instance, Segal showed in [19] that if \( f(n) = O(1) \) and \( f^{-1}(n) = O(1) \) as \( n \to \infty \), along with other assumptions, then \( \sum_{n \leq x} l(n) \sim x \), where \( l(n) \) represents the coefficients of the Dirichlet series

\[
-\frac{D'(s)}{D(s)} = \sum_{n=1}^{\infty} \frac{l(n)}{n^s} \quad \text{with} \quad D(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.
\]

This result can be seen as an analogue of the prime number theorem. On the other hand, Segal proposed in [18] the following generalization of Ingham’s summation method [11, 22]: a series \( \sum_{n=1}^{\infty} a_n \) is said to be \((D, f(n))-\text{summable} to \( A \in \mathbb{R} \) whenever

\[
\lim_{x \to \infty} D(x) = \lim_{x \to \infty} x \sum_{n \leq x} a_n f\left(\frac{n}{x}\right) = A.
\]

Properties of the \((D, f(n))-\text{summation method} crucially depend on the summability of \( f(n) \) and \( f^{-1}(n) \), and, as a consequence, on their asymptotic behaviour, see, e.g., [13, 18].

Assume now that \( f(n) \) is given by the Fourier coefficients of a function \( F \in L^2(0, 1) \) which is extended to the whole \( \mathbb{R} \) antiperiodically with period 1. In analogy with properties of the standard trigonometric system \( \{\sin(n\pi x)\} \) it is natural to ask which assumptions one should impose on \( F \) in order to guarantee that the system

\[
F(x), \ F(2x), \ F(3x), \ldots,
\]

forms a basis in \( L^2(0, 1) \) or at least is complete in the same space. Here, under completeness of (1.6) we mean that any function from \( L^2(0, 1) \) can be approximated in the \( L^2 \)-norm with an arbitrary precision by finite linear combinations of functions (1.6). This problem has been intensively studied, see, e.g., historical remarks in [8, 23]. In particular, the following result was obtained by Hedenmalm et al. in [7].

**Theorem 1.1** ([7, Theorem 5.7 (i)]). Let \( F \in L^2(0, 1) \) be such that

\[
\sum_{n=1}^{\infty} |f(n)|^2 \tau(n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |f^{-1}(n)|^2 \tau(n) < \infty,
\]

where \( \tau(n) \) is the number of divisors of \( n \). Then the system \( \{F(nx)\} \) is complete in \( L^2(0, 1) \).

It is well-known that \( \tau(n) = o(n^\delta) \) for any \( \delta > 0 \). Thus, the assumptions (1.7) can be easily verified provided

\[
|f(n)| \leq C_1 n^{-\frac{1}{2} - \varepsilon} \quad \text{and} \quad |f^{-1}(n)| \leq C_2 n^{-\frac{1}{2} - \eta}, \quad n \geq 1,
\]
for some \( C_1, C_2, \varepsilon, \eta > 0 \). See also [23, pp. 764-765] and [20] for similar assumptions guaranteeing that the system (1.6) forms a basis in \( L^2(0,1) \). We recognize that the asymptotics of \( f(n) \) and \( f^{-1}(n) \) can be used in the study of such kind of problems, as well.

Although the asymptotic behaviour of \( f(n) \) can be considered as given or relatively easy to obtain, the asymptotic behaviour of \( f^{-1}(n) \) is, in general, a hard issue and it can be drastically different from those of \( f(n) \). As an example, assume that \( f(2) = -1 \) and \( f(n) = 0 \) for all \( n \geq 3 \). Then we easily see from (1.3) that \( f^{-1}(2^k) = 1 \) for all \( k \geq 1 \), and \( f^{-1}(n) = 0 \) for any \( n \) with a prime factor different from 2. That is, \( |f^{-1}(n)| \) does not have to converge to 0 as \( n \to \infty \) even if \( |f(n)| \) decays arbitrarily fast. Clearly, this is due to the definition of \( f^{-1}(n) \) from which we see that the asymptotic behaviour of \( f^{-1}(n) \) depends on values of \( f(n) \) for all \( n \) rather than only for sufficiently large \( n \).

Nevertheless, under additional requirements, the asymptotic behaviour of \( f^{-1}(n) \) can be explicitly controlled by or compared with those of \( f(n) \). Perhaps, the simplest case of this type occurs if \( f(n) \) is assumed to be totally (completely) multiplicative, i.e.,

\[
f(m)f(n) = f(mn) \quad \text{for all } m, n \in \mathbb{N}.
\]

Then the Dirichlet inverse \( f^{-1}(n) \) has the following explicit form:

\[
f^{-1}(n) = \mu(n)f(n), \quad n \geq 1,
\]

where \( \mu(n) \) is the Möbius function defined by \( \mu(1) = 1 \) and

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ has a squared prime factor}, \\
(-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes}, 
\end{cases} \quad n \geq 2,
\]

see, e.g., [1, Theorem 2.17]. That is, \( |\mu(n)| \leq 1 \) and hence

\[
|f^{-1}(n)| \leq |f(n)| \quad \text{for all } n \geq 1.
\]

In other words, \( |f^{-1}(n)| \) cannot grow faster or decay slower than \( |f(n)| \).

However, in general position, \( f(n) \) is not totally multiplicative, and to the best of our knowledge there are not many results connecting the asymptotic behaviour of \( f^{-1}(n) \) with those of \( f(n) \) without assuming (1.8), see, e.g., [5, 14, 20] for some particular classes of \( f(n) \). The aim of the present article is thus to investigate the assumptions on \( f(n) \) under which the explicit control of the behaviour of \( f^{-1}(n) \) is possible. We will concentrate on the cases where \( f(n) \) has at most polynomial or exponential speed as \( n \to \infty \), that is,

\[
either |f(n)| \leq Cn^\gamma \quad \text{or} \quad |f(n)| \leq Ae^n, \quad n \geq 2,
\]

for some \( C > 0, \gamma \in \mathbb{R}, \) and \( A, e > 0 \).

This article is organized as follows. In Section 2, we obtain some estimates on \( H(n), H_k(n), \) and their generalizations which will be used in the sequel but also have an independent interest. In Section 3, we present and prove our main results concerning the asymptotic behaviour of \( f^{-1}(n) \). We consider the case of multiplicative functions \( f(n) \) in Section 3.1, the general case is studied in Section 3.2, and we conclude the article with some miscellaneous cases in Section 3.3.
2. Number of ordered factorizations

In this section, we give some upper bounds on the functions $H(n)$ and $H_k(n)$ defined by (1.5) and on their generalizations. Let $\mathcal{P}$ be a subset of $\mathbb{N}_2 = \{2, 3, \ldots \} \subset \mathbb{N}$. Denote by $H(n, \mathcal{P})$ the number of ordered factorizations of $n$ where each factor belongs to $\mathcal{P}$, and by $H_k(n, \mathcal{P})$ the corresponding number of ordered factorizations of $n$ into $k$ factors, that is,

$$H(n, \mathcal{P}) = \sum_{k=1}^{\Omega(n)} H_k(n, \mathcal{P}) = \sum_{k=1}^{\Omega(n)} \sum_{d_1 \cdot \cdots \cdot d_k = n} 1, \quad n \geq 2.$$ 

In particular, if $\mathcal{P} = \mathbb{N}_2$, then $H(n, \mathcal{P}) = H(n)$ and $H_k(n, \mathcal{P}) = H_k(n)$. The functions $H(n, \mathcal{P})$ and $H_k(n, \mathcal{P})$ have been intensively studied starting from the work of Kalmár [15], see, e.g., [3, 4, 9, 10] and overviews [16, 21].

We start with several standard observations. Notice that $H(n, \mathcal{P})$ is given by (2.1). We see that $H(n, \mathcal{P}) = H(n)$ decreases, $H(n, \mathcal{P}) \to 0$ as $s \to \infty$, and there exists $\sigma_0 \geq \max\{\sigma_\mathcal{P}, 0\}$ such that $\zeta_\mathcal{P}(s_0) \geq 1$. Hereinafter, we will denote by $\rho(\mathcal{P})$ the unique real root of $\zeta_\mathcal{P}(s) = 1$, $s \geq s_0$.

The functions $H(n, \mathcal{P})$ and $H_k(n, \mathcal{P})$ can be determined recursively by

$$H(1, \mathcal{P}) = 1 \quad \text{and} \quad H(n, \mathcal{P}) = \sum_{d|n} H\left(\frac{n}{d}, \mathcal{P}\right), \quad n \geq 2, \quad \text{(2.2)}$$

and

$$H_k(n, \mathcal{P}) = \sum_{d|n} H_{k-1}\left(\frac{n}{d}, \mathcal{P}\right), \quad k \geq 2, \quad \text{(2.3)}$$

where $H_1(n, \mathcal{P})$ is given by (2.1). We see that

$$H_k(n, \mathcal{P}) = H_{k-1}(n, \mathcal{P}) \ast H_1(n, \mathcal{P}) = H_{k-2}(n, \mathcal{P}) \ast H_1(n, \mathcal{P}) \ast H_1(n, \mathcal{P}) = \ldots$$

Thus, considering the Dirichlet series associated with $H_k(n, \mathcal{P})$, we obtain

$$\sum_{m=1}^{\infty} \frac{H_k(m, \mathcal{P})}{m^s} = \zeta_\mathcal{P}(s)^k, \quad k \geq 1. \quad \text{(2.4)}$$

Lemma 2.1. Let $n \geq 1$. Then

$$H(n, \mathcal{P}) \leq n^{\rho(\mathcal{P})}. \quad \text{(2.5)}$$
Proof. We will argue in much the same way as in [4, Section 2] and prove the result by induction. The base of induction is trivial. Take some \(n \geq 2\) and suppose that
\[
H(m, \mathcal{P}) \leq m^{\rho(\mathcal{P})}
\]
for all \(m < n\).

Let us show that the inequality remains valid for \(m = n\). Using (2.2), we deduce that
\[
H(n, \mathcal{P}) = \sum_{d|n} H\left(\frac{n}{d}, \mathcal{P}\right) \leq \sum_{d|n} \left(\frac{n}{d}\right)^{\rho(\mathcal{P})} \leq n^{\rho(\mathcal{P})} \sum_{d \in \mathcal{P}} \frac{1}{d^{\rho(\mathcal{P})}} = n^{\rho(\mathcal{P})} \zeta_P(\rho(\mathcal{P})) = n^{\rho(\mathcal{P})},
\]
which completes the proof. \(\square\)

Now we obtain an upper bound for \(H_k(n, \mathcal{P})\).

**Lemma 2.2.** Let \(s > \sigma_\mathcal{P}, n \geq 2\), and \(k \geq 1\). Then
\[
H_k(n, \mathcal{P}) \leq \frac{\zeta_{\mathcal{P}}(s)^{k-1}n^s}{\varrho^s},
\]
where \(\varrho\) is the minimal element of \(\mathcal{P}\).

**Proof.** Inequality (2.5) is trivial for \(k = 1\), see (2.1). Assuming \(k \geq 2\), we use (2.3) and (2.4) to deduce that
\[
\frac{H_k(n, \mathcal{P})}{n^s} = \sum_{d|n} \left(\frac{d}{n}\right)^s H_k(n, \mathcal{P}) \leq \frac{1}{\varrho^s} \sum_{m=1}^\infty H_k(m, \mathcal{P}) = \frac{\zeta_{\mathcal{P}}(s)^{k-1}}{\varrho^s}, \quad n \geq 2,
\]
where we applied the inequality \(d \geq \varrho\) for \(d \in \mathcal{P}\). \(\square\)

Let us provide two corollaries of Lemmas 2.1 and 2.2 where we treat the cases \(\mathcal{P} = \mathbb{N}_2\) and \(\mathcal{P} = \mathbb{N}_{3,\text{odd}} = \{3, 5, 7, \ldots\}\) which is the set of all odd natural numbers except 1. The latter case occurs naturally in the basisness and completeness problems, see Section 3.3.2 for further discussion. One can show (cf. [3, Proposition 1]) that
\[
\zeta_{\mathbb{N}_{3,\text{odd}}}(s) = \sum_{m \geq 3 \text{ odd}} \frac{1}{m^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} - 1 = \left(1 - \frac{1}{2^s}\right) \zeta(s) - 1,
\]
and find that \(\rho(\mathbb{N}_{3,\text{odd}}) = 1.37779\ldots\)

**Corollary 2.3.** Let \(n \geq 2\). Then \(H(n) \leq n^\rho\) for all \(n \geq 2\), where \(\rho = \rho(\mathbb{N}_2) = 1.72865\ldots\) If, in addition, \(n\) is odd, then \(H(n) \leq n^\eta\), where \(\eta = \rho(\mathbb{N}_{3,\text{odd}}) = 1.37779\ldots\)

**Remark 2.4.** In fact, the inequalities in Corollary 2.3 are strict, see [3, Theorem 5]. Furthermore, the growth rate \(n^\rho\) is optimal. For any \(\varepsilon > 0\) there exist infinitely many \(n\) such that \(H(n) > n^{\rho-\varepsilon}\), see [9] and also [4, Section 3] for an explicit construction.

**Corollary 2.5.** Let \(s > 1\), \(n \geq 2\), and \(k \geq 1\). Then
\[
H_k(n) \leq \frac{(\zeta(s) - 1)^{k-1}n^s}{2^s}.
\]
Moreover, if \(n\) is odd, then
\[
H_k(n) \leq \frac{(1 - \frac{1}{2^s}) \zeta(s) - 1)^{k-1}n^s}{3^s}.
\]
In this section, we state and prove our main results concerning the behaviour of $f^{-1}(n)$.

### 3. Asymptotics of $f^{-1}(n)$

#### 3.1 Multiplicative $f(n)$

Let us consider the following relaxation of the total multiplicativity assumption (1.8). We say that $f(n)$ is multiplicative if

$$f(m)f(n) = f(mn) \quad \text{for all coprime } m, n \in \mathbb{N}. \quad (3.1)$$

Note that the Dirichlet inverse $f^{-1}(n)$ of a multiplicative $f(n)$ is also multiplicative (see, e.g., [1, Theorem 2.16]). Hereinafter, we write arbitrary $n \in \mathbb{N}$ as its prime decomposition

$$n = p_1^{e_1} \cdots p_{\omega(n)}^{e_{\omega(n)}},$$

where $\omega(n)$ stands for the number of distinct prime factors of $n$. For such $n$, we get

$$f^{-1}(n) = f^{-1}\left(\prod_{j=1}^{\omega(n)} p_j^{e_j}\right) = \prod_{j=1}^{\omega(n)} f^{-1}(p_j^{e_j}). \quad (3.2)$$

In its turn, the Dirichlet inverse for prime powers can be found recursively as follows (see (1.2)):

$$f^{-1}(p) = -f(p) \quad \text{and} \quad f^{-1}(p^k) = -\sum_{m=0}^{k-1} f(p^{k-m})f^{-1}(p^m), \quad k \geq 2. \quad (3.3)$$

Let us also refer the reader to [6, (2.5)-(2.7)] for nonrecurrent expressions of $f^{-1}(p^k)$. Notice that all interim results for powers of primes that we present in this section are also valid for non-multiplicative $f(n)$.

We divide our results into two subsections according to polynomial and exponential behaviour of $f(n)$.

##### 3.1.1 Polynomial behaviour

Along this subsection, we will assume that $|f(n)| \leq Cn^\gamma$ for some $C > 0$, $\gamma \in \mathbb{R}$, and all $n \geq 2$. We start from the following general result.

**Proposition 3.1.** Let $f(n)$ be multiplicative. Assume that there exist $C > 0$ and $\gamma \in \mathbb{R}$ such that $|f(n)| \leq Cn^\gamma$ for all $n \geq 2$. Then

$$|f^{-1}(n)| \leq \left(\frac{C}{C+1}\right)^{\omega(n)}(C+1)^{\Omega(n)}n^\gamma, \quad n \geq 2.$$

**Proof.** Assume first the case $n = p^k$ for prime $p \geq 2$ and natural $k \geq 1$. Let us argue by induction with respect to $k$. To show the base of induction, we recall that $f^{-1}(p) = -f(p)$, which implies $|f^{-1}(p)| = |f(p)| \leq Cp^\gamma$, see (3.3). As the hypothesis of induction, we assume that $|f^{-1}(p^m)| \leq C(C+1)^{m-1}p^{m\gamma}$ for all $m \leq k-1$. We perform the inductive step using (3.3):

$$|f^{-1}(p^k)| \leq |f(p^k)| + \sum_{m=1}^{k-1} |f(p^{k-m})||f^{-1}(p^m)|$$

$$\leq Cp^{k\gamma} + C^2p^{k\gamma}\sum_{m=0}^{k-2} (C+1)^m = C(C+1)^{k-1}p^{k\gamma}.$$
Hence, we have $|f^{-1}(p^k)| \leq C(C + 1)^{k-1}p^{k\gamma}$ for all powers of primes. Therefore, since $f^{-1}(n)$ is multiplicative, we derive from (3.2) that

$$|f^{-1}(n)| = \prod_{j=1}^{\omega(n)} |f^{-1}(p_j^{e_j})| \leq \prod_{j=1}^{\omega(n)} C(C + 1)^{e_j-1}p_j^{e_j\gamma} = \left(\frac{C}{C + 1}\right)^{\omega(n)} (C + 1)^{\Omega(n)} n^{\gamma}$$

for arbitrary natural $n \geq 2$.

Since $\Omega(n)$ possesses the upper bound $\frac{\ln n}{\ln 2}$, we can simplify the statement of Proposition 3.1. The resulting corollary shows that the asymptotic behavior of the Dirichlet inverse $f^{-1}(n)$ depends directly on the value of the constant $C > 0$.

**Corollary 3.2.** Let $f(n)$ be multiplicative. Assume that there exist $C > 0$ and $\gamma \in \mathbb{R}$ such that $|f(n)| \leq Cn^{\gamma}$ for all $n \geq 2$. Then $|f^{-1}(n)| \leq n^{\gamma + \frac{\ln(1+C)}{\ln 2}}$ for all $n \geq 2$.

**Remark 3.3.** The upper bound obtained by Proposition 3.1 is optimal. Indeed, fix any $\gamma \in \mathbb{R}$ and $C > 0$, and define $f(n)$ by $f(2^k) = -C2^{k\gamma}$, $k \geq 1$, and $f(n) = 0$ if $n > 2$ is not a power of 2. We see that such $f(n)$ is multiplicative. Using (3.3), we obtain by induction that

$$f^{-1}(2^k) = C \sum_{m=0}^{k-1} 2^{(k-m)\gamma} f^{-1}(p^m) = C2^{k\gamma} + C \sum_{m=1}^{k-1} (C + 1)^{m-1} 2^{k\gamma} = C(C + 1)^{k-1} 2^{k\gamma}$$

for all $k \geq 1$, and $f^{-1}(n) = 0$ for all other natural numbers $n > 2$, which means the claimed optimality.

By imposing additional assumptions on $f(n)$, one can improve the upper bound in Proposition 3.1. For instance, assume that $f(n)$ is multiplicative and supported on the square-free numbers, i.e., $f(p^k) = 0$ for all primes $p \geq 2$ and naturals $k \geq 2$. We obtain recursively from (3.3) that $f^{-1}(p^k) = (-1)^k f(p)^k$ for $k \geq 1$. Therefore, for arbitrary natural $n \geq 2$, (3.2) implies

$$f^{-1}(n) = \prod_{j=1}^{\omega(n)} f^{-1}(p_j^{e_j}) = (-1)^{\Omega(n)} \prod_{j=1}^{\omega(n)} f(p_j)^{e_j},$$

which shows that $f^{-1}(n)$ is totally multiplicative, cf. [1, Exercise 26, p. 49]. Let us remark that this is a complementation to the totally multiplicative case discussed in Section 1. As a consequence of (3.4), we get the following result.

**Proposition 3.4.** Let $f(n)$ be multiplicative and $f(p^k) = 0$ for all prime powers $p^k$ with $k \geq 2$. Assume that there exist $C > 0$ and $\gamma \in \mathbb{R}$ such that $|f(n)| \leq Cn^{\gamma}$ for all $n \geq 2$. Then

$$|f^{-1}(n)| \leq C^{\Omega(n)} n^{\gamma}, \quad n \geq 2.$$ 

In particular, if $C \geq 1$, then

$$|f^{-1}(n)| \leq n^{\gamma + \frac{\ln C}{\ln 2}}, \quad n \geq 2.$$
3.1.2 Exponential behaviour. Along this subsection, we will assume that $|f(n)| \leq A c^n$ for some $A, c > 0$ and all $n \geq 2$. Let us denote by $\mathcal{P}(m)$ the set of all partitions of $m \in \mathbb{N}$ and assume, without loss of generality, that each entry of $\mathcal{P}(m)$ is arranged in the decreasing order, e.g.,

$$\mathcal{P}(5) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\}.$$ 

Using the recursive formula (3.3), one can derive the following upper bound for the Dirichlet inverse for prime powers:

$$|f^{-1}(p^k)| \leq \sum_{l, \{l_1, l_2, \ldots, l_m\} \in \mathcal{P}(k)} \left(\frac{l}{l_1, l_2, \ldots, l_m}\right) A^l \prod_{j=1}^l c^{\phi_j},$$

where $l, l_1, l_2, \ldots, l_m \geq 1$ are such that $l_1 + \cdots + l_m = l$ and

$$\phi_1 = \cdots = \phi_l > \phi_{l+1} = \cdots = \phi_{l_1} + l_2 > \cdots > \phi_{l_1 + \cdots + l_m - 1} = \cdots = \phi_{l_1 + \cdots + l_m} = \phi_l,$$

and $(l_1, l_2, \ldots, l_m)$ is the multinomial coefficient. Let us remark that the inequality (3.5) turns to equality for any function $f(n)$ satisfying $f(p^k) = -A c^k$ for prime powers $p^k$, $k \geq 1$. That is, (3.5) gives the optimal upper bound for the Dirichlet inverse for prime powers.

If we assume now that $f(n)$ is multiplicative, then (3.2) and (3.5) yield

$$|f^{-1}(n)| \leq \prod_{i=1}^{\omega(n)} \sum_{\{l_1, l_2, \ldots, l_m\} \in \mathcal{P}(\omega_i)} \left(\frac{l}{l_1, l_2, \ldots, l_m}\right) A^l \prod_{j=1}^l c^{\phi_j}, \quad n \geq 2. \quad (3.6)$$

Although this upper bound is optimal and explicit, its application to particular choices of $f(n)$ can be complicated. Let us provide a simpler upper bound for (3.6) in the case $c \in (0, 1)$.

**Proposition 3.5.** Let $f(n)$ be multiplicative. Assume that there exist $A > 0$ and $c \in (0, 1)$ such that $|f(n)| \leq A c^n$ for all $n \geq 2$. Then

$$|f^{-1}(n)| \leq \left(\frac{A}{A+1}\right)^{\omega(n)} (A+1)^{\Omega(n)} n^{\frac{\ln c}{\ln 3}}, \quad n \geq 2. \quad (3.7)$$

**Proof.** Taking any prime power $p^k$, a partition $(\phi_1, \phi_2, \ldots, \phi_l) \in \mathcal{P}(k)$ as above, and applying [12, Lemma 2.2], we get

$$\prod_{j=1}^l c^{\phi_j} = c^{\sum_{j=1}^l \phi_j} \leq c^{\frac{\ln p}{\ln 3}}.$$

On the other hand, we have

$$\sum_{(\phi_1, \phi_2, \ldots, \phi_l) \in \mathcal{P}(k)} \left(\frac{l}{l_1, l_2, \ldots, l_m}\right) A^l = \sum_{l=1}^k \left(\frac{k-1}{l-1}\right) A^l = A(A+1)^{k-1}$$

since the sums in (3.8) (considered without $A^l$) correspond to the number of compositions of $k$ into exactly $l$ parts. Therefore, (3.5) implies that $|f^{-1}(p^k)| \leq A(A+1)^{k-1} c^{\frac{\ln p}{\ln 3}}$ for all prime powers. Finally, using the multiplicativity of $f^{-1}(n)$, we derive the inequality (3.7) similarly as in the proof of Proposition 3.1. \hfill \Box

**Corollary 3.6.** Let $f(n)$ be multiplicative. Assume that there exist $A > 0$ and $c \in (0, 1)$ such that $|f(n)| \leq A c^n$ for all $n \geq 2$. Then $|f^{-1}(n)| \leq n^{\frac{3 \ln c}{\ln 3} + \frac{\ln (1+A)}{\ln 2}}$ for all $n \geq 2$. 

Remark 3.7. Even if $|f(n)|$ decays exponentially as $n \to \infty$, the same exponential decay of $|f^{-1}(n)|$ cannot be guaranteed, as was already discussed in Section 1. In certain cases, a polynomial upper bound for $|f^{-1}(n)|$ gives the best achievable asymptotic behaviour. To show this, we fix any $A > \rho$ where $\varsigma > n$ and consider the multiplicative $f(n)$ defined as $f(2) = -Ac^2$ and $f(n) = 0$ for all $n \geq 3$. Then we see that $f^{-1}(n) = 0$ provided $n \neq 2^k$, and

$$f^{-1}(2^k) = A^k c^{2k} = 2^{k \ln A + 2k \ln c}$$

which reads as $f^{-1}(n) = n^{\ln(\frac{Ac^2}{\ln n})}$ for $n = 2^k$. That is, we have an explicit polynomial decay.

Remark 3.8. In the case $c > 1$, the growth of $|f^{-1}(n)|$ cannot be expected to be slower than the exponential growth. Let us consider the multiplicative $f(n)$ defined by $f(2^k) = -Ac^2$ for $k \geq 1$ and $f(n) = 0$ for all other $n > 2$. Clearly, we have $f^{-1}(n) = 0$ if $n \neq 2^k$, $f^{-1}(2) = Ac^2$, $f^{-1}(4) = Ac^4 + A^2 c^4$, and

$$f^{-1}(2^k) = Ac^k + \text{lower order terms}, \quad k \geq 3.$$  

An exponential upper bound for $|f^{-1}(n)|$ will be given for general $f(n)$ in Section 3.2.2 below.

3.2 General case. In this section, we consider the asymptotic behaviour of $f^{-1}(n)$ regardless the multiplicativity assumptions (1.8) and (3.1). Again, we divide our results into two subsections according to polynomial and exponential behaviour of $f(n)$, respectively.

3.2.1 Polynomial behaviour. Along this subsection, we will assume that $|f(n)| \leq Cn^\gamma$ for some $C > 0$, $\gamma \in \mathbb{R}$, and all $n \geq 2$. Using this upper bound and Corollary 2.5, we readily deduce from (1.3) that

$$|f^{-1}(n)| \leq \sum_{k=1}^{\Omega(n)} C^k H_k(n) \leq \frac{Cn^{\gamma+c}}{2^c} \sum_{k=1}^{\Omega(n)} (C(\zeta(s) - 1))^{k-1}$$

(3.9)

where $\varsigma > 1$ is chosen in such a way that $C(\zeta(s) - 1) = 1$. On the other hand, if $C = 1$, then by Corollary 2.3 we obtain

$$|f^{-1}(n)| \leq H(n)n^\gamma \leq n^{\gamma + \rho}, \quad n \geq 2,$$

where $\rho = 1.72865 \ldots$ is the unique root of $\zeta(s) = 2$.

Let us provide the following improvement of (3.9).

Proposition 3.9. Assume that there exist $C > 0$ and $\gamma \in \mathbb{R}$ such that $|f(n)| \leq Cn^\gamma$ for all $n \geq 2$. Then

$$|f^{-1}(n)| \leq n^{\gamma+c}, \quad n \geq 2,$$

(3.10)

where $\varsigma > 1$ is the unique root of $\zeta(s) = \frac{1}{c} + 1$.

Proof. Let us prove (3.10) by induction. Notice that the recurrence formula (1.2) can be equivalently rewritten as

$$f^{-1}(1) = 1 \quad \text{and} \quad f^{-1}(n) = -\sum_{d|n} f(d)f^{-1}\left(\frac{n}{d}\right), \quad n \geq 2.$$  

(3.11)
The base of induction is trivial. Let us fix some \( n \geq 2 \) and suppose that \(|f^{-1}(m)| \leq m^{\gamma+\varepsilon}\) for all \( m < n \). Then we obtain from (3.11) that

\[
|f^{-1}(n)| \leq \sum_{d|n, d > 1} C d \Omega \left( \frac{n}{d} \right)^{\gamma+\varepsilon} = C \left( \sum_{d|n} \frac{1}{d^\varepsilon} - 1 \right) n^{\gamma+\varepsilon} \leq C'(\kappa(s) - 1)n^{\gamma+\varepsilon} = n^{\gamma+\varepsilon}
\]

since \( C'(\kappa(s) - 1) = 1 \) by definition, and hence the result follows.

\[\square\]

**Remark 3.10.** The upper bound for \(|f^{-1}(n)|\) obtained in Proposition 3.9 is optimal at least for \( C = 1 \). Let us take some \( \gamma \in \mathbb{R} \) and set \( f(n) = -n^\gamma \) for all \( n \geq 2 \). Then we see from (1.3) that \( f^{-1}(n) = H(n)n^\gamma \) for all \( n \geq 2 \), which is an extension of the example from [9] discussed in Section 1. Recall that \( H(n) < n^\rho \) for all \( n \geq 2 \), and for any \( \varepsilon > 0 \) there exist infinitely many \( n \) such that \( H(n) > n^{\rho-\varepsilon} \), see Remark 2.4. Thus, for any \( \varepsilon > 0 \) there exist infinitely many \( n \) such that

\[
n^{\gamma+\rho-\varepsilon} < f^{-1}(n) < n^{\gamma+\rho},
\]

which yields the optimality.

### 3.2.2 Exponential behaviour.

Along this subsection, we will assume that \(|f(n)| \leq Ac^n\) for some \( A, c > 0 \) and all \( n \geq 2 \). We easily see from (1.3) that, under this assumption,

\[
|f^{-1}(n)| \leq \sum_{k=1}^{\Omega(n)} A^k \sum_{d_1 \cdots d_k = n, d_1, \ldots, d_k \geq 2} |f(d_1)| \cdots |f(d_k)| \leq \sum_{k=1}^{\Omega(n)} A^k c^{d_k^{\text{min}}(n)} H_k(n) \quad \text{for } c \in (0, 1),
\]

and

\[
|f^{-1}(n)| \leq \sum_{k=1}^{\Omega(n)} A^k \sum_{d_1 \cdots d_k = n, d_1, \ldots, d_k \geq 2} |f(d_1)| \cdots |f(d_k)| \leq \sum_{k=1}^{\Omega(n)} A^k c^{d_k^{\text{max}}(n)} H_k(n) \quad \text{for } c > 1,
\]

where

\[
d_k^{\text{min}}(n) = \min \{d_1 + \cdots + d_k : d_1 \cdots d_k = n, d_i \geq 2, d_i \in \mathbb{N}\}
\]

and

\[
d_k^{\text{max}}(n) = \max \{d_1 + \cdots + d_k : d_1 \cdots d_k = n, d_i \geq 2, d_i \in \mathbb{N}\},
\]

respectively. Moreover, we set \( d_k^{\text{min}}(n) = \infty \) and \( d_k^{\text{max}}(n) = -\infty \), provided the corresponding feasible sets are empty.

In order to estimate \(|f^{-1}(n)|\) via (3.12) or (3.13), we obtain the following lower bounds for \( d_k^{\text{min}}(n) \) and upper bound for \( d_k^{\text{max}}(n) \).

**Lemma 3.11.** For all \( n \geq 1 \) and \( k \geq 1 \), there hold

\[
d_k^{\text{min}}(n) \geq kn^{\frac{1}{k}} \geq e \ln n \quad (3.14)
\]

and

\[
d_k^{\text{max}}(n) \leq 2(k - 1) + \frac{n}{2^k - 1}. \quad (3.15)
\]
Proof. First, let us obtain the lower bounds for $d_k^\min(n)$. If we assume that $n = 2^k$, then $d_k^\min(n) = 2^k$, and if $n < 2^k$, then $d_k^\min(n) = \infty$, i.e., the first inequality of (3.14) is satisfied for $n \leq 2^k$. Therefore, let us assume that $n > 2^k$. Notice that $d_k^\min(n) \geq D_k^\min(n)$, where

$$D_k^\min(n) = \min \{ d_1 + \cdots + d_k : d_1 \cdots d_k = n, \ d_i \geq 2, \ d_i \in \mathbb{R} \}.$$ 

That is, in the definition of $D_k^\min(n)$ we allow each $d_i$ to be non-natural. It is not hard to see that $D_k^\min(n)$ has a solution $(d_1, \ldots, d_k)$. Therefore, $(d_1, \ldots, d_k)$ satisfies the Karush–Kuhn–Tucker conditions, see, e.g., [17]. Namely, there exist $\lambda_0 \in \mathbb{R}$ and $\lambda_i \leq 0$ for $i = 1, \ldots, k$ such that

$$1 + \frac{\lambda_0 n}{d_i} + \lambda_i = 0, \ i = 1, \ldots, k,$$

and if some $d_i > 2$, then $\lambda_i = 0$.

Since each $d_i \geq 2$ and we assume that $n > 2^k$, there exists at least one $d_m > 2$. Thus,

$$\lambda_m = 0 \quad \text{and} \quad \lambda_0 = -\frac{d_m}{n}. \quad (3.17)$$

If there exists another $d_l > 2$, $l \neq m$, then, as above,

$$\lambda_l = 0 \quad \text{and} \quad \lambda_0 = -\frac{d_l}{n},$$

which yields $d_l = d_m$. Suppose now that there exists some $d_\kappa = 2$. Taking into account (3.17) and recalling that $d_m > 2$, we see from (3.16) that

$$\lambda_\kappa = -1 + \frac{d_m}{2} > 0,$$

which is impossible since $\lambda_i \leq 0$ for all $i = 1, \ldots, k$. Therefore, we conclude that each $d_i > 2$ and $d_i = d_j$ for $i \neq j$. Consequently, $n = d^k$ and $d_k^\min(n) \geq k n^{\frac{1}{k}}$.

Let us now estimate $k n^{\frac{1}{k}}$ from below by $e \ln n$. To this end, we fix some $n \geq 2$ and consider the function

$$G(x) = xn^{\frac{1}{x}}, \quad x \in \mathbb{R}, \ x > 0.$$ 

It is clear that $G(x)$ has exactly one global minimizer $x_0 = \ln n$ with $G(x_0) = e \ln n$. Hence, we conclude that

$$d_k^\min(n) \geq k n^{\frac{1}{k}} \geq e \ln n, \quad n \in \mathbb{N}.$$ 

Second, let us obtain the upper bound for $d_k^\max(n)$ by following the same strategy as above. Evidently, for $n \leq 2^k$ the inequality (3.15) holds true and we can restrict ourselves to $n > 2^k$. Denoting

$$D_k^\max(n) = \max \{ d_1 + \cdots + d_k : d_1 \cdots d_k = n, \ d_i \geq 2, \ d_i \in \mathbb{R} \},$$

we see that $d_k^\max(n) \leq D_k^\max(n)$. Every solution $(d_1, \ldots, d_k)$ to $D_k^\max(n)$ fulfils the Karush–Kuhn–Tucker conditions, i.e., there are $\lambda_0 \in \mathbb{R}$ and $\lambda_i \leq 0$ for $i = 1, \ldots, k$ satisfying

$$-1 + \frac{\lambda_0 n}{d_i} + \lambda_i = 0, \ i = 1, \ldots, k,$$

and if some $d_i > 2$, then $\lambda_i = 0$.

Recalling that $n > 2^k$, we can find $d_m > 2$. Arguing as above, we deduce that if there is another $d_l > 2$ with $l \neq m$, then $d_l = d_m$. Therefore, we can assume, without loss of
generality, that $d_1 = \cdots = d_r > 2$ for some $r \in \{1, \ldots, k\}$, and $d_{r+1} = \cdots = d_k = 2$ if $r < k$. This implies that $n = d_1^r 2^{k-r}$, and hence $d_1 = \left(\frac{n}{2^{k-r}}\right)^{\frac{1}{r}}$ and

$$D_k^{\text{max}}(n) = 2(k - r) + \frac{rn - 1}{2^{k-r}}. \quad (3.18)$$

To determine the actual value of $r \in \{1, \ldots, k\}$, let us maximize the right-hand side of (3.18) with respect to $r$. To this end, consider the function

$$G(x) = 2(k - x) + \frac{xn^{\frac{1}{r}}}{2^{k-x}}, \quad x \in \mathbb{R}, \ x > 0.$$ 

We see that

$$G'(x) = \frac{n^{\frac{1}{r}}}{2^{k-x}} \left(1 - \left(\frac{2k}{n}\right)^{\frac{1}{x}} + \ln \left(\left(\frac{2k}{n}\right)^{\frac{1}{x}}\right)\right) \leq 0, \quad x > 0,$$

and $G'(x) = 0$ if and only if $n = 2^k$. Since $n > 2^k$, $G(x)$ decreases with respect to $x > 0$, and we conclude that $r = 1$ is the unique maximizer for the right-hand side of (3.18), which yields the claimed upper bound (3.15).

Combining now (3.12) or (3.13) with Lemma 3.11, we obtain the following result.

**Proposition 3.12.** Assume that there exist $A, c > 0$ such that $|f(n)| \leq Ae^n$ for all $n \geq 2$. If $c \in (0, 1)$, then

$$|f^{-1}(n)| \leq \Omega(n) \cdot \frac{An^{\frac{\rho + e \ln c}{2}}} {2^n} \leq \frac{An^{\frac{\rho + e \ln c}{2}} \ln n} {2^n \ln 2}, \quad n \geq 2,$$ \quad (3.19)

where $\zeta > 1$ is the unique root of $\zeta(s) = \frac{1}{A} + 1$. If $c \in (0, 1)$ but $A \leq 1$, then

$$|f^{-1}(n)| \leq n^{\rho + e \ln c}, \quad n \geq 2,$$ \quad (3.20)

where $\rho = 1.72865\ldots$ is the unique root of $\zeta(\rho) = 2$.

If $c > 1$, then there exists $\tilde{A} > 0$ such that

$$|f^{-1}(n)| \leq Ae^n + \frac{(\Omega(n) - 1)An^{\frac{\rho + e \ln c}{2}}}{2^n} \leq \frac{\tilde{A}e^n}{2^n}, \quad n \geq 2,$$ \quad (3.21)

where $\rho > 1$ is the unique root of $\zeta(s) = \frac{1}{\tilde{A}e^x} + 1$.

**Proof.** First, assume that $c \in (0, 1)$. We estimate (3.12) by (2.7) and (3.14) as

$$|f^{-1}(n)| \leq \sum_{k=1}^{\Omega(n)} A_k^{e_{\text{min}}(n)} H_k(n) \leq \frac{An^{\frac{\rho + e \ln c}{2}} \Omega(n)}{2^n} \sum_{k=1}^{\Omega(n)} (A(\zeta(s) - 1))^{k-1}, \quad n \geq 2,$$ 

for every $s > 1$. Taking $s = \zeta$, we have $A(\zeta(s) - 1) = 1$ and (3.19) follows directly. Under the additional assumption $A \leq 1$, we easily get (3.20) from (3.12) and Corollary 2.3:

$$|f^{-1}(n)| \leq \sum_{k=1}^{\Omega(n)} A_k^{e_{\text{min}}(n)} H_k(n) \leq e^{\ln n} \sum_{k=1}^{\Omega(n)} H_k(n) = n^{e \ln c} H(n) \leq n^{e \ln c + \rho}, \quad n \geq 2.$$
Second, assume that \( c > 1 \). With \( d_1^{\max}(n) = n \), \( H_1(n) = 1 \), (2.7), and (3.15), we deduce

\[
|f^{-1}(n)| \leq Ac^n + \sum_{k=2}^{\Omega(n)} A^k c^{2(k-1)+\frac{n+1}{2k} + \frac{1}{2k}} (\zeta(s) - 1)^{k-1}n^s
\]

\[
\leq Ac^n + An^s c^s \sum_{k=2}^{\Omega(n)} (A^2(\zeta(s) - 1))^{k-1}, \quad n \geq 2,
\]

(3.22)

for every \( s > 1 \). For \( s = \nu \) with \( Ac^2(\zeta(\nu) - 1) = 1 \) we obtain the first inequality in (3.21). Obviously, the first term in (3.22) is the leading term as \( n \to \infty \), and hence the second inequality in (3.21) is valid, as well.

**Remark 3.13.** Comparing the upper bound (3.20) for a general \( f(n) \) with the upper bound (3.7) for a multiplicative \( f(n) \), we see that (3.7) provides the better asymptotic. On the other hand, (3.20) provides an improvement of the following upper bound obtained in the proof of [20, Theorem 3]: if \( |f(1)| \geq \frac{c}{2} \) and \( |f(n)| \leq c^n \) for \( n \geq 2 \) where \( c \leq \left( 2 \cdot 3^\frac{4}{5} + 3^\frac{3}{5} \right)^{-1} \), then

\[
|f^{-1}(n)| \leq \frac{2}{cn^2}, \quad n \geq 2.
\]

### 3.3 Miscellaneous cases

In this section, we consider the asymptotic behaviour of \( f^{-1}(n) \) for several special classes of \( f(n) \). We limit ourselves to the consideration of a polynomial bound for \( |f(n)| \).

**3.3.1 Truncated \( f(n) \).** First, let us assume that there exists \( N \geq 2 \) such that \( f(n) = 0 \) for all \( 2 \leq n \leq N \). We see from (1.3) that \( f^{-1}(n) = 0 \) for all \( 2 \leq n \leq N \), and \( f^{-1}(n) \) depends only on the values of \( f(d) \) with \( d > N \). Therefore, we can argue in much the same way as in Proposition 3.9 to obtain the following result.

**Proposition 3.14.** Assume that there exist \( N \geq 2 \), \( C > 0 \), and \( \gamma \in \mathbb{R} \) such that \( f(n) = 0 \) for all \( 2 \leq n \leq N \) and \( |f(n)| \leq Cn^\gamma \) for all \( n > N \). Then

\[
|f^{-1}(n)| \leq n^{\gamma + \varsigma}, \quad n > N,
\]

where \( \varsigma > 1 \) is the unique root of \( \zeta(s) = \frac{1}{\gamma} + \sum_{m=1}^{N} \frac{1}{m^s} \).

Second, let us assume that there exists \( N \geq 2 \) such that \( f(n) = 0 \) for all \( n \geq N + 1 \). Clearly, in this case \( f^{-1}(n) \) depends only on the values of \( f(d) \) with \( d < N \). Arguing along the same lines as in Proposition 3.9, we derive the following result.

**Proposition 3.15.** Assume that there exist \( N \geq 2 \), \( C > 0 \), and \( \gamma \in \mathbb{R} \) such that \( f(n) = 0 \) for all \( n \geq N + 1 \) and \( |f(n)| \leq Cn^\gamma \) for all \( n \leq N \). Then

\[
|f^{-1}(n)| \leq n^{\gamma + \varsigma}, \quad n \geq 2,
\]

where \( \varsigma > 0 \) is the unique root of \( \sum_{m=2}^{N} \frac{1}{m^s} = \frac{1}{\gamma} \).
3.3.2 \( f(n) \) supported on odd \( n \). Assume that \( f(n) \) represents the Fourier coefficients of a function \( F \in L^2(0, 1) \), that is, \( F(x) = \sum_{n=-\infty}^{\infty} f(n) \sin(n \pi x) \). If one is interested in the basisness or completeness of the system \( \{ F(nx) \} \) (see Section 1), then it seems natural to choose \( F \) such that it is symmetric with respect to the point \( x = 1/2 \), see, e.g., generalized trigonometric functions [2]. Under this symmetry assumption, we have \( f(n) = 0 \) for all even \( n \) and it is clear from (1.3) that \( f^{-1}(n) = 0 \) for all even \( n \), as well. Arguing as in the proof of Proposition 3.9 and using (2.6), we get the following result.

**Proposition 3.16.** Assume that \( f(n) = 0 \) for all even \( n \) and there exist \( C > 0 \) and \( \gamma \in \mathbb{R} \) such that \( |f(n)| \leq C n^{\gamma} \) for all \( n \geq 2 \). Then

\[
|f^{-1}(n)| \leq n^{\gamma + \varsigma}, \quad n \geq 2,
\]

where \( \varsigma > 1 \) is the unique root of \((1 - \frac{1}{2^s}) \zeta(s) = \frac{1}{C} + 1\). In particular, if \( C = 1 \), then \( \varsigma = \eta = 1.37779\ldots \)

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