Affine group dg-schemes and linear representations I

Basic theory and Tannakian reconstructions

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Abstract. We develop a basic theory of affine group dg-schemes, their Lie algebraic counterparts and linear representations. We prove Tannaka type reconstruction theorems that an affine group dg-scheme can be recovered from the dg-tensor category of its linear representations as well as from the rigid dg-tensor category of its finite dimensional linear representations along with the forgetful functors to the underlying dg-tensor category of cochain complexes.

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1. Introduction

The theory of affine group schemes and their representations is a classic subject in algebraic geometry—we refer to [15,9] for reviews. It has led to the notion of Tannakian category as envisioned by Grothendieck and established by Saavedra Rivano [11] and Deligne [4] realizing the Tannaka-Klein duality for the group objects in the category of affine scheme—see [5,12] for a reviews. Beside from its initially intended role in algebraic geometry (Tannakian fundamental group schemes and motives), the ideas of Tannakian category find applications in wider branches in mathematics—we refer to the book [6] and references therein.

We recall that an affine group scheme over a field $k$ is a representable functor $G : cAlg(k) \rightarrow Grp$ from the category $cAlg(k)$ of commutative algebras over $k$ to the category $Grp$ of groups, whose representing object is a commutative Hopf algebra $B^0$. The category $Rep(G)$ of linear representations of $G$ is isomorphic to the category $Comod_B(B^0)$ of right comodules over $B^0$. Then a Tannakian reconstruction theorem is that $G$ can be reconstructed from the category $Rep(G)_f$ of its finite dimensional linear representations via the forgetful functor $\omega : Rep(G)_f \rightarrow Vec(k)_f$ to the category of underlying vector spaces. The category $Rep(G)$ with the functor $\omega$ is the prototype of neutral Tannakian category, which is a rigid $k$-linear abelian tensor category with a fiber functor to the category $Vec(k)_f$. The other side of the duality is that any neutral Tannakian category $(T, \omega)$ is equivalent to the category of the finite dimensional linear representations of an affine group scheme $G$, which is called the Tannakian fundamental group scheme of $(T, \omega)$.

This paper is about a basic theory of affine group dg-schemes, their Lie algebraic counterparts, and their linear representations as well as Tannakian reconstruction theorems. This paper is a companion to our recent paper on the similar study of representable presheaves of groups on the homotopy category of cocommutative dg-coalgebras [7]. The prefix “dg” stands for “differential graded” and a dg-category is an enriched category whose morphism sets are endowed with the structure of a cochain complex, where cochain complexes over $k$ form the prototypical dg-category denoted by $CoCh(k)$.

In Section 3, we define an affine group dg-scheme over $k$ as a representable functor $\mathcal{G} : hocdgA(k) \rightarrow Grp$ from the homotopy category $hocdgA(k)$ of commutative dg-algebras over $k$ to the category $Grp$ of groups. A representing object $\mathcal{G}$ is a commutative dg-Hopf algebra $B$, and we use the notation $\mathcal{G}^B$ for it. The category of affine group dg-schemes is anti-equivalent to the homotopy category $hocdgH(k)$ of commutative dg-Hopf algebras. For the Lie theoretic counterpart to affine group dg-scheme, we construct a functor $T\mathfrak{g}^B : hocdgA(k) \rightarrow Lie(k)$ to the category $Lie(k)$ of Lie algebras over $k$ so that $T\mathfrak{g}^B$ and $T\mathfrak{g}^B$ are naturally isomorphic whenever $\mathcal{G}^B$ and $\mathfrak{g}^B$ are naturally isomorphic. For a pro-unipotent affine group dg-scheme $\mathcal{G}^B$, where $B$ is conilpotent, we show that the underlying set-valued functors $\mathcal{G}^a : hocdgA(k) \rightarrow Set$ and $T\mathfrak{g}^a : hocdgA(k) \rightarrow Set$ are naturally isomorphic—we can recover the group $\mathcal{G}^B(A)$ from the Lie algebra $T\mathfrak{g}^B(A)$ for every cdg-algebra $A$. 

In Sect. 4, we define a linear representation of $\mathfrak{S}^B$ via a linear representation of the associated representable functor $\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}$ from the category $\text{cdgA}(k)$ of commutative dg-algebras over $k$, which is represented by $B$ and induces $\mathfrak{S}^B$ on the homotopy category $\text{hodgA}(k)$. A linear representation of $\mathfrak{S}^B$ induces a linear representation of $\mathfrak{S}^B$. We shall form a dg-tensor category $\text{Rep}(\mathcal{G}^B)$ of linear representations of $\mathcal{G}^B$ and show that it is isomorphic to a dg-tensor category $\text{dgComod}_{R(B)}$ formed by right dg-comodules over $B$.

In Sect. 5, we reconstruct $\mathfrak{S}^B$ via the forgetful functor $\omega : \text{dgComod}_{R(B)} \to \text{CoCh}(k)$ to the underlying dg-tensor category $\text{CoCh}(k)$ of cochain complexes. Out of $\omega$, we shall construct two functors $G^\omega_0 : \text{cdgA}(k) \to \text{Grp}$ and $\mathfrak{S}^\omega_0 : \text{hodgA}(k) \to \text{Grp}$ and establish natural isomorphisms of functors

$$G^\omega_0 \cong \mathcal{G}^B : \text{cdgA}(k) \to \text{Grp} \quad \text{and} \quad \mathfrak{S}^\omega_0 \cong \mathfrak{S}^B : \text{hodgA}(k) \to \text{Grp},$$

which is our 1st reconstruction theorem of an affine group dg-scheme from the dg-tensor category of linear representations. We also consider the dg-tensor category $\text{Rep}(\mathcal{G}^B)$ of finite dimensional linear representations of $\mathcal{G}^B$ which is isomorphic to the dg-tensor category $\text{dgComod}_{R(B)}$ of finite dimensional right dg-comodules over $B$. From the forgetful functor $\omega_f : \text{dgComod}_{R(B)} \to \text{CoCh}(k)_f$, we shall construct two functors $G^\omega_f : \text{cdgA}(k) \to \text{Grp}$ and $\mathfrak{S}^\omega_f : \text{hodgA}(k) \to \text{Grp}$ and establish natural isomorphisms $G^\omega_f \cong \mathcal{G}^B$ and $\mathfrak{S}^\omega_f \cong \mathfrak{S}^B$, which is our 2nd reconstruction theorem of an affine group dg-scheme from the dg-tensor category of finite dimensional linear representations. The 2nd reconstruction theorem is obtained by the reductions of 1st reconstruction theorem to the finite dimensional cases, based on the basic fact the every right dg-comodule over $B$ is a filtered colimit of its finite dimensional subcomodules over $B$. We shall also give an independent proof using the dg-version of rigidity of $\text{dgComod}_{R(B)}$.

Our study of affine group dg-schemes is originally motivated from de Rham side of the rational homotopy theory of Sullivan [14] and Chen [2] as well as the adoption by Deligne for rational de Rham fundamental group scheme of algebraic varieties and their periods [3]. We like to have a natural extension of rational de Rham fundamental group scheme of a space to a rational de Rham fundamental group dg-scheme encoding more general invariants of rational homotopy types of those spaces, including all higher rational homotopy groups—but these and other applications will be the subjects of a sequel to this paper [8]. Another motivation for this paper is to do some groundworks toward the theory of "Tannakian dg-tensor categories" as many Tannakian categories come naturally with dg-tensor categories—see [13] for examples. We note that some progress along this line is reported in [10].

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1 We do not have the similar reduction to finite dimensional cases for the dg-tensor category of dg-modules over cocommutative dg-Hopf algebra studied in [7].
2. Notations

We use the similar notations and conventions as in our previous paper [7], except that we use the cohomological $\mathbb{Z}$-grading—every differential has degree 1 and a dg-category is a category enriched in the category of cochain complexes:

Throughout this paper $k$ is a ground field of characteristic 0. Unadorned tensor product $\otimes$ is over $k$. By an element in a $\mathbb{Z}$-graded vector space we shall usually mean a homogeneous element $x$ whose degree will be denoted $|x|$. Let $V = \bigoplus_{n \in \mathbb{Z}} V^n$ and $W = \bigoplus_{n \in \mathbb{Z}} W^n$ be $\mathbb{Z}$-graded vector spaces. Then $V \otimes W = \bigoplus_{i+j=n \in \mathbb{Z}} V^i \otimes W^j$, is also a $\mathbb{Z}$-graded vector space. Denote $\text{Hom}(V, W) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V, W)^n$ as the $\mathbb{Z}$-graded vector space of $k$-linear maps from $V$ to $W$. A cochain complex $(V, d_V)$ is often denoted by $V$ for simplicity. The ground field $k$ is a cochain complex with the zero differential. If $V$ and $W$ are cochain complexes $V \otimes W$ and $\text{Hom}(V, W)$ are also cochain complexes with the following differentials

$$\begin{align*}
&d_{V \otimes W} = d_V \otimes 1_W + 1_V \otimes d_W, \\
&d_{V, W} f = d_W \circ f - (-1)^{|f|} f \circ d_V, \quad \forall f \in \text{Hom}(V, W)^{|f|}.
\end{align*} \quad (2.1)$$

A cochain map $f : (V, d_V) \to (W, d_W)$ is an $f \in \text{Hom}(V, W)^0$ satisfying $d_{V, W} f = d_V \circ f - f \circ d_W = 0$. Two cochain maps $f$ and $\tilde{f}$ are homotopic, denoted by $f \sim \tilde{f}$, or have the same homotopy type, denoted by $[f] = [\tilde{f}]$, if there is a cochain homotopy $\lambda \in \text{Hom}(V, W)^{-1}$ such that $\tilde{f} - f = d_{V, W} \lambda$.

The set of morphisms from an object $C$ to another object $C'$ in a category $C$ is denoted by $\text{Hom}_C(C, C')$. We denote the set of natural transformations of functors $F \Rightarrow G : C \to D$ by $\text{Nat}(F, G)$. For any functor $F : C \to D$, where $D$ is small, we use the notation $F : C \to \text{Set}$ for the underlying set valued functor obtained by composing with the forgetful functor $\text{Forget} : D \to \text{Set}$. Such a functor $F : C \to D$ is called representable if $F$ is representable.

Remark that the ground field $k$ is an algebra $(k, u_k, m_k)$ where $u_k = 1_k$ and $m_k(a \otimes b) = a \cdot b$, and a coalgebra $k^\vee = (k, \epsilon_k, \Delta_k)$ with $\epsilon_k = 1_k$ and $\Delta_k(1) = 1 \otimes 1$. The canonical isomorphisms $k \otimes V \cong V$ and $V \otimes k \cong V$ will be denoted by $i_V : k \otimes V \to V$ and $i_V^\vee : V \to k \otimes V$, as well as $j_V : V \otimes k \to V$ and $j_V^\vee : V \to V \otimes k$.

A commutative dg Hopf algebra (cdg-Hopf algebra) is a tuple $B = (B, u_B, m_B, \epsilon_B, \Delta_B, \zeta_B, d_B)$, where

- $(B, d_B)$ is a cochain complex;
- $(B, u_B, m_B, d_B)$ is a commutative dg algebra (cdg-algebra) that both the unit $u_B : k \to B$ and the product $m_B : B \otimes B \to B$ are cochain maps satisfying the unit axiom and the associativity axiom:

$$\begin{align*}
&d_B \circ u_B = 0, \\
m_B \circ d_B \otimes B = d_B \circ m_B,
\end{align*} \quad (2.2)$$

and the commutativity $m_B = m_B \circ \tau$, where $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x, \forall x, y \in A$;
The antipode is a cdg-bialgebra. In other words a cdg-Hopf algebra is a cdg-bialgebra with an antipode. An antipode \( \varsigma \) is a morphism \( f \colon B \to B \otimes B \) of cdg-algebras such that the counit \( \epsilon_B : B \to k \) and the coproduct \( \Delta_B : B \to B \otimes B \) are algebra maps:

\[
\begin{align*}
\epsilon_B \circ u_B &= u_k, \\
\epsilon_B \circ m_B &= m_k \circ (\epsilon_B \otimes \epsilon_B), \\
\Delta_B \circ u_B &= u_{B \otimes B} \circ \Delta_k, \\
\Delta_B \circ m_B &= m_{B \otimes B} \circ (\Delta_B \otimes \Delta_B),
\end{align*}
\]

and \( \varsigma_B : B \to B \) satisfies the antipode axiom:

\[
\begin{align*}
m_B \circ (\varsigma_B \otimes I_B) \circ \Delta_B &= m_B \circ (I_B \otimes \varsigma_B) \circ \Delta_B = u_B \circ \epsilon_B.
\end{align*}
\]

**Remark 2.1.** We have used the structure \( (B \otimes B, u_{B \otimes B}, m_{B \otimes B}, d_{B \otimes B}) \) of cdg-algebra on \( B \otimes B \) induced from the cdg-algebra on \( B \) where

\[
u_{B \otimes B} = u_B \otimes u_B, \quad m_{B \otimes B} = (m_B \otimes m_B) \circ (I_B \otimes \tau \otimes I_B).
\]

We also have the structure \( (B \otimes B, \epsilon_{B \otimes B}, \Delta_{B \otimes B}, d_{B \otimes B}) \) of cdg-algebra on \( B \otimes B \) induced from the cdg-algebra on \( B \), where

\[
u_{B \otimes B} = \epsilon_B \otimes \epsilon_B, \quad \Delta_{B \otimes B} = (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B).
\]

The conditions in eq. (2.4) is equivalent to the conditions that the unit \( u_B : k \to B \) and the product \( m_B : B \otimes B \to B \) are algebra maps. A cdg-Hopf algebra after forgetting the antipode is a cdg-bialgebra. In other words a cdg-Hopf algebra is a cdg-bialgebra with an antipode. An antipode \( \varsigma_B \) of a cdg-bialgebra \( B \) is unique if exists, and is an algebra homomorphism and an anti-coalgebra homomorphism:

\[
\begin{align*}
\varsigma_B \circ u_B &= u_B, \\
\varsigma_B \circ m_B &= m_B \circ (\varsigma_B \otimes \varsigma_B), \\
\epsilon_B \circ \varsigma_B &= \epsilon_B, \\
\Delta_B \circ \varsigma_B &= \tau \circ (\varsigma_B \otimes \varsigma_B) \circ \Delta_B.
\end{align*}
\]

A morphism \( f : B \to B' \) of cdg-Hopf algebras is simultaneously

- a morphism \( f : (B, u_B, m_B, d_B) \to (B', u_{B'}, m_{B'}, d_{B'}) \) of cdg-algebras:

\[
f \circ d_B = d_{B'} \circ f, \quad f \circ u_B = u_{B'}, \quad f \circ m_B = m_{B'} \circ (f \otimes f),
\]

- and a morphism \( f : (B, \epsilon_B, \Delta_B, d_B) \to (B', \epsilon_{B'}, \Delta_{B'}, d_{B'}) \) of dg-coalgebras:

\[
f \circ d_B = d_{B'} \circ f, \quad \epsilon_{B'} \circ f = \epsilon_B, \quad \Delta_{B'} \circ f = (f \otimes f) \circ \Delta_B.
\]
Then it is automatic that \( f \) commutes with the antipodes:

\[
f \circ \zeta_B = \zeta_{B'} \circ f.
\]  

(2.9)

The composition of morphisms of cdg-Hopf algebras is defined by the composition as linear maps, which is obviously a morphism of cdg-Hopf algebras, and the category formed by cdg-Hopf algebras is denoted by \( \text{cdgH}(k) \). The category of cdg-algebras is denoted by \( \text{cdgA}(k) \), and the category of dg-coalgebras is denoted by \( \text{dgC}(k) \).

A homotopy pair on \( \text{Hom}_{\text{cdgH}(k)}(B, B') \) is a pair of one parameter families \( \{f(t), \xi(t)\} \in \text{Hom}(B, B')^{0}[t] \oplus \text{Hom}(B, B')^{-1}[t] \), parametrized by the time variable \( t \) with polynomial dependences, satisfying the homotopy flow equation \( \frac{d}{dt}f(t) = dB \cdot \xi(t) \) generated by \( \xi(t) \), subject to the following two types of conditions:

- **infinitesimal algebra map**: \( f(0) \in \text{Hom}_{\text{cdgA}(k)}(B, B') \) and

  \[
  \xi(t) \circ u_B = 0, \quad \xi(t) \circ m_B = m_{B'} \circ (f(t) \otimes \xi(t) + \xi(t) \otimes f(t));
  \]

- **infinitesimal coalgebra map**: \( f(0) \in \text{Hom}_{\text{dgC}(k)}(B, B') \) and

  \[
  \epsilon_{B'} \circ \xi(t) = 0, \quad \Delta_{B'} \circ \xi(t) = (f(t) \otimes \xi(t) + \xi(t) \otimes f(t)) \circ \Delta_{B}.
  \]

Let \( \{f(t), \xi(t)\} \) be a homotopy pair on \( \text{Hom}_{\text{cdgH}(k)}(B, B') \). By the homotopy flow equation, \( f(t) \) is determined uniquely by \( \xi(t) \) modulo an initial condition \( f(0) \) such that \( f(t) = f(0) + dB \cdot \int_{0}^{t} \xi(s)ds \), and we can check that \( f(t) \) is a family of morphisms of cdg-Hopf algebras. We say \( f(1) \) is homotopic to \( f(0) \) by the homotopy \( \int_{0}^{1} \xi(t)dt \), and denote \( f(0) \sim f(1) \), which is clearly an equivalence relation. In other words, two morphisms \( f \) and \( \tilde{f} \) of cdg-Hopf algebras are homotopic if there is a homotopy flow connecting them (by the time 1 map). Then, we also say that \( f \) and \( \tilde{f} \) have the same homotopy type, denoted by \( [f] = [\tilde{f}] \).

For any diagram

\[
\begin{array}{c}
B \\
\downarrow f \\
B' \\
\downarrow f'
\end{array}
\]

in the category \( \text{cdgH}(k) \), where \( f \sim \tilde{f} \) and \( f' \sim \tilde{f}' \), it is straightforward to check that \( f' \circ f \sim \tilde{f}' \circ \tilde{f} \), and the homotopy type of \( f' \circ f \) only depends on the homotopy types of \( f \) and \( f' \), so that we have the well-defined composition \( [f'] \circ_h [f] := [f' \circ f] \) of homotopy types. A morphism \( B \longrightarrow B' \) of cdg-Hopf algebras is a homotopy equivalence if there is a morphism \( B \longrightarrow B' \) of cdg-Hopf algebras from the opposite direction such that \( h \circ f \sim \mathbb{I}_B \) and \( f \circ h \sim \mathbb{I}_{B'} \).

The homotopy category \( \text{hodgH}(k) \) of cdg-Hopf algebras over \( k \) is defined such that the objects are cdg-Hopf algebras and morphisms are homotopy types of morphisms of cdg-Hopf algebras. Note that a homotopy equivalence of cdg-Hopf algebras is an isomorphism in the homotopy category \( \text{hodgH}(k) \).
We define a homotopy pair on the morphisms of cdg-algebras as the case of cdg-Hopf algebras but without imposing the condition for infinitesimal coalgebra map. Then, we have corresponding notions for homotopy types of morphisms of cdg-algebras and a homotopy equivalence of cdg-algebras. Thus we can form the homotopy category $\text{hodgA}(k)$ of cdg-algebras, whose morphisms are homotopy types of morphisms of cdg-algebras.

A dg-category $\mathcal{C}$ over $k$ is a category enriched in the category $\text{CoCh}(k)$ of cochain complexes over $k$. A dg-category shall be distinguished from an ordinary category by putting an "overline". We follows [13] for the notion of dg-tensor categories. Denoted by $\text{Hom}_{\mathcal{C}}(X, Y)$ with differential $d_{\text{Hom}_{\mathcal{C}}(X, Y)}$ is the cochain complex of morphisms from object $X$ to object $Y$ in a dg-category $\mathcal{C}$. A morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ between two objects $X$ and $Y$ in $\mathcal{C}$ is an isomorphism if $f \in \text{Hom}_{\mathcal{C}}(X, Y)^0$ and satisfies $d_{\text{Hom}_{\mathcal{C}}(X, Y)} f = 0$, with its inverse $g \in \text{Hom}_{\mathcal{C}}(Y, X)^0$ satisfying $d_{\text{Hom}_{\mathcal{C}}(Y, X)} g = 0$.

A dg-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor which induces cochain maps $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ for every pair $(X, Y)$ of objects in $\mathcal{C}$. The set $\text{Nat}(F, G)$ of natural transformations of dg-functors is a cochain complex $\left(\text{Nat}(F, G), \delta, \delta \right)$, where

- its degree $n$ element is a collection of morphisms $\eta = \{ \eta^X : F(X) \rightarrow G(X) | X \in \text{Ob}(\mathcal{C}) \}$ of degree $n$, where $\eta^X$ is called the component of $\eta$ at $X$, with the supercommuting naturalness condition, i.e. $G(f) \circ \eta^X = (-1)^{m \cdot n} \eta^Y \circ F(f)$ for every morphism $f : X \rightarrow Y$ of degree $m$.
- for every $\eta \in \text{Nat}(F, G)$ of degree $n$ we have $\delta \eta \in \text{Nat}(F, G)$ of degree $n + 1$, whose component at $X$ is defined by $(\delta \eta)^X := d_{\text{Hom}_{\mathcal{C}}(F(X), G(X))} \eta^X$, and $\delta \circ \delta = 0$.

The dg-functors from $\mathcal{C}$ to $\mathcal{D}$ form a dg-category, with morphisms as the above natural transformations. In particular, the set $\text{End}(F) := \text{Nat}(F, F)$ of natural endomorphisms has a canonical structure of dg-algebra. A natural transformation $\eta$ from a dg-functor $F$ to another dg-functor $G$ is often indicated by a diagram $\eta : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$. A natural transformation $\eta$ is an (natural) isomorphism if the component morphism $\eta^X : F(X) \rightarrow G(X)$ is an isomorphism in $\mathcal{D}$ for every object $X$ of $\mathcal{C}$.

The notion of tensor categories [11, 4] has a natural generalization to dg-tensor categories. For a dg-category $\mathcal{C}$ we have a new dg-category $\mathcal{C} \otimes \mathcal{C}$, whose objects are pairs denoted by $X \otimes Y$ and whose Hom complexes are the tensor products of Hom complexes of $\mathcal{C}$, i.e., $\text{Hom}_{\mathcal{C} \otimes \mathcal{C}}(X \otimes Y, X' \otimes Y') = \text{Hom}_{\mathcal{C}}(X, X') \otimes \text{Hom}_{\mathcal{C}}(Y, Y')$ with the natural composition operation and differentials. Then we have a natural equivalence of dg-categories $(\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C} \cong \mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C})$. A dg-category $\mathcal{C}$ is a dg-tensor category if we have dg-functor $\otimes : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$ and a unit object $1_\mathcal{C}$ satisfying the associativity, the commutativity and the unit axioms subject to coherence conditions. (See pp 40-41 in [13] for the details.)
The fundamental example of dg-tensor category over $k$ is the dg-category $\text{CoCh}(k)$ of cochain complexes, whose set of morphisms $\text{Hom}_{\text{CoCh}(k)}(V,W)$ from a cochain complex $V$ to a cochain complex $W$ is the $k$-linear Hom complex $\text{Hom}(V,W)$ with the differential $d_{\text{Hom}_{\text{CoCh}(k)}(V,W)} = d_{V,W}$. The dg-functor $\otimes: \text{CoCh}(k) \boxtimes \text{CoCh}(k) \to \text{CoCh}(k)$ sends $(V,d_V) \boxtimes (W,d_W)$ to the cochain complex $(V \otimes W, d_{V \otimes W})$ and an unit object is the ground field $k$ as a cochain complex $(k,0)$, where all coherence isomorphisms are the obvious ones.

A dg-tensor functor $F: \overline{C} \to \overline{D}$ between dg-tensor categories is a dg-functor satisfying $F(X \otimes Y) \cong F(X) \otimes F(Y)$ and $F(1_C) \cong 1_D$. A tensor natural transformation $\eta: F \Rightarrow G$ of dg-tensor functors is a natural transformation of degree 0 satisfying $\eta_X \otimes \eta^Y \cong \eta_{X \otimes Y}$ and $\eta_1 \cong 1_{1_D}$.

We use the notation $[\alpha]$ for the homotopy type of a morphism $\alpha$ as well as for the cohomology class of a cocycle $\alpha$, depending on the context.

3. Affine group dg-scheme

An affine group dg-scheme over $k$ is a representable functor $\mathcal{G}: \text{ho cdgA}(k) \to \text{Grp}$ from the homotopy category $\text{ho cdgA}(k)$ of cdg-algebras over $k$ to the category $\text{Grp}$ of groups. A representing object of $\mathcal{G}$ is a cdg-Hopf algebra $B$, and we use the notation $\mathcal{G}^B$ for it. The category formed by affine group dg-schemes over $k$ is anti-equivalent to the homotopy category $\text{ho cdgH}(k)$ of cdg-Hopf algebras over $k$.

For each cdg-Hopf algebra $B$, we also construct a functor $T\mathcal{G}^B: \text{ho cdgA}(k) \to \text{Lie}(k)$ to the category $\text{Lie}(k)$ of Lie algebras so that $T\mathcal{G}^B$ and $T\mathcal{G}^{B'}$ are naturally isomorphic whenever $\mathcal{G}^B$ and $\mathcal{G}^{B'}$ are naturally isomorphic.

3.1. Representable functor $\mathcal{G}: \text{ho cdgA}(k) \to \text{Grp}$ and cdg-Hopf algebra

The purpose of this subsection is to prove the following:

**Theorem 3.1 (Definition).** For every cdg-Hopf algebra $B = (B, u_B, m_B, \epsilon_B, \Delta_B, \zeta_B, d_B)$ we have a functor $\mathcal{G}^B: \text{ho cdgA}(k) \to \text{Grp}$ represented by $B$ defined as follows:

- for each cdg-algebra $(A, u_A, m_A, d_A)$ we have a group $\mathcal{G}^B(A)$ defined by
  \[ \mathcal{G}^B(A) := \left( \text{Hom}_{\text{ho cdgA}(k)}(B,A), \epsilon_B, \ast_B, A \right), \]

with the identity element $\epsilon_B := [u_A \circ \epsilon_B]$, the group operation $[g_1] \ast_B [g_2] := [m_A \circ (g_1 \otimes g_2) \circ \Delta_B]$, and the inverse $[g]^{-1} := [g \circ \zeta_B]$ of $[g]$, where $g \in \text{Hom}_{\text{ho cdgA}(k)}(B,A)$ is an arbitrary representative of the homotopy type $[g] \in \text{Hom}_{\text{ho cdgA}(k)}(B,A)$.
– for each morphism \([f] \in \text{Hom}_{\text{cdgA}(k)}(A, A')\) we have a group homomorphism
\[
\mathfrak{G}^B([f]) : \mathfrak{G}^B(A) \to \mathfrak{G}^B(A')
\]
defined by, \(\forall [g] \in \text{Hom}_{\text{cdgA}(k)}(B, A)\),
\[
\mathfrak{G}^B([f])([g]) := [f \circ g],
\]
where \(f \in \text{Hom}_{\text{cdgA}(k)}(A, A')\) and \(g \in \text{Hom}_{\text{cdgA}(k)}(B, A)\) are arbitrary representatives of the homotopy types \([f]\) and \([g]\), respectively, such that \(\mathfrak{G}^B([f])\) is an isomorphism of groups whenever \(f : A \to A'\) is a homotopy equivalence of cdg-algebras.

For each morphism \([\psi] \in \text{Hom}_{\text{cdgH}(k)}(B, B')\) in the homotopy category of cdg-Hopf algebras, we have a natural transformation \(\mathcal{N}_{[\psi]} \in \text{Nat}(\mathfrak{G}^B, \mathfrak{G}^B)\) defined as follows: for each cdg-algebra \(A\) and for every \([g'] \in \text{Hom}_{\text{cdgA}(k)}(B', A)\) we have \(\mathcal{N}_{[\psi]}^{\mathfrak{G}^B}([g']) := [g' \circ \psi]\), where \(\psi \in \text{Hom}_{\text{cdgH}(k)}(B, B')\) and \(g' \in \text{Hom}_{\text{cdgA}(k)}(B', A)\) are arbitrary representatives of the homotopy types \([\psi]\) and \([g']\).

We have \(\text{Hom}_{\text{cdgH}(k)}(B, B') \cong \text{Nat}(\mathfrak{G}^B, \mathfrak{G}^B)\) such that \(\mathcal{N}_{[\psi]}\) is a natural isomorphism whenever \(\psi : B \to B'\) is a homotopy equivalence of cdg-Hopf algebras.

We need the forthcoming three lemmas for the proof.

**Lemma 3.1.** For every cdg-Hopf algebra \(B\) we have a functor \(\mathcal{E}^B : \text{cdgA}(k) \to \text{dgA}(k)\), sending

– each cdg-algebra \(A\) to the dg-algebra \(\mathcal{E}^B(A) := (\text{Hom}(B, A), u_A \circ \epsilon_B, \ast_{B,A}, d_{B,A})\), where
\[
\alpha_1 \ast_{B,A} \alpha_2 := m_A \circ (\alpha_1 \otimes \alpha_2) \circ \Delta_B
\]
for all \(\alpha_1, \alpha_2 \in \text{Hom}(B, A)\), and

– each morphism \(f : A \to A'\) of cdg-algebras to a morphism \(\mathcal{E}^B(f) : \mathcal{E}^B(A) \to \mathcal{E}^B(A')\) of dg-algebras defined by \(\mathcal{E}^B(f)(\alpha) := f \circ \alpha\) for all \(\alpha \in \text{Hom}(B, A)\).

**Proof.** It is a standard fact that \(\mathcal{E}^B(A)\) is a dg-algebra. We check that \(\mathcal{E}^B(f)\) is a morphism of dg-algebras, as follows:

\[
\mathcal{E}^B(f)(u_A \circ \epsilon_B) = f \circ u_A \circ \epsilon_B = u_A' \circ \epsilon_B,
\]

\[
\mathcal{E}^B(f)(d_{B,A}) = f \circ d_A = f \circ \alpha - f \circ \alpha \circ d_B = d_{B'} \circ f \circ \alpha - f \circ \alpha \circ d_{B'} = d_{B,A} \circ (\mathcal{E}^B(f)(\alpha)),
\]

\[
\mathcal{E}^B(f)(\alpha_1 \ast_{B,A} \alpha_2) = f \circ m_A \circ (\alpha_1 \otimes \alpha_2) \circ \Delta_B = m_{A'} \circ (f \circ \alpha_1 \otimes f \circ \alpha_2) \circ \Delta_B = \mathcal{E}^B(f)(\alpha_1) \ast_{B,A} \mathcal{E}^B(f)(\alpha_2),
\]

The functoriality of \(\mathcal{E}^B\) is obvious. \(\square\)

**Lemma 3.2.** For every cdg-Hopf algebra \(B\) we have a functor \(\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}\) represented by \(B\), sending

– each cdg-algebra \(A\) to a group \(\mathcal{G}^B(A) := (\text{Hom}_{\text{cdgA}(k)}(B, A), u_A \circ \epsilon_B, \ast_{B,A})\), where the inverse \(g^{-1}\) of \(g \in \text{Hom}_{\text{cdgA}(k)}(B, A)\) is \(g^{-1} := g \circ \epsilon_B\), and
– each morphism \( A \xrightarrow{f} A' \) of cdg-algebras to a morphism \( G^B(f) : G^B(A) \to G^B(A') \) of groups defined by \( G^B(f)(g) := f \circ g \) for all \( g \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \).

**Proof.** 1. We show that \( G^B(A) \) is a group for every cdg-algebra \( A \). By Lemma 3.1 the tuple \( \langle \text{Hom}(B, A), u_A \circ e_B, s_{R_A}, d_{R_A} \rangle \) is a dg algebra, and

\[
\text{Hom}_{\text{cdgA}}\langle B, A \rangle := \left\{ g \in \text{Hom}(B, A) \right\} \text{d}_{B, A} g = 0, \ a \circ u_B = u_A, \ g \circ m_B = m_A \circ (g \otimes g) \}.
\]

Therefore, all we need to check are that \( e_{B, A}, g_1 \ast_{B, A} g_2, g^{-1} \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \) and \( g \ast_{B, A} g^{-1} = g^{-1} \ast_{B, A} g = u_A \circ e_B \) for all \( g_1, g_2, g \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \).

– \( u_A \circ e_B \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \): It is trivial that \( d_{B, A}(u_A \circ e_B) = d_A \circ u_A \circ e_B - u_A \circ e_B \circ d_B = 0 \) and \( (u_A \circ e_B) \circ u_B = u_A \). We check that \( u_A \circ e_B \) is an algebra map as follows:

\[
u_A \circ e_B \circ m_B = u_A \circ m_k \circ (e_B \otimes e_B),
\]

\[
m_A \circ ((u_A \circ e_B) \otimes (u_A \circ e_B)) = m_A \circ (u_A \otimes u_A) \circ (e_B \otimes e_B) = u_A \circ m_k \circ (e_B \otimes e_B).
\]

– \( g_1 \ast_{B, A} g_2 \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \): Note that \( d_{B, A}(g_1 \ast_{B, A} g_2) = d_{B, A} g_1 \ast_{B, A} g_2 + g_1 \ast_{B, A} d_{B, A} g_2 \). The property \( g_1 \ast_{B, A} g_2 \circ u_B = u_A \) can be checked as follows:

\[
(g_1 \ast_{B, A} g_2) \circ u_B = m_A \circ (g_1 \otimes g_2) \circ \Delta_B \circ u_B = m_A \circ (g_1 \otimes g_2) \circ (u_B \otimes u_B) \circ \Delta_k
\]

\[
= m_A \circ (u_A \otimes u_A) \circ \Delta_k = u_A.
\]

Now we check that \( g_1 \ast_{B, A} g_2 \) is an algebra map. Consider

\[
(g_1 \ast_{B, A} g_2) \circ m_B = m_A \circ (g_1 \otimes g_2) \circ \Delta_B \circ m_B
\]

\[
= m_A \circ (g_1 \otimes g_2) \circ (m_B \otimes m_B) \circ (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B)
\]

\[
= m_A \circ (m_B \otimes m_A) \circ (g_1 \otimes g_2 \otimes g_2) \circ (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B).
\]

From the commutativity \( m_A \circ \tau = m_A \) of \( m_A \) we have \( m_A \circ (m_A \otimes m_A) \circ (g_1 \otimes g_2 \otimes g_2) \circ (I_B \otimes \tau \otimes I_B) = m_A \circ (m_A \otimes m_A) \circ (g_1 \otimes g_2 \otimes g_1 \otimes g_2) \), so that

\[
(g_1 \ast_{B, A} g_2) \circ m_B = m_A \circ (m_A \otimes m_A) \circ (g_1 \otimes g_2 \otimes g_2 \otimes g_2) \circ (\Delta_B \otimes \Delta_B)
\]

\[
= m_A \circ (g_1 \ast_{B, A} g_2 \otimes (u_A \ast_{B, A} g_2)).
\]

– \( g^{-1} \in \text{Hom}_{\text{cdgA}}\langle B, A \rangle \): From the condition \( d_B \circ \zeta_B = \zeta_B \circ d_B \), we have \( d_{R_A} (g^{-1}) = 0 \). It is trivial that \( g^{-1} \circ u_B = g \circ \zeta_B \circ u_B = g \circ u_B = u_A \). We check that \( g^{-1} \) is an algebra map as follows:

\[
 g^{-1} \circ m_B = g \circ \zeta_B \circ m_B = g \circ m_B \circ (\zeta_B \otimes \zeta_B) \circ \tau = g \circ m_B \circ g \circ \zeta_B
\]

\[
= m_A \circ (g \otimes g) \circ (\zeta_B \otimes \zeta_B) = m_A \circ (g^{-1} \circ g^{-1}),
\]

where we have used the commutativity \( m_B \circ \tau = m_B \) of \( m_B \).

– \( g \ast_{B, A} g^{-1} = m_A \circ (g \otimes g) \circ (I_B \otimes \zeta_B) \circ \Delta_B = g \circ m_B \circ (I_B \otimes \zeta_B) \circ \Delta_B = g \circ u_B \circ e_B = u_A \circ e_B. \)
determined by some morphisms cdg-algebras. Applying the Yoneda lemma again, a plain calculation shows that

\[ g^{-1} \circ_{B,A} g = m_A \circ (g \otimes g) \circ (\zeta_B \otimes I_B) \circ \Delta_B = g \circ m_B \circ (\zeta_B \otimes I_B) \circ \Delta_B = g \circ u_B \circ \epsilon_B = u_A \circ \epsilon_B. \]

2. We show that \( \mathcal{G}^B(f) : \mathcal{G}^B(A) \to \mathcal{G}^B(A') \) is a group homomorphism. We first check that \( \mathcal{G}^B(f)(g) \in \text{Hom}_{\text{cdgA}}(k)(B, A') \) whenever \( g \in \text{Hom}_{\text{cdgA}}(k)(B, A) \):

\[
d_B \cdot \mathcal{G}^B(f)(g) = d_B \cdot (f \circ g) = d_A \circ f \circ g - f \circ d_B = (d_A f - f \circ d_A) \circ g = 0,
\]

\[
\mathcal{G}^B(f)(g) \circ m = f \circ g \circ m = f \circ m_A \circ (g \otimes g) = m_A \circ (f \otimes f) \circ (g \otimes g)
= m_A \circ (\mathcal{G}^B(f)(g) \otimes \mathcal{G}^B(f)(g)).
\]

Now we check that \( \mathcal{G}^B(f) \) is a group homomorphism:

\[
\mathcal{G}^B(f)(u_A \circ \epsilon_B) = f \circ u_A \circ \epsilon_B = u_A' \circ \epsilon_B,
\]

\[
\mathcal{G}^B(f)(g_1 \cdot_{B,A} g_2) = f \circ (g_1 \otimes g_2) = f \circ m_A \circ (g_1 \otimes g_2) \circ \Delta_C = m_A \circ (f \otimes f) \circ (g_1 \otimes g_2) \circ \Delta_C
= (f \circ g_1) \cdot_{B,A} (f \circ g_2) = \mathcal{G}^B(f)(g_1) \cdot_{B,A} \mathcal{G}^B(f)(g_2).
\]

The functoriality of \( \mathcal{G}^B \) is obvious. \( \square \)

Lemma 3.3. Suppose \( \mathcal{G} : \text{cdgA}(k) \to \text{Grp} \) is a representable functor. Then \( \mathcal{G} \cong \mathcal{G}^B \) for some cdg-Hopf algebra \( B \).

Proof. Since the functor \( \mathcal{G} : \text{cdgA}(k) \to \text{Set} \) is representable, we have an isomorphism \( \mathcal{G} \cong \text{Hom}_{\text{cdgA}}(k)(B, -) \) for some cdg-algebra \( B \). We show that \( B \) in fact a cdg-Hopf algebra. We can restate the condition of \( \mathcal{G} \) factoring through \( \text{Grp} \) as follows:

- For each cdg-algebra \( A \) there is a structure of group on \( \mathcal{G}(A) \), i.e., there are three functions \( \mu^A : \mathcal{G}(A) \times \mathcal{G}(A) \to \mathcal{G}(A) \), \( e^A : \{\ast\} \to \mathcal{G}(A) \) and \( i^A : \mathcal{G}(A) \to \mathcal{G}(A) \) satisfying the group axioms.

- For each morphism \( f : A \to A' \) of cdg-algebras, the function \( \mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(A') \) is a homomorphism of groups.

This is equivalent to the existence of three natural transformations \( \mu, e, i : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) and \( i : \mathcal{G} \to \mathcal{G} \) satisfying the group axioms. Here, \( \{\ast\} \) is a functor \( \text{cdgA}(k) \to \text{Set} \) sending every cdg-algebra \( A \) to a one-point set \( \{\ast\} \).

Let \( B \otimes B \) be the cdg-algebra obtained by the tensor product of the cdg-algebra \( B \). We claim that there are natural isomorphisms of functors

\[
\{\ast\} \cong \text{Hom}_{\text{cdgA}}(k)(B, -), \quad \mathcal{G} \times \mathcal{G} \cong \text{Hom}_{\text{cdgA}}(k)(B \otimes B, -). \tag{3.1}
\]

Then, by the Yoneda lemma, the natural transformations \( \mu, e, i \) are completely determined by some morphisms \( \Delta_B : B \to B \otimes B, \epsilon_B : B \to k \) and \( \zeta_B : B \to B \) of cdg-algebras. Applying the Yoneda lemma again, a plain calculation shows that

1. \( \mu \circ (\mu \times 1_B) = \mu \circ (1_B \times \mu) \) implies the coassociativity of \( \Delta_B \).
Lemma 3.4. For every morphism \( \psi : B \to B' \) of cdg-Hopf algebras we have a natural transformation \( \mathcal{N}_\psi : \mathcal{G}' \Rightarrow \mathcal{G} : \text{cdgA}(\mathbb{k}) \to \text{Grp} \) functors, whose component \( \mathcal{N}_\psi^A : \mathcal{G}'^A(B) \to \mathcal{G}^A(B) \) at each cdg-algebra \( A \) is defined by \( \mathcal{N}_\psi^A(g') := g' \circ \psi \) for all \( g' \in \text{Hom}_{\text{cdgA}(\mathbb{k})}(B, B') \). We also have \( \text{Hom}_{\text{cdgA}(\mathbb{k})}(B, B') \cong \text{Nat}(\mathcal{G}'^B, \mathcal{G}^B) \).

Proof. We show that \( \mathcal{N}_\psi^A \) is a natural transformation for every \( \psi \in \text{Hom}_{\text{cdgA}(\mathbb{k})}(B, B') \), which is a linear map \( \psi : B \to B' \) satisfying the following relations:

\[
\begin{align*}
d_{B'} \circ \psi &= \psi \circ d_B, \\
\psi \circ u_B &= u_{B'}, \\
\psi \circ m_B &= m_{B'} \circ (\psi \otimes \psi), \\
\epsilon_B &= \epsilon_{B'} \circ \psi, \\
\Delta_{B'} \circ \psi &= (\psi \otimes \psi) \circ \Delta_B,
\end{align*}
\]

as follows.
Affine group dg-schemes and linear representations I

Assume that

Then we have following homotopy pairs:

These can be checked by routine computations, which are omitted for the sake of space.

Proof. These can be checked by routine computations, which are omitted for the sake of space. □
Now we are ready for the proof of Theorem 3.1.

Proof (Theorem 3.1). After Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5, we just need to check few things to finish the proof.

1. We check that the group \( \mathcal{G}^B(A) \) is well defined for every cdg-algebra \( A \).
   - We have \( g_1 \ast_{B,A} g_2 \sim \tilde{g}_1 \ast_{B,A} \tilde{g}_2 \in \text{Hom}_{cdgA[k]}(B, A) \) whenever \( g_1 \sim \tilde{g}_1, g_2 \sim \tilde{g}_2 \in \text{Hom}_{cdgA[k]}(B, A) \); this follows from Lemma 3.5(a).
   - We have \( g^{-1} \sim \tilde{g}^{-1} \in \text{Hom}_{cdgA[k]}(B, A) \) whenever \( g \sim \tilde{g} \in \text{Hom}_{cdgA[k]}(B, A) \); this follows from Lemma 3.5(b).

Also followed is that the homotopy type \( [g_1 \ast_{B,A} g_2] \) of \( g_1 \ast_{B,A} g_2 \) depends only on the homotopy types \([g_1], [g_2] \in \text{Hom}_{hoedgA[k]}(B, A)\) of \( g_1 \) and \( g_2 \). Therefore the group \( \mathcal{G}^B(A) \) is well-defined.

2. We check that the homomorphism \( \mathcal{G}^B([f]): \mathcal{G}^B(A) \to \mathcal{G}^B(A') \) of groups is well defined for every \([f] \in \text{Hom}_{hoedgA[k]}(A, A')\). Let \( f \sim \tilde{f} \in \text{Hom}_{cdgA[k]}(A, A') \) and \( g \sim \tilde{g} \in \text{Hom}_{cdgA[k]}(B, A) \). Then, by Lemma 3.5(b), we have \( f \circ g \sim \tilde{f} \circ \tilde{g} \sim \tilde{f} \circ g \sim f \circ g \) that \( \mathcal{G}^B([f])([g]) = [f \circ g] \) depends only on the homotopy types \([f] \) and \([g]\). Therefore \( \mathcal{G}^B([f]) \) is a well defined group homomorphism. It is obvious that \( \mathcal{G}^B([f]) \) is an isomorphism of groups whenever \( f: A \to A' \) is a homotopy equivalence of cdg-algebras.

3. We check that the natural transformation \( \mathcal{N}_{[\psi]} : \mathcal{G}^{B'} \to \mathcal{G}^B : \text{hoedgA(k)} \to \text{Grp} \) is well-defined for every \([\psi] \in \text{Hom}_{hoedgH[k]}(B, B')\). Let \( \psi \sim \tilde{\psi} \in \text{Hom}_{cdgH[k]}(B, B') \) and \( g' \sim \tilde{g}' \in \text{Hom}_{cdgA[k]}(B', A) \). Then, by Lemma 3.5(d), we have \( g' \circ \psi \sim \tilde{g}' \circ \tilde{\psi} \sim \tilde{g}' \circ \psi \sim \tilde{\psi} \circ \tilde{\psi} \in \text{Hom}_{cdgA[k]}(B, A) \) so that \( \mathcal{N}_{[\psi]}([g']) = [g' \circ \psi] \) for every cdg-algebra \( A \) depends only on the homotopy types \([\psi] \) and \([g']\). Therefore the natural transformation \( \mathcal{N}_{[\psi]} : \mathcal{G}^{B'} \to \mathcal{G}^B \) is well-defined such that \( \mathcal{N}_{[\psi]} \in \text{Nat}(\mathcal{G}^{B'}, \mathcal{G}^B) \) whenever \([\psi] \in \text{Hom}_{hoedgH[k]}(B, B')\). Combined with the Yoneda lemma, we have

\[
\text{Nat}(\mathcal{G}^{B'}, \mathcal{G}^B) \cong \text{Hom}_{hoedgH[k]}(B, B').
\]

That is, the category of affine group dg-schemes over \( k \) is anti-equivalent to the homotopy category \( \text{hoedgH[k]} \) of cdg-Hopf algebras over \( k \). It is obvious that \( \mathcal{N}_{[\psi]} : \mathcal{G}^{B'}(A) \to \mathcal{G}^B(A) \) is an isomorphism of groups for every cdg-algebra \( A \) whenever \( \psi : B \to B' \) is a homotopy equivalence of cdg-Hopf algebras. Therefore \( \mathcal{N}_{[\psi]} \) is a natural isomorphism whenever \( \psi \) is a homotopy equivalence of cdg-Hopf algebra. \( \Box \)
3.2. \( \mathcal{G}(k) \) action on the affine dg-scheme \( \hat{\mathcal{G}} : \text{hodgA}(k) \to \text{Set} \).

Fix an affine group dg-scheme \( \mathcal{G} : \text{hodgA}(k) \to \text{Grp} \) over \( k \). Note that the ground field \( k \), as a cdg-algebra with zero differential, is an initial object in \( \text{hodgA}(k) \). Any cdg-algebra \( A = (A, u_A, m_A, d_A) \) comes with the cdg-algebra map \( u_A \in \text{Hom}_{\text{cdgA}}(k, A) \), which induces a canonical group homomorphism \( \mathcal{G}([u_A]) : \mathcal{G}(k) \to \mathcal{G}(A) \).

**Lemma 3.6.** For every affine group dg-scheme \( \mathcal{G} \) the underlying representable functor

\[
\mathcal{G} = \text{Hom}_{\text{hodgA}(k)}(B, -) : \text{hodgA}(k) \to \text{Set},
\]

is \( \mathcal{G}(k) \)-set valued, such that for each cdg-algebra \( A \), \( \mathcal{G}(A) \) is a \( \mathcal{G}(k) \)-set with the action \( r^A : \mathcal{G}(k) \times \mathcal{G}(A) \to \mathcal{G}(A) \), defined by, for all \( [g] \in \mathcal{G}(k) \) and \( [x] \in \mathcal{G}(A) \),

\[
r^A([g], [x]) := \mathcal{G}(B)((u_A)([g])) \ast_{B, A} [x] = [g \cdot x], \quad g \cdot x = (u_A \circ g) \ast_{B, A} x,
\]

where \( g \) and \( x \) are representatives of \( [g] \) and \( [x] \), respectively, and for every morphism \( [f] \in \text{Hom}_{\text{hodgA}(k)}(A, A') \) the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{G}(k) \times \mathcal{G}(A) & \xrightarrow{r^A} & \mathcal{G}(A) \\
\downarrow_{\text{id} \times \mathcal{G}([f])} & & \downarrow_{\mathcal{G}([f])} \\
\mathcal{G}(k) \times \mathcal{G}(A') & \xrightarrow{r^{A'}} & \mathcal{G}(A').
\end{array}
\]

**Proof.** This is a corollary of Theorem 3.1, whose details are omitted. \( \square \)

3.3. Reduction to an affine group scheme

Let \( H(B) \) be the cohomology of cdg-Hopf algebra \( B \). Then, \( H(B) \) has a uniquely induced structure of cdg-Hopf algebra with the zero-differential, since every structure in \( B \) is compatible with the differential \( d_B \). In particular the 0-th cohomology \( H^0(B) \) is a commutative Hopf algebra. Here we compare the affine group dg-scheme \( \mathcal{G} : \text{hodgA}(k) \to \text{Grp} \) with the affine group scheme \( \text{Grp}^{H^0(B)} : \text{cAlg}(k) \to \text{Grp} \) represented by \( H^0(B) \), where \( \text{cAlg}(k) \) is the category of commutative algebras over \( k \). Note that \( \text{cAlg}(k) \) is a full subcategory of \( \text{hodgA}(k) \).

The cochain complex \( (B, d_B) \) is over a field \( k \) so that it splits. A choice of splitting, which is an \( \chi \in \text{Hom}(B, B)^{-1} \) satisfying \( d_B = d_B \circ \chi \circ d_B \), provides us with a strong deformation retract:

\[
\begin{align*}
(H(B), 0) & \xrightarrow{p} (B, d_B), \\
& \quad \begin{cases}
p \circ p = \text{Id}_{H(B)}, \\
p \circ q = d_B \circ \chi - \chi \circ d_B,
\end{cases}
\end{align*}
\]
where \( p \) and \( q \) are homotopy equivalences. We can always choose \( \chi \) such that \( \chi \circ u_B = 0 \). Then, \( H(B) \) has uniquely determined, independent to the choice of splitting, structure of \( \mathbb{Z} \)-graded commutative Hopf algebra with the unit \( u_{H(B)} \), the counit \( \epsilon_{H(B)} \), the product \( m_{H(B)} \), the coproduct \( \Delta_{H(B)} \) and the antipode \( \zeta_{H(B)} \) defined as follows:

\[
\begin{align*}
    u_{H(B)} &= q \circ u_B, \\
    \epsilon_{H(B)} &= \epsilon_B \circ p, \\
    m_{H(B)} &= q \circ m_B \circ (p \otimes p), \\
    \Delta_{H(B)} &= (q \otimes q) \circ \Delta_B \circ p, \\
    \zeta_{H(B)} &= q \circ \zeta_B \circ p.
\end{align*}
\]

On the other hand the cochain maps \( p : H(B) \to B \) and \( q : B \to H(B) \) are morphisms of cdg-Hopf algebras only up to homotopy; it may preserve unit and counit but is an algebra map only up to homotopy, coalgebra map only up homotopy and commutes with antipodes only up to homotopy.

Let \( A \) is a commutative algebra. Note that \( \alpha \in \text{Hom}(B, A)^0 \) is a zero-map on \( B^k \) unless \( k = 0 \) since the degree of \( A \) is concentrated to zero. We compare \( \text{Hom}_{\text{cdgAlg}[k]}(B, A) \) with \( \text{Hom}_{\text{cdgAlg}[k]}(H(B), A) \equiv \text{Hom}_{\text{Alg}}(H^0(B), A) \). Consider \( g \in \text{Hom}_{\text{cdgAlg}[k]}(B, A) \). It is apparent that \( g \) induces a morphism of algebra \( H(g) \in \text{Hom}_{\text{Alg}}(H^0(B), A) \) and \( H(g) = H(g) \), whenever \( g \sim \tilde{g} \in \text{Hom}_{\text{cdgAlg}[k]}(B, A) \). However, we do not know if every element of \( \text{Hom}_{\text{Alg}}(H^0(B), A) \) is obtained in the above manner: for any \( a \in \text{Hom}_{\text{Alg}}(H^0(B), A) \) we have \( a = a \circ q \in \text{Hom}(B, A)^0 \) such that \( H(a) = a, d_B A a = 0, a \circ u_B = u_A \), while \( \alpha \) is an algebra map only up to homotopy since \( q_0 : B^0 \to H^0(B) \) is an algebra map only up to homotopy. Nevertheless, the following limited statement is valid.

**Lemma 3.7.** For every cdg-Hopf algebra \( B \) whose degrees are concentrated to non-negative integers, the group \( \mathcal{G}^B(A) \) is isomorphic to the group \( \mathcal{G}^{H^0(B)}(A) \) for every commutative algebra \( A \).

**Proof.** Because of the degree reason \( \chi \) is the zero map on \( B^0 \), i.e, \( \chi_0 = 0 \). Therefore we have \( q_0 \circ p_0 = 1_{H^0(B)} \) and \( p_0 \circ q_0 = 1_{B^0} - \chi_1 \circ d_{B^0} \), which imply that both \( p_0 : H^0(B) \to B^0 \) and \( q_0 : B^0 \to H^0(B) \) are Hopf algebra maps. \( \square \)

**Corollary 3.1.** For every cdg-Hopf algebra \( B \) with the degrees concentrated to non-negative integers, the Lie algebra \( T \mathcal{G}^B(A) \) is isomorphic to the Lie algebra \( T \mathcal{G}^{H^0(B)}(A) \) for every commutative algebra \( A \).

### 3.4. Functor \( T \mathcal{G} : \text{ho cdgAlg}(k) \to \text{Lie}(k) \)

For each cdg-Hopf algebra \( B \) we construct a functor \( T \mathcal{G}^B : \text{ho cdgAlg}(k) \to \text{Lie}(k) \) to the category \( \text{Lie}(k) \) of Lie algebras over \( k \), so that we have a natural isomorphism \( T \mathcal{G}^B \cong T \mathcal{G}^{B'} \) of functors whenever \( \mathcal{G}^B \cong \mathcal{G}^{B'} \). The assignment \( \mathcal{G}^B \to T \mathcal{G}^B \) for every cdg-Hopf algebra \( B \) is functorial – there is a functor \( T \) from the category of affine group dg-schemes to the category of functors from \( \text{ho cdgAlg}(k) \) to \( \text{Lie}(k) \).
For each cdg-algebra $A$, we define $\text{THom}_{\text{cdgA}[k]}(B, A)$ as the set of all tangential morphisms of cdg-algebras about the identity $e_{BA} := u_A \circ e_B$:

$$\text{THom}_{\text{cdgA}[k]}(B, A) = \left\{ u \in \text{Hom}(B, A) \mid \begin{array}{l} d_{BA} u = 0 \\ u \circ u_B = 0 \\ u \circ m_B = m_A \circ (e_{BA} \otimes u + u \otimes e_{BA}) \end{array} \right\}.$$  

A homotopy pair on $\text{THom}_{\text{cdgA}[k]}(B, A)$ is a pair of families

$$(u(t), \sigma(t)) \in \text{Hom}(B, A)^0[t] \oplus \text{Hom}(B, A)^{-1}[t],$$

satisfying the homotopy flow equation $\frac{d}{dt} u(t) = d_{BA} \sigma(t)$ generated by $\sigma(t)$ subject to the following conditions:

$$u(0) \in \text{THom}_{\text{cdgA}[k]}(B, A), \quad \left\{ \begin{array}{l} \sigma(t) \circ u_B = 0 \\ \sigma(t) \circ m_B = m_A \circ (e_{BA} \otimes \sigma(t) + \sigma(t) \otimes e_{BA}) \end{array} \right\}.$$

Then, $u(t)$ is a family of tangential morphisms of cdg-algebras about the identity. We say $u, \tilde{u} \in \text{THom}_{\text{cdgA}[k]}(B, A)$ are homotopic, $u \sim \tilde{u}$, or having the same homotopy type, $[u] = [\tilde{u}]$, if there is a homotopy flow connecting them. The set of homotopy types of all tangential morphisms of cdg-algebras about the identity is denoted by $\text{THom}_{\text{ho cdgA}[k]}(B, A)$.

**Theorem 3.2.** For every cdg-Hopf algebra $B$ we have a functor $T\mathfrak{g}^B : \text{ho cdgA}[k] \to \text{Lie}(k)$, sending

- each cdg-algebra $A$ to the Lie algebra

$$T\mathfrak{g}^B(A) := \left( \text{THom}_{\text{ho cdgA}[k]}(B, A), [-, -]_{B,A} \right)$$

with the Lie bracket defined by, $\forall [v_1], [v_2] \in (\text{THom}_{\text{ho cdgA}[k]}(B, A),$

$$[[v_1], [v_2]]_{B,A} := [m_A \circ (v_1 \otimes v_2 - v_2 \otimes v_1) \circ \Delta_B],$$

where $v_1, v_2 \in \text{THom}_{\text{cdgA}[k]}(B, A)$ are arbitrary representatives of the homotopy types $[v_1], [v_2]$, respectively.

- each morphism $[f] \in \text{Hom}_{\text{ho cdgA}[k]}(A, A')$ to the morphism $T\mathfrak{g}^B([f]) : T\mathfrak{g}^B(A) \to T\mathfrak{g}^B(A')$ of Lie algebras defined by, $\forall [v] \in \text{THom}_{\text{ho cdgA}[k]}(B, A),$

$$T\mathfrak{g}^B([f])([v]) := [f \circ v],$$

where $f$ and $v$ are arbitrary representatives of the homotopy types $[f]$ and $[v]$, respectively.
For every \( \psi \in \text{Hom}_{\text{cdgA}(k)}(B, B') \) we have a natural transformation

\[
T_N[\psi] : T \Theta^B' \implies T \Theta^B : \text{cdgA}(k) \to \text{Lie}(k),
\]
whose component \( T_N^A[\psi] : T \Theta^B'(A) \to T \Theta^B(A) \) at each cdg-algebra \( A \) is defined by, for all \( [\nu'] \in \text{THom}_{\text{cdgA}(k)}(B', A) \),

\[
T_N^A[\psi](\nu') := [\nu' \circ \psi],
\]
where \( \psi \in \text{Hom}_{\text{cdgA}(k)}(A, A') \) and \( \nu' \in \text{THom}_{\text{cdgA}(k)}(B', A') \) are arbitrary representatives of the homotopy types \( [\psi] \) and \( [\nu'] \), respectively.

The natural transformation \( T_N[\psi] : T \Theta^B' \implies T \Theta^B \) is an isomorphism whenever \( \psi : B \to B' \) is a homotopy equivalence of cdg-Hopf algebras.

The basic idea of proof is to define a functor \( T \mathcal{G}^B : \text{cdgA}(k) \to \text{Lie}(k) \) from \( \text{cdgA}(k) \) and show that \( T \mathcal{G}^B \) induces the functor \( T \Theta^B : \text{cdgA}(k) \to \text{Lie}(k) \) on \( \text{cdgA}(k) \). Our proof shall be a consequence of three lemmas, which will be stated without proofs.

**Lemma 3.8.** For every cdg-Hopf algebra \( B \) we have a functor \( T \mathcal{G}^B : \text{cdgA}(k) \to \text{Lie}(k) \), sending each cdg-algebra \( A \) to the Lie algebra

\[
T \mathcal{G}^B(A) := \left( \text{THom}_{\text{cdgA}(k)}(B, A), [-, -], \sigma_{BA} \right),
\]
where \( [\nu_1, \nu_2]_{BA} := \nu_1 \ast_B A \nu_2 - \nu_2 \ast_B A \nu_1 \) for all \( \nu_1, \nu_2 \in \text{THom}_{\text{cdgA}(k)}(B, A) \), and each morphism \( f : A \to A' \) of cdg-algebras to the Lie algebra homomorphism

\[
T \mathcal{G}^B(f) : T \mathcal{G}^B(A) \to T \mathcal{G}^B(A')
\]
defined by \( T \mathcal{G}^B(f)(\nu) := f \circ \nu \) for all \( \nu \in \text{THom}_{\text{cdgA}(k)}(B, A) \).

**Lemma 3.9.** For each morphism \( \psi : B \to B' \) of cdg-Hopf algebras we have a natural transformation \( T_N[\psi] : T \mathcal{G}^B' \implies T \mathcal{G}^B : \text{cdgA}(k) \to \text{Lie}(k) \), whose component \( T_N^A[\psi] : T \mathcal{G}^B'(A) \implies T \mathcal{G}^B(A) \) at each cdg-algebra \( A \) is defined by \( T_N^A[\psi](\nu') := \nu' \circ \psi \) for all \( \nu' \in \text{THom}_{\text{cdgA}(k)}(B, A) \).

**Lemma 3.10.** Assume that

- \((\nu(t), \sigma(t)) \in \text{THom}_{\text{cdgA}(k)}(B, A)\);
- \((f(t), \lambda(t)) \) is a homotopy pair on \( \text{Hom}_{\text{cdgA}(k)}(A, A') \);
- \((\psi(t), \xi(t)) \) is a homotopy pair on \( \text{Hom}_{\text{cdgH}(k)}(B, B') \);
- \((\nu'(t), \sigma'(t)) \) is a homotopy pair on \( \text{THom}_{\text{cdgA}(k)}(B', A) \).
Then we have following homotopy pairs

(a) $\left(\left[\left(v_1(t), v_2(t)\right)_*|_{\mathcal{A}} + \left[\left(\sigma_1(t), \sigma_2(t)\right)_*|_{\mathcal{A}}\right]\right)\right)$ on $\text{THom}_{\text{cdgA}}(k)(B, A)$.

(b) $\left\{f(t) \circ v(t), f(t) \circ \sigma(t) + \lambda(t) \circ v(t)\right\}$ on $\text{THom}_{\text{cdgA}}(k)(B, A')$.

(c) $\left\{v'(t) \circ \psi(t), v'(t) \circ \zeta(t) + \sigma'(t) \circ \psi(t)\right\}$ on $\text{Hom}_{\text{cdgA}}(k)(B, A)$.

Now we are ready for the proof of Theorem 3.2

Proof (Theorem 3.2). From Lemmas 3.8, 3.10(a) and 3.10(b), it is trivial to check that the Lie algebra $T \mathfrak{g}^B(A)$ is well-defined, and $T \mathfrak{g}^B([f]): T \mathfrak{g}^B(A) \to T \mathfrak{g}^B(A')$ is a well-defined Lie algebra homomorphism. Therefore $T \mathfrak{g}^B$ is a functor from the homotopy category $\text{ho cdgA}(k)$ of cdg-algebras as claimed, where the functoriality of $T \mathfrak{g}^B$ is obvious. It is also obvious that $T \mathfrak{g}^B([f])$ is an isomorphism of Lie algebras whenever $f: A \to A'$ is a homotopy equivalence of cdg-algebras.

From Lemmas 3.9 and 3.10(c), it is also trivial to check that the natural transformation $T \mathcal{N}_{[\psi]}: T \mathfrak{g}^B \Rightarrow T \mathfrak{g}^B: \text{ho cdgA}(k) \to \text{Lie}(k)$ is well-defined for every $[\psi] \in \text{Hom}_{\text{cdgH}(k)}(B, B')$. Finally it is obvious that $T \mathcal{N}_{[\psi]}$ is a natural isomorphism whenever $\psi: B \to B'$ is a homotopy equivalence of cdg-Hopf algebras. □

Remind that the category of affine group dg-scheme is anti-equivalent to the homotopy category $\text{ho cdgH}(k)$ of cdg-Hopf algebras—Theorem 3.1. Therefore, the assignments $\mathfrak{g}^B \to T \mathfrak{g}^B$ and $\mathcal{N}_{[\psi]} \to T \mathcal{N}_{[\psi]}$ define a contravariant functor $T$ from the category of affine group dg-schemes to the category of functors from $\text{ho cdgA}(k)$ to $\text{Lie}(k)$, where $T \mathcal{N}_{[\psi]}$ is a natural isomorphism whenever $\mathcal{N}_{[\psi]}$ is a natural isomorphism.

3.5. Pro-unipotent affine group dg-scheme

We say an affine group dg-scheme $\mathfrak{g}^B$ pro-unipotent if the cdg-Hopf algebra $B$ is conilpotent. Then, we shall show that the underlying set-valued functors

$$
\hat{\mathfrak{g}}^B: \text{ho cdgA}(k) \to \text{Set} \quad \text{and} \quad \hat{T}\mathfrak{g}^B: \text{ho cdgA}(k) \to \text{Set}
$$

are naturally isomorphic—we can recover the group $\mathfrak{g}^B(A)$ from the Lie algebra $T \mathfrak{g}^B(A)$ for every cdg-algebra $A$.

Consider a cdg-Hopf algebra $B = (B, u_B, m_B, \epsilon_B, m_B, \zeta_B, d_B)$. Let $\hat{B} = \ker \epsilon_B$ be the kernel of the counit $\epsilon_B: B \to k$. Then we have the splitting $B = k \cdot u_B(1) \oplus \hat{B}$ since $\epsilon_B \circ u_B = 1_k$. Define the linear map $\hat{\Delta}_B: B \to B \otimes B$ such that, $\forall x \in B$,

$$
\hat{\Delta}_B(x) := \Delta_B(x) - u_B(1) \otimes x - x \otimes u_B(1).
$$
From the properties of counit \( \epsilon_B \) we have, \( \forall x \in B \),
\[
(\epsilon_B \otimes \epsilon_B) \circ \Delta_B(x) := \Delta_B(\epsilon_B(x)) - 1 \otimes \epsilon_B(x) - \epsilon_B(x) \otimes 1,
\]
\[
\epsilon_B(d_B(x)) = d_B(\epsilon_B(x)).
\]
Therefore we have \( \Delta_B(B) \subset \bar{B} \otimes \bar{B} \) and \( d_B(\bar{B}) \subset \bar{B} \) so that \( (\bar{B}, \Delta_B, d_B) \) is a non-counital dg coalgebra.

Consider the \( n \)-fold iterated reduced coproduct \( \Delta_B^{(n)} : B \to \bar{B}^\otimes n \), \( n \geq 1 \), generated by \( \Delta_B \), where \( \Delta_B^{(1)} = I_B \) and \( \Delta_B^{(n+1)} = (\Delta_B^{(n)} \otimes I_B) \circ \Delta_B \). Let \( \bar{B}_n \subset \bar{B} \) be the kernel of \( \Delta_B^{(n)} \).

Then, we have the filtration
\[
0 = \bar{B}_0 \subset \bar{B}_1 \subset \bar{B}_2 \subset \bar{B}_3 \subset \cdots,
\]
satisfying \( \Delta_B(\bar{B}_n) \subset \sum_{i=0}^n \bar{B}_i \otimes \bar{B}_{n-i} \) and \( d_B(\bar{B}_n) \subset \bar{B}_n \).

**Definition 3.1.** A cdg-Hopf algebra \( B \) is conilpotent if \( \bar{B} \) is the union \( \cup_{n=0}^{\infty} \bar{B}_n \), i.e., for any \( x \in B \) there is some positive integer \( n \) such that \( \Delta_B^{(n+1)}(x) = 0 \). An affine group dg-scheme \( \mathfrak{G}^B \) is pro-unipotent if the cdg-Hopf algebra \( B \) is conilpotent.

The notion of conilpotent cdg-Hopf algebras is the straightforward dg-version of conilpotent commutative Hopf algebra [1].

We introduce some new notations. Let \( \Delta_B^{(n)} : B \to B^\otimes n \), \( n \geq 1 \), be the \( n \)-fold iterated coproduct generated by \( \Delta_B \), where \( \Delta_B^{(1)} := I_B \) and \( \Delta_B^{(n+1)} := (\Delta_B^{(n)} \otimes I_B) \circ \Delta_B \). Let \( \Delta_{B \otimes B} : B \otimes B \to (B \otimes B)^\otimes 2 \), \( n \geq 1 \), be the \( n \)-fold iterated coproduct generated by \( \Delta_{B \otimes B} = (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B) : B \otimes B \to (B \otimes B)^\otimes 2 \), where \( \Delta_B^{(1)} := I_{B \otimes B} \) and \( \Delta_B^{(n+1)} := (\Delta_B^{(n)} \otimes I_{B \otimes B}) \circ \Delta_{B \otimes B} \). Then it is trivial to show that, \( \forall n \geq 1 \),
\[
\Delta_B^{(n)} \circ m_B = (m_B \otimes \ldots \otimes m_B) \circ \Delta_{B \otimes B}^{(n)}.
\]

Let \( B \) be a conilpotent cdg-Hopf algebra, \( \mathfrak{G}^B : \text{hodg}A(k) \to \text{Grp} \) be the pro-unipotent affine group dg-scheme represented by \( B \) and \( T\mathfrak{G}^B : \text{hodg}A(k) \to \text{Lie}(k) \) be the associated Lie algebra valued functor. We shall show that one can recover \( \mathfrak{G}^B \) from \( T\mathfrak{G}^B \).

For the precise statement we need some notations. Let \( A = (A, u_A, m_A, d_A) \) be a cdg-algebra and \( m^{(n)}_\Omega : \Omega^\otimes n \to \Omega, n \geq 1 \), be the \( n \)-fold iterated product generated by \( m_\Omega : \Omega \otimes \Omega \to \Omega \) such that \( m^{(1)}_\Omega = I_\Omega \) and \( m^{(n+1)}_\Omega = m_\Omega \circ (m^{(n)}_\Omega \otimes I_\Omega) = m^{(n)}_\Omega \circ (I_{\Omega}^{n-1} \otimes m_\Omega) \).

Consider the underlying set-valued functors of the functors \( \mathfrak{G}^B \) and \( T\mathfrak{G}^B \):
\[
\mathfrak{G}^B = \text{Hom}_{\text{hodg}A(k)}(B, -) : \text{hodg}A(k) \to \text{Set},
\]
\[
T\mathfrak{G}^B = \text{THom}_{\text{hodg}A(k)}(B, -) : \text{hodg}A(k) \to \text{Set}.
\]
Theorem 3.3. For every conilpotent cdg-Hopf algebra $B$ we have an natural isomorphism $\exp\circ \ln : \mathbf{G}^B \cong \mathcal{T}^B$ of functors whose components $\exp_A : \mathcal{T}^B(A) \cong \mathbf{G}^B(A)$ for each cdg-algebra $A$ is given by

$$\exp_A([v]) := [u_A \circ \epsilon_B] + \sum_{n=1}^{\infty} \frac{1}{n!} [m_A^{(n)} \circ (v \otimes \ldots \otimes v) \circ \Delta_B^{(n)}],$$

$$\ln_A([g]) := - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} [m_A^{(n)} \circ (\bar{g} \otimes \ldots \otimes \bar{g}) \circ \Delta_B^{(n)}],$$

where $g \in \text{Hom}_{cdgA}(B, A)$ and $\nu \in \text{THom}_{cdgA}(B, A)$ are arbitrary representatives of the homotopy types $[g]$ and $[\nu]$, respectively; and $\bar{g} := g - u_A \circ \epsilon_B$.

The above theorem implies that the affine group dg-scheme $\mathbf{G}^B$ can be recovered from the Lie algebra valued functor $\mathcal{T} \mathbf{G}^B$ by Baker-Campbell-Hausdorff formula. We divide the proof into pieces, and begin with two technical lemmas.

Lemma 3.11. For every cdg-algebra $A$ we have

(a) $\alpha_1 \ast_B \cdots \ast_B \alpha_n = m_A^{(n)}(\alpha_1 \otimes \cdots \otimes \alpha_n) \circ \Delta_B^{(n)}$, $\forall \alpha_1, \ldots, \alpha_n \in \text{Hom}(B, A)$ and $n \geq 1$;

(b) $m_A \circ m_A^{(n)} = m_A^{(n)}(m_A \otimes \cdots \otimes m_A)$ for all $n \geq 1$.

Proof. The property (a) is trivial for $n = 1$. It is the definition of the convolution product $\ast_B$ for $n = 2$ and the rest can be checked easily by an induction from the associativity of $m_A$ and the coassociativity of $\Delta_B$. The property (b) is trivial for $n = 1$, since $m_A^{(1)} := I_{A \otimes A}$ and $m_A^{(1)} := I_A$. For $n = 2$, from $m_A^{(2)} := m_{A \otimes A} = (m_A \otimes m_A) \circ (I_A \otimes \tau \otimes I_A)$ and $m_A^{(2)} := m_A$ it becomes the identity

$$m_A \circ (m_A \otimes m_A) \circ (I_A \otimes \tau \otimes I_A) = m_A \circ (m_A \otimes m_A),$$

which is valid due to the commutativity of $m_A$. The rest can be checked easily by an induction. □

Lemma 3.12. For every $\beta \in \text{Hom}(B, A)^0$ satisfying $\beta \circ u_B = 0$ and for all $n \geq 1$, we have

$$m_A^{(n)}(\beta \otimes \cdots \otimes \beta) \circ \Delta_B^{(n)} = m_A^{(n)}(\beta \otimes \cdots \otimes \beta) \circ \Delta_B^{(n)}.$$  

Proof. This is an immediate consequence of the splitting $B = k \cdot u_B(1) \oplus \bar{B}$ and the identity $m_A^{(n)}(\beta \otimes \cdots \otimes \beta) \circ \Delta_B^{(n)} \circ u_B = 0$ for all $n \geq 1$, which follows from the properties that $\Delta_B^{(n)} \circ u_B = (u_B \otimes \cdots \otimes u_B) \circ \Delta_B^{(n)}$ and $\beta \circ u_B = 0$. □

Here is the main propositions for the proof.
Proposition 3.1. We have an isomorphism \( \THom_{\text{cdgA}(k)}(B, A) \xrightarrow{\exp_A} \Hom_{\text{cdgA}(k)}(B, A) \)
for every cdg-algebra \( A \), where \( \forall v \in \THom_{\text{cdgA}(k)}(B, A) \) and \( \forall g \in \Hom_{\text{cdgA}(k)}(B, A) \)
\[
\exp_C(v) := u_A \circ e_B + \sum_{n=1}^{\infty} \frac{1}{n!} m_A^{(n)} \circ (v \otimes \ldots \otimes v) \circ \Delta_B^{(n)},
\]
\[
\ln_A(g) := -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} m_A^{(n)} \circ (\tilde{g} \otimes \ldots \otimes \tilde{g}) \circ \Delta_B^{(n)}
\]
such that \( \exp_A(v) \sim \exp_A(\tilde{v}) \in \Hom_{\text{cdgA}(k)}(B, A) \) whenever \( v \sim \tilde{v} \in \THom_{\text{cdgA}(k)}(B, A) \),
and \( \ln_A(g) \sim \ln_A(\tilde{g}) \in \THom_{\text{cdgA}(k)}(B, A) \) whenever \( g \sim \tilde{g} \in \Hom_{\text{cdgA}(k)}(B, A) \).

Proof. We use some shorthand notations. We set \( e = u_A \circ e_B \). We also set \( \ast = \ast_{B,A} \),
\( \ast^0 = e \) and \( \ast^n = \underbrace{a \ast \ldots \ast a}_{n} \), \( n \geq 1 \), for all \( a \in \Hom(B, A)^0 \). Then, by Lemma 3.11(a),
we have
\[
\exp_A(v) = e + \sum_{n=1}^{\infty} \frac{1}{n!} v \ast^n = \sum_{n=0}^{\infty} \frac{1}{n!} v \ast^n, \quad \ln_A(g) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} g \ast^n. \quad (3.3)
\]
Remind that \( e \in \Hom_{\text{cdgA}(k)}(B, A) \), i.e., \( d_{B,A} e = 0 \), \( e \circ u_B = u_A \) and \( e \circ m_B = m_A \circ (e \otimes e) \).

1. We have to justify that the infinite sums in the definition of \( \exp_A \) and \( \ln_A \) make sense. Note the \( v \circ u_B = 0 \) by definition. We also have \( \tilde{g} \circ u_B = 0 \) since \( g \circ u_B = u_A \)
and \( \tilde{g} = g - e \). Then, by Lemma 3.12 and the conilpotency of \( \Delta_B \) the conilpotency of \( \Delta_B \), both \( \exp_A(v) \) and \( \ln_A(a) \) are finite sums.

2. We check that \( \exp_A(v) \in \Hom_{\text{cdgA}(k)}(B, A) \) for every \( v \in \THom_{\text{cdgA}(k)}(B, A) \):
\[
d_{B,A} \exp_A(v) = 0, \quad \exp_C(v) \circ u_B = u_A, \quad \exp_A(v) \circ m_B = m_A \circ (\exp_A(v) \otimes \exp_A(v)).
\]
The 1st relation is trivial since \( d_{B,A} \) is a derivation of \( \ast \) and \( d_{B,A} e = d_{B,A} v = 0 \). The 2nd relation is also trivial since \( u_B = u_A \) while \( v \ast^n \circ u_B = (v \circ u_B) \ast^n = 0 \) for all \( n \geq 1 \).
It remains to check the 3rd relation, which is equivalent to the following relations, \( \forall n \geq 0 \),
\[
v \ast^n \circ m_B = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} m_A \circ (v \ast^{n-k} \otimes v \ast k). \quad (3.4)
\]
For \( n = 0 \) the above becomes \( e \circ m_B = m_A \circ (e \otimes e) \), which is trivial.

For cases with \( n \geq 1 \), we adopt an additional notation. Remind that \( (B \otimes B, e_{B \otimes B}, \Delta_{B \otimes B}) \)
is a \( \mathbb{Z} \)-graded coassociative coalgebra and \( (A \otimes A, u_{A \otimes A}, m_{A \otimes A}) \) is a \( \mathbb{Z} \)-graded supercommutative associative algebra. Therefore we have a \( \mathbb{Z} \)-graded associative algebra \( \{ \Hom(B \otimes B, A \otimes A), e \otimes e, \ast \} \), where \( \chi_1 \ast \chi_2 := m_{A \otimes A} \circ (\chi_1 \otimes \chi_2) \circ \Delta_{B \otimes B}, \forall \chi_1, \chi_2 \in \Hom(B \otimes B, A \otimes A) \).
We also have, for all \( n \geq 1 \) and \( \chi_1, \ldots, \chi_n \in \Hom(B \otimes B, A \otimes A) \),
\[
\chi_1 \ast \ldots \ast \chi_n = m_{A \otimes A}^{(n)} \circ (\chi_1 \otimes \ldots \otimes \chi_n) \ast \Delta_{C \otimes C}^{(n)}. \quad (3.5)
\]
For example, consider $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \text{Hom}(B, A)$ so that $\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \in \text{Hom}(B \otimes B, A \otimes A)$. Then we have $(\alpha_1 \otimes \beta_1) \circ (\alpha_2 \otimes \beta_2) = \alpha_1 \ast \alpha_2 \otimes \beta_1 \ast \beta_2$.

It follows that $(v \otimes e) \circ (e \otimes v) = (e \otimes v) \circ (v \otimes e)$ since the both terms are $v \otimes v$. We also have $(v \otimes e)^{\otimes n} = v^{\ast n} \otimes e$ and $(e \otimes v)^{\otimes n} = e \otimes v^{\ast n}$. Combined with the binomial identity, we obtain that, $\forall n \geq 1$,

$$(v \otimes e + e \otimes v)^{\otimes n} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(v^{\ast n-k} \otimes v^{\ast k}).$$

Therefore the RHS of eq. (3.4) becomes

$$\text{RHS} := m_A \circ (v \otimes e + e \otimes v)^{\otimes n} = m_A \circ m_A^{(n)} \circ (v \otimes e + e \otimes v)^{\otimes n} \circ \Delta_{BBB}^{(n)}$$

$$= m_A^{(n)} \circ (m_A \otimes \ldots \otimes m_A) \circ (v \otimes e + e \otimes v)^{\otimes n} \circ \Delta_{BBB}^{(n)},$$

where we use Lemma 3.11(b) for the last equality. Consider the LHS of eq. (3.4):

$$\text{LHS} := v^{\ast n} \circ m_B = m_A^{(n)} \circ (v \otimes \ldots \otimes v) \circ \Delta_{BBB}^{(n)} \circ m_B$$

$$= m_A^{(n)} \circ (v \circ m_B \otimes \ldots \otimes v \circ m_B) \circ \Delta_{BBB}^{(n)}$$

$$= m_A^{(n)} \circ (m_A \otimes \ldots \otimes m_A) \circ (v \otimes e + e \otimes v)^{\otimes n} \circ \Delta_{BBB}^{(n)},$$

where we use eq. (3.2) for the 3rd equality and the property $v \circ m_B = m_A \circ (v \otimes e + e \otimes v)$ for the last equality. Therefore we have $\exp_A(v) \circ m_B = m_A \circ (\exp_A(v) \otimes \exp_A(v))$.

2. We check that $\ln_A(g) \in \text{THom}_{cdgA[k]}(B, A)$ for every $g \in \text{Hom}_{cdgA[k]}(B, A)$:

$$d_{BA} \ln_A(g) = 0, \quad \ln_A(g) \circ u_B = 0, \quad \ln_A(g) \circ m_B = m_A \circ (\ln_A(g) \otimes e + e \otimes \ln_A(g)).$$

The 1st relation is obvious since $d_{BA}(\bar{g}) = d_{BA}g = 0$ and $d_{BA}e = 0$ and $d_{BA}A$ is a derivation of $\ast$. The 2nd relation is also obvious since $g^{\ast n} \circ u_B = m_A^{(n)} \circ (\bar{g} \otimes u_B)^{\otimes n} \circ \Delta_k^{(n)} = 0$ for all $n \geq 1$. Therefore it remains to check the 3rd relation.

Define $\ln_A(\chi) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(\chi - e \otimes e)^{\otimes n}$ for all $\chi \in \text{Hom}(C \otimes C, \Omega \otimes \Omega)$ satisfying $\chi \circ (u_B \otimes u_B) = u_A \otimes u_A$. Then, we have

$$\ln_A(g \otimes e) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(g \otimes e - e \otimes e)^{\otimes n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(g \otimes e)^{\otimes n} = \ln_A(g) \otimes e,$$

and, similarly, $e \otimes \ln_A(g) = \ln_A(g \otimes e)$. Therefore, we have

$$\ln_A(g \otimes e) + e \otimes \ln_A(g) = \ln_A(g \otimes e) + \ln_A(e \otimes g) = \ln_A((g \otimes e) \ast (e \otimes g)) = \ln_A(g \otimes g).$$
On the other hand, we have
\[ g^n \circ m_B = m_A^n \circ (g \otimes e)^\otimes \circ g \circ m_B \]
\[ = m_A^n \circ (g \circ m_B \otimes \cdots \otimes g \circ m_B) \circ \Delta_B \otimes B \]
\[ = m_A^n \circ (m_A \otimes \cdots \otimes m_A) \circ (g \otimes g - e \otimes e)^\otimes \otimes \Delta_B \otimes B \]
\[ = m_A \circ m_A^n \circ (g \otimes g - e \otimes e)^\otimes \otimes \Delta_B \otimes B \]
\[ = m_A \circ m_A^n \circ (g \otimes g - e \otimes e)^\otimes \otimes \Delta_B \otimes B \]
where we have used eq. (3.2) for the 2nd equality, the condition \( g \circ m_B = g \circ m_B - e \circ m_B = m_A \circ (g \otimes g - e \otimes e) \) for the 3rd equality, Lemma 3.11(b) for the 4th equality, and eq. (3.5) for the last equality. Therefore, we obtain that
\[ \ln_A(g) \circ m_B = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} m_A \circ (g \otimes g - e \otimes e)^\otimes = m_A \circ m_A^n (g \otimes g) \]
\[ = m_A \circ \left( \ln_A(g) \otimes e + e \otimes \ln_A(g) \right). \]

3. It is obvious now that \( \ln_A(\exp_A(\nu)) = \nu \) and \( \exp_A(\ln_A(\nu)) = \nu \). Hence \( \exp_A, \ln_A \) is an isomorphism.

4. Let \( \nu \sim \tilde{\nu} \in \mathbf{THom}_{cdgA[k]}(B, A) \). Then we have a corresponding homotopy pair \( (\nu(t), \sigma(t)) \) on \( \mathbf{THom}_{cdgA[k]}(B, A) \) such that \( \nu(0) = \nu \) and \( \nu(1) = \tilde{\nu} \). Let
\[ g(t) := \exp_A(\nu(t)), \]
\[ \lambda(t) := \sum_{n=1}^{\infty} \frac{n}{n!} \nu(t)^{n-1} \otimes \nu(t)^n. \]

Then it is trivial to check that \( (g(t), \lambda(t)) \) is a homotopy pair on \( \mathbf{Hom}_{cdgA[k]}(B, A) \), so that \( \exp_A(\nu) = g(0) \sim g(1) = \exp_A(\tilde{\nu}) \in \mathbf{Hom}_{cdgA[k]}(B, A) \).

5. Let \( g \sim \tilde{g} \in \mathbf{Hom}_{cdgA[k]}(B, A) \) and \( (g(t), \lambda(t)) \) be the corresponding homotopy pair on \( \mathbf{Hom}_{cdgA[k]}(B, A) \) such that \( g(0) = g \) and \( g(1) = \tilde{g} \). Let
\[ \nu(t) := \ln_A(g(t)), \]
\[ \sigma(t) := - \sum_{n=1}^{\infty} \frac{n}{n!} \tilde{g}(t)^{n-1} \otimes \lambda(t) \otimes \tilde{g}(t)^n. \]

Then it is trivial to check that \( (\nu(t), \sigma(t)) \) is a homotopy pair on \( \mathbf{THom}_{cdgA[k]}(B, A) \), so that \( \ln_A(\nu) = \nu(0) \sim \nu(1) = \ln_A(\tilde{g}) \in \mathbf{THom}_{cdgA[k]}(B, A) \). □

**Proposition 3.2.** Let \( B \) be a conilpotent cdg-Hopf algebra. Then we have a natural isomorphism \( \mathbf{Tg}^B \xrightarrow{\exp} \mathbf{g}^B : \mathbf{cdgA(k)} \to \mathbf{Set \ of \ functors \ on \ cdgA(k)} \), whose component at each cdg-algebra \( A \) is \( (\exp_A, \ln_A) \) defined in Proposition 3.1.
Proof. Remind that $T\mathcal{G}^B = T\text{Hom}_{cdgA[k]}(B,-)$ and $\hat{\mathcal{G}}^B = \text{Hom}_{cdgA[k]}(B,-)$. In Proposition 3.1, we have shown that $\mathcal{G}^B(A) \xrightarrow{\exp_A} \hat{\mathcal{G}}^B(A)$ is an isomorphism for every cdg-algebra $A$. It remains to check the naturalness of $\exp$ and $\ln$ that for every morphism $f : A \to A'$ of cdg-algebras the diagrams are commutative:

$$
\begin{array}{ccc}
\mathcal{G}^B(A) & \xrightarrow{T\mathcal{G}^B} & \mathcal{G}^B(A') \\
\exp_A & \downarrow & \exp_{A'} \\
\hat{\mathcal{G}}^B(A) & \xrightarrow{\hat{\mathcal{G}}^B} & \hat{\mathcal{G}}^B(A')
\end{array}
$$

That is, $\hat{\mathcal{G}}^B(f) \circ \exp_A = \exp_A \circ T\mathcal{G}^B(f)$ and $T\mathcal{G}^B(f) \circ \ln_A = \ln_A \circ \hat{\mathcal{G}}^B(f)$. These are straightforward since for every $\nu \in T\mathcal{G}^B(A)$ we have

$$
\hat{\mathcal{G}}^B(f) (\exp_A(\nu)) = f \circ \exp_A(\nu) = f \circ u_A \circ e_B + \sum_{n=1}^{\infty} \frac{1}{n!} f \circ m_A^{(n)} \circ (\nu \otimes \ldots \otimes \nu) \circ \Delta_B^{(n)}
$$

$$
= u_A \circ e_B + \sum_{n=1}^{\infty} \frac{1}{n!} m_A^{(n)} \circ (f \circ \nu \otimes \ldots \otimes f \circ \nu) \circ \Delta_B^{(n)} = \exp_A(f \circ \nu)
$$

The naturalness of $\ln$ can be checked similarly. $\square$

Now we can finish our proof of Theorem 3.3.

Proof (Theorem 3.3). We note that the components $T\check{\mathcal{G}}^B(A) \xrightarrow{\exp_A} \check{\mathcal{G}}^B(A)$ of $\exp$ and $\ln$ at every cdg-algebra $A$ are defined such that $\exp_A([\nu]) = [\exp_A(\nu)]$ and $\ln_A([g]) = [\ln_A(g)]$. Due to Proposition 3.1, they are well defined, depending only on the homotopy types of $\nu$ and $g$, and are isomorphisms for every cdg-algebra $A$.

Remains to check the naturalness of $\exp$ and $\ln$ that for every $[f] \in \text{Hom}_{cdgA[k]}(A, A')$ the following diagrams commute

$$
\begin{array}{ccc}
\check{\mathcal{G}}^B(A) & \xrightarrow{T\check{\mathcal{G}}^B([f])} & \check{\mathcal{G}}^B(A') \\
\exp_A & \downarrow & \exp_{A'} \\
\check{\mathcal{G}}^B(A) & \xrightarrow{T\check{\mathcal{G}}^B([f])} & \check{\mathcal{G}}^B(A')
\end{array}
$$

$$
\begin{array}{ccc}
\check{\mathcal{G}}^B(A) & \xrightarrow{T\check{\mathcal{G}}^B([f])} & \check{\mathcal{G}}^B(A') \\
\ln_A & \downarrow & \ln_{A'} \\
\check{\mathcal{G}}^B(A) & \xrightarrow{T\check{\mathcal{G}}^B([f])} & \check{\mathcal{G}}^B(A')
\end{array}
$$

Here we will check the naturalness of $\exp$ only, since the proof is similar of $\ln$. 
Let $f \in \text{Hom}_{\text{ho cdgA}(k)}(A, A')$ be an arbitrary representative of $[f]$. Consider any $[v] \in \text{THom}_{\text{ho cdgA}(k)}(B, A)$ and let $v \in \text{THom}_{\text{cdgA}(k)}(B, A)$ be an arbitrary representative of $[v]$. Then it is straightforward to check that the homotopy type $[f \circ v]$ of $f \circ v \in \text{THom}_{\text{cdgA}(k)}(B, A')$ depends only on $[f]$ and $[v]$. From Proposition 3.2, it also follows that the homotopy type $[\exp_A(f \circ v)]$ of $\exp_A(f \circ v) \in \text{Hom}_{\text{cdgA}(k)}(B, A')$ depends only on $[f]$ and $[v']$. It is also obvious that the homotopy type $[f \circ \exp_A(v)]$ of $f \circ \exp_A(v) \in \text{Hom}_{\text{cdgA}(k)}(B, A')$ depends only on $[f]$ and $[v]$. Combined with the identity $f \circ \exp_A(v) = \exp_A(f \circ v)$ in the proof of Proposition 3.2, we have

$$\exp B ([f])(\exp A ([v])) = [f \circ \exp A (v)] = [\exp A (f \circ v)] = \exp A (T \exp B ([f])([v])).$$

Hence $\exp : T \exp B \Rightarrow \exp B : \text{ho cdgA}(k) \to \text{Set}$ is a natural isomorphism. □

4. Representations of an affine group dg-scheme

Throughout this section we fix a cdg-Hopf algebra $B = (B, u_B, m_B, \varepsilon_B, \Delta_B, \zeta_B, d_B)$. We define a linear representation of an affine group dg-scheme $\mathfrak{S}^B : \text{ho cdgA}(k) \to \text{Grp}$ via a linear representation of the associated functor $\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}$ on the category $\text{cdgA}(k)$ of cdg-algebras, which is represented by $B$ and induces $\mathfrak{S}^B$ on the homotopy category $\text{ho cdgA}(k)$.

The linear representations of $\mathcal{G}^B$ form a dg-tensor-category $\text{Rep}(\mathcal{G}^B)$, which is isomorphic to the dg-tensor-category $\text{dgComod}_B(B)$ of right dg-comodules over $B$. Working with the linear representations of $\mathcal{G}^B$ instead of the linear representations of $\mathfrak{S}^B$ will be a crucial step for a Tannakian reconstruction of $\mathfrak{S}^B$.

4.1. Preliminary

Our main concern here is a dg-tensor-category formed by free right dg-modules over a cdg-algebra $A = (A, u_A, m_A, d_A)$.

We shall need the following basic lemma, which is due to the defining properties of dg-algebra $A$, which is standard.

**Lemma 4.1.** For every pair $(M, N)$ of cochain complexes we have an exact sequence of cochain complexes

$$0 \to \text{Hom} (M, N \otimes A) \to \text{Hom} (M \otimes A, N \otimes A) \to \text{Hom} (M \otimes A \otimes A, N \otimes A),$$
Corollary 4.1. We have a bijection $q : \text{Hom}_{m_A}(M \otimes A, N \otimes A) \longrightarrow \text{Hom}(M, N \otimes A) : p$. Using Lemma 4.1, it is straightforward to check that $\text{dgMod}_R^P(A)$ is indeed a dg-category.
Lemma 4.3 (Definition). We have a tensor dg-functor \( \otimes A : \text{CoCh} (k) \to \text{dgMod}^{fr}_R (A) \) for every cdg-algebra \( A \), sending

1. each cochain complex \( M \) to the free right dg-module \( M \otimes A, I_M \otimes m_A \), and

2. each \( k \)-linear map \( M \to M' \) to the following morphism of right \( A \) dg-modules:

\[
(M \otimes A, I_M \otimes m_A) \xrightarrow{\varphi \otimes I_A} (M' \otimes A, I_{M'} \otimes m_A).
\]

Proof. Trivial.

For every cdg-algebra \( A \), we have the following constructions.

1. Let \( \text{End}_{m_A} (M \otimes A) := \text{Hom}_{m_A} (M \otimes A, M \otimes A) \), which is the \( \mathbb{Z} \)-graded vector space of linear maps \( \varphi : M \otimes A \to M \otimes A \) satisfying \( \varphi \circ (I_M \otimes m_A) = (I_M \otimes m_A) \circ (\varphi \otimes I_A) \). Then we have a dg-algebra

\[
\mathcal{E}^M (A) = \left( \text{End}_{m_A} (M \otimes A), I_{M \otimes A} \circ, d_{M \otimes A, M \otimes A} \right).
\]

2. Let \( Z^0 \text{Aut}_{m_A} (M \otimes A) \) be the subset of \( \text{End}_{m_A} (M \otimes A) \) consisting every degree zero element \( \varphi \) that has a composition inverse \( \varphi^{-1} \) and satisfies \( d_{M \otimes A, M \otimes A} \varphi = 0 \). Then we have a group

\[
\mathcal{G}^M (A) := \left( Z^0 \text{Aut}_{m_A} (M \otimes A), I_{M \otimes A} \circ \right).
\]
Lemma 4.4. Let $H^0$Aut$_{m_1}(M \otimes A)$ be the set of cohomology classes of elements in $Z^0$Aut$_{m_1}(M \otimes A)$ that $\varphi, \tilde{\varphi} \in Z^0$Aut$_{m_1}(M \otimes A)$ belongs to the same cohomology class $\varphi \sim \tilde{\varphi}$, i.e., $[\varphi] = [\tilde{\varphi}] \in H^0$Aut$_{m_1}(M \otimes A)$, if $\varphi - \tilde{\varphi} = d_{M \otimes \Lambda, M \otimes \Lambda} \lambda$ for some $\lambda \in \text{End}_m(M \otimes A)^{-1}$.

We can check that $\varphi_1 \circ \varphi_2 \sim \varphi_1 \circ \varphi_2 \in Z^0$Aut$_{m_1}(M \otimes A)$ whenever $\varphi_1 \sim \varphi_1, \varphi_2 \sim \varphi_2 \in Z^0$Aut$_{m_1}(M \otimes A)$ whenever $\varphi \sim \tilde{\varphi} \in Z^0$Aut$_{m_1}(M \otimes A)$. Let $[\varphi_1] \circ [\varphi_2] := [\varphi_1 \circ \varphi_2]$ and $[\varphi]^{-1} := [\varphi^{-1}]$. Then we have a group

$$\mathcal{G}^M(A) := \{H^0$Aut$_{m_1}(M \otimes A), [I]_{m_1, \Lambda, \circ}\}.$$ (4.5)

The above three constructions are functorial as described in the forthcoming three lemmas. We shall omit the proofs.

Lemma 4.4. For every cochain complex $M$ we have a functor $\mathcal{E}^M : \text{cdgA}(k) \rightarrow \text{dgA}(k)$, sending

- each cdg-algebra $A$ to the dg-algebra $\mathcal{E}^M(A) = \{\text{End}_m(M \otimes A), I_{M \otimes \Lambda, \circ}, d_{M \otimes \Lambda, M \otimes \Lambda}\}$;
- each $f \in \text{Hom}_{\text{cdgA}}(A, A')$ to a morphism $\mathcal{E}^M(f) : \mathcal{E}^M(A) \rightarrow \mathcal{E}^M(A')$ of dg-algebras defined by $\forall \varphi \in \text{End}_m(M \otimes A)$,

$$\mathcal{E}^M(f)(\varphi) := p\left(\underbrace{I_{M \otimes f} \circ q(\varphi)}_{\text{defined by } \phi} \right)$$

$$= \left[I_{M \otimes m_A} \circ \left(I_{M \otimes f \otimes I_{A'}} \circ (q(\varphi) \otimes I_{A'})\right) \right]$$

$$= \left[I_{M \otimes m_A} \circ \left(I_{M \otimes f \otimes I_{A'}} \circ (\varphi \circ (I_{M \otimes u_A}) \circ d_{M \otimes \Lambda, \circ})\right) = \left[I_{M \otimes m_A} \circ \left(I_{M \otimes f \otimes I_{A'}} \circ (\varphi \circ (I_{M \otimes u_A}) \circ d_{M \otimes \Lambda, \circ})\right)\right].$$

(See $\mathcal{E}^M(f)(\varphi) : M \otimes A' \xrightarrow{\varphi \otimes m_{A'}} M \otimes A \otimes A' \xrightarrow{I_{M \otimes f \otimes I_{A'}}} M \otimes A' \otimes A' \xrightarrow{I_{M \otimes m_{A'}}} M \otimes A'$.)

That is, we have $\mathcal{E}^M(f)(\varphi) \in \text{End}_{m_1}(M \otimes A')$, and

(a) $\mathcal{E}^M(f)(I_{M \otimes \Lambda}) = I_{M \otimes \Lambda}$;
(b) $\mathcal{E}^M(f)(\varphi_1 \circ \varphi_2) = \mathcal{E}^M(f)(\varphi_1) \circ \mathcal{E}^M(f)(\varphi_2)$;
(c) $\mathcal{E}^M(f) \circ d_{M \otimes \Lambda, M \otimes \Lambda} = d_{M \otimes \Lambda, M \otimes \Lambda} \circ \mathcal{E}^M(f)$;
(d) $\mathcal{E}^M(f') \circ \mathcal{E}^M(f) = \mathcal{E}^M(f' \circ f)$ holds for another morphism $f' : A' \rightarrow A''$ of cdg-algebras.

Lemma 4.5. For every cochain complex $M$ we have a functor $\mathcal{G}^M : \text{cdgA}(k) \rightarrow \text{Grp}$, sending

- each cdg-algebra $A$ to the group $\mathcal{G}^M(A) = \{Z^0$Aut$_{m_1}(M \otimes A), I_{M \otimes \Lambda, \circ}\}$;
- each $f \in \text{Hom}_{\text{cdgA}}(A, A')$ to a homomorphism $\mathcal{G}^M(f) : \mathcal{G}^M(A) \rightarrow \mathcal{G}^M(A')$ of groups defined by $\mathcal{G}^M(f) := \mathcal{E}^M(f)$,

such that
Lie

For every cochain complex $M$ we have a functor $\mathcal{G}^M : \text{ho} cdg A(k) \to \text{Grp}$, sending

- each cdg-algebra $A$ to the group $\mathcal{G}^M(A) = \{ H^0 \text{Aut}_{m_A}(M \otimes A), [\cdot], \cdot \};$
- each $[f] \in \text{Hom}_{\text{ho} cdg A(k)}(A, A')$ to a homomorphism $\mathcal{G}^M([f]) : \mathcal{G}^M(A) \to \mathcal{G}^M(A')$ of groups defined by, for all $[\varphi] \in H^0 \text{Aut}_{m_A}(M \otimes A),$

$$\mathcal{G}^M([f])([\varphi]) := [\mathcal{G}^M(f)(\varphi)],$$

where $f \in \text{Hom}_{\text{ho} cdg A(k)}(A, A')$ and $\varphi \in Z^0 \text{Aut}_{m_A}(M \otimes A)$ are arbitrary representatives of $[f]$ and $[\varphi]$, respectively.

As variants of the above three lemmas, we also have the following associated corollaries.

Remind that a dg-Lie algebra is both a cochain complex and a $\mathbb{Z}$-graded Lie algebra, whose differential is a derivation of the bracket. For any dg-algebra $(A, u_A, m_A, d_A)$ we have a dg-Lie algebra $(A, [-, -]_{m_A}, d_A)$ where $[x, y]_{m_A} := m_A(x \otimes y) - (-1)^{|x||y|} m_A(y \otimes x)$ for all $x, y \in A$. A morphism of dg-Lie algebras is simultaneously a cochain map and a Lie algebra map.

**Corollary 4.2.** For every cochain complex $M$ we have a functor $\mathcal{L}^M : \text{cdg A}(k) \to \text{dgl} (k)$, sending

- each cdg-algebra $A$ to the dg-Lie algebra $\mathcal{L}^M(A) = \{ \text{End}_{m_A}(M \otimes A), [-, -], d_{M \otimes A, M \otimes A} \};$
- each $f \in \text{Hom}_{	ext{cdg A}(k)}(A, A')$ to a morphism $\mathcal{L}^M(f) : \mathcal{L}^M(A) \to \mathcal{L}^M(A')$ of dg-Lie algebras defined by $\mathcal{L}^M(f) = E^M(f)$.

**Corollary 4.3.** For every cochain complex $M$ we have a functor $\mathfrak{g}^M : \text{cdg A}(k) \to \text{Lie}(k)$, sending

- each cdg-algebra $A$ to the Lie algebra $\mathfrak{g}^M(A) = \{ Z^0 \text{End}_{m_A}(M \otimes A), [-, -], \}$;
- each $f \in \text{Hom}_{\text{cdg A}(k)}(A, A')$ to a morphism $\mathfrak{g}^M(f) : \mathfrak{g}^M(A) \to \mathfrak{g}^M(A')$ of Lie algebras defined by $\mathfrak{g}^M(f) := \mathcal{L}^M(f)$, such that

$$(a) \mathfrak{g}^M(f)(\varphi) \sim \mathfrak{g}^M(f)(\varphi) \in Z^0 \text{End}_{m_A}(M \otimes A') \text{ for all } \varphi \in Z^0 \text{End}_{m_A}(M \otimes A) \text{ whenever } f \sim \tilde{f} \in \text{Hom}_{\text{cdg A}(k)}(A, A'),$$
(b) $\mathfrak{t}^M(f)(\varphi) \sim \mathfrak{t}^M(f)(\varphi) \in Z^0\text{End}_{m_A}(M \otimes A')$ for all $f \in \text{Hom}_{\text{cdgA}}(A, A')$ whenever $\varphi \sim \tilde{\varphi} \in Z^0\text{End}_{m_A}(M \otimes A)$.

**Corollary 4.4.** For every cochain complex $M$ we have a functor $\mathfrak{g}^M : \text{hocdgA}(k) \to \text{Lie}(k)$, sending

- each cdg-algebra $A$ to the group $\mathfrak{g}^M(A) = \left( \text{H}^0\text{End}_{m_A}(M \otimes A), [\cdot, \cdot]_A \right)$;
- each $[f] \in \text{Hom}_{\text{hocdgA}}(A, A')$ to a morphism $\mathfrak{g}^M([f]) : \mathfrak{g}^M(A) \to \mathfrak{g}^M(A')$ of Lie algebras defined by $\mathfrak{g}^M([f])[\varphi] := \left[ \mathfrak{g}^M(f)(\varphi) \right]$, for all $\varphi \in \text{H}^0\text{End}_{m_A}(M \otimes A)$, where $f \in \text{Hom}_{\text{cdgA}}(A, A')$ and $\varphi \in Z^0\text{End}_{m_A}(M \otimes A)$ are arbitrary representatives of $[f]$ and $[\varphi]$, respectively.

### 4.2. Linear representations of affine group dg-scheme

We are ready to define a linear representation of the functor $\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}$ represented by $B$.

**Definition 4.1.** A linear presentation of the functor $\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}$ is a pair $(M, \rho^M)$, where $M$ is a cochain complex and $\rho^M : \mathcal{G}^B \Rightarrow \mathcal{G}^M : \text{cdgA}(k) \to \text{Grp}$ is a natural transformation of the functors.

**Remark 4.1.** From the condition that a linear representation $\rho^M : \mathcal{G}^B \Rightarrow \mathcal{G}^M$ is a natural transformation of covariant functors, we have

- the component $\rho^M_A : \mathcal{G}^B(A) \to \mathcal{G}^M(A)$ of $\rho^M$ at each cdg-algebra $A$ is a homomorphism of groups, and
- for every morphism $f : A \to A'$ of cdg-algebras the diagram commutes

\[
\begin{array}{ccc}
\mathcal{G}^B(A) & \xrightarrow{\mathcal{G}^B(f)} & \mathcal{G}^B(A') \\
\rho^M_A & \downarrow & \rho^M_A' \\
\mathcal{G}^M(A) & \xrightarrow{\mathcal{G}^M(f)} & \mathcal{G}^M(A')
\end{array}
\]

(4.6)

Since $\mathcal{G}^B$ is representable, the Yoneda lemma implies that a natural transformation $\rho^M : \mathcal{G}^B \Rightarrow \mathcal{G}^M$ is determined completely by the universal element $\rho^M_B(\mathbb{I}_B) : M \otimes B \to M \otimes B$.

Indeed, the naturalness eq. (4.6) of $\rho^M$ imposes that for every morphism $B \xrightarrow{g} A$ of cdg-algebras we have $\rho^M_A(g) = \rho^M_M \left( \mathcal{G}^B(g)(\mathbb{I}_B) \right) = \mathcal{G}^M(g) \left( \rho^M_B(\mathbb{I}_B) \right)$. Explicitly, we obtain that

\[
\rho^M_A(g) = p \left( (\mathbb{I}_M \otimes g) \circ q \left( \rho^M_B(\mathbb{I}_B) \right) \right) \iff q \left( \rho^M_A(g) \right) = (\mathbb{I}_M \otimes g) \circ q \left( \rho^M_B(\mathbb{I}_B) \right),
\]

(4.7)

where $p$ and $q$ are defined in Lemma 4.1.
Lemma 4.7. A linear representation $\rho^M : G^B \rightarrow \mathcal{G}^M$ of $G^B$ induces a natural transformation $[\rho]^M : G^B \Rightarrow \mathcal{G}^M : \text{hodgA}(k) \rightarrow \text{Grp}$, whose component $[\rho]^M_A$ at each cdg-algebra $A$ is the homomorphism $[\rho]^M_A : \mathcal{G}^B(A) \rightarrow \mathcal{G}^M(A)$ of groups defined by $[\rho]^M_A([g]) = [\rho]^M_A(g) \in H_0\text{Aut}_{m_B}(M \otimes A)$ for all $[g] \in \text{Hom}_{\text{hodgA}(k)}(B, A)$, where $g \in \text{Hom}_{\text{cdgA}(k)}(B, A)$ is an arbitrary representative of $[g]$.

Proof. Let $g \sim \tilde{g} \in \text{Hom}_{\text{cdgA}(k)}(B, A)$. Then we have we have a homotopy pair $[g(t), \chi(t)]$ on $\text{Hom}_{\text{cdgA}(k)}(B, A)$ such that we have a family $g(t) = g + d_{B, A} \int_0^t \chi(s) ds$ of morphism of cdg-algebras satisfying $g(0) = g$ and $g(1) = \tilde{g}$. From eq. $(4.7)$ it follows that $\rho^M_A(g) \sim \rho^M_A(\tilde{g}) \in Z_0\text{Aut}_{m_B}(M \otimes A)$, since both $p$ and $q$ are cochain maps and $\rho^M_B([\rho]) \in Z_0\text{Aut}_{m_B}(M \otimes B)$. Therefore we have $[\rho^M_A(g)] = [\rho^M_A(\tilde{g})] \in H_0\text{Aut}_{m_B}(M \otimes A)$, so that $[\rho]^M_A : \mathcal{G}^B(A) \rightarrow \mathcal{G}^M(A)$ is well-defined homomorphism of groups for every $A$. The naturalness of $[\rho]^M_A$, i.e., for every $[f] \in \text{Hom}_{\text{cdgA}(k)}(A, A')$ we have $[\rho]^M_A \circ \mathcal{G}^B([f]) = \mathcal{G}^M([f]) \circ [\rho]^M_A$ due to the naturalness eq. $(4.6)$ of $\rho_A^M$ and by the definitions of $[\rho]^M_A$, $\mathcal{G}^B([f])$ and $\mathcal{G}^M([f])$. □

Definition 4.2. A linear representation of an affine group dg-scheme $\mathcal{G}^B$ is a pair $(M, \rho^M)$ of cochain complex $M$ and a natural transformation $\rho^M : \mathcal{G}^B \Rightarrow \mathcal{G}^M$, which is induced from a linear representation $\rho^M : G^B \Rightarrow \mathcal{G}^M$ of $G^B$.

Remark 4.2. Despite of the above definition we will work with linear representations of $\mathcal{G}^B$ rather than those of $\mathcal{G}^B$. Working with the dg-tensor-category of linear representations of $\mathcal{G}^B$ will be a crucial step for our Tannakian reconstructions of both $\mathcal{G}^B$ and $\mathcal{G}^B$ in the next section. The linear representations of $\mathcal{G}^B$ shall form a dg-tensor-category $\text{Rep}(\mathcal{G}^B)$. We regard $\text{Rep}(\mathcal{G}^B)$ as the dg-tensor-category of "linear representations of $\mathcal{G}^B".

Here are two basic examples of linear representations of $\mathcal{G}^B$.

Example 4.1 (The trivial representation). The ground field $k$ as a cochain complex $k = (k, 0)$ with zero differential defines the trivial representation $(k, \rho^k)$, where the component $\rho_A^k$ of $\rho^k : \mathcal{G}^B \Rightarrow \mathcal{G}^k$ at every cdg-algebra $A$ is the trivial homomorphism $\rho_A^k : \mathcal{G}^B(A) \rightarrow \mathcal{G}^k(A)$ of groups: $\forall g \in \text{Hom}_{\text{cdgA}(k)}(B, A)$,

$$\rho_A^k(g) := I_k \otimes I_A : k \otimes A \rightarrow k \otimes A.$$ (4.8)

Example 4.2 (The regular representation). Associated to the cdg-Hopf algebra $B$ as a cochain complex we have the regular representation $(B, \rho^B)$, where the component $\rho_A^B$ of $\rho^B : \mathcal{G}^B \Rightarrow \mathcal{G}^B$ at each cdg-algebra $A$ is the homomorphism $\rho_A^B : \mathcal{G}^B(A) \rightarrow \mathcal{G}^B(A)$ of groups defined by, $\forall g \in \text{Hom}_{\text{cdgA}(k)}(B, A)$,

$$\rho_A^B(g) = \begin{cases} I_B \otimes m_B & \text{if } A = B \otimes A \\ \Delta_B \otimes I_A & \text{if } B \otimes A \\ \imath_B \otimes \imath_A & \text{if } B \otimes A \end{cases}.$$ (4.9)

\[ B \otimes A \xrightarrow{\Delta_B \otimes I_A} B \otimes B \otimes A \xrightarrow{\imath_B \otimes \imath_A} B \otimes A \otimes A \xrightarrow{\imath_B \otimes m_A} B \otimes A \]
We can check that \((B, \rho^B)\) is a linear representation as follows

- We have \(d_{M\otimes A, M\otimes A}\rho^B_A(g) = p((d_{B,AG}\otimes I_A)\circ \Delta_B) = 0\) for all \(g \in \text{Hom}_{\text{cdgA[k]}}(B, A)\), since both \(p\) and \(\Delta_B\) are cochain maps;
- We have \(\rho^B_A(u_A \circ \epsilon_B) = (I_B \otimes m_A) \circ (I_B \otimes (u_A \circ \epsilon_B) \otimes I_A) \circ (\Delta_B \otimes I_A) = I_{B\otimes A}\);
- For all \(g_1, g_2 \in \text{Hom}_{\text{cdgA[k]}}(B, A)\):

\[
\rho^B_A(g_1 \ast_{B, A} g_2) := (I_B \otimes m_A) \circ (I_B \otimes (m_A \circ (g_1 \otimes g_2) \circ \Delta_B) \otimes I_A) \circ (\Delta_B \otimes I_A) \\
= (I_B \otimes m_A) \circ (I_B \otimes (g_1 \otimes I_A) \circ (\Delta_B \otimes I_A) \circ (I_B \otimes m_A) \circ (I_B \otimes g_2 \otimes I_A) \circ (\Delta_B \otimes I_A)) \\
= \rho^B_A(g_1) \circ \rho^B_A(g_2)
\]

The 2nd equality is due to coassociativity of \(\Delta_B\) and the associativity of \(m_A\). □

**Definition 4.3 (Lemma).** The dg-tensor-category \(\left[\text{Rep}\left(\mathcal{G}^B\right), \otimes, (k, \rho^k)\right]\) of linear representations of \(\mathcal{G}^B\) is defined as follows.

(a) An object is a linear presentation \((M, \rho^M)\) of \(\mathcal{G}^B\);

(b) A morphism \(\psi : (M, \rho^M) \to (M', \rho^{M'})\) of linear representations of \(\mathcal{G}^B\) is a linear map \(\psi : M \to M'\) making the following diagram commutative for every cdg-algebra \(A\) and every \(g \in \text{Hom}_{\text{cdgA[k]}}(B, A)\)

\[
\begin{array}{ccc}
M \otimes A & \xrightarrow{\rho^M_A(g)} & M \otimes A \\
\psi \otimes 1 & & \psi \otimes 1 \\
\downarrow & & \downarrow \\
M' \otimes A & \xrightarrow{\rho^{M'}_A(g)} & M' \otimes A,
\end{array}
\]

i.e., \((\psi \otimes I_A) \circ \rho^M_A(g) = \rho^{M'}_A(g) \circ (\psi \otimes I_A)\).

(c) The differential of a morphism \(\psi : (M, \rho^M) \to (M', \rho^{M'})\) of linear representations is the morphism \(d_{M, M'} : \psi : (M, \rho^M) \to (M', \rho^{M'})\) of linear representations.

(d) The tensor product \((M, \rho^M) \otimes (M', \rho^{M'})\) of two objects is the linear representation \((M \otimes M', \rho^{M \otimes M'})\), where \(M \otimes M' = (M \otimes M', d_{M, M'})\) is the tensor product of cochain complexes and \(\rho^{M \otimes M'} : \mathcal{G}^B \Rightarrow \text{Gr}^{M \otimes M'}\) is the natural transformation whose component \(\rho^{M \otimes M'}_A : \mathcal{G}^B(A) \to \text{Gr}^{M \otimes M'}(A)\) at each cdg-algebra \(A\) is the group homomorphism defined by, \(\forall g \in \text{Hom}_{\text{cdgA[k]}}(B, A)\),

\[
\rho^{M \otimes M'}_A(g) := \rho^M_A(g) \otimes_{m_A} \rho^{M'}_A(g)
\]

The unit object for the tensor product is the trivial representation \((k, \rho^k)\) in Example 4.1.
Proof. We check that $\overline{\text{Rep}}(G)$ is a dg-category as follows. Let $\psi : (M, \rho_M) \to (M', \rho_{M'})$ be a morphism of linear representations. Then, by definition, for every cdg-algebra $A$ we have

$$(\psi \otimes \mathbb{I}_A) \circ \varrho^M_A(g) = \varrho^M_A(g) \circ (\psi \otimes \mathbb{I}_A),$$

where the 2nd set of two relations is due to the conditions that $\varrho^M_A(g) \in \text{ZAut}_A(M \otimes A)$ and $\varrho^M_A(g) \in \text{ZAut}_A(M' \otimes A)$. Combining the relations in eq. (4.10) we can deduce that

$$d_{M,M'} \psi \otimes \mathbb{I}_A) \circ \varrho^M_A(g) = \varrho^M_A(g) \circ (d_{M,M'} \psi \otimes \mathbb{I}_A).$$

Therefore $d_{M,M'} \psi : (M, \rho_M) \to (M', \rho_{M'})$ is also a morphism of linear representations. It is obvious that $d_{M,M'} d_{M,M'} = 0$. Hence $\text{Hom}_{\overline{\text{Rep}}(G)}((M, \rho_M), (M', \rho_{M'}))$ is a cochain complex with the differential $d_{M,M'}$. For every consecutive morphisms $\psi' : (M', \rho_{M'}) \to (M'', \rho_{M''})$ of linear representations, it is straightforward to show that $\psi' \circ \psi : (M, \rho) \to (M'', \rho_{M''})$ is a morphism of linear representations and $d_{M,M''}(\psi' \circ \psi) = d_{M,M'} \psi' \circ \psi + (-1)^{|\psi'| |\psi'} d_{M,M'} \psi$. It is trivial to check that the tensor product and the unit object in $(d)$ endow $\overline{\text{Rep}}(G)$ with a structure of dg-tensor-category. □

**Definition 4.4.** A linear presentation of the functor $T\mathcal{G}^B : \text{cdgA}(k) \to \text{Lie}(k)$ is a pair $(M, e^M)$, where $M$ is a cochain complex and $e^M : T\mathcal{G}^B \Rightarrow \mathfrak{gl}^M : \text{cdgA}(k) \to \text{Lie}(k)$ is a natural transformation of the functors.

We can check that a linear representation $e^M : T\mathcal{G}^B \Rightarrow \mathfrak{gl}^M$ of $T\mathcal{G}^B$ induces a natural transformation $[e]^M : T\mathcal{E}^B \Rightarrow \mathfrak{gl}^M : \text{ho cdgA}(k) \to \text{Lie}(k)$, whose component $[e]^M_{A}$ at each cdg-algebra $A$ is the homomorphism $[e]^M_{A} : T\mathcal{E}^B(A) \to \mathfrak{gl}^M(A)$ of groups defined by $[e]^M_{A}([v]) = [e^M_{A}(v)] = \mathbb{H}_{0} \text{End}_{\mathcal{M}_{A}}(M \otimes A)$ for all $[v] \in \text{THom}_{\text{cdgA}(k)}(B, A)$, where $v \in \text{THom}_{\text{cdgA}(k)}(B, A)$ is an arbitrary representative of $[v]$.

**Definition 4.5.** A linear presentation of the functor $T\mathcal{B}^B$ is a pair $(M, [e]^M)$, where $M$ is a cochain complex and $[e]^M : T\mathcal{B}^B \Rightarrow \mathfrak{gl}^M$ is a natural transformation induced from a linear representation $e^M : T\mathcal{G}^B \Rightarrow \mathfrak{gl}^M$ of $T\mathcal{G}^B$.

**Remark 4.3.** We can also form a dg-tensor category of linear representations of $T\mathcal{G}^B$. It is not difficult to show that the dg-tensor categories formed by linear presentations of $T\mathcal{G}^B$ and linear representations of $G^B$ are isomorphic if $G^B$ is pro-unipotent, i.e., $B$ is conilpotent cdg-Hopf algebra.
4.3. The dg-tensor-category of dg-comodules over cdg-Hopf algebra

A right dg-comodule over a cdg-Hopf algebra $B$ is a pair $(M, \gamma^M)$ of a cochain complex $M = (M, d_M)$ and a right coaction $\gamma^M : M \to M \otimes B$, which is a cochain map making the following diagram commute:

That is, $\gamma^M \in \text{Hom}(M, M \otimes B)^0$ and satisfies

\[
\gamma^M \circ d_M = d_{M \otimes B} \circ \gamma^M, \quad \left\{ \begin{array}{l}
(\mu_B \otimes \eta_B) \circ \gamma^M = (\eta_M \otimes \Delta_B) \circ \gamma^M,
(\eta_M \otimes \varepsilon_B) \circ \gamma^M = \gamma_{M}^{-1}.
\end{array} \right.
\]

**Example 4.3.** Here are some standard examples.

1. The ground field $k$ as a cochain complex $(k, 0)$ is a right dg-comodule $(k, \gamma^k)$ over $B$ with the coaction $\gamma^k := t_B^{-1} \circ u_B : k \xrightarrow{u_B} B \xrightarrow{t_B^{-1}} k \otimes B$.

2. The cdg-Hopf algebra $B$ as a cochain complex $(B, d_B)$ is a right dg-comodule $(B, \Delta_B)$ over $B$ with the coaction $\Delta_B : B \to B \otimes B$. Note that $(B, \Delta_B)$ is also a left dg-comodule over $B$.

3. For every cochain complex $M = (M, d_M)$ we have a right dg-comodule $(M \otimes B, \eta_M \otimes \Delta_B)$ with the coaction $\eta_M \otimes \Delta_B : M \otimes B \to M \otimes B \otimes B$, called the cofree right dg-comodule over $B$ cogenerated by $M$.

A morphism $(M, \gamma^M) \xrightarrow{\psi} (M', \gamma'^{M'})$ of dg-comodules over $B$ is a linear map $\psi : M \to M'$ making the following diagram commute:

We can check that $d_{M, M'} : M \to M'$ is a morphism of dg-comodules whenever $\psi : M \to M'$ is a morphism of dg-comodules:

\[
\gamma'^{M'} \circ \psi = (\psi \otimes \eta_B) \otimes \gamma^M \quad \Rightarrow \quad \gamma'^{M'} \circ d_{M, M'} \psi = (d_{M, M'} \psi \otimes \eta_B) \otimes \gamma^M.
\]
The tensor product of right dg-comodules \((M, \gamma^M)\) and \((N, \gamma^N)\) over \(B\) is the right dg-comodule \((M \otimes N, \gamma^{M \otimes N})\) over \(B\), where \(M \otimes N = (M \otimes N, \gamma^{M \otimes N})\) is the tensor product of the underlying cochain complexes and the coaction \(\gamma^{M \otimes N} : M \otimes N \to M \otimes N \otimes B\) is defined by

\[
\gamma^{M \otimes N} \colon (\text{Id}_M \otimes \text{Id}_N) \circ (\tau \otimes \tau) \circ (\gamma^M \otimes \gamma^N),
\]

\[
M \otimes N \xrightarrow{\gamma^M \otimes \gamma^N} M \otimes B \otimes N \otimes B \xrightarrow{\text{Id}_M \otimes \text{Id}_B \otimes \text{Id}_N} M \otimes N \otimes B \otimes B \xrightarrow{\text{Id}_M \otimes \text{Id}_N \otimes m_B} M \otimes N \otimes B.
\]

Let \((M, \gamma^M) \xrightarrow{\psi} (M', \gamma^{M'})\) and \((N, \gamma^N) \xrightarrow{\phi} (N', \gamma^{N'})\) be morphisms of right dg-comodules over \(B\). Then the linear map \(\psi \otimes \phi : M \otimes N \to M' \otimes N'\) is a morphism \((M \otimes N, \gamma_{M \otimes N}) \xrightarrow{\psi \otimes \phi} (M' \otimes N', \gamma_{M' \otimes N'})\) of dg-comodules over \(B\), and we have

\[
d_{M \otimes N, M' \otimes N'}(\psi \otimes \phi) = d_{M, M'} \psi \otimes \phi + (-1)^{|\psi|} \psi \otimes d_{N, N'} \phi.
\]

**Definition 4.6 (Lemma).** The right dg-comodules over \(B\) form a dg-tensor-category \((\text{dgComod}_R(B), \otimes_{m_B}, [k, \gamma^k])\).

**Proof.** Exercise. \(\Box\)

Consider the functor \(\mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}\) represented by cdg-Hopf algebra \(B\).

**Theorem 4.1.** The dg-category \(\text{Rep}(\mathcal{G}^B)\) of linear representations of \(\mathcal{G}^B\) is isomorphic to the dg-category of \(\text{dgComod}_R(B)\) of right dg-modules over \(B\) as dg-tensor categories. Explicitly, we have an isomorphism of dg-tensor categories

\[
\begin{array}{c}
\mathcal{X} : \text{Rep}(\mathcal{G}^B) \\
\cong \\
\text{dgComod}_R(B) : \mathcal{Y}
\end{array}
\]

defined as follows.

- The functor \(\mathcal{X}\) sends each linear representation \((M, \rho^M)\) to the right dg-comodule \((M, \tilde{\gamma}^M)\), where

\[
\tilde{\gamma}^M := q(\rho^M_B(\text{Id}_B)) = \rho^M_B(\text{Id}_B) \circ (\text{Id}_M \otimes u_B) \circ J^{-1}_M : M \to M \otimes B
\]

and each morphism \(\psi : (M, \rho^M) \to (M', \rho^{M'})\) of linear representations to the morphism \(\psi : (M, \tilde{\gamma}^M) \to (M', \tilde{\gamma}^{M'})\) of right dg-comodules.

- The functor \(\mathcal{Y}\) sends each right dg-comodule \((M, \tilde{\gamma}^M)\) to the linear representation \((M, \tilde{\rho}^M)\), where the component \(\tilde{\rho}^M_A\) of \(\tilde{\rho}^M\) at a cdg-algebra \(A\) is defined by \(\forall g \in \text{Hom}_{\text{cdgA}(k)}(B, A)\),

\[
\tilde{\rho}^M_A(g) := p((\text{Id}_M \otimes g) \circ \gamma^M)
= (\text{Id}_M \otimes m_A) \circ (\text{Id}_M \otimes g \otimes \text{Id}_A) \circ (\gamma^M \otimes \text{Id}_A) : M \otimes A \to M \otimes A,
\]

\[\tag{4.13}
\]

\[\tag{4.14}
\]
and each morphism \( \psi : (M, \gamma^M) \to (M', \gamma^{M'}) \) of right dg-comodules to the morphism \( \psi : (M, \tilde{\rho}^M) \to (M', \tilde{\rho}^{M'}) \) of linear representations.

**Proof.** We need to check that both \( X \) and \( Y \) are dg-tensor functors and show that they are inverse to each other.

1. We check that \((M, \tilde{\gamma}^M) = X(M, \rho^M)\) is a right dg-comodule over \( B \) as follows.

From eq. (4.7) in Remark 4.1 and the definition eq. (4.13), we have the following relation for every morphism \( g : B \to A \) of cdg-algebras:

\[
q\left( \rho^M_A(g) \right) = (1_M \otimes g) \circ \tilde{\gamma}^M : M \to M \otimes A.
\]  

(4.15)

The component \( \rho^M_A \) of \( \rho^M \) at every cdg-algebra \( A \), by definition, is a morphism \( \rho^M_A : G^B(A) \to G^M(A) \) of groups, i.e., for every pair of morphisms \( g_1, g_2 : B \to A \) of cdg-algebras, we have

\[
\rho^M_A(u_A \circ e_B) = 1_{M \otimes A}, \quad \rho^M_A(g_1 \ast_{B, A} g_2) = \rho^M_A(g_1) \circ \rho^M_A(g_2).
\]  

(4.16)

– Applying \( q \) on the 1st equality of eq. (4.16) and using eq. (4.15), we have

\[(1_M \otimes u_A) \circ (1_M \otimes e_B) \circ \tilde{\gamma}^M = 1_M \otimes u_A.
\]

(4.17)

By putting \( A = k \), we obtain that \((1_M \otimes e_B) \circ \tilde{\gamma}^M = J_M^{-1} \).

– Applying \( q \) on the 2nd equality of eq. (4.16), we have

\[
(1_M \otimes m_A) \circ (1_M \otimes g_1 \otimes g_2) \circ (1_M \otimes \Delta_B) \circ \tilde{\gamma}^M = (1_M \otimes m_A) \circ (1_M \otimes g_1 \otimes g_2) \circ (\tilde{\gamma}^M \otimes 1_B) \circ \tilde{\gamma}^M.
\]

(4.18)

Consider the cdg-algebra \( B \otimes B \) and the inclusion maps \( i_1, i_2 : B \to B \otimes B \)

\[
i_1 := B \xrightarrow{\eta^{-1}_B} B \otimes k \xrightarrow{1_M \otimes u_B} B \otimes B, \quad i_2 := B \xrightarrow{\eta^{-1}_B} k \otimes B \xrightarrow{u_B \otimes 1_B} B \otimes B,
\]

which are morphisms of cdg-algebras. We can check that \( m_{B \otimes B} \circ (i_1 \otimes i_2) = 1_{B \otimes B} \) from an elementary calculation. Let \( A = B \otimes B \). By substituting \( g_1 = i_1, g_2 = i_2 \) in eq. (4.18) we obtain that \((1_M \otimes \Delta_B) \circ \tilde{\gamma}^M = (\tilde{\gamma}^M \otimes 1_B) \circ \tilde{\gamma}^M \).

– Finally we check that \( \tilde{\gamma}^M : M \to M \otimes B \) is a cochain map:

\[
d_{M, M \otimes B} \tilde{\gamma}^M = d_{M, M \otimes B} q(\rho^M_B(1_B)) = q(d_{M \otimes B, M \otimes B} \rho^M_B(1_B)) = 0,
\]

where we have used the facts that \( q \) is a cochain map defined in Lemma 4.1 and \( \rho^M_B(1_B) \in Z^0_{\text{Aut}_{m_B}(M \otimes B)} \).
2. We show that $X$ is a dg-tensor functor. Given a morphism $\psi : (M, \rho^M) \to (M', \rho^{M'})$ of representations, $X(\psi) = \psi : (M, \tilde{\rho}^M) \to (M', \tilde{\rho}^{M'})$ is a morphism of right dg-comodules over $B$, since the following diagram commutes

\[
\begin{array}{ccccccccc}
M & \xrightarrow{\psi} & M' & \xrightarrow{\tilde{\rho}^M} & M \otimes B & \xrightarrow{\rho_B^M(1_B)} & M \otimes B \\
\downarrow{j^M_1} & & \downarrow{\psi \otimes 1_B} & & \downarrow{1_M \otimes \rho_B} & & \downarrow{\psi \otimes 1_B} \\
M' & \xrightarrow{\tilde{\rho}'^{M'}} & M' \otimes B & \xrightarrow{\rho_B^{M'}(1_B)} & M' \otimes B
\end{array}
\]

where the very right square commutes since $\psi$ is a morphism of representations and the commutativity of the other squares are obvious. It is obvious that $X(d_{M,M'}\psi) = d_{M,M'}X(\psi)$. Therefore $X$ is a dg-functor. The tensor property of $X$ is checked as follows:

- From Example 4.1 we have $X(k, \rho^k) = (k, \gamma^k)$.
- For two representations $(M, \rho^M)$ and $(M', \rho^{M'})$ of $\mathcal{G}^B$, we have
  \[
  \gamma^{X(M, \rho^M) \otimes m_B} X(M', \rho^{M'}) = (\Pi_M \otimes m_B) \circ (\Pi_M \otimes \tau \otimes \Pi_B) \circ (\gamma^{X(M, \rho^M)} \otimes \gamma^{X(M', \rho^{M'})}) \\
  = \gamma^{X((M, \rho^M) \otimes (M', \rho^{M'}))}.
  \]

The 1st equality is from the definition of tensor product of right dg-comodules over $B$, and the 2nd equality is from the definition of tensor products of representations of $\mathcal{G}^B$. We conclude that $X((M, \rho^M) \otimes (M', \rho^{M'})) = X(M, \rho^M) \otimes m_B (X(M', \rho^{M'}))$.

3. We show that $(\tilde{M}, \tilde{\rho}^M) = Y(M, \gamma^M)$ is a representation of $\mathcal{G}^B$ as follows. We first show that $\tilde{\rho}^M_A : \mathcal{G}^B(A) \to \mathcal{G}^M(A)$ is a morphism of groups for every cdg-algebra $A$:
- We have $\tilde{\rho}^M_A(1_A \otimes m_A) = (\Pi_M \otimes m_A) \circ (\Pi_M \otimes 1_A) = \Pi_M \otimes 1_A$ where we have used the counity of $m_A$ and the property $(\Pi_M \otimes 1_A) \circ \gamma^M = j_M^{-1}$.
- For all $g_1, g_2 \in \Hom_{\text{cdg}A}[B, A]$:
  \[
  \tilde{\rho}^M_A(g_1 \cdot_B g_2) := (\Pi_M \otimes m_A) \circ (\Pi_M \otimes (m_A \circ (g_1 \otimes g_2) \circ \Delta_B) \otimes 1_A) \circ (\gamma^M \otimes 1_A)
  = (\Pi_M \otimes m_A) \circ (\Pi_M \otimes 1_A) \circ (\gamma^M \otimes 1_A) \circ (\Pi_M \otimes m_A) \circ (\Pi_M \otimes g_2 \otimes 1_A) \circ (\gamma^M \otimes 1_A)
  = \tilde{\rho}^M_A(g_1) \cdot_B \tilde{\rho}^M_A(g_2),
  \]
  where the 2nd equality is due to associativity of $m_A$ and the property $(\Pi_M \otimes \Delta_B) \circ \gamma^M = (\gamma^M \otimes 1_B) \circ \gamma^M$. 


4. We show that \( Y \) is a dg-tensor functor. Given a morphism \( \psi : (M, \gamma^M) \to (M', \gamma'^M) \) of right dg-comodules over \( B \), \( Y(\psi) = \psi : (M, \hat{\rho}^M) \to (M', \hat{\rho}'^M) \) is a morphism of representations, since the following diagram commutes for every morphism \( g : B \to A \) of cdg-algebras:

\[
\begin{array}{c}
M \otimes A \\
\psi \otimes I_A \\
M' \otimes A \\
\hat{\rho}'^M(g)
\end{array}
\longrightarrow
\begin{array}{c}
M \otimes B \otimes A \\
(\hat{\rho}^M(g)) \otimes I_A \\
M' \otimes B \otimes A \\
\hat{\rho}'^M(g)
\end{array}
\longrightarrow
\begin{array}{c}
M \otimes A \\
\hat{\rho}^M(g) \\
M' \otimes A \\
\hat{\rho}'^M(g)
\end{array}
\]

It is obvious that \( Y(d_{M,M'}, \psi) = d_{M,M'} \circ Y(\psi) = d_{M,M'}(Y(\psi)) \). Therefore \( Y \) is a dg-functor. The tensor property of \( Y \) is checked as follows:

- From Example 4.1, we have \( Y(\mathbb{k}, \gamma^\mathbb{k}) = \mathbb{k, \hat{\rho}^k} \).

- Let \((M, \gamma^M)\) and \((M', \gamma'^M)\) be right dg-comodules over \( B \), and \( g : B \to A \) be a morphism of cdg-algebras. Then by the definition of tensor product of representations, we have

\[
q\left( \rho_A \otimes Y(M', \gamma'^M) \right)(g) = (I_M \otimes g) \circ ((I_M \otimes g) \otimes g) \circ (I_M \otimes \tau \otimes I_A) \circ (\gamma^M \otimes \gamma'^M) = (I_M \otimes g) \circ (I_M \otimes m_B) \circ (I_M \otimes \tau \otimes I_A) \circ (\gamma^M \otimes \gamma'^M) = (I_M \otimes g) \circ \gamma^M \otimes \gamma'^M = q\left( \rho_A \otimes Y(M', \gamma'^M) \right)(g).
\]

Thus we conclude that \( Y \left[ (M, \gamma^M) \otimes_{m_B} (M', \gamma'^M) \right] = Y(M, \gamma^M) \otimes Y(M', \gamma'^M) \).

5. It is immediate from the constructions that \( X \) and \( Y \) are inverse to each other. \( \square \)
Every structure in a cdg-Hopf algebra $B$ are reflected faithfully in the dg-tensor category $\text{dgComod}_B(B) \cong \text{Rep}(G^B)$, and this phenomena will make our Tannakian reconstruction possible. To begin with, the following lemma reflects the cdg-bialgebra properties of $B$. Remind that $(B, \Delta_B)$ is a right dg-comodule over $B$.

**Lemma 4.8.** We have the following morphisms of right dg-comodules over $B$.

(a) The product $m_B : B \otimes B \to B$ induces a morphism $(B \otimes B, \gamma_{B \otimes m_B}) \to (B, \Delta_B)$.

(b) The unit $u_B : k \to B$ induces a morphism $(k, \gamma^k) \to (B, \Delta_B)$.

(c) The coaction $\gamma^M : M \to M \otimes B$ in each right dg-comodule $(M, \gamma^M)$ over $B$ induces a morphism $(M, \gamma^M) \to (M \otimes B, \gamma^{M \otimes_B B})$.

**Proof.** (a) is immediate from the comodule axiom that $\gamma^M$ satisfies. (b) follows from $u_B$ being a dg coalgebra map. Indeed, $(u_B \otimes I_B) \circ \gamma^k = (u_B \otimes u_B) \circ \Delta_k = \Delta_B \circ u_B$. Finally (c) follows from $m_B$ being a dg coalgebra map. Indeed, $(m_B \otimes I_B) \circ \gamma_{B \otimes m_B} = (m_B \otimes m_B) \circ (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B) = \Delta_B \circ m_B$. □

Now we turn to the roles of the antipode $\zeta_B : B \to B$. Since the antipode is an anti-morphism of dg-coalgebras, we have a right dg-comodule $(B^*, \gamma^{B^*})$ over $B$ where $B^* = B^\ast$ as cochain complexes but with the alternative coaction

$$\gamma^{B^*} := \tau \circ (\zeta_B \otimes I_B) \circ \Delta_B : B^* \to B^* \otimes B.$$

This is indeed a right $B$-comodule, since

$$(\gamma^{B^*} \otimes I_B) \circ \gamma^{B^*} = \sigma \circ (\tau \otimes I_B) \circ (\zeta_B \otimes I_B) \circ (I_B \otimes \Delta_B) \circ \Delta_B$$

$$= \sigma \circ (\tau \otimes I_B) \circ (\zeta_B \otimes \zeta_B \otimes I_B) \circ (\Delta_B \otimes I_B) \circ \Delta_B$$

$$= \sigma \circ (\Delta_B \otimes I_B) \circ (\zeta_B \otimes I_B) \circ \Delta_B$$

$$= (I_B \otimes \Delta_B) \circ \gamma^{B^*}.$$  

Here, $\sigma : B^{o3} \to B^{o3}$ is the permutation $\sigma(b_1 \otimes b_2 \otimes b_3) := (-1)^{|b_1|b_2|b_3} b_3 \otimes b_1 \otimes b_2$. We used $\zeta_B$ being an anti-morphism of dg-coalgebras on the 3rd equality. Using the unit $u_B : k \to B$, every cochain complex $M$ becomes a right dg-comodule $(M, \gamma^M)$ over $B$ with the coaction $\gamma^M := (I_M \otimes u_B) \circ j^{-1} : M \to M \otimes B$.

**Lemma 4.9.** We have the following morphisms of right dg-comodules over $B$:

(a) The product $m_B : B \otimes B \to B$ induces a morphism $(B^* \otimes B^*, \gamma^{B^* \otimes m_B B^*}) \to (B^*, \gamma^{B^*})$.

(b) The unit $u_B : k \to B$ induces a morphism $u_B : (k, \gamma^k) \to (B^*, \gamma^{B^*})$;

(c) The coaction $\gamma^M : M \to M \otimes B$ in each right dg-comodule $(M, \gamma^M)$ over $B$ induces a morphism $(M, \gamma^M) \to (M \otimes B^*, \gamma^{M \otimes_B B^*})$. 

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Proof. We shall see that (a) and (b) follows from \( \zeta_B \) being a dg-algebra map, and (c) follows from the antipode axiom of \( \zeta_B \).

For (a) we check that \((m_B \otimes I_B) \circ \gamma^{B \otimes m_B B^*} = \gamma^{B^*} \circ m_B:\)

\[
(m_B \otimes I_B) \circ \gamma^{B \otimes m_B B^*} = (m_B \otimes m_B) \circ (I_B \otimes \tau \otimes I_B) \circ (\tau \otimes \tau) \circ (\zeta_B \otimes I_B \otimes \zeta_B \otimes I_B) \circ (\Delta_B \otimes \Delta_B),
\]

\[
= (m_B \otimes m_B) \circ (I_B \otimes I_B \otimes \zeta_B \otimes \zeta_B) \circ \sigma'(\Delta_B \otimes \Delta_B),
\]

\[
\gamma^{B^*} \circ m_B = (\tau \circ (\zeta_B \otimes I_B) \circ \Delta_B \circ m_B)
\]

\[
= \tau \circ (\zeta_B \otimes I_B) \circ (m_B \otimes m_B) \circ (I_B \otimes \tau \otimes I_B) \circ (\Delta_B \otimes \Delta_B)
\]

\[
= (I_B \otimes \zeta_B) \circ (m_B \otimes m_B) \circ \sigma'(\Delta_B \otimes \Delta_B),
\]

where \( \sigma' := (I_B \otimes \tau \otimes I_B) \circ (\tau \otimes \tau) : B^{\otimes 4} \to B^{\otimes 4}. \) From the property \( m_B \circ (\zeta_B \otimes \zeta_B) = \zeta_B \circ m_B, \) we have \((m_B \otimes I_B) \circ \gamma^{B \otimes m_B B^*} = \gamma^{B^*} \circ m_B.\)

For (b) we check that \( \gamma^{B^*} \circ u_B = (u_B \otimes I_B) \circ \gamma^k: \)

\[
\gamma^{B^*} \circ u_B = (\tau \circ (\zeta_B \otimes I_B) \circ \Delta_B \circ u_B = \tau \circ (\zeta_B \otimes I_B) \circ (u_B \otimes u_B) \circ \Delta_k = (u_B \otimes u_B) \circ \Delta_k
\]

\[
= (u_B \otimes I_B) \circ \gamma^k,
\]

where we used the property \( \zeta_B \circ u_B = u_B \) and the cocommutativity of \( \Delta_k. \)

For (c) we check that \( \gamma^{M \otimes m_B B^*} \circ \gamma^M = (\gamma^M \otimes I_B) \circ \gamma^M: \)

\[
\gamma^{M \otimes m_B B^*} \circ \gamma^M
\]

\[
= (I_M \otimes m_B \otimes I_B) \circ (I_M \otimes \tau \otimes I_B) \circ (I_M \otimes \tau \otimes I_B) \circ (\gamma^M \otimes I_B) \circ \gamma^M
\]

\[
= (I_M \otimes \tau) \circ (I_M \otimes m_B \otimes I_B) \circ (I_M \otimes \tau \otimes I_B) \circ (\gamma^M \otimes I_B) \circ \gamma^M
\]

\[
= (I_M \otimes \tau) \circ (I_M \otimes m_B \otimes I_B) \circ (I_M \otimes \tau \otimes I_B) \circ (\gamma^M \otimes I_B) \circ \gamma^M
\]

\[
= (I_M \otimes \tau) \circ (I_M \otimes m_B \otimes I_B) \circ (I_M \otimes \tau \otimes I_B) \circ (\gamma^M \otimes I_B) \circ \gamma^M
\]

\[
= (I_M \otimes \tau) \circ (I_M \otimes u_B \otimes I_B) \circ (I_M \otimes \epsilon_B \otimes I_B) \circ (\gamma^M \otimes I_B) \circ \gamma^M
\]

\[
= (\gamma^M \otimes I_B) \circ (I_M \otimes u_B \otimes I_B) \circ (J^{-1}_{1} \otimes I_B) \circ \gamma^M
\]

\[
= (\gamma^M \otimes I_B) \circ (I_M \otimes u_B \otimes I_B) \circ J^{-1}_{1} \gamma^M
\]

\[
= (\gamma^M \otimes I_B) \circ \gamma^M.
\]

In the above, we used \((\gamma^M \otimes I_B) \circ \gamma^M = (I_M \otimes \Delta_B) \circ \gamma^M) \) on the 3rd and 6th equality, the coassociativity of \( \Delta_B \) on the 4th equality and the antipode axiom \( m_B \circ (\zeta_B \otimes I_B) \circ \Delta_B = u_B \circ \epsilon_B \) on the 5th equality. The rest equalities are straightforward. \( \square \)

The existence of those morphisms of right dg-comodules over \( B \) in Lemma 4.8 and Lemma 4.9 will play the crucial roles in our Tannaka type reconstruction theorem.
5. Tannakian reconstruction theorem for affine group dg-schemes

Throughout this section we fix a cdg-Hopf algebra $B = (B, u_B, m_B, ε_B, Δ_B, ς_B)$. Let $G_B : cdgA(k) → Grp$ be the functor represented by the cdg-algebra $B$, which induces the functor $ς_B : hocdgA(k) → Grp$ represented by $B$ on the homotopy category $hocdgA(k)$ — an affine group dg-scheme.

In Sect. 5.1, we consider the forgetful functor $ω : dgComod R(B) → CoCh(k)$ from the dg-category of right dg-comodules over $B$ to the dg-category of cochain complexes over $k$ and, out of $ω$, construct two functors

$$G_ω : cdgA(k) → Grp$$

$$ς_ω : hocdgA(k) → Grp.$$

Then we shall establishes natural isomorphisms of functors $G_ω ≅ G^B$ and $ς_ω ≅ ς^B$, which is our reconstruction theorem of an affine group dg-scheme from the dg-tensor category of linear representations.

In Sect. 5.2, we consider the dg-category $dgComod R(B)_f$ of finite dimensional right dg-comodules over $B$, which is isomorphic as dg-tensor categories to the dg-category $\overline{\text{Rep}}(G^B)_f$ of finite dimensional linear representations of $G^B$. From the forgetful functor $ω_f : dgComod R(B)_f → CoCh(k)_f$ to the dg-category of finite dimensional cochain complexes, we construct two functors

$$G_ω : cdgA(k) → Grp$$

$$ς_ω : hocdgA(k) → Grp,$$

and establishes natural isomorphisms of functors $G_ω ≅ G^B$ and $ς_ω ≅ ς^B$, which is our reconstruction theorem of an affine group dg-scheme from the dg-tensor category of finite dimensional linear representations.

Our proofs in Sect. 5.2 are based on the reductions of the constructions in Sect. 5.1 to the finite dimensional cases, and we do not use so called rigidity. In Sect. 5.3, we give an independent proof using the dg-version of rigidity of the dg-tensor category $dgComod R(B)_f$.

5.1. Reconstruction from the dg-category of linear representations

The forgetful functor $ω : dgComod R(B) → CoCh(k)$ is a dg-tensor functor. The functor $ω$ sends a right dg-comodule $(M, γ^M)$ over $B$ to its underlying cochain complex $M$, and a morphism $ψ : (M, γ^M) → (M', γ'M')$ of right dg-comodules over $B$ to its underlying $k$-linear map $ψ : M → M'$. It sends the unit object $(k, γ^k)$ to the unit object $k$ and the tensor product $(M, γ^M) ⊗_{m_B} (M', γ'M')$ of right dg-comodules over $B$ to the tensor product $M ⊗ M'$ of the underlying cochain complexes. It also sends the following isomorphisms

$$((M, γ^M) ⊗_{m_B} (M', γ'M')) ⊗_{m_B} (M'', γ'M'') ≅ (M, γ^M) ⊗_{m_B} ((M', γ'M') ⊗_{m_B} (M'', γ'M'')),$$

$$(M, γ^M) ⊗_{m_B} (k, γ^k) ≅ (M, γ^M) ≅ (k, γ^k) ⊗_{m_B} (M, γ^M).$$
of right dg-comodules over $B$ to the corresponding isomorphisms $(M \otimes M') \otimes M'' \cong M \otimes (M' \otimes M'')$ and $M \otimes k \cong M \cong k \otimes M$ of the underlying cochain complexes.

Out of $\omega$, we shall construct three functors $E^\omega : \text{cdgA}(k) \rightarrow \text{dgA}(k), G^\omega : \text{cdgA}(k) \rightarrow \text{Grp}$ and $\mathcal{G}^\omega : \text{hocdgA}(k) \rightarrow \text{Grp}$ in turns. Then we shall establish natural isomorphisms $E^\omega \cong E^B, G^\omega \cong G^B$ and $\mathcal{G}^\omega \cong \mathcal{G}^B$, which constitute our reconstruction theorem.

In Lemma 4.3, we have defined a dg-tensor functor $- \otimes A : \text{CoCh}(k) \rightarrow \text{dgMod}_{R}^{R}(A)$ for each cdg-algebra $A$. By composing it with $\omega$, we get a dg-tensor functor

$$\omega \otimes A := (- \otimes A) \circ \omega : \text{dgComod}_{R}(B) \rightarrow \text{dgMod}_{R}^{R}(A).$$

sending

$$- \text{ a right dg-comodule } (M, \gamma^M) \text{ over } B \text{ to a free right dg-module } (M \otimes A, \mathbb{I}_M \otimes m_A) \text{ over } A, \text{ and}$$

$$- \text{ a morphism } \psi : (M, \gamma^M) \rightarrow (M', \gamma^{M'}) \text{ of right dg-comodules over } B \text{ to a morphism } \psi \otimes \mathbb{I}_A : (M \otimes A, \mathbb{I}_M \otimes m_A) \rightarrow (M' \otimes A, \mathbb{I}_M' \otimes m_A') \text{ of right dg-modules over } A.$$

Let $\text{End}(\omega \otimes A) := \text{Nat}(\omega \otimes A, \omega \otimes A)$ be the set of natural endomorphisms of the dg-functor $\omega \otimes A$. We write an element in $\text{End}(\omega \otimes A)$ as $\eta_A$, and denote $\eta_A^M$ as its component at a right dg-comodule $(M, \gamma^M)$ over $B$. The component at the tensor product $(M, \gamma^M) \otimes_{m_B} (M', \gamma^{M'})$ is denoted by $\eta_A^{M \otimes_{m_B} M'}$. Be aware that the component at the cofree right dg-comodule $(M \otimes B, \mathbb{I}_M \otimes \Delta_B)$ over $B$ is denoted by $\eta_A^{M \otimes B}$. We have the following structure of dg-algebra on $\text{End}(\omega \otimes A)$:

$$E^\omega(A) := \{ \text{End}(\omega \otimes A), \mathbb{I}_{\omega \otimes A}, \circ, \delta_A \} \quad (5.1)$$

where $\mathbb{I}_{\omega \otimes A}$ is the identity natural transformation, $\circ$ is the composition and $\delta_A$ is the differential given by $(\delta_A \eta_A)^M := d_{M \otimes A, M \otimes A} \eta_A^M$.

**Lemma 5.1.** We have a functor $E^\omega : \text{cdgA}(k) \rightarrow \text{dgA}(k)$, sending

$$- \text{ each } \text{cdg-algebra } A \text{ to the dg-algebra } E^\omega(A), \text{ and}$$

$$- \text{ each morphism } f : A \rightarrow A' \text{ of cdg-algebras to a morphism } E^\omega(f) : E^\omega(A) \rightarrow E^\omega(A') \text{ of dg-algebras, where the image of } \eta_A \in \text{End}(\omega \otimes A) \text{ is defined by}$$

$$E^\omega(f)(\eta_A)^M := p \left( (f \otimes \mathbb{I}_A) \circ q(\eta_A^M) \right) = (\mathbb{I}_M \otimes m_{A'}) \circ (\mathbb{I}_M \otimes f \otimes \mathbb{I}_A) \circ (q(\eta_A^M) \otimes \mathbb{I}_A)$$

$$= (\mathbb{I}_M \otimes m_{A'}) \circ (\mathbb{I}_M \otimes f \otimes \mathbb{I}_A) \circ (q(\eta_A^M) \otimes (\mathbb{I}_M \otimes A) \circ J_M^{-1} \otimes \mathbb{I}_A')$$

for every right dg-comodule $(M, \gamma^M)$ over $B$. 

Proof. We first show that $E_\omega(f)(\eta_A)$ is an element in $\text{End}(\omega \otimes A')$ of degree $|\eta_A|$. Let $\psi : (M, \gamma^M) \to (M', \gamma^{M'})$ be a morphism of right dg-comodules over $B$. Since $\eta_A$ is a natural transformation, the following diagram commutes:

\[
\begin{array}{c}
M \otimes A \xrightarrow{\psi \otimes I_A} M' \otimes A, \\
\eta_A^M \downarrow \quad \eta_A^{M'} \downarrow
\end{array}
\]

It follows that

\[
(\psi \otimes I_A) \circ E_\omega(f)(\eta_A)^M = (\eta_A^M \circ (\eta_A^{M'} \circ (\psi \otimes I_A))).
\]

Therefore $E_\omega(f)(\eta_A)$ is an element in $\text{End}(\omega \otimes A')^{\eta_A}$. It remains to show that

- $E_\omega(f)$ is a morphism of $\text{End}(\omega \otimes A')$,
- $E_\omega(I_A) = I_{\omega \otimes A}$, and
- $E_\omega(f' \circ f) = E_\omega(f') \circ E_\omega(f)$ for another morphism $f' : A' \to A''$ of cdg-algebras.

These are immediate from the analogous properties of $E^M$ for cochain complexes $M = \omega(M, \gamma^M)$, as stated in Lemma 4.4. □

Later in this section we shall construct an isomorphism $E_\omega \cong E^B : \text{cdgA}(k) \to \text{dgA}(k)$ of functors, where $E^B$ is the functor defined in Lemma 3.1.

Now we turn to construct the functor $G_\omega : \text{cdgA}(k) \to \text{Grp}$ after some preparations.

**Definition 5.1.** We consider the following subsets of $\text{End}(\omega \otimes A)$:

(a) $Z^0 \text{End}(\omega \otimes A)$ consisting of every element $\eta_A \in \text{End}(\omega \otimes A)^0$ satisfying $\delta_A \eta_A = 0$.

(b) $\text{End}_\omega(\omega \otimes A)$ consisting of every element $\eta_A \in \text{End}(\omega \otimes A)^0$ satisfying the conditions

\[
\eta_A^k = I_{k \otimes A}, \quad \eta_A^M \otimes_{m_A} \eta_A^{M'} = \eta_A^M \otimes_{m_A} \eta_A^{M'}
\]

for all right dg-comodules $(M, \gamma^M)$ and $(M', \gamma^{M'})$ over $B$.

(c) $Z^0 \text{End}_\omega(\omega \otimes A) := Z^0 \text{End}(\omega \otimes A) \cap \text{End}_\omega(\omega \otimes A)$.

We say an $\eta_A \in \text{End}_\omega(\omega \otimes A)$ is a tensor natural transformation. We remind that

\[
\eta_A^M \otimes_{m_A} \eta_A^{M'} = (I_{M \otimes M'} \otimes m_A) \circ (I_M \otimes \tau \otimes I_A) \circ (q(\eta_A^M) \otimes q(\eta_A^{M'}))
\]

We also have
Lemma 5.2. If $\eta_A \in \text{End}_\omega(\omega \otimes A)$ then for right dg-comodules $(M, \gamma^M)$ and $(M', \gamma^{M'})$ over $B$, we have $(\delta_A \eta_A)^{M \otimes_{\omega} M'} = (\delta_A \eta_A)^M \otimes_{m_A} \eta_A^{M'} + \eta_A^M \otimes_{m_A} (\delta_A \eta_A)^{M'}$.

Proof. Since $\eta_A$ is a tensor natural transformation, we have
\[
(\delta_A \eta_A)^{M \otimes_{\omega} M'} := d_{M \otimes M' \otimes A, M \otimes M' \otimes A} \eta_A^{M \otimes_{\omega} M'} = d_{M \otimes M' \otimes A, M \otimes M' \otimes A} \left( \eta_A^M \otimes_{m_A} \eta_A^{M'} \right)
\]
\[
= (d_{M \otimes A, M \otimes A} \eta_A^M) \otimes \eta_A^{M'} + (-1)^{|\eta_A^M|} \eta_A^M \otimes (d_{M' \otimes A, M' \otimes A} \eta_A^{M'})
\]
\[
= (\delta_A \eta_A)^M \otimes_{m_A} \eta_A^{M'} + \eta_A^M \otimes_{m_A} (\delta_A \eta_A)^{M'}.
\]
\[\square\]

We say an $\eta_A \in Z^0 \text{End}_\omega(\omega \otimes A)$ is a dg-tensor natural transformation. It is clear that the set $Z^0 \text{End}_\omega(\omega \otimes A)$ is closed under composition and contains $\mathbb{I}_{\omega \otimes A}$ so that it has a structure of monoid for every cdg-algebra $A$ denoted by
\[
\mathcal{G}_\omega^*(A) := \left( Z^0 \text{End}_\omega(\omega \otimes A), \mathbb{I}_{\omega \otimes A}, \circ \right) \tag{5.4}
\]
We shall show that this is actually a group. We begin with a technical lemma.

Lemma 5.3. For every $\eta_A \in \text{End}(\omega \otimes A)$, its component $\eta_A^{M \otimes B}$ at the cofree right dg-comodule $(M \otimes B, \mathbb{I}_M \otimes \Delta_B)$ over $B$ cogenerated by a cochain complex $M$ is $\eta_A^{M \otimes B} = \mathbb{I}_M \otimes \eta_A^B$.

Proof. For each $z \in M$, define a linear map $f_z : B \rightarrow M \otimes B$ of degree $|z|$ by $f_z(b) := z \otimes b$ for all $b \in B$. Then $f_z : (B, \Delta_B) \rightarrow (M \otimes B, \mathbb{I}_M \otimes \Delta_B)$ is a morphism of right dg-comodules over $B$. Since $\eta_A$ is a natural transformation, the following diagram commutes:
\[
\begin{array}{ccc}
B \otimes A & \xrightarrow{f_z \otimes A} & M \otimes B \otimes A \\
\eta_A \downarrow & & \downarrow \eta_A^{M \otimes B} \\
B \otimes A & \xrightarrow{f_z \otimes A} & M \otimes B \otimes A
\end{array}
\]
i.e., $\eta_A^{M \otimes B} \circ (f_z \otimes \mathbb{I}_A) = (-1)^{|\eta_A^M|} |z| (f_z \otimes \mathbb{I}_A) \circ \eta_A^B$.

Then for every $b$ and $a$ in $A$, we have $\eta_A^{M \otimes B}(z \otimes b \otimes a) = \eta_A^{M \otimes B} \circ (f_z \otimes \mathbb{I}_A)(b \otimes a) = (-1)^{|\eta_A| |z|} (f_z \otimes \mathbb{I}_A) \circ \eta_A^B(b \otimes a) = (-1)^{|\eta_A| |z|} z \otimes \eta_A^B(b \otimes a) = \mathbb{I}_M \otimes \eta_A^B(z \otimes b \otimes a)$. Since this equality holds for all $b$ and $z$, we conclude that $\eta_A^{M \otimes B} = \mathbb{I}_M \otimes \eta_A^B$. \[\square\]

Proposition 5.1. For every natural endomorphism $\eta_A \in \text{End}(\omega \otimes A)$ we have another natural endomorphism $\zeta(\eta_A) \in \text{End}(\omega \otimes A)$, whose component $\zeta(\eta_A)^M$ at each right dg-comodule $(M, \gamma^M)$ over $B$ is
\[
\zeta(\eta_A)^M := (f_M \otimes \mathbb{I}_A) \circ (\mathbb{I}_M \otimes \epsilon_B \otimes \mathbb{I}_A) \circ (\mathbb{I}_M \otimes \eta_A^B) \circ (\gamma^M \otimes \mathbb{I}_A) : M \otimes A \rightarrow M \otimes A,
\]
such that $\zeta(\eta_A) \in Z^0 \text{End}_\omega(\omega \otimes A)$ whenever $\eta_A \in Z^0 \text{End}_\omega(\omega \otimes A)$.

The monoid $\mathcal{G}_\omega^*(A) = (Z^0 \text{End}_\omega(\omega \otimes A), \mathbb{I}_{\omega \otimes A}, \circ)$ is actually a group, where the inverse of an $\eta_A \in Z^0 \text{End}_\omega(\omega \otimes A)$ is $\zeta(\eta_A)$. 
Proof. 1. We first check that $\zeta(\eta_A) \in \text{End}(\omega \otimes A)$ whenever $\eta_A \in \text{End}(\omega \otimes A)$. For each morphism $\psi : (M, \gamma^M) \to (M', \gamma'^M)$ of right dg-comodules over $B$ the following diagram commutes:

This shows that $\zeta(\eta_A) \in \text{End}(\omega \otimes A)$.

2. We check that $\zeta(\eta_A) \in Z^0 \text{End}_q(\omega \otimes A)$ whenever $\eta_A \in Z^0 \text{End}_q(\omega \otimes A)$.

It is obvious that $\zeta(\eta_A)$ is in $Z^0 \text{End}_q(\omega \otimes A)$ since for every right dg-comodule $(M, \gamma^M)$ over $B$ all the maps $J_M \otimes I_A, J_M \otimes \epsilon_B \otimes I_A, J_M \otimes \eta_A^R, \gamma^M \otimes I_A$ are of degree 0 and are in the kernels of differentials.

It remains to show that $\zeta(\eta_A) \in \text{End}_q(\omega \otimes A)$ i.e., we have $\zeta(\eta_A)^{M \otimes m_B M'} = \zeta(\eta_A)^M \otimes m_A$ $\zeta(\eta_A)^{M'}$ for all right dg-comodules $(M, \gamma^M)$ and $(M', \gamma'^M)$ over $B$ and $\zeta(\eta_A)^k = I_{k \otimes A}$.

From Lemma 4.9(a), the product $m_B : B \otimes B \to B$ of $B$ is a morphism $m_B : (B^* \otimes B^*, \gamma^R \otimes m_B R^*) \to (B^*, \gamma^R)$ of right dg-comodules over $B$. Since $\eta_A$ is a tensor natural transformation, the following diagram commutes:

Using the above property we obtain that

$$
\zeta(\eta_A)^M \otimes m_A \zeta(\eta_A)^{M'} = (I_M \otimes J_{M'} \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (m_B \otimes I_A) \circ (\eta_A^R \otimes m_A, \eta_A^R) 
$$

$$
= (I_M \otimes J_{M'} \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (m_B \otimes I_A) 
$$

$$
= (I_M \otimes J_{M'} \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (m_B \otimes I_A) 
$$

$$
= \zeta(\eta_A)^{M \otimes m_B M'}. 
$$
From Lemma 4.9(b), the unit $u_B : k \to B$ of $B$ is also a morphism $u_B : (k, \gamma^k) \to (B^*, \gamma^{B^*})$ of right dg-comodules over $B$ so that the following diagram also commutes:

$$
\begin{array}{ccc}
    k \otimes A & \xrightarrow{u_B \otimes 1_A} & B \otimes A \\
\end{array}
$$

i.e., $\eta_A^{B^*} \circ (u_B \otimes \mathbb{I}_A) = u_B \otimes \mathbb{I}_A$.

It follows that $\zeta(\eta_A^k) = (\epsilon_B \otimes \mathbb{I}_A) \circ \eta_A^{B^*} \circ (u_B \otimes \mathbb{I}_A) = (\epsilon_B \otimes \mathbb{I}_A) \circ (u_B \otimes \mathbb{I}_A) = \mathbb{I}_{\otimes A}$.

Therefore we have shown that $\zeta(\eta_A) \in Z^0\mathcal{E}(\omega \otimes A)$.

3. Finally, we show that $\zeta(\eta_A)$ is the inverse of $\eta_A$. It suffices to show that $\zeta(\eta_A)$ is the right inverse of $\eta_A$—we have $\eta_A \circ \zeta(\eta_A) = \mathbb{I}_{\otimes A}$—since every monoid with all right inverses is a group. Equivalently, we should have $\eta_A^M \circ \zeta(\eta_A)^M = \mathbb{I}_{M \otimes A}$ for every right dg-comodule $(M, \gamma^M)$ over $B$. By the definition of $\zeta(\eta_A)^M$ and eq. (5.3), we obtain that

$$
\eta_A^M \circ \zeta(\eta_A)^M = (\eta_A^M \otimes m_A) \circ (\zeta(\eta_A)^M) = (\epsilon_M \otimes m_A) \circ (\zeta(\eta_A)^M)
$$

which immediately implies that $\eta_A^M \circ \zeta(\eta_A)^M = \mathbb{I}_{M \otimes A}$ since we have $(\epsilon_M \otimes \mathbb{I}_A) \circ (\eta_A^M \otimes m_A) \circ (\gamma^M \otimes \mathbb{I}_A) = \mathbb{I}_{M \otimes A}$ by the counit property of coaction $\gamma^M$. It remains to check the claim. Lemma 4.9(c) states that for each right dg-comodule $(M, \gamma^M)$ over $B$, the coaction $\gamma^M$ is a morphism $\gamma^M : (M, \gamma^M) \to (M \otimes B^*, \gamma^{M \otimes B^*})$ of right dg-comodules over $B$. Since $\eta_A^M$ is a tensor natural transformation, the following diagram commutes

$$
\begin{array}{ccc}
    M \otimes A & \xrightarrow{\gamma^M \otimes \mathbb{I}_A} & M \otimes B \otimes A \\
\end{array}
$$

i.e., $(\eta_A^M \otimes m_A) \circ (\gamma^M \otimes \mathbb{I}_A) = (\gamma^M \otimes \mathbb{I}_A) \circ \eta_A^M$.

Then the claimed identity eq. (5.6) amounts to $\eta_A^M = \mathbb{I}_{M \otimes A}$. Indeed, by Lemma 4.8, the coaction $\gamma_M : (M, \gamma_M) \to (M \otimes B, \mathbb{I}_M \otimes \Delta_B)$ and the unit $u_B : (k, \gamma^k) \to (B, m_B)$ are morphisms of right dg-comodules over $B$. Since $\eta_A$ is a tensor natural transformation, the following diagrams commute:

$$
\begin{array}{ccc}
    M \otimes A & \xrightarrow{\gamma_M \otimes \mathbb{I}_A} & M \otimes B \otimes A \\
\end{array}
$$

i.e., $\eta_A^M \circ (\gamma_M \otimes \mathbb{I}_A) = (\gamma_M \otimes \mathbb{I}_A) \circ \eta_A^M$.

$$
\begin{array}{ccc}
    k \otimes A & \xrightarrow{u_B \otimes \mathbb{I}_A} & B \otimes A \\
\end{array}
$$

i.e., $\eta_A^k \circ (u_B \otimes \mathbb{I}_A) = (u_B \otimes \mathbb{I}_A) \circ \eta_A^k$.
Equality on the left diagram is due to Lemma 5.3. Therefore, we have
\[ n^M_A = (J_M \circ (I_M \otimes \epsilon_B) \circ \gamma^M) \otimes I_A \circ n^M_A \]
\[ = (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (\gamma^M \otimes I_A) \circ n^M_A \]
\[ = (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes n^B_A) \circ (I_M \otimes u_B \otimes I_A) \circ (j^{-1}_M \otimes I_A) \]
\[ = (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes u_B \otimes I_A) \circ (j^{-1}_M \otimes I_A) = I_{M \otimes A} \]

This finish our proof that the monoid \( G^\omega(A) \) is actually a group. \( \Box \)

The following lemma shows that the above construction is functorial.

**Proposition 5.2.** We have a functor \( G^\omega : \text{cdgA}(k) \to \text{Grp} \), sending
- each cdg-algebra \( A \) to the group \( G^\omega(A) \), and
- each morphism \( f : A \to A' \) of cdg-algebras to a morphism \( G^\omega(f) : G^\omega(A) \to G^\omega(A') \)
of groups defined by \( G^\omega(f) : := \mathcal{E}^\omega(f) \).

**Proof.** In Proposition 5.2, we have already showed that \( G^\omega(A) \) is a group for every \( A \). To show that \( G^\omega(f) \) is a group homomorphism, it suffices to check that for every morphism \( f : A \to A' \) of cdg-algebras, we have \( G^\omega(f)(\eta_A) \in Z^0 \text{End}_k(\omega \otimes A') \) whenever \( \eta_A \in Z^0 \text{End}_k(\omega \otimes A) \), i.e.,
1. \( G^\omega(f)(\eta_A) \in Z^0 \text{End}_k(\omega \otimes A') \);
2. \( G^\omega(f)(\eta_A)^k = I_{k \otimes A} \);
3. \( G^\omega(f)(\eta_A)^M \otimes_{m_B} M' = G^\omega(f)(\eta_A)^M \otimes_{m_{A'}} G^\omega(f)(\eta_A)^{M'} \) for all right dg-comodules \( (M, \gamma^M) \) and \( (M', \gamma^{M'}) \) over \( B \).

Then, \( G^\omega(f) = \mathcal{E}^\omega(f) \) is a group homomorphism due to Lemma 5.1, which also implies the functoriality of \( G^\omega(f) \).

Property (1) is obvious since \( G^\omega(f) \) is a cochain map. Property (2) follows from \( \eta_A^k = I_{k \otimes A} \) and \( f \circ u_A = u_A \), since we have
\[ G^\omega(f)(\eta_A)^k = (I_k \otimes m_A) \circ (I_k \otimes f \otimes I_A) \circ (\eta_A^k \otimes I_A) \circ (I_k \otimes u_A \otimes I_A) \circ (\Delta_k \otimes I_A) \]
\[ = (I_k \otimes m_A) \circ (I_k \otimes u_A \otimes I_A) \circ (\Delta_k \otimes I_A) = I_{k \otimes A} \]

Note that Property (3) is equivalent to the condition
\[ q(G^\omega(f)(\eta_A)^M \otimes_{m_{A'}} G^\omega(f)(\eta_A)^{M'}) = q(G^\omega(f)(\eta_A)^M \otimes_{m_{A'}} G^\omega(f)(\eta_A)^{M'}) \]
which can be checked as follows:
\[ q\left(\mathcal{G}^\omega(f)(\eta_A)^M \otimes_{m_A} \mathcal{G}^\omega(f)(\eta_A)^M'\right) \]
\[ = (1_{M\otimes M'} \otimes m_A) \circ (1_M \otimes \tau \otimes 1_A) \circ q\left(\mathcal{G}^\omega(f)(\eta_A)^M \otimes q\left(\mathcal{G}^\omega(f)(\eta_A)^M'\right)\right) \]
\[ = (1_{M\otimes M'} \otimes m_A) \circ (1_M \otimes \tau \otimes 1_A) \circ (1_{M \otimes M'} \otimes f) \circ q\left(\eta_A^M \otimes q(\eta_A^M')\right) \]
\[ = (1_{M \otimes M'} \otimes f) \circ (1_{M \otimes M'} \otimes m_A) \circ (1_{M \otimes \tau \otimes 1_A}) \circ q\left(\eta_A^M \otimes q(\eta_A^M')\right) \]
\[ = (1_{M \otimes M'} \otimes f) \circ q\left(\eta_A^M \otimes_{m_A} \eta_A^M'\right) = (1_{M \otimes M'} \otimes f) \circ q\left(\eta_A^M \otimes_{m_B} \eta_A^M'\right). \]

We used \( f \circ m_A = m_A \circ (f \otimes f) \) on the 3rd equality, and used \( \eta_A^M \otimes_{m_A} \eta_A^M' = \eta_A^{M \otimes m_B} \eta_A^{M'} \) on the 5th equality. \( \square \)

Later in this section we shall construct an isomorphism \( \mathcal{G}^\omega \cong \mathcal{G}^B : cdgA(k) \rightarrow dgA(k) \) of functors, where \( \mathcal{G}^B \) is the functor represented by the cdg-Hopf algebra \( B \) as defined in Lemma 3.2.

We remind that the group \( \mathcal{G}^B(A) \) for each cdg-algebra \( A \) is the group formed by the set \( \text{Hom}_{cdgA(k)}(B, A) \) of morphisms of cdg-algebras. We also remind the functor \( \mathcal{G}^B : cdgA(k) \rightarrow dgA(k) \) induces the functor \( \mathcal{G}^B : h{}ocdgA(k) \rightarrow dgA(k) \) on the homotopy category \( h{}ocdgA(k) \), where \( \mathcal{G}^B(A) \) is the group formed by the set \( \text{Hom}_{h{}ocdgA(k)}(B, A) \) of homotopy types of elements in \( \text{Hom}_{cdgA(k)}(B, A) \). Likewise, we need to define homotopy types of elements in \( Z^0\text{End}_\omega(\omega \otimes A) \)—taking cohomology classes is not compatible with the tensor condition eq. (5.2): let \( \eta_A \in Z^0\text{End}_\omega(\omega \otimes A) \) and \( \tilde{\eta}_A = \eta_A + \delta_A \lambda_A \) for some \( \lambda_A \in \text{End}(\omega \otimes A) \) of degree \(-1\), then \( \tilde{\eta}_A \) and \( \eta_A \) belong to the same cohomology class but \( \tilde{\eta}_A \), in general, is not a tensor natural transformation.

**Definition 5.2.** A homotopy pair on \( Z^0\text{End}_\omega(\omega \otimes A) \) is a pair of one parameter families \((\eta(t)_A, \lambda(t)_A) \in \text{End}(\omega \otimes A)^0[t] \otimes \text{End}(\omega \otimes A)^{-1}[-t] \) parametrized by the time variable \( t \) with polynomial dependence, satisfying the homotopy flow equation \( \frac{d}{dt}\eta(t)_A = \delta_A \lambda(t)_A \) generated by \( \eta(t)_A \) subject to the following conditions:

\[ \eta(0)_A \in Z^0\text{End}_\omega(\omega \otimes A), \quad \left\{ \begin{array}{l}
\lambda_A^k = 0, \\
\lambda_A^M \otimes_{m_B} \eta_A^M' = \lambda_A^M \otimes_{m_A} \eta_A^M' + \eta_A^M \otimes_{m_A} \lambda_A^M'.
\end{array} \right. \]

Let \((\eta(t)_A, \lambda(t)_A)\) be a homotopy pair on \( Z^0\text{End}_\omega(\omega \otimes A) \). It follows from the homotopy flow equation that \( \eta(t)_A \) is uniquely determined by \( \eta(t)_A = \eta(0)_A + \delta_A \int_0^t \lambda(s)_A ds \), and we have \( \delta_A \eta(t)_A = 0 \) since \( \delta_A \eta(0)_A = 0 \). From the condition \( \eta_A^k = \eta_A^k = 1_{k \otimes A} \) and \( \lambda(t)_A = 0 \), we have \( \delta_A(t) = 1_{k \otimes A} \). Moreover, by applying Lemma 5.2, we can check
that
\[
\frac{d}{dt}(\eta(t)_A^M \otimes_m \eta(t)_A^{M'}) = \delta_A \left[ (\lambda(t)_A^M \otimes_m \eta(t)_A^{M'} - \eta(t)_A^M \otimes_m \lambda(t)_A^{M'})\right] = 0.
\]

It follows that we have \(\eta(t)_A^M \otimes_m \eta(t)_A^{M'} = \eta(t)_A^M \otimes_m \eta(t)_A^{M'}\) for all \(t\) since \(\eta(0)_A^M \otimes_m \eta(0)_A^{M'} = \eta(0)_A^M \otimes_m \eta(0)_A^{M'}\). Therefore \(\eta(t)_A\) is a family of elements in \(Z^0 \text{End}_\omega(\omega \otimes A)\). Then, we declare that \(\eta(1)_A\) is homotopic to \(\eta(0)_A\) by the homotopy \(\int_0^1 \lambda(t)_A dt\), and denote \(\eta(0)_A \sim \eta(1)_A\), which is clearly an equivalence relation. In other words, two elements \(\eta_A\) and \(\tilde{\eta}_A\) in the set \(Z^0 \text{End}_\omega(\omega \otimes A)\) are homotopic if there is a homotopy flow connecting them (by the time 1 map). Then, we also say that \(\eta_A\) and \(\tilde{\eta}_A\) have the same homotopy type, and denote it as \([\eta_A] = [\tilde{\eta}_A]\).

Let \(hoZ^0 \text{End}_\omega(\omega \otimes A)\) be the set of homotopy types of elements in \(Z^0 \text{End}_\omega(\omega \otimes A)\). It is a routine check that \(\eta_A' \circ \eta_A \sim \tilde{\eta}_A' \circ \tilde{\eta}_A \in Z^0 \text{End}_\omega(\omega \otimes A)\) whenever \(\eta_A' \sim \tilde{\eta}_A'\) in \(Z^0 \text{End}_\omega(\omega \otimes A)\) and \(\eta_A' \circ \eta_A\) and \(\tilde{\eta}_A' \circ \tilde{\eta}_A\) depend only on the homotopy types of \(\eta_A'\) and \(\eta_A\). Therefore we have a well-defined associative composition \( [\eta_A'] \circ [\eta_A] := [\eta_A' \circ \eta_A] \). This shows that \(hoZ^0 \text{End}_\omega(\omega \otimes A), [\omega_\otimes A], \circ \) is a group.

**Proposition 5.3.** We have a functor \(\mathfrak{C}_\omega^\omega: \text{hodgA}(k) \rightarrow \text{Grp}\), sending

- each cdg-algebra \(A\) to the group \(\mathfrak{C}_\omega^\omega(A) := (hoZ^0 \text{End}_\omega(\omega \otimes A), [\omega_\otimes A], \circ)\), and

- each morphism \([f] \in \text{Hom}_{\text{hodgA}(k)}(A, A')\) to a morphism \(\mathfrak{C}_\omega^\omega([f]) : \mathfrak{C}_\omega^\omega(A) \rightarrow \mathfrak{C}_\omega^\omega(A')\) of groups defined by \(\mathfrak{C}_\omega^\omega([f])([\eta_A]) := [G^\omega_\omega(f)([\eta_A])]\) for all \([\eta_A] \in hoZ^0 \text{End}_\omega(M \otimes A)\),

where \(f \in \text{Hom}_{\text{cdgA}(k)}(A, A')\) and \(\eta_A \in Z^0 \text{End}_\omega(M \otimes A)\) are arbitrary representatives of \([f]\) and \([\eta_A]\), respectively.

**Proof.** All we need to show is that \(G^\omega_\omega(f)([\eta_A]) \sim G^\omega_\omega(f)([\tilde{\eta}_A])\) in \(Z^0 \text{End}_\omega(\omega \otimes A')\) whenever \(f \sim \tilde{f}\) in \(\text{Hom}_{\text{cdgA}(k)}(A, A')\) and \(\eta_A \sim \tilde{\eta}_A\) in \(Z^0 \text{End}_\omega(\omega \otimes A)\). It suffices to show the following statement: Let \((f(t), s(t))\) be a homotopy pair on \(\text{Hom}_{\text{cdgA}(k)}(A, A')\) and \((\eta(t)_A, \lambda(t)_A)\) be a homotopy pair on \(Z^0 \text{End}_\omega(\omega \otimes A)\). Then the pair

\[
(\gamma(t)_A := \mathcal{E}^\omega(f(t))(\eta(t)_A), \quad \chi(t)_A := \mathcal{E}^\omega(f(t))(\lambda(t)_A) + \mathcal{E}^\omega(s(t))(\eta(t)_A))
\]

is a homotopy pair on \(Z^0 \text{End}_\omega(\omega \otimes A')\), i.e.,

1. \(\frac{d}{dt} \gamma(t)_A = \delta_N \chi(t)_A\);
2. \(\gamma(0)_A \in Z^0 \text{End}_\omega(\omega \otimes A')\);
3. \(\chi(t)_A|_t = 0\);
(4) \( \chi(t)_{A'}^{M \otimes_{m \cdot} M'} = \chi(t)_{A'}^M \circ \gamma'(t)_{A'}^{M'} + \chi(t)_{A'}^M \otimes_{m \cdot} \chi(t)_{A'}^{M'} \) holds for right dg-comodules \((M, \gamma^M)\) and \((M', \gamma'^M)\) over \(B\).

For property (1), let \((M, \gamma^M)\) be a right dg-comodule over \(B\). Then we have

\[
\frac{d}{dt} \gamma(t)_{A'}^M = \frac{d}{dt} \left[ p \left( [I_M \otimes f(t)] \circ q(\gamma(t)_{A'}^M) \right) \right] \\
= p \left( [I_M \otimes d_{A, A'} s(t)] \circ q(\gamma(t)_{A'}^M) + [I_M \otimes f(t)] \circ q(\delta_A \gamma(t)_{A'}^M) \right) \\
= \left( \delta_A \gamma(t)_{A'}^M \right),
\]

where we used \(d_{A, A'} f(t) = 0\) and \(\delta_A \gamma(t)_{A'}^M = 0\) on the 3rd equality. Property (2) is obvious since \(\eta(0)_A\) is in \(Z^0 \text{End}_\phi(\omega \otimes A)\) and \(f(0) : A \to A'\) is a morphism of cdg-algebras. Property (3) follows from \(\lambda(t)_{A}^k = 0, \eta(t)_{A}^k = \mathbb{I}_{k \otimes A}\) and \(s(t) \circ u_A = 0\), since we have

\[
\chi(t)_{A'}^k = p \left( [I_k \otimes f(t)] \circ q(\lambda(t)_{A}^k) + [I_k \otimes s(t)] \circ q(\eta(t)_{A}^k) \right) = p \left( [I_k \otimes s(t)] \circ (I_k \otimes u_A) \circ \Delta_k \right) \\
= 0.
\]

Note that Property (4) is equivalent to the condition

\[
q(\chi(t)_{A'}^{M \otimes_{m \cdot} M'}) = q(\chi(t)_{A'}^M \circ \gamma'(t)_{A'}^{M'} + \gamma(t)_{A'}^M \otimes_{m \cdot} \chi(t)_{A'}^{M'}), \quad (5.7)
\]

which can be checked as follows. We consider the 1st term in the RHS of eq. (5.7):

\[
q(\chi(t)_{A'}^M \circ \gamma'(t)_{A'}^{M'}) = (I_M \otimes_{m \cdot} m_{A'}) \circ (I_M \otimes \tau \otimes \mathbb{I}_{A'}) \circ q(\chi(t)_{A'}^M \circ q(\gamma(t)_{A'}^{M'})) \\
= (I_M \otimes_{m \cdot} m_{A'}) \circ (I_M \otimes \tau \otimes \mathbb{I}_{A'}) \\
\circ (I_M \otimes s(t) \otimes I_{M'} \otimes f(t)) \circ q(\eta(t)_{A}^M) \circ q(\eta(t)_{A}^{M'}) \\
+ (I_M \otimes_{m \cdot} m_{A'}) \circ (I_M \otimes \tau \otimes \mathbb{I}_{A'}) \\
\circ (I_M \otimes f(t) \otimes I_{M'} \otimes f(t)) \circ q(\lambda(t)_{A}^M) \circ q(\eta(t)_{A}^{M'}). \]
Combining with the similar calculation for the 2nd term in the RHS of eq. (5.7), we obtain that
\[
q(\chi(t)_A^M \otimes m_A \cdot \nu(t)_A^M + \nu(t)_A^M \otimes m_A \cdot \chi(t)_A^M)
\]
\[
= (I_{M \otimes M} \otimes m_A) \circ (I_M \otimes \tau \otimes I_A)
\]
\[
\circ (I_M \otimes s(t) \otimes I_M \otimes f(t) + I_M \otimes f(t) \otimes I_M \otimes s(t)) \circ (q(\nu(t)_A^M) \otimes q(\nu(t)_A^M))
\]
\[
+ (I_M \otimes M) \circ (I_M \otimes s(t) \otimes I_M \otimes f(t) + I_M \otimes f(t) \otimes I_M \otimes s(t)) \circ (q(\nu(t)_A^M) \otimes q(\nu(t)_A^M))
\]
\[
= (I_{M \otimes M} \otimes s(t)) \circ (I_{M \otimes M} \otimes m_A) \circ (I_M \otimes \tau \otimes I_A)\circ (q(\nu(t)_A^M) \otimes q(\nu(t)_A^M))
\]
\[
+ (I_{M \otimes M} \otimes f(t)) \circ (I_{M \otimes M} \otimes m_A) \circ (I_M \otimes \tau \otimes I_A)\circ (q(\nu(t)_A^M) \otimes q(\nu(t)_A^M))
\]
\[
= (I_{M \otimes M} \otimes s(t)) \circ q(\nu(t)_A^M \otimes m_A \cdot \eta(t)_A^M)
\]
\[
+ (I_{M \otimes M} \otimes f(t)) \circ q(\nu(t)_A^M \otimes m_A \cdot \eta(t)_A^M + \eta(t)_A^M \otimes m_A \cdot \lambda(t)_A^M)
\]
\[
= (I_{M \otimes M} \otimes s(t)) \circ q(\nu(t)_A^M \otimes m_A \cdot \eta(t)_A^M)
\]
\[
+ (I_{M \otimes M} \otimes f(t)) \circ q(\nu(t)_A^M \otimes m_A \cdot \eta(t)_A^M + \eta(t)_A^M \otimes m_A \cdot \lambda(t)_A^M)
\]

In the above, we used \( f(t) \circ m_A = m_A \cdot f(t) \) and \( s(t) \circ m_A = m_A \cdot f(t) \) on the 2nd equality, and used \( \eta(t)_A^M \otimes m_A \cdot \eta(t)_A^M = \eta(t)_A^M \otimes m_A \cdot \lambda(t)_A^M = \lambda(t)_A^M \otimes m_A \cdot \lambda(t)_A^M \) on the 4th equality. \( \square \)

Now we are ready to state the main theorem of this section.

**Theorem 5.1.** We have a natural isomorphism of functors
\[
\mathcal{E}^\omega \cong \mathcal{E}^B : \text{ho\textsc{cdgA}(k)} \to \text{Grp}.
\]

Equivalently, the functor \( \mathcal{E}^\omega \) is representable and is represented by the cdg-Hopf algebra \( B \).

The remaining part of this section is devoted to the proof of the above theorem, which is divided into several pieces.

**Proposition 5.4.** We have natural isomorphisms of functors
\[
\mathcal{E}^\omega \cong \mathcal{E}^B : \text{cdgA}(k) \to \text{dgA}(k), \quad \mathcal{G}^\omega \cong \mathcal{G}^B : \text{cdgA}(k) \to \text{Grp}.
\]

In particular, the functor \( \mathcal{G}^\omega \) is representable and is represented by the cdg-Hopf algebra \( B \).
The proof of this proposition is based on the forthcoming two lemmas. Remind that in Lemma 3.1, we defined the dg-algebra $E^B(A) = \{ \text{Hom}(B, A), u_A \circ e_B \cdot \ast, B, A, d_B, A \}$ for every cdg-algebra $A$.

**Lemma 5.4.** We have an isomorphism $\tilde{\eta}_A : E^B(A) \rightleftharpoons E^\omega(A) : \tilde{g}_A$ of dg-algebras for every cdg-algebra $A$, where

- for each $\alpha \in \text{Hom}(B, A)$, the component of $\tilde{\eta}_A(\alpha) \in \text{End}(\omega \otimes A)$ at a right dg-comodule $(M, \gamma^M)$ over $B$ is defined by

\[
\tilde{\eta}_A(\alpha)^M := p((I_M \otimes \alpha) \circ \gamma^M) = (I_M \otimes m_A) \circ (I_M \otimes \alpha \otimes I_A) \circ (\gamma^M \otimes I_A).
\]

- for each $\eta_A \in \text{End}(\omega \otimes A)$, the linear map $\tilde{g}_A(\eta_A) \in \text{Hom}(B, A)$ is defined by

\[
\tilde{g}_A(\eta_A) := I_A \circ (e_B \otimes I_A) \circ q(\eta_A^B) = I_A \circ (e_B \otimes I_A) \circ \eta_A^B \circ (I_B \otimes u_A) \circ j_B^{-1}.
\]

**Proof.** The map $\tilde{g}_A$ is well-defined, since $\tilde{g}_A(\eta_A)$ is obviously a $k$-linear map. The map $\tilde{\eta}_A$ is also well-defined. This is because for every morphism $\psi : (M, \gamma^M) \rightarrow (M', \gamma'^M)$ of right dg-comodules over $B$, we have the following commutative diagram:

This shows that $\tilde{\eta}_A(\alpha)$ is a natural transformation. We claim that $\tilde{g}_A$ and $\tilde{\eta}_A$ are inverse to each other.

- $\tilde{g}_A(\tilde{\eta}_A(\alpha)) = \alpha$ holds for all $\alpha \in \text{Hom}(B, A)$:

\[
\tilde{g}_A(\tilde{\eta}_A(\alpha)) = I_A \circ (e_B \otimes I_A) \circ q(\tilde{\eta}_A(\alpha)^B) = I_A \circ (e_B \otimes I_A) \circ (I_B \otimes \alpha) \circ \Delta_B = \alpha.
\]

- $\tilde{\eta}_A(\tilde{g}_A(\eta_A)) = \eta_A$ holds for all $\eta_A \in \text{End}(\omega \otimes A)$: Let $(M, \gamma^M)$ be a right dg-comodule over $B$. Lemma 4.3(a) states that $\gamma^M : (\Omega \otimes M, m_B \otimes I_M) \rightarrow (M, \gamma^M)$ is a morphism of right dg-modules over $B$. Since $\eta_A$ is a natural transformation, the following diagram commutes

\[
\begin{array}{c}
M \otimes A \xrightarrow{\gamma^M \otimes I_A} M \otimes B \otimes A \\
\downarrow \eta_A^M \quad \downarrow \eta_A^M \otimes I_A \quad \downarrow \eta_A^M \\
M \otimes A \xrightarrow{\gamma^M \otimes I_A} M \otimes B \otimes A
\end{array}
\]

i.e., $(\gamma^M \otimes I_A) \circ \eta_A^M = (I_M \otimes \eta_A^B) \circ (\gamma^M \otimes I_A)$. 

$\Box$
The equality on the diagram is due to Lemma 5.3. Thus we have
\[
\tilde{\eta}_A\left(\tilde{g}_A(\eta_A)^M\right) = (I_M \otimes m_A) \circ (I_M \otimes \left(t_A \circ (\epsilon_B \otimes I_A) \circ q(\eta_A^B)\right) \otimes I_A) \circ (\gamma^M \otimes I_A)
\]
\[
= (I_M \otimes t_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \left(p(\eta_A^B)\right)) \circ (\gamma^M \otimes I_A)
\]
\[
= (I_M \otimes t_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (\gamma^M \otimes I_A)
\]
\[
= (I_M \otimes t_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (\gamma^M \otimes I_A) \circ \eta_A^M = \eta_A^M.
\]

We are left to show that $\tilde{\eta}_A$ and $\tilde{g}_A$ are morphisms of dg-algebras. Since they are inverse to each other, it suffices to show that $\tilde{\eta}_A$ is a morphism of dg-algebras. Clearly, $\tilde{\eta}_A$ is a $k$-linear map of degree 0. Let $(M, \gamma^M)$ be a right dg-comodule over $B$.

- $\tilde{\eta}_A$ is a cochain map, i.e. $\tilde{\delta}_A \circ \tilde{\eta}_A = \tilde{\eta}_A \circ d_{B,A}$: For $\forall \alpha \in \text{Hom}(B, A)$,
  \[
  \delta_A(\tilde{\eta}_A(\alpha))^M = d_{M \otimes A, M \otimes A}(I_M \otimes m_A) \circ (I_M \otimes \alpha \otimes I_A) \circ (\gamma^M \otimes I_A)
  \]
  \[
  = (I_M \otimes m_A) \circ (I_M \otimes d_{B,A} \alpha \otimes I_A) \circ (\gamma^M \otimes I_A)
  \]
  \[
  = \tilde{\eta}_A(d_{B,A}(\alpha)^M).
  \]

- $\tilde{\eta}_A$ sends the identity to the identity, i.e. $\tilde{\eta}_A(u_A \circ \epsilon_B) = \tilde{\eta}_A$:
  \[
  \tilde{\eta}_A(u_A \circ \epsilon_B)^M := (I_M \otimes m_A) \circ (I_M \otimes (u_A \circ \epsilon_B) \otimes I_A) \circ (\gamma^M \otimes I_A) = I_{M \otimes A}.
  \]

- $\tilde{\eta}_A$ preserves the binary operations, i.e. $\tilde{\eta}_A(\alpha_1 \ast_{B,A} \alpha_2) = \tilde{\eta}_A(\alpha_1) \circ \tilde{\eta}_A(\alpha_2)$ for all $\alpha_1, \alpha_2 \in \text{Hom}(B, A)$:
  \[
  \tilde{\eta}_A(\alpha_1 \ast_{B,A} \alpha_2)^M := (I_M \otimes m_A) \circ (I_M \otimes (m_A \circ (\alpha_1 \otimes \alpha_2) \circ \Delta_B) \otimes I_A) \circ (\gamma^M \otimes I_A)
  \]
  \[
  = (I_M \otimes m_A) \circ (I_M \otimes A_1 \otimes I_A) \circ (\gamma^M \otimes I_A) \circ (I_M \otimes m_A) \circ (I_M \otimes \alpha_2 \otimes I_A) \circ (\gamma^M \otimes I_A)
  \]
  \[
  = \tilde{\eta}_A(\alpha_1)^M \circ \tilde{\eta}_A(\alpha_2)^M.
  \]

We used the associativity of $m_A$ and the coaction axiom of $\gamma^M$ on the 2nd equality. □

In Lemma 3.2, we showed that $G^B(A) = \{\text{Hom}_{\text{cdgA}}(B, A), u_A \circ \epsilon_B, \ast_{B,A}\}$ is a group for every cdg-algebra $A$. The inverse of $g \in \text{Hom}_{\text{cdgA}}(B, A)$ is given by $g^{-1} := g \circ \zeta_B$. Remind that $\text{Hom}_{\text{cdgA}}(B, A)$ is the subset of $\text{Hom}(B, A)$ consisting of morphisms of cdg-algebras:

\[
\text{Hom}_{\text{cdgA}}(B, A) = \left\{ g \in \text{Hom}(B, A) \mid \left. \begin{align*}
  d_{B,A}g &= 0, \\
  g \circ m_B &= m_A \circ (g \otimes g), \\
  g \circ u_B &= u_A \right\} \right\}.
\]

**Lemma 5.5.** For every cdg-algebra $A$, the isomorphism in Lemma 5.4 gives an isomorphism $\tilde{\eta}_A : G^B(A) \overset{\sim}{\longrightarrow} G^B_\circ(A) : \tilde{g}_A$ of groups.
Proof. It suffices to check that $\tilde{g}_A(Z^0\text{End}_\omega(\omega \otimes A))$ is contained in $\text{Hom}_{\text{cdga}}(B, A)$ and $\tilde{\eta}_A(\text{Hom}_{\text{cdga}}(B, A))$ is contained in $Z^0\text{End}_\omega(\omega \otimes A)$.

1. For every $\eta_A \in Z^0\text{End}_\omega(\omega \otimes A)$ we have $\tilde{g}_A(\eta_A) \in \text{Hom}_{\text{cdga}}(B, A)$.

- $\tilde{g}_A(\eta_A)$ is of degree 0 and $d_{B,A} \tilde{g}_A(\eta_A) = 0$: This is immediate since $\eta_A$ is of degree 0 with $d_A \eta_A = 0$, and $\tilde{g}_A$ is a cochain map by Lemma 5.4.

- $\tilde{g}_A(\eta_A) \circ u_B = u_A$: Lemma 4.8(b) states that the unit $u_B : k \to B$ is a morphism $(k, \gamma^k \mapsto (B, \Delta_B)$ of right dg-comodules over $B$. Since $\eta_A$ is a tensor natural transformation, the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{k} \otimes A & \xrightarrow{u_B \otimes \iota_A} & B \otimes A \\
\eta_A - \iota_{B,A} & \downarrow & \eta_A \\
\mathbb{k} \otimes A & \xrightarrow{u_B \otimes \iota_A} & B \otimes A,
\end{array}
$$

i.e., $\eta_A^B(\eta_A) \circ (u_B \otimes \iota_A) = (u_B \otimes \iota_A)$.

Therefore we have
$$
\tilde{g}_A(\eta_A) \circ u_B = t_A \circ (\iota_B \otimes \iota_A) \circ \eta_A^B(\eta_A) \circ (u_B \otimes \iota_A) \circ (\iota_k \otimes u_A) \circ \Delta_k
= t_A \circ (\iota_B \otimes \iota_A) \circ (u_B \otimes \iota_A) \circ (\iota_k \otimes u_A) \circ \Delta_k = u_A.
$$

- $\tilde{g}_A(\eta_A) \circ m_B = m_A \circ (\tilde{g}_A(\eta_A) \otimes \tilde{g}_A(\eta_A))$: Lemma 4.8(c) states that the product $m_B : B \otimes B \to B$ is a morphism $(B \otimes B, \gamma_{B \otimes B}^k) \mapsto (B, \Delta_B)$ of right dg-comodules over $B$. Since $\eta_A$ is a tensor natural transformation, the following diagram commutes:

$$
\begin{array}{ccc}
B \otimes B \otimes A & \xrightarrow{m_B \otimes \iota_A} & B \otimes A \\
\eta_A \circ m_B & = \eta_A \circ m_A \circ \eta_A^B & \downarrow \eta_A^B \\
B \otimes B \otimes A & \xrightarrow{m_B \otimes \iota_A} & B \otimes A,
\end{array}
$$

i.e., $(m_B \otimes \iota_A)(\eta_A) = \eta_A^B(\eta_A)$.

Therefore we obtain that
$$
\tilde{g}_A(\eta_A) \circ m_B = t_A \circ (\iota_B \otimes \iota_A) \circ \eta_A^B(\eta_A) \circ (m_B \otimes \iota_A) \circ (\iota_{B \otimes B} \otimes u_A) \circ \Delta_k^{-1}
= t_A \circ (\iota_B \otimes \iota_A) \circ (m_B \otimes \iota_A) \circ (\iota_{B \otimes B} \otimes u_A) \circ \Delta_k^{-1}
= m_A \circ (\tilde{g}_A(\eta_A) \otimes \tilde{g}_A(\eta_A)).
$$

2. For every $g \in \text{Hom}_{\text{cdga}}(B, A)$, we have $\tilde{\eta}_A(g) \in Z^0\text{End}_\omega(\omega \otimes A)$.

- $\tilde{\eta}_A(g)$ is of degree 0 and satisfies $d_A \tilde{\eta}_A(g) = 0$: This is immediate, since $g$ is of degree 0 with $d_{B,A} g = 0$, and $\tilde{\eta}_A$ is a cochain map.

- $\tilde{\eta}_A(g)^k = \iota_{B,A}$: Using $g \circ u_B = u_A$, we have

$$
\begin{align*}
\tilde{\eta}_A(g)^k &= (\iota_k \otimes m_A) \circ (\iota_k \otimes g \otimes \iota_A) \circ (\iota_k \otimes u_B \otimes \iota_A) \circ (\Delta_k \otimes \iota_A) \\
&= (\iota_k \otimes m_A) \circ (\iota_k \otimes u_A \otimes \iota_A) \circ (\Delta_k \otimes \iota_A) = \iota_{B,A}.
\end{align*}
$$
\[ \tilde{\eta}_A(g)^{M \otimes_B M'} = \tilde{\eta}_A(g)^M \otimes_{m_A} \tilde{\eta}_A(g)^{M'} \] holds for right dg-comodules \((M, \gamma^M)\) and \((M', \gamma^{M'})\) over \(B\). This is equivalent to the condition
\[ q(\tilde{\eta}_A(g)^{M \otimes_B M'}) = q(\tilde{\eta}_A(g)^M \otimes_{m_A} \tilde{\eta}_A(g)^{M'}) , \]
which can be checked as follows. Using \(m_A \circ (g \otimes g) = g \circ m_B\), we have
\[ q(\tilde{\eta}_A(g)^M \otimes_{m_A} \tilde{\eta}_A(g)^{M'}) = (\mathbb{I}_M \otimes m_A) \circ (\mathbb{I}_M \otimes (g \otimes I_A)) \circ (\tilde{\eta}_A(g)^M) \]
\[ = (\mathbb{I}_M \otimes m_A) \circ (\mathbb{I}_M \otimes (g \otimes I_B)) \circ (\tilde{\eta}_A(g)^M \otimes (g \otimes \gamma^{M'})) \]
\[ = (\mathbb{I}_M \otimes (g \otimes \gamma^{M'})) \circ (\gamma^{M \otimes_B M'}) = q(\tilde{\eta}_A(g)^{M \otimes_B M'}). \]
\[ \square \]

Now we finish the proof of Proposition 5.4.

**Proof (Proposition 5.4).** We claim that the isomorphisms \(\tilde{\eta}_A : \mathcal{E}^\mathbb{G}(A) \to \mathcal{E}^\omega(A)\) are natural in \(A \in \text{cdgA}(k)\). This will give us a natural isomorphism
\[ \tilde{\eta} : \mathcal{E}^\mathbb{G} \to \mathcal{E}^\omega : \text{cdgA}(k) \to \text{dgA}(k), \]
whose component at a cdg-algebra \(A\) is \(\tilde{\eta}_A\). Then \(\tilde{g} = \{ \tilde{g}_A \}\) automatically becomes a natural transformation, which is the inverse of \(\tilde{\eta}\). Moreover, \(\tilde{\eta}\) will canonically induce a natural isomorphism
\[ \tilde{\eta} : \mathcal{G}^\mathbb{G} \to \mathcal{G}^\omega : \text{cdgA}(k) \to \text{Grp}, \]
with its inverse, again, \(\tilde{g}\). We need to show that for every morphism \(f : A \to A'\) of cdg-algebras the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{E}^\mathbb{G}(A) & \xrightarrow{\tilde{\eta}_A} & \mathcal{E}^\omega(A) \\
\mathcal{E}^\mathbb{G}(f) \downarrow & & \downarrow \mathcal{E}^\omega(f) \\
\mathcal{E}^\mathbb{G}(A') & \xrightarrow{\tilde{\eta}_{A'}} & \mathcal{E}^\omega(A')
\end{array}
\]
i.e., \(\mathcal{E}^\omega(f) \circ \tilde{\eta}_A = \tilde{\eta}_{A'} \circ \mathcal{E}^\mathbb{G}(f)\).

It suffices to show that for every linear map \(g : B \to A\) and every right dg-comodule \((M, \gamma^M)\) over \(B\), we have
\[ q(\mathcal{E}^\omega(f)(\tilde{\eta}_A(g)^M)) = q(\tilde{\eta}_A(f \circ g)^M). \]
Indeed,
\[ q(\mathcal{E}^\omega(f)(\tilde{\eta}_A(g)^M)) = (\mathbb{I}_M \otimes f) \circ q(\tilde{\eta}_A(g)^M) \]
\[ = (\mathbb{I}_M \otimes f) \circ (\mathbb{I}_M \otimes g) \circ \gamma^M \]
\[ = (\mathbb{I}_M \otimes (f \circ g)) \circ \gamma^M = q(\tilde{\eta}_A(f \circ g)^M). \]
\[ \square \]

Finally we can finish the proof of Theorem 5.1.
Proof (Theorem 5.1). By Proposition 5.4 and the definitions of the functors $\mathfrak{S}^B$ and $\mathfrak{S}^c_\omega$ from the homotopy category $\mathop{hodgA(k)}$ to $\mathbf{Grp}$, it suffices to show that for each cdg-algebra $A$,

(a) $\widetilde{\eta}_A$ sends a homotopy pair $\{g(t), \chi(t)\}$ on $\mathop{Hom_{cdgA(k)}}(B, A)$ to a homotopy pair $\{\widetilde{\eta}_A(g(t)), \widetilde{\eta}_A(\chi(t))\}$ on $Z^0\mathop{End}_A(\omega \otimes A)$, and

(b) $\widetilde{g}_A$ sends a homotopy pair $(\eta(t)_A, \lambda(t)_A)$ on $Z^0\mathop{End}_A(\omega \otimes A)$ to a homotopy pair $(\widetilde{g}_A(\eta(t)_A), \widetilde{g}_A(\lambda(t)_A))$ on $\mathop{Hom_{cdgA(k)}}(B, A)$.

Then $\widetilde{\eta}_A$ and $\widetilde{g}_A$ will give an isomorphism of groups $\mathfrak{S}^c_\omega(A) \cong \mathfrak{S}^B(A)$. Moreover, this isomorphism is natural in $A \in \mathop{cdgA(k)}$ by Proposition 5.4 and Lemma 5.3.

We will prove (a) only since the proof of (b) is similar. We need to check the following properties:

1. $\frac{d}{d_t} \widetilde{\eta}_A(g(t)) = \delta_A \widetilde{\eta}_A(\chi(t));$

2. $\widetilde{\eta}_A(g(0)) \in Z^0\mathop{End}_A(\omega \otimes A);$

3. $\widetilde{\eta}_A(\chi(t))^k = 0;$

4. $\widetilde{\eta}_A(\chi(t))^M \otimes_{m_A} \widetilde{\eta}_A(\chi(t))^{M'} = \widetilde{\eta}_A(\chi(t))^M \otimes_{m_A} \widetilde{\eta}_A(\chi(t))^{M'}$ holds for all right cdg-comodules $(M, \gamma^M)$ and $(M', \gamma^{M'})$ over $B$.

Property (1) follows from the condition $\frac{d}{d_t} g(t) = d_{B,A} \chi(t)$, since we have $\frac{d}{d_t} \widetilde{\eta}_A(g(t)) = \delta_A \widetilde{\eta}_A(\chi(t))$. Property (2) follows from the condition $g(0) \in \mathop{Hom_{cdgA(k)}}(B, A)$. Property (3) follows from the condition $\chi(t)^k = 0$, since $\widetilde{\eta}_A(\chi(t))^k = 0$. Note that Property (4) is equivalent to the identity

$$q\left(\widetilde{\eta}_A(\chi(t))^{M} \otimes_{m_A} \widetilde{\eta}_A(g(t))^{M'}\right) = q\left(\widetilde{\eta}_A(\chi(t))^{M} \otimes_{m_A} \widetilde{\eta}_A(g(t))^{M'}\right),$$

which can be checked as follows. We begin with the 1st term in the RHS of eq. (5.8):

$$q\left(\widetilde{\eta}_A(\chi(t))^{M} \otimes_{m_A} \widetilde{\eta}_A(g(t))^{M'}\right)$$

$$= (I_{M \otimes M'} \otimes m_A) \circ (I_{M} \otimes \tau \otimes I_A) \circ \left(q\left(\widetilde{\eta}_A(\chi(t))^{M}\right) \otimes q\left(\widetilde{\eta}_A(g(t))^{M'}\right)\right)$$

$$= (I_{M \otimes M'} \otimes m_A) \circ (I_{M} \otimes \tau \otimes I_A) \circ (I_{M} \otimes \chi(t) \otimes I_M \otimes g(t)) \circ (\gamma^M \otimes \gamma^{M'}).$$
After the similar calculation for the 2nd term in the RHS of eq. (5.8), we have

\[ q(\tilde{\eta}_A(\chi(t))^M \otimes m_A \tilde{\eta}_A(g(t))^M + \tilde{\eta}_A(g(t))^M \otimes m_A \tilde{\eta}_A(\chi(t))^M) \]

\[ = (\mathbb{I}_{M \otimes M'} \otimes m_A) \circ (\mathbb{I}_{M \otimes M'} \otimes \{\chi(t) \otimes g(t) + g(t) \otimes \chi(t)\}) \circ (\mathbb{I}_M \otimes \tau \otimes \mathbb{I}_B) \circ (\gamma^M \otimes \gamma^{M'}) \]

\[ = (\mathbb{I}_{M \otimes M'} \otimes \chi(t)) \circ (\mathbb{I}_{M \otimes M'} \otimes m_B) \circ (\mathbb{I}_M \otimes \tau \otimes \mathbb{I}_B) \circ (\gamma^M \otimes \gamma^{M'}) \]

\[ = (\mathbb{I}_{M \otimes M'} \otimes \chi(t)) \circ \gamma^{M \otimes m_B} = q(\tilde{\eta}_A(\chi(t)))^{M \otimes m_B} \).

We used \( m_A \circ (\chi(t) \otimes g(t) + g(t) \otimes \chi(t)) = \chi(t) \circ m_B \) on the 2nd equality. \( \square \)

### 5.2. Reduction to the dg-category of finite dimensional linear representations

A finite dimensional linear representation of \( \mathcal{G}^B \) is a representation \((M, \rho^M)\) of \( \mathcal{G}^B \) whose underlying cochain complex \( M \) is finite dimensional over \( k \). Denote

\(- \mathcal{CoCh}(k)_f \) as the full sub dg-category of \( \mathcal{CoCh}(k) \) consisting of finite dimensional cochain complexes over \( k \),

\(- \mathcal{Rep}(\mathcal{G}^B)_f \) as the full sub dg-category of \( \mathcal{Rep}(\mathcal{G}^B) \) consisting of finite dimensional linear representations of \( \mathcal{G}^B \) and

\(- \mathcal{dgComod}_{R(B)}(\mathcal{G}^B)_f \) as the full sub dg-category of \( \mathcal{dgComod}_{R(B)}(\mathcal{G}^B) \) consisting of finite dimensional right dg-comodules over \( B \).

Then \( \left( \mathcal{CoCh}(k)_f, \otimes, k \right), \left( \mathcal{Rep}(\mathcal{G}^B)_f, \otimes, (k, \rho^B) \right) \) and \( \left( \mathcal{dgComod}_{R(B)}(\mathcal{G}^B)_f, \otimes_{m_B}, (k, \gamma^B) \right) \) are dg-tensor categories, and there is an isomorphism of dg-tensor categories

\[ \left( \mathcal{Rep}(\mathcal{G}^B)_f, \otimes, (k, \rho^B) \right) \cong \left( \mathcal{dgComod}_{R(B)}(\mathcal{G}^B)_f, \otimes_{m_B}, (k, \gamma^B) \right) \]

by the arguments in Theorem 4.1. We denote \( \omega_f : \mathcal{dgComod}_{R(B)}(\mathcal{G}^B)_f \rightarrow \mathcal{CoCh}(k)_f \) as the forgetful functor, which is also a dg-tensor functor. In this subsection, we define three functors

\[ \mathcal{E}^{\omega_f} : \mathcal{cdgA}(k) \rightarrow \mathcal{dgA}(k), \quad \mathcal{G}^{\omega_f} : \mathcal{cdgA}(k) \rightarrow \mathcal{Grp} \quad \text{and} \quad \mathcal{S}^{\omega_f} : \mathcal{hoA}(k) \rightarrow \mathcal{Grp}, \]

analogous to the three functors \( \mathcal{E}^{\omega}, \mathcal{G}^{\omega} \) and \( \mathcal{S}^{\omega} \) in the previous subsection. We shall show that there are natural isomorphisms of functors

\[ \mathcal{E}^{\omega_f} \cong \mathcal{E}^{\omega}, \quad \mathcal{G}^{\omega_f} \cong \mathcal{G}^{\omega} \quad \text{and} \quad \mathcal{S}^{\omega_f} \cong \mathcal{S}^{\omega}. \]

This will imply that the reconstructions of \( \mathcal{G}^B \) and \( \mathcal{S}^B \) are also valid if we work with their finite dimensional representations.

In this subsection we adapt the Einstein summation convention.

We begin with the following well-known lemma.
**Lemma 5.6.** Let $B$ be a dg-coalgebra, and $(M, \gamma^M)$ be a right dg-comodule over $B$. Then $(M, \gamma^M)$ is the union of its finite dimensional subcomodules over $B$.

**Proof.** It suffices to show that for each $m \in M$, there is a finite dimensional subcomodule of $M$ containing $m$. Fix a basis $\{b_i\}$ of $B$ over $k$ and write

$$\Delta_B(b_i) = \Delta^i_j b_j \otimes b_k, \quad \gamma^M(m) = m^i \otimes b_i \quad (5.9)$$

for some $\Delta^i_j \in k$ and $m^i \in M$. Note that the indices of the sums in eq. (5.9) run finite. Let $M^m_f$ be the $k$-subspace of $M$ spanned by the elements $m$, $m^i$, $d_M m$ and $d_M m^i$. Clearly, $(M^m_f, d_M)$ is a finite dimensional subcomplex of $(M, d_M)$ containing $m$. Moreover, $(M^m_f, \gamma^M)$ is a subcomodule of $(M, \gamma^M)$ over $B$ because

$$\gamma^M(m) = m^i \otimes b_i \in M_m \otimes B,$$

$$\gamma^M(d_M m) = d_{M \otimes B}(\gamma^M(m)) = d_M m^i \otimes b_j + (-1)^{|m^i|} m^i \otimes d_B b_j \in M_m \otimes B.$$  

- the comodule axiom of $\gamma^M$ implies that

$$\gamma^M(m^k) \otimes b_k = (\gamma^M \otimes \mathbb{I}_B)(m^k \otimes b_k) = (\gamma^M \otimes \mathbb{I}_B) \circ \gamma^M(m) = (\mathbb{I}_M \otimes \Delta_B) \circ \gamma^M(m) = (\mathbb{I}_M \otimes \Delta_B)(m^i \otimes b_i) = \Delta^i_j \cdot m^i \otimes b_j \otimes b_k.$$

Therefore we have $\gamma^M(m^k) = \Delta^i_j \cdot m^i \otimes b_j \in M_m \otimes B$.

$$\gamma^M(d_M m^k) = d_{M \otimes B}(\gamma^M(m^k)) = \Delta^i_j \cdot d_M m^i \otimes b_j + (-1)^{|m^i|} \Delta^i_j \cdot m^i \otimes d_B b_j \in M_m \otimes B.$$  

Thus for each $m \in M$, there exists a finite dimensional subcomodule $M^m_f$ of $M$ over $B$ containing $m$. We conclude that $M$ is the union of its finite dimensional subcomodules over $B$. \( \square \)

We may restate the result of Lemma 5.6 as follows: Every right dg-comodule $(M, \gamma^M)$ over $B$ is a filtered colimit of its finite dimensional subcomodules over $B$. Equivalently, we have $M \simeq \lim_{\alpha} M^m_f$ where the index $\alpha$ runs all finite dimensional subcomodules $M^m_f$ of $M$ over $B$.

Take an element $\xi_A \in \text{End}(\mathbb{O}_f \otimes A)$. Since colimits commute with tensor products, we also have $M \otimes A \simeq \lim_{\alpha} (M^m_f \otimes A)$. Hence there exists a unique morphism

$$(\lim_{\alpha} \xi_A)^M := \lim_{\alpha} \left( \xi^M_A \right): M \otimes A \to M \otimes A$$
of right dg-modules over $A$, making the following diagram commutative for all $\alpha$

\[
\begin{array}{ccc}
M \otimes A & \xrightarrow{(\text{lim}_A \xi_A)^M} & M \otimes A \\
\downarrow & & \downarrow \\
M_f^\alpha \otimes A & \xrightarrow{\xi_A^M} & M_f^\alpha \otimes A
\end{array}
\]

**Lemma 5.7.** For every cdg-algebra $A$, there is an isomorphism of dg-algebras

\[
\text{lim}_A : \text{End}(\omega \otimes A) \longrightarrow \text{End}(\omega \otimes A) : \text{res}_A,
\]

which sends

- each $\eta_A \in \text{End}(\omega \otimes A)$ to its restriction $\text{res}_A \eta_A \in \text{End}(\omega \otimes A)$ whose component at a finite dimensional right dg-comodule $(M_f, \gamma^M)$ over $B$ is given by

  \[
  (\text{res}_A \eta_A)^{M_f} := \eta_A^{M_f} : M_f \otimes A \rightarrow M_f \otimes A.
  \]

- each $\xi_A \in \text{End}(\omega \otimes A)$ to $\text{lim}_A \xi_A \in \text{End}(\omega \otimes A)$ whose component at a right dg-comodule $(M, \gamma^M)$ over $B$ is given by

  \[
  (\text{lim}_A \xi_A)^M := \text{lim}_{\alpha} (\xi_A^{M_{\alpha}^f}) : M \otimes A \rightarrow M \otimes A,
  \]

where the index $\alpha$ runs over all finite dimensional subcomodules $M_{\alpha}^f$ of $M$ over $B$.

Moreover, the above isomorphism is natural in $A \in \text{cdgA}(k)$.

**Proof.** For every $\xi_A \in \text{End}(\omega \otimes A)$, we claim that $\text{lim}_A \xi_A$ is a natural transformation. Given a morphism $\psi : (M, \gamma^M) \rightarrow (M', \gamma^{M'})$ of right dg-comodules over $B$, we need to show

\[
(\text{lim}_A \xi_A)^M \circ (\psi \otimes 1_A) = (-1)^{\|\psi\|}(\psi \otimes 1_A) \circ (\text{lim}_A \xi_A)^{M'} : M \otimes A \rightarrow M' \otimes A. \tag{5.10}
\]

Let $M_{\alpha}^f$ be a finite dimensional subcomodule of $M$ over $B$. Then by Lemma 5.6, there exists a finite dimensional subcomodule $M_{\alpha}^{f'}$ of $M'$ such that $\psi(M_{\alpha}^f) \subseteq M_{\alpha}^{f'}$. Since $\xi_A$ is a natural transformation, we have

\[
\xi_A^{M_{\alpha}^f} \circ (\psi \otimes 1_A) = (-1)^{\|\psi\|}(\psi \otimes 1_A) \circ \xi_A^{M_{\alpha}^{f'}} : M_{\alpha}^f \otimes A \rightarrow M_{\alpha}^{f'} \otimes A. \tag{5.11}
\]

Note that $M_{\alpha}^{f'}$ covers $M'$ as $\alpha$ vary. Thus we get eq. (5.10) by applying $\text{lim}_A$ on eq. (5.11).

Clearly the map $\text{res}_A$ is well-defined. It follows from the definitions that both $\text{res}_A$ and $\text{lim}_A$ are morphisms of dg-algebras, are natural in $A$ and are inverse to each other.

\square
Let us define $Z^0\End_\omega(\omega_f \otimes A)$, $\End_\omega(\omega_f \otimes A)$, $Z^0\End_\omega(\omega_f \otimes A)$ and $hoZ^0\End_\omega(\omega_f \otimes A)$ as we did in the previous subsection.

**Lemma 5.8.** The isomorphism in Lemma 5.7 gives an isomorphism

$$\lim_{\rightarrow A} : Z^0\End_\omega(\omega_f \otimes A) \xrightarrow{\sim} Z^0\End_\omega(\omega \otimes A) : \mathrm{res}_A$$

of monoids that are natural in $A \in \mathbf{cdgA}(k)$. In particular, $Z^0\End_\omega(\omega_f \otimes A)$ is a group.

**Proof.** It suffices to check that $\lim_{\rightarrow A} \left(Z^0\End_\omega(\omega_f \otimes A)\right)$ is contained in $Z^0\End_\omega(\omega \otimes A)$ and $\mathrm{res}_A \left(Z^0\End_\omega(\omega \otimes A)\right)$ is contained in $Z^0\End_\omega(\omega_f \otimes A)$.

- It is clear from the definitions that $\mathrm{res}_A \left(Z^0\End_\omega(\omega_f \otimes A)\right) \subseteq Z^0\End_\omega(\omega_f \otimes A)$.

- Let $\xi_A \in Z^0\End_\omega(\omega_f \otimes A)$. Clearly, we have $\lim_{\rightarrow A} \xi_A \in Z^0\End_\omega(\omega_f \otimes A)$ and $\lim_{\rightarrow A} \xi_A^k = \xi_A^k = 1_{k \otimes A}$. Moreover, for right dg-comodules $(M, \gamma^M)$ and $(M', \gamma^{M'})$ over $B$, we have

$$\lim_{\rightarrow A} \xi_A^{M \otimes m_{\beta} M'} = \left(\lim_{\rightarrow A} \xi_A^M \otimes m_{\alpha} \left(\lim_{\rightarrow A} \xi_A^\alpha\right)^{M'}\right) : M \otimes M' \otimes A \rightarrow M \otimes M' \otimes A. \quad (5.12)$$

For the proof, let $M^\alpha_f$ and $M'^{\beta}_f$ be finite dimensional subcomodules of $M$ and $M'$ over $B$, respectively. Then the tensor product $M^\alpha_f \otimes m_{\alpha} M'^{\beta}_f$ is a finite dimensional subcomodule of $M \otimes m_{\alpha} M'$ over $B$. Since $\xi_A$ is a tensor natural transformation, we have

$$\xi_A^{M^\alpha_f \otimes m_{\alpha} M'^{\beta}_f} = \xi_A^{M^\alpha_f} \otimes m_{\alpha} \xi_A^{M'^{\beta}_f} : M^\alpha_f \otimes M'^{\beta}_f \otimes A \rightarrow M^\alpha_f \otimes M'^{\beta}_f \otimes A. \quad (5.13)$$

Note that $M^\alpha_f \otimes m_{\alpha} M'^{\beta}_f$ covers $M \otimes m_{\alpha} M'$ as the indices $\alpha$ and $\beta$ vary. Thus we get eq. (5.12) by applying the colimit $\lim_{\rightarrow A}$ on eq. (5.13). $\square$

**Lemma 5.9.** The isomorphism in Lemma 5.8 induces an isomorphism

$$\lim_{\rightarrow A} : hoZ^0\End_\omega(\omega_f \otimes A) \xrightarrow{\sim} hoZ^0\End_\omega(\omega \otimes A) : \mathrm{res}_A$$

of monoids that is natural in $A \in ho\mathbf{cdgA}(k)$, where we define

$$\lim_{\rightarrow A} [\xi_A] := [\lim_{\rightarrow A} \xi_A], \quad \mathrm{res}_A [\eta_A] := [\mathrm{res}_A \eta_A]$$

for $[\xi_A] \in hoZ^0\End_\omega(\omega_f \otimes A)$ and $[\eta_A] \in hoZ^0\End_\omega(\omega \otimes A)$. In particular, $hoZ^0\End_\omega(\omega_f \otimes A)$ is a group.
Proof. It suffices to show that the maps $\lim_{A}$ and $\text{res}_{A}$ send homotopy pairs on each side to the other. Clearly, $\text{res}_{A}$ sends homotopy pairs on $\mathcal{Z}$ to homotopy pairs on $\mathcal{Z}$. Moreover, $\lim_{A}$ sends homotopy pairs on $\mathcal{Z}$ to homotopy pairs on $\mathcal{Z}$. This can be checked by taking colimits, analogous to the arguments introduced in Lemma 5.7 and Lemma 5.8. □

Theorem 5.2. We have functors
$\mathcal{E}^{\omega}/ : \mathcal{cdgA}(k) \rightarrow \mathcal{dgA}(k)$, $\mathcal{G}^{\omega}/ : \mathcal{cdgA}(k) \rightarrow \mathcal{Grp}$ and $\mathcal{S}^{\omega}/ : \mathcal{hcdgA}(k) \rightarrow \mathcal{Grp}$, sending each cdg-algebra $A$ to
$\mathcal{E}^{\omega}/(A) := \text{End}(\omega \otimes A)$, $\mathcal{G}^{\omega}/(A) := \mathcal{Z} \text{End}(\omega \otimes A)$ and $\mathcal{S}^{\omega}/(A) := \text{hoZ} \text{End}(\omega \otimes A)$.
Moreover, we have natural isomorphisms $\mathcal{E}^{\omega}/ \cong \mathcal{E}^{\omega}$, $\mathcal{G}^{\omega}/ \cong \mathcal{G}^{\omega}$ and $\mathcal{S}^{\omega}/ \cong \mathcal{S}^{\omega}$.

Proof. By Lemma 5.7, we have a natural isomorphism $\lim_{A} : \mathcal{E}^{\omega}/ \Rightarrow \mathcal{E}^{\omega}$ whose component at a cdg-algebra $A$ is $\lim_{A} : \mathcal{E}^{\omega}/(A) \rightarrow \mathcal{E}^{\omega}(A)$, with its inverse $\text{res}_{A} : \mathcal{E}^{\omega} \Rightarrow \mathcal{E}^{\omega}/$ defined by $\text{res}_{A} := \{ \text{res}_{A} \}$, where $\{ \text{res}_{A} \}$ is defined in Lemma 5.8 and Lemma 5.9 implies that $\lim_{A}$ and $\text{res}_{A}$ induce natural isomorphisms $\mathcal{G}^{\omega}/ \cong \mathcal{G}^{\omega}$ and $\mathcal{S}^{\omega}/ \cong \mathcal{S}^{\omega}$. □

5.3. Remark on rigidity

A more conventional way of recovering the antipode $\zeta_{B}$ of a cdg-Hopf algebra $B$ from the category of finite dimensional dg-comodules over $B$ may be by considering dg-versions of the rigidity, as introduced in [4, 11]—see also [5], [6] and [12].

In this subsection, we provide an independent proof that $\mathcal{G}^{\omega}/(A)$ is a group for every cdg-algebra $A$ adopting the dg-versions of the rigidity in tensor category. At the end of this subsection, we will directly show that the inverse $S(\xi_{A})$ of an element $\xi_{A} \in \mathcal{G}^{\omega}/(A)$ defined in this way agrees with our previous inverse $\zeta(\eta_{A})$ of $\eta_{A} \in \mathcal{G}^{\omega}(A)$ introduced in Proposition 5.1, via the isomorphism in Lemma 5.7. This implies that our Tannakian reconstruction restricted to commutative Hopf algebra and associated affine group scheme an alternative to but agrees with the well-known reconstruction theorem [4, 11] from the category of finite dimensional linear representations, which is a rigid tensor Abelian category. In our previous paper [7], we showed that our method of Tannakian reconstruction also works in the categorial dual version to affine group dg-scheme, where we cannot restrict to the finite dimensional linear representations.

In this subsection we adapt the Einstein summation convention.

Let $(M, \gamma^{M})$ be a finite dimensional right dg-comodule over $B$. Using the antipode $\zeta_{B}$, we can define a right dg-comodule structure on the dual cochain complex $M^{\vee}$ as follows. Consider the cochain isomorphism
$\text{Hom}(M, B) \cong \text{Hom}(M, k) \otimes B = M^{\vee} \otimes B$. 

We can explicitly describe the above isomorphism by taking a basis of $M$. Fix a basis \( \{ x_i \} \) of $M$ over $k$ and denote \( \{ e^i \} \) as the corresponding dual basis of $M^\vee$. We will write \((-1)^{|i|} = (-1)^{|x_i|} = (-1)^{|e^i|}\). Then the above isomorphism sends a linear map $l : M \to B$ to $(-1)^{|l|} l(e^i) \otimes l(x_i)$. Moreover, the isomorphism is independent of the choice of the basis.

Define a cochain map $\gamma^M : M^\vee \to \text{Hom}(M, k)$, which are cochain maps and make the following diagrams commute:

\[
\begin{align*}
\text{ev}_M : M^\vee \otimes M &\to k, \\
\text{cv}_M : k \to M \otimes M^\vee,
\end{align*}
\]

which are cochain maps and make the following diagrams commute:

\[
\begin{array}{ccc}
M & \xrightarrow{J_M^M} & M \otimes k \\
\downarrow I_M^M & & \downarrow I_M^M \\
k \otimes M & \xrightarrow{\text{cv}_M \otimes I_M^M} & M \otimes M^\vee \otimes M
\end{array}
\quad \quad
\begin{array}{ccc}
M^\vee & \xrightarrow{J_{M^\vee}^M} & k \otimes M^\vee \\
\downarrow I_{M^\vee}^M & & \downarrow I_{M^\vee}^M \\
M^\vee \otimes k & \xrightarrow{\text{ev}_M \otimes I_{M^\vee}^M} & M^\vee \otimes M \otimes M^\vee
\end{array}
\]

i.e.

\[
(I_M \otimes \text{ev}_M) \circ (\text{cv}_M \otimes I_M) \circ I_M^M = J_M^M, \quad \text{and} \quad (\text{ev}_M \otimes I_{M^\vee}) \circ (I_{M^\vee} \otimes \text{cv}_M) \circ J_{M^\vee}^M = I_{M^\vee}^M. \tag{5.14}
\]

Using the antipode axiom of $\zeta_B$, we can check that $\text{ev}_M$ and $\text{cv}_M$ are morphisms of right dg-comodules over $B$:

\[
(M^\vee, \gamma^M) \otimes_{mb} (M, \gamma^M) \xrightarrow{\text{ev}_M} (k, \gamma^k), \quad (k, \gamma^k) \xrightarrow{\text{cv}_M} (M, \gamma^M) \otimes_{mb} (M^\vee, \gamma^{M^\vee}).
\]

We call the triple \((M^\vee, \gamma^{M^\vee}), \text{ev}_M, \text{cv}_M\) the dual of \((M, \gamma^M)\) in the dg-tensor category \(\text{dgComod}_{R(B)_f} \otimes_{mb} (k, \gamma^k)\). For convenience, we will often write the dual of \((M, \gamma^M)\) as \((M^\vee, \gamma^{M^\vee})\). Let us fix the evaluation map $\text{ev}_M$ for each right dg-comodule \((M, \gamma^M)\) over $B$. Then the dual \((M^\vee, \gamma^{M^\vee})\) of \((M, \gamma^M)\) is unique up to a unique isomorphism, since it is the representing object of the dg-functor

\[
\text{Hom}_{\text{dgComod}_{R(B)_f}}((- \otimes M, k)) : \text{dgComod}_{R(B)_f} \to \text{Ch}(k)_f.
\]

This is analogous to [12, Lemma 6.3.2] for the non dg-version. For every morphism $\psi : (M, \gamma^M) \to (M', \gamma'^M)$ of right dg-comodules over $B$, we have a morphism $\psi^\vee : (M', \gamma^{M'}) \to (M^\vee, \gamma^M)$.

Lemma 5.10. For every cdg-algebra $A$ and $\xi_A \in \operatorname{End}(\omega_A)$, define the dual $(\xi^M_A)\vee$ of the component $\xi^M_A$ at a finite dimensional right dg-comodule $(M, \gamma^M)$ over $B$ by

$$(\xi^M_A)\vee := (\omega_A(\omega_M) \otimes \mathbb{I}_{M^\vee A}) \circ (\mathbb{I}_{M^\vee} \otimes \xi^M_A \otimes \mathbb{I}_{M^\vee A}) \circ (\mathbb{I}_{M^\vee} \otimes \omega_A(\omega_M)) : M^\vee A \to M^\vee A.$$
(a) the following diagram commutes:

\[
\begin{array}{ccc}
M^\vee A \otimes MA & \xrightarrow{\langle \xi_A^M \vee \otimes \rangle_M} & M^\vee A \otimes MA \\
\downarrow_{\otimes \xi_M^A} & & \downarrow_{\omega_A(ev_M)} \\
M^\vee A \otimes MA & \xrightarrow{\omega_A(ev_M)} & A
\end{array}
\]

(b) for any morphism \( \psi : (M, \gamma^M) \rightarrow (M', \gamma'^M) \) of finite dimensional right dg-comodules over \( B \), the following diagram commutes,

\[
\begin{array}{ccc}
M^\vee A & \xrightarrow{\langle \xi_A^M \rangle^\vee} & M^\vee A \\
\downarrow_{\omega_A(\psi^\vee)} & & \downarrow_{\omega_A(\psi^\vee)} \\
M^\vee A & \xrightarrow{\xi_A^M} & M^\vee A
\end{array}
\]

Proof. We can show (a) as follows:

\[
\omega_A(ev_M) \circ (\xi_A^M) \circ I_{MA} = (\omega_A(ev_M) \otimes \omega_A(ev_M)) \circ (I_{M^\vee A} \otimes \xi_A^M \otimes I_{M \otimes MA}) \circ (I_{M^\vee A} \otimes \omega_A(cv_M) \otimes I_{MA})
\]

\[
= \omega_A(ev_M) \circ (I_{M^\vee A} \otimes \xi_A^M) \circ (I_{M^\vee A} \otimes \omega_A(cv_M) \otimes I_{MA})
\]

\[
= \omega_A(\psi) \circ (I_{M^\vee A} \otimes \xi_A^M).
\]

Next, we show (b). Since \( \psi : (M, \gamma^M) \rightarrow (M', \gamma'^M) \) is a morphism of finite dimensional right dg-comodules over \( B \) and \( \xi_A \) is a natural transformation, we have

\[
\xi_A^{M'} \circ \omega_A(\psi) = (-1)^{|\xi_A|} \xi_A^{M} \circ \omega_A(\psi) \circ \xi_A^M.
\]  (5.16)

We first show that \( (\omega_A(ev_M)) \circ (I_{M^\vee A}) \circ (\xi_A^{M'} \circ \omega_A(\psi)) \circ (I_{M^\vee A} \otimes \omega_A(cv_M)) = \omega_A(\psi) \circ (\xi_A^{M'} \circ \omega_A(\psi)) \). This is essentially computing the dual of \( \xi_A^{M'} \circ \omega_A(\psi) \):

\[
\begin{align*}
(\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (I_{M^\vee A} \otimes (\xi_A^{M'} \circ \omega_A(\psi)) \otimes I_{M^\vee A}) & \circ (I_{M^\vee A} \otimes \omega_A(cv_M)) \\
= (\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (I_{M^\vee A} \otimes (\xi_A^{M'} \circ \omega_A(\psi))) \circ (I_{M^\vee A} \otimes I_{M^\vee A} \otimes \omega_A(cv_M)) \\
= (\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (\xi_A^{M'}) \circ (I_{M^\vee A} \otimes \omega_A(cv_M)) \\
= (\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (\xi_A^{M'}) \circ (\xi_A^{M'} \circ \omega_A(cv_M)) \\
= (\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (\xi_A^{M'}) \circ (\xi_A^{M'} \circ \omega_A(cv_M)) \\
= (\omega_A(ev_M) \otimes I_{M^\vee A}) \circ (\xi_A^{M'}) \circ (\xi_A^{M'} \circ \omega_A(cv_M)) \\
= \omega_A(\psi) \circ (\xi_A^{M'}) \circ \omega_A(cv_M) \\
= \omega_A(\psi) \circ (\xi_A^{M'}) \circ \omega_A(cv_M).
\end{align*}
\]
We used \((a)\) on the 2nd equality, the relation eq. \((5.15)\) on the 4th equality and the relation eq. \((5.14)\) on the last equality. Note that \(\omega_A\) is a \(dg\)-tensor functor. Similarly, we have
\[
(e_A(\text{ev}_{M^\vee}) \otimes I_{M^\vee A}) \circ (I_{M^\vee A} \otimes (e_A(\psi) \otimes e_A(M))) \circ (I_{M^\vee A} \otimes e_A(\text{ev}_M)) = e_A(^M \xi^\vee) \circ e_A(\psi^\vee).
\]
From eq. \((5.16)\), together with the above calculations, we conclude that \(e_A(\psi^\vee) \circ (e_A(M^\vee))^\vee = (-1)^{\|\psi\|} (e_A(M)^\vee)^\vee \circ e_A(\psi^\vee).\)

**Lemma 5.11.** For every cdg-algebra \(A\) and for each natural endomorphism \(\xi_A \in \text{End}(\omega_A)\) we have an another natural endomorphism \(S(\xi_A) \in \text{End}(\omega_A)\), whose component at a finite dimensional right dg-comodule \((M, \gamma^M)\) over \(B\) is given by
\[
S(\xi_A)^M := (\xi_A^M)^\vee : MA \to MA.
\]

Since \((M^{\vee\vee}, \gamma^{M^{\vee\vee}}) = (M, \gamma^M)\), the above Lemma says that we have a group \(G^\omega(A) = \{Z^0 \text{End}_\omega(\omega_A), \iota_{\omega_A}, \circ\}\), where the inverse of an element \(\xi_A \in Z^0 \text{End}_\omega(\omega_A)\) is \(S(\xi_A)\). Remark that the above Lemma when \(A = k\) and the degree of everything is concentrated to zero follows from \([6, \text{Prop. 5.3.1}]\).

**Proof.** Given \(\xi_A \in \text{End}(\omega_A)\), we first show \(S(\xi_A)\) is a natural transformation. Let \(\psi : (M, \gamma^M) \to (M', \gamma'^M)\) be a morphism of finite dimensional right dg-comodules over \(B\). Substituting \(\psi\) in Lemma 5.10 (2) by the dual \(\psi^\vee : (M'^{\vee}, \gamma'^{M'^{\vee}}) \to (M^\vee, \gamma^M^\vee)\) and using \(\psi^\vee \circ \psi = 1\), we obtain that \(e_A(\psi^\vee) \circ S(\xi_A)^M = (-1)^{\|\psi\|} S(\xi_A)^M \circ e_A(\psi^\vee)\). Therefore, \(S(\xi_A)\) is a natural transformation of degree \(|\xi_A|\).

Suppose further that \(\xi_A\) is in \(Z^0 \text{End}_\omega(\omega_A)\). Then, for every finite dimensional right dg-comodule \((M, \gamma^M)\) over \(B\) we have \(S(\xi_A)^M \circ \xi_A^M = \xi_A^M \circ S(\xi_A)^M = I_{MA}\), which can be checked by the commutativity of the following two diagrams:

The top horizontal composition in the 1st diagram and the bottom horizontal composition in the 2nd diagram are equal to \(S(\xi_A)^M\), and we used the relations in eq. \((5.14)\) twice. Since \(\xi_A\) is a tensor natural transformation, the left square on the 1st diagram and the right square on the 2nd diagram commute. Moreover, we have \(S(\xi_A) \in Z^0 \text{End}_\omega(\omega_A)\) since \(S(\xi_A)\) is the inverse of \(\xi_A\). \(\square\)
We return to our original notations. We can express the component $S(\xi_A)^M$ of $S(\xi_A)$ introduced in Lemma 5.11 as follows:

$$S(\xi_A)^M := (\xi_A^M)^V = \tau \circ_{A \otimes M} (ev_{M^V} \otimes I_{A \otimes M}) \circ (I_M \otimes \xi_A^M \otimes I_M)
\circ (I_{M \otimes M^V} \otimes \tau) \circ (I_M \otimes cv_{M^V} \otimes I_A) \circ (j_M^{-1} \otimes I_A) : M \otimes A \to M \otimes A.$$

**Lemma 5.12.** For every cdg-algebra $A$ and $\eta_A \in \text{End}(\omega \otimes A)$, we have

$$\zeta(\eta_A) = \lim_{\to} S(\text{res}_A \eta_A).$$

**Proof.** It suffices to show that the components of $\zeta(\eta_A)$ and $S(\text{res}_A \eta_A)$ at a finite dimensional right dg-comodule $(M, \gamma^M)$ over $B$ agree with each other. From Proposition 5.1, the component of $\zeta(\eta_A) \in \text{End}(\omega \otimes A)$ at a right dg-comodule $(M, \gamma^M)$ over $B$ is

$$\zeta(\eta_A)^M = (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^R) \circ (\gamma^M \otimes I_A).$$

Fix a basis $\{a_i\}$ of $A$ and a basis $\{x_i\}$ of $M$, as $\mathbb{Z}$-graded vector spaces. Let let $e^i$ be the basis of $M^V$ dual to $\{x_i\}$. Again, we will write $(-1)^{|i|} = (-1)^{|x_i|} = (-1)^{|e^i|}$. Note that the index $i$ runs finite. For each $x_i$, we have $\gamma^M(x_i) = x_f \otimes \gamma_f^i$ for some $\gamma_f^i \in B$. Fix elements $a \in A$ and $m \in M$, and denote $m' := e'(m) \in k$, which is zero if $|m| \neq |i|$. 

1. We calculate the value of $\zeta(\eta_A)^M(m \otimes a)$. To do so, we need to describe $\eta_A^R$ first. Using the isomorphism in Lemma 5.4, we get

$$\eta_A^R = (J_B \otimes I_A) \circ (I_B \otimes \epsilon_B \otimes I_A) \circ (I_B \otimes \eta_A^R) \circ (\gamma^M \otimes I_A)$$

where we defined $\gamma^M := \tau \circ (\zeta_B \otimes I_B) \circ \Delta_B$. Therefore we obtain that

$$\zeta(\eta_A)^M = (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^R) \circ (\gamma^M \otimes I_A)
= (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (\gamma^B \otimes I_A)
= (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \zeta_B \otimes I_A) \circ (I_M \otimes \Delta_B \otimes I_A) \circ (\gamma^M \otimes I_A)
= (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (I_M \otimes \zeta_B \otimes I_A) \circ (\gamma^M \otimes I_A).$$

We are ready to calculate $\zeta(\eta_A)^M(m \otimes a)$. Let us write $m = m^k x_k$ for the indices $k$ with $|k| = |m|$. Then we have

$$\zeta(\eta_A)^M(m \otimes a)
= (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (I_M \otimes \zeta_B \otimes I_A) \circ (\gamma^M \otimes I_A) \circ (m \otimes a)
= m^k \cdot (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (I_M \otimes \zeta_B \otimes I_A) \circ (m \otimes a)
= m^k \cdot (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (I_M \otimes \eta_A^B) \circ (x_i \otimes \gamma_f^i \otimes a)
= (-1)^{|i||a|} m^k \cdot (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (x_i \otimes \eta_A^B \circ B^\gamma_f^i \otimes a)
= (-1)^{|i||a|} m^k \cdot (J_M \otimes I_A) \circ (I_M \otimes \epsilon_B \otimes I_A) \circ (x_i \otimes b_k^{it} \otimes a_t)
= (-1)^{|i||a|} m^k \epsilon_B(b_k^{it}) \cdot (x_i \otimes a_t)$$
where \( b_k^{li} \in B \) is defined by the relation 
\[
\eta_A^R(\zeta_B(\gamma_k^i) \otimes a) = b_k^{li} \otimes a_i.
\]

2. We calculate the value of \( S(\text{res}_A \eta_A)M(\text{m} \otimes a) \). First, note that \( \tilde{\gamma}^M(e^i) = t_B \circ (e^i \otimes \zeta_B) \circ \gamma^M : M \to B \) is of degree \(|i| \) and 
\[
\tilde{\gamma}^M(e^i)(x_k) = t_B \circ (e^i \otimes \zeta_B) \circ \gamma^M(x_k) = t_B \circ (e^i \otimes \zeta_B)(x_j \otimes \gamma_k^i) = \zeta_B(\gamma_k^i).
\]

Therefore we have 
\[
\gamma^M(e^i) = (-1)^{|k||k+i||i|} \cdot e^k \otimes \tilde{\gamma}^M(e^i)(x_k) = (-1)^{|k||k+i||i|} \cdot e^k \otimes \zeta_B(\gamma_k^i).
\]

Next, we calculate \( \eta_A^{M^r}(e^i \otimes a) \):
\[
\eta_A^{M^r}(e^i \otimes a) = (I_{M^r} \otimes t_A) \circ (I_{M^r} \otimes \epsilon_B \otimes I_A) \circ (I_{M^r} \otimes \eta_A^r) \circ (\gamma^M \otimes I_A)(e^i \otimes a)
\]
\[= (-1)^{|k|+|i||i|} \cdot (I_{M^r} \otimes t_A) \circ (I_{M^r} \otimes \epsilon_B \otimes I_A) \circ (I_{M^r} \otimes \eta_A^r)(e^k \otimes \zeta_B(\gamma_k^i) \otimes a)
\]
\[= (-1)^{|k|+|i||i|+|k||i|} \cdot (I_{M^r} \otimes t_A) \circ (I_{M^r} \otimes \epsilon_B \otimes I_A)(e^k \otimes \eta_A^r(\zeta_B(\gamma_k^i) \otimes a))
\]
\[= (-1)^{|k|+|i||i|} \cdot (I_{M^r} \otimes t_A) \circ (I_{M^r} \otimes \epsilon_B \otimes I_A)(e^k \otimes b_k^{li} \otimes a_i)
\]

Here, we used the defining relation \( \eta_A^R(\zeta_B(\gamma_k^i) \otimes a) = b_k^{li} \otimes a_i \).

Finally, we calculate \( S(\text{res}_A \eta_A)M(\text{m} \otimes a) \):
\[
S(\text{res}_A \eta_A)M(\text{m} \otimes a) = \tau \circ t_{A \otimes M} \circ (\text{ev}_{M^r} \otimes I_{A \otimes M}) \circ (I_M \otimes \eta_A^{M^r} \otimes I_M)
\]
\[= (-1)^{|i|+|a||a|} \cdot \tau \circ t_{A \otimes M} \circ (\text{ev}_{M^r} \otimes I_{A \otimes M}) \circ (I_M \otimes \eta_A^{M^r} \otimes I_M)(m \otimes e^i \otimes \epsilon_B(b_k^{li}) \cdot (e^k \otimes \epsilon_B(b_k^{li}) \cdot (a_i \otimes x_i))
\]

Now we examine the sign factors.

- \(|m| = |k| \) since we calculated \( e^k(m) = m^k \) on the 6th equality.

- \(|b_k^{li}| = 0 \) since other degree terms vanish when we calculate \( \epsilon_B(b_k^{li}) \) on the 5th equality. Thus we have \( |a_i| = |a| + |\eta_A| + |k| + |i| \) from the defining relation \( \eta_A^R(\zeta_B(\gamma_k^i) \otimes a) = b_k^{li} \otimes a_i \).
Therefore, in modulo 2, we have

\[\equiv i + \|i\|\alpha + \|m\|\eta_A + 1\|m| + \|\|\eta_A| \equiv i + \|\|\eta_A| \equiv i\]

This shows that \(\varpi(\eta_A)^M(m \otimes a) = S(\text{res}_A \eta_A)^M(m \otimes a)\) holds for all \(m \in M\) and \(a \in A\).

We conclude that \(\varpi(\eta_A)^M = S(\text{res}_A \eta_A)^M\) holds for every finite dimensional right dg-comodule \((M, \gamma^M)\) over \(B\). \(\square\)

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