Global existence and asymptotic behavior of solutions to a nonlocal Fisher-KPP type problem

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Abstract

In this work, we consider a nonlocal Fisher-KPP reaction-diffusion problem with Neumann boundary condition and nonnegative initial data in a bounded domain in $\mathbb{R}^n (n \geq 1)$, with reaction term $u^\alpha(1 - m(t))$, where $m(t)$ is the total mass at time $t$. When $\alpha \geq 1$ and the initial mass is greater than or equal to one, the problem has a unique nonnegative classical solution. While if the initial mass is less than one, then the problem admits a unique global solution for $n = 1, 2$ with any $1 \leq \alpha < 2$ or $n \geq 3$ with any $1 \leq \alpha < 1 + 2/n$. Moreover, the asymptotic convergence to the solution of the heat equation is proved. Finally, some numerical simulations in dimensions $n = 1, 2$ are exhibited. Especially, for $\alpha > 2$ and the initial mass is less than one, our numerical results show that the solution exists globally in time and the mass tends to one as time goes to infinity.

1 Introduction

In this work we consider the following nonlocal initial boundary value problem,

\begin{align}
  u_t - \Delta u &= u^\alpha \left(1 - \int_\Omega u(x, t) \, dx\right), \quad x \in \Omega, t > 0, \\
  \nabla u \cdot \nu &= 0, \quad x \in \partial \Omega, \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \Omega,
\end{align}

where $u$ is the density, $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\alpha \geq 1$ and $\nu$ is the outer unit normal vector on $\partial \Omega$. Without loss of generality, throughout this paper we assume $|\Omega| = 1$

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(otherwise, rescale the problem by $|\Omega|$), let $m(t) = \int_{\Omega} u(x,t) \, dx$ and $m_0 = m(0)$. A damping term with $\sigma > 0$ can also be included to get $u_t - \Delta u + \sigma u = u^\alpha (1 - m(t))$. In this case, similar results to this paper can also be obtained. For simplicity, we assume that $\sigma = 0$.

In the 1930s, Fisher [16] and Kolmogorov, Petrovskii, Piskunov [24] in population dynamics and Zeldovich, Frank-Kamenetskii [47] in combustion theory started to study problems with this kind of reaction terms. Actually, they introduced the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u),$$

and studied the existence, stability and speed of propagation. In the theory of population dynamics, the function $F$ is considered as the rate of the reproduction of the population. It is usually of the form

$$F(u) = \beta u^\alpha (1 - u) - \gamma u.$$  

From the above model two cases emerge depending on the values of $\alpha$.

In the case of $\alpha = 1$, the reproduction rate is proportional to the density $u$ of the population and to available resources $(1 - u)$. The last term, $-\gamma u$, describes the mortality of the population.

The case $\alpha = 2$, which is the motivation for our work, considers the addition of sexual reproduction to the model with the reproduction rate proportional to the square of the density, see [42]. For more information on reaction-diffusion waves in biology, we refer to the review paper of Volpert and Petrovskii [41].

Next we pass to the relation between the local and the non-local consumption of resources. In the local reaction-diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta u^\alpha (1 - u) - \gamma u, \quad (2)$$

where $u$ is the population density, $\frac{\partial^2 u}{\partial x^2}$ describes the random displacement of the individuals of this population and the reaction term represents their reproduction and mortality. Moreover, the reaction term consists of the reproduction term which is represented by the population density to a power, $u^\alpha$, multiplied with the term $(1 - u)$ which stands for the local consumption of available resources.

The nonlocal version of the above problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta u^\alpha \left( 1 - \int_{-\infty}^{\infty} \phi(x - y)u(y,t) \, dy \right) - \gamma u, \quad (3)$$

where $\beta, \gamma > 0$ and $\int_{-\infty}^{\infty} \phi(y) \, dy = 1$. It can be seen as the case where the individual, located at a certain point, can consume resources in some area around that point. $\phi(x - y)$ represents the probability density function that describes the distribution of individuals around their average
positions and it depends on the distance from the average point \( x \) to the actual point \( y \). One can easily verify that if \( \phi \) is a Dirac \( \delta \)-function, then the nonlocal problem reduces to (\ref{2}).

In the current paper we will study problems with reaction terms similar to the above nonlocal reaction terms. There are some already known results on the reaction-diffusion equation with a nonlocal term,

\[
  u_t = \Delta u + F(t, u, I(u)), \quad I(u) = \int_{\Omega} u(y, t)dy,
\]

in a bounded domain \( \Omega \). However, compared to the local version, the results for the nonlocal reaction terms of Fisher-KPP type are relatively limited. Here we list some of the known recent results.

Anguiano, Kloeden and Lorenz considered \( F = f(u)I(u^4)(1 - I(u^4)) \) and proved the existence of a global attractor \cite{1}. Wang and Wo \cite{44} proved the convergence to a stationary solution with \( F = u^m - I(u^m), \ m > 1 \). For \( F = \alpha e^{\gamma u} + b I(e^{\gamma u}) \), Pao \cite{33} studied the existence or nonexistence of stationary solutions. Liu, Chen and Lu \cite{28} proved also the blow-up of solutions for a similar equation, see also \cite{14}. Rouchon obtained global estimates of solutions in \cite{37}. For more information on nonlocal KPP-Fisher type problems, we refer to a recent book by Volpert \cite{40}.

Nonlocal Fisher-KPP type reaction terms can describe also Darwinian evolution of a structured population density or the behavior of cancer cells with therapy as well as polychemotherapy and chemotherapy, we refer the interested reader to the models found in \cite{29, 30, 31}.

Bebernes and Bressan \cite{6} (see also Bebernes \cite{7}, Pao\cite{33}) considered the equation with reaction term

\[
  F(t, u, I(u)) = f(t, u(t, x)) + \int_{\Omega} g(t, u(t, y))dy, \quad t > 0, \ x \in \Omega.
\]

They considered the case when \( f(t, u) = e^u, \ g(t, u) = ke^u \ (k > 0) \), for which the above problem represents an ignition model for a compressible reactive gas, and proved that solutions blow-up.

Later, Wang and Wang \cite{45} considered a power-like nonlinearity, i.e.

\[
  F(t, u, I(u)) = \int_{\Omega} u^p(t, y)dy - ku^q(t, x), \quad t > 0, \ x \in \Omega,
\]

with \( p, q > 1 \), and proved the blow-up of the solutions.

Budd, Dold and Stuart \cite{11}, Hu and Yin \cite{22} considered a similar to the above problem in the case \( p = 2 \) and general \( p \) respectively,

\[
  F(t, u, I(u)) = u^p - \frac{1}{|\Omega|} \int_{\Omega} u^p(t, y)dy, \quad t > 0, \ x \in \Omega.
\]

With this typical structure, the energy of the solutions is conserved (under Neumann boundary conditions). For this kind of nonlocal problems it is known \cite{45} that there is no comparison principle
and they are the closest models to the ones we are considering in this work. For a general study on nonlocal problems, we refer to Quittner’s and Souplet’s book [35] as well as the paper by Souplet [36].

1.1 Preliminary discussion

In this article, we will focus on (1) which has a reaction term of the type

$$F(t, u, I(u)) = u^\alpha \left(1 - \int_{\Omega} u(x, t) \, dx\right)$$

for $\alpha \geq 1$. One additional fact that makes this problem more difficult to handle is the lack of comparison principle as one can see for example from [45].

For nonnegative $u$, formally by integrating (1) over $\Omega$, we get,

$$m'(t) = (1 - m(t)) \int_{\Omega} u^\alpha \, dx,$$

where $m(t) = \int_{\Omega} u(x, t) \, dx$ is the total mass at time $t$.

If we start at time $t_0$ such that $1 - m(t_0) < 0$, which means that $m(t_0) > 1$, we can see that $m'(t)$ is negative and therefore $m(t)$ decreases in time. In this case it is natural to expect the global existence of solutions.

On the other hand, if we start at time $t_0$ such that $1 - m(t_0) > 0$, we can see that $m(t)$ increases in time. However if $(1 - m(t))$ remains positive, the equation has a similar structure to the heat equation with a power-like reaction term for which we know that the problem might have no global solution for super-critical exponent $\alpha < 1 + 2/n$ (see for example [4, 5, 18, 19, 20, 23]). In this paper we give a negative answer to this observation. Our main results are the following two theorems:

**Theorem 1.** Let $n \geq 1$, $\alpha \geq 1$ and $\int_{\Omega} u_0(x) \, dx = m_0 > 0$. Assume $u_0$ is nonnegative and $u_0 \in L^k(\Omega)$ for any $1 < k < \infty$. Then for $m_0 < 1$ with $\alpha$ satisfying

$$1 \leq \alpha < 1 + 2/n, \quad n \geq 3,$$

or $m_0 \geq 1$ with arbitrary $\alpha \geq 1$, problem (1) has a unique nonnegative classical solution. Moreover, the following a priori estimates hold true. That’s for $m_0 < 1$,

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq C + C \, t^{-\frac{k-1}{\alpha-1}} \quad \text{for any } t > 0.$$
For $m_0 \geq 1$,
\begin{align}
\|u\|_{L^k(\Omega)}^k &\leq C + C t^{-\frac{n(k-1)}{2}}, \quad n \geq 3, 
\|u\|_{L^k(\Omega)}^k &\leq C + C t^{-(k-1)}, \quad n = 1, 2.
\end{align}
Here $C$ denote different constants depending on $m_0, k, \alpha$, but not depending on $\|u_0\|_{L^k(\Omega)}$.

**Theorem 2.** Let $u(x, t)$ be the unique nonnegative classical solution obtained from Theorem 1, $v$ be the solution to the heat equation with Neumann boundary condition and initial data $\int_\Omega v_0(x)dx = m_0$, then as $t \to \infty$,
\begin{align}
\|u(\cdot, t) - v(\cdot, t) - (m_0 - 1)\|_{L^2(\Omega)} \leq C_1 e^{-C_2 t},
\end{align}
where $C_1, C_2$ are constants depending on the initial mass $m_0$ and $\|u_0\|_{L^{2\alpha}(\Omega)}$.

This paper is organized as follows. In Section 2, we firstly present the dynamics of the mass. The global existence of the solutions and thus the proof the Theorem 1 are shown in Section 3. Section 4 is devoted to the proof of Theorem 2. Section 5 shows the numerical simulations of the problem which give the motivation for our future study for the case $\alpha \geq 1 + 2/n$. Finally, Section 6 concludes the main work of our paper, some open questions for problem (1) are also addressed.

## 2 Dynamics of the total mass

The evolution of the total mass plays the key role in our proof for the global existence of the classical solution. Therefore, we firstly give the evolution of mass $\int_\Omega u(t)dx$ in time.

**Lemma 3.** For $m_0 > 0$, the mass $\int_\Omega u(t)dx = m(t)$ satisfies
\begin{align}
\min\{1, m_0\} \leq m(t) \leq \max\{1, m_0\}.
\end{align}
Furthermore, we have the following decay estimates
\begin{align}
|1 - m(t)| \leq |1 - m_0| e^{-\min\{1, m_0^\alpha\} t}.
\end{align}

**Proof.** We return to the original problem (1) and integrate it over $\Omega$ to get:
\begin{align}
m'(t) = (1 - m(t)) \int_\Omega u^\alpha dx.
\end{align}
There are two possibilities depending on the initial mass.
• If we start at time $t_0$ where $1 - m(t_0) > 0$, we can see that $m'(t)$ is positive and therefore $m$ increases in time. Moreover, with the use of Jensen’s inequality we get by using $m(t) \geq m_0$,

$$m'(t) \geq (1 - m(t))m^\alpha(t) \geq (1 - m(t))m_0^\alpha.$$ 

By solving this inequality we get a lower bound on the speed with which $m(t)$ increases to 1 i.e.

$$m(t) \geq 1 - e^{-m_0^\alpha t}(1 - m_0).$$

• If $1 - m(t_0) < 0$, then $m$ decreases in time. By monotonicity, we get $m(t) \leq m_0$ and again by Jensen’s inequality,

$$m'(t) \leq (1 - m(t))m^\alpha(t) < 1 - m(t),$$

then we get that

$$m(t) \leq 1 + (m_0 - 1)e^{-t}.$$ 

By putting together the above two cases we have the expected results. □

3 Global Existence

This section mainly focuses on the global existence of the classical solution to (1). We will use the following ODE inequality from [9], which was also used in [10].

**Lemma 4.** Assume $y(t) \geq 0$ is a $C^1$ function for $t > 0$ satisfying

$$y'(t) \leq \alpha - \beta y(t)^a$$

for $a > 1$, $\alpha > 0$, $\beta > 0$, then $y(t)$ has the following hyper-contractive property

$$y(t) \leq (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}} \text{ for any } t > 0. \quad (13)$$

Furthermore, if $y(0)$ is bounded, then

$$y(t) \leq \max \left(y(0), (\alpha/\beta)^{1/a}\right). \quad (14)$$

The proof of global existence heavily depends on the following two a priori estimates, Proposition 5 and Proposition 6, and then we will use the compactness arguments to close the proof. Due to the preliminary discussion, we will divide the a priori estimates into two cases $m_0 < 1$ and $m_0 \geq 1$. Firstly, we focus on $m_0 < 1$. 

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Proposition 5. Let \( n \geq 1 \) and \( m_0 < 1 \). If \( \alpha \) satisfies

\[
1 \leq \alpha < 1 + 2/n, \quad n \geq 3, \\
1 \leq \alpha < 2, \quad n = 1, 2,
\]
then for any \( 1 < k < \infty \), the nonnegative solution of \( u \) satisfies

\[
\|u(\cdot, t)\|_{L^k(\Omega)} \leq C(m_0, k, \alpha) + C(m_0, k, \alpha) \ t^{-\frac{k-1}{\alpha-1}} \quad \text{for any } t > 0.
\] (15)

Moreover, if \( u_0(x) \in L^k(\Omega) \), then

\[
\|u(\cdot, t)\|_{L^k(\Omega)} \leq \max \left\{ \|u_0(x)\|_{L^k(\Omega)}, C(m_0, k, \alpha) \right\},
\] (16)

and for any \( 0 < T < \infty \)

\[
\nabla u^\frac{k}{2} \in L^2(0, T; L^2(\Omega)).
\]

Proof. Since \( m_0 < 1 \), by lemma 3 one has \( m_0 \leq m(t) \leq 1 \). Using \( ku^{k-1} \) as a test function for equation \( (1) \) and integrating it by parts

\[
\frac{d}{dt} \int_\Omega u^k \ dx = -\frac{4(k-1)}{k} \int_\Omega \nabla u^\frac{k}{2} \cdot \nabla u - k \int_\Omega u^k \ dx + k \int_\Omega u^{k+\alpha-1} \ dx \left( 1 - \int_\Omega u \ dx \right),
\]

\[
\frac{d}{dt} \int_\Omega u^k \ dx + \frac{4(k-1)}{k} \int_\Omega \nabla u^\frac{k}{2} \cdot \nabla u + km(t) \int_\Omega u^{k+\alpha-1} \ dx = k \int_\Omega u^{k+\alpha-1} \ dx.
\] (17)

Choosing \( 1 < k' < k + \alpha - 1 \), combining Hölder’s inequality and the Sobolev embedding theorem one has

\[
\int_\Omega u^{k+\alpha-1} \ dx = \int_\Omega u^{1-\lambda} u^{k} u^{2(k+\alpha-1)/k} \ dx \\
\leq \left\| u^{\frac{k}{2}} \right\|_{L^{2k'}(\Omega)} \left\| \nabla u^\frac{k}{2} \right\|_\infty \left\| u^{1-\lambda} \right\|_{L^{2k'/k}(\Omega)} \left\| u^{2(k+\alpha-1)/k} \right\|_{L^{2k'/k}(\Omega)} \left\| u^{\frac{k}{2}} \right\|_{L^{2k'/k}(\Omega)} \left\| u^{2(1-\lambda)(k+\alpha-1)} \right\|_{L^{2k'/k}(\Omega)}
\]

\[
\leq C(k) \left( \left\| \nabla u^\frac{k}{2} \right\|_{L^2(\Omega)} \left\| u^{1-\lambda} \right\|_{L^{2k'/k}(\Omega)} + \left\| u^\frac{k}{2} \right\|_{L^{2k'/k}(\Omega)} \left\| u^{2(1-\lambda)(k+\alpha-1)/k} \right\|_{L^{2k'/k}(\Omega)} \right),
\]

(18)

where \( \lambda \) is the exponent from Hölder’s inequality, i.e.

\[
\lambda = \frac{k}{2k'} - \frac{k}{2(k+\alpha-1)} \in (0, 1).
\] (19)
and \( p \) satisfies
\[
\begin{cases}
  p = \frac{2n}{n-2}, & n \geq 3, \\
  \frac{2(k+\alpha-1)}{k} < p < \infty, & n = 2, \\
  p = \infty, & n = 1.
\end{cases}
\]

(20)

Now we will divide the analysis into three cases \( n \geq 3, n = 2 \) and \( n = 1 \).

For \( n \geq 3 \), \( p = \frac{2n}{n-2} \) and then
\[
\lambda = \frac{kn}{2k^2} - \frac{kn}{2(2k+\alpha-1)} \in (0,1),
\]
with \( k > \max \left\{ \frac{(n-2)(\alpha-1)}{2}, 1 \right\} \). Taking \( k' > \frac{(\alpha-1)n}{2} \), simple computations arrive at
\[
\frac{2\lambda(k+\alpha-1)}{k} = \frac{kn}{2k^2} + 1 - \frac{n}{2} < 2.
\]

To sum up, for \( k' > \max \left\{ \frac{(\alpha-1)n}{2}, 1 \right\} \), thanks to the Young’s inequality, from (18) one has
\[
\int_{\Omega} u^{k+\alpha-1} dx \leq \frac{k-1}{k^2} \left\| \nabla u^\frac{k}{2} \right\|^2_{L^2(\Omega)} + C(k) \left\| u^\frac{k}{2} \right\|^2_{L^{2k'}(\Omega)}
\]
\[
+ C(k) \left\| u^\frac{2\lambda(k+\alpha-1)}{k} \right\|^2_{L^2(\Omega)} \left\| u^\frac{2(1-\lambda)(k+\alpha-1)}{k} \right\|^2_{L^{2k'}(\Omega)}.
\]

(22)

Letting
\[
r = (1-\lambda)\frac{2(k+\alpha-1)}{k} \frac{1}{1 - \frac{\lambda(k+\alpha-1)}{k}},
\]
recalling \( m(t) \geq m_0 \), together with (22) we arrive at
\[
\frac{d}{dt} \int_{\Omega} u^k dx + km_0 \int_{\Omega} u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \left\| \nabla u^\frac{k}{2} \right\|^2_{L^2(\Omega)}
\]
\[
\leq C(k) \left\| u \right\|^\frac{k}{k'}_{L^k(\Omega)} + C(k) \left\| u \right\|^\frac{\lambda(k+\alpha-1)}{k}_{L^k(\Omega)} \left\| u \right\|^\frac{(1-\lambda)(k+\alpha-1)}{k}_{L^{k'}(\Omega)}.
\]

(24)

On the other hand, using Hölder’s inequality with \( 1 < k' < k+\alpha-1 \) we have
\[
\left\| u \right\|^{\frac{\theta}{k'}}_{L^k(\Omega)} \leq C \left\| u \right\|^\theta_{L^{k+\alpha-1}(\Omega)} \left\| u \right\|^\frac{1-\theta}{L^k(\Omega)},
\]
\[
\left\| u \right\|^{\frac{\eta}{k}}_{L^k(\Omega)} \leq C \left\| u \right\|^\eta_{L^{k+\alpha-1}(\Omega)} \left\| u \right\|^\frac{1-\eta}{L^k(\Omega)},
\]

(25)

(26)

where
\[
\theta = \frac{(k+\alpha-1)(k'-1)}{k'(k+\alpha-2)} \in (0,1), \quad \eta = \frac{(k+\alpha-1)(k-1)}{k(k+\alpha-2)} \in (0,1).
\]

(27)
Hence
\[ \|u\|_{L_{k}^{r}(\Omega)}^{kr} \leq \left( C\|u\|_{L_{k+\alpha-1}^{\theta}(\Omega)}^{\theta} \|u\|_{L_{1}^{1}(\Omega)}^{1-\theta} \right)^{kr} \leq C(m_0, k)\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{kr\theta}. \tag{28} \]

Taking (24)-(28) into account we obtain that
\[
\frac{d}{dt} \int_{\Omega} u^k dx + km_0\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{k+\alpha-1} + \frac{3(k-1)}{k} \|\nabla u\|_{L_{2}(\Omega)}^{2} \leq C(m_0, k)\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{kr\theta} + C(m_0, k)\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{(k+\alpha-1)(\lambda\eta+(1-\lambda)\theta)}. \tag{29} \]

Here
\[ \frac{kr}{2} = (1-\lambda)(k+\alpha-1) - \frac{1}{1 - \frac{\lambda(k+\alpha-1)}{k}}, \]
\[ \frac{k+\alpha-1}{\theta} = \frac{k+\alpha-2}{1 - \frac{1}{k'}}. \]

Recalling the definition of \(\theta, \eta, \lambda\), direct computations show that \(\lambda\eta + (1-\lambda)\theta < 1\). For
\[ 1 \leq \alpha < 1 + \frac{2}{n}, \tag{30} \]
one can derive that
\[ \frac{kr\theta}{2} < k + \alpha - 1. \tag{31} \]

Next for \(n = 2\), \(\frac{2(k+\alpha-1)}{k} < p < \infty\), by proceeding the similar arguments to the case \(n \geq 3\) from (21) to (29), we obtain that for
\[ 1 \leq \alpha < 2 - \frac{2}{p}, \tag{32} \]
(31) holds true. When \(n = 1\), \(p = \infty\), then for \(1 \leq \alpha < 2\), (31) also holds true.

Therefore, combining the three cases \(n \geq 3\), \(n = 2\) and \(n = 1\), using Young’s inequality we obtain from (29) that
\[
\frac{d}{dt} \int_{\Omega} u^k dx + km_0\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{k+\alpha-1} + \frac{3(k-1)}{k} \|\nabla u\|_{L_{2}(\Omega)}^{2} \leq \frac{km_0}{4}\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{k+\alpha-1} + \frac{km_0}{4}\|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{k+\alpha-1} + C(m_0, k). \tag{33} \]

In addition, Hölder’s inequality yields that
\[ \left( \|u\|_{L_{k}^{1}(\Omega)}^{k} \right)^{1 + \frac{\alpha-1}{k}} \leq \|u\|_{L_{k+\alpha-1}^{1}(\Omega)}^{\frac{\alpha-1}{k}}. \tag{34} \]
Then using lemma 4, we solve the ODE inequality
\[
\frac{d}{dt} \int_\Omega u^k dx + \frac{km_0}{2\|u\|^\frac{\alpha}{2}} \left( \int_\Omega u^k dx \right)^{\frac{1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}} \leq C(m_0, k)
\]
(35)
to obtain that for any \(1 < k < \infty\),
\[
\|u\|_{L^k(\Omega)}^k \leq C(m_0, k, \alpha) + \left[ C(m_0, k, \alpha) \frac{k-1}{\alpha-1} \right] \frac{1}{t} \text{ for any } t > 0.
\]
(36)
Furthermore, if \(u_0(x) \in L^k(\Omega)\) for any \(1 < k < \infty\), then taking \(y(t) = \int_\Omega u^k dx\) in lemma 4 one has
\[
\|u\|_{L^k(\Omega)} 
\leq \max \left\{ \|u_0(x)\|_{L^k(\Omega)}, C(m_0, k, \alpha) \right\}.
\]
(37)
Now we integrate (33) from 0 to \(T\) in time, then we can obtain that for any \(T > 0\)
\[
\int_\Omega u^k(T) dx + \int_0^T \int_\Omega |\nabla u^k|^2 dx dt + \int_0^T \int_\Omega k+\alpha-1dx dt \leq \int_\Omega u_0^k dx + C(m_0, k)T,
\]
from which we derive
\[
\nabla u^k \in L^2(0, T; L^2(\Omega)) \text{ for any } T > 0.
\]
This completes the proof. \(\square\)

For \(m_0 \geq 1\), owing to lemma 3 we know \(m(t) \geq 1\) for any \(t > 0\), thus we have the following result

**Proposition 6.** Let \(n \geq 1\) and \(\alpha \geq 1\). Assume \(u_0 \in L^1_+(\Omega)\) and \(\int_\Omega u_0(x) dx = m_0 \geq 1\), Then for any \(1 < k < \infty\), the nonnegative solution of (7) satisfies that for any \(t > 0\)
\[
\|u\|_{L^k(\Omega)}^k \leq C(m_0, k) + \left[ C(m_0, k) \frac{1}{t} \right]^{\frac{n(k-1)}{2}} , \quad n \geq 3,
\]
(38)
\[
\|u\|_{L^k(\Omega)}^k \leq C(m_0, k) + \left[ C(m_0, k) \frac{1}{t} \right]^{k-1} , \quad n = 1, 2.
\]
(39)
Moreover, if \(u_0 \in L^k(\Omega)\), then
\[
\int_0^\infty \int_\Omega |\nabla u^k|^2 dx dt \leq \int_\Omega u_0^k dx.
\]
(40)
**Proof.** Recalling lemma 3, we know that if \(m_0 > 1\), then \(m(t) \geq 1\) for any \(t > 0\). Hence the \(L^k\) estimates (17) can be reduced to
\[
\frac{d}{dt} \int_\Omega u^k dx + \frac{4(k-1)}{k} \int_\Omega |\nabla u^k|^2 dx \leq 0.
\]
(41)
For $n\geq 1$, using Hölder’s inequality one has that for any $1<k<\infty$, the following estimate holds

$$
\|u\|_{L^k(\Omega)}^k \leq \|u\|_{L^2(\Omega)}^\theta \|u\|_{L^1(\Omega)}^{k(1-\theta)}
$$

where $\theta = \frac{k-1}{k-2}$ and

$$
p = \begin{cases} 
\frac{2n}{n-2}, & n \geq 3, \\
2 < p < \infty, & n = 2, \\
p = \infty, & n = 1.
\end{cases}
$$

Thanks to the Sobolev embedding theorem and Young’s inequality, from (42) one has

$$
\left(\|u\|_{L^k(\Omega)}^k\right)^{\frac{1}{\theta}} \leq C(n) \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^k\right) \|u\|_{L^1(\Omega)}^{k(1-\theta)}
$$

$$
\leq C(m_0, n)\|\nabla u\|_{L^2(\Omega)}^2 + C(m_0, n)\|u\|_{L^k(\Omega)}^k
$$

$$
\leq C(m_0, n)\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2} \left(\|u\|_{L^k(\Omega)}^k\right)^{\frac{1}{\theta}} + C(m_0, n, k). \quad (44)
$$

Plugging the above estimates into (41) yields that

$$
\frac{d}{dt} \int_\Omega u^k dx + C(m_0, n, k) \left(\int_\Omega u^k dx\right)^{\frac{1}{\theta}} + C(m_0, n, k)\|\nabla u\|_{L^2(\Omega)}^2 \leq C(m_0, k, n), \quad (45)
$$

solving the ODE inequality we have that for any $t > 0$

$$
\|u\|_{L^k(\Omega)}^k \leq C(m_0, k, n) + \left[\frac{C(m_0, k, n)}{t}\right]^{\frac{n(k-1)}{2}}, \quad n \geq 3, \quad (46)
$$

$$
\|u\|_{L^k(\Omega)}^k \leq C(m_0, k, n) + \left[\frac{C(m_0, k, n)}{t}\right]^{k-1}, \quad n = 1, 2. \quad (47)
$$

Moreover, if $u_0 \in L^k(\Omega)$, then (41) directly yields

$$
\|u\|_{L^k(\Omega)} \leq \|u_0\|_{L^k(\Omega)}. \quad (48)
$$

Next integrating (41) from 0 to $\infty$ in time we obtain that

$$
\int_0^\infty \int_\Omega |\nabla u|^2 dx dt \leq \int_\Omega u_0^k dx. \quad (49)
$$

This closes the proof. \(\square\)

**Remark 7.** In fact, (38) and (39) also hold true for heat equation, and the uniform boundedness in time of the $L^k$ norm depends only on the initial mass, not depends on the initial $L^k$ norm.
Now we have obtained the necessary a priori estimates to complete the proof of Theorem 1. It can be proved by standard methods and for the convenience of the reader we mention the key steps in the following. First of all, from the above estimates, we can take $k = 2$ and $k = 2\alpha$ in Proposition 5 and 6 to get the estimates for $\|\nabla u\|_{L^2(0,T)}$ and $\|u\|_{L^2(H^{-1}(0,T))}$ for any $T > 0$. By Aubin-Lions lemma [21, 43], we have the strong compactness of $u$ in $L^2$ so that the nonlinear terms can be handled. Therefore, the global existence of weak solutions (in the sense of distributions) can be obtained by standard compactness argument. Secondly, from the estimates of the weak solution in Proposition 5 and Proposition 6, the nonlinear term $u^\alpha(1 - m(t)) \in L^k([0,T] \times \Omega), \forall k > 1$ for any $T > 0$. The solution is a strong $W^{2,1}_k$ solution from classical parabolic theory, [26, 27]. By Sobolev embedding, we can bootstrap it to get that classical solution. In the end, the uniqueness can be obtained directly from comparison principle [26, 27] since $u^{\alpha-1}(1 - m(t))$ is bounded from below.

$\square$

4 The long time behavior of solutions

As we can see from the above arguments, equation (1) has a unique classical solution. In this section, we will detect the long time behavior of the global solution.

**Theorem 8.** Assume $u_0 \in L^k(\Omega)$ for any $1 < k < \infty$. Let $u$ be the classical solution to problem (7) and $v$ be the solution to the heat equation with Neumann boundary condition and initial data $v_0$ such that $\int_\Omega v_0(x)dx = m_0$, Then as $t \to \infty$

$$\|u(\cdot,t) - v(\cdot,t) - (m_0 - 1)\|_{L^2(\Omega)} \leq C_1 e^{-C_2 t},$$

where the constants $C_1, C_2$ depend on $m_0$ and $\|u_0\|_{L^{2\alpha}(\Omega)}$.

**Proof.** The difference between the two equations is

$$(u - v)_t + \Delta(u - v) = u^\alpha(1 - m(t)).$$

Let $\overline{u}(t) = \int_\Omega u(x,t)dx$ and $\overline{v}(t) = \int_\Omega v(x,t)dx$. By (12), we have $\overline{u}_t = m'(t) = (1 - m(t)) \int_\Omega u^\alpha dx.$ $\overline{v}_t(t) = 0$ because of $\overline{v}(t) = v_0$. Therefore,

$$(u - v)_t - (\overline{u} - \overline{v})_t + \Delta(u - v) = u^\alpha(1 - m(t)) - (1 - m(t)) \int_\Omega u^\alpha.$$

The standard $L^2$ estimate shows that,

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |(u-v)-(\overline{u} - \overline{v})|^2 dx + \int_\Omega |\nabla (u-v)|^2 dx = (1-m(t)) \int_\Omega \left[ (u^\alpha - \int_\Omega u^\alpha dy)(u-v) - (\overline{u} - \overline{v}) \right] dx.$$
By taking \( k = 2 \alpha \) in Proposition 5 and Proposition 6, we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx + \int_\Omega |\nabla (u - v)|^2 dx \\
\leq \frac{1}{2} |1 - m(t)| \int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx + C|1 - m(t)|,
\]

where \( C \) depend on \( m_0, \alpha \) and \( \|u_0\|_{L^2(\Omega)} \). Applying Poincaré inequality and lemma 3, we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx + C(\Omega) \int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx \\
\leq C e^{-C t} \int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx + C e^{-C t}.
\]

From the above ODE we get the following estimate,

\[
\int_\Omega |(u - v) - (\overline{u} - \overline{v})|^2 dx \leq C_1 e^{-C_2 t},
\]

where \( C_1, C_2 \) are constants depending on \( m_0 \) and \( \|u_0\|_{L^2(\Omega)} \). Thus completes Theorem 8. \( \square \)

5 Numerical Results

For \( 1 \leq \alpha < 2 \) (\( n = 1, 2 \)), if \( m_0 < 1 \), it has been shown in the previous sections that the solution will exist globally without any restriction on the initial data. An interesting question is whether the solution also exists globally for \( \alpha > 2 \). In the following, we will give a complete numerical study for \( n = 1, 2 \) with any \( \alpha \geq 1 \).

5.1 Numerical scheme

For the numerical simulation, we consider the 2-dimensional equation in \( \Omega = [0, b]^2 \) with any \( b > 0 \)

\[
\begin{cases}
    u_t = \Delta u + u^\alpha \left( 1 - \int_\Omega u dx dy \right), \quad (x, y) \in [0, b] \times [0, b], \quad t \geq 0, \\
    u(0, x, y) = u_0(x, y) \geq 0, \quad (x, y) \in [0, b] \times [0, b], \\
    \frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(b, y, t) = 0, \\
    \frac{\partial u}{\partial y}(x, 0, t) = \frac{\partial u}{\partial y}(x, b, t) = 0.
\end{cases}
\]

(50)

Denote

\[
f(u) = u^\alpha \left( 1 - \int_\Omega u dx dy \right).
\]

(51)
Here an alternating direction implicit difference scheme is applied to construct the numerical computations.

Let $h$ be the space step and $\tau$ be the time step, $N = b/h + 1$ is the number of the discrete points, $T$ is the final time and $K = T/\tau + 1$. Denote

\[
x(i) = (i - 1) * h, \quad i = 1, 2, 3, \ldots, N - 1, N,
\]
\[
y(j) = (j - 1) * h, \quad j = 1, 2, 3, \ldots, N - 1, N,
\]
\[
t(k) = (k - 1) * \tau, \quad k = 1, 2, \ldots, T/\tau + 1,
\]
\[
\Omega_h = \{(x(i), y(j))|1 \leq i, j \leq N\}.
\]

The discrete solution at each time is presented as a matrix $u^k_{i,j} \in \mathbb{R}^{N \times N}$, where

\[
u_{i,j} = u(x(i), y(j), \cdot), \quad 1 \leq i, j \leq N,
\]
\[
u^k = u(\cdot, \cdot, t(k)), \quad k = 1, 2, \ldots, K.
\]

Next we introduce some notations:

\[
\delta^2_x u_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2},
\]
\[
\delta^2_y u_{ij} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2}.
\]

We construct the ADI scheme with operator splitting method and using Taylor expansion linearizes $f(u)$ to approximate this semi-linear equation as follows

\[
\frac{u^{k+1}_{ij} - u^k_{ij}}{\tau} = \left(\delta^2_x + \delta^2_y\right)\left(\frac{u^{k+1}_{ij} + u^k_{ij}}{2}\right) - \frac{\tau}{4} \left(\delta^2_x + 1\right) \delta^2_y \left(u^{k+1}_{ij} - u^k_{ij}\right)
\]
\[
+ f\left(u^k_{ij}\right) + \frac{u^{k+1}_{ij} - u^k_{ij}}{2} f'\left(u^k_{ij}\right), \quad 2 \leq i, j \leq N - 1,
\]
\[
u^1_{i,j} = U_0\left(x(i), y(j)\right), \quad 1 \leq i, j \leq N.
\]

(52) can be rearranged into

\[
\left[1 - \frac{\tau}{2} \delta^2_x - \frac{\tau}{2} f'\left(u^k_{ij}\right)\right] \left[1 - \frac{\tau}{2} \delta^2_y\right] u^{k+1}_{ij} = h\left(u^k_{ij}\right),
\]
\[
h\left(u^k_{ij}\right) = \left[1 - \frac{\tau}{2} f'\left(u^k_{ij}\right)\right] u^k_{ij} + \tau f\left(u^k_{ij}\right) + \frac{\tau}{2} \delta^2_x u^k_{ij} + \frac{\tau}{4} \delta^2_x + \frac{\tau}{4} f'\left(u^k_{ij}\right) + \frac{\tau}{2} \delta^2_y u^k_{ij}.
\]
Figure 1: $u(x,y)$ with time evolution, initial mass $m_0 < 1$
(a) mass of $u(x, y)$ with time evolution

Figure 2: Mass $u$ with time evolution, initial mass $m_0 < 1$

(53) is equivalent to the following ADI scheme

$$
\left[1 - \frac{\tau}{2} \delta_x^2 - \frac{\tau}{2} f'(u_{ij}^k)\right] \tilde{u}_{ij} = h\left(u_{ij}^k\right), \quad 2 \leq i \leq N - 1, \quad j = 2, 3, \ldots, N - 1, \quad k = 1, 2, \ldots, K. \tag{54}
$$

$$
\tilde{u}_{1,j} = \frac{4\tilde{u}_{2,j} - \tilde{u}_{3,j}}{3}, \quad \tilde{u}_{N,j} = \frac{4\tilde{u}_{N-1,j} - \tilde{u}_{N-2,j}}{3}, \quad j = 2, 3, \ldots, N - 1, \tag{55}
$$

$$
\left(1 - \frac{\tau}{2} \delta_y^2\right) u_{ij}^{k+1} = \bar{u}, \quad 2 \leq j \leq N - 1, \quad i = 2, 3, \ldots, N - 1, \quad k = 1, 2, \ldots, K - 1, \tag{56}
$$

$$
u_{i,1}^k = \frac{4u_{i,2}^k - u_{i,3}^k}{3}, \quad u_{i,N}^k = \frac{4u_{i,N-1}^k - u_{i,N-2}^k}{3}, \quad i = 2, 3, \ldots, N - 1. \tag{57}
$$

5.2 Numerical examples

Let $\Omega = [0, 1]^2$. In the numerical study, we will consider three cases:

1. $n = 2, 1 < \alpha < 2$, we choose $m_0 < 1$ and $u_0 \in L^\alpha(\Omega)$.

2. $n = 2, \alpha > 2$, we choose $m_0 < 1$ and $u_0 \in L^\alpha(\Omega)$, $\int_{\Omega} u_0^{\alpha} dx$ is large enough.

3. $n = 1, \alpha = 2$, we choose $m_0 < 1$ and $u_0 \in L^\alpha(\Omega)$, $\int_{\Omega} u_0^{2} dx$ is large enough.

**Case 1:** $n = 2$, we choose $\alpha = 3/2$ and the initial data

$$
u_0(x, y) = (-2x^3 + 3x^2 + 0.5)(-2y^3 + 3y^2), \quad \int_{\Omega} \nu_0(x, y) dx dy = 0.5 < 1. \tag{58}
$$

Fig. 1 shows the evolution of the solutions with time. We notice that the solution converges to $1/|\Omega|$, where $|\Omega|$ is the area of the 2-n domain. Fig. 2 shows its corresponding total mass with time evolution. Eventually the mass converges to 1.
Figure 3: $u(x, y)$ with time evolution, initial mass $m_0 > 1$
For the initial mass is greater than 1, we choose
\[ u_0(x, y) = (-2x^3 + 3x^2 + 1)(-2y^3 + 3y^2 + 1), \quad \int \int_{\Omega} u_0(x, y) dx dy = 2.25 > 1. \quad (59) \]
the results are shown in Fig. 3 where the solution converges to \(1/|\Omega|\). The evolution of mass is shown in Fig. 4, we observe that it finally converges to 1.

**Case 2:** \( n = 2, \alpha = 3 \) and \( m_0 < 1 \), the initial data is chosen to be a characteristic function
\[ u = \begin{cases} \frac{1}{40h^2}, & 0.3 \leq x \leq 0.3 + 5h, \ 0.3 \leq y \leq 0.3 + 5h, \\ 0, & \text{other}, \end{cases} \]
where \( h = 0.01 \). Simple computations deduce \( \int_{\Omega} u_0 dx = m_0 = 0.625 < 1 \) and \( \int_{\Omega} u_0^2 dx = 3.9 \times 10^4 \).

Fig. 5 shows the evolution of the solutions with time. We can notice that the mass tends to one very quickly and \( u(x, y) \) goes to \(1/|\Omega|\) with time evolution.

**Case 3:** \( n = 1, \alpha = 2 \) and \( m_0 < 1 \), the initial data is chosen to be
\[ u = \begin{cases} 10, & 0.3 \leq x \leq 0.35, \\ 0, & \text{other}, \end{cases} \quad (60) \]
the initial mass \( m_0 = 0.5 \). It can be observed in Fig. 6 that \( u(x) \) converges to the constant \(1/|\Omega|\), Fig. 7 shows the mass tends to one finally.
Figure 5: $n = 2$, $u(x, y)$ with time evolution, $m_0 < 1$
(a) $u(x)$ with time evolution

(b) Max of $u(x)$ with time evolution

Figure 6: $n = 1$, $u(x)$ with time evolution, $m_0 < 1$

(a) $\int_{\Omega} u(x) dx$ with time evolution

Figure 7: $n = 1$, mass of $u(x)$ with time evolution, initial mass $m_0 < 1$
6 Conclusions

This paper concerns the nonlocal Fisher KPP problem with the reaction term’s power $\alpha$. For $\alpha \geq 1$, the global existence of a classical solution to (1) is analyzed. When the initial mass is less than one, $1 \leq \alpha < 2$ for $n = 1, 2$ or $1 \leq \alpha < 1 + 2/n$ for $n \geq 3$, there exists a global unique nonnegative classical solution. When the initial mass is greater than or equal to one, the Fisher KPP problem admits a unique classical solution for any $\alpha \geq 1$. Our numerical simulations show that when the initial mass is less than one and $\alpha > 2$ for $n = 1, 2$, the unique nonnegative classical solution will exist globally. Therefore, for $n = 1, 2$ with $\alpha \geq 2$ or $n \geq 3$ with $\alpha \geq 2/n$, our conjecture is that the problem also admits a global unique nonnegative classical solution. This is a challenging problem which we will study in the future.

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