Gamma N Delta Form Factors and Wigner Rotations

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Abstract

For more than 50 years the $\Delta N\gamma$ form factors have been studied experimentally, theoretically, and phenomenologically. Although there has been substantial progress in understanding their behavior, there remains much work to be done. A major tool used in many investigations is the Jones-Scadron $\Delta$ rest frame parametrization of the three $\Delta N\gamma$ form factors. We point out that many studies utilizing this parametrization may not account for Wigner rotations and the consequent helicity mixing that ensues when the $\Delta$ is not at rest.

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The study of the $\Delta N\gamma$ transition form factors $G_M^*(q^2)$, $G_E^*(q^2)$, and $G_C^*(q^2)$ associated with the $\Delta(1232)$ nucleon resonance isobar has engendered much experimental and theoretical research for many decades. As the most studied nucleon resonance—the $\Delta(1232)$ has proved to be very difficult in the determination of its physical properties vis-à-vis its relation to the nucleon. Although similar to the nucleon in valence quark content, it has spin and isospin of $3/2$ as opposed to $1/2$ for the nucleon and thus its interaction with other particles via form factors is much more complex than that of the nucleon. In addition, the $\Delta(1232)$ is unstable with a large width, making measurement of physical observables and theoretical modeling much more difficult as well. Of special interest is the $\Delta(1232)$–nucleon 4-vector electromagnetic current matrix element in momentum space $\langle N | j^{\mu}(0) | \Delta \rangle$ associated with the process $\Delta \leftrightarrow N + \gamma^*$ described covariantly by the form factors $G_M^*(q^2)$, $G_E^*(q^2)$, and $G_C^*(q^2)$, where $q^2$ is the photon 4-momentum transfer squared. This matrix element and associated form factors is important in pion photoproduction and electroproduction (i.e. $\pi N \rightarrow \Delta \rightarrow \pi N\gamma$ or $\pi N \rightarrow \Delta \rightarrow \pi N$ or $\pi N\gamma$). In a world with unbroken $SU_F(N)$ flavor symmetry, one expects that $G_E^*(q^2) = G_C^*(q^2) = 0$ and that $G_M^*(q^2)$ would exhibit the same $q^2$ behavior as does the Sachs nucleon form factor $G_M$ thus giving rise to pure magnetic dipole $\Delta N\gamma$ transitions. Instead, one finds that $G_M^*$ appears to decrease faster as a function of $Q^2 \equiv -q^2$ than does $G_M$, the ratio $-G_E^*/G_M^* \neq 0$, the magnitude of $G_E^*(Q^2)$ is small when compared to $G_M^*(q^2)$ near $Q^2 \approx 0$, and that $G_M^*$ possesses a complicated behavior as a function of $Q^2$.

Probably the most widely used parametrization for the study of the $\Delta N\gamma$ transition form factors is that to Jones and Scadron [1] (JS) followed by very closely related variations such as the helicity form factors of Devenish, Eisenschitz, and Korner [2] (DEK) or Bjorken and Walecka [3]. The JS parametrization is written explicitly in the the rest frame of the $\Delta$ and introduces covariant couplings $G_M^*(q^2)$, $G_E^*(q^2)$, and $G_C^*(q^2)$ analogous to the familiar Sachs nucleon form factors $G_M(q^2)$ and $G_E(q^2)$. The fact that the JS parametrization—though covariant—utilizes in a very explicit fashion the rest frame of the $\Delta$ is the subject of this work as significant modifications may be manifested due to the existence of Wigner rotations [4] of purely geometric origin.

Setting notation, we normalize physical states with $\langle \tilde{p}'|\tilde{p} \rangle = (2\pi)^3 \delta^3(\tilde{p}' - \tilde{p})$. Dirac spinors are normalized by $\bar{u}(p)u(p) = 2m$ . Our conventions for Dirac matrices are $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $\gamma_5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3$, where $g^{\mu\nu} = \text{Diag} (1, -1, -1, -1)$. The Ricci-Levi-Civita tensor is defined by $\varepsilon_{0123} = -\varepsilon_{0123}^* = 1 = \varepsilon_{123}$. As usual, we use natural units where $\hbar = c = 1$.

In general, one may write for the $\Delta N\gamma$ transition amplitude—and incorporating the DEK prescription for removal of kinematic singularities at threshold and pseudo-threshold via an explicit
$Q^+ Q^-$ factor (defined below)—we have the following JS-DEK expression where the $\Delta$ is at rest:

$$\langle N \bar{p}, \lambda_N | j_{\mu}(0) | \Delta p^*, \lambda_\Delta \rangle = e \bar{u}_N(\bar{p}, \lambda_N) [\Gamma_{\mu \beta}] u_\Delta^\beta(p^*, \lambda_\Delta)$$

where $e = \sqrt{4\pi \alpha}$, $\alpha$ is the fine structure constant, and $u_\Delta^\beta$ is a Rarita-Schwinger spinor.

$$\Gamma_{\mu \beta} = \frac{3(m^* + m)}{2m}(G_M^* - 3G_E^*)(Q^+ Q^-) \left[ im^* q_\beta \epsilon_\mu(qp) \gamma \right] + \frac{3(m^* + m)}{2m}(G_M^* + G_E^*)(Q^+ Q^-) \left[ im^* q_\beta \epsilon_\mu(qp) \gamma - 2\epsilon_\beta\sigma(p^* p)\epsilon_\mu^{\sigma}(p^* p)\gamma_5 \right] + \frac{3(m^* + m)}{m}G_C^*(Q^+ Q^-)q_\beta \left[ p \cdot q \mu - q^2 p_\mu \right] \gamma_5$$

$$= -h_2(Q^+ Q^-) \left[ im^* q_\beta \epsilon_\mu(qp) \gamma \right] - h_3(Q^+ Q^-) \left[ im^* q_\beta \epsilon_\mu(qp) \gamma - 2\epsilon_\beta\sigma(p^* p)\epsilon_\mu^{\sigma}(p^* p)\gamma_5 \right] + h_1(Q^+ Q^-)q_\beta \left[ p \cdot q \mu - q^2 p_\mu \right] \gamma_5,$$

where

$$h_2 = -\frac{3(m^* + m)}{2m}(G_M^* + G_E^*),$$

$$h_3 = -\frac{3(m^* + m)}{2m}(G_M^* - 3G_E^*),$$

$$h_1 = \frac{3(m^* + m)}{m}G_C^*.$$ (3)

In Eqs. (1) and (2), the electromagnetic current is denoted by $j_{\mu}$ which transforms like a Lorentz 4-vector, $q = p^* - p$, $p^*$ and $p$ are the four-momenta of the $\Delta$ and nucleon respectively with $p^* = (m^*, \hat{0})$ and $p = (p^0, \hat{p})$. $q^2$ is the invariant 4-momentum transfer, $m^*$ is the $\Delta$ mass, $m$ is the nucleon mass, and the magnitude of the three-momentum of the photon in the $\Delta$ rest frame is $|q_\gamma|$: $Q^+ Q^- = 4m^2 q_\gamma^2$ with $Q^\pm = (m^* \pm m)^2 - q^2$; $\lambda_N$ and $\lambda_\Delta$ are the helicities of the nucleon and the $\Delta$ respectively.

Note that the first (fourth $h_2$), second (fifth $h_3$), and third (sixth $h_1$) terms in Eq. (2) induce transverse $\frac{1}{2}$, transverse $\frac{3}{2}$, and longitudinal (scalar) helicity transitions, respectively, in the rest frame of the $\Delta$. $h_1$, $h_2$, and $h_3$ are the DEK helicity form factors and $G_M^*$, $G_E^*$, and $G_C^*$ induce magnetic, electric, and coulombic (scalar) multipole transitions respectively.

The magnetic, electric, and scalar multipole transition moments given by $M_{1+}(q^2)$, $E_{1+}(q^2)$, and $S_{1+}(q^2)$ can be written in terms of $G_M^*(q^2)$, $G_E^*(q^2)$, and $G_C^*(q^2)$:

$$M_{1+} = a_1 \sqrt{Q^-} G_M^*,$$ (4)
\[ E_{1+} = a_2 \sqrt{Q^-} G^*_E, \]
\[ S_{1+} = a_3 Q^- \sqrt{Q^+} G^*_C, \]

where \( a_1, a_2 = -a_1, \) and \( a_3 \) are dependent on the \( \Delta \) mass \( m^* \) and width at resonance and other parameters governing the process \( \Gamma(\Delta \to \pi N) \).

However, generally (theoretically, experimentally, and phenomenologically), one is confronted with the transition amplitude given by:

\[ \langle N\bar{s}, \lambda_N | j_\mu(0) |\Delta\bar{r}, \lambda_\Delta \rangle, \quad (5) \]

\( s \equiv s^\mu = (s^0, \vec{s}), \) \( t \equiv t^\mu = (t^0, \vec{t}) \) where the \( \Delta \) is not at rest and the nucleon and \( \Delta \) 3-momenta are not necessarily collinear. Then one finds that matrix elements Eq. (1) and Eq. (5) are related through a sequence of Lorentz transformations and consequent Wigner rotation angles [5, 6, 7, 8, 9, 10].

Let \( U(\Lambda_\chi) \) be an proper homogeneous Lorentz transformation operator which brings the \( \Delta \) represented by the single-particle helicity state \( |\Delta\bar{r}, \lambda_\Delta \rangle \) to rest \( |\hat{\Delta}\vec{0}, \lambda_\Delta \rangle \) and \( U(H(\Lambda_\chi\bar{r})) \) defines the transformation for the helicity state \( |N\bar{s}, \lambda_N \rangle = |Ns\theta\phi\lambda_N \rangle = U(R^N(\theta, \phi)) |Ns\lambda_N \rangle = U(R^N(\theta, \phi)Z_N(s)) |N\lambda_N \rangle \), thus \( H_N(\bar{s}) = R^N(\theta, \phi)Z_N(s) \) is the homogeneous Lorentz transformation which defines the single-particle helicity state \( |N\bar{s}, \lambda_N \rangle \rightarrow R^N(\theta, \phi) \) is a rotation with Euler angles \( \theta \) and \( \phi \) and \( Z_N(s) \) is a pure Lorentz boost such that \( (m, \vec{0}) \rightarrow (m, \vec{s} | \hat{\bar{\xi}}) \). Similarly, \( H_\Delta(\bar{r}) \) defines the state \( |\Delta(\bar{r}, \lambda_\Delta \rangle = U(H_\Delta(\bar{r})) |\Delta\lambda_\Delta \rangle \). \( |N\lambda_N \rangle \) and \( |\Delta\lambda_\Delta \rangle \) are rest frame states for the nucleon and \( \Delta \) respectively. Note that the quantization axis \( \hat{\bar{\xi}} \) is defined by \( \bar{r} = |\bar{r}| \hat{\bar{\xi}} \) and that \( p = \Lambda_\chi\bar{s} \).

\( \Lambda_\chi \) is a Lorentz transformation such that \((\Lambda_\chi)^\mu \nu r^\nu = r'^\mu = (r^0, \vec{r}') = (r^0, \Lambda_\chi \vec{r}) \), where \( r \) and \( r' \) are four-momenta ⇒

\[ U(\Lambda_\chi |N\bar{s}, \lambda) = \sum_{\lambda'} |N \Lambda_\chi \bar{s} \lambda' \rangle \langle N \Lambda_\chi \bar{s} \lambda'| U(\Lambda_\chi |N\bar{s}, \lambda) \]
\[ = \sum_{\lambda'} |N \Lambda_\chi \bar{s} \lambda' \rangle \langle N \lambda' | U(H_N^{-1}(\Lambda_\chi\bar{s}) \Lambda_\chi H_N(\bar{s})) |N, \lambda \rangle \]
\[ = \sum_{\lambda'} |N \Lambda_\chi \bar{s} \lambda' \rangle D^{(1/2)}_{\lambda', \lambda}(R^N_W), \]

where \( D^{(1/2)}_{\lambda', \lambda}(R^N_W) = \langle N \lambda' | U(R^N_W(\Lambda_\chi s, s)) |N, \lambda \rangle = \langle N \lambda' | U(R^N_W) |N, \lambda \rangle, \) and the Wigner rotation which connects the rest frame nucleon states is given by \( R^N_W(\Lambda_\chi s, s) = \)
The helicity representation transformation law for $\Delta N \gamma$ matrix elements involving an operator $A$ (for clarity, we suppress any contravariant or covariant Lorentz indices here) which transforms under the Lorentz transformation $\Lambda_x$ like $U(\Lambda_x)AU(\Lambda_x)^{-1} = A_{\Lambda_x}$ is then given by [10]

$$\{N\tilde{s}, \lambda_N|A|\Delta\tilde{t}, \lambda_{\Delta}\} = \sum_{\lambda', \lambda'} D_{\lambda', \lambda_N}^{(1/2)*}(R_W^N) < N\overrightarrow{\Lambda_{\Delta}\tilde{s}\lambda'}|A_{\Lambda_x}|\Delta \overrightarrow{\Lambda_{\Delta}\tilde{t}\lambda} > D_{\lambda', \lambda_{\Delta}}^{(3/2)}(R_W^N). \tag{7}$$

For the case where $U(\Lambda_x)$ is a pure homogeneous Lorentz transformation operator along the $-\hat{z}$ direction associated with a velocity $v_x$ and which brings the $\Delta$ represented by the single-particle helicity state $|\Delta(\tilde{t}, \lambda_{\Delta})\rangle$ to rest $|\Delta(0, \lambda_{\Delta})\rangle$, i.e. $\Lambda_x(\tilde{t}, \hat{z}) = \tilde{0}$ with $\tilde{t} = (0, 0, t_z)$, then Eq. (7) greatly simplifies since $H_{\Delta}^{-1}(\overrightarrow{\Lambda_x\tilde{t}})\Lambda_x|\Delta(\tilde{t})\rangle = H_{\Delta}^{-1}(0)\Lambda_x|\Delta(\tilde{t})\rangle = 1 \Rightarrow D_{\lambda', \lambda_{\Delta}}^{(3/2)}(R_W^N) = \delta_{\lambda', \lambda_{\Delta}}$ and we obtain

$$\{N\tilde{s}, \lambda_N|A|\Delta\tilde{t}, \lambda_{\Delta}\} = \sum_{\lambda'} D_{\lambda', \lambda_N}^{(1/2)*}(R_W^N) < N\overrightarrow{\Lambda_x\tilde{s}\lambda'}|A_{\Lambda_x}|\Delta \overrightarrow{0\lambda_{\Delta}} >. \tag{8}$$

Setting $A_{\Lambda_x} = \Lambda_x j^\mu \Lambda_x^{-1}$ one has

$$\{N\tilde{s}, \lambda_N|j^\mu|\Delta\tilde{t}, \lambda_{\Delta}\} = \sum_{\lambda'} D_{\lambda', \lambda_N}^{(1/2)*}(R_W^N) < N\overrightarrow{\Lambda_x\tilde{s}\lambda'}|\Lambda_x j^\mu \Lambda_x^{-1}|\Delta \overrightarrow{0\lambda_{\Delta}} > \tag{9}$$

$$= D_{\lambda', \lambda_N}^{(1/2)*}(R_W^N) < N\overrightarrow{\Lambda_x\tilde{s}} - \frac{1}{2}|\Lambda_x j^\mu \Lambda_x^{-1}|\Delta \overrightarrow{0\lambda_{\Delta}} > + D_{\lambda', \lambda_N}^{(1/2)*}(R_W^N) < N\overrightarrow{\Lambda_x\tilde{s}} + \frac{1}{2}|\Lambda_x j^\mu \Lambda_x^{-1}|\Delta \overrightarrow{0\lambda_{\Delta}} >.$$

Eq. (9) is the main result of this work and demonstrates explicitly the helicity mixing that occurs when the JS $\Delta N \gamma$ form factor construction is used. The Wigner rotation matrices are automatically brought into play and must be considered in most circumstances. There are exceptions as will be made clearer below.

Without loss of generality, we consider only $\tilde{x} - \tilde{z}$ plane dynamics, where for instance, for the transverse components $\mu = 1$ or $\mu = 2$, $\Lambda_x j^\mu \Lambda_x^{-1} = j^\mu$. The polar angles of $\tilde{s}$ are given by $(\theta, \phi = 0)$ and and the polar angles (all referred to the $\tilde{z}$ axis) of $\tilde{p} = \overrightarrow{\Lambda_x\tilde{s}}$ are given by $(\theta', \phi' = 0)$. Thus, the four-momentum vectors $s$ and $p$ are related by $p = \Lambda_x s$, whereas $p^* = (m^*, 0) = \Lambda_x t$. 

5
We calculate the Wigner rotation angle with:

\[
(\Lambda)^\mu_\nu(\vec{\sigma}, \vec{\theta}) = \begin{pmatrix}
\cosh(\vec{\sigma}) & 0 & 0 & \sinh(\vec{\sigma}) \\
\sin(\vec{\theta}) \sinh(\vec{\sigma}) & \cos(\vec{\theta}) & 0 & \cosh(\vec{\theta}) \sin(\vec{\sigma}) \\
0 & 0 & 1 & 0 \\
\cos(\vec{\theta}) \sinh(\vec{\sigma}) - \sin(\vec{\theta}) & 0 & \cosh(\vec{\theta}) & 0
\end{pmatrix}
\]  

(10)

\[
R_W(\omega) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\omega) & 0 & \sin(\omega) \\
0 & 0 & 1 & 0 \\
0 & -\sin(\omega) & 0 & \cos(\omega)
\end{pmatrix}
\]  

(11)

Thus, we find that—

\[
R_N^N(\Lambda_\chi s, s) = \Lambda_N^{-1}(\sigma', \theta')\Lambda_\chi \Lambda_N(\sigma, \theta)
\]

\[
\tan(\omega_N) = \frac{-\sin(\theta) \sinh(\chi)}{\cosh(\chi) \sinh(\sigma) - \cos(\theta) \cosh(\sigma) \sinh(\chi)} = \frac{-\sin(\theta) m \mid \vec{\tau} \mid}{t^0 |\vec{s}| - \cos(\theta) s^0 |\vec{\tau}|} 
\]

(12)

\[
\sin(\omega_N) = \frac{-\sin(\theta) \sinh(\chi)}{\sinh(\sigma')} = \frac{-\sin(\theta) m \mid \vec{\tau} \mid}{m^* \mid \vec{\rho} \mid} = \frac{-\sin(\theta) \mid \vec{\tau} \mid \sqrt{1 - v_p^2}}{\sqrt{1 - u_{\chi}^2} \mid \vec{\rho} \mid}.
\]

(13)

In Eq. (12) and Eq. (13), \(\sinh(\chi) = \mid \vec{\tau} \mid (1 - u_{\chi}^2)^{-1/2}\) where \(u_{\chi}\) is the velocity parameter which specifies the Lorentz boost \(\Lambda_{\chi}\), \(v_s\) is the velocity of the nucleon in the frame where \(\mid \vec{s} \mid = m \sinh(\sigma)\) and \(v_p\) is the velocity of the nucleon in the \(\Delta\) rest frame. Thus, \(\omega_N\) is independent of the nucleon mass and is a purely geometric phenomenon. Now \(D^{(1/2)}_{m, m'}(R_N^N) = e^{-im\alpha} D^{(1/2)}_{m, m'}(\omega_N)e^{-im'\gamma}\), so choosing for conciseness \(\alpha = \gamma = 0\), then

\[
D^{(1/2)}_{m, m'}(\omega_N) = \begin{pmatrix}
\cos\left(\frac{\omega_N}{2}\right) & -\sin\left(\frac{\omega_N}{2}\right) \\
\sin\left(\frac{\omega_N}{2}\right) & \cos\left(\frac{\omega_N}{2}\right)
\end{pmatrix}.
\]

(14)

We give an example (transverse 1/2 transition, with \(\mu = 1 - i 2, \lambda_\Delta = 1/2, \) and \(\lambda_N = -1/2\)) using Eq. (9):

\[
\langle N\vec{s}, -1/2 | j_{1-i/2}^1 | \Delta_{\vec{\tau}} = t_{\vec{\tau}}, +1/2 \rangle = \sum_{\lambda'} D^{(-1/2)*}_{\lambda', \lambda} (R_N^N) < N \Lambda_{\chi} \vec{s}' | j_{1-i/2}^1 | \Delta_{\vec{\tau}} = t_{\vec{\tau}}, +1/2 >
\]

(15)
\[
\begin{align*}
&= \cos \left( \frac{\omega N}{2} \right) < N \overrightarrow{\Lambda} s \frac{1}{2} \mid j^{1-i2} \mid \Delta 0 \frac{1}{2} > \\
&- \sin \left( \frac{\omega N}{2} \right) < N \overrightarrow{\Lambda} s \frac{1}{2} \mid j^{1-i2} \mid \Delta 0 \frac{1}{2} > .
\end{align*}
\]

We see that if \( \overrightarrow{\Lambda} s \) is not collinear with the \( \hat{z} \) axis, \( \sin \left( \frac{\omega N}{2} \right) \neq 0 \) and the matrix element \( < N \overrightarrow{\Lambda} s \frac{1}{2} \mid j^{1-i2} \mid \Delta 0 \frac{1}{2} > \) is non-vanishing as well. There exist cases besides collinearity where the Wigner angle need not be calculated and the JS parametrization can be used without change: A case example is when only the helicity-averaged quantity \( \frac{1}{2} \sum_{\lambda, N} \left| \langle N \overrightarrow{s}, \lambda N \mid j^\mu \mid \Delta \vec{i}, \lambda \Delta \rangle \right|^2 \) is utilized in one’s theoretical or experimental model. That is because

\[
\sum_{\lambda, N} \left| \langle N \overrightarrow{s}, \lambda N \mid j^\mu \mid \Delta \vec{i}, \lambda \Delta \rangle \right|^2 = \sum_{\lambda, N} \left| < N \overrightarrow{\Lambda} s \lambda N \mid \Lambda \vec{j}^\mu \Delta \vec{\Lambda} \lambda > \right|^2 \text{ since } \sum_{m''} D^{(j)\ast}_{m'' m} (R) D^{(j)}_{m'' m'} (R) = \delta_{m m'} \text{[11].}
\]

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