On Type II noncommutative geometry and the JLO character

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Abstract
The Jaffe-Lesniewski-Osterwalder (JLO) character [23] is a homomorphism from K-homology to entire cyclic cohomology. This paper extends the domain of the JLO character to include Type II noncommutative geometry, the geometry represented by Breuer-Fredholm modules; and shows that the JLO character defines the same cohomology class as the Connes-Chern character [2] in entire cyclic cohomology.

Contents
0 Introduction 1
1 Breuer-Fredholm modules and Connes-Chern character 3
  1.1 Breuer-Fredholm modules ........................................ 3
  1.2 Entire cyclic (co)homology ...................................... 4
  1.3 Connes-Chern character .......................................... 7
2 Unbounded Breuer-Fredholm modules and JLO character 10
  2.1 Unbounded Breuer-Fredholm modules .......................... 10
  2.2 JLO character ..................................................... 11
  2.3 Homotopy invariance of the JLO class ....................... 15
  2.4 Index pairing in (co)homology ................................. 19
3 Reduction from JLO character to Connes-Chern character 20
  3.1 JLO character for $p$-summable unbounded Breuer-Fredholm modules ........................................... 20
  3.2 From unbounded to bounded Breuer-Fredholm modules .................. 21
  3.3 From JLO character to Connes-Chern character ............ 23
A Appendix 28

0 Introduction
In non-commutative geometry, the guiding principle is that the topology of spaces is encoded in properties of their algebras of continuous functions. A theorem of Gelfand-Naimark [24] states that any commutative unital $C^*$-algebra is of the form $C(X)$ for some compact Hausdorff space $X$. Therefore, the category of $C^*$-algebras (or even more generally Banach $*$-algebras) is seen as an extension of the category of compact Hausdorff topological spaces, and a general $C^*$-algebra is sometimes referred to as a non-commutative topological space. The geometric features on a $C^*$-algebra $A$ are incorporated by the concept of an an unbounded Fredholm module $(\rho, B(\mathcal{H}), \mathcal{D})$ over $A$, where $\rho$ is a continuous representation of $A$ onto the Hilbert space $\mathcal{H}$, and $\mathcal{D}$ is an unbounded Fredholm operator on $\mathcal{H}$ that satisfies certain axioms. As the prototypical example, let $A$ be the algebra of continuous functions on a closed Riemannian manifold, $\mathcal{H}$ the square integrable sections of a spinor bundle with its natural action of $A$, and

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\( \mathcal{D} \) the associated Dirac operator. Geometric features on the manifold such as geodesics, dimension, integrations, and differential forms etc can be retrieved algebraically in terms of \( A, B(\mathcal{H}), \) and \( \mathcal{D} \) [21]. Connes gives a set of five axioms characterizing the Fredholm modules arising in this way [15]. Taking \( A \) to be non-commutative thus leads to a notion of a non-commutative manifold. This theory is summarized in Connes’ famous book [14], further details and newer developments are described in [21] and [24].

Each Fredholm module assigns an integer, the Fredholm index, to a given element in the K-theory of \( A \). The Fredholm index provides a \( \mathbb{Z} \)-valued pairing between the K-homology of \( A \) and the K-theory of \( A \). In the commutative setting, the index can be viewed as the index of the Dirac operator \( \mathcal{D} \), twisted by a vector bundle.

Suppose the unbounded Fredholm module \((\rho, B(\mathcal{H}), \mathcal{D})\) is finitely summable, a condition that models finite dimensionality according to Connes’ axioms, Jaffe-Lesniewski-Osterwalder [23] defined a cocycle \( \text{Ch}_{\text{JLO}}^\bullet \) in the entire cyclic cohomology \( \text{HE}^\bullet(A) \), now known as the JLO character. Together with the K-theory character \( \text{ch}_n : K_n(A) \to \text{HE}_n(A) \), they intertwine the K-theoretical pairing given by the Fredholm index with the cohomological pairing between \( \text{HE}^\bullet(A) \) and \( \text{HE}_n(A) \) [18, 20]. Such a result was originally established in a more general setting for weakly \( \theta \)-summable Fredholm modules, where weak \( \theta \)-summability can be thought of a suitable notion of infinite-dimensionality. Consequently, the JLO character provides a formula for the Fredholm index in terms of entire cyclic (co)homology for infinite dimensional non-commutative manifolds, which was the original motivation of JLO’s work [23]. Furthermore, the formula reduces to the index formula of Atiyah-Singer in the commutative setting [3, 19].

The operator \( \mathcal{D} \) of a Fredholm module plays the role of a Dirac operator, and is typically unbounded. However, there is a canonical way of passing from an unbounded Fredholm module to a bounded one \((\rho, B(\mathcal{H}), F)\), essentially by taking bounded functions of \( \mathcal{D} \), and the latter are often easier to work with in practice.

When the bounded Fredholm module \((\rho, B(\mathcal{H}), F)\) is finitely summable, there is a character \( \text{ch}_n^{A} \) due to Connes [13], which again is a cocycle in \( \text{HE}^n(A) \), and \( \text{ch}_n^{A} \) intertwines the K-theoretical pairing the same way as the JLO character [13, 16]. When \((\rho, B(\mathcal{H}), F)\) is the associated bounded module of a finitely summable unbounded Fredholm module \((\rho, B(\mathcal{H}), \mathcal{D})\), Connes-Moscovici proved that in fact the cocycle \( \text{ch}_n^{A}(F) \) of \((\rho, B(\mathcal{H}), F)\) defines the same cohomology class as the cocycle \( \text{Ch}_{\text{JLO}}^\bullet(\mathcal{D}) \) of \((\rho, B(\mathcal{H}), \mathcal{D})\) in \( \text{HE}^\bullet(A) \) [16].

Type II non-commutative geometry replaces the algebra \( B(\mathcal{H}) \) with a (possibly) Type II von Neumann algebra \( \mathcal{N} \subset B(\mathcal{H}) \), using Breuer’s Fredholm theory relative to the von Neumann algebra \( \mathcal{N} \) [5, 6]. A Type II non-commutative geometry on the algebra \( A \) is given by an unbounded Breuer-Fredholm module \((\rho, \mathcal{N}, \mathcal{D})\) over \( A \), where \( \mathcal{D} \) is a Breuer-Fredholm operator affiliated with \( \mathcal{N} \) that satisfies certain axioms. Examples of unbounded Breuer-Fredholm modules arise from foliations or geometry with degeneracies. A number of examples can be found in [2].

Parallel to the Type I setting, the Breuer-Fredholm theory provides an index pairing between Breuer-Fredholm modules and K-theory given by the Breuer-Fredholm index [5, 6, 11]. As a characteristic of the Type II von Neumann algebra \( \mathcal{N} \), the Breuer-Fredholm index now takes value in \( \mathbb{R} \) as opposed to \( \mathbb{Z} \) as in the Type I case. In this paper, we will develop the even JLO character for Type II non-commutative geometry. For completeness, we also include the odd case, which was developed by Carey-Phillips in [7]. Similar to the Type I case, we show how to pass unbounded Breuer-Fredholm modules to bounded ones. Extending the argument for the Type I case, we show that this correspondence takes the class of the (both even and odd) JLO character to that of the Connes character in \( \text{HE}^\bullet(A) \), as defined in Type II case by Benenmut-Fack [2].

The first section starts with background material on Breuer-Fredholm theory and the index pairing between K-homology and K-theory. Then following [2] we define the Connes-Chern character for K-homology and the Chern character for K-theory, and show that for \( p \)-summable Breuer-Fredholm modules, these two characters intertwine the index pairing with the pairing in entire cyclic (co)homology. In Section 2, the JLO character for \( \theta \)-summable unbounded Breuer-Fredholm modules is defined. We study its homotopy invariance as an entire cyclic cohomology class by following along the lines of Getzler and Szenes [20] and show that it preserves the index pairing. Section 3 connects the previous two sections by showing that a \( p \)-summable unbounded Breuer-Fredholm module canonically gives rise to a \( p \)-summable Breuer-Fredholm module. We then proceed using techniques from Connes and Moscovici [16] to show that the JLO character for the \( p \)-summable unbounded Breuer-Fredholm module and the Connes-Chern character for the \( p \)-summable Breuer-Fredholm module define the same entire cyclic cohomology class. In the Appendix we recall some definitions and inequalities needed for the discussion in our paper.
1 Breuer-Fredholm modules and Connes-Chern character

The section starts by stating the definition of Breuer-Fredholm modules from [8]. With the notion of \((e,f)\)-Fredholm from [12], we proceed to develop a suitable Fredholm theory by following [2]. Entire cyclic (co)homology will be introduced, followed by a discussion of the Chern character [20, 18] on K-theory and Connes-Chern character [2] on K-homology. The Section ends by showing that the characters intertwine the K-theoretical pairing given by the Fredholm index, with the cohomological pairing.

1.1 Breuer-Fredholm modules

For a given semi-finite von Neumann algebra \(\mathcal{N} \subset B(\mathcal{H})\) of bounded operators on a Hilbert space \(\mathcal{H}\), with a faithful semi-finite normal trace \(\tau\), we denote by \(K_{\mathcal{N}}\) the ideal of \(\tau\)-compact operators in \(\mathcal{N}\). A \(\tau\)-compact operator is a (densely defined closed) operator affiliated with \(\mathcal{N}\), such that its generalized singular number \(\mu_\tau(T)\) with respect to \(\tau\) converges to 0. The definitions and properties of \(K_{\mathcal{N}}\) and \(\mu_\tau(T)\) can be found in the Appendix.

**Definition 1.1.** An odd Breuer-Fredholm module over a unital Banach \(\ast\)-algebra \(A\) is a triple \((\rho,\mathcal{N},F)\) for which \(\mathcal{N}\) is a (separable) semi-finite von Neumann algebra with faithful semi-finite normal trace \(\tau\), \(\rho: A \to \mathcal{N}\) a continuous \(\ast\)-representation, and \(F \in \mathcal{N}\) an operator such that \(F^2 = 1\) and \([F,\rho(a)] \in K_{\mathcal{N}}\) for all \(a \in A\).

If \((\rho,\mathcal{N},F)\) is equipped with a \(\mathbb{Z}_2\) grading \(\chi \in \mathcal{N}\) such that all \(\rho(a)\) are even and \(F\) is odd, then we call \((\rho,\mathcal{N},F)\) an even Breuer-Fredholm module.

If \(\mathcal{N} = B(\mathcal{H})\) and \(\tau\) is the standard operator trace, we drop the prefix Breuer.

As Fredholm modules are representatives of K-homology classes in Kasparov’s sense [22], they are also referred to as K-cycles.

Technically speaking, Breuer-Fredholm modules do not define K-homology classes in the usual sense, however one can still consider its classes given by the equivalence relations in K-homology. i.e., up to degenerate modules, two such modules are equivalent if their Fredholm operators are connected by a norm continuous homotopy of Fredholm operators (in \(\mathcal{N}\)) (see for example [22] for a precise definition). We think of Breuer-Fredholm modules as representatives of elements in some semi-finite or Type II K-homology as [9, 11] did. Whenever we write \([(\rho,\mathcal{N},F)] \in K^*(A)\), we implicitly mean that the K-homology is in the semi-finite sense.

Recall that a densely defined closed operator \(T\) with polar decomposition \(T = U|T|\) is said to be affiliated with \(\mathcal{N}\) if \(U \in \mathcal{N}\) and also the spectral projections of \(|T|\) lie in \(\mathcal{N}\) (see Appendix). The only unbounded operators we are dealing with here are densely defined closed operators, hence the properties of an unbounded operator being densely defined and closed are automatically assumed throughout this paper. In particular, when we speak of an operator \(T\) affiliated with \(\mathcal{N}\), we demand that \(T\) is densely defined and closed.

**Definition 1.2.** Given two projections \(e,f \in \mathcal{N}\), a (possibly unbounded) operator \(T\) affiliated with \(\mathcal{N}\) is called \((e,f)\)-Fredholm if there is an operator \(S \in \mathcal{N}\), such that

\[
e - eSTe \in K_{e\mathcal{N}e}\quad \text{and}\quad f - fTeSf \in K_{f\mathcal{N}f},
\]

where \(K_{e\mathcal{N}e}\) denotes the set of \(\tau\)-compact operators in \(e\mathcal{N}e\), likewise for \(K_{f\mathcal{N}f}\). The operator \(S\) is called an \((e,f)\)-parametrix for \(T\).

**Example 1.1.**

- Let \((\rho,\mathcal{N},F)\) be a Breuer-Fredholm module. If \(u \in \mathcal{N}\) is a unitary, then \(u\) is \((\frac{F+1}{2}, \frac{F+1}{2})\)-Fredholm with \((\frac{F+1}{2}, \frac{F+1}{2})\)-parametrix \(u^{-1}\).

- Suppose that \((\rho,\mathcal{N},F)\) comes equipped with a \(\mathbb{Z}_2\) grading \(\chi\) and that the projection \(p \in \mathcal{N}\) is even with respect to \(\chi\), then \(F\) is \((p^+, p^-)\)-Fredholm with \((p^+, p^-)\)-parametrix \(F\) again.

The following Proposition can be found in [2]. We adopted it in the \((e,f)\)-parametrix case.

**Proposition 1.1.** Let \(T\) be a \((e,f)\)-Fredholm operator, and \(P_{kerT}\) and \(P_{ker(T^*)}\) be the projections onto the kernels of \(T\) and \(T^*\) respectively. Then \(eP_{kerT}\) and \(P_{ker(T^*)}f\) have finite trace with respect to \(\tau\).
Proof. Let \( S \) be a \((e, f)\)-parametrix of \( T \) as in Definition 1.2. We have \( (e - eSTe)P_{\ker T} = eP_{\ker T} \) and \( P_{\ker (T^*)} (f - fTeSf) = P_{\ker (T^*). f} \) and \( eP_{\ker T} \) and \( P_{\ker (T^*)} f \) projections onto \( \ker (T) \cap e(\mathcal{H}) = \ker (fTe|_{\mathcal{H}}) \) and \( \ker (T^*) \cap f(\mathcal{H}) = \ker (eT^* f|_{\mathcal{H}}) \) respectively. By the ideal property of \( \mathcal{K}_{\nu \nu} \), \( eP_{\ker T} \) is a \( \tau \)-compact projection. As projections only have eigenvalue \( \{0, 1\} \), \( \tau \)-compactness forces the singular values of projections to have support in a bounded region, hence \( \tau \) of any \( \tau \)-compact projection must be finite, and \( \tau (eP_{\ker T}) < \infty \). Likewise, \( \tau (P_{\ker (T^*)} f) < \infty \).

\[ \square \]

**Definition 1.3.** The \((e, f)\)-index \( \text{Ind}_\tau (fTe) \) of an \((e, f)\)-Fredholm operator \( T \) is defined to be

\[ \text{Ind}_\tau (fTe) := \tau (eP_{\ker T}) - \tau (P_{\ker (T^*)} f), \]

where \( P_{\ker T} \) and \( P_{\ker (T^*)} \) are the projections onto the kernel of \( T \) and \( T^* \) respectively.

Given an even Breuer-Fredholm module \((\rho, \mathcal{N}, F)\) over \( A \), and a projection \( p \in A \). It follows from Example 1.1 that \( F \) is a \((\rho(p)^+, \rho(p)^- )\)-Fredholm operator. Thus it has a well-defined \((\rho(p)^+, \rho(p)^- )\)-index, given by \( \text{Ind}_\tau (\rho(p)^- F \rho(p)^+) \).

For a given odd Breuer-Fredholm module \((\rho, \mathcal{N}, F)\), and a unitary \( u \in A \), then \( \rho(u) \) is a \((Q, Q)\)-Fredholm operator, where \( Q = \frac{F + 1}{2} \).

Thus it has a well-defined \((Q, Q)\)-index, given by \( \text{Ind}_\tau (Q \rho(u) Q) \).

Since the function \( \text{Ind}_\tau \) is locally constant \([12]\), the \((\rho(p)^+, \rho(p)^- )\)-index descends to a pairing between the K-homology class \([\rho, \mathcal{N}, F]\) \( \in \mathcal{K}^0 (A) \) and the K-theory class \([p]\) \( \in \mathcal{K}_0 (A) \). Likewise, the \((Q, Q)\)-index descends to a pairing between the classes \([\rho, \mathcal{N}, F]\) \( \in \mathcal{K}^1 (A) \) and \([u]\) \( \in \mathcal{K}_1 (A) \).

To simplify our notation, whenever we mention an element \( a \in A \), we think of it as an operator \( \rho(a) \in \mathcal{N} \) represented on \( \mathcal{H} \), and will stop writing \( \rho \).

Similarly, when we have \( a \in M_N (A) \), we think of it as an operator in \( M_N (\mathcal{N}) \) represented on \( \mathcal{H}^N = \mathcal{H} \otimes \mathbb{C}^N \) with the obvious representation extended from \( \rho \).

**Definition 1.4 \([8, 9, 11]\).**

1. Let \((\rho, \mathcal{N}, F)\) be an even Breuer-Fredholm module over \( A \), representing the K-homology class \([\rho, \mathcal{N}, F]\) \( \in \mathcal{K}^0 (A) \), and \( p \in M_N (A) \) be a projection , representing the K-theory class \([p]\) \( \in \mathcal{K}_0 (A) \). We define the \textbf{even index pairing to be:}

\[ \langle [\rho, \mathcal{N}, F], [p] \rangle := \text{Ind}_\tau (p^-(F \otimes 1_N) p^+) \]

where \( p^-(F \otimes 1_N) p^+ \) is an operator from \( p^+ \mathcal{H}^N \) to \( p^- \mathcal{H}^N \).

2. Let \((\rho, \mathcal{N}, F)\) be an odd Breuer-Fredholm module over \( A \), representing the K-homology class \([\rho, \mathcal{N}, F]\) \( \in \mathcal{K}^1 (A) \), \([u]\) \( \in M_N (\mathcal{N}) \) be a unitary, representing the K-theory class \([u]\) \( \in \mathcal{K}_1 (A) \). We define the \textbf{odd index pairing to be:}

\[ \langle [\rho, \mathcal{N}, F], [u] \rangle := \text{Ind}_\tau (QuQ) \]

where \( Q = \frac{F \otimes 1_N + 1}{2} \) is a projection in \( M_N (\mathcal{N}) \), and \( QuQ \) is an operator from \( Q \mathcal{H}^N \) to \( Q \mathcal{H}^N \).

### 1.2 Entire cyclic (co)homology

Our goal is to construct characters from K-homology to another cohomology theory that intertwine the above K-theoretical pairing with the cohomological pairing. The target space of both the Connes-Chern character and the JLO character to be introduced next section is the entire cyclic (co)homology.

Entire cyclic homology is not as well-known as its cohomology counterpart. We adopt the bicomplex construction from [18] and use the entire growth control given in [26]. Under this definition, the homology theory is precisely (pre-)dual to the cohomology counterpart [20] in the sense that their pairing produces a finite value.

If \( B \) is a topological unital algebra over \( \mathbb{C} \), define

\[ C_n (B) := B \otimes (B / \mathbb{C}) \otimes n \]
where \( \otimes \) denotes the projective tensor product. Denote the element \( a_0 \otimes \cdots \otimes a_n \) of \( C_n(\mathcal{B}) \) by \( (a_0, \ldots, a_n)_n \), when the context is clear we will omit the subscript \( n \). The operators \( b : C_n(\mathcal{B}) \to C_{n-1}(\mathcal{B}) \) and \( B : C_n(\mathcal{B}) \to C_{n+1}(\mathcal{B}) \) are given in terms of simple tensors by the formulas

\[
b(a_0, \ldots, a_n)_n := \sum_{j=0}^{n-1} (-1)^j (a_0, \ldots, a_j a_{j+1}, \ldots, a_n)_{n-1} + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1})_{n-1},
\]

\[
B(a_0, \ldots, a_n)_n := \sum_{j=0}^{n} (-1)^{nj} (1, a_j, \ldots, a_n, a_0, \ldots, a_j)_{n+1}.
\]

Simple calculations show that \( b^2 = 0 \), \( B^2 = 0 \), and \( Bb + bB = 0 \). Therefore \( (b + B)^2 = 0 \) and we get the following bicomplex:

\[
\begin{array}{ccccccc}
\cdots & b & C_3(\mathcal{B}) & b & C_2(\mathcal{B}) & b & C_1(\mathcal{B}) & b & C_0(\mathcal{B}) \\
& B & & B & & B & & B \\
\cdots & b & C_2(\mathcal{B}) & b & C_1(\mathcal{B}) & b & C_0(\mathcal{B}) & \\
& B & & B & & B & & \\
\cdots & b & C_1(\mathcal{B}) & b & C_0(\mathcal{B}) & & & \\
& B & & B & & & & \\
\cdots & b & C_0(\mathcal{B}) & & & & & \\
\end{array}
\]

The space \( C_\bullet(\mathcal{B}) := \prod_{n=0}^{\infty} C_n(\mathcal{B}) \) has a natural \( \mathbb{Z}_2 \) grading given by \( C_+(\mathcal{B}) = \prod_{k=0}^{\infty} C_{2k}(\mathcal{B}) \) and \( C_-(\mathcal{B}) = \prod_{k=0}^{\infty} C_{2k+1}(\mathcal{B}) \). We get a chain complex \( (C_\bullet(\mathcal{B}), b + B) \) with the odd boundary map \( b + B \). However, the homology of this chain complex is trivial [26]. In order to make it nontrivial, we need to control the growth of a chain as \( n \) varies. The following definition is taken from [18, 26].

**Definition 1.5.** Define the space of **entire chains**

\[
C_\omega(\mathcal{B}) := \left\{ A_\bullet \in C_\bullet(\mathcal{B}) : \sup_n \left( \frac{\|A_n\|}{\pi \Gamma\left(\frac{n+1}{2}\right)} \right) < \infty \text{ for some } \lambda > 0 \right\}
\]
where \(\|\cdot\|\) is the projective tensor norm. \((C^\omega_\bullet(B), b + B)\) forms a subcomplex of \((C_\bullet(B), b + B)\). The homology defined by \((C^\omega_\bullet(B), b + B)\) is the entire cyclic homology of \(B\), denoted \(\text{HE}_\bullet(B) = \text{HE}^+(B) \oplus \text{HE}^-(B)\). \(\text{HE}_\bullet(B)\) is equipped with the obvious group structure inherited from the addition on \(C_n(B)\).

We set \(C^n(B) := \text{Hom}(C_n(B), \mathbb{C})\) and let \((b + B) : C^+(B) \to C^+(B)\) be the transpose of the odd boundary map \((b + B) : C^\bullet_n(B) \to C^\bullet_n(B)\) where \(C^+(B) := \prod_{n=0}^\infty C^n(B)\), then we get a similar diagram as above with the arrows reversed. The space \(C^\bullet(B)\) has a natural \(\mathbb{Z}_2\) grading given by \(C^+(B) = \prod_{k=0}^\infty C^{2k}(B)\) and \(C^-(B) = \prod_{k=0}^\infty C^{2k+1}(B)\).

\((C^\bullet(B), b + B)\) forms a cochain complex with the odd boundary map \(b + B\), which gives trivial cohomology [26].

**Definition 1.6.** Define the space of **entire cochains**

\[
C^\bullet_\bullet(B) := \left\{ \phi_\bullet \in C^\bullet_\bullet(B) : \sum_{n=0}^\infty \Gamma\left(\frac{n}{2}\right) \|\phi_n\| z^n \text{ is an entire function in } z \right\}
\]

where \(\|\phi_n\| := \sup \{|\phi_n(a_0, \ldots, a_n)| : \|a_j\| \leq 1 \forall j\}\). \((C^\bullet_\bullet(B), b + B)\) forms a subcomplex of \((C^\bullet_\bullet(B), b + B)\). The cohomology defined by \((C^\bullet_\bullet(B), b + B)\) is the entire cyclic cohomology of \(B\), denoted \(\text{HE}^\bullet_\bullet(B) = \text{HE}^+(B) \oplus \text{HE}^-(B)\). \(\text{HE}^\bullet_\bullet(B)\) is equipped with the obvious group structure inherited from the addition on \(\text{Hom}(C_n(B), \mathbb{C})\).

It is known that the de Rham homology (over \(\mathbb{C}\)) on a closed manifold \(M\) is a summand of the entire cyclic cohomology of the algebra \(C^\infty(M)\). They are expected to be equal, however it is not proved except for the case when \(M\) is one-dimensional [19].

Let \(\text{Tr} : C_n(M_N(A)) \to C_n(A)\) be the map defined by

\[
\text{Tr}(m_0, m_1, \ldots, m_n) := \sum_{0 \leq n_0, \ldots, n_N \leq N} (m_{0_{n_01}}, m_{1_{n_12}}, \ldots, m_{n_{n_N0}})
\]

where \((m_k)_{ij}\) denotes the entries of the matrix \(m_k\).

**Definition 1.7.**

1. Let \(p \in M_N(A)\) be a projection. Define the **even Chern character** \(\text{ch}_+(p) \in C_+(A)\) of \(p\) to be

\[
\text{ch}_+(p) := \sum_{k=0}^\infty \text{ch}_{2k}(p)
\]

where

\[
\text{ch}_0(p) := \text{Tr}(p), \\
\text{ch}_{2k}(p) := (-1)^k \frac{(2k)!}{2^k k!} \text{Tr}(2p - 1, p, \ldots, p)_{2k}.
\]

2. Let \(u \in M_N(A)\) be a unitary. Define the **odd Chern character** \(\text{ch}_-(u) \in C_-(A)\) of \(u\) to be

\[
\text{ch}_-(u) := \sum_{k=0}^\infty \text{ch}_{2k+1}(u)
\]

where

\[
\text{ch}_{2k+1}(u) := \frac{1}{\Gamma\left(\frac{k+1}{2}\right)} (-1)^{k+1} k! \cdot \text{Tr}(u^{-1}, u, \ldots, u^{-1}, u)_{2k+1}.
\]

For convenience, we often write \(\text{Tr}(m_0, m_1, \ldots, m_n)\) simply as \((m_0, \ldots, m_n)\).
**Definition 1.9.** Let $A$ be a Banach $\star$-algebra. Breuer-Fredholm module $\mathfrak{F}$ is given by

\[
\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}_n,
\]

where $\mathfrak{F}_n$ is a Banach $\star$-algebra and $\mathfrak{F}_{n+1} \subseteq \mathfrak{F}_n$. The Connes-Chern character is a cohomological Chern character due to Connes that assigns to a Breuer-Fredholm module $\mathfrak{F}$ a cocycle in entire cyclic cohomology. However, not every Breuer-Fredholm module lies inside the domain, we need the following.

Furthermore, the homology classes $[\text{ch}_n(p)]$ and $[\text{ch}_n(u)]$ depend only on the K-theory classes of $[p] \in K_0(A)$ and $[u] \in K_1(A)$ respectively.

As a result of Lemmas 1.2, the Chern character $\text{ch}_\bullet$ descends to a map from $K_\bullet(A)$ to $HE_\bullet(A)$. It is easy to see that $\text{ch}_\bullet$ respects group additions, hence it is a group homomorphism.

### 1.3 Connes-Chern character

The Connes-Chern character is a cohomological Chern character due to Connes that assigns to a Breuer-Fredholm module a cocycle in entire cyclic cohomology. However, not every Breuer-Fredholm module lies inside the domain of the Connes-Chern character. To characterize those that are within the domain, we need the following summability condition.

For $0 < p < \infty$, let $L^p_N$ be the set of $p$-summable operators in $N$. That is, an operator $T$ is in $L^p_N$ if $T \in N$ and its $p$-norm $\|T\|_p$ with respect to $\tau$ is finite. More details can be found in the Appendix.

The Connes character for the Type II setting first appeared in the work of Benamour and Fack in [2]. Let $A$ be a Banach $\star$-algebra.

**Definition 1.8.** A Breuer-Fredholm module $(\rho,N,F)$ over $A$ is called $p$-summable for $[F,\rho(a)]$ is $p$-summable for all $a \in A$ (see Definition A.3).

**Definition 1.9.**

1. Recall that $\chi$ is the grading operator that anti-commutes with $F$. Define the even Connes character $\text{ch}^n(F)$ of an even $p$-summable Breuer-Fredholm module $(\rho,N,F)$ to be the linear functional on $C_n(A)$ given by

\[
(\text{ch}^n(F), (a_0, \ldots, a_n)_n) := \frac{\Gamma(\frac{n}{2} + 1)}{2 \cdot n!} \tau(F[a_0][F,a_1]\cdots[F,a_n]),
\]

where $n$ is an even integer greater than $p$ and $(\cdot, \cdot)$ denotes the pairing between cochains and chains.

2. Define the odd Connes character $\text{ch}^n(F)$ of an odd $p$-summable Breuer-Fredholm module $(\rho,N,F)$ to be the linear functional on $C_n(A)$ given by

\[
(\text{ch}^n(F), (a_0, \ldots, a_n)_n) := \frac{\Gamma(\frac{n}{2} + 1)}{2 \cdot n!} \tau(F[a_0][F,a_1]\cdots[F,a_n]),
\]

where $n$ is an odd integer greater than $p$ and $(\cdot, \cdot)$ denotes the pairing between cochains and chains.

**Theorem 1.3.** For $n > p$, the even/odd Connes character $\text{ch}^n(F)$ of an even/odd $p$-summable Breuer-Fredholm module $(\rho,N,F)$ defines an entire cyclic cocycle, its cohomology class is independent of $n$ with the same parity.

**Proof.** It is clear that $\text{ch}^n(F)$ entire. Since $\mathcal{B}\text{ch}^n(F) = 0$, we only need to show that $\text{bch}^n(F) = 0$, and that $\text{ch}^n(F) - \text{ch}^{n+2}(F)$ is exact. A short computation shows that

\[
\text{ch}^n(F) = \mathcal{B}\psi^{n+1}(F) \quad \text{and} \quad -\text{ch}^{n+2}(F) = b\psi^{n+1}(F)
\]

where the entire cochain $\psi^{n+1}(F)$ is given by

\[
(\psi^{n+1}(F), (a_0, \ldots, a_{n+1})_{n+1}) := \frac{\Gamma(\frac{n}{2} + 2)}{(n + 2)!} \times \begin{cases}
\tau(\chi a_0 F[F,a_1][F,a_2]\cdots[F,a_{n+1}]) & n \text{ even} \\
\tau(a_0 F[F,a_1][F,a_2]\cdots[F,a_{n+1}]) & n \text{ odd}
\end{cases}
\]
Thus,
\[ \text{bch}^{n+2}(F) = bb(-v^{n+1}(F)) = 0 \]
and
\[ \text{ch}^n(F) - \text{ch}^{n+2}(F) = (b + B)v^{n+1}(F), \]
which completes the proof.

Suppose that \( F_t \) is a norm continuous family of Fredholm operators parametrized by \( t \) so that \((\rho, \mathcal{N}, F_t)\) defines 1-parameter family of \( p \)-summable Breuer-Fredholm modules.

**Theorem 1.4.** The entire cyclic cohomology class defined by the Connes character \( \text{ch}^n(F_t) \) is independent of \( t \). More explicitly
\[
\frac{d}{dt} \text{ch}^n(F_t) = (b + B)\iota(\dot{F}_t)\text{ch}^{n-1}(F_t),
\]
where the entire cochain \( \iota(\dot{F}_t)\text{ch}^{n-1}(F_t) \) is given by
\[
\left( \iota(\dot{F}_t)\text{ch}^{n-1}(F_t), (b_0, \ldots, b_{n-1})_{n-1} \right) := \frac{\Gamma(\frac{n}{2} + 1)}{n!} \sum_{k=0}^{n-1} (-1)^{k+1} \tau(\chi F_t[F_t, b_0] \cdots [F_t, b_k][\dot{F}_t] \cdots [F_t, b_{n-1}])
\]
with the convention that \( \chi = 1 \) in the odd case.

**Proof.** \( \dot{F}_t \) being bounded implies that \( \iota(\dot{F}_t)\text{ch}^{n-1}(F_t) \) is entire. It is clear that
\[ B_t(\dot{F}_t)\text{ch}^{n-1}(F_t) = 0, \]
so we only need to show that
\[ \frac{d}{dt} \text{ch}^n(F_t) = b(\dot{F}_t)\text{ch}^{n-1}(F_t). \]
Let \( T_k \in C^{n-1}(A) \) be defined by the equation
\[ (T_k, (b_0, \ldots, b_{n-1})_{n-1}) := (-1)^k \tau(\chi F_t[F_t, b_0] \cdots [F_t, b_k][\dot{F}_t] \cdots [F_t, b_{n-1}]), \]
then
\[
(bT_k, (a_0, \ldots, a_n)) = (-1)^k \left( \sum_{j=0}^{k-1} (-1)^j \tau(\chi F_t \cdots [F_t, a_ja_{j+1}] \cdots [F_t, a_k][\dot{F}_t] \cdots [F_t, a_n]) \right.
\]
\[ + \sum_{j=0}^{n-1} (-1)^j \tau(\chi F_t \cdots [F_t, a_k][\dot{F}_t] \cdots [F_t, a_ja_{j+1}] \cdots [F_t, a_n]) \]
\[ + (\tau(\chi F_t[F_t, a_0a_0] \cdots [F_t, a_k][\dot{F}_t] \cdots [F_t, a_{n-1}]) \bigg). \]
By expanding the term \([F_t, a_ja_{j+1}] = [F_t, a_j] + [F_t, a_ja_{j+1}] \), we see that the above becomes a telescope sum and most of the terms cancel. Using the identity
\[ 0 = \frac{d}{dt} (F_t \cdot F_t) = \dot{F}_t \cdot F_t + F_t \cdot \dot{F}_t \]
we simplify further and obtain
\[
(bT_k, (a_0, \ldots, a_n)) = (-1)^k \left( \tau(\chi F_t a_0[F_t, a_1] \cdots [F_t, a_k][\dot{F}_t] \cdots [F_t, a_n]) 
\right.
\]
By construction,
\[ \iota(\hat{F}_t)\text{ch}^{n-1}(F_t) = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{n!} \sum_{k=1}^{n} T_k, \]
and hence
\[ \left(b_t(\hat{F}_t)\text{ch}^{n-1}(F_t), (a_0, \ldots, a_n)\right) = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{n!} \left( \sum_{k=1}^{n} \tau(\chi F_t[F_t, a_0] \cdots [\hat{F}_t, a_k] \cdots [F_t, a_n]) 
- \tau(\chi F_t a_0 \hat{F}_t [F_t, a_1] \cdots [F_t, a_n]) 
+ (-1)^n \tau(\chi a_0 F_t[F_t, a_1] \cdots [F_t, a_n] \hat{F}_t) \right) \]
\[ = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{n!} \left( \sum_{k=0}^{n} \tau(\chi F_t[F_t, a_0] \cdots [\hat{F}_t, a_k] \cdots [F_t, a_n]) 
+ \tau(\chi \hat{F}_t F_t a_0[F_t, a_1] \cdots [F_t, a_n]) 
- \tau(\chi a_0 [F_t, a_1] \cdots [F_t, a_n] \hat{F}_t F_t) \right) \]
\[ = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{n!} \left( \sum_{k=0}^{n} \tau(\chi F_t[F_t, a_0] \cdots [\hat{F}_t, a_k] \cdots [F_t, a_n]) 
+ \tau(\chi \hat{F}_t F_t a_0[F_t, a_1] \cdots [F_t, a_n]) \right) \]
\[ = \left( \frac{d}{dt} \text{ch}^{n}(F_t), (a_0, \ldots, a_n) \right). \]

The proof is complete. \(\square\)

In fact, the way we obtain the transgression formula in Theorem 1.4 is by taking limits of the transgression formula in Theorem 3.4 below.

**Proposition 1.5** ([2]). Suppose that \(T\) is a \((e, f)\)-Fredholm operator with parametrix \(S\) such that
\[ e - e S f T e \in \mathcal{L}_{e N e}^{p/2} \quad \text{and} \quad f - f T e S f \in \mathcal{L}_{f N f}^{p/2}, \]
where \(\mathcal{L}_{e N e}^{p/2}\) denote the set of \(\frac{p}{2}\)-summable operators in \(e N e\), likewise for \(\mathcal{L}_{f N f}^{p/2}\). Then
\[ \text{Ind}_e(f T e) = \tau((e - e S f T e)^n) - \tau((f - f T e S f)^n) \]
for \(2m > p\).

The following theorem shows that the characters \(\text{ch}^n\) and \(\chi_{\bullet}\) intertwine the K-theoretical pairing with the (co)homological pairing of entire cyclic (co)homology.

**Theorem 1.6** ([2]).

1. Let \((\rho, \mathcal{N}, F)\) be an even \(p\)-summable Breuer-Fredholm module and \(p \in M_N(A)\) be a projection, then for \(n > p\) even
\[ \langle (\rho, \mathcal{N}, F), [p] \rangle = \langle [\text{ch}^n(F)], [\text{ch}_+(p)] \rangle. \]

2. Let \((\rho, \mathcal{N}, F)\) be an odd \(p\)-summable Breuer-Fredholm module and \(u \in M_N(A)\) be a unitary, then for \(n > p\) odd
\[ \langle (\rho, \mathcal{N}, F), [u] \rangle = \langle [\text{ch}^n(F)], [\text{ch}_-(u)] \rangle. \]
2 Unbounded Breuer-Fredholm modules and JLO character

This section repeats the language in Section 1 for unbounded Breuer-Fredholm modules. It starts with the definition of unbounded Breuer-Fredholm modules from [8] and its pairing with K-theory. The JLO character is defined and a proof of its homotopy invariance is shown according to [20]. The section concludes by showing that the JLO character computes the index.

Much of the work in this section is taken directly from [20] with minor modifications. Nonetheless, we give full details to illustrate the changes made in this Type II setting.

2.1 Unbounded Breuer-Fredholm modules

Definition 2.1. An odd unbounded Breuer-Fredholm module over a unital Banach *-algebra $A$ is a triple $(\rho, \mathcal{N}, \mathcal{D})$ for which $\mathcal{N}$ is a (separable) semi-finite von Neumann algebra in $B(\mathcal{H})$ with a faithful semi-finite normal trace $\tau$, $\rho : A \to \mathcal{N}$ a continuous *-representation, and $\mathcal{D}$ is an unbounded self-adjoint operator on $\mathcal{H}$ such that

1. $\mathcal{D}$ is affiliated with $\mathcal{N}$,
2. For all $a \in A$, the commutator $[\mathcal{D}, \rho(a)]$ extends to an operator in $\mathcal{N}$ and there is a constant $C$ such that $\| [\mathcal{D}, \rho(a)] \| \leq C \| a \|$.
3. $(1 + \mathcal{D}^2)^{-1/2} \in K_{\mathcal{N}}$.

If $(\rho, \mathcal{N}, \mathcal{D})$ is equipped with a $\mathbb{Z}_2$ grading $\chi \in \mathcal{N}$ such that all $\rho(a)$ are even and $\mathcal{D}$ is odd, then we call $(\rho, \mathcal{N}, \mathcal{D})$ an even unbounded Breuer-Fredholm module.

If $\mathcal{N} = B(\mathcal{H})$ and $\tau$ is the standard operator trace, we drop the prefix Breuer.

To avoid confusion, we will sometimes refer to the Breuer-Fredholm module from Definition 1.1 as bounded. Similar to its bounded counterpart, an unbounded Fredholm module is sometimes called an unbounded K-cycle.

The term (semi-finite) spectral triple seems to be popular among physicists. It is a convenient term for the package consisting of the algebra $A$ and an unbounded (Breuer-)Fredholm module. In this thesis, our algebra $A$ is always fixed and we view the JLO character and Connes character as maps from K-homology classes to some cohomology classes that respect group additions. Hence, the term unbounded Breuer-Fredholm module is more convenient and suitable in our settings.

An example of an unbounded Breuer-Fredholm module is given by the semi-finite spectral triple over a space $A$ package consisting of the algebra $A$ and its pairing with K-theory. The JLO character is defined and a proof of its homotopy invariance is shown according to [20]. The section concludes by showing that the JLO character computes the index.

In Sections 3.2, we will explain in details how we would associate a bounded Breuer-Fredholm module to an unbounded one.

Definition 2.2.

1. For a given even unbounded Breuer-Fredholm module $(\rho, \mathcal{N}, \mathcal{D})$ over $A$, define its pairing with the even K-theory $K_0(A)$ of $A$ given by the index:

$$\langle (\rho, \mathcal{N}, \mathcal{D}), [p] \rangle := \text{Ind}_\tau (p^- (\mathcal{D} \otimes 1_N) p^+)$$

for a projection $p \in M_N(A)$ representing the class $[p] \in K_0(A)$, where $p^- (\mathcal{D} \otimes 1_N) p^+ : p^+ \mathcal{H}^N \to p^- \mathcal{H}^N$.

2. For a given odd unbounded Breuer-Fredholm module $(\rho, \mathcal{N}, \mathcal{D})$ over $A$, define its pairing with the odd K-theory $K_1(A)$ of $A$ given by the spectral flow:

$$\langle (\rho, \mathcal{N}, \mathcal{D}), [u] \rangle := \text{sf} (\mathcal{D} \otimes 1_N, u(\mathcal{D} \otimes 1_N) u^{-1})$$

for a unitary $u \in M_N(A)$ representing the class $[u] \in K_1(A)$, where $\text{sf} (\mathcal{D} \otimes 1_N, u(\mathcal{D} \otimes 1_N) u^{-1})$ is the spectral flow from $(\mathcal{D} \otimes 1_N) (1 + (\mathcal{D} \otimes 1_N)^2)^{-\frac{1}{2}}$ to $(u(\mathcal{D} \otimes 1_N) u^{-1}) (1 + (u(\mathcal{D} \otimes 1_N) u^{-1})^2)^{-\frac{1}{2}}$ defined in [8].
2.2 JLO character

The JLO character is a cohomological Chern character due to Jaffe, Lesnieswksi, and Osterwalder that assigns cocycles in entire cyclic cohomology to unbounded Breuer-Fredholm modules satisfying an appropriate summability condition. We begin by defining the summability conditions of main concern.

Definition 2.3. An unbounded Breuer-Fredholm module $(\rho, \mathcal{N}, \mathcal{D})$ over $A$ is:

(a) $p$-summable if $\tau((1 + D^2)^{-p/2}) < \infty$;

(b) $\theta$-summable if $\tau(e^{-tD^2}) < \infty$ for all $t > 0$;

(c) weakly $\theta$-summable if $\tau(e^{-tD^2}) < \infty$ for some $0 < t < 1$.

Observe that $p$-summability implies $\theta$-summability, which in turn implies weak $\theta$-summability.

Example 2.1. Let $\Gamma \hookrightarrow \hat{M} \hookrightarrow M$ be a Galois cover of a compact $p$-dimensional manifold $M$. Let $\mathcal{D}$ be the $\Gamma$ cover of a generalized Dirac operator on $M$. Consider the von Neumann algebra $\mathcal{N}$ of bounded $\Gamma$-invariant operators defined by Atiyah, with its natural trace $\text{Tr}_{\mathcal{N}}$. If the Hilbert space $\mathcal{H}$ represents on, then $(\rho, \mathcal{N}, \mathcal{D})$ is a $p$-summable unbounded Breuer-Fredholm module over $C^\infty(M)$ with $\rho$ given by point-wise multiplication [2].

Example 2.2. The unbounded Breuer-Fredholm module given by Aastrup-Grimstrup-Nest’s noncommutative space of connections is weakly $\theta$-summable if the sequence $\{a_j\}$ in its definition diverges sufficiently fast [1].

The following Lemma was proved in [16] in the Type I case.

Lemma 2.1. If $(\rho, \mathcal{N}, \mathcal{D})$ is $p$-summable for any finite $p$, then it is also $\theta$-summable, and $\tau(e^{-tD^2}) = O(t^{-p/2})$ as $t \searrow 0$.

Proof. We can write $e^{-tD^2} = (1 + D^2)^{p/2}e^{-tD^2}(1 + D^2)^{-p/2}$ with $\tau((1 + D^2)^{-p/2}) < \infty$ by hypothesis, and $(1 + D^2)^{p/2}e^{-tD^2}$ bounded by $\left\|(1 + x^2)^{p/2}e^{-tx^2}\right\|_{\infty} = \left(\frac{p}{2e}\right)^{p/2}t^{-p/2}e^t$ by functional calculus. Hence as a consequence of Proposition A.1 and Proposition A.2, we have

$$\tau(e^{-tD^2}) \leq \left(\frac{p}{2e}\right)^{p/2}t^{-p/2}e^t(1 + D^2)^{-p/2},$$

which proves the lemma. \hfill \Box

To make the JLO character and other useful formulas easier to write down, we will define the JLO character in two steps. We start with the following definition.

Let $\Delta_n := \{(t_1, \ldots, t_n) \in \mathbb{R}^n; 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$ be the standard $n$-simplex and $d^n t = dt_1 \cdots dt_n$ is the standard Lesbeque measure on $\Delta_n$ with volume $\frac{1}{n!}$.

Definition 2.4. Let $(\rho, \mathcal{N}, \mathcal{D})$ be a weakly $\theta$-summable unbounded Breuer-Fredholm module over $A$. Given $F_0, \ldots, F_n$ operators affiliated with $\mathcal{N}$, define

$$\langle F_0, F_1, \ldots, F_n \rangle_{\mathcal{D}} := \int_{\Delta_n} \tau(\chi_0 F_0 e^{-t_1 D^2} F_1 e^{-t_2 - t_1 D^2} \cdots F_n e^{-(1 - t_n) D^2}) d^n t,$$

where $\chi = 1$ when $\mathcal{D}$ is even.

Let $T$ be an operator affiliated with $\mathcal{N}$, denote by $[T]_\chi$ the degree of $T$ with respect to $\chi$. Any operators that we will consider will be either even or odd. From here and on, the commutator $[\ , \ ]$ is always graded with respect to $\chi$.

Lemma 2.2. Let $F_0, \ldots, F_n$ be operators affiliated with $\mathcal{N}$ that are either even or odd, then

1. $$\langle F_0, \ldots, F_n \rangle_{\mathcal{D}} = (-1)^{(\sum_{j=0}^n \deg|F_j|_\chi + \sum_{j=1}^n \deg|F_j|_\chi)} (\langle F_0, F_1, \ldots, F_n \rangle_{\mathcal{D}}).$$
Proof.

1. The statement follows from \( \tau(\chi[X,Y]) = 0 \) for \( X,Y \) operators affiliated with \( \mathcal{N} \).

2. The left hand side can be regarded as \( \int_0^1 \langle F_0, \ldots, F_n \rangle_D^n \) by introducing a trivial extra integration; the polyhedron \( \Delta_n \times [0,1] \) can be subdivided by the inequalities \( t_j \leq u \leq t_{j+1} \) into \( n+1 \) simplices, each of which is a copy of \( \Delta_{n+1} \); integration over these simplices yield the terms on the right hand side.

3. By observing the Leibniz property of \( [\mathcal{D}, \cdot] \) and

\[
0 = \tau \left( \chi[\mathcal{D}, F_0 e^{-D^2} F_1 e^{-(t_2-t_1)D^2} \ldots F_n e^{-(1-t_n)D^2}] \right),
\]
equality follows.

4. We first prove that

\[
0 = [e^{-D^2}, X] + \int_0^1 e^{-sD^2} [\mathcal{D}^2, X] e^{-(1-s)D^2} ds.
\]

It comes from

\[
[e^{-D^2}, X] = e^{-sD^2} X e^{-(1-s)D^2} \bigg|_{s=0}^{s=1} = \int_0^1 \frac{d}{ds} (e^{-sD^2} X e^{-(1-s)D^2}) ds
\]

\[
= \int_0^1 e^{-sD^2} (-D^2) X e^{-(1-s)D^2} + e^{-sD^2} X D^2 e^{-(1-s)D^2} ds
\]

\[
= - \int_0^1 e^{-sD^2} [\mathcal{D}^2, X] e^{-(1-s)D^2} ds.
\]

Replacing \( D^2 \) by \( (t_{j+1} - t_j)D^2 \) and using the substitution \( u = (t_{j+1} - t_j)s + t_j \), we obtain

\[
0 = [e^{-(t_{j+1}-t_j)D^2}, X] + \int_{t_j}^{t_{j+1}} e^{-(t_{j+1}-u)D^2} [\mathcal{D}^2, X] e^{-(u-t_j)D^2} du.
\]

Inserting this into the definition of \( \langle F_0, \ldots, [\mathcal{D}^2, F_j], \ldots, F_n \rangle^n_D \) gives the formula.

\[\square\]

Definition 2.5.

1. The odd JLO character \( \text{Ch}_{\text{JLO}}(\mathcal{D}) \in C^{-}(A) \) of a weakly \( \theta \)-summable odd unbounded Breuer-Fredholm module \((\rho, \mathcal{N}, \mathcal{D})\) is defined to be

\[
\text{Ch}_{\text{JLO}}(\mathcal{D}) := \sum_{k=0}^{\infty} \text{Ch}_{\text{JLO}}^{2k+1}(\mathcal{D}),
\]

12
2. The even JLO character \( \text{Ch}_{2}\times_{JLO}(D) \) of a weakly \( \theta \)-summable even unbounded Breuer-Fredholm module \((\rho, N, D)\) is defined to be
\[
\text{Ch}_{2}\times_{JLO}(D) := \sum_{k=0}^{\infty} \text{Ch}_{2}^{2k}(D),
\]
where
\[
(\text{Ch}_{2}^{2k}(D), (a_0, \ldots, a_n)) := (a_0, [D, a_1], \ldots, [D, a_n])_{D}^{n}.
\]

**Theorem 2.3.** The JLO character \( \text{Ch}_{\times_{JLO}}(D) \) is an entire cyclic cocycle in \( HE^\bullet(A) \).

More specifically,
\[
\text{Ch}_{\times_{JLO}}(D) \in C^\bullet_{\omega}(A) \quad \text{and} \quad (b + B)\text{Ch}_{\times_{JLO}}(D) = 0.
\]

The following norm estimate will show that \( \text{Ch}_{\times_{JLO}}(D) \) is entire.

Whenever we have an operator affiliated with \( \mathcal{N} \), we demand that it is either even or odd with respect to \( \chi \).

**Lemma 2.4.** Let \((\rho, N, D)\) be a weakly \( \theta \)-summable unbounded Breuer-Fredholm module over \( A \). If \( F_j \) and \( R_j \) are operators in \( \mathcal{N} \) for \( j = 0, \ldots, n \), and at most \( k \) of the operators \( F_j \) are non-zero, then for \( \varepsilon \in [0, 1) \),
\[
\left\| \langle F_0|D|^{1+\varepsilon} + R_0, \ldots, F_n|D|^{1+\varepsilon} + R_n \rangle_D \right\| \leq \left( \frac{2}{(1-\varepsilon)\delta} \right)^{k} \left( \frac{e^{-((1-\delta)D^2)}}{(n-k)!} \prod_{j=0}^{n} (\|F_j\| + \|R_j\|) \right),
\]
where \( 0 < \delta < \frac{1}{2\varepsilon} \).

For the purpose of future applications, Lemma 2.4 is slightly strengthened from the one in [20]. The proof in [20] carries through to our setting with minor modifications.

**Proof.** From the generalized Hölder’s inequality, Theorem A.4(1), the following estimate holds:
\[
|\tau(\chi T_0 \ldots T_n)| \leq \tau(|\chi T_0 \ldots T_n|) = \|\chi T_0 \ldots T_n\|_1 \leq \|T_0\|_{s_0}^{-1} \cdots \|T_n\|_{s_n}^{-1},
\]
if \( s_0 + \cdots + s_n = 1 \). Therefore,
\[
\left\| \langle F_0|D|^{1+\varepsilon} + R_0, \ldots, F_n|D|^{1+\varepsilon} + R_n \rangle \right\| \leq \int_{\Delta_n} \left\| (F_0|D|^{1+\varepsilon} + R_0)e^{-s_0D^2} \right\|_{s_0}^{-1} \cdots \left\| (F_n|D|^{1+\varepsilon} + R_n)e^{-s_nD^2} \right\|_{s_n}^{-1} \cdot d^n s.
\]

For each \( \left\| (F|D|^{1+\varepsilon} + R)e^{-sD^2} \right\|_{s-1} \), observe that by using Proposition A.3 and functional calculus
\[
\left\| F|D|^{1+\varepsilon}e^{-sD^2} \right\|_{s-1} \leq \|F\| \cdot \left\| |D|^{1+\varepsilon}e^{-sD^2} \right\| \cdot \left\| e^{-s(1-\delta)D^2} \right\|_{s-1} \leq \|F\| \cdot \sup_{x \in \mathbb{R}} \left( |x|^{1+\varepsilon}e^{-sD^2} \right) \cdot \left\| e^{-s(1-\delta)D^2} \right\|_{s-1},
\]
and that
\[
\left\| Re^{-sD^2} \right\|_{s-1} \leq \|R\| \cdot \left\| e^{-sD^2} \right\| \cdot \left\| e^{-s(1-\delta)D^2} \right\|_{s-1} \leq \|R\| \cdot \sup_{x \in \mathbb{R}} \left( e^{-sD^2} \right) \cdot \left\| e^{-s(1-\delta)D^2} \right\|_{s-1}.
\]

Since the function \( |x|^{1+\varepsilon}e^{-sD^2} \) is bounded by \( \left( \frac{1+\varepsilon}{2\varepsilon s} \right)^{s+1} \) and \( e^{-sD^2} \) is bounded by 1, we can put together the above terms using Theorem A.4(ii) and get that
\[
\left\| (F|D|^{1+\varepsilon} + R)e^{-sD^2} \right\|_{s-1} \leq \left( \frac{1+\varepsilon}{2\varepsilon s} \right)^{s+1} \|F\| + \|R\| \left( \tau(e^{-(1-\delta)D^2}) \right)^{s}.
\]
Keeping in mind that at most $k$ of the $F_j$'s are non-zero, we get

\[
| \langle F_0|D|^{1+\epsilon} + R_0, \ldots, F_n|D|^{1+\epsilon} + R_n \rangle^\ast_D^n | \\
\leq \tau(e^{-(1-\delta)D^2}) \prod_{j=0}^n (\|F_j\| + \|R_j\|) \left( \frac{1+\epsilon}{2d_1} \right)^k k \int_{\Delta_n} (s_0 \ldots s_{k-1})^{-\frac{1+\epsilon}{\delta}} d^n s
\]

Along with the estimates

\[
\left( \frac{1+\epsilon}{2d_1} \right)^k k \leq \left( \frac{1}{\delta e} \right)^k
\]

and

\[
\int_{\Delta_n} (s_0 \ldots s_{k-1})^{-\frac{1+\epsilon}{\delta}} d^n s \leq \left( \frac{2}{1-\epsilon} \right)^{k} \frac{1}{(n-k)!}.
\]

the proof is complete. \(\square\)

The above norm estimate immediately implies that \(\|Ch^*_{JLO}(D)\| < \frac{1}{n!}\tau(e^{-(1-\delta)D^2})C^n\). Therefore, \(Ch^*_{JLO}(D)\) is an entire cochain when \(\tau(e^{-(1-\delta)D^2}) < \infty\), which is exactly the weakly $\theta$-summable condition.

\textbf{Proof of Theorem 2.3.} Lemma 2.4 guarantees that \(Ch^*_{JLO}(D)\) is entire. What remains to check is that \(Ch^*_{JLO}(D)\) is $(b+B)$ closed. We adopted the computation in [23] to the Type II case.

We compute \((Ch^*_{JLO}(D), b(a_0, \ldots, a_{n+1})_{n+1}\)

\[
(Ch^*_{JLO}(D), b(a_0, \ldots, a_{n+1})_{n+1}) = \langle a_0 a_1, [D, a_2], \ldots, [D, a_{n+1}] \rangle^\ast_D^n + \sum_{j=1}^n (-1)^j \langle a_0, \ldots, [D, a_j a_{j+1}], \ldots \rangle^\ast_D^n \\
+ (-1)^{n+1} \langle a_{n+1} a_0, [D, a_1], \ldots, [D, a_n] \rangle^\ast_D^n
\]

\[
= \langle a_0 a_1, [D, a_2], \ldots \rangle^\ast_D^n - \langle a_0, a_1 [D, a_2], \ldots \rangle^\ast_D^n
\]

\[
+ \sum_{j=2}^n (-1)^{j-1} \langle a_0, \ldots, [D, a_{j-1} a_j], \ldots \rangle^\ast_D^n
\]

\[
+ \langle a_0, \ldots, a_2 [D, a_{j+1}], \ldots \rangle^\ast_D^n
\]

\[
+ (-1)^n \langle a_0, [D, a_1], \ldots, [D, a_n a_{n+1}] \rangle^\ast_D^n - \langle a_{n+1} a_0, [D, a_1], \ldots, [D, a_n] \rangle^\ast_D^n
\]

\[
= \frac{2}{2D(4)} \sum_{j=1}^{n+1} (-1)^{j-1} \langle a_0, \ldots, [D^2, a_j], \ldots \rangle^\ast_D^{n+1}.
\]

The last term forms a telescope sum and reduces to

\[
\langle a_0 D, [D, a_1], \ldots \rangle^\ast_D^{n+1} + (-1)^n \langle a_0, [D, a_1], \ldots, [D, a_{n+1}] \rangle^\ast_D^{n+1} = -\langle [D, a_0], \ldots, [D, a_{n+1}] \rangle^\ast_D^{n+1}.
\]

Now apply Lemma 2.2(1)(2), one checks that

\[
([D, a_0], \ldots, [D, a_{n+1}])^\ast_D^{n+1} = (Ch^*_{JLO}^{n+2}(D), B(a_0, \ldots, a_{n+1})_{n+1})
\]

Therefore, \(bCh^*_{JLO}(D) = -BCh^*_{JLO}^{n+2}(D)\) and \((b+B)Ch^*_{JLO}(D) = 0\). The proof is complete. \(\square\)

As a result, the JLO character defines an entire cyclic cohomology class called the JLO class.
2.3 Homotopy invariance of the JLO class

In this section, we will show that the cohomology class given by the JLO character is homotopy invariant. As a consequence, the JLO character descends to a well-defined map from (semi-finite) K-homology to entire cyclic cohomology. We follow closely to work by Getzler and Szenes [20].

**Definition 2.6.** Let \( V \) be an operator affiliated with \( \mathcal{N} \). Define the **contraction** \( \iota(V) \) by \( V \) to be

\[
\iota(V) (F_0, \ldots, F_n)_{\mathcal{D}} := \sum_{k=0}^{n} (F_0, \ldots, F_k, V, F_{k+1}, \ldots, F_n)_{\mathcal{D}}^{n+1}.
\]

**Definition 2.7.** Let \( V \) be an operator affiliated with \( \mathcal{N} \) such that it has the same degree as \( \mathcal{D} \), i.e. \( |\mathcal{D}|_x = |V|_x \). Define \( \text{Ch}^*_{\text{JLO}}(\mathcal{D}, V) \) to be given by the equation

\[
(\text{Ch}^n_{\text{JLO}}(\mathcal{D}, V), (a_0, \ldots, a_n)_n) := \sum_{j=1}^{n+1} (-1)^j \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}, a_j], V, \ldots, [\mathcal{D}, a_n] \rangle_{\mathcal{D}}^{n+1}.
\]

**Theorem 2.5.** Let \((\rho, \mathcal{N}, \mathcal{D})\) be a weakly \( \theta \)-summable unbounded Breuer-Fredholm module.

1. \( \text{Ch}^*_{\text{JLO}}(\mathcal{D}, V) \) is an entire cochain if \( V = F|\mathcal{D}|^{1+\varepsilon} + R \) where \( 0 \leq \varepsilon < 1 \), \( F \) and \( R \) are operators in \( \mathcal{N} \).
2. Let \( V \) be an operator affiliated with \( \mathcal{N} \) such that it has the same degree as \( \mathcal{D} \), i.e. \( |\mathcal{D}|_x = |V|_x \). Then

\[
b\text{Ch}^{n-1}_{\text{JLO}}(\mathcal{D}, V) + B\text{Ch}^{n+1}_{\text{JLO}}(\mathcal{D}, V) = -\iota(DV + VD)\text{Ch}^n_{\text{JLO}}(\mathcal{D}) + \alpha^n(\mathcal{D}, V),
\]

where \( \alpha^n(\mathcal{D}, V) \) is defined to be

\[
(\alpha^n(\mathcal{D}, V), (a_0, \ldots, a_n)_n) := \sum_{j=1}^{n} \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}, a_j], V, \ldots, [\mathcal{D}, a_n] \rangle_{\mathcal{D}}^{n}.
\]

**Proof.**

1. From Lemma 2.4 we have that

\[
\|\text{Ch}^n_{\text{JLO}}(\mathcal{D}, V)\| \leq \left( \frac{2}{(1 - \varepsilon)\delta e} \right) \frac{(n + 1)}{n!} \tau(e^{-(1-\varepsilon)\mathcal{D}}) C^n
\]

Therefore,

\[
\sum_{n=0}^{\infty} \Gamma\left( \frac{n}{2} \right) \|\text{Ch}^n_{\text{JLO}}(\mathcal{D}, V)\| z^n
\]

defines an entire function in \( z \) and \( \text{Ch}^*_{\text{JLO}}(\mathcal{D}, V) \) is entire.

2. Recall that

\[
(\text{Ch}^{n-1}_{\text{JLO}}(\mathcal{D}, V), (b_0, \ldots, b_{n-1})_{n-1})
\]

\[
= \sum_{j=1}^{n} (-1)^{j-1} \langle b_0, \ldots, [\mathcal{D}, b_{j-1}], V, [\mathcal{D}, b_j], \ldots, [\mathcal{D}, b_{n-1}] \rangle_{\mathcal{D}}^{n}.
\]

Denote by \( E_j \) the cochain

\[
(E_j, (b_0, \ldots, b_{n-1})_{n-1}) := \langle b_0, \ldots, [\mathcal{D}, b_{j-1}], V, [\mathcal{D}, b_j], \ldots, [\mathcal{D}, b_{n-1}] \rangle_{\mathcal{D}}^{n},
\]

so that

\[
\text{Ch}^{n-1}_{\text{JLO}}(\mathcal{D}, V) = \sum_{j=1}^{n} (-1)^j E_j.
\]
First we compute \( E_j \) paired with \( b(a_0, \ldots, a_n)_n \):

\[
(E_j, b(a_0, \ldots, a_n)_n) = \langle a_0 a_1, \ldots, [D, a_j], V, [D, a_{j+1}], \ldots, [D, a_n] \rangle^n_D \\
+ \sum_{k=1}^{j-1} (-1)^k \langle a_0, \ldots, [D, a_k a_{k+1}], \ldots, [D, a_j], V, \ldots, [D, a_n] \rangle^n_D \\
+ \sum_{k=j}^{n-1} (-1)^k \langle a_0, \ldots, V, [D, a_j], \ldots, [D, a_k a_{k+1}], \ldots, [D, a_n] \rangle^n_D \\
+ (-1)^n \langle a_n a_0, \ldots, V, [D, a_j], \ldots, [D, a_{n-1}] \rangle^n_D.
\]

By expanding the \([D, a_k a_{k+1}]\) terms using the Leibniz rule and re-ordering the sum, we get

\[
(E_j, b(a_0, \ldots, a_n)_n) = \langle a_0 a_1, \ldots, [D, a_j], V, \ldots \rangle^n_D - \langle a_0, a_1 [D, a_2], \ldots, [D, a_j], V, \ldots \rangle^n_D \\
+ \sum_{k=2}^{j} (-1)^{k-1} \langle \ldots, [D, a_{k-1}] a_k, \ldots, V, \ldots \rangle^n_D - \langle \ldots, a_k [D, a_{k+1}], \ldots, V, \ldots \rangle^n_D \\
+ (-1)^{j-1} \langle \ldots, a_j V, \ldots \rangle^n_D - (-1)^{j-1} \langle \ldots, V a_j, \ldots \rangle^n_D \\
+ \sum_{k=j}^{n-1} (-1)^{k-1} \langle \ldots, V, \ldots, [D, a_{k-1}] a_k, \ldots \rangle^n_D - \langle \ldots, V, \ldots, a_k [D, a_{k+1}], \ldots \rangle^n_D \\
+ (-1)^{n-1} \langle \ldots, V, \ldots, [D, a_{n-1}] a_n \rangle^n_D - \langle a_n a_0, \ldots, V, \ldots, [D, a_{n-1}] \rangle^n_D.
\]

We are now in the setting to apply Lemma 2.2(4) to obtain

\[
(E_j, b(a_0, \ldots, a_n)_n) = (-1)^j \langle \ldots, [V, a_j], \ldots \rangle^n_D \\
+ \sum_{k=1}^{j} (-1)^{k-1} \langle a_0, \ldots, [D^2, a_k], V, \ldots \rangle^n_{D^2} \\
+ \sum_{k=j}^{n} (-1)^{k-1} \langle a_0, \ldots, V, \ldots, [D^2, a_k] \rangle^n_{D^2}.
\]

From the facts that \([D^2, a_k] = D[D, a_k] + [D, a_k]D\) and \(D\) commutes with \(e^{-s_k D^2}\), one observes the above forms a telescope sum and reduces to the following:

\[
(E_j, b(a_0, \ldots, a_n)_n) = (-1)^j \langle \ldots, [V, a_j], \ldots \rangle^n_D + \langle a_0 D, \ldots, [D, a_j], V, \ldots \rangle^{n+1}_D \\
+ (-1)^{j-1} \langle \ldots, [D, a_j], DV, \ldots \rangle^{n+1}_{D^2} + (-1)^{j-1} \langle \ldots, V D, [D, a_j], \ldots \rangle^{n+1}_{D^2} \\
+ (-1)^{n-1} \langle \ldots, V[D, a_j], \ldots, [D, a_n] \rangle^{n+1}_D \\
= (-1)^j \langle \ldots, [V, a_j], \ldots \rangle^n_D + \langle a_0 D, \ldots, [D, a_j], V, \ldots \rangle^{n+1}_D \\
+ \langle D a_0, \ldots, V[D, a_j], \ldots, [D, a_n] \rangle^{n+1}_D \\
+ (-1)^{j-1} \langle \ldots, [D, a_j], DV, \ldots \rangle^{n+1}_{D^2} + (-1)^{j-1} \langle \ldots, V D, [D, a_j], \ldots \rangle^{n+1}_{D^2}.
\]
Now we sum over $j$ with the appropriate sign:

\[
\left( \sum_{j=1}^{n} (-1)^j E_j, b(a_0, \ldots, a_n)_n \right) = \sum_{j=1}^{n} \langle \ldots, [V, a_j], \ldots \rangle^n_D
\]

\[
- \sum_{j=0}^{n} \langle \ldots, [\mathcal{D}, a_j], D'V + V\mathcal{D}, \ldots \rangle^n_{D'}
\]

\[
- \sum_{j=0}^{n} (-1)^j \langle [\mathcal{D}, a_0], \ldots, [\mathcal{D}, a_j], V, \ldots, [\mathcal{D}, a_n] \rangle^n_{D'}.
\]

Equations (3) and (4) give $\alpha^n(\mathcal{D}, V)$ and $-\iota(\mathcal{D}V + V\mathcal{D}) \text{Ch}_{\text{JLO}}^n(\mathcal{D})$ respectively. By using Lemma 2.2(1)(2), each summand in Equation (5) can be written as

\[
(-1)^j \langle [\mathcal{D}, a_0], \ldots, [\mathcal{D}, a_j], V, \ldots \rangle^{n+1}_{D'}
\]

\[
2.2^{(2)} \equiv (-1)^j \sum_{k=0}^{j+1} \langle \ldots, [\mathcal{D}, a_{k-1}], 1, \ldots, V, \ldots \rangle^{n+2}_{D'}
\]

\[
+ (-1)^j \sum_{k=j+1}^{n} \langle \ldots, V, \ldots, 1, [\mathcal{D}, a_k], \ldots \rangle^{n+2}_{D'}
\]

\[
2.2^{(1)} \equiv \sum_{k=0}^{j+1} ((-1)^{j+2-k} E_{j+2-k}, (-1)^{nk}(1, a_k, \ldots, a_{k-1})_{n+1})
\]

\[
+ \sum_{k=j+1}^{n} ((-1)^{-k+j+3} E_{n-k+j+3}, (-1)^{nk}(1, a_k, \ldots, a_{k-1})_{n+1}) .
\]

Now we sum over $j = 0$ to $n$, and do a change of indices of $i = j - k + 1$ for Equation (9) and $i = n - k + j + 1$ for Equation (10), then Equation (5) becomes

\[
\sum_{i=0}^{n} \left( (-1)^{i+1} E_{i+1}, \sum_{k=0}^{n-i} (-1)^{nk}(1, a_k, \ldots, a_{k-1})_{n+1} \right)
\]

\[
+ \sum_{i=0}^{n} \left( (-1)^{i+2} E_{i+2}, \sum_{k=n-i}^{n} (-1)^{nk}(1, a_k, \ldots, a_{k-1})_{n+1} \right),
\]

which equals $(\text{Ch}_{\text{JLO}}^{n+1}(\mathcal{D}, V), B(a_0, \ldots, a_n)_n)$. Hence we have obtained

\[
b\text{ch}_{\text{JLO}}^{n-1}(\mathcal{D}, V) = -\iota(\mathcal{D}V + V\mathcal{D}) \text{Ch}_{\text{JLO}}^n(\mathcal{D}) + \alpha^n(\mathcal{D}, V) - B\text{Ch}_{\text{JLO}}^{n+1}(\mathcal{D}, V),
\]

which is the desired result.

\[\square\]

Suppose that $\mathcal{D}_t$ is a $t$-parameter family of operators so that it defines a differentiable family of weakly summable unbounded Breuer-Fredholm modules $(\rho, \mathcal{N}, \mathcal{D}_t)$. Namely, $\mathcal{D}_t$ is a $t$-parameter family of self-adjoint operators on $\mathcal{H}$ with common domain of definition so that the following is satisfied:

- $\mathcal{D}_t$ is affiliated with $\mathcal{N}$ for all $t$,
- For all $a \in A$, $[\mathcal{D}_t, \rho(a)]$ is a norm-differentiable family of operators in $\mathcal{N}$, and there is a constant $C$ for each compact interval such that $\| [\mathcal{D}_t, \rho(a)] \| \leq C \| a \|$. 

17
• \((1+D_t^2)^{-1/2}\) is a norm-differentiable family of operators in \(K_N\).

• There exists a \(u \in (0,1)\) such that \(\tau(e^{-uD_t^2})\) is bounded for each compact interval.

If \((\rho, N, D_t)\) is equipped with a \(\mathbb{Z}_2\) grading \(\chi \in N\) so that \(\rho(a)\) is even for all \(a \in A\) and \(D_t\) is odd for all \(t\), then similarly we call the family of Breuer-Fredholm modules \((\rho, N, D_t)\) even.

The differentiable families of unbounded operators in our discussion will often be “functions” of \(D\), hence we do not alter the spectral projections. For more general notions of differentiable family of unbounded operators, readers may refer to [25].

**Lemma 2.6** ([8]). Let \((\rho, N, D_t)\) be a differentiable family of weakly \(\theta\)-summable unbounded Breuer-Fredholm modules, and \(F_0, \ldots, F_n\) be operators affiliated with \(N\), then

\[
\frac{d}{dt} (F_0, \ldots, F_n)^n_{D_t} = -\sum_j^n \left( F_0, \ldots, F_j, D_t\dot{D}_t + \dot{D}_t D_t, F_{j+1}, \ldots, F_n \right)^{n+1}_{D_t}.
\]

**Theorem 2.7.** If \(\dot{D}_t = F_1|D_t|^{1+\varepsilon} + R_t\) for \(0 \leq \varepsilon < 1\) and \(F_t, R_t \in N\) are continuous families of operators that are uniformly bounded in \(t\) then \(\text{Ch}^n_{\text{JLO}}(D_t, \dot{D}_t)\) is an entire cochain and for every \(n\)

\[
\frac{d\text{Ch}^n_{\text{JLO}}(D_t)}{dt} = b\text{Ch}^{n-1}_{\text{JLO}}(D_t, \dot{D}_t) + B\text{Ch}^{n+1}_{\text{JLO}}(D_t, \dot{D}_t).
\]

**Proof.** By applying the Leibniz rule on \(\frac{d}{dt}\text{Ch}^n_{\text{JLO}}(D_t)\), we will get terms containing \(\frac{d}{dt}e^{-(t_j+1-t_i)D_t^2}\) and terms containing \(\frac{d}{dt}[D_t, a_j]\). By Lemma 2.6, the former collects into \(\alpha(D_t D_t + \dot{D}_t D_t)\text{Ch}^n_{\text{JLO}}(D_t)\), while the latter collects into \(\alpha^n(D_t, \dot{D}_t)\). Hence together with Theorem 2.5(3)

\[
\frac{d\text{Ch}^n_{\text{JLO}}(D_t)}{dt} = \alpha(D_t \dot{D}_t + \dot{D}_t D_t)\text{Ch}^n_{\text{JLO}}(D_t) + \alpha^n(D_t, \dot{D}_t)
\]

\[
= b\text{Ch}^{n-1}_{\text{JLO}}(D_t, \dot{D}_t) + B\text{Ch}^{n+1}_{\text{JLO}}(D_t, \dot{D}_t).
\]

The fact that \(\text{Ch}^n_{\text{JLO}}(D_t, \dot{D}_t)\) is entire follows from Lemma 2.4 and the uniform boundedness of \(F_t\) and \(R_t\). The result is obtained.

The following Proposition gives a stability of bounded perturbation of weakly \(\theta\)-summable unbounded Breuer-Fredholm modules. It is Theorem C in [20].

**Proposition 2.8.** For a weakly \(\theta\)-summable unbounded Breuer-Fredholm module \((\rho, N, D)\), and an operator \(V \in N\) such that \(V\) has the same degree as \(D\), i.e. \(|V|_{\chi} = |D|_{\chi}\). Then \((\rho, N, D + V)\) is again a weakly \(\theta\)-summable unbounded Breuer-Fredholm module and

\[
\tau \left( e^{-((1+\varepsilon/2)(D+V)^2)} \right) \leq e^{(1+2/\varepsilon)\|V\|^2} \cdot \tau (e^{-(1-\varepsilon)D^2})
\]

**Proof.** It is obvious that

\[
\|D + V, a\| \leq (C + 2\|V\|) \|a\|
\]

hence if we obtain \(\tau (e^{-((1+\varepsilon/2)(D+V)^2)} \leq e^{(1+2/\varepsilon)\|V\|^2} \cdot \tau (e^{-(1-\varepsilon)D^2})\), we are done.

Observe that if \(A\) and \(B\) are positive operators, then

\[
\tau (e^{-A-B}) \leq \tau (e^{-A})
\]

We proceed by introducing the operators

\[
A = (1-\varepsilon)D^2, \\
B = \frac{\varepsilon}{2}D^2 + (1-\frac{\varepsilon}{2}) (DV + VD + V^2) + (1 + \frac{2}{\varepsilon}) \|V\|^2.
\]

18
A is a positive operator, and to see that $B$ is also positive, we use the fact that

$$-(DV + VD) \leq \frac{\varepsilon}{2}D^2 + \frac{2}{\varepsilon}V^2 \leq \frac{\varepsilon}{2}D^2 + \frac{2}{\varepsilon}\|V\|^2.$$  

Therefore,

$$\tau\left(e^{-\left(1-\frac{\varepsilon}{2}\right)D^2-(1-\frac{\varepsilon}{2})(DV+VD+V^2)-(1+\frac{\varepsilon}{2})\|V\|^2}\right) = \tau\left(e^{-A-B}\right) \leq \tau\left(e^{-A}\right) = \tau\left(e^{(1-\varepsilon)D^2}\right)$$

$$\leq e^{(1+2/\varepsilon)\|V\|^2} \cdot \tau\left(e^{-(1-\varepsilon)D^2}\right),$$

and the result is obtained. \hfill \Box

### 2.4 Index pairing in (co)homology

This section will show that the JLO character for a weakly $\theta$-summable even unbounded Breuer-Fredholm module produces an index formula. For the odd case, we refer to a paper by Carey and Phillips [8], who developed the JLO character in the Type II setting.

**Theorem 2.9 ([8]).** Let $(\rho, N, D)$ be an odd weakly $\theta$-summable unbounded Breuer-Fredholm module over $A$ and $u \in M_N(A)$ be a unitary, then

$$[[\rho, N, D]], \langle u \rangle = \langle [\text{ch}_{\text{JLO}}(D)], [\text{ch}_-(u)] \rangle,$$

where the angle bracket on the left is the spectral flow pairing [8] and the round bracket on the right is the (co)homology pairing.

**Theorem 2.10.** Let $(\rho, N, D)$ be an even weakly $\theta$-summable unbounded Breuer-Fredholm module over $A$ and $p \in M_N(A)$ be a projection, then

$$[[\rho, N, D]], \langle p \rangle = \langle [\text{ch}_{\text{JLO}}(D)], [\text{ch}_+(p)] \rangle,$$

where the angle bracket on the left is the index pairing and the round bracket on the right is the (co)homology pairing.

**Proof.** It suffices to prove that

$$\text{Ind}_+(p^- (D \otimes 1_N) p^+) = \langle [\text{ch}_{\text{JLO}}(D)], [\text{ch}_+(p)] \rangle.$$

It follows from the definition of (co)homology that the above equality will descend to the result stated in the theorem.

For any projection $p \in A$, one can deform $D$ to $(pDp + (1-p)D(1-p))$ via the homotopy $\mathcal{D}_t = D + t(2p - 1)[D, p]$ where $t \in [0,1]$. As $\mathcal{D}_t = (2p - 1)[D, p]$ is odd and in $N$, by Proposition 2.8, $(\rho, N, \mathcal{D}_t)$ is a differentiable family of weakly $\theta$-summable unbounded Breuer-Fredholm modules. By Theorem 2.7, $\text{ch}_{\text{JLO}}^+(D)$ and $\text{ch}_{\text{JLO}}^+(pDp + (1-p)D(1-p))$ are cohomologous. Specifically,

$$\text{ch}_{\text{JLO}}^+(pDp + (1-p)D(1-p)) - \text{ch}_{\text{JLO}}^+(D) = \text{ch}_{\text{JLO}}^+(D_1) - \text{ch}_{\text{JLO}}^+(D_0) = (b + B) \int_0^1 \text{ch}_{\text{JLO}}^+(\mathcal{D}_t, \mathcal{D}_t) dt.$$  

Therefore,

$$\langle [\text{ch}_{\text{JLO}}(D)], [\text{ch}_+(p)] \rangle = \langle [\text{ch}_{\text{JLO}}^+(pDp + (1-p)D(1-p))], [\text{ch}_+(p)] \rangle - (b + B) \int_0^1 [\text{ch}_{\text{JLO}}^+(\mathcal{D}_t, \mathcal{D}_t), [\text{ch}_+(p)] \rangle$$

$$= \langle [\text{ch}_{\text{JLO}}^+(pDp + (1-p)D(1-p))], [\text{ch}_+(p)] \rangle.$$
where the last equality follows from the fact that \([D_1,p] = [pDP + (1 - p)D(1 - p),p] = 0\) and \(\text{ch}_c(p)\) is closed.

Hence the pairing \((\text{Ch}^{\text{JLO}}_n(D),\text{ch}_c(p))\) yields the McKean-Singer index formula

\[\tau(\chi p e^{-D^2}) .\]

The fact that the McKean-Singer index formula produces the desirable index \(\text{Ind}_d(p^{-1}DP^+)\) is proved in [12]. If \(p\) is a projection in \(M_N(A)\), one extends \(D\) to \(D \otimes 1_N\) and \(\tau\) to \(\tau \otimes \text{Tr}\), where \(\text{Tr}\) is the matrix trace from \(M_N(C) \to C\), the result follows. \(\square\)

### 3 Reduction from JLO character to Connes-Chern character

For every \(p\)-summable unbounded Breuer-Fredholm module there is a canonically associated \(p\)-summable bounded Breuer-Fredholm module. Using techniques from [16], this section connects the previous two sections by showing that the JLO character of a \(p\)-summable unbounded Breuer-Fredholm module and the Connes-Chern character of its associated \(p\)-summable Breuer-Fredholm module define the same class in entire cyclic cohomology.

Most of the work in this section is adapted from [16] and [21].

#### 3.1 JLO character for \(p\)-summable unbounded Breuer-Fredholm modules

As shown in Lemma 2.1, \(p\)-summable unbounded Breuer-Fredholm modules are also \(\theta\)-summable (in particular weakly \(\theta\)-summable). Therefore, the JLO character of a \(p\)-summable unbounded Breuer-Fredholm module is defined.

Given a \(p\)-summable unbounded Breuer-Fredholm module \((p,N,D)\), its JLO class defined by the JLO character \(\text{Ch}^{\text{JLO}}_n(D)\) has a representative which consists of only finitely many terms.

For convenience, we denote

\[\widetilde{\text{Ch}}^n(tD) := \text{Ch}^{\leq n}_{\text{JLO}}(tD) + B \int_0^t \text{Ch}^{n+1}_{\text{JLO}}(uD,D)du .\]  

Here \(\text{Ch}^{\leq n}_{\text{JLO}}(tD)\) means that we discard the terms greater than \(n\) in \(\text{Ch}^{\bullet}_{\text{JLO}}(tD)\). That is,

\[\text{Ch}^{\leq n}_{\text{JLO}}(tD) := \begin{cases} \sum_{k=1}^{2k \leq n} \text{Ch}^{2k}_{\text{JLO}}(tD) & \text{when } (p,N,D) \text{ is even} \\ \sum_{k=1}^{2k+1 \leq n} \text{Ch}^{2k+1}_{\text{JLO}}(tD) & \text{when } (p,N,D) \text{ is odd} \end{cases} .\]

**Proposition 3.1.** Given a \(p\)-summable unbounded Breuer-Fredholm module \((p,N,D)\), its JLO-cocycle \(\text{Ch}^{\bullet}_{\text{JLO}}(D)\) is cohomologous to \(\widetilde{\text{Ch}}^n(tD)\) for \(t \in [1,\infty), n > p\). When \(D\) is invertible, \(B \int_0^\infty \text{Ch}^{n+1}_{\text{JLO}}(uD,D)du\) is a well-defined entire cochain and is cohomologous to \(\text{Ch}^{\bullet}_{\text{JLO}}(D)\).

**Proof.** The proof will make use of Theorem 2.5(2) twice.

\[\text{Ch}^{\bullet}_{\text{JLO}}(D) = \text{Ch}^{\bullet}_{\text{JLO}}(tD) + (b + B) \int_1^t \text{Ch}^{\bullet}_{\text{JLO}}(uD,D)du \]

\[= \text{Ch}^{\leq n}_{\text{JLO}}(tD) + \text{Ch}^{n+1}_{\text{JLO}}(sD) + \left(\text{Ch}^{n+2}_{\text{JLO}}(tD) - \text{Ch}^{n+2}_{\text{JLO}}(sD)\right) \]

\[+ (b + B) \int_1^t \text{Ch}^{\bullet}_{\text{JLO}}(uD,D)du \]

\[= \text{Ch}^{\leq n}_{\text{JLO}}(tD) + \text{Ch}^{n+2}_{\text{JLO}}(sD) \]

\[+ \left(-b \int_s^t \text{Ch}^{n+1}_{\text{JLO}}(uD,D)du - B \int_s^t \text{Ch}^{n+3}_{\text{JLO}}(uD,D)du\right) \]

\[+ (b + B) \int_1^t \text{Ch}^{\bullet}_{\text{JLO}}(uD,D)du \]

\[= \text{Ch}^{\leq n}_{\text{JLO}}(tD) + B \int_s^t \text{Ch}^{n+1}_{\text{JLO}}(uD,D)du + \text{Ch}^{n+2}_{\text{JLO}}(sD) \]

\[+ (b + B) \left(\int_1^t \text{Ch}^{\bullet}_{\text{JLO}}(uD,D)du - \int_s^t \text{Ch}^{n+1}_{\text{JLO}}(uD,D)du\right) .\]
Since \( \tau(e^{-(1-\delta)t^2D^2}) = O(s^{-p}) \) by Lemma 2.1, it follows from Lemma 2.4 that for \( m > p \)

\[
\|Ch_{JLO}^n(sD)\| = O(s^{m-p})
\]

\[
\|Ch_{JLO}^m(uD, D)\| = O(u^{m-p})
\]

so that \( \lim_{s \to 0} Ch_{JLO}^m(sD) = 0 \) in norm and \( Ch_{JLO}^m(uD, D) \) is integrable from 0 to \( t \) for \( m > p \). Hence

\[
Ch_{JLO}^\bullet(D) = Ch_{JLO}^{\leq n}(tD) + B \int_0^t Ch_{JLO}^{n+1}(uD, D)du
\]

\[+(b + B) \left( \int_1^t Ch_{JLO}^n(uD, D)du - \int_0^t Ch_{JLO}^{\geq n+1}(uD, D)du \right).\]

Now suppose \( D \) is invertible. Then \( \lambda := \inf \{ \sigma(D^2) \} > 0 \) where \( \sigma(D^2) \) is the spectrum of \( D^2 \). From Lemma 2.4 and (the proof of) Lemma 2.1, we get that

\[
\|Ch_{JLO}^n(tD)\| \leq \frac{t^r}{r!} \tau(e^{-(1-\delta)t^2D^2}) C^r
\]

\[
\leq \frac{t^r}{r!} e^{-(1-\delta)t^2D^2/2} \tau(e^{-(1-\delta)t^2D^2/2}) C^r
\]

\[
\leq \left( \frac{t^r}{r!} e^{-(1-\delta)t^2D^2/2} \right) \tau(e^{-(1-\delta)t^2D^2/2}) C^r,
\]

and

\[
\|Ch_{JLO}^{n+1}(uD, D)\| \leq \frac{2u^{n+1}}{n!\sqrt{\epsilon\delta}} \tau(e^{-(1-\delta)u^2D^2}) C^{n+1}
\]

\[
\leq \frac{2u^{n+1}e^{-(1-\delta)u^2\lambda^2/2}}{n!\sqrt{\epsilon\delta}} \tau(e^{-(1-\delta)u^2D^2/2}) C^{n+1}
\]

\[
\leq \frac{2u^{n+1}e^{-(1-\delta)u^2\lambda^2/2}}{n!\sqrt{\epsilon\delta}} \left( \frac{p}{e(1-\delta)} \right)^{p/2} \|D\|^{-1} \|D\|^p C^{n+1}.
\]

The norm of \( \tau(e^{-(1-\delta)t^2D^2/2}) \) is uniformly bounded for \( t \in [1, \infty) \), and the term \( u^{n+1}e^{-(1-\delta)u^2\lambda^2/2} \) is integrable from 0 to \( \infty \), therefore the limit for \( t \to \infty \) exists in norm for \( Ch_{JLO}^n(tD) \) and \( \int_0^t Ch_{JLO}^{n+1}(uD, D)du \). In particular, \( \lim_{t \to \infty} Ch_{JLO}^{\leq n}(tD) = 0 \) in norm. Thus,

\[
Ch_{JLO}^\bullet(D) = B \int_0^\infty Ch_{JLO}^{n+1}(uD, D)du
\]

\[+(b + B) \left( \int_1^\infty Ch_{JLO}^n(uD, D)du - \int_0^\infty Ch_{JLO}^{\geq n+1}(uD, D)du \right).\]

From the proof of Proposition 3.1,

\[
\lim_{t \to \infty} Ch_{JLO}^m(D) = B \int_0^\infty Ch_{JLO}^{n+1}(uD, D)du.
\]

### 3.2 From unbounded to bounded Breuer–Fredholm modules

For every \( p \)-summable unbounded Breuer–Fredholm module there is a canonically associated \( p \)-summable bounded Breuer–Fredholm module. We will go through a concrete construction of such a bounded Breuer–Fredholm module from an unbounded one when \( D \) is invertible, and remove the invertibility assumption at the end of the section. Most of the work in this section is adopted from [16] and [28].

Given an unbounded Breuer–Fredholm module \( (\rho, \mathcal{N}, D) \) with \( D \) invertible, there is an associated bounded Breuer–Fredholm module \( (\rho, \mathcal{N}, F) \) by taking \( F = D|D|^{-1} \). We will follow a technique in [28, 21] to show that if \( (\rho, \mathcal{N}, D) \) is \( p \)-summable, then so is \( (\rho, \mathcal{N}, F) \).
Corollary 3.3. If \((\rho, N, D)\) is a \(p\)-summable unbounded Breuer-Fredholm module with \(D\) invertible, then its associated Breuer-Fredholm module \((\rho, N, F)\) is also \(p\)-summable.
3.3 From JLO character to Connes-Chern character

Let \( \mathcal{D}_\alpha := \mathcal{D}/\mathcal{D}^{-\alpha} \) for \( \alpha \in [0,1] \), then we have a homotopy between the unbounded Breuer-Fredholm module \((\rho, \mathcal{N}, \mathcal{D})\) when \( \alpha = 0 \) and its associated bounded Breuer-Fredholm module \((\rho, \mathcal{N}, \mathcal{D})\) when \( \alpha = 1 \). When \((\rho, \mathcal{N}, \mathcal{D})\) is \( p \)-summable and \( n - 1 > p \), we will see that \( \mathcal{C}_n^\alpha(\mathcal{D}_\alpha) \) (see Equation (11)) defines an \( \alpha \)-family of entire cyclic cocycles for \( t > 0 \). Moreover, these cocycles in fact live in the same entire cyclic cohomology class.

Note that our calculations include both the even and odd cases.

**Theorem 3.4.** The cochain \( \frac{d\mathcal{C}^n_\alpha(\mathcal{D}_\alpha)}{da} \) is exact for \( \alpha \in [0,1] \) and \( t \in [1, \infty] \). Explicitly, it is the \((b+1)\)-coboundary of the entire cochain

\[
\int_0^t b(\mathcal{D}_\alpha)\mathcal{C}^n_{\alpha}(u\mathcal{D}_\alpha, u\mathcal{D}_\alpha)du - \mathcal{C}^{\leq n-1}_{\alpha}(t\mathcal{D}_\alpha, t\mathcal{D}_\alpha).
\]

To prove this theorem, we need the following identities and estimate. They are nothing but elaborations of Theorem 2.5(2), Lemma 2.6, and Lemma 2.4.

**Definition 3.1.** Let \( V \) and \( W \) be operators affiliated with \( \mathcal{N} \) such that they have the same degree as \( \mathcal{D} \), i.e. \( [\mathcal{D}]_\chi = [V]_\chi = [W]_\chi \). Define \( \mathcal{C}^\alpha_{\alpha}(\mathcal{D}, V, W) \) to be given by the equation

\[
(\mathcal{C}^n_{\alpha}(\mathcal{D}, V, W), (a_0, \ldots, a_n))_n = \sum_{k=1}^j (-1)^k \sum_{j=1}^{n+1} (-1)^j \langle a_0, \ldots, [\mathcal{D}, a_{k-1}], V, \ldots, [\mathcal{D}, a_j], \ldots \rangle^2_{\mathcal{D}} + \sum_{k=j}^{n+2} (-1)^{k+1} \sum_{j=1}^{n+1} (-1)^j \langle a_0, \ldots, [\mathcal{D}, a_{j-1}], V, \ldots, [\mathcal{D}, a_k], \ldots \rangle^2_{\mathcal{D}}.
\]

**Lemma 3.5.** Let \( V \) and \( W \) be operators affiliated with \( \mathcal{N} \). Then we have

\[
b\mathcal{C}^{n-1}_{\alpha}(\mathcal{D}, V, W) + B\mathcal{C}^{n+1}_{\alpha}(\mathcal{D}, V, W) = \nu(V)(\nu(DW+W\mathcal{D})\mathcal{C}^\alpha_{\alpha}(\mathcal{D}) - \alpha^n(\mathcal{D}, W)) - \nu(W)(\nu(DV+V\mathcal{D})\mathcal{C}^\alpha_{\alpha}(\mathcal{D}) - \alpha^n(\mathcal{D}, V)).
\]

The above lemma can be found in [16]. Its proof is nothing but an elaboration of the proof to Lemma 2.5(2), which is a lengthy but straightforward calculation, so we decide to skip it here.

**Lemma 3.6.** Suppose that \( \mathcal{D}_a \) and \( \mathcal{V}_a \) are 1-parameter families of operators affiliated with \( \mathcal{N} \) so that \( \mathcal{D}_a \) and \( \mathcal{V}_a \) are defined and affiliated with \( \mathcal{N} \), then

\[
\frac{d}{da}\mathcal{C}^n_{\alpha}(\mathcal{D}_a, \mathcal{V}_a) = \mathcal{C}^n_{\alpha}(\mathcal{D}_a, \mathcal{V}_a) + \nu(V_a)\alpha^n(\mathcal{D}_a, \mathcal{V}_a) - \nu(V_a)\nu(\mathcal{D}_a, \mathcal{V}_a, \mathcal{D}_a, \mathcal{V}_a)\mathcal{C}^\alpha_{\alpha}(\mathcal{D}_a).
\]

**Proof.** By applying Leibniz rule on \( \frac{d}{da}\mathcal{C}^n_{\alpha}(\mathcal{D}_a, \mathcal{V}_a) \), we will obtain a sum of the term containing \( \frac{d}{da}\mathcal{V}_a \), terms containing the \( \frac{d}{da}[\mathcal{D}_a, a_k] \), and the terms containing \( \frac{d}{da}e^{-s\mathcal{D}_a^2} \). They collect into \( \mathcal{C}^n_{\alpha}(\mathcal{D}_a, \mathcal{V}_a) \). By Lemma 2.6, \( -\nu(V_a)\nu(\mathcal{D}_a, \mathcal{V}_a, \mathcal{D}_a, \mathcal{V}_a)\mathcal{C}^\alpha_{\alpha}(\mathcal{D}_a) \).

By using Lemma 3.5 and 3.6, we can establish the algebraic equality

\[
\frac{d\mathcal{C}^n_{\alpha}(\mathcal{D}_a)}{da} = (b + B) \left( \int_0^t b(\mathcal{D}_\alpha)\mathcal{C}^n_{\alpha}(u\mathcal{D}_\alpha, u\mathcal{D}_\alpha)du - \mathcal{C}^{\leq n-1}_{\alpha}(t\mathcal{D}_\alpha, t\mathcal{D}_\alpha) \right).
\]

However, the major task is to show that in fact

\[
\int_0^t b(\mathcal{D}_\alpha)\mathcal{C}^n_{\alpha}(u\mathcal{D}_\alpha, u\mathcal{D}_\alpha)du - \mathcal{C}^{\leq n-1}_{\alpha}(t\mathcal{D}_\alpha, t\mathcal{D}_\alpha)
\]

is entire. And the analysis required to prove its entirety is a little involved.

We begin by observing the operator \([F \ln|\mathcal{D}|, a] \) is bounded. To show this, we need the following lemma.
Lemma 3.7. Let $H$ be a positive operator. Then

$$H^{-\frac{\alpha}{2}} \ln H = \frac{1}{C_\alpha} \int_0^\infty (H + \lambda)^{-\frac{\alpha}{2}} (\ln \lambda) d\lambda - \frac{C'_\alpha}{C_\alpha} H^{-\frac{\alpha}{2}}$$

for $\alpha > 0$, where $C_\alpha = \int_0^\infty (1 + x)^{-1/2} x^{-\alpha/2} dx$ and $C'_\alpha = \int_0^\infty (1 + x)^{-1/2} (\ln x) dx$.

Proof. From changing variable $x = \lambda/y$, we obtain

$$\int_0^\infty (1 + x)^{-1/2} x^{-\alpha/2} dx = y^{\alpha/2} \int_0^\infty (y + \lambda)^{-1} \lambda^{-\alpha/2} d\lambda.$$ 

By differentiating both sides with respect to $\alpha$, the above turns into

$$\int_0^\infty (1 + x)^{-1} x^{-\alpha/2} (\ln x) dx = y^{\alpha/2} \int_0^\infty (y + \lambda)^{-1} \lambda^{-\alpha/2} (\ln \lambda) d\lambda - \ln y \int_0^\infty (1 + x)^{-1/2} x^{-\alpha/2} dx,$$

where the integrals converge as long as $\alpha > 0$. Now using functional calculus to substitute $H$ in $y$ to get

$$H^{-\frac{\alpha}{2}} \ln H = \frac{1}{C_\alpha} \int_0^\infty (H + \lambda)^{-1} \lambda^{-\frac{\alpha}{2}} (\ln \lambda) d\lambda - \frac{C'_\alpha}{C_\alpha} H^{-\frac{\alpha}{2}},$$

which is the desired equation.

\[\Box\]

Proposition 3.8. Let $D$ be invertible and $F = D|D|^{-1}$. For any $a \in A$, the commutator $[F \ln |D|, a]$ is bounded.

Proof. By applying Lemma 3.7 for $H = D^2$ and $\alpha = 1$, one obtains

$$2[F \ln |D|, b] = [D, b]|D|^{-1} \ln D^2 + D|D|^{-1} \ln D^2, b]$$

$$= [D, b] \left( \frac{1}{C_1} \int_0^\infty (D^2 + \lambda)^{-1} \lambda^{-1/2} (\ln \lambda) d\lambda - \frac{C'_1}{C_1} |D|^{-1} \right)$$

$$+ D \left( \frac{1}{C_1} \int_0^\infty [(D^2 + \lambda)^{-1}, b] \lambda^{-1/2} (\ln \lambda) d\lambda - \frac{C'_1}{C_1} |D|^{-1}, b] \right)$$

$$= \frac{1}{C_1} \int_0^\infty |D(D^2 + \lambda)^{-1}, b| \lambda^{-1/2} (\ln \lambda) d\lambda - \frac{C'_1}{C_1} [F, b]$$

for $b \in A$. Since $[F, b]$ is bounded, we see that $[F \ln |D|, b]$ is bounded if and only if $\frac{1}{C_1} \int_0^\infty |D(D^2 + \lambda)^{-1}, b| \lambda^{-1/2} (\ln \lambda) d\lambda$ is bounded. We compute

$$\frac{1}{C_1} \int_0^\infty |D(D^2 + \lambda)^{-1}, b| \lambda^{-1/2} (\ln \lambda) d\lambda$$

$$= \frac{1}{C_1} \int_0^\infty ([D, b](D^2 + \lambda)^{-1} + D((D^2 + \lambda)^{-1}, b)] \lambda^{-1/2} (\ln \lambda) d\lambda$$

$$= \frac{1}{C_1} \int_0^\infty ([D, b](D^2 + \lambda)^{-1}$$

$$- D(D^2 + \lambda)^{-1} [D^2 + \lambda, b](D^2 + \lambda)^{-1} \lambda^{-1/2} (\ln \lambda) d\lambda$$

$$= \frac{1}{C_1} \int_0^\infty ((D^2 + \lambda)(D^2 + \lambda)^{-1} [D, b](D^2 + \lambda)^{-1}$$

$$- D(D^2 + \lambda)^{-1} (D[D, b] + [D, b]D)(D^2 + \lambda)^{-1} \lambda^{-1/2} (\ln \lambda) d\lambda$$

$$= \frac{1}{C_1} \int_0^\infty (\lambda(D^2 + \lambda)^{-1} [D, b](D^2 + \lambda)^{-1}$$

$$- D(D^2 + \lambda)^{-1} [D, b]D(D^2 + \lambda)^{-1} \lambda^{-1/2} (\ln \lambda) d\lambda.$$ 

24
If $[D, b]$ is self-adjoint, then the estimate $-||[D, b]\| \leq ||D, b\| \leq ||[D, b]\|$ and the fact that $D$ is self-adjoint yield

$$-\lambda(\lambda + D^2)^{-1} ||[D, b]\| (\lambda + D^2)^{-1} \leq \lambda(\lambda + D^2)^{-1} ||[D, b]\| (\lambda + D^2)^{-1} \leq \lambda(\lambda + D^2)^{-1} ||[D, b]\| (\lambda + D^2)^{-1},$$

and

$$-||D(\lambda + D^2)^{-1}[[D, b]\| (\lambda + D^2)^{-1}||D]\| \leq -\lambda(\lambda + D^2)^{-1} ||[D, b]\| (\lambda + D^2)^{-1} D \leq ||D(\lambda + D^2)^{-1} ||[D, b]\| (\lambda + D^2)^{-1}||D|. $$

Hence,

$$-||[D, b]\| \left( ||D\|^{-1} \ln ||D\| + C^\prime_1 C^\prime_1 ||D\|^{-1} \right) \leq \frac{1}{C^\prime_1} \int_0^\infty [D(D^2 + \lambda)^{-1}, b] \lambda^{-1/2} (\ln \lambda) d\lambda \leq ||[D, b]\| \left( ||D\|^{-1} \ln ||D\| + C^\prime_1 C^\prime_1 ||D\|^{-1} \right).$$

Therefore, in the end we obtain

$$\|\{F \ln ||D\|, b\| \leq \frac{1}{2} ||[D, b]\| \left( ||D\|^{-1} \ln ||D\| + C^\prime_1 C^\prime_1 ||D\|^{-1} \right) + C^\prime_2 \ |\ |_{2C^\prime_2} ||[F, b]\|,$$

which is bounded. Since for any $a \in A$, $[F \ln ||D\|, a] = [F \ln ||D\|, \frac{a - a^*}{2}] + i [F \ln ||D\|, \frac{a + a^*}{2}]$ with $[F \ln ||D\|, \frac{a - a^*}{2}]$ and $[F \ln ||D\|, \frac{a + a^*}{2}]$ self-adjoint. By Equation 14, $[F \ln ||D\|, \frac{a - a^*}{2}], [F \ln ||D\|, \frac{a + a^*}{2}]$ are bounded. Hence, so is $\|\{F \ln ||D\|, a\|.$

**Proposition 3.9.** For a p-summable unbounded Breuer-Fredholm module $(\rho, N, D)$ with $D$ invertible, set $D_\alpha = D/|D|^{-\alpha}$. Then $\int_0^t b_t(D_\alpha)CH^\alpha_{A\alpha}(uD_\alpha, uD_\alpha)du$ for $n - 1 > p$, is a well-defined family of entire cyclic cocycles for $\alpha \in [0, 1]$ and $t \in [1, \infty]$, where $D_\alpha = -D_\alpha \ln ||D\|.$

**Proof.** The proof will go in two steps. First we estimate the norm of a generic term of $\int_0^t b_t(D_\alpha)CH^\alpha_{A\alpha}(uD_\alpha, uD_\alpha)\ln ||D|| du$ for $\alpha \in [0, 1)$. Set $\lambda = \inf (\sigma(D^2)) > 0$, we compute:

$$\left\|\left\{b_0, uD_\alpha, b_1, \ldots, uD_\alpha \ln ||D||, \ldots, D_\alpha, \ldots, uD_\alpha, b_n\right\}\right\|^2_{uD_\alpha}$$

$$\leq \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|b_0 e^{-s_0(uD_\alpha)^2} [D_\alpha, b_1] e^{-s_1(uD_\alpha)^2} \ldots [D_\alpha, b_{p}] e^{-s_p(uD_\alpha)^2} \ldots \ldots [D_\alpha, b_{p+1}] e^{-s_{p+1}(uD_\alpha)^2} \ldots \ldots [D_\alpha, b_n] e^{-s_n(uD_\alpha)^2} \right\|^2_1 d^m + 2 s du$$

$$\leq \left\|b_0\right\| \left\|D_\alpha, b_1\right\| \frac{d^m}{1 + s_0} \ldots \left\|D_\alpha, b_{p}\right\| \frac{d^m}{1 + s_p} \ldots \ldots \left\|D_\alpha, b_{p+1}\right\| \frac{d^m}{1 + s_{p+1}} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{d^m}{1 + s_n}$$

$$\leq \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|e^{-s_0(uD_\alpha)^2} \frac{1}{1 + s_0} \ldots \left\|D_\alpha, b_{p}\right\| \frac{1}{1 + s_p} \ldots \ldots \left\|D_\alpha, b_{p+1}\right\| \frac{1}{1 + s_{p+1}} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{1}{1 + s_n} \right\|^2_{\Delta_{n+2}} d^m + 2 s du$$

$$\leq \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|e^{-s_0(uD_\alpha)^2} \frac{1}{1 + s_0} \ldots \left\|D_\alpha, b_{p}\right\| \frac{1}{1 + s_p} \ldots \ldots \left\|D_\alpha, b_{p+1}\right\| \frac{1}{1 + s_{p+1}} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{1}{1 + s_n} \right\|^2_{\Delta_{n+2}} d^m + 2 s du$$

$$\leq \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|e^{-s_0(uD_\alpha)^2} \frac{1}{1 + s_0} \ldots \left\|D_\alpha, b_{p}\right\| \frac{1}{1 + s_p} \ldots \ldots \left\|D_\alpha, b_{p+1}\right\| \frac{1}{1 + s_{p+1}} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{1}{1 + s_n} \right\|^2_{\Delta_{n+2}} d^m + 2 s du$$

$$\leq \left\|b_0\right\| \left(\prod_{j=1}^{n} \left\|D_j, b_j\right\| \right) \left( \left\|D\right\|^{-1} \right)^{p\alpha} \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|e^{-s_0(1 - \delta)(uD_\alpha)^2} \frac{1}{1 + s_0(1 - \delta)} \ldots \ldots \left\|D_\alpha, b_{p}\right\| \frac{1}{1 + s_p(1 - \delta)} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{1}{1 + s_n(1 - \delta)} \right\|^2_{\Delta_{n+2}} d^m + 2 s du$$

$$\leq \left\|b_0\right\| \left(\prod_{j=1}^{n} \left\|D_j, b_j\right\| \right) \left( \left\|D\right\|^{-1} \right)^{p\alpha} \int_0^t u^{n+1} \int_{\Delta_{n+2}} \left\|e^{-s_0(1 - \delta)(uD_\alpha)^2} \frac{1}{1 + s_0(1 - \delta)} \ldots \ldots \left\|D_\alpha, b_{p}\right\| \frac{1}{1 + s_p(1 - \delta)} \ldots \ldots \left\|D_\alpha, b_n\right\| \frac{1}{1 + s_n(1 - \delta)} \right\|^2_{\Delta_{n+2}} d^m + 2 s du$$

25
we estimate the norm of a generic term of 

\( \|u_{\alpha}\|_\alpha \) entire for \( \alpha \) bounded by Proposition 3.8. Therefore, by the continuity, estimate techniques deployed in the proof of Proposition 3.9, it is easy to see that 

\[
\int_{\Delta_n+2} \left( \frac{1 + \varepsilon}{2e\delta s_j u^2} \right)^{1+\frac{\|D\|}{\|D\|}} \left( \frac{1}{2e\delta s_j u^2} \right)^{\phi} d^n+2 s \ du 
\]

\[
\leq C^n \left( \prod_{j=1}^n \|b_j\| \left( \frac{\|D\|^{-1}}{p} \right)^p \left( (1-\alpha)^{-\varepsilon} \ln x \right) \int_0^t u^{n+1} \|e^{-(1-\delta)(u_{\alpha})^2/2} \right) \left( \sum_{j=1}^n \left( \frac{1 + \varepsilon}{2e\delta s_j u^2} \right)^{1+\frac{\|D\|}{\|D\|}} \left( \frac{1}{2e\delta s_j u^2} \right)^{\phi} \right) d^n+2 s \ du 
\]

\[
\leq C^n \left( \prod_{j=1}^n \|b_j\| \left( \frac{\|D\|^{-1}}{p} \right)^p \frac{1 + \varepsilon}{2e\delta \ln x} \left( \frac{1 + \varepsilon}{2e\delta} \right)^{1+\frac{\|D\|}{\|D\|}} \right) \int_0^t u^{n+1} \|e^{-(1-\delta)(u_{\alpha})^2/2} \right) \left( \sum_{j=1}^n \left( \frac{1 + \varepsilon}{2e\delta s_j u^2} \right)^{1+\frac{\|D\|}{\|D\|}} \left( \frac{1}{2e\delta s_j u^2} \right)^{\phi} \right) d^n+2 s \ du 
\]

The integral

\[
\int_0^t u^{n+1} \|e^{-(1-\delta)(u_{\alpha})^2/2} \right) \left( \sum_{j=1}^n \left( \frac{1 + \varepsilon}{2e\delta s_j u^2} \right)^{1+\frac{\|D\|}{\|D\|}} \left( \frac{1}{2e\delta s_j u^2} \right)^{\phi} \right) d^n+2 s \ du 
\]

exists for \( t \in [1, \infty] \) as long as \( n - 1 - \varepsilon \geq p \). Thus, \( b \int_0^\infty (\tau s_{\alpha}^p \chi a uF, uF \ln|D|)du \) is entire for \( t \in [1, \infty] \), \( \alpha \in [0, 1] \), \( n - 1 > p \), and \( 0 < \delta, \varepsilon \) sufficiently small. Now we suppose that \( \alpha = 1 \). Since

\[
\int_0^t (\tau s_{\alpha}^p \chi a uF, uF \ln|D|)du = (\tau s_{\alpha}^p \chi a uF, uF \ln|D|) \int_0^t u^{n+1} e^{-u^2} du 
\]

to show \( \int_0^t b(F) \chi_n^p \chi a uF, uF \ln|D|)du \) is bounded, it suffices to know that \( b \chi_n^p \chi a uF, uF \ln|D|)du \) is bounded. Hence, we estimate the norm of a generic term of \( \chi_n^p \chi a uF, uF \ln|D|)du \) paired with \( b(a_0, \ldots, a_{n+1})u_{n+1} \).

\[
\left\| \sum_{j=0}^{n+1} (-1)^j \tau (\chi a_0 \cdots [F, a_j a_{j+1}] \cdots F \ln|D|) \cdots [F, a_{n+1}] \right\| 
\]

\[
\leq \left\| \tau (\chi a_0 [F, a_1] \cdots [F \ln|D|, a_k] \cdots [F, a_{n+1}]) \right\| 
\]

\[
\leq \left\| [F \ln|D|, a_k]\| |a_0| \prod_{j=1}^{n+1} \| [F, a_j] \|_\alpha 
\]

which is bounded by Proposition 3.8. Therefore, by the continuity,

\[
\int_0^t b(\tau s_{\alpha}^p \chi a uF, uF \ln|D|)du 
\]

is entire for \( \alpha \in [0, 1] \).
Proof of Theorem 3.4. We compute

\[
\frac{d}{dx} \tilde{\text{Ch}}_{n}^{\rho_n}(D_\alpha) = \frac{d}{dx} \text{Ch}_{n}^{\rho_n}(D_\alpha) + B \int_0^t \frac{d}{dx} \text{Ch}_{n}^{\rho_n+1}(uD_\alpha, D_\alpha)du \\
= -(b + B)\text{Ch}_{n}^{\rho_n-1}(D_\alpha, tD_\alpha) - B\text{Ch}_{n}^{\rho_n+1}(D_\alpha, tD_\alpha) \\
+ B \int_0^t \frac{d}{dx} \text{Ch}_{n}^{\rho_n+1}(uD_\alpha, D_\alpha)du \\
= -(b + B)\text{Ch}_{n}^{\rho_n-1}(D_\alpha, tD_\alpha) \\
+ B \int_0^t \left( \text{Ch}_{n}^{\rho_n+1}(uD_\alpha, D_\alpha) - \text{Ch}_{n}^{\rho_n+1}(uD_\alpha, uD_\alpha) \right)du \\
= -(b + B)\text{Ch}_{n}^{\rho_n-1}(D_\alpha, tD_\alpha) + B \int_0^t \left( b(uD_\alpha)\text{Ch}_{n}^{\rho_n}(uD_\alpha, uD_\alpha) \\
+ Bu(uD_\alpha)\text{Ch}_{n}^{\rho_n+2}(uD_\alpha, uD_\alpha) \right)du \\
= (b + B) \left( \int_0^t b(uD_\alpha)\text{Ch}_{n}^{\rho_n}(uD_\alpha, uD_\alpha)du - \text{Ch}_{n}^{\rho_n-1}(D_\alpha, tD_\alpha) \right)
\]

where the last equality follows from the identities \( b^2 = B^2 = 0 \).

Finally, \( \int_0^t b(uD_\alpha)\text{Ch}_{n}^{\rho_n}(uD_\alpha, uD_\alpha)du \) is entire by Proposition 3.9. \( \text{Ch}_{n}^{\rho_n-1}(D_\alpha, tD_\alpha) \) is also entire by the same proof as Proposition 3.9.

**Proposition 3.10.** For a \( p \)-summable unbounded Breuer-Fredholm module \( (\rho, N, D) \) with \( D \) invertible, set \( D_\alpha = D[D]^{-\alpha} \). Then \( \lim_{t \to \infty} \text{Ch}_{n}^{\rho_n}(D_\alpha) \) is a family of entire cocycles for \( \alpha \in [0, 1] \).

**Proof.** Same proof as Proposition 3.9.

**Theorem 3.11.** For a \( p \)-summable unbounded Breuer-Fredholm module \( (\rho, N, D) \) with \( D \) invertible, its JLO character is cohomologous to the Connes character of its associated Breuer-Fredholm module.

**Proof.** By Theorem 3.4,

\[
B \int_0^\infty \text{Ch}_{n}^{\rho_n+1}(uF, F)du - \lim_{t \to \infty} \text{Ch}_{n}^{\rho_n}(D) = (b + B) \int_0^1 \int_0^\infty b(uD_\alpha)\text{Ch}_{n}^{\rho_n}(uD_\alpha, D_\alpha)du \, dx.
\]

Together with Proposition 3.1, we conclude that \( B \int_0^\infty \text{Ch}_{n}^{\rho_n+1}(uF, F)du \) is cohomologous to the JLO character \( \text{Ch}_{n}^{\rho_n}(D) \), where \( F = D[D]^{-\alpha} \).

The map \( B : C_n(B) \to C_{n+1}(B) \) on chains can be decomposed into \( B = sN \) where

\[
N(a_0, \ldots, a_n) := \sum_{j=0}^n (-1)^j(a_j, \ldots, a_n, a_0, \ldots, a_{j-1}) \\
s(a_0, \ldots, a_n) := (1, a_0, \ldots, a_n).
\]
By observing the fact that $F[F,a] = -[F,a]F$ and combining Lemma 2.2(1)(2), the rest is straightforward calculation:

\[
B \int_{0}^{\infty} \chi_{\rho}^{n+1}(u,F)du, (a_{0}, \ldots, a_{n})
\]

\[
= N \int_{0}^{\infty} \langle F, [uF,a_{0}], \ldots, [uF,a_{n}] \rangle du
\]

\[
= \left( \int_{0}^{\infty} u^{n+1} e^{-u^{2}} du \int_{\Delta_{n+1}} ds \right) \tau (\chi F[a_{0}] \cdots [F,a_{n}])
\]

\[
= \left( \frac{1}{2} \int_{0}^{\infty} t^{n/2} e^{-t} dt \frac{1}{(n+1)!} \right) (n+1) \tau (\chi F[a_{0}] \cdots [F,a_{n}])
\]

\[
= \frac{\Gamma(n+1)}{2 \cdot n!} \tau (\chi F[a_{0}] \cdots [F,a_{n}])
\]

\[
= (\chi h^{n}(F), (a_{0}, \ldots, a_{n})).
\]

The invertibility assumption in this section can be removed as follows (see [10]). Given an unbounded Breuer-Fredholm module $(\rho, N, D)$, we can associate to it another unbounded Breuer-Fredholm module $(\rho', N', D')$ with $D'$ invertible. First we form the sum $(\rho \oplus 0, N \otimes M_{2}(\mathbb{C}), D \oplus -D)$ and equip it with the grading $\chi \oplus -\chi$, then perturb $D \oplus -D$ by the isometry

\[
\begin{pmatrix}
0 & K \\
K & 0
\end{pmatrix}
\in N \otimes M_{2}(\mathbb{C})
\]

that exchanges the two copies of $\mathcal{H}$. Here $K$ is made to be odd with respect to the grading by exchanging the $\mathcal{H}^{+}$ and $\mathcal{H}^{-}$ subspaces when $D$ is graded,

\[
K = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

We set

\[
(\rho', N', D') := (\rho \oplus 0, N \otimes M_{2}(\mathbb{C}), \begin{pmatrix}
D & K \\
K & -D
\end{pmatrix})
\]

then $(\rho', N', D')$ is an unbounded Breuer-Fredholm module, and $D'$ has the same summability as $D$. The identity

\[
D'^{2} = \begin{pmatrix}
D & K \\
K & -D
\end{pmatrix}^{2} = \begin{pmatrix}
D^{2} + 1 & 0 \\
0 & D^{2} + 1
\end{pmatrix} = \begin{pmatrix}
(D + i)(D - i) & 0 \\
0 & (D + i)(D - i)
\end{pmatrix}
\]

implies that $D'$ is a bijection (from its domain), and is invertible. Furthermore, it represents the same K-homology class as $(\rho, N, D)$. As the procedure of obtaining $(\rho', N', D')$ can be described by adding zero to $(\rho, N, D)$ and perturbing the sum by an isometry, which is the equivalence relation in K-homology.

A Appendix

The Appendix gives an account on basic definitions needed for the discussion of the paper, it includes affiliated operators, $\tau$-compact operators, $p$-summable operators etc., then followed by some basic properties of these operators. The ideals $K_{\mathcal{A}}$ and $Z_{\mathcal{A}}$ are then defined in terms of $\tau$-compactness and $p$-summability, and finally the Appendix ends by stating Hölder’s inequality, which is crucial in our work.

The presentation in this section follows closely [17] and [2] to which we refer to proofs and further details.

A von Neumann algebra with underlying Hilbert space $\mathcal{H}$ is a unital $*$-subalgebra of the algebra of bounded operators $B(\mathcal{H})$ on $\mathcal{H}$ that is closed under the weak operator topology.

A positive linear functional on a von Neumann algebra is said to be normal if it preserves sup of any increasing nets of positive operators in the von Neumann algebra; faithful if it is positive-definite on positive operators;
semi-finite if the $*$-subalgebra generated by positive elements with finite value under the functional is $\sigma$-weak dense in the von Neumann algebra [4]. A von Neumann algebra is called semi-finite if it admits a faithful, semi-finite normal trace. A von Neumann algebra is Type I if it is semi-finite and every projection contains a minimal sub-projection; Type II if it is semi-finite but not Type I [4].

Let $\mathcal{N}$ be a semi-finite von Neumann algebra with underlying Hilbert space $\mathcal{H}$ and a faithful semi-finite normal trace $\tau$.

**Definition A.1.** A densely defined closed operator $T$ on $\mathcal{H}$ with polar decomposition $T = U|T|$ [27] is said to be affiliated with $\mathcal{N}$ if $U \in \mathcal{N}$ and also the spectral projection $1_{[0,\lambda]}(|T|)$ of $|T|$ lies in $\mathcal{N}$ for all $\lambda$, where $1_{[0,\lambda]}$ is the characteristic function supported on the closed interval $[0, \lambda] \subseteq \mathbb{R}$.

For a positive self-adjoint operator $T = \int_0^\infty \lambda dE_\lambda$ affiliated with $\mathcal{N}$ with $E_\lambda = 1_{[0,\lambda]}(|T|)$, we define its semi-finite trace by

$$\tau(T) = \int_0^\infty \lambda d\tau(E_\lambda).$$

From now on, when we say that an operator $T$ is affiliated with $\mathcal{N}$, we implicitly demand that $T$ is densely defined and closed.

**Definition A.2.** For an operator $T$ affiliated with $\mathcal{N}$ and $x > 0$, the generalized singular number $\mu_x(T)$ with respect to $(\mathcal{N}, \tau)$ is defined to be

$$\mu_x(T) := \inf \{ \|TE\| : \tau(1-E) \leq x \},$$

where the infimum is taken over projections $E \in \mathcal{N}$.

**Definition A.3.** Let $T$ be an operator affiliated with $\mathcal{N}$, $0 < p < \infty$, and $x > 0$. Then $T$ is said to be

- **$p$-summable** if
  $$\|T\|^p := \tau(|T|^p)^{1/p} < \infty,$$

- **$\tau$-compact** if
  $$\lim_{x \to \infty} \mu_x(T) = 0,$$

- **$\tau$-measurable** if for each $\varepsilon > 0$ there exists a projection $E \in \mathcal{N}$ such that
  $$\text{Ran}(E) \subset \text{Dom}(T) \quad \text{and} \quad \tau(1-E) < \varepsilon.$$

**Remark A.4.** Anything in $\mathcal{N}$ is $\tau$-measurable. If a self-adjoint operator $T$ is affiliated with $\mathcal{N}$ and its resolvent is $\tau$-compact, then $T$ is $\tau$-measurable [2].

**Proposition A.1** ([17]). Let $T, S, R$ be $\tau$-measurable operators.

1. The map: $x \in (0,\infty) \to \mu_x(T)$ is non-increasing and continuous from the right. Moreover,\n
$$\lim_{x \searrow 0} \mu_x(T) = \|T\| \in [0,\infty].$$

2. $\mu_x(T) = \mu_x(|T|) = \mu_x(T^*)$ and $\mu_x(zT) = |z|\mu_x(T)$ for $x > 0$ and $z \in \mathbb{C}$.

3. $\mu_x(T) \leq \mu_x(S)$, $x > 0$, if $0 \leq T \leq S$.

4. $\mu_x(f(|T|)) = f(\mu_x(|T|))$, $x > 0$ for any continuous increasing function $f$ on $[0,\infty)$ with $f(0) \geq 0$.

5. $\mu_x(SR) \leq \|S\|\|R\|\mu_x(T)$, $x > 0$.

**Proposition A.2** ([17]). Let $T$ be a positive $\tau$-measurable operator. Then

$$\tau(T) = \int_0^\infty \mu_x(T)dx.$$
Proposition A.3 ([17]). Let $T$, $S$, and $R$ be operators in $\mathcal{N}$. Then for $0 < p < \infty$,
\[ \|STR\|_p \leq \|S\| \|R\| \|T\|_p. \]

Denote by $\mathcal{L}_p^p$ the space of all $p$-summable operators in $\mathcal{N}$. For $0 < p < \infty$, the space $\mathcal{L}_p^p$ forms a norm closed two-sided ideal in $\mathcal{N}$ with norm given by $\|\cdot\|_p + \|\cdot\|$. Denote by $\mathcal{K}_\tau$ the space of all $\tau$-compact operators in $\mathcal{N}$. The space $\mathcal{K}_\tau$ forms a norm closed two-sided ideal in $\mathcal{N}$.

Theorem A.4 ([17]). Let $T$, $S$ be $\tau$-measurable operators. Then
1. $\|TS\|_r \leq \|T\|_p \|S\|_q$ for $p, q, r > 0$ and $p^{-1} + q^{-1} = r^{-1}$.
2. $\|T + S\|_p \leq \|T\|_p + \|S\|_p$ for $p \geq 1$.

For $\mathcal{N} = B(\mathcal{H})$ with $\tau$ the operator trace, then $p$-summability and $\tau$-compactness are the usual notion of $p$-summability and compactness, and the ideals $\mathcal{L}_p^p$ and $\mathcal{K}_\tau$ are the usual ideal of Schatten $p$-class and the ideal compact operators.

References

[1] J. Aastrup, J. Grimstrup, and R. Nest. On Spectral Triples in Quantum Gravity II. *Journal of Noncommutative Geometry*, 3:47–81, 2009.
[2] M. Benameur and T. Fack. Type II non-commutative geometry. I. Dixmier trace in von Neumann algebras. *Advance in Mathematics*, 199:29–87, 2006.
[3] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Springer, 2004.
[4] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. Springer, 2002.
[5] M. Breuer. Fredholm Theories in von Neumann Algebras. I. *Math. Ann.*, 178:243–254, 1968.
[6] M. Breuer. Fredholm Theories in von Neumann Algebras. II. *Math. Ann.*, 180:313–325, 1969.
[7] A. Carey and J. Phillips. Unbounded Fredholm modules and spectral flow. *Can. J. Math.*, 50:673–718, 1998.
[8] A. Carey and J. Phillips. Spectral Flow in Fredholm Modules, Eta Invariant and JLO Cocycle. *K-Theory*, 31:135–194, 2004.
[9] A. Carey, J. Phillips, A. Rennie, and F. Sukochev. The Hochschild class of the Chern character for semifinite spectral triples. *J. Func. Anal.*, 213:111–153, 2004.
[10] A. Carey, J. Phillips, A. Rennie, and F. Sukochev. The Chern Character of semifinite spectral triples. *Journal of Noncommutative Geometry*, 2:141–193, 2006.
[11] A. Carey, J. Phillips, A. Rennie, and F. Sukochev. The local index formula in semifinite von Neumann algebras I: Spectral flow. *Adv. Math.*, 202:451–516, 2006.
[12] A. Carey, J. Phillips, A. Rennie, and F. Sukochev. The local index formula in semifinite von Neumann algebras II: The even case. *Adv. Math.*, 202:517–554, 2006.
[13] A. Connes. Non-commutative differential geometry. *Publ. Math. IHES*, 39:257–360, 1985.
[14] A. Connes. *Noncommutative Geometry*. Academic Press, 1994.
[15] A. Connes. On the spectral characterization of manifolds. arXiv:math-OA/0810.2088, 2008.
[16] A. Connes and H. Moscovici. Transgression and the Chern Character of Finite-Dimensional K-Cycles. *Commun. Math. Phys.*, 155:103–133, 1993.
[17] T. Fack and H. Kosaki. Generalized $s$-Numbers of $\tau$-Measurable Operators. *Pac. J. Math.*, 123:2, 1986.

[18] E. Getzler. The Odd Chern Character in Cyclic Homology and Spectral Flow. *Topology*, 32:489–507, 1993.

[19] E. Getzler. Cyclic Homology and the Atiyah-Patodi-Singer index theorem. *Contemporary Mathematics*, 00, 1997.

[20] E. Getzler and A. Szenes. On the Chern Character of a Theta-Summable Fredholm Module. *J. Func. Anal.*, 84:343–357, 1989.

[21] J. Gracia-Bondía, J. Várilly, and H. Figueroa. *Elements of Noncommutative Geometry*. Birkhäuser, 2000.

[22] N. Higson and J. Roe. *Analytic K-Homology*. Oxford University Press, 2000.

[23] A. Jaffe, A. Lesniewski, and K. Osterwalder. Quantum K-theory: the Chern character. *Commun. Math. Phys.*, 112:75–88, 1988.

[24] M. Khalkhali and M. Marcolli. *An Invitation to Noncommutative Geometry*. World Scientific Pub Co Inc, 2008.

[25] A. Kriegl and P. Michor. Differential Perturbation of Unbounded Operators. *Math. Ann.*, 327:192–201, 2003.

[26] R. Meyer. *Local and Analytic Cyclic Homology*. European Mathematical Society, 2007.

[27] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I: Functional Analysis*. Academic Press Inc., 1970.

[28] E. Schrohe, M. Walze, and J. Warzecha. Construction de triplets spectraux a partir de modules de Fredholm. *R. Acad. Sd. Paris*, 326:1195–1199, 1998.