FIXED POINT PROPERTIES AND SECOND BOUNDED COHOMOLOGY OF UNIVERSAL LATTICES ON BANACH SPACES

MASATO MIMURA

Abstract. Let $B$ be any $L^p$ space or any Banach space isomorphic to a Hilbert space, and $k \geq 0$ be integer. We show that if $n \geq 4$, then the universal lattice $\Gamma = SL_n(\mathbb{Z}[X_1, \ldots, X_k])$ has property $(F_B)$ in the sense of Bader–Furman–Gelander–Monod. Namely, any affine isometric action on $B$ has a global fixed point. The property of having $(F_B)$ for all $B$ is known to be strictly stronger than Kazhdan’s property (T). We also define the following generalization of property $(F_B)$: the boundedness property of all affine quasi-actions on $B$. We name it property $(FF_B)$ and prove that the group $\Gamma$ above also has this property for non-trivial linear part. The conclusion above implies that the comparison map $H^2_{bb}(\Gamma; B) \to H^2(\Gamma; B)$ from bounded to ordinary cohomology is injective, provided that the associated linear representation has no trivial part.

1991 Mathematics Subject Classification: primary 22D12; secondary 20F32

1. Introduction and main results

Kazhdan’s property (T), which was first introduced in [16], represents certain forms of rigidity of a group, and now it plays important role in wide range of mathematical fields: for instance, see [3]. Property (T) is initially defined in terms of unitary representations. Recall that a group $\Gamma$ is defined to have property $(T)$ if any unitary representation $(\pi, \mathcal{H})$ of $\Gamma$ does not admit almost invariant vectors in $\mathcal{H}_{\pi(\Gamma)}$. (The symbol $\mathcal{H}_{\pi(\Gamma)}$ means the orthogonal complement of the subspace of all $\pi(\Gamma)$-invariant vectors.) Here a representation $\rho$ of a group $\Gamma$ on a Banach space $B$ is said to admit almost invariant vectors if for any compact set $F \subset \Gamma$ and any $\varepsilon > 0$, there exists $x \in S(B)$ (namely, with $\|x\| = 1$) such that $\sup_{s \in F} \|\rho(s)x - x\| \leq \varepsilon$ holds. P. Delorme [9] and A. Guichardet [13] have shown that for any locally compact and second countable group $\Gamma$, property (T) is equivalent to Serre’s property (FH): a group $\Gamma$ is said to have (FH) if any affine isometric action on a Hilbert space has a global fixed point.

In 2007, Bader–Furman–Gelander–Monod [2] investigated similar properties in the broader framework of general Banach spaces. They named the Kazhdan type property and the fixed point property respectively $(T_B)$ and $(F_B)$. For convenience, we would like to use the following symbols for certain classes of Banach spaces.

- The symbol $L^p$ is defined as the class of all $L^p$ spaces (on arbitrary measures).

The author is supported by JSPS Research Fellowships for Young Scientists No.20-8313.
• The symbol $[\mathcal{H}]$ is defined as the class of all Banach spaces which are isomorphic to Hilbert spaces.

Bader–Furman–Gelander–Monod proved the following theorem and revealed that $(F_B)$ is stronger than $(T_B)$ in general. We note that the statement for $[\mathcal{H}]$ in (iii) in the theorem is due to Y. Shalom in his unpublished work. (We mention that the original statement requires $\sigma$-finiteness of measure spaces in (ii), but the argument in [2, §4.1] works for general cases.)

**Theorem 1.1.** ([2, Theorem A and Theorem B]) Let $G$ be a locally compact and second countable group.

(i) For any Banach space $B$, property $(F_B)$ implies property $(T_B)$.

(ii) Property $(T)$ is equivalent to property $(T_{L^p})$, where $p \in (1, \infty)$. It is also equivalent to property $(F_{L^p})$, where $p \in (1, 2]$.

(iii) Suppose that $G = \prod_{i=1}^m G_i(k_i)$, where $k_i$ are local fields and $G_i(k_i)$ are $k_i$-points of Zariski connected simple $k_i$-algebraic groups. If each simple factor $G_i(k_i)$ has $k_i$-rank $\geq 2$, then $G$ and the lattices in $G$ have property $(F_{L^p})$ for $1 < p < \infty$ and property $(F_{[\mathcal{H}]}$).

They also mentioned that both of the properties “$(F_{L^p})$ for all $2 < p < \infty$” and $(F_{[\mathcal{H}]}$) are stronger than $(T)$. Indeed, G. Yu [32] has proved that any hyperbolic group, including one with $(T)$, admits a proper affine isometric action on some $\ell^p$ space. Existence of a proper affine action represents opposite nature to rigidity of a group. Hence higher rank algebraic groups and lattices have stronger rigidity than hyperbolic $(T)$ groups do. We also note that Y. Shalom has announced that $Sp(n, 1)$ fails to have $(T_{[\mathcal{H}]}$). The author does not know whether there exists an infinite hyperbolic group with $(F_{[\mathcal{H}]}$) (or $(T_{[\mathcal{H}]}$)).

From the backgrounds above, it seems to be a significant problem to establish $(F_{L^p})$ $(1 < p < \infty)$ and $(F_{[\mathcal{H}]}$) for certain groups. However as far as the author knows, the only known examples were the groups in (iii) of Theorem 1.1. One of the main results of the paper is to provide a new example. The group $SL_n(\mathbb{Z}[X_1, \ldots, X_k])$ is called the universal lattice by Y. Shalom [25].

**Theorem 1.2.** Let $k \geq 0$ be an integer. Then for $n \geq 4$, the universal lattice $SL_n(\mathbb{Z}[X_1, \ldots, X_k])$ has property $(F_C)$. Here $C$ stands for either the class $L^p$ $(1 < p < \infty)$ or the class $[\mathcal{H}]$.

We note that this theorem particularly implies property $(T_{[\mathcal{H}]}$) of universal lattices with $n \geq 4$. It follows from (i) of Theorem 1.1.

For our proof of Theorem 1.2, we need to deduce $(F_B)$ from $(T_B)$. There are the following two well known cases in which the direction above is true: first, the case that $B = \mathcal{H}$ is due to P. Delorme [9] with the aid of conditionally negative definite functions. Second, the case of a higher rank algebraic group is treated in [2, §5]. In this case, the Howe–Moore property of simple algebraic groups is the key. By making use of the relative versions of $(T_B)$ and $(F_B)$, we have shown the following new implication:
Theorem 1.3. Let \( k \geq 0 \) be an integer and \( A_k = \mathbb{Z}[X_1, \ldots, X_k] \). Suppose \( B \) is a superreflexive Banach space. If \( EL_2(A_k) \ltimes A_k^2 \triangleright A_k^2 \) has relative property \((T_B)\), then \( SL_3(A_k) \ltimes A_k^3 \triangleright A_k^3 \) has relative property \((F_B)\).

To prove Theorem 1.2, we combine Theorem 1.3 with the following relative \((T_B)\), Shalom’s argument in [27], and Vaserstein’s bounded generation in [30].

Theorem 1.4. With the same notation as one in Theorem 1.3, \( EL_2(A_k) \ltimes A_k^2 \triangleright A_k^2 \) has relative property \((T_C)\), where \( C \) stands for \( L^p (1 < p < \infty) \) or \( [H] \).

Here for a unital ring \( A, EL_n(A) \) denotes the subgroup of \( M_n(A) \) generated by all elementary matrices. Suslin’s result [29] states that if \( n \geq 3 \), then \( EL_n(\mathbb{Z}[X_1, \ldots, X_k]) \) coincides with \( SL_n(\mathbb{Z}[X_1, \ldots, X_k]) \).

Further, we generalize property \((F_B)\) in a way similar to one by N. Monod [20]. Our next result is that the universal lattices have this property for \( B \in \bigcup_{1 < p < \infty} L^p \cup [H] \) for the case that we restrict linear representations to non-trivial parts. We define that a group has property \((FFF_B)\) if any quasi-action on \( B \) of the group has bounded orbits. We also define the property \((FF_B)\) for nontrivial linear part as a certain weaker form of \((FFF_B)\). More precisely, property \((FF_B)\) for non-trivial linear part is some stronger modification of the boundedness property for all of the quasi-actions on \( B \) whose associated linear representations do not have non-zero invariant vectors. The exact definition and precise arguments will be taken place in Section 6. In the case that one restricts groups to those with finite abelianizations, for any superreflexive Banach space \( B \), property \((FF_B)\) for non-trivial linear part is stronger than property \((F_B)\). We establish the following result. We mention that the result might be new even for the case that \( B = \mathcal{H} \).

Theorem 1.5. (Main Theorem) Let \( k \geq 0 \) be an integer and \( n \geq 4 \). Then universal lattice \( SL_n(\mathbb{Z}[X_1, \ldots, X_k]) \) has property \((FFF_{L^p})\) \( (1 < p < \infty) \) for non-trivial linear part and property \((FF_{[H]})\) for non-trivial linear part.

The author does not know whether our argument can be extended to the case that \( B \) is not superreflexive. We note that V. Lafforgue [18], [19, Corollaire 0.7, Proposition 5.2 and Proposition 5.6] has shown that \( SL_4(\mathbb{Q}_p) \) (where \( p \) is any prime number) and cocompact lattices therein, for instance, have \((F_B)\) for any Banach space \( B \) of non-trivial type. He has asked whether these group have \((FF_B)\) for a Banach space \( B \) of non-trivial type (or more generally, of non-trivial cotype). We note that the boundedness property \((FF_B)\) does not necessarily imply the fixed point property \((F_B)\) if \( B \) is not superreflexive.

We apply Theorem 1.2 and Theorem 1.3 to the following two objects: actions on the circle, and second bounded cohomology. We note that for the same class \( \mathcal{C} \) as in Theorem 1.2 \((FC)\) and \((FFC)\) (and \((FFC)\) for nontrivial linear part) pass to quotient groups and subgroups of finite indices. (For the heredity to subgroups of finite indices, one uses induction. See [2, §8]). Therefore the corollaries below hold for \( \Gamma = EL_n(A) \), where \( A \) stands for any unital, commutative and finitely generated ring.
Corollary 1.6. Let \( k \geq 0 \) be an integer and \( C \) be the same class as one in Theorem 1.2. Let \( n \geq 4 \), and \( \Gamma \) be any group of one of the following three forms:

(a) universal lattices \( SL_n(\mathbb{Z}[X_1, \ldots, X_k]) \),
(b) quotient groups of (a),
(c) subgroups of (a) or (b) with finite indices.

Then for any \( \alpha > 0 \), every homomorphism \( \Phi : \Gamma \to \text{Diff}^{1+\alpha}(S^1) \) has finite image.

Corollary 1.7. Let \( C \) and \( \Gamma \) be the same as in Corollary 1.6. Then the comparison map in degree 2

\[
\Psi^2 : H^2_\text{b}(\Gamma; B) \to H^2(\Gamma; B)
\]

is injective, for any \( B \in C \) and any isometric representation \( \rho \) on \( B \) without trivial part.

These applications above shall be discussed in Section 7. Corollary 1.6 states that the group \( \Gamma \) cannot act on the circle in non-trivial way, and it can be seen as an extension of Navas’ theorem [22] for Kazhdan groups for the case \( \alpha > 1/2 \). In the case of subgroups of finite indices in \( SL_n(\mathbb{Z}) \), stronger result [31, Corollary 2.4], which shows a similar rigidity for homeomorphisms on the circle for these groups, is proved by D. Witte. The proof needs Margulis’ normal subgroup theorem, and one cannot apply it straightforwardly to universal lattices. Corollary 1.7 can be seen as some generalization of [7, Theorem 21]. However, in the case of lattices \( \Gamma \) of higher rank algebraic groups \( G \), much stronger result is known. Indeed, N. Monod and Y. Shalom [21, Theorem 1.4] have proved that unless \( \pi_1(G) \) is infinite and the local field is \( \mathbb{R} \), the second bounded cohomology of \( \Gamma \) with separable coefficient Banach modules always vanishes. The author would like to conjecture the following: “Let \( \Gamma \) be the same as in Corollary 1.6. Then the second bounded cohomology of \( \Gamma \) vanishes for every separable coefficient Banach module.”

In addition, we have defined the relative Kazhdan constant for property \((T)_{[\mathfrak{H}]}\) and perform a certain estimate. (See Appendix.)

Proposition 1.8. With the same notation as one in Theorem 1.3, let \( G = EL_2(A_k) \rtimes A_k^2 \) and \( N = A_k^2 \). Set \( F \) be the set of all unit elementary matrices in \( G \subset SL_3(A_k) \).

Then the inequality

\[
\overline{\kappa}(G; N, F; M) > (15k + 100)^{-1}M^{-6}
\]

holds. In the case that \( k = 0 \), \( \overline{\kappa}(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2; \mathbb{Z}^2, F; M) > (21M^6)^{-1} \) holds. Here the symbol \( \overline{\kappa}(G; N, F; M) \) denotes the generalized relative Kazhdan constant for uniformly bounded representations, which is defined in Definition A.1.

Notation and conventions.

Throughout this paper, we assume all rings are associative, all representations of a topological group are strongly continuous, and all subgroups of a topological group are closed. We also assume all discrete groups are countable. We let \( \Gamma, G \) and \( N \) be topological groups, \( B \) be a Banach space, \( C \) be a class of Banach spaces, and
\( \mathcal{H} \) be an arbitrary Hilbert space. For a Banach space \( B \), we define \( S(B) \) as the unit sphere, \( \mathbb{B}(B) \) as the Banach algebra of all bounded linear operators on \( B \), and \( \langle \cdot, \cdot \rangle \) as the duality \( B \times B^* \to \mathbb{C} \). In this paper, we shall define the following properties in terms of \( B \):

- relative \((T_B)\), \((T_B)\);
- relative \((F_B)\), \((F_B)\); the Shalom property for \((F_B)\);
- relative \((FF_B), (FF_B), (FF_B)\) for non-trivial linear part; and the Shalom property for \((FF_B)\).

If we let \( (P_B) \) represent any of these properties, then we define the property \((P_C)\) in terms of \( C \) as follows: having \((P_C)\) stands for having \((P_B)\) for all \( B \in C \).

**Acknowledgments**

The author would like to thank his supervisor Narutaka Ozawa for introducing him to this topic, and Yasuyuki Kawahigashi for comments. He is grateful to Bachir Bekka for the symbol \( H \), to Andrés Navas for suggesting him stating Corollary 1.6 explicitly, and to Mamoru Tanaka for pointing out a mistake on stability under ultralimits in the previous version of this paper. He also thanks Alex Furman for the reference [31], Uzy Hadad for correcting some errors, Martin Kassabov for conversations on noncommutative universal lattices, and Vincent Lafforgue for arguments and drawing my attention to non-superreflexive cases. Finally, he would like to express his gratitude to the referee, whose comments have improved this paper.

2. Preliminaries

2.1. Superreflexivity and property \((T_B)\).

**Definition 2.1.** ([2]) Let \( B \) be a Banach space.

- A pair \((G \vartriangleright N)\) of groups is said to have relative property \((T_B)\) if for any isometric representation \( \rho \) of \( G \) on \( B \), the natural isometric representation \( \rho' \) on \( B/B^{\rho(N)} \) does not admit almost invariant vectors. Here \( B^{\rho(N)} \) stands for the subspace of \( B \) of all \( \rho(N) \)-invariant vectors.
- A group \( \Gamma \) is said to have property \((T_B)\) if \((\Gamma \vartriangleright \Gamma)\) has relative \((T_B)\).

In the case that \( B \) is superreflexive, we take the natural complement in \( B \) of \( B^{\rho(N)} \). We start with the definition of superreflexivity.

**Definition 2.2.** Let \( B \) be a Banach space.

- The space \( B \) is said to be uniformly convex (or uc) if for all \( 0 < \varepsilon < 2 \), \( d_{\| \cdot \|}(\varepsilon) > 0 \) holds. Here for \( 0 < \varepsilon < 2 \), we define
  \[
  d_{\| \cdot \|}(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \text{ and } \|x - y\| \geq \varepsilon \right\}.
  \]

- The space \( B \) is said to be uniformly smooth (or us) if \( \lim_{\tau \to 0} r_{\| \cdot \|}(\tau)/\tau = 0 \) holds. Here for \( \tau > 0 \), we define
  \[
  r_{\| \cdot \|}(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq \tau \right\}.
  \]
• The space $B$ is said to be ucus if $B$ is uc and us.
• The space $B$ is said to be superreflexive if it is isomorphic to some ucus Banach space.

We call $d$ and $r$ the \textit{modulus of convexity} and \textit{smoothness} respectively.

We refer to \cite{4} §A for details on ucus Banach spaces.

**Lemma 2.3.** (\cite{4}) Let $(B, \| \cdot \|)$ be a Banach space. Then for any $\tau > 0$, $r_{\| \cdot \|}(\tau) = \sup_{0 < \varepsilon < \tau} \{ \varepsilon \tau / 2 - d_{\| \cdot \|}(\varepsilon) \}$. In particular, $B$ is us if and only if $B^*$ is uc.

**Lemma 2.4.** (\cite{4}) Let $B$ be a us Banach space. Then for any $x \in S(B)$, there exists a unique element $x^* \in S(B^*)$ such that $\langle x, x^* \rangle = 1$. Moreover, the map $S(B) \to S(B^*); x \mapsto x^*$ is uniformly continuous. We call this map $x \mapsto x^*$ the duality mapping.

Any Hilbert space $\mathcal{H}$ is usc because $d_{\| \cdot \|_{\mathcal{H}}}(\varepsilon) = 1 - \sqrt{1 - (\varepsilon / 2)^2}$ and $r_{\| \cdot \|_{\mathcal{H}}}(\tau) = \sqrt{1 + \tau^2} - 1$. Any $L^p$ space is usc if $1 < p < \infty$, whereas the $L^1$ spaces and the $L^\infty$ spaces are not. (Here we assume dimensions $\geq 2$.)

**Remark 2.5.** A representation $\rho$ of $\Gamma$ on $B$ is said to be \textit{uniformly bounded} if $|\rho| = \sup_{g \in \Gamma} \|\rho(g)\|_{B(B)} < +\infty$. Bader–Furman–Gelander–Monod \cite{2} Proposition 2.3 have proved that any uniformly bounded representation $\rho$ on a superreflexive Banach space $B$ is isometric with respect to some ucus compatible norm.

For an isometric representation $\rho$ of $\Gamma$ on $B$, we define the \textit{contragredient representation} $\rho^!$ of $\Gamma$ on $B^*$ as follows: for any $g \in \Gamma$, $\phi \in B^*$ and $x \in B$, $\langle x, \rho^!(g)\phi \rangle = \langle \rho(g^{-1})x, \phi \rangle$. If $B$ is us, then the equality $(\rho(g)x)^* = \rho^!(g)x^*$ holds by definition.

**Proposition 2.6.** (\cite{2} Proposition 2.6) Suppose $G \actson N$ and $\rho$ is an isometric representation of $G$ on a us space $B$. Let $B_0$ be $B^{\rho(N)}$ and let $B_1 = B^{\rho(N)}_\alpha$ denote the annihilator of $(B^*)^{\rho(N)}$ in $B$. Then $B = B_0 \oplus B_1$ is a decomposition of $B$ into two $\rho(G)$-invariant subspaces. Furthermore, for any $x = x_0 + x_1$ ($x_0 \in B_0, x_1 \in B_1$), the inequality $\|x_0\| \leq \|x\|$ holds.

**Proposition 2.7.** (\cite{2} Proposition 2.10) Let $G \actson N$ be a group pair. Suppose a Banach space $B$ is superreflexive. Then for any isometric representation $\rho$ of $G$, $B_1 = B^{\rho(N)}_\alpha$ is isomorphic to $B/B^{\rho(N)}_\alpha$ as $G$-representations. Particularly, $G \actson N$ has relative $(T_B)$ if and only if no isometric representation $\rho$ admits almost invariant vectors in $B^{\rho(N)}_\alpha$.

2.2. \textbf{Property} ($F_B$). An \textit{affine isometric action} $\alpha$ of $\Gamma$ on $B$ is an action of the form $\alpha(g)x = \rho(g)x + c(g)$. Here $\rho$ is an isometric representation and $c(g) \in B$. We sometimes simply write $\alpha = \rho + c$. We call $\rho$ and $c$ respectively the \textit{linear part} and the \textit{transformation part} of $\alpha$. Because $\alpha$ is an action, the transition part $c$ satisfies the following condition, called the \textit{cocycle identity}:

For any $g, h \in \Gamma$, \quad $c(gh) = c(g) + \rho(g)c(h)$.

We also call $c$ the \textit{cocycle part} of $\alpha$. 
Definition 2.8. For an isometric representation $\rho$ on $B$, we call a map $c: \Gamma \to B$ a $\rho$-cocycle if it satisfies the cocycle identity. We say $c$ to be \textit{inner} if there exists $x \in B$ such that $c(g) = x - \rho(g)x$ for all $g \in \Gamma$. We let $Z^1(\Gamma, \rho)$ and $B^1(\Gamma, \rho)$ denote respectively the spaces of all $\rho$-cocycles and all inner $\rho$-cocycles. We define the \textit{first cohomology} of $\Gamma$ with $\rho$-coefficient as the additive group $H^1(\Gamma, \rho) = Z^1(\Gamma, \rho)/B^1(\Gamma, \rho)$.

The space $Z^1(\Gamma, \rho)$ is a Fréchet space with respect to its natural topology. Namely, the uniform convergence topology on compact subsets of $\Gamma$. However the coboundary $B^1(\Gamma, \rho)$ is not closed in general. We shall examine details in Section 5.

Definition 2.9. (\cite{2}, for the second case) Let $B$ be a Banach space.

- A pair $(G > N)$ of groups is said to have \textit{relative property} $(F_B)$ if any affine isometric action of $G$ on $B$ has an $N$-fixed point.
- A group $\Gamma$ is said to have \textit{property} $(F_B)$ if $(\Gamma > \Gamma)$ has relative $(F_B)$. Equivalently, if for any isometric representation $\rho$ of $\Gamma$ on $B$, $H^1(\Gamma, \rho) = 0$ holds.

2.3. Useful lemmas. Let $B$ be a superreflexive Banach space, $G > N$, and $F \subset G$ be a compact set. We define the \textit{Kazhdan constant} for property $(T_B)$ of $(G; N, F, \rho)$ by the following equality: $K(G; N, F, \rho) = \inf_{x \in S(B_1)} \sup_{s \in F} \|\rho(s)x - x\|$. Here $B_1 = B_{\rho(N)}^\prime$ as in Proposition 2.6. If $G > N$ have relative $(T_B)$ and $F$ generates $G$, then for any isometric representation $\rho$ on $B$, the constant $K(G; N, F, \rho)$ is strictly positive.

Lemma 2.10. Suppose $B$ is us, $G$ is a compactly generated group and $F$ is a compact generating set of $G$. Let $\rho$ be any isometric representation of $G$ on $B$, $x$ be any vector in $B$ and $\delta_x := \sup_{s \in F} \|\rho(s)x - x\|$. If a pair $G > N$ has relative $(T_B)$, then there exists a $\rho(N)$-invariant vector $x_0 \in B$ with $\|x - x_0\| \leq 2K^{-1}\delta_x$. Here $K$ stands for the Kazhdan constant $K(G; N, F, \rho)$ for $(T_B)$.

Proof. Decompose $B$ as $B = B_0 \oplus B_1 = B_{\rho(N)}^\prime \oplus B_{\rho(N)}^\prime$, and let $x = x_0 + x_1$ ($x_0 \in B_0, x_1 \in B_1$). Then $\rho(s)x - x = (\rho(s)x_0 - x_0) + (\rho(s)x_1 - x_1)$ is the decomposition of $\rho(s)x - x$. For a general decomposition $y = y_0 + y_1$, one has $\|y_1\| \leq \|y\| + \|y_0\| \leq 2\|y\|$ by applying Proposition 2.6. Hence the inequality $2\delta_x = 2\sup_{s \in F} \|\rho(s)x - x\| \geq \sup_{s \in F} \|\rho(s)x_1 - x_1\| \geq K\|x_1\|$ holds.

The following lemma and its corollary are well-known, and also important.

Lemma 2.11. (\textit{lemma of the Chebyshev center}) Let $B$ be a uc Banach space and $X$ be a bounded subset. Then there exists a unique closed ball with the minimum radius which contains $X$. We define the Chebyshev center of $X$ as the center of this ball.

Corollary 2.12. Let $B$ be a superreflexive Banach space and $N$ be a subgroup of $G$. Then for any affine isometric action of $G$ on $B$, the following are equivalent.

(i) The action has an $N$-fixed point.

(ii) Some (or equivalently, any) $N$-orbit is bounded.
2.4. Unit elementary matrices. Let $A$ be a unital ring and $n \geq 2$. Let $i, j$ be indices with $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$. For $a \in A$, we let $E_{i,j}(a)$ denote the matrix in $M_n(A)$ whose all diagonal entries are 1, $(i, j)$-th entry is $a$ and the other entries are 0.

Let $G = EL_n(A) \ltimes A^n > A^n = N$. Then we identify $G$ as

$$G \cong \left\{(R, v) : R \in EL_n(A), v \in A^n\right\} \subset EL_{n+1}(A).$$

We also identify $N$ with the additive group of all column vector $v$. Here we abbreviate $(I, v) \in N \subset G$ by omitting $I (= I_n)$.

In the case $A = A_k = \mathbb{Z}[X_1, \ldots, X_k]$, we define the elementary unit matrices as the matrices of the form $E_{i,j}(\pm X_l)$ $(0 \leq l \leq k)$. Here we set $X_0 = 1$. We also consider the case that $G = EL_n(A_k) \ltimes A^n_k > A^n_k = N$. In the case above, we define the finite generating set $F$ as follows: with the above identification $G \subset EL_{n+1}(A_k)$, we let $F$ be the set of all unit elementary matrices in $G$. We also let $F_1 = F \cap N$ and $F_2 = F \setminus F_1$.

3. Proof of Theorem 1.3

We keep the same notation and identifications as in Subsection 2.4 (with $n = 3$). We let $N_1$ be the subgroup of $N(\subset SL_4(A_k))$ of all elements whose $(2, 4)$-th and $(3, 4)$-th entries are 0. Take an arbitrary affine isometric action $\alpha$ on $B$ and fix one norm on $B$ as in Remark 2.5. We decompose $\alpha$ into the linear part $\rho$ and the cocycle part $c$. We also decompose $B$ as $B = B_0 \oplus B_1 = B_0^{o(N)} \oplus B_1^{\rho(N)}$ and obtain the associated decomposition $c = c_0 + c_1$. From the $\rho(G)$- invariance of $B_0$ and $B_1$, each $c_j, j \in \{0, 1\}$ is a $\rho$-cocycle. For any elements $g = (R, 0) \in G$ and $h = v \in N$, $ghg^{-1} = (I, Rv) =: Rv \in N$ holds. In particular, by noting that $\rho|_{N} = \text{id}$ on $B_0$, we have the following equality: for any $R \in SL_3(A_k)$ and $v \in N$, $c_0(Rv) = \rho((R, 0)) c_0(v)$. One can deduce the following observations from the equality above and the cocycle identity for $c_1$:

- The set $c_0(N)$ is bounded (and hence actually 0).
- If $c_1(N_1)$ is bounded, then $c_1(N)$ is bounded.

(The first part follows from the fact that any column vector in $N$ can be written as a sum of three columns whose entries contain at least one 1.)

Proof. (Theorem 1.3) Thanks to the two observations above and Corollary 2.12, it suffices to verify the boundedness of $c_1(N_1)$. We define a finite subset $F_0$ and two subgroups $G_1, G_2$ of $G$ by the following expressions respectively:

\[
\left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & * & 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 & t v' \ 0 \\ 0 & R' \ 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & R' & v' \\ 0 & 0 & 1 \end{pmatrix} \right\}.
\]
Then by applying Lemma 2.11, there must exist a \( B \) in the third expressions, \( R' \) moves all elements in \( EL_2(A_k) \) and \( v' \) moves all elements in \( A_k^2 \). We let \( D = \sup_{s \in F_0} \| c_1(s) \| \). We set \( N_2 \langle \sigma G_2 \rangle \) as the group of all elements in \( G_2 \) with \( R' = I \) and \( L \langle \sigma G_1 \rangle \) as the group of all elements in \( G_1 \) with \( R' = I \). A crucial point is that \( N_1 \) commutes with \( F_0 \). Therefore for any \( h \in N_1 \) and any \( s \in F_0 \), we have the following inequality:

\[
\| \rho(s)c_1(h) - c_1(h) \| = \| c_1(sh) - c_1(h) - c_1(s) \| \leq \| c_1(hs) - c_1(h) \| + \| c_1(s) \| = \| \rho(h)c_1(s) \| + \| c_1(s) \| = 2\| c_1(s) \| \leq 2D.
\]

We set a number \( \mathcal{K} \) as the minimum of the two numbers \( K(G_2; N_2, F_0 \cap G_2, \rho |_{G_2}) \) and \( K(G_1; L, F_0 \cap G_1, \rho |_{G_1}) \). Then from relative \( (T_B) \) of \( EL_2(A_k) \times A_k^2 \rtimes A_k^2 \), \( \mathcal{K} \) is strictly positive. From Lemma 2.10 for any \( x \in c_1(N_1) \) one can choose a \( \rho(N_2) \)-invariant vector \( y \) and a \( \rho(L) \)-invariant vector \( z \) with \( \| x - y \| \leq 4\mathcal{K}^{-1}D \) and \( \| x - z \| \leq 4\mathcal{K}^{-1}D \). Note that \( N_1 \) is obtained by single commutators between \( N_2 \) and \( L \): for any \( h \in N \), there exist \( h_1 \in N_1, h_2 \in N_2, h' \in N_2, \) and \( l \in L \) such that \( h = h_1h_2 \) and \( h_1 = h'lh'^{-1}l^{-1} \). Hence for any \( x \in c_1(N_1) \) and \( h \in N \), the following inequality holds:

\[
\| \rho(h)x - x \| = \| \rho(h'lh'^{-1}l^{-1}h_2)x - x \| \leq 6\| x - y \| + 4\| x - z \| \leq 40\mathcal{K}^{-1}D.
\]

Now suppose that \( c_1(N_1) \) is not bounded. We note that the upper bound of the inequality above is independent of the choices of \( x \in c_1(N_1) \) and \( h \in N \). Therefore one can choose \( x \in c_1(N_1) \) such that \( \| \rho(h)x - x \| < \| x \| \) holds for all \( h \in N \). Then by applying Lemma 2.11 there must exist a non-zero \( \rho(N) \)-invariant vector in \( B_1 = B_\rho(N) \), but it is a contradiction.

4. Proof of Theorem 1.4

We would like to concentrate on investigation for the case of relative \( (T_{\mathcal{H}}) \): the case of relative \( (T_{L_F}) \) directly follows from the original relative property \( (T) \) proved by Y. Shalom [25] and the relative version of \( (ii) \) in Theorem 1.1. We keep the same notation and identifications as in Subsection 2.4 (with \( n = 2 \)).

For any \( B \in \mathcal{H} \) and any isometric representation \( \rho \) on \( B \), one can regard \( \rho \) as a uniformly bounded representation on a Hilbert space \( \mathcal{H} \). The key to proving Theorem 1.4 is the following proposition by J. Dixmier, that states any uniformly bounded representation on a Hilbert space of an amenable group is unitarizable.

**Proposition 4.1.** ([10]) Let \( \Lambda \) be a locally compact group. Suppose \( \Lambda \) is amenable. Then for any uniformly bounded representation \( \rho \) on \( \mathcal{H} \) of \( \Lambda \), there exists an invertible operator \( T \in B(\mathcal{H}) \) such that \( \pi = \text{Ad}(T) \circ \rho \) is a unitary representation. In addition, one can choose \( T \) with \( \| T \|_{B(\mathcal{H})} \| T^{-1} \|_{B(\mathcal{H})} \leq |\rho|^2 \).

**Proof.** (Theorem 1.4 Outlined) For simplicity, we shall show the case \( k = 0 \). Namely, relative property \( (T_{\mathcal{H}}) \) of \( N = \mathbb{Z}^2 \ltimes SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = G \).
Suppose that there exist a ucs Banach space $B \in [\mathcal{H}]$ and an isometric representation $(\rho, B)$ of $G$ such that $\rho$ admits almost invariant vectors in $B^\prime_{\rho(N)}$. We may assume that $B^\prime_{\rho(N)} = B$ because $B^\prime_{\rho(N)} \in [\mathcal{H}]$. Thanks to the amenability of $N$ and Proposition 4.11, we may also assume $(\rho, \mathcal{H})$ is a unitary representation on $N$. We choose any element $x \in S(B)$ and fix it. We let $\delta_x = \sup_{s \in F} \|\rho(s)x - x\|_B$ and $\delta^\ast_x = \sup_{s \in F} \|\rho^1(s)x^* - x^*\|_{B^*}$. Here $x \mapsto x^*$ is the duality mapping defined in Lemma 2.3.

Then from chosen vector $x$ and the duality on $B$, we can construct a spectral measure $\mu = \mu_x$ on the Pontryagin dual $\hat{\mathbb{T}}^2 \cong \mathbb{T}^2 \cong \mathbb{R}/\mathbb{Z}$ as $\delta_x, \delta^\ast_x \to 0$. The method for constructing the measure is similar to one in the original relative (T) argument as in [25] or in its slightly different interpretation [5, Theorem 12.1.10]. Unlike the original case of relative (T), $\mu$ is complex-valued in general. Therefore we need to take the Hahn–Jordan decomposition of $\mu$ and obtain the positive part $\mu_+$. One can establish the following three facts:

- The inequality $\mu_+(\mathbb{T}^2) \geq 1$ holds.
- For any Borel set $D$ being far from the origin 0 of $\mathbb{T}^2$ (in certain quantitative sense), $\mu_+(D) = O(\delta_x \cdot \delta^\ast_x)$ as $\delta_x, \delta^\ast_x \to 0$.
- For any Borel subset $Z \subset \mathbb{T}^2$ and $g \in F_2$, $|\mu_+(gZ) - \mu_+(Z)| = O(\delta_x + \delta^\ast_x)$ as $\delta_x, \delta^\ast_x \to 0$. Here for $g \in SL_2(\mathbb{Z})$, $\hat{g} = (t \cdot g)^{-1}$ and $SL_2(\mathbb{Z})$ naturally acts on $\mathbb{T}^2$.

Let $x \in S(B)$ move almost invariant vectors with $\delta_x \to 0$. Then from (uniform) continuity of the duality mapping, $\delta^\ast_x$ also tends to 0. Hence there must exist some vector $x \in S(B)$ such that the associated positive measure $\mu_+$ has a non-zero value on $\{0\} \subset \mathbb{T}^2$. This contradicts our assumption that $B^\rho(N) = 0$. \(\square\)

We refer to the Appendix for details and a certain quantitative treatment.

5. Reduced cohomology, ultralimit, and Shalom’s machinery

Throughout this section, we let $\Gamma$ be a discrete and finitely generated group and $F$ be a finite generating subset of $\Gamma$. Y. Shalom [26] has defined the following property: an affine isometric action $\alpha$ of $\Gamma$ on a Banach space $B$ is said to be uniform if there exists $\varepsilon > 0$ such that $\inf_{x \in B} \sup_{s \in F} \|\alpha(s)x - x\| \geq \varepsilon$ holds. We note that this definition is independent of the choice of finite generating set $F$. The conception of uniformity of actions is closely related to the closure $\overline{B^1}(\Gamma, \rho)$ of the coboundary $B^1(\Gamma, \rho)$: for any isometric representation $\rho$, a $\rho$-cocycle $c$ is in $\overline{B^1}(\Gamma, \rho)$ if and only if the associated affine action $\alpha = \rho + c$ is not uniform.

**Definition 5.1.** The reduced first cohomology of $\Gamma$ with $\rho$-coefficient is defined as the additive group $\overline{H^1}(\Gamma, \rho) = Z^1(\Gamma, \rho)/\overline{B^1}(\Gamma, \rho)$.

In [26, Theorem 6.1], Y. Shalom has shown the following theorem: Suppose $G$ is a compactly generated topological group. If $G$ fails to have (FH), then there exists a unitary representation $(\pi, \mathcal{H})$ with $\overline{H^1}(G, \pi) \neq 0$. At least in the case of discrete
groups, one can extend this theorem to more general situations. One extension was essentially found by M. Gromov \cite{12}, and his idea is to take a scaling limit.

An ultralimit means a unital, positive and multiplicative *-homomorphism $\omega$: $\ell^\infty(\mathbb{N}) \to \mathbb{C}$ such that for any $(x_n)_{n=0}^\infty$ converging to some element, $\omega: \lim_{n \to \infty} x_n$ holds. Choose any ultralimit $\omega$-lim. Then one can define the ultralimit of Banach spaces $(B_{\omega}, \| \cdot \|_\omega, z_\omega)$ for any sequence $(B_n, \| \cdot \|_n, z_n)_n$ of Banach spaces, norms and base points. Moreover, let $(\alpha_n, B_n)_n$ be a sequence of affine isometric actions of $\Gamma$. If the condition $\sup_{s \in F} \sup_n \| \alpha_n(s) z_n - z_n \| < +\infty$ holds, then we can naturally define the ultralimit of actions $\alpha_\omega$ on $B_\omega$. We refer to Silberman’s website \cite{28} for details of above and for a proof of the following proposition.

**Proposition 5.2.** (proposition of scaling limit) Let $\alpha$ be an affine isometric action of $\Gamma$ on a Banach space $B$. Suppose $\alpha$ is not uniform but has no fixed point. Then there exist a sequence of base points and positive numbers $(z_n, b_n)$ with $\lim_n b_n = +\infty$ such that the ultralimit action $\alpha_\omega$ on $B_\omega = \omega:\lim (B, b_n \| \cdot \|, z_n)$ is uniform.

**Corollary 5.3.** Let $C$ be a class of Banach spaces which is stable under ultralimits. If a finitely generated discrete group $\Gamma$ does not have property $(F_C)$, then there exist $B \in C$ and an isometric representation $\rho$ of $\Gamma$ on $B$ such that $\overline{\mathcal{H}}^1((\Gamma, \rho)) \neq 0$.

We extend the conception of the Shalom property, which is found in \cite{5} Definition 12.1.13).

**Definition 5.4.** Let $B$ be a Banach space and $\Gamma$ be a finitely generated group. A triple of subgroups $(G, H_1, H_2)$ of $\Gamma$ is said to have the Shalom property for $(F_B)$ if all of the following four conditions hold:

1. The group $\Gamma$ is generated by $H_1$ and $H_2$ together.
2. The subgroup $G$ normalizes $H_1$ and $H_2$.
3. The group $\Gamma$ is boundedly generated by $G, H_1$, and $H_2$, namely, there exists $l \in \mathbb{N}$ such that $\Gamma = H_{i(1)} \cdots H_{i(l)} G$, where every $i(1), \ldots, i(l) \in \{1, 2\}$.
4. For both $i \in \{1, 2\}$, $H_i < \Gamma$ has relative $(F_B)$.

Now we shall introduce Shalom’s machinery. We refer to \cite{27} §4 for the original idea for the case that $C = \mathcal{H}$.

**Theorem 5.5.** (Shalom’s Machinery) Let $C$ be a class of superreflexive Banach spaces which is stable under ultralimits. Let $\Gamma$ be a finitely generated group with a finite abelianization. Suppose there exist subgroups $G, H_1,$ and $H_2$ of $\Gamma$ such that $(G, H_1, H_2)$ has the Shalom property for $(F_C)$. Then $\Gamma$ has property $(F_C)$.

**Proof.** Suppose the contrary. Then from Corollary 5.3 there must exist an affine isometric action $\alpha_0$ on some $B_0 \in C$ such that $\alpha_0$ is uniform. For simplicity, we may assume that $B_0$ is uc. Fix a finite generating set $F$ of $\Gamma$. We set $\mathcal{A}$ as the class of all pairs $(\alpha, E)$ of an affine isometric action and a uc Banach space with the following two conditions: first, for any $x \in E$, $\sum_{s \in F} |\alpha(s) x - x|_E \geq 1$ holds. Second, for all $0 < \varepsilon < 2$, the value of the modulus of convexity of $E$ at $\varepsilon$ is not less than that of $B_0$. \hfill \Box
(We refer to Definition 2.2). We note that this class $\mathcal{A}$ is non-empty. Furthermore, thanks to [11 §2, Theorem 4.4], $\mathcal{A}$ is stable under ultralimits.

Next we define a number $D$ as $\inf \{ \| x^1 - x^2 \| : (\alpha, E) \in \mathcal{A} \}$. Here for $i \in \{1, 2\}$, $x^i$ moves through all $\alpha(H_i)$-fixed point in $E$. We observe that the definition above makes sense with the aid of condition (4). By taking an ultralimit, one can show that $D$ is actually a minimum. Let $x^1_\infty$ and $x^2_\infty$ be vectors which attain the minimum $D$. Also let $(\alpha_\infty, E_\infty) \in \mathcal{A}$ be the associated affine action and $\rho_\infty$ be the linear representation for $\alpha_\infty$.

Decompose the action $\alpha_\infty$ into $\alpha_\infty^{\text{triv}}$ and $\alpha_\infty'$, where the former takes values in $E_{\rho_\infty(\Gamma)}$ and the latter tales values in $E_{\rho_\infty', \rho_\infty(\Gamma)}$. Then from the strict convexity of $E_\infty$ and condition (2), one can conclude that the $\alpha_\infty'(G)$-orbit of $E_{\rho_\infty', \rho_\infty(\Gamma)}$ component of each $x^i_\infty$, $i \in \{1, 2\}$ is one point (otherwise $E_{\rho_\infty', \rho_\infty(\Gamma)}$ must contain a non-zero $\rho_\infty(\Gamma)$-invariant vector) and hence bounded. Note that $\alpha_\infty^{\text{triv}}(G)$-orbits of the $E_{\rho_\infty(\Gamma)}$ components are also bounded because $\Gamma$ has a finite abelianization. Therefore in particular, $\alpha_\infty(\Gamma)$-orbit of $x^1_\infty$ must be bounded with the use of condition (3). However it contradicts the definition of $\mathcal{A}$.

Proof. (Theorem 1.2) Let $\Gamma = SL_n(A_k)$, $G \cong SL_{n-1}(A_k)$, and $H_1, H_2 \cong A_k^{n-1}$. Here in $\Gamma$ we realize $G$ as the $((1-(n-1)) \times (1-(n-1)))$-th parts, realize $H_1$ as the $((1-(n-1)) \times n)$-th parts, and realize $H_2$ as the $(n \times (1-(n-1)))$-th parts. One can directly check that $\Gamma$ has the trivial abelianization. We claim that $(G, H_1, H_2)$ has the Shalom property for $(F_C)$. Indeed, condition (1) and (2) are confirmed directly, and condition (4) follows from Theorem 1.3 and Theorem 1.4. Condition (3) is also ascertained from a deep theorem of L. Vaserstein [30].

Thanks to Theorem 5.5 it suffices to verify that $\mathcal{C}$ is stable under ultralimits. In the case that $\mathcal{C} = L^p$, it follows from [14] (we also refer to [17 §15,Theorem 3] and [11 §2]). In the case that $\mathcal{C} = [\mathcal{H}]$, it is not stable. However for any $M \geq 1$, the following class $\mathcal{B}_M$ is stable under ultralimits: we define $\mathcal{B}_M$ as the class of all elements $B$ in $[\mathcal{H}]$ which have compatible Hilbert norms with the norm ratio $\leq M$.

By noticing that $[\mathcal{H}] = \bigcup_{M \geq 1} \mathcal{B}_M$, one can accomplish the conclusion.

The author does not know whether the assertion of Theorem 1.2 is satisfied for the noncommutative universal lattice $EL_n(\mathbb{Z}[X_1, X_2, \ldots, X_k])$ ($n \geq 3$). In the case above, although most of the ingredients are still valid, the bounded generation property may fail. We note that M. Ershov and A. Jaikin-Zapirain [11] have proved property (T) of noncommutative universal lattices.

6. Quasi-actions and property (FF$_B$)

**Definition 6.1.** Let $B$ be a Banach space and $\Gamma$ be a group.

- A map $\beta$ from $\Gamma$ to the set of all affine isometries on $B$ is called a **quasi-action** if the expression $\sup_{g,h \in \Gamma} \sup_{x \in B} \| \beta(gh)x - \beta(g)\beta(h)x \|$ is finite.

- Let $\rho$ be an isometric representation. A map $b$ from $\Gamma$ to $B$ is called a **quasi-$\rho$-cocycle** if the expression $\sup_{g,h \in \Gamma} \| b(gh) - b(g) - \rho(g)b(h) \|$ is finite.
Remark 6.2. In the definition of quasi-actions, one can decompose the map $\beta$ into the linear part $\rho$ and the transition part $b$, namely, $\beta(g)x = \rho(g)x + b(g)$ for any $g \in \Gamma$ and $x \in B$. Then from a standard argument, the map $\beta$ is a quasi-action if and only if $\rho$ is a group representation and $b$ is a quasi-$\rho$-cocycle.

Next we define property (FF$_B$) as follows. We mention that the original terminology in [20] for (FF$_H$) is property (TT). We use the terminology (FF$_B$) because this property is more related to (F$_B$) than to (T$_B$).

Definition 6.3. Let $B$ be a Banach space.

- A pair $(G > N)$ of groups is said to have relative property (FF$_B$) if for any quasi-action on $B$, some (or equivalently, any) $N$-orbit is bounded. This is equivalent to the condition that for any isometric representation $\rho$ of $G$ on $B$ and any quasi-$\rho$-cocycle $b$, $\|b(N)\|$ is bounded.
- A group $\Gamma$ is said to have property (FF$_B$) if $(\Gamma > \Gamma)$ has relative (FF$_B$).
- Suppose $B$ is superreflexive in addition. Then a group $\Gamma$ is said to have property (FF$_B$) for non-trivial linear part if the following condition holds: for any isometric representation $\rho$ of $\Gamma$ on $B$ and any quasi-$\rho$-cocycle $b$, $b_1(\Gamma)$ is bounded. Here we decompose $b$ as $b_0 + b_1$ such that $b_0$ takes values in $B_0 = B^{\rho(\Gamma)}$ and $b_1$ takes values in $B_1 = B'_{\rho(\Gamma)}$.

By observing our proof of Theorem 1.3, one can extend the argument to the case that $c$ is a quasi-$\rho$-cocycle. Thus one obtains the following theorem.

Theorem 6.4. With the same notation as one in Theorem 1.3, let $B$ be any superreflexive Banach space. If $EL_2(A_k) \ltimes A_k^2$ has relative (T$_B$), then $SL_3(A_k) \ltimes A_k^3$ has relative (FF$_B$).

We define the following property to prove Theorem 1.3.

Definition 6.5. With the same notation as in Definition 5.4, the triple of subgroups $(G, H_1, H_2)$ of $\Gamma$ is said to have the Shalom property for (FF$_B$) if the following four conditions (1), (2), (3), and (4') hold: condition (1), (2), and (3) are same as in Definition 5.4. And we define a new condition (4') by replacing relative (F$_B$) with relative (FF$_B$) in condition (4) in Definition 5.4.

Proposition 6.6. Let $B$ be a superreflexive Banach space and $\Gamma$ be a group. Suppose $\Gamma$ has property (T$_B$) and there exist subgroups $G, H_1$, and $H_2$ of $\Gamma$ such that $(G, H_1, H_2)$ has the Shalom property for (FF$_B$). Then $\Gamma$ has property (FF$_B$) for non-trivial linear part.

Proof. For simplicity, we assume that $B$ is ucus. Let $\rho$ be an arbitrary isometric representation of $\Gamma$ on $B$ and $b$ be an arbitrary quasi-$\rho$-cocycle. We decompose $B = B_0 \oplus B_1 = B^{\rho(\Gamma)} \oplus B'_{\rho(\Gamma)}$ and $b = b_0 + b_1$. We set $C = \sup_{g,h \in \Gamma} \|b_1(g) + \rho(g)b_1(h) - b_1(gh)\|$. From condition (4') of the Shalom property for (FF$_B$), there exists a positive number $C'$ such that for any $h \in H_1 \cup H_2$, $\|b_1(h)\| \leq C'$ holds. By making use of
condition (2), one obtains the following inequality: for any $g \in G$ and $h \in H_1 \cup H_2$,
\[
\|\rho(h) b_1(g) - b_1(g)\| \leq \|b_1(hg) - h b_1(g)\| + \|b_1(g)\| + C
\]
\[
\leq \|b_1(hg^{-1}h)\| + C + C' = \|b_1(gg^{-1}hg) - b_1(g)\| + C + C'
\]
\[
\leq \|\rho(g) b_1(g^{-1}h)\| + 2C + C' \leq 2(C + C').
\]

Let $S$ be any finite subset of $\Gamma$. From the inequality above and condition (1),
\[
\sup_{s \in S} \|\rho(s)x - x\| \text{ is bounded independently of the choice of } x \in b_1(G).
\]

Now suppose that $b_1(G)$ is not bounded. Then $\rho$ must admit almost invariant vectors in $B_1$, but it contradicts property $(T_B)$ of $\Gamma$. Therefore $b_1(G)$ is bounded. Finally, one obtains the boundedness of $b_1(\Gamma)$ through use of the bounded generation (condition (3)). □

Proof. (Theorem 1.5) The conclusion follows from Theorem 1.2, Theorem 6.4, and Proposition 6.6. □

The author does not know whether similar boundedness property holds for trivial linear part.

Remark 6.7. Recently, N. Ozawa has strengthened property (TT)(= property (FFH)) and defined the concept of property (TTT). In [24], he has proved that $\text{EL}_2(A_k) \ltimes A_k \bowtie A_k$ has relative property (TTT), where $A_k$ means $\mathbb{Z}[X_1, \ldots, X_k]$. Hence from Proposition 6.6 it is established that the universal lattice $SL_3(A_k)$ has (FFH) for non-trivial linear part. Ozawa exploits Fock Hilbert spaces and positive definite kernels to obtain the relative property (TTT) in above. Therefore it may remain unknown whether $SL_3(A_k)$ possesses property (FFC) for non-trivial linear part or (more weakly, ) property (FC). Here $C$ stands for $L^p$ ($2 < p < \infty$) or $[\mathcal{H}]$.

7. Applications

7.1. Actions on the circle. Let $S^1$ be the unit circle in $\mathbb{R}^2$ and identify $S^1$ with $[-\pi, \pi)$. We denote by $\text{Diff}_+(S^1)$ the group of orientation preserving group diffeomorphisms of $S^1$.

Definition 7.1. Let $\alpha > 0$ be a real number. The group $\text{Diff}_+^{1+\alpha}(S^1)$ is defined as the class of all orientation preserving group diffeomorphisms $f$ of $S^1$ such that $f'$ and $(f^{-1})'$ are Hölder continuous with exponent $\alpha$. Here a function $g \in \text{Diff}_+(S^1)$ is said to be Hölder continuous with exponent $\alpha$ if
\[
\|g\|_\alpha = \sup_{\theta_1 \neq \theta_2} \frac{|g(\theta_1) - g(\theta_2)|}{|\theta_1 - \theta_2|^\alpha} < \infty
\]
holds.

A. Navas [22] has shown the following theorem: For any discrete group $\Gamma$ with property $(T)$, every homomorphism from $\Gamma$ into $\text{Diff}_+^{1+\alpha}(S^1)$ has finite image, for any $\alpha > 1/2$. He has also noted in [23, Appendix] that his theorem can be extended to general $L^p$ cases. (See also [2, §1.b].)
**Theorem 7.2.**  (23) Let $1 < p < \infty$ and $\Gamma$ be a discrete group with property (F$_{L^p}$). Then for any $\alpha > 1/p$, every homomorphism $\Gamma \to \operatorname{Diff}^{1+\alpha}(S^1)$ has finite image.

For the proof, Navas generalizes the argument in [22] by using the Liouville $L^p$ cocycle of $\operatorname{Diff}^{1+\alpha}(S^1)$ on $L^p(S^1 \times S^1)$, that is,

$$c_p(g^{-1})(\theta_1, \theta_2) = \frac{|g'(\theta_1)g'(\theta_2)|^{1/p}}{|2 \sin ((g(\theta_1) - g(\theta_2))/2)|^{2/p}} - \frac{1}{|2 \sin ((\theta_1 - \theta_2)/2)|^{2/p}}.$$

**Proof.** (Corollary 1.4) It is straightforward from Theorem 7.2 and Theorem 1.2. □

**7.2. Bounded cohomology.** We would like to refer to Monod’s book [20] for details on bounded cohomology. Throughout this subsection, we let $\Gamma$ be a discrete group. In the definitions below, we write $H_{\bullet}^A(\Gamma; B, \rho)$ for short.

**Definition 7.3.** (20) Let $(B, \rho)$ be a Banach $\Gamma$-module, namely, $B$ be a Banach space and $\rho$ be an isometric representation of $\Gamma$ on $B$.

- The bounded cohomology $H_{\bullet}^A(\Gamma; B, \rho)$ of $\Gamma$ with coefficients in $(B, \rho)$ is defined as the cohomology of the following cochain complex:

  $$0 \rightarrow \ell^\infty(\Gamma, B)^{\rho(\Gamma)} \rightarrow \ell^\infty(\Gamma^2, B)^{\rho(\Gamma)} \rightarrow \ell^\infty(\Gamma^3, B)^{\rho(\Gamma)} \rightarrow \cdots$$

- The comparison map is the collection of linear maps $\Psi_{\bullet}: H_{\bullet}^A(\Gamma; B, \rho) \rightarrow H_{\bullet}(\Gamma; B, \rho)$, where the maps above are naturally determined by the complex inclusion.

We note that in general the comparison map is neither injective nor surjective.

**Proof.** (Corollary 1.7) From an argument similar to one in [20, Proposition 13.2.5], one can show the following: The comparison map in degree 2 $\Psi^2: H^2_{\bullet}(\Gamma; B) \rightarrow H^2(\Gamma; B)$ is injective for any isometric representation without trivial part $\rho$ on $B$ if $\Gamma$ has (FF$_B$) for non-trivial linear part. □

**Appendix A. Relative Kazhdan constant for property (T) for uniformly bounded representations**

We define the Kazhdan constant for relative property (T$_{[\mathcal{H}]}$). For quantitative treatments, it is more convenient to focus on a Hilbert space $\mathcal{H}$. Therefore we define the extension of the Kazhdan constant in terms of uniformly bounded representations on $\mathcal{H}$.

**Definition A.1.** Let $\Gamma \triangleright N$ be a pair of groups, and $F$ be a compact subset of $\Gamma$. For $M \geq 1$, we define $\mathcal{A}_M$ as the class of all pairs $(\rho, \mathcal{H})$ with $|\rho| \leq M$. We define the generalized relative Kazhdan constant for uniformly bounded representations by

$$\overline{K}(\Gamma; N, F; M) = \inf_{(\rho, \mathcal{H}) \in \mathcal{A}_M} \inf_{x \in S(\mathcal{H}, \rho)} \sup_{s \in F} \|\rho(s)x - x\|_{\mathcal{H}}.$$

Our proof of Proposition 1.8 is a development of Kassabov’s work [13], originally by M. Burger [6] and Y. Shalom [25]. We make use of the following quantitative version of Lemma 2.4.
Lemma A.2. ([4 Proposition A.5], modified) Let $B$ be us. Suppose $0 < \kappa < 2$. Then for all $x, y \in S(B)$ with $\|x - y\| \leq \kappa$, the inequality $\|x^* - y^*\| \leq 2 \|x - y\| / \kappa$ holds.

Let $\Gamma$ be a group and $M \geq 1$. For any $(\rho, \mathcal{H}) \in \mathcal{A}_M$, we define the norm $\|\cdot\|_{\rho}$ on $\mathcal{H}$ as the dual norm of the following norm $\|\cdot\|_{\rho^*}$: for $\phi \in \mathcal{H}^*$, $\|\phi\|_{\rho^*} := \sup_{y \in \Gamma} \|\rho^i(y)\phi\|_{\mathcal{H}^*}$.

This norm $\|\cdot\|_{\rho}$ satisfies the following three properties: first, $\|\cdot\|_{\rho}$ is compatible with $\|\cdot\|_{\mathcal{H}}$ with the norm ratio $M$. Second, $\rho$ is isometric with respect to $\|\cdot\|_{\rho}$. Third, $(\mathcal{H}, \|\cdot\|_{\rho})$ is us. Indeed, thanks to Lemma 2.3 one has that for any $\tau > 0$, the inequality $\rho_{\|\cdot\|_\rho}(\tau) \leq \sqrt{1 + M^2 \tau^2} - 1 \leq M^2 \tau^2 / 2$ holds.

Proof. (Proposition 1.3) We stick to the notation and the identifications in Subsection 2.4 (with $n = 2$). Let $\varepsilon > 0$. Suppose that there exists $(\rho, \mathcal{H}) \in \mathcal{A}_M$ such that $\rho$ admits a non-zero vector $x$ in $\mathcal{H}_{\rho(N)}$ which satisfies $\sup_{s \in F} \|\rho(s)x - x\|_{\mathcal{H}} \leq \varepsilon \|x\|_{\mathcal{H}}$.

We may assume that $\mathcal{H}_{\rho(N)} = 0$. For this $(\rho, \mathcal{H})$, we take the us norm $\|\cdot\|$ defined in the paragraph above. Thus by applying Lemma A.2 we can assume that there exists $x \in \mathcal{H}$ with $\|x\|_{\rho} = 1$ such that

$$\sup_{s \in F} \|\rho(s)x - x\|_{\rho} \leq M \varepsilon \quad \text{and} \quad \sup_{s \in F} \|\rho^i(s)x^* - x^*\|_{\rho s} \leq 4M^3 \varepsilon.$$

Thanks to Dixmier’s unitarization, we have an invertible operator $T \in \mathbb{B}(\mathcal{H})$ with $\|T\|_{\mathbb{B}(\mathcal{H})} \leq M^2$ such that $\pi := \text{Ad}(T) \circ \rho |_N$ is usary. Let $\hat{N}$ denote the Pontrjagin dual of $N$. By general theory of Fourier analysis, one obtains a standard unital $\ast$-hom $\sigma : C(\hat{N}) \to \mathbb{B}(\mathcal{H})$ from the usary operators $\pi(N)$. From Riesz–Markov–Kakutani theorem, one obtains the complex-valued regular Borel measure $\mu$ on $\hat{N}$ satisfying the following: for any $f \in C(\hat{N})$, $\int_{\hat{N}} f d\mu = \langle T^{-1} \sigma(f)T, x^* \rangle$. (We note that $T = I$ in our proof of Theorem 1.4.) We take the Jordan decomposition of $\text{Re}\mu = \mu_+ - \mu_-$. Here $\mu_+ \perp \mu_-$ (this means they are singular) and both of them are positive regular Borel measure. Then the inequality $\mu_+(\hat{N}) \geq 1$ holds.

First, we discuss the case that $k = 0$. We take the following well known decomposition of $\hat{N} = T^2 \cong \left\{ \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) : t_1, t_2 \in [-\frac{1}{2}, \frac{1}{2}] \right\}$:

- $\{0\}$, $D_0 = \{ |t_1| \geq 1/4 \text{ or } |t_2| \geq 1/4 \}$,
- $D_1 = \{ |t_2| \leq |t_1| < 1/4 \text{ and } t_1t_2 > 0 \}$, $D_2 = \{ |t_1| < |t_2| < 1/4 \text{ and } t_1t_2 \geq 0 \}$,
- $D_3 = \{ |t_1| \leq |t_2| < 1/4 \text{ and } t_1t_2 < 0 \}$, $D_4 = \{ |t_2| \leq |t_1| < 1/4 \text{ and } t_1t_2 \leq 0 \}$.

We consider the natural $SL_2(\mathbb{Z})$-action on $T^2$ defined as follows: for any $g \in SL_2(\mathbb{Z})$, the action map $\hat{g}$ of $g : t \mapsto \hat{g} t$ is the left multiplication of the matrix $\hat{g} = (^tg)^{-1}$. This action naturally induces the $SL_2(\mathbb{Z})$-action on $C(T^2)$ as $\hat{g} f(t) = f(\hat{g} t)$. Then one can check the following equality: for any $g \in SL(2, \mathbb{Z})$ and any $f \in C(T^2)$, $\sigma(\hat{g} f) = T \rho(g) T^{-1} \sigma(f) T \rho(g^{-1}) T^{-1}$. With some calculation, one can also obtain the following two estimations:

- The inequality $\mu_+(D_0) \leq 4M^7 \varepsilon^2$ holds.
- For any Borel subset $Z \subset T^2$ and any $g \in F_2(\subset SL_2(\mathbb{Z}))$, the inequality $|\mu_+(\hat{g} Z) - \mu_+(Z)| \leq 5M^6 \varepsilon$ holds.
Indeed, for instance, the former inequality follows from the argument below. For $i = 1, 2$, set $D_0^i = \text{supp} \mu_+ \cap \{|t_1| \geq 1/4\} \subset D_0$. By approximating (pointwise) $\chi_{D_0^i}$ by continuous functions and obtaining an associated projection $P \in \mathcal{B}(\mathcal{H})$, one can make estimate as follows: for each $i \in \{1, 2\}$,

$$2 \mu_+(D_0^i) \leq \left| \int_{D_0^i} (1 - z_i)^2 \, d\mu \right| = \left| \int_{\mathbb{T}^2} (1 - z_i) \chi_{D_0^i} (1 - z_i) \, d\mu \right|$$

$$= \left| \left\langle T^{-1} \sigma (1 - z_i)^* P \sigma (1 - z_i) \right\rangle \right|$$

$$= \left| \left\langle T^{-1} (I - T \rho(h_i)T^{-1})^* P (I - T \rho(h_i)T^{-1}) \right\rangle \right|$$

$$= \left| \left\langle (I - \rho(h_i^{-1})) T^{-1} \right\rangle \left\langle (I - \rho(h_i)) \right\rangle \right| (\text{Recall Ad}(T) \circ \rho |_N \text{ is unitary}),$$

$$\leq \left\| T^{-1} \right\|_\rho \left\| x - \rho^j(h_i)x \right\|_{\rho^j} \left\| x - \rho^j(h_i)x \right\|_{\rho^j} \leq 4M^7 \varepsilon^2.$$

Here for $i \in \{1, 2\}$, $z_i \in C(\mathbb{T}^2)$ means $t \mapsto e^{2\pi \sqrt{-1} t_i}$ ($t_i$ is the $i$-th component of $t \in \mathbb{T}^2$) and $h_i \in F_1$ means $E_{i,3}(1)$ as in Subsection 2.4 In above, we mention that for $V \in \mathcal{B}(\mathcal{H})$, $V^*$ means the adjoint operator of $V$.

Thanks to these two estimations, one can verify $\mu_+(D_1) < 5M^6 \varepsilon + 4M^7 \varepsilon^2$ for $1 \leq i \leq 4$. (For instance, $\mathcal{g}_{1,-}(D_1 \cup D_2) \subset D_2 \cup D_0$, where $\mathcal{g}_{1,-} = E_{1,2}(-1).$) Hence the inequality $\mu_+(\mathbb{T}^2 \setminus \{0\}) \leq 20M^6 \varepsilon + 20M^7 \varepsilon^2$ holds. If $\varepsilon \leq (21M^6)^{-1}$, then there must exist a non-zero $\rho(N)$-invariant vector. It is a contradiction.

For the general case, let us recall Kassabov’s argument in [15]. We identify $\hat{\mathbb{A}}_k$ with the set of all formal power series of variables $X_{i}^{-1}$ $(1 \leq i \leq k)$ over $\hat{\mathbb{Z}} \cong \mathbb{T}$. Here the pairing is defined by

$$\langle aX_{1\cdot l} \cdots X_{k\cdot l} | \phi X_{1\cdot j}^{-1} \cdots X_{k\cdot j}^{-1} \rangle = \phi(a) \delta_{i_1, j_1} \cdots \delta_{i_k, j_k}.$$

We define the valuation $v$ on $\hat{\mathbb{A}}_k$ as the minimum of the total degrees of all terms. Here we naturally define $v(0) = +\infty$. We decompose $\hat{\mathbb{N}} \setminus \{0\} = A_k^2 \setminus \{0\}$ as follows:

$$A = \{(\chi_1, \chi_2) : v(\chi_1) > v(\chi_2) > 0\}, \quad B = \{(\chi_1, \chi_2) : v(\chi_1) = v(\chi_2) > 0\},$$

$$C = \{(\chi_1, \chi_2) : v(\chi_2) > v(\chi_1) > 0\}, \quad D = \{(\chi_1, \chi_2) : v(\chi_1)v(\chi_2) = 0\}.$$

Then from an argument similar to one in [15], we have the following inequalities: $\mu_+(A) \leq \mu_+(D) + 5(k + 1)M^6 \varepsilon$, $\mu_+(B) \leq \mu_+(D) + 5kM^6 \varepsilon$, and $\mu_+(C) \leq \mu_+(D) + 5(k + 1)M^6 \varepsilon$. We naturally define the restriction map $\text{res} : \hat{\mathbb{N}} \to \mathbb{Z}^2$ and obtain that $\mu_+(D) = \mu_+(\hat{\mathbb{N}} \setminus \text{res}^{-1}\{0\}) \leq 20M^6 \varepsilon + 20M^7 \varepsilon^2$. Finally, by combining these inequalities we conclude that

$$\mu_+(\hat{\mathbb{N}} \setminus \{0\}) \leq (15k + 90)M^6 \varepsilon + 80M^7 \varepsilon^2.$$

Hence in particular $\varepsilon$ must be more than $(15k + 100)^{-1}M^{-6}$. \[\square\]
References

[1] A.G. Aksoy, and M.A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, 1990

[2] U. Bader, A. Furman, T. Gelander and N. Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Math., **198** (2007), no.1, 57–105

[3] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, Vol. 11, Cambridge University Press, Cambridge, 2008

[4] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Vol. 1, Amer. Math. Sci, Providence, RI, 2000

[5] N. Brown and N. Ozawa, *C*-algebras and Finite-Dimensional Approximations, Graduate Studies in Mathematics, 88, Amer. Math. Sci, Providence, RI, 2008

[6] M. Burger, *Kazhdan constants for SL(3, Z)*, J. reine angew. Math., **413** (1991), 36–67

[7] M. Burger and N. Monod, *Continuous bounded cohomology and application to rigidity theory*, Geom. Funct. Anal., **12**(2) (2002), 219–280

[8] D. Carter and G. Keller, *Bounded elementary generation of SL_n(O)*, Amer. J. Math., **105**(1983), 673–687

[9] P. Delorme, *1-cohomologie des représentations unitaires des groupes de Lie semisimples et résolubles. Produits tensoriels continus de représentations*, Bull. Soc. Math. France **105** (1977), 281–336

[10] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications* Acta Sci. Math. Szeged. **12** (1950). Leopoldo Fejér et Frerico Riesz LXX anno natis dedicatus, Pars A, 213–227

[11] M. Ershov and A. Jaikin-Zapirain, *Property (T) for noncommutative universal lattices*, arXiv:0809.4095

[12] M. Gromov, *Random walks in random groups*, Geom. Funct. Anal., **13** (2003), 73–148

[13] A. Guichardet, *Sur la cohomologie des groupes topologiques. II*, Bull. Sci. Math., **96** (1972), 305–332

[14] S. Heinrich, *Ultraproduct in Banach space theory*, J. reine angew. Math., **313** (1980), 72–104

[15] M. Kassabov, *Universal lattices and unbounded rank expanders*, Invent. Math., **170** (2007), no. 2, 297–326

[16] D. A. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Anal. Appl., **1** (1967), 63–65

[17] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, **88**, 1974

[18] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke. Math. J., **143** (2008), 559–602

[19] V. Lafforgue, *Propriété (T) renforcée banachique et transformation de Fourier rapide*, preprint

[20] N. Monod, *Continuous bounded cohomology of locally compact groups*, Springer Lecture notes in Mathematics, 1758, 2001

[21] N. Monod and Y. Shalom, *Cocycle superrigidity and bounded cohomology for negatively curved space*, J. Differ. Geom., **67** (2004), 395–455

[22] A. Navas, *Actions de groupes de Kazhdan sur le cercle*, Ann. Sci. École Norm. Sup., **35** (2002), 749–758

[23] A. Navas, *Reduction of cocycles and groups of diffeomorphisms of the circle*, Bull. Belg. Math. Soc. Simon Steven **13** (2006), no. 2, 193–205

[24] N. Ozawa, *Quasi-homomorphism rigidity with noncommutative targets*, preprint

[25] Y. Shalom, *Bounded generation and Kazhdan’s property (T)*, Inst. Hautes Études Sci. Publ. Math., **90** (1999), 145–168

[26] Y. Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math., **141**(1) (2000), 1–54
[27] Y. Shalom, *The algebraization of Kazhdan’s property (T)*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich (2006), 1283–1310
[28] L. Silberman, *Scaling limit*. Note available at Silberman’s website
[29] A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izv., 11 (1977), 221–238
[30] L. Vaserstein, *Bounded reduction of invertible matrices over polynomial ring by addition operators*, preprint
[31] D. Witte, *Arithmetic groups of higher Q-rank cannot act on 1-manifolds*, Proc. Amer. Math. Soc., 122 (1994), 333–340
[32] G. Yu, *Hyperbolic groups admit proper affine isometric actions on ℓp-spaces*, Geom. Funct. Anal., 15 (2005), 1144–1151

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo, 153-8914, Japan

e-mail: mimurac@ms.u-tokyo.ac.jp