A REMARK ON THE NUMBER OF INVISIBLE DIRECTIONS FOR A SMOOTH RIEMANNIAN METRIC

MISHA BIALY

Abstract. In this note we give a construction of a smooth Riemannian metric on $\mathbb{R}^n$ which is standard Euclidean outside a compact set $K$ and such that it has $N = n(n + 1)/2$ invisible directions, meaning that all geodesics lines passing through the set $K$ in these directions remain the same straight lines on exit. For example in the plane our construction gives three invisible directions. This is in contrast with billiard type obstacles where a very sophisticated example due to A.Plakhov and V.Roshchina gives 2 invisible directions in the plane and 3 in the space.

We use reflection group of the root system $A_n$ in order to make the directions of the roots invisible.

1. THE PROBLEM OF INVISIBILITY

Consider a smooth Riemannian metric $g$ on $\mathbb{R}^n$ which is supposed to be standard Euclidean outside a compact set $K$. Geodesics of the metric outside the set $K$ are straight lines and are deformed somehow inside $K$. Following [7], we say that the obstacle $K$ is invisible in the direction $v$ if every geodesic in the direction $v$ passing the obstacle remains the same straight line. This direction $v$ is called the direction of invisibility in this case. It is important question how many invisible direction can exist for a non-flat smooth Riemannian metric. It was shown in [3] basing on [2] and generalizing previous results [1],[5] that the invisibility in all directions implies that the metric is isometric to Euclidean one. This is the so called lens rigidity phenomena (see also [8] for further developments). It is a natural question to ask how large the set of invisible direction can be. In particular can it be large or even infinite. In [7] this question is studied for an analogous model of perfect reflections. A very sophisticated construction of two invisible directions in the plane and three in the space is given in [7]. On the other hand there are non-smooth examples of Riemannian metrics with singularities which are perfect lenses (see [6] for further references).

Our remark is that for the smooth case one can construct a Riemannian metric with $N = n(n + 1)/2$ invisible directions in $\mathbb{R}^n$.

Theorem 1.1. There exists a family of smooth non-flat Riemannian metrics $g$ on $\mathbb{R}^n$ which are Euclidean outside a compact set $K$ and having $N = n(n + 1)/2$ invisible directions.

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The idea is that \( N \) is the number of components of the metric tensor and also is the number of positive roots of the root system \( \Lambda_n \). It is not clear if other reflection groups can be used in a similar manner.

**Remark 1.** A similar problem for conformally flat metrics can be posed also. Our construction in this case gives only one invisible direction. Analogously, one can construct metrics in the diagonal form with \( n \) invisible directions.

**Remark 2.** It is worth mentioning that there exist smooth Finsler non-flat metrics which are compactly supported and have all the directions invisible. This is a very well known observation related to Hopf rigidity. These metrics can be constructed by the action of small compactly supported symplectic diffeomorphism of \( T^*\mathbb{R}^n \) on the Lagrangian foliation corresponding to Minkowskii metric.

**Remark 3.** One can use this construction on other manifolds also. The most natural is to implant such a metric into a small ball of flat torus. Then the resulting geodesic flow has \( \frac{n(n+1)}{2} \) invariant Lagrangian torii. It would be interesting to understand geometry and dynamics of these examples further.

### 2. Using generating functions

In this section we use generating functions to create Lagrangian submanifolds in the energy level of the Riemannian metric in question.

Recall that the root system \( \Lambda_n \) can be realized as a set of the integer vectors \( e_i - e_j \) (of the length \( \sqrt{2} \)) in \( \mathbb{R}^{n+1} \), where \( \mathbb{R}^n \) is viewed as hyperplane of \( \mathbb{R}^{n+1} \) defined as \( \{x_1 + \ldots + x_{n+1} = 0\} \).

For our purposes it will be convenient to arrange the roots in the following order. Let \( v_1, \ldots, v_N \) are such that first \( n \) are defined by \( v_i = e_i - e_{n+1} \) and the rest are \( e_i - e_j \) for \( 1 \leq i < j \leq n \). So there are \( N = n(n+1)/2 \) of them and together with their negatives they form all the roots. Notice that \( (v_1, \ldots, v_n) \) form a basis of \( \mathbb{R}^n \) and the rest are are their differences \( v_i - v_j, \ 1 \leq i < j \leq n \).

For every \( i = 1, \ldots, N \) consider the Lagrangian sections \( L_i \) of \( T^*\mathbb{R}^n \) equipped with the standard symplectic structure which are defined by the generating function

\[
S_i(x) = (v_i, x) + \epsilon \phi_i(x), \quad L_i = \{ p = \nabla S_i = v_i + \epsilon \nabla \phi_i \},
\]

where \( \phi_i \) are any smooth functions on \( \mathbb{R}^n \) with the support in a ball \( B \). Here and later we denote by \( (,\) the standard scalar product. It is a well known fact that any root system determines the scalar product uniquely. Therefore we have the following

**Theorem 2.1.** If \( \epsilon > 0 \) is small enough then there exists and unique Riemannian metric \( g \) on \( \mathbb{R}^n \) such that all \( L_i \) lie in the level \( \{h = 1\} \) of the corresponding Hamiltonian function \( h \). Moreover this metric is standard Euclidean outside \( B \).

**Proof.** Let the Riemannian metric and the Hamiltonian function are given by the matrices \( G \) and \( H \), \( G = H^{-1} \):

\[
g = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j = (G dx, dx) ;
\]

\[
h = \frac{1}{2} \sum_{i,j=1}^{n} h_{ij} p_i p_j = \frac{1}{2} (H p, p).
\]
For a given choice of the functions $\phi_i, i = 1, \ldots, N$, the requirements $L_i \in \{ h = 1 \}$ form a linear system of $N$ inhomogeneous equations on the $N$ unknown coefficients $h_{ij}$. For $\epsilon = 0$ this system reads simply that the vectors $v_i$ have all length $\sqrt{2}$, which has unique solution namely the standard Euclidean metric, i.e $G = H = Id$. Therefore the determinant of the system is not zero and then for $\epsilon$ small enough there is a unique solution also which is a positive definite form. □

Remark 4. In principle one could find the solution $h_{ij}$ of this linear system explicitly in terms of derivatives of the functions $\phi_i$, thus determining the metric coefficients (see also Section 4).

Since by Theorem 2.1 all Lagrangian submanifolds $L_i$ lie in an energy level of $h$, then it follows that $L_i$ are invariant under the geodesic flow. Thus every geodesic straight line of the metric $g$ which enters the ball $B$ in one of the directions $v_i, i = 1, \ldots, N$ leaves the ball by a parallel straight line. Next we can use a simple symmetry idea which makes this line to be identical with the initial one.

3. Symmetrizing the metric

The symmetrization procedure is based on the following obvious

Lemma 3.1. Suppose a Riemannian metric $g$ is invariant under the reflection $s_v$, which is the reflection with respect to the hyperplane $P_v$ orthogonal to $v$ in $\mathbb{R}^n$. Then any geodesic which crosses $P_v$ orthogonally is symmetric with respect to $P_v$.

As a corollary we have the following. Take the metric $g$ constructed in a previous section, where the ball $B$ lies in one halfspace with respect to $P_v$, and reflect the metric to the other halfspace of $P_v$. By the lemma one gets a new metric supported on $B \cup s_v(B)$ with the property that the direction $v$ is not visible.

Using this observation we proceed as follows. Consider the Weyl group $W$ of the root system $A_n$ generated by the reflections $s_{v_i}, i = 1, \ldots, N$. Consider an arbitrary point $P_1$ lying inside the Weyl chamber $C$ together with a sufficiently small ball $B_1$ centered in $P_1$.

Use the construction (1) to define Riemannian metric $g_1$ on it. The Weyl group $W$ acts on the chambers simply transitively. We define the points $P_i$ and the balls $B_i$ together with the Riemannian metric on $B_i$ pushed forward from the initial one. Here $i$ ranges from 1 to $|W| = (n + 1)!$.

I claim that so constructed metric $g$ on $\mathbb{R}^n$ is invisible in the directions of every root $v_k$ of $A_n$. Indeed, by the construction every reflection $s_{v_k}$ is an isometry of the constructed metric $g$. Moreover by formula (1) any geodesic straight line passing every ball $B_i$ in the direction $v_k$ remains a parallel straight line and so crosses $P_{v_k}$ orthogonally. Therefore by the lemma the whole geodesic is symmetric and so the direction $v_k$ is invisible.

Moreover if the radius of the initial ball $B_1$ was chosen sufficiently small then every geodesic in the direction $v_k$ crosses in fact only two of the balls or non, where these two are symmetric with respect to reflection $s_{v_k}$. Indeed, let us arrange all the balls into symmetric pairs with respect to $P_{v_k}$. Since the
Weyl group acts by orthogonal transformations on $\mathbb{R}^n$, so no three centers $P_i$ of the balls can lie on a straight line. Then it is obvious that if the radii of the balls are small enough, then the convex hulls of these pairs are all disjoint.

4. Checking non-flatness

In this section we check that the constructed Riemannian metrics are in fact non-flat. Of course it is impossible to compute curvatures in finite time. Therefore we proceed by a different argument. Suppose on the contrary that the metric $g$ is flat. In such a case it must be isometric to Euclidean (see for example [1]) and therefore scalar product with respect to $g$ of the geodesic fields given by the sections $L_i$ and $L_j$ must be constant on $\mathbb{R}^n$ for all $i, j = 1, \ldots, N$. In particular, we can write the following identities.

\[(H(v_k + \epsilon \nabla \phi_k), v_k + \epsilon \nabla \phi_k) = 2;\]
\[(H(v_k - v_l + \epsilon \nabla \phi_{kl}), v_k + \epsilon \nabla \phi_k) = \text{const};\]
\[(H(v_k - v_l + \epsilon \nabla \phi_{kl}), v_l + \epsilon \nabla \phi_l) = \text{const};\]
\[(H(v_l + \epsilon \nabla \phi_l), v_l + \epsilon \nabla \phi_l) = 2.\]

Here $1 \leq k < l \leq n$ and the function $\phi_{kl}$ corresponds to the root $v_k - v_l$ in the formula (1). Let $H = Id + \epsilon H_1 + \ldots$ and extract in these equations terms of order $\epsilon$. We have

\[(2) \quad \langle H_1(v_k), v_k \rangle + 2(v_k, \nabla \phi_k) = 0;\]
\[(3) \quad \langle H_1(v_k - v_l), v_k \rangle + (\nabla \phi_{kl}, v_k) + (\nabla \phi_k, v_k - v_l) = \text{const};\]
\[(4) \quad \langle H_1(v_k - v_l), v_l \rangle + (\nabla \phi_{kl}, v_l) + (\nabla \phi_l, v_k - v_l) = \text{const};\]
\[(5) \quad \langle H_1(v_l), v_l \rangle + 2(v_l, \nabla \phi_l) = 0.\]

Subtract (2) from (3) and also add (4) and (5):

\[(6) \quad -(H_1(v_l), v_k) + (\nabla \phi_{kl}, v_k) - (\nabla \phi_k, v_k + v_l) = \text{const};\]
\[(7) \quad (H_1(v_k), v_l) + (\nabla \phi_{kl}, v_l) + (\nabla \phi_l, v_k + v_l) = \text{const}.\]

Since $H_1$ is symmetric matrix we can add the last two equations to get.

\[(8) \quad \langle \nabla(\phi_{kl} - (\phi_k - \phi_l)), v_k + v_l \rangle = \text{const}.\]

Outside the support $B$ the LHS of (8) is obviously zero, so

\[(9) \quad \langle \nabla(\phi_{kl} - (\phi_k - \phi_l)), v_k + v_l \rangle = 0.\]

But these are strong restrictions on the functions $\phi_i$'s of the construction. Thus if we choose functions $\phi_k, \phi_l, \phi_{kl}$ in [1] violating at least one of these identities then the corresponding metric is not flat.

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School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Israel
E-mail address: bialy@post.tau.ac.il