INFINITE HIERARCHIES OF POISSON STRUCTURES FOR INTEGRABLE SYSTEMS AND SPECTRAL CURVES.

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ABSTRACT. In this short survey we describe our approach for constructing hierarchies of Poisson brackets for classical integrable systems using its' spectral curves.

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1. INTRODUCTION.

Hamiltonian formalism is the main mathematical apparatus of the classical and quantum physics. Hamiltonian formalism of completely integrable systems was intensively studied in the last fifty years. Starting with pioneering work of work of Gardner, [3], and Zakharov and Fadeev, [30], different approaches were developed by Leningrad school of Faddeev, [2], Belavin and Drinfeld [10], Gelfand-Dickey, [8], Atiyah–Hitchin [1], Magri, [14], Novikov and Veselov [31], Krivechev and Phong, [12], with hundreds papers following these paths. Nevertheless the subject remains incomplete, there is no approach that covers all known examples integrable by the methods of spectral curves and algebraic geometry. Even the simplest questions have no answer. The ongoing work of the author in an attempt to fill this gap.

Let us describe a general framework. It is believed that on phase space $\mathcal{M}$ of completely integrable system there exists a finite or infinite set of commutative vector fields

$$X_1, X_2, \ldots$$

that are compatibility conditions for the commutator representation or Lax’s equations

$$X_k : \quad L = [L, A_k], \quad k = 1, 2, \ldots.$$
This vector fields can be written with the classical Poisson bracket \{ , \}π₀ and different Hamiltonians \( H_1, H_2, H_3, \ldots \); as

\[ X_k = \{ , H_k \}π_0, \quad k = 1, 2, \ldots. \]

It is believed that on the phase space \( \mathcal{M} \) there exists an infinite sequence of compatible Poisson brackets

\[ \{ , \}π_0 \quad \{ , \}π_1 \quad \{ , \}π_2 \quad \ldots \quad (1.1) \]

Any vector field of the hierarchy can be written using these brackets and different Hamiltonians \( H_0, H_1, H_2, \ldots \)

\[ X_k = \{ , H_{k-p} \}π_p, \quad k = 1, 2, \ldots; \quad 0 \leq p \leq k. \]

For each integrable system an explicit form of the Poisson brackets 1.1 on \( \mathcal{M} \) is different. The first few usually can be written down explicitly but the formulas quickly become unmanageable.

2. The main conjecture.

In our approach we construct a parametrization of the space \( \mathcal{M} \) in terms of the Hurwitz space

\[ \mathcal{M} \rightarrow (\Gamma, dp, dE, χ). \quad (2.1) \]

The Riemann surface \( Γ \) with two differentials \( dp \) and \( dE \) arises for the periodic spectral for the operator \( L \) or its two-dimensional analogs entering the commutator formalism, [27]. Moreover, on \( Γ \) there exist a meromorphic function \( χ(Q) = χ(x, Q) \) which we call the Weyl function on the Riemann surface. It is closely connected with the standard Weyl function which arises in classical spectral theory, [16], and to the classical Baker–Akhiezer function \( e(x, Q) \) by the formula (for the KdV equation)

\[ i\chi(x, Q) = \frac{∂}{∂x} \log e(x, Q). \]

The careful description of the image of this direct spectral transform is required but this can be done for the most interesting examples. The quadruples \( (Γ, dp, dE, χ) \) are the main object of our consideration.

Within our approach we write Poisson brackets in form

\[ \{ χ(P), χ(Q) \}^f = \sum \int_{O_k} ω^f_{PQ}, \quad (2.2) \]

where the evaluation map is defined as

\[ Q: (Γ, χ) \rightarrow χ(Q), \quad Q \in Γ. \]

The meromorphic one-form \( ω^f_{PQ} \) depends on the functional parameter \( f \). The one-form has poles at the poles of \( χ \), at the points \( P, Q \) and at infinities of \( Γ \). The small circles \( O_k \) surround poles of \( χ \). The sum can be finite or infinite depending on the total number of poles of \( χ \).
We conjecture that a single formula 2.2 with different choices of the meromorphic one–form $\omega_{PQ}$ describes infinite hierarchies 1.1 of Poisson brackets. This is not a theorem but a guiding principle which we checked for a few examples described below.

3. The Camassa–Holm equation.

The Camassa–Holm equation is an approximation to the Euler equation describing an ideal fluid

\[ X_1 : \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} R \left[ v^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \right] = 0 \]

in which $t \geq 0$ and $-\infty < x < \infty$, $v = v(x,t)$ is velocity, and $R$ is inverse to $L = 1 - d^2/dx^2$ i.e.

\[ R[f](x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} f(y) dy. \]

Introducing the function $m = L[v]$ one writes the equation in the form \(^1\)

\[ m^\bullet + (mD + Dm) v = 0. \]

The CH equation is a Hamiltonian system $m^\bullet + \{m, H_1\}_{\pi_0} = 0$ with Hamiltonian

\[ H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} mv dx = \text{energy} \]

and the bracket

\[ \{A, B\}_{\pi_0} = \{A, B\} = \int_{-\infty}^{+\infty} \frac{\delta A}{\delta m} (mD + Dm) \frac{\delta B}{\delta m} dx. \quad (3.1) \]

We consider the CH equation with nonnegative ($m \geq 0$) initial data and such decay at infinity that:

\[ \int_{-\infty}^{+\infty} m(x)e^{|x|} dx < \infty. \]

We denote this class of functions by $\mathcal{M}$. One can associate to the CH equation an auxiliary string spectral problem, \(^2\)

\[ f''(\xi) + \lambda g(\xi)f(\xi) = 0, \quad -2 \leq \xi \leq 2. \]

The background information for this spectral problem can be found in \([4, 7, 6]\). The variables $\xi$ and $x$ are related by

\[ x \rightarrow \xi = 2 \tanh \frac{x}{2}. \]

Also the potential $g(\xi)$ is related to $m(x)$ by the formula $g(\xi) = m(x) \cosh^4 \frac{\xi}{2}$. For initial data from $\mathcal{M}$ the total mass of the associated string is finite $\int_{-2}^{+2} g(\xi) d\xi < \infty$. For simplicity we consider $N$-peakon solutions of the CH equation. In this case $g(\xi)$ is a sum of finite number of point masses.

\(^1\)We use notation $D$ for the $x$-derivative and $\bullet$ for the $t$–derivative. We use $\delta$ for the Frechet derivative.

\(^2\)We use prime $'$ to denote $\xi$-derivative.
Figure 1. The reducible Riemann surface $\Gamma$ which consist of two components $\Gamma_-$ and $\Gamma_+$. These components are copies of the Riemann sphere attached to each other at the points $z_0, z_1, \ldots, z_{N-1}$.

Two important solutions $\varphi(\xi, \lambda)$ and $\psi(\xi, \lambda)$ of the string spectral problem are specified by initial data

\[
\begin{align*}
\varphi(-2, \lambda) &= 1 & \varphi(-2, \lambda) &= 0 \\
\varphi'(-2, \lambda) &= 0 & \psi'(-2, \lambda) &= 1.
\end{align*}
\]

The Weyl function is defined by the formula

\[
\chi(\lambda) = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)}.
\]

The Riemann surface $\Gamma$ associated with the string spectral problem consists of two components $\Gamma_+$ and $\Gamma_-$ two copies of the Riemann sphere. The points $\lambda$'s where two spheres are glued to each other are points of the Dirichlet spectrum. The points $\gamma$'s on $\Gamma_-$ are the points of the Newmann spectrum, see Figure 1. The pair $(\Gamma, \chi)$ provides a parametrization of the phase space $\mathcal{M}$, see [21]. Such reducible Riemann surfaces were introduced in [13].
Now we introduce a family of compatible Poisson brackets. Changing spectral variable \( \lambda \to z = -1/\lambda \) we have

\[
\chi(z) = -\frac{1}{4} + \sum_{m=0}^{N-1} \frac{\rho'_m}{z^m - z},
\]

where poles \( z_m \) accumulate near the origin.

\[
0 < z_0 < z_1 < \ldots < z_{N-1}.
\] (3.2)

The Weyl function belongs to the class \( \text{Rat}_N \) of the rational functions on the Riemann sphere which have \( N \) distinct poles 3.2 and vanish at infinity.

Consider a connected set of functions \( \chi(z) \) on \( \mathbb{CP}^1 \) with the property \( \chi(\infty) = 0 \) and \( N \) simple poles at \( z_0, z_1, \ldots, z_{N-1} \). We denote all such functions as \( \text{Rat}_N \). Apparently any function from \( \text{Rat}_N \) can be uniquely written as

\[
\chi(z) = \frac{q(z)}{p(z)}, \quad \text{where} \quad p(z) = \prod_{k=0}^{N-1} (z - z_k), \quad q(z) = q_0 \prod_{k=1}^{N-1} (z - \gamma_k).
\]

The space \( \text{Rat}_N \) has complex dimension \( 2N \) and \( z - q(z) \) complex coordinates

\[
z_0, \ldots, z_{N-1}; q(z_0), \ldots, q(z_{N-1}).
\]

Any such function can be represented as

\[
\chi(z) = \sum_{k=1}^{N} \frac{\rho_k}{z_k - z}, \quad \rho_k = -\text{res}_{z_k} \chi(z).
\] (3.3)

We have another set of \( z - \rho \) coordinates

\[
z_0, \ldots, z_{N-1}; \rho_0, \ldots, \rho_{N-1}.
\]

To introduce a formula for the hierarchy we consider a meromorphic differential \( \omega_{pq}^f \) on \( \Gamma \) which depends on the entire function \( f(z) \) and two points \( p \) and \( q \)

\[
\omega_{pq}^f = \frac{\epsilon_{pq}(z)}{p - q} \times f(z)\chi(z) (\chi(p) - \chi(q)), \quad (3.4)
\]

where

\[
\epsilon_{pq}(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - p} - \frac{1}{z - q} \right] dz;
\]

is the standard differential Abelian differential of the third kind with residues \( \pm 1 \) at the points \( p \) and \( q \).

**Theorem 3.1.** [15, 22] For any entire function \( f \) the Poisson bracket 2.2 satisfies the Jacobi identity

\[
\{\{\chi(p), \chi(q)\}, \chi(r)\} + cp(p, q, r) = 0.
\]

Finally, we can state the theorem

**Theorem 3.2.** [21] The family Poisson brackets for the Camassa–Holm is produced by 2.2 with the differential \( \omega_{pq}^f(z) \) given by 3.4. The infinite hierarchy 1.1 is produced by 2.2 for a particular choice of \( f(z) = z^n, \, n = 0, 1, \ldots \).
The map \( \mathcal{M} \rightarrow (\Gamma, \chi) \) induces Poisson structure on \( \mathcal{M} \).

By the Cauchy formula from 2.2 we have for any entire \( f(z) \)
\[
\{ \chi(p), \chi(q) \}^f = \text{res}_p \omega^f_{pq} + \text{res}_q \omega^f_{pq} + \text{res}_\infty \omega^f_{pq} = \frac{f(p)\chi(p) - f(q)\chi(q)}{p - q} (\chi(p) - \chi(q)) + \text{res}_\infty \omega^f_{pq}.
\]
If \( f(z) = z^n, n = 0, 1, \ldots \); then the residue at infinity vanishes identically only for \( n = 0 \) or 1. When \( f(z) = 1 \) we obtain quadratic Poisson algebra corresponding to the rational solution of CYBE or the Atiyah-Hitchin bracket
\[
\{ \chi(p), \chi(q) \}^1 = (\chi(p) - \chi(q)) \frac{\chi(p) - \chi(q)}{p - q}.
\]
Another quadratic Poisson algebra is obtained for \( f(z) = z \) and it corresponds to the trigonometric solution of CYBE
\[
\{ \chi(p), \chi(q) \}^z = (p\chi(p) - q\chi(q)) \frac{\chi(p) - \chi(q)}{p - q}.
\]

It can be verified directly that (3.5) and (3.6) satisfy Jacobi identity.

4. The open Toda lattice.

The open finite Toda lattice is a mechanical system of \( N \)–particles connected by elastic strings. The Hamiltonian of the system is
\[
\mathcal{H}_1 = \sum_{k=0}^{N-1} \frac{p_k^2}{2} + \sum_{k=0}^{N-2} e^{q_{k+1}-q_k}.
\]
Introducing the classical Poisson bracket
\[
\{ f, g \}_\pi = \sum_{k=0}^{N-1} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k},
\]
we write the equations of motion as
\[
X_1 : \begin{align*}
q_k^\bullet &= \{ q_k, \mathcal{H}_0 \} = p_k, \\
p_k^\bullet &= \{ p_k, \mathcal{H}_0 \} = -e^{q_{k+1}-q_k} + e^{q_{k-1}-q_k}, \quad k = 1, \ldots, N - 1.
\end{align*}
\]
We put \( q_{-1} = -\infty, q_N = \infty \) in all formulas. These equations define the vector field \( X_1 \).

Following [11], consider functions \( \chi(\lambda) \) with the properties i. analytic in the half–planes \( \Re z > 0 \) and \( \Re z < 0 \), ii. \( \chi(\bar{z}) = \overline{\chi(z)} \), if \( \Re z \neq 0 \) iii. \( \Re \chi(z) > 0 \), if \( \Re z > 0 \). All such function are called \( R \)–functions. They play central role in the spectral theory of selfadjoint operators. The Weyl function of a Jacobi matrix is an \( R \)–function.

We denote by \( \text{Rat}'_N \) the subset of all functions from \( \text{Rat}_N \) which satisfy the condition
\[
q_0 = \sum_{n=1}^{N} \rho_n = 1.
\]

The Weyl functions of a Jacobi matrix are exactly those \( R \)–functions that belong to \( \text{Rat}'_N \). This implies that all \( z_k \) are real and \( \rho_k > 0 \) in the representation 3.3.
We consider two functionals
\[ \Phi_1 = I_0 + I_1 + \ldots + I_N, \quad \Phi_2 = \log q_0; \]
where \( I_k \) are defined by
\[ I_k = \int_{\infty}^{\zeta_k} \frac{d\zeta}{f(\zeta)}, \quad k = 0, 1, \ldots, N. \]

**Theorem 4.1.** [22] The family of Poisson brackets for the open Toda lattice is produced by a Dirac restriction of the Poisson bracket 2.2 with the differential 3.4 on the sub-manifold \( \Phi_1 = c_1, \quad \Phi_2 = c_2 \); (4.2)
is given by formula 2.2 where the new modified differential \( \tilde{\omega}_{pq} \) is
\[ \tilde{\omega}_{pq} = \frac{\epsilon_{pq}(z)}{p - q} \times f(z) \chi(p) - \chi(q)) - \epsilon_{pq}(z) \times f(z) \chi(z) \chi(p) \chi(q) e^{-c_2}. \]
The infinite hierarchy of Poisson brackets 1.1 corresponds to the choice of \( f(z) = z^n, \ n = 0, 1, \ldots. \)

5. **The periodic solutions of the KdV equation.**

We consider the hierarchy of flows Korteweg de Vries equation \(^3\)
\[ u^* = \frac{3}{2} uu' - \frac{1}{4} u''; \quad u = u(x,t); \]
in the space \( \mathcal{M} \) of all infinitely differentiable \( 2l \)-periodic functions \( u(x,t) = u(x + 2l,t) \).
The KdV equation is a compatibility condition for the Lax representation
\[ L^* = [L, A]. \]
Where \( L \) is the Shrödinger operator
\[ L = -\partial_x^2 + u. \]
and
\[ A = 4\partial_x^3 - 6u\partial_x - 3ux. \]
The KdV equation is one in the infinite hierarchy of vector fields
\[ X_0 : \quad u^* = u'; \]
\[ X_1 : \quad u^* = \frac{3}{2} uu' - \frac{1}{4} u''; \]
\[ X_2 : \quad u^* = \frac{1}{16} u^{(4)} - \frac{5}{4} u' u'' - \frac{5}{8} uu''' + \frac{15}{8} u^2 u', \quad \text{etc.} \]

Each vector field \( X_k, \ k = 0, 1, 2, \ldots; \) is a compatibility condition for the commutator formalism \( L^* = [L, A_k] \) with some \( A_k \) and \( A_1 = A \). These vector fields produce an infinite hierarchy of commutative flows \( e^{tX_m}, \ m = 0, 1, 2, \ldots. \)

Gardner and Zakharov and Faddeev found that each flow \( X_n \) can be written as a Hamiltonian system
\[ u^* = \{u, \mathcal{H}_n\}_{\pi_0}; \]

\(^3\)Prime \(^\prime\) signifies the derivative in the variable \( x \) and dot \( \bullet \) the derivative with respect to time.
with the bracket
\[ \{A, B\}_\pi = \int \frac{\delta A}{\delta u(x)} D \frac{\delta B}{\delta u(x)} \, dx \]  
(5.2)
where \( D = \partial_x \) and corresponding Hamiltonian \( \mathcal{H}_n \). Here the first three Hamiltonians
\[
\begin{align*}
\mathcal{H}_0 &= \frac{1}{2} \int u^2 \, dx, \\
\mathcal{H}_1 &= \frac{1}{4} \int \left[ u^3 + \frac{1}{2} u'^2 \right] \, dx, \\
\mathcal{H}_2 &= \frac{1}{16} \int \left[ \frac{1}{2} u''^2 + 5uu'^2 + \frac{5}{2} u^4 \right] \, dx, \quad \text{etc.}
\end{align*}
\]
For each Hamiltonian \( \mathcal{H}_n = \int L_n(u(x), u'(x), u''(x), \ldots, u^{(n)}(x)) \, dx \)
the variational derivative can be computed by the formula
\[
\frac{\delta \mathcal{H}_n}{\delta u(x)} = \sum_{k=0}^{n} (-1)^k \partial_x^k \frac{\partial L_n}{\partial u^{(k)}(x)}.
\]
The Hamiltonians commute with respect to the Poisson bracket. Thus we have a hierarchy of commutative Hamiltonian flows.

For the periodic potentials the spectral curve is defined by
\[
\Gamma = \{ Q = (z, w) \in \mathbb{C}^2 : R(z, w) = \det[wI - T(z)] = w^2 - 2w\Delta(z) + 1 = 0 \},
\]
where \( 2\Delta(z) = \text{trace} \, T(z) \), and represents two sheets covering of \( \mathbb{C}P^1 \). In other words on \( \Gamma \) there exists two functions \( z = z(Q) \) and \( w = w(Q) \) which determine an embedding of the spectral curve into \( \mathbb{C}^2 \). In this case the two differentials are \( dE = dz \) and \( dp = \frac{1}{i} d\log w \).

The key role in our considerations is played by the scalar Weyl function \( \chi(x, Q) \) defined by
\[
i\chi(x, Q) = \frac{\partial}{\partial x} \log e(x, y, Q).
\]
(5.3)
where \( e(x, y, Q) \) is the standard Floquet solution normalized as \( e(y, y, Q) = 1 \) and
\[
i\chi(x, Q) = \frac{w(Q) - T_{11}(y, z)}{T_{12}(y, z)} = \frac{T_{21}(y, z)}{w(Q) - T_{22}(y, z)}.
\]
(5.4)
Now we defined all objects that constitute a quadruples \( \{ \Gamma, dp, dE, \chi \} \).

First we define the meromorphic differential \( \epsilon_{QP}(R) \) is of the third kind on \( \Gamma \) with poles at the points \( Q = (z_Q, w_Q) \) and \( P = (z_P, w_P) \) and the residues there equal to \( \pm 1 \)
\[
\epsilon_{QP}(R) = \left( \frac{w(R) - w^{-1}(Q)}{z(R) - z(Q)} - \frac{w(R) - w^{-1}(P)}{z(R) - z(P)} \right) \frac{dz(R)}{w(R) - w^{-1}(R)}.
\]
This differential has simple poles at the points \( P \) and \( Q \). The residues are \( \pm 1 \). Also singularities may arise from the factor
\[
\frac{dz(R)}{w(R) - w^{-1}(R)} \quad (5.5)
\]
when \( w(R) = \pm 1 \). This happens at the points \( z_k^- \leq z_k^+ \) of the periodic/anti periodic spectrum. If the spectrum is simple \( z_k^- < z_k^+ \) then at these branch points the differential \( dz(R) \) has simple zeros and these zeros annihilate the singularity of the denominator. If \( z_k^- = z_k^+ \), then the singularity is nodal. The function \( z \) serves as a local parameter in the vicinity of these points and the differential \( dz(R) \) does not vanish. The differential \( 5.5 \) changes sign under involution \( \tau_p \) permuting sheets of the curve and it has residues of opposite sign on different sheets.

The meromorphic differential \( \omega^f_{QP} \) is such that

\[
\omega^f_{QP}(R) = \frac{1}{2\pi i} f(z(R)) \times \epsilon_{QP}(R) \times \chi(R)(\chi(Q) - \chi(P)) \times \frac{\Omega(Q) + \Omega(P)}{2},
\]

where the function \( f(z) \) is an entire function, e.g. \( f(z) = z^n, n = 0, 1, 2, \ldots \).

The deformation factor \( \Omega(Q) \) is defined by

\[
\Omega(Q) = \frac{w(Q) + w^{-1}(Q)}{w(Q) - w^{-1}(Q)}.
\]

It becomes infinite at the branch and intersection points of the curve \( \Gamma \).

It is easy to state analytic properties of \( 5.6 \). Fix two points \( Q \) and \( P \) away from the branch or intersection points of \( \Gamma \). The differential \( 5.6 \) has poles at the points \( M_k = M_k(x) \) coming from \( \chi(R) = \chi(x, R) \) and it also has poles at the points \( P \) and \( Q \) and also at the points of the double spectrum (unopened zones) \( (z_k^\pm, (-1)^k) \). The last group of poles arises from \( \epsilon_{QP}(R) \). The residues of \( 5.6 \) on different sheets of the curve above the double spectrum are of the opposite sign and cancel each other if we integrate over the contour containing these points.

**Conjecture.** [23]. The family of Poisson brackets for KdV hierarchy is produced by formula \( 2.2 \) with the one-form \( \omega^f_{QP} \) defined by \( 5.6 \). The hierarchy of Poisson brackets \( 1.1 \) is produced for a particular choice of \( f(Q) = z(Q)^n, n = 0, 1, \ldots \).

Now we explain why we think our conjecture is true. The formula is modeled upon the simplest formula \( 3.4 \) for the CH hierarchy. Moreover, for a particular choice of \( f(z) = 1 \) by the residue theorem the conjectured formula produces the quadratic algebra for the Gardner-Zakharov-Faddeev bracket

\[
\{\chi(P), \chi(Q)\} = \frac{(\chi(P) - \chi(Q))^2}{z(P) - z(Q)} \times \frac{\Omega(P) + \Omega(Q)}{2}.
\]

The last formula was proved by direct computations for the first time in [20]. It corresponds to the rational solution for the PB on the entries of the monodromy matrix and we call it in [20] the deformed Attiah–Hitchin bracket, compare \( 3.5 \).

**6. The Poisson brackets associated with differentials of the second kind.**

In all previous examples the one-form \( \omega_{PQ} \) was constructed using differential of the third kind. Now we want to show an example of the Poisson bracket and the one form associated with a differential of the second kind, [24].
We consider the space $\text{Rat}_N$ of the meromorphic functions on the Riemann sphere. Let
\[
\epsilon_P^{(n+1)}(z) = \frac{n!}{2\pi i (z - z_P)^{n+1}} dz, \quad n = 1, 2, \ldots;
\]
be Abelian differential of the second kind with a pole of degree $n + 1$ at the point $p$. Note that for any function $f(z)$ which is holomorphic in the vicinity of the point $P$ we have
\[
\int_{\partial P} \epsilon_P^{(n+1)}(z)f(z) = f^{(n)}(P).
\]
Let us define the new meromorphic 1-form
\[
\omega_{\chi}^{fPQ} = \epsilon_P^{(2)}(\chi) f(z) \chi(Q) - \epsilon_Q^{(2)}(\chi) f(z) \chi(z_P).
\]
The new analytic Poisson brackets are defined by usual formula 2.2.

When $f(z) = 1$ or $z$ we can obtain closed expression in terms of $\chi$ and its first derivative. For $f(z) = 1$ we have
\[
\{\chi(P), \chi(qQ)\}^1 = \chi'(P) \chi(Q) - \chi'(Q) \chi(P).
\]
For $f(z) = z$ we have
\[
\{\chi(P), \chi(Q)\}^z = z_P \chi'(P) \chi(Q) - z_Q \chi'(Q) \chi(P).
\]
It can be proved that these formulas define genuine Poisson brackets e.g. they satisfy the Jacobi identity.
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