More Non Semigroup Lie Gradings

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Abstract. This note is devoted to the construction of two very easy examples, of respective dimensions 4 and 6, of graded Lie algebras whose grading is not given by a semigroup, the latter one being a semisimple algebra. It is shown that 4 is the minimal possible dimension.

Patera and Zassenhaus [PZ89] define a Lie grading as a decomposition of a Lie algebra into a direct sum of subspaces

$$\mathcal{L} = \oplus_{g \in G} \mathcal{L}_g,$$

indexed by elements $g$ from a set $G$, such that $\mathcal{L}_g \neq 0$ for any $g \in G$, and for any $g, g' \in G$, either $[\mathcal{L}_g, \mathcal{L}_{g'}] = 0$ or there exists a $g'' \in G$ such that $0 \neq [\mathcal{L}_g, \mathcal{L}_{g'}] \subseteq \mathcal{L}_{g''}$.

Then, in [PZ89 Theorem 1.(d)], it is asserted that, given a Lie grading, the set $G$ embeds in an abelian semigroup so that the following property holds:

(P) For any $g, g', g'' \in G$ with $0 \neq [\mathcal{L}_g, \mathcal{L}_{g'}] \subseteq \mathcal{L}_{g''}$, $g + g' = g''$ holds in the semigroup.

In [Eld06], a counterexample to this assertion was given. A nilpotent Lie algebra of dimension 16 was defined, with a grading not given by an abelian semigroup. This example came as a surprise (see [Svo08]), but its difficulty may give the impression that this is a rare phenomenon.

The purpose of this note is to give two more counterexamples that show that non semigroup gradings are not so rare. The first one will be a non semigroup grading on a four dimensional solvable (actually metaabelian) Lie algebra. It will be shown that there are no counterexamples in dimension $\leq 3$, so this is a counterexample of minimal dimension. On the other hand, another such non semigroup grading will be defined over the direct sum of two three dimensional simple Lie algebras. It has the interesting feature of being a coarsening of a group grading. Note that in the last paragraph in [Eld06] a grading on $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ was shown which is not a group grading, however it is a semigroup grading.

Anyway, the question posed in [Eld06] still remains open: Is any grading on a simple finite dimensional complex Lie algebra a group grading?

Some notation is in order before we start.

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Given a graded Lie algebra $L = \oplus_{g \in G} L_g$, a subspace $S$ of $L$ is said to be graded if it satisfies the condition $S = \oplus_{g \in G} \cap L$. In particular, the derived ideal $L' = [L, L]$ is always graded, as $L' = \sum_{g, h \in G} [L_g, L_h]$ is the sum of graded subspaces.

Another grading $L = \oplus_{\gamma \in \Gamma} L_{\gamma}$ is said to be a coarsening of the previous one in case for any $g \in G$ there is a $\gamma \in \Gamma$ such that $L_g \subseteq L_{\gamma}$. In other words, each $L_{\gamma}$ is a sum (necessarily direct) of homogeneous subspaces of the first grading. In this situation, the first grading is said to be a refinement of the second one.

In what follows, we will work over an arbitrary ground field $F$ (even characteristic 2 is allowed).

1. A counterexample of minimal dimension

The counterexample. Consider the four dimensional Lie algebra $L$ with a basis \{a, u, v, w\} and multiplication given by:

\[
[\, a, u \,] = u, \quad [\, a, v \,] = w, \quad [\, a, w \,] = v,
\]

all the other brackets being 0. Thus $L$ is the semidirect sum of the one dimensional subalgebra spanned by $a$ and the three dimensional abelian ideal spanned by $u$, $v$ and $w$. Then $L$ is graded as follows: $L = L_\alpha \oplus L_\beta \oplus L_\gamma$, with

\[
L_\alpha = F a + F u, \quad L_\beta = F v, \quad L_\gamma = F w.
\]

If this were a semigroup grading, there would be a semigroup $\Gamma$ with three different elements $\alpha$, $\beta$ and $\gamma$ satisfying the following conditions (the binary operation in the semigroup will be denoted by juxtaposition, and note that it is not even necessary to assume that the semigroup is commutative):

\[
\alpha^2 = \alpha, \quad \text{as} \quad [L_\alpha, L_\alpha] = Fu \subseteq L_\alpha,
\]

\[
\alpha \beta = \gamma, \quad \text{as} \quad [L_\alpha, L_\beta] = Fw = L_\gamma,
\]

\[
\alpha \gamma = \beta, \quad \text{as} \quad [L_\alpha, L_\gamma] = Fv = L_\beta.
\]

But then we would obtain:

\[
\gamma = \alpha \beta = \alpha^2 \beta = \alpha (\alpha \beta) = \alpha \gamma = \beta,
\]

a contradiction. \hfill \Box

The next result shows that the dimension of this counterexample is minimal.

Theorem. For any grading on a Lie algebra of dimension $\leq 3$ there exists a semigroup satisfying the property (P).

Proof. The result is trivial for Lie algebras of dimension 1 and very easy for dimension 2, so it will be assumed that $L$ is a three dimensional graded Lie algebra, with a grading with two or three homogeneous subspaces: $L = \oplus_{\gamma \in \Gamma} L_{\gamma}$. Several possibilities may occur:

1. If the dimension of the derived subalgebra $L' = [L, L]$ is 1 then, since $L'$ is a graded subalgebra, there is an element $\alpha \in \Gamma$ such that $L' \subseteq L_\alpha$. Then $\Gamma$ becomes a (commutative) semigroup by means of $\gamma \delta = \alpha$ for any $\gamma, \delta \in \Gamma$, and the grading is obviously a grading over this semigroup.
(2) If the dimension of \( L' \) is 2 there are several subcases. Note that, as shown in [Jac62] p. 12, \( L' \) is abelian and the center of \( L \) is trivial.

(a) There is an element \( \alpha \in \Gamma \) such that \( L' = L_\alpha \). But then \( L = L_\alpha \oplus L_\beta \) for some other element \( \beta \in \Gamma \), and \([L_\beta, L_\alpha] = L_\alpha \). This is then a grading over the integers, with \( L_0 = L_\beta \) and \( L_1 = L_\alpha \).

(b) \( L = L_\alpha \oplus L_\beta \) with \( \dim L_\alpha = 2 \) and \( L' = (L' \cap L_\alpha) \oplus L_\beta \). Here \( L' = [L_\alpha, L_\alpha] + [L_\alpha, L_\beta] \) and \([L_\alpha, L_\beta] \neq 0 \) as the center of \( L \) is trivial. Hence either \([L_\alpha, L_\alpha] = L_\beta \) and \([L_\alpha, L_\beta] = L' \cap L_\alpha \), in which case this is a \( \mathbb{Z}/2\mathbb{Z} \)-grading with \( L_\beta = L_0 \) and \( L_\alpha = L_1 \), or \([L_\alpha, L_\alpha] \subseteq L_\alpha \) and \([L_\alpha, L_\beta] = L_\beta \), and we get a \( \mathbb{Z} \)-grading with \( L_\alpha = L_0 \) and \( L_\beta = L_1 \).

(c) \( L = L_\alpha \oplus L_\beta \oplus L_\gamma \), with all the homogeneous subspaces of dimension 1 and \( L' = L_\beta \oplus L_\gamma \). Since \( L' \) is abelian, we get \( L' = [L_\alpha, L_\beta] \oplus [L_\alpha, L_\gamma] \), so either \([L_\alpha, L_\beta] = L_\beta \) and \([L_\alpha, L_\gamma] = L_\gamma \), in which case we have a \( \mathbb{Z} \)-grading with \( L_\alpha = L_0 \), \( L_\beta = L_1 \) and \( L_\gamma = L_{-1} \), or \([L_\alpha, L_\beta] = L_\gamma \) and \([L_\alpha, L_\gamma] = L_\beta \), which is a \( (\mathbb{Z}/2\mathbb{Z})^2 \)-grading with \( L_\alpha = L_{(1,0)} \), \( L_\beta = L_{(1,1)} \) and \( L_\gamma = L_{(0,1)} \).

(3) Finally, if \( L' = L \) then \( L \) is simple (see [Jac62] p. 12). Again there are two subcases:

(a) Assume that \( L = L_\alpha \oplus L_\beta \) with \( \dim L_\alpha = 1 \), \( \dim L_\beta = 2 \). Then \( L' = [L_\alpha, L_\beta] + [L_\beta, L_\beta] \). Note that the first summand is not contained in \( L_\alpha \), as this would force \( L_\alpha \) to be an ideal. Hence we have \([L_\alpha, L_\beta] = L_\beta \), \([L_\beta, L_\beta] = L_\alpha \), and this is clearly a \( \mathbb{Z}/2\mathbb{Z} \)-grading.

(b) Otherwise, \( L = L_\alpha \oplus L_\beta \oplus L_\gamma \), for one dimensional subspaces \( L_\alpha \), \( L_\beta \) and \( L_\gamma \). If there are two indices \( \mu, \nu \) such that \([L_\mu, L_\nu] \subseteq L_\nu \), we may assume without loss of generality that \([L_\alpha, L_\beta] = L_\beta \).

Hence \( L_\alpha = \mathbb{F} h \), \( L_\beta = \mathbb{F} x \) with \([h, x] = x \). Since the trace of the adjoint action by any element is 0 (as \( L = L' \)), it follows that there exists an element \( y \in L \) with \( L_\gamma = \mathbb{F} y \) and \([h, y] = -y \).

Also, since \( L = L' \), \([y, z] \) must belong to \( L_\alpha \), and this gives a \( \mathbb{Z} \)-grading with \( L_\alpha = L_0 \), \( L_\beta = L_1 \) and \( L_\gamma = L_{-1} \). We are left with the case in which \([L_\alpha, L_\beta] = L_\gamma \), \([L_\beta, L_\gamma] = L_\alpha \), and \([L_\gamma, L_\alpha] = L_\beta \), and this is a \( (\mathbb{Z}/2\mathbb{Z})^2 \)-grading with trivial zero homogenous space. \( \square \)

**Remark.** Consider the three dimensional Lie algebra \( L \) with a basis \( \{x, y, z\} \) and multiplication given by \([x, z] = [y, z] = z, [x, y] = 0\). This is a direct sum of the two dimensional non abelian Lie algebra spanned by \( y \) and \( z \) and the one dimensional center spanned by \( x - y \). The grading where the homogeneous spaces are the subspaces spanned by the basic elements is not a group grading (otherwise the indices of both \( x \) and \( y \) should correspond to the neutral element). This gives an example of minimal dimension of a non group grading on a Lie algebra.

2. A counterexample on a semisimple Lie algebra

**The counterexample.** Consider now the three dimensional simple Lie algebras \( J = \text{span} \{h, x, y\} \) and \( K = \text{span} \{e_1, e_2, e_3\} \), with multiplication
given by:
\[
[h, x] = x, \quad [h, y] = -y, \quad [x, y] = h,
\]
\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.
\]
If the characteristic of the ground field is \( \neq 2 \), then \( \mathcal{J} \) is isomorphic to \( \mathfrak{sl}(2) \), and \( \mathcal{K} \) to the orthogonal Lie algebra \( \mathfrak{so}(3) \). If, in addition, the ground field contains the square roots of \(-1\), then both Lie algebras are isomorphic, and its direct sum is isomorphic to the orthogonal Lie algebra \( \mathfrak{so}(4) \).

Let us take the Lie algebra \( \mathcal{L} = \mathcal{J} \oplus \mathcal{K} \) with the grading given by:
\[
\mathcal{L}_\alpha = Fh + Fe_1, \quad \mathcal{L}_\beta = Fx, \quad \mathcal{L}_\gamma = Fy, \quad \mathcal{L}_\delta = Fe_2, \quad \mathcal{L}_\mu = Fe_3.
\]
It is straightforward to check that this is indeed a Lie grading. But if this were a semigroup grading, we would have the following condition in the semigroup:
\[
\alpha = \beta \gamma = (\alpha \beta) \gamma = \alpha (\beta \gamma) = \alpha^2,
\]
and, therefore,
\[
\mu = \alpha \delta = \alpha^2 \delta = \alpha (\alpha \delta) = \alpha \mu = \delta,
\]
a contradiction. \( \square \)

Note that \( \mathcal{J} \) is \( \mathbb{Z} \)-graded with \( \mathcal{J}_0 = Fh, \mathcal{J}_1 = Fx, \) and \( \mathcal{J}_{-1} = Fy \). On the other hand, \( \mathcal{K} \) is \( (\mathbb{Z}/2\mathbb{Z})^2 \)-graded with \( \mathcal{K}_{(1,0)} = Fe_1, \mathcal{K}_{(0,1)} = Fe_2, \) and \( \mathcal{K}_{(1,1)} = Fe_3 \). Therefore, \( \mathcal{L} = \mathcal{J} \oplus \mathcal{K} \) is \( \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \)-graded, with all the homogeneous spaces of dimension 1. The grading in the counterexample above is a coarsening of this grading, obtained by joining together the homogeneous subspaces spanned by \( h \) and \( e_1 \).

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