A New Expansion of the Heisenberg Equation of Motion with Projection Operator

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We derive a new expansion of the Heisenberg equation of motion based on the projection operator method proposed by Shibata, Hashitsume and Shingū. In their projection operator method, a certain restriction is imposed on the initial state. As a result, one cannot prepare arbitrary initial states, for example a coherent state, to calculate the time development of quantum systems. In this paper, we generalize the projection operator method by relaxing this restriction. We explain our method in the case of a Hamiltonian both with and without explicit time dependence. Furthermore, we apply it to an exactly solvable model called the damped harmonic oscillator model and confirm the validity of our method.

I. INTRODUCTION

It is important to calculate the time evolution in quantum field theory in various fields of physics. However, there is no established method to carry out such calculations. In this work, we study a new expansion of the Heisenberg equation of motion based on the projection operator method. The projection operator method, which we use in this paper, was first proposed by Shibata, Hashitsume and Shingū. The projection operator method has several important characteristics. (i) It can be applied both in Schrödinger and Heisenberg pictures. (ii) There is an arbitrariness in the selection of projection operators. (iii) Equations with and without time convolution integrals are treated systematically. (iv) One can use it to calculate in the case of an explicitly time-dependent Hamiltonian. The method has been successfully applied to problems in quantum mechanics. The Mori and Nakajima-Zwanzig methods, which are well-known methods using the projection operator, are included in this projection operator method. We believe that the projection operator method is more convenient than other common formalisms, e.g., the closed time-path formalism and the Feynman-Vernon influence functional technique. In such formalisms, further approximation is needed to derive an equation of motion without a time convolution integral, for example, as performed in the physics of hadrons and the early universe. However, in the projection operator method, there is no such problem, because of the above characteristic (iii). Furthermore, since this method consists of canonical formalism, it is useful to discuss the meanings of time evolution in quantum field theory by comparing with usual scattering theory.

In a previous paper, we discussed the renormalization of ultraviolet divergences in a certain model, using an equation of motion without a time convolution integral derived with the projection operator method. However, there is still the problem that we cannot prepare arbitrary initial states, because a restriction is imposed on the initial state to carry out the systematic expansion of the Heisenberg equation of motion. In this work, we study a new expansion of the Heisenberg equation of motion based on the projection operator method without imposing the restriction. For this reason, we can prepare more general initial states, such as coherent states, which may be important to describe phase transitions.

This paper is organized as follows. In §2, we explain the formalism in the case that Hamiltonian has no explicit time dependence. In §3, the case in which Hamiltonian has explicit time dependence is discussed. Such a case is also important for studying physical phenomena, for example, pair creation in a strong external field. In §4, we apply our formalism to the damped harmonic oscillator model, which is exactly solvable, to confirm the validity of our method. Conclusions are given in §5.

II. THE HEISENBERG EQUATION WITH TIME-INDEPENDENT HAMILTONIAN

Our strategy is to study quantum field theory as an initial value problem. In this section, we derive our equation in the case that the Hamiltonian has no explicit time dependence. Our starting point is the Heisenberg equation of motion,

\[ \frac{d}{dt} O(t) = i[H, O(t)] \]
\[ = iLO(t) \]
\[ \rightarrow O(t) = e^{i\mathcal{L}(t-t_0)}O(t_0), \]

where \( \mathcal{L} \) is the Liouville operator.
where $L$ is the Liouville operator and $t_0$ is the time at which we prepare an initial state. The Heisenberg equation contains complete information of the time evolution of the operator, but in general, it is difficult to solve exactly when there are interactions. Therefore it is necessary to make some approximations. For this purpose, we introduce generic projection operators $P$ and $Q$ that have the following general properties:

\begin{align}
  P^2 &= P, \\
  Q &= 1 - P, \\
  PQ &= QP = 0.
\end{align}

By using these projection operators, we can carry out coarse-grainings in the time development. From Eq. (3), one can see that the time dependence of the operators is determined by $e^{iL(t-t_0)}$. This yields the equation

\begin{align}
  \frac{d}{dt}e^{iL(t-t_0)} &= e^{iL(t-t_0)}iL \nonumber \\
  &= e^{iL(t-t_0)}(P + Q)iL.
\end{align}

From this equation, we can derive the following two equations:

\begin{align}
  \frac{d}{dt}e^{iL(t-t_0)}P &= e^{iL(t-t_0)}P_iLP + e^{iL(t-t_0)}Q_iLP, \\
  \frac{d}{dt}e^{iL(t-t_0)}Q &= e^{iL(t-t_0)}P_iLQ + e^{iL(t-t_0)}Q_iLQ.
\end{align}

Equation (3) can be solved for $e^{iL(t-t_0)Q}$. We obtain

\begin{align}
  e^{iL(t-t_0)Q} &= Qe^{iLQ(t-t_0)} + \int_{t_0}^{t} ds e^{iL(s-t_0)} P_iLQ e^{iLQ(t-s)} \nonumber \\
  &= Qe^{iLQ(t-t_0)} + e^{iL(t-t_0)}(P + Q) \int_{t_0}^{t} ds e^{-iL(s-t_0)} P_iLQ e^{iLQ(t-s)} \\
  &= Qe^{iLQ(t-t_0)} \frac{1}{1 - \Sigma(t, t_0)} + e^{iL(t-t_0)}P\Sigma(t, t_0) \frac{1}{1 - \Sigma(t, t_0)},
\end{align}

where

\begin{align}
  \Sigma(t, t_0) &= \int_{t_0}^{t} ds e^{-iL(s-t_0)} P_iLQ e^{iLQ(t-s)}.
\end{align}

Substituting Eq. (11) into Eq. (3) and operating with $O(t_0)$ from the right, we obtain

\begin{align}
  \frac{d}{dt}O(t) &= e^{iL(t-t_0)}P_iLO(t_0) + e^{iL(t-t_0)}P\Sigma(t, t_0) \frac{1}{1 - \Sigma(t, t_0)}iLO(t_0) \\
  &\quad + Qe^{iLQ(t-t_0)} \frac{1}{1 - \Sigma(t, t_0)}iLO(t_0).
\end{align}

This equation is equivalent to the Heisenberg equation, and it has no time convolution integral. We demonstrate this point at the end of this section. Therefore, we call this the time-convolutionless (TCL) equation. It is the same equation as that used in the previous projection operator method.

Now we consider the situation that the total system can be divided into two parts: the system and the environment. We want to know the detailed behavior of the degrees of freedom of the system. Therefore, we carry out coarse-grainings to treat the environment degrees of freedom, as in the Feynman-Vernon influence functional technique and the projected effective Hamiltonian. Our next task is to specify the projection operator for this purpose. In this case, the total Hamiltonian can be divided into three parts, the system ($\Sigma$), the environment ($E$) and the interaction ($I$) between the system and the environment:

\begin{align}
  H &= H_S + H_E + H_I.
\end{align}
The self-interaction of the system and/or the environment can also be included in  $H_I$. Furthermore, we assume that the initial density matrix $\rho$ is given by the direct product of the system density matrix $\rho_S$ and the environment density matrix $\rho_E$, all given at the initial time $t_0$:

$$\rho = \rho_S \otimes \rho_E.$$  \hfill (15)

We then define the projection operator as

$$PO = \text{Tr}_E[\rho_E O] \equiv \langle O \rangle_E$$  \hfill (16)

for any operator $O$. With this projection, we can replace an operator acting on the environment with a c-number. From the nature of the projection operator, we obtain the following relations:

$$PL_S = LS P, \quad QL_S = LS Q,$$

$$L_E P = 0, \quad L_E Q = L_E,$$  \hfill (17)

where $L_a O = [H_a, O]$ for $a = S, E, I$. Using these properties of the projection operator, we have the following relations:

$$QL_0 Q = QL_0,$$

$$Q e^{iL_0 t} Q = Q e^{iQ L_0 Q t} Q = Q e^{iL_0 t},$$  \hfill (19)

$$Q e^{iL_0 t} Q = Q e^{iQ L_0 Q t} Q = Q e^{iL_0 t},$$  \hfill (20)

where $L_0 = L_S + L_E$.

In the previous projection operator method, the additional relation $PL_E = 0$ is assumed to derive a systematic expansion. This relation is satisfied only in restricted cases, for example, when $\rho_E$ is an eigenstate of $H_E$ or a mixed state which includes diagonal components in the basis of eigenstates of $H_E$. Therefore, this relation is not satisfied for general $\rho_E$. Now, we take a coherent state for $\rho_E$ as an example:

$$\rho_E = e^{-|\alpha|^2} a a^\dagger \langle 0 | \langle 0 | e^{a^\dagger a} \equiv | \alpha \rangle \langle \alpha |.$$  \hfill (21)

The state $| \alpha \rangle$ is an eigenstate of the annihilation operator $a$:

$$a | \alpha \rangle = \alpha | \alpha \rangle.$$  \hfill (22)

The quantity $PL_E O$ is calculated as

$$PL_E O = \text{Tr}_E[\rho_E L_E O]$$

$$= \langle \alpha | [H_E O - OH_E] | \alpha \rangle$$

$$= E \alpha^* \langle \alpha | O | \alpha \rangle - E \alpha \langle \alpha | O a \rangle | \alpha \rangle,$$  \hfill (23)

where we assume $H_E = E a a^\dagger$. The r.h.s. of the above equation does not become zero for an arbitrary operator $O$. Therefore, we can see that the relation $PL_E = 0$ is not satisfied. From these facts, we can conclude that the expansion used in the previous projection operator method is not applicable to general initial states. We will derive a systematic expansion of the Heisenberg equation of motion without imposing the relation $PL_E = 0$. With this, we can calculate the time evolution by using more general initial states, but with the assumption Eq. (23).

With the above properties of the projection operators (16) and (17), the function $\Sigma(t, t_0)$ can be expressed as

$$\Sigma(t, t_0) = \int_{t_0}^{t} ds e^{-iL(t-s)} P S Q L e^{iLQ(t-s)}$$

$$= Q - e^{-iL(t-t_0)} Q e^{iLQ(t-t_0)}$$

$$= Q - e^{-iL_0(t-t_0)} C(t, t_0) D(t, t_0) e^{iL_0(t-t_0)}$$

$$= Q - e^{-iL_0(t-t_0)} Q e^{iL_0(t-t_0)} - e^{-iL_0(t-t_0)} (C(t, t_0) - 1) Q e^{iL_0(t-t_0)}$$

$$- e^{-iL_0(t-t_0)} (Q (D(t, t_0) - 1) e^{iL_0(t-t_0)}$$

$$- e^{-iL_0(t-t_0)} (C(t, t_0) - 1) Q (D(t, t_0) - 1) e^{iL_0(t-t_0)}.$$  \hfill (24)

Here, the functions $C(t, t_0)$ and $D(t, t_0)$ are expressed as
The operator \( C(t, t_0) \) derived in Appendix A. Then, we have
\[
C(t, t_0) = e^{iL_0(t-t_0)}e^{-iL(t-t_0)}
\]
\[
= 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{L}_I(t_1 - t_0)\hat{L}_I(t_2 - t_0) \cdots
\]
\[
\times \hat{L}_I(t_n - t_0),
\]
\[
(25)
\]
\[
D(t, t_0) = e^{iQL(t-t_0)}e^{-iQL_0(t-t_0)}
\]
\[
= 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{Q} \hat{L}_I(t_1 - t_0)\hat{L}_I(t_2 - t_0) \cdots
\]
\[
\times \hat{Q} \hat{L}_I(t_n - t_0),
\]
\[
(26)
\]
where
\[
\hat{L}_I(t - t_0) \equiv e^{iL_0(t-t_0)}L_I e^{-iL_0(t-t_0)},
\]
\[
(27)
\]
\[
\hat{L}_I(t - t_0) \equiv e^{iL_0(t-t_0)}L_I e^{-iL_0(t-t_0)}.
\]
\[
(28)
\]
The operator \( C(t, t_0) \) [\( D(t, t_0) \)] is a time [an anti-time] ordered function of Liouville operators. These expressions are derived in Appendix A. Then, we have
\[
P \Sigma(t, t_0) \frac{1}{1 - \Sigma(t, t_0)} = P \Sigma(t, t_0) \frac{1}{1 - Q \Sigma(t, t_0)}
\]
\[
= -P e^{-iL_0(t-t_0)}C(t, t_0)Q e^{iL_0(t-t_0)}
\]
\[
\times \frac{1}{1 + e^{-iL_0(t-t_0)}(C(t, t_0) - 1)Q e^{iL_0(t-t_0)}}.
\]
\[
(29)
\]
Note that the second line on the r.h.s. does not include \( D(t, t_0) \). The detailed derivation of Eq. \( (29) \) appears in Appendix B. Substituting Eq. \( (29) \) into Eq. \( (25) \), we obtain the final expression of the Heisenberg equation:
\[
\frac{d}{dt} O(t) = e^{iL(t-t_0)} P \hat{L} O(t_0)
\]
\[
- e^{iL(t-t_0)} P e^{-iL_0(t-t_0)}C(t, t_0)Q \frac{1}{1 + (C(t, t_0) - 1)Q e^{iL_0(t-t_0)}} e^{iL_0(t-t_0)}iLO(t_0)
\]
\[
+ Q e^{iLQ(t-t_0)} \frac{1}{1 - \Sigma(t, t_0)}iLO(t_0).
\]
\[
(30)
\]
When we expand \( P \Sigma(t, t_0)/(1 - \Sigma(t, t_0)) \) up to first order in the interaction \( H_I \), we have
\[
\frac{d}{dt} O(t) = e^{iL(t-t_0)} P e^{-iL_0(t-t_0)} P e^{iL_0(t-t_0)}iLO(t_0)
\]
\[
+ e^{iL(t-t_0)} P e^{-iL_0(t-t_0)} P \int_{t_0}^{t} ds e^{iL_0(s-t_0)}iL_I e^{-iL_0(s-t_0)} Q e^{iL_0(t-t_0)}iLO(t_0)
\]
\[
+ Q e^{iLQ(t-t_0)} \frac{1}{1 - \Sigma(t, t_0)}iLO(t_0).
\]
\[
(31)
\]
Here, we do not expand the third term on the r.h.s. of Eq. \( (31) \). This term becomes zero when we take the expectation value, and therefore we do not expand it. This expanded equation is used in §4.

It can be seen that Eq. \( (31) \) does not contain a time convolution integral, because of the form of the full time-evolution operator, \( e^{iL(t-t_0)} \), which operates from the left in the second term on the r.h.s. of the equation. If this did contain a time convolution integral, the form of the full time-evolution operator must be \( e^{iL(t-s)} \), where \( s \) is an integral variable. Such a time-convolution (TC) equation is discussed in Appendix C. In other formulations, e.g., the closed time-path formalism and the Feynman-Vernon influence functional technique, the derived equation of motion has a time convolution integral in general. Such a time convolution term is called a “memory term”. The derived
equation of motion is often solved using the Markov approximation. However, in our improved projection operator method (and the previous projection operator method), the equation without a time convolution integral, that is, the TCL equation is automatically obtained and it is not necessary to make the Markov approximation.

We would like to make some remarks regarding the operator $O(t_0)$. First, the choice of the operator $O(t_0)$ is not restricted to an operator of the system. We can choose an environment operator or a product of system and environment operators as the operator $O(t_0)$. This is different from the case of the previous projection operator method and the Uchiyama-Shibata (U-S) projection operator method, which was proposed recently, because those method use the condition $PO(t_0) = O(t_0)$, which is satisfied only when $O(t_0)$ is a system operator. Therefore, in our improved projection operator method, it is possible to examine the time evolution of conserved quantities which are composed of not only system operators but also environment operators. Any operator $O(t_0)$ which commutes with the total Hamiltonian is conserved, for any order of expansion, because $LO(t_0) = 0$ in Eqs. (13) and (30). This is a reasonable result, because conserved quantities should be time independent.

Until now, we have used the projection operator defined in Eq. (16). However, to derive Eq. (30), only the condition (19) is needed. Therefore, we can use a more general projection operator to derive Eq. (30) in the case that the condition (19) is met.

III. THE HEISENBERG EQUATION FOR A TIME-DEPENDENT HAMILTONIAN

In this section, we consider the case of a Hamiltonian with explicit time dependence. We distinguish the time dependence of operators from explicit time dependence. To do this, the time dependent Hamiltonian $H(t)$ is written with two time arguments:

$$H(t) = H(t, t),$$

(32)

where the first argument represents the explicit time dependence, and the second that of operators.

The time development of any operator $O$ which has no explicit time dependence can be expressed as

$$O(t) = e^{i \int_{t_0}^{t} ds L(s, t_0)} O(t_0),$$

(33)

where $t_0$ is the time at which we prepare an initial state. The operator $e^{i \int_{t_0}^{t} d\tau L(\tau, t_0)}$ is defined as follows:

$$e^{i \int_{t_0}^{t} d\tau L(\tau, t_0)} = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n L(t_n, t_0) L(t_{n-1}, t_0) \cdots L(t_1, t_0).$$

(34)

Similarly, $e^{-i \int_{t_0}^{t} d\tau L(\tau, t_0)}$ is defined as

$$e^{-i \int_{t_0}^{t} d\tau L(\tau, t_0)} = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n L(t_n, t_0) L(t_{n-1}, t_0) \cdots L(t_1, t_0).$$

(35)

Therefore, the time evolution of the operator included in the Hamiltonian is expressed as

$$H(t, t) = e^{i \int_{t_0}^{t} ds L(s, t_0)} H(t, t_0).$$

(36)

By using this relation, it becomes clear that the operator $O(t)$ given in Eq. (33) satisfies the Heisenberg equation

$$\frac{d}{dt} O(t) = e^{i \int_{t_0}^{t} ds L(s, t_0)} iL(t, t_0) O(t_0)$$

$$= e^{i \int_{t_0}^{t} ds L(s, t_0)} i[H(t, t_0), O(t_0)]$$

$$= i[H(t, t), O(t)].$$

(37)

Note the following properties:
\[
\frac{d}{ds} e^{\int_0^s dt L(s,t_0)} = -i L(s, t_0) e^{\int_0^s dt L(s,t_0)},
\]
\[
\frac{d}{ds} e^{\int_0^t dt L(s,t_0)} = i \int_0^t dt L(s, t_0) \left( e^{\int_0^s dt L(s,t_0)} \right),
\]
\[
\frac{d}{dt} e^{\int_0^t ds L(s,t_0)} = -i \int_0^t ds L(s, t_0) e^{\int_0^t ds L(s,t_0)} i \int_0^t dt L(s, t_0) e^{\int_0^t ds L(s,t_0)} = 0.
\]

From Eq. (40), we can derive the identities
\[
\frac{d}{dt} e^{\int_0^t ds L(s,t_0)} e^{-i \int_0^t ds L(s,t_0)} = -i \int_0^t ds L(s, t_0) i \int_0^t dt L(s, t_0) e^{\int_0^t ds L(s,t_0)} e^{-i \int_0^t ds L(s,t_0)} = 1.
\]

The following equations are derived similarly to those derived in the previous section. Corresponding to Eqs. (8) and (9), we obtain
\[
\frac{d}{dt} e^{\int_0^t ds L(s,t_0)} P = e^{\int_0^t ds L(s,t_0)} P i L(t, t_0) P + e^{\int_0^t ds L(s,t_0)} Q i L(t, t_0) P,
\]
\[
\frac{d}{dt} e^{\int_0^t ds L(s,t_0)} Q = e^{\int_0^t ds L(s,t_0)} P i L(t, t_0) Q + e^{\int_0^t ds L(s,t_0)} Q i L(t, t_0) Q.
\]

Equation (42) gives us
\[
\frac{d}{dt} e^{\int_0^t ds L(s,t_0)} Q = Q e^{\int_0^t ds L(s,t_0)} + \int_0^t ds e^{\int_0^s dt L(s,t_0)} P i L(t, t_0) e^{\int_0^t ds L(s,t_0)} Q e^{\int_0^t dt L(t, t_0)} Q
\]
\[
= Q e^{\int_0^t ds L(s,t_0) Q}
\]
\[
+ e^{\int_0^t ds L(s,t_0)} (P + Q) \int_0^t ds e^{\int_0^s dt L(s,t_0)} i \int_0^t dt L(t, t_0) e^{\int_0^t ds L(s,t_0)} i \int_0^t dt L(t, t_0) e^{\int_0^t ds L(s,t_0)} Q e^{\int_0^t dt L(t, t_0) Q}
\]
\[
= \{Q e^{\int_0^t ds L(s,t_0) Q} + e^{\int_0^t ds L(s,t_0)} P \Sigma_{ex}(t, t_0) \} \frac{1}{1 - \Sigma_{ex}(t, t_0)},
\]
where
\[
\Sigma_{ex}(t, t_0) = \int_0^t ds e^{\int_0^s dt L(s,t_0)} i \int_0^t dt L(t, t_0) P i L(s, t_0) Q e^{\int_0^t dt L(t, t_0) Q}.
\]

Therefore, we can obtain the following equation:
\[
\frac{d}{dt} O(t) = e^{\int_0^t ds L(s,t_0)} P i L(t, t_0) O(t_0)
\]
\[
+ e^{\int_0^t ds L(s,t_0)} P \Sigma_{ex}(t, t_0) \frac{1}{1 - \Sigma_{ex}(t, t_0)} i L(t, t_0) O(t_0)
\]
\[
+ Q e^{\int_0^t ds L(s,t_0) Q} \frac{1}{1 - \Sigma_{ex}(t, t_0)} i L(t, t_0) O(t_0).
\]

This equation is equivalent to the Heisenberg equation, and corresponds to Eq. (13) in §2. When we ignore the explicit time dependence of the Hamiltonian, this equation agrees with Eq. (13).

Now we define the projection operator. As in the previous section, we consider the case in which the total Hamiltonian can be divided into three parts, the system (S), the environment (E), and the interaction (I) between the system and the environment:
By using the above functions and mathematical induction, we obtain

\[ H(t, t) = H_S(t, t) + H_E(t, t) + H_I(t, t). \]  (47)

The self-interaction of the system and/or the environment can also be included in \( H_I(t, t) \). Furthermore, we assume the same form of the initial density matrix, given at the initial time \( t_0 \), as in Eq. (15). We then define the projection operator as

\[ PO = \text{Tr}_E[\rho_E O] \equiv \langle O \rangle_E \]  (48)

for any operator \( O \). From the nature of the projection operator, we obtain the relations

\[ PL_S(t, t_0) = L_S(t, t_0)P, \quad QL_S(t, t_0) = L_S(t, t_0)Q, \]  (49)

\[ L_E(t, t_0)P = 0, \quad L_E(t, t_0)Q = L_E(t, t_0), \]  (50)

where \( L_a(t, t_0)O = [H_a(t, t_0), O] \) for \( a = S, E, I \). Using these properties of the projection operator, we have

\[ QL_0(t, t_0)Q = QL_0(t, t_0), \]

\[ Qe^{-i\int_{t_0}^{t}d\tau L(\tau, t_0)} = Qe^{-i\int_{t_0}^{t}d\tau L(\tau, t_0)}Q. \]  (51)

where \( L_0(t, t_0) = L_S(t, t_0) + L_E(t, t_0) \).

Next, \( \Sigma_{ex}(t, t_0) \) is expressed as

\[ \Sigma_{ex}(t, t_0) = \int_{t_0}^{t}dse^{-i\int_{t_0}^{\tau}d\tau L(\tau, t_0)}P_iL(s, t_0)Qe^{-i\int_{\tau}^{t}d\tau L(\tau, t_0)}Q \]

\[ = Q - e^{-i\int_{t_0}^{t}d\tau L(\tau, t_0)}Q \]

\[ = Q - U_0^{-1}(t, t_0)C_{ex}(t, t_0)QD_{ex}(t, t_0)U_0(t, t_0) \]

\[ = Q - U_0^{-1}(t, t_0)QU_0(t, t_0) - U_0^{-1}(t, t_0)(C_{ex}(t, t_0) - 1)QU_0(t, t_0) \]

\[ - U_0^{-1}(t, t_0)(C_{ex}(t, t_0) - 1)QD_{ex}(t, t_0)U_0(t, t_0) \]  (52)

Here, we have introduced the following functions:

\[ C_{ex}(t, t_0) = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 \cdots \int_{t_0}^{t_{n-1}}dt_n \]

\[ \times \hat{L}_{\tau}^{ex}(t_1, t_0)\hat{L}_{\tau}^{ex}(t_2, t_0) \cdots \hat{L}_{\tau}^{ex}(t_n, t_0), \]  (54)

\[ D_{ex}(t, t_0) = 1 + \sum_{n=1}^{\infty}i^n \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 \cdots \int_{t_0}^{t_{n-1}}dt_nQ\hat{L}_{\tau}^{Q ex}(t_n, t_0) \]

\[ \times Q\hat{L}_{\tau}^{Q ex}(t_{n-1}, t_0) \cdots \hat{L}_{\tau}^{Q ex}(t_1, t_0), \]  (55)

where

\[ U_0(t, t_0) = e^{-i\int_{t_0}^{t}dsL_0(s, t_0)} \]  (56)

\[ \hat{L}_{\tau}^{ex}(t, t_0) = U_0(t, t_0)L_I(t, t_0)U_0^{-1}(t, t_0), \]  (57)

\[ \hat{L}_{\tau}^{Q ex}(t, t_0) = U_0(t, t_0)L_I(t, t_0)QU_0^{-1}(t, t_0). \]  (58)

By using the above functions and mathematical induction, we obtain

\[ \frac{d}{dt}O(t) \]

\[ = e^{-i\int_{t_0}^{t}d\tau L(s, t_0)}PiL(t, t_0)O(t_0) \]

\[ - e^{-i\int_{t_0}^{t}d\tau L(s, t_0)}PQU_0^{-1}(t, t_0)C_{ex}(t, t_0)Q \]

\[ \times \frac{1}{1 + (C_{ex}(t, t_0) - 1)Q}U_0(t, t_0)L(t, t_0)O(t_0) \]

\[ + Qe^{-i\int_{t_0}^{t}d\tau L(s, t_0)}Q \]

\[ \frac{1}{1 - \Sigma_{ex}(t, t_0)}iL(t, t_0)O(t_0). \]  (59)
The projection operator is defined in Eq. (16). When we substitute the harmonic oscillator model, which has no explicit time dependence. The Hamiltonian is divided into three parts:

\[ H = \sum_k \left\{ \Omega_k (A_k^\dagger A_k - B_k^\dagger B_k) + i\Gamma_k (A_k^\dagger B_k^\dagger - A_k B_k) \right\}. \]  

The \( k \) dependence of \( \Gamma_k \) is such that \( \sum_k \Gamma_k \) is finite. We impose the following commutation relation:

\[ [A_i, A_j^\dagger] = [B_i, B_j^\dagger] = \delta_{i,j}, \quad [A_i, B_j] = 0. \]  

The time development of the operator \( A_k^\dagger(t) \) is exactly solved as

\[ A_k^\dagger(t) = e^{i\Omega_k t} \left( \cosh(\Gamma_k t)A_k^\dagger(0) + \sinh(\Gamma_k t)B_k(0) \right). \]  

This is a solution of the following equation of motion:

\[ \frac{d}{dt} \langle A_k^\dagger(t) \rangle = i\Omega_k \langle A_k^\dagger(t) \rangle + \Gamma_k \tanh(\Gamma_k t) \langle A_k^\dagger(t) \rangle + e^{i\Omega_k t} \frac{1}{\cosh(\Gamma_k t)} \langle B_k(0) \rangle. \]  

Here, \( \langle \cdot \rangle \) represents the expectation value taken with respect to an arbitrary density matrix \( \rho \) that satisfies the condition (35).

Now, we apply our method to this model, and compare the result with the exact equation (64). We consider the degree of freedom of the system to be \( A_k \) and that of the environment to be \( B_k \). The Hamiltonian is divided into three parts:

\[ H_S = \sum_k \Omega_k A_k^\dagger A_k, \]  
\[ H_E = -\sum_k \Omega_k B_k^\dagger B_k, \]  
\[ H_I = \sum_k i\Gamma_k (A_k^\dagger B_k^\dagger - A_k B_k). \]  

The projection operator is defined in Eq. (40). When we substitute \( t_0 = 0 \) and \( O(0) = A_k^\dagger(0) \) into Eq. (31), we have

\[ \frac{d}{dt} \langle A_k^\dagger(t) \rangle = i\Omega_k \langle A_k^\dagger(t) \rangle + \Gamma_k^2 t \langle A_k^\dagger(t) \rangle + e^{i\Omega_k t} \Gamma_k \langle B_k(0) \rangle. \]  

Here, we do not expand the third term on the r.h.s. of Eq. (60). This equation is the TCL equation in the case of a time-dependent Hamiltonian, and it becomes the same equation as Eq. (71) when we ignore the explicit time dependence in the Hamiltonian. The TC equation can be obtained as in Appendix C.
This equation is identical with Eq. (64) when we expand \( \tanh(\Gamma_k t) \) and \( 1/\cosh(\Gamma_k t) \) to lowest order in \( \Gamma_k \). Furthermore, higher-order contributions are easily calculable. By using Eq. (D2) in Appendix D instead of Eq. (31), we can obtain the equation of motion with a higher-order term:

\[
\frac{d}{dt} \langle A_k(t) \rangle = i\Omega_k \langle A_k(t) \rangle + \Gamma_k^2 t \langle A_k(t) \rangle + e^{i\Omega_k t} \Gamma_k \left( 1 - \frac{1}{2} \Gamma_k^2 t^2 \right) \langle B_k(0) \rangle.
\] (69)

This equation also coincides with Eq. (64) with the appropriate perturbative expansion of \( \Gamma_k \).

Now, we apply this model to the previous projection operator method. The approximate equation in that method which corresponds to Eq. (31) is

\[
\frac{d}{dt} O(t) = e^{iL(t-t_0)} P_i L_0(t_0) + e^{iL(t-t_0)} \int_{t_0}^t ds e^{-iL(s-t_0)} P_i L_0 e^{iL_0(s-t_0)} Q_i L_0(t_0) + Q e^{iLQ(t-t_0)} L_0(t_0).
\] (70)

Here, we use the same initial density matrix and projection operator. Now, the equation of motion for \( A_k(t) \) is

\[
\frac{d}{dt} \langle A_k(t) \rangle = i\Omega_k \langle A_k(t) \rangle + \Gamma_k^2 t \langle A_k(t) \rangle + \Gamma_k \langle B_k(0) \rangle.
\] (71)

It seems that the factor \( e^{i\Omega_k t} \) in the third term on the r.h.s. of Eq. (64) cannot be reproduced. However, Eq. (71) gives consistent results only when we use a restricted density matrix that satisfies the condition \( P L_E = 0 \). This condition implies

\[ P L_E B_k(0) = 0 \rightarrow \langle B_k(0) \rangle = 0. \] (72)

Therefore, the above equation is consistent with the exact result. From this argument, we can see that the previous projection operator method is valid only for a restricted initial state.

V. CONCLUSIONS AND DISCUSSION

We have given a new expansion of the Heisenberg equation of motion with a projection operator in the two cases of Hamiltonian both with and without explicit time dependence. In our method, one can prepare more general initial states which are forbidden in the previously applied projection operator method. Until now, the projection operator has been chosen as Eq. (16). However, if the condition \( Q L_0 Q = Q L_0 \) (in other words, \( P L_0 P = L_0 P \)) is satisfied, one can use Eqs. (30) and (59) for any kind of projection operators.

Recently, a similar expansion method with a projection operator was proposed by Uchiyama and Shibata. In their method, one can expand the Heisenberg equation without such a restriction on projection operators as \( Q L_0 Q = Q L_0 \), which is needed in our method. However, they impose some conditions on the operator \( O(t_0) \) whose time evolution we want to calculate. The first condition is that the time evolution with the unperturbed Hamiltonian should be solved as

\[ e^{iL_0(t-t_0)} O(t_0) = f(t, t_0) O(t_0), \] (73)

where \( f(t, t_0) \) is a c-number. The second condition is

\[ P O(t_0) = O(t_0). \] (74)

This condition implies that we cannot calculate the time evolution of environment operators, when we use the definition (14). However, it is possible to formulate the method without the above conditions. This extended Uchiyama-Shibata projection operator method is explained in Appendix E.
APPENDIX A: THE DEFINITION OF THE OPERATORS $C$ AND $D$

We define the operators $C(t, t_0)$ and $D(t, t_0)$ as

$$
e^{-iL(t-t_0)} = e^{-iL_0(t-t_0)}C(t, t_0),$$

$$C(t, t_0) = e^{iL_0(t-t_0)}e^{-iL(t-t_0)},$$

$$e^{iQL(t-t_0)} = D(t, t_0)e^{iQL_0(t-t_0)},$$

$$D(t, t_0) = e^{iQL(t-t_0)}e^{-iQL_0(t-t_0)}.$$  \hfill (A1)

These operators satisfy the following differential equations:

$$\frac{d}{dt}C(t, t_0) = e^{iL_0(t-t_0)}(iL_0 - iL)e^{-iL(t-t_0)}$$

$$= -i\tilde{L}_I(t, t_0)C(t, t_0),$$

$$\frac{d}{dt}D(t, t_0) = e^{iQL(t-t_0)}Q(iL - iL_0)Qe^{-iQL_0(t-t_0)}$$

$$= D(t, t_0)Q\tilde{L}_I^Q(t, t_0),$$  \hfill (A3)

where

$$\tilde{L}_I(t, t_0) = e^{iL_0(t-t_0)}L_Ie^{-iL_0(t-t_0)},$$

$$\tilde{L}_I^Q(t, t_0) = e^{iL_0(t-t_0)}L_IQe^{-iL_0(t-t_0)}.$$  \hfill (A4)

From these differential equations, we obtain

$$C(t, t_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \tilde{L}_I(t_1 - t_0)\tilde{L}_I(t_2 - t_0)$$

$$\times \cdots \tilde{L}_I(t_n - t_0),$$  \hfill (A7)

$$D(t, t_0) = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n Q\tilde{L}_I^Q(t_1 - t_0)Q\tilde{L}_I^Q(t_2 - t_0)$$

$$\times \cdots Q\tilde{L}_I^Q(t_n - t_0).$$  \hfill (A8)

Similarly, $C_{ex}(t, t_0)$ and $D_{ex}(t, t_0)$ are defined as

$$e^{-i\int_{t_0}^{s} dsL(s, t_0)} = U_0^{-1}(t, t_0)C_{ex}(t, t_0),$$

$$C_{ex}(t, t_0) = U_0(t, t_0)e^{-i\int_{t_0}^{s} dsL(s, t_0)},$$

$$i\int_{t_0}^{s} dsQL(s, t_0)Q = D_{ex}(t, t_0)e^{-i\int_{s}^{t} dsQL_0(s, t_0)Q},$$

$$D_{ex}(t, t_0) = e^{-i\int_{t_0}^{s} dsQL(s, t_0)Q - i\int_{t_0}^{s} dsQL_0(s, t_0)Q}.$$  \hfill (A9)

Differential equations can be constructed again, and are solved to yield

$$C_{ex}(t, t_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \tilde{L}_{I_{ex}}(t_1, t_0)\tilde{L}_{I_{ex}}(t_2, t_0)$$

$$\times \cdots \tilde{L}_{I_{ex}}(t_n, t_0),$$  \hfill (A11)

$$D_{ex}(t, t_0) = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n Q\tilde{L}_{I_{ex}}^Q(t_1, t_0)Q\tilde{L}_{I_{ex}}^Q(t_2, t_0)$$

$$\times \cdots Q\tilde{L}_{I_{ex}}^Q(t_n, t_0),$$  \hfill (A12)

where

$$\tilde{L}_{I_{ex}}(t, t_0) = U_0(t, t_0)L_I(t, t_0)U_0^{-1}(t, t_0),$$

$$\tilde{L}_{I_{ex}}^Q(t, t_0) = U_0(t, t_0)L_I(t, t_0)QU_0^{-1}(t, t_0).$$  \hfill (A13)

(A14)
APPENDIX B: THE TRANSFORMATION OF THE OPERATOR $\Sigma(T, T_0)$

The operator $\Sigma(t, t_0)$ in §2 can be expressed as

$$\Sigma(t, t_0) = \int_{t_0}^{t} ds e^{-iL(t-s)} P_iLQ e^{iL(t-s)}Q$$

$$= -\int_{t_0}^{t} ds e^{-iL(t-s)} P \frac{d}{ds} e^{iL(t-s)}Q$$

$$= \left\{ P - e^{-iL(t-t_0)} Pe^{iL(t-t_0)}Q - \int_{t_0}^{t} ds e^{-iL(t-s)} Pe^{iL(t-s)}Q \right\}$$

$$= \left\{ P - e^{-iL(t-t_0)} Pe^{iL(t-t_0)}Q - 1 + e^{-iL(t-t_0)} e^{iL(t-t_0)}Q \right\}$$

$$= Q - e^{-iL_0(t-t_0)} Q e^{iL_0(t-t_0)}Q$$

Similarly, $\Sigma_{ex}(t, t_0)$ in §3 is given by

$$\Sigma_{ex}(t, t_0) = \int_{t_0}^{t} ds e^{-i\int_{t_0}^{t} d\tau L(\tau, t_0) + \int_{t_0}^{t} d\tau L(\tau, t_0)} P_iL(s, t_0)Q e^{i\int_{t_0}^{t} d\tau L(\tau, t_0)Q}$$

$$= -\int_{t_0}^{t} ds e^{-i\int_{t_0}^{t} d\tau L(\tau, t_0) + \int_{t_0}^{t} d\tau L(\tau, t_0)} P \frac{d}{ds} e^{i\int_{t_0}^{t} d\tau L(\tau, t_0)Q}$$

$$= -\int_{t_0}^{t} ds e^{-i\int_{t_0}^{t} d\tau L(\tau, t_0) + \int_{t_0}^{t} d\tau L(\tau, t_0)} P - Pe^{-i\int_{t_0}^{t} d\tau L(\tau, t_0)Q}$$

$$= -\int_{t_0}^{t} ds e^{-i\int_{t_0}^{t} d\tau L(\tau, t_0) + \int_{t_0}^{t} d\tau L(\tau, t_0)} Q - Q e^{-i\int_{t_0}^{t} d\tau L(\tau, t_0)Q}$$

$$= Q - U_{0}^{-1}(t, t_0)C_{ex}(t, t_0)Q D_{ex}(t, t_0)e^{iQL_{0}(t-t_0)}Q.$$  \hspace{1cm} (B1)

To allow for simultaneous discussion, we introduce the following notation:

$$C(t) = \begin{cases} e^{-iL_{0}(t-t_0)} C(t, t_0) e^{iL_{0}(t-t_0)} & \text{for the case of § 2} \\ U_{0}^{-1}(t, t_0) C_{ex}(t, t_0) U_{0}(t, t_0) & \text{for the case of § 3}, \end{cases} \hspace{1cm} (B3)$$

$$D(t) = \begin{cases} e^{-iL_{0}(t-t_0)} D(t, t_0) e^{iL_{0}(t-t_0)} & \text{for the case of § 2} \\ U_{0}^{-1}(t, t_0) D_{ex}(t, t_0) U_{0}(t, t_0) & \text{for the case of § 3}, \end{cases} \hspace{1cm} (B4)$$

$$Q(t) = \begin{cases} e^{-iL_{0}(t-t_0)} Q e^{iL_{0}(t-t_0)} & \text{for the case of § 2} \\ U_{0}^{-1}(t, t_0) Q U_{0}(t, t_0) & \text{for the case of § 3}. \end{cases} \hspace{1cm} (B5)$$

Using the mathematical induction, we confirm the following relation:
\[ P\Sigma(t, t_0) [Q\Sigma(t, t_0)]^n = \left[ (-1)^{n-1} P\{Q(t)(C(t) - 1)\}^n Q(t) + (-1)^{n-1} P\{(C(t) - 1)Q(t)\}^{n+1} \right] \\
= \sum_{l=0}^{n-1} (-1)^l \{(C(t) - 1)Q(t)\}^l Q(t) (D(t) - 1)(Q\Sigma(t, t_0))^{n-1-l} \\
+ P \sum_{l=0}^{n} \{(C(t) - 1)Q(t)\}^{l+1} (D(t) - 1)(Q\Sigma(t, t_0))^{n-1-l} \\
- P \sum_{l=0}^{n} (-1)^l \{(C(t) - 1)Q(t)\}^l Q(t) (D(t) - 1)(Q\Sigma(t, t_0))^{n-l} \\
- P \sum_{l=0}^{n} (-1)^l \{(C(t) - 1)Q(t)\}^l (D(t) - 1)(Q\Sigma(t, t_0))^{n-l}, \tag{B6} \]

where \( n \) is integer and \( n \geq 1 \). The second and third terms in \( P\Sigma(t, t_0)(Q\Sigma(t, t_0))^n \) and the fourth and fifth terms in \( P\Sigma(t, t_0)(Q\Sigma(t, t_0))^{n-1} \) cancel. The fourth and fifth terms in \( P\Sigma(t, t_0)(Q\Sigma(t, t_0))^n \) and the second and third terms in \( P\Sigma(t, t_0)(Q\Sigma(t, t_0))^{n+1} \) also cancel. Therefore, only the first term survives. As a result, all the terms including \( D(t) \) disappear. Finally, noting the relation \( \Sigma(t, t_0) = \Sigma(t_0)Q \), we find that \( P\Sigma(t, t_0) \frac{1}{1 - Q\Sigma(t_0)} \) can be expressed as follows:

\[
P\Sigma(t, t_0) \frac{1}{1 - \Sigma(t, t_0)} = P\Sigma(t, t_0) \frac{1}{1 - Q\Sigma(t, t_0)} \\
= P\Sigma(t, t_0) \sum_{n=0}^{\infty} (Q\Sigma(t, t_0))^n \\
= -P \sum_{n=0}^{\infty} [(-Q(t)(C(t) - 1))^n Q(t) - (-Q(t)(C(t) - 1))^n] \\
= -PQ(t) \frac{1}{1 + (C(t) - 1)Q(t)} - P \frac{(C(t) - 1)Q(t)}{1 + (C(t) - 1)Q(t)} \\
= -PC(t)Q(t) \frac{1}{1 + (C(t) - 1)Q(t)}. \tag{B7} \]

**APPENDIX C: THE DERIVATION OF THE TIME-CONVOLUTION EQUATION**

Here, we derive the equation with a time-convolution integral. In this appendix, we consider the case of a Hamiltonian without explicit time dependence. The discussion, however, can also be applied to the case of a Hamiltonian with explicit time dependence. We substitute Eq. (11), (instead of (11)) into Eq. (9), and we obtain

\[
\frac{d}{dt} O(t) = e^{iL(t-t_0)} P_i LO(t_0) + \int_{t_0}^{t} ds e^{iL(s-t_0)} P_i LQ e^{iLQ(s-t_0)} iLO(t_0) \\
+ Q e^{iLQ(t-t_0)} iLO(t_0) \\
= e^{iL(t-t_0)} P_i LO(t_0) + \int_{t_0}^{t} ds e^{iL(t-s)} P_i LQ e^{iLQ(t-s-t_0)} iLO(t_0) \\
+ Q e^{iLQ(t-t_0)} iLO(t_0). \tag{C1} \]

This equation is equivalent to the Heisenberg equation. For the existence of the operator \( e^{iL(t-s)} \), the second term on the r.h.s. of the equation has a time-convolution integral. Therefore, this equation is a called a time-convolution (TC) equation. This can be rewritten by using \( D(t, t_0) \) as

\[
\frac{d}{dt} O(t) = e^{iL(t-t_0)} P_i LO(t_0) + \int_{t_0}^{t} ds e^{iL(t-s)} P_i LQ D(s, t_0) e^{iQLQ(s-t_0)} iLO(t_0) \\
+ Q e^{iLQ(t-t_0)} iLO(t_0). \tag{C2} \]
By expanding $\mathcal{D}(t,t_0)$ to lowest order, we have

$$\frac{d}{dt}O(t) = e^{iL(t-t_0)}P\overline{LO}(t_0) + \int_{t_0}^{t}dse^{iL(t-s)}P_iLQe^{iQL_0Q(s-t_0)}iLO(t_0)$$

$$+ Qe^{iLQ(t-t_0)}iLO(t_0). \quad (C3)$$

At this level of expansion, the difference between the equation in the improved projection operator method and that in the previous projection operator method cannot be observed, because this difference is included in the higher-order term in $\mathcal{D}(t,t_0)$. In the case of the TC equation, the expansion (C2) is always valid, because no restriction is imposed on the projection operator. This is different from the case of the TCL equation, for which the condition $QL_0Q = QL_0$ must be satisfied.

**APPENDIX D: THE DERIVATION OF HIGHER-ORDER TERMS**

We expand $\mathcal{C}(t,t_0)/\left\{1 + (\mathcal{C}(t,t_0) - 1)\right\}$ to second order in the interaction:

$$\mathcal{C}(t,t_0)Q \frac{1}{1 + (\mathcal{C}(t,t_0) - 1)}Q$$

$$= \mathcal{C}(t,t_0)Q\left\{1 - (\mathcal{C}(t,t_0) - 1)Q + (\mathcal{C}(t,t_0) - 1)Q(\mathcal{C}(t,t_0) - 1)Q + \cdots \right\}$$

$$\sim \left\{1 - i \int_{t_0}^{t}dt_1 \tilde{L}_1(t_1 - t_0) + (-i)^2 \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 \tilde{L}_2(t_1 - t_0)\tilde{L}_1(t_2 - t_0)\right\}Q$$

$$\times \left\{1 + i \int_{t_0}^{t}dt_1 \tilde{L}_1(t_1 - t_0)Q - (-i)^2 \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 \tilde{L}_2(t_1 - t_0)\tilde{L}_1(t_2 - t_0)Q\right\}$$

$$(-i)^2 \int_{t_0}^{t}dt_1 \tilde{L}_1(t_1 - t_0)Q \int_{t_0}^{t}dt_2 \tilde{L}_2(t_2 - t_0)Q$$

$$\sim Q - P \int_{t_0}^{t}dt_1 i\tilde{L}_1(t_1 - t_0)Q + P \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 i\tilde{L}_2(t_1 - t_0)P\tilde{L}_1(t_2 - t_0)Q$$

$$- P \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 i\tilde{L}_1(t_2 - t_0)Q i\tilde{L}_1(t_1 - t_0)Q. \quad (D1)$$

In substituting this into Eq. (B3), we have

$$\frac{d}{dt}O(t)$$

$$= e^{iL(t-t_0)}P e^{-iL_0(t-t_0)}P e^{iL_0(t-t_0)}iLO(t_0)$$

$$+ e^{iL(t-t_0)}P e^{-iL_0(t-t_0)}P \int_{t_0}^{t}ds i\tilde{L}_1(s - t_0)Q e^{iL_0(t-t_0)}iLO(t_0)$$

$$- e^{iL(t-t_0)}P e^{-iL_0(t-t_0)}P \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 i\tilde{L}_1(t_1 - t_0)P i\tilde{L}_1(t_2 - t_0)Q e^{iL_0(t-t_0)}iLO(t_0)$$

$$+ e^{iL(t-t_0)}P e^{-iL_0(t-t_0)}P \int_{t_0}^{t}dt_1 \int_{t_0}^{t_1}dt_2 i\tilde{L}_1(t_2 - t_0)Q i\tilde{L}_1(t_1 - t_0)Q e^{iL_0(t-t_0)}iLO(t_0)$$

$$+ Q e^{iLQ(t-t_0)} \frac{1}{1 - \Sigma(t,t_0)} iLO(t_0). \quad (D2)$$

The third and fourth lines on the r.h.s. of this equation are higher-order contributions.

---

1 It may seem difficult to solve $e^{iQL_0PQ(s-t_0)}iLO(t_0)$, because of the operator $Q$. However, it can be solved using the following differential equation: $\frac{d}{dt} e^{iQL_0PQ(s-t_0)}iLO(t_0) = e^{iQL_0PQ(s-t_0)}iQL_0PQ_iLO(t_0)$. 

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APPENDIX E: THE EXTENDED UCHIYAMA-SHIBATA PROJECTION OPERATOR METHOD

Here we explain the extended Uchiyama-Shibata (U-S) projection operator method. The time evolution of an operator can be written as

\[
O(t) = e^{\frac{i}{\hbar} \int_{t_0}^{t} ds L(s, t_0)} O(t_0)
\]

\[
= \hat{U}_-(t, t_0)(U_0^{-1}(t, t_0))^{\dagger}O(t_0),
\]

where

\[
\hat{U}_-(t, t_0) = e^{\frac{i}{\hbar} \int_{t_0}^{t} ds \hat{L}_I(s, t_0)}.
\]

(E2)

Here \(\hat{L}_I(t, t_0)\) is the same as that defined in §2. Now, we expand all the fields with creation and annihilation operators at \(t_0\), and prepare the initial state at \(t_0\). The time evolution is determined by the following equation:

\[
\frac{d}{dt}O(t) = \frac{d}{dt} \hat{U}_-(t, t_0) \cdot (U_0^{-1}(t, t_0))^{\dagger}O(t_0) + \hat{U}_-(t, t_0) \frac{d}{dt}(U_0^{-1}(t, t_0))^{\dagger}O(t_0).
\]

(E3)

Let us now discuss the time evolution of \(\hat{U}_-(t, t_0)\). We introduce a generic projection operator which has the following properties:

\[
P^2 = P, \quad Q = 1 - P, \quad PQ = QP = 0.
\]

(E4)

(E5)

(E6)

These are the same as those introduced in §2. By using the projection operators, we obtain the following differential equation:

\[
\frac{d}{dt} \hat{U}_-(t, t_0) = \hat{U}_-(t, t_0)Pi\hat{L}_I(t, t_0) + \hat{U}_-(t, t_0)Qi\hat{L}_I(t, t_0).
\]

(E7)

We can construct the differential equation for \(\hat{U}_-(t, t_0)Q\) and solve it as we have done in §2:

\[
\hat{U}_-(t, t_0)Q = Q\hat{u}_-(t, t_0) + \int_{t_0}^{t} ds \hat{U}_-(s, t_0)Pi\hat{L}_I(s, t_0)Q\hat{u}_-(t, s)
\]

\[
= [Q\hat{u}_-(t, t_0) - \hat{U}_-(t, t_0)P(\hat{\Theta}_-^{-1}(t, t_0) - 1)]\hat{\Theta}_-(t, t_0),
\]

(E8)

(E9)

where

\[
\hat{u}_-(t, t_0) = e_{\rightarrow t_0}^{\int_{t_0}^{t} ds \hat{L}_I(s, t_0)}Q,
\]

(E10)

\[
\hat{U}_+(t, t_0) = e_{\leftarrow t_0}^{\int_{t_0}^{t} ds \hat{L}_I(s, t_0)},
\]

(E11)

\[
\hat{\Theta}_-(t, t_0) = [1 - \int_{t_0}^{t} ds \hat{U}_+(t, s)Pi\hat{L}_I(s, t_0)Q\hat{u}_-(t, s)]^{-1}.
\]

(E12)

When we substitute Eq. (E8) into (E7), we can obtain the TC equation. On the other hand, substituting Eq. (E9) into (E7), the TCL equation can be derived. Here we discuss only the TCL equation. We obtain

\[
\frac{d}{dt} \hat{U}_-(t, t_0) = \hat{U}_-(t, t_0)Pi\hat{L}_I(t) - \hat{U}_-(t, t_0)P(1 - \hat{\Theta}_-(t))i\hat{L}_I(t) + Q\hat{u}_-(t, t_0)\hat{\Theta}_-(t)i\hat{L}_I(t).
\]

(E13)

When we substitute this result into Eq. (E3), we obtain
\[
\frac{d}{dt} O(t) = e^{i \int_{t_0}^{t} dsL(s,t_0)} iL_0 O(t_0) + e^{i \int_{t_0}^{t} dsL(s,t_0)} (U_0(t,t_0))^{\dagger} P i \tilde{L}_I(t,t_0) (U_0^{-1}(t,t_0))^{\dagger} O(t_0) \\
- e^{i \int_{t_0}^{t} dsL(s,t_0)} (U_0(t,t_0))^{\dagger} P (1 - \tilde{\Theta}_-(t_0)) i \tilde{L}_I(t)(U_0^{-1}(t,t_0))^{\dagger} O(t_0) \\
+ Q \hat{u}_(t,t_0) \hat{\Theta}_-(t_0) i \tilde{L}_I(t,t_0) (U_0^{-1}(t,t_0))^{\dagger} O(t_0). 
\]

(E14)

This equation is equivalent to the Heisenberg equation of motion and corresponds to Eq. (13). Up to second order, we have

\[
\frac{d}{dt} O(t) = e^{i \int_{t_0}^{t} dsL(s,t_0)} iL_0 O(t_0) + e^{i \int_{t_0}^{t} dsL(s,t_0)} (U_0(t,t_0))^{\dagger} P i \tilde{L}_I(t,t_0) (U_0^{-1}(t,t_0))^{\dagger} O(t_0) \\
+ Q \hat{u}_(t,t_0) \hat{\Theta}_-(t_0) i \tilde{L}_I(t,t_0) (U_0^{-1}(t,t_0))^{\dagger} O(t_0). 
\]

(E15)

Here, we do not expand the third term on the r.h.s. of Eq. (E13).

The derivation given here is not the exactly same as that proposed by Uchiyama and Shibata. We have modified the following two points. In the original paper of Uchiyama and Shibata, they imposed the condition

\[(U_0^{-1}(t,t_0))^{\dagger} O(t_0) = f(t,t_0) O(t_0),\]

(E16)

where \(f(t,t_0)\) is a c-number function, and \(O(t_0)\) is the operator seen on the r.h.s. of Eq. (E14). However, we have not used this condition in the derivation given here. Furthermore, they imposed the condition

\[PO(t_0) = O(t_0).\]

(E17)

It is easily seen that no condition needs to be imposed to derive the extended equation in the extended U-S method, while in the method given in this paper the condition \(QL_0Q = QL_0\) is required. However, there is a merit to our method: All quantities which commute with the total Hamiltonian are conserved in the time evolution regardless of the order of the expansion of the equation. It is not clear that this property is satisfied in the extended U-S method.

This can be shown concretely. We consider the following Hamiltonian:

\[H = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a^2 b^2 + a^{12} b^2).\]

(E18)

The operator \(a\) is the degree of freedom of the system, and the operator \(b\) is that of the environment. The total Hamiltonian can be divided into the following three parts:

\[H_S = \omega_a a^\dagger a,\]

(E19)

\[H_E = \omega_b b^\dagger b,\]

(E20)

\[H_I = g(a^2 b^2 + a^{12} b^2).\]

(E21)

We use the same projection operator as in Eq. (14). Now, we calculate the time evolution of the total Hamiltonian \(H\) in the extended U-S method. We expand \((1 - \tilde{\Theta}_-(t))\) to zeroth order in the interaction in Eq. (E14). This means that we ignore the third term on the r.h.s. of the equation. The equation of motion is expressed as

\[
\frac{d}{dt} \langle H(t) \rangle = \langle e^{iLt} iL_0 H + e^{iLt} e^{-iL_0 t} P e^{iL_0 t} iL_I H \rangle \\
= 2i g(\omega_a - \omega_b)(-a^2(t)b^2(t)) + \langle a^{12}(t)b^2(t) \rangle \\
+ e^{2i\omega_b t}(a^2(t)b^2(t) - e^{-2i\omega_b t} a^{12}(t)b^2(t)), 
\]

(E22)

where we take \(t_0 = 0\). The total Hamiltonian has no time dependence originally. However, it is not obvious whether the r.h.s. of the equation becomes zero for arbitrarily states.

On the other hand, the time dependence of the Hamiltonian calculated using our method vanishes: \(\frac{d}{dt} H(t) = 0\). This can be easily shown. To begin, we substitute \(O = H\) into Eq. (31). Then, we must calculate the commutation relation \(LO = LH\) first. This quantity clearly becomes zero. This is satisfied regardless of the order of the expansion and Hamiltonian considered. Similarly, other quantities that commute with the total Hamiltonian are also conserved.
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