A NOTE ABOUT THE NOWICKI CONJECTURE ON WEITZENBÖCK DERIVATIONS

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Abstract. We reduce the Nowicki conjecture on Weitzenböck derivations of polynomial algebras to a well known problem of classical invariant theory.

1. Let $\mathbb{K}$ be a field of characteristic 0. A linear locally nilpotent derivation $\mathcal{D}$ of the polynomial algebra $\mathbb{K}[Z] = \mathbb{K}[z_1, z_2, \ldots, z_m]$ is called a Weitzenböck derivation. It is well known that the kernel

$$\ker \mathcal{D} := \{ f \in \mathbb{K}[Z] \mid \mathcal{D}(f) = 0 \}$$

of the linear locally nilpotent derivation $\mathcal{D}$ is a finitely generated algebra, see [13, 11, 12].

Let $\mathbb{K}[X,Y] = \mathbb{K}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the polynomial $\mathbb{K}$-algebra in $2n$ variables. Consider the following Weitzenböck derivation $\mathcal{D}_1$ of $\mathbb{K}[X,Y]$:

$$\mathcal{D}_1(x_i) = 0, \mathcal{D}_1(y_i) = x_i, \quad i = 1, 2, \ldots, n.$$
In [8] Nowicki conjectured that $\ker D_1$ is generated by the elements $x_1, x_2, \ldots, x_n$ and the determinants
\[
\begin{vmatrix}
x_i & x_j \\
y_i & y_j
\end{vmatrix}, \quad 1 \leq i < j \leq n.
\]
The conjecture was confirmed by several authors, see [6, 1, 7].

In this note we show that the Nowicki conjecture is equivalent to a well-known problem of classical invariant theory, namely, to the problem to describe the algebra of joint covariants of $n$ linear binary forms. Using the same idea we present an explicit set of generators of the kernel of the derivation $D_2$ of
\[
\mathbb{K}[X, Y, Z] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n]
\]
defined by
\[
D_2(x_i) = 0, D_2(y_i) = x_i, D_2(z_i) = y_i, \quad i = 1, \ldots, n.
\]

2. It is well known that there is a one-to-one correspondence between the $G_a$-actions on an affine algebraic variety $V$ and the locally nilpotent $\mathbb{K}$-derivations on its algebra of polynomial functions. Let us identify the algebra $\mathbb{K}[X, Y]$ with the algebra $\mathcal{O}[\mathbb{K}^{2n}]$ of polynomial functions of the algebraic variety $\mathbb{K}^{2n}$. Then, the kernel of the derivation $D_1$ coincides with the invariant ring of the induced via $\exp(t D_1)$ action:
\[
\ker D_1 = \mathcal{O}[\mathbb{K}^{2n}]^{G_a} = \mathbb{K}[X, Y]^{G_a}.
\]

Now, let
\[
V_1 := \{\alpha X + \beta Y \mid \alpha, \beta \in \mathbb{K}\}
\]
be the vector $\mathbb{K}$-space of linear binary forms endowed with the natural action of the group $SL_2$. Consider the induced action of the group $SL_2$ on the algebra of polynomial functions $\mathcal{O}[n V_1 \oplus \mathbb{K}^2]$ on the vector space $n V_1 \oplus \mathbb{K}^2$, where
\[
n V_1 := \bigoplus_{n \text{ times}} V_1.
\]
Let
\[
U_2 = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{K} \right\}
\]
be the maximal unipotent subgroup of the group $SL_2$. The application of the Grosshans principle, see [5, 10], gives
\[
\mathcal{O}[n V_1 \oplus \mathbb{K}^2]^{SL_2} \cong \mathcal{O}[n V_1]^{U_2}.
\]
Thus,
\[ \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{sl_2} \cong \mathcal{O}[nV_1]^{u_k}. \]

Since \( U_2 \cong (\mathbb{K}, +) \), it follows that
\[ \ker D_1 \cong \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{sl_2} \cong \mathcal{O}[nV_1]^{u_k}. \]

In the language of classical invariant theory the algebra \( C_1 := \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{sl_2} \) is called the algebra of joint covariants of \( n \) linear binary forms and the algebra \( S_1 := \mathcal{O}[nV_1]^{u_k} \) is called the algebra of joint semi-invariants of \( n \) linear binary forms. Algebras of joint covariants of binary forms were an object of research in the classical invariant theory of the 19th century.

3. Let us consider the set of \( n \) linear binary forms \( f_i = x_iX + y_iY, \) \( i = 1, \ldots, n. \) Then any element of \( \mathcal{O}[nV_1 \oplus \mathbb{K}^2] \) can be considered as a polynomial from \( \mathbb{K}[X, Y, X^r, Y^r]. \) Gordan’s famous theorem, see [3, 2], implies:

**Theorem 1** (A weak form of Gordan’s theorem). If \( T \) is a subalgebra of \( C_1 \) with the property that \( (f, z)^r \in T \) whenever \( r \in \mathbb{N}, \) \( z \in T, \) then \( T = C_1. \)

Here \( (u, v)^r \) denotes the \( r \)-transvectants of the binary forms \( u, v \in \mathbb{K}[X, Y, X^r, Y^r]: \)
\[ (u, v)^r := \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r u}{\partial X^{r-i} \partial Y^i} \frac{\partial^r v}{\partial X^i \partial Y^{r-i}}. \]

Observe, that \( (u, v)^0 = uv \) and \( (u, v)^1 \) is exactly the Jacobian \( J(u, v) \) of the forms \( u, v. \) The above theorem yields:

**Theorem 2.** The algebra of joint covariants \( C_1 \) of \( n \) linear binary forms \( f_i, i = 1, \ldots, n, \) is generated by the forms \( f_1, f_2, \ldots, f_n \) and their Jacobians \( J(f_i, f_j), 1 \leq i < j \leq n. \)

**Proof.** All forms \( f_i, i = 1, \ldots, n, \) belong to the algebra of covariants \( C_1. \) By direct calculations we get
\[ (f_i, f_j)^1 = J(f_i, f_j) = \begin{vmatrix} \frac{\partial f_i}{\partial X} & \frac{\partial f_i}{\partial Y} \\ \frac{\partial f_j}{\partial X} & \frac{\partial f_j}{\partial Y} \end{vmatrix} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \]
and
\[ (f_i, f_j)^r = 0 \quad \text{for} \quad r > 1. \]
Let us consider the subalgebra $T$ of $C_1$ generated by the linear forms $f_1, f_2, \ldots, f_n$ and theirs jacobians $J(f_i, f_j), i < j$. Since $(f_i, J(f_j, f_k))^r = 0$ for all $r \geq 1$, it follows that $T = C_1$.  

Let us show that the result is equivalent to the Nowicki conjecture:

Identify the algebra of semi-invariants $S_1$ with ker $D_1$. The isomorphism $\tau : C_1 \to S_1$ takes each homogeneous covariant of degree $m$ (with respect to the variables $X, Y$) to its coefficient of $X^m$. The proof proceeds in the same manner as the proof in the case of a unique binary form, see [9, Proposition 9.45].

Thus, the following statement holds:

**Theorem 3.** The algebra of joint semi-invariants $S_1 = \ker D_1$ of $n$ linear binary forms $f_i, i = 1, \ldots, n$, is generated by the elements $x_1, x_2, \ldots, x_n$ and the determinants

\[
\begin{vmatrix}
  x_i & x_j \\
  y_i & y_j
\end{vmatrix}, \quad 1 \leq i < j \leq n.
\]

**Proof.** The algebra $S_1$ is generated by the images of the generating elements of the algebra $C_1$ under the homomorphism $\tau$. We have $\tau(f_i) = x_i$ and

\[
\tau(J(f_i, f_j)) = \begin{vmatrix}
  x_i & x_j \\
  y_i & y_j
\end{vmatrix}.
\]

Theorem 3 is exactly the Nowicki conjecture.

4. Other ways to prove Theorem 3 were suggested by Dersken and Pan-nyushev, see the comments in [1]. Taking into account $K^2 \cong_{sl_2} V_1$ we get

\[
\ker D_1 \cong O[nV_1 \oplus K^2]^{sl_2} \cong O[(n+1)V_1]^{sl_2}.
\]

But then the invariant algebra $O[(n+1)V_1]^{sl_2}$ is well known because of the First fundamental theorem of invariant theory for $SL_2$, see [14].

5. A natural generalization of the above problem looks as follows. Let

\[
K[X, Y, Z] = K[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n]
\]

be the polynomial $K$-algebra in $3n$ variables. Consider the following derivation $D_2$ of the algebra $K[X, Y, Z]$:

\[
D_2(x_i) = 0, \quad D_2(y_i) = x_i, \quad D_2(z_i) = y_i, \quad i = 1, 2, \ldots, n.
\]
The following theorem holds.

**Theorem 4.** The kernel of the derivation \(D_2\) is generated by the elements of the following types:

1. \(x_1, x_2, \ldots, x_n\),
2. \(J_{1,2}, J_{1,3}, \ldots, J_{n-1,n}\),
3. \(H_{1,2}, H_{1,3}, \ldots, H_{n-1,n}\),
4. \(\Delta_{1,2,3}, \Delta_{1,2,4}, \ldots, \Delta_{n-2,n-1,n}\),

where

\[ J_{i,j} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad 1 \leq i < j \leq n, \]

\[ H_{i,j} = x_i z_j - y_i y_j + z_i x_j, \quad 1 \leq i \leq j \leq n, \]

and

\[ \Delta_{i,j,k} := \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix}, \quad 1 \leq i < j < k \leq n. \]

The proof follows from the description of the generating elements of the algebra of covariants for \(n\) quadratic binary forms, see [4, page 162].

6. Any Weitzenböck derivation of the polynomial algebra is completely determined by its Jordan normal form. Denote by \(D_k\) the Weitzenböck derivation with Jordan normal form which consists of \(n\) Jordan blocks of size \(k + 1\).

**Problem.** Find a generating set of \(\ker D_k\).

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