CHIRAL RANDOM MATRIX THEORY AND QCD

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ABSTRACT

As was shown by Leutwyler and Smilga, the fact that chiral symmetry is broken and the existence of a effective finite volume partition function leads to an infinite number of sum rules for the eigenvalues of the Dirac operator in QCD. In this paper we argue these constraints, together with universality arguments from quantum chaos and universal conductance fluctuations, completely determine its spectrum near zero virtuality. As in the classical random matrix ensembles, we find three universality classes, depending on whether the color representation of the gauge group is pseudo-real, complex or real. They correspond to $SU(2)$ with fundamental fermions, $SU(N_c), N_c \geq 3$, with fundamental fermions, and $SU(N_c), N_c \geq 3$, with adjoint fermions, respectively.

1. Introduction

It is widely believed that QCD is the theory of strong interactions which describes the hadronic world as we know it. On the other hand, at low energies, the same physics can be described much more economically in terms of a partition function involving effective degrees of freedom. This raises the question to which extent the low energy effective theory put constraints on the underlying microscopic theory.

This question can be answered if we restrict ourselves to a space time volume $V_4 = L^4$ with $L \ll m^{-1}_r$. In that case only the static modes of the Goldstone fields contribute to the partition function. If, at the same time, $L \gg \Lambda_{QCD}^{-1}$, the low-energy effective partition function coincides with the QCD partition function. Its mass dependence is given by

$$Z^{eff}(m, \theta) = \int_{U \in G/H} DU \exp(V_4 \Sigma \text{ReTr} m e^{i\theta/N_f} U)$$ (1)

where the integral is over the coset $G/H$. For the standard scheme of chiral symmetry breaking we have $G/H = SU(N_f) \times SU(N_f)/SU(N_f)$. In agreement with the Vafa-Witten theorem, this corresponds to a condensate $\Sigma$ that is flavor symmetric (see also refs. 4). For simplicity we take the mass matrix $m$ diagonal in this paper. As dictated by QCD Ward identities, the masses always occur in the combination $me^{i\theta/N_f}$, where $\theta$ is the vacuum angle.

The mass dependence of the QCD partition function is given by

$$Z^{QCD}(m, \theta) = \sum_{\nu} e^{i\nu\theta} \langle \prod_f m_f^\nu \prod_{\lambda_n > 0} (\lambda_n^2 + m_f^2) \rangle_{\nu},$$ (2)
where the average $\langle \cdots \rangle_{\nu}$ is over all gauge configurations with topological charge $\nu$ weighted by the gluonic action.

As was argued correctly by Leutwyler and Smilga\footnote{1}, the finite volume partition function, imposes constraints on the eigenvalues of the Dirac operator. More precisely, sum rules are obtained by equating the coefficients of the expansion in powers of the quark masses of both sides of the equation

$$
\frac{Z_{\nu}^{\text{QCD}}(m)}{Z_{\nu}^{\text{QCD}}(0)} = \frac{Z_{\nu}^{\text{eff}}(m)}{Z_{\nu}^{\text{eff}}(0)},
$$

where $Z_{\nu}(m)$ is the $\nu$'th Fourier coefficient of $Z(m, \theta)$. Since this leads to an infinite set of sum rules, the following question should be raised: to what extent is the spectrum of the Dirac operator determined by the finite volume partition function? Because we study a region where only the static modes contribute to the finite volume partition function, we expect that it only determines the part of the spectrum near zero virtuality. To make this statement more precise let us consider the simplest sum rule obtained by equating the $O(m^2)$ terms in (3)

$$
\frac{1}{V_4^2} \left( \sum' \frac{1}{\lambda_n^2} \right)_{\nu} = \frac{\Sigma^2}{4(N_f + \nu)} ,
$$

where the sum is over the positive nonzero eigenvalues of the Dirac operator. If the spectral density is defined by

$$
\rho(\lambda) = \langle \sum' \delta(\lambda - \lambda_n) \rangle
$$

we can rewrite the l.h.s of (4) as

$$
\frac{1}{V_4^2} \left( \sum' \frac{1}{\lambda_n^2} \right) = \int dz \frac{1}{V_4} \rho\left( \frac{z}{V_4} \right) \frac{1}{z^2} ,
$$

where we have introduced a new integration variable by

$$
z = \lambda V_4.
$$

This leads to the definition of the microscopic limit of the spectral density\footnote{3}

$$
\rho_S(z) = \lim_{V_4 \to \infty} \frac{1}{V_4} \rho\left( \frac{z}{V_4} \right).
$$

It is clear that information of the static finite volume partition function is contained in this limiting function of the spectral density. Since $Z^{\text{eff}}$ is based on symmetries and chiral symmetry breaking only we arrive at the conjecture that the microscopic spectral density is completely determined by the symmetries of the QCD partition function. The same applies to microscopic correlation functions of the spectral density which appear in higher order sum rules.
It is our program to construct a spectral density that contains nothing else but the symmetries of the QCD partition function as input. Such question has been phrased in general terms by Balian, who introduced the maximal entropy approach to obtain the probability distribution of the ensemble that describes a system that obeys a set of well defined constraints. This approach has been applied successfully to the theory of statistical $S$-matrix fluctuations\cite{8,9,10} and universal conductance fluctuations\cite{11,12,13}. For example, in the case of the classical random matrix ensembles, a Gaussian probability distribution of the matrix elements is obtained in this way.

Along these lines we can construct a random matrix model with the symmetries of the QCD partition function as input. As in the classical random matrix ensembles\cite{14}, we find three different universality classes\cite{15}: 1. The chiral orthogonal ensemble (ch-GOE) for $SU(2)$ with fundamental fermions when the Dirac operator is real. The chiral unitary ensemble (chGUE) for $SU(N_c), N_c \geq 3$, with fundamental fermions (in this case the Dirac operator is complex). 3. The chiral symplectic ensemble (chGSE) for $SU(N_c), N_c \geq 2$, with adjoint fermions (in this case the Dirac operator can be regrouped into real quaternions). In all three models the scale is set by the chiral condensate, which, via the Banks-Casher formula\cite{16}, is related to the average level density. Further they depend on the number of flavors and the total topological charge. Precursors of the chGUE discussed in this paper can be found in refs.\cite{17,18}.

Of course, it makes only sense to use random matrix theory for observables that are not sensitive to the detailed dynamics of the system. From studies of nuclear level resonances\cite{19,20}, universal conductance fluctuations\cite{21,22}, quantum chaos\cite{23,24,25,26,27}, and general arguments\cite{28} we know that correlations between eigenvalues on the scale of a finite number level spacings are such observables. One of the best known examples is the nearest neighbor spacing distribution. Because the microscopic spectral density\cite{8} describes correlations on the scale of a finite number of levels away from zero, we expect that this quantity is universal. It is our conjecture that the exact QCD microscopic spectral density belongs to this universality class.

What are the arguments in favor of universality? In the ideal case one would like to analyze the spectrum of the QCD-Dirac operator as obtained in numerical lattice calculations. However, technically it is not possible to work close enough to the chiral limit, and the best one can do in this direction is to work with cooled configurations which are represented by a partition function of a liquid of instantons and anti-instantons\cite{29} (see however\cite{30}). In this paper we summarize the main numerical results (see section 6). Important support in favor universality comes from a different branch of physics, namely from the theory of universal conductance fluctuations. In that context, the microscopic spectral density of the eigenvalues of the transmission matrix, was calculated for the Hofstadter\cite{31} model, and, to a high degree of accuracy, it agrees with the random matrix prediction\cite{32}. The Hofstadter model, which has the symmetries of the chGUE for zero flavors, is therefore in the same universality class as the corresponding random matrix model. Because, this model differs in all other respects from the random matrix model, this result illustrates the size of the basin of attraction of the random matrix model and greatly increases the hope that the exact QCD microscopic spectral density also belongs to this universality class.
Other arguments in favor of the universality come from the finite volume partition function. Using general arguments used before in ref. for SU($N_c$), $N_c \geq 3$, with fundamental fermions, it is possible to write down the effective partition function for SU(2) gauge theories with fundamental fermions and for SU($N_c$) theories with adjoint fermions. This allows us to derive sum rules for the inverse eigenvalues of the Dirac operator for an arbitrary number of flavors, and for a given value of the topological charge. All results obtained this way coincide with sum-rules derived from random matrix theory.

Questions related to the universality of correlations in quantum spectra are discussed in section 2. The symmetries of the QCD partition function will be analyzed in section 3. The main part of our program is to construct a random matrix model that has nothing else but these symmetries as input. This so called chiral random matrix model will be introduced in section 4. A summary of the analytical results will be presented in section 5, and numerical results for the microscopic spectral density in a liquid of instantons are shown in section 6. Concluding remarks are made in section 7.

2. Universality of Eigenvalue Correlations in Quantum spectra

More than three decades ago it was realized that correlations between the energy levels of compound nuclei show a universal behavior (see for example ref.19). First, it was discovered that the nearest neighbor spacing distribution follows the so called Wigner surmise. Later it was found that not only short range correlations but also correlations over many level spacing do not depend on the specific dynamics of the system. One of the most outstanding characteristics is the stiffness or rigidity of the spectrum. For instance, if an energy interval contains on an average $N$ levels, the variance of the distribution, $\Sigma_2(N)$, is not $N$ (as for a random sequence), but rather $(\log N)/\pi^2$. Both results can be derived from a random matrix theory with only the symmetries of the system as input. This provides a strong argument in favor of the universality of such level correlations. In particular through the work of Dyson, we know that there are three distinct universality classes corresponding to real symmetric, complex hermitean or quaternion real hamiltonian matrices. They correspond to systems with time reversal invariance and integral spin or rotational invariance, with broken time reversal invariance, and half-integer spin systems with time reversal invariance and no rotational invariance.

Much more recently, it was realized that random matrix correlations do not only appear in complex systems with many degrees of freedom, but also in systems with as few as two degrees of freedom. This was first shown by pioneering studies with quantum billiards. Later it was found that complete chaos is not only a sufficient ingredient but also a necessary ingredient. As an illustration, we show in Fig. 1 the $\Delta_3(L)$ statistic (an integral transform of the number variance $\Sigma_2(N)$) versus the length of an interval containing on an average $L$ levels, and the nearest neighbor spacing distribution, $P(S)$, calculated from the (approximately) hundred lowest levels
of the Hamiltonian

$$H = \frac{1}{2}((p_x - \frac{B}{2} y)^2 + (p_y - \frac{B}{2} x)^2) + \frac{1}{2}((x/\sigma_1)^6 + (y/\sigma_2)^6 + g(x - y)^6). \quad (9)$$

For a suitable choice of the parameters $\sigma_1$, $\sigma_2$ and the magnetic field, this Hamiltonian is completely chaotic for $g = 0$. Although in this case the time reversal symmetry is broken an additional symmetry allows us to choose a basis in which the Hamiltonian is real. As shown in Fig. 1 the spectral correlations are indeed described by the GOE. Only when the coupling constant is large enough to destroy all remnants of this symmetry, while $B \neq 0$ is kept at its original value (broken time reversal invariance), the spectral correlations are given by the GUE (see Fig. 2).

Fig. 1. Numerical results for the $\Delta_3$ statistic (squares) and the nearest neighbor spacing distribution (histogram) for a nonzero magnetic field and $g = 0$. The full and the dashed curves show the corresponding results for the GUE and the GOE, respectively.

Fig. 2. Numerical results for a nonzero magnetic field and $g \neq 0$. See the caption of Fig. 1 for further explanation.

Two more cases of universal correlations should be mentioned. First, cross-section fluctuations in compound nuclei are determined by the average $S$ matrix\footnote{9}. Second, universal conductance fluctuations: the variance of the conductivity is a given by a pure number times the quantum conductance $2e^2/\hbar$. This phenomenon is closely related to $S$–matrix fluctuations\footnote{22}, in particular to Ericson fluctuations\footnote{34} which involve correlations between two $S$ matrices at different energies.

What did we learn from the above studies? One very important implicit ingredient in all cases is the separation of scales: a slow varying or average scale and a rapid
varying or microscopic scale. This allows us to express all correlations in terms of the slow varying scale, the average level spacing, the average $S$-matrix, or the average conductance. This would not be possible if fluctuations on all scales were present. The central point is that only correlations expressed in terms of the rescaled variables are universal and can be described by random matrix theory. In particular, for level correlations it means that only correlations of the unfolded spectrum $\{\lambda_n^U\}$ defined by ($\overline{\rho}$ is the average level density)

$$\lambda_n^U = \int_{-\infty}^{\lambda_n} \overline{\rho}(\lambda) d\lambda,$$

(10)

are described by random matrix theory.

Second, the microscopic correlations are completely determined by the symmetries of the system. In the QCD partition function, the eigenvalues show fluctuations over the ensemble of gauge field configurations. From what is said above we expect that only the microscopic correlations are universal and can be described by random matrix theory. In the next section we will analyze the symmetries of the QCD-Dirac operator and use that as an input for the random matrix model to be constructed later.

3. Symmetries of the Dirac operator

In this section we consider the symmetries of the QCD Dirac operator for a fixed external gauge field $A_\mu$. The masses of the quarks are taken to be zero (chiral limit). In that case the Dirac operator $D$ is chirally symmetric

$$\{\gamma_5, D\} = 0.$$

(11)

As a consequence the eigenvalues occurs in pairs $\pm \lambda_n$, and the eigenfunctions have opposite chirality. The only exception are the zero eigenvalues. For field configurations with topological charge $\nu$ we have exactly $\nu$ zero modes, all of the same chirality.

A less obvious symmetry, which plays an important role in random matrix theory, is the symmetry that dictates the Wigner type of color representation of the gauge group. As suggested by the classical random matrix ensembles, the reality type of the matrix elements determines the corresponding universality class. In QCD, with three colors and fundamental fermions, there is no additional symmetry and the color representation of the gauge group is complex. Its Lagrangian is invariant under $U(N_f) \times U(N_f)$ (a subgroup $U_A(1)$ is broken by the quantum fluctuations). As is well known this symmetry group is enlarged to $U(2N_f)$ for $SU(2)$ with fundamental fermions, and to $SU(N_f)$ for $SU(N_c)$ in the adjoint representation with $N_f$ Majorana fermions. This fact is closely related to the Wigner type of the representation of the color group.

Indeed, for $SU(2)$ with fundamental fermions the Dirac operator,

$$D = i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu^a \frac{\tau^a}{2}$$

(12)
has an additional symmetry
\[ [i \gamma_2 \gamma_4 \tau_2^\text{color} K, D] = 0. \] (13)

Here, \( \tau_k \) are the Pauli matrices acting in color space, and \( K \) is the complex conjugation operator. The symmetry operator satisfies the property
\[ (i \gamma_2 \gamma_4 \tau_2^\text{color} K)^2 = 1, \] (14)

which allows us to choose a basis in which the Dirac operator is real. Note, however, that the representation of the color group is pseudo-real. Because of the symmetry (13) the symmetry group is enlarged to \( U(2N_f) \).

For gauge theories with adjoint fermions the Dirac operator is given by
\[ D = i \gamma_\mu \partial_\mu + if_{k\ell m} \gamma_\mu A_\mu^{m}, \] (15)

where the \( f_{k\ell m} \) are the structure constants of the gauge group. In this case the additional symmetry follows from the commutator
\[ [i \gamma_2 \gamma_4 K, D] = 0. \] (16)

Now, the symmetry operator satisfies
\[ (i \gamma_2 \gamma_4 K)^2 = -1, \] (17)

which allows us to choose a basis in which the Dirac matrix can be regrouped such that its matrix elements are quaternion real. In this case the representation of the color group is real. Because adjoint fermions are formulated in terms of Majorana fermions, the symmetry (16) results in an \( SU(N_f) \) flavor symmetry.

In each of the three cases discussed above the scheme of chiral symmetry breaking is different. For \( SU(N_c), N_c \geq 3 \), with fundamental fermions the chiral symmetry is broken according to
\[ SU(N_f) \times SU(N_f) \supset SU(N_f). \] (18)

For \( SU(2) \) with fundamental fermions the symmetry group is enlarged to \( SU(2N_f) \) and spontaneously broken according to
\[ SU(2N_f) \supset Sp(2N_f). \] (19)

In the case of adjoint fermions and an \( SU(N_c), N_c \geq 2 \), color group the flavor symmetry group is \( SU(2N_f) \). In this case the symmetry breaking pattern is
\[ SU(N_f) \supset O(N_f). \] (20)

The different symmetry breaking schemes are determined by the dynamics of the theory. They have to obey general constraints due to the Vafa-Witten theorem, and,
in agreement with general QCD inequalities, they should give rise to pseudoscalar Goldstone bosons. More discussion of this point can be found in ref. 5.

### 4. Chiral Random Matrix Theory

The structure of the Dirac operator in QCD is much richer than that of a Hamiltonian of a completely chaotic non-relativistic quantum system. Nevertheless, it is possible to write down a random matrix theory that includes all symmetries discussed in previous section. For \( N_f \) flavors the partition function for the sector of topological charge \( \nu \) is defined by

\[
Z_\nu = \int \mathcal{D}T \prod_{f=1}^{N_f} \det \left( \begin{array}{cc} m_f & iT \\ iT & m_f \end{array} \right) \exp\left( -\frac{n\beta}{2\sigma^2} \text{Tr}T^\dagger T \right)
\]  

(21)

Here, the integral is over \( n \times m \) matrices \( T \) with the Haar measure. The determinant in this equation plays the role of the fermion determinant in QCD. We want to remark that the matrix

\[
\left( \begin{array}{cc} 0 & iT \\ iT & 0 \end{array} \right)
\]  

(22)

has exactly \( \nu = |n - m| \) zero eigenvalues. All other eigenvalues occur in pairs \( \pm \lambda_n \). We will identify \( N = n + m \) as the volume of space time. For definiteness we take \( m > n \), and, as in QCD, we always take \( \nu \ll N \). The connection with the real world goes through the parameter \( \Sigma \). In this model it is given by

\[
\Sigma = \lim_{m \to 0} \lim_{N \to \infty} \frac{\pi \rho(0)}{N}.
\]  

(23)

According to the Banks-Casher formula, \( \Sigma \) can be identified as the chiral condensate.

The representation of the color group is implemented via the the matrix elements \( T \) and the integration measure. The number of independent variables per matrix element is denoted by \( \beta \) (called Dyson parameter)*. For \( SU(2) \) with fundamental fermions the color representation is pseudoreal and the Dirac operator is real. The integral in Eq. (21) is over real matrices and we have \( \beta = 1 \). For \( SU(N_c) \), \( N_c \geq 3 \), both the color representation and the Dirac operator are complex. The matrix \( T \) in Eq. (21) is complex and \( \beta = 2 \). Finally, for \( SU(N_c) \), \( N_c \geq 3 \), with adjoint fermions, the color representation is real, and the Dirac operator is quaternion real. The matrix elements of \( T \) are also quaternion real, and we have \( \beta = 4 \).

An explicit realization of this partition function is for a liquid of instantons. In that case the Dirac operator is evaluated in the space of fermionic zero modes leading to the same matrix structure as in Eq. (21). However, instead of the integral over the

*This parameter has also been introduced in the exponent, so that the average level density becomes independent of \( \beta \).
matrix elements, we have an integral over the collective coordinates of the instantons. For a discussion and a numerical evaluation of this partition function we refer to section 7.

5. Analytical Results

The theoretical analysis of the partition function Eq. (21) is straightforward. The eigenvalue distribution is obtained by changing the integration variables according to

\[ T = U \Lambda V^{-1}, \]  

where \( \Lambda \) is a diagonal matrix with positive real matrix elements. The Jacobian of this transformation is

\[ J(\Lambda) = \prod_{k<l} |\lambda_k^2 - \lambda_l^2|^{\beta} \prod_k \lambda_k^{2\nu+\beta-1}. \]  

This Jacobian can be obtained by an explicit calculation (see for a discussion of the necessary techniques). It also follows from the fact that the Jacobian vanishes when two eigenvalues coincide or one eigenvalue equals zero, and that it should depend symmetrically on all eigenvalues. The correct powers then follow from dimensional arguments.

The integration over \( U \) and \( V \) only contributes a constant factor, so that the partition function is given by

\[ Z_v = C_{\beta,n} \int d\lambda_1 \cdots d\lambda_n \prod_{k<l} |\lambda_k^2 - \lambda_l^2|^{\beta} \prod_k \lambda_k^{2\nu+\beta-1} \exp \left( -\frac{n\beta \Sigma^2}{2} \sum_k \lambda_k^2 \right), \]  

where the constant \( C_{\beta,n} \) is determined by the normalization. The joint eigenvalue distribution \( \rho(\lambda_1, \cdots, \lambda_n) \) is just the integrand of Eq. (26). We have

\[ Z = \int d\lambda_1 \cdots d\lambda_n \rho(\lambda_1, \cdots, \lambda_n). \]  

The eigenvalue distribution follows by integrating over all eigenvalues except one

\[ \rho(\lambda_1) = \int d\lambda_2 \cdots d\lambda_n \rho(\lambda_1, \cdots, \lambda_n). \]  

Integrals of this type have been widely studied in the context of random matrix theory. One very powerful method is the orthogonal polynomial method Dyson, Wigner and Mehta (see the book of Mehta for references). This method is particularly suited for \( \beta = 2 \), when the integrals follow immediately from the orthogonality relations of the orthogonal polynomials. For \( \beta = 1 \) and \( \beta = 4 \) it is still possible to do the integrals, but the resulting expressions are much more complicated. Instead of classical orthogonal polynomials, one has to invoke skew orthogonal polynomials. The general method was developed by Dyson and by Mahoux and Mehta, and
applied to many different ensembles by Nagao, Slevin and Wadati\cite{41}. Unfortunately, the generalized Laguerre ensemble for $\beta = 1$ was not analyzed before, so we performed the analysis in ref.\cite{42}. The symplectic generalized Laguerre ensemble has not yet been studied, but using similar methods it is possible to obtain an explicit expression for the level density.

First we discuss the simplest case, $\beta = 2$. This case is also known as the unitary generalized Laguerre ensemble and was studied in\cite{43,44,45,46,41}. The latter authors also evaluated the microscopic limit. This ensemble was put in the context of QCD in ref.\cite{47}, where the microscopic spectral density for $\nu = 0$ was derived. The microscopic spectral density for arbitrary $\nu$ was first given in ref.\cite{42}.

The result for the microscopic spectral density\cite{8} for the chGUE ($\beta = 2$) is

$$\rho_S(z) = \frac{\Sigma^2 z}{2} (J_{N_f + \nu}^2(\Sigma z) - J_{N_f + \nu + 1}(\Sigma z)J_{N_f + \nu - 1}(\Sigma z)). \quad (29)$$

It depends only on the combination $N_f + \nu$ which agrees with the sum rules derived by Leutwyler and Smilga\cite{1}. It is also possible to obtain explicit expressions for the two level correlation functions. For results we refer to the literature\cite{47}.

The expressions for $SU(2)$ with fundamental fermions ($\beta = 1$) are much more complicated. In this case we find the microscopic spectral density\cite{42}

$$\rho_S(z) = \frac{\Sigma}{4} J_{2a+1}(z\Sigma) + \frac{\Sigma}{2} \int_0^\infty dw (zw)^{2a+1} \epsilon(z - w) \left( \frac{1}{w} \frac{d}{dw} - \frac{1}{z} \frac{d}{dz} \right) \times \frac{wJ_{2a}(z\Sigma)J_{2a-1}(w\Sigma) - zJ_{2a-1}(z\Sigma)J_{2a}(w\Sigma)}{(zw)^{2a}(z^2 - w^2)}, \quad (30)$$

where $a$ is the combination

$$a = N_f - \frac{1}{2} + \frac{\nu}{2}. \quad (31)$$

Eqs. (29) and (30) can be used to derive Leutwyler-Smilga sum rules. However, there is a more direct way to obtain these results. Exactly, these type of integrals follow from the Selberg formula\cite{48,38}. Sum rules have been obtained\cite{33} for the quantities

$$S_p \equiv \left\langle \sum_{n_1 \neq n_2 \neq \cdots \neq n_p} \frac{1}{\lambda_{n_1}^2 \cdots \lambda_{n_p}^2} \right\rangle, \quad (32)$$

and

$$S_{pq} \equiv \left\langle \sum_{\text{all } \lambda_i \text{ different}} \frac{1}{\lambda_{m_1}^4 \cdots \lambda_{m_p}^4 \lambda_{n_1}^2 \cdots \lambda_{n_q}^2} \right\rangle. \quad (33)$$

The integrals can be calculated by deriving a recursion relation. It is also possible to derive recursion relations for more complicated sum rules. However, then the recursion branches off and cannot be solved analytically any longer.
The results for the simplest family of sum rules are given by
\[ S_p = \frac{1}{\prod_{k=0}^{p-1}(\beta + \beta \nu + 2(N_f - 1) + \beta k)} \binom{n}{p} (\beta n \Sigma^2)^p, \] (34)
which, in the limit \( N \to \infty \), simplifies to
\[ S_p = \frac{(N^2 \Sigma^2)^p}{2^{2p} p!} \frac{\Gamma(\nu + 1 + 2(N_f - 1)/\beta)}{\Gamma(\nu + p + 1 + 2(N_f - 1)/\beta)}. \] (35)

This result for \( \beta = 2 \), and for \( \beta = 4, p = 1 \) and \( N_f = 1 \) or \( N_f = 2 \), were derived previously with the help of the finite volume static partition function. Results for all other case, in particular for \( SU(2) \) with fundamental fermions, were first derived with the help of random matrix theory. Presently, we have shown that the \( p = 1 \) sum rules can also be obtained starting from the finite volume partition function.

Note, that for \( N_f = 1 \) the parameter \( \beta \) drops from the equation. Indeed, in this case the finite volume partition function is solely determined by the \( U_A(1) \) anomaly and is not dependent on the Wigner type of the color representation. There are no additional degrees of freedom that, for more than one flavor, contain information on the color representation. This implies that the microscopic limit of the spectral density is not determined uniquely by the finite volume partition function.

The results for \( S_{pq} \) can be derived in a similar fashion. We only quote the result
\[ S_{pq} = N_{pq} \frac{\Gamma(\alpha + \frac{2}{\beta}) \Gamma(\alpha + p) \Gamma(\alpha + \frac{2}{\beta} + p + n)}{\Gamma(\alpha + \frac{2}{\beta} + q + 2p) \Gamma(\alpha) \Gamma(\alpha + \frac{2}{\beta} + n)} (n^2 \Sigma^2)^{q+2p}, \] (36)
where the factor \( N_{pq} \) is defined by
\[ N_{pq} = \frac{n!}{p!q!(n-p-q)!}, \] (37)
and
\[ \alpha = \nu + 1 + \frac{2(N_f - 2)}{\beta}. \] (38)

These sum rules depend on \( \beta \). Indeed, the effective low energy field theories are different for different values of \( \beta \). Even for \( N_f = 1 \) they depend on \( \beta \) (the integrals should be convergent). Since such sum rules cannot be derived from the finite volume partition function these results are completely consistent.

The result (36) was derived from the finite volume partition function only for \( \beta = 2, p = 1 \) and \( q = 0 \).

6. Numerical Results

It would be very interesting to check the results of section 5 by lattice QCD simulations. However, in practise, it is not possible to obtain a sufficient number of
eigenvalues of the Dirac operator close to the chiral limit (see however ref. [24], where the spectrum near zero virtuality has been obtained for quark masses that are much larger than the smallest eigenvalues, and in the present context, corresponds to $N_f = 0$).

Now, remember that the correlations discussed above are based a random Dirac operator. When we consider the fluctuations of the eigenvalues of the Dirac operator over the ensemble of the gauge field configurations, it seems clear that if the random matrix results are reproduced by cooled configurations, they are certainly found for the original uncooled configurations. The sum over all cooled configurations is well known: it is a dilute gas or liquid of instantons. Although, it does not have confinement, the bulk of hadronic physics can be reproduced by this model [50].

The partition function for a liquid of instantons is given by

$$Z = \int d\Omega \prod_{i=1}^N \prod_{f=1}^{N_f} \det(T^\dagger T + m_f^2) \exp(-S_{\text{glue}}), \quad (39)$$

where the integral is over the collective coordinates of each instanton, and the Dirac operator is calculated in the space of fermionic zero modes with matrix elements denoted by $T_{ij}$. The gluonic action, $S_{\text{glue}}$, contains the interaction between the instantons approximated by a sum over all pairs. The total number of instantons in $N$. In general, the number of instantons is not equal to the number of anti-instantons. However, in the numerical calculations to be presented below, this will be the case. Further details of this model and the calculations presented in this section can be found in ref. [24].
Fig. 3. A histogram of the total spectral density of the Dirac operator in a liquid of instantons

In the partition function (39) it is straightforward to change the number of colors from $N_c = 2$ to $N_c = 3$. In fact, we use the same computer program that covers both cases. At present it is not yet clear how to write down an instanton partition function for adjoint fermions. For a discussion of the difficulties related to this issue we refer to ref. 1.

The numerical results have been obtained for 32 instantons and 32 anti-instantons confined to a box of $2.37^3 \times 4.74\Lambda_{QCD}^{-4}$. For $N_f = 0$, the average was over 100,000 configurations. For all other flavors the average was over only 20,000 (correlated) configurations. A histogram of the results for the total spectral density is shown in Fig. 3. The results are for two and three colors and one, two and three flavors (see label of the figure). In Fig. 4 the region of the spectrum near zero virtuality is put under the microscope. We show a histogram of the unfolded spectrum in terms of the microscopic variable $z$, so that the average spacing between the eigenvalues is one.

Fig. 4. The microscopic spectral density. The full line represents the numerical results. Random matrix result of the chGOE and the chGUE is shown by the dotted and the dashed curves, respectively.

In the same figures we also show the random matrix results for the chGOE (dotted curve) and the chGUE (dashed curve). The only deviation is that the oscillations in the chGUE random matrix results seem not to be completely reproduced. At the moment it is not clear whether that this is a real effect, of that it is a consequence of the finite size of the system.
7. Conclusions

We have argued that the average microscopic spectral density of the QCD Dirac operator follows from its symmetries and a maximum entropy principle. The latter principle uniquely determines the random matrix theory obeying all constraints imposed by the symmetries. At present, it is a conjecture that the fluctuations of the eigenvalues near zero virtuality over the ensemble of gauge field configurations as weighted by the QCD partition function belong to the same universality class as the corresponding chiral random matrix theory. We have presented several strong arguments in favor of this conjecture. First, results from the Hofstadter model show that the random matrix model has a wide basin of attraction. Second, the triality of the classical random matrix ensemble also occurs in the chiral random matrix ensembles and coincides with the different schemes of chiral symmetry breaking. Third, numerical calculations for instantons agree with the random matrix result. Fourth, all spectral sum rules that can be obtained from the finite volume static partition function, which coincides with the full QCD partition function in its region of applicability, are also given by the chiral random matrix theory.

The finite volume static partition function alone does not determine the microscopic spectral density uniquely. The most striking example is that of one flavor. In that case the effective theory does not depend on the color representation, whereas the microscopic spectral density is different in each of the three universality classes.

To address the question which effective field theory contains the information of the full microscopic spectral density, remember that the resolvent is defined by

\[ G(z) = \left\langle \sum_{\lambda_n} \frac{1}{z - \lambda_n} \right\rangle. \]  
(40)

It can be expressed as follows in terms of the partition function

\[ iG(iz) = \left. \frac{d}{dz} \right|_{u=z} \left\langle \frac{\prod_{n} (\lambda_n^2 + z^2)}{\prod_{n} (\lambda_n^2 + u^2)} \prod_{f=1}^{N_f} (\lambda_n^2 + m_{f}^2) \right\rangle. \]  
(41)

In (40) and (41) the sum and product is over the positive eigenvalues only. Therefore, the spectral density for \( N_f \) flavors can be obtained from a partition function of \( N_f + 1 \) fermionic flavors and one complex scalar flavor.

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