MULTIPLICITY, ASYMPTOTICS AND STABILITY OF STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATION WITH ROTATION

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Abstract. In this article, we study the multiplicity, asymptotics and stability of standing waves with prescribed mass $c > 0$ for nonlinear Schrödinger equation with rotation in the mass-supercritical regime arising in Bose-Einstein condensation. Under suitable restriction on the rotation frequency, by searching critical points of the corresponding energy functional on the mass-sphere, we obtain a local minimizer $u_c$ and a mountain pass solution $\tilde{u}_c$. Furthermore, we show that $u_c$ is a ground state for small mass $c > 0$ and describe a mass collapse behavior of the minimizers as $c \to 0$, while $\tilde{u}_c$ is an excited state. Finally, we prove that the standing wave associated with $u_c$ is stable. Notice that the pioneering works [2, 6] imply that finite time blow-up of solutions to this model occurred in the mass-supercritical setting, therefore, we in the present paper obtain a new stability result. The main contribution of this paper is to extend the main results in [4, 16] concerning the same model from mass-subcritical and mass-critical regimes to mass-supercritical regime, where the physically most relevant case is covered.

Key words: Bose-Einstein condensation; Rotation; Multiplicity; Asymptotics; Stability.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the multiplicity, asymptotics and stability of standing waves with prescribed mass for the nonlinear Schrödinger equation with rotation

$$\begin{cases}
i\partial_t \psi = -\frac{1}{2} \Delta \psi + V(x)\psi - \Omega \cdot L\psi - a|\psi|^{p-2}\psi, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\psi(0, x) = \psi_0(x),
\end{cases}$$

where $a > 0$, $N = 2, 3$, $V(x) = \frac{|x|^2}{2}$ and $2 + \frac{4}{N} \leq p < 2^* := \frac{2N}{(N-2)^*}$. The rotation term $\Omega \cdot L$ reads

$$\Omega \cdot L := -i\Omega \cdot (x \wedge \nabla) = -i(\Omega \wedge x) \cdot \nabla = -i|\Omega| (x_1 \partial_{x_2} - x_2 \partial_{x_1}),$$

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where \( i = \sqrt{-1} \), \( \Omega = (0, 0, |\Omega|) \in \mathbb{R}^3 \) is a given angular velocity vector, \( L = -ix \wedge \nabla \) is the quantum mechanical angular momentum operator and \( \wedge \) is the wedge product of the two vectors.

Problem (1.1) with \( p = 4 \) arises in Bose-Einstein condensation (BEC), which describes the quantum effects in macro scope. Physically, the BEC is set into rotation by a stirring of the condensate particles and the parameter \( a > 0 \) (\( a < 0 \)) characterizes the strengthening of attractive (repulsive) interactions between the cold atoms. Vortices are believed to be unstable for \( a > 0 \) (see [11, 13, 29]), but they form stable lattice configurations for \( a < 0 \) (see [1, 12, 15]). We mainly consider the existence and stability of standing waves of (1.1) in the general case \( 2 + \frac{4}{N} \leq p < 2^* \), where the physically most relevant case \( p = 4 \) is covered.

Throughout this paper, we denote the norm of \( L^p(\mathbb{R}^N) \) by \( \|u\|_p := (\int_{\mathbb{R}^N} |u|^p)^{\frac{1}{p}} \) for any \( 1 \leq p < \infty \). Our working space \( \Sigma \) is defined as

\[
\Sigma := \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |x|^2 |u|^2 < +\infty \right\},
\]

which is a Hilbert space with the inner product and norm

\[
(u, v)_\Sigma := \text{Re} \int_{\mathbb{R}^N} (\nabla u \nabla \bar{v} + |x|^2 u \bar{v} + u \bar{v}) \, dx,
\]

where \( \|u\|_\Sigma^2 := \|\nabla u\|_2^2 + \|xu\|_2^2 \), “Re” stays for the real part and \( \bar{v} \) denotes the conjugate of \( v \). We use “\( \rightarrow \)” and “\( \rightharpoonup \)” respectively to denote the strong and weak convergence in the related function spaces. \( C \) will denote a positive constant unless specified. \( a_n(1) \) and \( O_n(1) \) mean that \( |a_n(1)| \to 0 \) as \( n \to +\infty \) and \( |O_n(1)| \leq C \) as \( n \to +\infty \), respectively. \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of real and complex numbers respectively.

A standing wave of (1.1) with a prescribed mass \( c > 0 \) is a solution having the form \( \psi(t, x) = e^{-i\omega t} u(x) \) for some \( (u, \omega) \in \Sigma \times \mathbb{R} \) such that \( \|u\|_2^2 = \|\psi\|_2^2 = c \) and \( u \) weakly solves

\[
\left( -\frac{1}{2} \Delta + \frac{1}{2} |x|^2 - (\Omega \cdot L) \right) u - a|u|^{p-2}u = \omega u, \quad x \in \mathbb{R}^N
\]

in the following sense

\[
\text{Re} \left[ \frac{1}{2} \int_{\mathbb{R}^N} \nabla u \nabla \bar{\varphi} + \int_{\mathbb{R}^N} \frac{1}{2} |x|^2 u \bar{\varphi} - \int_{\mathbb{R}^N} \varphi (\Omega \cdot L) u - a \int_{\mathbb{R}^N} |u|^{p-2} u \bar{\varphi} - \omega \int_{\mathbb{R}^N} u \bar{\varphi} \right] = 0, \quad \forall \varphi \in \Sigma.
\]

To study the time-dependent equation (1.1), we shall concern firstly the stationary equation (1.2). Physicists usually call this type of solution \( u \) a “normalized solution” to (1.2). This fact implies that \( \omega \) cannot be determined a priori, but is part of the unknown. Normalized
solutions to (1.2) can be obtained by searching critical points of the energy functional

\[ I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |x|^2 |u|^2 - \frac{2a}{p} \int_{\mathbb{R}^N} |u|^p - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L) u dx \]  

on the constraint

\[ S(c) = \{ u \in \Sigma : \|u\|_2^2 = c \} \]  

with Lagrange multipliers \( \omega \).

Recently, J. Arbunich et al. in [4] concerned the existence, stability and instability properties of the standing waves to (1.1) with \( V(x) = \sum_{j=1}^N \frac{\gamma_j x_j^2}{2} \). They assumed that

\[ \text{(i)} \ a < 0, \ |\Omega| < \min_{1 \leq j \leq N} \gamma_j, \ 2 < p < 2^*; \quad \text{(ii)} \ a > 0, \ |\Omega| < \min_{1 \leq j \leq N} \gamma_j, \ 2 < p < 2 + \frac{4}{N}. \]

Under these hypotheses, the authors in [4] studied the global minimization problem

\[ E(c) := \inf_{u \in S(c)} I(u). \]  

They proved the relative compactness of any minimizing sequence for (1.5) and hence obtained the existence of minimizers and stability of \( \mathcal{M}_c \), where

\[ \mathcal{M}_c := \{ u \in S(c) : I(u) = E(c) \}. \]  

Whether the minimizer of (1.5) is radially symmetric or not is still unknown. When \( |\Omega| = 0 \) and \( V(x) = \frac{|x|^2}{2} \), the results of [28, 17] indicated that the minimizer of (1.5) is radially symmetric. In [30, 31, 18], a symmetry breaking result for the energy minimizers was proved for \( |\Omega| \) above a certain critical speed \( \Omega_{\text{crit}} > 0 \). Furthermore, an estimate for \( \Omega_{\text{crit}} \) in \( N = 2 \) can be found in [18].

Notice that \( \bar{p} := 2 + \frac{4}{N} \) is the \( L^2 \)-critical or mass-critical exponent for problem (1.5) since \( E(c) > -\infty \) if \( p \in (2, \bar{p}) \) and \( E(c) = -\infty \) if \( p \in (\bar{p}, 2^*) \). Indeed, for fixed \( u \in S(c) \), we have \( u_\tau(x) = \tau^\frac{N}{2} u(\tau x) \in S(c) \) and

\[ I(u_\tau) = \frac{\tau^2}{2} \|\nabla u\|_2^2 + \frac{1}{2 \tau^2} \|xu\|_2^2 - \frac{2a}{p} \tau^{p \delta_p} \|u\|_p^p - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L) u \to -\infty \]  

as \( \tau \to +\infty \), where

\[ \delta_p = \frac{N(p - 2)}{2p}. \]  

Y. J. Guo et al. in [16] considered the two-dimensional attractive BEC (i.e. \( N = 2, a > 0 \) and \( p = 4 \) in (1.2)) in a general trap \( V(x) \) satisfying \( 0 \leq V(x) \in L^\infty_0(\mathbb{R}^2) \) and \( \lim_{|x| \to \infty} \frac{V(x)}{|x|^2} > 0 \), which falls in the \( L^2 \)-critical case. They proved that there exists \( \Omega^* > 0 \) and \( a^* > 0 \) such that

\[ \text{(i)} \ \text{if} \ 0 \leq |\Omega| < \Omega^* \text{ and } 0 \leq a < a^*, \text{there exists at least one global minimizer}; \quad \text{(ii)} \ \text{if} \ 0 \leq |\Omega| < \Omega^* \text{ and } a \geq a^*, \text{there is no global minimizer}; \quad \text{(iii)} \ \text{if} \ |\Omega| > \Omega^* \text{ and } a > 0, \text{there is no} \]
global minimizer. The authors also analyzed the limit behavior and mass concentration of the global minimizers as a $a \not\to a^*$ if $0 < \Omega < \Omega^*$.

To our best knowledge, the existence and stability of standing waves to (1.1) with $\bar{p} < p < 2^*$ is still unknown. Much attention should be paid to this case since it contains the physically most relevant case $p = 4$, $N = 3$.

Since $E(c) = -\infty$ for $p \in (\bar{p}, 2^*)$, the global minimization method adopted in (1.5) does not work. Furthermore, due to the existence of the trapping potential $\frac{1}{2}|x|^2$, it seems not applicable to find a critical point of $I|_{S(c)}$ by minimizing $I$ on a constructed submanifold of $S(c)$ as [21] does. Motivated by [7, 8], we study a local minimization problem: for any given $r > 0$, define

$$m_c^r := \inf_{u \in S(c) \cap B(r)} I(u),$$

where

$$B(r) = \left\{ u \in \Sigma : \|u\|^2_{\Sigma} = \|\nabla u\|^2_2 + \|xu\|^2_2 \leq r \right\}.$$

For any fixed $r > 0$, it is clear that $m_c^r > -\infty$ if $S(c) \cap B(r) \neq \emptyset$. We will claim that $S(c) \cap B(r) \neq \emptyset$ (see Lemma 2.3) and $m_c^r$ is achieved. Once the claim is true, we have

$$M_c^r := \left\{ u \in S(c) \cap B(r) : I(u) = m_c^r \right\} \neq \emptyset.$$

After excluding the possibility of the minimizers locating on the boundary of $S(c) \cap B(r)$, then the minimizer of $m_c^r$ is indeed a critical point of $I|_{S(c)}$ as well as a normalized solution to (1.2). The main results in this aspect are stated as follows.

**Theorem 1.1.** Let $a > 0$, $N = 2, 3$, $2 + \frac{4}{N} \leq p < 2^*$ and $0 < |\Omega| < 1$. For any fixed $r > 0$, we could find some $c_0 := c_0(r, a, |\Omega|) > 0$ such that for any $c < c_0$, there exist $(u_c, \omega_c) \in \Sigma \times \mathbb{R}$ such that $u_c \in M_c^r$ and $u_c$ weakly solves (1.2) with $\omega = \omega_c$. Furthermore,

$$N\left(\frac{1 - |\Omega|^2}{2(1 + 3|\Omega|)} - ac_{N,p}\frac{c_{N,p} - 2}{c_{N,p}}\right) \leq \omega_c < \frac{N}{2},$$

and

$$\sup_{u \in M_c^r} \|u - l_0\psi_0\|^2_{\Sigma} = O(c + c^\frac{p(1-\delta_p)}{2}),$$

where $\delta_p = \frac{N(p-2)}{2p}$, $c_{N,p}$ is some positive constant, $\psi_0$ is the unique normalized positive eigenvector of the harmonic oscillator $-\Delta + |x|^2$ and $l_0 = \int_{\mathbb{R}^N} u\psi_0$.

Next, we show that $u_c$ is a normalized ground state if $c > 0$ is sufficiently small and concern the asymptotic behavior of $u_c$ as $c \to 0^+$. Following [8], we say that $u_c \in S(c)$ is a normalized ground state solution to problem (1.2) if $I'|_{S(c)}(u_c) = 0$ and $I(u_c) = \inf\{I(u) : u \in S(c), I'|_{S(c)}(u) = 0\}$.

**Theorem 1.2.** Assume that $a > 0$, $N = 2, 3$, $2 + \frac{4}{N} < p < 2^*$ and $0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p_0^*}\right)^2}$. Let $(u_c, \omega_c) \in M_c^r \times \mathbb{R}$ be given by Theorem 1.1 and $I(u_c) = m_c^r$. Then, $u_c$ is a normalized
ground state to (1.2) provided $c > 0$ is sufficiently small. Furthermore, $u_c \to 0$ in $\hat{\Sigma}$ as $c \to 0^+$, $\lim_{c \to 0^+} \frac{m_r^c}{c} = \lim_{c \to 0^+} \omega_c = \omega$ for some $\omega \in \left[\frac{(1-|\Omega|^2)N}{2(1+3|\Omega|)}, \frac{N}{2}\right]$ and
\[
\lim_{c \to 0^+} \frac{\|\nabla u_c\|_2^2 - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c}{c} = \lim_{c \to 0^+} \frac{\|xu_c\|_2^2 - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c}{c} = \omega.
\]

**Remark 1.1** Theorem 1.1 implies that the standing wave $\psi_c(t, x) = e^{-i\omega ct}u_c(x)$ of (1.1) behaves like the first eigenvector of the harmonic oscillator for small $c > 0$. Theorem 1.2 describes a mass collapse behavior of the minimizers $u_c \in \mathcal{M}_c^r$. It indicates that $u_c \to 0$ in $\hat{\Sigma}$ with $\|\nabla u_c\|_2^2 - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c$ and $\|xu_c\|_2^2 - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c$ converging to $0$ at the same rate, and the corresponding frequency $\omega_c$ converges to some $\omega$ as $c \to 0^+$. Due to the existence of the rotation term in (1.1), the limit $\omega$ is inaccurate. When the rotation frequency $|\Omega|$ vanishes, we could get
\[
\lim_{c \to 0^+} \frac{m_r^c}{c} = \lim_{c \to 0^+} \omega_c = \frac{N}{2}, \quad \lim_{c \to 0^+} \frac{\|\nabla u_c\|_2^2}{c} = \lim_{c \to 0^+} \frac{\|xu_c\|_2^2}{c} = \frac{N}{2},
\]
where we notice that $\frac{N}{2}$ is the first eigenvalue of $-\frac{1}{2}\Delta + \frac{1}{2}|x|^2$.

P. Antonelli et al. in [2] proved the local well-posedness of (1.1) in $\Sigma$ (See Lemma 3.1 of [2]), which states that for any $u_0 \in \Sigma$, there exists a $T > 0$ and a unique solution $u \in C([0, T], \Sigma)$ of (1.1) with $u(0, \cdot) = u_0$. In addition, they proved that the mass and energy are preserved for all $t \in [0, T)$, where either $T = +\infty$ or $T < +\infty$ and $\lim_{t \to T^-} \|\nabla u\|_2 = +\infty$. We say that a set $Y \subset \Sigma$ is stable under the flow associated with problem (1.1) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $u_0 \in \Sigma$ satisfying
\[
\text{dist}_\Sigma(u_0, Y) < \delta,
\]
the solution $u(t, \cdot)$ of problem (1.1) with $u(0, \cdot) = u_0$ satisfies
\[
\sup_{t \in [0, T]} \text{dist}_\Sigma(u(t, \cdot), Y) < \varepsilon,
\]
where $T$ is the existence time for $u(t, \cdot)$. With these preliminaries, we can study the stability of $\mathcal{M}_c^r$.

**Theorem 1.3.** Let $a > 0$, $N = 2, 3, 2 + \frac{4}{N} < p < 2^*$ and $0 < |\Omega| < 1$. Then, $\mathcal{M}_c^r$ is stable under the flow corresponding to problem (1.1).

**Remark 1.2** In [2], P. Antonelli et al. proved the global existence of solutions to (1.1) with $V(x) = \sum_{j=1}^N \frac{\gamma_j^2 x_j^2}{2}$ provided either $a < 0$ and $2 < p < 2^*$ or $a > 0$ and $2 < p < 2 + \frac{4}{N}$. On the contrary, finite time blow-up of the solutions to (1.1) occurred in two cases: (i) $a > 0$, $(\Omega \cdot L)V(x) = 0$ and $2 + \frac{4}{N} \leq p < 2^*$; (ii) $a > 0$, $(\Omega \cdot L)V(x) \neq 0$, $|\Omega| < \gamma$ and $2 + \frac{4}{N} \sqrt{\frac{\gamma^2}{\gamma^2 - |\Omega|^2}} \leq p < 2^*$, where $\gamma := \min_{1 \leq j \leq N} \gamma_j$. More recently, N. Basharat et al. in [6]
also studied (1.1). They obtained a sharp condition on the global existence and blowup of solutions to (1.1) for $p = \tilde{p}$ and some blowup conditions for $\tilde{p} < p < 2^*$. Moreover, similar results were extended to (1.1) with an inhomogeneous nonlinearity. Compared with the results in [2, 6], we obtain a new stability result in Theorem 1.3.

Recall that the solutions obtained in Theorem 1.1 are local minimizers of $I|_{S(c)}$ and $I|_{S(c)}$ is unbounded from below for $p \in (\tilde{p}, 2^*)$. Motivated by [8], by using the local minimizers in $\mathcal{M}^c_r$, we obtain a mountain pass critical point of $I|_{S(c)}$.

**Theorem 1.4.** Let $a > 0$, $N = 2, 3$, $2 + \frac{4}{N} < p < 2^*$ and $0 < |\Omega| < \sqrt{1 - \frac{2}{2^*}}$. For any $c < c_0$, there exist $(\hat{u}_c, \hat{\omega}_c) \in \Sigma \times \mathbb{R}$ such that $\hat{u}_c$ weakly solves (1.2) with $\omega = \hat{\omega}_c$ and $I(\hat{u}_c) > m^c_r$, where $c_0$ is defined by Theorem 1.1.

We give the outline of the proof for our main results. Theorem 1.1 is proved by searching minimizers of $I|_{S(c) \cap B(r)}$. Once $m^c_r > -\infty$ is proved, each minimizing sequence of $m^c_r$ is bounded in $\Sigma$. Observing that $\frac{1}{2}\|u\|_\Sigma^2 - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)udx$ is an equivalent norm in $\Sigma$ provided $0 < |\Omega| < 1$, we deduce that $I$ is weakly lower semi-continuous (see (3.3)). Moreover, Lemma 2.5 gives the compactness of the embedding $\Sigma \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C})$ for $q \in [2, 2^*)$, then the existence of minimizer to $m^c_r$ follows. The rest is to show that the minimizer is not on the boundary of $S(c) \cap B(r)$, then it is indeed a critical point of $I|_{S(c)}$. To this end, we find a suitable constant $c_0 = c_0(r, a, |\Omega|)$ such that for $c \leq c_0$, it holds that

$$\inf_{u \in S(c) \cap B(\nu r)} I(u) < \inf_{u \in S(c) \cap (B(r) \setminus B(\nu r))} I(u),$$

where $\nu = \frac{1 - |\Omega|}{2}$ and $\mu = \frac{1 + |\Omega|}{2}$. This local minima structure (1.10) guarantees that all minimizing sequences of $m^c_r$ shrink and results in $\mathcal{M}^c_r \subset B(\nu r)$, which leads to the minimizer of (1.8) is bounded away from the boundary of $S(c) \cap B(r)$.

The proof of Theorem 1.2 mainly comes from [7]. The key point is to prove that

$$v \in S(c) \text{ such that } \int_{S(c)} \bar{v}(\Omega \cdot L)vdx \quad \text{within } I(v) \text{ is sign indefinite. In fact, } \int_{S(c)} \bar{v}(\Omega \cdot L)vdx = 0 \text{ gives the Pohozaev identity}$$

$$Q(v) := \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}\|xv\|_2^2 - a_\delta p \|v\|_p^p = 0,$$

and hence $I(v)$ can be rewrite as $I(v) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\|\nabla v\|_2^2 + \left(\frac{1}{2} + \frac{1}{2^*}\right)\|xv\|_2^2 - \int_{\mathbb{R}^N} \bar{v}(\Omega \cdot L)v$, then the extra condition $0 < |\Omega| < \sqrt{1 - \frac{2}{2^*}}$ indicates that

$$C\|v\|^2_2 \leq I(v) < m^c_r < \frac{Nc}{2} \to 0 \text{ as } c \to 0$$

for some constant $C > 0$. So $v \in B(r)$ as $c \to 0$ follows.
To prove Theorem 1.3, we use the fact that any minimizing sequence of $m_c^r$ is precompact and $\mathcal{M}_c^r \neq \emptyset$ (see the proof of Theorem 1.1). By a contradiction argument, we obtain the stability of $\mathcal{M}_c^r$.

Theorem 1.4 is proved by a variant of mountain pass theorem. Let $\gamma(c)$ be the mountain pass level, we will construct a special Palais-Smale sequence $\{v_n\}$ at energy level $\gamma(c)$ with $Q(v_n) \to 0$. When $|\Omega| = 0$, the property $Q(v_n) \to 0$ is sufficient to derive the boundedness of $\{v_n\}$ (see [8]). However, the term $\int_{\mathbb{R}^N} \bar{v}_n(\Omega \cdot L)v_n dx$ within $I(v_n)$ is sign indefinite if $0 < |\Omega| < 1$ and we can not proceed as in [8]. Under the stronger condition $0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p\delta_p}\right)^2}$, we can prove that

$$C\|v_n\|_{\Sigma}^2 + o_n(1) \leq I(v_n) \leq \gamma(c) + 1$$

for some constant $C > 0$. Then $\{v_n\}$ is bounded in $\Sigma$. The rest is standard as in [8].

**Remark 1.3** The conditions $0 < |\Omega| < 1$ in Theorem 1.1 and $0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p\delta_p}\right)^2}$ in Theorem 1.4 are necessary. In fact, the essence of the restrictions on $|\Omega|$ is that,

$$\|u\|_{\Omega_1}^2 \approx \|u\|_{\Sigma}^2 \text{ if } 0 < |\Omega| < 1, \quad \|u\|_{\Omega_2}^2 \approx \|u\|_{\Sigma}^2 \text{ if } 0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p\delta_p}\right)^2}, \quad \forall u \in \Sigma,$$

where

$$\|u\|_{\Omega_1}^2 := \frac{1}{2}\|u\|_{\Sigma}^2 - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)u, \quad \|u\|_{\Omega_2}^2 := \left(\frac{1}{2} - \frac{1}{p\delta_p}\right)\|\nabla u\|_2^2 + \left(\frac{1}{2} + \frac{1}{p\delta_p}\right)\|\cdot u\|_2^2 - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)u,$$

and $\| \cdot \|_A \approx \| \cdot \|_B$ means $\| \cdot \|_A \approx \| \cdot \|_B$ and $\| \cdot \|_A$ and $\| \cdot \|_B$ are two equivalent norms. As pointed out above, $0 < |\Omega| < 1$ guarantees $\| \cdot \|_{\Omega_1} \approx \| \cdot \|_{\Sigma}$ and hence the weakly lower semi-continuity of $I$ in proving Theorem 1.1, and $0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p\delta_p}\right)^2}$ guarantees $\| \cdot \|_{\Omega_2} \approx \| \cdot \|_{\Sigma}$ and hence the boundedness of the corresponding Palais-Smale sequence in proving Theorem 1.4. Alternatively, we can obtain Theorem 1.1 by studying

$$m_c^r := \inf_{u \in S(c) \cap B(r)} I(u) \text{ for } B(r) = \left\{ u \in \Sigma : \|u\|_{\Omega_1}^2 \leq r \right\}.$$

**Remark 1.4** Our main results in the present paper can be extended from $V(x) = \frac{|x|^2}{2}$ to $V(x) = \sum_{j=1}^{N} \frac{\gamma_j^2 x_j^2}{2}$, with the rotation frequency satisfying $0 < |\Omega| < \min_{1 \leq j \leq N} \gamma_j$. Here $\gamma_j > 0$ ($j = 1, \cdots, N$) is the trapping frequencies in each spatial direction, see [2, 4, 6] for details.

The paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we prove Theorems 1.1-1.3. In Section 4, we prove Theorem 1.4.
2. Preliminary Results

In this section, we give some preliminary results. Firstly, we give the Gagliardo-Nirenberg inequality (See [32]).

**Lemma 2.1.** Let \( N \geq 2 \) and \( p \in (2, 2^*) \). Then there exists a constant \( C_{N,p} > 0 \) such that

\[
||u||_p \leq C_{N,p} \|\nabla u\|_2^{\delta_p} ||u||_2^{(1-\delta_p)}, \quad \forall u \in H^1(\mathbb{R}^N, \mathbb{R})
\]

where \( C_{N,p} = \left( \frac{p}{2|W_p|_2^2} \right)^{\frac{1}{p}} \), \( W_p \) is the ground state solution of \(-\Delta W + \frac{1}{\delta_p} - 1)W = \frac{2}{p\delta_p}|W|^{p-2}W\) and \( \delta_p = \frac{N(p-2)}{2p} \).

**Lemma 2.2.** ([32]) Let \( |x|u \) and \( \nabla u \) belong to \( L^2(\mathbb{R}^N, \mathbb{R}) \). Then, \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \) and

\[
||u||_2^2 \leq \frac{2}{N} ||\nabla u||_2^2 ||x||_2^2,
\]

with equality holding for the functions \( u(x) = e^{-\frac{1}{2}|x|^2} \).

**Remark 2.1** Lemma 2.1 remains true for any \( u \in H^1(\mathbb{R}^N, \mathbb{C}) \) and Lemma 2.2 remains true for any \( u \in \Sigma \) since \( |\nabla u| \leq |\nabla u| \), see Theorem 6. 17 in [20].

**Lemma 2.3.** For any \( r > 0 \), \( S(c) \cap B(r) \neq \emptyset \) iff \( c \leq \frac{\epsilon}{N} \).

**Proof.** Let \( r > 0 \) be fixed. For any \( u \in S(c) \cap B(r) \neq \emptyset \), Lemma 2.2 and Remark 2.1 indicate that

\[
c = ||u||_2^2 \leq \frac{2}{N} ||\nabla u||_2^2 ||x||_2^2 \leq \frac{2}{N} \left( \frac{1}{2} ||\nabla u||_2^2 + \frac{1}{2} ||x||_2^2 \right) = \frac{1}{N} ||u||_2^2 \leq \frac{r}{N},
\]

On the other hand, let \( \phi(x) = e^{-\frac{1}{2}|x|^2} \) and \( \psi_0 = \pi^{-\frac{N}{2}} e^{-\frac{1}{2}|x|^2} \), then we have

\[
||\nabla \phi||_2^2 = ||x\phi||_2^2 = \frac{N}{2} ||\phi||_2^2 = \frac{N}{2} \pi^\frac{N}{2}, \quad ||\nabla \psi_0||_2^2 = ||x\psi_0||_2^2 = \frac{N}{2}, \quad ||\psi_0||_2^2 = 1.
\]

For any \( c \leq \frac{\epsilon}{N} \), we have \( \sqrt{c}\psi_0 \in S(c) \cap B(r) \). \( \square \)

By Young’s inequality and the fact that \( \Omega \cdot L := -i\Omega \cdot (x \wedge \nabla) = -i(\Omega \wedge x) \cdot \nabla \), we obtain the following interpolation inequality.

**Lemma 2.4.** ([4], Inequality (2.3)) Let \( \Omega \cdot L := -i\Omega \cdot (x \wedge \nabla) \). For any \( \epsilon > 0 \), it holds that

\[
|\langle u, (\Omega \cdot L)u \rangle| \leq \|\Omega \wedge x\|_2 \|\nabla u\|_2 \leq \frac{\|\Omega\|_2^2}{2\epsilon} ||x||_2^2 + \frac{\epsilon}{2} ||\nabla u||_2^2, \quad \forall u \in \Sigma.
\]

We recall the following compactness result:

**Lemma 2.5.** ([28, 34]) For \( N \geq 2 \) and \( q \in [2, 2^*) \), the embedding \( \Sigma \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C}) \) is compact.
3. Proof of Theorems 1.1-1.3

In this section, we prove Theorems 1.1-1.3. To begin with we show that $I|_{S(c)}$ presents a local minima structure by the previous lemmas. This fact guarantees that the minimizer of $m^*_c$ is indeed a critical point of $I|_{S(c)}$.

**Proposition 3.1.** Let $a > 0$, $N = 2, 3$, $2 + \frac{4}{N} \leq p < 2^*$ and $0 < |\Omega| < 1$. For any $r > 0$, if $S(c) \cap (B(\mu r) \setminus B(\nu r)) \neq \emptyset$, then there exists $c_0 := c_0(r, a, \Omega) > 0$ such that for any $c < c_0$,

$$\inf_{u \in S(c) \cap B(\nu r)} I(u) < \inf_{u \in S(c) \cap (B(\nu r) \setminus B(\mu r))} I(u),$$

(3.1)

where $\nu = \frac{1 - |\Omega|}{4}$ and $\mu = \frac{1 + |\Omega|}{2}$. 

**Proof.** Let $\nu = \frac{1 - |\Omega|}{4}$ and $\mu = \frac{1 + |\Omega|}{2}$ be fixed, then $0 < \nu < \mu < 1$ and $\frac{\nu}{\mu} < \frac{1 - |\Omega|}{1 + |\Omega|} < \frac{1}{\sqrt{\Omega}}$ as $0 < |\Omega| < 1$. By Lemma 2.3, $S(c) \cap B(\nu r) \neq \emptyset$ iff $c \leq \frac{\nu}{\mu}$. If $S(c) \cap B(\nu r) \neq \emptyset$ and $S(c) \cap (B(r) \setminus B(\mu r)) \neq \emptyset$, we will prove (3.1). Since $\frac{\nu}{\mu} < \frac{1 - |\Omega|}{1 + |\Omega|}$, we can choose $\varepsilon_0 \in (\frac{\mu + \nu}{\mu - \nu}|\Omega|^2, \frac{\mu - \nu}{\mu + \nu})$ and denote

$$C_s(\Omega) := \min \left\{ \left( \frac{1 - \varepsilon_0}{2}, \frac{1}{2} \right) \right\}, C^*(\Omega) := \max \left\{ \left( 1 + \varepsilon_0, \frac{1 + |\Omega|^2}{2\varepsilon_0} \right) \right\}.$$

We deduce from $\frac{\nu}{\mu} < \frac{1 - |\Omega|}{1 + |\Omega|}$ that $\frac{\mu + \nu}{\mu - \nu}|\Omega|^2 < \frac{\mu + \nu}{\mu - \nu}|\Omega| < \frac{\mu - \nu}{\mu + \nu}$. From now on, let

$$\varepsilon_0 = \frac{\mu + \nu}{\mu - \nu}|\Omega| = \frac{3 + |\Omega|}{1 + 3|\Omega|} = \frac{3}{1 + 3|\Omega|}.$$

be fixed. Direct calculations imply that

$$\frac{1 - |\Omega|^2}{2(1 + 3|\Omega|)} = C_s(\Omega) < C^*(\Omega) = \frac{1 + 6|\Omega| + |\Omega|^2}{2(1 + 3|\Omega|)} < C^*(\Omega) = \frac{\mu}{\nu}C_s(\Omega).$$

(3.2)

Applying inequality (2.2) in Lemma 2.4 with $\varepsilon = \varepsilon_0$, we have

$$C^*(\Omega)||u||_S^2 \leq \frac{1}{2}||u||_S^2 - \int_{R^N} \hat{u}(\Omega \cdot L)udx \leq C^*(\Omega)||u||_S^2.$$

(3.3)

We see that $\frac{1}{2}||u||_S^2 - \int_{R^N} \hat{u}(\Omega \cdot L)udx$ is a new norm which is equivalent to $||u||_S^2$. This fact is also observed by N. Basharat et al. in [6].

For any $u \in S(c) \cap (B(r) \setminus B(\mu r))$, by (2.1) in Lemma 2.1 and (3.3), we have

$$I(u) = \frac{1}{2}||\nabla u||_2^2 + ||xu||_2^2 - \frac{2a}{p}||u||_p^p - \int_{R^N} \hat{u}(\Omega \cdot L)udx$$

$$\geq C^*(\Omega)||u||_S^2 - \frac{2a}{p}C_{N,p}C_{p\delta}^p \left( \frac{p(1-\delta p)}{2} \right) \geq C^*(\Omega)||u||_S^2 - \frac{2a}{p}C_{N,p}C_{p\delta}^p \left( \frac{p(1-\delta p)}{2} \right)$$

$$= ||u||_S^2 \left( C^*(\Omega) - \frac{2a}{p}C_{N,p}C_{p\delta}^p \left( \frac{p(1-\delta p)}{2} \right) \right) \geq \mu r \left( C^*(\Omega) - \frac{2a}{p}C_{N,p}C_{p\delta}^p r \left( \frac{p(1-\delta p)}{2} \right) \right).$$

(3.4)
where we restrict $c < \left[ -\frac{pC_*(\Omega)}{2aC_{N,p}^p - p^2} \right] \frac{2}{p(1 - sp)}$ in the last inequality.

On the other hand, for any $u \in S(c) \cap B(\nu r)$, we deduce from (3.3) that

$$I(u) = \frac{1}{2} \|u\|_p^2 - \frac{2a}{p} \|u\|_p^p - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)u \, dx \leq C^*(\Omega)\|u\|_\Sigma^2 \leq \nu r C^*(\Omega). \quad (3.5)$$

Hence by (3.4) and (3.5), we have (3.1) holds provided

$$c < c_0 = c_0(r, a, \Omega),$$

where

$$c_0 := \min \left\{ \frac{(1 - |\Omega|)r}{4N}, \frac{p(1 - |\Omega|)^3}{16(1 + 3|\Omega|)aC_{N,p}^p \frac{p^2 - 2}{2} \frac{2}{p - 2}}, \frac{1 - |\Omega|^2}{2(1 + 3|\Omega|)aC_{N,p}^p \frac{p^2 - 2}{2} \frac{2}{p - 2}} \right\}. \quad (3.6)$$

We also need the following Pohozaev identity.

**Proposition 3.2.** Let $a, \lambda \in \mathbb{R}$, $N = 2, 3$, $2 < p \leq 2^*$ and $0 < |\Omega| < 1$. If $v \in \Sigma$ weakly solves

$$-\frac{1}{2} \Delta v + \frac{1}{2} |x|^2 v - (\Omega \cdot L)v - a|v|^{p-2}v = \lambda v, \quad (3.7)$$

then the Pohozaev identity

$$Q(v) := \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{2} \|xv\|_2^2 - a\delta_p \|v\|_p^p = 0 \quad (3.8)$$

holds, where $\delta_p = \frac{N(p-2)}{2p}$.

**Proof.** Multiply (3.7) by $x \cdot \nabla \bar{v}$, integrate by parts and take real parts, we obtain

$$\frac{2 - N}{4} \|\nabla v\|_2^2 - \frac{N + 2}{4} \|xv\|_2^2 + \frac{N\lambda}{2} \|v\|_2^2 + \frac{Na}{p} \|v\|_p^p - \Re \int_{\mathbb{R}^N} [(\Omega \cdot L)v](x \cdot \nabla \bar{v}) = 0. \quad (3.9)$$

To eliminate $\lambda$, we multiply (3.7) by $\bar{v}$ and get

$$\frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \|xv\|_2^2 - \lambda \|v\|_2^2 - a\delta_p \|v\|_p^p = \int_{\mathbb{R}^N} \bar{v}(\Omega \cdot L)v = 0. \quad (3.10)$$

Combine (3.9) and (3.10), we have the following Pohozaev identity

$$\frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{2} \|xv\|_2^2 - a\delta_p \|v\|_p^p - \Re \int_{\mathbb{R}^N} [(\Omega \cdot L)v](x \cdot \nabla \bar{v}) - \frac{N}{2} \int_{\mathbb{R}^N} \bar{v}(\Omega \cdot L)v = 0. \quad (3.11)$$

The facts $\Re \int_{\mathbb{R}^N} [(\Omega \cdot L)v](x \cdot \nabla \bar{v}) = -\Re \int_{\mathbb{R}^N} [(\Omega \cdot L)\bar{v}](x \cdot \nabla v)$ and

$$\int_{\mathbb{R}^N} [(\Omega \cdot L)v](x \cdot \nabla \bar{v}) = \int_{\mathbb{R}^N} [(\Omega \cdot L)\bar{v}](x \cdot \nabla v) = N \int_{\mathbb{R}^N} \bar{v}(\Omega \cdot L)v$$

imply that

$$\Re \int_{\mathbb{R}^N} [(\Omega \cdot L)v](x \cdot \nabla \bar{v}) = -\frac{N}{2} \int_{\mathbb{R}^N} \bar{v}(\Omega \cdot L)v.$$
Therefore, (3.11) can be reduced to

\[ Q(v) := \frac{1}{2} \| \nabla v \|_2^2 - \frac{1}{2} \| xv \|_2^2 - a\delta_p \| v \|_p^p = 0. \]

We now prove the existence of a local minimizer.

**Proof of Theorem 1.1.** First, we show the existence of a local minimizer. It is sufficient to prove \( \mathcal{M}_c^r \neq \emptyset \). Let \( \{ u_n \} \subset S(c) \cap B(r) \) be a minimizing sequence for \( m_c^r = \inf_{u \in S(c) \cap B(r)} I(u) \), then \( \{ u_n \} \) is bounded in \( \Sigma \). By the compactness of the embedding \( \Sigma \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C}) \) for \( q \in [2, 2^*) \), see Lemma 2.5, there exists \( u \in \Sigma \) such that

\[
\begin{aligned}
&u_n \to u \text{ in } \Sigma, \\
u_n \to u \text{ in } L^q(\mathbb{R}^N, \mathbb{C}), \\
u_n \to u \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
\]

Consequently, we have \( u \in S(c) \cap B(r) \). Moreover, we deduce from (3.3) that the energy functional \( I \) is weakly lower semi-continuous. Therefore, we have

\[ I(u) \leq \lim_{n \to \infty} I(u_n) = m_c^r \leq I(u), \]

which gives \( I(u) = m_c^r \) and \( u_n \to u \) in \( \Sigma \). This implies that any minimizing sequence for \( m_c^r \) is precompact and \( \mathcal{M}_c^r \neq \emptyset \). For any \( u_c \in \mathcal{M}_c^r \), Proposition 3.1 implies that \( u_c \not\in S(c) \cap \partial B(r) \) as \( u_c \in B(\nu r) \), where \( \partial B(r) = \{ u \in \Sigma : \| u \|_2^2 = r \} \). Then \( u_c \) is indeed a critical point of \( I\vert_{S(c)} \). So, there exists a Lagrange multiplier \( \omega_c \in \mathbb{R} \) such that \( (u_c, \omega_c) \) is a couple of weak solution to problem (1.2).

Next, we estimate the bound of the Lagrange multiplier \( \omega_c \). Notice that the pure point spectrum of the harmonic oscillator is

\[ \sigma_p(-\Delta + |x|^2) = \{ \lambda_k = N + 2k : k \in \mathbb{N} \} \]

and the corresponding eigenfunctions are given by Hermite functions (denoted by \( \psi_k \), associated to \( \lambda_k \)), which form an orthonormal basis of \( L^2(\mathbb{R}^N, \mathbb{R}) \) (see [3]). Let \( \psi_0 \in S(1) \) be an eigenfunction with respect to the first eigenvalue \( \lambda_0 = N \) and \( \psi = \sqrt{\omega} \psi_0 \in S(c) \). Then \( \psi \in B(r) \) if \( c \leq \frac{r}{\sqrt{\omega}} \). As \( \psi \) is real valued, we have \( \int_{\mathbb{R}^N} \tilde{\psi}(\Omega \cdot L) \psi \, dx = 0 \) and

\[ m_c^r \leq I(\psi) = \frac{1}{2} \| \psi \|_\Sigma^2 - \frac{2a}{p} \| \psi \|_p^p - \int_{\mathbb{R}^N} \tilde{\psi}(\Omega \cdot L) \psi \, dx < \frac{1}{2} \| \psi \|_\Sigma^2 = \frac{1}{2} N c. \quad (3.12) \]

Since \( (u_c, \omega_c) \in \mathcal{M}_c^r \times \mathbb{R} \) weakly solves problem (1.2), we learn from (3.12) that

\[ \omega_c \| u_c \|_2^2 = \frac{1}{2} \| u_c \|_\Sigma^2 - a \| u_c \|_p^p - \int_{\mathbb{R}^N} \tilde{u}_c(\Omega \cdot L) u_c \, dx \]

\[ = I(u_c) + \frac{a(2 - p)}{p} \| u_c \|_p^p < I(u_c) = m_c^r < \frac{1}{2} N c, \quad (3.13) \]
which implies that $\omega_c < \frac{N}{2}$. On the other hand, by (2.1) and (3.3), we have

$$\omega_c\|u_c\|^2 = \frac{1}{2}\|u_c\|^2 - a\|u_c\|^p_{\Sigma} - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c\,dx$$

$$\geq C_s(\Omega)\|u_c\|^2 - a\mathcal{C}_{N,p}^a\|\nabla u_c\|^2_{\Sigma} e^{\frac{p(1-\delta_p)}{2}} \geq C_s(\Omega)\|u_c\|^2 - a\mathcal{C}_{N,p}^a\|u_c\|^{p\delta_p}_{\Sigma} e^{\frac{p(1-\delta_p)}{2}}$$

$$= \|u_c\|^2_{\Sigma}\left(C_s(\Omega) - a\mathcal{C}_{N,p}^a\|u_c\|^{p\delta_p-2}_{\Sigma} e^{\frac{p(1-\delta_p)}{2}}\right) \geq NC_s(\Omega) - a\mathcal{C}_{N,p}^a r\frac{p\delta_p-2}{2} e^{\frac{p(1-\delta_p)}{2}},$$

which implies that $\omega_c \geq N\left(\frac{1-|\Omega|^2}{2(1+3|\Omega|)} - a\mathcal{C}_{N,p}^a r\frac{p\delta_p-2}{2} e^{\frac{p(1-\delta_p)}{2}}\right) > 0$ as $c < c_0$, see (3.6).

Finally, we show that

$$\sup_{u \in \mathcal{M}_c^r} \|u - l_0\psi_0\|_{\Sigma}^2 = O\left(c + c^\frac{p(1-\delta_p)}{2}\right).$$

For any $u \in \mathcal{M}_c^r$, we rewrite $u = u_1 + iu_2$, it results to

$$u = \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^N} u_1 \psi_k(\cdot) + i \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^N} u_2 \psi_k(\cdot) \psi_k = \sum_{k=0}^{\infty} l_k \psi_k \right) \right) \text{ with } l_k = \int_{\mathbb{R}^N} u \psi_k,$$

where $u_1$ is the real part and $u_2$ is the imaginary part of $u$, $\{\psi_k\}$ is an orthonormal basis of $L^2(\mathbb{R}^N, \mathbb{R})$. Thus

$$c = \|u\|^2 = \sum_{k=0}^{\infty} l_k \bar{l}_k \int_{\mathbb{R}^N} |\psi_k|^2 = \sum_{k=0}^{\infty} |l_k|^2,$$

where $\bar{l}_k$ is the conjugate of $l_k$. Since $u \in \mathcal{M}_c^r \subset B(r)$, by (2.1) and (3.3), we have

$$\frac{Nc}{2} > I(u) = \frac{1}{2}\|u\|^2_{\Sigma} - \frac{2a}{p}\|u\|^p_{\Sigma} - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)u\,dx$$

$$\geq C_s(\Omega)\|u\|^2_{\Sigma} - \frac{2a}{p}\mathcal{C}_{N,p}^a\|\nabla u\|^2_{\Sigma} e^{\frac{p(1-\delta_p)}{2}} \geq C_s(\Omega)\|u\|^2_{\Sigma} - \frac{2a}{p}\mathcal{C}_{N,p}^a\|u\|^{p\delta_p}_{\Sigma} e^{\frac{p(1-\delta_p)}{2}}$$

$$\geq C_s(\Omega)\|u\|^2_{\Sigma} - \frac{2a}{p}\mathcal{C}_{N,p}^a r\frac{p\delta_p}{2} e^{\frac{p(1-\delta_p)}{2}} = C_s(\Omega) \sum_{k=0}^{\infty} \lambda_k |l_k|^2 = \frac{2a}{p}\mathcal{C}_{N,p}^a r\frac{p\delta_p}{2} e^{\frac{p(1-\delta_p)}{2}},$$

which implies that

$$\sum_{k=1}^{\infty} |l_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k |l_k|^2 \leq \sum_{k=0}^{\infty} \lambda_k |l_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k |l_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k |l_k|^2 \leq \frac{Nc}{2C_s(\Omega)} + \frac{2a}{p} \mathcal{C}_{N,p}^a r\frac{p\delta_p}{2} e^{\frac{p(1-\delta_p)}{2}},$$

by using (3.12). Thus, we have

$$\|u - l_0\psi_0\|_{\Sigma}^2 = \left\| \sum_{k=1}^{\infty} l_k \psi_k \right\|^2 \leq \sum_{k=1}^{\infty} \lambda_k |l_k|^2 \leq \frac{Nc}{2C_s(\Omega)} + \frac{2a}{p} \mathcal{C}_{N,p}^a r\frac{p\delta_p}{2} e^{\frac{p(1-\delta_p)}{2}},$$

and

$$\|u - l_0\psi_0\|_{\Sigma}^2 = \left\| \sum_{k=1}^{\infty} l_k \psi_k \right\|^2 = \sum_{k=1}^{\infty} |l_k|^2 \leq \frac{c}{2C_s(\Omega)} + \frac{2a}{pN} \mathcal{C}_{N,p}^a r\frac{p\delta_p}{2} e^{\frac{p(1-\delta_p)}{2}}.$$
Thus, we deduce from (3.12) that \[ \sup_{u \in M_c} \|u - l_0 \psi_0\|^2_\Sigma \leq (N+1) \left( \frac{4(1+3|\Omega|)nC_\Omega}{pN(1-|\Omega|^2)} + \frac{4|\Omega|^2}{pN(1-|\Omega|^2)} \right). \]

So we have \[ \sup_{u \in M_c} \|u - l_0 \psi_0\|^2_\Sigma = O(c + c^{\frac{p(1-\delta_p)}{2}}). \]

Next, we show that \( u_c \) is a normalized ground state if \( c > 0 \) is sufficiently small. We also concern the asymptotic behavior of \( u_c \) obtained by Theorem 1.1 as \( c \to 0^+ \).

**Proof of Theorem 1.2.** This is motivated by [7]. On the contrary, we assume that there exists a \( v \in S(c) \) such that

\[ I'|_{S(c)}(v) = 0 \text{ and } I(v) < m_c^r. \]

Since \( I'|_{S(c)}(v) = 0 \), then \( v \) satisfies

\[ \left( -\frac{1}{2} \Delta + \frac{1}{2} |x|^2 - (\Omega \cdot L) \right)v - \alpha|v|^{p-2}v = \lambda v, \quad x \in \mathbb{R}^N \] (3.14)

for some \( \lambda \in \mathbb{R} \). It follows from Proposition 3.2 that \( Q(v) := \frac{1}{2}\|\nabla v\|^2_{2} - \frac{1}{2}\|xv\|^2_{2} - a\delta_p \|v\|^p_{p} = 0 \). Therefore, we have

\[ I(v) = \left( \frac{1}{2} - \frac{1}{p\delta_p} \right)\|\nabla v\|^2_{2} + \left( \frac{1}{2} + \frac{1}{p\delta_p} \right)\|xv\|^2_{2} - \int_{\mathbb{R}^N} \tilde{u}(\Omega \cdot L)v. \]

Since \( 0 < |\Omega| < \sqrt{1 - \left( \frac{2}{p\delta_p} \right)^2} \), we can choose \( \varepsilon_1 \in \left( \frac{p\delta_p}{p\delta_p + 2|\Omega|^2}, 1 - \frac{2}{p\delta_p} \right) \) and denote

\[ C_1(\Omega) := \frac{1}{2} - \frac{1}{p\delta_p} - \frac{\varepsilon_1}{2}, \quad C_2(\Omega) := \frac{1}{2} + \frac{1}{p\delta_p} - \frac{|\Omega|^2}{2\varepsilon_1}, \quad C_\Omega := \min \{C_1(\Omega), C_2(\Omega)\}. \] (3.15)

It’s easy to see that \( C_1(\Omega) > 0, C_2(\Omega) > 0 \). Applying inequality (2.2) with \( \varepsilon = \varepsilon_1 \), we have

\[ I(v) \geq C_1(\Omega)\|\nabla v\|^2_{2} + C_2(\Omega)\|xv\|^2_{2} \geq \min \{C_1(\Omega), C_2(\Omega)\} \|v\|^2_{\Sigma} = C_\Omega \|v\|^2_{\Sigma}. \]

Thus, we deduce from (3.12) that

\[ C_\Omega \|v\|^2_{\Sigma} \leq I(v) < m_c^r < \frac{Nc}{2} \to 0 \text{ as } c \to 0. \]

If \( c \) is sufficiently small, we have \( v \in B(r) \) and \( I(v) \geq m_c^r \), which contradicts to \( I(v) < m_c^r \).

Next, we show that \( u_c \to 0 \) in \( \tilde{\Sigma} \) as \( c \to 0^+ \). Since \( (u_c, \omega_c) \in M_c^r \times \mathbb{R} \) weakly solves (1.2), Proposition 3.2 indicates that \( Q(u_c) = 0 \). Then, \( I(u_c) \) can be rewrite as

\[ I(u_c) = \left( \frac{1}{2} - \frac{1}{p\delta_p} \right)\|\nabla u_c\|^2_{2} + \left( \frac{1}{2} + \frac{1}{p\delta_p} \right)\|xu_c\|^2_{2} - \int_{\mathbb{R}^N} \bar{u}_c(\Omega \cdot L)u_c. \]

Applying inequality (2.2) with \( \varepsilon = \varepsilon_1 \), we also have

\[ I(u_c) \geq C_1(\Omega)\|\nabla u_c\|^2_{2} + C_2(\Omega)\|xu_c\|^2_{2} \geq \min \{C_1(\Omega), C_2(\Omega)\} \|u_c\|^2_{\Sigma} = C_\Omega \|u_c\|^2_{\Sigma}. \]

Therefore, it holds that \( C_\Omega \|u_c\|^2_{\Sigma} \leq I(u_c) = m_c^r < \frac{Nc}{2} \to 0 \text{ as } c \to 0^+. \)
Let \( v_c := \frac{u_n}{\|u_n\|_2} = \frac{u_n}{\sqrt{c}}, \) then we have the following estimates
\[
C_\Omega \|v_c\|_\Sigma^2 \leq \frac{I(u_c)}{\|u_c\|_2^2} = \frac{m^r_c}{c} < \frac{N}{2}. \tag{3.16}
\]
By using (2.1) and (3.16), we deduce that
\[
0 < \frac{C^p_{N,p} \|\nabla u_c\|_2^{p(1-\delta_p)}}{\|u_c\|_2^2} = \frac{C^p_{N,p} \|\nabla v_c\|_2^{p(1-\delta_p)}}{\|u_c\|_2^2} \leq \frac{C^p_{N,p} \|\nabla v_c\|_2^{p-2}}{\|u_c\|_2^2} \leq C c^{-\frac{p-2}{p}} \to 0
\]
as \( c \to 0^+ \), where \( C = C(p, N, |\Omega|) > 0 \) is some constant. From Proposition 3.2, we have
\[
0 = \frac{Q(u_c)}{\|u_c\|_2^2} = \frac{1}{2} \|\nabla v_c\|_2^2 - \frac{a}{2} \|x v_c\|_2^2 - a \delta_p \|u_c\|_2^p.
\]
which gives \( \lim_{c \to 0^+} \|\nabla v_c\|_2^2 = \lim_{c \to 0^+} \|x v_c\|_2^2 \). Since \( N \left( \frac{1-|\Omega|^2}{2(1+|\Omega|)} - aC^p_{N,p} r^{p-2} c \frac{\rho(1-\delta_p)}{2} \right) \leq \omega_c < \frac{N}{2} \), then there exists an \( \omega \in (\frac{1-|\Omega|^2}{2(1+|\Omega|)}, \frac{N}{2}) \) such that \( \lim_{c \to 0^+} \omega_c = \omega \) as \( c \to 0^+ \). By these facts and (3.13), we have
\[
\lim_{c \to 0^+} \omega_c = \lim_{c \to 0^+} \left[ \frac{1}{2} \|v_c\|_\Sigma^2 - \int_{\mathbb{R}^N} \tilde{v}_c(\Omega \cdot L) v_c dx - a \|u_c\|_2^p \right] = \lim_{c \to 0^+} \left[ \frac{1}{2} \|v_c\|_\Sigma^2 - \int_{\mathbb{R}^N} \tilde{v}_c(\Omega \cdot L) v_c dx \right],
\]
\[
\lim_{c \to 0^+} \frac{m^r_c}{c} = \lim_{c \to 0^+} \left[ \frac{1}{2} \|v_c\|_\Sigma^2 - \int_{\mathbb{R}^N} \tilde{v}_c(\Omega \cdot L) v_c dx - \frac{2a}{p} \|u_c\|_2^p \right] = \lim_{c \to 0^+} \left[ \frac{1}{2} \|v_c\|_\Sigma^2 - \int_{\mathbb{R}^N} \tilde{v}_c(\Omega \cdot L) v_c dx \right].
\]
Finally, we deduce that \( \lim_{c \to 0^+} \frac{m^r_c}{c} = \omega \) and
\[
\lim_{c \to 0^+} \frac{\|\nabla u_c\|_2^2 - \int_{\mathbb{R}^N} \tilde{u}_c(\Omega \cdot L) u_c dx}{c} = \lim_{c \to 0^+} \frac{\|x u_c\|_2^2 - \int_{\mathbb{R}^N} \tilde{u}_c(\Omega \cdot L) u_c dx}{c} = \omega.
\]

At the end of this Section, we prove Theorem 1.3, i.e. the stability of \( \mathcal{M}^r_c \).

**Proof of Theorem 1.3.** Just suppose that there exists an \( \varepsilon_0 > 0 \), a sequence of initial data \( \{u_n^0\} \subset \Sigma \) and a sequence \( \{t_n\} \subset \mathbb{R}^+ \) such that the unique solution \( u_n \) of problem (1.1) with initial data \( u_n(0, \cdot) = u_n^0(\cdot) \) satisfies
\[
\text{dist}_\Sigma(u_n^0, \mathcal{M}_c) < \frac{1}{n} \quad \text{and} \quad \text{dist}_\Sigma \left( u_n(t_n, \cdot), \mathcal{M}_c \right) \geq \varepsilon_0.
\]
Without loss of generality, we may assume that \( \{u_n^0\} \subset S(c) \). Since \( \text{dist}_\Sigma(u_n^0, \mathcal{M}_c) \to 0 \) as \( n \to \infty \), the conservation laws of the energy and mass imply that \( \{u_n(t_n, \cdot)\} \) is a minimizing sequence for \( m^r_c = \inf_{u \in S(c) \cap B_r} I(u) \) provided \( \{u_n(t_n, \cdot)\} \subset B(r) \). Indeed, if \( \{u_n(t_n, \cdot)\} \subset (\Sigma \setminus B(r)) \), then by the continuity there exists \( \tilde{t}_n \in [0, t_n) \) such that \( \{u_n(\tilde{t}_n, \cdot)\} \subset \partial B(r) \), where \( \partial B(r) = \left\{ u \in \Sigma : \|u\|_\Sigma^2 = r \right\} \). Hence by Proposition 3.1,
\[
I(u_n(\tilde{t}_n, \cdot)) \geq \inf_{u \in S(c) \cap \partial B(r)} I(u) > \inf_{u \in S(c) \cap B(r)} I(u) = \inf_{u \in S(c) \cap B_r} I(u) = m^r_c,
\]
which is a contradiction. Therefore, \( \{u_n(t_n, \cdot)\} \) is a minimizing sequence for \( m^r_c \). Then there exists \( v_0 \in \mathcal{M}^r_c \) such that \( u_n(t_n, \cdot) \to v_0 \) in \( \Sigma \), which contradicts to

\[
\text{dist}_\Sigma(u_n(t_n, \cdot), \mathcal{M}^r_c) \geq \varepsilon_n.
\]

\( \square \)

4. Proof of Theorems 1.4

In this section, we prove Theorem 1.4, i.e. the existence of a mountain pass solution. Let us fix \( u_c \in \mathcal{M}^r_c \) and \( v_c(x) = t^r u_c(lx) \) for \( l \gg 1 \) such that \( v_c \in S(c) \setminus B(r) \) and \( I(v_c) < 0 \) (\( \mathcal{M}^r_c \) is defined in (1.9)). First, we introduce a min-max class

\[
\Gamma(c) := \{g \in C([0, 1], S(c)) : g(0) = u_c \text{ and } g(1) = v_c\} \tag{4.1}
\]

and a min-max value

\[
\gamma(c) := \inf_{g \in \Gamma(c)} \max_{0 \leq t \leq 1} I(g(t)). \tag{4.2}
\]

Notice that \( \Gamma(c) \neq \emptyset \), for \( g(t) = (1 + tl - t)^2 u_c(x + t(l - 1)x) \in \Gamma(c) \). By (4.1) and Proposition 3.1, we have

\[
\gamma(c) > \max\{I(u_c), I(v_c)\} > 0. \tag{4.3}
\]

Next, we introduce an auxiliary functional \( \tilde{I} : S(c) \times \mathbb{R} \to \mathbb{R}, (u, \theta) \to I(\kappa(u, \theta)) \) for \( \kappa(u, \theta) := e^{\frac{k}{2} \theta} u(e^{\theta}x) \). To be precise, we have

\[
\tilde{I}(u, \theta) = I(\kappa(u, \theta)) = \frac{e^{2\theta}}{2} \|\nabla u\|^2_2 + \frac{1}{2e^{2\theta}} \|xu\|^2_2 - \frac{2a}{p} e^{\theta p} \|u\|^p_p - \int_{\mathbb{R}^N} \bar{u}(\Omega \cdot L)u.
\]

Define a set of paths

\[
\tilde{\Gamma}(c) := \{\tilde{g} \in C([0, 1], S(c) \times \mathbb{R}) : \tilde{g}(0) = (u_c, 0) \text{ and } \tilde{g}(1) = (v_c, 0)\} \tag{4.4}
\]

and a minimax value

\[
\tilde{\gamma}(c) := \inf_{\tilde{g} \in \tilde{\Gamma}(c)} \max_{0 \leq t \leq 1} \tilde{I}(\tilde{g}(t)),
\]

we claim that \( \tilde{\gamma}(c) = \gamma(c) \). In fact, it follows immediately from the definition of \( \tilde{\gamma}(c) \) and \( \gamma(c) \) along with the fact that the maps

\[
\varphi : \Gamma(c) \to \tilde{\Gamma}(c), \ g \to \varphi(g) := (g, 0) \text{ and } \psi : \tilde{\Gamma}(c) \to \Gamma(c), \ \tilde{g} \to \psi(\tilde{g}) := \kappa \circ \tilde{g}
\]

satisfy

\[
\tilde{I}(\varphi(g)) = I(g) \text{ and } I(\psi(\tilde{g})) = \tilde{I}(\tilde{g}).
\]

Denote \( |r|_\mathbb{R} = |r| \) for \( r \in \mathbb{R} \), \( E := \Sigma \times \mathbb{R} \) endowed with the norm \( \|.|^2_E = \|.|^2_{\Sigma} + |.|^2_\mathbb{R} \) and \( E^{-1} \) the dual space of \( E \). We give two useful Lemmas.
Lemma 4.1. ([19], Lemma 2.3) Let \( \varepsilon > 0 \). Suppose that \( \tilde{g}_0 \in \tilde{\Gamma}(c) \) satisfies
\[
\max_{0 \leq t \leq 1} \tilde{I}(\tilde{g}_0(t)) \leq \tilde{\gamma}(c) + \varepsilon.
\]
Then there exists a pair of \( (u_0, \theta_0) \in S(c) \times \mathbb{R} \) such that:

1. \( \tilde{I}(u_0, \theta_0) \in [\tilde{\gamma}(c) - \varepsilon, \tilde{\gamma}(c) + \varepsilon] \);
2. \( \min_{0 \leq t \leq 1} \| (u_0, \theta_0) - \tilde{g}_0(t) \|_E \leq \sqrt{\varepsilon} \);
3. \( \tilde{I} \bigg|_{S(c) \times \mathbb{R}}(u_0, \theta_0) \bigg|_{E^{-1}} \leq 2\sqrt{\varepsilon} \), i.e. \( \left| \tilde{I}(u_0, \theta_0), z \right|_{E^{-1} \times E} \leq 2\sqrt{\varepsilon} \| z \|_E \) holds, for all \( z \in \tilde{T}(u_0, \theta_0) := \{ (z_1, z_2) \in E, \langle u_0, z_1 \rangle_{L^2} = 0 \} \).

Lemma 4.2. ([5], Lemma 3) Let \( I \in C^1(\Sigma, \mathbb{R}) \). If \( \{ v_n \} \subset S(c) \) is bounded in \( \Sigma \), then
\[
I'(v_n) \to 0 \text{ in } \Sigma^{-1} \iff I'(v_n) - \frac{1}{c} \langle I'(v_n), v_n \rangle v_n \to 0 \text{ in } \Sigma^{-1} \text{ as } n \to \infty.
\]

Then, we construct a special Palai-Smale sequence for \( \gamma(c) \) defined by (4.2) and show the compactness of the corresponding Palai-Smale sequence.

Proposition 4.3. Let \( a > 0, N = 2, 3, 2 + \frac{4}{N} < p < 2^*, 0 < |\Omega| < 1 \) and \( c < c_0 \) for \( c_0 \) obtained by Theorem 1.1. Then, there exists a sequence \( \{ v_n \} \subset S(c) \) such that
\[
\begin{cases}
I(v_n) \to \gamma(c), \\
I' \bigg|_{S(c)}(v_n) \to 0, \\
Q(v_n) \to 0
\end{cases}
\]
as \( n \to +\infty \), where \( Q(v_n) = \frac{1}{2} \| \nabla v_n \|_2^2 - \frac{1}{2} \| x v_n \|_2^2 - a \delta_p \| v_n \|_p^p \).

Proof. By the definition of \( \gamma(c) \), there exists a \( g_n \in \Gamma(c) \) such that
\[
\gamma(c) \leq \max_{0 \leq t \leq 1} I(g_n(t)) \leq \gamma(c) + \frac{1}{n}, \forall n \in \mathbb{N}^+.
\]
Since \( \tilde{\gamma}(c) = \gamma(c) \), \( \tilde{g}_0 = (g_n, 0) \in \tilde{\Gamma}(c) \), we have \( \max_{0 \leq t \leq 1} \tilde{I}(\tilde{g}_n(t)) \leq \tilde{\gamma}(c) + \frac{1}{n} \). Therefore, Lemma 4.1 indicates the existence of a sequence \( \{ (u_n, \theta_n) \} \subset S(c) \times \mathbb{R} \) such that

(i) \( \tilde{I}(u_n, \theta_n) \in [\gamma(c) - \frac{1}{n}, \gamma(c) + \frac{1}{n}] \);
(ii) \( \min_{0 \leq t \leq 1} \| (u_n, \theta_n) - (g_n(t), 0) \|_E \leq \sqrt{\frac{1}{n}} \);
(iii) \( \tilde{I} \bigg|_{S(c) \times \mathbb{R}}(u_n, \theta_n) \bigg|_{E^{-1}} \leq 2\sqrt{\frac{1}{n}} \), i.e. \( \left| \tilde{I}(u_n, \theta_n), z \right|_{E^{-1} \times E} \leq 2\sqrt{\frac{1}{n}} \| z \|_E \) holds for all \( z \in \tilde{T}(u_n, \theta_n) := \{ (z_1, z_2) \in E, \langle u_n, z_1 \rangle_{L^2} = 0 \} \).

Let \( v_n = \kappa(u_n, \theta_n), \forall n \in \mathbb{N}^+ \), then we prove that \( \{ v_n \} \subset S(c) \) satisfies (4.5). Firstly, from (i) and the fact that \( I(v_n) = I(\kappa(u_n, \theta_n)) = \tilde{I}(u_n, \theta_n) \), we have \( I(v_n) \to \gamma(c) \) as \( n \to +\infty \).
Secondly, direct calculation implies that
\[
2Q(v_n) = \|\nabla v_n\|^2 - \|xv_n\|^2 - 2a\delta_p \|v_n\|^p
= e^{2\eta_n} \|\nabla u_n\|^2 - e^{-2\theta_n} \|xu_n\|^2 - 2a\delta_p e^{\rho_p \theta_n} \|u_n\|^p
\]
(4.6)

Thus (iii) yields $Q(v_n) \to 0$ as $n \to \infty$, for $(0, 1) \in \bar{T}_{(u_n, \theta_n)}$. Finally, we prove that
\[
I'_{S(c)}(v_n) \to 0 \text{ as } n \to \infty.
\]
We claim that for $n \in \mathbb{N}$ sufficiently large, it holds that
\[
|\langle I'(v_n), \eta \rangle| \leq \frac{2\sqrt{2}}{\sqrt{n}} \|\eta\|, \forall \eta \in T_{v_n} = \{\eta \in \Sigma, \langle v_n, \eta \rangle_{L^2} = 0\}.
\]
In fact, for any $\eta \in T_{v_n}$, let $\tilde{\eta} = \kappa(\eta, -\theta_n)$, we have
\[
\langle I'(v_n), \eta \rangle = \langle \tilde{I}(u_n, \theta_n), (\tilde{\eta}, 0) \rangle.
\]
(4.7)
Since $\int_{\mathbb{R}^N} u_n\tilde{\eta} = \int_{\mathbb{R}^N} v_n\eta$, we obtain $(\tilde{\eta}, 0) \in \bar{T}_{(u_n, \theta_n)} \Leftrightarrow \eta \in T_{v_n}$. It follows from (ii) that
\[
|\theta_n| = |\theta - 0| \leq \min_{0 \leq t \leq 1} \|(u_n, \theta_n) - (g_n(t), 0)\|_E \leq \frac{1}{\sqrt{n}}.
\]
Consequently, for $n$ large enough, we have
\[
\|(\tilde{\eta}, 0)\|^2_E = \|\tilde{\eta}\|_\Sigma^2 = \|\eta\|^2 + e^{-2\theta_n} \|\nabla \eta\|^2 + e^{2\theta_n} \|x\eta\|^2_2 \leq 2\|\eta\|^2_\Sigma.
\]
Thus, (iii) implies that
\[
\left|\langle I'(v_n), \eta \rangle\right| = \langle \tilde{I}(u_n, \theta_n), (\tilde{\eta}, 0) \rangle \leq \frac{2}{\sqrt{n}} \|(\tilde{\eta}, 0)\|_E \leq \frac{2\sqrt{2}}{\sqrt{n}} \|\eta\|_\Sigma.
\]
It results to
\[
\left\|I'_{S(c)}(v_n)\right\|_{\Sigma}^{-1} = \sup_{\eta \in T_{v_n}, \|\eta\| \leq 1} \left|\langle I'(v_n), \eta \rangle\right| \leq \frac{2\sqrt{2}}{\sqrt{n}} \to 0 \text{ as } n \to +\infty.
\]

\[
\Box
\]

**Proposition 4.4.** Assume that $a > 0$, $N = 2, 3$, $2 + \frac{4}{N} < p < 2^*$, $0 < |\Omega| < \sqrt{1 - \left(\frac{2}{p\rho_p}\right)^2}$ and $c < c_0$ for $c_0$ obtained by Theorem 1.1. Let $\{v_n\} \subset S(c)$ be a sequence such that
\[
\left\{\begin{array}{l}
I(v_n) \to \gamma(c), \\
I'_{S(c)}(v_n) \to 0, \\
Q(v_n) \to 0
\end{array}\right.
\]
(4.8)
as $n \to +\infty$. Then there exist a $v \in \Sigma$, a sequence $\{\omega_n\} \subset \mathbb{R}$ and a $\tilde{\omega} \in \mathbb{R}$ such that
(i) $v_n \to v$ in $\Sigma$, up to a subsequence, as $n \to +\infty$;
(ii) $\omega_n \to \tilde{\omega}$ in $\mathbb{R}$, up to a subsequence, as $n \to +\infty$;
(iii) \( \Re \left[ I'(v_n) - \omega_n v_n \right] \to 0 \) in \( \Sigma^{-1} \), up to a subsequence, as \( n \to +\infty \);
(iv) \( \Re\left[ I'(v) - \bar{\omega} v \right] = 0 \) in \( \Sigma^{-1} \).

**Proof.** We first show that \( \{v_n\} \) is bounded in \( \Sigma \). Notice that \( p\delta_p > 2 \) as \( p \in (\bar{p}, 2^*) \). By using \( Q(v_n) \to 0 \), we have

\[
I(v_n) = \left( \frac{1}{2} - \frac{1}{p\delta_p} \right) \| \nabla v_n \|^2 + \left( \frac{1}{2} + \frac{1}{p\delta_p} \right) \| x v_n \|^2 + \int_{\mathbb{R}^N} \bar{v}_n(\Omega \cdot L)v_n + o_n(1) \leq \gamma(c) + 1.
\]

Since \( 0 < |\Omega| < \sqrt{1 - \left( \frac{2}{p\delta_p} \right)^2} \), we can choose \( \varepsilon_1 \in \left( \frac{p\delta_p}{p\delta_p + 2}|\Omega|^2, 1 - \frac{2}{p\delta_p} \right) \) and denote

\[
C_1(\Omega) := \frac{1}{2} - \frac{1}{p\delta_p} - \frac{\varepsilon_1}{2} > 0, \quad C_2(\Omega) := \frac{1}{2} + \frac{1}{p\delta_p} - \frac{|\Omega|^2}{2\varepsilon_1} > 0, \quad C_\Omega = \min\{C_1(\Omega), C_2(\Omega)\}.
\]

Applying inequality (2.2) with \( \varepsilon_1 = \varepsilon_1 \), we have

\[
\gamma(c) + 1 \geq I(v_n) \geq C_1(\Omega) \| \nabla v_n \|^2 + C_2(\Omega) \| x v_n \|^2 + o_n(1) \geq C_\Omega \| v_n \|_{L_\Sigma^2}^2 + o_n(1).
\]

Thus, \( \{v_n\} \) is bounded in \( \Sigma \). Then, up to a subsequence, there exists a \( v \in \Sigma \) such that

\[
\begin{cases}
  v_n \rightharpoonup v & \text{in } \Sigma, \\
v_n \to v & \text{in } L^2(\mathbb{R}^N, \mathbb{C}), \\
v_n \to v & \text{in } L^p(\mathbb{R}^N, \mathbb{C}), \\
v_n \to v & \text{a.e. in } \mathbb{R}^N.
\end{cases}
\]

By Lemma 4.2, we know that

\[
I'\bigg|_{\Sigma(c)}(v_n) \to 0 \text{ in } \Sigma^{-1} \iff I'(v_n) - \frac{1}{c} \langle I'(v_n), v_n \rangle v_n \to 0 \text{ in } \Sigma^{-1} \text{ as } n \to +\infty.
\]

Therefore, we have \( \Re \left[ \frac{1}{2} \int_{\mathbb{R}^N} \left( \nabla v_n \cdot \nabla \varphi + |x|^2 v_n \varphi \right) - \int_{\mathbb{R}^N} \varphi(\Omega \cdot L)v_n - a \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi - \omega_n \int_{\mathbb{R}^N} v_n \varphi \right] \to 0,
\]

(4.9)

where

\[
\omega_n = \frac{1}{c} \langle I'(v_n), v_n \rangle = \frac{1}{c} \left( \frac{1}{2} \| \nabla v_n \|^2 + \frac{1}{2} \| x v_n \|^2 - \int_{\mathbb{R}^N} \bar{v}_n(\Omega \cdot L)v_n - a \| v_n \|^p \right).
\]

(4.10)

Thus (iii) is proved. By Lemma 2.1 and inequality (2.2), each term in the right hand of (4.10) is bounded. So there exists \( \bar{\omega} \in \mathbb{R} \) such that, up to a subsequence, \( \omega_n \to \bar{\omega} \) as \( n \to +\infty \). Thus (ii) is proved and (iv) follows from (iii). By (ii) (iii) and (iv) we have

\[
\Re \left[ I'(v_n) - \bar{\omega} v_n, v_n - v \right] = o_n(1) \quad \text{and} \quad \Re \left[ I'(v) - \bar{\omega} v, v_n - v \right] = 0.
\]

(4.11)

We deduce from (4.11) and (3.3) that

\[
o_n(1) = \frac{1}{2} \| (v_n - v) \|^2_{L_\Sigma^2} - \int_{\mathbb{R}^N} (v_n - v)(\Omega \cdot L)(v_n - v) dx \geq C_\Omega \| (v_n - v) \|^2_{L_\Sigma^2}.
\]
It results to $v_n \to v$ in $\hat{\Sigma}$ as $n \to \infty$. As $v_n \to v$ in $L^2(\mathbb{R}^N, \mathbb{C})$, we see that $v_n \to v$ in $\Sigma$ and (i) is proved.

**Proof of Theorem 1.4.** Propositions 4.3-4.4 guarantee the existence of a couple of weak solution $(\hat{u}_c, \hat{\omega}_c) \in \Sigma \times \mathbb{R}$ to problem (1.2) with $\|\hat{u}_c\|_2^2 = c$. By using (4.3), we have

$$I(\hat{u}_c) = \gamma(c) > I(u_c) = m^r_c.$$

□

**References**

[1] A. Aftalion, Vortices in Bose-Einstein Condensates, Progress in Nonlinear Differential Equations and their Applications, 67, Birkhäuser Boston, Inc., Boston, MA, 2006.

[2] P. Antonelli, D. Marahrens, C. Sparber, On the Cauchy problem for nonlinear Schrödinger equations with rotation, Discrete Cont. Dyn. Syst., 32(3), 703-715 (2012).

[3] P. Antonelli, R. Carles, J. D. Silva, Scattering for nonlinear Schrödinger equation under partial harmonic confinement, Commun. Math. Phys., 334, 367-396 (2015).

[4] J. Arbunich, I. Nenciu, C. Sparber, Stability and instability properties of rotating Bose-Einstein condensates, Lett. Math. Phys., 109, 1415-1432 (2019).

[5] H. Berestycki, P. L. Lions, Nonlinear scalar field equations, II existence of infinitely many solutions, Arch. Ration. Mech. Anal., 82, 347-375 (1983).

[6] N. Basharat, H. Hajaiej, Y. Hu, S. J. Zheng, Threshold for Blowup and Stability for Nonlinear Schrödinger Equation with Rotation, Preprint, arXiv: 2002.04722.

[7] J. Bellazzini, N. Boussaid, L. Jeanjean, N. Visciglia, Existence and stability of standing waves for supercritical NLS with a partial confinement, Comm. Math. Phys., 353, 229-251 (2017).

[8] J. Bellazzini, L. Jeanjean, On dipolar quantum gases in the unstable regime, SIAM J. Math. Anal., 48, 2028-2058 (2016).

[9] W. Bao, Q. Du, Y. Z. Zhang, Dynamics of rotating Bose-Einstein condensates and its efficient and accurate numerical computation, SIAM J. Appl. Math., 66, 758-786(2006).

[10] W. Bao, H. Wang, P. Markowich, Ground, symmetric and central vortex states in rotating Bose-Einstein condensates, Commun. Math. Sci., 3(1), 57-88 (2005).

[11] L. D. Carr, C. W. Clark, Vortices in attractive Bose-Einstein condensates in two dimensions, Phys. Rev. Lett., 97, 010403 (2006).

[12] N. R. Cooper, Rapidly rotating atomic gases, Adv. Phys., 57, 539-616 (2008).

[13] A. Collin, E. Lundh, K.-A. Suominen, Center-of-mass rotation and vortices in an attractive Bose gas, Phys. Rev. A, 71, 023613 (2005).

[14] F. Dalfovo, S. Stringari, Bosons in anisotropic traps: ground state and vortices, Phys. Rev. A, 53, 2477-2485 (1996).

[15] A. Fetser, Rotating trapped Bose-Einstein condensates, Rev. Mod. Phys., 81, 647 (2009).

[16] Y. J. Guo, Y. Luo, W. Yang, Refined Mass Concentration of Rotating Bose-Einstein Condensates with Attractive Interactions, arXiv:1901.09619.

[17] M. Hirose, M. Ohta, Uniqueness of positive solutions to scalar field equation with harmonic potential, Funk. Ekvac., 50, 67-100 (2007).

[18] R. Ignat, V. Millot, The critical velocity for vortex existence in a two-dimensional rotating Bose-Einstein condensate, J. Funct. Anal., 233(1), 260-306 (2006).

[19] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlin. Anal., 28, 1633-1659 (1997).

[20] E. H. Lieb, M. Loss, Analysis, Second edition, Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
[21] G. B. Li, X. Luo, Normalized solutions for the Chern-Simons-Schrödinger equation in $\mathbb{R}^2$, Ann. Acad. Sci. Fenn. Math, 42, 405-428 (2017).
[22] E. H. Lieb, R. Seiringer, J. Yngvason, Bosons in a trap: a rigorous derivation of the Gross-Pitaevskii energy functional, Phys. Rev. A, 61, 043602 (2000).
[23] K. W. Madison, F. Chevy, W. Wohlleben, J. Dalibard, Vortex formation in a stirred Bose-Einstein condensate, Phys. Rev. Lett, 84, 806-809 (2000).
[24] K. W. Madison, F. Chevy, V. Bretin, J. Dalibard, Stationary states of a rotating Bose-Einstein condensate: Routes to vortex nucleation, Phys. Rev. Lett, 86, 4443-4446 (2001).
[25] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, E. A. Cornell, Vortices in a Bose-Einstein condensate, Phys. Rev. Lett, 83, 2498-2501 (1999).
[26] F. Mehats, C. Sparber, Dimension reduction for rotating Bose-Einstein condensates with anisotropic confinement, Discrete Contin. Dyn. Syst, 36(9), 5097-5118 (2016).
[27] S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, J. Dalibard, Bose-Einstein condensates in fast rotation, Laser Phy. Lett, 2, 275-284(2005).
[28] F. H. Selem, H. Hajaiej, P. A. Markowich, S. Trabelsi, Variational approach to the orbital stability of standing waves of the Gross-Pitaevskii equation, Milan J. Math, 84(2), 273-295 (2014).
[29] H. Saito, M. Ueda, Split-merge cycle, fragmented collapse, and vortex disintegration in rotating Bose-Einstein condensates with attractive interactions, Phys. Rev. A, 69, 013604 (2004).
[30] R. Seiringer, Gross-Pitaevskii theory of the rotating gas, Commun. Math. Phys, 229, 491-509 (2002).
[31] R. Seiringer, Ground state asymptotics of a dilute, rotating gas, J. Phys. A, 36(37), 9755 (2003).
[32] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys, 87, 567-576 (1983)
[33] J. F. Yang, J. G. Yang, Normalized solutions and mass concentration for supercritical nonlinear schrödinger equations, Preprint, arXiv: 1905.09422v1.
[34] J. Zhang, Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials, Z. Angew. Math. Phys, 51(3), 498-503 (2000).