Measure Concentration of Hidden Markov Processes

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Abstract

We prove what appears to be the first concentration of measure result for hidden Markov processes. Our bound is stated in terms of the contraction coefficients of the underlying Markov process, and strictly generalizes the Markov process concentration results of Marton (1996) and Samson (2000). Somewhat surprisingly, the hidden Markov process is at least as “concentrated” as its underlying Markov process; this property, however, fails for general hidden/observed process pairs.

1 Introduction

Recently several general techniques have been developed for proving concentration results for nonproduct measures [1, 4, 6] (see the references cited in [1] for a brief overview). Let \((\mathcal{S}, \mathcal{F})\) be a Borel-measurable space, and consider the probability space \((\mathcal{S}^n, \mathcal{F}^n, \mu)\) with the associated random process \((X_i)_{1 \leq i \leq n}, X_i \in \mathcal{S}\). Suppose further that \(\mathcal{S}^n\) is equipped with a metric \(d\). For our purposes, a concentration of measure result is an inequality stating that for any 1-Lipschitz (with respect to \(d\)) function \(f : \mathcal{S}^n \to \mathbb{R}\), we have

\[
\mathbb{P}\{|f(X) - \mathbb{E}f(X)| > t\} \leq 2 \exp(-Kt^2) \tag{1}
\]

where \(K\) may depend on \(n\) but not on \(f\).\(^1\) The quantity \(\bar{\eta}_{ij}\), defined below, has proved useful for obtaining concentration results. For \(1 \leq i < j \leq n\), \(y \in \mathcal{S}^{i-1}\) and \(w \in \mathcal{S}\), let

\[
\mathcal{L}(X_j^n | X_1^{i-1} = y, X_i = w)
\]

be the law of \(X_j^n\) conditioned on \(X_1^{i-1} = y\) and \(X_i = w\). Define

\[
\eta_{ij}(y, w, w') = \|\mathcal{L}(X_j^n | X_1^{i-1} = y, X_i = w) - \mathcal{L}(X_j^n | X_1^{i-1} = y, X_i = w')\|_{TV}
\]

and

\[
\bar{\eta}_{ij} = \sup_{y \in \mathcal{S}^{i-1}} \sup_{w, w' \in \mathcal{S}} \eta_{ij}(y, w, w') \tag{2}
\]

where \(\|\|_{TV}\) is the total variation norm (see §2.3 to clarify notation).

\(^1\)See §4 for a much more general notion of concentration.
Let $\Gamma$ and $\Delta$ be upper-triangular $n \times n$ matrices, with $\Gamma_{ii} = \Delta_{ii} = 1$ and
$$
\Gamma_{ij} = \sqrt{\eta_{ij}}, \quad \Delta_{ij} = \bar{\eta}_{ij}
$$
for $1 \leq i < j \leq n$.

For the case where $S = [0, 1]$ and $d$ is the Euclidean metric on $\mathbb{R}^n$, Samson \cite{samson6} showed that if $f : [0, 1]^n \rightarrow \mathbb{R}$ is convex and Lipschitz with $\|f\|_{\text{Lip}} \leq 1$, then
$$
\mathbb{P} \{|f(X) - \mathbb{E}f(X)| > t\} \leq 2 \exp \left( \frac{-t^2}{2 \|\Gamma\|_2^2} \right)
$$
where $\|\Gamma\|_2$ is the $\ell_2$ operator norm of the matrix $\Gamma$; Marton \cite{marton} has a comparable result.

For the case where $S$ is countable and $d$ is the (normalized) Hamming metric on $S^n$,
$$
d(x, y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x_i \neq y_i\}},
$$
Kontorovich and Ramanan \cite{kontorovich} showed that if $f : S^n \rightarrow \mathbb{R}$ is Lipschitz with $\|f\|_{\text{Lip}} \leq 1$, then
$$
\mathbb{P} \{|f(X) - \mathbb{E}f(X)| > t\} \leq 2 \exp \left( \frac{-nt^2}{2 \|\Delta\|_\infty^2} \right)
$$
where $\|\Delta\|_\infty$ is the $\ell_\infty$ operator norm of the matrix $\Delta$, also given by
$$
\|\Delta\|_\infty = \max_{1 \leq i < n} (1 + \bar{\eta}_{i,i+1} + \ldots + \bar{\eta}_{i,n}).
$$
This is a strengthening of the Markov measure concentration result in Marton \cite{marton}.

These two results provide ample motivation for bounding $\bar{\eta}_{ij}$ as a means of obtaining a concentration result for a process. For Markov processes, Samson gives bounds on $\|\Gamma\|_2$, while Kontorovich and Ramanan bound $\|\Delta\|_\infty$.

In this paper, we extend the technique in \cite{kontorovich} to the case of hidden Markov processes. If $(X_i)_{1 \leq i \leq n}$ is a hidden Markov process whose underlying Markov process has contraction coefficients $(\theta_i)_{1 \leq i < n}$, we will show that
$$
\bar{\eta}_{ij} \leq \theta_i \theta_{i+1} \ldots \theta_{j-1}.
$$
To our knowledge, this is the first concentration result for hidden Markov processes. In light of the discussion in \cite{samson6} the form of the bound – identical to the one in \cite{kontorovich} and \cite{samson6} for the simple Markov case – should be at least somewhat surprising. Our result may be summarized by the statement that a hidden Markov process is at least as concentrated as its underlying Markov process.

## 2 Bounding $\bar{\eta}_{ij}$ for hidden Markov processes

### 2.1 Definition of hidden Markov process

Consider two countable sets, $\hat{S}$ (the “hidden state” space) and $S$ (the “observed state” space), equipped with $\sigma$-algebras $\hat{\mathcal{F}} = 2^{\hat{S}}$ and $\mathcal{F} = 2^S$, respectively. Let $(\hat{S}^n, \mathcal{F}^n, \mu)$ be a probability space, where $\mu$ is a Markov measure with transition kernels $p_t(\cdot | \cdot)$. Thus for $\hat{x} \in \hat{S}^n$, we have
$$
\mu(\hat{x}) = p_0(\hat{x}_1) \prod_{k=1}^{n-1} p_{\hat{K}_k}(\hat{x}_{k+1} | \hat{x}_k).
$$
Suppose $(\hat{S}^n \times S^n, \hat{F}^n \times F^n, \nu)$ is a probability space whose measure $\nu$ is defined by
\[
\nu(\hat{x}, x) = \mu(\hat{x}) \prod_{\ell=1}^{n} q_{\ell}(x_{\ell} | \hat{x}_{\ell}),
\]
where $q_{\ell}(\cdot | \hat{x})$ is a probability measure on $(S, F)$ for each $\hat{x} \in \hat{S}$ and $1 \leq \ell \leq n$. On this product space we define the random process $(\hat{X}_i, X_i)_{1 \leq i \leq n}$, which is clearly Markov since
\[
\begin{align*}
\mathbb{P} \left\{ (\hat{X}_{i+1}, X_{i+1}) = (\hat{x}, x) \mid (\hat{X}_i, X_i) = (\hat{y}, y) \right\} &= \sum_{\hat{x} \in \hat{S}^n} \mathbb{P} \left\{ (\hat{X}_{i+1}, X_{i+1}) = (\hat{x}, x) \mid (\hat{X}_i, X_i) = (\hat{y}, y) \right\}.
\end{align*}
\]

The (marginal) projection of $(\hat{X}_i, X_i)$ onto $X_i$ results in a random process on the probability space $(S^n, F^n, \rho)$, where
\[
\rho(x) = \mathbb{P} \{ X_i = x \} = \sum_{\hat{x} \in \hat{S}^n} \nu(\hat{x}, x).
\]

The random process $(X_i)_{1 \leq i \leq n}$ (or measure $\rho$) on $(S^n, F^n)$ is called a hidden Markov process (resp., measure); it is well known that $(X_i)$ need not be Markov to any order\(^2\). We will refer to $(\hat{X}_i)$ as the underlying process; it is Markov by construction.

### 2.2 Statement of result

**Theorem 2.1.** Let $(X_i)_{1 \leq i \leq n}$ be a hidden Markov process, whose underlying process $(\hat{X}_i)_{1 \leq i \leq n}$ is defined by the transition kernels $p_i(\cdot | \cdot)$. Define the $k$th (Doeblin) contraction coefficient $\theta_k$ by
\[
\theta_k = \sup_{\hat{x}, \hat{x}' \in \hat{S}} \| p_k(\cdot | \hat{x}) - p_k(\cdot | \hat{x}') \|_{TV}.
\]

Then for the hidden Markov process $X$, we have
\[
\bar{\eta}_{ij} \leq \theta_i \theta_{i+1} \cdots \theta_{j-1},
\]
for $1 \leq i < j \leq n$.

**Remark 2.2.** Modulo measurability issues, a hidden Markov process may be defined on continuous hidden and observed state spaces; the definition of $\bar{\eta}_{ij}$ is unchanged (we may weaken the sup in (2) to ess sup; see [2]). For convenience, the proof of Theorem 2.1 is given for the countable case, but can straightforwardly be extended to the continuous case.

The bounds in (3) and (4) are for different metric spaces and therefore not readily comparable (the result in (3) has the additional convexity assumption; see [2] for a discussion). In the special case where the underlying Markov process is contracting, i.e., $\theta_i \leq \theta < 1$ for $1 \leq i < n$, Theorem 2.1 yields
\[
\bar{\eta}_{ij} \leq \theta^{j-i}.
\]

In this case, Samson gives the bound
\[
\| \Theta \|_2 \leq \frac{1}{1 - \theta^2}.
\]

\(^2\)One can easily construct a hidden Markov process over $\hat{S} = \{0, 1, 2\}$ and $S = \{a, b\}$ where, with probability 1, consecutive runs of $b$ will have even length. Such a process cannot be Markov.
and the bound
\[ \| \Delta \|_\infty \leq \sum_{k=0}^{\infty} \theta^k = \frac{1}{1 - \theta} \]
holds trivially via (9).

### 2.3 Notational conventions

Since the calculation is notationally intensive, we emphasize readability, sometimes at the slight expense of formalistic precision.

The probability spaces in the proof are those defined in [2.1]. We will consistently distinguish between hidden and observed state sequences, indicating the former with a ˆ. Random variables are capitalized (\( X \)), specified state sequences are written in lowercase (\( x \)), and brackets denote sequence concatenation: \( [x^i_j x^j_{i+1}] = x^i_t \).

Sums will range over the entire space of the summation variable; thus \( \sum_{x^i_j \in S^j_{i+1}} f(x^i_j) \) with an analogous convention for \( \sum_{\hat{x}^i_j} f(\hat{x}^i_j) \).

The **total variation** norm \( \| \cdot \|_{TV} \) is defined here, for any signed measure \( \tau \) on a countable set \( \mathcal{X} \), by
\[
\| \tau \|_{TV} = \frac{1}{2} \sum_{x \in \mathcal{X}} |\tau(x)|.
\]

The probability operator \( \mathbb{P} \{ \cdot \} \) is defined with respect to \( (\mathcal{S}^\ell, \mathcal{F}^\ell, \rho) \) whose measure \( \rho \) is given in (8). Lastly, we use the shorthand
\[
\mu(\hat{u}^\ell_k) = p_0(\hat{u}_k) \prod_{t=k}^{\ell-1} p_t(\hat{u}_{t+1} | \hat{u}_t),
\]
\[

\nu(u^\ell_k | \hat{u}^\ell_k) = \prod_{t=k}^{\ell} q_t(\hat{u}_t | \hat{u}_t),
\]
\[
\rho(\hat{u}^\ell_k) = \mathbb{P} \{ X^\ell_k = u^\ell_k \}.
\]

### 2.4 Proof of main result

The proof of Theorem 2.1 is elementary – it basically amounts to careful bookkeeping of summation indices, rearrangement of sums, and probabilities marginalizing to 1. At the core is a basic contraction result for Markov operators, which we quote as Lemma B.1 of [1], though it has been known for quite some time (see references cited ibid.):

**Lemma 2.3.** For a countable set \( \mathcal{X} \), let \( \mathbf{u} \in \mathbb{R}^\mathcal{X} \) be such that \( \sum_{x \in \mathcal{X}} u_x = 0 \), and \( A \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}} \) be a column-stochastic matrix: \( A_{xy} \geq 0 \) for \( x, y \in \mathcal{X} \) and \( \sum_{x \in \mathcal{X}} A_{xy} = 1 \) for all \( y \in \mathcal{X} \). Then
\[
\| A\mathbf{u} \|_{TV} \leq \theta_A \| \mathbf{u} \|_{TV},
\]
where \( \theta_A \) is the (Doeblin) contraction coefficient of \( A \):
\[
\theta_A = \frac{1}{2} \sup_{y,y' \in \mathcal{X}} \sum_{x \in \mathcal{X}} |A_{xy} - A_{xy'}|.
\]
Proof of Theorem 2.7. For $1 \leq i < j \leq n$, $y_i^{i-1} \in S^{i-1}$ and $w_i, w'_i \in S$, we use (9) to expand
\[\eta_{ij}(y_i^{i-1}, w_i, w'_i) = \frac{1}{2} \sum_{x_j^n} \left| \mathbb{P} \left\{ X_j^n = x_j^n \mid X_1^i = [y_i^{i-1} w_i] \right\} - \mathbb{P} \left\{ X_j^n = x_j^n \mid X_1^i = [y_i^{i-1} w'_i] \right\} \right| \]
\[= \frac{1}{2} \sum_{x_j^n} \left| \sum_{z_{i+1}^n} \left( \mathbb{P} \left\{ X_j^n = [z_{i+1}^{j-1} x_j^n] \mid X_1^i = [y_i^{i-1} w_i] \right\} - \mathbb{P} \left\{ X_j^n = [z_{i+1}^{j-1} x_j^n] \mid X_1^i = [y_i^{i-1} w'_i] \right\} \right) \right| \]
\[= \frac{1}{2} \sum_{x_j^n} \left| \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \sum_{\tilde{x}_j^n} \mu(\hat{y}_i) \left( \frac{\nu([y_i^{i-1} w_i, z_{i+1}^{j-1} x_j^n])}{\rho([y_i^{i-1} w_i])} - \frac{\nu([y_i^{i-1} w'_i, z_{i+1}^{j-1} x_j^n])}{\rho([y_i^{i-1} w'_i])} \right) \right| \]
\[= \frac{1}{2} \sum_{x_j^n} \left| \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \sum_{\tilde{x}_j^n} \mu([y_i^{i-1} w_i]) \nu(x_j^n \mid \tilde{x}_j^n) \rho([y_i^{i-1} w_i]) \right|, \]
where
\[\delta(\hat{y}_i) = \frac{q_i(w_i \mid \hat{y}_i)}{\rho([y_i^{i-1} w_i])} - \frac{q_i(w'_i \mid \hat{y}_i)}{\rho([y_i^{i-1} w'_i])}.\]
Since $|\sum_{i} a_i b_j| \leq \sum_i a_i |\sum_j b_j|$ for $a_i \geq 0$ and $b_i \in \mathbb{R}$, we may bound
\[\eta_{ij}(y_i^{i-1}, w_i, w'_i) \leq \frac{1}{2} \sum_{x_j^n} \left| \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \mu(\hat{y}_i) \nu(x_j^n \mid \tilde{x}_j^n) \rho([y_i^{i-1} w_i]) \right| \]
\[= \frac{1}{2} \sum_{x_j^n} \left| \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \nu(y_i^{i-1} \mid y_i^{i-1}) \rho([y_i^{i-1} w_i]) \right|. \]
where
\[\zeta(\hat{x}_j) = \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \mu([y_i^{i-1} \hat{y}_i]) \nu(y_i^{i-1} \mid y_i^{i-1}) \delta(\hat{y}_i).
= \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \mu([y_i^{i-1} \hat{y}_i]) \nu(y_i^{i-1} \mid y_i^{i-1}) \delta(\hat{y}_i).\]
Define the vector $h \in \mathbb{R}^{\hat{S}}$ by
\[h_{\hat{y}_i} = \delta(\hat{v}) \sum_{y_i^{i-1}} \mu([y_i^{i-1} \hat{y}_i]) \nu(y_i^{i-1} \mid y_i^{i-1}). \]
Then
\[\zeta(\hat{x}_j) = \sum_{z_{i+1}^{j-1}} \sum_{\hat{y}_i} \mu([y_i^{i-1} \hat{y}_i]) h_{\hat{y}_i}.
Define the matrix $A^{(k)} \in \mathbb{R}^{\hat{S} \times \hat{S}}$ by $A^{(k)}_{\hat{y}_i \hat{y}_j} = p_k(\hat{u} \mid \hat{v})$, for $1 \leq k < n$. With this notation, we have $\zeta(\hat{x}_j) = z_{\hat{x}_j}$, where $z \in \mathbb{R}^{\hat{S}}$ is given by
\[z = A^{(j-1)} A^{(j-2)} \ldots A^{(i+1)} A^{(i)} h. \]
\[\text{Note that all the sums are absolutely convergent, so exchanging the order of summation is justified.} \]
In order to apply Lemma 2.3 to (13), we must verify that
\[ \sum_{\hat{v} \in \hat{S}} h_{\hat{v}} = 0, \quad ||h||_{TV} \leq 1. \] (14)

From (12) we have
\[ h_{\hat{v}} = \left( \frac{q_i(w_i | \hat{v})}{\rho([y_{i-1}^i w_i])} - \frac{q_i(w_i' | \hat{v})}{\rho([y_{i-1}^i w_i'])} \right) \sum_{\hat{y}_{i-1}} \mu([\hat{y}_{i-1}^i \hat{v}]) \nu([\hat{y}_{i-1}^i | \hat{v}]). \]

Summing over \( \hat{v} \), we get
\[ \sum_{\hat{v} \in \hat{S}} \left( \frac{q_i(w_i | \hat{v})}{\rho([y_{i-1}^i w_i])} \right) \sum_{\hat{y}_{i-1}} \mu([\hat{y}_{i-1}^i \hat{v}]) \nu([\hat{y}_{i-1}^i | \hat{v}]) = \frac{1}{\mathbb{P} \{ X_1^i = [y_{i-1}^i w_i] \} } \sum_{\hat{y}_{i-1}} \mu(\hat{y}_{i-1}^i) \nu([y_{i-1}^i w_i | \hat{y}_{i-1}^i]) = 1; \]
an analogous identity holds for the \( \frac{q_i(w_i' | \hat{v})}{\rho([y_{i-1}^i w_i'])} \) term, which proves (14).

Therefore, combining (11), (13), and Lemma 2.3 we have
\[ \eta_{ij}(y_{i-1}^i, w_i, w_i') \leq \frac{1}{2} \sum_{\hat{x}_j^n} \mu(\hat{x}_j^n) \nu(\hat{x}_j^n) \]
\[ = \frac{1}{2} \sum_{\hat{x}_j} \|z_{\hat{x}_j}^{i+1} \| \sum_{\hat{x}_j^{i+1}} \mu(\hat{x}_j^{i+1}) \nu([y_{i-1}^i w_i | \hat{y}_{i-1}^i]) \]
\[ \leq \|z\|_{TV} \leq \theta_i \theta_{i+1} \cdots \theta_{j-1}. \]

3 Discussion

The relative ease with which we were able to bound \( \bar{\eta}_{ij} \) is encouraging; it suggests that the technique used in [2] and here – namely, matrix algebra combined with the Markov contraction lemma – could be applicable to other processes.

We noted in §1 that the bound for the hidden Markov process is identical to the one in [1] and [6] for the simple Markov case. One might thus be tempted to pronounce Theorem 2.1 as “obvious” in retrospect, based on the intuition that the observed sequence \( X_i \) is an independent process conditioned the hidden sequence \( \hat{X}_i \). Thus, the reasoning might go, all the dependence structure is contained in \( \hat{X}_i \), and it is not surprising that the underlying process alone suffices to bound \( \bar{\eta}_{ij} \) – which, after all, is a measure of the dependence in the process.

Such an intuition, however, would be wrong, as it fails to carry over to the case where the underlying process is not Markov. As a numerical example, take \( n = 4 \), \( \hat{S} = S = \{0, 1\} \) and define the probability measure \( \mu \) on \( \hat{S}^4 \) as given in Figure 1. Define the conditional probability
\[ q(x | \hat{x}) = \frac{1}{4} \mathbb{1}_{\{x = \hat{x}\}} + \frac{3}{4} \mathbb{1}_{\{x \neq \hat{x}\}}. \]

Together, \( \mu \) and \( q \) define the measure \( \rho \) on \( S^4 \):
\[ \rho(x) = \sum_{\hat{x} \in \hat{S}^4} \mu(\hat{x}) \prod_{\ell=1}^4 q(x_{\ell} | \hat{x}_{\ell}). \]
Associate to \((\hat{S}^4, \mu)\) the “hidden” process \((\hat{X}_i)_{i=1}^4\) and to \((S^4, \rho)\) the “observed” process \((X_i)_{i=1}^4\). A straightforward numerical computation (whose explicit steps are given in the proof of Theorem 2.1) shows that the values of \(\mu\) can be chosen so that \(\bar{\eta}_{24}(X) > 0.06\) while \(\bar{\eta}_{24}(\hat{X})\) is arbitrarily small.

Thus one cannot, in general, bound \(\tilde{\eta}_{ij}(X)\) by \(c\tilde{\eta}_{ij}(\hat{X})\) for some universal constant \(c\); we were rather fortunate to be able to do so in the hidden Markov case.

| \(\hat{x}_n^1\) | \(\mu(\hat{x}_n^1)\) |
|-----------------|-----------------|
| 0000            | 0.0000000       |
| 0001            | 0.0000000       |
| 0010            | 0.288413        |
| 0011            | 0.0000000       |
| 0100            | 0.0000000       |
| 0101            | 0.0000000       |
| 0110            | 0.176290        |
| 0111            | 0.0000000       |
| 1000            | 0.0000000       |
| 1001            | 0.010514        |
| 1010            | 0.0000000       |
| 1011            | 0.139447        |
| 1100            | 0.0000000       |
| 1101            | 0.024783        |
| 1110            | 0.0000000       |
| 1111            | 0.360553        |

Figure 1: The numerical values of \(\mu\) on \(\hat{S}^4\)

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