Inverse Problem for the Wave Equation with a White Noise Source

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Abstract: We consider a smooth Riemannian metric tensor $g$ on $\mathbb{R}^n$ and study the stochastic wave equation for the Laplace-Beltrami operator $\partial_t^2 u - \Delta_g u = F$. Here, $F = F(t, x, \omega)$ is a random source that has white noise distribution supported on the boundary of some smooth compact domain $M \subset \mathbb{R}^n$. We study the following formally posed inverse problem with only one measurement. Suppose that $g$ is known only outside of a compact subset of $M^{int}$ and that a solution $u(t, x, \omega_0)$ is produced by a single realization of the source $F(t, x, \omega_0)$. We ask what information regarding $g$ can be recovered by measuring $u(t, x, \omega_0)$ on $\mathbb{R}_+ \times \partial M$? We prove that such measurement together with the realization of the source determine the scattering relation of the Riemannian manifold $(M, g)$ with probability one. That is, for all geodesics passing through $M$, the travel times together with the entering and exit points and directions are determined. In particular, if $(M, g)$ is a simple Riemannian manifold and $g$ is conformally Euclidian in $M$, the measurement determines the metric $g$ in $M$.

1. Introduction

We consider the wave equation

$$
\partial_t^2 u(t, x) - \Delta_g u(t, x) = F(t, x) \quad \text{on } (0, \infty) \times \mathbb{R}^n,
$$

$$
u|_{t=0} = \partial_t u|_{t=0} = 0, \quad (1)
$$

where $n \geq 2$ and $\Delta_g$ is the Laplace–Beltrami operator corresponding to a smooth time-independent Riemannian metric $g(x) = [g_{jk}]_{j,k=1}^n$, that is,

$$
\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial x^j} \left(|g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),
$$
where \(|g| = \det(g_{jk})\) and \(g_{jk} = g(x)^{-1}\). Let \(M \subset \mathbb{R}^n\) be a compact domain with smooth boundary. We suppose that \(g\) is known only outside of a compact subset \(K \subset M^{\text{int}}\) and that the source \(F\) is a realization of a random variable with the Gaussian white noise distribution on \((0, \infty) \times \partial M\). Moreover, we assume that the Riemannian manifold \((M, g)\) is non-trapping and that \(\partial M\) is strictly convex with respect to the metric \(g\). We show that the scattering relation of \((M, g)\) is determined by \(F\) and the trace of \(u\) on \((0, \infty) \times \partial M\) almost surely, see Theorem 1 below for the precise formulation.

In particular, if the Riemannian manifold \((M, g)\) is simple, then the pair \((F, u)\) on \((0, \infty) \times \partial M\) determines \((M, g)\) almost surely in each of the following cases:

(i) The dimension \(n = 2\) or
(ii) \(n \geq 3\) and the metric is conformally Euclidean, that is, \(g_{jk} = a(x)\delta_{jk}\) for a strictly positive function \(a\) or
(iii) \(n \geq 3\) and the metric is close to the Euclidean metric.

Indeed, by Theorem 1 below, the case (i) follows from [52], (ii) follows from [49], and (iii) from [17]. Moreover, there is a conjecture by Gunther Uhlmann [65] that the scattering relation determines any non-trapping compact manifold with boundary. We refer to [59] for work toward resolving the conjecture.

If the source \(F\) in (1) can be controlled, that is, if we can measure the trace of \(u\) on \((0, \infty) \times \partial M\) for all \(F \in C^\infty((0, \infty) \times \partial M)\), then the problem to determine \((M, g)\) is equivalent with Gel’fand’s inverse problem, whence it has unique solution [7,8]. Contrary to the problem with a single measurement as considered in the present paper, Gel’fand’s problem is overdetermined. Indeed, the dimension of the data in Gel’fand’s problem is \(2n - 1\) which is strictly greater than the dimension \(n \geq 2\) of the unknown \(g|_M\). Notice that \(2n - 1\) is the number of free variables of the kernel of the map \(F \mapsto u|_{(0, \infty) \times \partial M}\), since \(F\) and the trace of \(u\) are defined on the \(n\) dimensional manifold \((0, \infty) \times \partial M\) and the translation invariance in time accounts for the reduction of the dimension by one. The dimension \(n\) of the single-measurement data is equal to the dimension of the trace of \(u\).

In fact, most of the thoroughly studied inverse boundary value problems are overdetermined. Calderon’s inverse problem is overdetermined in dimensions \(n \geq 3\), see [62] for the isotropic and [23,44–46] for the anisotropic case. Likewise, the inverse boundary value problems for the wave, heat, and the dynamical Schrödinger equations with Dirichlet-to-Neumann map as data are all equivalent with the Gel’fand’s inverse problem [40], and they are overdetermined in dimensions \(n \geq 2\). However, the two dimensional Calderon’s inverse problem (see [1,15,50] for the isotropic and [2,31,61] for the anisotropic case) is an example of a formally determined inverse problem, that is, the dimension of the data equals to that of the unknown.

In [32], we solved a formally determined inverse problem for the wave equation with a single measurement. Although satisfactory in terms of the dimensions, the result [32] relies on the use of a source \(F = F_\delta\) given as a weighted sum of point sources,

\[
F_\delta(t, x) = \sum_{j=1}^\infty 2^{-2j} \delta_{x_j}(x) \delta(t).
\]

As the weights vanish superexponentially, the source \(F_\delta\) may be hard to realize in practice with sufficient precision. On the other hand, random noise sources, as considered in the present paper, appear in many applications. In seismology, cross-correlations of signal amplitudes generated by ambient seismic noise source are used to study travel times...
inside the Earth [28, 67]. Moreover, in one-dimensional radar imaging models, white noise signals are considered to be optimal sources when imaging a stationary scatterer [64]. Such models correspond mathematically to the one-dimensional deconvolution problem, and the present problem can be seen as a multi-dimensional analogue that is translation invariant in only one direction.

Although we are ultimately concerned with (1) for just a single realization of the white noise as the source, we consider (1) to be a stochastic partial differential equation. In particular, we show that the pair \((F, u)\) on \((0, \infty) \times \partial M\) is a Gaussian random variable and study its ergodicity properties. The literature on the stochastic hyperbolic equation and the direct problem is extensive. To our knowledge, the earliest existence and uniqueness results were given in [18] for a one-dimensional setting. The research has then extended to higher dimensions and more generalized settings (e.g. [51, 53]), geometrical wave equations [13] and to non-gaussian sources [41], to name a few directions. The pathwise properties have been studied e.g. in [48, 57]. In our problem formulation we have a white noise source which is supported on the boundary of a manifold. Closely related results with boundary supported white noise have been introduced in the work by Dalang and Lévéque [21, 22]. For hyperbolic equations with random boundary conditions, see [14].

Inverse problems related to stochastic wave equations have been mostly studied in the framework of random media. For this imaging setting we refer to [6, 11, 27] and the extensive research by their authors. Let us also mention the interesting approaches to stochastic inverse problems taken in [19, 43, 58].

In addition to our approach, we are aware of two other methods to solve formally determined hyperbolic inverse problems. First, the adaptation of the Gelfand-Levitan method to multidimensional problems, see [54, 55], assumes that the problem is close to being symmetric in all but one direction. Second, the Carleman estimates based approach, see [9, 16, 38, 39, 60], assumes that the initial data is non-zero and satisfies certain conditions.

2. Statement of the Results

Let us begin by introducing the scattering relation. Below, the tangent space of \(M\) is denoted by \(TM\) and \(\dot{\gamma}\) denotes the tangent vector of a smooth curve \(\gamma: [a, b] \to M\). We set \(SM = \{(x, \xi) \in TM; \|\xi\|_g = 1\}\) to be the unit sphere bundle on \(M\) and write

\[
\partial_\pm SM = \{(x, \xi) \in SM; x \in \partial M, \pm (v, \xi) > 0\}
\]

where \(v\) is the interior normal vector of \(\partial M\). Further, we denote by \(\gamma(t; x, \xi)\) the geodesic with initial data \((x, \xi) \in TM\) and we set

\[
\tau(x, \xi) = \inf\{t \in (0, \infty); \gamma(t; x, \xi) \notin M\}.
\]

We say that \(M\) is trapping if the set in the definition of \(\tau\) is empty for some \((x, \xi) \in TM\). We make the standing assumption on the metric \(g\):

(A1) All geodesics \(\gamma\) of \((\mathbb{R}^n, g)\) move eventually off to infinity, that is, \(|\gamma(t)| \to \infty\) as \(t \to \infty\).

The assumption (A1) implies, in particular, that \(M\) is non-trapping. Then the scattering relation of \((M, g)\) is defined by

\[
\Sigma_{M, g}: \partial_+ SM \to (0, \infty) \times \partial_- SM,
\]

\[
\Sigma_{M, g}(x, \xi) = (\tau(x, \xi), \gamma(\tau(x, \xi); x, \xi), \dot{\gamma}(\tau(x, \xi); x, \xi))
\]

We write \(\Sigma = \Sigma_{M, g}\) when considering a fixed Riemannian manifold.
Let us now consider the Eq. (1) more carefully. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \(W\) be a random variable with the Gaussian white noise distribution supported on \(\mathbb{R} \times \partial M\). For a construction of the distribution, see Appendix A. To avoid technicalities arising from compatibility conditions between the source term and the vanishing initial conditions we consider the stochastic wave equation

\[
\partial_t^2 u - \Delta_g u = \chi_+ W \quad \text{on } (0, \infty) \times \mathbb{R}^n, \\
u|_{t=0} = \partial_t u|_{t=0} = 0,
\]

where \(\chi_+ \in C^\infty(\mathbb{R})\) is a fixed cut-off satisfying \(\chi_+ = 0\) and \(\chi_+ = 1\) in neighborhoods of \((-\infty, 0]\) and \([1, \infty)\), respectively.

Notice that the wave front set \(WF(u)\) of the solution of (2) intersects the conormal bundle of \((0, \infty) \times \partial M\) almost surely since the support of a realization of \(W\) coincides with \(\mathbb{R} \times \partial M\) almost surely. Thus the trace \(u|_{(0, \infty) \times \partial M}\) cannot be defined in the sense of distributions [24]. Instead, in Sect. 3.2 we define the trace in a scattering sense. Let us suppose that \(K \subset M^\text{int}\) is compact and that \(g|_{\mathbb{R}^n \setminus K}\) is known. We choose a smooth Riemannian metric tensor \(g_0\) such that \(g_0 = g\) in \(\mathbb{R}^n \setminus K\) and consider the wave equation

\[
\partial_t^2 u_{in} - \Delta_{g_0} u_{in} = \chi_+ W(\omega_0) \quad \text{on } (0, \infty) \times \mathbb{R}^n, \\
u_{in}|_{t=0} = \partial_t u_{in}|_{t=0} = 0
\]

for some fixed \(\omega_0 \in \Omega\), that is, \(W(\omega_0)\) is a realization of the random process \(W\). The measurement operator \(L_{M,g}\) is defined by

\[
L_{M,g}(W(\omega_0)) = u_{sc}|_{(0, \infty) \times \partial M}
\]

for the scattered wave \(u_{sc} = u - u_{in}\). We abbreviate \(L = L_{M,g}\) when considering a fixed Riemannian manifold. The Sobolev regularity properties of the random variable \(L(W)\) are reviewed in Sect. 3.

To avoid technicalities, we consider the following standing assumptions:

(A2) \(\partial M\) is strictly convex with respect to the metric \(g\) and

(A3) \(g\) coincides with the Euclidean metric outside a compact set.

We believe that (A2) is not an essential assumption. In fact, it is not needed in our previous work [32]. We are now ready to formulate our main result.

**Theorem 1.** Let \(M \subset \mathbb{R}^n\) be a compact domain with smooth boundary. Suppose that two smooth Riemannian metrics \(g\) and \(\widetilde{g}\) satisfy the assumptions (A1)–(A3). Furthermore, let \(K \subset M^\text{int}\) be compact and assume that \(g|_{\mathbb{R}^n \setminus K} = \widetilde{g}|_{\mathbb{R}^n \setminus K}\). Then for \(\mathbb{P}\)-almost every \(\omega_0 \in \Omega\), the identity \(L_{M,g}(W(\omega_0)) = L_{M,\widetilde{g}}(W(\omega_0))\) implies that \(\Sigma_{M,g} = \Sigma_{M,\widetilde{g}}\).

**2.1. Outline of the proof.** Let us summarize how Theorem 1 is obtained. First, for an open set \(B \subset \mathbb{R}^k\) denote the dual pairing of a generalized function \(f \in \mathcal{D}'(B)\) and a test function \(g \in C^\infty_0(B)\) by \((f, g)_{\mathcal{D}' \times C^\infty_0}((0, T) \times \partial M)\). Using integration by parts we show in Sect. 3.3 that for almost every \(\omega_0\), the pairing \((W(\omega_0), \chi_+ w)_{\mathcal{D}' \times C^\infty_0}((0, T) \times \partial M)\) is determined by the data pair \((W(\omega_0), L W(\omega_0))\) for fixed \(T > 0\) and for such a smooth solution \(w\) of \((\partial_t^2 - \Delta_g) w = 0\) that the state \((w(T), \partial_t w(T))\) is supported outside \(M\).

We choose \(w\) to be a Gaussian beam solution that is sent backwards in time from the exterior of \(M\). Section 4.1 is devoted to the introduction of Gaussian beams. A backward Gaussian beam is concentrated on a geodesic \(\gamma\) and is essentially determined by the end
point \( \gamma(T) \) and direction \( \dot{\gamma}(T) \) and a scaling parameter \( \epsilon > 0 \). We assume that a backward Gaussian beam \( w_\epsilon \) enters in \( M \) at time \( T - r \) and write \((x, \xi) = (\gamma(T - r), -\dot{\gamma}(T - r))\). In Sect. 4.2 we construct an oscillating test function \( \psi_\epsilon \) that imitates a Gaussian beam passing through a point \((y, \eta) \in \partial_s SM \) at time \( s > 1 \). The crux of the method lies in studying the limit

\[
\beta = \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} (\psi_\epsilon, w_\epsilon)_{L^2((0,T) \times \partial M)}.
\]

(3)

Namely, in Theorem 4 we show that

\[
\beta \neq 0, \quad \text{if and only if } (T - r - s, y, \eta) = (x, \xi).
\]

(4)

This can be interpreted as follows: the limit (3) is non-zero if and only if the test function and the Gaussian beam exit \( M \) through \((y, \eta)\) simultaneously.

In Sect. 5 we consider the correlations of two random variables \( X_\epsilon = (W, \epsilon^{-n/4} \psi_\epsilon)_{L^2((0,T) \times \partial M)} \) and \( Y_\epsilon = (W, \epsilon^{-n/4} \chi_{+} w_\epsilon)_{L^2((0,T) \times \partial M)} \), and in Sect. 6 we combine energy decay and ergodicity arguments to show that

\[
\lim_{N \to \infty} \frac{1}{N^3} \sum_{j=1}^{N^3} X_j^\epsilon Y_j^\epsilon = \mathbb{E} X_\epsilon Y_\epsilon = \epsilon^{-\frac{n}{2}} (\psi_\epsilon, w_\epsilon)_{L^2((0,T) \times \partial M)}
\]

(5)

for time-translated variables \( X_j^\epsilon \) and \( Y_j^\epsilon \). Since the variables \( X_j^\epsilon \) and \( Y_j^\epsilon \) are determined by the pair \((W, LW)\) almost surely, we find out if \( w_\epsilon \) and \( \phi_\epsilon \) coincide in the sense of Eq. (4). By repeating the argument for a dense numerable set of initial data, we obtain the scattering relation by continuity results.

### 3. The Stochastic Direct Problem

#### 3.1. White noise and generalized solutions.

We recall that a random variable with the Gaussian white noise distribution supported on \( \mathbb{R} \times \partial M \) can be defined in the local Sobolev spaces,

\[
W : \Omega \to H_{locl}^{-(n+1)/2-\epsilon} (\mathbb{R}^{1+n}),
\]

where \( \epsilon > 0 \), see Appendix A for more details. The characterizing property of the white noise on \( \mathbb{R} \times \partial M \) is the following equation that holds for any \( \phi, \psi \in C_0^\infty (\mathbb{R}^{1+n}) \)

\[
\mathbb{E} \left( (W, \phi)_{D'(\mathbb{R}^{1+n})} (W, \psi)_{D'(\mathbb{R}^{1+n})} \right) = (\phi, \psi)_{L^2(\mathbb{R} \times \partial M)}.
\]

(6)

Here and throughout the paper we are using real inner products.

Let us recall that the equation

\[
\partial_t^2 u - \Delta_x u = \chi_{+} f \quad \text{on } (0, \infty) \times \mathbb{R}^n,
\]

\[
u|_{t=0} = 0, \quad u|_{t=0} = 0,
\]

(7)

has a unique solution \( u \in D'((0, \infty) \times \mathbb{R}^n) \) for all \( f \in D'((0, \infty) \times \mathbb{R}^n) \), see e.g. [24, Lem. 5.1.5] for uniqueness. The existence follows by transposing the smooth case, see e.g. [26, Th. 7.2.7]. Moreover, the parametrix construction [36, Th. 26.1.14] implies that the solution map \( K_\epsilon : f \mapsto u \) is continuous

\[
K_\epsilon : H_{locl}^{s}((0, \infty) \times \mathbb{R}^n) \to H_{locl}^{s+1}((0, \infty) \times \mathbb{R}^n), \quad s \in \mathbb{R}.
\]

(8)
Due to [10, Prop. 3.7.2] we have that
\[ U = K_g W : \Omega \to H^{-\frac{n-1}{2}-\epsilon}_{loc}((0, \infty) \times \mathbb{R}^n) \] (9)
is a well-defined Gaussian random variable. Moreover, \( U \) satisfies
\[ \partial_t^2 U - \Delta_g U = \chi + W \text{ on } (0, \infty) \times \mathbb{R}^n, \]
\[ U|_{t=0} = \partial_t U|_{t=0} = 0. \] (10)
almost surely in the sense of distributions.

We point out that the solution \( U \) can be shown to have stronger pathwise properties, see e.g. [21]. However, such results are not crucial in this treatise, whereas the random variable formalism provides us some flexibility for the analysis of the inverse problem.

3.2. Trace of the solution in the scattering sense. In below, we will use regularity properties of the traces
\[ \text{Tr}_{\partial M} K_g f = u|_{(0, \infty) \times \partial M}, \quad \text{and} \quad \text{Tr}_\Omega^T K_g f = u|_{[T] \times \Omega}, \] (11)
where \( T > 0 \) and \( \Omega \subset \mathbb{R}^n \) is open. The above traces are not well defined in the sense of distributions if the wavefront set \( \text{WF}(f) \) intersects the conormal bundles of the sets
\[ (0, \infty) \times \partial M, \quad [T] \times \Omega. \] (12)
However, the traces are well defined by [36, Th. 26.1.14] if \( \text{supp}(f) \) does not intersect the sets (12).

Let \( B \subset \mathbb{R}^n \) be open and let us define the closed subspace
\[ H^s_{loc}((0, \infty) \times B) \subset H^s_{loc}((0, \infty) \times \mathbb{R}^n) \]
consisting of distributions supported in \([0, \infty] \times \overline{B}\). If \( \overline{B} \cap \overline{\Omega} = \emptyset \), then we have the regularity
\[ \text{Tr}_\Omega^T K_g : H^s_{loc}((0, \infty) \times B) \to H^{s+1}_{loc}(\Omega), \quad s \in \mathbb{R}. \] (13)
By [35] the mapping \( \text{Tr}_\Omega^T K_g \) is a Fourier integral operator of order \(-5/4\) with the canonical relation \( C \) consisting of the points
\[ (t, x, |\xi|, \xi, \gamma(T - t; x, \hat{\xi}), \hat{\gamma}(T - t; x, \hat{\xi})), \quad (x, \xi) \in T^* B \setminus \emptyset, \quad t > 0, \]
such that \( \gamma(T - t; x, \hat{\xi}) \in \Omega \). Here \( \hat{\xi} = \xi/|\xi| \) and we have identified the cotangent and the tangent space using the metric \( g \). In particular, the canonical relation is parametrized by \((t, x, \xi)\) and the projection \( C \to T^*(0, \infty) \times B \) has the rank \( 2(n+1) - 1 \). We may apply [35, Th. 4.3.2] to get the continuity (13).

Analogously, if \( \overline{B} \cap \partial M = \emptyset \) and \( \partial M \) is strictly convex with respect to the metric \( g \), then we have the regularity
\[ \text{Tr}_{\partial M} K_g : H^s_{loc}((0, \infty) \times B) \to H^{s+1}_{loc}((0, \infty) \times \partial M), \quad s \in \mathbb{R}. \] (14)
Indeed, \( \text{Tr}_{\partial M} K_g \) is a Fourier integral operator of order \(-5/4\) with the canonical relation consisting of the points
\[ (t, x, |\xi|, \xi, T, \gamma(T - t; x, \hat{\xi}), |\xi|, \hat{\gamma}(T - t; x, \hat{\xi})), \]
where \( \hat{\xi} = \xi/|\xi| \) and we have identified the cotangent and the tangent space using the metric \( g \).
such that \( \gamma(T - t; x, \hat{\xi}) \in \partial M, (x, \xi) \in T^*B \setminus 0 \) and \( t, T > 0 \). Here \( v \mapsto v^\top \) is the projection \( T^*\mathbb{R}^n \to \mathcal{T} \). The strict convexity of \( \partial M \) implies that \( T \) is locally a function of \( (t, x, \hat{\xi}) \). As above, the continuity (14) follows from \([35, \text{Th. 4.3.2}]\). The convexity assumption is not essential here since we could use the result \([29]\) as in \([63]\). However, we will use the strict convexity assumption also for other purposes in below.

Let \( K \subset M^\text{int} \) be compact and let \( g_0 \) be a smooth Riemannian metric tensor such that \( g_0 = g \) in \( \mathbb{R}^n \setminus K \). We consider the incoming wave defined by \( u_{in} = \mathcal{K}_{g_0} f \). Then the scattered wave \( u_{sc} = u - u_{in} \) satisfies

\[
u_{sc} = (\mathcal{K}_g - \mathcal{K}_{g_0}) f = \mathcal{K}_g \tilde{f}
\]

where \( \tilde{f} = (\Delta_g - \Delta_{g_0}) u_{in} \) is supported in \((0, \infty) \times K\). Thus the measurement operator

\[
L f = \text{Tr}_{\partial M} \mathcal{K}_g (\Delta_g - \Delta_{g_0}) \mathcal{K}_{g_0} f = u_{sc}|_{(0,\infty) \times \partial M}
\]

is continuous

\[
L : H^s_{\text{locc}} ((0, \infty) \times B) \to H^s_{\text{loc}} ((0, \infty) \times \partial M),
\]

where \( B \subset \mathbb{R}^n \) is a neighborhood of \( \partial M \) satisfying \( K \cap \overline{B} = \emptyset \).

From these considerations it follows that we can factorize

\[
U = U_{in} + U_{sc}, \quad \text{almost surely},
\]

where \( U_{in} = \mathcal{K}_{g_0} W \) and \( U_{sc} = \mathcal{K}_g (\Delta_g - \Delta_{g_0}) \mathcal{K}_{g_0} W \). Moreover, the measurement

\[
L W : \Omega \to H^{-\left(\frac{n+1}{2} - \epsilon\right)}_{\text{loc}} ((0, \infty) \times \partial M)
\]

is a well-defined Gaussian random variable.

### 3.3. Integration by parts

Following \([32]\) we have for a smooth source \( f \in C^\infty_0 ((0, \infty) \times \mathbb{R}^n) \) and \( T > 0 \) that

\[
(\chi_+ f, w)_{L^2((0, T) \times M)} = (\partial_t u(T), w(T))_{L^2(\mathbb{R}^n)} - (u(T), \partial_t w(T))_{L^2(\mathbb{R}^n)},
\]

where \( u \) is the solution of (7) and \( w \in C^\infty([0, T] \times \mathbb{R}^n) \) satisfies the wave equation \((\partial^2_t - \Delta_g) w = 0\). We let

\[
w(T), \partial_t w(T) \in C^\infty_0 (\mathbb{R}^n \setminus M)
\]

and denote \( \Omega = \text{supp}(w(T)) \cup \text{supp}(\partial_t w(T)) \). Let \( B \subset \mathbb{R}^n \) be a neighborhood of \( \partial M \) such that \( \overline{B} \cap \Omega = \emptyset \). The density of the embedding

\[
C^\infty_0 ((0, \infty) \times B) \subset H^s_{\text{locc}} ((0, \infty) \times B)
\]

and the continuity (13) imply that the identity

\[
(\chi_+ W, w)_{D' \times C^\infty_0 ((0, T) \times \partial M)} = (\partial_t U(T), w(T))_{D' \times C^\infty_0 (\mathbb{R}^n)} - (U(T), \partial_t w(T))_{D' \times C^\infty_0 (\mathbb{R}^n)},
\]

holds almost surely.
Now, we insert (17) to the right-hand side of (18) whence it splits into four terms. The two terms corresponding to $U_{in}$ are almost surely determined by $W$ and $g_0$. We show next that the traces $\text{Tr}_{\Omega}^T \partial_t^j U_{sc}$, $j = 0, 1$ [see (11)] are almost surely determined from the solutions of the exterior problem

$$\begin{align*}
\partial_t^2 v - \Delta_g v &= 0, \quad \text{in} \ (0, T) \times \mathbb{R}^n \setminus M, \\
v|_{(0,T) \times \partial M} &= h, \\
v|_{t=0} &= 0, \quad \partial_t v|_{t=0} = 0.
\end{align*}$$

Let us write $K_{ex}: h \mapsto v$ and recall, see e.g. [32], that the maps $\text{Tr}_{\Omega}^T \partial_t^j K_{ex}$, $j = 0, 1$, are continuous $E'(\mathbb{R}) \to D'(\Omega)$.

Notice that $U_{sc}|_{(0,\infty) \times \partial M}$ vanishes almost surely for small $t > 0$. Moreover, we may introduce a cut-off function $\chi_T \in C^\infty(0, T)$ satisfying $\chi_T = 0$ near $t = T$ and $\chi_T = 1$ away from a neighborhood of $T$. By choosing the neighborhood small enough, we have by finite speed of propagation that $\text{Tr}_{\Omega}^T \partial_t^j K_{ex} h = \text{Tr}_{\Omega}^T \partial_t^j K_{ex}(\chi_T LW)$, $j = 0, 1$, since the distance between $\Omega$ and $\partial M$ is strictly positive. Hence the traces of the scattered waves are determined as

$$\text{Tr}_{\Omega}^T \partial_t^j U_{sc} = \text{Tr}_{\Omega}^T \partial_t^j K_{ex}(\chi_T LW), \quad j = 0, 1,$$

almost surely.

In particular, a realization of the measurement $LW$ almost surely determines the realization of the distribution pairing via

$$\begin{align*}
(W, \chi+w)_{D' \times C_0^\infty((0,T) \times \partial M)} &= (\partial_t U_{in}(T) + \text{Tr}_{\Omega}^T \partial_t K_{ex}(\chi_T LW), w(T))_{D' \times C_0^\infty(\mathbb{R}^n)} \\
&= -(U_{in}(T) + \text{Tr}_{\Omega}^T K_{ex}(\chi_T LW), \partial_t w(T))_{D' \times C_0^\infty(\mathbb{R}^n)},
\end{align*}$$

where $w$ satisfies the wave equation $(\partial_t^2 - \Delta_g)w = 0$ with the initial conditions as in (19).

4. Gaussian Beams

4.1. Gaussian beam solutions. In what follows we will choose the test function $w$ in (23) to be a Gaussian beam solution to the wave equation $(\partial_t^2 - \Delta_g)w = 0$. A formal Gaussian beam of order $N_U \in \mathbb{N}$ propagating on a geodesic $\gamma_U$ is a function of the form

$$U_\epsilon(t, x) = e^{i\epsilon^{-1}\theta(t, x)} \sum_{n=0}^{N_U} e^n u_n(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n.$$

Here the phase function $\theta$ is of the form

$$\theta(t, x) = p(t)(x - \gamma_U(t)) + \frac{1}{2}(x - \gamma_U(t))H(t)(x - \gamma_U(t))$$

$$+ \sum_{2<|\alpha| \leq N} \frac{\theta_\alpha(t)}{\alpha!} (x - \gamma_U(t))^\alpha,$$

(24)
where all the coefficients are smooth functions, $H$ is a symmetric matrix with positive definite imaginary part and $p$ is the covariant representation of the velocity $\dot{\gamma}_U$. That is, using the notation
\[ \xi^b = \xi^k g_{jk}(p)dx^j, \quad \xi \in T_pM, \ p \in M, \] (25)
we have $p(t) = \dot{\gamma}_U^b(t)$. Moreover, the principal amplitude function $u_0$ satisfies $u_0(t, \gamma_U(t)) \neq 0$.

Let $T > 0$. Then the phase and the amplitude functions are constructed so that there is a neighborhood $V_U$ of the trajectory
\[ \{(t, \gamma_U(t)); \ t \in [0, T]\} \subset \mathbb{R} \times \mathbb{R}^n \]
and $C > 0$ such that
\[ \| (\partial^2_t - \Delta_g) U_\epsilon \|_{C^k(V_U)} \leq C e^{N_U - k}, \quad k < N_U. \]

See [40, Cor. 2.63] for a proof of this estimate and [3–5,56] for earlier references of Gaussian beams. Moreover, we may choose $V_U$ so that there is a constant $C_\theta > 0$ dependent on the phase $\theta$ satisfying
\[ \text{Im} \theta(t, x) \geq C_\theta d^2(x, \gamma(t)), \ (t, x) \in \overline{V_U}. \] (26)

Notice that $H(t)$ is non-degenerate since
\[ \text{Im} H(t) = \frac{H(t) - \overline{H(t)}}{2i} > 0. \] (27)
Indeed, by symmetry of $H(t)$ we have for nonzero $\xi \in \mathbb{C}^n$ that
\[ \text{Im}(\xi \overline{H(t)} \xi) = \frac{1}{2i} (\xi \overline{H(t)} \xi - \overline{\xi} H(t) \xi) = \frac{1}{2i} (\xi \overline{H(t)} \xi - \xi H(t) \overline{\xi}) = \xi (\text{Im} H(t)) \overline{\xi} > 0, \]
whence $H(t) \xi \neq 0$.

A Gaussian beam solution is given by the following theorem, see [40, Th. 2.64] for a proof.

**Theorem 2.** Let $T > 0$, $\chi_U \in C_0^\infty(V_U)$ and consider the wave equation
\[ \partial^2_t w_\epsilon - \Delta_X w_\epsilon = 0 \quad \text{on} \ (-\infty, T) \times \mathbb{R}^n, \]
\[ w_\epsilon(T) = (\chi_U U_\epsilon)(T), \ \partial_t w_\epsilon(T) = \partial_t(\chi_U U_\epsilon)(T). \]
Then there is $C > 0$ such that
\[ \| w_\epsilon - \chi_U U_\epsilon \|_{C^k([0,T] \times \mathbb{R}^n)} \leq C e^{N_U - k}, \quad k < N_U. \]

We will also need to estimate $w_\epsilon|_{(-\infty,-S) \times \partial M}$ for $S > 0$. This is obtained from an energy decay estimate by Vainberg [66] that follows from the assumptions (A1) and (A3).
Theorem 3. Let $\hat{M} \subset \mathbb{R}^n$ be a compact set. There is $T = T_{ED} > 0$ such that if $w$ is a smooth solution to the wave equation

$$\partial_t^2 w - \Delta_g w = 0 \text{ on } (-\infty, T) \times \mathbb{R}^n,$$

satisfying $\text{supp}(w(T)) \cup \text{supp}(\partial_t w(T)) \subset \hat{M}$, then for all $S > 0$

$$\|w\|^2_{L^2((-\infty,-S) \times \partial M)} \leq C \eta(S) \left( \|w(T)\|^2_{L^2(\hat{M})} + \|\partial_t w(T)\|^2_{L^2(\hat{M})} \right),$$

where

$$\eta(S) = \begin{cases} S^{-2n+3} & \text{for even } n, \\ \exp(-\delta S) & \text{for odd } n. \end{cases} \tag{29}$$

Above, $C, \delta > 0$ are constants which depend only on $\hat{M}$ and on the Riemannian manifold $(\mathbb{R}^n, g)$.

Proof. According to [25, Th. 2.104] we have for some $T > 0$ and $C > 0$ the estimate

$$|w(x, -t)| \leq C \eta'(t) \left( \|w(T)\|^2_{L^2(\hat{M})} + \|\partial_t w(T)\|^2_{L^2(\hat{M})} \right),$$

where $x \in \hat{M}, t > 0$ and

$$\eta'(t) = \begin{cases} t^{-n+1} & \text{for even } n, \\ \exp(-\delta t) & \text{for odd } n. \end{cases} \tag{31}$$

The claim follows after integration. \(\Box\)

4.2. Dual pairing of beam-like functions. In the following we study dual pairing of Gaussian beams with oscillating functions having a phase of the form

$$\theta(t, x) = p(t)(x - \gamma(t)) + \frac{1}{2}(x - \gamma(t))H(t)(x - \gamma(t)) + O(|x - \gamma(t)|^3).$$

where $\gamma$ and $p$ are smooth paths in $\mathbb{R}^n$ and $H \in \mathbb{C}^{n \times n}$ is symmetric.

Moreover, we assume that there is $t_0 \in (0, T)$ such that

(B1) $\gamma(t_0) \in \partial M$ and $\dot{\gamma}(t_0) \notin T_{\gamma(t_0)} \partial M$,

(B2) $p(t_0) = \dot{\gamma}(t_0)\flat$ and

(B3) $\text{Im } H(t_0)$ is positive definite.

Recall that $\dot{\gamma}\flat$ is the covariant representation of the vector $\dot{\gamma}$, see (25). We employ the method of stationary phase to get the following lemma.

Lemma 1. Let $\theta_1$ and $\theta_2$ be two phase functions of the form (32) such that for some $t_0 \in (0, T)$ both phases satisfy assumptions (B1)-(B3). We use notation $\gamma_j, p_j$ and $H_j$ for the paths and the matrix for the phase $\theta_j$, $j = 1, 2$, respectively. Suppose the paths $\gamma_j$ collide at $t_0$, that is,

$$\gamma_1(t_0) = \gamma_2(t_0) \quad \text{and} \quad \dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0). \tag{33}$$
It follows that there exists an open neighbourhood \( V \) of \((t_0, x_0)\) such that if \( u \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n) \) and \( \text{supp } u \subset V \) then
\[
\int_0^\infty \int_{\partial M} e^{i\epsilon^{-1}\theta_1(t, x)} e^{i\epsilon^{-1}\theta_2(t, x)} u \, dSdt = \beta \epsilon^{n/2} + O(\epsilon^{n/2+1}),
\]
where \( \beta \neq 0 \) if and only if \( u(t_0, x_0) \neq 0 \).

**Proof.** Let us choose such coordinates in a neighborhood of \( x_0 \) that \( \partial M = \{ x \in \mathbb{R}^n; \ x^n = 0 \} \).

We write \( x = (x^T, x^n) \in \mathbb{R}^n \) where \( x^T = (x^1, \ldots, x^{n-1}) \in \mathbb{R}^{n-1} \). Then we have for a phase function of the form (32) that
\[
\partial_{x^T} \theta = p(t)^T + (H(t)(x - \gamma(t)))^T + O\left(|x - \gamma(t)|^2\right),
\]
\[
\partial_t \theta = -p(t)^T \dot{\gamma}(t) + \partial_t p(t)(x - \gamma(t)) - \dot{\gamma}(t)H(t)(x - \gamma(t)) + O\left(|x - \gamma(t)|^2\right),
\]
\[
\partial^2_{x^T} \theta = H(t)^T + O\left(|x - \gamma(t)|\right),
\]
\[
\partial_t \partial_{x^T} \theta = \partial_t p(t)^T - (H(t) \dot{\gamma}(t))^T + O\left(|x - \gamma(t)|\right),
\]
\[
\partial_t^2 \theta = -2\partial_t p(t)^T \dot{\gamma}(t) - p(t)\partial_t \dot{\gamma}(t) + \dot{\gamma}(t)H(t)\dot{\gamma}(t) + O\left(|x - \gamma(t)|\right),
\]
where \( H(t)^T = (H(t))^{n-1}_{j,k=1} \).

Let us consider the map \( \Theta(t, x^T) = (\theta_1 - \theta_2)(t, x^T, 0) \). Assumptions (B1) and (B2) imply that \((t_0, x_0^T)\) is a critical point of \( \Theta \). We will show next that the critical point is non-degenerate. As the Hessian \( (\partial^2_{x^T} \Theta)(t_0, x_0^T) \) is symmetric, it is enough to show that its imaginary part is positive definite. We denote \( \xi_0 = \dot{\gamma}_1(t_0) \) and
\[
H_0 = \text{Im } H_1(t_0) + \text{Im } H_2(t_0).
\]

Let \((\alpha, v) \in \mathbb{R} \times \mathbb{R}^{n-1}\) be nonzero. We denote \( w = (v, 0) \in \mathbb{R}^n \). Notice that \( \xi_0^T \neq 0 \) since \( \xi_0 \notin T \partial M \). Thus \( w - \alpha \xi_0 \neq 0 \) and as \( H_0 \) is positive definite by assumption (3), we have
\[
0 < (w - \alpha \xi_0)H_0(w - \alpha \xi_0) = v(H_0)^Tv - 2\alpha (H_0 \xi_0)^Tv + \alpha^2 \xi_0H_0 \xi_0 = (\alpha, v)(\text{Im } \partial^2_{x^T} \Theta(t_0, x_0^T))(\alpha, v).
\]

As \((t_0, x_0^T)\) is non-degenerate, there are no other critical points in \( \text{supp}(u) \) if it is small enough. Moreover, for small \( \text{supp}(u) \), we have that \( \text{Im } \Theta \geq 0 \) in \( \text{supp}(u) \) since the first order term in \( x - \gamma_j(t) \) is real and the imaginary part of the second order term is non-negative and dominates the higher order terms.

The method of stationary phase [37, Th.7.7.5] gives
\[
\left| \int_0^\infty \int_{\partial M} e^{i\epsilon^{-1}\Theta(t, x)} u \, dSdt - \epsilon^{n/2} \left( \det(\partial^2_{x^T} \Theta/2\pi i)^{-1/2} u \sqrt{g} \right) \right|_{t=t_0, x=x_0} \leq C \epsilon^{n/2+1} \|u\|_{C^{2s}},
\]
where $dS = \sqrt{g}dx^T$ is the Riemannian volume measure of $\partial M$ and $s$ is the smallest integer satisfying $s \geq n/2 + 1$. □

Let us consider the test functions of the form
\[
\psi_e(t, x; s, y, \eta) = e^{i e^{-1}(t, x)} \chi_{\psi}(t, x),
\]
\[
\tilde{\theta}(t, x) = q(x - \mu(t)) + i|x - \mu(t)|^2,
\]
\[
\mu(t) = (t - s)\eta + y,
\]
where $y \in \partial M$, $\eta \in \partial_\nu S_y M$, $s > 0$, $\chi_{\psi} \in C_0^\infty(\mathbb{R}^{1+n})$ and $q = \eta^b$. We assume that $\chi_{\psi}(s, y) = 1$ and that the support of $\chi_{\psi}$ is small enough so that $\mu(t)$ intersects $\partial M$ only at $t = s$ for $t$ such that $(t, x) \in \text{supp}(\chi_{\psi})$ for some $x \in \mathbb{R}^n$.

**Theorem 4.** Let $T > 0$ and suppose that $w_\epsilon$ is a Gaussian beam solution as in Theorem 2, and that the corresponding formal Gaussian beam $U_\epsilon$ has the order $N_U \geq n/4 + 1$ and propagates along the geodesic $\gamma_U$. Suppose, furthermore, that $\gamma_U$ is transversal to $\partial M$. Then
\[
\|w_\epsilon\|_{L^2((0, T) \times \partial M)} = O(\epsilon^{n/2}).
\]
Furthermore, let $\psi_\epsilon$ be as in (34).

Then
\[
\|\psi_\epsilon\|_{L^2((0, T) \times \partial M)} = O(\epsilon^{n/2}),
\]
\[
(\psi_\epsilon, w_\epsilon)_{L^2((0, T) \times \partial M)} = \beta \epsilon^{n/2} + O(\epsilon^{n/2+1}),
\]
where $\beta$ has the following two properties
\[
\beta \neq 0 \quad \text{if } \gamma_U(s) = y \text{ and } p(s) = q,
\]
\[
\beta = 0 \quad \text{if } \gamma_U(s) \neq y \text{ or } p(s)^T \neq q^T.
\]
Here $\xi^T$ denotes the projection of $\xi \in T^*_y M$ on $T^*_y \partial M$ and $p = \dot{\gamma}_U^b$.

**Remark 1.** Notice that if $w_\epsilon$ exits $M$ at time $s$, then $\beta \neq 0$ if and only if $\gamma_U(s) = y$ and $\dot{\gamma}_U(s) = \eta$. In particular, if $w_\epsilon$ enters $M$ at time $T - r$ then by inspecting $\beta$ as a function of the parameters $(s, y, \eta)$, see (34), we can determine the largest $s < T - r$ such that $\gamma_U(s) \in \partial M$, that is, the first time exit time of $w_\epsilon$. Thus we see that $\beta$ determines the first exit time of $w_\epsilon$ and also the exit point $\gamma_U(s)$ and the direction $\dot{\gamma}_U(s)$.

**Proof of Theorem 4.** Notice that
\[
\|w_\epsilon - \chi_U U_\epsilon\|_{L^2((0, T) \times \partial M)} = O(\epsilon^{n/4+1}),
\]
whence it is enough to prove the claimed estimates with $w_\epsilon$ replaced by $\chi_U U_\epsilon$. Let us write $\gamma = \gamma_U$. As $\gamma$ is transversal to $\partial M$, there are finitely many points in $\gamma((0, T)) \cap \partial M$, say $x_1 = \gamma(t_1), \ldots, x_J = \gamma(t_J)$. If $V_j \subset (0, T) \times \partial M$ is a sufficiently small neighborhood of $(t_j, x_j)$, then Lemma 1 yields that
\[
\|\chi_U U_\epsilon\|_{L^2(V_j)}^2 = O(\epsilon^{n/2}).
\]
Let us write \( R = ((0, T) \times \partial M) \setminus \bigcup_{j=1}^{J} V_j \). By (26) and [37, Th. 7.7.1] we have that
\[
\| \chi_U U_e \|_{L^2(R)}^2 = \mathcal{O}(\epsilon^N),
\]
for any \( N \in \mathbb{N} \).

Let us now consider the asymptotic estimates involving \( \psi_\epsilon \). By the definition of \( \psi_\epsilon \) we have
\[
\text{supp}(\psi_\epsilon) \cap ((0, T) \cap \partial M) \cap \{(t, \mu(t)); t \in (0, T)\} = \{(s, y)\}.
\]
Let \( V \subset (0, T) \times \partial M \) be a neighborhood of \( (s, y) \) and define the set \( R = ((0, T) \cap \partial M) \setminus V \). Then \( \text{Im}(\theta + \bar{\theta}) \geq \text{Im} \bar{\theta} > 0 \) in \( R \), whence
\[
\| \psi_\epsilon \|_{L^2(R)}^2 = \mathcal{O}(\epsilon^N), \quad (\psi_\epsilon, w_\epsilon)_{L^2(R)} = \mathcal{O}(\epsilon^N),
\]
for any \( N \in \mathbb{N} \). Thus it is enough to consider the norm and the inner product on \( V \).

Lemma 1 gives immediately the claimed asymptotic behaviour of the norm of \( \psi_\epsilon \) and also that of the inner product if \( \gamma(s) = y \) and \( p(s) = q \). Under this assumption \( \beta \neq 0 \) since the principal amplitude function of the Gaussian beam satisfies \( u_0(s, \gamma(s)) \neq 0 \).

Let us consider the case \( \gamma(s) \neq y \). If \( V \) is small enough, then we have that \( \text{Im}(\theta + \bar{\theta}) \geq \text{Im} \theta > 0 \) in \( V \) and
\[
(\psi_\epsilon, \chi_U U_e)_{L^2(V)} = \mathcal{O}(\epsilon^N),
\]
for any \( N \in \mathbb{N} \).

Let us consider the case \( \gamma(s) = y \) and \( p(s)^T \neq q^T \). Then
\[
\partial_{x^T} \text{Re}(\theta - \bar{\theta})(s, y) = p(s)^T - q^T \neq 0.
\]
If \( V \) is small enough, then the function \( \text{Re}(\theta - \bar{\theta}) \) has no critical points in \( V \) and [37, Th. 7.7.1] implies
\[
(\psi_\epsilon, \chi_U U_e)_{L^2(V)} = \mathcal{O}(\epsilon^N),
\]
for any \( N \in \mathbb{N} \). \( \square \)

5. Correlations Generated by the White Noise Source

In the following two sections we extract the travel time data by studying the correlation of certain white noise related random processes on the boundary \( \partial M \).

5.1. Definition of two random processes. Let us choose a compact set \( \hat{M} \subset \mathbb{R}^n \) such that \( M \subset \hat{M}^{\text{int}} \) and a constant \( T > T_{ED} \) such that \( T - 1 \) is greater than the maximum length of a geodesic of \( (\hat{M}, g) \). Here the constant \( T_{ED} \) is as in Theorem 3.

Let us introduce the notation \( \mathbf{p} \) and \( \mathbf{q} \) for the points
\[
\mathbf{p} = (s, y, \eta), \quad (s, y, \eta) \in (1, T) \times \partial_{+} SM \quad (36)
\]
\[
\mathbf{q} = (T, z, \zeta), \quad (z, \zeta) \in S(\hat{M} \setminus M). \quad (37)
\]
In what follows, we are only interested in the points \( \mathbf{q} \) such that the geodesic \( \gamma \) from \((z, \zeta)\) enters in \( M \).
We define our test process

\[ X_\epsilon(p) = (W, \epsilon^{-n/4} \psi_\epsilon(p))_{D' \times C^\infty_0((0,T) \times \partial M)}. \]  

(38)

where \( \psi_\epsilon(t, x; p) \) is the test function defined in (34). We assume that the cut-off function \( \chi_\psi \) in (34) is chosen so that

\[ \text{supp}(\psi_\epsilon(p)) \subset (1, T) \times \mathbb{R}^n. \]

Moreover, we denote by \( w_\epsilon(t, x; q) \) the Gaussian beam solution of Theorem 4, where we have written out explicitly the dependence on the parameters \( z = \gamma_U(T) \) and \( \zeta = -\dot{\gamma}_U(T) \). Let us write \( \gamma(t) = \gamma_U(T - t) \). The assumption (A2) implies that if the geodesic \( \gamma \) enters in \( M \) then it is transversal to \( \partial M \). Let us define the process

\[ Y_\epsilon(q) = (W, \epsilon^{-n/4} \chi w_\epsilon(q))_{D' \times C^\infty_0((0,T) \times \partial M)}. \]  

(39)

We remind the reader that the measurement \( L(W(\omega_0)) \) determines the realization \( Y_\epsilon(q; \omega_0) \) almost surely via Eq. (23).

Notice that the functions \( p \mapsto X_\epsilon(p) \) and \( q \mapsto Y_\epsilon(q) \) are complex-valued zero-centered Gaussian random processes parametrized by sets (36) and (37), respectively.

Let us define

\[ \beta_\epsilon(p, q) = \mathbb{E} X_\epsilon(p) Y_\epsilon(q), \quad \beta(p, q) = \lim_{\epsilon \to 0} \beta_\epsilon(p, q). \]

Then (6) implies that

\[ \beta_\epsilon(p, q) = \epsilon^{-n/2}(\psi_\epsilon(p), w_\epsilon(q))_{L^2((0,T) \times \partial M)}, \]

since \( \chi_\epsilon = 1 \) in \( \text{supp}(\psi_\epsilon(p)) \). Moreover, Theorem 4 implies that \( \beta(p, q) = \beta \) satisfies (35), and that, for fixed \( p \) and \( q \), the expected values \( \mathbb{E}|X_\epsilon(p)|^2 \) and \( \mathbb{E}|Y_\epsilon(q)|^2 \) stay bounded as \( \epsilon \to 0 \).

5.2. Time translations of the processes. Let us introduce time translated variables to our analysis. We let \( j \in \mathbb{N} \), define

\[ \psi^j_\epsilon(t, x; p) = \psi_\epsilon(t - (j - 1)T, x; p) \]

and denote by \( w^j_\epsilon(t, x; q) \) the solution of

\[
\begin{align*}
\partial_t^2 w - \Delta_g w &= 0 \quad \text{on } (-\infty, jT) \times \mathbb{R}^n, \\
w(jT) &= w_\epsilon(T; q), \\
\partial_t w_\epsilon(jT) &= \partial_t w_\epsilon(T; q) \cdot \partial_t^j w_\epsilon(T; q).
\end{align*}
\]

In the following, we abbreviate \( \psi^j_\epsilon(p) = \psi^j_\epsilon(\cdot, \cdot; p) \) and \( w^j_\epsilon(q) = w^j_\epsilon(\cdot, \cdot; q) \), respectively. Moreover, we write

\[ X^j_\epsilon(p) = (W, \epsilon^{-n/4} \psi^j_\epsilon(p))_{D' \times C^\infty_0((0,jT) \times \partial M)}, \]

\[ Y^j_\epsilon(q) = (W, \epsilon^{-n/4} \chi w^j_\epsilon(q))_{D' \times C^\infty_0((0,jT) \times \partial M)}. \]
Notice that for fixed \( p \) and \( \epsilon \) the Gaussian random variables \( X^j_\epsilon(p) \), \( j = 1, 2, \ldots \), are independent and identically distributed (i.i.d). Also for fixed \( q \) and \( \epsilon \) the Gaussian random variables

\[
(W, \epsilon^{-n/4} w^j_\epsilon(q))_{\mathcal{D}^c \times C^\infty_0((j-1)T, jT) \times \partial M}, \quad j = 1, 2, \ldots,
\]

are i.i.d. Further, we have

\[
\mathbb{E}(X^j_\epsilon(p) Y^j_\epsilon(q)) = \mathbb{E}(\psi^j_\epsilon(p), X^j w^j_\epsilon(q))_{L^2((0,jT) \times \partial M)} = \mathbb{E}(\psi_\epsilon(p), w_\epsilon(q))_{L^2((0,jT) \times \partial M)} = \beta_\epsilon(p, q), \tag{40}
\]

since \( \psi^j_\epsilon \) is supported on \(((j-1)T + 1, jT) \times \partial M\). Theorem 3 guarantees that \( w_\epsilon(q) \in L^2((-\infty, T) \times \partial M) \). Hence for fixed \( q \) and \( \epsilon > 0 \) the random variables \( Y^j_\epsilon(q) \) are uniformly bounded with respect to \( j \), that is,

\[
\mathbb{E}|Y^j_\epsilon(q)|^2 = \left\| X^j w^j_\epsilon(q) \right\|^2_{L^2((0,jT) \times \partial M)} \leq \left\| w_\epsilon(q) \right\|^2_{L^2((-\infty, T) \times \partial M)} < \infty. \tag{41}
\]

Furthermore, Theorem 3 implies the following estimate

\[
\left\| X^j w^j_\epsilon(q) \right\|^2_{L^2((0,(j-1)T) \times \partial M)} \leq \left\| w_\epsilon(q) \right\|^2_{L^2((-\infty, -(j-1)T) \times \partial M)} \leq C J^{-2n+3}, \quad J \geq 1, \tag{42}
\]

where the constant \( C > 0 \) does not depend on \( j \) and \( J \). Notice that if the dimension is odd we obtain even better estimates. However, in this treatise they are not needed.

**Lemma 2.** Suppose \( X, Y, Z, V : \Omega \to \mathbb{C} \) are zero-mean Gaussian random variables. Then it holds that

\[
|\mathbb{E}(XYZV)| \leq 3\sqrt{|\mathbb{E}|X|^2|Y|^2|Z|^2|V|^2} \tag{43}
\]

**Proof.** Let us begin by pointing out that if \( X \) and \( Y \) are zero-mean real valued Gaussian random variables, then it holds (see e.g. [43]) that

\[
\mathbb{E}(XY)^2 = 2(\mathbb{E}XY)^2 + \mathbb{E}X^2 \mathbb{E}Y^2.
\]

It follows by a straightforward calculation that if instead \( X \) and \( Y \) are complex-valued, we have

\[
|\mathbb{E}XY|^2 = |\mathbb{E}XY|^2 + |\mathbb{E}XY|^2 + \mathbb{E}|X|^2 \mathbb{E}|Y|^2 \tag{44}
\]

and, consequently, \(|\mathbb{E}XY|^2 \leq 3 \mathbb{E}|X|^2 \mathbb{E}|Y|^2 \) by the Cauchy–Schwarz inequality. It remains to note that

\[
|\mathbb{E}(XYZV)|^2 \leq \mathbb{E}|XYZV|^2 \leq 9 \mathbb{E}|X|^2 \mathbb{E}|Y|^2 \mathbb{E}|Z|^2 \mathbb{E}|V|^2
\]

and the claim follows. \( \square \)

**Lemma 3.** Let \( \epsilon > 0 \), \( p \) and \( q \) be fixed and let us denote \( X^j = X^j_\epsilon(p) \) and \( Y^j = Y^j_\epsilon(q) \) for \( j = 1, 2, \ldots \). Suppose that \( |j-k| \geq 1 \). Then

\[
|\mathbb{E}(X^j Y^j - \mathbb{E}X^j Y^j)| (X^k Y^k - \mathbb{E}X^k Y^k) = O(|j-k|^{-n+2}). \tag{45}
\]
Proof. We may assume without loss of generality that \( j > k \). Let us split \( Y_j \) in two parts
\[
Y_j = (W, e^{-n/4} \chi_k \chi_+ w_j^e(q)) + (W, e^{-n/4} (1 - \chi_k) \chi_+ w_j^e(q)) = Y_{j,\text{large}} + Y_{j,\text{small}},
\]
where \( \chi_k \in C^\infty(\mathbb{R}) \) satisfies \( \chi_k(t) = 0 \) for \( t \in (0, kT) \) and \( \chi_k(t) = 1 \) outside a neighborhood of \([0, kT]\). Notice that \((X_j, Y_{j,\text{large}})\) and \((X_k, Y_k)\) are independent Gaussian random variables from \( \Omega^1 \) to \( C^2 \). Thus also the products \( X_j Y_{j,\text{large}} \) and \( X_k Y_k \) are independent random variables. In particular,
\[
\mathbb{E} X_j Y_{j,\text{large}} X_k Y_k = \mathbb{E} X_j Y_{j,\text{large}} \mathbb{E} X_k Y_k.
\]
Hence it follows that
\[
\left| \mathbb{E} \left( X_j Y_{j,\text{large}} - \mathbb{E} X_j Y_{j,\text{large}} \right) \left( X_k Y_k - \mathbb{E} X_k Y_k \right) \right| = \left| \mathbb{E} X_j Y_{j,\text{large}} X_k \mathbb{E} X_k Y_k - \mathbb{E} X_j Y_{j,\text{large}} \mathbb{E} X_k Y_k \right| \leq 4 \sqrt{\mathbb{E} |Y_{j,\text{small}}|^2 \mathbb{E} |X_j|^2 \mathbb{E} |X_k|^2 \mathbb{E} |Y_k|^2},
\]
where we have used Lemma 2 and the Cauchy–Schwarz inequality. The claim follows as \( \sqrt{\mathbb{E} |Y_{j,\text{small}}|^2} = \mathcal{O}((j - k)^{-n+\frac{3}{2}}) \) and the norms of \( X_j, X_k \) and \( Y_k \) are uniformly bounded with respect to \( j \) and \( k \).

6. Ergodicity by Time Translations

In our problem formulation the source is an infinitely long realisation of a white noise process. However, the random processes introduced in Sect. 5.1 are mostly affected by the source on some finite interval in time. Similarly, the correlation decays fast between the source at other time instances and the variables (39) and (38), respectively. In Sect. 5.2 we introduced time-translations to sample the full time domain. Below we utilize the zero correlation length of this process in time in order to apply ergodicity arguments. We begin by recording the following lemma.

Lemma 4. Let \((a_j)_{j=1}^\infty\) and \((b_k)_{k=1}^\infty\) be sequences of complex numbers and let us write
\[
S_N = \frac{1}{N^2} \sum_{1 \leq j < k \leq N} a_j b_k.
\]
Suppose that \(|a_j b_k| \leq |j - k|^{-p}\) for \( p \geq 1/2 \). Then \( |S_N| = \mathcal{O}(N^{-1/2}) \).

Proof. We notice \( \sum_{1 \leq j < k \leq N} |j - k|^{-p} = \sum_{\ell=1}^{N-1} (N - \ell) \ell^{-p} \) and conclude that
\[
N^2 |S_N| \leq \sum_{\ell=1}^{N-1} (N - \ell) \ell^{-p} \leq N + \int_1^{N-1} (N - s) s^{-p} ds = \mathcal{O}(N^{3/2}).
\]
Now the claim follows.
Let us now show that $\beta_\epsilon$ and, consequently, $\beta$ can be recovered from the measurement data. We consider the convergence of the following series

$$\beta_{\epsilon,N}(p, q; \omega) = \frac{1}{N} \sum_{j=1}^{N} X^j_\epsilon(p; \omega) Y^j_\epsilon(q; \omega). \quad (46)$$

**Theorem 5.** Let $p$, $q$ and $\epsilon > 0$ be fixed. Then

$$\lim_{N \to \infty} \beta_{\epsilon,N^3}(p, q; \omega) = \beta_\epsilon(p, q)$$

almost surely.

**Proof.** By (40) we have that $E\beta_{\epsilon,N}(p, q; \omega) = \beta_\epsilon(p, q)$. Let us compute

$$\text{Var}(\beta_{\epsilon,N}) = \frac{1}{N^2} \sum_{j=1}^{N} \left( E|X_j^j Y_j^j|^2 - |E X_j^j Y_j^j|^2 \right)$$

$$+ \frac{2}{N^2} \sum_{1 \leq j < k \leq N} \text{Re} \ E \left( (X_j^j Y_j^j - E X_j^j Y_j^j)(X_k^k Y_k^k - E X_k^k Y_k^k) \right).$$

Using the identity (44) and the Cauchy–Schwarz we obtain

$$E|X_j^j Y_j^j|^2 - |E X_j^j Y_j^j|^2$$

which is bounded by (41). Consequently, Lemmas 3 and 4 guarantee that $\text{Var}(\beta_{\epsilon,N}) = O(N^{-1/2})$. It remains to notice that the sum $\sum_{N=1}^{\infty} \text{Var}(\beta_{\epsilon,N^3})$ converges, which implies that $\sum_{N=1}^{\infty} |\beta_{\epsilon,N^3} - \beta_{\epsilon,N^3}|^2$ converges almost surely. Therefore we have almost surely

$$\lim_{N \to \infty} \beta_{\epsilon,N^3}(p, q) = \lim_{N \to \infty} E\beta_{\epsilon,N^3}(p, q) = \beta_\epsilon(p, q). \quad \Box$$

Finally, the information obtained by ergodicity arguments with probability one yields the values of the indicator function $\beta$ by continuity arguments.

**Theorem 6.** A single realization of the measurement $L(F)$ determines the function $(p, q) \mapsto \beta(p, q)$ with probability one.

**Proof.** For a fixed $\epsilon > 0$, the function $\beta_\epsilon$ is continuous. Indeed, the inner product $(\psi_\epsilon(p), w_\epsilon(q))_{L^2((0, T) \times \partial M)}$ is clearly continuous with respect to the parameter $p$. Moreover, the formal Gaussian beam $U_\epsilon$ is a smooth function of the variables $(t, x, q)$, see e.g. [32]. In particular, the initial data in Theorem 2 depends smoothly on $q$, whence the map

$$q \mapsto w_\epsilon(q) : S(\hat{M} \setminus M) \to C((0, T); H^1(\mathbb{R}^n))$$

is continuous by the energy estimates for the wave equation.

Let us fix a countable dense set $D_0 \subset D = ((1, T) \times \partial_\epsilon S M) \times (S(\hat{M} \setminus M))$. Then with probability one, the Eq. (47) is satisfied simultaneously for all $(p, q, \epsilon) \in D_0 \times \{ \frac{1}{j} \mid j \in \mathbb{N} \}$. This together with the continuity of $\beta_\epsilon : D \to \mathbb{R}$ implies that mappings

$$(p, q) \mapsto \beta_{1/j}(p, q), \quad j \in \mathbb{N},$$

are recovered. Now, for fixed parameters $p \in (1, T) \times \partial_\epsilon S M$ and $q \in S(\hat{M} \setminus M)$ we have $\beta(p, q) = \lim_{j \to \infty} \beta_{1/j}(p, q)$ by Theorem 4. \quad \Box

Theorem 1 follows since the function $\beta$ determines the scattering relation as explained after Theorem 4.
Appendix A. White Noise as a $H_{loc}^{s}$ -Valued Random Variable

Gaussian white noise has been studied extensively in the literature, see e.g. [34], and there are different procedures to construct the corresponding probability measure. Let us mention the classical approaches by Hida [33] and the abstract Wiener space construction by Gross [30]. It is well-known [42] that the white noise measure is supported on any $H_{loc}^{-s}(\mathbb{R})$, $s > \frac{1}{2}$. However, to see whether the classical constructions extend to a Borel measure in this space requires some work. For clarity, we provide a direct construction below.

Let $D = (1 - \partial f^2)^{1/2}$ and write

$$H^{-s}(\mathbb{R}, m) = \{ f \in D'(\mathbb{R}) \mid f = D^s(m^{-\frac{1}{2}}g), \ g \in L^2(\mathbb{R}) \},$$

where $m(t) = \frac{1}{1+t^2}$. Clearly, $H^{-s}(\mathbb{R}, m)$ is a separable Hilbert space with the inner product $(f, g)_{H^{-s}(\mathbb{R}, m)} = (m^{1/2}D^{-s}f, m^{1/2}D^{-s}g)_{L^2(\mathbb{R})}$. We refer to [47] for further properties of $H^{-s}(\mathbb{R}, m)$.

We define $C = D^{-s}mD^{-s}$. Let $(e_j)_{j \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R})$. Since $D^s m^{-1/2} : L^2(\mathbb{R}) \rightarrow H^{-s}(\mathbb{R}, m)$ is an isometry we have that functions $f_j = D^s m^{-1/2} e_j$, $j \in \mathbb{N}$, form an orthonormal basis in $H^{-s}(\mathbb{R}, m)$. By a direct calculation we obtain

$$(Cf_j, f_k)_{H^{-s}(\mathbb{R}, m)} = (Se_j, e_k)_{L^2(\mathbb{R})},$$

where $S = m^{1/2} D^{-2s} m^{1/2}$. The operator $S$ is bounded in $L^2(\mathbb{R})$ and has a Schwartz kernel

$$k(x, y) = (1 + x^2)^{-\frac{1}{2}} (1 + y^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(-i(y-x)\xi)(1 + \xi^2)^{-s} d\xi. \quad (48)$$

Clearly, we have $\int_{\mathbb{R}} k(x, x) dx < \infty$ when $s > 1/2$ and thus $S$ is trace-class by [12]. In consequence, also $C$ is trace-class and it follows by [20, Prop. 2.18] that there exists a zero-centered Gaussian measure $\mu_{\mathbb{R}}$ on $H^{-s}(\mathbb{R}, m)$ with the covariance $C$.

Suppose that $W_{\mathbb{R}} : \Omega \rightarrow H^{-s}(\mathbb{R}, m)$ is a random variable with the distribution $\mu_{\mathbb{R}}$. For $\phi, \psi \in C_0^\infty(\mathbb{R})$ we obtain

$$\mathbb{E}(W_{\mathbb{R}}, \phi)_{D' \times C_0^\infty(\mathbb{R})}(W_{\mathbb{R}}, \psi)_{D' \times C_0^\infty(\mathbb{R})} = \mathbb{E}(W_{\mathbb{R}}, C^{-1}\phi)_{H^{-s}(\mathbb{R}, m)}(W_{\mathbb{R}}, C^{-1}\psi)_{H^{-s}(\mathbb{R}, m)} = (CC^{-1}\phi, C^{-1}\psi)_{H^{-s}(\mathbb{R}, m)} = (\phi, \psi)_{L^2(\mathbb{R})}.$$  

Hence, $W_{\mathbb{R}}$ is called the white noise.

Let us now consider the source studied in this paper. From a construction analogous to the above one, we obtain the white noise measure $\mu_{\partial M}$ on $H^{-s_1}(\partial M)$, $s_1 > (n - 1)/2$. We denote by $\iota$ the embedding of $\partial M$ into $\mathbb{R}^n$ and use the notation from [37] for the push-forward $\iota_*$. We extend $\mu_{\partial M}$ to $\mathbb{R}^n$ by defining $\mu_{\mathbb{R}^n} = \iota_* \mu_{\partial M}$. Notice that $\iota_*$ is continuous $H^{-s_1}(\partial M) \rightarrow H^{-s_2}(\mathbb{R}^n)$ where $s_2 = s_1 + 1/2$. Thus the measure $\mu_{\mathbb{R}^n}$ is Borel on $H^{-s_2}(\mathbb{R}^n)$. We have the continuous embedding of the tensor product

$$H^{-s}(\mathbb{R}, m) \otimes H^{-s_2}(\mathbb{R}^n) \hookrightarrow H^{-s_3}(\mathbb{R}^{1+n}, m) \hookrightarrow H_{loc}^{-s_3}(\mathbb{R}^{1+n})$$

where $\tilde{m}(t, x) = m(t)$ and $s_3 = s + s_2$. Thus $\mu_{\mathbb{R}^{1+n}} = \mu_{\mathbb{R}} \otimes \mu_{\mathbb{R}^n}$ extends to a Borel measure on $H_{loc}^{-(n+1)/2-\epsilon}(\mathbb{R}^{1+n})$ for any $\epsilon > 0$. Let us point out that if $W : \Omega \rightarrow H_{loc}^{-(n+1)/2-\epsilon}(\mathbb{R}^{1+n})$ is a random variable with the distribution $\mu_{\mathbb{R}^{1+n}}$ then it satisfies (6).
Finally, we remind the reader that supp$(W) \subset \mathbb{R} \times \partial M$. In particular, $\chi_+ W$ is almost surely in the space $H^s_{locc}((0, \infty) \times B)$ for any open set $B$ containing $\partial M$. The space $H^s_{locc}((0, \infty) \times B)$ is defined in Sect. 3.2.

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