SOME HEREDITARILY JUST INFINITE SUBGROUPS OF \( J \)

CORNELIUS GRIFFIN

School of Mathematical Sciences
University of Nottingham
Nottingham
NG7 2RQ

10th September 2002

Abstract. This work examines the commutator structure of some closed subgroups of the wild group of automorphisms of a local field with perfect residue field, a group we call \( J \). In particular, we establish a new approach to evaluating commutators in \( J \) and using this method investigate the normal subgroup structure of some classes of index subgroups of \( J \) as introduced by Klopsch. We provide new proofs of Fesenko’s results that lead to a proof that the torsion free group \( T = \{ t + \sum_{k \geq 1} a_k t q^{k+1} : a_k \in \mathbb{F}_p \} \) is hereditarily just infinite, and by extending his work, we also demonstrate the existence of a new class of hereditarily just infinite subgroups of \( J \) which have non-trivial torsion.

Index Subgroups of \( J \)

Section 1: Introduction

Although as a result of the work of Camina [C1] it is known that \( J \) has an extremely rich subgroup structure, it was generally considered difficult to write down arbitrary subgroups of \( J \). This problem recieved some attention from Klopsch, who in his thesis [K] considered a particular class of subgroups in \( J \) which are easier to describe than others. His class was defined as follows: take a subset \( \Lambda \subset \mathbb{N} \) and look at the equivalent subset

\[
J(\Lambda) := \left\{ t + \sum_{\lambda \in \Lambda} a_{\lambda} t^{q^{\lambda+1}} : a_{\lambda} \in \mathbb{F}_p \right\}
\]
of $\mathcal{J}$. Under certain arithmetic conditions on $\Lambda$ it was already known that this subset actually formed a subgroup of $\mathcal{J}$. Klopsch managed to give a complete description of sets $\Lambda$ for which $\mathcal{J}(\Lambda)$ is a subgroup of $\mathcal{J}$. He proved

**Theorem** [K]. *Let $\Lambda$ and $\mathcal{J}(\Lambda)$ be as above. Then $\mathcal{J}(\Lambda) \leq \mathcal{J}$ if and only if*

$$\forall \lambda, \mu \in \Lambda, \forall k \leq \lambda + 1: \left(\frac{\lambda + 1}{k}\right) \equiv 0 \mod p \text{ or } \lambda + k\mu \in \Lambda.$$  

Given this arithmetic condition he went on to describe four classes of subsets which are easily demonstrable to satisfy the condition in the Theorem:

1. $A: = d\mathbb{N}$ for some $d \in \mathbb{N}$;
2. $B: = p\mathbb{N} \cup p\mathbb{N} - 1$;
3. $C: = p^i\mathbb{N} - 1$ for some $i \in \mathbb{N}$;
4. $D: = \{p^i - 1 : i \in \mathbb{N}\}$.

Klopsch went on to prove some elementary results about the groups arising from these subsets of $\mathbb{N}$. He considered properties such as finite generation, centralizers, normalizers etc., and also used these subgroups to calculate part of the Hausdorff Spectrum for the Nottingham Group.

Independently of Klopsch’s work, Fesenko has studied the group he called $T := \mathcal{J}(q\mathbb{N})$ for some power $q = p^r$ of $p$. It transpires that this group has some interesting properties, both as a group in its own right, and also as the Galois group of a particular type of field extension. Coates and Greenberg had asked the following

**Question** [CG]. *Is it true that for every finite extension $K$ of $\mathbb{Q}_p$ there exists a deeply ramified Galois $p$-extension $M$ of $K$ so that no subfield $M'$ of $M$ is an infinitely ramified Galois extension of a finite extension $Q$ of $\mathbb{Q}_p$ with $\text{Gal}(M'/Q)$ being a $p$-adic Lie group?*

The terms need not concern us greatly here; I would just point out that an arithmetically profinite field extension is a particular type of a deeply ramified field extension. In particular an arithmetically profinite field extension is one whose non-trivial upper ramification groups are all open — this is the property we need. Fesenko managed to give an affirmative answer to this question by taking a field extension whose Galois Group is equal to $T$. Then to answer the question, the problem is reduced to a need to demonstrate some properties of the group $T$. Fesenko proved

**Theorem** [F1]. *Let $q$ be a fixed power of the odd prime $p$ and let $T$ be the group

$$\left\{t + \sum_{k=1}^{\infty} a_k t^{qk+1} : a_k \in \mathbb{F}_p, \forall k\right\}.$$
Then

(1) If $\sigma \in T_i \setminus T_{i+1}$ then $\sigma^p \in T_{pi} \setminus T_{pi+1}$ and so $T$ is torsion-free;
(2) $[T_i, T_i] \leq T_{(q+1)i+1}$ and the group $T_i/T_{i+1}$ is abelian of exponent $p$;
(3) $[T, T]^p > T_{q+2}$; the number of generators of $T$ is at most $q + 1$;
(4) $T$ is not $p$-adic analytic;
(5) $T$ is hereditarily just infinite; i.e., non-trivial closed normal subgroups of open subgroups are open.

By using this Theorem and some ramification theory it is possible to give an answer to the Coates and Greenberg Question. For instance, that $T$ is hereditarily just infinite allows one to show that the corresponding field extension must be arithmetically profinite.

This Theorem of Fesenko can be seen in other contexts as well. Hereditarily just infinite groups, and their ancestors the just infinite groups play the same role in pro-$p$ group theory as the simple groups play in finite group theory. So it is natural to try to classify the hereditarily just infinite groups as one does for the finite simple groups. In the advent of Fesenko’s work, the current state of this classification is as follows:

**Partition of JI Groups.** The class of just infinite pro-$p$ groups consists of four types:

(1) Solvable ones that are linear over $\mathbb{Z}_p$;
(2) Nonsolvable ones that are linear over $\mathbb{Z}_p$;
(3) Nonsolvable ones that are linear over $\mathbb{F}_p[[t]]$;
(4) The rest — namely groups of “Nottingham type”, “Grigorchuk type”, “Fesenko type”, and so on.

Apart from aesthetic concerns there are strong practical reasons why it would be useful to complete this classification. For instance, Boston has reduced the Fontaine-Mazur Conjecture concerning $p$-adic Galois representations to a question about hereditarily just infinite pro-$p$ groups [B]. As the Fontaine-Mazur Conjecture implies amongst other things an alternative proof of Fermat’s Theorem, one can see the strength of such a classification of HJI groups.

To prove that a group is HJI, the most important thing a person must do is to understand properly the commutator structure of the group in question. The main thrust of Fesenko’s argument is a series of highly technical Lemmas in which sufficiently many commutators are evaluated to leave him able to deduce his final results.

The aim of this paper is twofold. Firstly, we introduce a new approach to calculating commutators in groups of formal power series. To illustrate the usefulness of this approach, we provide new proofs of the Lemmas contained in the work of Fesenko about commutators in $T$ and in the interest of completeness, we indicate how from here one may deduce that $T$ is HJI.
Secondly, we use the same methods, and indeed Fesenko’s proof that $T$ is HJI to evaluate commutators in $\mathcal{J}(B)$ and as a result, we are able to prove the following

**Theorem 6.3.** Let $B := p\mathbb{N} \cup p\mathbb{N} - 1$ and let $S := \mathcal{J}(B)$. Then $S$ is a hereditarily just infinite pro-$p$ group with non-trivial torsion.

The method employed could also be used to study the normal subgroup structure of subgroups of the Nottingham Group in more generality. It may be possible to extend the methods used here to calculate, for instance, the obliquity/width of various index subgroups. This has been done for the group $S$ in a recent preprint of Barnea and Klopsch [BK] which also contains an alternative proof of Theorem 6.3.

This work is part of the author’s PhD thesis, Nottingham 2002, supported by an EPSRC studentship and under the supervision of Professor Ivan Fesenko. The author thanks EPSRC and also B. Klopsch, I. Fesenko for their interest and helpful comments.

**Section 2: Evaluating Commutators in Groups of Formal Power Series**

Throughout this text, by $[v, u]$ we will mean $v \circ u \circ v^{-1} \circ u^{-1}$. Here $\circ$ denotes the group operation; to indicate formal products of two power series $v$ and $u$ we will merely write $vu$.

So suppose we are given two formal power series $u$ and $v$ in some group of formal power series. Then we may write

$$[v, u] := t + \sum_{k \geq l} a_k t^{k+1}$$

for some $a_k \in \mathbb{F}_p$ and some $l \in \mathbb{N}$. We want to understand the series $(a_k)$ for values $k \in \mathbb{N}$. Now $[v, u] = v \circ u \circ v^{-1} \circ u^{-1}$ and so

$$v \circ u \circ v^{-1} \circ u^{-1} = t + \sum_{k \geq l} a_k t^{k+1}$$

$$\iff v \circ u \circ v^{-1} \circ u^{-1} \circ u \circ v = (t + \sum_{k \geq l} a_k t^{k+1}) \circ u \circ v$$

$$\iff v \circ u - u \circ v = \sum_{k \geq l} a_k (u \circ v)^{k+1}.$$  

So we have reduced understanding commutators of elements to the simpler problem of understanding products of elements. Now to evaluate commutators of formal
power series, we need only solve some recurrence relations on the \( a_k \). To illustrate this process I will now give a new proof of the Lemmas used by Fe senko evaluating particular commutators that were necessary to show that \( T \) is HJI. In all the work that follows, we will implicitly use the following two Lemmas.

**Crucial Lemma 2.1.** Let \( m \geq n \in \mathbb{N} \), let \( l_1, \ldots, l_n \in \mathbb{N} \) be so that \( \sum_{i=1}^{n} l_i = m \). Write \( A_{m,n}(l_1, \ldots, l_n) \) for the number of maps

\[
f : \{1, \ldots, m\} \to \{1, \ldots, n\} \text{ so that } |f^{-1}(i)| = l_i.
\]

Then

\[
A_{m,n}(l_1, \ldots, l_n) = \prod_{i=1}^{n} \left( m - \sum_{k=1}^{l_i-1} l_k \right).
\]

**Proof.** When \( n = 1 \) the result is obvious.

Now count the number of maps \( f : \{1, \ldots, m\} \to \{1, \ldots, n+1\} \) so that \( |f^{-1}(i)| = l_i \).

To do this, take \( l_1 \) elements in \( \{1, \ldots, m\} \) that map onto 1. This can be done in \( \binom{m}{l_1} \) ways and so the number of maps having the required property is \( \binom{m}{l_1} \) times the number of maps \( f : \{1, \ldots, m-l_1\} \to \{1, \ldots, n\} \) so that \( |f^{-1}(i)| = l_i \) where we have done some relabelling. The result now follows by induction on \( n \).

**Lemma 2.2** [Lu]. Let \( n \in \mathbb{N} \) and let \( a = \sum_{i=0}^{n} a_i p^i, b = \sum_{i=0}^{n} b_i p^i \in \mathbb{N}, \) where \( a_i, b_i \in \{0, \ldots, p-1\} \) for all \( i \in \{0, \ldots, p-1\} \). Then

\[
\left( \begin{array}{c} a \\ b \end{array} \right) \not\equiv 0 \mod p \iff a_i \geq b_i \ \forall i.
\]

Throughout the following work we will use the following notation; for a given element \( u = t + \sum_{k \geq 1} u_{k+1} t^{k+1} \) we will set

\[
E_i(u) := u_i \text{ and } S(u) := \{ i \in \mathbb{N} : E_i(u) \neq 0 \}.
\]

**Section 3: Commutators in \( T \)**

\( T \) is the group consisting of formal power series of the form

\[
t + \sum_{k \geq 1} a_{qk+1} t^{qk+1}, \quad a_l \in \mathbb{F}_p
\]
for \( q \) some power of the prime \( p \). We can define a filtration on \( T \) by setting

\[
T_i := \left\{ t + \sum_{k \geq i} a_{qk+1} t^{qk+1} \right\}.
\]

In order to investigate the commutator structure of this group we want to calculate \([v,u]\) for arbitrary elements \( v \in T_j, u \in T_i \) for \( i \geq j \).

For convenience set \( i = j + e \) and given elements \( u,v \in T \) write \((u \circ v)_{nc} = u \circ v - u - v + t\). Notice this means that \((v \circ u)_{nc} - (u \circ v)_{nc} = v \circ u - u \circ v\).

Recall the reasoning outlined that will allow us to calculate commutators of this form. We have that \([v,u] = t + \sum_{k \geq K} a_{qk+1} t^{qk+1} \) if and only if

\[
v \circ u - u \circ v = \sum_{k \geq K} a_{qk+1} (u \circ v)^{qk+1}
\]

and so we must evaluate the compositions \( v \circ u, u \circ v \). It is a simple exercise to verify that we may write

\[
u \circ v = u + v - t + \sum_{s \geq 2} \left( \sum_{k=j}^{s-j} u_{q(k+e)+1} f_{s,k} \right) t^{q(s+e)+1}
\]

and

\[
v \circ u = v + u - t + \sum_{s \geq 2} \left( \sum_{k=j}^{s-j} v_{qk+1} g_{s,k} \right) t^{q(s+e)+1}
\]

where now

\[
f_{s,k} = \sum_{j(1)+\cdots+j(q(k+e)+1)=q(s+e)+1} v_{j(1)} \cdots v_{j(qk+1)}
\]

and

\[
g_{s,k} = \sum_{j(1)+\cdots+j(q(k+1)+1)=q(s+e)+1} u_{j(1)} \cdots u_{j(q(k+1))}.
\]

In the first sum we sum over integers \( j(l) \) so that \( j(l) > 1 \Rightarrow j(l) \geq qj + 1 \) whereas the second sum is over integers \( j(l) \) so that \( j(l) > 1 \Rightarrow j(l) \geq q(j + e) + 1 \). It should be clear now why we have identified Lemmas 2.1, 2.2 as crucial if we want to evaluate these products, and hence also commutators in \( T \). The parts of the sums above requiring analysis are the \( f_{s,k}, g_{s,k} \). Let us first consider \( f_{s,k} \); we collect information into a
Lemma 3.1. Fix \( s \) and \( k \) and take notation as above.

(1) Suppose there exists a unique \( l \) so that \( j(l) > 1 \). Set \( j(l) = qm + 1 \) and \( j(n) = 1 \) for all \( n \neq l \). Then \( m + k = s \) and so \( s \geq 2j \);

(2) Suppose there exists \( l_1, \ldots, l_q \) so that \( j(l_i) > 1 \). Set \( j(l_i) = qm + 1 \). Then \( qm + k = s \) and so \( s \geq qj + j \);

(3) Suppose there exists \( l_1, \ldots, l_{q+1} \) so that \( j(l_i) > 1 \). Set \( j(l_i) = qm + 1 \) for \( i = 1, \ldots, q \) and \( j(l_{q+1}) = qm + 1 \). Then \( qm + n + k = s \) and so \( s \geq qj + 2j \);

(4) Suppose in general there exist \( b_1 \) so that \( j(l) = qm_1 + 1 \), \( b_2 \) so that \( j(l) = qm_2 + 1 \) and so on to \( b_d \) so that \( j(l) = qm_d + 1 \). Then \( s = b_1 m_1 + \cdots + b_d m_d + k \) and so \( s \geq \Delta(b_1 + \cdots + b_d)j + j \) where \( \Delta := |\{ i : m_i > 0 \}| \).

This simple Lemma allows one to write down \( f_{s,k} \) for small values of \( s \) and so to write down \( u \circ v \).

Proposition 3.2.

(1) Let \( 2j \leq s < qj + j \). Then \( f_{s,k} = v_{q(s-k)+1} \) and so the coefficient of \( t^{q(s+e)+1} \) in \( (u \circ v)_{nc} \) is

\[
\sum_{k=j}^{s-j} u_{q(k+e)+1} v_{q(s-k)+1};
\]

(2) Let \( qj + j \leq s < qj + 2j \). Then the coefficient of \( t^{q(s+e)+1} \) in \( (u \circ v)_{nc} \) is

\[
\sum_{k=j}^{s-j} u_{q(k+e)+1} v_{q(s-k)+1} + \sum_{m,k \geq j, \quad qm+k=s} (k+e) u_{q(k+e)+1} v_{qm+1};
\]

(3) Let \( s = qj + 2j \). In addition to the terms described above we also get in \( (u \circ v)_{nc} \) the term

\[
(j + e) u_{q(j+e)+1} v_{qj+1}^2;
\]

(4) This process continues to expand in a uniform way as \( s \) increases in size.

Proof. Everything follows straightforwardly from Lemmas 2.1, 2.2, 3.1. To illustrate the process I will evaluate \( f_{qj+j,j} \). We have

\[
f_{qj+j,j} = \sum_{j(1)+\cdots+j(q(j+e)+1)=q(j+j+e)+1} v_{j(1)} \cdots v_{j(qj+1)}.
\]

If there exists a unique \( l \) so that \( j(l) > 1 \) then this \( j(l) \) can be chosen in \( \binom{q(j+e)+1}{1} \) ways and we have \( j(l) + q(j+e) = q(j+1+e) + 1 \) from which it follows that \( j(l) = q(j) + 1 \) and we get a contribution to \( f_{qj+j,j} \) of \( \binom{q(j+e)+1}{1} v_{q(j)+1} \).

If there exists \( d \) say values of \( l \) so that \( j(l) > 1 \) then these \( d \) values can be chosen in \( \binom{q(j+e)+1}{d} \) ways. Thus Lucas’ Lemma 2.2 tells us that \( d \) is either 0 or 1 mod \( q \).
So the size of $s$ tells us that we must have $d = q$ from which we may deduce that
$qj(l) + q(j + e) + 1 - q = q(qj + j + e) + 1$. Thus $j(l) = qj + 1$ and we get a
contribution to $f_{qj + j, j}$ of $(q_{(j+e)+1})v_{qj+1}$.

The result follows.

We can do exactly the same for $v \circ u$; omitting the details, one may prove

**Proposition 3.3.**

1. Let $2j \leq s < qi + j - e$. Then $g_{s,k} = u_{q(s+e-k)+1}$ and so the coefficient of
   $t^{q(s+e)+1}$ in $(v \circ u)_nc$ is
   $$\sum_{k=j}^{s-j} v_{k+1}u_{q(s-e-k)+1};$$

2. Let $qi + j - e \leq s < qi + 2j$. Then the coefficient of $t^{q(s+e)+1}$ in $(v \circ u)_nc$ is
   $$\sum_{k=j}^{s-j} v_{k+1}u_{q(s-e-k)+1} + \sum_{k \geq j, l \geq j+e \atop ql+k=s+e} kv_{k+1}u_{q+l+1};$$

3. Let $s = qi + 2j$. In addition to the terms described above we also get in
   $(v \circ u)_nc$ the term
   $$ju_{q(j+e)+1}v_{qj+1};$$

4. This process continues to expand in a uniform way as $s$ increases in size.

So we now have all the tools we will require to evaluate commutators in $T$. The
first thing to notice is that the leading coefficients of $[v, u]$ and $t + v \circ u - u \circ v$ are
the same and so we may immediately deduce

**Proposition 3.4.** Let $u, v$ be as above. Then

1. $[v, u] = t - iu_{qj+1}v_{qj+1}t^{q(qj+i)+1} + \ldots$;
2. $[T_j, T_i] \leq T_{qj+i}$ and if $p$ divides $i$ then furthermore $[T_j, T_i] \leq T_{qj+i+1}$.

**Proof.** We have

$$t + v \circ u - u \circ v = t + \left( \sum_{s=2j}^{\infty} \sum_{k=j}^{s-j} (u_{q(k+e)+1}v_{q(s-k)+1} - v_{qk+1}u_{q(s+e-k)+1}) \right) t^{q+1}$$

$$+ \sum_{s=qj+j}^{\infty} \left( \sum_{m, k \geq j, \atop qm+k=s} -(k + e)u_{q(k+e)+1}v_{qm+1} \right) t^{q(s+e)+1} + \ldots$$
where $r = s + e$ in the first sum. Notice now that the first sum gives an identically zero expression and so in actual fact
\[
t + v \circ u - u \circ v = t + \sum_{s=q_j+j}^{\infty} \left( \sum_{m,k \geq j, \atop qm+k=s} -(k+e)u_{q(k+e)+1}v_{qm+1} \right) t^{q(s+e)+1} + \ldots
\]
\[
= t - iu_{q_i+1}v_{q_j+1}t^{q(j+i)+1} + \ldots
\]
and the Proposition is proved.

**Section 4: T is Hereditarily Just Infinite**

Now that we have established some basic commutator relationships, we are in a position to give an alternative proof to Fesenko’s Lemmas about the nature of commutators of particular elements in $T$. For future reference, we include a combinatorial Lemma that is proved in [F1] and will be useful to us. We use the following notation: \( j = j'p^{n(j)} \) where \( j' \) is coprime to \( j \). Recall also that \( q = p^r \).

**Lemma 4.1** [F1, Lemma 1]. Fix \( s \) so that \( 1 \leq s \leq r \). Let \( i > j \geq q^2 \) and let \( i \) be coprime to \( p \). Let \( i_m, j_m \) satisfy the following conditions:

1. \( i_m \geq i, j_m \geq j - q \);
2. \( (i_m,i) = 1 \);
3. \( j_m \geq j \) if \( i_m = i \); \( qj_m + p^s\i_m > qj + p^s i \) if \( i_m > i \);
4. \( qj + qm < j + q, \) then \( i_m = r_m i + s_m q \) for integers \( r_m \geq 1, s_m \geq 0 \);

Let \( v_m, w_m, x_m, y_m, z_m \) be non-negative integers so that \( v_m > 0 \iff w_m > 0, x_m > 0 \iff y_m > 0 \) and \( z_m > 0 \) only if \( x_m > 0 \). Let \( q \) divide \( z_m \) if \( z_m \geq qj \).

Then the equality
\[
\sum(v_{i_m} + w_{qj} + x_{qj} + y_{qi} + z_m) + \sum(x_{j_m} + y_{qj}p^{j_m} + z) = I + qj, \quad p^{s-1}i < I \leq p^s I, p^s|I
\]
implies that
\[
I = p^s i;
\]

Furthermore:

if \( p^s < q \) then up to renumbering we have \( v_1 = p^s, w_1 = 1, i_1 = i, j_1 = j \) and \( v_m = w_m \) for \( m > 1, x_m = y_m = z_m = 0 \) for \( m \geq 1 \);
if \( p^s = q \) then either up to renumbering \( v_1 = q, w_1 = 1i_1 = i, j_1 = j \) and everything
else is zero, or up to renumbering $x_1 = q, y_1 = 1, i_1 = i, j_1 = j$ and $v_m = w_m = z_m = 0$ for $m \geq 1, x_m = y_m = 0$ for $m > 1$.

The relevance of this result will become apparent in due course.

Fesenko considers elements $u, v \in T$ whose coefficients satisfy particular arithmetic requirements. He takes an element $v \in T_j \setminus T_{j+1}$ for some $j \geq q^2$ as

$$v = t + \sum_{k \geq j} v_{qk+1} t^{qk+1}$$

where $v_{qk+1} = 0$ if $j + 1 \leq k \leq qj$ is not divisible by $q$. Also

$$u = t + u_{qi+1} t^{qi+1}$$

for some non-zero $u_{qi+1}$ and $i > j$ relatively prime to $p$. He proves

**Lemma 4.2** [F1, Lemma 3]. With the same notation as above $[v, u]$ is congruent modulo $t^{1+q^2(i+j)+q}$ to

1. $t + \sum_{j_m \geq j} c_m t^{1+q(v_m i+w_m q j_m)} + \sum_{j_m \geq j} d_m t^{1+q(x_m j_m+y_m q i p^{n(j_m)}+z_m)}$;
2. $t + \sum_{i \geq 1} e_v t^{1+q(qj+v)}$.

In (1) the $v_m, \ldots, z_m$ together with $i_m = i, j_m \geq j$ satisfy the conditions of Lemma 4.1. In (2) the $e_v$ satisfy

1. if $v + q j < j + q i$ and $e_v \neq 0$, then $v = s_v i + r_v q$ for $s_v \geq 1, r_v \geq 0$;
2. $e_{p^s i} = -i u_{qi+1} v_{qj+1}$ for $0 \leq s < r$, and $e_{qi} = (j-i) u_{qi+1} v_{qj+1}$.

Using the methods we have developed above we are now able to give an alternative proof of this result. We prove the results about the nature of the $e_v$, the other results follow in a similar way.

**Proof.** We demonstrate that the second description of the commutator is correct. Essentially to prove this result we merely solve recurrence relations on the coefficients of the commutator, which we evaluate using the methods outlined previously. In this special case that $u, v$ have a particularly simple form $v \circ u, u \circ v$ become easier to describe when we work modulo $t^{q(qj+qi)+q+1}$:

$$v \circ u = v \circ (t + u_i t^{qi+1})$$

$$= t + u_i t^{qi+1} + \sum_{k \geq j} v_{qk+1} (t + u_i t^{qi+1}) q^{k+1}$$

$$= t + u_i t^{qi+1} + \sum_{k \geq j} v_{qk+1} t^{qk+1} + \sum_{k \geq j} v_{qk+1} u_i t^{q(k+i)+1}$$

$$\quad + \binom{qj+1}{q} v_{qj+1} u_i t^{q(qi+j)+1} + \binom{qj+1}{q+1} v_{qj+1} u_i q t^{q(qi+j)+1}$$

$$\quad + \binom{qj+1}{2q} v_{qj+1} u_i 2 q t^{q(2qi+j)+1} + \ldots$$
and
\[ u \circ v = u \circ \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right) \]
\[ = t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right)^{q_i+1} \]
\[ = t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t^{q_i+1} + \sum_{k \geq j} v_{qk+1} t^{q(k+i)+1} \right) \]
\[ + \sum_{k \geq j} \left( \frac{q_i + 1}{q} \right) v_{qk+1} t^{q(qk+i+1)} + \left( \frac{qj + 1}{q} \right) v_{qj+1} t^{q(qj+i+j+1)} \]
\[ + \left( \frac{qj + 1}{2q} \right) v_{qj+1} t^{2q(qj+i+j)+1} + \ldots - u_i \left( \sum_{k \geq j} \left( \frac{q_i + 1}{q} \right) v_{qk+1} t^{q(qk+i)+1} \right) \]
\[ + \sum_{k \geq j} \left( \frac{q_i + 1}{q} \right) \left( \frac{q(i-1) + 1}{1} \right) v_{qk+1} v_{ql+1} t^{q(qk+i+l)+1} + \ldots \]
\[ = \sum_{n \geq K} \alpha_n \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right)^{q_i+1} \right)^{qn+1}. \]

So as before we have \( v \circ u - u \circ v = \sum \alpha_n (u \circ v)^{qn+1} \) where \([v, u] = t + \sum \alpha_n t^{qn+1}\) and so
\[
\left( \frac{qj + 1}{q} \right) v_{qj+1} u_i t^{q(qj+i)+1} + \left( \frac{qj + 1}{q} \right) v_{qj+1} u_i t^{q(qj+i+j)+1} \]
\[ + \left( \frac{qj + 1}{2q} \right) v_{qj+1} u_i t^{2q(qj+i+j)+1} + \ldots - u_i \left( \sum_{k \geq j} \left( \frac{qj + 1}{q} \right) v_{qk+1} t^{q(qk+i)+1} \right) \]
\[ + \sum_{k \geq j} \left( \frac{qj + 1}{q} \right) \left( \frac{q(i-1) + 1}{1} \right) v_{qk+1} v_{ql+1} t^{q(qk+i+l)+1} + \ldots \]
\[ = \sum_{n \geq K} \alpha_n \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right)^{q_i+1} \right)^{qn+1}. \]

In order to prove the first claim of the Lemma we work modulo \( t^{q(qi+qj)+1} \) and so this expression simplifies to
\[
- u_i \left( \sum_{k \geq j} \left( \frac{qj + 1}{q} \right) v_{qk+1} t^{q(qk+i)+1} \right) \]
\[ + \sum_{k \geq j} \left( \frac{qj + 1}{q} \right) \left( \frac{q(i-1) + 1}{1} \right) v_{qk+1} v_{ql+1} t^{q(qk+i+l)+1} + \ldots \]
\[ = \sum_{n \geq K} \alpha_n \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right)^{q_i+1} \right)^{qn+1}. \]
Now we merely equate coefficients. We can trivially confirm what we already know, namely that $\alpha_n = 0$ for all $n < qj + i$ and that $\alpha_{qj+i} = -iu_i v_{qj+1}$. It is also immediate that $\alpha_n = 0$ for all $qj + i < n < qj + i + q$. Comparing coefficients of $t^{q(qj+i+q)+1}$ we see that

$$-iu_i v_{q(j+1)+1} = \alpha_{qj+i+q};$$

this process continues until we reach $t^{q(qj+i+j)+1}$. Here we have

$$-iu_i v_{qj+1}^2 = \alpha_{qj+i+j} + v_{qj+1} \alpha_{qj+i}$$

and so $\alpha_{qj+i+j} = 0$ as required.

At this stage it is worth tidying up what we know. The previous large expression now simplifies to:

$$-u_i \left( \sum_{k \geq j} \binom{q_i + 1}{q} v_{qk+1} t^{q(qk+i)+1} \right) + \sum_{\substack{k \geq j \\ell \geq j}} \binom{q_i + 1}{q} \binom{q(i-1) + 1}{1} v_{qk+1} v_{q\ell+1} t^{q(qk+i+1)+1} + \ldots$$

$$= \alpha_{qj+i} \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + u_i \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} \right) q^{i+1} \right) q^{q(qj+i)+1}$$

$$+ \alpha_{qj+i+q} \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + \ldots \right) q^{q(qj+i+q)+1}$$

$$+ \alpha_{qj+i+2q} \left( t + \sum_{k \geq j} v_{qk+1} t^{qk+1} + \ldots \right) q^{q(qj+i+2q)+1}$$

$$+ \ldots$$

Compare coefficients of $t^{q(qj+2i)+1}$:

$$0 = \alpha_{qj+2i} + u_i \alpha_{qj+i}$$

and so $\alpha_{qj+2i} = iu_i^2 v_{qj+1}$.

Now using the nature of the non-zero coefficients of $v$ and continuing in this way we may deduce that $\alpha_{\Delta} \neq 0 \implies \Delta = qj + s_v i + r_v q$ for some $s_v \geq 1, r_v \geq 0$ whenever $\Delta < qi + j$. We can also see that for all $p^s < q$, we have

$$\alpha_{qj+p^s i} = -iu_i^{p^s} v_{qj+1} = -iu_i v_{qj+1}.$$  

We finally need to look at $\alpha_{q(i+j)}$. Notice that

$$\alpha_{qi+j} = j u_i v_{qj+1}$$
and so the same reasoning as above tells us that

\[ \alpha_{qi+p^s} = j u_i v_{qj+1}^{p^s}. \]

The value of \( \alpha_{q(i+j)} \) follows and the result is proved.

The following Lemma, amended from [F1] is the final result needed in order to
demonstrate the main thrust of Fesenko’s reasoning.

**Lemma 4.3** [F1, Lemma 5]. *Let \( 1 \neq H \triangleleft U <_o T \). Then for all sufficiently large \( j \) coprime to \( p \), for all non-zero \( a \in \mathbb{F}_p \), there exists a series*

\[ t + \sum_{k \geq j} a_k t^{qk+1} \in H, \quad a_j = a \]

*so that for \( j + 1 \leq k \leq qj + q^2 \), \( a_k = 0 \) if \( q \) does not divide \( k \).*

In the interests of completeness we now outline how Fesenko completed his proof
that \( T \) is hereditarily just infinite.

**Theorem 4.4** [F1]. *\( T \) is a hereditarily just infinite pro-\( p \) group for \( p > 2 \).*

**Proof Outline.** Let \( i, j, i - j \) be coprime to \( p \). Then by Lemma 4.3, given any closed
normal subgroup \( H \) of an open subgroup \( U \) of \( T \) there exists an element \( v \) of the
form described in Lemma 4.2. So applying Lemma 4.2 twice, once to \((j, i)\) and once
to \((j - p, i + qp)\) gives us modulo a high power of \( t \) elements

\[ t + \sum_{v \geq i} e_v t^{1+q(vj+v)} \quad \text{and} \quad t + \sum_{v \geq i+qp} f_v t^{1+q(vj+v-qp)} \]

in \( H \) for arbitrary non-zero elements \( e_i, f_{i+qp} \). Thus we may pick these elements in
such a way that the composition gives us, again modulo a high power of \( t \)

\[ [v(j), u(i)] \circ [v(j - p), u(i + qp)] = t + \sum_{v > i} g_v t^{1+q(vj+v)}. \]

Fesenko shows that the coefficients of this power series also satisfies the conditions
of Lemma 4.2. Using this fact one can continue in this way to produce an element

\[ t + \sum_{v \geq pi} h_v t^{1+q(vj+v)} \]

in \( H \) where again \( h_{pi} \neq 0 \) and the \( h_v \) satisfy the conditions of Lemma 4.2. By induction, one may now produce an element of the form \( t + t^{1+q(vj+qi)} + \ldots \). Combined
with Proposition 3.4 it follows that we may, for all $\lambda$ sufficiently large produce the element $t + t^{\lambda+1} + \ldots$ in $H$.

Thus it follows that $T$ is HJI as claimed.

**Remark.** The stumbling block to a proof that $T$ is hereditarily just infinite for $p = 2$ is the first arithmetic condition that $i, j, i - j$ are all coprime to $p$. Lemmas 4.1, 4.2, 4.3 remain valid as they stand for $p = 2$. It is entirely likely that one will be able to remove this arithmetic condition on $i, j$ and prove $T$ is HJI for $p = 2$ without adopting that different a method to the one outlined above. At the moment this remains out of reach but I feel sure a closer investigation of the methods used in this chapter would bear fruit.

**Section 5: Commutators in $S$**

$S$ is the group consisting of formal power series of the form

$$u(t) = t + a_1 t^p + a_2 t^{p+1} + \ldots.$$ 

We may define a filtration on $S = S_1 > S_2 > \ldots$ where now

$$S_{2n} := \{ t + a_{np+1} t^{np+1} + \cdots : a_i \in \mathbb{F}_p \}$$

and

$$S_{2n-1} := \{ t + a_{np} t^{np} + \cdots : a_i \in \mathbb{F}_p \}.$$ 

Suppose without loss of generality we are given two elements $u = t + u_{pi} t^{pi} + \ldots$ and $v = t + v_{pj} t^{pj} + \ldots$ with $i = j + \epsilon$. Then as above one can easily verify that

$$u \circ v(t) = u + v - t + \sum_{s \geq i-1+jp} \left( \sum_{k \geq j+\epsilon} u_{kp} \sum_{j(1)+\cdots+j(kp)=sp} v_{j(1)} \cdots v_{j(kp)} \right) t^{sp}$$

$$+ \sum_{\Delta \geq (i+j)p} \left( \sum_{k \geq j+\epsilon} u_{kp+1} \sum_{j(1)+\cdots+j(kp+1)=\Delta} v_{j(1)} \cdots v_{j(kp+1)} \right) t^{\Delta},$$

and similarly

$$v \circ u(t) = u + v - t + \sum_{s \geq j-1+ip} \left( \sum_{k \geq j} v_{kp} \sum_{j(1)+\cdots+j(kp)=sp} u_{j(1)} \cdots u_{j(kp)} \right) t^{sp}$$

$$+ \sum_{\Delta \geq (i+j)p} \left( \sum_{k \geq j} v_{kp+1} \sum_{j(1)+\cdots+j(kp+1)=\Delta} u_{j(1)} \cdots u_{j(kp+1)} \right) t^{\Delta}.$$
In the first of these two expressions \( j(l) > 1 \implies j(l) \geq jp \) and \( j(l) \equiv 0, 1 \mod p \) whereas in the second of the two expressions \( j(l) > 1 \implies j(l) \geq ip \).

In order to prove that \( S \) is HJI we must evaluate some commutators. The following Lemmas are simple to verify using the same method as before.

**Lemma 5.1.** Let \( i > j \).

1. Let \( u = t + u_i t^{p_i+1}, v = t + v_j t^{p_j+1} + \ldots \) Then \([v, u] = t - iu_i v_j t^{p(p_j+i)+1} + \ldots\);
2. Let \( u = t + u_i t^{p_i+1}, v = t + v_j t^{p_j} + \ldots \) Then \([v, u] = t - u_i v_j t^{(i+j)} + \ldots\);
3. Let \( u = t + u_i t^{p_i}, v = t + v_j t^{p_j+1} + \ldots \) Then \([v, u] = t + u_i v_j t^{(i+j)} + \ldots\).

This Lemma is proved exactly as Proposition 3.4 was proved and so we omit the details. The following is less obvious and more important to the proof of the main result.

**Lemma 5.2.** Let \( i > j \) and let \( i \) be coprime to \( p \). Take elements \( u = t + u_i t^{p_i+1}, v = t + v_j t^{p_j} + \ldots \) with \( u_i v_j \neq 0 \). Then the first power of \( t \) in \([v, u]\) that has non-zero coefficient and is congruent to \( 1 \mod p \) is

\[-iu_i v_j t^{p(p_j+i-1)+1}.

**Proof.** We proceed exactly as we have done previously. A simple exercise in combinatorics enables us to evaluate

\[
u \circ v = t + v_{p_j} t^{p_j} + v_{p_j+1} t^{p_j+1} + \cdots + u(t + v_{p_j} t^{p_j} + v_{p_j+1} t^{p_j+1} + \cdots) = u + v - t + \sum_{\lambda \geq p_j} u_i v_{\lambda} t^{p_i+\lambda} + \sum_{\lambda \equiv 0, 1 \mod p, \lambda \equiv 0, 1 \mod p} (p_i + 1) u_i v_{\lambda} t^{\lambda + p + 1 - p} + \ldots
\]

and similarly

\[
v \circ u = (t + u_i t^{p_i+1}) + v_{p_j} (t + u_i t^{p_i+1}) t^{p_j} + \ldots = u + v - t + \sum_{\mu \geq p_j} u_i v_{\lambda} t^{p_i + \mu} + \sum_{\mu \equiv 0, 1 \mod p} (\mu/p) u_i v_{\mu} t^{p^2 \mu + \mu - p} + \ldots
\]

Thus it follows that

\[
v \circ u - u \circ v = - \sum_{\lambda \geq p_j} u_i v_{\lambda} t^{p_i + \lambda} - iu_i v_{p_j} t^{p(p_j+i-1)+1} + \ldots := \sum_{k \in \mathbb{N}} \alpha_k t^k
\]
where again \([v, u] = t + \sum_{k>1} \alpha_k t^k\).

The result is now self evident, once it is appreciated that \(\alpha_k = 0\) unless we have that \(k\) is 0 or 1 modulo \(p\).

In order to prove the main results, the last piece of the jigsaw we need is a description of some of the torsion in \(B\). The following Lemma is simple and indeed is contained in [C2]:

**Lemma 5.3.** Write \(v^p\) for group composition of \(v\) with itself \(p\) times.

1. Let \(v = t + v_1 t^{p^n} + \cdots \in S_{2n-1} \setminus S_{2n}\). Then

   \[ v^p \in S_{2np-1}; \]

2. Let \(v = t + v_1 t^{p^{n+1}} + \cdots \in S_{2n} \setminus S_{2n+1}\). Then

   \[ v^p \in S_{2np}. \]

In his thesis [Y] York gives a complete description of the elements of order \(p\) in the Nottingham Group. The next two results are those contained therein that are of relevance to the situation here.

**Theorem 5.4.** [Y, Theorem 5.5.3]. Let \(\alpha = t + \sum_{k \geq n} a_{pk} t^k \in J(F_p)\). Then \(\alpha\) has order \(p\) if and only if

\[ a_{(2n+s-np+1)p-1} = f_s(a_{(2n+s-np+1)p-2}, \ldots, a_{np}) \]

for some given polynomial \(f_s\) dependent upon \(s, \alpha\) and \(n + s \geq np\).

**Theorem 5.5** [Y, Theorem 5.5.4]. Let \(\alpha = t + at^{pk+1} + \cdots \in J(F_p)\). Then \(\alpha\) has infinite order and \(\alpha^p = t + at^{p(pk)+1} + \ldots\)

These results follow merely from a close examination of the coefficients of the power series. Notice also that Theorem 5.5 strengthens Lemma 5.3(2) above to show that given \(v \in S_{2n} \setminus S_{2n+1}\) then \(v^p \in S_{2pn} \setminus S_{2pn+1}\).

Given these results we are in a position to prove the final Proposition we will need in order to prove that \(B\) is hereditarily just infinite.

**Proposition 5.6.** Let \(1 \neq H \triangleleft_c U \triangleleft_a B\). Then \(H\) is infinite and furthermore, \(H\) contains elements of arbitrarily large depth.

**Proof.** As \(H\) is non-trivial there is an element in \(H\) different from \(t\). Take such an element \(v\). If \(v = t + v_j t^{p^j+1} + \ldots\) then by Theorem 5.5 \(v\) has infinite order and \(H\) is infinite. Thus taking powers of \(v\) gives elements of arbitrary depth in \(H\). Thus in this case we’re done.
Suppose instead that \( v = t + v_j t^{pj} + \ldots \). Then Lemma 5.1(2) implies that for any \( i > j \),
\[
[t + ut^{pi+1}, v] = t + uv_j t^{p(i+j)} + \ldots
\]
and so it follows that \( H \) is infinite.

The same commutator relation tells us that we may alter coefficients of powers of \( t \) in \( v \) occurring after the \( p(2j + 1) \)-th power of \( t \) by composing \( v \) with
\[
[t + u_j t^{pj+1}, v] = t + u_j v_j t^{p(2j+1)} + \ldots
\]
Thus by Theorem 5.4, \( H \) contains an element not of order \( p \) where the coefficients of later powers of \( t \) are determined by the previous ones. Continue this process with the \( p \)-th power of this element. The result follows.

Section 6: \( S \) is Hereditarily Just Infinite

The proof mirrors Fesenko’s proof outlined earlier. In particular, we take a non-trivial (closed) normal subgroup \( H \) of an open subgroup \( U \) of \( S \). Then \( H \) is infinite and contains elements of arbitrary depth. We take such an element \( v \) and show that by taking appropriate commutators of \( v \), and of powers of \( v \) with arbitrary elements \( u \in S_i \) for sufficiently large \( i \) that we may realise, for \( \lambda \) sufficiently large in \( p\mathbb{N} \cup p\mathbb{N} - 1 \), any element of the form \( t + t^\lambda + \ldots \) in \( H \). This will be sufficient to complete the proof.

**Proposition 6.1.** Suppose that \( H \) is a non-trivial closed normal subgroup of an open subgroup \( U \) of \( S \). Suppose also that \( H \) contains an element of infinite order of the form
\[
v = t + v_1 t^{pj+1} + \ldots
\]
for some non-zero \( v_1 \in \mathbb{F}_p \). Then \( H \) is open.

**Proof.** If \( v = t + v_1 t^{pj+1} + \ldots \) has infinite order, then taking a sufficiently large power of \( v \) we may assume that \( j \) is arbitrarily large. Then by Lemma 5.1(3), we can commutate \( v \) in such a way that \( v \) approximates an element of the group \( T \) arbitrarily closely. \( T \) is hereditarily just infinite, and so it follows that for all sufficiently large \( \lambda \) we may realise \( t + t^{p\lambda+1} + \ldots \) as an element of \( H \).

Also by Lemma 5.1(3) we may realise \( t + t^{p\lambda} + \ldots \) as an element of \( H \) for all sufficiently large \( \lambda \). Thus the result follows.

In order to prove that \( S \) is hereditarily just infinite, it is now sufficient to establish the next
Proposition 6.2. Let $H$ be a non-trivial normal closed subgroup of an open subgroup $U$ of $S$. Then there exists in $H$ an element of the form

$$v = t + v_1 t^{pj+1} + \ldots, \quad v_1 \neq 0$$

necessarily of infinite order.

Proof. Suppose that this is not the case. By the previous results we have proved, $H$ must contain an element of the form

$$v = t + v_j t^{pj} + \ldots$$

not of order $p$. Then from Lemma 5.2 we have

$$[v, t + u't^{pi+1}] = t + u'vt^{p(i+j)} + \ldots + (-iu'v)t^{p(pj+i-1)+1} + \ldots$$

where $t^{p(pj+i-1)+1}$ is the first power of $t$ that is $1 \mod p$ and has non-zero coefficient.

Also by Lemma 5.3 we may deduce that

$$v^p = t + \alpha t^{p\lambda} + \ldots$$

for some non-zero $\alpha$ and sufficiently large $\lambda$. Thus

$$[v^p, t + ut^{pi+1}] = t + \alpha ut^{p(\lambda+i)} + \ldots + (-iu\alpha)t^{p(\lambda+i-1)+1} + \ldots$$

Similarly

$$[v, t + u't^{p(\lambda+i-j-1)+1}] = t + vu't^{p(\lambda+i)} + \ldots + (j-i)vu't^{p(pj+\lambda+i-j-1)+1} + \ldots$$

where now $t^{p(pj+\lambda+i-j-1)+1}$ is the first power of $t$ that is $1 \mod p$ and has non-zero coefficient. Thus noticing taking the composition of these two commutators for an appropriate choice of $u, u'$ gives us the element

$$t + \gamma t^{p(\lambda+i+1)} + \ldots + \delta t^{p(pj+\lambda+i-j-1)+1} + \ldots$$

where $\delta$ is non-zero, and this is the coefficient of the lowest power of $t$ that is $1 \mod p$.

We can repeat this process as often as we need, arriving in a finite number of steps at the element

$$t + \delta t^{p(pj+\lambda+i-j-1)+1} + \ldots$$

The result follows.
Corollary 6.3. $S$ is a hereditarily just infinite subgroup of $J$.

Proof. Simply combine Propositions 6.1, 6.2.

Remark. Notice that this result seems to display the characteristics that one would expect in attempting to prove a Theorem of this nature. It is generally harder to produce as a commutator elements of the form $t + t^{p+1} + \ldots$ than anything else. It was this problem that caused difficulties when people tried to prove the Nottingham Group is HJI for $p = 2$. Notice that when $p = 2$ we actually have that $J = S$. These difficulties have been overcome for $J$ in a recent article by Hegedus [H], although it is worth pointing out that the first proof that $J$ is HJI for $p = 2$ was communicated by Fesenko to Leedham-Green in 1999 and in fact the result follows from reasoning in [F1]. In some sense Fesenko’s calculations do the hard work for us in this case, and it is merely necessary to piece all the various fragments together as we have done here.

It was only to ease difficulties of notation, and to keep the calculations as simple as possible that we took $B = p\mathbb{N} \cup p\mathbb{N} - 1$. The same calculations will lead with no great complications to a proof that $S_{m,n} := J(B_{m,n})$ is hereditarily just infinite, for $B_{m,n} := p^n\mathbb{N} \cup p^m\mathbb{N} - 1$. However in the light of the method of proof, this reasoning is only valid for odd primes as we do not as yet have a full even characteristic equivalent to Theorem 4.4.

Lastly note that the group $S$ has the property that every open subgroup has non-trivial torsion. Thus the group $S$ remains as a potential candidate to be the Galois group of a just infinite unramified field extension of $\mathbb{Q}$. See [B] for more details.

Appendix

Since the completion of this work, a preprint of Barnea and Klösch [BK] has appeared which also considers various properties of index subgroups in $J$. The main results contained in this paper are summarised in this

Theorem 7 [BK]. Let $S_{r,s} := J(p^r\mathbb{N} \cup p^s\mathbb{N} - 1)$. Then

1. $S_{r,s}$ is hereditarily just infinite;
2. $S_{r,r}$ is of finite width and infinite obliquity;
3. The width of $S_{r,r}$ is $\leq p + p^r - 1$;
4. The groups $S_{r,r}$ are pairwise non-commensurable.

They also consider implications for the Hausdorff Spectrum of the Nottingham Group, and indeed calculate a large part of this spectrum.
To prove results about the normal subgroup structure of a substitution group of formal power series, the only real possible approach to this problem is to evaluate commutators. In [BK] the authors adopt a different method to do this than the one contained here, and it is reassuring to see that our conclusions are the same.

In the light of [BK] a major problem to tackle would still seem to be to calculate the width of the Fesenko Group $T$. It does not appear possible to use the calculations of Barnea and Klopsch for the width of $S_{r,r}$ to do this. Indeed it is still far from clear to me whether or not $T$ will have finite width. Fesenko believes that $T$ will have finite width based on some lengthy preliminary calculations he performed several years ago but he emphasizes this has still to be confirmed. However that $S$ has infinite obliquity would certainly suggest that $T$ will do too.

**Bibliography**

[B] N Boston, *Some cases of the Fontaine-Mazur Conjecture. II*, Alg. Number Th. Arch. **129** (1998).

[BK] Y Barnea, B Klopsch, *Index subgroups of the Nottingham Group*, Preprint, (2002).

[C1] R Camina, *Subgroups of the Nottingham Group*, J. Algebra **196** (1997), 101–113.

[C2] R Camina, *The Nottingham Group*, New Horizons in Pro-$p$ Groups, Birkhauser Publ., 2000.

[CG] J Coates, R Greenberg, *Kummer theory for the abelian varieties over local fields*, Invent. Math. **124** (1996), 129–174.

[duSF] M du Sautoy, I Fesenko, *Where the wild things are: ramification groups and the Nottingham Group*, New Horizons in Pro-$p$ Groups, Birkhauser Publ., 2000, 285–326.

[F1] I Fesenko, *On just infinite pro-$p$ groups and arithmetically profinite extensions of local fields*, J. reine angew. Math. **517** (1999), 61–80.

[F2] I Fesenko, *On deeply ramified extensions*, J. LMS (2) **57** (1998), 325–335.

[FV] I Fesenko, S Vostokov, *Local fields and their extensions*, AMS, Providence, R.I. 1993.

[H] P Hegedus, *The Nottingham group for $p = 2$*, preprint, 2001.

[J] S A Jennings, *Substitution groups of formal power series under substitution*, Canad. J. Math. **6** (1954), 325–340.

[Jo] D Johnson, *The group of formal power series under substitution*, Austral. Math. Soc. **45** (1988), 296–302.

[KLP] G Klaas, C R Leedham-Green, W Plesken, *Linear pro-$p$ groups of finite width*, preprint, Aachen-London, 1997.
[K] B Klopsch, *Substitution groups, subgroup growth and other topics*, D.Phil Thesis, Oxford, 1999.

[Lu] E Lucas, *Sur les congruences des nombres eulerians et des coefficients différentiels des fonctions trigonométriques, suivant un module premier*, Bull. Soc. Math. France 6 (1878), 49–54.

[Y] I York *The group of formal power series under substitution*, PhD Thesis, Nottingham, 1990.