Vector-Like Pairs and Brill–Noether Theory

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Abstract

How likely is it that there are particles in a vector-like pair of representations in low-energy spectrum, when neither symmetry nor anomaly consideration motivates their presence? We address this question in the context of supersymmetric and geometric phase compactification of F-theory and Heterotic dual. Quantisation of the number of generations (or net chiralities in more general term) is also discussed along the way. Self-dual nature of the fourth cohomology of Calabi–Yau fourfolds is essential for the latter issue, while we employ Brill–Noether theory to set upper bounds on the number \( \ell \) of vector-like pairs of chiral multiplets in the SU(5)_{GUT} \( (5+\bar{5}) \) representations. For typical topological choices of geometry for F-theory compactification for SU(5) unification, the range of \( 0 \leq \ell \lesssim 4 \) for perturbative unification is not in immediate conflict with what is already understood about F-theory compactification at this moment.
1 Introduction

“Who ordered that?” The Standard Model of particle physics contains three generations of quarks and leptons. Particle theorists have long been wondering what can be read out from the number of generations, $N_{\text{gen}} = 3$. If the Standard Model as a low-energy effective theory is obtained as a consequence of compactification of a high-energy theory in higher dimensional space-time, $N_{\text{gen}}$ is often determined by index theorem (or an equivalent topological formula) on some internal geometry. Historically, it was first considered to be $\chi(Z; T^*Z) = \chi(Z)_{\text{top}}/2$, the Euler characteristic of the cotangent bundle of a Calabi–Yau threefold $Z$, in a (2,2) compactification of Heterotic string theory [1]. Its generalisation in Heterotic string (0,2) compactifications is $\chi(Z; V)$, where $V$ is a vector bundle on $Z$. In Type IIB / F-theory language, $N_{\text{gen}}$ is given by $\chi(\Sigma; K^{1/2}_\Sigma \otimes \mathcal{L}) = c_1(\mathcal{L})$, where $\mathcal{L}$ is a line bundle on a holomorphic curve $\Sigma$ in a complex threefold $M_{\text{int}}$. In any one of those implementations, the fact that $N_{\text{gen}} = 3$ only means that one number characterising topology of compactification data happens to be 3.

Study of string phenomenology in the last three decades provides a dictionary of translation between the data of effective theory models and those for compactifications. An important question, then, is whether such a dictionary is useful. The former group of data have direct connection with experiments, while we need to be lucky to have an experimental access to the latter in a near future; this means that the dictionary may not be testable. Compactification data may still provide correlations/constraints through the dictionary among various pieces of information in the effective theory model data—that is the remaining hope. From this perspective, it is crucial which observable parameter constrains compactification data more. This letter shows, in section 2, that the value of $N_{\text{gen}}$ brings virtually no constraint on the topology of the curve $\Sigma$, threefold $M_{\text{int}}$ or $Z$; this is due to the self-dual nature of the middle dimensional cohomology group of Calabi–Yau fourfolds, in F-theory language. This is a good news for those who seek for existence proof of appropriate compactifications, and a bad news for those who seek for profound meaning in $N_{\text{gen}} = 3$.

In section 3, we focus on the number of matter fields in a vector-like pair of representations, as in the title of this article. It has often been adopted as a rule of game in bottom-up model building that vector-like pairs of matter fields are absent unless their mass terms are forbidden

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1Such a dictionary is important for those who ask whether string theory is able to reproduce the Standard Model at low-energy (if the answer is no, we should rule out string theory as the theory of quantum gravity!), even if the dictionary may not come with practical benefits (usefulness) in understanding the low-energy Lagrangian of this universe better.
by some symmetry. Papers from string phenomenology community, on the other hand, often end up with such vector-like pairs in low-energy spectrum; difficulty of removing them from the spectrum is reflected best in the heroic effort the U. Penn group had to undertake until they find a Heterotic compactification with just one pair of Higgs doublets. We will see, in section 3 that there is no reason to trust the bottom up principle based on the current understanding of F-theory/Heterotic string compactification; in the meanwhile, there is a good reason to believe (cf [5]) that generic vacua of F-theory compactification (and Heterotic dual) will predict smaller number of vector-like pairs than in papers (such as [2, 3]) that have been written. Brill–Noether theory sets upper bounds on the number of vector-like pairs \( \ell \) for a given genus \( g \) of a relevant curve \( \Sigma \); given the typical range \( \mathcal{O}(10) - \mathcal{O}(100) \) for \( g(\Sigma) \) for the matter fields in the SU(5)\(_{\text{GUT}}\)-(5 + \bar{5}) representations, the range of \( 0 \leq \ell \lesssim 4 \) for perturbative unification are not in immediate conflict with most of internal geometry for F-theory / Heterotic string compactifications.

Discussions in section 2 and section 3 are mutually almost independent. Despite many math jargons, logic of section 3 will be simple enough for non-experts to follow. Observations in both sections will have been known to stringpheno experts already to some extent (e.g. section 7 of [5]), but have not been written down as clearly and in simple terms as in this article, to the knowledge of the author. So, there will be a non-zero value in writing up an article like this.

Language of supersymmetric and geometric phase F-theory compactification is used in most of discussions in this article. Heterotic string compactification on elliptic fibred Calabi–Yau threefolds is also covered by the same discussion, due to the Heterotic–F-theory duality. It is worth noting that large fraction of Calabi–Yau threefolds admit elliptic fibration [6].

\(^2\)In this article, we are concerned about vector-like pairs in string compactification that are not associated in any way with symmetry or anomaly (and its flow). In compactifications that have an extra U(1) symmetry (which may be broken spontaneously or at non-perturbative level), low-energy spectrum tends to be richer, partially due to the 6D box anomaly cancellation of U(1) (cf [4]). This article is concerned about more conservative set-ups, where there may or may not be an extra U(1) symmetry; matter parity is enough for SUSY phenomenology.

\(^3\)M-theory compactification on \( G_2 \)-holonomy manifolds is not discussed here, because the author is not a big fan of it. It is difficult to obtain realistic flavour pattern in SU(5) GUT in that framework [7], and a solution to this problem has not been known so far. If SU(5) unification is not used as a motivation, however, almost all kinds of string vacua (including IIA, IIB, Type I and those in non-geometric phase) will be just as interesting.
2 Quantisation of the Number of Generations

Self-dual Lattice

Let $X$ be a compact real $2n$-dimensional oriented manifold. Combination of the Poincare duality and the universal coefficient theorem implies that the middle dimensional homology group $[H_n(X;\mathbb{Z})]_{\text{free}}$ forms a self-dual lattice\footnote{This is not necessarily an even lattice. In the 2nd homology group of del Pezzo surfaces, for example, exceptional curves have odd self-intersection numbers. The 4th homology group of the sextic complex fourfold $((6) \subset \mathbb{P}^5)$ is not an even lattice either; the signature of this lattice is $(1754, 852)$, where the difference $1754 - 852 = 902$ is not divisible by 8.} the intersection pairing matrix in

$$[H_n(X;\mathbb{Z})]_{\text{free}} \times [H_n(X;\mathbb{Z})]_{\text{free}} \rightarrow \mathbb{Z}$$

is symmetric and integer-valued, and its determinant is $\pm 1$.

There are several useful properties of self-dual lattices. Let $L$ be a self-dual lattice, $M$ a primitive non-degenerate sublattice of $L$, and $M' := [M^\perp \subset L]$ its orthogonal complement in $L$. The dual lattices of $M$ and $M'$ are denoted by $M^\vee$ and $(M')^\vee$, respectively. The intersection pairing of $L$ induces a homomorphism $L \rightarrow M^\vee$. When $L$ is self-dual, this homomorphism is surjective, and the kernel is $M'$. This homomorphism induces an isomorphism between $L/(M \oplus M')$ and a finite group $M^\vee/M$ (cf \[8\]).

F-theory Applications

Warming-up We begin with the simplest example imaginable. Consider using the sextic fourfold $X = (6) \subset \mathbb{P}^5$ for M-theory compactification. We have an effective theory of 2+1-dimensions then.

For a generic complex structure of $X$, algebraic two-cycles (real four-cycles) generate a rank-1 sublattice $M := \mathbb{Z}\langle H^2|_X \rangle$ of $L := H_4(X;\mathbb{Z})$; the generator\footnote{An element $x$ of a lattice is a primitive element, if its self-intersection $(x, x)$ is not divisible by the square of an integer. Thus, in particular, $H^2|_X$ is a primitive element in $H^2 \in H_4(X;\mathbb{Z})$.} is $H^2|_X$, where $H$ is the hyperplane divisor of $\mathbb{P}^5$, and $(H^2, H^2) = 6$. Let $M' := [M^\perp \subset L]$ be the orthogonal complement of $M$. Since the dimension of the primary horizontal and primary vertical components of $H^{2,2}(X;\mathbb{R})$—$h^{2,2}_H(X)$ and $h^{2,2}_V(X)$—add up to be $h^{2,2}(X)$ in this example, $M' \otimes \mathbb{R} \subset L \otimes \mathbb{R}$ corresponds to the primary horizontal component of $X = (6) \subset \mathbb{P}^4$. $M'$ must be a lattice of rank-$(b^4(X) - 1)$ whose intersection form is given by a matrix with the determinant 6. Due to the property of self-dual lattices stated earlier, $L/(M \oplus M') \cong M^\vee/M \cong \mathbb{Z}_6$; we can choose $(1/6) \times H^2|_X \mod M$ as a generator of $M^\vee/M$. 

\[\]
When a fourform is restricted within a class

\[ G = \left( \frac{15}{2} + n \right) H^2|_X \in \frac{c_2(TX)}{2} + M \quad \forall n \in \mathbb{Z}, \tag{2} \]

it is guaranteed to be purely of (2,2) Hodge component for any complex structure of the sextic fourfold. Its integral over the algebraic cycle \( H^2|_X \) can take a value in

\[ \int_{H^2|_X} G = \left( \frac{15}{2} + n \right) \times 6 = 45 + 6n \in 3 + 6\mathbb{Z}; \tag{3} \]

the value is quantised in units of 6, and cannot be 0, 1, 2, 4 or 5 modulo 6. When we allow the flux to be in \( G \in c_2(TX) \), however, the self-dual nature of the lattice \( L = H_4(X; \mathbb{Z}) \) indicates that the integral \( \int_{H^2|_X} G = (H^2|_X, G) \) can take any integer value. Such a flux \( G \) is not purely of (2,2) Hodge component in an arbitrary complex structure of \( X \), but the Gukov–Vafa–Witten superpotential drives the complex structure of \( X \) to an F-term minimum, where the (1,3) and (3,1) Hodge components of the flux \( G \) vanish (see also a comment later).

**SU(5) GUT models:** Let us consider F-theory compactification on a fourfold \( X_4 \) so that there is a stack of 7-branes along a divisor \( S \) in \( B_3 \). This means that there is an elliptic fibration \( \pi : X_4 \rightarrow B_3 \), there is a section \( \sigma : B_3 \rightarrow X_4 \), and \( X_4 \) has a locus of codimension-2 \( A_4 \) singularity in \( \pi^{-1}(S) \). Let \( \hat{X}_4 \) be a non-singular Calabi–Yau fourfold obtained by resolving singularities of \( X_4 \) (see [10, 16] for conditions to impose on \( \hat{X}_4 \)).

For concreteness of presentation, we choose the base threefold to be a \( \mathbb{P}^1 \)-fibration over \( \mathbb{P}^2 \),

\[ B_3 = \mathbb{P} \left[ \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-n) \right], \quad -3 < n < 3, \tag{5} \]

and the 7-brane locus \( S \) to be the zero locus of the fibre of \( \mathcal{O}_{\mathbb{P}^2} \). \( H^{1,1}(\hat{X}_4) \) is generated by \( \sigma, S, H_{\mathbb{P}^2} \) and \( E_{1,2,3,4} \), where \( H_{\mathbb{P}^2} \) is the hyperplane divisor of \( \mathbb{P}^2 \), and \( E_{1,2,3,4} \) are the Cartan divisors—the exceptional divisors emerging in the resolution of the \( A_4 \) singularity in \( X_4 \). The vertical component of \( H^{2,2}(\hat{X}_4) \) is of 9 dimensions. We choose the following cycles as a set of generators of the vertical component of \( H^{2,2}(\hat{X}_4) \): the first four are

\[ \sigma \cdot S, \quad \sigma \cdot H_{\mathbb{P}^2}, \quad S \cdot H_{\mathbb{P}^2}, \quad H_{\mathbb{P}^2} \cdot H_{\mathbb{P}^2}. \tag{6} \]

Gravitino mass (the (4,0) and (0,4) Hodge components) does not vanish automatically, though. This is a well-known problem of supersymmetric compactification of string theory. This article has nothing more to say about this. In this article, vanishing gravitino mass is not imposed, when we refer to supersymmetric compactifications.

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and the remaining five

\[ E_i \cdot H_{p2} \quad (i = 1, \cdots, 4) \quad \text{and} \quad E_2 \cdot E_4. \quad (7) \]

These 9 cycles generate a rank-9 sublattice \( M_{\text{vert}} \) of a self-dual lattice \( L = H_4(\hat{X}_4; \mathbb{Z}) \). The intersection form is given by

\[
\begin{pmatrix}
-n(3 + n) & -(n + 3) & n & 1 \\
-(n + 3) & 2 & 1 & 0 \\
n & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \oplus
\begin{pmatrix}
-2 & 1 & 1 & (3 + n) \\
1 & -2 & 1 & -(3 + n) \\
1 & -2 & 1 & -(3 + n) \\
(3 + n) & -(3 + n) & (3 + n) & -(3 + n) & -n(3 + n)
\end{pmatrix}
\]

in the basis of those 9 cycles; the determinant of this \( 9 \times 9 \) matrix is \( \text{discr}(M_{\text{vert}}) = (3 + n)(18 + n) \), which does not vanish in the range \(-3 < n < 3\) of our interest.

It is not obvious whether the lattice \( M_{\text{vert}} \) generated by the nine elements above is a primitive sublattice of \( L \); since \( L \) is not necessarily an even lattice, we have a limited set of tools to address this question. When it is not, however, we just have to replace the nine generators appropriately, so that \( M_{\text{vert}} \) becomes the primitive sublattice of \( L \). Arguments in the following needs to be modified accordingly, but not in an essential way. \( \text{discr}(M_{\text{vert}}) \) may not be the same as \( (3 + n)(18 + n) \) after the replacement, but the sublattice \( M_{\text{vert}} \) still remains non-degenerate.

Let \( M' \) be the orthogonal complement, \([M^\perp \subset L]\), in the lattice \( L \). In the examples considered here, \( M' \) corresponds to the horizontal components, \( M_{\text{horz}} \), because \( M \otimes \mathbb{Q} = M_{\text{vert}} \otimes \mathbb{Q} \) and the non-vertical non-horizontal component is empty [5]. The quotient

\[ L/(M \oplus M') \cong H_4(\hat{X}; \mathbb{Z})/(M_{\text{vert}} \oplus M_{\text{horz}}) \quad (9) \]

is a finite group isomorphic to \( M' / M = M'_{\text{vert}} / M_{\text{vert}} \).

For a flux \( G \) to preserve the SO(3,1) and SU(5) symmetry, it has to satisfy all of [11]

\[ (G, x) = 0 \quad \text{for} \quad x = \sigma \cdot S, \quad \sigma \cdot H_{p2}, \quad S \cdot H_{p2}, \quad H_{p2}^2, \quad E_i \cdot H_{p2} \quad (i = 1, 2, 3, 4). \quad (10) \]

When we choose a fourform flux \( G \) from \( c_2(T\hat{X}_4)/2 + M \), the conditions above leave

\[ G_{\text{FMW}} = \lambda_{\text{FMW}} (5E_2 \cdot E_4 + (3 + n)H_{p2} \cdot (2E_1 - E_2 + E_3 - 2E_4)), \quad \lambda_{\text{FMW}} \in \frac{1}{2} + \mathbb{Z}, \quad (11) \]

as the only possible choice. This flux is always of pure \((2,2)\) Hodge component for any complex structure of \( \hat{X}_4 \), and hence defines a supersymmetric vacuum. This is the flux constructed
in [12]; see [13, 14, 15, 16]. Within this class of choice of the fourform flux, the number of
generations is quantised as follows [17]:

\[ N_{\text{gen}} = (E_2 \cdot E_4, \ G) = \lambda_{\text{FMW}} (3 + n)(18 + n); \]  

(12)

although \(\lambda_{\text{FMW}}\) can change its value by \(\pm 1\), \(N_{\text{gen}}\) cannot change by \(\pm 1\). This would serve as
a tight constraint in search of a geometry with “right topology” for the real world; the value
of \(|\lambda_{\text{FMW}} (3 + n)(18 + n)|\) would never be as small as 3 for the choice of \((B_3, S)\) we made here.

In fact, we do not have to choose the flux from \(c_2(T\hat{X}_4)/2 + M\). The condition of [9]
does not rule out choice of flux from a broader class \(c_2(T\hat{X}_4)/2 + L\). Because of the self-dual
nature of \(L\), the homomorphism \(L^{-} \rightarrow M^\lor\) is surjective. This means that we can change
the flux by \(\Delta G \in L\) whose image in \(M^\lor\) is anything one likes. In particular, there exists
a change \(\Delta G \in L\) so that \((\Delta G, x) = 0\) for all the eight generators in \(\{10\}\), while \(N_{\text{gen}}\) is
changed by \((\Delta G, E_2 \cdot E_4) = \pm 1\). Therefore, the flux \(G\) can be chosen within \(c_2(T\hat{X}_4)/2 + L\)
so that \(N_{\text{gen}} = 3\), and the SO(3,1) and SU(5) symmetry is preserved. Certainly such a flux is
not purely of \((2,2)\) Hodge component for generic complex structur e of \(\hat{X}_4\), but the complex
structure of \(\hat{X}_4\) is driven to an F-term minimum of the Gukov–Vafa–Witten superpote ntial,
where the \((1,3) + (3,1)\) Hodge component of the flux is absent automatically, and the moduli
are stabilised (cautionary remark follows shortly, however).

To put it from a slightly different perspective, the surjectivity of the homomorphism \(L \rightarrow M^\lor\) means that we can choose the \(M^\lor \subset M \otimes \mathbb{Q}\) component of the flux in \(L \otimes \mathbb{Q}\) arbitrarily,
to suit the need from phenomenology (such as symmetry preservation and choosing \(N_{\text{gen}}\));
this is, in effect, to relax the condition \(\lambda_{\text{FMW}} \in (1/2) + \mathbb{Z}\) and allow the overall coefficient
(denoted \(\lambda\) instead of \(\lambda_{\text{FMW}}\)) to take any value in \([1/(3 + n)(18 + n)] \times \mathbb{Z}\). Once the \(M^\lor\) component is chosen, then one can always find some element in \((M')^\lor\) so that their sum fits
within \(L \subset (M^\lor \oplus (M')^\lor)\). Depending upon phenomenological input, such as \(N_{\text{gen}} = 3\), we
may not be able to choose the flux so that the \((M')^\lor\) component vanishes, but that is an
advantage rather than a problem, since complex structure moduli of \(\hat{X}_4\) tend to be stabilised
then.

One can see that the \(M^\lor\)-component of the flux, \((11)\) with a relaxed quantisation in \(\lambda\),
satisfies the primitiveness condition \(J \wedge G = (t_S S + t_{p_2} H_{p_2}) \cdot G = 0\), where \(J\) is the Kähler
form on \(B_3\). This is enough to conclude that the primitiveness condition is satisfied, because
the non-vertical component does not contribute to \(J \wedge G\).

A cautionary remark is in order here. First, the \((M')^\lor = M^\lor_{\text{horz}}\) component of the flux
\(G_{\text{horz}}\) needs to be chosen so that \((G_{\text{horz}})^2 > 0\), or otherwise there is no chance of finding a
supersymmetric vacuum. This condition is not hard to satisfy, because we can change $G_{\text{horz}}$ freely by $+M_{\text{horz}}$ without changing the value of $N_{\text{gen}}$ or breaking the SO(1,3) and SU(5) symmetry, and the lattice $M_{\text{horz}}$ is not negative definite. An open question is, for a given $[G_{\text{horz}}] \in M_{\text{horz}}^\vee / M_{\text{horz}}$, how one can find out whether there is a choice of Hodge structure of $\hat{X}_4$ so that there exists $G_{\text{horz}} \in M_{\text{horz}}^\vee$ with the vanishing negative component; note that a choice of Hodge structure introduces a decomposition of $M_{\text{horz}} \otimes \mathbb{R}$ into $(2h^{4,0} + h^{2,2}_H)$-dimensional positive definite directions and $2h^{3,1}$-dimensional negative definite directions. Due to the absence of a convenient Torelli theorem for general Calabi–Yau fourfolds, the author does not have a good idea how to address this problem.

**Generalisation:** The argument above can be used in set-ups where more phenomenological requirements are implemented. One can impose an extra U(1) symmetry (for spontaneous R-parity violation scenario instead of $\mathbb{Z}_2$ parity), and a flux for SU(5) → SU(3)$_C \times$ SU(2)$_L \times$ U(1)$_Y$ symmetry breaking can be introduced in the non-vertical non-horizontal component of $H_4(\hat{X}_4)$ \[19\]. One just has to take the lattice $M \subset L = H_4(\hat{X}_4; \mathbb{Z})$ so it contains all the cycles relevant to symmetry (symmetry breaking) and the net chiralities of various matter representations in the low-energy spectrum. The self-dual nature of $H_4(\hat{X}_4; \mathbb{Z})$ is the only essential ingredient in the argument above, and hence the same argument applies to more general cases.\[8\]

**Heterotic Dual**

The same story should hold true, when the argument above in F-theory language is translated into the language of Heterotic string. $N_{\text{gen}}$ can be chosen as we want it to be, by choosing the value of $\lambda_{\text{PMW}}$ characterising the vector bundle for Heterotic compactification not necessarily in $(1/2) + \mathbb{Z}$. Supersymmetry can still be preserved, presumably by choosing the complex structure of a Calabi–Yau threefold $Z$ and vector bundle moduli appropriately and introducing a threeform flux and non-Kählerity of the metric on $Z$. It is hard to verify this statement directly in Heterotic string language, but that must be true, if we believe that

\[7\] Even when such $G_{\text{horz}} \in M_{\text{horz}}^\vee$ and an appropriate Hodge structure is present, too large a positive value of $(G_{\text{horz}})^2$ would violate the D3-tadpole condition. So, this is another physics condition to be imposed.

\[8\] The algebraic cycles $S$ to be used in $\chi = \int_S G$ to determine net chiralities need to be primitive elements of the primitive sublattice $M \subset L$, for the argument to apply. If some cycle $S$ were an integer multiple of another topological cycle, $mS'$ for some $m \in \mathbb{Z}$, then the net chirality on $S$ is always divisible by $m$, no matter how we choose a flux. The Madrid quiver $[18]$—fractional D3-branes at $\mathbb{C}^3/\mathbb{Z}_3$ singularity—is the best known example of that kind. The matter curve is effectively the canonical divisor of the vanishing cycle $\mathbb{P}^2$ at $\mathbb{C}^3/\mathbb{Z}_3$. $-3H$ is not primitive.
there is one-to-one dual correspondence (even at the level of flux compactification) between elliptic fibred Calabi–Yau threefold compactification of Heterotic string and elliptic fibred K3-fibred Calabi–Yau fourfold compactification of F-theory.

3 Number of Vector-Like Pair Multiplets

We often encounter in supersymmetric string compactification with SU(5)$_{\text{GUT}}$ unification that there are multiple pairs of chiral multiplets in the SU(5)$_{\text{GUT}}$-$\bar{5} + 5$ representations left in the low-energy spectrum and no perturbation in moduli can provide large masses to those vector-like multiplets. A good example is the one in [2], where the low-energy spectrum has $34 + N'$ chiral multiplets in the $5$ representation and $34 + N' + N_{\text{gen}}$ of those in the $\bar{5}$ representation. The $N' > 0$ copies of chiral multiplets in the $5 + \bar{5}$ representations have $\Delta W = \phi \cdot \bar{5} \cdot 5$ coupling with moduli fields $\phi$, but 34 other vector-like pairs remain in the low-energy spectrum (at least without supersymmetry breaking) in the example studied in [2]. It is likely that those 34 vector-like pairs have nothing to do with some symmetry in the 4D effective theory.

Symmetry has been one of the most important guiding principles in bottom-up effective theory model building for more than three decades. It has often been assumed in model building papers that matter fields in a vector-like pair of representations are absent in low-energy spectrum, unless their mass terms are forbidden by some symmetry principle. Does the bottom-up guiding principle overlook something in string theory, or is there something yet to be understood in string phenomenology?

This guiding principle in bottom-up model building corresponds to the following statement in mathematics. Let us first note that the number of SU(5)$_{\text{GUT}}$-$5$ and $\bar{5}$ chiral multiplets

9 The author does not make a bet on whether the same statement applies to Heterotic string compactification on Calabi–Yau’s that do not admit an elliptic fibration morphism.

10 In the example studied in [2], there are $N_{\text{gen}} = 3$ chiral multiplets in the SU(5)$_{\text{GUT}}$-10 representation, while there is none in the $\underline{10}$ representation.

11 Translation for bottom-up model builders: roughly speaking, the holomorphic curve here is a real 2-dimensional submanifold within a real 6-dimensional internal space $M_{\text{int}}$. It corresponds, in Type IIB language, to intersection of a 7-brane and another 7-brane, each one of which is wrapped on a 4-dimensional submanifold of $M_{\text{int}}$ (and $\mathbb{R}^{3,1}$). The line bundle or flux here means gauge field configuration on the 4-dimensional submanifold. Even in large fraction of Heterotic string compactifications, the number of the vector-like pairs $\ell$ can be discussed essentially with the same language, due to the duality between Heterotic string and F-theory.
are given by
\[ h^0(\Sigma, \mathcal{O}(D)) \quad \text{and} \quad h^1(\Sigma, \mathcal{O}(D)), \]  
(13)
respectively, for some holomorphic curve \( \Sigma \) and a line bundle \( \mathcal{O}(D) \) on \( \Sigma \), quite often in supersymmetric and geometric phase compactifications of F-theory for SU(5) unification models \( [12, 17, 2, 20, 14, 15] \). We assume that the flux (i.e., \( \mathcal{O}(D) \)) is chosen to realise the appropriate net chirality (cf discussion in section 2)
\[ \chi := h^0(\Sigma, \mathcal{O}(D)) - h^1(\Sigma, \mathcal{O}(D)), \]  
(14)
which may be \(-N_{\text{gen}} = -3\) or 0, depending on whether or not the vector-like pairs are on the same complex curve as in the Standard Model \( 5 \)'s. The number of extra vector-like pairs is
\[ \ell := h^0(\Sigma, \mathcal{O}(D)). \]  
(15)
Now, it is known in mathematics \( [22] \) that \( \ell = 0 \) if

(a) the complex structure \( \tau \) of \( \Sigma \) is a generic element of the moduli space of the genus \( g \) curve \( \mathcal{M}_g \), and

(b) the flux configuration \( \mathcal{O}(D) \) is a generic element in \( \text{Pic}^{\chi+g-1}(\Sigma_g) \).

Thus, this general statement in math is in line with the bottom-up principle. The gap between the bottom-up guiding principle and the predictions of multiple vector-like pairs as in \( [2, 3] \) must be due to non-genericity of the complex structure of the holomorphic curve, of the flux configuration, or of both, in the math moduli space \( \mathcal{M}_g \) and \( \text{Pic}^{\chi+g-1}(\Sigma_g) \).

Most of papers for spectrum computation in F-theory or Heterotic string compactification so far employed the flux \( (11) \) or something similar. With more general type of flux configuration (as discussed in section 2), however, more general elements of \( \mathcal{O}(D) \in \text{Pic}^{\chi+g-1}(\Sigma_g) \) can be realised than, for example, in \( [2, 3] \). Smaller number of vector-like pairs may be predicted in F-theory and elliptic fibred Heterotic string compactifications then \( (5) \).

The question is how general \( \tau \in \mathcal{M}_g \) and \( \mathcal{O}(D) \in \text{Pic}^{\chi+g-1}(\Sigma) \) can be in such string compactifications. It is easy to see that the complex structure of the holomorphic curve \( \Sigma \) for the \( 5 + 5 \) matter cannot be fully generic. Let us take the example \( [5] \) for illustration purpose. The genus \( g \) of \( \Sigma \) is given by \( [21, 15] \)
\[ 2g - 2 = (3n + 24)(3n + 21) - 2(3 + n)(9 + n) = 7n^2 + 111n + 450, \]  
(16)
and the dimension of $\mathcal{M}_g$ is $3g - 3$. On the other hand, the defining equation of the curve $\Sigma$ involves
\[
\binom{5 + n}{2} + \binom{8 + n}{2} + \binom{11 + n}{2} + \binom{14 + n}{2} + \binom{21 + n}{2} - 9 = \frac{5n^2 + 113n + 770}{2}
\] (17)
complex parameters; the first five terms correspond to $h^0(\mathbb{P}^2; L)$ for line bundles $L = \mathcal{O}(3+n)$, $\mathcal{O}(6+n)$, $\mathcal{O}(9+n)$, $\mathcal{O}(12+n)$ and $\mathcal{O}(18+n)$; the last term accounts for the isometry of $\mathbb{P}^2$ and the overall scaling of the defining equation. The freedom (17) available for the complex structure of $\Sigma$ in F-theory compactification remains to be smaller than the $3g - 3$ dimensions of the moduli space $\mathcal{M}_g$, as long as $-3 \leq n$, which allows SU(5) GUT models. The condition (a) necessary for the general math statement $\ell = 0$ (and absence of vector-like pairs) is not satisfied in string compactifications. We will also find more direct evidence for this in footnote 14.

To summarise, predictions of multiple vector-like pairs in string compactifications, such as those in [2, 3], do not have to be taken at face value, because only purely vertical flux was considered in those works; more generic choice (that involves horizontal components) would predict smaller number of vector-like pairs. But, the bottom-up guiding principle does not have to be trusted too seriously either, because the holomorphic curve $\Sigma$ for SU(5)$_G$-$5 + \bar{5}$ matter fields is not expected to have a generic complex structure.

Brill–Noether theory [22] tells us a little more than the general math statement quoted above. Let $\Sigma$ be a genus $g$ curve and $\mathcal{O}(D)$ a line bundle on $\Sigma$ whose degree is $d = \chi + g - 1$. First of all,
\[
\ell = 0 \quad \text{if } d < 0.
\] (18)
When $0 \leq d \leq g - 1$, there are soft upper bound and hard upper bound. Clifford’s theorem provides the hard upper bound,
\[
\ell \leq \frac{d}{2} + 1 = \frac{\chi + g + 1}{2},
\] (19)
which holds for any complex structure of smooth curve $\Sigma$. When the complex structure of $\Sigma$ is not non-generic, there is a stronger upper bound\(^{14}\)
\[ \ell \leq \frac{\chi + \sqrt{\chi^2 + 4g}}{2}, \]  
(20)

because the Brill–Noether number $\rho := g - \ell(\ell - \chi)$ becomes negative for $\ell$ beyond this upper bound. Due to the Serre duality, it is enough to focus on the cases with $d \leq g - 1$.

In the case of SU(5)$_{GUT}$-$\overline{10}$ + 10 matter fields, string compactification often ends up with $g \leq -\chi = N_{\text{gen}} = 3$ (though not always), and hence the $d < 0$ case applies. The vector-like pair of $\overline{10} + 10$ is absent then. In the case of SU(5)$_{GUT}$-$5 + \overline{5}$ matter fields, however, $g$ often takes a much larger value (as in the example (16)), and hence the $\ell = 0$ result does not apply. Typical values of $g$ listed in Table 1 and 2 of [3] are in the range of $\mathcal{O}(10) - \mathcal{O}(100)$.

For such large values of $g$, $d = \chi + g - 1$ is close to $g - 1$ for $\chi = -N_{\text{gen}} = -3$ or $\chi = 0$. The upper bounds (19, 20) for those $g$ and $d$ have no conflict with vector-like pairs in the range of $0 \leq \ell \lesssim 4$ for perturbative gauge coupling unification\(^{15}\).

It requires much more dedicated study to go beyond. One could try to characterise what the physically realised subspace—one with the dimension given in (17)—in $\mathcal{M}_g$ would be like, or to work out the image of not necessarily purely vertical fluxes mapped into $\text{Pic}^{\chi+g-1}(\Sigma)$; the cautionary remark in page 6 also needs to be taken care of along the way. They are way too beyond the scope of this article, however. It is also worth studying how discussion in this article needs to be modified, when spontaneous R-parity violation scenario is at work (where an off-diagonal 4D scalar field breaking a U(1) symmetry to absorb a non-zero Fayet–Iliopoulos parameter (cf section 5 of [12] and [25, 26, 27, 28, 29, 30])).

\(^{14}\) This upper bound is not always satisfied (hence this is a soft upper bound), when the complex structure of $\Sigma$ is somewhat special. A good example is found in [3]. There, a flux is chosen as in (11), including the quantisation condition on $\lambda_{FMW}$, so that $\chi = -N_{\text{gen}} = -17$. In addition to this net chirality in the SU(5)$_{GUT}$-$5 + 5$ sector, non-removable $\ell = 11$ vector-like pairs are predicted in that example. In this case, $g = 174$, and hence $d = 156$. The hard upper bound $\ell \leq d/2 + 1 = 79$ is satisfied, but the stronger upper bound for $\Sigma$ with a generic complex structure, $\ell \leq 7.15$, is not satisfied. So, this computation is a direct evidence that the curve $\Sigma$ for the $5 + 5$ matter in F-theory does have a special complex structure (even after choosing the complex structure of $X_4$ completely generic). The dimension counting argument using (17) is not the only evidence for the non-genericity of $\tau \in \mathcal{M}_g$. It will be possible to carry out similar study for the examples in [29].

\(^{15}\) The $H^{2,1}$ moduli of F-theory compactification (and also presumably their Heterotic dual) do not receive large supersymmetric mass terms from the Gukov–Vafa–Witten superpotential, and are likely to change $\mathcal{O}(D) = K^{1/2}_\Sigma \otimes \mathcal{L}$ in Pic$^{\chi+g-1}(\Sigma)$. So they are good candidates of a singlet field $S$ that have a coupling $\Delta W = S \cdot 5 \cdot \overline{5}$; some of the $H^{3,1}$ moduli may also remain unstabilized supersymmetrically (i.e., in the low-energy spectrum) and play the same role. There is nothing new in that observation, but there will be some value to leave such a footnote in this article as a reminder.
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