SOME IRREDUCIBLE REPRESENTATIONS OF THE BRAID GROUP $\mathbb{B}_n$ OF DIMENSION GREATER THAN $n$

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Abstract. For any $n \geq 3$, we construct a family of finite dimensional irreducible representations of the braid group $\mathbb{B}_n$. Moreover, we give necessary conditions for a member of this family to be irreducible. In particular we give a explicitly irreducible subfamily $(\phi_m, V_m)$, $1 \leq m < n$, where $\dim V_m = \left( \begin{array}{c} n \\ m \end{array} \right)$. The representation obtained in the case $m = 1$ is equivalent to the standard representation.

1. Introduction

The braid group of $n$ strings $\mathbb{B}_n$, is defined by generators and relations as follows

$$\mathbb{B}_n = \langle \tau_1, \ldots, \tau_{n-1} \rangle/\sim$$

$$\sim = \{ \tau_k \tau_j = \tau_j \tau_k, \text{ if } |k - j| > 1; \tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1} \ 1 \leq k \leq n - 2 \}$$

We will consider finite dimensional complex representations of $\mathbb{B}_n$; that is pairs $(\phi, V)$ where

$$\phi : \mathbb{B}_n \to \text{Aut}(V)$$

is a morphism of groups and $V$ is a complex vector space of finite dimension.

In this paper, we will construct a family of finite dimensional complex representations of $\mathbb{B}_n$ that contains the standard representations. Moreover, we will give necessary conditions for a member of this family to be irreducible. In this way, we can find explicit families of irreducible representations. In particular, we will define a subfamily of irreducible representations $(\phi_m, V_m)$, $1 \leq m < n$, where $\dim V_m = \left( \begin{array}{c} n \\ m \end{array} \right)$ and the corank of $\phi_m$ is equal to $2(n-2)! / [m-1] [(n-m-1)]!$.

This family of representations can be useful in the progress of classification of the irreducible representations of $\mathbb{B}_n$. As long as we known, there are only few contributions in this sense, some known results are the following ones. Formanek classified all the irreducible representations of $\mathbb{B}_n$ of dimension lower than $n$ [2]. Sysoeva did it for dimension equal to $n$ [5]. Larsen and Rowell gave some results for unitary representation of $\mathbb{B}_n$ of dimension multiples of $n$. In particular, they prove there are not irreducible representations of dimension $n+1$. Levaillant proved when the Lawrence-Krammer representation is irreducible and when it is reducible [4].

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2. Construction and Principal Theorems

In this section, we will construct a family of representations of $\mathbb{B}_n$ that we believe to be new, and we will obtain a subfamily of irreducible representations. We choose $n$ non negative integers $z_1, z_2, \ldots, z_n$, not necessarily different. Let $X$ be the set of all the possible $n$-tuples obtained by permutation of the coordinates of the fixed $n$-tuple $(z_1, z_2, \ldots, z_n)$. For example, if the $z_i$ are all different, then the cardinality of $X$ is $n!$. Explicitly, if $n = 3$,

$$X = \{(z_1, z_2, z_3), (z_1, z_3, z_2), (z_2, z_1, z_3), (z_2, z_3, z_1), (z_3, z_1, z_2), (z_3, z_2, z_1)\}$$

Or if $z_1 = z_2 = 1$ and $z_i = 0$ for all $i = 3, \ldots, n$, then the cardinality of $X$ is $\binom{n}{2} = \frac{n(n-1)}{2}$. Explicitly, for $n = 3$

$$X = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

Let $V$ be a complex vector space with orthonormal basis $\beta = \{v_x : x \in X\}$. Then the dimension of $V$ is the cardinality of $X$.

We define $\phi : \mathbb{B}_n \to \text{Aut}(V)$, such that

$$\phi(\tau_k)(v_x) = q_{x_k,x_{k+1}} v_{\sigma_k(x)}$$

where $q_{x_k,x_{k+1}}$ is a non-zero complex number that depends on $x = (x_1, \ldots, x_n)$, but, it only depends on the places $k$ and $k+1$ of $x$; and

$$\sigma_k(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_n)$$

With this notations, we have the following theorem,

**Theorem 2.1.** $(\phi, V)$ is a representation of the braid group $\mathbb{B}_n$.

**Proof.** We need to check that $\phi(\tau_k)$ satisfy the relations of the braid group. We have for $j \neq k, k, k+1$ that

$$\phi(\tau_k)\phi(\tau_j)(v_x) = \phi(\tau_k)(q_{x_j,x_{j+1}} v_{\sigma_j(x)}) = q_{x_j,x_{j+1}} q_{x_k,x_{k+1}} v_{\sigma_k \sigma_j(x)}$$

On the other hand

$$\phi(\tau_j)\phi(\tau_k)(v_x) = \phi(\tau_j)(q_{x_k,x_{k+1}} v_{\sigma_k(x)}) = q_{x_k,x_{k+1}} q_{x_j,x_{j+1}} v_{\sigma_j \sigma_k(x)}$$

As $\sigma_k \sigma_j(x) = \sigma_j \sigma_k(x)$, if $|j - k| > 1$, then $\phi(\tau_k)\phi(\tau_j) = \phi(\tau_k)\phi(\tau_j)$ if $|j - k| > 1$.

In the same way, we have

$$\phi(\tau_k)\phi(\tau_{k+1})\phi(\tau_k)(v_x) = \phi(\tau_k)\phi(\tau_{k+1}) (q_{x_k,x_{k+1}} v_{\sigma_k(x)})$$

$$= \phi(\tau_k) (q_{x_k,x_{k+1},x_{k+2}} v_{\sigma_{k+1} \sigma_k(x)})$$

$$= q_{x_k,x_{k+1},x_{k+2}} q_{x_k,x_{k+1}} v_{\sigma_{k+1} \sigma_k(x)}$$

Similarly,

$$\phi(\tau_{k+1})\phi(\tau_k)\phi(\tau_{k+1})(v_x) = \phi(\tau_{k+1})\phi(\tau_k) (q_{x_{k+1},x_{k+2}} v_{\sigma_{k+1} \sigma_k(x)})$$

$$= \phi(\tau_{k+1}) (q_{x_{k+1},x_{k+2},x_{k+3}} v_{\sigma_{k+2} \sigma_{k+1} \sigma_k(x)})$$

$$= q_{x_{k+1},x_{k+2},x_{k+3}} q_{x_{k+1},x_{k+2}} q_{x_k,x_{k+1}} v_{\sigma_{k+2} \sigma_{k+1} \sigma_k(x)}$$

As $\sigma_k \sigma_{k+1} \sigma_k(x) = \sigma_{k+1} \sigma_k \sigma_{k+1}(x)$, for all $k$ and $x \in X$, then $\phi(\tau_k)\phi(\tau_{k+1})\phi(\tau_k) = \phi(\tau_{k+1})\phi(\tau_k)\phi(\tau_{k+1})$ for all $k$. □
As $\beta$ is an orthonormal basis, we have that,

$$\langle \phi(\tau_k)v_y, v_x \rangle = \langle q_{y_k, y_{k+1}} v_{\sigma_k(y)}, v_x \rangle = \langle v_y, \overline{q_{x_{k+1}, x_k}} v_{\sigma_k(x)} \rangle$$

then,

$$(\phi(\tau_k))^*(v_x) = \overline{q_{x_{k+1}, x_k}} v_{\sigma_k(x)}$$

therefore, $\phi(\tau_k)$ is self-adjoint if and only if $q_{x_{k+1}, x_k} = \overline{q_{y_{k+1}, y_k}}$ for all $x \in X$. In particular, if $x_k = x_{k+1}$ then $q_{x_k, x_{k+1}}$ is a real number. In the same way, $\phi(\tau_k)$ is unitary if and only if $|q_{x_k, x_{k+1}}|^2 = 1$ for all $x \in X$.

Now, we will give a subfamily of irreducible representations.

**Theorem 2.2.** If $\phi(\tau_k)$ is a self-adjoint operator for all $k$, and for any pair $x, y \in X$, there exists $j, 1 \leq j \leq n - 1$, such that $|q_{x_j, x_{j+1}}|^2 \neq |q_{y_j, y_{j+1}}|^2$, then $\phi(V)$ is an irreducible representation of the braid group $\mathbb{B}_n$.

**Proof.** Let $W \subset V$ be a non-zero invariant subspace. It is enough to prove that $W$ contains one of the basis vectors $v_x$. Indeed, given $y \in X$, there exists a permutation $\sigma$ of the coordinates of $x$, that sends $x$ to $y$. This happens because the elements of $X$ are $n$-tuples obtained by permutation of the coordinates of the fixed $n$-tuple $(z_1, \ldots, z_n)$. Suppose that $\sigma = \sigma_1 \ldots \sigma_i$, then $\tau := \tau_1 \ldots \tau_i$ satisfies that $\phi(\tau)(v_x) = \lambda v_y$, for some non-zero complex number $\lambda$. Then $W$ contains $v_y$ and therefore, $W$ contains the basis $\beta = \{v_x : x \in X\}$.

As $\phi(\tau_k)$ is a self-adjoint operator, it commutes with $P_W$, the orthogonal projection over the subspace $W$. Therefore, $(\phi(\tau_k))^2$ commute with $P_W$. On the other hand, note that $(\phi(\tau_k))^2(v_x) = |q_{x_{x_{k}, x_{k+1}}}|^2 v_x$, hence, $(\phi(\tau_k))^2$ is diagonal in the basis $\beta = \{v_x : x \in X\}$. Then, the matrix of $P_W$ has at least the same blocks than $(\phi(\tau_k))^2$ for all $k, 1 \leq k \leq n - 1$.

If for some $k$, the matrix of $(\phi(\tau_k))^2$ has one block of size $1 \times 1$, then the matrix of $P_W$ has one block of size $1 \times 1$. In other words, there exists $x \in X$ such that $v_x$ is an eigenvector. If the eigenvalue associated to $v_x$ is non-zero, then $v_x \in W$.

It rest to see that the matrix of $(\phi(\tau_k))^2$ has all its blocks of size $1 \times 1$. By hypothesis, for each pair of vectors in the basis $\beta$, $v_x$ and $v_y$, there exists $k, 1 \leq k \leq n - 1$, such that $|q_{x_{x_{k}, x_{k+1}}}|^2 \neq |q_{y_{y_{k+1}, y_k}}|^2$. Fix any order in $X$ and let $x$ and $y$ the first and second element of $X$. Then there exists $k$ such that $v_x$ and $v_y$ are eigenvectors of $(\phi(\tau_k))^2$ of different eigenvalue. Hence $(\phi(\tau_k))^2$ has the first block of size $1 \times 1$. As $(\phi(\tau_j))^2$ commute with $(\phi(\tau_k))^2$ for all $j$, $(\phi(\tau_j))^2$ also has this property.

By induction, suppose that for all $j$ $(\phi(\tau_j))^2$ has its $r - 1$ first blocks of size $1 \times 1$. Let $x'$, $y'$ the elements $r$ and $r + 1$ of $X$, then there exists $k'$ such that $v_{x'}$ and $v_{y'}$ are eigenvectors of $(\phi(\tau_{k'}))^2$ of different eigenvalue. Hence, $(\phi(\tau_{k'}))^2$ has the $r$ block of size $1 \times 1$. Therefore $(\phi(\tau_{k}))^2$ too because it commute with $(\phi(\tau_{k'}))^2$, for all $j$. Then we obtain that all the blocks are of size $1 \times 1$.

Note that if the numbers $q_{x_{x_{k}, x_{k+1}}}$ are all equal and $|X| > 1$, then $\phi$ is not irreducible because the subspace $W$, generated by the vector $v = \sum_{x \in X} v_x$, is an invariant subspace.

2.1. **Examples.** We are going to compute some explicit examples of this family of representations. We will show that the standard representation $(\mathbb{H}, \mathbb{H})$ is a member of this family.
2.1.1. Standard Representation. Let $z_1 = 1$ and $z_j = 0$ for all $j = 2, \ldots, n$. Then the cardinality of $X$ is $n$ and $\dim V = n$ too. For each $x \in X$, let $q_{x_k, x_{k+1}} = 1 + (t - 1)x_{k+1}$, where $t \neq 0, 1$ is a complex number. Therefore $\phi : B_n \to \text{Aut}(V)$, given by $\phi(\tau_k)v_x = q_{x_k, x_{k+1}}v_{\sigma_k(x)}$, is equivalent to the standard representation $\rho$, given by

$$
\rho(\tau_k) = \begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & 0 & t \\
& & & 1 & 0 \\
& & & & \ddots \\
& & & & & 1 \\
\end{pmatrix}
$$

where $t$ is in the place $(k, k+1)$. In fact, if $\{\beta_j : j = 1, \ldots, n\}$ is the canonical basis of $\mathbb{C}^n$, and if $x_j$ is the element of $X$ with 1 in the place $j$ and zero elsewhere, define

$$
\alpha : \mathbb{C}^n \to V \\
\beta_j \mapsto v_{x_j}
$$

Then $\alpha(\rho(\tau_k)(\beta_j)) = \phi(\tau_k)(\alpha(\beta_j))$ for all $j = 1, \ldots, n$. Hence the representations are equivalent.

2.1.2. Example. Let $z_1, \ldots, z_n \in \{0, 1\}$, such that $z_1 = z_2 = \cdots = z_m = 1$ and $z_m+1 = \cdots = z_n = 0$. Then the cardinality of $X$ is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. If $V_m$ is the vector space with basis $\beta_m = \{v_x : x \in X\}$, then $\dim V_m = \frac{n!}{m!(n-m)!}$.

For each $x := (x_1, \ldots, x_n) \in X$, let

$$
q_{x_k, x_{k+1}} = \begin{cases}
1 & \text{if } x_k = x_{k+1} \\
t & \text{if } x_k \neq x_{k+1}
\end{cases}
$$

where $t$ is a real number, $t \neq 0, 1, -1$.

We define $\phi_m : B_n \to \text{Aut}(V_m)$, given by

$$
\phi_m(\tau_k)v_x = q_{x_k, x_{k+1}}v_{\sigma_k(x)}
$$

For example, fixing the lexicographic order in $X$, if $n = 5$ and $m = 3$, then $\dim V_m = 10$, the ordered basis is

$$
\beta := \{v((0,0,1,1,1), v(0,1,0,1,1), v(0,1,1,1,0), v(0,1,1,0,1), v(1,0,0,1,1)), \\
v(1,0,1,0,1), v(1,0,1,1,0), v(1,1,0,0,1), v(1,1,0,1,0), v(1,1,1,0,0)\}
$$

and the matrices in this basis are

$$
\phi_3(\tau_1) = \begin{pmatrix}
1 & & & & \\
0 & t & & & \\
0 & 0 & 0 & t & \\
0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 \\
& t & 0 & 0 & 0 \\
& & t & 0 & 0 \\
& & & 1 & 1 \\
\end{pmatrix}
$$
Theorem 2.3. Let \( x \neq y \in X \), then there exists \( j, 1 \leq j \leq n \), such that \( x_j \neq y_j \). If \( j > 1 \), we may suppose that \( x_{j-1} = y_{j-1} \), then \( q_{x_{j-1},x_j} \neq q_{y_{j-1},y_j} \), therefore \( |q_{x_{j-1},x_j}|^2 \neq |q_{y_{j-1},y_j}|^2 \). If \( j = 1 \), and \( n \neq 2m \), there exists \( l = 2, \ldots, n \) such that \( x_{l-1} \neq y_{l-1} \) and \( x_l = y_l \), then \( |q_{x_{l-1},x_l}|^2 \neq |q_{y_{l-1},y_l}|^2 \). Then, by theorem 2.2, \( \phi_m \) is an irreducible representation.

Proof. We analyze two cases, \( n \neq 2m \) and \( n = 2m \). Suppose that \( n \neq 2m \). Let \( x \neq y \in X \), then there exists \( j, 1 \leq j \leq n \), such that \( x_j \neq y_j \). If \( j > 1 \), we may suppose that \( x_{j-1} = y_{j-1} \), then \( q_{x_{j-1},x_j} \neq q_{y_{j-1},y_j} \), therefore \( |q_{x_{j-1},x_j}|^2 \neq |q_{y_{j-1},y_j}|^2 \). If \( j = 1 \), and \( n \neq 2m \), there exists \( l = 2, \ldots, n \) such that \( x_{l-1} \neq y_{l-1} \) and \( x_l = y_l \), then \( |q_{x_{l-1},x_l}|^2 \neq |q_{y_{l-1},y_l}|^2 \). Then, by theorem 2.2, \( \phi_m \) is an irreducible representation.

Note that if \( n = 2m \), \( x_0 = (1,\ldots,1,0,\ldots,0) \) and \( y_0 = (0,\ldots,0,1,\ldots,1) \) satisfy \( x_0 \neq y_0 \) but \( q_{x_{j-1},x_j} = q_{y_{j-1},y_j} \) for all \( j \). So, we can not use theorem 2.2. But in the proof of the theorem, we really use that \( x \) and \( y \) are consecutive in some order. Considering the lexicographic order, \( x_0 \) and \( y_0 \) are not consecutive. In general, for each \( x \in X \), there exists \( y_x \in X \) such that \( q_{x_j,x_{j+1}} = q_{y_j,y_{j+1}} \) for all \( j = 1, \ldots, n-1 \). We define \( y_x \) changed in \( x \) the zeros by ones and the ones by zeros. For example, if \( x = (1,0,0,1,0,1) \), then \( y_x = (0,1,1,0,1,0) \). However, only \( x = (0,1,\ldots,1,0,\ldots,0) \) satisfies that \( y_x \) is consecutive to \( x \). Therefore \( P_W \), the
Theorem 2.4. Let $\tau_k$ be a representation of dimension $k$, $1 \leq k \leq n$. The number does not depend on $\tau_k$. Therefore, $\dim V_m = \binom{n}{m}$ is the cardinality of $X$, then

$$\dim V_m = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

We compute the corank of $\phi_m$. Let $x \in X$ such that $\sigma_k(x) = x$, then $x_k = x_{k+1}$ and $q_{k,k+1} = 1$. Therefore $\phi_m(\tau_k)(v_x) = v_x$. Hence the corank of $\phi_m$ is equal to the cardinality of $Y = \{x \in X : \sigma_k(x) \neq x\}$. But it is equal to the cardinality of $X$ minus the cardinality of $\{x \in X : x_k = x_{k+1} = 0 \text{ or } x_k = x_{k+1} = 1\}$. Therefore

$$\text{cork}(\phi_m) = rk(\phi_m(\tau_k) - 1) = \frac{n!}{m!(n-m)!} - \frac{(n-2)!}{m!(n-m-2)!} - \frac{(n-2)!}{(m-2)!(n-m)!}$$

$$= \frac{2(n-2)!}{(m-1)!(n-m-1)!}$$

In the example $n = 5$ and $m = 3$, we have that $\text{cork}(\phi_m) = 6$.

Note that if $m = 1$, the dimension of $\phi_m$ is $n$ and the corank is 2. Therefore $\phi_1$ is equivalent to the standard representation, because this is the unique irreducible representations of $\mathbb{B}_n$ of dimension $n$ [5].

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