Bosonic symmetry protected topological phases with reflection symmetry

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We study two-dimensional bosonic symmetry protected topological (SPT) phases which are protected by reflection symmetry and local symmetry \([Z_N \times R, Z_N \times R, U(1) \times R, \text{or } U(1) \times R]\), in the search for two-dimensional bosonic analogs of topological crystalline insulators in integer-S spin systems with reflection and spin-rotation symmetries. To classify them, we employ a Chern-Simons approach and examine the stability of edge states against perturbations that preserve the assumed symmetries. We find that SPT phases protected by \(Z_N \times R\) symmetry are classified as \(Z_2 \times Z_2\) for even \(N\) and \(0\) (no SPT phase) for odd \(N\) while those protected by \(U(1) \times R\) symmetry are \(Z_2\). We point out that the two-dimensional Affleck-Kennedy-Lieb-Tasaki state of \(S = 2\) spins on the square lattice is a \(Z_2\) SPT phase protected by reflection and \(\pi\)-rotation symmetries.

I. INTRODUCTION

Topological insulators and superconductors are gapped phases of noninteracting fermions in which the ground state wave functions have topologically nontrivial structures in the presence of certain symmetry constraints. The topological structures in the bulk wave functions guarantee the presence of gapless excitations at the boundary which are robust against any perturbation respecting the symmetry constraints. These gapless excitations lead to novel transport properties. For example, time-reversal-invariant topological insulators in three dimensions are characterized by \(Z_2\) topological invariants in the bulk.1,2 Correspondingly, stable gapless Dirac fermions emerge at the surface, which lead to the topological magnetoelectric effect.3–5 Topological insulators and superconductors of noninteracting fermions have been classified, in terms of the time-reversal, particle-hole, and chiral symmetries, into the “periodic table.”6–8

Interacting bosons can also host gapped phases with gapless boundary excitations that are stable against any perturbation as far as certain symmetries are preserved. These phases are dubbed bosonic symmetry protected topological (SPT) phases. One of representative examples in one dimension is the Haldane phase in the spin-1 antiferromagnetic Heisenberg chain.9,10 At each end of the Haldane spin chain emerge zero-energy spin-

\(\frac{1}{2}\) degrees of freedom which are protected by \(\pi\)-rotation about two orthogonal axes in the spin space or by the time-reversal symmetry.11 Bosonic SPT phases in two dimensions are also studied in two-component boson gas with \(U(1)\) symmetry.12–15 The SPT phases with local symmetries have been classified with various methods including matrix product state representation,16–19 group cohomology,18,20 Chern-Simons theory,21–24 nonlinear sigma models,25 and cobordisms.26–28

Recently, the concept of topological phases of noninteracting fermions has been further extended to topological crystalline insulators by including spatial symmetries.29 Experimental realization of topological crystalline insulator with reflection symmetry is achieved in SnTe.30–32

Surfaces of this material have an even number of Dirac cones, and the strong \(Z_2\) index for topological insulators is trivial. Thus, the time-reversal symmetry does not protect these Dirac cones. Instead, reflection symmetry is responsible for the symmetry protection of Dirac cones, allowing for a nontrivial mirror Chern number defined on the reflection invariant plane in the three-dimensional Brillouin zone. Now, a natural question we may ask is whether analogs of topological crystalline insulators of fermions exist in bosonic SPT phases. The bosonic SPT phases protected by spatial symmetries have not been fully explored so far, while there are several recent attempts in terms of their classification.24,25,33,34 In particular, few realistic models are known for such two-dimensional (2D) SPT phases with spatial symmetries.

Motivated by these, we explore topological crystalline insulators in interacting bosonic systems. Specifically, focusing on integer-spin systems, we study 2D bosonic SPT phases protected by reflection symmetry as well as spin-rotation symmetry. In order to classify SPT phases, we employ the Chern-Simons approach which is well suited for classifying Chern-Simons approach which is well suited for classifying 2D SPT phases with reflection symmetry. In this approach, SPT phases are classified by analyzing the stability of gapless edge modes against any perturbation allowed under given symmetry constraints. If all gapless modes are gapped out without symmetry breaking, the system is in a trivial phase; otherwise it is an SPT phase. We apply this method to integer-spin systems protected by reflection and discrete spin-rotation symmetry \((Z_N \times R)\). Our classification results show that SPT phases form \(Z_2 \times Z_2\) group for even \(N\), while no SPT phase is allowed for odd \(N\). Furthermore, the Affleck-Kennedy-Lieb-Tasaki (AKLT) state of \(S = 2\) spins on the square lattice is shown to be a bosonic SPT phase characterized by a \(Z_2\) invariant. This can be regarded as a topological crystalline insulator in spin systems.

The rest of this paper is organized as follows. Section III summarizes the classification scheme based on the Chern-Simons approach. Our main results are presented in Sec. III. We focus on SPT phases protected by \(Z_N \times R\) symmetry and discuss the AKLT state on the square lattice. Classification of 2D bosonic SPT phases
under other related symmetries \([Z_N \times R, U(1) \times R, \text{and} U(1) \times R]\) is discussed in Appendix.

II. A BRIEF REVIEW OF THE CHERN-SIMONS APPROACH

In this section, we briefly review the classification scheme of 2D bosonic SPT phases using the Chern-Simons approach. In this scheme, SPT phases are classified through stability analysis of gapless edge states. The group structure of SPT phases is obtained by studying the equivalence class of a direct sum of two phases.

A. Classification scheme

In this paper we consider a class of 2D bosonic SPT phases with nonchiral gapless edge modes. We assume that SPT phases have \(N_0\) pairs of helical edge states and that their bulk low-energy effective theory is given by the Chern-Simons action

\[
S_{\text{CS}} = \int dt dx \mathcal{L}_{\text{bulk}}^0, 
\]

\[
\mathcal{L}_{\text{bulk}}^0 = \frac{\epsilon_{\mu \nu \rho}}{4\pi} K_{I,J} a_{I\mu}(t, x) \partial_\nu a_{J\rho}(t, x), 
\]

where \(\epsilon_{\mu \nu \rho}\) is the totally anti-symmetric Levi-Civita tensor, \((\partial_\mu, \partial_\nu, \partial_\rho) = (\partial_t, \partial_{x_1}, \partial_{x_2})\), and summation is assumed over repeated indices \(\mu, \nu, \rho = 0, 1, 2\) and \(I, J = 1, \ldots, 2N_0\) (we adopt this convention throughout this paper). The Chern-Simons gauge fields \(a_{I\mu}\) describe low-energy dynamics in the gapped phase, and \(K\) is a symmetric matrix in \(GL(2N_0, \mathbb{Z})\) with \(\det K = (-1)^{N_0}\). In bosonic systems every diagonal element of \(K\)-matrices is an even integer.

The gapless boundary modes along the boundary (say, at \(x_2 = 0\)) of the SPT phases are described by the boundary action

\[
S_{\text{edge}} = \int dt dx_1 \mathcal{L}_{\text{edge}}^0, 
\]

\[
\mathcal{L}_{\text{edge}}^0 = \frac{1}{4\pi} \left[ K_{I,J} (\partial_1 \phi_I)(\partial_{x_1} \phi_J) - V_{I,J} (\partial_{x_1} \phi_I)(\partial_{x_1} \phi_J) \right],
\]

where \(\phi_I\) (\(I = 1, \ldots, 2N_0\)) are scalar bosonic fields satisfying the equal-time commutation relation

\[
[\phi_I(t, x_1), \partial_{x_1} \phi_J(t, x_1')] = 2\pi i (K^{-1})_{I,J} \delta(x_1 - x_1').
\]

The matrix \(V_{I,J}\) in Eq. (2b) is a nonuniversal positive definite matrix. The operator \(\partial_{x_1} \phi_I\) and the vertex operator :e^{i\phi_I}: are the density and the creation operators of excitations in the \(I\)th edge mode. Here colons denote normal ordering.

We assume that the boundary action \(S_{\text{edge}}\) and the bulk action \(S_{\text{CS}}\) are invariant under a symmetry group \(G\). For the boundary action this means that

\[
\mathcal{G} S_{\text{edge}} \mathcal{G}^{-1} = S_{\text{edge}},
\]

for any \(\mathcal{G} \in G\) which induces linear transformation of the bosonic fields \(\phi_I\)

\[
\mathcal{G} \phi \mathcal{G}^{-1} = U_{\mathcal{G}} \phi + \delta \phi_{\mathcal{G}},
\]

where \(\delta \phi_{\mathcal{G}}\) is a constant \(2N_0\)-dimensional vector with \((\delta \phi_{\mathcal{G}})_I \in [0, 2\pi)\) \((I = 1, \ldots, 2N_0)\) and the matrix \(U_{\mathcal{G}} \in GL(2N_0, \mathbb{Z})\). The transformation [5] for reflection and other local transformations will be discussed in the following section.

A ground state with gapped excitations in the bulk is in an SPT phase protected by a symmetry group \(G\), if it has gapless edge states that cannot be gapped without symmetry breaking by any perturbation allowed by the symmetry \(G\). Thus 2D SPT phases are classified by examining the stability of their edge states against perturbations of the form

\[
\mathcal{L}_{\text{edge}}^{\text{int}} = \sum_{j=1}^{N_0} C_j : \cos(I_j \cdot \phi + \alpha_j) : ,
\]

where \(C_j\) and \(\alpha_j\) are real constants, and \(\{I_1, \ldots, I_{N_0}\}\) is a set of linearly independent vectors from \(\mathbb{Z}^{2N_0}\). The cosine terms in Eq. (6) are normal-ordered as indicated by the colons. However, we will omit colons for normal-ordered vertex operators in the rest of this paper to simplify the notation.

The perturbations in \(\mathcal{L}_{\text{edge}}^{\text{int}}\) are assumed to fulfill the following conditions. First, any pair of vectors, \((I_j, I_k)\), from the set \(\{I_1, \ldots, I_{N_0}\}\) satisfies the Haldane’s null vector condition [22]

\[
I_j^T K^{-1} I_k = 0 \quad (j, k = 1, \ldots, N_0),
\]

so that the linearly independent combinations of the bosonic fields \(I_j \cdot \phi\) (\(j = 1, \ldots, N_0\)) can be simultaneously pinned by the cosine potentials in Eq. (6).

Second, \(\mathcal{L}_{\text{edge}}^{\text{int}}\) must be invariant under the symmetry group \(G\),

\[
\mathcal{G} \mathcal{L}_{\text{edge}}^{\text{int}} \mathcal{G}^{-1} = \mathcal{L}_{\text{edge}}^{\text{int}},
\]

Since \(\mathcal{G}(I_j \cdot \phi) \mathcal{G}^{-1} = I_j \cdot (U_{\mathcal{G}} \phi + \delta \phi_{\mathcal{G}})\), the invariance of \(\mathcal{L}_{\text{edge}}^{\text{int}}\) imposes the condition

\[
I_j^T K^{-1} U_{\mathcal{G}}^T I_k = 0 \quad (j, k = 1, \ldots, N_0).
\]

The third condition is concerned with the absence of spontaneous symmetry breaking. The symmetry \(G\) can be spontaneously broken in the ground state when \(N_0\) linearly independent fields \(I_j \cdot \phi\) (\(j = 1, \ldots, N_0\)) are pinned, even if the interactions in \(\mathcal{L}_{\text{edge}}^{\text{int}}\) respect the symmetry. To see this, let us define from the vectors \(\{I_1, \ldots, I_{N_0}\}\) a set of vectors

\[
L := \left\{ l \mid l = \sum_{n=1}^{N_0} j_n l_n, \; j_n \in \mathbb{Z} \; (n = 1, \ldots, N_0) \right\}.
\]
Any field $l \cdot \phi$ $(l \in L)$ takes a constant expectation value in the ground state. We then define from $L$ another set of vectors
\begin{equation}
\tilde{L} := \left\{ l \mid l = \frac{l}{\gcd(l_1, \ldots, l_{2N_0})}, l = (l_1, \ldots, l_{2N_0}) \in L \right\},
\end{equation}
where $\gcd$ denotes the greatest common divisor of the integers in the parentheses. Note that $\tilde{L} \supseteq L$. The set $\tilde{L}$ is a Bravais lattice, whose primitive lattice vectors are denoted by $v_1, \ldots, v_{N_0}$. The elementary bosonic fields
\begin{equation}
v_n \cdot \phi \quad (n = 1, \ldots, N_0)
\end{equation}
take constant expectation values in the ground state. If the expectation values are invariant, i.e.,
\begin{equation}
\langle v_n \cdot (U_G \phi + \delta \phi_G) \rangle = \langle v_n \cdot \phi \rangle \quad (n = 1, \ldots, N_0)
\end{equation}
modulo $2\pi$ for any $G \in G$, then the edge modes can be gapped without any symmetry breaking. Otherwise, the ground state breaks the symmetry $G$ spontaneously.

If there exists a set of vectors $\{l_1, \ldots, l_{N_0}\}$ that satisfies the above three conditions, then the edge states can be gapped out without symmetry breaking [with strong enough $C_j$ even when $\cos(l_j \cdot \phi)$ is an irrelevant operator in the renormalization-group sense], and the resulting gapped phase is a trivial phase. On the other hand, if we cannot find such a set of vectors $\{l_1, \ldots, l_{N_0}\}$, then the edge states are stable, and the bulk ground state realizes a 2D SPT phase.

SPT phases form an Abelian group as follows. Elements of the Abelian group are equivalence classes of phases that are connected without a closing of a gap under adiabatic deformation of the action (Hamiltonian) while preserving the symmetry $G$. The SPT phases described by the action (2) and the transformation (3) under the symmetry group $G$ are denoted by $\Psi_G[K, \{U_G, \delta \phi_G\}]$, while a trivial phase is denoted by “0”. The summation of two phases is defined as a direct sum of the two phases,
\begin{equation}
\Psi_G[K, \{U_G, \delta \phi_G\}] \oplus \Psi_G[K', \{U_{G}', \delta \phi_{G}'\}] = \Psi_G[K \oplus K', \{U_G \oplus U_{G}', \delta \phi_G \oplus \delta \phi_{G}'\}].
\end{equation}
The bosonic fields in the direct sum of two phases, $\phi = (\phi_1, \ldots, \phi_{4N_0})^T$, are transformed by $G \in G$ as
\begin{equation}
G \phi G^{-1} = (U_G \oplus U_{G}') \phi + \delta \phi_G \oplus \delta \phi_{G}'.
\end{equation}
with $G \in G$. The inverse element of an SPT phase $\Psi_G[K, \{U_G, \delta \phi_G\}]$ is found from the relation
\begin{equation}
\Psi_G[-K, \{U_G, \delta \phi_G\}] \oplus \Psi_G[K, \{U_G, \delta \phi_G\}] = 0,
\end{equation}
which is understood by noting that the fields $(\phi_1 - \phi_{2N_0+1}, \phi_2 - \phi_{2N_0+2}, \ldots, \phi_{2N_0} - \phi_{4N_0})$ can be pinned without symmetry breaking by the pinning potential
\begin{equation}
\mathcal{L}_{\text{edge}} = \sum_{j=1}^{2N_0} C_j \cos(\phi_j - \phi_{2N_0+j}).
\end{equation}
Thus, the equivalence relation between $\Psi_G[K, \{U_G, \delta \phi_G\}]$ and $\Psi_G[K, \{U'_{G}, \delta \phi'_{G}\}]$ is identical to the equivalence relation between $\Psi_G[K, \{U_G, \delta \phi_G\}] \oplus \Psi_G[-K, \{U'_{G}, \delta \phi'_{G}\}]$ and a trivial phase “0”.

The addition rule of SPT phases allows us to construct SPT phases of a larger number of degrees of freedom from small building blocks. Following Lu and Vishwanath, we consider a minimal model of SPT phases with a pair of helical edge states, described by the edge theory $\mathcal{S}_{\text{edge}}$ of two bosonic fields $\phi_l$ $(l = 1, 2)$ with a $2 \times 2$ $K$-matrix of $\det K = -1$. SPT phases with multiple pairs of helical edge states are obtained by combining minimal SPT phases, i.e., taking a direct sum of minimal models. Thus, we take
\begin{equation}
K = \sigma^x,
\end{equation}
unless otherwise noted, since $2 \times 2$ $K$-matrices for bosonic systems can be reduced to $K = \sigma^x$.

Finally we note that there is a redundancy in the representation in Eq. (5). The action $S_{\text{edge}}^0 = \int dx dt \mathcal{L}_{\text{edge}}^0$ is unchanged by substituting the fields $\phi' = X \phi + \Delta \phi$ with $\Delta \phi \in \mathbb{R}^2$ and $X \in GL(2, \mathbb{Z})$ satisfying $X^T K X = K$. Two representations $\{U_G, \delta \phi_G\}$ and $\{U_G', \delta \phi_G'\}$ are therefore equivalent if they satisfy
\begin{equation}
\delta \phi_G' = X[\delta \phi_G + (I - U_G) X^{-1} \Delta \phi],
\end{equation}
\begin{equation}
U_G' = X U_G X^{-1}.
\end{equation}

III. SPT PHASES PROTECTED BY REFLECTION SYMMETRY AND $Z_N$ SYMMETRY

In this section, we classify bosonic SPT phases protected by reflection symmetry $R$ and discrete local symmetry $Z_N$. There are two possible group structures for these symmetries: (i) $Z_N \times R$ and (ii) $Z_N \times R$. When the $Z_N$ symmetry corresponds to $2\pi/N$ rotation in integer-spin systems, these two group structures are realized (a) when the spin-rotation axis is parallel to the reflection plane and (b) when the spin-rotation axis is perpendicular to the reflection plane, respectively (see Fig. 1). In this section we focus on the $Z_N \times R$ symmetry and show that an example of the SPT phases protected by this symmetry is given by the $S = 2$ Affleck-Kennedy-Lieb-Tasaki (AKLT) state on the square lattice. The classification of SPT phases protected by (ii) $Z_N \times R$ symmetry, (iii) $U(1) \times R$ symmetry, and (iv) $U(1) \times R$ symmetry is discussed in Appendix. The results of the classification are summarized in Table II.

A. $Z_N \times R$

In this subsection, we apply the Chern-Simons approach to 2D bosonic SPT phases protected by the symmetry group $Z_N \times R$. 
First, we determine the transformation laws [Eq. (5)] of bosonic fields $\phi_i$’s under $R$ and $Z_N$. The invariance of the Lagrangian $\int dx_1 L_{\text{edge}}^0$ under the reflection $R$,

$$\int dx_1(U^T_R K U_R)_{I,J} \partial_t \phi_I(t,-x_1) \partial_{x_1} \phi_J(t,-x_1)$$

$$= \int dx_1 K_{I,J} \partial_t \phi_I(t,x_1) \partial_{x_1} \phi_J(t,x_1), \quad (20)$$

is guaranteed if $\phi_i$’s obey the transformation

$$R \phi(t,x_1) R^{-1} = U_R \phi_J(t,-x_1) + \delta \phi_R, \quad (21)$$

where the matrix $U_R \in GL(2,\mathbb{Z})$ satisfies

$$U^T_R K U_R = -K. \quad (22)$$

This condition is satisfied by $U_R = \pm \sigma^z$ for $K = \sigma^z$. Since the two representations $U_R = \sigma^z$ and $U_R = -\sigma^z$ are related by the transformation with $X = \sigma^z$ [Eq. (12)], it suffices to take the representation

$$U_R = -\sigma^z. \quad (23)$$

Similarly, the Lagrangian $L_{\text{edge}}^0$ is invariant if the bosonic fields $\phi_i$’s are transformed by $g \in Z_N$ as

$$g \phi(x) g^{-1} = U_g \phi(x) + \delta \phi_g \quad (24)$$

with a matrix $U_g \in GL(2,\mathbb{Z})$ satisfying

$$U^T_g K U_g = K. \quad (25)$$

Any choice from $U_g = \pm \mathbb{1}, \pm \sigma^z$ fulfills this condition. However, we discard the representations $U_g = \pm \sigma^z = \pm K$, since they are not compatible with Eq. (9) for $j = k$. Here we take the representation

$$U_g = \mathbb{1}, \quad (26)$$

since it is realized in spin models in which the $Z_N$ symmetry corresponds to spin-rotation symmetry (see Sec. III). We do not consider the other case $U_g = -\mathbb{1}$ in this paper. Incidentally, this case with $N = 2$ can be relevant to bosonic systems with charge conjugation symmetry, where creation operators of quasiparticles $e^{i\phi_i}$ are transformed to annihilation operators.

The representation $\{U_g, \delta \phi_g\}$ of symmetry operation $G \in G$ is constrained by the group structure of the symmetry group $G$. For the symmetry group $G = Z_N \times R$, the generators of the group satisfy the relation $R^2 = e = g_N$, where $e$ denotes the identity element of $G$. Accordingly, the representation $\{U_g, \delta \phi_g\}$ must satisfy the conditions

$$U^2_R \phi + (\mathbb{1} + U_R) \delta \phi_R = \phi, \quad (27)$$

$$U^N_g \phi + \sum_{k=0}^{N-1} U^k_g \delta \phi_g = \phi, \quad (28)$$

where $\mathbb{1}$ is a $2 \times 2$ unit matrix. Furthermore, the algebraic relation $RgRg = e$ obeyed by the generators of the symmetry group $G = Z_N \times R$ leads to the additional condition

$$U_g U_R U_g \phi + (\mathbb{1} + U_g U_R) (U_g \delta \phi_R + \delta \phi_g) = \phi. \quad (29)$$

In the following we discuss cases where $N$ is even and odd separately.
1. Even $N$

Given the representation

$$U_R = -\sigma^z, \quad U_g = 1,$$  \hfill (30)

we deduce from Eqs. (27)-(29) the transformation laws for bosonic fields

$$g\phi g^{-1} = \phi + \frac{2\pi}{N} \left[ k_g \begin{pmatrix} 0 \\ 0 \end{pmatrix} R \right]$$ \hfill (31a)

$$R\phi R^{-1} = -\sigma^x \phi + \pi \begin{pmatrix} 0 \\ n_R \end{pmatrix},$$ \hfill (31b)

where

$$n_g, n_R = 0, 1, \quad k_g = 0, \ldots, N - 1.$$ \hfill (31c)

In Eq. (31a) the phase shift $\delta \phi$ caused by the reflection $R$ is set equal to zero by the basis transformation in Eq. (19) with $X = 1$ and $\Delta \phi$ chosen appropriately.

Let us label by a set of integers $[k_g, n_g, n_R]$ a topological phase in which bosonic fields are transformed as in Eqs. (31). We will show that the SPT phases form an Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ by proving the following three properties:

(a) Phases $[0, n_g, n_R]$ and $[k_g, 0, 0]$ are trivial \( ([0, n_g, n_R] = [k_g, 0, 0] = 0) \).

(b) Any phase is generated from $[1, 0, 1]$ and $[1, 1, 0]$, which satisfy $[1, 0, 1] \oplus [1, 0, 1] = [1, 1, 0] \oplus [1, 1, 0] = 0$.

(c) The two phases $[1, 0, 1]$ and $[1, 1, 0]$ are independent generators of SPT phases.

Proof of (a): The null vector condition \([\text{Eq. (7)}]\) with $K = \sigma^x$ allows only pinning potentials of the form $\cos(l\phi_1 + \alpha_l)$ or $\cos(l\phi_2 + \alpha_l)$ with $l \in \mathbb{Z}$ and $\alpha_l \in \mathbb{R}$. When $k_g = 0$, the pinning potential

$$H_{\text{int}} = C \int dx_1 \cos(\phi_1),$$ \hfill (32)

is invariant under the transformations in Eq. (31) and can pin the field $\phi_1$ at $\langle \phi_1 \rangle = 0$ or $\pi$ depending on the sign of $C$. No symmetry is broken by the pinning. Thus, the phase $[0, n_g, n_R]$ is reduced to a trivial insulator. When $n_g = n_R = 0$, the pinning potential

$$H_{\text{int}} = C \int dx_1 \cos(\phi_2 + \alpha),$$ \hfill (33)

is invariant under the transformations in Eq. (31) and can pin the field $\phi_2$ at $\langle \phi_2 + \alpha \rangle = 0$ or $\pi$ without symmetry breaking. Thus, the phase $[k_g, 0, 0]$ is a trivial insulator.

Proof of (b): We first show the following addition relations of SPT phases:

$$[k_g, n_g, n_R] \oplus [k_g, n_g', n_R'] = [k_g, n_g + n_g', n_R + n_R'],$$ \hfill (34a)

$$[k_g, n_g, n_R] \oplus [k_g', n_g, n_R'] = [k_g + k_g', n_g, n_R].$$ \hfill (34b)

The composition of two phases $[k_g, n_g, n_R]$ and $[k_g', n_g', n_R']$ has bosonic fields $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and a $K$-matrix $K = \sigma^x \oplus \sigma^y$. The fields obey the commutation relations

$$[\phi_i(x_1), \partial_{x_1} \phi_j(x_1')] = 2\pi i (\sigma^x \oplus \sigma^y)_{i,j} \delta(x_1 - x_1'),$$ \hfill (35)

and the transformation laws

$$g\phi g^{-1} = \phi + \frac{2\pi k_g}{N}(e_1 + e_3) + \pi n_g e_2 + \pi n_g' e_4,$$ \hfill (36a)

$$R\phi R^{-1} = - (\sigma^x \oplus \sigma^y) \phi + \pi n_R e_2 + \pi n_R' e_4,$$ \hfill (36b)

with

$$k_g = 0, 1, \ldots, N - 1, \quad n_g, n_g', n_R, n_R' = 0, 1.$$ \hfill (36c)

Here, $e_j$ ($j = 1, \ldots, 4$) denotes the $j$th unit vector, $(e_j)_i = \delta_{j,i}$. We now make a basis transformation and define a new set of bosonic fields

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T = (\phi_1 - \phi_3, \phi_2, \phi_3, \phi_2 + \phi_4)^T,$$ \hfill (37)

which have the same $K$-matrix and commutators

$$[\psi_i(x), \partial_x \psi_j(x')] = 2\pi i (\sigma^x \oplus \sigma^y)_{i,j} \delta(x - x').$$ \hfill (38)

Without pinning potentials, there are two pairs of gapless helical edge modes: $(\psi_1, \psi_2)$ and $(\psi_3, \psi_4)$. A potential of the form

$$H_{\text{int}} = C \int dx_1 \cos(\psi_1)$$ \hfill (39)

can pin the $\psi_1$ field and gap out the $(\psi_1, \psi_2)$ sector without symmetry breaking. The helical edge states in the $(\psi_3, \psi_4)$ sector remains gapless and correspond to the phase $[k_g, n_g', n_R + + n_R']$. Equation (31b) follows.

In a similar way, we obtain Eq. (31b) by making basis transformation

$$\psi' = (\psi_1', \psi_2', \psi_3', \psi_4')^T = (\phi_1 + \phi_3, \phi_2, \phi_3, \phi_2 - \phi_4)^T$$ \hfill (40)

and adding a potential of the form

$$H_{\text{int}} = C \int dx_1 \cos(\psi_1' + \alpha).$$ \hfill (41)

In this case the $(\psi_3', \psi_4')$ sector is a trivial gapped state and can be discarded. The edge states in the remaining $(\psi_1', \psi_2')$ sector corresponds to the phase $[k_g + k_g', n_g, n_R]$, and thus we obtain Eq. (31b).

We find from Eqs. (31b) that

$$[1, 1, 0] \oplus [1, 1, 0] = [1, 2, 0] = [1, 0, 0] = 0,$$ \hfill (42a)

$$[1, 0, 1] \oplus [1, 0, 1] = [1, 0, 2] = [1, 0, 0] = 0,$$ \hfill (42b)

since phase shifts are defined modulo $2\pi$. Furthermore, using Eqs. (31b) successively, we can reduce any phase $[k_g, n_g, n_R]$ to four phases:

$$[k_g, n_g, n_R] = \begin{cases} 
0, & (k_g n_g, k_g n_R) = (e, e), \\
[1, 1, 0], & (k_g n_g, k_g n_R) = (e, e), \\
[1, 0, 1], & (k_g n_g, k_g n_R) = (e, e), \\
[1, 1, 0] \oplus [1, 0, 1], & (k_g n_g, k_g n_R) = (e, e),
\end{cases}$$ \hfill (43)
where “e” and “o” stand for “even” and “odd”, respectively.

Proof of (c): We show that the two phases $[1, 0, 1]$ and $[1, 1, 0]$ are neither equivalent to each other nor connected to the trivial phase $0$. To this end, we show that edge modes of the phase $[k_g, n_g, n_R] \oplus [k'_g, n'_g, n'_R]^{-1}$ with $k_g, k'_g = 0, 1$ cannot be gapped out, unless $(k_g, n_g, n_R) = (k'_g, n'_g, n'_R)$ or $(k_g, 0, 0) = (0, n'_g, n'_R)$. It follows from Eq. 13 that the phase $[k_g, n_g, n_R] \oplus [k'_g, n'_g, n'_R]^{-1}$ has edge modes described by the bosonic fields $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ with a $K$-matrix $K = \sigma^X \oplus (-\sigma^X)$. The bosonic fields $(\phi_1, \phi_2)$ and $(\phi_3, \phi_4)$ obey the transformation laws of $[k_g, n_g, n_R]$ and $[k'_g, n'_g, n'_R]^{-1}$, respectively.

Gapping the bosonic fields $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ requires two pinning potentials $\cos(l_1 \cdot \phi + \alpha_1)$ and $\cos(l_2 \cdot \phi + \alpha_2)$, whose integer vectors $l_1$ and $l_2$ must satisfy Eqs. 7 and 9, or equivalently,

$$I_1^T [\sigma^X \oplus (-\sigma^X)] I_2 = 0,$$

$$I_1^T [\sigma^Y \oplus (-\sigma^Y)] I_2 = 0,$$

for $i, j = 1, 2$. Solutions to these equations are given by

$$l_1 = (\alpha p, \beta q, \gamma p')^T,$$

$$l_2 = (\alpha' p, \beta' q, \gamma' p')^T,$$

with $\alpha, \beta, \alpha', \beta', p, q \in \mathbb{Z}$. If $pq = 0$ and $p \neq q$, then the elementary bosonic fields defined in Eq. 12 are given by $(v_1 \cdot \phi, v_2 \cdot \phi) = (\phi_1, \phi_2)$ or $(\phi_1, \phi_3)$. If $pq \neq 0$, we can assume $\gcd(p, q) = 1$ and obtain the elementary bosonic fields $v_1 \cdot \phi = p\phi_1 + q\phi_3$ and $v_2 \cdot \phi = q\phi_1 + p\phi_3$. In either case the fields are transformed as

$$g(v_1 \cdot \phi) g^{-1} = v_1 \cdot \phi + \frac{2\pi}{N} (pk_g + qk'_g),$$

$$R(v_1 \cdot \phi) R^{-1} = -v_1 \cdot \phi,$$

$$g(v_2 \cdot \phi) g^{-1} = v_2 \cdot \phi + \pi (qn_g + pm'_g),$$

$$R(v_2 \cdot \phi) R^{-1} = v_2 \cdot \phi + \pi (qn_R + pm'_R),$$

where we assume $(p, q) = (1, 0)$ or $(0, 1)$ if $pq = 0$. When $(p, q) = (odd, odd)$, the phase shifts in Eqs. 46a, 46c, and 46d are equal to zero (mod 2\pi) only when $(k_g, n_g, n_R) = (k'_g, n'_g, n'_R)$. This means that the edge modes cannot be gapped out without symmetry breaking unless $(k_g, n_g, n_R) = (k'_g, n'_g, n'_R)$. Similarly, when $(p, q) = (even, odd)$, the edge modes can be gapped out without symmetry breaking only if $n_g = n_R = k'_g = 0$, i.e., $(k_g, n_g, n_R) = (k'_g, n'_g, n'_R) = 0$. Thus, the two phases $[1, 0, 1]$ and $[1, 1, 0]$ are inequivalent, and both of them are distinct from the trivial phase.

From (a), (b), and (c), we conclude that the Abelian group of the SPT phases protected by $Z_N \times R = Z_2 \times Z_2$ generated by $[1, 0, 1]$ and $[1, 1, 0]$.

Finally, we note that the SPT phase $[1, 1, 0]$ is stable even in the absence of the reflection symmetry, while the other two SPT phases, $[1, 0, 1]$ and $[1, 1, 0] \oplus [1, 0, 1]$, are SPT phases that are stable only in the presence of both the reflection symmetry and the $Z_N$ symmetry.

2. Odd $N$

When $N$ is odd, the bosonic fields $\phi = (\phi_1, \phi_2)^T$ are transformed under symmetry operations as

$$g \phi g^{-1} = \phi + \frac{2\pi}{N} \begin{pmatrix} k_g \\ 0 \end{pmatrix},$$

$$R \phi R^{-1} = -\sigma^X \phi + \pi \begin{pmatrix} 0 \\ n_R \end{pmatrix},$$

with

$$n_R = 0, 1, \quad k_g = 0, \ldots, N - 1.$$  

We note that the above transformation rules are obtained from Eqs. 31 by setting $n_g = 0$. The vanishing phase shift of $\phi_2$ under the $g$ transformation is a consequence of the conditions 23 and 24 for odd $N$.

There is no SPT phase when $N$ is odd. This conclusion is obtained by using the classification for even $N$ discussed above. Let us label phases by a set of integers $[k_g, n_R]$. Imposing $n_g = 0$ in Eq. 43, we find

$$[k_g, n_R] = \begin{cases} 0, & k_g n_R = even, \\ [1, 1], & k_g n_R = odd. \end{cases}$$

Next we prove that two phases $[0, 1](= 0)$ and $[1, 1]$ are equivalent by showing that the edge modes of the phase $[0, 1] \oplus [1, 1]^{-1}$ can be gapped without symmetry breaking (i.e., $[0, 1] \oplus [1, 1]^{-1} = 0$). The phase $[0, 1] \oplus [1, 1]^{-1}$ has edge modes described by bosonic fields $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ with a $K$-matrix $K = \sigma^X \oplus (-\sigma^X)$. The bosonic fields $(\phi_1, \phi_2)$ and $(\phi_3, \phi_4)$ obey the transformation laws of $[0, 1]$ and $[1, 1]$, respectively. These fields are gapped by the pinning potential of the form

$$H_{int} = C_1 \int dx_1 \cos(l_1 \cdot \phi) + C_2 \int dx_1 \cos(l_2 \cdot \phi + \alpha),$$

where $l_1 \cdot \phi = \phi_1 + N\phi_3$ and $l_2 \cdot \phi = N\phi_2 + \phi_4$. These integer vectors $l_1$ and $l_2$ satisfy the conditions in Eq. 44. The corresponding elementary bosonic variables,

$$v_1 \cdot \phi = l_1 \cdot \phi = \phi_1 + N\phi_3,$$

$$v_2 \cdot \phi = l_2 \cdot \phi = N\phi_2 + \phi_4,$$

are transformed as

$$g(v_1 \cdot \phi) g^{-1} = v_1 \cdot \phi + 2\pi,$$

$$R(v_1 \cdot \phi) R^{-1} = -v_1 \cdot \phi,$$

$$g(v_2 \cdot \phi) g^{-1} = v_2 \cdot \phi,$$

$$R(v_2 \cdot \phi) R^{-1} = v_2 \cdot \phi + (N + 1)\pi.$$  

Since the phase shift $(N + 1)\pi$ is a multiple of $2\pi$ for odd $N$, the edge modes can be gapped out without symmetry breaking by the pinning potential. We thus obtain $[0, 1] \oplus [1, 1]^{-1} = 0$, thereby $[1, 1] = [0, 1] = 0$. Hence we conclude that $[k_g, n_R] = 0$ for any $k_g$ and $n_R$. 


It is interesting to note that, as long as the \( K \)-matrix is fixed to be the two-dimensional matrix \( \sigma^z \), a helical edge mode of the \([0,1]\) phase cannot be gapped. The addition of a helical edge mode of the trivial \([1,1]^{-1}\) phase leads to the gapping of the edge of the composite phase \([0,1] \oplus [1,1]^{-1}\) with \( K = \sigma^z \oplus (-\sigma^z) \). Since we define SPT phases as topologically stable phases against addition of an arbitrary number of trivial phases, we concluded that the \([0,1]\) phase is trivial. This situation is somewhat reminiscent of the classification of three-dimensional free-fermion insulators of class A, in which there are topologically stable two-band insulators (with a Hopf invariant) that are reduced to trivial insulators upon addition of extra trivial band insulators.\textsuperscript{6,47}

**B. The \( S = 2 \) AKLT state as an SPT phase with \( Z_2 \times R \) symmetry**

We show that the \( S = 2 \) AKLT model on the 2D square lattice realizes a nontrivial SPT phase protected by reflection symmetry \( R \) and spin \( \pi \)-rotation symmetry, \( Z_2 \times R \). In the \( S = 2 \) AKLT model, \( S = 2 \) spins are placed on the square lattice and interact with each other according to the Hamiltonian:\textsuperscript{36,39}

\[
H = \sum_{(i,j)} P_j(i,j). \tag{52}
\]

Here, \( P_j(i,j) \) is a projection operator acting on the Hilbert space spanned by the two spins \( S_i \) and \( S_j \) at sites \( i \) and \( j \) onto the subspace of total spin \( J \). The summation is taken over all pairs of neighboring sites \( (i,j) \). The AKLT Hamiltonian is written in terms of spin operators as

\[
H = \frac{1}{40320} \sum_{(i,j)} \prod_{j=0,1,2,3} [(S_i + S_j)^2 - J(J + 1)]
\]

\[
= \sum_{(i,j)} \left[ \frac{1}{2520} (S_i \cdot S_j)^4 + \frac{1}{180} (S_i \cdot S_j)^3 + \frac{1}{40} (S_i \cdot S_j)^2 + \frac{1}{28} (S_i \cdot S_j) \right]. \tag{53}
\]

The ground state of this model is a valence bond solid or the AKLT state. In this state each \( S = 2 \) spin is decomposed into four \( S = 1/2 \) spins, and at every link a spin singlet is formed by a pair of \( S = 1/2 \) spins from the sites connected by the link, as schematically shown in Fig. 2. The ground state is obtained by projecting the wave function built out of \( S = 1/2 \) spins onto the original Hilbert space spanned by \( S = 2 \) spins. The bulk excitations are gapped due to the singlet formation. Along the edge of the square lattice, however, an unpaired \( S = 1/2 \) spin appears at every site as shown in Fig. 2. These free effective \( S = 1/2 \) spins at the boundary form dispersionless zero-energy edge states.

Now we are going to show that the \( S = 2 \) AKLT state is in the SPT phase \([k_x, n_g, n_R] = [1, 0, 1]\) defined in Eq. 31 with \( N = 2 \). The AKLT model respects the symmetry group \( Z_2 \times R \). The Hamiltonian is invariant under the reflection with respect to the \( x \)-direction (which is parallel to the edge) and under the \( \pi \)-rotation around the \( S^z \) axis as shown in Fig. 2. Under these transformations the effective \( S = 1/2 \) spins at the boundary are transformed as

\[
g \begin{pmatrix} S^x(x) \\ S^y(x) \end{pmatrix} g^{-1} = \begin{pmatrix} -S^x(x) \\ -S^y(x) \end{pmatrix}, \quad (54a)
\]

\[
R \begin{pmatrix} S^x(x) \\ S^y(x) \end{pmatrix} R^{-1} = \begin{pmatrix} S^x(-x) \\ S^y(-x) \end{pmatrix}, \quad (54b)
\]

where \( g \) denotes the operator for \( \pi \)-rotation around the \( S^z \) axis. Actually, the group \( Z_2 \times R \) is identical to \( Z_2 \times R \). Here we opted to write \( Z_2 \times R \) because we utilize results from Sec. III A and because the group structure is \( Z_N \times R \) for \( N > 2 \) in the case when the \( Z_N \) rotation axis is parallel to the reflection plane (Fig. 1). Incidentally, we present the classification of SPT phases protected by \( Z_N \times R \) symmetry in Appendix. As we mentioned above, the AKLT state has free \( S = 1/2 \) spins at the boundary that respects the \( Z_2 \times R \) symmetry. When the Hamiltonian is perturbed away from the AKLT point given by Eq. 33, we expect that the free spins should interact with each other antiferromagnetically. Indeed, this picture is supported by the recent numerical study\textsuperscript{46,41} in which the entanglement spectrum
of the AKLT model on the square lattice is shown to correspond to the low-energy spectrum of the spin-1/2 antiferromagnetic Heisenberg chain, i.e., a conformal field theory with central charge \( c = 1 \). Thus, the low-energy effective theory for the boundary \( S = 1/2 \) spins below the energy scale of the bulk spin gap should be the conformal field theory that describes the low-energy excitations of the spin-\( 1/2 \) antiferromagnetic Heisenberg model. In this low-energy theory the boundary \( S = 1/2 \) spins can be bosonized:

\[
S^i(x) = -\frac{1}{2\pi} \partial_x \varphi(x) + A(-1)^x \cos \varphi(x), \tag{55a}
\]

\[
S^+(x) = e^{-i\partial(x)}[B(-1)^x + C \cos \varphi(x)], \tag{55b}
\]

where \( A, B, \) and \( C \) are real constants, and the bosonic fields \( \varphi(x) \) and \( \partial(x) \) obey the commutation relation

\[
[\varphi(x), \partial(x')] = -i\pi + i\pi \text{sgn}(x - x'). \tag{55c}
\]

Here \( \text{sgn}(x) \) equals 1, 0, and -1 for \( x > 0, x = 0, \) and \( x < 0 \), respectively. Substituting the bosonization formula (55) into Eq. (54), we find that the bosonic fields defined by \( \phi(x) = (\phi_1(x), \phi_2(x))^T = (-\varphi(x), \varphi(x))^T \) are transformed as

\[
\begin{align}
g\phi(x)g^{-1} &= \phi(x) + \pi e_1, \tag{56a} \\
R\phi(x)R^{-1} &= -\sigma^z \phi(-x) + \pi e_2, \tag{56b}
\end{align}
\]

and obey the commutation relation \( [\phi_I(x), \partial_J\phi_{I'}(x')] = 2\pi i(\sigma^z)_{IJ} \delta(x - x') \) with \( I, J = 1, 2 \). The transformation of bosonic fields in Eq. (56) corresponds to the one for the SPT phase \([1, 0, 1]\) that we discussed in Sec. III A. Therefore, the gapless edge modes are stable against perturbations as long as the \( Z_2 \times R \) symmetry is preserved. The \([1, 0, 1] \oplus [1, 0, 1] = 0 \), two coupled copies of the \( S = 2 \) AKLT model will have a trivial gapped ground state where edge spins are gapped by couplings between the copies.

The symmetry protection of edge modes can be intuitively understood as follows. In principle, the edge modes can be gapped out by perturbations that cause singlet formation or magnetization. However, the ground states gapped by such perturbations break the symmetry; singlet formation or out-of-plane magnetization (magnetization along the \( S^z \) axis) breaks the reflection symmetry, and in-plane magnetization (magnetization along the \( S^x \) or \( S^y \) axis) breaks the \( \pi \)-rotation symmetry.

IV. DISCUSSIONS

So far we have studied 2D SPT phases protected by \( Z_N \times R \) symmetry. The \( Z_N \) symmetry can be considered as \( 2\pi/N \) rotation about a fixed axis in the spin space in integer-spin systems (note that \( g^N = e \) for any \( g \in Z_N \) for integer-S spins). In this case the semi-direct product group structure \( Z_N \times R \) is realized when the spin-rotation axis is parallel to the reflection plane.

As we mentioned at the beginning of Sec. III, for integer-spin systems we can also consider other combinations of reflection symmetry and spin-rotation symmetry: \( Z_N \times R, U(1) \times R, \) and \( U(1) \times R. \) The symmetry group \( Z_N \times R \) is realized in spin systems with a spin-rotation axis perpendicular to a reflection plane. The continuous \( U(1) \) rotation symmetry can be viewed as \( N \to \infty \) limit of the discrete \( Z_N \) symmetry. The classification of SPT phases for those symmetries is obtained in a similar manner as in the case of \( Z_N \times R; \) the results are summarized in Table I (For details, see Appendix).

Table I indicates that the Abelian group structures of SPT phases are different depending on whether the spin-rotation symmetry is discrete \( Z_N \) or continuous \( U(1). \) For example, the Abelian group of SPT phases for \( U(1) \times R \) is \( Z_2 \), while it is \( Z_2 \times Z_2 \) for \( Z_N \times R \) with even \( N. \) Furthermore, the \( Z_N \times R \) symmetry supports SPT phases of \( Z_2 \times Z_2 \) while the \( U(1) \times R \) symmetry allows only the trivial gapped phase without gapless edge modes. These differences in the Abelian group structures of SPT phases arise from an additional constraint present in the case of \( U(1) \) symmetry. That is, any \( U(1) \) transformation is continuously connected to the identity transformation as the rotation angle goes to zero; this poses a further constraint on SPT phases realized in the presence of the \( U(1) \) symmetry compared to the discrete symmetries.

We finish this section with a comment on the effect of disorder on SPT phases with reflection symmetry. Considering the stability of surface Dirac fermions in weak topological insulators\(^{43-45}\) and topological crystalline insulators,\(^{46-49}\) we can expect that edge modes remain gapless and metallic in the presence of disorder that keeps the reflection symmetry on average. The disorder effect is experimentally relevant and an important issue. Detailed analysis is left for a future work.

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Appendix: Other bosonic SPT phases

1. \( U(1) \times R \) symmetry

In this appendix, we obtain the classification of SPT phases protected by \( U(1) \times R \) symmetry by making use of the similar analysis for the \( Z_N \times R \) case presented in Sec. III A. The obtained result agrees with the one reported in Ref. 24.

U(1) rotations are generated from an infinitesimal rotation, which can be thought of as the \( N \to \infty \) limit of the \( Z_N \) transformation with \( n_g = 0 \) in Eq. (31a). Thus, the transformation law of the bosonic fields \( \phi := (\phi_1, \phi_2)^T \)
under $U(1)$ rotation by a finite angle $\theta$ is given by

$$u_\theta \phi u_\theta^{-1} = \phi + \theta \begin{pmatrix} k \\ 0 \end{pmatrix} \quad \text{(A.1a)}$$

Here $u_\theta$ is an element of the $U(1)$ group whose rotation angle is $\theta$, and $k$ is an integer. As in the $Z_N \times R$ case, the transformation under reflection $R$ is

$$R\phi R^{-1} = -\sigma^z \phi + \pi \begin{pmatrix} 0 \\ n_R \end{pmatrix} \quad \text{(A.1b)}$$

with $n_R = 0, 1$. As in Sec. III A 1 topological phases are labeled by the integer indices $[k, n_R]$. Now we can follow the argument employed for proving the properties (a) and (b) in Sec. III A 1 with setting $n_g = 0$ in the relevant equations. In this way we find

$$[k, n_R] = \begin{cases} 0, & k n_R = \text{even}, \\
[1,1], & k n_R = \text{odd}. \end{cases} \quad \text{(A.2)}$$

Next, we prove that the phase $[1,1]$ is not connected to the trivial phase by showing that the edge modes of $[\tilde{k}, n_{R}] \oplus [k', n'_{R}]$ where $\tilde{k}, k', n_R, n'_R = 0$ or 1, can be gapped out without symmetry breaking only when $(\tilde{k}, n_R) = (k', n'_{R})$. For this purpose, we need to see how the elementary bosonic fields $v_1 \cdot \phi = p \phi_1 + q \phi_3$ and $v_2 \cdot \phi = q \phi_2 + p \phi_4$ are transformed, where $p$ and $q$ are coprime integers, $\text{gcd}(p,q) = 1$. In the phase $[\tilde{k}, n_R] \oplus [k', n'_R]$, the fields are transformed as

$$u_\theta (v_1 \cdot \phi) u_\theta^{-1} = v_1 \cdot \phi + \theta (pk + qk') \quad \text{(A.3a)}$$

$$R(v_1 \cdot \phi) R^{-1} = -v_1 \cdot \phi \quad \text{(A.3b)}$$

$$u_\theta (v_2 \cdot \phi) u_\theta^{-1} = v_2 \cdot \phi \quad \text{(A.3c)}$$

$$R(v_2 \cdot \phi) R^{-1} = -v_2 \cdot \phi + \theta (qn_R + pn'_R) \quad \text{(A.3d)}$$

If $(p, q) = (\text{odd}, \text{odd})$, then the edge modes can be gapped out without symmetry breaking only when $(\tilde{k}, n_R) = (k', n'_{R})$. If $(p, q) = (\text{even, odd})$, the edge modes can be gapped out without symmetry breaking only when $\tilde{k} = k' = n_R = 0$.

From the above arguments, we conclude that the Abelian group of SPT phases protected by $U(1) \times R$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $[1,1]$.

2. $Z_N \times R$

In this section we classify 2D bosonic SPT phases protected by the symmetry group $G = Z_N \times R$, using the same approach as in Sec. III A.

The invariance of the Lagrangian (2) of edge modes under symmetry transformation imposes the transformation rules of $\phi$ in Eqs. (22) and (23). As in Sec. III A we take the representation

$$U_R = -\sigma^z, \quad U_g = 1 \quad \text{(A.4)}$$

for the $K$-matrix $K = \sigma^z$. The generators $R$ and $g$ of the symmetry group $G = Z_N \times R$ obey the relation $R^2 = e = g^N$, where $e$ denotes the identity element of $G$. Accordingly, the representation $\{U_g, \delta \phi_g\}$ must satisfy the conditions (27) and (28). Furthermore, the algebraic relation $R g R^{-1} = e$ for the generators leads to the additional condition

$$U_R^{-1} R U_g U_R \phi + U_R^{-1} (1 + U_R U_g) \delta \phi_R + U_R^{-1} (U_R - 1) \delta \phi_g = \phi \quad \text{(A.5)}$$

where we have used the relation $g^{-1} \phi g = U_R^{-1} (\phi - \delta \phi_g)$. We discuss cases where $N$ is even and odd separately.

2a. Even $N$

It follows from Eqs. (27), (28), and (A.5) that the bosonic fields $\phi = (\phi_1, \phi_2)^T$ are transformed as

$$g \phi g^{-1} = \phi + \pi \begin{pmatrix} n_g \\ 0 \end{pmatrix} + \frac{2\pi}{N} \begin{pmatrix} 0 \\ k_g \end{pmatrix} \quad \text{(A.6a)}$$

$$R \phi R^{-1} = -\sigma^z \phi + \pi \begin{pmatrix} 0 \\ n_R \end{pmatrix} \quad \text{(A.6b)}$$

with

$$n_g, n_R = 0, 1, \quad k_g = 0, \ldots, N - 1. \quad \text{(A.6c)}$$

In Eq. (A.6), the phase shift $\delta \phi_1$ under the reflection $R$ is set equal to zero by the basis transformation in Eq. (19) with $X = 1$ and $\delta \phi_1$ chosen appropriately.

As in Sec. III A 1 we label SPT phases by the set of integers $[n_g, k_g, n_R]$ that appear in the transformation (A.6). We show below that the SPT phases form an Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ by proving the following three properties:

(a) Phases $[0, k_g, n_R]$ and $[n_g, 0, 0]$ are trivial $([0, k_g, n_R] = [n_g, 0, 0] = 0)$.

(b) Any phase is generated from $[1, 0, 1]$ and $[1, 1, 0]$, which satisfy $[1, 0, 1] \oplus [1, 0, 1] = [1, 1, 0] \oplus [1, 1, 0] = 0$.

(c) Two phases $[1, 0, 1]$ and $[1, 1, 0]$ are independent generators of SPT phases.

Proof of (a): The null vector condition [Eq. (2)] with $K = \sigma^z$ allows only pinning potentials of the form, $\cos(l \phi_1 + \alpha_1)$ or $\cos(l \phi_2 + \alpha_1)$, with $l \in \mathbb{Z}$. When $n_g = 0$, the pinning potential

$$H_{\text{int}} = C \int dx \cos(\phi_1) \quad \text{(A.7)}$$

is invariant under the transformations and can pin the field $\phi_1$ at $\langle \phi_1 \rangle = 0$ or $\pi$ depending on the sign of $C$. No symmetry is broken by pinning. Thus, $[0, k_g, n_R]$ is connected to a trivial insulator. When $k_g = n_R = 0$, the pinning potential

$$H_{\text{int}} = C \int dx \cos(\phi_2 + \alpha) \quad \text{(A.8)}$$

is trivial for all $\alpha$. Therefore, $[0, k_g, n_R]$ is trivial for all $n_g$ and $k_g$. The result follows for $[n_g, 0, 0]$.
is invariant under the transformation in Eqs. (A.6) and can pin the field \( \phi \) at \( \langle \phi_2 + \alpha \rangle = 0 \) or \( \pi \) without symmetry breaking. Thus, \([g, 0, 0]\) is a trivial insulator.

**Proof of (b):** We first show the following addition relation of SPT phases:

\[
[g, k_g, n_R] \oplus [g, k'_g, n'_R] = [g, k_g + k'_g, n_R + n'_R].
\]

(A.9a)

\[
[g, k_g, n_R] \oplus [n'_g, k_g, n_R] = [n_g + n'_g, k_g, n_R].
\]

(A.9b)

The composition of two phases \([g, k_g, n_R]\) and \([n_g, k'_g, n'_R]\) has bosonic fields \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \) and a \( K \)-matrix \( K = \sigma^x \oplus \sigma^y \). These fields obey the commutation relations

\[
[\phi_I(x_1), \partial_{x_1} \phi_J(x_1')] = 2\pi i(\sigma^x \oplus \sigma^y)_{I,J} \delta(x_1 - x_1'),
\]

(A.10)

and are transformed as

\[
g \phi g^{-1} = \phi + \pi n_g(e_1 + e_3) + \frac{2\pi}{N}(k_g e_2 + k'_g e_4),
\]

(A.11a)

\[
R \phi R^{-1} = - (\sigma^x \oplus \sigma^y) \phi + \pi (n_R e_2 + n'_R e_4),
\]

(A.11b)

where

\[
n_g, n_R, n'_R = 0, 1, \quad k_g, k'_g = 0, \ldots, N - 1.
\]

(A.11c)

Here, \( e_j \) \((j = 1, \ldots, 4)\) denotes the \( j \)-th unit vector in which \((e_j)_I = \delta_{j,I} \). We now make basis transformation and define a new set of bosonic fields

\[
\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T = (\phi_1 - \phi_3, \phi_2, \phi_3, \phi_2 + \phi_4)^T,
\]

(A.12)

which have the same \( K \)-matrix and commutators

\[
[\psi_I(x), \partial_{x} \psi_J(x')] = 2\pi i(\sigma^x \oplus \sigma^y)_{I,J} \delta(x - x').
\]

(A.13)

In the absence of pinning potentials there are two pairs of gapless helical edge modes: \((\psi_1, \psi_2)\) and \((\psi_3, \psi_4)\). A potential of the form

\[
H_{\text{int}} = C \int dx \cos(\psi_1)
\]

(A.14)

can pin the field and gap out the edge modes in the \((\psi_1, \psi_2)\) sector without symmetry breaking. The helical edge states in the \((\psi_3, \psi_4)\) sector remain gapless and correspond to the phase \([n_g, k_g + k'_g, n_R + n'_R]\). Equation (A.9b) follows. In a similar way, we obtain Eq. (A.9c) by making basis transformation

\[
\psi' = (\psi'_1, \psi'_2, \psi'_3, \psi'_4)^T = (\phi_1 + \phi_3, \phi_2, \phi_3, \phi_2 - \phi_4)^T
\]

(A.15)

and adding a potential of the form

\[
H_{\text{int2}} = C \int dx \cos(\psi'_4 + \alpha).
\]

(A.16)

In this case the \((\psi'_3, \psi'_4)\) sector is gapped and can be discarded. The edge states in the remaining \((\psi'_1, \psi'_2)\) sector correspond to the phase \([n_g + n'_g, k_g, n_R]\), and thus we obtain Eq. (A.9d). We find from Eqs. (A.9e)

\[
\begin{align*}
[1, 0, 1] &\oplus [1, 0, 1] = [0, 0, 1] = 0, \quad (A.17a) \\
[0, 1, 1] &\oplus [0, 1, 1] = [1, 1, 1] = 0, \quad (A.17b)
\end{align*}
\]

since phase shifts are defined modulo \(2\pi\). Furthermore, using Eqs. (A.9c) successively, we can reduce any phase \([n_g, k_g, n_R]\) to four phases:

\[
[n_g, k_g, n_R] = \begin{cases} 0, & (k_g n_g, n_g n_R) = (e, e), \\
[1, 1, 0], & (k_g n_g, n_g n_R) = (o, e), \\
[1, 0, 1], & (k_g n_g, n_g n_R) = (e, o), \\
[1, 1, 0] \oplus [1, 0, 1], & (k_g n_g, n_g n_R) = (o, o),
\end{cases}
\]

where “e” and “o” stand for “even” and “odd”, respectively.

**Proof of (c):** We show that two phases \([1, 0, 1]\) and \([1, 1, 0]\) are neither equivalent to each other nor connected to the trivial phase 0. To this end, we first show that the edge modes of the phase \([n_g, \bar{k}_g, n_R] \oplus [n'_g, \bar{k}'_g, n'_R]\) with \(k_g, k'_g = 0, 1\) cannot be gapped out completely unless \((n_g, \bar{k}_g, n_R) = (n'_g, \bar{k}'_g, n'_R)\) or \((n_g, 0, 0) = (0, \bar{k}_g, n'_R)\). It follows from Eq. (16) that the phase \([n_g, \bar{k}_g, n_R] \oplus [n'_g, \bar{k}'_g, n'_R]\) has edge modes described by the bosonic fields \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \) with a \( K \)-matrix \( K = \sigma^x \oplus (-\sigma^y) \). The bosonic fields \((\phi_1, \phi_2)\) and \((\phi_3, \phi_4)\) obey the transformation laws of \([n_g, \bar{k}_g, n_R]\) and \([n'_g, \bar{k}'_g, n'_R]\), respectively.

Gapping out the edge modes \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \) requires two pinning potentials \( \cos(l_1 \cdot \phi + \alpha_1) \) and \( \cos(l_2 \cdot \phi + \alpha_2) \), whose integer vectors \( l_1 \) and \( l_2 \) must satisfy Eqs. (7) and (9), or equivalently, Eqs. (11). Solutions to these equations are given by

\[
\begin{align*}
I_1 &= (\alpha p, \beta q, \alpha q, \beta p)^T, \\
I_2 &= (\alpha' p, \beta' q, \alpha' q, \beta' p)^T,
\end{align*}
\]

(A.18)

(A.19)

with \(\alpha, \beta, \alpha', \beta', p, q \in \mathbb{Z}\). If \(pq = 0\) and \(p \neq q\), then the elementary bosonic fields defined in Eq. (12) are given by \((v_1 \cdot \phi, v_2 \cdot \phi) = (\phi_3, \phi_2)\) or \((\phi_1, \phi_4)\). If \(pq \neq 0\), we can assume \(\gcd(p, q) = 1\) and obtain the elementary bosonic fields \(v_1 \cdot \phi = p \phi_1 + q \phi_3\) and \(v_2 \cdot \phi = q \phi_2 + p \phi_4\). In either case, the fields are transformed as

\[
\begin{align*}
g(v_1 \cdot \phi) g^{-1} &= v_1 \cdot \phi + \pi (p n_g + q n'_g), \\
R(v_1 \cdot \phi) R^{-1} &= -v_1 \cdot \phi, \\
g(v_2 \cdot \phi) g^{-1} &= v_2 \cdot \phi + \frac{2\pi}{N} (q k_g + p k'_g), \\
R(v_2 \cdot \phi) R^{-1} &= v_2 \cdot \phi + \pi (q n_R + p n'_R).
\end{align*}
\]

(A.20a)

(A.20b)

(A.20c)

(A.20d)

We assume \((p, q) = (1, 0)\) or \((0, 1)\) if \(pq = 0\). When \((p, q) = (\text{odd}, \text{odd})\), the phase shifts in Eqs. (A.20a), (A.20b), (A.20c), and (A.20d) can be equal to zero (mod 2\pi) only when \((n_g, \bar{k}_g, n_R) = (n'_g, \bar{k}'_g, n'_R)\). This means that the edge modes cannot be gapped out without symmetry breaking unless \((n_g, \bar{k}_g, n_R) = (n'_g, \bar{k}'_g, n'_R)\). Similarly, when \((p, q) = (\text{even}, \text{odd})\), the edge modes can be gapped out.
out without symmetry breaking only if $\tilde{k}_g = n_R = n'_g = 0$, i.e., $[n_g, \tilde{k}_g, n_R] = [n'_g, \tilde{k}'_g, n'_R] = 0$. Thus, the two phases $[1, 0, 1]$ and $[1, 1, 0]$ are inequivalent, and both of them are distinct from the trivial phase.

From (a), (b), and (c), we conclude that, when $N$ is even, the Abelian group of the SPT phases protected by $Z_N \times R$ is $Z_2 \times Z_2$ generated by $[1, 0, 1]$ and $[1, 1, 0]$.

### b. Odd $N$

When $N$ is odd, the transformation law of the bosonic fields are determined from Eqs. (27), (28), and (A.5) to be

\[
g g^{-1} = \phi + \frac{2\pi}{N} n_R, \tag{A.21a}
\]

\[
R\phi R^{-1} = -\sigma^z \phi + \pi \left( \begin{array}{c} 0 \\ n_R \end{array} \right), \tag{A.21b}
\]

with

\[
n_R = 0, 1, \quad k_g = 0, \ldots, N - 1. \tag{A.21c}
\]

In Eqs. (A.21b), the phase shift $\delta \phi$ caused by the reflection $R$ is set equal to zero by the basis transformation in Eq. (19) with $X = 1$ and appropriately chosen $\Delta \phi$. The phase shift of $\phi$ vanishes in Eq. (A.21) because of the conditions (28) and (A.6), i.e., $g^N = e$ and $RgR^{-1} = e$.

No SPT phase is allowed in this case because $\phi$ acquires no phase shift under the symmetry transformations. The potential

\[
H_{int} = C \int dx \cos(\phi_1 + \alpha) \tag{A.22}
\]

is invariant under the transformations in Eqs. (A.21) and can pin the field $\phi_1$ at $\langle \phi_1 + \alpha \rangle = 0$ or $\pi$ depending on the sign of $C$. No symmetry is broken by the pinning. Thus, there is only a topologically trivial gapped phase.

### 3. $U(1) \times R$

The classification of SPT phases under $U(1) \times R$ symmetry is obtained by taking the limit $N \to \infty$ in the classification for the $Z_N \times R$ symmetry. The result is in agreement with Ref. [24]. Under the $U(1)$ rotation by a finite angle $\theta$ the bosonic fields $\phi = (\phi_1, \phi_2)^T$ are transformed as

\[
u_\theta \phi \nu_\theta^{-1} = \phi + \theta \left( \begin{array}{c} 0 \\ k \end{array} \right), \tag{A.23}
\]

which is obtained by replacing $2\pi/N$ with $\theta$ in Eq. (A.21b) [or further setting $n_g = 0$ in Eq. (A.6)]. The transformation under reflection is given by Eq. (A.21). Obviously we can employ the same argument as the one for the $Z_N \times R$ symmetry with odd $N$. The potential in Eq. (A.22) can pin the $\phi_1$ field, and the resulting gapped state is a trivial state.
This is obtained as follows. The generic solution to the equation $l^T K^{-1} l = 0$ with $K = \sigma^x \oplus (-\sigma^x)$ is given by

$$l = (pm, qn, qm, pn)^T = p(m, 0, 0, n) + q(0, n, m, 0) = m(p, 0, q, 0) + n(0, q, 0, p),$$

where $n, m, p, q \in \mathbb{Z}$. Thus the two linearly independent vectors $l_1$ and $l_2$ satisfying Eq. (44b) have the form

$$\alpha a + \beta b, \quad \alpha^\prime a + \beta^\prime b,$$

where $\alpha, \beta, \alpha^\prime, \beta^\prime$ are integers with $\alpha\beta^\prime - \beta\alpha^\prime \neq 0$. Here we have two possible choices for the vectors $a$ and $b$:

$$a = (m, 0, 0, n)^T, \quad b = (0, n, m, 0)^T,$$