Going from the huge to the small: Efficient succinct representation of proofs in Minimal implicational logic

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Abstract

A previous article shows that any linear height bounded normal proof of a tautology in the Natural Deduction for Minimal implicational logic (M⊃) is as huge as it is redundant. More precisely, any proof in a family of super-polynomially sized and linearly height bounded proofs have a sub-derivation that occurs super-polynomially many times in it. In this article, we show that by collapsing all the repeated sub-derivations we obtain a smaller structure, a rooted Directed Acyclic Graph (r-DAG), that is polynomially upper-bounded on the size of α and it is a certificate that α is a tautology that can be verified in polynomial time. In other words, for every huge proof of a tautology in M⊃, we obtain a succinct certificate for its validity. Moreover, we show an algorithm able to check this validity in polynomial time on the certificate’s size. Comments on how the results in this article are related to a proof of the conjecture NP = CoNP appears in conclusion.

1 Introduction

In [13] and [12] we discuss the correlation between the size of proofs and how redundant they can be. A proof or logical derivation is redundant whenever it has
sub-proofs that are repeated many times inside it. The articles \[13\] and \[12\] focus on Natural Deduction (ND) proofs in the purely implicational minimal logic \(M \supset\). This logic, here called \(M \supset\), is \(\text{PSPACE}\)-complete and simulates polynomially any proof in Intuitionistic Logic and full minimal logic, being hence an adequate representative to study questions regarding computational complexity. The fact that \(M \supset\) has straightforward syntax and ND system is worthy of notice. Moreover, compressing proofs in \(M \supset\) can provide very good glues to compress proofs in any one of these mentioned systems, even for the Classical Propositional Logic. In \[8\] and \[7\], we prove that for every \(M \supset\) tautology \(\alpha\) there is a two-fold certificate for the validity of \(\alpha\) in \(M \supset\). The certificate is polynomially sized on the length of \(\alpha\) and verifiable in polynomial time on this length too. This is the general approach of our proof that \(NP = \text{PSPACE}\). It is well-known that \(NP = \text{PSPACE}\) implies \(\text{CoNP} = NP\), so we can conclude also that \(\text{CoNP} = NP\). This article, together with the articles \[13\], \[12\], and one of the appendixes of \[13\], aim to show an alternative and more intuitive proof that \(NP = \text{CoNP}\). Recently, we deposited in arxiv a note, see \[9\], that explains how to have a third alternative proof of \(NP = \text{CoNP}\), this time with a double certificate on linear height normal proofs, without to use Hudelmaier result. This paper aims to show how to use the inherent redundancy in huge proofs to have polynomial and polytime certificates by removing this redundancy from the proofs. This is made by collapsing all the redundant sub-proofs into only one occurrence in the proof, by type. We start with tree-like Natural Deduction proofs and end up with a labeled r-DAG (rooted Directed Acyclic Graph). In the sequel we explain and show what is said in the last two phrases, previously stated. Currently, the use of the redundancy theorem and corollary shown in \[12\], that is the essence of this proof’s approach, is not easily adaptable to a proof of \(NP = \text{PSPACE}\). In \[8\] the linearly height upper-bounded proofs of the tautologies in \(M \supset\) are do not have have to be normal.

In \[13\], we identify sets of huge proofs with sets of proofs that, when viewed as strings, have their length lower-bounded by some exponential function. Moreover, we can consider, without loss of generality, proof/deductions, which are linearly height bounded, as stated in \[7\]. We prove that the exponentially lower-bounded \(M \supset\) proofs are redundant, in the sense that there is at least one sub-proof for each proof that occurs exponentially many times in it. In \[12\], we show that this result extends to super-polynomial proofs, i.e., proofs that are lower-bounded by any polynomial. We consider huge proofs/derivations as super-polynomial sized proofs. Thus, we prove that, in any set of super-polynomially lower-bounded proofs in \(M \supset\), of tautologies, all proof are redundant. Redundancy means that there is a sub-proof that occurs super-polynomially many times in it for almost
every proof in this set of huge proofs.

In section 2, we present a brief presentation and explanation on the proof-theoretical terminology used here and the results from our previous articles. For a more detailed and comprehensive reference, see [13], [12], [8] and [7]. Section 5 is the section where we show how to remove redundancies using a recursive method adequately. Section 5 shows that the certificate obtained in section 3 is of polynomial size and can be checked in polynomial time too. We conclude this article in section 9. Finally, due to have a self-contained article, and the background and previous results that this paper uses, we advise the reader that there is an essential superposition of this article with [12] and [13].

2 Background on proof-theory and brief presentation of previous results

One reason to study redundancy in proofs is to obtain a compressing method based on redundancy removals. According to the redundancy theorem (theorem 12, page 17 in [12]) and its corollary (corollary 13, page 18) proofs belonging to a family of super-polynomial proofs are super-polynomial redundant. The reader can find all the content of this section in [12] in more detail. This section contains excerpts of [12].

The Natural Deduction system, defined in [5]), is taken as a set of inference rules that settle the concept of a logical deduction. Natural Deduction does not have axioms. It implements in the level of the logical calculus the (meta)theorem of deduction, \( \Gamma, A \vdash A \supset B \), via the discharging mechanism. The \( \supset \)-introduction rule, for example, uses this discharging mechanism in the logic calculus level.

\[
\begin{align*}
[A] \\
\Pi \\
& \vdash B \supset \text{Intro}
\end{align*}
\]

We embrace formulas occurrence \( A \) in the derivation \( \Pi \) of \( B \) from \( A \) with a pair of [ ] to indicate the discharge of them. An embraced formula occurrence \( [A] \) means that from the \( \supset \)-Intro rule discharging it down to the conclusion of the derivation, the inferred formulas do not depend anymore on this occurrence \( A \). The choice of formulas to be discharged in an \( \supset \)-Intro rule is arbitrary and liberal. The range of this choice goes from every occurrence of \( A \), the discharged formula until none of them.
The following derivations show two different ways of deriving \( A \supset (A \supset A) \). Observe that in both deductions or derivations, we use numbers to indicate the application of the \( \supset \)-Intro that discharged the marked formula occurrence. For example, in the right derivation, the upper application discharges the marked occurrences of \( A \), while in the left derivation, it is the lowest application that discharges the formula occurrences \( A \). There is a third derivation that both applications do not discharge any \( A \), and the conclusion \( A \supset (A \supset A) \) keep depending on \( A \). This third alternative appears in figure 1. Natural Deduction systems can provide logical calculi without any need to use axioms. In this article, we focus on the system formed only by the \( \supset \)-Intro rule and the \( \supset \)-Elim rule, as shown below, also known by modus ponens. The logic behind this logical calculus is the purely minimal implicational logic, \( M_\supset \).

\[
\frac{A}{A \supset B} \quad \frac{A \supset B}{B} \quad \supset \text{-Elim}
\]

Without loss of generality, we substitute the liberal discharging mechanism by a greedy one that discharges every possible formula occurrence whenever the \( \supset \)-Intro is applied. As stated and proved in [12], if there is an N.D. proof of \( \alpha \) in \( M_\supset \), then there is a proof of \( \alpha \) in \( M_\supset \) that has all applications of \( \supset \)-Intro as greedy ones.

\[
\frac{[A]^1}{A \supset A} \quad \frac{[A]^1}{A \supset A} \\
\frac{A \supset (A \supset A)}{1} \quad \frac{A \supset (A \supset A)}{1}
\]

Figure 1: Two vacuous \( \supset \)-Intro applications

In our previous articles, we consider Natural Deduction as trees with the sake of having simpler proofs of our results. There is a binary tree with nodes labelled by the formulas and edges linking premises to the conclusion for any ND derivation. The tree’s root is the conclusion of the derivation, and the leaves are its assumptions. Figure 2 has the tree in figure 3 representing it. In a proof-tree, the
set of formulas that the label of \( u \) depends on \( v \) labels the edge from \( v \) to \( u \). This set of formulas is called the dependency set of \( u \) from \( v \). The greedy version of the \( \supset \)-intro removes the discharged formula from the corresponding dependency sets, as shown in figure 3. We need one more extra edge and the root node. The dependency set of the conclusion labels this edge. That is why the edge links the conclusion to the dot in figure 3.

\[
\frac{[A] \quad A \supset B}{B} \quad B \supset C \quad 1 \quad A \supset C
\]

Figure 2: A derivation in \( M \supset \)

![Diagram](image)

Figure 3: The tree representing the derivation in figure 2

Finally, we use bitstrings induced by an arbitrary linear ordering of formulas to have a more compact representation of the dependency sets. Considering that only subformulas of the conclusion can be in any dependency set, we only need bitstrings of the size of the conclusion of the proof. Figure 4 shows this final form of tree representing the derivation in figure 2 and 3 when the linear order \( \prec \) is \( A \prec B \prec C \prec A \supset B \prec B \supset C \prec A \supset C \). This explanation is an excerpt from [12].
Without loss of generality, we consider the additional hypothesis on the linear bound on height of the proof of $M_\supset$-tautologies. In [7], we show that any tautology in $M_\supset$ has a Natural Deduction normal proof of height bound by the size of this tautology. However, proof of the tautology does not need to be normal. On the other hand, if we consider the complexity class $CoNP$ (see the appendix in [13]) we are naturally limited to linearly height-bounded proofs. The proofs, in $M_\supset$, of the non-hamiltonianicity of graphs, are linearly height bounded.

We consider the usual definition of the syntax tree for $M_\supset$-formulas. Given a formula $\phi_1 \supset \phi_2$ in $M_\supset$, we call $\phi_2$ its right-child and $\phi_1$ its left-child. These formulas label the respective right and left child vertexes. A right-ancestral of a vertex $v$ in a syntax-tree $T_\alpha$ of a formula $\alpha$ is any vertex $u$, such that, either $v$ is the right-child of $u$, or, there is a vertex $w$, such that $v$ is the right-child of $w$ and $u$ is right-ancestral of $w$.

The left premise of a $\supset$-Elim rule is called a minor premise, and the right premise is called the major premise. We should note that the conclusion of this rule and its minor premise, are sub-formulas of its major premise. A derivation is a tree-like structure built using $\supset$-Intro and $\supset$-Elim rules. We have some examples depicted in the last section. The derivation conclusion is the root of this tree-like structure, and the leaves are what we call top-formulas. A proof is a derivation that has every top-formula discharged by a $\supset$-Intro application in it. The top-formulas are also called assumptions. An assumption that it is not discharged by any $\supset$-Intro rule in a derivation is called an open assumption. If $\Pi$ is a derivation with conclusion $\alpha$ and $\delta_1, \ldots, \delta_n$ as all of its open assumptions then we say that $\Pi$ is a derivation of $\alpha$ from $\delta_1, \ldots, \delta_n$. 
**Definition 1** (Branch). A branch in a derivation or proof \( \Pi \) is any sequence \( \beta_1, \ldots, \beta_k \) of formula occurrences in \( \Pi \), such that:

- \( \delta_1 \) is a top-formula, and;
- For every \( i = 1, k-1 \), either \( \beta_i \) is a \( \supset - \text{Elim} \) major premise of \( \beta_{i+1} \) or \( \beta_i \) is a \( \supset - \text{Intro} \) premise of \( \beta_{i+1} \), and;
- \( \delta_k \) either is the conclusion of the derivation or the minor premise of a \( \supset - \text{Elim} \).

A normal derivation/proof in \( M_\supset \) is any derivation that does not have any formula occurrence simultaneously a major premise of a \( \supset - \text{Elim} \) and the conclusion of a \( \supset - \text{Intro} \). A formula occurrence that is a conclusion of a \( \supset - \text{Intro} \) and a major premise of \( \supset - \text{Elim} \) is called a maximal formula. In [18], Dag Prawitz proves the following theorem for the Natural Deduction system for the full propositional fragment of minimal logic.

**Theorem 1** (Normalization). Let \( \Pi \) be a derivation of \( \alpha \) from \( \Delta = \{ \delta_1, \ldots, \delta_n \} \). There is a normal proof \( \Pi' \) of \( \alpha \) from \( \Delta' \subseteq \Delta \).

In any normal derivation/proof, a branch’s format provides worth information on why huge proofs are redundant, as we will see in the next sections. Since no formula occurrence can be a major premise of \( \supset - \text{Elim} \) and conclusion of a \( \supset - \text{Intro} \) rule in a branch we have that the conclusion of a \( \supset - \text{Intro} \) can only be the minor premise of a \( \supset - \text{Elim} \) or it is not a premise of any rule application at all in the same branch. In this last case, it is the derivation’s conclusion or the minor premise of a \( \supset - \text{Elim} \) rule. In any case, it is the last formula in the branch. Thus, for any branch, every conclusion of a \( \supset - \text{Intro} \) has to be a premise of a \( \supset - \text{Intro} \). Hence, any branch in a normal derivation splits into two parts (possibly empty). The E-part starts it with the top-formula, and, every formula occurrence in it is the major premise of a \( \supset - \text{Elim} \). We may have then a formula occurrence that is the conclusion of a \( \supset - \text{Elim} \) and premise of a \( \supset - \text{Intro} \) rule that is called minimal formula of the branch. The minimal formula starts the I-part of the branch, where every formula is the premise of a \( \supset - \text{Intro} \), excepted the last formula of the branch. From the branches’ format, we can conclude that the sub-formula principle holds for normal proofs in Natural Deduction for \( M_\supset \), in fact, for many extensions. A branch in \( \Pi \) is said to be a principal branch if its last formula is the conclusion of

\[1\text{The full propositional fragment is } \{ \lor, \land, \supset, \neg, \bot \} \]
A secondary branch is a branch that is not principal. The primary branch is called a 0-branch. Branches, where the last formula is the minor premise of a rule in the E-part of a \( n \)-branch, is a \( n + 1 \)-branch.

**Corollary 2** (Sub-formula principle). Let \( \Pi \) be a normal derivation of \( \alpha \) from \( \Delta = \{ \delta_1, \ldots, \delta_m \} \). It is the case that for every formula occurrence \( \beta \) in \( \Pi \), \( \beta \) is a sub-formula of either \( \alpha \) or of some of \( \delta_i \).

To facilitate the presentation, we only handle normal proofs in expanded form.

**Definition 2.** A normal proof/derivation is in expanded form, if and only if, all of its minimal formulas are atomic.

Without loss of generality, we can consider that formula in \( M \supset \) is a tautology if and only if there is a normal proof in expanded form that proves it. Of course, if it is a tautology, it has proof, and so it has normal proof by normalization. We use the following fact to obtain the expanded form from normal proof. In [12] we prove that all tautologies have normal proofs in expanded form. See the first appendix of [12].

In [12], we observed correspondence between each minimal formula of a branch, in a normal and expanded proof \( \Pi \), employing a one-to-one correspondence to the respective top-formula occurrence of its branch. Figure 5 illustrates the mapping from the proof into the syntax-tree of the proved formula according to this correspondence. Note that the two positions of the atomic formula \( q \) in the syntax tree uniquely indicates the top-formula in the E-part of the Natural Deduction proof/derivation to which it belongs. We can consider that the two \( q \)'s are in fact, different. The top-formula of each \( q \) is the biggest in the inverse path (upwards) following the reverse of the right child edge. Definition 5 in the sequel, has the purpose of setting this correspondence in the representation of proofs formally. With the sake of a more precise presentation, we provide below the definition of a formula’s syntax tree.

**Definition 3** (Syntax tree of a formula). Let \( \alpha \) be a \( M \supset \) formula. The syntax tree of \( \alpha \) is the triple \( \langle V, E_{\text{left}}, E_{\text{right}}, L \rangle \) where \( V \) is a set, of vertexes, \( E_s \subseteq V \times V \), \( s = \text{left}, \text{right} \), the corresponding left and right edges, such that \( \langle V, E_{\text{left}}, E_{\text{right}} \rangle \) is an ordered full binary tree, and, \( L \) is a bijective function from \( V \) onto the sub-formulas of \( \alpha \), such that:

- \( L(r) = \alpha \), where \( r \in V \) is the root of the tree \( \langle V, E_{\text{left}}, E_{\text{right}} \rangle \), and;
Figure 5: A mapped N.D. proof
• For every formula \( \varphi_1 \supset \varphi_2 \in \text{Sub}(\alpha) \), if \( L(v) = \varphi_1 \supset \varphi_2 \), \( \langle v, v_1 \rangle \in E_{\text{left}} \) and \( \langle v, v_2 \rangle \in E_{\text{right}} \) then \( L(v_1) = \varphi_1 \) and \( L(v_2) = \varphi_2 \).

**Definition 4** (Partially mapped ND-proofs). Let \( \alpha \) be a \( M \supset \)-formula and \( T_\alpha = \langle V, E_{\text{left}}, E_{\text{right}}, L \rangle \) its syntax tree. Let \( \Pi \) be a \( M \supset \)-ND normal derivation of \( \alpha \). A partially mapped ND-proof of \( \alpha \) is a structure \( \langle \Pi, T_\alpha, l \rangle \), where \( l \) is a partial function from the formula occurrences in \( \Pi \) to \( V \), such that, the following conditions hold.

- If \( \gamma \) is the minimal formula of a branch \( \xrightarrow{\beta} \) in \( \Pi \) then if \( l(\gamma) \) is defined then \( L(l(\gamma)) = \gamma \);

- If \( \gamma \) is the minimal formula of a branch \( \xrightarrow{\beta} = \langle b_0, \ldots, b_j = \gamma, \ldots, b_k \rangle \) and \( l(\gamma) \) is defined then either \( l(b_{j-1}) \) or \( l(b_{j+1}) \) are defined, and;

- If \( \varphi_2 \) is the conclusion of a \( \supset \)-Elim rule in \( \Pi \), that has premises \( \varphi_1 \supset \varphi_2 \) and \( \varphi_1 \), and \( l(\varphi_2) = v_2 \) then there are \( v \) and \( v_1 \), such that \( \langle v, v_2 \rangle \in E_{\text{right}} \), \( \langle v, v_1 \rangle \in E_{\text{left}} \), \( l(\varphi_1) = v_1 \) and \( l(\varphi_1 \supset \varphi_2) = v \);

- If \( \varphi_1 \supset \varphi_2 \) is the conclusion of a \( \supset \)-Intro rule in \( \Pi \), that has premise \( \varphi_2 \) and \( l(\varphi_1 \supset \varphi_2) = v \) then there is \( v' \in V \), \( \langle v, v' \rangle \in E_{\text{right}} \) and \( l(v') = \varphi_2 \).

**Definition 5** (E-mapped Natural Deduction Normal Expanded proofs). Let \( \alpha \) be a \( M \supset \)-formula, \( T_\alpha = \langle V, E_{\text{left}}, E_{\text{right}}, L \rangle \) be the syntax tree of \( \alpha \) and \( \Pi \) a normal and expanded proof of \( \alpha \). The triple \( \langle \Pi, T_\alpha, l \rangle \) is an E-mapped Natural Deduction proof, if and only if, \( l \) is defined on all formula occurrences that take part in the E-parts of branches in \( \Pi \), including the minimal formulas. Moreover the following condition must hold:

- For every branch \( \xrightarrow{\beta} \), if \( q \) is the minimal formula of \( \xrightarrow{\beta} \), \( l(q) = v \in V \) and \( \beta \) is the top-formula (occurrence) of \( \xrightarrow{\beta} \) then \( l(\beta) = u \), where \( u \) is the right-ancestral of \( v \) that is left-child of some \( w \in V \).

In [12] we show that the above definition of E-mapped Natural Deduction Normal Expanded proof, \( \text{EmND} \) for short, is well-defined. Moreover, we have the following proposition. We consider a branch as a sequence of formula occurrences numbered from top-formula down to the branch’s last formula. The proof of this proposition is in [12].
Proposition 3. Let $\langle \Pi, T_\alpha, l \rangle$ be a EmND of $\alpha$. We have that to each branch $\vec{b} = \langle \beta_0, \ldots, \beta_k \rangle$ in $\Pi$ that has the minimal formula occurrence $q = \beta_j$, such that $\ell(\beta_j) = u \in V$, there exists one and only one path $p = \langle u_0, \ldots, u_j \rangle$ in $T_\alpha$, with $u_j = u$, and $u_0$ such that, $\ell(\beta_i) = u_i$, $i = 0, \ldots, j$.

We point out that in proposition 3 above, $\langle \beta_0, \ldots, \beta_j = q \rangle$ is the E-part of $\vec{b}$. This proposition 3 states that any given E-part $\langle \beta_0, \ldots, \beta_j \rangle$ of a branch in an EmND is an instance of at most one path $p = \langle u_0, \ldots, u_j \rangle$ in $T_\alpha$, such that $L(u_i) = \beta_i$, $i = 0, \ldots, j$. Moreover, this path $p$ is as stated in the definition of EmND in its only item. Given a EmND $\Pi$, for each E-part, in an EmND, exists a path of the form stated in definition of EmND, in the syntax tree of the conclusion of the EmND. The number of such paths in the syntax tree is upper-bounded by its size, then the number of different E-parts types in any EmND is at most of the size of the conclusion of this EmND. We have the following lemma:

Lemma 4 (Linear upper-bound on types of E-parts). Let $\Pi$ be an EmND proving the $M_\succsim$ formula $\alpha$. The number of different types of E-parts occurring in this EmND is at most the size of the $T_\alpha$.

We remark, as also observed in [12], that we can label the nodes of a Natural Deduction proof-tree with the nodes (not the labels) of the syntax tree of the conclusion of the proof-tree. In doing that we will have the same effect on counting different types of E-parts that is stated by lemma 4.

2.1 Redundancy in huge $M_\succsim$ mapped derivations

Due to the linear speedup theorem, see [19] page 63-64, Theorem 3.10, we can consider, w.l.o.g a linear height bounded proof of $\alpha$ is a proof which height is upper-bounded by the length of $\alpha$. In fact, in this article, because of counting details, we consider that the upper-bounded is the size of the syntax tree $|T_\alpha|$. Since $|T_\alpha| = |\alpha|$, the definition is equivalent. From [12], we have the following lemmas. In [12] the reader can fund both proofs.

Lemma 5 (Spreading Branchs Repetitions). Let $\langle \Pi, T_\alpha, l \rangle$ be a linearly height bounded EmND proof of $\alpha$, $0 < p \in \mathbb{N}$ and $m = |\alpha|$. If there is a branch $\vec{b}$ that has more than $m^p$ instances occurring in $\Pi$ then there is a level $\mu$, such that, at least $m^{p-1}$ instances of $\vec{b}$ have the minimal formula $q_\vec{b}$ of $\vec{b}$ occurring in level $\mu$. 

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Lemma 6 (Branchs and sub-derivations). Let $\Pi$ be a proof of $\alpha$, and $\vec{b}$ a branch in $\Pi$ under the same conditions of the lemma 5 above. Then there is a (sub) derivation $\Pi_{\vec{b}}$ of $\Pi$, such that, $\Pi_{\vec{b}}$ has at least $m^{p-1}$ instances occurring in $\Pi$.

Definition 6 (Linearly height-bounded EmND proofs). Let $\Lambda$ be the set of mapped linearly height-bounded ND $M \supset \neg$ proofs. We use the notation $c(\Pi)$ to denote the formula that is the conclusion of $\Pi$. Note that we can consider $\Lambda$ as a predicate $\Lambda(x)$ that is true if and only if $x$ is assigned to a mapped linearly height-bounded ND proof.

$$S_\Lambda = \{ \Pi \in \Lambda : \forall p \in \mathbb{N}, p > 0, \exists n_0, \forall n > n_0, |T(c(\Pi))| = n \text{ and } |\Pi| > n^p \}$$

As explained in [12], $S_\Lambda$ contains all huge or hard linearly height upper-bounded proofs in $M \supset \neg$. Of particular interest is the following set. Let $Taut_{M \supset \neg}$ be the set of all ND mapped proofs of $M \supset \neg$ tautologies. The following set:

Definition 7. Let $\Theta$ be the following set:

$$\Theta_{M \supset \neg} = \{ \Pi \in Taut_{M \supset \neg} : \forall p \in \mathbb{N}, p > 0, \exists n_0, \forall n > n_0, |T(c(\Pi))| = n \text{ and } |\Pi| > n^p \}$$

$\Theta_{M \supset \neg}$ is the set of super-polynomially sized $M \supset \neg$ ND mapped proofs.

In [12], we show that every $\Pi \in S_\Lambda$ is redundant. This means that there is at least one sub-proof $\Pi_s$ of $\Pi$ that repeats as many times as it is the size of $\Pi$. We have the following theorem[7] proved in [12]. We emphasise that the proofs in $S_\Lambda$ are linearly heigh-bounded.

Theorem 7. For all $p \in \mathbb{N}$, $p > 3$, and for all $\Pi \in S_\Lambda$, such that, $|T(c(\Pi))| = m$ and $|\Pi| > m^p$, then there is a sub-derivation $\Pi_s$ of $\Pi$ and a level $\mu$ in $\Pi$, such that, $\Pi_s$ has at least $m^{p-3}$ instances occurring in the level $\mu$ in $\Pi$.

From theorem[7] we can roughly state the corollary[8]

Corollary 8. All, but finitely many, proofs belonging to an arbitrary family of super-polynomial and linearly height upper-bounded proofs, are super polynomial redundant.

3 Removing redundancies from huge $M \supset \neg$ linearly height bounded proofs

Corollary[8] says that any proof in an unbounded set of super-polynomial proofs, linearly bounded on the height, is almost as redundant as it is huge. We can show
that there are level $\mu$ and a derivation $\Pi$ that occurs as many times in $\mu$ as it is the size of the proof. This section shows a polynomial sized certificate of validity for any huge tautology that belongs to this set of super-polynomial proofs. Our argumentation to prove this is to remove all redundancies in the original derivation, preserving logical consequence.

Given a non-empty finite set $S$, $\text{card}(S) = n$, and a total order $\mathcal{O}_S = \{s_1, \ldots, s_n\}$ on $S$, the set $B(\mathcal{O}_S)$ is $\{b_1 \ldots b_n : b_i = 0 \text{ or } b_i = 1, i = 1 \ldots n\}$. There is a bijection $F$ from $B(\mathcal{O}_S)$ onto the powerset of $S$ given by $F(b_1 \ldots b_n) = \{s_i : b_i = 1\}$. $B(\mathcal{O}_S)$ is also called the set of bitstrings over $\mathcal{O}_S$.

Given a $M \supset \alpha$ formula $\alpha$, $\text{sub}(\alpha)$ is the set of all sub-formulas of $\alpha$.

**Definition 8 (r-DagProof).** A pre $r$-DagProof for a $M \supset \alpha$ formula $\alpha$ is a structure $\mathcal{C} = \langle V, E_d, E_A, r, \ell, L, \delta, \rho, O_\alpha \rangle$

1. $V$ is a non-empty set of nodes;
2. $E_d \subseteq V \times V$, deduction edges;
3. $E_A \subseteq V \times V$, ancestrality edges;
4. $O_\alpha$ is a total order on $\text{sub}(\alpha)$;
5. $r \in V$ is the root of the $\mathcal{C}$;
6. $\ell : V \to \text{sub}(\alpha)$, for $v \in V$, $\ell(v)$ is the (formula) label of $v$, $\text{sub}(\alpha)$ is the set of all sub-formulas of $\alpha$;
7. $L : E_d \to B(\mathcal{O})$ is a function, such that, for every $\langle u, v \rangle \in E_d$, $L(\langle u, v \rangle) \in B(\mathcal{O}_S)$;
8. $\delta : E_A \to \mathbb{N}$, a total function;
9. $\rho : E_d \to \mathbb{N}$, a partial function;

Subjected to the following conditions:

(Global) $\langle V, E_d, r \rangle$ is a connected DAG with unique root $r$ and for each node $v \in V$, if $\langle r = u_1, \ldots, u_m = v \rangle$ and $\langle r = v_1, \ldots, v_n = v \rangle$ are two inverse paths from $r$ to $v$ then $m = n$, i.e., for any $v$, any longest path from $r$ to $v$ has the same length of a shortest path from $r$ to $v$. Moreover, for each $v \in V$, if $v \neq r$ then there is a path from $v$ to $r$ and for every $u$, $\langle u, u \rangle \notin E_d$. 

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(E_d – (1/L)_1 consistency) For every \( \langle u, v \rangle, \langle v, w \rangle \in E_d \), if there is no \( \langle u', v \rangle \in E_d \), such that, \( u' \neq u \) and \( \rho(\langle v, w \rangle) = \uparrow \), then \( \ell(u) = \varphi_1 \), \( \ell(v) = \varphi_2 \supset \varphi_1 \), \( L(\langle u, v \rangle) = L(\langle u, v \rangle) \neq \overline{b}_{\varphi_2} \);

(E_d – (1/L)_2 consistency) For every \( \langle u_1, v \rangle, \langle u_2, v \rangle, \langle v, w \rangle \in E_d \), there is no \( \langle u', v \rangle \in E_d \), such that, \( u' \neq u_i \), \( i = 1, 2 \), and \( \rho(\langle v, w \rangle) = \uparrow \), then \( \ell(u_1) = \varphi_1 \), \( \ell(u_2) = \varphi_1 \supset \varphi_2 \), \( \ell(v) = \varphi_2 \), \( L(\langle v, w \rangle) = L(\langle u_1, v \rangle) \oplus L(\langle u_2, v \rangle) \);

(E_A - target consistency) If \( \langle u, v \rangle \in E_A \), then either there is no \( \langle w, v \rangle \in E_d \) or there is \( \langle w, v \rangle \in E_A \), \( w \neq u \);

(E_A - source consistency) If \( \langle u, v \rangle \in E_A \), then there is \( \langle w, u \rangle \in E_d \), \( w \neq v \) and \( \delta(\langle u, v \rangle) = \rho(\langle w, u \rangle) \);

(E_A - ancestraly irreflexivity) For every \( u, \langle u, u \rangle \notin E_A \).

A pre r-DagProof, satisfying definition 8 above, is a r-DagProof, or a certificate for \( \ell(r) \), if it is sound, or equivalently, if algorithm 3 answers “Correct” when executes on it. In section 5 we define soundness of a r-DagProof and we define the algorithm 3 in the meanwhile we deal with pre r-DagProofs. Specifically, in this section, this difference between pre r-DagProofs and r-DagProofs is not relevant.

We provide a bit of terminology in this part of the article. Regarding theorem 7 we denote by matrix the sub-derivation of the ambient derivation, i.e. the huge one, that has many instances repeated. Definition 11 define the set of nodes in an r-DagProof that is formed by top-formulas or their representatives after collapsing. We need the definition of top-formulas and representative top-formulas.

**Definition 9 (Top-formula of a r-DagProof).** A node \( v \in V \), of a r-DagProof \( C = \langle V, E_d, E_A, r, l, L, L_E, O_A, E_d \rangle \), is a top-formula, if and only if, there is no \( w \in V \), such that \( \langle w, v \rangle \in E_d \) or \( \langle w, v \rangle \in E_A \).

**Definition 10 (Representative top-formula of a r-DagProof).** Given a r-DagProof \( C = \langle V, E_d, E_A, r, l, L, \rho, \delta, O_A \rangle \), \( v \in V \) is a representative top-formula, if and only if, there is no \( w \in V \), such that \( \langle w, v \rangle \in E_A \). Moreover, there is a sequence \( v = w_1, \ldots, w_n \), \( w_i \in V \), \( i = 1, n \), such that, for every \( i = 1, n - 1 \), \( \langle w_i, w_{i+1} \rangle \in E_A \), and there is no \( w \in V \), such that, \( \langle w, w_n \rangle \in E_d \).

**Definition 11.** Given a r-DagProof \( C = \langle V, E_d, E_A, r, l, L, \rho, \delta, O_A \rangle \), its set of initials, \( I(C) \) is the set of representative top-formulas together with its top-formulas.
The following mapping defines the detachment of an instance of the matrix $C$, defined by its root $k$, from $D$ and link its position to the matrix $C$ accordingly. In the following definitions, if $C$ is a r-DagProof $\langle V, E_d, E_A, r, l, L, \rho, \delta, O_\alpha \rangle$, we use the notations $E_A(C)$, $E_d(C)$, $V(C)$, $r(C)$, $\ell(C)$, etc., to denote $E_A$, $E_d$, $V$, $r$, $l$, etc., respectively. Given a set of nodes $V' \subseteq V$, such that $r \notin V'$, the restriction of $C$ to the set of nodes $V'$ is $C|_{V'}$ and it is defined below:

$$\langle V', E_d|_{V'}, E_A|_{V'}, r, l|_{V'}, L|_{(E_d|_{V'})}, \rho|_{(E_d|_{V'})}, \delta|_{(E_A|_{V'})}, O_\alpha \rangle$$

where if $E \subseteq V \times V$ is a set of edges and $V' \subseteq V$ then $E|_{V'} = \{\langle v, u \rangle : v \in V' \text{ and } u \in V'\}$.

**Definition 12 (Difference of r-DagProofs).** Let $C$ and $D$ be two r-DagProofs, the graph difference $D - C$ is $D|_{(V(D) - V(C))}$.

The difference of r-DagProofs is not a r-DagProof itself. In the definition below we use the notation $D^{\uparrow}(k)$, where $D$ is a r-DagProof and $k \in V(D)$, to denote the biggest sub r-DagProof of $D$ that has $k$ as root. In the particular case that $D$ is a tree, $D^{\uparrow}(k)$ is the sub-tree of $D$ that has root $k$.

**Definition 13 (Detach and link sub r-DagProofs function).** Let $D$ be a r-DagProof of $\alpha$ and $k$ a node of $D$ that it is the root of an instance of the r-DagProof $C$ that is its matrix, as given by theorem $[\square]$. Let $i \in \mathbb{N}$, a label, and $C' = D^{\uparrow}(k)$. Define $D' = (D - C') \cup C$. We define $\text{DetachLink}(D, k, C, i)$ as following:

$$\langle V(D'), E_d(D'), E_A(D'), r(D'), L(D'), \rho(D'), \delta(D'), O_\alpha \rangle$$

Figures $[\circ]$ and $[\square]$ illustrate what happens when we apply three $\text{DetachLink}$ operations in an ambient r-DagProof. When we repeat this operation to every instance of a matrix $C$ occurring in a fixed level $\mu$, we say that we performed a collapse of the instances of $C$ in $D$ in level $\mu$. This operation is described by algorithm $[\square]$ and is denoted by $\text{Collapse}(C, D, \mu)$. 

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Consider a **EmND** proof that has more than one matrix $C$ with instances occurring super-polynomially many times in it. Thus, Lemma 9 shows that we can use theorem 7 to obtain the list of all matrices having a super-polynomially many instances occurring in a fixed level $\mu$ of this **EmND** proof. We remember that a matrix derivation/proof, in our terminology, is nothing but a sub-derivation/sub-proof that has at least one instance in other proof. This lemma obtains a set of Matrices having all instances occurring in the lowest level in a specific **EmND** proof. Moreover, no matrix instance is a proper sub-derivation of any other matrix instance in the set. We call this an independent set of matrices from $\Pi$ to this set.

**Definition 14** (Independent set of matrices in a proof/derivation). Let $\Pi$ be an **EmND** proof/derivation. A set of matrices

$$S = \{ \Pi_\nu : \Pi_\nu \text{ is a matrix in } \Pi \text{ which only has instance in a level } \nu \}$$

is an independent set of matrices, if and only if, there is no $\nu_1$ and $\nu_2$ levels in $\Pi$, $\nu_1 \neq \nu_2$, and instances $\Pi_1$ and $\Pi_2$ of $\Pi_{\nu_1}$ and $\Pi_{\nu_2}$, respectively, $\Pi_{\nu_1} \in S, 1 = 1, 2$, such that, $\Pi_1$ is a sub-derivation of $\Pi_2$, or $\Pi_2$ is a sub-derivation of $\Pi_1$.

Given a set $S$ of independent matrices of a proof/derivation, we can prove that if $\Pi_\nu \in S$ then no instance of $\Pi_\nu$ is sub-derivation of $\Pi_\mu \in S$, unless $\Pi_\mu = \Pi_\nu$. Thus, if $\Pi_\nu \in S$ then there is no level $\mu < \nu$ with instances that are super-derivations of instances of $\Pi_\nu$. In a certain sense, $\nu$ is a local lowest level.

**Lemma 9** (List of super-polynomially repeated matrices). For all $p \in \mathbb{N}, p > 3,$ and for all $\Pi \in S_\Lambda$, such that, $|T(c(\Pi))| = m$ and $|\Pi| > m^p$, then there is a set $M$ of independent matrices for sub-derivations instances of $\Pi$, such that, for every $\Pi_\xi \in M$, $\Pi_\xi$ has at least $m^{p-3}$ instances occurring in some level $\xi$ in $\Pi$.

**Proof.** of lemma 9. Using the conditions on lemma, Theorem 7 provides at least one level $\mu$ and a matrix $\Pi_s$ that has at least $m^{p-3}$ instances occurring in $\mu$. Thus, the sets

$$S_\nu = \{ \Pi_s : \Pi_s \text{ is a matrix having at least } m^{p-3} \text{ instances in } \nu \text{ in } \Pi \}$$

where $\nu$ is a level in $\Pi$, form a family. The family $(S_\nu)_{\nu \in \text{Lev}(\Pi)}$ has at least the non-empty set $S_\mu$, and hence, by defining the set $L$ of levels $\xi$, such that $S_\xi$ has

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$^2$The set $S_\Lambda$ is defined in definition 6
no instance in $\Pi$ in level $\xi$ that is a super-derivation of an instance of $\Pi_s \in S_\mu$, $\mu \neq \xi$. The set $\{\Pi_\xi : \xi \in L\}$ is an independent set of matrices. It is the biggest one, indeed. The inclusion of sub-derivations orders the levels ensures that the family’s member’s ordering is well-defined.

Lemma 9 is used to provide the initial list of instances to collapse in the algorithm.

Figure 6: Some instances of the matrix $C$ in the ambient r-DagProof $D$
We remember that \( r_C \) is the root of the r-DagProof \( C \) and \( \text{Starts}(C) \) is the set of initials of \( C \) as stated by definition [11].

We observe that the resulting (pre) r-DagProof yielded from the collapse in level \( \mu \) can be bigger than \( m^p \) yet. With the collapse of \( m^{p-3} \) sub-proofs/derivations, the size of the resulting r-DagProof is at least \( \frac{|D|}{|C| \times m^p-m} \). If the mentioned size of the resulting r-DagProof is bigger than \( m^p \), then there must be two main reasons: (1) The collapsed sub-proof/derivation is bigger than \( m^p \) by itself, or; (2) There must be more matrices in level \( \mu \) that we consider. The second alternative dealt proceeds by collapsing all instances of all matrices occurring at the lowest level. This is addressed in lines 3 to 11 of algorithm 2. For the first alternative, we only have to recursively find more redundant parts in the sub-proof/derivation that must exist by theorem 7 in the matrix that had all instances collapsed in the ambient r-DagProof. Since theorem 7 works on EmND proofs, as opposed to derivations with open assumptions, we need lemma 10 below.

**Lemma 10.** Let \( p \in \mathbb{N}, p > 3, \Pi \) be an EmND proof of a \( M_\alpha \)-tautology \( \alpha \), such that, \( |T(c(\Pi))| = m \) and \( |\Pi| > m^p \). According theorem 7 let \( \Pi_s \) be a matrix that has at least \( m^{p-3} \) instances occurring in the level \( \mu \) in \( \Pi \). If \( |\Pi_s| > m^p \) then there is a matrix \( \Pi'_s \), that is a sub-derivation of \( \Pi_s \), such that it has at least \( m^{p-3} \) instances in \( \Pi_s \) in some, fixed, level \( \nu > \mu \).

**Proof.** of 10. The main feature of theorem 7 is the fact that it works for EmND proofs and \( \Pi_s \) does not need to be a proof, it is instead a derivation with open

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Figure 7: Three DetachLink were applied in the ambient r-DagProof \( D \) of fig. 6.
assumptions. We can discharge in the correct order, dictated by the syntax tree of \( \alpha \) and the mapping \( l \) that defines the EmND \( \Pi_s \). To do this, we restrict \( l \) to \( \Pi_s \). This restriction does not yield an EmND yet, and we have to consider the syntax tree of the conclusion of \( \Pi_s \), i.e. the syntax tree \( T_{c(\Pi_s)} \). However, as we have already said, we do not have a tautology as the conclusion of \( \Pi_s \). We discharge all open assumptions in \( \Pi_s \), obtaining a conclusion \( \beta \) that is a tautology. The result is then an EmND that proves \( \beta \). The introduction part of the main branch does not disrupt the condition of definition 5. Finally, since \( \beta \) is smaller than \( \alpha \) we choose a new propositional variable \( q_{\text{new}} \) and define a new formula “\( q_{\text{new}} \supset q_{\text{new}} \supset \ldots \beta \)” with as many \( q_{\text{new}} \) repetitions as it is enough to have a formula of the same size as \( \alpha \). The EmND adjusted to this new formula is a proof of a tautology, and we can apply Theorem 7 to obtain a matrix \( \Pi_s' \) that has \( m^p - 3 \) instances in \( \Pi_s \). Note that \(|T_{c(\Pi_s)}| = |T_\alpha|\). Since there is only one main branch, the many instances of \( \Pi_s' \) occur above the original conclusion of \( \Pi_s \). Finally, the I-part of the main branch, from \( c(\Pi_s) \) down to “\( q_{\text{new}} \supset q_{\text{new}} \supset \ldots \beta \)” can be eliminated by merely deleting the introduction rules applied to draw “\( q_{\text{new}} \supset q_{\text{new}} \supset \ldots \beta \)”, including the rules used to prove \( \beta \). In this way we obtain the original matrix \( \Pi_s \) and the desired \( m^p - 3 \) instances of \( \Pi_s' \) that occurs in it. Finally, as \( \Pi_s' \) is sub-derivation of \( \Pi_s \) then the level \( \nu \) of its root is strictly above \( \mu \).

By the proof of Lemma 10, we can see that even in the case that \( \Pi_s \) is not an EmND proof, if \(|\Pi_s| > m^p\), with \( m = c(\Pi_s) \), then there is a sub-derivation of \( \Pi_s \) that it is repeated at least \( m^p - 3 \) many times in \( \Pi_s \).

**Corollary 11.** Let \( p \in \mathbb{N}, p > 3 \), \( \Pi \) be an EmND derivation of \( \alpha \), such that, \(|T(c(\Pi))| = m \) and \( \Pi_s \) a sub-derivation of \( \Pi \) that occurs in \( \Pi \) in level \( \mu \). If \(|\Pi_s| > m^p\) then there is a sub-derivation \( \Pi_s' \) of \( \Pi_s \) that has at least \( m^p - 3 \) instances occurring in a level \( \nu \) \( \mu \).

The above corollary is used to ensure the termination and correctness of algorithm in the section 4. Algorithm below, defines the operation of collapsing a list \( \mathcal{Y} \) of instances of the r-DagProof matrix \( \mathcal{I} \) occurring in level \( \mu \).

**Definition 15 (Collapse operation inside pre r-DagProofs).** Let \( \mathcal{D} = \langle V, E_d, E_A, r, l, l, p, \delta, \mathcal{O}_\alpha \rangle \)

\(^3\text{We remember that a matrix that occurs in a level } \mu \text{ in an r-DagProof } \mathcal{D} \text{ is any sub-r-DagProof of } \mathcal{D} \text{ that has root in level } \mu \text{ and may have some other instances in level } \mu \text{ too.}\)
be a pre r-DagProof for a $M \supset$ formula $\alpha$ and $C$ be a pre sub r-DagProof of $D$ and $Y$ the set of roots of the instances of $C$, occuring in level $\mu$ that will be collapsed. Algorithm[7] defines and computes the result of collapsing all instances of $C$ in only one in $D$.

Algorithm 1

Precondition: $D, C$, r-DagProofs, $C \prec D$, and the list $Y$ containing the roots of the instances of $C$
Ensure: a r-DagProof $D'$ having all instances of $C$ collapsed in a unique r-DagProof

1: Function $\text{Collapse}(D, Y, C)$
2: $D' \leftarrow D$
3: $j \leftarrow l_C(\text{root}(C))$
4: $i \leftarrow 1$
5: $Y \leftarrow \text{rest}(Y)$
6: for $k \in Y$ do
7: $D' \leftarrow DetachLink(C, k, D', j \oplus i); i \leftarrow i + 1$
8: end for
9: Return $D$

Examining algorithm[1], we have the Lemma[12] that provides an upper-bounded for the resulting r-DagProof after the collapses of all instances of the matrix in the ambient r-DagProof $D$.

Lemma 12. $|D'| \leq \frac{|D|}{\text{length}(Y) \times |C|}$

4 r-Dags as succinct certificates for $M \supset$ tautologies

In this section, we show how to use the Collapse operation defined in previous section[3] Algorithm[2] defines the operation of compression that collapses all redundancies that occur in any huge EmND proof. In line[3] the function $\text{Lemma}[9](T)$ returns the set of independent matrices in $T$ that exists by Lemma[9]. Since it is an independent set, by collapsing all of its instances in only one instance to each matrix, we do not need to collapse the upper levels that are not local lowest levels, the roots of the matrices’ elements in the independent set.
Algorithm 2 Compress an EmND proof $T$ using corol.

Precondition: Uses the global variable $m$ with value $|c(T)|$

Precondition: $3 < p \in \mathbb{N}$, $T$ is an EmND, $h(T) \leq |c(T)|$

Ensure: a r-DagProof $D$ proving $c(T)$ of size smaller than $|c(T)|^{p-3}$

1: Function Compress($T$, $p$)
2: if $(m^p < |T|)$ then
3: $\text{LocalLowestLevels} \leftarrow \text{MinLevel}(\text{Lemma 9}(T))$
4: $D \leftarrow T$
5: for $\text{lev} \in \text{LocalLowestLevels}$ upwards $h(T)$ do
6: $L \leftarrow \text{SuperPolySubProofs}(T, \text{lev})$
7: for $(\mathcal{Y}, C_{\mathcal{Y}}) \in L$ do
8: $D_{\mathcal{Y}} \leftarrow \text{Compress}(C_{\mathcal{Y}}, p)$
9: $D \leftarrow \text{Collapse}(D, \mathcal{Y}, D_{\mathcal{Y}})$
10: end for
11: end for
12: Return $D$
13: else
14: Return $T$
15: end if

Below we have some lemmas that are proved easily by inspecting the code of algorithms 1, 2 and definition 13.

**Lemma 13.** In algorithm 2 above, the number of recursive calls after an initial invocation of $\text{Compress}(T, p)$, with $m = |c(T)|$, is at most $|T|$.  

**Proof.** Since $m^p$ is constant during all the recursive calls, and the size of the first argument of $\text{Compress}$ is strictly smaller than each previous recursive call. Thus, there must be a call such that $|T| < m^p$. When $T$ is of this size, it is the recursive basis case, anyway. Taking the (worst) case into account, the number of recursive calls is upper-bounded by the size of the EmND $T$ itself. If any recursive call, on the first argument $T_1$, is such that $|T_1| > m^p$, then it obtains the sub-derivations that occurs at least $m^p - 3$, collapsing $\text{Compress}(T_1)$ into a unique compressed version of $T_1$. This is done with lesser than $|T|$ recursive calls. In fact, there is at least $m^p - 3$ collapses, so the total amount of recursive calls is at most $\frac{|T|}{|T_1| \times m^p}$. Lastly, the compression of $T_1$ is obtained in a unique call to $\text{Compress}$. 

Below we find a useful Lemma used to prove that after the compression any linear height bounded EmND super-polynomially bounded proof becomes a polynomial in size.

**Definition 16.** Given a EmND linearly height bounded proof $T$, such that, $|T| > |\alpha|^p$, $p \geq 3$. We denote by $\text{LRI}(T)$ the list of all lowest instances that occur at least $m^{p-3}$ times, for each of the matrices, as provided by Lemma 9.

$LRI$ is an acronym to LowestInstancesRedundant

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Definition 17 (EmND proofs difference operation). Let $T$ and $T_1$ EmND derivations, such that, $T_1$ is sub-derivation of $T$ occurring in level $\nu$. The difference derivation $T' = T - T_1$ obtains by removing all nodes of $T_1$ from $T$, but $c(T_1)$. Moreover, $l_{T'}$ is the restriction of $l_T$ to this $T'$ new derivation.

The difference of $T$ and the set of derivations $S$ is the iterating the difference of $T$ to each member of $S$.

The following Lemma[14] shows that for every proof of size bigger than $m^p$, when removing its redundant part, that exists in virtue of Theorem[7], what remains is a rDagProof of size less than $m^p$.

Lemma 14. For any EmND proof $T$ of a tautology $\alpha$ and $p > 3$, if $|T| \geq |\alpha|^p$ and $(T_1, \ldots, T_n) \in LRI(T)$ then

$$|T - \bigcup_{i=1}^{n} T_i| < m^p$$

Proof. Let $T_0 = T - \bigcup_{i=1}^{n} T_i$ and $k$ the lowest level corresponding to $LRI(T)$ and suppose that

$$|T_0| = \left|T - \bigcup_{i=1}^{n} T_i\right| \geq m^p$$

For each top-formula of $T_0$ that is the conclusion of any of $T_i$ we add the $\supset$-Intro part of its main branch including the minimal atomic formula. After adjusting the $l$ mapping appropriately, this yields a derivation of the minimal formula of the main branch of $T_0$ from the open atomic assumptions formed by the minimal atomic formula. From this minimal formula, we apply successive of $\supset$-I rules to discharge all minimal atomic top-formulas in $T_0$. We obtain an EmND proof of a formula $\alpha'$ that is $\alpha$ when we replace all initially discharged top-formula $\delta_i$ by $q_i$, the minimal formula of the branch that has $\delta_i$ as top-formula in $T_0$. We call $T_0'$ the proof of $\alpha'$ that we have just described above. We observe that

$$m^p < |T_0| < |T_0| + m \times m^{p-3} < T_0'$$

We also note that $|\alpha'| < |\alpha|$. Hence $|c(T_0')|^p < |T_0'|$ we apply Theorem[7] to $T_0'$ obtaining a list of instances of matrices that occurs at least $|c(T_0')|^{p-3}$ in $T_0'$. Finally we reconstruct $T$ putting back all instances removed, obtaining a contradiction, since this list of instances occurs below the level of Lowest Redundant Instances of $T$. We have then to conclude that $T_0$ is smaller than $m^p$. $\square$
Lemma 15. For any EmND proof $T$ of a tautology $\alpha$ and $p > 3$, if $|T| \geq |\alpha|^p$ then $|\text{Compress}(T, p)| < |\alpha|^p$.

Proof. We prove by induction on the number of recursive calls that $|\text{Compress}(T, p)| < |\alpha|^p$, for in Lemma 13 we have already proven that the algorithm stops for any valid input pair $\langle T, p \rangle$.

- Basis No recursive call: In this case we already have $|\text{Compress}(T, p)| < |\alpha|^p$. The “else” of the “if” in line 2 of algorithm 2 is used.

- I.H. Suppose $|T| \geq |\alpha|^p$ holds. Thus, a call to Lemma 9($T$), in line 3 return the list of all occurrences of sub-derivations, instances of the independent set of matrices given by Lemma 9 that occurs more than $mp^3$ in the lowest levels of $T$. By inductive hypothesis, there is less one recursive call, $\text{Compress}(T_i)$ of each instance $T_i$ of the list returns a r-Dag of size less than $|\alpha|^p$. By Lemma 14 the part of $T$ that it is not collapsed is less than $|\alpha|^p$ and by Lemma 12 we obtain that $|\text{Compress}(T, p)| < |\alpha|^p$.

The above upper-bound is not tight. A tighter one obtains by counting the number of matrices for each level, according to Theorem 7 and corollary 11. However, in this article, we do not need a tighter upper-bound than what we state in lemma 13.

5 Checking r-DagProofs in Polynomial Time

The following definitions are central in proving that r-DagProofs are certificates for $M \models$ formulas validity. Let $C = \langle V, E_d, E_A, r, l, L, \rho, \delta, \mathcal{O}_\alpha \rangle$ be a pre r-DagProof of $\alpha$ from $M \models$. We associate to each $v \in V$ an entailment relation. The entailment represents the logical consequence relation carried within $C$ from the DAG’s leaves downwards until $v$. Due to the collapse operation, and the many downward detours that the collapses introduce, we use an environment function that keeps track of the entailment relation related to each detour. We use the notation $\text{Emt}(\alpha)$ to denote the set $\{\Delta \vdash \beta : \Delta \subseteq \text{Sub}(\alpha) \text{ and } \beta \in \text{Sub}(\alpha)\}$ of all possible entailments between sets of (sub)formulas of $\alpha$, $\Delta$, and (sub)formulas of $\alpha$.

Definition 18. Let $\Delta \vdash \delta \in \text{Emt}(\alpha)$, and $\mathcal{O}$ be a total order on the subformulas of alpha. We define $\text{Ant}_\mathcal{O}(\Delta \vdash \delta) = b_\mathcal{O}(\Delta)$.
In what follows, consider a pre r-DagProof $C = \langle V, E_d, E_A, r, \ell, L, \rho, \delta, O_{\alpha} \rangle$ of $\alpha$. A node $v \in V$ is called a deductive leaf, iff, it has no incoming deductive edge, otherwise we call it as deductive (internal) node. The nodes of $C$ that have mode than one different Deductive Edges outcoming from it are called divergent nodes.

**Definition 19** (Local Entailment). Given a pre r-DagProof $C = \langle V, E_d, E_A, r, \ell, L, \rho, \delta, O_{\alpha} \rangle$ of $\alpha$, we define the mapping $M^C : V \times \mathbb{N} \rightarrow Emt(\alpha) \cup \{\vdash'\}$, for each $v \in V$ recursively as follows:

**Deductive Leaf, no Ancestrality** If $v \in V$ and, there is no $u \in V$, such that $\langle u, v \rangle \in E_d$ and, there is no $w \in V$, such that $\langle w, v \rangle \in E_A$ then $M^C(v, 0) = \{\ell(v)\} \vdash \ell(v)$, and $M^C(v, j) \vdash' i$, for $j \in \mathbb{N}$, $j \neq 0$, and;

**Deductive Leaf with Ancestrality** If $v \in V$ and, there is no $u \in V$, such that $\langle u, v \rangle \in E_d$ and, there is $w \in V$, such that $\langle w, v \rangle \in E_A$ then $M^C(v, i) = \{\ell(v)\} \vdash \ell(v)$, for each $i$, such that there is $\langle w, v \rangle \in E_A$ with $\delta(\langle w, v \rangle) = i$, and $M^C(v, j) \vdash' i$, for every $j \in \mathbb{N}$, such that, there is no $\langle w, v \rangle \in E_A$ with $\delta(\langle w, v \rangle) = j$, and;

**Deductive Internal Node, no Ancestrality** If $v \in V$ and, there is $u \in V$, such that $\langle u, v \rangle \in E_d$ and, there is no $w \in V$, such that $\langle w, v \rangle \in E_A$ then we have two cases:

1. There are only two $u_1, u_2 \in V$, such that, $\langle u_i, v \rangle \in E_d$, for $i = 1, 2$, $\ell(u_1) = \delta_1$, $\ell(u_2) = \delta_1 \supset \delta_2$ and, $\ell(v) = \delta_2$. Moreover, let $I_i = \{j : M^C(u_i, j) \neq \vdash' i\}$. Thus, we have that:

   **If** $I_1 = I_2$ **then** for every $j \in I_1 = I_2$, we have that:
   
   $M^C(v, j) = \begin{cases} \Delta_1 \cup \Delta_2 \vdash \delta_2 & \text{if } M^C(u_i, j) = \Delta_i \vdash \ell(u_i) \\ \text{otherwise} & \end{cases}$

   and, every $j \in \mathbb{N} - (I_1 \cup I_2)$, $M^C(v, j) \vdash' i$, Moreover, if $L(\langle u_i, v \rangle) \downarrow$, $i = 1, 2$, then $L(\langle u_i, v \rangle) = \text{Ant}_{\alpha}(M^C(u_i))$, and;

   **If** $I_1 \neq I_2$ **then** for every $j \in \mathbb{N}$, $M^C(v, j) \vdash' i'$

2. There is only one $u \in V$, such that, $\langle u, v \rangle \in E_d$, $\ell(u) = \delta_2$ and $\ell(v) = \delta_1 \supset \delta_2$. Moreover, let $I = \{j : M^C(u, j) \neq \vdash' i\}$. If $I \neq \emptyset$, and hence, for every $j \in I$, we have that:

   $M^C(v, j) = \begin{cases} \Delta - \{\delta_1\} \vdash \delta_1 \supset \delta_2 & \text{if } M^C(u, j) = \Delta \vdash \ell(u) \text{ and } \ell(u) = \delta_2 \\ \text{otherwise} & \end{cases}$

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If every $j \in \mathbb{N} - I$, $M^c_{L}(v, j) \models \vdash \ell'$, and; For every $j \in \mathbb{N}$, $M^c_{L}(v, j) \models \vdash \ell'$. Moreover, if $L(\langle u, v \rangle) \downarrow$ then $L(\langle u, v \rangle) = \text{Ant}_{O_0}(M^c_r(u))$. If $I = \emptyset$ then $M^c_{L}(v, j) \models \vdash \ell'$, for every $j \in \mathbb{N}$.

**Deductive Internal Node with Ancestrality** If $v \in V$ and, there is $u \in V$, such that $\langle u, v \rangle \in E_d$ and, there is $w \in V$, such that $\langle w, v \rangle \in E_A$ then we have two cases:

1. There are only two $u_1, u_2 \in V$, such that, $\langle u_i, v \rangle \in E_d$, for $i = 1, 2$, $\ell(u_1) = \delta_1$, $\ell(u_2) = \delta_1 \supset \delta_2$ and, $\ell(v) = \delta_2$. Moreover, let $I_i = \{ j : M^c_{L}(u_i, j) \neq \vdash \ell' \}$. Thus, we have that:
   
   **If $I_1 = I_2$ then** for every $j \in I_1 = I_2$, we have that:
   
   $$M^c_{L}(v, j) = \begin{cases} \Delta_1 \cup \Delta_2 \models \vdash \ell(u_i) & \text{if } M^c_{L}(u_i, j) = \Delta_i \models \ell(u_i) \\ \text{otherwise} & \end{cases}$$
   
   , and every $j \in \mathbb{N} - (I_1 \cup I_2)$, $M^c_{L}(v, j) \models \vdash \ell'$, and;

   **If $I_1 \neq I_2$ then** for every $j \in \mathbb{N}$, $M^c_{L}(v, j) \models \vdash \ell'$

2. There is only one $u \in V$, such that, $\langle u, v \rangle \in E_d$, $\ell(u) = \delta_2$ and, $\ell(v) = \delta_1 \supset \delta_2$. Moreover, let $I = \{ j : M^c_{L}(u, j) \neq \vdash \ell' \}$. If $I \neq \emptyset$, and hence, for every $j \in I$, we have that:

   $$M^c_{L}(v, j) = \begin{cases} \Delta - \{ \delta_1 \} \models \vdash \delta_2 & \text{if } M^c_{L}(u, j) = \Delta \models \ell(u) \text{ and } \ell(u) = \delta_2 \\ \text{otherwise} & \end{cases}$$
   
   , and every $j \in \mathbb{N} - I$, $M^c_{L}(v, j) \models \vdash \ell'$, and; For every $j \in \mathbb{N}$, $M^c_{L}(v, j) \models \vdash \ell'$. If $I = \emptyset$ then $M^c_{L}(v, j) \models \vdash \ell'$, for every $j \in \mathbb{N}$.

**Divergent Deductive Internal Node** In this case $v$ should not be the target of an ancestry edge, otherwise $C$ is not a valid pre rDagProof and $M^c_{L}(v, j) \models \vdash \ell'$, for every $j \in \mathbb{N}$. Moreover, the set $S = \{ \langle w : \langle v, w \rangle \in E_d \}$ has at least two node.\footnote{This is just the case for divergent deductive nodes} We have two cases to consider:

1. There are only two $u_1, u_2 \in V$, such that, $\langle u_i, v \rangle \in E_d$, for $i = 1, 2$, $\ell(u_1) = \delta_1$, $\ell(u_2) = \delta_1 \supset \delta_2$ and, $\ell(v) = \delta_2$. Moreover, let $I_i = \{ j : M^c_{L}(u_i, j) \neq \vdash \ell' \}$ and $T = \{ \rho(\langle v, w \rangle) : w \in S \}$. Thus, we have that:
If \( I_1 = I_2 = I \) then for every \( j \in I_1 = I_2 = T \), we have that:

\[
M^C(v, j) = \begin{cases} 
\Delta_1 \cup \Delta_2 \vdash \delta_2 & \text{if } M^C(u_i, j) = \Delta_i \vdash \ell(u_i) \\
\text{otherwise} & \end{cases}
\]

, and every \( j \in \mathbb{N} - (I_1 \cup I_2) \), \( M^C(v, j) = \vdash \ell' \), and;

If \( I_1 \neq I_2 \) or \( I_i \neq T \), \( i = 1 \) or \( i = 2 \) then for every \( j \in \mathbb{N} \),

\[
M^C(v, j) = \vdash \ell'
\]

2. There is only one \( u \in V \), such that, \( \langle u, v \rangle \in E_d \), \( \ell(u) = \delta_2 \) and \( \ell(v) = \delta_1 \supset \delta_2 \). Moreover, let \( I = \{ j : M^C(u, j) = \vdash \} \), \( T = \{ \rho(\langle v, w \rangle) : w \in S \} \). If \( T = \emptyset \) then for every \( j \in I \), we have that:

\[
M^C(v, j) = \begin{cases} 
\Delta - \{ \delta_1 \} \vdash \delta_1 \supset \delta_2 & \text{if } M^C(u, j) = \Delta \vdash \ell(u) \text{ and } \ell(u) = \delta_2 \\
\text{otherwise} & \end{cases}
\]

, and every \( j \in \mathbb{N} - I \), \( M^C(v, j) = \vdash \ell' \), and; For every \( j \in \mathbb{N} \), \( M^C(v, j) = \vdash \ell' \). Moreover, if \( L(\langle u, v \rangle) \downarrow \) then \( L(\langle u, v \rangle) = \text{Ant}_{\alpha}(M^C(u)) \).

If \( I = \emptyset \) then \( M^C(v, j) = \vdash \ell' \), for every \( j \in \mathbb{N} \).

**Target of Divergent Deductive Internal Node** In this case \( v \) is such that there is a divergent deductive node \( u \) with \( \langle u, v \rangle \in E_d \) and \( \rho(\langle u, v \rangle) \) is defined. Thus, we have two cases:

- **\( v \) is not target of an ancestry edge** \( M^C(v, 0) = M^C(u, \rho(\langle u, v \rangle)) \) and \( M^C(v, j) = \vdash \ell' \), for every \( j \neq 0 \);

- **\( v \) is target of an ancestry edge** There is \( \langle w, v \rangle \in E_A \). Hence we set \( M^C(v, \delta(\langle w, v \rangle)) = M^C(u, \rho(\langle u, v \rangle)) \) and \( M^C(v, j) = \vdash \ell' \), for every \( j \neq \delta(\langle w, v \rangle) \);

**Obs:** In what follows, sometimes we use the notation \( M^C(v, i) \) instead of \( M^C(v, i) \), whenever \( C \) can be easily inferred from the context.

A full subgraph of a graph \( A = \langle V_A, E_A \rangle \) is any graph \( B = \langle V_A, E \rangle \), with \( E \subseteq E_A \). The labelled version of full subgraph keeps all labels that label the elements of \( V_A \) and \( E \) with the same value they have in \( A \).

**Definition 20** (Underlying-deductive-structure of an rDagProof). Given a pre rDagProof \( C = \langle V, E_d, E_A, r, \ell, L, \rho, \delta, \mathcal{O}_\alpha \rangle \). The full sub-graph of \( C \) when we consider all and only all of the edges in \( E_d \) is denoted by \( C|_{E_d} \). It is called the underlying deductive struture of \( C \).
Definition 21 (maximal-path). Given a pre rDagProof $C = \langle V, E_d, E_A, r, \ell, L, \rho, \delta, O, \alpha \rangle$ and $v_k, \ldots, v_1, v_1 \in V$, such that, $\langle v_{i+1}, v_i \rangle \in E_d$, $i = 1, \ldots, k - 1$. We say that $v_1, \ldots, v_k$ is a maximal path in $C|E_d$, if and only if, $v_k$ is a top-formula. We say that the maximal-path starts in $v_1$.

The length of the sequence of nodes $v_k, \ldots, v_1$, is $k$.

Definition 22 (reverse-deductive height). Given pre r-DagProof $C = \langle V, E_d, E_A, r, \ell, L, \rho, \delta, O, \alpha \rangle$ of $\alpha$. Let $C|E_d$ the sub-graph of $C$ restricted to deductive edges only ($E_d$). The reverse deductive height of a node $v \in V$ in $C|E_d$, named $rdh(v)$ is defined as:

$$rdh(v) = \max \{k : v_1, \ldots, v_k \text{ is a maximal-path with } v_1 = v\}$$

The above definition 19 of $M_C \vdash$ is recursive. Given a pre rDagProof $C$, we can assign to each node $v \in V_C$ the value of $rhd(v)$. By the recursion theorem, from set theory, we have that the function $M_C \vdash$ is well-defined for every node $v$ and natural number $i$ in any rDagProof $C$. According to this assignment of values, we have that the value assigned to the root of $C|E_d$ is $h(C)$, the value of all of its leaves is 0 (zero) and the value of the children of any node is smaller than the value of their respective parent. Thus, $M_C \vdash$ is well-defined and unique for any $C$.

We note the following well-known facts, regarding usual Kripke semantics for $M_\supset$, denoted by $|\vdash_M\supset$.

Fact 1 (Soundness of ND $M_\supset$ rules). Consider $\Delta_1$ and $\Delta_2$ two sets of $M_\supset$ formulas, and, $\delta_1$ and $\delta_2$ two $M_\supset$ formulas. We have that:

1. If $\Delta_1 \vdash_M \supset \delta_1$ and $\Delta_2 \vdash_M \supset \delta_2$ then $\Delta_1 \cup \Delta_2 \vdash_M \supset \delta_2$, and;

2. If $\Delta_1 \vdash_M \supset \delta_2$ then $\Delta_1 - \{\delta_1\} \vdash_M \supset \delta_2$

Note that the above items are just the $\supset$-Intro and $\supset$-Elim rules. Concerning Item[2] we have both cases $\delta_1 \in \Delta_1$ and $\delta_1 \notin \Delta_1$, as it is the case with the $\supset$-Intro rule.

In what follows we omit the symbol $M_\supset$ in the notation $|\vdash_M\supset$ whenever its meaning as the minimal entailment is made clear.

Definition 23 (rDagProof correctness). Let $C = \langle V, E_d, E_A, r, l, L, \rho, \delta, O, \alpha \rangle$ be a pre r-DagProof. We say that $C$ is correct iff $M_r(r, 0) \neq \lnot'\lnot'$ and, for each $i \neq 0$, $i \in \mathbb{N}$, $M_r(r, i) = \lnot'\lnot'$.
When a pre rDagProof C is correct we simply call it rDagProof. Given a correct rDagProof C, such that, \( M^C_r_C, 0) = \Delta \vdash \beta \), we have that C is a certificate that \( \beta \) as logical consequence of \( \Delta \) in \( M_\Sigma \). This is what we state in Lemma 16 below.

**Lemma 16 (Local entailment sounds).** Let \( C = \langle V, E_d, E_A, r, l, L, \rho, \delta, O_a \rangle \) be a correct rDagProof of \( \alpha \). Thus, for every \( v \in V \), for every \( i \in \mathbb{N} \), such that \( M_C(v, i) \neq \vdash \), then, if \( M^C_v(v, i) = \Delta \vdash \ell(v), \ell(v) = \beta \), then \( \Delta \models_{M_\Sigma} \beta \).

**Proof.** By induction on the definition of \( M_\Sigma \) definition and using Lemma 1.

**Corollary 17.** If in the stating of Lemma 16 above, we consider that:

- There is a formula \( \delta \), subformula of \( \alpha \), such that, it is top-formula in \( C \) and, there is no deductive path from this top-formula to the root \( r \) of \( C \) that applies an \( \supset \) introduction rule having \( \delta \) as the antecedent of the formula that it is the conclusion of this application.

Then \( M^C_v(v, i) = \Delta \vdash \ell(v), \ell(v) = \beta \), with \( \delta \in \Delta \).

**Proof.** This corollary is a consequence of the proof of Lemma 16. Its proof is an extension of the induction proof of the lemma, by the inclusion the condition in the statement and verify that the formula \( \delta \) is not removed during the evaluation of \( M^C_v(v, i) \). Since it is a top formula, \( \delta \) is included in the local entailment antecedent, in the basic step of the induction and, we do not remove it anymore.

**Theorem 18 (Completeness of rDagProofs).** For any \( M_\Sigma \) formula \( \alpha \) and set of subformulas \( \Delta \) of \( \alpha \) and subformula \( \beta \) of \( \alpha \), we have that if \( \Delta \models_{\Sigma} \beta \) holds then there is a correct rDagProof \( C \), such that, \( M^C_r_C, 0) = \Delta' \vdash \beta \), with \( \Delta' \subseteq \Delta \).

**Proof.** The system of Natural Deduction for \( M_\Sigma \) is sound and complete regarded the usual Kripke semantics for \( M_\Sigma \). Since \( ND_{M_\Sigma} \) proofs are particular cases of rDagProofs we have completeness of rDagProofs. Thus, if \( \Delta \models \beta \) then there is a derivation \( \Pi \) having \( \beta \) as conclusion and a set \( \Delta' \) of open assumptions, with \( \Delta' \subseteq \Delta \). Taking \( \Pi \) as a rDagProof and by Corollary 17 we have that \( M^C_r_C, 0) = \Delta' \vdash \beta \).

**Theorem 19 (Soundness of rDagProofs).** If \( C \) is a correct rDagProof and \( M^C_r_C, 0) = \Delta \vdash \beta \) then \( \Delta \models_{\Sigma} \beta \).

**Proof.** This theorem is an immediate consequence of Lemma 16.
Corollary 20. If $C$ is a correct $rDagProof$ of $\alpha$ then $\alpha$ is a $M \supset$ tautology

We have the following lemmata that help us to show that a correct $rDagProof$, compressed by the technique that algorithm 2 implements, is sound. Moreover, in the next section, we show an algorithm that checks whether a pre $rDagProof$ is correct or not. This verification is efficient (linear) on the size of the pre $rDagProof$.

In the next section, we show that for any Natural Deduction $\Pi$ that has its height linearly bounded by the size of its conclusion, $Compress(\Pi)$ is a correct $rDagProof$ of $\alpha$. From this result, we can conclude that any $M \supset$ tautology has a succinct (polynomial) and correct $rDagProof$. In conclusion, we describe how to use this result to show that $NP = CoNP$.

6 The compression of linearly height-bounded $rDagProof$s preserves soundness

This section contains some lemmata that help us to prove that the correctness of $rDagProof$s are preserved by the compression of $rDagProof$s.

Definition 24 (A-consistent sub-$rDagProof$). Let $D$ be a pre $rDagProof$ of $\alpha$ and $C$ a sub-$rDagProof$ of $D$. We say that $C$ is an A-consistent sub-$rDagProof$ of $D$, if and only if, $D \uparrow$ is graph-isomorphic to $C$ and for every $\langle u, v \rangle \in E_A$, we have that $v \in V_C$ and $u \notin V_C$, if and only if, $v \in V_{D \uparrow k}$ and $u \notin V_{D \uparrow k}$.

Lemma 21 (Local Entailment preservation under DetachLink). Let $D$ be a pre $rDagProof$ and $k$ a node of $D$ that is the root of an instance of a A-consistent sub-$rDagProof$, $C$ of $D$. Let $i \in \mathbb{N}$ be such that $i$ does not label any of the $E_d$ edges going out of $k$. Moreover, consider $D' = DetachLink(D, k, C, i)$ as defined in Definition 13. We have that for every $v \in V_{D'}$ and $j \in \mathbb{N}, j \neq i$, the following conditions hold:

- If $v \notin V(D) \uparrow k$ then $M^C_D(v, j) = M^{D'}_C(v, j)$, and;
- If $v \in V(D) \uparrow k$ then $M^C_L(h^{-1}(v), j) = M^{D'}_L(v, j)$, and;
- If $v \in V(D) \uparrow k$ then $M^C_L(h^{-1}(v), 0) = M^{D'}_L(v, i)$;

Where $h$ is the (full) labeled graph-isomorphism from $C$ into $D \uparrow k$. 
Proof. By inspecting Definition 13, we observe that \( D' = (D - C') \cup C \), where \( D \uparrow k = C' = h(C) \). We note that \( M_{D'} \) definition is then either on the complement of \( D \uparrow k \), or on \( C \). The former agrees with \( M_D \) and the later agrees with \( M_{D \uparrow k} = M_{h(C)} \). Finally, the third item in the statement of the lemma is related to the new edge that links the root of \( C \) to the former target of the \( E_d \) edge that linked \( k \) with this target. The top-formulas that are related to the basis steps of \( M \), recursion are accordingly accordingly associated to each respective part of \( D' \), the same can be said about the recursive steps. So we have the desired result.

\[ \square \]

**Corollary 22** (Soundness of DetachLink). Consider the conditions of Lemma 27 above. We have that if \( C \) is correct and \( D \) is also correct then DetachLink\((D, k, C, i)\) is correct. Moreover, the local entailment is preserved, modulo the isomorphism between \( D \uparrow k \) and \( C \).

An immediate consequence of Corollary 22 is that the DetachLink operation can be repeated many times, without disturbing the soundness of the yielded rDagProof. Due to this, we have the following lemma.

**Lemma 23** (Soundness of Collapse). Let \( D \) and \( C \) be r-DagProofs. Let \( C \) be A-consistent subgraph of \( D \). Let \( Y \) be a list containing the roots of the instances of \( C \) in a fixed level \( \mu \). We have that if \( D \) and \( C \) are correct then Collapse\((D, Y, C)\) is correct too. Moreover the local entailment is preserved as stated in Lemma 27.

The above Lemma 23 is the correctness proof of algorithm 1.

The following theorem proves the correctness of algorithm 2.

**Theorem 24** (Soundness of Compressed rDagProofs). Let \( D = \langle V, E_d, E_A, r, l, L, \rho, \delta, O_\alpha \rangle \) be a pre rDagProof for a \( M \)-formula \( \alpha \). If \( D \) is correct then for every \( 3 < p \) Compress\((D, p)\) is correct. Moreover, the local entailment is preserved, i.e.,

\[ \square \]
7 On the complexity of verifying that a pre rDag-Proof is correct or not

The following algorithm performs a top-down sweeping in any pre rDagProof to check whether it is correct or not. It is an iterative implementation of $M \vdash$ that prints can check whether the pre rDagProof is correct, and certifies a tautology or not. In the case, it is not a tautology it prints “DERIVATION”. Finally, it prints “INCORRECT” if the rDagProof is not correct. The definition of $M$ points out the correctness of the algorithm. We analyse its computational complexity in the sequel. We have to note that the update – and – check, inside the iteration structures in lines 17 and 35 is responsible by updating the local entailment data-structure (the $Reg$ indexed structure) with ′ $\vdash$ ′ to indicate that the checking algorithm detected an incorrect pre rDagProof.
Algorithm 3 Verifies whether a r-DagProof is valid

1: Function Check-rDagProof(C)
2: for k = height(C) downto 0 do
3: L ← TopFormulas(k)
4: for top ∈ L do
5: Reg(top) ← ∅
6: for edge ∈ AncestorEdges(top) do
7: Reg(top) ← Reg(top) ∪ {δ(edge) → ℓ(top) ⊢ ℓ(top)}
8: end for
9: Reg(top) ← Reg(top) ∪ {0 → ℓ(top) ⊢ ℓ(top)}
10: end for
11: I ← InternalNodes(k)
12: for v ∈ I do
13: q ← Premisses(v)
14: if Divergent(v) then
15: Lv ← {⟨v, w⟩ : ⟨v, w⟩ ∈ EA}
16: if {i : Defined(Reg(q, i))} = {ρ(e) : e ∈ Le} then
17: for i ∈ Reg(q) do
18: Reg(v, i) ← update – and – check(v, q, i)
19: end for
20: else
21: Reg(v, 0) ← ‘ ⊢ ’
22: end if
23: else
24: if TargetDivergent(v) then
25: u ← {u : (u, v) ∈ Ed}
26: if Defined(ρ(u, v)) ∧ ¬∃(w, v) ∈ EA then
27: Reg(v, 0) ← Reg(u, ρ((u, v)))
28: Reg(v, j) ← ‘ ⊢ ’ ∀j ≠ 0
29: end if
30: if Defined(ρ(u, v)) ∧ ∃(w, v) ∈ EA then
31: Reg(v, δ((w, v))) ← Reg(u, ρ((u, v)))
32: Reg(v, j) ← ‘ ⊢ ’ ∀j ≠ δ((w, v))
33: end if
34: else
35: for i ∈ Reg(q) do
36: Reg(v, i) ← update – and – check(v, q, i)
37: end for
38: end if
39: end if
40: end for
41: if Reg(r) ≠ ‘ ⊢ ’ then
42: CORRECT
43: if Reg(r) == ⊢ ℓ(r) then
44: TAUTOLOGY
45: else
46: DERIVATION
47: end if
48: else
49: INCORRECT
50: end if
51: end if

Let C = ⟨V, Ed, EA, r, l, L, ρ, δ, Oα⟩ be a pre rDagProof. We set n_v = |V|, n_A = |EA|, m = |Oα| = |Ta| ≤ len(α) and h = height(C) = height(⟨V, Ed⟩).

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In the sequel, all the line references are in algorithm 3. We proceed to a worst case analysis to find an upper-bounded for the number of steps to check whether \( C \) is correct or not. The loop that starts in line 2 consumes \( h \) steps, for each of these steps, we have at most \( n_v \) possible nodes that are top-formulas, this is what line 4 sweeps using the “for” statement. Inside this “for” there is other nested iteration on the set of ancestor edges that target the top-formulas, and a consequent updating in the list of local entailments stored in the \( Reg \) data-structure. Line 9 is responsible by the update of the main local-entailment (indexed by 0). After that, the loop that starts in line 12 takes care of the internal nodes, and for each internal node, we have two possible cases, either it is a target of a divergent node and the needed update on the local indexed by the index 0 is made, or, the update of all indexes, including the 0, of the local entailment structure is made. The choice is made by the “if” statement in line 24. The cost of with steps that takes care of the top-formulas is \( h \times n_v \times n_A \). To this we have to add the cost of processing the internal nodes that is \( h \times n_v \times n_A \) too, due to the cost of updating the indexes related to each Ancestor edge in \( E_A \). However, as top-formulas and internal nodes are disjoint then we have that algorithm 3 when applied on \( C \) performs the number of steps upper-bounded by disequality 1:

\[
\text{Steps}(C) \leq h \times n_v \times n_A \quad (1)
\]

\[
\text{Steps}(C) \leq n_v \times n_v^3 = n_v^4 \quad (2)
\]

Observing that \( n_A \leq n_v^2 \) and \( h \leq n_v \) we have the upper-bound in disequality 2. Since we are counting steps, we can say that the time complexity to check whether a \texttt{prDagProof} \( C \) is correct or not is polynomial, 4th power indeed, on the size of \( C \).

8 A brief argument towards \( CoNP = NP \)

We have already discussed in the introduction of this text that when considering the complexity class \( CoNP \), we are naturally limited to linearly height-bounded proofs. The proofs, in \( M_C \), of the non-hamiltonianicity of graphs, are linearly height bounded. See the appendix in [13] or [9] for a detailed explanation on this. If \( NP \neq CoNP \) then the set of non-hamiltonian graphs there is no polynomially sized and verifiable in polynomial time certificate for each of its elements. Thus, the set \( S \) of all formulas that have Normal Natural Deduction proofs linear height-bounded contains the valid for the non-hamiltonian graphs. Hence, by assuming
that $NP \neq CoNP$, we have to conclude that $S$ is a family of normal super-polynomial proofs with linear height. If we consider any proof in $S$, either it is polynomially sized, and we have nothing to prove, or it is bigger than $m^p$, for some $p > 3$, where $m$ is the size of the proof’s conclusion. We observe that the case $p \leq 3$ is subsumed by $p > 3$, anyway. Thus, we can apply Theorem 7 to show that this big proof is redundant, so we can apply the compression algorithm 2 to obtain a correct rDagProof of size smaller than $m^p$, according to Lemma 15. Finally, the algorithm 3 can check the correctness of this polynomially sized rDagProof in time upper-bounded by $m^{4p}$.

The last paragraph provided a precise argumentation showing polynomial certificates for each non-hamiltonicity of each non-hamiltonian graph. We can check each of them is a (correct) certificate in polynomial time too. We can conclude that $CoNP \subseteq NP$, since non-hamiltonicity of graphs is a $CoNP$-complete problem. Having proved that $CoNP \subseteq NP$ we have proof that $NP = CoNP$, as the following reasoning shows, where $\overline{L}$ is the set-theoretical complement of $L$. We have used the logically simplest definition of the class $CoNP$ class as $\{ L : L \in NP \}$.

\[
\Rightarrow CoNP \subseteq NP \\
\downarrow \\
L \in NP, \text{ iff, } \overline{L} \in CoNP, CoNP \subseteq NP \text{ so } \overline{L} \in NP, \text{ iff, } L \in CoNP \\
\downarrow \\
CoNP = NP
\]

More details and mathematical precision on this argument contain an alternative proof to the conjecture $CoNP = NP$ and are a matter for a further article.

9 Conclusion

This article shows that for any huge proof of a tautology in $M$, we obtain a succinct certificate for its validity. Moreover, we offer an algorithm able to check this validity in polynomial time on the certificate’s size. We can use this result to provide a compression method to propositional proofs. Moreover, we can efficiently check the compressed proof without uncompressing it. Thus, we have many advantages over traditional compression methods based on strings. The compression ratio of techniques based on collapsing redundancies seems to be bigger, as shown in [22] that reports some experiments with a variation of the
Horizontal Compression method compared with Huffman compression. The second and more important advantage is the possibility to check for the validity of the compressed proof without having to uncompress it. In general, the original proof is huge, super-polynomial and hard check computationally.

Another application of the results in this article is to provide an alternative proof of $NP = CoNP$. In [8] we have a proof that $NP = NPSPACE$. An immediate consequence of this equality is that $NP = CoNP$. The approach that arises from the results we have shown here does not need Hudelmaier [15] linearly bounded sequent calculus for M ⊃ logic. The proof reported in [8], on the other hand, needs Hudelmaier Sequent Calculus and a translation to Natural Deduction proofs that preserves the linear upper-bound. However, the resulted translation is not normal, and it is well-known that normalization does not preserve upper-bounds in general. Thus, we cannot apply our approach to the whole class of $M \supset$ tautologies to prove that $NPSPACE \subseteq NP$, for the use of normal proofs is essential to obtain the redundancy lemma, i.e., Lemma 7. However, the compression method reported in this article, due to the redundancy lemma, provides knowledge to prove $M \supset$ short tautologies automatically. It seems easier than the use of the double certificate approach in [8].

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