Erratum: Initial fields and instability in the classical model of the heavy-ion collision

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We correct a mistake in the analytical expression for the energy density given in Phys. Rev. C 76, 021902 (2007) [arXiv:0704.3625 [hep-ph]]. The expression should be multiplied by 16. One question then arises; how could it be possible to explain this difference between the analytical and numerical results in the same model if both are correct? We find a subtle problem in the treatment of the randomness of the color source along the longitudinal direction and the treatment of the longitudinal extent of the color source.

The initial energy density given in Ref. [1] should be multiplied by 16, that is, Eq. (6) should be corrected as

$$ \frac{g^2}{(g^2 \mu)^4} \cdot 2 \langle \text{tr}(B_n^0)^2 \rangle = \frac{1}{2} N_c (N_c^2 - 1) \sigma^2, \quad (1) $$

and, accordingly, Eq. (8) should be corrected as

$$ \frac{g^2}{(g^2 \mu)^4} \cdot \varepsilon_{(0)} = 12 \sigma^2. \quad (2) $$

Also, not only the $\tau^0$-order terms but the $\tau^2$-order terms missed the same factor 16. Equation (11), therefore, should be

$$ \frac{g^2}{(g^2 \mu)^4} \cdot 2 \langle \text{tr}(B_n^{(2)} B_n^{(0)}) \rangle = \frac{g^2}{(g^2 \mu)^4} \cdot 2 \langle \text{tr}(E_n^{(2)} E_n^{(0)}) \rangle $$

$$ = - \frac{1}{2} N_c (N_c^2 - 1) \sigma \cdot \chi, \quad (3) $$

and Eq. (13) should be

$$ \frac{g^2}{(g^2 \mu)^4} \cdot \langle \text{tr}(B_n^{(2)} B_n^{(2)}) \rangle = \frac{g^2}{(g^2 \mu)^4} \cdot \langle \text{tr}(E_n^{(2)} E_n^{(2)}) \rangle $$

$$ = \frac{1}{4} N_c (N_c^2 - 1) \sigma \cdot \chi. \quad (4) $$

As a result, the estimate for the initial energy density in Eq. (14) (which had a typo; $\pi$ in the square brackets should be $\frac{1}{2}$), its resummed form in Eq. (15), and the compact formula in Eq. (16) in the continuum limit should be corrected respectively as

$$ \frac{g^2}{(g^2 \mu)^4} \cdot \varepsilon \simeq \frac{1}{2} N_c (N_c^2 - 1) \sigma \left[ \sigma - \frac{1}{2} \left( \frac{g^2 \tau^2}{g^2 \mu^2} \right)^2 \right], \quad (5) $$

$$ \simeq 12 \left\{ \frac{1}{4 \pi} \ln \left[ \frac{c^2 \left( g^2 \mu L \right)^2}{(g^2 \mu a)^2 + \pi (g^2 \mu \tau)^2} \right] \right\}^2, \quad (6) $$

and

$$ \varepsilon = \frac{3 (g^2 \mu)^4}{\pi^2 g^2} \left[ \ln \left( \Lambda_{QCD} / \tau \right) \right]^2. \quad (7) $$

We remark that the first term of above Eq. (6) has the same overall factor as given in Eqs. (14) and (16) in Ref. [2]. Our calculations [2] make use of the technique developed in Ref. [3] (the necessary correlation function is obtained from $\mathcal{N}^{(b)}$ derived as Eq. (75) in Ref. [3]), while Eqs. (16) in Ref. [2] seems to be based on Eq. (2.23) in Ref. [1]. These are independent calculations with different cut-off prescriptions, but of course, the coefficients in front of the logarithmic singularity should coincide with each other.

In a quantitative sense, in fact, the analyses on the Glasma instability should be affected by the missing factor 16 through Eqs. (22) and (23) in Ref. [1]. Our discussion on instability is not beyond the qualitative level, however, and the essential idea for the possible instability mechanism still works. Also, in the first part of Ref. [1], no modification is required in the essential ideas; the energy density at $\tau = 0$ has a logarithmic divergence and the expansion in terms of $\tau$ is singular around $\tau = 0$. The resummed form in the logarithmic ansatz is a natural choice to remove the singularity at finite $\tau$.

One question arises immediately, however. The initial energy density has been evaluated in the same model and the same cut-off prescription in different two methods. If both methods equally work well, two results must be identical apart from a small discrepancy originating from the lattice and continuum formulations. Why can one differ from the other?

This problem is so interesting on its own that it may deserve another publication [4], but we shall briefly see where the difference stems from. In short, the important point is that the McLerran-Venugopalan (MV) model implemented in the numerical calculation is not faithful to the analytical formulation. The subtlety originates from how to define the ill-defined expression of Eq. (4) in Ref. [1]. We should introduce some regularization to write it in a form of

$$ V_m^{(\perp)}(\mathbf{x}_\perp) = \mathcal{P} \exp \left[ -ig \int dz \frac{1}{\partial^2_{\perp}} \rho_0(\mathbf{x}_\perp, z^-) \right]. \quad (8) $$

Here we defined the regularized color source as

$$ \lim_{\tau \to 0} \rho_0(\mathbf{x}_\perp, z^-) = \rho(\mathbf{x}_\perp) \delta(z^-). \quad (9) $$

Another delta function in the longitudinal direction appears in the correlation function in terms of the sources,

$$ \langle \rho_a(\mathbf{x}_\perp, x^-) \rho_b(\mathbf{y}_\perp, y^-) \rangle \approx \frac{g^2 \mu^2}{\delta(\mathbf{x}_\perp - \mathbf{y}_\perp) \delta(x^- - y^-)} \delta(x^- - y^-). \quad (10) $$
where we replaced the delta function by the regularized one in the longitudinal direction such that,
\[
\lim_{\zeta \to 0} \delta_\zeta(x^-) = \delta(x^-) .
\] (11)

The question we are addressing here is whether we are allowed to adopt the following simplification:
\[
V^\dagger(x_\perp) \to V^\dagger(x_\perp) = \exp\left[ -ig \frac{1}{\partial_\perp} \rho_\perp(x_\perp) \right],
\] (12)

to prevent the delta function from appearing at all. The numerical calculations commonly make use of Eq. [12] for practical reasons with hope that the final answer would not depend on this replacement (see, e.g. Eq. (36) in Ref. [3], Eq. (4) in Ref. [6], and so on).

Using the notations we introduced above, we can reiterate the question more rigorously. That is, we shall check,
\[
\lim_{\zeta \to 0} \mathcal{O}[V_\epsilon] = \lim_{\epsilon \to 0} \frac{1}{\zeta} \lim_{\epsilon \to 0} \mathcal{O}[V_\epsilon] = \frac{1}{\zeta} \lim_{\epsilon \to 0} \mathcal{O}[V_\epsilon].
\] (13)

The left-hand side corresponds to the numerical implementation and the right-hand side to the analytical formulation of the MV model.

We treat more general cases in a separate literature [2] and will limit the current discussion only to the tadpole calculation, namely, \( \mathcal{O}[V] = V^\dagger \). This simplest choice is, as we will see, enough to exemplify two limits in Eq. (13) and will limit the current discussion only to the tadpole simulation and the right-hand side to the analytical formulation of the MV model.

We already know the analytical answer for the right-hand side of Eq. (13). That is by [3,8]
\[
\lim_{\epsilon \to 0} \frac{1}{\zeta} \lim_{\epsilon \to 0} \langle V^\dagger_\epsilon \rangle = \exp\left[ -g^4 \mu^2 \frac{N_c}{4N_c} - \frac{1}{\eta} \right],
\] (14)

where \( \mu^2 = \int dx x^2 (\bar{\psi} \psi) \) and
\[
\eta = \frac{a^2}{4L^2} \sum_{n_1=-L/2}^{L/2} \frac{1}{\left[ 2 - \cos(2\pi n_1/L) - \cos(2\pi n_2/L) \right]^2}.
\] (15)

To evaluate the left-hand side, we have to perform the following Gaussian integral,
\[
\lim_{\zeta \to 0} \lim_{\epsilon \to 0} \langle V^\dagger_\epsilon \rangle = \int [d\rho] \exp\left[ -i g \frac{1}{\partial_\perp} \rho_\perp(x_\perp) \right] \times \exp\left[ - \int d^2 x_\perp \frac{\rho_\perp(x_\perp) \rho_\perp(x_\perp)}{2g^2 \mu^2} \right],
\] (16)

where \( t^a \)'s are the SU\((N_c)\) algebra. It is hard to do this integration for arbitrary SU\((N_c)\) group, while the SU\((2)\) case is immediately doable.

In the case of SU\((2)\) (i.e. \( N_c = 2 \)), the above Gaussian integral leads to
\[
\lim_{\epsilon \to 0} \lim_{\zeta \to 0} \langle V^\dagger_\epsilon \rangle = \left( 1 - g^4 \mu^2 - \frac{1}{4} \right) \exp\left[ -g^4 \mu^2 \frac{3}{8} \right],
\] (17)

which is obviously different from the right-hand side with \( N_c = 2 \) substituted, that is,
\[
\lim_{\epsilon \to 0} \frac{1}{\zeta} \lim_{\epsilon \to 0} \langle V^\dagger_\epsilon \rangle = \exp\left[ -g^4 \mu^2 \frac{3}{8} \right],
\] (18)

from Eq. (14).

Interestingly, though Eqs. (17) and (18) look quite different, the curvature near \( \eta \approx 0 \) is the same. As we report in Ref. [7], the discrepancy between the left-hand and right-hand sides in Eq. (13) becomes significant when the singlet component survives unsuppressed by the system size. In fact, the expectation value of the gauge fields, \( \langle V_\epsilon \partial_\epsilon V^\dagger_\epsilon \cdot V_\epsilon \partial_\epsilon V^\dagger_\epsilon \rangle \), leads to different energy densities by a factor.

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