Non Trivial Saddle Points and Band Structure of Bound States of the Two Dimensional $\mathcal{O}(N)$ Vector Model

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Abstract

We discuss $\mathcal{O}(N)$ invariant scalar field theories in 0+1 and 1+1 space-time dimensions. Combining ordinary “Large $N$” saddle point techniques and simple properties of the diagonal resolvent of one dimensional Schrödinger operators we find exact non-trivial (space dependent) solutions to the saddle point equations of these models in addition to the saddle point describing the ground state of the theory. We interpret these novel saddle points as collective $\mathcal{O}(N)$ singlet excitations of the field theory, each embracing a host of finer quantum states arranged in $\mathcal{O}(N)$ multiplets, in an analogous manner to the band structure of molecular spectra. We comment on the relation of our results to the classical work of Dashen, Hasslacher and Neveu and to a previous analysis of bound states in the $\mathcal{O}(N)$ model by Abbott.

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I Introduction

Field theories involving a very large number \( N \) of components have turned out to be an extremely useful tool in addressing many non-perturbative aspects of quantum field theory and statistical mechanics \([1, 2, 3]\). In such theories the action is proportional to \( N \) (or in some cases, to \( N^2 \)) which plays the role of \( 1/\hbar \). Therefore, as \( N \to \infty \), the path integral of such field theories is dominated by the saddle points of the action, which are protected against being washed out by quantum fluctuations in this limit.

An important subclass of such field theories are the \( \mathcal{O}(N) \) vector models whose dynamical variables are \( N \) real scalar fields arranged into an \( \mathcal{O}(N) \) vector \( \Phi \) with \( \mathcal{O}(N) \) invariant self interactions of the form \( (\Phi^2)^n \) which have been studied intensively in the past \([4, 5, 6, 11]\). More recently, \( \mathcal{O}(N) \) vector models have attracted interest \([12, 13, 14, 15, 16]\) in relation to the so called “double scaling limit” \([3]\). The cut-off dependent effective field theory has also been discussed very recently \([17]\). The main object of these studies were the true vacuum state of the \( \mathcal{O}(N) \) model, and the small quantum fluctuations around it. In the case of quartic interactions, of interest to us here, the theory is best analyzed by introducing into the lagrangian an auxiliary field \( \sigma \) that can be integrated out by its equation of motion \( \sigma = g^2 N \Phi^2 \) (where \( g^2/N \) is the quartic coupling in the lagrangian)\(^2\). In this way the original fields \( \Phi \) may be integrated out exactly, yielding an effective non-local action \( S_{\text{eff}}[\sigma] \) in terms of \( \sigma \) alone.

The vacuum and the low lying states above it are described by the small fluctuations of the \( \sigma \) field around a certain local minimum of \( S_{\text{eff}}[\sigma] \). In this minimum the \( \sigma \) field configuration is homogeneous (i.e. space-time independent) \( \sigma = \sigma_c \), where the constant \( \sigma_c \) (which is the true mass of the \( \Phi \) “mesons” around that vacuum)\(^1\)

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\(^1\)We concentrate on the case of quartic interactions \( n = 2 \)

\(^2\)For higher interactions we need an additional lagrange multiplier field.

\(^3\)This minimum may not be the absolute minimum of \( S_{\text{eff}}[\sigma] \) in certain circumstances \([10, 11]\) but the decay rate of the metastable vacuum is suppressed by a very small tunnelling factor of the order of \( e^{-NV} \), where \( V \) is the volume of space-time.
is obtained from the gap equation $\frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma}\big|_{\sigma=\sigma_c} = 0$. Homogeneity of the vacuum $\sigma$ configuration is expected due to Poincaré invariance of the latter.

$S_{\text{eff}}[\sigma]$ is a rather complicated functional of $\sigma$ and in addition to $\sigma = \sigma_c$ it possesses yet many other extremal points at which the solution to the extremum condition $\frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma} = 0$ are space-time dependent $\sigma$ configurations $\sigma(x)$. These extrema cannot describe the ground state (and indeed, neither a metastable ground state) and are thus saddle points of $S_{\text{eff}}[\sigma]$ rather than (local) minima. Indeed, it can be shown by direct calculation\(^4\) that the second variation around them has an infinite number of negative eigenvalues. They correspond to collective “heavy” $\mathcal{O}(N)$ singlet excitations of the $\Phi$ field, where its length squared oscillates in isotopic space as a function of space and time.

In this work we find explicit exact formulae for these $\sigma(x)$ configurations in $0 + 1$ and in $1 + 1$ space time dimensions. In the two dimensional case we discuss only time independent $\sigma$ configurations. In either dimensionalities they have the general form $\sigma(x) = \psi(x - x_0)$, where $\psi$ is an elliptic function and $x_0$ is an arbitrary parameter responsible for translational invariance. In $0 + 1$ dimensions if the amplitude of $\psi(t)$ along the real axis is finite it corresponds to a finite $S_{\text{eff}}[\sigma]$ (after regularizing the contribution of the fluctuations of the $\mathcal{O}(N)$ vector field into a finite expression), while if the amplitude is infinite (i.e. the pole of $\psi$ lies on the real axis) it is a potential source of an infinite $S_{\text{eff}}[\sigma]$, which can suppress this mode enormously relative to the homogeneous $\sigma = \sigma_c$ configuration. Moreover, such infinite amplitude $\sigma(t)$ configurations, imply very large fields, a fact that might throw us away from the validity domain of the large $N$ approximation. We will always keep this in mind when addressing such configurations, but in $0 + 1$ dimensions we argue that they might be relevant in certain circumstances. In the two dimensional field theory case most $\sigma(x) = \psi(x - x_0)$ configurations give rise to infinite $S_{\text{eff}}[\sigma]$ values (whose divergences are inherent and cannot be removed by regularization) and are therefore highly suppressed. Infinite amplitude $\sigma(x)$ configurations fall obviously into this

\(^4\) This will not be carried out here.
category, but so do finite amplitude $\sigma(x)$ configurations. Some of the latter yield (regularized) finite $S_{\text{eff}}[\sigma]$ values only in the limit in which their real period (as elliptic functions) becomes infinite. Similar $\sigma(x)$ configurations were found in the two dimensional $\mathcal{O}(N)$ model in [22] by using inverse scattering techniques. Nevertheless, they turn out to be quite distinct from those discussed here. Our finite amplitude periodic $\sigma(x)$ configurations might become important if we put the field theory in a spatial “box” of a finite length $L$, but as $L \to \infty$ they will be highly suppressed due to their diverging $S_{\text{eff}}[\sigma]$. These $\sigma(x)$ saddle point configurations are hybrid objects, in the sense that they are static solutions of the classical equations of motion $\frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma} = 0$ of the action $S_{\text{eff}}[\sigma]$ that embodies all quantal effects of the $\Phi$ field. Therefore, due to all their properties enumerated above, the finite $S_{\text{eff}}[\sigma]$ $\sigma(x)$ configurations are reminiscent, in some sense, of soliton solutions in other field theories, but this analogy is only a remote one, since the $\sigma(x)$’s carry no topological charge of any sort and they are not local minima of $S_{\text{eff}}[\sigma]$. They are more similar to the so called “non-topological solitons” [24], [25]. In this paper we ignore the gaussian fluctuations of the $\sigma$ field around the saddle points $\sigma(x)$, but explain how to find the spectrum of “mesons” (i.e. $\Phi$ quanta) coupled to the $\sigma(x)$ configuration. It is clear that each $\sigma(x)$ configuration gives rise to a tower of “meson” bound states as well as to “meson” scattering states, arranged in highly reducible $\mathcal{O}(N)$ multiplets. The infinite negative eigenvalues of the second variation of $S_{\text{eff}}[\sigma]$ around these configurations correspond to the infinite number of decay modes of such excited states into all lower ones. This form of the spectrum of quantum states is reminiscent of the vibrational-rotational band structure of molecular spectra [1]. Since we ignore fluctuations of the $\sigma$ field around the saddle point configuration $\sigma(x)$, the “molecular analog” to this would be to have the molecule only at one of its vibrational ground states and consider the rotational excitations around it. As a matter of fact, in the $0+1$ dimensional case, which is nothing but the quantum mechanics of a single unharmonic oscillator in $N$ euclidean dimensions this analogy is actually the correct physical picture. In this case the $\sigma(t)$
configurations are actual vibrations of the $N$ dimensional vector, changing its length. As the vector stretches or contracts, it might rotate as well, changing its orientation in $N$ dimensional space, which is energetically much cheaper. The latter gives rise to the rich $O(N)$ structure of the spectrum.

Our approach to the specific problem of bound state spectrum in the $1+1$ dimensional $O(N)$ model has been inspired by the seminal papers of Dashen, Hasslacher and Neveu (hereafter abbreviated as DHN) [18], [19], [20], and especially, by their analysis of the bound state spectrum in the Gross-Neveu model [21]. In [20] DHN present a detailed construction of a sector of the bound state spectrum of the Gross-Neveu model whose corresponding $\sigma(x)$ configurations differ only slightly from the homogeneous vacuum configuration $\sigma = \sigma_c$ of the Gross-Neveu model, at least as far as states with a principal quantum number well below $N$ are concerned. For such states, the corresponding $\sigma(x)$ configuration looks like a pair of interacting (very close) kink and anti-kink, where the kink amplitude is of the order $1/N$. This might be thought of as the back-reaction of the fermions in the bound state on the vacuum $\sigma = \sigma_c$ polarizing it into a $\sigma(x)$ configuration. Despite the smallness of $\sigma(x) - \sigma_c$, it has a dramatic consequence: the production of its corresponding bound state. Indeed, DHN find the explicit form of $\sigma(x)$ by performing an inverse scattering analysis of the one dimensional Schrödinger operator $\partial_x^2 + \sigma(x)^2 - \sigma_c^2 + \sigma'(x)$ obtained straightforwardly from the Dirac operator $i\partial_x - \sigma$. The potential term in this Schrödinger operator vanishes identically and cannot bind for $\sigma = \sigma_c$, but for the $\sigma(x)$ configuration of DHN it becomes an attractive reflectionless potential (of depth of the order $1/N^2$). Thus, despite the small magnitude of the “back-reaction” of the fermions on the vacuum, it is enough in order to create a very shallow dip in the one dimensional Schrödinger potential, causing it to produce one bound state. This is the reason why DHN attribute the appearance of fermionic bound states in the Gross-Neveu model to its intrinsic infra-red instabilities (the distortion of $\sigma_c$ into $\sigma(x)$) which result from the asymptotic freedom of the model. Indeed, the same infra-red instabilities generate

\footnote{Here $\sigma(x)$ stands for the $\bar{\psi}\psi$ bilinear fermionic condensate. We refer here only to their static $\sigma(x)$ configurations.}
non-perturbatively the fermion mass $g\sigma_c$ in this model in the first place.

The $1 + 1$ $O(N)$ vector model with quartic interactions being trivially “asymptotically free” due to its super-renomalizability, exhibits for negative quartic coupling a sector of bound states (whose $\sigma(x)$ configuration was mentioned above) that is very similar to the bound states of the Gross-Neveu model. It is probably created by the same mechanism of vacuum infra-red instabilities. Their energy spectrum and corresponding reflectionless kink like $\sigma(x)$ configurations were found in \cite{22} by following the DHN prescription step by step.

In addition to the description above, DHN mention briefly yet another sector of bound states \cite{20} in the Gross-Neveu model, namely, those built around the Callan-Coleman-Gross-Zee kink. The latter is a local minimum $\sigma(x)$ configuration very different from the homogeneous vacuum one. It connects the two true degenerate vacua of the Gross-Neveu model. Here all fermions are bound at zero binding energy in the fermionic zero mode of the kink (which is the only bound state supported by the kink), with no back-reaction at all on the kink independently of the number of fermions trapped in that state. The $\sigma(x)$ configurations we find in the $O(N)$ model are precisely of this type. However, the latter cannot be associated with degenerate vacua, since the $O(N)$ model lack such a structure. Due to this fact and since we do not use inverse scattering techniques, our results are complementary to those of \cite{20,22}. Our method could be used also to find analogues to the Callan-Coleman-Gross-Zee kink in other two dimensional theories, where inverse scattering methods become practically useless \cite{38}. As far as we know this is the first use made of the method described below. The idea we use is very simple. A generic saddle point condition for the $O(N)$ model (in any dimension) $\frac{\delta S_{eff}[\sigma]}{\delta \sigma} = 0$ relates $\sigma$ to the diagonal resolvent of a Klein-Gordon operator (for scalar fields) with $\sigma$ as its potential term. In the static case the Klein-Gordon operator reduces into a Schrödinger operator. In $1 + 1$ dimensions the resulting diagonal resolvent of the one dimensional Schrödinger operator is known to obey a certain differential equation, known as the Gelfand-Dikii equation \cite{23}. Since the resolvent is essentially $\sigma(x)$ due to the saddle point condition,
the latter must obey an ordinary differential equation induced from the Gelfand-Dikii equation.

The differential equation imposed on $\sigma(x)$ is of second order. Solving it for $\sigma(x)$ we encounter two integration constants. One constant is trivial and insures translational invariance of the solution for $\sigma(x)$ while the other acts as a parameter that distorts the shape of $\sigma(x)$ and parametrises its period along the $x$ axis. When $\sigma(x)$ has a finite amplitude we can always use this parameter to drive $\sigma(x)$ into an infinite period behaviour. In this limit $\sigma(x)$ reproduces the reflectionless configuration analogous to the Callan-Coleman-Gross-Zee kink in the case of negative quartic coupling.

At this point it is worth mentioning that some bound states in $O(N)$ models in various dimensions were found around the homogeneous vacuum $\sigma = \sigma_c$ configuration in the singlet $\Phi \cdot \Phi$ channel of “meson-meson” scattering, by simply analyzing the poles of the $\sigma - \sigma$ propagator [8], [9], [10]. Specific conditions for the existence of such bound states and resonances in two space-time dimensions to four may be found in [8], [9]. In certain theories in three space-time dimensions such bound states may become massless (to leading order in $1/N$) and are identified as goldstone poles (a dilaton [20] or a dilaton and a dilatino [27]) associated with spontaneous scale invariance breakdown. Occurance of such massless bound states is responsible for the double scaling limit in vector models [10] and the consequences of obstructions for this to happen in two space-time dimensions were addressed in [14].

The paper is organized as follows: In section (II) we analyze the $0+1$ dimensional model. This is done mainly to introduce notations and gain confidence in our method. In particular, we show that our method for obtaining the time dependent saddle point $\sigma$ configuration is completely equivalent to a JWKB analysis of the radial Schrödinger equation of the unharmonic oscillator where $\hbar = 1/N$ establishing the validity of our method. We analyze both cases of real and imaginary time qualitatively, without giving explicit formulae for the $\sigma(t)$ configurations.

In section (III) we turn to the $1+1$ dimensional case. Here as well we analyze both

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6See also the third reference in [7].
Minkowsky and Euclidean signatures of space-time. Our static $\sigma(x)$ configurations are obtained in a certain adiabatic approximation to the saddle point equation. Non-zero frequency modes are strongly suppressed and the only meaningful result of the saddle point equation concerns the zero frequency mode. In this way we effectively reduce the theory from $1 + 1$ dimensions to $0 + 1$ dimensions. We give an explicit expression for some of the $\sigma(x)$ configuration that is important in the infinite volume case and construct the spectrum of “mesonic” bound states in its background.

We draw our conclusions in section (IV). A simple proof of the Gelfand-Dikii equation, emphasizing its elementarity in the theory of one dimensional Schrödinger operators is given in an appendix.
II The Unharmonic Oscillator

We begin our investigation of the bound state spectrum in the two dimensional $\mathcal{O}(N)$ vector model by considering the simpler case of the quantum mechanics of a single $\mathcal{O}(N)$ invariant unharmonic oscillator. Clearly, the most straightforward way of analyzing the quantum state spectrum of this system when $N \to \infty$ is to apply the large $N$ approximation directly to the hamiltonian of this oscillator. This procedure is entirely equivalent to the JWKB approximation to the corresponding radial Schrödinger equation, where $\hbar = \frac{1}{N}$. Comparing our method of calculating the spectrum, which is an analysis of the action in the large $N$ limit, to the JWKB approximated Schrödinger equation reveals their equivalence, which is not surprising, since the large $N$ approximation to the path integral is equivalent to the semiclassical approximation with the role of $\hbar$ played by $1/N$.

Therefore, our purpose in this section is to better understand the way our method works in this simpler case and gain confidence in it before generalizing to the field theoretic case. Such an understanding is achieved precisely by demonstrating its equivalence to the JWKB approximated Schrödinger equation. This reasoning is completely analogous to the one made in [28, 29, 1], in order to understand the role of instantons as tunnelling configurations in the path integral. Indeed, tunnelling effects in quantum mechanics are calculated straightforwardly by using JWKB approximated Schrödinger equations rather than path integral instanton methods, but it is the instanton calculus that is generalizable to quantum field theory and not the JWKB approximated Schrödinger equation. Since our main interest in this paper is the $1 + 1$ dimensional field theoretic case (discussed in the next section) we will demonstrate the consistency of our method in the $0 + 1$ dimensional case without giving the resulting $\sigma(t)$ configurations explicitly in terms of elliptic functions of $t$. Evaluation of these functions may be done straightforwardly from our discussion.
The $O(N)$ invariant unharmonic oscillator is described by the action

$$S = \int_0^T dt \left[ \frac{1}{2} \dot{x}^2 - \alpha \left( \frac{1}{2} m^2 x^2 + \frac{1}{4N} \beta g^2 (x^2)^2 \right) \right]$$  \hspace{1cm} (2.1)$$

where $x$ is an $N$ dimensional vector, $g^2$ is the quartic coupling and $m$ is the pure harmonic frequency (as $g = 0$) of the oscillator. Here $\alpha$ and $\beta$ are discrete parameters whose values are either $+1$ or $-1$. $\alpha = 1$ corresponds to the real time case, while $\alpha = -1$ corresponds to imaginary (euclidean) time. Likewise, $\beta = 1$ corresponds to the well defined stable case where the quartic potential goes to $+\infty$ as $|x| \to \infty$, while $\beta = -1$ corresponds to the upside down potential. We have introduced these parameters for efficiency reasons, allowing us to discuss all different possibilities compactly. In the large $N$ approximation $g^2$ is finite, hence the quartic interactions in Eq. (2.1) are in the weak coupling regime.

Rescaling

$$x^2 = N y^2$$  \hspace{1cm} (2.2)$$

Eq. (2.1) may be rewritten as

$$S = N \int_0^T dt \left[ \frac{1}{2} \dot{y}^2 - \alpha \left( \frac{m^2}{2} y^2 + \frac{\beta g^2}{4} (y^2)^2 \right) \right].$$  \hspace{1cm} (2.3)$$

Introducing an auxiliary variable $\sigma$, the action is finally recast into

$$S = N \int_0^T dt \left[ \frac{1}{2} \dot{y}^2 - \alpha \frac{m^2}{2} y^2 + \alpha \beta (\sigma^2 - g \sigma y^2) \right]$$  \hspace{1cm} (2.4)$$

which is the starting point of our calculations. Note that $\sigma$ has no kinetic energy and thus may be eliminated from Eq. (2.4) via its equation of motion

$$\sigma = \frac{1}{2} g y^2.$$  \hspace{1cm} (2.5)$$

We assume throughout this work that $m^2 \geq 0$. A sign flip of $m^2$ is equivalent to a simultaneous sign flip of the parameters $\alpha$ and $\beta$.

Taking the scale dimension of $m$ to be canonically $1$ the other dimensions are $[t] = -1$, $[x] = -\frac{1}{2}$, $[g^2] = 3$. The dimensionless ratio $g^2/N m^2$ which is much smaller than $1$ for finite $g^2$, $m^2$ clearly implies weak coupling.
Throughout this paper we adopt the convention $g > 0$ and therefore $\sigma$ is a non-negative variable.

Since in principle we are interested in calculating the spectrum of bound state of the oscillator we consider the partition function

$$\text{Tr} \ e^{\frac{1}{\gamma} HT} = \int d^n y_0 \ W_P(T)$$

where $W_P(T)$ is the transition amplitude of the oscillator to start at $y_0$ and return there after time-lapse $T$

$$\langle y_0; t=T|y_0; t=0 \rangle \equiv W_P(T) = \int \mathcal{D} y \mathcal{D} \sigma \ e^{\gamma S}. \ (2.6)$$

Here the subscript “pbc” denotes integration over paths obeying the obvious periodic boundary conditions implied by Eq. (2.6) and $\gamma = i$ for the real time case, while $\gamma = -1$ for imaginary time. Clearly, the periodic boundary conditions on $\sigma$ are dictated by Eq. (2.5).

Eq. (2.4) is quadratic in $y$ which may be therefore integrated completely out of the action leading to

$$W_P(T) = \int \mathcal{D}\sigma \ \det_{\text{pbc}}^{-N/2} \left[ -\partial_t^2 - \alpha m^2 - 2\alpha\beta g\sigma \right] e^ {\gamma \int_0^T \sigma^2 dt} \ (2.7)$$

where the operator whose determinant is taken in Eq. (2.7) is defined on the interval $[0,T]$ with periodic boundary conditions. Exponentiating the determinant in Eq. (2.7) we may express $W_P(T)$ as

$$W_P(T) = \int \mathcal{D}\sigma \ e^{N\gamma S_{eff}[\sigma]} \ (2.8)$$

where

$$S_{eff}[\sigma] = \alpha\beta \int_0^T \sigma^2 - \frac{1}{2\gamma} \text{Tr}_{\text{pbc}} \ln \left[ -\partial_t^2 - \alpha m^2 - 2\alpha\beta g\sigma \right] \ (2.9)$$

is the exact effective non-local action in terms of the $\sigma$ field.

Due to the explicit factor of $N$ in the exponent of Eq. (2.8), $W_P(T)$ is dominated by the extremal points of $S_{eff}[\sigma]$, and to leading order in $1/N$ it is given by the

\[\text{Thus ignoring quadratic fluctuations of } \sigma \text{ around these extrema.}\]
coherent sum over all such extremal configurations $\sigma(t)$ of the integrand in Eq. (2.8) evaluated at these configurations.

These extremum configurations are solutions of the functional condition

$$\delta S_{\text{eff}}[\sigma]/\delta \sigma = 0 \ ; \ \sigma(0) = \sigma(T) \quad (2.10)$$

which is equivalent to

$$2\sigma(t) = -\frac{g}{\gamma} \langle t| \frac{1}{-\partial_t^2 - \alpha m^2 - 2\alpha \beta g \sigma} |t\rangle; \ \sigma(0) = \sigma(T) . \quad (2.11)$$

Eq. (2.11) is a rather complicated functional equation for generic $\sigma$ configurations. However, it simplifies enormously for constant $\sigma = \sigma_c$ configurations which obey the periodicity boundary condition automatically for any $T$. In this case the diagonal matrix element is given by a simple Fourier integral (for $T \to \infty$), and Eq. (2.11) descends into the cubic equation

$$32\beta g \sigma_c^3 + 16m^2 \sigma_c^2 - g^2 = 0 . \quad (2.12)$$

Eq. (2.12) has been used in [11, 12, 13] to determine the optimal frequency squared $\omega_c^2 = m^2 + 2\beta g \sigma_c$ of the harmonic approximation to Eq. (2.1) for low lying excitations around the vacuum (or metastable vacuum, for $\beta = -1$) of the unharmonic oscillator.

Following Eq. (2.5) one should obviously consider only positive real roots of Eq. (2.12). Analyzing the left hand side of Eq. (2.12) it is clear that for $\beta = +1$ there is only one such root, while for $\beta = -1$ there are two, but only the smaller one corresponds to the metastable vacuum. These assertions will be demonstrated more clearly when we will examine the JWKB approximation to the radial Schrödinger equation of the unharmonic oscillator in the last part of this section.

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10 In case of finite dimensional integrals, not all saddle points necessarily contribute. Only those that are located along the steepest contour contribute. In the functional integral (Eq. (2.8)) the situation is not so clear. However all $t$ dependent extrema $\sigma(t)$ we find have simple classical meaning, and therefore must be all important.

11 See also the third reference in [7].

12 In this case $\omega_c^2 = m^2 - 2g \sigma_c$ and $\omega_c^2 < 0$ for the larger root.
We now turn to generic time dependent solutions of Eq. (2.11). This equation relates the diagonal resolvent \( R(t) = \langle t | (h - \alpha m^2)^{-1} | t \rangle \) of the Schrödinger operator

\[
h = -\partial_t^2 - 2\alpha \beta g \sigma \tag{2.13}
\]

whose potential term is \( U(t) = -2\alpha \beta g \sigma \), to \( \sigma(t) \). This diagonal resolvent obeys the Gelfand-Dikii equation (Eq. (A.12)) discussed in the appendix. Thus, eqs. (2.11) and (A.12) imply

\[
2\sigma \sigma'' - (\sigma')^2 + 4\alpha (m^2 + 2\beta g \sigma) \sigma^2 = \frac{g^2}{4\alpha} ; \quad \sigma(0) = \sigma(T) \tag{2.14}
\]

where \( \sigma' = d\sigma/dt \) and we have used the relation \( \alpha = -\gamma^2 \).

Eq. (2.14) is one of the main results of this paper. It implies that the \textit{functional} equation Eq. (2.11) is equivalent to an \textit{ordinary differential} equation that we readily solve below, obtaining the required extremal \( \sigma(t) \) configuration.

Note that Eq. (2.14) admits a time independent solution \( \sigma = \sigma_c \) for which it reproduces Eq. (2.12) exactly. Regarding time dependent solutions, first integration of Eq. (2.14) may be done straightforwardly by substituting

\[
\sigma'(t) = f(\sigma) \tag{2.15}
\]

which leads to

\[
f^2(\sigma) \equiv \left( \frac{d\sigma}{dt} \right)^2 = \alpha \left[ \frac{g^2}{4} \left( \frac{\sigma}{\sigma_0} - 1 \right) - 4 \left( m^2 \sigma^2 + \beta g \sigma^3 \right) \right] ; \quad \sigma(0) = \sigma(T) \tag{2.16}
\]

where \( \sigma_0 \) is an integration constant (and we have used \( \alpha^2 = 1 \)). Eq. (2.16) has the form of the equation of motion of a point particle in one dimension along the ray \( \sigma \geq 0 \) in the potential \( V(\sigma) \) given by

\[
-V(\sigma) = \frac{g^2}{4} \left( \frac{\sigma}{\sigma_0} - 1 \right) - 4 \left( m^2 \sigma^2 + \beta g \sigma^3 \right) ; \quad \sigma \geq 0 \tag{2.17}
\]

\footnote{Here we assume that the constraint \( \sigma \geq 0 \) stemming from the classical equation of motion of \( \sigma \) (Eq. (2.5)) remains in tact quantum mechanically. We will see that this is indeed the case. Since Eq. (2.5) could be enforced as an exact \textit{functional} constraint by introducing a lagrange multiplier field, this conclusion should not be of any surprise.}
(we have shifted the constant piece of $V$ such that Eq. (2.16) corresponds to a motion at energy zero). Note that $\alpha$ has been factored out in front of $-V$ in Eq. (2.16), as it should, since $\alpha = +1$ corresponds to motion in real time while $\alpha = -1$ corresponds to motion in imaginary time. The classical turning points are the non-negative roots of the cubic equation

$$V(\sigma) = 0. \quad (2.18)$$

For real time motions, the classical $\sigma$ paths will be confined to regions where $-V \geq 0$, while for imaginary time motions, to regions where $-V \leq 0$. In addition to the turning points given by Eq. (2.18), there is a turning point at $\sigma = 0$ due to the constraint $\sigma \geq 0$.\textsuperscript{14}

Denoting the roots of Eq. (2.18) by $\sigma_1, \sigma_2, \sigma_3$\textsuperscript{15} the following formulae are useful for listing the various solutions of Eq. (2.16).

\begin{align*}
-V(0) & = -\frac{g^2}{4} < 0 \quad (2.19a) \\
-V(\pm\infty) & = \mp \beta \cdot \infty \quad (2.19b) \\
\sigma_1 \sigma_2 \sigma_3 & = -\frac{\beta g}{16}. \quad (2.19c)
\end{align*}

Assuming $m$ and $g$ are given, $\sigma_0$ is the only arbitrary parameter, arising as an integration constant of Eq. (2.16). We now show that equation (2.16) is equivalent to the (one dimensional) radial equation of motion of the oscillator in a quantum effective potential obtained as $N \to \infty$, whose energy parameter $E$ is inversely proportional to $\sigma_0$.

This observation is conceptually an important one, assuring the validity of our calculation of the time dependent extremal $\sigma$ configurations of $S_{\text{eff}}[\sigma]$, identifying them as corresponding to collective radial excitations of the oscillator, as we have discussed at the beginning of this section. Moreover, holding $m$ and $g$ fixed, $\sigma_1, \sigma_2$ and $\sigma_3$ are functions of $\sigma_0$, and in the most generic case two of them are complex

\textsuperscript{14}$\sigma$ trajectories that reach the endpoint $\sigma = 0$ always occur in imaginary time. They are undesirable since they have no quantal counterpart. We comment on this point in the last part of this section.

\textsuperscript{15}We adopt the convention that when all three roots are real they are ordered as $\sigma_1 \leq \sigma_2 \leq \sigma_3$. 

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(conjugate). Therefore, the best way to analyze the various turning point configurations of the solutions to Eq. (2.16) is to consider the (large $N$) effective radial potential of the oscillator giving rise to these radial motions. Varying the energy parameter of the latter will show us the way the turning points of Eq. (2.16) change as functions of $\sigma_0$. Recall that $W_P(T)$ defined in Eq. (2.3) is given as a sum over all saddle points $\sigma(t)$ found from Eq. (2.11). These are essentially parametrized by $\sigma_0$. Thus a crude estimate of $W_P(T)$ would be to insert these saddle point configurations into the integrand of Eq. (2.8) and sum over all allowed values of $\sigma_0$. Since $\sigma_0$ turns out to be inversely proportional to the energy parameter $\mathcal{E}$ of the associated radial Schrödinger equation this is just the natural thing to do, since by definition $W_P(T)$ is a Laplace transform of the transition amplitude $W_P(\mathcal{E})$ in the energy plane. In what follows, however, we will not calculate $W_P(T)$ but rather concentrate on a specific saddle point $\sigma(t)$ with its particular $\sigma_0$ parameter.

Before turning to the hamiltonian formulation we depict qualitatively in figures (1) and (2) the turning point structure of Eq. (2.16) in the case where $\sigma_1, \sigma_2$ and $\sigma_3$ are all real. Figure (1) corresponds to the case $\beta = +1$ while Figure (2) describes the various possibilities for $\beta = -1$. When $\beta = +1$, Eqs. (2.19) imply $-V(0) < 0$, $-V(\mp \infty) = \pm \infty$ and $\sigma_1 \sigma_2 \sigma_3 < 0$. Therefore, if $\sigma_1, \sigma_2$ and $\sigma_3$ are all real, either $\sigma_1 < \sigma_2 < \sigma_3 < 0$ or $\sigma_1 < 0 < \sigma_2 < \sigma_3$. These two possibilities are shown in figures (1a) and (1b), respectively. For $\beta = -1$ we have $-V(0) < 0$, $-V(\pm \infty) = \pm \infty$ and $\sigma_1 \sigma_2 \sigma_3 > 0$. Thus, if $\sigma_1, \sigma_2$ and $\sigma_3$ are all real either $\sigma_1 < \sigma_2 < 0 < \sigma_3$ or $0 < \sigma_1 < \sigma_2 < \sigma_3$. These two possibilities appear in figures (2a) and (2b), respectively.
FIGURE 1: The $\beta = 1$ case of Eq. (2.16). Either $\sigma_1 < \sigma_2 < \sigma_3 < 0$ (a) or $\sigma_1 < 0 < \sigma_2 < \sigma_3$ (b). Bold faced dots correspond to turning points of the $\sigma(t)$ motions. The empty circle denotes the reflecting infinite potential wall at $\sigma = 0$. The regions along the $\sigma$ axis accessible to real time motions are marked by the zig-zag line and those accessible to motions in imaginary time-by double lines.

FIGURE 2: The $\beta = -1$ case of Eq. (2.16). Either $\sigma_1 < \sigma_2 < 0 < \sigma_3$ (a) or $0 < \sigma_1 < \sigma_2 < \sigma_3$ (b). For various notations see caption to Fig. (1).
The Schrödinger hamiltonian corresponding to the action of Eq. (2.1) is

\[ H = -\frac{\hbar^2}{2} \Delta + U(r) \]  

(2.20)

where \( r^2 = x^2 \), \( \Delta \) is the \( N \) dimensional laplacian and

\[ U(r) = \frac{m^2}{2} r^2 + \frac{\beta g^2}{4N} r^4. \]  

(2.21)

Separating out the radial Schrödinger equation for a state carrying angular momentum \( \ell \) we find

\[ \left\{ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell + N - 2)}{r^2} \right\} R_{n,\ell}(r) = \mathcal{E}_{n,\ell} R_{n,\ell}(r), \]

\[ R_{n,\ell}(r) \sim r^\ell \]  

(2.22)

where \( n \) is the principal quantum number. Following [4] we perform the similarity transformation

\[ H \rightarrow \tilde{H} = r^{-\nu} H r^\nu, \quad R_{n,\ell}(r) \rightarrow \chi_{n,\ell}(r) = r^{-\nu} R_{n,\ell}(r), \quad \nu = -\frac{N-1}{2} \]  

(2.23)

which disposes of the \( \frac{\partial}{\partial r} \) term in Eq. (2.22).\[16\] We finally obtain

\[ \tilde{H} \equiv N\hbar = N \left[ -\frac{\hbar^2}{2N^2} \frac{\partial^2}{\partial \rho^2} + \mathcal{V}(\rho) + w_{n,\ell}(\rho) \right] \]  

(2.24)

where \( \rho^2 = y^2 = r^2/N \) and

\[ \mathcal{V}(\rho) = \frac{\hbar^2}{8\rho^2} + \mathcal{U}(\rho) = \frac{\hbar^2}{8\rho^2} + \frac{m^2}{2} \rho^2 + \frac{\beta g^2}{4} \rho^4 \]

\[ w_{n,\ell}(\rho) = \hbar^2 \left[ \frac{\ell - 1}{2N} + \frac{(\ell - \frac{1}{2})(\ell - \frac{3}{2})}{2N^2} \right] \frac{1}{\rho^2}. \]  

(2.25)

Eqs. (2.22) and (2.23) imply

\[ \chi_{n,\ell}(\rho) \sim \rho^{\ell + \frac{N-1}{2}} \]  

(2.26)

\[16\]This transformation is familiar from elementary quantum mechanics. What it does, is essentially swallowing a square root of the radial jacobian (the \( \rho^{N-1} \) factor) into a wave function in an inner product: \( R_{n,\ell}(r) \rightarrow \chi_{n,\ell}(r) \), inducing the corresponding transformation Eq. (2.23) on H.
assuring the self adjointness of $\tilde{H}$.$^{17}$ For a later reference let us recall that the singular solution $\tilde{\chi}_{n,\ell}$ to the equation $\tilde{H}\tilde{\chi}_{n,\ell} = \mathcal{E}_{n,\ell}\tilde{\chi}_{n,\ell}$ that is definitely not in the Hilbert space at hand, blows up at $\rho = 0$ as$^{18}$

$$
\tilde{\chi}_{n,\ell}(\rho) \sim \rho^{1-\ell-\frac{N-1}{2}}.
$$

(2.27)

The explicit $\hbar$ dependence of $V$ and $w_{N,\ell}$ in Eq. (2.23) is obvious- its origins are the centrifugal barrier in Eq. (2.22) and the similarity transformation we performed on the operator $H$. To leading order in $1/N$ the only $\hbar$ dependent term that survives there is the $\frac{1}{\hbar^2}$ piece in $V$. This is consistent with Langer’s modification of the centrifugal term $\frac{1}{2}\ell(\ell + N - 2) \rightarrow \frac{1}{2}\ell(\ell + \frac{N-2}{2})^2$ in the limit $N \rightarrow \infty$ $^{12}$. It is therefore the leading $1/N$ quantum contribution to the effective potential. Thus, it must be equivalent to the leading $1/N$ correction to the action from the determinant in Eq. (2.4). We show now that this is indeed the case, namely, as we have stated earlier the solutions of Eq. (2.16) correspond exactly to classical motions in the potential $V(\rho)$. From now on we set $\hbar = 1$ for convenience in eqs. (2.24)-(2.25). Clearly, from the point of view of the large $N$ approximation, $1/N$ will play the role of a small redefined Planck constant. Therefore $1/N$ analysis of Eq.(2.24) is equivalent to its JWKB analysis. But this is clearly not the ordinary case considered in a JWKB approximation, namely, the potential in Eq.(2.24) depends explicitly on the small parameter $1/N$. While this may complicate the $1/N$ expansion of the actual eigenvalues of $h$ in Eq.(2.24), as far as the $1/N$ expansion of wavefunctions away from the turning points is concerned, the situation is much simpler. Indeed, substituting the expansion

$$
\chi_{\ell}(\rho) = \exp \left\{ iN \left[ \sum_{n=0}^{\infty} N^{-n} \varphi_n(\rho) \right] \right\}
$$

(2.28)

into the eigenvalue equation for $h$ in Eq.(2.24) (assuming the eigenvalue is independent of $N$ up to an overall scaling) yields to first subleading order in $1/N$

$$
\chi_{\ell}(\rho) = [2(\mathcal{E} - V(\rho))]^{-1/4}
$$

$^{17}$Alternatively, Eq. (2.26) may be derived directly from Eqs. (2.24)-(2.25).

$^{18}$This can be seen immediately from the fact that the wronskian $W(\chi,\tilde{\chi})$ is necessarily a non vanishing constant.
\[ \exp \left\{ \pm iN \int^\rho [2 (\mathcal{E} - \mathcal{V}(\rho))]^{1/2} \left( 1 - \frac{\ell - 1}{2N\rho^2} \frac{1}{2(\mathcal{E} - \mathcal{V})} + \mathcal{O}(N^{-2}) \right) d\rho \right\}. \] (2.29)

Therefore, the classical momentum associated with Eq. (2.29) is

\[ p(\rho) = [2 (\mathcal{E} - \mathcal{V}(\rho))]^{1/2} \left( 1 - \frac{\ell - 1}{2N\rho^2} \frac{1}{2(\mathcal{E} - \mathcal{V})} + \mathcal{O}(N^{-2}) \right) \]

\[ = [2 (\mathcal{E} - \mathcal{V}(\rho) - w_{N,\ell}(\rho))]^{1/2} + \mathcal{O}(N^{-2}). \] (2.30)

Thus, to leading order in \( 1/N \) we may safely throw away all non leading terms in the potential of Eq. (2.24) (which include all \( \ell \) dependent terms) obtaining

\[ \frac{p^2}{2} = \frac{1}{2} \alpha \left( \frac{d\rho}{dt} \right)^2 = \mathcal{E} - \mathcal{V}(\rho), \quad \rho(T) = \rho(0) \] (2.31)

Using Eqs. (2.25), (2.26) we find that Eq. (2.31) may be written as

\[ \alpha \left( \frac{d\sigma}{dt} \right)^2 = \frac{g^2}{4} \left( \frac{16\mathcal{E}}{g} \sigma - 1 \right) - 4(m^2\sigma^2 + \beta g^2) \sigma, \quad \sigma(T) = \sigma(0) \] (2.32)

which is nothing but Eq. (2.16) upon the identification

\[ \frac{16\mathcal{E}}{g} = \frac{1}{\sigma_0}. \] (2.33)

Thus, we have proven that our calculation of time dependent saddle point configurations \( \sigma(t) \) (Eqs. (2.14)-(2.16)) of \( S_{eff}[\sigma] \) is equivalent completely to the time dependent classical solutions in the effective potential \( \mathcal{V} \) (Eqs. (2.25)-(2.31)). Note from Eqs. (2.24)-(2.25) that angular momentum effects will show up in \( S_{eff}[\sigma] \) only in higher terms of the \( 1/N \) expansion. Post factum the equivalence of Eqs. (2.16) and (2.31) seems self evident, but we have presented an explicit proof of it. Moreover, in passing from Eq. (2.31) to (2.32) we have used the classical \( \sigma \) equation of motion (Eq. (2.3)). This equation must hold also quantum mechanically because we can enforce it as a constraint by introducing a lagrange multiplier field in an equivalent formulation. Indeed, note that Eq. (2.3) used in passing from Eq. (2.31) to (2.32) relates the slow radial semiclassical motions \( \rho(t) \) of \( y \) in the effective potential \( \mathcal{V} \) (that already includes quantal corrections from \( y \) fluctuations) to the \( \sigma \) fields.

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The equivalence of Eqs. (2.14) and (2.29) holds for their time independent solutions as well. Indeed, from Eq. (2.24) the extremum condition for $\mathcal{V}(\rho)$ reads

$$\frac{\partial \mathcal{V}}{\partial \rho} = -\frac{1}{4\rho^3} + m^2 \rho + \beta g^2 \rho^3 = 0$$

which is equivalent to Eq. (2.12), the equation determining constant $\sigma = \sigma_c$ extremal configurations of $S_{eff}[\sigma]$. The ground state of the unharmonic oscillator correspond to the local minimum of $\mathcal{V}(\rho)$. For $\beta = -1$ this is only a metastable ground state. It will decay at a rate suppressed by a factor of $e^{-N}$ to $-\infty$. $\mathcal{V}(\rho)$ has been drawn schematically in figures 3(a) and 3(b) (for $\beta = 1$ and $\beta = -1$, respectively) where we have also denoted the various regions along the $\rho$ axis accessible to motions either in real or imaginary times for some energy parameter $\mathcal{E}$. $\mathcal{E}_0 = \mathcal{V}(\rho_0)$ in Fig. (3a) (Fig. (3b)) corresponds to the absolute (metastable) ground state of the oscillator, thus, using Eqs. (2.3), (2.34) we have

$$\sigma_c = \frac{1}{2} g \rho_0^2.$$  

(2.35)

For $\beta = -1$ Eqs. (2.12), (2.34) have yet another solution, corresponding to the local maximum in fig. (3b).

**FIGURE 3:** Radial motions from Eqs. (2.24) and (2.31) as $N \to \infty$ for $\beta = +1$ (a) and $\beta = -1$ (b). For the various notions see the caption to Fig. (1).
The consistency of Figs. (1a,b) and (3a) and of Figs. (2a,b) and (3b) is apparent. Clearly Fig. (1b) corresponds to the case $E > E_0$ in Fig. (3a) while Fig. (1a) corresponds to $E < E_0$. Similarly, Fig. (2a) corresponds to the cases $E > E_c$ or $E < E_0$ in Fig. (3b) while Fig. (2b) corresponds to $E_0 < E < E_c$.

Having figure (3) at hand it is clear now how the turning point structure of Eq. (2.16) and figures (1) and (2) change under variations of the parameter $\sigma_0$ holding $m$ and $g$ fixed. Consider a classical trajectory in Fig. (3a) with a very large and positive energy parameter $E$. By virtue of Eq. (2.33) this corresponds to a very small positive $\sigma_0$ parameter in Fig. (1b). Clearly in this situation we have $0^+ \sim \sigma_2 \ll \sigma_3$. Reducing $E$ in Fig. (3a) (equivalently, increasing $\sigma_0$), there are no qualitative changes in Fig. (1b). It is clear that as $E$ decreases, $\sigma_2$ increases while $\sigma_3$ decreases, diminishing the amplitude of the real-time motion, and in addition, reducing the value of $-V(\sigma)$ at its maximum $A$. These changes persist until $E$ hits $E_0 = V(\rho_0)$, where $\sigma_2$ and $\sigma_3$ coincide, shrinking the real time trajectory to a point, corresponding to the fact that the oscillator is frozen at its ground state. In this case the common value of $\sigma_2$ and $\sigma_3$ is the constant configuration $\sigma = \sigma_c$ mentioned above. Decreasing $E$ further, we flip from Fig (1b) to Fig. (1a), where only imaginary time motions are allowed. A similar behaviour occurs in figures (3b) and (2a,b).

Figures (1)-(3) exhibit imaginary time solutions to Eqs. (2.16) and (2.31) that hit the boundary point $\sigma = 0$ for any value of $E$ or $\sigma_0$. Such solutions describe situations in which the particle may be close to the end-point $\sigma = 0$ during a finite time period. This is in obvious contradiction with the small $\rho$ behavior of the square integrable wave functions $\chi_{n,\ell}(\rho)$ in Eq. (2.20) which suppress the probability of having the oscillator oscillating at $0 \sim \rho \ll 1$ completely (for any $\ell$) as $N \to \infty$. We will not discuss these solutions further.

As was mentioned at the beginning of this section, a flip in the sign of $m^2$ is

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19With the exception that $\sigma_1, \sigma_2$ there may become complex (conjugate).
20The same problem arises for a free particle in $N$ dimensions and imaginary time.
21It might be though that these classical trajectories are associated with the functions $\tilde{\chi}_{n,\ell}$ in Eq. (2.27) that are solutions of the Schrödinger equation, but do not belong to the Hilbert space.
equivalent to a simultaneous flip of \( \alpha \) and \( \beta \), i.e. a change from real (imaginary) time motions in Fig. (3a) to imaginary (real) time motions in Fig. (3b). Therefore, if one starts with \( m^2 > 0 \), a flip in the sign of \( m^2 \) will not induce a local minimum in addition to the one that already exists in Figs. (3a) and (3b) at \( \rho = \rho_0 \). Alternatively, one can check explicitly, that the local minimum of \( U(\rho) \) in Eq. (2.21) at \( \rho = -m^2/g^2 \) (for \( m^2 < 0 \)) is distorted away by the \( 1/8\rho^2 \) term in in Eq. (2.25).

We end this section with some general observations and remarks on the possible solutions \( \sigma(t) \) to Eq. (2.16). Following our discussion of the various possible trajectories, a straightforward integration of Eq. (2.16) yields the latter as

\[
\sigma(t) = \psi(t - t_0) \tag{2.36}
\]

where \( \psi \) is an elliptic function \([34, 33]\) and \( t_0 \) is an integration constant assuring time translational invariance. As should be clear from Figs. (1)-(3) the amplitudes of the oscillator are either bounded (the poles of \( \psi \) in Eq.(2.36) are away from the real \( t \) axis) or unbounded (\( \psi \) has a double pole on the real \( t \) axis). In the latter case \( \psi(t - t_0) \) is essentially a Weierstrass \( P \) function, while in the former case, it may be written most simply as a rational expression in terms of the Jacobi \( sn \) function. These finite amplitude solutions descend into the expected harmonic oscillations in real (imaginary) time as \( \mathcal{E} \rightarrow \mathcal{E}_0^+ \) (\( \mathcal{E} \rightarrow \mathcal{E}_c^- \)) in Figs. (3a) and (3b), where the oscillator executes small oscillations around the relevant minimum. The unbounded oscillations on the other hand are potential sources to an infinite \( S_{eff}[\sigma] \), which can suppress such modes completely. Clearly, as they stand, they are not even periodic in time, since there is nothing to reflect \( \sigma(t) \) back from infinity. Moreover, these modes are associated with very large values of \( x \), which might throw us out of the validity domain of the large \( N \) approximation. However, regardless of all these problems, the Weierstrass \( P \) function \( \sigma(t) \) configurations are very interesting since for such \( \sigma \)'s the Schrödinger hamiltonian \( h \) in Eq.(2.13), which is essentially the inverse propagator of the \( y \) field in Eq. (2.4), is one of the completely integrable one dimensional hamiltonians of the Calogero type \([35]\).

For real time motions, such trajectories occur only for \( \beta = -1 \) (see Fig. (3)),
i.e. only in the case of negative quartic coupling. In such a case the system, strictly speaking, does not have a ground state and is meaningless beyond large $N$ saddle point considerations. However, due to the very fast decrease of the potential to $-\infty$, the time of flight to infinity is finite, and the unharmonic oscillator attains in this case a one parameter self adjoint extension [36] that might be used to redefine the quantal system into a well behaved form that will render these interesting $\mathcal{P}$ function shape trajectories meaningful. In this way we actually compactify the $\sigma$ (or $\rho$) semiaxis, making these trajectories effectively periodic, in accordance with the periodicity of the $\mathcal{P}$ function.

Any of the semiclassical trajectories $\rho(t)$ with bounded amplitude from fig. (3) or their corresponding $\sigma(t)$ configurations in figs. (1) and (2) defines a vibrational mode of the $\mathcal{O}(N)$ vector. In such a vibrational mode, the unharmonic oscillator is described effectively by an harmonic oscillator that is coupled to a time dependent potential $v(t) = \frac{m^2}{2} + \beta \sigma(t)$ (Eq. (2.4)). Each vibrational mode corresponds to a unique $\sigma(t)$ configuration found from Eqs. (2.16) or (2.32). On top of these vibrations, the $\mathcal{O}(N)$ vector rotates as well, giving rise to a host of rotational states that form a band superimposed on top of the vibrational mode. To leading order in $1/N$, this rotational band collapses into a single energy value as should be clear from Eq. (2.25).

Each vibrational mode $\sigma(t)$ is an unstable saddle point of $S_{\text{eff}}[\sigma]$. Indeed it can be shown straightforwardly that the second variation of $S_{\text{eff}}[\sigma]$ around each such $\sigma(t)$ configuration has an infinite number of negative directions, corresponding to the infinite number of decay modes of this excited mode as $N \to \infty$. Nevertheless, we must include these unstable saddle point configurations in calculating $W_p(T)$ in Eq. (2.3) for finite $T$, since excited states of the oscillator do contribute to it. We will not pursue the case of the unharmonic oscillator any further in this paper, but rather turn to the two dimensional field theoretic case in the next section.
III The $\mathcal{O}(N)$ Vector Model in Two Dimensions

The action for the two dimensional $\mathcal{O}(N)$ model reads

$$S = \int_0^T dt \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\partial_{\mu} \Phi)^2 - \frac{\alpha m^2}{2} \Phi^2 - \frac{\alpha \beta g^2}{4N} (\Phi^2)^2 \right]$$

(3.1)

where $\Phi$ is an $N$ component scalar field transforming as an $\mathcal{O}(N)$ vector, $m^2$ is the bare “meson” mass and $g^2$ is the finite quartic coupling. $\alpha, \beta$ and $\gamma$ (that appears below) have the same meaning as in the previous section, hence $(\partial_{\mu} \Phi)^2 = (\partial_0 \Phi)^2 - \alpha (\partial_1 \Phi)^2$. Rescaling

$$\Phi^2 = N \phi^2$$

(3.2)

and introducing the auxiliary field $\sigma(x)$ as before, Eq. (3.1) turns into

$$S = N \int_0^T dt \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{\alpha m^2}{2} \phi^2 + \alpha \beta (\sigma^2 - g \sigma \phi^2) \right].$$

(3.3)

Note again, that by its equations of motion $\sigma = \frac{\phi}{\sqrt{N}} \Phi^2$ is a non-negative field (recall our convention $g \geq 0$). This positivity should persist in the quantal domain as well, since this equation of motion may be enforced equivalently as a constraint by introducing a Lagrange multiplier.

In complete analogy with Eqs. (2.6)-(2.9) we define the amplitude

$$W_P(T) = \langle \phi_0 ; t = T | \phi_0 ; t = 0 \rangle =$$

$$\int \mathcal{D}_{pbc} \sigma \ e^{N \gamma S_{eff}[\sigma]}$$

(3.4)

where $\phi$ has been integrated out yielding formally

$$S_{eff}[\sigma] = \alpha \beta \int_0^T dt \int_{-\infty}^{\infty} dx \sigma^2 - \frac{1}{2 \gamma} \text{Tr}_{pbc} \ln \left[ -\Box - \alpha m^2 - 2 \alpha \beta g \sigma \right]$$

(3.5)

in which $\Box = \partial_0^2 - \alpha \partial_1^2$.

As was mentioned in the introduction the ground state of the $\mathcal{O}(N)$ model is governed by the homogeneous constant $\sigma = \sigma_c$ configuration, where $\sigma_c$ found from
by the extremum condition
\[ \frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma(x)} \bigg|_{\sigma = \sigma_c} = 0, \]
the so called “gap equation” of the model, and is a _local_ minimum of \( S_{\text{eff}}[\sigma] \). This gap equation expressed in terms of the bare quantities \( m \) and \( g \) in Eq. (3.1) contains logarithmic divergences that can be swallowed into the bare \( m^2 \) parameter, replacing it by a renormalized finite \( m^2_R \) parameter, that depends on an arbitrary finite mass scale, while \( g^2 \) remains unchanged \[7, 9, 13, 14\].

In general, this “gap equation” might have more than one solution \( \sigma = \sigma_c \), but only one such solution corresponds to the ground state of the theory. The latter is \( \mathcal{O}(N) \) invariant \[9\] as we are discussing a two dimensional field theory with a continuous symmetry \[37\].

The local minimum \( \sigma = \sigma_c \) of \( S_{\text{eff}}[\sigma] \) supports small fluctuations of the fields in Eq. (3.3) around it giving rise to a low energy spectrum of “free mesons” with mass squared \( m^2_R + 2\beta g \sigma_c \). There exist also bound states and resonances in the \( \mathcal{O}(N) \) singlet \( \phi \cdot \phi \) channel \[9\]. This picture is intuitively clear in the case of positive quartic interactions \( \beta = 1 \), but it turns out to persist also in the \( \beta = -1 \) case of negative quartic coupling \[9\], provided \( g^2 \) is not too large. What happens in the latter case is that despite the fact that for \( \beta = -1 \) Eq. (3.1) has no ground state, \( S_{\text{eff}}[\sigma] \) still possesses a local minimum that support small oscillations of the fields in Eq. (3.1). To leading order in \( 1/N \) these fluctuations cannot spill over the adjacent local maximum of \( S_{\text{eff}}[\sigma] \) towards infinite field values, provided \( g^2 \) does not exceed a certain maximal critical value. It was found in \[9\] that the \( \sigma - \sigma \) propagator is indeed free of tachyons even in the \( \beta = -1 \) case, provided \( g^2 \) is smaller than that critical bound.

We now turn to the central topic in this work, namely, the static \( \sigma(x) \) extremal configurations of \( S_{\text{eff}}[\sigma] \), and the bands of bound \( \phi \) states associated with them. Note that such static \( \sigma(x) \) configurations satisfy the periodic boundary condition along the time direction
\[ \sigma(x, 0) = \sigma(x, T) \]
(3.6)
implied by Eq. (3.4) automatically, for any value of \( T \).
Our approach to these static \( \sigma(x) \) configurations will be indirect. Namely, we assume that the extremal \( \sigma(x,t) \) configuration we are looking for has a very slow time variation, such that \( [\partial_0, \sigma] \approx 0 \) will be a very good approximation in calculating the trace in Eq. (3.3). In this "adiabatic" approximation we effectively carry a smooth dimensional reduction of the extremum condition \( \frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma(x,t)} = 0 \) from 1 + 1 to 1 + 0 space-time dimensions. Therefore, we use this "adiabatic" limit merely as an infra-red regulator of the \( t \)-integrations that occur in matrix elements in the trace of Eq. (3.3) which enables us to extract the static \( \sigma(x) = \sigma_0(x) \) piece. We do not know if the slowly time varying \( \sigma(x,t) \) configurations we encounter in this approximation are relevant as genuine space-time dependent fields, or just artifacts of the approximation.

Carrying the "adiabatic" approximation we expand the real field \( \sigma \) in frequency modes

\[
\sigma(x,t) = \sum_{n=-\infty}^{\infty} \sigma_n(x)e^{i\omega_n t} \tag{3.7}
\]

where by virtue of Eq. (3.6) and the fact that \( \sigma^* = \sigma \) we have

\[
\omega_n = \frac{2\pi}{T} P(n) \quad ; \quad P(-n) = -P(n) \quad ; \quad \sigma^*_n(x) = \sigma_{-n}(x) \tag{3.8}
\]

in which \( P(n) \) is any monotonously increasing map from the integers to themselves. The precise form of \( P(n) \) will be immaterial to us, since we are interested only in the zero frequency mode \( \sigma_0(x) \) in Eq. (3.7).

Using Eqs. (3.7) and (3.8) \( S_{\text{eff}}[\sigma] \) in Eq. (3.5) becomes

\[
S_{\text{eff}}[\sigma] = \alpha \beta T \int_{-\infty}^{\infty} dx \left[ \sigma_0^2(x) + 2 \sum_{n=1}^{\infty} \sigma_{-n}(x)\sigma_n(x) \right] \\
-\frac{1}{2\gamma} \int_{-\infty}^{\infty} dx \sum_{n=\infty}^{\infty} \langle x,n | \ln \left[ \omega_n^2 + \alpha \partial_x^2 - \alpha m^2 - 2\alpha \beta g \sigma_n \right] | x,n \rangle \tag{3.9}
\]

where \( \{|x,n\} \) is a complete orthonormal basis of positions \( x \) and frequency modes \( \omega_n \) and we have used the adiabatic approximation \( [\partial_0, \sigma] \approx 0 \).

Varying Eq. (3.9) with respect to \( \sigma_{-k} \) \((k \geq 0)\) the extremum condition reads
\[ \sigma_k(x) = \frac{g}{2\alpha \gamma T} \langle x \rvert \frac{1}{-\partial_x^2 + 2\beta g \sigma_{-k} + m^2 - \alpha \omega_k^2} \rvert x \rangle \]  

(3.10)

for the \( k \)-th mode in Eq. (3.7). Here we have used the relations \( \langle n\rvert n \rangle = 1 \) and \( \alpha^2 = 1 \). Eq. (3.10) is of the general form of Eq. (2.11) and thus shows explicitly that our adiabatic approximation has reduced the \( 1 + 1 \) dimensional extremum condition into a \( 0 + 1 \) dimensional problem, as was stated above. Indeed, Eq. (3.10) for \( k = 0 \) could have been obtained equivalently by ignoring completely any time dependence in the extremum condition

\[ \frac{\delta S_{\text{eff}}[\sigma]}{\delta \sigma(x,t)} = \alpha \beta \left[ 2\sigma(x,t) + \frac{g}{\gamma} \langle x, t \rvert \frac{1}{-\partial_0^2 + \alpha \partial_1^2 - \alpha m^2 - 2\alpha \beta g \sigma} \rvert x, t \rangle \right] = 0, \]  

(3.11)

namely discarding \( t, \rvert t \rangle \) and \( \partial_0 \) in this equation from the very beginning. In order to maintain the correct scale dimensions of the remaining objects, one simply replaces \( g \) by \( g/T \) in Eq. (3.11), obtaining Eq. (3.10) again.

The advantage in performing the adiabatic approximation (Eqs. (3.7)-(3.10) ) rather than the simpler derivation of the extremum condition for \( \sigma_0(x) \) we have just describe is that it enables us to see explicitly what happens to non-zero frequency modes in Eqs. (3.7)-(3.10).

At this point it is very important to note that we must exclude space independent solutions to Eq. (3.10) (as far as \( k = 0 \) is concerned). For such solutions Eq. (3.10) is equivalent to an algebraic cubic equation similar to Eq. (2.12). However, the homogeneous \( \sigma = \sigma_0 \) vacuum condensate satisfies the “gap-equation” which is a transcendental equation \[ \]  

rather than an algebraic one. For this reason it is very clear that our resulting static \( \sigma_0(x) \) configuration will be very different from the one found in \[ \] , which is only a mild distortion of the vacuum \( \sigma_0 \) condensate. Thus, the band of bound states associated with our \( \sigma_0(x) \) complements those discussed in \[ \] .

Since we are interested essentially in the zero frequency mode of Eq. (3.7), we are free to assume that \( \sigma(x,t) \) is an even function of time, namely

26
\[ \sigma_{-k}(x) = \sigma_k(x) = \sigma_k^*(x) \quad (3.12) \]

In this case, we see that Eq. (3.10) relates the diagonal resolvent \( R \) of the Schrödinger operator

\[ h = -\partial_x^2 + 2\beta g \sigma_k(x) \quad (3.13) \]

at energy parameter \( \alpha \omega_k^2 - m^2 \), to its potential term \( \sigma_k(x) \):

\[ R(x) = \frac{2\alpha \gamma T g}{g} \sigma_k(x), \quad (3.14) \]

in complete analogy with Eq. (2.11).

Following the same steps as in the previous section, we find from the Gelfand-Dikii equation (Eq. A.12) that \( \sigma_k \) must satisfy the equation

\[ 2\sigma_k \sigma_k'' - (\sigma_k')^2 - 4(2\beta g \sigma_k + m^2 - \alpha \omega_k^2) \sigma_k^2 = \alpha \left( \frac{g}{2T} \right)^2 \quad (3.15) \]

where we have used the relations \( \alpha^2 = 1 \) and \( \gamma^2 = -\alpha \). Note from Eq. (3.10) that Eq. (3.15) holds for the zero frequency mode \( \sigma_0(x) \) independently of whether the adiabatic extension \( \sigma(x,t) \) of the static configuration has any definite parity properties under \( t \to -t \) or not.

Naturally, Eq. (3.16) coincides with Eq. (2.14) in the case of euclidean signature \( \alpha = -1 \) upon the replacements \( m^2 \to m^2 + \omega^2, \ g \to g/T \).

As in the previous section, first integration of Eq. (3.15) is done by substituting

\[ \sigma_k'(x) = f_k(\sigma_k) \quad (3.16) \]

leading to

\[ f_k^2 = \left( \frac{d\sigma_k}{dx} \right)^2 = 4 \left[ \beta g \sigma_k^3 + (m^2 - \alpha \omega_k^2) \sigma_k^2 \right] + \alpha \left( \frac{g}{2T} \right)^2 \left( \frac{\sigma_k}{\sigma_k} - 1 \right) \quad (3.17) \]
where $\bar{\sigma}$ is an integration constant. Note that unlike the $0 + 1$ dimensional model discussed in the previous section, in the field theoretic case we lack a simple physical interpretation of Eq. (3.17) as an equation of motion in one dimension allowing us to discard “radial” trajectories (in euclidean signature) that hit the origin $\sigma = 0$.

Simple analysis of Eq. (3.17) for very high frequencies $\omega_k$, reveals that the corresponding modes $\sigma_k(x)$ with bounded amplitudes are highly suppressed. Namely, one finds that $\sigma_k$ is of the order $\frac{g^2}{\sigma(\omega_k T)^2} \ll 1$. Therefore, if we choose $P(n)$ in Eq. (3.8) such that already $2\pi P(1) \gg 1$ (e.g. $P(n) = ne^{(n^2+1)^{1000}}$), then for large enough $T$ and generic values of $\bar{\sigma}$ all non zero frequency modes will be highly suppressed in the resulting extremum condition, assuring self consistency of the adiabatic approximation. As $\sigma_0(x)$ is decoupled from non zero frequency modes within the framework of the approximation made, we are free to make such choices of the $P(n)$’s.

Concentrating on the zero-frequency mode, Eq. (3.17) becomes

\[
\left( \frac{d\sigma}{dx} \right)^2 \equiv f^2(\sigma) = 4 \left[ \beta g \sigma^3 + m^2 \sigma^2 \right] + \alpha \left( \frac{g}{2T} \right) \left( \frac{\sigma}{\bar{\sigma}} - 1 \right) \tag{3.18}
\]

where we have dropped the sub-index $k = 0$ everywhere. Note the explicit $T$ dependence of the right hand side of Eq. (3.18). This implies that $\sigma_0(x)$, which is our time-independent saddle point configuration, will have a peculiar dependence on $T$.

However, this paradox will be resolved by the fact that solutions to Eq. (3.18) in which the $T$ dependence is “important” (in a sense that will become clear when we consider these solutions) always give rise to infinite $S_{eff}[\sigma]$ values, and are thus highly suppressed in the path integral (Eq. (3.4)), unless we put the field theory in a finite “spatial” box of length $L$. In the latter case, these $T$ dependent $\sigma_0(x)$ solutions are not suppressed. In such cases having explicit $T$ dependence is consistent with having explicit $L$ dependence. We will not discuss the interesting finite size behavior of this field theory and its corresponding thermodynamic behaviour in this paper.

Thus, if $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the three roots of the cubic equation $f^2(0) = 0$, we have
\[
\begin{align*}
  f^2(0) &= -\alpha \left( \frac{g}{2T} \right)^2 \\
  f^2(\pm \infty) &= \pm \beta \infty \\
  \sigma_1 \sigma_2 \sigma_3 &= \frac{\alpha \beta g}{16T^2} \\
  \sigma_1 + \sigma_2 + \sigma_3 &= -\frac{\beta m^2}{g}.
\end{align*}
\] (3.19a, 3.19b, 3.19c, 3.19d)

Here we also adopt the convention that if \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are all real, then \( \sigma_1 \leq \sigma_2 \leq \sigma_3 \). Unlike Eq. (2.10), \( \alpha \) is not an overall factor in (3.18) and we shall have to analyze all four possible \((\alpha, \beta)\) combinations separately. We discuss below only cases in which \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are all real.

Clearly, then, (3.19c) implies that each \( \alpha, \beta \) combination gives rise to two possibilities, according to the arithmetic signs of \( \sigma_1, \sigma_2 \) and \( \sigma_3 \), yielding eight possibilities in all. These eight \( \sigma_0(x) \) configurations are depicted schematically in figs. (4a)-(4d). However, two of these (figs. (4a.i) and (4b.i) are explicitly ruled out by (3.19d), since it implies that when \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) have all the same sign, the latter must be \(-\beta\).

In the remaining configurations of fig. (4), the physically allowed domains for \( \sigma \) are segments on the semiaxis \( \sigma > 0 \) along which \( f^2(\sigma) \geq 0 \).

As in the previous section, it is clear from (3.18) and figs. (4a-d) that all static \( \sigma \) configurations are of the form \( \sigma(x) = \psi(x - x_0) \) where \( x_0 \) is an integration parameter and \( \psi \) is either (essentially) a Weierstrass \( \mathcal{P} \) function (unbounded \( \sigma \) configurations) or a simple rational function in terms of Jacobi functions, e.g. the \( sn \) or \( cn \) functions (bounded \( \sigma \) configurations). All these \( \sigma \) configurations are periodic in \( x \). In euclidean space-time signature we have again the \( \sigma \) configurations that hit the endpoint \( \sigma = 0 \).

Unlike the \( 0 + 1 \) dimensional case discussed in the previous section, we have no a priori reason to discard such solutions, but note that as functions of \( x \), their first derivative suffers a finite jump discontinuity whenever \( \sigma \) vanishes. This fact makes such \( \sigma(x) \) configurations rather suspicious.
Fig. 4a  \( \alpha = \beta = 1 \)

Fig. 4b  \( \alpha = -\beta = 1 \)

Fig. 4c  \( \alpha = -\beta = -1 \)

Fig. 4d  \( \alpha = -\beta = -1 \)

FIGURE 4
We will discuss explicit solutions of Eq. (3.18) in the last part of this section. At this point we make a small digression to recall the way the spectrum of “mesonic” bound states is calculated, given an extremal static $\sigma(x)$ configuration (i.e., a specific solution to Eq. (3.18) in our case). In this digression we follow [20, 22] very closely. Given a static $\sigma$ configuration and ignoring the gaussian fluctuations of $\sigma$ around it, dynamics of the “meson” $\phi$ field in this background is governed by the action in Eq. (3.3) upon substituting into it the particular $\sigma(x)$ configuration we are discussing. Clearly then the fluctuations of $\phi$ satisfy

$$[\square + \alpha m^2 + 2\alpha \beta g\sigma(x)]\phi_a = 0; \quad a = 1, \ldots N$$

with the periodic boundary conditions

$$\phi_a(t + T, x) = \phi_a(t, x).$$

(3.21)

Upon a simple separation of variables $\phi_a(x, t) = e^{i\lambda t} \phi_a(x)$ Eq. (3.20) turns into the eigenvalue problem of the time independent Schrödinger equation:

$$\left[-\partial_x^2 + m^2 + 2\beta g\sigma(x)\right]\phi_a(x) = \lambda^2[\sigma]\phi_a(x)$$

(3.22)

where the functional dependence of the eigenvalue $\lambda^2$ on the Schrödinger potential $\sigma(x)$ has been written explicitly.

Following Eq. (3.22) we expand $\phi_a(x)$ as

$$\phi_a(x, t) = \sum_i A_i(t) \psi_i(x)$$

(3.23)

where $\{\psi_i(x)\}$ are the complete orthonormal set of eigenfunctions of the Schrödinger hamiltonian in Eq. (3.22) with eigenvalues $\{\lambda_i^2\}$, and $A_i(t)$ are arbitrary amplitudes to be integrated over in the functional integral. The latter are subjected to the periodic boundary condition $A_i(t + T) = A_i(t)$.

Using Eqs. (3.20)-(3.23) the $\phi$ dependent part of Eq. (3.3) becomes

$$S = \frac{N}{2} \int_0^T dt \sum_{a=1}^N \sum_i \left[ (\partial_0 A_i^a)^2 - \alpha \lambda_i^2 (A_i^a)^2 \right]$$

(3.24)

\footnote{We have omitted an overall factor $\alpha$ from one of the sides of Eq. (3.21). This will show up later.}
which is nothing but the action $N$ identical (infinite) sets of harmonic oscillators oscillating at frequencies $\{\lambda_i\}$.

The natural place to look for bound state energies is the partition function associated with Eq. (3.24), i.e. with the particular static $\sigma(x)$ configuration we are considering.

Using well known results [39, 20, 22] to integrate over the $A_i(t)$ in Eq. (3.24), the desired partition function associated with $\sigma(x)$

$$Z[\sigma] = \text{Tr} e^{H[\sigma]T/\gamma} = \int \mathcal{D}_{phc} \phi \ W_P (T; [\sigma])$$

(3.25)

(where $W_P(T; [\sigma])$ is the contribution of our particular $\sigma(x)$ configuration to Eq. (3.4)) may be expressed straightforwardly as

$$Z[\sigma] = \left[ \prod_i e^{\lambda_i T/2\gamma} \right]^N \cdot e^{\frac{\lambda}{\gamma} \int_{-\infty}^{\infty} \sigma^2 dx} \cdot \sum_{\{\lambda\}} C(\{n\}) e^{\frac{T}{\gamma} \sum_j n_j \lambda_j}.$$  

(3.26)

The sum in Eq. (3.26) runs over all ordered sets $\{n\}$ of non negative integers $n_i$ and

$$C(\{n\}) = \prod_{n_i \epsilon \{n\}} \frac{(N+n_i-1)!}{(N-1)! n_i!}.$$  

(3.27)

By definition we have (Eq. (3.25))

$$Z[\sigma] = \sum_{\{n\}} C(\{n\}) e^{E(\{n\}) T/\gamma}$$

(3.28)

where

$$E(\{n\}) = \left[ -\beta N \int_{-\infty}^{\infty} \sigma^2 dx + \frac{N}{2} \sum_i \lambda_i \right] + \sum_{n_j \epsilon \{n\}} n_j \lambda_j.$$  

(3.29)

is the energy of the state corresponding to the set $\{n\}$ and $C(\{n\})$ is its degeneracy.

---

23We use the relations $\gamma^2 = -\alpha$, $\alpha^2 = 1$.

24That is, inserting $n_k$ “mesons” into energy level $\lambda_k$, $k = 1, 2, \ldots$.

25Clearly $E(\{n\})$ corresponds to a “mesonic” bound state only if all quantum numbers $n_k$ that correspond to scattering states in Eq. (3.22) vanish.
Therefore each $\sigma(x)$ saddle point configuration gives rise to a band of bound states (as well as scattering states) of $\phi$ “mesons” with a very rich $O(N)$ structure. Generically, the degeneracy factors of energy states in the band are very large and each $E(\{n\})$ multiplet forms a highly reducible representation of $O(N)$. The lowest state in a band has no “mesons” in it at all ($n_i = 0$ for all $i$’s) and its energy is given by

$$E_0[\sigma] = -\beta N \int_{-\infty}^{\infty} \sigma^2 dx + \frac{N}{2} \sum_i \lambda_i[\sigma]$$

(3.30)

where the first term is the classical energy of the $\sigma(x)$ field and the other term is the zero point contribution of the oscillators in Eq. (3.24).

In general both terms in Eq. (3.30) are divergent and must be regularized. In certain cases where the period of $\sigma(x)$ as a function of $x$ becomes infinite regularization of Eq. (3.30) may be achieved simply by subtracting from it the zero point contribution of oscillators in the background of the vacuum configuration $\sigma = \sigma_c$, namely the quantity $\sum_i \lambda_i[\sigma_c]$ \[20, 22\]. We elaborate on such cases below. In other cases $E_0[\sigma]$ might be inherently divergent, suppressing contributions from such $\sigma(x)$ configurations enormously relative to the vacuum. Such cases include the Weierstrass $\mathcal{P}$ function configurations as well as finite amplitude $\sigma(x)$’s with a finite period along the $x$ axis. However, the latter may become important if we put the field theory into a spatial “box” of finite length $L$, that is an integral multiple of period of $\sigma(x)$. As $L \to \infty$ these modes are suppressed as was stated above.

In \[20, 22\] Eqs. (3.29) and (3.30) were the starting point in a calculation determining extremal $\sigma(x)$ configurations. In these papers one extremises the regularized form of Eq. (3.29) with respect to $\sigma(x)$. In order to account for the functional dependence of the $\lambda_i$ on $\sigma(x)$ one has to invoke inverse scattering techniques. In these papers, the $\sigma(x)$ configurations that resulted from the inverse scattering analysis were generically very mild distortions of the vacuum condensation $\sigma = \sigma_c$ obtained from the gap equation, expressed in terms of reflectionless Schrödinger potentials. Since $\sigma = \sigma_c$ is a local minimum of the appropriate $S_{eff}[\sigma]$, it is clear from the self consistency of

\[26\] Note in this respect that our $2g\sigma_c$ corresponds to $(\chi - \overline{\chi})/N$ in \[22\].
these analyses that such $\sigma(x)$ configurations must be also local minima of $S_{\text{eff}}[\sigma]$, since they are just the exact result (as $N \to \infty$) for the back-reaction of either bosons or fermions on the vacuum, which is in turn the local minimum of $S_{\text{eff}}[\sigma]$ trapping these particles.

As was mentioned above briefly, our static extremal $\sigma(x)$ configurations are very different from those found in [20, 22]. Moreover, contrary to the latter, they are generically not local minima of $S_{\text{eff}}[\sigma]$ but rather saddle points of this object such that the second variation of $S_{\text{eff}}[\sigma]$ around them is not positive definite. We have seen in the previous section that despite this fact the contribution of these saddle points to the path integral representation of the partition function $\text{Tr} \ e^{HT/\gamma}$ is important, because they correspond to (collective) excited states of the quantized theory. Clearly, only the extremal configuration that describes the ground state of the quantum theory must be a local minimum of $S_{\text{eff}}[\sigma]$. We may strengthen our conclusion of the importance of these $\sigma(x)$ configurations by making an analogy with a simpler elementary example. Namely, consider a variational calculation of the spectrum of a Schrödinger hamiltonian (for simplicity we consider the one dimensional case)

$$h = -\partial_x^2 + V(x).$$  \hspace{1cm} (3.31)

The energy functional is

$$S [\psi^\dagger, \psi; E] = \int_a^b \psi^\dagger (h - E) \psi \ dx + E$$  \hspace{1cm} (3.32)

where $E$ is a Lagrange multiplier enforcing the normalization condition $\langle \psi | \psi \rangle = 1$ and Neumann boundary conditions are assumed at the end points.

The extremum condition $\frac{\delta S}{\delta \psi^\dagger(x)} = 0$ reads

$$(h - E) \psi = 0 \ ; \ \psi'(a) = \psi'(b) = 0.$$  \hspace{1cm} (3.33)

Obviously, Eq. (3.33) is nothing but the Schrödinger equation of $h$, and thus has an infinite number of solutions $\{\psi_n(x), E_n\}$ (n = 0, 1, 2...) – the spectrum of $h$. Clearly, only one extremum is an (absolute) minimum of $S$ - this is the ground state
\{\psi_0(x), E_0\}. The other extrema corresponding to excited states are only saddle points of \(H\). Indeed, the \(n\)-th state, \(\psi_n(x)\), has exactly \(n\) unstable directions in its saddle point (recall we are considering the one dimensional case where no degeneracy occurs), corresponding to the \(n\) states below it. This is clear from the second variation of \(S\) around \{\psi_n(x), E_n\}

\[
S\left[\psi_n^\dagger + \delta\psi^\dagger, \psi_n + \delta\psi, E_n\right] = \sum_{m=0}^\infty |C_m|^2 (E_m - E_n) + E_n
\]  

(3.34)

where \(\delta\psi = \sum_{m=0}^\infty C_m \psi_m(x)\) and we have used the orthonormality of the \{\psi_n\}. These saddle points are important as they correspond to all excited states of the spectrum and must be taken into account.

Having established the importance of our static saddle point configurations we now turn to the explicit calculation of these. Out of eight cases depicted in fig. (4) only two yield a solution to Eq. (3.18) that can produce a finite \(S_{\text{eff}}[\sigma]\) value. Therefore, these are the only cases relevant for our discussion of bound states and resonances. These cases are described by figs. (4b.ii) and (4d.i), which have the same behaviour around \(\sigma_2\) and \(\sigma_3\). As will be shown below, the finite \(S_{\text{eff}}[\sigma]\) value for these cases is obtained in the limit \(\sigma_2 \to \sigma_1 + \to 0\). Two of the other cases in fig. (4) have already been ruled out by Eq. (3.19d) and the remaining four give rise to infinite \(S_{\text{eff}}[\sigma]\)'s or contain \(\sigma(x)\) configurations in euclidean time that hit the \(\sigma = 0\) boundary point.

In both cases of interest to us here the quartic interaction is unbounded from below (\(\beta = -1\)), but as was discussed at the beginning of this section, the field theory is perfectly defined to leading order in \(1/N\) in terms of the fluctuations around the vacuum of the theory which is a local minimum of \(S_{\text{eff}}[\sigma]\). Though our resulting \(\sigma(x)\) configurations are quite distinct from the vacuum condensation itself, they are of a finite amplitude, and are therefore protected against being driven to infinity by the bottomless quartic interaction. The solution of Eq. (3.18) for both these cases read

\[
\sigma(x) = \sigma_2 + (\sigma_3 - \sigma_2)cn^2 \left[\sqrt{g(\sigma_3 - \sigma_1)} (x - x_0)\right]|\mu|
\]  

(3.35)
where \( cn \) is a Jacobi elliptic function with parameter \([34, 33]\)

\[
\mu = \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1}.
\] (3.36)

For generic values of \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) we have \( 0 < \mu < 1 \) and \( \sigma(x) \) has a finite real period \( L \) along the \( x \) axis given by

\[
L = 2K(\mu) \tag{3.37}
\]

where \( K(\mu) \) is a complete elliptic integral of the first kind. In this case Eq. (3.37) yields obviously an infinite \( S_{eff}[\sigma] \) value, unless the field theory is defined in a “box” of finite volume. However, for \( \mu \to 1^- \), \( L \) in Eq. (3.37) diverges as \( \frac{1}{2} \ln \frac{1}{1-\mu} \) and \( \sigma(x) \) degenerates into

\[
\sigma(x) \approx \sigma_2 + \sigma_{32}\text{sech}^2 \left[ \sqrt{g\sigma_{31}}(x-x_0) \right] + \mathcal{O}(1-\mu) \tag{3.38}
\]

where \( \sigma_{3k} = \sigma_3 - \sigma_k; \ k = 1, 2 \). Eq. (3.38) leads evidently to a finite \( S_{eff}[\sigma] \) value.

The desired limit \( \mu \to 1^- \) is obtained as \( \sigma_1 \to \sigma_2^- \). In the Minkowsky space theory (fig. (4b.ii)) this limit is possible only as both \( \sigma_1 \) and \( \sigma_2 \) vanish, since they have opposite signs. This is possible only when \( \left( \frac{g}{T^2} \right)^2 \) in Eq. (3.18) becomes very small. In such a case we have from Eq. (3.18)

\[
\begin{aligned}
\sigma_3 &= \frac{m^2}{g} + \mathcal{O}\left(\frac{g}{T^2}\right) \approx \frac{m^2}{g} \\
\sigma_2 &= -\sigma_1 = \mathcal{O}\left(\frac{\sqrt{g}}{T}\right) \approx 0.
\end{aligned} \tag{3.39}
\]

Using Eq. (3.39), Eq. (3.38) becomes (for \( \mu = 1 \))

\[
g\sigma(x) = m^2 \text{sech}^2[m(x-x_0)]. \tag{3.40}
\]

Strictly speaking the euclidean space theory approaches this limit through complex \( \sigma_2 = \sigma_1^* \) values, and fig. (4d.i) does not hold in this case, but the end result, Eqs. (3.38) and (3.40) are the same, as is obvious from Eq. (3.18).

Substituting Eq. (3.40) into Eq. (3.22) (for \( \beta = -1, \ \alpha = \pm 1 \)) we find that \( \phi \) is coupled to the one dimensional potential \( V(x) \) given by

\[
V(x) = m^2 - 2g\sigma(x) = m^2 - 2m^2 \text{sech}^2[m(x-x_0)]. \tag{3.41}
\]
$V(x)$ is a reflectionless Schrödinger potential with a single bound state $\psi_0(x)$ at zero energy $\lambda_0 = 0$, where

$$
\psi_0(x) = \sqrt{\frac{m}{2}} \sech \left[ m(x - x_0) \right].
$$

In addition to $\psi_0(x)$, $V(x)$ has a continuum of scattering states

$$
\psi_q(x) = \{iq - m \tanh [m(x - x_0)]\} e^{iqx}
$$

with eigenvalues $\lambda_q^2 = m^2 + q^2$ in which the reflectionless nature of $V(x)$ is explicit.

In this respect, our $\sigma(x)$ configuration is reminiscent of the Callan-Coleman-Gross-Zee kink in the Gross-Neveu model [20]. As in that kink solution, the profile of $\sigma(x)$ is independent of the number of particles ($\phi$ “mesons”) trapped in it. This is contrary to the behaviour of $\sigma(x)$ configurations found in [20, 22] by inverse scattering techniques, that are all small distortions of the vacuum configuration $\sigma = \sigma_c$. We would like to stress at this point that unlike the Callan-Coleman-Gross-Zee kink, our configuration (Eq. (3.41)) does not connect degenerate vacua, since the $\mathcal{O}(N)$ model lacks such structures.

As far as “mesonic” bound states are concerned, we have only one relevant quantum number $n_0$ in Eqs. (3.26)-(3.29), corresponding to Eq. (3.42). In this case the degeneracy factor in Eq. (3.27) reads

$$
C(n_0) = \frac{(N + n_0 - 1)!}{(N - 1)! \ n_0!}
$$

which implies that the dimension of the multiplet with eigenvalue $E(n_0)$ is that of a symmetric tensor of rank $n_0$ which is a highly reducible representation of $\mathcal{O}(N)$, the irreducible components of which are all traceless symmetric tensors of lower ranks $n'_0$ such that $n_0 - n'_0 \equiv 0 \ (\text{mod } 2)$.

Up to this point we have not specified the value of $m$ in any of our equations. Explicit comparison with Eq. (4.38) of [22] as well as the analogy with the Callan-Coleman-Gross-Zee kink indicate that we must identify $m$ as the finite (renormalized) mass of the $\phi$ “mesons” obtained from the “gap-equation” [9]. Moreover, from Eq.
(4.14) of [22] it seems that our “kink-like” \( \sigma(x) \) configuration corresponds to the parameter \( \theta \) in that equation having the value \( \theta = \frac{\pi}{2} \). Ref. [22] considered only the range \( 0 \leq \theta \leq \frac{\pi}{4} \), discarding \( \theta = \frac{\pi}{2} \). The reason for this was that only in that region did \( E(n_0, \theta) \) obtained extrema which were actually local minima of \( E(n_0, \theta) \) as a function of \( \theta \). But since we are discussing excited states of the field theory, it is clear that these minima prevail only in a subspace of the total Hilbert space of states that is orthogonal to all lower energy states (recall Eqs. (3.31)-(3.34)). Therefore there is nothing wrong with our \( \sigma(x) \) configurations lying outside the domain considered by [22]. It has been shown in [22] that in the case of stable quartic interactions \( (\beta = +1) \) no saddle points \( \sigma(x) \) configurations that are minor distortions of the vacuum configuration \( \sigma = \sigma_c \) could be found. We complement this conclusion by proving that in this case “big” kink like \( \sigma(x) \) configurations do not occur as well.

We close this section by actually calculating the spectrum given by Eq. (3.29) and (3.30). The regularized lowest energy state in the bound is given by

\[
E_{0}^{\text{Reg}}[\sigma] = E_{0}[\sigma] - E_{0}[m], \quad (3.45)
\]

Namely, following Eq. (3.30) (for \( \beta = -1 \))

\[
\frac{1}{N} E_{0}^{\text{Reg}}[\sigma] = \int_{-\infty}^{\infty} \sigma^2 \, dx + \frac{1}{2} \sum_{i} \left( \lambda_i[\sigma] - \lambda_i[m^2] \right). \quad (3.46)
\]

Note that \( V(x) \) in Eq. (3.41) is isospectral to the “potential” \( \tilde{V}(x) = m^2 \), except for the single zero energy bound state of \( V(x) \) [40]. Therefore, all terms in the sum over \( \lambda_i \) in Eq. (3.46) cancel identically except for \( \lambda_0 \), which is zero anyway. Thus, only the integral in Eq. (3.46) contributes to \( E_{0}^{\text{Reg}} \), yielding

\[
E_{0}^{\text{Reg}}[\sigma] = \frac{4Nm^3}{3g^2}. \quad (3.47)
\]

Since the only discrete eigenvalue of \( V(x) \) in Eq. (3.41) is zero, Eq. (3.47) yields the common value of all bound state energies

\[
E(n_0) = \frac{4Nm^3}{3g^2}, \quad (3.48)
\]

independently of \( n_0 \). In this case the whole band degenerates into a single energy level.
Conclusion

In this work we have shown how simple properties of a one dimensional Schrödinger Hamiltonian can be used to obtain time or space dependent saddle point configurations contributing to the path integral in the large \(N\) limit. As a specific model we have analyzed the \(\mathcal{O}(N)\) vector model in \(0+1\) and in \(1+1\) space-time dimensions.

In the quantum mechanical case our results are equivalent *completely* to those obtained directly from the radial Schrödinger equation in the large \(N\) limit. The time dependent \(\sigma(t)\) configurations we have found are nothing but the semiclassical radial trajectories of that equation. In this respect our calculation is to the radial Schrödinger equation exactly the same as what the instanton calculus is to JWKB calculations of tunnelling effects.

In the two dimensional field theoretic case our method has been applied to find static saddle point configurations. In this case, the static case was approached through an “adiabatic” approximation that discarded all time dependence. This led to static saddle points that were quite distinct from the homogeneous vacuum one and from other static configurations obtained as minor distortions of the vacuum by use of inverse scattering techniques. Many of our novel static saddle points are important only in finite volume cases. We did not elaborate on these in this paper. Only one homogeneous saddle point configuration turned out to be relevant in the case of infinite volume. The latter gave rise to “mesonic” bound states at zero binding energy and is reminiscent of the Callan-Coleman-Gross-Zee kink found in the Gross-Neveu model.

All extremal \(\sigma(x)\) configuration we have found in the \(\mathcal{O}(N)\) vector model turn out to be saddle points of \(S_{\text{eff}}[\sigma]\) rather than local minima of that object. This, however should not be of any surprise, since our saddle points correspond to excited states of the quantal system.

Our method could be applied straightforwardly to other two dimensional field theoretic models.
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Appendix

The differential equation obeyed by the diagonal resolvent of a Schrödinger operator in one dimension

Consider the spectral problem for the one dimensional Schrödinger operator

\[ \hat{h} = -\partial_x^2 + U(x) \]  

(A.1)

on the segment \( a \leq x \leq b \) (where any of the end points may be at infinity) with boundary conditions

\[ \alpha \psi(z) + \beta \psi'(z) = 0; \quad z = a, b \]  

(A.2)

on the wave functions.

The eigenfunctions \( \psi_n(x) \) and the eigenvalues \( E_n \) of \( \hat{h} \) are therefore the solutions of

\[ \left[ -\partial_x^2 + U(x) \right] \psi(x) = E \psi(x) \]  

(A.3)

subjected to the boundary condition of Eq. (A.2). Let \( E \) be any real number, and let \( \psi_a(x) \) (\( \psi_b(x) \)) be the solution to Eq. (A.3) that satisfy the boundary condition Eq. (A.2) only at \( x = a \) (\( x = b \)), but not at the other boundary point.

The Wronskian of \( \psi_a \) and \( \psi_b \)

\[ W_{ab} = \psi_a(x)\psi_b'(x) - \psi_a'(x)\psi_b(x) \equiv C(E) \]  

(A.4)

is \( x \) independent, and clearly vanishes (as a function of \( E \)) only on the spectrum of \( \hat{h} \). For a generic \( E \), the Green function corresponding to Eq. (A.3) is the symmetric function.

\[ G_E(x, y) = \frac{1}{C(E)} \left\{ \Theta(x - y)\psi_a(y)\psi_b(x) + \Theta(y - x)\psi_a(x)\psi_b(y) \right\} \]  

(A.5)

\footnote{Indeed, if \( a \) and \( b \) are finite, the spectrum of \( \hat{h} \) is non-degenerate. If either \( a \) and/or \( b \) become infinite, the continuum part of the spectrum (if exists) is only two-fold degenerate. Explicitly, if \( E \) is an eigenvalue, \( \psi_a \) must satisfy the boundary condition at \( x = b \) as well, leading to \( C(E) = 0 \). Conversely, if \( C(E) = 0 \), \( \psi_a(x) \) satisfies the boundary condition at \( x = b \), hence \( E \) is in the spectrum.}
where $\Theta(x)$ is the step function.  

It is easy to check that $G_E(x, y)$ satisfies its defining equation

$$\left[-\partial_x^2 + U(x)\right] G_E(x, y) = \left[-\partial_y^2 + U(y)\right] G_E(x, y) = \delta(x - y), \quad (A.6)$$

and therefore

$$G_E(x, y) = \langle x | \frac{1}{-\partial^2 + U - E} | y \rangle. \quad (A.7)$$

The diagonal resolvent of $\hat{h}$ at energy $E$

$$R_E(x) = \langle x | \frac{1}{-\partial^2 + U - E} | x \rangle \quad (A.8)$$

is obtained from $G_E$ as

$$R_E(x) = \lim_{\epsilon \to 0^+} \frac{1}{2} \left[ G_E(x, x + \epsilon) + G_E(x + \epsilon, x) \right] = \frac{\psi_a(x)\psi_b(x)}{C(E)}. \quad (A.9)$$

Using Eq. (A.3) we find

$$R_E''(x) = 2 \left[ (U - E) + \frac{\psi_a' \psi_b'}{\psi_a \psi_b} \right] R \quad (A.10)$$

leading to

$$R_E'^2 - 2 R_E R_E'' = -4 R_E^2 (U - E) + \left( \frac{W_{ab}}{\psi_a \psi_b} \right)^2 R^2. \quad (A.11)$$

Using eqs. (A.4) and (A.9) we finally arrive at

$$-2 R_E R_E'' + R_E'^2 + 4 R_E^2 (U - E) = 1 \quad (A.12)$$

which is nothing but the “Gelfand-Dikii” equation [23], alluded to in sections II and III.
References

[1] S. Coleman, *Aspects of Symmetry* (selected Erice Lectures), Chs. 6-8, Cambridge Univ. Press, (1985), and references therein;
R. Rajaraman, *Solitons and Instantons*, North-Holland Publ., Amsterdam (1982).

[2] E. Brézin and S.R. Wadia (eds.) *The Large N Expansion in Quantum Field Theory and Statistical Physics*, World Scientific, Singapore (1993).

[3] E. Brézin, in *Proc. 8th Jerusalem Winter School for Theoretical Physics*, D.J. Gross et. al (eds.), World Scientific, Singapore (1992). See also other contributions to these proceedings.

[4] E. Witten, in *Recent Developments in Gauge Theories*, Cargèse 1979, G.’t Hooft et al. (eds.), Plenum Press, New-York, (1980) (and references therein).

[5] Sumit R. Das, *Rev. Mod. Phys.* 59 (1987) 235 , (and references therein).

[6] L. Yaffe, *Rev. Mod. Phys.* 54 (1982) 407 , (and references therein).

[7] L. Dolan and R. Jackiw, *Phys. Rev.* D9 (1974) 3320 ;
   H. J. Schnitzer, *Phys. Rev.* D10 (1974) 1800, 2042 ;
   S. Coleman, R. Jackiw and H.D. Politzer, *Phys. Rev.* D10 (1974) 2491 ; R.G. Root, *Phys. Rev.* D10 (1974) 3322 ;
   M. Kobayashi and T. Kugo, *Prog. Theor. Phys.* 54 (1975), 1537.

[8] L.F. Abbott, J.S. Kang and J.J. Schnitzer, *Phys. Rev.* D13 (1976) 2212 ; R.W. Haymaker, *Phys. Rev.* D13 (1976) 968 .

[9] H.J. Schnitzer, *Nucl. Phys.* B109 (1976) 297 .

[10] W.A. Bardeen and M. Moshe, *Phys. Rev.* D28 (1983) 1372 ; *Phys. Rev.* D34 (1986) 1229 .

43
[11] C.M. Bender, G.S. Guralnik, R.W. Keener and K. Olaussen, *Phys. Rev.* D14 (1976) 2590 ; C.M. Bender, F. Cooper and G.S. Guralnik, *Ann. Phys. (NY)* 109 (1977) 165.

[12] J.Zinn-Justin, *Phys. Lett.* B257 (1991) 335.

[13] P. DiVecchia, M. Kato and N. Ohta, *Int. J. Mod. Phys.* A7 (1992) 1391.

[14] P. DiVecchia and M. Moshe, *Phys. Lett.* B300 (1993) 49.

[15] H.J. Schnitzer, *Mod. Phys. Lett.* A7 (1992) 2449.

[16] M. Moshe, in Proc. of “Renormalization Group-91”, Dubna 1991, D.V. Shirkov et. al (eds.), World Scientific (1992). See also contribution to the 3rd Workshop on Thermal Field Theories, Banff, Canada (1993) [hep-th/9312111] and references therein.

[17] J.P. Nunes and J.J. Schnitzer, Brandeis preprint BRX-TH-351 (hep-ph-9311319), and references therein.

[18] R.F. Dashen, B. Hasslacher and A. Neveu, *Phys. Rev.* D10 (1974) 4114, ibid 4130, 4138.

[19] R.F. Dashen, B. Hasslacher and A. Neveu, *Phys. Rev.* D11 (1975) 3424.

[20] R.F. Dashen, B. Hasslacher and A. Neveu, *Phys. Rev.* D12 (1975) 2443.

[21] D.J. Gross and A. Neveu, *Phys. Rev.* D10 (1974) 3235.

[22] L.F. Abbott, *Phys. Rev.* D14 (1976) 552.

[23] I.M. Gelfand and L.A. Dikii, *Russian Math. Surveys* 30 (1975) 77.

[24] R. Friedberg and T.D. Lee, *Phys. Rev.* D15 (1977) 1694, ibid D16 (1977) 1096, ibid D18 (1978) 2623.
[25] A.G. Williams and R.T. Cahill, Phys. Rev. D28 (1983) 1966 ; ibid D30 (1984) 391.

[26] W.A. Bardeen, M. Moshe and M. Bander, Phys. Rev. Lett. 52 (1984) 1188.

[27] W.A. Bardeen, K. Higashijima and M. Moshe, Nucl. Phys. B250 (1985) 437.

[28] A.M. Polyakov, Nucl. Phys. B120 (1977) 429.

[29] A.M. Polyakov, Gauge Fields and Strings, ch. 4, Harwood Academic Publishers, Chur, Switzerland, (1987).

[30] L.S. Schulman Techniques and Applications of Path Integrations, chs. 13-20, John Wiley and Sons, New York, 1981.

[31] C.M. Bender and A. Orszag Advanced Mathematical Methods for Scientists and Engineers, Ch. 6, McGraw-Hill, New York 1978.

[32] C. Lovelace, Nucl. Phys. B190 (1987) 45.

[33] M. Abramowitz and I.A. Stegun (eds.) Handbook of Mathematical Functions, chs. 16-18, Dover, New York 1972.

[34] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Chs. 20-22, Cambridge Univ. Press, Cambridge 1988 (fourth ed.).

[35] F. Calogero, Lett. Nuovo Cimento 13 (1975) 507; I.M. Krichever, Funct. Anal. Appl. 14 (1981) 282;
M. A. Olshanetsky and A.M. Perelomov, Phys. Rep. 71 (1981) 313, ibid 94 (1983) 313.

[36] M. Carreau, E. Farhi, S. Gutmann and P.F. Mende, Ann. Phys. (NY) 204 (1990) 186;
J. Ambjorn and C.F. Kristjansen, Int. J. Mod. Phys. A8 (1993) 1259;
J. Feinberg, Nucl. Phys. B405 (1993) 389.
[37] N.C. Mermin and H. Wagner, *Phys. Rev. Lett.* **17**(1966) 1133 ;
S. Coleman, *Commun. Math. Phys.* **31** (1973), 259.

[38] Shung-Sheng Shei, *Phys. Rev.* **D14** (1976) 535 .

[39] R.P. Feynman and A.R. Hibbs *Quantum Mechanics and Path Integrals*, ch. 8,
McGraw-Hill, New York 1965;
I.M. Gel’fand and A.M. Yaglom, *J. Mat. Phys.* **1** (1960), 48.

[40] J. Goldstone and R. Jackiw, *Phys. Rev.* **D11** (1975) 1486 ;
W. Kwong and J.L. Rosner, *Prog. Theor. Phys.* **86** (suppl.) (1986) 366.