KPZ Analysis for $W_3$ Gravity

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Abstract

Starting from the covariant action for $W_3$ gravity constructed in [1], we discuss the BRST quantization of $W_3$ gravity. Taking the chiral gauge the BRST charge has a natural interpretation in terms of the quantum Drinfeld–Sokolov reduction for $Sl(3,\mathbb{R})$. Nilpotency of this charge leads to the KPZ formula for $W_3$. In the conformal gauge, where the covariant action reduces to a Toda action, the BRST charge is equivalent to the one recently constructed in [2].

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1. Introduction

Consider some matter fields coupled to gravity in a diffeomorphism and Weyl invariant way. Integrating out the matter from such a theory one obtains a gravitational induced action $\Gamma_{[g_{\mu\nu}]}$. If the theory has no anomalies, $\Gamma_{[g_{\mu\nu}]}$ reduces to an action on moduli space, since on the classical level the number of degrees of freedom of the metric equals the number of invariances. However, as is well known, the procedure of integrating out the matter cannot be done in both a Weyl and diffeomorphism invariant way, leading to a non-trivial $g$ dependence of $\Gamma_{[g_{\mu\nu}]}$. Using a diffeomorphism invariant regulator to handle the matter integration one obtains the following unique expression for the induced action

$$\Gamma_{[g_{\mu\nu}]} = \frac{d}{96\pi} \int R \frac{1}{\Box} R, \quad (1.1)$$

a result first obtained by Polyakov [3]. Note that $d$, the central charge of the matter system, is the only remnant of the matter system we used to define the induced action. Parametrizing the metric by $ds^2 = e^{-2\phi}|dz + \mu d\bar{z}|^2$, the induced action $\Gamma_{[g_{\mu\nu}]}$ decomposes into a sum of three terms [4]:

$$\Gamma_{[g_{\mu\nu}]} = \Gamma_{[\mu]} + \Delta S_{[\mu, \bar{\mu}, \phi]} + \Gamma_{[\bar{\mu}]}, \quad (1.2)$$

where $\Gamma_{[\mu]}$ and $\Gamma_{[\bar{\mu}]}$ are non-local actions, known as the chiral actions, while $\Delta S$ is a local counterterm, which is such that the sum of the three terms is invariant under diffeomorphisms. Using the operator product expansion of the stress-energy tensor $T_{\text{mat}}$ with itself, the chiral action $\Gamma_{[\mu]}$ can be computed from

$$\exp(-\Gamma_{[\mu]}) = \left\langle \exp \frac{1}{\pi} \int \mu T_{\text{mat}} \right\rangle_{\text{OPE}}. \quad (1.3)$$

A similar definition holds for $\Gamma_{[\bar{\mu}]}$. In [3, 4] it was shown that in the semi-classical limit $d \to \infty$ the chiral action $\Gamma_{[\mu]}$ is related to the WZW action based on $SL(2,\mathbb{R})$

$$\exp(-\Gamma_{[\mu]}) = \int Dh \delta(k\mu - (J_+ + 2zJ_0 - z^2J_-)) \exp(-kS_{\text{wzw}}^+(h)), \quad (1.4)$$

with $J = kh^{-1}\partial h$ and $k \sim d/6$.

In [3, 4] we generalized the above results to $W_N$ gravity. Starting with the chiral action

$$\Gamma_{[\mu_i]} = \left\langle \exp \frac{1}{\pi} \int \sum_{i=2}^{N} \mu_i W_{\text{mat}}^i \right\rangle_{\text{OPE}}, \quad (1.5)$$

In [3, 4] we generalized the above results to $W_N$ gravity. Starting with the chiral action
where the $\mu_i$ are the $W_N$ generalizations of the Beltrami differential $\mu_2$, we constructed the local counterterm $\Delta S$ such that

$$S_{\text{cov}} = \Gamma[\mu_i] + \Delta S[\mu_i, \bar{\mu}_i, G] + \Gamma[\bar{\mu}_i], \quad (1.6)$$

is invariant under left and right $W_N$ transformations. Here $G$ is a $\text{Sl}(N, \mathbb{R})$ valued field needed to make the action invariant. Taking a Gauss decomposition for $G$ it turns out that some of the components of $G$ are auxiliary fields that can be integrated out \cite{6, 7}. For instance, for $\text{Sl}(2, \mathbb{R})$ only the Cartan subgroup labels a true degree of freedom, which is the Liouville field $\phi$. Furthermore, it was shown that in the limit $d \to \infty$ the chiral actions are related to the WZW action based on $\text{Sl}(N, \mathbb{R})$ in a way similar as (1.4), and that the covariant action $S_{\text{cov}}$ is nothing but a Legendre transform of the WZW action \cite{7}.

In this paper we will discuss the quantization of this theory for the case of $\text{Sl}(3, \mathbb{R})$, i.e. $W_3$ gravity. Previously, there have been attempts to quantize $W_3$ gravity from the chiral point of view \cite{8} (see also \cite{9, 10, 11}). For example Schoutens et. al. \cite{8} computed the one-loop contributions to the effective action for $W_3$ gravity, using only the chiral action $\Gamma[\mu, \nu]$ (where we denoted the $W_3$ analogue of the Beltrami differential by $\nu$). Here we take a different point of view and discuss the covariant quantization of $W_3$ gravity. Starting from the covariant action $S_{\text{cov}}$ specialized to $\text{Sl}(3, \mathbb{R})$, we will impose the chiral gauge using the BRST formalism. This procedure gives us a classical BRST charge, whose nilpotency can be checked using the Poisson brackets of the fields contained in the BRST charge. Quantization of the theory is subsequently done in the following way: the Poisson brackets are replaced by OPE’s, the classical BRST charge is replaced by a normal-ordered one, and we allow for multiplicative renormalizations of the fields contained in the BRST charge. Nilpotency of this quantum BRST charge will lead to the KPZ formula for $W_3$.

We will illustrate our method in section 2 for the case of ordinary gravity. We show that BRST quantization of ordinary gravity in the chiral gauge $\bar{\mu} = \phi = 0$ leads to a BRST charge of the form

$$Q = Q_{ds} + Q_{FF}, \quad (1.7)$$

with

$$Q_{ds} = \oint c(J_\gamma - k),$$

$$Q_{FF} = \oint : \bar{c} \left( T_{mat} + T_{FF} + \frac{1}{2} T_{gh} \right) :,$$  \quad (1.8)

with $\oint = \frac{1}{2\pi i} \oint d\bar{z}$. Here $Q_{ds}$ follows from the gauge-fixing of $\bar{\phi}$; it is the BRST charge that is naturally associated with the Drinfeld–Sokolov reduction for $\text{Sl}(2, \mathbb{R})$. $Q_{FF}$ follows from
the gauge-fixing of $\bar{\mu}$, and $T_{FF}$ appearing in $Q_{FF}$ is the generator of the cohomology of $Q_{ds}$. Nilpotency of $Q$, and in fact

$$Q_{ds}^2 = Q_{ds}Q_{FF} + Q_{FF}Q_{ds} = Q_{FF}^2 = 0,$$  \hspace{1cm} (1.9)

holds iff

$$d = 13 + 6 \left( (k + 2) + \frac{1}{(k + 2)} \right),$$  \hspace{1cm} (1.10)

which is the well-known KPZ result for ordinary gravity [12].

In section 3 these results are generalized to the case of $W_3$ gravity in the following way. Starting with the covariant action and subsequently imposing the chiral gauge we will derive the BRST charge for $W_3$. It again takes the form $Q = Q_{ds} + Q_{FF}$, with $Q_{ds}$ the BRST charge associated with the Drinfeld–Sokolov reduction for $Sl(3, \mathbb{R})$, and

$$Q_{FF} = \oint : \bar{c}_1 \left( T_{mat} + T_{FF} + \frac{1}{2} T_{gh} \right) + \bar{c}_2 \left( \pm i \tilde{W}_{mat} + \tilde{W}_{FF} + \frac{1}{2} \tilde{W}_{gh} \right) :,$$  \hspace{1cm} (1.11)

where $T_{FF}, \tilde{W}_{FF}$ are the generators of the $Q_{ds}$ cohomology. Nilpotency of $Q$ now holds iff

$$d = 50 + 24 \left( (k + 3) + \frac{1}{(k + 3)} \right),$$  \hspace{1cm} (1.12)

which is the KPZ result for $W_3$ gravity, previously conjectured in [13, 14, 15].

It should be stressed that the existence of a BRST charge for $W_3$ matter coupled to $W_3$ gravity is somewhat surprising, since one of the basic requirements for the construction of a BRST charge is that the generators contained in the BRST charge form a closed algebra. (The structure coefficients of the algebra may be field dependent. As long as the algebra closes one can in general construct the BRST charge.) But for $W_3$ matter coupled to $W_3$ gravity this is not the case. Although $W_{mat}$ and $W_{FF}$ form a closed $W_3$ algebra separately, a linear combination of them does not, due to the non-linearity of the $W_3$ algebra. Nevertheless, there exists a BRST charge for this coupled system. Its existence can be justified as follows: since there is a covariant action describing $W_3$ matter coupled to $W_3$ gravity, we know that the BRST quantization of this action should lead to the BRST charge for $W_3$ matter coupled to $W_3$ gravity.

Recently, there have been other constructions of the BRST charge [3, 16] for $W_3$ matter coupled to $W_3$ Toda theory. In section 4 we will discuss the BRST quantization of the covariant action in the conformal gauge, where it reduces to a Toda theory, and show that in this gauge our BRST charge equals that of [3, 16].
2. Covariant Quantization of Ordinary Gravity

In this section we will perform the BRST quantization of ordinary gravity. We start with a covariant formulation of 2D gravity and subsequently impose the chiral gauge using the BRST formalism. We argue that the corresponding BRST charge can be understood from the point of view of (quantum) Drinfeld–Sokolov reductions [17, 18, 14]. Using this connection we will in the next section discuss the quantization of $W_3$ gravity. Previous studies concerning the BRST quantization of 2D gravity have appeared in [19, 20, 21].

The covariant action for ordinary gravity, given by the general formula (1.6) specialized to the case of $\text{Sl}(2, \mathbb{R})$, reads:

$$S_{\text{cov}} = \Gamma[\mu] + \Delta S[\mu, \bar{\mu}, G] + \Gamma[\bar{\mu}]. \quad (2.1)$$

If we take the following Gauss decomposition for $G$

$$G = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\omega} \\ 0 & 1 \end{pmatrix}, \quad (2.2)$$

the local counterterm $\Delta S$ becomes [4, 6]

$$\Delta S = \frac{k}{\pi} \int d^2z \left[ \partial \phi \partial \bar{\phi} + \omega (2 \partial \phi + \partial \mu) + \bar{\omega} (2 \partial \bar{\phi} + \partial \bar{\mu}) + \mu \omega^2 + \bar{\mu} \bar{\omega}^2 + 2 \omega \bar{\omega} - (1 - \mu \bar{\mu}) e^{-2\phi} \right]. \quad (2.3)$$

As we already mentioned in the Introduction, the relation between $\Gamma[\mu]$ and the $\text{Sl}(2, \mathbb{R})$ WZW model reads [3, 4, 7]:

$$\exp(-\Gamma[\mu]) = \int Dh \delta(k\mu - (J_+ + 2zJ_0 - z^2 J_-)) \exp(-kS_{wzw}^+(h)), \quad (2.4)$$

where we decomposed the $\text{Sl}(2, \mathbb{R})$ current $J = kh^{-1} \partial h$ as

$$J = J_aT^a = \begin{pmatrix} J_0 & J_+ \\ J_- & -J_0 \end{pmatrix}. \quad (2.5)$$

The definitions of the WZW actions $S_{wzw}^{\pm}$ can be found in Appendix A. Note that the equations of motion $\partial J_a = 0$ for the currents $J_a$, that follow from the WZW action, lead to the correct equation of motion for $\mu$, i.e. $\partial^3 \mu = 0$. Classically, the $J_a$ satisfy the Poisson brackets

$$\{J_a(z), J_b(\bar{w})\} = k \eta_{ab} \delta'(z - \bar{w}) + f_{ac}^b J_c(\bar{w}) \delta(z - \bar{w}), \quad (2.6)$$

*Our conventions are: $\eta^{ab} = \text{Tr}(T^a T^b)$, $\eta_{ab}$ is the inverse of $\eta^{ab}$, $f_{ac}^b = [T^a, T^b]$, and indices are raised using $\eta^{ab}$.\]
where the prime means derivation w.r.t. $\bar{w}$. These Poisson brackets should be replaced by the Operator Product Algebra (OPA)

$$J_a(\bar{z})J_b(\bar{w}) = \frac{k\eta_{ab}}{(\bar{z} - \bar{w})^2} + \frac{f^c_{ab}J_c(\bar{w})}{(\bar{z} - \bar{w})},$$

(2.7)

when quantizing the theory.

In (2.3) $\omega, \bar{\omega}$ are auxiliary fields that can be integrated out. If one does so \[4, 6\], one recovers Polyakov’s result for the (induced) gravitational action \[3\]:

$$S_{cov} \sim \int R \frac{1}{\Box} R,$$

with $\Box$ the covariant Laplacian and $R$ the scalar curvature corresponding to the metric defined by $ds^2 = e^{-2\phi}|dz + \mu d\bar{z}|^2$. We prefer not to do this, because in the case of $W$ gravity the auxiliary fields appear in general in more than second order and can thus not be integrated out \[6\]. The covariant action (2.1) is invariant under general two-dimensional diffeomorphisms, whose action on the fields is given by \[6, 7\]:

$$\begin{align*}
\delta_{\epsilon, \bar{\epsilon}} \mu &= \bar{\partial}\epsilon + \epsilon \partial \mu - \mu \partial \epsilon, \\
\delta_{\epsilon, \bar{\epsilon}} \bar{\mu} &= \partial \bar{\epsilon} + \epsilon \partial \bar{\mu} - \bar{\mu} \partial \bar{\epsilon}, \\
\delta_{\epsilon, \bar{\epsilon}} \phi &= -\frac{1}{2} \partial \epsilon - \frac{1}{2} \partial \bar{\epsilon} - \omega \epsilon - \bar{\omega} \bar{\epsilon}, \\
\delta_{\epsilon, \bar{\epsilon}} \omega &= \frac{1}{2} \partial^2 \epsilon + \partial (\epsilon \omega) - \bar{\mu} \epsilon e^{-2\phi} + \bar{\epsilon} e^{-2\phi}, \\
\delta_{\epsilon, \bar{\epsilon}} \bar{\omega} &= \frac{1}{2} \partial^2 \bar{\epsilon} + \partial (\bar{\epsilon} \bar{\omega}) - \mu \bar{\epsilon} e^{-2\phi} + \epsilon e^{-2\phi}.
\end{align*}$$

(2.8)

### 2.1. BRST Quantization

We now study this theory in the chiral gauge $\bar{\mu} = \phi = 0$. This can be done most easily using the BRST formalism. We replace the above transformation rules of the fields by BRST transformations, i.e. we replace $\epsilon, \bar{\epsilon}$ in (2.8) by ghost fields $c, \bar{c}$, supplemented with the rule $\delta_{brst}c = c\partial c$ and $\delta_{brst}\bar{c} = \bar{c}\partial \bar{c}$. In order to impose the gauge we introduce two more anti-ghost fields $b, \bar{b}$, whose BRST transformation rules are $\delta_{brst}b = B$ and $\delta_{brst}\bar{b} = \bar{B}$. The gauge-fixed action reads

$$S_{gf} = S_{cov} - \delta_{brst} \left( \frac{1}{\pi} \int d^2z \left[ \bar{b}\bar{\mu} - 2b\bar{\phi} \right] \right).$$

(2.9)

Obviously, this action is invariant under the above BRST transformations, since the classical action was BRST invariant from the start and for the gauge fixing term we have\[7\]

\[\text{Note that it is not true that $\delta_{brst}^2$ vanishes off-shell on all the fields. In particular $\delta_{brst}^2 \omega$ and $\delta_{brst}^2 \bar{\omega}$ are proportional to the equations of motion of $\omega, \bar{\omega}$. Since these fields do not appear in the gauge fixing term, this does not cause any trouble.}\]
\( \delta^2_{\text{brst}}(\text{gauge fixing term}) = 0 \). Integrating out the Nakanishi–Lautrup fields \( B, \bar{B} \) imposes the gauge fixing condition \( \bar{\mu} = \phi = 0 \), and changes the BRST transformation of \( b, \bar{b} \) into

\[
\begin{align*}
\delta_{\text{brst}} b &= \frac{-\pi}{2} \frac{\delta S_{gf}}{\delta \phi} = k(\bar{\omega} + \partial \bar{\omega} - 1), \\
\delta_{\text{brst}} \bar{b} &= \frac{\pi}{2} \frac{\delta S_{gf}}{\delta \bar{\mu}} = T_{\text{mat}} + T_{\Delta} + T_{gh},
\end{align*}
\]

where we defined

\[
\begin{align*}
T_{\text{mat}} &= \frac{\pi}{2} \frac{\delta \Gamma[\bar{\mu}]}{\delta \bar{\mu}}, \\
T_{\Delta} &= \frac{\pi}{2} \frac{\delta \Delta S}{\delta \bar{\mu}} = k(\bar{\omega}^2 - \bar{\partial} \bar{\omega} + \mu), \\
T_{gh} &= -b \partial \bar{c} - \bar{\partial}(b \bar{c}).
\end{align*}
\]

The gauge-fixed action becomes:

\[
S_{gf} = \Gamma[\mu] + \frac{k}{\pi} \int d^2 z \left[ \omega \partial \mu + \mu \omega^2 + 2 \omega \bar{\omega} \right] + \frac{1}{\pi} \int d^2 z \left[ \bar{b} \partial \bar{c} + b(\partial c + \bar{\partial} \bar{c} + 2c \omega + 2\bar{c} \bar{\omega}) \right]. \tag{2.12}
\]

At this point we would like to integrate out the auxiliary fields \( \omega, \bar{\omega} \). Notice that the term quadratic in \( \bar{\omega} \) has disappeared. This is something which generalizes to \( W_3 \) gravity: after taking the ‘chiral’ gauge, whose definition for \( W_3 \) gravity is given in the next section, the troublesome terms of higher than quadratic order in the auxiliary fields disappear making it possible to integrate them out. In this case the equations of motion for \( \omega, \bar{\omega} \) are: \( \omega = -\frac{1}{2} \bar{b} \bar{c} \) and \( \bar{\omega} = -\frac{1}{2} \partial \mu + \frac{1}{k}(\mu \bar{b} \bar{c} - bc) \). Substituting these equations of motion leaves us with the following action:

\[
S_{gf} = \Gamma[\mu] + \frac{1}{\pi} \int d^2 z \left[ b \partial \bar{c} + b \partial c + b \bar{\partial} \bar{c} - b \bar{c} \partial \mu \right], \tag{2.13}
\]

which is of course still BRST invariant. The ghost action, which is off-diagonal in this form, can be diagonalized by making a ghost field redefinition

\[
\begin{align*}
\gamma &= c - \mu \bar{c} + z \bar{\partial} \bar{c}, & \bar{\gamma} &= \bar{c}, \\
\bar{\beta} &= \bar{b} + \mu b + z \partial b, & \beta &= b.
\end{align*}
\]

In terms of these new variables the final action reads

\[
S_{\text{final}} = \Gamma[\mu] + \frac{1}{\pi} \int d^2 z \left[ \bar{\beta} \partial \bar{\gamma} + \beta \partial \gamma \right]. \tag{2.15}
\]

Note that the new ghost fields have the same ‘diagonal’ Poisson brackets

\[
\{ \bar{\beta}(\bar{z}), \bar{\gamma}(\bar{w}) \} = \delta(\bar{z} - \bar{w}), & \quad \{ \beta(z), \gamma(w) \} = \delta(z - w) \tag{2.16}
\]
as the old ones. Also these Poisson brackets should be replaced by OPE’s when quantizing the theory.

If we use the correspondence (2.4) between $\mu$ and the currents $J_a$ of the $Sl(2, \mathbb{R})$ current algebra, we arrive at the following on-shell BRST transformation rules$^1$ [21]:

\begin{align*}
\delta_{\text{brst}} J_- &= \bar{\partial} J_\gamma \gamma, \\
\delta_{\text{brst}} J_0 &= -J_- \gamma + \bar{\partial} (J_0 \gamma) - \frac{k}{2} \bar{\partial}^2 \gamma, \\
\delta_{\text{brst}} J_+ &= 2J_0 \gamma + k \bar{\partial} \gamma + (\bar{\partial} J_+) \bar{\partial} \gamma + 2J_+ \bar{\partial} \gamma, \\
\delta_{\text{brst}} \bar{\gamma} &= \bar{\gamma} \bar{\partial} \gamma, \\
\delta_{\text{brst}} \gamma &= -\bar{\partial} (\gamma \bar{\gamma}), \\
\delta_{\text{brst}} \beta &= J_- - k - (\bar{\partial} \beta) \gamma, \\
\delta_{\text{brst}} \bar{\beta} &= T_{\text{mat}} + T_{\text{sug}} + T_{gh},
\end{align*}

(2.17)

where $T_{\text{sug}}$ is an improved version of the classical Sugawara stress-energy tensor

\[ T_{\text{sug}} = \frac{1}{2k} \eta^{ab} J_a J_b + \bar{\partial} J_0, \]

(2.18)

and $T_{gh} = T_{gh}^{\delta, \bar{\gamma}} + T_{gh}^{3, \gamma} = -\bar{\beta} \bar{\partial} \bar{\gamma} - \bar{\partial} (\beta \bar{\gamma}) + (\bar{\partial} \beta) \gamma$.

It is straightforward to show that these BRST transformations are generated by the BRST charge

\[ Q = \oint \gamma (J_- - k) + \bar{\gamma} \left( T_{\text{mat}} + T_{\text{sug}} + \frac{1}{2} T_{gh}^{\delta, \bar{\gamma}} + T_{gh}^{3, \gamma} \right). \]

(2.19)

Gauge independence of the theory is guaranteed by nilpotency of the BRST charge $Q$. Classically, using the Poisson brackets (2.6) and (2.16) to evaluate $Q^2$, this gives the following relation between the matter central charge $d$ and the level $k$ of the $Sl(2, \mathbb{R})$ current algebra: $d - 6k = 0$. At the quantum level things are a bit more complicated: first, we should replace the BRST operator by a normal ordered version, and second, we allow for possible renormalizations of the stress-energy tensor $T_{\text{sug}}$ appearing in the BRST charge:

\[ Q_{\text{quant}} = \oint : \gamma (J_- - k) + \bar{\gamma} \left( T_{\text{mat}} + T_{\text{sug}} + \frac{1}{2} T_{gh}^{\delta, \bar{\gamma}} + T_{gh}^{3, \gamma} \right) :, \]

(2.20)

with

\[ T_{\text{sug}} = \frac{1}{2} N_T \eta^{ab} : J_a J_b : + x \bar{\partial} J_0. \]

(2.21)

$^1$Note that on-shell fields are anti-holomorphic, i.e. $\partial (\text{fields}) = 0$. 

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Here $N_T, x$ should follow from the requirement of nilpotency of the quantum BRST charge. Using the operator product expansions for the ghosts and currents one easily verifies that the BRST charge is only nilpotent when the following criteria are satisfied

$$x = 1, \quad N_T = \frac{1}{k+2}, \quad d + \frac{3k}{k+2} - 6k - 28 = 0. \quad (2.22)$$

Thus the requirement of diffeomorphism invariance at the quantum level leads to a renormalization of the level $k$ of the $\text{Sl}(2, \mathbb{R})$ current algebra from $k = d/6$ to the one following from (2.22), a result first obtained by KPZ [12]. Note that the relation between $d$ and $k$ can also be written as:

$$d = 13 + 6 \left( (k + 2) + \frac{1}{(k + 2)} \right). \quad (2.23)$$

2.2. Relation with Quantum Drinfeld–Sokolov Reductions

In this section we want to reinterpret the previous results in terms of the quantum Drinfeld–Sokolov reduction of $\text{Sl}(2, \mathbb{R})$. Recall that (quantum) Drinfeld–Sokolov reductions provide a very efficient way to obtain (quantum) $W$ algebras [17, 18, 14, 22]. Let us illustrate this for the $\text{Sl}(2, \mathbb{R})$ case. Classically the approach boils down to the following: one starts with the dual space $M$ of the level-$k$ extended loop algebra of $\text{Sl}(2, \mathbb{R})$, on which we have the Poisson structure (2.6). Subsequently, one defines the reduced phase space $M_{\text{red}}$ by imposing the (first-class) constraint $J_+ = \xi$, with $\xi$ some constant, on the $\text{Sl}(2, \mathbb{R})$ current $J$. We will take $\xi = k$ to be in agreement with the previous subsection, although one could leave $\xi$ a free parameter. As usual, such a first-class constraint generates gauge invariances on the reduced phase space. The physical phase space $M_{\text{phys}}$ is the set of gauge invariant quantities built on elements of the constrained phase space $M_{\text{red}}$, i.e. $M_{\text{phys}} = M_{\text{red}}/\mathcal{G}$, where $\mathcal{G}$ is the symmetry group generated by the first class constraint, acting on $M_{\text{red}}$ via gauge transformations. In the case of $\text{Sl}(2, \mathbb{R})$ we have $\mathcal{G} = U^+$, i.e. the group of upper triangular matrices, which can be used to bring the elements of $M_{\text{red}}$ in the form:

$$\left( \begin{array}{cc} 0 & T \\ k & 0 \end{array} \right), \quad (2.24)$$

where $T$ is related to the original currents by: $T = \frac{1}{k} J_0^2 + J_+ + \bar{\partial} J_0$ (one easily checks that this $T$ is indeed invariant under $\mathcal{G}$ gauge transformations). The Poisson bracket on $M$ induces a Poisson bracket on $M_{\text{phys}}$. Using the Poisson brackets (2.6) an easy computation shows that in this case $T$ is the generator of the Virasoro algebra with central charge $-6k$

$$\{T(\bar{z}), T(\bar{w})\} = -\frac{1}{2} k \delta'''(\bar{z} - \bar{w}) + 2T(\bar{w})\delta'(\bar{z} - \bar{w}) + T'(\bar{w})\delta(\bar{z} - \bar{w}). \quad (2.25)$$
In the quantum analogue of this construction the constraint $J_- = k$ is implemented using the BRST procedure. For this one introduces the (anti)ghost $\beta, \gamma$ and the BRST charge

$$Q_{ds} = \oint \gamma (J_- - k).$$

(2.26)

Physical operators $O_i$ are now characterized as follows: $Q_{ds}(O_i) = 0$, and $O_i$ and $O_j$ are equivalent if their difference is the $Q_{ds}$ of something. $(Q(O)$ is computed by performing the contour integration $\frac{1}{2\pi i} \oint_C d\bar{z}$ of the OPE of $\gamma(\bar{z})(J_-(\bar{z}) - k)$ with $O(\bar{w})$. Here $C$ is a contour around $\bar{w}$.) So we are naturally interested in the cohomology of $Q_{ds}$

$$H_{Q_{ds}} = \frac{\ker Q_{ds}}{\text{im} Q_{ds}}.$$ (2.27)

Turning back to the quantum BRST charge (2.20) we derived in the previous subsection, we see that part of this charge is precisely $Q_{ds}$. In fact, if we split the BRST charge in (2.20) as $Q = Q_{ds} + Q_{sug}$, we have a stronger result than nilpotency of the total BRST charge $Q$, namely:

$$Q_{ds}^2 = Q_{ds}Q_{sug} + Q_{sug}Q_{ds} = Q_{sug}^2 = 0.$$ (2.28)

Here nilpotency of $Q_{ds}$ is obvious, $Q_{ds}$ and $Q_{sug}$ anti-commute due to the fact that

$$Q_{ds}(T_{sug} + T_{gh}^{\beta,\gamma}) = 0,$$ (2.29)

($Q_{ds}$ obviously vanishes on $T_{\text{mat}}$ and $T_{gh}^{\beta,\gamma}$) and $Q_{sug}$ is nilpotent iff $T_{ds} = T_{sug} + T_{gh}^{\beta,\gamma}$ generates a Virasoro algebra with central charge $26 - d$. This last requirement leads of course to the relation (2.24) between the level $k$ and the matter central charge $d$. Because $T_{ds}$ generates a Virasoro algebra with non-vanishing central charge it cannot be $Q_{ds}$ exact, which implies that $T_{ds}$ is a generator of $H_{Q_{ds}}$. In fact, it was proven by Feigin and Frenkel [18] that $T_{ds}$ generates the whole $Q_{ds}$ cohomology.

The above makes it clear that the (quantum) Drinfeld–Sokolov reduction for the case of $\text{Sl}(2, \mathbb{R})$ gives rise to a (quantum) Virasoro algebra. Furthermore, we see that quantization of 2D gravity in the chiral gauge actually resolves the cohomological problem posed in the Drinfeld–Sokolov approach; by quantizing the covariant action (2.1) we found the explicit form for the generator of $H_{Q_{ds}}$. Conversely, we could use the Drinfeld–Sokolov approach as follows: find the generator of the cohomology of $Q_{ds}$ and use this to construct $Q_{sug}$. Nilpotency of $Q_{sug}$ then gives the desired KPZ result for the renormalization of the level $k$.

This latter observation can in principle be applied directly to the case of $W_3$ gravity. The main difficulty now is to find the spin-three analogue of $T_{ds}$. Here we run into trouble.
since (to our knowledge) there is no such spin-three field that decomposes into a current and a ghost part, and that forms a closed quantum $W_3$ algebra together with the $SL(3,\mathbb{R})$ generalization of $T_{ds}$. It was suggested by Feigin and Frenkel \[18\] that a better starting point for the generalization of the above to the case of $W_3$ is provided by another stress-energy tensor, namely

$$T_{FF} = \frac{1}{k+2} \left( :\hat{J}_0\hat{J}_0 : + k\hat{J}_+ + (k+1)\partial\hat{J}_0 \right),$$

(2.30)

where we defined $\hat{J}_0 = J_0 + \beta \gamma$. In the next section we will show that for the $SL(3,\mathbb{R})$ generalization of this stress-energy tensor there is a spin three field such that they form a quantum $W_3$ algebra.

The following two questions arise: what is the relation between $T_{FF}$ and $T_{ds}$, and can $T_{FF}$ also be derived from the quantization of the covariant action? The answer to the first question is straightforward: $T_{FF}$ differs from $T_{ds}$ by a $Q_{ds}$ exact term, so it is simply a different representative of $H_{Q_{ds}}$. In formula we have

$$T_{FF} = T_{ds} + Q_{ds}(\beta J_+).$$

(2.31)

The answer to the second question is in the affirmative. To see this we have to reexamine the ghost redefinitions \([2.14]\).

2.3. GHOST REDEFINITIONS

Recall that we had to introduce new ghost fields in section 2.1 (see \((2.14)\)) in order to diagonalize the ghost action and obtain a recognizable form for the BRST charge $Q$. These ghost redefinitions might have seem an ad hoc trick, whose systematics is unclear. In this subsection we follow a different procedure to construct the BRST charge, making it clear where the ghost redefinitions stem from. Obviously, ghost redefinitions lead to different expressions for the BRST charge $Q$. We show that a particular ghost redefinition brings the BRST charge into the form $Q = Q_{ds} + Q_{FF}$, with

$$Q_{FF} = \oint :\hat{\gamma} \left( T_{\text{mat}} + T_{FF} + \frac{i}{2} T_{\text{gh}}^{\beta\gamma} \right) :.$$  

(2.32)

First, let us review the construction of the BRST charge for general algebras, linear and non-linear, due to Fradkin and Fradkina \([23]\). Suppose we have a set of constraint equations

\[\text{In fact, the stress-energy tensor considered by Feigin and Frenkel equals } T_{FF} \text{ without the } kJ_+ \text{ term. This does not alter the algebra formed by } T_{FF}, \text{ but it should be noted that without the } kJ_+ \text{ term } T_{FF} \text{ is not } Q_{ds} \text{ closed.}\]
\(G_a = 0\), where the \(G_a\) have Poisson brackets

\[
\{G_a, G_b\} = C_{ab}^c G_c. \quad (2.33)
\]

Here the \(C_{ab}^c\) are structure functions, i.e. they can be field dependent. If they are field independent, the constraints form a true algebra, otherwise the algebra is called ‘soft’. Introducing for every constraint \(G_a\) a ghost \(c^a\) and antighost \(b^a\), which satisfy the Poisson brackets

\[
\{b^a, b^b\} = \{c^a, c^b\} = 0, \quad \{b^a, c^b\} = \delta^b_a, \quad (2.34)
\]

the classical BRST charge takes the form \[24\]

\[
Q = \sum_{n \geq 0} (-)^n c^{b_{n+1}} \ldots c^{b_1} U^{(n)a_1 \ldots a_n}_{b_1 \ldots b_{n+1}} b_{a_n} \ldots b_{a_1}, \quad (2.35)
\]

where the \(U^{(n)a_1 \ldots a_n}_{b_1 \ldots b_{n+1}}\) are higher-order structure functions, which can be determined from the following two requirements: \(i\) \(U^{(0)} a = G_a\), and \(ii\) \(\{Q, Q\} = 0\). Working out the latter condition, one obtains the following recursive formula for the higher order structure functions \[24\]:

\[
(n + 1) U^{(n+1)a_1 \ldots a_{n+1}}_{b_1 \ldots b_{n+2}} G_{a_{n+1}} = \frac{1}{2} \sum_{p=0}^{n} (-)^{np+1} \{U^{(p)a_1 \ldots a_p}_{b_1 \ldots b_{p+1}}, U^{(n-p)p+1 \ldots a_n}_{b_{p+2} \ldots b_{n+2}} \} + \sum_{p=0}^{n-1} (-)^{np+n(p+1)(n-p+1)} U^{(p+1)a_1 \ldots a_p}_{b_1 \ldots b_{p+2}} U^{(n-p)p+1 \ldots a_n}_{b_{p+3} \ldots b_{n+2}}, \quad (2.36)
\]

where one should antisymmetrize the r.h.s. in both the \(a\) and \(b\) indices. The first order term follows trivially: \(U^{(1)} a c = - \frac{1}{2} C_{ab}^c\). In the case of a true algebra the \(C_{bc}^a\) are constants, from which it follows that the higher order structure functions vanish. The BRST charge then takes the familiar form

\[
Q = c^a G_a - \frac{1}{2} b_c C_{ab}^c c^a b^b. \quad (2.37)
\]

For ‘soft’ algebras the higher order structure functions are in general non-vanishing. Below we will see an example of such an algebra.

Redefining the ghost fields \(c^a\) changes the BRST charge as follows: transforming \(c^a \rightarrow \tilde{c}^a = V_b^a c^b\), with \(V_b^a\) some invertible matrix which may depend on the fields, is equivalent to defining new generators \(\tilde{G}_a = (V^{-1})^b_c G_b\). (So that the lowest order contribution to the BRST

\footnote{For notational simplicity the sub and superscripts denote generalized indices in the following; besides labeling the set of constraints they also label the points in space-time. This suppresses delta-functions and corresponding integrals in what follows.}
charge remains the same.) The new antighost fields $\tilde{b}_a$ follow from the requirement that they have ‘diagonal’ Poisson brackets with the $\tilde{c}^a$. Denoting the structure functions corresponding to the $\tilde{G}_a$ by $\tilde{U}^{(n)}$, the new BRST charge becomes:

$$Q = \sum_{n \geq 0} (-)^n \tilde{c}^{b_{n+1}} \ldots \tilde{c}^{b_1} \tilde{U}^{(n)}_{b_1 \ldots b_{n+1}} \tilde{b}_{a_n} \ldots \tilde{b}_{a_1}. \quad (2.38)$$

Note that only the lowest-order contribution to the BRST charge $c^a G_a = \tilde{c}^a \tilde{G}_a$ is invariant under the ghost redefinitions.

Let us apply the above considerations to the quantization of two-dimensional gravity. If we impose the chiral gauge $\bar{\mu} = \phi = 0$, we have the following constraints:

$$\theta = -\frac{\pi}{2} \frac{\delta S_{\text{cov}}}{\delta \phi} = J_\mu - k,$$

$$T = \frac{\pi}{2} \frac{\delta S_{\text{cov}}}{\delta \bar{\mu}} = T_{\text{mat}} + T_{\text{sug}} - \mu \theta - z \bar{\partial} \theta,$$  

(2.39)

where we used the equations of motion $\omega = 0$ and $\bar{\omega} = -\frac{1}{2} \partial \mu$, and $T_{\text{sug}}$ is given by (2.18). To make contact with the previous subsection, we want (instead of $T$) $T_{\text{mat}} + T_{\text{sug}}$ as one of the generators. The BRST charge to lowest order in the ghost fields reads: $Q = \oint \gamma \theta + \bar{\gamma} (T_{\text{tot}} + T_{\text{mat}} + T_{\text{sug}})$.

As we explained the corresponding change for the antighost variables $b \rightarrow \beta$, $\bar{b} \rightarrow \bar{\beta}$ as in (2.14) simply follow by demanding that they have ‘diagonal’ Poisson brackets with $\gamma, \bar{\gamma}$. The complete BRST charge follows from the algebra formed by $\theta$ and $T_{\text{tot}} = T_{\text{mat}} + T_{\text{sug}}$:

$$\{ \theta(\bar{w}), \theta(\bar{w}) \} = 0, \quad \{ T_{\text{tot}}(\bar{z}), \theta(\bar{w}) \} = \theta'(\bar{w}) \delta(\bar{z} - \bar{w}), \quad \{ T_{\text{tot}}(\bar{z}), T_{\text{tot}}(\bar{w}) \} = \frac{1}{12} (d - 6k) \delta'''(\bar{z} - \bar{w}) + 2T_{\text{tot}}(\bar{w}) \delta'(\bar{z} - \bar{w}) + T_{\text{tot}}'(\bar{w}) \delta(\bar{z} - \bar{w}).$$

(2.40)

Note that in this case we are dealing with a true algebra, i.e. the $C_{\text{bc}}^a$ are field independent, and using (2.37) we can construct the full BRST charge $Q$ as

$$Q = \oint \gamma \theta + \bar{\gamma} \left( T_{\text{tot}} + \frac{1}{2} T_{\text{gh}}^{\beta, \bar{\gamma}} + T_{\text{gh}}^{\beta, \gamma} \right),$$

(2.41)

which is the result we found before, see (2.19).

The above ghost redefinition occurred because we insisted on having $T_{\text{mat}} + T_{\text{sug}}$ as one of the generators. We can also take a different redefinition such that our BRST charge becomes
\[ Q = Q_{ds} + Q_{FF}. \] For this define yet another set of ghost fields:

\[
\begin{align*}
\zeta &= \gamma + J_+ \bar{\gamma}, & \bar{\zeta} &= \bar{\gamma}, \\
\bar{\eta} &= \bar{\beta} - J_+ \beta, & \eta &= \beta.
\end{align*}
\] (2.42)

This redefinition implies that the new generators are given by \( \theta \) and \( T_{\text{new}} \), where

\[
T_{\text{new}} = T_{\text{mat}} + \frac{1}{k} J_0^2 + J_+ + \bar{\partial}J_0.
\] (2.43)

The symmetry algebra formed by \( T_{\text{new}} \) and \( \theta \) is slightly different from the previous one, namely

\[
\begin{align*}
\{ \theta(z), \theta(\bar{w}) \} &= 0, \\
\{ T_{\text{new}}(z), \theta(\bar{w}) \} &= -\theta(\bar{w}) \delta'(\bar{z} - \bar{w}) + \frac{2}{k} J_0(\bar{w}) \theta(\bar{w}) \delta(\bar{z} - \bar{w}), \\
\{ T_{\text{new}}(\bar{z}), T_{\text{new}}(\bar{w}) \} &= \frac{1}{12} (d - 6k) \delta'''(\bar{z} - \bar{w}) + 2T_{\text{new}}(\bar{w}) \delta'(\bar{z} - \bar{w}) + T_{\text{new}}'(\bar{w}) \delta(\bar{z} - \bar{w}).
\end{align*}
\] (2.44)

From this the full classical BRST charge can be constructed: we find

\[
Q = Q_{ds} + Q_{FF} = \oint \zeta \theta + \bar{\zeta} \left( T_{\text{mat}} + T_{FF} + \frac{1}{2} T_{gh}^\theta \bar{\zeta} \right),
\] (2.45)

where \( T_{FF} \) is the classical version of \( T_{FF} \): \( T_{FF} = \frac{1}{k} J_0^2 + J_+ + \bar{\partial}J_0 \), with \( \hat{J}_0 = J_0 + \eta \zeta \). To extend this result to the quantum level we should normal order the BRST charge and again allow for possible renormalizations of \( T_{FF} \), i.e. we define

\[
T_{FF} = N_T \left( \hat{J}_0 \hat{J}_0 : +kJ_+ + x\bar{\partial}\hat{J}_0 \right).
\] (2.46)

Using the OPE’s of the currents and the ghosts, we find that if we choose

\[
x = k + 1, \quad N_T = \frac{1}{k + 2}, \quad d = 13 + 6 \left( k + 2 + \frac{1}{(k + 2)} \right),
\] (2.47)

the quantum BRST charge \( Q = Q_{ds} + Q_{FF} \) is nilpotent; in fact

\[
Q_{ds}^2 = Q_{ds} Q_{FF} + Q_{FF} Q_{ds} = Q_{FF}^2 = 0.
\] (2.48)
3. Quantization of $W_3$ Gravity

In this section we apply the above machinery to the $W_3$ case. As before we start with the covariant action and then impose the chiral gauge. We derive the constraints in this gauge, and construct (making use of ghost field redefinitions) the BRST charge corresponding to these constraints. This BRST charge again has a natural interpretation in terms of the Drinfeld–Sokolov reduction for $Sl(3, \mathbb{R})$. Nilpotency of this charge will lead to the desired KPZ formula for $W_3$ gravity.

The covariant action for $W_3$ gravity is again given by the general formula (1.6), now specialized to the case of $Sl(3, \mathbb{R})$:

$$S_{\text{cov}} = \Gamma[\mu, \nu] + \Delta S[\mu, \nu, \bar{\mu}, \bar{\nu}, G] + \Gamma[\bar{\mu}, \bar{\nu}].$$

Taking the following Gauss decomposition for $G$

$$G = \begin{pmatrix} 1 & 0 & 0 \\ \omega_1 & 1 & 0 \\ \omega_3 & \omega_2 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{\phi_2 - \phi_1} & 0 \\ 0 & 0 & e^{-\phi_2} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\omega}_1 & -\bar{\omega}_3 \\ -\bar{\omega}_1 & 0 & 1 \\ -\bar{\omega}_3 & 1 & 0 \end{pmatrix}.$$  

(3.2)

the local counterterm becomes

$$\Delta S = \frac{k}{\pi} \int d^2 z \left[ A^{ij}(\omega_i + \bar{\partial} \phi_i)(\bar{\omega}_j + \bar{\partial} \phi_j) - \sum_{i} e^{-A^{ij} \phi_j} - \frac{1}{2} A^{ij} \bar{\partial} \phi_i \bar{\partial} \phi_j \right.$$

$$+ e^{\phi_1 - 2\phi_2}(\mu - \frac{1}{2} \partial \nu - \nu \omega_1)(\bar{\mu} + \frac{1}{2} \bar{\partial} \bar{\nu} + \bar{\nu} \bar{\omega}_1) + e^{-\phi_1 - 2\phi_2} \nu \bar{\nu}$$

$$+ e^{\phi_2 - 2\phi_1}(\mu + \frac{1}{2} \partial \nu + \nu \omega_2)(\bar{\mu} - \frac{1}{2} \bar{\partial} \bar{\nu} - \bar{\nu} \bar{\omega}_2) + \mu T + \nu W + \bar{\mu} \bar{T} + \bar{\nu} \bar{W},$$

(3.3)

where $A^{ij}$ is the Cartan matrix of $Sl(3, \mathbb{R})$, $k$ is related to the matter central charge $d$ via $d = 24k$, and we defined $T, \bar{T}, W$ through the Fateev-Lyukhanov construction:

$$(\bar{\partial} - \omega_2)(\bar{\partial} - \bar{\omega}_1 + \omega_2)(\bar{\partial} + \omega_1) = \bar{\partial}^3 - T \bar{\partial} - W - \frac{1}{2} \bar{\partial} T,$$

$$(\bar{\partial} - \omega_2)(\bar{\partial} - \bar{\omega}_1 + \omega_2)(\bar{\partial} + \bar{\omega}_1) = \bar{\partial}^3 - \bar{T} \bar{\partial} + \bar{W} - \frac{1}{2} \bar{T}.$$

(3.4)

Note that the auxiliary fields $\omega_i, \bar{\omega}_i$ appear in higher than quadratic form in $S_{\text{cov}}$, making it impossible to integrate them out at this level. $S_{\text{cov}}$ is invariant under left and right $W_3$ transformations whose precise form are given in Appendix B. The chiral action $\Gamma[\mu, \nu]$ is related to a $Sl(3, \mathbb{R})$ WZW model by

$$\exp(-\Gamma[\mu, \nu]) = \int Dh \, \delta(k \mu - F_\mu(J)) \delta(k \nu - F_\nu(J)) \exp(-k S^+_{wzw}(h)), $$

(3.5)

*In this counterterm the auxiliary fields $\omega_i, \bar{\omega}_i$ are already integrated out.

†In fact to make contact with the WZW model, we should define $\langle \exp\left(\frac{1}{k} \int \mu T_{\text{mat}} + \alpha \nu W_{\text{mat}}\right)\rangle$, with $\alpha^2 = -5/2$. Of course, a similar correction should then be taken into account in the definition of $\Gamma[\bar{\mu}, \bar{\nu}]$. 
with \([7]\)

\[
F_\mu(J) = \frac{1}{2}(J_1 + J_2) + \frac{1}{2}z(H_0 + H_1) - \frac{1}{4}z^2(K_1 + K_2),
\]

\[
F_\nu(J) = J_3 + z(J_1 - J_2) + \frac{3}{2}z^2(H_0 - H_1) - \frac{1}{2}z^2(K_1 - K_2) + \frac{1}{4}z^4K_3, \quad (3.6)
\]

where we decomposed the \(Sl(3, \mathbb{R})\) current \(J = kh^{-1}\tilde{\partial}h\) as

\[
J = J_a T^a = 
\begin{pmatrix}
H_0 & J_1 & J_3 \\
K_1 & H_1 - H_0 & J_2 \\
K_3 & K_2 & -H_1
\end{pmatrix}, \quad (3.7)
\]

### 3.1. The chiral gauge

Let us now impose the chiral gauge \(\mu = \nu = \phi_1 = \phi_2 = 0\). In this gauge the cubic term in the auxiliary fields \(\tilde{\omega}_i\) disappears, and the equations of motion for the auxiliary fields can be solved,

\[
\omega_1 = 0, \quad \tilde{\omega}_1 = -\partial\mu - \frac{1}{6}\partial^2\nu,
\]

\[
\omega_2 = 0, \quad \tilde{\omega}_2 = -\partial\mu + \frac{1}{6}\partial^2\nu. \quad (3.8)
\]

Taking the functional derivative of \(S_{\text{cov}}\) w.r.t. \(\phi_1\) and \(\phi_2\) we obtain the constraints corresponding to the gauge-fixing of the \(\phi_i\)s:

\[
\pi \frac{\delta S_{\text{cov}}}{\delta \phi_1} = K_2 - 2K_1 + k - 3zK_3 = 0,
\]

\[
\pi \frac{\delta S_{\text{cov}}}{\delta \phi_2} = K_1 - 2K_2 + k + 3zK_3 = 0, \quad (3.9)
\]

where we used the above equations of motion \((3.8)\) and the relation \((3.6)\) between \(\mu, \nu\) and the currents \(J_a\). Since the currents \(K_i\) are anti-holomorphic, the above two constraints lead in fact to the following three constraints on the currents \(K_i\)

\[
K_1 = K_2 = k \quad \text{and} \quad K_3 = 0. \quad (3.10)
\]

So the \(Sl(3, \mathbb{R})\) current \(J\) is constrained to

\[
J_{fix} = J + k\Lambda = 
\begin{pmatrix}
H_0 & J_1 & J_3 \\
k & H_1 - H_0 & J_2 \\
0 & k & -H_1
\end{pmatrix}, \quad (3.11)
\]
where $J$ is the matrix containing the currents $H^i, J^i$ and $\Lambda$ is the constant matrix with zeroes everywhere except $\Lambda_{21} = \Lambda_{32} = 1$.

The constraints corresponding to the gauge-fixing of $\bar{\mu}, \bar{\nu}$ read

\[
T \equiv \pi \frac{\delta S_{\text{cov}}}{\delta \bar{\mu}} = T_{\text{mat}} + T_{\Delta},
\]
\[
W \equiv \frac{1}{\sqrt{-\beta_{\text{mat}}}} \pi \frac{\delta S_{\text{cov}}}{\delta \bar{\nu}} = \pm i \frac{W_{\text{mat}}}{\sqrt{\beta_{\text{mat}}}} + \frac{W_{\Delta}}{\sqrt{\beta_{\text{mat}}}}, \tag{3.12}
\]

where the factor $\alpha^{-1} = \sqrt{-2/5}$ is explained in the footnote above. Following [2] we divided by an extra factor $\sqrt{-\beta_{\text{mat}}}$, which has proven to be very useful in the construction of the BRST charge for $W_3$. Since classically we have $\beta_{\text{mat}} = 16/5c_{\text{mat}}$ and $c_\Delta = -c_{\text{mat}}$, as we shall see shortly, the matter spin-three field contains an extra factor $\pm i$ when compared to $W_{\Delta}$. From now on the rescaled fields $W/\sqrt{\beta}$, are denoted as $\tilde{W}$. Discarding terms proportional to the $\phi_1, \phi_2$ constraints (3.10), since they can be absorbed by making suitable ghost redefinitions, we find that $T_{\Delta}$ and $W_{\Delta}$ are given by

\[
T_{\Delta} = \frac{1}{2k} \text{Tr}(J_{f}^2) + \text{Tr}(PJ_{f})',
\]
\[
\alpha W_{\Delta} = \frac{1}{3k^2} \text{Tr}(J_{f}^3) + \frac{1}{2k} \text{Tr}(PJ_{f}^2)' + \frac{1}{2} \text{Tr}(P^2 J_{f}^2)' + \frac{1}{4k} \left[ \text{Tr}(PJ_{f}) \text{Tr}(P^2 J_{f})' - \text{Tr}(P^2 J_{f}) \text{Tr}(PJ_{f})' \right], \tag{3.13}
\]

where the prime denotes derivation w.r.t. $\bar{z}$, and we defined $P = \text{diag}(1, 0, -1)$, satisfying $[P, \Lambda] = \Lambda$. They form a classical $W_3$ algebra with central charge $c_\Delta = -24k = -c_{\text{mat}}$. (Note that the factor in front of the $\frac{1}{3} \text{Tr} J_{f}^3$ term equals $\sqrt{3\beta_\Delta/k^2}$, and that the lower order terms pick up a factor $k$ for each derivative.)

At this point we are in the position to determine the BRST charge. We have five generators: $G_1 = K_1 - 1$, $G_2 = K_2 - 1$ and $G_3 = K_3$ following from the $\phi_1, \phi_2$ gauge-fixing, and $G_4 = T$, $G_5 = W$ following from the $\bar{\mu}, \bar{\nu}$ gauge-fixing respectively. Using the general scheme of Fradkin and Fradkina explained in the previous section we can now compute the BRST charge. The result takes the form

\[
Q = Q_{ds} + Q_{W_3}, \tag{3.14}
\]

where

\[
Q_{ds} = \oint c_1 (K_1 - k) + c_2 (K_2 - k) + c_3 K_3 - c_1 c_2 b_3,
\]
\[
Q_{W_3} = \oint \bar{c}_1 \left( T_{\text{mat}} + T_{ff} + \frac{1}{2} T_{gh} \right) + \bar{c}_2 \left( \pm i \tilde{W}_{\text{mat}} + \tilde{W}_{ff} + \frac{1}{2} \tilde{W}_{gh} \right). \tag{3.15}
\]
This can be established as follows. The last term in \( Q_{ds} \) results from the commutator between \( G_1 \) and \( G_2 \). From the sub-algebra formed by \( T \) and \( G_1, G_2, G_3 \)

\[
\{T(\bar{z}), G_1(\bar{w})\} = -G_1 \delta'(\bar{z} - \bar{w}) + \frac{1}{k}(2H_0 - H_1)G_1 \delta(\bar{z} - \bar{w}),
\]

\[
\{T(\bar{z}), G_2(\bar{w})\} = -G_2 \delta'(\bar{z} - \bar{w}) + \frac{1}{k}(2H_1 - H_0)G_2 \delta(\bar{z} - \bar{w}),
\]

\[
\{T(\bar{z}), G_3(\bar{w})\} = -2G_3 \delta'(\bar{z} - \bar{w}) + \left(\frac{1}{k}(H_0 + H_1)G_3 + G_2 - G_1\right) \delta(\bar{z} - \bar{w}),
\]

one deduces the first-order structure functions \( U^{(1)}_{ba} \). We see that the algebra of the constraints is ‘soft’. Using the recursion formula (2.36) one computes the second-order structure functions \( U^{(2)}_{ab} \) as:

\[
U^{(2)}_{12} = U^{(2)}_{13} = U^{(2)}_{23} = 1/6k;
\]

and the other vanish. From these first and second-order structure functions we deduce that \( T_{ff} \) appearing in (3.13) is given by

\[
T_{ff} = \frac{1}{2k} \text{Tr}(\hat{J}^2_{fix}) + \text{Tr}(P \hat{J}_{fix}'),
\]

i.e. it is the simply the formula we found before (3.13) for \( T_\Delta \), with \( J_{fix} \) replaced by \( \hat{J}_{fix} \),

\[
\hat{J}_{fix} = \begin{pmatrix}
\hat{H}_0 & \hat{J}_1 & \hat{J}_3 \\
k & \hat{H}_1 - \hat{H}_0 & \hat{J}_2 \\
0 & k & -\hat{H}_1
\end{pmatrix},
\]

with \( \hat{H}_0 = H_0 + b_1 c_1 + b_3 c_3 \), \( \hat{H}_1 = H_1 + b_2 c_2 + b_3 c_3 \),

\[
\hat{J}_1 = J_1 + b_3 c_2, \quad \hat{J}_2 = J_2 - b_3 c_1, \quad \hat{J}_3 = J_3.
\]

In a similar way one finds that the spin-three field \( \tilde{W}_{ff} \) occuring in (3.17) is of the same form as \( W_\Delta \), again with \( J_{fix} \) replaced by \( \hat{J}_{fix} \).

The only problem left is the determination of \( T_{gh} \) and \( \tilde{W}_{gh} \). Due to the fact that \( T \) and \( W \) as defined in (8.12) do not form a closed \( W_3 \) algebra (even though \( T_{mat} \) and \( W_{mat} \), and \( T_\Delta \) and \( W_\Delta \) separately do), we can not use the scheme of Fradkin and Fradkina to compute these ghost fields. Instead we have to make use of the transformation rules for \( \bar{\mu}, \bar{\nu} \) (3.23) that leave the covariant action invariant. Differentiating these rules w.r.t. \( \bar{\mu}, \bar{\nu} \) gives us the ghost currents

\[
T_{gh} = \frac{\delta}{\delta \bar{\mu}}(\bar{b}_1 \delta_{\bar{e}_1, \bar{e}_2} \bar{\mu}) + \frac{\delta}{\delta \bar{\mu}}(\bar{b}_2 \delta_{\bar{e}_1, \bar{e}_2} \bar{\nu}),
\]

\[
\tilde{W}_{gh} = \kappa \frac{\delta}{\delta \bar{\nu}}(\bar{b}_1 \delta_{\bar{e}_1, \bar{e}_2} \bar{\mu}) + \frac{\delta}{\delta \bar{\nu}}(\bar{b}_2 \delta_{\bar{e}_1, \bar{e}_2} \bar{\nu}),
\]

(3.21)
where $\delta c_1, c_2$ are the transformations (3.8) with $\tilde{c}$ replaced by $\bar{c}_1$ and $\bar{\lambda}$ by $\bar{c}_2$, and we inserted a factor
\[
\kappa = -\frac{1}{\alpha^2 \beta_{\text{mat}}} = \frac{1}{5} (\beta_{\text{mat}}^{-1} - \beta_{\Delta}^{-1}),
\] (3.22)
to account for the fact that we divided the spin-three generator by a factor $\alpha \sqrt{-\beta_{\text{mat}}}$. This yields the following expressions for $T_{gh}$ and $\tilde{W}_{gh}$
\[
T_{gh} = -\bar{\partial} b_1 \bar{c}_1 - 2 b_1 \bar{\partial} \bar{c}_1 - 2 \bar{\partial} b_2 \bar{c}_2 - 3 b_2 \bar{\partial} \bar{c}_2,
\]
\[
\tilde{W}_{gh} = -\bar{\partial} b_2 \bar{c}_1 - 3 \bar{\partial} b_2 \bar{c}_1 + 3 \kappa^{-1} \bar{b}_2 \bar{c}_2 (T_{ff} - T_{mat}) + 3 b_1 \bar{\partial} \bar{c}_2 (W_{ff} - W_{mat})
+ \frac{1}{3} b_1 \bar{c}_2 (3 \partial T_{mat} - \partial T_{ff}) + \frac{1}{2} \bar{\partial} b_1 \bar{c}_2 (3 T_{ff} - T_{mat}) + 2 b_1 \bar{\partial} \bar{c}_2 (T_{mat} - T_{ff})
+ \kappa \left( \frac{5}{3} b_1 \bar{c}_2 + \frac{2}{3} \bar{\partial} b_1 \bar{c}_2 + \frac{2}{3} \bar{\partial} b_1 \bar{\partial} \bar{c}_2 + \frac{1}{3} \partial T_{mat} \right).
\]
(3.23)

The reason for rewriting $\kappa$ as the difference of $1/\beta_{\text{mat}}$ and $1/\beta_{\Delta}$ is that then $\tilde{W}_{gh}$ can be easily generalized to the quantum case. One should simply the classical expression for $\beta = 16/5c$ by its quantum analogue $\beta = 16/(5c + 22)$. The spin three ghost current is different from the one considered in [4, 10]. However, this difference will disappear after multiplying the ghost current with $\bar{c}_2$, so it gives the same contribution to the BRST charge. Having determined all the ingredients that go into the BRST charge, one can check the following properties of $Q$ using the Poisson brackets of the currents and ghosts
\[
Q_{ds}^2 = Q_{ds} Q_{W_3} + Q_{W_3} Q_{ds} = Q_{W_3}^2 = 0,
\] (3.24)
provided that (classically) $c_{\text{mat}} + c_{ff} = 0$, or equivalently $d = 24k$.

To extend these results to the quantum level we should again normal-order the BRST charge and redefine $T_{ff}$ and $W_{ff}$ to
\[
T_{FF} = N_T \left( \frac{1}{2} \text{Tr}(\hat{J}_{fix}^2) + x \text{Tr}(P \hat{J}_{fix})' \right),
\]
\[
W_{FF} = N_W \left( \frac{1}{3} \text{Tr}(\hat{J}_{fix}^3) + \frac{x}{2} \text{Tr}(P \hat{J}_{fix}^2)' + \frac{x^2}{2} \text{Tr}(P^2 \hat{J}_{fix}'')
+ \frac{x}{4} \left[ \text{Tr}(P \hat{J}_{fix}) \text{Tr}(P^2 \hat{J}_{fix})' - \text{Tr}(P^2 \hat{J}_{fix}) \text{Tr}(P \hat{J}_{fix}') \right] \right).
\] (3.25)

where we have to normal order the r.h.s. Note that we scale every derivative with a factor $x$. Using the OPE’s of the currents and ghosts, one verifies that $T_{FF}$ and $W_{FF}$ form a quantum $W_3$ algebra with central charge
\[
c_{FF} = \frac{8k}{k + 3} - 24k - 30 = 50 - 24 \left( (k + 3) + \frac{1}{(k + 3)} \right),
\] (3.26)
if we take
\[ x = k + 2, \quad N_T = \frac{1}{k + 3}, \quad N_W = \sqrt{\frac{3\beta_{FF}}{(k + 3)^3}}, \] (3.27)
where \( \beta_{FF} = 16/(5c_{FF} + 22) \).

The BRST charge thus reads \( Q = Q_{ds} + Q_{FF} \), with
\[ Q_{FF} = \oint : \tilde{c}_1 \left( T_{mat} + T_{FF} + \frac{1}{2} T_{gh} \right) + \tilde{c}_2 \left( \pm i \tilde{W}_{mat} + \tilde{W}_{FF} + \frac{1}{2} \tilde{W}_{gh} \right) : \] (3.28)
where we changed the classical value for \( \beta = 16/5c \) to its quantum value \( 16/(5c + 22) \) in the definition of \( \tilde{W}_{mat} \) and \( \tilde{W}_{gh} \). We find that
\[ Q_{ds}^2 = Q_{ds} Q_{FF} + Q_{FF} Q_{ds} = Q_{FF}^2 = 0 \] (3.29)
still holds provided we have \( c_{mat} + c_{FF} = 100 \), or
\[ d = 50 + 24 \left( (k + 3) + \frac{1}{(k + 3)} \right), \] (3.30)
which is the desired KPZ result for \( W_3 \).

4. The Conformal Gauge

In this section we compute the BRST charge in the conformal gauge \( \mu = \bar{\mu} = \nu = \bar{\nu} = 0 \). In this gauge the covariant action reduces to the Toda action for \( Sl(3, \mathbb{R}) \)
\[ S_{toda} = -\frac{1}{\pi} \int d^2 z \left[ \frac{1}{2} A^{ij} \partial \phi_i \bar{\partial} \phi_j + \sum_i e^{-A^{ij} \phi_j} \right], \] (4.1)
when we substitute the equations of motion for \( \omega_i, \bar{\omega}_i \), which in this gauge read \( \omega_i = -\partial \phi_i \) and \( \bar{\omega}_i = -\bar{\partial} \bar{\phi}_i \). The constraints following from the gauge-fixing of \( \mu, \nu \) are
\[ T \equiv \frac{\pi S_{cov}}{\delta \mu} = T_{mat} + T_{toda}, \]
\[ W \equiv \frac{1}{\sqrt{-\beta_{mat}}} \frac{\pi S_{cov}}{\delta \nu} = \pm i \frac{W_{mat}}{\sqrt{\beta_{mat}}} + \frac{W_{toda}}{\sqrt{\beta_{toda}}}. \] (4.2)
Similar constraints follow of course from the gauge-fixing of \( \bar{\mu}, \bar{\nu} \). We will consider only one chiral sector, since the other sector gives identical results. So the total BRST charge will be
given by $Q_{\text{total}} = Q + \bar{Q}$, where $\bar{Q}$ is the conjugate of $Q$. They trivially anti-commute with each other. Using the equations of motion for the $\omega_i, \bar{\omega}_i$ we find

\begin{align*}
T_{\text{toda}} &= \frac{1}{2} A_{ij} \partial \phi_i \partial \phi_j + \sqrt{k} (\partial^2 \phi_1 + \partial^2 \phi_2), \\
\alpha W_{\text{toda}} &= \frac{1}{\sqrt{k}} \partial \phi_1 (\partial \phi_2)^2 + \frac{x}{2} \partial \phi_1 \partial^2 \phi_2 - \partial \phi_1 \partial^2 \phi_1 - \frac{x}{2} \sqrt{k} \partial^3 \phi_1 - (1 \leftrightarrow 2), \tag{4.3}
\end{align*}

where we rescaled the fields $\phi_i \rightarrow \phi_i / \sqrt{k}$. The Poisson brackets for the $\phi_i$ following from (4.1) are

\begin{align*}
\{ \partial \phi_i(z), \partial \phi_j(w) \} &= A_{ij}^{-1} \delta'(z - w). \tag{4.4}
\end{align*}

Using these Poisson brackets one easily verifies that $\{T_{\text{toda}}, W_{\text{toda}}\}$ form a classical $W_3$ algebra with central charge $c_{\text{toda}} = -24k$. The BRST charge is given by

\begin{align*}
Q_{\text{toda}} &= \oint \bar{c}_1 \left( T_{\text{mat}} + T_{\text{toda}} + \frac{1}{2} T_{\text{gh}} \right) + \bar{c}_2 \left( \pm i \bar{W}_{\text{mat}} + \bar{W}_{\text{toda}} + \frac{1}{2} \bar{W}_{\text{gh}} \right), \tag{4.5}
\end{align*}

where the ghost currents are still given by (3.23), with $T_{ff}, W_{ff}$ replaced by $T_{\text{toda}}, W_{\text{toda}}$. Nilpotency of this classical charge holds due to $d = 24k$. This BRST charge is precisely the same as the one recently constructed in [2] (see also [11]).

The above results are generalized to the quantum level following the approach stated in the introduction. We should normal-order the BRST charge and allow for multiplicative renormalizations of $T_{\text{toda}}$ and $W_{\text{toda}}$. So we take

\begin{align*}
T_{\text{toda}} &= N_T \left( \frac{1}{2} A_{ij} \partial \phi_i \partial \phi_j + x (\partial^2 \phi_1 + \partial^2 \phi_2) \right), \\
W_{\text{toda}} &= N_W \left( \partial \phi_1 ((\partial \phi_2)^2 + \frac{x}{2} \partial^2 \phi_2 - x \partial^2 \phi_1) - \frac{x^2}{2} \partial^3 \phi_1 - (1 \leftrightarrow 2) \right), \tag{4.6}
\end{align*}

where the r.h.s. should be normal ordered. These Toda fields form a quantum $W_3$ algebra of central charge

\begin{align*}
c_{\text{toda}} &= 2 - 24 \frac{(k + 2)^2}{k + 3}, \tag{4.7}
\end{align*}

provided we take

\begin{align*}
x &= \frac{k + 2}{\sqrt{k + 3}}, \quad N_T = 1, \quad N_W = \sqrt{3 \beta_{\text{toda}}}, \quad \beta_{\text{toda}} = 16 / (5c_{\text{toda}} + 22). \quad (4.8)
\end{align*}

where $\beta_{\text{toda}} = 16/(5c_{\text{toda}} + 22)$. The quantum BRST charge

\begin{align*}
Q_{\text{toda}} &= \oint : \bar{c}_1 \left( T_{\text{mat}} + T_{\text{toda}} + \frac{1}{2} T_{\text{gh}} \right) + \bar{c}_2 \left( \pm i \bar{W}_{\text{mat}} + \bar{W}_{\text{toda}} + \frac{1}{2} \bar{W}_{\text{gh}} \right): \tag{4.9}
\end{align*}
is nilpotent iff the KPZ result for $W_3$ holds, \textit{i.e.}

\[ d + c_{Toda} = 100. \]  

(4.10)

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Appendix A

The WZW actions $S_{wzw}^\pm$ are given by:

$$S_{wzw}^\pm(g) = \frac{1}{4\pi} \int_\Sigma d^2z \Tr g^{-1} \partial gg^{-1} \partial g \pm \frac{1}{12\pi} \int_B Tr g^{-1} \partial^3 g.$$  \hspace{1cm} (A.1)

They satisfy the following Polyakov–Wiegman identities [25]:

$$S_{wzw}^+(gh) = S_{wzw}^+(g) + S_{wzw}^+(h) + \frac{1}{2\pi} \int d^2z \Tr g^{-1} \partial g \partial h h^{-1},$$

$$S_{wzw}^-(gh) = S_{wzw}^-(g) + S_{wzw}^-(h) + \frac{1}{2\pi} \int d^2z \Tr g^{-1} \partial g \bar{\partial} h h^{-1}.$$ \hspace{1cm} (A.2)

Appendix B

The covariant is constructed such that it is invariant under $\bar{W}, \bar{W}$ transformations. In this Appendix we give the $\bar{W}_3$ transformations that leave the covariant action describing $W_3$ gravity invariant [6, 7]. One should realize that there are similar transformations of opposite chirality that also leave the covariant action invariant. Let $\text{ad}_\Lambda$ denote the adjoint action of $\Lambda$ on $sl_3(\mathbb{R})$, i.e. $\text{ad}_\Lambda(Y) = [\Lambda, Y]$. Its kernel $K$ is spanned by $\{\Lambda, \Lambda^2\}$, and $sl_3(\mathbb{R})$ can be decomposed as $K \oplus \text{Im} (\text{ad}_\Lambda)$. Let $L$ be the linear operator from $sl_3(\mathbb{R})$ to $sl_3(\mathbb{R})$, defined as the inverse of $\text{ad}_\Lambda$ on $\text{Im}(\text{ad}_\Lambda)$ and extended to the zero map on the rest of $sl_3(\mathbb{R})$. Define the ‘field-matrices’ $F_\Delta = T_\Delta \Lambda + W_\Delta \Lambda^2$ and $F_{\text{mat}} = T_{\text{mat}} \Lambda + W_{\text{mat}} \Lambda^2$, with $\Lambda$ the transpose of $\Lambda$, and

$$T_{\text{mat}} = \pi \frac{\delta \Gamma[\mu, \nu]}{\delta \mu}, \hspace{1cm} T_\Delta = \pi \frac{\delta \Delta S}{\delta \mu},$$

$$W_{\text{mat}} = \pi \frac{\delta \Gamma[\mu, \nu]}{\alpha \delta \nu}, \hspace{1cm} W_\Delta = \frac{\pi}{\alpha} \frac{\delta \Delta S}{\delta \nu}. \hspace{1cm} (B.1)$$

Let $M(\bar{\mu}, \bar{\nu})$ be the matrix which has zero-curvature together with $F_{\text{mat}}$. In [3, 7] it was shown that $M(\bar{\mu}, \bar{\nu})$ can be expressed in the following way:

$$M(\bar{\mu}, \bar{\nu}) = \frac{1}{1 + L(\bar{\partial} + k^{-1} \text{ad}_{F_{\text{mat}}})} M_0(\bar{\mu}, \bar{\nu}), \hspace{1cm} (B.2)$$

where $M_0(\bar{\mu}, \bar{\nu}) = \bar{\mu} \Lambda + \bar{\nu} \Lambda^2$, and $k = d/24$. (Note that $M_0 \in K$.) In fact the curvature of $M(\bar{\mu}, \bar{\nu})$ and $F_{\text{mat}}$ is not identically zero, but of the form

$$\begin{pmatrix}
0 & \gamma_1 + \frac{1}{2} \bar{\partial} \gamma_2 & \gamma_2 \\
0 & 0 & \gamma_1 - \frac{1}{2} \bar{\partial} \gamma_2 \\
0 & 0 & 0
\end{pmatrix}.$$ \hspace{1cm} (B.3)
where the $\gamma$’s are the (chiral) Ward identities for $W_3$, i.e.
\[
\begin{align*}
\gamma_1 &= \partial T_{\text{mat}} - D_1 \bar{\mu} - [3 W_{\text{mat}} \bar{\phi} + 2 (\bar{\partial} W_{\text{mat}})] \bar{\nu}, \\
\gamma_2 &= \partial W_{\text{mat}} - [3 W_{\text{mat}} \bar{\phi} + (\bar{\partial} W_{\text{mat}})] \bar{\mu} - D_2 \bar{\nu},
\end{align*}
\]
with $D_1$ and $D_2$ the third and fifth order Gelfand–Dickey operators
\[
\begin{align*}
D_1 &= -\frac{d}{12} \bar{\partial}^3 + 2 T_{\text{mat}} \partial + \bar{\partial} T_{\text{mat}}, \\
D_2 &= -\frac{d}{360} \bar{\partial}^5 + \frac{1}{3} T_{\text{mat}} \bar{\partial}^3 + \frac{1}{2} \bar{\partial} T_{\text{mat}} \bar{\partial}^2 + \frac{3}{10} \bar{\partial}^2 T_{\text{mat}} \bar{\partial} + \frac{1}{15} \bar{\partial}^3 T_{\text{mat}} - \beta_{\text{mat}} (T^2_{\text{mat}} \bar{\partial} + T_{\text{mat}} \bar{\partial} T_{\text{mat}}).
\end{align*}
\]
Next define $X(\bar{\epsilon}, \bar{\lambda})$ to be
\[
X(\bar{\epsilon}, \bar{\lambda}) = \frac{1}{1 + L(\partial + k^{-1} \text{ad}_{\Delta})} X_0(\bar{\epsilon}, \bar{\lambda}),
\]
with $X_0(\bar{\epsilon}, \bar{\lambda}) = \bar{\epsilon} \Lambda + \bar{\lambda} \Lambda^2$. Recall that the fields $\omega_i, \bar{\omega}_i, \phi_i$ appearing in $\Delta S$ originated from one $Sl(3, \mathbb{R})$ valued field $G$
\[
G = \begin{pmatrix} 1 & 0 & 0 \\ \omega_1 & 1 & 0 \\ \omega_3 & \omega_2 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{\phi_2 - \phi_1} & 0 \\ 0 & 0 & e^{-\phi_2} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\omega}_1 & -\bar{\omega}_3 \\ 0 & 1 & -\bar{\omega}_2 \\ 0 & 0 & 1 \end{pmatrix}.
\] (B.7)

The transformation of the $\omega_i, \bar{\omega}_i, \phi_i$ and $\bar{\mu}, \bar{\nu}$ can now be extracted from the following formula’s
\[
\begin{align*}
\delta_{\bar{\epsilon}, \bar{\lambda}} G &= -G X(\bar{\epsilon}, \bar{\lambda}), \\
\delta_{\bar{\epsilon}, \bar{\lambda}} M(\bar{\mu}, \bar{\nu}) &= \partial X(\bar{\epsilon}, \bar{\lambda}) + [M(\bar{\mu}, \bar{\nu}), X(\bar{\epsilon}, \bar{\lambda})].
\end{align*}
\] (B.8)

For $\bar{\mu}, \bar{\nu}$ these transformation rules become:
\[
\begin{align*}
\delta_{\bar{\epsilon}, \bar{\lambda}} \bar{\mu} &= \partial \bar{\epsilon} - \bar{\mu} \bar{\partial} \bar{\epsilon} + \bar{\epsilon} \bar{\partial} \bar{\mu} - \frac{1}{6} \bar{\lambda} \partial^3 \bar{\nu} + \frac{1}{4} \bar{\partial} \bar{\lambda} \partial^2 \bar{\nu} - \frac{1}{4} \partial^2 \bar{\lambda} \partial \bar{\nu} + \frac{1}{6} \partial^3 \bar{\lambda} \bar{\nu} + \frac{1}{3} \bar{\lambda} \bar{\nu} (\bar{\partial} T_{\Delta} - \bar{\partial} T_{\text{mat}}) + \frac{1}{k} \bar{\lambda} \bar{\nu} (W_{\Delta} - W_{\text{mat}}), \\
\delta_{\bar{\epsilon}, \bar{\lambda}} \bar{\nu} &= \partial \bar{\lambda} + \bar{\nu} \partial \bar{\epsilon} - 2 \bar{\nu} \partial \bar{\epsilon} + 2 \lambda \bar{\partial} \bar{\mu} - \bar{\mu} \partial \bar{\lambda} + \frac{1}{k} \bar{\lambda} \bar{\nu} (T_{\Delta} - T_{\text{mat}}).
\end{align*}
\] (B.9)
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