Sampling basis in reproducing kernel Banach spaces

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Abstract

We present necessary and sufficient conditions to hold true a Kramer type sampling theorem over semi-inner product reproducing kernel Banach spaces. Under some sampling-type hypotheses over a sequence of functions on these Banach spaces it results necessary that such sequence must be a \( X_2 \)-Riesz basis and a sampling basis for the space. These results are a generalization of some already known sampling theorems over reproducing kernel Hilbert spaces.

Keywords: Sampling basis, Non-uniform sampling, Reproducing kernel Hilbert spaces, Reproducing kernel Banach spaces, frames, Riesz basis, Kramer sampling theorems, semi-inner product.

1 Introduction

The celebrated sampling theorem of Whittaker-Shannon-Kotel’nikov (1933) [3, 15] establishes that all finite energy function \( f \in L^2(\mathbb{R}) \) band-limited to \([−\sigma, \sigma]\), i.e., the Fourier transform of \( f \) is supported on the interval \([−\sigma, \sigma]\), can be completely recovered through samples in the integers \( \{f(n)\}_{n \in \mathbb{Z}} \), obtaining in this way the following representation

\[
f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (2\sigma t - n)}{\pi (2\sigma t - n)} \quad t \in \mathbb{R}
\]

with the series being absolutely and uniformly convergent on compact subsets of \( \mathbb{R} \). By writing it a bit different, we note that the band-limited functions can be given by

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(x)e^{-it\omega} d\omega \quad t \in \mathbb{R},
\]
being \( \{e^{in(\cdot)}\}_{n \in \mathbb{Z}} \) an orthonormal basis of \( L^2[-\sigma, \sigma] \). By noting this, later in the 1959, Kramer [3, 9] extended this result to functions defined by another integral operator \( TF = f \), now with kernel \( \kappa \) instead of the exponentials:

\[
f(t) = \int_I F(x)\kappa(t, x)dx \quad t \in \mathbb{R}
\]

where \( I \) is a compact interval of \( \mathbb{R} \) and \( \kappa(t, \cdot) \in L^2(I) \ \forall t \in \mathbb{R} \). The existence of a sequence \( \{t_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) such that \( \kappa(t_n, \cdot) \) is an orthogonal and complete sequence in \( L^2(I) \) was the hypothesis used by Kramer for this result to hold. Thanks to this he obtained the sampling expansion for such functions:

\[
f(t) = \sum_{n=-\infty}^{\infty} f(t_n)S_n(t) \quad \text{with} \quad S_n(t) = \frac{\int_I \kappa(t_n, x)\kappa(t, x)dx}{\int_I |\kappa(t_n, x)|^2dx} \quad t \in \mathbb{R}
\]

as before, the series is absolutely convergent. This result allow us to work in non uniform sampling problems in contrast to the Whittaker-Shannon-Kotel’nikov sampling theorem. Both of the integral operators could be written by using the usual inner product of \( L^2(I) \) and then we obtain a possible direction to where it can be generalized this Kramer sampling theorem.

Thanks to the theory of reproducing kernel Hilbert spaces (written RKHS for short) by Aronszajn [1] in the 1950 and its particular case of functions which are image by an integral operator (Saitoh 1988, [12]), the previous sampling results can be naturally viewed inside this framework. Thanks to a new generalization (again by Saitoh), it can be considered like particular cases of the so-called Abstract Kramer sampling theorem (García, Hernández-Medina & Muñoz-Bouzo, 2014 [6]), where the functions now have the form:

\[
f(t) = \langle x, \Phi(t) \rangle \quad t \in \Omega,
\]

where \( \Omega \) is an arbitrary set, \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) is a Hilbert space and \( \Phi : \Omega \to \mathcal{H} \) is an arbitrary function. Under the hypotheses of the existence of sequences \( \{t_n\}_{n \in \mathbb{N}} \subset \Omega, \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\} \) and \( \{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) a Riesz basis such that the sequence \( \{\Phi(t_n)\}_{n \in \mathbb{N}} \) satisfies the interpolation condition \( \Phi(t_n) = \overline{a_n}x_n \ \forall n \in \mathbb{N} \), they were able to prove that

\[
f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{S_n(t)}{a_n} \quad \text{with} \quad S_n(t) = \langle y_n, \Phi(t) \rangle \quad t \in \Omega
\]

where \( \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) is the biorthogonal Riesz basis of \( \{x_n\}_{n \in \mathbb{N}} \) and the series is convergent in the RKHS-norm that contains such functions, also, the convergence is absolute and uniform on subsets of \( \Omega \) where the map \( t \mapsto \|\Phi(t)\| \) is bounded.

Due to the recent theory of reproducing kernel Banach spaces (written RKBS for short) developed by Zhang, Xu & Zhang [16] and the subsequent theory of \( X_d \)-Bessel sequences, \( X_d \)-frames and \( X_d \)-Riesz basis by Zhang & Zhang [17], García & Portal (2013, [5]) were able to extend the last result (stated in Section 3) to the Banach spaces setting. By using these recent concepts we state and prove a generalization of the following possible “converse” of the Kramer sampling theorem:

**Theorem 1.1 (A converse of the Kramer sampling theorem [4]).** Let \( \mathcal{H} \) be the range of the integral linear transform \( T : L^2(I) \ni F \to f \in \mathcal{H} \) considered as a RKHS with the kernel \( k \) defined by \( k(t, s) := \langle K(\cdot, t), K(\cdot, s) \rangle_{L^2(I)} \). Let \( \{S_n\}_{n=0}^{\infty} \) be a sequence in \( \mathcal{H} \) such that \( \sum_{n=0}^{\infty} |S_n(t)|^2 < +\infty, t \in \Omega \) and let \( \mathcal{H}_{\text{samp}} \) be a RKHS corresponding to the kernel \( K_{\text{samp}}(s, t) := \sum_{n=0}^{\infty} S_n(s)S_n(t) \). Then, we have the following results:
1°) Suppose that the sequence \( \{S_n\}_{n=0}^{\infty} \) satisfies the condition that for each sequence \( \{\alpha_n\}_{n=0}^{\infty} \in l_2(\mathbb{N}_0) \) such that \( \sum_{n=0}^{\infty} \alpha_n S_n(t) = 0 \) implies \( \alpha_n = 0 \) for all \( n \). Then, \( H_{samp} \subset H \) and \( \{S_n\}_{n=0}^{\infty} \) is an orthonormal basis in \( H_{samp} \).

2°) Suppose in addition to 1°) the existence of sequences \( \{t_n\}_{n=0}^{\infty} \) in \( \Omega \) and \( \{a_n\}_{n=0}^{\infty} \) in \( \mathbb{C} \setminus \{0\} \) such that

\[
\left\{ \frac{f(t_n)}{a_n} \right\}_{n\in\mathbb{N}_0} \in l_2(\mathbb{N}_0) \quad \text{and} \quad f(t) = \sum_{n=0}^{\infty} f(t_n) \frac{S_n(t)}{a_n} \quad \text{for any} \quad f \in H
\]

where the sampling series is pointwise convergent in \( \Omega \). Then

- \( H_{samp} = H \).
- The norms of \( H_{samp} \) and \( H \) are equivalent, i.e., for some constants \( 0 < a \leq b \)

\[ a\|f\|_{samp} \leq \|f\|_{H} \leq b\|f\|_{samp} \]

Consequently \( \{S_n\}_{n=0}^{\infty} \) is a Riesz basis for \( H \).

- The sequences \( \{a_i^{-1}K(\cdot, t_i)\}_{i=1}^{\infty} \) and \( \{\sum_{n=0}^{\infty} \langle S_j, S_n \rangle \eta K(\cdot, t_n)\}_{j=0}^{\infty} \) as well as the sequences \( \{S_i\}_{i=0}^{\infty} \) and \( \{\sum_{n=0}^{\infty} k_i(t_n) a_i^{-1} S_n\}_{j=0}^{\infty} \) are biorthonormal in \( L^2(I) \) and \( H \) respectively.

- If \( a = b \) then \( a^2 k(s, t) = k_{samp}(s, t) \) for all \( s, t \in \Omega \) and the sequence \( \{S_n\}_{n=0}^{\infty} \) is a complete and orthogonal set in \( L^2(I) \).

Recently, in [8] is obtained another possible converse with different choices of hypotheses. In the next section we give the preliminaries needed for the extension of this theorem to the Banach space setting. We only list the results and invite to the reader to see [2, 7, 10, 16, 17] for much more details.

## 2 Definitions and basic results

### 2.1 The normalized duality mapping and semi-inner products

Let \((E, \| \cdot \|)\) be a normed space over \( \mathbb{C} \) and \((E^*, \| \cdot \|_*)\) its corresponding dual space formed by the \| \cdot \| \text{-continuous} \mathbb{C}\text{-linear functional}. We have defined the bilinear form \((\cdot, \cdot)_E : E \times E^* \to \mathbb{C}\) given by \((f, f^*)_E = f^*(f), \ f \in X \text{ and } f^* \in E^*\). The mapping \(J : E \to 2^{E^*}\) given by

\[
J(f) = \{f^* \in E^* : f^*(f) = \|f\| \|f^*\|_*, \ |f| = \|f^*\|_*\} \quad f \in E
\]

will be called the normalized duality mapping of the normed space \(E\) or shortly the dual map of \(E\). For our purposes, here and henceforth \(E\) will be a uniform Banach space, i.e., uniformly Fréchet differentiable and uniformly convex space [11]. In this way, given \(f \in E\) there exists a unique \(f^* \in E^*\) such that \(J(f) = \{f^*\}\) and so we have an isometric bijection \(f \mapsto f^*\) between \(E\) and \(E^*\). For the proofs of these statements and more about the dual map see for example [2] and the references therein. We introduce the semi-inner products (s.i.p. for short), these share almost all properties of the inner products.

**Definition 2.1.** Let \( \mathcal{V} \) be a \( \mathbb{C}\)-vector space, a map \([\cdot, \cdot] : \mathcal{V} \times \mathcal{V} \to \mathbb{C}\) is called a semi-inner product (in Lumer’s sense [10]) if \( \forall \alpha \in \mathbb{C} \text{ and } \forall x, y, z \in \mathcal{V} \) satisfies:

- \([\alpha x + y, z] = \alpha[x, z] + [y, z]\).
- \([x, \alpha y] = \overline{\alpha}[x, y]\).
- \([x, x] > 0\) if \(x \neq 0\).
- \(\|x, y\|^2 \leq [x, x][y, y]\).
2.2 Bessel sequences, Frames and Riesz bases via s.i.p.

The following are included in [17]. A BK-space $X_d$ on a countable well-ordered index set $\mathbb{I}$ is a Banach space of sequences indexed by $\mathbb{I}$ where the canonical vector forms a Schauder basis. We impose the following additional conditions over $X_d$: it is a reflexive space, which guarantees its dual $X_d^*$ is also a BK-space and the duality between them is given by $(c,d)_{X_d} = \sum_{j \in \mathbb{I}} c_j d_j, \forall c = \{c_j\}_{j \in \mathbb{I}} \in X_d, d = \{d_j\}_{j \in \mathbb{I}} \in X_d^*$; if the series $\sum_{j \in \mathbb{I}} c_j d_j$ converges in $C$ for all $c \in X_d$ then $d \in X_d^*$ and vice versa; finally the series $\sum_{j \in \mathbb{I}} c_j d_j$ converges absolutely in $C$ for all sequences $c \in X_d, d \in X_d^*$. For another types of sequence spaces we refer to [13, 14].

Given a sequence $\{f_j\}_{j \in \mathbb{I}}$ in $E$ we note by $\{f_j^*\}_{j \in \mathbb{I}}$ its dual sequence in $E^*$. A sequence $\{f_j\}_{j \in \mathbb{I}}$ in $E$ is called minimal, if $f_k \notin \text{span}\{f_j : k \neq j\} \forall k \in \mathbb{I}$ and is called complete, if $\text{span}\{f_j : j \in \mathbb{I}\} = E$. We have the following characterizations:

Proposition 2.2. Let $\{f_j\}_{j \in \mathbb{I}}$ be a sequence in $E$, then:

a) $\{f_j\}_{j \in \mathbb{I}}$ is minimal if and only if $\exists \{g_j\}_{j \in \mathbb{I}}$ in $E$ such that $[f_j, g_k] = \delta_{j,k} \forall j,k \in \mathbb{I}$.

b) $\{f_j\}_{j \in \mathbb{I}}$ is complete if and only if $f \in E$ is such that $[f_j, f] = 0 \forall j \in \mathbb{I}$ then $f = 0$.

Where $\delta_{j,k}$ denotes the Kronecker’s delta. The sequence $\{g_j\}_{j \in \mathbb{I}}$ in a) is called a biorthogonal sequence of $\{f_j\}_{j \in \mathbb{I}}$ and when $\{f_j\}_{j \in \mathbb{I}}$ is also a complete sequence in $E$, then $\{g_j\}_{j \in \mathbb{I}}$ is unique.

We give first the definition of $X_d$-Riesz-Fischer sequences and then introduce $X_d$-Bessel sequences, $X_d$-frames and $X_d$-Riesz basis at the same time as its characterizations, these will be used in the main result in Section 4. See [17, Proposition 2.3 – 2.13].

Definition 2.3 (X_d-Riesz-Fischer sequences). $\{f_j\}_{j \in \mathbb{I}} \subset E$ is a $X_d$-Riesz-Fischer sequence for $E$ if

$$\forall c = \{c_j\}_{j \in \mathbb{I}} \in X_d, \exists f \in E \text{ such that } [f, f_j] = c_j \forall j \in \mathbb{I}. \quad (1)$$

Proposition 2.4 (X_d-Bessel sequences). Let $\{f_j\}_{j \in \mathbb{I}}$ be a sequence in $E$, are equivalent:

i) ($X_d$-Bessel definition) There exists a constant $B > 0$ such that

$$\|\{f, f_j\}_{j \in \mathbb{I}}\|_{X_d} \leq B\|f\|_E \quad \forall f \in E \quad (2)$$

ii) $U : E \to X_d$ given by $Uf = \{[f, f_j]\}_{j \in \mathbb{I}} f \in E$ is a well-defined bounded operator.

iii) $U^* : X_d^* \to E^*$ given by

$$U^* d = \sum_{j \in \mathbb{I}} d_j f_j^* \quad \forall d = \{d_j\}_{j \in \mathbb{I}} \in X_d^* \quad (3)$$

is a bounded operator and the series $\sum_{j \in \mathbb{I}} d_j f_j^*$ converges unconditionally in $E^*$.

Proposition 2.5 (X_d-frames). Let $\{f_j\}_{j \in \mathbb{I}}$ be a sequence in $E$, are equivalent:

i) ($X_d$-frame definition) There exists constants $B \geq A > 0$ such that

$$A\|f\|_E \leq \|\{f, f_j\}_{j \in \mathbb{I}}\|_{X_d} \leq B\|f\|_E \quad \forall f \in E \quad (4)$$

ii) $U : E \to X_d$ is bounded and bounded below.

iii) $U^* : X_d^* \to E^*$ is bounded and surjective.
It is clear that an $X_d$-frame for $E$ is an $X_d$-Bessel sequence for $E$.

**Proposition 2.6** ($X_d$-Riesz basis). Let $\{f_j\}_{j \in \mathbb{I}}$ be a sequence in $E$, are equivalent:

i) ($X_d$-Riesz basis definition) $\{f_j\}_{j \in \mathbb{I}}$ is complete and $\exists B \geq A > 0$ such that

$$A\|c\|_{X_d} \leq \left\| \sum_{j \in \mathbb{I}} c_j f_j \right\|_E \leq B\|c\|_{X_d} \quad \forall c = \{c_j\}_{j \in \mathbb{I}} \in X_d,$$

(5)

ii) $\{f_j^*\}_{j \in \mathbb{I}}$ is an $X_d^*$-frame for $E^*$ and $\{f_j\}_{j \in \mathbb{I}}$ is a minimal sequence in $E$.

iii) $\{f_j\}_{j \in \mathbb{I}}$ is complete and $V : E^* \to X_d^*$ is bounded and surjective.

iv) $\{f_j\}_{j \in \mathbb{I}}$ is complete and $V^* : X_d \to E$ is bounded and bounded below.

### 2.3 Reproducing kernel Banach spaces

A reproducing kernel Hilbert space on a set $\Omega$ is a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ of $\mathbb{C}$-valued functions on $\Omega$ and the point evaluations in $t \in \Omega$ are continuous linear functionals on $\mathcal{H}$. The second condition is equivalent to the existence of a function $K : \Omega \times \Omega \to \mathbb{C}$ such that $K(t, \cdot) \in \mathcal{H}$ for each $t \in \Omega$, and for each $f \in \mathcal{H}$ there holds the reproducing property:

$$f(t) = \langle f, K(t, \cdot) \rangle \quad t \in \Omega.$$  

(6)

where the choice of the first variable of $K$ is simply by convenience (the second one is usually used). $K$ is unique and is called the reproducing kernel for $\mathcal{H}$. For our main purpose of doing sampling theory, we adopt the next definition of reproducing kernel Banach space [16] to extend these Hilbert spaces to the Banach space setting.

**Definition 2.7.** A reproducing kernel Banach space on a set $\Omega$ is a reflexive Banach space $(\mathcal{B}, \|\cdot\|)$ of $\mathbb{C}$-valued functions on $\Omega$ for which $\mathcal{B}^*$ is isometrically isomorphic to a Banach space $\mathcal{B}^\#$ of $\mathbb{C}$-valued functions on $\Omega$ and the point evaluations in $t \in \Omega$ are continuous linear functionals on both $\mathcal{B}$ and $\mathcal{B}^\#$.

Since we want to use the results of the previous section, we are going to work with a special class of reproducing kernel Banach spaces. We call a uniform reproducing kernel Banach space by a semi-inner product reproducing kernel Banach space (s.i.p. RKBS for short). While it is true that a RKBS possesses some sort of function that resembles to the reproducing kernel for a RKHS, in a s.i.p. RKBS we have a function with those same attributes of the reproducing kernel for a RKHS.

**Proposition 2.8.** Let $\mathcal{B}$ be a RKBS on $\Omega$, then there exists a unique function (reproducing kernel) $K : \Omega \times \Omega \to \mathbb{C}$ such that:

1. For all $t \in \Omega$, $K(\cdot, t) \in \mathcal{B}^*$ and $f(t) = \langle f, K(\cdot, t) \rangle_\mathcal{B}$ for all $f \in \mathcal{B}$.
2. For all $t \in \Omega$, $K(t, \cdot) \in \mathcal{B}$ and $f^*(t) = \langle K(t, \cdot), f^* \rangle_\mathcal{B}$ for all $f^* \in \mathcal{B}^*$.
3. $\mathcal{B}^* = \overline{\text{span}} \{K(\cdot, t) : t \in \Omega\}$ and $\mathcal{B} = \overline{\text{span}} \{K(t, \cdot) : t \in \Omega\}$.
4. $K(s, t) = \langle K(s, \cdot), K(t, \cdot) \rangle_\mathcal{B}$ for all $s, t \in \Omega$.

Moreover, if $\mathcal{B}$ is also a s.i.p. RKBS on $\Omega$, then there exists another unique function (s.i.p. kernel) $G : \Omega \times \Omega \to \mathbb{C}$ such that:

5. $G(t, \cdot) \in \mathcal{B}$ and $K(\cdot, t) = (G(t, \cdot))^* \in \mathcal{B}^*$ for all $t \in \Omega$.
6. $f(t) = [f, G(t, \cdot)]$ and $f^*(t) = [K(t, \cdot), f]$ for all $f \in \mathcal{B}$, $t \in \Omega$.

When $K = G$, we call it the s.i.p. reproducing kernel for $\mathcal{B}$.

An important result in a RKHS is that norm convergence implies pointwise convergence, the same is true in a RKBS (therefore in a s.i.p. RKBS). An another one is about how it can be constructed a s.i.p. RKBS by using an isometric operator. The following construction appears in [5, 16].
Remark 2.9 (s.i.p. RKBS construction by using an operator). Let \((E, [\cdot, \cdot]_E)\) be a uniform Banach space; let \(\Phi : \Omega \rightarrow E\) be a function and let \(T_\Phi : E \rightarrow \mathbb{C}^\Omega\) be an operator defined by \(T_\Phi x = f_x\) with \(f_x(t) = [x, \Phi(t)]_E\), \(t \in \Omega\). It follows that \(T_\Phi\) is linear, and it is injective if we suppose further that \(\{\Phi(t) : t \in \Omega\}\) is a complete set in \(E\). Let \(\mathcal{B} = \mathcal{R}(T_\Phi)\) be the range of \(T_\Phi\) and define the \(\mathcal{B}\)-norm by \(\|f_x\|_\mathcal{B} := \|[x]_E\) this turns \(T_\Phi\) into an isometric isomorphism between \(E\) and \(\mathcal{B}\), therefore \(\mathcal{B}\) is a uniform Banach space of \(\mathbb{C}\)-valued functions on \(\Omega\). Moreover, \([\cdot, \cdot]_\mathcal{B}\) defined by \([f_x, f_y]_\mathcal{B} := [x, y]_E\), \(x, y \in E\) is the unique (norm compatible) s.i.p. on \(\mathcal{B}\). For each \(t \in \Omega\) the point evaluations over \(\mathcal{B}\) are continuous but, for being continuous over \(\mathcal{B}^*\) we need some extra hypotheses. We consider the function \(\Phi^* : \Omega \rightarrow E^*\) given by \(\Phi^*(t) = (\Phi(t))^*, t \in \Omega\), and impose that \(\overline{\text{span}}\{\Phi^*(t) : t \in \Omega\} = E^*\). In this way (see [16, Theo. 10]) \(\mathcal{B}^* = \{f^*_x := [\Phi(t), x]_E : x \in E\}\) endowed with \([f^*_x, f^*_y]_{\mathcal{B}^*} := [f_x, f_y]_\mathcal{B}\), \(x, y \in E\) is the dual of \(\mathcal{B}\) with the bilinear form \([f_x, f^*_y]_{\mathcal{B}^*} := (x, y^*)_E\) \(x, y \in E\) which s.i.p. reproducing kernel \(G\) for \(\mathcal{B}\) is given by \(G(s, t) = [\Phi(s), \Phi(t)]_E\), \(s, t \in \Omega\).

If it is necessary to distinguish each characteristic component of a s.i.p. RKBS on \(\Omega\) constructed as before, then we write it as \((\mathcal{B}, [\cdot, \cdot]_\mathcal{B}, G, E, \Phi)\).

In the spirit of Zayed’s book [15, Def. 10.1.3] we have the following definition which is our main objective when we talk about reconstruction in sampling theory.

**Definition 2.10 (Sampling Basis).** A basis \(\{S_j\}_{j \in I}\) of a reproducing kernel Banach space \(\mathcal{B}\) on a subset \(\Omega\) is called a **sampling basis** if there exists a sequence \(\{t_j\}_{j \in I} \subset \Omega\) such that

\[
f(t) = \sum_{j \in I} f(t_j) S_j(t) \quad \forall f \in \mathcal{B}, \ t \in \Omega
\]  

(7)

Again, by similarity with the Hilbert space sampling theory, we need to restrict to work in a s.i.p. RKBS.

**Proposition 2.11 (Sampling basis).** Let \(\{S_j\}_{j \in I}\) be a basis of a s.i.p. RKBS \(\mathcal{B}\) on a set \(\Omega\) with s.i.p. reproducing kernel \(G\). Then, \(\{S_j\}_{j \in I}\) is a sampling basis if and only if its biorthogonal basis \(\{F_j\}_{j \in I}\) is given by

\[
F_j(t) = G(t_j, t) := G_{t_j}(t) \quad j \in I, \ t \in \Omega
\]  

(8)

**Proof.** \((\Rightarrow)\) Since \(\{S_j\}_{j \in I}\) is a sampling basis for \(\mathcal{B}\) its (unique) biorthogonal Schauder basis, for each \(f \in \mathcal{B}\), \(\{F_j\}_{j \in I}\) satisfies:

\[
\sum_{j \in I} [f, F_j] S_j(t) = f(t) = \sum_{j \in I} f(t_j) S_j(t) = \sum_{j \in I} [f, G_{t_j}] S_j(t) \quad t \in \Omega
\]

In the first equality is used the Schauder basis property of both sequences, while the second equality follows by the sampling basis hypothesis over \(\{S_j\}_{j \in I}\) and the last one is due to the reproducing property of \(G\). Thus, by uniqueness, it must be \([f, F_j] = [f, G_{t_j}]\) \(\forall j \in I\), whence \(F_j = G_{t_j} \ \forall j \in I\).

\((\Leftarrow)\) If the biorthogonal Schauder basis to \(\{S_j\}_{j \in I}\) is given by \(F_j(t) = G(t_j, t) := G_{t_j}(t)\), then for each \(f \in \mathcal{B}\):

\[
f(t) = \sum_{j \in I} [f, F_j] S_j(t) = \sum_{j \in I} [f, G_{t_j}] S_j(t) = \sum_{j \in I} f(t_j) S_j(t) \quad t \in \Omega
\]

therefore \(\{S_j\}_{j \in I}\) is a sampling basis for \(\mathcal{B}\). \(\square\)

Of course, the definition of sampling basis as well as the last proposition are valid in a RKHS because every RKHS is a s.i.p. RKBS.
3 Kramer-Type Sampling Theorems

The next procedure for obtaining a s.i.p. RKBS version of the Kramer sampling theorem is due to García, Hernández-Medina & Muñoz-Bouzo [6], they use a BK-space instead of the ℓ₂ space, an X_d*-Riesz basis instead of a Riesz basis and a s.i.p. RKBS instead of a RKHS. Let (E, [·, ·]_E), Φ : Ω → E and T_Φ : E → ℂΩ be as in Remark 2.9. First we suppose there exists a sequence {x_j}j∈I ⊂ E such that {x_j*}j∈I is an X_d*-Riesz basis for E*, then, there exists an unique biorthogonal sequence {y_j}j∈I which is an X_d*-Riesz basis for E (see [16]). In second place, suppose the existence of sequences {t_j}j∈I ⊂ Ω and {a_j}j∈I ⊂ ℂ \ {0} such that the interpolation condition (Φ(t_k))* = a_kx_k* k ∈ I or, equivalently, S_j(t_k) := [y_j, Φ(t_k)]_E = a_jδ_j,k j, k ∈ I holds true, where for fixed t ∈ Ω we have (Φ(t))* = ∑_j∈I S_j(t) x_j* ∈ E* with S_j(t) := [y_j, Φ(t)]_E, being the sequence {S_j(t)}j∈I ⊂ X_d* for each t ∈ Ω as can be checked by straightforward calculations. Under these hypotheses is obtained the s.i.p. RKBS on Ω explicitly given by B = {f_ξ(·) = [x, Φ(·)]_E : x ∈ E}, with norm ∥f_ξ∥_B := ∥x∥_E and s.i.p. reproducing kernel G(s, t) = [Φ(s), Φ(t)]_E s, t ∈ Ω. We now state the before mentioned s.i.p. RKBS version of the Kramer sampling theorem.

**Theorem 3.1** (s.i.p. RKBS Kramer sampling theorem [6, p. 19]). Let B be a s.i.p. RKBS on Ω as before. Then, the sequence {S_j(t)}j∈I ⊂ B is an X_d*-Riesz basis for B and for any f ∈ B we have the sampling expansion

\[
 f(t) = \sum_{j∈I} f(t_j) \frac{S_j(t)}{a_j} \quad t ∈ Ω
\]

The series converges in the B-norm sense and also, absolutely and uniformly on subsets of Ω where the function t ↦ ∥Φ(t)∥_E is bounded.

**Corollary 2.2.** Under hypotheses of Theorem 3.1, {a_j⁻¹ S_j}j∈I is a sampling basis for B.

**Proof.** Indeed, for each j, k ∈ I we have

\[
 [a_j⁻¹ S_j, G_{tk}]_B = \frac{S_j}{a_j}, G_{tk}]_B = \frac{1}{a_j} [S_j, G_{tk}]_B = \frac{1}{a_j} S_j(t_k) = \frac{a_j}{a_j} δ_j,k = δ_j,k. \]

4 The main result: A converse of the Kramer Sampling Theorem in s.i.p. RKBS

Keeping in mind the statement as well as the proof of the Theorem 1.1 ([4, pp. 55–58]), we consider (B, [·, ·]_B, G, E, Φ) a s.i.p. RKBS on Ω that it has been built by an isometric operator (Remark 2.9) and we assume the existence of a sequence {S_j}j∈I ⊂ B such that

\[
 \{S_j(t)\}_{j∈I} ⊂ X_d \quad \text{and} \quad \{S_j(t)\}^*_j{j∈I} ⊂ X_d^* \quad \forall t ∈ Ω
\]

with X_d a uniform BK-space (for instance ℓ₂(ℤ)). We define two functions

\[
 \phi : Ω \quad \longrightarrow \quad X_d \quad t \quad \longmapsto \quad \{S_j(t)\}_{j∈I} \quad \text{and} \quad \phi^* : Ω \quad \longrightarrow \quad X_d^* \quad t \quad \longmapsto \quad \{S_j(t)\}^*_j_{j∈I}
\]

using the notation \(\phi^*(t) = (\phi(t))^*, t ∈ Ω\). We also assume that holds true:

\[
 \begin{cases} 
 \text{if } c ∈ X_d \text{ is such that } \sum_{j∈I} c_j(S_j(t))^* = 0 \quad \forall t ∈ Ω \Rightarrow c = 0. \\
 \text{if } d ∈ X_d^* \text{ is such that } \sum_{j∈I} d_j S_j(t) = 0 \quad \forall t ∈ Ω \Rightarrow d = 0. 
\end{cases}
\]  

(9)
This requirement is similar to that in the item 1°) of Theorem 1.1, and it is equivalent to the completeness statement:

$$\overline{\text{span}}\{\phi(t) : t \in \Omega\} = X_d \quad \text{and} \quad \overline{\text{span}}\{\phi^*(t) : t \in \Omega\} = X_d^*$$

(10)

which is necessary for the definition itself of the s.i.p. RKBS $$(\mathcal{B}_{\text{samp}}, [\cdot, \cdot]_{\text{ samp}}, G_{\text{ samp}}, X_d^*, \phi^*)$$ on $\Omega$. By the way, its s.i.p. reproducing kernel $G_{\text{ samp}}$ is given by

$$G_{\text{ samp}}(s, t) := [\phi^*(s), \phi^*(t)]_{X_d^*} = \sum_{j \in I} (S_j(s))^* S_j(t) \quad s, t \in \Omega$$

(11)

where the reflexivity of $X_d$ was used to the identification of $(\phi(t))^*$ with $\phi(t)$.

We have taken $X_d^*$ instead of $X_d$ in the definition of $\mathcal{B}_{\text{ samp}}$ because we want the similarity between the s.i.p. reproducing kernel $G_{\text{ samp}}$ and the reproducing kernel $K_{\text{ samp}}$, where the last one was used in Theorem 1.1. We are going to prove three propositions that will be used in the demonstration of the main result, the first two are interesting on their own.

**Proposition 4.1.** Let $$(\mathcal{B}, [\cdot, \cdot]_B, G, E, \Phi)$$ and $$(\mathcal{B}_{\text{ samp}}, [\cdot, \cdot]_{\text{ samp}}, G_{\text{ samp}}, X_d^*, \phi^*)$$ be two s.i.p. RKBS on $\Omega$ as before. If the sets $\{\{S_j(t)\}_{j \in I} : t \in \Omega\} \subset X_d$ and $\{\{S_j(t)\}_{j \in I} : t \in \Omega\} \subset X_d^*$ are complete, then $\{S_j^*\}_{j \in I}$ is an $X_d$-Bessel sequence for $\mathcal{B}^*$.

**Proof.** The completeness conditions (equivalent to (9)) are stated because it is necessary for the definition of $\mathcal{B}_{\text{ samp}}$. We must to show there exist $B > 0$ such that

$$\left\| \sum_{j \in I} d_j S_j^* \right\|_B \leq B \|d\|_{X_d^*} \quad \forall \ d \in X_d^*$$

For Proposition 2.4 it is equivalent to show the associated analysis operator given by

$$V : \mathcal{B}^* \rightarrow X_d$$

$$f^* \rightarrow \{[f^*, S_j^*]_B\}_{j \in I}$$

is bounded. The operator $T : E^* \rightarrow B^*$ defined by $T x^* = [x^*, \Phi^*(\cdot)]_{E^*} := f_x^*$ is an isometric isomorphism, therefore it sends dense subspaces on $E^*$ in dense subspaces on $\mathcal{B}^*$. We know it suffices to prove the Bessel condition of $\{S_j^*\}_{j \in I}$ on a dense subset of $\mathcal{B}^*$. Since the set $\text{span}\{\Phi^*(s) : s \in \Omega\}$ is dense in $E^*$ and

$$T \Phi^*(s) = [\Phi^*(s), \Phi^*(\cdot)]_{E^*} = [\Phi(\cdot), \Phi(s)]_{E} = G(\cdot, s) = (G(s, \cdot))^* := G_s^* \quad s \in \Omega$$

the set $\mathcal{B}_0^* = \text{span}\{G_s^* : s \in \Omega\}$ is dense in $\mathcal{B}^*$. Now, we consider for each $N \in \mathbb{N}$

$$V_N : \mathcal{B}_0^* \rightarrow X_d$$

$$f^* \rightarrow \{1_{\mathbb{N}}(j)[f^*, S_j^*]_B\}_{j \in I}$$

and

$$V' : \mathcal{B}_0^* \rightarrow X_d$$

$$f^* \rightarrow \{[f^*, S_j^*]_B\}_{j \in I}$$

where $1_{\mathbb{N}}$ denotes the characteristic function of $\mathbb{N}$ (the first $N$ elements of $\mathbb{I}$). For each $s \in \Omega$, $j \in \mathbb{I}$ there holds:

$$[G_s^*, S_j^*]_B = [S_j, G_s]_B = S_j(s)$$

and since $\{S_j(s)\}_{j \in I} \subset X_d \ \forall \ s \in \Omega$, the operators $V_N$ are well defined and also they are bounded for each $N \in \mathbb{N}$, since

$$\|V_N f^*\|_{X_d} = \sup_{d \in S_{X_d}} \left| \sum_{j \in \mathbb{N}} d_j [f^*, S_j^*]_B \right| \leq \sup_{d \in S_{X_d^*}} \left( \sum_{j \in \mathbb{N}} d_j \|S_j^*\|_B \right) \|f^*\|_B$$
Furthermore, they converge pointwise to $V'$ since

$$
\|V_Nf^*-V'f\|_{X_d} = \sup_{d \in S_{X_d}} \left| \sum_{j \in \mathbb{N}} d_j[f^*, S_j^*]_{B^*} \right| \leq \|\{1_{\mathbb{N} \setminus \{j\}}[f^*, S_j^*]_{B^*}\}_{j \in \mathbb{N}}\|_{X_d} \xrightarrow{N \to \infty} 0
$$

Thus, by Banach-Steinhaus theorem, $V'$ is a bounded operator, therefore $V$ it is, and $\{S_j^*\}_{j \in \mathbb{N}}$ is an $X_d$-Bessel sequence for $B^*$.

An infinite-dimensional vector space can be endowed with various norms which turns it in a Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals). Of course, this phenomenon does not occur in a finite-dimensional Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals). Of course, this phenomenon does not occur in a finite-dimensional Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals). Of course, this phenomenon does not occur in a finite-dimensional Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals). Of course, this phenomenon does not occur in a finite-dimensional Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals). Of course, this phenomenon does not occur in a finite-dimensional Banach space, but being non-equivalent between them (by the existence of unbounded linear functionals).

**Proposition 4.2.** Let’s suppose that $B$ is a s.i.p. RKBS on $\Omega$ endowed with the norm $\| \cdot \|_B$ either the norm $\| \cdot \|$. Then, the norms are equivalent.

**Proof.** We show the identity operator $id : (B, \| \cdot \|_B) \to (B, \| \cdot \|)$ is bounded, and then by the open mapping theorem will result bi-continuous. By the closed graph theorem, we only need to check:

$$
f_j \to f \text{ in } \| \cdot \|_B \text{ and } f_j \to g \text{ in } \| \cdot \|	ext{ then } f = g
$$

and this is clear due to the convergence property in a s.i.p. RKBS.

**Proposition 4.3.** Let $(B, [\cdot, \cdot]_B, G, E, \Phi)$ and $(B_{samp}, [\cdot, \cdot]_{samp}, G_{samp}, X_d^*, \phi^*)$ be two s.i.p. RKBS on $\Omega$ as before. Let’s suppose that:

1°) The sets $\{ \{S_j(t)\}_{j \in \mathbb{N}} : t \in \Omega \} \subset X_d$ and $\{ \{S_j(t)\}_{j \in \mathbb{N}}^* : t \in \Omega \} \subset X_d^*$ are complete.

2°) There exists sequences $\{t_j\}_{j \in \mathbb{N}} \subset \Omega$, $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ such that there holds the following sampling conditions:

$$
\left\{ \frac{f(t_j)}{a_j} \right\}_{j \in \mathbb{N}} \in X_d^* \quad \forall f \in B \quad (12)
$$

and

$$
f(t) = \sum_{j \in \mathbb{N}} f(t_j) \frac{S_j(t)}{a_j} \quad \forall f \in B 
$$

where the series converges absolutely on $\Omega$.

If we call:

$$
M_j(\cdot) := a_j^{-1}G_{samp}(t_j, \cdot) \quad \text{and} \quad M_j^* = a_j^{-1}G_{samp}(\cdot, t_j) \in B_{samp}^* \quad j \in \mathbb{I} \quad (14)
$$

Then:

a) $\{M_j^*\}_{j \in \mathbb{I}}$ is a complete sequence in $B_{samp}^*$.

b) $\{M_j\}_{j \in \mathbb{I}}$ is an $X_d^*$-Bessel sequence for $B_{samp}$.

c) $\{M_j^*\}_{j \in \mathbb{I}}$ is a minimal sequence in $B_{samp}^*$ with biorthogonal sequence $\{S_j\}_{j \in \mathbb{N}}$. Also, $\{M_j^*\}_{j \in \mathbb{I}}$ is a minimal sequence in $B_{samp}$ with biorthogonal sequence $\{S_j^*\}_{j \in \mathbb{N}}$. 

Because of this we also obtain the well-definition and boundedness of the analysis operator then $f$. The main result: A converse of the Kramer sampling theorem in s.i.p. RKBS

Proof. a) We assume there exists $f \in B_{\text{samp}}$ such that $0 = [f, M_j]_{\text{samp}} = [M_j^*, f^*]_{\text{samp}} \forall j \in \mathbb{I}$. But, since $f = 0$.

b) This is immediate since $\{f(t_j)a_j^{-1}\}_{j \in \mathbb{I}} \in X_d^*$ \forall $f \in B_{\text{samp}}$ and, $\forall j \in \mathbb{I}$, holds

$$a_j^{-1}f(t_j) = a_j^{-1}[f, G_{\text{samp}}(t_j, \cdot)]_{\text{samp}} = [f, \overline{\omega_j}^{-1}G_{\text{samp}}(t_j, \cdot)]_{\text{samp}} = [f, M_j]_{\text{samp}}$$

Because of this we also obtain the well-definition and boundedness of the analysis operator $U : B_{\text{samp}} \rightarrow X_d^*$ associated to the sequence $\{M_j\}_{j \in \mathbb{I}}$ as well as its adjoint $U^* : X_d \rightarrow B_{\text{samp}}^*$, in particular $U^*c = \sum_{j \in \mathbb{I}} c_jM_j^*$ converges (unconditionally) in $B_{\text{samp}}^*$ for all $c \in X_d$.

c) Due to $[M_k^*, S_j]_{\text{samp}} = [S_j, M_k]_{\text{samp}}$, we only need to show that $[S_j, M_k]_{\text{samp}} = \delta_{j,k} \forall j, k \in \mathbb{I}$. In one hand we have

$$S_k(t) = \sum_{j \in \mathbb{I}} \delta_{j,k}S_j(t) \quad k \in \mathbb{I}, t \in \Omega$$

and by other hand

$$S_k(t) = \sum_{j \in \mathbb{I}} \frac{S_k(t_j)}{a_j}S_j(t) = \sum_{j \in \mathbb{I}} [S_j, M_k]_{\text{samp}}S_j(t) \quad k \in \mathbb{I}, t \in \Omega$$

then

$$0 = \sum_{j \in \mathbb{I}} ([S_j, M_k]_{\text{samp}} - \delta_{j,k})S_j(t) \quad k \in \mathbb{I}, t \in \Omega$$

where the coefficients are in $X_d^*$, therefore we obtain $[S_j, M_k]_{\text{samp}} = \delta_{j,k} \forall j, k \in \mathbb{I}$ as we needed.

d) Given a sequence $d = \{d_j\}_{j \in \mathbb{I}} \in X_d^*$ we must see there exist $f \in B_{\text{samp}}$ such that $Uf = d$. By considering $f = \sum_{j \in \mathbb{I}} d_jS_j$ (it belongs to $B_{\text{samp}}$) it leads to

$$[f, M_k]_{\text{samp}} = \left[\sum_{j \in \mathbb{I}} d_jS_j, M_k\right]_{\text{samp}} = \sum_{j \in \mathbb{I}} d_j[S_j, M_k]_{\text{samp}} = d_k \quad k \in \mathbb{I}$$

therefore $\{Uf\}_k = d_k \forall k \in \mathbb{I}$ and $U$ is surjective.

e) It follows by items b) and d) due to Proposition 2.5.

f) It follows by items e) and c) due to Proposition 2.6.

We now prove the main result of this paper.

**Theorem 4.4** (A Converse of the Kramer sampling theorem - s.i.p. RKBS Version).

Under hypotheses of Proposition 4.3 we have:

a) $B_{\text{samp}} = B$.

b) The norms $\|\cdot\|_{B_{\text{samp}}}$ and $\|\cdot\|_B$ are equivalent and consequently $\{S_j\}_{j \in \mathbb{I}}$ is an $X_d^*$-Riesz basis for $B$. 



4. The main result: A converse of the Kramer sampling theorem in s.i.p. RKBS

\[ \left\{ \sum_{k \in I} \left[ \frac{\phi^*(t_j)}{\alpha_j}, \frac{\phi^*(t_k)}{\alpha_k} \right] X_{d} \right\}_{j \in I} \]

(15)

d) The biorthogonal sequence of \( \{S_j\}_{j \in I} \) in \( B \) is given by
\[ \left\{ \sum_{k \in I} \left[ \Phi(t_j), \Phi(t_k) \right] E \right\}_{j \in I} \]

(16)

**Proof.**
a) We first prove that \( B_{\text{samp}} \subset B \) by only assuming the item \(^1\). Due to \( B_{\text{samp}} \) comprises functions of the form
\[ \sum_{j \in I} \alpha_j S_j \quad \text{with} \quad \alpha = \{\alpha_j\}_{j \in I} \subset X_d^* \]

by definition, it follows that \( \| \sum_{j \in I} \alpha_j S_j \|_{\text{samp}} < \infty \quad \forall \alpha \in X_d^* \). To see \( B_{\text{samp}} \subset B \) we must to show \( \| \sum_{j \in I} \alpha_j S_j \|_B < \infty \quad \forall \alpha \in X_d \). By Proposition 4.1 \( \{S_j^*\}_{j \in I} \) is an \( X_d \)-Bessel sequence for \( B^* \), therefore the analysis operator associated to \( \{S_j^*\}_{j \in I} \) is bounded and so it is the synthesis operator, i.e.,
\[ \| \sum_{j \in I} \alpha_j S_j \|_B \leq B \| \alpha \|_{X_d} < \infty \quad \text{for some} \ B > 0 \]

For the other inclusion we also assume to hold true the sampling conditions (12) and (13). We pick \( f \in B \), then \( \{f(t)\alpha_j^{-1}\}_{j \in I} \subset X_d^* \) by (12) and the series \( \sum_{j \in I} f(t)\alpha_j^{-1} S_j \) converges in \( \| \cdot \|_{\text{samp}} \), we say to \( g \in B_{\text{samp}} \), therefore it converges pointwise to \( g \in B_{\text{samp}} \), but the series also converges pointwise to \( f \) by (13), whence \( g(t) = f(t) \quad \forall t \in \Omega \) and hence \( f \in B_{\text{samp}} \).

b) As we have \( B = B_{\text{samp}} \), the equivalence between the norms \( \| \cdot \|_B \) and \( \| \cdot \|_{\text{samp}} \) follows by Proposition 4.2 and since \( \{S_j\}_{j \in I} \) is the biorthogonal sequence to \( \{M_j\}_{j \in I} \) (Proposition 4.3, item c)), it is an \( X_d \)-Riesz basis for \( B^* \) [17, Theo. 2.14 and 2.15] as well as an \( X_d \)-Riesz basis for \( B \) by norm equivalence.

We recall the notations (14) and now we add a new one: \( G_j(\cdot) := \alpha_j^{-1} G(t, \cdot) \quad j \in I \).

c) We have already seen that \( \{S_j\}_{j \in I} \) and \( \{M_j\}_{j \in I} \) are biorthogonal sequences in \( B_{\text{samp}} \), so we are going to see there holds (15), indeed for \( k \in I, t \in \Omega \) we have
\[ M_k(t) = \sum_{j \in I} \frac{M_k(t_j)}{a_j} S_j(t) = \sum_{j \in I} [M_k, M_j]_{\text{samp}} S_j(t) = \sum_{j \in I} \left[ \frac{\phi^*(t_k)}{\alpha_k}, \frac{\phi^*(t_j)}{\alpha_j} \right] X_{d^*} S_j(t) \]

d) Again, we have already seen that \( S_j(t_k) = a_k \delta_{j,k} \quad \forall j, k \in I \), whence
\[ \delta_{j,k} = \frac{S_j(t_k)}{a_k} = \left[ S_j, \frac{G_{t_k}}{\alpha_k} \right]_B = [S_j, G_{t_k}]_B \quad j, k \in I \]

and therefore \( \{S_j\}_{j \in I} \) and \( \{G_j\}_{j \in I} \) are biorthogonal sequences in \( B \). Finally, \( \{G_j\}_{j \in I} \) satisfies (16), since
\[ G_k(t) = \sum_{j \in I} \frac{G_k(t_j)}{a_j} S_j(t) = \sum_{j \in I} [G_k, G_j]_B S_j(t) = \sum_{j \in I} \left[ \frac{\phi(t_k)}{\alpha_k}, \frac{\phi(t_j)}{\alpha_j} \right] E S_j(t) \]

for all \( k \in I, t \in \Omega \). This finishes the proof. \( \square \)
Corollary 4.5. Under hypotheses of Theorem 4.4, \( \{a_j^{-1}S_j\} \in \mathbb{I} \) is a sampling basis for \( B_{\text{samp}} \).

**Proof.** By Proposition 2.11 we only need to check \( [a_j^{-1}S_j, G_{tk}]_{\text{samp}} = \delta_{j,k} \forall j, k \in \mathbb{I} \) since \( \{a_j^{-1}S_j\} \in \mathbb{I} \) is a Schauder basis. We have

\[
[a_j^{-1}S_j, G_{tk}]_{\text{samp}} = \left[ \frac{S_j}{a_j} \frac{G_{tk}}{a_k} \right]_{\text{samp}} = \frac{a_k}{a_j} [S_j, M_k]_{\text{samp}} = \frac{a_k}{a_j} \delta_{j,k} = \delta_{j,k}
\]

as we needed. □

We finish with a classical example.

**Example.** We consider \( \Omega = \mathbb{R} \), \( I = [\frac{-1}{2}, \frac{1}{2}] \), \( 1 < p, q < +\infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), the “Time-limited” uniform Banach spaces:

\[
L^p(I, \mathbb{R}) := \{ f \in L^p(\mathbb{R}) : f \equiv 0 \text{ a.e. on } \mathbb{R} \setminus I \}
\]

and the “band-limited” uniform Banach spaces:

\[
B^p := \{ f \in C(\mathbb{R}) \cap L^q(\mathbb{R}) : \hat{f} \equiv 0 \text{ a.e. on } \mathbb{R} \setminus I \}
\]

We define \( \Phi : \Omega \to L^p(\mathbb{R}) \) and \( \Phi^* : \Omega \to L^q(\mathbb{R}) \) by

\[
\Phi(\omega)(t) := e^{-2\pi i t \omega} \quad \text{and} \quad \Phi^*(\omega)(t) := e^{2\pi i t \omega} \quad \omega \in \mathbb{R}, \ t \in \mathbb{R}
\]

It is well-known that

\[
\text{span}\{ \chi_I(\cdot)e^{-2\pi i(\cdot)\omega} : \omega \in \mathbb{R} \} = L^p(I)
\]

Let \( \mathcal{F} \) be the Fourier transform, \( \mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R}) \) given by

\[
\hat{f}(\omega) := \mathcal{F}[f](\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i t \omega} dt
\]

and we note by \( f^\vee \) the Fourier inversion of \( f \) given by

\[
f^\vee(\omega) := \mathcal{F}^{-1}[f](\omega) = \int_{\mathbb{R}} f(t)e^{2\pi i t \omega} dt
\]

Clearly, \( \hat{f}(\omega) = f^\vee(-\omega) \forall \omega \in \mathbb{R} \), and by Fourier Analysis we know that if \( f, \hat{f} \in L^1(\mathbb{R}) \) then \( f \) and \( \hat{f} \) are continuous and we have the inversion formulae:

\[
f(t) = \hat{f}(\omega)^\vee(\omega) = \int_{\mathbb{R}} f^\vee(\omega)e^{-2\pi i t \omega} d\omega
\]

\[
f(t) = \hat{f}(\omega)^\vee(\omega) = \int_{\mathbb{R}} \hat{f}(\omega)e^{2\pi i t \omega} d\omega
\]

where the equality is pointwise \( t \in \mathbb{R} \). Also there holds

\[
\int_{\mathbb{R}} f(\omega)g(\omega) d\omega = \int_{\mathbb{R}} \hat{f}(\omega)g(\omega) d\omega
\]

\[
\int_{\mathbb{R}} \hat{f}(\omega)g^\vee(\omega) d\omega = \int_{\mathbb{R}} f(x)g(x) dx.
\]
Let \([\cdot, \cdot]_p\) be the semi-inner product of \(L_p(I)\) given by

\[
[f, g]_p = \int_I f(\cdot)g(\cdot)\left|\frac{g'}{\|g\|_p^{p-2}}\right| \, dm.
\]

Then, we can write the “band-limited” uniform Banach spaces and their duals as:

\[
\mathcal{B}_p := \{ f = [\hat{f}, \Phi(\cdot)]_p \in C(\mathbb{R}) : \hat{f} \in L_p(I) \} \\
\mathcal{B}'_p := \{ h = [\Phi(\cdot), \hat{h}]_p \in C(\mathbb{R}) : \hat{h} \in L_p(I) \}.
\]

By norming them with \(\|f\|_{\mathcal{B}_p} = \|\hat{f}\|_{L_p(I)}\) and \(\|h\|_{\mathcal{B}'_p} = \|\hat{h}\|_{L_p(I)}\) respectively, indeed, we obtain two uniform Banach spaces. The duality between them can be written as

\[
(f, h)_{\mathcal{B}_p} = (\hat{f}, (\hat{h})^*)_p = [\hat{f}, \hat{h}]_p \quad f \in \mathcal{B}_p, \ h \in \mathcal{B}'_p.
\]

In these terms, \(\mathcal{B}_p\) is a s.i.p. RKBS on \(\mathbb{R}\) with the semi-inner product given by

\[
[f, g]_{\mathcal{B}_p} = \left[ [\hat{f}, \Phi(\cdot)]_p, [\hat{g}, \Phi(\cdot)]_p \right]_{\mathcal{B}_p} = [\hat{f}, \hat{g}]_p \quad f, g \in \mathcal{B}_p
\]

and the s.i.p. reproducing kernel \(G\) has the form

\[
G(\omega, t) := [\Phi(\omega), \Phi(t)]_p = \int_I e^{-2\pi i \omega x} e^{2\pi it x} \, dx = \frac{\sin \pi(t - \omega)}{\pi(t - \omega)} = \text{sinc}(t - \omega) \quad t, \omega \in \mathbb{R}.
\]

The reproducing property is satisfied, since

\[
[f, G(t, \cdot)]_{\mathcal{B}_p} = \left[ [\hat{f}, \Phi(\cdot)]_p, [\Phi(t), \Phi(\cdot)]_p \right]_{\mathcal{B}_p} = [\hat{f}, \Phi(t)]_p = \int_{\mathbb{R}} \hat{f}(\omega)e^{2\pi i t \omega} \, d\omega = f(t)
\]

Consequently, we have \((\mathcal{B}_p, [\cdot, \cdot]_{\mathcal{B}_p}, G, L_p(I), \Phi)\) a s.i.p. RKBS on \(\mathbb{R}\).

Now, we consider the sequence \(\{G_j(\cdot)\}_{j \in \mathbb{Z}} \in \mathcal{B}_p\), where

\[
G_j(t) = G(j, t) = \text{sinc}(t - j) \quad j \in \mathbb{Z}, \quad t \in \mathbb{R},
\]

being the integers ordered by \(\mathbb{Z} = \{0, -1, 1, \cdots\}\). That \(\{G_j(t)\}_{j \in \mathbb{Z}} \in \ell_q(\mathbb{Z}) \quad \forall \ t \in \mathbb{R}\) is due to

\[
\left( \sum_{j \in \mathbb{Z}} |G_j(t)|^q \right)^{\frac{1}{q}} \leq \frac{1}{\pi} \left( \sum_{j \in \mathbb{Z}} \frac{1}{|t - j|^q} \right)^{\frac{1}{q}} < +\infty.
\]

Of course, when \(t \in \mathbb{Z}\), \(G_j(t) = 1\). The last calculation shows that \(\{G_j(t)\}_{j \in \mathbb{Z}} \in \ell_q(\mathbb{Z}) \quad \forall \ t \in \mathbb{R}\) (and \(1 < q < +\infty\) in fact), therefore \(\{G_j(t)\}_{j \in \mathbb{Z}}^* \in \ell_p(\mathbb{Z}) \quad \forall \ t \in \mathbb{R}\).

By calling \(\phi : \mathbb{R} \rightarrow \ell_q(\mathbb{Z})\) to the map \(t \mapsto \{G_j(t)\}_{j \in \mathbb{Z}}\) and \(\phi^* : \mathbb{R} \rightarrow \ell_p(\mathbb{Z})\) to the map \(t \mapsto \{G_j(t)\}_{j \in \mathbb{Z}}^*\), follows immediately that

\[
\text{span}\{\{G_j(t)\}_{j \in \mathbb{Z}} : t \in \mathbb{R}\} = \ell_q(\mathbb{Z}) \\
\text{span}\{\{G_j(t)\}_{j \in \mathbb{Z}}^* : t \in \mathbb{R}\} = \ell_p(\mathbb{Z}),
\]

since \(G_j(k) = \delta_{j,k} \quad \forall \ j, k \in \mathbb{Z}\), thus the canonical unconditional basis for \(\ell_q(\mathbb{Z})\) (whenever \(1 < q < +\infty\)) is contained in both sets. At this point, we already can define the s.i.p. RKBS on \(\mathbb{R}\) \((B_{\text{samp}}, [\cdot, \cdot]_{\text{samp}}, G_{\text{samp}}), \ell_q(\mathbb{Z}), \phi)\), which s.i.p. reproducing kernel \(G_{\text{samp}}\) is given by

\[
G_{\text{samp}}(s, t) = \sum_{j \in \mathbb{Z}} (G_j(s))^* G_j(t) = \frac{1}{\|\{G_j(s)\}_{j \in \ell_p(\mathbb{Z})}\|^{p-2}} \sum_{j \in \mathbb{Z}} G_j(t)\overline{G_j(s)}|G_j(s)|^{p-2}.
\]
The main result: A converse of the Kramer sampling theorem in s.i.p. RKBS

Also it holds true that \( B_{\text{samp}} \subseteq B_p \) (by the last two completeness conditions).

We are going to see that \( \{ G_j(\cdot) \}_{j \in \mathbb{Z}} \) satisfies the hypotheses of the “Converse Sampling Theorem” so, we choose the sequences \( \{ t_j := j \}_{j \in \mathbb{Z}} \subset \mathbb{R} \) and \( \{ a_j := 1 \}_{j \in \mathbb{Z}} \subset \mathbb{C} \setminus \{ 0 \} \).

In the first place, we need to show that the sequence \( \{ f(j) \}_{j \in \mathbb{Z}} \) belongs to \( \ell_p(\mathbb{Z}) \) for all \( f \in B_p \). If \( f \in B_p \) and \( j \in \mathbb{Z} \), then

\[
\begin{align*}
f(j) &= [f, G_j(\cdot)]_{B_p} = [\widehat{f}, \Phi(t)]_p = \int_I \widehat{f}(\omega) e^{2\pi ij\omega} d\omega = \int_I \widehat{f}(\omega) \widehat{G}_j(\omega) d\omega
\end{align*}
\]

so, for \( j \neq 0 \):

\[
\begin{align*}
|f(j)|^p &= \left| \int_I \widehat{f}(\omega) \widehat{G}_j(\omega) d\omega \right|^p \\
&= \left| \int_I \widehat{f}(\omega) e^{2\pi ij\omega} d\omega \right|^p \\
&= \left| \left[ \frac{\widehat{f}(\omega) e^{2\pi ij\omega}}{2\pi i j} \right]^{1/2} - \frac{1}{2\pi i j} \int_I \widehat{f}'(\omega) e^{2\pi ij\omega} d\omega \right|^p \\
&\leq \frac{C(p)}{|j|^p}.
\end{align*}
\]

while for \( j = 0 \) we have \( |f(j)|^p \leq \|f\|_p^p \). Therefore, taking the \( \ell_p \)-norm results:

\[
\| \{ f(j) \}_{j \in \mathbb{Z}} \|_{\ell_p(\mathbb{Z})} \leq C(p, \widehat{f}, \widehat{f}') \left( 1 + 2 \sum_{j \in \mathbb{N}} \frac{1}{j^p} \right)^{1/2} < +\infty.
\]

In the second place, we want the sampling representation \( f(t) = \sum_{j \in \mathbb{Z}} f(j) G_j(t) \), \( t \in \mathbb{R} \) for all \( f \in B_p \), being the series pointwise convergent at least. The series in fact is absolutely convergent since \( \{ f(j) \}_{j \in \mathbb{Z}} \in \ell_p(\mathbb{Z}) \) and \( \{ G_j(t) \}_{j \in \mathbb{Z}} \in \ell_q(\mathbb{Z}) \) for all \( f \in B_p \), \( t \in \mathbb{R} \). In this way only remains to check the pointwise convergence to \( f(t) \). If \( f \in B_p \), we have the following representation:

\[
\begin{align*}
f(t) &= [f, G(t, \cdot)]_{B_p} = \int_I \widehat{f}(\omega) \widehat{G}_t(\omega) d\omega \quad t \in \mathbb{R},
\end{align*}
\]

We consider the sequence of functions \( \{ f_N \}_{N \in \mathbb{N}_0} \) in \( B_p \) given by

\[
\begin{align*}
f_N(t) = \sum_{|j| \leq N} f(j) G_j(t) \quad N \in \mathbb{N}_0, \ t \in \mathbb{R}.
\end{align*}
\]

By fixing \( f \in B_p \) and \( t \in \mathbb{R} \), for \( N \in \mathbb{N}_0 \) we have:

\[
\begin{align*}
|f(t) - f_N(t)| &= \left| \int_I \widehat{f}(\omega) \widehat{G}_t(\omega) d\omega - \sum_{|j| \leq N} \left( \int_I \widehat{f}(\omega) \widehat{G}_j(\omega) d\omega \right) G_j(t) \right| \\
&\leq \int_I |\widehat{f}(\omega)| \left| \widehat{G}_t(\omega) - \sum_{|j| \leq N} \widehat{G}_j(\omega) G_j(t) \right| d\omega \\
&= \int_I |\widehat{f}(\omega)| \left| \sum_{j \in \mathbb{Z}} G_t(j) \widehat{G}_j(\omega) - \sum_{|j| \leq N} \widehat{G}_j(\omega) G_j(t) \right| d\omega \\
&= \int_I |\widehat{f}(\omega)| \left| \sum_{|j| > N} \widehat{G}_j(\omega) G_j(t) \right| d\omega \\
&\leq \| \widehat{f} \|_{L_p(t)} \left( \int_I \left| \sum_{|j| > N} \widehat{G}_j(\omega) G_j(t) \right|^q d\omega \right)^{1/q}.
\end{align*}
\]
Where in the third equality we used that $G_t(j) = G_j(t) \; \forall t \in \mathbb{R}, j \in \mathbb{Z}$. Then, by Lebesgue’s Dominated convergence Theorem, follows that $\{f_N\}_{N \in \mathbb{N}_0}$ converges pointwise to $f$ in $\mathbb{R}$.

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