CURVATURE GROWTH OF SOME 4-DIMENSIONAL GRADIENT RICCI SOLITON SINGULARITY MODELS

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Abstract. In this note we discuss estimates for the curvature of 4-dimensional gradient Ricci soliton singularity models by applying Perelman’s point selection, a fundamental result of Cheeger and Naber, and topological lemmas.

1. Introduction

1.1. Definition of a singularity model. For a finite-time singular solution to the Ricci flow on a closed oriented manifold \((\mathcal{M}^n, g(t)), t \in [0, T), T < \infty\), we have \(\sup_{\mathcal{M} \times [0, T)} |Rm| = \infty\). An associated singularity model \((\mathcal{M}^n, g_\infty(t)), t \in (-\infty, 0]\), is a complete ancient solution which is a limit of pointed rescalings. More precisely, there exists a sequence of space-time points \((x_i, t_i)\) in \(\mathcal{M} \times [0, T)\) with \(K_i \equiv |Rm|(x_i, t_i) \to \infty\) such that the sequence of pointed solutions \((\mathcal{M}, g_i(t), (x_i, 0))\), where \(g_i(t) = K_i g(K_i^{-1} t + t_i)\) and \(t \in [-K_i t_i, 0]\), converges in the \(C^\infty\) pointed Cheeger–Gromov sense to the complete ancient solution \((\mathcal{M}_\infty, g_\infty(t), (x_\infty, 0)), t \in (-\infty, 0]\), for some \(x_\infty \in \mathcal{M}_\infty\). Note that \(g_\infty(t)\) is not assumed to have bounded curvature on each time slice.

A folklore conjecture is that any singularity model must have bounded curvature. In dimension 3, this is true by the work of Perelman [21]. Observe that it is not obvious that singularity models are necessarily of finite (topological) type. Neither is it obvious that singularity models are embeddable in the compact manifold from which they arise. However, if a singularity model has finite type, then it is embeddable in the original compact manifold and thus has an orientation induced by the embedding.

1.2. Classification of 3-dimensional singularity models. There is now a complete classification of 3-dimensional singularity models. We shall use GRS as an abbreviation for gradient Ricci soliton. Firstly, 3-dimensional noncollapsed shrinking GRS with bounded curvature have been classified by Hamilton [11] and by Perelman [22], who proved nonexistence in the noncompact positive sectional curvature case. Naber [19] showed that shrinking GRS with bounded curvature must be noncollapsed. It was shown that 3-dimensional shrinking GRS must have bounded curvature via the works of Cao, Chen, and Zhu [5] and Ni and Wallach [20], with related works by Chen.

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and Petersen and Wylie [23]. Secondly, Brendle [3] proved the assertion by Perelman that the only 3-dimensional nonflat noncollapsed steady GRS is the rotationally symmetric Bryant soliton. Thirdly, the works of Hamilton [11], Perelman [21], and Brendle [4], prove that any 3-dimensional singularity model must be either a shrinking GRS or a steady GRS. In particular, Brendle [4] proved Perelman’s conjecture that the any 3-dimensional noncollapsed ancient solution with positive bounded sectional curvature must be a steady GRS. Finally, one can conclude from the results above that the possible 3-dimensional oriented singularity models are classified as: a shrinking spherical space form $S^3/\Gamma$, a round cylinder $S^2 \times \mathbb{R}$ or its $\mathbb{Z}_2$-quotient, or the Bryant soliton.

There has been much progress on the understanding of higher-dimensional shrinking GRS, largely due to the works of Munteanu and Wang (see e.g. [17, 18] and the references therein), with the strongest results in dimension 4. Kotschwar and Wang proved an important uniqueness result [12]. For progress on steady GRS, see Deng and Zhu [10] and Munteanu, Sung, and Wang [16] and the references therein.

In the case where the potential function is constant, a steady GRS is a Ricci-flat manifold. Assuming also that this manifold is a singularity model, by Perelman’s no local collapsing theorem it is necessarily $\kappa$-noncollapsed on all scales. In particular, it has Euclidean volume growth. In dimension 4, by a fundamental result of Cheeger and Naber [7], it must be a Ricci-flat asymptotically locally Euclidean (ALE) manifold. We remark that singularity models of 4-dimensional singular solutions with bounded scalar curvature have been shown to be such ALE manifolds by Bamler and Zhang [1] and Simon [24].

1.3. Main results. In this paper we study the curvature growth of 4-dimensional GRS singularity models.

**Theorem 1.** Any 4-dimensional steady GRS singularity model $(M,g,f)$ must have bounded curvature, that is, there exists a constant $C$ depending on the GRS such that $|\text{Rm}| \leq C$ on $M$.

For similar reasons, we have:

**Theorem 2.** Any 4-dimensional shrinking GRS singularity model must have curvature which grows at most quadratically. That is, for any $o \in M$ there exists a constant $C$ depending only on the GRS and $o$ such that $|\text{Rm}|(x) \leq C(d(x,o) + 1)^2$ for all $x \in M$.

The paper is organized as follows. In Section 2 we first recall Perelman’s point selection method on Riemannian manifolds, which is generally used to obtain limits. We then discuss under what conditions the local derivative of curvature estimates of Shi yield instantaneous estimates for GRS. Next, we prove the main Theorems 1 and 2 modulo the topological lemmas proved in Section 3.
For a survey of 4-dimensional Ricci flow, see [9] by Gompf, Hillman, and two of the authors.

2. Curvature estimates for GRS singularity models

2.1. Point selection. The following is Perelman’s point selection method; see [21]. Since the method is crucial to our results, we include its proof for the sake of completeness.

Lemma 3. Let \((M^n, g)\) be a complete Riemannian manifold. For any \(y_0 \in M\) (let \(P_0 \doteq |\text{Rm}|(y_0)\)) and \(A_0 \in \mathbb{R}^+\) there exists \(x_0 \in B_{2A_0P_0^{-1/2}}(y_0)\) such that \(Q_0 \doteq |\text{Rm}|(x_0) \geq P_0\) and

\[
|\text{Rm}| \leq 4Q_0 \quad \text{in} \quad B_{A_0Q_0^{-1/2}}(x_0).
\]

For example, we may choose \(A_0 = \frac{1}{3}P_0^{1/2}\), in which case we also have \(A_0Q_0^{-1/2} < \frac{1}{3}\), so that \(x_0 \in B_{2/3}(y_0)\) and \(B_{A_0Q_0^{-1/2}}(x_0) \subset B_1(y_0)\).

Proof. If (1) holds for \(x_0 = y_0\), then we are done. So suppose (1) does not hold for \(x_0 = y_0\). Let \(O_0 \doteq |\text{Rm}|(y_0) = P_0\). Since (1) does not hold for \(x_0 = y_0\), there exists \(y_1 \in B_{A_0O_0^{-1/2}}(y_0)\) such that

\[
O_1 \doteq |\text{Rm}|(y_1) > 4O_0.
\]

By induction, suppose that \(y_0, \ldots, y_j\) have been chosen such that (1) does not hold for \(x_0 = y_k\) for all \(0 \leq k \leq j - 1\) and with \(O_k \doteq |\text{Rm}|(y_k)\) for \(0 \leq k \leq j\) we have \(y_{k+1} \in B_{A_0O_k^{-1/2}}(y_k)\) satisfies \(O_{k+1} > 4O_k\) for \(0 \leq k \leq j - 1\). Then, for \(0 \leq k \leq j\) we have

\[
O_k \geq 4^kO_0 = 4^kP_0
\]

and

\[
d(y_k, y_0) \leq \sum_{\ell=0}^{k-1} d(y_{\ell+1}, y_{\ell}) < \sum_{\ell=0}^{k-1} A_0O_\ell^{-1/2}
\]

\[
\leq A_0\sum_{\ell=0}^{k-1} O_0^{-1/2}2^{-\ell}
\]

\[
< 2A_0O_0^{-1/2}.
\]

Hence \(y_k \subset B_{2A_0O_k^{-1/2}}(y_0)\) and \(B_{A_0O_k^{-1/2}}(y_k) \subset B_{3A_0O_k^{-1/2}}(y_0)\) for \(0 \leq k \leq j\).

For the sequence \(y_0, y_1, \ldots\) there exists a first \(j_0\) for which (1) holds for \(x_0 = y_{j_0}\), for otherwise we would have an infinite sequence of points \(\{y_k\}_{k=0}^{\infty}\)
for which $|Rm| (y_k) \geq 4^k P_0$ and $y_k \in B_{2A_0 r_0^{-1/2}} (y_0)$ for all $k \geq 0$, a contradiction. The lemma follows from taking $x_0 = y_{j_0}$ since $|Rm| (y_{j_0}) \geq 4^{j_0} P_0 \geq P_0$ and since (3) implies $d(y_{j_0}, y_0) < 2A_0 P_0^{-1/2}$.

To summarize, given any $A_0 > 0$ and point $y_0$, Perelman’s point selection method finds a nearby point $x_0$ such that $|Rm|$ in the ball centered at $x_0$ of scaled radius $A_0$ is bounded by $4$ times its value at $x_0$. This is effective since $A_0$ is arbitrary. So, given a sequence $\{y_i\}$ with $|Rm|(y_i) \to \infty$, we may choose $\{A_i\}$ so that $A_i \to \infty$. In particular, as an immediate consequence of the lemma, we have:

**Proposition 4.** Let $(M^n, g)$ be a complete Riemannian manifold. For any sequence $\{y_i\}_{i=1}^{\infty}$ in $M$ with $p_i \in |Rm|(y_i) \to \infty$ there exists a sequence $\{x_i\}_{i=1}^{\infty}$ in $M$ such that for each $i \geq 1$ we have $x_i \in B_{2/3} (y_i)$, $Q_i \in |Rm|(x_i) \geq p_i$, and

$$|Rm| \leq 4Q_i \text{ in } B_{A_i Q_i^{-1/2}} (x_i), \text{ where } A_i = \frac{1}{3} B_i^{1/2} \to \infty.$$  

2.2. **Instantaneous local derivative estimates.** We have the following (instantaneous in time) local derivative estimates. This is useful for rescalings about points where $|Rm|$ is bounded below by a positive constant.

**Lemma 5.** Let $(M^n, g, f)$ be a complete steady or shrinking GRS. Suppose that $p \in M$, $r > 0$, and $C$ are such that in $B_{2r} (p)$ we have $|Rm| \leq C r^{-2}$ and $|\nabla f| \leq r^{-1}$. Then $|\nabla^m Rm| \leq C_m r^{-2-m}$ in $B_r (p)$, where $C_m$ depends only on $m$, $C$, and $n$.

**Proof.** Let $(M, g(t), f(t))$, $\lambda t < 1$, be the canonical form associated to the steady GRS; here $\lambda = 0$ in the steady case and $\lambda = 1$ in the shrinking case, so that $\lambda \geq 0$. By hypothesis, $|Rm|(x, 0) \leq C r^{-2}$ for $x \in B_{2r} (p)$. Since

$$\frac{\partial}{\partial t} \varphi_t (x) = \frac{1}{1 - \lambda t} (\nabla g(0) f(0)) (\varphi_t (x))$$

and by the assumption that $|\nabla f(0)|_{g(0)} \leq r^{-1}$ in $B_{2r} (p)$, we have the inequality

$$\left| \frac{\partial}{\partial t} \varphi_t (x) \right|_{g(0)} \leq r^{-1}$$

whenever $\varphi_t (x) \in B_{2r} (p)$ and $t \leq 0$. Hence, if $x \in B_{3r/2} (p)$ and $t \in [-r^2/2, 0]$, then

$$d_{g(0)}(\varphi_t (x), x) \leq \int_t^0 \left| \frac{\partial}{\partial t} \varphi_t (x) \right|_{g(0)} d \bar{t} \leq \frac{r}{2}.$$ 

Thus $\varphi_t (x) \in B_{2r} (p)$, so that $|Rm|(x, t) = |Rm|(\varphi_t (x), 0) \leq C r^{-2}$. By Shi’s local derivative of curvature estimates, $|\nabla^m Rm|(x, 0) \leq C_m r^{-2-m}$ for $x \in B_r (p)$. \qed
The condition $|\nabla f| \leq r^{-1}$ puts some restriction on the scales $r$ for which we may apply this lemma. For instance, on a steady GRS it is only known that $|\nabla f|^2 = 1 - R \leq 1$; hence this results works naturally for $0 < r \leq 1$ (or less than any fixed constant). On the other hand, on a shrinking GRS, since we only have $|\nabla f| \leq \sqrt{f} - R \leq \frac{1}{4}d(o,x) + C$, this lemma works naturally for $0 < r \lesssim d^{-1}(o,x)$. In particular, for the steady case, we have:

**Corollary 6.** Let $(M^n, g, f)$ be a complete steady GRS with $|Rm| \leq Cr^{-2}$ in $B_{2r}(p)$, where $r \in (0, 1]$. Then $|\nabla^m Rm| \leq Cm r^{-2-m}$ in $B_r(p)$, where $C_m$ depends only on $m$, $C$, and $n$.

### 2.3. Four-dimensional Ricci-flat ALE manifolds.

We say that a complete noncompact oriented Riemannian 4-manifold $(M^4, g)$ is asymptotically locally Euclidean (ALE) if there exists $\tau > 0$, a compact subset $K$, a finite subgroup $\Gamma$ of $SO(4)$ acting freely on $S^3$, and an orientation-preserving diffeomorphism $\Phi : \{\mathbb{R}^4 - BC(0)\}/\Gamma \to M - K$ for some $C$ such that the pullback $\tilde{h}$ to the cover $\mathbb{R}^4 - BC(0)$ of $h = \Phi^*g$ satisfies $\partial_i(\tilde{h}_{jk} - \delta_{jk}) = O(|x|^{-\tau - |I|})$ for each multi-index $I = (i_1, i_2, i_3, i_4)$. By Bando, Kasue, and Nakajima [2, Theorem 1.5], if an ALE manifold is Ricci flat, then there exists $\Phi$ with $\tau = 4$.

It is conjectured that any simply-connected Ricci-flat ALE 4-manifold must be hyperkähler, the latter of which has been classified by Kronheimer [13, 14]. For progress in this direction, see Lock and Viaclovsky [15].

### 2.4. Steady GRS singularity models.

**Proposition 7.** If $(M^4, g, f)$ is a steady GRS which is also a singularity model, then $|Rm|$ is bounded.

**Proof.** Suppose $|Rm|$ is not bounded. Then there exists $\{y_i\}_{i=1}^\infty$ in $M$ with $P_i \downarrow |Rm|(y_i) \to \infty$. By Proposition 4, there exists a sequence $\{x_i\}_{i=1}^\infty$ in $M$ such that for each $i \geq 1$ we have $x_i \in B_{2/3}(y_i)$, $Q_i \downarrow |Rm|(x_i) \geq P_i \geq 100$, and

\[(4) \quad |Rm| \leq 4Q_i \quad \text{in} \quad B_{A_iQ_i^{-1/2}}(x_i), \quad \text{where} \quad A_i = \frac{1}{3} P_i^{1/2} \to \infty.
\]

Since $(M^4, g, f)$ is a singularity model, it is $\kappa$-noncollapsed on all scales for some $\kappa > 0$. Hence (4) implies

$$\text{Vol} B_{Q_i^{-1/2}}(x_i) \geq \kappa Q_i^{-2} \quad \text{for all} \quad i \geq 1.$$ 

Let $(M^4, g(t), f(t))$, $t \in (-\infty, \infty)$, be the associated canonical form, where $g(t) = \varphi_t^*g$ and $f(t) = f \circ \varphi_t$. Since $\frac{\partial}{\partial t} \varphi_t(x) = (\nabla g(0)f(0)) \cdot (\varphi_t(x))$, we have $\left|\frac{\partial}{\partial t} \varphi_t(x)\right|_{g(0)} \leq 1$. Hence, if $x \in B_{Q_i^{-1/2}}(x_i)$ and $t \in [-Q_i^{-1}, 0]$, then

$$d_{g(0)}(\varphi_t(x), x) \leq |t| \leq Q_i^{-1} \leq Q_i^{-1/2},$$
so that $\varphi_t(x) \in B_{A_i Q_i^{-1/2}}(x_i)$ and hence $|Rm|(x, t) = |Rm|(\varphi_t(x), 0) \leq 4Q_i$. Hence, by Shi’s local derivative estimates, we have

$$|\nabla^m Rm| \leq C_m Q_i^{1+\frac{m}{2}} \text{ in } B_{A_i Q_i^{-1/2}}(x_i) \times [-\frac{1}{2} Q_i^{-1}, 0].$$

By the above and by Hamilton’s Cheeger–Gromov compactness theorem, there exists a subsequence such that $(\mathcal{M}_i^4, Q_i g(Q_i^{-1} t), x_i)$ converges in the $C^\infty$ pointed Cheeger–Gromov sense to a complete solution to the Ricci flow $(\mathcal{M}_\infty^4, g_\infty(t), x_\infty)$, $t \in [-\frac{4}{3}, 0]$, with $|Rm_{g_\infty}| \leq 4$ and $|Rm_{g_\infty}|(x_\infty, 0) = 1$. In particular, $g_\infty(t)$ is nonflat. Since $(\mathcal{M}_4^4, g, f)$ is a steady GRS, we have $0 < R \leq 1$. This and $Q_i \to \infty$ imply that $R_{g_\infty} \equiv 0$, which in turn implies that $Rc_{g_\infty} \equiv 0$ by the Ricci flow equation $\frac{\partial R_{g_\infty}}{\partial t} = \Delta_{g_\infty} R_{g_\infty} + 2 |Rc_{g_\infty}|^2$. By Perelman’s no local collapsing theorem and since $R_{g_\infty}(0) = 0$, there exists $\kappa > 0$ such that $Vol_{g_\infty}(0) B_{g_\infty}^{\infty}(x_\infty) \geq \kappa t^4$ for all $t > 0$; hence $g_\infty(0)$ has Euclidean volume growth. By Cheeger and Naber’s theorem (Corollary 8.86 of [7]), $(\mathcal{M}_\infty^4, g_{\infty}(0))$ is asymptotically locally Euclidean (ALE).

Since $(\mathcal{M}_\infty^4, g_{\infty}(0))$ is a nonflat Ricci-flat ALE 4-manifold, there exists a finite subgroup $\Gamma$ of $SO(4)$ such that the end of $\mathcal{M}_\infty$ is diffeomorphic to $(S^3/\Gamma) \times (0, 1)$.\footnote{A nonflat Ricci-flat ALE 4-manifold has only one end since otherwise it will split as the product of a line and a Ricci flat three-manifold, which in turn implies that it is flat.} Let $S^3/\Gamma \subset \mathcal{M}_\infty$ be embedded so that the noncompact region in $\mathcal{M}_\infty$ bounded by $S^3/\Gamma$ is diffeomorphic to $(S^3/\Gamma) \times (0, 1)$. Let $\Omega$ denote the compact region in $\mathcal{M}_\infty$ bounded by $S^3/\Gamma$. Then $\Omega^c$ is diffeomorphic to $\mathcal{M}_\infty$. By the definition of Cheeger–Gromov convergence, there exist an exhaustion $U_i$ of $\mathcal{M}_\infty$ and embeddings $\psi_i : U_i \to B_{A_i Q_i^{-1/2}}(x_i)$.

In particular, $\psi_i(\Omega)$ are embeddings of $\Omega$ with $\psi_i(\Omega) \subset B_{A_i Q_i^{-1/2}}(x_i)$ for each $i$ sufficiently large. So we may assume that the $\psi_i(\Omega)$ are disjoint from each other in the steady GRS $\mathcal{M}$.

By hypothesis, there exists a solution to the Ricci flow on a closed 4-manifold $(\mathcal{N}_4^4, h(t))$, $t \in [0, T)$, $T < \infty$, for which the steady GRS $(\mathcal{M}, g, f)$ is a singularity model. Hence there exist an exhaustion $\{V_j\}$ of $\mathcal{M}$ and embeddings $\phi_j : V_j \to \mathcal{N}$. In particular, for any $I \in \mathbb{N}$ there exists $j(I)$ such that $V_{j(I)}(I)$ contains $\psi_i(\Omega)$ for all $1 \leq i \leq I$. Then the $\Omega_i \equiv (\phi_{j(I)} \circ \psi_i)(\Omega)$ are disjoint embeddings into $\mathcal{N}$ for $1 \leq i \leq I$. Since the $\Omega_i$ are each diffeomorphic to $\mathcal{M}_\infty$ and are pairwise disjoint and since $I$ is arbitrary, we obtain a contradiction to Theorem 11 below.

### 2.5. Shrinking GRS singularity models

By a similar argument we can prove the following.

**Proposition 8.** If $(\mathcal{M}_4^4, g, f)$ is a shrinking GRS which is also a singularity model and $o \in \mathcal{M}$, then there exists a constant $C$ such that

$$|Rm|(x) \leq C(d(x, o) + 1)^2 \text{ for all } x \in \mathcal{M}.$$
Proof. Suppose there exists a sequence of points in the shrinking GRS \( \{y_i\}_{i=1}^{\infty} \) in \( \mathcal{M} \) with \( P_i(d(y_i, o) + 1)^{-2} \to \infty \), where \( P_i \doteq |\mathrm{Rm}|(y_i) \). By a previous proposition, there exists a sequence \( \{x_i\}_{i=1}^{\infty} \) in \( \mathcal{M} \) such that for each \( i \geq 1 \) we have \( x_i \in B_2/3(y_i) \), \( Q_i \doteq |\mathrm{Rm}|(x_i) \geq P_i \), and
\[
|\mathrm{Rm}| \leq 4Q_i \quad \text{in} \quad B_{A_iQ_i^{-1/2}}(x_i), \quad \text{where} \quad A_i = \frac{1}{3}P_i^{1/2} \to \infty.
\]
Let \( (\mathcal{M}^4, g(t), f(t)), \quad t \in (-\infty, 1) \), be the associated canonical form, where \( g(t) = (1-t)\varphi_+^*g \) and \( f(t) = f \circ \varphi_t \). Since \( \frac{\partial}{\partial t}\varphi_t(x) = \frac{1}{1-t} (\nabla_{g(0)}f(0)) (\varphi_t(x)) \), we have
\[
\left| \frac{\partial}{\partial t}\varphi_t(x) \right|_{g(0)} \leq \frac{1}{1-t} |\nabla g(0)| (\varphi_t(x)) \\
\leq \frac{1}{1-t} f^{1/2} (\varphi_t(x), 0) \\
\leq \frac{1}{1-t} (d_{g(0)}(\varphi_t(x), o) + C) \\
\leq d_{g(0)}(\varphi_t(x), o) + C
\]
for \( t \leq 0 \) since \( |\nabla (f^{1/2}(0))|_{g(0)} \leq 1 \) by \( R \geq 0 \). Hence, if \( x \in B_{\frac{1}{4}A_iQ_i^{-1/2}}(x_i) \) and \( t \in [-Q_i^{-1}, 0] \), then
\[
d_{g(0)}(\varphi_t(x), x) \leq \int_0^t \left| \frac{\partial}{\partial t}\varphi_t(x) \right|_{g(0)} dt \\
\leq \int_0^t (d_{g(0)}(\varphi_t(x), o) + C) dt \\
\leq CQ_i^{-1} (d_{g(0)}(x_i, o) + C) \\
\ll Q_i^{-1/2}
\]
since \( Q_i(d_{g(0)}(x_i, o) + 1)^{-2} \gtrsim P_i(d_{g(0)}(y_i, o) + 1)^{-2} \to \infty \). Thus \( \varphi_t(x) \in B_{\frac{1}{4}A_iQ_i^{-1/2}}(x_i) \) and hence \( |\mathrm{Rm}|(x_i, t) = |\mathrm{Rm}|(\varphi_t(x), 0) \leq 4Q_i \). Hence we may apply Shi’s local derivative estimates and Hamilton’s Cheeger–Gromov compactness theorem to obtain that there exists a subsequence such that \( (\mathcal{M}^4, Q_i g(Q_i^{-1}t), x_i) \) converges in the \( C^\infty \) pointed Cheeger–Gromov sense to a complete solution to the Ricci flow \( (\mathcal{M}^4_\infty, g_\infty(t), x_\infty) \), \( t \in [-\frac{1}{2}, 0] \), with \( |\mathrm{Rm}|_g \leq 4 \) and \( |\mathrm{Rm}|_g(x_\infty, 0) = 1 \). Since \( (\mathcal{M}^4, g, f) \) is a shrinking GRS, we have \( 0 < R \leq f \). This and \( Q_i f^{-1}(x_i) \to \infty \) imply that \( R_{g_\infty} \equiv 0 \), which in turn implies that \( R_{g_\infty} \equiv 0 \) and so \( g_\infty(0) \) is ALE. The remainder of the proof will be the same in the steady GRS case. \( \square \)

2.6 Shrinking GRS models with quadratic curvature growth. The following result is proved using similar methods.
Proposition 9. Let \((\mathcal{M}^4, g, f)\) be a shrinking GRS which is also a singularity model and let \(o \in \mathcal{M}\). If there exist a constant \(c > 0\) and a sequence \(y_i \to \infty\) such that
\[
|Rm| (y_i) \geq c (d(y_i, o) + 1)^2 \quad \text{for all} \; i,
\]
then there exists \(x_i \to \infty\) with associated rescalings limiting to a non-Ricci-flat steady GRS with bounded curvature.

Proof. Let \(\{x_i\}, Q_i\), and \(A_i\) be as in Proposition 4. By (5), we have that if \(x \in B_{A_i Q_i^{-1/2}}^g(x_i)\) and \(t \in [-Q_i^{-1}, 0]\), then
\[
d_g(\varphi_t(x), x) \leq C Q_i^{-1} \left( d_g(x_i, o) + C \right) \leq C Q_i^{-1/2}.
\]
Since \(A_i \to \infty\), we have \(\varphi_t(x) \in B_{A_i Q_i^{-1/2}}^g(x_i)\) for \(i\) sufficiently large.

As in the proof of Proposition 8, there exists a subsequence such that \((\mathcal{M}, Q_i g(Q_i^{-1} t), x_i)\) converges in the \(C^\infty\) pointed Cheeger–Gromov sense to a complete solution to the Ricci flow \((\mathcal{M}^4, g_\infty(t), x_\infty)\), \(t \in [-\frac{1}{2}, 0]\), with \(|Rm_{g_\infty}| \leq 4\) and \(|Rm_{g_\infty}|(x_\infty, 0) = 1\). Let \(g_i = Q_i g\) and let \(f_i(x) = f(x) - f(x_i)\). We have
\[
\Re_{g_i} + \nabla^2_{g_i} f_i = \frac{1}{2} Q_i^{-1} g_i.
\]
For any \(A > 0\) and for any \(x \in B_A^g(x_i)\), we have
\[
(6) \quad |\nabla f_i|^2_{g_i} + R_{g_i} = \inf \left\{ \frac{f(y)}{Q_i} : y \in B_A^g \frac{1}{4} A Q_i^{-1/2}(x_i) \right\} \geq \int \frac{1}{C (d_g(x_i, o) + 1)^2} \geq c > 0,
\]
for all \(i\) large enough, where \(c > 0\) is independent of \(A\) and where we have used the lower bound for the potential function of Cao and Zhou [6]. Furthermore, we have
\[
|\nabla f_i|_{g_i} (x) = Q_i^{-1/2} |\nabla f| (x) \leq Q_i^{-1/2} f^{1/2}(x) \leq C,
\]
\[
|\nabla^2_{g_i} f_i|_{g_i} = |\Re_{g_i} - \frac{1}{2} Q_i^{-1} g_i|_{g_i} \leq C,
\]
and
\[
|\nabla^k_{g_i} f_i|_{g_i} = |\nabla^k_{g_i} \Re_{g_i}|_{g_i} \leq C_k \quad \text{for} \; k \geq 3.
\]
Hence, by taking \(i \to \infty\) we obtain \(f_i \to f_\infty\) in \(C^\infty\) with respect to the Cheeger–Gromov convergence of \(g_i\) to \(g_\infty = g_\infty(0)\), where
\[
|\nabla_{g_\infty} f_\infty|^2_{g_\infty} + R_{g_\infty} = c_0 > 0,
\]
\[
\Re_{g_\infty} + \nabla^2_{g_\infty} f_\infty = 0.
\]
Notice that the left-hand-side of the first equation is constant because the steady GRS equation holds by the second equation; it follows from (6) that this constant is positive. According to our point selection procedure, the
limit steady GRS \((\mathcal{M}_4^4, g_\infty, f_\infty)\) is nonflat and has bounded curvature. To see that it is non-Ricci-flat, we assume the contrary and obtain
\[
|\nabla g_\infty f_\infty|^2_{g_\infty} = c_0 > 0,
\]
\[
\nabla^2 g_\infty f_\infty = 0.
\]
It follows immediately that \((\mathcal{M}_\infty^4, g_\infty)\) splits as the product of a line and a three-dimensional Ricci-flat manifold and hence \(g_\infty\) must be flat; this is a contradiction. □

One would like to show that steady GRS limits as in the proposition above are not possible, in which case it would follow for a 4-dimensional shrinking GRS singularity model that \(|\text{Rm}| = o(d^2)\) (this would imply finite topological type).

2.7. Steady GRS models with curvature not decaying.

**Proposition 10.** Let \((\mathcal{M}_4^4, g, f)\) be a steady GRS which is also a singularity model. If there exist a constant \(c > 0\) and a sequence \(y_i \to \infty\) such that
\[
|\text{Rm}|(y_i) \geq c \quad \text{for all } i,
\]
then there exists \(x_i \to \infty\) with associated rescalings limiting to a non-Ricci-flat steady GRS with bounded curvature.

**Proof.** Observe that we may assume that \((\mathcal{M}_4^4, g)\) is not Ricci flat, since otherwise there does not exist a sequence \(\{y_i\}\) as in the hypothesis by Cheeger and Naber’s aforementioned theorem. Since the singularity model has bounded curvature by Proposition 7, it follows from Shi’s local derivative estimates (or Lemma 5) that the covariant derivative of curvature of each order is uniformly bounded. Consider the sequence \(\{(\mathcal{M}, g, f_i, y_i)\}_{i=1}^\infty\), where \(f_i(x) = f(x) - f(y_i)\). We have \(|\nabla f_i| = |\nabla f| = \sqrt{1 - R}\) and \(|\nabla^2 f| = |\text{Rc}|\) are both uniformly bounded. By applying Shi’s Bernstein-type estimates and by the covariant derivatives of curvature bounds, it is not hard to show that the covariant derivatives of \(f_i\) of each order are bounded independent of \(i\). Taking into account the noncollapsing condition, we may extract a subsequence from \(\{(\mathcal{M}, g, f_i, y_i)\}_{i=1}^\infty\) which converges in the pointed smooth Cheeger–Gromov sense to \((\mathcal{M}_\infty^4, g_\infty, f_\infty, y_\infty)\). It then follows from the smooth convergence that
\[
\text{Rc}_\infty + \nabla^2 f_\infty = 0,
\]
\[
|\nabla f_\infty|^2 + R_\infty = 1.
\]
The limit is evidently a steady GRS. It is also nonflat since
\[
c \leq |\text{Rm}|(y_i) \to |\text{Rm}_\infty|(y_\infty) \geq c > 0.
\]
To see that it is not Ricci flat, we assume for a contradiction that \(g_\infty\) is Ricci flat. Then immediately we have
\[
|\nabla^2 f_\infty| = 0,
\]
\[
|\nabla f_\infty|^2 = 1.
\]
Hence $(\mathcal{M}_\infty, g_\infty)$ splits as the product of a line and a three-dimensional Ricci-flat manifold, which is flat; a contradiction.

3. Excluding an unbounded number of copies of a Ricci-flat ALE 4-manifold

3.1. Statement of the result. In this section we prove the result which we applied in the previous section to exclude there existing an unbounded number of disjoint copies of a Ricci-flat ALE 4-manifold in a closed 4-manifold.

**Theorem 11.** If $\mathcal{N}$ is a closed 4-manifold and if $\mathcal{A}$ is a Ricci-flat ALE 4-manifold, then there can exist at most a bounded number of disjoint copies of $\mathcal{A}$ embedded in $\mathcal{N}$.

3.2. Spherical space form ends of 4-manifolds having an unbounded number of disjoint embeddings. We first prove the following.

**Proposition 12.** Let $\mathcal{M}_0$ be a noncompact 4-manifold without boundary and with a single end diffeomorphic to $S^3/\Gamma \times [0, \infty)$, where $S^3/\Gamma$ is a spherical space form. If there exists a compact 4-manifold $\mathcal{N}$ containing an unbounded number of disjoint copies of $\mathcal{M}_0$, then $H_1(S^3/\Gamma, \mathbb{Z})$ is a direct double, i.e., isomorphic to $A \oplus A$ for some abelian group $A$.

Let $\mathcal{M}_0$ be as in the hypotheses of the proposition. Add $S^3/\Gamma$ to $\mathcal{M}_0$ as its boundary to obtain a compact manifold $\mathcal{M}$ with boundary $\partial \mathcal{M} = S^3/\Gamma \cong \partial$.

**Lemma 13.** Under the hypotheses of Proposition 12, the homomorphism $H_i(\partial; R) \to H_i(\mathcal{M}; R)$ induced by $\partial \hookrightarrow \mathcal{M}$ is onto for $i = 0, 1, 2, 3$ and any coefficient ring $R$. The cases $R \cong \mathbb{Z}$ and $R \cong \mathbb{Z}_p$ will be useful for us. We suppress writing the coefficients, $R$, in the proof.

**Proof.** This is clear for $i = 0$. Applying Mayer–Vietoris to one copy of $\mathcal{M}$ and to the complement of its interior: $\mathcal{N} - \mathcal{M}_0$, which have intersection $\partial$, we obtain

$$\cdots \to H_i(\partial) \xrightarrow{(i_1^*, i_2^*)} H_i(\mathcal{M}) \oplus H_i(\mathcal{N} - \mathcal{M}) \xrightarrow{j_1^* - j_2^*} H_i(\mathcal{N}) \xrightarrow{\partial} H_{i-1}(\partial) \to \cdots.$$ 

Suppose $i_1^* : H_i(\partial) \to H_i(\mathcal{M})$ is not onto. Then there exists $\alpha \in H_i(\mathcal{M})$ which is not in the image of $i_1^*$. Thus, for any $\beta \in H_i(\mathcal{N} - \mathcal{M}_0)$, for example $\beta = 0$, we have $j_1^*(\alpha) - j_2^*(\beta) \neq 0$. That is, the cokernel of $i_1^*$ injects in $H_i(\mathcal{N})$. Furthermore, replacing $\mathcal{M}$ by $\bigsqcup_{j=1}^I \mathcal{M}_j$ with $I$ increasing without bound in the preceding exact sequence, the same argument shows that $\bigoplus_j \text{coker}(i_{j,1})$ injects into $H_i(\mathcal{N})$. That is, there can be no relations between cokernels from disjoint copies of $\mathcal{M}$. We conclude that $H_i(\mathcal{N})$ is infinitely generated, which is a contradiction. \qed
By Lemma 13, the long exact sequence of the pair \((\mathcal{M}, \partial)\) yields the three short exact sequences. Setting \(R \cong \mathbb{Z}\), we have:

\[
\begin{align*}
(7) & \quad 0 \to H_2(\mathcal{M}, \partial; \mathbb{Z}) \to H_1(\partial; \mathbb{Z}) \to H_1(\mathcal{M}; \mathbb{Z}) \to 0, \\
(8) & \quad 0 \to H_3(\mathcal{M}, \partial; \mathbb{Z}) \to H_2(\partial; \mathbb{Z}) \to H_2(\mathcal{M}; \mathbb{Z}) \to 0, \\
(9) & \quad 0 \to H_4(\mathcal{M}, \partial; \mathbb{Z}) \to H_3(\partial; \mathbb{Z}) \to H_3(\mathcal{M}; \mathbb{Z}) \to 0.
\end{align*}
\]

Since \(\partial\) is a spherical space form, \(H_3(\partial; \mathbb{Z}) \cong H^1(\partial; \mathbb{Z}) \cong 0\). Thus (8) implies

\[
H_2(\mathcal{M}; \mathbb{Z}) \cong 0.
\]

Since \(H_1(\partial; \mathbb{Z})\) is torsion, so is \(H_1(\mathcal{M}; \mathbb{Z})\). We also have that \(H_3(\partial; \mathbb{Z}) \cong \mathbb{Z}\), but maps to zero in \(H_3(\mathcal{M}; \mathbb{Z})\). Thus \(H_3(\mathcal{M}; \mathbb{Z}) \cong 0\).

Applying (7) now with coefficients \(R = \mathbb{Z}_p\), we have

\[
0 \to H_2(\mathcal{M}, \partial; \mathbb{Z}_p) \to H_1(\partial; \mathbb{Z}_p) \to H_1(\mathcal{M}; \mathbb{Z}_p) \to 0.
\]

By Lefschetz duality, \(H_2(\mathcal{M}, \partial; \mathbb{Z}_p) \cong H^2(\mathcal{M}; \mathbb{Z}_p)\). By the cohomology universal coefficient theorem,

\[
0 \to \operatorname{Ext}^1_\mathbb{Z}(H_1(\mathcal{M}; \mathbb{Z}), \mathbb{Z}_p) \to H^2(\mathcal{M}; \mathbb{Z}_p) \to \operatorname{Hom}_\mathbb{Z}(H_2(\mathcal{M}; \mathbb{Z}), \mathbb{Z}_p) \cong 0 \to 0.
\]

We also have

\[
H^2(\mathcal{M}; \mathbb{Z}_p) \cong \operatorname{Ext}^1_\mathbb{Z}(H_1(\mathcal{M}; \mathbb{Z}), \mathbb{Z}_p) \cong \operatorname{Hom}_\mathbb{Z}(H_2(\mathcal{M}; \mathbb{Z}), \mathbb{Z}_p) \cong H_1(\mathcal{M}; \mathbb{Z}_p).
\]

(The final isomorphism is not functorial, but still suffices.)

These observations applied to (11) yield:

\[
H_1(\partial; \mathbb{Z}_p) = H_1(\mathcal{M}; \mathbb{Z}_p) \oplus H_1(\mathcal{M}; \mathbb{Z}_p).
\]

To bring this information back towards integer coefficients, apply the homology universal coefficient theorem, for \(X = \mathcal{M}\) and \(\partial\):

\[
0 \to H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p \to H_1(X; \mathbb{Z}_p) \to \operatorname{Tor}(H_0(X; \mathbb{Z}), \mathbb{Z}_p) \cong 0 \to 0
\]

to get

\[
H_1(\partial; \mathbb{Z}_p) \cong H_1(\mathcal{M}; \mathbb{Z}_p) \oplus H_1(\mathcal{M}; \mathbb{Z}_p) \otimes \mathbb{Z}_p \quad \text{for all primes } p.
\]

This still does not quite prove Proposition 12. We need to exclude possibilities like \(H_1(\partial; \mathbb{Z}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2\) and \(H_1(\mathcal{M}; \mathbb{Z}) \cong \mathbb{Z}_2\). This can be done by returning to line (7): order \(H_1(\partial; \mathbb{Z}) = \text{(order } H_2(\mathcal{M}, \partial; \mathbb{Z})) \times \text{(order } H_1(\mathcal{M}; \mathbb{Z}))\), but \(H_2(\mathcal{M}, \partial; \mathbb{Z}) \cong H^2(\mathcal{M}; \mathbb{Z}) \cong \operatorname{Ext}^1_\mathbb{Z}(H_1(\mathcal{M}; \mathbb{Z}), \mathbb{Z}) \cong \text{torsion } H_1(\mathcal{M}; \mathbb{Z}) \cong H_1(\partial; \mathbb{Z})\), so order \(H_1(\partial; \mathbb{Z}) = \text{(order } H_1(\mathcal{M}; \mathbb{Z}))^2\). Proposition 12 follows. \(\square\)

3.3. Proof of Theorem 11. We may now prove Theorem 11 using Bamber’s argument in §6 of [25]. For a contradiction, suppose that there exists a nonflat Ricci-flat ALE 4-manifold \(\mathcal{A}\) that admits unbounded collections of disjoint embeddings in a compact 4-manifold \(\mathcal{N}\). By Proposition 12, the spherical space form \(S^3/\Gamma\), \(\Gamma\) nontrivial, of the end of \(\mathcal{A}\) satisfies \(H_1(S^3/\Gamma; \mathbb{Z}) \cong \mathcal{A} \oplus \mathcal{A}\) for some abelian group \(\mathcal{A}\). This implies (see e.g. Lemma 6.3 in [25]) that either \(\Gamma\) is the binary dihedral group \(D_n^*\) with \(n\)
even or $\Gamma$ is the binary icosahedral group of order 120. Now, by Lemma 6.5 in [25], $b_2(A) \geq 1$, which contradicts (10). □

**Remark 14.** We may also rule out the case where $\Gamma$ is the binary icosahedral group $I^*$ of order 120, that is, where $\partial = S^3/I^*$ is the Poincaré homology sphere, as follows. By Rochlin’s theorem, if $\partial M = S^3/I^*$, then $H_*(M; \mathbb{Z}) \not\cong H_*(pt; \mathbb{Z})$. As in Lemma 13, the facts that $H_*(S^3/I^*; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ and $H_*(M; \mathbb{Z}) \not\cong H_*(pt; \mathbb{Z})$ imply, via Mayer–Vietoris, that $H_*(\mathcal{N}; \mathbb{Z})$ is infinitely generated, a contradiction.

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