Solving High-dimensional Linear Stochastic Partial Differential Equations via A Kernel-based Approximation Method

Qi Ye
Mathematics Department, Syracuse University, Syracuse, NY 13244
E-mail address: qiye@syr.edu

Abstract
In this paper, we improve and complete the theoretical results of the kernel-based approximation (collocation) method for solving the high-dimensional stochastic partial differential equations (SPDEs) given in our previous papers. According to the extended theorems, we can use more general positive definite kernels to construct the kernel-based estimators to approximate the numerical solutions of the SPDEs. Because a parabolic SPDE driven by Lévy noises can be discretized into several elliptic SPDEs by the implicit Euler scheme at time. We mainly focus on how to solve a system of elliptic SPDEs driven by various kinds of right-hand-side random noises. The kernel-based approximate solution of the elliptic SPDEs is a linear combination of the positive definite kernel with the differential and boundary operators of the SPDEs centered at the chosen collocation points, and its random coefficients are obtained by solving a system of random linear equations, whose random parts are simulated by the elliptic SPDEs. Moreover, we introduce the error bounds – confident intervals – of the kernel-based approximate solutions of the elliptic (parabolic) SPDEs in terms of fill distances (or possible time distances) in the probability sense. We also give a well coding algorithm to compute the kernel-based solutions of the second-order parabolic SPDEs driven by time and space Poisson noises. The two-dimensional numerical experiments show that the approximate probability distributions of the kernel-based solutions are well-behave for the Sobolev-spline kernels and the compact support kernels.

Keywords: Kernel-based approximation (collocation) method; meshfree approximation method; high dimension; stochastic partial differential equation; positive definite kernel; Gaussian field; Lévy noise; Poisson noise.
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1 Introduction

Recently, stochastic partial differential equations (SPDEs) provide a quantitative description for a lot of mathematical models in areas such as physics, engineering, biology, geography and finance. Many specialists exert a strong interest in SPDEs and develop their mathematical theories and analytical techniques. However, it is a difficult target to obtain the analytical solutions of SPDEs.
Thus, how to estimate the numerical solutions of SPDEs becomes a fast growing research area. A lot of modern numerical tools are devised to solve SPDEs such as the stochastic collocation method [2], the Itô Taylor expansions method [17] and the finite element method combining with Monte Carlo and Quasi-Mote Carlo method [18]. But there are still great many unsolved problems of the numerical solutions of SPDEs.

The books [4, 7, 25] show that the kernel-based approximation method to solve deterministic high-dimensional PDEs, and moreover, this method can be also applied into the stochastic models mentioned in [3, 24]. It offers us a new idea to apply the kernel-based approximation method (kernel-based collocation method, meshfree approximation method) to obtain numerical solutions of high-dimensional SPDEs given in our recent paper and doctoral thesis [5, 26]. Its approximate basis induced by the positive definite kernel (reproducing kernel) is different from the polynomial basis of the finite element method, which means that the construction of the kernel basis does not need an underlying triangular mesh. Furthermore, the data points can be flexibly chosen for the use of either deterministic or random design, e.g., Halton points or Sobol points.

After the discussions of this fresh numerical method with many mathematicians and engineers, we determine to renew and improve the analytical results and numerical algorithms of our previous papers by their nice and helpful suggestions. We want to let this numerical tool for SPDEs be well readable in the interdisciplinary fields of both computational mathematics and statistics. In Section 2, we extend [Theorem 3.1, 5] and [Theorem 7.2, 26] into Theorem 2.1 such that we can replace integral-type kernels by more general positive definite kernels to construct Gaussian fields in the classical $L_2$-based Sobolev spaces instead of the reproducing kernel Hilbert spaces. We will employ these Gaussian fields to introduce the kernel-based approximate solutions of SPDEs similar as the techniques of [5, 26]. Their approximate kernel bases are set up by the positive definite kernels with the related differential and boundary operators defined as in the equation (2.6), and the covariance matrixes of the Gaussian fields at the collocation points given in the formula (2.3) are used to compute their expansion random coefficients. The blocks of the covariance matrixes are corresponding to the pairwise of collocation points. The covariance matrixes can be seen as the generalization forms of the traditional kernel-based interpolation matrixes discussed in [7, 25].

Section 3 complements the construction processes and the proofs of the kernel-based approximation method, and we even give the new kernel-based approximate results for solving a system of high-dimensional linear elliptic SPDEs driven by various kinds of right-hand-side random noises, which are unsolved in the final remark section of [5]. The kernel-based approximate solution is obtained for fitting the observation values simulated by the elliptic SPDEs. The kernel-based approximate solution of the elliptic SPDEs is a linear combination of the kernel basis, and moreover, its expansion random coefficients are solved by a random linear system whose random parts are simulated by the elliptic SPDEs (see the equations (3.3, 3.4)). Proposition 3.1 shows that the errors of the kernel-based estimators can be bounded by the fill distances in the probability sense. The fill distance denotes the radius of the largest ball which is completely contained in the space domain and which does not contain the chosen collocation points. This means that the kernel-based approximate solutions are convergent to the exact solutions of the elliptic SPDEs in probabilities and distributions. We present more detail of the error analysis for the elliptic SPDEs than in [5], and we
discuss the convergent analysis by the knowledge of statistical learning – replacing the maximum error bound to the confident interval – in a different way of the classical kernel-based approximation method for the deterministic PDEs.

In Section 4, we use the implicit Euler scheme to discretize the parabolic SPDE driven by time and space Lévy noises at time in order to transform it to several elliptic SPDEs at each discretization time step. Next we solve these elliptic SPDEs by the kernel-based approximation method. We also briefly discuss the convergence of the kernel-based approximate solutions of the parabolic SPDEs undone in [5]. We will consider many other time-stepping schemes and their convergent rates in our future research. The numerical examples for the Sobolev-spline kernels and the compact support kernels show that the approximate probability distributions are well-behave for the second-order parabolic SPDEs driven by the time and space Poisson noises (see Figure 5.1). Since the covariance matrixes of the compact support kernels are sparse, we can solve the related linear systems as fast as the finite element method. More numerical examples will be posed in the author’s personal webpage.

In our next paper, we will use the extended theoretical results given in this article to set up the kernel-based estimators for the nonlinear SPDEs driven by Lévy noises.

Remark 1.1. In our previous papers and doctoral thesis of SPDEs, the kernel-based approximation method is also called the kernel-based collocation method. But some people may confuse the original name with another different method as stochastic collocation. In the same way as in the book [7] we recall this numerical method its general name, and moreover, its estimator is said the kernel-based approximate solution or the kernel-based solution in this article. The kernel-based approximation method, the kernel-based collocation method and the meshfree approximation method are the same in all our SPDEs papers.

1.1 Examples and Algorithms

We want to make a convenience for the engineers and the computer scientists to understand the kernel-based approximation method and avoid its technical details and proofs. In the beginning we give a traditional example of the parabolic SPDE (4.1) to explain the kernel-based approximate processes and algorithms in a simple way.

Let $\mathcal{D}$ be a regular bounded open domain of $\mathbb{R}^d$, and $\mathcal{H}^m(\mathcal{D})$ be a $L_2$-based Sobolev space of degree $m > d/2$. Suppose that $N_t$ is a Poisson noise in $\mathcal{H}^m(\mathcal{D})$ with the form

$$N_{t,x} := \sum_{k=1}^{n_l} N_{k,t}^n \alpha_k \phi_k(x), \quad x \in \mathcal{D}, \ t \geq 0,$$

where $\{\phi_k\}_{k=1}^{n_l} \subseteq \mathcal{H}^m(\mathcal{D})$ is an orthonormal subset of $L_2(\mathcal{D}), \{\alpha_k\}_{k=1}^{n_l}$ is a positive sequence and $N_{k,t}$ are the independent scalar Poisson processes with parameter $\lambda > 0$ for all $k = 1, \ldots, n_l$.

We consider the second-order parabolic SPDE driven by $N_t$

$$\begin{aligned}
\frac{dU_t}{dt} &= \Delta U_t dt + \Phi(U_t) dN_t, \quad \text{in } \mathcal{D}, \quad 0 < t < T, \\
U_t &= 0, \quad \text{on } \partial \mathcal{D}, \quad 0 < t < T, \\
U_0 &= u_0 \in \mathcal{H}^m(\mathcal{D}),
\end{aligned}$$
where $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is a Laplace differential operator and $\Phi \in C^2(\mathbb{R})$. Suppose that this parabolic SPDE is well-posed and its solution $U_t \in \mathcal{H}^m(\mathcal{D})$ almost surely such that $\int_0^T E\|U_t\|_{\mathcal{H}^m(\mathcal{D})}^2 < \infty$.

The proposed numerical method for solving the SPDE (1.1) can be described as follows:

(S1) Discretize the SPDE (1.1) in time by the implicit Euler scheme at equally space time points, i.e.,

$$U_t - U_{t-1} = \Delta U_t \delta t + \Phi(U_{t-1}) \delta N, \quad i = 1, \ldots, n,$$

where $\delta t := T/n$, $t_i := t_{i-1} + \delta t$ and $\delta N := N_t - N_{t-1} \sim N_{\delta t}$. Let $P_{\delta t} := I - \delta t \Delta$.

(S2) We choose a finite collection of predetermined pairwise distinct collocation points

$$X_{\mathcal{D}} := \{x_1, \ldots, x_N\} \subseteq \mathcal{D}, \quad X_{\partial\mathcal{D}} := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial\mathcal{D},$$

and a symmetric positive definite kernel $K \in C^{2m}(\overline{\mathcal{D}} \times \overline{\mathcal{D}})$ to construct the basis

$$k_{PB,X} := (P_{\delta t,2}K(\cdot, x_1), \ldots, P_{\delta t,2}K(\cdot, x_N), K(\cdot, x_{N+1}), \ldots, K(\cdot, x_{N+M}))^T$$

and the covariance matrix (interpolating matrix)

$$K_{PB,X} := \begin{pmatrix} \left(\sum_{z_1=1}^{N} \sum_{z_2=1}^{N} \delta_{z_1} K(x_k, x_{N+k}) z_1, \delta_{z_2} K(x_k, x_{N+k}) z_2 \right)_{z_1, z_2=1}^{N, N} & \left(\sum_{z_1=1}^{N} \sum_{z_2=1}^{N} \delta_{z_1} K(x_k, x_{N+k}) z_1, \delta_{z_2} K(x_k, x_{N+k}) z_2 \right)_{z_1, z_2=1}^{N, M} \\ \left(\sum_{z_1=1}^{N} \sum_{z_2=1}^{N} \delta_{z_1} K(x_k, x_{N+k}) z_1, \delta_{z_2} K(x_k, x_{N+k}) z_2 \right)_{z_1, z_2=1}^{M, N} & \left(\sum_{z_1=1}^{N} \sum_{z_2=1}^{N} \delta_{z_1} K(x_k, x_{N+k}) z_1, \delta_{z_2} K(x_k, x_{N+k}) z_2 \right)_{z_1, z_2=1}^{M, M} \end{pmatrix}.$$  \hspace{1cm} (1.4)

Here $P_{\delta t,1}$ and $P_{\delta t,2}$ mean that we differentiate the kernel function $K$ with respect to its first and second arguments, respectively, i.e., $P_{\delta t,1}P_{\delta t,2}K(x, y) := P_{\delta t,1}P_{\delta t,2}K(z_1, z_2)|_{z_1=x, z_2=y}$. We can also compute that $P_{\delta t,1}P_{\delta t,2}K(x, y) = K(x, y) - \delta t \Delta_1 K(x, y) - \delta t \Delta_2 K(x, y) + \delta^2 \Delta_1 \Delta_2 K(x, y)$ and $P_{\delta t,1}K(y, x) = P_{\delta t,2}K(x, y) = K(x, y) - \delta t \Delta_2 K(x, y)$.

(S3) Because the white noise increment $\delta N$ at each time instance $t_i$ is independent of the solution $U_{t_{i-1}}$. The noise term $\xi := \delta N \sim N_{\delta t}$ is well-defined and we can simulate $\xi$ at $X_{\mathcal{D}}$, i.e.,

$$\xi_{x_j} := \sum_{k=1}^{n_t} \xi_{k}\alpha_k \phi_k(x_j), \quad j = 1, \ldots, N, \quad \xi_k \sim \text{i.i.d. Poisson}(\lambda \delta t), \quad k = 1, \ldots, n_t.$$

Let $\Gamma(v, w) := v + w\Phi(v)$ for $v, w \in \mathbb{R}$. Combing the equation (1.2) with the Dirichlet boundary condition we obtain the elliptic SPDE

$$\begin{cases} \frac{\partial \bar{u}^i}{\partial t} = \Gamma(f^i, \xi), & \text{in } \mathcal{D}, \\ \bar{u}^i = 0, & \text{on } \partial\mathcal{D}, \end{cases}$$

\hspace{1cm} (1.5)

where $\bar{u}^i := U_t$ is seen as an unknown part, and $f^i := U_{t_{i-1}}$ and $\xi$ are viewed as given parts. The kernel-based solution of the SPDE (1.5) can be written as

$$\bar{u}^i(x) := \sum_{k=1}^{N} c_k p_{\delta t,2}K(x, x_k) + \sum_{k=1}^{M} b_k K(x, x_{N+k}), \quad x \in \mathcal{D},$$

and the covariance matrix (interpolating matrix)
and its random expansion coefficients \( c := (c_1, \cdots, c_N)^T \) and \( b := (b_1, \cdots, b_M)^T \) are computed by the random linear equation

\[
K_{PB,X} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} y^i_{\xi,X_D} \\ 0 \end{pmatrix} \in \mathbb{R}^{N+M},
\]

where \( y^i_{\xi,X_D} := \left( \Gamma(f^i(x_1), \xi_{x_1}), \cdots, \Gamma(f^i(x_N), \xi_{x_N}) \right)^T. \)

This means that \( U_{t,x_j} \) is approximated by \( \hat{u}^i(x_j) \) for all \( j = 1, \ldots, N. \)

(S4) Repeat (S3) for all \( i = 1, \ldots, n. \)

We can also create an algorithm to obtain the sample paths of the SPDE (1.1):

**INITIALIZE**

- \( k_{PB,X} \) and \( K_{PB,X} \) are given in the equations (1.3) and (1.4) for \( \delta t := T/n. \)
- \( \hat{u}^0 := (u_0(x_1), \cdots, u_0(x_N))^T \in \mathbb{R}^N. \)
- \( B_{PB,X} := (k_{PB,X}(x_1), \cdots, k_{PB,X}(x_N)) \in \mathbb{R}^{(N+M) \times N}. \)
- \( A_{PB,X} := B_{PB,X}^T K_{PB,X}^{-1} \in \mathbb{R}^{N \times (N+M)}. \)

**REPEAT** \( i = 1, \ldots, n \)

- Simulate \( \zeta_k \sim \text{i.i.d. Poisson}(\lambda \delta t) \) for all \( k = 1, \ldots, n_l. \)
- \( \xi_{x_j} := \sum_{k=1}^{n_l} \xi_k \alpha_k \phi_k(x_j) \) for all \( j = 1, \ldots, N. \)
- \( y^i_{\xi,X_D} := \left( \hat{u}^i_1 + \xi_{x_1} \Phi(\hat{u}^i_1), \cdots, \hat{u}^i_N + \xi_{x_N} \Phi(\hat{u}^i_N) \right)^T \in \mathbb{R}^N. \)
- \( \hat{u}^i := A_{PB,X} \begin{pmatrix} y^i_{\xi,X_D} \\ 0 \end{pmatrix} \in \mathbb{R}^N. \)

2 **Constructing Gaussian Fields by Positive Definite Kernels with Differential and Boundary Operators**

In this section we want to extend the theoretical results of the integral-type kernels and the related reproducing kernel Hilbert spaces given in [Lemma 2.2 and Theorem 3.1, [5]] and [Lemma 7.1 and Theorem 7.2, [26]] so that we can apply many various kinds of the positive definite kernels and the related Sobolev spaces to create Gaussian fields. These Gaussian fields and positive definite kernels are used to introduce the kernel-based solutions of SPDEs in the following sections.

**Definition 2.1** ([Definition 6.24, [25]]). A continuous symmetric kernel \( K : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R} \) is called **positive definite** on \( \mathcal{D} \subseteq \mathbb{R}^d \) if, for all \( N \in \mathbb{N} \) and all sets of pairwise distinct centers \( X_{\mathcal{D}} := \{x_1, \ldots, x_N\} \subseteq \mathcal{D}, \) the quadratic form

\[
\sum_{j=1}^N \sum_{k=1}^N c_j c_k K(x_j, x_k) = e^T K_{X_D} e > 0, \quad \text{for all } e \in \mathbb{R}^N \setminus \{0\},
\]
which is equivalent to the matrix \( K_{x_{ij}} := (K(x_i, x_j))_{i,j=1}^{N,N} \) is positive definite.

Let \( H^m(D) \) be the \( L_2 \)-based Sobolev space of degree \( m > d/2 \) defined as in Section \( A \) and \( \mathcal{B}(H^m(D)) \) be the Borel \( \sigma \)-algebra on \( H^m(D) \). Suppose that \( D \) is a regular bounded open domain of \( \mathbb{R}^d \) for \( d \in \mathbb{N} \), and the symmetric kernel function \( K \) is a positive definite kernel on \( D \). Further suppose that \( K \in C^{2m}(\overline{D} \times D) \). According to the discussions in Section \( B \), the kernel function \( K \) has the positive eigenvalues \( \{ \lambda_n \}_{n=1}^{\infty} \) and the continuous eigenfunctions \( \{ e_n \}_{n=1}^{\infty} \) such that \( \int_D K(x, y)e_n(y)dy = \lambda_n e_n(x) \) and it possesses the absolutely and uniformly convergent representation

\[
K(x, y) = \sum_{n=1}^{\infty} \lambda_n e_n(x)e_n(y), \quad x, y \in D.
\]

For simplifying the notations, we denote

\[
P_1P_2K(x, y) := \sum_{n=1}^{\infty} \lambda_n e_n(x)e_n(y), \quad B_1B_2K(x, y) := B_{\sum_{n=1}^{\infty} \lambda_n} e_n(x)e_n(y),
\]

for the linear differential and boundary operators \( P \) and \( B \) denoted in the equations (A.1-A.2). Here \( P_1, B_1 \) and \( P_2, B_2 \) mean that we differentiate the kernel function \( K \) with respect to its first and second arguments, respectively.

The Kolmogorov’s extension theorem guarantees the existence of countable independent standard normal random variables \( \{ \xi_n \}_{n=1}^{\infty} \) on some probability space \( (\Omega_0, \mathcal{F}_0, \mathbb{P}_0) \), i.e., \( \xi_n \sim \text{i.i.d.} \mathcal{N}(0, 1) \).

Since \( \sum_{k=1}^{N} \lambda_n |e_n(x)|^2 = K(x, x) < \infty \) for all \( x \in D \), the stochastic process

\[
\xi_x(\omega) := \sum_{n=1}^{\infty} \xi_n(\omega) \sqrt{\lambda_n} e_n(x), \quad \text{for all } x \in D \text{ and all } \omega \in \Omega_0,
\]

(2.1)
is well-defined on \( (\Omega_0, \mathcal{F}_0, \mathbb{P}_0) \). According to [Theorem A.19, (21)], the random variable \( \xi_x \) is normal for all \( x \in D \). We can also compute the mean \( E(\xi_x) = \sum_{n=1}^{\infty} E(\xi_n) \sqrt{\lambda_n} e_n(x) = 0 \) and the covariance \( \text{Cov}(\xi_x, \xi_y) = E(\xi_x \xi_y) = \sum_{n=1}^{\infty} \text{Var}(\xi_n) \lambda_n e_n(x)e_n(y) = K(x, y) \) for all \( x, y \in D \). Therefore \( \xi \) is a Gaussian field with mean 0 and covariance kernel \( K \) (see Definition C.1).

**Lemma 2.1.** Almost all sample paths of the Gaussian field \( \xi \) defined in the formula (2.1) belong to \( H^m(D) \), i.e., \( x \mapsto \xi_x(\omega) \in H^m(D) \) for almost all \( \omega \in \Omega_0 \).

**Proof.** Denote that \( \xi_n := \sum_{k=1}^{n} \xi_k \sqrt{\lambda_k} \). Since \( K \in C^{2m}(\overline{D} \times D) \), we have \( \lim_{n \to \infty} \xi_n = \xi \) in the \( L_2(\Omega_0, \mathcal{F}_0, \mathbb{P}_0; L_2(\mathcal{D})) \)-norm. We fix any \( 0 \leq |\alpha| \leq m \). According to Lemma B.1, the eigenfunctions \( \{ e_n \}_{n=1}^{\infty} \subseteq C^{2m}(\overline{D}) \subseteq H^m(\mathcal{D}) \) and the convergent representation \( D_1^\alpha D_2^\beta K(x, y) = \sum_{n=1}^{\infty} \lambda_n D_1^\alpha e_n(x)D_2^\beta e_n(y) \) is absolute and uniform on \( \overline{D} \times \mathcal{D} \). Since \( D \) is bounded, the map \( x \mapsto D_1^\alpha D_2^\beta K(x, x) \in L_1(D) \) and the representation \( D_1^\alpha D_2^\beta K(x, x) = \sum_{n=1}^{\infty} \lambda_n |D_1^\alpha e_n(x)|^2 \) is also convergent in the \( L_1(\mathcal{D}) \)-norm. Thus, the sequence \( \{ D_1^\alpha D_2^\beta e_n \}_{n=1}^{\infty} \) is a Cauchy sequence in the Hilbert space \( L_2(\Omega_0, \mathcal{F}_0, \mathbb{P}_0; L_2(D)) \) because

\[
\int_D E \left| D_1^\alpha e_n(x) - D_1^\beta e_n(x) \right|^2 dx = \int_D \sum_{k=1}^{n_1+n_2} \lambda_k \left| D_1^\alpha e_k(x) \right|^2 dx \to 0,
\]

when \( n_1, n_2 \to \infty \). This ensures that there exists a \( \xi_\alpha \in L_2(\Omega_0, \mathcal{F}_0, \mathbb{P}_0; L_2(D)) \) such that \( \lim_{n \to \infty} D_1^\alpha e_n = \xi_\alpha \) in the \( L_2(\Omega_0, \mathcal{F}_0, \mathbb{P}_0; L_2(D)) \)-norm. Combining the above results with Lemma A.1, we can conclude that \( \xi \in L_2(\Omega_0, \mathcal{F}_0, \mathbb{P}_0; H^m(D)) \) which indicates that \( x \mapsto \xi_x(\omega) \in H^m(D) \) for almost all \( \omega \in \Omega_0 \). \( \square \)
Let \( \mathcal{H}_{K,m} (\mathcal{D}) \) be the reproducing kernel Hilbert space of a Sobolev spline kernel \( K_{m,\theta} \) with degree \( m \) and shape parameter \( \theta = 1 \) defined in Example [B.1]. Since \( \mathcal{H}_{K,m} (\mathcal{D}) \) is equivalent to the Sobolev space \( \mathcal{H}^m (\mathcal{D}) \), the measurable spaces \( (\mathcal{H}_{K,m} (\mathcal{D}), \mathcal{B}(\mathcal{H}_{K,m} (\mathcal{D}))) = (\mathcal{H}^m (\mathcal{D}), \mathcal{B}(\mathcal{H}^m (\mathcal{D}))) \), where \( \mathcal{B}(\mathcal{H}_{K,m} (\mathcal{D})) \) and \( \mathcal{B}(\mathcal{H}^m (\mathcal{D})) \) are the Borel \( \sigma \)-algebras on \( \mathcal{H}_{K,m} (\mathcal{D}) \) and \( \mathcal{H}^m (\mathcal{D}) \). Thus \( x \mapsto \xi_x (\omega) \) belongs to \( \mathcal{H}_{K,m} (\mathcal{D}) \cong \mathcal{H}^m (\mathcal{D}) \) for almost all \( \omega \in \Omega_0 \).

**Lemma 2.2** (Generalization of [Lemma 2.2, [5]] and [Lemma 7.1, [26]]). Suppose that \( \mathcal{D} \) is a regular bounded open domain of \( \mathbb{R}^d \) and the symmetric positive definite kernel \( K \in \mathcal{C}^{2m} (\mathcal{D} \times \mathcal{D}) \) for \( m > d/2 \). Then, for any fixed \( \mu \in \mathcal{H}^m (\mathcal{D}) \), there exists a probability measure \( \mathbb{P}^\mu_K \) defined on the measurable space \( (\Omega_m, \mathcal{F}_m) := (\mathcal{H}^m (\mathcal{D}), \mathcal{B}(\mathcal{H}^m (\mathcal{D}))) \) such that the stochastic process

\[
S_x (\omega) := \omega(x), \quad \text{for all } x \in \mathcal{D} \text{ and all } \omega \in \Omega_m = \mathcal{H}^m (\mathcal{D}),
\]

is a Gaussian field with mean \( \mu \) and covariance kernel \( K \) on the probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K) \).

**Proof.** According to Lemma 2.1, almost all the sample paths of the Gaussian field \( \xi \) defined in the formula (2.1) belong to \( \mathcal{H}_{K,m} (\mathcal{D}) \cong \mathcal{H}^m (\mathcal{D}) \). Lemma C.1 provides that the probability measure induced by \( \xi \)

\[
\mathbb{P}_\xi (A) := \mathbb{P}_0 (\xi^{-1} (A)), \quad \text{for all } A \in \mathcal{B}(\mathcal{H}_{K,m} (\mathcal{D})) = \mathcal{B}(\mathcal{H}^m (\mathcal{D})) = \mathcal{F}_m,
\]

is well-defined on \( (\mathcal{H}_{K,m} (\mathcal{D}), \mathcal{B}(\mathcal{H}_{K,m} (\mathcal{D}))) = (\mathcal{H}^m (\mathcal{D}), \mathcal{B}(\mathcal{H}^m (\mathcal{D}))) = (\Omega_m, \mathcal{F}_m) \), and \( S \) has the same probability distribution as \( \xi \). This shows that \( S \) is a Gaussian field with mean \( \mu \) and covariance kernel \( K \) on the probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}_\xi) \).

Because of the fact that \( \mu \in \mathcal{H}^m (\mathcal{D}) \), we have \( \mu + \Omega_m = \Omega_m \) and \( \mu + A \in \mathcal{F}_m \) for all \( A \in \mathcal{F}_m \). We can set up the another probability measure \( \mathbb{P}^\mu_K \) by shifting \( \mathbb{P}_\xi \) for \( \mu \), i.e.,

\[
\mathbb{P}^\mu_K (A) := \mathbb{P}_\xi (A - \mu), \quad \text{for all } A \in \mathcal{F}_m.
\]

Since \( \mathbb{E} (\mu(x) + S_x ) = \mu(x) \) and \( \text{Cov}(\mu(x) + S_x , \mu(y) + S_y ) = \text{Cov}(S_x , S_y ) = K(x, y) \) on \( (\Omega_m, \mathcal{F}_m, \mathbb{P}_\xi) \) for \( x, y \in \mathcal{D} \), the stochastic process \( S \) is a Gaussian field with mean \( \mu \) and covariance kernel \( K \) on the probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K) \). \( \square \)

Using Lemma 2.2, we can set up many other kinds of Gaussian fields with respect to differential and boundary operators on the above probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K) \).

**Theorem 2.1** (Generalization of [Theorem 3.1, [5]] and [Theorem 7.2, [26]]). Suppose that \( \mathcal{D} \) is a regular bounded open domain of \( \mathbb{R}^d \) and the symmetric positive definite kernel \( K \in \mathcal{C}^{2m} (\mathcal{D} \times \mathcal{D}) \) for \( m > d/2 \). Let \( P \) and \( B \) be the linear differential and boundary operators of orders \( O(P) \leq m \) and \( O(B) \leq m - 1 \), respectively, (see the equations (A.1) (A.2)). Then, for any fixed \( \mu \in \mathcal{C}^m (\mathcal{D}) \), there exists a probability measure \( \mathbb{P}^\mu_K \) (independent of \( P \) and \( B \)) defined on the measurable space \( (\Omega_m, \mathcal{F}_m) := (\mathcal{H}^m (\mathcal{D}), \mathcal{B}(\mathcal{H}^m (\mathcal{D}))) \) such that the stochastic processes

\[
PS_x (\omega) := P \omega(x), \quad \text{for all } x \in \mathcal{D} \text{ and all } \omega \in \Omega_m = \mathcal{H}^m (\mathcal{D}),
\]

\[
BS_x (\omega) := B \omega(x), \quad \text{for all } x \in \partial \mathcal{D} \text{ and all } \omega \in \Omega_m = \mathcal{H}^m (\mathcal{D}),
\]

are the Gaussian fields with means \( P \mu \), \( B \mu \) and covariance kernels \( P_1 P_2 K, B_1 B_2 K \) on the probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K) \), respectively.
Proof. Denote that \( \tilde{S} := S - \mu \) where \( S \) is the Gaussian field given in Lemma 2.2. Because \( PS = P\mu + P\tilde{S} \) and \( BS = B\mu + B\tilde{S} \). If we can verify that \( P\tilde{S} \) and \( B\tilde{S} \) are the Gaussian fields with means 0 and covariance kernels \( \hat{P}\hat{K} \) and \( \hat{B}\hat{K} \), then the proof is completed.

Since \( \tilde{S} \) is a Gaussian field with mean 0 and covariance kernel \( K \) defined on the probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}_m^{K}) \), the Karhunen representation theorem provides that \( \tilde{S} = \sum_{n=1}^{\infty} \zeta_n \sqrt{\lambda_n} e_n \), where the random variables \( \zeta_n \sim \text{i.i.d. } N(0, 1) \) on \( (\Omega_m, \mathcal{F}_m, \mathbb{P}_m^{K}) \). According to Lemma A.1, \( P\tilde{S} \) and \( B\tilde{S} \) have the representations \( P\tilde{S} = \sum_{n=1}^{\infty} \zeta_n \sqrt{\lambda_n} Pe_n \) and \( B\tilde{S} = \sum_{n=1}^{\infty} \zeta_n \sqrt{\lambda_n} Be_n \). Combining the expansions of \( P\tilde{S} \) and \( B\tilde{S} \) with Lemma B.1 we can solve the means and covariance kernels of \( P\tilde{S} \) and \( B\tilde{S} \), i.e.,

\[
E(P\tilde{S}_x) = \sum_{n=1}^{\infty} E(\zeta_n) \sqrt{\lambda_n} Pe_n(x) = 0, \quad E(B\tilde{S}_x) = \sum_{n=1}^{\infty} E(\zeta_n) \sqrt{\lambda_n} Be_n(x) = 0
\]

and

\[
\text{Cov}(P\tilde{S}_x, P\tilde{S}_y) = \text{Var}(P\tilde{S}_x) = \sum_{n=1}^{\infty} \text{Var}(\zeta_n) \lambda_n Pe_n(x)Pe_n(y) = P_1 P_2 K(x, y),
\]

\[
\text{Cov}(B\tilde{S}_x, B\tilde{S}_y) = \text{Var}(B\tilde{S}_x) = \sum_{n=1}^{\infty} \text{Var}(\zeta_n) \lambda_n Be_n(x)Be_n(y) = B_1 B_2 K(x, y).
\]

Since \( K \in C^{2m}(\partial D \times \partial D) \), we have \( P_1 P_2 K \in C(\partial D \times \partial D) \) and \( B_1 B_2 K \in C(\partial D \times \partial D) \). According to the above deductions, we can conclude that \( P\tilde{S} \) and \( B\tilde{S} \) are Gaussian with means 0 and covariance kernels \( P_1 P_2 K \) and \( B_1 B_2 K \), respectively. \( \square \)

Remark 2.1. The construction of Gaussian fields in Theorem 2.1 is analogous to the form of Wiener measure defined on the measurable space \( (C[0, \infty), \mathcal{B}(C[0, \infty])) \), called canonical space, such that the coordinate mapping process \( W(\omega) = \omega(t) \) is a Brownian motion. Why do we need these kinds of Gaussian fields? Because they help us to produce the normal random variables associated to the differential and boundary operators of SPDEs and the given collocation points located in the space domain and its boundary. Using their joint probability density functions and conditional probability density functions, we can obtain the kernel-based solutions to fit the observation values which are simulated by the SPDEs.

We choose any linear differential and boundary operators \( P_1, P_2 \) and \( B_1, B_2 \) of orders \( O(P_1) \leq m, O(P_2) \leq m \) and \( O(B_1) \leq m-1, O(B_2) \leq m-1 \), respectively. In the same manner of the proof of Theorem 2.1, we can compute their covariances: \( \text{Cov}(P_1 S_x, P_2 S_y) = P_{1,1} P_{2,2} K(x, y), \) \( \text{Cov}(B_1 S_x, B_2 S_y) = B_{1,1} B_{2,2} K(x, y) \) and \( \text{Cov}(P_1 S_x, B_1 S_y) = P_{1,1} B_{1,2} K(x, y) \).

Next we consider the vector linear differential and boundary operators

\[
P := (P_1, \cdots, P_{n_p})^T, \quad B := (B_1, \cdots, B_{n_b})^T,
\]

composed of the finite linear differential and boundary operators of the orders \( O(P) \leq m \) and \( O(B) \leq m-1 \), respectively, where their orders are denoted by

\[
O(P) := \max \{ O(P_1), \cdots, O(P_{n_p}) \}, \quad O(B) := \max \{ O(B_1), \cdots, O(B_{n_b}) \}
\]
Given the pairwise distinct collocation points

\[ X_D := \{x_1, \ldots, x_N\} \subseteq \mathcal{D}, \quad X_{\partial\mathcal{D}} := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial \mathcal{D}, \quad X := X_D \cup X_{\partial\mathcal{D}}, \]

we can use the Gaussian fields \( P_1, \ldots, P_{n_P} S \) and \( B_1, \ldots, B_{n_B} S \) to create the related multi-normal vector

\[
S_{PB,X} := \begin{pmatrix} PS_{X_D} \\ BS_{X_{\partial\mathcal{D}}} \end{pmatrix}
\]

(2.2)
on \((\Omega_m, \mathcal{F}_m, \mathbb{P}_K^\mu)\) by Theorem 2.1, where

\[
PS_{X_D} := \left( P_1 S_{x_1}, \ldots, P_{n_P} S_{x_1}, \ldots, P_1 S_{x_N}, \ldots, P_{n_P} S_{x_N} \right)^T,
\]

\[
BS_{X_{\partial\mathcal{D}}} := \left( B_1 S_{x_{N+1}}, \ldots, B_{n_B} S_{x_{N+1}}, \ldots, B_1 S_{x_{N+M}}, \ldots, B_{n_B} S_{x_{N+M}} \right)^T.
\]

The mean and covariance matrix of \( S_{PB,X} \) can be computed by the same method as above.

**Corollary 2.1.** The multi-normal vector \( S_{PB,X} \) given in the equation (2.2) defined on the probability space \((\Omega_m, \mathcal{F}_m, \mathbb{P}_K^\mu)\) has the mean

\[
\mu_{PB,X} := \left( P_1 \mu(x_1), \ldots, P_{n_P} \mu(x_1), \ldots, B_1 \mu(x_{N+M}), \ldots, B_{n_B} \mu(x_{N+M}) \right)^T,
\]

and the covariance matrix

\[
K_{PB,X} := \begin{pmatrix}
(K_{PP}(x_{k_1}, x_{k_2}))_{k_1,k_2=1}^{N,M} & (K_{PB}(x_{k_1}, x_{N+k_2}))_{k_1,k_2=1}^{N,M} \\
(K_{BB}(x_{N+k_1}, x_{k_2}))_{k_1,k_2=1}^{N,M} & (K_{BB}(x_{N+k_1}, x_{N+k_2}))_{k_1,k_2=1}^{N,M}
\end{pmatrix},
\]

(2.3)

where

\[
K_{PP}(x_{k_1}, x_{k_2}) := \left( P_{j_1,1} P_{j_2,2} K(x_{k_1}, x_{k_2}) \right)_{j_1,j_2=1}^{n_P,n_P},
\]

\[
K_{BB}(x_{N+k_1}, x_{N+k_2}) := \left( B_{j_1,1} B_{j_2,2} K(x_{N+k_1}, x_{N+k_2}) \right)_{j_1,j_2=1}^{n_B,n_B},
\]

\[
K_{PB}(x_{k_1}, x_{N+k_2}) = K_{BP}(x_{N+k_1}, x_{k_2})^T := \left( P_{j_1,1} B_{j_2,2} K(x_{k_1}, x_{N+k_2}) \right)_{j_1,j_2=1}^{n_P,n_B}.
\]

We let the linear operator \( L \) be a differential operator of order \( O(P) \leq m \) or a boundary operator \( B \) of order \( O(B) \leq m - 1 \), i.e., \( L := P \) or \( L := B \). We fix any data point \( x \in \mathcal{D} \) or \( x \in \partial \mathcal{D} \) corresponding to the operator \( L \), i.e., when \( L = P \) then \( x \in \mathcal{D} \), or when \( L = B \) then \( x \in \partial \mathcal{D} \). Since Theorem 2.1 shows that the random variable \( L_X \) and the random vector \( S_{PB,X} \) are both normal on \((\Omega_m, \mathcal{F}_m, \mathbb{P}_K^\mu)\), the conditional probability density function \( p_{LS,X|SPB,X}^{\mu}(\cdot) \) of \( L_X \) given \( S_{PB,X} \) has the explicit form

\[
p_{LS,X|SPB,X}^{\mu}(v) = p_{LS,X,SPB,X}^{\mu}(v) / p_{SPB,X}^{\mu}(v),
\]

where \( p_{LS,X,SPB,X}^{\mu} \) and \( p_{SPB,X}^{\mu} \) are the joint probability density functions of \((L_X, S_{PB,X})\) and \( S_{PB,X} \).

**Corollary 2.2.** The conditional probability density function of the random variable \( L_X \) given the random vector \( S_{PB,X} \) defined on the probability space \((\Omega_m, \mathcal{F}_m, \mathbb{P}_K^\mu)\) (discussed as in the above paragraph) has the form

\[
p_{LS,X|SPB,X}^{\mu}(v) := \frac{1}{\sigma_{L,SPB,X}(x) \sqrt{2\pi}} \exp \left( -\frac{(v - m_{L,SPB,X}^{\mu}(x))^2}{2\sigma_{L,SPB,X}(x)^2} \right),
\]

(2.4)
for \( v \in \mathbb{R} \) and \( v \in \mathbb{R}^{n_p N + n_0 M} \), with the mean

\[
m_{L,x, PB,X}^\mu(v) := L\mu(x) + k_{L, PB,X}(x)^T K_{PB,X}^\dagger (v - \mu_{PB,X})
\]

and the variance

\[
\sigma_{L, PB,X}^2(x) := L_1 L_2 K(x, x) - k_{L, PB,X}(x)^T K_{PB,X}^\dagger k_{L, PB,X}(x), \tag{2.5}
\]

where \( k_{L, PB,X}(x) := L_2 k_{PB,X}(z)|_{z=x} \) and

\[
k_{PB,X}(x) := \left( P_{1,2} K(x, x_1), \cdots, P_{n_p, 2} K(x, x_1), \cdots, B_{1,2} K(x, x_{N+M}), \cdots, B_{n_b, 2} K(x, x_{N+M}) \right)^T. \tag{2.6}
\]

(Here \( \mu_{PB,X} \) and \( K_{PB,X} \) are the mean and covariance matrix of \( S_{PB,X} \) solved in Corollary 2.1.) In particular, for the given observation values \( \hat{v} \in \mathbb{R}^{n_p N + n_0 M} \), the probability density function of \( L S_x \)
given \( S_{PB,X} = \hat{v} \) is equal to \( p_{L S_x | S_{PB,X}}^{\mu}(\cdot | \hat{v}) \).

Since the covariance matrix \( K_{PB,X} \) is always semi-positive definite, its pseudo-inverse \( K_{PB,X}^\dagger \) is well-behaved. We observe that \( k_{PB,X} \) is the kernel basis of kernel-based solutions, and the variance function \( \sigma_{L, PB,X}^2 \), which is equal to the power function, is used to estimate the error bound.

According to Theorem 2.1 and Fubini’s Theorem we can use the Gaussian field \( S \) defined on \((\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K)\) to obtain that

\[
E \left( \int_{\mathcal{D}} \left| D^\alpha S_x - D^\alpha \mu(x) \right|^2 \, dx \right) = \int_{\mathcal{D}} E \left| D^\alpha S_x - D^\alpha \mu(x) \right|^2 \, dx = \int_{\mathcal{D}} D^\alpha_1 D^\alpha_2 K(x, x) \, dx,
\]

for all \( 0 \leq |\alpha| \leq m \) which indicates that

\[
E \left( \| S - \mu \|_{H^m(\mathcal{D})}^2 \right) = \sum_{|\alpha| \leq m} \int_{\mathcal{D}} D^\alpha_1 D^\alpha_2 K(x, x) \, dx.
\]

The Markov’s inequality provides that

\[
\mathbb{P}^\mu_K \left( \| S - \mu \|_{H^m(\mathcal{D})} \geq r \right) \leq \frac{E \left( \| S - \mu \|_{H^m(\mathcal{D})}^2 \right)}{r^2}.
\]

According to the construction of the Gaussian field \( S \) defined on \( \Omega_m = H^m(\mathcal{D}) \), we can introduce the following corollary.

Corollary 2.3. Suppose that the probability space \((\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K)\) defined same as in Lemma 2.2. For any \( r > 0 \), the subsets

\[
\mathcal{N}_r^\mu := \{ \omega \in \Omega_m = H^m(\mathcal{D}) : \| \omega - \mu \|_{H^m(\mathcal{D})} \geq r \},
\]

have the probabilities

\[
\mathbb{P}^\mu_K \left( \mathcal{N}_r^\mu \right) \leq \frac{1}{r^2} \sum_{|\alpha| \leq m} \int_{\mathcal{D}} D^\alpha_1 D^\alpha_2 K(x, x) \, dx.
\]
3 Solving Elliptic Stochastic Partial Differential Equations

In this section, we employ the similar techniques as in [12, 13, 26] to introduce the kernel-based solutions of the elliptic SPDEs and their convergent rates.

Let \( D \) be regular bounded open domain of \( \mathbb{R}^d \), \( m > d/2 \) and the vector linear differential and boundary operators \( P := (P_1, \ldots, P_{n_p})^T \) and \( B := (B_1, \ldots, B_{n_p})^T \) of the orders \( O(P) \leq m \) and \( O(B) \leq m-1 \), whose elements defined as in the equations \( A.1 \) with the coefficients \( c_\alpha \in C^{2(m-\alpha)}(\Omega) \) and \( b_\alpha \in C^{2(m-\alpha)}(\partial \Omega) \) are linear independent. Denote that \( \Gamma_1, \ldots, \Gamma_{n_p} \in C^2(\mathbb{R}^2) \) such that \( \Gamma_j \) is bijective for any fixed \( v \in \mathbb{R} \) and \( 1 \leq j_p \leq n_p \). Suppose that the vector noise \( \xi := (\xi_1, \ldots, \xi_{n_p}) \) such that \( \xi_j \) are independent and \( \xi_j, \ldots, \xi_{n_p} \) are linear independent. Denote that \( \Omega \) is the probability space \( \Omega = (\mathbb{R}^d, \mathcal{F}_w, P_w) \) for \( j_p = 1, \ldots, n_p \).

We consider a system of elliptic SPDEs driven by \( \xi \) as follow:

\[
\begin{align*}
P_{j} u &= \Gamma_j (f_j, \xi_j), \quad \ldots, \quad P_{n_p} u &= \Gamma_{n_p} (f_{n_p}, \xi_{n_p}), \quad \text{in } D, \\
B_1 u &= g_1, \quad \ldots, \quad B_{n_b} u &= g_{n_b}, \quad \text{on } \partial D,
\end{align*}
\]

where \( f_j, \ldots, f_{n_p}, g_1, \ldots, g_{n_b} \) are the deterministic functions for \( j_p = 1, \ldots, n_p \) and \( j_b = 1, \ldots, n_b \).

Suppose that the SPDE (3.1) is well-posed for its differential and boundary operators. Thus, when \( P_{j} \bar{u}, \ldots, P_{n_p} \bar{u}, B_1 \bar{u}, \ldots, B_{n_b} \bar{u} \) are convergent to the left-hand sides of the SPDE (3.1), then the estimator \( \bar{u} \) is convergent to the exact solution \( u \) of the SPDE (3.1) in the same rate, e.g., the maximum principle for the Laplace differential operator gives

\[
||u - \bar{u}||_{L^2(\Omega)} \leq C ||u - \Delta \bar{u}||_{L^2(\Omega)} + ||u - \bar{u}||_{L^2(\partial \Omega)}
\]

for the heat SPDEs with Dirichlet boundary conditions. (Here the notation means that \( f = O(g) \) if there is a positive constant \( C \) such that \( |f| \leq C |g| \).) Further suppose that the solution \( u \) belongs to the Sobolev space \( H^m(\Omega) \) almost surely.

We choose the sets of pairwise distinct collocation points \( X := X_D \cup X_{\partial D} \) from the domain \( D \) and its boundary \( \partial D \), i.e.,

\[
X_D := \{x_1, \ldots, x_N\} \subseteq D, \quad X_{\partial D} := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial D.
\]

The fill distance of \( X \) for \( D \) and \( \partial D \) is denoted by

\[
h_X := \max \{h_{X_D, D}, h_{X_{\partial D}, \partial D}\},
\]

where

\[
h_{X_D, D} := \sup_{x \in X_D} \min_{k=1, \ldots, N} ||x - x_k||_2, \quad h_{X_{\partial D}, \partial D} := \sup_{x \in X_{\partial D}} \min_{k=1, \ldots, M} ||x - x_{N+k}||_2.
\]

Let \( p_{\xi_j, x_j} : \mathbb{R}^N \to [0, \infty) \) be the joint probability density functions of the random vectors \( \xi_{j_p, x_j} := (\xi_{j_p, x_1}, \ldots, \xi_{j_p, x_N}) \) for all \( j_p = 1, \ldots, n_p \). Same as the techniques in Section 3, we can be simulated \( \xi_{1, x_1}, \ldots, \xi_{n_p, x_{n_p}} \) by their joint probability density functions \( p_{\xi_1, x_1}, \ldots, p_{\xi_{n_p, x_{n_p}}} \). Since \( \xi_{1, x_1}, \ldots, \xi_{n_p, x_{n_p}} \) are independent and \( \Gamma_j (\varphi(x_k), \cdot) \) is a bijective function for any fixed \( 1 \leq k \leq N \)
and $1 \leq j_p \leq n_p$, we can use $p_{\xi_1,x_D}, \ldots, p_{\xi_{np},x_D}$ to obtain the joint probability density function of $
abla(f(x_1), \xi_{x_1}), \ldots, \nabla(f(x_1), \xi_{x_1}), \ldots, \nabla(f(x_N), \xi_{x_N}), \ldots, \nabla(p_{j_p}(f(x_N), \xi_{x_N})$.

Next we find an appropriate positive definite kernel $K : D \times D \to \mathbb{R}$ to introduce the kernel-based solutions of the SPDEs (3.1). The covariance matrix $K_{PB,X}$ given in the equation (2.3) (see Corollary 2.1) will be used to evaluate the coefficients of the kernel-based solutions later. Because we need $K_{PB,X}$ to be nonsingular. We suppose that the symmetric positive definite kernel $K \in C^{2m}(D \times D)$ satisfies the condition
\[
\left\{ \delta_{x_p} \circ P_{j_p}, \delta_{x_b} \circ B_{j_b} : x_p \in D, x_b \in \partial D \right\}_{j_p, j_b=1}^{n_p, n_b}
\]
are linearly independent over $\mathcal{H}_K(D)$, (3.2)

where $\delta_x$ is the point evaluation functional at $x$. According to [Theorem 16.8, (25)], the condition (3.2) ensures that all covariance kernels $P_{j_p}K$ and $B_{j_b}K$ are positive definite on $D$ and $\partial D$, respectively, for $j_p = 1, \ldots, n_p$ and $j_b = 1, \ldots, n_b$. The linearly independent condition (3.2) even indicates that the covariance matrix $K_{PB,X}$ is positive definite. So the inverse of $K_{PB,X}$ exists and $\tilde{K}_{PB,X} = K_{PB,X}^{-1}$.

Remark 3.1. [10, 11, 26] show that that $\mathcal{H}_K(D)$ can contain enough more polynomials. One technique to verify the condition (3.2) is to find a polynomial $q_X \in \mathcal{H}_K(D)$ such that $\sum_{j_p=1}^{n_p} \sum_{j_b=1}^{n_b} c_{j_p} P_{j_p} q_X(x_k)+ \sum_{k=1}^{M} \sum_{j=1}^{n_p} b_{jk} B_{j_b} q_X(x_{N+k}) = 0$ if and only if $c_{1,1} = \cdots = c_{n_p,N} = b_{1,1} = \cdots = b_{n_b,M} = 0$ for any finite pairwise distinct collocation points $X$. As a special case, the kernel function $K$ has the form $K(x, y) = G(x - y)$ for a positive definite function $G \in C^{2m}(\mathbb{R}^d)$ and all coefficients of the differential and boundary operators of the SPDE (3.1) are scalars, then [Corollary 16.12, (25)] provides that the condition (3.2) is always true.

The main reason of the condition (3.2) is to let the covariance matrix $K_{PB,X}$ be always nonsingular so that the system of linear equations (3.4) is uniquely solvable for any choice of collocation points $X$. This means that the condition (3.2) can be replaced to choose well-distributed collocation points $X$ dependent of the differential and boundary operators of the SPDE (3.4) such that $K_{PB,X}$ is nonsingular. Actually, we may not need the condition (3.2) to obtain the kernel-based solutions because we could solve the linear system (3.4) by the least square method for the semi-positive definite matrix $K_{PB,X}$. Because we want to employ the theorems of the power functions given in [7, 25] to introduce the convergent rate directly. In this paper we always assume that the kernel function $K$ satisfies the condition (3.2) in order to avoid the technical discussions and reproof of the similar theorems in [7, 25]. But we can still introduce the similar convergent rate as the techniques of the proof of [Theorem 14.4 and 14.5, (7)] without the condition (3.2).

Since $K \in C^{2m}(D \times D)$, [Theorem 10.45, (25)] shows that the reproducing kernel Hilbert space $\mathcal{H}_K(D) \subseteq C^{2m}(\mathbb{R}^d)$. Using Theorem 2.1, for any fixed $\mu \in \mathcal{H}_K(D)$, we can create a probability measure $\mathbb{P}_K^\mu$ on the measurable space $(\Omega_m, \mathcal{F}_m) = (\mathcal{H}_m(D), \mathcal{B}((\mathcal{H}_m(D)))$ such that the stochastic processes
\[
\begin{align*}
P_{j_p} S_{X_p}(\omega_1) = P_{j_p} \omega_1(x) \quad \text{and} \quad B_{j_b} S_{X_b}(\omega_1) = B_{j_b} \omega_1(x), \quad x_p \in D, x_b \in \partial D, \quad \omega_1 \in \Omega_m,
\end{align*}
\]
are the Gaussian fields with means $P_{j_p} \mu, B_{j_b} \mu$ and covariance kernels $P_{j_p} K, B_{j_b} K$ defined on the probability space $(\Omega_m, \mathcal{F}_m, \mathbb{P}_K^\mu)$, for $j_p = 1, \ldots, n_p$ and $j_b = 1, \ldots, n_b$. 

Since we have two different kinds of probability spaces \((\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K)\) and \((\Omega_w, \mathcal{F}_w, \mathbb{P}_w)\), it is necessary for us to combine them into a new product probability space. We define the tensor product probability space

\[
\Omega_{mw} := \Omega_m \times \Omega_w, \quad \mathcal{F}_{mw} := \mathcal{F}_m \otimes \mathcal{F}_w, \quad \mathbb{P}_K^{\mu} := \mathbb{P}^\mu_K \otimes \mathbb{P}_w,
\]

and let all the original random variables be extended in the natural way: if the random variables \(V_1 : \Omega_m \to \mathbb{R}\) and \(V_2 : \Omega_w \to \mathbb{R}\) are defined on \((\Omega_m, \mathcal{F}_m, \mathbb{P}^\mu_K)\) and \((\Omega_w, \mathcal{F}_w, \mathbb{P}_w)\), respectively, then their extensions

\[
V_1(\omega_1 \times \omega_2) := V_1(\omega_1) \quad \text{and} \quad V_2(\omega_1 \times \omega_2) := V_2(\omega_2), \quad \text{for} \ \omega_1 \in \Omega_m \quad \text{and} \quad \omega_2 \in \Omega_w.
\]

The extensions \(V_1\) and \(V_2\) preserve the original probability distributions, and they are also independent on the product probability space \((\Omega_{mw}, \mathcal{F}_{mw}, \mathbb{P}_K^{\mu})\). This means that the Gaussian fields induced by the chosen positive definite kernels and the noise terms of SPDEs can be extended to the product probability space while preserving the original probability distributional properties, and moreover, their extensions are independent, e.g., the extensions of the Gaussian field \(S(\omega_1 \times \omega_2) = S(\omega_1)\) and the noise \(\xi(\omega_1 \times \omega_2) = \xi(\omega_2)\) for \(\omega_1 \times \omega_2 \in \Omega_{mw}\) are independent on \((\Omega_{mw}, \mathcal{F}_{mw}, \mathbb{P}_K^{\mu})\). In addition, since the solution \(u\) of SPDEs can be seen as a mapping from \(\Omega_w\) into \(\mathcal{H}^m(D) = \Omega_m\), we have \(u(\cdot, \omega_2) \in \Omega_{mw}\) for all \(\omega_2 \in \Omega_w\).

For any \(\omega_1 \in \mathcal{H}^m(D)\), we let

\[
\mathbf{P}_{\omega_1}(X_D) := (P_1 \omega_1(x_1), \ldots, P_n \omega_1(x_1), \ldots, P_1 \omega_1(x_N), \ldots, P_n \omega_1(x_N))^T,
\]

\[
\mathbf{b}_{\omega_1}(X_D) := (B_1 \omega_1(x_{N+1}), \ldots, B_n \omega_1(x_{N+1}), \ldots, B_1 \omega_1(x_{N+M}), \ldots, B_n \omega_1(x_{N+M}))^T.
\]

### 3.1 Constructing Kernel-based Approximate Solutions

We simulate the observation values of the right-hand side of the SPDE (3.1) at the collocation points \(X\), i.e.,

\[
\hat{y}_{1,1} := \Gamma_1(f_1(x_1), \xi_{1,x_1}), \quad \ldots, \quad \hat{y}_{1,N} := \Gamma_1(f_1(x_N), \xi_{1,x_N}),
\]

\[
\ldots,
\]

\[
\hat{y}_{n,1} := \Gamma_n(f_n(x_1), \xi_{n,x_1}), \quad \ldots, \quad \hat{y}_{n,N} := \Gamma_n(f_n(x_N), \xi_{n,x_N}),
\]

and

\[
\hat{z}_{1,1} := g_1(x_{N+1}), \quad \ldots, \quad \hat{z}_{1,N} := g_1(x_{N+M}),
\]

\[
\ldots,
\]

\[
\hat{z}_{n,1} := g_n(x_{N+1}), \quad \ldots, \quad \hat{z}_{n,N} := g_n(x_{N+M}).
\]

Denote that

\[
y_{\xi,X_D} := (\hat{y}_{1,1}, \ldots, \hat{y}_{n,1}, \ldots, \hat{y}_{1,N}, \ldots, \hat{y}_{n,N})^T,
\]

\[
z_{X_D} := (\hat{z}_{1,1}, \ldots, \hat{z}_{n,1}, \ldots, \hat{z}_{1,N}, \ldots, \hat{z}_{n,N})^T,
\]

\[
y_{\xi,X} := \begin{pmatrix} y_{\xi,X_D} \\ z_{X_D} \end{pmatrix}.
\]
We let
\[ \mathcal{A}_{PB,X}(y, z) := \{ \omega_1 \in \Omega_{m} : \mathbf{P} \omega_1(X_D) = y \text{ and } \mathbf{B} \omega_1(X_{D^2}) = z \}, \]
\[ = \{ \omega_1 \in \Omega_{m} : S_{PB,X}^T(\omega_1) = (y^T, z^T) \}, \quad y \in \mathbb{R}^{p_N}, \ z \in \mathbb{R}^{n_b M}, \]
where the random vector \( S_{PB,X} \) induced by the Gaussian fields \( \{ P_{j_p} S \}_{j_p=1}^{n_p} \) and \( \{ B_{j_k} S \}_{j_k=1}^{n_b} \) at the collocation points \( X \) is defined in the equation (2.2).

The approximate probability measure \( \mathbb{P}_{KW}^{\hat{u}} \) is used to set up the kernel-based solution \( \hat{u} \) to approximate the exact solution \( u \) of the SPDE (3.1). Let
\[ \mathcal{E}_x(v) := \{ \omega_1 \times \omega_2 \in \Omega_{m w} : \omega_1(x) = v \} = \{ \omega_1 \times \omega_2 \in \Omega_{m w} : S_x(\omega_1) = v \}, \]
for \( v \in \mathbb{R} \) and \( x \in \mathcal{D} \). Using the same techniques as in [5, 26], the kernel-based approximate solution \( \hat{u}(x) \) is a global maximizer of the conditional probability, i.e.,
\[ \hat{u}(x, \omega_2) := \arg \max_{v \in \mathbb{R}} \sup_{\mu \in \mathcal{H}_k(\mathcal{D})} \mathbb{P}_{KW}^{\mu}(\mathcal{E}_x(v) | \mathcal{A}_{PB,X}(y_{\xi,X_D}(\omega_2), z_{X_{D^2}})) \]
\[ = \arg \max_{v \in \mathbb{R}} \sup_{\mu \in \mathcal{H}_k(\mathcal{D})} \mathbb{P}_{KW}^{\mu}(S_x = v | S_{PB,X} = v_{\xi,X}(\omega_2)) \]
\[ = \arg \max_{v \in \mathbb{R}} \sup_{\mu \in \mathcal{H}_k(\mathcal{D})} p_{S_x | S_{PB,X}}^\mu(v | v_{\xi,X}(\omega_2)) \]
\[ = k_{PB,X}(x)^T K_{PB,X}^{-1} v_{\xi,X}(\omega_2), \quad x \in \mathcal{D}, \ \omega_2 \in \Omega_{w}, \]
where \( p_{S_x | S_{PB,X}}^\mu(\cdot) \) is the conditional probability density function of the random variable \( S_x \) given \( S_{PB,X} \) defined on \( (\Omega_{m w}, \mathcal{F}_{m w}, \mathbb{P}_{KW}^{\mu}) \) (see Corollary 2.2), and the kernel basis \( k_{PB,X} \) is given in the equation (2.6). Here we can think \( y_{\xi,X_D} \) and \( z_{X_{D^2}} \) as the fixed values to find the approximate mean.

The kernel-based solution \( \hat{u} \) can be also written as the linear combination of the kernel basis
\[ \hat{u}(x) = \sum_{k=1}^{N} \sum_{j=1}^{n_p} c_{j} P_{j_k} K(x, x_k) + \sum_{k=1}^{M} \sum_{j=1}^{n_b} b_{j} B_{j_k} K(x, x_{N+k}), \quad x \in \mathcal{D}, \]
and its random expansion coefficients
\[ c := (c_{1,1}, \ldots, c_{n_p,1}, \ldots, c_{1,N}, \ldots, c_{n_p,N})^T \]
\[ b := (b_{1,1}, \ldots, b_{n_b,1}, \ldots, b_{1,M}, \ldots, b_{n_b,M})^T \]
are solved by the random linear equations
\[ K_{PB,X}(c, b) = \left( y_{\xi,X_D}, z_{X_{D^2}} \right) = v_{\xi,X}. \]
deterministic vector \( z_{\mathcal{X}_D} \). We can formally rewrite \( \hat{u}(x, \omega_2) \) into \( \hat{u}(x, y_{\mathcal{X}_D}, z_{\mathcal{X}_D}) \). This means that \( \hat{u}(x) \) can be seen as a random variable defined on the finite-dimensional probability space \( (\mathbb{R}^{n_p}, \mathcal{B}(\mathbb{R}^{n_p}), m_{y_{\mathcal{X}_D}}) \) in order that its probability distribution is the same as the original version, where the probability measure \( m_{y_{\mathcal{X}_D}}(dy) := p_{y_{\mathcal{X}_D}}(y)dy \) and \( p_{y_{\mathcal{X}_D}} \) is the joint probability density function of \( y_{\mathcal{X}_D} \).

### 3.2 Error Analysis

Now we propose to describe the convergence of the kernel-based solutions. Let \( x \) belong to \( \mathcal{D} \) or \( \partial \mathcal{D} \) corresponded to the linear operator \( L \), which is equal to the differential operator \( P_j \) for \( 1 \leq j \leq n_b \) or the boundary operator \( B_{j_b} \) for \( 1 \leq j_b \leq n_p \), i.e.,

\[
\text{when } L = P_j, \text{ then } x \in \mathcal{D}, \text{ or when } L = B_{j_b} \text{ then } x \in \partial \mathcal{D}.
\]

We define

\[
\mathcal{E}_{L, x, p_B}(\varepsilon) := \left\{ \omega_1 \times \omega_2 \in \Omega_{mv} : \left| L \omega_1(x) - L \hat{u}(x, \omega_2) \right| \geq \varepsilon \right\},
\]

where the kernel-based solution \( \hat{u} \) and the observation values \( v_{x}^{T} = (y_{\mathcal{X}_D}^{T}, z_{\mathcal{X}_D}^{T}) \) induced by the linear SPDE (3.1) are the same as in Section [3.1](#).

We fix any \( \mu \in \mathcal{H}_x(\mathcal{D}) \subseteq C^m(\overline{\mathcal{D}}) \). According to Section [B.1](#) the conditional mean \( m_{L, x, p_B}^{\mu}(v) \) of the conditional probability density function \( p_{L, x, p_B}^{\mu}(\cdot | v) \) stated in Corollary 2.2 has the uniformly representation independent of \( x \), i.e.,

\[
m_{L, x, p_B}^{\mu}(y, z) = k_{L, x, p_B}^{\mu}(x)^{T} K_{p_B}^{-1}(y) + O(h_x^{-O(L)})
\]

\[
= L \hat{u}(x, y, z) + O(h_x^{-O(L)}), \quad \text{for all } y \in \mathbb{R}^{n_p} \text{ and all } z \in \mathbb{R}^{n_b}.\]

When the fill distance \( h_x \) is enough small, then the Chebyshev’s inequality provides

\[
\text{Chebyshev’s inequality provides}\n\]

\[
\text{Chebyshev’s inequality provides}\n\]

\[
\text{Chebyshev’s inequality provides}\n\]

\[
\text{Chebyshev’s inequality provides}\n\]
where δ_{z_{X,D}} is the Dirac delta function at z_{X,D} and p_{X_{D}} is the joint probability density function of y_{X_{D}}. Since the variance function \sigma_{L,PB,X}^2 is equal to the power function P_{PB,X} (\delta_{x} \circ L) stated in Section 3.1, we have

\[ P_{KW}^\mu \left( E_{L,X,PB,X}(\varepsilon) \right) = \frac{O\left( h_X^{m-O(P)} \right)}{h_X}, \quad \text{when } h_X \text{ is enough small.} \]

Because |Lu(x, \omega_2) - \hat{u}(x, \omega_2)| \geq \varepsilon if and only if u(\cdot, \omega_2) \in E_{L,X,PB,X}(\varepsilon) for all \omega_2 \in \Omega_u. We can find a x_{jp} \in D and a x_{jb} \in \partial D such that

\[
\left\{ u(\cdot, \omega_2) \in \Omega_{mw} : \left\| P_{jp} u(\cdot, \omega_2) - P_{jp} \hat{u}(\cdot, \omega_2) \right\|_{L^2(D)} \geq \varepsilon \right\} \subseteq E_{P_{jp}, x_{jp}, PB,X} \left( \frac{\varepsilon}{2} \right),
\]

\[
\left\{ u(\cdot, \omega_2) \in \Omega_{mw} : \left\| B_{jb} u(\cdot, \omega_2) - B_{jb} \hat{u}(\cdot, \omega_2) \right\|_{L^2(\partial D)} \geq \varepsilon \right\} \subseteq E_{B_{jb}, x_{jb}, PB,X} \left( \frac{\varepsilon}{2} \right),
\]

for all j_p = 1, \ldots, n_p and j_b = 1, \ldots, n_b. This indicates that

\[
P_{KW}^\mu \left( \left\| P_{jp} u - P_{jp} \hat{u} \right\|_{L^2(D)} \geq \varepsilon \right) = \frac{O\left( h_X^{m-O(P_{jp})} \right)}{\varepsilon}, \quad j_p = 1, \ldots, n_p,
\]

\[
P_{KW}^\mu \left( \left\| B_{jb} u - B_{jb} \hat{u} \right\|_{L^2(\partial D)} \geq \varepsilon \right) = \frac{O\left( h_X^{m-O(B_{jb})} \right)}{\varepsilon}, \quad j_b = 1, \ldots, n_b.
\]

Since the SPDE (3.1) is well-posed for its differential and boundary operators, we conclude that:

**Proposition 3.1.** The kernel-based approximate solution \( \hat{u} \) given in the equations (3.11, 3.12) is convergent to the exact solution \( u \) of the SPDE (3.1) in the probability \( P_{KW}^\mu \) for all \( \mu \in H_K(D) \) when the fill distance \( h_X \) tends to 0, i.e., for any \( \mu \in H_K(D) \) and any \( \varepsilon > 0 \),

\[
P_{KW}^\mu \left( \left\| u - \hat{u} \right\|_{L^2(D)} \geq \varepsilon \right) = \frac{O\left( h_X^{m-\ell} \right)}{\varepsilon} \to 0, \quad \text{when } h_X \to 0,
\]

where \( \ell := \max \{O(P), O(B)\} \).

**Remark 3.2.** More precisely, the convergent rate can be represented as \( O\left( h_X^{m-O(P)} + O\left( h_X^{m-O(B)} \right) \right) \) which indicates that one choice of the optimal designs of \( X_D \) and \( x_{\partial D} \) has the form \( h_X^{m-O(P)} \approx h_X^{m-O(B)} \). Actually the error bounds of the kernel-based estimators can be also described in terms of the number \( N + M \) of the collocation points \( X \) and the dimension \( d \) of the domain space \( D \), and moreover, the worst-case errors for some special kernel functions can be even dimension-independent and decay as a polynomial in terms of \( N + M \) as done in [8, 9].

For the deterministic problem, the maximum error of the kernel-based estimator is bounded by the power function. For the stochastic problem, we discuss the convergence of the kernel-based estimators by using probability measures, which means that the deterministic error bound is replaced by the confidence interval. The confidence interval can be computed by the variance function and it is employed to predict the error in the probability sense. The power function and the variance function have the same forms but they represent different mathematical meanings. We can also obtain the analogous stable error estimate for the both deterministic and stochastic problems.
Because the convergence in the probability implies the convergence in the distribution, i.e., $F_{\hat{u}(x)}$ is pointwise convergent to $F_u(x)$ when $h \to 0$, where $F_{\hat{u}(x)}$ and $F_u(x)$ are the cumulative distribution functions of $\hat{u}(x)$ and $u(x)$. According to Proposition 3.1 the distributions of $u$ can be estimated by the distributions of $\hat{u}$.

**Corollary 3.1.** Let $\hat{u}$ and $u$ be the same as discussed in Proposition 3.1. If $G$ is a continuous and bounded function defined on $\mathbb{R}$, then

$$\lim_{h \to 0} E(G(\hat{u}(x))) = E(G(u(x))), \quad x \in \mathcal{D}.$$

In particular, if $E\|u\|^2_{L^2(\mathcal{D})} < \infty$, then

$$\lim_{h \to \infty} E(\hat{u}(x)) = E(u(x)), \quad \lim_{h \to \infty} \text{Var}(\hat{u}(x)) = \text{Var}(u(x)), \quad x \in \mathcal{D}.$$

### 4 Solving Parabolic Stochastic Partial Differential Equations

Let $\mathcal{D}$ be a regular bounded open domain of $\mathbb{R}^d$, and $\mathcal{L}$ be a Lévy process in the Sobolev space $\mathcal{H}^m(\mathcal{D})$ for $m > d/2$ defined on the probability space $(\Omega_w, \mathcal{F}_w, (\mathcal{F}_w^t)_{t \geq 0}, \mathbb{P}_w)$ with the form

$$\mathcal{L}_{i,x} := \sum_{k,l=1}^{n_l} L_{k,l} \phi_k(x), \quad x \in \mathcal{D}, \quad t \geq 0,$$

where $\{\phi_k\}_{k=1}^{n_l} \subseteq \mathcal{H}^m(\mathcal{D})$ is an orthonormal subset of $L^2(\mathcal{D})$, $\{\alpha_k\}_{k=1}^{n_l}$ is a positive sequence and $L_{k,l}$ are the independent scalar Lévy processes with triples $(0, t, \nu_k)$ for all $k = 1, \ldots, n_l$ such that $\sum_{k=1}^{n_l} \int |\zeta|^2 \nu_k(\,d\zeta) < \infty$ (see [6, 19, 22]). When $\mathcal{L}$ is a Wiener process, then $\mathcal{L}$ can be constructed by infinite countable orthonormal bases and independent standard scalar Brownian motions, e.g., $\mathcal{L}$ is a Wiener process in $\mathcal{H}^m(\mathcal{D})$ with mean zero and spatial covariance kernel function $G$ given by $E(\mathcal{L}_{i,x}\mathcal{L}_{j,y}) = \min\{t, s\}G(x,y)$. Suppose that the linear differential operator $P$ of order $O(P) \leq m$ and the linear boundary operators $B_j_b$ of orders $O(B_{j_b}) \leq m - 1$ for all $j = 1, \ldots, n_b$ defined as in the equations A.1 A.2 with the coefficients $c_{\alpha} \in C^{2(m-|\alpha|)}(\overline{\mathcal{D}})$ and $b_{\alpha} \in C^{2(m-|\alpha|)}(\partial\mathcal{D})$ are linear independent. Given an initial condition $u_0 \in \mathcal{H}^m(\mathcal{D})$, we consider a parabolic SPDE driven by $\mathcal{L}$

$$\begin{aligned}
\frac{dU_t}{dt} &= PU_t + \Phi(U_t)\,d\mathcal{L}_t, &\quad &\text{in } \mathcal{D}, &\quad 0 < t < T, \\
B_1 U_t &= g_1(t, \cdot), \ldots, B_{n_b} U_t = g_{n_b}(t, \cdot), &\quad &\text{on } \partial\mathcal{D}, &\quad 0 < t < T, \\
U_0 &= u_0,
\end{aligned}$$

where $\Phi \in C^2(\mathbb{R})$ and $g_{j_b} : [0, T] \times \partial\mathcal{D} \to \mathbb{R}$ are the deterministic functions for $j_b = 1, \ldots, n_b$.

Suppose that the SPDE (4.1) is well-posed and that its exact solution $U_t$ belongs to $L_2(\Omega_w, \mathcal{F}_w^T, \mathbb{P}_w; \mathcal{H}^m(\mathcal{D}))$ for all $0 < t < T$ such that $\int_0^T E\|U_t\|_{\mathcal{H}^m(\mathcal{D})}^2 \,dt < \infty$.

We transform the parabolic SPDE into several elliptic SPDEs by the implicit Euler schemes and solve these elliptic SPDEs using the kernel-based approximation method as in Section 3.
(S1) We discretize the equation (4.1) by the implicit Euler schemes at time $0 = t_0 < t_1 < \cdots < t_n = T$, i.e.,

$$U_{t_i} - U_{t_{i-1}} = P_{t_i} U_{t_{i-1}} + \Phi(U_{t_{i-1}}) \delta L_i , \quad i = 1, \ldots, n ,$$

(4.2)

where $\delta t_i := t_i - t_{i-1}$ and $\delta L_i := L_{t_i} - L_{t_{i-1}}$ for all $i = 1, \ldots, n$.

(S2) Because the Le"vy noise increment $\xi_t^i := \delta L_i$ at each time instance $t_i$ is independent of the solution $U_{t_{i-1}}$. We can view that $u^i := U_{t_i}$ is an unknown part and $f^i := U_{t_{i-1}}$ is a given part computed at the previous step. The equation (4.2) together with the corresponding moving boundary condition becomes a well-posed elliptic SPDE of the form

$$\begin{cases}
    P_{\delta t_i} u^i = \Gamma(f^i, \xi^i), & \text{in } \mathcal{D}, \\
    B_1 u^i = g_1(t_i, \cdot), \ldots, B_{n_b} u^i = g_{n_b}(t_i, \cdot), & \text{on } \partial \mathcal{D},
\end{cases}$$

(4.3)

where $\Gamma(v, w) := v + w \Phi(v)$ and $P_{\delta t_i} := I - \delta t_i P$.

For solving the SPDE (4.3), we select the well-distributed pairwise distinct collocation points $X_{\mathcal{D}} := \{x_1, \ldots, x_N\} \subseteq \mathcal{D}$ and $X_{\partial \mathcal{D}} := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial \mathcal{D}$. In addition, we simulate the noise $\xi^i$ at $X_{\mathcal{D}}$, i.e.,

$$\xi^i_{x_j} := \sum_{k=1}^{n_t} \xi^i_k \phi_k(x_j), \quad j = 1, \ldots, N, \quad \xi^i_k \sim \mathcal{L}_{k, \delta t_i}, \quad k = 1, \ldots, n_t.$$

Next we choose a positive definite kernel $K \in C^2_m(\mathcal{D} \times \mathcal{D})$ which satisfies the condition (3.2) related to $P_{\delta t_i}$ and $B_1, \ldots, B_{n_b}$ defined same as in Section 3. The kernel-based solution $\hat{u}^i$ of the SPDE (4.3) is computed by the equations (3.3-3.4), i.e.,

$$\hat{u}^i(x) := k_{PB,X}(x)^T K_{PB,X}^{-1} \left( y_{\xi_{X_{\mathcal{D}}}}^i \right), \quad x \in \mathcal{D},$$

(4.4)

where

$$y_{\xi_{X_{\mathcal{D}}}}^i := \begin{bmatrix}
    \Gamma(f^i(x_1), \xi^i_{x_1}) \\
    \vdots \\
    \Gamma(f^i(x_N), \xi^i_{x_N})
\end{bmatrix} \in \mathbb{R}^N, \quad z_{X_{\mathcal{D}}}^i := \begin{bmatrix}
    g_1(t_i, x_{N+1}) \\
    \vdots \\
    g_{n_b}(t_i, x_{N+M})
\end{bmatrix} \in \mathbb{R}^{n_b M}.$$

Here the kernel base $k_{PB,X}$ and the covariance matrix $K_{PB,X}$ induced by the kernel $K$ with $P := P_{\delta t_i}$ and $B := (B_1, \ldots, B_{n_b})^T$ are defined in the equations (2.3) and (2.6).

This means that $U_{t_i}$ is approximated by the kernel-based solution $\hat{u}^i$.

(S3) Repeat (S2) for all $i = 1, \ldots, n$.

We briefly discuss the convergence of the above algorithm for the SPDE (4.1) similar as done in [26]. Suppose that the distances of all discretization time steps are equal to $\delta t_i$ and that the collocation points $X$ and the positive definite kernel $K$ are chosen to be the same at each time step $t_i$. According to Theorem 2.1, we firstly set up a probability measure $\mathbb{P}^\mu_K$ on the measurable space
By induction we can deduce that
\[ \text{Combining the both local errors given in the equations (4.5) and (4.6), we have} \]
\[ (4.5) \]
for all \( i = 1, \ldots, n \), where \( \gamma > 0 \) is dependent of the scalar Lévy processes \( \{\mathcal{L}_k\}_{k=1}^n \). Moreover, Proposition 3.1 provides the errors for each local elliptic SPDE (4.3)
\[ (4.6) \]
where \( \ell := \max \{O(P), O(B)\} \). Denote the global errors
\[ e^0 := 0, \quad e^i := \begin{pmatrix} e^i_1 \\ \vdots \\ e^i_N \end{pmatrix} := \begin{pmatrix} U_{t_i,x_1} \\ \vdots \\ U_{t_i,x_N} \end{pmatrix} - \begin{pmatrix} \hat{u}^i(x_1) \\ \vdots \\ \hat{u}^i(x_N) \end{pmatrix}, \quad i = 1, \ldots, n. \]
Combining the both local errors given in the equations (4.5) and (4.6), we have
\[ \left( e^i \right)^{\mathbb{P}_W} A_{PB,X} \left( e^{i-1} \right) + O(\delta t^\gamma) + O(h_X^{m-\ell}), \quad i = 1, \ldots, n, \]
where
\[ A_{PB,X} := (k_{PB,X}(x_1), \ldots, k_{PB,X}(x_N), 0, \ldots, 0)^T K_{PB,X}^{-1}. \]
By induction we can deduce that
\[ \left( e^n \right)^{\mathbb{P}_W} \left( I + A_{PB,X} + \cdots + A_{PB,X}^{n-1} \right) \left( O(\delta t^\gamma) + O(h_X^{m-\ell}) \right). \]
If \( \delta t = O(h_X^\rho) \) then the spectral radius \( \rho \) of the matrix \( A_{PB,X} \) satisfies \( \rho < 1 \), where \( p > 0 \) is dependent of \( K \) and \( P \) (see [15]). Thus
\[ \frac{1}{\sqrt{N}} \left\| e^n \right\|_2^{\mathbb{P}_W} \equiv \frac{1 - \rho^n}{1 - \rho} \left( O(\delta t^\gamma) + O(h_X^{m-\ell}) \right) \rightarrow 0 \quad \text{when } \delta t \rightarrow 0 \text{ and } h_X \rightarrow 0. \]
This indicates that \( \hat{u}^i(x_k) \) is convergent to \( U_{t_i,x_k} \) in the probability \( \mathbb{P}_W^\mu \) when both \( \delta t \) and \( h_X \) tend to 0, for all \( k = 1, \ldots, N \).

**Remark 4.1.** In this paper, we focus mainly on the step (S2) to solve the elliptic SPDE (4.3) same as in [5]. The numerical analysis of deterministic parabolic PDEs for the kernel-based approximation method is a delicate and nontrivial question, only recently solved in [15]. We will address this question in the case of many other parabolic SPDEs in our future research.
5 Numerical Examples

Now we do a numerical test for the two-dimensional parabolic SPDE driven by the Poisson noises with Dirichlet boundary conditions – a typical case of the SPDE (1.1). Let the domain \( D := (0, 1)^2 \subseteq \mathbb{R}^2 \), and the Poisson noise \( N_t \) have the form

\[
N_{t,x} := \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} N_{k_1, k_2, t} \left( \frac{4}{k_1^2 + k_2^2 \pi^2} \sin(k_1 \pi x_1) \sin(k_2 \pi x_2) \right), \quad x = (x_1, x_2)^T \in D,
\]

where \( N_{k_1, k_2, t} \) are the independent scalar Poisson process with parameter \( \lambda > 0 \) for all \( k_1, k_2 = 1, \ldots, n \).

We use this Poisson noise \( N_t \) to create the parabolic SPDE

\[
\begin{cases}
    dU_t = \Delta U_t \, dt + \sigma U_t \, dN_t, & \text{in } D, \quad 0 < t < T = 1, \\
    U_t = 0, & \text{on } \partial D, \\
    U_0 = u_0,
\end{cases}
\]

where \( \sigma > 0 \) and \( u_0(x) := \sin(\pi x_1) \sin(\pi x_2) + \frac{1}{4} \sin(2\pi x_1) \sin(2\pi x_2) \). According to the same algorithm given in Section 1.1 we can compute the kernel-based solutions of the SPDE (5.1) by two kinds of positive definite kernels: compact support kernel induced by \( C^4 \)-compact support radial basis function

\[
K_{C, \vartheta}(x, y) := \left( 3 + 18 \theta \|x - y\|_2 + 35 \theta^2 \|x - y\|_2^2 \right) \left( 1 - \theta \|x - y\|_2 \right)^6, \quad x, y \in D,
\]

and Sobolev-spline kernel induced by \( C^4 \)-Matérn function

\[
K_{S, \vartheta}(x, y) := \left( 3 + 3 \theta \|x - y\|_2 + \theta^2 \|x - y\|_2^2 \right) \exp\left( -\theta \|x - y\|_2 \right), \quad x, y \in D,
\]

where \( \theta > 0 \) and the cutoff function \( (r)_+ := r \) when \( r \geq 0 \) otherwise \( (r)_+ := 0 \). As collocation points we select Halton points \( X_D \) in \( D \) and evenly space points \( X_{\partial D} \) on \( \partial D \). Using the kernel-based approximation method, we can obtain thousand numerical sample paths \( \hat{u}(\omega_k) \approx U(\omega_k) \) for \( k = 1, \ldots, s = 10,000 \) by the algorithm given in Section 1.1. Moreover, we can compute its sample means and sample standard deviations by these estimate sample paths, i.e.,

\[
E(U) \approx \frac{1}{s} \sum_{k=1}^{s} \hat{u}(\omega_k), \quad \text{Dev}(U) \approx \sqrt{\frac{1}{s-1} \sum_{i=1}^{s} \left( \hat{u}(\omega_i) - \frac{1}{s} \sum_{k=1}^{s} \hat{u}(\omega_k) \right)^2}.
\]

Observing Figure 5.1, we find that the approximate means and the approximate standard deviations for both kernels are symmetric with the line \( x_1 = x_2 \) because the time and space Poisson noises \( N_t \) are symmetric with \( x_1 \) and \( x_2 \) in space.

Remark 5.1. In our current numerical experiments, the distribution of collocation points and the shape parameter are chosen empirically and based on the authors’ experiences. Actually, different choices of the shape parameters will affect the convergent rate and the stability of the algorithms. The convergent rate will decrease when the shape parameter becomes large, but the algorithm will be unstable when the shape parameter becomes small. How to select the best shape parameter is still an open problem. We will try to solve this problem using the probability measures of the kernel functions in our future research.

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Compact Support Kernel, $\theta = 1.2$

Sobolev-spline Kernel, $\theta = 6.8$

Figure 5.1: Numerical experiments of the probability distributions for the SPDE (5.1) for $\sigma = 1$, $n_l = 10$ and $\lambda = 10$ by the kernel-based collocation method using $X_D$-Halton points of $N = 81$ and $X_{\partial D}$-evenly space points of $M = 36$, time step $\delta t = 0.00167$.

6 Final Remarks

In this paper we present how to employ the kernel-based approximation method to estimate the numerical solutions of SPDEs driven by the Lévy noises. We transform the parabolic SPDE into the elliptic SPDE by the implicit Euler time scheme at each time step. The kernel-based solution of the elliptic SPDE is a linear combination of the kernel basis with the related differential and boundary operators centered at the chosen collocation points. Here we only consider the elliptic SPDEs with right-hand-side random noises. Actually, the kernel-based approximation method can be even applied into solving the SPDE driven by the random differential and boundary operators as done in [13]. The main idea of this paper is the same as in our recent paper [5] but we give the new contents and extensions to improve the previous theoretical results as follow:

- We extend [Theorem 3.1, [5]] into Theorem 2.1 in order that we can apply more general positive definite kernels to set up the kernel-based approximate solutions of SPDEs instead of the integral-type kernels, and we only need to assume that the exact solutions of SPDEs belong to the classical Sobolev spaces rather than the reproducing kernel Hilbert spaces.

- We obtain the kernel-based approximate solutions of a system of linear elliptic SPDEs driven by various kinds of random noises undone before.

- We complete the discussions of the error analysis of the kernel-based approximation method for SPDEs and provide its precise convergent rate in terms of the fill distances (or possible the time distances).
Comparing the kernel-based approximation method for solving the deterministic and stochastic elliptic PDEs similar as in [23]:

- The kernel-based solution for the deterministic case is to minimize the reproducing kernel norm for interpolating the data values induced by the PDEs, and the kernel-based solution for the stochastic case is to maximize the probability conditioned on the observation values simulated by the SPDEs.

- The estimate error is bounded by the power function for the deterministic case, and the confidence interval is computed by the variance function for the stochastic case, and moreover, the formulas of the power function and the variance function are equal when the PDE and the SPDE have the same differential and boundary operators.

We also discuss the side-by-side differences of the kernel-based approximation method and the finite element method for the elliptic SPDE stated as in [12]:

- For the kernel-based approximation method we transfer the original SPDE probability space to the tensor product probability space such that the extension of the noise preserves the same probability distributions. For the finite element method, we approximate the noise by its truncated noise which means that we truncate the original SPDE probability space to the finite dimensional probability space.

- The bases of the kernel-based solution are set up by the positive definite kernels with the differential and boundary operators of the SPDE and the collocation points, while the bases of the finite element solution are the finite element polynomials induced by the triangular meshes.

- We can simulate the noise at the collocation points by its probability structure to compute the random coefficients of the kernel-based solution, but we can simulate the random part of the finite element solution on the truncated probability space.

- The convergent rate of the kernel-based solution is only dependent of the fill distances. However, the convergent rate of the finite element solution depends on the maximum mesh spacing parameter of the triangulation and the truncation dimension of the original probability space.

In our future work we will try to solve the open problems of the kernel-based approximation method for SPDEs:

- We will solve the kernel-based estimators of the nonlinear elliptic SPDEs based on the theoretical results given in this paper.

- We will try many other time-stepping schemes to create the kernel-based solutions of the parabolic SPDEs, and introduce the precise rates of their convergence.

- We will design the best choice of the collocation points and the optimal kernel function by maximizing the conditional probability measure dependent of the observation values simulated by the SPDEs analogous as the maximum likelihood estimation method.
A Differential and Boundary Operators

In this section, we review some classical materials of linear differential and boundary operators mentioned in [1, 16].

Let $\mathcal{D}$ be a regular bounded open domain of $\mathbb{R}^d$, e.g., it satisfies the uniform $C^m$-regularity condition which implies the strong local Lipschitz condition and the uniform cone condition. This means that $\mathcal{D}$ has a regular boundary $\partial \mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$. We call the partial derivative of order $\alpha := (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}},$$

and denote its degree $|\alpha| := \sum_{j=1}^d \alpha_j$. The test function space is chosen to be $\mathcal{C}^\infty_0(D)$ composed of all functions $\gamma \in \mathcal{C}^\infty(D)$ with compact support in $D$. For any fixed $f \in L^1_{loc}(D)$, if there exist a function $f^\alpha \in L^1_{loc}(D)$ such that

$$\int_D f^\alpha(x) \gamma(x)dx = (-1)^{|\alpha|} \int_D f(x) D^\alpha \gamma(x)dx, \quad \text{for all } \gamma \in \mathcal{C}^\infty_0(D),$$

then $f^\alpha$ is said the weak derivative $D^\alpha f$ of $f$. The $L^2$-based Sobolev space of degree $m \in \mathbb{N}_0$ is defined by

$$\mathcal{H}^m(D) := \{ f \in L^1_{loc}(D) : D^\alpha f \in L^2(D), \text{ for all } 0 \leq |\alpha| \leq m \},$$

equipped with the inner product

$$(f, g)_{\mathcal{H}^m(D)} := \sum_{|\alpha| \leq m} \int_D D^\alpha f(x) D^\alpha g(x)dx, \quad f, g \in \mathcal{H}^m(D).$$

Because the weak derivative $D^\alpha$ can be seen as a linear bounded operator from $\mathcal{H}^m(D) \to L^2(D)$ when $|\alpha| \leq m$ or a linear bounded operator from $\mathcal{H}^m(D) \to L^2(\partial D)$ when $|\alpha| \leq m - 1$ according to the boundary trace embedding theorem. In this paper all linear differential and boundary operators are the linear combinations of weak derivatives with uniformly continuous coefficients. We define that the linear differential operator

$$P := \sum_{|\alpha| \leq m} c^\alpha D^\alpha : C^m(\overline{D}) \subseteq \mathcal{H}^m(D) \to C(\overline{D}) \subseteq L^2(D), \quad (A.1)$$

for all $c^\alpha \in C(D)$ has the order

$$O(P) := \max \{ |\alpha| : c^\alpha \neq 0 \},$$

and the linear boundary operator

$$B := \sum_{|\alpha| \leq m-1} b^\alpha \frac{\partial^{\alpha}}{\partial \nu} : C^m(\overline{D}) \subseteq \mathcal{H}^m(D) \to C(\partial D) \subseteq L^2(\partial D), \quad (A.2)$$

for all $b^\alpha \in C(\partial D)$ has the order

$$O(B) := \max \{ |\alpha| : b^\alpha \neq 0 \}.$$ 

It is easy to check that $P$ and $B$ are linear and bounded.
Lemma A.1. Suppose that a sequence \( \{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}^m(\mathcal{D}) \) and a function \( f := \lim_{n \to \infty} f_n \) in the \( L_2(\mathcal{D}) \)-norm. If there exists a function \( f_\alpha \in L_2(\mathcal{D}) \) such that \( \lim_{n \to \infty} D^\alpha f_n = f_\alpha \) in the \( L_2(\mathcal{D}) \)-norm for any fixed \( 0 \leq |\alpha| \leq m \), then \( f \in \mathcal{H}^m(\mathcal{D}) \). This indicates that \( Pf = \lim_{n \to \infty} P f_n \) in the \( L_2(\mathcal{D}) \)-norm and \( Bf = \lim_{n \to \infty} B f_n \) in the \( L_2(\partial \mathcal{D}) \)-norm for any differential and boundary operators \( P \) and \( B \) of orders \( O(P) \leq m \) and \( O(B) \leq m - 1 \), respectively.

Proof. Fixing \( 0 \leq |\alpha| \leq m \), we have

\[
\int_{\mathcal{D}} f_n(x) \gamma(x) dx = \lim_{n \to \infty} \int_{\mathcal{D}} D^\alpha f_n(x) \gamma(x) dx = (-1)^{|\alpha|} \lim_{n \to \infty} \int_{\mathcal{D}} f_n(x) D^\alpha \gamma(x) dx
\]

\[
= (-1)^{|\alpha|} \int_{\mathcal{D}} f(x) D^\alpha \gamma(x) dx, \quad \text{for all } \gamma \in \mathcal{K}(\mathcal{D}),
\]

which shows that \( f_\alpha \) is equal to the weak derivative \( D^\alpha f \) of \( f \). Therefore \( f \in \mathcal{H}^m(\mathcal{D}) \) and \( f = \lim_{n \to \infty} f_n \) in the \( \mathcal{H}^m(\mathcal{D}) \)-norm.

\( \square \)

B Reproducing Kernels and Reproducing Kernel Hilbert Spaces

Most of the detail presented in this section can be found in the monograph [1, 25]. For the reader’s convenience we repeat here what is essential to the kernel-based approximation method.

Definition B.1 ([Definition 10.1, [25]]). A Hilbert space \( \mathcal{H}_K(\mathcal{D}) \) consisting of functions \( f : \mathcal{D} \subseteq \mathbb{R}^d \to \mathbb{R} \) is called a reproducing kernel Hilbert space and a kernel function \( K : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \) is called a reproducing kernel for \( \mathcal{H}_K(\mathcal{D}) \) if

(i) \( K(\cdot, y) \in \mathcal{H}_K(\mathcal{D}) \) and (ii) \( f(y) = (f, K(\cdot, y))_{\mathcal{H}_K(\mathcal{D})} \),

for all \( f \in \mathcal{H}_K(\mathcal{D}) \) and \( y \in \mathcal{D} \), where \( (\cdot, \cdot)_{\mathcal{H}_K(\mathcal{D})} \) is used to denote the inner product of \( \mathcal{H}_K(\mathcal{D}) \).

According to [Theorem 10.4, [25]] the reproducing kernel \( K \) is always semi-positive definite. Moreover, [Theorem 10.10, [25]] guarantees the existence of the reproducing kernel Hilbert space with the positive definite kernel \( K \).

Suppose that the symmetric positive definite kernel \( K \in C(\overline{\mathcal{D}} \times \mathcal{D}) \) and \( \mathcal{D} \) is a regular bounded open domain of \( \mathbb{R}^d \). Since \( \overline{\mathcal{D}} \) is compact, the Mercer’s theorem shows that there exists a countable set of positive eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots > 0 \) and continuous eigenfunctions \( \{e_n\}_{n=1}^{\infty} \subseteq L_2(\mathcal{D}) \) such that \( \int_{\mathcal{D}} K(x, y)e_n(y) dy = \lambda_n e_n(x) \) and the kernel \( K \) has the absolutely and uniformly convergent representation \( K(x, y) = \sum_{n=1}^{\infty} \lambda_n e_n(x)e_n(y) \). Furthermore, \( \{e_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( L_2(\mathcal{D}) \) and \( \sum_{n=1}^{\infty} \lambda_n < \infty \).

Lemma B.1. If the symmetric positive definite kernel \( K \in C^{2m}(\overline{\mathcal{D}} \times \mathcal{D}) \), then its eigenfunctions \( \{e_n\} \subseteq C^{2m}(\overline{\mathcal{D}}) \). Moreover, the convergent representations \( P_1 P_2 K(x, y) = \sum_{n=1}^{\infty} \lambda_n Pe_n(x) Pe_n(y) \) and \( B_1 B_2 K(x, y) = \sum_{n=1}^{\infty} \lambda_n Be_n(x) Be_n(y) \) are absolute and uniform on \( \overline{\mathcal{D}} \times \overline{\mathcal{D}} \) and \( \partial \mathcal{D} \times \partial \mathcal{D} \) for any differential and boundary operators \( P \) and \( B \) of orders \( O(P) \leq m \) and \( O(B) \leq m - 1 \) defined as in Equations (A.1)-(A.2), respectively.
Proof. Because of $K \in C^{2m}(\overline{D} \times \overline{D})$,

$$D^\alpha e_n = \frac{1}{\lambda_n} \int_D D^\alpha K(\cdot, y)e_n(y)dy \in C(\overline{D}), \quad n \in \mathbb{N},$$

for any $0 \leq |\alpha| \leq 2m$. This indicates that the convergent representation

$$D^\alpha D^\beta K(x, y) = \sum_{n=1}^{\infty} \lambda_n D^\alpha e_n(x) D^\beta e_n(y)$$

is absolute and uniform on $\overline{D} \times \overline{D}$ for any $0 \leq |\alpha| \leq m$. □

Now we show a special class of reproducing kernels whose reproducing kernel Hilbert spaces are equivalent to the Sobolev spaces.

Example B.1 ([Example 5.7, [10]] and [Example 4.4, [26]]). We consider the Matérn function of degree $m > d/2$ and shape parameter $\theta > 0$

$$G_{m, \theta}(x) := \frac{2^{-1-m-d/2}}{\pi^{d/2} \Gamma(m) \theta^{d-m} \Gamma(m-d/2-m)} \left( \theta \|x\|_2 \right)^{m-d/2} K_{d/2-m}(\theta \|x\|_2), \quad x \in \mathbb{R}^d,$$

where $t \mapsto K_\nu(t)$ is the modified Bessel function of the second kind of order $\nu$.

According to the theoretical results given in [10, 26], we can check that the Sobolev spline kernel induced by the Matérn function $G_{m, \theta}$

$$K_{m, \theta}(x, y) := G_{m, \theta}(x - y), \quad x, y \in \mathbb{R}^d,$$

is a positive definite kernel on $\mathbb{R}^d$ and its reproducing kernel Hilbert space $H_{K_{m, \theta}}(\mathbb{R}^d)$ is equivalent to the Sobolev space $H^m(\mathbb{R}^d)$, i.e.,

$$H_{K_{m, \theta}}(\mathbb{R}^d) \equiv H^m(\mathbb{R}^d).$$

Let $D$ be a regular domain of $\mathbb{R}^d$. According to [Theorem 6, [3]], the reproducing kernel Hilbert space $H_{K_{m, \theta}}(D)$ of the Sobolev spline kernel $K_{m, \theta}$ restricted on $D$ is endowed with the reproduction norm

$$\|f\|_{H_{K_{m, \theta}}(D)} = \inf_{g \in H_{K_{m, \theta}}(\mathbb{R}^d)} \left\{ \|g\|_{H_{K_{m, \theta}}(\mathbb{R}^d)} : g|_D = f \right\}, \quad f \in H_{K_{m, \theta}}(D).$$

and it is also equivalent to the Sobolev space $H^m(D)$, i.e.,

$$H_{K_{m, \theta}}(D) \equiv H^m(D).$$

The papers [10, 11, 26] also show many other general kinds of reproducing kernels and their related reproducing kernel Hilbert spaces are introduced by Green functions and generalized Sobolev spaces.
B.1 Power Functions

Let $P_1, \ldots, P_{n_p}$ be linear differential operators of orders no more than $m$, whose coefficients $c_{\alpha} \in C^{2(m-|\alpha|)}(\overline{D})$, and $B_1, \ldots, B_{n_b}$ be linear boundary operators of orders no more than $m - 1$, whose coefficients $b_{\alpha} \in C^{2(m-|\alpha|)}(\partial D)$. Denote that $P := (P_1, \ldots, P_{n_p})^T$ and $B := (B_1, \ldots, B_{n_b})^T$. We choose the set of pairwise distinct collocation points $X = X_D \cup X_{\partial D}$ from a regular bounded open domain $D$ and its boundary $\partial D$, i.e., $X_D := \{x_1, \ldots, x_N\} \subseteq D$ and $X_{\partial D} := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial D$.

Suppose that the symmetric positive definite kernel $K \in C^2m(D \times \overline{D})$ satisfies the condition (3.2) related to $P$ and $B$ denoted as in Section 3. According to [Theorem 16.8, [25]], $\delta_x \circ L$ belongs to the dual space $H_k(D)'$ of the reproducing kernel Hilbert space $H_k(D)$, where $L$ is a differential or boundary operator of order $O(L) \leq m$ and $O(L) \leq m - 1$ and $\delta_x$ is the point evaluation functional at $x \in D$ or $x \in \partial D$.

The power function induced by the positive definite kernel with the differential and boundary operators at the collocation points is defined by

$$
\mathcal{P}_{PB,X}(\delta_x \circ L)^2 := \min_{c_{k,j},b_{j} \in \mathbb{R}} \left\| \delta_x \circ L - \sum_{k=1}^{N} \sum_{j=1}^{n_p} c_{k,j} \delta_{x_k} \circ P_j - \sum_{k=1}^{M} \sum_{j=1}^{n_b} b_{k,j} \delta_{x_{N+k}} \circ B_j \right\|^2_{H_k(D)'}
$$

$$
= L_1 L_2 K(x, x) - k_{PB,X}^T(x)^T K_{PB,X}^1 k_{LPB,X}(x),
$$

where the matrix $K_{PB,X}$ and the vector $k_{LPB,X}(x)$ are denoted in the equations (2.3) and (2.6) respectively. We can observe that the power function $\mathcal{P}_{PB,X}(\delta_x \circ L)$ is equal to the formula of the variance $\sigma_{PB,X}(x)$ defined in the equation (2.5). [Theorem 16.11, [25]] provides that

$$
\mathcal{P}_{PB,X}(\delta_x \circ L) \leq \min_{j_p=1, \ldots, n_p, j_b=1, \ldots, n_b} \left\{ \mathcal{P}_{PB,X,\partial D}(\delta_x \circ L), \mathcal{P}_{B_{j_b},X_{\partial D}}(\delta_x \circ L) \right\}.
$$

As the discussion of [Section 16.3, [25]] we have

$$
|L_{\mu}(x) - L s_{\mu,X}(x)| \leq \mathcal{P}_{PB,X}(\delta_x \circ L) \|\mu\|_{H_k(D)}, \quad \mu \in H_k(D),
$$

where $s_{\mu,X} := k_{PB,X}^T \mu_{PB,X}$ and $\mu_{PB,X} := (P_{\mu}(X_D)^T, B_{\mu}(X_{\partial D})^T)$. Combining [Theorem 11.3 and 16.9, [25]] the power functions can be also bounded by the fill distances of the collocation points $X_D$ and $X_{\partial D}$, respectively,

$$
h_{X_D, D} := \sup_{x \in D} \min_{k=1, \ldots, N} \|x - x_k\|_2, \quad h_{X_{\partial D}, \partial D} := \sup_{x \in \partial D} \min_{k=1, \ldots, N} \|x - x_{N+k}\|_2,
$$

i.e.,

$$
\mathcal{P}_{PB,X}(\delta_x \circ P_{j_p}) \leq C_{P_{j_p}} h_{X_D, D}^{m-O(\|P_{j_p}\|)}, \quad \text{when } x \in D, \quad j_p = 1, \ldots, n_p,
$$

$$
\mathcal{P}_{PB,X}(\delta_x \circ B_{j_b}) \leq C_{B_{j_b}} h_{X_{\partial D}, \partial D}^{m-O(\|B_{j_b}\|)}, \quad \text{when } x \in \partial D, \quad j_b = 1, \ldots, n_b,
$$

where $C_{P_{j_p}}$ and $C_{B_{j_b}}$ are the positive constants independent of $x$.

**Remark B.1.** The precise form of the fill distance of $\partial D$ is equal to

$$
\rho_{X_{\partial D}, \partial D} := \sup_{x \in \partial D} \min_{k=1, \ldots, M} \text{dist}_{\partial D}(x, x_{N+k})
$$

using the distance defined on the manifold surface $\partial D$. Since the boundary $\partial D$ is regular, we have $\rho_{X_{\partial D}, \partial D} = O(h_{X_{\partial D}, \partial D})$. For convenience we does not consider the manifold distance in this article.
C Gaussian Fields

In this section we discuss the basic relationship of Gaussian fields and reproducing kernel Hilbert spaces given in [3] [20].

**Definition C.1** ([Definition 3.28, [3]]) Let \( \mathcal{D} \subseteq \mathbb{R}^d \). A stochastic process \( S : \mathcal{D} \times \Omega \rightarrow \mathbb{R} \) is said to be Gaussian with mean \( \mu : \mathcal{D} \rightarrow \mathbb{R} \) and covariance kernel \( K : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) if, for any pairwise distinct points \( x_D := \{x_1, \ldots, x_N\} \subseteq \mathcal{D} \), the random vector \( S_{x_D} := (S_{x_1}, \cdots, S_{x_N})^T \) is a multi-normal random variable on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with mean \( \mu_{x_D} \) and covariance matrix \( K_{x_D} \), i.e.,

\[
S_{x_D} \sim \mathcal{N}(\mu_{x_D}, K_{x_D}),
\]

where \( \mu_{x_D} := (\mu(x_1), \cdots, \mu(x_N))^T \) and \( K_{x_D} := (K(x_j, x_k))_{j,k=1}^{N,N} \).

Suppose that \( \mathcal{H}_K(\mathcal{D}) \) is a reproducing kernel Hilbert space and \( \mathcal{B}(\mathcal{H}_K(\mathcal{D})) \) is the Borel \( \sigma \)-algebra on \( \mathcal{H}_K(\mathcal{D}) \). Let \( \xi : \mathcal{D} \times \Omega \rightarrow \mathbb{R} \) be a Gaussian field defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \). If the sample paths \( x \mapsto \xi(x, \omega) \) belong to \( \mathcal{H}_K(\mathcal{D}) \) for almost all \( \omega \in \Omega \), then \( \xi \) can be seen as a mapping from \( \Omega \) into \( \mathcal{H}_K(\mathcal{D}) \).

**Lemma C.1** ([Theorem 3.91, [3]] and [Lemma 2.1, [20]]). Suppose that \( \xi : \mathcal{D} \times \Omega \rightarrow \mathbb{R} \) is a Gaussian field on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with almost all sample paths in a reproducing kernel Hilbert space \( \mathcal{H}_K(\mathcal{D}) \). Then the probability measure given by

\[
\mathbb{P}_\xi(A) := \mathbb{P}(\xi^{-1}(A)), \quad \text{for all } A \in \mathcal{B}(\mathcal{H}_K(\mathcal{D})),
\]

is well-defined on the measurable space \( (\mathcal{H}_K(\mathcal{D}), \mathcal{B}(\mathcal{H}_K(\mathcal{D})) ) \) such that the stochastic process defined by

\[
S_x(\omega) := \omega(x), \quad \text{for all } x \in \mathcal{D} \text{ and all } \omega \in \mathcal{H}_K(\mathcal{D}),
\]

is a Gaussian field on the probability space \( (\mathcal{H}_K(\mathcal{D}), \mathcal{B}(\mathcal{H}_K(\mathcal{D})), \mathbb{P}_\xi) \) and \( S \) has the same probability distribution as \( \xi \), i.e., the both Gaussian fields \( S \) and \( \xi \) have the same mean and covariance kernel. (Here the reproducing kernel \( K \) may be different from the covariance kernel.)

**Lemma C.1** shows that we can transfer the original probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) into the new probability space \( (\mathcal{H}_K(\mathcal{D}), \mathcal{B}(\mathcal{H}_K(\mathcal{D})), \mathbb{P}_\xi) \) so that the original Gaussian field \( \xi \) has the invariant element \( S \) defined on the new probability space.

D Simulation of Random Variables

For the kernel-based collocation methods, we need to simulate the SPDE noise term \( \xi : \mathcal{D} \times \Omega \rightarrow \mathbb{R} \) defined on the SPDE probability space \( (\Omega_w, \mathcal{F}_w, \mathbb{P}_w) \) at the collocation points \( X_D := \{x_1, \cdots, x_N\} \subseteq \mathcal{D} \subseteq \mathbb{R}^d \). For example, if the noise \( \xi \) is equal to the product of a Poisson random variable \( \zeta \) with parameter \( \lambda > 0 \) and a deterministic function \( \varphi : \mathcal{D} \rightarrow \mathbb{R} \), i.e., \( \xi := \zeta \varphi \), then \( \xi_{x_j} \sim \text{Poisson}(\lambda) \cdot \varphi(x_j) \) for all \( j = 1, \ldots, N \), which can be simulated by the Monte Carlo methods (see [14]).
Furthermore, we can simulate many other kinds of noises by the Monte Carlo methods if the joint probability density function $p_{\xi_{XD}} : \mathbb{R}^N \to [0, \infty)$ of the random vector $\xi_{XD} := (\xi_1, \ldots, \xi_N)^T$ is known. If there is an one-to-one function $\phi : (0, 1)^N \to \mathbb{R}^N$ such that

$$p_{\xi_{XD}}(v) = \left| \det \left( D\phi^{-1}(v) \right) \right|, \quad v \in \mathbb{R}^N,$$

where $D\phi^{-1}(v)$ is the Jacobian matrix of the inverse $\phi^{-1}$ of $\phi$ evaluated at $v$, then we can simulate $\xi_{XD} \sim \phi(U_1, \ldots, U_N)$ by the independent strand uniform random variables $U_1, \ldots, U_N$, i.e., $U_k \sim \text{i.i.d.Unif}(0, 1)$ for all $k = 1, \ldots, N$. The vector function $\phi := (\phi_1, \ldots, \phi_N)^T$ can be computed by

$$\phi_1(u) := F_{\xi_1}^{-1}(u_1), \quad u := (u_1, \ldots, u_N)^T \in (0, 1)^N,$$

$$\phi_2(u) := F_{\xi_2|\xi_1}(u_2|\phi_1(u)), \quad u \in (0, 1)^N,$$

$$\ldots$$

$$\phi_N(u) := F_{\xi_N|\xi_1, \ldots, \xi_{N-1}}(u_N|\phi_1(u), \ldots, \phi_{N-1}(u)), \quad u \in (0, 1)^N,$$

where $F_{\xi_1}$ is the cumulative distribution function of $\xi_1$ and $F_{\xi_2|\xi_1}(\cdot|\phi_1(u))$ is the cumulative distribution function of $\xi_2$ given $\xi_1 = \phi_1(u)$, etc., i.e.,

$$F_{\xi_1}(z_1) := \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \cdots \int_{-\infty}^{z_N} p_{\xi_{XD}}(v_1, t_2, \ldots, t_N) dt_N \cdots dt_2 dv_1,$$

$$F_{\xi_2|\xi_1}(z_2|z_1) := \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \cdots \int_{-\infty}^{z_N} \frac{p_{\xi_{XD}}(z_1, v_2, t_3, \ldots, t_N) dt_N \cdots dt_3}{p_{\xi_{XD}}(z_1, t_2, \ldots, t_N) dt_N \cdots dt_2} dv_2,$$

$$\ldots$$

$$F_{\xi_N|\xi_1, \ldots, \xi_{N-1}}(z_N|z_1, \ldots, z_{N-1}) := \int_{-\infty}^{z_N} \frac{p_{\xi_{XD}}(z_1, \ldots, z_{N-1}, v_N)}{\int_{-\infty}^{z_N} p_{\xi_{XD}}(z_1, \ldots, z_{N-1}, t_N) dt_N} dv_N.$$

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