RANK 1 ABELIAN NORMAL SUBGROUPS OF 2-KNOT GROUPS

JONATHAN A. HILLMAN

Abstract. If a 2-knot group \( \pi \) other than \( \mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z} \) is almost coherent and has a torsion free abelian normal subgroup \( A \) of rank 1 which is not finitely generated then \( A \) meets nontrivially every subgroup which is not locally free, and \( A/A \cap \pi'' \) is finite cyclic, of odd order.

If \( A \) is an abelian normal subgroup of a 2-knot group \( \pi \) then either \( A \) is finitely generated or the Hirsch-Plotkin radical \( \sqrt{\pi} \) is torsion free and abelian of rank 1 and is not finitely generated. The only known example of the latter type is the group \( \Phi \) with presentation \( \langle t, a \mid tat^{-1} = a^2 \rangle \), which is the group of Fox’s Example 10 and its reflection \( \Phi \), and for which \( \sqrt{\pi} = \pi' \) (the commutator subgroup) and \( \pi' = \mathbb{Z}[\frac{1}{2}] \).

If \( \sqrt{\pi} \) is not finitely generated and \( \pi \not\cong \Phi \) then \( \pi \) is a PD4-group. It remains an open question whether there are any such 2-knot groups. Indeed, it is not known whether the centre of a PD4-group can fail to be finitely generated.

If there is such a 2-knot group \( \pi \) then we may ask whether \( \sqrt{\pi} \) is finitely generated as a \( \mathbb{Z}[\pi] \)-module. (Since \( \sqrt{\pi} \) is torsion free and abelian of rank 1, this is equivalent to it being the normal closure in \( \pi \) of one element.) Since \( Aut(\sqrt{\pi}) \) is abelian, \( \pi \) acts on \( \sqrt{\pi} \) through \( \pi/\pi' \cong \mathbb{Z} \), and so the module structure is determined by the action of a meridian. If \( \sqrt{\pi} \) is not finitely generated as a \( \mathbb{Z}[\pi/\pi'] \)-module, is it at least a minimax group? Although we cannot yet answer these questions, we shall use homological algebra to give some restrictions.

It is easily seen that if \( \pi \) has an infinite cyclic normal subgroup \( A < \pi' \) then \( A/A \cap \pi'' \) is finite of odd order. In §1 we show that each odd order is realized by \( \sqrt{\pi} \), for some fibred 2-knot. The main results consider the possibly empty class of 2-knots with \( \pi \not\cong \Phi \) and \( \sqrt{\pi} \) not finitely generated. In §2 we show that no such knot can be fibred or have group of deficiency 1. In §3 we show that if \( \pi \) is almost coherent (finitely generated subgroups are FP2) and these conditions hold then \( \sqrt{\pi}/\sqrt{\pi} \cap \pi'' \) is again finite of odd order, \( \sqrt{\pi} \) meets nontrivially every subgroup which is not locally free, and \( c.d.\pi'' = 3 \). If \( \pi \) has a normal subgroup \( N \) such that \( N < \pi' \) and \( c.d.N = 2 \) then \( \pi \) acts on \( \sqrt{\pi} \) through \( \pm 1 \).

1. INFINITE CYCLIC NORMAL SUBGROUPS

Let \( \pi \) be a 2-knot group with an abelian subgroup \( A \) of rank 1 such that \( A \leq \pi' \). Then \( \pi \) acts on \( A \) through \( \pi/\pi' \), since \( Aut(A) \) is abelian, and so \( A \leq \zeta\pi' \) (the centre of \( \pi' \)). The choice of a meridian \( t \in \pi \) determines an isomorphism \( \mathbb{Z}[\pi/\pi'] \cong \Lambda = \mathbb{Z}[t, t^{-1}] \), and \( tat^{-1} = \frac{2}{p}a \), for some \( \frac{2}{p} \in \mathbb{Q}^\times \) and all \( a \in A \). If \( A \) is finitely generated
as a \( \Lambda \)-module then it is cyclic, and is isomorphic to \( \mathbb{Z}[\frac{1}{pq}] \) as an abelian group. Hence it is normally generated in \( \pi \) by one element, and \( \pi/A \) is finitely presentable.

If \( A \) is not finitely generated as a \( \Lambda \)-module then every \( a \neq 1 \) in \( A \) is divisible by arbitrarily large integers relatively prime to \( pq \). Moreover, for any \( a \neq 1 \) in \( A \) the quotient \( \pi/(a) \) is finitely presentable and has an infinite abelian normal subgroup, and so has one end.

The group \( A \) has infinite cyclic subgroups which are normal in \( \pi \) if and only if \( \frac{\pi}{\pi} = \pm 1 \), in which case the centralizer \( C_\pi(A) \) has index \( \leq 2 \) in \( \pi \). If \( \pi \) is a PD4-group and has a normal subgroup \( C \cong \mathbb{Z} \) then \( G = \pi/C \) is finitely presentable, and an LHSSS argument shows that \( H^3(G; \mathbb{Z}[G]) \cong \mathbb{Z} \), while \( H^q(G; \mathbb{Z}[G]) = 0 \) if \( q \neq 3 \).

It is possible that these conditions imply that \( G \) is virtually a PD3-group. It would then follow that the abelian normal torsion subgroup \( A/C \) must be finite. We can confirm this in certain cases. We may write \( G = HNN(B; \phi : J \to K) \) with \( B, J \) and \( K \) finitely generated, by the Bieri-Strebel Theorem [2], since \( G \) is finitely presentable and \( G/G' \cong \mathbb{Z} \). The associated Mayer-Vietoris sequence for cohomology with coefficients \( \mathbb{Z}[G] \) gives an exact sequence

\[
0 \to H^2(B; \mathbb{Z}[G]) \to H^2(J; \mathbb{Z}[G]) \to H^3(G; \mathbb{Z}[G]) = \mathbb{Z} \to 0.
\]

If \( B \) is FP\( \infty \) and the HNN extension is ascending then it follows from [3] Lemma 3.4 that \( B = G' \), and so \( \pi' \) is finitely generated. Hence \( H^2(B; \mathbb{Z}[B]) \cong \mathbb{Z} \), and so \( B \) is virtually a PD2-group [3]. The group \( G \) is then virtually a PD3-group.

In particular, if \( \pi \) is almost coherent, has a normal subgroup \( C \cong \mathbb{Z} \) and has no non-cyclic free subgroup then \( G \) is such an HNN extension. Since the base has no non-cyclic free subgroup it is virtually \( \mathbb{Z}^2 \). Hence \( G \) and \( \pi \) are polycyclic, and so all subgroups are finitely generated.

**Lemma 1.** If \( \pi \) has a normal subgroup \( A \cong \mathbb{Z} \) such that \( A < \pi' \) then either \( A \leq \zeta \pi \cap \pi'' \) or \( [\pi : C_\pi(A)] \leq 2 \) and \( A/A \cap \pi'' \) is finite cyclic of odd order.

**Proof.** This follows easily from the facts that \( \text{Aut}(A) = \{ \pm 1 \} \) and that \( t - 1 \) acts invertibly on \( \pi'/\pi'' \). \( \square \)

For example, we may take the group \( \pi \) with presentation

\[
\langle a, b, c, d, e, f, z \mid [a, b][c, d][e, f] = z^m, \ az = za, \ cz = zc, \ ez = ze, \\
tat^{-1} = f, \ tbd^{-1} = ec, \ tct^{-1} = d, \ tdt^{-1} = cd, \ tet^{-1} = b, \ tft^{-1} = ab, \\
tzt^{-1} = z^{-1} \rangle,
\]

with \( m \) odd. It is easily seen that \( \pi \) is the normal closure of \( t \), while \( \sqrt{\pi} = \langle z \rangle \) and \( \sqrt{\pi}/\sqrt{\pi} \cap \pi'' \cong \mathbb{Z}/m\mathbb{Z} \). The corresponding knot manifold is the total space of an \( S^1 \)-bundle over a non-orientable 3-manifold, and is also the mapping torus of a self-homeomorphism of a \( \overline{\text{SL}} \)-manifold.

2. **NON-FINITELY GENERATED ABELIAN NORMAL SUBGROUP**

The most familiar constructions of 2-knots are perhaps

1. Artin spins of classical knots;
2. ribbon immersions of \( D^3 \) in \( S^4 \);
3. surgery on finite sets of loops in connected sums of \( S^3 \times S^1 \);
4. twist-spinning classical knots; and
5. surgery on the mapping torus of a self-homeomorphism of a 3-manifold.
These constructions are not disjoint; in fact (1) ⊆ (2) ⊆ (3) and (4) ⊆ (5). Knots arising from the first three have groups of deficiency 1, while those arising from the fourth and fifth are fibred, and so have finitely generated commutator subgroup. The groups which can be realized by either construction are the knot groups π with π' = F(r) free of finite rank r.

Just one of the knot groups arising from any of these constructions has an abelian normal subgroup which is not finitely generated.

**Theorem 2.** Let K be a 2-knot whose group π = πK has an abelian normal subgroup A which is not finitely generated. Then π' is not finitely generated. If π has deficiency 1 then π ≃ Φ and K is topologically isotopic to Fox’s Example 10 (up to reflection).

**Proof.** Let M = M(K) be the knot manifold obtained by elementary surgery on K. Then π1(M) ≃ π and χ(M) = 0. If π' is finitely generated then it has one end, since clearly A ≤ ζπ'. Hence M is aspherical and π' is a PD3-group [11, Theorem 4.5]. But the centres of PD3-groups are finitely generated [3].

If π has deficiency 1 then c.d.π = 2 [11, Theorem 2.5]. (Note that β1(2)(π) = 0 since π has an infinite amenable normal subgroup.) Since A is not finitely generated, A ≤ π', and since c.d.A ≤ c.d.π = 2, it must have rank 1, and so Aut(A) is abelian. Hence A ≤ ζπ'. Since c.d.π' ≤ 2 also and ζπ' is not finitely generated, π' must be abelian of rank 1 [11, Theorem 8.8]. Since π is finitely presentable and has abelianization Z, it is an HNN extension with finitely generated base B < π' and associated subgroups [2]. Since π is solvable, the extension must be ascending, and it follows easily that π ≃ Φ = Z*2.

In [8] it is shown that any knot with group Φ is TOP isotopic to Example 10 of [6], up to reflection.

In [11, Theorem 15.7] it is shown that if a 2-knot group π other than Φ has an abelian normal subgroup A which is not finitely generated then π is a PD4-group and √π is torsion-free abelian of rank 1.

**3. Abelian Normal Subgroups of PD4-groups**

We assume henceforth that π is a 2-knot group which is a PD4-group, and shall use the fact that subgroups of infinite index in PDn-groups have cohomological dimension < n [12] to derive our constraints. If a group is not finitely generated then its cohomology is related to the cohomology of its finitely generated subgroups through sequences involving lim and its derived functor. We summarize briefly the relevant material. (See [13].)

Let \{p_n : B_n \rightarrow B_{n-1} ; n \geq 1\} a sequence of homomorphisms of abelian groups. Then

\[ \lim_{\leftarrow}^0(B_n, p_n) = \text{Ker}(\Delta) \quad \text{and} \quad \lim_{\leftarrow}^1(B_n, p_n) = \text{Coker}(\Delta), \]

where \( \Delta : \Pi B_i \rightarrow \Pi B_i \) is the homomorphism given by \( \Delta(b_\bullet) = b_n - p_{n+1}(b_{n+1}) \), for all \( b_\bullet \in \Pi B_i \) and \( n \geq 0 \). (We shall usually write \( \lim_{\leftarrow}^0 B_n \) and \( \lim_{\leftarrow}^1 B_n \) for simplicity.) The sequence \( \{p_n\} \) satisfies the Mittag-Leffler condition if for each \( i \geq 0 \) the decreasing sequence of images of the successive composites \( q_{i,j} : B_{i+j} \rightarrow B_i \) with \( j > 0 \) stabilizes. If this is so then \( \lim_{\leftarrow}^1 B_n = 0 \). Conversely, if the groups \( B_i \) are all countable and \( \lim_{\leftarrow}^1 B_n = 0 \) then \( \{p_n\} \) satisfies the Mittag-Leffler condition [4, Satz V.2.9].
Let $G = \cup G_n$ be a group which is an increasing union of a sequence of subgroups. Then there are short exact sequences

$$0 \to \lim_{\to} H^{s-1}(G_k; M) \to H^s(G; M) \to \lim_{\to} H^s(G_k; M) \to 0,$$

for all $\mathbb{Z}[G]$-modules $M$ and all $s \geq 0$. (This may be obtained from the Mayer-Vietoris sequence associated with the direct limit $G = \cup G_n$, together with the above descriptions of $\lim_{\to}^0$ and $\lim_{\to}$).

**Lemma 3.** Let $G$ be an $FP_2$ group with one end and such that $c.d.G = 2$, and let $p$ be a prime. Then $M = H^2(G; \mathbb{Z}[G])$ is countable, and $p^{k+1}M$ is a proper submodule of $p^k M$, for all $k \geq 0$.

**Proof.** The module $M = H^2(G; \mathbb{Z}[G])$ is a submodule of a finitely generated free module, since $G$ is $FP_2$, and so is countable. Let $R_k = \mathbb{Z}/p^k \mathbb{Z}$, for $k \geq 1$. Then there are short exact sequences of $\mathbb{Z}[G]$-modules

$$(1)\quad 0 \to \mathbb{Z}[G] \xrightarrow{p^k} \mathbb{Z}[G] \to R_k[G] \to 0$$

and

$$(2)\quad 0 \to R_k[G] \xrightarrow{p} R_{k+1}[G] \to R_k[G] \to 0.$$

The exact sequence of cohomology associated to sequence (1) gives $H^2(G; R_k[G]) \cong M/M_k$, since $c.d.G = 2$. Since $G$ has one end, $H^i(G; R_k[G]) = 0$ for all $k \geq 1$, and so the exact sequence of cohomology associated to sequence (2) reduces to

$$0 \to M/M_k \to M/M_{k+1} \to M/M_k \to 0,$$

where the righthand map is the canonical epimorphism. Therefore if $M_{k+1} = M_k$ then $M/M_k = 0$, i.e., $M = pM$. Since $G$ is $FP_2$ and has one end it follows that $H^i(G; F) = 0$ for any free $F_p[G]$-module and $i \leq 2$. Hence $F_p$ is itself projective, and so the augmentation homomorphism $\varepsilon : F_p[G] \to F_p$ splits. It follows immediately that $G$ is finite, of order $\equiv 1 \mod (p)$, contradicting the hypotheses that $G$ have one end and $c.d.G = 2$. \hfill \Box

**Lemma 4.** Let $G$ be an $FP_2$ group with one end and such that $c.d.G = 2$, and let $A \cong \mathbb{Z}[\frac{1}{m}]$, where $m > 1$. Then $c.d.A \times G = 4$.

**Proof.** Clearly $c.d.A \times G \leq c.d.A + c.d.G = 4$.

We may write $A = \cup_{k \geq 0} A_k$, where $A_k \cong \mathbb{Z}$ and $A_k = mA_{k+1}$ for all $k \geq 0$. Let $G_k = A_k \times G$. Then $c.d.G_k = 3$.

Let $p$ be a prime factor of $m$, and let $F = \mathbb{Z}[G]$. There are natural isomorphisms $H^3(G_k; F) \cong H^2(G; F)$, for all $k \geq 0$. The homomorphism $p_k : H^3(G_k; F) \to H^3(G_{k-1}; F)$ induced by the inclusion $G_k \to G_{k+1}$ is given by multiplication by $m$ on $H^3(G; F)$. Then $H^4(A \times G; F) \cong \lim_{\to} H^3(G_k; F)$, since $H^4(G_k; F) = 0$ for all $k \geq 0$. Since $H^2(G; F)$ is countable, so are all the terms $H^3(G_k; F)$. Since the the sequence of subgroups $m^j H^2(G; F)$ with $j \geq 0$ is properly descending, by Lemma 3, the sequence $\{p_k\}$ does not satisfy the Mittag-Leffler condition. Hence $\lim_{\to} H^3(G_k; F) \neq 0$, and so $c.d.A \times G = 4$. \hfill \Box

Does the conclusion of this lemma hold if $FP_2$ is weakened to finitely generated?

**Theorem 5.** Let $\pi$ be a 2-knot group which is an almost coherent $PD_1$-group with an abelian normal subgroup $A$ which is torsion free and of rank 1, but not finitely generated. Then $A \leq \pi'$, we may assume that $p = 1$ (i.e., $tat^{-1} = a^q$ for all $a \in A$) and $A/A \cap \pi''$ is finite cyclic of odd order.
Proof. If $A$ is torsion-free of rank 1 and has non-trivial image in $\pi/\pi'$ then $A \cong \mathbb{Z}$, and so must be finitely generated.

Let $t$ be a meridian for $\pi$ and $x$ a nontrivial element of $A$. Then the subgroup $B$ generated by $\{t, x\}$ has the presentation

$$\langle t, x \mid tx^pt^{-1} = x^q, t^kxt^{-k} = x, \forall k \rangle.$$ 

Since $\pi$ is almost coherent, $B$ is $FP_2$, and so is an HNN extension with finitely generated base $[2]$. Since $B$ is solvable, the HNN extension must be ascending, and so (after replacing $t$ by $t^{-1}$, if necessary) we may assume that $p = 1$.

If $A \cap \pi'' = 1$ then $A$ embeds in the knot module $\pi'/\pi''$, and so is a direct summand of a submodule of finite index. Hence $A \cong \Lambda/(t - q)\Lambda$, and $A \cong \mathbb{Z}[\frac{1}{r}]$ as an abelian group. Moreover, $\pi'$ has a subgroup $\rho$ of finite index such that $A\rho \leq \rho$ and $\rho \cong A \times \sigma$, where $\sigma = \rho/A$. Since $\pi$ is a PD$_1$-group and $\pi/\pi' \cong \mathbb{Z}$, $h.d.\pi' \geq 3$, while $c.d.\pi' \leq 3$ by $[12]$. Hence $h.d.\pi' = c.d.\pi' = 3$. Since $\mathbb{Z} \times \sigma \leq \pi'$, we see that $c.d.\sigma \leq 2$. Since $h.d.\pi = 1$, we have $1 + h.d.\sigma = h.d.\rho = h.d.\pi' = 3$, and so $h.d.\sigma = 2$. Therefore $c.d.\sigma = h.d.\sigma = 2$, so $\sigma$ is not locally free. Let $\nu$ be a finitely generated subgroup of $\sigma$ which is not free, and is indecomposable as a free product. Then $\nu$ has one end and $c.d.\nu = 2$, and so $c.d.\pi \times \pi = 4$, by Lemma 4. But this contradicts $c.d.\pi' = 3$. Hence $A/A \cap \pi'' \neq 1$.

Since $A/A \cap \pi''$ is a $\mathbb{Z}$-torsion submodule of the knot module $\pi'/\pi''$, it is finite, and hence cyclic. It is of odd order, since $t - 1$ acts invertibly on $\pi'/\pi''$.

The order of $A/A \cap \pi''$ is relatively prime to $q$ and $q - 1$, since $q$ and $t - 1$ act invertibly on it.

**Corollary 6.** The characteristic class in $H^2(\pi'/A; A)$ for $\pi'$ as an extension of $\pi'/A$ by $A$ has infinite order.

**Proof.** The image of this characteristic class in $\text{Hom}(H_2(\pi'/A; \mathbb{Z}), A)$ is the connecting homomorphism $\delta$ in the five-term exact sequence of low degree

$$H_2(\pi'; \mathbb{Z}) \to H_2(\pi'/A; \mathbb{Z}) \xrightarrow{\delta} A \to \pi'/\pi'' \to \pi'/A\pi'' \to 0.$$ 

(See $[9]$ Theorem 4.) The corollary follows since $A$ is infinite and torsion-free, and its image in $\pi'/\pi''$ is finite. □

In particular, $\pi'$ does not have a subgroup of finite index which splits as a direct product $A \times \sigma$. In fact it is clear from the proof of Theorem 7 that no subgroup $\tau \leq \pi'$ which contains $A$ and with $h.d.\tau = 3$ can split as a direct product.

If $\pi \not\cong \Phi$ then it is not elementary amenable $[11]$ Theorem 15.14. Hence $\pi'/A$ is not locally finite. Must it have an element of infinite order? Since $h.d.\pi' = c.d.\pi' = 3$, we may write $\pi' = \bigcup_{k \geq 0} P_k$ as a union of finitely generated subgroups $P_k$ with $h.d.P_k = c.d.P_k = 3$.

We may also use considerations of cohomological dimension to show that $A$ interacts strongly with subgroups which are not locally free.

**Lemma 7.** If a PD$_1$-group $\pi$ has a finitely generated normal subgroup $N$ such that $N \leq \pi'$, $c.d.N = 2$ and $\zeta N \neq 1$ then $h(\sqrt{\pi}) \geq 2$.

**Proof.** Since $N$ is finitely generated, $c.d.N = 2$ and $\zeta N \neq 1$, either $N$ is $\mathbb{Z}^2$ or $N'$ is free of finite rank $[11]$ Theorem 8.8. If $N'$ is free of rank $r > 0$ then

$$H^3(\pi/N'; H^1(N'; \mathbb{Z}[\pi])) \cong H^4(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z},$$
by an LHSSS corner argument. Hence

\[ H^3(\pi/N'; Z[\pi/N']) \otimes H^1(N'; Z[N']) \cong \mathbb{Z}, \]

and so \( N' \) has rank 1. Hence \( N' \cong \mathbb{Z} \times \mathbb{Z} \).

In either case, \( \sqrt{N} \cong \mathbb{Z}^2 \) and so \( h(\sqrt{\pi}) \geq 2 \). \( \square \)

**Theorem 8.** Let \( \pi \) be a 2-knot group which is an almost coherent PD\(_4\)-group with an abelian normal subgroup \( A \) which is not finitely generated. Let \( N \) be a subgroup of \( \pi' \) which is not locally free. Then \( A \cap N \neq 1 \). If \( N \) is normal in \( \pi \) and \( c.d. N = 2 \) then \( N' \) is free of rank \( > 1 \), \( [\pi : C_\pi(A)] \leq 2 \) and \( N \) is not finitely generated.

**Proof.** If \( A \cap N = 1 \) then \( AN \cong A \times N \), since \( A \) is central in \( \pi' \). If \( c.d. N = 3 \) then \( c.d. Z \times N = 4 \), and so \( A \times N \) would have finite index in \( \pi \). Therefore we may assume that \( c.d. N = 2 \). Since \( N \) is not locally free it has a finitely generated subgroup \( \nu \) with one end. Since \( c.d. A \times \nu = 4 \), by Lemma 4, while \( c.d. A \nu \leq 3 \), we must have \( A \cap N \neq 1 \). Hence \( \zeta N \neq 1 \). Since \( c.d. N = 2 \) and \( N \) is not locally cyclic, either \( N \cong \mathbb{Z}^2 \) or \( N \cong \mathbb{Z} \times \mathbb{Z} \), or \( N' \) is free of rank \( > 1 \) and \( \zeta N \cong \mathbb{Z} \), by Theorem 8.8 of [1]. In each case, \( A \cap N \cong \mathbb{Z} \). If \( N \) is normal in \( \pi \) then so is \( A \cap N \), and so \( \frac{A}{\pi} \cong \mathbb{Z} \). Hence \( [\pi : C_\pi(A)] \leq 2 \). Moreover \( N' \) must then be free of rank \( > 1 \), since \( \sqrt{N} \leq \sqrt{\pi} = A \). The final observation follows from Lemma 7, since \( \sqrt{\pi} \) is torsion-free abelian of rank 1, by [7] Theorem 15.7]. \( \square \)

If \( N \) is a locally free subgroup which is not abelian then it has no nontrivial abelian normal subgroup, and so \( A \cap N = 1 \). (Note also that \( \Phi' \) is locally free.)

**Corollary 9.** If \( \pi'' \) is finitely generated then it is FP\(_2\), and \( \pi \) is an ascending HNN extension over a finitely generated, one ended base. However \( \pi'' \) is not FP\(_3\). In all cases, \( c.d. \pi'' = 3 \).

**Proof.** If \( \pi'' \) is finitely generated then it is FP\(_2\), since \( \pi \) is almost coherent. Moreover \( \pi/\pi'' \) is finitely presentable, and so is an HNN extension over a finitely generated base [2]. Since \( \pi/\pi'' \) is metabelian the extension is ascending and the base is finitely presentable. The HNN structure lifts to an ascending HNN structure for \( \pi \), and the base is again finitely generated, since \( \pi'' \) is finitely generated. The HNN base has one end since it has an infinite abelian normal subgroup.

If \( \pi'' \) were FP\(_3\) then it would be FP. Let \( \tau \) be the preimage in \( \pi \) of the torsion subgroup of \( \pi'/\pi'' \). Then \( \tau \) is also FP, since it is torsion-free and \( [P : \pi'' \pi'] \) is finite, and \( \pi/\tau \) is torsion-free metabelian of finite Hirsch length. Hence \( c.d. \pi = c.d. \pi + c.d. \pi/\tau \) [1] Theorem 5.5], and so \( c.d. \pi/\tau = 1 \). Hence \( \tau = \pi' \). But \( \pi' \) is not finitely generated, so \( \pi'' \) cannot be FP\(_3\).

Since \( \pi \) is a PD\(_4\)-group, \( \pi'' \) is not abelian, for otherwise \( \pi \) would be polycyclic and all subgroups would be finitely generated. Since \( A \cap \pi'' \) is not finitely generated, by Theorem 5, and is central in \( \pi'' \), it follows from Theorem 8 that \( c.d. \pi'' \geq 2 \). Hence \( c.d. \pi'' = c.d. \pi' = 3 \). \( \square \)

There are uncountably many torsion-free abelian groups of rank 1. Only countably many can occur as \( \sqrt{G} \) for some finitely presentable group \( G \). A countable abelian group is the centre of some finitely presentable group if and only if it is recursively presentable [11]. What else can be said in the present context? Must \( \sqrt{\pi} \) be minimax? (The minimax subgroups of \( \mathbb{Q} \) are isomorphic to \( Z[\frac{1}{m}] \), for some \( m \geq 1 \)?) Every subgroup of \( \mathbb{Q} \) which is not finitely generated contains such a group, for some \( m > 1 \).
References

[1] Bieri, R. Homological Dimension of Discrete Groups, Queen Mary College Lecture Notes (1976).
[2] Bieri, R. and Strebel, R. Almost finitely presentable soluble groups, Comment. Math. Helvetici 53 (1978), 258–278.
[3] Bowditch, B. H. Planar groups and the Seifert conjecture, J. Reine Angew. Math. 576 (2004), 11–62.
[4] Bröcker, T. and tom Dieck, T. Kobordismentheorie, Lecture Notes in Mathematics 178, Springer-Verlag, Berlin – Heidelberg – New York (1970).
[5] Brown, K. S. and Geoghegan, R. Cohomology with free coefficients of the fundamental group of a graph of groups, Comment. Math. Helvetici 60 (1985), 31–45.
[6] Fox, R. H. A quick trip through knot theory, in Topology of 3-Manifolds and Related Topics (edited by M.K.Fort, Jr), Prentice-Hall, Englewood Cliffs, N.J.(1962), 120–167.
[7] Hillman, J. A. Four-Manifolds, Geometries and Knots, Geometry and Topology Monographs 5, Geometry and Topology Publications (2002). (Revisions 2007 and 2014).
[8] Hillman, J. A. Strongly minimal PD4-complexes, Top. Appl. 156 (2009), 1565–1577.
[9] Hillman, J. A. Sections of surface bundles, in Interactions between low dimensional topology and mapping class groups, Max Planck Institut Conference (2013), Geometry and Topology Monographs, vol. 19, 1–19, Mathematical Sciences Publications, Berkeley, Cal. (2015).
[10] Jensen, C. U. Les foncteurs dérivées de lim et ses applications a la théorie des modules, Lecture Notes in Mathematics 254, Springer-Verlag, Berlin – Heidelberg – New York (1972).
[11] Ould Houcine, A. Embeddings in finitely presentable groups which preserve the centre, J. Alg. 307 (2007), 1–23.
[12] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups, Comment. Math. Helv. 52 (1977), 317–324.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: jonathan.hillman@sydney.edu.au