MOTIVIC CONNECTIVE $K$-THEORIES AND THE COHOMOLOGY OF $A(1)$

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ABSTRACT. We make some computations in stable motivic homotopy theory over $\text{Spec} \mathbb{C}$, completed at 2. Using homotopy fixed points and the algebraic $K$-theory spectrum, we construct a motivic analogue of the real $K$-theory spectrum $KO$. We also establish a theory of connective covers to obtain a motivic version of $ko$. We establish an Adams spectral sequence for computing motivic $ko$-homology. The $E_2$-term of this spectral sequence involves $\text{Ext}$ groups over the subalgebra $A(1)$ of the motivic Steenrod algebra. We make several explicit computations of these $E_2$-terms in interesting special cases.

1. INTRODUCTION

In classical homotopy theory, there is a strong and useful relationship between the connective $K$-theories $ku$ and $ko$ and algebraic computations involving the subalgebras $E(1)$ and $A(1)$ of the Steenrod algebra. More specifically, the $\mathbb{F}_2$-cohomology $H^*(ku)$ of $ku$ is equal to the quotient $A//E(1)$ of the Steenrod algebra by the augmentation ideal of the subalgebra $E(1)$. For formal algebraic reasons, it follows that the Adams spectral sequence for computing $ku_*(X)$ has $\text{Ext}_{E(1)}(H^*X, \mathbb{F}_2)$ as its $E_2$-term. Thus, computations in $ku$-homology are essentially the same as algebraic computations of $E(1)$-modules. Similarly, $H^*(ko)$ is equal to $A//A(1)$, and the Adams spectral sequence for computing $ko_*(X)$ has $\text{Ext}_{A(1)}(H^*X, \mathbb{F}_2)$ as its $E_2$-term.

The goal of this paper is to describe similar phenomena in 2-complete motivic stable homotopy theory over $\text{Spec} \mathbb{C}$. Over $\text{Spec} \mathbb{C}$, the motivic Steenrod algebra is completely understood [20], and detailed algebraic computations are possible [5]. After 2-completion, we know enough about the motivic homotopy groups of spheres [5] [11] in order to carry out the necessary homotopical arguments.

In motivic homotopy theory, the algebraic $K$-theory spectrum $KGL$ plays the role of the classical complex $K$-theory spectrum $KU$. We are able to construct a connective version, which we call $kgl$, and we compute that the motivic cohomology $H^{**}(kgl)$ of $kgl$ with $\mathbb{F}_2$-coefficients is equal to $A//E(1)$, where $A$ is the motivic Steenrod algebra and $E(1)$ is the subalgebra generated by classes $Q_0 = Sq^1$ and $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$ of degrees $(1,0)$ and $(3,1)$ respectively. It follows that over $\text{Spec} \mathbb{C}$ and working 2-complete, the motivic Adams spectral sequence [11] [5] for computing $kgl_*(X)$ has $\text{Ext}_{E(1)}(H^{**}X, \mathbb{M}_2)$ as its $E_2$-term, where $\mathbb{M}_2 = \mathbb{F}_2[\tau]$ is the motivic cohomology of a point.

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There are two possible approaches to constructing a motivic spectrum that is analogous to classical $KO$. The first approach is to use Hermitian $K$-theory, as in [8], but we do not adopt this viewpoint. Rather, we use a $\mathbb{Z}/2$-action on $KGL$ and apply homotopy fixed points to obtain a new motivic spectrum $KGL^{h\mathbb{Z}/2}$. This is analogous to the classical fact that $KO$ is equivalent to $KU^{h\mathbb{Z}/2}$. We shall call this spectrum $KO$ for convenience.

Although we believe that $KGL^{h\mathbb{Z}/2}$ is equivalent to Hermitian $K$-theory over $\text{Spec } \mathbb{C}$, we have not proved this result here because the methods would take us too far afield from our main point. Philosophically, we find that the homotopy fixed points description of our motivic spectrum tells us everything that we would like to know about it. It turns out that we really don’t need a more concrete geometric description.

Having constructed the motivic version of $KO$, we can consider the connective cover $ko$. We compute that the motivic cohomology $H^{**}(ko)$ of $ko$ is equal to $A/A(1)$, where $A(1)$ is the subalgebra of the motivic Steenrod algebra generated by $Sq^1$ and $Sq^2$. It follows that over $\text{Spec } \mathbb{C}$ and working 2-complete, the motivic Adams spectral sequence for computing $ko_{*,*}(X)$ has $\text{Ext}_{A(1)}(H^{**}X, M_2)$ as its $E_2$-term.

The result is that motivic $ko$-homology is effectively computable. To demonstrate this point, we compute the groups $\text{Ext}_{A(1)}(M, M_2)$ for various $A(1)$-modules $M$ of interest. In this paper, we do not provide any calculations over $E(1)$ because the motivic calculations are essentially identical to the classical ones.

Our algebraic computations over motivic $A(1)$ are similar to the classical computations over $A(1)$. One of the fundamental differences between classical homotopy theory and motivic homotopy theory is that $\eta^4$ is zero classically, but $\eta^k$ is non-zero motivically for all $k \geq 0$. Our calculations over $A(1)$ detect this difference.

Along the way to computing the cohomology of $kgl$ and $ko$, we need a collection of technical results about cellular motivic spectra over $\text{Spec } \mathbb{C}$. These results are likely to be useful in other contexts as well.

Ultimately, one would like to compute as much as possible about $\text{Ext}_A(M_2, M_2)$, which is the $E_2$-term of the Adams spectral sequence that converges to motivic stable homotopy groups. Some progress on this has been made [9], and our computations over $A(1)$ also contribute to this larger program. We have focused on $kgl$-homology and $ko$-homology because of their relationship to motivic stable homotopy groups, but it is possible to study $kgl$-cohomology and $ko$-cohomology within the same framework.

Since we are working over $\text{Spec } \mathbb{C}$, there is a realization functor from motivic homotopy theory to classical homotopy theory [2] [16, Section 3.3]. Our motivic computations must be compatible under this functor with the corresponding classical computations. We strive to provide proofs that are internal to motivic homotopy theory wherever possible, but realization will be a useful tool for us from time to time.

In this paper, we will use the notion of motivic ring spectra, but only in the naive sense. For our purposes, we have no need for modern theories of highly structured ring spectra, although such theories do exist for motivic spectra [7] [9] [10] [14].

1.1. Organization. We begin with a review of the algebraic objects that we will study. Then we provide some background on 2-complete motivic stable homotopy theory over $\text{Spec } \mathbb{C}$. Here we are collecting the homotopical tools necessary for
our later calculations. We then compute the homotopy of $ko$ using a homotopy fixed points spectral sequence. We also compute the homotopy of $kgl^{h\mathbb{Z}/2}$; this calculation contains a curious difference to the analogous classical calculation of the homotopy of $ku^{h\mathbb{Z}/2}$. Our next calculation is the ordinary motivic $\mathbb{F}_2$-cohomology of $kgl$ and $ko$. This allows us to describe the $E_2$-terms of Adams spectral sequences for computing $kgl_{*,*}(X)$ and $ko_{*,*}(X)$. Finally, we conclude the paper with some specific calculations related to $ko$-homology.

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2. Algebraic Definitions

In this section, we introduce the basic algebraic objects that we will study, and we remind the reader of their relevance to motivic homotopy theory.

Throughout the paper, we will primarily be working with bigraded objects.

Definition 2.1. An element of bidegree $(a, b)$ is said to have topological degree $a$ and weight $b$.

Definition 2.2. Let $\mathcal{M}_2$ be the bigraded polynomial ring $\mathbb{F}_2[\tau]$, where $\tau$ has bidegree $(0, 1)$.

The importance of $\mathcal{M}_2$ is that it is the $\mathbb{F}_2$-motivic cohomology ring of Spec $C$ [22].

Recall that the motivic Steenrod algebra $A$ is an $\mathcal{M}_2$-algebra generated by elements $Sq^{2k}$ and $Sq^{2k+1}$ of bidegrees $(2k, k)$ and $(2k + 1, k)$, subject to a motivic version of the Adem relations [21].

Notation 2.3. We write $Sq^{i_1, \ldots, i_n}$ for the product $Sq^{i_1} \cdots Sq^{i_n}$.

Definition 2.4.

1. Let $E(0)$ be the $\mathcal{M}_2$-subalgebra of $A$ generated by $Sq^1$.
2. Let $E(1)$ be the $\mathcal{M}_2$-subalgebra of $A$ generated by $Q_0 = Sq^1$ and $Q_1 = Sq^{1,2} + Sq^{2,1}$.
3. Let $A(1)$ be the $\mathcal{M}_2$-subalgebra of $A$ generated by $Sq^1$ and $Sq^2$.

The following lemma is a straightforward calculation with low-dimensional Adem relations.

Lemma 2.5.

1. $E(0)$ is equal to $\mathcal{M}_2[Sq^1]/Sq^{1,1}$.
2. $E(1)$ is equal to $\mathcal{M}_2[Q_0, Q_1]/Q_0^2, Q_1^2, Q_0 Q_1 + Q_1 Q_0$, i.e., the exterior $\mathcal{M}_2$-algebra on $Q_0$ and $Q_1$.
3. $A(1)$ is equal to $\mathcal{M}_2[Sq^1, Sq^2]$: $Sq^{1,1} = 0, Sq^{2,2} = \tau Sq^{1,2,1}, Sq^{1,2,1,2} = Sq^{2,1,2,1}$.

Remark 2.6. Classically, the Adem relation $Sq^{2,2} = Sq^{1,2,1}$ implies that $Sq^{1,2,1,2} = Sq^{2,1,2,1}$. However, in the motivic situation, the Adem relation $Sq^{2,2} = \tau Sq^{1,2,1}$ only implies that $\tau Sq^{1,2,1,2} = \tau Sq^{2,1,2,1}$. Therefore, we must include the relation $Sq^{1,2,1,2} = Sq^{2,1,2,1}$ in the description of $A(1)$.
Figure 2.1 is a pictorial representation of $A(1)$, where each circle at height $a$ stands for a copy of $M_2$ in topological degree $a$. Multiplications by $Sq^1$ are represented by straight lines, and multiplications by $Sq^2$ are represented by curved lines. The dashed line indicates that $Sq^2$ on the generator in bidegree $(2, 1)$ equals $\tau$ times the generator in bidegree $(4, 1)$.

For a subalgebra $B$ of the motivic Steenrod algebra $A$, we write $A//B$ for the quotient of $A$ by the augmentation ideal of $B$. In this paper, $B$ will be $E(0)$, $E(1)$, or $A(1)$.
Lemma 2.7.

1. The kernel of right multiplication by \( Sq^1 \) on \( A \) equals the image of right multiplication by \( Sq^1 \).
2. The kernel of right multiplication by \( Q_1 \) on \( A//E(0) \) equals the image of right multiplication by \( Q_1 \).
3. The kernel of right multiplication by \( Sq^2 \) on \( A//E(1) \) equals the image of right multiplication by \( Sq^2 \).

Proof. All three claims follow in the motivic case for the same combinatorial reasons as in the classical case. \( \square \)

Definition 2.8. For any \( A(1) \)-module \( M \), let \( \Sigma^{k,l} M \) denote the \( A(1) \)-module obtained by increasing the bidegree of each element of \( M \) by \((k, l)\). Let \( \Sigma \) be \( \Sigma^{1,0} \).

Lemma 2.9. There are short exact sequences

\[
0 \to \Sigma A//E(0) \xrightarrow{-Sq^1} A \to A//E(0) \to 0
\]

\[
0 \to \Sigma^{3,1} A//E(1) \xrightarrow{-Q_1} A//E(0) \to A//E(1) \to 0
\]

\[
0 \to \Sigma^{2,1} A//A(1) \xrightarrow{-Sq^2} A//A(1) \to A//A(1) \to 0.
\]

Proof. This follows immediately from Lemma 2.7. \( \square \)

Definition 2.10. For \( n = 2k \), let \( Q_n \) be the quadric hypersurface of \( \mathbb{CP}^{n+1} \) defined by \( x_0 x_1 + \cdots + x_{2k} x_{2k+1} = 0 \). For \( n = 2k + 1 \), let \( Q_n \) be the quadric hypersurface of \( \mathbb{CP}^{n+1} \) defined by \( x_0 x_1 + \cdots + x_{2k} x_{2k+1} + x_{2k+2}^2 = 0 \). Let \( DQ_n \) be the open complement \( \mathbb{CP}^n - Q_{n-1} \) of \( Q_{n-1} \) in \( \mathbb{CP}^n \).

Consider the inclusions

\[
\mathbb{CP}^{2k} \to \mathbb{CP}^{2k+1} : [x_0 : \cdots : x_{2k}] \mapsto [x_0 : \cdots : x_{2k-1} : x_{2k} : x_{2k+1}]
\]

and

\[
\mathbb{CP}^{2k-1} \to \mathbb{CP}^{2k} : [x_0 : \cdots : x_{2k-1}] \mapsto [x_0 : \cdots : x_{2k-1} : 0].
\]

These maps restrict to inclusions \( DQ_n \to DQ_{n+1} \). We will implicitly assume that \( DQ_n \) is a subvariety of \( DQ_{n+1} \).
Definition 2.11. Let $DQ_{\infty}$ be $\operatorname{colim}_n(DQ_n)$.

All cohomology groups throughout the paper are taken with $\mathbb{F}_2$-coefficients.

Proposition 2.12. \cite{3} \cite{21}

\[
\begin{align*}
H^{*,*}(DQ_n) &= \bigg\{ 
\begin{array}{ll}
\mathbb{M}_2[a, b]/a^2 - \tau b, b^{k+1} & \text{if } n = 2k + 1 \\
\mathbb{M}_2[a, b]/a^2 - \tau b, b^{k+1}, ab^k & \text{if } n = 2k
\end{array}
\bigg\} \\
H^{*,*}(DQ_{\infty}) &= \mathbb{M}_2[a, b]/a^2 - \tau b,
\end{align*}
\]

where $a$ has bidegree $(1, 1)$ and $b$ has bidegree $(2, 1)$.

Proposition 2.13. The $A(1)$-module structure on $H^{*,*}(DQ_n)$ is given by:

\[
\begin{align*}
\text{Sq}^1ab^k &= b^{k+1} \\
\text{Sq}^1b^k &= 0 \\
\text{Sq}^2b^k &= \bigg\{ 
\begin{array}{ll}
0 & \text{if } k \text{ is even} \\
b^{k+1} & \text{if } k \text{ is odd}
\end{array}
\bigg\} \\
\text{Sq}^2ab^k &= \bigg\{ 
\begin{array}{ll}
0 & \text{if } k \text{ is even} \\
ab^{k+1} & \text{if } k \text{ is odd}
\end{array}
\bigg\}.
\end{align*}
\]

Proof. As proved in \cite{3} Lemma 4.7 and \cite{21}, we have that $\text{Sq}^1a = b$. Since $\text{Sq}^2$ has degree $(2, 1)$ and $a$ has degree $(1, 1)$, properties of the motivic Steenrod algebra given in \cite{21} Lemma 9.9 imply that $\text{Sq}^2a = 0$.

From the definition of $A(1)$, we have $\text{Sq}^1b = \text{Sq}^{-1}a = 0$. Also, results in \cite{21} Lemma 9.8 imply that $\text{Sq}^2b = b^2$.

The $A(1)$-action on the other elements of $H^{*,*}(DQ_n)$ follows from the Cartan formula \cite{21} Proposition 9.6].

Remark 2.14. The $A(1)$-module structure on $H^{*,*}(DQ_n)$ is easily derived from the above proposition, using the map $DQ_n \to DQ_{\infty}$.

Figure 2.2 is a pictorial representation of the $A(1)$-module $H^{*,*}(DQ_{\infty})$, where a circle at height $p$ stands for a copy of $\mathbb{M}_2$ in topological degree $p$. Multiplications by $\text{Sq}^1$ are indicated by straight lines, and multiplications by $\text{Sq}^2$ are indicated by curved lines.

3. BACKGROUND ON MOTIVIC SPECTRA

In this paper, we work with stable motivic homotopy theory over $\text{Spec } \mathbb{C}$, after completion at the Eilenberg-Mac Lane spectrum $HF_2$ in the sense of \cite{5}. We suppress the completion from the notation. (For example, $\pi_{p,q}$ is the $HF_2$-completed motivic stable homotopy group.) The results of \cite{11} imply that $HF_2$-completion is the same as 2-completion.

We refer to \cite{6} Part 3] for background on motivic stable homotopy theory. There are several well-behaved model structures for motivic stable homotopy theory. Although we will use model theoretic techniques occasionally, we will not be precise about these technical details. For example, we shall implicitly assume that all motivic spectra are cofibrant and fibrant.

If $X$ is a based motivic space, then we abuse notation and write $X$ also for the motivic suspension spectrum of $X$.

The following calculation is essential to our results.
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Figure 2.2. Action of A(1) on H^*(DQ_∞)

Proposition 3.1.

1. The subring π_0,∗ = ⊕_k π_0,k of the motivic stable homotopy ring is equal to Z_2[τ], where τ has bidegree (0, −1).

2. The motivic stable homotopy group π_p,q is zero if p < 0 or q > p.

Proof. This follows from motivic versions of the Adams spectral sequence [5] or the Adams-Novikov spectral sequence [11]. □

For any X and any j, observe that π_j,∗X = ⊕_k π_j,k X is a module over Z_2[τ].

We recall the notion of cellular spectra from [4, Definition 2.10].

Definition 3.2. The class of cellular motivic spectra is the smallest class such that:

1. Every sphere S^{p,q} is cellular;
2. If X is weakly equivalent to a cellular motivic spectrum, then it is cellular;
3. If X is a homotopy colimit of cellular motivic spectra, then it is cellular.

We will need the following Whitehead theorem for cellular motivic spectra.

Proposition 3.3 (DDDI2, Corollary 7.2). A map between cellular motivic spectra is a weak equivalence if and only if it induces an isomorphism on π_p,q for all p and q.

Recall that the motivic Eilenberg-Mac Lane spectrum H\mathbb{F}_2 represents motivic cohomology with F_2-coefficients.

Proposition 3.4 ([11]). The motivic spectrum H\mathbb{F}_2 is cellular.

The following proposition is proved in [12] and [14].

Proposition 3.5. The functor π_p,q commutes with filtered colimits of motivic spectra.

Finally, we recall that there is a topological realization functor from motivic spectra to ordinary spectra [2] [16, Section 3.3]. It is built from the functor that takes a complex algebraic variety to its space of \mathbb{C}-valued points.

Topological realization preserves homotopy colimits, and in particular preserves cofiber sequences. As a result, all of our motivic calculations must be compatible
with analogous classical calculations. For example, there are maps of long exact sequences and spectral sequences from the motivic setting to the classical setting, but we will not make this precise.

3.1. Equivariant motivic spectra. Let \( G \) be a group. (In our application, we are concerned only with the group \( \mathbb{Z}/2 \).) For general reasons, there is a convenient homotopy theory of \( G \)-equivariant motivic spectra, where weak equivalences are detected by the underlying non-equivariant motivic spectra. The homotopy theory of \( G \)-equivariant motivic spectra has an internal function object. This means that for any equivariant motivic spectra \( X \) and \( Y \), there is an equivariant motivic spectrum \( \mathcal{F}_G(X,Y) \) with appropriate adjointness properties. There are other possible homotopy theories for equivariant motivic spectra, but this is the one that is relevant to homotopy fixed point constructions.

We apply the standard notion of a homotopy fixed point spectrum to our motivic setting. Among many other places in the literature, homotopy fixed point spectra are treated in [17, Section 2.1].

The following definition is lifted straight from the classical setting.

**Definition 3.6.** Let \( X \) be a motivic spectrum with an action by a group \( G \). Define the homotopy fixed point spectrum \( X^{hG} \) of \( X \) to be \( \mathcal{F}_G(EG_+, X) \).

**Lemma 3.7.** If \( X \) is a ring spectrum with an action by \( G \), then \( X^{hG} \) is also a ring spectrum.

**Proof.** The multiplication map \( X^{hG} \wedge X^{hG} \rightarrow X^{hG} \) is the composition

\[
\mathcal{F}_G(EG_+, X) \wedge \mathcal{F}_G(EG_+, X) \rightarrow \mathcal{F}_G(EG_+ \wedge EG_+, X) \rightarrow \mathcal{F}_G(EG_+, X),
\]

where the first map is smash product of maps, and the second map is induced by the diagonal \( EG_+ \rightarrow EG_+ \wedge EG_+ \) and the multiplication \( X \wedge X \rightarrow X \).

The following theorem can be proved in the same way as the analogous classical result.

**Theorem 3.8.** Let \( G \) be a finite group, and let \( X \) be a \( G \)-equivariant motivic spectrum. The motivic homotopy fixed point spectral sequence is conditionally convergent and takes the form

\[
E_2^{a,p,u} = H^p(G; \pi_{n+p,u}X) \Rightarrow \pi_{n+u}(X^{hG}),
\]

where \( H^p(G; -) \) is group cohomology.

3.2. Motivic connective covers. We continue to work in \( HF_2 \)-complete stable motivic homotopy theory over \( \text{Spec } \mathbb{C} \).

**Proposition 3.9.** For any map \( f : X \rightarrow Y \) of motivic spectra, there is a factorization

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Z \\
\p & \rightarrow & Y
\end{array}
\]

of \( f \) such that:

1. \( p \) is an isomorphism on \( \pi_{a,b} \) if \( a \geq 1 \) and \( a - b \geq 1 \).
2. \( i \) is an isomorphism on \( \pi_{a,b} \) if \( a < 0 \) or \( a - b < 0 \).
3. \( \mathcal{F} \) is injective on \( \pi_{0,b} \) if \( b \leq 0 \) and on \( \pi_{a,0} \) if \( a \geq 0 \).
4. \( i \) is surjective on \( \pi_{0,b} \) if \( b \leq 0 \) and on \( \pi_{a,0} \) if \( a \geq 0 \).
Proof. It is easiest to work with model categories. We may assume that $X$ and $Y$ are cofibrant and fibrant.

Let $S^{p,q} \to D^{p,q}$ be a cofibration whose target is contractible. Construct $Z$ by applying the small object argument to the set of generating acyclic cofibrations together with all maps of the form $S^{p,q} \to D^{p,q}$, with $p \geq 0$ and $p - q \geq 0$. Proposition 3.1(2) ensures that $Z$ has the desired motivic homotopy groups.

Remark 3.10. The factorizations produced in Proposition 3.9 are functorial in $f$.

Corollary 3.11. Given a motivic spectrum $X$, there exists a motivic spectrum $PX$ and a map $X \to PX$ such that:

(1) $\pi_{p,q}X \to \pi_{p,q}PX$ is an isomorphism for $p < 0$ or $p - q < 0$, and
(2) $\pi_{p,q}PX = 0$ if $p \geq 0$ and $p - q \geq 0$.

Proof. Apply Proposition 3.9 to the map $X \to \ast$.

Corollary 3.12. Given a motivic spectrum $X$, there exists a motivic spectrum $CX$ and a map $CX \to X$ such that:

(1) $\pi_{p,q}CX \to \pi_{p,q}X$ is an isomorphism for $p \geq 0$ and $p - q \geq 0$, and
(2) $\pi_{p,q}CX = 0$ if $p < 0$ or $p - q < 0$.

Proof. Apply Proposition 3.9 to the map $\ast \to X$.

Definition 3.13. The motivic spectrum $CX$ of Corollary 3.12 is the connective cover of $X$.

Remark 3.14. Let $G$ be a group. The proof of Proposition 3.9 works just as well in the category of $G$-equivariant motivic spectra. Therefore, we can construct connective covers of $G$-equivariant motivic spectra.

3.3. Positive cellular motivic spectra.

Definition 3.15. The class of positive cellular motivic spectra is the smallest class of motivic spectra such that:

(1) $S^{p,q}$ is positive cellular if $p \geq 0$ and $p - q \geq 0$.
(2) If $f : X \to Y$ is any map such that $X$ and $Y$ are both positive cellular, then the cofiber of $f$ is also positive cellular.
(3) If $X$ is weakly equivalent to a positive cellular motivic spectrum, then $X$ is positive cellular.
(4) The filtered colimit of a diagram of positive cellular motivic spectra is positive cellular.

In essence, a positive cellular motivic spectrum is one that can be built by attaching cells along the spheres $S^{p,q}$ with $p \geq 0$ and $p - q \geq 0$.

Proposition 3.16. The class of positive cellular motivic spectra is equal to the class of cellular motivic spectra $X$ such that $\pi_{a,b}X$ is zero if $a < 0$ or $a - b < 0$.

Proof. Consider the class of motivic spectra $X$ such that $\pi_{a,b}X$ is zero if $a < 0$ or $a - b < 0$. This class satisfies the four properties of Definition 3.15. Also, the class of cellular motivic spectra satisfies these four properties. This shows that if $X$ is a positive cellular motivic spectrum, then $X$ is cellular and $\pi_{a,b}X$ is zero if $a < 0$ or $a - b < 0$. 

Now suppose that $X$ is a cellular motivic spectrum such that $\pi_{a,b}X$ is zero if $a < 0$ or $a - b < 0$. Apply Proposition 3.9 to the map $\ast \to X$ to obtain a map $f : Z \to X$. By construction, $Z$ is a positive cellular motivic spectrum, and Proposition 3.9 guarantees that $f$ induces an isomorphism on $\pi_{a,b}$ for all $a$ and $b$. Since $Z$ and $X$ are both cellular, it follows from Proposition 3.9 that $f$ is a weak equivalence. Hence $X$ is positive. □

Lemma 3.17. Let $p \geq 0$ and $p - q \geq 0$. If $X$ is a positive cellular motivic spectrum, then so is $\Sigma^{p,q}X$.

Proof. Consider the class of all motivic spectra such that $\Sigma^{p,q}X$ is positive cellular. We would like to show that this class contains the positive cellular motivic spectra, so it is enough to show that the class satisfies the four properties of Definition 3.15. These properties are easy to check. □

Proposition 3.18. If $X$ and $Y$ are both positive cellular motivic spectra, then so is $X \wedge Y$.

Proof. Fix a positive cellular motivic spectrum $X$, and consider the class of all motivic spectra $Z$ such that $X \wedge Z$ is a positive cellular motivic spectrum. We would like to show that this class contains all positive motivic cellular spectra. Thus, we only need to show that the class satisfies the four properties of Definition 3.15.

Property (1) is Lemma 3.17. Property (2) follows from the fact that smash product with $X$ preserves cofiber sequences. Property (3) follows from the fact that smash product with $X$ preserves weak equivalences. Property (4) follows from the fact that smash product with $X$ commutes with filtered colimits. □

Lemma 3.19. Let $X$ be a positive cellular motivic spectrum, and let $Y$ be a motivic spectrum such that $\pi_{a,b}Y$ is zero if $a \geq 0$ and $a - b \geq 0$. Then $[\Sigma Y, X]$ equals zero.

Proof. Recall that the category of motivic spectra is enriched over simplicial sets. This means that for all motivic spectra $X$ and $Y$, there is a simplicial set $\text{Map}(X,Y)$ such that $[\Sigma^n X, Y]$ equals $\pi_n \text{Map}(X,Y)$.

Consider the class of motivic spectra $W$ such that $\text{Map}(W,Y)$ is contractible. It suffices to show that this class satisfies the four properties of Definition 3.15.

Property (1) is satisfied by the assumption on $Y$. Property (2) is satisfied by the fact that the functor $\text{Map}(Y,-)$ takes cofiber sequences of motivic spectra to fiber sequences of simplicial sets. Property (3) follows from the fact that $\text{Map}(Y,-)$ takes weak equivalences of motivic spectra to weak equivalences of simplicial sets. Property (4) follows from the fact that $\text{Map}(Y,-)$ takes filtered colimits, which are homotopy colimits, to homotopy limits. □

Corollary 3.20. Let $X$, $Y$, and $Z$ be cellular motivic spectra. Suppose that $X$ is positive, and suppose that there is a map $Y \to Z$ that is:

1. injective on $\pi_{a,b}$ if $a \geq 0$ and $a - b \geq 0$;
2. surjective on $\pi_{a,b}$ if $a \geq -1$ and $a - b \geq -1$.

Then $[X,Y] \to [X,Z]$ is an isomorphism.

Proof. Let $F$ be the fiber of $Y \to Z$. By the long exact sequence

$$\cdots \to \pi_{a+1,b} Y \to \pi_{a+1,b} Z \to \pi_{a,b} F \to \pi_{a,b} Y \to \pi_{a,b} Z \to \cdots,$$
\( \pi_{a,b} F \) is zero if \( a \geq 0 \) and \( a - b \geq 0 \). It follows from Lemma 3.19 that \( \text{Map}(X, F) \) is contractible. By consideration of the fiber sequence

\[
\text{Map}(X, F) \to \text{Map}(X, Y) \to \text{Map}(X, Z)
\]

of simplicial sets, it follows that \( \text{Map}(X, Y) \to \text{Map}(X, Z) \) is a weak equivalence. \( \square \)

**Corollary 3.21.** Let \( X, Y, \) and \( Z \) be cellular motivic spectra. Suppose that \( \pi_{a,b} X \) is zero if \( a \geq 0 \) and \( a - b \geq 0 \). Also suppose that there is a map \( Y \to Z \) that is:

1. injective on \( \pi_{a,b} \) if \( a < 1 \) or \( a - b < 1 \);
2. surjective on \( \pi_{a,b} \) if \( a < 0 \) or \( a - b < 0 \).

Then \( [Y, X] \to [Z, X] \) is an isomorphism.

**Proof.** Let \( C \) be the cofiber of \( Y \to Z \). By the long exact sequence

\[
\cdots \to \pi_{a,b} Y \to \pi_{a,b} Z \to \pi_{a,b} C \to \pi_{a-1,b} Y \to \pi_{a-1,b} Z \to \cdots ,
\]

\( \pi_{a,b} C \) is zero if \( a < 0 \) or \( a - b < 0 \). In other words, \( C \) is positive cellular. It follows from Lemma 3.19 that \( \text{Map}(C, X) \) is contractible. By consideration of the fiber sequence

\[
\text{Map}(C, X) \to \text{Map}(Y, X) \to \text{Map}(Z, X)
\]

of simplicial sets, it follows that \( \text{Map}(Y, X) \to \text{Map}(Z, X) \) is a weak equivalence. \( \square \)

The following result is a straightforward motivic version of [15, Lemma 2.11].

**Proposition 3.22.** If \( X \) is a cellular motivic ring spectrum, then \( CX \) has a unique (up to homotopy) multiplication such that the map \( CX \to X \) is a map of motivic ring spectra.

**Proof.** First note that \( CX \) is a positive cellular motivic spectrum. By Proposition 3.18 \( CX \wedge CX \) is a positive cellular motivic spectrum, so the map

\[
[ CX \wedge CX, CX ] \to [ CX \wedge CX, X ]
\]

is an isomorphism by Corollary 3.20. Therefore, there is a unique homotopy class of maps \( CX \wedge CX \to CX \) such that the diagram

\[
\begin{array}{ccc}
CX \wedge CX & \to & X \wedge X \\
\downarrow & & \downarrow \\
CX & \to & X
\end{array}
\]

commutes. \( \square \)

Recall that classical Eilenberg-Mac Lane spectra are unique in the following sense. If \( H \) and \( H' \) are any two spectra such that \( \pi_0 H \) and \( \pi_0 H' \) are isomorphic while \( \pi_k H \) and \( \pi_k H' \) are zero if \( k \neq 0 \), then \( H \) and \( H' \) are weakly equivalent. We next prove a motivic version.

**Proposition 3.23.** Suppose that \( H \) and \( H' \) are cellular motivic spectra such that \( \pi_{0,*} H \) and \( \pi_{0,*} H' \) are isomorphic as \( \mathbb{Z}_2[\tau] \)-modules. Suppose also that \( \pi_{a,b} H \) and \( \pi_{a,b} H' \) are zero unless \( a = 0 \) and \( b \leq 0 \). Then \( H \) and \( H' \) are weakly equivalent.
Proof. Let $N$ be the $\mathbb{Z}_2[\tau]$-module $\pi_{0,*}H = \pi_{0,*}H'$. Consider the motivic Moore spectrum $C$ constructed by the cofiber sequence

$$\vee \alpha S^{0,k} \to \vee \beta S^{0,j} \to C,$$

where the second wedge is indexed by a set of generators of $N$ and the first wedge is indexed by a set of defining relations for $N$. A straightforward calculation shows that $\pi_{0,*}C$ is isomorphic to $N$ and $\pi_{a,b}C$ is zero if $a < 0$. Since $N$ is concentrated in degrees $(0,k)$ with $k \leq 0$, it follows that $\pi_{a,b}C$ is zero if $a - b < 0$. There are obvious maps $C \to H$ and $C \to H'$ that induce isomorphisms on $\pi_{0,*}$. Let $F$ be the fiber of $C \to H$. From the long exact sequence in homotopy groups, it follows that $\pi_{a,b}F$ is zero if $a < 1$ or $a - b < 0$. Thus $\pi_{a,b}(\Sigma^{-1,-1}F)$ is zero if $a < 0$ or $a - b < 0$. There exists a map $H \to H'$ making the diagram

$$
\begin{array}{ccc}
C & \to & H \\
\downarrow & & \downarrow \\
H' & \to & \Sigma^{-1,-1}F
\end{array}
$$

commute. This map $H \to H'$ is an isomorphism on $\pi_{*,*}$, so it is a weak equivalence by Proposition 3.3.

Remark 3.24. Proposition 3.23 generalizes in a straightforward way to cellular motivic spectra $H$ such that $\pi_{0,*}H$ is bounded above in the sense that $\pi_{0,k}H$ is zero for $k$ greater than some fixed $n$.

4. Motivic $K$-theory spectra

We remind the reader that we are working with $HF_2$-complete stable motivic homotopy theory over $\text{Spec } C$.

4.1. $KGL$. Recall that $KGL$ is the algebraic $K$-theory spectrum [6, Part 3, Section 3.2]. This is a ring spectrum; in fact, it has a strictly associative and commutative multiplication [18]. In [7, Theorem 6.2] it is shown that $KGL$ is cellular.

The group schemes $GL_n$ are equipped with an involution given by inverse-transpose. These involutions extend to an involution on $GL$, and then to $BGL$. In this way, the motivic spectrum $KGL$ is a $\mathbb{Z}/2$-equivariant motivic spectrum.

Proposition 4.1. The ring $\pi_{*,*}KGL$ is equal to $\mathbb{Z}_2[\tau, \beta^{\pm 1}]$, where $\tau$ has bidegree $(0,-1)$ and $\beta$ has bidegree $(2,1)$. The involution on $\pi_{*,*}KGL$ fixes $\tau$ but takes $\beta$ to $-\beta$.

Proof. The first part follows from the calculation of 2-complete algebraic $K$-theory of $C$ [19], together with the fact that $\pi_{p,q}KGL$ is isomorphic to $K_{p-2q}(C)$ [10, Section 4, Theorem 3.13].

The second part follows from the fact that the involution on $KGL$ takes the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ to $\mathcal{O}(1)$. □
4.2. *KO*.

**Definition 4.2.** Let *KO* be the homotopy fixed points spectrum \( KGL^h\mathbb{Z}/2 \).

**Remark 4.3.** Recall that in the classical case, the real *K*-theory spectrum *KO* is weakly equivalent to \( KU^h\mathbb{Z}/2 \). Our terminology is chosen to emphasize this analogy.

**Remark 4.4.** We warn the reader that we are not claiming that *KO* represents Hermitian *K*-theory, although we suspect that this is true.

Our next goal is to compute the homotopy of *KO*. We will start with the homotopy of \( KGL \) and apply the homotopy fixed points spectral sequence.

**Proposition 4.5.** The \( E_2 \)-page of the homotopy fixed points spectral sequence for *KO* is \( \mathbb{Z}_2 [\tau, h_1, c^{\pm 1}] / 2h_1 \), where the degree of \( \tau \) is \((0, 0, -1)\), the degree of \( h_1 \) is \((1, 1, 1)\), and the degree of \( c \) is \((4, 0, 2)\).

**Proof.** This is a straightforward calculation, using the classical computations of \( H^*(\mathbb{Z}/2; \mathbb{Z}) \) and \( H^*(\mathbb{Z}/2; \mathbb{Z}(-1)) \), where \( \mathbb{Z}(-1) \) is the \( \mathbb{Z}/2 \)-module whose involution is multiplication by \(-1\). \( \square \)

Figure 4.1 is a pictorial representation of the computation in Proposition 4.5. Here, and in the figures following it, \( E_2^{n,p,u} \) is located at coordinates \((n, p)\), and the weight is not shown. Copies of \( \mathbb{Z}_2 [\tau] \) are represented by open boxes. Copies of \( \mathbb{F}_2 [\tau] \) are represented by solid circles. Copies of \( \mathbb{F}_2 [\tau] / \tau \) are represented by open circles. Lines of slope 1 represent multiplications by \( h_1 \).

![Figure 4.1](image)

**Figure 4.1.** The \( E_2 \)-page of the homotopy fixed points spectral sequence for *KO*

For dimension reasons, the \( d_2 \)-differential is zero.

**Lemma 4.6.** In the homotopy fixed points spectral sequence for *KO*, \( d_3(\tau) = 0 \), \( d_3(h_1) = 0 \), and \( d_3(c) = \tau h_1^3 \).

**Proof.** This follows immediately by degree considerations and topological realization. Classically, \( d_3 \) takes \( h_1 \) to 0 and takes \( c \) to \( h_1^3 \). \( \square \)
Proposition 4.7. The $E_{\infty}$-page of the homotopy fixed points spectral sequence for $KO$ is

$$\mathbb{Z}_2[\tau, h_1, a, b^{\pm 1}] \overline{2h_1, \tau h_1^1, a^2 = 4b, h_1a},$$

where the degree of $\tau$ is $(0,0,-1)$, the degree of $h_1$ is $(1,1,1)$, the degree of $a$ is $(4,0,2)$, and the degree of $b$ is $(8,0,4)$.

Proof. The $E_4$-page can be computed from the description of $d_3$ given in Lemma 4.6. The element $a$ corresponds to $2c$, while $b$ corresponds to $c^2$.

Then observe that all higher differentials vanish for dimension reasons. \qed

Figure 4.2 is a pictorial representation of the $E_{\infty}$-page of the spectral sequence. The notation is the same as in Figure 4.1.

![Figure 4.2. The $E_{\infty}$-page of the homotopy fixed points spectral sequence for $KO$.](image)

Theorem 4.8. The ring $\pi_{\ast,\ast}(KO)$ is

$$\mathbb{Z}_2[\tau, h_1, a, b^{\pm 1}] \overline{2h_1, \tau h_1^1, a^2 = 4b, h_1a},$$

where the degree of $\tau$ is $(0,-1)$, the degree of $h_1$ is $(1,1)$, the degree of $a$ is $(4,2)$, and the degree of $b$ is $(8,4)$.

Proof. For dimension reasons, there are no extensions to resolve in the $E_{\infty}$-page described in Proposition 4.7. \qed

4.3. $ko$.

Definition 4.9. Let $ko$ be the connective cover of $KO$ in the sense of Definition 3.13.

Theorem 4.10. The ring $\pi_{\ast,\ast}(ko)$ is

$$\mathbb{Z}_2[\tau, h_1, a, b] \overline{2h_1, \tau h_1^1, h_1a, a^2 = 4b}.$$

By direct computation of group cohomology, we find that the degree of $\tau$ is $(0, -1)$, the degree of $h_1$ is $(1, 1)$, the degree of $a$ is $(4, 2)$, and the degree of $b$ is $(8, 4)$.

Proof. This follows from Theorems 4.8 and Corollary 3.12.

4.4. $kgl$ and $kgl^{hZ/2}$.

Definition 4.11. Let $kgl$ be the connective cover of $KGL$, in the sense of Definition 3.13.

It follows from Proposition 4.22 that $kgl$ is a ring spectrum, and the homotopy of $kgl$ is easily described from Proposition 4.1 and Corollary 3.12.

Proposition 4.12. The ring $\pi_{*,*}(kgl)$ is isomorphic to $\mathbb{Z}[[\tau, \beta]]$, where the degree of $\tau$ is $(0, -1)$ and the degree of $\beta$ is $(2, 1)$.

Since we may construct $kgl$ from $KGL$ equivariantly as in Remark 3.14, it follows that $kgl$ has a $\mathbb{Z}/2$-action. Next we study the homotopy fixed points spectrum $kgl^{hZ/2}$.

Theorem 4.13. The ring $\pi_{*,*}(kgl^{hZ/2})$ is

$$\mathbb{Z}_2[\tau, h_1, a, b, x]/2h_1, \tau h_1, \tau h_1 x, h_1 a, a^2 = 4b, bx = h_1^4),$$

where the degree of $\tau$ is $(0, -1)$, the degree of $h_1$ is $(1, 1)$, the degree of $a$ is $(4, 2)$, the degree of $b$ is $(8, 4)$, and the degree of $x$ is $(-4, 0)$.

Proof. We use the homotopy fixed points spectral sequence

$$E_2^{n,p} = HP(\mathbb{Z}/2; \pi_{n+p,u}(KGL)) \Rightarrow \pi_{n,u}(kgl^{hZ/2}).$$

By direct computation of group cohomology, we find that

$$E_2 = \mathbb{Z}_2[\tau, h_1, c, z]/2h_1, cz = h_1^2,$$

where the degree of $\tau$ is $(0, 0, -1)$, the degree of $h_1$ is $(1, 1, 1)$, the degree of $c$ is $(4, 0, 2)$, and the degree of $z$ is $(-2, 2, 0)$. A pictorial representation of $E_2$ is shown in Figure 4.3.

As in Lemma 4.6 $d_3(c) = \tau h_1^3$. From this, it follows that $E_4$ is equal to

$$\mathbb{Z}_2[\tau, h_1, a, b, x]/2h_1, \tau h_1, \tau h_1 x, h_1 a, a^2 = 4b, bx = h_1^4),$$

where the degree of $\tau$ is $(0, 0, -1)$, the degree of $h_1$ is $(1, 1, 1)$, the degree of $a$ is $(4, 0, 2)$, and the degree of $b$ is $(8, 0, 4)$. Here $a$ corresponds to $2c$, $b$ corresponds to $c^2$, and $x$ corresponds to $z^2$.

For dimension reasons, there are no higher differentials, and $E_\infty$ is equal to $E_4$. Also for dimension reasons, there are no extensions to resolve in passing from $E_\infty$ to $\pi_{*,*}(kgl^{hZ/2})$. A pictorial representation of $E_\infty$ is shown in Figure 1.4.

Remark 4.14. We draw the reader’s attention to the relation $bx = \tau h_1^4$. This is a curious difference to the classical case, where the element $x$ of $\pi_{-2}ku^{hZ/2}$ annihilates all of the other generators of $\pi_{*,*}ku^{hZ/2}$.
5. The cohomology of $kgl$ and $ko$

In this section, we compute the motivic $\mathbb{F}_2$-cohomology of $kgl$ and $ko$.

5.1. The cohomology of $HZ_2$.

**Definition 5.1.** Let $HZ_2$ be the cofiber of the map $\beta : \Sigma^{2,1}kgl \to kgl$.

**Remark 5.2.** As its name suggests, $HZ_2$ represents motivic cohomology with $\mathbb{Z}_2$-coefficients. We will not use this fact. Everything that we need to know about $HZ_2$ comes from its definition as a cofiber.

**Lemma 5.3.** The cofiber of the map $2 : HZ_2 \to HZ_2$ is $HF_2$. 

---

**Figure 4.3.** The $E_2$-page of the homotopy fixed points spectral sequence for $kgl^{h\mathbb{Z}/2}$

**Figure 4.4.** The $E_\infty$-page of the homotopy fixed points spectral sequence for $kgl^{h\mathbb{Z}/2}$

Proof. As $\mathbb{Z}[\tau]$-modules, the homotopy groups of the cofiber are isomorphic to the homotopy groups of $HF_2$. By Proposition 3.23, the cofiber is weakly equivalent to $HF_2$. □

We will write $p : H\mathbb{Z}_2 \to HF_2$ for the map from $H\mathbb{Z}_2$ to the cofiber of 2. Also, we will write $\delta : HF_2 \to \Sigma H\mathbb{Z}_2$ for the boundary map of the cofiber sequence of 2. We will make use of the cofiber sequence

$$H\mathbb{Z}_2 \xrightarrow{2} H\mathbb{Z}_2 \xrightarrow{p} HF_2 \xrightarrow{\delta} \Sigma H\mathbb{Z}_2.$$ 

We shall freely substitute the motivic Steenrod algebra $A$ for $H^*HF_2$, since they are equal by definition.

Lemma 5.4. The composition $\delta^*p^* : A \to \Sigma^{-1}A$ is equal to right multiplication by $Sq^1$.

Proof. Since $\delta^*p^*$ is an $A$-module map, it suffices to compute $\delta^*p^*(1)$.

Consider the diagram

$$
\begin{array}{ccc}
S^{0,0} & \xrightarrow{p'} & M2 \\
\downarrow & & \downarrow \\
H\mathbb{Z}_2 & \xrightarrow{p} & HF_2 & \xrightarrow{\delta} & \Sigma H\mathbb{Z}_2,
\end{array}
$$

where both rows are cofiber sequences induced by the map 2. Here $M2$ is the motivic mod 2 Moore spectrum. Recall that $H^{**}M2$ is free over $M2$ on two generators $x$ and $y$ of degrees $(0,0)$ and $(1,0)$, and $Sq^1x = y$ because $Sq^1$ is the same as the Bockstein [21, Section 9]. It follows that the map $\delta^*p^* : H^{**}M2 \to H^{**}\Sigma^{-1}M2$ takes $x$ to $y$.

The elements 1 and $Sq^1$ of $A$ map to $x$ and $y$ respectively in $H^{**}M2$. A diagram chase now shows that $\delta^*p^*$ takes 1 to $Sq^1$. □

Theorem 5.5. The motivic $F_2$-cohomology $H^{**}H\mathbb{Z}_2$ of $H\mathbb{Z}_2$ is equal to $A//E(0)$.

Proof. The composition $\delta p$ is null-homotopic, so $p^*\delta^*$ is zero. This implies that $p^*\delta^*p^*$ is zero. By Lemma 5.3, this shows that $p^*$ annihilates the left ideal generated by $Sq^1$. Hence $p^*$ extends to a map $\overline{p} : A//E(0) \to H^{**}H\mathbb{Z}_2$. This gives us a commutative diagram

$$
\begin{array}{cccccc}
0 & \xrightarrow{1} & H^{**}\Sigma H\mathbb{Z}_2 & \xrightarrow{\delta^*} & H^{**}HF_2 & \xrightarrow{p^*} & H^{**}H\mathbb{Z}_2 & \xrightarrow{0} \\
\overline{p^*} & \cong & \cong & \cong & \cong & \cong & \\
0 & \xrightarrow{1} & \Sigma A//E(0) & \xrightarrow{1} & A & \xrightarrow{1} & A//E(0) & \xrightarrow{1} 0.
\end{array}
$$

The top row is exact because the map $H^{**}H\mathbb{Z}_2 \to H^{**}H\mathbb{Z}_2$ induced by 2 is zero. The bottom row is exact because of Lemma 2.9.

The right square commutes by definition of $\overline{p}$, and the left square commutes by Lemma 5.4. It follows that $\overline{p}$ is an isomorphism. □
5.2. **The cohomology of $kgl$.** Having computed the cohomology of $HZ_2$, we will now exploit the cofiber sequence

$$\Sigma^{2,1}kgl \xrightarrow{\beta} kgl \xrightarrow{p} HZ_2 \xrightarrow{\delta} \Sigma^{3,1}kgl$$

in order to compute the cohomology of $kgl$. Beware that $p$ and $\delta$ are different than (but analogous to) the maps of the same name in Section 5.1.

Because of Theorem 5.5 we shall freely substitute $A/E(0)$ for $H^{*,*}HZ_2$.

**Lemma 5.6.** The composition $\delta^*p^* : A/E(0) \to \Sigma^{-1}A/E(0)$ is equal to right multiplication by $Q_1$.

**Proof.** Since $\delta^*p^*$ is an $A$-module map, it suffices to show that $\delta^*p^*(1) = Q_1$.

Consider the classical case. We have a cofiber sequence

$$\Sigma^2ku \xrightarrow{\beta} ku \xrightarrow{p'} HZ \xrightarrow{\delta'} \Sigma^3ku.$$

It follows from the arguments of [1, p. 366] that $\delta^*p^*(1) = Q_1$. Since our motivic computation must be compatible with the classical computation under topological realization, it follows that $\delta^*p^*$ takes 1 to $Q_1$. □

**Theorem 5.7.** The motivic $\mathbb{F}_2$-cohomology $H^{*,*}kgl$ of $kgl$ is equal to $A/E(1)$.

**Proof.** The composition $\delta p$ is null-homotopic, so $p^*\delta^*$ is zero. This implies that $p^*\delta^*p^*$ is zero. By Lemma 5.6 this shows that $p^*$ annihilates the left ideal generated by $Q_1$. Hence $p^*$ extends to a map $p^* : A/E(1) \to H^{*,*}kgl$. This gives us a commutative diagram

$$
\begin{array}{ccccccccc}
H^{*,*}\Sigma kgl & \xrightarrow{\Sigma \delta^*} & H^{*,*}\Sigma^{3,1}kgl & \xrightarrow{\delta^*} & H^{*,*}HZ_2 & \xrightarrow{p^*} & H^{*,*}kgl & \xrightarrow{\beta^*} & H^{*,*}\Sigma^{2,1}kgl \\
\Sigma^{3,1}p^* & \xrightarrow{\cong} & p^* & \xrightarrow{\cong} & \cong & \cong & \cong & \cong & \cong \\
0 & \xrightarrow{\cong} & \Sigma^{3,1}A/E(1) & \xrightarrow{\cong} & A/E(0) & \xrightarrow{\cong} & A/E(1) & \xrightarrow{\cong} & 0.
\end{array}
$$

The bottom row is exact because of Lemma 2.4. The right square commutes by definition of $p^*$, and the left square commutes by Lemma 5.6.

We already know that $p^*$ is surjective because the right square commutes. We shall prove by induction that $p^*$ is an isomorphism. The base case occurs in bidegrees $(a, b)$ with $a < 0$. If $a < 0$, then $H^{a+b}kgl$ is zero by Lemma 3.19 and $A/E(1)$ is also zero by definition.

Now suppose that $p^*$ is an isomorphism in bidegrees $(a, b)$ for $a < n$. Then $\Sigma^{3,1}p^*$ is an isomorphism in bidegrees $(a, b)$ for $a < n + 3$. This implies that $\delta^*$ is injective in bidegrees $(a, b)$ for $a < n + 3$, so $\Sigma^3p^*$ is zero in the same bidegrees. In other words, $\beta^*$ is zero in bidegrees $(a, b)$ with $a < n + 4$. We have now shown that the top sequence in the above diagram splits in bidegrees $(a, b)$ with $a < n + 3$. It follows that $p^*$ is an isomorphism in bidegrees $(a, b)$ with $a < n + 3$. This completes the induction step. □

5.3. **The cohomology of $ko$.** In order to compute the cohomology of $ko$, we will use an argument that is very similar to the argument of Section 5.2.

Recall the motivic Hopf map $\eta$ in $\pi_{1,0}$. The shortest description of $\eta$ is the projection $\mathbb{A}^2 - 0 \to \mathbb{P}^1$, because $\mathbb{A}^2 - 0$ is a model for $S^{3,2}$ and $\mathbb{P}^1$ is a model for $S^{2,1}$. 
Lemma 5.8. The cofiber of the map $\Sigma^{1,1}ko \to ko$ induced by $\eta$ is weakly equivalent to $kgl$.

Proof. Let $C$ be the cofiber of $\Sigma^{1,1}ko \to ko$. There is a map $f : ko \to kgl$ because of formal properties of homotopy fixed points spectra. Consider the diagram

$$
\begin{array}{ccc}
ko \wedge S^{1,1} & \xrightarrow{id \wedge \eta} & ko \wedge ko \\
\downarrow & & \downarrow \\
ko & \xrightarrow{\eta \wedge f} & kgl \wedge kgl \\
\downarrow & & \downarrow \\
ko & \xrightarrow{f} & kgl,
\end{array}
$$

where the vertical maps are the multiplication maps of the motivic ring spectra $ko$ and $kgl$. The composition across the top is the smash product of $f$ with a map $S^{1,1} \to kgl$, which must be zero since $\pi_{1,1}kgl$ is zero. This shows that the composition $\Sigma^{1,1}ko \to ko \to kgl$ is zero, so $f$ extends to a map $C \to kgl$.

Having constructed the map $C \to kgl$, it remains to show that it is an isomorphism on homotopy groups. Then we can finish with Proposition 5.3.

Use the computation of $\pi_*, \pi_* ko$ and the long exact sequence in homotopy groups to compute $\pi_*, \pi_* C$. Analysis of the long exact sequence leaves one ambiguity, which can be resolved by showing that the element $\beta$ of $\pi_{2,1}kgl$ is contained in the Toda bracket $\langle i, \eta, 2 \rangle$, where $i : S^{0,0} \to kgl$ is the unit map. This computation can be made by comparing to the classical computation via topological realization. □

Remark 5.9. The same argument shows that $KGL$ is the cofiber of $\eta : KO \to KO$.

Having computed the cohomology of $kgl$, we will now exploit the cofiber sequence

$$
\begin{array}{ccc}
\Sigma^{1,1}ko & \xrightarrow{\eta} & ko \\
& & \downarrow p \\
kgl & \xrightarrow{\delta} & \Sigma^{2,1}ko,
\end{array}
$$

in order to compute the cohomology of $ko$. Beware that $p$ and $\delta$ are different than (but analogous to) the maps of the same name in Sections 5.1 and 5.2.

Because of Theorem 5.7 we shall freely substitute $A//E(1)$ for $H^{**}kgl$.

Lemma 5.10. The composition $\delta^*p^* : A//E(1) \to \Sigma^{-1}A//E(1)$ is equal to right multiplication by $Sq^2$.

Proof. Since $\delta^*p^*$ is an $A$-module map, it suffices to show that $\delta^*p^*(1) = Sq^2$.

Consider the diagram

$$
\begin{array}{ccc}
S^{0,0} & \xrightarrow{p'} & C\eta \\
\downarrow & & \downarrow \\
kko & \xrightarrow{p} & kgl \\
\downarrow & & \downarrow \\
\Sigma^{2,1}ko,
\end{array}
$$

where both rows are cofiber sequences induced by the map $\eta$. Here $C\eta$ is the suspension spectrum of $\mathbb{P}^2$. Recall that $H^{**}C\eta$ is free over $\mathbb{Z}_2$ on two generators $x$ and $y$ of degrees $(0,0)$ and $(2,1)$, and $Sq^2x = y$ because of the cup product structure on $H^{**}\mathbb{P}^2$. It follows that the map $\delta^*p^* : H^{**}C\eta \to H^{**}C\eta$ takes $x$ to $y$. 
The elements 1 and $\text{Sq}^2$ of $A$ map to $x$ and $y$ respectively in $H^*C\eta$. A diagram chase now shows that $\delta^p\rho^*$ takes 1 to $\text{Sq}^2$.

\[\text{Theorem 5.11.}\] The motivic $\mathbb{F}_2$-cohomology $H^{**}\text{ko}$ of $\text{ko}$ is equal to $\Lambda/\Lambda(1)$.

\textbf{Proof.} The proof is essentially identical to the proof of Theorem 5.7. In the diagram,

\[
\begin{array}{c}
H^{**}\Sigma\text{ko} \xrightarrow{\Sigma^p} H^{**}\Sigma^1\text{ko} \xrightarrow{\delta^*} H^{**}\text{ko} \xrightarrow{\rho^*} H^{**}\Sigma^1\text{ko} \\
\Sigma^2\eta \downarrow \cong \downarrow \eta \\
0 \xrightarrow{} \Sigma^2\Lambda/\Lambda(1) \xrightarrow{} \Lambda/\Lambda(1) \xrightarrow{} \Lambda/\Lambda(1) \xrightarrow{} 0
\end{array}
\]

one can prove by induction that $\eta$ is an isomorphism. \qed

6. COMPUTATIONS OF $\text{ko}$-HOMOLOGY

The first goal of this section is to establish an Adams spectral sequence for computing $\text{ko}$-homology. We identify the $E_2$-term of this spectral sequence in terms of Ext groups over $\Lambda(1)$. The remainder of the section is dedicated to computing these Ext groups over $\Lambda(1)$ for various $\Lambda(1)$-modules of interest, i.e., $E_2$-terms of the spectral sequence. In all of the cases that we study below, all differentials are trivial for simple algebraic reasons. The interested reader can reconstruct $\text{ko}$-homology groups from our computations.

\textbf{Lemma 6.1.} For $0 \leq m \leq n \leq \infty$, there is an isomorphism

\[H^{*,*}\left(\text{ko} \wedge \frac{DQ_n}{DQ_m}\right) \cong H^{*,*}\text{ko} \otimes_{\mathcal{M}_2} H^{*,*}\left(\frac{DQ_n}{DQ_m}\right).\]

\textbf{Proof.} By a Künneth theorem for motivic cohomology [1, Theorem 8.6], the lemma holds for $n < \infty$ because $DQ_n/DQ_m$ is a finite complex. Note that the higher Tor terms

\[\text{Tor}_{\mathcal{M}_2}^{*,*}\left(\text{ko} \wedge \frac{DQ_n}{DQ_m}\right)
\]

vanish because $H^{**}\text{ko}$ is free over $\mathcal{M}_2$ by Theorem 5.11.

It remains to consider the case $n = \infty$. Recall that $DQ_\infty/DQ_m$ is equal to $\text{colim}_k \frac{DQ_k}{DQ_m}$ and that $\text{ko} \wedge \frac{DQ_k}{DQ_m}$ is equal to $\text{colim}_k \text{ko} \wedge \frac{DQ_k}{DQ_m}$. Observe that both $\lim^1 H^{*,*}\left(\frac{DQ_k}{DQ_m}\right)$ and $\lim^1 H^{*,*}\left(\text{ko} \wedge \frac{DQ_k}{DQ_m}\right)$ vanish. This follows from the fact that for fixed $p$ and $q$, the groups $H^{p,q}\left(\frac{DQ_k}{DQ_m}\right)$ and $H^{p,q}\left(\text{ko} \wedge \frac{DQ_k}{DQ_m}\right)$ do not depend on $k$, as long as $k$ is sufficiently large.

Using $\lim^1$ short exact sequences and the first paragraph, we have the chain

\[H^{*,*}\left(\text{ko} \wedge \frac{DQ_\infty}{DQ_m}\right) \cong \lim_k H^{*,*}\left(\text{ko} \wedge \frac{DQ_k}{DQ_m}\right) \cong \lim_k H^{*,*}\text{ko} \otimes_{\mathcal{M}_2} H^{*,*}\left(\frac{DQ_k}{DQ_m}\right) \cong H^{*,*}\text{ko} \otimes_{\mathcal{M}_2} \lim_k H^{*,*}\left(\frac{DQ_k}{DQ_m}\right) \cong H^{*,*}\text{ko} \otimes_{\mathcal{M}_2} H^{*,*}\left(\frac{DQ_\infty}{DQ_m}\right)
\]

of isomorphisms. \qed
Remark 6.2. It is possible to prove that $H^{*,*}(ko \wedge X)$ is isomorphic to $H^{*,*}ko \otimes_{M_2} H^{*,*}X$ for a larger class of $X$ than in Lemma 6.1. We have avoided this generality for sake of simplicity.

**Theorem 6.3.** Let $0 \leq m \leq n \leq \infty$. There is a spectral sequence

$$\text{Ext}_{A(1)} \left( H^{*,*} \left( \frac{DQ_n}{DQ_m} \right), M_2 \right) \Rightarrow ko_{*,*}X.$$

**Proof.** By [5, Lemma 7.13], we have an Adams spectral sequence

$$\text{Ext}_A \left( H^{*,*} \left( ko \wedge \frac{DQ_n}{DQ_m} \right), H^{*,*}S^0 \right) \Rightarrow ko_{*,*}X.$$

By Lemma 6.1 and Theorem 5.11, $H^{*,*} \left( ko \wedge \frac{DQ_n}{DQ_m} \right)$ is isomorphic to $A/\langle (1) \otimes_{M_2} H^{*,*}S^0 \rangle$. A standard change of rings finishes the proof. \hfill \square

**Remark 6.4.** Similarly, because of Theorems 5.5 and 5.7, there are Adams spectra lsequences

$$\text{Ext}_{E(0)} \left( H^{*,*}X, M_2 \right) \Rightarrow (HZ_2)_{*,*}X$$

and

$$\text{Ext}_{E(1)} \left( H^{*,*}X, M_2 \right) \Rightarrow kgl_{*,*}X.$$

The rest of this section is dedicated to computing Ext groups over $A(1)$ for various $A(1)$-modules of interest.

**Definition 6.5.** For $A(1)$-modules $M$ and $N$, an element in $\text{Ext}_{A(1)}^n(M, N)$ of internal bidegree $(a, b)$ has Adams tridegree $(a-n, n, b)$ and Adams bidegree $(a-n, n)$.

**Theorem 6.6.** The ring $\text{Ext}_{A(1)}^*(M_2, M_2)$ is the $M_2$-algebra given by the following generators and relations:

| generator | Adams tridegree | relations |
|-----------|----------------|-----------|
| $h_0$     | $(0, 1, 0)$    | $h_0h_1 = 0$ |
| $h_1$     | $(1, 1, 1)$    | $\tau h_1^2 = 0$ |
| $\alpha$  | $(4, 3, 2)$    | $h_1\alpha = 0$ |
| $\beta$   | $(8, 4, 4)$    | $\alpha^2 = h_2^2\beta$ |

**Proof.** The shortest proof is to use the motivic May spectral sequence, as in [5] and [13].

Alternatively, one can construct an explicit $A(1)$-resolution of $M_2$ to compute the additive structure of the Ext groups, and then use explicit cocycles to find Yoneda products. \hfill \square

Figure 6.1 is a pictorial representation of the previous theorem.

**Remark 6.7.** Recall the calculation of $\pi_{*,*}(ko)$ from Theorem 4.10. The calculation in Theorem 6.6 is the associated graded of $\pi_{*,*}(ko)$, filtered by powers of 2.

In order to simplify notation, we write $E$ for $\text{Ext}_{A(1)}(M_2, M_2)$. We will describe Ext groups of various $A(1)$-modules as modules over the ring $\text{Ext}_{A(1)}(M_2, M_2)$.

From now on, we will use reduced motivic cohomology with coefficients in $F_2$. The point is that unreduced cohomology splits as an $A(1)$-module into reduced cohomology plus a copy of $M_2$. From the perspective of $A(1)$-module theory, the extra copy of $M_2$ clutters the calculations needlessly.
Definition 6.8. Let $R$ be the $A(1)$-module generated by $x_i$ for $i \geq 0$, where the degree of $x_i$ is $(4i - 1, 2i)$, subject to the relations $Sq^{1,2} x_i = Sq^{1} x_{i+1}$.

Figure 6.1 gives a pictorial representation of $R$, with notation as in the previous figures.

Figure 6.1. The $A(1)$-module $R$

Theorem 6.9. As an $E$-module, $\text{Ext}^*_{A(1)}(R, M_2)$ is given by the following generators and relations:

| generator $r_{4i-1}$ for $i \geq 0$ | Adams tridegree $(4i - 1, 0, 2i)$ |
|--------------------------------------|----------------------------------|
| relations $h_1 r_{4i-1} = 0$ for $i \geq 0$ |
| $\alpha r_{4i-1} = h_0^3 r_{4i+3}$ for $i \geq 0$ |
| $\beta r_{4i-1} = h_0^5 r_{4i+7}$ for $i \geq 0$ |

Proof. It is straightforward to write down a free $A(1)$-resolution of $R$. The $E$-module structure comes from explicit computations with cocycles in low dimensions. □

Figure A.2 is a pictorial representation of $\text{Ext}^*_{A(1)}(R, M_2)$.

Theorem 6.10. As an $E$-module, $\text{Ext}^*_{A(1)}(\tilde{H}^{*,*}(DQ_{\infty}), M_2)$ is given by the following generators and relations:

| generator $r_{4i-1}$ for $i \geq 1$ | Adams tridegree $(4i - 1, 0, 2i)$ |
|--------------------------------------|----------------------------------|
| relations $h_0 k = 0$ |
| $\alpha k = 0$ |
| $h_0^2 r_{3} = \tau h_1^2 k$ |
| $h_0^5 r_{7} = 0$ |
| $\alpha r_{4i-1} = h_0^3 r_{4i+3}$ for $i \geq 1$ |
| $\beta r_{4i-1} = h_0^5 r_{4i+7}$ for $i \geq 1$ |

Proof. There is a short exact sequence

$$H^{*,*}(DQ_{\infty}) \to R \to \Sigma^{-1,0} M_2,$$

which yields a long exact sequence in $\text{Ext}$ groups. The maps of the long exact sequence are completely determined by $E$-linearity and direct calculation in homological degree 0. The long exact sequence tells us most of what we want to know, but it leaves one ambiguity in the $E$-module structure of $\text{Ext}^*_{A(1)}(\tilde{H}^{*,*}(DQ_{\infty}), M_2)$. Namely, it is not immediately clear whether $\tau h_1^2 k$ equals zero or $h_0^3 r_{3}$. This ambiguity can be resolved by explicit calculations with cocycles of homological degree 2. □
Theorem 6.11. Let $n$ be a positive integer that is congruent to 0 modulo 4. As an $E$-module, $\text{Ext}_{\mathbb{A}(1)}^*(\tilde{H}^{*,*}(DQ_n), M_2)$ is given by the following generators and relations:

| generator | Adams tridegree |
|-----------|----------------|
| $k$       | $(1, 0, 1)$     |
| $r_{4i-1}$ for $1 \leq i \leq \frac{n}{4}$ | $(4i-1, 0, 2i)$ |

Relations:

- $h_{0}k = 0$
- $\alpha k = 0$
- $h_{0}^{2}r_{3} = \tau h_{1}^{2}k$
- $h_{0}^{2}r_{7} = 0$
- $h_{1}r_{4i-1} = 0$ if $1 \leq i < \frac{n}{4}$
- $\alpha r_{4i-1} = h_{0}^{2}r_{4i+3}$ if $1 \leq i < \frac{n}{4}$
- $\beta r_{4i-1} = h_{0}^{2}r_{4i+7}$ if $1 \leq i < \frac{n}{4} - 1$
- $\beta r_{n-5} = h_{0}\alpha r_{n-1}$

Proof. Consider the short exact sequence

$$\tilde{H}^{*,*}(\frac{DQ_{\infty}}{DQ_{n}}) \rightarrow \tilde{H}^{*,*}(DQ_{\infty}) \rightarrow \tilde{H}^{*,*}(DQ_{n}).$$

Since $n$ is congruent to 0 modulo 4, the first term is isomorphic to $\Sigma^{n} \tilde{H}^{*,*}(DQ_{\infty})$. Therefore, as an $E$-module, $\text{Ext}_{\mathbb{A}(1)}^*(\tilde{H}^{*,*}(DQ_{\infty}), M_2)$ is equal to a shifted copy of $\text{Ext}_{\mathbb{A}(1)}^*(\tilde{H}^{*,*}(DQ_{\infty}), M_2)$, which we computed in Theorem 6.10.

The maps in the associated long exact of Ext groups are entirely determined by $E$-linearity and explicit computation in homological degree 0. The long exact sequence tells us most of what we need. The only ambiguity is whether $h_{1}r_{n-1}$ is zero or not zero. Computations with explicit cocycles in homological degree 1 show that it is not zero. \hfill $\square$

Figure A.3 is a pictorial representation of the above result.

Theorem 6.12. As an $E$-module, $\text{Ext}_{\mathbb{A}(1)}^*(\tilde{H}^{*,*}(DQ_{2}), M_2)$ is given by the following generators and relations:

| generator | Adams tridegree |
|-----------|----------------|
| $x$       | $(1, 0, 1)$     |
| $y$       | $(3, 1, 2)$     |

Relations:

- $h_{0}x = 0$
- $h_{0}y = \tau h_{1}^{2}x$
- $\alpha x = \tau h_{1}^{2}y$
- $\alpha y = 0$

Proof. The short exact sequence

$$\Sigma^{2,1}M_2 \rightarrow \tilde{H}^{*,*}(DQ_{2}) \rightarrow \Sigma^{1,1}M_2$$

yields a long exact sequence in Ext groups. The maps in the long exact sequence are determined by $E$-linearity, together with explicit computations in homological degrees 0 and 1. The long exact sequence tells us most of what we want. The only ambiguity is whether $\tau h_{1}^{2}y$ is zero or non-zero. Explicit computations with cocycles in low dimensions shows that it is non-zero. \hfill $\square$

Figure A.4 is a pictorial representation of the above result.

Theorem 6.13. Let $n$ be congruent to 2 modulo 4, and let $n > 2$. As an $E$-module, $\text{Ext}_{\mathbb{A}(1)}^*(\tilde{H}^{*,*}(DQ_{n}), M_2)$ is given by the following generators and relations:
Proof. The quotient $\tilde{H}^{*,*}(DQ_n)$ is isomorphic to $\Sigma^n \tilde{H}^{*,*}(DQ_2)$. The short exact sequence

$$\Sigma^n \tilde{H}^{*,*}(DQ_2) \to \tilde{H}^{*,*}(DQ_{n+2}) \to \tilde{H}^{*,*}(DQ_n)$$

yields a long exact sequence in Ext groups. The maps in this sequence are determined by $E$-linearity and explicit computations in homological degree 0.

The long exact sequence gives us most of what we want. The only ambiguity is whether $h_1 z$ is zero or non-zero. Computations with explicit cocycles in homological dimension 2 shows that $h_1 z$ is non-zero. □

Figure [A.6] is a pictorial representation of the previous result.

Remark 6.14. When $n$ is congruent to 3 modulo 4, the $A(1)$-module $\tilde{H}^{*,*}(DQ_n)$ splits as

$$\tilde{H}^{*,*}(DQ_{n-1}) \bigoplus \Sigma^n M_2.$$ 

Therefore this case reduces to the case when $n$ is congruent to 2 modulo 4.

Theorem 6.15. Let $n$ be congruent to 1 modulo 4, and let $n > 1$. As an $E$-module, $\text{Ext}_{A(1)}^{*,*}(\tilde{H}^{*,*}(DQ_n), M_2)$ is given by the following generators and relations:
**MOTIVIC CONNECTIVE K-THEORIES AND THE COHOMOLOGY OF A(1)**

| generator | Adams tridegree |
|-----------|-----------------|
| \( k \)   | \((1,0,1)\)      |
| \( r_{4i-1} \) for \(1 \leq i \leq \frac{n-1}{4}\) | \((4i-1,0,2i)\) |
| \( v \)   | \((n,1,\frac{n+1}{2})\) |
| \( u \)   | \((n+2,2,\frac{n+3}{2})\) |
| \( t \)   | \((n+4,3,\frac{n+5}{2})\) |

**relations**

- \( h_0 k = 0 \)
- \( \alpha k = 0 \)
- \( h_0^3 r_3 = \tau h_1^2 k \)
- \( h_0^4 r_7 = 0 \)
- \( h_1 r_{4i-1} = 0 \) if \(1 \leq i \leq \frac{n-1}{4}\)
- \( \alpha r_{4i-1} = h_0^3 r_{4i+3} \) if \(1 \leq i < \frac{n-1}{4}\)
- \( \beta r_{4i-1} = h_0^4 r_{4i+7} \) if \(1 \leq i < \frac{n-5}{4}\)
- \( \alpha r_{n-2} = h_0 u \)
- \( \beta r_{n-6} = h_0^3 u \)
- \( h_1 v = 0 \)
- \( h_1 u = 0 \)
- \( h_1 t = 0 \)
- \( \alpha v = h_0 t \)
- \( \alpha u = h_0 \beta r_{n-2} \)
- \( \alpha t = h_0 \beta v \)

**Proof.** The short exact sequence

\[
\Sigma^{n+1} M_2 \rightarrow \widetilde{H}^*, * (DQ_{n+1}) \rightarrow \widetilde{H}^*, * (DQ_n)
\]

yields a long exact sequence in Ext groups. The maps in this sequence are determined by \(E\)-linearity and explicit computations in homological degrees 0 and 1.

The long exact sequence gives us most of what we want to know. The only ambiguity is whether \(h_0 t\) is zero or non-zero. As usual, this can be resolved by explicit computations with cocycles in low dimensions. The result is that \(h_0 t\) is non-zero. \(\square\)
APPENDIX A. ADAMS CHARTS

This appendix contains Adams charts for many of the computations made earlier in Section 6.

We use the following notation in all of the charts. Solid dots represent copies of $M_2$, while open circles represent copies of $M_2/\tau$.

Vertical lines indicate multiplications by $h_0$, and lines of slope 1 indicate multiplications by $h_1$. Dotted vertical lines indicate that $h_0$ times a generator equals $\tau$ times a generator. In Figure A.5, the diagonal dotted lines indicate that $\alpha$ times a generator equals $\tau$ times a generator.

The horizontal and vertical coordinates indicate the Adams bidegree. The weights of some elements are shown in parentheses.

Figure A.4 is representative of the computation of $\text{Ext}^*_{A(1)}(\tilde{H}^{*,*}(DQ_n), M_2)$ for $n$ congruent to 0 modulo 4. The interested reader can easily construct charts for values of $n$ other than 16. Similarly, Figures A.6 and A.7 are representative of the computation of $\text{Ext}^*_{A(1)}(\tilde{H}^{*,*}(DQ_n), M_2)$ for $n$ congruent to 2 and 1 modulo 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{\text{Ext}^*_{A(1)}(M_2, M_2)}
\end{figure}
MOTIVIC CONNECTIVE K-THEORIES AND THE COHOMOLOGY OF $A(1)$  

**Figure A.2.** $\text{Ext}^*_A(R,M_2)$

**Figure A.3.** $\text{Ext}^*_A(\tilde{H}^{*,*}(DQ_\infty),M_2)$
Figure A.4. $\text{Ext}_{A(1)}^*(\tilde{H}^{*,*}(DQ_{16}), \mathbb{M}_2)$

Figure A.5. Adams Chart for $\text{Ext}_{A(1)}^*(\tilde{H}^{*,*}(DQ_2), \mathbb{M}_2)$
Figure A.6. $\text{Ext}_{A(1)}^{*}(\overline{H}^{*,*}(DQ_{14}), M_{2})$

Figure A.7. $\text{Ext}_{A(1)}^{*}(\overline{H}^{*,*}(DQ_{17}), M_{2})$
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