Gap Formation Probability of the $\alpha-$ Ensemble

Yang Chen and Kasper Juel Eriksen†

Department of Mathematics, Imperial College

180 Queen’s Gate, London SW7 2BZ, U. K.

† Ørsted Laboratory, Niels Bohr Institute

H.C.Ø. Universitetsparken 5, 2100 Kbh. Ø, Denmark

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Abstract

In this paper we employ the continuum approximation of Dyson to determine the asymptotic gap formation probability in the spectrum of $N \times N$ Hermitean matrices associated with orthogonal polynomials that are solutions of indeterminate moment problems.
I Introduction

Random matrix ensembles originally proposed by Wigner as a phenomenological description of the statistical properties of the energy spectra \[1\],\[2\] has recently seen applications in other area of physics such as Quantum Chaos\[3\], transport in disordered systems\[4\] and 2-d quantum gravity\[5\]. From the random matrix point of view\[6\],\[7\], the quantities of interest (in increasing order of refinement) are the eigenvalue density, the density-density correlation function and \( E_\beta[n, J] \)—the probability that an interval, \( J \), contains exactly \( n \) eigenvalues. [This interval can be generalized to the case where it is the union of several mutually disjoint intervals\[8\],\[9\]].

This paper shall focus on \( E_\beta[0, J] \), henceforth denoted as \( E_\beta[J] \). The probability that there is no eigenvalue in a scaled interval \((-t, t)\), is \( \ln E_\beta(-t, t) \sim -\frac{\pi^2}{4} \beta t^2 - (1 - \beta/2) t \); first determined by Dyson\[10\], using the methods of classical electrostatics, potential theory and thermodynamics, for random ensembles with unitary (\( \beta = 2 \)), orthogonal (\( \beta = 1 \)) and symplectic (\( \beta = 4 \)) symmetries. The missing constant and \( O(\ln t) \) terms were later found in \[11\],\[12\] and \[13\].

For Hermitean matrices (\( \beta = 2 \)) a fundamental result of Gaudin and Mehta \[6\] shows that

\[
E_2[J] = \det(1 - K_J),
\]

where \( K_J \) is the kernel of an integral operator \( K(x, y) \) over \( J \);

\[
K(x, y) = e^{-u(x)/2} e^{-u(y)/2} \sum_{n=0}^{N-1} p_n(x)p_n(y),
\]

\[
\int_P dx \, w(x) p_m(x) = \delta_{mn}, \quad w(x) := e^{-u(x)},
\]

where \( P \) is the interval available to the eigenvalues. Here \( u(x) \) in the Coulomb gas interpretation is an external (confining) potential that holds together a gas of repelling
particles. An alternative representation for the Fredholm determinant is

\[ E_\beta[J] = \frac{Z(J^c)}{Z(J \cup J^c)} = e^{-(F[J] - F[J \cup J^c])}, \] (4)

recognized as the ratio of two partition functions, where

\[ Z[J] = \left( \prod_{a=1}^{N} \int_J dx_a \right) e^{-W} \] (5)

and

\[ W = -\beta \sum_{0 \leq a < b \leq N} \ln |x_a - x_b| + \sum_{a=0}^{N} u(x_a). \] (6)

A Fredholm determinant of this kind also appears in context of the physics of interacting bosons in one dimension\[14\] where it gives the probability of emptiness formation of a segment of the real line.

For sufficiently large \(N\) it is expected that the discrete Coulomb gas can be well approximated by a continuum Coulomb Fluid\[15\] with an equilibrium density \(\sigma(x)\) of charged particles that satisfies a Hückel-like self consistent equation;

\[ u(x) - \beta \int_J dy \sigma(y) \ln |x - y| + \left( 1 - \frac{\beta}{2} \right) \ln [\sigma(x)] = A = \text{constant}, \] (7)

and gives the free energy, at equilibrium, with precisely \(N\) charges (eigenvalues) contained in an interval \(J\);

\[ F[J] = \frac{1}{2} AN + \frac{1}{2} \int_J dx u(x) \sigma(x) - \frac{1}{2} \left( 1 - \frac{\beta}{2} \right) \int_J dx \sigma(x) \ln [\sigma(x)], \] (8)

subject to \(\int_J dx \sigma(x) = N\). This approach has the advantage of being more general and has been shown to supply accurate asymptotics of the level-spacing distribution for the Laguerre ensemble at the hard edge\[16\] for all \(\beta > 0\). We expect that the continuum
Coulomb Fluid description will also be useful in determining $E_\beta[J]$ of orthogonal polynomial random matrix ensembles associated with non-classical weight, $w(x) = e^{-u(x)}$, where an explicit form of the polynomials are not known.

This paper is organized as follows: In section II, we shall introduce and discuss the properties of Hermitian random ensembles (hence forth called the $\alpha-$ensembles) with $u(x) = x^\alpha/\alpha, \ (\alpha > 0, \ x \in (0, \infty))$. It will be shown in the continuum approach the parameter $\alpha$ plays the role of a coupling constant with a critical value ($\alpha_c$), above which the density is universal (i.e. $\alpha$ independent) at both the hard and soft edges [18]. For $\alpha > \alpha_c$, the density at the soft edge remains universal, however, an explicit $\alpha$ dependence is found at the hard edge.

By using a heuristic argument based on the leading asymptotics found by Dyson[10] followed by an un-folding transformation of the density, we obtained an asymptotic formula for, $E_2[J]$, the probability that the interval $J$ contains no eigenvalue. This is first used to test against known results [18],[16] and later make predictions on the gap formation probability of the $\alpha$ ensembles.

In section III a screening theory of the Coulomb Fluid is presented to justify the results obtained in section II. Level Spacing Distribution for $\alpha = 1/2$ is case presented in detail in section IV. This paper concludes with section V, in which we give $E_2[J]$ for certain random matrix ensemble arising from the double scaling limit of two-dimensional quantum gravity.

II Behaviour of density at edges
As we shall be interested exclusively in $\beta = 2$ case, Eq.(7) becomes linear and reads;

$$u(x) - \beta \int_0^b dy \sigma(y) \ln |x - y| = A,$$

where $b$, the upper band edge, is related to the total number of particle (i.e. eigenvalues) through $\int_0^b dy \sigma(y) = N$. By taking a derivative with respect to $x$, Eq.(9) is converted into an integral equation with Cauchy kernel;

$$u'(x) = \beta P \int_0^b dy \frac{\sigma(y)}{x - y},$$

whose solution can be obtained by a standard technique\[19]\;:

$$\sigma(x) = \frac{1}{\beta \pi^2} \sqrt{\frac{b - x}{x}} P \int_0^b dy \frac{y}{y - x} \sqrt{\frac{y}{b - y}} u'(y).$$

Since

$$P \int_0^b dy \frac{1}{x - y} \sqrt{\frac{1}{y(b - y)}} = 0,$$

the general solution of Eq.(10) is found by adding to Eq.(11) the solution of the homogeneous equation, \frac{\text{constant}}{\sqrt{x(b-x)}}. However, it is clear that by including the homogeneous solution, the equilibrium free energy will be larger. The homogeneous solution will not be included.

It can also be verified by direct computation that the density obtained without the homogeneous solution compares well with the density obtained from orthogonal polynomials.

A generalization to the case of finitely many non-overlapping has been studied by Akhiezer\[19]\;:

$$u'(x) = \beta P \int_J dy \frac{\sigma(y)}{x - y}, \quad x \in J$$

where

$$J = \cup_{k=1}^p J_k, \quad J_k = (a_k, b_k), \quad a_k < b_k.$$
The solution to Eq. (13) is

\[ \sigma(x) = \frac{1}{\pi^{2}\beta} \sqrt{\prod_{k=1}^{p} \left( \frac{b_{k} - x}{x - a_{k}} \right)} \mathcal{P} \int_{J} \frac{dy}{y - x} \sqrt{\prod_{l=1}^{p} \left( \frac{b_{l} - y}{y - a_{l}} \right)} u'(y), \quad x \in J. \] (15)

This integral representation can be useful in determining the level spacing distribution in the multiple interval case.

If we now take \( u(x) = \frac{1}{\alpha} x^{\alpha}, \alpha > 0 \) then

\[ \sigma(x) = \frac{1}{\pi^{2}\beta} \sqrt{x} \int_{0}^{b} \frac{dy}{y - x} \sqrt{\frac{y}{b - y}} y^{\alpha - 1} \]

\[ = \frac{b^{\alpha - 1} 2\Gamma(1/2)\Gamma(\alpha + 1/2)}{\pi^{2}\beta} \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)} \sqrt{\frac{b - x}{x}} F(1 - \alpha, 1, 3/2, 1 - x/b), \] (16)

where \( F(\alpha, \beta, \gamma, x) = _2F_1(\alpha, \beta; \gamma; x) \) is the Hypergeometric function. Note that for \( \alpha = 1 \), the result for the Laguerre Ensemble is recovered; \( \sigma(x) = \frac{1}{\pi \beta} \sqrt{\frac{b - x}{x}} \). We have in the Hypergeometric function the following parameters, \( a = 1 - \alpha, \quad b = 1, \quad c = 3/2, \quad \) and \( a + b - c = \frac{1}{2} - \alpha \). From this the behaviour of the density at the hard edge \( (x \sim 0) \) and soft edge \( (x \sim b) \) can be determined. Consider \( \alpha > 1/2, \) and \( x \ll b \), we find from the Gauss summation formula,

\[ F(1 - \alpha, 1, 3/2, 1) = \frac{\Gamma(3/2)\Gamma(\alpha - 1/2)}{\Gamma(1/2)\Gamma(\alpha + 1/2)}, \]

\[ \sigma(x) \sim \frac{1}{\pi^{3/2}\beta} \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)} b^{\alpha - 1} \sqrt{\frac{b}{x}}, \quad x \ll b, \] (17)

The density has a square root singularity at \( x = 0 \), independent of \( \alpha \) and is therefore universal. Near the soft edge, \( x \sim b \),

\[ \sigma(x) \sim \frac{1}{\pi^{3/2}\beta} \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)} b^{\alpha - 1} \sqrt{\frac{b - x}{b}}, \quad b - x \ll b, \] (18)

which shows that the density is universal near \( b \). However, for \( |\alpha| < 1/2 \), a completely different behaviour at the hard edge occurs, this can be seen as follows: A translation, \( x \rightarrow \)
1 − x gives,

\[ F(1 - \alpha, 1, 3/2, 1 - x) = \frac{\Gamma(3/2)\Gamma(\alpha - 1/2)}{\Gamma(\alpha + 1/2)\Gamma(1/2)} F(1 - \alpha, 1, 3/2 - \alpha, x) \]

\[ + x^{\alpha-1/2} \frac{\Gamma(3/2)\Gamma(1/2 - \alpha)}{\Gamma(1 - \alpha)} F(\alpha + 1/2, 1/2, \alpha + 1/2, x). \]  

(19)

In view of the combination of the parameters \((a, b, c)\), the second term of the right hand side of Eq.(19) dominates as \(x \to 0\). Putting \(x = 0\) in the second Hypergeometric function; we find \(F(a, b, c, 0) = 1, \ 0 < a + b - c < 1\). Therefore,

\[ \sigma(x) \sim \frac{\tan \pi \alpha}{\pi \beta} \frac{1}{x^{1-a}}, \ x \ll b. \]  

(20)

Eq.(20) has two features that are worth noting; 1. \(\sigma(x)\) near the hard edge is independent of the macroscopic parameter \(b\) [which increases with \(N\)]. This type of behaviour at the hard edge is also be observed in the unitary \(q\)– Laguerre Ensemble with \(u(x; q) = \sum_{n=0}^{\infty} \ln[1 + (1 - q)xq^n], \ 0 < q < 1 \ x \geq 0\);

\[ \sigma_N(0) = \frac{1 - q^N}{\ln (\frac{1}{q})}, \]

which rapidly becomes \(N\) independent as \(N\) increases\[20\]. 2. The hard edge density has a explicit \(\alpha\) dependence.

We remark here that the kernel in the unitary \(q\)– Laguerre Ensemble reads,

\[ K(x, y) = \frac{\text{constant}}{x - y} \left[ \left( \frac{x}{y} \right)^{1/4} \sin \left( \frac{\pi}{2\gamma} \ln x \right) \cos \left( \frac{\pi}{2\gamma} \ln y \right) - \left( \frac{y}{x} \right)^{1/4} \sin \left( \frac{\pi}{2\gamma} \ln y \right) \cos \left( \frac{\pi}{2\beta} \ln x \right) \right], \]

(21)

in which \(\gamma := \ln(1/q) \gg 1\) and \(N \to \infty\)[21]. Therefore the so-called universal density-density correlation function at the hard edge \(< \rho(x)\rho(y) > - < \rho(x) > < \rho(y) >= \delta(x - \)
\[ y < \rho(x) > - [K(x, y)]^2 = (xy)^{-1/2}(x^{1/2} + y^{1/2})/(x - y)^2 \] is seen to be violated. This is due to the weakly confining nature of the \( q - \) Laguerre potential; \( u(x; q) \sim [\ln x]^2, \ x \to \infty \), producing a heavily depleted density at the origin. We expect in the \( \alpha - \) ensemble a similar phenomena to occur for \( \alpha < 1/2 \). In principle, the gap formation probability of the unitary \( \alpha - \) ensemble at the hard edge can be computed directly from the Fredholm determinant in Eq.(1), however, the associated orthogonal polynomials being related to an indeterminate moment problem \([23, 24]\) is not known, unlike the unitary \( q - \) Laguerre Ensemble \(24\). A direct evaluation of the Fredholm determinant would be difficult. To extract the leading term in \(- \ln E_2(0, s)\) we shall adopt the Coulomb Fluid approximation mentioned in section I.

Before we give results on the \( \alpha - \) ensemble, it is interesting to note that for \( \alpha = 1/2 \),

\[
\sigma(x) = \frac{1}{\pi^2 \beta} \frac{1}{\sqrt{x}} \ln \left[ \frac{1 + \sqrt{1 - x/b}}{1 - \sqrt{1 - x/b}} \right], \quad b = \pi^2 N^2
\]

(22)

where we have made use of the identity

\[
F(1/2, 1, 3/2, x) = \frac{1}{2\sqrt{x}} \ln \left[ \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right],
\]

to arrive at Eq.(22).

Thus for \( x \ll b \)

\[
\sigma(x) \sim \frac{1}{\pi^2 \beta} \frac{\ln(4b/x)}{\sqrt{x}} + o(\sqrt{x}),
\]

(23)

while for \( x \sim b \)

\[
\sigma(x) \sim \frac{1}{\pi^2 \beta} \left[ (1 - x/b)^{1/2} + \frac{5}{3} (1 - x/b)^{3/2} + \cdots \right],
\]

remains to be universal. We may therefore interpret \( \alpha = 1/2 \) to be the critical point at which there is a logarithmic correction to the density.
In order to gain insight into the possible leading terms in the gap formation probability, we examine the results first obtained by Dyson for the Circular ensemble\cite{10} where by construction the density is uniform over the circle, \( N/2\pi \). It was shown that
\[-\ln E_2(0, s) \sim \text{constant} \ s^2, \quad s \gg 1.\]
Here \( s \) being expressed in terms of the average distance between the charges is dimensionless. The leading term is simply proportional to the square of the number of particles excluded in an interval of length \( s \). Using the unfolding transformation in the density, we can always define a new variable \( y \), such that
\[\sigma(x) dx = dy, \quad \text{or equivalently} \quad \frac{dy}{dx} = \sigma(x).\]
The density in \( y \) is unity. At this point, we may make use of Dyson’s asymptotic and conclude that the leading term is proportional to the square of the number of particles excluded in that interval. The probability that the interval \( J \) contains no eigenvalues is
\[-\ln E_\beta[J] \sim \left[ \int_J dx \sigma(x) \right]^2. \quad (24)\]
Based on Eq.(24), the probability that an interval of length \( a \) (measured from the origin) contains no eigenvalue is,
\[E_\beta[a] \sim e^{-[\mathcal{N}(a)]^2},\]
where
\[\mathcal{N}(a) := \int_0^a dx \sigma(x),\]
is the number of charges excluded in \((0, a)\). Taking \( \sigma(x) = \frac{1}{\pi\beta} \sqrt{\frac{b-x}{x}} \), we find
\[\int_0^a dx \sigma(x) = \frac{b}{\pi\beta} \int_0^{a/b} \sqrt{\frac{1-x}{x}} \sim \frac{2}{\pi\beta} \left[ \sqrt{ab} - \frac{1}{6} \frac{(ab)^{3/2}}{b^2} + \cdots \right]. \quad (25)\]
In the limit \( b \to \infty \) (or \( N \to \infty \), since \( b \propto N \)) and \( a \to 0 \) such that the combination \( ab = s \) is finite, gives
\[-\ln E_\beta(0, s) \sim s, \quad (26)\]
agrees with the known leading behaviour\[18], \[26], \[16]. By taking $J = (a, b), a << b$ and scaling $b - a$ with respect to the true soft edge density $\sigma_N(N) \sim N^{-1/3} \sim b^{-1/3}$, we find,

$$- \ln E_\beta(0, s) \sim s^3,$$
which agrees with that found in\[18], \[27]. Observe that although the density in the continuum approach vanishes at $b$, the correct dependence at the soft edge, i.e., $\sigma(b) \sim b^{-1/3}$, is found\[26].

For the $\alpha$-ensembles,

$$- \ln E_\beta^{(\alpha)}(0, a) \sim a^{2\alpha}, \quad 0 < \alpha < 1/2,$$
while $\alpha = 1/2$,

$$\sqrt{- \ln E_\beta^{(1/2)}(0, a)} \sim \int_0^a dx \frac{\ln(2\beta b/x)}{\sqrt{x}} \sim \sqrt{a} \ln \left( \frac{2\beta b}{a} \right).$$

II Screening theory

In this section we shall consider the unitary case ($\beta = 2$). With a slight change of notation, the free energy of $N$ particles restricted in an interval $I = (0, b)$ under the influence of an external potential $u(x)/\beta$ is

$$F[\sigma, I, N] = \int_I dx u(x) \sigma(x, I, N) - \int_I dx \int_I dy \sigma(x, I, N) \ln |x - y| \sigma(y, I, N),$$
where $\sigma(x, I, N)$ is the equilibrium density that minimizes $F[\sigma, I, N]$, subject to

$$\int_I \sigma(x) dx = N.$$
According to Eq.(4) the quantity of interest is the change in free energy, i.e. the free energy when the charges are excluded from the sub-interval $J = (0, a), a < b$ of $I$ minus
the free energy when the charges are in $I$:

$$\Delta F[J, N] = F[J^c, N] - F[I, N].$$

(31)

in the limit $N \to \infty$. Here the interval $J^c$ is the the interval where the charges are distributed, when there are no charges in $J$. For later use we introduce,

$$\Delta \sigma(x, J, N) := \sigma(x, J^c, N) - \sigma(x, I, N),$$

(32)

and require that all the $\sigma$’s are zero outside their respective supports. The support of $\Delta \sigma(x, J, N)$ is $L = I \cup J^c$.

Using the definitions of $\Delta F$ and $\Delta \sigma$ in Eqs.(30) and (31), a simple calculation gives,

$$\Delta F[J, N] = \int_L dx \left[ u(x) - 2 \int_I dy \ln |x - y| \sigma(y, I, N) \right] \Delta \sigma(x, J, N)$$

$$- \frac{1}{2} \int_L dx \int_L dy \Delta \sigma(x, J, N) \ln |x - y| \Delta \sigma(y, J, N).$$

(33)

Since $\sigma(x, I, N)$ is the density that minimizes the free energy; $\left[ \cdots \right]$, in Eq.(33) is equal to the chemical potential, $\mu(I, N)$, for $x \in I$ and is a function of of $x$, for $x \in L - I = J^c - I$.

Charge neutrality states that $\int_L dx \Delta \sigma(x, J, N) = 0$. Therefore the first term in Eq.(33)

can be rewritten as

$$\int_L dx \left[ u(x) - 2 \int_I dy \ln |x - y| \sigma(y, I, N) \right] \Delta \sigma(x, J, N)$$

$$= \int_L dx \left[ u(x) - 2 \int_I dy \ln |x - y| \sigma(y, I, N) - \mu(I, N) \right] \Delta \sigma(x, J, N)$$

$$= \int_{L - I} dx \left[ u(x) - 2 \int_I dy \ln |x - y| \sigma(y, I, N) - \mu(I, N) \right] \Delta \sigma(x, J, N).$$

The term $\left[ \cdots \right]$ under $\int_{L - I} dx$ is of the same order of magnitude as the change in the chemical potential when the particles are excluded from $J$: $\Delta \mu(J, N) = \mu(J^c, N) - \mu(I, N)$. 

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\( \mu(I, N) \). \( \Delta \mu(J, N) \) decreases as \( N \) grows and in the limit, \( N \to \infty \), it is proportional to the number of particles excluded from \( J \). Therefore the first term in Eq.(33) is approximately \( \Delta \mu(J, N) \) times the number of particles in \( J^c - I \), which is the number of particles that spilled over when all particles are excluded from \( J \). In the bulk and at the hard edge this number goes to zero when \( N \to \infty \) and therefore the first term in Eq.(33) can be neglected. At the soft edge the first term contributes a term quadratic in the number of particles excluded from \( J \). This is of the expected form from the unfolding transformation and will not affect our final result. We therefore have,

\[
\Delta F[J, N] = -\frac{1}{2} \int_L dx \int_L dy \Delta \sigma(x, J, N) \ln |x - y| \Delta \sigma(y, J, N),
\]

(34)

here \( \Delta \sigma(x, J, N) \) is simply \( -\sigma(x, I, N) \) for \( x \in J \). For \( x \notin J \), the expression is more complicated. From now on we shall work exclusively with rescaled quantities such that lengths are measured in the units of \( |J| (= a) \). We also translate the origin of the real axis to the left end of \( J \). With the scaling, \( x \to ax \), and \( y \to ay \), and the introduction of \( \rho(x) \) and \( \lambda(x) \),

\[
\rho(x) := \rho(x, J, N) := -a \sigma(ax), \quad x \in (0, 1)
\]

(35)

\[
\lambda(x) := \lambda(x, J, N) = a \Delta(ax, J, N), \quad x \in [1, b/a).
\]

(36)

The change in free energy when expressed in terms of scaled quantities reads,

\[
\Delta F[J, N] = -\frac{a^2}{2} \int_L dx \int_L dy \Delta \sigma(ax, a) \ln |ax - ay| \Delta \sigma(ay, a)
\]

\[
= -\frac{\ln a}{2} \left[ \int_L dx \Delta \sigma(x, J) \right]^2 - \frac{a^2}{2} \int_{L/a} dx \int_{L/a} dy \Delta \sigma(ax, a) \ln |x - y| \Delta(ay, a)
\]

\[
= \int_0^1 dx \int_{J^c/a} dy \rho(x) \ln |x - y| \lambda(y)
\]

\[
- \frac{1}{2} \int_{J^c/a} dx \int_{J^c/a} dy \lambda(x) \ln |x - y| \lambda(y) - \frac{1}{2} \int_0^1 dx \int_0^1 dy \rho(x) \ln |x - y| \rho(y).
\]

(37)
Note that the first integral in the second equality of Eq. (37) vanishes due to charge neutrality, $\int_L dx \Delta \sigma(x, J) = 0$. The problem of finding $\Delta \sigma(x, J, N)$ in the limit $N \to \infty$ can be reformulated in physical terms. The interval $I$ is force (or field) free since $I$ is a conducting region. Outside $I$ we have a potential that grows when $I$ tends to infinity. We introduce negative charges in $J$ with density $-\sigma(x, I, N)$ and precisely the same numbers of positive charges outside $J$. [The is the charge neutrality condition stated above.] The charges redistribute themselves in such a way that the interval $L - J$ is force free again.

The charges in $J$ are screened by those outside $J$. This screening process is not as efficient as in three dimension since the charges are only allowed to be built up along a line while the force extends out in a plane. $\lambda(x, J)$ is then determined by the condition that the total force in $J^c/a$ is zero.

The force produced by the charges in $J$ is

$$f(x) = \int_0^1 dy \frac{\rho(y)}{x - y}.$$  \hfill (38)

The force produced by the charges in $J^c$ is

$$f_{J^c}(x) = \int_{J^c/a} dy \frac{\lambda(y)}{x - y}.$$  \hfill (39)

Therefore the total force is

$$f_{J^c}(x) + f(x) + f_I(x) = 0, \quad x \in J^c/a,$$  \hfill (40)

where $f_I$ is the combined force in $I$, produced by the external potential $u(x)/2$ and the charge distribution $\Delta \sigma(x, I, N)$.

Two constraints are required to solve Eq.(40): 1. The number of charges in $J^c/a$ equals that in $J/a$, 2. $\Delta \sigma(x, J^c)$ is positive semi-definite for $x \in J^c$, and minimizes the free energy.
We shall also require that at an end point of \( J^c/a \) the density vanishes. The reason is simply that the charges would otherwise distribute in a larger interval. This argument can be generalized to the case where \( J^c = \bigcup_{k=1}^{p} I_k \), with the additional requirement that the chemical potentials to be the same in each intervals \( I_k \). The above formulation can be applied to \( \lambda(x) \). In the above scheme we make the following approximation: If \( J \) is not at the end of \( I \) where new charges gather, the total potential can be approximated by a constant all the way up to infinity and not only inside \( I \). Observe that in this approximation \( \lambda \) is supported everywhere on the real axis outside \( J \); except in the hard edge case where it is only a half line. Call this interval \( P \). In the rescaled units we have as an example in the hard edge problem, \( P = (1, \infty) \), while in the bulk problem, \( P = (-\infty, 0) \cup (1, \infty) \). In terms of the interval \( P \), the change in free energy reads,

\[
\Delta F[J, N] = + \int_0^1 dx \int_P dy \rho(x) \ln |x - y| \lambda(y) - \frac{1}{2} \int_P dx \int_P dy \lambda(x) \ln |x - y| \lambda(x) - \frac{1}{2} \int_0^1 dx \int_0^1 dy \rho(x) \ln |x - y| \rho(y).
\]

To proceed further, we can express \( \lambda \) in terms of \( \rho \) by the use of the condition that the total force is 0 in \( P \). As above we find that the total force is

\[
\int_P dx \frac{\lambda(x)}{x - y} + f(y) = 0. \tag{42}
\]

The solution of Eq.(42) depends only on \( P \) and \( \lambda|_{x \in \partial P} \). For, \( P = (1, \infty) \), the solution is

\[
\lambda(x) = \frac{1}{\pi^2} \int_{1}^{\infty} \frac{dy}{x - y} \sqrt{y - 1} f(y). \tag{43}
\]

The most important feature about this solution is that \( \lambda \) is a linear functional of \( f \) and therefore also of \( \rho \), (see Eq.(40)), which does not depend on \( a \) and \( N \). To proceed further,
we now develop $f$ in a “multipole” (or moment) expansion;

$$f(x) = \int_0^1 dy \frac{\rho(y)}{x-y}, \quad x \in (1, \infty)$$

$$= \sum_{m=0}^{\infty} \frac{1}{x^{m+1}} \int_0^1 dy y^m \rho(y). \quad (44)$$

The moments

$$A_m := \int_0^1 dx x^m \rho(x), \quad (45)$$

will play an important role in a later development. Since $\lambda$ is a linear functional of $f$ and $f$ is a linear functional of $A'_m$s, we conclude that $\lambda$ is a linear functional of the $A'_m$s. By expanding $\sqrt{x} \rho(x), \ x \in [0, 1]$, in terms of Legendre polynomials $P_n(2x - 1),$

$$\sqrt{x} \rho(x) = \sum_{m=0}^{\infty} r_m P_m(2x - 1), \ x \in [0, 1] \quad (46)$$

we find also that $\sqrt{x} \rho(x)$ is a linear functional of the moments $A_{m + \frac{1}{2}}$. In summary, the densities can be expressed as linear functional of $A_m$ and $A_{m + \frac{1}{2}}$. Since the change in the free energy is quadratic in the densities, it can be expressed as a sum quadratic in the $A'_m$s. A simple estimate gives,

$$|A_m| \leq |A_0|, \quad (47)$$

from which we deduce that the change in free energy is bounded from above by $A_0^2$. This we recognized as the square of the number of charges excluded in $(0, a)$. Precisely the same analysis can be carried through in the bulk scaling case.

The upshot of the above analysis is that $\Delta F$ is quadratic in the moments of the local densities, when we adopt the Coulomb fluid approximation. From this we may infer two consequences: 1. The probability of gap formation depends only on the local densities. In the bulk scaling case the density is always a constant and the asymptotics level spacing
distribution found by Dyson is reproduced, 2. Higher moments are nominally linear functions of the number of excluded charges [a logarithmic dependence is in general missed in the continuum approach, because $\Delta F(J)$ disappears when the length of $J$ tends to zero]. Therefore, $\Delta F$ for large spacing is proportional to the square of the number of excluded charges in the appropriate interval. On the other hand it is shown that the gap formation distribution of the Laguerre Ensemble at the hard edge, the large interval formula involves quadratic, linear and a logarithm of the number of excluded charges ($\propto \sqrt{s}$). The breakdown of the present theory can be traced back to the fact that the density of the Laguerre Ensemble contains a point charge at the origin and therefore can not be adequately described by its moments. We may be conclude that leading term of $-\ln E_\beta[J]$, is proportional to $N^2[J]$, where $N[J]$ is the number of charges excluded in $J$. This justifies the un-folding transformation discussed in section II.

We now proceed to express the change in free energy in terms of $A_m'$s; substitute Eq.(43) into Eq.(42), we have

$$\lambda(x) = -\frac{1}{\pi^2} \frac{1}{\sqrt{x-1}} \int_1^\infty \frac{dy}{x-y} \sqrt{y-1} \sum_{m=0}^{\infty} \frac{2A_m}{(2y-1)^{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{2A_m}{\pi} \frac{B(1/2, m + 1/2)}{(2x-1)} \frac{F(-m, 1, 1/2, 1/(2x-1))}{\sqrt{2x-2}}$$

$$= \sum_{m=0}^{\infty} \frac{2A_m}{\pi} \frac{1}{(2x-1)\sqrt{2x-2}} P_m^{(-1/2,1/2-m)} \left(1 - \frac{2}{2x-1}\right)$$

$$= \sum_{m=0}^{\infty} \frac{2A_m}{m!\pi} \frac{[1 - 1/(2x-1)]]^{m-1}}{(2x-1)^2} \frac{d^m}{dz^m} \left[\sqrt{1 - z} z^{m-1/2}\right] \bigg|_{z=1/(2x-1)}, \quad (48)$$

where $P_n^{\mu,\nu}(x)$ are the Jacobi polynomials.

With this expression for $\lambda(x)$ the second integral in the third equality of Eq.(37) reads;

$$\frac{1}{2} \int_1^\infty dx \int_1^\infty dy \lambda(x) \ln |x-y| \lambda(y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m A_n L_{mn}, \quad (49)$$
where the matrix elements $L_{mn} = L_{nm}$ are

$$L_{mn} = \frac{1}{2\pi^2} \int_0^1 ds \int_0^1 dt \sqrt{\frac{t}{1-t}} P_m^{(-1/2, 1/2-m)}(1-2t) \ln \left| \frac{1}{2t} - \frac{1}{2s} \right| \sqrt{\frac{s}{1-s}} P_n^{(-1/2, 1/2-n)}(1-2s).$$

(50)

The third integral, with the aid of Eq.(45), reads

$$\frac{1}{2} \int_0^1 dx \int_0^1 dy \rho(x) \ln |x-y| \rho(y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_m r_n R_{mn},$$

(51)

where

$$R_{mn} := \frac{1}{2} \int_0^1 dx \int_0^1 dy \ln |x-y| \frac{P_m(2x-1) P_n(2y-1)}{\sqrt{x} \sqrt{y}},$$

(52)

The first integral is,

$$\int_0^1 dx \int_1^\infty dy \rho(x) \ln |x-y| \lambda(y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_m A_n U_{mn},$$

(53)

where

$$U_{mn} := \int_0^1 ds \int_0^1 dt \frac{P_m(2t-1)}{\sqrt{t}} \sqrt{\frac{s}{1-s}} P_n^{(-1/2, 1/2-n)}(1-2s) \ln \left[ \frac{1}{2} \left( \frac{1}{s} - (2t - 1) \right) \right].$$

(54)

As an example we determine the leading term of $-\ln E_{\beta[J]}$ of the $\frac{1}{2}$ ensemble. The density in this case is given by Eq.(23) with $\beta = 2$. We find the number of excluded charges in the interval $(0, a)$,

$$N(a) = \int_0^a dx \sigma(x) \sim \frac{2}{\pi^2} \sqrt{a} \ln N, \ N \to \infty.$$  

(55)

In the limit $N \to \infty$ and $a \to 0$ such that $\sqrt{a} \ln N$ is finite, we find according to the screening theory that

$$\Delta F \sim s, \ N \to \infty, \ s := a [\ln N]^2 \text{ finite},$$

(56)

which agrees with the results obtained from the un-folding transformation argument.
IV Solution via the continuum approach for $\alpha = 1/2$

For $\alpha = 1/2$ the solution of the integral equation reads

$$\sigma(x) = \left(1 + \beta \int_a^b \frac{dy}{y-x} \sqrt{\frac{y-a}{y-b}} \frac{1}{\sqrt{y}} \right)^{-1}, \quad x \in (a,b).$$

where

$$K(p) := \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-p^2y^2)}}, \quad p^2 := \frac{b-a}{b}$$

is the complete elliptic function of the first kind, and

$$\Pi(n,p) := \int_0^1 \frac{dy}{1-ny^2} \sqrt{(1-y^2)(1-p^2y^2)}, \quad n := \frac{b-a}{b-x}$$

is the complete elliptic function of the third kind. Note that the $\sigma(x)$ vanishes at $x = b$ and is positive for $x \in (a,b)$. The upper band edge $b$ is related to $a$ and $N$ via the normalization condition; $\int_a^b dx \sigma(x) = N$, as

$$N = \frac{2}{\pi \beta} \sqrt{a} (K(p) - D(p)),$$

where

$$D(x) := \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{x(t^2/(1-t)(1-tx^2)}} = \frac{\pi}{4} F(1/2,3/2,2,x^2)$$

is an elliptic function. The derivation of Eq. (58) is placed in Appendix A. In order to determine the free energy in the interval $(a,b)$, we require the chemical potential $A$ and the interaction energy $\frac{1}{2} \int_a^b dx u(x) \sigma(x)$. The chemical potential reads

$$A = 2\sqrt{b} - \beta \int_a^b dx \sigma(x) \ln |b-x| = \beta N \left(2 \ln 2 - \ln b - \ln p^2 \right) + \frac{4}{\pi \sqrt{b} p^2} E(p),$$

where a derivation of the above is placed in appendix B. The interaction energy reads

$$\frac{1}{2} \int_a^b dx u(x) \sigma(x) = \frac{2}{\pi^2 \beta} (b-a) + \frac{2}{\pi^2 \beta} (b-a) \int_0^{a/b} dt \left[ D \left(\sqrt{1-t} \right) \right]^2.$$
The derivation of Eq.(60) is placed in appendix C.

The free energy for \( N \) charges residing in \((a, b)\) is recapitulated as follows:

\[
F(a, b) = \frac{\beta}{2} \left[ 2 \ln 2 - \ln (b - a) \right] + \frac{2N\sqrt{b}p}{\pi} E(p) + \frac{2}{\beta \pi^2} [b - a] \left[ 1 + \int_0^{a/b} dt D^2 \left( \sqrt{1 - t} \right) \right],
\]

where

\[
N = \frac{2\sqrt{b}}{\beta \pi} \left[ E(p) - \frac{a}{b} K(p) \right].
\]

We now compute the change in the free energy, by expressing \( b \) as function of \( N \) and \( a \) in the limits \( N \rightarrow \infty \) and \( a \rightarrow 0 \), after some straightforward but tedious calculations (not reproduced here), we find

\[
\Delta F \sim \frac{1}{\pi^2} \left[ \sqrt{a} \ln N \right]^2
\]

which substantiates the result obtained from the screening theory.

V Conclusion

We have shown that applying the screening theory to the Coulomb fluid model of random matrix ensemble, gap formation distribution, which is the probability that there are no eigenvalue in a certain interval, \( J \), of the spectrum is, in the limit of large interval

\[
E_\beta[J] \sim e^{-\mathcal{N}[J]},
\]

where

\[
\mathcal{N}[J] = \int_J dx \sigma(x) = \text{number of eigenvalues excluded in } J.
\]

This result shows that the determination of the leading term in \(- \ln E_\beta[J]\), requires only the integral of the local density \( \sigma(x) \), which enables us to make prediction on the gap formation probability of the \( \alpha - \) ensemble at the hard edge.
It is shown that for $\alpha > 1/2$, the leading term in the asymptotic expansion is independent of $\alpha$ with the scaling variable, $s \propto Na$. For $\alpha = 1/2$, due to the logarithmic correction of the hard edge density the scaling variable, $s \propto [\ln N]^2 a$. However, for $\alpha < 1/2$, the hard edge density acquires $\alpha$ dependence, and gives $s \propto N^0 a$. Universality is violated for $\alpha < 1/2$. Similar behaviour is found in the $q-$Hermite [16] and $q-$Laguerre ensembles[20]. The advantage of this approach is that it by-passes the requirement of the knowledge of the asymptotic expansion of non-classical orthogonal polynomials, especially those associated with the indeterminate classical moment problems.

As a side product of the screening theory, we make prediction on the level spacing distribution in the soft edge of Hermitian random matrix models that arises in double-scaling limit of the two-dimensional quantum gravity. For example, taking $u(x) = N(x + \frac{g}{4N}x^4)$, it can be shown that[28] tuning $g$ to $g_c (< 0)$, the critical point density reads,

$$\sigma_c(x) = \frac{1}{24\pi} (8 - x^2)^{3/2},$$

where now $\int_{-\sqrt{8}}^{\sqrt{8}} dx \sigma(x) = 1$. Note that in this formulation,

$$\frac{F}{N^2} = -\frac{1}{2} \int_{-b}^{b} dx \int_{-b}^{b} dy \sigma(x) \ln |x - y| \sigma(y) + \int_{-b}^{b} dx \sigma(x) \left( x + \frac{g}{4N} x^4 \right).$$

Using our theory, a simple calculation gives,

$$- \ln E_2(s) \sim N^2 \left( \int_{a}^{\sqrt{8}} dx \sigma_c(x) \right)^2 \sim s^5$$

where the scaling variable is $s := N^{2/5}(\sqrt{8} - a)$, in the limit $N \to \infty$, $a \to \sqrt{8}$ such that $s$ is finite. The detail exposition for this and other multi-critical point models can be found in[29]. Work is underway to use this approach to determine, $E_\beta(s)$, of the $q-$Laguerre ensembles.
Appendix A The normalization integral

By definition,

\[ N = \int_{a}^{b} dx \sigma(x) = \frac{1}{\pi^2 \beta} \sqrt{b} \int_{0}^{1} dy \sqrt{\frac{1 - y}{y(1 - p^2 y)}} \int_{0}^{1} dx \sqrt{\frac{x}{1 - x}} \]

\[ = \frac{1}{\pi \beta} \sqrt{b} \int_{0}^{1} dy \sqrt{\frac{1 - y}{y(1 - p^2 y)}} = \frac{1}{\pi \beta} \sqrt{b} B(1/2, 3/2)F(1/2, 1/2, 2, p^2) \]

\[ = \frac{2}{\pi \beta} \sqrt{b} p^2 [K(p) - D(p)], \quad (A.1) \]

where we have made of

\[ F(1/2, 1/2, 2, p^2) = \frac{4}{\pi} [K(p) - D(p)]. \quad (A.2) \]
Appendix B Evaluation of the chemical potential

We have

\[ A = 2\sqrt{b} - \beta N \ln(b-a) + I(a,b), \quad (B.1) \]

where

\[ I(a,b) := -\beta(b-a) \int_0^1 dx \ln (b-x(b-a)) \]

\[ = -\frac{2\sqrt{b}}{\pi^2 p^2} \int_0^1 dy \frac{\sqrt{1-y}}{1-p^2 y} \int_0^1 dx \frac{x}{1-x} \ln x \]

\[ = -\frac{2\sqrt{b}}{\pi^2 p^2} \int_0^1 dy \left[ \frac{1-y^2}{1-p^2 y^2} \frac{\partial}{\partial \nu} \int_0^1 dx \frac{x^{\nu-1}}{1-x} \right] \]

\[ = \frac{2\sqrt{b}}{\pi^2 p^2} \int_0^1 dy \sqrt{\frac{1-y^2}{1-p^2 y^2}} \frac{\partial}{\partial \nu} B(-1/2, \nu) F(3/2 - \nu, 1, 3/2, 1 - y^2) \bigg|_{\nu=3/2}. \quad (B.2) \]

We now focus on the derivative with respect to \( \nu \),

\[ \frac{\partial}{\partial \nu} B(-1/2, \nu) F(3/2 - \nu, 1, 3/2, 1 - y^2) \bigg|_{\nu=3/2} \]

\[ = \frac{\partial}{\partial \nu} B(-1/2, \nu) \bigg|_{\nu=3/2} F(0, 1, 3/2, 1 - y^2) + B(-1/2, 3/2) \frac{\partial}{\partial \nu} F(3/2 - \nu, 1, 3/2, 1 - y^2) \bigg|_{\nu=3/2} \]

\[ = -2\pi (1 - \ln 2) + \pi \frac{\partial}{\partial z} F(z, 1, 3/2, 1 - y^2) \bigg|_{z=0}. \quad (B.3) \]

Note that we can make use of

\[ \frac{\partial}{\partial z} \frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{\Gamma(z+a)}{\Gamma(z+b)} \left[ \psi(z+a) - \psi(z+b) \right], \]

to compute the derivative of the Beta function with respect to its argument. We now only have to compute the derivative of the Hypergeometric function with respect to its first parameter. As \( z+1 - 3/2 < 0 \), for \( z \) sufficiently close to 0, we may employ the power series representation of \( F(z,1,3/2,1-y^2) \) which is valid in 0 \( \leq |1-y^2| < 1 \) and differentiate term by term resulting in

\[ \frac{\partial}{\partial z} F(z,1,3/2,1-y^2) \bigg|_{z=0} = \sum_{n=1}^{\infty} B(n,3/2)(1-y^2)^n \]
\[ I(a, b) = 2 \frac{\sqrt{b}}{\pi^2} \left[ -\frac{\pi^2}{p^2} + \int_0^1 dy \sqrt{\frac{1 - y^2}{1 - p^2 y^2}} \left( \frac{2\pi(1 - p^2)}{p^2(1 - y^2)} + 2\pi(1 + \ln 2) \right) \right], \quad (B.5) \]

The integrals in Eq.(B.5) are recognized as the standard elliptic integrals \( K \) and \( D \), and we find,

\[ I(a, b) = 2 \sqrt{b} \left[ -1 + \left( \frac{2(1 - p^2)}{p^2} + \frac{2p^2(1 + \ln 2)}{\pi} \right) K(p) - \frac{2p^2(1 + \ln 2)}{\pi} D(p) \right]. \quad (B.6) \]

Observe that \( D(p) \) can be expressed in terms of \( N \) and \( K(p) \) [see Eq. (A.1)]. Further more using

\[ p^2 D(p) = K(p) - E(p), \]

where

\[ E(p) = \int_0^1 dy \sqrt{\frac{1 - p^2 y^2}{1 - y^2}}, \]

we have,

\[ I(a, b) = 2\beta N \ln 2 - 2\sqrt{b} + \frac{4\sqrt{b}}{\pi} p^2 E(p) \]

and therefore

\[ A = \beta N(2 \ln 2 - \ln b - \ln p^2) + \frac{4\sqrt{b}}{\pi} p^2 E(p). \]
Appendix C Evaluation of the interaction energy

In this appendix we determine the interaction energy for \( \alpha > 0 \), thus,

\[
\frac{1}{2} \int_a^b u(x) \sigma(x) = I_1(a, b, \alpha) + I_2(a, b, \alpha),
\]

(C.1)

where

\[
I_1(a, b, \alpha) := -\frac{b^{2\alpha-1}(b-a)}{2\pi^2 \alpha \beta} \int_0^1 dx \int_0^1 dy \frac{\left(\frac{1-p^2x}{1-p^2y}\right)^\alpha - 1}{y-x} (1-p^2y)^{2\alpha-1} \sqrt{\frac{x}{y}} \sqrt{\frac{1-y}{1-x}},
\]

(C.2)

and

\[
I_2(a, b, \alpha) := \frac{b^{2\alpha-1}(b-a)}{2\pi^2 \alpha \beta} \int_0^1 dx \int_0^1 dy (1-p^2y)^{2\alpha-1} \sqrt{\frac{1-y}{y}} \sqrt{\frac{1-x}{1-y}}.
\]

(C.3)

We first compute \( I_2(a, b, \alpha) \),

\[
I_2(a, b, \alpha) = \frac{a^{2\alpha-1}(b-a)}{2\pi \alpha \beta} \int_0^1 dy (1-p^2y)^{2\alpha-1} \sqrt{\frac{1-y}{y}} = \frac{b-a}{2\beta} \cdot \frac{a^{2\alpha-1}}{2\alpha} F(1-2\alpha, 1/2, 2, p^2),
\]

and find

\[
I_2(a, b, 1/2) = \frac{b-a}{2\beta}.
\]

(C.4)

We now compute \( I_1(a, b, \alpha) \). Observe that,

\[
\frac{\left(\frac{1-p^2x}{1-p^2y}\right)^\alpha - 1}{y-x} = \frac{\left[1 - \frac{p^2(x-y)}{1-p^2y}\right]^\alpha - 1}{y-x} = -\int_0^\frac{p^2}{1-p^2y} \frac{d\epsilon}{x-y} \frac{\partial}{\partial \epsilon} [1-\epsilon(x-y)]^\alpha
\]

\[
= \alpha \int_0^\frac{p^2}{1-p^2y} d\epsilon (1+\epsilon y)^{\alpha-1} \left(1 - \frac{\epsilon x}{1+\epsilon y}\right)^{\alpha-1}.
\]

With the change of variable

\[
\frac{\epsilon}{1+\epsilon y} = t,
\]
we get
\[
\left( \frac{1-p^2x}{y-x} \right)^\alpha - 1 = \alpha \int_0^{p^2} \frac{dt}{(1-ty)^{1+\alpha}(1-tx)^{1-\alpha}}. \tag{C.5}
\]

Using Eq. (C.5) in Eq. (C.2),
\[
I_1(a,b,\alpha) = -\frac{b^{2\alpha-1}(b-a)}{2\pi^2\beta} \int_0^1 dx \sqrt{\frac{x}{1-x}} \int_0^1 dy \sqrt{\frac{1-y}{y}} \int_0^{p^2} \frac{dt}{(1-ty)^{1+\alpha}(1-tx)^{1-\alpha}}
\]
\[
= -\frac{b^{2\alpha-1}(b-a)}{2\pi^2\beta} B(1/2,3/2) \int_0^{p^2} dt F(1-\alpha,3/2,2,t) \int_0^1 dy \frac{(1-p^2y)^{2\alpha-1}}{1-ty^{1+\alpha}} \sqrt{\frac{1-y}{y}}. \tag{C.6}
\]

For \(\alpha = 1/2\), Eq. (C.6) becomes
\[
I_1(a,b,1/2) = -\frac{b-a}{2\pi^2\beta} B^2(1/2,3/2) \int_0^{p^2} dt F(1/2,3/2,2,t) F(3/2,1/2,2,t)
\]
\[
= -\frac{b-a}{8\beta} \int_0^{p^2} dt \left[ D\left(\sqrt{t}\right) \right]^2, \tag{C.7}
\]
where we have used \( F(3/2,1/2,2,t) = F(1/2,3/2,2,t) = \frac{4}{\pi} D\left(\sqrt{t}\right) \), to get Eq. (C.7).

From Eqs. (C.4) and (C.7), the interaction energy reads
\[
I_1(a,b,1/2) + I_2(a,b,1/2) = \frac{b-a}{2\beta} \left(1 - \frac{4}{\pi^2} \int_0^{p^2} dt \left[ D\left(\sqrt{t}\right) \right]^2\right). \tag{C.8}
\]

Note that
\[
\int_0^{p^2} dt \left[ D\left(\sqrt{t}\right) \right]^2 = \int_0^1 dt \left[ D\left(\sqrt{t}\right) \right]^2 - \int_0^{a/b} dt \left[ D\left(\sqrt{t}\right) \right]^2, \tag{C.9}
\]
since \( p^2 = 1 - a/b \). With the aid of
\[
\int_0^1 dt \left[ D\left(\sqrt{t}\right) \right]^2 = \frac{\pi^2}{4} - 1,
\]
and Eq. (C.9), we arrive at Eq. (60) in the text.
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