Queue layouts and nonrepetitive colouring of planar graphs and powers of trees

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Abstract

Dujmović, Joret, Micek, Morin, Ueckerdt and Wood recently in [Planar graphs have bounded queue-number, Journal of the ACM, Volume 67, Issue 4, Article No.: 22, August 2020] showed some attractive graph product structure theorems for planar graphs. By using the product structure, they proved that planar graphs have bounded queue-number 48; in [Planar graphs have bounded nonrepetitive chromatic number, Advances in Combinatorics, 5, 11 pp, 2020], the authors proved that planar graphs have bounded nonrepetitive chromatic number 768.

In this paper, still by using some product structure theorem, we improve the upper bound of queue-number of planar graphs to 27 and the non-repetitive chromatic number to 320. We also study powers of trees. We show a graph product structure theorem of the \( k \)-th power \( T^k \) of tree \( T \), then use it giving an upper bound of the nonrepetitive chromatic number of \( T^k \). We also give an asymptotically tight upper bound of the queue-number of \( T^k \).

Key words and phrases: Queue number; Nonrepetitive chromatic number; Linear ordering; Strong product; Decomposition.

1 Introduction

We consider finite undirected graphs with no loops or parallel edges. The strong product of graphs \( A \) and \( B \), denoted by \( A \boxtimes B \), is the graph with vertex set \( V(A) \times V(B) \), where distinct vertices \( (v, x), (w, y) \in V(A) \times V(B) \) are adjacent if (1) \( v = w \) and \( xy \in E(B) \), or (2) \( x = y \) and \( vw \in E(A) \), or (3) \( vw \in E(A) \) and \( xy \in E(B) \).

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In a recent breakthrough, Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [9] showed that every planar graph is a subgraph of the strong product of a graph of bounded treewidth and a path. This graph product structure theorem is attractive since it describes planar graphs in terms of graphs of bounded treewidth, which are considered much simpler than planar graphs. Then it has been used to solve two longstanding open problems: For planar graphs, Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [9] proved that planar graphs have bounded queue-number 48, solving a 27 year old problem of Heath, Leighton, and Rosenberg [11]. Dujmović, Esperet, Joret, Walczak and Wood [8] proved that planar graphs have bounded nonrepetitive chromatic number 768, solving a 17 year old problem of Alon, Grytczuk, Hałuszczak, and Riordan [3].

In a very recent paper [6], Dvořák, Huynh, Joret, Liu and Wood surveyed generalisations of this result for graphs on surfaces, minor-closed classes, various non-minor-closed classes, and graph classes with polynomial growth, and explored how graph product structure might be applicable to more broadly defined graph classes.

In this paper, we shall present an observation of the original [9, Theorem 16], this will be our Theorem 2.2. Then we shall use this graph product structure Theorem 2.2 to improve the upper bound of queue-number of planar graphs to 27 and the non-repetitive chromatic number to 320. These will be presented in Section 2.

In Section 3, we shall study the \( k \)-th power \( T^k \) of trees \( T \). We shall give a graph product structure theorem of \( T^k \) (Theorem 3.1), which will describe \( T^k \) by using some simpler graphs than \( T^k \). By using this graph product structure theorem, we give an upper bound of the nonrepetitive chromatic number of \( T^k \). We shall also give an upper bound of the queue-number of \( T^k \), this will be our Theorem 1.6 and our bound is asymptotically tight.

1.1 Main results

Heath, Leighton and Rosenberg [11] extended the data structure of queue to graph theory, and gave the concepts of queue-number. The queue number of a graph \( G \) are defined as follows.

Let \( V(G) \) and \( E(G) \) respectively denote the vertex and edge set of \( G \). A vertex (linear) ordering of graph \( G \) is a bijection \( \sigma : V(G) \to \{1, 2, \ldots, |V(G)|\} \). Consider disjoint edges \( uv, xy \in E(G) \) and a linear ordering \( \sigma \) of \( V(G) \). Without loss of generality, \( \sigma(v) < \sigma(w) \) and \( \sigma(x) < \sigma(y) \) and \( \sigma(v) < \sigma(x) \). Then \( uv \) and \( xy \) are said to cross if \( \sigma(v) < \sigma(x) < \sigma(w) < \sigma(y) \) and are said to nest if \( \sigma(v) < \sigma(x) < \sigma(y) < \sigma(w) \). A queue (with respect to \( \sigma \)) is a set of pairwise non-nested edges.

For an integer \( k \geq 0 \), a \( k \)-queue layout of \( G \) consists of a linear ordering \( \sigma \) of \( V(G) \) and a partition \( E_1, E_2, \ldots, E_k \) of \( E(G) \) into queues with respect to \( \sigma \). The queue-number of a graph \( G \), denoted by \( qn(G) \), is the minimum integer \( k \) such that \( G \) has a \( k \)-queue layout. Note that \( k \)-stack layouts are equivalent to \( k \)-page book embeddings, first introduced by Ollmann [16], and stack-number is also called page-number, book thickness, or fixed outer-thickness.
A *rainbow* in a vertex ordering $\sigma$ of a graph $G$ is a set of pairwise nested edges (and thus a matching). Heath and Rosenberg [12] proved the following proposition:

**Proposition 1.1** ([12]) A vertex ordering $\sigma$ of any graph admits a $k$-queue layout if and only if every rainbow in $\sigma$ has size at most $k$.

For the definition of strict $k$-queue layout, let $\sigma$ be a vertex ordering of a graph $G$, let $L_e$ and $R_e$ denote the endpoints of each edge $e \in E(G)$ such that $\sigma(L_e) < \sigma(R_e)$. We say an edge $e$ is *inside* a distinct edge $f$, and $e$ and $f$ *overlap*, if

$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f).$$

A set of edges $Q \subseteq E(G)$ is a *strict queue* in $\sigma$ if no edge in $Q$ is inside another edge in $Q$. Alternatively, $Q$ is a strict queue in $\sigma$ if

$$\sigma(L_e) < \sigma(L_f) \text{ and } \sigma(R_e) < \sigma(R_f),$$

or

$$\sigma(L_f) < \sigma(L_e) \text{ and } \sigma(R_f) < \sigma(R_e).$$

A strict $k$-queue layout of $G$ is a pair $(\sigma, Q_1, Q_2, \ldots, Q_k)$ where $\sigma$ is a vertex ordering of $G$, and $\{Q_1, Q_2, \ldots, Q_k\}$ is a partition of $E(G)$, such that each $Q_i$ is a strict queue in $\sigma$. The *strict-queue-number* of a graph $G$, denoted by $sqn(G)$, is the minimum $k$ such that there is a strict $k$-queue layout of $G$.

A *weak rainbow* in a vertex ordering $\sigma$ of a graph $G$ is a set of edges $R$ such that for every pair of edges $e, f \in R$, $e$ is inside $f$ or $f$ is inside $e$. Wood [22] proved the following proposition:

**Proposition 1.2** ([22]) A vertex ordering $\sigma$ of any graph admits a strict $k$-queue layout if and only if every weak rainbow in $\sigma$ has size at most $k$.

A linear forest is a graph in which every component is a path. The *linear arboricity* of a graph $G$, denoted by $la(G)$, is the minimum integer $k$ such that $E(G)$ can be partitioned in $k$ linear forests; see [2, 4, 23]. Wood [22] gave the following lower bounds on $sqn(G)$.

**Lemma 1.3** ([22]) The strict queue-number of every graph $G$ satisfies

$$sqn(G) \geq la(G) \geq \Delta(G)/2.$$
repetitively coloured by \( \phi \). A walk in a graph \( G \) is a sequence \((v_1, v_2, \ldots, v_t)\) of vertices in \( G \) such that \( v_i v_{i+1} \in E(G) \) for each \( i \in \{1, \ldots, t-1\} \). A path in a graph \( G \) is a walk \((v_1, v_2, \ldots, v_t)\) in \( G \) such that \( v_i \neq v_j \) for all distinct \( i, j \in \{1, \ldots, t\} \). A colouring \( \phi \) of a graph \( G \) is path-nonrepetitive, or simply nonrepetitive, if no path of \( G \) is \( \phi \)-repetitive. The (path-)nonrepetitive chromatic number \( \pi(G) \) is the minimum integer \( k \) such that \( G \) admits a nonrepetitive \( k \)-colouring. Thue’s theorem \cite{19} says that paths are nonrepetitively 3-colourable. Every path-nonrepetitive colouring is proper, as otherwise like-coloured adjacent vertices would form a repetitively coloured path on 2 vertices. Moreover, every nonrepetitive colouring has no 2-coloured \( P_4 \) (a path on four vertices). A proper colouring with no 2-coloured \( P_4 \) is called a star colouring since each bichromatic subgraph is a star forest. The star chromatic number \( \chi_s(G) \) is the minimum number of colours in a proper colouring of \( G \) with no 2-coloured \( P_4 \). Thus
\[
\chi(G) \leq \chi_s(G) \leq \pi(G).
\]

In Section 2 of this paper, we improve the upper bounds of queue-numbers of planar graphs (Theorem 1.4) and graphs with Euler genus \( g \) (Theorem 2.6), and the non-repetitive chromatic number of planar graphs.

**Theorem 1.4** For any planar graph \( G \), the queue-number of \( G \) satisfies \( qn(G) \leq 27 \).

**Theorem 1.5** For any planar graph \( G \), the nonrepetitive chromatic number of \( G \) satisfies \( \pi(G) \leq 320 \).

For two vertices \( x \) and \( y \) in the same component of a graph \( G = (V, E) \), the distance \( \text{dist}_G(x, y) \) between \( x \) and \( y \) is the length of a shortest \( x, y \)-path in \( G \). The \( k \)-th power of \( G \) is the graph \( G^k = (V, E^k) \), where \( E^k = \{xy : 1 \leq \text{dist}_G(x, y) \leq k\} \). In this paper, we shall also study the queue-number and the nonrepetitive chromatic number of the \( k \)-th power of trees.

Queue layouts are inherently related to breadth-first search (BFS). For example, a BFS ordering of the vertices of a tree has no two nested edges, and thus defines a 1-queue layout. So every tree has queue-number 1.

In \cite{22}, Wood proved that the \( k \)-th power of a path \( P_n(n \geq k + 1) \) has queue-number \( qn(P_k^n) = \left[ \frac{k}{2} \right] \) and the \( k \)-th power of a cycle \( C_n(n \geq 2k) \) has queue-number \( \frac{k}{2} < qn(C_k^n) \leq k \). In Section 3, we shall study the \( k \)-th power of trees, we shall prove the following theorems.

**Theorem 1.6** For any tree \( T \) with maximum degree \( \Delta \), the queue-number of its \( k \)-th power \( qn(T^k) \) satisfies:
\[
qn(T^k) \leq \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 1}{\Delta - 2} + \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 1}{\Delta - 2} + \left[ \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}}{2} \right] =: Z.
\]
**Theorem 1.7** For any tree $T$ with maximum degree $\Delta$, the nonrepetitive chromatic number of its $k$-th power $\pi(T^k)$ satisfies:

$$\pi(T^k) \leq \frac{16\Delta}{\Delta - 2}(\Delta - 1)^{\lceil k/2 \rceil+1} - \frac{32}{\Delta - 2}(\Delta - 1)^{\lceil k/2 \rceil+1}. $$

If $T$ is a maximum tree of height $h = \lfloor \frac{k}{2} \rfloor$ and maximum degree $\Delta$, then the $k$-th power of $T$ is a complete graph $K_p$, where

$$p = 1 + \Delta \sum_{i=0}^{\lfloor k/2 \rfloor-1} (\Delta - 1)^i = 1 + \Delta \frac{(\Delta - 1)^{\lfloor k/2 \rfloor} - 1}{\Delta - 2} = \frac{\Delta}{\Delta - 2}(\Delta - 1)^{\lfloor k/2 \rfloor} - \frac{2}{\Delta - 2}. \quad (1)$$

The queue-number of complete graph $K_p$ was first given by Heath and Rosenberg in [12], we shall need the queue-number, the strict-queue-number, and the nonrepetitive chromatic number of $K_p$ in our study of powers of trees. For the completeness of this paper, we give the following lemma.

**Lemma 1.8** For complete graph $K_p$, $q(K_p) = \lfloor \frac{p}{2} \rfloor$, $sqn(K_p) = p - 1$, $\pi(K_p) = p$.

**Proof.** We use Proposition 1.4 to show $qn(K_p) = \lfloor \frac{p}{2} \rfloor$: since every rainbow of $K_p$ has size at most $\lfloor \frac{p}{2} \rfloor$, thus $qn(K_p) \leq \lfloor \frac{p}{2} \rfloor$; for every vertex ordering $\sigma$ of vertices of $K_p$, there exists a rainbow with size $\lfloor \frac{p}{2} \rfloor$, this gives $qn(K_p) \geq \lfloor \frac{p}{2} \rfloor$.

Fix a vertex ordering $\sigma$ of vertices of $K_p$, suppose the vertex ordering $\sigma$ is from 1 to $p$. We define a partition $\{Q_1, \ldots, Q_{p-1}\}$ of $E(K_p)$ in the following way: for an edge $e \in E(K_p)$, if $\sigma(R_e) - \sigma(L_e) = i$, then put $e$ into $Q_i$.

For each $Q_i$ ($i \in \{1, \ldots, p-1\}$) and $e, f \in Q_i$, if $\sigma(L_e) < \sigma(L_f)$, then $\sigma(R_e) = \sigma(L_e) + i < \sigma(R_f) + i = \sigma(R_f)$. This shows $Q_i$ is a strict queues in $\sigma$. Thus $sqn(K_p) \leq p - 1$. For the lower bound, consider the first vertex $v_1$ in the ordering $\sigma$, all the edges incident with $v_1$ will go to different queues in any strict-queue layout of $K_p$, thus $sqn(K_p) \geq p - 1$.

It is obvious that $\pi(K_p) = p$. □

Note that by Lemma 1.8 and Equation (1), the bound $Z$ in Theorem 1.6 is asymptotically tight.

In the remaining part of this section, we introduce some tools that will be used in our presentation, they are so-called tree-decompositions and treewidth, strongly nonrepetitive colouring, and strong products.

### 1.2 Treewidth

A *tree-decomposition* of a graph $G$ consists of a collection $\{B_x \subseteq V(G) : x \in V(T)\}$ of subsets of $V(G)$, called bags, indexed by the vertices of a tree $T$, and with the following properties:

1. For every vertex $v$ of $G$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of $T$;
2. For every edge $vw$ of $G$, there is a vertex $x \in V(T)$ for which $v, w \in B_x$.

The width of such a tree-decomposition is $\max\{|B_x| : x \in V(T)\} - 1$. The treewidth $\text{tw}(G)$ of a graph $G$ is the minimum width of a tree-decomposition of $G$. Tree-decompositions were introduced by Robertson and Seymour [18]. Treewidth measures how similar a given graph is to a tree, and is particularly important in structural and algorithmic graph theory.

Dujmović, Morin and Wood [10] first proved that graphs of bounded treewidth have bounded queue-number. Their bound on the queue-number was doubly exponential in the treewidth. Wiechert [20] improved this bound to singly exponential.

**Lemma 1.9** ([20]) Every graph with treewidth $k$ has queue-number at most $2^k - 1$.

Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [1] also improved the bound in the case of planar 3-trees. (A $k$-tree is an edge-maximal graph of treewidth $k$.) The following lemma that will be useful later is implied by this result and the fact that every planar graph of treewidth at most 3 is a subgraph of a planar 3-tree [14].

**Lemma 1.10** ([1, 14]) Every planar graph with treewidth at most 3 has queue-number at most 5.

### 1.3 Strongly nonrepetitive colouring

A walk $(v_1, v_2, \ldots, v_{2t})$ in a graph is boring if $v_i = v_{t+i}$ for each $i \in \{1, \ldots, t\}$. Every colouring of a boring walk is repetitive. So Barát and Wood [7] defined a colouring to be walk-nonrepetitive if every repetitively coloured walk is boring. For a graph $G$, the walk-nonrepetitive chromatic number $\theta(G)$ is the minimum number of colours in a walk-nonrepetitive colouring of $G$.

A stroll in a graph $G$ is a walk $(v_1, v_2, \ldots, v_{2t})$ such that $v_i \neq v_{t+i}$ for each $i \in \{1, \ldots, t\}$. A colouring of $G$ is stroll-nonrepetitive if no stroll is repetitively coloured. For a graph $G$, the stroll-nonrepetitive chromatic number $\rho(G)$ is the minimum number of colours in a stroll-nonrepetitive colouring of $G$. Every walk-nonrepetitive colouring is stroll-nonrepetitive and every stroll-nonrepetitive colouring is path-nonrepetitive. Thus every graph $G$ satisfies

$$\pi(G) \leq \rho(G) \leq \theta(G).$$

Barát and Varjú [5] and Kündgen and Pelsmajer [15] independently proved that graphs of bounded treewidth have bounded nonrepetitive chromatic number. Specifically, Kündgen and Pelsmajer [15] proved that every graph with treewidth $k$ is non-repetitively $4^k$-colourable, which is the best known bound. Dujmović, Esperet, Joret, Walczak and Wood strengthened this result. The proof is almost identical to that of Kündgen and Pelsmajer [15] and depends on the following lemma which showed that paths are walk-nonrepetitively 4-colourable.
Lemma 1.11 ([15]) For every path $P$,
\[ \theta(P) \leq 4 \]
with equality if $|V(P)| \geq 6$.

Lemma 1.12 ([8, 21]) For every graph $G$ of treewidth at most $k \geq 0$, we have $\rho(G) \leq 4^k$.

1.4 Strong products

Nonrepetitive colourings of graph products have been studied in [7, 13, 15, 17]. Stroll-
nonrepetitive colourings of graph products have been studied in [8, 21], the following lemma is very useful in the proof of Theorem 1.5 and Theorem 1.7.

Lemma 1.13 ([8, 21]) For all graphs $G$ and $H$,
\[ \pi(G \boxtimes H) \leq \rho(G \boxtimes H) \leq \rho(G)\theta(H). \]

Wood [22] gave an upper bound on the queue-number of strong products. This is used in our proof of Theorem 1.4.

Lemma 1.14 ([22]) For all graphs $G$ and $H$,
\[ qn(G \boxtimes H) \leq 2sqn(G) \cdot qn(H) + sqn(G) + qn(H). \]

2 Planar graphs

A partition of a graph $G$ is a set $\mathcal{P}$ of non-empty sets of vertices in $G$ such that each vertex of $G$ is in exactly one element of $\mathcal{P}$. The quotient $G/\mathcal{P}$ is the graph with vertex set $\mathcal{P}$ where distinct parts $X, Y \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in $X$ is adjacent in $G$ to some vertex in $Y$.

If $T$ is a tree rooted at a vertex $r$, then a non-empty path $(x_1, \ldots, x_p)$ in $T$ is vertical if for some $d \geq 0$ for all $i \in \{1, \ldots, p\}$ we have $\text{dist}_T(x_i, r) = d + i$. The vertex $x_1$ is called the upper endpoint of the path and $x_p$ is its lower endpoint.

In a rooted spanning tree $T$ of a graph $G$, a tripod consists of up to three pairwise disjoint vertical paths in $T$ whose lower endpoints form a clique in $G$.

We shall use the following result, which is the original [9 Theorem 16].

Theorem 2.1 ([9 Theorem 16]) Let $T$ be a rooted spanning tree in a plane triangulation $G$. Then $G$ has a vertex partition $\mathcal{P}$ into tripods in $T$ such that $G/\mathcal{P}$ has treewidth at most 3.

From above theorem, we derive the following key theorem in this section.
Theorem 2.2  For every planar graph $G$, there exists a tripod $R$ and graph $H$ with treewidth at most 3, such that $G$ is a subgraph of $R \boxtimes H$.

Proof. Without loss of generality, suppose that $G$ is a plane triangulation. Following Theorem 2.1 ([9, Theorem 16]), suppose the vertex partition $\mathcal{P} = \{R_1, \ldots, R_s\}$, where each $R_i$ $(1 \leq i \leq s)$ is a tripod in $T$. Given $\mathcal{P}$, we define a tripod $R$ such that each $R_i$ $(1 \leq i \leq s)$ is a subgraph of $R$: $R$ is defined as a triangle with vertices $v_1, v_2, v_3$, plus three (vertex disjoint) paths $P_1, P_2, P_3$ which end up with $v_1, v_2, v_3$ respectively, and have lengths $l$ such that $l$ is longer than any longest path in $R_i \in \mathcal{P}$. Then, for any tripod $R_i \in \mathcal{P}$, let its lower endpoints be in $\{v_1, v_2, v_3\}$, note that since paths $P_1, P_2, P_3$ have lengths $l$ (which is long enough for all $R_i \in \mathcal{P}$), $R_i$ is a subgraph of $R$.

Define $H := G/\mathcal{P}$, where $G$ and $\mathcal{P} = \{R_1, \ldots, R_s\}$ are as defined in Theorem 2.1. Then $H$ has treewidth at most 3; since $R_i \subseteq R \ (1 \leq i \leq s)$, we have $G$ is a subgraph of $R \boxtimes H$.  

Next, for the tripod $R$ in above theorem, we give its strict-queue-number.

Lemma 2.3  For tripod $R$, the strict-queue-number of $R$ satisfies that $sqn(R) = 2$.

Proof. Since $\Delta(R) = 3$, Lemma 1.3 implies that $sqn(R) \geq \frac{3}{2}$. Thus $sqn(R) \geq 2$.

Now we prove the upper bound. Suppose tripod $R$ is defined as a triangle with vertices $v_1, v_2, v_3$, plus three (vertex disjoint) paths $P_1, P_2, P_3$ which end up with $v_1, v_2, v_3$ respectively. Consider $P_1v_1$ is a vertex ordering along $P_1$ to $v_1, v_3P_3$ is a vertex ordering beginning with $v_3$ followed by $P_3$. Suppose $P_2 = v_2v'_2P'_2$ and a vertex ordering along $P_2$ is $v_2v'_2P'_2$. Define a vertex ordering $\sigma$ of tripod $R$ as $\sigma := P_1v_1, v_2, v_3P_3, v'_2P'_2$.

Next we define a partition $\{Q_1, Q_2\}$ of $E(R)$, such that both $Q_1$ and $Q_2$ are strict queues in $\sigma$. For this, let $Q_1 := P_1 + v_1v_2 + v_2v_3 + P_3$, $Q_2 := v_3P_3 + P_2$. It is straightforward to verify that $Q_1$ and $Q_2$ are strict queues in $\sigma$. This proves $sqn(R) \leq 2$.

Theorem 1.4  For any planar graph $G$, the queue-number of $G$ satisfies $qn(G) \leq 27$.

Proof. For planar graph $G$, by Theorem 2.2, $G$ is a subgraph of $R \boxtimes H$, where $R$ is a tripod and $H$ has treewidth at most 3. By Lemma 2.3, $sqn(R) = 2$; by Lemma 1.10, $qn(H) \leq 5$. Then, by Lemma 1.14 we have

$$qn(G) \leq qn(R \boxtimes H) \leq 2sqn(R) \cdot qn(H) + sqn(R) + qn(H) \leq 2 \times 2 \times 5 + 2 + 5 = 27.$$ 

Dujmović, Joret, Micek, Morin, Ueckerdt and Wood gave the following lemma [9, Lemma 21] and proved that the upper bound for the queue-number of graphs with Euler genus $g$ is $4g + 49$.

Lemma 2.4 ([9, Lemma 21]) Let $G$ be a connected graph with Euler genus $g$. For every BFS layering $(V_0, V_1, \ldots)$ of $G$, there is a set $Z \subseteq V(G)$ with at most $2g$ vertices in each layer $V_i$, such that $G - Z$ is planar. Moreover, there is a connected planar graph $G^+$ containing $G - Z$ as a subgraph, such that there is a BFS layering $(W_0, W_1, \ldots)$ of $G^+$ and $W_i \cap V(G - Z) = V_i \setminus Z$ for all $i \geq 0$. 

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Theorem 2.5 ([9, Theorem 2]) Every graph with Euler genus $g$ has queue-number at most $4g + 49$.

We use the proof of [9, Theorem 2] and Theorem 1.4 to improve the upper bound.

Theorem 2.6 Every graph with Euler genus $g$ has queue-number at most $4g + 27$.

Proof. Let $G$ be a graph $G$ with Euler genus $g$. We may assume that $G$ is connected. Let $(V_0, V_1, \ldots, V_t)$ be a BFS layering of $G$. By Lemma 2.4 ([9, Lemma 21]), there is a set $Z \subseteq V(G)$ with at most $2g$ vertices in each layer $V_i$, such that $G - Z$ is planar, and there is a connected planar graph $G'$ containing $G - Z$ as a subgraph, such that there is a BFS layering $(W_0, W_1, \ldots, W_t)$ of $G'$ such that $W_i \cap V(G - Z) = V_i \setminus Z$ for all $i \in \{0, 1, \ldots, t\}$.

By Theorem 1.4 and the proof of [9, Theorem 2], there is a 27-queue layout of $G'$ and 4g-queue layout of the edges incident with vertices in $Z$. Hence there are $4g + 27$ queues in total. ■

To prove Theorem 2.6, we give an upper bound for the walk-nonrepetitive chromatic number of tripod $R$ in Theorem 2.2.

Lemma 2.7 For tripod $R$, its walk-nonrepetitive chromatic number $\theta(R) \leq 5$.

Proof. By Lemma 1.11, path $P_1v_1v_2P_2$ has a walk-nonrepetitively colouring $\phi_1$ using colours $C$ with $|C| \leq 4$. Suppose path $P_3 = v_3v'_3P'_3$. Apply Lemma 1.11 again, for path $v'_3P'_3$, there exists a walk-nonrepetitively colouring $\phi_2$ using colours $C$. To prove the lemma, we use $\phi_1, \phi_2$, and a new colour (than $C$) for the remaining vertex $v_3$. Then, we find a walk-nonrepetitively colouring $\phi$ using at most 5 colours for tripod $R$. ■

Theorem 1.5 For any planar graph $G$, the nonrepetitive chromatic number of $G$ satisfies $\pi(G) \leq 320$.

Proof. For planar graph $G$, by Theorem 2.2, $G$ is a subgraph of $R \boxtimes H$, where $R$ is a tripod and $H$ has treewidth at most 3. By Lemma 2.7, $\theta(R) \leq 5$; by Lemma 1.12, $\rho(H) \leq 4^3 = 64$. Then, by Lemma 1.13, we have

$$\pi(G) \leq \rho(G) \leq \rho(R \boxtimes H) \leq \theta(R)\rho(H) \leq 5 \times 4^3 = 320.$$ ■

3 Powers of trees

In this section, we study powers of trees $T$. In Subsection 3.1, we shall give a graph product structure theorem of the $k$-th power $T^k$ of $T$ (Theorem 3.1). By using this theorem, we give an upper bound of the nonrepetitive chromatic number of $T^k$. In
Subsection 3.2, we shall prove some asymptotically tight upper bounds for the queue-number and the strict-queue-number of $T^k$, these will be Theorems 1.6 and 3.6.

We say a tree $T$ is a maximum tree if all the vertices of $T$ except the leaves have the maximum degree of $T$. Note that any tree $T'$ is a subgraph of a maximum tree $T$ with the same maximum degree as $T'$. For this reason, throughout this section, we suppose that $T$ is a maximum tree with maximum degree $\Delta$.

For our proofs, we suppose that $T$ is a rooted tree with root $r$, and the level of $r$ is 0. We denote such a tree as $T_r$. Note that the rooted tree $T_r$ has a natural family-tree-hierarchy: For each vertex $v \in V(T_r)$, there is a unique path $P(v, r)$ from $v$ to $r$, the hierarchy of $v$ is defined by $\text{dist}_{T_r}(v, r) = |P(v, r)|$. For vertex $w \in P(v, r)$, and $\text{dist}_{T_r}(w, r) = \text{dist}_{T_r}(v, r) - i$ (then $\text{dist}_{T_r}(v, w) = |vP(v, r)w| = i$), we say $w$ is the $i$-ancestor of $v$, and $v$ is an $i$-descendant of $w$.

### 3.1 Product structure and nonrepetitive colouring of powers of trees

In this subsection, we shall give a graph product structure theorem of $T^k_r$ (Theorem 3.1), then use it to give some upper bound of the nonrepetitive chromatic number and queue-number of $T^k$. In our proofs of this subsection, we suppose the root $r$ of rooted tree $T_r$ has degree $\Delta$.

We shall need the following concept. In a rooted tree $R_s$, add all the edges between vertices of $R$ and their 2-ancestors, we say the resulting graph is the similar tree of $R$, and denote it as $2R$. Suppose $\sigma$ is a BFS vertex ordering of $R_s$. For each vertex $v \in 2R$, there are at most two neighbors before $v$ in $\sigma$, and they form a clique in $2R$. Thus $\text{tw}(2R) \leq 2$.

**Theorem 3.1** For tree $T_r$ with maximum degree $\Delta$, the $k$-th power of $T^k_r$ is a subgraph of $2T \boxtimes K_t$, where $2T$ is the similar tree of a tree $T$, and complete graph $K_t$ has order $t = \frac{\Delta - 2}{\Delta - 2}((-1)^{\frac{j}{2}} - \frac{\Delta - 2}{\Delta - 2}((-1)^{\frac{j}{2}})$.

**Proof.** The proof relies on a partition $B$ of $V(T_r)$ that we are going to construct, we say the elements of $B$ are bags of $T_r$, and bags will satisfy some well defined properties. The construction of $B$ will be done gradually through adding series of bags to $B$ one series after another. When we add a new series of bags $Z$ to $B$, the vertices in bags $Z \in Z$ will be removed from the remaining graph of $T_r$ simultaneously. We shall repeat the above process until all the vertices of $T_r$ are added to $B$ eventually.

The construction of $B$ is described using the following inductive process. During this process, we shall also construct a rooted tree $T(B)$ which each vertex of $T(B)$ will be a bag of $B$ (its vertex set will be $B$), its root and edges will be defined in the process.

For a rooted subtree $Q$ of $T_r$, define the level of $Q$ is the level of the root of $Q$ in $T_r$. For integers $j \geq 0$, define the $j$-top-tree $Q_j$ of $T_r$ is the subgraph of $T_r$ which is induced by the vertices of $V(T_r)$ from 0-th level to $j$-th level.
To construct $B$, we use the $\lfloor \frac{k}{2} \rfloor$-top-tree vertices $V(Q_{\lfloor \frac{k}{2} \rfloor})$ of $T_r$ as our first bag: the first series adding to $B$ is defined as $Z_0 := \{V(Q_{\lfloor \frac{k}{2} \rfloor})\}$, and we name it as $0$-th level series. Also add the bag of $Z_0$ to the vertex set of $T(B)$, designate this bag as the root of $T(B)$.

For the induction hypothesis, suppose we just added $(i - 1)$-th level series of bags $Z_{i-1}$ to $B$, and also added the bags of $Z_{i-1}$ to the vertex set of $T(B)$. Suppose that all the subtrees contained in the series $Z_{i-1}$ are in the same level of $T_r$.

Then we remove the vertices in these bags of series $Z_{i-1}$ from the remaining graph of $T_r$, every component left is a subtree of $T_r$, suppose this set of left subtrees is $Q$. For every $Q \in Q$, we consider $Q$ as a rooted subtree: the root of $Q$ is the vertex $q$ whose $1$-ancestor $p$ is in some bag of $Z_{i-1}$ (indeed, $p$ is a leaf in a just removed subtree in $Z_{i-1}$), we shall use this relationship later to construct the edges of $T(B)$.

Applying the induction hypothesis, we know that all the roots of subtrees in $Q$ are in the same level of $T_r$, therefore their $\lfloor \frac{k}{2} \rfloor$-ancestors are in the same level of $T_r$. We partition $Q$ into subtree classes in following way: $Q \in Q$ go to the same class $S$, if the roots of these $Q$ are descendants of a common $\lfloor \frac{k}{2} \rfloor$-ancestor. Note that the root $r$ can not be such an ancestor, the number of $\lfloor \frac{k}{2} \rfloor$-descendants of any vertex different than the root $r$ is at most $(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}$. Thus each class $S$ has at most $(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}$ subtrees as its elements.

For each $S$ in $Q$, we define a bag $Z(S)$, where $Z(S)$ consists of all vertices of the $\lfloor \frac{k}{2} \rfloor$-top-tree of all subtrees $Q \in S$. Series $Z_i$ is then defined as all bags $Z(S)$, where $S$ in $Q$. We name $Z_i$ as $i$-th level series.

Add $i$-th level series of bags $Z_i$ to $B$, and also add the bags of $Z_i$ to the vertex set of $T(B)$. For new bag $Z \in Z_i$, if there is a subtree $Q \subseteq Z$ such that the root $q$ of $Q$ has its $1$-ancestor in some bag $Z'$ of $Z_{i-1}$, then add an edge $ZZ'$ in $T(B)$. This finishes the induction step of the construction.

The above process repeats until all the vertices of $T_r$ are added to $B$, this finishes the construction of $B$. Eventually, suppose $B = \{Z_0, \ldots, Z_w\}$, each $Z_i$ ($0 \leq i \leq w$) is a series of bags of $T_r$. To prove our theorem, in the $k$-th power of $T_r$ (denote it as $T_r^k$), we contract each bag in $B$ to a single vertex.

**Claim 3.2** Suppose $H := T_r^k/B$, then $H$ is a subgraph of the similar tree of $T(B)$, i.e. $H \subseteq 2T(B)$.

**Proof.** First we show that, for any $Z_i$ ($0 \leq i \leq w$), if two different bags $B_1, B_2 \in Z_i$, then there is no edge between $B_1$ and $B_2$ in $T(B)$. To see this, suppose $Q_1$ and $Q_2$ are two subtrees in $B_1$ and $B_2$ respectively, and suppose $q_1$ and $q_2$ are the roots of $Q_1$ and $Q_2$ respectively. Since $B_1$ and $B_2$ are different bags in $Z_i$, we know the $\lfloor \frac{k}{2} \rfloor$-ancestors of $q_1$ and $q_2$ in $T_r$ are different vertices, and they are indeed in the same level. This shows dist$_{T_r}(q_1, q_2) \geq 2(\lfloor \frac{k}{2} \rfloor + 1) \geq k + 1$. Thus no vertices between $B_1$ and $B_2$ can be joined by an edge in $T_r^k$. This shows that the bags in $Z_i$ are independent with each other in $T(B)$. 


Now we show that, for any two bags $B_1$ and $B_2$, $B_1 \in Z_i$, $B_2 \in Z_j$ ($0 \leq i < j \leq w$), if $B_1$ is not the 1-ancestor or 2-ancestor of $B_2$ in $T(B)$, then there is no edge between $B_1$ and $B_2$ in $T(B)$.

If $j = i + 1$, suppose that $B'_1$ is the 1-ancestor of $B_2$ in $T(B)$. Since $B_1$ is not the 1-ancestor of $B_2$ in $T(B)$, we know that $B_1$ and $B'_1$ are two different bags in $Z_i$, and note that those paths connecting vertices between $B_1$ and $B_2$ have to go through some vertices of $B'_1$. As showed above, $B'_1$ and $B_1$ are independent in $T(B)$, this shows that $B_1$ and $B_2$ are independent in $T(B)$.

For all the left cases, it suffices for us to check the case that $j = i + 3$, and $B_1$ is the 3-ancestor of $B_2$ in $T(B)$. For this case, note that for any vertices $b_1 \in B_1$ and $b_2 \in B_2$, $\text{dist}_{T_r}(b_1, b_2) \geq 2 \lfloor \frac{k}{2} \rfloor + 3 \geq k + 2$, thus no vertices between $B_1$ and $B_2$ can be joined by an edge in $T_r$.

Equation (1) tells us that every $\lfloor \frac{k}{2} \rfloor$-top-tree has at most $\frac{\Delta}{\Delta - 2}(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - \frac{2}{\Delta - 2}$ vertices, combining $|S| \leq (\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}$, we know the number of vertices in each bag is at most $t := \frac{\Delta}{\Delta - 2}(\Delta - 1)^{2\lfloor \frac{k}{2} \rfloor} - \frac{2}{\Delta - 2}(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}$, thus the induced graph of every bag in $T_r$ is a subgraph of $K_t$.

By Claim 3.2, we have $T_r^k / B \subseteq 2T(B)$. Let $T = T(B)$, then $T_r^k \subseteq 2T \boxtimes K_t$. This proves the theorem.

As some applications of Theorem 3.1, we use it to prove Theorem 1.7 and give an upper bound of the queue-number $\text{qn}(T^k)$ (Theorem 3.3).

**Theorem 1.7** For any tree $T$ with maximum degree $\Delta$, the nonrepetitive chromatic number of its $k$-th power $\pi(T^k)$ satisfies:

$$\pi(T^k) \leq \frac{16\Delta}{\Delta - 2}(\Delta - 1)^{2\lfloor \frac{k}{2} \rfloor + 1} - \frac{32}{\Delta - 2}(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor + 1}.$$  

**Proof.** By Theorem 3.1 $\text{tw}(H) \leq 2$; by Lemma 1.12 $\rho(H) \leq 16$. Combining Lemma 1.13 and Theorem 3.1 we have

$$\pi(T^k) \leq \rho(T^k) \leq \rho(H \boxtimes K_t) \leq \rho(H)\theta(K_t) \leq 16t$$

$$\leq \frac{16\Delta}{\Delta - 2}(\Delta - 1)^{2\lfloor \frac{k}{2} \rfloor} - \frac{32}{\Delta - 2}(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}.$$  

**Theorem 3.3** For any tree $T$ with maximum degree $\Delta$, the queue-number of its $k$-th power $\text{qn}(T^k)$ satisfies:

$$\text{qn}(T^k) \leq \frac{5\Delta}{\Delta - 2}(\Delta - 1)^{2\lfloor \frac{k}{2} \rfloor + 1} - \frac{10}{\Delta - 2}(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor + 1} - 3.$$  


Proof. For the similar tree $H$, after removing $E(T_r)$ from $H$, in the left graph, even-level vertices induce a tree and odd-level vertices induce some vertex-disjoint trees, and there are no edges between each other. Denote this left graph as $F$, then $F$ is a forest.

Suppose $\sigma$ is a breadth-first search (BFS) vertex ordering of $T_r$. Then $T_r$ (also $F$) is a queue in $\sigma$. Thus $qn(H) \leq 2$. By Lemma 1.8, we have $sqn(K_t) \leq t - 1$. Combining Lemma 1.14 and Theorem 3.1, we have

$$qn(T^k) \leq qn(H \boxtimes K_t) \leq 2sqn(K_t) \cdot qn(H) + sqn(K_t) + qn(H)$$

$$\leq 2(t - 1) \cdot 2 + (t - 1) + 2 \leq 5t - 3$$

$$\leq \frac{5\Delta}{\Delta - 2} (\Delta - 1)^{\lceil \frac{k}{2} \rceil} - \frac{10}{\Delta - 2} (\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 3.$$

3.2 Queue and strict-queue layouts of powers of trees

In this subsection, we shall prove some asymptotically tight upper bounds of the queue-number and strict-queue-number of $T^k$. Our proofs will use Propositions 1.1 and 1.2, and some counting techniques. To count the elements, we shall do a partition of them. The partition relies on the following level function.

In our proofs of this subsection, we suppose the root $r$ of rooted tree $T_r$ is a leaf-vertex. Note that this is different than the previous subsection. In rooted tree $T_r$, denote the level of a vertex $v \in V(T_r)$ by $l(v)$ (then $l(r) = 0$). In the $k$-th power $T^k_r$ of $T_r$, we say an edge $uv \in E(T^k_r)$ is an $i$-level edge, if $|l(v) - l(u)| = i$.

For convenience of our presentation, for vertices $u$ and $v$, if $l(u) = j$, $l(v) = j + i$, and the $h$-ancestor of $u$ and $(h+i)$-ancestor of $v$ are the same vertex $w$, then for $h' \geq h$, we say the $h'$-ancestor of $u$ and $(h'+i)$-ancestor of $v$ are the same, even if this common ancestor may not exist in $T_r$ (but $w$ exists).

**Theorem 1.6** For any tree $T$ with maximum degree $\Delta$, the queue-number of its $k$-th power $qn(T^k)$ satisfies:

$$qn(T^k) \leq \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 1}{\Delta - 2} + \frac{(\Delta - 1)^{\lceil \frac{k}{2} \rceil} - 1}{\Delta - 2} + \left\lfloor \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}}{2} \right\rfloor =: Z.$$

Proof. Suppose $\sigma$ is a breadth-first search (BFS) vertex ordering of $T_r$. By Proposition 1.1, it suffices to prove every rainbow in $\sigma$ has size at most $Z$.

We say a rainbow is an $i$-level rainbow if it consists of only $i$-level edges. We consider those maximum $i$-level rainbows in $\sigma$. We begin with a maximum 0-level rainbow $R_0$ in $\sigma$, this part also serves as a warm-up for the rest of the proofs.

Note the following fact: If $uv$ is a 0-level edge of $T^k_r$, then vertices $u$ and $v$ share a common $\left\lfloor \frac{k}{2} \right\rfloor$-ancestor. Otherwise, suppose the $\left\lfloor \frac{k}{2} \right\rfloor$-ancestors of $u$ and $v$ in $T_r$ are different vertices. Then $\text{dist}_{T_r}(u, v) \geq 2\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \geq k + 1$. But this contradicts that $uv$ is an edge in $T^k_r$. 

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Claim 3.4 Suppose edge $xy \in R_0$, $l(x) = l(y) = j$, vertex $z$ is the common $\lfloor \frac{k}{2} \rfloor$-ancestor of $x$ and $y$. If a different edge $uv$ also belongs to $R_0$, then $l(u) = l(v) = j$, and the common $\lfloor \frac{k}{2} \rfloor$-ancestor of $u$ and $v$ is $z$.

Proof. Since edge $uv \in R_0$, $l(u) = l(v)$. Suppose $l(u) = l(v) = j'$, and $j' < j$ ($j' > j$), then, following from the BFS ordering $\sigma$, both $\sigma(u)$ and $\sigma(v)$ are smaller (larger) than both $\sigma(x)$ and $\sigma(y)$. This contradicts that both $uv, xy \in R_0$. This shows $l(u) = l(v) = j$.

Suppose the common $\lfloor \frac{k}{2} \rfloor$-ancestor of $u$ and $v$ is $w$, and $w \neq z$. Then either $\sigma(w) < \sigma(z)$ or $\sigma(w) > \sigma(z)$. It follows from the BFS ordering $\sigma$, both $\sigma(u)$ and $\sigma(v)$ are either smaller or larger than both $\sigma(x)$ and $\sigma(y)$. This contradicts that both $uv, xy \in R_0$. ■

Note that those vertices in level $j$ and sharing a common $\lfloor \frac{k}{2} \rfloor$-ancestor $z$, induce a clique $Q$ in $T^k$. By Claim 3.4 and Lemma 1.8 this maximum 0-level rainbow $R_0$ has size $\lfloor \frac{|Q|}{2} \rfloor$, and notice that $|Q| \leq (\Delta - 1)\lfloor \frac{k}{2} \rfloor$.

Then we consider a maximum $i$-level rainbow $R_i$ ($1 \leq i \leq k$) in $\sigma$.

Note the following fact: Suppose $uw$ is an $i$-level edge of $T^k$, $l(u) = j$ and $l(v) = j'$. Then the $\lfloor \frac{k-i}{2} \rfloor$-ancestor of $u$ and the $(\lfloor \frac{k-i}{2} \rfloor + i)$-ancestor of $v$ are the same vertex. Otherwise, suppose they are different vertices, then $\text{dist}_{T_i}(u, v) \geq 2(\lfloor \frac{k-i+2}{2} \rfloor) + i \geq k + 1$. But this contradicts that $uw$ is an edge in $T^k$.

Claim 3.5 Suppose edge $xy \in R_i$, $l(x) = j$ and $l(y) = j + i$, vertex $z$ is the $\lfloor \frac{k-i}{2} \rfloor$-ancestor of $x$, and the $(\lfloor \frac{k-i}{2} \rfloor + i)$-ancestor of $y$. If a different edge $uv$ also belongs to $R_i$, then $l(u) = j$ and $l(v) = j + i$, $z$ is the $\lfloor \frac{k-i}{2} \rfloor$-ancestor of $u$, and the $(\lfloor \frac{k-i}{2} \rfloor + i)$-ancestor of $v$.

Proof. Suppose $l(u) = j'$, since edge $uv \in R_i$, $l(v) = j'$. Suppose $j' < j$ ($j' > j$), then, following from the BFS ordering $\sigma$, both $\sigma(u)$ and $\sigma(v)$ are smaller (larger) than both $\sigma(x)$ and $\sigma(y)$. This contradicts that both $uv, xy \in R_i$. This shows $l(u) = j$ and $l(v) = j + i$.

Suppose vertex $w$ is the $\lfloor \frac{k-i}{2} \rfloor$-ancestor of $u$, and the $(\lfloor \frac{k-i}{2} \rfloor + i)$-ancestor of $v$, and $w \neq z$. Then either $\sigma(w) < \sigma(z)$ or $\sigma(w) > \sigma(z)$. It follows from the BFS ordering $\sigma$, both $\sigma(u)$ and $\sigma(v)$ are either smaller or larger than both $\sigma(x)$ and $\sigma(y)$. This contradicts that both $uv, xy \in R_i$. ■

By Claim 3.5 $u$ is a $\lfloor \frac{k-i}{2} \rfloor$-descendant of vertex $z$. Note that in the $j$-th level, the number of $\lfloor \frac{k-i}{2} \rfloor$-descendants of vertex $z$ is at most $(\Delta - 1)^{\lfloor \frac{k-i}{2} \rfloor}$. This shows

$$|R_i| = |\{u : uv \in R_i, \text{ and } u \text{ is the left endpoint of } uv\}| \leq (\Delta - 1)^{\lfloor \frac{k-i}{2} \rfloor}.$$  

Therefore a maximum rainbow $R$ in $\sigma$ satisfies that

$$|R| \leq \sum_{i=0}^{k} |R_i| \leq \sum_{i=1}^{k} (\Delta - 1)^{\lfloor \frac{k-i}{2} \rfloor} = \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 1}{\Delta - 2} + \frac{(\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} - 1}{\Delta - 2} + \lfloor (\Delta - 1)^{\lfloor \frac{k}{2} \rfloor} \rfloor.$$
This finishes the proof of this theorem. □

The following result on the strict-queue-number $sqn(T^k)$ is similar to Theorem 1.6.

**Theorem 3.6** For any tree $T$ with maximum degree $\Delta$, the strict-queue-number of its $k$-th power $sqn(T^k)$ satisfies:

$$sqn(T^k) \leq \frac{(\Delta - 1)^{k+1} - 1}{\Delta - 2} + \frac{(\Delta - 1)^k - 1}{\Delta - 2} - (k + 1) =: X.$$

**Proof.** Suppose $\sigma$ is a breadth-first search (BFS) vertex ordering of $T_r$. By Proposition 1.2, it suffices to prove every weak rainbow in $\sigma$ has size at most $X$.

We say a weak rainbow is an $i$-level weak rainbow if it consists of only $i$-level edges. We consider those maximum $i$-level weak rainbows in $\sigma$.

Suppose $R_0$ is a maximum 0-level weak rainbow in $\sigma$. Note that Claim 3.4 and its proof can be applied to this 0-level weak rainbow $R_0$. Then apply Lemma 1.8, we know $R_0$ has size $|Q| - 1$, where $Q$ is a clique in $T_r^k$ induced by vertices in the same level and sharing a common $\lfloor \frac{k}{2} \rfloor$-ancestor in $T_r$, therefore $|Q| \leq (\Delta - 1)^{\lfloor \frac{k}{2} \rfloor}$.

Then we consider a maximum $i$-level weak rainbow $R_i$ ($1 \leq i \leq k$) in $\sigma$. Note that Claim 3.5 and its proof can be applied to this $i$-level weak rainbow $R_i$.

Apply Claim 3.5 to this $i$-level weak rainbow $R_i$, we know that $u$ is a $\lfloor \frac{k-i}{2} \rfloor$-descendant of vertex $z$, and $v$ is a $\lfloor \frac{k+i}{2} \rfloor$-descendant of vertex $z$. Note that in the $j$-th level, the number of $\lfloor \frac{k-i}{2} \rfloor$-descendants of vertex $z$ is at most $(\Delta - 1)^{\lfloor \frac{k+i}{2} \rfloor}$; in the $(j+i)$-th level, the number of $\lfloor \frac{k+i}{2} \rfloor$-descendants of vertex $z$ is at most $(\Delta - 1)^{\lfloor \frac{k+i}{2} \rfloor}$.

Thus $R_i$ is a weak rainbow in a subgraph $H$ of $T_r^k$, and the number of vertices in $H$ is at most $N := (\Delta - 1)^{\lfloor \frac{k+i}{2} \rfloor} + (\Delta - 1)^{\lfloor \frac{k-i}{2} \rfloor}$. Since $H \subseteq K_N$, by using Lemma 1.8 we have $|R_i| \leq N - 1 = (\Delta - 1)^{\lfloor \frac{k+i}{2} \rfloor} + (\Delta - 1)^{\lfloor \frac{k-i}{2} \rfloor} - 1$.

Therefore a maximum weak rainbow $R$ in $\sigma$ satisfies that

$$|R| \leq \sum_{i=0}^{k} |R_i| \leq \sum_{i=0}^{2k} (\Delta - 1)^{\lfloor \frac{i}{2} \rfloor} - (k + 1) = \frac{(\Delta - 1)^{k+1} - 1}{\Delta - 2} + \frac{(\Delta - 1)^k - 1}{\Delta - 2} - (k + 1).$$

This finishes the proof of this theorem. □

If $T_r$ is a maximum tree of height $h = k$ with maximum degree $\Delta$, then the maximum degree of $k$-th power of $T$ is

$$\Delta(T^k) = \Delta \sum_{i=0}^{k-1} (\Delta - 1)^i = \Delta \frac{(\Delta - 1)^k - 1}{\Delta - 2}. \quad (2)$$

Note that by Lemma 1.3 and Equation (2), the bound $X$ in Theorem 3.6 is asymptotically tight.
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