Cosmological Einstein-Maxwell Instantons and Euclidean Supersymmetry: Beyond Self-Duality

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Abstract

We construct new supersymmetric solutions to the Euclidean Einstein-Maxwell theory with a non-vanishing cosmological constant, and for which the Maxwell field strength is neither self-dual or anti-self-dual. We find that there are three classes of solutions, depending on the sign of the Maxwell field strength and cosmological constant terms in the Einstein equations which arise from the integrability conditions of the Killing spinor equation. The first class is a Euclidean version of a Lorentzian supersymmetric solution found in \cite{1, 2}. The second class is constructed from a three dimensional base space which admits a hyper-CR Einstein-Weyl structure. The third class is the Euclidean Kastor-Traschen solution.
1 Introduction

A considerable amount of work has been done in recent years towards the classification of solutions admitting supersymmetry in various supergravity theories. The classification of supersymmetric solutions was initiated in the work of [3, 4]. There, Einstein-Maxwell theory was considered as the bosonic sector of $N = 2$ supergravity in four dimensions. In [4], solutions with time-like and null Killing vectors, admitting a supercovariantly constant spinor were determined.

More recently there has been some work on the classification of solutions of Euclidean Einstein-Maxwell theory in the case with a zero [5, 6], and non-zero [7] cosmological constant. The motivation for studying such Euclidean solutions is their possible relevance to the non-perturbative analysis in the theory of quantum gravity. In the early studies of gravitational instantons, the Einstein equations of motion, which are in general hard to solve, were simplified by assuming a self-dual Riemann tensor. This is analogous to the condition of self-dual Yang-Mills field strengths imposed in the study of instantons in [8].

The analysis of [7] was mainly focused on the cases where the Maxwell field is anti-self-dual, where it was shown that the field equations of supersymmetric solutions reduce to a sub-class of Einstein-Weyl equations in three dimensions [9] which is integrable by twistor transform [10, 11]. For supersymmetric Euclidean solutions, it turns out that the anti-selfduality of the Maxwell field implies the conformal anti-selfduality (ASD) of the Weyl tensor. The Killing spinor equations used in the analysis of [7] contain a continuous parameter. For a particular value of this parameter, the solutions constructed have a Killing vector and are related to solutions of the $SU(\infty)$ Toda equation.

In our present work, the anti-self-duality condition on the Maxwell field is relaxed and we classify solutions admitting Killing spinors using spinorial geometry techniques which were first used to analyse supersymmetric solutions in ten and eleven dimensions in [12], partly based on [13]. Other applications of spinorial geometry techniques are the classification of solutions in lower dimensions [14]; the first systematic classification of supersymmetric extremal black hole near-horizon geometries in ten-dimensional heterotic supergravity [15]; and the classification of supersymmetric solutions of Euclidean $N = 4$ super Yang-Mills theory [16].

We plan our work as follows. In section two, we write down a Killing spinor equation for the Euclidean Einstein-Maxwell theory with a cosmological constant. The Killing spinor equation is fixed by considering the associated integrability condition and comparing with the Einstein equations of motion. Four possibilities for the Killing spinor equation are obtained, corresponding to the different possible choices for signs of the Maxwell and cosmological constant terms in the Einstein field equations. In section two we also write down the essential equations needed for our analysis based on spinorial geometry. In sections three and four we present the solutions obtained through the analysis of the Killing spinor equation. The solutions in section three are a Euclidean version of the “timelike” class of solutions found in [11, 12]. The solutions in section four consist of the Euclidean Kastor-Traschen solution, and a set of solutions constructed from a 3-dimensional base space which admits a hyper-CR Einstein-Weyl structure. In section 5 we present our conclusions.
2 The Killing Spinor Equations

Let \( g \) be a positive definite metric on a Riemannian four manifold, and let \( \Gamma_\mu \) be the generators of the Clifford algebra, \( \{ \Gamma_\mu, \Gamma_\nu \} = 2\delta_{\mu\nu} \). Unless stated otherwise, we shall work in an orthonormal frame, with frame indices \( \mu, \nu \) in which \( g_{\mu\nu} = \delta_{\mu\nu} \). To begin, we consider a generalized Killing spinor equation:

\[
\nabla_\mu \epsilon = (c_1 + d_1 \gamma_5) F_{\nu_1 \nu_2} \Gamma^{\nu_1 \nu_2} \Gamma_\mu \epsilon + (c_2 + d_2 \gamma_5) \Gamma_\mu \epsilon + (c_3 + d_3 \gamma_5) A_\mu \epsilon
\]

for complex constants \( c_1, c_2, c_3, d_1, d_2, d_3 \), and \( \Gamma_\mu = \Gamma_{[\mu} \Gamma_{\nu]} \). In addition, \( \nabla_\mu \epsilon \equiv (\partial_\mu + \frac{1}{2} \Omega_{\mu \nu_1 \nu_2} \Gamma^{\nu_1 \nu_2}) \epsilon \) is the supercovariant derivative with spin connection \( \Omega \), and \( F = dA \) is the gauge field strength satisfying

\[
dF = 0, \quad d \ast F = 0.
\]

We proceed to evaluate the integrability conditions associated with (2.1). By examining the terms which are linear in the gauge potential \( A \), one must have

\[
d_3 = 0.
\]

The integrability conditions further imply that

\[
R_{\mu\nu} + 32(c_1^2 - d_1^2) F_{\mu\sigma} F_{\nu}{}^{\sigma} + (12(c_1^2 - d_2^2) - 8(c_1^2 - d_1^2) F_{\nu_1 \nu_2} F^{\nu_1 \nu_2}) g_{\mu\nu} = 0,
\]

with \( c_3 \) fixed by

\[
c_3 = 8(c_3 c_2 - d_1 d_2),
\]

and

\[
c_1 d_2 - d_1 c_2 = 0.
\]

Next, consider a re-definition of the spinor via

\[
\epsilon = (a + b \gamma_5) \epsilon'
\]

for complex constants \( a, b \), such that \( a^2 - b^2 \neq 0 \). Substitute this spinor into (2.1) and multiply on the left by \( (a + b \gamma_5)^{-1} \) to obtain

\[
\nabla_\mu \epsilon' = (c_1' + d_1' \gamma_5) F_{\nu_1 \nu_2} \Gamma^{\nu_1 \nu_2} \Gamma_\mu \epsilon' + (c_2' + d_2' \gamma_5) \Gamma_\mu \epsilon' + 8(c_3 c_2 - d_1 d_2) A_\mu \epsilon'
\]

where

\[
\begin{align*}
\frac{1}{(a^2 - b^2)}((a^2 + b^2)c_1 - 2abd_1), & \quad d_1' = \frac{1}{(a^2 - b^2)}(-2abc_1 + (a^2 + b^2)d_1), \\
\frac{1}{(a^2 - b^2)}((a^2 + b^2)c_2 - 2abd_2), & \quad d_2' = \frac{1}{(a^2 - b^2)}(-2abc_2 + (a^2 + b^2)d_2).
\end{align*}
\]

Note in particular that

\[
\begin{align*}
(c_1')^2 - (d_1')^2 &= (c_1)^2 - (d_1)^2, & (c_2')^2 - (d_2')^2 &= (c_2)^2 - (d_2)^2 \\
(c_1' d_2' - d_1' c_2') &= c_1 c_2 - d_1 d_2, & (c_1' d_2' - d_1' c_2') &= c_1 d_2 - d_1 c_2.
\end{align*}
\]
In order for the integrability condition (2.4) to correspond to the Einstein-Maxwell equations

\[ R_{\mu\nu} + 6\Lambda g_{\mu\nu} + c(4F_{\mu\sigma}F^\nu_{\sigma} - g_{\mu\nu}F_{\nu 1\nu 2}F^{\nu 1\nu 2}) = 0 \]  

for non-zero constants \( \Lambda, c \), one must have \( c_1 \neq \pm d_1 \). Hence, it is straightforward to show that, without loss of generality, one can choose \( a, b \) with \( a \neq \pm b \) such that \( d'_1 = 0 \). On dropping the primes throughout, one then obtains \( d_2 = 0 \) from (2.6). After making this transformation, the Killing spinor equation becomes:

\[ \nabla_\mu \epsilon = c_1 F_{\nu 1\nu 2} \Gamma^{\nu 1\nu 2} \Gamma_\mu \epsilon + c_2 \Gamma_\mu \epsilon + 8c_1 c_2 A_\mu \epsilon . \]  

We shall call backgrounds which admit a spinor \( \epsilon \) satisfying this equation supersymmetric. The Killing spinor equation (2.12) has the associated integrability condition

\[ R_{\mu\nu} + 32c_1^2 F_{\mu\nu} = 0 \]  

for \( c_1, c_2 \) are then fixed, up to a sign, by comparison with the Einstein-Maxwell equations (2.11). We consider four cases, corresponding to \( (c_1, c_2) = (-\frac{i}{4}, -\frac{1}{2}), (c_1, c_2) = (-\frac{i}{4}, -\frac{1}{2}), (c_1, c_2) = (-\frac{1}{4}, -\frac{i}{2}) \) for \( \ell \in \mathbb{R} \). Note that the cosmological constant \( \Lambda \) appearing in the Einstein-Maxwell equations (2.11) is given by \( \Lambda = 2c_2^2 \), and so \( \ell^{-2} = \pm 2\Lambda \), depending on whether \( c_2 \) is real or imaginary. We remark that in [7] a similar analysis was carried out in the case for which \( F \) is anti-self-dual. It was shown that when this restriction is made, the conditions imposed on the constants appearing in the Killing spinor equation, which one obtains by comparing the integrability conditions with the Einstein-Maxwell equations, are weaker. In particular, in addition to the cosmological constant, there is an extra free real parameter in the Killing spinor equations. However, when one drops the anti-self-dual condition on \( F \), the conditions obtained from matching the integrability conditions to the Einstein-Maxwell equations are stronger, and as we have shown, the only free parameter remaining, after re-scaling of \( F \) is taken into account, is the cosmological constant.

In order to analyse the solutions of the Killing spinor equation (2.8), it will be convenient to work with a Hermitian basis \( e^1, e^2, e^1, e^2 \), with \( e^1 = e^1, e^2 = e^2 \), for which the spacetime metric is

\[ ds^2 = 2\left(e^1 e^1 + e^2 e^2\right). \]  

The metric has signature \((+, +, +, +)\). The space of Dirac spinors is the complexified space of forms on \( \mathbb{R}^2 \), with basis \( \{1, e_1, e_2, e_{12} = e_1 \wedge e_2\} \); a generic Dirac spinor \( \epsilon \) is a complex linear combination of these basis elements. In this basis, the action of the Gamma matrices on the Dirac spinors is given by

\[ \Gamma_m = \sqrt{2}e_m \mathcal{J}, \quad \Gamma_m = \sqrt{2}e_m \wedge \]  

for \( m = 1, 2 \), where \( \mathcal{J} \) denotes contraction. We define

\[ \gamma_5 = \Gamma_{12} \]  

for \( m = 1, 2 \).
which acts on spinors via
\[ \gamma_5 = 1, \quad \gamma_5 e_{12} = e_{12}, \quad \gamma_5 e_m = -e_m \quad m = 1, 2. \] (2.17)

To proceed we examine the orbits of \( \text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1) \) acting on the space of Dirac spinors. In particular, as observed (for example) in [17], there are three non-trivial orbits. In our notation, following the reasoning set out in [18], one can use \( SU(2) \) transformations to rotate a generic spinor \( \epsilon \) into the canonical form
\[ \epsilon = \lambda + \sigma e_1 \] (2.18)

where \( \lambda, \sigma \in \mathbb{R} \). The three orbits correspond to the cases \( \lambda = 0, \sigma \neq 0; \lambda \neq 0, \sigma = 0 \) and \( \lambda \neq 0, \sigma \neq 0 \).

### 3 Solutions with \( c_1 = -\frac{i}{4}, c_2 = -\frac{1}{2\ell} \)

In this case, the Killing spinor equation is
\[ \left( \partial_\mu + \frac{1}{4} \Omega_{\mu \nu \rho} \Gamma^{\nu \rho} + \frac{i}{4} F_{\nu \rho} \Gamma^{\nu \rho} \Gamma_\mu + \frac{1}{2\ell} \Gamma_\mu - \frac{i}{\ell} A_\mu \right) \epsilon = 0. \] (3.1)

To proceed, one evaluates the Killing spinor equation (3.1) acting on the spinor (2.18). First we consider the special cases for which \( \epsilon = \lambda 1 \) (\( \lambda \neq 0 \)) or \( \epsilon = \sigma e_1 \) (\( \sigma \neq 0 \)). In the former case, on substituting \( \sigma = 0 \) into the Killing spinor equation, we obtain the following conditions
\[ \lambda \left( \frac{i}{\sqrt{2}} (F_{11} - F_{22}) + \frac{1}{\sqrt{2}} \ell^{-1} \right) = 0, \]
\[ \lambda \left( -\frac{i}{\sqrt{2}} (F_{11} - F_{22}) + \frac{1}{\sqrt{2}} \ell^{-1} \right) = 0, \] (3.2)

which admit no solution. In the latter case, on substituting \( \lambda = 0 \), we obtain
\[ \sigma \left( \frac{i}{\sqrt{2}} (F_{11} + F_{22}) + \frac{1}{\sqrt{2}} \ell^{-1} \right) = 0, \]
\[ \sigma \left( -\frac{i}{\sqrt{2}} (F_{11} + F_{22}) - \frac{1}{\sqrt{2}} \ell^{-1} \right) = 0, \] (3.3)

which again admit no solution. It follows that there are no supersymmetric solutions corresponding to these cases. Note that this analysis has not made use of any reality conditions, and hence these types of solutions are excluded for all choices of \( (c_1, c_2) \).

It remains to analyse (3.1) in the case for which \( \lambda \neq 0 \) and \( \sigma \neq 0 \). In this case, one obtains the following geometric conditions:
\[ 2\partial_1 \lambda - \lambda \Omega_{2,12} + \sqrt{2} \ell^{-1} \sigma = 0, \]
\[ 2\partial_1 \sigma + \sigma \Omega_{2,12} + \sqrt{2} \ell^{-1} \lambda = 0, \]
\[ 2\partial_2 \lambda + \lambda \Omega_{1,12} = 0, \]
\[ 2\partial_2 \sigma - \sigma \Omega_{1,12} = 0, \]
\[ \Omega_{1,21} = \Omega_{1,12} = \Omega_{2,12} = \Omega_{2,21} = 0, \]
\[ 2\Omega_{2,11} + \Omega_{1,21} + \Omega_{1,12} = 0, \]
\[ \lambda \sigma (-2\Omega_{1,11} - \Omega_{2,12} + \Omega_{2,21}) + \sqrt{2} \ell^{-1} (\lambda^2 + \sigma^2) = 0, \] (3.4)
the following conditions on the gauge potential

\[
A_1 = \frac{i\ell}{2} \left( \Omega_{1,22} + \frac{1}{2} \Omega_{2,12} + \frac{1}{2} \Omega_{2,21} + \frac{1}{\sqrt{2}\ell\lambda\sigma}(\lambda^2 - \sigma^2) \right),
\]
\[
A_2 = \frac{i\ell}{2} \left( \Omega_{2,22} + \frac{1}{2} \Omega_{1,12} - \frac{1}{2} \Omega_{1,21} \right),
\]  

(3.5)
and on the gauge field strength

\[
\lambda \Omega_{1,12} + \sqrt{2}i\sigma F_{12} = 0,
\]
\[
\sigma \Omega_{1,21} + \sqrt{2}i\lambda F_{12} = 0,
\]
\[
\sigma \Omega_{2,21} + \frac{\lambda}{\sqrt{2}} \left( -i(F_{11} - F_{22}) + \frac{1}{\ell} \right) = 0,
\]
\[
\lambda \Omega_{2,12} - \frac{\sigma}{\sqrt{2}} \left( iF_{qq} + \frac{1}{\ell} \right) = 0.
\]  

(3.6)
To proceed with the analysis, observe that (3.4) and (3.6) imply that

\[
W = i\lambda\sigma(e^1 - e^\dagger)
\]  

(3.7)
defines a Killing vector. Furthermore, one can without loss of generality make a $U(1)$
transformation (combined with appropriately chosen $SU(2)$ gauge transformations
which leave the Killing spinor invariant), and take

\[
W \cdot A = \frac{1}{\sqrt{2}}(\lambda^2 - \sigma^2).
\]  

(3.8)
It is then straightforward to show that (3.4) implies that

\[
\mathcal{L}_W e^1 = \mathcal{L}_W e^2 = 0,
\]  

(3.9)
and furthermore

\[
\mathcal{L}_W \lambda = \mathcal{L}_W \sigma = 0.
\]  

(3.10)
Observe also that (3.4) implies that

\[
d\left( \lambda\sigma(e^1 + e^\dagger) \right) = 0,
\]  

(3.11)
and

\[
d(\lambda\sigma e^2) = -\ell^{-1} \left( 2iA + \sqrt{2}\left( \frac{\lambda}{\sigma}e^1 + \frac{\sigma}{\lambda}e^\dagger \right) \right) \wedge (\lambda\sigma e^2).
\]  

(3.12)
Further simplification can be obtained by noting that one can apply a $U(1) \times SU(2)$
transformation $e^{-i\Theta_1}e^{i\Theta_2}$ for $\Theta \in \mathbb{R}$ such that $\mathcal{L}_W \Theta = 0$ and work in a gauge for
which

\[
A_1 + A_1 = 0
\]  

(3.13)while preserving the gauge condition (3.8). So, in this gauge

\[
A_1 = \frac{i}{2\sqrt{2}\lambda\sigma}(\lambda^2 - \sigma^2)
\]  

(3.14)
and on substituting this into (3.15) one obtains the following extra constraint on the spin connection
\[ \Omega_{1,2} + \frac{1}{2} \Omega_{2,12} + \frac{1}{2} \Omega_{2,21} = 0. \] (3.15)

Next we introduce co-ordinates, and re-write all the conditions in these co-ordinates. First, introduce a real local co-ordinate \( \psi \) such that
\[ W = \frac{\partial}{\partial \psi}. \] (3.16)

Note that all components of the spin connection, gauge potential and \( \lambda, \sigma \) are independent of \( \psi \). Note that (3.11) implies there is a real co-ordinate \( x \) such that
\[ \lambda \sigma (e^1 + e^\dagger) = dx \] (3.17)
and (3.12) implies that
\[ \lambda \sigma e^2 = Hz \] (3.18)
for complex \( H, z \). In fact, working in the gauge given by (3.14), it is straightforward to see that (3.12) implies that, without loss of generality, one can take \( H \in \mathbb{R} \). Furthermore, we write
\[ e^1 - e^\dagger = -2i\lambda \sigma (d\psi + \phi) \] (3.19)
where \( \phi = \phi_x dx + \phi_z dz + \phi_{\bar{z}} d\bar{z} \) is a real 1-form. The functions \( \lambda, \sigma, H \) and the 1-form \( \phi \) are all independent of \( \psi \). To proceed, note that the geometric conditions (3.4) and (3.15) are equivalent to:
\[ \partial_x \log H = -\frac{1}{\sqrt{2\ell}} (\lambda^{-2} + \sigma^{-2}) \] (3.20)
and
\[ d\phi = -\frac{i}{(\lambda \sigma)^2} (\partial_z \log \frac{\lambda}{\sigma}) dx \wedge dz + \frac{i}{(\lambda \sigma)^2} (\partial_{\bar{z}} \log \frac{\lambda}{\sigma}) dx \wedge d\bar{z} \]
\[ + \left( \frac{2iH^2}{(\lambda \sigma)^2} \partial_x \log \frac{\sigma}{\lambda} \right) dz \wedge d\bar{z} + i\sqrt{2\ell^{-1}} \frac{H^2}{(\lambda \sigma)^2} \left( \frac{1}{\sigma^2} - \frac{1}{\lambda^2} \right) dz \wedge d\bar{z}. \] (3.21)

The integrability condition associated with (3.21) is given by
\[ \partial_x \left( \frac{H^2}{(\lambda \sigma)^2} \left( -2\sigma \partial_x \lambda + 2\lambda \partial_x \sigma + \sqrt{2\ell^{-1}} \left( \frac{\lambda}{\sigma} - \frac{\sigma}{\lambda} \right) \right) \right) \]
\[ - \frac{2}{(\lambda \sigma)^3} (\sigma \partial_x \partial_{\bar{z}} \lambda - \lambda \partial_x \partial_{\bar{z}} \sigma) + \frac{6}{(\lambda \sigma)^4} (\sigma^2 \partial_x \lambda \partial_{\bar{z}} \lambda - \lambda^2 \partial_x \sigma \partial_{\bar{z}} \sigma) = 0. \] (3.22)

Next, consider the gauge potential \( A \) fixed by (3.14) and (3.5); we find
\[ A = \frac{1}{\sqrt{2}} (\lambda^2 - \sigma^2) (d\psi + \phi) - \frac{i\ell}{2} (\partial_z \log Hz - \partial_{\bar{z}} \log Hz). \] (3.23)
On comparing $dA$ with $F$ given in (3.6), we find that the previous constraints imply that all components of $dA$ agree with $F$ with no further constraint, with the exception of the $2\bar{2}$ component, from which we find

$$H^{-2}\partial_{x}\partial_{\bar{z}}\log H = \frac{\sqrt{2}}{2}\ell^{-1}\partial_{x}\left(\frac{1}{\lambda^2} + \frac{1}{\sigma^2}\right) - \frac{1}{\ell^2\lambda^2\sigma^2}\left(\left(\frac{\lambda}{\sigma}\right)^2 + \left(\frac{\sigma}{\lambda}\right)^2 - 1\right). \quad (3.24)$$

Finally, it remains to compute the gauge field equations. Note that it is most straightforward to impose these by requiring that $F - \ast F$ is closed. From this condition one obtains the final constraint

$$(\partial_{x}\partial_{\bar{z}} + H^2\partial_{x}\partial_{\bar{z}})\lambda^{-2} - \frac{3\sqrt{2}H^2}{\ell}\lambda^{-2}\partial_{x}(\lambda^{-2}) + \frac{2H^2}{\ell^2\lambda^6} = 0. \quad (3.25)$$

To summarize, on setting $H = e^u$, one finds that the metric and gauge potential are given by

$$ds^2 = 2\lambda^2\sigma^2(d\psi + \phi)^2 + \frac{1}{\lambda^2\sigma^2}\left(\frac{1}{2}dx^2 + 2e^2udzd\bar{z}\right)$$

$$A = \frac{1}{\sqrt{2}}(\lambda^2 - \sigma^2)(d\psi + \phi) - \frac{i\ell}{2}\partial_{x}udz + \frac{i\ell}{2}\partial_{z}ud\bar{z} \quad (3.26)$$

where $\lambda, \sigma, u$ are functions, and $\phi = \phi_{x}dx + \phi_{z}dz + \phi_{\bar{z}}d\bar{z}$ is a 1-form. All components of the metric and gauge potential are independent of $\psi$, and $u, \lambda, \sigma, \phi$ must satisfy

$$\partial_{x}u = \frac{1}{\sqrt{2}\ell}(\lambda^{-2} + \sigma^{-2}) \quad (3.27)$$

and

$$\partial_{x}\partial_{\bar{z}}(\lambda^{-2} - \sigma^{-2}) + e^{2u}\left(\partial_{\bar{z}}^2(\lambda^{-2} - \sigma^{-2}) + 3(\lambda^{-2} - \sigma^{-2})\partial_{x}^2u\right) + 3(\lambda^{-2} - \sigma^{-2})(\partial_{x}u)^2 + 3\partial_{x}u\partial_{\bar{z}}(\lambda^{-2} - \sigma^{-2}) + \frac{1}{2}\ell^{-2}(\lambda^{-2} - \sigma^{-2})^2 = 0 \quad (3.28)$$

and

$$\partial_{x}\partial_{\bar{z}}u + e^{2u}\left(\partial_{\bar{z}}^2u + \frac{1}{2}(\partial_{x}u)^2 + \frac{3}{4}\ell^{-2}(\lambda^{-2} - \sigma^{-2})^2\right) = 0 \quad (3.29)$$

and

$$d\phi = -\frac{i}{(\lambda\sigma)^2}(\partial_{x}\log\frac{\lambda}{\sigma})dx \wedge dz + \frac{i}{(\lambda\sigma)^2}(\partial_{\bar{z}}\log\frac{\lambda}{\sigma})dx \wedge d\bar{z}$$

$$+ \frac{ie^{2u}}{(\lambda\sigma)^2}\left(2\partial_{x}\log\frac{\sigma}{\lambda} + \sqrt{2}\ell^{-1}\frac{\lambda^2 - \sigma^2}{(\lambda\sigma)^2}\right)dz \wedge d\bar{z}. \quad (3.30)$$

In order to recover the anti-self-dual solutions found in (7) corresponding to this class of solutions, note first that $F$ is anti-self-dual if and only if

$$\partial_{x}\lambda = 0, \quad \partial_{x}\lambda = -\frac{1}{2\sqrt{2}\ell}\lambda^{-1}. \quad (3.31)$$

1Positive orientation is fixed by $\epsilon_{1122} = -1$.
In this case, (3.28) is implied by (3.27) and (3.29). In order to simplify (3.29), it is useful to set

\[ x = \frac{1}{y}, \quad u = -2 \log y + \frac{w}{2}. \]  

Then (3.29) is equivalent to

\[ \partial_z \partial_{\bar{z}} w + \partial_y \partial_y e^w = 0 \]  

i.e. \( w \) is a solution of the \( SU(\infty) \) toda equation. Furthermore, if one defines \( V \) via

\[ \sigma^{-2} = -\sqrt{2} \ell^{-1} y V \]  

then (3.27) relates \( V \) to \( w \) by

\[ -2 \ell^{-2} V = y \partial_y w - 2 \]  

and the metric can be written as

\[ ds^2 = \frac{1}{y^2} \left( V^{-1} (d\psi + \phi)^2 + V (dy^2 + 4e^w dz d\bar{z}) \right) \]  

where (3.21) implies that

\[ d\phi = i \partial_z V dy \wedge dz - i \partial_{\bar{z}} V dy \wedge d\bar{z} + 2i \partial_y (e^w V) dz \wedge d\bar{z}. \]  

This solution corresponds to one of the anti-self-dual solutions found in [7].

We remark that the solutions for which \( c_1 = -\frac{1}{4}, c_2 = -\frac{1}{2\ell} \) are found using an essentially identical analysis as given above. The metric and gauge potential are given by (3.26), on making the replacement \( \sigma \to i \sigma \), and taking \( ds^2 \to -ds^2 \) (to restore the metric signature to \((+,+,+:+)\)). The functions \( u, \lambda, \sigma \) and the 1-form \( \phi \) then satisfy the conditions (3.27), (3.28), (3.29) and (3.30), again making the replacement \( \sigma \to i \sigma \).

4 Solutions with \( c_1 = -\frac{1}{4}, c_2 = -\frac{1}{2\ell} \)

For these solutions, one can make a gauge transformation of the \( U(1) \) connection \( A \), which acts on spinors as \( \epsilon \to e^h \epsilon \), where \( h \) is a real function, and take, without loss of generality

\[ \epsilon = 1 + \sigma e_1. \]  

(4.1)
On evaluating the linear system, one finds the following geometric constraints

\[
\frac{\partial_1 \sigma - \sigma \Omega_{1,22}}{\partial_2 \sigma + \sigma \Omega_{2,11}} = 0,
\]

\[
\Omega_{1,21} = \Omega_{1,12} = \Omega_{2,12} = \Omega_{2,21} = 0,
\]

\[
\Omega_{2,11} + \Omega_{2,22} - \Omega_{1,12} = 0,
\]

\[
-\Omega_{2,11} + \Omega_{1,21} + \Omega_{2,22} = 0,
\]

\[
\Omega_{1,11} - \Omega_{1,22} + \Omega_{2,21} = 0,
\]

\[
\Omega_{2,12} + \Omega_{1,11} + \Omega_{1,22} - \sqrt{2} \epsilon^{-1} \sigma = 0,
\]

\[
\Omega_{1,22} - \Omega_{1,21} = \frac{1}{\sqrt{2}} \epsilon^{-1} (\sigma - \sigma^{-1}),
\]

\[
\Omega_{1,11} - \Omega_{1,12} = \frac{1}{\sqrt{2}} \epsilon^{-1} (\sigma + \sigma^{-1}),
\]

(4.2)

together with the following constraints on the gauge potential and field strength

\[
\ell^{-1} A_1 = \frac{1}{2} (\Omega_{1,11} + \Omega_{1,22}), \quad \ell^{-1} A_2 = -\frac{1}{2} (\Omega_{2,11} + \Omega_{2,22})
\]

(4.3)

and

\[
F_{11} = \ell^{-1} - \frac{1}{\sqrt{2}} ((\sigma + \sigma^{-1}) \Omega_{1,11} - (\sigma - \sigma^{-1}) \Omega_{1,22}),
\]

(4.4)

\[
F_{22} = \frac{1}{\sqrt{2}} ((\sigma - \sigma^{-1}) \Omega_{1,11} - (\sigma + \sigma^{-1}) \Omega_{1,22}),
\]

\[
F_{12} = -\frac{1}{\sqrt{2} \sigma} \Omega_{1,12},
\]

\[
F_{\bar{1}2} = \frac{1}{\sqrt{2}} \sigma \Omega_{1,21}.
\]

We remark that (4.3) relates the $U(1)$ connection $A$ to the spin connection $\Omega$, and leads to a partial cancellation of these two connections in the Killing spinor equation. This is similar to what happens in twisted field theories.

To proceed, it is useful to define

\[
V = e^1 + e^{\bar{1}}
\]

(4.5)

and denote the vector field dual to $V$ by $V = \frac{\partial}{\partial \psi}$. Then (4.2) implies that

\[
\frac{\partial \sigma}{\partial \psi} = \frac{1}{\sqrt{2} \epsilon} (\sigma^2 - 1).
\]

(4.6)

There are therefore two subcases, corresponding to $\sigma^2 \neq 1$ and $\sigma^2 = 1$.

### 4.1 Solutions with $\sigma^2 = 1$

The solutions with $\sigma = -1$ are gauge-equivalent to those with $\sigma = 1$, so it suffices to take $\sigma = 1$. In this case, one has

\[
i (e^1 - e^{\bar{1}}) = \sqrt{2} e^{-\sqrt{2} \epsilon^{-1} \psi} e^1, \quad e^2 = e^{-\sqrt{2} \epsilon^{-1} \psi} e^2
\]

(4.7)
where
\[ \mathcal{L}_V \hat{e}^1 = \mathcal{L}_V \hat{e}^2 = 0. \] (4.8)

On setting
\[ e^1 + e^\bar{1} = 2d\psi + \Phi, \]
(4.9)
one then finds that
\[ d\hat{e}^1 = \Psi \wedge \hat{e}^1, \quad d\hat{e}^2 = \Psi \wedge \hat{e}^2 \] (4.10)

where
\[ \Psi = -\frac{1}{\sqrt{2}}\ell^{-1}\Phi - \sqrt{2}ie^{-\sqrt{2}\ell^{-1}\psi}(\Omega_{1,1\bar{1}} + \frac{1}{\sqrt{2}}\ell^{-1})\hat{e}^1 \]
\[ -e^{-\sqrt{2}\ell^{-1}\psi}(\Omega_{2,2\bar{2}}\hat{e}^2 - \Omega_{2,2\bar{2}}\hat{e}^2) \] (4.11)
satisfies
\[ \mathcal{L}_V \Psi = 0. \] (4.12)

It follows that one can introduce further local co-ordinates \( x, z \), where \( x \) is real and \( z \) is complex, and a real function \( H = H(x, z, \bar{z}) \) such that
\[ \hat{e}^1 = Hdx, \quad \hat{e}^2 = Hz \] (4.13)
such that
\[ \Psi = d\log H. \] (4.14)

Furthermore, the geometric conditions imply
\[ \mathcal{L}_V \Phi = \sqrt{2}\ell^{-1}\Phi + 2d\log H \] (4.15)
and
\[ \check{d}\Phi = -d\log H \wedge \Phi \] (4.16)
where \( \check{d} \) denotes the exterior derivative restricted to hypersurfaces of constant \( \psi \). It then follows that
\[ \Phi = -\sqrt{2}\ell H^{-1}dH + e^{\sqrt{2}\ell^{-1}\psi}H^{-1}d\chi \] (4.17)
where \( \chi \) is a function of \( x, z, \bar{z} \). The gauge potential is then given by
\[ \ell^{-1}A = \frac{1}{\sqrt{2}}\ell^{-1}d\psi - \frac{1}{2}d\log H + \frac{1}{\sqrt{2}}\ell^{-1}e^{\sqrt{2}\ell^{-1}\psi}H^{-1}d\chi. \] (4.18)

The solution can then be simplified further by making the co-ordinate transformation
\[ \psi = \psi' + \frac{1}{\sqrt{2}}\ell \log H \] (4.19)
and dropping the prime on \( \psi' \) to obtain
\[ ds^2 = \frac{1}{2} \left( 2d\psi + e^{\sqrt{2}\ell^{-1}\psi}d\chi \right)^2 + e^{-2\sqrt{2}\ell^{-1}\psi}ds^2 (\mathbb{R}^3) \] (4.20)
with
\[ F = d \left( \frac{1}{\sqrt{2}}e^{\sqrt{2}\ell^{-1}\psi}d\chi \right). \] (4.21)

Imposing the gauge field equations \( d \times F = 0 \) implies that
\[ \Box_3 \chi = 0 \] (4.22)
where \( \Box_3 \) denotes the Laplacian on \( \mathbb{R}^3 \). This is the Euclidean analogue of the Kastor-Traschen solution \[7, 19\].
4.2 Solutions with $\sigma^2 \neq 1$

For these solutions,

$$\sigma = -\tanh \left( \frac{\psi}{\sqrt{2}\ell} + h \right)$$  \hspace{1cm} (4.23)

where $h$ is a function such that $\frac{\partial h}{\partial \psi} = 0$. By making a re-definition of $\psi$, one can, without loss of generality, set $h = 0$. Next, observe that (4.2) implies that

$$L_V(i(e^1 - e^\dagger)) = -\frac{1}{\sqrt{2}\ell} (\sigma^{-1} + \sigma) i(e^1 - e^\dagger),$$

$$L_V e^2 = -\frac{1}{\sqrt{2}\ell} (\sigma^{-1} + \sigma) e^2.$$  \hspace{1cm} (4.24)

It follows that

$$i(e^1 - e^\dagger) = \sqrt{2\sigma} \hat{e}^1, \quad e^2 = \frac{\sigma}{1 - \sigma^2} \hat{e}^2$$  \hspace{1cm} (4.25)

where $\hat{e}^1$ is a real 1-form, and $\hat{e}^2$ is a complex 1-form such that

$$L_V \hat{e}^1 = L_V \hat{e}^2 = 0.$$  \hspace{1cm} (4.26)

It is convenient to define the $\psi$-independent 3-metric on the 3-manifold $GT$ corresponding to the space of orbits of $V$ by

$$ds_{GT}^2 = (\hat{e}^1)^2 + 2\hat{e}^2 \hat{e}^\dagger,$$  \hspace{1cm} (4.27)

then (4.2) implies that $GT$ admits a $\psi$-independent real orthonormal basis $E^i (i = 1, 2, 3)$ such that

$$dE^i = \ell^{-1} \star_3 E^i + \mathcal{B} \wedge E^i$$  \hspace{1cm} (4.28)

where $\star_3$ is the Hodge dual on $GT$ and

$$\mathcal{B} = \frac{i}{\sqrt{2}} \frac{\sigma^2}{(1 - \sigma^2)^2} \left[ (\sigma^{-1} + \sigma) (\Omega_{1,22} + \Omega_{1,22}) - (\sigma^{-1} - \sigma) (\Omega_{1,11} + \Omega_{1,11}) \right] \hat{e}^1$$

$$+ \frac{\sigma^2}{(1 - \sigma^2)^2} [ -(\sigma^{-1} - \sigma) \Omega_{2,22} + (\sigma^{-1} + \sigma) \Omega_{2,11} ] \hat{e}^2$$

$$+ \frac{\sigma^2}{(1 - \sigma^2)^2} [ (\sigma^{-1} - \sigma) \Omega_{2,22} + (\sigma^{-1} + \sigma) \Omega_{2,11} ] \hat{e}^\dagger.$$  \hspace{1cm} (4.29)

It follows that $GT$ admits a hyper-CR Einstein-Weyl structure [20]. Note that (4.28) implies that

$$\mathcal{L}_V \mathcal{B} = 0$$  \hspace{1cm} (4.30)

and

$$d\mathcal{B} = -\ell^{-1} \star_3 \mathcal{B}.$$  \hspace{1cm} (4.31)

Next, write

$$e^1 + e^\dagger = 2d\psi + \Phi$$  \hspace{1cm} (4.32)

where $\Phi$ is a 1-form on $GT$; note that (4.2) implies that

$$\mathcal{L}_V \Phi = 2\mathcal{B} + \frac{1}{\sqrt{2}\ell} (\sigma + \sigma^{-1}) \Phi.$$  \hspace{1cm} (4.33)
Hence
\[ \Phi = -\frac{\ell}{\sqrt{2}}(\sigma^{-1} + \sigma)B + (\sigma^{-1} - \sigma)\xi \] (4.34)
where \( \xi \) is a \( \psi \)-independent 1-form on \( GT \), with
\begin{align*}
\xi &= \frac{i\ell \sigma^2}{2(1 - \sigma^2)^2} \left[ (\sigma^{-1} - \sigma) \left( \Omega_{1,22} + \Omega_{1,22} \right) - (\sigma^{-1} + \sigma) \left( \Omega_{1,1\bar{1}} + \Omega_{1,1\bar{1}} \right) \right] \hat{e}^1 \\
&+ \frac{\ell \sigma^2}{\sqrt{2}(1 - \sigma^2)^2} \left[ (\sigma^{-1} - \sigma) \Omega_{2,1\bar{1}} - (\sigma + \sigma^{-1}) \Omega_{2,2\bar{2}} \right] \hat{e}^2 \\
&+ \frac{\ell \sigma^2}{\sqrt{2}(1 - \sigma^2)^2} \left[ - (\sigma^{-1} - \sigma) \Omega_{2,2\bar{2}} + (\sigma + \sigma^{-1}) \Omega_{2,2\bar{2}} \right] \hat{e}^2.
\end{align*}
(4.35)
The remaining content of (4.2) implies that
\[ d\xi + B \wedge \xi = -\ell^{-1} *_3 \xi. \] (4.36)
The gauge potential is determined by (4.3) as
\begin{align*}
\ell^{-1} A &= \frac{1}{2} d\log \sigma + \frac{1}{2\sqrt{2}\ell}(\sigma^{-1} + \sigma) d\psi - \frac{1}{4} (\sigma^2 + \sigma^{-2})B + \frac{1}{2\sqrt{2}\ell} (\sigma^2 - \sigma^{-2})\xi. \quad (4.37)
\end{align*}
On taking the exterior derivative of this expression, and using the geometric constraints described above, one obtains the components of \( F \) given in (4.3); moreover, the gauge field equations hold with no additional constraints.
To summarize, the metric is given by
\[ ds^2 = \frac{1}{2} \left( \frac{2\sqrt{2}\ell}{\sigma^2 - 1} d\sigma - \frac{\ell}{\sqrt{2}} (\sigma^{-1} + \sigma) B + (\sigma^{-1} - \sigma) \xi \right)^2 + \frac{\sigma^2}{(1 - \sigma^2)^2} ds^2_{GT} \] (4.38)
where we have changed co-ordinates from \( \psi \) to \( \sigma \). \( GT \) is a 3-manifold, which admits a hyper-CR Einstein-Weyl structure; in particular, there is a \( \sigma \)-independent real basis \( E^i \) such that
\[ dE^i = \ell^{-1} *_3 E^i + B \wedge E^i \] (4.39)
where \( B \) is a \( \sigma \)-independent 1-form on \( GT \) satisfying
\[ dB = -\ell^{-1} *_3 B \] (4.40)
and \( \xi \) is a \( \sigma \)-independent 1-form on \( GT \) satisfying
\[ d\xi + B \wedge \xi = -\ell^{-1} *_3 \xi. \] (4.41)
The gauge field strength is given by
\[ F = d \left[ -\frac{1}{4} \ell (\sigma^2 + \sigma^{-2}) B + \frac{1}{2\sqrt{2}} (\sigma^2 - \sigma^{-2}) \xi \right]. \] (4.42)
The field strength \( F \) is anti-self-dual if and only if \( \xi = \frac{\ell}{\sqrt{2}} B \).
We remark that the solutions for which $c_1 = -\frac{i}{4}, c_2 = -\frac{i}{2}$ are determined using an almost identical analysis to that given in this section, so we again simply summarize the result. In this case, the metric is given by

$$ds^2 = \frac{1}{2} \left( \frac{2\sqrt{2} \ell}{1 + \sigma^2} d\sigma - \frac{1}{\sqrt{2}} \ell (\sigma^{-1} - \sigma) B + (\sigma^{-1} + \sigma) \xi \right)^2 + \frac{\sigma^2}{(1 + \sigma^2)^2} ds_{GT}^2 , \quad (4.43)$$

$GT$ is a 3-manifold, which admits a hyper-CR Einstein-Weyl structure; there is a $\sigma$-independent real basis $E^i$ satisfying (4.39), where $B$ is a $\sigma$-independent 1-form on $GT$ satisfying (4.40), and $\xi$ is a $\sigma$-independent 1-form on $GT$ satisfying (4.41). Furthermore, the gauge field strength is given by (4.42).

## 5 Conclusions

In this work, we have classified all local forms of supersymmetric solutions to four-dimensional Einstein-Maxwell Euclidean supergravity, for which the Maxwell field strength is neither anti-self-dual or self-dual. The backgrounds we have considered admit a Killing spinor which satisfies a Killing spinor equation, whose structure was fixed by requiring that the integrability conditions of the Killing spinor equation should be compatible with the Einstein field equations of an Einstein-Maxwell Lagrangian. We have therefore extended the earlier classification of [7], in which the solutions for which the Maxwell field is anti-self-dual were classified.

The solutions which we have found fall into three classes. The first class of solutions corresponds to the solution given in equations (3.26)-(3.30) of section 3. This solution is a Euclidean version of the solution originally found in the “timelike” class of the Lorentzian $N = 2, D = 4$ supergravity in [1, 2]. In particular, we remark that the equations (3.27)-(3.29), which determine the solution are, under a rescaling of $\lambda, \sigma$, identical to equations (2.9) and (2.10) of [2], with the exception of the sign of the term involving $(\lambda^{-2} - \sigma^{-2})^2$ in (3.29), which differs from the sign of $B^2$ in equation (2.10) of [2]. All of the solutions in section 3 preserve two (real) supersymmetries. The remaining two classes of solution are obtained in section 4. One of these, derived in section 4.1, is the Euclidean version of the Kastor-Traschen solution. The remaining class of solutions obtained in section 4.2 does not have an analogous solution in the Lorentzian theory, and is constructed in (4.38)-(4.42) from a 3-dimensional base space which is a 3-manifold admitting a hyper-CR Einstein-Weyl structure.

All of the solutions in section 4 automatically preserve four (real) supersymmetries; this can be seen by observing that if the constants $c_1, c_2$ appearing in the Killing spinor equation (2.12) are both real, and $\epsilon$ is a Killing spinor, then so is $C \ast \epsilon$, where $C$ is an appropriately constructed charge conjugation operator. If, however, $c_1, c_2$ are purely imaginary and $\epsilon$ solves (2.12), then $C \ast \gamma_5 \epsilon$ also solves (2.12). This type of automatic supersymmetry enhancement does not occur for the cases described in section 3, where one of $c_1, c_2$ is real, and the other imaginary. The analysis of solutions of Euclidean four-dimensional supergravity with enhanced supersymmetry is currently work in progress.

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