ABSOLUTE CONTINUITY AND LARGE-SCALE GEOMETRY OF POLISH GROUPS

JAKE HERNDON

ABSTRACT. We apply the theory of large-scale geometry of Polish groups to groups of absolutely continuous homeomorphisms. Let \( M \) be either the compact interval or circle. We prove that the Polish group \( AC_+^b(M) \) of orientation-preserving homeomorphisms \( f : M \to M \) such that \( f \) and \( f^{-1} \) are absolutely continuous has a trivial quasi-isometry type. We also prove that the Polish group \( AC_+^{loc}Z(R) \) of homeomorphisms \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) commutes with integer translations and both \( f \) and \( f^{-1} \) are locally absolutely continuous is quasi-isometric to the group of integers. To study \( AC_+^b(S^1) \) and \( AC_+^{loc}Z(R) \) we use the observation that these groups are Zappa-Szép products.

1. Introduction

Throughout this paper \( I, S^1, \mathbb{R}, \) and \( \mathbb{Z} \) denote the compact interval, the circle, the real line, and the integers, respectively. For \( M = I \) and \( M = S^1 \) we let \( AC_+(M) \) denote the group of orientation-preserving homeomorphisms \( f : M \to M \) such that both \( f \) and \( f^{-1} \) are absolutely continuous, and we let \( AC_+^{loc}(\mathbb{R}) \) denote the group of homeomorphisms \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) commutes with integer translations and both \( f \) and \( f^{-1} \) are locally absolutely continuous. A Polish group is, as usual, a topological group whose underlying topology is Polish, i.e. separable and completely metrizable, and the underlying topology is called a Polish group topology. In \[6\] Solecki defines a Polish group topology on \( AC_+(I) \), and here we define a Polish group topology on \( AC_+(S^1) \) and another Polish group topology on \( AC_+^{loc}(\mathbb{R}) \). Henceforth we also use \( AC_+(I), AC_+(S^1), \) and \( AC_+^{loc}(\mathbb{R}) \) to denote these Polish groups.

The goal of this paper is to study \( AC_+(I), AC_+(S^1), \) and \( AC_+^{loc}(\mathbb{R}) \) from the perspective of large-scale geometry. There is a general theory of large-scale geometry of Polish groups which is due to Rosendal and detailed in \[4\]. The main results of this paper are summarized by the following theorem.

Theorem 1. Each of the Polish groups \( AC_+(I), AC_+(S^1), \) and \( AC_+^{loc}(\mathbb{R}) \) has a well-defined quasi-isometry type. For both \( AC_+(I) \) and \( AC_+(S^1) \) the quasi-isometry type is trivial and for \( AC_+^{loc}(\mathbb{R}) \) the quasi-isometry type is that of \( \mathbb{Z} \).

The theory of large-scale geometry of Polish groups generalizes the classical theory of large-scale geometry of finitely generated groups by viewing every finitely generated group as a discrete topological group, and therefore, it is helpful to first review the situation for finitely generated groups before making Theorem 1 precise. Recall, for a group \( G \) generated by a subset \( S \) there is the associated left-invariant word metric \( \rho_S \) defined by

\[ \rho_S(x, y) = \min \left\{ k \geq 0 | x^{-1} y \in (S \cup S^{-1})^k \right\} \]
for all \( x, y \in G \). If \( G \) is finitely generated then the word metrics obtained from finite generating sets are all mutually quasi-isometric, and so we say the quasi-isometry type of \( G \) is the quasi-isometry equivalence class of the metric space \((G, \rho_S)\) for any finite generating set \( S \). A running theme in the subject of geometric group theory is the relation of algebraic properties of a finitely generated group with large-scale metric properties of its quasi-isometry type.

A crucial part of the general theory for Polish groups is to identify a collection of distinguished generating sets that, when present, give mutually quasi-isometric word metrics. Following [4], a subset \( S \) of a Polish group \( G \) is coarsely bounded in \( G \) if \( S \) is bounded in every metric on \( G \) which is topologically compatible and left-invariant. For a Polish group \( G \) which is generated by a coarsely bounded subset the quasi-isometry type of \( G \) is the quasi-isometry equivalence class of the metric space \((G, \rho_S)\) for any coarsely bounded and closed generating set \( S \). Indeed, by the Baire Category Theorem, any word metrics obtained from generating sets which are coarsely bounded and closed are mutually quasi-isometric. We note that the collection of all coarsely bounded subsets of a Polish group is closed under taking topological closures, and so any Polish group which is generated by a coarsely bounded subset is also generated by one which is closed. We also note that the property of a subset being coarsely bounded is not a homeomorphism invariant as the definition quantifies over metrics on the ambient group. However, if it does not create ambiguity we omit reference to the ambient group. A Polish group which is a coarsely bounded subset of itself is a coarsely bounded group. For Polish groups \( G \) and \( H \), a function \( G \to H \) is a quasi-isometry of Polish groups if \( G \) and \( H \) are generated by subsets which are coarsely bounded and \((G, d_G) \to (H, d_H)\) is a quasi-isometry whenever \( d_G \) and \( d_H \) represent the quasi-isometry types of \( G \) and \( H \), respectively. In this case \( G \) and \( H \) are quasi-isometric Polish groups. The coarsely bounded Polish groups are those which are quasi-isometric to the trivial group.

In the context of the above general theory we prove Theorem [I] by showing \( AC_+ (I) \) and \( AC_+ (S^1) \) are coarsely bounded groups and by showing the identification of \( Z \) with the subgroup of \( AC_{loc}^Z (R) \) consisting of integer translations defines a quasi-isometry of Polish groups \( Z \to AC_{loc}^Z (R) \). Our exposition repeatedly makes use of the observation that for both the case of \( G = AC_+ (S^1) \) and \( G = AC_{loc}^Z (R) \) the group \( G \) is a Zappa-Szép product of subgroups \( H \) and \( K \) with \( K \) isomorphic to \( AC_+ (I) \). For this reason Section [II] is a standalone section which contains information relevant to Zappa-Szép products. In Section [II] we recall the definition of absolute continuity and its local counterpart and we give a compatible, right-invariant metric on each of the Polish groups \( AC_+ (I) \), \( AC_+ (S^1) \), and \( AC_{loc}^Z (R) \). In Section [III] we prove Solecki’s topology on \( AC_+ (I) \) makes it a coarsely bounded group, and in Section [IV] we verify that the topologies defined on \( AC_+ (S^1) \) and \( AC_{loc}^Z (R) \) are, in fact, Polish group topologies and then we complete the proof of Theorem [I].

Let us mention some other groups of homeomorphisms where the large-scale perspective is relevant. For a compact manifold \( M \) the group \( \text{Homeo}(M) \) of homeomorphisms of \( M \) is a Polish group with the compact-open topology. Let \( M \) be a compact manifold and let \( \text{Homeo}_0 (M) \) denote the identity component in \( \text{Homeo}(M) \). In [II] Mann and Rosendal study the large-scale geometry of \( \text{Homeo}_0 (M) \). The main results there include proving \( \text{Homeo}_0 (M) \) is generated by a coarsely bounded subset and so has a well-defined quasi-isometry type. They show if \( M \) is the \( n \)-sphere then
Homeo₀(M) is a coarsely bounded group and if dim(M) > 1 and π₁(M) is non-trivial. Let k > 0 be an integer or possibly ∞ and let M = I or M = S¹. The group Diff⁺(M) of orientation-preserving Cᵏ−diffeomorphisms of M is a Polish group with its Cᵏ-topology. In [1] Cohen proves Diff⁺(M) is generated by a coarsely bounded subset if and only if k < ∞, and in this case the quasi-isometry type of Diff⁺(M) is non-trivial.

2. Preliminaries

We identify I with the unit interval [0, 1] and S¹ with the group of complex numbers with unit norm, and we let π : R → S¹ be the covering map x → e²πix.

We use dₘ to denote the metric on S¹ which is defined by taking the minimum distance between π⁻¹(x) and π⁻¹(y) in R for all x, y ∈ S¹.

A homeomorphism f : R → R commutes with integer translations if

\[ f(x + n) = f(x) + n \]

for all x ∈ R and all n ∈ Z. The group of all homeomorphisms of R that commute with integer translations is denoted Homeo₂(R). For G = Homeo⁺(I), G = Homeo⁺(S¹), and G = Homeo₂(R) we use d∞ to denote a right-invariant metric on G which induces the compact-open topology. Hence G is a Polish group with the topology induced by d∞. For concreteness we set

\[ d_∞(f, g) = \sup_{x \in I} |f(x) - g(x)| \]

for all f, g ∈ Homeo⁺(I) and for all f, g ∈ Homeo₂(R), and we set

\[ d_∞(f, g) = \sup_{x \in S¹} dₘ(f(x), g(x)) \]

for all f, g ∈ Homeo⁺(S¹).

A function f : J → R whose domain is an interval is absolutely continuous if for every ε > 0 there is a δ > 0 such that for every finite sequence (a₁, b₁),..., (aₙ, bₙ) of disjoint subintervals of J, if \( \sum_{i=1}^{n} |b_i - a_i| < \delta \) then \( \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon \). A function f : J → R whose domain as an interval is locally absolutely continuous if the restriction of f to every compact subinterval of J is absolutely continuous. For every homeomorphism f : S¹ → S¹ there is a unique homeomorphism \( \hat{f} : R \rightarrow R \) with \( f(0) \in [0, 1) \) and \( \pi \circ \hat{f} = f \circ \pi \). A homeomorphism f : S¹ → S¹ is absolutely continuous if \( \hat{f} \) is locally absolutely continuous.

Suppose J is a compact interval and f : J → R is continuous and nondecreasing. The Fundamental Theorem of Lebesgue Integration states that f is absolutely continuous if and only if f has a derivative f' almost everywhere on J with respect to Lebesgue measure, f' ∈ L¹(J), and for all x ∈ J

\[ f(x) = f(a) + \int_a^x f'(t) \, dt. \]

See, for instance, [2] Theorem 7.18.

We set

\[ d_*(f, g) = \int_0^1 |f'(t) - g'(t)| \, dt \]

for all f, g ∈ AC⁺(I). In the proof of [2] Lemma 2.4 Solecki shows d_* defines a right-invariant metric on AC⁺(I) which induces a Polish group topology.
Lemma 2. On $AC_+(I)$ the metrics $d_\infty$ and $d_*$ satisfy $d_\infty \leq d_*$. Consequently, inclusion $AC_+(I) \to \text{Homeo}_+(I)$ is continuous.

Proof. Let $f, g \in AC_+(I)$ and let $x \in I$. Then

$$f(x) - g(x) = \int_0^x f'(t) - g'(t) \, dt \leq \int_0^x |f'(t) - g'(t)| \, dt \leq d_*(f, g)$$

and likewise $g(x) - f(x) \leq d_*(f, g)$. So $d_\infty \leq d_*$ on $AC_+(I)$. This says inclusion $(AC_+(I), d_*) \to \text{(Homeo}_+(I), d_\infty)$ is a contraction mapping and so it is continuous. \qed

We set

$$d_*(f, g) = \int_0^1 |f'(t) - g'(t)| \, dt$$

for all $f, g \in AC^{loc}_+(I)$. In other words, on $AC^{loc}_+(\mathbb{R})$ the quantity $d_*$ is given by the same formula which defines the metric $d_*$ on $AC_+(I)$. It is straightforward to check that $d_*$ defines a pseudometric on $AC^{loc}_+(\mathbb{R})$. For $f, g \in AC^{loc}_+(\mathbb{R})$ we have $d_*(f, g) = 0$ if and only if $f - g$ is a constant function, so $d_*$ does not define a metric on $AC^{loc}_+(\mathbb{R})$.

For each $r \in \mathbb{R}$ we let $\tau_r : \mathbb{R} \to \mathbb{R}$ be translation $x \mapsto x + r$.

Proposition 3. On $AC^{loc}_+(\mathbb{R})$ the pseudometric $d_*$ is right-invariant and satisfies

$$d_*(\tau_r \circ f, \tau_s \circ g) = d_*(f, g)$$

for all $r, s \in \mathbb{R}$ and all $f, g \in AC^{loc}_+(\mathbb{R})$.

Proof. For each $f \in AC^{loc}_+(\mathbb{R})$ we have

$$f' = (\tau_1 \circ f)' = (f \circ \tau_1)' = f' \circ \tau_1$$

so $f'$ is periodic with period 1. From this it follows that the integrand $|f'(t) - g'(t)|$ that appears in the definition of $d_*$ is periodic in $t$ with period 1 and so

$$d_*(f, g) = \int_a^b |f'(t) - g'(t)| \, dt$$

for all $f, g \in AC^{loc}_+(\mathbb{R})$ and all $a, b \in \mathbb{R}$ with $b - a = 1$. Now for any $f, g, u \in AC^{loc}_+(\mathbb{R})$

$$d_*(f \circ u, g \circ u) = \int_0^1 |(f \circ u)'(t) - (g \circ u)'(t)| \, dt$$

$$= \int_{u(0)}^{u(1)} |f'(t) - g'(t)| \, dt$$

$$= d_*(f, g)$$

by integration by substitution and because $u(1) - u(0) = 1$, and so $d_*$ is right-invariant. For all $r \in \mathbb{R}$ and $f \in AC^{loc}_+(\mathbb{R})$ we have $(\tau_r \circ f)' = f'$ which implies the equality in the proposition. \qed

We seek a right-invariant metric on $AC^{loc}_+(\mathbb{R})$ which induces a Polish group topology and there is an obvious candidate for such a metric. The sum of a right-invariant metric and right-invariant pseudometric is always a right-invariant metric, so $d_\infty + d_*$ defines a right-invariant metric on $AC^{loc}_+(\mathbb{R})$. We also set

$$d_*(f, g) = \int_0^1 |\tilde{f}'(t) - \tilde{g}'(t)| \, dt$$
for all \( f, g \in \text{AC}_+ (S^1) \). Following similar reasoning, \( d_* \) defines a right-invariant pseudometric and \( d_\infty + d_* \) defines a right-invariant metric on \( \text{AC}_+ (S^1) \).

We prove for both \( G = \text{AC}_+ (S^1) \) and \( G = \text{AC}^\text{loc}_+ (\mathbb{R}) \) the metric \( d_\infty + d_* \) is compatible with a Polish group topology on \( G \). See Propositions 19 and 20. We note here that Lemma 2 implies the metrics \( d_* \) and \( d_\infty + d_* \) induce the same topology on \( \text{AC}_+ (I) \).

3. \( \text{AC}_+ (I) \) is a Coarsely Bounded Group

To prove that \( \text{AC}_+ (I) \) is a coarsely bounded group we find a suitably uniform deformation retract of \( \text{AC}_+ (I) \) onto its trivial subgroup. Lemma 4 serves to isolate the uniformity condition.

**Lemma 4.** Suppose \( G \) is a Polish group and \( F : G \times I \to G \) is a deformation retract of \( G \) onto its trivial subgroup. Also suppose for every identity neighborhood \( U \subset G \) there exists \( \epsilon > 0 \) such that for all \( g \in G \) and all \( r, s \in I \), if \( |r - s| \leq \epsilon \) then \( F(g, r)F(g, s)^{-1} \in U \). Then \( G \) is a coarsely bounded group.

**Proof.** Let \( U \subset G \) be an identity neighborhood, let \( \epsilon > 0 \) be given by the condition in the lemma, and let \( n > 0 \) be a positive integer with \( n \geq \epsilon^{-1} \). For every \( g \in G \) we have

\[
g = F(g, 0)F(g, 1)^{-1} = \prod_{i=1}^{n} F\left(g, \frac{i-1}{n}\right) F\left(g, \frac{i}{n}\right)^{-1} \in U^n
\]

so \( G \subset U^n \), and so the condition implies the following: For every identity neighborhood \( U \) there is an integer \( n > 0 \) such that \( G = U^n \).

Let \( d \) be a compatible, left-invariant metric on \( G \). We must show \( G \) is bounded in \( d \). So let \( B \) denote the open unit \( d \)-ball centered at the identity in \( G \). By the condition at the end of the previous paragraph there is an integer \( n > 0 \) such that \( G = B^n \). By left invariance and the triangle inequality this implies the \( d \)-diameter of \( G \) is no greater than \( n \). \( \square \)

**Proposition 5.** \( \text{AC}_+ (I) \) is a coarsely bounded group.

**Proof.** For every \( f \in \text{AC}_+ (I) \) and \( r \in I \) let \( F(f, r) \) be the element of \( \text{AC}_+ (I) \) defined by

\[
F(f, r)(x) = (1 - r)f(x) + rx
\]

for all \( x \in I \). Every \( f \in \text{AC}_+ (I) \) is increasing so \( f' \geq 0 \). It follows

\[
d_* (F(f, r), F(f, s)) = |r - s| \int_0^1 |f'(t) - 1| \, dt
\]

\[
\leq |r - s| \left( \int_0^1 |f'(t)| \, dt + \int_0^1 1 \, dt \right)
\]

\[
= |r - s| \left( \int_0^1 f'(t) \, dt + 1 \right)
\]

\[
= |r - s| 2
\]

for all \( f \in \text{AC}_+ (I) \) and \( r, s \in I \).

Now apply Lemma 4 with \( G = \text{AC}_+ (I) \). It is true that \( F \) is a deformation retract of \( G \) onto its trivial subgroup but we omit verifying this. Note that the proof of Lemma 4 does not involve continuity of \( F \) and so it is enough to check that the
uniformity condition on $F$ is satisfied. Let $U$ be any identity neighborhood in $G$, let $B$ be the open ball of radius $\delta$ about the identity in $G$ with $\delta$ chosen sufficiently small so that $B \subset U$, and take $\epsilon = \delta/2$. From the above inequality it follows that for any $r, s \in I$, if $|r - s| \leq \epsilon$ then $d_*(f(r), f(s)) \leq \delta$, and so by right invariance of $d_*$ we have $f(r)F^{-1}(f(s)) \in U$. So Lemma 4 applies and $AC_+(I)$ is a coarsely bounded group.

\[ \square \]

**Figure 1.** The deformation retract $F$ of $AC_+(I)$ onto its trivial subgroup that appears in the proof of Proposition 5 deforms the graph of a homeomorphism towards the diagonal. The graphs of $f : I \to I, x \mapsto x^{1/2}$ and the identity $I \to I$ are shown in black and the graphs of $F(f, r)$ for $r = 1/4, 1/2, 3/4$ are shown in gray.

A Polish group $G$ is **Roelcke precompact** if for every identity neighborhood $U \subset G$ there is a finite set $F \subset G$ so that $G = UFU$. As noted in [3] the Polish group $\text{Homeo}_+(I)$ is Roelcke precompact.

**Question 6.** Is $AC_+(I)$ Roelcke precompact?

### 4. Topology and Geometry of the Zappa-Szép Product

A group $G$ is a **Zappa-Szép product** of subgroups $H$ and $K$ if $H \cap K = \{1\}$ and $G = HK$ [7, 8] or equivalently, if the group operation $G \times G \to G$ restricts to a bijection $H \times K \to G$. Every semidirect product is a Zappa-Szép product, and as with the semidirect product there is both an internal and external definition for the Zappa-Szép product. Suppose $H$ and $K$ are any groups and $\alpha : K \times H \to H$ and $\beta : K \times H \to K$ are functions. On $H \times K$ define a binary operation by

$$(h_1, k_1)(h_2, k_2) = (h_1\alpha(k_1, h_2), \beta(k_1, h_2)k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K$. If (and only if) this operation makes $H \times K$ a group and also makes the injections

$H \to H \times K, h \mapsto (h, 1_K)$

and

$K \to H \times K, k \mapsto (1_H, k)$
group homomorphisms, then the **external Zappa-Szép product** of $H$ and $K$ with respect to $\alpha$ and $\beta$ is $H \times K$ equipped with this operation. The identity element in the external product is $(1_H, 1_K)$ and the inverse of $(h, k)$ is

$$(\alpha(k^{-1}, h^{-1}), \beta(k^{-1}, h^{-1}))$$

for all $h \in H$ and all $k \in K$. Of course, the external product of $H$ and $K$ is an internal product of the subgroups $H \times \{1_K\}$ and $\{1_H\} \times K$.

Given an internal Zappa-Szép product $G$ of subgroups $H$ and $K$ there are functions $\alpha : K \times H \to H$ and $\beta : K \times H \to K$ which are uniquely determined by the equation

$$kh = \alpha(k, h)\beta(k, h)$$

for all $h \in H$ and $k \in K$. The binary operation defined above on $H \times K$ makes it the external Zappa-Szép product with respect to $\alpha$ and $\beta$ and makes the bijection $H \times K \to G, (h, k) \mapsto hk$ a group isomorphism. It follows that there is a natural correspondence between internal and external Zappa-Szép products which is analogous to the correspondence which holds for semidirect products.

For a group $G$ with subgroups $H$ and $K$ the subset $HK$ of $G$ is a subgroup if and only if $HK = KH$, so in the definition of the Zappa-Szép product the two factor subgroups play a symmetric role. We set aside some notation that will help us study our main examples in which the two factor subgroups play very different roles.

For any set $X$ we let $\text{Bij}(X)$ denote the group of all bijections of $X$ with composition as the group operation. For any $S, T \subset \text{Bij}(X)$ and $Y \subset X$ we let $S \circ T = \{f \circ g | f \in S, g \in T\}$ and $S(Y) = \{f(x) | f \in S, x \in Y\}$. So if $X = H$ is a group then $\text{Bij}(H)$ denotes the group of all bijections of the underlying set $H$. Similarly if $H$ is a topological group then $\text{Homeo}(H)$ denotes the group of all homeomorphisms of the underlying space $H$. For a group $H$ and an element $h \in H$ we let $\lambda_h : H \to H$ be left translation $h \mapsto hx$, and we let

$$\Lambda_H = \{\lambda_h | h \in H\}$$

denote the subgroup of $\text{Bij}(H)$ consisting of left translations.

**Notation 7.** Suppose $H$ is a group and $G \leq \text{Bij}(H)$ is a subgroup such that $\Lambda_H \leq G$. We let

$$K_G = \{k \in G | k(1_H) = 1_H\}$$

denote the isotropy subgroup of $1_H$ in $G$. We let

$$\Omega : H \times K_G \to G$$

be the function $(h, k) \mapsto \lambda_h \circ k$.

The assumption $\Lambda_H \leq G$ in Notation 7 ensures that $G$ is a Zappa-Szép product of $\Lambda_H$ and $K_G$. It is clear that $\Lambda_H \cap K_G$ is the trivial subgroup of $G$. To see $G = \Lambda_H \circ K_G$ note that any $g \in G$ may be decomposed

$$g = \lambda_{g(1_H)} \circ (\lambda_{g(1_H)}^{-1} \circ g)$$

and the composition in parentheses is an element of $K_G$. Similarly the assumption $\Lambda_H \leq G$ ensures that $\Omega$ is a bijection.
In Section 5 we take \( G = \text{AC}_+ (\mathbb{S}^1) \) (with \( H = \mathbb{S}^1 \)) and \( G = \text{AC}^\text{loc}\_c (\mathbb{R}) \) (with \( H = \mathbb{R} \)). In both of these cases the isotropy subgroup \( K_G \) is isomorphic to \( \text{AC}_+ (I) \), and as there is already a known Polish group topology on \( \text{AC}_+ (I) \) the bijection \( \Omega \) suggests one should consider the product topology \( H \times \text{AC}_+ (I) \) on \( G \). Before moving on to these examples we take a few moments to record a number of observations about the topology and geometry of the Zappa-Szép product.

Suppose \( G \) is a topological group which is a Zappa-Szép product of subgroups \( H \) and \( K \), then the group operation restricts to a continuous bijection \( H \times K \to G \) on the product space \( H \times K \). The following is [4, Theorem A.3].

**Theorem 8** (Rosendal). Suppose \( G \) is a Polish group which is a Zappa-Szép product of closed subgroups \( H \) and \( K \). Then the group operation is a homeomorphism \( H \times K \to G \).

**Corollary 9.** Suppose \( G \) is a Polish group which is a Zappa-Szép product of closed subgroups \( H \) and \( K \). Then \( S \mathcal{T} = \mathcal{T} S \) for any subsets \( S \subset H \) and \( T \subset K \).

**Proof.** Let \( S \subset H \) and \( T \subset K \). Continuity of the group operation \( H \times K \to G \) implies \( S \mathcal{T} \subset \mathcal{T} S \), and by Theorem 8 \( S \mathcal{T} \) is a closed subset of \( G \) which contains \( ST \), so \( S \mathcal{T} \subset \mathcal{T} S \). \( \square \)

**Corollary 10.** Suppose \( H \) is a Polish group and \( G \leq \text{Homeo}(H) \) is a subgroup such that \( \Lambda_H \leq G \). Let \( \Omega : H \times K_G \to G \) be the bijection from Notation 7. Suppose \( G \) is equipped with a Polish group topology such that \( H \to \Lambda_H, h \mapsto \lambda_h \) is a homeomorphism onto the subspace \( \Lambda_H \) and \( K_G \) is closed. Then \( \Omega \) is a homeomorphism.

**Proof.** Let \( \Phi : H \times K_G \to \Lambda_H \times K_G \) be \( (h,k) \mapsto (\lambda_h,k) \) and let \( \Psi : \Lambda_H \times K_G \to G \) be composition of functions.

By assumption \( H \to \Lambda_H, h \mapsto \lambda_h \) is a homeomorphism and so \( \Phi \) is a homeomorphism as well. By virtue of being Polish it follows that \( \Lambda_H \) is a closed subgroup of \( G \) and so applying Theorem 8 we get that \( \Psi \) is also a homeomorphism. As \( \Omega = \Psi \circ \Phi \) this implies \( \Omega \) is a homeomorphism. \( \square \)

Without the topological assumptions in Corollary 10 it is possible to have a subgroup \( G \leq \text{Homeo}_+(H) \) and a Polish group topology on \( G \) for which \( \Omega \) is not a homeomorphism. If \( H \) is a group that supports multiple Polish group topologies then considering one topology on \( H \) and another topology on \( G = \Lambda_H \) yields such a counterexample.

**Proposition 11.** Suppose \( H \) is a topological group and \( G \leq \text{Homeo}(H) \) is a subgroup such that \( \Lambda_H \leq G \). Let \( \Omega : H \times K_G \to G \) be the bijection from Notation 7. Also suppose there is a topology on \( K_G \) which makes it a topological group. Then there is a unique topology on \( G \) which makes \( \Omega \) a homeomorphism. This topology makes \( G \) a topological group if and only if evaluation \( K_G \times H \to H, (k,h) \mapsto k(h) \)

and the function \( K_G \times H \to K_G, (k,h) \mapsto \lambda_{k(h)}^{-1} \circ k \circ \lambda_h \)

are continuous. If \( d_H \) and \( d_K \) are compatible metrics on \( H \) and \( K_G \), respectively, then \( d \) defined

\[
d(f,g) = d_H (f(1_H), g(1_H)) + d_K (\lambda_{f(1_H)}^{-1} \circ f, \lambda_{g(1_H)}^{-1} \circ g)
\]

is a compatible metric on \( G \).
Proof. The unique topology on $G$ which makes $\Omega$ a homeomorphism is clearly the one obtained by declaring $U \subset G$ open if and only if $\Omega^{-1}(U) \subset H \times K_G$ is open.

For all $h \in H$ and all $k \in K_G$ let $\phi(k, h) = k(h)$ and let $\psi(k, h) = \lambda^{-1}_{k(h)} \circ k \circ \lambda_h$. Let $\otimes$ be the binary operation on $H \times K_G$ defined by

$$(h_1, k_1) \otimes (h_2, k_2) = (h_1 \phi(k_1, h_2), \psi(k_1, h_2) \circ k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$. Then

$$\Omega(h_1, k_1) \circ \Omega(h_2, k_2) = \lambda_{h_1 \circ k_1} \circ \lambda_k \circ k_2$$

$$= \lambda_{h_1, k_1(h_2)} \circ (\lambda_{k_1(h_2)} \circ k_1 \circ \lambda_{h_2} \circ k_2)$$

$$= \Omega(h_1 \phi(k_1, h_2), \psi(k_1, h_2) \circ k_2)$$

$$= \Omega((h_1, k_1) \otimes (h_2, k_2))$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$. This says $\Omega : (H \times K_G, \otimes) \to (G, \circ)$ is an operation-preserving bijection and thus a group isomorphism. Indeed, $(H \times K_G, \otimes)$ is the external Zappa-Szép product of $H$ and $K_G$ with respect to $\phi$ and $\psi$.

The proposition states the equivalence between (1) and (3) among the following three equivalent conditions.

(1) $G$ is a topological group with the product topology from $\Omega$.
(2) $(H \times K_G, \otimes)$ is a topological group with the product topology.
(3) The functions $\phi$ and $\psi$ are continuous.

The equivalence between (1) and (2) is immediate because $\Omega$ is a group isomorphism and a homeomorphism.

Suppose for a moment that $H$ and $K$ are some arbitrary topological groups and $G = H \times K$ is an external Zappa-Szép product with respect to functions $\alpha : K \times H \to H$ and $\beta : K \times H \to K$. The claim is that the product topology $H \times K$ makes $G$ a topological group if and only if $\alpha$ and $\beta$ are continuous. For one direction, if $\alpha$ and $\beta$ are continuous then the group operation and inversion in $G$ have continuous coordinate functions and so are continuous themselves. For the reverse direction, if the group operation in $G$ is continuous with respect to the product topology $H \times K$ then

$$(k, h) \mapsto (1_H, k)(h, 1_K) = (\alpha(k, h), \beta(k, h))$$

defines a continuous function $K \times H \to H \times K$ and so $\alpha$ and $\beta$ are continuous. Now returning to the setting of the current proposition, $(H \times K_G, \otimes)$ is the external Zappa-Szép product with respect to $\phi$ and $\psi$ so (2) and (3) are equivalent.

If $d_H$ and $d_K$ are compatible metrics on $H$ and $K_G$, respectively, then $D$ defined

$$D((h_1, k_1), (h_2, k_2)) = d_H(h_1, h_2) + d_K(k_1, k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$ is a metric on $H \times K$ which is compatible with the product topology. With $d$ as in the proposition we have

$$D((h_1, k_1), (h_2, k_2)) = d(\Omega(h_1, k_1), \Omega(h_2, k_2))$$

so $d$ is a metric on $G$ and $\Omega : (H \times K_G, D) \to (G, d)$ is an isometry. Hence $d$ is compatible with the topology on $G$.  

The product of two Polish spaces is a Polish space, and so in our applications of Proposition 11 once we know $G$ is a topological group it becomes obvious it is a Polish group.
We now turn our attention to the large-scale geometry of the Zappa-Szép product. Recall that a subset of a group is symmetric if it is closed under inversion.

**Proposition 12.** Suppose $G$ is a Zappa-Szép product of subgroups $H$ and $K$. Also suppose $H$ is generated by a symmetric subset $S \subset H$ and $K$ is generated by a symmetric subset $T \subset K$ with $1 \in S \cap T$ and $ST = TS$. Then the word metric $\rho_{ST}$ is defined on $G$ and

$$\rho_{ST}(hk, 1) = \max\{\rho_S(h, 1), \rho_T(k, 1)\}$$

for all $h \in H$ and $k \in K$. Consequently, the inclusions of the two factor subgroups $(H, \rho_S) \to (G, \rho_{ST})$ and $(K, \rho_T) \to (G, \rho_{ST})$ are isometric embeddings.

**Proof.** For any integer $n \geq 0$ we have $S^n T^n = (ST)^n$ by repeatedly applying the assumption $ST = TS$.

Let $h \in H$ and $k \in K$ and set $M = \max\{\rho_S(h, 1), \rho_T(k, 1)\}$. Because $1 \in S \cap T$ we have $hk \in S^M T^M = (ST)^M$ so $\rho_{ST}$ is defined on $G$ and $\rho_{ST}(hk, 1) \leq M$. If $n \geq 0$ is an integer with $\rho_{ST}(hk, 1) \leq n$ then $hk \in (ST)^n = S^n T^n$ so there exists $s_1, \ldots, s_n \in S$ and $t_1, \ldots, t_n \in T$ with

$$hk = s_1 \cdots s_n t_1 \cdots t_n$$

and because the group operation $H \times K \to G$ is injective this implies $h = s_1 \cdots s_n$ and $k = t_1 \cdots t_n$, so $\rho_S(h, 1) \leq n$ and $\rho_T(k, 1) \leq n$. This holds for any $n \geq 0$ so $\rho_{ST}(hk, 1) \geq \rho_S(h, 1)$ and $\rho_{ST}(hk, 1) \geq \rho_T(k, 1)$, and hence $\rho_{ST}(hk, 1) \geq M$. This proves the equality in the proposition.

Now

$$\rho_{ST}(h_1^{-1} h_2, 1) = \max\{\rho_S(h_1^{-1} h_2, 1), \rho_T(1, 1)\} = \rho_S(h_1^{-1} h_2, 1)$$

for all $h_1, h_2 \in H$. By left invariance it follows that inclusion $(H, \rho_S) \to (G, \rho_{ST})$ is an isometric embedding. Similarly $(K, \rho_T) \to (G, \rho_{ST})$ is an isometric embedding.

\[\square\]

**Theorem 13.** Suppose $G$ is a Polish group which is a Zappa-Szép product of closed subgroups $H$ and $K$. Also suppose $H$ is generated by a subset $S \subset H$ which is coarsely bounded in $H$, $K$ is a coarsely bounded group when equipped with the subspace topology, and $SK = KS$. Then inclusion $H \to G$ is a quasi-isometry of Polish groups.

**Proof.** Set $\mathcal{F} = S \cup \{1\} \cup S^{-1}$. As $K = K^{-1}$ and $SK = KS$ we have

$$S^{-1} K = (KS)^{-1} = (SK)^{-1} = KS^{-1}$$

and

$$\mathcal{F} K = SK \cup K \cup S^{-1} K = KS \cup K \cup KS^{-1} = K \mathcal{F}$$

so by Corollary 2

$$\overline{\mathcal{F}} K = \overline{\mathcal{F} K} = \overline{K \mathcal{F}} = K \overline{\mathcal{F}}$$

because $\overline{\mathcal{F}}$ and $K$ are closed. Now by Proposition 14 inclusion $(H, \rho_{\mathcal{F}}) \to (G, \rho_{\mathcal{F}} K)$ is an isometric embedding.

For every $g \in G$ there is $h \in H$ and $k \in K$ with $g = hk$, and by left invariance

$$\rho_{\mathcal{F}} K(g, h) = \rho_{\mathcal{F}} K(k, 1) \leq 1$$

so inclusion $(H, \rho_{\mathcal{F}}) \to (G, \rho_{\mathcal{F}} K)$ is coarsely onto. Hence inclusion $(H, \rho_{\mathcal{F}}) \to (G, \rho_{\mathcal{F}} K)$ is a quasi-isometry of metric spaces.
As $\mathcal{F}$ is a symmetric generating set for $H$ which is closed and coarsely bounded in $H$ the quasi-isometry type of $H$ is that of $(H, \rho_{\mathcal{F}})$. It remains to show that the quasi-isometry type of $G$ is that of $(G, \rho_{\mathcal{F}K})$. We know $\mathcal{F}K = \mathcal{F}K$ is a generating set for $G$ which is closed, so we must show $\mathcal{F}K$ is coarsely bounded in $G$. Let $d$ be a compatible, left-invariant metric on $G$. Then $d$ restricts to a compatible, left-invariant metric on both $H$ and $K$. The subsets $\mathcal{F}$ and $K$ are coarsely bounded in $H$ and $K$, respectively, so these subsets are bounded in $d$. By left invariance and the triangle inequality it follows that $\mathcal{F}K$ is bounded in $d$, and thus $\mathcal{F}K$ is coarsely bounded in $G$. So the quasi-isometry type of $G$ is that of $(G, \rho_{\mathcal{F}K})$ as required to make inclusion $H \to G$ a quasi-isometry of Polish groups. □

Theorem 14. Suppose $H$ is a Polish group and $G \leq \text{Homeo}(H)$ is a subgroup such that $\Lambda_H \leq G$. Let $K_G \leq G$ be the isotropy subgroup from Notation 4. Suppose $G$ is equipped with a Polish group topology which satisfies the assumptions of Corollary 10. Also suppose $H$ is generated by a subset $S \subset H$ which is coarsely bounded in $H$, $K_G$ is a coarsely bounded group when equipped with the subspace topology, $K_G(S) \subset S$, and $K_G(S^{-1}) \subset S^{-1}$. Then $H \to G, h \mapsto \lambda_h$ is a quasi-isometry of Polish groups.

Proof. First note that for all $h \in H$ and $k \in K_G$

$$k \circ \lambda_h = \lambda_{k(h)} \circ \left(\lambda^{-1}_{k(h)} \circ k \circ \lambda_h \right)$$

and

$$\lambda_h \circ k = \left(\lambda_h \circ k \circ \lambda_{k^{-1}(h^{-1})} \right) \circ \lambda^{-1}_{k^{-1}(h^{-1})}$$

and in both equations the composition in parentheses is an element of $K_G$.

Set $\Lambda_S = \{\lambda_s \mid s \in S\}$. By assumption $k(s) \in S$ and $k^{-1}(s^{-1}) \in S^{-1}$ for every $s \in S$ and $k \in K_G$, so the first equation above implies $K_G \circ \Lambda_S \subset \Lambda_S \circ K_G$ and the second equation implies $\Lambda_S \circ K_G \subset K_G \circ \Lambda_S$, and thus $\Lambda_S \circ K_G = K_G \circ \Lambda_S$.

By Theorem 13 with $\Lambda_S$ in place of $S$ it follows that inclusion $\Lambda_H \to G$ is a quasi-isometry of Polish groups. As $H \to \Lambda_H, h \mapsto \lambda_h$ is an isomorphism of Polish groups it is a quasi-isometry of Polish groups, and so Theorem 13 follows by composing the quasi-isometry $H \to \Lambda_H$ with the quasi-isometry $\Lambda_H \to G$. □

We make note of a few examples of Theorems 13 and 14. First, suppose $G$ is a Polish group which is a semidirect product of closed subgroups $H$ and $K$ with $K$ normal, so $SK = KS$ for every subset $S \subset G$. If $H$ is generated by a subset which is coarsely bounded in $H$ and $K$ is a coarsely bounded group when equipped with the subspace topology, then Theorem 13 says inclusion $H \to G$ is a quasi-isometry of Polish groups. The Zappa-Szép products of Example 15 need not be semidirect products.

Example 15. Let $H$ be a Polish group which admits a complete, compatible, left-invariant metric $d$ whose closed unit ball generates $H$, and let $G$ be the Polish group obtained by equipping the isometry group $\text{Isom}(H, d)$ with the topology of pointwise convergence. In this case $H \to \Lambda_H, h \mapsto \lambda_h$ is a homeomorphism onto the subspace $\Lambda_H$ and $K_G$ is closed, so the conditions of Corollary 10 are satisfied. Take $S$ to be the closed unit ball in $(H, d)$. If $S$ is coarsely bounded in $H$ and $K_G$ is a coarsely bounded group with the subspace topology, then Theorem 13 says
$H \to G, h \mapsto \lambda h$ is a quasi-isometry of Polish groups. The condition relating $S$ and $K_G$ in the theorem is satisfied trivially: For every $h \in H$ and $k \in K_G$
\[d(k(h), 1_H) = d(k(h), k(1_H)) = d(h, 1_H)\]
which implies $K_G(S) \subset S$, and since $S = S^{-1}$ this also says $K_G(S^{-1}) \subset S^{-1}$.

If $H$ is locally compact then the isotropy subgroup of $\text{Isom}(H,d)$ is compact and thus is a coarsely bounded group. So for instance Example 15 describes the situation when $H$ is a finitely generated group and $d = \rho_S$ is the word metric with respect to a finite generating set $S$.

For $f \in \text{Homeo}_Z(\mathbb{R})$ we let $f|_I$ denote the restriction of $f$ to a homeomorphism $I \to f(I)$ and for $f \in \text{Homeo}_+(I)$ we let $\hat{f}$ denote the homeomorphism $\mathbb{R} \to \mathbb{R}$ which is defined by
\[
\hat{f}(x) = f(x - n) + n
\]
for all $n \in \mathbb{Z}$ and all $x \in [n, n+1]$. It is straightforward to check that $k \mapsto k|_I$ defines an isomorphism of topological groups from the isotropy subgroup of $0$ in $\text{Homeo}_Z(\mathbb{R})$ to $\text{Homeo}_+(I)$ with inverse isomorphism given by $k \mapsto \hat{k}$. See Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{The isotropy subgroup of $0$ in $\text{Homeo}_Z(\mathbb{R})$ is isomorphic to $\text{Homeo}_+(I)$. The graph of $f : I \to I, x \mapsto x^4$ is shown in black and the graph of $\hat{f} : \mathbb{R} \to \mathbb{R}$ is shown in gray.}
\end{figure}

**Example 16.** We apply Theorem 14 with $H = \mathbb{R}$, $G = \text{Homeo}_Z(\mathbb{R})$, and $S = [-1, 1]$. It is straightforward to verify the topological assumptions of Corollary 10. The subset $S$ generates $H$ and because it is compact it is also coarsely bounded. By [2, Lemma 8] the Polish group $\text{Homeo}_0(\mathbb{B}^n)$ of homeomorphisms of the compact ball of dimension $n$ which fix the boundary is a coarsely bounded group for any integer $n > 0$. As noted above $K_G$ and $\text{Homeo}_+(I)$ are isomorphic groups and because $\text{Homeo}_+(I)$ and $\text{Homeo}_0(\mathbb{B}^1)$ are isomorphic it follows that $K_G$ is a coarsely bounded group as well. For every $s \in S$ and $k \in K_G$ the Intermediate Value Theorem gives $k(s) \in S$ so $K_G(S) \subset S$, and $S = S^{-1}$ so this also says $K_G(S^{-1}) \subset S^{-1}$.

Applying Theorem 14 shows $\mathbb{R} \to \text{Homeo}_Z(\mathbb{R}), r \mapsto \tau_r$ is a quasi-isometry of Polish groups.

5. $\text{AC}_+^1(\mathbb{S}^1)$ and $\text{AC}_Z^{\text{loc}}(\mathbb{R})$

In this final section we study the Zappa-Szép products $\text{AC}_+^1(\mathbb{S}^1)$ and $\text{AC}_Z^{\text{loc}}(\mathbb{R})$. In particular, we verify the claim from Section 2 that for $G = \text{AC}_+^1(\mathbb{S}^1)$ and
G = AC^{loc}_{\infty}(\mathbb{R}) the right-invariant metric $d_{\infty} + d_*$ on $G$ is compatible with a Polish group topology. Then we use Theorem 14 to describe the quasi-isometry types of these Polish groups.

We set

$$K_* = \left\{ k \in AC^{loc}_{\infty}(\mathbb{R}) | k(0) = 0 \right\}$$

and note that $k \mapsto k|_I$ defines an isomorphism of groups $K_* \to AC_+(I)$ which preserves $d_*$. As $d_*$ is a right-invariant metric on $AC_+(I)$ which induces a Polish group topology it follows that $d_*$ is also a right-invariant metric on $K_*$ which induces a Polish group topology, and the Polish groups $K_*$ and $AC_+(I)$ are isomorphic.

**Lemma 17.** Evaluation

$$K_* \times \mathbb{R} \to \mathbb{R}, \ (k, r) \mapsto k(r)$$

is continuous.

*Proof.* Let

$$K_\infty = \left\{ k \in \text{Homeo}_{\infty}(\mathbb{R}) | k(0) = 0 \right\},$$

let

$$\Phi : K_* \times H \to K_\infty \times H$$

be inclusion, and let

$$\Psi : K_\infty \times H \to H$$

be evaluation. By Lemma 2 inclusion $AC_+(I) \to \text{Homeo}_+(I)$ is continuous so also $\Phi$ is continuous, and by Proposition 1 (with $H = \mathbb{R}$ and $G = \text{Homeo}_{\infty}(\mathbb{R})$) $\Psi$ is continuous. Evaluation $K_* \times H \to H$ is the composition $\Psi \circ \Phi$, and so it is continuous. \(\square\)

**Lemma 18.** The function

$$K_* \times \mathbb{R} \to K_*, \ (k, r) \mapsto \tau_{-1}^{-1} k \circ \tau_r$$

is continuous.

*Proof.* For all $r \in \mathbb{R}$ and $k \in K_*$ let $\psi(k, r) = \tau_{-1}^{-1} k \circ \tau_r$. Now fix $(k, r) \in K_* \times \mathbb{R}$ and let $\epsilon > 0$ be given. For any compact interval $J$ the collection of continuous functions $\mathcal{C}(J)$ is a dense subset of $L^1(J)$, so there exists some continuous function $\gamma : [-\epsilon, 1 + \epsilon] \to \mathbb{R}$ with

$$\int_{-\epsilon}^{1+\epsilon} |k'(t) - \gamma(t)| \, dt < \frac{\epsilon}{4}$$

and by uniform continuity of $\gamma$ there exists some $\delta > 0$ so that

$$|\gamma(x) - \gamma(y)| < \frac{\epsilon}{4}$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Using the properties of $d_*$ from Proposition 3 for all $s \in \mathbb{R}$ and $l \in K_*$

$$d_* (\psi(k, r), \psi(l, s)) = d_*(k, l \circ \tau_{s-r})$$
and if $d_*(k, l) + |r - s| < \min\{\delta, \epsilon/4\}$ then

$$d_*(k, l \circ \tau_{s-r}) \leq \int_0^1 |k'(t) - \gamma(t)| \, dt$$

$$+ \int_0^1 |\gamma(t) - \gamma(t + s - r)| \, dt$$

$$+ \int_0^1 |\gamma(t + s - r) - k'(t + s - r)| \, dt$$

$$+ \int_0^1 |k'(t + s - r) - \gamma'(t + s - r)| \, dt$$

$$< \epsilon$$

so $\psi$ is continuous at $(k, r)$. Because the argument given works for an arbitrary point $(k, r) \in K_* \times \mathbb{R}$ it follows that $\psi$ is continuous on $K_* \times \mathbb{R}$.

Now we extend the topology on $K_*$ to a Polish group topology on $AC^\text{loc}_Z(\mathbb{R})$.

**Proposition 19.** Set $H = \mathbb{R}$ and $G = AC^\text{loc}_Z(\mathbb{R})$ and let $\Omega : H \times KG \to G$ be the bijection from Notation 7. Then the unique topology on $G$ which makes $\Omega$ a homeomorphism also makes $G$ a Polish group, and the metric $d_\infty + d_*$ is compatible with this topology.

**Proof.** In the present notation $K_* = KG$. Evaluation $KG \times H \to H$ and the function

$$K_G \times H \to KG; (k, h) \mapsto \lambda_{(h)}^{-1} \circ k \circ \lambda_h$$

are continuous by Lemmas 17 and 18. Now Proposition 11 applies and so the topology that makes $\Omega$ a homeomorphism also makes $G$ a Polish group. A compatible metric $d$ on $G$ is given by

$$d(f, g) = |f(0) - g(0)| + d_* \left( \tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g \right) = |f(0) - g(0)| + d_*(f, g)$$

for all $f, g \in G$.

For all $f, g \in G$

$$d(f, g) = |f(0) - g(0)| + d_* (f, g) \leq d_\infty (f, g) + d_*(f, g)$$

and

$$d_\infty (f, g) \leq |f(0) - g(0)| + d_* \left( \tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g \right)$$

$$\leq |f(0) - g(0)| + d_* \left( \tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g \right)$$

$$= |f(0) - g(0)| + d_*(f, g)$$

so $d_\infty (f, g) + d_*(f, g) \leq 2 \, d(f, g)$. This implies $d$ and $d_\infty + d_*$ induce the same topology on $G$.

As an aside we note that the topology induced by the pseudometric $d_*$ on $AC^\text{loc}_Z(\mathbb{R})$ is not a group topology. To see this, let

$$\mathcal{T} = \{ \tau_r \mid r \in \mathbb{R} \}$$

be the subgroup of $AC^\text{loc}_Z(\mathbb{R})$ consisting of real translations and for $k \in K_*$ consider the cosets $\mathcal{T} \circ k$ and $k \circ \mathcal{T}$. By Proposition 8 the right coset $\mathcal{T} \circ k$ has $d_*$-diameter 0. On the other hand, if $k \circ \mathcal{T}$ has $d_*$-diameter 0 then the Fundamental Theorem implies the homeomorphism $k$ is also a homomorphism of $(\mathbb{R}, +)$, and so $k$ must be
the identity $\mathbb{R} \to \mathbb{R}$. This says $k \circ \mathcal{T}$ has $d_*-$diameter 0 if and only if $k$ is the identity. From this it follows that inversion in $AC^\infty_{\mathbb{Z}}(\mathbb{R})$ exchanges subsets with $d_*-$diameter 0 and subsets with positive $d_*-$diameter, and so inversion is not a homeomorphism with the topology induced by $d_*$. We set

$$K_o = \{ k \in AC_+ (S^1) | k(1) = 1 \}$$

and note that $k \mapsto \tilde{k}|_I$ defines an isomorphism of groups $K_o \to AC_+(I)$ which preserves $d_*$. It follows that $d_*$ is a right-invariant metric on $K_o$ which induces a Polish group topology, and the Polish groups $K_o$ and $AC_+(I)$ are isomorphic. We extend the topology on $K_o$ to a Polish group topology on $AC_+(S^1)$. Alternatively, one may define the same Polish group topology on $AC_+(S^1)$ by identifying this group with the quotient of $AC^\infty_{\mathbb{Z}}(\mathbb{R})$ by the closed normal subgroup consisting of integer translations.

**Proposition 20.** Set $H = S^1$ and $G = AC_+(S^1)$ and let $\Omega : H \times K_G \to G$ be the bijection from Notation 7. Then the unique topology on $G$ which makes $\Omega$ a homeomorphism also makes $G$ a Polish group, and the metric $d_\infty + d_*$ is compatible with this topology.

**Proof.** In the present notation $K_o = K_G$. Let

$$K_\infty = \{ k \in \text{Homeo}_+ (S^1) | k(1) = 1 \}$$

let

$$\Phi : K_G \times H \to K_\infty \times H$$

be inclusion, and let

$$\Psi : K_\infty \times H \to H$$

be evaluation. Lemma 2 implies inclusion $K_G \to K_\infty$ is continuous so also $\Phi$ is continuous, and by Proposition 11 (with $H = S$ and $G = \text{Homeo}_+(S^1)$) $\Psi$ is continuous. Evaluation $K_G \times H \to H$ is the composition $\Psi \circ \Phi$, and so it is continuous. The function in Lemma 18 is continuous and descends to a continuous function $K_* \times S^1 \to K_*$, so

$$K_G \times H \to K_G, (k, r) \mapsto \lambda_k^{-1} \circ k \circ \lambda_r$$

is continuous. By Proposition 11 the topology that makes $\Omega$ a homeomorphism also makes $G$ a Polish group. A compatible metric $d$ on $G$ is given by

$$d(f, g) = d_{\infty} (f(1), g(1)) + d_* (f, g)$$

for all $f, g \in G$.

The argument that $d$ and $d_\infty + d_*$ induce the same topology on $G$ works similarly as in the proof of Proposition 19. □

**Proposition 21.** $AC_+(S^1)$ is a coarsely bounded group.

**Proof.** Apply Theorem 14 with $H = S = S^1$ and $G = AC_+(S^1)$. By Proposition 20 the assumptions of Corollary 10 are satisfied. As $S$ is compact it is coarsely bounded. The isotropy subgroup $K_G$ is isomorphic to $AC_+(I)$ and so is a coarsely bounded group. The condition $K_G(S) \subset S$ and $K_G(S^{-1}) \subset S^{-1}$ is satisfied trivially because $S = H$. By Theorem 14 $H \to G, h \mapsto \lambda_h$ is a quasi-isometry of Polish groups. Since $H$ is a coarsely bounded group so is $G$. □

**Proposition 22.** $AC^\infty_{\mathbb{Z}}(\mathbb{R})$ is quasi-isometric to $\mathbb{Z}$. 
Proof. Apply Theorem 14 with $H = \mathbb{R}$, $S = [-1,1]$, and $G = AC^{\text{loc}}_Z(\mathbb{R})$. By Proposition 19 the assumptions of Corollary 10 are satisfied. As $S$ is compact it is a coarsely bounded subset of $H$. The isotropy subgroup $K_G$ is isomorphic to $AC_+(I)$ and so is a coarsely bounded group. The condition $K_G(S) \subset S$ and $K_G(S^{-1}) \subset S^{-1}$ is satisfied by the Intermediate Value Theorem just the same as in Example 16. By Theorem 14 $H \to G, h \mapsto \lambda_h$ is a quasi-isometry of Polish groups. Now $\mathbb{Z} \to AC^{\text{loc}}_Z(\mathbb{R}), n \mapsto \tau_n$ is a quasi-isometry of Polish groups because it is the composition of inclusion $\mathbb{Z} \to H$ and $H \to G, h \mapsto \lambda_h$. \hfill\Box

Theorem 1 then just collects the statements of Propositions 5, 21, and 22. In the general theory of [4], if $G$ is a Polish group which is generated by a coarsely bounded subset then there is always a metric on $G$ which is simultaneously compatible with the topology, right-invariant, and realizes the quasi-isometry type of $G$. In closing we complete the proof that $d_\infty + d_*$ is such a metric on $AC^{\text{loc}}_Z(\mathbb{R})$.

**Proposition 23.** The metric $d_\infty + d_*$ on $AC^{\text{loc}}_Z(\mathbb{R})$ is a representative of the quasi-isometry type of $AC^{\text{loc}}_Z(\mathbb{R})$.

**Proof.** For all $r, s \in \mathbb{R}$

$$d_\infty(\tau_r, \tau_s) + d_*(\tau_r, \tau_s) = |r - s|$$

so $r \mapsto \tau_r$ defines an isometric embedding of $\mathbb{R}$ with its standard metric into $AC^{\text{loc}}_Z(\mathbb{R})$ with the metric $d_\infty + d_*$. As $AC^{\text{loc}}_Z(\mathbb{R}) = T \circ K_{*}$ and $K_*$ is bounded in $d_\infty + d_*$ it follows that the isometric embedding of $\mathbb{R}$ into $AC^{\text{loc}}_Z(\mathbb{R})$ is coarsely onto, and so the metric $d_\infty + d_*$ on $AC^{\text{loc}}_Z(\mathbb{R})$ represents the quasi-isometry of type of $AC^{\text{loc}}_Z(\mathbb{R})$. \hfill\Box

**References**

[1] Michael P. Cohen. On the large-scale geometry of diffeomorphism groups of 1-manifolds.
[2] Kathryn Mann and Christian Rosendal. Large scale geometry of homeomorphism groups. 2016.
[3] W. Roelcke and S. Dierolf. *Uniform Structures on Topological Groups and Their Quotients*. McGraw-Hill International Book Co., 1981.
[4] Christian Rosendal. *Coarse Geometry of Topological Groups*. 2017.
[5] Walter Rudin. *Real & Complex Analysis*, MHHE, 1987.
[6] Slawomir Solecki. *Polish Group Topologies*, in *Sets and Proofs* (London Mathematical Society Lecture Note Series 258). Cambridge University Press, 1999. pp 339–364.
[7] J. Szép. On the structure of groups which can be represented as the product of two subgroups. *Acta Sci. Math. Szeged*, 1950.
[8] G. Zappa. Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili traloro. *Atti Secondo Congresso Un. Mat. Ital.*, Bologna; Edizioni Cremonense, Rome, 1942.