The imposition of Cauchy data to the Teukolsky equation III: The rotating case

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We solve the problem of expressing the Weyl scalars $\psi$ that describe gravitational perturbations of a Kerr black hole in terms of Cauchy data. To do so we use geometrical identities (like the Gauss-Codazzi relations) as well as Einstein equations. We are able to explicitly express $\psi$ and $\partial_t \psi$ as functions only of the extrinsic curvature and the three-metric (and geometrical objects built out of it) of a generic spacelike slice of the spacetime. These results provide the link between initial data and $\psi$ to be evolved by the Teukolsky equation, and can be used to compute the gravitational radiation generated by two orbiting black holes in the close limit approximation. They can also be used to extract waveforms from spacetimes completely generated by numerical methods.

I. INTRODUCTION

In Ref. [1] the question was raised of how to impose initial data to the Teukolsky equation (that describe perturbations around a rotating black hole). We noted that the expressions of Chrzanowsky [2] for the Weyl scalars $\psi_4$ and $\psi_0$ in terms of metric perturbations were written as second order operators on the four-metric and appeared inconvenient at the moment to use them for building up the initial values needed to start the integration of the Teukolsky equation.

The work of reference [1] showed how to solve the problem for a non rotating background, i.e. perturbations around a Schwarzschild hole by relating Weyl scalars $\psi$, to the Moncrief waveforms $\phi_M$, an alternative description of metric perturbations explicitly built up out of the three-metric $g_{ij}$ and the extrinsic curvature $K_{ij}$ of the hypersurface $t = \text{constant}$. In Ref. [3] the $\psi - \phi_M$ relations were successfully tested with a program for integration of the Teukolsky equation.

It is not obvious how to extend the above techniques to the rotating case. Thus, in the present paper we turned to a more geometrical approach that lead us to the desired relations for rotating holes. In Sec. II we collect the results of the 3+1 decomposition reviewed in Ref. [4] relevant for our derivation. This has the advantage that makes $\psi$ to be automatically independent of the shift, so our task is reduced to prove that terms depending on the first perturbative order lapse vanish. This is made in Sec. III, where we also build up $\partial_t \psi$ in terms only of $\bar{g}_{ij}$ and $K_{ij}$. This results allow to compare, given the initial data, evolution through integration of the full Einstein equations and Teukolsky equation (linearization around a Kerr hole), and test, for instance, the close limit approximation for orbiting holes.

Notation: We use Ref. [5] conventions. An overbar on geometric quantities means that they are three-dimensional quantities, i.e. defined on the $t = \text{constant}$ hypersurfaces $\Sigma_t$ (an exception to this rule is the complex conjugation of the vector $m^\alpha$, i.e. $\bar{m}^\alpha$). $(\alpha, \beta)$ and $[\alpha, \beta]$ on indices $\alpha, \beta$ represent the usual symmetric and antisymmetric parts respectively. Greek letters indices run from 0 to 3 while latin letters indices run from 1 to 3. Subindexes (0) and (1) mean pieces of exclusively zeroth and first order respectively.

II. GEOMETRIC STRUCTURE AND GRAVITATION

Following Ref. [4] we write the metric as

$$ds^2 = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,$$  \hfill (2.1)

with $\theta^0 = dt$ and $\theta^i = dx^i + N^i dt$, where $N^i$ is the shift vector and $N$ the lapse.

The cobasis $\theta^\alpha$ satisfies

$$d\theta^\alpha = -\frac{1}{2}C^\beta_{\gamma\alpha} \theta^\beta \wedge \theta^\gamma,$$  \hfill (2.2)

with $C^i_{0j} = -C^i_{j0} = \partial_j N^i$ and all other structure coefficients zero. Note that $\bar{g}_{ij} = g_{ij}$ and $\bar{g}^{ij} = g^{ij}$.

The spacetime connection one-forms are defined by

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The Ricci tensor another important relation in three dimensions is given by
\[ R_{\alpha\beta} = \partial_{\alpha} \omega_{\beta}^\sigma - \partial_{\beta} \omega_{\alpha}^\sigma + \omega_{\alpha}^{\lambda} \omega_{\beta\lambda} - \omega_{\alpha}^{\lambda} \omega_{\lambda\beta} - \omega_{\alpha}^{\lambda} \omega_{\beta\lambda} - \omega_{\alpha\beta}^C \delta^\lambda_{\rho\sigma} \] (2.6)

For rewriting in the next section the Weyl scalars in terms of hypersurface quantities only, we relate the spacetime quantities to the 3-dimensional Riemann and the extrinsic curvature tensors

\[ R_{ijkl} = R_{iklj} + 2K_{ik}K_{lj} - 2K_{ij}K_{kl}, \] (2.7)

\[ R_{0ijkl} = N(\partial_0 K_{ij} + N K_{ip} K_{jq} + \nabla_i \nabla_j N). \] (2.8)

Another important relation in three dimensions is

\[ R_{ijkl} = 2g_{[ik}R_{lj]} + 2g_{[j}R_{ik]} + \tilde{R}_{ij}g_{kl}. \] (2.10)

The Ricci tensor \( R_{\alpha\beta} = R^\gamma_{\alpha\beta\gamma} \) is given by

\[ R_{ij} = \tilde{R}_{ij} - N^{-1} \partial_0 K_{ij} + KK_{ij} - 2K_{ip} K_{jq} - N^{-1} \nabla_i \nabla_j N, \] (2.11)

\[ R_{0i} = N \nabla^j (K g_{ij} - K_{ij}), \] (2.12)

\[ R_{00} = N \nabla^2 N - N^2 K_{pq} K^{pq} + N \tilde{\partial}_0 K. \] (2.13)

In order to incorporate the source terms we consider the Einstein equations as \( R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta} T. \) For instance, the “energy constraint” is defined by

\[ G^0_0 = \frac{1}{2} \left( K_{mk} K^{mk} - K^2 - \tilde{R} \right) = T^0_0. \] (2.14)

Finally, from its definitions

\[ \tilde{\partial}_0 \tilde{R}_{ij} = \nabla_k (\tilde{\partial}_0 \Gamma^k_{ij}) - \nabla_j (\tilde{\partial}_0 \Gamma^k_{ik}), \] (2.15)

where

\[ \tilde{\partial}_0 \Gamma^k_{ij} = -2\nabla_i (NK^k_{jq}) + \nabla^k (NK_{ij}). \] (2.16)

Note that writing equations in terms of \( \tilde{\partial}_0 \) instead of \( \partial_t \) allowed us to get rid of the shift dependence. This is because \( \tilde{\partial}_0 \) is orthogonal to the spacelike hypersurface \( \Sigma_0 \).

**III. WEYL SCALARS FOR KERR PERTURBATIONS**

For the computation of gravitation radiation from astrophysical sources it is convenient to work with the Weyl scalar

\[ \psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha m^\beta n^\gamma m^\delta, \]
since it is directly related to the outgoing gravitational waves. For perturbations around a Kerr hole we have

\[-\psi_4 = R_{ijkl} n^i \overline{m}^j n^k \overline{m}^l + 4R_{ijkl} n^{[0} \overline{m}^{l]} n^i \overline{m}^j + 4R_{ijkl} n^{[0} \overline{m}^{l]} n^{0} \overline{m}^{i}].\]

Eqs. (2.5) and (2.8) directly give us the two first terms in the above sum in terms of hypersurface geometrical objects \((g_{ij}, K_{ij})\). In the last term we have to make use of Einstein equation (2.11) to eliminate \(\partial_0 K_{ij}\). If one now considers first order perturbations around a Kerr hole, one would have to consider in \(\psi_4\) two types of terms: terms that involve first order perturbative Riemann tensors contracted with the background tetrads and terms that involve the Riemann tensor of the background contracted with three background and one perturbative tetrads. It is not difficult to see that the latter terms vanish for the Kerr background. For the Kerr geometry the only non-vanishing Weyl scalar is \(\psi_{(1)}\) given by Eq. (2.14)

\[\psi_{(1)} = \frac{2}{3} \partial_0 - \frac{4}{3} \partial_0 \partial_0 T - N_{(0)} T \hat{K}_{ij} \]

This term vanishes because it is contracted with two \(\overline{m}\) vectors, and any contraction with a repeated tetrad vector of the Riemann tensor vanishes for the Kerr spacetime. Similar arguments apply to the other terms.

Let us turn our attention to the terms that involve the first order Riemann tensors contracted with the background tetrads. Taking a look at equations (2.7)-(2.9) we see that if one considers first order perturbations, we will have expressions involving the first order extrinsic curvature, metric, and lapse. We do not want our final expression to depend on the perturbative lapse. It is easy to see that it actually does not depend on it. For \(R_{0ijk}\) if we rewrite it using the Einstein equation (2.11) again the lapse appears as an overall factor. So the expression evaluated for the perturbative lapse is proportional to the expression evaluated in the background, which vanishes. For \(R_{00ij}\) if we rewrite it using the Einstein equation (2.11)

\[\psi_{(0)} = \frac{2}{3} \partial_0 - \frac{4}{3} \partial_0 \partial_0 T - N_{(0)} T \hat{K}_{ij} \]

where \(N_{(0)} = (g^{\mu\nu}_{\text{Kerr}})^{-1/2}\) is the zeroth order lapse, \(n^i, \overline{m}^j\) are two of the null vectors of the (zeroth order) tetrad (see Ref. [7]), latin indices run from 1 to 3, and the brackets are computed to only first order (zeroth order excluded).

To obtain \(\partial_0 \psi_4\), the other relevant quantity in order to start the integration of the Teukolsky equation, we can operate with \(\partial_0\) on \(\psi_4\) given by Eq. (3.3) to find

\[\partial_0 \psi_4 = \frac{2}{3} \partial_0 T - \frac{4}{3} \partial_0 \partial_0 T - N_{(0)} T \hat{K}_{ij}.\]

where we made use of the equality

\[g_{ip} \partial_0 g_{pj} = 2 N K^p_i.\]

The derivatives appearing in Eq. (3.2) can be obtained from Eq. (2.13)

\[\partial_0 T = N_{(0)} K_{pq} K^{pq} - \hat{K}^2 N_{(0)} - N_{(0)}^{-1} T_{00},\]

from Eq. (2.14)

\[\partial_0 \hat{K} = 2 K^{pq} \hat{K}_{pq} + 4 N_{(0)} K_{pq} K^{pq} K^{pq} - 2 K \hat{K}_{0} - 2 \partial_0 T_{00},\]

and from Eqs. (2.10) and (2.4)

\[\partial_0 \hat{R} = 2 K^{pq} \hat{K}_{pq} + 4 N_{(0)} K_{pq} K^{pq} K^{pq} - 2 K \hat{K}_{0} - 2 \partial_0 T_{00},\]
\[
\tilde{\partial}_0 R_{ijkl} = -4N(0) \left\{ K_{i[l} \tilde{R}^{l]}_{j]} - K_{j[l} \tilde{R}^{l]}_{i]} - \frac{1}{2} \tilde{R} \left( K_{i[l} K_{j]} - K_{j[l} K_{i]} \right) \right\} + 2g_{i[l} \tilde{\partial}_0 \tilde{R}^{l]}_{j]} - 2g_{j[l} \tilde{\partial}_0 \tilde{R}^{l]}_{i]} - g_{i[l} K_{j]} - 2K_{i[l} \tilde{\partial}_0 K_{j]} - 2K_{j[l} \tilde{\partial}_0 K_{i]}. \tag{3.5}
\]

Note that in the last three equations we have taken explicitly the lapse to the zeroth perturbative order. This is so because in building up \( \tilde{\partial}_0 \psi_4 \) explicitly all dependence on \( N(1) \) cancels out. To prove this one can do the explicit calculation for the Kerr background using computer algebra. An alternative is to notice that \( \tilde{\partial}_0 \psi_4 = \mathcal{L}_t \psi_4 \) where \( t^\mu \) is a vector that includes the background and first order perturbations of the lapse and shift. If one now expands out this expression one gets \( \tilde{\partial}_0 \psi_4 = \mathcal{L}_{t(0)} \psi_4(0) + \mathcal{L}_{t(1)} \psi_4(1) \). Now, since \( \psi_4(0) \) vanishes identically for all time, the only contribution one has is \( \tilde{\partial}_0 \psi_4 = \mathcal{L}_{t(1)} \psi_4(1) \). Therefore the time derivative of \( \psi_4 \) does not depend on the perturbative lapse and shift, since neither \( \mathcal{L}_{t(0)} \) (by construction) nor \( \psi_4(1) \) (due to the proof we gave above), do.

The other pieces needed to build up \( \tilde{\partial}_0 \psi_4 \) only out of hypersurface data are \( \tilde{\partial}_0 K_{ij}, \tilde{\partial}_0 \Gamma^k_{ij}, \) and \( \tilde{\partial}_0 \tilde{R}_{ij} \) that are given by Eqs. (2.13), (2.16) and (2.11) respectively. As before, we have to consider the zeroth order lapse only, for instance

\[
\tilde{\partial}_0 K_{ij} = N(0) \left[ \tilde{R}_{ij} + K K_{ij} - 2K_{ip} K_{pj} - N(0) \nabla_i \nabla_j N(0) - T_{ij} + \frac{1}{2} T g_{ij} \right]_{(1)}. \tag{3.6}
\]

This completes our proof. A check of the relations (3.1) and (3.2) can be made in the Schwarzschild background for close limit initial data where \( \tilde{\partial}_0 \psi = -\frac{2 \Delta}{\Sigma} \psi \).

### IV. DISCUSSION

The issue of expressing \( \psi \) explicitly in terms of hypersurface data only appears as of a purely technical character, but it is of great practical use. Especially when one thinks of the important role played by first order perturbations in writing them in terms only of objects on the slice \( \Sigma \), but we did not said why. In fact it is not warranted that one can do that with any object defined on the spacetime. Is this because they are first order gauge invariant objects? This shouldn’t be enough since we checked that for \( \psi_3 \) (and the same for \( \psi_1 \)), we do not succeed in writing them in terms only of objects on the slice \( t = \) constant. The key point here seems to be that \( \psi_3 \) and \( \psi_0 \) are also invariant under tetrad rotations and then directly connected to physical quantities, while \( \psi_3 \) and \( \psi_1 \) are not.

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### APPENDIX A: ALTERNATIVE EQUATIONS

We can put all this together to yield the following expression of the first order perturbation in \( \psi_4 \) in terms of perturbations in the 3-metric \( \delta g_{ij} \), perturbations in the extrinsic curvature \( \delta K_{ij} \), and several quantites from the the
background (Kerr) geometry, the spatial metric \(^{(3)}g_{ij}^{(0)}\), the extrinsic curvature \(K_{ij}^{(0)}\), the lapse \(N^{(0)}\) and the shift \(N_i^{(0)}\). We have already argued that first order perturbations of the principal null vectors \(n^\mu\) and \(\bar{m}^\nu\) will not contribute to \(\delta \psi_4\) so we have

\[
\delta \psi_4 = \delta A_{ijkl} n^i \bar{m}^j n^k \bar{m}^l + 2 \delta B_{ijk} n^i \bar{m}^k [n^0 \bar{m}^j + n^j \bar{m}^0] + \delta C_{ij} [n^0 \bar{m}^i n^0 \bar{m}^j + n^i \bar{m}^0 n^j \bar{m}^0 - n^i \bar{m}^i n^j \bar{m}^j - n^0 \bar{m}^j n^i \bar{m}^j]
\]

where

\[
\delta A_{ijkl} = \delta (3) R_{ijkl} + \left[ K_{jl}^{(0)} \delta K_{ik} + K_{ik}^{(0)} \delta K_{jl} - (k \leftrightarrow l) \right]
\]

\[
\delta B_{ijk} = N^{(0)} \left[ D_j \delta K_{ik} - \frac{1}{2} \left[ D_k \delta (3) g_{mi} + D_i \delta (3) g_{mk} - D_m \delta (3) g_{ik} \right] (3) g_{ijl}^{(0)} \right] (3) g_{lm}^{(0)} K_{ij}^{(0)} - (k \leftrightarrow j)
\]

\[
\delta C_{ij} = N^{(0)} [A_{ij}^{(0)} + \frac{1}{2} D_i \delta (3) g_{j}^{(0)m} + D_j \delta (3) g_{i}^{(0)m} - \left[ \delta B_{ijl} N^{(0)} + \delta A_{ljm}^{(0)} + \delta A_{jlm}^{(0)} \right] + (3) g_{lm}^{(0)} N_i^{(0)} m + A_{ijl}^{(0)} \delta (3) g_{lm}^{(0)} N_i^{(0)} m + A_{ijm}^{(0)} d_{jm}^{(0)} N_i^{(0)} m + \frac{1}{2} (3) g_{lm}^{(0)} N_i^{(0)} k]
\]

and

\[
\delta (3) R_{ijkl} = \frac{1}{2} D_k^{(3)} g_{ij}^{(0)m} (D_l \delta (3) g_{mij} + D_j \delta (3) g_{mj} - D_m \delta (3) g_{jli}) - (k \leftrightarrow l)
\]

To calculate \(\partial_t \delta \psi_4\) we use the above expression for \(\delta \psi_4\) and plug in \(\partial_t \delta (3) g_{ij}\) and \(\partial_t \delta K_{ij}\) for \(\delta (3) g_{ij}\) and \(\delta K_{ij}\) in the above, respectively. Where, \(\partial_t \delta (3) g_{ij}\) and \(\partial_t \delta K_{ij}\) can be obtained from Einstein’s equations as follows:

\[
\partial_t \delta (3) g_{ij} = -2 N^{(0)} \delta K_{ij} + N^{(0)} \delta (3) g_{ij, k} + N^{(0)} \delta (3) g_{ij, k} + \delta (3) g_{ik} N^{(0)} k, j
\]

\[
\delta \partial_t K_{ij} = \frac{1}{2} \left[ D_j \delta (3) g_{mi} + D_i \delta (3) g_{mj} - D_m \delta (3) g_{ij} \right] (3) g_{mk}^{(0)} N^{(0)} k
\]

\[
+ N^{(0)} \delta (3) R_{ij} - 2 K^{(0)}_{ij} \delta K_{ik} - 2 \delta K_{j}^{(0)} K_{ik}^{(0)} + K^{(0)}_{ij} \delta K + K^{(0)} \delta K_{ij}
\]

\[
+ N^{(0)} k \delta K_{ij, k} + \delta K_{ik} N^{(0)} k, j + \delta K_{ij} N^{(0)} k, i + K^{(0)} l \left[ \delta (3) g_{kl}^{(0)} N^{(0)} k, j
\right]
\]

\[
+ K^{(0)} l \delta (3) g_{kl}^{(0)} N^{(0)} k, i + N^{(0)} l \delta (3) g_{kl}^{(0)} K_{ij, k}
\]

where \(\delta K = (3) g^{(0)}_{ij} \delta K_{ij} + K^{(0)}_{ij} (3) g^{(0)}_{ij} \delta K_{ij} + K^{(0)}_{ij} \delta K_{ij} \).

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