Nontrivial solutions of a class of fractional differential equations with $p$-Laplacian via variational methods

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Abstract
In this paper, a class of boundary value problems for fractional differential equations with a parameter is studied via the variational methods. Firstly, we present a result that the boundary value problems have at least one weak solution under the quadratic condition and the superquadratic condition, respectively. Additionally, we obtain the existence of at least one nontrivial solution by using the famous mountain pass lemma without the Ambrosetti–Rabinowitz condition. Finally, by a recent critical points theorem of Bonanno and Marano, the existence of at least three solutions is established.

MSC: 34A08; 35D30
Keywords: Fractional differential equations; Boundary value problem; Critical point theorem; Nontrivial solutions; Existence and multiplicity

1 Introduction
Fractional-order derivatives and integrals are more suitable to describe the properties of real materials than those of integer-order, so fractional-order differential equations are more and more widely used in simulating the mechanical and electrical characteristics of real materials, dynamic system control theory, rock rheological properties, and many other fields (see [1–10] and their references). The existence of solutions for boundary value problems of fractional differential equations is also studied in various ways, such as some fixed point theorems, the fixed point index theory in cones, methods of upper and lower solutions, coincidence degree theory, topological degree theory, etc. (see [11–16]). Later, with the introduction of the mountain pass theorem, variational methods have become a new and effective tool to study the existence of solutions for boundary value problems (see [17–19] and their references). In addition, nonlinear terms are often required to satisfy the Ambrosetti–Rabinowitz (A-R for short) condition when variational methods are applied. For example, in [20], Zhao and Tang studied a class of boundary value problems for fractional differential equations and obtained the existence of at least one weak solution by using the critical point theory under satisfying the A-R condition.
Chen and Liu [21] considered the following boundary value problems of fractional differential equations and obtained the existence of at least one weak solution under the A-R condition:

\[
\begin{cases}
\mathcal{D}_a^\alpha \phi_p(\mathcal{D}_a^\alpha u(t)) = f(t, u(t)), & t \in [0, T], \\
u(0) = u(T) = 0,
\end{cases}
\]

(1.1)

where \(\mathcal{D}_a^\alpha\) and \(\mathcal{D}_r^\alpha\) are the left and right Riemann–Liouville fractional derivatives of order \(\alpha \in (0, 1]\) respectively. \(f : [0, T] \times \mathbb{R} \to \mathbb{R}\). The result is as follows.

**Theorem 1.1** ([21], Theorem 4.2) Let \(\frac{1}{p} < \alpha \leq 1\). Assume that

(V\(_1\)) \(f \in C([0, T] \times \mathbb{R}, \mathbb{R})\).

(V\(_2\)) There exist constants \(\mu \in (0, \frac{1}{p})\) and \(M > 0\) such that

\[0 < F(t, x) \leq \mu x f(t, x), \quad \forall t \in [0, T], x \in \mathbb{R} \text{ with } |x| \geq M.\]

(V\(_3\)) For \(t \in [0, T]\) and \(x \in \mathbb{R}\), we have

\[
\limsup_{|x| \to 0} \frac{F(t, x)}{|x|^p} < \frac{(\Gamma(\alpha+1))^p}{p \Gamma(\alpha p)}.
\]

Then BVP (1.1) admits at least one nontrivial weak solution.

Motivated by the above mentioned works, we are interested in the following fractional differential boundary value problem:

\[
\begin{cases}
\mathcal{D}_a^\alpha \phi_p(\mathcal{D}_r^\alpha x(t)) = \lambda f(t, x(t)), & \text{a.e. } t \in [0, T], \\
x(0) = x(T) = 0
\end{cases}
\]

(1.2)

where \(0 < \alpha \leq 1\), \(\mathcal{D}_a^\alpha\) and \(\mathcal{D}_r^\alpha\) denote the left and right Riemann–Liouville fractional derivatives of order \(\alpha\) respectively. \(\lambda > 0\) is a parameter. \(\phi_p : \mathbb{R} \to \mathbb{R}\) is the \(p\)-Laplacian defined by \(\phi_p(s) = |s|^{p-2}s\) if \(s \neq 0\), \(\phi_p(0) = 0\), \(1 < p < \infty\). \(f : [0, T] \times \mathbb{R} \to \mathbb{R}\) and \(F(t, x) = \int_0^x f(t, s) \, ds\). The existence and multiplicity of solutions for boundary value problem (BVP) for short (1.2) will be derived by the critical point theory.

In the following proofs, we assume that \(f(t, x)\) and \(F(t, x)\) satisfy some of the following conditions:

(A\(_0\)) \(F(t, x)\) is measurable in \(t\) for every \(x \in \mathbb{R}\) and continuously differentiable in \(x\) for a.e. \(t \in [0, T]\), and there exist \(k_1 \in C(\mathbb{R}^+, \mathbb{R}^+)\) and \(k_2 \in L^1([0, T], \mathbb{R}^+)\) such that

\[
|F(t, x)| \leq k_1(|x|)k_2(t), |f(t, x)| \leq k_1(|x|)k_2(t)
\]

for all \(x \in \mathbb{R}\) and a.e. \(t \in [0, T]\).

(A\(_1\)) There exist a constant \(0 \leq \theta < p\) and a function \(\eta_1(t) \in C([0, T])\) with \(\text{ess inf} \eta_1 > 0\) such that

\[
\limsup_{|x| \to 0} \frac{F(t, x)}{|x|^\theta} < \eta_1(t)
\]

uniformly for a.e. \(t \in [0, T]\).
There is a function $\eta_2(t) \in C([0, T])$ with $\text{ess inf} \, \eta_2 > 0$ such that

$$\limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^p} < \eta_2(t)$$

uniformly for a.e. $t \in [0, T]$.

(A3) $\limsup_{|x| \to 0} \frac{F(t, x)}{|x|^p} = 0$ uniformly for a.e. $t \in [0, T]$.

(A4) $F(t, x) \geq 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}$.

(A5) There is a constant $\vartheta > p$ such that

$$\liminf_{|x| \to \infty} \vartheta F(t, x) - f(t, x)|x|^\vartheta \\ \leq 0$$

uniformly for a.e. $t \in [0, T]$.

(A6) There exists a subset $\Omega$ of $[0, T]$ with $\text{meas}(\Omega) > 0$ such that

$$\liminf_{|x| \to \infty} \frac{F(t, x)}{|x|^p} > 0$$

uniformly for a.e. $t \in \Omega$.

**Remark 1.1** By (A1), there exists a constant $L_1 > 0$ such that

$$F(t, x) \leq \eta_1(t)|x|^\theta + L_1$$

for a.e. $t \in [0, T]$ and $x \in \mathbb{R}$.

Similarly, (A2) implies that there is a constant $L_2 > 0$ such that

$$F(t, x) \leq \eta_2(t)|x|^p + L_2$$

for a.e. $t \in [0, T]$ and $x \in \mathbb{R}$.

The following arrangement of the article is as follows. In the second part, the basic definitions, some properties and lemmas of fractional calculus are given. Additionally, the related fractional derivative space and the variational structure of BVP (1.2) is established. In the third part, we give the existence and multiplicity theorems of nontrivial solutions of BVP (1.2) under some appropriate conditions.

### 2 Preliminaries

In order to obtain the existence of solutions to BVP (1.2), we need to recall some necessary definitions and related properties of fractional calculus, which will be used in the proofs later in this paper.

**Definition 2.1** ([22, 23]) Let $u$ be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional integrals of order $\gamma$ for the function $u$ denoted by $aD_t^{-\gamma}u(t)$ and $tD_b^{\gamma}u(t)$, respectively, are defined by

$$aD_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1}u(s) \, ds, \quad t \in [a, b], \gamma > 0$$

$$tD_b^{\gamma}u(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (t-s)^{\gamma-1}u(s) \, ds, \quad t \in [a, b], \gamma > 0$$
and

\[ aD_t^{-\gamma} u(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (t - s)^{\gamma - 1} u(s) \, ds, \quad t \in [a, b], \gamma > 0, \]

provided the right-hand sides are pointwise defined on \([a, b]\), where \(\Gamma > 0\) is the gamma function.

**Definition 2.2** ([22, 23]) Let \(u\) be a function defined on \([a, b]\). The left and right Riemann–Liouville fractional derivatives of order \(\gamma\) for the function \(u\) denoted by \(aD_t^{\gamma} u(t)\) and \(tD_k^{\gamma} u(t)\), respectively, are defined by

\[ aD_t^{\gamma} u(t) = \left( aD_t^{\gamma - n} u(t) \right)^{(n)} = \frac{1}{\Gamma(n - \gamma)} \frac{d^n}{dt^n} \left( \int_a^t (t - s)^{n - \gamma - 1} u(s) \, ds \right) \]

and

\[ tD_k^{\gamma} u(t) = (-1)^n \left( tD_k^{\gamma - n} u(t) \right)^{(n)} = \frac{1}{\Gamma(n - \gamma)} (-1)^n \frac{d^n}{dt^n} \left( \int_t^b (s - t)^{n - \gamma - 1} u(s) \, ds \right), \]

where \(t \in [a, b]\), \(n - 1 \leq \gamma < n\), and \(n \in \mathbb{N}\).

**Remark 2.1** ([22, 23]) If \(0 \leq \gamma < 1\), then

\[ aD_t^{\gamma} u(t) = \left( aD_t^{\gamma - 1} u(t) \right)' = \frac{1}{\Gamma(1 - \gamma)} \frac{d}{dt} \left( \int_a^t (t - s)^{-\gamma} u(s) \, ds \right), \quad t \in [a, b] \]

and

\[ tD_k^{\gamma} u(t) = -\left( tD_k^{\gamma - 1} u(t) \right)' = -\frac{1}{\Gamma(1 - \gamma)} \frac{d}{dt} \left( \int_t^b (s - t)^{-\gamma} u(s) \, ds \right), \quad t \in [a, b]. \]

**Remark 2.2** ([19]) For \(n \in \mathbb{N}\), if \(\gamma = n - 1\), according to Definition 2.2, we find that

\[ aD_t^{n-1} u(t) = u^{(n-1)}(t), \]

\[ tD_k^{n-1} u(t) = (-1)^{n-1} u^{(n-1)}(t), \]

where \(u^{(n-1)}(t)\) is the usual derivative of order \(n - 1\).

Now, we present the rule for fractional integration by parts.

**Property 2.1** ([23]) If \(u \in L^p([a, b], \mathbb{R}^N)\) and \(v \in L^q([a, b], \mathbb{R}^N)\) and \(p \geq 1, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \gamma\) or \(p \neq 1, q \neq 1, \frac{1}{p} + \frac{1}{q} = 1 + \gamma\). Then

\[ \int_a^b \left[ aD_t^{\gamma} u(t) \right] v(t) \, dt = \int_a^b \left[ tD_k^{\gamma} v(t) \right] u(t) \, dt, \quad \gamma > 0. \]

In order to deal with BVP (1.2), we introduce appropriate function spaces and the related variational structure.
Let $L^p([0, T], \mathbb{R}) (1 \leq p < \infty)$ and $C([0, T], \mathbb{R})$ be the $p$-Lebesgue space and the continuous function space, respectively, with the norms

$$
\| u \|_{L^p} = \left( \int_0^T |u(t)|^p \, dt \right)^{\frac{1}{p}}, \quad u \in L^p([0, T], \mathbb{R}) (1 \leq p < \infty)
$$

and

$$
\| u \|_{\infty} = \max_{t \in [0, T]} |u(t)|, \quad u \in C([0, T], \mathbb{R}).
$$

To set up the variational structure of BVP (1.2), we need to construct the appropriate function space first.

**Definition 2.3** Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_{\alpha, p}^0$ is defined as the closure of $C_0^\infty([0, T], \mathbb{R})$, that is, $E_{\alpha, p}^0 = \overline{C_0^\infty([0, T], \mathbb{R})}$ with the norm

$$
\| x \|_{\alpha, p} = \left( \int_0^T \left| 0 D_{\alpha}^p x(t) \right|^p \, dt + \int_0^T |x(t)|^p \, dt \right)^{\frac{1}{p}}. \quad (2.1)
$$

**Remark 2.3** The fractional derivative space $E_{\alpha, p}^0$ is the space of functions $x(t) \in L^p([0, T], \mathbb{R})$ with an $\alpha$-order Riemann–Liouville fractional derivative $0 D_{\alpha}^p x(t) \in L^p([0, T], \mathbb{R})$ and $x(0) = x(T) = 0$.

**Remark 2.4** From [19], the fractional derivative space $E_{\alpha, p}^0 (0 < \alpha \leq 1, 1 < p < \infty)$ is a reflexive and separable Banach space.

**Lemma 2.1** ([19]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For any $x \in E_{\alpha, p}^0$, we have

$$
\| x \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \| 0 D_{\alpha}^p x \|_{L^p}. \quad (2.2)
$$

If $\alpha > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\| x \|_{\infty} \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(\alpha - 1)q + 1} \| 0 D_{\alpha}^p x \|_{L^p}. \quad (2.3)
$$

We denote $G = \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(\alpha - 1)q + 1}^\frac{1}{p}$.

**Remark 2.5** According to formula (2.2), let $x \in E_{\alpha, p}^0$, the norm $\| x \|_{\alpha, p}$ is equivalent to

$$
\| x \|_{\alpha, p} = \left( \int_0^T \left| 0 D_{\alpha}^p x(t) \right|^p \, dt \right)^{\frac{1}{p}}. \quad (2.4)
$$

In the following discussion, we denote $\| x \| = \| x \|_{\alpha, p}$.

**Lemma 2.2** ([19]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{ x_n \}$ converges weakly to $x$ in $E_{\alpha, p}^0$, that is, $x_n \rightharpoonup x$ as $n \to \infty$. Then $x_n \to x$ in $C([0, T], \mathbb{R})$ as $n \to \infty$, that is, $\| x_n - x \|_{\infty} \to 0$ as $n \to \infty$. 

Definition 2.4 The weak solution of BVP (1.2) is a function  in $E_0^{a,p}$ satisfying the following equation:

$$
\int_0^T \phi_p (D^a_t x(t) , D^p_t y(t)) \, dt = \lambda \int_0^T f(t, x(t)) y(t) \, dt, \quad \forall y \in E_0^{a,p}.
$$

For the sake of convenience, the solution of BVP (1.2) mentioned later is the weak solution of BVP (1.2).

For $\forall x \in E_0^{a,p}$, we consider the functional $I : E_0^{a,p} \to \mathbb{R}$ as follows:

$$
I(x) = \frac{1}{p} \int_0^T |D^a_t x(t)|^p \, dt - \lambda \int_0^T F(t, x(t)) \, dt
$$

From [25], we can know that $I \in C^1(E_0^{a,p}, \mathbb{R})$ and

$$
I'(x)y = \int_0^T \phi_p (D^a_t x(t) , D^p_t y(t)) \, dt - \lambda \int_0^T f(t, x(t)) y(t) \, dt, \quad \forall y \in E_0^{a,p}.
$$

It follows from Definition 2.4 and (2.6) that if $x \in E_0^{a,p}$ is a solution of $I'(x) = 0, x$ is a weak solution of BVP (1.2).

3 Existence and multiplicity of solutions of BVP (1.2)

Definition 3.1 Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$. If any sequence $\{x_n\} \subset E$ for which $(J(x_n))$ is bounded and $J'(x_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence, then we say that $J$ satisfies the Palais–Smale condition (P.S. condition for short).

Lemma 3.1 ([24]) Let $E$ be a real reflexive Banach space. If the functional $J : E \to \mathbb{R}$ is weakly lower semicontinuous and coercive, i.e., $\lim_{\|x\| \to \infty} J(x) = +\infty$, then there exists $x_* \in E$ such that $J(x_*) = \inf_{x \in E} J(x)$. Moreover, if $J$ is also Fréchet differentiable on $E$, then $J'(x_*) = 0$.

Lemma 3.2 ([25]) Mountain pass theorem Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the P.S. condition. Suppose that

1. $J(0) = 0$.
2. There exist $\rho > 0$ and $\sigma > 0$ such that $J(x_0) \geq \sigma$ for all $x_0 \in E$ with $\|x_0\| = \rho$.
3. There exists $x_1 \in E$ with $\|x_1\| \geq \rho$ such that $J(x_1) < \sigma$.

Then $z = \inf_{h \in \Delta} \max_{t \in [0,1]} J(h(t)) \geq \sigma$ is a critical value of $J$, where

$$
\Delta = \{ h \in C([0,1], E) \mid h(0) = x_0, h(1) = x_1 \}.
$$

Lemma 3.3 ([26]) Let $E$ be a real reflexive Banach space, $\Phi : E \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $E^*$, $\Psi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that

$$
\inf_{x \in E} \Phi(x) = \Phi(0) = \Psi(0) = 0.
$$

Assume that there exist $r > 0$ and $\omega \in E$, with $r < \Phi(\omega)$ such that
First we prove that

\[ \text{sup} (|x| : \Phi (x) \leq r) < \frac{r}{\Phi_i (x)} \]

(2) for all \( \lambda \in (\Phi_i (x), \sup (|x| : \Phi (x) \leq r)) \), the functional \( \Phi - \lambda \Phi \) is coercive.

Then, for each \( \lambda \in (\Phi_i (x), \sup (|x| : \Phi (x) \leq r)) \), the functional \( \Phi - \lambda \Phi \) has at least three distinct critical points in \( E \).

Lemma 3.4 Suppose that \((A_0)\) and \((A_1 - A_2)\) hold, then \( I \) satisfies the P.S. condition, i.e.,

for every sequence \( \{x_n\} \in E_0^{\alpha, p} \), \( \{x_n\} \) has a convergent subsequence if

\[ \{ I(x_n) \} \text{ is bounded and } I'(x_n) \to 0 \text{ as } n \to \infty. \]  

(3.1)

Proof First we prove that \( \{x_n\} \) is a bounded sequence in \( E_0^{\alpha, p} \).

Without loss of generality, in the next proof of the theorem, the sequence \( \{x_n\} \) and its subsequences are all denoted as \( x_n \).

Suppose that \( \{x_n\} \) is unbounded. Passing to a subsequence, we may assume that \( \|x_n\| \to \infty \). Let \( \alpha_n = \frac{x_n}{\|x_n\|} \), so that \( \|\alpha_n\| = 1 \). By Lemma 2.2, also passing to a subsequence, we can suppose that

\[ \alpha_n \to \alpha_0 \text{ weakly in } E_0^{\alpha, p}, \]

\[ \alpha_n \to \alpha_0 \text{ strongly in } C([0, T], \mathbb{R}) \]

as \( n \to \infty \). Moreover, we have

\[ \bar{\sigma}_n = \frac{1}{T} \int_0^T \sigma_n(t) \, dt \to \frac{1}{T} \int_0^T \sigma_0(t) \, dt = \bar{\sigma}_0 \]  

(3.2)

as \( n \to \infty \). By (3.1) there exists a constant \( K_1 > 0 \) such that

\[ \left( \frac{\partial}{\partial t} - 1 \right) \|x_n\|^p = \partial I(x_n) - \partial' (x_n) x_n + \lambda \int_0^T \left( \partial F(t, x_n(t)) - f(t, x_n(t)) x_n(t) \right) \, dt \]

\[ \leq K_1 (1 + \|x_n\|) + \lambda \int_0^T \left( \partial F(t, x_n(t)) - f(t, x_n(t)) x_n(t) \right) \, dt. \]

Notice that \( \|x_n\| \to \infty \), we have

\[ \left( \frac{\partial}{\partial t} - 1 \right) \|\sigma_n\|^p \leq K_1 (1 + \|x_n\|) + \lambda \int_0^T \frac{\partial F(t, x_n(t)) - f(t, x_n(t)) x_n(t)}{\|x_n\|^p} \, dt. \]  

(3.3)

In view of \((A_0)\) and \((A_2)\), there exists \( \Omega_0 \subset [0, T] \) with \( \text{meas}(\Omega_0) = 0 \) such that

\[ |F(t, x)| \leq k_1 (|x|) k_2 (t), \quad |f(t, x)| \leq k_1 (|x|) k_2 (t) \]  

(3.4)

for all \( x \in \mathbb{R} \) and \( t \in [0, T] \setminus \Omega_0 \) and

\[ \lim_{|x| \to \infty} \frac{\partial F(t, x) - f(t, x)x}{|x|^p} \leq 0 \]

uniformly for \( t \in [0, T] \setminus \Omega_0 \). This yields

\[ \lim_{n \to \infty} \frac{\partial F(t, x_n(t)) - f(t, x_n(t)) x_n(t)}{\|x_n\|^p} \leq 0 \]  

(3.5)
for \( t \in [0, T] \setminus \Omega_0 \). Otherwise, there exist \( t_0 \in [0, T] \setminus \Omega_0 \) and a subsequence of \( \{x_n\} \) such that

\[
\limsup_{n \to \infty} \frac{\partial F(t_0, x_n(t_0)) - f(t_0, x_n(t_0))x_n(t_0)}{\|x_n\|^p} > 0. \tag{3.6}
\]

If \( \{x_n(t_0)\} \) is bounded, then there exists a positive constant \( K_2 \) such that \( |x_n(t_0)| \leq K_2 \) for all \( n \in \mathbb{N} \). By (3.4) we find

\[
\frac{\partial F(t_0, x_n(t_0)) - f(t_0, x_n(t_0))x_n(t_0)}{\|x_n\|^p} \leq \frac{(\theta + K_2) \max_{s \in [0,K_2]} k_1(s)k_2(t_0)}{\|x_n\|^p} \to 0
\]

as \( n \to \infty \), which contradicts (3.6). So, there is a subsequence of \( \{x_n(t_0)\} \) such that \( |x_n(t_0)| \to \infty \) as \( n \to \infty \).

This contradicts (3.6). Thus, (3.5) holds. From (3.3) and (3.5) we obtain

\[
\limsup_{n \to \infty} \left( \frac{\partial}{p} - 1 \right) \|\sigma_n\|^p \leq 0.
\]

Since \( \partial > p \), we get

\[
\|\sigma_n\|^p \to 0 \quad \text{as} \quad n \to \infty, \tag{3.7}
\]

but \( \|\sigma_n\| = 1 \). This is a contradiction. Hence, \( \{x_n\} \) is a bounded sequence in \( E_0^{\alpha,p} \).

Since \( E_0^{\alpha,p} \) is a reflexive space, there exists a weakly convergent subsequence such that \( x_n \rightharpoonup x \) in \( E_0^{\alpha,p} \). Hence, we have

\[
\langle I'(x_n) - I'(x), x_n - x \rangle \\
= \langle I'(x_n), x_n - x \rangle - \langle I'(x), x_n - x \rangle \\
\leq \|I'(x_n)\|_p \|x_n - x\| - \|I'(x), x_n - x\| \\
\to 0 \tag{3.8}
\]

as \( n \to \infty \). Combining Lemma 2.2 with (2.3), we obtain that \( \{x_n\} \) converges to \( x \) strongly in \( C([0,T], \mathbb{R}) \), i.e., \( \|x_n - x\| \to 0 \) as \( n \to \infty \), which implies that

\[
\int_0^T (f(t, x_n(t)) - f(t, x(t)))(x_n(t) - x(t)) \, dt \to 0 \tag{3.9}
\]

as \( n \to \infty \). Note that

\[
\langle I'(x_n) - I'(x), x_n - x \rangle
\]
\[
I_n := \int_0^T \left( \phi_p \left( D_t^n x(t) \right) - \phi_p \left( D_t^n y(t) \right) \right) \left( D_t^n x_n(t) - D_t^n y(t) \right) \, dt
\]

Thus, noting that (3.8) and (3.9) we have

\[
\int_0^T \left( \phi_p \left( D_t^n x(t) \right) - \phi_p \left( D_t^n y(t) \right) \right) \left( D_t^n x_n(t) - D_t^n y(t) \right) \, dt \to 0
\]

(3.10)

as \( n \to \infty \). For any \( s_1, s_2 \in \mathbb{R} \), it is well known that (see Lemma 4.2 in [27]) there exists \( a_1 > 0 \) such that

\[
|s_1|^{p-2}s_1 - |s_2|^{p-2}s_2, s_1 - s_2 | \geq \begin{cases} 
\alpha_1 |s_1 - s_2|^p, & p \geq 2, \\
\alpha_1 |s_1 - s_2|^p, & 1 < p \leq 2.
\end{cases}
\]

(3.11)

Following (3.11), we obtain that there exist \( c_1, c_2 > 0 \) such that

\[
\int_0^T \left( \phi_p \left( D_t^n x(t) \right) - \phi_p \left( D_t^n y(t) \right) \right) \left( D_t^n x_n(t) - D_t^n y(t) \right) \, dt
\]

\[
\geq \begin{cases} 
c_1 \int_0^T |D_t^n x(t) - D_t^n y(t)|^p \, dt, & p \geq 2, \\
c_2 \int_0^T \frac{|D_t^n x_n(t) - y(t)|^p}{(1 + |D_t^n x(t)|)^{p-2}} \, dt, & 1 < p < 2.
\end{cases}
\]

(3.12)

When \( 1 < p < 2 \), one has

\[
\int_0^T |D_t^n x(t) - D_t^n y(t)|^p \, dt
\]

\[
\leq \left( \int_0^T \frac{|D_t^n x(t) - D_t^n y(t)|^2}{(1 + |D_t^n x(t)|)^{p-2}} \, dt \right)^{\frac{p}{2}} \left( \int_0^T \left( |D_t^n x(t)| + |D_t^n y(t)| \right)^p \, dt \right)^{\frac{p-2}{2}}
\]

Thus, noting that \((b_1 + b_2)^\gamma \leq 2^{\gamma-1}(b_1^\gamma + b_2^\gamma)\), where \( b_1, b_2 \geq 0, \gamma \geq 1 \) (see [28]), we have

\[
\int_0^T \left( |D_t^n x(t)| + |D_t^n y(t)| \right)^p \, dt
\]

\[
\leq c_3 \left( \alpha_1 |s_1 - s_2|^p \right) \left( \int_0^T \frac{|D_t^n x(t) - D_t^n y(t)|^2}{(1 + |D_t^n x(t)|)^{2-p}} \, dt \right)^{\frac{2-p}{2}},
\]

where \( c_3 = 2^{(\gamma-1)(\gamma-2)} \), which, together with (3.12), implies

\[
\int_0^T \left( \phi_p \left( D_t^n x(t) \right) - \phi_p \left( D_t^n y(t) \right) \right) \left( D_t^n x_n(t) - D_t^n y(t) \right) \, dt
\]

\[
\geq c_2 c_3^{-\frac{\gamma}{2}} \left( \alpha_1 |s_1 - s_2|^p \right) \left( \int_0^T \frac{|D_t^n x(t) - D_t^n y(t)|^2}{(1 + |D_t^n x(t)|)^{2-p}} \, dt \right)^{\frac{p-2}{2}},
\]

(3.13)

When \( p \geq 2 \), by (3.12) we get

\[
\int_0^T \left( \phi_p \left( D_t^n x(t) \right) - \phi_p \left( D_t^n y(t) \right) \right) \left( D_t^n x_n(t) - D_t^n y(t) \right) \, dt
\]
\[ \geq c_1 \|x_n - x\|^p, \quad p \geq 2. \quad (3.14) \]

It follows from (3.10), (3.13), and (3.14) that

\[ \|x_n - x\| \to 0 \]

as \( n \to \infty \), that is, \( \{x_n\} \) converges strongly to \( x \) in \( L^{\alpha,p}_0 \). The proof is completed. \( \square \)

**Theorem 3.1** Assume that \((A_0)\) and one of the following conditions hold, then BVP (1.2) admits at least one nontrivial solution.

(1) \((A_1)\) holds and \( \lambda \in (0, \infty) \).

(2) \((A_2)\) holds and \( \lambda \in (0, \frac{1}{\mu G^\theta \int_0^T \eta_2(t) \, dt}) \).

**Proof** We shall prove this theorem by Lemma 3.1.

Notice that

\[ \liminf_{n \to \infty} \|x_n\| \geq \|x\| \]

Let \( \{x_n\} \) converge weakly to \( x \) in \( L^{\alpha,p}_0 \), then \( x_n \) converges uniformly to \( x \) in \([0, T]\). Therefore

\[ \liminf_{n \to \infty} I(x_n) = \liminf_{n \to \infty} \left\{ \frac{1}{p} \|x_n\|^p - \lambda \int_0^T F(t, x_n(t)) \, dt \right\} \]

\[ \geq \frac{1}{p} \|x\|^p - \lambda \int_0^T F(t, x(t)) \, dt \]

\[ = I(x), \]

which means that \( I(x) \) is weakly lower semicontinuous.

If \((A_1)\) holds, it follows from (1.3) and (2.5) that

\[ I(x) = \frac{1}{p} \|x\|^p - \lambda \int_0^T F(t, x(t)) \, dt \]

\[ \geq \frac{1}{p} \|x\|^p - \lambda G^\theta \|x\|^{\theta} \int_0^T \eta_2(t) \, dt - \lambda L_1 T. \]

Because of \( 0 \leq \theta < p \), we obtain that \( I(x) \to \infty \) as \( \|x\| \to \infty \), which indicates that \( I \) is coercive. Therefore, by Lemma 3.1, there is \( x_0 \in L^{\alpha,p}_0 \) such that \( I(x_0) = \inf_{x \in L^{\alpha,p}_0} I(x) \) and \( I'(x_0) = 0 \).

In the same way, if \((A_2)\) holds, according to (1.4) and (2.5), we have

\[ I(x) \geq \frac{1}{p} \|x\|^p - \lambda G^\theta \|x\|^{\theta} \int_0^T \eta_2(t) \, dt - \lambda L_2 T. \]

On account of \( 0 < \lambda < \frac{1}{\mu G^\theta \int_0^T \eta_2(t) \, dt} \), then \( I(x) \to \infty \) as \( \|x\| \to \infty \) is the natural result. The proof is finished. \( \square \)
Remark 3.1
(1) By (A₃), we know that, for any \(0 < \epsilon < \frac{1}{\sqrt{p + 1}}\), there exists \(\delta \in (0, \epsilon)\) such that
\[
|F(t, x)| \leq \epsilon |x|^p
\]
for a.e. \(t \in [0, T]\) and \(x \in \mathbb{R}\) with \(|x| \leq \delta\).

(2) By (A₃) and (A₆), there exist constants \(K_3 > 1, \varrho > \frac{(\vartheta - p)G}{2K_p}\) and a subset of \(\Omega\), still denoted by \(\Omega\), with \(|\Omega| = \text{meas}(\Omega) > 0\) such that
\[
\vartheta F(t, x) - f(t, x)x \leq \varrho |x|^p
\]
and
\[
F(t, x) > \frac{2\varrho}{\vartheta - p} |x|^p
\]
for all \(|x| \geq K_3\) and \(t \in \Omega\).

It follows from (A₆) that
\[
F(t, x) \leq \max_{s \in [0, K_3]} k_1(s)k_2(t)
\]
for all \(|x| \leq K_3\) and a.e. \(t \in [0, T]\).

Therefore, we obtain
\[
F(t, x) > \frac{2\varrho}{\vartheta - p} (|x|^p - K_3^p) - \max_{s \in [0, K_3]} k_1(s)k_2(t)
\]
for all \(|x| \in \mathbb{R}\) and \(t \in \Omega\).

Theorem 3.2 If (A₅) and (A₃ - A₆) hold, then BVP (1.2) admits at least one nontrivial weak solution for each \(\lambda \in (0, \frac{1}{\sqrt{p + 1}})\).

Proof Obviously, it follows from the definition of \(I\) that \(I(0) = 0\). Therefore, (1) of Lemma 3.2 is true.

Let
\[
0 < \rho = \frac{\delta}{G}\quad (\delta \text{ is defined by Remark 3.1}),
\]
then it follows from (2.3) that
\[
\|x\|_\infty \leq G\|x\| = G\rho = \delta
\]
for all \(x \in E_0^{\alpha, p}\) with \(\|x\| = \rho\).

Therefore, for all \(x \in E_0^{\alpha, p}\) with \(\|x\| = \rho = \frac{\delta}{G}\), from Remark 3.1, we have
\[
I(x) = \frac{1}{p} \|x\|^p - \lambda \int_0^T F(t, x(t)) \, dt
\]
\[ \geq \frac{1}{p} \|x\|^p - \lambda \epsilon \kappa \|x\|^p \]
\[ \geq \left( \frac{1}{p} - \lambda \epsilon \right) \frac{\partial p}{\partial p} \]
\[ = \sigma. \]

In view of \(0 < \lambda < \frac{1}{\kappa \epsilon \kappa} \), we obtain that \(\sigma > 0\). This implies that (2) in Lemma 3.2 is satisfied.

Choose \(x_0(t) \in L_0^{\alpha, p}\) with \(\|x_0\| \leq G\) and \(\int_\Omega |x_0(t)|^p \, dt = 1\).

And for \(\varsigma > 0\), we have
\[ I(\varsigma u_0) = \frac{1}{p} \|\varsigma x_0\|^p - \lambda \int_0^T F(t, \varsigma x_0(t)) \, dt \]
\[ \leq \frac{1}{p} \|\varsigma x_0\|^p - \lambda \int_0^T |x_0(t)|^p \, dt \]
\[ \leq \frac{\varsigma^p}{p} \|x_0\|^p - \frac{2 \lambda \varrho}{\vartheta - p} \int_0^T |x_0(t)|^p \, dt + \frac{2 \lambda \varrho |\Omega|}{\vartheta - p} K_3^p + \lambda \max_{\alpha \in [0, K_3]} k_1(s) \int_\Omega k_2(t) \, dt \]
\[ \leq \left( \frac{G^p}{p} - \frac{2 \lambda \varrho}{\vartheta - p} \right) \varsigma^p + \frac{2 \lambda \varrho |\Omega|}{\vartheta - p} K_3^p + \lambda \max_{\alpha \in [0, K_3]} k_1(s) \int_\Omega k_2(t) \, dt. \]

Notice that \(\varrho > \frac{(\vartheta - p)\kappa}{\vartheta - p}\), we can get that \(I(\varsigma x_0) \to -\infty\) as \(\varsigma \to \infty\). Hence (3) of Lemma 3.2 holds.

Finally, note that \(I(0) = 0\), while for critical point \(x\), \(I(x) \geq \sigma > 0\). So \(x\) is a nontrivial weak solution of BVP (1.2). The proof is completed. \(\square\)

We then use Lemma 3.3 to show the multiplicity of solutions to BVP (1.2).

**Theorem 3.3** Assume that \((A_0)\) and \((A_1)\) hold. Suppose also that there exist \(r > 0\) and \(\omega \in L_0^{\alpha, p}\) such that \(\frac{1}{p} \|\omega\|^p > r\) and
\[ \lambda_l = \frac{1}{r} \int_0^T |D_\omega^p \omega(t)|^p \, dt < \frac{r}{\int_0^T \max_{x \in L_0^{\alpha, p}} F(t, x) \, dt} = \lambda_r. \]

Then, for \(\lambda \in (\lambda_l, \lambda_r)\), BVP (1.2) admits at least three weak solutions, which are critical points of the functional \(I : L_0^{\alpha, p} \to \mathbb{R}\).

**Proof** As far as we know, \(L_0^{\alpha, p}\) is a real reflexive Banach space. We consider the functionals
\( \Phi : L_0^{\alpha, p} \to \mathbb{R} \) and \( \Psi : L_0^{\alpha, p} \to \mathbb{R} \) defined, respectively, by
\[ \Phi(x) = \frac{1}{p} \int_0^T |D_\alpha^p x(t)|^p \, dt \]
and
\[ \Psi(x) = \int_0^T F(t, x(t)) \, dt. \]
According to the condition of \((A_0)\), we can know that \(\Phi, \Psi\) are well defined and Gâteaux differentiable functionals whose Gâteaux derivatives at the point \(x \in E_0^{α, p}\) are the functionals \(\Phi'(x), \Psi'(x) \in (E_0^{α, p})^*\), given by

\[
\Phi'(x)y = \int_0^T \phi_p(\partial D_α^x x(t)) \partial D_α^x y(t) \, dt
\]

and

\[
\Psi'(x)y = \int_0^T f(t, x(t)) y(t) \, dt
\]

for every \(y \in E_0^{α, p}\). For \(\{x_n\}, x \in E_0^{α, p}, x_n \to x\) in \(E_0^{α, p}\) as \(n \to \infty\), by Lemma 2.2, we get that \(\{x_n\}\) converges uniformly to \(x\) in \(C([0, T], \mathbb{R})\). Hence

\[
\limsup_{n \to \infty} \Psi(x_n) \leq \int_0^T \limsup_{n \to \infty} F(t, x_n) \, dt
\]

\[
= \int_0^T F(t, x) \, dt
\]

\[
= \Psi(x),
\]

which implies that \(\Psi\) is sequentially weakly upper semicontinuous.

By \((A_0)\), one can get \(F(t, x_n) \to F(t, x)\) as \(n \to \infty\). By the Lebesgue control convergence theorem, \(\Psi'(x_n) \to \Psi'(x)\) strongly, which implies that \(\Psi' : L_0^{α, p} \to L_0^{α, p}\) is strongly continuous on \(L_0^{α, p}\), that is, \(\Psi'\) is a compact operator.

From \((A_0)\), it is standard to see that \(\Phi : E_0^{α, p} \to \mathbb{R}\) is sequentially weakly lower semicontinuous, coercive, and its derivative \(\Phi'\) admits a continuous inverse on \((E_0^{α, p})^*\).

Moreover, it is clear that \(\inf_{x \in E_0^{α, p}} \Phi(x) = \Phi(0) = \Psi(0) = 0\).

Put \(r > 0\), for \(x \in E_0^{α, p}\) with \(\Phi(x) = \frac{1}{p}\|x\|^p \leq r\), then \(\|x\|^p \leq Gp\|x\|^p \leq pGpr\), we can get

\[
\{x \in E_0^{α, p} : \Phi(x) \leq r\} \subseteq \left\{x : \frac{1}{pGp}\|x\|^p \leq \frac{1}{r}\right\} \\
= \left\{x : \max_{t \in [0, T]} |x(t)| \leq G\sqrt[p]{pr}\right\}.
\]

Hence, using \(\lambda < \lambda_r\), we have

\[
\sup_{\Phi(x) \leq r} \Psi(x) = \sup_{\Phi(x) \leq r} \int_0^T F(t, x(t)) \, dt
\]

\[
\leq \int_0^T \max_{|x| \leq G\sqrt[p]{pr}} F(t, x) \, dt
\]

\[
< \frac{r}{\lambda}.
\]

On the other hand, by \(\lambda > \lambda_l\), we get

\[
\Psi(\omega) = \int_0^T F(t, \omega(t)) \, dt > \frac{\Phi(\omega)}{\lambda}.
\]
Thus
\[
\frac{\Psi(\omega)}{\Phi(\omega)} > \frac{1}{\lambda} > \sup_{r} \frac{\Psi(x)}{\Phi(x)} \leq r
\]
and (1) of Lemma 3.3 holds.

Then, let us verify that, for \( \lambda \in (\lambda_l, \lambda_r) \), \( \Phi - \lambda \Psi \) is coercive. For \( x \in L_0^{a,p} \), we obtain
\[
\Phi(x) - \lambda \Psi(x) = \frac{1}{p} \int_0^T \left| \frac{d}{dt} x(t) \right|^p dt - \lambda \int_0^T F(t, x(t)) dt \\
\geq \frac{1}{p} \|x\|^p - \lambda \left( \int_0^T \eta_1(t) |x|^p dt + \int_0^T L_1 dt \right) \\
\geq \frac{1}{p} \|x\|^p - \lambda G \|x\|^\theta \int_0^T \eta_1(t) dt - \lambda L_1 T.
\]
Notice that \( 0 < \theta < p \), so \( \Phi(x) - \lambda \Psi(x) \to +\infty \) as \( \|x\| \to +\infty \), that is, \( \Phi - \lambda \Psi \) is coercive. Obviously, (2) in Lemma 3.3 holds.

Then, by Lemma 3.3, we claim that, for each \( \lambda \in (\lambda_l, \lambda_r) \), \( I : L_0^{a,p} \to \mathbb{R} \) given by \( I = \Phi - \lambda \Psi \) has at least three distinct critical points in \( L_0^{a,p} \), that is, BVP (1.2) has at least three solutions. \( \square \)

Acknowledgements
The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions.

Funding
The article is supported by the National Natural Science Foundation of China (Grant No.: 11572148, 11872201).

Availability of data and materials
This paper focuses on theoretical analysis, not involving experiments and data.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 February 2020 Accepted: 19 March 2020 Published online: 30 March 2020

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