Weighted 2-Motzkin Paths

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Abstract. This paper is motivated by two problems recently proposed by Coker on combinatorial identities related to the Narayana polynomials and the Catalan numbers. We find that a bijection of Chen, Deutsch and Elizalde can be used to provide combinatorial interpretations of the identities of Coker when it is applied to weighted plane trees. For the sake of presentation of our combinatorial correspondences, we provide a description of the bijection of Chen, Deutsch and Elizalde in a slightly different manner in the form of a direct construction from plane trees to 2-Motzkin paths without the intermediate step involving the Dyck paths.

AMS Classification: 05A15, 05A19

Keywords: Plane tree, Narayana number, Catalan number, 2-Motzkin path, weighted 2-Motzkin path, multiple Dyck path, bijection.

Suggested Running Title: Weighted 2-Motzkin Paths

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1. Introduction

The structure of 2-Motzkin paths, introduced by Barcucci, Lungo, Pergola and Pinzani [1], has proved to be highly efficient in the study of plane trees, Dyck paths, Motzkin paths, noncrossing partitions, RNA secondary structures, Devenport-Schinzel sequences, and combinatorial identities, see [6, 8, 12]. While it is most natural to establish a correspondence between Dyck paths of length $2n$ and 2-Motzkin paths of length $n - 1$, Deutsch and Shapiro came to the realization that direct correspondences between plane trees and 2-Motzkin paths can have many applications. Recently, Chen, Deutsch and Elizalde [2] found bijections between plane trees and 2-Motzkin paths for the enumeration of plane trees by the numbers of old and young leaves [2]. The main result of this paper is to show that the bijection of Chen, Deutsch and Elizalde, presented in a slightly different manner, can be applied to weighted plane trees in order to give combinatorial interpretations of two identities involving the Narayana numbers and Catalan numbers due to Coker [4]. This leads to the solutions of the two open problems left in the paper [4].

Recall that a 2-Motzkin path is a lattice path starting at $(0,0)$ and ending at $(n,0)$ but never going below the $x$-axis, with possible steps $(1,1), (1,0)$ and $(1,-1)$, where the level steps $(1,0)$ can be either of two kinds: straight and wavy. The length of the path is defined to the number of its steps. Deutsch and Shapiro [6] presented a bijection between plane trees with $n$ edges and 2-Motzkin paths of length $n - 1$. So the number of 2-Motzkin paths of length $n - 1$
equals the Catalan number
\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]

Recently, Coker [4] established very interesting combinatorial identities ((1.1) and (1.2) below) by using generating functions and the Lagrange inversion formula based the study of multiple Dyck paths. A multiple Dyck path is a lattice path starting at \((0,0)\) and ending at \((2n,0)\) with big steps that can be regarded as segments of consecutive up steps or consecutive down steps in an ordinary Dyck path. Note that the notion of multiple Dyck path is formulated by Coker in different coordinates. The main ingredients in Coker’s identities are the Catalan number and the Narayana numbers:

\[ N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}, \]

which counts the number of all plane trees with \(n\) edges and \(k\) leaves [12, 13]. It is sequence A001263 in [10]. Coker [4] left the following two open problems:

**Problem 1.** Find a bijective proof of the following identity

\[ \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} 4^{n-k} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} 4^k 5^{n-2k-1}. \] (1.1)

**Problem 2.** Find a combinatorial explanation for the following identity

\[ \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{2k}(1+x)^{2n-2k} = x^2 \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (1+x)^k, \] (1.2)

which is equation (6.2) in [4]. The above identity (1.1) is a special case of the following identity:

\[ \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} t^{n-k} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} t^k (1+t)^{n-2k-1}, \] (1.3)

where the left hand side of (1.3) is the Narayana polynomial, as denoted by \(N_n(t)\) in [4]. The identity (1.3) is the relation (4.4) in [4], which can be derived as an identity on the Narayana numbers and the Catalan numbers due to Simion and Ullman [9], see also [3]. Remarkably, (1.3) has many consequences as pointed by Coker [4]. For example, it implies the classical identity of Touchard [14], and the formula on the little Schröder numbers in terms of the Catalan numbers [7]. The reason for the evaluation of \(N_n(t)\) at \(t = 4\) lies in the fact that \(N_n(4)\) equals the number \(d_n\) of multiple Dyck paths of length \(2n\). The first few values of \(d_n\) for \(n = 0, 1, 2, 3, 4, 5, 6, 7\) are

1, 1, 5, 29, 185, 1257, 8925, 65445,

which is the sequence A059231 in [10]. From the interpretation of Narayana numbers in terms of Dyck paths of length \(2n\) and of \(k\) peaks, it is not difficult to show that \(d_n = N_n(4)\). However, the right hand side of (1.1) does not seems to be obvious, which is obtained by establishing a functional equation and by using the Lagrange inversion formula. The natural question as raised by Coker [4] is to find a combinatorial interpretation of (1.1). Note that the enumeration of multiple Dyck paths has also been studied independently by Sulanke [11] and Woan [15].
The relation (1.2) was established from the enumeration of multiple Dyck paths of length $2n$ with a given number of steps. Let $\lambda_{n,j}$ be the number of multiple Dyck paths of length $2n$ and $j$ steps, $P_n(x)$ be the polynomial

$$P_n(x) = \sum_{j=2}^{2n} \lambda_{n,j} x^j.$$  

It was shown that

$$P_n(x) = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} \left( \binom{n}{k-1} x^{2k} (1 + x)^{2n-2k} \right),$$  

which can be restated as

$$P_n(x) = x^{2n} N_n((1 + x^{-1})^2).$$  

Coker [4] discovered the connection between $P_n(x)$ and the polynomial $R_n(x)$ introduced by Denise and Simion [5] in their study of the number of exterior pairs of Dyck paths of length $2n$. The polynomials $R_n(x)$ have the following expansion:

$$R_n(x) = \sum_{k=0}^{n-1} (-1)^k C_{k+1} \binom{n-1}{k} x^k (1 - x)^k.$$  

It now becomes clear that the identity (1.2) can be rewritten as

$$P_n(x) = x^2 R_n(-x).$$  

To give combinatorial interpretations of both (1.1) and (1.2), we apply the bijection of Chen, Deutsch and Elizalde [2] to weighted plane trees to get weighted 2-Motzkin paths. Then we use weight-preserving operations on 2-Motzkin paths to derive the desired combinatorial identities. More precisely, these weight-preserving operations are essentially the reductions from weighted 2-Motzkin paths to Dyck paths and 2-Motzkin paths. It would be interesting to find a direct correspondence on Dyck paths which leads to a combinatorial interpretation of (1.7).

2. Weighted 2-Motzkin Paths

Let us review a bijection between plane trees and 2-Motzkin paths due to Chen, Deutsch and Elizalde [2], which is devised for the enumeration of plane trees with $n$ edges and a fixed number of old leaves and a fixed number of young leaves. Such a consideration of old and young leaves reflects to the four types of steps of 2-Motzkin paths. For the purpose of this paper, we present a slightly modified version of the nonrecursive bijection in [2]. Our terminology is also somewhat different.

For a plane tree $T$, a vertex of $v$ is called a leaf if it does not have any children. An internal vertex is a vertex that has at least one child. An edge is denoted as a pair $(u, v)$ of vertices such that $v$ is a child of $u$. Let $u$ be an internal vertex, and $v_1, v_2, \ldots, v_k$ be the children of $u$ listed from left to right. Then we call $v_k$ an exterior vertex and $(u, v_k)$ an exterior edge. If $k > 1$, then the edges $(u, v_1), (u, v_2) \ldots, (u, v_{k-1})$ are called interior edges and $v_1, v_2, \ldots, v_{k-1}$ are called interior vertices. An edge containing a leaf vertex is called a terminal edge. Let $u$ be the root of $T$, $(u, u_1)$ be the exterior edge of $u$, $(u_1, u_2)$ be the exterior edge of $u_1$, and so on, finally $(u_{k-1}, u_k)$ be the exterior edge of $u_{k-1}$ such that $u_k$ is a leaf. The exterior edge $(u_{k-1}, u_k)$ is called the critical edge of $T$. To summarize, the edges of a plane tree $T$ are classified into five categories.
• Non-terminal interior edges.
• Non-terminal exterior edges.
• Terminal interior edges.
• Terminal exterior edges (which do not include the critical edge).
• The critical edge.

Note that the critical edge of $T$ is the last encountered edge when we traverse the edges of $T$ in preorder. From the above classification on the edges of a plane tree, it is easy to describe the Chen-Deutsch-Elizalde bijection between plane trees and 2-Motzkin paths by the preorder traversal of the edges of $T$. To be precise, let $u$ be the root of $T$, $v_1, v_2, \ldots, v_k$ be the children of $u$, and $T_1, T_2, \ldots, T_k$ be the subtrees of $T$ rooted at $v_1, v_2, \ldots, v_k$, respectively. Then the preorder traversal of the edges of $T$, denoted by $P(T)$, is a linear order of the edges of $T$ recursively defined by

$$ (u, v_1)P(T_1)(u, v_2)P(T_2)\cdots(u,v_k)P(T_k). $$

The Bijection of Chen, Deutsch and Elizalde [2]: Let $T$ be any nonempty plane tree. At each step of the traversal of the edges of $T$ in preorder,

(i) draw an up step for a non-terminal interior edge;
(ii) draw a straight level step for a non-terminal exterior edge;
(iii) draw a wavy level step for a terminal interior edge;
(iv) draw a down step for a terminal exterior edge;
(v) do nothing for the critical edge.

![Figure 1: The Bijection $\Phi$](image)

It is easy to see that we have obtained a 2-Motzkin path. More precisely, a plane tree with $n$ edges corresponds to a 2-Motzkin path of length $n - 1$. The above bijection is denoted by $\Phi$. As a hint to why the above bijection works, one may check that for any plane tree $T$,

$$ \# \text{non-terminal interior edges} = \# \text{terminal exterior edges}. $$

We are now ready to assign weights to the edges of a plane tree $T$ in order to obtain combinatorial interpretations of the identities (1.1) and (1.2). The weights of the edges of a plane tree will translate into weights of steps of the corresponding 2-Motzkin path. In fact, we will take slightly different formulations of (1.3) and (1.2).
Theorem 2.1 For $n \geq 1$, we have
\[
\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{k-1} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1 + x)^{n-2k-1}.
\] (2.1)

Proof. Let $T$ be a plane tree with $n$ edges. We assign the weights to the edges of $T$ by the following rule: All the terminal edges except the critical edge are given the weight $x$ and all other edges are given the weight 1. The weight of $T$ is the product of the weights of its edges. Then the left hand side of (2.1) is the sum of the weights of all plane trees with $n$ edges.

By the above bijection $\Phi$, the set of weighted plane trees with $n$ edges is mapped to the set of 2-Motzkin paths of length $n - 1$ in which all the down steps and wavy level steps are given the weight $x$, and other steps are given the weight 1. Consider the weighted 2-Motzkin paths of length $n - 1$ that have $k$ up steps and $k$ down steps. These $k$ up steps and $k$ down steps form a Dyck path of length $2k$. The binomial coefficient $\binom{n-1}{2k}$ comes from the choices of the $2k$ positions for these up steps and $k$ down steps. The remaining $n - 2k - 1$ steps are either wavy level steps or straight level steps. Since only a wavy level step carries the weight $x$, the total contributions of the weights of $n - 2k - 1$ level steps amount to $(1 + x)^{n-2k-1}$. The $k$ up steps would contribute $x^k$. Therefore, the right hand side of (2.1) equals the total contributions of all the weighted 2-Motzkin paths of length $n - 1$, as desired.

Theorem 2.2 For $n \geq 1$, we have
\[
\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{2(k-1)} (1 + x)^{2(n-k)} = \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (1 + x)^k.
\] (2.2)

Proof. Given a plane tree $T$ with $n$ edges, we assign the weights to the edges of $T$ by the following rule: All the terminal edges except the critical edge are assigned the weight $x^2$, all the non-terminal edges are given the weight $(1 + x)^2$, and the critical edge is assigned the weight 1. Then the bijection $\Phi$ transform $T$ into a 2-Motzkin path in which all the down steps and wavy level steps have the weight $x^2$ and all the up steps and straight level steps have the weight $(1 + x)^2$.

By the above weight assignment, the left hand side of (2.2) is then the sum of the weights of all plane trees with $n$ edges. We now proceed to show that the right hand side of (2.2) is the sum of weights of all 2-Motzkin paths of length $n - 1$. Consider a 2-Motzkin path that has $k$ up steps and $k$ down steps. Since the up steps have weight $x^2$ and the down steps have weight $(1 + x)^2$, it makes no difference with respect to the sum of weights if one changes the weights of both up steps and down steps to $x(1 + x)$. In other words, such an operation on the change of weights is a weight-preserving bijection on the set of 2-Motzkin paths.

Note that for any weight assignment, we may transform the sum of weights of all 2-Motzkin paths of length $n - 1$ to the sum of weights of all Motzkin paths of the same length by the following weight assignment: the up steps and down steps carry the same weight, and the horizontal steps in the Motzkin paths carry the weight as the sum of the weights of a straight level step and a wavy level step in the 2-Motzkin path. Therefore, for our weight assignment the sum of weights of 2-Motzkin paths of length $n - 1$ equals the sum of Motzkin paths of length $n - 1$ given the following weight assignment: up steps and down steps are given the weight $x(1 + x)$, and the horizontal steps are given the weight $x^2 + (1 + x)^2 = 1 + x(1 + x) + x(1 + x)$.
So we have transformed the sum of weights of 2-Motzkin paths of length $n - 1$ to the sum of weights of all Motzkin paths of length $n - 1$ in which all the up steps, down steps have weights $x(1 + x)$, and the horizontal steps can be regarded as either a straight level step with weight $x(1 + x)$, or a wavy level step with weight $x(1 + x)$ or a special dotted step with weight 1.

We now get the desired sum as on the right hand side of (2.2) by considering the distribution of the special dotted steps, because the remaining steps (up, down, straight level, wavy level) all have the weight $x(1 + x)$ and they form a 2-Motzkin path.

Setting $x = 1/4$ in (2.1) we obtain (1.1).

Acknowledgments. This work was done under the auspices of the National Science Foundation, the Ministry of Education, and the Ministry of Science and Technology of China.

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