Reevaluating the performance of the Double Exponential Smoothing filter and its Control Parameters

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Abstract—Double Exponential Smoothing (DES) has broad application in various fields primarily as a forecasting tool. The values of the two smoothing parameters $\alpha$ and $\beta$, involved in DES, are traditionally chosen by the users which yield minimum MSE. In this work the authors endeavor to assess the performance of the DES as a filter and tried to suggest the suitable values of the $\alpha$ and $\beta$ for which DES perform best as a filter. In this regard along with the conventional MSE method, the dependency of the stability and other aspects associated with the frequency response of the filter like transfer function, cutoff frequency, bandwidth and center frequency on the smoothing parameters are also studied. The values of the parameters close to 0.5 are found to be most appropriate when DES acts as a filter.

Index Terms—Double Exponential Smoothing, smoothing parameters, mean square error (MSE), Infinite Impulse Response (IIR), Finite Impulse Response (FIR)

I. INTRODUCTION

In the various field of research different time series filters are used to reduce the effects of data irregularities, trends, cyclicity. [1] As all these filters are the linear combination of the previous and current data, are casual filters. The sequence length of the coefficients of these data may be finite or infinite and according they are categorized as Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) respectively. [2] One such widespread filter is Simple Exponential Smoothing (SES). [3] SES is good for the stationary signals where it effectively smooths the data irregularities or outliers. [4] If the moving averages of a time series signal remain constant, the signal is generally judged as stationary. The presence of any trend may drift the mean averages and hence such signals are non-stationary. Since most of the real world signals are nonstationary, SES is not a fruitful tool for the majority of the applications. To address this issue, Charles Holt has modified the SES to Holt’s Exponential Smoothing or Double Exponential Smoothing (DES) or Second-Order Exponential Smoothing which can handle these linear trends. [5] [6] The DES is a bit complicated than SES and involves two equations

\[ y_i = \alpha x_i + (1 - \alpha)(y_{i-1} + b_{i-1}) \quad 0 < \alpha < 1 \]  

\[ b_i = \beta(y_i - y_{i-1}) + (1 - \beta)b_{i-1} \quad 0 < \beta < 1 \]

Where $y_i$ is the output and $x_i$ is the input at the $i^{th}$ time instant. [7] [8]

It is evident from these equations, the DES is cascaded with two feedback loops and hence it may be treated as Recursive System with Infinite Impulse Response, i.e., IIR system or filter. Upon meticulously thinking it will be appropriate to say that DES is essentially the IIR approximation of FIR linear least square filter.

In SES one control parameter $\alpha$ determines the smoothing ability of the filter whereas one more parameter namely $\beta$ is introduced in DES to take care of the trend. DES primarily involves two SES filters connected in series. However, the word "double" does not mean two SES; instead, it signifies that exponential filtering is operated both on the input time series and estimated trend in the input (i.e., derivative of the input). [9] The first filter, as in Eq. 1, which operates on the input yields the smoothed estimate of the value of the data at each time instant. The second filter, as in Eq. 2, generates the smoothed estimate of average growth, i.e., the trend at each time. Trend or incremental changes in the input is slowly varying and hence is of very low frequency. So a raw signal with trend includes three types of frequencies; high frequencies due to noise, low frequencies due to the trend and the signal frequency.

There are three types of time series filters; low-pass filters (LPF) to obstruct the high-frequency noises; high-pass filters (HPF) to block low frequencies and the band-pass filter (BPF) that permit frequencies within a particular user-defined band. As DES is meant to reduce the low-frequency trend effect and high-frequency noise effect, it ought to be both high pass and low pass filters simultaneously, i.e., bandpass filter. The two parameters $\alpha$ and $\beta$ control the attributes of the BPF like the lower and upper cutoff frequencies, bandwidth, center frequency. As $\beta$ is related to low-frequency trend, it will have more control on the HPF activity of the DES filter whereas $\alpha$ will have more command on the LPF performance. So there is a need to optimize these two control factors, $\alpha$ and $\beta$, to get the best combination yielding the most desirable output from the DES filter.

In this paper, an initiative has been taken to investigate the behavior of the DES for the different combination of the $\alpha$ and $\beta$. Many initiatives have already been taken by many researchers to estimate the optimized values of $\alpha$ and $\beta$, but most of them are by using the traditional error function MSE

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only. [10] [11] [12] Here in this work the behavioral aspects of
the DES filter which are investigated with respect to the
variation of α and β are its transfer function stability, cut-off
frequencies, center frequency, bandwidth along with the long-
established Minimum Square Error (MSE).

II. STABILITY OF THE DES FILTER

Transforming the Eq. 1 & 2 from the time domain to z-
domain the transfer function of the DES filter obtained is

\[ H(z) = \frac{Y(Z)}{X(Z)} = \frac{\alpha}{1 - (1 - \alpha)z^{-1} - (1 - \alpha)z^{-1} \beta Z^{-1}} \]

\[ = \frac{\alpha}{z^2 + (\alpha + \beta - 2)z + (1 - \alpha)} \]  \hspace{1cm} (3)

From Eq.3 the zeros of the DES obtained are 0 and \((1 - \beta)\) whereas its characteristics equation is

\[ Z^2 + Z(\alpha + \beta - 2) + (1 - \alpha) = 0 \] \hspace{1cm} (4)

The roots of the characteristic equation which are the poles of the DES filter are

\[ Z = \frac{-(\alpha + \beta - 2) \pm \sqrt{(\alpha + \beta - 2)^2 - 4(1 - \alpha)}}{2} \]

\[ = \frac{-\alpha \pm \frac{1}{2} \sqrt{4\alpha^2(1 + \beta)^2 - 4\alpha\beta}}{2} \]  \hspace{1cm} (5)

For a linear system to be stable, all of its poles must lie within
the unit circle. Now three cases may arise from Eq. (5).

Case I: \(\alpha^2(1 + \beta)^2 - 4\alpha\beta = 0\)

This case arises when \(\alpha = \frac{4\beta}{(1 + \beta)^2}\), considering \(\alpha \neq 0\).

Since \(0 < \alpha < 0.828\) (explained in Appendix), it can be
claimed that there will be mapped value of \(\beta\) for every value
of \(\alpha\) and those mapped values will be in the range of \(0 < \beta <
0.41\). Here only one real pole at \(Z = \frac{-(\alpha + \beta - 2)}{2}\) is available.

For the sake of stability, the magnitude of the pole must be
less than one i.e.

\[ \left|\frac{-(\alpha + \beta - 2)}{2}\right| \leq 1 \Rightarrow \alpha(1 + \beta) \leq 4 \] \hspace{1cm} (6)

Putting the range of \(0 < \alpha < 0.828\) and \(0 < \beta < 0.41\) in Eq.6 it is found to satisfy the condition of stability of the DES
expressed.

Case II: \(\alpha^2(1 + \beta)^2 - 4\alpha\beta > 0\)

Two real poles will now be obtained in this case. Here the
value of \(\alpha\) needs to be more than \(\frac{4\beta}{(1 + \beta)^2}\) i.e.

\[ \alpha > \frac{4\beta}{(1 + \beta)^2} \] [considering \(\alpha \neq 0\)] \hspace{1cm} (7)

As the maximum value of \(\alpha\) is less than 0.828, the value of
\(\beta\) will remain confined within the magnitude of 0.41. So any
combination of \(\alpha\) and \(\beta\) within the span of \(0 < \alpha < 0.828\)
and \(0 < \beta < 0.41\) respectively will generate two real poles
except at the mapped combinations discussed in Case I.

In both the case I and II, as the poles are real, the DES filter is
over-damped and hence may be little sluggish.

Case III: \(\alpha^2(1 + \beta)^2 - 4\alpha\beta < 0\)

Under this case, the DES will have two complex conjugate
poles and \(\alpha > \frac{4\beta}{(1 + \beta)^2}\) \hspace{1cm} [considering \(\alpha \neq 0\)] \hspace{1cm} (8)

So here it is noticed that \(\beta\) can go up even beyond the value
of 0.41, though the value of \(\alpha\) remains limited within 0.828.
Here also the stability of the DES filter will remain intact for
the entire range of \(0 < \alpha < 0.828\) and \(0 < \beta < 1\). For the
combination of lower values of \(\alpha\) (close to zero) and higher
values of \(\beta\) (close to unity), the poles will move outwards
towards the unit circle, and hence the degree of stability gets
reduced. So, it is recommendable to choose \(\beta\) close 0.41.

However, the DES filter under this situation will be in an
under-damped state which will make fast but at the cost of
initial oscillation. Effectively it can be said that within \(0 < \beta < 0.41\) the DES
filter may be overdamped or underdamped depending upon
the values of \(\alpha\) whereas for \(\beta > 0.41\) the filter is always
under-damped.

III. TRANSFER FUNCTION, CUT OFF FREQUENCY, CENTER
FREQUENCY AND BANDWIDTH

For any filter, the attributes like the magnitude of the
transfer function, Cutoff frequency, Center frequency and
bandwidth are essential to determine the performance of the
filter. Fig.1 gives an illustration of these attributes. Depending
on the values of the coefficients of the filter these attributes
get changed. For DES filter the coefficients \(\alpha\) and \(\beta\) are the
factors to control the performance attributes. This section
investigates the effect of \(\alpha\) and \(\beta\) on performance attributes
of the DES filter.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{bandpass_filter.png}
  \caption{Center frequency, Cut-off frequency and Passband of a typical Band-pass filter}
  \label{fig:bandpass_filter}
\end{figure}

A. Effect of \(\alpha\) and \(\beta\) on DES Transfer Function

Putting \(z = e^{i\omega}\) in the Eq. 3 the transfer function \(H(z)\) can be
expressed as \(H(z) = \frac{X + Y}{Z}\) where

\[ X = 3\alpha + 2a^2\cos \omega - 4a\cos \omega + a\cos 2\omega - \alpha^2 \cos 2\omega - \alpha^2 + 2a\cos \omega + a^2 = 2a \]

\[ Y = [2a\sin \omega + 2a^2\sin \omega - 2a^2\sin \omega\cos \omega + 2a\sin \omega\cos \omega] \]

\[ Z = [(\cos 2\omega + a\cos \omega + a\cos \omega - 2\cos \omega + 1 - \alpha^2 - a\sin \omega + a\sin \omega - 2\sin \omega + 2a\sin \omega\cos \omega)] \]

Therefore \(H(e^{i\omega})\) is a function of \(\alpha\), \(\beta\) and \(\omega\). In Fig 1 the
nature of variation of the magnitude of the transfer function
\(H(e^{i\omega})\) with \(\alpha\), \(\beta\) and \(\omega\) is illustrated. The magnitude of
\(H(e^{i\omega})\) is represented by the colors depicted in the color bar.
As is evident from Fig. 1, when \( \alpha \) is low, the amplitude is low at the higher frequency for all values of \( \beta \), but it is more prominent at low \( \beta \). It implies that high-frequency components of the signal are attenuated for low \( \alpha \) and it is more effective for lower \( \beta \). However, for high \( \alpha \), amplitudes of all the frequency (high or low) components remain high for all values of \( \beta \), particularly at high \( \beta \). No significant change in the amplitude is observed. At almost zero frequency, whatever will be the value of \( \alpha \) and \( \beta \) amplitude will remain constant. It means that for DC signal the value of \( \alpha \) and \( \beta \) does not affect the filter. So, the values of \( \alpha \) determine the effectiveness of the LPF. Lesser the value of \( \alpha \), LPF activity will be more stringent.

The figure indicates that with the increase in \( \beta \) the cutoff frequency increases whereas the change in \( \alpha \) has less effect (particularly for \( \alpha < 0.5 \)) on the change of the cutoff frequency.

**B. Effect of \( \alpha \) and \( \beta \) on DES magnitude response**

The magnitude response of a filter is characterized by its cut-off frequency, bandwidth and center frequency. To realize the effect of the \( \alpha \) and \( \beta \) on the cut-off frequency, center frequency and bandwidth of DES filter, a non-stationary signal contaminated with noise are being operated by the filter. The values of the \( \alpha \) and \( \beta \) are varied with step size 0.01 and the corresponding cut-off frequencies, center frequency and the bandwidth is noted from the magnitude response of the DES filter for each combination of \( \alpha \) and \( \beta \).

**i. Cut-off frequency (\( \omega_c \))**

The energy of the signal flowing through a filter varies with the frequency of the signal. The energy may increase to its maximum or decrease from it. The frequency at which energy is 0.707 times of its maximum is the Cut-off frequency, corner frequency, or break frequency. At this frequency the energy is 3dB down from 0dB and hence referred as -3dB down point. For the different combination of \( \alpha \) and \( \beta \) the frequency where the magnitude of the transfer function in frequency domain attains -3dB point is noted as the cut-off frequency (\( \omega_c \)) for that particular combination of \( \alpha \) and \( \beta \). Since DES is a band-pass filter, it ought to have two cut-off frequencies (lower and upper). Lower cutoff frequency is associated with the high pass filter (to filter out the trend). As the trend is a slow varying pattern with large periods, it has a negligible frequency. So, to remove the effect of the trend, the lower cutoff frequency has been taken close to zero whereas the variation of the upper cut-off frequency with \( \alpha \) and \( \beta \) are noted and plotted in Fig.3. The upper cut-off frequency is denoted here by \( \omega_u \).

**ii. Bandwidth (BW)**

By definition, Bandwidth is merely the width between the higher and lower cut-off frequencies. It gives the measure of the filter capacity to allow the maximum range of the frequencies. The values of the BW of the DES filter is calculated from the upper and lower cut-off frequencies at different combinations of the smoothing parameters \( \alpha \) and \( \beta \) and are plotted in Fig. 4. Moreover, to detect which of these two parameters has more impact on the DES BW, the variation of BW with a single parameter is plotted keeping the other one constant. Fig. 5 illustrates the change in BW with \( \beta \) (keeping \( \alpha \) constant at different values) and with \( \alpha \) (keeping \( \beta \) constant at different values).
From Fig. 4 & Fig. 5 it is seen that the bandwidth of the DES filter increases with increase in $\alpha$. The change in $\beta$ does not affect the bandwidth significantly for any particular value of $\alpha$.

### iii. Center Frequency

The center frequency of a filter gives the measure of the location of the passband in the frequency spectrum. By changing the center frequency, the passband of a filter can be shifted to the desired frequency band. The average of the lower and upper cut-off frequencies gives the measure of the central frequency. It is of interest to find the effect of $\alpha$ and $\beta$ on the shifting of the passband by noting the change of the center frequency at their various combinations.

Fig. 6 demonstrates that at higher $\alpha$ the plot nearly resembles that of the BW in Fig. 4 but at lower $\alpha$ the center frequency changes with $\beta$. This can be more distinctly observed in Fig. 7.

As a whole, it can be said that the pass-band of the DES filter shifts to the right with the increase of both $\alpha$ and $\beta$.

### IV. Mean Square Error

Mean Square Error (MSE) [8] is a signal fidelity measure. The goal of a signal fidelity measure is to compare two signals by providing a quantitative score that describes the degree of similarity between them.

Let $X = \{x_i | i = 1,2,\ldots,N\}$ and $Y = \{y_i | i = 1,2,\ldots,N\}$ are the two finite length discrete signals of length $N$; $x_i$ and $y_i$ are the value of the $i^{th}$ samples in $X$ and $Y$ respectively then the MSE between the signals is

$$MSE(x, y) = \frac{1}{N} \sum_{i=1}^{N} (x_i - y_i)^2 \quad (12)$$

Let a stationary signal of the form $X = A \sin(\omega t)$, is sampled and contaminated with noise. The frequency of this noisy signal is varied and made to pass through the DES filter. For every frequency, the values $\alpha$ and $\beta$ of the DES filter are changed, and its output is obtained as a vector $Y$. The MSEs for these combinations of $\alpha$ and $\beta$ for a particular frequency are calculated using Eq. 12. Repetition of this operation for all the frequencies $0 \leq \omega \leq \pi$ generates a $n \times m \times l$ matrix of MSE which is plotted in Fig. 8. For the better realization of the variation of MSE, the matrix is illustrated in the Fig. 8 from different views.

From Fig. 8 it can be seen that MSE is almost immune to the changes of $\beta$ and $\omega$ rather it only depends on $\alpha$ inversely. As the value of $\alpha$ increases or decreases, MSE decreases or increases accordingly.

### V. Conclusion

Underdamped filters are preferable and commonly used filters. Though it faces an initial overshoot, it is faster than the overdamped one. So to get a steady and fast DES filter response, we should select $\beta > 0.41$ for any value of $\alpha$. On the other hand, the effectiveness of the LPF is determined by $\alpha$. For a lower value of $\alpha$, the filter will act as a low pass one at high frequency. A filter’s cutoff frequency is influenced by
β significantly. An enhancement in β increases the cutoff frequency of the filter, but the change in α (particularly for α < 0.5) is not a factor. Smoothing constant α plays a big roll on filter's scaling property, i.e., bandwidth. The bandwidth of a filter will increase with the increase in α whereas the change in β has almost no impact on it. However, the shifting property of a filter, i.e., change in center frequency is influenced by both smoothing constants α and β. Hence it can be said that the pass-band of the DES filter shifts to the right with increasing values of both α and β. Finally, the MSE inversely varies with the smoothing parameter α, and it is independent with the change of β and ω.

Therefore from the above studies, it can be concluded that to get an underdamped filter with suitable cutoff frequency, bandwidth, center frequency and MSE the selection of smoothing parameters should be near 0.5.

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Appendix:

The transfer function of SES in FIR form is given by

$$H(z) = \sum_{k=1}^{N} h(k) z^{-k} = \sum_{k=1}^{N} \alpha(1-\alpha)^{N-k} z^{-k}$$

Let $A(\alpha) = \sum_{k=1}^{N} \alpha(1-\alpha)^{N-k} \cos \omega k$ and $B(\alpha) = \sum_{k=1}^{N} \alpha(1-\alpha)^{N-k} \sin \omega k$.

Therefore magnitude response is $|H(e^{j\omega})| = \sqrt{A^2 + B^2}$ and phase response: $\angle H(e^{j\omega}) = \tan^{-1} \frac{B}{A}$.

From the definition of Cut off frequency or corner frequency or break frequency ($\omega_c$) it is to be said that at 3 dB point the power of the filter gets halved to that of the passband i.e. $|H(z)|^2 \approx 0.5$.

From equation (c) and (d) we get Cut off frequency-$\omega_c = \cos^{-1} \left[1 - \left\{\alpha^2 / 2(1-\alpha)\right\} \right]$

$1 - \left\{\alpha^2 / 2(1-\alpha)\right\} \geq -1$

$\Rightarrow \alpha \leq -0.82843$

Now $\alpha$ cannot be negative as this will force the pole of the transfer function of Eq. (c) to move out of the unit circle making the system unstable. So, $\alpha \leq +0.82843$.