Exponentially Stable Adaptive Control. Part I. Time-Invariant Plants

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Abstract—We propose a new controller parameter adaptive law that guarantees the exponential stability of the classical dynamic model of the tracking error without using its coordinates in the adaptive law and relaxes some classical assumptions and requirements of adaptive control theory (the need to know the sign/value of the control input gain, the need for an experimental choice of the adaptive law gain, and the requirement to the tracking error transfer function to be strictly positive real considering the output feedback control). The applicability of the proposed law to adaptive state and output feedback control problems is shown. The advantages of developed approach over the existing ones are demonstrated mathematically and experimentally.

Keywords: adaptive control, output control, relative degree, time-invariant parameters, parametric error, finite excitation, exponential stability

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1. INTRODUCTION

In the literature on adaptive control, the common form of representation of a plant with parametric uncertainty is the differential error equation [1, 2]

\[ \dot{e}_{\text{ref}} = A_{\text{ref}} e_{\text{ref}} + B \bar{\theta}^T \omega, \quad e_{\text{ref}}(0) = e_{0\text{ref}}, \] (1.1)

where \( e_{\text{ref}} \in \mathbb{R}^q \) is the vector of errors in tracking a reference model by a plant, \( e_{\text{ref}}(0) \in \mathbb{R}^q \) is the vector of initial conditions, \( A_{\text{ref}} \in \mathbb{R}^{q \times q} \) is the Hurwitz matrix of the reference model, \( B \in \mathbb{R}^q \) is the plant input vector, \( \bar{\theta} = \hat{\theta} - \theta \in \mathbb{R}^m \) is the parametric error between adjustable and ideal unknown but constant (\( \dot{\theta} = 0 \)) control law parameters, and \( \omega \in \mathbb{R}^m \) is a measurable regressor. The matrix \( A_{\text{ref}} \) is known, and the pair \((A_{\text{ref}}, B)\) is completely controllable.

In case, when only the output tracking error is measurable, Eq. (1.1) acquires the form

\[ \dot{e}_{\text{ref}} = A_{\text{ref}} e_{\text{ref}} + B \bar{\theta}^T \omega, \quad e_{\text{ref}}(0) = e_{0\text{ref}}, \] (1.2)

where \( c \in \mathbb{R}^q \) is a vector that forms the error \( \varepsilon \) available for direct measurement.

Usually, the error model (1.1) is obtained by parametrization carried out to solve the state feedback adaptive control problems, and the model (1.2) occurs in the output feedback adaptive control problems (see [1–3] for details).
It is well known [1–3] that when the error vector $e_{\text{ref}}$ is measurable, then, for the model (1.1) to be asymptotically stable, it suffices to choose the law to adjust the controller parameters in the form

$$\dot{\hat{\theta}} = -\Gamma \omega e_{\text{ref}}^T PB,$$

(1.3)

where $\Gamma$ is an adaptive gain matrix of appropriate size and $P \in \mathbb{R}^{q \times q}$ is a solution of the Lyapunov equation $A_{\text{ref}}P + PA_{\text{ref}}^T = -Q$, where $Q \in \mathbb{R}^{q \times q}$ is a positive definite matrix.

It is also well known that if the transfer function $A^T (pI - A_{\text{ref}})^{-1} B$ is real and strictly positive, then there simultaneously exists a solution of the following two equations [1–3]

$$A_{\text{ref}}P + PA_{\text{ref}}^T = -Q,$$

$$PB = e^T,$$

(1.4)

and to ensure the asymptotic stability of the error model (1.2), one can use an adaptive law in the form

$$\dot{\hat{\theta}} = -\Gamma \omega \varepsilon.$$

(1.5)

If the requirement of persistent excitation of the regressor $\omega \in \text{PE}$ is additionally met, then the adaptive laws (1.3) and (1.5) ensure [1, 2] the exponential stability of the error $\xi$.

The main disadvantages of the adaptive laws (1.3) and (1.5) are the exponential stability of the error $\xi$ only when the restrictive condition $\omega \in \text{PE}$ is satisfied, the low quality of transients with respect to the tracking errors $e_{\text{ref}}$, $\varepsilon$ and the adjustable parameters $\hat{\theta}$. To eliminate these shortcomings, various modifications [4–12] of the adaptive laws (1.3) and (1.5) were proposed in the literature, which allow one to relax the persistent excitation condition and/or improve the quality of transients of the tracking errors $e_{\text{ref}}$ and $\varepsilon$ and the adjustable parameters $\hat{\theta}$.

In addition, the adaptive law (1.5) for the output feedback control problem can be implemented if and only if the requirement of strict positive realness of the transfer function $C^T (pI - A_{\text{ref}})^{-1} B$ is satisfied [1–3, 13]. The augmented error method [15], the high-order tuners [16], the shunt compensator [17], iterative synthesis procedures [18], and their various modifications [3] were developed [3, 13, 14]. However, most of these approaches are noise-sensitive, cumbersome, and difficult to be implemented in practice [14]. The higher the relative degree of the plant, the more evident the last two properties are.

In the present paper, to overcome the restrictions on the feasibility of the law (1.5), as well as to improve the quality of the controller parameter adjustment for the state and output feedback adaptive control problems, a new adaptive law is proposed that does not use the states of the error equation (1.1) and is equally applicable to the error models with measurable state (1.1) and output (1.2); it ensures the exponential stability of the error $\xi$ under the condition of regressor finite excitation and the elementwise monotonicity of adjustable controller parameters $\hat{\theta}$ and additionally solves a number of classical problems of adaptive control theory (the need to know the sign/value of the plant input vector $B$ and the need to manually select the values of the adaptive gain matrices $\Gamma$).

The paper is organized as follows. Section 2 states a theorem on the existence of a new parameter adjustment law $\hat{\theta}$ ensuring the exponential stability of the error $\xi$ under the condition of regressor finite excitation and proposes simple algorithms for its implementation for the state and output feedback adaptive control problems. In Sec. 3, we give a brief discussion of the results obtained in the paper, and Sec. 4 presents the results of mathematical modeling of the developed adaptive law.
In the proof of theorems and propositions in this paper, we use the following definition of regressor finite excitation and a corollary of the Kalman–Yakubovich–Popov lemma.

**Definition.** A regressor $\omega$ is said to be *finitely exciting* ($\omega \in FE$) on the interval $[t^*_e; t^*_e]$ if there exist $t^*_e \geq 0$, $t^*_e \geq t^*_e$, and $\alpha$ such that the inequality holds

$$\int_{t^*_e}^{t_e} \omega(\tau) \omega^T(\tau) d\tau \geq \alpha I, \quad (1.6)$$

where $\alpha > 0$ is the level of excitation and $I$ is the identity matrix.

**Corollary.** A scalar transfer function $H(p) = d$, where $d > 0$, is real and strictly positive if and only if there exists a Hurwitz matrix $A$, a matrix $P = P^T > 0$, a vector $q$, a vector $B$, and a constant $\mu > 0$ such that

$$A^T P + PA = -qq^T - \mu P, \quad PB = \sqrt{2dq}. \quad (1.7)$$

2. MAIN RESULT

Assume that for the vector $\theta$ of unknown parameters in the models (1.1) and (1.2) there exists a linear regression model of the form

$$\Upsilon = \Omega(\Delta) \theta, \quad (2.1)$$

where $\Omega \in R$, $\Upsilon \in R^n$, and $\Delta \in R$ are a measurable regressor, a regression function, and a regressor argument independent of the elements of the vector $e_{ref}$ of the error equation (1.1).

Then, based on the regression (2.1), we can prove a theorem on the existence of an adaptive law for the parameters $\hat{\theta}$ that does not use the states of the error equation (1.1) and ensures the exponential stability of the error $\xi$.

**Theorem.** If for $\Delta \in FE$ the regressor $\Omega$ is such that

(a) $\forall t \geq t^*_e \quad \Omega(t) \in L_\infty$, $\Omega(t) \geq 0$,
(b) $\forall t \geq t^*_e \quad 0 < \Omega_{LB} \leq \Omega(t) \leq \Omega_{UB}$,

then there exists an adaptive law

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = -\gamma \Omega \left( \Omega \hat{\theta} - \Omega \theta \right) = -\gamma \Omega^2 \tilde{\theta}, \quad (2.2)$$

ensuring the following properties:

1. $\forall t_a \geq t_b \quad |\tilde{\theta}(t_a)| \leq |\tilde{\theta}(t_b)|$.
2. $\forall t \geq t^*_e \quad \xi \in L_\infty$.
3. $\forall t \geq t^*_e$ the error $\xi$ exponentially converges to zero at a rate which minimum value is directly proportional to the parameters $\gamma_0 \geq 1$ and $\gamma_1 \geq 0$. 


The proof of the Theorem is given in the Appendix.

According to the results in the Theorem, the adaptive law (2.2) is equally applicable to dynamic models based on both measurable state vector (1.1) and measurable output (1.2). Compared with the adaptive laws (1.3) and (1.5), the law (2.2) ensures the elementwise monotonicity of the adjustable controller parameters \( \hat{\theta} \), for the exponential convergence of error \( \xi \) it requires that the condition \( \Delta \in \mathbb{F}E \) is satisfied, which is strictly weaker than \( \omega \in \mathbb{P}E \), it does not require the sign/value of the vector \( B \) to be known, it is less demanding on the manual selection of the coefficient \( \gamma \), and, additionally, for the output feedback adaptive control problem, in contrast to the existing solutions [15, 16], it allows one to overcome the condition of strict positive realness without the sufficient increase in complexity of the control system when the value of the plant relative degree improves.

Remark 1. The adaptive law (2.2) is less demanding on the manual selection of the coefficient \( \gamma \), because, first, it contains the dynamic term \( \gamma_0 \lambda_{\max} (\omega \omega^T) \), which does not require manual selection, and second, using the value of \( \gamma_1 \), the minimum rate of convergence of error \( \xi \) to zero can be set at a required level regardless of \( \Omega_{LB} \).

The fundamental practical difficulty in implementing the adaptive law (2.2) is obtaining a linear regression model (2.1) with a regressor \( \Omega \) that satisfies the requirements of the Theorem for each specific adaptive control problem.

In the following subsections, we show how such a regression model can be obtained in the most general classical state feedback and output feedback adaptive control problems leading to the error models (1.1) and (1.2), respectively.

2.1. State Feedback Control

Consider the problem of state feedback adaptive control of the LTI SISO plants [1–3],

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0, \]  

(2.1.1)

where \( x \in \mathbb{R}^n \) is the measurable state vector, \( x(0) \in \mathbb{R}^n \) is the unknown vector of initial conditions, \( u \in \mathbb{R} \) is the control signal, \( A \in \mathbb{R}^{n \times n} \) is the system state matrix, and \( B \in \mathbb{R}^n \) is the input vector. The pair \( (A, B) \) is controllable, and the values of the elements of \( A \) and \( B \) are time-invariant and unknown. The state vector \( x \) and the control \( u \) are assumed to be available for direct measurement.

The reference model determining the required quality of control for the closed-loop system with the control signal \( u \) and the plant (2.1.1) has the form

\[ \dot{x}_{\text{ref}} = A_{\text{ref}} x_{\text{ref}} + B_{\text{ref}} r, \quad x_{\text{ref}}(0) = x_{0\text{ref}}, \]  

(2.1.2)

where \( x_{\text{ref}} \in \mathbb{R}^n \) is the state vector of the reference model, \( x_{\text{ref}}(0) \in \mathbb{R}^n \) is the vector of initial conditions, \( r \in \mathbb{R} \) is the reference signal, \( A_{\text{ref}} \in \mathbb{R}^{n \times n} \) is the Hurwitz state matrix of the reference model, and \( B_{\text{ref}} \in \mathbb{R}^n \) is the input vector of the reference model.

The control law for the plant (2.1.1) is chosen in the form

\[ u = \hat{k}_x x + \hat{k}_r r, \]  

(2.1.3)

where \( \hat{k}_x^T \in \mathbb{R}^n \) and \( \hat{k}_r \in \mathbb{R} \) are the adjustable parameters and \( \hat{k}_r(0) \neq 0 \).
Substituting the control law (2.1.3) into (2.1.1), we obtain the closed-loop system

$$\dot{x} = (A + B\hat{k}_x)x + B\hat{k}_r r.$$  \hspace{1cm} (2.1.4)

The Erzberger condition [19] is assumed to be satisfied for the plant (2.1.4).

**Assumption 1.** There exist ideal control law parameters \( k_x^T \in R^n \) and \( k_r \in R \) such that

$$A + Bk_x = A_{\text{ref}},$$

$$Bk_r = B_{\text{ref}}.$$ \hspace{1cm} (2.1.5)

Then the error equation between the equation of the closed-loop system (2.1.4) and the reference model (2.1.2) has the form

$$\dot{e}_{\text{ref}} = A_{\text{ref}}e_{\text{ref}} + B\hat{\theta}^T \omega,$$

$$e_{\text{ref}}(0) = x_0 - x_{0\text{ref}}.$$ \hspace{1cm} (2.1.6)

Here \( e_{\text{ref}} = x - x_{\text{ref}}, \hat{k}_x = \hat{k}_x - k_x, \) and \( \hat{k}_r = \hat{k}_r - k_r. \) The following notation is introduced in (2.1.6):

$$\omega = \begin{bmatrix} x^T \\ r \end{bmatrix}, \quad \hat{\theta}^T = \begin{bmatrix} \hat{k}_x \\ \hat{k}_r \end{bmatrix} = \hat{\theta}^T - \theta^T,$$ \hspace{1cm} (2.1.7)

where \( \omega \in R^{n+1} \) is the regressor and \( \hat{\theta} \in R^{n+1} \) is the vector of adjustable control law parameters.

Then Eq. (2.1.6) with the reference to (2.1.7) and the initial conditions can be written in the form (1.1),

$$\dot{e}_{\text{ref}} = A_{\text{ref}}e_{\text{ref}} + B\hat{\theta}^T \omega, \quad e_{\text{ref}}(0) = x_0 - x_{0\text{ref}}.$$ \hspace{1cm} (2.1.8)

Considering the state feedback control problem, the derivation of the adaptive law (2.2) consists of two main stages:

1. To obtain the linear regression equation \( Y(t) = \Delta(t)\theta \) with measurable regressor \( \Delta \in R \) and function \( Y \in R^{n+1}. \)

2. To transform this equation with the aim to obtain the regression (2.1) with a regressor \( \Omega \) that functionally depends on \( \Delta \) and has the properties necessary in accordance with the Theorem.

To obtain the regression equation \( Y(t) = \Delta(t)\theta \), we write the plant equation (2.1.1) in the form of the linear regression equation

$$\dot{x} = \theta^T_{AB}\Phi,$$

$$\theta^T_{AB} = \begin{bmatrix} A \\ B \end{bmatrix}, \quad \Phi = \begin{bmatrix} x^T \\ u \end{bmatrix}^T,$$ \hspace{1cm} (2.1.9)

where \( \Phi \in R^{n+1} \) is a measurable regressor and \( \theta^T_{AB} \in R^{n \times (n+1)} \) is the matrix of unknown parameters.

Despite the unavailability of \( \dot{x} \) for direct measurement, linear stable filters for all dynamic variables in Eq. (2.1.9) are introduced:

$$\dot{\bar{\mu}} = -l\bar{\mu} + \dot{x}, \quad \bar{\mu}(0) = 0_n,$$

$$\dot{\bar{\Phi}} = -l\bar{\Phi} + \Phi, \quad \bar{\Phi}(0) = 0_{n+1}.$$ \hspace{1cm} (2.1.10)

where \( l > 0 \) is the filter constant.

The regressor \( \bar{\Phi} \) is determined by the solution of the second differential equation in (2.1.10), and the function \( \bar{\mu} \), according to [20], can be calculated without knowing \( \dot{x} \) in the following way:

$$\bar{\mu}(t) = e^{-lt}\bar{\mu}(0) + x(t) - e^{-lt}x(0) - l\bar{x}(t) + le^{-lt}\bar{x}(0),$$ \hspace{1cm} (2.1.11)

where \( \bar{x} \) an element of the vector \( \bar{\Phi} \).
where $\bar{\Delta}$ is a measurable function, $\bar{\varphi} \in \mathbb{R}^{n+2}$ is a measurable regressor, and $\bar{\theta}_{AB}^T \in \mathbb{R}^{n \times (n+2)}$ is the augmented vector of unknown parameters.

Using the DREM procedure [21, 22], Eq. (2.1.12) with the vector regressor $\bar{\varphi}$ is transformed into the equation with a scalar regressor. Following [21, 22], we introduce the $n + 2$ minimum-phase dynamic filters

$$
(\cdot)_{f_i(t)} := [H_i(\cdot)](t), \quad H_i(p) = \frac{\alpha_i^f}{p + \beta_i^f}, \quad i \in \{1, 2, \ldots, n + 2\},
$$

where $\alpha_i^f, \beta_i^f > 0$ and $\alpha_i^f \neq \alpha_j^f$ for all $i \neq j$.

Passing the function $\bar{z}$ and the regressor $\bar{\varphi}$ through (2.1.13), the extended regression equation is formed:

$$
\bar{z}_f(t) = \bar{\varphi}_f(t) \bar{\theta}_{AB},
$$

$$
\bar{z}_f(t) := [\bar{z}(t) \; \bar{z}_f(t) \; \ldots \; \bar{z}_{f_n(t)}]^T, \quad \bar{\varphi}_f(t) := [\bar{\varphi}(t) \; \bar{\varphi}_f(t) \; \ldots \; \bar{\varphi}_{f_{n+2}}(t)]^T,
$$

where $\bar{z}_f \in \mathbb{R}^{(n+3) \times n}$ and $\bar{\varphi}_f \in \mathbb{R}^{(n+3) \times (n+2)}$.

Premultiplying Eq. (2.1.14) by $\text{adj} \{ \bar{\varphi}_f^T \bar{\varphi}_f \} \bar{\varphi}_f^T$ and using the equality

$$
\text{adj} \{ \bar{\varphi}_f^T \bar{\varphi}_f \} \bar{\varphi}_f^T \bar{\varphi}_f = \det \{ \bar{\varphi}_f^T \bar{\varphi}_f \} I,
$$

an equation with scalar regressor is obtained:

$$
z = \varphi \bar{\theta}_{AB},
$$

$$
z = \text{adj} \{ \varphi_f^T \varphi_f \} \varphi_f^T \bar{z}_f(t), \quad \varphi = \det \{ \varphi_f^T \varphi_f \},
$$

where $z \in \mathbb{R}^{(n+2) \times n}$ and $\varphi \in \mathbb{R}$.

Taking into account the definition of $\bar{\theta}_{AB}$ and $\varphi \in \mathbb{R}$ in (2.1.15), one can readily write the regression equations

$$
z_A = z^T H = \varphi A,
$$

$$
z_B = z^T e_{n+1} = \varphi B,
$$

where $z_A \in \mathbb{R}^{n \times n}$, $z_B \in \mathbb{R}^n$, $H = [I_{n \times n} \; 0_{n \times 2}]^T \in \mathbb{R}^{(n+2) \times n}$, and $e_{n+1} = [0_{1 \times n} \; 1 \; 0]^T \in \mathbb{R}^{n+2}$.

Now it is possible to transform the regressions (2.1.16) to the regression with respect to the parameters $\theta$ of the control law (2.1.3). To this end, we multiply the Erzberger condition (2.1.5) by the regressor $\varphi \in \mathbb{R}$ and substitute the functions (2.1.16), thus obtaining

$$
\bar{Y} = \theta \bar{\Delta},
$$

$$
\bar{\Delta} = \varphi B^T = z_B^T, \quad \bar{Y} = \varphi \left[ A_{ref} - A \; B_{ref} \right]^T = \left[ \varphi A_{ref} - z_A \; \varphi B_{ref} \right]^T,
$$

where $\bar{Y} \in \mathbb{R}^{(n+1) \times n}$ and $\bar{\Delta} \in \mathbb{R}^{1 \times n}$.
Let us obtain a scalar regressor $\Delta$ from the vector regressor $\bar{\Delta}$. To this end, we multiply Eq. (2.1.17) by $\bar{\Delta}^T$

$$
Y(t) = \Delta(t) \theta,
Y = \bar{Y}(t) \bar{\Delta}^T(t), \quad \Delta(t) = \bar{\Delta} \bar{\Delta}^T,
$$

where $Y \in \mathbb{R}^{n+1}$ and $\Delta \in \mathbb{R}$.

**Remark 2.** According to the results in Lemma 6.8 [2] and by virtue of stability of the filters (2.1.10), if $\Phi \in \mathbb{F}E$, then also $\bar{\varphi} \in \mathbb{F}E$. In [22], the implication $\bar{\varphi} \in \mathbb{F}E \Rightarrow \varphi \in \mathbb{F}E$ was proved for the extension scheme (2.1.13)–(2.1.15). Then, since the regressor $\bar{\Delta}$ depends on one dynamic variable $\varphi$ and it does not become singular when $\varphi = \text{const}$ by Assumption 1, the implications $\varphi \in \mathbb{F}E \Rightarrow \bar{\Delta} \in \mathbb{F}E \Rightarrow \Delta \in \mathbb{F}E$ also hold true. Thus, when using the procedure (2.1.9)–(2.1.18), the excitation of the original regressor $\Phi$ does not vanish, and it holds that $\Phi \in \mathbb{F}E \Rightarrow \Delta \in \mathbb{F}E$.

The next aim is to transform the equation (2.1.18) to obtain the regression (2.1) with the regressor $\Omega$, which meets all the requirements of Theorem.

Using the results in the papers [20, 23, 24], the filter with exponential forgetting is introduced:

$$
\begin{align*}
\dot{\beta} &= \sigma, \quad \beta(0) = 0 \\
\dot{v}_f &= \exp(-\beta) v, \quad v_f(0) = 0,
\end{align*}
$$

where $\sigma > 0$ is the arbitrary parameter and $v$ and $v_f$ are the input and output of the filter, respectively.

The extended regressor $\Delta^2$ and the function $\Delta Y$ are passed through the filter (2.1.19),

$$
\begin{align*}
\Upsilon &= \Omega \theta, \\
\Omega(t) &= \int_{t_f^+}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau, \\
\Upsilon(t) &= \int_{t_f^+}^{t} e^{-\sigma \tau} \Delta(\tau) Y(\tau) d\tau,
\end{align*}
$$

where $\Upsilon \in \mathbb{R}^{n+1}$ and $\Omega \in \mathbb{R}$.

The following proposition was obtained for the resulting regressor $\Omega$ in [23].

**Proposition 1.** If $\Delta \in \mathbb{F}E$ on the interval $[t_f^+; t_e]$ and $\Delta \in L_\infty$, then

1. $\forall t \geq t_f^+ \Omega(t) \in L_\infty$, $\Omega(t) \geq 0$.
2. $\forall t \geq t_e \Omega(t) > 0$, $\int_{t_f^+}^{t_e} e^{-\sigma \tau} \Delta^2(\tau) d\tau \leq \Omega(t) \leq \delta_{\Delta}^2 \sigma^{-1},$

where $\delta_{\Delta} = \sup_{t \geq 0} \max |\Delta(t)|$.

The proof of Proposition 1 can be found in [23].

Proposition 1 considers the case in which $\Delta \in L_\infty$, but from the viewpoint of practical application of adaptive control systems, it is impossible to claim a priori that $\Delta \in L_\infty$, because $\Delta$ depends (see (2.1.9)–(2.1.18)) on the state vector $x$. Therefore, in the following proposition we consider the case in which $\Delta \notin L_\infty$ but increases no faster than some known exponential.

**Proposition 2.** If $\Delta \in \mathbb{F}E$ on the interval $[t_f^+; t_e]$ and $\forall t \geq t_f^+$ one has $|\Delta| \leq c_1 e^{c_2 t}$ and $\sigma > 2 c_2$, then
1. $\forall t \geq t^+_e \Omega(t) \in L_\infty \Rightarrow \Omega(t) \geq 0$.

2. $\forall t \geq t_e \Omega(t) > 0, \quad \frac{c_2^2}{c_3} \left( e^{-c_3 t e} - e^{-c_3 t} \right) \leq \Omega(t) \leq \frac{c_1^2}{c_3}$,

where $c_1 > 0$, $c_2 > 0$, and $c_3 = \sigma - 2c_2 > 0$.

The proof of Proposition 2 is given in the Appendix.

According to Proposition 2, one can always choose a value of the parameter $\sigma$ such that even for $\Delta \notin L_\infty$ the regressor $\Omega$ lies in $L_\infty$ and satisfies the conditions of the Theorem (in practice, for the regressor $\Delta$ of any unstable plant, it is possible to set a majorant function in the form of the conservative exponent $|\Delta| \leq c_1 e^{c_2 t}$ with known $c_1$ and $c_2$, and therefore, the requirements of Proposition 2 are not restrictive).

The results of the second statements of Propositions 1 and 2 are summarized by introducing an inequality for $\Omega$ that holds for all $t \geq t_e$ and for any $\Delta \in FE$ such that $\Delta \in L_\infty$ or $|\Delta| \leq c_1 e^{c_2 t}$:

$$0 < \min_{\Omega_{LB}} \left\{ \int_{t^+_e}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau; \frac{c_2^2}{c_3} \left( e^{-c_3 \tau} - e^{-c_3 t} \right) \right\} \leq \Omega(t) \leq \max_{\Omega_{UB}} \left\{ \delta_{\Delta}^2 \sigma^{-1}; \frac{c_1^2}{c_3} \right\}.$$  \hspace{1cm} (2.1.21)

According to the equation (2.1.21), under the condition $\Delta \in FE$, the regressor $\Omega$ has the properties necessary from the viewpoint of the Theorem, and hence if the parameters of the control law (2.1.3) are adjusted in accordance with (2.2), then the error $\xi$ of the state feedback adaptive control problem is exponentially stable.

Thus, the state feedback adaptive control system with the developed adaptive law (2.2) consists of the control law (2.1.3) as well as the procedures (2.1.9)–(2.1.18) to obtain the linear regression $Y(t) = \Delta(t) \theta$ and (2.1.19)–(2.1.20) to transform it.

Compared with the law (1.3), the proposed adaptive law (2.2) ensures the exponential convergence of the error $\xi$ to zero under the weaker condition $\Delta \in FE$, and compared with various composite adaptive laws [5, 8–10], which have become widespread in recent years, the proposed law (2.2) ensures the elementwise monotonicity of the adjustable controller parameters $\theta$, does not require the knowledge of the sign/value of the vector $B$, and is less demanding on manual selection of the adaptive law gain $\gamma$. The result admits a straightforward generalization to the case of MIMO LTI plants.

### 2.2. Output Feedback Control

Consider an output feedback adaptive control problem for a class of linear time-invariant plants [1–3] of the form

$$y(t) = b_m Z(p) R(p)^{-1} u(t),$$  \hspace{1cm} (2.2.1)

where $p = d/dt$ is the differentiation operator, $y$ is the output variable, $u$ is the control signal, $b_m$ is the plant gain, and $Z(p) = p^n + \sum_{i=0}^{m-1} b_i b_m^{-1} p^i$ and $R(p) = p^n + \sum_{i=0}^{n-1} a_i p^i$ are the characteristic polynomials with quasistationary unknown parameters ($b_i \approx 0$ and $a_i \approx 0$).

The desired control quality for the plant (2.2.1) is determined by the reference model

$$y_{ref}(t) = b_{ref} Z_{ref}(p) R_{ref}(p)^{-1} r(t),$$  \hspace{1cm} (2.2.2)
where $y_{ref}$ is the output variable of the reference model, $r$ is the reference signal, $b_{ref}$ is the reference model gain, and $Z_{ref} (p)$ and $R_{ref} (p)$ are Hurwitz characteristic polynomials of degrees $m^*$ and $n^*$, respectively. The relative degree of the reference model $\rho^* = n^* - m^*$ is assumed to be equal to the relative degree $\rho = n - m$ of the plant.

The control problem for the plant (2.2.1) is to ensure that the output of the plant (2.2.1) asymptotically tracks the output of the reference model (2.2.2),

$$\lim_{t \to \infty} (y(t) - y_{ref}(t)) = \lim_{t \to \infty} \varepsilon(t) = 0. \quad (2.2.3)$$

This problem is considered under the following classical assumptions [1–3, 13, 14].

**Assumption 2.** The numerator polynomial $Z(p)$ is a Hurwitz one.

**Assumption 3.** The degrees $n$ and $m$ are known, so $\rho = n - m \geq 1$ is also known.

**Assumption 4.** Only $y$ and $u$ but not their derivatives are available for direct measurement.

To parametrize the adaptive control problem and eventually obtain the error equation (1.2) for the plant (2.2.1), in accordance with [1–3], we introduce the state filters

$$\dot{v}_1 = \Lambda v_1 + hu, \quad v_1 (0) = 0,$$

$$\dot{v}_2 = \Lambda v_2 + hy, \quad v_2 (0) = 0, \quad (2.2.4)$$

where $v_1 \in \mathbb{R}^{n-1}$, $v_2 \in \mathbb{R}^{n-1}$, $h = [0, 0, ..., 0, 1]^T \in \mathbb{R}^{n-1}$, and $\Lambda$ is the companion matrix of the Hurwitz polynomial $\Lambda (p) = \Lambda_0 (p) Z_{ref} (p)$.

Then, taking into account the filters (2.2.3), the plant (2.2.1) can be reduced to the form [1–3]

$$y = \frac{Z_{ref} (p)}{R_{ref} (p)} b_m \left[ u - k_1^T v_1 - k_2^T v_2 - k_3 y \right] + \varepsilon_y, \quad (2.2.5)$$

where $k_1 \in \mathbb{R}^{n-1}$, $k_2 \in \mathbb{R}^{n-1}$, $k_3 \in \mathbb{R}$, and $\varepsilon_y$ is an exponentially decaying disturbance due to a mismatch in the initial conditions.

Based on (2.2.5), to achieve the objective (2.2.3), the control law with adjustable parameters is chosen in the form

$$u = \tilde{k}_1^T v_1 + \tilde{k}_2^T v_2 + \tilde{k}_3 y + \tilde{k}_4 r, \quad (2.2.6)$$

where $\tilde{k}_4 (0) \neq 0$.

Taking into account (2.2.6), the reference model equation (2.2.2) is subtracted from (2.2.5) to obtain

$$\varepsilon = \frac{Z_{ref} (p)}{R_{ref} (p)} b_m \left[ \left( \tilde{k}_4 - b_m^{-1} b_{ref} \right) r + \tilde{k}_1^T v_1 + \tilde{k}_2^T v_2 + \tilde{k}_3 y \right] + \varepsilon_y$$

$$= \frac{Z_{ref} (p)}{R_{ref} (p)} b_m \left[ \tilde{k}_4 r + \tilde{k}_1^T v_1 + \tilde{k}_2^T v_2 + \tilde{k}_3 y \right] + \varepsilon_y, \quad (2.2.7)$$

where $k_4 = b_m^{-1} b_{ref}$.

Transforming the input–output description (2.2.7) to the state space one, a differential error equation with measurable output (1.2) is obtained,

$$\dot{e}_{ref} = A_{ref} e_{ref} + B \dot{\theta}^T \omega, \quad e_{ref} (0) = e_{0ref},$$

$$\varepsilon = c^T e_{ref}, \quad (2.2.8)$$
where $A_{\text{ref}} \in R^{n\times n}$ is the companion matrix of the polynomial $R_{\text{ref}}(p)$, $B \in R^{n}$ is the gain vector, $c \in R^{n}$ is the output vector, $\omega = [r \ v_1 \ v_2 \ y]^T \in R^{2n}$ is the regressor vector, and $\hat{\theta} = [\hat{k}_4 \hat{k}_1 \hat{k}_2 \hat{k}_3]^T \in R^{2n}$ is the parametric error vector.

Considering the output feedback adaptive control problem, the derivation of the adaptive law (2.2) also includes two stages:

1. To obtain the linear regression equation $Y(t) = \Delta(t) \theta$ with a measurable regressor $\Delta \in R$ and a function $Y \in R^{2n}$.
2. To transform this equation with the aim to obtain the regression (2.1) with a regressor $\Omega$ functionally depending on $\Delta$ and having the properties necessary in accordance with the Theorem.

To obtain the linear regression equation $Y(t) = \Delta(t) \theta$, a control law with ideal parameters is introduced

\[ u^* = k_1 v_1 + k_2 v_2 + k_3 y + k_4 r = k_1^T \alpha(p) u + k_2^T \alpha(p) y + k_3 y + k_4 r, \tag{2.2.9} \]

where $\alpha(p)$ is the differentiation operator defined as follows:

\[ \alpha(p) = \begin{cases} [p^{n-2}, p^{n-3}, \ldots, p, 1]^T & \text{if } n \geq 2 \\ 0 & \text{if } n = 1. \end{cases} \tag{2.2.10} \]

Substituting (2.2.9) into (2.2.1), the equation of the closed-loop is obtained:

\[ y(t) = \frac{k_4 b_m Z(p) \Lambda(p)}{[\Lambda(p) - k_1^T \alpha(p)] R(p) - b_m Z(p) [k_2^T \alpha(p) + k_3 \Lambda(p)]} r(t). \tag{2.2.11} \]

Considering $\Lambda(p) = \Lambda_0(p) Z_{\text{ref}}(p)$, the equations (2.2.11) and (2.2.2) are set equal to obtain a kind of the Erzberger's conditions (2.1.5):

\[ b_{\text{ref}} k_1^T \alpha(p) R(p) + b_{\text{ref}} b_m Z(p) [k_2^T \alpha(p) + k_3 \Lambda(p)] \]

\[ + k_4 b_m Z(p) \Lambda_0(p) R_{\text{ref}}(p) = b_{\text{ref}} \Lambda(p) R(p). \tag{2.2.12} \]

**Remark 3.** According to the results in the papers [1, 2, 25, 26], under the conditions of Assumption 3, Eq. (2.2.12) is solvable with respect to the parameters $k_1$, $k_2$, $k_3$, and $k_4$ and has exactly one solution.

Now, to obtain the equation $Y(t) = \Delta(t) \theta$ based on (2.2.12), the plant (2.2.1) is represented in the observability canonical form:

\[ \begin{cases} \dot{x} = A_o x + B_o u, & x(0) = x_0 \\ y = C^T x \end{cases} \tag{2.2.13} \]

\[ A_o = [-a I_{(n-1)\times(n-1)} 0_{1\times(n-1)}]; \quad B_o = \begin{bmatrix} 0_{n-(m+1)} \\ b \end{bmatrix}; \quad C^T = \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}, \]

where $x \in R^n$ is the state vector unavailable for direct measurement, $a \in R^n = [a_{n-1} \ldots a_0]^T$, and $b \in R^{m+1} = [b_m \ldots b_0]^T$. 

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Let us introduce a Hurwitz matrix \( \Psi_c \in \mathbb{R}^{n \times n} \) of the canonical form and, taking into account the equality \((\dot{\psi} - a)C^T = A_o - \Psi_c\), add the expression \( \pm \Psi_A x \) to (2.2.13):

\[
\dot{x} = \Psi_c x + (\psi - a) y + B_a u,
\]

where \( \psi \in \mathbb{R}^n \) is the coefficient vector of the characteristic polynomial of the matrix \( \Psi_c \).

To form the regression with respect to the parameter vectors \( a \) and \( b \) from (2.2.13), the Kreeselmeier parameterization [27] is used. By analogy with (2.2.4), the set of linear state filters is introduced

\[
\dot{\eta}_f = \Psi_c^T \eta_f + C u, \quad \eta_f(0) = 0_n,
\]

\[
\dot{\eta}_g = \Psi_c^T \eta_g + C y, \quad \eta_g(0) = 0_n,
\]

where \( \eta_f, \eta_g \in \mathbb{R}^n \).

Then, according to the results in [27], the regression equation is obtained:

\[
x = \Phi(t-(\psi-a)B_a) + e^{\Psi_c t} x(0), \quad \Phi = [\Phi_{f_1}, \ldots, \Phi_{f_{2n}}], \quad \Phi_i = T_i \eta_{f_1}, \quad \Phi_{i+n} = T_i \eta_{f_2}, \quad i = 1, \ldots, n,
\]

where \( \Phi \in \mathbb{R}^{n \times 2n} \) is the measurable regressor, \( \theta_{(\psi-a)B_a} \in \mathbb{R}^{2n} = [\psi^T - a^T B_a^T]^T \) is the vector of unknown parameters, \( T_i \in \mathbb{R}^{n \times n} \) is the transformation matrix consisting of the coefficients of the polynomial in the numerator of the vector function \((sI - \Psi_c)^{-1} e_i\), and \( e_i \) is a vector, which is filled with zeros, but its \( i \)th element is equal to one.

Having multiplied Eq. (2.2.16) by the matrix \( C^T \), the measurable regression function is obtained

\[
\ddot{z} = y - C^T \Phi \theta_\psi = C^T \Phi \theta_{(\psi-a)B_a} - C^T \Phi \theta_\psi + C^T e^{\Psi_c t} x(0) = \ddot{\theta}_{-aB_a} \ddot{\varphi},
\]

\[
\ddot{\varphi} = \left[ C^T \Phi \ C^T e^{\Psi_c t} \right]^T, \quad \ddot{\theta}_{-aB_a} = \left[ -a^T B_a^T x(0) \right]^T, \quad \theta_\psi = \left[ \psi^T \ 0_{1 \times n} \right]^T,
\]

where \( \ddot{\varphi} \in \mathbb{R}^{3n} \) and \( \ddot{\theta}_{-aB_a} \in \mathbb{R}^{3n} \).

To obtain a regression with a scalar regressor \( \Delta \) from (2.2.17), the procedure from [22, 28] is applied, according to which the following extension scheme is introduced:

\[
\dot{\ddot{z}}_f = -l \ddot{z}_f + \ddot{\varphi}_f^T, \quad \ddot{z}_f(0) = 0,
\]

\[
\ddot{\varphi}_f = -l \dot{\ddot{\varphi}}_f + \ddot{\varphi}_f^T, \quad \ddot{\varphi}_f(0) = 0,
\]

where \( l > 0 \).

Passing the function \( \ddot{z} \) and the regressor \( \ddot{\varphi} \) through (2.2.18), the extended regression equation is obtained:

\[
\ddot{z}_f = \ddot{\varphi}_f \ddot{\theta}_{-aB_a},
\]

where \( \ddot{z}_f \in \mathbb{R}^{3n} \) and \( \ddot{\varphi}_f \in \mathbb{R}^{3n \times 3n} \).

Premultiplying Eq. (2.2.19) by \( \text{adj} \{ \ddot{\varphi}_f \} \) and using the equality \( \text{adj} \{ \ddot{\varphi}_f \} \ddot{\varphi}_f = \text{det} \{ \ddot{\varphi}_f \} I \), it is obtained

\[
z(t) = \varphi \ddot{\theta}_{-aB_a},
\]

\[
z(t) = \text{adj} \{ \ddot{\varphi}_f \} \ddot{z}_f,
\]

\[
\varphi(t) = \text{det} \{ \ddot{\varphi}_f \},
\]

where \( \varphi \) and \( z \in \mathbb{R}^{3n} \) are a measurable regressor and a function, respectively.
Remark 4. One can also use the scheme (2.1.13)–(2.1.14) to obtain the extended regression (2.2.19); this adds one degree of freedom to the algorithm (2.2.20). Compared with (2.1.13)–(2.1.14), the advantage of the scheme (2.2.18)–(2.2.19) is the need to select lower number of parameters. It is also admissible to use the extension scheme (2.2.18)–(2.2.19) instead of (2.1.13)–(2.1.14) for the state feedback control problem.

Taking into account the definition of $\bar{\phi} - aB_0$ and $\phi \in R$ in (2.2.20), one can readily obtain the regression equations

\begin{align*}
z_a &= H_1 z = \varphi a, \quad z_b = H_2 z = \varphi b, \\
H_1 &= \begin{bmatrix} -I_{n \times n} & 0_{n \times 2n} \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0_{(m+1) \times (2n-(m+1))} & I_{(m+1) \times (m+1)} & 0_{(m+1) \times n} \end{bmatrix},
\end{align*}

(2.2.21)

where $z_a \in R^n$ and $z_b \in R^{m+1}$ are measurable functions and $H_1 \in R^{n \times 3n}$ and $H_2 \in R^{(m+1) \times 3n}$ are transformation matrices.

Then, multiplying the condition (2.2.12) by $\varphi$, matching the coefficients of the same powers of the differentiation operator $p$ on the left- and right-hand sides of the resulting equation, and substituting the corresponding scalar equations for $a_{n-1} \ldots a_0$ and $b_m \ldots b_0$ from (2.2.21), we write the matrix regression equation for the unknown parameters $\theta$ of the ideal control law (2.2.9),

\begin{align*}
N &= M\theta,
\end{align*}

(2.2.22)

where $M \in R^{2n \times 2n}$ and $N \in R^{2n}$ are a measurable regressor and a function, respectively.

Now the matrix regressor $M$ is transformed again into a scalar one $\Delta$. To this end, Eq. (2.2.22) is multiplied by the matrix $\text{adj} \{M\}$ and also the equality $\text{adj} \{M\} M = \det \{M\} I$ is applied

\begin{align*}
Y(t) &= \Delta (t) \theta, \\
Y(t) &= \text{adj} \{M(t)\} N(t), \\
\Delta(t) &= \det \{M(t)\},
\end{align*}

(2.2.23)

where $Y \in R^{2n}$ and $\Delta \in R$ is a scalar regressor.

Remark 5. According to the results in [28], when using the extension scheme (2.2.18)–(2.2.20), the implication $\bar{\varphi} \in \text{FE} \Rightarrow \varphi \in \text{FE}$ holds. Since the regressor $M$ is determined by the single dynamic variable $\varphi$ and does not become singular for $\varphi = \text{const}$ in accordance with Remark 3, one has $\varphi \in \text{FE} \Rightarrow M \in \text{FE} \Rightarrow \Delta \in \text{FE}$. Thus, the excitation of the original regressor $\bar{\varphi}$ does not vanish for the resulting parametrization (2.2.13)–(2.2.23), and one has $\bar{\varphi} \in \text{FE} \Rightarrow \Delta \in \text{FE}$.

The next aim is to transform the regression (2.2.23) into the regression (2.2.1) with a regressor $\Omega$ that has properties necessary from the viewpoint of the Theorem. To this end, we use the results from [20, 23, 24] and introduce the filter (2.1.19) with exponential forgetting.

The extended regressor $\Delta^2$ and the function $\Delta Y$ are passed through the filter (2.1.19),

\begin{align*}
\Upsilon &= \Omega \theta, \quad \Omega(t) = \int_{t^+}^{t} e^{-\sigma \tau} \Delta^2(\tau) \, d\tau, \quad \Upsilon(t) = \int_{t^+}^{t} e^{-\sigma \tau} \Delta(\tau) Y(\tau) \, d\tau,
\end{align*}

(2.2.24)

where $\Upsilon \in R^{2n}$ and $\Omega \in R$.  

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According to Propositions 1 and 2, for $\Delta \in \text{FE}$ the regressor $\Omega$ has the properties required by the Theorem.

**Proposition 3.** If $\Delta \in \text{FE}$ on the interval $[t^+_r; t_e]$ and $\Delta \in L_\infty$ or $|\Delta| \leq c_1 e^{c_2 t}$ and $\sigma > 2c_2$, then

1. $\forall t \geq t^+_r \Omega(t) \in L_\infty$, $\Omega(t) \geq 0$.
2. $\forall t \geq t_e \Omega(t) > 0$, $0 < \Omega_{LB} \leq \Omega(t) \leq \Omega_{UB}$,

where $\delta_\Delta = \sup_{t \geq 0} \max_{t \geq 0} |\Delta(t)|$, $c_3 = \sigma - 2c_2 > 0$.

The correctness of Proposition 3 follows from the proofs of Propositions 1 and 2, and the estimates of $\Omega_{LB}$ and $\Omega_{UB}$ coincide with the estimates given in (2.1.21).

Then, if the parameters of the control law (2.2.6) are adjusted in accordance with (2.2), then the error $\xi$ for the output feedback control problem is exponentially stable.

Thus, the output feedback adaptive control system with the adaptive law (2.2) consists of the control law (2.2.6) as well as the procedures (2.2.13)–(2.2.23) to obtain computing the linear regression $Y(t) = \Delta(t) \theta$ and (2.2.24) to transform it.

Compared with the law (1.5), the proposed adaptive law (2.2) does not require the transfer function $C^T (pI - A_{\text{ref}})^{-1} B$ to be strictly positive real and ensures the exponential convergence of the error $\xi$ to zero under the weaker condition $\Delta \in \text{FE}$. Compared with other approaches [3, 13, 14], in addition to overcoming the condition of strict positive realness, the proposed law (2.2)

1. Ensures the elementwise monotonicity of the adjustable parameters $\hat{\theta}$.
2. Is much easier to be implemented in practice.
3. Does not require the sign/value of the plant gain $b_m$ to be known.
4. Is less demanding on the manual selection of the gain $\gamma$.

### 3. RESULTS AND DISCUSSION

#### 3.1. Comparison with Existing Adaptive Control Concepts

Traditionally, adaptive control systems are divided into direct [1–3, 26], indirect [1–3, 26], and combined/composite [4, 5] ones. As for direct adaptive control systems, the control law parameters are adjusted directly, for example, in accordance with the laws (1.3) or (1.5); in indirect adaptive control systems, the plant parameters are first identified, and then they are recalculated into the control law parameters in accordance with the matching conditions (2.1.5) or (2.2.12). So far, there is no well-established definition of combined/composite adaptive control systems, and therefore, this class of adaptive systems includes modifications of direct adaptive laws based on the combination (summation) of various adaptation laws and robust modifications [4–12].

The law (2.2) proposed in the present paper is direct according to the above classification, because it directly adjusts the control law parameters without identifying the plant parameters. However, the parametrization (2.1.9)–(2.1.12) or (2.2.13)–(2.2.17) resulting in (2.2) was previously used only in systems with indirect adaptation. In the present paper, owing to the DREM regressor scalarization procedure, it proved possible to propose a new type of parametrization using (2.1.11) and (2.2.17) based on the analytical transition from scalar regressions with respect to the plant parameters $\hat{\theta}_{AB}$ or $\hat{\theta}_{A-B_o}$ to matrix regression with respect to the unknown controller parameters $\theta$. This parametrization method permits one to synthesize a direct adaptive law that does not require knowledge of the sign/value of the vector $B$ or the high-frequency coefficient $b_m$ of the plant transfer.
function and, unlike the recent papers [20, 29], does not require the use of nonlinear operations for protecting the adjustable parameter in the feedforward part of the control law from crossing zero.

Earlier, it was generally accepted in the literature on adaptive control [1–3] that it is possible to strictly guarantee the asymptotic stability of the classical error equation (1.1) only by using its states in the adaptive law. As for the output feedback adaptive control problem, this led to the requirement of the strict positive realness, which demands that the measurable output of the differential error equation is described by a strictly positive real transfer function

\[ H(p) = C^T (pI - A_{\text{ref}})^{-1} B. \]

In the present paper, using a corollary of the Kalman–Yakubovich–Popov lemma, it is shown that the asymptotic stability of the differential error equation under condition (1.6) can be guaranteed by an adaptive law that does not use the error vector or its measurable elements; this allowed us too relax the strict positive realness condition. In fact, the goal of the baseline adaptive laws (1.3) and (1.5) is to asymptotically stabilize the output \( e_{\text{ref}} \) or \( \varepsilon \) of the differential error equation, while the goal of the proposed adaptive law is the asymptotic stabilization of the input \( \tilde{\theta}^T \omega \), thus making it possible to replace the requirement of strict positive realness of the transfer function \( H(p) = d \) with respect to the output of the error equation by the requirement that the virtual transfer function \( H(p) = \gamma \) is strictly positive real with respect to its input, a condition that is always satisfied.

An interesting result is given by the condition obtained for the exponential stability of the developed closed-loop adaptive control system, which can be extracted from (A.6),

\[ \tilde{\theta}^T \omega \omega^T \tilde{\theta} - \tilde{\theta}^T \gamma \Omega^2 \tilde{\theta} \leq 0. \]  

Condition (3.1.1) can be interpreted as the requirement to select the gain \( \gamma \) to be sufficiently high so as to cover the system uncertainty concentrated in the term \( \tilde{\theta}^T \omega \omega^T \tilde{\theta} \). In the present paper, to satisfy condition (3.1.1), we suggest to determine the gain \( \gamma \) in accordance with (2.2), but such a choice is conservative and nonunique. Let us discuss one option with a less conservative selection of the coefficient \( \gamma \),

\[ \gamma = \begin{cases} 
1 & \text{if } \Omega = 0 \\
\frac{\gamma_0 \omega \omega^T + \gamma_1}{\Omega^2} & \text{otherwise.} 
\end{cases} \]  

(3.1.2)

For \( \gamma_0 \geq 1, \gamma_1 = 0, \) and \( \Omega \neq 0, \) choice of the coefficient \( \gamma \) according to (3.1.2) permits one to compensate for the influence of the uncertainty of the system \( \tilde{\theta}^T \omega \omega^T \tilde{\theta} \) with minimum nonpositive margin \( (1 - \gamma_0) \tilde{\theta}^T \omega \omega^T \tilde{\theta} \). However, when using definition (3.1.2), the elementwise monotonicity of parametric errors \( \tilde{\theta} \) is violated, being replaced by the monotonicity of the norm of the vector \( \tilde{\theta} \); this degrades the quality of the resulting estimates and the adaptive control quality in general. Therefore, one direction for improving the proposed adaptive system is the search for a scalar \( \gamma \) that is a solution of the optimization problem

\[ \gamma_{\text{opt}} = \arg \min_{\gamma \geq 1 \gamma_0} \left[ \tilde{\theta}^T \omega \omega^T \tilde{\theta} - \tilde{\theta}^T \gamma \Omega^2 \tilde{\theta} \right]. \]  

(3.1.3)

It should also be noted here that the operation of division by the square of the regressor \( \Omega \) used in definition (2.2) or (3.1.2) of the coefficient \( \gamma \) is a “safe” and justified procedure, because the regressor \( \Omega \) is a positive semidefinite function vanishing only at the time \( t_e^+ \) of the beginning of the excitation interval \( [t^+_e; t_e] \).

Thus, the main differences between the proposed direct adaptive law (2.2) and those known in the literature are:
1. Elementwise monotonicity of the controller parameters $\hat{\theta}$ adjustment process.

2. A relaxed condition for the exponential convergence of the adjustable controller parameters $\hat{\theta}$ to their true values $\theta$.

3. The sign/value of the vector $B$ or the gain $b_m$ does not need to be known.

4. The possibility of an a priori choice of the convergence minimum rate of the augmented tracking error $\xi$ to zero in the absence of disturbances.

It should be noted that properties 1–4 and all the properties stated in the Theorem are provided by the law (2.2) only if $\Delta \in \text{FE}$; in accordance with Remarks 2 and 5, this requires that $\Phi \in \text{FE}$ for the state feedback control problem and $\bar{\varphi} \in \text{FE}$ for the output feedback control problem. If condition (1.6) is not satisfied for $\Phi$ and $\bar{\varphi}$, then the law (2.2) (unlike the laws (1.3) and (1.5)) does not even guarantee that the tracking error $e_{\text{ref}}$ is bounded, because $\Omega \equiv 0$ for $\Delta \in \text{FE}$ and the control law parameters are not adjusted according to (2.2). The finite excitation condition for the regressor (1.6) is very weak from the practical point of view, and for the linear plants (2.1.1) and (2.2.1) of the considered class it is ad hoc satisfied in tracking problems (for $r \neq 0$) but may not be satisfied in stabilization ones (for $r = 0$). Therefore, if one needs to solve the adaptive stabilization problem using the law (2.2), it must be reduced to a tracking problem, for example, by setting $r = r_0 e^{-r_1 t}$, $r_1 > 0$. Stating more strict conditions under which condition (1.6) is satisfied for the regressors $\Phi$, $\bar{\varphi}$ of the plants (2.1.1) and (2.2.1) is beyond the scope of the present paper and is a promising topic for a separate study.

### 3.2. System Robustness under Initial Conditions and Disturbances

In practice, the state vector $x$, the output $y$, and the control $u$ are usually measured with some bounded noise. So the regression equation (2.1) acquires the form

$$\Upsilon = \Omega(\Delta) \theta + w,$$

where $w$ is the disturbance due to measurement noise and a mismatch between the initial conditions of the filters and of the plant. It was shown in [30] that, when using the integral filtering (2.1.19) with exponential forgetting, the disturbance $w$ is bounded, $\|w\| \leq w_{\text{max}}$.

In view of Eq. (3.2.1), the parameter adjustment law for the controller (2.1) becomes

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = -\gamma \Omega \left(\Omega \hat{\theta} - \Omega \theta - w\right) = -\gamma \Omega^2 \tilde{\theta} + \gamma \Omega w.$$

Based on (3.2.2), one can readily show that if the regressor $\Omega$ satisfies the conditions of the Theorem, then the parametric error $\tilde{\theta}$ exponentially converges to the bounded set $\|\tilde{\theta}\| \leq \frac{w_{\text{max}}}{\Omega_{\text{LB}}}$. Then, according to (1.1), the steady-state value of the tracking error $e_{\text{ref}}$ can be reduced by increasing $\Omega_{\text{LB}}$ or decreasing $w_{\text{max}}$. It is possible to decrease the value of $w_{\text{max}}$ by choosing the parameters of the filters (2.1.10), (2.1.13) or (2.2.15), (2.2.18), in particular, by choosing them so that all filters are low-pass filters. It is possible to increase $\Omega_{\text{LB}}$ by decreasing the value of the parameter $\sigma$ of the integral filter (2.1.19) with exponential forgetting. Thus, in the developed adaptive control system, there is a direct relationship between the controller parameter identification quality and the control quality.

A more extensive and rigorous analysis of the robustness of the developed adaptive system under measurement noise and other exogenous disturbances is beyond the scope of the present paper and is the subject of a separate study.
3.3. Computational Elimination of Regressor Excitation

In accordance with Remarks 2 and 3, the excitation of the original regressor is not mathematically eliminated in the proposed algorithms for implementing the adaptive law (2.2) for the state and output feedback adaptive control problems. However, when the algorithms (2.1.9)–(2.1.18) and (2.2.13)–(2.2.23) to obtain the regression (2.1) are implemented numerically, it is possible to eliminate the excitation of the regressor by a numerical procedure.

It is well known that modern modeling environments and programming tools for controllers operate with values of limited dimension. For example, in MATLAB/Simulink, numbers less than $10^{-324}$ and more than $10^{308}$ are considered equal to zero and infinity, respectively. The DREM procedure used at the stage (2.1.15) and (2.2.20), depending on the degree of excitation of the original regressor, the parameters, and the number of selected filters (2.1.13), can generate a regressor $\phi$ of a sufficiently small amplitude (for the sake of concreteness of further reasoning, let $\phi = 10^{-81} = \text{const}$). Further, the resulting scalar regressor $\phi$ is transformed by a number of proposed algebraic operations to obtain a regressor $\Omega$ with the properties required by the Theorem. As a result of these operations, the regressor $\Delta$ calculated on the basis of $\phi$ can take values less than $10^{-324}$, which are equivalent to zero for MATLAB/Simulink, which is the computational elimination of regressor excitation.

For a numerical description of this problem, let us reveal the functional dependence for the quantity $\Delta$ in the definition of the regressor $\Omega$. For the case of state feedback control, substituting $\Delta = \bar{\Delta} \bar{\Delta}^T = \phi^2 B^T B$ into (2.1.19), we obtain

$$
\Omega (t) = \int_{t^\sigma}^{t} e^{-\sigma \tau} \Delta^2 \, d\tau = (B^T B)^2 \int_{t^\sigma}^{t} e^{-\sigma \tau} \phi^4 \, d\tau. \tag{3.3.1}
$$

In the considered special case of $\phi = 10^{-81} = \text{const}$, in (3.3.1) we obtain $\phi^4 = 10^{-324}$, this means that $\phi^4 \equiv 0$ and $\Omega \equiv 0$ when implementing (3.3.1) in MATLAB/Simulink.

For the case of output feedback control, substituting $\Delta = \det \{M\} = \phi^{2n} \det \{M_{ss}\}$ into (2.2.19), it is obtained

$$
\Omega (t) = \int_{t^\sigma}^{t} e^{-\sigma \tau} \Delta^2 \, d\tau = \det^2 \{M_{ss}\} \int_{t^\sigma}^{t} e^{-\sigma \tau} \phi^{4n} \, d\tau, \tag{3.3.2}
$$

where $M_{ss} = \phi^{-1} M$ is the matrix obtained from Eq. (2.2.12) by writing the equations of static rather than dynamic (2.2.17) regression with respect to the parameters $\theta$.

In the considered special case of $\phi = 10^{-81} = \text{const}$, we have $\phi^{4n} = 10^{-324n}$; when implementing (3.3.1) in MATLAB/Simulink, this is equivalent to $\phi^{4n} \equiv 0$ and $\Omega \equiv 0$.

Thus, from the viewpoint of analytical calculations there are no barriers to adjust the control law parameters when $\phi = 10^{-81} = \text{const}$, but from the viewpoint of a specific numerical implementation in MATLAB/Simulink, adjustment of the control law parameters is impossible for $\phi = 10^{-81} = \text{const}$, because $\Omega = 0 \forall t$.

Having not a static regressor $\phi = 10^{-81} = \text{const}$ but a dynamic one such that $\forall t \in [t^\sigma; t_0]$ $\phi < 10^{-81}$, we are able to adjust the controller parameters starting from some point $t_0$ in time. Then the time $t_0$ bounds the maximum achievable rate of convergence of the estimates for the unknown parameters to their true values and hence the adaptive control quality.
The problem of computational elimination of the regressor excitation can be overcome by choosing sufficiently large gains $\alpha_f$ of the filters (2.1.13) or by using the procedure for normalizing the excitation of the scalar regressor $\varphi$ proposed in [31]. In accordance with [31], we introduce the normalizing function

$$f(\varphi) := \text{sat}(\varphi) = \begin{cases} 
\text{sgn}(\varphi) \varphi_{\text{min}} & \text{if } |\varphi| \leq \varphi_{\text{min}} \\
\varphi & \text{otherwise.}
\end{cases} \quad (3.3.3)$$

The regressions (2.1.15) and (2.2.20) are divided by the normalizing function (3.3.3),

$$z_n(t) = \varphi_n(t)\bar{\theta}_{AB}, \quad z_n(t) = \varphi_n(t)\bar{\theta}_{-A\theta}, \quad z_n := \frac{z}{f(\varphi)}; \quad \varphi_n := \frac{\varphi}{f(\varphi)} = \begin{cases} 
|\varphi|\varphi_{\text{min}}^{-1} & \text{if } |\varphi| \leq \varphi_{\text{min}} \\
1 & \text{otherwise,}
\end{cases} \quad (3.3.4)$$

where $\varphi_n$ is the normalized scalar regressor.

Then, if we define the admissible values of the parameter $\varphi_{\text{min}}$ as

$$\varphi_{\text{min}} \leq 10^{-81}, \quad (3.3.5)$$

then, by choosing the values of the parameter $\varphi_{\text{min}}$, one can set the time $t_0$ and hence also the maximum attainable rate of convergence of the unknown parameters to their true values.

Thus, for example, for the previously considered static regressor $\varphi = 10^{-81} = \text{const}$ with $\varphi_{\text{min}} = 10^{-81}$ we obtain $\varphi_n = 1$. Substituting the value $\varphi_n = 1$ for $\varphi$ into (3.3.1) and (3.3.2), one can readily verify that, when using the normalization (3.3.3)–(3.3.5), the regressor excitation is not eliminated by the computational procedure in MATLAB/Simulink.

Thus, in the numerical implementation of the proposed adaptive law by means of programming controller or in simulation environments, it is necessary to take into account the possibility of computational elimination of the regressor excitation.

### 3.4. Loss of Identification Alertness and Adaptive Control of Plants with Piecewise Constant Unknown Parameters

The present paper discusses the problem of adaptive control of plants with time-invariant unknown parameters; however, in practice, the problem of adaptive control of plants with time-varying parameters is of greater interest. Such parameters can be piecewise-constant or change their values continuously. Therefore, any solution of the adaptive control problem for plants with time-invariant parameters must admit extension to at least one of the classes of time-varying parameters.

A straightforward application of the adaptive law (2.2) to solving the adaptive control problem for plants with time-varying parameters encounters two problems:

1. The loss of sensitivity of the filter (2.1.19) in $(3 \div 5)\sigma^{-1}$ to the new data [24] and the superpositional mixing of information about regressions with different parameters by the filter (2.1.19) [23].

2. Generating scalar perturbed regression equations by the DREM procedure (2.1.14), (2.1.15) or (2.2.19), (2.2.20) when applied to regressions with time-varying parameters [32].
To solve these problems in the case of piecewise constant unknown plant parameters and piecewise constant reference signal $r$, it was proposed in [32, 33] to use a new extension procedure based on integral filtering with exponential forgetting and periodic reset instead of (2.1.14) and (2.2.19),

$$
\bar{\varphi}_f (t_k) = \bar{\varphi}_f (t_r^+) = 0; \quad \bar{z}_f (t_k) = \bar{z}_f (t_r^+) = 0,
$$

$$
t_r^+ (0) = 0,
$$

$$
t_r^+ = \begin{cases} 
  t & \text{if } r(t) \neq r(t - T_d) \\
  t_r^+ & \text{otherwise.}
\end{cases}
$$

$$
\bar{\varphi}_f^T (t) = \int_{t_k}^{t} \exp \left( - \int_0^{\tau} \sigma d\tau_1 \right) \bar{\varphi} (\tau) \bar{\varphi}^T (\tau) d\tau; \quad \bar{z}_f^T (t) = \int_{t_k}^{t} \exp \left( - \int_0^{\tau} \sigma d\tau_1 \right) \bar{z} (\tau) \bar{z}^T (\tau) d\tau,
$$

and add a periodic reset procedure to the filter (2.1.19) on the interval $[t_r^+; t_e]$ of excitation caused by a change of the piecewise-constant reference signal $r$,

$$
\Omega (t_k) = \Omega (T^+) = \Omega (t_r^+) = \bar{\varphi} (t_k) = \bar{\varphi} (T^+) = \bar{\varphi} (t_r^+) = 0,
$$

$$
\Omega (t) = \begin{cases} 
  \int_{t_k}^{t} e^{-\int_0^\tau \sigma d\tau_1} \Delta^2 (\tau) d\tau & \text{if } t < T^+ \\
  \int_{T^+}^{t} e^{-\int_0^\tau \sigma d\tau_1} \Delta^2 (\tau) d\tau & \text{if } t \geq T^+,
\end{cases}
$$

$$
\bar{\varphi} (t) = \begin{cases} 
  \int_{t_k}^{t} e^{-\int_0^\tau \sigma d\tau_1} Y (\tau) \Delta (\tau) d\tau & \text{if } t < T^+ \\
  \int_{T^+}^{t} e^{-\int_0^\tau \sigma d\tau_1} Y (\tau) \Delta (\tau) d\tau & \text{if } t \geq T^+.
\end{cases}
$$

where $t_k = T \cdot \text{floor} (t/T)$ is the start time of the new filtering interval ($\text{floor}(\cdot)$ is the integer rounding function), $0 < T < t_e - t_r^+$ is the filter window width, $T_r = t_r^+ + T_\Delta$ is the final time instant of the filtering interval, which belongs to the excitation time range, $T_d \to 0$ is the constant of the algorithm for detecting a change in the piecewise constant reference signal $r$, and $0 < T_\Delta < t_e - t_r^+$ is a constant.

Using the filterings (3.4.1) and (3.4.2), each excitation interval $[t_r^+; t_e]$ caused by a change in the piecewise constant reference signal $r$ is divided into small filtering intervals $[t_k; t_{k+1}]$, and as a result the problems indicated earlier are solved:

1. The filter (2.1.19) becomes sensitive to new data, and at each time $t_r^+$ or $t_k$ possible superpositional mixing of information about regressions with different parameters is eliminated [33].

2. When using the extension algorithm (3.4.1), the DREM procedure generates scalar regression equations with an adjustable disturbance level [32].

The main properties of the proposed adaptive control system, which is based on the regression $Y(t) = \Delta(t)\theta$ generated with the use of (3.4.1) and (3.4.2), are formulated as the following propositions [33]. If the time $t_j$ of change in the plant parameters belongs to the filtering interval $[t_k; t_{k+1}]$, then there exists a $T_{0k} \in [t_k; t_{k+1}]$ such that the adaptive law (2.2) guarantees the interval boundedness of the parametric error $\tilde{\theta}$ for all $t \in [T_{0k}; t_{k+1}]$. If, in addition, $t_{j+1} \notin [t_{k+1}; t_{k+2}]$, then there
exists a \( T_{0(k+1)} \in [t_{k+1}; t_{k+2}] \) such that for all \( t \in [T_{0(k+1)}; t_{k+2}] \) the adaptive law (2.2) guarantees the exponential boundedness of the error \( \hat{\theta} \). If \( t_j \in [T_+; t_\epsilon] \), then for all \( t \geq t_\epsilon \) the adaptive law (2.2) guarantees the uniform ultimate boundedness of the error \( \hat{\theta} \). If \( t_j \notin [T_+; t_\epsilon] \), then for all \( t \geq t_\epsilon \) the adaptive law (2.2) ensures the exponential convergence of \( \hat{\theta} \) to zero.

Thus, according to the results in [32, 33], the described modification (3.4.1), (3.4.2) of the procedure to obtain the regression \( \Upsilon(t) = \Omega(t) \theta \) permits one to derive an adaptive law, which provides interval exponential boundedness of the error when identifying the parameters \( \theta \) that are piecewise constant on the excitation interval \([t_\tau; t_\epsilon]\) and time-invariant beyond it. However, the interval exponential boundedness of the error \( \hat{\theta} \) only is proved in [33], and therefore, despite the direct relationship between the estimation quality of ideal controller parameters and the control quality, a promising direction for further research, which is beyond the scope of the present paper, is the analysis of interval properties of the error \( \xi \) when using the modifications (3.4.1), (3.4.2) and solving the state and output feedback adaptive control problems for plants with piecewise constant unknown parameters.

4. NUMERICAL EXAMPLE

We simulate the state and output feedback adaptive control systems proposed in Sec. 2 of this paper in the MATLAB/Simulink environment. The simulation is carried out using numerical integration by the Euler method with constant discretization step \( \tau_s = 10^{-4} \) s.

4.1. State Feedback Adaptive Control

As for the state feedback control problem, the unstable second-order aperiodic link has been chosen as a plant:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\] (4.1.1)

The desired control quality was determined by the reference model

\[
\dot{x}_{\text{ref}} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} x_{\text{ref}} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} r, \quad x_{\text{ref}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\] (4.1.2)

The value of the reference \( r \), the parameters of the filters (2.1.10), (2.1.13), (2.1.18), the parameters \( \gamma_0 \) and \( \gamma_1 \), and the initial values of the parameters of the control law (2.1.3) were set in accordance with the expression

\[
\begin{align*}
& r = 1; \quad l = 1; \quad \alpha_i^f = \beta_i^f = i, \quad i \in 1, \ldots, 5; \quad \sigma = 5/10; \\
& \gamma_0 = 1; \quad \gamma_1 = 0; \quad \hat{\theta}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T.
\end{align*}
\] (4.1.3)

Figure 1 shows the transient processes of (a) the states of the plant (4.1.1) and the reference model (4.1.2), (b) the adjustable parameters of the control law \( \hat{\theta} \), (c) the regressor \( \Omega \), and (d) the variable \( \lambda_{\text{max}}(\omega \omega^T) \).

The transient curves in Fig. 1 validate the theoretical conclusions and confirm the exponential stability of the augmented tracking error \( \xi \) when using the proposed adaptive law (2.2).

Then, it was demonstrated that the adaptive law (2.2) was invariant to the sign of the vector \( B \). To this end, we set the initial values of the parameters of the control law as follows:
Fig. 1. Transient processes of (a) the states of the plant (4.1.1) and the reference model (4.1.2), (b) the adjustable control law parameters, (c) the regressor $\Omega$, and (d) the variable $\lambda_{\text{max}}(\omega^\top\omega)$. 

$\hat{\theta}(0) = [0 0 -1]^\top$. Figure 2 shows the transient processes of (a) the states of the plant (4.1.1) and the reference model (4.1.2) and (b) the adjustable parameters of the control law.

As can be seen from the curves in Fig. 2, the control law (2.2) demonstrated effectiveness when the sign of the gain vector $B$ was unknown.
Then the developed system was applied to solve the stabilization problem, in the stabilization mode, which is important for practice. To this end, we set $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and, in accordance with the recommendations given in Sec. 3.1, chose $r = e^{-t}$. The initial conditions of the reference model and the remaining loop parameters were set in accordance with (4.1.2) and (4.1.3).

The simulation results shown in Fig. 3 demonstrate the efficiency of the proposed adaptive law (2.2) in solving the problem of adaptive state feedback stabilization.
Figure 4 presents the norms of the error $\xi$ for various values of the coefficient $\gamma_0$.

Figure 4 confirms the conclusions obtained in the proof of the Theorem and demonstrates the possibility of increasing the rate of convergence of the error $\xi$ to zero by increasing the coefficient $\gamma_0$.

The next aim was to check whether the value of the minimum rate of convergence of the error $\xi$ to zero in the developed adaptive control system could be set by choosing the parameter $\gamma_1$. To this end, we simulated the system for $\gamma_1 = 0$, $\gamma_1 = 10$, and different values of the reference $r$. Figure 5 shows the transients of $\|\xi\|$ for $\gamma_1 = 0$ and various $r$ (Fig. 5a) as well as for the same $r$ and $\gamma_1 = 10$ (Fig. 5b).

The results of the experiment shown in Fig. 5 confirmed that the value of the minimal rate of the error $\|\xi\|$ convergence to zero could be adjusted by change of the $\gamma_1$ value. In this experiment, over the time range $t \in [0; 0.04]$, the errors $\|\xi\|$ for various values of $r$ did not decrease due to the problem of computational elimination of the regressor excitation noted in Sec. 3.3.
4.2. Output Feedback Adaptive Control

The plant and reference model to solve the output feedback adaptive control problem were chosen as (4.1.1) and (4.2.2), but they were written as transfer functions:

\[ y = \frac{2}{p^2 - 2p - 4}u; \quad y_\text{ref} = \frac{8}{p^2 + 4p + 8}r. \]  

(4.2.1)

The only directly measurable signals were the output \( y = x_1 \) and the control signal \( u \).

The value of the reference \( r \), the parameters of the filters (2.2.4), (2.2.15), (2.2.16), (2.2.18), and (2.2.24), the parameters \( \gamma_0, \gamma_1 \), and the initial values of the output of the plant and the reference model, as well as the coefficients (2.1.3), were set as follows:

\[ r = 1; \quad \Lambda = -1; \quad \psi = \begin{bmatrix} 20 & 100 \end{bmatrix}^T; \quad l = 0.1; \]

\[ T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -100 \end{bmatrix}; \quad T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 20 \end{bmatrix}; \quad y(0) = y_{\text{ref}}(0) = 0; \]

\[ \sigma = 5/10; \quad \gamma_0 = 1; \quad \gamma_1 = 0; \quad \hat{\theta}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T. \]  

(4.2.2)

Figure 6 shows the transient processes of (a) the plant output \( y \) and the reference model \( y_{\text{ref}} \), (b) the adjustable control law parameters \( \hat{\theta} \), (c) the regressor \( \Omega \), and (d) the value of \( \lambda_{\text{max}}(\omega \omega^T) \).

The simulation results validated the theoretical conclusions and confirmed the exponential stability of the augmented tracking error \( \xi \) for the output feedback control problem using the proposed adaptive law (2.2).

Then, it was demonstrated that the adaptive law (2.2) was invariant to the sign of the gain \( b_m \).

To this end, the initial values of the control law coefficients were set as \( \hat{\theta}(0) = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T \).

Figure 7 shows the transients of (a) \( y \) and \( y_{\text{ref}} \) and (b) the adjustable parameters \( \hat{\theta} \).

The transient processes shown in Fig. 7 confirm the performance of the adaptive law (2.2) even with an unknown sign of the gain \( b_m \).

Then the developed system was simulated in the practically important stabilization mode. To this end, we set \( y(0) = 1 \) and, in accordance with the recommendations given in Sec. 3.1, chose \( r = e^{-1t} \).

The initial conditions of the reference model and the remaining parameters of the scheme were taken to be equal to the values in (4.2.1) and (4.2.2).

The transients in Fig. 8 demonstrate the efficiency of the developed adaptive control system when solving the output feedback adaptive stabilization problem.

The next experiment was to compare the performance of the proposed output feedback system for different values of \( \gamma_0 \). Figure 9 shows the transients in \( \|\xi\| \) obtained for various values of the coefficient \( \gamma_0 \).

The transient curves in Fig. 9 confirm the conclusions obtained in the proof of the Theorem and demonstrate the possibility of increasing the rate of convergence of the error \( \xi \) to zero by increasing the coefficient \( \gamma_0 \).

Finally, it was checked whether the developed output feedback adaptive control system allowed one to set the value of the minimal rate of the error \( \xi \) convergence to zero with the help of \( \gamma_1 \) value choice, as it was noted in Remark 1. To this end, the system was simulated with \( \gamma_1 = 0, \gamma_1 = 10, \) and various reference \( r \) values. Figure 10 shows the transients in \( \|\xi\| \) for (a) \( \gamma_1 = 0 \) and various \( r \) and (b) for the same \( r \) and \( \gamma_1 = 10 \).
Fig. 6. Transient processes in (a) $y$ and $y_{ref}$ and (b) the adjusted parameters $\hat{\theta}$.

The results of the experiment shown in Fig. 10 confirmed that the value of the minimal rate of the error $\|\xi\|$ convergence to zero could be adjusted by change of $\gamma_1$ value.

In this experiment, on the interval $t \in [0; 0.1]$ the errors $\|\xi\|$ for various values of $r$ did not decrease due to the problem of computational elimination of the regressor excitation noted in Sec. 3.3.
5. CONCLUSIONS

The new adaptive law to adjust the controller parameters was proposed, which was equally applicable to the linear error models with a measurable state (1.1) and output (1.2). Under the conditions that the regressor was finitely exciting $\Delta \in \text{FE}$ and the unknown controller ideal parameters $\theta$ were time-invariant, the law provided the exponential stability of the error $\xi$, the elementwise monotonicity of the adjustable parameters $\hat{\theta}$ of the controller, and, additionally, the solution of
Fig. 9. Dependence of $\|\xi\|$ on the values of the coefficient $\gamma_0$.

Fig. 10. Dependence of $\|\xi\|$ on the reference $r$ and the coefficient $\gamma_1$.

Another aim is to extend these results to control plants with time-varying parameters. The second part of the paper will be devoted to the problem of state and output feedback adaptive control of linear plants with piecewise-constant unknown parameters, and the third part – linear plants with time-varying parameters of a certain type.

**APPENDIX**

**Proof of the Theorem.** The solution of Eq. (2.2) has the form

$$\tilde{\theta}(t) = e^{i\tilde{\omega}(t)}\tilde{\theta}(t^+).$$

(A.1)
Since \( \text{sgn}(\gamma \Omega^2) = \text{const} > 0 \), it follows from (A.1) that \( |\dot{\theta}_i(t_a)| \leq |\dot{\theta}_i(t_b)| \forall t_a \geq t_b \), which was to be proved in the first part of the Theorem. Let us proceed to proving its second and third parts.

Since the matrix \( A_{\text{ref}} \) is Hurwitz, according to a corollary from the Kalman–Yakubovich–Popov lemma, one can always find some number \( d > 0 \) and the corresponding virtual output

\[
y_e = d\dot{\theta}^T \omega,
\]

which cannot be measured, such that the transfer function \( H(s) = d \) is strictly positive real and the following equations hold

\[
A_{\text{ref}}^T P + PA_{\text{ref}}^T = -qq^T - \mu P,
\]

\[
PB = \sqrt{2d}q.
\]

Then, to analyze the stability of the differential error equation (1.1) when using the adaptive law (2.2), the quadratic function is used

\[
V = \xi^T H \xi = e_{\text{ref}}^T P e_{\text{ref}} + \frac{1}{2} \hat{\theta}^T \hat{\theta}, \quad H = \text{blockdiag} \left\{ P, \frac{1}{2} I \right\},
\]

\[
\lambda_{\min}(H) \| \xi \|^2 \leq V (\| \xi \|) \leq \lambda_{\max}(H) \| \xi \|^2,
\]

where the matrix \( P \) corresponds to the solution of Eqs. (A.3).

The derivative of the quadratic function (A.4) according to Eqs. (1.1) and (2.2) has the form

\[
\dot{V} = e_{\text{ref}}^T (A_{\text{ref}}^T P + PA_{\text{ref}}) e_{\text{ref}} + 2\hat{\theta}^T \omega e_{\text{ref}} P B - \hat{\theta}^T \gamma \Omega^2 \hat{\theta}
\]

\[
= -\mu e_{\text{ref}}^T P e_{\text{ref}} - e_{\text{ref}}^T q q^T e_{\text{ref}} + 2\hat{\theta}^T \omega e_{\text{ref}} P B - \hat{\theta}^T \gamma \Omega^2 \hat{\theta}.
\]

Following (A.2), as \( d > 0 \), then it is acceptable to consider \( d = 0.5 \) for certainty. In this case, from (A.5) it is obtained

\[
\dot{V} = -\mu e_{\text{ref}}^T P e_{\text{ref}} - e_{\text{ref}}^T q q^T e_{\text{ref}} + 2\hat{\theta}^T \omega e_{\text{ref}} P B - \hat{\theta}^T \gamma \Omega^2 \hat{\theta}
\]

\[
= -\mu e_{\text{ref}}^T P e_{\text{ref}} - \left( e_{\text{ref}}^T q - \hat{\theta}^T \omega \right)^2 + \hat{\theta}^T \omega \omega^T \hat{\theta} - \hat{\theta}^T \gamma \Omega^2 \hat{\theta}
\]

\[
\leq -\mu e_{\text{ref}}^T P e_{\text{ref}} + \hat{\theta}^T \omega \omega^T \hat{\theta} - \hat{\theta}^T \gamma \Omega^2 \hat{\theta}.
\]

Let two different cases be considered: \( t < t_c \) and \( t \geq t_c \). As for the first one, in the worst case scenario, according to (2.1) and (2.2) \( \Omega = 0 \) and \( \| \dot{\theta} \| = \| \dot{\theta}(0) \| \). Then for each \( t < t_c \) we can rewrite Eq. (A.6) in the form

\[
\dot{V} \leq -\mu e_{\text{ref}}^T P e_{\text{ref}} + \hat{\theta}^T (0) \omega \omega^T \hat{\theta} (0) - \hat{\theta}^T \hat{\theta}
\]

\[
\leq -\mu e_{\text{ref}}^T P e_{\text{ref}} - \hat{\theta}^T \hat{\theta} + \hat{\theta}^T (0) \omega \omega^T \hat{\theta} (0) + \hat{\theta}^T (0) \hat{\theta} (0).
\]

The notion of maximum eigenvalue of the matrix \( \omega \omega^T \) on the time interval \([0; t_c]\) is introduced:

\[
\delta = \sup_{\forall t < t_c} \lambda_{\max} \left( \omega \omega^T \right).
\]

In view of (A.8), Eq. (A.7) can be rewritten for \( t < t_c \) in the form

\[
\dot{V} \leq -\mu \lambda_{\min}(P) \| e_{\text{ref}} \|^2 - \| \dot{\theta} \|^2 + (\delta + 1) \| \dot{\theta}(0) \|^2 \leq -\eta_1 V + r_B,
\]

where \( \eta_1 = \min \left\{ \frac{\mu \lambda_{\min}(P)}{\lambda_{\max}(P)}, 2 \right\} \) and \( r_B = (\delta + 1) \| \dot{\theta}(0) \|^2 \).
Having solved the differential equation \( (A.9) \), we obtain
\[
\forall t < t_e : V = e^{-n t} V(0) + \frac{r_B}{\eta_1}.
\] (A.10)

Taking into account \( \lambda_m \| \xi \|^2 \leq V \) and \( V(0) \leq \lambda_M \| \xi(0) \|^2 \), from (A.10) the following estimate is obtained for all \( t < t_e \) for the augmented tracking error vector:
\[
\| \xi \| \leq \sqrt{\frac{\lambda_M}{\lambda_m} e^{-n t} \| \xi(0) \|^2 + \frac{r_B}{\lambda_m \eta_1}} \leq \sqrt{\frac{\lambda_M}{\lambda_m} \| \xi(0) \|^2 + \frac{r_B}{\lambda_m \eta_1}}.
\] (A.11)

This implies the boundedness of \( \xi \) for all \( t < t_e \).

Then the second situation is considered. Taking into account the fact that the inequality \( 0 < \Omega_{LB} \leq \Omega \leq \Omega_{UB} \) holds for all \( t \geq t_e \) and the definition of the gain \( \gamma \), from (A.6) for \( t \geq t_e \) we obtain
\[
\dot{V} = -\frac{\mu e^T P e_{\text{ref}} + \bar{\theta}^T \omega \gamma_0 \lambda_{\max} (\omega \omega^T) + \gamma_1 \Omega^2}{\Omega^2} \bar{\theta} - \frac{\mu e^T e_{\text{ref}}}{\lambda_{\max}(P)} \| e_{\text{ref}} \|^2 - (\bar{\theta}^T \gamma_0 \lambda_{\max} (\omega \omega^T) I) \bar{\theta} \leq 0.
\] (A.12)

The following inequality holds for all \( \omega \):
\[
\bar{\theta}^T \omega \bar{\theta} - \bar{\theta}^T \gamma_0 \lambda_{\max} (\omega \omega^T) \bar{\theta} = \bar{\theta}^T \left( \frac{\omega \omega^T - \gamma_0 \lambda_{\max} (\omega \omega^T) I}{\lambda_{\max}(P)} \right) \bar{\theta} \leq -\kappa \bar{\theta}.
\] (A.13)

So (A.12) is rewritten as
\[
\dot{V} \leq -\frac{\mu e^T P e_{\text{ref}}}{\lambda_{\max}(P)} \| e_{\text{ref}} \|^2 - (\bar{\theta}^T \gamma_0 \lambda_{\max} (\omega \omega^T) I) \bar{\theta} \leq -\eta_2 V,
\] (A.14)

where
\[
\eta_2 = \min \left\{ \frac{\mu \lambda_{\min}(P)}{\lambda_{\max}(P)} ; 2(\kappa + \gamma_1) \right\}.
\]

Having solved the differential inequality (A.14), for \( t \geq t_e \) it is obtained:
\[
V \leq e^{-n t} V(t_e).
\] (A.15)

Taking into account \( \lambda_m \| \xi \|^2 \leq V(t_e) \leq \lambda_M \| \xi(t_e) \|^2 \), and the expression (A.11), from (A.15) for \( t \geq t_e \) we obtain the following estimate for the augmented tracking error vector:
\[
\| \xi \| \leq \sqrt{\frac{\lambda_M}{\lambda_m} e^{-n t} \| \xi(t_e) \|^2} \leq \sqrt{\frac{\lambda_M}{\lambda_m} \left( \frac{\lambda_M}{\lambda_m} \| \xi(0) \|^2 + \frac{r_B}{\lambda_m \eta_1} \right)}.
\] (A.16)

This, together with (A.11), implies that \( \xi \in L_\infty \) and that the error \( \xi \) exponentially converges to zero at a rate directly proportional to the parameters \( \gamma_0 \) and \( \gamma_1 \) for all \( t \geq t_e \); this is what was to be proved in the second and third statements of the Theorem.

**Proof of Proposition 2.** To prove the first part in Proposition 2, we take into account the fact that \( |\Delta| \leq c_t e^{c z t} \) and substitute the estimate \( \Delta^2 \leq c_t^2 e^{2 c z t} \) into the definition of the regressor \( \Omega \) to obtain
\[
\Omega(t) = \int_{t_\sigma}^t e^{-\sigma \tau} \Delta^2(\tau) d\tau \leq c_t^2 \int_{t_\sigma}^t e^{(2 c - \sigma) \tau} d\tau.
\] (A.17)
Taking into account the fact that $\sigma > 2c_2$, it is obtained from (A.17) that
\[
\Omega(t) \leq c_1^2 \int_{t^*_r}^{t} e^{-c_3 \tau} d\tau = \frac{c_1^2}{c_3} (1 - e^{-c_3 t}) \leq \frac{c_1^2}{c_3}.
\] (A.18)

Therefore, it follows that $\Omega(t) \in L_{\infty} \forall t \geq t^*_r$, as was to be proved in the first part of the proposition.

To prove the second part of the proposition, following the definition in (1.6), the finite excitation condition is written for the regressor $\Delta$ on the interval $[t^*_r; t_e]$,
\[
\int_{t^*_r}^{t_e} \Delta^2(\tau) d\tau \geq \alpha.
\] (A.19)

Then for $t \geq t_e$ the inequality is obtained
\[
\int_{t^*_r}^{t} \Delta^2(\tau) d\tau > 0.
\] (A.20)

Since $\forall t < t_e$ $e^{-\sigma t} > 0$ and inequality (A.20) holds, we have
\[
\Omega(t) = \int_{t^*_r}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau > 0, \quad \forall t \geq t_e.
\] (A.21)

For further proof, based on the definition of $\Omega$ the notation is introduced
\[
\Omega(t) = \int_{t^*_r}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau + \int_{t_e}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau.
\] (A.22)

Since $\Delta^2 \leq c_1^2 e^{2c_3 t}$, it follows that the first integral in (A.22) is bounded
\[
\int_{t^*_r}^{t} e^{-\sigma \tau} \Delta^2(\tau) d\tau \leq c_1^2 \int_{t^*_r}^{t_e} e^{-c_3 \tau} d\tau = \frac{c_1^2}{c_3} \left( e^{-c_3 t^*_r} - e^{-c_3 t_e} \right).
\] (A.23)

Taking into account (A.18) and (A.22), (A.23), for $t \geq t_e$ the inequality is obtained
\[
0 < \frac{c_1^2}{c_3} \left( e^{-c_3 t^*_r} - e^{-c_3 t_e} \right) \leq \Omega(t) \leq \frac{c_1^2}{c_3}.
\] (A.24)

The proof of Proposition 2 is complete.
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