A UNIFIED FINITE DIFFERENCE CHEBYSHEV WAVELET METHOD FOR NUMERICALLY SOLVING TIME FRACTIONAL BURGERS’ EQUATION

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Abstract. In this paper, we developed a unified method to solve time fractional Burgers’ equation using the Chebyshev wavelet and L1 discretization formula. First we give the preliminary information about Chebyshev wavelet method, then we describe time discretization of the problems under consideration and then we apply Chebyshev wavelets for space discretization. The performance of the method is shown by three test problems and obtained results compared with other results available in literature.

1. Introduction. Fractional calculus have been used by mathematicians since early 1700s for only pure theoretical calculations, however it has been gained more attention lately from all other areas of science and engineering after it has been recognized that fractional calculus can be used for the description of properties of real materials such as, chemical analysis of aqueous solutions, quantum mechanical calculations and dissemination of atmospheric pollutants etc. [9, 15, 10]. It has been showed that fractional calculus has more advantages than integer order derivatives that is, fractional derivatives can be used to describe memory and hereditary properties of physical materials [1, 2, 14]. This property was not recognized in integer order derivatives.

Wavelet methods have been widely used to obtain numerical solutions of differential equations because of properties such as simplicity, accuracy. Legendre and Chebyshev wavelets studied by many authors [7, 16, 17, 21, 18, 20, 8, 19, 4, 11, 12, 3] for obtaining numerical solutions of differential and integral equations. In this paper we combine Chebyshev wavelet method with so-called L1 discretization formula to obtain a numerical solution of time fractional Burgers’ equation. This method has a significant advantage over operational matrix with fractional order of Chebyshev wavelets because of its simplicity.

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For the application of the present method we use Burgers’ equation which is given as follows,

\[ D_\alpha^t u + uu_x - \nu u_{xx} = F(x,t) \quad 0 \leq x \leq 1, \quad 0 < \alpha \leq 1, \]

where \( F(x,t) \) is prescribed function and fractional derivatives are in the Caputo’s sense which is given as

\[ D_\alpha^t f(t) = \frac{\partial^n f(t)}{\partial t^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial \tau^n} d\tau, \]

\[ n - 1 \leq \alpha < n \]

The outline of this study is as follows:

In Section 2, we give preliminaries about the materials that are used in the study. In Section 3, we describe method of solution. In Section 4, we give the numerical results of the problems at the hand to demonstrate the performance of the proposed method. Finally we conclude the paper in Section 5.

2. Preliminaries and notations. In this section, we give some necessary definitions and mathematical preliminaries of the fractional derivative and Chebyshev wavelets.

**Definition 1.** Let \( n \) be the smallest integer exceeding \( \alpha \), the Caputo time fractional derivative operator of order \( \alpha > 0 \) is defined

\[ \frac{\partial^n u(x,t)}{\partial t^n} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n - 1 \leq \alpha < n \\ \frac{\partial^n u(x,t)}{\partial t^n} & \alpha = n. \end{cases} \]

**Definition 2.** L1 formula for the discretization of the fractional derivative of the Caputo sense given as [10]:

\[ \left. \frac{\partial^n f(t)}{\partial t^n} \right|_{t_n} = \frac{(\Delta t)^{n-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k^n [f(t_{n-k}) - f(t_{n-1-k})] + O(\Delta t) \]

where \( \Delta t \) is step size of the variable \( t \), \( 0 < \alpha < 1 \) and

\[ b_k^n = (k+1)^{1-\alpha} - k^{1-\alpha} \]

2.1. Chebyshev wavelets. Wavelets constitute a family of functions generated from dilation and translation of a single function which is known as mother wavelet \( \psi(x) \). If the dilation parameter \( a \) and the translation parameter \( b \) vary continuously we have the following family of continuous wavelets [5]:

\[ \psi_{a,b}(x) = |a|^{-1/2} \psi \left( \frac{x-b}{a} \right), \]

where \( a, b \in \mathbb{R} \) and \( a \neq 0 \). Chebyshev wavelets \( \psi_{lm} = \psi(s,l,m,x) \) have four arguments, here \( l = 1, 2, ..., 2^{s-1} \), \( s \) can take any positive integer, \( m \) is the degree of Chebyshev polynomials of first kind and \( x \) is the normalized time.

\[ \psi_{lm}(x) = \begin{cases} \gamma_m \frac{2^{(s-1)/2}}{\sqrt{\pi}} T_m \left( 2^s x - 2l + 1 \right), & \frac{l-1}{2^{s-1}} \leq x < \frac{l}{2^{s-1}} \\ 0, & \text{else} \end{cases} \]

where

\[ \gamma_m = \begin{cases} \sqrt{2}, & m = 0 \\ 2, & m = 1, 2, ... \]
\[ l = 1, 2, \ldots, 2^{s-1}, \quad m = 0, 1, \ldots, M - 1 \quad \text{and} \quad x \text{ is the normalized time.} \] 
Here \( T_m(x) \) are Chebyshev polynomials of the first kind of degree \( m \) and satisfy the following recursive formula:
\[ T_0(x) = 1, T_1(x) = x, \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x). \]
which are orthogonal with respect to the weight function \( \omega(x) = 1/\sqrt{1-x^2} \).

We should remind that Chebyshev wavelets are orthogonal with respect to the weight function \( \omega_l(x) = \omega(2^s x - 2l + 1) \). We also denote the first integral of the Eq. (2) as \( p_{lm}(x) = \int_0^x \psi_{lm}(x')dx' \) and the second integral of the Eq. (2) as \( q_{lm}(x) = \int_0^x p_{lm}(x')dx' \).

2.2. Function approximation. Any function \( u(x) \in L_2^2[0,1] \) can be expanded into Chebyshev wavelets as follows:
\[ u(x) = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} c_{lm} \psi_{lm}(x). \quad (3) \]

Here wavelet coefficients \( c_{lm} \) are determined by the operation \( \langle u(x), \psi_{lm}(x) \rangle \), where \( \langle \cdot, \cdot \rangle \) represents the inner product with respect to \( \omega_l(x) \).

Practically, one needs truncated version of the Eq. namely:
\[ u(x) = \sum_{l=1}^{2^{s-1}M-1} \sum_{m=0}^{\infty} c_{lm} \psi_{lm}(x) = C^T \Psi(x), \quad (4) \]

where \( C \) and \( \Psi(x) \) are \( 2^{s-1}M \times 1 \) matrices given as
\[
C = [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, c_{21}, \ldots, c_{2(M-1)}, \ldots, \nonumber \\
\quad c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}(M-1)}]^{T}, 
\]
\[
\Psi(x) = [\psi_{10}(x), \psi_{11}(x), \ldots, \psi(x)_{1(M-1)}, \psi_{20}(x), \psi_{21}(x), \ldots, \nonumber \\
\quad \psi(x)_{2(M-1)}, \ldots, \psi_{2^{s-1}0}(x), \psi_{2^{s-1}1}(x), \ldots, \psi(x)_{2^{s-1}(M-1)}]^{T}. 
\]

3. Method of solution. In this section firstly we describe time discretization of the problems under consideration and then we apply Chebyshev wavelets for space discretization.

3.1. Time fractional Burgers’ equation. Consider time fractional Burgers’ equation in the following form
\[ D_t^\alpha u + uu_x - \nu u_{xx} = F(x, t) \quad (5) \]
subjected to the initial condition
\[ u(x, 0) = \phi(x), \quad t \geq 0, \]
and boundary conditions
\[ u(0, t) = f_1(t), \quad u(1, t) = f_2(t), \quad 0 \leq x \leq 1. \]

To discretize fractional derivative in Eq. (5), we use (1) formula and we use time average for \( u_{xx} \) and \( uu_x \) terms as follows:
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} (u_{n+1} - u_n) + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} b_k^0 [u_{n-k+1} - u_{n-k}] 
\]
\[
+ \frac{[(uu_x)_{n+1} + (uu_x)_n]}{2} - \nu \frac{[(u_{xx})_{n+1} + (u_{xx})_n]}{2} = F(x, t) 
\]
linearizing the nonlinear term \((u u_x)_n+1\) by using \(u_{n+1}(u_x)_n + u_n (u_x)_{n+1} - (u u_x)_n\) linearization technique [13] and making algebraic manipulations on the above equation yields

\[
Au_{n+1} + \frac{[u_{n+1}(u_x)_n + u_n (u_x)_{n+1}]}{2} - \nu \frac{1}{2} (u_{xx})_{n+1} = \\
Au_n + \nu \frac{1}{2} (u_{xx})_n - A \sum_{k=1}^{n-1} b_k \left[ u(t_{n-k+1}) - u(t_{n-k}) \right] + F(x, t_{n+1}).
\]

(6)

with the initial condition

\[u_0 = \phi(x)\]

and boundary conditions

\[u_{n+1}(0) = f_1(t_{n+1}), \quad u_{n+1}(1) = f_2(t_{n+1}), \quad n = 0, 1, \ldots N-1\]

(7)

where \(A = \frac{(\Delta t)^{-\alpha}}{1(2-\alpha)}\) and \(u_{n+1}\) is the solution of the Eq. (6) at the \((n+1)\)th time step.

3.2. Space discretization with Chebyshev wavelets. We expand the highest space derivatives in the problem that we consider into Chebyshev wavelets. So we discretize highest space derivative meanly \((u_{xx})_{n+1}\) by Chebyshev wavelets as follows:

\[ (u_{xx})_{n+1}(x) = \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} \psi_{lm}(x) \]

(8)

Integrating Eq. (8) with respect to \(x\) from 0 to \(x\), we get the following equation

\[ (u_x)_{n+1}(x) = (u_x)_{n+1}(0) + \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} \psi_{lm}(x). \]

(9)

In Eq. (9), the term \((u_x)_{n+1}(0)\) is unknown so to find it, we first integrate Eq. (9) from 0 to 1 and then we use boundary conditions (7). Hence we get

\[ (u_x)_{n+1}(0) = f_2(t_{n+1}) - f_1(t_{n+1}) - \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} q_{lm}(1). \]

(10)

Plugging (10) into Eq. (9) we have

\[ (u_x)_{n+1}(x) = \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} \psi_{lm}(x) + f_2(t_{n+1}) - f_1(t_{n+1}) - \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} q_{lm}(1). \]

(11)

Finally, integrating Eq. (11) from 0 to \(x\), we get

\[ u_{n+1}(x) = \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} q_{lm}(x) + f_1(t_{n+1}) + x (f_2(t_{n+1}) - f_1(t_{n+1})) - x \sum_{l=1}^{2_{n-1}} \sum_{m=0}^{M-1} c_{lm} q_{lm}(1) \]

(12)
For the fractional Burgers equation we substitute (12), (11) and (8) into (6), and discretize the results at the collocation points 
\[ x_l = l - 0.5 \frac{m^\prime}{m}, \quad l = 1, 2, ..., m^\prime \]
where 
\[ m^\prime = 2^s - 1 \cdot M. \]
Hence we get a system of algebraic equations:
\[
\begin{align*}
\sum_{l=1}^{2^{s-1} M - 1} & \sum_{m=0}^{m^\prime} c_{lm} \left[ A (q_{nm}(x_l) - x_l q_{nm}(1)) + \frac{(u_x)_n}{2} (q_{nm}(x_l) - x_l q_{nm}(1)) \\
+ \frac{u_n}{2} (p_{nm}(x_l) - q_{nm}(1)) - \frac{\nu}{2} \psi_{nm}(x_l) \right] = \\
Au_n + \frac{\nu}{2} (u_{xx})_n - A \sum_{k=1}^{n-1} b_k^\alpha [u(t_{n-k+1}) - u(t_{n-k})] \\
- A [f_1(t_{n+1}) + x_l (f_2(t_{n+1}) - f_1(t_{n+1}))] \\
- \frac{(u_x)_n}{2} [f_1(t_{n+1}) + x_l (f_2(t_{n+1}) - f_1(t_{n+1}))] \\
- \frac{u_n}{2} [f_2(t_{n+1}) - f_1(t_{n+1})] + F(x_l, t_{n+1}).
\end{align*}
\]
By solving the above system, the wavelet coefficients \( c_{lm} \) can be obtained. Then substituting \( c_{lm} \) into Eqs. (8)-(12), the numerical solutions can be extracted successively.

4. **Numerical experiments.** For measuring accuracy of the proposed method we use \( L_2 \) and \( L_\infty \) which are defined as follows.
\[
L_2 = \sqrt{\Delta x \sum_{i=1}^{m^\prime} |u_i^{\text{exact}} - u_i^{\text{num}}|^2}
\]  
(13)
and
\[
L_\infty = \max_i |u_i^{\text{exact}} - u_i^{\text{num}}|
\]
(14)
respectively. Also we denote the number of collocation points of the present method by \( m^\prime = 2^{s-1} M \).

4.1. **Numerical results of fractional Burgers’ equation.**

4.1.1. **Problem 1.** For the fractional Burgers’ equation, we consider Eq. (5) with following initial condition
\[ u(x, 0) = 0, \quad t \geq 0, \]
and boundary conditions
\[ u(0, t) = t^2, \quad u(1, t) = et^2, \quad 0 \leq x \leq 1. \]
The exact solution of the problem is given by
\[ u(x, t) = t^2 e^x. \]
The \( F(x, t) \) is in the following form
\[ F(x, t) = \frac{2t^{2-\alpha} e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - \nu t^2 e^x. \]
In Table 1, we gave error norms of the numerical results for various values of \( \alpha \) and \( \Delta t = 0.00025 \) at \( t = 1 \) and also compared with Cubic B-spline finite element method [6]. We take the number of collocation points as \( m^\prime = 10 \) meanly we choose \( s = 2 \) and \( M = 5 \). It can be seen from the table that present method give better results than [6] for \( \alpha = 0.1 \) and \( \alpha = 0.25 \). We also present error norms for decreasing
\( \alpha = 0.1 \)

| \( \Delta t = 0.002 \) | \( \Delta t = 0.001 \) |
|-----------------|-----------------|
| \([6]\) Present | \([6]\) Present |
| \( N = 40 \) | \( N = 40 \) |
| \( m' = 10 \) | \( m' = 10 \) |
| \( L_2 \times 10^3 \) | \( 0.434586 \) | \( 0.176195 \) |
| \( L_\infty \times 10^3 \) | \( 0.642003 \) | \( 0.265419 \) |

\( \alpha = 0.25 \)

| \( \Delta t = 0.00025 \) | \( \Delta t = 0.0005 \) |
|-----------------|-----------------|
| \([6]\) Present | \([6]\) Present |
| \( N = 40 \) | \( N = 40 \) |
| \( m' = 10 \) | \( m' = 10 \) |
| \( L_2 \times 10^3 \) | \( 0.906733 \) | \( 0.606733 \) |
| \( L_\infty \times 10^3 \) | \( 0.106340 \) | \( 0.104141 \) |

Table 1. Error norms for various values of \( \alpha \) and for \( \Delta t = 0.00025 \) at \( t = 1 \).

\( \Delta t = 0.002 \)

| \( \Delta t = 0.001 \) | \( \Delta t = 0.0005 \) |
|-----------------|-----------------|
| \([6]\) Present | \([6]\) Present |
| \( N = 40 \) | \( N = 40 \) |
| \( m' = 10 \) | \( m' = 10 \) |
| \( L_2 \times 10^3 \) | \( 0.345448 \) | \( 0.069536 \) |
| \( L_\infty \times 10^3 \) | \( 0.124569 \) | \( 0.098312 \) |

Table 2. Error norms for various values of \( \Delta t \) and for \( \nu = 1, \alpha = 0.5 \) at \( t = 1 \)

For the second problem of the fractional Burgers’ equation we consider the following initial condition

\( u(x, 0) = 0, \quad t \geq 0, \)

and boundary conditions

\( u(0, t) = t^2, \quad u(1, t) = -t^2, \quad 0 \leq x \leq 1. \)

The exact solution of the problem is given by

\( u(x, t) = t^2 \cos(\pi x). \)

The \( F(x, t) \) is in the following form

\[
F(x, t) = \frac{2t^{2-\alpha} \cos(\pi x)}{\Gamma(3 - \alpha)} - \pi t^4 \cos(\pi x) \sin(\pi x) + \nu \pi^2 t^2 \cos(\pi x).
\]

This problem is also compared with the Ref. [6]. In Table 3 we demonstrate the error norms for various values of \( \nu \) and for \( \Delta t = 0.0005, \alpha = 0.5 \) at \( t = 0.1 \). We take \( s = 2 \) and \( M = 5 \). The comparison of the error norms for the various values of \( \nu \) shows that the error norms get smaller as the \( \nu \) decreases.

The error norms are compared for the various collocation points in Table 4. We see that the present method has real superiority over the method given in Ref. [6].
Figure 1. Numerical solution and exact solution for $\alpha = 0.5$, $\Delta t = 0.0025$, $m' = 10$ and $\nu = 1$ at $t = 1$.

|          | $\nu = 1$                         | $\nu = 0.5$                         |
|----------|-----------------------------------|-------------------------------------|
|          | [6] Present                        | [6] Present                         |
| $N = 80$ | $m' = 10$                          | $N = 80$                            |
| $m' = 10$|                                   | $m' = 10$                          |
| $L_2 \times 10^3$ | 0.006528     | 0.005835                           |
| $L_\infty \times 10^3$ | 0.009164    | 0.009547                           |
|          |                                   |                                     |
|          | $\nu = 0.1$                         |                                     |
|          | [6] Present                        | [6] Present                         |
| $N = 80$ | $m' = 10$                          | $N = 80$                            |
| $m' = 10$|                                   | $m' = 10$                          |
| $L_2 \times 10^3$ | 0.003105     | 0.004847                           |
| $L_\infty \times 10^3$ | 0.004847    | 0.005714                           |

Table 3. Error norms for various values of $\nu$ and for $\Delta t = 0.0005$, $\alpha = 0.5$ at $t = 0.1$.

We show the numerical and exact solutions in Figure 2 for $\alpha = 0.5$, $\Delta t = 0.0005$ and $\nu = 1$ at $t = 0.1$.

4.1.3. Problem 3. For the last problem of the fractional Burgers’ equation we consider following initial condition

$$u(x, 0) = 0, \quad t \geq 0,$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq x \leq 1.$$
The exact solution of the problem is given by
\[ u(x,t) = t^2 \sin(2\pi x). \]

The \( F(x,t) \) is in the following form
\[ F(x,t) = \frac{2t^{2-\alpha} \sin(2\pi x)}{\Gamma(3-\alpha)} + 2\pi t^4 \sin(2\pi x) \cos(2\pi x) + 4\nu \pi^2 t^2 \sin(2\pi x). \]
We choose \( s = 3 \) and \( M = 4 \). We plot the numerical and exact solutions in Figure 3 for \( \alpha = 0.5 \), \( \Delta t = 0.005 \) and \( \nu = 1 \) at \( t = 0.5 \).

5. **Conclusion.** In this study, for the solution of time fractional Burger’s equation we used Chebyshev wavelet method and L1 discretization formula. We applied the
Δt = 0.002  Δt = 0.001

|       | [6] Present |       | [6] Present |
|-------|-------------|-------|-------------|
| N     | 120         | m'    | 16          |
| L2 × 10^3 | 1.220123  | L2 × 10^3 | 0.532436  |
| L∞ × 10^3 | 1.725765  | L∞ × 10^3 | 0.753171  |

Table 5. Error norms for various values of Δt and for ν = 1, α = 0.5 at t = 1

Figure 3. Numerical solution and exact solution for α = 0.5, Δt = 0.005 and ν = 1 at t = 0.5

The proposed method to three problems and compare the results with the other methods in the literature. For the performance of the method we calculated $L_2$ and $L_\infty$ error norms. When compared with the other methods in the literature, the proposed method gave better results in all three problems. As a result, this method is simple, easy to use and it can be easily extended to other time fractional problems.

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