Cotangent Microbundle Category, I

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Abstract

We define a local version of the extended symplectic category, the cotangent microbundle category, \( \mathcal{M} \mathcal{I} \mathcal{C} \), which turns out to be a true monoidal category. We show that a monoid in this category induces a Poisson manifold together with the local symplectic groupoid integrating it. Moreover, we prove that monoid morphisms produce Poisson maps between the induced Poisson manifolds in a functorial way. This gives a functor between the category of monoids in \( \mathcal{M} \mathcal{I} \mathcal{C} \) and the category of Poisson manifolds and Poisson maps. Conversely, the semi-classical part of the Kontsevich star-product associated to a real-analytic Poisson structure on an open subset of \( \mathbb{R}^n \) produces a monoid in \( \mathcal{M} \mathcal{I} \mathcal{C} \).

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1 Introduction

There is a category \textbf{Sympl} whose objects are finite-dimensional symplectic manifolds \((M, \omega)\) and whose morphisms are symplectomorphisms \( \Psi : (M, \omega_M) \rightarrow (N, \omega_N) \). In attempting
to understand the quantization procedure of physicists from a mathematical perspective, one may think of it as a functor from this symplectic category, where classical mechanics takes place, into the category of Hilbert spaces and unitary operators, which is the realm of quantum mechanics. It is well known that this symplectic category is too large, since there are “no-go” theorems which show that the group of all symplectomorphisms on \((M,\omega)\) does not act in a physically meaningful way on a corresponding Hilbert space. One standard remedy for this is to replace Sympl by a smaller category, replacing the symplectomorphism groups by certain finite-dimensional subgroups. Another is to replace the Hilbert spaces and operators by objects depending on a formal parameter.

But there is also a sense in which the category Sympl is too small, since it does not contain morphisms corresponding to operators such as projections and the self-adjoint (or skew-adjoint) operators which play the role of observables in quantum mechanics, nor can it encode the algebra structure itself on the space of observables. (This collection of observables is not actually a Hilbert space, but certain spaces of operators do carry a vector space structure, with the inner product associated to the Hilbert-Schmidt norm.)

To enlarge the symplectic category, we look to the “dictionary” of quantization, following, for example, [1]. In this dictionary, the cartesian product of symplectic manifolds corresponds to the tensor product of Hilbert spaces, and replacing a symplectic manifold \((M,\omega)\) by \((M, -\omega)\) (which we denote by \(\overline{M}\) when we omit the symplectic structure from the notation for a given symplectic manifold) corresponds to replacing a Hilbert space \(\mathcal{H}\) by its conjugate, or dual, space \(\mathcal{H}^*\). Thus, if symplectic manifolds \(M_1\) and \(M_2\) correspond to Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\) the product \(\overline{M}_1 \times \overline{M}_2\) corresponds to \(\mathcal{H}_1^* \otimes \mathcal{H}_2\), which, with a suitable definition of the tensor product, is the space \(L(\mathcal{H}_1, \mathcal{H}_2)\) of all linear operators from \(\mathcal{H}_1\) to \(\mathcal{H}_2\).

Another entry in the dictionary says that lagrangian submanifolds in symplectic manifolds (perhaps carrying half-densities) correspond to vectors or lines in Hilbert space. Combining this idea with that in the paragraph above, we conclude that lagrangian submanifolds in \(\overline{M} \times N\) should correspond to linear operators from \(\mathcal{H}_1\) to \(\mathcal{H}_2\).

This suggests that, if the space of observables \(\mathcal{H}\) for a quantum system corresponds to a symplectic manifold \(\mu\) in \(\overline{M} \times \overline{M} \times M\), then the algebra structure on \(\mathcal{H}\) should be given by a lagrangian submanifold \(\mu\) in \(\overline{M} \times \overline{M} \times M\). The algebra axioms of unitality and associativity should be encoded by monoidal properties of \(\mu\) in an extended symplectic category, ExtSympl, where the morphisms from \(M\) to \(N\) are the canonical relations, i.e. all the lagrangian submanifolds of \(\overline{M} \times N\) (not just those which are the graphs of symplectomorphisms) and where the morphism composition is the usual composition of relations\(^1\). However, a problem immediately occurs: the composition of canonical relations may yield relations which are not submanifolds anymore and thus, not canonical relations! ExtSympl is then not a true category, as the morphisms can not be always composed. It is thus rather uncomfortable to speak about a quantization functor in this context.

\[\text{In the context of symplectic geometry, the composition of canonical relations may be seen as a special instance of symplectic reduction. Consider } C := \overline{M} \times \Delta_N \times P, \text{ where } \Delta_N \text{ is the diagonal subset of } \overline{N} \times N.\]

\(C\) is a coisotropic submanifold of \(\overline{M} \times N \times \overline{N} \times P\) and \(L_2 \circ L_1\) happens to be the reduction of the lagrangian submanifold \(L_1 \times L_2\) with respect to \(C\). Thus, if \(L_1 \subset \overline{M} \times N\) and \(L_2 \subset \overline{N} \times P\) are lagrangian submanifolds, then \(L_2 \circ L_1\) is a lagrangian submanifold of \(\overline{M} \times P\) whenever it is a submanifold.
There have already been several approaches to remedy this defect. One, by Guillemin and Sternberg [8], is to consider only symplectic vector spaces and linear canonical relations. Another, by Wehrheim and Woodward [14], is to enlarge the category still further by allowing arbitrary “formal” products of canonical relations, and equating them to actual products when the latter exist as manifolds.

In this paper, we take another approach. We define a local version of the extended symplectic category which is a true category. We restrict ourselves to cotangent bundles with their canonical symplectic structures and define \( \text{Hom}(T^*M, T^*N) \) to be germs near the zero section of canonical relations which are suitably close to the conormal bundles of graphs of diffeomorphisms from \( N \) to \( M \). We call the resulting category the cotangent microbundle category. We choose this name for the category since the objects involved are symplectic version of the microbundles introduced by Milnor in [11].

In Section 2, we express, in terms of transversality, the condition that germs of lagrangian submanifolds are somehow close to the conormal bundle of the graph of a map between the bases.

In Section 3, we define the cotangent microbundle category \( \text{MiC} \), by allowing the morphisms \( \text{Hom}(T^*M, T^*N) \) to be the transverse lagrangian germs in \( T^*M \times T^*N \) as defined in Section 2. We show that the composition is always well-defined and that the resulting category is a true monoidal category. Let us note here that the lagrangian operads considered in [2] and in [3] are closely related to the endomorphism operad associated to any object in \( \text{MiC} \) in the usual way. This will be the subject of future work.

In Section 4, we describe each morphism locally in terms of a single function: the generating function of the transverse lagrangian germ. We derive a composition formula for generating functions and show how they behave under changes of charts.

In Section 5, we prove that a monoid \( (T^*M, \mu, e) \) in the cotangent microbundle category induces a Poisson structure on the base \( M \) together with a local symplectic groupoid \( (s, t) : T^*M \rightrightarrows M \) integrating it. All the induced structures are described explicitly in terms of generating functions. We show that isomorphisms of monoids produce Poisson diffeomorphisms between the induced Poisson structures and local groupoid isomorphisms between the induced local symplectic groupoids. This gives a functor from the category of monoids in \( \text{MiC} \) to the category \( \text{Poiss} \) of Poisson manifolds. These results are very much in the line of the “categories” introduced by Zakrzewski in [15] and studied by Crainic and Fernandes in [6].

Section 6 is devoted to explicit examples of monoids in \( \text{MiC} \), their induced Poisson structures and local symplectic groupoids. In particular, we give the generating function that induces the symplectic Poisson structure, the generating function that induces the Kirillov–Kostant Poisson structure on the dual of a Lie algebra - the generating function is the Baker–Campbell–Hausdorff formula in this case - and the generating function attached to an analytical Poisson structure on open subset of \( \mathbb{R}^d \). The latest generating function encompasses the two previous ones. It is given by the semi-classical part of Kontsevich’s star-product. This last example supports the hope that the cotangent microbundle category is the right framework to construct a quantization functor.

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2 The transversality condition

The extended symplectic category is not a true category, as morphisms can not always be composed. In order to obtain a true category, we may restrict to special classes of symplectic manifolds and special classes of lagrangian submanifolds $L \subset M \times N$ so that the composition is always well-defined. Guillemin and Sternberg used linear symplectic spaces and lagrangian linear subspaces, but this is too restrictive for most purposes. In this article, we consider the cotangent bundle category and modify it carefully. The objects in the cotangent bundle category are cotangent bundles $T^*M$ over smooth manifolds $M$ endowed with their canonical symplectic structure. Naively, a morphism $\Psi : T^*M \to T^*N$ is a symplectomorphism that respects the zero sections. For further generalization purposes, we will rename the zero-section of a cotangent bundle $T^*M$, the core of $T^*M$ and we will refer to it as either $Z_M$ or simply as $M$. Thus, a morphism in the cotangent bundle category preserves the core.

In this section, we reformulate this property (preserving the core) in a way that it may be applied to general lagrangian submanifolds $L$ of $T^*M \times T^*N$ and not only to the graphs of symplectomorphisms. We call this condition the transversality condition. As this condition is a local one (it concerns only a neighborhood of $M \times N$ in $T^*M \times T^*N$ exactly as the condition $\Psi(M) = N$), we are led to consider germs of lagrangian submanifolds.

Definition 2.1. We say that a diffeomorphism $\Psi : T^*M \to T^*N$ covers a map $\phi : N \to M$ if $\Psi(0, \phi(x)) = (0, x)$ for all $x \in M$.

Note that this seems to say that $\Psi$ extends $\phi^{-1}$. Later, we will need to allow situations where $\phi$ not invertible.

Lemma 2.2. Let $\Psi : T^*M \to T^*N$ be a symplectomorphism preserving the cores, i.e., such that $\Psi(M) = N$. Then:

(1) There exists a unique map $\phi : N \to M$ such that $\psi$ covers $\Phi$.

(2) In any local chart of the form $U = T^*U_1 \times T^*U_2$ of $T^*M \times T^*N$, there exists a neighborhood $V$ of $U_1 \times U_2$ in $U$ where the graph of $\Psi$ is of the form:

$$\text{graph } \Psi \cap V = \left\{ \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right) : (p_1, x_2) \in W \right\}.$$
where \( p_1, x_1 \) and \( p_2, x_2 \) are the local coordinates of \( T^*U_1 \) and \( T^*U_2 \) respectively, \( W \) is a neighborhood of \( \{0\} \times U_2 \) in \((\mathbb{R}^d)^* \times U_2\) and \( G : W \rightarrow U_1 \), and \( H : W \rightarrow (\mathbb{R}^d)^* \) are smooth maps such that \( G(0, x_2) = \phi(x_2) \) and \( H(0, x_2) = 0 \).

**Proof.** As \( \Psi \) respects the core, its restriction to \( M \) induces a map \( g := \Psi|_M \) from \( M \) to \( N \). Since \( \Psi \) is a diffeomorphism, the induced map \( g \) is invertible. We denote by \( \phi \) the inverse of \( g \). Clearly, \( \phi \) is a diffeomorphism covered by \( \Psi \). In a local chart \( U = T^*U_1 \times T^*U_2 \) of \( T^*M \times T^*N \), let us write \( \Psi \) as:

\[
\Psi(p_1, x_1) = (U(p_1, x_1), V(p_1, x_1)).
\]

Then, for fixed \( p_1 \), consider the equation

\[
V(p_1, x_1) = x_2.
\]

If \( p_1 = 0 \), then \( V(0, x_1) = \phi^{-1}(x_1) \) and \( \nabla_{x_1} V(0, x_1) = \nabla \phi^{-1}(x_1) \) is invertible as \( \phi \) is a diffeomorphism. The implicit function Theorem tells us that we may invert equation (1), i.e., we may find a function \( G \) such that:

\[
x_1 = G(p_1, x_2) \quad \text{s.t.} \quad V(p_1, G(p_1, x_2)) = x_2,
\]

for \((p_1, x_2)\) in a neighborhood \( W \) of \( \{0\} \times U_2 \) in \((\mathbb{R}^d)^* \times U_2\). Thus, in a neighborhood \( V \) of \( U_1 \times U_2 \) in \( T^*U_1 \times T^*U_2 \), we have that:

\[
\text{graph } \Psi \cap V = \left\{ \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right) : (p_1, x_2) \in W \right\},
\]

where \( H(p_1, x_2) = U(p_1, G(p_1, x_2)) \). Now, by definition, we have that,

\[
x_2 = V(0, G(0, x_2)) = \phi^{-1}(G(0, x_2)),
\]

and thus \( G(0, x_2) = \phi(x_2) \). On the other hand,

\[
H(0, x_2) = U(0, G(0, x_2)) = 0.
\]

Let us express the content of Lemma 2.2 in a geometrical way. Consider two cotangent bundles \( T^*M \) and \( T^*N \) and a map \( \phi \) from \( N \) to \( M \). Let \( B_\phi \) be the pullback of \( T^*M \) by \( \phi \), i.e.,

\[
B_\phi := \left\{ (p_1, x_2) : p_1 \in T^*_{\phi(x_2)} M, \quad x_2 \in N \right\}
\]

and \( Z_\phi \) its zero section. Define \( G_\phi : Z_\phi \rightarrow T^*M \times T^*N \) to be the map taking \( Z_\phi \) to the graph of \( \phi \) in \( M \times N \) considered as a submanifold of \( T^*M \times T^*N \):

\[
G_\phi(0, x) = ((0, \phi(x)), (0, x)).
\]
**Definition 2.3.** A lagrangian embedding germ around $G_\phi$ is an equivalence class of lagrangian embeddings $i_\phi : B_\phi \hookrightarrow T^* M \times T^* N$ such that $i_\phi |_{Z_\phi} = G_\phi$, where two such lagrangian embeddings are equivalent if there exists a neighborhood $U$ of $Z_\phi$ in $B_\phi$ where their images coincide. We denote the class of $i_\phi$ by $[i_\phi]$. When the context is clear, we will use the $i_\phi$ to denote its class.

The tangent bundle $T(T^* M \times T^* N)$, restricted to the product of the bases $M \times N$, has a natural subbundle over $M \times N$:

$$\Lambda := T(Z_M) \times V(T^* N),$$

where $T(Z_M)$ is the tangent space to the zero section $Z_M$ in $T^* M$ and $V(T^* N)$ is the tangent space to the vertical fibers in $T^* N$. We may pull back this bundle via the map $G_\phi$ to a bundle $G_\phi^* \Lambda$ over $Z_\phi$, the zero section of $B_\phi$. Figure 1 represents a fiber of this bundle $G_\phi^* \Lambda$ over a point in graph $\phi = G_\phi(Z_\phi)$.

![Figure 1: The distribution $G_\phi^* \Lambda$ at the point $(0,\phi(x_2)),(0,x_2)$ in $T^* M \times T^* N$.](image)

**Definition 2.4.** We call transverse lagrangian germ a germ $[i_\phi]$ of a lagrangian embedding $i_\phi : B_\phi \hookrightarrow T^* M \times T^* N$ around $G_\phi$ such that one (and thus any) of its representatives $i_\phi$ is transverse to $G_\phi^* \Lambda$.

Figure 2 represents such transverse germs around the same core map $\phi$.

Let us see how this transversality condition translates in local charts. Take $U_1$ a local chart of $M$ and $U_2$ a local chart of $N$. Then $U = T^* U_1 \times T^* U_2$ is a local chart of $T^* M \times T^* N$, and $B_\phi^U = \phi^*(T^* U_1)$ is a local chart of $B_\phi$. Observe that these special local charts cover a neighborhood of $M \times N$ in $T^* M \times T^* N$ and, thus, are enough to describe completely germs of lagrangian embedding $i_\phi : B_\phi \hookrightarrow T^* M \times T^* N$ around $G_\phi$. Let us denote by $i_\phi^U : B_\phi^U \hookrightarrow U$ the representation of $i_\phi$ in $U$. If the local coordinates on $U$ are $p_1, x_1, p_2, x_2$, then the local coordinates on $B_\phi^U$ are $p_1, x_2$. 
Figure 2: Two transverse lagrangian germs $i_\phi$ and $j_\phi$ around $G_\phi$.

**Lemma 2.5.** A germ of lagrangian embedding $i_\phi : B_\phi \hookrightarrow T^*M \times T^*N$ is transverse to $G_\phi^*\Lambda$ iff, for any local chart $U$ as above, we have that:

$$i_\phi^U(W) = \left\{ \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right) : (p_1, x_2) \in W \right\},$$

where $W$ is a neighborhood of the zero section of $B_\phi^U$ and $G : W \to U_1$ and $H : W \to (\mathbb{R}^k)^*$ are smooth maps such that $G(0, x_2) = \phi(x_2)$ and $H(0, x_2) = 0$.

**Proof.** In a local chart $U = T^*U_1 \times T^*U_2$ of $T^*M \times T^*N$, the bundle $G_\phi^*\Lambda$ is the restriction of

$$K := \bigcup_{(p_1, x_1), (p_2, x_2) \in T^*U_1 \times T^*U_2} \left\{ \left( (0, v_1), (\mu_2, 0) \right) : v_1 \in \mathbb{R}^k, \mu_2 \in (\mathbb{R}^l)^* \right\}$$

to $\text{graph } \phi = G_\phi(Z_\phi^U)$. The transversality condition tells us that the tangent space of $i_\phi^U(W)$ is transverse to $K$ on $G_\phi(Z_\phi^U)$. Now, by continuity, there exists a neighborhood $V$ of $G_\phi(Z_\phi^U)$ in $T^*U_1 \times T^*U_2$ where the tangent space $i_\phi^U(W)$ is transverse to $K$ on $i_\phi^U(W) \cap V$. Observe that $K$ is the bundle transverse to the $p_1, x_2$ fibers. This means that $i_\phi^U(W) \cap V$ is projectable on the $p_1, x_2$ fibers and, thus, must be of the form [2]. This situation is illustrated in Figure 3. Considering that $i_\phi^U|_{Z_\phi^U} = G_\phi^U$, we get immediately that $G(0, x_2) = \phi(x_2)$ and $H(0, x_2) = 0$. \hfill $\Box$

**Proposition 2.6.** Let $\Psi : T^*M \to T^*N$ be a symplectomorphism sending the zero section $Z_M$ to the zero section $Z_N$. Then there exists a neighborhood $V$ of $Z_M \times Z_N$ in $T^*M \times T^*N$, and a transverse lagrangian germ $i_\phi : B_\phi \hookrightarrow T^*M \times T^*N$ such that

$$\text{graph } \Psi \cap V = i_\phi(W),$$

where $W$ is a neighborhood of $Z_\phi$ in $B_\phi$.

**Proof.** Let us prove the proposition in a local chart $U = T^*U_1 \times T^*U_2$ of $T^*M \times T^*N$. Lemma 2.2 tells us that $\Psi$ covers a map $\phi : N \to M$. In the local chart $U$, we have that:

$$B_\phi^U = \phi^*(T^*U_1) = (\mathbb{R}^d)^* \times U_1.$$
Lemma 2.2 gives a neighborhood $W$ of the zero section in $B^U_\phi$ and a neighborhood $V$ of $U_1 \times U_2$ in $T^*U_1 \times T^*U_2$ where

$$
\text{graph } \Psi \cap V = \left\{ \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right) : (p_1, x_2) \in W \right\}.
$$

such that $G(0, x_2) = \phi(x_2)$ and $H(0, x_2) = 0$. Thus, there is a lagrangian germ $i^U_\phi : B^U_\phi \to T^*U_1 \times T^*U_2$ around $G_\phi$ given by

$$
i^U_\phi(p_1, x_2) = \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right)
$$

such that $i^U_\phi(W) = \text{graph } \Psi \cap V$. Lemma 2.5 tells us that $i^U_\phi$ is transverse. \qed

**Example 2.7.** Let $\phi$ be a map from $N$ to $M$ and consider $d\phi^* : T^*M \to T^*N$ its cotangent lift. In a local chart $U = T^*U_1 \times T^*U_2$, the graph of $d\phi^*$ is:

$$
\text{graph } d\phi^* = \left\{ \left( (p_1, \phi(x_2)), (d\phi^*(x_2)p_1, x_2) \right) : x_2 \in U_2, \ p_1 \in (\mathbb{R}^d)^* \right\}.
$$

The induced transverse germ $i_\phi : B_\phi \hookrightarrow \overline{T^*M} \times T^*N$ is given by

$$
i^U_\phi(p_1, x_2) = \left( (p_1, \phi(x_2)), (d\phi^*(x_2)p_1, x_2) \right),
$$
in the local chart $U$.

The next proposition is a local converse of Proposition 2.6.

**Proposition 2.8.** Let $\phi$ be a map from $N$ to $M$ and $i_\phi : B_\phi \hookrightarrow \overline{T^*M} \times T^*N$ be a transverse lagrangian germ around $G_\phi$. If $\phi$ is invertible, then there exists a germ of symplectomorphism $\Psi$ which covers $\phi$ and such that:

$$
\text{graph } \Psi \cap V = i_\phi(W),
$$

where $V$ is a neighborhood of $Z_M \times Z_N$ in $T^*M \times T^*N$ and $W$ is a neighborhood of $Z_\phi$ in $B_\phi$. 

Figure 3: Transversality condition and projectability on the $(p_1, x_2)$–fibers.
Proof. We prove the proposition in a local chart $U = T^*U_1 \times T^*U_2$. Lemma \ref{lemma:local_chart} tells us that:

$$i_\phi^U(W) = \left\{ \left( (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) \right) : (p_1, x_2) \in W \right\}$$

where $W$ is a neighborhood of $Z_\phi^U$ in $B_\phi^U$. Now, consider the equation

$$G(p_1, x_2) = x_1.$$ \hfill (3)

Remark that $G(0, x_2) = \phi(x_2)$ and $\nabla_{x_2} G(0, x_2) = \nabla \phi(x_2)$ which is invertible as $\phi$ is a diffeomorphism. Then the implicit function Theorem tells us that, for $(p_1, x_2)$ in a neighborhood $W$ of the zero section in $B_\phi^U$, we may invert equation (3), i.e., we may find a function $K$ such that

$$x_2 = K(p_1, x_1) \quad \text{s.t.} \quad G(p_1, K(p_1, x_1)) = x_1.$$ 

Thus, we get that:

$$i_\phi^U(W) = \left\{ \left( (p_1, x_1), (H(p_1, K(p_1, x_1)), K(p_1, x_1)) : (p_1, x_1) \in T \right) \right\},$$

where $T$ is a neighborhood of $U_1$ in $T^*U_1$. Thus, setting

$$\Psi_U(p_1, x_1) = \left( H(p_1, K(p_1, x_1)), K(p_1, x_1) \right),$$

and remarking that $\Psi_U(p_1, \phi(x)) = (0, x)$, one gets a local description of a symplectomorphism germ $\Psi : T^*M \rightarrow T^*N$ which covers $\phi$ and which sends a neighborhood of $M$ in $T^*M$ to a neighborhood of $N$ in $T^*N$ preserving the bases. \hfill \qed

3 Definition of the category

In this Section, we construct a new monoidal category, the cotangent microbundle category MiC. Our goal is to extend (i.e. to replace maps by relations) the category of cotangent bundles so that the resulting “category” is a true category. The key observation is the following. A morphism $\Psi : T^*M \rightarrow T^*N$ in the cotangent bundle category is a differentiable map which satisfies the two following properties:

1. $\Psi$ is a symplectomorphism,
2. $\Psi$ preserves the zero sections (or cores).

The idea is to reformulate these two properties in terms of the graph of $\Psi$ so that they will still make sense for general differentiable relations $L \subset T^*M \times T^*N$. It is well known that $\Psi$ is a symplectomorphism if and only if its graph is a lagrangian submanifold of $\overline{T^*M} \times T^*N$. Now, in the previous section, we have seen that asking that $\Psi$ preserves the cores is equivalent to ask that its graph satisfies the transversality condition of Definition \ref{definition:transversality}, which makes sense.
for general lagrangian submanifolds of $T^*M \times T^*N$. However, this transversality condition is a local condition. It concerns only the geometry of the graph of $\Psi$ around a neighborhood of

$$\text{graph } \Psi \cap (Z_M \times Z_N) = \text{graph } \phi,$$

in $T^*M \times T^*N$, where $\phi = \Psi^{-1}_M$. We are thus led to the following definition for the morphism in $\text{MiC}$.

We keep the same notations as introduced in Section 2.

**Definition 3.1.** In $\text{MiC}$, a morphism from $T^*M$ to $T^*N$ is given by a pair $(i_\phi, \phi)$ where $\phi$ is a map from $N$ to $M$ and $i_\phi$ is a transverse lagrangian germ as in Definition 2.4.

**Remark 3.2.** In the same way, we may define a “micro” version of the extended symplectic category, the microsymplectic category, whose objects are pairs $(M, L)$ of a symplectic manifold and a lagrangian submanifold $L \subset M$, its core, and whose morphisms from $(M_1, L_1)$ to $(M_2, L_2)$ are pairs $(i_\phi, \phi)$ of a smooth map $\phi : L_2 \to L_1$ and a transverse germ of lagrangian embeddings

$$[i_\phi] : \phi^*(T^*L_1) \hookrightarrow \overline{M_1} \times M_2,$$

along $\phi$. In this context, we say that a germ $[i_\phi]$ is transverse if for a representative $i_\phi$ (and hence any) of $[i_\phi]$ and for any identifications $\Psi_i$ of a neighborhood $U_i$ of $L_i$ in $M_i$ with a neighborhood $V_i$ of $L_i$ in $T^*L_i$, $i = 1, 2$, the induced germ

$$[i_\phi^\Psi] : \phi^*(T^*L_1) \hookrightarrow T^*L_1 \times T^*L_2$$

is transverse in the sense of Definition 2.4. We will not pursue this point of view this the article.

**Example 3.3.** The category $\text{MiC}$ has a distinguished object, the cotangent bundle of the one point manifold $E := T^*\{\ast\}$. There is a unique morphism $e$ in $\text{Hom}(E, T^*M)$; it is given by $e = (i_M, \text{pr})$ where $i_M$ is the inclusion of $M$ as the zero section of $T^*M$ and $\text{pr}$ is the projection of the whole manifold $M$ onto $\ast$.

**Example 3.4.** The base maps $\phi$ of morphisms $(i_\phi, \phi) \in \text{Hom}(T^*M, E)$ are indexed by points of $M$, namely $\phi : \{\ast\} \to M$ sends the unique point $\ast$ to a point $x$ of $M$. The transversality condition in this context tells that images of transverse germs of lagrangian embeddings $i_\phi : B_\phi = T^*_xM \hookrightarrow T^*M \times E$ are lagrangian submanifolds through $x \in M$ which are transverse to the zero section of $T^*M$.

**Example 3.5.** We may identify $\overline{T^*M} \times T^*N$ with the cotangent bundle $T^*(M \times N)$ via the Schwartz transform:

$$S((p_1, x_1), (p_2, x_2)) = ((-p_1, p_2), (x_1, x_2)).$$

Let $\phi$ be a smooth map from the manifold $N$ to the manifold $M$. The normal bundle $N^*(\text{graph } \phi)$ of the graph of $\phi$ in $T^*(M \times N)$ induces, via the Schwartz transform, a transverse germ of lagrangian embeddings:

$$d\phi^*: B_\phi \longrightarrow \overline{T^*M} \times T^*N.$$

We denote it by $d\phi^*$ as it comes from the cotangent lift $d\phi^*: T^*M \to T^*N$ of $\phi$ if $\phi$ is invertible. We call $(d\phi^*, \phi)$ the generalized cotangent lift of $\phi$.  

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Example 3.6. For any cotangent bundle $T^*M$, there is an identity morphism. It is given by $id = (\Delta_{T^*M}, \text{id}_M) \in \text{Hom}(T^*M, T^*M)$ where $\text{id}_M$ is the identity map on $M$ and $\Delta_{T^*M}$ the germ induced by the diagonal in $T^*M \times T^*M$.

Consider a morphism $(i_{\phi_1}, \phi_1)$ from $T^*M$ to $T^*N$ and a morphism $(i_{\phi_2}, \phi_2)$ from $T^*N$ to $T^*P$. For two neighborhoods $N_1$ of $Z_{\phi_1}$ and $N_2$ of $Z_{\phi_2}$, we may compose $i_{\phi_2}(N_2) \circ i_{\phi_1}(N_1)$ via composition of canonical relations. The following proposition describes this composition.

Proposition 3.7. In the above notation, there exists a transverse germ of lagrangian embeddings $i_{\phi_1 \circ \phi_2} : B_{\phi_1 \circ \phi_2} \to T^*U \to T^*P$ such that we may find neighborhoods $N_k$ of $Z_{\phi_k}$, $k = 1, 2$, and a neighborhood $N_3$ of $Z_{\phi_1 \circ \phi_2}$ for which:

$$i_{\phi_1 \circ \phi_2}(N_3) = i_{\phi_2}(N_2) \circ i_{\phi_1}(N_1).$$

Proof. We check the proposition in local coordinates. Let $U_1$, $U_2$ and $U_3$ be local charts of $M$, $N$ and $P$ respectively. Denote by $V_1 = T^*U_1 \times T^*U_2$ and $V_2 = T^*U_2 \times T^*U_3$ the local charts of $T^*M \times T^*N$ and $T^*N \times T^*P$. In these charts, we have, thanks to Lemma 2.5 that:

$$i_{\phi_1}(N_1) = \left\{ (p_1, F(p_1, x_2), (G(p_1, x_2), x_2)) : (p_1, x_2) \in N_1 \right\}$$

$$i_{\phi_2}(N_2) = \left\{ (p_2, H(p_2, x_3), (L(p_2, x_3), x_3)) : (p_2, x_3) \in N_2 \right\}.$$

The implicit function Theorem tells us that there exists a neighborhood $N_3$ of $Z_{\phi_1 \circ \phi_2}$ in $B_{\phi_1 \circ \phi_2}^U$ such that for $(p_1, x_3) \in N_3$, we can always find a unique couple $(p_2, x_2)$ such that:

$$(G(p_1, x_2), x_2) = (p_2, H(p_2, x_3)).$$

Namely, consider the function:

$$I(p_1, x_3, p_2, x_2) = \begin{pmatrix} p_2 - G(p_1, x_2) \\ x_2 - H(p_2, x_3) \end{pmatrix}.$$

Thanks to the fact that $G(0, x_1) = 0$ and $H(0, x_3) = \phi_2(x_3)$, we get

$$I(0, x_3, 0, \phi_2(x_3)) = 0.$$

Moreover, the Jacobi matrix of $I$ at this point

$$\frac{\partial I}{\partial (p_2, x_2)}((0, x_3, 0, \phi(x_3)) = \begin{pmatrix} \text{id} & 0 \\ -\nabla_p H(0, x_3) & \text{id} \end{pmatrix}$$

is invertible. This shows that there exists a neighborhood $N_3$ of the zero section in $B_{\phi_1 \circ \phi_2}^U$ and a unique solution $p_2 = U(p_1, x_3)$ and $x_2 = V(p_1, x_3)$ such that:

$$U(p_1, x_3) = H(V(p_1, x_3), x_3), \quad V(p_1, x_3) = G(p_1, U(p_1, x_3)).$$

(4)
for \((p_1, x_3) \in N_3\). Then we have that composition of canonical relations yields:

\[
V_2^V(N_2) \circ V_1^V(N_1) = \left\{ \left( (p_1, R(p_1, x_3)), (T(p_1, x_3), x_3) \right) : (p_1, x_3) \in N_3 \right\},
\]

where \(R(p_1, x_3) = F(p_1, U(p_1, x_3))\) and \(T(p_1, x_3) = L(V(p_1, x_3), x_3)\). Setting \(p_1 = 0\) in \(4\), we get that \(U(0, x_3) = \phi_2(x_3)\) and \(V(0, x_3) = 0\) and then \(R(0, x_3) = \phi_1 \circ \phi_2(x_3)\) and \(T(0, x_3) = 0\). In conclusion, \(V_2^V(N_2) \circ V_1^V(N_1)\) is a true lagrangian submanifold and defines a germ of lagrangian embeddings

\[
i_{\phi_1 \circ \phi_2} : B_{\phi_1 \circ \phi_2} \hookrightarrow T^*M \times T^*N.
\]

around \(G_{\phi_1 \circ \phi_2}\). By Lemma 2.5 this germ is transverse to \(G^*\Lambda\).

**Definition 3.8.** Let \((i_{\phi_1}, \phi_1) \in \text{Hom}(T^*M, T^*N)\) and \((i_{\phi_2}, \phi_2) \in \text{Hom}(T^*N, T^*P)\) be two morphisms in \(\text{MiC}\). We define the composition between them by

\[
(i_{\phi_2}, \phi_2) \circ (i_{\phi_1}, \phi_1) := (i_{\phi_1 \circ \phi_2}, \phi_1 \circ \phi_2),
\]

where \(i_{\phi_1 \circ \phi_2}\) is the germ obtained in Proposition 3.7 by the usual composition of canonical relations and \(\phi_1 \circ \phi_2\) is the usual composition of maps.

**Example 3.9.** Suppose we have a map \(\phi_1\) from \(N\) to \(M\) and a map \(\phi_2\) from \(M\) to \(Q\). Consider the generalized cotangent lifts \((d\phi_1^*, \phi_1)\) and \((d\phi_2^*, \phi_2)\) as defined in Example 3.3. Then we have that \((d\phi_2^*, \phi_2) \circ (d\phi_1^*, \phi_1) = (d(\phi_1 \circ \phi_2)^*, \phi_1 \circ \phi_2)\).

We may also define a bifunctor:

\[
\otimes : \text{MiC} \times \text{MiC} \longrightarrow \text{MiC}
\]

in the following way. Take two cotangent bundles \(T^*M, T^*N\). We define the product between objects as \(T^*M \otimes T^*N = T^*(M \times N)\). Take two morphisms \((i_{\phi_1}, \phi_1) \in \text{Hom}(T^*M, T^*N)\) and \((i_{\phi_2}, \phi_2) \in \text{Hom}(T^*P, T^*Q)\). The product between morphisms is given by \((i_{\phi_1}, \phi_1) \otimes (i_{\phi_2}, \phi_2) = (i_{\phi_1} \times i_{\phi_2}, \phi_1 \times \phi_2)\). The bifunctoriality of \(\text{MiC}\) follows trivially from the bifunctoriality of the Cartesian product on sets. Let us summarize the results obtained so far in the following theorem.

**Theorem 3.10.** \(\text{MiC}\) is a monoidal category.

### 4 Generating functions

In this Section, we describe morphisms \((i_\phi, \phi)\) from \(T^*M\) to \(T^*N\) in local charts in terms of a single function: the generating function \(S\) of the lagrangian embedding \(i_\phi\). We derive then a composition formula for these generating functions which represents the composition of morphisms. At last, we see how these generating functions behave under change of coordinates.
Observe first that, for any manifold $M$, we may always find a system of star-shaped charts $\{U_a\}_{a \in A}$ which covers $M$. In the sequel, we always assume that the charts of the base manifolds are of this sort. In particular, we consider the induced charts of $T^*M \times T^*N$ of the type $U = T^*U_1 \times T^*U_2$ where $U_1$ and $U_2$ are star-shaped charts of $M$ and $N$ respectively. Now, if $i_\phi : B_\phi \hookrightarrow T^*M \times T^*N$ is a transverse germ of lagrangian embeddings around $G_\phi$, Lemma 2.5 tells us that there exists a neighborhood $W$ of $Z_\phi^U$ such that:

$$i_\phi^U(W) = \left\{ (p_1, G(p_1, x_2)), (H(p_1, x_2), x_2) : (p_1, x_2) \in W \right\}.$$ 

The fact that $i_\phi^U(W)$ is a lagrangian submanifold and that $U$ is topologically trivial implies that there exists a function $S_U : W \to \mathbb{R}$ such that

$$G(p_1, x_2) = \nabla_{p_1} S_U(p_1, x_2) \quad \text{and} \quad H(p_1, x_2) = \nabla_{x_2} S_U(p_1, x_2).$$

The fact that $i_\phi|_{Z_\phi^U}^U = G_\phi^U$ imposes that:

$$\nabla_{x_2} S_U(0, x_2) = 0 \quad \text{and} \quad \nabla_{p_1} S_U(0, x_2) = \phi(x_2). \quad (5)$$

This implies that $S_U(0, x_2)$ is equal to a constant. We may normalize $S_U$ by choosing this constant to be zero. From now on, we consider only normalized generating functions.

**Definition 4.1.** We call $S_U$ as above the generating function of the morphism $(i_\phi, \phi) \in \text{Hom}(T^*M, T^*N)$ in the local chart $U = T^*U_1 \times T^*U_2$. Notice that $S_U$ may be considered as a germ of smooth functions on $B_\phi^U$ around $Z_\phi^U$.

We provide now the generating functions of the morphisms given in Examples 3.3 to 3.6.

**Example 4.2.** Consider the unique morphism $(i_M, pr)$ of $\text{Hom}(E, T^*M)$. Then, in a local chart, $B_{pr} = M$. The unique normalized function $e : M \to \mathbb{R}$ is the zero function, $e(x) = 0$.

**Example 4.3.** Take a morphism $(i_\phi, \phi) \in \text{Hom}(T^*M, E)$. In this case, $B_\phi = T^*xM$ where $x = \phi(*)$. Then the generating function is a germ of functions $F : T^*xM \to \mathbb{R}$ around 0 such that $F(0) = 0$ and $\nabla F(0) = x$. Let $V = T^*U$ be local chart of $T^*M$. Then

$$i_\phi^V(N) = \left\{ (p, \nabla F(p)) \times \{\ast\} : p \in T^*xM \right\}.$$ 

**Example 4.4.** Consider $(d\phi^*, \phi) \in \text{Hom}(T^*M, T^*N)$. In a local chart $U := T^*U_1 \times T^*U_2$, we have that $B_\phi^U = (\mathbb{R}^m)^* \times U_2$ and the generating function of $d\phi^*$ is the function $G_\phi : B_\phi^U \to \mathbb{R}$ given by $G_\phi(p_1, x_2) = \langle \phi(x_2), p_1 \rangle$.

**Example 4.5.** As a special instance of Example 4.4, the generating function of the identity morphism $id = (\Delta_{T^*M}, id_M) \in \text{Hom}(T^*M, T^*M)$ is $G_\Delta(p_1, x_2) = p_1 x_2$.

Let us see how composition of morphisms reflects locally on their generating functions. For that consider some local charts $U_1, U_2$ and $U_3$ of respectively $M$, $N$ and $P$. Let $(i_{\phi_1}, \phi_1) \in \text{Hom}(T^*M, T^*N)$ and $(i_{\phi_2}, \phi_2) \in \text{Hom}(T^*N, T^*P)$. We denote by $G$ and $F$ the generating functions of $(i_{\phi_1}, \phi_1)$ and $(i_{\phi_2}, \phi_2)$ in the local charts $V_1 = T^*U_1 \times T^*U_2$ and $V_2 = T^*U_2 \times T^*U_3$. Let also be $V_3 = T^*U_1 \times T^*U_3$ the local chart of $T^*M \times T^*P$.
Definition 4.6. Let $f \in C^\infty(\mathbb{R}^k)$ be a function which has only one critical point on $\mathbb{R}^k$. We denote by $\text{Stat}_{(x)}\{f\}$ the value of $f$ at its critical point $x_0$, i.e., at the point $x_0$ such that $\nabla_x f(x_0) = 0$. If $f$ depends on the variables $x, y$, we denote by $\text{Stat}_{(x)}\{f\}(y)$ the function depending on $y$ defined by $f(x_0(y), y)$ where $x_0(y)$ is the implicit function solution of the equation $\nabla_x f(x_0(y), y) = 0$.

Lemma 4.7. There is a neighborhood $N$ of the zero section of $B_{\phi_1 \circ \phi_2}^{V_3}$ such that, for all $(p_1, x_3) \in N$, the function

$$H(\bar{p}, \bar{x}) := F(\bar{p}, x_3) + G(p_1, \bar{x}) - \bar{p}\bar{x}$$

has only one critical point with respect to the $\bar{p}, \bar{x}$ variables.

Proof. The critical points $\bar{p}$ and $\bar{x}$ are the solution of the following system of implicit equations: $\bar{p} = \nabla_x G(p_1, \bar{x})$ and $\bar{x} = \nabla_{p_2} F(\bar{p}, x_3)$. The implicit function theorem tells us that this system has always a unique solution for small enough $(p_1)$. Namely, set

$$H(p_1, x_3, \bar{p}, \bar{x}) = \begin{pmatrix} \bar{p} - \nabla_{x_2} G(p_1, \bar{x}) \\ \bar{x} - \nabla_{p_2} F(\bar{p}, x_3) \end{pmatrix}.$$ 

Thanks to the fact that $G(0, x_1) = 0$ and $\nabla_{p_2} F(0, x_3) = \phi_2(x_3)$, we get

$$H(0, x_3, 0, \phi_2(x_3)) = 0$$

which means that for $p_1 = 0$, the critical points are $\bar{p} = 0$ and $\bar{x} = \phi_2(x_3)$. Moreover, the Jacobi matrix of $H$ at this point with respect to the $\bar{p}, \bar{x}$ variables

$$\frac{\partial H}{\partial (\bar{p}, \bar{x})}((0, x_3, 0, \phi(x_3)) = \begin{pmatrix} \text{id} & 0 \\ -\nabla_{p} \nabla_{p} F(0, x_3) & \text{id} \end{pmatrix}$$

is invertible. This shows that, for $(p_1, x_3)$ in a neighborhood $N$ of the zero section in $B_{\phi_1 \circ \phi_2}^{V_3}$, $H$ always possesses unique critical points $\bar{p}$ and $\bar{x}$. \hfill $\square$

Definition 4.8. Let $F$ and $G$ be as above. We define the composition of generating function as:

$$F \circ G(p_1, x_3) := \text{Stat}_{(\bar{p}, \bar{x})}\{F(\bar{p}, x_3) + G(p_1, \bar{x}) - \bar{p}\bar{x}\}.$$ 

(6)

Note that Lemma 4.7 guarantees that the composition is well-defined.

Lemma 4.9. In the above notation, we have that

$$\nabla_p (F \circ G)(p_1, x_3) = \nabla_p G(p_1, \bar{x})$$

$$\nabla_x (F \circ G)(p_1, x_3) = \nabla_p F(\bar{p}, x_3)$$

where $\bar{p}$ and $\bar{x}$ are solutions of the implicit system:

$$\bar{p} = \nabla_x G(p_1, \bar{x}),$$

(7)

$$\bar{x} = \nabla_p F(\bar{p}, x_3).$$

(8)
Proof. From Definition 4.8 we have that
\[ F \circ G(p_1, x_3) := F(\bar{p}, x_3) + G(p_1, \bar{x}) - \bar{p}\bar{x}, \]
where \( \bar{p} \) and \( \bar{x} \) is the unique solution of the system \((7) - (8)\). Deriving \( F \circ G \) with respect to \( p_1 \), we get that:
\[
\nabla_p(F \circ G)(p_1, x_3) = \nabla_pG(p_1, \bar{x}) + \nabla_pF(\bar{p}, x_3) \frac{d\bar{p}}{dp_1} + \nabla_xG(p_1, \bar{x}) \frac{d\bar{x}}{dp_1} - \frac{d(\bar{p}\bar{x})}{dp_1}.
\]
Noticing that,
\[
\frac{d(\bar{p}\bar{x})}{dp_1} = \nabla_pF(\bar{p}, x_3) \frac{d\bar{p}}{dp_1} + \nabla_xG(p_1, \bar{x}) \frac{d\bar{x}}{dp_1},
\]
we get that:
\[
\nabla_p(F \circ G)(p_1, x_3) = \nabla_pG(p_1, \bar{x}).
\]
Similarly, we get that:
\[
\nabla_x(F \circ G)(p_1, x_3) = \nabla_xF(\bar{p}, x_3).
\]

Lemma 4.10. Let \( F \) and \( G \) be as above, then we have that:
\[
F \circ G(0, x) = 0 \quad \text{and} \quad \nabla_{p_1} F \circ G(0, x) = \phi_1 \circ \phi_2(x).
\]

Proof. The critical points are given by the equations \( \bar{p} = \nabla_xG(p_1, \bar{x}) \) and \( \bar{x} = \nabla_xF(\bar{p}, x_3) \). If \( p_1 = 0 \), we get that \( \bar{p} = 0 \) and \( \bar{x} = \phi_2(x_3) \). Thus, we have immediately that \( F \circ G(0, x_3) = 0 \).

Lemma 4.9 tells us that:
\[
\nabla_{p_1}(F \circ G(p_1, x_3)) = \nabla_{p_1}G(p_1, \bar{x}) \quad \text{and thus, we have that:}
\]
\[
\nabla_{p_1}(F \circ G(0, x_3)) = \nabla_{p_1}G(0, \phi_2(x_3)) = \phi_1 \circ \phi_2(x_3).
\]

Proposition 4.11. Let \( F \) and \( G \) be as above, then \( F \circ G \) is the generating function of \((i_{p_2}, \phi_2) \circ (i_{p_1}, \phi_1)\) in the local chart \( T^*U_1 \times T^*U_3 \).

Proof. In the local charts \( V_1 \) and \( V_2 \), we have that:
\[
i_{\phi_1}(N_G) = \left\{ (p_1, \nabla_pG(p_1, x_2), (\nabla_xG(p_1, x_2), x_2) : (p_1, x_2) \in N_G \right\}
\]
\[
i_{\phi_2}(N_F) = \left\{ (p_2, \nabla_pF(p_2, x_3), (\nabla_xF(p_2, x_3), x_3) : (p_2, x_3) \in N_F \right\}
\]
where \( N_F \) and \( N_G \) are the neighborhood of the zero section in respectively \( B_{p_1}^{V_1} \) and \( B_{p_2}^{V_2} \).

The composition of canonical relations yields:
\[
i_{\phi_2}(N_F) \circ i_{\phi_1}(N_G) = \left\{ (p_1, \nabla_pG(p_1, x_2), (\nabla_xF(p_2, x_3), x_3) : x_2 = \nabla_pF(p_2, x_3), \quad p_2 = \nabla_xG(p_1, x_2), \quad (p_1, x_3) \in N \right\}
\]
where $N$ is a neighborhood of the zero section in $B_{\phi_1 \phi_2}^V$ where the system,

\begin{align*}
p_2 &= \nabla_2 G(p_1, x_2), \\
x_2 &= \nabla_p F(p_2, x_3),
\end{align*}

has a unique solution $(p_2, x_2)$ for $(p_1, x_3) \in N$. Lemma 4.7 tells us that $F \circ G$ is exactly defined on $N$ and induces a lagrangian germ described by

$$i_{F \circ G}(N) = \left\{ (p_1, \nabla_p (F \circ G)(p_1, x_3)), (\nabla_x (F \circ G)(p_1, x_3), x_3) : (p_1, x_3) \in N \right\}.$$ 

An inspection of Lemma 4.3 shows that $F \circ G$ is the generating function of $(i_{\phi_1}, \phi_1) \circ (i_{\phi_2}, \phi_2)$ in the local chart $T^*U_1 \times T^*U_3$.

Suppose we have a morphism $T = (i_\phi, \phi)$ from $T^*M$ to $T^*N$ and a morphism $L = (i_\psi, \psi)$ from $T^*P$ to $T^*Q$. The tensor product $T \otimes L$ is then a morphism from $T^*(M \times P)$ to $T^*(N \times Q)$. Let $p_1, x_1, p_2, x_2$ be local coordinates on $T^*M \times T^*N$ and $\bar{p}_1, \bar{x}_1, \bar{p}_2, \bar{x}_2$ local coordinates on $T^*P \times T^*Q$ and let $F$ and $G$ be the generating functions of $T$ and $L$. The generating function of $T \otimes L$ in these charts is a germ of a smooth function $F \otimes G$ on $B_{\phi \times \psi} = B_\phi \times B_\psi$ around the zero section. Note that the induced local coordinates on $B_{\phi \times \psi}$ are $p_1, \bar{p}_1, x_2, \bar{x}_2$. The following lemma gives us the form of $F \otimes G$.

**Lemma 4.12.** In the above notation, the generating function $F \otimes G$ of $T \otimes L$ is a germ of smooth functions $F \otimes G : B_{\phi \times \psi} \to \mathbb{R}$ around the zero section given by:

$$F \otimes G(p_1, \bar{p}_1, x_2, \bar{x}_2) := F(p_1, x_2) + G(\bar{p}_1, \bar{x}_2).$$

**Proof.** One sees that directly on the graph of $T \otimes L$ written in the local coordinates as above:

$$\left\{ \left( p_1, \bar{p}_1, \nabla_{p_1} F(p_1, x_2), \nabla_{p_1} G(\bar{p}_1, \bar{x}_2) \right), \left( \nabla_{x_2} F(p_1, x_2), \nabla_{x_2} G(\bar{p}_1, \bar{x}_2), x_2, \bar{x}_2 \right) \right\}.$$

Before ending this Section, we describe how the generating functions behave locally when changing coordinates. Suppose we have two local charts $W_\alpha = T^*U_\alpha \times T^*V_\alpha$ and $W_\beta = T^*U_\beta \times T^*V_\beta$ of $T^*M \times T^*N$. Let us denotes by respectively $S_\alpha$ and $S_\beta$ the generating functions of the local restriction $(i_{\phi_\alpha}, \phi_\alpha)$ and $(i_{\phi_\beta}, \phi_\beta)$ of a morphisms $(i_{\phi}, \phi) \in \text{Hom}(T^*M, T^*N)$ in these local charts. If $g : U_\alpha \to U_\beta$ and $h : V_\beta \to V_\alpha$ are the changes of coordinates on the base manifolds, then $(dg^*, g) \in \text{Hom}(T^*U_\beta, T^*U_\alpha)$ and $(dh^*, h) \in \text{Hom}(T^*V_\alpha, T^*V_\beta)$ (see Example 3.3). Let us denote by $p_1, x_1, p_2, x_2$ the local coordinates on $W_\alpha$ and by $\bar{p}_1, \bar{x}_1, \bar{p}_2, \bar{x}_2$ the local coordinates on $W_\beta$. In these coordinates, the generating function of $(dg^*, g)$ is $G(\bar{p}_1, x_1) = g(x_1)\bar{p}_1$ and the generating function of $(dh^*, h)$ is $H(p_2, \bar{x}_2) = h(\bar{x}_2)p_2$.

**Lemma 4.13.** In the above notation, we have that:

$$H \circ S_\alpha \circ G(\bar{p}_1, \bar{x}_2) = S_\alpha \circ G(\bar{p}_1, h(\bar{x}_2)).$$

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Proof. By definition, we have that:

\[ H \circ (S_\alpha \circ G)(\bar{p}_1, \bar{x}_2) = H(\bar{p}, \bar{x}_2) + (S_\alpha \circ G)(\bar{p}_1, \bar{x}) - \bar{p}_x, \]

where the critical point computation yields:

\[ \bar{p} = \nabla_x (S_\alpha \circ G)(\bar{p}_1, \bar{x}) \quad \text{and} \quad \bar{x} = \nabla_p H(\bar{p}, \bar{x}_2). \]

Computing \( S_\alpha \circ G(\bar{p}_1, \bar{x}) \), we get:

\[ S_\alpha \circ G(\bar{p}_1, \bar{x}) = S_\alpha(\bar{p}, \bar{x}) + G(\bar{p}_1, \bar{x}) - \bar{p}_x, \]

where the critical points are given by:

\[ \bar{p} = \nabla_x G(\bar{p}_1, \bar{x}) \quad \text{and} \quad \bar{x} = \nabla_p S_\alpha(\bar{p}, \bar{x}). \]

Remarking that \( \bar{x} = h(\bar{x}_2) \) and that \( \bar{p}_x = H(\bar{p}, \bar{x}_2) \), we get that:

\[ H \circ (S_\alpha \circ G)(\bar{p}_1, \bar{x}_2) = S_\alpha(\bar{p}, h(\bar{x}_2)) + G(\bar{p}_1, \bar{x}) - \bar{p}_x, \]

where

\[ \bar{p} = \nabla_x G(\bar{p}_1, \bar{x}) \quad \text{and} \quad \bar{x} = \nabla_p S_\alpha(\bar{p}, h(\bar{x}_2)). \]

Thus, we get that:

\[ H \circ S_\alpha \circ G(\bar{p}_1, \bar{x}_2) = \text{Stat}_{(\bar{p}, \bar{x})}\{ S_\alpha(\bar{p}, h(\bar{x}_2)) + G(\bar{p}_1, \bar{x}) - \bar{p}_x, \}, \]

which ends the proof.

Suppose that we are given a collection of morphisms \((i_{\phi_\gamma}, \phi_\gamma) \in \text{Hom}(T^*U_\gamma, T^*V_\gamma)\) on local charts \(\{T^*U_\gamma \times T^*V_\gamma\}_{\gamma \in A}\) of \(T^*M \times T^*N\) whose generating functions are denoted by \(S_\gamma\). Suppose further that the \(\phi_\gamma : V_\gamma \to U_\gamma\) are the restrictions of a global morphism \(\phi : N \to M\). The following proposition tells us when this collection \(C := \{(i_{\phi_\gamma}, \phi_\gamma)\}_{\gamma \in A}\) of local morphisms comes from a global morphism \((i_\phi, \phi) \in \text{Hom}(T^*M, T^*N)\).

**Proposition 4.14.** Let \(C := \{(i_{\phi_\gamma}, \phi_\gamma)\}_{\gamma \in A}\) be a collection of local morphisms corresponding to local charts \(T^*U_\gamma \times T^*V_\gamma\) of \(T^*M \times T^*N\) as above. The following statements are equivalents:

1. The collection \(C\) comes from the restrictions of a global morphism \((i_\phi, \phi) \in \text{Hom}(T^*M, T^*N)\) to the local charts.

2. For any two morphisms \((i_{\phi_\alpha}, \phi_\alpha), (i_{\phi_\beta}, \phi_\beta) \in C\) we have, on overlapping domains, that:

\[ (i_{\phi_\beta \gamma}, \phi_\beta) = (dh^*, h) \circ (i_{\phi_\alpha}, \phi_\alpha) \circ (dg^*, g), \]

where \(g : U_\alpha \to U_\beta\) and \(h : V_\beta \to V_\alpha\) are the change of coordinates.
(3) For any two morphisms \((i_{\phi_\alpha}, \phi_\alpha), (i_{\phi_\beta}, \phi_\beta)\) \(\in C\) we have, on overlapping domains, that:

\[ S_\beta = H \circ S_\alpha \circ G, \]

where \(S_\alpha\) and \(S_\beta\) are the generating function of the local morphisms and \(H\) and \(G\) are the generating functions of respectively \((dh^*, h)\) and \((dg^*, g)\).

Proof. By definition, (2) and (3) are equivalent. We show here that (3) and (1) are also equivalent. To simplify the notation, we suppose that \(T^*U_\alpha \times T^*V_\alpha\) and \(T^*U_\beta \times T^*V_\beta\) describe the same open subset of \(T^*M \times T^*N\). The graph of \((i_{\phi_\alpha}, \phi_\alpha)\) in \(T^*U_\alpha \times T^*V_\alpha\) is given by:

\[ L_\alpha = \left\{ \left( (p_1, \nabla_p S_\alpha(p_1, x_2)), (\nabla_x S_\alpha(p_1, x_2), x_2) \right) : (p_1, x_2) \in N_\alpha \right\} \]

where \(N_\alpha\) is a neighborhood of the zero section in \(B^\alpha_\phi\). Similarly, the graph of \((i_{\phi_\beta}, \phi_\beta)\) in \(T^*U_\beta \times T^*V_\beta\) is given by:

\[ L_\beta = \left\{ \left( (\bar{p}_1, \nabla_p S_\beta(\bar{p}_1, \bar{x}_2)), (\nabla_x S_\beta(\bar{p}_1, \bar{x}_2), \bar{x}_2) \right) : (\bar{p}_1, \bar{x}_2) \in N_\beta \right\} \]

where \(N_\beta\) is a neighborhood of the zero section in \(B^\beta_\phi\). Now, \(L_\alpha\) and \(L_\beta\) describe the same submanifold of \(T^*M \times T^*N\) iff

\[
\begin{align*}
p_1 &= dg^*_{\beta \alpha} \left( \nabla_p S_\alpha(p_1, x_2) \right) \bar{p}_1, \\
\nabla_x S_\alpha(p_1, x_2) &= dh^*_\beta(\bar{x}_2) \left( \nabla_x S_\beta(\bar{p}_1, \bar{x}_2) \right), \\
\nabla_x S_\beta(\bar{p}_1, \bar{x}_2) &= x_2 = h_{\alpha \beta}(\bar{x}_2).
\end{align*}
\]

This is equivalent to have that:

\[
\begin{align*}
\nabla_x S_\beta(\bar{p}_1, \bar{x}_2) &= dh^*_\beta(\bar{x}_2) \left( \nabla_x S_\alpha(\hat{p}, h_{\alpha \beta}(\bar{x}_2)) \right) \quad (11) \\
\nabla_p S_\beta(\bar{p}_1, \bar{x}_2) &= g_{\beta \alpha}(\hat{x}) \quad (12)
\end{align*}
\]

where \(\hat{x} = \nabla_p S_\alpha(\hat{p}, h_{\alpha \beta}(\bar{x}_2))\) and \(\hat{p} = dg^*_{\beta \alpha}(\hat{x})\bar{p}_1\). Now, thanks to Lemma 4.13, we have that:

\[
H \circ S_\alpha \circ G(\bar{p}_1, \bar{x}_2) = S_\alpha(\hat{p}, h_{\alpha \beta}(\bar{x}_2)) + G(\bar{p}_1, \hat{x}) - \hat{p} \hat{x},
\]

(13)

where we also have that \(\hat{x} = \nabla_p S_\alpha(\hat{p}, h_{\alpha \beta}(\bar{x}_2))\) and \(\hat{p} = dg^*_{\beta \alpha}(\hat{x})\bar{p}_1\). Applying Lemma 4.9 to \(H \circ S_\alpha \circ G\) we get equations (11)–(12). Thus, this shows that (3) implies (1). On the other hand, (1) implies that the derivative of the generating function \(S_\beta\) have the form given by equations (11)–(12). The only normalized generating function which has these derivatives is \(H \circ S_\alpha \circ G\).

5 The Poisson functor

This Section is devoted to showing that a monoid structure on an object \(T^*M\) of the cotangent microbundle category induces a Poisson structure on the base manifold \(M\) together with a local symplectic groupoid \((s, t) : T^*M \rightrightarrows M\) integrating it. The description of both the Poisson structure and the local symplectic groupoid are given explicitly in terms of the
generating function of transverse lagrangian germs. We also prove that morphisms of monoid structures produce Poisson morphisms on the base. This yields, in particular, a contravariant functor

$$D : \text{Mon}(\text{MiC}) \to \text{Poiss},$$

form the category $\text{Mon}(\text{MiC})$ of monoid objects and monoid maps in $\text{MiC}$ to the category $\text{Poiss}$ of Poisson manifolds and Poisson maps.

**Definition 5.1.** In a monoidal category $C$ with neutral object $E$, a monoid is a triple $(M, \mu, e)$ made of an object $M$, a morphism $\mu \in \text{Hom}(M \otimes^2, M)$ called the product and a morphism $e \in \text{Hom}(E, M)$ called the unit. These morphisms should satisfy the two following relations:

$$\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu) \tag{14}$$

$$\mu \circ (e \otimes \text{id}) = \mu \circ (\text{id} \otimes e) = \text{id} \tag{15}.$$  

We call the couple $(\mu, e)$ a monoid structure on $M$.

**Definition 5.2.** Let $C$ be a monoidal category and let $(M, \mu_M, e_M)$ and $(N, \mu_N, e_N)$ be two monoids in $C$. We say that a morphism $T : M \to N$ is a monoid morphism if $T \circ \mu_M = \mu_N \circ (T \otimes T)$ and $T(e_M) = e_N$.

It is easy to see that the monoid object in a monoidal category $C$ together with their monoid morphisms form a category, which we denote by $\text{Mon}(C)$.

**Example 5.3.** A monoid $(V, \mu, e)$ in the category of complex vector spaces is a usual unital algebra. The morphism $\mu : V \otimes V \to V$ is the associative product and the unit morphism $e : \mathbb{C} \to V$ is given by $e(\lambda) = \lambda \cdot 1$ where $1$ is the unit of the algebra.

The following Proposition tells us that a monoid $(T^*M, \mu, e)$ in $\text{MiC}$ is completely determined by its product $\mu \in \text{Hom}(T^*M \otimes^2, T^*M)$ whose base map must be the diagonal map $\Delta : M \to M \times M$.

**Proposition 5.4.** Let $(T^*M, \mu, e)$ be a monoid in $\text{MiC}$. Then the unit morphism is the unique morphism of $\text{Hom}(E, T^*M)$, i.e., $e = (i_M, \text{pr})$, and the product $\mu \in \text{Hom}(T^*M \otimes^2, T^*M)$ is of the form $\mu = (i_\Delta, \Delta)$ where $\Delta : M \to M \times M$ is the diagonal map $\Delta(x) = (x, x)$.

**Proof.** As $\text{Hom}(E, T^*M)$ possesses only one element given by $(i_M, \text{pr})$, this imposes that $e = (i_M, \text{pr})$. Suppose now that $\mu = (i_\phi, \phi)$ satisfies $[15]$, i.e.,

$$\begin{align*}
(\Delta_{T^*M}, \text{id}_M) &= (i_\phi \circ (\Delta_{T^*M} \times i_M), (\text{id}_M \times \text{pr}) \circ \phi) \\
&= (i_\phi \circ (i_M \times \Delta_{T^*M}), (\text{pr} \times \text{id}_M) \circ \phi).
\end{align*}$$

If we set $\phi(x) = (\phi_1(x), \phi_2(x))$, then $(\text{id}_M \times \text{pr}) \circ \phi = \text{id}_M$ translates into $\phi_1(x) = x$ and $(\text{pr} \times \text{id}_M) \circ \phi = \text{id}$ into $\phi_2(x) = x$. Thus, $\phi(x) = (x, x)$. \qed
Proposition 5.3 tells us that monoid structures on an object $T^*M$ in $\text{MiC}$ are entirely determined by germs of lagrangian embedding,

$$i_\Delta : B_\Delta \hookrightarrow \underbar{T^*M} \times \underbar{T^*M} \times T^*M$$

around $G_\Delta$ which satisfy the conditions:

$$i_\Delta \circ (i_\Delta \times \Delta_{T^*M}) = i_\Delta \circ (\Delta_{T^*M} \times i_\Delta) \quad (16)$$

$$i_\Delta \circ (i_M \times \Delta_{T^*M}) = i_\Delta \circ (\Delta_{T^*M} \times i_M) = \Delta_{T^*M}. \quad (17)$$

We call such germs monoid structures on $T^*M$. We will omit the reference to the unit morphism $e \in \text{Hom}(E, T^*M)$ in the notation of a monoid $(T^*M, \mu, e)$ as we have no choice for it.

As $\Delta$ is the diagonal map, $i_\Delta$ is a lagrangian germ around

$$\left\{ ((0, x), (0, x), (0, x)) : x \in M \right\}$$

in $\underbar{T^*M} \times \underbar{T^*M} \times T^*M$. Thus, the local charts $\underbar{T^*U} \times \underbar{T^*U} \times T^*U$ induced by locals charts $U$ of the base $M$ are enough to describe $i_\Delta$ completely. In the remaining of this section, we consider only such charts and we denote by $p_1, x_1, p_2, x_2, p_3, x_3$ the local coordinates on them. In a local chart $V = \underbar{T^*U} \times \underbar{T^*U} \times T^*U$, the generating function of a monoid structure $i_\Delta$ is a germ of a smooth function,

$$S : B_\Delta^V = (\mathbb{R}^d)^* \times (\mathbb{R}^d)^* \times U \rightarrow \mathbb{R},$$

around the zero section which vanishes on it and such that:

$$\nabla_{p_1} S(0, 0, x) = \nabla_{p_2} S(0, 0, x) = x.$$ 

In terms of the generating function $S$, conditions (16)–(17) read:

$$S \circ (S \otimes I) = S \circ (I \otimes S) \quad (18)$$

$$S \circ (e \otimes I) = S \circ (I \otimes e) = I, \quad (19)$$

where by $e$ and $I$ stand for the generating functions of $(i_M, pr)$ and $(\Delta_{T^*M}, \text{id}_M)$ respectively. Recall from Example 4.2 and Example 4.5 that $e(x) = 0$ and $I(p, x) = px$ in local charts.

We reformulate now Equations (18)–(19) for the generating function $S$ of $(i_\Delta, \Delta)$ in a local chart.

**Lemma 5.5.** The identity $S \circ (I \otimes e) = S \circ (e \otimes I) = I$ is equivalent to $S$ satisfying the following condition:

$$S(p, 0, x) = S(0, p, x) = px.$$ 

**Proof.** We have that

$$S \circ (e \otimes I)(p, x) = \text{Stat}_{(p_1, p_2, x_1, x_2)} \left\{ S(p_1, p_2, x) + e(x_1) + I(p, x_2) - p_1 x_1 - p_2 x_2 \right\}$$

The critical points are $p_1 = 0, p_2 = p, x_1 = \nabla_{p_1} S(0, p, x)$ and $x_2 = \nabla_{p_2} S(p, 0, x)$. Thus, we get that $S \circ (e \otimes I)(p, x) = S(0, p, x) = I(p, x) = px$. Similarly, we obtain that $S(p, 0, x) = px$. $\square$
Lemma 5.6. The identity $S \circ (S \otimes I) = S \circ (I \otimes S)$ is equivalent to the existence of a neighborhood $N$ of $\{0\}^3 \times U$ in $(\mathbb{R}^d)^3 \times U$ where, for all $(p_1, p_2, p_3, x) \in N$, the generating function $S$ satisfies
\[
S(\bar{p}, p_3, x) + S(p_1, p_2, \bar{x}) - \bar{x} \bar{p} = S(\bar{p}, p_3, x) + S(p_2, p_3, \bar{x}) - \bar{x} \bar{p}, \tag{20}
\]
where $\bar{x}, \bar{p}, \bar{p}$ and $\bar{p}$ are solution of the following implicit equations
\[
\bar{x} = \nabla_{p_1} S(\bar{p}, p_3, x), \quad \bar{p} = \nabla_{p_2} S(p_1, \bar{p}, x) \quad \bar{p} = \nabla_x S(p_1, p_2, \bar{x}), \quad \bar{p} = \nabla_x S(p_2, p_3, \bar{x}).
\]

Proof. We have that
\[
S \circ (S \otimes I)(p_1, p_2, p_3, x) = \text{Stat}_{(p_1, p_2, \bar{x}, x_2)} \left\{ S(\bar{p}_1, p_2, x) + S(p_1, p_2, \bar{x}_1) + I(p_3, \bar{x}_2) - \bar{p}_1 \bar{x}_1 - \bar{p}_2 \bar{x}_2 \right\}
\]
The critical points computation yields:
\[
\bar{p}_1 = \nabla_x S(p_1, p_2, \bar{x}_1), \quad \bar{x}_1 = \nabla_{p_1} S(\bar{p}_1, p_2, x)
\]
\[
\bar{p}_2 = \nabla_x I(p_3, \bar{x}_2) = p_3, \quad \bar{x}_2 = \nabla_{p_2} S(\bar{p}_1, p_2, x)
\]
Thus, we get that $S \circ (S \otimes I)(p_1, p_2, p_3, x) = S(\bar{p}, p_3, x) + S(p_1, p_2, \bar{x}) - \bar{x} \bar{p}$, where $\bar{p} = \nabla_x S(p_1, p_2, \bar{x})$ and $\bar{x} = \nabla_{p_1} S(\bar{p}, p_3, x)$. Similarly, one computes $S \circ (I \otimes S)$ directly to obtain the right-hand side of (20).

The next proposition tells us that a monoid $(T^* M, \mu)$ in MiC induces a Poisson structure on each local chart $U \subset M$ together with the local symplectic groupoid integrating it. Let us first recall the definition of Poisson structures and local symplectic groupoids.

Definition 5.7. Let $M$ be a smooth manifold. A Poisson structure on $M$ is a Lie bracket $\{ , \}$ on the algebra of smooth functions $C^\infty(M)$ on $M$ which is a derivation in both of its arguments.

A Poisson structure may be represented by a bivector field $\alpha \in \Gamma(\wedge^2 T M)$ in the following way:
\[
\{ f, g \}(x) = \langle f \otimes g, \alpha \rangle.
\]
In a local chart $U$ of $M$, the bivector field $\alpha$ is represented by a matrix $(\alpha_{ij}(x))_{i,j=1}^{\dim M}$ whose coefficients depend on the point $x \in U$ and which satisfies the Jacobi identity:
\[
\alpha^i_k \partial_k \alpha^{jl} + \alpha^l_k \partial_l \alpha^{ij} + \alpha^j_k \partial_k \alpha^{il} = 0.
\]
In local coordinates, the bracket of two functions reads:
\[
\{ f, g \}(x) = \alpha_{ij}^i(x) \partial_i f(x) \partial_j g(x).
\]
Definition 5.8. A Poisson map \( \phi : (N, \alpha_N) \to (M, \alpha_M) \) is a smooth map which preserves the Poisson bracket, i.e., such that for \( f, g \in C^\infty(M) \):

\[
\phi^*\{f, g\}_M = \{\phi^*f, \phi^*g\}_N.
\]

In local coordinates, the condition that \( \phi : (N, \alpha_N) \to (M, \alpha_M) \) is a Poisson map reads:

\[
\alpha_M^{ij}(\phi(x)) = \frac{\partial \phi^i(x)}{\partial x^k} \alpha_N^{kl}(x) \frac{\partial \phi^j(x)}{\partial x^l}.
\]

The Poisson manifolds together with their Poisson maps form a category, which we denote by \( \text{Poiss} \).

Example 5.9. Let \((M, \omega)\) be a symplectic manifold with symplectic form \( \omega \in \Omega(M) \). For each function \( f \in C^\infty(M) \), we associate a Hamiltonian vector field \( X_f \) by the equation \( \iota(X_f)\omega = df \). The Poisson bracket associated to \( \omega \) is

\[
\{f, g\}_\omega = \omega(X_f, X_g).
\]

If \( J \) is the symplectic matrix of \( \omega \) in Darboux coordinates, the Poisson bivector of \( \{, \}_\omega \) is \( J^{-1} \).

Definition 5.10. A local symplectic groupoid over a Poisson manifold \((M, \alpha)\) is a symplectic manifold \((G, \omega)\) together with an lagrangian embedding \( \epsilon : M \to G \) and two submersions \( s, t : U \to M \), defined in a neighborhood \( U \) of \( \epsilon(M) \) in \( G \), such that

1. \( s \) and \( t \) are projection on \( M \), i.e., \( s \circ \epsilon = \text{id}_M \) and \( t \circ \epsilon = \text{id}_M \),
2. \( s \) and \( t \) are Poisson and anti-Poisson maps respectively,
3. \( s \) and \( t \) commute, i.e., we have that \( \{s^*f, t^*g\}_\omega = 0 \), for all \( f, g \in C^\infty(M) \) and where \( \{, \}_\omega \) is the Poisson bracket associated to the symplectic form \( \omega \).

The map \( s \) is called the source and the map \( t \) is called the target. We write sometimes a local symplectic groupoid over \( M \) as \((s, t) : G \rightrightarrows M\). We also say that \((s, t) : G \rightrightarrows M\) integrates (in a local context) the Poisson manifold \( M \).

Remark 5.11. Usually, the definition of symplectic groupoid \( G \rightrightarrows M \) includes a partially defined associative product on \( G \) (the product of two elements \( g_1 \) and \( g_2 \) in \( G \) is defined only when \( s(g_1) = t(g_2) \)) whose graph is a lagrangian submanifold of \( G \times G \). In the local case (i.e. when one requires the source and target domains to be only a neighborhood of \( \epsilon(M) \) in \( G \) and not the whole space \( G \)), it has been shown, in [5] and in [9] for instance, that it is possible to recover the partially defined product from the data of the source and the target maps. For this reason and for the sake of simplicity, we prefer not to mention the partially defined product in the definition of a local symplectic groupoid.
Proposition 5.12. Suppose that $\mu \in \text{Hom}(T^*M \otimes^2, T^*M)$ is a monoid structure on $T^*M$ whose generating function in a local chart $U$ is $S$. Define the bivector field $\alpha \in \Gamma(\wedge^2 TU)$ by the following matrix:

$$\alpha(x) := \left( \frac{\partial^2 S}{\partial p_k \partial p_l} (0, 0, x) - \frac{\partial^2 S}{\partial p_k \partial p_l} (0, 0, x) \right)_{kl=1}^d$$

and the maps $s, t : T^*U \to U$ by the formulas:

$$s(p, x) := \nabla_{p_2} S(p, 0, x)$$
$$t(p, x) := \nabla_{p_1} S(0, p, x).$$

Then $\alpha \in \Gamma(\wedge^2 U)$ is a Poisson bivector on $U$, and $(s, t) : T^*M \rightarrow M$ is a local symplectic groupoid integrating $\alpha$.

Proof. We have to show that

$$\{s^i, s^j\}_\omega(p, x) = \alpha^{ij}(s(p, x))$$
$$\{t^i, t^j\}_\omega(p, x) = -\alpha^{ij}(t(p, x))$$
$$\{s^i, t^j\}_\omega(p, x) = 0.$$  (23)

Notice that equation (21) implies that $\alpha$ is a Poisson bivector field. Namely, Equation (21) means that, for any function $f, g \in C^\infty(U)$, we have that

$$\{s^* f, s^* g\}_\omega(p, x) = s^* \{f, g\}_\omega(p, x),$$

which yields that

$$s^* \{f, \{g, h\}_\omega\} = \{s^* f, \{s^* g, s^* h\}_\omega\}.$$  (24)

As $\{, \}_\omega$ fulfills the Jacobi identity and as $s^* f(0, x) = f(x)$, we obtain that $\{, \}_\alpha$ also satisfies the Jacobi identity.

Let us check that (21) holds. Derive Equation (20) two times, first with respect to $p_3$ and then with respect to $p_2$. We obtain

$$\frac{\partial^2 S}{\partial p_1 \partial p_2} (\bar{p}, p_3, x) \frac{dp_j}{dp_k} = \frac{\partial^2 S}{\partial p_1 \partial p_2} (p_2, p_3, \bar{x}) + \frac{\partial^2 S}{\partial x_j \partial p_1} (p_2, p_3, \bar{x}) \frac{d\bar{x}}{dp_k}.$$  (25)

If we set $p_1 = p, p_2 = p_3 = 0$, the critical points computation together with Lemma 5.5 yields $\bar{p} = p, \bar{p} = 0, \bar{x} = x, \bar{x} = s(p, x)$, and

$$\frac{dp_j}{dp_k} = \frac{\partial^2 S}{\partial x_j \partial p_k} (p, 0, x), \quad \frac{d\bar{x}}{dp_k} = \frac{\partial^2 S}{\partial p_j \partial p_k} (p, 0, x).$$

Thus, we get

$$\frac{\partial^2 S}{\partial p_1 \partial p_2} (p, 0, x) \frac{\partial^2 S}{\partial p_1 \partial x_j} (p, 0, x) = \frac{\partial^2 S}{\partial p_1 \partial p_k} (0, 0, s(p, x)) + \frac{\partial^2 S}{\partial p_2 \partial p_k} (0, 0, x).$$

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Taking the difference between this last equation and itself but with the indices \( k \) and \( i \) interchanged we obtain

\[
\frac{\partial s^k}{\partial x^j}(p, x) \frac{\partial s^i}{\partial p^j}(p, x) - \frac{\partial s^i}{\partial x^j}(p, x) \frac{\partial s^k}{\partial p^j}(p, x) = \alpha^k(s(p, x))
\]

which is exactly (21).

The same strategy works for (22) and (23). However, for (22) we have to differentiate (20) with respect to \( p_2 \) and \( p_1 \) setting \( p_1 = p_2 = 0 \) and \( p_3 = p \). To check (23), we have to differentiate (20) with respect to \( p_1 \) and \( p_3 \) setting \( p_1 = p_3 = 0 \) and \( p_2 = p \). \( \square \)

**Proposition 5.13.** Let \( \mu \in \text{Hom}(T^*M \otimes 2, T^*M) \) be a monoid structure on \( T^*M \). The Poisson bivector field \( \alpha \) as well as the source map \( s \) and the target maps \( t \) defined in local charts \( U \subset M \) in Proposition 5.12 glue well together on overlapping charts and thus induce a local symplectic groupoid on \( T^*M \).

**Proof.** Suppose two \((U_\gamma, \phi_\gamma)\) and \((U_\beta, \phi_\beta)\) are two overlapping charts of \( M \). Set \( V_\gamma := \phi_\gamma(U_\gamma \cap U_\beta) \) and \( V_\beta := \phi_\beta(U_\gamma \cap U_\beta) \). We denote by \( p, x \) the coordinates on \( T^*V_\gamma \) and by \( \bar{p}, \bar{x} \) the coordinates on \( T^*V_\beta \). \( S_\gamma \) (resp. \( S_\beta \)) is the generating function of \((i_\Delta, \Delta)\) in \( U_\gamma \cap U_\beta \) expressed in the \( p, x \) (resp. \( \bar{p}, \bar{x} \)) coordinates. Denote by \( G_{\gamma \beta}(\bar{p}, x) = g_{\beta \gamma}(x) \bar{p} \) the generating function of the induced coordinate change \( dg^* \) from \( T^*V_\beta \) to \( T^*V_\gamma \) by the coordinate change on the base \( g := g_{\beta \gamma} \) from \( V_\gamma \) to \( V_\beta \). We know, from Lemma 4.12, Lemma 4.13 and Proposition 4.14 that

\[
S_\beta(\bar{p}_1, \bar{p}_2, \bar{x}) = G_{\beta \gamma} \circ S_\gamma \circ (G_{\gamma \beta} \otimes G_{\gamma \beta})(\bar{p}_1, \bar{p}_2, \bar{x})
\]

\[
= S_\gamma \circ (G_{\gamma \beta} \otimes G_{\gamma \beta})(\bar{p}_1, \bar{p}_2, g_{\gamma \beta}(\bar{x}))
\]

\[
= S_\gamma(\bar{p}_1, \bar{p}_2, g_{\gamma \beta}(\bar{x}))) + g_{\beta \gamma}(x_1)\bar{p}_1 + g_{\beta \gamma}(x_2)\bar{p}_2 - \bar{p}_1 x_1 - \bar{p}_2 x_2,
\]

where \( \bar{p}_1, \bar{x}_1, \bar{p}_2 \) and \( \bar{x}_2 \) are the critical points given by the following implicit equations:

\[
\bar{p}_1 = dg_{\beta \gamma}^*(x_1)\bar{p}_1, \quad \bar{x}_1 = \nabla_{\bar{p}_1} S_\gamma(\bar{p}_1, \bar{p}_2, g_{\gamma \beta}(\bar{x}))
\]

\[
\bar{p}_2 = dg_{\beta \gamma}^*(x_2)\bar{p}_2, \quad \bar{x}_2 = \nabla_{\bar{p}_2} S_\gamma(\bar{p}_1, \bar{p}_2, g_{\gamma \beta}(\bar{x}))
\]

Using Lemma 4.19 we get that:

\[
\nabla_{\bar{p}_2} S_\beta(\bar{p}_1, \bar{p}_2, \bar{x}) = g_{\beta \gamma}(\bar{x}_2).
\]

Now, setting \( \bar{p}_2 = 0 \), we get immediately that \( \bar{p}_2 = 0 \), Lemma 5.5 gives that \( \bar{x}_1 = g_{\gamma \beta}(\bar{x}) \) and thus

\[
\bar{p}_1 = dg_{\beta \gamma}^*(g_{\gamma \beta}(\bar{x}))\bar{p}_1, \quad \bar{x}_2 = \nabla_{\bar{p}_2} S_\gamma(dg^*(\bar{x})\bar{p}_1, 0, g_{\gamma \beta}(\bar{x})).
\]

Then we have that:

\[
s_\beta(\bar{p}, \bar{x}) = \nabla_{\bar{p}_2} S_\beta(\bar{p}, 0, \bar{x})
\]

\[
= g(\nabla_{\bar{p}_2} S_\gamma((dg^*\bar{p}, 0, g^{-1}(\bar{x}))))
\]

\[
= g(s_\gamma((dg^*(\bar{p}, \bar{x}))).
\]
Similarly, we get that \( t_\beta(\tilde{p}, \tilde{x}) = g(t_\gamma(dg^*((\tilde{p}, \tilde{x})))) \). Thus, the \( s_\gamma \)'s and the \( t_\gamma \)'s define a global source and target on a neighborhood of \( M \) in \( T^*M \). Now, let us check the invariance of the Poisson structure \( \alpha_\gamma \). Using Lemma 4.9 we get that:

\[
\nabla_{p_k^I} S_\beta(\tilde{p}_1, \tilde{p}_2, \tilde{x}) = \nabla_{\tilde{p}_k} G_{\gamma\beta}(\tilde{p}_1, \tilde{x}_1)
\]

and then

\[
\nabla_{p_k^I} \nabla_{\tilde{p}_k} S_\beta(\tilde{p}_1, \tilde{p}_2, \tilde{x}) = \nabla_{\tilde{p}_k} \nabla_{x^a} G_{\gamma\beta}(\tilde{p}_1, \tilde{x}_1) \frac{d\tilde{x}^{a}_1}{dp_{\tilde{l}}^2}.
\]

Now, if \( \tilde{p}_1 = \tilde{p}_2 = 0 \) then \( \tilde{p}_1 = \tilde{p}_2 = 0 \) and \( \tilde{x}_1 = \tilde{x}_2 = g^{-1}(\tilde{x}) \) and

\[
\left( \frac{d\tilde{x}^{a}_1}{dp_{\tilde{l}}^2} \right)_{|\tilde{p}_1 = \tilde{p}_2 = 0} = \alpha^{uv}_\gamma (g^{-1}(\tilde{x})) \left( \frac{dp_{\tilde{l}}^2}{dp_{\tilde{l}}^2} \right)_{|\tilde{p}_1 = \tilde{p}_2 = 0}
\]

Finally, we obtain the invariance of the Poisson structure, i.e.,

\[
\alpha^{ kl}_{\beta}(\tilde{x}) = \frac{\partial g^k}{\partial x^u} (g^{-1}(\tilde{x})) \alpha^{uv}_\gamma (g^{-1}(\tilde{x})) \frac{\partial g^l}{\partial x^v} (g^{-1}(\tilde{x})).
\]

\[\square\]

**Proposition 5.14.** Let \( (T^* M, \mu_M) \) and \( (T^* N, \mu_N) \) be two monoids and \( \alpha_M, \alpha_N \) their induced Poisson structure on the base \( M \) and \( N \) respectively. Suppose \( T = (i_\phi, \phi) \in \text{Hom}(T^* M, T^* N) \) is a monoid morphism. Then the base map \( \phi \) is a Poisson map from \( (N, \alpha_N) \) to \( (M, \alpha_M) \).

**Proof.** Consider \( U_1 \) and \( U_2 \) two local charts of \( M \) and \( N \) respectively. Denote by \( S_M, S_N \) and \( F \) the generating functions, in the induced local charts, of \( \mu_M, \mu_N \) and \( T \) respectively. Then we have that

\[
F \circ S_M = S_N \circ (F \otimes F).
\]  

(24)

Denote the local coordinates on \( T^* M \times T^* M \) by \( p_1, p_2, x_1, x_2 \) and the local coordinates on \( T^* N \) by \( \tilde{p}, \tilde{x} \). The, the left hand side of Equation (24) is:

\[
F \circ S_M(p_1, p_2, \tilde{x}) = F(\tilde{p}, \tilde{x}) + S_M(p_1, p_2, \tilde{x}) - \tilde{p}\tilde{x}
\]

where \( \tilde{p} \) and \( \tilde{x} \) are given by the following implicit equations:

\[
\tilde{p} = \nabla_x S_M(p_1, p_2, \tilde{x}) \quad \tilde{x} = \nabla_p F(\tilde{p}, \tilde{x}).
\]

By Lemma 4.9 we obtain that:

\[
\nabla_{p_1} (F \circ S_M)(p_1, p_2, \tilde{x}) = \nabla_{p_1} S_M(p_1, p_2, \tilde{x}).
\]
If we derive this equation again with respect to $p_2$, we get:

$$\nabla_{p_1} \nabla_{p_2} (F \circ S_M)(0, 0, \bar{x}) = \nabla_{p_1} \nabla_{p_2} S_M(p_1, p_2, \bar{x}) + \nabla_x S_M(p_1, p_2, \bar{x}) \frac{d \bar{x}}{dp_2}.$$  

Setting $p_1 = p_2 = 0$, we get that $\tilde{p} = 0$ and $\bar{x} = \phi(\bar{x})$ and thus:

$$\nabla_{p_1} \nabla_{p_2} (F \circ S_M)(0, 0, \bar{x}) = \nabla_{p_1} \nabla_{p_2} S_M(0, 0, \phi(\bar{x})) = \alpha_M(\phi(\bar{x})).$$

Now, the right hand side of Equation (24) yields:

$$S_N \circ (F \otimes F)(p_1, p_2, \bar{x}) = S_N(\tilde{p}_1, \tilde{p}_2, \bar{x}) + F(p_1, \bar{x}_1) + F(p_2, \bar{x}_2) - \bar{p}_1 \bar{x}_1 - \bar{p}_2 \bar{x}_2,$$

where $\tilde{p}_1, \tilde{x}_1, \tilde{p}_2$ and $\tilde{x}_2$ are given by the following implicit equations:

$$\begin{align*}
\tilde{p}_1 &= \nabla_x F(p_1, \tilde{x}_1) \\
\tilde{x}_1 &= \nabla_{p_1} S_N(\tilde{p}_1, \tilde{p}_2, \bar{x}) \\
\tilde{p}_2 &= \nabla_x F(p_2, \tilde{x}_2) \\
\tilde{x}_2 &= \nabla_{p_2} S_N(\tilde{p}_1, \tilde{p}_2, \bar{x}).
\end{align*}$$

Again, Lemma 4.9 gives us:

$$\nabla_{p_1} (S_N \circ (F \otimes F))(p_1, p_2, \bar{x}) = \nabla_{p_1} F(p_1, \bar{x}_1).$$

Deriving another times with respect to $p_2$, we obtain:

$$\frac{\partial^2 S_N \circ (F \otimes F)}{\partial p_1^i \partial p_2^j}(p_1, p_2, \bar{x}) = \frac{\partial^2 F}{\partial x^k \partial p_1^i}(p_1, \bar{x}_1) \frac{d \bar{x}_1^k}{dp_2^j}. $$

Setting $p_1 = p_2 = 0$, then $\tilde{p}_1 = \tilde{p}_2 = 0$ and $\bar{x}_1 = \bar{x}_2 = \bar{x}$ and

$$\frac{\partial^2 S_N \circ (F \otimes F)}{\partial p_1^i \partial p_2^j}(0, 0, \bar{x}) = \frac{\partial \phi^i}{\partial x^k}(\bar{x}) \left( \frac{d \bar{x}_1^k}{dp_2^j} \right) \bigg|_{p_1=p_2=0}.$$  

Now, we have that:

$$\frac{d \bar{x}_1^k}{dp_2^j} = \frac{\partial^2 S_N}{\partial p_1^i \partial p_2^j}(\tilde{p}_1, \tilde{p}_2, \bar{x}) \frac{d \tilde{p}_1^i}{dp_2^j} + \frac{\partial^2 S_N}{\partial p_1^i \partial p_2^j}(\tilde{p}_1, \tilde{p}_2, \bar{x}) \frac{d \tilde{p}_2^i}{dp_2^j}. $$

By Lemma 5.5, the first term of the last equation vanishes when $p_1 = p_2 = 0$ and we obtain that:

$$\left( \frac{d \bar{x}_1^k}{dp_2^j} \right) \bigg|_{p_1=p_2=0} = \alpha_{N(\bar{x})}^{ku}( \frac{d \tilde{p}_u^2}{dp_2^j} ) \bigg|_{p_1=p_2=0}. $$

In turns, we get:

$$\frac{d \tilde{p}_u^2}{dp_2^j} = \frac{\partial^2 F}{\partial x^u \partial p_2^j}(p_2, \bar{x}_2) + \frac{\partial^2 F}{\partial x^u \partial x^v}(p_2, \bar{x}_2) \frac{d \bar{x}_v^2}{dp_2^j},$$  

26
which yields:

\[ \left. \frac{d^2 p_j^2}{dp_i^2} \right|_{p_1=p_2=0} = \frac{\partial \phi^j}{\partial x^u}(\bar{x}). \]

Finally, we obtain:

\[ \frac{\partial^2 (S_N \circ (F \otimes F))}{\partial p_i^1 \partial p_j^2}(0,0,\bar{x}) = \frac{\partial \phi^i}{\partial x^k}(\bar{x}) \alpha_k^u_N(\bar{x}) \frac{\partial \phi^j}{\partial x^u}(\bar{x}). \]

As \( F \circ S_M = S_N \circ (F \otimes F) \), we conclude that:

\[ \alpha_i^j_M(\phi(\bar{x})) = \frac{\partial \phi^i}{\partial x^k}(\bar{x}) \alpha_k^u_N(\bar{x}) \frac{\partial \phi^j}{\partial x^u}(\bar{x}), \]

which means that \( \phi \) is a Poisson map form \((N, \alpha_N) \) to \((M, \alpha_M)\).

We may now define the Poisson functor

\[ D : \text{Mon(MiC)} \to \text{Poiss}, \]

by assigning to each monoid \((T^*M, \mu_M)\) the Poisson manifold \((M, \alpha_M)\) as in Proposition 5.12 and by assigning to each monoid morphism

\[ T = (i_\phi, \phi) : (T^*M, \mu_M) \to (T^*N, \mu_N) \]

the map \( \phi : (N, \alpha_N) \to (M, \alpha_M) \). Proposition 5.14 guarantees that \( \phi \) is a Poisson map. The functoriality of \( D \) follows directly from the properties of map composition.

**Definition 5.15.** Let \((s_M, t_M) : G_M \Rightarrow M\) and \((s_N, t_N) : G_N \Rightarrow N\) be two local symplectic groupoids. An isomorphism between local symplectic groupoids is a germ of symplectomorphisms \( \psi : G_M \to G_N \) around \( M \) which sends \( M \) to \( N \) and such that:

\[ s_N \circ \Psi = \Psi \circ s_M \tag{25} \]
\[ t_N \circ \Psi = \Psi \circ t_M. \tag{26} \]

**Proposition 5.16.** Under the same assumption as in Proposition 5.14, suppose further that \( T = (i_\phi, \phi) \) is invertible. Then the transverse lagrangian germ \( i_\phi \) comes from the graph of a germ \( \Psi \) of symplectomorphisms around \( Z_M \), and preserving the bases. Moreover, \( \Psi \) is an isomorphism between the induced local symplectic groupoids.

**Proof.** Let \( \Psi \) be the germ of symplectomorphism induced by \( T \) as in Proposition 2.8. Denote by \( S_M, S_N \) and \( F \) the generating function of \( \mu_M, \mu_N \) and \( T \) in a local chart and denote by \( p_1, x_1, p_2, x_2 \) the local coordinates on \( T^*M \times T^*M \). By definition, we have that:

\[ \Psi\left(p_1, \nabla_p F(p_1, x_2)\right) = \left(\nabla_x F(p_1, x_2), x_2\right). \]

Verifying Equation (25) is then equivalent to verifying that:

\[ s_N\left(\nabla_x F(p_1, x_2), x_2\right) = \Psi\left(0, s_M(p_1, \nabla_p F(p_1, x_2))\right). \]
This is equivalent to see that:
\[
\nabla_{p_2} S_N \left( \nabla_x F(p_1, x_2), 0, x_2 \right) = \phi^{-1} \left( \nabla_{p_2} S_M \left( p_1, 0, \nabla_p F(p_1, x_2) \right) \right).
\] (27)

Now, Lemma 4.9 gives us that:
\[
\nabla_{p_2} (F \circ S_M)(p_1, p_2, \tilde{x}) = \nabla_{p_2} S_M(p_1, p_2, \tilde{x})
\]
where \( \tilde{x} \) is defined by the implicit equations for \( \tilde{p} \) and \( \tilde{x} \):
\[
\begin{align*}
\tilde{p} &= \nabla_x S_M(p_1, p_2, \tilde{x}) \\
\tilde{x} &= \nabla_p F(\tilde{p}, \tilde{x}).
\end{align*}
\]
Setting \( p_2 = 0 \), we get, by Lemma 5.5, that \( \tilde{p} = p_1 \) and \( \tilde{x} = \nabla_p F(p_1, \tilde{x}) \). Thus,
\[
\nabla_{p_2} (F \circ S_M)(p_1, p_2, \bar{x}) = \nabla_{p_2} S_M(p_1, 0, \nabla_p F(p_1, \bar{x})).
\]

On the other hand, Lemma 4.9 tells us that:
\[
\nabla_{p_2} (S_N \circ (F \otimes F))(p_1, p_2, \tilde{x}) = \nabla_{p_2} F(p_2, \tilde{x}_2),
\]
where \( \tilde{x}_2 \) comes from the solution of the implicit system for \( \tilde{p}_1, \tilde{x}_1, \tilde{p}_2 \) and \( \tilde{x}_2 \):
\[
\begin{align*}
\tilde{p}_1 &= \nabla_x F(p_1, \tilde{x}_1) \\
\tilde{x}_1 &= \nabla_{p_1} S_N(\tilde{p}_1, \tilde{p}_2, \tilde{x}) \\
\tilde{p}_2 &= \nabla_x F(p_2, \tilde{x}_2) \\
\tilde{x}_2 &= \nabla_{p_2} S_N(\tilde{p}_1, \tilde{p}_2, \tilde{x}).
\end{align*}
\]
Setting \( p_2 = 0 \), we get that \( \tilde{p}_2 = 0, \tilde{x}_1 = \bar{x}, \tilde{p}_1 = \nabla_x F(p_1, \bar{x}) \) and thus:
\[
\tilde{x}_2 = \nabla_{p_2} S_N(\nabla_x F(p_1, \bar{x}), 0, \bar{x}).
\]
Thus we get that:
\[
\nabla_{p_2} (S_N \circ (F \otimes F))(p_1, 0, \bar{x}) = \phi(\tilde{x}_2) = \phi \left( \nabla_{p_2} S_N(\nabla_x F(p_1, \bar{x}), 0, \bar{x}) \right).
\]

Finally, the fact that \( F \circ S_M = S_N \circ (F \otimes F) \) implies (27). \( \square \)

Let us summarize the content of this section in the following theorem.

**Theorem 5.17.** In \( \text{MiC} \), a monoid \( (T^*M, \mu) \) induces a Poisson structure on the base \( M \) together with a local symplectic groupoid on \( (s,t) : T^*M \rightrightarrows M \) integrating it. Isomorphisms of monoids induce Poisson diffeomorphisms between the induced Poisson structures and local symplectic groupoid isomorphisms between the induced local symplectic groupoids.

### 6 Examples

In this Section, we describe explicitly some examples of monoid structures on cotangent bundles. We provide formulas for their induced Poisson structures and local symplectic groupoids.
6.1 Symplectic manifolds

Let \((\mathbb{R}^{2n}, J)\) be the standard symplectic manifold. We consider its cotangent bundle \(T^*\mathbb{R}^{2n}\) as an object in the cotangent microbundle category. We construct a monoid structure \(\mu = (i_S, \Delta) \in \text{Hom}((T^*\mathbb{R}^{2n})^{\otimes 2}, T^*\mathbb{R}^{2n})\) on it thanks to the symplectic matrix \(J\). The transverse lagrangian germ

\[
i_S : B_\Delta \hookrightarrow T^*\mathbb{R}^{2n} \times T^*\mathbb{R}^{2n} \times T^*\mathbb{R}^{2n}
\]

is given by the following generating function

\[
S(p_1, p_2, x) = (p_1 + p_2)x + \frac{1}{2} p_1^T J^{-1} p_2
\]  

(28)

where \(J^{-1}\) is the inverse of \(J\). A straightforward computation yields:

\[
M = \left( S \circ (S \otimes I) - S \circ (I \otimes S) \right)(p_1, p_2, p_3, x) = \frac{1}{2} (p_1^T J^{-1} p_2 + (p_1 + p_2)^T J^{-1} p_3) - \frac{1}{2} (p_1^T J^{-1} (p_2 + p_3) + p_2^T J^{-1} p_3) = 0,
\]

which means that \((i_S, \Delta)\) is a monoid structure on \(T^*\mathbb{R}^{2n}\). Note that the induced Poisson structure,

\[
\left( \frac{\partial^2 S}{\partial p_i \partial p_j} (0, 0, x) - \frac{\partial^2 S}{\partial p_i \partial p_j} (0, 0, x) \right) = (J^{-1})^{ij},
\]

is the inverse \(J^{-1}\) of the original symplectic form \(J\). The induced source and target \((s, t) : T^*\mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n}\) are given by the formulas:

\[
s(p, x) = x + \frac{1}{2} J p \quad \text{and} \quad t(p, x) = x - \frac{1}{2} J p.
\]

There is a nice geometric interpretation of both the generating function (28) and the associativity equation reminiscent of [12] and [13]. Let us consider \(\mathbb{R}^2\) with its standard symplectic form \(J\) for simplicity. To each point \((p_1, p_2, x) \in B_\Delta\), we may associate a triangle \(T(p_1, p_2, x)\) in \(\mathbb{R}^2\) in the following way. Consider the Hamilton flows on \(\mathbb{R}^2\), \(\Psi^t_1\) and \(\Psi^t_2\), of the linear Hamiltonians \(l_{p_1}(x) = p_1 x\) and \(l_{p_2}(x) = p_2 x\) respectively. The three vertices of the triangle are given by \(x_1 = x\), \(x_2 = \Psi^1_t \mid_{t=1}(x)\) and \(x_3 = \Psi^2_t \mid_{t=1}(x)\). The edge joining \(x_1\) to \(x_2\) is the trajectory of \(x_1\) under \(\Psi^1_t\) and the edge joining \(x_2\) to \(x_3\) is the trajectory of \(x_2\) under \(\Psi^2_t\). One can verify that the Hamilton flow of the Hamiltonian \(l_{p_1 + p_2}\) carries \(x_1\) to \(x_3\) along the third edge of the triangle. An alternative description of \(T(p_1, p_2, x)\) is the triangle with vertex \(x\) and defined by the two vectors \(J^{-1} p_1\) and \(J^{-1} p_2\) as in Figure 4.

The area \(A(p_1, p_2, x)\) of \(T(p_1, p_2, x)\) is given by the formula:

\[
\frac{1}{2} \text{det}(J^{-1} p_1, J^{-1} p_2) = \frac{1}{2} p_1^T J^{-1} p_2.
\]

The generating function \(S\) may then be written as:

\[
S(p_1, p_2, x) = (p_1 + p_2)x + \text{Area} \left( T(p_1, p_2, x) \right).
\]

The associativity equation may be interpreted as an equality between areas as shown in Figure 5.
6.2 Lie algebras

We consider the cotangent bundle $T^*\mathbb{R}^d$ and look for monoid structures

$$i_\Delta : B_\Delta \rightarrow T^*\mathbb{R}^d \times T^*\mathbb{R}^d \times T^*\mathbb{R}^d$$

whose generating function $S_\Delta : B_\Delta \rightarrow \mathbb{R}$ is linear in $x$:

$$S(p_1, p_2, x) = \langle x, A(p_1, p_2) \rangle.$$

Note that $S$ being a germ of functions around the zero section and which vanishes on it implies that

$$A : (\mathbb{R}^d)^* \times (\mathbb{R}^d)^* \rightarrow (\mathbb{R}^d)^*$$

must be a germ of a map around $(0, 0)$ and such that $A(0, 0) = 0$. The equation

$$S \circ (e \otimes I) = I = S \circ (I \otimes e)$$

implies by Lemma 5.5 that

$$A(p, 0) = A(0, p) = p.$$ 

A straightforward computation tells us that the associativity equation,

$$M = (S \circ (S \otimes I) - S \circ (I \otimes S))(p_1, p_2, p_3, x)$$

$$= \langle x, A(p_1, A(p_2, p_3) - A(A(p_1, p_2), p_3) \rangle$$

$$= 0,$$

is equivalent to the associativity of the map $A$. The induced Poisson structure is given by:

$$\alpha^{ij}(x) = \left( \frac{\partial^2 A_k}{\partial p_i \partial p_j^*}(0, 0) - \frac{\partial^2 A_k}{\partial p_j \partial p_i^*}(0, 0) \right) x^k,$$
which is a linear Poisson structure on \( \mathbb{R}^d \). This implies, in particular, that

\[
C^{ij}_k = \left( \frac{\partial^2 A_k}{\partial p^i_1 \partial p^j_2}(0,0) - \frac{\partial^2 A_k}{\partial p^j_1 \partial p^i_2}(0,0) \right)
\]

are the structure constants of a Lie algebra structure on \( \mathbb{R}^d \). We denote this Lie algebra by \( \mathcal{G} \). The source and target are given by the formulas:

\[
s(p,x) = \langle x, \nabla_{p_2} A(p,0) \rangle \quad \text{and} \quad t(p,x) = \langle x, \nabla_{p_1} A(0,p) \rangle.
\]

Conversely, if we start from a Lie algebra \( (\mathcal{G}, [\ , \ ] ) \), consider the Baker-Campbell-Hausdorff map:

\[
BCH : \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O}
\]

defined in a neighborhood \( \mathcal{O} \) of 0 in \( \mathcal{G} \) by

\[
BCH(p_1, p_2) = \log( \exp(p_1) \exp(p_2)),
\]

where \( \exp \) is the usual diffeomorphism one can construct between sufficiently small neighborhoods of 0 in \( \mathcal{G} \) and neighborhoods of the unit element \( e \) in the corresponding Lie group \( G \) and where \( \log \) stands for its inverse. The \( BCH \) map provides a generating function \( S : \mathcal{G} \oplus \mathcal{G} \oplus \mathcal{G}^* \rightarrow \mathbb{R} \) of the above form, i.e.,

\[
S(p_1, p_2, x) = \langle x, BCH(p_1, p_2) \rangle. \tag{29}
\]

This gives a monoid structure on \( T^* \mathcal{G}^* \). The induced Poisson structure on \( \mathcal{G}^* \) is the Kirillov-Kostant Poisson structure associated to the Lie bracket of \( \mathcal{G} \).

### 6.3 Kontsevich’s star-product

Consider an open subset \( U \) of \( \mathbb{R}^d \) endowed with an analytic Poisson structure \( \alpha \). We will describe here a monoid structure on \( T^* \mathbb{R}^d \) which induces the Poisson structure \( \alpha \) and encompasses the two previous examples, i.e., when \( \alpha \) comes from a symplectic structure \( J \) and when \( \alpha \) comes from a Lie algebra. Consider the following formal power series in \( \epsilon \):

\[
S(\alpha)(p_1, p_2, x) = (p_1 + p_2)x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in T_{n,2}} W_{\Gamma} \hat{B}_{\Gamma}(\alpha)(p_1, p_2, x) , \tag{30}
\]

where \( T_{n,2} \) are the Kontsevich trees of type \((n, 2)\) and \( W_{\Gamma} \) their associated Kontsevich weights. The \( \hat{B}_{\Gamma} \) are the symbols of the Kontsevich bidifferential operators \( B_{\Gamma} \), defined by the formula:

\[
B_{\Gamma}(e^{p_1 x}, e^{p_2 x}) = \hat{B}_{\Gamma}(p_1, p_2, x)e^{(p_1 + p_2)x},
\]

where \( p_1, p_2 \in (\mathbb{R}^d)^* \) and \( x \in \mathbb{R}^d \). We refer the reader to [4] and [10] for more details concerning the construction of formula \( \text{30} \). In [7], it has been shown that \( \text{30} \) converges in
a neighborhood of $Z_\Delta$ in $B_\Delta$ for $\epsilon \in (0, 1)$ for analytic Poisson structures and thus produces a transverse lagrangian germ

$$i_{S(\alpha)} : B_\Delta \hookrightarrow T^*U \times T^*U \times T^*U.$$ 

In [4], it has been shown, although not in the same language, that $S(\alpha)$ satisfies both:

$$S \circ (S \otimes I) = S \circ (I \otimes S)$$
$$S \circ (e \otimes I) = S \circ (I \otimes e) = I.$$ 

Thus, the associated germ $i_{S(\alpha)}$ produces a monoid structure on $T^*U$. The induced Poisson structure is the original one times $\epsilon$, i.e., $\epsilon \alpha$. When $\alpha$ is the inverse of a symplectic structure $J$, one verifies that we get back (29). When $\alpha$ comes from a Lie algebra, one gets back (28). The generating function (30), may be considered as the semi-classical part of Kontsevich’s star-product as constructed in [10] as it involved only the tree-level part of the star-product. Namely, Kontsevich star-product may be put into the following form (see [4]). For $f, g \in C^\infty(\mathbb{R}^d)$,

$$f \ast g(x) = \exp \left( \frac{1}{\epsilon} \sum_{l=0}^{\infty} \epsilon^l K_l(\epsilon \frac{\partial}{\partial y}, \epsilon \frac{\partial}{\partial z}, x) f(y)g(z) \bigg|_{y=z=x} \right),$$

where $K_l = \sum_{T \in G_l} W_T \hat{B}_T$ is a sum over the Kontsevich graphs with two ground vertices and with $l$ loops. $K_0$ is exactly the generating function in (30).

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