DYNAMICS OF ANNULUS MAPS II:
PERIODIC POINTS FOR COVERINGS

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Abstract. Let $f$ be a covering map of the open annulus $A = S^1 \times (0,1)$ of degree $d$, $|d| > 1$. Assume that $f$ preserves an essential (i.e. not contained in a disk of $A$) compact subset $K$. We show that $f$ has at least the same number of periodic points in each period as the map $z^d$ in $S^1$.

1. Introduction.

Existence of periodic orbits for orientation preserving annulus homeomorphisms has been extensively studied. One of the motivations is a celebrated theorem of dynamical systems, the so-called “last geometric theorem of Poincaré”. Roughly this result says that an area-preserving homeomorphism of the closed annulus which rotates one boundary component clockwise and the other counterclockwise possesses at least two fixed points. This result was conjectured and proved in special cases by Poincaré [P], and was finally proved by Birkhoff [B]. This problem has been considered by many authors and was actually the trigger for a great deal of research (see the paper [DR] for a historical review on the subject). Since Franks’ paper [Fr2], where he generalized and proved the statement for homeomorphisms of the open annulus, interest on the problem of existence of periodic orbits for non compact surface homeomorphisms arose (see, for example, [Fr3], [Fr4], [FH], [LC2]).

More recently, in [PS] the problem of existence and growth rate of periodic orbits for degree two surface endomorphism was considered. In this paper, they deal with a particular case of Problem 3 posed in [S]: let $S$ be the 2-sphere, and $f : S \to S$ a continuous map of degree 2; is the growth rate inequality

$$\limsup_{n \to \infty} \frac{1}{n} \ln(\#\{\text{Fix}(f^n)\}) \geq \ln(2)$$

true? The answer is no, as the map $(r, \theta) \to (2r, 2\theta)$ has only the poles as periodic points. However, in [PS] it is shown that the growth inequality holds in a particular case: if $f$ is $C^1$ and preserves the latitude foliation, then for each $n$, $f^n$ has at least $2^n$ fixed points.

In this paper, we study the existence of periodic orbits for covering maps of the open annulus $f : A \to A$ of degree $d$, $|d| > 1$. Note that the growth inequality holds trivially for the closed annulus $\overline{A}$ as each connected component of the boundary of the annulus must be invariant by $f$ or $f^2$, and we are assuming $|d| > 1$. On the other hand, the covering map $(r, \theta) \to (2r, 2\theta)$ provides a periodic point free example in the open annulus $\mathbb{C} \setminus \{0\}$. Our result relates both to the theory of annulus homeomorphisms, and to the work in [PS].

Let us introduce some preliminary definitions. If $f : A \to A$ is a continuous function, then the homomorphism $f_*$ induced by $f$ on the first homology group...
$H_1(A, Z) \simeq Z$, is $n \mapsto d n$, for some integer $d$. This number $d$ is called the degree of $f$.

We say that an open subset $U \subset A$ is essential, if $i_*(H_1(U, Z)) = Z$, where $i_* : H_1(U, Z) \to H_1(A, Z)$ is the induced map in homology by the inclusion $i : U \to A$. We say that a subset $X \subset A$ is essential if any neighbourhood of $X$ in $A$ is essential. We say that a subset is inessential if it is not essential, or equivalently, if it is contained in a disk of $A$. If $x$ is a periodic point for $f$, its period is the number $\min \{n \geq 1 : f^n(x) = x\}$. We write $\text{Per}_n(f)$ for the set of periodic points of period $n$ of a given map $f$, and $\text{Fix}(f) = \text{Per}_1(f)$.

Let $A^*$ be the compactification of the annulus $A$ with two points so that it is homeomorphic to the two-sphere. Each connected component of $A^* \setminus A$ is called an end of $A$. Note that if $f$ is a proper mapping, then $f$ extends continuously to $A^*$, and either fixes both ends or interchanges them.

We need one last definition: what it means for $f$ to be complete; we postpone this until Section 2 because the definition involves some Nielsen theory. We recall that $x, y \in \text{Fix}(f)$ are Nielsen equivalent if there exist an arc $\gamma$ joining $x$ and $y$ such that $\gamma$ is homotopic to $f(\gamma)$ with fixed endpoints. The notion of completeness implies for instance that for each $n$, $f^n$ has exactly $|d^n - 1|$ Nielsen classes of fixed points. But as will be explained later, it is a finer concept.

We prove the following:

**Theorem 1.** Let $f : A \to A$ be a covering map of degree $d$, $|d| > 1$. Suppose there exists an essential continuum $K \subset A$ such that $f(K) \subset K$. Then $f$ is complete.

Note that this result is strictly a consequence of degree; an irrational rotation in the open annulus has no periodic points, and has every essential circle as a compact invariant subset. Results in the same line of work have been obtained by Boronski in [Bo1] and [Bo2].

The periodic points given by Theorem 1 do not necessarily belong to $K$ (see Section 5 example 5.1). The problem of whether or not the fixed points of a given map with a compact invariant set $K$ belong to $K$ is known in the literature as Cartwright-Littlewood theory. In a seminal paper, M. Cartwright and J. Littlewood [CL] proved that if $K$ is a nonseparating continuum of the plane, invariant under an orientation preserving homeomorphism $h$, then $h$ has a fixed point in $K$. Existence of a fixed point under such hypothesis was already known on account of Brouwer’s plane translation theorem ([Brou]), the novelty being that the fixed point must belong to the set $K$. An easy proof or Cartwright-Littlewood’s theorem can be found in the extraordinary single-page paper of M. Brown [Brow].

In [Be], H. Bell proved Cartwright-Littlewood’s theorem for orientation reversing plane homeomorphisms, and his results were later generalized by K. Kuperberg in [K] for arbitrary plane continua (not necessarily nonseparating). In 5.1 Section 5 we construct a degree two covering map $f$ of the annulus with a totally invariant essential continuum $K$ ($f^{-1}(K) = K$), such that $\text{Fix}(f) \cap K = \emptyset$. However, $K$ is not filled, that is, its complement has bounded connected components. If $K \subset A$ is any compact set, the set $\text{Fill}(K)$ is defined as the union of $K$ with the bounded connected components of its complement. The definition of Nielsen classes of periodic points is contained in Section 2. We prove the following:
Theorem 2. Let \( f : A \rightarrow A \) be a covering map of degree \( d \), \(|d| > 1\). Suppose there exists an essential continuum \( K \subset A \) such that \( f(K) \subset K \). Then, there exists a representative \( x \in \text{Fill}(K) \) for each Nielsen class of periodic points of \( f \).

Notations. Throughout this article, \( A = S^1 \times (0, 1) \) is the open annulus, \( \tilde{A} = \mathbb{R} \times (0, 1) \) its universal covering space, and \( \pi : \tilde{A} \rightarrow A \) the universal covering projection. We will denote by \( F \) any lift of \( f : A \rightarrow A \) to the universal covering, that is, \( F \) is a map satisfying \( f \pi = \pi F \). Note that \( F(x + 1, y) = F(x, y) + (d, 0) \) if \( f \) has degree \( d \). To lighten notation, if \( z \in \tilde{A} \), we write \( z + k \) for the point \( z + (k, 0), k \in \mathbb{Z} \). The map \( m_d : S^1 \rightarrow S^1 \) is defined as \( m_d(z) = z^d \).

2. Nielsen theory background.

In this section, we gather the necessary information on Nielsen Theory. For what follows, \( f : A \rightarrow A \) is any continuous map. If \( p, q \in \text{Fix}(f) \), then \( p \) and \( q \) are said to be Nielsen equivalent if there exists a curve \( \gamma \) from \( p \) to \( q \) such that \( f(\gamma) \) and \( \gamma \) are homotopic with fixed endpoints. If \( p \) and \( q \) are periodic points of \( f \), then \( p \) and \( q \) are Nielsen equivalent if they are equivalent as fixed points of some \( f^k \), \( k \geq 1 \). The definition of Nielsen equivalence does not depend on the choice of \( k \).

Lemma 1. Let \( p, q \in \text{Fix}(f) \) and let \( \gamma \) be a curve from \( p \) to \( q \). If \( \gamma \sim f^k \gamma \) for some \( k > 1 \), then \( \gamma \sim f \gamma \).

Proof. Let \( \tilde{p} \) be a lift of \( p \) and let \( F \) be the lift of \( f \) that fixes \( \tilde{p} \). Let \( \tilde{\gamma} \) be the lift of \( \gamma \) starting at \( \tilde{p} \), and let \( \tilde{q} \) be the endpoint of \( \tilde{\gamma} \). Assume that \( \gamma \) is not homotopic to \( f(\gamma) \). This implies that \( F(\tilde{q}) \neq \tilde{q} \), and so there exists \( l \in \mathbb{Z}, l \neq 0 \) such that \( F(\tilde{q}) = \tilde{q} + l \). So, \( F^k(\tilde{q}) = \tilde{q} + \sum_{i=0}^{k-1} d^i \neq \tilde{q} \) and so \( f^k \gamma \) is not homotopic to \( \gamma \).

Lemma 2. Let \( p \) and \( q \) be fixed points of \( f \). The following conditions are equivalent:

1. \( p \) and \( q \) are Nielsen equivalent.
2. If \( F \) is any lift of \( f \), and \( \tilde{p} \) is a lift of \( p \), there exist \( \tilde{q} \) a lift of \( q \) such that \( F(\tilde{p}) = F(\tilde{q}) \).

Proof. (1) \( \Rightarrow \) (2): Let \( F \) be a lift of \( f \), and \( \tilde{p} \) any lift of \( p \). Then, as \( p \in \text{Fix}(f) \), there exists \( l \in \mathbb{Z} \) such that \( F(\tilde{p}) = \tilde{p} + l \). Let \( \tilde{q} \) be the endpoint of \( \tilde{\gamma} \), the lift of \( \gamma \) starting at \( \tilde{p} \), where \( \gamma \) is the arc given by the Nielsen equivalence. As \( \gamma \sim f(\gamma) \), the lift of \( f(\gamma) \) starting at \( \tilde{p} + l \) must end at \( \tilde{q} + l \). On the other hand, this lift must coincide with \( F(\tilde{\gamma}) \), which gives \( F(\tilde{q}) = \tilde{q} + l \).

(2) \( \Rightarrow \) (1): Let \( \tilde{p} \) be a lift of \( p \), and let \( F \) be the lift of \( f \) such that \( F(\tilde{p}) = \tilde{p} \). There exist \( \tilde{q} \) a lift of \( q \) such that \( F(\tilde{q}) = \tilde{q} \). Take any arc \( \tilde{\gamma} \) joining \( \tilde{p} \) and \( \tilde{q} \). Then, \( F(\tilde{\gamma}) \) is obviously homotopic to \( \tilde{\gamma} \). So, \( \gamma = \pi(\tilde{\gamma}) \) realizes the Nielsen equivalence between \( p \) and \( q \).

Note that the number of Nielsen classes of fixed points for the map \( m_d \) coincides with the number of fixed points of \( m_d \) which is \(|d - 1|\). We will give a simple proof of the following fact:

Theorem 3. If \( f \) is a covering map of degree \( d \), \(|d| > 1 \), of the annulus, then the number of equivalent classes of fixed points of \( f \) is less than or equal to \(|d - 1|\).
Lemma 4. Suppose that for every lift $F : \tilde{A} \to \tilde{A}$ of $f$, $\text{Fix}(F) \neq \emptyset$. Then, $f$ has $|d - 1|$ different Nielsen classes of fixed points.

Proof. Fix a lift $F_0 = F$ of $f$, and for $k \in \mathbb{Z}$ define $F_k(x) = F(x) + k$. Note that every lift of $f$ belongs to the family $(F_k)_{k \in \mathbb{Z}}$. For every $k \in \mathbb{Z}$, let $x_k \in \tilde{A}$ be such that $F_k(x_k) = x_k$. We want to show that there are $|d - 1|$ different Nielsen classes of fixed points.

Suppose there exists $i \neq 0$ such that $\pi(x_i)$ is Nielsen equivalent to $\pi(x_0)$. As $F(x_0) = x_0$, then by Lemma 2 there exists $l \in \mathbb{Z}$ such that $F(x_l + l) = x_l + l$. As $F_i(x_i) = x_i$ and $F_i(x_i) = F(x_i) + i$ then $F(x_i) = x_i - i$. So,

$$x_l + l = F(x_l + l) = F(x_l) + ld = x_l - i + ld$$

Then $i = l(d-1)$. As $i \neq 0$, then $l \neq 0$ and the points $\pi(x_0), \pi(x_1), \ldots, \pi(x_{|d-1|-1})$ are all in different Nielsen classes.
In most cases we will prove completeness by means of the following corollary:

**Corollary 1.** If for every $k$ every lift of $f^k$ has a fixed point, then $f$ is complete.

**Remarks.**

1. Note that the fact that a lift of $f$ has fixed points does not imply that every lift does (see Example 5.3).
2. The definition of completeness can also be applied to circle maps. If $f$ is a degree $d$ map of the circle, then no matter if it is a covering or not, the number of Nielsen classes of fixed points of $f$ is $|d - 1|$. In general, $N_k(f) = N_k(m_d)$ for every $k$. It follows that every circle map is complete.
3. Theorem 3 implies that $N_1(f) \leq |d - 1|$, and obviously $N_1(f^k) \leq |d^k - 1|$. However, it is not true in general that $N_j(f)$ must be less than or equal to $N_j(m_d)$ as the example in Figure 1 shows. It is a map of degree $-2$ that has two fixed points $r_1$ and $r_2$, the rays $S_1$ and $S_2$ verify $f(S_1) = S_2$, $f(S_2) = S_1$ and $\{p, q\}$ is a two periodic cycle. The open invariant region bounded by $S_1$ and $S_2$ contains no fixed point. Note that it has two fixed points (against three of $m_{-2}$) and has one two periodic cycle formed by $p$ and $q$ (against zero of $m_{-2}$).

It holds that $f^2$ has exactly four fixed points, but it has just three Nielsen classes, one of them contains the two periodic cycle. According to our definition, this map is not complete and consequently it cannot leave invariant an essential compact set.

### 3. Proof of Completeness

In this section, we prove Theorem 1. To find periodic orbits we proceed in the standard fashion: we lift to the universal covering space $\mathbb{R} \times (0, 1) \sim \mathbb{R}^2$ and try to use some fixed-point theorems for self-maps of the plane. If $f$ happens to be an orientation preserving covering map, then the lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving plane homeomorphism, and existence of fixed points is guaranteed by any kind of recurrence:

**Theorem 4.** [Brou] If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a fixed point free orientation preserving homeomorphism, then every point is wandering.

For a modern exposition of this theorem in its maximum expression, see [LC]. Although this technique is quite useful in the case that the map $f$ is isotopic to the identity, recurrence in the lift for maps of degree $d$, $|d| > 1$ is not so easy
to get. Indeed, the lifted map $F$ satisfies $F(x + 1, y) = F(x, y) + d$ and so every point wants to escape to infinity exponentially fast. Of course we may impose some strong hypothesis implying immediately recurrence for the lift:

**Lemma 5.** If $f : A \to A$ is an orientation preserving covering map of degree $d \neq 0$ preserving a contractible continuum $K \subset A$, then $\text{Fix}(f) \neq \emptyset$.

A continuum is **contractible** if it is contained in a disk of $A$. The proof is immediate from Brouwer’s Theorem [1] as the hypothesis implies that there exists a lift of $f$ that preserves a compact subset of the plane (namely, a connected component of the preimage of $K$ by the covering projection).

However, if $K$ is essential, no connected component of its lift to the universal covering space is compact. The proof of Theorem 1 in the orientation preserving case is based on a simple (though key) observation that was already made in [IPRX]. If $f : A \to A$ is a continuous map of degree $d$, $|d| > 1$ and $K \subset A$ is a compact set such that $f(K) \subset K$, then $f|_K$ is semiconjugate to the restriction of $m_d$ to an invariant subset. Existence and properties of the semiconjugacy $h : K \to S^1$ are contained in Lemma [6]. Using Brouwer’s Theorem, existence of fixed points is proved in Lemma [7] if $h^{-1}(1) \neq \emptyset$, as this guarantees existence of a compact invariant set for the lift. We prove completeness of $f$ using standard Nielsen theory if $K$ is essential (note Lemma [8] only gives a fixed point, not completeness); this is done in Lemmas [3] and [8].

If $f$ reverses orientation, we use Kuperberg’s theorem in [K] to find fixed points for orientation reversing plane homeomorphisms.

**Theorem 5.** [K] Let $f$ be an orientation reversing homeomorphism of the plane, and $X$ a continuum of the plane invariant under $f$. Then, $f$ has at least one fixed point in $X$.

The following lemma, which is essentially the Shadowing Lemma for expanding maps is key for the purposes of this paper. A proof can be found in [IPRX] lemmas 1 and 2.

**Lemma 6.** Let $f : A \to A$ be a continuous map of degree $d$, $|d| > 1$, and let $F : A \to A$ be a lift of $f$. Let $K$ be a compact $f$-invariant ($f(K) \subset K$) subset of the annulus, and $K = \pi^{-1}(K)$. Then there exists a continuous map $H_F : \hat{K} \to \mathbb{R}$ such that:

1. $H_F(x + 1, y) = H_F(x, y) + 1$,
2. $H_F F = dH_F$,
3. $|H_F(x, y) - x|$ is bounded on $\hat{K}$,
4. $H_F(x) = \lim_{n \to \infty} \frac{F^n(x)}{d^n}$, where $(\cdot)_1$ denotes projection over the first coordinate.

The previous lemma gives us:

**Corollary 2.** Let $f : A \to A$ be a continuous map of degree $d$, $|d| > 1$, and $K \subset A$ be compact and forward invariant. Then there exists a semiconjugacy from the restriction of $f$ to $K$ to the restriction of $m_d$ to an invariant subset.

**Proof.** Just note that for any lift $F$ of $f$, the quotient function of $H_F$ is well defined because of item [1] in the previous lemma.
Remark 1. Note that we have not yet assumed that $f$ is a covering and thus Lemma 2 and Corollary 3 are valid for continuous maps of degree $d$, $|d| > 1$.

From now on, we assume that $f$ is a covering map and that $K \subset A$ is a compact subset such that $f(K) \subset K$. If $F: \tilde{A} \to \tilde{A}$ is any lift of $f$, $H_F$ is the map given by Lemma 5. Note that $H_F \neq H_{F'}$, if $F$ and $F'$ are different lifts of $f$. If there is no place to confusion, we will write $H$ instead of $H_F$.

The proof of Theorem 4 will be divided in two cases.

3.1. The orientation preserving case.

Lemma 7. If $f$ preserves orientation, and there exists $F: \tilde{A} \to \tilde{A}$ a lift of $f$ such that $H^{-1}(0) \neq \emptyset$, then $\text{Fix}(F) \neq \emptyset$ (and so $\text{Fix}(f) \neq \emptyset$).

Proof. Note that as $f: A \to A$ is a covering, $F: \tilde{A} \to \tilde{A}$ is a homeomorphism.

Moreover, $\tilde{A}$ is homeomorphic to $\mathbb{R}^2$ and $F$ preserves orientation because $f$ does. So, by Brouwer’s Theorem 4, it is enough to prove that $H^{-1}(0)$ is a compact $F$-invariant set. Invariance follows from the equality $HF = dH$ (Lemma 6, item (2)).

To see it is compact, recall from Lemma 6 that the function $(x,y) \mapsto H(x,y) - x$ defined on $K$ is bounded. So, we may take $C \in \mathbb{R}$ such that $|H(x,y) - x| < C$ on $K$.

Then, $(x,y) \in H^{-1}(0)$ implies $x \in [-C,C]$, proving that $H^{-1}(0)$ is compact. □

Remark 2. The fixed point found in the previous lemma does not necessarily belong to $K$ (see Example 5.7 in Section 5).

The following remark resembles rotation theory for surface homeomorphisms.

Remark 3. The previous lemma can be restated as follows: if there exists $x \in K$, and a lift $F$ of $f$ such that $\lim_{n \to \infty} \frac{F^n(x)}{d^n} = 0$ for a lift $\tilde{x}$ of $x$, then $\text{Fix}(F) \neq \emptyset$.

Note, however, that the mere existence of a point $\tilde{x} \in \tilde{A}$ such that $\lim_{n \to \infty} \frac{F^n(\tilde{x})}{d^n} = 0$ for some lift $F$ of $f$ does not imply the existence of a fixed point; the set $K$ is key. An example is given in Example 5.2 in Section 5.

The following lemma is Lemma 3 in [IPRX].

Lemma 8. If $K$ is an essential subset of $A$, then for any lift $F$ of $f$, the function $H_F: K \to \mathbb{R}$ is surjective.

Lemma 9. Let $g: A \to A$ be an orientation preserving covering map of degree $d$, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset K$. Then, every lift of $g$ has a fixed point.

Proof. Lemma 8 states that $H_G$ is surjective for any lift $G$ of $g$. In particular, for any lift $G$ of $g$, $H_G^{-1}(0) \neq \emptyset$. Then, Lemma 7 implies that $\text{Fix}(G) \neq \emptyset$. □

3.2. The orientation reversing case.

Lemma 10. Let $g: A \to A$ be an orientation reversing covering map of degree $d$, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset K$. Then, every lift of $g$ to the universal covering $\tilde{A}$ has a fixed point.

Proof. There are two options:

Case 1: $g$ has negative degree and fixes both ends of $A$.

Let $G$ be a lift of $g$ and note that the map $G$ can be modified without changing...
its restriction to \( \pi^{-1}(K) \), in order to obtain a map that extends to the closure of \( \mathbb{R} \times (0,1) \). This extension (also denoted \( G \)) induces a homeomorphism defined in the compactification with two points \( \{-\infty, \infty\} \) of \( \mathbb{R} \times [0,1] \). Note that \( G \) carries \(-\infty\) to \( \infty \) and vice versa. The map \( G \) induces a homeomorphism \( G \) defined in the compactification with two points \( \{-\infty, \infty\} \) of \( \mathbb{R} \times [0,1] \). Note that this fixed point cannot be \(-\infty\) nor \( \infty \), as this constitutes a two periodic orbit. So, every lift of \( g \) has a fixed point. Note, moreover, that in this case, the fixed point belongs to the set \( \pi^{-1}(K) \).

**Case 2:** \( g \) has positive degree and interchanges both ends of \( A \). As the ends of the annulus are interchanged, the map \( g \) can be modified in a neighbourhood of the ends of \( A \) without changing the fixed point set and in order that it can be extended to a covering of the closed annulus. So, by Corollary 2 with \( K = \overline{\mathbb{A}} \), we may assume that \( g \) is semiconjugate to \( m_d \). This in its turn implies that \( g \) has an invariant connector, meaning an inessential continuum of the closed annulus connecting both boundary components (see [IPRX], Corollary 9). Moreover, \( g \) has an invariant connector contained in each of the preimages under the semiconjugacy of a fixed point of \( m_d \). Then any lift \( G \) of \( g \) must fix one of the lifts of these invariant connectors. Then extend \( G \) as a homeomorphism of the whole plane and apply Kuperberg’s Theorem to conclude that \( G \) has a fixed point in that invariant connector.

We are now ready to prove Theorem 1.

**Proof.** By Corollary 1, it is enough to prove that for all \( n \), every lift of \( f^n \) has a fixed point. Note that for all \( n \), \( f^n(K) \subset K \). If \( f \) is orientation preserving, so is \( f^n \) for all \( n \), so applying Lemma 9 we obtain the result. If \( f \) is orientation reversing, we obtain the result by applying Lemma 9 to even powers of \( f \), and Lemma 10 to odd powers of \( f \).

**4. Location of periodic orbits**

In this section, we prove Theorem 2. That is, that the periodic points given by Theorem 1 can be found in \( \text{Fill}(K) \). We assume throughout this section that \( K \) is an essential continuum such that \( f(K) \subset K \). Recall that \( f \) is complete because of Theorem 1.

We will denote \( \pi^{-1}(\text{Fill}K) = \hat{K} \). Note that \( \hat{K} \) is not necessarily connected (see Figure 2). However, as the set is filled, there exists a unique connected component of \( \hat{K}, \hat{K}_1 \), that separates both boundary components of \( \hat{A} \). Indeed, it is easy to see that for all \( n \geq 1 \) there exists a connected component of \( \hat{K} \) that intersects both \( x = -n \) and \( x = n \). These components must accumulate on a connected component of \( \hat{K}, \hat{K}_1 \), that separates boundary components of \( \hat{A} \). As \( \hat{K} \) is filled, this component is unique, and then \( F(K_1) \subset \hat{K}_1 \). Anyhow, \( \hat{K}_1 \) does not necessarily verify \( F^{-1}(\hat{K}_1) = \hat{K}_1 \). Taking \( \hat{K}_2 = \cap_{n \geq 0} F^n(\hat{K}_1) \), we obtain a connected, filled set that separates both boundary components of \( \hat{A} \) and such that \( F^{-1}(\hat{K}_2) = \hat{K}_2 \).

Because of the previous remark, we may assume that \( \pi^{-1}(\text{Fill}K) = \hat{K} \) is an invariant, connected, filled set that separates both boundary components of \( \hat{A} \). To
prove Theorem 2 it is enough to prove that for all $n \in \mathbb{N}$, every lift of $f^n$ has a fixed point in $\hat{K}$ (see Corollary 1).

**Lemma 11.** Let $g : A \to A$ be a covering map of degree $d$, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset \hat{K}$. Then, every lift of $g$ to the universal covering $\tilde{A}$ has a fixed point in $\hat{K}$.

**Proof.** The proof will be subdivided in three cases.

**Case 1:** $g$ is orientation preserving.

Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $g$ and let $H_G : \hat{K} \to \mathbb{R}$ be the map given by Lemma 6. Let $K_0 = H_G^{-1}(0)$ and recall from Lemmas 6 and 8 that $K_0 \neq \emptyset$, $K_0 \subset \hat{K}$, $G(K_0) \subset K_0$ and that $K_0$ is a compact subset of $\mathbb{R}^2$. Suppose for a contradiction that $\text{Fix}(G) \cap \hat{K} = \emptyset$. Let $U$ be the connected component of $\mathbb{R}^2 \setminus \text{Fix}(G)$ containing $\hat{K}$, and let $(\hat{U}, p)$ be the universal covering of $U$. Note that $U$ is $G$-invariant, and we claim that there exist a lift $\hat{G} : \hat{U} \to \hat{U}$ of $G|_U$ having a compact invariant set. To see this, take an open, connected and simply connected neighborhood $V \subset U$ of $\hat{K}$ (whose existence is guaranteed as the set is filled), and note that each connected component of $p^{-1}(V)$ is mapped homeomorphically onto $V$ by $p$. Moreover, as we are assuming that $\hat{K}$ is connected, there is only one connected component of $p^{-1}(\hat{K})$ in each connected component of $p^{-1}(V)$. Fix a connected component $K'$ of $p^{-1}(\hat{K})$ and take the lift $\hat{G}$ of $G$ such that $\hat{G}(K') = K'$. Note that $p^{-1}(H_G^{-1}(0)) \cap K'$ is $\hat{G}$-invariant and compact. So, $\Omega(\hat{G}) \neq \emptyset$ and as $\hat{G}$ is orientation preserving, Brouwer’s Theorem gives us $\text{Fix}(\hat{G}) \neq \emptyset$. This is a contradiction because by definition, $U \cap \text{Fix}(G) = \emptyset$.

**Case 2:** $g$ is orientation reversing, and $d < -1$.

Note that this has already been proved in Case 1 of the proof of Theorem 1.

**Case 3:** $g$ is orientation reversing, and $d > 1$.

As we explained at the beginning of this section, we may assume that $\hat{K}$ verifies $G^{-1}(\hat{K}) = \hat{K}$, $\hat{K}$ is connected, and its complement has exactly two connected components, each one containing one end of $\hat{A}$. Let $U_1$ and $U_2$ be the connected components of $\hat{A}\setminus\hat{K}$. Note that our hypothesis implies that $G(U_1) = U_2$ and $G(U_2) = U_1$. So, $\text{Fix}(G) \subset \hat{K}$. It is enough then to prove that $\text{Fix}(G) \neq \emptyset$. This has already been proved in Case 2 of the proof of Theorem 1.

□
5. Examples

In this section we exhibit a series of examples illustrating all the ideas in this article. Examples 5.1, 5.4 and 5.6 are particularly interesting, regardless of their connection to the theorems presented in this paper.

5.1. Location of periodic orbits. Our first example shows that the periodic points given by Theorem 1 do not necessarily belong to $K$.

We will show that there exists a degree two covering map $f$ of the annulus having an essential continuum $K$, totally invariant, which does not contain fixed points of $f$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3.}
\end{figure}

We construct an isotopy from $f_0 = p_2$, $p_2(z) = z^2$, to $f_1 = f$ in the annulus $A = \mathbb{C} \setminus \{0\}$. For every $t$, $f_t(z) = f_0(z)$ for every $z$ outside a neighborhood $V$ of the fixed point 1. Every $f_t$ will be a homeomorphism from $V$ to $f_0(V)$. For points in $V$, the restriction of $f_t$ to $V$ will have a unique fixed point at 1. Around this point $f_t$ performs a Hopf bifurcation (see Figure 3). That is, for $t$ close to 0, the eigenvalues at the fixed point 1 of $f_t$ have nonzero imaginary part; the modulus is decreasing, and for $t$ equal to 1/2 the Hopf bifurcation takes place: the modulus of the eigenvalues is equal to 1, while the imaginary part is different from 0. Then let $f_t$ for $t > 1/2$ be a generic family through the Hopf bifurcation. The following facts hold for $f_1$: 1 is an attracting fixed point, there is a repeller simple closed curve $C$ where $f$ is conjugate to a rotation with nonzero rotation number, and every point $z \in V$ has a preorbit in $V$ which converges to $C$.

Now let $K$ be the boundary of the basin of $\infty$. Then $K$ is a totally invariant essential continuum. It is clear that $K$ does not contain a fixed point of $f = f_1$.

5.2. A fixed point free example having a point with zero rotation number.

It may happen that \(\lim_{n \to \infty} \frac{(F^n(x))_1}{d^n} = 0\) for some lift $F$ of $f$ and $x \in \tilde{A}$, but
Fix$(F) = \text{Fix}(f) = \emptyset$. Just consider a degree 2 map preserving a ray of the annulus in which the dynamics is north-south, and lift it preserving a lift of that ray.

5.3. Changing the lift. This example shows that the map $f$ may have a lift with fixed points and another lift which is fixed-point free.

Let $f : [0, 2\pi] \times (0, 1) \to [0, 2\pi] \times (0, 1)$ con $f(\theta, r) = (3\theta, \phi(r, \theta))$, where $\phi$ fixes the rays $\theta = 0$ and $\theta = \pi$. On the ray $\theta = 0$, the dynamics of $\phi$ is as in Figure 5, and on the ray $\theta = \pi$, $\phi$ is as in Figure 4 (b). So, $(0, 1/2)$ is fixed by $f$ and you can lift $f$ by fixing any of the preimages of $(0, 1/2)$. However, if you take a lift $F$ of $f$ fixing any preimage of $\theta = \pi$, then $\text{Fix}(F) = \emptyset$.

![Figure 4](image.png)

5.4. Recurrence and periodic orbits. As in the fixed-point free degree 2 covering example $(r, \theta) \to (2r, 2\theta)$ every point is wandering, one may ask if the existence of a non-wandering point is enough to assure the existence of a fixed point. The next example shows that this is not the case.

We will construct a degree 2 covering $f : (0, +\infty) \times S^1 \to (0, +\infty) \times S^1$ such that there is a compact set $K$ satisfying $f(K) = K$ and $\text{Per}(f) = \emptyset$. Of course, $K$ must be inessential and not connected (see Theorem 1 and Proposition 2 item 2).

In fact, in this example $K$ is a Cantor set. We recall that in [IPRX] we showed that for a degree $d > 1$ covering $g$ of the circle $\text{Per}(g) = \Omega(g)$. This example also shows that this is no longer the case for annulus coverings.

We start with a degree 2 circle covering having a wandering interval. Let $g_1 : S^1 \to S^1$ be a Denjoy homeomorphism with a wandering interval $I$. Take an open interval $I_0 \subset I$ and an increasing function $h : I \to S^1$ such that $h(I_0) = S^1$ and $h|_{I \setminus I_0} \equiv g_1$ (see Figure 5(a)). Let $g : S^1 \to S^1$ be the map

$$g(x) = \begin{cases} g_1(x) & \text{if } x \notin I \\ h(x) & \text{if } x \in I \end{cases}$$

So, $g$ is a degree 2 covering of the circle and $g_1(I)$ is a wandering interval for $g$.

Besides, if $x_0 \in g_1(I)$ then $K_1 = \omega_g(x_0)$ is a Cantor set and $K_1 \cap \text{Per}(g) = \emptyset$.

Our example $f : (0, +\infty) \times S^1 \to (0, +\infty) \times S^1$ has the form $f(\theta, r) = (\phi(r, \theta), g(\theta))$, where $\phi$ is to be constructed. Let $\psi : S^1 \to \mathbb{R}$, $\psi(\theta) = \text{dist}(\theta, K_1)$ and let $\varphi : (0, +\infty) \to (0, +\infty)$ be as in Figure 5(b). Define $\phi(r, \theta) = \varphi(r) + r \psi(\theta)$.

Note that $f$ has the following properties:

(1) For fixed $\theta$, let $\phi_0(r) = \varphi(r, \theta)$. Then, $\phi_0$ has fixed points if and only if $\theta \in K_1$, and for $\theta \in K_1$, $\phi_0$ has a unique fixed point at $r = 1$. 

(2) \( K = \{1\} \times K_1 \) is compact and \( f(K) = K \).

Furthermore, \( \text{Per}(f) = \emptyset \). Indeed, if \((r_0, \theta_0)\) is \( f \)-periodic, then \( \theta_0 \) must be \( g \)-periodic. So, \( \theta_0 \notin K_1 \). But this is impossible, as dynamics in the lines \( \{(r, \theta) : r > 0, \theta \notin K_1\} \) is wandering.

Note also that this example can be made \( C^1 \), if we use the square of the distance in the definition of \( \psi \) and enough regularity for the rest of the functions.

5.5. **Non-essential totally invariant subset.** We give an example of a degree 2 covering of the annulus with a Cantor set \( K \subset A \) such that \( f^{-1}(K) = K \).

This implies that there is a continuous surjective function \( h : K \to S^1 \) such that \( hf|_K = m_d h \) (see Lemma 1 item 4).

Let \( g : S^1 \to S^1 \) be as in Figure 6 (a). Note that \( \Omega(g) = \{0\} \cup K_1 \) where 0 is an attracting fixed point and \( K_1 \) is an expanding Cantor set with \( g^{-1}(K_1) = K_1 \). Let \( f : (0, 1) \times S^1 \to (0, 1) \times S^1 \), \( f(r, \theta) = (\varphi(r), g(\theta)) \), where \( \varphi \) is as in Figure 6 (b). Then, \( K = 1/2 \times K_1 \) is a Cantor set and \( f^{-1}(K) = K \).

5.6. **Fail of Brouwer’s Theory.** We construct a self-map of the plane \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) of degree 1 such that \( \Omega(F) \neq \emptyset \) and \( \text{Per}(F) = \emptyset \) (compare with Brouwer’s Theorem 3).

Let \( g : S^1 \to S^1 \) be the degree 2 covering map of example 5.4 (Figure 5 (a)) and let \( K \) be the Cantor set such that \( g(K) = K \) and \( K \cap \text{Per}(g) = \emptyset \). Let \( h : S^1 \to S^1 \).
be an increasing semiconjugacy between $g$ and $q(z) = z^2$, that is $hq = qh$. Then, $h_1(K)$ is compact, $q$-invariant and $\text{Per}(q) \cap h_1(K) = \emptyset$. Now, consider the maps $h_2: S^1 \to [-2, 2]$, $h_2(z) = z + \frac{1}{2}$ and $p: [-2, 2] \to [-2, 2]$, $p(z) = z^2 - 2$. Note that $h_2$ is continuous, surjective and $h_2q = ph_2$. Moreover:

- $K_1 = h_2(h_1(K))$ is compact and $p(K_1) \subset K_1$.
- $\text{Per}(p) \cap K_1 = \emptyset$.

We need one more auxiliary function to make our example $F: \mathbb{R}^2 \to \mathbb{R}^2$ have degree 1. Let $f: \mathbb{R} \to \mathbb{R}$ be as in Figure 7(a), so that $f|_{[-2, 2]} = z^2 - 2$. Now we proceed in the same fashion than in example 5.3. Let $\psi: \mathbb{R} \to \mathbb{R}$, $\psi(x) = \text{dist}(x, K_1)$ and let $\phi: \mathbb{R} \to \mathbb{R}$ be as in Figure 7(b).

Define $\phi(x, y) = \varphi(y) + \psi(x)$ and $F: \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$F(x, y) = (f(x), \phi(x, y)).$$

Note that $K_2 = K_1 \times \{0\}$ is compact and $F$-invariant, so $\Omega(F) \neq \emptyset$. However, $\text{Per}(f) = \emptyset$ as the lines $\{(x, y) : x \notin K_1\}$ have wandering dynamics, and no line $\{(x, y) : x \in K_1\}$ is periodic.

6. **Applications**

We devote this section to applications of Theorem 4 and Lemma 7 to dynamics. Throughout this section, $f: A \to A$ is a degree $d$, $|d| > 1$, covering.

By **attracting set** we mean a proper open subset $U$ such that $f(U) \subset U$. By **repelling set** a proper open subset $U$ such that $f^{-1}(U) \subset U$.

A subset $X \subset A$ is **totally invariant** if $f^{-1}(X) = X$.

**Proposition 1.** Any of the following hypothesis imply that $f$ is complete.

1. There is an essential attracting set.
2. Both ends of $A$ are either attracting or repelling.
3. $f$ extends to a map of the two-point compactification of $A$ in such a way that it is $C^1$ at the poles.
(4) There exists a compact totally invariant subset.
(5) There exists an invariant continuum $K$ such that $h(K)$ is not reduced to a point.

Proof. \(^1\) Let $U \subset A$ be an essential open set such that $f(U) \subset U$. Then, $K = \cap_{n \geq 0} f^n(U)$ is an invariant continuum. Besides, it is essential, because $f^n(U)$ is essential for each $n$ as $|d| > 1$. The result now follows from Theorem \(^1\).

\(^2\) If both ends are attracting, then the complement of both basins of attraction is an essential invariant continuum, and we may apply Theorem \(^1\). If both ends are repelling, and $U_1, U_2$ are neighbourhoods of each end such that $f^{-1}(U_1) \subset U_1$ and $f^{-1}(U_2) \subset U_2$, then the sets of points $\{ x : f^{-n}x \subset U_1, \text{for some } n > 0 \}$ and $\{ x : f^{-n}x \subset U_2, \text{for some } n > 0 \}$ are open, disjoint and the complement is closed, invariant and essential.

\(^3\) Note that in this case both ends must be attracting, as the derivative in the compactification must be 0 at the poles.

\(^4\) Let $X$ be a compact set such that $f^{-1}(X) = X$. We claim that $m_d^{-1}(h(X)) = h(X)$. To see this, it is enough to show that if $x \neq y$ and $f(x) = f(y)$, then $h(x) \neq h(y)$. Let $\tilde{X} = \pi^{-1}(X)$, $F$ a lift of $f$ and $\tilde{x}$ a lift of $x$. Define $\tilde{y}_j = F^{-1}(\tilde{x} + j), j = 0, \ldots, |d| - 1$. Then, $\pi(\cup_{j=0}^{d-1} \tilde{y}_j) = f^{-1}(x)$. Moreover, $dH(\tilde{y}_j) = HF(\tilde{y}_j) = H(\tilde{x} + j) = H(\tilde{x}) + j$, and so $H(\tilde{y}_j) = \frac{H(\tilde{x})}{d} + \frac{j}{d}$, proving the claim.

Now, $m_d^{-1}(h(X)) = h(X)$ implies that $h(X)$ is dense in $S^1$. As $h(X)$ is also compact, $h(X) = S^1$. So, $H_F$ (from Lemma \(^4\)) is surjective for any lift $F$ of $f$. Using the same argument, $H_G$ is surjective for any lift $G$ of $f^n$, and we are done by Corollary \(^4\).

\(^5\) Note that $h(K)$ is an invariant interval that is not reduced to a point, and so $h(K) = S^1$, which implies that $H_F$ is surjective for any lift $F$ of $f$, and we conclude as in the previous item. \(\square\)

The following application shows how the existence of a periodic orbit can imply existence of infinitely many of them. The proof is immediate from item (3) in the previous proposition.

**Corollary 3.** Let $f : S^2 \to S^2$ be a $C^1$ degree $d$ map, $|d| > 1$, and $p,q$ a two-periodic totally invariant orbit $(f^{-1}(\{p,q\}) = \{p,q\}, f(p) = q, f(q) = p)$. If $f : S^2\setminus\{p,q\} \to S^2\setminus\{p,q\}$ is a covering, then $f$ has periodic points of arbitrarily large period.

We will make some calculations that will be used in the following lemma.

Fix a lift $F = F_0$ of $f$, and for any $k \in \mathbb{Z}$, define the maps $F_k(x) = F(x) + k$. Then, for any $m \in \mathbb{N}$ and $x \in A$,

\[ F^m_k(x) = F^m(x) + \sum_{i=0}^{m-1} kd^i = F^m(x) + \frac{k(1-d^m)}{1-d}. \]

The computation is straightforward, following from the fact that for any $k \in \mathbb{Z}$ and $x \in A$, $F(x + k) = F(x) + dk$.

**Lemma 12.** Suppose that there exists a compact set $K \subset A$ such that $f(K) \subset K$. If there exists $x \in K$, and a lift $F$ of $f$ such that $\lim_{m \to \infty} \frac{F^m(x)}{d^m} = \frac{k}{d^m-1}$, $k \in$
for a lift \( \tilde{x} \) of \( x \), then there exists \( z \in \tilde{A} \) such that \( F^n(z) = z + k \). In particular, \( \text{Per}(f) \neq \emptyset \).

Proof. We have to show that the map \( F^n - k \) has a fixed point. By Remark\(^3\) it is enough to show that \( \lim_{m \to \infty} \left( \frac{(F^m - k)(\tilde{x}))}{d^m} \right) = 0 \), where \( G = F^n - k \) (see Remark\(^3\) and note that \( G \) is a lift of \( f^n \)).

\[
\lim_{m \to \infty} \left( \frac{(G^m)(\tilde{x}))}{d^m} \right) = \lim_{m \to \infty} \left( \frac{F^m(\tilde{x}))}{d^m} - \sum_{i=0}^{m-1} \frac{k}{d^i} \right) = \frac{k}{d^m - 1} - \lim_{m \to \infty} \frac{k}{d^m} \sum_{i=0}^{m-1} d^i.
\]

Now,

\[
\lim_{m \to \infty} \frac{k}{d^m} \sum_{i=0}^{m-1} d^i = \lim_{m \to \infty} \frac{k}{d^m} \frac{1 - d^m}{1 - d} = \frac{k}{d^m - 1}.
\]

\[\square\]

Proposition 2. Any of the following hypothesis imply that \( f \) has periodic points.

1. There exists \( x \in A \) with bounded forward orbit such that \( \lim_{m \to \infty} \left( \frac{(F^m)(\tilde{x}))}{d^m} \right) = \frac{k}{d^m - 1} \), for some lift \( \tilde{x} \) of \( x \), \( k \in \mathbb{Z}, n \geq 1 \).

2. There exists an invariant continuum.

Proof. \[\square\] Let \( K \) be the closure of the forward orbit of \( x \). Then, \( K \) is compact, \( f(K) \subset K \), and \( \lim_{m \to \infty} \left( \frac{(F^m)(\tilde{x}))}{d^m} \right) = \frac{k}{d^m - 1} \), for a lift \( \tilde{x} \) of \( x \), \( k \in \mathbb{Z}, n \geq 1 \). Then, by Lemma\(^1\) \( \text{Per}(f) \neq \emptyset \).

\[\square\] If the continuum happens to be essential, then \( f \) is complete by Theorem\(^3\) Otherwise, if \( f \) preserves orientation, this follows from Lemma\(^5\). If \( f \) reverses orientation, one applies Kuperberg’s Theorem\(^7\). \[\square\]

7. Final Comments

There are essentially two examples of covering maps of the annulus without periodic points: The first one, given in the introduction, is conjugate to \( p_d(z) = z^d \) acting in the punctured unit disc. The second one was given in example 5.4, in this case the map has nonempty nonwandering set. As in example 5.3, examples of covering maps with any finite number of periodic points can be constructed, the question is if there exist examples of covering maps which are not complete but satisfy the growth rate inequality for periodic points. Note also that if both ends are attracting or both repelling, then the map is complete. In all the examples of non-complete maps it holds that one end is attracting and the other repelling. Is this necessary? For example, consider the following concrete question whose answer would be a great step: Let \( f : (z,x) \in S^1 \times (0,1) \to (z^3, \varphi_z(x)) \), where \( \varphi_z \) is an increasing homeomorphism of the interval \( (0,1) \) for each \( z \), such that \( \varphi_z(z) \to 1 \) and \( \varphi_z(1) \to 0 \) for every \( x \). This implies that the ends are neither attracting nor repelling and that the map is not complete (it has no fixed points). It can be shown that a map satisfying this hypothesis has nonempty nonwandering set. However we do not know if it must have periodic points, and in this case, if the growth rate inequality holds.
Finally, if the covering assumption is dropped and we remain with a degree \(d\) map of the annulus (\(|d| > 1\)). If \(K\) is an invariant essential continuum, is it necessarily complete?

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