**BOHR INEQUALITIES FOR FREE HOLOMORPHIC FUNCTIONS ON POLYBALLS**

GELU POPESCU

**Abstract.** Multivariable operator theory is used to provide Bohr inequalities for free holomorphic functions with operator coefficients on the regular polyball $B_n$, $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, which is a noncommutative analogue of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$. The Bohr radius $K_{mh}(B_n)$ (resp. $K_{h}(B_n)$) associated with the multi-homogeneous (resp. homogeneous) power series expansions of the free holomorphic functions are the main objects of study in this paper. We extend a theorem of Bombieri and Bourgain for the disc $D := \{z \in \mathbb{C} : |z| < 1\}$ to the polyball, and obtain the estimations

$$\frac{1}{3\sqrt{k}} < K_{mh}(B_n) < \frac{2 \log \sqrt{k}}{\sqrt{k}}$$

if $k > 1$, extending Boas-Khavinson result for the scalar polydisk $D^k$.

With respect to the homogeneous power series expansion, we prove that $K_{h}(B_n) = 1/3$, extending the classical result, and obtain analogues of Carathéodory, Fejér, and Egerváry-Szász inequalities for free holomorphic functions with operator coefficients and positive real parts on the polyball. These results are used to provide multivariable analogues of Landau’s inequality and Bohr’s inequality when the norm is replaced by the numerical radius of an operator.

When specialized to the regular polydisk $D^k$ (which corresponds to the case $n_1 = \cdots = n_k = 1$), we obtain new results concerning Bohr, Landau, Fejér, and Harnack inequalities for operator-valued holomorphic functions and $k$-pluriharmonic functions on the scalar polydisk $D^k$. The results of the paper can be used to obtain Bohr type inequalities for the noncommutative ball algebra $\mathcal{A}_n$, the Hardy algebra $\mathcal{F}_\infty^n$, and the $C^*$-algebra $C^*(S)$, generated by the universal model $S$ of the polyball $B_n$.

**Contents**

**Introduction**

1. Preliminaries on Berezin transforms on noncommutative polyballs
2. Bohr inequalities for free holomorphic functions on polyballs
3. Bohr inequalities for free holomorphic functions with $F(0) \geq 0$ and $\Re F \leq 1$
4. The Bohr radius $K_{mh}(B_n)$ and Bombieri-Bourgain theorem for the polyball
5. The Bohr radius $K_{h}(B_n)$ for the Hardy space $H^\infty(B_n)$
6. Fejér and Bohr inequalities for multivariable polynomials with operator coefficients
7. Harnack inequalities for free $k$-pluriharmonic functions

**References**

**Introduction**

Bohr’s inequality [5] asserts that if $f(z) := \sum_{k=0}^\infty a_k z^k$ is an analytic function on the open unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ such that $\|f\|_\infty \leq 1$, then

$$\sum_{k=0}^\infty r^k|a_k| \leq 1 \quad \text{for} \quad 0 \leq r \leq \frac{1}{3}.$$  

M. Riesz, Schur, and Weiner showed, independently, that $\frac{1}{3}$ is the best possible constant. Other proofs were later obtained by Sidon [33] and Tomic [35]. Dixon [11] used Bohr’s inequality in connection with

---

*Date: February 24, 2017.*

2000 *Mathematics Subject Classification.* Primary: 47A56; 46L52; Secondary: 32A38; 47A63.

*Key words and phrases.* Multivariable operator theory; Bohr’s inequality; Noncommutative polyball; Free holomorphic function; Free pluriharmonic function; Berezin transform; Fock space.

Research supported in part by NSF grant DMS 1500922.
the long-standing problem of characterizing Banach algebras satisfying the von Neumann inequality \([36]\) (see also \([10]\) and \([19]\)).

In 1997, Boas and Khavinson \([4]\) introduced the Bohr radius \(K_N\) for the Hardy space \(H^\infty(\mathbb{D}^N)\) of bounded holomorphic function on the \(N\)-dimensional polydisc and showed that, if \(N > 1\), then
\[
\frac{1}{3\sqrt{N}} < K_N < \frac{2\sqrt{\log N}}{\sqrt{N}}.
\]

Inspired by this result, several authors (see \([1, 2, 3, 5, 6, 8, 9, 10, 17, 23, 25]\) and the references therein) have considered multivariable analogues of Bohr’s inequality. Due to the remarkable work by Defant, Frerick, Ortega-Cerd, Ounaïes, and Seip \([8]\), and by Bayart, Pellegrino, and Seoane-Sepúlveda \([3]\), we know now the asymptotic behaviour of the Bohr radius \(K_N\), i.e.
\[
\lim_{N \to \infty} \frac{K_N}{\sqrt{(\log N)/N}} = 1.
\]

In \([9]\), Defant, Maestre, and Schwarting obtained upper and lower estimates for the Bohr radius in the setting of holomorphic functions defined on \(\mathbb{D}^N\) with values in Banach spaces. Noncommutative multivariable analogues of Bohr’s inequality were obtained in \([17]\) and \([23]\) for the class of noncommutative holomorphic functions on the open unit ball
\[
[B(H)^k]_1 := \left\{ (X_1, \ldots, X_k) \in B(H)^k : \|X_1X_1^* + \cdots + X_kX_k^*\|^{1/2} < 1 \right\},
\]
where \(k \in \mathbb{N} := \{1, 2, \ldots\\}\) and \(B(H)\) is the algebra of all bounded linear operators on a Hilbert space \(\mathcal{H}\).

The main goal of the present paper is to study the Bohr phenomenon in the setting of free holomorphic functions on the noncommutative polyball \(B_n, n = (n_1, \ldots, n_k) \in \mathbb{N}^k\), which is a noncommutative analogue of the scalar polyball \((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1\), where \((\mathbb{C}^n)_1 := \{z \in \mathbb{C}^n : \|z\|_2 < 1\}\).

To present our results, we need some definitions. We denote by \(B(H)^{n_1} \times_c \cdots \times_c B(H)^{n_k}\), where \(n_i \in \mathbb{N}\), the set of all tuples \(X := (X_1, \ldots, X_k)\) in \(B(H)^{n_1} \times \cdots \times B(H)^{n_k}\) with the property that the entries of \(X_s := (X_{s,1}, \ldots, X_{s,n_s})\) are commuting with the entries of \(X_t := (X_{t,1}, \ldots, X_{t,n_t})\) for any \(s, t \in \{1, \ldots, k\}\), \(s \neq t\). Note that the operators \(X_{s,1}, \ldots, X_{s,n_s}\) are not necessarily commuting. Define the polyball
\[
P_n(H) := [B(H)^{n_1}]_1 \times_c \cdots \times_c [B(H)^{n_k}]_1.
\]
If \(A\) is a positive invertible operator, we write \(A > 0\). The regular polyball on the Hilbert space \(\mathcal{H}\) is defined by
\[
B_n(H) := \{X \in P_n(H) : \Delta_X(I) > 0\},
\]
where the defect mapping \(\Delta_X : B(H) \to B(H)\) is given by \(\Delta_X := (id - \Phi_{X_1}) \circ \cdots \circ (id - \Phi_{X_k})\), and \(\Phi_X : B(H) \to B(H)\) is the completely positive linear map defined by
\[
\Phi_{X_i}(Y) := \sum_{j=1}^n X_{i,j}YX_{i,j}^*, \quad Y \in B(H).
\]
Note that if \(k = 1\), then \(B_n(H)\) coincides with the noncommutative unit ball \([B(H)^{n_1}]_1\). We remark that the scalar representation of the (abstract) regular polyball \(B_n(H) : \mathcal{H}\) is \(B_n(\mathbb{C}) = P_n(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1\). A multivariable operator model theory and a theory of free holomorphic functions on polydomains which admit universal operator models have been recently developed in \([26]\) and \([27]\). An important feature of these theories is that they are related, via noncommutative Berezin transforms, to the study of the operator algebras generated by the universal models associated with the domains, as well as to the theory of functions in several complex variable \([32, 33]\). These results played a crucial role in our work on the curvature invariant \([28]\), the Euler characteristic \([29]\), and the group of free holomorphic automorphisms on noncommutative regular polyballs \([30]\), and will play an important role in the present paper.

For each \(i \in \{1, \ldots, k\}\), let \(\mathbb{F}_n^+\) be the free semigroup with generators \(g_{i,1}^1, \ldots, g_{i,n_i}^1\) and identity \(g_0^i\). Let \(Z_i := (Z_{i,1}, \ldots, Z_{i,n_i})\) be an \(n_i\)-tuple of noncommuting indeterminates and assume that, for any \(p, q \in \{1, \ldots, k\}\), \(p \neq q\), the entries in \(Z_p\) are commuting with the entries in \(Z_q\). We set \(Z_{i,\alpha} := Z_{i,1} \cdots Z_{i,p}\) if \(\alpha \in \mathbb{F}_n^+\) and \(\alpha = g_{j,1}^1 \cdots g_{j,p}^1\), and \(Z_{i,\alpha} := 1\). If \(\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+\), we denote
$Z_{\alpha} := Z_{\alpha_1, \ldots, \alpha_k}$. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}^+$ be the set of all nonnegative integers. A formal power series $\varphi = \sum_{\alpha \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_k^+} a_{(\alpha)} \, Z_{\alpha}$ in indeterminates $Z = \{Z_{i,j}\}$ and scalar coefficients $a_{(\alpha)} \in \mathbb{C}$ is called free holomorphic function on the abstract polyball $B_n := \{B_n(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$ if the series

$$\varphi(X) := \sum_{p \in \mathbb{Z}_k^+} \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha} \quad \text{(multi-homogeneous expansion)}$$

is convergent in the operator norm topology for any $X = \{X_{i,j}\} \in B_n(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. Here we use the notation $\Lambda_p := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+ : |\alpha_1| + \cdots + |\alpha_k| = p\}$, where $p = (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ and $|\alpha_i|$ is the length of $\alpha_i$. In this case, we proved [30] that

$$\varphi(X) = \sum_{q=0}^\infty \sum_{\alpha \in \Gamma_q} a_{(\alpha)} X_{\alpha} \quad \text{(homogeneous expansion)}$$

where $\Gamma_q := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+ : |\alpha_1| + \cdots + |\alpha_k| = q\}$ and the convergence of the series is in the operator norm topology. In fact, this result holds true for free holomorphic functions with operator coefficients.

The Bohr radius $K_{mh}(B_n)$ for the Hardy space $H^\infty(B_n)$ of all bounded free holomorphic functions on $B_n$, with respect to the multi-homogeneous expansion of its elements, is the largest $r \geq 0$ such that

$$\sum_{p \in \mathbb{Z}_k^+} \left\| \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha} \right\| \leq \|\varphi\|_{\infty}, \quad X \in r B_n(\mathcal{H})^-,$$

for any $\varphi \in H^\infty(B_n)$. Similarly, we define the Bohr radius $K_h(B_n)$ for the Hardy space $H^\infty(B_n)$ with respect to the homogeneous expansion of its elements. Note that when $k = 1$ the two definitions coincide. When the Hardy space $H^\infty(B_n)$ is replaced by the subspace $H^\infty_\infty(B_n) := \{f \in H^\infty(B_n) : f(0) = 0\}$ the corresponding Bohr radii are denoted by $K_{mh}^\infty(B_n)$ and $K_h^0(B_n)$, respectively.

In Section 2, we obtain Weiner type inequalities for the coefficients of bounded free holomorphic functions on polyballs, which are used to obtain Bohr inequalities for bounded free holomorphic functions with operator coefficients. As a consequence, we show that the Bohr radius $K_{mh}(B_n)$ satisfies the inequalities

$$1 - \left(\frac{2}{3}\right)^{1/k} \leq K_{mh}(B_n) \leq \frac{1}{3}, \quad k \geq 1.$$

Let $F \in H^\infty(B_n)$ have the representation $F(X) := \sum_{p \in \mathbb{Z}_k^+} \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha}$ and let

$$D(F, r) := \sum_{p \in \mathbb{Z}_k^+} r^{\|p\|} \left\| \sum_{\alpha \in \Lambda_p} a_{(\alpha)} S_{\alpha} \right\|$$

be the associated majorant series, where $S = \{S_{i,j}\}$ is the universal model of the polyball (see Section 1). Define

$$d_{B_n}(r) := \sup_{\|F\|_{\infty}} \frac{D(F, r)}{\|F\|_{\infty}}, \quad r \in [0, 1),$$

where the supremum is taken over all $F \in H^\infty(B_n)$ with $F$ not identically 0. The results of this section show that $d_{B_n}(r) = 1$ if $0 \leq r \leq 1 - (2/3)^{1/k}$. While we obtain upper bounds for $mh_{B_n}(r)$, the precise value of $d_{B_n}(r)$ as $1 - (2/3)^{1/k} < r < 1$ remains unknown, in general. Progress on this problem was made, in the classical case of the disc $\mathbb{D}$, by Bombieri and Bourgain in [10], where they proved that $\lim_{r \to 1} \frac{d_{\mathbb{D}}(r)}{\sqrt{1-r^2}} = 1$ as $r \to 1$. We extend their result proving that $d_{B_n}(r)$ behaves asymptotically as $\left(\frac{1}{\sqrt{1-r}}\right)^k$ if $r \to 1$, i.e.

$$\lim_{r \to 1} \frac{d_{B_n}(r)}{\left(\frac{1}{\sqrt{1-r}}\right)^k} = 1.$$

In particular, the result applies to the scalar polydisc $\mathbb{D}^k$. 
In Section 3, we obtain an analogue of Landau’s inequality \[15\] for bounded free holomorphic functions with operator coefficients on the polyball (see Theorem \[3.1\]). This is used to obtain Bohr inequalities for free holomorphic functions \( F : B_{n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H}) \) such that \( F(0) \geq 0 \) and \( \Re F(X) \leq I \) for any \( X \in B_{n}(\mathcal{H}) \). The result plays a crucial role in Section 4, where we prove that the Bohr radius \( K_{mh}(B_n) \) satisfies the inequalities
\[
\frac{1}{3\sqrt{k}} < K_{mh}(B_n) < \frac{2\sqrt{\log k}}{\sqrt{k}}, \quad k > 1,
\]
and obtain the asymptotic upper bound
\[
\limsup_{k \to \infty} \frac{K_{mh}(B_n)}{\sqrt{(\log k)/k}} \leq 1.
\]

Section 5 concerns the Bohr radius \( K_{h}(B_n) \) for the Hardy space \( H^\infty(B_n) \), with respect to the homogeneous expansion of its elements. Using the results of Section 3, we prove that
\[
K_{h}(B_n) = \frac{1}{3},
\]
which extends the classical result to our multivariable noncommutative setting. Let \( F \in H^\infty(B_n) \) with representation \( F(X) := \sum_{q=0}^{\infty} \sum_{\alpha \in \Gamma_q} a(\alpha)X_{\alpha} \) and let
\[
\mathcal{M}(F,r) := \sum_{q=0}^{\infty} r^q \left\| \sum_{\alpha \in \Gamma_q} a(\alpha)S_{\alpha} \right\| \quad \text{and} \quad m_{B_n}(r) := \sup \frac{\mathcal{M}(F,r)}{\|F\|_{\infty}}, \quad r \in [0,1),
\]
where the supremum is taken over all \( F \in H^\infty(B_n) \) with \( F \) not identically 0. The results of this section show that \( m_{B_n}(r) = 1 \) if \( r \in [0,\frac{1}{2}] \). While we obtain upper bounds for \( m_{B_n}(r) \), the precise value of \( m_{B_n}(r) \) as \( \frac{1}{3} < r < 1 \) remains unknown, in general.

Concerning the Bohr radius \( K_{0}^{0}(B_n) \), we show that it satisfies the inequalities
\[
\left(1 - \left(\frac{1}{2}\right)^{1/k}\right)^{1/k} \leq K_{0}^{0}(B_n) \leq \frac{1}{\sqrt{2}}, \quad \text{if} \quad k \geq 1,
\]
and
\[
\frac{1}{2\sqrt{k}} < K_{0}^{0}(B_n) < \frac{2\sqrt{\log k}}{\sqrt{k}}, \quad \text{if} \quad k > 1.
\]
Estimations for the Bohr radius \( K_{0}^{0}(B_n) \) are also obtained.

In Section 6, we obtain analogues of Carathéodory’s inequality \[7\], and Fejér \[13\] and Egerváry-Százs inequalities \[12\] for free holomorphic functions with operator coefficients and positive real parts on the polyball (see Theorem \[6.1\]). These results are used to provide (see Theorem \[6.4\]) an analogue of Landau’s inequality \[12\] and Bohr type inequalities when the norm is replaced by the numerical radius of an operator, i.e.
\[
\omega(T) := \sup \{|\langle Th, h \rangle| : h \in \mathcal{H}, \|h\| = 1\}, \quad T \in B(\mathcal{H}).
\]
In particular, we obtain the following radius versions of Landau’s inequality and Bohr’s inequality for free holomorphic functions on polyballs. If \( m \in \mathbb{N} \cup \{\infty\} \) and \( f(X) := \sum_{q=1}^{m} \sum_{\alpha \in \Gamma_q} a(\alpha)X_{\alpha} \) is a free holomorphic function with \( f(0) \geq 0 \) and \( \Re f(X) \leq I \) on the polyball \( B_n \), then
\[
\omega \left( \sum_{\alpha \in \Gamma_q} a(\alpha)S_{\alpha} \right) \leq 2(1 - a_0) \cos \left( \frac{\pi}{\left[\frac{m}{q}\right] + 2} \right), \quad q \in \{1, \ldots, m\},
\]
and
\[
\sum_{q=0}^{m} \omega \left( \sum_{\alpha \in \Gamma_q} a(\alpha)S_{\alpha} \right)^{r^q} \leq 1, \quad r \in [0, t_m],
\]
where $t_m \in (0, 1)$ is the solution of the equation

$$\sum_{q=1}^{m} t^q \cos \frac{\pi}{\left\lfloor \frac{m}{q} \right\rfloor + 2} = \frac{1}{2},$$

and $[x]$ is the integer part of $x$. Moreover, the sequence $\{t_m\}$ is strictly decreasing and converging to $\frac{1}{2}$. When $m = \infty$, we have $t_\infty = \frac{1}{3}$. As a consequence of these results, we deduce that if $f$ is a holomorphic function on the polydisc $\mathbb{D}^k$ such that $\Re f(z) \leq 1$ for $z \in \mathbb{D}^k$ and $f(a) \geq 0$ for some $a = (a_1, \ldots, a_k) \in \mathbb{D}^k$, then

$$\sum_{i=1}^{k} (1 - |a_i|^2) \left| \left( \frac{\partial f}{\partial z_i} \right) (a) \right| \leq 2(1 - f(a)),$$

which is an extension of Landau’s inequality to the polydisc.

In Section 7, we provide Harnack type inequalities for positive free $k$-pluriharmonic function with operator coefficients on polyballs. In particular, if $F$ is a positive free $k$-pluriharmonic function with scalar coefficients and of degree $m_i \in \mathbb{N} \cup \{\infty\}$ with respect to the variables $X_{i,1}, \ldots, X_{i,n_i}$, then we prove that

$$F(X) \leq F(0) \prod_{i=1}^{k} \left( 1 + 2 \sum_{p_i=1}^{m_i} \rho_i^{p_i} \cos \frac{\pi}{\left\lfloor \frac{m_i}{p_i} \right\rfloor + 2} \right) \leq F(0) \prod_{i=1}^{k} \frac{1 + \rho_i}{1 - \rho_i}$$

for any $X \in \rho B_n(\mathcal{H})^{-}$ and $\rho := (\rho_1, \ldots, \rho_k) \in [0,1)^k$, where $[x]$ is the integer part of $x$.

We remark that when $n_1 = \cdots = n_k = 1$ the free holomorphic functions on the regular polydisc $\mathbb{D}^k := B_n$ can be identified with the holomorphic functions on the scalar polydisc $\mathbb{D}$, and the Hardy space $H^\infty(B_n)$ can be identified with the Hardy space $H^\infty(\mathbb{D})$. As a consequence all the results of the present paper hold true in this particular setting. In this way, we recover some known results but at the same time we provide new results concerning Bohr, Landau, Fejér, and Harnack inequalities for operator-valued holomorphic functions and $k$-pluriharmonic functions on the polydisc.

We should also mention that our results can be used to obtain Bohr type inequalities for the noncommutative ball algebra $\mathcal{A}_n$, the Hardy algebra $F_n^\infty$, and the $C^*$-algebra $C^*(S)$, generated by the universal model $S = \{S_{i,j}\}$ of the polyball $B_n$.

1. **Preliminaries on Berezin transforms on noncommutative polyballs**

Let $H_n$ be an $n$-dimensional complex Hilbert space with orthonormal basis $e_1, \ldots, e_n$. We consider the full Fock space of $H_n$, defined by $F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{p \geq 2} H_n^{\otimes p}$, where $H_n^{\otimes p}$ is the (Hilbert) tensor product of $p$ copies of $H_n$. Let $\mathbb{F}_n^+$ be the unital free semigroup on $n_i$ generators $g_1, \ldots, g_{n_i}$ and the identity $g_0$. Set $e_\alpha := e_{j_1} \cdots e_{j_p}$ if $\alpha = g_{j_1} \cdots g_{j_p} \in \mathbb{F}_n^+$ and $e_{g_0} := 1 \in \mathbb{C}$. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := p$ if $\alpha = g_{j_1} \cdots g_{j_p}$, where $j_1, \ldots, j_p \in \{1, \ldots, n_i\}$. We define the left creation operator $S_{i,j}$ acting on the Fock space $F^2(H_n)$ by setting $S_{i,j} e_\alpha := e_{j} \otimes e_\alpha$, $\alpha \in \mathbb{F}_n^+$, and the operator $S_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$S_{i,j} := I \otimes \cdots \otimes I \otimes S_{i,j} \otimes I \otimes I \otimes \cdots \otimes I,$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We denote $S := (S_1, \ldots, S_k)$, where $S_i := (S_{i,1}, \ldots, S_{i,n_i})$, or $S := \{S_{i,j}\}$. The noncommutative Hardy algebra $F_n^\infty$ (resp. the polyball algebra $\mathcal{A}_n$) is the weakly closed (resp. norm closed) non-selfadjoint algebra generated by $\{S_{i,j}\}$ and the identity. Similarly, we define the right creation operator $R_{i,j} : F^2(H_n) \rightarrow F^2(H_n)$ by setting $R_{i,j} e_\alpha := e_\alpha \otimes e_j$ for $\alpha \in \mathbb{F}_n^+$, and the operator $R_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$R_{i,j} := I \otimes \cdots \otimes I \otimes R_{i,j} \otimes I \otimes I \otimes \cdots \otimes I,$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. The polyball algebra $R_n$ is the norm closed non-selfadjoint algebra generated by $\{R_{i,j}\}$ and the identity.
We recall (see [22, 27]) some basic properties for the noncommutative Berezin transforms associated with regular polyballs. Let \( X = (X_1, \ldots, X_k) \in B_n(\mathcal{H})^- \) with \( X_i := (X_{i,1, \ldots, X_{i,n_i}}) \). We use the notation \( X_{i,\alpha} := X_{i,j_1} \cdots X_{i,j_p} \) if \( \alpha = j_1 \cdots j_p \in F_{n_i} \). The \( \alpha \) noncommutative Berezin kernel defined by \( \phi \) functions on completely positive \((\text{resp. bounded})\) maps we refer the reader to [16] and [19].

The topology. Moreover, if \( \bigoplus_{i,j} \bigoplus_{\alpha} \mathbf{1} \) is the defect operator. A very important property of the Berezin kernel is that \( \Phi \bigoplus_{i,j} \bigoplus_{\alpha} \mathbf{1} \) is in the operator norm topology. In this case, the Berezin transform at \( X \in B_n(\mathcal{H})^- \) by

\[
\mathcal{B}_X[g] := \mathcal{K}_X \cdot g \otimes f \mathcal{H} \mathcal{K}_X, \quad g \in B(\bigotimes_{i=1}^k F^2(H_{n_i})).
\]

If 

\[
\Delta := \sum_{i,1}^{n_i} \alpha \beta \in F^+_{n_i} 
\]

holomorphic function \((\text{with scalar coefficients})\) on the abstract polyball \( H \in \mathbb{C} \). More precisely, we proved that the map \( \Phi := \sum_{\alpha, \beta} a_{\alpha} Z_{\alpha} \) is a noncommutative Berezin transforms, we proved in [30] that a formal power series \( \varphi := \sum_{\alpha} a_{\alpha} r_1^{p_1} \cdots r_k^{p_k} S_{\alpha} \| \) converges for any \( r_i \in [0,1) \), which is equivalent to the condition that the series \( \sum_{q=0}^{\infty} \| \sum_{\alpha \in F^+_{n_1} \times \cdots \times F^+_{n_k}} a_{\alpha} r_1^{p_1} \cdots r_k^{p_k} S_{\alpha} \| \) is convergent for any \( r \in [0,1) \). In [27], we identified the noncommutative Hardy subalgebra \( H^\infty(B_n) \) with the Hardy subalgebra \( H^\infty(B_n) \) of bounded free holomorphic functions on \( B_n \) with scalar coefficients. More precisely, we proved that the map \( \Phi : H^\infty(B_n) \to F^\infty_n \) defined by \( \Phi(\varphi) := \text{SOT-lim}_{r \to 1} \varphi(rS) \), is a completely isometric isomorphism of operator algebras, where \( \varphi(rS) := \sum_{q=0}^{\infty} \sum_{\alpha \in F^+_{n_1} \times \cdots \times F^+_{n_k}} r^q a_{\alpha} S_{\alpha} \) and the convergence of the series is in the operator norm topology. Moreover, if \( \varphi \) is a free holomorphic function on the abstract polyball \( B_n \), then the following statements are equivalent:

(i) \( \varphi \in H^\infty(B_n) \);

(ii) \( \sup_{0 \leq r < 1} \| \varphi(rS) \| < \infty \);

(iii) there exists \( \psi \in F^\infty_n \) with \( \varphi(X) = \mathcal{B}_X[\psi] \), where \( \mathcal{B}_X \) is the noncommutative Berezin transform associated with the abstract polyball \( B_n \). Moreover, \( \psi \) is uniquely determined by \( \varphi \), namely, \( \psi = \text{SOT-lim}_{r \to 1} \varphi(rS) \) and

\[
\| \psi \| = \sup_{0 \leq r < 1} \| \varphi(rS) \| = \lim_{r \to 1} \| \varphi(rS) \| = \| \varphi \| .
\]

We use the notation \( \hat{\varphi} := \psi \) and call \( \hat{\varphi} \) the \((\text{model})\) boundary function of \( \varphi \) with respect to the universal model \( \mathcal{S} \) of the regular polyball. Similar results hold for free holomorphic functions with operator coefficients. More information on noncommutative Berezin transforms and multivariable operator theory on noncommutative balls and polydomains can be found in [22, 23, and 27]. For basic results on completely positive (resp. bounded) maps we refer the reader to [19] and [19].

If \( z := (z_1, \ldots, z_k) \in \mathbb{C}^k \) and \( p := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k \), we write \( Z \) for the monomial \( z_1^{p_1} \cdots z_k^{p_k} \) and use the notation \( |p| := p_1 + \cdots + p_k \). Similarly, if \( X := (X_1, \ldots, X_k) \in B(H)^k \), then \( X^p := X_1^{p_1} \cdots X_k^{p_k} \).
When \( n = (n_1, \ldots, n_k) \) with \( n_1 = \cdots = n_k = 1 \), we call \( D^k := B_n \) the regular polydisk. Note that the scalar representation \( D^k(\mathbb{C}) \) coincides with the scalar polydisk \( \mathbb{D}^k \).

**Proposition 1.1.** If \( \{a_p\}_{p \in \mathbb{Z}_+^k} \subset \mathbb{C} \), then \( f(z) := \sum_{p \in \mathbb{Z}_+^k} a_p z^p \) is a holomorphic function on the scalar polydisk \( \mathbb{D}^k \) if and only if \( F(X) := \sum_{p \in \mathbb{Z}_+^k} a_p X^p \) is a free holomorphic function on the regular polydisk \( D^k \). Moreover, the following statements hold.

1. \( f \in H^\infty(\mathbb{D}^k) \) if and only if \( F \in H^\infty(D^k) \), in which case \( \|f\|_\infty = \|F\|_\infty \).
2. \( \Re f(z) \leq 1 \) for any \( z \in \mathbb{D}^k \) if and only if \( \Re F(rS) \leq 1 \) for any \( r \in [0, 1) \), where \( S = (S_1, \ldots, S_k) \) is the universal model of the regular polydisk \( D^k \). Conversely, if we assume that \( F \in H^\infty(D^k) \), then \( \sup_{0 \leq r < 1} \|F(rS)\| \leq \sup_{z \in \mathbb{D}^k} |f(z)| < \infty \), which due to the remarks preceding the proposition shows that \( F \in H^\infty(\mathbb{D}^k) \).
3. The Banach algebra \( H^\infty(\mathbb{D}^k) \) is isometrically isomorphic to \( H^\infty(D^k) \), which is isometrically embedded in \( H^\infty(B_n) \).

**Proof.** Note that \( f(z) := \sum_{p \in \mathbb{Z}_+^k} a_p z^p \) is a holomorphic function on the scalar polydisk \( \mathbb{D}^k \) if and only if \( \sum_{p \in \mathbb{Z}_+^k} r^{|p|} |a_p| < \infty \) for any \( r \in [0, 1) \) (see [22]). Due to the remarks preceding the proposition, the latter condition is equivalent to the fact that \( F(X) := \sum_{p \in \mathbb{Z}_+^k} a_p X^p \) is a free holomorphic function on the regular polydisc \( D^k \). Let \( S = (S_1, \ldots, S_k) \) be the universal model of the regular polydisk \( D^k \) and note that \( S_1, \ldots, S_k \) are unitarily equivalent to the multiplication operators by the coordinate functions \( z_1, \ldots, z_k \) on the Hardy space \( H^2(\mathbb{D}^k) \). If \( f \in H^\infty(\mathbb{D}^k) \), then standard arguments imply \( \sup_{p \leq r < 1} \|F(rS)\| \leq \sup_{z \in \mathbb{D}^k} |f(z)| < \infty \), which due to the remarks preceding the proposition shows that \( F \in H^\infty(D^k) \).

To prove part (ii), assume that \( \Re f(z) \leq 1 \) for any \( z \in \mathbb{D}^k \). We use the natural identification of the Hardy spaces \( H^2(\mathbb{D}^k) \) with \( H^2(\mathbb{T}^k) \), and \( H^\infty(D^k) \) with \( H^\infty(\mathbb{T}^k) \). Under this identification, the shifts \( S_1, \ldots, S_n \) are the multiplications by the coordinate functions \( \xi_1, \ldots, \xi_k \) on \( H^2(\mathbb{T}^k) \). Note that, for each \( h \in H^2(\mathbb{T}^k) \), we have

\[
\langle [2I - F(rS)^* - F(rS)]h, h \rangle_{H^2(\mathbb{T}^k)} = \int_{\mathbb{T}^k} \left| 2 - \overline{f(r\xi)} - f(r\xi) \right|^2 dm_k(\xi) \geq 0, \quad \xi \in \mathbb{T}^k,
\]

where \( m_k \) is the normalized Lebesgue measure on \( \mathbb{T}^k \). Therefore, \( \Re F(rS) \leq 1 \) for any \( r \in [0, 1) \). Conversely, using the Berezin transform at \( z \in \mathbb{D}^k \), we have

\[
\overline{f(rz)} + f(rz) = B_z[F(rS)^* - F(rS)] \leq 2,
\]

for any \( r \in [0, 1] \) and \( z \in \mathbb{D}^k \), which completes the proof of part (ii).

The fact that the Banach algebra \( H^\infty(\mathbb{D}^k) \) is isometrically isomorphic to \( H^\infty(D^k) \) follows from item (i). Note that if \( F(Y) := \sum_{p \in \mathbb{Z}_+^k} a_p Y^p \), \( Y \in D^k \), is a free holomorphic function in \( H^\infty(D^k) \), then the function \( G(X) := \sum_{p=(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} a_p X_{1,1}^{p_1} X_{2,1}^{p_2} \cdots X_{k,1}^{p_k} \) is in the noncommutative Hardy algebra \( H^\infty(B_n) \) and \( \|F\|_\infty = \|G\|_\infty \). Hence, part (iii) follows. The proof is complete.

We remark that a result similar to Proposition 1.1 holds for free holomorphic functions with coefficients bounded linear operators acting on a separable Hilbert space.

2. **Bohr inequalities for free holomorphic functions on polyballs**

We define the right (resp. minimal) sets in \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \), mention some of their properties and give several examples. The exhaustive sets of \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \) by right minimal sets or by orthogonal sets will play an important role throughout the paper. We associate with each such an exhaustion, a Bohr type inequality for the Hardy space \( H^\infty(B_n) \). We obtain Wiener type inequalities for the coefficients of bounded free holomorphic functions on polyballs, which are used to obtain Bohr inequalities for bounded free holomorphic functions with operator coefficients. We also extend a theorem of Bombieri and Bourgain for the disc to the polyball and obtain estimations for the Bohr radii \( K_{mh}(B_n) \) and \( K_{mh}'(B_n) \).
implies that \( \gamma < \omega \) if there is \( \sigma \in F^+_n \setminus \{g_0\} \) such that \( \omega = \sigma \gamma \). In this case, we set \( \omega \setminus \gamma = \sigma \). We also use the notation \( \gamma \leq \omega \) when \( \gamma < \omega \) or \( \gamma = \omega \). Note that if \( \gamma = \omega \), then \( \omega \setminus \gamma := g_0 \). Similarly, we say that \( \gamma < \omega \) if there is \( \sigma \in F^+_n \setminus \{g_0\} \) such that \( \omega = \sigma \gamma \) and set \( \omega \setminus \gamma := \sigma \). The notation \( \gamma \leq \omega \) is clear. We denote by \( \overline{\alpha} \) the reverse of \( \alpha \in F^+_n \), i.e. \( \overline{\alpha} = \alpha_{i_k} \cdots \alpha_{i_1} \) if \( \alpha = \alpha_{i_1} \cdots \alpha_{i_k} \in F^+_n \). Note that \( \gamma \leq \omega \) if and only if \( \overline{\gamma} \leq \overline{\omega} \). In this case, we have \( \omega \setminus \gamma = \overline{\omega} \setminus \overline{\gamma} \). Let \( \omega = (\omega_1, \ldots, \omega_k) \) and \( \gamma = (\gamma_1, \ldots, \gamma_k) \) be in \( F^+_1 \times \cdots \times F^+_k \). We say that \( \omega \leq \gamma \), if \( \omega_i \leq \gamma_i \) for each \( i \in \{1, \ldots, k\} \). Similarly, we say that \( \omega \leq \gamma \), if \( \omega_i \leq \gamma_i \), where \( \overline{\omega} = (\overline{\omega_1}, \ldots, \overline{\omega_k}) \).

**Definition 2.1.** A subset \( \Lambda \) of \( F^+_{n_1} \times \cdots \times F^+_{n_k} \) is called right minimal if, for any \( \omega, \gamma \in \Lambda \), \( \omega \leq \gamma \) implies \( \omega = \gamma \). We say that \( \Lambda \) is left minimal if, for any \( \omega, \gamma \in \Lambda \), \( \omega \leq \gamma \) implies \( \omega = \gamma \). If a set \( \Lambda \) is both right and left minimal, we call it minimal.

Note that \( \Lambda \) is right minimal if and only if \( \overline{\Lambda} := \{ \overline{\omega} : \omega \in \Lambda \} \) is left minimal. Here are a few characterizations of right minimal sets. Since the proof is straightforward, we shall omit it.

**Proposition 2.2.** Let \( S := \{S_{i,j}\} \) be the universal model of the polyball \( B_n \), and let \( \{e^i_n\}_{\alpha \in F^+_n} \) be the orthogonal basis for the Fock space \( F^2(H_{n_i}) \). Then the following statements hold.

(i) For \( \alpha = g_{i_1} \cdots g_{i_p} \in F^+_n \), we denote by \( D^r_\alpha \) the set of all right divisors of \( \alpha \), i.e.

\[
D^r_\alpha := \{g_0, g_{i_p}, g_{i_{p-1}} g_{i_p}, \ldots, g_{i_1} \cdots g_{i_p}\}.
\]

If \( \beta \in F^+_n \setminus D^r_\beta \) and \( \alpha \in F^+_n \setminus D^r_\alpha \), then \( \{\alpha, \beta\} \) is a right minimal set. Moreover, a set \( \Lambda \subset F^+_n \) is right minimal if and only if, for any \( \alpha, \beta \in \Lambda \) with \( \alpha \neq \beta \), we have \( \beta \in F^+_n \setminus D^r_\alpha \) and \( \alpha \in F^+_n \setminus D^r_\beta \).

(ii) \( \Lambda \subset F^+_1 \times \cdots \times F^+_n \) is a right minimal set if and only if, for any \( \alpha, \gamma \in \Lambda \) and \( \alpha \in F^+_1 \times \cdots \times F^+_n \),

\[
\langle e^1_\alpha \otimes e^2_\beta, e^3_\gamma \rangle = 1
\]

if and only if \( \alpha = (g_{i_1}, \ldots, g_{i_p}) \) and \( \beta = \gamma \), where \( e^i_\beta := \frac{e^i_{{\beta}_{i_1}} \otimes \cdots \otimes e^i_{{\beta}_{i_p}}} {\sqrt{\det(\overline{\beta})}} \) if \( \beta = (\beta_1, \ldots, \beta_k) \).

**Example 2.3.** The following statements hold.

(i) If \( p \in \mathbb{N} \cup \{0\} \), then \( \Lambda_p := \{\alpha \in F^+_n : |\alpha| = p\} \) is a minimal set in \( F^+_n \).

(ii) If \( p_1, \ldots, p_k \in \mathbb{N} \cup \{0\} \) and \( \Lambda_{p} := \{\alpha \in F^+_n : |\alpha| = p\} \), then \( \Lambda_{p_1} \times \cdots \times \Lambda_{p_k} \) is a minimal set in \( F^+_1 \times \cdots \times F^+_n \).

(iii) If \( p \in \mathbb{N} \cup \{0\} \), then

\[
\Gamma_p := \{(\alpha_1, \ldots, \alpha_k) \in F^+_1 \times \cdots \times F^+_n : |\alpha_1| + \cdots + |\alpha_k| = p\}
\]

is a minimal set in \( F^+_1 \times \cdots \times F^+_n \).

(iv) Any subset of a (left, right) minimal set is a (left, right) minimal set.

(v) If \( i \in \{1, \ldots, k-1\} \) and \( \Lambda \subset F^+_1 \times \cdots \times F^+_n \) and \( \Gamma \subset F^+_{n_i+1} \times \cdots \times F^+_n \) are (left, right) minimal sets, then so if \( \Lambda \times \Gamma \).

**Definition 2.4.** Let \( S := \{S_{i,j}\} \) be the universal model of the polyball \( B_n \). A set \( \Lambda \subset F^+_1 \times \cdots \times F^+_n \) is called orthogonal if the isometries \( S_\alpha, \alpha \in \Lambda \), have orthogonal ranges in \( F^2(H_{n_i}) \otimes \cdots \otimes F^2(H_{n_k}) \), i.e. \( S_\beta S_\alpha = 0 \) for any \( \alpha, \beta \in \Lambda \) with \( \alpha \neq \beta \).

Here are a few properties for the orthogonal sets. Since the proof is straightforward, we shall omit it.

**Proposition 2.5.** Let \( \Lambda \) be a subset of \( F^+_1 \times \cdots \times F^+_n \).

(i) \( \Lambda \) is an orthogonal set if and only if, for any \( \alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k) \in \Lambda \) with \( \alpha \neq \beta \), there is \( i \in \{1, \ldots, k\} \) such that \( \alpha_i \neq g_0 \) and \( \beta_i \) is not a left divisor of \( \alpha_i \), i.e. there is no \( \gamma_i \in F_{n_i} \) such that \( \alpha_i = \beta_1 \gamma_i \).

(ii) If \( \Lambda \) is an orthogonal set, then \( \Lambda \) is left minimal.

(iii) If \( \Lambda \) is an orthogonal set and \( \Lambda = \Lambda \), then \( \Lambda \) is minimal.

**Example 2.6.** The following statements hold.

(i) If \( p \in \mathbb{N} \), then \( \Lambda_p := \{\alpha \in F^+_n : |\alpha| = p\} \) is an orthogonal set in \( F^+_n \).

(ii) Let \( \Lambda \subset F^+_n \) with \( \Lambda \neq \{g_0\} \). Then \( \Lambda \) is orthogonal if and only if it is left minimal.
(iii) If \( p_1, \ldots, p_k \in \mathbb{N} \cup \{0\} \) and \( \Lambda_{p_i} := \{ \alpha \in \mathbb{F}_{n_i}^+ : |\alpha| = p_i \} \), then \( \Lambda_{p_1} \times \cdots \times \Lambda_{p_k} \) is an orthogonal set in \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) if at least one \( p_i \geq 1 \).

(iv) Any subset of an orthogonal set is orthogonal.

(v) If \( i \in \{1, \ldots, k - 1\} \) and \( \Lambda \subset \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_i}^+ \) and \( \Gamma \subset \mathbb{F}_{n_{i+1}}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) are orthogonal sets, then so if \( \Lambda \times \Gamma \).

We remark that there are minimal and orthogonal sets in \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) which are infinite. Indeed, it is easy to see that

\[
\Lambda = \{ (g_2g_1, g_2g_1^2, g_2g_1^3, \ldots) \} \subset \mathbb{F}_n^+
\]
is such a set in \( \mathbb{F}_n^+ \). Taking cartesian products of this kind of sets, we obtain minimal and orthogonal sets in \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \), which are infinite.

In what follows we obtain a Weiner type inequality for the coefficients of bounded free holomorphic functions on the regular polyball. Without loss of generality, we assume throughout this paper that \( \mathcal{H} \) and \( \mathcal{K} \) are separable Hilbert spaces.

**Proposition 2.7.** Let \( \Lambda \) be a right minimal subset of \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) that does not contain the neutral element \( g_0 = (g_0^1, \ldots, g_0^k) \) and let \( F : B_n(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H}) \) be a bounded free holomorphic function with representation

\[
F(X) = \sum_{\alpha \in \mathbb{F}_{n_1}^+ \cdots \times \mathbb{F}_{n_k}^+} A_{(\alpha)} \otimes X_\alpha
\]
such that \( \|F\|_\infty \leq 1 \) and \( F(0) \) is a scalar operator, i.e. \( F(0) = a_0 I \) for some \( a_0 \in \mathbb{C} \). Then

\[
\left\| \sum_{\alpha \in \Lambda} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \leq 1 - |a_0|^2.
\]

**Proof.** Let \( \{e_\alpha\}_{\alpha \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \) be the orthonormal basis of the Hilbert space \( F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \) and let \( \mathcal{E} \) be the closed linear span of the vectors \( 1, e_\beta \), where \( \beta \in \Lambda \). If \( \widehat{F} := \text{SOT}-\lim_{r \to 1} F(rS) \) is the boundary function of \( F \) with respect to \( S \), we have

\[
\langle P_{\mathcal{K} \otimes \mathcal{E}} \widehat{F} |_{\mathcal{K} \otimes \mathcal{E}} (x \otimes 1), y \otimes 1 \rangle = \lim_{r \to 1} \langle F(rS) x \otimes 1, y \otimes 1 \rangle = a_0 \langle x, y \rangle,
\]

\[
\langle P_{\mathcal{K} \otimes \mathcal{E}} \widehat{F} |_{\mathcal{K} \otimes \mathcal{E}} (x \otimes e_\beta), y \otimes 1 \rangle = \lim_{r \to 1} \langle F(rS) (x \otimes e_\beta), y \otimes 1 \rangle = 0,
\]

and

\[
\langle P_{\mathcal{K} \otimes \mathcal{E}} \widehat{F} |_{\mathcal{K} \otimes \mathcal{E}} (x \otimes 1), y \otimes e_\beta \rangle = \lim_{r \to 1} \langle F(rS) (x \otimes 1), y \otimes e_\beta \rangle = \langle A(\beta) x, y \rangle,
\]

for any \( \beta \in \Lambda \) and \( x, y \in \mathcal{K} \). Since \( \Lambda \) is a right minimal subset of \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \), Proposition 2.7 part (ii), implies

\[
\langle P_{\mathcal{K} \otimes \mathcal{E}} \widehat{F} |_{\mathcal{K} \otimes \mathcal{E}} (x \otimes e_\beta), y \otimes e_\gamma \rangle = \lim_{r \to 1} \langle F(rS) (x \otimes e_\beta), y \otimes e_\gamma \rangle = \sum_{\alpha \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \langle A(\alpha) x, y \rangle \langle e_\alpha \otimes e_\beta, e_\gamma \rangle = a_0 \langle x, y \rangle \delta_{\beta, \gamma}
\]

for any \( \beta, \gamma \in \Lambda \) and \( x, y \in \mathcal{K} \). Consequently, the matrix representation of the contraction \( P_{\mathcal{K} \otimes \mathcal{E}} \widehat{F} |_{\mathcal{K} \otimes \mathcal{E}} \) with respect to the decomposition

\[
\mathcal{K} \otimes \mathcal{E} = (\mathcal{K} \otimes 1) \oplus \bigoplus_{\beta \in \Lambda} (\mathcal{K} \otimes e_\beta)
\]
is

\[
\begin{pmatrix}
\begin{bmatrix}
a_0 I_K & 0 & \cdots & 0 \\
A(\alpha) & a_0 I_K & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha \in \Lambda & 0 & \cdots & a_0 I_K
\end{bmatrix}
\end{pmatrix}
\]
According to Lemma 3.4 from [24], if $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, $B$ is a bounded linear operator from $\mathcal{M}$ to $\mathcal{G}$, and $\lambda \in \mathbb{C}$, then 
$$
\begin{pmatrix}
\lambda I & 0 \\
B & \lambda I_g
\end{pmatrix}
$$
is a contraction if and only if $\|B\| \leq 1 - |\lambda|^2$. Applying this result to our setting, we deduce that

$$
\left\| \sum_{\alpha \in \Delta} A^*_\alpha A_\alpha \right\|^{1/2} \leq 1 - |a_0|^2.
$$

The proof is complete.

In what follows, we need some notation. Let $\rho = (\rho_1, \ldots, \rho_k)$, $\rho_i = (\rho_{i,1}, \ldots, \rho_{i,n_i})$, with $\rho_{i,j} \geq 0$. We also use the abbreviation $\rho = (\rho_{ij})$. We denote $\rho_{i,a_i} := \rho_{i,j_1} \cdots \rho_{i,j_p}$ if $a_i = g_1^{j_1} \cdots g_p^{j_p} \in \mathbb{F}_n^+$, and $\rho_{i,g_n} := 1$, and $\rho_\alpha := \rho_{1,a_1} \cdots \rho_{k,a_k}$ if $\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+$. If $X = \{X_{i,j}\} \in B(\mathcal{H})^{n_1+\cdots+n_k}$, set $\rho X := (\rho_{i,j}X_{i,j})$. Moreover, if $\gamma := (\gamma_1, \ldots, \gamma_k)$, $\gamma_i \geq 0$, we set $\gamma X := (\gamma_1X_{1}, \ldots, \gamma_kX_k)$, where $\gamma_iX_i := (\gamma_iX_{i,1}, \ldots, \gamma_iX_{i,j})$. When $r \geq 0$, the notation $r X = (rX_{i,j})$ is clear.

An exhaustion of $\mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+$ by right minimal sets is a countable collection $\{\Sigma_m\}_{m=0}^\infty$ of non-empty, disjoint, right minimal sets $\Sigma_m$ with $\Sigma_0 = \{g\}$ and the property that $\cup_{m=0}^\infty \Sigma_m = \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+$.

**Theorem 2.8.** Let $F : B_\rho(\mathcal{H}) \rightarrow B(\mathcal{K}) \cap \min B(\mathcal{H})$ be a bounded free holomorphic function with representation

$$
F(X) = \sum_{\alpha \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+} A_\alpha \otimes X_\alpha
$$
such that $\|F\| \leq 1$ and $F(0)$ is a scalar operator. If $\{\Sigma_m\}_{m=0}^\infty$ is an exhaustion of $\mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+$ by right minimal sets and $\rho = (\rho_{ij}) \in [0,1]^{n_1+\cdots+n_k}$ is such that

$$
\sum_{m=1}^\infty \left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha^2 S_\alpha S_\alpha^* \right\|^{1/2} \leq \frac{1}{2},
$$

then

$$
\sup_{m=0}^\infty \left\| \sum_{\alpha \in \Sigma_m} A_\alpha \otimes X_\alpha \right\| \leq 1.
$$

**Proof.** Since $\Sigma_m$ is a right minimal set, we can use Proposition 2.7 and deduce that

$$
\left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha A_\alpha \otimes S_\alpha \right\| \leq \left\| \rho_\alpha S_\alpha \otimes I : \alpha \in \Sigma_m \right\| \left\| I \otimes A_\alpha \right\| \leq \left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha^2 S_\alpha S_\alpha^* \right\|^{1/2} \left\| \sum_{\alpha \in \Sigma_m} A^*_{\alpha} A_\alpha \right\|^{1/2} \leq (1 - |a_0|^2) \left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha^2 S_\alpha S_\alpha^* \right\|^{1/2}.
$$

Now, using the noncommutative von Neumann inequality for regular polyballs (see [21], [22]), we deduce that

$$
\sum_{m=0}^\infty \sup_{X \in \rho B_\rho(\mathcal{H})} \left\| \sum_{\alpha \in \Sigma_m} A_\alpha \otimes X_\alpha \right\| \leq \sum_{m=0}^\infty \left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha A_\alpha \otimes S_\alpha \right\| \leq |a_0| + (1 - |a_0|^2) \sum_{m=1}^\infty \left\| \sum_{\alpha \in \Sigma_m} \rho_\alpha^2 S_\alpha S_\alpha^* \right\|^{1/2} \leq |a_0| + \frac{1 - |a_0|^2}{2} \leq 1.
$$

The proof is complete.

□
According to Example 2.14 part (iii), if \( p_1, \ldots, p_k \in \mathbb{N} \cup \{0\} \) and \( \Lambda_{p_i} := \{ \alpha \in \mathbb{F}_n^+ \mid |\alpha| = p_i \} \), then \( \Lambda_{p} := \Lambda_{p_1} \times \cdots \times \Lambda_{p_k} \) is an orthogonal set in \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \) if at least one \( p_i \geq 1 \). On the other hand, due to Example 2.14 part (ii), the set \( \Lambda_{p} \) is minimal. Therefore, \( \{ \Lambda_{p} \}_{p \in \mathbb{Z}^+} \) is an exhaustion of \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \) by minimal and orthogonal sets. The following definition concerns the polyball \( B_n \), the Hardy space \( H^\infty(B_n) \) of all bounded free holomorphic functions on the polyball with scalar coefficients, and the representation of its elements \( F \) in terms of multi-homogeneous polynomials, i.e.

\[
F(X) = \sum_{\alpha \in \Lambda_p} \left( \sum_{\alpha \in \Lambda_p} a_\alpha X_\alpha \right), \quad X \in B_n(H),
\]

for any Hilbert space \( H \), where the convergence is in the operator norm topology. In this case, the Bohr radius for the polyball \( B_n \) is denoted by \( K_{mh}(B_n) \) and is the largest \( r \geq 0 \) such that

\[
\sum_{p \in \mathbb{Z}^+} \left\| \sum_{\alpha \in \Lambda_p} a_\alpha X_\alpha \right\| \leq \|F\|_\infty, \quad X \in rB_n(H)^-,
\]

for any \( F \in H^\infty(B_n) \). Due to the noncommutative von Neumann inequality, the latter inequality is equivalent to \( \sum_{p \in \mathbb{Z}^+} \left\| \sum_{\alpha \in \Lambda_p} a_\alpha r^{\alpha_0} S_\alpha \right\| \leq \|F\|_\infty \), where \( |\alpha| := |\alpha_1| + \cdots + |\alpha_k| \), if \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is in \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \). Now, we obtain the first estimations for the Bohr radius \( K_{mh}(B_n) \).

**Theorem 2.9.** Let \( F : B_n(H) \to B(K) \otimes_{min} B(H) \) be a bounded free holomorphic function with representation

\[
F(X) = \sum_{\alpha \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+} A_\alpha \otimes X_\alpha
\]

such that \( \|F\|_\infty \leq 1 \) and \( F(0) \) is a scalar operator.

(i) If \( \rho := (\rho_{i,j}) \) with \( \rho_{i,j} = r_i \in [0,1) \) for \( j \in \{1, \ldots, n_i\} \), then

\[
\sum_{p \in \mathbb{Z}^+} \left\| \sum_{\alpha \in \Lambda_p} A_\alpha \otimes X_\alpha \right\| \leq |a_0| + (1 - |a_0|^2) \left[ \prod_{i=1}^k (1 - r_i)^{-1} - 1 \right]
\]

for any \( X \in \rho B_n(H)^- \). Consequently, if \( \prod_{i=1}^k (1 - r_i) \geq \frac{2}{3} \), then

\[
\sum_{p \in \mathbb{Z}^+} \left\| \sum_{\alpha \in \Lambda_p} A_\alpha \otimes X_\alpha \right\| \leq 1, \quad X \in \rho B_n(H)^-.
\]

(ii) The Bohr radius \( K_{mh}(B_n) \) satisfies the inequalities

\[
1 - \left( \frac{2}{3} \right)^{1/k} \leq K_{mh}(B_n) \leq \frac{1}{3}.
\]

**Proof.** Let \( \{\Sigma_m\}_{m=0}^\infty \) be an exhaustion of \( \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+ \) by minimal and orthogonal sets and let \( \rho = (\rho_{i,j}) \in [0,1]^{n_1+\cdots+n_k} \) be such that \( \sum_{m=1}^\infty \sup_{\alpha \in \Sigma_m} \rho_\alpha \leq \frac{1}{2} \). Since \( \Sigma_m \) is an orthogonal set, the set \( \{S_\alpha\}_{\alpha \in \Sigma_m} \) consists of isometries with orthogonal ranges. Consequently, we have

\[
\sum_{\alpha \in \Sigma_m} \rho_\alpha S_\alpha S_\alpha^* = \sup_{\alpha \in \Sigma_m} \rho_\alpha^2.
\]
As in the proof of Theorem 2.8, in the particular case when the exhaustion \{\Sigma_m\} coincides with \{\Lambda_p\}_{p \in \mathbb{Z}^+} and \rho_{i,j} = r_1 \in [0, 1), we deduce that

\[
\sum_{p \in \mathbb{Z}^+_+} \left\| \sum_{\alpha \in \Lambda_p} A(\alpha) \otimes X_\alpha \right\| \leq |a_0| + (1 - |a_0|^2) \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^+_+} r_1^{p_1} \cdots r_k^{p_k}
= |a_0| + (1 - |a_0|^2) \left[ \prod_{i=1}^k (1 - r_i)^{-1} - 1 \right] .
\]

Consequently, if \(\prod_{i=1}^k (1 - r_i) \geq \frac{2}{3}\), then

\[
\sum_{p \in \mathbb{Z}^+_+} \left\| \sum_{\alpha \in \Lambda_p} A(\alpha) \otimes X_\alpha \right\| \leq 1, \quad X \in \rho B_n(\mathcal{H})^{-}.
\]

To prove part (ii), take \(r_1 = \cdots = r_k = r\) with the property that \(\prod_{i=1}^k (1 - r_i) = (1 - r)^k \geq \frac{2}{3}\). Applying part (i) of the theorem to \(F \in H^\infty(B_n)\), we deduce that \(1 - (\frac{2}{3})^{1/k} \leq K_{nm}(B_n)\). The second inequality in (ii) is due to the classical result and the fact, due to Proposition 1.1, that each function in \(H^\infty(D)\) can be seen as a function in \(H^\infty(B_n)\). Indeed, we have \(K_{nm}(B_n) \leq K_{nm}(D) = \frac{4}{3}\). The proof is complete. \(\square\)

The next result for the polydisc is a consequence of Proposition 1.1 and Theorem 2.9.

**Corollary 2.10.** Let

\[
f(z) = \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^+_+} A_{p_1, \ldots, p_k} z_1^{p_1} \cdots z_k^{p_k}, \quad z = (z_1, \ldots, z_k) \in \mathbb{D}^k,
\]

be an operator-valued analytic function on the polydisc \(\mathbb{D}^k\) such that \(\|f\|_\infty \leq 1\) and \(f(0) = a_0 I, a_0 \in \mathbb{C}\). Then

\[
\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^+_+} \|A_{p_1, \ldots, p_k}\| r_1^{p_1} \cdots r_k^{p_k} \leq 1
\]

for any \(r_i \in [0, 1)\) with \(\prod_{i=1}^k (1 - r_i) \geq \frac{2}{3}\).

Here is an extension of the classical result of Bohr to operator-valued bounded analytic function in the disc.

**Corollary 2.11.** Let \(f(z) = \sum_{n=0}^\infty A_n z^n, A_n \in B(\mathcal{H})\), be an operator-valued bounded analytic function in \(\mathbb{D}\) such that \(A_0 = a_0 I, a_0 \in \mathbb{C}\). Then

\[
\sum_{n=0}^\infty \|A_n\| r^n \leq \|f\|_\infty
\]

for any \(r \in [0, \frac{1}{3}]\), and \(\frac{1}{3}\) is the best possible constant. Moreover, the inequality is strict unless \(f\) is a constant.

**Proof.** The first part of the corollary is due to Corollary 2.10 and the classical result. To prove the second part, assume that \(\|f\|_\infty = 1\). As in the proof of Theorem 2.9, we have

\[
\sum_{n=0}^\infty \|A_n\| r^n \leq |a_0| + \frac{1 - |a_0|^2}{2} \leq 1 = \|f\|_\infty.
\]

If the equality holds, then \(|a_0| = 1\). Due to Proposition 2.7, we have \(\|A_n A_n\|^{1/2} \leq 1 - |a_0|^2\) if \(n \geq 1\). Consequently \(A_n = 0\) for \(n \geq 1\). This completes the proof. \(\square\)
Proposition 2.12. Let \( F : B_n(\mathcal{H}) \to B(K) \otimes_{\min} B(\mathcal{H}) \) be a bounded free holomorphic function with representation
\[
F(X) = \sum_{p \in \mathbb{Z}_+^k} \sum_{\alpha \in \Lambda_p} A_{(\alpha)} \otimes X_{\alpha}
\]
such that \( \|F\|_\infty \leq 1 \) and \( F(0) \) is a scalar operator. If \( \rho := (\rho_{i,j}) \) with \( \rho_{i,j} = r_i \) for \( j \in \{1, \ldots, n_i\} \), then
\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho B_n(\mathcal{H})^n} \left\| \sum_{\alpha \in \Lambda_p} A_{(\alpha)} \otimes X_{\alpha} \right\| \leq C(\rho) \|F\|_\infty,
\]
where
\[
C(\rho) := \begin{cases} 
1 & \text{if } c \leq \frac{1}{2} \\
2(1 + \frac{1}{2}) & \text{if } c > \frac{1}{2}
\end{cases}
\]
and \( c := \prod_{i=1}^k (1 - r_i)^{-1} - 1 \).

Proof. Without loss of generality, we may assume that \( \|F\|_\infty \leq 1 \) and, consequently \( |a_0| \leq 1 \). Note that \( c := \prod_{i=1}^k (1 - r_i)^{-1} - 1 \geq 0 \). It is easy to see that, if \( 0 \leq c \leq \frac{1}{2} \), then \( \sup \{x + (1 - x^2) c : 0 \leq x \leq 1\} \leq 1 \). On the other hand, if \( c > \frac{1}{2} \), then \( \sup \{x + (1 - x^2) c : 0 \leq x \leq 1\} = c + \frac{1}{2c} \). Consequently, using Theorem 2.13 part (i), we deduce that
\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho B_n(\mathcal{H})^n} \left\| \sum_{\alpha \in \Lambda_p} A_{(\alpha)} \otimes X_{\alpha} \right\| \leq |a_0| + (1 - |a_0|^2)c \leq C(\rho),
\]
which completes the proof. \( \square \)

Let \( F \in H^\infty(B_n) \) have the representation \( F(X) := \sum_{p \in \mathbb{Z}_+^k} \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha} \) and let
\[
D(F,r) := \sum_{p \in \mathbb{Z}_+^k} r^{|p|} \left\| \sum_{\alpha \in \Lambda_p} a_{(\alpha)} S_{\alpha} \right\|
\]
be the associated majorant series, where \( S = \{S_{ij}\} \) is the universal model of the polyball. Define
\[
d_{B_n}(r) := \sup \frac{D(F,r)}{\|F\|_\infty}, \quad r \in [0,1),
\]
where the supremum is taken over all \( F \in H^\infty(B_n) \) with \( F \) not identically 0. For the open unit disk, Bombieri and Bourgain (see [6]) proved that \( d_0(r) \sim \frac{1}{\sqrt{1-r^2}} \) as \( r \to 1 \). In what follows we extend their result to the regular polyball \( B_n \). In particular, the following result holds for the scalar polydisk \( \mathbb{D}^k \).

Theorem 2.13. Let
\[
F(X) = \sum_{p \in \mathbb{Z}_+^k} \left( \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha} \right), \quad X \in B_n(\mathcal{H}),
\]
be any bounded free holomorphic function on the polyball. If \( \rho := (\rho_{i,j}) \) with \( \rho_{i,j} = r_i \) for \( j \in \{1, \ldots, n_i\} \), then
\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho B_n(\mathcal{H})^n} \left\| \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_{\alpha} \right\| \leq K(\rho) \|F\|_\infty, \quad F \in H^\infty(B_n),
\]
where
\[
K(\rho) := \min \left\{ C(\rho), \prod_{i=1}^k (1 - r_i^2)^{-1/2} \right\}
\]
and \( C(\rho) \) is defined in Proposition 2.12. Moreover, \( d_{B_n}(r) \) has the following properties:
(i) \(1 \leq d_{B_n}(r) \leq \min \left\{ c(r), \left( \frac{1}{\sqrt{1-r^2}} \right)^k \right\} \), where

\[ c(r) := \begin{cases} 
1 & \text{if } 0 \leq r \leq 1 - \left( \frac{2}{3} \right)^{1/k} \\
\frac{1}{c} & \text{if } 1 - \left( \frac{2}{3} \right)^{1/k} < r < 1 
\end{cases} \]

and \(c := (1-r)^{-k} - 1\).

(ii) \(d_{B_n}(r)\) behaves asymptotically as \(\left( \frac{1}{\sqrt{1-r}} \right)^k\) if \(r \to 1\), i.e.

\[ \lim_{r \to 1} \frac{d_{B_n}(r)}{\left( \frac{1}{\sqrt{1-r}} \right)^k} = 1. \]

Proof. Let \(p := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k\) be such that at least one \(p_i \neq 0\). Since \(A_p\) is an orthogonal set, the set \(\{S_\alpha\}_{\alpha \in A_p}\) consists of isometries with orthogonal ranges. Consequently, we have

\[ \sup_{\alpha \in A_p} \rho_\alpha^2 S_\alpha S_\alpha^* = \sup_{\alpha \in A_p} \rho_\alpha^2 = r_1^{2p_1} \cdots r_k^{2p_k}, \]

where \(S = \{S_{i,j}\}\) is the universal model of the polyball. Due to the noncommutative von Neumann inequality and using the Cauchy Schwarz inequality, we obtain

\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho B_n(\mathcal{H})^*} \left\| \sum_{\alpha \in A_p} a(\alpha) X_{\alpha} \right\| \leq \sum_{p \in \mathbb{Z}_+^k} \left\| \sum_{\alpha \in A_p} a(\alpha) r_1^{2|\alpha_1|} \cdots r_k^{2|\alpha_k|} S_\alpha S_\alpha^* \right\|^{1/2} \left( \sum_{\alpha \in A_p} |a(\alpha)|^2 \right)^{1/2} \\
\leq \left( \sum_{p \in \mathbb{Z}_+^k} \left\| \sum_{\alpha \in A_p} r_1^{2|\alpha_1|} \cdots r_k^{2|\alpha_k|} S_\alpha S_\alpha^* \right\| \right)^{1/2} \left( \sum_{\alpha \in A_p} \sum_{\beta \in A_p} |a(\alpha)|^2 \right)^{1/2} \\
\leq \left( \sum_{p \in \mathbb{Z}_+^k} r_1^{2p_1} \cdots r_k^{2p_k} \right)^{1/2} \|F\|_2 \\
\leq \prod_{i=1}^k (1 - r_i^2)^{-1/2} \|F\|_\infty.
\]

Using now Proposition 2.12 we deduce the inequality in the theorem.

Now, we prove the second part of the theorem. First, note that \(d_{B_n}(r) \geq 1\) for any \(r \in [0, 1)\). Part (i) is due to the first part of the theorem when one takes \(r_1 = \cdots = r_k = r\). Note that if \(0 \leq r \leq 1 - \left( \frac{2}{3} \right)^{1/k}\), then \(d_{B_n}(r) = 1\). On the other hand, it is easy to see that if \(r\) is sufficiently close to 1, then we have

\[ c(r) > \left( \frac{1}{\sqrt{1-r^2}} \right)^k. \]

To prove part (ii), note that due to the result of Bombieri and Bourgain, for any \(\epsilon > 0\), there is \(\delta > 0\) such that

\[ \left| \frac{d_p(r)}{\left( \frac{1}{\sqrt{1-r^2}} \right)^k} - 1 \right| < \epsilon \]

for any \(r \in [0, 1)\) such that \(|r-1| < \delta\). Fix such an \(r\). For each \(i \in \{1, \ldots, k\}\), there exists \(f_i \in H^\infty(\mathbb{D})\) such that

\[ \frac{D(f_i, r)}{\|f_i\|_\infty} > (1 - \epsilon) \frac{1}{\sqrt{1-r^2}}. \]
Hence, we deduce that

\[
\prod_{i=1}^k \frac{D(f_i, r)}{\|f_i\|_\infty} > (1 - \epsilon)^k \left( \frac{1}{\sqrt{1 - r^2}} \right)^k.
\]

Note that the function \( f(z_1, \ldots, z_k) := f_1(z_1) \cdots f_k(z_k) \) is in the Hardy space of the polydisk \( H^\infty(\mathbb{D}^k) \).

Due to Proposition \([1, 3]\) the Banach algebra \( H^\infty(\mathbb{D}^k) \) is isometrically isomorphic to \( H^\infty(\mathbb{D}^k) \), which is isometrically embedded in \( H^\infty(\mathbb{B}_n) \). Let \( G \in H^\infty(\mathbb{B}_n) \) be the corresponding element to \( f \), via this embedding, and note that \( G(rS) := f_1(rS_1, 1) \cdots f_k(rS_{k, 1}) \). A careful examination reveals that

\[
D(G, r) = D(f_1, r) \cdots D(f_k, r) \quad \text{and} \quad \|G\|_\infty = \|f_1\|_\infty \cdots \|f_k\|_\infty.
\]

Hence, using inequality (2.1) and part (i) of the theorem, we obtain

\[
\left( \frac{1}{\sqrt{1 - r^2}} \right)^{\frac{k}{2}} \geq d_{\mathbb{B}_n}(r) \geq \frac{D(G, r)}{\|G\|_\infty} = \prod_{i=1}^k \frac{D(f_i, r)}{\|f_i\|_\infty} > (1 - \epsilon)^k \left( \frac{1}{\sqrt{1 - r^2}} \right)^k
\]

for any \( r \in [0, 1) \) such that \( |r - 1| < \delta \). This completes the proof. \( \square \)

Now, we consider the case when \( F(0) = 0 \).

**Corollary 2.14.** Let \( F : \mathbb{B}_n(\mathcal{H}) \to B(\mathcal{H}) \) be a bounded free holomorphic function with \( F(0) = 0 \) and representation

\[
F(X) = \sum_{p \in \mathbb{Z}_+^k} \left( \sum_{\alpha \in \Lambda_p} a_{\alpha} X_{\alpha} \right), \quad X \in \mathbb{B}_n(\mathcal{H}),
\]

and let \( \{\Sigma_m\}_{m=0}^\infty \) be an exhaustion of \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) by minimal and orthogonal sets.

(i) If \( \rho = (\rho_{i, j}) \in [0, 1)^{n_1 + \cdots + n_k} \), then

\[
\sup_{m=0}^{\infty} \left\| \sum_{\alpha \in \Sigma_m} a_{\alpha} X_{\alpha} \right\| \leq \left( \sum_{m=1}^{\infty} \sup_{\alpha \in \Sigma_m} \rho_{\alpha} \right)^{1/2} \|F\|_\infty.
\]

(ii) If \( \rho := (\rho_{i, j}) \) with \( \rho_{i, j} = r_i \in [0, 1) \) for \( j \in \{1, \ldots, n_i\} \), then

\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho \mathbb{B}_n(\mathcal{H})} \left\| \sum_{\alpha \in \Lambda_p} a_{\alpha} X_{\alpha} \right\| \leq K(\rho) \|F\|_\infty
\]

where

\[
K(\rho) := \min \left\{ \left[ \prod_{i=1}^k (1 - r_i)^{-1} - 1 \right], \left[ \prod_{i=1}^k (1 - r_i^2)^{-1} - 1 \right] \right\}^{1/2}.
\]

(iii) If \( \rho := (\rho_{i, j}) \) with \( \rho_{i, j} = r_i \in [0, 1) \) for \( j \in \{1, \ldots, n_i\} \) and \( \prod_{i=1}^k (1 - r_i^2) \geq \frac{1}{2} \), then

\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho \mathbb{B}_n(\mathcal{H})} \left\| \sum_{\alpha \in \Lambda_p} a_{\alpha} X_{\alpha} \right\| \leq \|F\|_\infty.
\]

In particular, the inequality holds if \( r_1 = \cdots = r_k \leq \sqrt{1 - \frac{1}{2^{1/k}}} \).

**Proof.** The proof of (i) and (ii) is very similar to that of Theorem 2.13 but we have to take into account that \( F(0) = 0 \). Part (iii) follows from part (ii). \( \square \)

If we replace the Hardy space \( H^\infty(\mathbb{B}_n) \) with the subspace

\[
H^\infty_0(\mathbb{B}_n) := \{ f \in H^\infty(\mathbb{B}_n) : f(0) = 0 \},
\]

the corresponding Bohr radius is denoted by \( K^0_{\text{nh}}(\mathbb{B}_n) \).
Corollary 2.15. The Bohr radius $K_{mh}^0(B_n)$ satisfies the inequalities
\[ \sqrt{1 - \left(\frac{1}{2}\right)^{1/k}} \leq K_{mh}^0(B_n) \leq \frac{1}{\sqrt{2}}. \]

Proof. The left hand inequality is due to part (iii) of Corollary 2.14. On the other hand, Proposition 1.1 shows that $H_0^\infty(D)$ is isometrically embedded in $H_0^\infty(B_n)$ via the map
\[ \sum_{n=1}^{\infty} a_n z^n \mapsto \sum_{n=1}^{\infty} a_n X^n_{1,1}. \]
Consequently, $K_{mh}^0(B_n) \leq K_{mh}^0(D)$. Since $K_{mh}^0(D) = \frac{1}{\sqrt{2}}$ (e.g. [17]), the result follows. \qed

Now, we consider the general case of arbitrary free holomorphic functions on polyballs, with operator coefficients. Our Bohr type result in this setting is the following.

Theorem 2.16. Let $F : B_n(H) \rightarrow B(K) \otimes_{\min} B(H)$ be a free holomorphic function with $\|F\|_\infty \leq 1$ and representation
\[ F(X) = \sum_{\alpha \in \mathbb{F}_+^{n_1} \times \cdots \times \mathbb{F}_+^{n_k}} A(\alpha) \otimes X_\alpha. \]
If $\Sigma_{m=0}^\infty$ is an exhaustion of $\mathbb{F}_+^{n_1} \times \cdots \times \mathbb{F}_+^{n_k}$ by right minimal sets, then
\[ \sum_{\alpha \in \Sigma_m} A(\alpha)^* A(\alpha) \leq I - A_0^* A_0 \]
for any $m \in \mathbb{N}$. If, in addition, each $\Sigma_m$ is an orthogonal set, then the following statements hold.

(i) For any $\rho = (\rho_{i,j}) \in [0,1]^{n_1+\cdots+n_k}$,
\[ \sup_{X \in \rho B_n(H)} \left\| \sum_{\alpha \in \Sigma_m} A(\alpha) \otimes X_\alpha \right\| \leq \|(I - A_0^* A_0)^{1/2}\| \sup_{\alpha \in \Sigma_m} \rho_\alpha. \]

(ii) If $\rho := (\rho_{i,j})$ with $\rho_{i,j} = r_i \in [0,1]$ for $j \in \{1, \ldots, n_i\}$, then
\[ \sum_{p \in \mathbb{Z}_+^k} \sup_{X \in \rho B_n(H)} \left\| \sum_{\alpha \in A_p} A(\alpha) \otimes X_\alpha \right\| \leq \|A_0\| + \|(I - A_0^* A_0)^{1/2}\| \left[ \prod_{i=1}^k (1 - r_i)^{-1} - 1 \right]. \]

Proof. Let $E$ be the closed linear span of the vectors $1, e_\beta$, where $\beta \in \Sigma_m$. As in the proof of Proposition 2.7, taking into account that $\Sigma_m$ is a right minimal subset of $\mathbb{F}_+^{n_1} \times \cdots \times \mathbb{F}_+^{n_k}$, one can show that the operator matrix of $P_{K \otimes E} F(rS)_{K \otimes E}$, $r \in [0,1)$, with respect to the decomposition
\[ K \otimes E = (K \otimes 1) \oplus \bigoplus_{\beta \in \Sigma_m} (K \otimes e_\beta) \]
is
\[ \begin{pmatrix} A_0 & [0 \cdots 0] \\ r^{[\alpha]} A(\alpha) & [A_0 \cdots 0] \\ \vdots & \vdots & \ddots & \vdots \\ [0 \cdots 0] & 0 & \cdots & A_0 \end{pmatrix}. \]
Since $P_{K \otimes E} F(rS)_{K \otimes E}$ is a contraction, so is the operator matrix
\[ \begin{pmatrix} A_0 \\ r^{[\alpha]} A(\alpha) \\ \vdots \\ [0 \cdots 0] \end{pmatrix}. \]
Hence, it is clear that
\[ \sum_{\alpha \in \Sigma_m} A(\alpha)^* A(\alpha) \leq I - A_0^* A_0. \]
Now we assume, in addition, that each $\Sigma_m$ is an orthogonal set, i.e. \{ $S_\alpha$ \}_{\alpha \in \Sigma_m} \) is a sequence of isometries with orthogonal ranges. As in the proof of Theorem \[\text{A0}\] we can use part (i) of the theorem to deduce the inequality of part (ii). \[\square\]

3. Bohr inequalities for free holomorphic functions with $F(0) \geq 0$ and $\Re f \leq I$

In this section, we obtain an analogue of Landau’s inequality \[\text{[15]}\] for bounded free holomorphic functions with operator coefficients on the polyball and use it to obtain Bohr inequalities for free holomorphic functions on polyballs with operator coefficients such that $F(0) \geq 0$ and $\Re F(X) \leq I$. The results play an important role in the next sections.

Theorem 3.1. Let $\Lambda$ be a right minimal subset of $\mathbb{F}^*_k \times \cdots \times \mathbb{F}^*_n$ that does not contain the neutral element $(g_1^0, \ldots, g_n^0)$ and let $F : \mathcal{B}_n(\mathcal{H}) \rightarrow B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$ be a free holomorphic function with representation

$$F(X) = \sum_{\alpha \in \mathbb{F}^*_k \times \cdots \times \mathbb{F}^*_n} A_{(\alpha)} \otimes X_{\alpha}$$

such that $F(0) \geq 0$ and $\Re F(X) \leq I$ for any $X \in \mathcal{B}_n(\mathcal{H})$. Then the operator matrix

$$P_\Lambda := \begin{bmatrix} (2I - A_0) & [A_{(\alpha)}^* : \alpha \in \Lambda] \\ \vdots & \ddots & \ddots & \vdots \\ \alpha \in \Lambda \\ 0 & \cdots & 2(I - A_0) \end{bmatrix}$$

is positive and

$$\sum_{\alpha \in \Lambda} A_{(\alpha)}^* A_{(\alpha)} \leq 4\|I - A_0\| (I - A_0).$$

If, in addition, $\Lambda$ is an orthogonal set, then

$$\sup_{X \in \mathcal{B}_n} \left\| \sum_{\alpha \in \Lambda} A_{(\alpha)} \otimes X_{\alpha} \right\| \leq \left\| \sum_{\alpha \in \Lambda} A_{(\alpha)} \otimes S_{\alpha} \right\| \leq 2\|I - A_0\|.$$

Proof. First, we recall that $F$ is a free holomorphic function on the polyball $\mathcal{B}_n$ if and only if the series

$$\sum_{q = 0}^{\infty} \sum_{\alpha \in \mathbb{F}^*_k \times \cdots \times \mathbb{F}^*_n} A_{(\alpha)} \otimes r^q S_{\alpha}$$

is convergent in the operator norm topology for any $r \in [0, 1)$. This shows that $F(rS)$ exists and it is in the polyball algebra $\mathcal{A}_n$. In particular, this implies for each $r \in [0, 1)$, the series $\sum_{\alpha \in \mathbb{F}^*_k \times \cdots \times \mathbb{F}^*_n} r^{|\alpha|} A_{(\alpha)}^* A_{(\alpha)}$ is convergent in the strong operator topology. Note that $\Re F(X) \leq I$ for any $X \in \mathcal{B}_n(\mathcal{H})$ if and only if

$$2I - F(rS) - F(rS)^* \geq 0$$

for any $r \in [0, 1)$. Indeed, using the fact that the noncommutative Berezin transform on the polyball $\mathcal{B}_n$ is a completely positive map and

$$2I - F(X) - F(X)^* = (id \otimes \mathcal{B}_n(\mathcal{H})) [2I - F(rS) - F(rS)^*],$$

where $r \in [0, 1)$ is such that $\mathcal{B}_n(\mathcal{H})$, the result follows. Note also that $F(0) = A_0 \otimes I \geq 0$, thus $A_0 \geq 0$.

Let $\mathcal{E}$ be the closed linear span of the vectors $1, e_\beta$, where $\beta \in \Lambda$. Setting $G(X) := 2I - F(X) - F(X)^*$ and taking into account that \{ $e_\alpha$ \}_{\alpha \in \mathbb{F}^*_k \times \cdots \times \mathbb{F}^*_n} \) is an orthonormal basis for the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$, we have

$$\langle P_{K \otimes \mathcal{E}} G(rS) | K \otimes \mathcal{E} (x \otimes 1), y \otimes 1 \rangle = \langle 2(I - A_0)x, y \rangle,$$

$$\langle P_{K \otimes \mathcal{E}} G(rS) | K \otimes \mathcal{E} (x \otimes e_\beta), y \otimes 1 \rangle = -r^{|eta|} \langle A_{(\beta)}^* x, y \rangle,$$
and

\[
\langle P_{\mathcal{K}} \otimes G(rS)|_{\mathcal{K} \otimes \mathcal{E}}(x \otimes 1), y \otimes e_{\beta}\rangle = -r^{\beta} \langle A_{(\beta)}x, y \rangle
\]

for any \( \beta \in \Lambda \), where \(|\beta| := |\beta_1| + \cdots + |\beta_k|\), if \( \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+ \). Since \( \Lambda \) is a right minimal subset of \( \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+ \), Proposition 2.2, part (ii), implies

\[
\langle P_{\mathcal{K}} \otimes G(rS)|_{\mathcal{K} \otimes \mathcal{E}}(x \otimes e_{\beta}), y \otimes e_{\gamma}\rangle = 2\delta_{\beta\gamma} \langle x, y \rangle - \sum_{\alpha \in \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+} r^{\alpha} \langle A_{(\alpha)}x, y \rangle \langle e_{\alpha} \otimes e_{\beta}, e_{\gamma}\rangle
\]

\[
- \sum_{\alpha \in \mathbb{F}_1^+ \times \cdots \times \mathbb{F}_k^+} r^{\alpha} \langle A_{(\alpha)}^*x, y \rangle \langle e_{\alpha} \otimes e_{\beta}, e_{\gamma}\rangle
\]

\[
= 2\delta_{\beta\gamma} \langle x, y \rangle - \delta_{\beta\gamma} \langle A_0x, y \rangle - \delta_{\beta\gamma} \langle A_0^*x, y \rangle
\]

\[
= \langle (2I - A_0)x, y \rangle \delta_{\beta\gamma}
\]

for any \( \beta, \gamma \in \Lambda \). Consequently, the matrix of \( G(rS) \) with respect to the decomposition

\[
\mathcal{K} \otimes \mathcal{E} = (\mathcal{K} \otimes 1) \oplus \bigoplus_{\beta \in \Lambda}(\mathcal{K} \otimes e_{\beta})
\]

is

\[
Q_{\Lambda}(r) := \begin{pmatrix} 2(I - A_0) & [-r^{\alpha}]_{\beta}A_{(\alpha)} : \alpha \in \Lambda \\ -r^{\alpha}\overline{A}_{(\alpha)} & 2(I - A_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \in \Lambda & 0 & \cdots & 2(I - A_0) \end{pmatrix}
\]

and it is positive for any \( r \in [0, 1) \). Consider the unitary operator \( U := \begin{pmatrix} I_{\mathcal{K}} & 0 \\ 0 & -I_{\mathcal{K}'} \end{pmatrix} \), where \( \mathcal{K}' := \bigoplus_{\beta \in \Lambda}(\mathcal{K} \otimes e_{\beta}) \), and note that \( P_{\Lambda}(r) := U^*Q_{\Lambda}(r)U \geq 0, \; r \in [0, 1) \). Let \( \{V_{(\alpha)}\}_{\alpha \in \Lambda} \) be a sequence of isometries with orthogonal ranges acting on a Hilbert space \( \mathcal{G} \). Applying Lemma 2.2 from [23] to our setting, we deduce that \( P_{\Lambda}(r) \geq 0 \) if and only if the operator matrix

(3.1)

\[
\left( \sum_{\alpha \in \Lambda} V_{(\alpha)}^* \otimes r^{\alpha}\overline{A}_{(\alpha)} + \sum_{\alpha \in \Lambda} V_{(\alpha)} \otimes r^{\alpha}A_{(\alpha)} \right)
\]

is positive. It is well-known (see eg. [10]) that if \( P, M \) are bounded operators on a Hilbert space \( \mathcal{H} \) such that

\[
\begin{pmatrix} P & M^* \\ M & P \end{pmatrix} \geq 0,
\]

then \( M^*M \leq \|P\|P \). Applying this result to the matrix (3.1), we deduce that

\[
\left( \sum_{\alpha \in \Lambda} V_{(\alpha)}^* \otimes r^{\alpha}\overline{A}_{(\alpha)} + \sum_{\alpha \in \Lambda} V_{(\alpha)} \otimes r^{\alpha}A_{(\alpha)} \right) \leq \|I_{\mathcal{G}} \otimes 2(I - A_0)\|I_{\mathcal{G}} \otimes 2(I - A_0),
\]

which, taking into account that \( V_{(\alpha)}^*V_{(\beta)} = \delta_{\alpha\beta}I \) for any \( \alpha, \beta \in \Lambda \), is equivalent to

\[
I_{\mathcal{G}} \otimes \sum_{\alpha \in \Lambda} r^{2\alpha}\overline{A}_{(\alpha)}A_{(\alpha)} \leq I_{\mathcal{G}} \otimes 4\|I - A_0\|(I - A_0).
\]

Hence, taking \( r \to 1 \) we deduce that

(3.2)

\[
\sum_{\alpha \in \Lambda} A_{(\alpha)}^*A_{(\alpha)} \leq 4\|I - A_0\|(I - A_0)
\]

and \( P_{\Lambda} = U^*Q_{\Lambda}(1)U \geq 0 \), which proves the first part of the theorem. To prove the last part of the theorem, we assume that \( \Lambda \) is, in addition, an orthogonal set, i.e. \( \{S_{(\alpha)}\}_{\alpha \in \Lambda} \) is a sequence of isometries with orthogonal ranges. Due to the von Neumann type inequality for regular polyballs [22] and using
According to Theorem 3.1, we have

\[ \left\| \sum_{\alpha \in \Lambda_0} A(\alpha) \otimes X_\alpha \right\| \leq \left\| \sum_{\alpha \in \Lambda_0} A(\alpha) \otimes r^{\alpha}|S_\alpha| \right\| \\
= \left\| \left[ I \otimes r^{\alpha}|S_\alpha| : \alpha \in \Lambda_0 \right] \right\| \\
= \left\| \inf_{r \in [0,1)} \left\{ \sum_{\alpha \in \Lambda_0}(r^{2\alpha}|I) \right\}^{1/2} \right\| \left\| \sum_{\alpha \in \Lambda_0} A^*_\alpha A(\alpha) \right\|^{1/2} \\
\leq 2\|I - A_0\| \inf_{r \in [0,1)} |r\alpha|, \\
\] for any \( X \in rB_n(\mathcal{H}) \), \( r \in [0,1) \), and any finite subset \( \Lambda_0 \subset \Lambda \). Consequently, 

\[ \left\| \sum_{\alpha \in \Lambda} A(\alpha) \otimes X_\alpha \right\| \leq 2\|I - A_0\| \\
\] for any \( X \in B_n(\mathcal{H}) \), which completes the proof.

We use Theorem 3.1 to the following Bohr type result in a very general setting.

**Theorem 3.2.** Let \( F : B_n(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H}) \) be a free holomorphic function with representation \( F(X) = \sum_{\alpha \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} A(\alpha) \otimes X_\alpha \) such that \( F(0) \geq 0 \) and \( RF(X) \leq I \) for any \( X \in B_n(\mathcal{H}) \). If \( \{\Sigma_m\}_{m=0}^\infty \) is an exhaustion of \( \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \) by minimal and orthogonal sets, then the following statements hold.

(i) If \( \{C(\alpha)\}_{\alpha \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \) is a sequence of bounded linear operators on a Hilbert space \( \mathcal{E} \) such that

\[ \|C_0\| \leq 1 \quad \text{and} \quad \sum_{m=1}^\infty \sup_{\alpha \in \Sigma_m} \|C(\alpha)\| \leq \frac{1}{2}, \]

then

\[ \sum_{m=0}^\infty \sup_{X \in B_n(\mathcal{H})} \left\| \sum_{\alpha \in \Sigma_m} C(\alpha) \otimes A(\alpha) \otimes X_\alpha \right\| \leq \|A_0\| + \|1 - A_0\|. \]

(ii) If

\[ \|C_0\| \leq 1 \quad \text{and} \quad \sum_{m=1}^\infty \left\| \sum_{\alpha \in \Sigma_m} C^*(\alpha) C(\alpha) \right\|^{1/2} \leq \frac{1}{2}, \]

then

\[ \left\| \sum_{m=0}^\infty \sum_{\alpha \in \Sigma_m} C(\alpha) \otimes A(\alpha) \otimes S_\alpha \right\| \leq 1. \]

**Proof.** Let \( \mathcal{F} \) be a finite subset of \( \Sigma_m \). According to Theorem 3.1, we have

\[ \sum_{\alpha \in \mathcal{F}} A^*_\alpha A(\alpha) \leq 4\|I - A_0\|(I - A_0). \]
Hence, and using the noncommutative von Neumann inequality, we deduce that
\[
\left\| \sum_{\alpha \in \mathcal{F}} C_\alpha \otimes A_{\alpha} \otimes X_\alpha \right\| \leq \left\| [C_{\alpha} \otimes I \otimes S_\alpha : \alpha \in \mathcal{F}] \begin{bmatrix} I \otimes A_{\alpha} \otimes I \\ \vdots \\ \alpha \in \mathcal{F} \end{bmatrix} \right\| \\
\leq \left\| \text{diag}_\mathcal{F}(C_{\alpha}^* C_{\alpha}) \right\|^{1/2} \left\| \sum_{\alpha \in \mathcal{F}} A_{\alpha}^* A_{\alpha} \right\|^{1/2} \\
= \left\| \sum_{\alpha \in \mathcal{F}} A_{\alpha}^* A_{\alpha} \right\|^{1/2} \sup_{\alpha \in \mathcal{F}} \| C_{\alpha} \| \\
\leq 2\|I - A_0\| \sup_{\alpha \in \mathcal{F}} \| C_{\alpha} \|.
\]
Consequently, \( \left\| \sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes X_\alpha \right\| \leq 2\|I - A_0\| \sup_{\alpha \in \Sigma_m} \| C_{\alpha} \| \) for any \( m \in \mathbb{N} \). Now, using the hypothesis, we have
\[
\sum_{m=0}^{\infty} \sup_{x \in B_{\mathfrak{H}}(\mathbb{N})} \left\| \sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes X_\alpha \right\| \leq \| A_0 \| + 2\|I - A_0\| \sum_{m=1}^{\infty} \sup_{\alpha \in \Sigma_m} \| C_{\alpha} \| \\
\leq \| A_0 \| + \|I - A_0\|.
\]
To prove part (ii), denote \( d_{\Sigma_m} := \left\| [C_{\alpha}^* : \alpha \in \Sigma_m] \right\| \) for \( m \in \mathbb{N} \), and note that
\[
Q_{\Lambda_m} := \begin{pmatrix} d_{\Sigma_m} I_\mathcal{E} \\ C_{\alpha}^* : \alpha \in \Sigma_m \end{pmatrix} \begin{pmatrix} C_{\alpha}^* : \alpha \in \Sigma_m \\ \vdots \\ \alpha \in \Sigma_m \end{pmatrix} \begin{pmatrix} I_\mathcal{E} \otimes 2d_{\Sigma_m}(I - A_0) \\ \vdots \\ 0 \\ \vdots \\ \alpha \in \Sigma_m \end{pmatrix} \begin{pmatrix} I_\mathcal{E} \otimes 2d_{\Sigma_m}(I - A_0) \\ \vdots \\ 0 \\ \vdots \\ \alpha \in \Sigma_m \end{pmatrix}
\]
is a positive operator matrix. Since \( P_{\Sigma_m} \geq 0 \), due to Theorem 4.1, we have \( P_{\Sigma_m} \otimes Q_{\Sigma_m} \geq 0 \). Compressing the operator matrix \( P_{\Sigma_m} \otimes Q_{\Sigma_m} \) to the appropriate entries, we obtain that the operator matrix
\[
(3.3)
\begin{pmatrix} I_\mathcal{E} \otimes 2d_{\Sigma_m}(I - A_0) \\
C_{\alpha} \otimes A_{\alpha} \\
\vdots \\
\alpha \in \Sigma_m \\
\end{pmatrix} \begin{pmatrix} I_\mathcal{E} \otimes 2d_{\Sigma_m}(I - A_0) \\
C_{\alpha} \otimes A_{\alpha} \\
\vdots \\
\alpha \in \Sigma_m \\
\end{pmatrix} \begin{pmatrix} C_{\alpha}^* \otimes A_{\alpha}^* : \alpha \in \Sigma_m \\
\vdots \\
\alpha \in \Sigma_m \\
\vdots \\
\alpha \in \Sigma_m \\
\end{pmatrix}
\]
is positive. Since \( \Sigma_m \) is an orthogonal set, the isometries \( \{S_\alpha\}_{\alpha \in \Sigma_m} \) have orthogonal ranges. Applying now Lemma 2.2 from [23] to the operator matrix \( (3.3) \), we deduce that
\[
(3.4)
\begin{pmatrix} I_\mathcal{E} \otimes 2d_{\Sigma_m}(I - A_0) \otimes I \\
\sum_{\alpha \in \Sigma_m} C_{\alpha}^* \otimes A_{\alpha}^* \otimes S_\alpha \\
\sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \\
\sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \\
\end{pmatrix} \geq 0,
\]
is positive. Since \( 0 \leq I - A_0 \leq I \), we deduce that
\[
\begin{pmatrix} 2d_{\Sigma_m} I_\mathcal{E} \otimes I_\mathcal{H} \otimes I \\
\sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \\
\sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \\
\sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \\
\end{pmatrix} \geq 0,
\]
which implies
\[
\left\| \sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \right\| \leq d_{\Sigma_m}, \quad m \in \mathbb{N}.
\]
Due to the hypothesis, we have \( \sum_{m=1}^{\infty} \sum_{\alpha \in \Sigma_m} d_{\Sigma_m} \leq \frac{1}{2} \), which shows that the series
\[
\sum_{m=1}^{\infty} \left\| \sum_{\alpha \in \Sigma_m} C_{\alpha} \otimes A_{\alpha} \otimes S_\alpha \right\|
\]
is convergent. Since $A_0 \geq 0$ and $\left( I_\mathcal{E} \otimes C_0 \right) \geq 0$ we also have $\left( I_\mathcal{E} \otimes A_0 \otimes I \right) \geq 0$. Taking the sum of the matrices and the latter one, we obtain

$$\left( I_\mathcal{E} \otimes \left[ A_0 + \sum_{m=1}^{\infty} 2d_{\Sigma_m} (I - A_0) \right] \otimes I \right) \geq 0,$$

since $0 \leq A_0 \leq I$ and completes the proof.

Since $0 \leq A_0 + \sum_{m=1}^{\infty} 2d_{\Sigma_m} (I - A_0) \leq I$, we deduce that

$$\left( I_\mathcal{E} \otimes I \otimes I \right) \geq 0,$$

which implies

$$\left\| \sum_{m=0}^{\infty} C_{(\alpha)} \otimes A_{(\alpha)} \otimes S_{(\alpha)} \right\| \leq 1$$

and completes the proof.

**Corollary 3.3.** Let $\{\Sigma_m\}_{m=0}^{\infty}$ be an exhaustion of $\mathbb{R}^+ \times \cdots \times \mathbb{R}^+$ by minimal and orthogonal sets, and let $F : \mathcal{B}_n(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$ be a free holomorphic function with representation

$$F(X) = \sum_{\alpha \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} A_{(\alpha)} \otimes X_{(\alpha)},$$

such that $F(0) \geq 0$ and $\Re F(X) \leq I$ for any $X \in \mathcal{B}_n(\mathcal{H})$. Then the following statements hold.

(i) For any $\rho = (\rho_{i,j}) \in [0,1]^{n_1 + \cdots + n_k},$

$$\sup_{X \in \rho \mathcal{B}_n(\mathcal{H})} \left\| \sum_{\alpha \in \Sigma_m} A_{(\alpha)} \otimes X_{(\alpha)} \right\| \leq 2\|I - A_0\| \sup_{\alpha \in \Sigma_m} \rho_{\alpha}.$$

(ii) If $\sum_{m=1}^{\infty} \sup_{\alpha \in \Sigma_m} \rho_{\alpha} \leq \frac{1}{2},$ then

$$\sum_{m=0}^{\infty} \sup_{X \in \rho \mathcal{B}_n(\mathcal{H})} \left\| \sum_{\alpha \in \Sigma_m} A_{(\alpha)} \otimes X_{(\alpha)} \right\| \leq \|A_0\| + \|I - A_0\|.$$

(iii) If $\rho := (\rho_{i,j})$ with $\rho_{i,j} = r_i \in [0,1]$ for $j \in \{1, \ldots, n_i\}$ and $\prod_{i=1}^{k} (1 - r_i) \geq \frac{2}{3},$ then

$$\sum_{p \in \mathbb{Z}_+^k} \left\| \sum_{\alpha \in \Lambda_p} A_{(\alpha)} \otimes X_{(\alpha)} \right\| \leq \|A_0\| + \|I - A_0\|, \quad X \in \rho \mathcal{B}_n(\mathcal{H})$$

In particular, the inequality holds when $r_1 = \cdots = r_k \leq 1 - \left( \frac{2}{3} \right)^{1/k}$.

**Proof.** Items (i) and (ii) are particular cases of Theorem 3.2. It is easy to see that part (iii) follows from part (ii).

We remark that under the conditions of Corollary 3.3 we have $0 \leq A_0 \leq I$. Due to the spectral theorem, one can easily see that $\|A_0\| + \|I - A_0\| = 1$ if and only $A_0 = a_0I$ for some scalar $a_0 > 0$. In the particular case when $n_1 = \cdots = n_k = 1$, we obtain the following result for the scalar polydisc $\mathbb{D}_k$.

**Corollary 3.4.** Let

$$f(z) = \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} A_{p_1, \ldots, p_k} z_1^{p_1} \cdots z_k^{p_k}, \quad z = (z_1, \ldots, z_k) \in \mathbb{D}_k,$$

be an operator-valued analytic function on the polydisk such that $\Re f(z) \leq I$ and $f(0) \geq 0$. Then

$$\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} \|A_{p_1, \ldots, p_k}\| r_1^{p_1} \cdots r_k^{p_k} \leq \|A_0\| + \|I - A_0\|$$
for any \( r_i \in [0, 1) \) with \( \prod_{i=1}^{k}(1 - r_i) \geq \frac{1}{2} \).

We should mention that, in the particular case when \( k = 1 \), we recover the corresponding result for the disc obtained by Paulsen and Singh in [18].

4. THE BOHR RADIUS \( K_{mh}(B_n) \) FOR THE HARDY SPACE \( H^\infty(B_n) \)

In this section, using the results of Section 3, we obtain estimations for the Bohr radii \( K_{mh}(B_n) \) and \( K_{mh}^0(B_n) \) which extend Boas-Khavinson results for the scalar polydisk to the polyball.

**Theorem 4.1.** Let \( F : B_n(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H}) \) be a free holomorphic function with representation

\[
F(X) = \sum_{\alpha \in F^+_{k}} A(\alpha) \otimes X_\alpha
\]

such that \( F(0) = a_0 I \), \( a_0 \geq 0 \), and \( \Re F(X) \leq I \) for any \( X \in B_n(\mathcal{H}) \). If \( k > 1 \), then

\[
\sum_{p \in \mathbb{Z}^k_+} \left\| \sum_{\alpha \in A_p} A(\alpha) \otimes X_\alpha \right\| \leq 1, \quad X \in rB_n(\mathcal{H})^-.
\]

for any \( r \in [0, \gamma_k] \), where \( \gamma_k \in \left( \frac{1}{3\sqrt{k}}, 1 \right) \) is the solution of the equation

\[
\sum_{m=1}^{\infty} \frac{(m+k-1)}{k-1} r^m = \frac{1}{2}.
\]

Moreover, if \( r \geq \frac{3 \log k}{\sqrt{k}} \), then the inequality above fails.

**Proof.** Assume that \( F \) has the representation \( F(X) = \sum_{p \in \mathbb{Z}^k_+} \left( \sum_{\alpha \in A_p} A(\alpha) \otimes Y_\alpha \right) \) and the properties stated in the theorem. Note also that \( a_0 \leq 1 \). Let \( Y = (Y_1, \ldots, Y_k) \in B_n(\mathcal{H})^- \) with \( Y_i = (Y_{i,1}, \ldots, Y_{i,n_i}) \) and let \( z := (z_1, \ldots, z_k) \in \mathbb{D}^k \). Since \( B_n(\mathcal{H}) \) is noncommutative complete Reinhardt domain (see Proposition 1.3 from [30]), we have \( zY := (z_1Y_1, \ldots, z_kY_k) \in B_n(\mathcal{H}) \). Consequently,

\[
g(z_1, \ldots, z_k) := F(zY) = \sum_{p \in \mathbb{Z}^k_+} \left( \sum_{\alpha \in A_p} A(\alpha) \otimes Y_\alpha \right) z_1^{p_1} \cdots z_k^{p_k}
\]

is an operator-valued analytic function on the polydisk \( \mathbb{D}^k \) with the properties that \( g(0) = a_0 \) and \( \Re g(z) \leq I \) for \( z \in \mathbb{D}^k \). Denote \( B_{p_1, \ldots, p_k} := \sum_{\alpha \in A_p} A(\alpha) \otimes Y_\alpha \) for all \( p = (p_1, \ldots, p_k) \in \mathbb{Z}^k_+ \). Applying Theorem 3.1 to \( g \) and the right minimal set

\[
\Lambda = \Gamma_m := \{ \alpha = (\alpha_1, \ldots, \alpha_k) \in F^+_{n_1} \times \cdots \times F^+_{n_k} : |\alpha| := |\alpha_1| + \cdots + |\alpha_k| = m \},
\]

when \( n_1 = \cdots = n_k = 1 \), we obtain

\[
\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^k_+} B_{p_1, \ldots, p_k}^* B_{p_1, \ldots, p_k} \leq 4\| I - B_0 \| \| I - B_0 \| = 4(1 - a_0)^2.
\]
Consequently, if \( z := (z_1, \ldots, z_k) \in \mathbb{D}^k \), we deduce that
\[
\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} \| B_{p_1,\ldots,p_k} z_1^{p_1} \cdots z_k^{p_k} \| \\
\leq \| B_0 \| + \sum_{m=1}^{\infty} \left( \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k : p_1 + \cdots + p_k = m} \| B_{p_1,\ldots,p_k} z_1^{p_1} \cdots z_k^{p_k} \| \right)^{1/2} \\
\leq \| B_0 \| + \sum_{m=1}^{\infty} \left( \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k : p_1 + \cdots + p_k = m} B_{p_1,\ldots,p_k}^k B_{p_1,\ldots,p_k} \right)^{1/2} \left( \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k : p_1 + \cdots + p_k = m} |z_1|^{2p_1} \cdots |z_k|^{2p_k} \right)^{1/2} \\
\leq a_0 + 2(1-a_0) \sum_{m=1}^{\infty} \left( \sum_{i=1}^{k} |z_i|^2 \right)^{m/2}.
\]
Hence, if \( \sum_{m=1}^{\infty} \left( \sum_{i=1}^{k} |z_i|^2 \right)^{m/2} \leq \frac{1}{2} \), then \( \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} \| B_{p_1,\ldots,p_k} z_1^{p_1} \cdots z_k^{p_k} \| \leq 1 \). In particular, if we take \( z_1 = \cdots = z_k = r \in \left[0, \frac{1}{3\sqrt{k}}\right] \), then
\[
\sum_{m=1}^{\infty} \left( \sum_{i=1}^{k} |z_i|^2 \right)^{m/2} = \sum_{m=1}^{\infty} (kr^2)^{m/2} \leq \sum_{m=1}^{\infty} \frac{1}{3^m} = \frac{1}{2}.
\]
The results above show that
\[
\sum_{p=(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} \left\| \left( \sum_{\alpha \in \Lambda_p} A(\alpha) \otimes Y_{\alpha} \right) r^{p_1+\cdots+p_k} \right\| \leq 1,
\]
which is equivalent to
\[
\sum_{p=(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} \left\| \sum_{\alpha \in \Lambda_p} A(\alpha) \otimes X_{\alpha} \right\| \leq 1, \quad X \in rB_n(\mathcal{H})^-,
\]
for any \( r \in \left[0, \frac{1}{3\sqrt{k}}\right] \). Now, note that
\[
\varphi(r_1, \ldots, r_k) := \sum_{m=1}^{\infty} \left( \sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k : p_1 + \cdots + p_k = m} r_1^{2p_1} \cdots r_k^{2p_k} \right)^{1/2} \leq \sum_{m=1}^{\infty} (r_1^2 + \cdots + r_k^2)^{m/2},
\]
whenever \( k > 1 \). Taking \( r_1 = \cdots = r_k = r \), the inequality above becomes
\[
\sum_{m=1}^{\infty} \left( \frac{m+k-1}{k-1} \right)^{1/2} r^m < \sum_{m=1}^{\infty} (kr^2)^{m/2}.
\]
In particular, if \( r = \frac{1}{3\sqrt{k}} \), we obtain
\[
\sum_{m=1}^{\infty} \left( \frac{m+k-1}{k-1} \right)^{1/2} \left( \frac{1}{3\sqrt{k}} \right)^m < 1.
\]
Hence, there is a unique solution \( \gamma_k > \frac{1}{3\sqrt{k}} \) of the equation
\[
\sum_{m=1}^{\infty} \left( \frac{m+k-1}{k-1} \right)^{1/2} r^m = \frac{1}{2}.
\]
Consequently, \( \varphi(\gamma_k, \ldots, \gamma_k) = \frac{1}{2} \). Using the first part of the proof when \( z_1 = \cdots = z_k = \gamma_k \), we deduce that the inequality in the theorem holds for any \( r \in [0, \gamma_k] \).
On the other hand, according to a result of Boas and Khavinson [4], if \( r \geq \frac{2\sqrt{\log k}}{\sqrt{k}} \), then there is a function \( f(z) = \sum_{p=(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} a_p z^p \), \( z = (z_1,\ldots,z_k) \in \mathbb{D}^k \), in the Hardy space \( H^\infty(\mathbb{D}^k) \) such that \( \|f\|_\infty \leq 1 \) and \( \sum_{p \in \mathbb{Z}_+^k} r^{\|p\|} |a_p| > 1 \). We can assume without loss of generality that \( f(0) \geq 0 \). We remark that the free holomorphic function \( F(X) := \sum_{p=(p_1,\ldots,p_k) \in \mathbb{Z}_+^k} a_p X_{1,1}^{p_1} X_{2,1}^{p_2} \cdots X_{k,1}^{p_k} \), where \( X = (X_1,\ldots,X_k) \in \mathcal{B}_n(\mathcal{H}) \) and \( X_i := (X_{i,1},\ldots,X_{i,n_i}) \), is in the noncommutative Hardy algebra \( H^\infty(\mathcal{B}_n) \) and \( \|F\|_\infty = \|f\|_\infty \). Therefore, \( F(0) = f(0)I \) and \( \Re F(X) \leq I \) for any \( X \in \mathcal{B}_n(\mathcal{H}) \). Since

\[
\sum_{p \in \mathbb{Z}_+^k} \sup_{X \in r\mathcal{B}_n(\mathcal{H})} \left\| a_p X_{1,1}^{p_1} X_{2,1}^{p_2} \cdots X_{k,1}^{p_k} \right\| = \sum_{p \in \mathbb{Z}_+^k} r^{\|p\|} \|S_{1,1}^{p_1} S_{2,1}^{p_2} \cdots S_{k,1}^{p_k}\| = \sum_{p \in \mathbb{Z}_+^k} r^{\|p\|} |a_p| > 1,
\]

the last part of the theorem holds. The proof is complete. \( \square \)

We remark that the hypothesis of Theorem 4.1 are satisfied, in particular, for any bounded free holomorphic function \( F : \mathcal{B}_n(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K}) \otimes_{min} \mathcal{B}(\mathcal{H}) \) with \( \|F\|_\infty \leq 1 \) and \( F(0) = a_0 I \) for some \( a_0 \geq 0 \). As a consequence, we obtain the following result concerning the Bohr radius for the Hardy space \( H^\infty(\mathcal{B}_n) \).

**Corollary 4.2.** If \( k > 1 \), then the Bohr radius for the Hardy space \( H^\infty(\mathcal{B}_n) \), with respect to the multihomogeneous expansion, satisfies the inequalities

\[
\frac{1}{3\sqrt{k}} < K_{mh}(\mathcal{B}_n) < \frac{2\sqrt{\log k}}{\sqrt{k}}
\]

and

\[
\limsup_{k \rightarrow \infty} \frac{K_{mh}(\mathcal{B}_n)}{\sqrt{\log k}} \leq 1.
\]

Moreover, if \( \gamma_k \in [0,1) \) is the solution of the equation

\[
\sum_{m=1}^{\infty} \left( \frac{m+k-1}{k-1} \right)^{1/2} r^m = \frac{1}{2}
\]

then

\[
\frac{1}{3\sqrt{k}} < \gamma_k \leq K_{mh}(\mathcal{B}_n).
\]

**Proof.** Let \( F \in H^\infty(\mathcal{B}_n) \) have the representation \( F(X) = \sum_{p \in \mathbb{Z}_+^k} \left( \sum_{\alpha \in \Lambda_p} a_\alpha X_\alpha \right) \) and assume that \( \|F\|_\infty \leq 1 \). Without loss of generality, we can assume that \( a_0 \geq 0 \). Note also that \( a_0 \leq 1 \) and \( \Re F(X) \leq I \) for any \( X \in \mathcal{B}_n(\mathcal{H}) \). Applying Theorem 4.1 to \( F \) one can obtain most of the results of the corollary. The only thing remaining to show is that the inequality with the lim sup holds. To this end, note that, due to Proposition 1.1 each function in \( H^\infty(\mathbb{D}^k) \) can be seen as a function in \( H^\infty(\mathcal{B}_n) \). Consequently, we have \( K_{mh}(\mathcal{B}_n) \leq K_{mh}(\mathbb{D}^k) \). Using the result from [4] (see also [5]) we have

\[
\limsup_{k \rightarrow \infty} \frac{K_{mh}(\mathcal{B}_n)}{\sqrt{\log k}} \leq \limsup_{k \rightarrow \infty} \frac{K_{mh}(\mathbb{D}^k)}{\sqrt{\log k}} \leq 1.
\]

The proof is complete. \( \square \)

**Theorem 4.3.** Let \( F : \mathcal{B}_n(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K}) \otimes_{min} \mathcal{B}(\mathcal{H}) \) be a bounded free holomorphic function with representation \( F(X) = \sum_{\alpha \in \mathbb{F}_n^+ \times \cdots \times \mathbb{F}_n^+} A_\alpha \otimes X_\alpha \) such that \( F(0) = 0 \). If \( k > 1 \), then

\[
\sum_{p \in \mathbb{Z}_+^k} \left\| \sum_{\alpha \in \Lambda_p} A_\alpha \otimes X_\alpha \right\| \leq \|F\|_\infty, \quad X \in r\mathcal{B}_n(\mathcal{H})^-
\]

for any \( r \in [0,t_k] \), where \( t_k \in \left( \frac{1}{\sqrt{k}},1 \right) \) is the solution of the equation

\[
\sum_{m=1}^{\infty} \left( \frac{m+k-1}{k-1} \right)^{1/2} r^m = 1.
\]
Moreover, if \( r \geq \max \{ \frac{2 \log k}{\sqrt{k}}, \frac{1}{\sqrt{2}} \} \), then the inequality above fails.

**Proof.** The proof is very similar to that of Theorem 4.1. We point out the differences. We use Proposition 2.7 (instead of Theorem 3.1) to obtain

\[
\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^+} B^{p_1 \cdots p_k} B^{p_1 \cdots p_k} \leq 1 - |a_0|^2 = 1.
\]

Consequently, we deduce that

\[
\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}^+} \|B^{p_1 \cdots p_k} z_1^{p_1} \cdots z_k^{p_k}\| \leq \sum_{m=1}^{\infty} \left( \sum_{i=1}^{k} |z_i|^2 \right)^{m/2}.
\]

In particular, if we take \( z_1 = \cdots = z_k = r \in [0, \frac{1}{2\sqrt{k}}] \), then

\[
\sum_{m=1}^{\infty} \left( \frac{1}{2m} \right)^{m/2} = \sum_{m=1}^{\infty} (kr^2)^{m/2} \leq \sum_{m=1}^{\infty} \frac{1}{2m} = 1.
\]

This leads to

\[
\sum_{p \in \mathbb{Z}^+} \left\| \sum_{\alpha \in \Lambda_p} A_\alpha \otimes X_\alpha \right\| \leq \|F\|_{\infty}, \quad X \in rB_n(\mathcal{H}),
\]

for any \( r \in \left[0, \frac{1}{2\sqrt{k}}\right] \). The rest of the proof is very similar to that of Theorem 4.1. We leave it to the reader. \(\square\)

A simple consequence of Theorem 4.3 is the following.

**Corollary 4.4.** If \( k > 1 \), the Bohr radius \( K_{mh}^0(B_n) \) satisfies the inequalities

\[
\frac{1}{2\sqrt{k}} < K_{mh}^0(B_n) < \frac{2\log k}{\sqrt{k}}.
\]

### 5. The Bohr radius \( K_n(B_n) \) for the Hardy space \( H^\infty(B_n) \)

With respect to the homogeneous power series expansion of the elements in the Hardy space \( H^\infty(B_n) \), we prove that \( K_n(B_n) = 1/3 \), extending the classical result to our multivariable noncommutative setting. We also obtain estimations for the Bohr radius \( K_n^0(B_n) \).

According to Example 2.3, if \( q \in \mathbb{Z}^+ \), then

\[
\Gamma_q := \{ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^+_n \times \cdots \times \mathbb{F}^+_n : |\alpha| := |\alpha_1| + \cdots + |\alpha_k| = q \}
\]

is a minimal set in \( \mathbb{F}^+_n \times \cdots \times \mathbb{F}^+_n \) and \( \{ \Gamma_q \}_{q=0}^\infty \) is an exhaustion of \( \mathbb{F}^+_n \times \cdots \times \mathbb{F}^+_n \) by minimal sets. Any function \( F \) in the Hardy algebra \( H^\infty(B_n) \) has a power series expansion in terms of homogeneous polynomials, i.e.

\[
F(X) = \sum_{q=0}^\infty \left( \sum_{\alpha \in \Gamma_q} a_\alpha X_\alpha \right), \quad X \in B_n(\mathcal{H}),
\]

for any Hilbert space \( \mathcal{H} \), where the convergence is in the operator norm topology.

**Definition 5.1.**

(i) The Bohr radius for the polyball \( B_n \) is denoted by \( K_n(B_n) \) and is the largest \( r \geq 0 \) such that

\[
\sum_{q=0}^\infty \left\| \sum_{\alpha \in \Gamma_q} a_\alpha X_\alpha \right\| \leq \|F\|_{\infty}, \quad X \in rB_n(\mathcal{H}),
\]

for any \( F \in H^\infty(B_n) \).
(ii) The Bohr scaling set for the polyball $B_n$ is denoted by $S_h(B_n)$ and consists of all $\rho = (\rho_{i,j})$ with $\rho_{i,j} \geq 0$ and such that
\[
\sum_{\alpha \in \Gamma_q} a(\alpha) X_\alpha \leq \|F\|_{\infty}, \quad X \in \rho B_n(H),
\]
for any $F \in H^\infty(B_n)$.

(iii) The Bohr part of the polyball $B_n$ is defined by $B_h(B_n) := \bigcup_{\rho \in S_h(B_n)} \rho B_n$.

(iv) The Bohr positive scalar part of the polyball $B_n$ is the set $\Omega_h(B_n)$ of all $r := \{r_{i,j}\}$ with $r_{i,j} \geq 0$ and such that
\[
\sum_{\alpha \in \Gamma_q} a(\alpha) r^{|\alpha|} \leq \|F\|_{\infty}
\]
for any $F \in H^\infty(B_n)$.

We remark that, due to the noncommutative von Neumann inequality for the polyball [22], we have
\[
\sup_{X \in r B_n(H)} \left\| \sum_{\alpha \in \Gamma_q} a(\alpha) X_\alpha \right\| = \left\| \sum_{\alpha \in \Gamma_q} a(\alpha) r^{|\alpha|} S_\alpha \right\|
\]
and the inequality in part (i) of the definition above is equivalent to
\[
\sum_{\alpha \in \Gamma_q} a(\alpha) r^{|\alpha|} \leq \|F\|_{\infty}. \quad A
\]
Similar observation should be made for the inequality in definition (ii). We should mention that due to the fact that
\[
\sum_{\alpha \in \Gamma_q} a(\alpha) X_\alpha \leq \sum_{p \in \mathbb{Z}_+^n} \sum_{\alpha \in \Lambda_p} a(\alpha) X_\alpha, \quad X \in B_n(H).
\]
we always have $K_h(B_n) \geq K_{mh}(B_n)$ and $K_{h}^0(B_n) \geq K_{mh}^0(B_n)$. As we will see soon, the inequalities are strict, in general.

**Theorem 5.2.** If $F : B_n(H) \to B(K \otimes \min B(H))$ is any bounded free holomorphic function with representation $F(X) := \sum_{m=0}^{\infty} \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes X_\alpha$ such that $F(0) = a_0 I$ for some $a_0 \in \mathbb{C}$, then
\[
\sum_{m=0}^{\infty} \left\| \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes X_\alpha \right\| \leq \|F\|_{\infty}
\]
for any $X \in \frac{1}{2} B_n(H)$. Moreover, $\frac{1}{2}$ is the best possible constant and the inequality is strict unless $F$ is a constant.

**Proof.** Without loss of generality, we can assume that $\|F\|_{\infty} = 1$. Let $Y \in B_n(H)$ and $z \in D$. According to Proposition 1.3 from [30], $B_n(H)$ is a noncommutative complete Reinhardt domain. Consequently, $z Y \in B_n(H)$ for any $z \in D$, and $\sup_{z \in \mathbb{D}} \|F(zY)\| \leq 1$ for any $Y \in B_n(H)$. Since $F$ is free holomorphic on $B_n(H)$, we have (see [30])
\[
\sum_{m=0}^{\infty} \left\| \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes Y_\alpha \right\| < \infty, Y \in B_n(H).
\]
Hence, we deduce that
\[
g(z) := F(zY) = \sum_{m=0}^{\infty} \left( \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes Y_\alpha \right) z^m, \quad z \in D,
\]
is an operator-valued analytic function on $D$ with $\|g\|_{\infty} \leq 1$ and $g(0) = a_0 I_H$, $a_0 \in \mathbb{C}$. Applying Theorem 2.9 part (i), when $k = 1$, to $g$ and taking into account that $0 \leq |a_0| \leq 1$, we obtain
\[
\sum_{m=0}^{\infty} \left\| \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes Y_\alpha \right\| r^m \leq |a_0| + (1 - |a_0|^2) \frac{r}{1 - r} \leq 1,
\]
for any $r \in [0, \frac{1}{2}]$, which is equivalent to the inequality in the theorem.
On the other hand, due to Proposition 5.3, we have \( K_h(B_n) \leq K_h(D) \leq \frac{1}{2} \). Therefore, \( \frac{1}{2} \) is the best possible constant. To prove the last part of the theorem, note that if \( \|F\|_\infty = 1 \), then using the inequality above, we deduce that
\[
|a_0| + \frac{k}{2}(1 - |a_0|^2) = 1,
\]
which implies \( a_0 = 0 \). Employing Proposition 5.3, we deduce that \( A(\alpha) = 0 \) for any \( \alpha \in \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_k} \) different from the identity \( g_0 \). This shows that \( F = a_0 I \). The proof is complete.

**Corollary 5.3.** If \( F : B_n(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H}) \) is any bounded free holomorphic function with representation \( F(X) := \sum_{m=0}^\infty \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes X_\alpha \) such that \( F(0) = 0 \), then
\[
\sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes X_\alpha \right\| \leq \|F\|_\infty
\]
for any \( X \in \frac{1}{2}B_n(\mathcal{H}) \).

**Proof.** Following the proof of Theorem 5.2 we have
\[
\sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} A(\alpha) \otimes Y_\alpha \right\| r^m \leq |a_0| + (1 - |a_0|^2) \frac{r}{1-r} \leq 1,
\]
for any \( r \in [0, \frac{1}{2}] \). This completes the proof. \( \square \)

**Corollary 5.4.** The Bohr radius \( K_h^0(B_n) \) satisfies the inequalities
\[
\max \left\{ \frac{1}{2} \sqrt{1 - \left( \frac{1}{2} \right)^{1/k}} \right\} \leq K_h^0(B_n) \leq \frac{1}{\sqrt{2}}.
\]

**Proof.** Due to Corollary 5.3, we have \( \frac{1}{2} \leq K_h^0(B_n) \). Using Corollary 5.4 and the fact that \( K_{mh}^0(B_n) \leq K_h^0(B_n) \), we deduce that \( \sqrt{1 - \left( \frac{1}{2} \right)^{1/k}} \leq K_h^0(B_n) \). On the other hand, employing Proposition 5.3, we deduce the inequality \( K_h^0(B_n) \leq K_h^0(D) \). Since \( K_h^0(D) = \frac{1}{\sqrt{2}} \) (see e.g. [17]), we conclude that \( K_h^0(B_n) \leq \frac{1}{\sqrt{2}} \) and complete the proof. \( \square \)

**Corollary 5.5.** Let \( F : B_n(\mathcal{H}) \to B(\mathcal{H}) \) be a bounded free holomorphic function with representation
\[
F(X) := \sum_{m=0}^\infty \sum_{\alpha \in \Gamma_m} a(\alpha) X_\alpha,
\]
Then the following statements hold:

(i) For any \( X \in \frac{1}{2}B_n(\mathcal{H}) \)
\[
\sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} a(\alpha) X_\alpha \right\| \leq \|F\|_\infty,
\]
and \( \frac{1}{3} \) is the best possible constant. Moreover, the inequality is strict unless \( F \) is a constant.

(ii) \( K_h(B_n) = \frac{1}{7} \).

(iii) \( S_h(B_n) = \left[ 0, \frac{2}{3} \right] \cup \cdots \cup \left[ 0, \frac{2}{3} \right] \setminus \left[ \frac{1}{2}, 1 \right] \) and \( B_h(B_n) = \frac{1}{3}B_n \).

(iv) The Bohr positive scalar part for the polyball \( \Omega_h(B_n) \) consists of all tuples of positive numbers \( \rho := (\rho_1, \ldots, \rho_k) \), \( r_i := (r_{i,1}, \ldots, r_{i,n_i}) \) with the property that \( r_{i,j} \geq 0 \) and \( \|r_i\|_2 \leq \frac{1}{2} \) for any \( i \in \{1, \ldots, k\} \).

**Proof.** Note that items (i) and (ii) hold due to Theorem 5.2. We need to prove (iii). If \( \rho = (\rho_{i,j}) \) is in \([0, \frac{1}{3}]^{n_1 + \cdots + n_k} \) and \( X \in \rho B_n(\mathcal{H}) \), then the noncommutative von Neumann inequality and item (i) imply
\[
\sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} a(\alpha) X_\alpha \right\| \leq \sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} a(\alpha) \rho \alpha S_\alpha \right\| \leq \sum_{m=0}^\infty \left\| \sum_{\alpha \in \Gamma_m} a(\alpha) S_\alpha \right\| \frac{1}{3^m} \leq \|F\|_\infty.
\]
This shows that \([0, \frac{1}{3}]^{n_1 + \cdots + n_k} \subseteq S_h(B_n) \). If we assume that there is \( \rho = (\rho_{i,j}) \in S_h(B_n) \) with at least one \( \rho_{i,j} > \frac{1}{3} \), then, due to the fact that \( H^\infty(D) \) is isometrically embedded in \( H^\infty(B_n) \) (see Proposition 5.1), we deduce that \( K_h(D) > \frac{1}{7} \), which is a contradiction. Therefore, we must have \( S_h(B_n) = \left[ 0, \frac{1}{3} \right]^{n_1 + \cdots + n_k} \) which clearly implies \( B_h(B_n) = \frac{1}{3}B_n \).
To prove item (iv), note that applying item (i) to $X = \{X_{ij}\}$ with $X_{ij} = r_{i,j}I_H$, where $r_{i,j} \geq 0$ and $\|r_i\|_2 \leq \frac{1}{3}$ for any $i \in \{1, \ldots, k\}$, we deduce that $r \in \Omega_h(B_n)$. Now, assume that $r := (r_1, \ldots, r_k)$, $r_i := (r_{i,1}, \ldots, r_{i,n_i})$ with the property that $r_{i,j} \geq 0$ and at least one $r_i$ satisfies the inequality $\|r_i\|_2 > \frac{1}{3}$. For simplicity, we assume that $\|r_1\| > \frac{1}{3}$. As in the proof of Theorem 4.5 from [17], we can construct a bounded free holomorphic function on the unit ball $[B(H)^n]_1$, $f(X_1) = \sum_{m=0}^{\infty} \sum_{\alpha \in F^+_n, |\alpha| = m} c_{\alpha} X_{1, \alpha}$, with the property that $\|f\|_\infty = 1$ and

$$\sum_{m=0}^{\infty} \left| \sum_{\alpha \in F^+_n, |\alpha| = m} c_{\alpha} r_{1, \alpha} \right| > 1.$$  

Define $F(X_1, \ldots, X_k) := f(X_1)$ on the polyball $B_n$, and note that $F \in H^\infty(B_n)$ and $\|F\|_\infty = 1$. The free holomorphic function $F$ has the representation $F(X) := \sum_{m=0}^{\infty} \sum_{\alpha \in \Gamma_m} b_{(\alpha)} X_\alpha$, where $b_{(\alpha)} = 0$ unless $(\alpha) = (\alpha_1, g_0^2, \ldots, g_0^k)$, in which case $b_{(\alpha_1, g_0^2, \ldots, g_0^k)} = c_{\alpha_1}$. Since

$$\sum_{m=0}^{\infty} \left| \sum_{\alpha \in \Gamma_m} b_{(\alpha)} r_{\alpha} \right| = \sum_{m=0}^{\infty} \left| \sum_{\alpha \in F^+_n, |\alpha| = m} a_{\alpha} r_{1, \alpha} \right| > 1,$$

we deduce that $r \notin \Omega_h(B_n)$. The proof is complete. \hfill \Box

**Theorem 5.6.** If $F : B_n(H) \rightarrow B(H)$ is a bounded free holomorphic function with representation

$$F(X) := \sum_{m=0}^{\infty} \sum_{\alpha \in \Gamma_m} a_{(\alpha)} X_\alpha,$$

then

$$\sum_{m=0}^{\infty} \left| \sum_{\alpha \in \Gamma_m} a_{(\alpha)} X_\alpha \right| \leq \Omega(r) \|F\|_\infty, \quad X \in rB_n(H)^-,$$

where $\Omega(r) := \min \left\{ M(r), \left( \frac{1}{\sqrt{1-r}} \right)^k \right\}$ and

$$M(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{3}, \\ \frac{4r^2 + (1-r) \sqrt{1-r}}{4r(1-r)} & \text{if } \frac{1}{3} < r < 1. \end{cases}$$

**Proof.** We can assume that $\|F\|_\infty \leq 1$. We saw in the proof of Theorem 5.2 that

$$\sum_{m=0}^{\infty} \left| \sum_{\alpha \in \Gamma_m} a_{(\alpha)} X_\alpha \right| \leq |a_0| + (1 - |a_0|^2) \frac{r}{1-r}$$

for any $X \in rB_n(H)^-$. Since $0 \leq |a_0| \leq 1$, as in the proof of Proposition 2.12 when $c = \frac{r}{1-r}$, we can prove that

$$\sup \{x + (1 - x^2) c : 0 \leq x \leq 1\} \leq M(r),$$

where $M(r)$ is given in the theorem. On the other hand, due to Theorem 2.13 if $\rho := (\rho_{i,j})$ with $\rho_{i,j} = r_i \in [0, 1)$ for $j \in \{1, \ldots, n_i\}$, then

$$\sum_{m=0}^{\infty} \left| \sum_{\alpha \in \Gamma_m} a_{(\alpha)} X_\alpha \right| \leq \sum_{p \in \mathbb{Z}_+^k} \left| \sum_{\alpha \in \Lambda_p} a_{(\alpha)} X_\alpha \right| \leq \prod_{i=1}^{k}(1 - r_{i,j}^2)^{-1/2} \|F\|_\infty,$$

for any $X \in \rho B_n(H)^-$. Taking $r_1 = \cdots = r_k = r$, we complete the proof. \hfill \Box

Let $F \in H^\infty(B_n)$ have the representation $F(X) := \sum_{m=0}^{\infty} \sum_{\alpha \in \Gamma_m} a_{(\alpha)} X_\alpha$ and let

$$M(F, r) := \sum_{m=0}^{\infty} \left| \sum_{\alpha \in \Gamma_m} a_{(\alpha)} S_\alpha \right|$$

be the associated majorant series. Define

$$m_{B_n}(r) := \sup \frac{M(F, r)}{\|F\|_\infty}, \quad r \in [0, 1),$$
where the supremum is taken over all $F \in H^\infty(B_n)$ with $F$ not identically 0. Due to Corollary 5.5 we have $m_{B_n}(r) = 1$ if $r \in [0, \frac{1}{2}]$. On the other hand, Theorem 5.4 shows that $m_{B_n}(r) \leq \Omega(r)$ for $r \in [0, 1)$.

The precise value of $m_{B_n}(r)$ as $\frac{1}{2} < r < 1$ remains unknown, in general.

6. Fejér and Bohr inequalities for multivariable polynomials with operator coefficients

In this section, we obtain analogues of Carathéodory’s inequality, and Fejér and Egerváry-Százs inequalities for free holomorphic functions with operator coefficients and positive real parts on the polyball. These results are used to provide an analogue of Landau’s inequality and Bohr type inequalities when the norm is replaced by the numerical radius of an operator.

The classical numerical radius of an operator $T \in B(\mathcal{H})$ is defined by

$$\omega(T) := \sup\{|\langle Th, h \rangle| : h \in \mathcal{H}, \|h\| = 1\}.$$ 

Here are some of its basic properties that will be used in what follows.

(i) $\omega(T_1 + T_2) \leq \omega(T_1) + \omega(T_2)$ for any $T_1, T_2 \in B(\mathcal{H})$.
(ii) $\omega(\lambda T) = |\lambda|\omega(T)$ for any $\lambda \in \mathbb{C}$.
(iii) $\omega(U^*TU) = \omega(T)$ for any unitary operator $U$.
(iv) $\omega(X^*TX) \leq \|X\|^2\omega(T)$ for any operator $X : \mathcal{K} \to \mathcal{H}$.
(v) $\omega(T \otimes I_\mathcal{E}) = \omega(T)$ for any separable Hilbert space $\mathcal{E}$.
(vi) The numerical radius is continuous in the operator norm topology.

For each $p \in \mathbb{Z}$, we set $p^+ := \max\{p, 0\}$ and $p^- := \max\{-p, 0\}$. A function $F$ with operator-valued coefficients in $B(\mathcal{K})$, where $\mathcal{K}$ is separable Hilbert space, is called free $k$-pluriharmonic on the abstract polyball $B_n$ if it has the form

$$F(X) = \sum_{p_i \in \mathbb{Z}} \cdots \sum_{p_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in \mathbb{N}_+^k, i \in \{1, \ldots, k\}} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \otimes X_{1, \alpha_1} \cdots X_{k, \alpha_k} X_{1, \beta_1}^* \cdots X_{k, \beta_k}^*,$$

where the multi-series converge in the operator norm topology for any $X = (X_1, \ldots, X_k) \in B_n(\mathcal{H})$, with $X_i := (X_{i,1}, \ldots, X_{i,n_i})$, and any Hilbert space $\mathcal{H}$. According to [31], the order of the series in the definition above is irrelevant. Note that any free holomorphic function on the polyball is $k$-pluriharmonic and so is the real part of a free holomorphic function.

In what follows, we obtain an analogue of Carathéodory’s inequality [7], and Fejér [13] and Egerváry-Százs inequalities [12] for free holomorphic functions with positive real parts on the polyball and operator coefficients.

**Theorem 6.1.** Let $m \in \mathbb{N} \cup \{\infty\}$ and let

$$f(X) := \sum_{q=1}^m \sum_{\alpha \in \Gamma_q^+} A_{(\alpha)}^* \otimes X_\alpha^* + A_0 \otimes I + \sum_{q=1}^m \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes X_\alpha, \quad X \in B_n(\mathcal{H}),$$

be a positive $k$-pluriharmonic function on the polyball $B_n$ with coefficients in $B(\mathcal{K})$. If $m \in \mathbb{N}$, then for each $q \in \{1, \ldots, m\}$,

$$\omega\left(\sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes X_\alpha\right) \leq \omega\left(\sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes S_\alpha\right) \leq \|A_0\| \cos \frac{\pi}{m} + 2, \quad X \in B_n(\mathcal{H}),$$

where $S$ is the universal model of the polyball and $[x]$ is the integer part of $x$. If $m = \infty$, then

$$\omega\left(\sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes S_\alpha\right) \leq \|A_0\|.$$
Proof. According to the multivariable dilation theory for regular polyballs [27], each element \( X = \{X_i\} \) in the polyball \( B_n(H) \), has a dilation \( \{S_{ij} \otimes I_M\} \) for some separable Hilbert space \( M \) with the property that \( X_i^* = (S_{ij} \otimes I_M)_{ji} \) for any \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, n_i\} \). Using the properties of the numerical radius, we obtain

\[
\omega \left( \sum_{\alpha \in \Gamma} A(\alpha) \otimes X_\alpha \right) \leq \omega \left( \sum_{\alpha \in \Gamma} A(\alpha) \otimes S_\alpha \right).
\]

Assume now that \( m \in \mathbb{N} \). Due to Theorem 2.29 from [24], we have the following operatorial version of the Fejér and Egerváry-Szász inequalities. If \( \{C_p\}_{p=0}^m \subset B(K) \), \( m \geq 1 \), and

\[
\sum_{1 \leq p \leq m} C_p^* z^p + C_0 + \sum_{1 \leq p \leq m} C_p z^p \geq 0, \quad z \in \mathbb{D},
\]

then

\[
\omega(C_p) \leq \|C_0\| \cos \frac{\pi}{\left\lfloor \frac{m}{p} \right\rfloor + 2}, \quad p \in \{1, \ldots, m\}.
\]

Since \( B_n \) is a noncommutative complete Reinhardt domain (see [30]), we deduce that the tuple \( zS \) is in \( B_n(F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})) \) for any \( z \in \mathbb{D} \). Consequently,

\[
\psi(z) := \sum_{q=1}^m \left( \sum_{\alpha \in \Gamma_q} A^*_\alpha \otimes S_\alpha \right) z^q + A_0 \otimes I + \sum_{q=1}^m \left( \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes S_\alpha \right) z^p
\]

is a positive operator-valued harmonic function on the disc \( \mathbb{D} \). Applying the above-mentioned result to \( \psi \), we deduce the inequality in the theorem.

Now, we consider the case \( m = \infty \). Setting

\[
M(r, N) := 2 \sum_{q=N+1}^{\infty} \left\| \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes r^{|\alpha|} S_\alpha \right\|, \quad r \in [0, 1), N \in \mathbb{N},
\]

and using the noncommutative von Neumann inequality for the polyball [22], we deduce that

\[
\varphi(z) := \sum_{q=1}^N \left( \sum_{\alpha \in \Gamma_q} A^*_\alpha \otimes S_\alpha \right) r^q z^q + (A_0 \otimes I + M(r, N)I) + \sum_{q=1}^N \left( \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes S_\alpha \right) r^q z^p
\]

is a positive operator-valued harmonic function on the disc \( \mathbb{D} \). As in the first part of the proof, if \( q \in \mathbb{N} \) and \( N \geq q \), we obtain

\[
\omega \left( \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes r^q S_\alpha \right) \leq \|A_0 \otimes I + M(r, N)I\| \cos \frac{\pi}{\left\lfloor \frac{N}{q} \right\rfloor + 2}.
\]

Since \( M(r, N) \to 0 \) as \( N \to \infty \), passing to the limit in the inequality above, we deduce that

\[
\omega \left( \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes r^q S_\alpha \right) \leq \|A_0\|, \quad r \in [0, 1).
\]

Since the numerical radius is continuous in the operator norm topology, taking \( r \to 1 \), we complete the proof. \( \square \)

When the positive \( k \)-pluriharmonic functions on the polyball \( B_n \) have scalar coefficients, we obtain the following consequence of Theorem 6.1.

**Corollary 6.2.** Let \( m \in \mathbb{N} \cup \{\infty\} \) and let

\[
g(X) := \sum_{q=1}^m \sum_{\alpha \in \Gamma_q} a(\alpha)_r X_\alpha + a_0 I + \sum_{q=1}^m \sum_{\alpha \in \Gamma_q} a(\alpha)_r X_\alpha, \quad X \in B_n(H),
\]

then \( g(X) \) is a positive operator-valued harmonic function on the disc \( \mathbb{D} \).
be a positive k-pluriharmonic function on the polyball. Then for each \( q \in \{1, \ldots, m\} \),

\[
\sup_{\lambda = (\lambda_{i,j}) \in B_n(\mathbb{C})^-} \left| \sum_{\alpha \in \Gamma_q} a_{(\alpha)} \lambda_{\alpha} \right| \leq a_0 \cos \frac{\pi}{m+2}.
\]

In particular, when \( q = 1 \), we obtain

\[
\frac{k}{\sum_{i=1}^{k} \left( \sum_{j=1}^{n_i} \left| \frac{\partial \tilde{g}}{\partial z_{i,j}} \right| (0) \right)^2} \leq a_0 \cos \frac{\pi}{m+2},
\]

where \( \tilde{g} \) is the scalar representation of \( g \), i.e. \( \tilde{g}(z) := g(z) \) for \( z = (z_{i,j}) \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1 \).

A simple consequence of Theorem 6.1 and Corollary 6.2 is the following result concerning holomorphic functions with positive real parts on the polydisk. In particular cases, we recover Fejér and Egerváry-Szász inequalities, as well as Carathéodory’s inequality.

**Corollary 6.3.** If \( m \in \mathbb{N} \cup \{\infty\} \) and \( f(z) = \sum_{p \in \mathbb{Z}_+^{n} : |p| \leq m} a_p z^p \) is a holomorphic function on the polydisk and \( \Re f(z) \geq 0, z \in \mathbb{D}^k \), then

\[
\sup_{\xi \in T^k} \left| \sum_{p \in \mathbb{Z}_+^{n} : |p| = q} a_p \xi^p \right| \leq 2 \Re a_0 \cos \frac{\pi}{m+2}, \quad q \in \{1, \ldots, m\}.
\]

In particular,

(i) if \( m \in \mathbb{N} \) and \( k = 1 \), then we recover Fejér and Egerváry-Szász inequalities, i.e.

\[
|a_{p_1}| \leq 2 \Re a_0 \cos \frac{\pi}{m+2}, \quad 1 \leq p_1 \leq m;
\]

(ii) if \( m = \infty \) and \( k = 1 \), then we recover the Carathéodory’s inequality, i.e.

\[
|a_{p_1}| \leq 2 \Re a_0, \quad p_1 \in \mathbb{N}.
\]

We refer the reader to the paper by Korányi and Pukánszky \[14\] for a characterization of the holomorphic functions with positive real parts on the polydisk.

In the next theorem, the first inequality can be seen as an analogue of Landau’s inequality \[15\] for free holomorphic functions on the polyball and operator coefficients, while the second inequality is a Bohr type result where the norms are replaced by the numerical radius of operators.

**Theorem 6.4.** Let \( m \in \mathbb{N} \cup \{\infty\} \) and let \( F(X) := \sum_{q=1}^{m} \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes X_{\alpha} \) be a free holomorphic function on the polyball with \( F(0) \geq 0 \) and \( \Re F(X) \leq I \) for \( X \in B_n \). Then, for each \( q \in \{1, \ldots, m\} \),

\[
\omega \left( \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes X_{\alpha} \right) \leq \omega \left( \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes S_{\alpha} \right) \leq 2 \Re \langle A_0 \rangle \cos \frac{\pi}{m+2}
\]

and

\[
\sum_{q=0}^{m} \sup_{X \in B_n(H)^-} \omega \left( \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes X_{\alpha} \right) = \sum_{q=0}^{m} \omega \left( \sum_{\alpha \in \Gamma_q} A_{(\alpha)} \otimes S_{\alpha} \right) r^q \leq \|A_0\| + \|I - A_0\|
\]

for any \( r \in [0, t_m] \), where \( t_m \in (0,1) \) is the solution of the equation

\[
\sum_{q=1}^{m} t^q \cos \frac{\pi}{m+2} = \frac{1}{2}.
\]

Moreover, the sequence \( \{t_m\} \) is strictly decreasing and converging to \( \frac{1}{2} \). When \( m = \infty \), we have \( t_\infty = \frac{1}{2} \).
Proof. The Landau type inequality is a consequence of Theorem 6.1. We prove the second part of the theorem. Let \( r \in [0, t_m] \), where \( t_m \in (0, 1) \) is the solution of the equation mentioned above. Note that the first part of the theorem implies

\[
\sum_{q=0}^{m} \omega \left( \sum_{\alpha \in \Gamma_q} A(\alpha) \otimes S_{\alpha} \right) r^q \leq \omega(A_0) + 2\|I - A_0\| \sum_{q=1}^{m} t^q \cos \frac{\pi}{\left\lfloor \frac{m}{q} \right\rfloor + 2} \leq \|A_0\| + \|I - A_0\|.
\]

A close look at the function \( \varphi_m(t) := \sum_{q=1}^{m} t^q \cos \frac{\pi}{\left\lfloor \frac{m}{q} \right\rfloor + 2} \), \( t \in [0, 1] \), reveals that \( \varphi_m \) is strictly increasing and, consequently, there is a unique solution \( t_m \in (0, 1) \) of the equation \( \varphi_m(t) = \frac{1}{2} \). Moreover, since \( \varphi_m(t) < \varphi_{m+1}(t) < \sum_{q=1}^{q_{\infty}} t^q \) for \( t \in [0, 1] \) and \( \sum_{q=1}^{q_{\infty}} \frac{1}{3^q} = \frac{1}{2} \), we have \( t_m > \frac{1}{3} \) and the sequence \( \{t_m\} \) is strictly decreasing. Since the sequence \( \varphi_m \) is uniformly convergent to \( \xi(t) := \sum_{q=1}^{q_{\infty}} t^q \) on any interval \([0, \delta] \) with \( \delta \in (0, 1) \), one can easily see that \( t_m \to \frac{1}{3} \) as \( m \to \infty \). Therefore, when \( m = \infty \), we have \( t_{\infty} = \frac{1}{3} \). The proof is complete.

Here is the numerical radius version of Bohr’s inequality for free holomorphic functions on polyballs.

**Corollary 6.5.** Let \( m \in \mathbb{N} \cup \{\infty\} \) and let \( f(X) := \sum_{q=1}^{m} \sum_{\alpha \in \Gamma_q} a(\alpha) X_{\alpha} \) be a free holomorphic function with \( f(0) \geq 0 \) and \( \Re f(X) \leq 1 \) on the polyball \( B_n \). Then

\[
\sum_{q=0}^{m} \sup_{z \in B_n(C)} \left| \sum_{\alpha \in \Gamma_q} a(\alpha) z_{\alpha} \right| \leq 1
\]

for any \( r \in [0, t_m] \), where the sequence \( \{t_m\} \) is defined in Theorem 6.4. In particular, the result holds if \( f \) is a free holomorphic function with \( \|f\|_{\infty} \leq 1 \). Moreover, we have

\[
\sum_{q=0}^{m} \left| a(\alpha) z_{\alpha} \right| \leq \left| f(0) \right|
\]

for any \( r \in [0, t_m] \). When \( m = \infty \), we have \( t_{\infty} = \frac{1}{3} \).

Due to Proposition 1.1, Theorem 6.4, and Corollary 6.5, we obtain the following result for the polydisk.

**Corollary 6.6.** Let \( m \in \mathbb{N} \cup \{\infty\} \) and let \( f(z) = \sum_{p \in \mathbb{Z}_n^k, |p| \leq m} a_p z^p \) be a holomorphic function on the polydisk such that \( f(0) \geq 0 \) and \( \Re f(z) \leq 1 \) for \( z \in \mathbb{D}_k \). Then, for each \( q \in \{1, \ldots, m\} \),

\[
\sup_{\xi \in \mathbb{T}^k} \left| \sum_{p \in \mathbb{Z}_n^k, |p| = q} a_p \xi^p \right| \leq 2(1 - |a_0|) \cos \frac{\pi}{\left\lfloor \frac{m}{q} \right\rfloor + 2}
\]

and

\[
\sum_{q=0}^{m} \sup_{\xi \in \mathbb{T}^k} \left| \sum_{p \in \mathbb{Z}_n^k, |p| = q} a_p \xi^p \right| r^q \leq 1
\]

for any \( r \in [0, t_m] \), where the sequence \( \{t_m\} \) is defined in Theorem 6.4.

We remark the following particular cases of Corollary 6.6.

(i) The result holds for any holomorphic function \( f \) on the polydisk with \( f(0) \geq 0 \) and \( \|f\|_{\infty} \leq 1 \).

(ii) If \( m = \infty \), then \( t_{\infty} = \frac{1}{3} \) is the best positive constant in the Bohr type inequality (see [5]).

(iii) If \( m = \infty \), \( k = 1 \), and \( n_1 = 1 \), we recover Bohr’s inequality (see [5]).

**Corollary 6.7.** Let \( f(z) = \sum_{p \in \mathbb{Z}_n^k, |p| \leq m} a_p z^p \) be a holomorphic function of degree \( m \in \mathbb{N} \cup \{\infty\} \) on the polydisc \( \mathbb{D}_k \) such that \( \Re f(z) \leq 1 \) for \( z \in \mathbb{D}_k \).
(i) If \( f(0) \geq 0 \), then
\[
\left| \left( \frac{\partial f}{\partial z_1} \right)(0) \right| + \cdots + \left| \left( \frac{\partial f}{\partial z_k} \right)(0) \right| \leq 2(1 - f(0)) \cos \frac{\pi}{m + 2}.
\]

(ii) If \( m = \infty \) and \( a = (a_1, \ldots, a_k) \in \mathbb{D}^k \) is such that \( f(a) \geq 0 \), then
\[
\sum_{i=1}^{k} (1 - |a_i|^2) \left| \left( \frac{\partial f}{\partial z_i} \right)(a) \right| \leq 2(1 - f(a)).
\]

**Proof.** Part (i) is obtained from Corollary 6.3 when \( q = 1 \). When \( m = \infty \), the inequality becomes
\[
(6.2) \quad \left| \left( \frac{\partial f}{\partial z_1} \right)(0) \right| + \cdots + \left| \left( \frac{\partial f}{\partial z_k} \right)(0) \right| \leq 2(1 - f(0)).
\]

Part (ii) is an extension of this inequality and can be obtained as follows. Assume that \( a = (a_1, \ldots, a_k) \in \mathbb{D}^k \) is such that \( f(a) \geq 0 \). For each \( i \in \{1, \ldots, k\} \), let \( \varphi_{a_i} \) be the automorphism of \( \mathbb{D} \) given by \( \varphi_{a_i}(z) := \frac{z - a_i}{1 - \overline{a}_i \cdot z} \) for \( z \in \mathbb{D} \). Define
\[
g(z) := f(-\varphi_{a_1}(z_1), \ldots, -\varphi_{a_k}(z_k)), \quad z = (z_1, \ldots, z_k) \in \mathbb{D}^k.
\]
Note that \( g \) is holomorphic on the polydisc such that \( g(0) = f(a) \geq 0 \) and \( \Re g(z) \leq 1 \) for \( z \in \mathbb{D}^k \). Applying inequality (6.2) to \( g \), we obtain
\[
\sum_{i=1}^{k} \left| \left( \frac{\partial f}{\partial z_i} \right)(a) \varphi'_{a_i}(0) \right| \leq 2(1 - g(0)).
\]
Now, taking into account that \( \varphi'_{a_i}(0) = 1 - |a_i|^2 \), we complete the proof. \( \square \)

## 7. Harnack inequalities for free \( k \)-pluriharmonic functions

In this section, we provide Harnack type inequalities for positive free \( k \)-pluriharmonic function with operator coefficients on polyballs.

A bounded linear operator \( A \in B(K \otimes \bigotimes_{i=1}^{k} F^2(H_{n_i})) \) is called \( k \)-multi-Toeplitz with respect to the universal model \( R := (R_1, \ldots, R_k) \), where \( R_i := (R_{i,1}, \ldots, R_{i,n_i}) \), if
\[
(I_K \otimes R_{i,a}^*)A(I_K \otimes R_{i,t}) = \delta_{s,t}A, \quad s, t \in \{1, \ldots, n_i\},
\]
for every \( i \in \{1, \ldots, k\} \). Let \( \mathcal{T}_n \) be the set of all \( k \)-multi-Toeplitz operators on \( K \otimes \bigotimes_{i=1}^{k} F^2(H_{n_i}) \). In [31], we proved that
\[
\mathcal{T}_n = \text{span}\{ f^*g : f, g \in B(K) \otimes_{\text{min}} \mathcal{A}_n \}^{\text{SOT}},
\]
where \( \mathcal{A}_n \) is the polyball algebra. A function \( F \) with operator-valued coefficients in \( B(K) \), where \( K \) is separable Hilbert space, is called free \( k \)-pluriharmonic on the abstract polyball \( B_n \) if it has the form (6.1). In particular, for any \( r \in [0, 1) \),
\[
F(rS) := \sum_{p_1 \in \mathbb{Z}} \cdots \sum_{p_k \in \mathbb{Z}} \sum_{\tilde{a}_{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k} \in \mathbb{F}_n^{\mathbb{Z}}} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \otimes r^{i|\alpha| + j|\beta|} S_1^{\alpha_1} \cdots S_k^{\alpha_k} S_1^{*\beta_1} \cdots S_k^{*\beta_k}
\]
is convergent in the operator norm topology to a \( k \)-multi-Toeplitz operator, with respect to the right universal model \( R = \{R_{i,j}\} \), which is in the operator space \( \text{span}\{ f^*g : f, g \in B(K) \otimes_{\text{min}} \mathcal{A}_n \}^{\text{WOT}} \).

In [24], we introduced a joint numerical radius for \( n \)-tuples of operators \( (T_1, \ldots, T_n) \in B(H)^n \) which turned out to be related to the classical numerical radius by the relation
\[
w(T_1, \ldots, T_n) = \omega(T_1^* \otimes S_1 + \cdots + T_n^* \otimes S_n),
\]
where \( S_1, \ldots, S_n \) are the left creation operators on the full Fock space \( F^2(H_n) \). The joint numerical radius has similar properties to those mentioned, in Section 6, for the classical numerical radius.
Theorem 7.1. Let $F$ be a positive free $k$-pluriharmonic function with operator coefficients and of degree $m_i \in \mathbb{N} \cup \{\infty\}, i \in \{1, \ldots, k\}$, with respect to the variables $(X_{i,1}, \ldots, X_{i,n_i})$. If $\rho := (\rho_1, \ldots, \rho_k) \in [0,1]^k$, then

$$
\|F(X)\| \leq \|F(0)\| \prod_{i=1}^{k} \left(1 + 2 \sum_{\rho_i \neq 0} \rho_i^p \cos \frac{\pi}{m_i} + 2\right) \leq \|F(0)\| \prod_{i=1}^{k} \frac{1 + \rho_i}{1 - \rho_i}
$$

for any $\rho \in \rho B_n(H)^{-}$, where $[x]$ is the integer part of $x$.

Proof. First assume that $m_i \in \mathbb{N}$. Let $A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \in B(K)$, where $\alpha_i, \beta_i \in \mathbb{F}_{m_i}^{+}$, $|\alpha_i| = q_i^{-}, |\beta_i| = q_i^{+}$, and $-m_i \leq q_i \leq m_i$ be the coefficients in the representation of $F$. Then, for any $r \in [0,1)$,

$$
F(\rho) = \sum_{i \leq \|\alpha\| \leq m_k} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \bigotimes_{i=1}^{k} \rho_i^{\alpha_i - \beta_i} S_{i,1} S_{i,2} \cdots S_{i,n_i}
$$

where $S_{i,j}$ are the left creation operators acting on the full Fock space $F^{2}(H_{n_i})$, as defined in Section 1. We also use the notation $S_i = (S_{i,1}, \ldots, S_{i,n_i})$. We remark that $F(\rho)$ is a positive $k$-multi-Toeplitz operator and

$$
F(\rho) = \sum_{\alpha_k \in \mathbb{F}_{m_k}^{+}, 1 \leq |\alpha_k| \leq m_k} C_{(\alpha_k)} \bigotimes_{i=1}^{k} \rho_i^{\alpha_i} S_{k,a_k} + C_0 \bigotimes_{i=1}^{k} \rho_i^{\alpha_i} S_{k,a_k}
$$

Note that $F(\rho)$ is also a positive 1-multi-Toeplitz operator with respect to the right creation operators $R_k = (R_{k,1}, \ldots, R_{k,n_k})$, with coefficients in $B(K) \otimes \min B(F^{2}(H_{n_1})) \otimes \min B(F^{2}(H_{n_2}) \cdots \otimes \min B(F^{2}(H_{n_k-1}))$.

According to Theorem 2.29 from [24], for each $p_k \in \{1, \ldots, m_k\}$, the joint numerical radius satisfies the inequality

$$
\rho_k^{p_k} w (C_{(\alpha_k)} : |\alpha_k| = p_k) \leq \langle C_0 \rangle \cos \frac{\pi}{m_k} + 2
$$

Note that $C_0 \bigotimes_{i=1}^{k} \rho_i^{\alpha_i} S_{i,a_k} = F(p_1 S_{i,1}, \ldots, p_k S_{i,n_k}, 0)$ is a positive $(k-1)$-multi-Toeplitz operator. Using the properties of the numerical radius, we deduce that

$$
\omega (F(\rho)) \leq \omega (C_0 \bigotimes_{i=1}^{k} \rho_i^{\alpha_i} S_{i,a_k}) + 2 \sum_{p_k = 1}^{m_k} \rho_k^{p_k} \omega \left( \sum_{\alpha_k \in \mathbb{F}_{m_k}^{+}, |\alpha_k| = p_k} C_{(\alpha_k)} \bigotimes_{i=1}^{k} S_{i,a_k} \right).
$$

Let $Y_1, \ldots, Y_{n_k}^{p_k}$ be an enumeration of the operators $C_{(\alpha_k)}^{*}$, where $\alpha_k \in \mathbb{F}_{n_k}^{+}$ and $|\alpha_k| = p_k$, and let $\tilde{S}_1, \ldots, \tilde{S}_{n_k}^{p_k}$ be the left creation operators on the full Fock space $F^{2}(H_{n_k})$ with $n_k^{p_k}$ generators. According to relation [24], we have

$$
w \left( C_{(\alpha_k)} : |\alpha_k| = p_k \right) = \omega \left( \sum_{j=1}^{n_k^{p_k}} Y_{j}^{*} \otimes \tilde{S}_j \right).
$$

On the other hand, since $[S_{k,a_k} : \alpha_k \in \mathbb{F}_{n_k}^{+}, |\alpha_k| = p_k]$ is a pure row isometry, it is unitarily equivalent to $[\tilde{S}_j \otimes I_{\mathcal{G}} : j \in \{1, \ldots, n_k^{p_k}\}]$ for some separable Hilbert space $\mathcal{G}$, due to the Wold type decomposition [20] for row isometries. Consequently, we obtain

$$
w \left( \sum_{j=1}^{n_k^{p_k}} Y_{j}^{*} \otimes \tilde{S}_j \right) = \omega \left( \sum_{\alpha_k \in \mathbb{F}_{n_k}^{+}, |\alpha_k| = p_k} C_{(\alpha_k)} \bigotimes_{i=1}^{k} S_{k,a_k} \right).
$$

Hence and using relations [24] and [24], we deduce that

$$
\omega (F(p_1 S_{i,1}, \ldots, p_k S_{i,k})) \leq \omega (F(p_1 S_{i,1}, \ldots, p_k-1 S_{i,k-1}, 0)) \left[ 1 + 2 \sum_{p_k = 1}^{m_k} \rho_k^{p_k} \cos \frac{\pi}{m_k} + 2 \right].
$$
Let us show that this inequality remains true even when $m_k = +\infty$. Indeed, in this case we have $0 \leq F(\rho S) \leq G(\rho S)$, where

$$G(\rho S) := \sum_{\alpha_k \in \mathbb{F}_k^+, 1 \leq |\alpha_k| \leq q_k} C_{(\alpha_k)}^* \rho_k^{\alpha_k} |S_{k, \alpha_k}^*| + C_{(\alpha_k)} \rho_k^{\alpha_k} |S_{k, \alpha_k}|$$

and

$$M_{q_k}(\rho) := \left\| \sum_{\alpha_k \in \mathbb{F}_k^+, |\alpha_k| > q_k} C_{(\alpha_k)}^* \rho_k^{\alpha_k} |S_{k, \alpha_k}^*| + \sum_{\alpha_k \in \mathbb{F}_k^+, |\alpha_k| > q_k} C_{(\alpha_k)} \rho_k^{\alpha_k} |S_{k, \alpha_k}| \right\|$$

Note that $M_{q_k}(\rho) \to 0$ as $q_k \to \infty$. As in the proof above, but written for $G(\rho S)$, and taking $q_k \to \infty$, we obtain

$$\omega(F(p_1 S_1, \ldots, p_k S_k)) \leq \omega(F(p_1 S_1, \ldots, p_{k-1} S_{k-1}, 0)) \left[ 1 + 2 \sum_{p_k = 1} \rho_k^{p_k} \right] \leq \omega(F(p_1 S_1, \ldots, p_{k-1} S_{k-1}, 0)) \frac{1 + p_k}{1 - p_k}$$

which can be obtained from (1.2) passing to the limit as $m_k \to \infty$. Therefore, the inequality (1.2) is valid if $m_k \in \mathbb{N} \cup \{\infty\}$. Since $F(p_1 S_1, \ldots, p_{k-1} S_{k-1}, 0)$ is a positive $k-1$ multi-Toeplitz operator with coefficients in $B(\mathcal{E}) \otimes_{\min} B(F^2(H_{n_1})) \otimes_{\min} \cdots \otimes_{\min} B(F^2(H_{n_{k-1}}))$, as above, we obtain the inequality

$$\omega(F(p_1 S_1, \ldots, p_{k-1} S_{k-1}, 0)) \leq \omega(F(p_1 S_1, \ldots, p_{k-2} S_{k-2}, 0, 0)) \left[ 1 + 2 \sum_{p_k = 1} \rho_k^{p_k-1} \cos \frac{\pi}{m_k - 2p_k - 1} \right].$$

Continuing this process, and putting together the inequalities obtained, we deduce that

$$\omega(F(\rho S)) = \omega(F(p_1 S_1, \ldots, p_k S_k)) \leq \omega(F(0)) \prod_{i=1}^k \left( 1 + 2 \sum_{p_i = 1} \rho_i^{p_i} \cos \frac{\pi}{m_i - 2p_i - 1} \right).$$

According to the multivariable dilation theory [27], each element $X = \{X_{i,j}\}$ in the polyball $B_n(\mathcal{H})$, has a dilation $\{S_{i,j} \otimes I_M\}$ for some separable Hilbert space $M$ such that $X_{i,j} = (S_{i,j} \otimes I_M)|_M$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Using to the properties of the numerical radius, we obtain

$$\omega(F(\rho X)) = \omega(F(p_1 X_1, \ldots, p_k X_k)) \leq \omega(F(p_1 S_1, \ldots, p_k S_k)) = \omega(F(\rho S)),$$

which combined with the latter inequality and the fact that the numerical radius of a positive operator coincides with its norm, implies the inequality of the theorem. The proof is complete.

**Corollary 7.2.** Let $F$ be a positive free $k$-pluriharmonic function with scalar coefficients and of degree $m_i \in \mathbb{N} \cup \{\infty\}$, $i \in \{1, \ldots, k\}$, with respect to the variables $(X_{i,1}, \ldots, X_{i,n_i})$, then

$$F(X) \leq F(0) \prod_{i=1}^k \left( 1 + 2 \sum_{p_i = 1} \rho_i^{p_i} \cos \frac{\pi}{m_i - 2p_i - 1} \right) \leq F(0) \prod_{i=1}^k \frac{1 + \rho_i}{1 - \rho_i}$$

for any $X \in \rho B_n(\mathcal{H})^-$ and $\rho := (\rho_1, \ldots, \rho_k) \in [0,1]^k$, where $[x]$ is the integer part of $x$.

**References**

[1] L. Aizenberg, Multidimensional analogues of Bohr’s theorem on power series, *Proc. Amer. Math. Soc.* **128** (2000), 1147–1155.

[2] L. Aizenberg and A. Vidyas, On the Bohr radius of two classes of holomorphic functions. (Russian) ; translated from *Sibirsk. Mat. Zh.*, **45** (2004), no. 4, 734–746 *Siberian Math. J.* **45** (2004), no. 4, 606–617.

[3] F. Bayart, D. Pellegrino, J.B. Seoane-Sepúlveda, The Bohr radius of the $n$-dimensional polydisk is equivalent to $\sqrt{\log n}/n$, *Adv. Math.* **264** (2014), 726–746.

[4] H.P. Boas and D. Khavinson, Bohr’s power series theorem in several variables, *Proc. Amer. Math. Soc.* **125** (1997), 2975–2979.
[5] H. Bohr, A theorem concerning power series, Proc. London Math. Soc. 2 (13) (1914), 1–5.
[6] E. Bombieri and J. Bourgain, A remark on Bohr’s inequality, Int. Math. Res. Not. 2004, no. 80, 4307–4330.
[7] Carathéodory, Über den Variabilitätsbereich der Koeffizienten der Potenzreihen, die gegebene Werte nicht annehmen, Math. Ann. 64 (1907), 95–115.
[8] A. Defant, L. Frerick, J. Ortega-Cerd, M. Ounaïes, and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, Ann. of Math. (2) 174 (2011), no. 1, 485–497.
[9] A. Defant, M. Maestre, and U. Schwarting, Bohr radii of vector valued holomorphic functions, Adv. Math. 231 (2012), no. 5, 2837–2857.
[10] S. Dineen and R.M. Timoney, On a problem of H. Bohr, Bull. Soc. Roy. Sci. Liege 60 (1991), 401–404.
[11] P.G. Dixon, Banach algebras satisfying the non-unital von Neumann inequality, Bull. London Math. Soc. 27 (1995), 359–362.
[12] E.V. Egerváry and O. Szász, Einige Extremalprobleme im Bereiche der trigonometrischen Polynome, Math. Zeitschrift 27 (1928), 641–652.
[13] L. Fejér, Über trigonometrische Polynome, J. Reine Angew. Math. 146 (1916), 53–82.
[14] A. Korányi and L. Pukánszky, Holomorphic functions with positive real part on polycylinders, Trans. Amer. Math. Soc. 108 (1963), No. 3, 449–456.
[15] E. Landau and D. Gaier, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, Springer-Verlag, 1986.
[16] V.I. Paulsen, Completely Bounded Maps and Dilations, Pitman Research Notes in Mathematics, Vol. 146, New York, 1986.
[17] V.I. Paulsen, G. Popescu, and D. Singh, On Bohr’s inequality, Proc. London Math. Soc. 85 (2002), 493–512.
[18] V.I. Paulsen and D. Singh, Extensions of Bohr’s inequality, Bull. London Math. Soc. 38 (2006), no. 6, 991–999.
[19] G. Pisier, Similarity Problems and Completely Bounded Maps, Springer Lect. Notes Math., Vol.1618, Springer-Verlag, New York, 1995.
[20] G. Popescu, Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc. 316 (1989), 523–536.
[21] G. Popescu, Von Neumann inequality for \( (B(H)^n)_{1} \), Math. Scand. 68 (1991), 292–304.
[22] G. Popescu, Poisson transforms on some \( C^* \)-algebras generated by isometries, J. Funct. Anal. 161 (1999), 27–61.
[23] G. Popescu, Multivariable Bohr inequalities, Trans. Amer. Math. Soc. 359 (2007), no. 11, 5283–5317.
[24] G. Popescu, Unitary invariants in multivariable operator theory, Mem. Amer. Math. Soc., 200 (2009), No.941, vi+91 pp.
[25] G. Popescu, Operator theory on noncommutative domains, Mem. Amer. Math. Soc. 205 (2010), no. 964, vi+124 pp.
[26] G. Popescu, Berezin transforms on noncommutative varieties in polydomains, J. Funct. Anal. 265 (2013), no. 10, 2500–2552.
[27] G. Popescu, Berezin transforms on noncommutative polydomains, Trans. Amer. Math. Soc. 368 (2016), 4357–4416.
[28] G. Popescu, Curvature invariant on noncommutative polyballs, Adv. Math. 279 (2015), 104–158.
[29] G. Popescu, Euler characteristic on noncommutative polyballs, J. Reine Angew. Math. 728 (2017), 195–236.
[30] G. Popescu, Holomorphic automorphisms of noncommutative polyballs, J. Operator Theory 76 (2016), no. 2, 387–448.
[31] G. Popescu, Free pluriharmonic functions on noncommutative polyballs, Analysis & PDE 9 (2016), 1185–1234.
[32] W. Rudin, Function theory in polydiscs, W. A. Benjamin, Inc., New York-Amsterdam 1969 vii+188 pp.
[33] W. Rudin, Function theory in the unit ball of \( C^n \), Springer-verlag, New-York/Berlin, 1980.
[34] S. Sidon, Über einen Satz von Herrn Bohr, Math. Z. 26 (1927), 731–732.
[35] M. Tomić, Sur un theoreme de H. Bohr, Math. Scand. 1 (1962), 103–106.
[36] J. von Neumann, Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes, Math. Nachr. 4 (1951), 258–281.

Department of Mathematics, The University of Texas at San Antonio, San Antonio, TX 78249, USA
E-mail address: gelu.popescu@utsa.edu