EXTREMAL SPECTRAL PROPERTIES OF LAWSON TAU-SURFACES AND THE LAMÉ EQUATION

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Abstract. Extremal spectral properties of Lawson tau-surfaces are investigated. The Lawson tau-surfaces form a two-parametric family of tori or Klein bottles minimally immersed in the standard unitary three-dimensional sphere. A Lawson tau-surface carries an extremal metric for some eigenvalue of the Laplace-Beltrami operator. Using theory of the Lamé equation we find explicitly these extremal eigenvalues.

INTRODUCTION

Let $M$ be a closed surface and $g$ be a Riemannian metric on $M$. Let us consider the associated Laplace-Beltrami operator $\Delta : C^\infty(M) \rightarrow C^\infty(M)$,

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right).$$

It is well-known that the eigenvalues

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \lambda_3(M, g) \leq \ldots$$

of $\Delta$ possess the following rescaling property,

$$\forall t > 0 \quad \lambda_i(M, tg) = \frac{\lambda_i(M, g)}{t}.$$

Hence, it is not a good idea to look for a supremum of the functional $\lambda_i(M, g)$ over the space of Riemannian metrics $g$ on a fixed surface $M$. But the functional

$$\Lambda_i(M, g) = \lambda_i(M, g) \text{Area}(M, g)$$

is invariant under rescaling transformations $g \mapsto tg$.

It turns out that the question about the supremum $\sup \Lambda_i(M, g)$ of the functional $\Lambda_i(M, g)$ over the space of Riemannian metrics $g$ on a fixed surface $M$ is very difficult and only few results are known.

In 1980 it was proven by Yang and Yau in the paper $[1]$ that for an orientable surface $M$ of genus $\gamma$ the following inequality holds,

$$\Lambda_1(M, g) \leq 8\pi(\gamma + 1).$$

A generalization of this result for an arbitrary $\Lambda_i$ was found in 1993 by Korevaar. It is proven in the paper $[2]$ that there exists a constant $C$ such that for any $i > 0$ and any compact surface $M$ of genus $\gamma$ the functional $\Lambda_i(M, g)$ is bounded,

$$\Lambda_i(M, g) \leq C(\gamma + 1)i.$$

It should be remarked that in 1994 Colbois and Dodziuk proved in the paper $[3]$ that for a manifold $M$ of dimension $\dim M \geq 3$ the functional $\Lambda_i(M, g)$ is not bounded on the space of Riemannian metrics $g$ on $M$.

2000 Mathematics Subject Classification. 58E11, 58J50.

Key words and phrases. Lawson minimal surfaces, extremal metric, Lamé equation, Magnus-Winkler-Ince equation.
The functional $\Lambda_i(M, g)$ depends continuously on the metric $g$, but this functional is not differentiable. However, it was shown in 1973 by Berger in the paper \cite{4} that for analytic deformations $g_t$ the left and right derivatives of the functional $\Lambda_i(M, g_t)$ with respect to $t$ exist. This led to the following definition, see the paper \cite{5} by Nadirashvili (1986) and the paper \cite{6} by El Soufi and Ilias (2000).

**Definition 1.** A Riemannian metric $g$ per \cite{4} that for analytic deformations $g_t$ the left and right derivatives of the functional $\Lambda_i(M, g_t)$ with respect to $t$ exist.

The list of surfaces $M$ and values of index $i$ such that the maximal or at least extremal metrics for the functional $\Lambda_i(M, g)$ are known is quite short.

- $\Lambda_1(S^2, g)$ Hersch proved in 1970 in the paper \cite{7} that $\sup \Lambda_1(S^2, g) = 8\pi$ and the maximum is reached on the canonical metric on $S^2$. This metric is the unique extremal metric.

- $\Lambda_1(\mathbb{R}P^2, g)$ Li and Yau proved in 1982 in the paper \cite{8} that $\sup \Lambda_1(\mathbb{R}P^2, g) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$. This metric is the unique extremal metric.

- $\Lambda_1(T^2, g)$ Nadirashvili proved in 1996 in the paper \cite{9} that $\sup \Lambda_1(T^2, g) = \frac{4\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus. El Soufi and Ilias proved in 2000 in the paper \cite{10} that the only extremal metric for $\Lambda_1(T^2, g)$ different from the maximal one is the metric on the Clifford torus.

- $\Lambda_1(\mathbb{K}, g)$ Jakobson, Nadirashvili and Polterovich proved in 2006 in the paper \cite{11} that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde{\tau}_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the same year in the paper \cite{12} that this metric is the unique extremal metric and the maximal one. Here $\sup \Lambda_1(\mathbb{K}, g) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where $E$ is a complete elliptic integral of the second kind, we recall its definition in the end of Introduction.

- $\Lambda_2(S^2, g)$ Nadirashvili proved in 2002 in the paper \cite{13} that $\sup \Lambda_2(S^2, g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together.

- $\Lambda_1(T^2, g), \Lambda_1(\mathbb{K}, g)$ Let $r, k \in \mathbb{N}$, $0 < k < r$, $(r, k) = 1$. Lapointe studied bipolar surfaces $\tilde{\tau}_{r, k}$ of Lawson tau-surfaces $\tau_{r, k}$ and proved the following result published in 2008 in the paper \cite{14}.

  (1) If $rk \equiv 0 \mod 2$ then $\tilde{\tau}_{r, k}$ is a torus and it carries an extremal metric for $\Lambda_{4r-2}(T^2, g)$.

  (2) If $rk \equiv 1 \mod 4$ then $\tilde{\tau}_{r, k}$ is a torus and it carries an extremal metric for $\Lambda_{2r-2}(T^2, g)$.

  (3) If $rk \equiv 3 \mod 4$ then $\tilde{\tau}_{r, k}$ is a Klein bottle and it carries an extremal metric for $\Lambda_{r-2}(\mathbb{K}, g)$.

We should also mention the paper \cite{15} published in 2005 by Jakobson, Levitin, Nadirashvili, Nigam and Polterovich. It is shown in this paper using a combination of analytic and numerical tools that the maximal metric for the first eigenvalue on the surface of genus two is the metric on the Bolza surface $P$ induced from the canonical metric on the sphere using the standard covering $P \rightarrow S^2$. In fact, the authors state this result as a conjecture, because a part of the argument is based on a numerical calculation.

The goal of the present paper is to study extremal spectral properties of metrics on Lawson tau-surfaces $\tau_{m, k}$. 
Definition 2. A Lawson tau-surface \( \tau_{m,k} \subseteq S^3 \) is defined by the doubly-periodic immersion \( \Psi_{m,k} : \mathbb{R}^2 \rightarrow S^3 \subseteq \mathbb{R}^4 \) given by the following explicit formula,

\[
\Psi_{m,k}(x,y) = (\cos(mx) \cos y, \sin(mx) \cos y, \cos(kx) \sin y, \sin(kx) \sin y).
\]

This family of surfaces is introduced in 1970 by Lawson in the paper [14]. He proved that for each unordered pair of positive integers \((m, k)\) with \((m, k) = 1\) the surface \( \tau_{m,k} \) is a distinct compact minimal surface in \( S^3 \). Let us impose the condition \((m, k) = 1\). If both integers \(m\) and \(k\) are odd then \( \tau_{m,k} \) is a torus. We call it a Lawson torus. If one of integers \(m\) and \(k\) is even then \( \tau_{m,k} \) is a Klein bottle. We call it a Lawson Klein bottle. Remark that \( m \) and \( k \) cannot both be even due to the identity \((m, k) = 1\). The torus \( \tau_{1,1} \) is the Clifford torus.

Since \( \tau_{m,k} \cong \tau_{k,m} \), we can fix a convenient order of \( m \) and \( k \). If \( \tau_{m,k} \) is a Lawson torus, i.e. \( m \) and \( k \) are both odd, \((m, k) = 1\), let us suppose that always \( m > k > 0 \) except the special case of the Clifford torus \( \tau_{1,1} \). If \( \tau_{m,k} \) is a Lawson Klein bottle, i.e. one of numbers \( m \) and \( k \) is even, \((m, k) = 1\), let us suppose that always \( m \) is even and \( k \) is odd.

It is clear that the map \( \Psi \) has periods \( T_1 = (2\pi, 0) \) and \( T_2 = (0, 2\pi) \). However, in the case of a Lawson torus \( \tau_{m,k} \) the smallest period lattice is generated by \( T_3 = (\pi, \pi) \) and \( T_4 = (\pi, -\pi) \). Hence, a Lawson torus \( \tau_{m,k} \) is isometric to the torus \( \mathbb{R}^2/\{aT_1 + bT_2 | a, b \in \mathbb{Z}\} \) with the metric induced by the immersion \( \Psi \). We identify these tori.

The torus \( \mathbb{R}^2/\{aT_1 + bT_2 | a, b \in \mathbb{Z}\} \) with the metric induced by the immersion \( \Psi \) is a double cover of the Lawson torus \( \tau_{m,k} \). We denote this double cover by \( \tilde{\tau}_{m,k} \). When it is necessary to have uniquely defined coordinates of a point on \( \tilde{\tau}_{m,k} \), we consider coordinates \((x, y) \in [0, 2\pi) \times [-\pi, \pi)\). Functions on a Lawson torus \( \tau_{m,k} \) are in one-to-one correspondence with functions on the double cover \( \tilde{\tau}_{m,k} \) invariant with respect to the translation by \( T_3 \).

In the case of a Lawson Klein bottle \( \tau_{m,k} \) the immersion \( \Psi \) value is invariant under transformations

\[
(x, y) \mapsto (x + \pi, -y), \quad (x, y) \mapsto (x, y + 2\pi).
\]

When it is necessary to have uniquely defined coordinates of a point on a Lawson Klein bottle \( \tau_{m,k} \), we consider coordinates \((x, y) \in [0, \pi) \times [-\pi, \pi)\).

The main result of the present paper is the following Theorem. Here \([\alpha]\) denotes the integer part of a real number \( \alpha \) and \( E \) is the complete elliptic integral of the second kind, we recall its definition in the end of the Introduction.

**Theorem.** Let \( \tau_{m,k} \) be a Lawson torus. We can assume that \( m, k \equiv 1 \mod 2 \), \((m, k) = 1\). Then the induced metric on its double cover \( \tilde{\tau}_{m,k} \) is an extremal metric for the functional \( \Lambda_j(\mathbb{T}^2, g) \), where

\[
j = 2 \left( \left[ \sqrt{m^2 + k^2} \right] + m + k \right) - 1.
\]

The corresponding value of the functional is

\[
\Lambda_j(\tilde{\tau}_{m,k}) = 16\pi m E \left( \frac{\sqrt{m^2 - k^2}}{m} \right).
\]

The induced metric on \( \tau_{m,k} \) is an extremal metric for the functional \( \Lambda_j(\mathbb{T}^2, g)\), where

\[
j = 2 \left( \frac{\sqrt{m^2 + k^2}}{2} \right) + m + k - 1.
\]
The corresponding value of the functional is
\[ \Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right). \]

Let \( \tau_{m,k} \) be a Lawson Klein bottle. We can assume that \( m \equiv 0 \mod 2, k \equiv 1 \mod 2, (m,k) = 1 \). Then the induced metric on \( \tau_{m,k} \) is an extremal metric for the functional \( \Lambda_j(\mathbb{K}, g) \), where
\[ j = 2 \left[ \frac{\sqrt{m^2+k^2}}{2} \right] + m + k - 1. \]

The corresponding value of the functional is
\[ \Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right). \]

We investigate the case of a double cover \( \tilde{\tau}_{m,k} \) since in this case we have separation of variables in the corresponding spectral problem. We reduce the case of Lawson tori to the case of their double covers.

We already described above the result by Lapointe from the paper \([12]\). It implies that if \( rk \equiv 0 \) or \( rk \equiv 1 \mod 4 \) then the surfaces \( \tilde{\tau}_{r,k} \) bipolar to Lawson tau-surfaces are tori and the metrics on these tori are extremal for a functional \( \Lambda_j \) with an even \( j \). In contrast with this result, we prove that the metrics on the Lawson tori \( \tau_{m,k} \) are extremal for a functional \( \Lambda_j \) with an odd \( j \). Hence we provide in this paper extremal metrics on the torus for eigenvalues such that extremal metrics for them were not known before. A similar situation with Klein bottles, Lapointe provided extremal metrics on the Klein bottle for eigenvalues \( \Lambda_j \) with an even \( j \) and we provide extremal metrics on the Klein bottle for eigenvalues \( \Lambda_j \) with an odd \( j \).

In the same time we should remark that we do not know at this moment extremal metrics for any \( \Lambda_j \) and even an extremal metric for \( \Lambda_2(\mathbb{T}^2, g) \) is yet unknown.

The proof of the Theorem consists of several steps. We start by using a beautiful result relating extremal metrics to minimal immersions in spheres proved by El Soufi and Ilias in the paper \([15]\). This result reduces calculating \( j \) to counting the eigenvalues of the Laplace-Beltrami operator \( \Delta \) on a Lawson tau-surface \( \tau_{m,k} \). Then we reduce this problem to counting eigenvalues of a periodic Sturm-Liouville problem. A crucial idea of this counting is based on the relation to the Lamé equation \((20)\).

We use on several occasions the complete elliptic integral of the first kind \( K(k) \) and the complete elliptic integral of the second kind \( E(k) \) defined by formulae
\[ K(k) = \int_0^1 \frac{da}{\sqrt{1-a^2} \sqrt{1-k^2 a^2}}, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2 a^2}}{\sqrt{1-a^2}} da. \]

1. Minimal submanifolds of a sphere and extremal spectral property of their metrics

Let us recall two important results about minimal submanifolds of a sphere. Let \( N \) be a \( d \)-dimensional minimal submanifold of the sphere \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \) of radius \( R \). Let \( \Delta \) be the Laplace-Beltrami operator on \( N \) equipped with the induced metric.

The first result is a classical one. Its proof can be found e.g. in the book \([16]\).

**Proposition 1.** The restrictions \( x^1|_N, \ldots, x^{n+1}|_N \) on \( N \) of the standard coordinate functions of \( \mathbb{R}^{n+1} \) are eigenfunctions of \( \Delta \) with eigenvalue \( \frac{d}{R^2} \).
Let us numerate the eigenvalues of $\Delta$ as in formula (1) counting them with multiplicities

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$$

Proposition 1 implies that there exists at least one index $i$ such that $\lambda_i = \frac{d}{R^2}$. Let $j$ denote the minimal number $i$ such that $\lambda_i = \frac{d}{R^2}$. Let us introduce the eigenvalues counting function

$$N(\lambda) = \# \{ \lambda_i \mid \lambda_i < \lambda \}.$$  

We see that $j = N(\frac{d}{R^2})$. Remark that we count the eigenvalues starting from $\lambda_0 = 0$.

The second result was published in 2008 by El Soufi and Ilias in the paper [15]. It could be written in the following form.

**Proposition 2.** The metric $g_0$ induced on $N$ by immersion $N \subset S^n$ is an extremal metric for the functional $\Lambda_N(\frac{d}{R^2})(N, g)$.

We should remark that isometric immersions by eigenfunctions of the Laplace-Beltrami operator were studied in the paper [17] by Takahashi. The results of Takahashi are used by El Soufi and Ilias in the paper [15].

Proposition 2 implies immediately the following one.

**Proposition 3.** The metric $g_0$ induced on a Lawson torus or Klein bottle $\tau_{m,k}$ by its immersion $\tau_{m,k} \subset S^3$ is an extremal metric for the functional $\Lambda_N(\frac{d}{R^2})(T^2, g)$ or $\Lambda_N(\frac{d}{R^2})(K, g)$, respectively. The similar statement holds also for the double cover $\tilde{\tau}_{m,k}$.

**Proof.** As we mentioned in the Introduction, Lawson proved in the paper [14] that $\tau_{m,k}$ is a complete minimal surface in the sphere $S^3$ of radius 1. The statement follows immediately from Proposition 2, where $R = 1$ and $d = 2.$ □

Proposition 3 is crucial for this paper. It reduces investigation of extremal spectral properties of the Lawson tau-surfaces to counting eigenvalues $\lambda_i$ of the Laplace-Beltrami operator such that $\lambda_i < 2$.

## 2. Eigenvalues of the Laplace-Beltrami operator on Lawson tau-surfaces and auxiliary periodic Sturm-Liouville problem

**Proposition 4.** Let $\tau_{m,k} \subset S^3$ be a Lawson tau-surface and

$$p(y) = \sqrt{k^2 + (m^2 - k^2) \cos^2 y}.$$  

Then the induced metric is equal to

$$p^2(y) dx^2 + dy^2$$

and the Laplace-Beltrami operator is given by the following formula,

$$\Delta f = -\frac{1}{p^2(y)} \frac{\partial^2 f}{\partial x^2} - \frac{1}{p(y)} \frac{\partial}{\partial y} \left( p(y) \frac{\partial f}{\partial y} \right).$$

The same holds for a double cover $\tilde{\tau}_{m,k}$.

**Proof.** is by direct calculation. □

Counting eigenvalues is known to be a difficult problem. Fortunately, we can reduce this problem to a one-dimensional one. Let us recall that a function $\varphi(y)$ is called $\pi$-antiperiodic if $\varphi(y + \pi) = -\varphi(y)$.

**Proposition 5.** Let $\varphi(l, y)$ be a solution of a periodic Sturm-Liouville problem

(3) $$-\frac{1}{p(y)} \frac{d}{dy} \left( \frac{p(y) \frac{d\varphi(y)}{dy}}{p(y)} \right) + \left( \frac{l^2}{p(y)^2} - \lambda \right) \varphi(y) = 0,$$

(4) $$\varphi(y + 2\pi) \equiv \varphi(y).$$
Let \( \hat{\tau}_{m,k} \) be a double cover of a Lawson torus. Then functions
\[
\varphi(l, y) \cos(l x), \quad l = 0, 1, 2, \ldots,
\]
and
\[
\varphi(l, y) \sin(l x), \quad l = 1, 2, 3, \ldots,
\]
form a basis in the space of eigenfunctions of the Laplace-Beltrami operator \( \Delta \) with eigenvalue \( \lambda \).

Let \( \tau_{m,k} \) be a Lawson torus. Then functions
\[
\varphi(l, y) \cos(l x), \quad l = 0, 2, 4, \ldots,
\]
and
\[
\varphi(l, y) \sin(l x), \quad l = 2, 4, 6, \ldots,
\]
where \( \varphi(l, y) \) is a \( \pi \)-periodic solution of the Sturm-Liouville problem \((3)\), and functions
\[
\varphi(l, y) \cos(l x), \quad l = 1, 3, 5, \ldots,
\]
and
\[
\varphi(l, y) \sin(l x), \quad l = 1, 3, 5, \ldots,
\]
where \( \varphi(l, y) \) is a \( \pi \)-antiperiodic solution of the Sturm-Liouville problem \((3)\), form a basis in the space of eigenfunctions of the Laplace-Beltrami operator \( \Delta \) with eigenvalue \( \lambda \).

Let \( \tau_{m,k} \) be a Lawson Klein bottle. Then functions
\[
\varphi(l, y) \cos(l x), \quad l = 0, 2, 4, \ldots,
\]
and
\[
\varphi(l, y) \sin(l x), \quad l = 2, 4, 6, \ldots,
\]
where \( \varphi(l, y) \) is an even solution of the periodic Sturm-Liouville problem \((3), (4)\), and functions
\[
\varphi(l, y) \cos(l x), \quad l = 1, 3, 5, \ldots,
\]
and
\[
\varphi(l, y) \sin(l x), \quad l = 1, 3, 5, \ldots,
\]
where \( \varphi(l, y) \) is an odd solution of the periodic Sturm-Liouville problem \((3), (4)\), form a basis in the space of eigenfunctions of the Laplace-Beltrami operator \( \Delta \) with eigenvalue \( \lambda \).

**Proof.** Let us remark that \( \Delta \) commutes with \( \frac{\partial}{\partial y} \). It follows that \( \Delta \) has a basis of eigenfunctions of the form \( \varphi(l, y) \cos(l x) \) and \( \varphi(l, y) \sin(l x) \). Substituting these eigenfunctions into the formula \( \Delta f = \lambda f \), we obtain equation \((3)\). However, these solutions should be invariant under transformations

\[
(x, y) \mapsto (x + 2\pi, y), \quad (x, y) \mapsto (x, y + 2\pi)
\]

in the case of a double cover \( \hat{\tau}_{m,k} \), the same transformations plus

\[
(x, y) \mapsto (x + \pi, y + \pi)
\]

in the case of a Lawson torus, and

\[
(x, y) \mapsto (x + \pi, -y), \quad (x, y) \mapsto (x, y + 2\pi)
\]

in the case of a Lawson Klein bottle.

These conditions imply the periodicity conditions \((4)\) and the conditions on \( l \) and parity or (anti-)periodicity of \( \varphi \). □

We consider \( l \) as a parameter in equation \((3)\) and in its solutions \( \varphi(l, y) \). For example, we assume \( l \) fixed and consider \( y \) as an independent variable when we discuss zeroes of the function \( \varphi(l, y) \).
Let us rewrite equation (3) in the standard form of a Sturm-Liouville problem,
\begin{equation}
(p(y)\varphi'(y))' + \left(\lambda p(y) - \frac{l^2}{p(y)}\right)\varphi(y) = 0.
\end{equation}
Since \( p(y) > 0 \), the following proposition holds, see e.g. the book [18].

**Proposition 6.** There are an infinite number of eigenvalues \( \lambda_i(l) \) of the periodic Sturm-Liouville problem (3), (4). The eigenvalues \( \lambda_i(l) \) form a sequence such that
\[ \lambda_0(l) < \lambda_1(l) \leq \lambda_2(l) < \lambda_3(l) \leq \lambda_4(l) < \lambda_5(l) \leq \lambda_6(l) < \ldots \]
For \( \lambda = \lambda_0(l) \) there exists a unique (up to a multiplication by a non-zero constant) eigenfunction \( \varphi_0(l, y) \). If \( \lambda_{2i+1}(l) < \lambda_{2i+2}(l) \) for some \( i \geq 0 \) then there is a unique (up to a multiplication by a non-zero constant) eigenfunction \( \varphi_{2i+1}(l, y) \) with eigenvalue \( \lambda = \lambda_{2i+1}(l) \) of multiplicity one and there is a unique (up to a multiplication by a non-zero constant) eigenfunction \( \varphi_{2i+2}(l, y) \) with eigenvalue \( \lambda = \lambda_{2i+2}(l) \) of multiplicity one. If \( \lambda_{2i+1}(l) = \lambda_{2i+2}(l) \) then there are two independent eigenfunctions \( \varphi_{2i+1}(l, y) \) and \( \varphi_{2i+2}(l, y) \) with eigenvalue \( \lambda = \lambda_{2i+1}(l) \) of multiplicity two.

The eigenfunction \( \varphi_0(l, y) \) has no zeroes on \([0, 2\pi]\). The eigenfunctions \( \varphi_{2i+1}(l, y) \) and \( \varphi_{2i+2}(l, y) \), \( i \geq 0 \), each have exactly \( 2i + 2 \) zeroes on \([0, 2\pi]\).

As usual, we can rewrite our periodic Sturm-Liouville problem as a periodic problem for a Hill equation.

**Proposition 7.** Equation (3) is equivalent to a Hill equation
\begin{equation}
-z''(y) + V(l, y)z(y) = \lambda z(y),
\end{equation}
where
\[ V(l, y) = \frac{l^2}{p(y)^2} + \frac{1}{4} \left( \frac{p'(y)}{p(y)} \right)^2 + \frac{1}{2} \left( \frac{p'(y)}{p(y)} \right)' \]
Periodic boundary condition (4) for equation (8) is equivalent to the periodic boundary condition
\begin{equation}
z(y + 2\pi) = z(y).
\end{equation}

**Proof.** A direct calculation shows that the change of variable \( z(y) = \sqrt{p(y)}\varphi(y) \) transforms equation (3) into equation (8). Since the function \( p(y) \) is \( 2\pi \)-periodic, boundary conditions (4) and (3) are equivalent. \( \Box \)

Since \( p(y) \) is an even \( \pi \)-periodic function, the following propositions hold.

**Proposition 8.** The solution \( \varphi_0(l, y) \) is even. If \( \lambda_i(l) \), \( i > 0 \), is of multiplicity one then the solution \( \varphi_i(l, y) \) is even or odd. If \( \lambda_{2i+1}(l) = \lambda_{2i+2}(l) \) then two independent eigenfunctions \( \varphi_{2i+1}(l, y) \) and \( \varphi_{2i+2}(l, y) \) with eigenvalue \( \lambda_{2i+1}(l) = \lambda_{2i+1}(l) \) of multiplicity two could be chosen in such a way that one of them is even and another is odd.

**Proof.** Applying Theorem 1.1 from the book [19] to equation (8) with periodic condition (9) and returning then back from \( z(y) \) to \( \varphi(y) \), we obtain the statement. \( \Box \)

**Proposition 9.** The solution \( \varphi_0(l, y) \) is \( \pi \)-periodic. The solutions \( \varphi_{2i+1}(l, y) \) and \( \varphi_{2i+2}(l, y) \) are \( \pi \)-periodic if \( i \) is odd and \( \pi \)-antiperiodic if \( i \) is even.

**Proof.** Applying Theorem 3.1 from Chapter VIII of the book [18] with period \( \pi \) and \( 2\pi \) and comparing, we immediately obtain the statement. \( \Box \)

It is easy now to establish a relation between the multiplicities of the eigenvalues of the operator \( \Delta \) and the eigenvalues \( \lambda_i(l) \) of the periodic Sturm-Liouville problem (3), (4). This relation permits us to express the quantity \( N(2) \) in terms of the eigenvalues \( \lambda_i(l) \).
Proposition 10. In the case of a double cover \( \tilde{\tau}_{m,k} \) of a Lawson torus we have

\[
N(2) = \#\{\lambda_0(0)|\lambda_i(0) < 2\} + 2\#\{\lambda_i(l)|\lambda_i(l) < 2, l > 0, l \in \mathbb{Z}\}.
\]

In the case of a Lawson torus \( \tau_{m,k} \) we have

\[
N(2) = 1 + 2\#\{\lambda_0(0)|\lambda_0(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is even}\} + \\
+ 2\#\{\lambda_{2i+1}(0)|\lambda_{2i+1}(0) < 2, i \text{ is odd}\} + \\
+ 2\#\{\lambda_{2i+2}(0)|\lambda_{2i+2}(0) < 2, i \text{ is even}\} + \\
+ 2\#\{\lambda_{2i+1}(l)|\lambda_{2i+1}(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is even}, i \text{ is odd}\} + \\
+ 2\#\{\lambda_{2i+2}(l)|\lambda_{2i+2}(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is odd}, i \text{ is even}\} + \\
+ 2\#\{\lambda_{2i+1}(l)|\lambda_{2i+1}(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is odd}, i \text{ is even}\}.
\]

In the case of a Lawson Klein bottle \( \tau_{m,k} \) we have

\[
N(2) = \#\{\lambda_0(0)|\lambda_i(0) < 2, \varphi_i(0, y) \text{ is even}\} + \\
+ 2\#\{\lambda_i(l)|\lambda_i(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is even, } \varphi_i(l, y) \text{ is even}\} + \\
+ 2\#\{\lambda_i(l)|\lambda_i(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is odd, } \varphi_i(l, y) \text{ is odd}\}.
\]

Proof. Let \( \tau_{m,k} \) be a Lawson torus. The eigenvalue \( \lambda_i(0) \) gives exactly one basis eigenfunction

\[
\varphi_i(0, y) \cos(0x) = \varphi_i(0, y)
\]

of the operator \( \Delta \). It follows that each \( \lambda_i(0) \) corresponds to one eigenvalue \( \lambda_q = \lambda_i(0) \)

of the Laplace-Beltrami operator \( \Delta \). The eigenvalue \( \lambda_i(l), l > 0 \), gives exactly two basis eigenfunctions

\[
\varphi_i(l, y) \cos(lx) \quad \text{and} \quad \varphi_i(l, y) \sin(lx)
\]

of the Laplace-Beltrami operator \( \Delta \). It follows that for \( l > 0 \) each \( \lambda_i(l) \) corresponds to two eigenvalues \( \lambda_q = \lambda_{q+1} = \lambda_i(l) \) of the Laplace-Beltrami operator \( \Delta \). This implies formula (10).

The term 1 in formula (11) counts \( \lambda_0(0) \).\( \square \)

Propositions 5 and 10 permit us to investigate the simplest case of the Lawson torus \( \tau_{1,1} \), i.e. the Clifford torus.

Proposition 11. The metric induced on the Lawson torus \( \tau_{1,1} \) (i.e. the Clifford torus) is an extremal metric for the functional \( \Lambda_1(T^2, g) \). The corresponding value of this functional is \( \Lambda_1(\tau_{1,1}) = 4\pi^2 = 39.479 \ldots \)

The metric induced on the double cover \( \tilde{\tau}_{1,1} \) of the Clifford torus \( \tau_{1,1} \) is an extremal metric for the functional \( \Lambda_5(T^2, g) \). The corresponding value of this functional is \( \Lambda_5(\tilde{\tau}_{1,1}) = 8\pi^2 = 78.957 \ldots \)

Proof. We have \( m = k = 1 \), hence \( p(y) \equiv 1 \). Equation (3) becomes the equation

\[
-(\varphi(y))'' + (l^2 - \lambda)\varphi(y) = 0.
\]

It is well-known that the eigenvalues \( \lambda_i(l) \) of this equation with the periodic boundary conditions (4) satisfy the relation

\[
\lambda_i(l) - l^2 = n^2, \quad n \in \mathbb{Z}.
\]

Hence we have the following eigenvalues of the periodic Sturm-Liouville problem that are less than 2,

\[
\lambda_0(0) = 0, \quad \lambda_1(0) = 1, \quad \lambda_2(0) = 1,
\]

\[
\lambda_0(1) = 1.
\]
It follows from formula (11) that \( N(2) = 1 \) and the induced metric is extremal for the functional \( \Lambda_1(T^2, g) \). Since \( \lambda_1 = 2 \) and \( \text{Area}(\tau_{1,1}) = 2\pi^2 \), the value of this functional is \( \Lambda_1(\tau_{1,1}) = 4\pi^2 \).

The case of the double cover \( \tau_{1,1} \) is similar. \( \square \)

This result is well-known and the example of \( \tau_{1,1} \) is a "toy example" because in this simplest case the corresponding equation (13) is exactly solvable. It is not the case for other Lawson tau-surfaces. However, we can find \( N(2) \) by investigating the structure of the eigenvalues \( \lambda_i(l) \) of the auxiliary Sturm-Liouville problems [3], [4].

3. Magnus-Winkler-Ince equation and eigenvalues of multiplicity 2

It turns out that it is important to investigate eigenvalues of multiplicity 2 of the periodic Sturm-Liouville problem [3], [4]. The problem of existence of such eigenvalues is usually called a coexistence problem since this means that for such an eigenvalue two independent \( 2\pi \)-periodic solutions exist.

Since the Lawson torus \( \tau_{1,1} \) is already investigated, we assume that \( m \neq k \) in this section.

Let us remark that our equation (9) can be written as

\[
(1 + a \cos(2y))\varphi''(y) + b \sin(2y)\varphi'(y) + (c + d \cos(2y))\varphi(y) = 0, 
\]

where

\[
\begin{align*}
 a &= \frac{m^2 - k^2}{m^2 + k^2}, & b &= -\frac{m^2 - k^2}{m^2 + k^2}, & c &= \lambda_i(l) - \frac{2l^2}{m^2 + k^2}, & d &= \lambda_i(l)\frac{m^2 - k^2}{m^2 + k^2}.
\end{align*}
\]

Equation (14) is called a Magnus-Winkler-Ince (MWI) equation. Theory of this equation is interesting for us because the coexistence problem for the MWI equation was intensively studied, see the book [19], and solved completely in some terms by Volkmer in 2003, see the paper [20]. We need the following result [19] Theorem 7.1.

**Proposition 12.** If the MWI equation (14) has two linearly independent solutions of period \( 2\pi \) then the polynomial

\[
Q^*(\mu) = a(2\mu - 1)^2 - b(2\mu - 1) - d
\]

vanishes for one of the values of \( \mu = 0, \pm 1, \pm 2, \ldots \).

This proposition implies immediately the following Proposition.

**Proposition 13.** Let \( m \neq k \). The only possible eigenvalue \( \lambda_i(l) \) such that \( \lambda_i(l) < 6 \) and \( \lambda_i(l) \) has multiplicity 2 is \( \lambda_i(l) = 2 \).

**Proof.** If \( \lambda_i(l) \) is an eigenvalue of multiplicity 2 then by Proposition [12] the polynomial \( Q^*(\mu) \) has integer root \( \mu_0 \). Substituting formulae (15) into equation (16), we obtain

\[
Q^*(\mu) = \frac{m^2 - k^2}{m^2 + k^2} \left[ (2\mu - 1)^2 + (2\mu - 1) - \lambda_i(l) \right].
\]

It follows that \( Q^*(\mu) = 0 \) is equivalent to

\[
\lambda_i(l) = 2\mu(2\mu - 1).
\]

Then \( \lambda_i(l) = 2\mu_0(2\mu_0 - 1), \mu_0 \in \mathbb{Z} \). The only possible eigenvalues \( 0 \leq \lambda_i(l) < 6 \) of this form are \( \lambda_i(l) = 0 \) and \( \lambda_i(l) = 2 \). We know that \( \lambda = 0 \) is the eigenvalue of multiplicity 1 of \( \Delta \) because there is no harmonic functions on a compact surface except constants. Hence, \( \lambda_i(l) = 0 \) is excluded and \( \lambda_i(l) = 2 \) is the only possibility. \( \square \)

We know from Proposition [1] that the restrictions on \( \tau_{m,k} \) of the coordinate functions \( x^1, \ldots, x^4 \) are eigenfunctions of \( \Delta \) with eigenvalue 2. The explicit formula of the immersion (2) gives these functions,

\[
\cos(mx) \cos y, \sin(mx) \cos y, \cos(kx) \sin y, \sin(kx) \sin y.
\]
Proposition 14. Let \( m \neq k \).

The function \( \cos y \) is an eigenfunction of the periodic Sturm-Liouville problem (3), (4) with \( l = m \) and the corresponding eigenvalue is equal to 2. This eigenvalue has multiplicity 1.

The function \( \sin y \) is an eigenfunction of the periodic Sturm-Liouville problem (3), (4) with \( l = k \) and the corresponding eigenvalue is equal to 2. This eigenvalue has multiplicity 1.

Proof. The eigenfunctions (17) are of the form (5) and (6). This implies that \( \cos y \) is an eigenfunction of the periodic Sturm-Liouville problem (3), (4). By the Wronskian shows that a linearly independent with \( \cos y \). The corresponding eigenvalue is equal to 2. In the same way, \( \sin y \) is an eigenfunction of the periodic Sturm-Liouville problem (3), (4) with \( l = k \) and the corresponding eigenvalue is equal to 2.

Let us consider the case \( l = m \) and \( \varphi(y) = \cos y \). The standard argument with the Wronskian shows that a linearly independent with \( \cos y \) solution of equation (3) can locally be presented in the form

\[
\cos y \int_{y_0}^{y} \frac{d\xi}{\cos^2 \xi \sqrt{k^2 + (m^2 - k^2) \cos^2 \xi}}.
\]

This integral has singularities at \( y = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \) and it is not possible to present a second solution in this form globally. Let us define a function \( F: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \) by the formula

\[
F(y) = \cos y \int_{0}^{y} \frac{d\xi}{\cos^2 \xi \sqrt{k^2 + (m^2 - k^2) \cos^2 \xi}}
\]

and a function \( G: (\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R} \) by the formula

\[
G(y) = \cos y \int_{\pi}^{y} \frac{d\xi}{\cos^2 \xi \sqrt{k^2 + (m^2 - k^2) \cos^2 \xi}}
\]

It is easy to see that the function \( F \) is an odd function. Let us remark that

\[
\lim_{y \to -\frac{\pi}{2}^-} F(y) = \frac{1}{k}, \lim_{y \to -\frac{\pi}{2}^-} F'(y) = \frac{1}{k} \left( E \left( \frac{\sqrt{k^2 - m^2}}{k} \right) - K \left( \frac{\sqrt{k^2 - m^2}}{k} \right) \right) \neq 0.
\]

The expression for \( \lim_{y \to -\frac{\pi}{2}^-} F'(y) \) is long, let us denote it by \( Z \) in order to shorten the notation.

The identity \( G(y) = F(\pi - y) \) implies

\[
\lim_{y \to -\frac{\pi}{2}^+} G(y) = \lim_{y \to -\frac{\pi}{2}^+} F(y) = \frac{1}{k}, \lim_{y \to -\frac{\pi}{2}^+} G'(y) = -\lim_{y \to -\frac{\pi}{2}^-} F'(y) = -Z \neq 0.
\]

It follows that we can define a linearly independent with \( \cos y \) solution on \( [-\frac{\pi}{2}, \frac{3\pi}{2}] \) by the formula

\[
\psi(y) = \begin{cases} 
\frac{1}{k}, & \text{if } x = -\frac{\pi}{2}, \\
F(y), & \text{if } x \in (-\frac{\pi}{2}, \frac{\pi}{2}); \\
\frac{1}{k}, & \text{if } x = \frac{\pi}{2}; \\
-2Z \cos y + G(y), & \text{if } x \in (\frac{\pi}{2}, \frac{3\pi}{2}); \\
\frac{1}{k}, & \text{if } x = \frac{3\pi}{2}.
\end{cases}
\]

This is a smooth solution. However, \( \psi(y) \) is not periodic because

\[
\lim_{y \to -\frac{\pi}{2}^+} \psi(y) = \lim_{y \to -\frac{\pi}{2}^+} F'(y) = \lim_{y \to -\frac{\pi}{2}^-} F'(y) = Z \neq 0
\]
It is clear that \( \lambda \) is a root of equation (18) if and only if \( \lambda \) is an eigenvalue of multiplicity 1. This implies that the eigenvalue \( \lambda_i(m) \) is a root of polynomial (19) if and only if \( \lambda \) is an eigenvalue of multiplicity 1. This implies that the eigenvalue \( \lambda \) has multiplicity 1.

The case \( l = k \) and \( \sin y \) can be investigated in the same way. □

4. Properties of Eigenvalue \( \lambda_i(l) \) as a Function of \( l \)

Let us now consider the periodic Sturm-Liouville problem \([3], [4]\) not only for integer values of \( l \) but also for real values of \( l \).

Let us recall a little bit of the general theory of a Sturm-Liouville problem, see e.g. the textbook \([18]\). Let us rewrite equation (7) as

\[
\left( \frac{p(y)}{m} \varphi'(y) \right)' + \left( \lambda \frac{p(y)}{m} - \frac{l^2}{mp(y)} \right) \varphi(y) = 0.
\]

This form is more standard because \( \frac{p(0)}{m} = 1 \). Let \( \Phi(\lambda, y) \) and \( \Psi(\lambda, y) \) denote two solutions of equation (18) such that

\[
\Phi(\lambda, 0) = 1, \quad \Phi'(\lambda, 0) = 0, \quad \Psi(\lambda, 0) = 0, \quad \Psi'(\lambda, 0) = 1.
\]

They form a basis in the space of solutions of equation (18). The matrix of the shift operator \((T \varphi)(y) = \varphi(y + 2\pi)\) in this basis is equal to

\[
\hat{T}(\lambda) = \begin{pmatrix} \Phi(\lambda, 2\pi) & \Psi(\lambda, 2\pi) \\ \Phi'(\lambda, 2\pi) & \Psi'(\lambda, 2\pi) \end{pmatrix}.
\]

The conservation law for the Wronskian implies \( \det \hat{T}(\lambda) = 1 \). Then the eigenvalues \( \mu \) of the matrix \( \hat{T}(\lambda) \) are roots of the polynomial

\[
\mu^2 - \text{tr} \hat{T}(\lambda) \mu + \det \hat{T}(\lambda) = \mu^2 - \text{tr} \hat{T}(\lambda) \mu + 1.
\]

It is clear that \( \lambda \) is an eigenvalue of the periodic Sturm-Liouville problem for equation (18) if and only if \( \mu = 1 \) is a root of polynomial (19). This is equivalent to the equation \( \text{tr} \hat{T}(\lambda) = 2 \).

Let us denote \( \text{tr} \hat{T}(\lambda) \) by \( f(l, \lambda) \), i.e.

\[
f(l, \lambda) = \Phi(\lambda, 2\pi) + \Psi'(\lambda, 2\pi).
\]

Then \( \lambda_i(l) \) is defined implicitly by the equation

\[
f(l, \lambda) = \lambda_i(l) = 2.
\]

It is known (see e.g. the textbook [18]) that if \( \lambda_i(l) \) is an eigenvalue of multiplicity 1 then \( \frac{\partial f}{\partial \lambda}(l, \lambda_i(l)) \neq 0 \).

**Proposition 15.** Let us fix \( i \). If \( \lambda_i(l) \) has multiplicity 1 for all \( l \in (0, l_1) \) then \( \lambda_i(l) \) is a strictly increasing function on \((0, l_1)\).

**Proof.** Let us introduce a new parameter \( \varkappa = l^2 \). We use \( \varkappa \) since equation (18) depends on \( \varkappa \) in a linear way. If \( \lambda_i(l) \) has multiplicity 1 then \( \frac{\partial f}{\partial \lambda}(\varkappa, \lambda_i(\varkappa)) \neq 0 \). This implies that \( \frac{\partial f}{\partial \varkappa}(\varkappa, \lambda_i(\varkappa)) \neq 0 \) because \( \frac{\partial f}{\partial \varkappa} = \frac{\partial f}{\partial \lambda} \cdot 2l \neq 0 \). It follows from the implicit function theorem that

\[
\frac{\partial \lambda_i(\varkappa)}{\partial \varkappa} = -\frac{\partial f}{\partial \lambda}(\varkappa, \lambda_i(\varkappa)),
\]

It is known (see e.g. the textbook [18]) that

\[
\frac{\partial f}{\partial \lambda}(\varkappa, \lambda_i(\varkappa)) = \int_0^{2\pi} \left[ \Phi^2(\tau) \Phi'(2\pi) + \Psi(\tau) \Phi'(2\pi) \Phi(2\pi) - \Psi'(\tau) \Psi(2\pi) \right] \frac{p(\tau)}{m} d\tau
\]
and \( \frac{\partial f}{\partial x}(\lambda_i(x)) > 0 \) for odd \( i \) and \( \frac{\partial f}{\partial y}(\lambda_i(x)) < 0 \) for even \( i \).

One can prove in a similar way that

\[
\frac{\partial f}{\partial x}(\lambda_i(x)) = - \int_0^{2\pi} \left[ \Psi(\tau)\Phi(2\pi) + \Psi(\Phi(2\pi)) - \Phi(\Psi(2\pi)) \right] \frac{d\tau}{mp(\tau)}
\]

and \( \frac{\partial f}{\partial y}(\lambda_i(x)) < 0 \) for odd \( i \) and \( \frac{\partial f}{\partial y}(\lambda_i(x)) > 0 \) for even \( i \).

It follows that

\[
\frac{\partial \lambda_i(x)}{\partial x} = -\frac{\partial f}{\partial y}(\lambda_i(x)) > 0
\]

for any parity of \( i \). □

5. The Lamé equation

In this Section we recall some properties of the Lamé equation usually written as

\[
d^2\varphi \over dz^2 + (h - n(n + 1)(\kappa \sin z)^2)\varphi = 0.
\]

see e.g. the book [21] or the book [22]. We denote the modulus of the elliptic function \( \text{sn} \) by \( \hat{k} \) since we already use a letter \( k \) in \( \tau_{m,k} \).

The Lamé equation could be written in different forms, we will use the trigonometric form of the Lamé equation

\[
[1 - (\kappa \sin y)^2] \frac{d^2\varphi}{dy^2} - \kappa \sin y \cos y \frac{d\varphi}{dy} + [h - n(n + 1)(\kappa \sin y)^2]\varphi = 0.
\]

Equation (21) could be obtained from equation (20) using the change of variable

\[
\text{sn} \ y = \sin y \quad \iff \quad y = \text{am} \ z,
\]

where \( \text{am} \ z \) is Jacobi amplitude function, see e.g. the book [21] Section 13.9. This trigonometric form of the Lamé equation is used in the book [22]. The change of variable \( \text{sn} \ y = \cos y \) leads to another trigonometric form used in the book [21], we use it later.

We are interested in \( 2\pi \)-periodic solutions of the Lamé equation (21). Usually \( 0 < \hat{k} < 1 \) and \( n \) are fixed parameters and \( h \) plays the role of an eigenvalue. The following Proposition holds.

Proposition 16. Given \( 0 < \hat{k} < 1 \) and \( n \), there exist an infinite sequence of values \( h_0 < h_1 \leq h_2 < h_3 \leq h_4 < \ldots \) of the parameter \( h \) such that if \( h = h_1 \) then the the Lamé equation (21) has a \( 2\pi \)-periodic solution \( \varphi_1(y) \neq 0 \).

For \( h = h_0 \) a solution \( \varphi_0(y) \) is unique up to a multiplication by a non-zero constant.

If \( h_{2i+1}(l) < h_{2i+2}(l) \), then solutions \( \varphi_{2i+1}(y) \) and \( \varphi_{2i+2}(y) \) are unique up to a multiplication by a non-zero constant.

If \( h_{2i+1}(l) = h_{2i+2}(l) \), then there exist two independent solutions \( \varphi_{2i+1}(y) \) and \( \varphi_{2i+2}(y) \) corresponding to \( h = h_{2i+1} = h_{2i+1} \).

The solution \( \varphi_0(y) \) has no zero on \( [0, 2\pi] \). For \( i \geq 0 \) both solutions \( \varphi_{2i+1}(y) \) and \( \varphi_{2i+2}(y) \) have exactly \( 2i + 2 \) zeroes on \( [0, 2\pi] \).

Our main interest is the case \( n = 1 \). In this case three wonderful solutions of the Lamé equation (21) are known,

\[
\text{Ec}_0^1(z) = \text{dn} \ z, \quad \text{Ec}_1^1(z) = \text{cn} \ z, \quad \text{Es}_1^1(z) = \text{sn} \ z.
\]
where we use the notation used by Ince in the paper [23]. Using standard properties
of the Jacobi elliptic functions and change of variable (22) we obtain three solutions
of the Lamé equation in the trigonometric form (21),
\[ Ec_0^0(y) = \sqrt{1 - \hat{k}^2 \sin^2 y}, \quad Ec_0^1(y) = \cos y, \quad Es_0^1(y) = \sin y. \]

**Proposition 17.** If \( n = 1 \) then we have
\[ \varphi_0(y) = Ec_0^0(y) = \sqrt{1 - \hat{k}^2 \cos^2 y}, \quad h_0 = \hat{k}^2, \]
\[ \varphi_1(y) = Ec_0^1(y) = \cos y, \quad h_1 = 1, \]
\[ \varphi_2(y) = Es_0^1(y) = \sin y, \quad h_2 = 1 + \hat{k}^2. \]

**Proof.** The function \( Ec_0^0(y) = \sqrt{1 - \hat{k}^2 \cos^2 y} \) has no zeroes, hence by Proposition 16 it is \( \varphi_0(y) \). Direct check by substitution shows that \( h_0 = \hat{k}^2 \). The same argument works for \( \varphi_1(y) \) and \( \varphi_2(y) \). \( \square \)

We should remark that in general \( h_i \) are roots of a very complicated transcendental equation with parameters \( n \) and \( \hat{k} \) and cannot be found explicitly.

Using the same approach as in Proposition 15 we can prove the following Proposition.

**Proposition 18.** Let us fix \( n = 1 \) and consider \( h_3 \) as a function of \( \hat{k}^2 \), where \( 0 < \hat{k}^2 \leq 1 \). Then \( h_3(\hat{k}^2) \) is a decreasing function.

When \( \hat{k} = 1 \) the Lamé equation (20) is called degenerate because in this case we have \( \text{sn} z = \tanh z \).

**Proposition 19.** Let \( n = 1 \) and \( \hat{k} = 1 \). Then we have
\[ h_0 = h_1 = 1, \quad h_2 = h_3 = 2. \]

**Proof** follows immediately from the explicit formulae for \( h_i \) in the paper [23, Section 9]. \( \square \)

Propositions 18 and 19 imply the following Proposition.

**Proposition 20.** Let \( n = 1 \). Then for \( 0 < \hat{k}^2 < 1 \) we have
\[ h_3 > 2. \]

### 6. PROOF OF THE THEOREM

#### 6.1. Case of a double cover \( \hat{\tau}_{m,k} \) of a Lawson torus.

It is easy to see from Proposition 11 that for the double cover \( \hat{\tau}_{1,1} \) of the Clifford torus the statement of the Theorem holds. Hence we can exclude this case from future considerations and suppose that \( m > k > 1, m \equiv k \equiv 1 \mod 2, (m, k) = 1 \).

The eigenvalues \( \lambda_0(l) \) of the periodic Sturm-Liouville problem (13), (14) are always of multiplicity one, see Proposition 6. Hence Proposition 13 implies that \( \lambda_0(l) \) is a strictly increasing function of \( l \). Let us denote by \( l_c \) the solution of the equation
\[ \lambda_0(l) = 2. \]

Then
\[ \# \{ \lambda_0(l) | \lambda_0(l) < 2, l > 0, l \in \mathbb{Z} \} = [l_c] - 1, \]
where \([\cdot]\) denotes the ceiling function, i.e. \([x] = \min \{ a \in \mathbb{Z} | a \geq x \}\).

Let us make now a crucial observation. One can check by a direct calculation that equation (3) could be written as the Lamé equation in the trigonometric form (21) with
\[ \hat{k} = \sqrt{m^2 - k^2} \quad \text{and} \quad h = \lambda - \frac{l^2}{m^2}, \quad n(n + 1) = \lambda. \]
Let us remark that $0 < \hat{k} < 1$ since $m > k > 1$.

It follows from (23) that $\lambda = 2$ corresponds to $n = 1$. Propositions 6 and 16 imply that $\lambda_0$ corresponds to $h_0$. Hence we obtain from (23) and Proposition 17 the identities

$$\frac{m^2 - k^2}{m^2} = \hat{k}^2 = h_0 = \lambda_0(l) - \frac{l^2}{m^2}. \tag{24}$$

We denoted the solution of the equation $\lambda_0(l) = 2$ by $l_c$ and we obtain from formula (24) the equation

$$\frac{m^2 - k^2}{m^2} = 2 - \frac{l^2}{m^2}.$$ 

It follows that

$$l_c = m \sqrt{2 - \frac{m^2 - k^2}{m^2}} = \sqrt{m^2 + k^2}.$$ 

It is easy to see that $\sqrt{m^2 + k^2}$ is not integer because $m$ and $k$ are both odd. It follows that

$$\# \{\lambda_0(l)|\lambda_0(l) < 2, l > 0, l \in \mathbb{Z}\} = \lfloor l_c \rfloor - 1 = \lfloor l_c \rfloor = \lfloor \sqrt{m^2 + k^2} \rfloor.$$ 

Let us now consider $\lambda_1(l)$ and $\lambda_2(l)$. We proved in Proposition 14 that $\cos y$ is an eigenfunction of the periodic Sturm-Liouville problem (3), (4) for $l = m$ and $\sin y$ is an eigenfunction of the same problem for $l = k$. We know that $\cos y$ has 2 zeroes on $[0, 2\pi)$. Hence $\cos y$ could be either $\varphi_1(m, y)$ or $\varphi_2(m, y)$. A similar argument shows that either $\varphi_1(k, y) = \sin y$ or $\varphi_2(k, y) = \sin y$.

Let us suppose that $\varphi_2(m, y) = \cos y$. Then $\lambda_2(m) = 2$ and this eigenvalue has multiplicity 1 by Proposition 14. It follows from Propositions 13 and 15 that for $l \in (0, m]$ the function $\lambda_2(l)$ is strictly increasing. Then $\lambda_2(k) < \lambda_2(m) = 2$ and by Proposition 6 we have the inequality $\lambda_1(k) \leq \lambda_2(k) < \lambda_2(m) = 2$. This contradicts the fact that either $\lambda_1(k) = 2$ or $\lambda_2(k) = 2$. The obtained contradiction shows that $\lambda_1(m) = 2$ and $\varphi_1(m, y) = \cos y$. A similar argument shows that $\lambda_2(k) = 2$ and $\varphi_2(k, y) = \sin y$.

It is easy to see that these solutions $\varphi_1(m, y) = \cos y$ and $\varphi_2(k, y) = \sin y$ of the periodic Sturm-Liouville problem (3), (4) correspond to the periodic solutions $\varphi_1(y) = E\mathcal{C}_1(y) = \cos y$ with $h_1 = 1$ and $\varphi_2(y) = E\mathcal{S}_1(y) = \sin y$ with $h_2 = 1 + k^2$ of the Lamé equation, see Proposition 17.

It follows from Propositions 13 and 15 that for $l \in (0, m]$ the function $\lambda_1(l)$ is strictly increasing from $\lambda_1(0)$ to $\lambda_1(m) = 2$. This implies that

$$\# \{\lambda_1(l)|\lambda_1(l) < 2, l > 0, l \in \mathbb{Z}\} = m - 1.$$ 

In a similar way we obtain

$$\# \{\lambda_2(l)|\lambda_2(l) < 2, l > 0, l \in \mathbb{Z}\} = k - 1.$$ 

Let us suppose that $\lambda_3(0) \leq 2$. We know that $\lambda_3(k) > \lambda_2(k) = 2$. It follows that there exists some value $l_3 \geq 0$ such that $\lambda_3(l_3) = 2$. We know that $\lambda = 2$ corresponds to $n = 1$ and $\lambda_3$ corresponds to $h_3$ and we see from formulae (23) and Proposition 20 that

$$2 - \frac{l_3^2}{m^2} = \lambda_3 - \frac{l_3^2}{m^2} = h_3 > 2.$$ 

This implies

$$\frac{l_3^2}{m^2} < 0,$$

but this is impossible. Hence, $\lambda_3(0) > 2$. It follows from Proposition 20 that $\lambda_1(l) > 2$ for $i \geq 3$ and $l \geq 0$. 

We are ready now to compute \( N(2) \). Using formula (11) from Proposition [11] we obtain
\[
N(2) = \# \{ \lambda_0(0), \lambda_1(0), \lambda_2(0) \} + 2\# \{ \lambda_0(l) | \lambda_0(l) < 2, l > 0, l \in \mathbb{Z} \} + 2\# \{ \lambda_1(l) | \lambda_1(l) < 2, l > 0, l \in \mathbb{Z} \} + 2\# \{ \lambda_2(l) | \lambda_2(l) < 2, l > 0, l \in \mathbb{Z} \} = 3 + 2 \left( \sqrt{m^2 + k^2} \right) + 2(m - 1) + 2(k - 1) = 2 \left( \sqrt{m^2 + k^2} + m + k \right) - 1.
\]

The statement of the Theorem follows now from Proposition [3] and the following formula,
\[
\Lambda_{N(2)}(\hat{\tau}_{m,k}) = \lambda_{N(2)}(\hat{\tau}_{m,k}) \text{Area}(\hat{\tau}_{m,k}) = 2 \int_0^{2\pi} dx \int_{-\pi}^{\pi} p(y)dy = 2 \cdot 2\pi \cdot 4kE\left( i \frac{\sqrt{m^2 - k^2}}{k} \right) = 16\pi k \frac{m}{k} E\left( \frac{\sqrt{m^2 - k^2}}{m} \right) = 16\pi m \left( \sqrt{m^2 - k^2} \right).
\]

6.2. Case of a Lawson torus \( \tau_{m,k} \). In this case the value of \( N(2) \) follows immediately from the case of its double cover \( \hat{\tau}_{m,k} \) and formula (11) from Proposition [11]. The area of \( \tau_{m,k} \) is just half of the area of its double cover \( \hat{\tau}_{m,k} \). This gives the answer.

6.3. Case of a Lawson Klein bottle \( \tau_{m,k} \). As we already discussed, we assume \( m \equiv 0 \mod 2, k \equiv 1 \mod 2, (m, k) = 1 \).

Let us consider the case \( m > k \). Then we have the same argument as for the double cover \( \hat{\tau}_{m,k} \) of a Lawson torus, but we have to take into account parity of solutions.

As we know from Proposition [3] the solutions \( \varphi_0(l, y) \) are even. This implies that
\[
\# \{ \lambda_0(l) | \lambda_0(l) < 2, l > 0, l \in \mathbb{Z}, \lambda_0(l) \text{ is even}, \varphi_0(l, y) \text{ is even} \} = \left\lceil \frac{l}{2} \right\rceil - 1.
\]

Since \( m \) is even, \( k \) is odd, \( l_\epsilon = \sqrt{m^2 + k^2} \) cannot be an even integer, and we obtain
\[
\# \{ \lambda_0(l) | \lambda_0(l) < 2, l > 0, l \in \mathbb{Z}, \lambda_0(l) \text{ is even}, \varphi_0(l, y) \text{ is even} \} = \left\lceil \frac{\sqrt{m^2 + k^2}}{2} \right\rceil.
\]

As we know from Proposition [3] a solution \( \varphi_1(l, y) \) is even or odd and parity is preserved on an interval \( a \leq l \leq b \) where \( \lambda_1(l) \) is of multiplicity one. It follows that \( \varphi_1(l, y) \) is even for \( 0 \leq l \leq m \) since \( \varphi_1(m, y) = \cos y \) is even and \( \varphi_2(l, y) \) is odd for \( 0 \leq l \leq k \) since \( \varphi_2(k, y) = \sin y \) is odd. This implies that
\[
\# \{ \lambda_1(l) | \lambda_1(l) < 2, l > 0, l \in \mathbb{Z}, \lambda_1(l) \text{ is even}, \varphi_1(l, y) \text{ is even} \} = \{ \lambda_1(2), \lambda_1(4), \ldots, \lambda_1(m - 2) \} = \frac{m}{2} - 1
\]
and
\[
\# \{ \lambda_2(l) | \lambda_2(l) < 2, l > 0, l \in \mathbb{Z}, \lambda_2(l) \text{ is odd}, \varphi_2(l, y) \text{ is odd} \} = \{ \lambda_2(1), \lambda_2(3), \ldots, \lambda_2(k - 2) \} = \frac{k - 1}{2}.
\]
Finally, using formula (12) from Proposition 10 we obtain

\[ N(2) = \#\{ \lambda_0(0), \lambda_1(0) \} + 2 \# \{ \lambda_0(l) | \lambda_0(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is even}, \varphi_0(l, y) \text{ is even} \} + 2 \# \{ \lambda_1(l) | \lambda_1(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is even}, \varphi_1(l, y) \text{ is even} \} + 2 \# \{ \lambda_2(l) | \lambda_2(l) < 2, l > 0, l \in \mathbb{Z}, l \text{ is odd}, \varphi_2(l, y) \text{ is odd} \} =
\]

\[ = 2 + 2 \left( \sqrt{\frac{m^2 + k^2}{2}} - \frac{1}{2} \right) + 2 + 2 \left( \sqrt{\frac{m^2 + k^2}{2}} - \frac{1}{2} \right) = 2 + 2 \left( \sqrt{\frac{m^2 + k^2}{2}} + \frac{m + k - 1}{2} \right).\]

The statement of the Theorem follows now from Proposition 3 and the following formula,

\[ \Lambda_{N(2)}(\tau_{m,k}) = \lambda_{N(2)}(\tau_{m,k}) \text{Area}(\tau_{m,k}) = 2 \int_0^\pi dx \int_{-\pi}^\pi p(y)dy =
\]

\[ = 2 \cdot \pi \cdot 4kE \left( \frac{i \sqrt{\frac{m^2 - k^2}{k}}}{k} \right) = 8 \pi mE \left( \frac{\sqrt{\frac{m^2 - k^2}{m}}}{m} \right).\]

The case \( k > m > 0 \) is similar. In this case we should use another trigonometric form of the Lamé equation

\[ [1 - (\hat{k} \cos y)^2] \frac{d^2 \varphi}{dy^2} + \hat{k}^2 \sin y \cos y \frac{d \varphi}{dy} + [h - n(n + 1)(\hat{k} \cos y)^2] \varphi = 0
\]

used e.g. in the book [21]. Equation (25) could be obtained from equation (20) using the change of variable

\[ \sin z = \cos y \iff y = \frac{\pi}{2} - \operatorname{am} z.
\]

Equation (3) could be written as the Lamé equation in the trigonometric form (25) with

\[ \hat{k} = \sqrt{\frac{k^2 - m^2}{k}}, \quad h = \lambda - \frac{l^2}{k^2}, \quad n(n + 1) = \lambda.
\]

Let us remark that \( 0 < \hat{k} < 1 \) since \( k > m > 0 \). The rest of the proof is similar to the proof in the case \( m > k > 0 \) and the resulting formulae are the same. □

**Acknowledgments**

The author is very indebted to I. V. Polterovich who inspired the interest to spectral geometry. The author is also grateful to H. Volkmer for providing the paper [20].

The author thanks A.P.Veselov and P. Winternitz for fruitful discussions.

This work was partially supported by Russian Federation Government grant no. 2010-220-01-077, ag. no. 11.G34.31.0005, by the Russian Foundation for Basic Research (grant no. 08-01-00541 and grant no. 11-01-12067-ofi-m-2011), by the Russian State Programme for the Support of Leading Scientific Schools (grant no. 5413.2010.1) and by the Simons-IUM fellowship.

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