On the Energy Equality for Distributional Solutions to Navier-Stokes Equations

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Abstract

A classical result of J.-L. Lions asserts that if a solution to the Navier-Stokes equations is such that: (i) it is in the Leray-Hopf class, and (ii) belongs to $L^4(0,T;L^4)$, then it must satisfy the energy equality in the time interval $[0,T]$. In this note we show that assumption (i) is not necessary.

1 Introduction

Consider the three-dimensional Cauchy problem for the Navier-Stokes equations:

$$\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v &= \Delta v - \nabla p \\
\text{div } v &= 0
\end{align*}$$

in $\mathbb{R}^3 \times (0,\infty)$

$$v(x,0) = v_0(x) \quad x \in \mathbb{R}^3,$$

where $v : \mathbb{R}^3 \times [0,\infty) \to v(x,t) \in \mathbb{R}^3$ is the flow velocity field and $p$ the associated pressure field.

It is well known since the work of Leray [8] and Hopf [7], that for any $v_0 \in L^2(\mathbb{R}^3)$ one can construct a global weak solutions to (1.1), namely, a function $v$ that, for each $T > 0$, is in the class

$$v \in L^\infty(0,T;L^2_0(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$$

and solves (1.1) in a distributional sense. In addition, such a $v$ satisfies the so-called energy inequality:

$$\|v(t)\|^2 + 2 \int_0^t \|\nabla v(\tau)\|^2 \leq \|v_0\|^2, \quad \text{all } t \geq 0,$$

(1.3)

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(1) Generalizations to other domains are discussed in Remark 2.1. We also assume, for simplicity, zero body force and, without loss of generality, take the kinematic viscosity coefficient to be 1.

(2) $L^2_0(\mathbb{R}^3)$ is the subspace of $L^2(\mathbb{R}^3)$ of divergence-free vector functions. Other notations are standard, like $H^{m,q}$, for Sobolev spaces, with corresponding norm $\|\cdot\|_{m,q}$, $L^r(I;X)$, $I$ real interval, $X$ Banach space, for Bochner spaces, etc.

(3) Actually, $v$ obeys the strong energy inequality [8 §§27–28], but this is irrelevant to the aim of our paper.
where $\| \cdot \|_q$, denotes the $L^q(\mathbb{R}^3)$-norm. The inequality sign in this relation is another indication of poor regularity of a weak solution. Actually, all sufficiently regular solutions to (1.1) satisfy (1.3) with the equality sign (energy equality), which provides the precise energy balance for the given flow. As a matter of fact, it is still an outstanding open question whether there exist global solutions to (1.1) satisfying the energy equality for arbitrary $v_0 \in L^2_\sigma$.

In this regard, a famous result of J.-L. Lions [10] states that if $v$, in addition to be in the class (1.2), satisfies also
\begin{equation}
 v \in L^4(0,T; L^4(\mathbb{R}^3)), \tag{1.4}
\end{equation}
then necessarily $v$ obeys the energy equality throughout the interval $[0,T]$ [4]. An important corollary to this finding is that the map $t \mapsto v(t)$ is continuous with values in $L^2_\sigma$. Notice that the two classes (1.2) and (1.4) are not comparable, in the sense that a generic function in (1.2) need not satisfy (1.4) and vice versa.

Objective of this note is to prove that, actually, for the validity of Lions result the requirement (1.2) is entirely redundant: it is just enough that $v$ satisfies (1.4), along with the (necessary) condition $v_0 \in L^2_\sigma(\mathbb{R}^3)$. More precisely, setting
\[ D_T := \{ \varphi \in C^\infty_0(\mathbb{R}^3 \times [0,T)) : \text{div} \varphi = 0 \} \]
we will show the following.

**Theorem 1.1** Let $v \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ be such that
\begin{align*}
 &\int_0^T \int_{\mathbb{R}^3} v \cdot (\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi) = -\int_{\mathbb{R}^3} v_0 \cdot \varphi(0) \\
 &\int_0^T \int_{\mathbb{R}^3} v \cdot \phi = 0
\end{align*}
for some $v_0 \in L^2_\sigma(\mathbb{R}^3)$ and all $\varphi \in D_T, \phi \in C^\infty_0(\mathbb{R}^3 \times (0,T))$. Then, if $v$ satisfies (1.4), necessarily $v$ is in the class (1.2) and thus obeys the energy equality.

The proof of this result, which we give in the following section, is surprisingly elementary, and is based on a mollifying procedure coupled with a simple duality argument.

## 2 Proof of Theorem 1.1

For a given $g: \mathbb{R}^3 \times (0,T) \mapsto \mathbb{R}^3$ we set $\bar{g} = g(\cdot, T-t)$. If $g$ is locally integrable, by $g^{(n)}$, and $g^{(\eta)}, \eta$ positive and sufficiently small, we denote space and space-time mollifiers of $g$:
\begin{align*}
 g^{(\eta)}(x, \cdot) &= \int_{\mathbb{R}^3} k_\eta(x-y)g(y, \cdot)dy, \quad g^{(\eta)}(x, t) = \int_0^T j_\eta(t-s)g^{(\eta)}(x, s)ds,
\end{align*}
where
\begin{align*}
 j_\eta(\tau) := \eta^{-1}j(\tau/\eta), \quad k_\eta(\xi) := \eta^{-1}k(\xi/\eta), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3,
\end{align*}

\[ (4) \] For further sufficient conditions other than (1.2), see II and the references therein.
with \( j \in C^\infty_0(-1, 1) \), and \( k \in C^\infty_0(\mathbb{R}^3) \). We also write, as customary, \((f, g) := \int_{\mathbb{R}^3} f \cdot g \), and set, for simplicity, \(L^{p,q} := L^p(0, T; L^q(\mathbb{R}^3))\). Finally, we define

\[
\mathcal{W}^{1,p} := \{ u \in L^1_{\text{loc}}(\mathbb{R}^3 \times (0, T)) : u \in H^{1,p}(0, T; L^p(\mathbb{R}^3)) \cap L^p(0, T; H^{2,p}(\mathbb{R}^3)) \}.
\]

Before proving the theorem, we need a preparatory result.

**Lemma 2.1** Let \( v \) and \( v_0 \) be as in Theorem 1.1 and \( f \in C^\infty_0(\mathbb{R}^3 \times (0, T)) \). Then the Cauchy problem

\[
\begin{align*}
\partial_t w + \overline{v}(\eta) \cdot \nabla w &= \Delta w - \nabla p + \tilde{f} \\
\text{div } w &= 0 \\
\end{align*}
\]

in \( \mathbb{R}^3 \times (0, \infty) \)

\[ w(x, 0) = (v_0)(\eta)(x), \quad x \in \mathbb{R}^3, \]

has one (and only one) solution \((w_\eta, p_\eta)\) such that\(^5\)

\[ w_\eta \in \mathcal{W}^{1,2} \subset C([0, T]; H^1), \quad \nabla p_\eta \in L^{2,2}, \]

and satisfying the uniform bound

\[
\max_{t \in [0, T]} \|w_\eta(t)\|_2^2 + \int_0^T \|\nabla w_\eta(t)\|_2^2 \leq \|v_0\|_2^2 + c_2 \int_0^T \|f(t)\|_2^2. \tag{2.3}
\]

In addition, if \( v_0 \equiv 0 \), we have also \((w_\eta, \nabla p_\eta) \in \mathcal{W}^{1,\frac{4}{3}} \times L^{\frac{4}{3},\frac{4}{3}}\).

**Proof.** The existence of a solution in the class \((2.2)\) is easily established by the “invading domains” technique coupled with the classical Galerkin method \[6\]. We will sketch a proof here. Let \( B_R \subset \mathbb{R}^3 \) be the ball of radius \( R \) centered at the origin, and consider the following problem:

\[
\begin{align*}
\partial_t w + \overline{v}(\eta) \cdot \nabla w &= \Delta w - \nabla p + \tilde{f} \\
\text{div } w &= 0 \\
\end{align*}
\]

in \( B_R \times (0, \infty) \)

\[ w(x, t) = 0, \quad (x, t) \in \partial B_R \times (0, \infty), \quad w(x, 0) = v_R(x), \quad x \in B_R, \]

where \( v_R \in H^1_0(B_R) \cap L^2_0(B_R) \) satisfies\(^6\)

\[
\lim_{R \to \infty} \|v_R - (v_0)(\eta)\|_{1,2} = 0. \tag{2.5}
\]

The existence of \( v_R \) is known \[5\] Theorem III.4.2]. If we dot-multiply both sides of \((2.4)_1\) by \( w \), formally integrate by parts over \( B_R \) and take into account \((2.4)_2\) and \( \text{div } \overline{v}(\eta) = 0 \), we get

\[
\|w(t)\|_2^2 + 2 \int_0^t \|\nabla w(\tau)\|_2^2 = \|v_R\|_2^2 + \int_0^t (\tilde{f}(\tau), w(\tau))
\]

\(^5\) The continuous embedding in \((2.2)\) is a classical interpolation result \[9\].

\(^6\) Since \( v_0 \in L^2_\sigma \), we have \((v_0)(\eta) \in H^{m,q} \), for all \( m \geq 0 \) and all \( q \in [2, \infty] \).
The latter, in turn, by (2.5), Sobolev inequality \( \|w\|_6 \leq c_1 \|\nabla w\|_2 \), and Cauchy–Schwartz inequality furnishes
\[
\|w(t)\|_2^2 + \int_0^t \|\nabla w(\tau)\|_2^2 \leq \|v_0\|_2^2 + c_2 \int_0^t \|f(\tau)\|_2^2 \cdot . \tag{2.6}
\]
Furthermore, since \( \tilde{w}_{(t)} \in L^{\infty, \infty} \), if we dot-multiply both sides of (2.4) a first time by \( P\Delta w \), with \( P : L^2 \rightarrow L^2 \) Helmholtz projector, and then a second time by \( \partial_t w \), and then integrate by parts the resulting relations over \( B_R \), we get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla w(t)\|_2^2 + \|P\Delta w\|_2^2 = (\tilde{w}_{(t)} \cdot \nabla w, P\Delta w) + (f, w) \\
\leq \left( \|\tilde{w}_{(t)}\|_{L^{\infty, \infty}} \|\nabla w\|_2 + \|f\|_2 \right) \|P\Delta w\|_2 ,
\]
and
\[
\frac{1}{2} \frac{d}{dt} \|\nabla w(t)\|_2^2 + \|\partial_t w\|_2^2 = (\tilde{w}_{(t)} \cdot \nabla w, \partial_t w) + (\tilde{f}, \partial_t w) \\
\leq \left( \|\tilde{w}_{(t)}\|_{L^{\infty, \infty}} \|\nabla w\|_2 + \|f\|_2 \right) \|\partial_t w\|_2 ,
\]
respectively. Therefore, by Cauchy–Schwartz inequality, (2.6), (2.5), and the well-known estimate \( \|w\|_{2,2} \leq c (\|P\Delta w\|_2 + \|\nabla w\|_2) \) valid with a constant \( c \) independent of \( R \) [6, Lemma 1], we show
\[
\int_0^t (\|\partial_t w(\tau)\|_2^2 + \|w(\tau)\|_{2,2}^2) \leq C , \quad t \in [0, T] , \text{ all } T > 0 , \tag{2.7}
\]
where the constant \( C \) depends only on \( T \), \( \|\tilde{w}_{(t)}\|_{L^{\infty, \infty}} \), and \( \|(v_0)^{(t)}\|_{1,2} \), and is therefore independent of \( R \). Thus, coupling the classical Galerkin method together with the estimate (2.7), we show the existence of a solution \((w_R, p_R)\) to problem (2.1) in the class (2.2) (with \( \mathbb{R}^3 \) replaced by \( B_R \)). Note that this solution continues to satisfy the uniform bounds (2.6) and (2.7). As a result, we may let \( R \rightarrow \infty \) along a sequence and use (2.7), to prove that \((w_R, p_R)\) converges (in suitable topology) to the desired solution \((w_\infty, p_\infty)\) for which (2.6) and (2.7) hold [6, p. 660 and ff]. Next, take \( v_0 \equiv 0 \). By Hölder inequality, (2.6) and (1.3), we have
\[
\|\tilde{w}_{(t)} \cdot \nabla w_\eta\|_{L^\frac{4}{3}, \frac{4}{3}} \leq \|\tilde{w}_{(t)}\|_{L^{4,4}} \|\nabla w_\eta\|_{L^{2,2}} \leq c_3 \|v\|_{L^{4,4}} \|f\|_{L^{2,\frac{4}{3}}} , \tag{2.8}
\]
which implies, in particular,
\[
\tilde{w}_{(t)} \cdot \nabla w_\eta \in L^{\frac{4}{3}, \frac{4}{3}} .
\]
Therefore, from classical results (e.g. [5 Theorem VIII.4.1]) the problem
\[
\partial_t w = \Delta w - \nabla \chi + F , \quad \text{div} \ w = 0 \quad \text{in} \ \mathbb{R}^3 \times (0, T) , \quad w(x, 0) = 0 \quad x \in \mathbb{R}^3 , \tag{2.9}
\]
with \( F := \tilde{w}_{(t)} \cdot \nabla w_\eta + \tilde{f} \) has at least one solution \((w_*, \chi_*)\) such that
\[
(w_*, \nabla \chi_*) \in H^{\frac{1}{3}, \frac{4}{3}} \times L^{\frac{4}{3}, \frac{4}{3}} .
\]
However, by uniqueness [5, Lemma VIII.4.2] we must have \((w_\eta, \nabla p_\eta) \equiv (w_*, \nabla \chi_*),\) which completes the proof of the lemma.

\[ \square. \]

**Proof of Theorem 1.1.** Let \(u_\eta\) be the solution of Lemma 2.1 corresponding to \(v\). From (1.5) and (2.1) we infer, for arbitrary \(\varphi \in \mathcal{D}_T,\)

\[
\int_0^T ((v - u_\eta), \partial_t \varphi + \Delta \varphi + v(\eta) \cdot \nabla \varphi) = \int_0^T ((v - v(\eta)) \cdot \nabla \varphi, v) - (v_0 - (v_0)^\eta, \varphi(0)).
\]

(2.10)

Next, let \(\Phi_\eta(x, t) := \psi_\eta(x, T - t), \Xi_\eta(x, t) := p_\eta(x, T - t), (x, t) \in \mathbb{R} \times [0, T],\) with \((\psi_\eta, p_\eta)\) solution constructed in Lemma 2.1 corresponding to \(v_0 \equiv 0\). It then follows that \((\Phi_\eta, \nabla \Xi) \in \mathcal{W}^{1,2} \times L^{4,2} \cap L^{2,2},\) and that \((\Phi, \Xi)\) solves the final-value problem

\[
\partial_t \Phi + v(\eta) \cdot \nabla \Phi + \Delta \Phi = -\nabla \Xi + f, \quad \text{div} \Phi = 0 \quad \text{in} \mathbb{R}^3 \times (0, T),
\]

(2.11)

\[
\Phi(x, T) = 0 \quad x \in \mathbb{R}^3.
\]

Since \(u_\eta \in \mathcal{W}^{1,2},\) by embedding \(u_\eta \in L^{4,4}\). Moreover, the trilinear form \(\mathcal{T} := \int_0^T (v_1 \cdot \nabla v_2, v_3)\) is continuous in \(L^{4,4} \times L^{2,2} \times L^{4,4}\). Thus, by (1.4), and with the help of a density result proved in Lemma A.1 in the Appendix, we can replace \(\Phi_\eta\) for \(\varphi\) in (2.10) and use (2.11) to show

\[
\int_0^T ((v - u_\eta), f) = \int_0^T ((v - v(\eta)) \cdot \nabla \Phi_\eta, v) - (v_0 - (v_0)^\eta, \Phi_\eta(0)),
\]

(2.12)

where we also observed that, by another density argument based on (1.5) and Lemma A.1, it is \(\int_0^T ((v - u_\eta), \nabla \Xi) = 0.\) We next pass to the limit \(\eta \to 0\) in (2.12). By (2.3), \(\|\Phi_\eta(0)\|_2\) is bounded by a constant independent of \(\eta\). Therefore, since \(v_0 \in L^2_p,\)

\[
\lim_{\eta \to 0} (v_0 - (v_0)^\eta, \Phi_\eta(0)) = 0.
\]

(2.13)

Next, by the continuity of the trilinear form \(\mathcal{T}, (1.4)\) and (2.3) it immediately follows that

\[
\lim_{\eta \to 0} \int_0^T ((v - v(\eta)) \cdot \nabla \Phi_\eta, v) = 0.
\]

(2.14)

Finally, Lemma 2.1 and in particular (2.3), entails the existence of an element \(u \in L^{\infty,2} \cap L^2(0; T; H^1)\) such that, along a sequence \(\{\eta_n\},\)

\[
u_{\eta_n} \to u, \quad \text{weak} - \ast \text{ in } L^{\infty,2} \text{ and weakly in } L^2(0, T; H^1).
\]

As a consequence, recalling that \(f \in C_0^\infty(\mathbb{R}^3 \times (0, T))\) we infer

\[
\lim_{n \to \infty} \int_0^T ((v - u_{\eta_n}), f) = \int_0^T ((v - u), f).
\]

(2.15)

Collecting (2.12)–(2.15) we thus conclude \(\int_0^T ((v - u), f) = 0,\) which, by the arbitrariness of \(f\) completes the proof of the theorem.

\[ \square. \]
Remark 2.1  (a) By a slight modification of the proof just given, one can show the same result
with the assumption (1.4) replaced by $v \in L^{2s,s}_{2,s}$, $s > 3$, thus reobtaining (by a different
argument) a well known result of Fabes et al. [2, Theorems (2.1) and (5.3)]. In fact, by our
method one could cover also the borderline case $v \in C([0,T];L^3)$ that is excluded in [2].

(b) The theorem continues to hold with $\mathbb{R}^3$ replaced by a sufficiently smooth bounded or
exterior domain, $\Omega$, and with $v$ vanishing at $\partial \Omega$ in the sense of “very weak” solutions; see, for
the latter, [3, 4]. In such a case, the proofs of Lemma 2.1 (especially for $\Omega$ exterior) and Lemma
A.1 in the Appendix may become technically (but not conceptually) more involved. This is also
a reason why we preferred to treat here the Cauchy problem.

(c) As an immediate corollary to Theorem 1.1, we obtain the following Liouville-type
result: If $v$ satisfies (1.4), (1.5) with $v_0 \equiv 0$, then $v \equiv 0$.

Appendix

Lemma A.1 Let $\mathcal{W}^{1,q} = \{ \psi \in \mathcal{W}^{1,q} \cap C([0,T];H^1) : \psi(T) = 0 \}$, $1 < q < \infty$. Then $D_T$ is dense
in $\mathcal{W}^{1,q}$.
Proof. Let $\chi_R \in C^\infty(\mathbb{R}^3)$ with $\chi_R(x) = 1$ for $|x| \leq R$, $\chi_R(x) = 0$ for $|x| \geq 2R$ such that
\[
|\nabla \chi_R(x)| \leq C R^{-1}, \quad |D^2 \chi_R(x)| \leq C R^{-2},
\] (A.1)
with $C$ independent of $R$. Likewise, for small $h > 0$, let $\zeta = \zeta_h(t)$ be a $C^\infty$ function such that
\[
\zeta_h(t) = \begin{cases} 0 & \text{for } t \geq T - h \\ 1 & \text{for } t \leq T - 2h \end{cases}, \quad |\zeta'_h(t)| \leq C h^{-1} \quad \text{(A.2)}
\]
with $C > 0$ independent of $h$. Set $\psi_{\eta_1,\eta_2}(x,t) = \int_0^T j_{\eta_1}(t-s)\psi(\eta_2)(x,s)ds$, and denote by
$w_R = w_R(x,t)$ a solution to the problem
\[
\div w_R = -\nabla \chi_R(\zeta_h \psi)_{\eta_1,\eta_2}, \quad w_R \in C^\infty(\mathbb{R} \times [0,T]),
\] (A.3)
satisfying the estimates
\[
R^{-1} \|w_R\|_q + \|\nabla w_R\|_q \leq C \|\nabla \chi_R(\zeta_h \psi)_{\eta_1,\eta_2}\|_q,
\]
\[
\|\nabla(\nabla w_R)\|_q \leq C \|\nabla(\nabla \chi_R(\zeta_h \psi)_{\eta_1,\eta_2})\|_q, \quad \text{(A.4)}
\]
\[
\|\partial_t w_R\|_q \leq C R \|\nabla \chi_R \partial_t(\zeta_h \psi)_{\eta_1,\eta_2}\|_q,
\]
where the constant $C$ is independent of $R$. The existence of a function $w_R$ is known [5, Theorem
III.3.3 and Exercise III.3.6]. We shall show that, for any $\varepsilon > 0$ there are sufficiently large $R$ and
small $\eta$ such that, for all $R \geq \overline{R}$ and all $h, \eta_1, \eta_2 \leq \overline{\eta},$
\[
\|w_R\|_{\mathcal{W}^{1,q}} + \max_{t \in [0,T]} \|w_R(t)\|_{1,2} < c_0 \varepsilon. \quad \text{(A.5)}
\]
To this end, set \( B(R) := \{ R < |x| < 2R \} \). By \((A.4)_3\) and \((A.1)\) we have
\[
\int_0^T \| \partial_t w_R \|_q \leq C \int_0^T \left[ \| (\zeta_h \psi)_{t_1, t_2} \|_{q, B(R)}^q + \| (\zeta_h \partial_t \psi)_{t_1, t_2} \|_{q, B(R)}^q \right] := C (I_1 + I_2),
\]
where the subscript \( B(R) \) means that the spatial integration is restricted to \( B(R) \). By the triangle inequality, the properties of the mollifier and \((A.2)_1\), we get
\[
I_2 \leq \int_0^T \left[ \| (\zeta_h \partial_t \psi)_{t_1, t_2} - \zeta_h \partial_t \psi \|_q^q + \| \partial_t \psi \|_{q, B(R)}^q \right] < \varepsilon,
\]
for all sufficiently large \( R \) and small \( \eta_1, \eta_2 \). Furthermore, also with the help of \((A.2)_2\), we show
\[
I_1 \leq c_1 \varepsilon.
\]
Next, by the properties \((A.1)\) of \( \chi_R \), for all sufficiently large \( R \) and small \( \eta_1, \eta_2 \), it easily follows that
\[
\int_0^T \| w_R(t) \|_q \leq C \left[ \| (\zeta_h \psi)_{t_1, t_2} - \zeta_h \psi \|_{L^q} + \int_0^T \| \psi(t) \|_{q, B(R)}^q \right] \leq \varepsilon
\]
(A.9)
In analogous (and simpler) fashion we can show
\[
\int_0^T \left( \| \nabla w_R(t) \|_q^q + \| D^2 w_R(t) \|_q^q \right) < \varepsilon.
\]
(A.10)
Finally, from \((A.3)_{1,2}\), \((A.1)\) and again the triangle inequality we get
\[
\| w_R(t) \|_{1,2} \leq C \left[ \| (\zeta_h \psi)_{t_1, t_2} - \zeta_h \psi \|_{1,2} + \| \psi(t) \|_{1,2, B(R)} \right], \quad t \in [0,T],
\]
which, in turn, since \( \psi \in C([0,T]; H^1) \), for sufficiently large \( R \) and small \( \eta_1, \eta_2 \), implies
\[
\max_{t \in [0,T]} \| w_R(t) \|_{1,2} < \varepsilon.
\]
As a result, \((A.5)\) follows from the latter and \((A.6) - (A.10)\). Set
\[
\Psi_{R,h,\eta_1, \eta_2} := \chi_R(\zeta_h \psi)_{t_1, t_2} - w_R \equiv U_{R,h,\eta_1, \eta_2} - w_R.
\]
(A.11)
It is at once checked that \( \Psi_{R,h,\eta_1, \eta_2} \in C^\infty_0 (\mathbb{R}^3 \times [0,T]) \) and, in view of \((A.3)\), also that \( \text{div} \, \Psi_{R,h,\eta_1, \eta_2} = 0 \). Consequently, \( \Psi_{R,h,\eta_1, \eta_2} \in D_T \). In view of \((A.11)\) and what we have already established in \((A.5)\), to complete the proof, it remains to show that, for a given \( \varepsilon > 0 \), it holds
\[
\| U_{R,h,\eta_1, \eta_2} - \psi \|_{\mathcal{W}^{1, q}} + \max_{t \in [0,T]} \| U_{R,h,\eta_1, \eta_2}(t) - \psi(t) \|_{1,2} < \varepsilon,
\]
for all sufficiently large \( R \) and small \( h, \eta_1, \eta_2 \). However, the proof of the latter is quite straightforward since, under the given assumptions on \( \psi \), it only requires the use of classical properties of mollifiers and, therefore, it will be omitted.
References

[1] Cheskidov, A., Friedlander, S., and Shvydkoy, R., On the energy equality for weak solutions of the 3D Navier-Stokes equations. *Advances in mathematical fluid mechanics*, 171–175, Springer, Berlin, 2010

[2] Fabes, E.B., Jones, B.F., and Rivi`ere, N.M., The initial value problem for the Navier-Stokes equations with data in $L^p$, *Arch. Rat. Mech. Anal.* **45** (1972), 222–240

[3] Farwig, R., Galdi, G.P., and Sohr, H., A new class of weak solutions of the Navier–Stokes equations with nonhomogeneous data. *J. Math. Fluid Mech.* **8** (2006), 423–444

[4] Farwig, R., Kozono, H. and Sohr, H., Very weak solutions of the Navier-Stokes equations in exterior domains with nonhomogeneous data. *J. Math. Soc. Japan* **59** (2007), 127–150

[5] Galdi, G.P., An introduction to the mathematical theory of the Navier–Stokes equations. Steady-state problems. Second edition. Springer Monographs in Mathematics. *Springer, New York*, 2011

[6] Heywood, J.G., The Navier–Stokes equations: on the existence, regularity and decay of solutions. *Indiana Univ. Math. J.* **29** (1980), 639–681

[7] Hopf, E., Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.* **4**, (1951), 213–231

[8] Leray, J., Sur le mouvements d’un liquide visqueux emplissant l’espace, *Acta Math.*, **63** (1934), 193–248

[9] Lions, J.L., Espaces intermédiaires entre espaces hilbertiens et applications, *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.)* **2** (1958), 419–432

[10] Lions, J.L., Sur la régularité et l’unicité des solutions turbulentes des équations de Navier Stokes, *Rend. Sem. Mat. Univ. Padova* **30** (1960), 16–23