MATHEMATICAL VINDICATIONS OF THE “JEANS SWINDLE”

Michael K.-H. Kiessling
Department of Mathematics, Rutgers University
110 Frelinghuysen Rd., Piscataway, N.J. 08854

ABSTRACT: The original Jeans dispersion relation and instability criterion are derived by a mathematically well-defined limiting procedure. The procedure highlights Jeans’ physical reasoning and vindicates the (in)famous “Jeans swindle.” A second, independent procedure is stated which yields the same result.

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1. INTRODUCTION

In 1902, J.H. Jeans [1] derived his celebrated dispersion relation

$$\omega^2 = |k|^2 c_s^2 - 4\pi G \rho_0 \tag{1.1}$$

governing the evolution of infinitesimal disturbances of a fictitious infinitely extended, homogeneous and isotropic, static fluid of mass density $\rho_0$ that is coupled to Newtonian gravity. According to (1.1), an initial disturbance whose wavelength $2\pi/|k|$ is much smaller than the Jeans length

$$\lambda_J = \sqrt{\frac{\pi c_s^2}{G \rho_0}} \tag{1.2}$$

behaves essentially like a classical sound wave of sound speed $c_s$, with self-gravity contributing only slight corrections, but, as Jeans noted [1], perturbations whose wavelengths surpass the Jeans length can grow exponentially in time. The dispersion relation (1.1) has a counterpart in the kinetic theory of encounter-less stellar dynamics; see, for instance, the monograph by Fridman and Polyachenko [2]. A related yet conceptually somewhat different question is the evolution of infinitesimal disturbances of an expanding homogeneous, isotropic general relativistic Friedman-Lemaître cosmology. E. Lifshitz [3] found that disturbances with wavelengths larger than the ‘dynamical’ cosmological Jeans length

$$\Lambda_J = \sqrt{\frac{\pi c_s^2}{G (\rho + p)}} \tag{1.3}$$

grow like a power law, as compared to the background evolution, cf. [4,5].

The Jeans instability is generally regarded as forming the basis of our understanding of gravitational condensation. In particular, the non-relativistic criterion is invoked in astrophysical theories of the formation of stars and some smaller stellar systems, the relativistic one in cosmological theories about the formation of galaxies.

Interestingly, while the analysis for the evolution of infinitesimal disturbances of a homogeneous and isotropic dynamical relativistic universe proceeds in an orderly manner (see in particular [4]), Jeans’ original analysis for the evolution of infinitesimal static non-relativistic universe, which is reproduced in pertinent textbooks and monographs on the subject, e.g. [4-7], enjoys a rather questionable reputation. The following quotation is taken from the excellent monograph by James Binney and Scott Tremaine [7, p. 287ff.] (emphasis in the original):

“\text{We construct our fictitious infinite homogeneous equilibrium by perpetrating what we shall call the \textbf{Jeans swindle} after Sir James Jeans, who studied this problem in 1902 (Jeans 1929). Mathematically, the difficulty we must overcome is that if the density and pressure of the medium $\rho_0, p_0$ are constant, and the mean velocity $\mathbf{v}_0$ is zero, it follows from Euler’s equation (5-8) that $\nabla \Phi_0 = 0$. On the other hand, Poisson’s equation (5-9) requires that $\nabla^2 \Phi_0 = 4\pi G \rho_0$. These two requirements are inconsistent unless $\rho_0 = 0$. Physically, there are no pressure gradients in a homogeneous medium to balance gravitational attraction. A similar inconsistency arises in an infinite homogeneous stellar system whose DF is independent of position. We remove the inconsistency}
by the *ad hoc* assumption that Poisson’s equation describes only the relation between
the perturbed density and the perturbed potential, while the unperturbed potential is
zero. This assumption constitutes the Jeans swindle; it is a swindle, of course, because
in general there is no formal justification for discarding the unperturbed gravitational
field.”

Since modern cosmological applications do not any more rely on the original Jeans
analysis but operate with the unproblematical relativistic dynamical analysis instead, and
since a procedure for which “in general there is no formal justification” does not make
much mathematical sense, one might want to conclude that the original Jeans analysis is
best avoided at all. However, the importance of the stability analysis of an idealized infinite
homogeneous and isotropic static universe lies not in real world physics applications but
in the prospect of simplifying the analysis of certain physically relevant questions while
retaining some essential features of a real system. As such it has been taught, tongue in
cheek of course, to generations of students and newcomers to the field. But if we cannot
backup the Jeans swindle by a formally correct analysis, then we face the dilemma of
rendering a questionable service, surely to the novice in the field, but ultimately also to
ourselves.

As way out of this dilemma, Binney and Tremaine suggest that in special situations a
formal justification of the “Jeans swindle” might exist. They list two examples [7, p. 288]:

“However, there are circumstances in which the swindle is justified. For example,
(i)... [if] ... the wavelength ... is much smaller than the scale over which the equilibrium
density and pressure vary ... the Jeans swindle should be valid for the analysis of small-
scale instabilities.
(ii) ... a uniformly rotating, homogeneous system ... can be in static equilibrium in
the rotating frame and no Jeans swindle is necessary (although the stability properties
are somewhat modified from those of the non-rotating medium because of Coriolis
forces...)”

Unfortunately, upon closer inspection neither point (i) nor (ii) really justifies the Jeans
swindle. Thus, (i) would work if the effective Jeans length of some self-gravitating equi-
librium were much smaller than the scale of non-uniformity of that equilibrium. However,
the typical scale of non-uniformity of a self-gravitating equilibrium is precisely the effec-
tive Jeans length, as emphasized in [2], so that any hypothetical “small scale instability”
would have to be small in scale compared to the effective Jeans length, hence would *not* be
analyzable by (1.1). In case of point (ii) the situation appears to be somewhat better, but
also here there is a catch. In fact, while the introduction of uniform rotation with angular
frequency vector $\Omega$ does regularize the homogeneous gravitational problem in such a way
that an analysis in the spirit of Jeans can be carried out without invoking any ‘swindle’
[8], and while the resulting instability criterion for wave vectors satisfying $k \cdot \Omega \neq 0$ is pre-
cisely $|k|^2 c_s^2 - 4\pi G \rho_0 < 0$, in agreement with (1.1), for this to justify the ‘Jeans swindle’
we would now have to be able to pass to the limit of a genuine ‘Jeans-swindle-situation’
from this ‘no-Jeans-swindle-situation.’ But exactly that is not possible, because the angu-
lar frequency of a uniformly rotating equilibrium and the equilibrium mass density $\rho_0$ are
related by $|\Omega|^2 = 2\pi G \rho_0$. As a consequence, the mass density $\rho_0$ has to vanish in the non-
rotating limit, and therefore a uniformly rotating system does not have a non-rotating limit in which the dispersion relation of the rotating system, which is different from (1.1) due to the presence of Coriolis forces [8], would go over into the dispersion relation discovered by Jeans using his ‘swindle.’

In this paper we will present a mathematically clean vindication of the genuine “Jeans swindle,” and a simple one at that, which emphasizes the physics underlying Jeans’ reasoning. We also mention a second, independent method which gives the same result.

To begin with the physics, the first point to realize is that what counts dynamically are the forces, not the potentials. Hence, as long as we obtain a sensible dynamics in some sensible limit, we should not worry too much if some potential ceases to exist in the same limit. The second point to realize is that in Jeans’ originally conceived system, three infinities are involved: (i) the infinite amount of matter, (ii) the infinite extend in space, and (iii) the infinite range of the Newtonian gravity. Of course, (i) and (ii) are not independent because the assumption that the putative equilibrium is homogeneous, and the perturbations only infinitesimal, couples these two infinities. However, (iii) is an independent source of trouble. Therefore, we are well advised to attempt the construction of the infinite system through a double limit. There are two mathematically natural possibilities:

(A) We first let the mass and size go to infinity in a system with ‘screened’ gravitational forces, and subsequently ‘switch on Newtonian gravity’ by removing the screening. This treatment is borrowed from the class of infrared problems well known in quantum field theory. The standard procedure of handling infrared divergences is to apply an infrared regularization, to solve the regularized problem, and to remove the regularization at the end of the calculation, perhaps involving a ‘renormalization.’ As we will see in a moment, such a procedure works also here in an orderly manner.

(B) We study the system from the beginning with classical gravity, though not in $\mathbb{R}^3$ but in $S^3_R$, which is $S^3$ scaled to have radius $R$. Now the space is finite but without boundary, yet static. We have no problem in defining a homogeneous and isotropic static universe for classical gravity on $S^3_R$, and also not in studying the dynamics in its neighborhood. Letting $R \to \infty$ subsequently, using an obvious renormalization, we arrive at the Jeans dispersion relation for a system in $\mathbb{R}^3$, again in an orderly manner.

In the rest of the paper, we will explain procedure (A) explicitly, leaving (B) as (easy) exercise for the interested reader. We first introduce the screened gravitational interactions and explain that this does not lead to physically unreasonable conclusions. Then we introduce the fluid-dynamical equations of an asymptotically homogeneous system with gravitational interactions by taking the no-screening limit of the corresponding model with screened interactions. The linearized version is precisely the system discussed by Jeans, and his dispersion relation (1.1) follows at once.

The equations of encounter-less stellar dynamics are treated analogously as limit of an asymptotically homogeneous system with screened gravitational interactions, yielding the corresponding Jeans dispersion relation for that model.
2. SCREENED GRAVITATIONAL INTERACTIONS

Consider first a sufficiently well behaved mass density function $\rho(x)$ which, for the moment, shall have finite mass, thus $\int_{\mathbb{R}^3} \rho(x) \, d^3x = M$. Replace the familiar Newtonian potential of $\rho$ at $x \in \mathbb{R}^3$, given by

$$\Phi(x) = -G \int_{\mathbb{R}^3} \frac{1}{|x-y|} \rho(y) \, d^3y, \quad (2.4)$$

by the screened Newtonian potential

$$\Psi(x) = -G \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} \rho(y) \, d^3y \quad (2.5)$$

where $\kappa^{-1}$ is the screening length. In the limit of vanishing screening, i.e. $\kappa \to 0$, (2.5) reduces to the Newtonian potential (2.4). Therefore, if $\kappa$ is tiny enough, for instance $\kappa^{-1} = \text{size of visible universe}$ say, then for all concrete physical situations where one can apply the Newtonian gravitational potential (e.g., non-relativistic planetary motion, stellar dynamics and even some galactic dynamics) one can as well apply the above screened gravitational potential.†

Now drop the requirement that $\int \rho \, d^3x = M$ and consider a monotone sequence of mass densities that converges to a constant mass density $\rho_0 > 0$ (for instance, in sup norm, which means that $\sup_{x \in \mathbb{R}^3} |\rho_0 - \rho(x)| \to 0$). Then (2.5) converges (likewise in sup norm) to a constant limit as well, given by

$$\Psi_0 = -G \rho_0 \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} \, d^3y = -4\pi G \rho_0 \frac{1}{\kappa^2} \quad (2.6)$$

while the Newtonian potential diverges,‡‡ $\Phi \to -\infty$, as $\rho \to \rho_0$. The divergence of $\Phi$ as $\rho \to \rho_0$ is not yet bad news, though, for we know what counts are not the potentials but the forces derived from them, viz. their gradients. Since a homogeneous system has no gradients, we would be in an acceptable limit situation if only we could guarantee that the gradient $\nabla \Phi$ would converge to zero as $\rho \to \rho_0$. Unfortunately, not only does the gradient $\nabla \Phi$ not converge to zero, its limit (whenever it exists) does not just depend on the limit

† Of course, this is not to say that a screened interaction with tiny screening enjoys an equal physical status as the Newtonian interaction – quite the contrary is true. By applying Occam’s razor, there is no point in introducing screened interactions for the discussion of actual finite astrophysical gravitating systems in the non-relativistic regime.

‡‡ In fact, $\Phi = -\infty$ even before $\rho$ has converged to $\rho_0$, which is a consequence of considering convergence in sup norm and the fact that $|x|^{-1}$ is not integrable over $\mathbb{R}^3$. If instead of sup norm convergence one uses the weaker concept of sup norm convergence on all compact sets (for instance by considering a mass density which equals $\rho_0$ inside a ball of radius $R$ centered at $y$ and which vanishes outside the ball, and then letting $R \to \infty$) then once again $\Phi \to -\infty$ though this time $\Phi$ stays finite for all finite $R$. Of course, $\Psi \to \Psi_0$ also now, this time on compact sets.
density \( \rho_0 \) but on the particular limiting sequence \( \rho \to \rho_0 \) (a fact which is well known in astrophysics and which has contributed to the general belief that Jeans’ analysis is a ‘swindle.’)

The existence of the constant limiting screened potential \( \Psi_0 \) on the other hand guarantees, in conjunction with the definition (2.5), that \( \nabla \Psi \to \nabla \Psi_0 = 0 \) when \( \rho = \rho_0 \). In other words, the screened gravitational interactions cancel themselves out when \( \rho = \rho_0 \). Since no pressure gradients are needed to counterbalance these self-balanced screened gravitational forces, it follows that such an infinite, homogeneous and isotropic fluid is automatically in equilibrium. The infinite homogeneous and isotropic equilibrium fluid with self-balanced gravitational forces can now be defined by simply taking the limit \( \kappa \to 0 \) of this equilibrium family.

In an analogous manner we can treat non-uniform mass density functions which differ from \( \rho_0 \) by the displacement of only a finite amount of mass. We could be more general, but this is certainly a reasonable class of mass densities to study. Hence, writing \( \rho(x) = \rho_0 + \sigma(x) \), the density disturbance \( \sigma(x) \) must be sufficiently integrable, satisfy

\[
\int_{\mathbb{R}^3} \sigma(x) \, d^3x = 0 ,
\]  
and be bounded below by \( -\rho_0 \) (for \( \rho_0 + \sigma(x) \) is a mass density and must therefore not be negative). For technical convenience, we actually demand that \( \sigma \) be smooth and decay rapidly to zero at spatial infinity.

The screened gravitational potential \( \Psi \) for such a mass density \( \rho(x) = \rho_0 + \sigma(x) \) is readily computed. By the linearity of the integral formula (2.5), we have

\[
\Psi(x) = \Psi_0 + \psi(x)
\]  
where \( \Psi_0 = -4\pi G \rho_0 / \kappa^2 \) as before, and

\[
\psi(x) = -G \int_{\mathbb{R}^3} \frac{e^{-\kappa |x-y|}}{|x-y|} \sigma(y) \, d^3y
\]  
Since \( \sigma \neq \text{constant} \), the gradient of \( \Psi \) in general now does not vanish, but is given by

\[
\nabla \Psi(x) = \nabla \psi(x)
\]  
Unless the force density \( -\rho(x) \nabla \psi(x) \) is counterbalanced by pressure gradients, \( \rho(x) \) will not be a stationary density. The ensuing time-evolution will be studied in the next section using the conventional Euler, respectively Vlasov equations for such a system.

The important point now is that because of the finite amount of mass involved in the density disturbance \( \sigma \), the limit \( \kappa \to 0 \) of \( \nabla \psi \) exists and is given by

\[
\lim_{\kappa \to 0} \nabla \psi(x) = \nabla \phi(x)
\]  
where

\[
\phi(x) = -G \int_{\mathbb{R}^3} \frac{1}{|x-y|} \sigma(y) \, d^3y
\]
is precisely the expression for the Newtonian potential of the density disturbance invoked by Jeans. Therefore our nonlinear dynamical equations for a spatially \textit{asymptotically} homogeneous and isotropic fluid (respectively, stellar system) with screened interactions turn – without any swindle – into nonlinear dynamical equations for an infinitely extended, asymptotically homogeneous and isotropic fluid (stellar system) with Newtonian gravitational interactions. Their linearization gives the equations originally derived – in his heuristic manner – by Jeans.

We conclude this section with a few remarks regarding Poisson’s equation. We notice that the screened gravitational potential \( \Psi \) given in (2.5) satisfies the inhomogeneous Helmholtz equation

\[
\Delta \Psi - \kappa^2 \Psi = 4\pi G \rho \tag{2.13}
\]

Formally, as \( \kappa \to 0 \), the Helmholtz equation (2.13) for \( \Psi \) goes over into the Poisson equation

\[
\Delta \Phi = 4\pi G \rho \tag{2.14}
\]

for the gravitational potential \( \Phi \), provided the solution \( \Psi \) to Helmholtz’s equation converges to a proper \( \Phi \) when \( \kappa \to 0 \). Now, for the asymptotically homogeneous system where \( \rho = \rho_0 + \sigma \) and \( \Psi = \Psi_0 + \psi \), we have \( \Psi_0 \to -\infty \) when \( \kappa \to 0 \), as obvious from (2.6). Hence, since \( \Psi \not\to \Phi \) in this case, Helmholtz’ equation (2.13) for \( \Psi \) does correspondingly not turn into Poisson’s equation (2.14) for some \( \Phi \). However, once again this is not a problem because a constant potential is dynamically irrelevant. Since we are entitled to ‘renormalize’ the potential by subtracting an overall \((\kappa\text{-dependent})\) constant, we study \( \Psi - \Psi_0 \). The Helmholtz equation for \( \Psi - \Psi_0 \) now does converge to a Poisson equation, namely the one which relates the density disturbance \( \sigma(x) \) to its gravitational potential disturbance \( \phi \). This is readily seen by recalling that \( \Psi(x) - \Psi_0 = \psi(x) \) is the screened gravitational potential disturbance associated with the density disturbance \( \rho(x) - \rho_0 = \sigma(x) \), so that the Helmholtz equation for \( \Psi - \Psi_0 \) simply reads

\[
\Delta \psi - \kappa^2 \psi = 4\pi G \sigma \tag{2.15}
\]

As \( \kappa \to 0 \), \( \psi \to \phi \), defined earlier in (2.9), and (2.15) turns into the Poisson equation

\[
\Delta \phi = 4\pi G \sigma \tag{2.16}
\]

for the gravitational potential disturbance \( \phi \). All the \textit{ad hoc} steps of the “Jeans swindle” have materialized in a mathematically clean way.
3. THE FLUID-DYNAMICAL JEANS DISPERSION RELATION

After our discussion of the screened gravitational interactions, the derivation of the Jeans dispersion relation (1.1) is now straightforward. The model considered in this section is the Euler model of the inviscid motion of a fluid with screened self-gravitation. The variables of the model are the fluid mass density $\rho$, pressure $p$, temperature $T$, fluid velocity $u$, and the screened gravitational potential $\Psi$. The model equations comprise the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$

the force balance equation

$$\rho \partial_t u + \rho u \cdot \nabla u = -\nabla p - \rho \nabla \Psi,$$

an equation of state, for which (for simplicity) we choose the law of the classical perfect gas at constant temperature $T_0$,

$$p = \frac{1}{m} \rho k_B T_0,$$

and the Helmholtz equation for the screened gravitational potential $\Psi$,

$$\Delta \Psi - \kappa^2 \Psi = 4\pi G \rho$$

displayed here again for the sake of completeness. These equations have to be supplemented by asymptotic conditions at spatial infinity, and initial conditions at some initial time, say $t_0 = 0$. In particular, we demand that asymptotically at spatial infinity the dynamical variables approach the values of the stationary, infinite, homogeneous and isotropic equilibrium fluid in which the screened gravitational forces balance themselves. It is a trivial matter to verify that the set of constant variables, $\rho(x) = \rho_0$, $p(x) = p_0 = (\rho_0/m)k_B T_0$, $u(x) = u_0 = 0$, and $\Psi(x) = \Psi_0 = -4\pi G \rho_0/\kappa^2$ for all $x$, forms such a stationary solution of (3.17-2.13). This constant solution will be our reference point.

To inquire into the dynamics in the mathematical neighborhood of this constant solution, we write $\rho(x,t) = \rho_0 + \sigma(x,t)$, and demand that the initial $\sigma(x,0)$ is smooth, rapidly decaying to zero at spatial infinity, and satisfies

$$\int_{\mathbb{R}^3} \sigma(x,0) \, d^3x = 0.$$  

Then $\int_{\mathbb{R}^3} \sigma(x,t) \, d^3x = 0$ for all $t \in (0, \tau)$, where $\tau$ is the mathematical lifespan of the solution. Pressure and screened potential are represented accordingly, thus $p(x,t) = p_0 + \sigma(x,t)k_B T_0/m$, and $\Psi(x,t) = \Psi_0 + \psi(x,t)$. We also write $u(x,t) = u_0 + v(x,t)$. Although for our choice of reference equilibrium we have $u_0 = 0$, whence $u(x,t) = v(x,t)$, we prefer to introduce a new symbol for the deviation from the equilibrium velocity field simply as a reminder that more general equilibrium velocity fields can be handled.

Inserting the above representation of the dynamical variables into our fluid-dynamical equations, and already implementing our equation of state into the force balance equation, as well as using the fact that derivatives of constant functions vanish and that $\Psi_0$ cancels
versus $\rho_0$ from Helmholtz’s equation, we obtain the dynamical equations for the unknowns $\sigma, v, \psi$,

$$\partial_t \sigma + \rho_0 \nabla \cdot v + \nabla \cdot (\sigma v) = 0 \tag{3.22}$$

$$(\rho_0 + \sigma) \partial_t v + (\rho_0 + \sigma) v \cdot \nabla v = -\frac{k_B T_0}{m} \nabla \sigma - (\rho_0 + \sigma) \nabla \psi \tag{3.23}$$

$$\Delta \psi - \kappa^2 \psi = 4\pi G \sigma \tag{3.24}$$

All deviation variables are equipped with the asymptotic conditions that they vanish asymptotically as $|x| \to \infty$, all $t$.

At this point already we can let $\kappa \to 0$ in (3.22-3.24), thereby obtaining the nonlinear dynamical equations for the evolution of the disturbances of an infinitely extended fluid with Newtonian gravity. The continuity equation remains unchanged,

$$\partial_t \sigma + \rho_0 \nabla \cdot v + \nabla \cdot (\sigma v) = 0 \tag{3.25}$$

while force balance and Helmholtz equation become

$$(\rho_0 + \sigma) \partial_t v + (\rho_0 + \sigma) v \cdot \nabla v = -\frac{k_B T_0}{m} \nabla \sigma - (\rho_0 + \sigma) \nabla \phi \tag{3.26}$$

$$\Delta \phi = 4\pi G \sigma \tag{3.27}$$

All deviation variables are equipped with the asymptotic conditions that they vanish at $|x| \to \infty$. Notice that no linearization has been invoked so far.

Proceeding on to the linearization of (3.25-3.27), we write

$$\sigma = \sigma_1 + \sigma_2 + ... \tag{3.28}$$

$$v = v_1 + v_2 + ... \tag{3.29}$$

$$\phi = \phi_1 + \phi_2 + ... \tag{3.30}$$

where the index $k = 1, 2, 3, ...$ indicates the ‘level of smallness.’ Thus, $\sigma_2$ is treated as one level smaller than $\sigma_1$; $\sigma_1 \nabla \phi_1$ is at the same level of smallness as $\rho_0 \nabla \phi_2$; etc. We are only interested in the first level of the hierarchy. Retaining only level 1 terms in (3.25-3.27), we obtain the dynamical equations at level 1,

$$\partial_t \sigma_1 + \rho_0 \nabla \cdot v_1 = 0 \tag{3.31}$$

$$\rho_0 \partial_t v_1 = -\frac{k_B T_0}{m} \nabla \sigma_1 - \rho_0 \nabla \phi_1 \tag{3.32}$$

$$\Delta \phi_1 = 4\pi G \sigma_1 \tag{3.33}$$

supplemented by initial conditions for $\sigma_1, v_1$, and the asymptotic condition of vanishing at infinity for $\sigma_1, v_1$, and $\phi_1$. The solution of these equations is carried out in the standard way using Fourier and Laplace transforms, denoted by $\hat{\cdot}$ and $\tilde{\cdot}$, respectively. For the density perturbation in particular we find

$$\tilde{\sigma}_1(k, \omega) = \frac{\omega \tilde{\sigma}_1(k, 0) - \rho_0 k \cdot \tilde{v}_1(k, 0)}{m^{-1} k_B T_0 |k|^2 - 4\pi G \rho_0 - \omega^2} \tag{3.34}$$

from which we read off the original Jeans dispersion relation (1.1) for the disturbances of an infinite, homogeneous fluid with isothermal equation of state and Newtonian gravitational interactions. We have done so without invoking a ‘swindle,’ or anything illegitimate of that sort.
4. THE STELLAR-DYNAMICAL JEANS DISPERSION RELATION

In complete analogy, except that one has to be careful with the analytic continuations, one derives the stellar dynamical Jeans dispersion relation. It suffices to summarize the main steps.

The dynamical variables of the model are, the particle density function \( f(x, v, t) \) on the one-particle phase space \( \mathbb{R}^3 \times \mathbb{R}^3 \) at time \( t \in \mathbb{R} \), and the screened gravitational potential \( \Psi(x, t) \). They satisfy Vlasov's dynamical equations, which comprise the kinetic equation

\[
\partial_t f + v \cdot \nabla f - \nabla \Psi \cdot \partial_v f = 0
\]

and the inhomogeneous Helmholtz equation for \( \Psi \),

\[
\Delta \Psi - \kappa^2 \Psi = 4\pi G \int_{\mathbb{R}^3} f \, d^3v
\]

Our reference solution consists of a phase space density that is homogeneous in physical space with mass density \( \rho_0 \) and Maxwellian in velocity space with temperature \( T_0 \), thus \( f = f_0(v) = \frac{\rho_0}{(2\pi k_B T_0/m)^{3/2}} \exp\left(-m|v|^2/2k_B T_0\right) \), and the Helmholtz potential \( \Psi = \Psi_0 = -4\pi G \rho_0/\kappa^2 \), as before. Deviations from the stationary reference solution, written as \( f(x, v, t) = f_0(v) + g(x, v, t) \) and \( \Psi(x, t) = \Psi_0 + \psi(x, t) \), are required to approach reference values at spatial and velocital infinity. The dynamical equations for the unknowns \( g \) and \( \psi \) are

\[
\partial_t g + v \cdot \nabla g - \nabla \psi \cdot \partial_v g = \nabla \psi \cdot \partial_v f_0
\]

\[
\Delta \psi - \kappa^2 \psi = 4\pi G \int_{\mathbb{R}^3} g \, d^3v
\]

Taking the limit \( \kappa \to 0 \) gives the nonlinear Vlasov-Poisson equations of an infinitely extended, asymptotically (in space) uniform encounter-less stellar-dynamical system,

\[
\partial_t g + v \cdot \nabla g - \nabla \phi \cdot \partial_v g = \nabla \phi \cdot \partial_v f_0
\]

\[
\Delta \phi = 4\pi G \int_{\mathbb{R}^3} g \, d^3v
\]

Expanding with respect to the levels of smallness,

\[
g = g_1 + g_2 + \ldots
\]

\[
\phi = \phi_1 + \phi_2 + \ldots
\]

and retaining only level 1 terms, we find the linearized Vlasov-Poisson equations,

\[
\partial_t g_1 + v \cdot \nabla g_1 = \nabla \phi_1 \cdot \partial_v f_0
\]

\[
\Delta \phi_1 = 4\pi G \int_{\mathbb{R}^3} g_1 \, d^3v
\]
whose solution in terms of Fourier and Laplace transformation is standard. In particular, the phase space density perturbation, or rather its Fourier-Laplace transformed expression, for $\Im m(\omega) < 0$, reads

$$\tilde{g}(k,v,\omega) = \frac{-i\tilde{g}(k,v,0)}{(\omega + k \cdot v) \left(1 - \frac{4\pi G m \rho_0}{k_B T_0 |k|^2} \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} \frac{\xi e^{-\xi^2}}{\xi + \frac{\omega}{|k|} \sqrt{\frac{m}{2k_B T_0}}} d\xi\right)}$$

which has to be analytically continued to $\Im m(\omega) \geq 0$. Apart from the ballistic term, absent in fluid theory, we immediately read off the stellar-dynamical Jeans dispersion relation for $\Im m(\omega) < 0$,

$$1 - \frac{4\pi G m \rho_0}{k_B T_0 |k|^2} \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} \frac{\xi e^{-\xi^2}}{\xi + \frac{\omega}{|k|} \sqrt{\frac{m}{2k_B T_0}}} d\xi = 0$$

which has to be analytically continued to $\Im m(\omega) \geq 0$. In particular, using Plemilj’s formula, we find that for $\Im m(\omega) = 0$ the dispersion relation can be fulfilled only if $\Re e(\omega) = 0$ as well, which gives as stability boundary the familiar formula of Jeans,

$$|k|_J = \sqrt{\frac{4\pi G m \rho_0}{k_B T_0}}$$

once again without any ‘swindle.’

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