Kinematic model of the parallel convergence method in space

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Abstract. In this article, the implementation of the method of parallel convergence in space in a computer mathematics system is considered and discussed. In this method, the pursuer’s velocity vector is directed arbitrarily. The pursuer’s trajectory gradually approaches movement in the plane formed by the line connecting the initial positions of the pursuer and the target, and the velocity vector. In this task, the target moves uniformly and rectilinearly. The pursuer moves evenly. The points of the pursuer’s trajectory are calculated sequentially. They are being the result of the intersection of the plane containing the line of sight, sphere and cone. As we approach the plane where the target is moving, the algorithm for calculating the trajectory points changes. Now the point of the pursuer’s trajectory is the result of the intersection of the sphere, the plane of movement of the target and the plane containing the line of sight.

1. Introduction
According to the method of parallel convergence on the plane and in space, the movement directions of the pursuer and the target intersect at point K of the Apollonius circle (Figure 1).

Let the pursuer be at point P, the target at point T, at some point in time t (Figure 1). By definition, the Apollonius circle is a set of points \{K\} for which the ratio of distances to two fixed points is constant:

$$\frac{|PK|}{|TK|} = \frac{|VP|}{|VT|} = \text{const},$$

where \(V_P\) is the speed modulus of uniform movement of the pursuer, \(V_T\) is the speed modulus of uniform movement of the target. If the direction of target’s movement is fixed, then there is a single point \(K\) on the Apollonius circle and a single direction of pursuer’s velocity \(V_P\).

![Figure 1. Characteristic points of the Apollonius circle](image)
The pursuer's velocity vector \( V_p \) and the target's velocity vector \( V_T \) in the parallel approach method determine the plane in which the pursuit takes place. Then the following iterative scheme takes place (Figure 2):

\[
P_{i+1} = P_i + \frac{P_iK_i}{|P_iK_i|} \cdot \Delta T,
\]

where \( \Delta T \) is the time interval into which the time of the discrete pursuit process is divided.

If the velocity vector of the target \( T \) is \( V_T \), then the position of the next step of the target \( T_{i+1} \) will be as follows:

\[
T_{i+1} = T_i + V_T \cdot \Delta T.
\]

The coordinates of the point \( K_i \) are the solution of the system of equations with respect to the parameter \( t \):

\[
\begin{cases}
(K_i - Q_i)^2 = R_i^2 \\
K_i = T_i + V_T \cdot \frac{T_{i+1} - T_i}{|T_{i+1} - T_i|} \cdot \bar{t}
\end{cases}
\]

The center of the Apollonius circle \( Q_i \) and its radius \( R_i \) are calculated as follows:

\[
R_i = \frac{V_T^2}{V_p^2 - V_T^2} \cdot |T_i - P_i|, Q_i = T_i + \frac{V_T^2}{V_p^2 - V_T^2} \cdot (T_i - P_i).
\]

\[\text{Figure 2. Calculation of the pursuer's next step}\]

The next step of the trajectory of the pursuer \( P_{i+1} \) in the parallel approach method satisfies the system of equations (1), with respect to the parameter \( h \):

\[
\begin{cases}
(P_{i+1} - P_i)^2 = (V_p \cdot \Delta T)^2 \\
P_{i+1} = T_{i+1} + h \cdot \frac{P_i - T_i}{|P_i - T_i|}
\end{cases}
\]

2. **Problem statement**

The purpose of this article is to calculate the pursuer's trajectory in space when the initial velocity vectors of the pursuer and the target are directed arbitrarily (Figure 3), in addition, the vectors \( V_p \) and \( V_T \) applied to the points \( P \) and \( T \), respectively, do not lie in the same plane.

In the test program, written based on the materials of the article, the goal moves rectilinearly and evenly.
3. Theory

3.1 Calculation of the pursuer’s trajectory in space

In the modification of the kinematic model of parallel approach in space, the calculation of the pursuer’s trajectory is calculated for two locations in space. In the first case, the segment of the pursuer’s trajectory is located in space. In the second case, the segment of the pursuer’s trajectory belongs to the plane, the task turns into a pursuit on the plane.

The smooth transition from space to the plane is calculated separately.

The pursuit model considered in the article is discrete, therefore, a time interval \( \Delta T \) is introduced, during which all participants take a step in the iterative process.

The pursuer, located at point \( P_t \) (Figure 3), takes a step within a sphere of radius \( V_P \cdot \Delta T \) centered at point \( P_t \). \( V_P \) is the pursuer’s module of the uniform movement speed. This possibility is bounded by a regular cone with a solution angle \( \alpha \) and a vertex at point \( P_t \), the axis of the cone is directed along the pursuer's velocity vector. The parameters of the cone follow from the restrictions on the curvature of the pursuer’s trajectory.

The angle at a conical surface (Figure 3) is equal to \( \alpha = \omega \cdot \Delta T \), \( \omega \) is the maximum angular rotation frequency of the pursuer equal to \( \omega = V_P / R_{\text{min}} \), where \( R_{\text{min}} \) is the minimum radius of curvature of the pursuer's trajectory.

The next point \( P_{t+1} \) of the pursuer's position belongs to the plane \( \Sigma_t \). When moving to the plane \( \Pi \) (Figure 3), the pursuit model is transformed into the iterative scheme shown in Figure 2.
The axis of the cone is directed along the current pursuer's velocity vector $V_P$, leaving the point $P_i$. At the stage when calculating the trajectory segment in space, we calculate the intersection point of three surfaces: a sphere, a cone and a plane.

Replace the regular cone with the plane $\Phi_i$ (Figure 3). The line of intersection of the regular cone and the sphere forms a circle and belongs to the plane $\Phi_i$ (Figure 3).

Plane parameters $\Phi_i$ will be as follows: $a_i = \frac{V_{P_i}}{V_P}$ - unit normal vector to the plane $\Phi_i$, $V_{P_i}$ - velocity vector for the current pursuer's position $P_i$, $V_P$ - the speed module of uniform pursuer's movement, $A_i = P_i + a_i \cdot R \cdot \cos(\alpha)$, where $R$ is the radius of the sphere equal to the pursuer's pitch $V_P \cdot \Delta T$ (Figure 3).

The points of intersection of the sphere, the cone and the plane is equivalent to the calculation of the intersection point of the planes $\Phi_i$ and $\Sigma_i$ with the sphere of radius $R$ with center at the point $P_i$.

The plane $\Sigma_i$ passes through the point $T_{i+1}$, the normal is the vector $b_i$ (Figure 3): $b_i = \left((P - T) \times [0, 0, 1]\right)$, where $P$ and $T$ are the initial positions of the pursuer and the target.

The test program defines a straight line (2), where the planes $\Phi_i$ and $\Sigma_i$ intersect:

$$L_i(t) = K_i + h \cdot \frac{[a_i \times b_i]}{||a_i \times b_i||},$$

where $K_i$ is the intersection point of the planes $\Phi_i$, $\Sigma_i$ and $\Pi$ (Fig. 3). The coordinates of the point $P_{i+1}$ from the first equation of the system (3) are substituted into the second equation of the system (3):

$$\begin{cases} P_{i+1} = K_i + h \cdot \frac{[a_i \times b_i]}{||a_i \times b_i||} \\ (P_{i+1} - P_i)^2 = (V_P \cdot \Delta T)^2 \end{cases}$$

The system of equations (3) is solved with respect to the parameter $h$. The value of $h$ is substituted into the first equation of the system (3). This is how the next point $P_{i+1}$ of the pursuer's trajectory is determined.

The iterative process of calculating the trajectory of the pursuer's movement in space is fully formed.

### 3.2 Calculation of the pursuer's trajectory on the plane

If the migration process, prosecution on the plane $\Pi$, the movement directions of the pursuer and target are focused to a point on the Apollonius circle (Figure 1), then use the iterative scheme presented by the system of equations (1).

In the case when the speed is not aimed at a point on the circle of Apollonius, the model uses the following iterative scheme (Figure 4).

![Figure 4](image)

**Figure 4.** Calculation of the pursuer's next step

Instead of a one-parameter set of parallel lines connecting the pursuer and the target $(P_iT_i)$, a set of composite parallel lines $\{l_i(t)\}$ is proposed, which is formed as follows:

$$l_{i+1}(t) = l_i(t) + (T_{i+1} - T_i).$$

The pursuer's next step $P_{i+1}$ is the intersection point of a circle of radius $V_P \cdot \Delta T$ centered at the point $P_i$ and line $l_{i+1}(t)$ (Figure 4).
The first line of one-parameter subgroups of the set of lines \( \{L_i(t)\} \) is formed from a circle of minimum radius \( R_{\text{min}} \) and the tangent of the straight line passing through the point \( T \) (Figure 5).

![Figure 5. Positions of the pursuer, the target and the circle](image)

The center of the circle with which the straight line \((P_{\text{tan}}T)\) is conjugated (Figure 5) is located at the point \( C = P + R_{\text{min}} \cdot n \), where P is the initial position of the pursuer, \( R_{\text{min}} \) is the minimum radius of curvature of the pursuer's trajectory, \( n \) is a unit vector perpendicular to the pursuer velocity vector \( V_P \).

The line consists of an arc \( PP_{\text{tan}} \) and the segment \([P_{\text{tan}}T]\), where T is the initial position of the target, and \( P_{\text{tan}} \) is the point of contact with the circle.

### 3.3 Criterion of transition to a plane

If we look at the calculated points of the pursuer’s trajectory without taking into account the smooth transition to the plane (Figure 6), we can assume that it would be possible to implement the transition in many ways.

The model of the article implements a method that fixes the moment when the pursuer's trajectory crosses the plane \( \Pi \), and returns to the pursuer's previous position.

![Figure 6. Calculation of the pursuer's trajectory in space without transition to a plane](image)

The screen displays (Figure 6) the points \( K_i \) of the intersection of the plane \( \Pi \) (the plane to which the pursuit process tends), the plane \( \Sigma \) and the plane \( \Phi \) (Figure 3) and the intersection points of the cone, sphere and plane \( \Sigma \).

When calculating the smooth transition of the pursuer to the plane \( \Pi \) (Figure 7), the iterative process analyzes the relative position of the pursuer \( P_1 \) and the plane \( \Pi \) of the target movement.
The plane $\Pi$ of the target movement coincides with the coordinate plane $XY$ (the coordinate system was transferred), then it is enough to analyze the applicate of the pursuer's trajectory to the sign. If the sign of the applicate changes, then come back to the previous point of the trajectory and the calculation is performed according to a different iterative scheme.

Consider the case when the applicate of the point $P_{i-1}$ has a positive value, and the applicate of the point $P^*_i$ has a negative value.

Point $P^*_i$ (Figure 8) is obtained as a result of the intersection of a sphere $S_i(P_i, V_P \cdot \Delta T)$, a cone with an axis of rotation along the vector $V_{P_{i-1}}$ with a solution angle $\alpha = \omega \cdot \Delta T$ (Figure 3), and a plane of parallel motion $\Sigma_i$.

The algorithm returns to the point $P_{i-1}$ and searches for $P_i$ as the intersection point of the sphere $S_{i-1}(P_{i-1}, V_P \cdot \Delta T)$ with the plane of parallel motion $\Sigma_i$ and the plane $\Pi$ of the target movement.

4. Experimental results
As a result of the test program, the points of the pursuer's trajectory are obtained when the target moves uniformly and rectilinearly (Figure 8). The trajectory passes from space to a plane.
The plane $\Pi$ in this case serves as a surface, which is not allowed to fall below. The program implements a model of pursuit, when the pursuer moves from movement in space to movement in a plane without crossing the plane. When the pursuer moves, curvature restrictions are observed.

5. Conclusion
The article proposes a kinematic model for constructing the trajectories of the pursuer in space, in which the transition to movement on the plane is made. With the development of artificial intelligence systems, satellite positioning technologies for moving objects, the simulation of pursuit tasks has acquired a new meaning. The research results may be in demand by developers of unmanned aerial vehicles with elements of artificial intelligence.

The article uses the theoretical results of the founders of game theory R. Isaacs, L.S. Pontryagin, N. N. Krasovsky, L.A. Petrosyan [1-4]. The program can be found on the resource [5]. The article uses the results of [6-9]. Animated images are posted on the resource [10]. According to the results of the article, a certificate of registration of computer programs was obtained [11].

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