EXISTENCE AND UNIQUENESS OF STRONG SOLUTION FOR SHEAR THICKENING FLUIDS OF SECOND GRADE

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Abstract. In this paper we study the equations governing the unsteady motion of an incompressible homogeneous generalized second grade fluid subject to periodic boundary conditions. We establish the existence of global-in-time strong solutions for shear thickening flows in the two and three dimensional case. We also prove uniqueness of such solution without any smallness condition on the initial data or restriction on the material moduli.

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1. INTRODUCTION

The theory and the applications of non-Newtonian fluids have attracted the attention of many scientists for a long time since they are more appropriate than Newtonian fluids in many areas of engineering sciences such as geophysics, glaciology, colloid mechanics, polymer mechanics, blood and food rheology, etc. Fluids with shear dependent viscosity, which can exhibit shear thinning and shear thickening and include the power-law fluids as a special case, constitute a large class of non-Newtonian fluids. For instance, see [16], [17] and [18] for more detailed discussions.

There are many models of non-Newtonian fluids which have recently become to be of great interest. Among these models one can cite fluids of differential type. The
second grade fluids, which are a subclass of them, have been successfully investigated in various kinds of flows of materials such as oils, greases, blood, polymers, suspensions, and liquid crystals. In the classical incompressible fluids of second grade, it is customary to assume that the Cauchy stress tensor $T$ is related to the velocity gradient $\nabla u$ and its symmetric part $D u$ through the relation

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where $p$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mu$ is the kinematic viscosity, and $\alpha_1$ and $\alpha_2$ are material moduli which are usually referred to as the normal stress coefficients. The kinematical tensor $A_1$ and $A_2$ are the first and the second Rivlin-Ericksen tensor, respectively, and they are defined through

$$A_1 = \nabla u + \nabla u^t = 2 D u = \nabla u + (\nabla u)^t,$$

where $u = (u_1, ..., u_d)$ and $\nabla u := (\partial_j u_i)_{1 \leq i,j \leq d}$ ($d = 2$ or $3$ is the space dimension) denote the velocity vector field and its gradient. Here $\partial_j u_i$ stands for the partial space derivative of $u_i$ with respect to $x_j$, $i,j = 1, ..., d$. The material derivative is given by

$$\frac{d}{dt}(.) = \partial_t (.) + (u.\nabla)(.),$$

where $\partial_t$ is the partial derivative with respect to time and $(u.\nabla)$ the differential operator with respect to the spatial variable defined by $u.\nabla = \sum_{i=1}^{d} u_i \partial_i$.

Indeed, in [9], Dunn studied the thermodynamics and stability of second-grade fluids with viscosity $\mu$ depending on the shear rate $|D u|$ (i.e. the Euclidean norm of the symmetric part of the velocity gradient defined by $|D u| = \sqrt{\text{Tr}(D u)^2}$) and showed that if the fluid is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid be minimum in equilibrium, then $\mu$, $\alpha_1$ and $\alpha_2$ in (1) must verify

$$\sqrt{\frac{3}{2}} \frac{\mu(|D u|)}{|D u|} \leq \alpha_1 + \alpha_2 \leq \sqrt{\frac{3}{2}} \frac{\mu(|D u|)}{|D u|}.$$

In the 1980s, Man, Kjartanson and coworkers [22] and [12] showed that polycrystalline ice in creeping flow under pressuremeter tests can be modeled as an incompressible second-grade fluid with the viscosity depending on the shear rate. The constitutive equation proposed by Man and coworkers, which leads to well-posed initial-boundary-value problems in nonsteady channel flow [20] and can also model the flow of ice in triaxial creep tests [23], is

$$T = -pI + \mu(|D u|) A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$
where $\mu(|Du|) = \nu |Du|^m$, with $-1 < m < \infty$ and $\nu$ is a material constant.

Man and Massoudi [21] conducted thermodynamic studies on some classes of generalized second-grade fluids, which include the class defined by (6). For the case $m > 1$ of this class, they showed that the viscosity function $\mu$ and the normal-stress coefficients must verify the inequality

$$\mu \geq 0, \quad \alpha_1 + \alpha_2 = 0, \quad \text{and} \quad \alpha_1 \geq 0$$

as restrictions imposed by thermodynamics and the assumption that the specific Helmholtz free energy of the fluid be minimum in equilibrium.

In this paper we study a class of generalized second-grade fluids with constitutive equation given by (6), which has its normal-stress coefficients satisfy the equation $\alpha_1 + \alpha_2 = 0$ and has a viscosity function $\mu(|Du|)$ different than that proposed by Man and coworkers. The mathematical assumptions on the viscosity function that we adopt will be given in Section 3. For the class of generalized second-grade fluid in question, the thermodynamic restrictions (7) and (7) are valid [9].

The interested reader can find more about the literature of the generalized fluids of second grade and their applications, for example in [21], [25] and [26]. Henceforth we will adopt (7) and (7) as assumptions. Based on these assumptions and the relations (2)-(3), one can deduce that the stress tensor (6) for generalized fluids of second grade could be written in the following form

$$T = -pI + 2\mu(|Du|)Du + \alpha_1[D_t(2Du) + 2(u,\nabla)Du + (\nabla u)^t \nabla u - \nabla u(\nabla u)^t].$$

Let us mention that by “generalized second grade fluid” we mean a fluid of second grade of type (6) (i.e. whose viscosity is a non linear function of the shear rate) as it will be considered in this study, and by word “ classical second grade fluid” a second grade fluid (1) (i.e. whose viscosity is constant).

This work is devoted to the mathematical analysis of the equations governing the flow of a homogeneous incompressible fluid which occupies all space $\mathbb{R}^d$ ($d = 2; 3$), the Cauchy stress of which is given by formula (8), and the velocity of which is $2\pi$-periodic. We will also assume that the kinematic viscosity $\mu(|Du|)$ is a nonlinear function of $|Du|$. The stress tensor $2\mu(|Du|)Du$ will be denoted by $S(Du)$. Next, we derive the equations modeling our studied system.

Let $T$ be an arbitrary positive real number. We set $[0, T]$ as the time interval and $\Omega := (0, 2\pi)^d$ the periodic box of dimension $d$. Since the fluid body is taken as incompressible and homogeneous in material, the density of the fluid can be put equal to 1. Then the momentum equation is expressed as

$$\frac{du}{dt} = div T + f,$$

with the condition of incompressibility

$$div u = 0.$$
Therefore, with the help of relation (4), inserting the expression (8) and expanding, equation (9) becomes

\[
\partial_t v + (u \cdot \nabla)v - \text{div}(S(Du)) + \sum_{j=1}^{d} v_j \nabla u_j = -\nabla p + f, \quad v = u - \alpha_1 \Delta u.
\]

One can look at [4] and more precisely at [5] to verify that the computations in (11), holds true. See also [6] and [11] where (11) is the adopted form for their analysis.

These equations come with the initial data prescription

\[
u(0, \cdot) = u_0 \in \mathbb{R}^d, \quad \text{and} \quad \text{div} u_0 = 0,
\]

where \( u_0 \) denotes the initial velocity vector field. We also require that the velocity and the pressure satisfy

\[
u, p : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R} \quad \text{are} \quad 2\pi - \text{periodic with respect to} \ x_i,
\]

\[
\int_{\Omega} u_i \, dx = 0, \quad \int_{\Omega} p \, dx = 0, \quad i = 1, \ldots, d.
\]

Recall that the zero mean value request is to be assumed for the Poincaré's inequality.

Note that we are looking for \( u : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \) and \( p : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R} \) that solve the system of equations made up of (10), (11) (12) and (13). For an explanation of the phenomena of unsteadiness of the considered flow, see [24].

Before proceeding further, let us clarify some expressions involved in the momentum equation (11). Recall that \( u \) is divergence free. Then, in the distributional sense, the \( i \) component of the convective term \((u \cdot \nabla)v\) is defined by

\[
((u \cdot \nabla)v)_i = \sum_{j=1}^{d} u_j \partial_j v_i = \sum_{j=1}^{d} u_j \partial_j u_i - \alpha_1 \sum_{j,k=1}^{d} u_j \partial_j \partial_{kk} u_i
\]

\[
= \sum_{j=1}^{d} u_j \partial_j u_i - \alpha_1 \sum_{j,k=1}^{d} \partial_j (u_j \partial_{kk} u_i)
\]

\[
= \sum_{j=1}^{d} u_j \partial_j u_i - \alpha_1 \sum_{j,k=1}^{d} \partial_j (u_j \partial_k u_i) + \alpha_1 \sum_{j,k=1}^{d} \partial_j (\partial_k u_j \partial_k u_i).
\]

Analogously the \( i \) component (i.e. the partial derivative) of \( \sum_{j=1}^{d} v_j \nabla u_j \) can explicitly be written as

\[
(\sum_{j=1}^{d} v_j \nabla u_j)_i = \sum_{j=1}^{d} u_j \partial_j u_j - \alpha_1 \sum_{j,k=1}^{d} \partial_{kk} u_j \partial_j u_j
\]

\[
= \sum_{j=1}^{d} \frac{1}{2} \partial_i |u_j|^2 - \alpha_1 \sum_{j,k=1}^{d} \partial_k (\partial_k u_j \partial_j u_j) + \alpha_1 \sum_{j,k=1}^{d} \partial_k u_j \partial_{kk} u_j
\]

\[
= \sum_{j=1}^{d} \frac{1}{2} \partial_i (|u_j|^2 + \alpha_1 |\nabla u_j|^2) - \alpha_1 \sum_{j,k=1}^{d} \partial_k (\partial_k u_j \partial_j u_j).
\]
Therefore (11) can be differently expressed in the following form

\[ \partial_t v + (u, \nabla) u - \text{div}(S(Du)) - \alpha_1 \sum_{j,k=1}^d \partial_{j,k}(u_j \partial_k u) + \alpha_1 \sum_{j,k=1}^d \partial_j(\partial_k u_j \partial_k u) \]

\[ - \alpha_1 \sum_{j,k=1}^d \partial_k(\partial_k u_j \nabla u_j u) = -\nabla(p + \frac{1}{2}|u|^2 + \frac{\alpha_1}{2}|\nabla u|^2)) + f. \]

The rest of the paper is planned as follows. In section 2, we introduce notations and functions spaces. We also outline some technical results that will be used later. In section 3, we set up the assumptions imposed on the stress tensor S. Moreover, we present the resulted properties that will allow us to prove our main result. We also give a brief explanation of the studied problem and the considered type of viscosities. Later, we state the main theorem and relate it with some previous results that are concerned with the classical second grade fluids and those with shear-rate dependent viscosities. In section 4, we give the proof of the existence part of our result. Finally, in section 5 we show the uniqueness of the constructed solution.

The main feature of the proof is based on a construction of a sequence of approximated solutions using a discretization in space of Galerkin’s type and a limit process using compactness arguments in order to control the nonlinear terms. In addition, we will also make sense of the weak formulation introduced in the main theorem and conclude by showing that such a solution is unique via Gronwall’s lemma.

2. Notations and auxiliary results

Since we deal with a spatial periodic problem on \( \mathbb{R}^d \), then functions defined on \([0,T] \times \mathbb{R}^d\) can be considered as defined on \([0,T] \times \Omega\). So, we introduce the following spaces. The space \( C^{\infty}_{\text{per}}(\Omega) \) consists of all smooth \( 2\pi \)-periodic functions at each direction \( x_i, i = 1, \ldots, d \). For \( k \in \mathbb{N} \) and \( 1 \leq q < \infty \), the Lebesgue space \( L^q_{\text{per}}(\Omega) \) (respectively \( W^{k,q}_{\text{per}}(\Omega) \)) are introduced as the closure of \( C^{\infty}_{\text{per}}(\Omega) \) in the norm \( \| \cdot \|_{L^q} \) (respectively \( \| \nabla^k \cdot \|_{L^q} \)) and having zero mean value \( \int_{\Omega} f(x) \, dx = 0 \), where \( \| f \|_{L^q} = \left( \int_{\Omega} |f(x)|^q \, dx \right)^{1/q} \) and \( \| \nabla^k f \|_{L^q} = \left( \int_{\Omega} |\nabla^k f(x)|^q \, dx \right)^{1/q} \) are the respective norms. Here \( \nabla^k \) denotes the space gradient of order \( k \). The spaces \( L^q_{\text{per,div}}(\Omega) \) (respectively \( W^{k,q}_{\text{per,div}}(\Omega) \)) stand for the set of functions belonging to \( L^q_{\text{per}}(\Omega) \) (respectively \( W^{k,q}_{\text{per}}(\Omega) \)) with zero divergence. Note that these spaces may also be defined as the closure of \( C^{\infty}_{\text{per}}(\Omega) \) with zero divergence and zero mean value with respect to the corresponding norms.

The scalar product in \( L^2_{\text{per}} \) will be denoted by \((.,.)\) and that in \( W^{k,2}_{\text{per}} \) by \((.,.)_k\). The scalar product of vectors \( u = (u_1, \ldots, u_d) \) and \( v = (v_1, \ldots, v_d) \) is denoted by \( u.v = \sum_{i=1}^d u_i v_i \) while that of tensors \( B = (B_{ij})_{1 \leq i,j \leq d} \) and \( D = (D_{ij})_{1 \leq i,j \leq d} \) is denoted by \( B : D = \sum_{i,j=1}^d B_{ij}D_{ij} \).

Given a Banach space \( X(\Omega) \) then \( X^*(\Omega) \) stands for its dual space. We will not distinguish between spaces of scalar, vector and tensor valued functions as one can easily make difference of sense between them. We denote by \( L^q(0;T;X(\Omega)) \) the
usual Bochner space consisting of functions which values are in $X$ and are $L^q$ time-integrable over $(0,T)$ and by $L^q([0,T];X^*(\Omega))$ its dual where $q'$ is the conjugate exponent of $q$ given by $q' = \frac{q}{q-1}$. On the other hand, $C([0,T];X(\Omega))$ stands for the space of continuous in time functions in $[0,T]$ and with values in $X$. We also denote by $C_w(0,T;X(\Omega))$ the space of functions $u$ which are in $L^\infty(0,T;X(\Omega))$ and continue for almost every $t \in [0,T]$ for the weak topology of $X(\Omega)$.

Throughout the paper, if we denote by $c$ a positive constant with neither any subscript nor superscript then $c$ is considered as a generic constant whose value can change from line to line in the inequalities and depends on the parameters in question but has no effect on the solution. On the other hand, we will denote in a bold character tensor functions and in the usual one vector valued and scalar functions.

In the sequel of this section, we review some technical results and classical lemmas on compactness arguments and limiting process.

Since we are dealing with space periodic and divergence-free vector fields, one can enjoy some special identities that will be summarized in the following lemma.

**Lemma 2.1.** Let $f : \Omega \rightarrow \mathbb{R}$ and $u, v : \Omega \rightarrow \mathbb{R}^d$ be periodic functions. Suppose that $u$ and $v$ are divergence-free, then the following identities hold

\begin{align}
\int_\Omega \nabla f \cdot u \, dx &= 0, \\
\int_\Omega (u \cdot \nabla) v \cdot v \, dx &= 0, \\
\int_\Omega (u \cdot \nabla) u \cdot v \, dx &= -\int_\Omega (u \otimes u) \cdot \nabla v \, dx.
\end{align}

**Proof.** The proof is based on the use of Green’s formula, the divergence-free property and the fact that boundary terms vanish due to the periodicity. For details, one can see Lemma 2.9 in [19].

The famous Aubin-Lions lemma will play an important role in the sequel.

**Lemma 2.2** (Aubin-Lions). Let $1 < \alpha < \infty$, $1 \leq \beta \leq \infty$ and $X_0, X_1, X_2$ be Banach reflexive separable spaces such that

$$X_0 \hookrightarrow X_1 \quad \text{and} \quad X_1 \hookrightarrow X_2.$$

Then

$$\left\{ u \in L^\alpha(0,T;X_0); \partial_t u \in L^\beta(0,T;X_2) \right\} \hookrightarrow \hookrightarrow L^\alpha(0,T;X_1).$$

Here, the symbol $\hookrightarrow \hookrightarrow$ stands for the compact imbedding while $\hookrightarrow$ for the continuous one. Lemma 2.2 is a general version of the Aubin-Lions lemma valid under the fulfillment of the assumption $\beta = 1$ and is proved in [27] and [29], separately.

**Lemma 2.3** (Vitali). Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ and $f^n : \Omega \rightarrow \mathbb{R}$ be a sequence of functions. Suppose that

$$\lim_{n \to \infty} f^n(x) \text{ exists and is finite for all } x \in \Omega,$$
- for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\sup_{n \in \mathbb{N}} \int_Q |f^n(x)| \, dx < \varepsilon \quad \forall Q \subset \Omega \quad \text{with} \quad |Q| < \delta.
\]

Then

\[
\lim_{n \to \infty} \int_{\Omega} f^n(x) \, dx = \int_{\Omega} \lim_{n \to \infty} f^n(x) \, dx.
\]

We would like to note that the second assumption stated in the lemma above expresses the equi-integrability of the sequence \( \{f^n\} \). Recall that if \( \{f^n\} \) is uniformly bounded in some Lebesgue spaces \( L^q(\Omega) \) for \( q > 1 \) then they are equi-integrable functions.

We finish this section by the Korn’s inequality whose a proof can be found in [19] (see Theorem 1.10 page 196).

\[\text{Lemma 2.4 (Korn’s Inequality). Let } q \in (1, \infty), \text{ then there exists a positive constant } c_k \text{ depending only on } \Omega \text{ and } q \text{ such that for all } u \in W^{1,q}_{\text{per}}(\Omega) \]

\[c_k \|\nabla u\|_{L^q} \leq \|D u\|_{L^q}.
\]

3. Assumptions and main results

Since we consider a variable viscosity depending on the shear rate, this makes the problem strongly nonlinear and brings some complications to its analysis. To deal with this situation, it seems naturally to impose further restrictions on the structure of the stress tensor \( S \).

We denote by \( S_{ij}(D) \) the \( ij \) entry of the matrix \( S(D) \). We shall assume that the stress tensor function \( S : M^d_{\text{sym}} \to M^d_{\text{sym}} \) is continuously differentiable and satisfy \( S(0) = 0 \).

In addition, we suppose that, for a given fixed parameter \( r \in (2, \infty) \), there are positive constants \( c_0, c_1 \) and \( c_2 \), such that for all \( B, D \in M^d_{\text{sym}} \)

\[c_0(1 + |D|)^{r-2}|B|^2 \leq \frac{\partial S_{ij}(D)}{\partial D_{kl}} B_{ij} B_{kl} \leq c_1(1 + |D|)^{r-2}|B|^2,
\]

\[|\frac{\partial S_{ij}(D)}{\partial D_{kl}}| \leq c_2(1 + |D|)^{r-2}.
\]

It is worth noticing that the nonlinearity of the tensor \( S \) has some useful properties that follows from (20) and (21).

\[\text{Lemma 3.1. Let } r \in (2, \infty) \text{ be a fixed real number and the tensor } S \text{ satisfy the assumptions (20) and (21). Then, there exist positive constants } c_3 \text{ and } c_4 \text{ such that, for all } B, D \in M^d_{\text{sym}},
\]

\[S(D) : D \geq c_3(1 + |D|)^{r-2}|D|^2,
\]

\[|S(D)| \leq c_4|D|(1 + |D|)^{r-2},
\]

\[|S(B) - S(D)| : |B - D| \geq c_3|B - D|^2(1 + |B| + |D|)^{r-2}.
\]

\[\text{Proof. See Lemma 1.19 page 198 in [19].}\]
According to this lemma, the inequalities (22) and (23) express respectively the $r$-coercivity and the polynomial growth of the tensor $S$ while (24) describes the strict monotonicity of this operator. To be more precise, one can give a typical example of stress tensors, where the above structure is satisfied, for example

$$(25)\quad S(D) = (\mu_0 + \mu_1 |D|^{r-2})D,$$

where $\mu_0$ and $\mu_1$ are positive constants. The models (25) constitute a large family of stress tensors but do not include the proposed model $(\mu|D|^{r-2})D$ by Man and Sun in [23] and later by Man [20]. The main results of our paper are summarized in the following theorem.

**Theorem 3.2.** Let $r \in [3, \infty)$ and let the stress tensor $S$ verify the assumptions (20) and (21). Assume that the forcing term $f \in L^2(0,T;W^{1,2}_{\text{per}}(\Omega))$ and that $u_0 \in W^{2,2}_{\text{per,div}}(\Omega)$. Then there exists a strong solution $u$ to the problem (11)-(13) such that

$u \in L^\infty(0,T;W^{2,2}_{\text{per,div}}(\Omega)) \cap L^r(0,T;W^{1,r}_{\text{per,div}}(\Omega))$

$u \in C([0,T];W^{1,\sigma}_{\text{per,div}}(\Omega)), \quad \text{with} \quad \sigma \in [0,2),$

$\partial_t u \in L^2(0,T;L^2_{\text{per,div}}(\Omega)), \quad \text{if} \quad d = 2,$

$\partial_t u \in L^2(0,T;L^2_{\text{per,div}}(\Omega)) \cap L^{r-1}(0,T;W^{1,r-1}_{\text{per,div}}(\Omega)), \quad \text{if} \quad d = 3,$

and satisfying the following weak formulation

$$(26)\quad \int_0^T (\partial_t u, \varphi) \, d\tau - \alpha_1 \int_0^T (\partial_t \Delta u, \varphi) \, d\tau + \int_0^T ((u \nabla)u, \varphi) \, d\tau + \alpha_1 \int_0^T ((u \nabla)D\varphi, u) \, d\tau + \int_0^T (S(Du), D\varphi) \, d\tau + \alpha_1 \int_0^T \int_\Omega \sum_{j,k=1}^d \partial_k u_j \partial_i u_j \partial_k \varphi_i \, dx \, d\tau = \int_0^T (f, \varphi) \, d\tau,$$

for all $\varphi \in L^2(0,T;W^{2,2}_{\text{per,div}}(\Omega)) \cap L^r(0,T;W^{1,r}_{\text{per,div}}(\Omega))$. Moreover, the solution $u$ is unique and attain the initial conditions in the following sense

$$(27)\quad \lim_{t \to 0^+} \|u(t) - u_0\|_{W^{1,2}} = 0.$$

**Remark 3.3.**

i) The principal reason for which we have considered the flow with shear thickening behavior and not with a shear thinning one comes from the following: when getting the energy inequality, we realize that $\nabla u \in L^r(0,T;L^r_{\text{per}}(\Omega))$ and since we have in our system terms with gradients products, namely $\nabla u(\nabla u)^t$ and its transpose, then to ensure the space integrability of these terms (eventually in $L^2_{\text{per}}(\Omega)$) and in accordance with the definition of Sobolev spaces, the parameter $\frac{r}{2}$ must be greater than 1, and hence $r \geq 2$. 
The existence of the pressure can be deduced, with the help of the De Rham’s
theorem, from the weak formulation (26) and one can assert that there is
some distribution \( p \) such that (11) holds in \( [0, T] \times \mathbb{R}^d \).

To the best of our knowledge, the question of existence of weak or strong solutions
has not been raised for the class of generalized second grade fluids. Furthermore,
this work seems to be the first to prove global existence and uniqueness of strong
solution without restrictive smallness assumptions on the initial data.

Let us come back to some related results concerning classical second grade fluids
whose Cauchy stress is described by the formula (1). In [10], Galdi and Sequeira
proved existence and uniqueness of global in time classical solution when the initial
data is very small in a bounded regular domain of \( \mathbb{R}^3 \) and subject to Dirichlet
boundary conditions. Later, Le Roux in [14] extended these results under suitable
regularity and growth restrictions on the initial data with nonlinear partial slip
boundary conditions in a bounded simply-connected domain. We also mention that
Cioranescu and Ouazar [7] proved, for the Dirichlet boundary conditions case, existence
and uniqueness of \( W^{3,2} \) solutions that are global in two dimensions of spaces
and local for small data in three dimensions. Further, Cioranescu and Girault [5]
improved the three dimensional result by showing global existence under further
appropriate assumptions on the data. On the other hand, Bresch and Lemoine [2]
established existence and uniqueness of \( W^{2,r} \) (\( r > 3 \)) stationary solution for three
dimensional bounded domain of class \( C^2 \) with smallness restrictions on the kinematic
viscosity \( \mu \) and the forcing term.

The first results concerning unsteady incompressible flows of Navier-Stokes equa-
tions with shear rate dependent material coefficients go back to Ladyzhenskaya [13]
and Lions [15], who proved for \( r \geq \frac{3d}{d+2} \) the existence of weak solutions by using the
monotone operator theory and compactness arguments. In the last two decades, the
mathematical analysis of fluids with shear rate dependent viscosities have known
a lot of relevant works dealing with existence of weak and strong solutions and
regularity results. One can cite some leaders in this field such as M. Bulíček, E.
Feireisl, J. Frehse, J. Málek, J. Nečas, K.R. Rajagopal, M. Růžička and many oth-
ers and we refer to [3], [18] and [17] where overviews of theses results are established.

In this work, we prove global in time existence of strong solution (in \( W^{1,r} \), \( r \geq 3 \)),
for spatially periodic two and three-dimensional flows and for a large class of fluids
of second grade with shear rate dependent viscosities. Moreover, in a second part
we prove uniqueness of such solution. All our results hold without restricting on the
size of the initial data.

The scheme of the proof is based on the use of Galerkin method. First, we construct
the Galerkin approximations (based on the eigenfunctions of the Stokes operator)
of the velocity and derive a priori \( W^{1,2} \)-estimates. But this is not enough to ensure
the convergence of nonlinear terms. That is why we have to look for new estimates
of higher derivatives order. More precisely, we derive the crucial estimates that are
sufficient to establish the compactness of the velocity gradient for the approximations.

Unlike the classical fluids of third grade, where in the \( W^{1,2} \) and \( W^{2,2} \) a priori estimates (established in a good way), one benefits from the presence of the term \( \text{div}(|Du|^2Du) \) which has a regularizing effect, this term is absent from the momentum equation of the classical second grade fluids. For this reason there are no existence results of \( W^{2,2} \) solutions for the last cited fluid but only \( W^{3,2} \) solutions.

In our case, one takes advantage from the coercivity of the stress tensor which will allow us to get \( W^{1,r} \) a priori estimates in a first step and help us in the absorption operation in order to obtain \( W^{2,2} \) estimates.

4. Existence of strong solutions

4.1. Existence of approximate solutions. In a first step, we start with an approximate problem, whose solutions have sufficient regularity properties. This will be done by performing a Galerkin scheme. So, let \( \{\omega^k\}_{k=1}^\infty \) be the set consisting of the 2π-periodic eigenvectors of the Stokes operator (denoted by \( A \)) and \( \lambda_k \) be the corresponding eigenvalues. Note that \( \int_\Omega \omega^k \, dx = 0 \).

We set the Galerkin approximations \( u^n \) of \( u \) of the form \( u^n = \sum_{k=1}^n c^n_k(t)\omega^k \), where the coefficients \( c^n_k(t) \) solve the the \( n \) coupled ordinary differential equations

\[
(\partial_tv^n, \omega^k) = -(u^n, \nabla)v^n, \omega^k) - (S(Du^n), D\omega^k) - \left( \sum_{j=1}^d v_j \nabla u_j, \omega^k \right) + (f, \omega^k), \quad v^n = u^n - \alpha_1 \Delta u^n, \tag{28}
\]

for all \( k = 1, \ldots, n \).

To close the system (28), we prescribe the initial data as

\[
(29) \quad u^0 = \mathbb{P}_n u_0 = \sum_{k=1}^n (u_0, \omega^k)\omega^k \quad \text{in} \quad \Omega,
\]

where \( \mathbb{P}_n \) denotes the orthogonal projection operator onto the subspace \( \mathbb{H}_n := \text{span}\{\omega^1, \ldots, \omega^n\} \) defined by \( \mathbb{P}_n u = \sum_{k=1}^n (u, \omega^k)\omega^k \). Furthermore, we have

\[
(30) \quad \lambda_k(u^n, \omega^k) = (Au^n, \omega^k) = (\nabla u^n, \nabla \omega^k).
\]

We would like to mention that \( \mathbb{P}_n : W^{s,2}_{\text{per,div}}(\Omega) \rightarrow \mathbb{H}_n \) is uniformly continuous for all \( s \in [0, 3] \) (see [19] or [28] for a proof).

Due to the continuity of the right hand side of (28), the local existence of a solution to (28)-(29) on a short time interval \( (0, t^*) \) follows from the Carathéodory theory (see [30] for instance). In order to extend the solution to the whole time interval \( [0, T] \), we need to show that the solution remains finite for all positive times which consequently implies that \( t^* = T \). To achieve this goal, we will derive some uniform estimates.
Multiplying the k-th equation in (28) by $c_k^n(t)$ and summing over $k = 1, \ldots, n$, we obtain the following energy equality

$$
\frac{1}{2} \frac{d}{dt} \|u^n\|_{L^2}^2 + \alpha_1 \|\nabla u^n\|_{L^2}^2 + \int_{\Omega} S(Du^n) : Du^n \, dx = \int_{\Omega} f \cdot u^n \, dx
$$

(31)

All the other nonlinear terms vanish, it is due to the incompressibility constraint $\text{div} \omega_k = 0$, see the identities (17) and (18).

Now integrating the identity (31) over time on $[0, t]$ and using (22) and the Korn’s inequality, we deduce that

$$
\|u^n(t)\|_{L^2}^2 + \alpha_1 \|\nabla u^n(t)\|_{L^2}^2 + c_3 c_k \int_0^t \|\nabla u^n\|_{L^r}^r \, d\tau
$$

$$
\leq \|u^n_0\|_{L^2}^2 + \alpha_1 \|\nabla u^n_0\|_{L^2}^2 + c(\varepsilon) \int_0^t \|f\|_{L^{r'}}_{L^{r'}} \, d\tau + \varepsilon c_s \int_0^t \|\nabla u\|_{L^r}^r \, d\tau
$$

(32)

where $c_s$ is the Sobolev constant of the embedding $W^{1,r}_{\text{per}}(\Omega) \hookrightarrow L^r_{\text{per}}(\Omega)$.

Choosing $\varepsilon$ small enough, we obtain the following uniform bounds for all $t \in (0, T)$

(33) $\{u^n\}$ is uniformly bounded in $L^\infty(0, t; W^{1,2}_{\text{per,div}}(\Omega))$

(34) $\{u^n\}$ is uniformly bounded in $L^r(0, t; W^{1,r}_{\text{per,div}}(\Omega))$.

From (33) it follows that

$$
|c_k^n(t)|^2 < \infty \quad \text{for all} \quad t \in [0, T], \quad \forall \quad k = 1, \ldots, n.
$$

(35)

Taking into account the continuity of $c_k^n$ on $[0, t^*]$, one can shift $t^*$ to $T$. Thus we realize that there exists $u$ such that for a certain subsequence (still denoted by $\{u^n\}$), we have

(36) $u^n \rightharpoonup u$ weakly in $L^\infty(0, T; W^{1,2}_{\text{per,div}}(\Omega))$

(37) $u^n \rightharpoonup u$ weakly in $L^r(0, T; W^{1,r}_{\text{per,div}}(\Omega))$.

Moreover, using the growth property (23), we infer that the sequence $\{S(Du^n)\}$ is uniformly bounded in $L^{r'}(0, T; L^{r'}_{\text{per,div}}(\Omega))$ where $r' = \frac{r}{r-1}$ is the conjugate exponent of $r$. Therefore, we can deduce that, up to subsequence extraction, there exists a tensor $\chi$ for which

(38) $S(Du^n) \rightharpoonup \chi$ weakly in $L^{r'}(0, T; L^{r'}_{\text{per,div}}(\Omega))$.

The main difficulty in the limiting process consists in the terms which are nonlinear in $\nabla u^n$ and more precisely those with product of spatial derivative of first order. So we are urged to obtain stronger regularity results on the sequence $\{u^n\}$ in order to have compactness of the sequence $\{\nabla u^n\}$, and thus consequently the pointwise convergence. The next paragraph will focus on.
4.2. **Compactness and limiting process.** The usual method to establish strong solutions for similar problems consists in considering $-\Delta u^k$ as a test function in the equation \((28)\) and trying to establish a priori estimates for the spatial derivative of second order of the velocity field. This is only allowed if $-\Delta u^k$ is sufficiently regular (say in $L^2_{per,div}(\Omega)$), but until now this is not verified. Fortunately, the periodic boundary setting enable us this operation since $Au^k = -\Delta u^k$ and the task will simplify to multiplying the equation in interest by the corresponding eigenvalues \(\{\lambda^k\}\) of the Stokes operator.

Multiplying the equations \((28)\) by $\lambda_k c_k^i(t)$ and summing the resulted equations over $k = 1, \ldots, n$, we obtain by means of integrations by parts

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla u^n\|_{L^2}^2 + \alpha_1 \|\nabla^2 u^n\|_{L^2}^2 \right) + (div(S(Du^n)), \Delta u^n) = \left( (u^n \cdot \nabla) u^n, \Delta u^n \right) + \left( \sum_{j=1}^d v_j \nabla u_j, \Delta u^n \right) - (f, \Delta u^n).
\]

In the following, we will estimate appropriately the different terms appearing in \((39)\). Using \((20)\) and \((22)\) we get from the stress tensor

\[
(div(S(Du^n)), \Delta u^n) = (\partial_D S(Du^n) D\nabla u^n, D\nabla u^n)
\]

\[
= \sum_{i,j,k,l=1}^d \frac{\partial S_{ij}(Du^n)}{\partial D_{kl}} D_{kl}(\nabla u^n) D_{ij}(\nabla u^n) dx
\]

\[
\geq c_0 \int_\Omega (1 + \|Du^n\|^{r-2} \|D\nabla u^n\|^2) dx = c_0 I_r(u^n)
\]

By Korn’s inequality \((2.4)\), we deduce that

\[
(div(S(Du^n)), \Delta u^n) \geq c_0 I_r(u^n)
\]

\[
\geq c_0 \int_\Omega |Du^n|^{r-2} |D\nabla u^n|^2 dx + c_0 c_k \|\nabla^2 u^n\|_{L^2}^2.
\]

Furthermore, we recall that the quantity

\[
((u^n \cdot \nabla) u^n, \Delta u^n) = \sum_{i,j,k=1}^d \int_\Omega \partial_k u^n_j \partial_j u^n_i \partial_k u^n_i dx,
\]

vanishes in the two dimensional space-periodic setting due to the fact that $\partial_1 u^n_1 = -\partial_2 u^n_2$. Using integration by parts and the divergence-free condition of $u^n$, we see from \((14)\) and \((17)\) that we have

\[
((u^n \cdot \nabla) u^n, \Delta u^n) = (u^n \cdot \nabla) u^n, \Delta u^n) - \alpha_1 ((u^n \cdot \nabla) \Delta u^n, \Delta u^n)
\]

\[
= \sum_{i,j,k=1}^d \int_\Omega \partial_k u^n_j \partial_j u^n_i \partial_k u^n_i dx - \alpha_1 \sum_{i,j=1}^d \int_\Omega u^n_j \partial_j \Delta u^n_i \Delta u^n_i
\]
\[
= \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_k u^n_i \partial_j u^n_i \partial_k u^n_i \, dx - \frac{\alpha_1}{2} \int_{\Omega} u^n_j \partial_j |\Delta u^n_i|^2 \, dx
\]
\[
\leq \|\nabla u^n\|_{L^3}^3 \leq c\|\nabla u^n\|_{L^r} \|\nabla^2 u^n\|_{L^2}^2.
\]

On the other hand, one can deduce from (15) that
\[
\left( \sum_{j=1}^{d} v^n_j \nabla u^n_j, \Delta u^n \right) = \alpha_1 \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_k(\partial_k u^n_j \partial_j u^n_i) \Delta u^n_i \, dx
\]
\[
= \alpha_1 \sum_{i,j,k=1}^{d} \int_{\Omega} \Delta u^n_j \partial_j u^n_i \Delta u^n_i \, dx + \alpha_1 \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_k u^n_j \partial_k u^n_i \Delta u^n_i \, dx
\]
\[
= \alpha_1 \sum_{i,j=1}^{d} \int_{\Omega} \Delta u^n_j \left[ \frac{1}{2}(\partial_i u^n_j + \partial_j u^n_i) \right] \Delta u^n_i \, dx
\]
\[
+ \frac{\alpha_1}{2} \sum_{i=1}^{d} \int_{\Omega} (\partial_i |\nabla u^n|^2) \Delta u^n_i \, dx
\]
\[
\leq \alpha_1 \int_{\Omega} |\Delta u^n|^2 |D u^n| \, dx \leq \alpha_1 \int_{\Omega} |D \nabla u^n|^2 |D u^n| \, dx,
\]

where in the third line of the last estimate we have used the symmetry of the scripts \(i\) and \(j\). If \(r > 3\) then by means of the Young’s inequality, one have
\[
|D u^n| = |D u^n| 1 \leq \varepsilon |D u^n| |r - 2| + c(\varepsilon) 1 \frac{|r - 2|}{2},
\]
which yields that
\[
\sum_{j=1}^{d} v^n_j \nabla u^n_j, \Delta u^n \leq \alpha_1 \varepsilon \int_{\Omega} |D \nabla u^n|^2 |D u^n| |r - 2| \, dx + \alpha_1 c(\varepsilon) \int_{\Omega} |D \nabla u^n|^2 \, dx.
\]

Putting all the estimates (10), (14) and (46) together, the energy inequality (39) becomes
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u^n\|^2_{L^2} + \frac{\alpha_1}{2} \frac{d}{dt} \|\nabla^2 u^n\|^2_{L^2} + c_0 \int_{\Omega} |D u^n| |r - 2| \|D \nabla u^n\|^2 \, dx + c_0 c_k \|\nabla^2 u^n\|^2_{L^2}
\]
\[
\leq \alpha_1 \varepsilon \int_{\Omega} |D u^n| |r - 2| \|D \nabla u^n\|^2 \, dx + c(\alpha_1, \varepsilon)(\|\nabla u^n\|_{L^r} + 1 + \|f\|_{L^2}) \|\nabla^2 u^n\|^2_{L^2}.
\]

Choosing \(\varepsilon << 1\) small enough in such a way that \(\alpha_1 < c_0\) one can absorb the first term of the right hand side of (47) and find that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u^n\|^2_{L^2} + \frac{\alpha_1}{2} \frac{d}{dt} \|\nabla^2 u^n\|^2_{L^2} + c \int_{\Omega} |D u^n| |r - 2| \|D \nabla u^n\|^2 \, dx
\]
\[
+ c_0 c_k \|\nabla^2 u^n\|^2_{L^2} \leq \tilde{c} \|\nabla u^n\|_{L^r} \|\nabla^2 u^n\|^2_{L^2},
\]
where \(c\) and \(\tilde{c}\) are some positive fixed constants. Note that for \(r = 3\), we do not need to use (15) and inequality (18) remains valid. Consequently, since \(\|\nabla u^n\|_{L^r} + \|f\|_{L^2}\)
is uniformly bounded in $L^1(0,T)$, the Gronwall’s lemma enables us to conclude for both cases $d = 2$ and $d = 3$ that
\begin{equation}
\{u^n\} \text{ is uniformly bounded in } L^\infty(0,T; W^{2,2}_{\text{per,div}}(\Omega)),
\end{equation}
and that for all $n$
\begin{equation}
\int_0^T \int_\Omega (1 + |Du^n|^2)|D\nabla u^n|^2 \, dx \, d\tau \leq c.
\end{equation}
Thus, we have
\begin{equation}
u^n \rightharpoonup u \ \text{ weakly in } L^\infty(0,T; W^{2,2}_{\text{per,div}}(\Omega)),
\end{equation}
\begin{equation}
u^n \rightharpoonup u \ \text{ weakly in } L^q(0,T; W^{2,2}_{\text{per,div}}(\Omega)), \ \forall q \in [1, \infty).
\end{equation}
In order to prove a compactness for the velocity field in some Sobolev spaces, we need uniform estimates for the time derivative $\partial_t u^n$. To do this, observe that for test function $\varphi \in L^2(0,T; W^{2,2}_{\text{per,div}}(\Omega)) \cap L^r(0,T; W^{1,r}_{\text{per,div}}(\Omega))$, if $d = 3$
\begin{equation}
\varphi \in L^2(0,T; W^{2,2}_{\text{per,div}}(\Omega)), \text{ if } d = 2,
\end{equation}
we have
\begin{equation}
((\partial_t v^n, \varphi))_2 = ((\partial_t v^n, \mathbb{P}^n \varphi))_2 = -((u^n \cdot \nabla)v^n, \mathbb{P}^n \varphi) + (\text{div}(\mathbf{S}(Du^n)), \mathbb{P}^n \varphi) - \sum_{j=1}^d v_j \nabla u_j, \mathbb{P}^n \varphi) + (f, \mathbb{P}^n \varphi).
\end{equation}
Let us now estimate the right hand side of (53). Recall that the projection operator $\mathbb{P}^n$ is continuous from $W^{q,2}_{\text{per,div}}(\Omega)$ to $\mathbb{H}^n$ for all $q \in [0,3]$. Recall also the usual Sobolev embeddings [1]
\begin{equation}
W^{1,2}_{\text{per}}(\Omega) \hookrightarrow L^q_{\text{per}}(\Omega), \ \forall q \in [1, \infty), \ \text{ if } d = 2
\end{equation}
\begin{equation}
W^{1,2}_{\text{per}}(\Omega) \hookrightarrow L^q_{\text{per}}(\Omega), \ \forall q \in [1, 6], \ \text{ if } d = 3.
\end{equation}
Moreover, it is obvious that
\[
\frac{2r}{r - 2} \in \begin{cases} 
(2, \infty), & \text{if } r \in (2, \infty), \\
(2, 6], & \text{if } r \in [3, \infty).
\end{cases}
\]
Therefore, by (13) and the Hölder’s inequality, we obtain
\[
\int_0^T ((u^n \cdot \nabla)v^n, \mathbb{P}^n \varphi) \, d\tau \leq \int_0^T \|u^n\|_{L^\infty} \|\nabla u^n\|_{L^2} (\|\mathbb{P}^n \varphi\|_{L^2} + \alpha_1 \|\nabla^2 \mathbb{P}^n \varphi\|_{L^2}) \, d\tau + \alpha_1 \int_0^T \|\nabla u^n\|_{L^r} \|\nabla^2 u^n\|_{L^2} \|\nabla \mathbb{P}^n \varphi\|_{L^{\frac{2r}{r-2}}} \, d\tau \leq c \|u^n\|_{L^\infty(W^{2,2})} \|\nabla u^n\|_{L^2(L^r)} \|\mathbb{P}^n \varphi\|_{L^2(W^{2,2})} \leq c.
\]
Similarly, using (13) and taking into account the divergence-free condition, we get
\[
\int_0^T \left( \sum_{j=1}^d v_j \nabla u_j, \mathbb{P}^n \varphi \right) d\tau \leq \alpha_1 \int_0^T \| \nabla u^n \|_{L^r} \| \nabla u^n \|_{L^2} \| \nabla \mathbb{P}^n \varphi \|_{L^{\frac{2r}{r-2}}} d\tau \\
\leq c \| \nabla u^n \|_{L^{\infty}(L^2)} \| u^n \|_{L^2(W^{1,r})} \| \mathbb{P}^n \varphi \|_{L^2(W^{2,2})} \leq c.
\]

To conclude, we consider now the stress tensor term. Thanks to the growth property (23) we have
\[
\int_0^T \left( \text{div}(S(Du^n)), \mathbb{P}^n \varphi \right) d\tau = \int_0^T \left( S(Du^n), D\mathbb{P}^n \varphi \right) d\tau \\
\leq c_4 \int_0^T \int_{\Omega} |(1 + |Du^n|)^{r-1}|D\mathbb{P}^n \varphi| \, dx \, d\tau \\
\leq c \int_0^T \int_{\Omega} |D\mathbb{P}^n \varphi| + |Du^n|^{r-1} |D(\mathbb{P}^n \varphi)| \, dx \, d\tau \\
\leq c \| \varphi \|_{L^1(W^{1,2})} + \int_0^T \| Du^n \|_{L^{r-1}} \| D(\mathbb{P}^n \varphi) \|_{L^r} \, d\tau \\
\leq c \| \varphi \|_{L^1(W^{1,2})} + \| u^n \|_{L^{r-1}(W^{1,r})} \| \varphi \|_{L^1(W^{2,2})} \leq c.
\]

In dimension two we can improve, thanks to (54), the last estimate as follows
\[
\int_0^T \left( \text{div}(S(Du^n)), \mathbb{P}^n \varphi \right) d\tau \leq c \| \varphi \|_{L^1(W^{1,2})} + \| u^n \|_{L^{r-1}(W^{2,2})} \| \varphi \|_{L^1(W^{2,2})} \leq c.
\]

Now, denoting by \( X^{2,r} = L^2(0, T; (W^{2,2}_{\text{per, div}}(\Omega))^*) \cap L^{\frac{r}{r-1}}(0, T; W^{1,r}_{\text{per, div}}(\Omega))^* \), keeping in mind the last estimates, we infer that
\[
\{ \partial_t v^n \} \quad \text{is uniformly bounded in} \quad L^2(0, T; (W^{2,2}_{\text{per, div}}(\Omega))^*), \quad \text{if} \quad d = 2,
\]
\[
\{ \partial_t v^n \} \quad \text{is uniformly bounded in} \quad X^{2,r} \quad \text{if} \quad d = 3,
\]
and hence by the classical elliptic regularity we deduce that
\[
\{ \partial_t u^n \} \quad \text{is uniformly bounded in} \quad L^2(0, T; L^2_{\text{per, div}}(\Omega)), \quad \text{if} \quad d = 2,
\]
\[
\{ \partial_t u^n \} \quad \text{is uniformly bounded in} \quad Y^{2,r} \quad \text{if} \quad d = 3,
\]
where \( Y^{2,r} = L^2(0, T; L^2_{\text{per, div}}(\Omega)) \cap L^{\frac{r}{r-1}}(0, T; W^{1,r}_{\text{per, div}}(\Omega)) \). Consequently, we infer that
\[
\{ \partial_t u^n \} \to \partial_t u \quad \text{in} \quad L^2(0, T; L^2_{\text{per, div}}(\Omega)), \quad \text{if} \quad d = 2,
\]
\[
\{ \partial_t u^n \} \to \partial_t u \quad \text{in} \quad L^2(0, T; L^2_{\text{per, div}}(\Omega)) \cap L^{\frac{r}{r-1}}(0, T; W^{1,r}_{\text{per, div}}(\Omega)) \quad \text{if} \quad d = 3.
\]

Since the power \( r \) lies in \([3, \infty)\) then \( \frac{r}{r-1} \in (1, \frac{3}{2}) \). So, by the Aubin-Lions lemma (2.2) (34) and (58), we deduce that
\[
u^n \to u \quad \text{strongly in} \quad L^r(0, T; W^{1,r}_{\text{per, div}}(\Omega)),
\]
which implies that
\[
u^n \to u \quad \text{almost everywhere in} \quad [0, T] \times \Omega,
\]
(61) \( \nabla u^n \rightarrow \nabla u \) almost everywhere in \( [0, T] \times \Omega \),
and by symmetry
(62) \( \mathbf{D} u^n \rightarrow \mathbf{D} u \) almost everywhere in \( [0, T] \times \Omega \).

Since the stress tensor \( \mathbf{S} \) is continuous
(63) \( \mathbf{S}(\mathbf{D} u^n) \rightarrow \mathbf{S}(\mathbf{D} u) \) almost everywhere in \( [0, T] \times \Omega \).

Consequently, by means of Vitali’s lemma we deduce that \( \chi = \mathbf{S}(\mathbf{D} u) \) and one have
(64) \( \mathbf{S}(\mathbf{D} u^n) \rightarrow \mathbf{S}(\mathbf{D} u) \) weakly in \( L^r(0, T; L^r_{per,div}(\Omega)) \),
and almost everywhere in \([0, T] \times \Omega\).

Now we outline the passage to the limit in the weak formulation (60). Using (56) and (57), we can pass to the limit in the terms corresponding to the time derivative for \( \Delta u \) and \( u \), respectively. On the other hand, the limiting process in the convective terms involving (14) and (15) and the stress tensor are ensured by (59), (59) and (61).

Finally, the continuity property follows from the fact that \( u \in L^2(0, T; L^2_{per,div}(\Omega)) \) and \( \partial_t u \in L^2(0, T; L^2_{per,div}(\Omega)) \) which implies that \( u \in \mathcal{C}([0, T]; L^2_{per,div}(\Omega)) \). To improve this regularity property, we see that for a fixed time \( t_0 \in (0, T) \) we have the following interpolation inequality
\[
\|u(t, \cdot) - u(t_0, \cdot)\|_{W^{2,2}} \leq c \|u(t, \cdot) - u(t_0, \cdot)\|_{L^2}^2 \|\nabla u(t, \cdot) - \nabla u(t_0, \cdot)\|_{W^{2,2}}, \quad \forall \sigma \in [0, 2),
\]
and therefore we deduce the strong continuity in \( W^{2,2}_{per,div}(\Omega) \).

Concerning the weak continuity, it is easy to show
\[
(u(t), \psi) + \alpha_1 (\nabla u(t), \nabla \psi) \rightarrow (u(t_0), \psi) + \alpha_1 (\nabla u(t_0), \nabla \psi),
\]
for all \( t_0 \in [0, T] \) and \( \psi \in W^{2,2}_{per,div}(\Omega) \).

To finish this section, we can easily handle the attainment of the initial data. For more details one can consult paragraph 3.10 in [17].

5. Uniqueness of strong solution

Consider \( u \) and \( \bar{u} \) (with initial data \( u_0 \) and \( \bar{u}_0 \) respectively) two strong solutions to the problem consisting of equations (10)-(13) as defined in Theorem 3.2. We set \( w := u - \bar{u} \) their difference. Subtracting the equations relatively to \( u \) and \( \bar{u} \) we obtain the following system
(65) \( \partial_t w - \alpha_1 \partial_t \Delta w + (u \cdot \nabla)(u - \alpha_1 \Delta u) - (\bar{u} \cdot \nabla)(\bar{u} - \alpha_1 \Delta \bar{u}) \)
\[-\operatorname{div}(\mathbf{S}(\mathbf{D} u) - \mathbf{S}(\mathbf{D} \bar{u})) + \sum_{j=1}^d (u_j - \alpha_1 \Delta u_j) \nabla u_j - \sum_{j=1}^d (\bar{u}_j - \alpha_1 \Delta \bar{u}_j) \nabla \bar{u}_j = 0.\]

Multiplying (65) by \( w \) and integrating over \( \Omega \), we obtain by (24) the following energy inequality
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \alpha_1 \frac{d}{dt} \|
abla w(t)\|_{L^2}^2 + \int_{\Omega} (1 + |\mathbf{D} u| + |\mathbf{D} \bar{u}|)^{r-2} |\mathbf{D} u - \mathbf{D} \bar{u}|^2 \, dx
\]
The first term of the right hand side of (66) can be handled in the following way

\[ - \int_{\Omega} ((u, \nabla) u - (\bar{u}, \nabla) \bar{u}) \cdot w \, dx - \alpha_1 \int_{\Omega} ((u, \nabla) D u - (\bar{u}, \nabla) D \bar{u}) : D w \, dx \]

(66)–(67)

\[ \sum_{j=1}^{d} (u_j - \alpha_1 \Delta u_j) \nabla u_j - (\bar{u}_j - \alpha_1 \Delta \bar{u}_j) \nabla \bar{u}_j \cdot w \, dx := I_1 + I_2 + I_3. \]

The first term of the right hand side of (66) can be handled in the following way

\[ I_1 = - \int_{\Omega} (w, \nabla) u \cdot w + (\bar{u}, \nabla) w \cdot w \, dx = - \int_{\Omega} (w, \nabla) u \cdot w \, dx \]

(67) \[ \leq \| w \|_{L^2}^2 \| \nabla u \|_{L^2} \leq \| \nabla u \|_{L^2} \| \nabla w \|_{L^2}^2. \]

Note that \( \int_{\Omega} (\bar{u}, \nabla) w \cdot w \, dx = 0 \) (see (68)) due to the fact that \( w \) is a divergence-free vector field. Thanks to (68) and Hölder’s inequality, we have

\[ I_2 = - \alpha_1 \int_{\Omega} ((u, \nabla) D w + (w, \nabla) D \bar{u}) : D w \, dx \]

\[ = - \alpha_1 \int_{\Omega} (w, \nabla) D \bar{u} : D w \, dx \]

\[ \leq \alpha_1 \int_{\Omega} \| w \|_{L^2} \| \nabla w \| \, dx \]

\[ \leq \alpha_1 \| w \|_{L^\infty} \left( \int_{\Omega} (1 + \| D \bar{u} \|_{L^2}^r \| D \nabla \bar{u} \|_{L^2}^2) \right) \| \nabla w \|_{L^2}. \]

Next, we deal with the last term in (66). Using (65), we have

\[ I_3 := \int_{\Omega} \sum_{j=1}^{d} (u_j - \alpha_1 \Delta u_j) \nabla u_j - (\bar{u}_j - \alpha_1 \Delta \bar{u}_j) \nabla \bar{u}_j \cdot w \, dx \]

(68) \[ = - \alpha_1 \int_{\Omega} \left[ \sum_{i,j,k=1}^{d} \partial_k (\partial_i u_j \partial_l u_j) - \partial_k (\partial_i \bar{u}_j \partial_l \bar{u}_j) \right] w_i \, dx \]

\[ = \alpha_1 \int_{\Omega} \sum_{i,j,k=1}^{d} (\partial_k w_j \partial_l u_j \partial_l w_i + \partial_k \bar{u}_j \partial_l w_j \partial_l w_i) \, dx \]

(69)

By symmetry and integration by parts, we get

\[ I_3 = - \alpha_1 \int_{\Omega} \sum_{i,j,k=1}^{d} \partial_k w_j \partial_l u_j \partial_l w_i + \partial_j \bar{u}_j \partial_k w_i \partial_j w_i \, dx \]

\[ \leq \alpha_1 \int_{\Omega} \| \nabla^2 w \| \| \nabla \| \| u \| \, dx + \alpha_1 \int_{\Omega} \| \nabla^2 \bar{u} \| \| \nabla \| \| w \| \, dx \]

(70) \[ \leq \alpha_1 \| \nabla^2 w \|_{L^2} \| \nabla \|_{L^2} \| u \|_{L^\infty} + \alpha_1 \| \nabla^2 \bar{u} \|_{L^2} \| \nabla \|_{L^2} \| w \|_{L^\infty}. \]
In view of all these estimates, we have

\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_{L^2} + \frac{\alpha_1}{2} \frac{d}{dt} \|\nabla w(t)\|^2_{L^2} + c_1 c_k \|\nabla w(t)\|_{L^2}^2 \\
\leq F(u, \bar{u}, w)\|\nabla w(t)\|^2_{L^2},
\]

(71)

where \( F(u, \bar{u}, w) \) is a function incorporating

\[
\|\nabla u(t)\|^2_{L^2}, \|\nabla^2 \bar{u}(t)\|^2_{L^2}, \|\nabla^2 \bar{u}(t)\|^2_{L^2}, \|\nabla w(t)\|^2_{L^2}, \|\nabla^2 w(t)\|^2_{L^2},
\]

and belonging to \( L^\infty(0, T) \). Therefore, we obtain

\[
\frac{d}{dt} \|w(t)\|^2_{L^2} + \alpha \frac{d}{dt} \|\nabla w(t)\|^2_{L^2} \leq cF(u, \bar{u}, w)(\|w(t)\|^2_{L^2} + \alpha \|\nabla w\|^2_{L^2}).
\]

(72)

Finally, by Gronwall’s inequality we infer that

\[
\|w(t)\|_{L^2} = \|\nabla w(t)\|_{L^2} = 0 \quad \text{for almost every } t \in (0, T).
\]

(73)

Consequently \( u = \bar{u} \), and the proof is complete.

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