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Oscillatory Properties of Solutions of Even-Order Differential Equations

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Abstract: This work is concerned with the oscillatory behavior of solutions of even-order neutral differential equations. By using Riccati transformation and the integral averaging technique, we obtain a new oscillation criteria. This new theorem complements and improves some known results from the literature. An example is provided to illustrate the main results.

Keywords: oscillatory solutions; even-order; neutral delay differential equations

1. Introduction

Neutral differential equations are used in numerous applications in technology and natural science. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [1], and therefore their qualitative properties are important.

Very recently, some scholars have been attracted by the problems of the oscillations of differential equations and made relative advances therein, as in [2–11].

Delay differential equations are often studied in one of two cases

\[ \int_{b}^{\infty} r^{-\frac{1}{\alpha_1}}(s) \, ds = \infty \]  

or

\[ \int_{b}^{\infty} r^{-\frac{1}{\alpha_1}}(s) \, ds < \infty, \]

which are said to be in the canonical or noncanonical form, respectively, see [12]. For the canonical form, many authors in [13–16] studied the asymptotic behavior of the solutions of equation

\[ \left( r(t) \left( z^{(n-1)}(t) \right)^{\alpha_1} \right)' + q(t) x^{\beta(\tau(t))} = 0. \]  

In the noncanonical form, Li and Rogovchenko [17] studied the asymptotic properties of solutions of higher-order neutral differential (2) under the assumptions that allow applications to even- and odd-order equations with delayed and advanced arguments.
This paper is motivated by several recent studies [3,7,9,10] of such higher order equations. Using the integral averaging technique and the Riccati transformation, we study the asymptotic properties of solutions of even order neutral delay differential equations of the form

\[
\left( r(t) \left| z^{(n-1)}(t) \right|^{ \alpha_1 - 1 } z^{(n-1)}(t) \right)^{ \frac{1}{ \alpha_1 } } + \sum_{i=1}^{m} q_i(t) \left| y(\sigma_i(t)) \right|^{ \alpha_i - 1 } y(\sigma_i(t)) = 0, \tag{3}
\]

where \( n \) is an even natural number and \( m \) is a natural number. In this paper, we assume that \( \alpha_i \) are positive integers, \( \alpha_i + 1 > \alpha_i \), \( r(t) > 0, r'(t) \geq 0, a \in C([t_0, \infty), [0,1)), \lim_{t \to \infty} a(t) = \infty, q_i, \sigma_i \in C([t_0, \infty), \mathbb{R}), q_i > 0, \sigma_i(t) \leq t, \) and \( \lim_{t \to \infty} \sigma_i(t) = \infty \) for all \( i = 1, ..., m \).

During the following results, for clarity of presentation, we study only the case where \( m = 3 \). Moreover, we denote, for convenience, that

\[
z(t) : = y(t) + a(t) y(\tau(t)), \quad \sigma(t) : = \min \{ \sigma_i(t), i = 1,2,3 \}, \quad B(t) : = \int_{t_0}^{t} \frac{1}{r^{1/\alpha_1}(t)} dt, \quad F_+(t) := \max \{ F(t), 0 \}, \quad A_i(t) : = q_i(t) \left( 1 - a(\sigma_i(t)) \right)^{\alpha_i}, \quad \text{for all } i = 1,2,3.
\]

\[
m_1 : = \frac{\alpha_3 + \alpha_2 - 2\alpha_1}{\alpha_2 - \alpha_1}, \quad m_2 : = \frac{\alpha_3 + \alpha_2 - 2\alpha_1}{\alpha_3 - \alpha_1}
\]

and

\[
A(t) = A_1(t) + (m_1A_2(t))^{1/m_1} (m_2A_3(t))^{1/m_2}
\]

We say that a function, \( y \), is a solution of (3), we mean a non-trivial real function \( z(t) \in C^{n-1}(\mathbb{I}_{y, \infty}), t_y \geq t_0, \) satisfying (3) on \( \mathbb{I}_{y, \infty} \) and which has the property \( r(t) \left( z^{(n-1)}(t) \right)^{\alpha_1} \in C(\mathbb{I}_{y, \infty}) \). We consider only those solutions \( y \) of (3) which satisfy \( \sup \{ |y(t)| : t \geq T \} > 0, \) for any \( T \geq t_y \). A solution of (3) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Studying the functional differential equations, the continuity of all functions, and \( r(t) > 0 \) are sufficient conditions for the existence of one or more solutions of the equation.

To establish our main results, we make use of the following lemmas:

**Lemma 1** ([18]). Let \( C \) and \( D \) nonnegative real numbers. Then

\[
C^h + (\mu - 1) D^\mu - \mu C D^{(\mu - 1)} \geq 0, \quad \mu > 1,
\]

where the equality holds, if and only if, \( C = D \).

**Lemma 2** ([19]). Let \( h \in C^n([t_y, \infty), (0, \infty)) \). If \( h^{(n)}(t) \) is eventually of one sign for all large \( t \), then there exist a \( t_y > t_1 \) for some \( t_1 > t_0 \) and an integer \( m, 0 \leq m \leq n \) such that \( h^{(n)}(t) \geq 0 \) or \( n + m \) odd for \( h^{(n)}(t) \leq 0 \) such that \( m > 0 \) implies that \( h^{(k)}(t) > 0 \) for \( t > t_y, k = 0,1, ..., m - 1 \) and \( m \leq n - 1 \) implies that \( (-1)^{m-k} h^{(k)}(t) > 0 \) for \( t > t_y, k = m, m + 1, ..., n - 1 \).

**Lemma 3** ([20]). Let \( h(t) \in C^n([t_0, \infty), (0, \infty)) \). If \( h^{(n-1)}(t) h^{(n)}(t) \leq 0 \) for all \( t \geq t_y \), then for every \( \theta \in (0,1) \), there exists a constant \( M > 0 \) such that

\[
h'(\theta t) \geq M \theta^{n-2} h^{(n-1)}(t),
\]

for all sufficiently large \( t \).

This paper is concerned with the oscillatory behavior of a class of even-order neutral differential equations with multi-delays. Firstly, by using the Riccati transformations, we obtain a new oscillation
criteria for this equation. Secondly, using the integral averaging technique, we establish a Philos type oscillation criterion. This new theorem complements and improves some known results in the literature. Finally, an example is provided to illustrate the main results.

2. Oscillation Criteria

In this section, we establish new oscillation results for Equation (3) using the Riccati transformation.

Theorem 1. Assume that (1) holds. If there exists a positive function \( \rho \in C^1 ([t_0, \infty), (0, \infty)) \) such that

\[
\int_{t_0}^{\infty} \left( \rho(s) A(s) - \frac{r^{a_1+1}(t)}{(\alpha_1 + 1)^{a_1+1} (\theta M\sigma'(s)\sigma''(s))^{a_1+1}} \right) ds = \infty,
\]

then all solutions of (3) are oscillatory.

Proof. Assume that (3) has a nonoscillatory solution \( y \). Without loss of generality, we may assume that there exists a \( t_1 \in [t_0, \infty) \) such that \( y(t) > 0, y(\tau(t)) > 0 \) and \( y(\sigma_i(t)) > 0 \) for all \( i = 1, 2, 3 \) and \( t \in [t_1, \infty) \). It follows from Lemma 2 that

\[
z(t) > 0, \quad z'(t) > 0, \quad z^{(n-1)}(t) > 0 \quad \text{and} \quad z^{(n)}(t) < 0,
\]

for \( t \geq t_1 \). Since \( \tau(t) \leq t \) and \( z'(t) > 0 \), we find

\[
y(t) = z(t) - a(t) y(\tau(t)) \geq z(t) - a(t) z(\tau(t)) \geq z(t) - a(t) z(t)
\]

Hence,

\[
y(\sigma_i(t)) \geq (1 - a(\sigma_i(t))) z(\sigma_i(t)), \quad i = 1, 2, 3,
\]

which, with (3), gives

\[
\left( r(t) \left( z^{(n-1)}(t) \right)^{a_1} \right)' = -q_1(t) y^{a_1}(\sigma_1(t)) - q_2(t) y^{a_2}(\sigma_2(t)) - q_3(t) y^{a_3}(\sigma_3(t)) \leq -A_1(t) z^{a_1}(\sigma_1(t)) - A_2(t) z^{a_2}(\sigma_2(t)) - A_3(t) z^{a_3}(\sigma_3(t)).
\]

Using Lemma 3, we obtain

\[
z'(\theta \sigma(t)) \geq M\sigma''(t) z^{(n-1)}(\sigma(t)) \geq M\sigma''(t) z^{(n-1)}(t).
\]

Now, we define a generalized Riccati substitution \( \omega \) by

\[
\omega(t) := \rho(t) r(t) \left( \frac{z(n-1)(t)}{\theta \sigma(t)} \right)^{a_1}.
\]

Then, \( \omega(t) > 0 \). By differentiating (7), we obtain

\[
\omega'(t) = \rho'(t) r(t) \left( \frac{z(n-1)(t)}{\omega(t)} \right)^{a_1} + \rho(t) \left( \frac{r(t) \left( \frac{z(n-1)(t)}{\omega(t)} \right)^{a_1}}{\left( \frac{z(n-1)(t)}{\theta \sigma(t)} \right)^{a_1}} \right)' - \alpha_1 \beta \rho(t) r(t) \left( \frac{z(n-1)(t)}{\omega(t)} \right)^{a_1} z'(\theta \sigma(t)) \sigma'(t).
\]
Since \( z'(t) > 0 \) and \( \sigma(t) \leq \sigma_i(t) \), we have
\[
z(\sigma_i(t)) \geq z(\sigma(t)), \quad i = 1, 2, 3.
\]
Hence, from (5), (6), and (8), we see that
\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) \left( A_1(t) + A_2(t) z^{a_2-a_1} (\sigma_2(t)) + A_3(t) z^{a_3-a_1} (\sigma_3(t)) \right) \]
\[
- \alpha_1 \theta M \omega'(t) \sigma^{n-2} (t) \rho(t) r(t) \omega^{(n-1) - \frac{1}{a_1}}(t) + \frac{1}{(\sigma_i(t))^{\frac{1}{a_1}}}. \tag{9}
\]
This implies that
\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) \left( A_1(t) + A_2(t) z^{a_2-a_1} (\sigma_2(t)) + A_3(t) z^{a_3-a_1} (\sigma_3(t)) \right) \]
\[
- \alpha_1 \theta M \omega'(t) \sigma^{n-2} (t) \rho(t) r(t) \omega^{(n-1) - \frac{1}{a_1}}(t) \]
\[
(\rho(t) r(t))^{1/a_1} \omega^{(a_1+1)/a_1}(t). \tag{10}
\]
Using Youngs inequality
\[
|uv| \leq \frac{1}{c_1} |u|^{c_1} + \frac{1}{c_2} |v|^{c_2}, \quad c_1 > 1, \quad c_2 > 1, \quad \frac{1}{c_1} + \frac{1}{c_2} = 1,
\]
with \( u = (m_1 A_2(t) z^{a_2-a_1} (\sigma(t)))^{1/\mu} \), \( v = (m_2 A_3(t) z^{a_3-a_1} (\sigma(t)))^{1/\mu} \) and \( c_i = m_i \), we obtain
\[
A_2(t) z^{a_2-a_1} (\sigma(t)) + A_3(t) z^{a_3-a_1} (\sigma(t)) \geq (m_1 A_2(t) )^{1/\mu} (m_2 A_3(t))^{1/\mu}. \tag{10}
\]
Combining (9) and (10), we have
\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) A(t) - \frac{\alpha_1 \theta M \omega'(t) \sigma^{n-2} (t)}{(\rho(t) r(t))^{1/a_1}} \omega^{(a_1+1)/a_1}(t). \]
If we set \( \mu = (a_1 + 1)/a_1 \), then we find
\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) A(t) - \frac{\alpha_1 \theta M \omega'(t) \sigma^{n-2} (t)}{(\rho(t) r(t))^{1/a_1}} \omega^{(a_1+1)/a_1}(t). \tag{11}
\]
Now, using Lemma 1 with
\[
C = \left( \frac{\alpha_1 \theta M \omega'(t) \sigma^{n-2} (t)}{(\rho(t) r(t))^{1/a_1}} \right)^{1/\mu} \omega(t)
\]
and
\[
D = \left( \frac{r(t) \rho'(t)}{\mu} \left( \frac{\alpha_1 \theta M \omega'(t) \sigma^{n-2} (t)}{(\rho(t) r(t))^{1/a_1}} \right)^{-1/\mu} \right)^{1/(\mu-1)},
\]
we obtain
\[
\frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\alpha_1 \theta M \omega'(t) \sigma^{n-2} (t)}{(\rho(t) r(t))^{1/a_1}} \omega^{(a_1+1)/a_1}(t) \leq \frac{1}{(a_1 + 1)^{a_1+1}} \left( \frac{r(t) \rho'(t) \alpha_1}{(\alpha_1 + 1)^{a_1+1} \theta M \sigma^{n-2} (t) \rho(t) r(t))^{a_1+1}} \right). \tag{12}
\]
Hence, from (11) and (12), we have
\[
\omega' (t) \leq \frac{1}{(a_1 + 1)^{a_1+1}} r^{a_1+1} (t) (\rho_+ (t))^{a_1+1} - \rho (t) A (t).
\]
Integrating from \( t_1 \) to \( t \) we find
\[
\int_{t_1}^{t} \left( \rho (s) A (s) - \frac{1}{(a_1 + 1)^{a_1+1}} r^{a_1+1} (t) (\rho_+ (t))^{a_1+1} \right) ds \leq \omega (t_1) - \omega (t) < \omega (t_1).
\]
which contradicts (4).
This completes the proof. \( \square \)

In Theorem 1, we can obtain different conditions for oscillation of all solutions of Equation (3) with different choices of \( \rho (t) \). If we set \( \rho (t) := B^{a_1} (s (t)) \), then we obtain the following corollary.

**Corollary 1.** Assume that (1) holds. If
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left( B^{a_1} (s (t)) A (s) - \frac{1}{(a_1 + 1)^{a_1+1}} r^{a_1+1} (t) (\rho_+ (t))^{a_1+1} \right) ds = \infty,
\]
then all solutions of (3) are oscillatory.

### 3. Kamenev-Type Criteria

In the section theorem, we establish new oscillation results for Equation (3) using the integral averaging technique to establish the Philos-type.

**Definition 1.** Let
\[
D = \{ (t, s) \in \mathbb{R}^2 : t \geq s \geq t_0 \} \quad \text{and} \quad D_0 = \{ (t, s) \in \mathbb{R}^2 : t > s > t_0 \}.
\]
A kernel function \( H \in \mathbb{C} (D, \mathbb{R}) \) is said to belong to the function class \( \mathfrak{A} \), written by \( H \in \mathfrak{A} \), if
\[
\begin{align*}
(1_1) & \quad H (t, t) = 0 \quad \text{and} \quad H (t, s) > 0, \quad (t, s) \in D_0 \quad \text{for} \quad t \geq t_0, \\
(1_2) & \quad H \quad \text{has a nonpositive continuous partial derivative} \quad \partial H / \partial s \quad \text{on} \quad D_0 \quad \text{with respect to the second variable,}
\end{align*}
\]
and there exist functions \( H \in \mathbb{C} (D, \mathbb{R}) \) and \( \delta \in \mathbb{C}^1 ([t_0, \infty), (0, \infty)) \) such that
\[
- \frac{\partial}{\partial s} (H (t, s) \delta (s)) = H (t, s) A (s) \frac{\rho' (t)}{\rho (t)} + h (t, s). \tag{13}
\]

**Theorem 2.** Assume that (1) holds. If there exist functions \( \rho, \delta \in \mathbb{C}^1 ([t_0, \infty), (0, \infty)) \) such that (13) and
\[
\limsup_{t \to \infty} \int_{t_1}^{t} (H (t, s) \delta (s) \rho (s) A (s) - \Theta (s)) ds = \infty, \tag{14}
\]
hold, where
\[
\Theta (s) := \left( h (t, s) \right)^{a_1+1} \frac{r (s) \rho (s)}{(\theta MH (t, s) \delta (s) \rho (s))^{a_1-1}},
\]
then every solution of (3) is oscillatory.
Proof. Proceeding as in the proof of Lemma 1, we obtain (11). Multiplying (11) by $H(t,s)\delta(s)$ and integrating from $t_1$ to $t$, we find

$$
\int_{t_1}^{t} H(t,s)\delta(s)\rho(s)A(s)\,ds \leq \int_{t_1}^{t} H(t,s)\delta(s)\frac{\rho'(t)}{\rho(t)}\omega(s)\,ds - \int_{t_1}^{t} H(t,s)\delta(s)\omega(s)\,ds
$$

$$
- \int_{t_1}^{t} H(t,s)\delta(s)\frac{\alpha_1\theta M\sigma'(s)\sigma^{n-2}(s)}{\rho(s)r(s)^{n-1}}\omega(s)\,ds.
$$

This implies that

$$
\int_{t_1}^{t} H(t,s)\delta(s)\rho(s)A(s)\,ds \leq -\int_{t_1}^{t} |h(t,s)|\omega(s)\,ds + H(t,t_1)\delta(t_1)\omega(t_1)
$$

$$
- \int_{t_1}^{t} H(t,s)\delta(s)\frac{\alpha_1\theta M\sigma'(s)\sigma^{n-2}(s)}{\rho(s)r(s)^{n-1}}\omega(s)\,ds.
$$

(15)

Using Lemma 1 with

$$
C = \left(\frac{\alpha_1\theta M\sigma'(s)\sigma^{n-2}(s)H(t,s)\delta(s)}{\rho(s)r(s)^{n+1}}\right)^{1/\mu}\omega(s);
$$

$$
D = \left(\frac{|h(t,s)|^{\alpha_1}\rho(s)r(s)}{\mu^{\alpha_1}(\alpha_1\theta M\sigma'(s)\sigma^{n-2}(s)H(t,s)\delta(s))^{\alpha_1}}\right)^{1/\mu},
$$

we have

$$
\int_{t_1}^{t} (H(t,s)\delta(s)\rho(s)A(s) - \Theta(s))\,ds \leq H(t,t_1)\delta(t_1)\omega(t_1),
$$

which contradicts (14). Theorem 2 is proved. \(\square\)

Corollary 2. If the condition (14) in Theorem 2 is replaced by the following conditions:

$$
\limsup_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)\delta(s)\rho(s)A(s)\,ds = \infty
$$

(16)

and

$$
\limsup_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \Theta(s)\,ds < \infty
$$

(17)

then every solution of (3) is oscillatory.

Example 1. Consider the differential equation

$$
\left(t \left(y(t) + \frac{1}{2}y\left(t\frac{t}{3}\right)\right)'\right)' + y\left(t\frac{t}{2}ight) + y^2(t) + y^2\left(t\frac{t}{4}\right) = 0,
$$

(18)
where \( t \geq 1 \). Note that \( r(t) = t, n = 2, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, a = 1/2, \sigma_1(t) = t/2, \sigma_2(t) = t, \sigma_3(t) = t/4, \) and \( q_i(t) = 1 \). Hence, we have \( m_1 = m_2 = 3, A_k(t) = 2^{-k} \)

\[
A(s) = \frac{1}{4} \sqrt{18} + \frac{1}{2}
\]

and

\[
\int_0^\infty \frac{1}{r^{1/\sigma_1}(t)} dt = \int_0^\infty \frac{1}{t} dt = \infty.
\]

If we set \( \rho(t) = 1 \), then condition (4) is satisfied. Therefore, from Theorem 1, every solution of Equation (18) is oscillatory.

4. Conclusions

In this work, by using the generalized Riccati transformation technique and the integral averaging technique, we establish a new oscillation criteria for (3) under (1). Further, in future work, we can attempt to find some oscillation criteria of Equation (3), if \( z(t) = y(t) - a(t) y(\tau(t)) \).

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