LEFT-EXACT MITTAG-LEFFLER FUNCTORS OF MODULES

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Abstract. Let $R$ be an associative ring with unit. This paper deals with various aspects of the category of functors of $R$-modules; that is, the category of additive and covariant functors from the category of $R$-modules to the category of abelian groups. We give several characterizations of left-exact Mittag-Leffler functors of $R$-modules.

1. Introduction

Various types of modules are defined or determined via certain functors associated with them: flat modules, projective modules, injective modules, etc. The aim of this paper is to study the functors of modules associated with flat Mittag-Leffler modules, much in the spirit as in the theory developed in [2] and [12] (for small categories).

In a more precise manner, if $R$ is an associative ring with unit, we will say that $M$ is an $R$-module (right $R$-module) if $M$ is a covariant additive functor from the category of $R$-modules (respectively, right $R$-modules) to the category of abelian groups.

Any right $R$-module $M$ produces an $R$-module. Namely, the quasi-coherent $R$-module $M$ associated with a right $R$-module $M$ is defined by

$$M(N) = M \otimes_R N,$$

for any $R$-module $N$. It is significant to note that the category of (right) $R$-modules is equivalent to the category of quasi-coherent $R$-modules (Cor. 2.7).

On the other hand, given an $R$-module $M$, $M^*$ is the right $R$-module defined as follows:

$$M^*(N) := \text{Hom}_R(M, N),$$

for any right $R$-module $N$.

If we consider an $R$-module $M$, then we will call $M^*$ the module scheme associated with $M$—by analogy with the geometric framework of functors defined on algebras [2]. Module schemes are projective $R$-modules and left-exact functors and a module scheme $M^*$ is quasi-coherent if and only if $M$ is a finitely generated projective $R$-module (Prop. 2.9 and 3.1, respectively).

A relevant fact is that quasi-coherent modules and module schemes are reflexive, that is, the canonical morphism of $R$-modules $M \to M^{**}$ is an isomorphism (Thm. 2.16; see also [2] for a slightly different version of this statement).

The category of $R$-modules is not a locally small category. Then, it is interesting to consider the following full subcategory.
Definition 1.1. Let \( \langle \text{ModSch} \rangle \) be the full subcategory of the category of \( \mathcal{R} \)-modules whose objects are those \( \mathcal{R} \)-modules \( M \) for which there exists an exact sequence of \( \mathcal{R} \)-module morphisms
\[
\bigoplus_{i \in I} N_i^* \to \bigoplus_{j \in J} N_j^* \to M \to 0.
\]

\( \langle \text{ModSch} \rangle \) is a bicomplete, locally small and abelian category (Thm 6.11). Besides, \( \langle \text{ModSch} \rangle \) is the smallest full subcategory of the category of \( \mathcal{R} \)-modules containing the \( \mathcal{R} \)-module \( \mathcal{R} \) that is stable by limits, colimits and isomorphisms (that is, if an \( \mathcal{R} \)-module is isomorphic to an object of the subcategory then it belongs to the subcategory).

From now on, we will assume that there exists an epimorphism \( \bigoplus_{j \in J} N_j^* \to M \) for the \( \mathcal{R} \)-modules \( M \) considered. If \( J \) is countable, we will say that \( M \) is countably generated.

We prove that \( M \) is a left-exact \( \mathcal{R} \)-module if and only if \( M \) is a direct limit of module schemes (Thm. 3.4). This is a version of Grothendieck’s representability theorem ([9, A. Prop. 3.1]) and its proof follows from standard categorical arguments. Likewise we prove that \( M \) is an exact functor if and only if \( M \) is a direct limit of module schemes \( F_i^* \), where \( F_i \) are free modules (Thm. 3.7). In particular, if \( M = M \) is quasi-coherent, Lazard’s Theorem follows: any flat \( \mathcal{R} \)-module is a direct limit of free \( \mathcal{R} \)-modules.

We then focus on the question of determining those \( \mathcal{R} \)-modules \( M \) that can be expressed as a union of module schemes. We see that this question is closely related to \( M \) being a left-exact Mittag-Leffler module.

Definition 1.2. An \( \mathcal{R} \)-module \( M \) is an ML module if the natural morphism
\[
M(\prod_{i \in I} N_i) \to \prod_{i \in I} M(N_i),
\]
is injective, for any set of \( \mathcal{R} \)-modules \( \{N_i\} \).

An \( \mathcal{R} \)-module \( M \) is a flat Mittag-Leffler module if and only if \( M \) is a left-exact ML \( \mathcal{R} \)-module (see [15 II 2.1.5])). Inspired by previous work of Grothendieck ([8]), the class of Mittag-Leffler modules was first introduced by Gruson and Raynaud ([10], [15]) in their study of flat and projective modules. Soon after that, Mittag-Leffler modules were studied in relation with different functorial properties: flat strict Mittag-Leffler modules are “universally torsionless modules” ([3]), “trace modules” ([14]) or “locally projective modules” (in the sense of [19]). More recently, there is a renewed interest in these modules, as they have been proposed as a generalized notion of vector bundle ([4]) and also have appeared to play a role in several different problems of Algebra ([11], [17]).

Theorem. Let \( M \) be an \( \mathcal{R} \)-module. \( M \) is a left-exact ML \( \mathcal{R} \)-module if and only if it is a direct limit of submodule schemes. If \( M \) is countably generated, \( M \) is a left-exact ML \( \mathcal{R} \)-module if and only if it is projective.

In particular, if \( M \) is a countably generated \( \mathcal{R} \)-module, \( M \) is a flat Mittag-Leffler module if and only if there exists a sequence of submodule schemes of \( M \),
\[
N_0^* \subseteq N_1^* \subseteq \cdots \subseteq N_i^* \subseteq \cdots \subseteq M,
\]
such that \( M = \bigcup_{i \in \mathbb{N}} N_i^* \), and \( M \) is a flat Mittag-Leffler module if and only if it is projective ([10 2.2.2]).
Kaplansky’s theorem about projective $R$-modules can be generalized to projective $\mathcal{R}$-modules:

**Theorem.** Every projective $\mathcal{R}$-module is a direct sum of projective countably generated $\mathcal{R}$-modules.

Finally, we study SML $\mathcal{R}$-modules.

**Definition 1.3.** An $\mathcal{R}$-module $\mathcal{M}$ is an SML $\mathcal{R}$-module if the natural morphism

$$\mathcal{M}(N) \rightarrow \text{Hom}_{\mathcal{R}}(\mathcal{M}^*(R), N), \ m \mapsto \tilde{m},$$

where $\tilde{m}(w) = w_N(m)$, is injective, for any $R$-module $N$.

$M$ is a flat strict Mittag-Leffler module if and only if $M$ is an SML module (see [6], Thm. 3.2). An $\mathcal{R}$-module $\mathcal{M}$ is an SML module iff there exists a monomorphism

$$\mathcal{M} \hookrightarrow \prod_{I} \mathcal{R} \quad \text{(Prop 5.1)}.$$

In particular, $M$ is a flat strict Mittag-Leffler module if and only if there exists a monomorphism $\mathcal{M} \rightarrow \prod_{I} \mathcal{R}$.

**Theorem.** An $\mathcal{R}$-module $\mathcal{M}$ is a left-exact SML $\mathcal{R}$-module if and only if any of the following, equivalent conditions are met:

1. $\mathcal{M}$ is a direct limit of submodule schemes, $N_i^* \subseteq M$ and the dual morphism $M^* \rightarrow N_i^*$ is an epimorphism, for any $i$.
2. If $\mathcal{M}$ is reflexive, every $\mathcal{R}$-module morphism $f : M^* \rightarrow N$ factors through the quasi-coherent module associated with $\text{Im} \ f_R$.

Now, assume $\mathcal{M} = \mathcal{M}$ is quasi-coherent. We can specify more in (1): Let $\{M_i\}_{i \in I}$ be the set of all finitely generated submodules of $M$, and $M_i^* := \text{Im}[M^* \rightarrow M_i^*]$. $M$ is a flat strict Mittag-Leffler module if and only if $\mathcal{M} = \lim_{\rightarrow} M_i'^*$, that is to say, the natural morphism

$$N \otimes_R M \rightarrow \lim_{\rightarrow} \text{Hom}_{\mathcal{R}}(M_i', N)$$

is an isomorphism, for any right $R$-module $N$ (Cor. 5.9). Besides, if $R$ is a local ring we prove that $M$ is a flat strict Mittag-Leffler module if and only if it is equal to the direct limit of its finite free direct summands. (2) means that $M$ is a locally projective $\mathcal{R}$-module (Prop 5.17) and it is equivalent to say that the cokernel of any morphism $f : \mathcal{M}^* \rightarrow \mathcal{R}$ is quasi-coherent, that means that $M$ is a trace module (Prop. 5.20).

In Algebraic Geometry it is usual to consider the category of covariant additive functors from the category of $\mathcal{R}$-algebras to the category of abelian groups, whereas in this work we consider the category of functors defined on the category of $\mathcal{R}$-modules. There is an adjunction between these categories (arXiv:1811.11487) and many of the results presented in this paper are also true for functors defined on the category of $\mathcal{R}$-algebras, although the direct proof is usually more difficult. Finally, let us mention that a similar functorial study of (non-left-exact) ML modules has been carried out in [10]. An effort has been made to make this paper as self-contained as possible.
2. The category of $\mathcal{R}$-modules

Let $R$ be an associative ring with unit.

**Definition 2.1.** A (left) $\mathcal{R}$-module is a covariant, additive functor $\mathbb{M}$ from the category of (left) $R$-modules to the category of abelian groups.

A morphism of $\mathcal{R}$-modules $f: \mathbb{M} \to \mathbb{M}'$ is a morphism of functors such that the morphisms $f_N: \mathbb{M}(N) \to \mathbb{M}'(N)$ are morphisms of groups, for any $R$-module $N$.

Throughout this paper, many definitions or statements are given with one module structure (left or right); we leave to the reader the task of producing the corresponding definitions or statements by interchanging the left and right structures.

Observe that, if $\mathbb{M}$ is a left $\mathcal{R}$-module, then $\mathbb{M}(R)$ is naturally a right $\mathcal{R}$-module: for any $r \in R$, consider the morphism of $\mathcal{R}$-modules $\cdot r: R \to R$, $r' \mapsto r' \cdot r$, and define

$$m \cdot r := \mathbb{M}(\cdot r)(m),$$

for any $m \in \mathbb{M}(R)$.

If $f: \mathbb{M} \to \mathbb{M}'$ is a morphism of $\mathcal{R}$-modules, then $f_R: \mathbb{M}(R) \to \mathbb{M}'(R)$ is a morphism of right $\mathcal{R}$-modules.

Let us write $\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{M}')$ to denote the family of morphisms of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$.

**Definition 2.2.** The dual of an $\mathcal{R}$-module $\mathbb{M}$ is the right $\mathcal{R}$-module $\mathbb{M}^*$ defined, for any right $\mathcal{R}$-module $N$, as follows:

$$\mathbb{M}^*(N) := \text{Hom}_\mathcal{R}(\mathbb{M}, N).$$

The $\mathcal{R}$-modules $\mathbb{M}$ considered in this paper verify that $\mathbb{M}^*(N)$ is a set. It can be proved that $\mathbb{M}^*(N)$ is a set for any $\mathcal{R}$-module $\mathbb{M}$ (see [16, 7.4]).

Kernels, cokernels and images of morphisms of $\mathcal{R}$-modules will always be regarded in the category of $\mathcal{R}$-modules, and it holds:

$$\text{Ker} f(N) = \text{Ker} f_N, \quad \text{Coker} f(N) = \text{Coker} f_N, \quad \text{Im} f(N) = \text{Im} f_N.$$

Besides, for any upward directed system $\{\mathbb{M}_i\}_{i \in I}$ and any downward directed system $\{\mathbb{M}_j\}_{j \in J}$,

$$(\lim_{\to} \mathbb{M}_i)(N) = \lim_{\to} (\mathbb{M}_i(N)),$$

$$(\lim_{\leftarrow} \mathbb{M}_i)(N) = \lim_{\leftarrow} (\mathbb{M}_i(N)).$$

2.1. Quasi-coherent modules.

**Definition 2.3.** The quasi-coherent (left) $\mathcal{R}$-module $\mathcal{M}$ associated with a right $\mathcal{R}$-module $M$ is defined by

$$\mathcal{M}(N) := M \otimes_R N.$$

Any morphism of (right) $\mathcal{R}$-modules $f_R: M \to M'$ induces the morphism of $\mathcal{R}$-modules $f: \mathcal{M} \to \mathcal{M}'$ defined by $f_N(m \otimes n) := f_R(m) \otimes n$, for any $R$-module $N$, $m \in M$ and $n \in N$.

**Definition 2.4.** Let $\mathbb{M}$ be an $\mathcal{R}$-module. The quasi-coherent module associated with the right $\mathcal{R}$-module $\mathbb{M}(R)$ will be denoted by $\mathbb{M}_{qc}$

$$\mathbb{M}_{qc}(N) := \mathbb{M}(R) \otimes_R N.$$
If \( N \) is an \( R \)-module, for any elements \( m \in M(R) \) and \( n \in N \), let us define \( m \cdot n \in M(N) \) in the following way:

\[
m \cdot n := M(n)(m),
\]

where \( M(n) : M(R) \to M(N) \) is the morphism of groups induced by the morphism of \( R \)-modules \( \cdot n : R \to N, r \mapsto r \cdot n \).

There exists a natural morphism \( \mathcal{M}_{qc} \to M \) defined by:

\[
\mathcal{M}_{qc}(N) = M(R) \otimes_R N \to M(N), \quad m \otimes n \mapsto m \cdot n.
\]

for any \( R \)-module \( N \).

**Proposition 2.5.** Let \( \mathcal{M} \) be an \( R \)-module and let \( M \) be a right \( R \)-module. The assignment \( f \mapsto f_R \) establishes a bijection:

\[
\text{Hom}_R(M, M) = \text{Hom}_R(M, M(R)).
\]

**Proof.** Given a morphism of \( R \)-modules \( M \to M(R) \), consider the induced morphism \( \mathcal{M} \to \mathcal{M}_{qc} \) and the composition of the morphisms \( \mathcal{M} \to \mathcal{M}_{qc} \to M \).

Any morphism of \( R \)-modules \( f : \mathcal{M} \to M \) is determined by \( f_R \): if \( N \) is an \( R \)-module, let us consider \( n \in N \) and the morphism of \( R \)-modules \( \cdot n : R \to N, r \mapsto r \cdot n \). The commutativity of the following diagrams imply \( f_N(m \otimes n) = M(n)(f_R(m)) \), for any \( m \in M \),

\[
\begin{array}{ccc}
M = M(R) & \xrightarrow{f_R} & M(R) \\
\downarrow M(n) & & \downarrow M(n) \\
M \otimes_R N = M(N) & \xrightarrow{f_N} & M(N) \\
\end{array}
\]

\[
\begin{array}{ccc}
m & \xrightarrow{f_R} & f_R(m) \\
\downarrow M(n) & & \downarrow M(n) \\
m \otimes n & \xrightarrow{f_N} & f_N(m \otimes n) = M(n)(f_R(m))
\end{array}
\]

\[ \square \]

**Corollary 2.6.** If \( \mathcal{M} \) is an \( R \)-module, there exists a functorial equality, for any quasi-coherent \( R \)-module \( N \),

\[
\text{Hom}_R(N, \mathcal{M}) = \text{Hom}_R(N, \mathcal{M}_{qc}).
\]

**Proof.** The last equality follows from a repeated use of Proposition 2.5:

\[
\text{Hom}_R(N, M) = \text{Hom}_R(N, M(R)).
\]

\[ \square \]

**Corollary 2.7.** The functors \( \mathcal{M} \cong M(R) \) and \( M \cong \mathcal{M} \) establish an equivalence of categories

\[
\text{Category of quasi-coherent } R\text{-modules } \equiv \text{Category of right } R\text{-modules }.
\]

In particular,

\[
\text{Hom}_R(M, M') = \text{Hom}_R(M, M').
\]

If \( f : \mathcal{M} \to \mathcal{N} \) is a morphism of \( R \)-modules, then \( \text{Coker } f \) is the quasi-coherent module associated with \( \text{Coker } f_R \).
2.2. Module schemes.

Definition 2.8. The $\mathcal{R}$-module scheme associated with an $R$-module $M$ is the $\mathcal{R}$-module $\mathcal{M}^*$. 

Observe that the module scheme $\mathcal{M}^*$ is precisely the functor of points of the $R$-module $M$: for any $R$-module $N$, in virtue of Corollary 2.7, 
\[ \mathcal{M}^*(N) = \text{Hom}_R(\mathcal{M}, N) = \text{Hom}_R(M, N). \]

Proposition 2.9. Module schemes $\mathcal{N}^*$ are projective $\mathcal{R}$-modules and left-exact functors.

Proof. By Yoneda’s Lemma, $\text{Hom}_\mathcal{R}(\mathcal{N}^*, M) = M(\mathcal{N})$, for any $R$-module $M$. Therefore, the functor $\text{Hom}_\mathcal{R}(\mathcal{N}^*, -)$ is exact. By Corollary 2.7, $\mathcal{N}^* = \text{Hom}_\mathcal{R}(\mathcal{N}, -)$, that is a left-exact functor. □

Proposition 2.10. Let $\{\mathcal{M}_i\}_{i \in I}$ be a directed system of $\mathcal{R}$-modules. Then, for any $R$-module $N$,
\[ \text{Hom}_\mathcal{R}(\mathcal{N}^*, \lim \to \mathcal{M}_i) = \lim \to \text{Hom}_\mathcal{R}(\mathcal{N}^*, \mathcal{M}_i). \]

Proof. It is a consequence of Yoneda’s Lemma,
\[ \text{Hom}_\mathcal{R}(\mathcal{N}^*, \lim \to \mathcal{M}_i) = (\lim \to \mathcal{M}_i)(N) = \lim \to (\mathcal{M}_i(N)) = \lim \to \text{Hom}_\mathcal{R}(\mathcal{N}^*, \mathcal{M}_i). \]

Proposition 2.11. Let $\mathcal{M}$ be a $\mathcal{R}$-module and $\mathcal{N}$ be a right $\mathcal{R}$-module. There exists an isomorphism of groups
\[ \text{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{N}^*) = \text{Hom}_\mathcal{R}(\mathcal{N}, \mathcal{M}^*). \]

Proof. Any morphism of $\mathcal{R}$-modules $f: \mathcal{M} \to \mathcal{N}^*$ defines a morphism $\tilde{f}: \mathcal{N} \to \mathcal{M}^*$ as follows: $(\tilde{f}_S(n))_S(m) := (f_S(m))_S(n)$, for any $n \in \mathcal{N}(S')$, $m \in \mathcal{M}(S)$ and $\mathcal{R}$-modules $S'$, $S$. This assignment is an isomorphism because, for any $\mathcal{R}$-module $S$ and any right module $S'$, 
\[ \text{Hom}_{\text{grp}}(\mathcal{M}(S), \mathcal{N}(S')) = \text{Hom}_{\text{grp}}(\mathcal{N}(S'), \mathcal{M}(S)). \]

Definition 2.12. If $\mathcal{M}$ is an $\mathcal{R}$-module, let $\mathcal{M}_{sch}$ be the module scheme defined as follows:
\[ \mathcal{M}_{sch} := (\mathcal{M}^*)^*. \]

Proposition 2.13. If $\mathcal{M}$ is an $\mathcal{R}$-module there exists a natural morphism
\[ \mathcal{M} \to \mathcal{M}_{sch}, \]
and a functorial equality, for any module scheme $\mathcal{N}^*$:
\[ \text{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{N}^*) = \text{Hom}_\mathcal{R}(\mathcal{M}_{sch}, \mathcal{N}^*). \]

Proof. The morphism $\mathcal{M} \to \mathcal{M}_{sch}$ is defined, on any $\mathcal{R}$-module $S$, as follows: an element $m \in \mathcal{M}(S)$ defines a morphism $\tilde{m}: \mathcal{M}^*(R) \to S$ via the formula $\tilde{m}(w) := w_S(m)$, so that there exists a map
\[ \mathcal{M}(S) \to \text{Hom}_\mathcal{R}(\mathcal{M}^*(R), S) = \text{Hom}_\mathcal{R}(\mathcal{M}^*_{qc}, S) = \mathcal{M}_{sch}(S). \]
The last equality is a consequence of both Proposition 2.11 and Corollary 2.6: \( \text{Hom}_R(M, N^*) = \text{Hom}_R(N, M^*) \) and \( \text{Hom}_R(\mathcal{M}, \mathcal{N}) = \text{Hom}_R(\mathcal{N}, \mathcal{M}) \).

2.3. Reflexivity of quasi-coherent modules and module schemes.

**Theorem 2.14.** Let \( M \) be a right \( R \)-module and let \( M' \) be an \( R \)-module. Then, the map \( m \otimes m' \mapsto \tilde{m} \otimes m' \) establishes an isomorphism

\[
M \otimes_R M' = \text{Hom}_R(M^*, M'),
\]

where \( (m \otimes m')(w) := w_R(m) \otimes m' \), for any \( w \in M^*(N) \).

**Proof.** As \( M^* \) is a functor of points, the statement readily follows applying Yoneda’s Lemma:

\[
\text{Hom}_R(M^*, M') = M'(M) = M \otimes_R M'.
\]

**Note 2.15.** On the other hand, it is not difficult to prove that the morphism

\[
f = \sum_{i=1}^{n} m_i \otimes m'_i \in \text{Hom}_R(M^*, M') = M \otimes_R M',
\]

coincides with the composition of the morphisms of \( R \)-modules \( M^* \otimes_R L \overset{h}{\rightarrow} M' \), where \( L \) is the free module with basis \( \{l_1, \ldots, l_n\} \), \( h_R(l_i) := m'_i \) for any \( i \), and \( g := \sum_i m_i \otimes l_i \in \text{Hom}_R(M^*, L) = M \otimes_R L \).

With these notations, observe that \( h \) factors through the quasi-coherent module associated with the finitely generated \( R \)-module \( \text{Im} h_R \subseteq M' \), and, hence, so does \( f \).

If \( M \) is an \( R \)-module, there exists a natural morphism

\[
M \rightarrow M^{**},
\]

that maps an element \( m \in M(N) \) to \( \tilde{m} \in M^{**}(N) = \text{Hom}_R(M^*, N) \), that is defined as \( \tilde{m}_N(w) := w_N(m) \), for any \( w \in M^*(N) = \text{Hom}_R(M, N') \).

**Theorem 2.16.** For any right \( R \)-module \( M \), the natural morphism

\[
\mathcal{M} \rightarrow \mathcal{M}^{**}
\]

is an isomorphism.

**Proof.** It is a consequence of Theorem 2.14:

\[
\mathcal{M}^{**}(N) = \text{Hom}_R(M^*, N) \cong M \otimes_R N = \mathcal{M}(N).
\]

**Proposition 2.17.** An \( R \)-module \( \mathcal{M} \) is a module scheme iff it is reflexive, projective and \( \text{Hom}_R(\mathcal{M}, \lim_{\rightarrow} M_i) = \lim_{\rightarrow} \text{Hom}_R(M_i, M_i) \), for any directed system \( \{M_i\}_{i \in I} \).

**Proof.** \( \Rightarrow \) It follows from Proposition 2.10, Proposition 2.14 and Theorem 2.16.

\( \Leftarrow \) The functor \( M^* \) is right-exact and commute with direct sums. Then, \( M^* \) is quasi-coherent by a theorem of Watts, \[18\] Th. 1., and \( \mathcal{M} = \mathcal{M}^{**} \) is a module scheme.
3. Left-exact $\mathcal{R}$-modules

**Proposition 3.1.** An $R$-module $M$ is a finitely generated projective $R$-module if and only if the quasi-coherent module $\mathcal{M}$ is a module scheme.

**Proof.** If $\mathcal{M} \cong \mathcal{N}^*$, then $M$ is a finitely generated $R$-module by Note 2.15. The functor, $\text{Hom}_R(M, -) \cong \text{Hom}_R(\mathcal{N}^*, -)$ is exact since $\mathcal{N}^*$ is a projective $\mathcal{R}$-module, by Proposition 2.9. Given an epimorphism of $R$-modules $S \to T$ then the associated morphism $S \to T$ is an epimorphism and the map

$$
\text{Hom}_R(M, S) \to \text{Hom}_R(M, S) \to \text{Hom}_R(M, T) \to \text{Hom}_R(M, T)
$$

is surjective. Therefore, $M$ is a projective $R$-module.

Conversely, there exist an $R$-module $M'$ and an isomorphism $M + M' \cong R^n$. Hence, there exists an isomorphism $\mathcal{M} \oplus \mathcal{M'} \cong \mathcal{R}^n$. The natural morphism $\mathcal{M} \to \mathcal{M}_{sch}$ is an isomorphism since the diagram

$$
\begin{array}{ccc}
\mathcal{M}_{sch} \oplus \mathcal{M'}_{sch} & \longrightarrow & (\mathcal{M} \oplus \mathcal{M'})_{sch} \\
\downarrow & & \downarrow \\
\mathcal{M} \oplus \mathcal{M'} & \longrightarrow & \mathcal{R}^n 
\end{array}
$$

is commutative.

**Lemma 3.2.** Let $\mathcal{M}$ be an $\mathcal{R}$-module. If $\mathcal{M}$ is left-exact, then $\text{Hom}_R(-, \mathcal{M})$ is a left-exact functor from the category of module schemes to the category of abelian groups. If $\mathcal{M}$ is right-exact, then $\text{Hom}_R(-, \mathcal{M})$ is a right-exact functor from the category of module schemes to the category of abelian groups.

**Proof.** Observe that $\text{Hom}_R(\mathcal{N}^*, \mathcal{M}) = \mathcal{M}(N)$ and the sequence of $\mathcal{R}$-module morphisms $N_1 \to N_2 \to N_3$ is exact in the category of module schemes iff the sequence of $R$-module morphisms $N_3 \to N_2 \to N_1$ is exact.

We will say that $\mathcal{M}$ is a left-exact $\mathcal{R}$-module if it is an $\mathcal{R}$-module and a left-exact functor.

**Lemma 3.3.** Let $\mathcal{M}$ be a left-exact $\mathcal{R}$-module and let $f: \mathcal{N}^* \to \mathcal{M}$ be an $\mathcal{R}$-module morphism. If $0 \neq m \in \text{Ker} f_S \subseteq \mathcal{M}(S)$, then there exists a submodule $N' \subseteq N$, such that $f$ (uniquely) factors through the induced morphism $\pi: \mathcal{N}^* \to N'^*$ and $\pi_S(m) = 0$.

**Proof.** We can consider $m \in \text{Ker} f_S \subseteq \mathcal{N}^*(S) = \text{Hom}_R(\mathcal{S}^*, \mathcal{N}^*)$ as a morphism $\tilde{m}: \mathcal{S}^* \to \mathcal{N}^*$, and it holds $f \circ \tilde{m} = 0$ and $\tilde{m}_S(\text{Id}_S) = m$ (where $\text{Id}_S \in \mathcal{S}^*(S)$ is the identity morphism). By Lemma 3.2 $f$ uniquely factors through the module scheme associated with $N' := \text{Ker} \tilde{m}_R^* \subseteq N$, which is different from $N$ since $\tilde{m}_R^* \neq 0$ (since $\tilde{m} \neq 0$). Besides, the composite morphism $\mathcal{S}^* \to \mathcal{N}^* \to \mathcal{N'}^*$ is zero, hence $\pi_S(m) = \pi_S(\tilde{m}_S(\text{Id}_S)) = 0$.

Unfortunately the category of $R$-modules is not small. If it were small then any $\mathcal{R}$-module would be a quotient $\mathcal{R}$-module of a direct sum of module schemes.

**Theorem 3.4.** Let $\mathcal{M}$ be an $\mathcal{R}$-module and assume that there exists an epimorphism $\pi: \bigoplus_{i \in I} \mathcal{W}^*_i \to \mathcal{M}$. Then, $\mathcal{M}$ is a left-exact $\mathcal{R}$-module iff $\mathcal{M}$ is a direct limit of module schemes.
Proof. \(\Leftarrow\) \(\mathcal{M}\) is left-exact since it is a direct limit of left-exact functors.

\(\Rightarrow\) Let \(J\) be the set of all finite subsets of \(I\). For each, \(j \in J\), put \(W_j := \oplus_{i \in j} W_i\) and let \(\pi_j\) be the composition \(W^*_j = \oplus_{i \in j} W^*_i \hookrightarrow \oplus_{i \in I} W^*_i \xrightarrow{\pi} \mathcal{M}\). Let \(K\) be the set of all the pairs \((j, V^*)\), where \(j \in J\) and \(V^*\) is a module scheme quotient of \(W^*_j\), in the category of module schemes, such that \(\pi_j\) (uniquely) factors through the natural morphism \(W^*_j \to V^*\). Given \((j, V^*), (j', V'^*) \in K\), we say that \((j, V^*) \leq (j', V'^*)\) if \(j \subseteq j'\) and \(\text{Ker}[W^*_j \to V^*] \subseteq \text{Ker}[W^*_j \to V'^*]\), then we have the obvious commutative diagram

\[
\begin{array}{ccc}
W^*_j & \xrightarrow{\pi_j} & \mathcal{M} \\
\downarrow & & \\
V^* & \xrightarrow{\pi_j'} & V'^*
\end{array}
\]

Given \((j, V^*), (j', V'^*) \in K\), put \(j'' = j \cup j'\), \(V'_1 := \text{Ker}[W^*_j \to V^*]\) (that is, \(V_1 = W_j/V\)) and \(V'_1 := \text{Ker}[W^*_j \to V'^*]\) (that is, \(V'_1 = W'_j/V'\)) and let \(V'^*\) be the cokernel in the category of module schemes of the obvious morphism \(V'_1 \oplus V'^*_j \to W'_j\). By Lemma \(3.2\) \((j, V^*), (j', V'^*), (j'', V'^*) \leq (j'', V'^*)\). Hence, \(K\) is an upward directed set. Let us prove that \(\lim_{(j, V) \in K} V^* \to \mathcal{M}\) is an epimorphism: Given \(m \in \mathcal{M}(S)\) there exist \(m' \in \oplus_{i \in I} W^*_i(S)\) such that \(\pi(m') = m\). Obviously, \(m' \in W^*_j(S)\), for some \(j \in J\). Hence, if \(V = W_j\), \(m \in \text{Im}[V^*(S) \to \mathcal{M}(S)]\).

The natural morphism \(\lim_{(j, V) \in K} V^* \to \mathcal{M}\) is a monomorphism: Given \(\lim_{(j, V) \in K} V^* \to \mathcal{M}\) there exist \((j, V) \in K\) and \(m \in \text{Ker}[V^*(S) \to \mathcal{M}(S)]\), such that the equivalence class of \(m\) is \(m\). By Lemma \(3.3\) there exists a submodule \(V' \subseteq V \subseteq W_j\) such that the morphism \(V^* \to \mathcal{M}\) factors through \(V'^*\) and \(m \in \text{Ker}[V^*(S) \to V'^*(S)]\). Hence, \(\tilde{m} = 0\).

For a characterization of left-exact functors in abstract categories, see [1].

**Corollary 3.5.** An \(R\)-module \(M\) is flat if and only if the quasi-coherent module \(\mathcal{M}\) is a direct limit of \(R\)-module schemes.

**Observation 3.6.** Given \(f = \sum_{i=1}^r n_i \otimes m_i \in \text{Hom}_R(\mathcal{N}^*, \mathcal{M}^*) = N \otimes_R M\), put \(N' := \langle n_1, \ldots, n_r \rangle \subseteq N\). Then, \(f\) is the composite morphism of the natural morphism \(\mathcal{N}^* \to N'^*\) and \(g = \sum_{i=1}^r n_i \otimes m_i \in \text{Hom}_R(\mathcal{N}'^*, \mathcal{M}^*) = N' \otimes_R M\).

Then, in Theorem \(3.4\), if \(\mathcal{M} = \mathcal{M}\) is quasi-coherent, we can suppose in the proof of this theorem that \(V, V'\), etc. are finitely generated modules. Then, \(\mathcal{M}\) is a direct limit of module schemes \(V^*_j\), where \(V_j\) are finitely generated \(R\)-modules.

**Theorem 3.7.** Let \(\mathcal{M}\) be an \(R\)-module and assume that there exists an epimorphism \(\pi: \oplus_{i \in I} W^*_i \to \mathcal{M}\). Then, \(\mathcal{M}\) is an exact \(R\)-module if and only if \(\mathcal{M} = \lim_{j \in J} L^*_j\), where \(L_j\) are free \(R\)-modules.
Proof. \( \Rightarrow \) By Theorem 3.4, \( \mathcal{M} \) is the direct limit of a directed system of module schemes \( \{ N_j^\ast, f_{jk} \}_{j \leq k \in J} \). Denote \( f_j \) the natural morphism \( N_j^\ast \to \mathcal{M} \). For any \( j \in J \), there exist a free module \( L_j \) and an epimorphism \( L_j \to N_j^\ast \). Let \( g_j' : N_j^\ast \to L_j^\ast \) be the associated morphism. By Lemma 3.2, there exists a morphism \( g_j : L_j^\ast \to \mathcal{M} \) such that the diagram

\[
\begin{array}{ccc}
N_j^\ast & \xrightarrow{g_j'} & L_j^\ast \\
\downarrow{f_j} & & \downarrow{g_j} \\
\mathcal{M} & & 
\end{array}
\]

is commutative. By Proposition 2.10 there exist \( j \leq \phi(j) \in J \) and a morphism \( g''_j : L_j^\ast \to N_{\phi(j)}^\ast \), such that \( f_{\phi(j)} \circ g''_j = g_j \). Again by Proposition 2.10, taking a greater \( \phi(j) \), if it is necessary, we can suppose that \( f_{\phi(j)} \) is equal to the composite morphism \( N_j^\ast \xrightarrow{g_j'} L_j^\ast \xrightarrow{g''_j} N_{\phi(j)}^\ast \). Let \( J' \) be the upward directed set defined by \( J' = J \) and \( j_1 < j_2 \) if \( \phi(j_1) < \phi(j_2) \). Consider the directed system \( \{ L_j^\ast, g_{jj'} = g_j' \circ f_{\phi(j)} \circ g''_{j'} \} \). Reader can easily check that \( \mathcal{M} = \lim_{j \in J} L_j^\ast \).

\[ \Box \]

Observation 3.8. If \( \mathcal{M} = \mathcal{M} \) is quasi-coherent, in the proof of Theorem 3.7 we can suppose that \( N_j \) are finitely generated \( \mathcal{R} \)-modules, by Observation 3.6. Then, we can suppose that \( L_j \) are finite free \( \mathcal{R} \)-modules. Hence, any flat \( \mathcal{R} \)-module \( \mathcal{M} \) is a direct limit of finite free \( \mathcal{R} \)-modules (Lazard’s theorem, [5 A6.6]).

4. LEFT-EXACT ML \( \mathcal{R} \)-MODULES

Recall the definition of ML \( \mathcal{R} \)-module (Definition 1.2). Module schemes \( \mathcal{M}^\ast \) are obviously left-exact ML \( \mathcal{R} \)-modules.

Proposition 4.1. Every \( \mathcal{R} \)-submodule of an ML \( \mathcal{R} \)-module is an ML \( \mathcal{R} \)-module. Infinite direct products of ML \( \mathcal{R} \)-modules are ML \( \mathcal{R} \)-modules. A direct limit of ML \( \mathcal{R} \)-submodules of an \( \mathcal{R} \)-module is an ML \( \mathcal{R} \)-module.

Proof. Let us only check the last sentence. Put \( \mathcal{M} = \lim_{i \in I} \mathcal{M}_i \), where \( \mathcal{M}_i \subseteq \mathcal{M} \) is an ML \( \mathcal{R} \)-module, for any \( i \in I \). Let \( \{ N_j \}_{j \in J} \) be a set of \( \mathcal{R} \)-modules. Then, the composite morphism \( \mathcal{M}_i(\prod_j N_j) \to \prod_j \mathcal{M}_i(N_j) \to \prod_j \mathcal{M}(N_j) \) is injective and taking \( \lim_{i \in I} \) the morphism \( \mathcal{M}(\prod_j N_j) \to \prod_j \mathcal{M}(N_j) \) is injective. Hence, \( \mathcal{M} \) is an ML \( \mathcal{R} \)-module.

\[ \Box \]

Proposition 4.2. Let \( \mathcal{M} \) be a left-exact ML \( \mathcal{R} \)-module and \( f : N^\ast \to \mathcal{M} \) a morphism of \( \mathcal{R} \)-modules. Then, \( f \) factors through a submodule scheme of \( \mathcal{M} \). Moreover, there exists the smallest submodule scheme of \( \mathcal{M} \) containing \( \text{Im} f \).

Proof. Let \( \{ N_i \}_{i \in I} \) be the set of submodules \( N_i \subseteq N \) such that \( f \) factors through the module scheme \( N_i^\ast \) associated with \( N_i \). Consider the obvious exact sequence of morphisms

\[ 0 \to \bigcap_{i \in I} N_i \to N \to \prod_{i \in I} N/N_i. \]
Therefore, we have the exact sequence of morphisms
\[ 0 \to \mathcal{M}(N'') \to \mathcal{M}(N') \to \mathcal{M}(N'). \]
Observe that \( M \) factors through the module scheme associated with a proper submodule of \( N' \).

Lemma 3.3, the morphism \( f \). Hence, \( N'' \)

Put \( N'_i := N/N_i \), for any \( i \in I \), \( N' := \prod_{i \in I} N'_i \) and \( N'' := \cap_{i \in I} N_i \). Then, we have the exact sequence
\[ 0 \to \mathcal{M}(N'') \to \mathcal{M}(N) \to \mathcal{M}(N'). \]
Observe that \( \mathcal{M}(N') = \mathcal{M}(\prod_{i \in I} N'_i) \subseteq \prod_{i \in I} \mathcal{M}(N'_i) \), then we have the exact sequence of morphisms of groups
\[ 0 \to \mathcal{M}(N'') \to \mathcal{M}(N) \to \prod_{i \in I} \mathcal{M}(N'_i). \]

Therefore, we have the exact sequence of morphisms
\[ 0 \to \text{Hom}_R(N''^*, \mathcal{M}) \to \text{Hom}_R(N^*, \mathcal{M}) \to \prod_{i \in I} \text{Hom}_R(N_i^*, \mathcal{M}). \]

Hence, \( f \) factors through a morphism \( g: N''^* \to \mathcal{M} \), since \( f \in \text{Hom}_R(N_i^*, \mathcal{M}) = \text{Ker}[\text{Hom}_R(N_i^*, \mathcal{M}) \to \text{Hom}_R(N_i^*, \mathcal{M})] \), for every \( i \). Obviously, \( g \) does not factor through the module scheme associated with a proper submodule of \( N'' \).

By Lemma 3.3, the morphism \( g: N''^* \to \mathcal{M} \) is a monomorphism. \( N''^* \) is the smallest submodule scheme of \( \mathcal{M} \) containing \( \text{Im} f \). If \( \text{Im} f \subseteq W^* \subseteq \mathcal{M} \), then the morphisms \( N^* \to W^* \) again factors through \( N''^* \), since \( W^* \) is a left-exact ML \( R \)-module.

\[ \square \]

**Theorem 4.3.** Let \( \mathcal{M} \) be an \( R \)-module and assume there exists an epimorphism \( \pi: \oplus_{i \in I} W_i^* \to \mathcal{M} \). The following statements are equivalent

1. \( \mathcal{M} \) is a left-exact ML module.
2. Every morphism of \( R \)-modules \( N^* \to \mathcal{M} \) factors through an \( R \)-submodule scheme of \( \mathcal{M} \), for any right \( R \)-module \( N \).
3. \( \mathcal{M} \) is equal to a direct limit of \( R \)-submodule schemes.

**Proof.** The implication (1) \( \Rightarrow \) (2) is precisely Proposition 4.2.

(2) \( \Rightarrow \) (3) Given a morphism \( f: N^* \to \mathcal{M} \), let \( W^* \) be a submodule scheme of \( \mathcal{M} \) containing \( \text{Im} f \). By Proposition 4.2, there exists the smallest submodule scheme \( V^* \) of \( W^* \) containing \( \text{Im} f \). If \( W''^* \) is another submodule scheme of \( \mathcal{M} \) containing \( \text{Im} f \), consider a submodule scheme \( W''''^* \) of \( \mathcal{M} \) containing \( V^* \) and \( W''^* \) (observe we have a natural morphism \( W^* \oplus W''^* \to \mathcal{M} \)). Hence, the smallest submodule scheme of \( W''^* \) containing \( \text{Im} f \) is equal to the smallest submodule scheme of \( W''''^* \) containing \( \text{Im} f \), which is equal to \( V^* \). Therefore, \( V^* \) is the smallest submodule scheme of \( \mathcal{M} \) containing \( \text{Im} f \).

Let \( J \) be the set of all finite subsets of \( I \), and \( V_j^* \) the smallest submodule scheme of \( \mathcal{M} \) containing \( \pi(\oplus_{i \in J} W_i^*) \), for any \( j \in J \). Then, \( \lim_{\to j \in J} V_j^* = \mathcal{M} \).

(3) \( \Rightarrow \) (1) \( \mathcal{M} \) is left-exact since it is a direct limit of left-exact functors. By Proposition 4.1, \( \mathcal{M} \) is an ML \( R \)-module.

\[ \square \]

**Definition 4.4.** An \( R \)-module \( \mathcal{M} \) is called countably generated if there exists an epimorphism \( \oplus_{n \in \mathbb{N}} N_i^* \to \mathcal{M} \).

By Note 2.15, \( \mathcal{M} \) is countably generated iff \( M \) is countably generated.

**Proposition 4.5.** An \( R \)-module \( \mathcal{M} \) is a left-exact ML \( R \)-module of countable type if and only if there exists a chain of submodule schemes of \( \mathcal{M} \)
\[ N_1^* \subseteq N_2^* \subseteq \cdots \subseteq N_r^* \subseteq \cdots \subseteq \mathcal{M} \]
such that \( \mathcal{M} = \cup_{r \in \mathbb{N}} N_r^* \).
Corollary 4.6. Let $M$ be a countably generated $R$-module. $M$ is a left-exact $R$-module if and only if it is projective.

Proof. $\iff$ Consider an epimorphism $\oplus_{i \in \mathbb{N}} \mathcal{N}_i^* \to M$. $M$ is a direct summand of $\oplus_{i \in \mathbb{N}} \mathcal{N}_i^*$. Hence, $M$ is left-exact and it is an ML $R$-module by Proposition 4.1.

$\Rightarrow$ By Proposition 4.5, there exists a chain of submodule schemes of $M$

$$\mathcal{N}_1^* \subseteq \mathcal{N}_2^* \subseteq \cdots \subseteq \mathcal{N}_n^* \subseteq \cdots \subseteq M$$

such that $M = \bigcup_{i \in \mathbb{N}} \mathcal{N}_i^*$. We have the exact sequence of morphisms

$$0 \to \oplus_{i \in \mathbb{N}} \mathcal{N}_i^* \xrightarrow{f} \oplus_{i \in \mathbb{N}} \mathcal{N}_i^* \xrightarrow{g} M \to 0,$$

where $f(n_0, n_1, n_2, \ldots) := (n_0, n_1 - n_0, n_2 - n_1, \ldots)$ and $g((n_i)) = \sum_i n_i$. The morphism $r: \oplus_{i \in \mathbb{N}} \mathcal{N}_i^* \to \oplus_{i \in \mathbb{N}} \mathcal{N}_i^*$, $r(n_0, n_1, n_2, \ldots) := (n_0, n_0 + n_1, n_0 + n_1 + n_2, \ldots)$ is a retract of $f$. Therefore, $M$ is a direct sum of $\oplus_{i \in \mathbb{N}} \mathcal{N}_i^*$ and it is projective. \hfill $\Box$

In particular, if $M$ is a countably generated, it is a Mittag-Leffler module if it is projective.

Theorem 4.7. Let $M$ be an $R$-module and assume there exists an epimorphism $\pi: \oplus_{i \in I} \mathcal{N}_i^* \to M$. If $M$ is projective then it is a direct sum of (projective) countably generated $R$-modules.

Proof. Let us repeat the arguments given by Kaplansky in [13]. $M$ is a direct summand of $N := \oplus_{i \in I} \mathcal{N}_i^*$. Put $N = M \oplus M'$. Let us inductively construct a well ordered increasing sequence of $R$-submodules $\{N_\alpha\}$ of $N$ such that

1. If $\alpha$ is a limit ordinal, $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$.
2. $N_{\alpha+1}/N_\alpha$ is countably generated.
3. Each $N_\alpha$ is the direct sum of a subset of the $\mathcal{N}_i^*$’s.
4. $N_\alpha = M_\alpha \oplus M'_\alpha$, where $M_\alpha = N_\alpha \cap M$ and $M'_\alpha = N_\alpha \cap M'$.

In this situation, $M_\alpha$ is a direct summand of $N$, since it is a direct summand of $N_\alpha$, which is a direct summand of $N$. Therefore, $M_\alpha$ is a direct summand of $M$. Besides,

$$N_{\alpha+1}/N_\alpha = M_{\alpha+1}/M_\alpha \oplus M'_{\alpha+1}/M'_\alpha$$

Hence, $M_{\alpha+1}/M_\alpha$ is countably generated. Now it is clear that $M = \cup_{\alpha} M_\alpha$ and $M = \oplus_{\alpha} (M_{\alpha+1}/M_\alpha)$.

Now, we proceed to construct the $N_\alpha$. Consider the obvious projections $\pi, \pi': N \to M, M'$. Given $N_\alpha$ let us construct $N_{\alpha+1}$. Choose $N_{\alpha+1} \subseteq N_\alpha$ and put $N_\alpha = N_{\alpha+1}$. By Proposition 2.10, there exists $j_1, \ldots, j_r \in I$, such that $\pi(N_{j_1}^*) \oplus \pi'(N_{j_1}^*) \subseteq N_{j_1}^* \oplus \cdots \oplus N_r^*$. Put $N_2 := N_{j_1}^* \subseteq \cdots \subseteq N_{r+1}^* := N_r^*$. Now we repeat on $N_2$ the treatment just given to $N_1^*$. The result will be a new finite set $N_{\alpha+2}^*, \ldots, N_n^*$. We proceed successively in this way. Finally, $N_{\alpha+1}$ is taken to be the $R$-submodule generated by $N_\alpha$ and all the $N_n^*$’s. That $N_{\alpha+1}$ has all the properties we desire is plain. \hfill $\Box$
Observation 4.8. If \( f : N^* \to M \) is a monomorphism, then \( N \) is a finitely generated module: The dual morphism \( f^* : M^* \to N^* \) factors through the quasi-coherent module associated with a finitely generated submodule \( N' \subseteq N \), by Note 2.13. The morphism \( N^* \to N'^* \) is a monomorphism since the composition \( N^* \to N'^* \to M \) is a monomorphism. By Lemma 5.5, the inclusion morphism \( N' \subseteq N \) is an epimorphism, that is, \( N = N' \).

5. LEFT-EXACT SML \( \mathcal{R} \)-MODULES

Recall the definition of SML \( \mathcal{R} \)-module (Definition 1.3). Module schemes \( M^* \) are obviously left-exact SML \( \mathcal{R} \)-modules.

Proposition 5.1. Let \( M \) be an \( \mathcal{R} \)-module. The following statements are equivalent:

1. \( M \) is an SML \( \mathcal{R} \)-module.
2. The natural morphism \( M \to M_{\text{sch}} \) is a monomorphism.
3. There exists a monomorphism \( M \hookrightarrow V^* \).
4. There exists a monomorphism \( M \hookrightarrow \prod_I \mathcal{R} \).

Proof. (1) \( \iff \) (2) It is an immediate consequence of the definition of \( M_{\text{sch}} \).

(2) \( \iff \) (3) It is obvious.

(3) \( \Rightarrow \) (4) Consider an epimorphism \( \oplus_I \mathcal{R} \to V \). Taking dual \( \mathcal{R} \)-modules, we have a monomorphism \( V^* \hookrightarrow \prod_I \mathcal{R} \). The composition of the monomorphisms \( M \hookrightarrow V^* \hookrightarrow \prod_I \mathcal{R} \) is a monomorphism.

(4) \( \Rightarrow \) (2) By Proposition 2.13, the monomorphism \( M \hookrightarrow \prod_I \mathcal{R} \) factors through a morphism \( M \to M_{\text{sch}} \), that has to be a monomorphism.

□

Corollary 5.2. [6, 5.3] If \( M \) is a flat strict Mittag-Leffler module, then \( M \) is a pure submodule of an \( \mathcal{R} \)-module \( \prod_I \mathcal{R} \).

Proof. There exists a monomorphism \( M \hookrightarrow \prod_I \mathcal{R} \). For any right \( \mathcal{R} \)-module \( S \), the morphism \( S \otimes_{\mathcal{R}} M \to S \otimes_{\mathcal{R}} (\prod_I \mathcal{R}) \) is injective, since the composition \( S \otimes_{\mathcal{R}} M \to S \otimes_{\mathcal{R}} (\prod_I \mathcal{R}) \to \prod_I S \) is injective.

□

Corollary 5.3. Any \( \mathcal{R} \)-submodule of an SML \( \mathcal{R} \)-module is an SML \( \mathcal{R} \)-module.

Proposition 5.4. If \( M_i \) is a left-exact SML \( \mathcal{R} \)-module, for any \( i \in I \), then \( \prod_{i \in I} M_i \) is a left-exact SML \( \mathcal{R} \)-module.

Proof. Obviously, \( \prod_{i \in I} M_i \) is a left-exact functor. Put \( M_i \subseteq V_i^* \). Then, \( \prod_{i \in I} M_i \subseteq \prod_i V_i^* = V^* \), where \( V = \oplus V_i \). By Proposition 5.1, \( \prod_{i \in I} M_i \) is an SML \( \mathcal{R} \)-module.

□

Lemma 5.5. If an \( \mathcal{R} \)-module morphism \( N^* \to M^* \) is a monomorphism, then the dual morphism \( M \to N \) is an epimorphism.

Proof. The category of modules schemes is anti-equivalent to the category of quasi-coherent modules. Hence, the morphism \( M \to N \) is an epimorphism in the category of quasi-coherent modules, therefore it is an epimorphism.

□

Proposition 5.6. Let \( M \) be an SML \( \mathcal{R} \)-module. If \( i : N^* \to M \) is a monomorphism, then \( i^* : M^* \to N^* \) is an epimorphism.
Proof.  By Proposition 4.1, there exists a monomorphism $M \hookrightarrow \prod_i \mathcal{R}$. Consider the monomorphisms $N^* \hookrightarrow M \hookrightarrow \prod_i \mathcal{R}$. Dually, the composite morphism
$$\oplus_i R \rightarrow M^* \hookrightarrow N$$
is an epimorphism, by Lemma 5.3. Hence, $i^*$ is an epimorphism.

\[ \square \]

**Theorem 5.7.** An $\mathcal{R}$-module $M$ is a left-exact SML $\mathcal{R}$-module iff $M$ is a direct limit of submodule schemes, $N_i^* \subseteq M$ and the dual morphism $M^* \rightarrow N_i$ is an epimorphism, for any $i$. In particular, an $\mathcal{R}$-module $M$ is a left-exact SML $\mathcal{R}$-module iff it is a left-exact ML $\mathcal{R}$-module and for any submodule scheme $N^* \subseteq M$ the dual morphism $M^* \rightarrow N$ is an epimorphism.

**Proof.** $\Rightarrow$) By Proposition 5.1, there exists a monomorphism $M \hookrightarrow V^*$. By Proposition 4.1, $M$ is an ML $\mathcal{R}$-module since $V^*$ is an ML $\mathcal{R}$-module. By Proposition 4.2, any morphism $N^* \rightarrow M$ factors through a submodule scheme of $M$. Recall $M(N) = \text{Hom}_{\mathcal{R}}(N^*, M)$, then given, $m \in M(N)$, there exists a morphism $f: N^* \rightarrow M$ such that $f_N(Id_N) = m$ (where $Id_N \in N^*(N)$ is the identity morphism). Hence, there exists a submodule scheme $N_i^*$ of $M$ such that $m \in N_i^*(N)$. The family of submodule schemes of $V^*$ is a set because the family of quotient modules of $V$ is a set. Then, the family of submodule schemes of $M$ is a set and $M$ is equal to the direct limit of its submodule schemes.

$\Leftarrow$) Put $M = \lim N_i^*$, where $N_i^* \subseteq M$ and the dual morphism $M^* \rightarrow N_i$ is an epimorphism, for any $i \in I$. $M$ is a left-exact $\mathcal{R}$-module since it is a direct limit of left-exact $\mathcal{R}$-modules. Observe that the morphism $M_i^*(R) \rightarrow N_i$ is surjective, for any $i$, then the morphism $\text{Hom}_{\mathcal{R}}(N_i, N) \rightarrow \text{Hom}_{\mathcal{R}}(M_i^*(R), N)$ is injective and the morphism $\lim \text{Hom}_{\mathcal{R}}(N_i, N) \rightarrow \text{Hom}_{\mathcal{R}}(M_i^*(R), N)$ is injective. Then, the composition
$$M(N) = \lim N_i^*(N) = \lim \text{Hom}_{\mathcal{R}}(N_i, N) \hookrightarrow \text{Hom}_{\mathcal{R}}(M_i^*(R), N)$$
is injective. Hence, $M$ is an SML $\mathcal{R}$-module.

\[ \square \]

**Theorem 5.8.** Let $M$ be a reflexive $\mathcal{R}$-module. Then, $M$ is a left-exact SML module iff every morphism $f: M^* \rightarrow N$ factors through the quasi-coherent module associated with $\text{Im} f_R$, for any right $\mathcal{R}$-module $N$.

**Proof.** $\Rightarrow$) The dual morphism $f^*: N^* \rightarrow M$ factors through a submodule scheme $N_i^* \subseteq M$, by Proposition 4.2. Dually, we have the morphisms $M^* \rightarrow N' \rightarrow N$ and $M^* \rightarrow N'$ is an epimorphism, by Proposition 5.6. Hence, $\text{Im} f_R = \text{Im}[N' \rightarrow N]$ and $M^*$ factors through the quasi-coherent modules associated to $\text{Im} f_R$ since the morphism $N' \rightarrow N$ factors through it.

$\Leftarrow$) A morphism $g: N^* \rightarrow M$ is zero iff the dual morphism $g^*: M^* \rightarrow N$ is zero, and this last morphism is zero iff $\text{Im} g_R = 0$, that is, $g_R = 0$. Therefore, the morphism $M(N) = \text{Hom}_{\mathcal{R}}(N^*, M) \rightarrow \text{Hom}_{\mathcal{R}}(M^*(R), N)$ is injective and $M$ is an SML module. Let us check that $M$ is left-exact. If $N_1 \subseteq N_2$, then the morphism $M(N_1) \rightarrow M(N_2)$ is injective, since $\text{Hom}_{\mathcal{R}}(M^*(R), N_1) \rightarrow \text{Hom}_{\mathcal{R}}(M^*(R), N_2)$ is injective. Consider an exact sequence of $\mathcal{R}$-module morphisms $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3$ and let $N_1 \rightarrow N_2 \rightarrow N_3$ be the associated morphisms. It remains to prove that
Corollary 5.9. Let \( M \) be an \( R \)-module and let \( \{M_i\}_{i \in I} \) be the set of all finitely generated \( R \)-submodules of \( M \), and \( M_i' := \text{Im}[M^* \to M_i] \). Then, \( M \) is a flat strict Mittag-Leffler module iff the natural morphism

\[
N \otimes_R M \to \lim_{\to} \text{Hom}_R(M_i', N)
\]

is an isomorphism, for any right \( R \)-module \( N \).

Proof. \( \Rightarrow \) Let \( \{M_i\}_{i \in I} \) be the set of finitely generated submodules of \( M \). Let \( L_i \) be a finite free module, \( L_i \to M_i \) an epimorphism and \( M_i \to M \) the inclusion morphism, for any \( i \in I \). Let \( \pi_i : L_i \to M_i \) and \( f_i : M_i \to M \) be the induced morphisms. Taking dual \( R \)-modules, we have the morphisms

\[
\mathcal{M}^* \xrightarrow{f^*} \mathcal{M}_i^* \xrightarrow{\pi_i^*} L_i^*.
\]

\( \text{Im}(\pi_i^* \circ f_i^*)_R = \text{Im}(f_i^*)_R = M_i' \). By Theorem 5.8, the composite morphism \( \pi_i^* \circ f_i^* \) factors through the natural morphism \( \mathcal{M}_i^* \to L_i^* \), which factors through the natural morphism \( \mathcal{M}_i' \to \mathcal{M}_i^* \). Then, we have the morphisms

\[
\mathcal{M}^* \to \mathcal{M}_i' \to \mathcal{M}_i^*.
\]

Taking dual \( R \)-modules, we have the morphisms \( \mathcal{M}_i \to \mathcal{M}_i'^* \to \mathcal{M} \) and \( \mathcal{M}_i'^* \to \mathcal{M} \) is a monomorphism. Hence, \( \mathcal{M} = \lim_{\to} \mathcal{M}_i'^* \). Therefore,

\[
N \otimes_R M = \mathcal{M}(N) = \lim_{\to} \mathcal{M}_i'^*(N) = \lim_{\to} \text{Hom}_R(M_i', N),
\]

for any right \( R \)-module \( N \).

\( \Leftarrow \) \( \mathcal{M} = \lim_{\to} \mathcal{M}_i'^* \) and the morphisms \( \mathcal{M}^* \to \mathcal{M}_i' \) are epimorphisms. By Theorem 5.7, \( M \) is a flat strict Mittag-Leffler module. \qed

5.1. Other characterizations of flat strict Mittag-Leffler modules.

Lemma 5.10. Let \( \mathbb{M} \) be left-exact and a reflexive \( R \)-module. The cokernel of an \( R \)-module morphism \( f : \mathbb{M}^* \to \mathcal{N} \) is quasi-coherent iff \( f \) factors through the quasi-coherent module associated with \( \text{Im} f_R \).

Proof. \( \mathcal{N}' := \text{Im} f_R \) is the kernel of the morphism \( \mathcal{N} \to \text{Coker} f =: \mathcal{N}'' \). Let \( \pi : \mathcal{N} \to \mathcal{N}'' \) be the associated morphism. Observe that \( \mathbb{M}(\mathcal{N}) = \mathbb{M}^{**}(\mathcal{N}) = \text{Hom}_R(\mathbb{M}^*, \mathcal{N}) \). Then, \( f \) factors through \( \mathcal{N}' \) iff \( \pi \circ f = 0 \), since \( \mathbb{M} \) is left-exact. The natural morphism \( \mathcal{N}'' \to \text{Coker} f \) is an epimorphism. Then, \( \pi \circ f = 0 \) iff \( \mathcal{N}'' = \text{Coker} f \). Therefore, \( f \) factors through \( \mathcal{N}' \) iff \( \mathcal{N}'' = \text{Coker} f \). \qed

Theorem 5.11. Let \( M \) be an \( R \)-module. Then \( M \) is a flat strict Mittag-Leffler module iff the cokernel of any morphism \( f : \mathcal{M}^* \to \mathcal{R} \) is quasi-coherent.
Proof. \( \Rightarrow \) It is a consequence of Theorem 5.8 and Lemma 5.10

\( \Leftarrow \) Let \( f : M^* \to N^* \) be a morphism of \( R\)-modules such that \( f_R = 0 \). We have to prove that \( f = 0 \). By Note 2.14 \( f \) factors through the quasi-coherent module associated with a finitely generated submodule of \( N \). Hence, we can suppose that \( N \) is finitely generated. We proceed by induction on the number of generators of \( N \). Suppose \( N = \langle n \rangle \). Let \( \pi : R \to N \) be an epimorphism. By Theorem 2.14 there exists a morphism \( g : M^* \to R \) such that \( f = \pi \circ g \). Observe that \( N' := \ker g_R \subseteq N \) is zero. By the hypothesis and Lemma 5.10 \( g \) factors through a morphism \( h : M^* \to N' \), then \( f = \pi \circ g = \pi \circ i \circ h = 0 \circ h = 0 \).

Suppose \( N = \langle n_1, \ldots, n_r \rangle \). Put \( N_1 := \langle n_1 \rangle \) and \( N_2 := N/N_1 \) and let \( i : N_1 \to N \) and \( \pi : N \to N_2 \) be the induced morphisms. By Theorem 2.14 the sequence of morphisms

\[
\text{Hom}_R(M^*, N_1) \to \text{Hom}_R(M^*, N) \to \text{Hom}_R(M^*, N_2) \to 0
\]

is exact. The morphism \( f \circ \pi \) is zero since \( (\pi \circ f)_R = \pi_R \circ f_R = \pi_R \circ 0 = 0 \) and the Induction Hypothesis. Hence, there exists a morphism \( g : M^* \to N_1 \) such that \( f = i \circ g \). The morphism \( g_R \) is zero, since \( 0 = f_R = (i \circ g)_R = i_R \circ g_R \) and \( i_R \) is injective. By the Induction Hypothesis, \( g = 0 \) and \( f = i \circ g = 0 \).

Lemma 5.12. Let \( f : N \to M \) be a morphism of \( R\)-modules. If the induced morphism \( M^* \to N^* \) is an epimorphism, then \( f \) has a retraction of \( R\)-modules.

Proof. The epimorphism \( M^* \to N^* \) has a section since \( N^* \) is a projective \( R\)-module by Proposition 2.9. Hence, \( f \) has a retraction since the category of \( R\)-modules is anti-equivalent to the category of \( R\)-module schemes.

Proposition 5.13. Let \( R \) be a local ring (that is, a ring where the non-units form a two-sided ideal). An \( R\)-module \( M \) is a flat strict Mittag-Leffler module if and only if it equals the direct limit of its finite free direct summands.

Proof. Let \( M \) be a flat strict Mittag-Leffler module, so that we can write \( M = \lim \limits_{i \in I} N_i^* \), where the morphisms \( M^* \to N_i^* \) are epimorphisms. By Note 2.14 \( N_i \) is finitely generated, for any \( i \in I \). There exist a finite free (right) \( R\)-module \( L_i \) and an epimorphism \( L_i \to N_i \) such that the induced morphism \( L_i/N_i : m \to N_i/N_i : m \) is an isomorphism of \( K\)-modules. By Theorem 2.14 the epimorphism \( M^* \to N_i^* \) factors through a morphism \( M^* \to L_i^* \). The morphism \( M^* \to L_i^* \) is an epimorphism by Nakayama’s lemma, since the morphism \( M^*/M^* : m \to L_i/L_i : m \simeq N_i/N_i : m \) is an epimorphism. Hence, the morphism \( M^* \to L_i^* \) is an epimorphism. By Lemma 5.12 the morphism \( L_i^* \to M \) has a retraction, that is, \( L_i^* \) is a direct summand of \( M \). Finally, any finitely generated submodule of \( M \) is included in some submodule \( N_i^* \) and \( N_i^* \subseteq L_i^* \). Hence, \( M = \lim \limits_{i \in I} L_i^* \).

Conversely, let \( \{ L_i \}_{i \in I} \) be the set of finite free summands of \( M \). Thus, \( M = \lim \limits_{i \in I} L_i \) and the morphisms \( M^* \to L_i^* \) are epimorphisms, so that \( M \) is a flat strict Mittag-Leffler module.

Proposition 5.14 ([7] Th. 3.29). Let \( R \) be a principal ideal domain. An \( R\)-module \( M \) is a flat strict Mittag-Leffler module if and only if it equals the direct limit of its finite free direct summands.
Proof. Assume $M$ is a flat strict Mittag-Leffler module, so that we can write $M = \lim_{\to} N_i^*$, where the morphisms $M^* \to N_i$ are epimorphisms. By Note 2.13, $N_i$ is finitely generated, for any $i \in I$. There exist a finite free $R$-module $L_i$ and an epimorphism $L_i \to N_i$. By Theorem 2.14 the epimorphism $M^* \to N_i$ factors through a morphism $\pi_i: M^* \to L_i$. By Theorem 5.11 and Lemma 5.10, $\pi_i$ factors through the quasi-coherent module associated with $L_i' = \text{Im} \pi_i$. Observe that $L_i'$ is a finite free module since it is a submodule of a finite free module and $R$ is a principal ideal domain. By Lemma 5.12 the morphism $L_i'^* \to M$ has a retraction, that is, $L_i'^*$ is a direct summand of $M$. Finally, any finitely generated submodule of $M$ is included in some submodule $N_i^*$ and $N_i^* \subseteq L_i'^*$. Hence, $M = \lim_{\to} L_i'^*$.

Conversely, proceed as in the previous proof.

\[ \square \]

**Corollary 5.15** (13). Let $R$ be a local ring or a principal ideal domain, then, any projective $R$-module of countable type is free.

**Proof.** Write $M = \langle m_i \rangle_{i \in \mathbb{N}}$. By Proposition 5.13 (or 5.14), there exists a chain of finite free direct summands of $M$

$$L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots \subseteq M$$

such that $\langle m_1, \ldots, m_n \rangle \subseteq L_n$, for any $n \in \mathbb{N}$, and $M = \cup_{n \in \mathbb{N}} L_n$. As $L_i$ is a direct summand of $L_{i+1}$, let us write $L_{i+1} = L_i \oplus L_i'$, with $L_0 = L_0$. It is then easy to check that $M \simeq \oplus_{n \in \mathbb{N}} L_n$.

\[ \square \]

Let us recall that Kaplansky also proved that any projective module over a local ring or a principal ideal domain is free, using that any projective module is a direct sum of countably generated projective modules (14).

**5.2. Other definitions of flat strict Mittag-Leffler modules.** Finally, let us prove some well-known characterizations of flat strict Mittag-Leffler modules.

**Definition 5.16.** An $R$-module $M$ is said to be locally projective if for any epimorphism $\pi: N \to M$ and any morphism $f: R^n \to M$ (for any $n \in \mathbb{N}$) there exists a morphism $s: M \to N$ such that $f = \pi \circ s \circ f$.

**Proposition 5.17** (13 Prop. 6). An $R$-module $M$ is locally projective iff it is a flat strict Mittag-Leffler module.

**Proof.** \( \Rightarrow \) Consider an $\mathcal{R}$-module morphism $f^*: M^* \to \mathcal{R}$ (or equivalently, an $\mathcal{R}$-module morphism $f: \mathcal{R} \to M$). Let $L$ be a free $\mathcal{R}$-module and $\pi: L \to M$ an epimorphism. There exists an $\mathcal{R}$-module morphism $s: M \to L$ such that $f = \pi \circ s \circ f$. Then, $f^* = f^* \circ s^* \circ \pi^*$ and $\text{Im} f^* = \text{Im}(f^* \circ s^* \circ \pi^*)$ since

$$\text{Im} f^* \supseteq \text{Im}(f^* \circ s^*) \supseteq \text{Im}(f^* \circ s^* \circ \pi^*) = \text{Im} f^*.$$ 

Hence, $\text{Coker} f^* = \text{Coker}(f^* \circ s^*)$, which is quasi-coherent by Theorem 5.11. Again by Theorem 5.11, $M$ is a flat strict Mittag-Leffler module.

\( \Leftarrow \) Let $\pi: \mathcal{N} \to M$ be an $\mathcal{R}$-module epimorphism and $f: \mathcal{R}^n \to M$ an $\mathcal{R}$-module morphism. The morphism $f$ factors through an $\mathcal{R}$-submodule scheme $i: W^* \subseteq \mathcal{M}$ and a morphism $f': \mathcal{R}^n \to W^*$, since $M$ is a flat Mittag-Leffler module. There exists a morphism $s': W^* \to \mathcal{N}$ such that $\pi \circ s' = i$ since $W^*$ is a projective $\mathcal{R}$-module, by Proposition 2.9. The map

$$\text{Hom}_R(M, \mathcal{N}) = M^*(\mathcal{N}) \to W(\mathcal{N}) = N \otimes_R W = \text{Hom}_R(W^*, \mathcal{N})$$
is surjective, by Proposition 5.6. Hence, there exists an ℛ-module morphism 
\( s: \mathcal{M} \rightarrow \mathcal{N} \) such that \( s \circ i = s' \). Therefore,
\[
\pi \circ s \circ f = \pi \circ s \circ i \circ f' = \pi \circ s' \circ f' = i \circ f' = f
\]
and \( \mathcal{M} \) is locally projective.

\[\square\]

**Proposition 5.18** ([3] Prop 7.). Let \( \mathcal{M} \) be a flat strict Mittag-Leffler module and \( \mathcal{N} \subseteq \mathcal{M} \) a pure submodule. Then, \( \mathcal{N} \) is a flat strict Mittag-Leffler module and it is locally split in \( \mathcal{M} \), that is, for any finitely generated submodule \( \mathcal{N}' \subseteq \mathcal{N} \) there exists an \( \mathcal{R} \)-module morphism \( r: \mathcal{M} \rightarrow \mathcal{N} \) such that \( r(n') = n' \) for any \( n' \in \mathcal{N}' \).

**Proof.** The induced morphism \( i: \mathcal{N} \rightarrow \mathcal{M} \) is a monomorphism. Then, by Corollary 5.3, \( \mathcal{N} \) is a flat strict Mittag-Leffler module. Given the submodule \( \mathcal{N}' \subseteq \mathcal{N} \), there exists a submodule scheme \( i': \mathcal{W}^* \subseteq \mathcal{N} \) such that \( \mathcal{N}' \subseteq \mathcal{W}^* \). The map
\[
\text{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{N}) = \mathcal{M}^*(\mathcal{N}) \rightarrow \mathcal{W}(\mathcal{N}) = \mathcal{W} \otimes_\mathcal{R} \mathcal{N} \xrightarrow{\text{(6.2.14)}} \text{Hom}_\mathcal{R}(\mathcal{W}^*, \mathcal{N})
\]
is surjective, by Proposition 5.6. Then, there exists an \( \mathcal{R} \)-module morphism \( t: \mathcal{M} \rightarrow \mathcal{N} \) such that \( t \circ i \circ i' = i' \). It is easy to check that \( t_R(n') = n' \) for any \( n' \in \mathcal{N}' \).

\[\square\]

**Definition 5.19** ([6]). A module \( \mathcal{M} \) is a trace module if every \( m \in \mathcal{M} \) holds
\[
m \in \mathcal{M}^*(m) \cdot \mathcal{M},
\]
where \( \mathcal{M}^*(m) := \{ w(m) \in R : w \in \mathcal{M}^* \} \).

**Proposition 5.20** ([10] II 2.3.4). \( \mathcal{M} \) is a trace module iff it is a flat strict Mittag-Leffler module.

**Proof.** Consider the canonical isomorphism \( \mathcal{M} \xrightarrow{\text{can}} \text{Hom}_\mathcal{R}(\mathcal{M}^*, \mathcal{R}) \), \( m \mapsto \hat{m} \) (where \( \hat{m}(w) := w(m) \)). Obviously, \( \text{Im} \hat{m}_R = \mathcal{M}^*(m) \). Let \( I \subseteq \mathcal{R} \) be an ideal, \( \hat{m} \) factors through \( I \) iff \( m \in I \cdot \mathcal{M} \), as it is easy to see taking into account the following diagram
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{R}(\mathcal{M}^*, I) & \longrightarrow & \text{Hom}_\mathcal{R}(\mathcal{M}^*, \mathcal{R}) \\
\downarrow \text{(2.1.14)} & & \downarrow \text{(2.1.14)} \\
I \otimes_\mathcal{R} \mathcal{M} & \longrightarrow & \mathcal{M}
\end{array}
\]
Then, \( \hat{m} \) factors the quasi-coherent module associated with \( \text{Im} \hat{m}_R \) if and only if \( m \in \mathcal{M}^*(m) \cdot \mathcal{M} \). We are done, by Lemma 5.10 and Theorem 5.11.

\[\square\]

6. **Appendix: Abelian subcategory generated by module schemes**

Let \( \mathcal{M} \) be an \( \mathcal{R} \)-module and \( \mathcal{P} = \oplus_{i \in I} \mathcal{N}_i^* \), then
\[
\text{Hom}_\mathcal{R}(\mathcal{P}, \mathcal{M}) = \prod_{i \in I} \text{Hom}_\mathcal{R}(\mathcal{N}_i^*, \mathcal{M}) = \prod_{i \in I} \mathcal{M}(\mathcal{N}_i).
\]
Hence, \( \mathcal{P} \) is a projective \( \mathcal{R} \)-module. Observe that \( \mathcal{P} \) is a left-exact functor and \( \mathcal{P} = \oplus_{i \in I} \mathcal{N}_i^* \subseteq \mathcal{N}^* \), where \( \mathcal{N} = \oplus \mathcal{N}_i \). Therefore, \( \mathcal{P} \) is a left-exact SML \( \mathcal{R} \)-module.

**Notation 6.1.** An infinite direct sum of modules schemes will be often denoted by \( \mathcal{P}, (\mathcal{P}', \mathcal{P}_1, \text{etc.}) \)
Recall the definition of \( \langle \text{ModSch} \rangle \) (Definition 1.1). Given \( M, M' \in \langle \text{ModSch} \rangle \), consider an epimorphism \( P \to M \), then
\[
\text{Hom}_R(M, M') \subseteq \text{Hom}_R(P, M')
\]
and \( \text{Hom}_R(M, M') \) is a set, that is, \( \langle \text{ModSch} \rangle \) is a locally small category.

**Examples 6.2.** Quasi-coherent modules and module schemes belong to \( \langle \text{ModSch} \rangle \).

If \( M \) is a left-exact functor and there exists an epimorphism \( P \to M \), then \( M \) belongs to \( \langle \text{ModSch} \rangle \), by Theorem 3.4.

Left-exact SML \( R \)-modules belong to \( \langle \text{ModSch} \rangle \), by Theorem 5.7.

**Proposition 6.3.** Let \( f : M_1 \to M_2 \) be an \( R \)-module morphism. If \( M_1, M_2 \in \langle \text{ModSch} \rangle \), then \( \text{Coker} f \in \langle \text{ModSch} \rangle \).

**Proof.** Consider the exact sequences
\[
P_1 \xrightarrow{\pi_1} M_1 \to 0, \quad P_2 \xrightarrow{i_2} P_2 \xrightarrow{\pi_2} M_2 \to 0.
\]
There exists an \( R \)-module morphism \( g : P_1 \to P_2 \) such that \( \pi_2 \circ g = f \circ \pi_1 \), since \( P_1 \) is a projective \( R \)-module. It is easy to check that
\[
\text{Coker}[P_1 \oplus P_2 \xrightarrow{g \oplus i_2} P_2] = \text{Coker} f.
\]
Therefore, \( \text{Coker} f \in \langle \text{ModSch} \rangle \).

**Proposition 6.4.** Let \( f : P \to P' \) be an \( R \)-module morphism. Then, \( \text{Ker} f \) is a left-exact SML module and \( \text{Im} f \in \langle \text{ModSch} \rangle \).

**Proof.** \( \text{Ker} f \) is left-exact since \( P \) and \( P' \) are left-exact. By Corollary 5.3, \( \text{Ker} f \) is an SML \( R \)-module, since \( P \) is an SML \( R \)-module. By Proposition 6.3, \( \text{Im} f = \text{Coker}[\text{Ker} f \to P] \) belongs to \( \langle \text{ModSch} \rangle \).

**Proposition 6.5.** Let \( 0 \to M_1 \to M_2 \xrightarrow{f} M_3 \to 0 \) be an exact sequence of \( R \)-module morphisms. If \( M_1, M_3 \in \langle \text{ModSch} \rangle \), then \( M_2 \in \langle \text{ModSch} \rangle \).

**Proof.** Consider exact sequences \( P_1 \xrightarrow{f_1} P_1' \xrightarrow{f_1'} M_1 \to 0 \) and \( P_3 \xrightarrow{f_3} P_3' \xrightarrow{f_3'} M_3 \to 0 \).

There exits a morphism \( g : P_3' \to M_2 \) such that \( \pi \circ g = f_3' \), since \( P_3' \) is projective. Consider the exact sequence
\[
0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{\pi} M_3 \longrightarrow 0
\]
\[
0 \longrightarrow P_1' \xrightarrow{f_1' \oplus g} P_1' \oplus P_3' \xrightarrow{f_3'} P_3' \longrightarrow 0
\]
By the snake lemma, \( f_1' \oplus g \) is an epimorphism and we have the exact sequence of morphisms
\[
0 \to \text{Ker} f_1' \to \text{Ker}(f_1' \oplus g) \to \text{Ker} f_3' \to 0,
\]
where \( \text{Ker} f_1' = \text{Im} f_1 \) and \( \text{Ker} f_3' = \text{Im} f_3 \) belong to \( \langle \text{ModSch} \rangle \) by Proposition 6.4.

Then, again there exists and epimorphism \( P'' \to \text{Ker}(f_1' \oplus g) \) and \( M_2 \in \langle \text{ModSch} \rangle \).

**Proposition 6.6.** Let \( f : P \to M \) be an \( R \)-module epimorphism. If \( M \in \langle \text{ModSch} \rangle \), then \( \text{Ker} f \in \langle \text{ModSch} \rangle \).
Proof. Consider an exact sequence of $\mathcal{R}$-module morphisms

$$P_1' \xrightarrow{i_1} P_1 \xrightarrow{\pi_1} M \to 0.$$  

There exists a morphism $g: P \to P_1$ such that $\pi_1 \circ g = f$, since $P$ is a projective $\mathcal{R}$-module. Consider the exact sequence of $\mathcal{R}$-module morphisms

$$0 \to \ker g \to \ker \pi_1 \to \ker f \to 0 \to 0.$$  

By the snake lemma, we have the exact sequence of $\mathcal{R}$-module morphisms

$$0 \to \ker f \to M \to 0.$$  

By Proposition 6.6, $\ker f, \ker(g \circ \pi_1) \in \langle \text{ModSch} \rangle$. By the snake lemma, we have the exact sequence

$$0 \to \ker \pi \to \ker(f \circ \pi) \to \ker f \to 0.$$  

By the Proposition 6.3, $\ker f \in \langle \text{ModSch} \rangle$.

Proposition 6.7. Let $f: M_i \to M'_i$ be an $\mathcal{R}$-module epimorphism. If $M_i, M'_i \in \langle \text{ModSch} \rangle$, then $Ker f \in \langle \text{ModSch} \rangle$.

Proof. Consider an epimorphism $\pi: P \to M$ and the commutative diagram

$$\begin{array}{ccc}
0 & \to & \ker f \\
\downarrow & & \downarrow \\
\ker i_1 & \to & \ker \pi_1 \\
\downarrow & & \downarrow \\
0 & \to & \ker (f \circ \pi) \\
\downarrow & & \downarrow \\
P & \xrightarrow{f} & M' \\
\downarrow & & \downarrow \\
\ker (f \circ \pi) & \to & \ker f \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}$$

Proposition 6.8. Let $M_i, M'_i \in \langle \text{ModSch} \rangle$ and let $f: M_i \to M'_i$ be an $\mathcal{R}$-module morphism. Then, $\ker f \in \langle \text{ModSch} \rangle$.

Proof. By Proposition 6.3, $\ker f \in \langle \text{ModSch} \rangle$. By Proposition 6.7, $\text{Im } f = \ker[M'_i \to \text{Coker } f] \in \langle \text{ModSch} \rangle$. By Proposition 6.7, $\ker f = \ker[M \to \text{Im } f] \in \langle \text{ModSch} \rangle$.

Proposition 6.9. If $M_i \in \langle \text{ModSch} \rangle$ for any $i \in I$, then $\bigoplus_{i \in I} M_i \in \langle \text{ModSch} \rangle$.

Proof. It is obvious.

Proposition 6.10. If $M_i \in \langle \text{ModSch} \rangle$ for any $i \in I$, then $\prod_{i \in I} M_i \in \langle \text{ModSch} \rangle$.

Proof. It is sufficient to prove that $\prod_{i \in I} P_i \in \langle \text{ModSch} \rangle$. $\prod_{i \in I} P_i$ is a left-exact SML $\mathcal{R}$-module by Proposition 5.3. We are done.

Given $\mathcal{M}^*$, consider a free presentation of $M, \bigoplus_I R \to \bigoplus_I R \to M \to 0$. Then, we have an exact sequence $0 \to \mathcal{M}^* \to \prod_I \mathcal{R} \to \prod_I \mathcal{R}$. Now, the following theorem is immediate.
Theorem 6.11. \((\text{ModSch})\) is a bicomplete, locally small and abelian category. Besides, \((\text{ModSch})\) is the smallest full subcategory of the category of \(\mathcal{R}\)-modules containing \(\mathcal{R}\) that is stable by kernels, cokernels, direct limits, inverse limits and isomorphisms (that is, if an \(\mathcal{R}\)-module is isomorphic to an object of the subcategory then it belongs to the subcategory).

Notation 6.12. Let \((\text{RModSch})\) be the full subcategory of the category of right \(\mathcal{R}\)-modules whose objects are those right \(\mathcal{R}\)-modules \(\mathcal{M}\) for which there exists an exact sequence of \(\mathcal{R}\)-module morphisms

\[
P' \to P \to M \to 0
\]

where \(P = \bigoplus_{i \in I} N_i^*\) and \(P' = \bigoplus_{j \in J} N_j^*\) (and \(N_i, N_j\) are right \(\mathcal{R}\)-modules).

Proposition 6.13. If \(M \in (\text{ModSch})\), then \(M^* \in (\text{RModSch})\).

Proof. Consider an exact sequence \(P' \to P \to M \to 0\). Dually, \(0 \to M^* \to P^* \to P'^*\) is exact. It is enough to prove that \(P^*, P'^* \in (\text{RModSch})\). Put \(P = \bigoplus_{i \in I} N_i^*\). By Proposition 6.10, \(P^* = \prod_{i \in I} N_i^* \in (\text{RModSch})\). \(\square\)

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