The Angular-Diameter-Distance-Maximum and Its Redshift as Constraints on $\Lambda \neq 0$ FLRW Models

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Abstract

The plethora of recent cosmologically relevant data has indicated that our universe is very well fit by a standard Friedmann-Lemaître-Robertson-Walker (FLRW) model, with $\Omega_M \approx 0.27$ and $\Omega_\Lambda \approx 0.73$ – or, more generally, by nearly flat FLRW models with parameters close to these values. Additional independent cosmological information, particularly the maximum of the angular-diameter (observer-area) distance and the redshift at which it occurs, would improve and confirm these results, once sufficient precise Supernovae Ia data in the range $1.5 < z < 1.8$ become available. We obtain characteristic FLRW closed functional forms for $C = C(z)$ and $\hat{M}_0 = \hat{M}_0(z)$, the angular-diameter distance and the density per source counted, respectively, when $\Lambda \neq 0$, analogous to those we have for $\Lambda = 0$. More importantly, we verify that for flat FLRW models $z_{\text{max}}$ – as is already known but rarely recognized – the redshift of $C_{\text{max}}$, the maximum of the angular-diameter-distance, uniquely gives $\Omega_\Lambda$, the amount of vacuum energy in the universe, independently of $H_0$, the Hubble parameter. For non-flat models determination of both $z_{\text{max}}$ and $C_{\text{max}}$ gives both $\Omega_\Lambda$ and $\Omega_M$, the amount of matter in the universe, as long as we know $H_0$.
independently. Finally, determination of $C_{\text{max}}$ automatically gives a very simple observational criterion for whether or not the universe is flat – presuming that it is FLRW.

1 Introduction

Over the last 10 or 12 years a great deal of outstanding observational work has indicated that the best fit model of our universe is a nearly flat Friedmann-Lemaître-Robertson-Walker (FLRW) model with $\Omega_M \approx 0.27$ and $\Omega_\Lambda \approx 0.73$ (Riess et al. 1998; Perlmutter et al. 1999; Bennett et al 2003 (WMAP results); Peacock et al. 2001; Percival et al. 2001; Efstathiou et al. 2002; Spergel et al. 2003, and references therein), where $\Omega_M$ and $\Omega_\Lambda$ are the usual density parameters for matter, including nonbaryonic dark matter, and dark energy, modelled here as vacuum energy (the cosmological constant $\Lambda$), respectively. Here and throughout this paper $\Omega_M$ and $\Omega_\Lambda$ refer to these quantities as evaluated at our time now. This remarkable concordance is based on WMAP cosmic microwave background (CMB) anisotropy measurements, a large number of Supernovae Ia data (see Riess et al. 2004), and large scale structure studies, and has been confirmed by other more recent work. Riess and his collaborators (Riess et al. 2004), for instance, have recently found a best-fit cosmology having $\Omega_M = 0.29$ and $\Omega_\Lambda = 0.71$ for their sample of 16 distant ($z > 1$) SN Ia, including 6 with $z > 1.25$, assuming the universe is exactly flat. Within the errors this is consonant with the “concordance” model given above.

Despite the strength of these results, they will obviously have to undergo gradual revision and continual verification, as more precise data from higher redshifts are acquired. When $\Lambda \neq 0$, there are at present, from a strictly mathematical consideration of the Einstein field equations, not yet enough completely independent observables to constrain all the free parameters of the cosmological model (Hellaby, 2006; Stoeger & Hellaby, in preparation).

Assuming that the universe is spherically symmetric on the largest scales (FLRW or, more generally, Lemaître-Tolman-Bondi (LTB)), one generally needs redshifts, luminosity distances (or angular-diameter distances), and galaxy number counts, together with a reliable galaxy evolution model, or an equivalent set of measurements, to constrain the model fully (see Ellis, et
al. 1985). If $\Lambda \neq 0$, however, or if there is some other form of dark energy, these data are not enough. We need at least one other independent parameter – that is, independent of the observables we have just mentioned and therefore of those which depend upon them. And, strictly speaking, this is what we have not had. Thus, the impressive fittings that have led to the concordance model are still model-dependent in some sense.

There is another pair of such independent observables. These would improve and verify our cosmological fitting, when we are able to obtain an adequate number of precise luminosity distances – or angular-diameter distances – and redshifts for SN Ia, or for other standard candles or standard rods, out to $z \approx 1.8$. These observables are the maximum of the angular-diameter distance (or observer-area distance) $C_{\text{max}}$ and the redshift $z_{\text{max}}$ at which it occurs. It has been realized for many years (McCrea 1935, Hoyle 1961, Ellis & Tivon 1985) that this distance reaches a maximum for relatively low redshifts in FLRW universes. For an Einstein-deSitter ($\Omega = 1$) universe filled with matter, for instance, the observer area distance $C$ has a maximum $C_{\text{max}}$ at $z_{\text{max}} = 1.25$. This effect is due to the global gravitational focusing of light rays caused by the matter in the universe – in effect the entire universe, filled with homogeneously distributed matter, acts like a gravitational lens.

Krauss and Schramm (1993) recognized that, for flat FLRW universes, determination of $z_{\text{max}}$ would give us $\Omega_\Lambda$. They plotted and provided a table giving this unique correspondence (see their Table 1), and proposed the possibility of using the measurement of compact parsec-scale radio jets to observationally exploit it, if the source-evolution problem can be tamed. Since then, there has been little development or discussion of this potentially important connection – except for Hellaby’s (2006) recent closely connected exploration of such measurements within the more general context of LTB universes (see below). Certainly, it is implicit in the Friedmann equation – most clearly in Refsdal, et al.’s (1967) numerical results of general cosmological models, in the brief treatment of cosmic distances by Carroll, et al., 1992 (see pages 510-512, and their Figure 5), and in Peeble’s treatment of angular diameters in cosmology (Peebles 1993), but not pointed out or discussed further, until Hellaby’s more general treatment. This may be partially due to the difficulty of obtaining reliable data at the redshifts where we would expect to locate $C_{\text{max}}$ (see below). Now, however, there is the very real prospect of obtaining angular diameter distances (indirectly, by measuring luminosity distances of
SN Ia) out to \( z \approx 1.8 \) using telescopes in space. Thus, it is important to point out again and stress this promising connection, which could eventually be incorporated in the Bayesian-Fisher matrix (see, for example, Albrecht, et al., 2006) fitting of models to data, or be used as an independent consistency check on such fittings.

Recently, as already mentioned, Hellaby (2006) emphasized the importance of such a measurement within a more general framework. He points out that in any LTB cosmology with \( \Lambda = 0 \) (which includes all \( \Lambda = 0 \) FLRW cosmologies as special cases) the measurement of \( C_{\text{max}} \) is equivalent to a measurement of the total mass \( M_{\text{max}} \) within the sphere defined by \( C_{\text{max}} \). For \( \Lambda \neq 0 \) we have for any LTB model, instead, a simple relationship between the \( \Lambda, C_{\text{max}} \) and \( M_{\text{max}} \) (see equation (11) below). So a measurement of \( M_{\text{max}} \), or its equivalent, and \( C_{\text{max}} \) determines \( \Lambda \). What becomes apparent is that \( C_{\text{max}} \) and the redshift \( z_{\text{max}} \) at which it occurs constitute independent cosmological observables – directly constraining \( \Lambda \) and \( \Omega_{M} \) (see Hellaby’s Figure A1 in his Appendix, which shows how different cosmological parameters vary with \( z_{\text{max}} \)).

Applying this directly to flat FLRW models, like those we have good evidence represent our universe, we quickly see that, since we implicitly have a relation between the total mass-energy density and the matter density, or equivalently between the matter density and \( \Omega_{\Lambda} \) — i.e. \( \Omega_{M} = 1 - \Omega_{\Lambda} \) — observational determination of \( z_{\text{max}} \) will directly determine \( \Omega_{\Lambda} \) in a very simple and straightforward way, supporting Krauss and Schramm’s results (1993). In this paper we shall integrate and generalize these results, first of all verifying Krauss and Schramm’s results for flat FLRW universes and writing down that relationship as an algebraic equation in closed form (they presented their results numerically), and then generalizing those results to non-flat FLRW universes, using the relationship Hellaby (2006) noticed. In this case, \( C_{\text{max}} \) and \( z_{\text{max}} \) directly determine both \( \Omega_{\Lambda} \) and \( \Omega_{M} \), if we know \( H_{0} \) independently. In the course of doing this, we shall, as useful and important by-products, obtain the FLRW \( C = C(z) \) and \( M_{0} = M_{0}(z) \) closed-form functional relationships for \( \Lambda \neq 0 \) universes, parallel to those which are well-known for \( \Lambda = 0 \) FLRW models (Ellis and Stoeger 1987; Stoeger, et al. 1992), as well as a very simple observational criterion for flatness in terms of \( C_{\text{max}} \). Here \( C(z) \), of course, is simply the angular-diameter distance as a function of the redshift \( z \), and \( M_{0}(z) \) is the mass density per source counted as a function
of $z$, which is closely related to the differential galaxy number counts $dN/dz$ (see Stoeger, et al. 1992). To our knowledge, these more general results, along with the closed-form expressions and the flatness criterion are new.

We have already indicated that these measurements will be able to be implemented once we have luminosity distances and redshifts for SN Ia, or for other standard candles or standard rods, in the interval $1.5 < z < 1.8$. As we shall show, it is precisely in this region that a flat FLRW universe will have a maximum in its angular-diameter distance, if $0.59 \leq \Omega_\Lambda < 0.82$. For the best fit FLRW given by Riess et al. (2004) with $\Omega_M = 0.29$ and $\Omega_\Lambda = 0.71$, $z_{\text{max}} = 1.62$. Another potential way of obtaining such precise measurements is – following Krauss and Schramm’s (1993) idea – the use of VLBI to determine the angular-size/redshift relation for ultra-compact (milliarcsecond) radio sources. These have been argued to be standard rods (Jackson and Dodgson 1997; Jackson 2004). If we actually do find the maximum angular-diameter distance near this value of the redshift, this would be independent confirmation of the concordance model. If we do not, but find the maximum angular-diameter distance $C$ at some other value of $z$, this will require further work at reconciling the models, and possibly modifying them. In that case, either the universe may still be flat, but the relative amounts of matter and dark energy would be quite different from that given by the concordance, or there is a significant deviation from flatness that must be taken into account, or possibly there are significant deviations from FLRW on the largest scales which must be included – or all three! At the very least, this would be a good consistency check on our cosmological fitting so far. Alternatively, as we have already mentioned, we could simply include both $C_{\text{max}}$ and $z_{\text{max}}$ data in our over-all fitting scheme – which would further improve the reliability of our results.

It is important to point out that this redshift range is already attracting special attention. That is because there have been preliminary indications (Gilliland et al. 1999) from an SN Ia at $z \approx 1.7$ that the universe was decelerating at that time! Further studies (Riess et al. 2001; Mortsell et al. 2001; Benítez et al. 2002) have confirmed the plausibility of that conclusion, but were unable to strengthen it without further SN Ia measurements in that interval. Thus, we now have two strong motivations for pursuing precise SN Ia searches and measurements in this redshift range.
Finally, one might wonder how measurements of the luminosity distances of SN Ia can reveal maxima in the angular-diameter (or observer-area) distances. The luminosity distances themselves will not have such maxima. The answer to this question is simple, though rarely adverted to. According to Etherington (1933; see also Ellis 1971), the luminosity distance $d_L$ is very generally related to the angular-diameter, or observer-area, distance by

$$d_L = (1 + z)^2 C. \quad (1)$$

This simple but important relationship holds for all cosmologies, even very inhomogenous ones. Thus, with observed luminosity distances and redshifts in the above mentioned crucial redshift range, we can very quickly convert to angular-diameter distances, and determine whether the maximum for those distances lies within that range.

Now we shall go on to work out the simple relationship between $z_{max}$ and $\Omega_\Lambda$ for flat FLRW.

### 2 The Maximum Angular-Diameter Distance in Flat FLRW with $\Lambda \neq 0$

The basic equations relating $z_{max}$ and $\Omega_\Lambda$ in flat FLRW with $\Lambda \neq 0$ are not difficult, but require some effort to obtain and check, because they involve elliptic integrals. As we have already mentioned, this represents the simplest and clearest example of a more general relationship between the redshift of the maximum of the angular-diameter distance (in LTB models this is often referred to as the “areal radius”) and the matter and vacuum-energy content of the universe for all FLRW and LTB models (Hellaby 2006). Furthermore, neither Krauss and Schramm (1993) nor Hellaby (2006) illustrate the actual calculation. Their results were obtained numerically, and presented in plotted or table form.

In flat FLRW, the angular-diameter (or observer-area) distance $C(\eta, y)$ is given by

$$C(\eta, y) = R(\eta)y = \frac{R_0y}{1 + z}, \quad (2)$$
where $R(\eta)$ is the scale factor, $\eta$ is the conformal time, $R_0$ is the scale factor now, $y$ is the comoving radial coordinate, and $z$ is the redshift of signals from distant sources. Here we have used the important FLRW relationship

$$1 + z = \frac{R_0}{R(\eta)}. \tag{3}$$

Clearly, if we differentiate equation (2) with respect to $y$ and set the result equal to zero, we shall have the equation for the maximum of $C(\eta, y)$, subject to the usual condition that $d^2C/dy^2 < 0$ for $dC/dy = 0$. We have then from equation (2)

$$\frac{dC}{dy} = \frac{R_0}{1 + z} - \frac{R_0 y}{(1 + z)^2} \frac{dz}{dy} = 0, \tag{4}$$

which becomes

$$\frac{R_0}{1 + z} - \frac{R_0 y}{(1 + z)^2} R_0 H_0 \sqrt{\Omega_\Lambda + (1 - \Omega_\Lambda)(1 + z)^3} = 0, \tag{5}$$

since the Friedmann equation in this case yields

$$\frac{dz}{dy} = R_0 H_0 \sqrt{\Omega_\Lambda + (1 - \Omega_\Lambda)(1 + z)^3}. \tag{6}$$

Thus, from solving equation (5) for $y$, we obtain the equation for $y_{max}$, the comoving radial coordinate distance to the point down the observer’s past light cone at which the angular-diameter distance is a maximum, as a function of $z_{max}$, the redshift there, and of $\Omega_\Lambda$:

$$y_{max} = \frac{1 + z_{max}}{R_0 H_0 \sqrt{\Omega_\Lambda + (1 + z_{max})^3(1 - \Omega_\Lambda)}}. \tag{7}$$

This is the first and most essential step in finding the equation we are looking for.

The second step involves finding the explicit solution to the Friedmann equation, essentially equation (6), to give us another expression for $y_{max}$ at $z_{max}$. Substituting this expression into left-hand-side of equation (7) gives a unique implicit equation for $\Omega_\Lambda$ as a function simply of $z_{max}$. This is the relationship we have been looking for.
So, what is the solution of equation (6)? Normally, we might want to simply do a numerical integration. However, this would not be very useful in our case. It turns out, as is well known (Byrd & Friedman (1954), pp. 8-10 and formula 260.00 (p. 135); see also Jeffrey (1995), pp. 225-226), that, since this equation involves the square root of a cubic polynomial, it has an analytic solution in terms of elliptic integrals. In our case the most useful form of the solution is:

\[ y = \frac{g}{R_0 H_0 \Omega_{\Lambda}^{1/2}} \left[ F(\phi, k) \big|_{(1+z)^{-1}} - F(\phi, k) \big|_{(1+z)^{-1}} \right], \]  

(8)

where the \( F(\phi, k) \) are standard elliptic integrals of the first kind, for the angle \( \phi \), which is a function of \( 1 + z \), and \( k \) is the modulus. More explicitly

\[
\begin{align*}
\phi &= \cos^{-1} \left[ \frac{-m(1 + z) + (\sqrt{3} - 1)}{-m(1 + z) - (\sqrt{3} + 1)} \right], \\
m &= \left[ \frac{1 - \Omega_{\Lambda}}{\Omega_{\Lambda}} \right]^{1/3}, \\
k^2 &= \frac{1}{2} + \frac{\sqrt{3}}{4}, \\
g &= \frac{1}{3^{1/4}} \left[ \frac{\Omega_{\Lambda}}{1 - \Omega_{\Lambda}} \right]^{1/3}.
\end{align*}
\]

This solution was obtained and checked using elliptic integral tables in Byrd & Friedman (1954) (entry 260.00, p. 135) in conjunction with MAPLE.

With equation (8) being substituted for \( y \), equation (2) is the characteristic FLRW relationship for the angular-diameter distance \( C = C(z) \) in terms of \( z \). It turns out (see below) that this same form of the relationship holds in the general (non-flat) FLRW cases – with the parameters \( \phi \), \( k \), and \( g \) being more complicated functions, involving \( \Omega_{\Lambda} \), either \( \Omega_M \) or \( C_{\text{max}} \), and \( H_0 \). We shall explicitly write these down in the next section. Similarly, we quickly can write down the complementary characteristic \( \Lambda \neq 0 \) mass density per source counted as a function of \( z \) (see Ellis and Stoeger 1987 and Stoeger, et al. 1992):

\[
\dot{M}_0(z) = \frac{\mu_{\text{ma}}(1 + z)^2}{R_0 H_0 \sqrt{\Omega_{\Lambda} + \Omega_M(1 + z)^3 - (\Omega_0 - 1)(1 + z)^2}}, \]

(9)
where $\mu_{m0}$ is the mass-energy density now and $\Omega_0 \equiv \Omega_\Lambda + \Omega_M$, and the last term under the radical sign in the denominator is zero when the universe is flat (see below). These characteristic FLRW relationships for $C(z)$ and for \( \dot{M}_0(z) \) are very useful to know (Ellis and Stoeger 1987; Stoeger, et al. 1992). If the universe is FLRW and $\Lambda = 0$, then these relationships inevitably follow. If, on the other hand, the data can be put into these functional forms, then it can be shown by solving the field equations with this data (Stoeger, et al. 1992; Araújo, Stoeger, et al., in preparation) that the universe must be FLRW. Thus, being able to fit the data to these forms, assures us that the universe is FLRW. Not being able to do so, assures us that it is not FLRW.

Returning to the main object of our derivation, substituting equation (8) into the left-hand-side of equation (7), we have simply:

\[
\frac{g}{\Omega_\Lambda^{1/2}} \left[ F(\phi, k) \big|_{(1+z)^{-1}=1} - F(\phi, k) \big|_{(1+z_{\text{max}})^{-1}} \right] = \frac{1 + z_{\text{max}}}{\sqrt{\Omega_\Lambda + (1 + z_{\text{max}})^3(1 - \Omega_\Lambda)}}. \tag{10}
\]

This is a transcendental relationship for $\Omega_\Lambda$ as a function of $z_{\text{max}}$. It is worth noticing that it does not involve any other parameters! This is the relationship which represents the numerical results obtained by Krauss and Schramm (1993).

The solutions to this implicit algebraic equation were obtained using MAPLE, and were checked by hand for values of $\Omega_\Lambda$ near the concordance model value of $\Omega_\Lambda = 0.73$. They are given in Table 1 and Figure 1 below. We can immediately see, that for the concordance model we should find...
For the nearby best fit model of Riess, et al. (2004) we have already mentioned, \( z_{\text{max}} = 1.62 \). Values of \( z_{\text{max}} \) for many other values of \( \Omega_{\Lambda} \) are given, as well. These verify the values presented in Krauss and Schramm (1992), and those evident in the plots of Refsdal, et al. (1967), Carroll, et al. (1992), and Hellaby (2006).

3 Non-Flat FLRW Universes

If the universe is not flat, a slight generalization of these same equations obtains, with the solution for \( y \) taking the same general form as given in equation (8). The generalization of equation (10) in this case will, however, include – as is intuitively clear – a dependence on \( \Omega_M \) as well as on \( \Omega_{\Lambda} \). Using the general relationship emphasized by Hellaby (2006)

\[
\Lambda C_{\max}^3 - 3C_{\max}^2 + 6M_{\max} = 0, \tag{11}
\]

we can determine \( \Omega_M \) through \( M_{\max} \) in terms of \( C_{\max} \) and \( \Lambda \). It is important to stress that equation (11) holds for these quantities as measured at \( z_{\text{max}} \), or \( y_{\text{max}} \), down the observer’s past light cone. From Hellaby’s (2006) results, we easily find that, for FLRW,

\[
M_{\max} = \frac{4}{3} \pi \rho_M C_{\max}^3, \tag{12}
\]

where \( \rho_M = \rho(t_{\max}) = \rho_0 (1 + z_{\max})^3 \). Here \( \rho_0 \) is the density at our time now, \( t_0 \). Using this together with the definition of \( \Omega_M \equiv 8\pi \rho_0 / 3H_0^2 \) and equation (11), we easily obtain\(^2\)

\[
\Omega_M = \frac{2}{H_0^2} \frac{1}{(1 + z_{\max})^3} [C_{\max}^2 - \Omega_{\Lambda} H_0^2]. \tag{13}
\]

This can be substituted into the non-flat versions of equations (6) and (7),

\[
dz/dy = R_0 H_0 \sqrt{\Omega_{\Lambda} + \Omega_M (1 + z)^3 - (\Omega_0 - 1)(1 + z)^2}, \tag{14}
\]

\(^2\) As in Hellaby (2006), we also use units such that \( G = c = 1 \).
and

\[ y_{\text{max}} = \frac{1 + z_{\text{max}}}{R_0 H_0 \sqrt{\Omega_\Lambda + \Omega_M (1 + z_{\text{max}})^3 - (\Omega_0 - 1)(1 + z_{\text{max}})^2}}, \quad (15) \]

In passing, we immediately see from equation (13) that we have a useful observational criterion for flatness of an FLRW universe:

\[ \Omega_0 = 1 \Rightarrow (1 + z_{\text{max}})^{-3} \left[ \frac{1}{H_0^2 C_{\text{max}}^2} - \Omega_\Lambda \right] + \Omega_\Lambda - 1 = 0, \quad (16) \]

Thus, if already know that the universe is flat, or nearly so, and we know both \( z_{\text{max}} \) and \( C_{\text{max}} \), we can directly determine \( \Omega_\Lambda \), and therefore \( \Omega_M \) itself from equation (16).

Proceeding on, then, equation (13) can therefore be substituted into the non-flat version of equation (10), which is the same as equation (10), except that its right-hand-side is identical to right-hand-side of equation (15) without the \( R_0 H_0 \) factors in the denominator (these have cancelled out, as before). Thus, we have, finally, the resulting algebraic relationship involving \( C_{\text{max}}, z_{\text{max}}, H_0 \) and \( \Omega_\Lambda \) as the general FLRW relationship corresponding to the flat case given in equation (10):

\[ \frac{g}{\Omega_\Lambda^{1/2}} \left[ F(\phi, k) \left|_{(1+z)^{-1}=1} \right. - F(\phi, k) \left|_{(1+z_{\text{max}})^{-1}} \right. \right] \]

\[ = \frac{1 + z_{\text{max}}}{\sqrt{\Omega_\Lambda + \Omega_M (1 + z_{\text{max}})^3 - (\Omega_0 - 1)(1 + z_{\text{max}})^2}}. \quad (17) \]

Here and in the solution of the Friedmann equation for the general FLRW case, the parameters associated with that solution are now given by:

\[ \phi_{(1+z)^{-1}} = \cos^{-1} \left[ \frac{(A - B) - (A + B)A(1 + z)}{(A + B) - (A + B)A(1 + z)} \right], \]

\[ k^2 = \frac{(A + B)^2 - (a - b)^2}{4AB}, \]

\[ g = \frac{1}{\sqrt{AB}}, \]

with \( a \equiv -\frac{\Omega_0 - 1}{\Omega_\Lambda} \), \( b \equiv \frac{\Omega_M}{\Omega_\Lambda} \), and
\[ A^2 = \bar{A}^2 + \bar{B}^2 - \bar{A}\bar{B}, \]
\[ B^2 = 3(\bar{A}^2 + \bar{B}^2) + 3\bar{A}\bar{B}. \]

Here, further,

\[
\bar{A} = \left\{ \frac{\Omega_M}{2\Omega_\Lambda} + \left[ \frac{\Omega_M^2}{4\Omega_\Lambda^2} - \frac{(\Omega_0 - 1)^3}{27\Omega_\Lambda^3} \right]^{1/2} \right\}^{1/3},
\]
\[
\bar{B} = \left\{ \frac{\Omega_M}{2\Omega_\Lambda} - \left[ \frac{\Omega_M^2}{4\Omega_\Lambda^2} - \frac{(\Omega_0 - 1)^3}{27\Omega_\Lambda^3} \right]^{1/2} \right\}^{1/3}.
\]

In these equations, remember that \( \Omega_M \) is given by equation (13), so that relationship given by equation (17) is indeed an algebraic relationship involving \( C_{max}, z_{max}, H_0 \) and \( \Omega_\Lambda \). Thus, if both \( C_{max} \) and \( z_{max} \), together with \( H_0 \), are all known from data, then equation (17) will determine \( \Omega_\Lambda \), the only unknown. Using that result in equation (13) will also determine \( \Omega_M \). Thus, observational determination of both \( C_{max} \) and \( z_{max} \), will determine both \( \Omega_M \) and \( \Omega_\Lambda \), as long as we also know \( H_0 \).
| $\Omega_\Lambda$ | $z_{\text{max}}$ | $\Omega_\Lambda$ | $z_{\text{max}}$ | $\Omega_\Lambda$ | $z_{\text{max}}$ | $\Omega_\Lambda$ | $z_{\text{max}}$ |
|----------------|-----------------|-----------------|-----------------|----------------|-----------------|----------------|----------------|
| 0.59           | 1.50            | 0.65            | 1.55            | 0.71           | 1.62            | 0.77           | 1.71           |
| 0.60           | 1.51            | 0.66            | 1.56            | 0.62           | 0.68            | 1.58            | 0.74           | 1.66           | 0.80           | 1.76           |
| 0.61           | 1.51            | 0.67            | 1.57            | 0.73           | 1.64            | 0.79           | 1.74           |
| 0.62           | 1.52            | 0.69            | 1.58            | 0.75           | 1.67            | 0.81           | 1.78           |
| 0.63           | 1.53            | 0.70            | 1.61            | 0.76           | 1.69            | 0.82           | 1.81           |

Table 1: List of pairs $(\Omega_\Lambda, z_{\text{max}})$ for $0.59 \leq \Omega_\Lambda \leq 0.82$ and $1.5 \leq z_{\text{max}} \leq 1.81$. 
Figure 1: Plot of $\Omega_\Lambda - z_{\text{max}}$, given by equation (10), which is for a flat FLRW universe. Here $z_{\text{max}}$ is the redshift at which the maximum of the angular diameter distance, $C_{\text{max}}$, occurs.
4 Observational Prospects and Conclusion

What are the prospects for actually determining $C_{\text{max}}$ and $z_{\text{max}}$ from observations? We would certainly need precise SN Ia luminosity-distance, or ultra-compact radio-source angular-diameter distance, and redshift data out to $z \approx 1.8$ or so. In the SN Ia case this would require careful, long-range programs using space-telescopes. However, as already mentioned, we already have detected and measured SN Ia out to $z \approx 1.7$, and in a recent assessment (Davis, Schmidt and Kim 2006), precision SN Ia measurements to $z \approx 1.8$ are considered attainable. This is already considered an important goal, in order to confirm at what redshift (and cosmic epoch) the universe made the transition from deceleration to acceleration. It is certainly fortuitous that the same redshift range promises to provide a strong independent test of the concordance FLRW model we have derived from CMB, SN Ia, and large-scale structure measurements.

Here we have provided a brief presentation of the straightforward relationship (first found in numerical form by Krauss and Schramm (1992)) between the present value of $\Omega_\Lambda$ and the redshift $z_{\text{max}}$ at which the angular-diameter (or observer area) distance $C$ occurs in a flat FLRW cosmology, like that which apparently models our universe. Furthermore, we have generalized this to non-flat FLRW cases, adding the $C_{\text{max}}$ measurements themselves. In doing this we have derived the characteristic FLRW observational relationships in closed form for $C(z)$ and $M_0(z)$ in the $\Lambda \neq 0$ case, and found a very simple and potentially useful observational criterion for flatness. These results promise to provide improved determination of the best fit cosmological model, or a strong consistency test of it, (depending on how the relationship and the data supporting it are used), once we have enough precise high-redshift luminosity-distance (or angular-diameter distance) data. That should be possible in the near future with the rapid progress being made in SN Ia measurements from space. If the concordance model – a nearly flat universe with $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$ – is approximately correct, we should find observationally that $z_{\text{max}} \approx 1.64$.

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References

Albrecht, A., et al., 2006, Report of the Dark Energy Task Force, astro-ph/0609591.
Benitez, N., Riess, A. G., Nugent, P., Dickinson, M., Chornook, R., & Filippenko, A. V. 2002, ApJ, 577, L1.
Bennett, C. L., et al. 2003, ApJS, 148, 1.
Byrd, P. F. & Friedman, M. D. 1954, Handbook of Elliptic Integrals for Engineers and Physicists, Springer Verlag.
Carroll, S. M., Press, W. H., & Turner, E. L., 1992, “The Cosmological Constant,” Ann. Rev. Astron. & Astrophys. 30, 499-542.
Efstathiou, G., et al. 2002, MNRAS, 330, L29.
Davis, T. M., Schmidt, B. P. & Kim, A. G. 2006, PASP, 118, 205.
Ellis, G. F. R. 1971, “Relativistic Cosmology,” in General Relativity and Cosmology, Proc. Int. School Phys. “Enrico Fermi,” R. K. Sachs, editor (New York: Academic Press), pp. 104-182 (see especially pp. 153-1540.
Ellis, G. F. R., Nel, S. D., Maartens, R., Stoeger, W. R., & Whitman, A. P. 1985, Phys. Reports, 124 (No. 5 and 6), 315.
Ellis, G. F. R. & Tivon, G. 1985, Observatory, 105, 189.
Ellis, G. F. R. & Stoeger, W. R. 1987. Class. Quantum Grav., 4, 1697.
Etherington, I. M. H. 1933, Phil. Mag., 15, 761.
Gilliland, R. L., Nugent, P. E., & Phillips, M. M. 1999, ApJ, 521, 30.
Hellaby, C. W. 2006, MNRAS, 370, 239 (astro-ph/0603637).
Hoyle, F., 1961, in Moller, C., ed., Proc. Enrico Fermi School of Physics, Course XX, Varenna, Evidence for Gravitational Theories, Academic Press, New York, p. 141.
Jackson, J. C. & Dodgson, M. 1997, Mon. Not. R. Astron. Soc., 285, 806.
Jackson, J. C. 2004, JCAP, 11, 007.
Jeffrey, A., 1995, Handbook of Mathematical Formulas and Integrals, Academic Press, Inc., pp. 225-234.
Krauss, L. M., and Schramm, D. N. 1993, ApJ, 405, L43.
McCrea, W. H., 1935, Z. Astrophys., 9, 290.
Mortsell, E., Gunnarson, C., & Goobar, A. 2001, ApJ 561, 106.
Peacock, J. A., et al. 2001, Nature, 410, 169.
Peebles, P. J. E., 1993, Principles of Physical Cosmology, Princeton University Press, Princeton, NJ, pp. 325-329.
Percival, W. J., et al. 2001, MNRAS, 327, 1297.
Perlmutter, S., et al. 1999, ApJ, 517, 565.
Refsdal, S., Stabell, R., & de Lange, F. G. 1967, Mem. R. Astron. Soc., 71, 143.
Riess, A. G., et al. 1998, AJ, 116, 1009.
Riess, A. G., et al. 2001, ApJ, 560, 49.
Riess, A. G., et al. 2004, ApJ, 607, 665.
Spergel, D. N., et al. 2003, ApJS, 148, 175.
Stoeger, W. R., Ellis, G. F. R. & Nel, S. D. 1992, Class. Quantum Grav., 9, 509.