FIELD ANTI-FIELD DUALITY, P-FORM GAUGE FIELDS
AND
TOPOLOGICAL FIELD THEORIES

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Abstract  We construct a framework which unifies in dual pairs the fields and
anti-fields of the Batalin and Vilkovisky quantization method. We consider gauge
theories of p-forms coupled to Yang-Mills fields. Our algorithm generates many
topological models of the Chern-Simon type or of the Donaldson-Witten type. Some
of these models can undergo a partial breaking of their topological symmetries.
1 Introduction

The field and anti-field formalism which has been developed by Batalin and Vilkovisky (B-V) attracts more and more attention \[1\] \[2\]. Anti-fields were first introduced in the context of the renormalization of the Lagrangian of Yang-Mills gauge theories as the sources of the local operators representing the BRST variations of the propagating fields, with a master equation which controls their renormalization \[3\]. Later on, after earlier developments in Hamiltonian formalism, Batalin and Vilkovisky have built a Lagrangian formalism, where one associates for each one of the classical and ghost fields, collectively denoted as \(\phi\), an anti-field \(\phi^*\) \[1\].

In the construction of Batalin and Vilkovisky, the local action which determines the quantum theory is a local functional \(S[\phi, \phi^*]\) satisfying the B-V graded master equation

\[
\frac{\delta S}{\delta \phi} \frac{\delta S}{\delta \phi^*} \pm \frac{\delta S}{\delta \phi} \frac{\delta S}{\delta \phi^*} = 0 \quad (1.1)
\]

The definition of the graded differential BRST operator \(s\) is

\[
s\phi = \frac{\delta S[\phi, \phi^*]}{\delta \phi^*} \quad s\phi^* = \frac{\delta S[\phi, \phi^*]}{\delta \phi} \quad (1.2)
\]

\(S[\phi, \phi^*]\) has ghost number zero by assumption. Thus, if the field \(\phi\) has ghost number \(g\), its anti-field \(\phi^*\) has ghost number \(-g - 1\).

The B-V equation takes its most transparent form if one defines the following graded bracket acting in the space of functionals of \(\phi\) and \(\phi^*\)

\[
\{X, Y\} = \frac{\delta X}{\delta \phi} \frac{\delta Y}{\delta \phi^*} \pm \frac{\delta Y}{\delta \phi} \frac{\delta X}{\delta \phi^*} \quad (1.3)
\]

where \(X\) and \(Y\) are functionals. Then one has

\[
s = \{S, \quad \} \quad (1.4)
\]

and the B-V equation is

\[
sS = \{S, S\} = 0 \quad (1.5)
\]

With this notation, the nilpotency property

\[
s^2\phi = s^2\phi^* = 0 \quad (1.6)
\]

is an obvious consequence of the B-V master equation and of the graded Jacobi identity of the bracket \(\{\quad , \quad \}\).
Reciprocally, if there is a way to define directly an operation $s$ acting on a set of fields and anti-fields $\phi$ and $\phi^*$ with the property $s^2 = 0$, and if a local functional $S[\phi, \phi^*]$ exists with $sS = 0$, then one has $s = \{ S, \}$, that is, eq. (1.2) holds, and $S[\phi, \phi^*]$ can be identified with the B-V action. This property will be used systematically in this paper: we will construct from an algebraic principle the BRST symmetry for large classes of gauge theories of forms coupled to a Yang-Mills field and deduce only afterwards the B-V action $S[\phi, \phi^*]$ and the classical action $S[\phi, 0]$.

One of the great advantages of the B-V formalism is of permitting a consistent quantization of actions invariant under gauge symmetries whose transformation laws close only modulo some of the equations of motion. Examples of such symmetries include supergravities, models with non-abelian form gauge fields with degree larger than one and open string field theory \cite{5} [6].

From the point of view of quantum field theory, fields and anti-fields seem to play dissymmetric roles. In our present understanding, anti-fields are not quantum fields: they are to be eliminated from the action through the choice of a local gauge function $Z(\phi)$ with ghost number minus one by mean of the constraint

$$\phi^* = \frac{\delta Z[\phi]}{\delta \phi}$$  \hspace{1cm} (1.7)

With appropriate choices of $Z$, $S(\phi, \phi^* = \frac{\delta Z[\phi]}{\delta \phi})$ becomes a consistently gauge fixed action which contains generally higher-order ghost interactions. Formal proofs which are based on the nilpotency of the operation $s$ before the elimination of anti-fields show that physical quantities do not depend on the choice of the functional $Z(\phi)$ \cite{1} \cite{2}. The classical action $S_0[\phi] = S[\phi, 0]$ is invariant under the restricted part $s_0$ of the BRST symmetry operator defined by $s_0 \phi = s\phi|_{\phi^* = 0}$. In general, the nilpotency of $s_0$ is broken by terms proportional to the equations of motion, a property which originates in the fact that the BRST variations of the anti-fields involve the equations of motion of $S_0[\phi]$.

In a previous attempt to incorporate the B-V formalism in a geometrical picture, some kind of unification between fields and anti-fields has been shown to exist in particular cases \cite{3} \cite{4}.

In this paper we will obtain a more general result. By considering the gauge theories of forms, including Yang-Mills and scalar fields, we will show that a sort of duality exists between the fields and the anti-fields of the B-V quantization in the framework of a beautiful algebraic structure. More precisely, given a $p$-form gauge field in $D$-dimensional space, valued in a given
Lie group representation, we will show that it has a natural “dual” companion which is a
$D - 1 - p$ gauge field. The argument is that the anti-fields of $B_p$ and of its ghosts and ghosts
of ghosts are contained in an expansion which includes negative ghost number components for
$B_{D-1-p}$, and vice-versa. This implies in particular that the natural companion of a Yang-Mills
field in 4 dimensions is a 2-form gauge field. Furthermore, we will show that quite simple
algebraic formulae determine the BRST equations as constraints on curvatures and, eventually,
the B-V actions.

The formalism that we shall present generates in a systematic way many topological actions
functions of $p$-form gauge fields. They are generalizations of the Chern-Simon action and/or
of the Donaldson-Witten action [10]. We get therefore theories which are either defined from
classical Hamiltonians which vanish up to gauge transformations, or from the gauge-fixing of
classical actions which are equivalent to topological terms.

There is a simple explanation for this possibility of unifying the fields and the anti-fields
presented in this paper. Indeed, the BRST formalism has to do with a superfield formalism
in a superspace $\{x^\mu, \theta\}$, where $\theta$ is a scalar Grassman variable and $x^\mu$ are the ordinary real
coordinates of the D-dimensional space. Forms should be expanded on monomials products
of $dx^\mu$ and $d\theta$. Since $dx^\mu$ is odd, the ordinary form degree of any given form can only take
integer values between 0 and $D$. On the other hand, $d\theta$ is a commuting object, and we have the
freedom to consider monomials of the type $(d\theta)^g$ with no restriction on the possible values of $g$.
In particular, $g$ can be a negative integer. Our proposal is that anti-fields must be identified as
forms with a negative ghost number (which should not be confused with the antighost number).

To be more precise, let us consider the tangent plane defined above the point with local
coordinates $(x^\mu, \theta = 0)$. One has the following decomposition for a $p$-form $\tilde{B}_p(x, \theta = 0)$ living
in this space

$$\tilde{B}_p(x, \theta = 0) = \sum_{q=0}^{D} B_{q}^{p-q}(x)$$

with

$$B_{q}^{p-q}(x) = \frac{1}{q!} B_{\mu_1 \cdots \mu_q}(x) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \, (d\theta)^{p-q}$$

The physical interpretation of this equation is that $B_0^p(x)$ is a classical $p$-form gauge field
and that the fields $B_q^{p-q}(x)$, with $0 \leq g \leq p - 1$, are the ghosts and ghosts of ghosts of $B_p(x)$.
The fields $B^{p}_{q}(x)$, for $p - D \leq p - q \leq -1$, have a negative ghost number, counted by the negative power of $(d\theta)^{p-q}$. The aim of this paper is to identify these new fields as the anti-fields of a “dual” $(D-1-p)$-form gauge field and of its ghosts and ghosts of ghosts. This is a consistent identification because (i) the anti-field of a field with ghost number $g$ has ghost number $-g-1$ and (ii) forms with ordinary form degree $q$ or $D-q$ contain as many independent Lorentz components in $D$-dimensional space.

Before giving the details of our general construction and some examples of interest, we will briefly list a few properties of $p$-form gauge fields.

## 2 Properties of $p$-form gauge fields

Consider a classical $p$-form gauge field valued in a given Lie Algebra $\mathcal{G}$

$$B_{p}(x) = \frac{1}{p!} B_{\mu_{1} \cdots \mu_{p}}(x) dx^{\mu_{1}} \wedge \cdots \wedge dx^{\mu_{p}}$$  \hspace{1cm} (2.1)

This form contains $C_{D}^{p}$ independent components in $D$-dimensional space. It is expected that a gauge field is massless and truly lives in a $(D-2)$-dimensional space, that is, in the hyperplane transverse to its propagation. Thus, by generalizing Feynman argument, one must introduce ghosts and anti-ghosts to add up positive and negative degrees of freedom and obtain a system of fields counting for an effective number of degrees of freedom equal to $C_{D-2}^{p}$. This can be achieved by extracting the ghosts and antighosts from the following expansion

$$B_{p} \rightarrow \tilde{B}_{p} = \sum_{g=0}^{p} \sum_{q=0}^{g} B_{p-g}^{q-g,q}$$  \hspace{1cm} (2.2)

The upper indices $g - q$ and $q$ are respectively the ghost number and the anti-ghost number of the form $B_{p-g}^{q-g,q}$, which has ordinary form degree $p - g$.

Let us briefly justify this decomposition, the goal of which is to subtract unwanted degrees of freedom. One defines the total degree of a field as the sum of its usual form degree and of its ghost and anti-ghost numbers. In this sense, each term of the series of fields which defines $\tilde{B}_{p}$ is a $p$-form. Moreover, one defines the statistics of the field $B_{\mu_{1} \cdots \mu_{p-g}}^{q-g,q}$ as even (resp. odd) if $g$ is even (resp. odd). This definition would become a tautology in the superspace notation of eq. (1.9) with an additional $\theta$ direction to accommodate for the anti-ghost components.
With these definitions, the number of propagating “physical” degree of freedom $N_D(p)$ of the $p$-form gauge field in $D$-dimensional space is obtained by adding the degrees of freedom of each field occurring in the definition of $\tilde{B}_p$ with an algebraic weight 1 for the field components with even statistics and $-1$ for those with odd statistics. It is rather simple to find the following formula for $0 \leq p \leq D - 2$

$$N_D(p) = \sum_{g=0}^{p} (-)^g(1 + g)C_{p-g}^g = C_{D-2}^p$$

and

$$N_D(D - 1) = N_D(D) = 0.$$  

This is the wanted property which ensures that we have a system of fields which amounts to a $p$-form existing in the transverse plane with $D - 2$ dimensions. Obviously $(D - 1)$- and $D$-forms carry no degree of freedom in $D - 2$ dimension space, which explains physically the result $N_D(D - 1) = N_D(D) = 0$.

In what follows, we will forget the antighost components, since the non trivial sector of the BRST symmetry is only for the fields with no anti-ghost component, of the type $B_{g,0}^p$. The extension to the anti-ghost sector of the formulae that we will derive would be obvious by introducing Nakanishi-Lautrup type auxiliary fields and an anti-BRST operation, but it is not the subject of this paper.

We shall therefore focus on the possible ways of defining the BRST operator on the following object

$$\hat{B}_p = \sum_{g=0}^{p} B_{p-g}^g.$$  

We will find equations which determine directly $s\hat{B}_p$, the remaining obvious task being the decomposition in ghost number of these equations.

In \cite{4}, it was shown that, given a collection of forms $B_{p_i}^{(i)}$, the determination of the BRST operator can be cast into the construction of curvatures

$$ \hat{G}_{p+1}^{(i)} = (d + s)\hat{B}_{p+1}^{(i)} + K_{p+1}^{(i)}(\hat{B})$$

upon which one puts "horizontality" constraints

$$ \hat{G}_{p+1}^{(i)} = \frac{1}{(p + 1)!} G_{\mu_1 \ldots \mu_{p+1}}^{(i)}(x) \ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}$$
provided that the form of the field polynomials $K_{p+i}^{i}$ is compatible with the Bianchi identity stemming from

$$s^2 = 0 \quad sd + ds = 0 \Leftrightarrow (s + d)^2 = 0.$$  \hspace{1cm} (2.8)

This construction can be applied to the ordinary Yang-Mills theory and also to less trivial examples such that the theories of forms coupled to Yang-Mills fields through a Chern-Simon term as in the Chapline-Manton symmetry \[8\] as well as to theories invariant under reparametrization and local supersymmetry \[9\]. However, there are cases for which this approach is not fully satisfying, because it provides a BRST symmetry whose nilpotency is broken by terms proportional to the gauge covariant equations of motions. Such cases require that one uses the B-V formalism.

The content of the next sections is thus to show that the freedom of introducing in the expansion of a $p$-form gauge fields objects with negative ghost numbers permits one to reconcile the B-V approach and the algebraic framework.

3 Field anti-fields unification of Yang-Mills fields and (D-2)-forms

Let us consider $D$-dimensional space. The basic object that we must introduce is a Yang-Mills field $A = A_{\mu}dx^{\mu}$ valued in a Lie algebra $G$.

According to the first section of this paper, one should consider the following generalized $G$-valued one-form:

$$\tilde{A}(x) = A_{D}^{-D} + A_{D-1}^{-D} + \cdots + A_{2}^{-2} + A_{1}^{-1} + A + c$$  \hspace{1cm} (3.1)

The field $c$ is the Faddeev-Popov ghost of $A$. Since the anti-field of a field with ghost number $g$ has ghost number $-g - 1$ and since $A_{p}^{-p+1}$ has the same number of Lorentz components as a form with ordinary form degree $(D-p)$, one can identify the fields with negative ghost numbers $A_{2}^{-1}, A_{3}^{-2}, A_{4}^{-3}, \ldots, A_{D}^{-D+1}$ as the anti-fields of a $G$-valued $(D-2)$-form $B_{D-2}$, of its ghost $B_{D-3}^{1}$ and of its ghosts of ghosts $B_{D-4}^{2}, \ldots, B_{0}^{D-2}$.

It is therefore natural to introduce another fundamental form which is a generalized $G$-valued
(D-2)-form based on a (D-2)-form gauge field $B_{D-2}$

$$\tilde{B}_{D-2}(x) = B_0^{D-2} + B_1^{D-3} + \cdots + B_{D-3}^{1} + B_{D-2}^{1} + B_{D-1}^{-1} + B_{D}^{-2}$$  \hspace{1cm} (3.2)$$

with

$$(B_{D-2-p}^{p})^* = A_{p+2}^{p-1} \quad p \geq 0 \hspace{1cm} (3.3)$$

It is rewarding that the objects with negative ghost numbers in the expansion of the 2-form $\tilde{B}_{D-2}$, namely $B_{D-1}^{-1}$ and $B_{D}^{-2}$, can be considered respectively as the anti-fields of the Yang-Mills one form $A = \tilde{A}_1^1$ and of its Faddeev-Popov ghost $c = \tilde{A}_0^1$

$$(B_{D}^{-2})^* = c \quad (B_{D-1}^{-1})^* = A \hspace{1cm} (3.4)$$

One has therefore the following field anti-field relations between $\tilde{A}$ and $\tilde{B}_{D-2}$

$$(A_{p}^{1-p})^* = B_{D-p}^{p-2} \hspace{1cm} (3.5)$$

and

$$A_{p}^{1-p} = (B_{D-p}^{p-2})^* \hspace{1cm} (3.6)$$

for $0 \leq p \leq D$. This relation is an involution, and we find it appropriate to call it a duality relation.

We desire to find algebraic equations generalizing the algorithm of [4] which only involve $\tilde{A}$ and $\tilde{B}$ and determine the action of the possible BRST operators $s$ on all fields and anti-fields. For this purpose, we define

$$D\tilde{A} = d + [\tilde{A}, ]$$

$$F\tilde{A} = d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] \hspace{1cm} (3.7)$$

and

$$D = s + D\tilde{A} = d + s + [\tilde{A}, ]$$

$$F = (s + d)\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = s\tilde{A} + F\tilde{A} \hspace{1cm} (3.8)$$

The relation

$$(s + d)^2 = 0 \hspace{1cm} (3.9)$$

amounts to the equation

$$DD = [F, ] \quad DF = 0 \hspace{1cm} (3.10)$$
Therefore, to determine consistently the action of \( s \), that is with \((s + d)^2 = 0\), we must simply put constraints on the generalized curvatures \( \mathcal{F} \) and \( \mathcal{D} \mathcal{B}_{D-2} \) of \( \tilde{A} \) and \( \tilde{B}_{D-2} \) which are compatible with eq. (3.10).

For generic values of the space dimension, we have the solution

\[
\begin{align*}
\mathcal{F} &= 0 \\
\mathcal{D} \mathcal{B}_{D-2} &= 0
\end{align*}
\] (3.11)

that is

\[
\begin{align*}
-s \tilde{A} &= F \tilde{A} \\
-s \tilde{B}_{D-2} &= D \tilde{A} \tilde{B}_{D-2}
\end{align*}
\] (3.12)

The expansion in ghost number of \( F \tilde{A} \) is

\[
F \tilde{A} = \frac{1}{2} [c, c] + D^{A} c + (dA + \frac{1}{2} [A, A] + [A^{-1}_2, c]) \\
+ (D^{A} A^{-1}_2 + [c, A^{-1}_3]) + \cdots + [c, A^{-D+1}_D]
\] (3.13)

Thus, eqs. (3.11) give the following expression for the nilpotent BRST transformations of all fields and anti-fields

\[
\begin{align*}
sc &= -\frac{1}{2} [c, c] \\
\quad \quad \quad sA &= -D^{A} c \\
\quad \quad \quad sA^{-1}_2 &= s(B_{D-2})^* = -F^{A} - [c, A^{-1}_2] \\
\quad \quad \quad \vdots \\
\quad \quad \quad sA^{-D+1}_D &= s(B_{D-2}^{D-2})^* = -D^{A} A^{-D+2}_D - [c, A^{-D+1}_D] - [A^{-1}_2, A^{-D+2}_D] - \cdots
\end{align*}
\]

and

\[
\begin{align*}
\quad \quad \quad sB^{-D-2}_0 &= -[c, B^{-D-2}_0] \\
\quad \quad \quad sB^{-3}_1 &= -[c, B^{-3}_1] - D^{A} B^{-2}_0 \\
\quad \quad \quad sB^{-1}_2 &= -[c, B^{-1}_2] - D^{A} B^{-3}_1 - [A^{-1}_2, B^{-2}_0] \\
\quad \quad \quad \vdots
\end{align*}
\]
\[ sB_{D-2} = -[c, B_{D-2}] - D^A B_{D-3}^1 - [A_2^{-1}, B_{D-4}^2] - \cdots + [A_{D-2}^{-D+3}, B_{0}^{D-2}] \]

\[ sB_{D-1}^{-1} = s(A^*) = -[c, B_{D-1}^{-1}] - D^A B_{D-2} - \cdots \]

\[ sB_{D}^{-2} = s(c^*) = -[c, B_{D}^{-2}] - D^A B_{D-1}^{-1} - \cdots \] (3.14)

If one sets equal to zero the anti-fields one gets the intuitive BRST transformation of a 2-form gauge field \( s_0 B_{D-2} = -[c, B_{D-2}] - D^A B_{D-1}^{D-3} \). \( s_0 \) is nilpotent only for \( F^A = 0 \).

To find the B-V action corresponding to the symmetry defined in eqs. (3.14), we observe that eqs. (3.12) imply the following cocycle equation

\[ (d + s)T_r(\tilde{F} \wedge \tilde{B}_{D-2}) = 0 \] (3.15)

Thus the invariant B-V action is

\[ S[\tilde{A}, \tilde{B}_{D-2}] = \int T_r \left[ \tilde{B}_{D-2} \wedge F^A \right]_D \] (3.16)

By expansion in field components of the forms \( \tilde{A} \) and \( \tilde{B}_{D-2} \), one gets

\[ S[\tilde{A}, \tilde{B}_{D-2}] = \int T_r \left( B_{D-2} \wedge F^A - \frac{1}{2} B_{D-2}^{-2}[c, c] - B_{D-1}^{-1} D^A c \right. \]

\[ + A_2^{-1} \wedge (-[c, B_{D-2}] - D^A B_{D-3}^1 \]

\[ \left. - [A_2^{-1}, B_{D-4}^2] - \cdots + [A_{D-2}^{-D+3}, B_{0}^{D-2}] \right) \] (3.17)

By setting all anti-fields equal to zero, one can verify in particular that the classical action is a BF action [11].

One can add to the B-V action \( S[\tilde{A}, \tilde{B}_{D-2}] \) a gauge invariant action \( S_{cl}[A] = \int L_{cl}(A) \), for instance \( \int F_{\mu\nu}^2 d^4x \) or a Chern-Simon action (for odd values of the space dimension D). Then one should replace the BRST symmetry defined in eqs. (3.11) by

\[ \mathcal{F} = 0 \]

\[ \mathcal{D} \tilde{B}_{D-2} = \frac{\delta S_{cl}}{\delta A} \] (3.18)

This modification is apparently spurious, since the equation of motion of the field \( B_{D-2} \) implies the vanishing curvature equation \( F^A = 0 \). We will come back on this point in the last section and see how the vanishing curvature condition could be mildened.
4 Coupling to p-form gauge fields: Chern-Simon type actions

Let us now introduce a $p$-form gauge field $X_p$ in addition to the dual pair $(\tilde{A}, \tilde{B}_{D-2})$. The integer $p$ is such that $0 \leq p \leq D - 1$. We must consider the following field anti-field decomposition comparable to eq. (1.8)

$$\tilde{X}_p = X_D^{p-D} + X_{D-1}^{p+1-D} + \cdots + X_{p+1}^{-1} + X_p + X_{p-1}^1 + \cdots + X_0^p$$

(4.1)

To interpret the anti-fields occurring in eq. (4.1), we introduce the dual form $\tilde{Y}_{D-1-p}$, such that

$$\tilde{Y}_{D-1-p} = Y_0^{D-1-p} + Y_1^{D-2-p} + \cdots + Y_{D-p-2} + Y_{D-p-1} + Y_{D-p} + \cdots + Y_D^{1-p}$$

(4.2)

The BRST equations are defined by

$$F = s\tilde{A} + d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = 0$$

$$\mathcal{D}\tilde{B}_{D-2} = s\tilde{B}_{D-2} + D\tilde{A}\tilde{B}_{D-2} = [\tilde{X}_p, \tilde{Y}_{D-1-p}]$$

$$\mathcal{D}\tilde{X}_p = s\tilde{X}_p + D\tilde{A}\tilde{X}_p = 0$$

$$\mathcal{D}\tilde{Y}_{D-1-p} = s\tilde{Y}_{D-1-p} + D\tilde{A}\tilde{Y}_{D-1-p} = 0$$

(4.3)

The fact that these equations define $s$ with $s^2 = 0$ is easily verified from the identities obtained by applying $\mathcal{D}$ on both sides of eqs. (4.3), using the equations $F = \mathcal{D}\mathcal{D} = 0$ and $\mathcal{D}F = 0$.

The existence of a B-V equation follows from the equation

$$(s + d)\tilde{L}_D = 0$$

(4.4)

with

$$\tilde{L}_D = T_r(\tilde{B}_{D-2} \wedge F\tilde{A} + \tilde{X}_p D\tilde{A}\tilde{Y}_{D-1-p})$$

(4.5)

The B-V action is thus

$$S[\phi, \phi^*] = \int \tilde{L}_D^0 = \int T_r \left[ B_{D-2} \wedge F\tilde{A} + X_p \wedge D\tilde{A} Y_{D-p-1} \right. + \sum_{q \neq 0} B_{D-2-q}^q \wedge F\tilde{A}^{1-q}_{q+2} + \sum_{q \neq 0} X_{p-q}^q \wedge (D\tilde{A}\tilde{Y}_{D-p-1})^{1-q}_{D+q-p}]$$

(4.6)
It is a simple exercise to verify that the BRST transformations stemming from this B-V action are identical to those following from the constraints defined in eqs. (4.3).

To understand the nature of the model let us consider the classical action

$$S_{cl} = S[\phi, \phi^* = 0] = \int T_r \left( B_{D-2} \wedge F^A + X_p \wedge D^A Y_{D-2} \right)$$

(4.7)

$S_{cl}$ is invariant under the following gauge symmetry, obtained by equating to zero all anti-fields and by replacing in the BRST transformations of the classical fields the primary ghosts $c, B_{D-3}, X_{p-1}, Y_{D-p-2}$ by infinitesimal parameters $\epsilon, \epsilon_{D-3}, \epsilon_{p-1}, \epsilon_{D-p-2}$

$$\delta A = D^A \epsilon$$

$$\delta B_{D-2} = D^A \epsilon_{D-3} + [\epsilon, B_{D-2}] + [\epsilon_{p-1}, Y_{D-p-1}] + [X_p, \epsilon_{D-p-2}]$$

$$\delta X_p = D^A \epsilon_{p-1} + [\epsilon, X_p]$$

$$\delta Y_{D-1-p} = D^A \epsilon_{D-2-p} + [\epsilon, Y_{D-1-p}]$$

(4.8)

The classical equations of motion are

$$F^A = 0$$

$$D^A X_p = 0$$

$$D^A Y_{D-1-p} = 0$$

(4.9)

The model is thus quite similar to the Chern-Simon theory.

There is of course the possibility of adding other actions made from several pairs $(\tilde{X}_{p_i}, \tilde{Y}_{D-p_i-1})$, with all possible values of $p_i$. The B-V action is in this case

$$S = \int T_r \left[ \tilde{B}_{D-2} \wedge F^{\tilde{A}} + \sum_i \tilde{X}_{p_i} \wedge D^{\tilde{A}} \tilde{Y}_{D-p_i-1} \right]_D^0$$

(4.10)

5 Topological actions stemming from d-exact Lagrangian

The Yang-Mills topological BRST symmetry is based on the following BRST transformations of the Yang-Mills fields and of its Faddev-Popov ghost

$$s A = -Dc + \Psi$$

$$sc = -\frac{1}{2} [c, c] + \Phi$$

(5.1)
\[ \Psi = \Psi dx^\mu \] is a one-form with ghost number one and \( \Phi \) is a commuting scalar ghost with ghost number two. We will show that this type of symmetry enters naturally in the field anti-field dual framework explained in this paper.

We first consider the case \( D \neq 4 \). In addition to the dual pair \((\tilde{A}, \tilde{B}_{D-2})\), we introduce another \( \mathcal{G} \)-valued dual pair \((\tilde{X}_2, \tilde{Y}_{D-3})\). The expansions in form components of \( \tilde{X}_2 \) and \( \tilde{Y}_{D-3} \) are similar to the expression given in eq. (4.1) and contain all possible fields and anti-fields compatible with the form degrees 2 and \( D - 3 \).

We then define the nilpotent \( s \)-operation by the following constraints compatible with Bianchi identities

\[
\begin{align*}
\mathcal{F} &= s\tilde{A} + d\tilde{A} + \tilde{A}\tilde{A} = \tilde{X}_2 \\
\mathcal{D}\tilde{X}_2 &= s\tilde{X}_2 + D\tilde{A}\tilde{X}_2 = 0 \\
\mathcal{D}\tilde{B}_{D-2} &= s\tilde{B}_{D-2} + D\tilde{A}\tilde{B}_{D-2} = [\tilde{X}_2, \tilde{Y}_{D-3}] \\
\mathcal{D}\tilde{Y}_{D-3} &= s\tilde{Y}_{D-3} + D\tilde{A}\tilde{Y}_{D-3} = 0
\end{align*}
\]

The first equation gives the BRST topological symmetry defined in eq. (5.1) with \( \Psi = \tilde{X}_1 \) and \( \Phi = \tilde{X}_2 \).

Furthermore, the above constraints imply

\[
(s + d) \, T_r(\tilde{B}_{D-2} \wedge (F\tilde{A} + \tilde{X}_2) + \tilde{X}_2 \wedge D\tilde{A}\tilde{Y}_{D-3}) = 0
\]

It follows that the B-V action of the system is

\[
S(\phi, \phi^*) = \int \tilde{T}_r \left[ B_{D-2}(F\tilde{A} + \tilde{X}_2) + \tilde{X}_2 D\tilde{A}\tilde{Y}_{D-3} \right]_D^0
\]

and the classical action is

\[
S_{cl}(\phi, \phi^* = 0) = \int \tilde{T}_r \left( B_{D-2} \wedge (F^A + X_2) + X_2 \wedge D^A\tilde{Y}_{D-3} \right)
\]

The field \( X_2 \) can be eliminated by its equation of motion, with

\[
S_{cl}(\phi, \phi^* = 0) \sim \int \tilde{T}_r \left( D^A\tilde{Y}_{D-3} \wedge F^A \right) = \int d \, T_r \left( \tilde{Y}_{D-3} \wedge F^A \right)
\]

This shows that the B-V action, after the replacement of the anti-fields by antighosts via the standard procedure \cite{2}, should be the gauge fixing of a topological term \( \int d \, T_r \tilde{Y}_{D-3} \wedge F^A \). We
are thus considering theories of the Donaldson-Witten type. As an example, for D=3, one gets
the topological field theory based on the Bogolmony equations \[12\].

Let us consider now the exceptional case \(D = 4\). In this important particular case, \(\tilde{B}_{D-2}\) is
a 2-form and can be used as a substitute for \(\tilde{X}_2\). The system becomes simplest and one has
\[
\mathcal{F} = s\tilde{A} + F\tilde{A} = \tilde{B}_2
\]
\[
\mathcal{D}\tilde{B}_2 = s\tilde{B}_2 + D\tilde{A}\tilde{B}_2 = 0
\]  
(5.7)
One has \((s + d) T_r[\tilde{B}_2 \wedge (F\tilde{A} + \tilde{B}_2)] = 0\). Thus the B-V action is
\[
S[\phi, \phi^*] = \int T_r \left[\tilde{B}_2 \wedge (\tilde{F} + \frac{1}{2}\tilde{B}_2)\right]_4^0
\]  
(5.8)
and the classical action is
\[
S[\phi, \phi^* = 0] = \int T_r \left(\tilde{B}_2 \wedge (F + \frac{1}{2}\tilde{B}_2)\right)
\]  
(5.9)
By eliminating the field \(B_2\) by its equation of motion, one finds \(S[\phi, 0] \sim \int T_r F \wedge F\). This
coincides with the fact that the 4-D Yang-Mills topological action is the gauge fixing of the
second Chern-class \[12\].

To conclude this section, let us remark that we have not considered the possibility that
\(p = D\). This is indeed a very special case, since in the decomposition \(\tilde{X}_D = \sum_{p=0}^D X^p_{D-p}\) one finds
only terms with positive ghost numbers. No anti-field occurs in the decomposition of such a
\(D\)-form. One must therefore introduce a form made of all the anti-fields \(X_{p-1-p}\) of the fields
\(X^p_{D-p}\). Such a generalized differential form deserves the name of a \(-1\)-form gauge field, since
it has the following decomposition
\[
\tilde{Y}_{-1} = X_{-1}^0 + X_{1}^{-1} + \cdots + X_{D-1}^{-D}
\]  
(5.10)
The BRST equations are now
\[
\mathcal{F} = 0
\]
\[
\tilde{D}\tilde{B}_{D-2} = [\tilde{X}_D, \tilde{Y}_{-1}]
\]
\[
\tilde{D}\tilde{X}_D = 0
\]
\[
\tilde{D}\tilde{Y}_{-1} = 0
\]  
(5.11)
and the B-V action action is still \(\int T_r (\tilde{F} \wedge \tilde{B}_{D-2} + \tilde{X}_D \wedge D\tilde{A}\tilde{Y}_{-1})_0^0\). We see however that no
equation of motion exists for \(\tilde{X}_D\), which is in fact absent from the classical action.


6 The case of three-dimensional space

It is worth mentioning the case of $D = 3$ dimensions. In this case, one can consider the components with negative ghost number of $\tilde{A}$ as the anti-fields of its components with positive or zero ghost number. In this sense, $\tilde{A}$ is a “self-dual” potential

$$\tilde{A} = A_3^{-2} + A_2^{-1} + A + c \quad (6.1)$$

with

$$A_3^{-2} = c^* \quad A_2^{-1} = A^* \quad (6.2)$$

The BRST symmetry is defined by

$$\tilde{F} = s\tilde{A} + d\tilde{A} + \tilde{A}\tilde{A} = 0 \quad (6.3)$$

and one has

$$(s + d) T_r \left( \tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}\tilde{A}\tilde{A} \right) = 0 \quad (6.4)$$

Thus, the Batalin-Vilkovisky action is simply

$$S(A, A^*) = \int T_r \left[ \tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}\tilde{A}\tilde{A} \right]_3^0 = \int T_r \left( AdA + \frac{2}{3}A^3 + A_2^{-1}Dc + A_3^{-2}cc \right) \quad (6.5)$$

which is the standart result for the B-V action for the Chern-Simon theory including its invariance under the ordinary Yang-Mills symmetry [3].

It is quite natural to introduce a $G$-valued scalar field $\tilde{\varphi} = \varphi + \varphi_1^{-1} + \varphi_2^{-2} + \varphi_3^{-3}$ with its dual 2-form $\tilde{Y}_2 = Y_3^{-1} + Y_2 + Y_1 + Y_0^2$. To do so we must relax the condition that the Yang-Mills field is "self-dual". We introduce another $G$-valued one-form $\tilde{a}$, distinct from the Yang-Mills one-form $\tilde{A}$, such that $\tilde{a}$ is the dual of $\tilde{A}$. The symmetry is defined now as

$$\mathcal{F} = s\tilde{A} + d\tilde{A} + \tilde{A}\tilde{A} = 0$$

$$\mathcal{D}\tilde{a} = s\tilde{a} + D\tilde{\varphi} = F^A + [\tilde{Y}_2, \tilde{\varphi}]$$

$$\mathcal{D}\tilde{\varphi} = s\tilde{\varphi} + D\tilde{\varphi} = 0$$

$$\mathcal{D}\tilde{Y}_2 = s\tilde{Y}_2 + D\tilde{\varphi} = 0$$

1 For interesting results about the quantization of this action and the correspondance with the unification that we have found here, see ref. [3]

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The corresponding B-V action is

\[
S[\phi, \phi^*] = \int T_r \left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \tilde{a} \wedge F^\tilde{A} + \tilde{Y}_2 \wedge D^\tilde{A} \varphi \right]^0_3
\]

(6.7)

It is instructive enough to write the classical action

\[
\int T_r \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A + a \wedge F^A + Y_2 \wedge D^A \varphi \right)
\]

(6.8)

This action is interesting as a generalized Chern-Simon type action involving couplings of the Yang-Mills to a scalar field and a 2-form gauge field. The equations of motion are \( F^A = 0 \) as in the genuine Chern-Simon theory and \( D^A a = D^A \varphi = D^A Y = 0 \). The BRST symmetry operator \( s_0 \) for the classical action is

\[
\begin{align*}
 s_0 A &= -D^A c \\
 s_0 a &= -D^A a_0 - [c, a] - [Y_1^1, \varphi] \\
 s_0 \varphi &= -[c, \varphi] \\
 s_0 Y_2 &= -D^A Y_1^1 - [c, Y_2]
\end{align*}
\]

(6.9)

The quantization and the gauge-fixing of this action would necessitates that one uses the full symmetry stemming from eq. (6.6), including the anti-fields with \( sY_2 = -DY_1^1 - [c, Y_2] - [A_2^{-1}, Y_2^2] \) and \( sA_2^{-1} = F^A - [c, A_2^{-1}] \).

7 Possible breaking of the topological invariance toward the creation of physical excitations

We have shown in the previous sections a rather general way to produce actions which are of the topological type in the sense that they have vanishing Hamiltonians (up to gauge transformations) or are of the Donaldson-Witten type. In this section we intend to sketch possible scenarios which could break at least partially the topological symmetries of these models and possibly provide models with physical excitations.

From now on, we restrict to \( D = 4 \) dimensions. In a quite generic way, we have been led to consider actions of the type

\[
S_4 = \int T_r \left( \tilde{B}_2 \wedge F^\tilde{A} + \tilde{\varphi} D^\tilde{A} \tilde{C}_3 \right)^0_4
\]

(7.1)
Here $\tilde{\varphi} = \varphi_{-4} + \varphi_{-3} + \varphi_{-2} + \varphi_{-1} + \varphi$ is a generalized 0-form, and $\tilde{C}_3 = C_{-4} + C_{-3} + C_{-2} + C_{-1} + C_0$ is its dual. The action (7.1) determines a theory with a Yang-Mills field coupled to a scalar $\varphi$, a 2-form $B_2$ and a 3-form $C_3$.

This model could be useful for the purpose of computing mathematical quantities from the path integral point of view. However, the addition of gauge invariant terms like $F^2_{\mu\nu}$ and $(D_\mu \varphi)^2$ seems of no relevance, since the equations of motion of $B_2$ and $c_3$ would imply $F_{\mu\nu} = 0$ and $D_\mu \varphi = 0$.

There is a first possibility of getting out of this situation. It consists in freezing the Yang-Mills symmetry, while keeping all other local symmetries. Indeed, everywhere in our formula, we can put $c = 0$, provided one has also $s A = 0$. By doing so, one can add to the Lagrangian the term $A^2$, which yields actions as in Freedman-Townsend model [7]. By eliminating the field $B_2$, the constraint $F^4 = 0$ arises. It can be solved with $A$ equal to a pure gauge, which gives a Lagrangian term $A^2 = (g^{-1} \partial_{\mu\nu} g)^2$. One gets a non-linear sigma model, with possible couplings to $X_p$ and $Y_{D-1-p}$.

The second possibility is to introduce a symmetry breaking mechanism, by adding to the action from an ordinary Higgs potential

$$V(\varphi) = -\mu^2 \varphi^2 + \lambda \varphi^4$$

(7.2)

The symmetry of the action is

$$F = s A + F^A \neq 0$$
$$D_\mu B_2 = s B_2 + D_\mu \tilde{B}_2 = [\tilde{C}_3, \varphi]$$
$$D_\mu \tilde{C}_3 = s \tilde{C}_3 + D_\mu \tilde{C}_3 = \delta^* V / \delta \varphi$$
$$D_\mu \tilde{\varphi} = s \tilde{\varphi} + D_\mu \tilde{\varphi} = 0$$

(7.3)

If the potential $V$ is chosen such that $\langle \varphi \rangle \neq 0$, we get from these equations

$$s B_2 = [\langle \varphi \rangle, c_2] + \cdots$$

(7.4)

This implies that we can gauge fix to zero certain components of the 2-form gauge field $B_2$ along group directions. This might relax the constraints that the Yang-Mills curvature vanishes along these directions. Our claim is thus that one can consider actions of the type

$$S = \int \left( \left[ \tilde{B}_2 \wedge F + \tilde{C}_3 \wedge D_\mu \tilde{\varphi} \right]_4 + d^4x \left( F^2_{\mu\nu} + D_\mu \varphi + V(\varphi) \right) \right)$$

(7.5)
and that after gauge-fixing and symmetry breaking, one obtains effectively

\[ S = \int d^4x \left( \text{tr}\left( F_{\mu\nu}^2 \right) + \ldots + \text{supersymmetric term} \right) \] (7.6)

where \( \text{tr}F_{\mu\nu}^2 \) means the trace of \( F_{\mu\nu}^2 \) in the broken gauge directions and the supersymmetric terms stand for the ghost interactions coming from the gauge-fixing.

A third more elementary possibility for softening the topological invariance is to consider a pure \( B_2 F \) model coupled to a Higgs field

\[ S_4 = \int Tr \left( [\tilde{B}_2 \wedge F^4]_4^0 + d^4x \left( D_\mu \varphi D^\mu \varphi + V(\varphi) \right) \right) \] (7.7)

By symmetry breaking due to the Higgs field, one obtains mass terms for the Yang-Mills field, and thus a Freedman-Townsend model yielding a non-linear sigma model in the broken directions and a topological BF model in the unbroken directions.

8 conclusion

We have shown that the B-V formalism for the gauge theories forms coupled to Yang-Mills forms can be formulated in a unifying algebraic framework. The main idea is to group all relevant fields and anti-fields for the B-V quantization of a p-form gauge field as the components of differential forms which are graded by the sum of the ghost number and ordinary form degree. This suggests that a p-form gauge field comes in a "dual" pair with a (D-p-1)-form gauge field. In this way, we have obtained an algorithm which generates topological actions function of such p-form gauge fields which are of the Chern-Simons and/or Donaldson-Witten type in any given space-time dimension, on the basis of vanishing curvature conditions. We have indicated that some of the models which arise in this straightforward construction could undergo a symmetry breaking mechanism. The latter would soften the requirement that all components of the classical field strenghts vanish classically and possibly determine actions with physical excitations. In a separate publication, we will show how to generalize our observations to the case of 2D reparametrization invariance.

Acknowledgments: The author would like to express his deep gratitude to RIMS for the hospitality extended to him during his stay in Japan.
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