AN UNCOUNTABLE VERSION OF
PTÁK’S COMBINATORIAL LEMMA

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Abstract. In this note we are concerned with the validity of an uncountable analogue of a combinatorial lemma due to Vlastimil Pták. We show that the validity of the result for \( \omega_1 \) cannot be decided in ZFC alone. We also provide a sufficient condition, for a class of larger cardinals.

1. Introduction

In his 1959’s paper [33], Vlastimil Pták distilled a combinatorial lemma aimed at the investigation of weak compactness in Banach spaces. An interesting application of the lemma, and partial motivation for the result itself, was an elementary proof of the fact that if a uniformly bounded sequence of continuous functions \((f_n)_{n=1}^\infty \subseteq C(K)\) converges pointwise to a continuous function \(f\), then \(f\) may be uniformly approximated by convex combinations of the \(f_n\)’s ([35, §2.1]). It is actually a standard exercise in Functional Analysis to understand this assertion as a particular case of Mazur’s theorem that a closed and convex subset of a Banach space is weakly closed. However, this approach requires the Riesz representation theorem for \(C(K)^*\) and Lebesgue’s dominated convergence theorem; therefore, it relies on much deeper principles than the assertion itself.

In later papers, Pták also applied and extended the same combinatorial ideas to include a treatment of weak compactness in terms of inverting the order of two limit processes, cf. [34, 36]. This approach, already present in [35], is also followed in the monograph [25, §24.6], for the proof of Krein’s theorem.

This interesting lemma attracted the attention of the mathematical community, as witnessed by several papers dedicated to its different proofs or extensions; let us mention, among them, [24, 40, 41, 42]. More recently, it was also used—and given a different, Banach space theoretic, proof—in the paper [9]. This proof is also included in [12, Exercise

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14.28], where Pták’s elementary proof of Mazur theorem is also outlined (cf. Exercise 14.29). One further introduction to this result may be found in §I.3, or in the systematic survey by Pták himself.

Let us now proceed to recall the statement of the result under investigation; we shall require a piece of terminology, and we follow Pták’s notation from §I.3. Given a set $S$ and a function $\lambda: S \to \mathbb{R}$ for the support of $\lambda$ we understand the set $\text{supp}(\lambda) := \{ s \in S : \lambda(s) \neq 0 \}$; in the case that $\text{supp}(\lambda)$ is a finite set, we shall say that $\lambda$ is finitely supported.

A convex mean is a finitely supported function $\lambda: S \to [0, \infty)$ such that

$$\sum_{s \in S} \lambda(s) = 1.$$ 

Plainly, a convex mean can also be naturally interpreted as a finitely supported probability measure on $(S, 2^S)$ via the definition $\lambda(A) := \sum_{s \in A} \lambda(s)$, for $A \subseteq S$. In what follows, we shall profit from this notation, whenever convenient.

Given a set $S$, we shall also denote by $[S]^{<\omega}$ the collection comprising all finite subsets of $S$. All the necessary notation being set forth, we are now in position to recall the original statement of Pták’s lemma.

**Lemma 1.1** (Pták’s combinatorial lemma, [33]). Let $S$ be an infinite set and $\mathcal{F} \subseteq [S]^{<\omega}$ be a collection of finite subsets of $S$. Then the following conditions are equivalent:

(i) there exist an infinite subset $H$ of $S$ and $\delta > 0$ such that for every convex mean $\lambda$ with $\text{supp}(\lambda) \subseteq H$ one has

$$\sup_{F \in \mathcal{F}} \lambda(F) \geq \delta;$$

(ii) there exist a strictly increasing sequence of finite sets $(B_n)_{n=1}^{\infty} \subseteq [S]^{<\omega}$ and a sequence $(F_n)_{n=1}^{\infty} \subseteq \mathcal{F}$ such that $B_n \subseteq F_n$, for every $n \in \mathbb{N}$.

Let us observe that the proof of the implication (ii) $\implies$ (i) is immediate, as witnessed by the choice of the set $H := \bigcup_{n=1}^{\infty} B_n$. Therefore, the actual content of the lemma lies in the validity of the implication (i) $\implies$ (ii) and it is precisely this implication to appear in the result as devised in [9].

In order to state this second formulation, we need one more definition. A family $\mathcal{F} \subseteq 2^S$ is said to be hereditary if whenever $F \in \mathcal{F}$ and $G \subseteq F$, then $G \in \mathcal{F}$ too.

**Lemma 1.2** (Pták’s lemma, second formulation, [9]). Let $S$ be an infinite set and let $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family. Assume that there exists $\delta > 0$ such that for every convex mean $\lambda$ on $S$ one has

$$\sup_{F \in \mathcal{F}} \lambda(F) \geq \delta.$$ 

Then there exists an infinite subset $M$ of $S$ such that every finite subset of $M$ is in $\mathcal{F}$.

Observe that, for an hereditary family $\mathcal{F}$, condition (ii) of Lemma 1.1 is equivalent to the conclusion of Lemma 1.2 as a simple verification shows. More precisely, the condition $\mathcal{F}$ being hereditary is only used in the verification that (ii) implies the thesis of Lemma 1.2. Moreover, the assumption in Lemma 1.2 immediately implies (i) with $H = S$ and,
conversely, under the validity of (i), the assumption of Lemma 1.2 is satisfied for the infinite set $H$ and the hereditary family $\{F \cap H : F \in \mathcal{F}\} = \{F \in \mathcal{F} : F \subseteq H\}$.

Consequently, the two statements are formally equivalent. The advantage of the second formulation, from our perspective, is that it immediately suggests its possible generalisations to larger cardinalities, that we shall consider in our note.

In order to have a more succinct formulation of our results, the following definition seems appropriate.

**Definition 1.3.** Let $\kappa$ be an infinite cardinal number. We say that Pták’s lemma holds true for $\kappa$ if for every set $S$ with $|S| = \kappa$ and every hereditary family $\mathcal{F} \subseteq [S]^{<\omega}$ such that

$$\delta := \inf \left\{ \sup_{F \in \mathcal{F}} \lambda(F) : \lambda \text{ is a convex mean on } S \right\} > 0,$$

there exists a subset $M$ of $S$, with $|M| = \kappa$, such that every finite subset of $M$ belongs to $\mathcal{F}$.

Let us now proceed to state our main results.

**Theorem A.** The validity of Pták’s lemma for $\omega_1$ is independent of ZFC. More precisely:

(i) (MA$_{\omega_1}$) Pták’s lemma holds true for $\omega_1$;

(ii) (CH) Pták’s lemma fails to hold for $\omega_1$.

**Theorem B.** Let $\kappa$ be a regular cardinal number such that $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$. Then Pták’s lemma is true for $\kappa$.

Note that, if $\mu$ is any infinite cardinal number, then $\kappa := (2^\mu)^+$ satisfies the assumptions of Theorem B. Consequently, there are in ZFC arbitrarily large cardinal numbers for which Pták’s lemma is true. Moreover, the smallest cardinal Theorem B applies to is $\mathfrak{c}^+$. The next corollary is also a consequence of the theorem.

**Corollary 1.4.** (CH) Pták’s lemma is true for $\omega_2$.

(GCH) If $\tau$ is a cardinal number with $\text{cf}(\tau) > \omega$, then Pták’s lemma is true for $\kappa = \tau^+$.

The paper is organised as follows: in Section 2 we shall present some general observations concerning the condition appearing in Pták’s lemma. These considerations will, in particular, allow us to present the proof of Pták’s original result and are based on the proof given in [9]. In Section 3 we shall prove Theorem A, while Section 4 is dedicated to the proof of Theorem B.

In conclusion to this section, we mention that when the notation $\lambda^\omega$ is used, it will be cardinal exponentiation that is intended. Moreover, all the topological spaces relevant for this note will be Hausdorff and we will tacitly assume this property throughout the paper. We refer, e.g., to [20] for our notation on set theory and to [11, 12, 16] for unexplained notation concerning Banach spaces.
2. General remarks

If \( S \) is any set, a subset \( A \) of \( S \) can be naturally identified, via the correspondence \( A \mapsto \chi_A \), with an element of the compact topological space \( \{0,1\}^S \), endowed with the canonical product topology. Let us recall that, under this identification, if \( A \in 2^S \), a basis of neighborhoods of \( A \) is given by the collection of sets

\[
\{ B \in 2^S : F \subseteq B \subseteq S \setminus G \},
\]

where \( F \) and \( G \) are finite subsets of \( A \) and \( S \setminus A \) respectively. Through all our note, we shall make this identification and we shall not distinguish between the set \( A \) and its characteristic function \( \chi_A \). Therefore, when \( F \subseteq 2^S \), we may consider the closure \( \overline{F} \) of \( F \), in the product topology of \( \{0,1\}^S \); henceforth, whenever we use the notation \( F \) it will be the product topology the one under consideration.

In the case that \( \mathcal{F} \) is an hereditary family, it is easily seen that \( \overline{\mathcal{F}} \) is an adequate compact, in the sense of the following definition, first introduced by Talagrand [43, 46]. A family \( G \subseteq 2^S \) is said to be adequate if:

(i) whenever \( G \subseteq \mathcal{F} \) and \( F \in G \), then \( G \in \mathcal{F} \), i.e., \( G \) is hereditary;

(ii) if every finite subset of \( G \) belongs to \( \mathcal{G} \), then \( G \in \mathcal{G} \) too.

Conversely, every adequate family \( \mathcal{G} \) can be expressed as \( \overline{\mathcal{F}} \), for some hereditary family of finite sets, namely \( \mathcal{F} = \{ F \in \mathcal{G} : |F| < \omega \} \); in particular, every adequate family is a closed subset of \( \{0,1\}^S \). As it turns out, compact sets that originate from adequate families of sets are very fascinating objects in Functional Analysis and have been exploited in several important examples; let us refer, e.g., to [4, 7, 8, 10, 28, 29, 32, 43, 46] for a sample of some of these constructions.

Our interest in adequate families originates from the following fact, a particular case of the observation that \( \overline{\mathcal{F}} \) is adequate, whenever \( \mathcal{F} \) is hereditary.

**Fact 2.1.** Let \( \mathcal{F} \) be an hereditary family and \( M \in \overline{\mathcal{F}} \). Then every finite subset of \( M \) belongs to \( \mathcal{F} \).

The next proposition is the non-separable counterpart to the argument in [9, Lemma 3.1], with the same proof, which we include for the sake of completeness.

**Proposition 2.2.** Let \( S \) be an infinite set and \( \mathcal{F} \subseteq [S]^{<\omega} \) be an hereditary family such that (†) holds. Then \( C(\overline{\mathcal{F}}) \) contains an isomorphic copy of \( \ell_1(S) \).

**Proof.** Let us preliminarily note that if \( \lambda \) is any convex mean on \( S \), then \( \sup_{F \in \mathcal{F}} \lambda(F) \geq \delta \); since this supremum is actually over the finite set consisting of all \( F \subseteq \text{supp}(\lambda) \), it follows that there exists \( F \in \mathcal{F} \) with \( \lambda(F) \geq \delta \). Consequently, for every finitely supported function \( \lambda : S \to [0,\infty) \) there exists \( F \in \mathcal{F} \) such that

\[
\sum_{s \in F} \lambda(s) \geq \delta \cdot \sum_{s \in S} \lambda(s).
\]
For an element $x = (x(s))_{s \in S} \in c_{00}(S)$, let us define

$$
\|x\| := \sup \left\{ \left| \sum_{s \in F} x(s) \right| : F \subseteq \mathcal{F} \right\};
$$

we claim that $\|\cdot\|$ is a norm on $c_{00}(S)$, equivalent to the $\|\cdot\|_1$ norm. In order to prove this, fix $x \in c_{00}(S)$ and let $P$ be the finite set $P := \{ s \in S : x(s) > 0 \}$; up to replacing $x$ with $-x$, we may assume without loss of generality that

$$
\sum_{s \in P} x(s) \geq \frac{1}{2} \sum_{s \in S} |x(s)|.
$$

Moreover, our assumption implies the existence of $F \in \mathcal{F}$, with $F \subseteq P$, such that

$$
\delta \cdot \sum_{s \in P} x(s) \leq \sum_{s \in F} x(s).
$$

Consequently, we obtain

$$
\frac{\delta}{2} \cdot \sum_{s \in S} |x(s)| \leq \delta \cdot \sum_{s \in P} x(s) \leq \sum_{s \in F} x(s) \leq \|x\|,
$$

which proves our claim. In particular, the completion $X$ of $(c_{00}(S), \|\cdot\|)$ is isomorphic to $\ell_1(S)$.

Associated to $F \in 2^S$ there is a naturally defined functional $F^* \in X^*$, given by $F^* x := \sum_{s \in F} x(s)$; note that $F^*$ is well defined for every $F \subseteq S$ in light of the fact that $X$ is isomorphic to $\ell_1(S)$. It is also clear from the definition of $\|\cdot\|$ that $F^* \in \mathcal{B}_{X^*}$, whenever $F \in \mathcal{F}$. Moreover, the correspondence $F \mapsto F^*$ defines a function $\Phi : \{0,1\}^S \rightarrow (X^*, w^*)$, which is easily seen to be continuous and, of course, injective. It readily follows that $\Phi$ establishes an homeomorphism between $\mathcal{F} \subseteq \{0,1\}^S$ and $\mathcal{F}^* w^* \subseteq \mathcal{B}_{X^*}$, where $\mathcal{F}^* := \Phi(\mathcal{F})$.

Finally, it is a standard fact that $X$ isometrically embeds into $C\left(\mathcal{F}^* w^*\right) = C \left(\overline{\mathcal{F}}\right)$, as a consequence of $\mathcal{F}^*$ clearly being 1-norming for $X$. The fact that $X$ is isomorphic to $\ell_1(S)$ then allows us to conclude the proof.

Remark 2.3. Since the argument is completely direct, it is actually possible to keep track of the various embeddings and localise precisely the position of $\ell_1(S)$ into $C \left(\overline{\mathcal{F}}\right)$; more precisely, it is possible to describe the vectors in $C \left(\overline{\mathcal{F}}\right)$ that correspond to the canonical basis of $\ell_1(S)$.

For $s \in S$, let us denote by $\pi_s : \{0,1\}^S \rightarrow \{0,1\}$ the canonical projection and let $V_s$ be the clopen set

$$
V_s := \pi_s^{-1}(\{1\}) \cap \overline{\mathcal{F}} = \{ F \in \overline{\mathcal{F}} : s \in F \}.
$$

Inspection of the proof of the previous proposition shows that, assuming ($\dagger$), the collection $(\chi_{V_s})_{s \in S} \subseteq C \left(\overline{\mathcal{F}}\right)$ is equivalent to the canonical basis of $\ell_1(S)$.

A simple modification of an argument given in the course of the proof of Theorem A(ii) will also prove the validity of the converse implication.
Remark 2.4. From the appearance of Rosenthal’s celebrated paper [38], a well known criterion to prove that a family \((f_\alpha)_{\alpha<\tau} \subseteq B_{C(K)}\) is equivalent to the canonical basis of \(\ell_1(\tau)\) consists in showing that, for some reals \(r\) and \(\delta > 0\), the collection of sets

\[
(\{f_\alpha \leq r\}, \{f_\alpha \geq r + \delta\})_{\alpha<\tau}
\]

is independent. (We refer to [38, Proposition 4] for the definition of the notion of independence and for the simple proof of this claim.) It is perhaps of interest to note that the copy of \(\ell_1(S)\) obtained in Proposition 2.2 does not originate from such criterion, unless we are in the trivial case that \(F = [S]^{<\omega}\).

In fact, if there were reals \(r\) and \(\delta > 0\) such that

\[
(\{\chi_{V_s} \leq r\}, \{\chi_{V_s} \geq r + \delta\})_{s \in S}
\]

is independent, then this would imply that \((V^c_s, V_s)_{s \in S}\) is an independent family. As a consequence, for distinct \(s_1, \ldots, s_n \in S\) we would have

\[
\emptyset \neq V_{s_1} \cap \cdots \cap V_{s_n} = \{F \in \mathcal{F} : \{s_1, \ldots, s_n\} \subseteq F\}.
\]

it would follow from this and \(\mathcal{F}\) being hereditary that \(\{s_1, \ldots, s_n\} \in \mathcal{F}\), hence \(\mathcal{F} = [S]^{<\omega}\).

In conclusion to this section, let us record how the results presented so far imply the validity of the original statement of Pták’s lemma.

**Proof of Lemma 1.2.** Proposition 2.2 yields that \(C(\mathcal{F})\) contains a copy of \(\ell_1\), which in turn implies that \(C(\mathcal{F})\) is not an Asplund space. As a consequence of this, \(\mathcal{F}\) is necessarily uncountable and it can not be a subset of the countable set \([S]^{<\omega}\). Fact 2.1 leads us to the desired conclusion. \(\Box\)

3. Pták’s lemma for \(\omega_1\)

This section is dedicated to the proof of Theorem A; both clauses will heavily depend on results from [7]. The proof of claim (i) is essentially the same argument as in the proof of Lemma 1.2 given above, but with the Asplund property being replaced by the WLD one. Let us recall a bit of terminology, in order to explain this.

A compact space \(K\) has property \((M)\) if every regular Borel measure on \(K\) has separable support. Here, for the support of a Borel measure \(\mu\) on \(K\), we understand the closed set

\[
\text{supp}(\mu) := \{x \in K : |\mu|(U) > 0 \text{ for every neighborhood } U \text{ of } x\}.
\]

A topological space \(K\) is a Corson compact whenever it is homeomorphic to a compact subset \(C\) of the product space \([-1, 1]^\Gamma\) for some set \(\Gamma\), such that every element of \(C\) has only countably many non-zero coordinates. A Banach space \(X\) is weakly Lindelöf determined (hereinafter, WLD) if the dual ball \(B_{X^*}\) is a Corson compact in the relative \(w^*-\)topology.

We shall need the following topological characterisation of WLD Banach spaces of continuous functions, due to Argyros, Mercourakis, and Negrepontis [7, Theorem 3.5] (we also refer to the same paper, or to [16, 22, 23], for more on Corson compacta and WLD spaces).
**Theorem 3.1** (Argyros, Mercourakis, and Negrepontis, [7]). Let $K$ be a compact topological space. Then $C(K)$ is WLD if and only if $K$ is a Corson compact with property (M).

The rôle of Martin’s axiom $\text{MA}_{\omega_1}$ in connection with the above result is that, under $\text{MA}_{\omega_1}$, every Corson compact has property (M) (cf. [7, Remark 3.2.3] or [16, Theorem 5.62]). More precisely, recall that a compact space $K$ satisfies the countable chain condition (ccc, for short) if every collection of non-empty disjoint open sets in $K$ is at most countable. It is clear that if $\mu$ is a regular Borel measure on $K$, then the support $\text{supp}(\mu)$ of $\mu$ is ccc.

The previous claim then follows from the fact that, under $\text{MA}_{\omega_1}$, every ccc Corson compact is separable ([11, p. 201, Theorem (b)], or [13, p. 207, Exercise (i)]).

The combination of the above considerations assures us that, assuming $\text{MA}_{\omega_1}$, a compact space $K$ is Corson if and only if $C(K)$ is WLD; by means of this equivalence, we may now readily prove the first part of Theorem A.

**Proof of Theorem A(i).** According to Proposition 2.2, $C(\mathcal{F})$ contains an isomorphic copy of $\ell_1(\omega_1)$ and, therefore, it fails to be WLD. Consequently, $\mathcal{F}$ is not Corson and it follows immediately that there exists $M \in \mathcal{F}$ with $|M| \geq \omega_1$; in fact, if this were false, we may use the inclusion map $\mathcal{F} \subseteq [0,1]^{\omega_1}$ as a witness that $\mathcal{F}$ is Corson. We may therefore apply Fact 2.1 and conclude the proof. □

As it turns out, assuming some additional set-theoretic axioms is necessary for the validity of the results described above. In particular, the Continuum Hypothesis allows for the construction of Corson compacta failing property (M). The first such example was constructed by Kunen in [26] and one its generalisation, the Kunen–Haydon–Talagrand example, is described in [30, §5], combining Kunen’s construction with Haydon’s and Talagrand’s examples, [18, 44]. Such compact $K$ also has the property that $C(K)$ fails to contain an isomorphic copy of $\ell_1(\omega_1)$. One simpler example, still under CH, is based upon the Erdős’ space and may be found in [7, Theorem 3.12] or [16, Theorem 5.60]. Interestingly, if the Corson compact $K$ is an adequate compact, then $K$ fails to have property (M) if and only if $C(K)$ contains an isomorphic copy of $\ell_1(\omega_1)$ [7, Theorem 3.13]; in other words, $C(K)$ contains $\ell_1(\omega_1)$, whenever it fails to be WLD.

The proof of claim (ii) in Theorem A that we shall give presently will also implicitly depend on the Erdős space and is based on a combination of arguments from [7, Theorems 3.12, 3.13].

**Proof of Theorem A(ii).** The assumption of the validity of the Continuum Hypothesis allows us to enumerate in an $\omega_1$-sequence $(K_\alpha)_{\alpha < \omega_1}$ the collection of all compact subsets of $[0,1]$ with positive Lebesgue measure (which, in what follows, we shall denote $\mathcal{L}$). We may also let $(x_\beta)_{\alpha < \omega_1}$ be a well ordering of the interval $[0,1]$. The set $K_\alpha \cap \{x_\beta\}_{\alpha < \beta < \omega_1}$ having positive measure, the regularity of $\mathcal{L}$ allows us to select a compact subset $C_\alpha$ of $K_\alpha \cap \{x_\beta\}_{\alpha < \beta < \omega_1}$ such that $\mathcal{L}(C_\alpha) > 0$, for each $\alpha < \omega_1$. Note that if $A \subseteq \omega_1$ is any
uncountable set, then \( \text{sup } A = \omega_1 \) and it follows that
\[
\bigcap_{\alpha \in A} C_\alpha \subseteq \bigcap_{\alpha \in A} \{ x_\beta \}_{\alpha \leq \beta < \omega_1} = \emptyset.
\]

We are now in position to define a Corson compact that fails property (M). Consider the set
\[
A := \{ A \subseteq \omega_1 : \bigcap_{\alpha \in A} C_\alpha \neq \emptyset \};
\]
if every finite subset of a given set \( A \) belongs to \( A \), then the collection of closed sets \( \{ C_\alpha \}_{\alpha \in A} \) has the finite intersection property and \( A \in A \) follows by compactness. Consequently, \( A \) is an adequate compact. Moreover, the previous consideration shows that every \( A \in A \) is a countable subset of \( \omega_1 \), whence \( A \) is a Corson compact. The proof that \( A \) fails to have property (M) may be found in [7, Theorem 3.12], or [16, Theorem 5.60] and we shall not reproduce it here. Therefore, we may fix a positive regular Borel measure \( \mu \) on \( A \), whose support is not separable.

We now consider again the clopen subsets of \( A \) (cf. Remark 2.3)
\[
V_\alpha := \pi^{-1}_\alpha(\{1\}) \cap A = \{ A \in A : \alpha \in A \} \quad (\alpha < \omega_1)
\]
and we shall consider the set \( I := \{ \alpha < \omega_1 : \mu(V_\alpha) > 0 \} \). Plainly, for \( A \in \text{supp}(\mu) \), we have \( \mu(V_\alpha) > 0 \) whenever \( \alpha \in A \); consequently, \( A \subseteq I \) and we obtain that \( \text{supp}(\mu) \subseteq 2' \). In light of the fact that the support of \( \mu \) is not separable, it follows that \( I \) is uncountable.

(Here, we are using the fact that every subspace of a separable Corson compact is separable, immediate consequence of the easy observation that separable Corson compacta are indeed metrisable.) In turn, we also obtain the existence of an uncountable subset \( S \) of \( I \) and a real \( \delta > 0 \) such that \( \mu(V_\alpha) > \delta \) for \( \alpha \in S \).

We may now define the desired hereditary family of finite sets: let us consider \( F_0 := \{ F \in A : F \text{ is a finite set} \} \) and set \( F := \{ F \in F_0 : F \subseteq S \} \). Clearly, \( \overline{F} \subseteq \overline{F_0} = A \); we infer, in particular, that \( \overline{F} \) contains no uncountable set and the conclusion of Pták’s lemma for the cardinal number \( \omega_1 \) fails to hold for \( F \).

On the other hand, for every convex mean \( \lambda \) on \( S \) we have
\[
\left\| \sum_{s \in S} \lambda(s) \chi_{V_s} \right\|_{C(A)} \geq \mu \left( \sum_{s \in S} \lambda(s) \chi_{V_s} \right) = \sum_{s \in S} \lambda(s) \mu(V_s) > \delta \cdot \sum_{s \in S} \lambda(s) = \delta.
\]
From this strict inequality and \( \overline{F_0} = A \), we conclude the existence of \( F \in F_0 \) such that
\[
\sum_{s \in S} \lambda(s) \chi_{V_s}(F) > \delta;
\]
therefore,
\[
\delta < \sum_{s \in S} \lambda(s) \chi_{V_s}(F) = \sum_{s \in F \cap S} \lambda(s) = \lambda(F \cap S) \leq \sup_{G \in F} \lambda(G)
\]
and we see that \( F \) satisfies (†).
4. Larger cardinals

In this section we are going to prove Theorem B; before entering in the core of the proof, it will be convenient to recall some results that we shall make use of in the course of the argument.

A topological space is *totally disconnected* if every its non-empty connected subset is a singleton. Clearly, topological products and subspaces of totally disconnected spaces are totally disconnected.

For a topological space \((X, \mathcal{T})\) and a point \(x \in X\), a *local \(\pi\)-basis* for \(x\) (cf. [21, §1.15]) is a family \(B\) of non-empty open subsets of \(X\) such that for every neighborhood \(V\) of \(x\) there exists \(B \in B\) with \(B \subseteq V\) (note that it is not required \(B\) to contain \(x\)). Every local basis is a local \(\pi\)-basis, *a fortiori*. The pseudo-weight of \((X, \mathcal{T})\) at \(x\) is the minimal cardinality of a local \(\pi\)-basis for \(x\).

The first ingredient we need is the following result, due to Šapirovskiǐ, [39] (see, e.g., [21, §3.18] or [30, Theorem 2.11]).

**Theorem 4.1** (Šapirovskiǐ). Let \(K\) be a totally disconnected compact topological space and \(\kappa\) be an infinite cardinal number. Then there exists a continuous function from \(K\) onto \(\{0,1\}^{\kappa}\) if and only if there exists a non-empty closed subset \(F\) of \(K\) such that the pseudo-weight of \(F\) at \(x\) is at least \(\kappa\), for every \(x \in F\).

The second building block for our proof is a characterisation, due to Richard Haydon, of those compact spaces whose associated Banach space of continuous functions contains an isomorphic copy of \(\ell_1(\kappa)\), for a certain cardinal number \(\kappa\). Let us, preliminarily, shortly review some results in this area.

Pełczyński [31] and Hagler [15] proved that a Banach space \(X\) contains an isomorphic copy of \(\ell_1(\kappa)\) (let us write \(\ell_1(\kappa) \hookrightarrow X\), for short) if and only if \(L_1[0,1] \hookrightarrow X^*\). Pełczyński also demonstrated that, for an infinite cardinal \(\kappa\), \(L_1\{0,1\}^{\kappa} \hookrightarrow X^*\) whenever \(\ell_1(\kappa) \hookrightarrow X\) and he conjectured the validity of the converse implication. (Here, by \(L_1\{0,1\}^{\kappa}\) we understand the Lebesgue space corresponding to \(\{0,1\}^{\kappa}\), the Borel \(\sigma\)-algebra and the Haar measure on the compact group \(\{0,1\}^{\kappa}\).)

The complete solution to Pełczyński’s conjecture follows from a combination of results due to Argyros and Haydon: Haydon [18] proved that the conjecture is false for \(\kappa = \omega_1\) and assuming CH. On the other hand, Argyros [2] proved the correctness of the conjecture for \(\kappa \geq \omega_2\) (in ZFC) and for \(\kappa = \omega_1\), assuming \(\text{MA}_{\omega_1}\). A different proof of Argyros’ result can be obtained from [5]; let us also refer to [6, 17, 19, 30] for a discussion of these and related results.

In a related direction, Talagrand [45] (also see [3], for a simplified proof) proved that, for a cardinal number \(\kappa\) with \(\text{cf}(\kappa) \geq \omega_1\), \(\ell_1(\kappa) \hookrightarrow X\) if and only if there exists a continuous function from \((B_{X^*}, \sigma_{X^*})\) onto \([0,1]^{\kappa}\). The result we shall need is a similar statement in the case that \(X\) is a \(C(K)\) space (cf. [17, Remark 2.5]).

**Theorem 4.2** (Haydon). Let \(\kappa\) be a regular cardinal number such that \(\lambda^\omega < \kappa\) whenever \(\lambda < \kappa\) and let \(K\) be a compact topological space. Then \(\ell_1(\kappa) \hookrightarrow C(K)\) if and only if there exists a continuous function from \(K\) onto \([0,1]^{\kappa}\).
Having recorded all the results we shall build on, we can now approach the proof of Theorem B.

**Proof of Theorem B.** Let $\kappa$ be a regular cardinal number such that $\lambda\omega < \kappa$ whenever $\lambda < \kappa$, let $S$ be a set with $|S| = \kappa$ and $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family such that $|\mathcal{F}| < \kappa$. Whenever $\lambda < \kappa$, let $S$ be a set with $|S| = \kappa$ and $F \subseteq \mathcal{P}(S)$ be an hereditary family such that $\lambda < \kappa$. We combine Proposition 2.2 with Haydon’s result, we obtain the existence of a continuous surjection from $F$ to $[0, 1]^\kappa$. Consequently, there exists a closed subset $K_1$ of $F$ that continuously maps onto $\{0, 1\}^\kappa$; note that, being a subspace of $\{0, 1\}^\kappa$, $K_1$ is totally disconnected. In light of Šapirovskiǐ’s theorem, we conclude that there exists a closed subspace $K$ of $(K_1, \mathcal{K})$, hence of $F$, such that the pseudo-weight of $K$ at $x$ is at least $\kappa$, for every $x \in K$. In particular, if $B$ is any local basis for the topology of $K$ at any $x \in K$, then $|B| \geq \kappa$.

Before we proceed, introducing a bit of notation is in order. If $A \in K$ and $F$ and $G$ are finite subsets of $A$ and $S \setminus A$ respectively, then we shall denote

$$U_A(F, G) := \{B \in K : F \subseteq B \subseteq S \setminus G\},$$

a neighborhood of $A$ in $K$. A particular case of this piece of notation is that $U_A(F, \emptyset) := \{B \in K : F \subseteq B\}$.

Plainly, $$\{U_A(F, G) : F \in [A]^{<\omega}, G \in [S \setminus A]^{<\omega}\}$$

is a local basis for the topology of $K$ at $A$.

We now note that $K$ may also be considered as a partially ordered set, with respect to set inclusion. When such partial order is considered, then every chain in $K$ admits an upper bound in $K$; in fact, the union of the chain belongs to $K$, $K$ being closed in the pointwise topology. Appeal to Zorn’s lemma allows us to deduce that there exist in $K$ maximal elements with respect to inclusion.

**Claim.** If $M \in K$ is any maximal element, then a local basis for the topology of $K$ at $M$ is given by

$$B := \{U_M(F, \emptyset) : F \in [M]^{<\omega}\}.$$  

Since clearly $|B| \leq |M|$, our previous considerations allow us to conclude that, if $M \in K$ is any maximal element, then $|M| \geq \kappa$. If we select any such maximal element—whose existence we noted above—then Fact 2.1 assures us that $M$ is the set we were looking for. Therefore, in order to conclude the proof, we only need to establish the claim.

**Proof of the claim.** Assume by contradiction that $B$ is not a local basis. Then there exist finite sets $\bar{F}$ and $\bar{G}$ with $\bar{F} \subseteq M$ and $\bar{G} \subseteq S \setminus M$ such that no element of $B$ is contained in $U_\bar{F}(\bar{F}, \bar{G})$. In particular, for every finite set $I$ with $\bar{F} \subseteq I \subseteq M$ there exists an element $A_I \in K$ with $A_I \in U_M(I, \emptyset) \setminus U_\bar{F}(\bar{F}, \bar{G})$. Being $\bar{F} \subseteq I$, this is equivalent to the fact that $I \subseteq A_I$ and $A_I \cap \bar{G} \neq \emptyset$.

Let us denote by $I$ the directed set $I := \{I \in [M]^{<\omega} : \bar{F} \subseteq I\}$; we therefore have a net $(A_I)_{I \in I}$ in $K$ such that $I \subseteq A_I$ and $A_I \cap \bar{G} \neq \emptyset$, for every $I \in I$. By compactness of $K$,
such a net clusters at some $\tilde{A} \in \mathcal{K}$. A simple argument, whose details we include below for the sake of completeness, then implies that $M \subseteq \tilde{A}$ and $\tilde{A} \cap \tilde{G} \neq \emptyset$. Consequently, $\tilde{A}$ is a proper extension of $M$ (recall that $M \cap \tilde{G} = \emptyset$), thereby contradicting the maximality of $M$ and thus concluding the proof.

In order to check that $M \subseteq \tilde{A}$, fix any finite set $F$ with $\tilde{F} \subseteq F \subseteq M$ and consider the neighborhood $U_{\tilde{A}}(F \cap \tilde{A}, F \setminus \tilde{A})$ of $\tilde{A}$. By definition, there must exist $I \in \mathcal{I}$ with $F \subseteq I$ such that $A_I \in U_{\tilde{A}}(F \cap \tilde{A}, F \setminus \tilde{A})$; it follows, in particular, that $A_I \subseteq (F \setminus \tilde{A})^c = \tilde{A} \cup F^c$. Therefore, $F \subseteq I \subseteq A_I \subseteq \tilde{A} \cup F^c$ yields $F \subseteq \tilde{A}$ and $M \subseteq \tilde{A}$ follows.

Finally, for the second assertion, we consider the neighborhood $U_{\tilde{A}}(\tilde{G} \cap \tilde{A}, \tilde{G} \setminus \tilde{A})$ of $\tilde{A}$. By definition, some $A_I$ belongs to such neighborhood and it follows that $A_I \subseteq \tilde{A} \cup \tilde{G}^c$. Consequently, $\emptyset \neq A_I \cap \tilde{G} \subseteq (\tilde{A} \cup \tilde{G}^c) \cap \tilde{G} = \tilde{A} \cap \tilde{G}$, and we are done. □

In conclusion to our note, we shall add a few comments on Haydon’s result, Theorem 4.2. At the appearance of [17] it was unknown whether the equivalence stated in Theorem 4.2 (or, more generally, the equivalence between the assertions in [17, Remark 2.5]) could possibly hold under more general assumptions on $\kappa$. The sufficient condition holding true for every cardinal $\kappa$, Haydon himself (unpublished) later noted that the necessary condition fails to hold for $\kappa = \omega_1$, under the Continuum Hypothesis. One such example was also obtained by N. Kalamidas, in his Doctoral dissertation (cf. [30, Example 1.3]).

Incidentally, this is also a consequence of the results presented in our note, since the unique point where the proof of Theorem B depends on some cardinality assumption is the appeal to Theorem 4.2; in particular, Theorem B actually holds true for every cardinal number for which the equivalence in Theorem 4.2 holds. Theorem A(ii) then yields the desired counterexample.

In accordance to Argyros’ results on Pelczyński’s conjecture that we mentioned above, it is natural to conjecture that Haydon’s equivalence may actually be valid for every cardinal number $\kappa \geq \omega_2$. This would, of course, imply the validity of Pták’s lemma for every $\kappa \geq \omega_2$.

In case that the conjecture were true, it would also lead to a negative answer to the following question.

**Problem 4.3.** Is the existence of a Corson compact $K$ such that $\ell_1(\omega_2) \hookrightarrow C(K)$ consistent with ZFC?

Let us just note that, under CH, such a compact space cannot exist, in light of Theorem 4.2 and the fact that continuous images of Corson compacta are Corson compacta ([14, 27]), while $[0,1]^{\omega_2}$ is not Corson. Such a compact space also fails to exist under $\text{MA}_{\omega_1}$, according to the results we recorded at the beginning of Section 3.

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