FINITE-AMPLITUDE ACOUSTICS UNDER THE CLASSICAL THEORY OF PARTICLE-LADEN FLOWS

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Abstract. We consider acoustic propagation in a particle-laden fluid, specifically, a perfect gas, under a model system based on the theories of Marble (1970) and Thompson (1972). Our primary aim is to understand, via analytical methods, the impact of the particle phase on the acoustic velocity field. Working under the finite-amplitude approximation, we investigate singular surface and traveling wave phenomena, as admitted by both phases of the flow. We show, among other things, that the particle velocity field admits a singular surface one order higher than that of the gas phase, that the particle-to-gas density ratio plays a number of critical roles, and that traveling wave solutions are only possible for sufficiently small values of the Mach number.

1. Introduction. The basis of the present study is Marble’s [27] theory of particle-laden flows (PLF)s, which he put forth in 1970. Here, as in Ref. [27], the fluid phase is taken to be a perfect gas; i.e., one that obeys what is usually referred to as the “ideal gas law”

\[ \varphi = (c_p - c_v) \varrho \vartheta, \]  

subject to the proviso that \( c_p > c_v > 0 \) are both constant [34]. In Eq. (1), \( \varphi(>0) \) is the thermodynamic pressure, \( \varrho(>0) \) is the mass density of the gas, \( \vartheta(>0) \) is the absolute temperature of the gas, and \( c_p \) and \( c_v \) denote the specific heats at constant pressure and volume, respectively, of the gas.

Our focus here shall be on finite-amplitude acoustic phenomena in such “dusty gases”, specifically, traveling waves and singular surfaces generated by compressive piston motion. Employing the tools and methods of classical analysis, our primary aim is to understand how the particle phase impacts the acoustic field, as modeled under the finite-amplitude approximation, vis-à-vis these two classes of waveforms. To simplify the analysis, however, we do not consider Marble’s [27] theory in its most general form; instead, the governing system adopted in the present study stems from the (1D) special case of the former whose distinguishing assumption is

\[ c_{pp} \simeq 0, \]

i.e., the particle specific heat (at constant pressure) is negligibly small; see Thompson [34, pp. 553–556]. The immediate (practical) consequence of this assumption is \( \tau_T = 0 \) (⇒ the absence of thermal inertia), where \( \tau_T \propto c_{pp} \) is the thermal relaxation time.
It is appropriate to mention that the model formulated below is also a special case of the (1D) weakly-nonlinear dusty gas model developed by Crighton [15], who took the fluid phase to be a classical \textit{thermoviscous} perfect gas, which we note is a class of (perfect) gases more general than that considered in the present study. In Ref. [15], however, the focus is limited to the gas phase and solutions are presented for only a single special case of the relaxation parameters, specifically (see Ref. [15, p. 75]),

$$\tau_M = \tau_T := \tau > 0,$$ (3)

where $\tau_M$ is the momentum relaxation time (see Eq. (7) below). Hence, the present investigation will be seen to complement Crighton’s analysis by providing solutions/results corresponding to the case wherein $\tau_M > 0$, $\tau_T = 0$ hold simultaneously, which of course is not a special case of Eq. (3), and for both phases of the flow. Here, we observe that our assumption $c_{pp} \simeq 0$ corresponds to $c \simeq 0$ in Ref. [15].

The remainder of this exposition is organized as follows. In Sect. 2, what we term the \textit{Marble–Thompson model-1}\(^1\) is stated and the corresponding finite-amplitude system of equations (see Sys. (11), below) is presented. In Sect. 3, a singular surface analysis of the both the bi-directional-linear and right-running weakly-nonlinear versions this system is carried out. Then, in Sect. 4, we seek traveling wave solutions (TWS) of the aforementioned system and examine their analytical properties. And lastly, in Sect. 5, some implications of our findings are reviewed and connections to other fields are noted.

**Remark 1.** When the fluid phase involves a thermoviscous perfect gas, the thermal relaxation time can be expressed as

$$\tau_T = 3 \left( \frac{Pr}{Nu} \right) \left( \frac{c_{pp}}{c_p} \right) \tau_M,$$ (4)

where $Pr$ is the Prandtl number and $Nu$ is the Nusselt number. It is noteworthy that in his dusty gas model, Crighton assumes $Nu = 2$ from the outset; see Ref. [15, p. 71]. This “unforced” limitation is, however, easily circumvented through the use of Eq. (4). Hence, for the general case $Nu > 0$, the assumption stated in Eq. (3) implies that

$$c_{pp} := \frac{1}{3} \left( \frac{Nu}{Pr} \right) \left( \tau_M = \tau_T \right),$$ (5)

assuming a thermoviscous perfect gas; see also the footnote on Ref. [34, p. 553].

2. \textit{The Marble–Thompson model-1}. Assuming that the fluid phase is a perfect gas of negligible thermal conductivity and that the particles are tiny, uniform, rigid spheres that are composed of a material for which Eq. (2) is a valid assumption, 1D acoustic propagation in such a dual-phase medium is approximately described by the following system:

\begin{align*}
\rho_t + u \rho_x + \rho u_x &= 0, \quad \text{(6a)} \\
\rho(u_t + u u_x) + \varphi_x &= -m_p n(u - v)/\tau_M, \quad \text{(6b)} \\
\rho \vartheta_t (\eta_t + u \eta_x) &= m_p n(u - v)^2 / \tau_M, \quad \text{(6c)}
\end{align*}

\(^1\)What we would term the \textit{Marble–Thompson model-2} is the generalization of Sys. (6) (see below) in which the assumption $c_{pp} \simeq 0$ is removed; see Ref. [34, p. 559] for a discussion of the linearized version of this more realistic model, wherein the effects of both thermal and momentum relaxation are present.
\[
\varphi = \gamma^{-1} A \left( \varrho / \varrho_0 \right)^{\gamma} \exp \left( \frac{\eta - \eta_0}{c_v} \right),
\]

\(6d\)

\[
v_1 + u v_x = (u - v) / \tau_M,
\]

\(6c\)

\[
n_t + u n_x + n v_x = 0,
\]

\(6f\)

where the absence of both external body forces and internal heat sources has been assumed. Here, \(\mathbf{u} = (u(x,t), 0, 0)\) and \(\mathbf{v} = (v(x,t), 0, 0)\) are the gas and particle velocity vectors, respectively; \(\eta = \eta(x,t)\) is the specific entropy of the gas; \(n = n(x,t)\) is the number of particles per unit volume; \(m_p(> 0)\), the mass of each particle, is assumed constant; \(A = \varrho_0 c_0^2\) denotes the adiabatic bulk modulus \([31]\) of the clean gas, where \(c_0 = \sqrt{\gamma \varrho_0 / \varrho_0}\) is the sound speed in the undisturbed (clean) gas; and \(\tau_M\), which we recall is the momentum relaxation time, is given by

\[
\tau_M = \frac{2 \varrho_p r_p^2}{9 \mu} = \frac{m_p}{6 \pi \mu r_p},
\]

\(7\)

where the density of the material that constitutes the dust particles, \(\varrho_p\), the shear viscosity of the (clean) gas, \(\mu\), and the particle radius, \(r_p\), are all positive and constant-valued.

Also, in this communication the ambient state of both phases is assumed to be homogeneous-quiescent \([31, p. 14]\); i.e., while \(\varrho_0, \vartheta_0, \eta_0, \) and \(n_0\) are constant-valued and positive, \(v_0 = u_0 = 0\), where a “0” subscript attached to a dependent variable denotes the ambient state value of that variable.

At this point the following observations regarding Sys. (6) are in order: (a) From Eq. (6b), the momentum equation for the gas phase, we see that the viscous contribution to the total stress tensor has been neglected; however, viscous effects still impact the flow, in the form of Stokes drag, via the parameter \(\tau_M\). (b) Since it implies a flow free of the effects of thermal inertia (i.e., \(\tau_T \simeq 0\)), Eq. (2) also implies that

\[
\vartheta_p(x,t) = \vartheta(x,t) \quad (c_{pp} \simeq 0),
\]

\(8\)

where \(\vartheta_p(x,t)\) denotes the particle absolute temperature; see Ref. [34, p. 556]. (c) The right-hand side (RHS) of Eq. (6c) has been simplified using Eq. (8); see Remark 2 below. (d) Eq. (6d), the non-isentropic equation of state (EoS) for a perfect gas, stems, in part, from Eq. (1); see Ref. [20, p. 129]. And (e), our assumption of a homogeneous-quietescent ambient state is consistent with the requirement that the ambient state values of the field variables must, themselves, satisfy Sys. (6).

Observing that the 1D nature of the flow renders it irrotational, it follows that \(u(x,t) = \phi_x(x,t)\), where \(\phi\) is the scalar velocity potential\(^2\) corresponding to the gas phase. Also, anticipating our use of the finite-amplitude approximation, we now neglect both the RHS and the term \(u n_x\) in Eq. (6c); consequently, the flow is now homentropic\(^3\) and our EoS is reduced to a barotropic expression.

\(^2\) In the multi-dimensional case, \(\mathbf{u} = \nabla \phi\) would require that \(\nabla \times \mathbf{u} = (0,0,0)\) be assumed.

\(^3\) For the definition of this special case of isentropic flow, see Ref. [34, p. 60].
Thus, on making the substitution $u = \phi_x$, observing that homentropic flow implies $\eta(x,t) = \eta_0$, and introducing the following dimensionless variables:

$$
\begin{align*}
  u^o &= u/U_{\text{piston}}, & \phi^o &= \phi/(LU_{\text{piston}}), & s &= (\rho - \rho_0)/\rho_0, & v^o &= v/U_{\text{piston}}, \\
  \epsilon &= (\eta - \eta_0)/c_v, & n &= (n - n_0)/n_0, & \hat{x} &= x/L, & \hat{t} &= t(c_0/L),
\end{align*}
$$

Sys. (6) is reduced to

$$
\begin{align*}
  s_t &= -\epsilon \partial_x[(1 + s)\phi_x], & (10a) \\
  (1 + s)\partial_x[\phi_t + \frac{1}{2}\epsilon(\phi_x)^2] + \epsilon^{-1}(\varphi/A)_x &= -(\kappa/\hat{\tau})(1 + n)(\phi_x - v), & (10b) \\
  \epsilon &= 0, & (10c) \\
  \varphi/A &= \gamma^{-1}(1 + s)^\gamma, & (10d) \\
  \hat{\tau}v_t + \epsilon\hat{\tau}v\phi_x &= (\phi_x - v), & (10e) \\
  n_t &= -\epsilon \partial_x[(1 + n)v]. & (10f)
\end{align*}
$$

In Sys. (10), $s = s(x,t)$ is the condensation; we term $n = n(x,t)$ the particle condensation; $U_{\text{piston}}(>0)$ and $L(>0)$, both constant, respectively denote the piston speed and a characteristic (macro-scale) length associated with the particular problem/flow under consideration; $\epsilon = c_0^{-1}U_{\text{piston}}$ denotes the acoustic Mach number; $\hat{\tau} = c_0L^{-1}\tau_M$ is the dimensionless momentum relaxation time; $\kappa$, the mass fraction of particles, is given by $\kappa = m_p\eta_0/\rho_0$ [34, Eq. (11.78)]; and all diamond ($\diamond$) superscripts are here and henceforth suppressed, but should remain understood.

Under the finite-amplitude approximation, the basis of which are the assumptions $\epsilon \ll 1$ and $|s| \sim O(\epsilon)$ (see, e.g., Jordan [20, §2.3]), Sys. (10) can be reduced to a weakly-nonlinear system involving only two PDEs. Omitting the details4, it is a straightforward matter to show that this approximating system reads

$$
\begin{align*}
  \hat{\tau}\phi_{tt} + \hat{c}_e^{-2}\phi_{tt} - \hat{c}_0^2[1 - \epsilon(\gamma - 1)\phi]\phi_{xx} + \epsilon\partial_x(\phi_x)^2 = \hat{\tau}\hat{c}_0^2\phi_{txx}, & (11a) \\
  \hat{\tau}v_t + v &= \phi_x, & (11b)
\end{align*}
$$

where $\hat{c}_e = \hat{c}_0/\sqrt{1 + \kappa}$ is the dimensionless version of the “equilibrium sound speed” [34, p. 557], $\hat{c}_0(=1)$ is the dimensionless version of $c_0$, and we observe that $\hat{c}_e < \hat{c}_0 = 1$. In deriving this system the following were also assumed:

$$
|u - v| \sim O(\epsilon), & |n| \sim O(\epsilon), & \hat{\tau} \sim O(\epsilon), & \kappa \sim O(\epsilon),
$$

and, in accordance with the finite-amplitude scheme, only nonlinear terms of $O(\epsilon^2)$ have been neglected.

**Remark 2.** It is the following, i.e., not Thompson’s [34, Eq. (11.80)], that is the “exact” form of the entropy production equation vis-à-vis the assumptions of Ref. [34, §11.5]:

$$
\varrho\partial_t \left( \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \right) = m_p \left[ c_{p\varrho}\tau_T^{-1}(\varrho_p - \varrho) + \tau_M^{-1}(u - v)^2 \right] n,
$$

\[\text{Ref. [19, 21], each of which presents the application of the finite-amplitude approximation to a particular acoustic system.} \]
which we have expressed in terms of the dimensional variables and parameters of the present communication. Here, we note that \( \tau_T \) (i.e., \( \tau_i \) in terms of Thompson’s notation) is given, most generally, by

\[
\tau_T = \frac{1}{3} c_p \rho_p r_p / h,
\]

where \( h (>0) \) denotes the convective heat transfer coefficient. (In the thermoviscous case, this expression can, of course, be re-expressed as Eq. (4).)

3. **Singular surface results: The piston problem in the half-space** \( x > 0 \).

3.1. **Mathematical preliminaries.** In this section our primary object is to investigate how the particle phase impacts shock propagation in the gas phase, in particular, the evolution of \([ u ]\). Employing the usual notation of singular surface theory, we define the amplitude of the jump discontinuity in a function \( \mathfrak{F} = \mathfrak{F}(x,t) \) across a propagating surface \( x = \Sigma(t) \) as

\[
[\mathfrak{F}] := \mathfrak{F}^- - \mathfrak{F}^+,
\]

where \( \mathfrak{F}^- := \lim_{x \to \Sigma(t)^-} \mathfrak{F}(x,t) \) are assumed to exist, and where a “+” superscript corresponds to the region into which \( \Sigma \) is advancing while a “−” superscript corresponds to the region behind \( \Sigma \). If \( \mathfrak{F} \) is a velocity component and \([ \mathfrak{F} ] \neq 0\), then \([ \mathfrak{F} ]\) represents the shock amplitude and \( x = \Sigma(t) \) the wave-front, or, in this case, the shock-front, across which \( \mathfrak{F} \) is discontinuous.

The half-space version of the piston problem considered in this section has been investigated for over a century now; see, e.g., Lamb [25, §63] and Moran and Shen [28] for variants of this problem involving dust-free gases. Under the formulation considered, the piston, located at \( x = 0 \), impulsively begins “moving” to the right at time \( t = 0 \), thus compressing the dusty gas that fills the half-space \( x > 0 \); here, however, as in most of the previous studies of this version of the problem, the displacement of the piston in not explicitly taken into account. Known in the acoustics literature as a signaling problem [16, p. 189], this version of the piston problem is, since it involves initial conditions (ICs), also a start-up problem.

As Eq. (11a) is a quasilinear PDE, it does not directly lend itself to the study of wave phenomena in which \([ u ] \neq 0\), at least not by analytical methods. As such, we are left with two options. The first is to consider the linearized (i.e., \( \epsilon := 0\)) version of Sys. (11), which we do in the next subsection. The second is to turn to what is generally referred to as the unidirectional approximation. As its name implies, this scheme is based on the assumption of unidirectional propagation; it is the methodology used to derive the KdV equation from the Boussinesq equations (see, e.g., Whitham [38, §8]) and the well known Burgers equation from the PDE which some authors refer to as the “Blackstock–Lesser–Seebeass–Crighton” (BLSC) acoustic model (see, e.g., Crighton [14, p. 16]). While its disadvantage is clear (i.e., reflected waves are not allowed), the advantage of the unidirectional approximation is that it yields a simplified EoM whose highest-order time derivative is one order less than that of the original EoM.

As our attention in this section is limited to right-running waves, propagating along the \(+x\)-axis, with no possibility of reflection, reducing Eq. (11a) to a second

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\(^5\)That is, the problems considered in this section do not involve a moving boundary; see, e.g., Ref. [17, p. 107], wherein this simplification and its justifications are discussed.
order PDE via the unidirectional approximation is readily accomplished\(^6\). Thus, in terms of this new (simplified) EoM for the gas phase Sys. (11) becomes

\[
\frac{1}{2}c_x^2\dot{\tau}(u_{tt} - u_{xx}) + u_t + c_x u_x + \epsilon\beta c_x^2 uu_x = 0, \quad (16a)
\]

where we have made use of both the relation \(u = \phi_x\) and the fact that \(c_0 = 1\). Here, we have introduced \(\beta > 1\), the coefficient of nonlinearity \([4]\), where \(\beta = (\gamma + 1)/2\) in the case of perfect gases. It is noteworthy that Eq. (16a) is a special case of the PDE known today as the hyperbolic Burgers equation (HBE), which first arose as a kinematic wave-based traffic flow model; see, e.g., Refs. [12, 19] and those cited therein.

Anticipating the singular surface analysis that shall be performed in Sect. 3.3, we close the present subsection by recasting Eq. (16a) as the kinematic-type system

\[
\begin{pmatrix}
    u \\
    j
\end{pmatrix}_t + \begin{pmatrix}
    0 & 1 \\
    0 & 0
\end{pmatrix}
\begin{pmatrix}
    u \\
    j
\end{pmatrix}_x = \frac{2}{c_x^2} \left( c_x u (1 + \frac{1}{2} \beta c_x u) - j \right), \quad (17)
\]

where, as in Ref. [19, §4.3], \(j\) has been introduced here to serve in the role of a flux. As its characteristics, which are defined by \(dx/dt = \pm 1\), are real-valued and unequal, Sys. (17), and thus Eq. (16a), are actually strictly hyperbolic \([26]\), meaning that their solutions satisfy the requirements of causality. Equally important, vis-à-vis analyzing shock phenomena using singular surface theory, is the following: Since its eigenvalues are constants (i.e., \(\pm 1\)) and all nonlinearity is confined to its source vector, Sys. (17), and thus Eq. (16a), are also semilinear.

Remark 3. Since we have assumed only right-running waveforms, it is clear that \(\Sigma(t) = t + x_0\) under Sys. (17), where the constant \(x_0\) denotes the initial location of \(\Sigma(t)\) on the \(x\)-axis. That is, in the case of Sys. (17) \(\Sigma(t)\) is a planar wave-front, across which \(u = u(x,t)\) may suffer a jump discontinuity, that propagates (to the right) along, and perpendicular to, the \(x\)-axis with speed \(d\Sigma(t)/dt = 1\).

3.2. Velocity field shocks: Linearized system. In this subsection we consider the following initial-boundary value problem (IBVP) involving the linearized version of Sys. (11):

\[
\begin{align*}
\hat{\tau}u_{ttt} + \hat{c}_x^2 u_{tt} - u_{xx} &= \hat{\tau}u_{txx}, \quad (x,t) \in (0,\infty) \times (0,\infty); \quad \text{(18a)} \\
\hat{\tau}u_t + v &= u, \quad (x,t) \in (0,\infty) \times (0,\infty); \quad \text{(18b)} \\
u(0,t) &= \Theta(t), \quad u(\infty,t) = 0, \quad t > 0; \quad \text{(18c)} \\
u(x,0) = 0, \quad u_t(x,0) = 0, \quad u_{tt}(x,0) = 0, \quad v(x,0) = 0, \quad x > 0. \quad \text{(18d)}
\end{align*}
\]

Here, \(\Theta(\zeta)\) denotes the Heaviside unit step function; we have made use of both the relation \(u = \phi_x\) and the fact that \(c_0 = 1\); and we note that the first and fourth ICs imply that \(u^+\) and \(v^+\), the values of \(u\) and \(v\) immediately ahead of \(\Sigma\), are both zero.

On applying the Laplace transform \([11]\) to Eqs. (18a), (18b), and the boundary conditions, IBVP (18) is reduced to, after making use of the ICs, the following system of subsidiary equations:

\[
\begin{align*}
(\hat{\tau}s + 1)\ddot{u} + (\hat{\tau}s^3 + \hat{c}_x^{-2}s^2)\dot{u} &= 0, \quad x > 0, \quad \text{(19a)} \\
(\hat{\tau}s + 1)\ddot{v} &= \ddot{u}, \quad x > 0, \quad \text{(19b)}
\end{align*}
\]

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\(^6\)See, e.g., the derivation of Ref. [19, Eq. (74)].
which is to be solved subject to
\[ u(0, s) = \frac{1}{s}, \quad \bar{u}(\infty, s) = 0. \] (20)

Here, \( s \) is the Laplace transform parameter and a bar over a quantity denotes the image of that quantity in the Laplace transform domain. Solving Sys. \((19)\) is not difficult; after simplifying, the exact (transform-domain) solutions can be expressed as
\[ \bar{u}(x, s) = \frac{1}{s} \exp \left( -sx \sqrt{\frac{s + \sigma}{s + 1/\tau}} \right), \] (21)
\[ \bar{v}(x, s) = \frac{\tau^{-1}}{s(s + 1/\tau)} \exp \left( -sx \sqrt{\frac{s + \sigma}{s + 1/\tau}} \right), \] (22)
where we have set \( \sigma := (\tau c_e^2)^{-1} \). We now expand Eqs. \((21)\) and \((22)\) for large-\(s\):
\[ \bar{u}(x, s) \sim \frac{1}{s} e^{-sx} \exp \left[ -x(\sigma - 1/\tau)/2 \right] \times \left[ 1 + \frac{x(\sigma^2 \tau^2 + 2\sigma \tau - 3)}{8\tau^2 s} + O(s^{-2}) \right] \quad (s \to \infty), \] (23)
\[ \bar{v}(x, s) \sim \frac{1}{\tau} \left\{ \frac{1}{s^2} e^{-sx} \exp \left[ -x(\sigma - 1/\tau)/2 \right] \times \left[ 1 - \frac{1}{\tau} \left( 1 - \frac{x(\sigma^2 \tau^2 + 2\sigma \tau - 3)}{8\tau} \right) \right] \frac{1}{s} + O(s^{-2}) \right\} \quad (s \to \infty). \] (24)

Since the coefficient of \(sx\) in the factor \(e^{-sx}\), which each of these expansions contains, is \(-1\), it follows\(^7\) that \(d\Sigma(t)/dt = 1\); therefore, in the case of IBVP \((18)\)
\[ \Sigma(t) = t, \] (25)
where the fact that the signal is inserted at the boundary \(x = 0\) \((\Rightarrow \Sigma(0) = 0)\) means that the resulting constant of integration is zero.

Now applying the theorem given in Ref. \[7, \S4\] to Eq. \((23)\), it is readily established that, across \(x = t\),
\[ [u] = \exp \left[ -(\kappa/\tau) t/2 \right], \] (26)
where we note that \(\sigma - 1/\tau = \kappa/\tau\), and it follows that Eq. \((25)\) is a \textit{shock-front} in the case of the gas phase.

In contrast, applying the aforementioned theorem to Eq. \((24)\) reveals that, across \(x = t\),
\[ [v] = 0, \quad [v_1] = \hat{\tau}^{-1} \exp \left[ -(\kappa/\tau) t/2 \right], \quad [v_2] = -\hat{\tau}^{-1} \exp \left[ -(\kappa/\tau) t/2 \right]. \] (27)

Here, the expression for \([v_2]\) was obtained from that of \([v_1]\) using the fact that if \([\delta]\) = 0, then
\[ [\delta_x] = -(d\Sigma(t)/dt)^{-1} [\delta_t] \quad (d\Sigma(t)/dt \neq 0); \] (28)
see, e.g., Ref. \[6, p. 182\]. From Eqs. \((27)\) we find that while \(v(x,t) \in C^0(\mathbb{R}^+, \mathbb{R}^+)\), its first derivatives suffer jump discontinuities across \(x = t\); i.e., with respect to the particle phase Eq. \((25)\) describes an \textit{acceleration wave}\(^8\) in the case of IBVP \((18)\).

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\(^7\)See Theorem V in Ref. \[11, p. 7\].
\(^8\)Also known as a “discontinuity wave”; see, e.g., Refs. \[5, 30\].
In closing this subsection we compute and plot the solutions of IBVP (18) by numerically inverting the Laplace domain solutions (i.e., Eqs. (21) and (22)) using the following formula:

\[
\mathfrak{F}(x, t) \approx \frac{\exp(4.7)}{t} \left\{ \frac{1}{2} \mathfrak{F} \left( x, \frac{4.7}{t} \right) + \Re \left[ \sum_{m=1}^{M} (-1)^m \mathfrak{F} \left( x, \frac{4.7 + im\pi}{t} \right) \text{sinc} \left( \frac{m\pi}{M} \right) \right] \right\} \quad (t > 0),
\]

where \( M \gg 1 \) is an integer and

\[
\text{sinc}(\zeta) := \begin{cases} 
\zeta^{-1} \sin(\zeta), & \zeta \neq 0, \\
1, & \zeta = 0.
\end{cases}
\]

Eq. (29), we note, is a modified version of Tzou’s [35] Riemann-sum inversion approximation, which Keiffer et al. [24] recently put forth, wherein the \( \text{sinc}(m\pi/M) \)
factors have been introduced in order to reduce the Gibbs phenomenon that occurs when approximating our (discontinuous) solution $u(x, t)$ using Tzou’s formula.

The sequences shown in Fig. 1 clearly illustrate the analytical results presented in this subsection, e.g., the fact that $d\Sigma(t)/dt = 1$. Comparing the (discontinuous) $u$ vs. $x$ profiles with those of the Type II case in Ref. [3, Fig. 3] reveals that the most obvious effect of the particle phase (i.e., taking $\gamma > 0$) on the acoustic velocity field is to “round-off” the corner that would otherwise exist on the leading side of the propagating Heaviside function; in contrast, comparison with those of the Type III case in Ref. [3, Fig. 3] makes clear that without the term $u_{\tau\tau\tau}$, solutions of Eq. (18a) do not satisfy causality. Lastly, Fig. 1 also illustrates the fact that the acceleration wave amplitude (of the particle phase) is simply the slope of the tangent to the (continuous) $u$ vs. $x$ profile at the acceleration wave(-front), which is located at the point $(t, 0)$.

**Remark 4.** Eq. (18a), which appears to have been first derived by Stokes [32] in his 1851 paper on acoustic propagation in radiating gases, is a special case of the 1D version of the PDE that has been termed the “Moore–Gibson–Thompson” (MGT) equation [23]; see also Ref. [22, Eq. (64)]. In this regard it is noteworthy that Eq. (18a) satisfies the exponential stability condition discussed by Kaltenbacher [22, p. 464]; specifically, her critical parameter (i.e., $\gamma$) is equal to $\kappa$, and $\kappa > 0$.

### 3.3. Velocity field shocks: HBE-based system

In this subsection we consider the following IBVP involving Sys. (16):

\[ \frac{1}{2} \hat{\epsilon}^2 \hat{\gamma} ( u_{tt} - u_{xx} ) + u_t + \hat{c}_e u_x + \epsilon \hat{c}_e^2 u u_x = 0, \quad (x, t) \in (0, \infty) \times (0, \infty); \]
\[ \hat{\gamma} u_t + v = u, \quad (x, t) \in (0, \infty) \times (0, \infty); \]
\[ u(0, t) = \Theta(t), \quad u(\infty, t) = 0, \quad t > 0; \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad v(x, 0) = 0, \quad x > 0. \]

Since here too we have $\Sigma(0) = 0$, it follows that $x_0 = 0$; i.e., $\Sigma(t)$ is given by Eq. (25) in the case of this IBVP (see Remark 3), and the ICs imply that, once again, $u^\pm = v^\pm = 0$.

Returning now to Sys. (17) and employing the machinery of singular surface theory (see, e.g., Bland [6, §6.9] and Refs. [5, 33]), along with the *Rankine–Hugoniot conditions*\(^9\), it is readily established that $\Sigma(t)$ satisfies a (quadratic) Bernoulli equation, viz.:

\[ \hat{\gamma} \frac{D\hat{\Sigma}}{Dt} = - \left( \frac{1 - \hat{c}_e}{\hat{c}_e^2} \right) \hat{\Sigma} + \frac{1}{2} \epsilon \beta \hat{\Sigma}^2. \]

Here, we have set $\Sigma(t) = [u]$ for convenience and $D/Dt$, the 1D displacement derivative, gives the time-rate-of-change measured by an observer traveling with $\Sigma$.

A phase plane analysis of Eq. (32) reveals the stability characteristics of its (two) equilibria, which we denote using a superposed tilde (i.e., $\tilde{\Sigma}$). As $\alpha^\bullet > 0$, since $u^+ = 0$, it follows that $\tilde{\Sigma} = \{0, \alpha^\bullet\}$ are stable and unstable, respectively. Here,

\[ \alpha^\bullet = 2 \left( \frac{1 + \kappa - \sqrt{1 + \kappa}}{\epsilon \beta} \right), \tag{33} \]

where we note for later reference the approximation

\[ \alpha^\bullet \approx \frac{\kappa}{\epsilon \beta} + \frac{\kappa^2}{4 \epsilon \beta}, \tag{34} \]

\(^9\)See, again, Bland [6, p. 184], who does not use this terminology.
which follows from our assumption that $\alpha \sim \mathcal{O}(\epsilon)$.

Eq. (32) is, of course, readily integrated and yields the exact solution

$$S(t) = \frac{\alpha}{1 - (1 - \alpha) \exp \left[ \frac{1}{2} \beta \alpha (t/\hat{t}) \right]},$$

where we observe that $S(0) = 1$. Eq. (35) indicates that, from the mathematical standpoint, the evolution of $S(t)$ can occur in any one of the following three ways:

- (I) If $\alpha^* > 1$, then $S(t) \to 0$ (from above) as $t \to \infty$.
- (II) If $\alpha^* = 1$, then $S(t) = \alpha^* = 1$ for all $t \geq 0$.
- (III) If $\alpha^* \in (0, 1)$, then $S(t) \to \infty$ as $t \to t_\infty$.

Here, $t_\infty$, the time at which $S(t)$ exhibits blow-up, is given by

$$t_\infty = \frac{2 \hat{t}}{\epsilon \beta \alpha^*} \ln \left[ \frac{S(0)}{S(0) - \alpha^*} \right],$$

where we observe that $t_\infty > 0$ holds only under Case (III). From these results it is clear that Sys. (16) breaks down if $\epsilon > \epsilon_\infty$, where

$$\epsilon_\infty := 2 \left( 1 + \kappa - \sqrt{1 + \kappa} \right) = \frac{\kappa}{\beta} \left[ 1 + \frac{\kappa}{4} + \mathcal{O}(\kappa^2) \right].$$

Turning our attention to the velocity field of the particle phase, we observe that taking jumps of Eq. (16b) yields

$$\hat{t} \left[ u_1 \right] + [u] = [u] = S(t).$$

On setting $[u] = 0$, based on the findings of the previous subsection, it is easy to show that

$$\left[ u_1 \right] = \hat{t}^{-1} S(t) \quad \text{and} \quad [u_x] = -\hat{t}^{-1} S(t),$$

where we have once again made use of Eq. (28).

**Remark 5.** Eq. (26) is recovered from Case (I) by first replacing $\alpha^*$ with the first term of the approximation given in Eq. (34) and then letting $\epsilon \to 0$.

4. **Traveling wave analysis: The piston problem on the real line.** We begin our study of this well known problem\footnote{See, e.g., von Neumann and Richtmyer [36, §IV] who, it should be noted, formulated their system in Lagrangian coordinates.} with the following observation: since Eq. (11a) is invariant under the transformation $x \mapsto -x$, we shall, without loss of generality, hereafter restrict our attention to only right-running waves. That is, waveforms that travel with constant speed, $a > 0$, and whose single argument is the similarity variable $\xi$, which takes the specific form $\xi := x - at$.

The formulation of this version of the piston problem has the piston located at $x = -\infty$, and moving to the right with constant speed $U_{piston}$; it also involves the assumption that at $x = +\infty$ the medium is in its equilibrium state, i.e., at rest with all thermodynamic variables assuming their ambient state values.

And lastly, in this study we employ the following definition for the waveform metric known as the **shock thickness**:

$$\text{shock thickness} := \frac{\text{max}_{\xi \in \mathbb{R}} |\hat{\mathcal{S}}'(\xi)|}{\hat{\mathcal{S}}(-\infty) - \hat{\mathcal{S}}(+\infty)},$$

(40)
which, as Morduchow and Libby [29, p. 680] point out, can be traced back to Prandtl. Here, it should be noted that Eq. (40) has meaning only for those traveling wave profiles that take the form of a kink [2].

### 4.1. Velocity field of gas phase.

Introducing the ansatz $\phi(x,t) = \mathcal{P}(\xi)$ and substituting it into Eq. (11a), the latter is reduced to, after integrating once with respect to $\xi$, the second order ODE

$$\dot{\tau}a(1-a^2)\mathcal{P}'' - (1-a^2/\hat{c}_e^2)\mathcal{P}' - \epsilon\beta a(\mathcal{P}')^2 = K_1,$$

where a prime denotes $d/d\xi$ and $K_1$ is the constant of integration. On setting $f(\xi) = \mathcal{P}'(\xi)$ and then imposing and enforcing the asymptotic conditions of the piston problem, i.e., $f \to 1, 0$ as $\xi \to \mp\infty$, the following results are readily established: $K_1 = 0$, which follows directly from the right-asymptotic condition, and

$$a = \frac{1}{2}\hat{c}_e^2\epsilon\beta + \hat{c}_e\sqrt{1 + \frac{1}{4}\epsilon^2\beta^2c_e^2},$$

which we note is the positive root of $\hat{c}_e^{-2}a^2 - 1 = \epsilon\beta a$; and thus, our associated ODE is found to be

$$\dot{\tau}(1-a^2)\frac{df}{d\xi} + \epsilon\beta(1-f) = 0.$$

On integrating this (Bernoulli type) ODE, which we do subject to the usual condition $f(0) = 1/2$, we find that our TWS assumes the form of a Taylor shock\(^{11}\), specifically,

$$f(\xi) = \frac{1}{2}\left\{1 - \tanh\left[\frac{\epsilon\beta\xi}{2\dot{\tau}(1-a^2)}\right]\right\} \quad (0 < \epsilon < \epsilon^*),$$

where $\epsilon^* := \kappa/\beta$ is a critical value of the Mach number (recall Eq. (37)), with shock thickness

$$\ell = \frac{4\dot{\tau}(1-a^2)}{\epsilon\beta} \quad (0 < \epsilon < \epsilon^*).$$

Here, we observe that this restriction on the Mach number must be satisfied if $f(\xi)$ is to represent a physically admissible solution; in particular, $f(\xi)$ does not satisfy the imposed asymptotic conditions, because it becomes an increasing function, when $\epsilon > \epsilon^*$ is taken.

### 4.2. Particle velocity field.

Under the traveling wave assumption, i.e., on setting $v(\xi) = v(x,t)$, Eq. (11b) becomes

$$\frac{dv}{dZ} - \lambda v = -\lambda f,$$

where in simplifying we have set $Z = 4\xi/\ell$ and defined

$$\lambda := \frac{\ell}{4a^2} = \frac{1-a^2}{\epsilon\beta a} \quad (0 < \epsilon < \epsilon^*).$$

As we expect $v(Z)$ to be bounded for all $Z \in \mathbb{R}$, we see from Eq. (46) that this implies

$$\lim_{Z \to \mp\infty} v = \lim_{Z \to \mp\infty} f = 1, 0,$$

respectively.

\(^{11}\)Note that a Taylor shock is a special case of a kink-type traveling wave profile.
With \( \exp(-\lambda Z) \) as the integrating factor, and employing the identity
\[
\tanh\left(\frac{\zeta}{2}\right) = 1 - \frac{2}{1 + \exp(\zeta)},
\]
Eq. (46) yields the quadrature
\[
v(Z) = -\lambda \exp(\lambda Z) \int Z \exp(-\lambda \zeta) d\zeta.
\]
Here, \( K_3 \), the constant of integration, has been set to zero to suppress this ODE’s complementary solution, which is given by \( v_c(Z) = K_3 \exp(\lambda Z) \). As shown in the Appendix, this integral can be evaluated for arbitrary values of \( \lambda > 0 \). For the purposes of the present section, however, the following series representation of \( v(Z) \), which we obtained by repeatedly applying integration by parts to Eq. (50), shall prove adequate:
\[
v(Z) = f(Z) - \frac{\exp(Z)}{\lambda(1 + \exp(Z))^2} - \frac{\exp(Z)(1 - \exp(Z))}{\lambda^2(1 + \exp(Z))^3} + O(\lambda^{-3}).
\]
From Eq. (51) it is clear that \( v \) is a perturbation of \( f \). As such, satisfying the assumption given in Eq. (12) requires that we now impose the restriction \( \lambda \gg 1 \).

Note, however, that satisfying both \( \lambda \gg 1 \) and \( 0 < \epsilon < \epsilon^* \) (see Eq. (44)) requires us to impose the following (comprehensive) restriction:
\[
0 < \epsilon \ll \epsilon^*.
\]
where
\[
\epsilon^* := \frac{\epsilon^*}{2\sqrt{1 + \kappa/2}} = \frac{\kappa}{2\beta\sqrt{1 + \kappa/2}} < \epsilon^*.
\]
That is, if the inequality in Eq. (52) is satisfied, then the velocity TWSs of both phases are physically admissible.

From Eq. (51) we see that, as is true in the case of the gas phase, the particle velocity profile assumes the form of a kink, to which a shock thickness value can be assigned. Denoting by \( \ell_p \) the shock thickness corresponding to the particle velocity field, we now revert back to the (independent) variable \( \xi \) and introduce the function \( v_1(\xi) \), where
\[
v(\xi) = v_1(\xi) + O(\lambda^{-2}).
\]
Noting that \( v_1''(\xi_m) = 0 \), where
\[
\xi_m := \frac{1}{4} \ell \ln \left( \frac{-2 + \sqrt{3 + \lambda^2}}{\lambda - 1} \right),
\]
it is not difficult to establish, based on Eqs. (40) and (48), that
\[
\ell_p \approx \ell \left( 1 - \frac{1}{4\lambda^2} \right) \quad (0 < \epsilon \ll \epsilon^*),
\]
From this it is clear that \( \ell_p < \ell \). Here, we observe that \( \xi_m < 0 \); more specifically,
\[
\xi_m \approx -\frac{\ell}{4\lambda} \left( 1 - \frac{2}{3\lambda^2} \right) \quad (0 < \epsilon \ll \epsilon^*).
\]
And lastly, from Eq. (51) it is easily shown that
\[
v(0) = \frac{1}{2} \left[ 1 - \frac{1}{2\lambda} + O(\lambda^{-3}) \right].
\]
from which we see that \(0 < v(0) < f(0) = 1/2\). For completeness we note that Eqs. (A.66) and (A.71) (both) give
\[
    v(0) = \frac{\lambda}{2} \left[ \Psi \left( \frac{1}{2} (2 + \lambda) \right) - \Psi \left( \frac{1}{2} (1 + \lambda) \right) \right] \quad (\lambda > 0),
\]
which is an exact result. Here, \(\Psi(\zeta)\) is the digamma function [1, §6] and we observe that
\[
    1 - \ln(2) < v(0) < 1/2 \quad (1 < \lambda < \infty),
\]
for \(v(0)\) given by Eq. (59), where \(1 - \ln(2) \approx 0.3069\).

5. Observations and final remarks.

(i) For both singular surface and traveling wave phenomena, the model considered demands that we restrict the value of the Mach number; in both cases, the ratio \(\epsilon^* := \kappa/\beta\) appears as a factor in the corresponding upper bound.

(ii) Observing that \(\epsilon^*\) can be expressed as
\[
    \epsilon^* = \frac{m_p n_0}{\varrho_0 \beta},
\]
where it should be noted that the numerator factors relate only to the particle phase while the denominator factors relate only to the gas phase, the condition \(\epsilon < \epsilon^*, \, \text{e.g.,}\), is seen to imply that
\[
    U_{\text{piston}} < c_0 \left( \frac{m_p n_0}{\varrho_0 \beta} \right);
\]
i.e., we have placed an upper bound on a parameter over which an experimentalist has full and direct control. Such relations can, in principal, allow us to experimentally determine properties of the particle phase when the type of (perfect) gas that constitutes the gas phase is known.

(iii) Under the present model, the particle velocity field admits a singular surface one order higher than that of the gas phase.

(iv) On letting \(m_p \to 0\), Eq. (11a) reduces to the 1D (dimensionless) case of the PDE some authors refer to as the lossless BLSC equation (see, e.g., Ref. [20, Eq. (51)]) while Eq. (11b) becomes simply \(v = u\); i.e., in this limit the particles move with the same velocity as the gas, but their vanishing mass does not allow them to influence the motion of the latter.

(v) With regard to the gas phase, \(1/\tau\) plays the role of a Reynolds number, specifically, one based on Lighthill's diffusivity of sound, \(\delta(> 0)\) [34, §4.13]. What is more, within this analog to thermoviscous gas dynamics we find that \(m_p\) corresponds to \(\delta\), as highlighted by the following observation: Since \(m_p \to 0 \quad (\Rightarrow \tau \to 0)\) causes \(\ell \to 0\), the \(f(\xi)\) vs. \(\xi\) profile “shocks-up” in the same way that the (traveling wave) profile described by Ref. [16, Eq. (22)] does when we let \(\delta \to 0\).

(vi) The fact that \(\ell_p < \ell\) indicates that, in the context of traveling waves, the particle velocity field experiences less dissipation than that of the gas phase; it should be noted, however, that \(\ell_p \to \ell \quad (\Rightarrow \text{particle phase dissipation increases})\) as \(\lambda \to \infty\).

(vii) With the second term omitted, the approximation to \(a^*\) given in Eq. (34) is the dusty gas shock analog of the critical acceleration wave amplitude value
exhibited by the “Darcy–Jordan model”\textsuperscript{12} of poroacoustics; see, e.g., Ref. [20, Eq. (96)], wherein the dimensionless Darcy coefficient divided by two, i.e., $\delta/2$ (not to be confused with the diffusivity of sound divided by two), corresponds to $\kappa$ here.

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**Appendix — Evaluation of particle velocity field integral.** When evaluating the integral in Eq. (50), the cases $\lambda \notin \mathbb{N}$ and $\lambda \in \mathbb{N}$ must be treated separately; here, we consider the latter first.

**The case $\lambda \in \mathbb{N}$.** On setting $\varpi = \exp(Z)$, Eq. (50) is reduced to

$$v(\varpi) = -\lambda \varpi^\lambda \int \frac{d\varpi}{\varpi^{\lambda+1}(1 + \varpi)},$$

which under a partial fraction decomposition becomes

$$v(\varpi) = -\lambda(-1)^\lambda \varpi^\lambda \int \left[ \frac{1}{\varpi} - \frac{1}{1 + \varpi} - \sum_{m=2}^{\lambda+1} \frac{(-1)^m}{\varpi^m} \right] d\varpi.$$

Now integrating term-by-term, we find that

$$v(\varpi) = -\lambda(-1)^\lambda \varpi^\lambda \left[ \ln \left( \frac{\varpi}{1 + \varpi} \right) + \sum_{m=2}^{\lambda+1} \frac{(-1)^m}{m-1} \varpi^{m-1} \right],$$

from which we immediately obtain

$$v(Z) = \lambda(-1)^\lambda \exp(\lambda Z)$$

$$\times \left\{ \ln[1 + \exp(-Z)] - \sum_{m=2}^{\lambda+1} \frac{(-1)^m \exp[-(m-1)Z]}{m-1} \right\} \quad (\lambda \in \mathbb{N}).$$

**The case $\lambda \notin \mathbb{N}$.** Assuming, for the moment, $Z > 0$ and multiplying the numerator and denominator of its integrand by $\exp(-\zeta)$, Eq. (50) can be re-expressed as

$$v(Z) = \lambda \exp(\lambda Z) \int_Z^\infty \frac{\exp[-(1 + \lambda)\zeta] d\zeta}{1 + \exp(-\zeta)},$$

where we have also multiplied by $-1$ to compensate for the fact that Eq. (50) gives $v$ as a function of the upper limit of integration. Since $\zeta \in (Z, \infty)$, we can now expand the integrand factor $1/(1 + \exp(-\zeta))$ in a binomial series. Doing so and then integrating term-by-term yields

$$v(Z) = \lambda \exp(-Z) \sum_{m=0}^\infty \frac{(-1)^m \exp(-mZ)}{m + (1 + \lambda)},$$

\textsuperscript{12}Also known as the “Jordan–Darcy model” (of poroacoustics); see Ciarletta and Straughan [13], as well as Straughan [33, §8.1].
which we can recast as

$$\nu(Z) = \lambda \exp(-Z) \Phi[-\exp(-Z), 1, (1 + \lambda)], \quad (A.69)$$

where the Lerch transcendent \([37]\) is defined as

$$\Phi(\zeta, \nu, \chi) := \sum_{m=0}^{\infty} \frac{\zeta^m}{(m + \chi)^\nu} \quad (|\zeta| < 1, \chi \neq 0, -1, -2, \ldots). \quad (A.70)$$

Since \(\nu(Z)\) must be physically well-defined for all \(Z \in \mathbb{R}\), we now employ the analytic continuation\([13]\) of \(\Phi\) given by Guillera and Sondow \([18, \text{Thm } 2.1]\). This gives, after simplifying,

$$\nu(Z) = \lambda \sum_{m=0}^{\infty} \left\{ \frac{\exp[-(m + 1)Z]}{[1 + \exp(-Z)]^{m+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \right\} \quad (\lambda \notin \mathbb{N}), \quad (A.71)$$

where Eq. (A.71) is convergent for all \(\exp(-Z) > -1/2\), i.e., for all \(Z \in \mathbb{R}\).

Lastly, we observe that, according to the software package MATHEMATICA (ver. 11.2), Eq. (A.71) can also expressed as

$$\nu(\xi) = \frac{\lambda}{1 + \lambda} \left\{ \frac{F[1, 1; 2 + \lambda; 1 + \zeta]}{1 + \exp(Z)} \right\} \quad (\lambda \notin \mathbb{N}), \quad (A.72)$$

where \(F(c_1, c_2; c_3; \zeta)\) denotes the Gauss hypergeometric series \([1, \S 15]\).

REFERENCES

[1] M. Abramowitz and I. A. Stegun (eds.), Handbook of Mathematical Functions, Dover, 1966.

[2] J. Angulo, Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions, Mathematical Surveys and Monographs, vol. 156, American Mathematical Society, 2009.

[3] S. Bargmann, P. Steinmann and P. M. Jordan, On the propagation of second-sound in linear and nonlinear media: Results from Green–Naghdi theory, Phys. Lett. A, 372 (2008), 4418–4424.

[4] R. T. Beyer, The parameter \(B/A\) in: Nonlinear Acoustics (eds. M. F. Hamilton and D. T. Blackstock), Academic Press, 1998, 25–39.

[5] J. Bissell and B. Straughan, Discontinuity waves as tipping points: Applications to biological & sociological systems, Discrete Cont. Dyn. Sys., Ser. B (DCDS-B), 19 (2014), 1911–1934.

[6] D. R. Bland, Wave Theory and Applications, Oxford University Press, 1988.

[7] B. A. Boley and R. B. Hetnarski, Propagation of discontinuities in coupled thermoelastic problems, J. Appl. Mech. (ASME), 35 (1968), 489–494.

[8] J. P. Boyd, A proof, based on the Euler sum acceleration, of the recovery of an exponential (geometric) rate of convergence for the Fourier series of a function with Gibbs phenomenon, Spectral and High Order Methods for Partial Differential Equations, 131–139, Lect. Notes Comput. Sci. Eng., 76, Springer, Heidelberg, 2011, (https://arxiv.org/abs/1003.5263v1).

[9] J. P. Boyd, Dynamics of the Equatorial Ocean, Springer–Verlag, 2018, §§A.13, A.14.

[10] J. P. Boyd, Private communication, 24 February 2018.

[11] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics, Dover, 1963.

[12] I. Christov and P. M. Jordan, Shock bifurcation and emergence of diffusive solitons in a nonlinear wave equation with relaxation, New J. Phys., 10 (2008), 043027.

[13] M. Carletta and B. Straughan, Poroacoustic acceleration waves, Proc. R. Soc. A, 462 (2006), 3493–3499.

[14] D. G. Crighton, Model equations of nonlinear acoustics, Ann. Rev. Fluid Mech., 11 (1979), 11–33.

[15] D. G. Crighton, Nonlinear waves in aerosols and dusty gases, in: Nonlinear Waves in Real Fluids (ed. A. Kluwick), Springer–Verlag, 1991, 69–82.

\(^{13}\)Boyd [10] has pointed out that Eq. (A.71) is more accurately described as the “Euler acceleration” (or “Abel–Euler summability” [9]) of Eq. (A.70); see also Ref. [8] and those cited therein.
[16] D. G. Crighton, Propagation of finite-amplitude waves in fluids, in: Handbook of Acoustics (ed. M. J. Crocker), Wiley, 1998, Chap. 17.
[17] D. G. Crighton and J. T. Scott, Asymptotic solutions of model equations in nonlinear acoustics, Phil. Trans. R. Soc. London A, 292 (1979), 101–134.
[18] J. Guillera and J. Sondow, Double integrals and infinite products for some classical constants via analytic continuations of Lerch’s transcendent, Ramanujan J., 16 (2008), 247–270, (http://arxiv.org/abs/math.NT/0506319v3).
[19] P. M. Jordan, Second-sound phenomena in inviscid, thermally relaxing gases, Discrete Cont. Dyn. Sys., Ser. B (DCDS-B), 19 (2014), 2189–2205.
[20] P. M. Jordan, A survey of weakly-nonlinear acoustic models: 1910–2009, Mech. Res. Commun., 73 (2016), 127–139.
[21] P. M. Jordan, R. S. Keiffer and G. Saccomandi, Anomalous propagation of acoustic traveling waves in thermoviscous fluids under the Rubin–Rosenau–Gottlieb theory of dispersive media, Wave Motion, 51 (2014), 382–388.
[22] B. Kaltenbacher, Mathematics of nonlinear acoustics, Evol. Eq. Control Theory (EECT), 4 (2015), 447–491.
[23] B. Kaltenbacher, I. Lasiecka and R. Marchand, Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound, Control and Cybernetics (C&C), 40 (2011), 971–988.
[24] R. S. Keiffer, P. M. Jordan and I. C. Christov, Acoustic shock and acceleration waves in selected inhomogeneous fluids, Mech. Res. Commun., 93 (2018), 80–88.
[25] H. Lamb, The Dynamical Theory of Sound, 2nd ed. Dover Publications, Inc., New York, 1960.
[26] J. D. Logan, An Introduction to Nonlinear Partial Differential Equations, Wiley, 1994.
[27] F. E. Marble, Dynamics of dusty gases, Ann. Rev. Fluid Mech., 2 (1970), 397–446.
[28] J. P. Moran and S. F. Shen, On the formation of weak plane shock waves by impulsive motion of a piston, J. Fluid Mech., 25 (1966), 705–718.
[29] M. Morduchow and P. A. Libby, On a complete solution of the one-dimensional flow equations of a viscous, heat-conducting, compressible gas, J. Aeronaut. Sci., 16 (1949), 674–684, 704.
[30] A. Morro, Jump relations and discontinuity waves in conductors with memory, Math. Comput. Modelling, 43 (2006), 138–149.
[31] A. D. Pierce, Acoustics: An Introduction to its Physical Principles and Applications, Acoustical Society of America, 1989.
[32] G. G. Stokes, An examination of the possible effect of the radiation of heat on the propagation of sound, Phil. Mag. (Ser. 4), 1 (1851), 305–317.
[33] B. Straughan, Stability and Wave Motion in Porous Media, Applied Mathematical Sciences, vol. 165, Springer, 2008, Chap. 8.
[34] P. A. Thompson, Compressible-Fluid Dynamics, McGraw–Hill, 1972.
[35] D. Y. Tzou, Macro- to Microscale Heat Transfer: The Lagging Behavior, Taylor & Francis, 1997, §2.5.1.
[36] J. von Neumann and R. D. Richtmyer, A method for the numerical calculation of hydrodynamic shocks, J. Appl. Phys., 21 (1950), 232–237.
[37] E. W. Weisstein, Lerch Transcendent, From MathWorld—A Wolfram Web Resource (http://mathworld.wolfram.com/LerchTranscendent.html).
[38] G. B. Whitham, Non-linear dispersive waves, Proc. R. Soc. London A, 283 (1965), 238–261.