ELLIPITC RUIJSENAARS DIFFERENCE OPERATORS ON BOUNDED PARTITIONS

JAN FELIPE VAN DIEJEN AND TAMÁS GÖRBE

Abstract. By means of a truncation condition on the parameters, the elliptic Ruijsenaars difference operators are restricted onto a finite lattice of points encoded by bounded partitions. A corresponding orthogonal basis of joint eigenfunctions is constructed in terms of polynomials on the joint spectrum. In the trigonometric limit, this recovers the diagonalization of the truncated Macdonald difference operators by a finite-dimensional basis of Macdonald polynomials.

1. Introduction

Ruijsenaars’ commuting difference operators constitute the quantum integrals for a relativistic deformation of the elliptic Calogero-Moser-Sutherland system [R87]. For integral values of the coupling parameter, the spectral problem for these difference operators has been fruitfully analyzed within the framework of the algebraic Bethe Ansatz [B98, HSY04, K01a, K01b]; the underlying solutions of the Yang-Baxter equation incorporating Ruijsenaars’ difference operators within the algebraic Bethe Ansatz formalism had been developed for this purpose in [H97, KH97, PV97]. For arbitrary positive values of the coupling parameter, an orthogonal basis of eigenfunctions for the Ruijsenaars difference operators can be generated by means of a Cauchy-type Hilbert-Schmidt kernel [R09]. Very recently, remarkably explicit formulas for these eigenfunctions were presented in [LNS20] (cf. also [L14] and references therein for a direct construction of the eigenfunctions from the Hilbert-Schmidt kernel in the nonrelativistic limit). It is expected that a complete solution of the eigenvalue problem for the elliptic Ruijsenaars difference operators gives rise to an intricate elliptic counterpart [EK95, LNS20, MMZ21] of Macdonald’s ubiquitous theory of symmetric orthogonal polynomials [M95, Chapter VI].

Motivated by the state of the art sketched above, we will restrict the elliptic Ruijsenaars difference operators in the present note to functions supported on a finite lattice in the center-of-mass configuration space (inside the Weyl alcove of the $\mathfrak{sl}(n+1; \mathbb{C})$ Lie algebra). The corresponding classical integrable particle system has a compact phase space given by the complex projective space and has been studied in [FG16]. In the trigonometric limit, the lattice quantum particle model of interest was investigated in [DV98, GH18] as the quantization of a corresponding classical integrable system from [R95, FKL1]. The hyperbolic counterpart of our quantum model involves particles moving on an infinite lattice subject to a dynamics that is governed by a scattering matrix that factorizes into two-particle contributions [R02]:
when placed in an integrability-preserving external field, extra factors encoding one-particle contributions emerge in the scattering matrix [DE16]. At the elliptic level of present concern, a detailed mathematical study of the quantum eigenvalue problem for the two-particle Ruijsenaars difference operator on the finite lattice was performed recently in [DG21].

Specifically, below we will adjust the real period of the elliptic functions in terms of the positive coupling parameter so as to truncate the elliptic Ruijsenaars operators. This entails a corresponding commuting system of discrete difference operators, which are normal in a Hilbert space of functions supported in a finite lattice of shifted dominant weights associated with the \(\mathfrak{sl}(n + 1; \mathbb{C})\) Lie algebra. With the aid of a standard labelling of the pertinent dominant weights by means of (bounded) partitions, we build an orthogonal basis of joint eigenfunctions in terms of polynomials evaluated on the spectrum; the polynomials in question are uniquely determined by a recurrence stemming from the eigenvalue equations. In this approach, the spectral theorem for commuting normal operators in finite dimension entails the orthogonality and Pieri rules for these polynomials. In the trigonometric limit, our construction recovers the unitary diagonalization of finitely truncated Macdonald operators by means of Macdonald polynomials from [DV98].

The material is organized as follows. In Section 2 the elliptic Ruijsenaars quantum lattice model restricted to bounded partitions is presented; details on how to retrieve this lattice model from Ruijsenaars’ commuting difference operators by discretization are supplied in Appendix A (at the end of this note). Section 3 promotes the corresponding discrete elliptic Ruijsenaars operators to commuting normal operators in an appropriate Hilbert space. An orthogonal basis of joint eigenfunctions given by polynomials evaluated on the spectrum is constructed in Section 4. In Section 5 we compare the trigonometric limit of the present construction with the corresponding diagonalization in terms of Macdonald polynomials stemming from [DV98]. This comparison reveals some salient features of the joint spectrum at the elliptic level, which we briefly highlight in the form of an epilogue in Section 6.

Note. Throughout elliptic functions will be expressed in terms of the following rescaled Jacobi theta function

\[
[z] = [z; p] = \frac{\theta_1 \left( \frac{z}{2}; p \right)}{2 \theta_1 \left( 0; p \right)} \quad (z \in \mathbb{C}, \quad \alpha > 0, \quad 0 < p < 1),
\]

with

\[
\vartheta_1(z; p) = 2 \sum_{l \geq 0} (-1)^l p^{(l+\frac{1}{2})^2} \sin(2l + 1)z,
\]

\[
= 2p^{1/4} \sin(z) \prod_{l \geq 1} (1 - p^{2l})(1 - 2p^{2l} \cos(2z) + p^{4l}).
\]

Hence \(\vartheta_1' \left( 0; p \right) = 2p^{1/4}(p^2; p^2)^3\) \(\infty\), where

\[
(z; p)_\infty = \lim_{k \to \infty} (z; p)_k \quad \text{with} \quad (z; p)_k = \prod_{0 \leq l < k} (1 - zp^l)
\]

(and the convention \((z; p)_0 = 1\)). The relation to the Weierstrass sigma function associated with the period lattice \(\Omega = 2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}\) reads (cf. e.g. [OLBC10 §23.6(i)]):

\[
[z] = \sigma(z)e^{-\frac{\zeta(\omega_1)}{2\omega_1}z^2},
\]
where \( \alpha = \frac{\pi}{\omega_1}, \) \( p = e^{i\pi \tau} \) with \( \tau = \frac{2\pi}{\omega_1} \), and \( \zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \).

2. Lattice Ruijsenaars models on partitions

2.1. Discrete Ruijsenaars operators on partitions. Let us recall that a partition

\[ \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{\ell}) = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{\ell}, 0, 0, \ldots) \]

of weight \( |\lambda| = \sum_{j \in \mathbb{N}} \lambda_j \) consists of a weakly decreasing sequence of nonnegative integers of which only a finite number are allowed to be positive. These positive terms are called parts and their number gives the length \( \ell = \ell(\lambda) = |\{ j \in \mathbb{N} \mid \lambda_j > 0 \}| \) of the partition (where \(| \cdot |\) refers to the cardinality of the set in question). Following standard conventions one writes for two partitions that \( \lambda \subset \mu \) if \( \lambda_j \leq \mu_j \) for all \( j \in \mathbb{N} \); in this situation \( \mu \) and \( \lambda \) are said to differ by a vertical \( r \)-strip \( \theta \) if \( |\mu| = |\lambda| + r \) and \( \theta_j = \mu_j - \lambda_j \in \{ 0, 1 \} \) for all \( j \in \mathbb{N} \). It is important to emphasize that a vertical \( r \)-strip \( \theta \) is only a partition if \( \theta = 1^r \), where

\[ m^r = (m_1, \ldots, m_r) \]

refers to the rectangular partition with \( r \) parts of size \( m \) each. For \( n \in \mathbb{N} \), we will single out the set of partitions with at most \( n \) parts:

\[ \Lambda^{(n)} = \{ \lambda \in \Lambda \mid \ell(\lambda) \leq n \}, \quad (2.1) \]

where \( \Lambda \) stands for the set of all partitions. Moreover, occasionally we will indicate for a vertical strip \( \theta \) (committing a slight abuse of notation) that \( |\theta| = r \) and \( \theta \subset 1^n \) (even if \( \theta \) is not a partition) so as to specify that one deals with an \( r \)-strip for which \( \theta_j = 0 \) when \( j > n \).

After these preliminaries, we are now in the position to define the following discrete Ruijsenaars operators acting in the space of complex lattice functions

\[ C(\Lambda^{(n)}) = \{ \lambda \overset{\sim}{\lambda_r} f_{\lambda} \in \mathbb{C} \mid \lambda \in \Lambda^{(n)} \} \]

for \( 1 \leq r \leq n \):

\[ (D_r f)_{\lambda} = \sum_{\lambda \subset \mu \subset \Lambda^{(n+1)}} B_{\mu/\lambda} f_{\mu} \quad (f \in C(\Lambda^{(n)}), \ \lambda \in \Lambda^{(n)}), \quad (2.2a) \]

where

\[ B_{\mu/\lambda} = \prod_{1 \leq j < k \leq n + 1} \frac{[\lambda_j - \lambda_k + g(k - j + \theta_j - \theta_k)]}{[\lambda_j - \lambda_k + g(k - j)]} \quad \text{with} \ \theta = \mu - \lambda. \quad (2.2b) \]

Here \( g \in \mathbb{R} \) denotes a coupling parameter that is momentarily assumed to be generic such that

\[ \forall \lambda \in \Lambda^{(n)} : \prod_{1 \leq j < k \leq n + 1} \sin \frac{\pi}{2}(\lambda_j - \lambda_k + g(k - j)) \neq 0 \quad (2.3) \]

(which is the case e.g. if we pick \( g \in \mathbb{R} \) such that \( j g \notin \mathbb{Z} + \frac{2\pi}{\omega_1} \mathbb{Z} \) for \( j = 1, \ldots, n \)). Notice that the genericity condition in (2.3) ensures that the denominator of \( B_{\mu/\lambda} \) (2.2a) does not vanish (cf. Eqs. (1.1a), (1.1d)).
Proposition 1 (Commutativity). The discrete Ruijsenaars operators $D_1, \ldots, D_n$ commute in $\mathcal{C}(\Lambda^{(n)})$.

Proof. The operators $D_1, \ldots, D_n$ arise as discretizations of Ruijsenaars' commuting difference operators and therewith inherit their commutativity (cf. Appendix A for further details).

2.2. Finite-dimensional truncation on bounded partitions. From now on we fix $n, m \in \mathbb{N}$ and pick

\[ \alpha = \frac{2\pi}{(n+1)g+m} \quad \text{with } g > 0. \quad (2.4) \]

Let

\[ \Lambda^{(n,m)} = \{ \lambda \in \Lambda^{(n)} \mid \lambda \subset m^n \}. \quad (2.5) \]

Notice that $|\Lambda^{(n,m)}| = \binom{n+m}{n}$.

Lemma 2 (Truncation). Let $\lambda \in \Lambda^{(n,m)}$ and $\mu \in \Lambda^{(n+1)}$ such that $\theta = \mu - \lambda$ is a vertical $\tau$-strip. Then—for parameters in accordance with Eq. (2.4)—the coefficient $B_{\mu/\lambda}$ is positive iff $\mu \in \Lambda^{(n,m)}$ and vanishes iff $\mu \notin \Lambda^{(n,m)}$.

Proof. For $\lambda \in \Lambda^{(n,m)}$ and $1 \leq j < k \leq n + 1$, one has that

\[ 0 < g \leq \lambda_j - \lambda_k + (k-j)g \leq m + ng < \frac{2\pi}{\alpha}. \]

Hence, it is clear from the product formula in Eqs. (1.1a), (1.1b) that the corresponding factors in the denominator of $B_{\mu/\lambda}$ are all positive. In the same way one deduces that

\[ 0 \leq \lambda_j - \lambda_k + g(k-j + \theta_j - \theta_k) \leq \frac{2\pi}{\alpha} \]

in this situation. The corresponding factors in the numerator of $B_{\mu/\lambda}$ are therefore all nonnegative. Vanishing factors in the numerator occur iff the bounds of the interval are reached. One reaches the lower bound zero iff $k = j + 1$ with $\theta_j = 0$, $\theta_{j+1} = 1$ and $\lambda_j = \lambda_{j+1}$ for some $j \in \{1, \ldots, n\}$; this can only happen iff $\mu_j - \mu_{j+1} = \theta_j - \theta_{j+1} = 0$, which contradicts our assumption that $\mu \in \Lambda^{(n+1)}$.

The upper bound $\frac{2\pi}{\alpha}$ is reached on the other hand iff $j = 1$, $k = n + 1$, with $\theta_1 = 1$, $\theta_{n+1} = 0$ and $\lambda_1 = m$ (since $\lambda_{n+1} = 0$); this happens iff $\mu_1 - \mu_{n+1} = \lambda_1 + \theta_1 - \theta_{n+1} = m + 1$, i.e. iff $\mu \notin \Lambda^{(n,m)}$. \qed

It is immediate from Lemma 2 that—for parameters subject to the truncation condition in (2.4)—the $(n+m)$-dimensional subspace

\[ \mathcal{C}(\Lambda^{(n,m)}) = \{ \lambda \mathcal{D}_f \lambda, f \in \mathbb{C} \mid \lambda \in \Lambda^{(n,m)} \} \]

of functions in $\mathcal{C}(\Lambda^{(n)})$ supported on the finite lattice $\Lambda^{(n,m)}$ of bounded partitions is stable with respect to the action of $D_r$:

\[ (D_r f)\lambda = \sum_{\lambda \subset \mu \subset \lambda + 1^{n+1}, |\mu| = |\lambda| + r} B_{\mu/\lambda} f \mu \quad (f \in \mathcal{C}(\Lambda^{(n,m)}), \lambda \in \Lambda^{(n,m)}). \quad (2.6) \]

Notice in this connection that Lemma 2 ensures that all coefficients $B_{\mu/\lambda}$ in $D_r$ remain regular for parameter values in the domain (2.4); as a consequence, we will drop the transitory genericity condition in Eq. (2.3) from now on (by analytic continuation) unless explicitly stated otherwise.
Corollary 3 (Commutativity). For parameters in accordance with Eq. (2.4), the finite discrete Ruijsenaars operators $D_1, \ldots, D_n$ commute in $C(\Lambda^{(n,m)})$.

3. Hilbert space

3.1. Inner product. We consider elliptic weights on $\Lambda^{(n,m)}$ of the form

$$\Delta_{\lambda} = \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j - \lambda_k + (k-j)g]}{[(k-j)g]} \frac{[(k-j)g+1]_{\lambda_j - \lambda_k}}{[1+(k-j)g]_{\lambda_j - \lambda_k}}, \quad (3.1)$$

where $[z]_k$, $k = 0, 1, 2, \ldots$ denotes the elliptic factorial

$$[z]_k = \prod_{0 \leq l < k} [z+l] \quad \text{with} \quad [z]_0 = 1.$$

Lemma 4 (Positivity). For parameters in accordance with Eq. (2.4) and any $\lambda \in \Lambda^{(n,m)}$, the elliptic weight $\Delta_{\lambda}$ (3.1) is positive.

Proof. In view of the product formula for $[z]_k$ (1.1a), (1.1b) it is immediate from the assumptions that all factors in $\Delta_{\lambda}$ remain positive, since

$$0 < (k-j)g \leq \lambda_j - \lambda_k + (k-j)g < \frac{2\pi}{\alpha}$$

if $1 \leq j < k \leq n+1$, and

$$0 < l + (k-j+1)g < \frac{2\pi}{\alpha}, \quad 0 < l + 1 + (k-j-1)g < \frac{2\pi}{\alpha}$$

if $0 \leq l < \lambda_j - \lambda_k$.

The positivity of the weights promotes $C(\Lambda^{(n,m)})$ to a $(^m_n)$-dimensional Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ via the inner product

$$\langle f, g \rangle_{\Delta} = \sum_{\lambda \in \Lambda^{(n,m)}} f_{\lambda} \bar{g}_{\lambda} \Delta_{\lambda} \quad (f, g \in \ell^2(\Lambda^{(n,m)}, \Delta)). \quad (3.2)$$

3.2. Self-adjointness. The elliptic weights $\Delta_{\lambda}$ (3.1) obey a recurrence relation governed by the coefficients $B_{\mu/\lambda}$ (2.2c).

Lemma 5 (Recurrence for Elliptic Weights). The elliptic weights $\Delta_{\lambda}$ (3.1) satisfy the following recurrence relation:

$$B_{\lambda+\theta/\mu} \Delta_{\lambda} = B_{\mu+\theta/\mu} \Delta_{\mu}, \quad (3.3)$$

for any $\lambda \in \Lambda^{(n,m)}$ and $\mu \in \Lambda^{(n+1)}$ such that $\theta = \mu - \lambda$ is a vertical $r$-strip and $\mu \in \Lambda^{(n,m)}$. Here $\theta^c$ denotes the vertical $(n+1-r)$-strip such that $\theta + \theta^c = 1^{n+1}$.

Proof. An elementary computation reveals that

$$\Delta_{\mu} = \prod_{1 \leq j < k \leq n+1} \frac{[(k-j+1)g]_{\lambda_j - \lambda_k + \theta_j - \theta_k}}{[(k-j)g]_{\lambda_j - \lambda_k + \theta_j - \theta_k}} \frac{[(k-j)g+1]_{\lambda_j - \lambda_k + \theta_j - \theta_k}}{[1+(k-j)g]_{\lambda_j - \lambda_k + \theta_j - \theta_k}}$$

$$= \Delta_{\lambda} \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j - \lambda_k + (k-j+1)g]}{[\lambda_j - \lambda_k + (k-j)g]} \frac{[\lambda_j - \lambda_k + (k-j)g+1]}{[\lambda_j - \lambda_k + (k-j)g]}$$

$$\times \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j - \lambda_k + (k-j-1)g]}{[\lambda_j - \lambda_k + (k-j)g]} \frac{[\lambda_j - \lambda_k + (k-j)g+1]}{[\lambda_j - \lambda_k + (k-j)g]} \Delta_{\mu}.$$
Multiplication by
\[ B_{\mu+\theta/\mu} = \prod_{1 \leq j < k \leq n+1} \begin{array}{c} \frac{[\mu_j-\mu_j+(k-j+\theta_k-\theta_j)r)}{[\mu_j-\mu_j+(k-j)r)} \\
\end{array} \]
\[ = \prod_{1 \leq j < k \leq n+1} \begin{array}{c} \frac{[\lambda_j-\lambda_j+(k-j+1)r]}{[\lambda_j-\lambda_j+(k-j)r]} \\
\end{array} \prod_{1 \leq j < k \leq n+1} \begin{array}{c} \frac{[\lambda_j-\lambda_j+(k-j+1)r]}{[\lambda_j-\lambda_j+(k-j)r]} \\
\end{array} \]
yields
\[ \Delta\lambda \prod_{1 \leq j < k \leq n+1} \begin{array}{c} \frac{[\lambda_j-\lambda_j+(k-j+1)r]}{[\lambda_j-\lambda_j+(k-j)r]} \\
\end{array} \prod_{1 \leq j < k \leq n+1} \begin{array}{c} \frac{[\lambda_j-\lambda_j+(k-j+1)r]}{[\lambda_j-\lambda_j+(k-j)r]} \\
\end{array} \]
\[ = \Delta\lambda \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j-\lambda_j+(k-j+1)r]}{[\lambda_j-\lambda_j+(k-j)r]} = \Delta\lambda B_{\lambda+\theta/\lambda}. \]

With the aid of the recurrence in Lemma 5 one readily computes the adjoint of \( D_r \) in \( \ell^2(\Lambda^{(n,m)}, \Delta) \). To this end it is convenient to rewrite the action of \( D_r \) in the form
\[ (D_r f)_{\lambda} = \sum_{\theta \leq n+1, |\theta| = r} B_{\lambda+\theta/\lambda} f_{\lambda+\theta} \quad (f \in \ell^2(\Lambda^{(n,m)}, \Delta), \lambda \in \Lambda^{(n,m)}). \quad (3.4) \]
In this formula the sum on the RHS is meant to be over all \( r \)-strips \( \theta \) with \( \theta_j = 0 \) for \( j > n + 1 \), where \( B_{\lambda+\theta/\lambda} = 0 \) unless \( \lambda + \theta \in \Lambda^{(n,m)} \) by (the proof of) Lemma 2.

**Proposition 6 (Adjoint).** For parameters in accordance with Eq. (2.4), the operators \( D_r \) and \( D_{n+1-r} \) are each others adjoints in the Hilbert space \( \ell^2(\Lambda^{(n,m)}, \Delta) \), i.e.
\[ \forall f, g \in \ell^2(\Lambda^{(n,m)}, \Delta) : \quad (D_r f, g)_\Delta = (f, D_{n+1-r}g)_\Delta. \quad (3.5) \]

**Proof.** Successive manipulations hinging on Lemma 2 Eq. (5.3) and Lemma 5 sustain that
\[ (D_r f, g)_\Delta \leq \sum_{\lambda \in \Lambda^{(n,m)}} (D_r f)_{\lambda} g_{\lambda} \Delta\lambda = \sum_{\lambda \in \Lambda^{(n,m)}, \theta \leq n+1, |\theta| = r} (B_{\lambda+\theta/\lambda} f_{\lambda+\theta} g_{\lambda} \Delta\lambda = \sum_{\mu \in \Lambda^{(n,m)}, \theta \leq n+1, |\theta| = n+1-r} f_{\mu} (B_{\mu+\theta/\mu} g_{\mu+\theta} \Delta\mu = (f, D_{n+1-r}g)_\Delta. \]

**Corollary 7 (Self-adjointness).** The discrete difference operators
\[ C_r = \frac{1}{2}(D_r + D_{n+1-r}) \quad (r = 1, \ldots, \lfloor n+1 \rfloor) \quad (3.6a) \]
and
\[ S_r = \frac{1}{2}(D_r - D_{n+1-r}) \quad (r = 1, \ldots, \lfloor n/2 \rfloor) \quad (3.6b) \]
provide \( n \) commuting Ruijsenaars operators that are self-adjoint in \( \ell^2(\Lambda^{(n,m)}, \Delta) \).
Note. If one replaces the truncation condition in Eq. (2.4) with the genericity condition from Eq. (2.3), then the definition of the elliptic weight $\Delta_\lambda$ (3.1) actually makes sense for any $\lambda \in \Lambda^{(n)}$ (as the zeros of the factors in the denominator are avoided). The recurrence in Eq. (3.3) holds in this situation for any $\lambda \in \Lambda^{(n)}$ and $\mu \in \Lambda^{(n+1)}$ such that $\theta = \mu - \lambda$ is a vertical $r$-strip, even though the positivity is now no longer guaranteed.

4. Eigenfunctions

4.1. Diagonalization. The main result of this note consists of the following theorem, which describes the construction of an orthogonal basis of joint eigenfunctions for the Ruijsenaars model on the finite lattice of bounded partitions; the (values of the) eigenfunctions in question are expressed by means of polynomials in terms of the corresponding eigenvalues. To describe these polynomials we recur to following partial order on $\Lambda^{(n)}$ (which stems from the dominance ordering of the $\mathfrak{sl}(n+1;\mathbb{C})$ dominant weights via the bijection in Eq. (A.8)):

$$\forall \lambda, \mu \in \Lambda^{(n)} : \quad \lambda \leq \mu \iff \sum_{1 \leq j \leq r} (\lambda_j - \mu_j) - \frac{r(\lambda_1 - |\mu|)}{n+1} \in \mathbb{Z} \setminus \mathbb{N} \quad \text{for} \quad r = 1, \ldots, n \quad (4.1)$$

(while $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$).

Theorem 8 (Diagonalization). The following statements hold for parameters from the regime in Eq. (2.3).

(i) The discrete Ruijsenaars operators $D_1, \ldots, D_n$ (2.6) are simultaneously diagonalized in $l^2(\Lambda^{(n,m)}, \Delta)$ by an orthogonal basis of joint eigenfunctions.

(ii) Upon fixing the normalization such that its value at $\mu = 0$ is equal to 1, an element $p(e) \in l^2(\Lambda^{(n,m)}, \Delta)$ of the joint eigenbasis, satisfying

$$D_r p(e) = e_r p(e) \quad (r = 1, \ldots, n), \quad (4.2a)$$

is uniquely determined by the corresponding eigenvalues collected in $e = (e_1, \ldots, e_n)$.

(iii) The value of the joint eigenfunction $p(e)$ at $\mu \in \Lambda^{(n,m)}$ is given by

$$p_\mu(e) = c_\mu P_\mu(e) \quad \text{with} \quad c_\mu = \prod_{1 \leq j < k \leq n+1} \frac{[(k-j)\mu_j - \mu_k]}{[(k-j+1)\mu_j - \mu_k]}, \quad (4.2b)$$

Here $P_\mu(e)$ denotes a polynomial in the eigenvalues $e_1, \ldots, e_n$ of the form

$$P_\mu(e) = e_\mu + \sum_{\nu \in \Lambda^{(n,m)}, \nu < \mu} u_{\mu,\nu} e_\nu \quad \text{with} \quad e_\mu = \prod_{1 \leq j \leq n} e^{|j-j+1|}, \quad (4.2c)$$

whose expansion coefficients $u_{\mu,\nu} = u_{\mu,\nu}(g;p) \in \mathbb{R}$ are uniquely determined by the recurrence

$$P_\mu(e) = e_r P_\lambda(e) - \sum_{\lambda \subset \nu, \lambda + 1 \subset \mu, \nu = |\mu|} \psi_{\nu/\lambda} P_\nu(e), \quad (4.2d)$$

where $\lambda = \mu - 1^r$,

$$r = r_\mu = \min\{1 \leq j \leq n \mid \mu_j - \mu_{j+1} > 0\}, \quad (4.2e)$$

and

$$\psi_{\nu/\lambda} = \prod_{1 \leq j < k \leq n+1} \frac{|\nu_j - \nu_k + g(k-j+1)|}{|\nu_j - \nu_k + g(k-j)|} \frac{|\lambda_j - \lambda_k + g(k-j-1)|}{|\lambda_j - \lambda_k + g(k-j)|} \quad \text{with} \quad \theta = \nu - \lambda. \quad (4.2f)$$
(iv) The polynomials $P_{\lambda}(e)$, $\lambda \in \Lambda^{(n,m)}$ obey the following Pieri rule on the spectrum
\[
P_{\lambda}(e)P_{\lambda}(e) = \sum_{\lambda \subset \mu \subseteq \lambda + 1^{n+1}, |\mu| = |\lambda| + r, \text{s.t. } \mu \in \Lambda^{(n,m)}} \psi'_{\mu/\lambda}P_{\mu} \quad \text{for } r = 1, \ldots, n. \tag{4.2g}
\]

(v) The joint eigenfunctions $p(e)$ and $\hat{p}(\hat{e})$ satisfy the orthogonality relation
\[
\langle p(e), p(\hat{e}) \rangle_\Delta = 0 \quad \text{if } e \neq \hat{e}. \tag{4.2h}
\]

4.2. **Proof of Theorem** \[8\] (i) By Proposition \[6\] the commuting difference operators $D_1, \ldots, D_n$ (2.6) are normal in the $(n+m)$-dimensional space $\ell^2(\Lambda^{(n,m)}, \Delta)$.

Invoking of the spectral theorem for commuting normal operators in finite dimension (cf. e.g. \[G98, Chapter IX.15\] or \[HJ13, Chapter 2.5\]) thus suffices to establish the existence of an orthogonal basis of joint eigenfunctions.

(ii) & (iii) Let $p \in \ell^2(\Lambda^{(n,m)}, \Delta)$ be a joint eigenfunction of $D_1, \ldots, D_n$, which we assume to be normalized such that $p_{\lambda} = 1$ at $\lambda = 0$. In other words, we have that $D_i p = e_i p$ for some eigenvalue $e_i \in \mathbb{C}$ ($r = 1, \ldots, n$). It is immediate from the explicit product formulas in Eqs. (2.2f), (4.2b) and (4.2f) that for all $\lambda \in \Lambda^{(n,m)}$ and $\lambda \subset \mu \subseteq \lambda + 1^{n+1}$ such that $\mu \in \Lambda^{(n,m)}$:
\[
\psi'_{\mu/\lambda} = B_{\mu/\lambda} e_{\mu/\lambda}
\]

(where the parameter restriction (2.4) guarantees that $e_{\lambda} > 0$, cf. the proofs of Lemmas \[2\] and \[4\]). The eigenvalue equations for $p$ thus give rise to the following identities for $P_{\lambda} = p_{\lambda}/c_{\lambda}$ ($\lambda \in \Lambda^{(n,m)}$):
\[
e_i P_{\lambda} = \sum_{\lambda \subset \mu \subseteq \lambda + 1^{n+1}, |\mu| = |\lambda| + r, \text{s.t. } \mu \in \Lambda^{(n,m)}} \psi'_{\mu/\lambda}P_{\mu} \quad (r = 1, \ldots, n). \tag{4.3}
\]

We will now show that these identities imply that for any $\mu \in \Lambda^{(n,m)}$ the value of $P_{\mu}$ can be computed uniquely (thus proving (ii)) in terms of the polynomial in the eigenvalues $e_1, \ldots, e_n$ generated by the recurrence from (iii). To this end we perform lexicographical induction in $(d_\mu, r_\mu)$, where $d_\mu = \mu_1 - \mu_{n+1}$ refers to the degree and (recall) $r_\mu = \min\{1 \leq j \leq n \mid \mu_j - \mu_{j+1} > 0\}$ denotes the minimal column size (with the convention that $r_0 = 0$), starting from the trivial case that $d_\mu = 0$ governed by the initial condition ($d_\mu = 0 \Rightarrow \mu = 0$, so $P_{\mu} = e_{\mu} = 1$ in this trivial situation). Assuming now $d_\mu > 0$, we can write $\mu = \lambda + 1^r$ with $r = r_\mu > 0$ and $\lambda \in \Lambda^{(n,m)}$, which implies that $d_\lambda = d_{\mu} - 1$ and $\psi'_{\mu/\lambda} = 1$. The induction hypothesis now ensures that on the LHS of the rth relation in Eq. (4.3) the product $e_i P_{\lambda}$ expands as $e_i e_{\lambda} = e_{\mu}$ plus a linear combination of monomials of the form $e_i e_{\nu} = e_{\nu+1^r}$ with $\nu < \lambda$, i.e. $\nu + 1^r < \mu$; the coefficients in this expansion stem from $P_{\lambda}$ which is generated by the recurrence from (iii) (by the induction hypothesis). The terms on the RHS of the rth relation in Eq. (4.3) consist on the other hand of $P_{\mu}$ plus a linear combination of $P_{\hat{\mu}}$ with $d_{\hat{\mu}} \leq d_\mu$ and $\hat{\mu} < \mu$. Notice that for the latter terms either one has that $d_{\hat{\mu}} < d_\mu$ or one has that $d_{\hat{\mu}} = d_\mu$ and $\hat{\mu} = \hat{\lambda} + 1^r$ with $\hat{\lambda} \in \Lambda^{(n,m)}$, $d_{\hat{\lambda}} = d_{\mu} - 1$, and $1 \leq r < r_\mu$. In both cases it follows from the induction hypothesis that $P_{\hat{\mu}}$ is generated by the recurrence from (iii) and that its monomial expansion is given by $e_{\hat{\mu}}$ perturbed by a linear combination of $e_{\nu}$ with $\nu < \hat{\mu}$. Hence, by comparing the expressions on the LHS and the RHS of the
we can express $P_\nu$ as $e_\mu$ perturbed by a linear combination of monomials $e_r$ with $\nu < \mu$.

(iv) The asserted Pieri formula is now immediate from Eq. 4.3 and the observation that $P_r = e_r$ for $r = 1, \ldots, n$.

(v) It follows from Proposition 6 that

$$e_r = \frac{\langle D_r p(e), p(e) \rangle_\Delta}{\langle p(e), p(e) \rangle_\Delta} = \frac{\langle p(e), D_{n+1-r} p(e) \rangle_\Delta}{\langle p(e), p(e) \rangle_\Delta} = \bar{e}_{n+1-r},$$

and thus

$$e_r \langle p(e), p(\bar{e}) \rangle_\Delta = \langle D_r p(e), p(\bar{e}) \rangle_\Delta = \langle p(e), D_{n+1-r} p(\bar{e}) \rangle_\Delta = \bar{e}_r \langle p(e), p(\bar{e}) \rangle_\Delta.$$

Since $e_r \neq \bar{e}_r$ for some $r \in \{1, \ldots, n\}$ if $e \neq \bar{e}$, the latter identity requires that in this situation $\langle p(e), p(\bar{e}) \rangle_\Delta = 0$.

5. Trigonometric limit

5.1. Macdonald difference operators. From Eqs. 1.1a, 1.1b it is immediate that the scaled theta function $[z;p]$ is analytic in the elliptic nome $p$ for $|p| < 1$, while $[z;0] = \frac{\sin(\alpha z^2)}{\alpha^z}$ At $p = 0$ the operator $D_r$ reduces to a finite-dimensional reduction of Macdonald’s difference operator [M95, M99] governed by trigonometric coefficients of the form:

$$B_{\mu/\lambda} = \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j - \lambda_k + g(k-j+\theta) - \theta_k]_q}{[\lambda_j - \lambda_k + g(k-j)]_q}$$

with $\theta = \mu - \lambda$, (5.1a)

where

$$[z]_q = \frac{\sin(\alpha z^2)}{\sin(\alpha^2)} = \frac{q^{z^2} - q^{-z^2}}{q^2 - q^{-2}} \quad \text{with} \quad q = e^{i\alpha}. \quad (5.1b)$$

For parameters given by Eq. (2.4), the latter commuting difference operators are normal in $\ell^2(\Lambda^{(m,n)}, \Delta)$ with

$$\Delta_\lambda = \prod_{1 \leq j < k \leq n+1} \frac{[\lambda_j - \lambda_k + (k-j+\theta)\theta_j]_q}{[\lambda_j - \lambda_k + (k-j)]_q} \frac{[(k-j+1)\theta_j]_q}{[(k-j+1)\theta_j]_q} \frac{[\lambda_j - \lambda_k]_q}{[\lambda_j - \lambda_k]_q}, \quad (5.2)$$

where $[z]_{q,k} = \prod_{0 \leq l < k} [z + l]_q$ and $[z]_{q,0} = 1$. Their spectral decomposition in $\ell^2(\Lambda^{(m,n)}, \Delta)$ by means of an orthogonal basis of joint eigenfunctions constructed in terms of Macdonald polynomials goes back to [DV98, Section 4]. It is instructive to compare the eigenfunctions in Theorem 5 for $p \to 0$ with the ones from [DV98] given by Macdonald polynomials.

5.2. Macdonald polynomials. For $\lambda \in \Lambda^{(n+1)}$ let $P_\lambda(z_1, \ldots, z_{n+1}; q, t)$ denote the Macdonald polynomial [M95, Chapter VI] with a leading monomial given by

$$m_\lambda(z_1, \ldots, z_{n+1}) = \sum_{\nu \in S_{n+1}(\lambda)} z_{\nu_1}^{\nu_1} \cdots z_{n+1}^{\nu_{n+1}}. \quad (5.3)$$

Here the summation is over all compositions reordering the parts of $\lambda$ (i.e. we sum over the orbit of $\lambda$ with respect to the action of the permutation-group $S_{n+1}$ of permutations $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1})$ on $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$). The following proposition computes the eigenvalues $e = (e_1, \ldots, e_n)$ in Theorem 5 explicitly for $p \to 0$ in terms of elementary symmetric polynomials and expresses the corresponding eigenfunctions $p(e)$ in terms of Macdonald polynomials.
Proposition 9 (Diagonalization at p = 0). Let $t = q^e$ and $q = e^{i\alpha}$ with $\alpha, g$ taken from Eq. (2.4). The diagonalization and orthogonality from Theorem 5 can then be rewritten for $p \to 0$ in the following explicit form:
\[
D_r p(e_\nu) = e_{r, p}(e_\nu) \quad (r = 1, \ldots, n, \; \nu \in \Lambda^{(n, m)})
\]
and
\[
\langle p(e_\nu), p(e_\nu) \rangle_\Delta = 0 \quad \text{if } \nu \neq \nu' \quad (\nu, \nu' \in \Lambda^{(n, m)}).
\]
Here the eigenvalues collected in $e_\nu = (e_{1, \nu}, \ldots, e_{n, \nu})$ are expressed explicitly in terms of the elementary symmetric polynomials $m_{1r}$, $r = 1, \ldots, n$:
\[
e_{r, \nu} = q^{-r\left(\frac{\nu_1 + \nu_2}{2}\right)} m_{1r} (q^{\nu_1 + ng}, q^{\nu_2 + (n-1)g}, \ldots, q^{\nu_n + g}, 1),
\]
and the value of the corresponding joint eigenfunction $p(e_\nu) \in \ell^2(\Lambda^{(n, m)}, \Delta)$ at $\mu \in \Lambda^{(n, m)}$ is given by the normalized Macdonald polynomial:
\[
p_\mu(e_\nu) = c_\mu q^{-|\mu|\left(\frac{\nu_1 + \nu_2}{2}\right)} P_\mu (q^{\nu_1 + ng}, q^{\nu_2 + (n-1)g}, \ldots, q^{\nu_n + g}, 1; q, q^g)
\]
with
\[
c_\mu = \prod_{1 \leq j < k \leq n+1} \frac{[(k-j)\rho_{\nu_1, \nu_2} - \nu_k]}{[(k-j+1)\rho_{\nu_1, \nu_2} - \nu_k]}.
\]

Proof. The orthogonality (5.4b) follows from a reformulation of the finite-dimensional orthogonality relation for the Macdonald polynomials in [DV98, Eq. (4.15)] by means of the bijection from Eq. (A.3) in the appendix below (cf. also [DV98 Appendix B]). The eigenvalue equation (5.4a) amounts in turn to a corresponding reformulation of [DV98, Eq. (4.14)] (cf. also Eqs. (4.10b), (4.12) and Appendix B of [DV98]).

Epilogue

Because the vector of joint eigenvalues $e = (e_1, \ldots, e_n)$ in Theorem 8 is multiplicity free (by part (iii)), Proposition 9 entails a natural labelling of the corresponding basis of joint eigenfunctions for the finite elliptic Ruijsenaars operators $D_1, \ldots, D_n$ (2.0) in terms of bounded partitions.

Corollary 10 (Joint Spectrum). The joint spectrum in Theorem 8 is given by
\[
e_\nu = (e_{1, \nu}, \ldots, e_{n, \nu}) \in \mathbb{C}^n \quad (\nu \in \Lambda^{(n, m)})
\]
that extend analytically to $-1 < p < 1$, such that
\[
e_{r, \nu}|_{p=0} = q^{-r\left(\frac{\nu_1 + \nu_2}{2}\right)} m_{1r} (q^{\nu_1 + ng}, q^{\nu_2 + (n-1)g}, \ldots, q^{\nu_n + g}, 1)
\]
\[
(r = 1, \ldots, n).
\]

Notice at this point that since the operators in question are normal in $\ell^2(\Lambda^{(n, m)}, \Delta)$ by Proposition 8, the analyticity of $e_\nu$ (6.1a) in $p \in (-1, 1)$ is inherited from the analyticity of (the coefficients of) $D_r$ (2.0) and of $\Delta$ (8.1) (cf. [K95 Chapter 2, Theorem 1.10]).

It is now immediate from the orthogonality in Theorem 8 that the matrix
\[
[\Delta_\mu^{1/2} \Delta_\nu^{-1/2} p_\mu(e_\nu)]_{\mu, \nu \in \Lambda^{(n, m)}}
\]
is unitary, where
\[
\Delta_\nu = 1/\langle p(e_\nu), p(e_\nu) \rangle_\Delta \quad (\nu \in \Lambda^{(n, m)}).
\]
Corollary 11 (Orthogonality). The polynomials $P_\mu(e), \mu \in \Lambda^{(n,m)}$ form an orthogonal basis for the $(n+m)$-dimensional Hilbert space of (complex) functions on the joint spectrum \{e_\nu | \nu \in \Lambda^{(n,m)}\} associated with the weights $\Delta_\nu$ \eqref{6.2}:
\[
\forall \lambda, \mu \in \Lambda^{(n,m)} : \sum_{\nu \in \Lambda^{(n,m)}} P_\lambda(e_\nu)P_\mu(e_\nu) \Delta_\nu = \begin{cases} \frac{1}{c_\lambda^\Delta_\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases} \tag{6.3}
\]

APPENDIX A. DISCRETIZATION OF RUIJSENAARS OPERATORS

In this appendix it is outlined how the lattice quantum Ruijsenaars model of Section 2 is retrieved from Ruijsenaars’ commuting difference operators by discretization. To this end let us start by recalling that the $\mathfrak{sl}(n+1; \mathbb{C})$ Ruijsenaars operators are commuting difference operators with coefficients built from the Weierstrass $\sigma$-function (cf. \cite{R87, R99}):
\[
D_{r,\sigma} = \sum_{J \subset \{1, \ldots, n+1\} \atop |J|=r} \left( \prod_{j \in J \atop k \notin J} \frac{\sigma(x_j - x_k + g)}{\sigma(x_j - x_k)} \right) T_J, \quad r = 1, \ldots, n. \tag{A.1}
\]

Here $g$ denotes a real coupling parameter and $T_J$ acts by translation on complex functions $f(x) = f(x_1, \ldots, x_{n+1})$:
\[
(T_J f)(x) = f(x + \varepsilon_J) \quad \text{with } \varepsilon_J = \sum_{j \in J} \varepsilon_j,
\]

where $\varepsilon_j = e_j - \frac{1}{n+1}(e_1 + \cdots + e_{n+1})$ and $e_1, \ldots, e_{n+1}$ refers to the standard unit basis of $\mathbb{C}^{n+1}$. A straightforward similarity transformation governed by an appropriate Gaussian
\[
D_r = c_r G(x)^{-1} D_{r,\sigma} G(x),
\]
with
\[
c_r = e^{\frac{2\xi}{r(n+1-r)}g(1-g)} \quad \text{and } \quad G(x) = \exp\left( -\frac{g}{2\omega_1} \sum_{1 \leq j < k \leq n+1} (x_j - x_k)^2 \right),
\]
recasts these difference operators into the form (cf. Eq. \eqref{163}):
\[
D_r = \sum_{J \subset \{1, \ldots, n+1\} \atop |J|=r} V_J(x) T_J \quad \text{with } V_J(x) = \prod_{j \in J \atop k \notin J} \frac{|x_j - x_k + g|}{|x_j - x_k|}. \tag{A.2}
\]

We will now discretize the Ruijsenaars operator \eqref{A.2} on a translate of the $\mathfrak{sl}(n+1; \mathbb{C})$ dominant weight lattice
\[
\Lambda^{(n)} = \{ l_1 \omega_1 + \cdots + l_n \omega_n | l_1, \ldots, l_n \in \mathbb{Z}_{\geq 0} \}, \tag{A.3}
\]
which is generated by the corresponding fundamental weights $\omega_r = \varepsilon_1 + \cdots + \varepsilon_r$ ($r = 1, \ldots, n$). The pertinent translation is over a $g$-deformation of the Weyl vector
\[
\rho_g = g(\omega_1 + \cdots + \omega_n) = g \sum_{1 \leq j \leq n+1} \left( \frac{g}{2} + 1 - j \right) \varepsilon_j. \tag{A.4}
\]

To avoid singularities stemming from the denominator of $V_J(x)$ \eqref{A.2}, it will from now on be assumed that $g \in \mathbb{R}$ is generic such that
\[
\forall \lambda \in \Lambda^{(n)} : \prod_{1 \leq j < k \leq n+1} \sin \frac{g}{2} (x_j - x_k) \big|_{x = \rho_g + \lambda} \neq 0. \tag{A.5}
\]
The next lemma confirms that $D_r$ restricts in this situation to a discrete difference operator acting on lattice functions $f : (\rho_\varepsilon + \Lambda^{(n)}) \to \mathbb{C}$.

**Lemma 12.** For $\lambda \in \Lambda^{(n)}$ and $J \subset \{1, \ldots, n+1\}$ one has that

$$V_J(\rho_\varepsilon + \lambda) = 0 \quad \text{if} \quad \lambda + \varepsilon_J \notin \Lambda^{(n)}.$$  

**Proof.** Let us recall that the dominant cone $\Lambda^{(n)}$ constitutes a fundamental domain for the $\mathfrak{sl}(n+1; \mathbb{C})$ weight lattice $\{l_1 \omega_1 + \cdots + l_n \omega_n \mid l_1, \ldots, l_n \in \mathbb{Z}\}$ with respect to the action of the permutation group $S_{n+1}$ on the unit basis $e_1, \ldots, e_{n+1}$. More specifically, a vector $\lambda = \sum_{1 \leq j \leq n+1} \lambda_j e_j$ in the weight lattice belongs to $\Lambda^{(n)}$ iff $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}$. Hence, if $\lambda \in \Lambda^{(n)}$ and $\mu = \lambda + \varepsilon_J \notin \Lambda^{(n)}$ then $\mu_j - \mu_{j+1} < 0$ for some $j \in \{1, \ldots, n\}$, which implies that $j \notin J$, $j+1 \in J$ and $\lambda_j - \lambda_{j+1} = 0$. We thus pick up a zero of $V_J(x)$ at $x = \rho_\varepsilon + \lambda$ from the factor $[x_{j+1}-x_j+g]$ in this situation. \hfill \Box

With the aid of Lemma 12 we see that the discretized Ruijsenaars operator

$$(D_r f)(\rho_\varepsilon + \lambda) = \sum_{J \subset \{1, \ldots, n+1\}} V_J(\rho_\varepsilon + \lambda) f(\rho_\varepsilon + \lambda + \varepsilon_J)$$  

(A.7)

gives rise to commuting difference operators $D_1, \ldots, D_n$ in the space of lattice functions $f : (\rho_\varepsilon + \Lambda^{(n)}) \to \mathbb{C}$.

The discrete Ruijsenaars operator $D_r$ boils down to a reformulation of $D_r$ in terms of partitions, via the bijection

$$\lambda = l_1 \omega_1 + \cdots + l_n \omega_n \leftrightarrow (l_1 + \cdots + l_n, l_2 + \cdots + l_{n-1}, l_n) = \lambda$$  

(A.8)

identifying the dominant weight lattice $\Lambda^{(n)}$ with the lattice $\Lambda^{(n)}$ of partitions of length at most $n$. This bijection maps the action of the discrete Ruijsenaars operator $D_r$ on $f : (\rho_\varepsilon + \Lambda^{(n)}) \to \mathbb{C}$ to that of $D_r$ on the lattice function $\lambda \mapsto f_\lambda$, $\lambda \in \Lambda^{(n)}$ via the dictionary

$$\lambda \mapsto f_\lambda \quad \text{and} \quad \theta_j = \begin{cases} 1 & \text{if} \ j \in J \\ 0 & \text{if} \ j \notin J \end{cases}.$$  

(A.9)

Indeed, with these identifications one has that

$$\lambda + \varepsilon_J \leftrightarrow \mu \quad \text{and} \quad V_J(\rho_\varepsilon + \lambda) = B_{\mu/\lambda} \quad \text{where} \quad \mu - \lambda = \theta.$$  

(A.10)

The lattice operators $D_r$ thus inherit the commutativity from $D_r$. \hfill \Box

**ACKNOWLEDGEMENTS**

The work of JFvD was supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grant # 1210015. TG was supported in part by the NKFIH Grant K134946.

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 795471.
References

B98. E. Billey, Algebraic nested Bethe ansatz for the elliptic Ruijsenaars model, arXiv:math/9806068

DE16. J.F. van Diejen and E. Emsiz, Spectrum and eigenfunctions of the lattice hyperbolic Ruijsenaars-Schneider system with exponential Morse term, Ann. Henri Poincaré 17 (2016), 1615–1629.

DG21. J.F. van Diejen and T. Görbe, Elliptic Kac-Sylvester matrix from difference Lamé equation, Ann. Henri Poincaré (2021) https://doi.org/10.1007/s00023-021-01063-y

DV98. J.F. van Diejen and L. Vinet, The quantum dynamics of the compactified trigonometric Ruijsenaars-Schneider model, Comm. Math. Phys. 197 (1998), 33–74.

EK95. P.I. Etingof and A. Kirillov Jr., On the affine analogue of Jack and Macdonald polynomials, Duke Math. J. 78 (1995), 229–256.

FG16. L. Fehér and T.F. Görbe, Trigonometric and elliptic Ruijsenaars-Schneider systems on the complex projective space, Lett. Math. Phys. 106 (2016), 1429–1449.

FK14. L. Fehér and T.J. Kluck, New compact forms of the trigonometric Ruijsenaars-Schneider system, Nuclear Phys. B 882 (2014), 97–127.

FV97. G. Felder and A. Varchenko, Elliptic quantum groups and Ruijsenaars models, J. Statist. Phys. 80 (1997), 963–980.

G98. F.R. Gantmacher, The Theory of Matrices, vol. 1, Reprint of the 1959 translation, AMS Chebea Publishing, Providence, RI, 1998.

GH18. T.F. Görbe and M. Hallnäs, Quantization and explicit diagonalization of new compactified trigonometric Ruijsenaars-Schneider systems, J. Integrable Syst. 3 (2018), no. 1, xyy015, 29 pp.

H97. K. Hasegawa, Ruijsenaars’ commuting difference operators as commuting transfer matrices, Comm. Math. Phys. 187 (1997), 289–325.

HJ13. R.A. Horn and C.R. Johnson, Matrix Analysis, Second Edition, Cambridge University Press, Cambridge, 2013.

HSY04. B.Y. Hou, R. Sasaki, W.-L. Yang, Eigenvalues of Ruijsenaars-Schneider models associated with $A_{n-1}$ root system in Bethe ansatz formalism, J. Math. Phys. 45 (2004), 559–575.

K95. T. Kato, Perturbation Theory for Linear Operators, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

KH97. Y. Komori and K. Hikami, Quantum integrability of the generalized elliptic Ruijsenaars models, J. Phys. A 30 (1997), 4341–4364.

K01a. Y. Komori, Essential self-adjointness of the elliptic Ruijsenaars models, J. Math. Phys. 42 (2001), 4523–4553.

K01b. Y. Komori, Ruijsenaars’ commuting difference operators and invariant subspace spanned by theta functions, J. Math. Phys. 42 (2001), 4503–4522.

L14. E. Langmann, Explicit solution of the (quantum) elliptic Calogero-Sutherland model, Ann. Henri Poincaré 15 (2014), 755–791.

LNS20. E. Langmann, M. Noumi, and J. Shiraishi, Construction of eigenfunctions for the elliptic Ruijsenaars difference operators, arXiv:2012.05564

M95. I.G. Macdonald, Symmetric Functions and Hall Polynomials, Second Edition, Clarendon Press, Oxford, 1995.

M00. I.G. Macdonald, Orthogonal polynomials associated with root systems, Sém. Lothar. Combin. 45 (2000/01), Art. B45a.

MMZ21. A. Mironov, A. Morozov, Y. Zenkevich, Duality in elliptic Ruijsenaars system and elliptic symmetric functions, Eur. Phys. J. C 81 (2021), 461.

OLBC10. F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark. (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

R87. S.N.M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Comm. Math. Phys. 110 (1987), 191–213.

R95. S. Ruijsenaars, Action-angle maps and scattering theory for some finite-dimensional integrable systems. III. Sutherland type systems and their duals, Publ. Res. Inst. Math. Sci. 31 (1995), 247–353.
R99. S.N.M. Ruijsenaars, Systems of Calogero-Moser type. In: *Particles and Fields* (Banff, AB, 1994), G.W. Semenoff and L. Vinet (eds.), CRM Ser. Math. Phys., Springer, New York, 1999, 251–352.

R02. S.N.M. Ruijsenaars, Factorized weight functions vs. factorized scattering, Comm. Math. Phys. **228** (2002), 467–494.

R09. S.N.M. Ruijsenaars, Hilbert-Schmidt operators vs. integrable systems of elliptic Calogero-Moser type. I. The eigenfunction identities, Comm. Math. Phys. **286** (2009), 629–657; II. The $A_{N-1}$ case: first steps, Comm. Math. Phys. **286** (2009), 659–680.

Instituto de Matemáticas, Universidad de Talca, Casilla 747, Talca, Chile

*Email address*: diejen@inst-mat.utalca.cl

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK

*Email address*: T.Gorbe@leeds.ac.uk