MASTERING THE MASTER FIELD

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Abstract

The basic concepts of non-commutative probability theory are reviewed and applied to the large $N$ limit of matrix models. We argue that this is the appropriate framework for constructing the master field in terms of which large $N$ theories can be written. We explicitly construct the master field in a number of cases including QCD$_2$. There we both give an explicit construction of the master gauge field and construct master loop operators as well. Most important we extend these techniques to deal with the general matrix model, in which the matrices do not have independent distributions and are coupled. We can thus construct the master field for any matrix model, in a well defined Hilbert space, generated by a collection of creation and annihilation operators—one for each matrix variable—satisfying the Cuntz algebra. We also discuss the equations of motion obeyed by the master field.

*This work was supported in part by the National Science Foundation under grant PHY90-21984.
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1 Introduction

Theories invariant under $U(N)$ (or $O(N)$), in which the basic dynamical variables are $N^2$ dimensional matrices in the adjoint representation of the group, simplify greatly in the limit of large $N$. In some cases the simplification is so great that the $N = \infty$ theory is solvable. Large $N$ matrix models are of great interest for many reasons \[1\]. First, QCD is such a theory, if we regard the number of colors as a free parameter. There is much evidence that the large $N$ expansion of QCD correctly captures the essence of confinement and asymptotic freedom and that $1/3^2$ is a good expansion parameter. Second, matrix models have proved useful as devices for constructing string theories. Thus the control of the large $N$ expansion of simple matrix models led some years ago to the non-perturbative solution of toy string models in dimensions less than or equal to two \[2\]. In fact, QCD itself might be such an example, the large $N$ expansion of QCD might be described by a string theory, a goal which has been realized in two dimensions \[3\]. Therefore it is important to explore and develop all available methods for controlling large $N$ matrix models.

One of the most appealing ideas to emerge in the study of the large $N$ is that of the master field \[4\]. The idea is that there exists a particular classical matrix field such that the large $N$ limit of all $U(N)$ invariant Green’s functions are given by their values at the master field. Thus the master field is analogous to the classical field, in terms of which all correlation functions are determined in the classical, $\hbar \to 0$, limit; $1/N^2$ playing the role of $\hbar$. The argument for the existence of such a master field is simple. Consider a general matrix model. By this we mean a theory in which the dynamical variables are $N \times N$ dimensional Hermitian matrices $M_i$ with an action $S[M_i]$ that is invariant under the global $U(N)$ transformation $M_i \to U M_i U^\dagger$. Consider the correlation functions of $U(N)$ invariant observables. If we have a denumerable set of variables then the most general invariant is a function of the traces of products of the matrices, i.e., $O = \frac{1}{N} \text{Tr} [M_{i_1} M_{i_2} \ldots M_{i_n}]$ (normalized so as to have a finite limit as $N \to \infty$.) For a field theory with continuum fields, such as QCD,

\[3\]In the large $N$ limit there is no difference between $SU(N)$ (or $SO(N)$) and $U(N)$ (or $O(N)$).
we also consider continuous products of the matrix fields, such as Wilson loops. Denote the expectation value of $O$ as

$$
\langle O \rangle \equiv Z^{-1} \int \prod_i dM_i e^{-S[M_i]} \frac{1}{N} \text{Tr} [M_{i_1} M_{i_2} \ldots M_{i_n}],
$$

(1.1)

where $Z = \int \prod_i dM_i \exp(-S[M_i])$. The important property of the large $N$ limit is that the expectation value of a product of such invariant observables factorizes:\[4, 5, 6\]:

$$
\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle + O(1/N^2).
$$

(1.2)

This can be proved in perturbation theory by the analysis of the Feynman graphs. In lattice QCD it can also be proved order by order in the strong coupling expansion. Consequently, the variance of any invariant observable vanishes in the large $N$ limit, namely the probability that $O$ differs from its expectation value is of order $1/N^2$

$$
\langle (O - \langle O \rangle)^2 \rangle = \langle O^2 \rangle - \langle O \rangle^2 = O(1/N^2).
$$

(1.3)

This must mean that the path integral measure is localized on a particular set of matrices—the master field—up to a $U(N)$ transformation; just as in the classical limit the path integral measure is localized, infinitely sharply as $\hbar \to 0$, on the classical solution of the field equation.

Given the master field, i.e., a set of “$\infty \times \infty$” matrices $\bar{M}_i$, all the correlation functions of the invariant observables are then calculable as

$$
\langle O \rangle = \text{tr} [\bar{M}_{i_1} \bar{M}_{i_2} \ldots \bar{M}_{i_n}],
$$

(1.4)

where no functional integral need be done, we simply evaluate the trace of the product of master fields.

In a gauge theory, in addition to the global $U(N)$ symmetry that we used above we have a local $U(N)$ gauge symmetry. In that case, when considering gauge invariant Green’s functions, we can only conclude that the path integral is localized as $N \to \infty$ on a single gauge orbit of the gauge group. In other words, if $\bar{A}_\mu(x)$ is the master gauge field then an equivalent master field is $\bar{A}^U_\mu(x) = U(x)A_\mu(x)U^\dagger(x) - iU(x)\partial_\mu U^\dagger(x)$. 

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Given the master field for a pure gauge theory, say QCD in four dimensions, one could then calculate the meson spectrum very directly. Since in the large $N$ limit the quarks play no dynamical role, quark loops being suppressed by $1/N$, the quarks are spectators and can be integrated out. Thus, for example, the meson propagator $G(x, y) = \langle \bar{\Psi}(x) \Gamma \Psi(x) \bar{\Psi}(y) \Gamma \Psi(y) \rangle$, where $\Gamma$ is a matrix in flavor space, is given by

$$G(x, y) = \int \mathcal{D}(\bar{\Psi} \Psi) e^{-\int d^4x \left[ \bar{\Psi}(i\not\partial - m) \Psi + \frac{N}{4g^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right]} \Psi(x) \Gamma \Psi(x) \bar{\Psi}(y) \Gamma \Psi(y) = \text{tr} \langle x| \Gamma \frac{1}{i\not\partial - \bar{\mathcal{A}} - m} \Gamma \frac{1}{i\not\partial - \bar{\mathcal{A}} - m} |y\rangle. \quad (1.5)$$

Thus, if we knew the master field, $\bar{\mathcal{A}}(x)$, we could calculate the meson spectrum for $N = \infty$. In a gauge theory we can argue further that the master field can be chosen, by a choice of gauge, to be independent of space and time! This is reasonable if we think of the master field as the field configuration that yields the large $N$ saddlepoint of the path integral. Since the action and measure are translationally invariant we might expect that the saddlepoint is translationally invariant, so that $\bar{\mathcal{A}}(x)$ and $\bar{\mathcal{A}}(0)$ are equivalent up to a similarity transformation,

$$\bar{\mathcal{A}}(x) = e^{iP \cdot x} \bar{\mathcal{A}}(0) e^{-iP \cdot x}, \quad (1.6)$$

where $P_{\mu}$ plays the role of the momentum operator. If so, then we can perform a gauge transformation $\bar{\mathcal{A}}(x) \rightarrow \bar{\mathcal{A}}^U(x) = U(x) \bar{\mathcal{A}}(x) U^\dagger(x) - iU(x) \partial_{\mu} U^\dagger(x)$, with $U(x) = \exp(iP \cdot x)$, to derive a gauge equivalent, $x_{\mu}$-independent master field

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}(0) + P_\mu, \quad \bar{F}_{\mu\nu} = [\bar{\mathcal{A}}, \bar{\mathcal{A}}]. \quad (1.7)$$

Thus the complete solution of large $N$ QCD would be determined if we could write down four “$\infty \times \infty$” matrices $\bar{\mathcal{A}}_\mu$!

But what kind of matrix is $\bar{\mathcal{A}}_\mu$? What does an “$\infty \times \infty$” matrix mean? What is $U(\infty)$? To make these questions sharper let us consider a solvable example of a large $N$ matrix model, a model of $n$ independent Hermitian matrices, where the vacuum-to-vacuum amplitude is given by

$$Z = \int \prod_i \mathcal{D} M_i e^{-N \text{Tr} V_i(M_i)}, \quad (1.8)$$
where $V_i(M_i)$ is an arbitrary polynomial function of $M_i$.

The case of $n = 1$ is the one-matrix-model, which is easily solved for $N = \infty$. The invariant observables are class functions of $M$, determined by the eigenvalues $m_i, i = 1 \ldots N$, which for $N = \infty$ yield a continuous function $m(x = i/N)$. The matrix integral can be reduced to an integral over $m_i$, by diagonalizing $M = \Omega m \Omega^{-1}$ (where $m = \text{diag}(m_1, \ldots, m_n)$), and using the fact that $\mathcal{D}M = \mathcal{D}\Omega \prod_i dm_i \Delta(m_i)^2$, where $\mathcal{D}\Omega$ is the invariant Haar measure on $U(N)$ and $\Delta(m_i) = \prod_{i \leq j} (m_i - m_j)$. The eigenvalue density, $\rho(m) = \frac{1}{N} \frac{\partial}{\partial m}$ is determined for $N = \infty$ by the saddlepoint equation,

$$\frac{1}{2} V'(x) = \int \frac{d\rho(y)}{x - y}. \quad (1.9)$$

For the simplest Gaussian potential, $V(M) = \frac{1}{2} M^2$ the eigenvalues have the famous Wigner semi-circular distribution

$$\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}. \quad (1.10)$$

Thus, in the one matrix model we can say that the master matrix is an $\infty \times \infty$ matrix with eigenvalues $m_i$, where the $m_i$ are determined by $\rho(x)$, the solution of (1.9). If we now return to the $n$-matrix model, since the matrices are independent the eigenvalues of each are determined, so that we can say that the master matrices are

$$\tilde{M}_i = \Omega_i m^{(i)} \Omega_i^\dagger, \quad \text{where} \quad m^{(i)} = \text{diag}(m_{1}^{(i)}, m_{2}^{(i)}, \ldots, m_{N}^{(i)}), \quad (1.11)$$

and the $\Omega_i$ are undetermined unitary matrices. These master matrices are perfectly adequate to calculate decoupled observables such as $\langle \text{tr} M_i^\Gamma \rangle$, in which the $\Omega_i$’s do not appear. However the general invariant observable in this theory is a trace of an arbitrary product of different $M_i$’s, namely $\langle O_\Gamma \rangle$, where $\Gamma$ denotes an arbitrary word, i.e., a free product of $M_i$’s:

$$O_\Gamma = \frac{1}{N} \text{Tr} [M_{i_1} M_{i_2} \ldots M_{i_k} \ldots] \quad (1.12)$$

Here the product does depend on the $\Omega_i$’s and if we choose any particular $\Omega_i$’s in (1.11) we would not get the correct answer. Of course if we integrate over all values of the $\Omega_i$’s with Haar measure then we get the right result, however this would not be a master field description.
There is a direct, but rather ugly, way of dealing with this problem. Consider the case $n = 2$, with two independent matrices. Write each $N \times N$ dimensional master matrix as a block diagonal matrix of $K M \times M$-dimensional matrices, where $N = M \cdot K$,

$$\tilde{M}_i = \begin{pmatrix} \Omega_1^{(i)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Omega_K^{(i)} \end{pmatrix} \begin{pmatrix} m^{(i)} \\ \vdots \\ m^{(i)} \end{pmatrix} \begin{pmatrix} \Omega_1^{(i)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Omega_K^{(i)} \end{pmatrix}. \quad (1.13)$$

The $\Omega_j^{(i)}$, $j = 1 \ldots K$ are specific $M \times M$ unitary matrices chosen at random from the group (with Haar measure) and $m^{(i)} = \text{diag}(m_1^{(i)}, m_2^{(i)}, \ldots, m_M^{(i)})$, with the $m_j^{(i)}$ determined (as $M \to \infty$) by the saddlepoint eigenvalue distribution. The expectation value of arbitrary words of $M_1$ and $M_2$ will now be correctly given by the trace of these master matrices when we take both $K \to \infty$ and $M \to \infty$. For example consider

$$\frac{1}{N} \langle \text{Tr} \left[ M_1 M_2 M_1 M_2 \right] \rangle \equiv \lim_{M,K \to \infty} \frac{1}{N} \text{Tr}_M \left[ \tilde{M}_1 \tilde{M}_2 \tilde{M}_1 \tilde{M}_2 \right] = \lim_{M,K \to \infty} \frac{1}{M K} \sum_{j=1}^{K} \text{Tr}_M [V_j m^{(1)} V_j^\dagger m^{(2)} V_j m^{(1)} V_j^\dagger m^{(2)}], \quad (1.14)$$

where $V_j = \Omega_j^{(1)} \Omega_j^{(2)\dagger}$ are a set of unitary matrices chosen at random from $U(N)$. In the limit of $K \to \infty$, the average of the product of the $V$’s, $\lim_{K \to \infty} \frac{1}{K} \sum_{j=1}^{K} (V_j)_{a,b} (V_j^\dagger)_{c,d} (V_j)_{e,f} (V_j^\dagger)_{g,h}$, is equal to the integral of the product over Haar measure, which for $M \to \infty$ is:

$$\int \mathcal{D}V (V_j)_{a,b} (V_j^\dagger)_{c,d} (V_j)_{e,f} (V_j^\dagger)_{g,h} = \frac{1}{M^2} \left[ \delta_{ad} \delta_{be} \delta_{eh} \delta_{fg} + \delta_{ah} \delta_{bg} \delta_{ec} \delta_{gf} \right] - \frac{1}{M^3} \delta_{ad} \delta_{bg} \delta_{eh} \delta_{fc}. \quad (1.15)$$

This is simply the law of large numbers. Inserting this into (1.14) yields

$$\frac{1}{N} \langle \text{Tr} \left[ M_1 M_2 M_1 M_2 \right] \rangle = \frac{1}{N^3} \left[ \text{Tr} \left[ \tilde{M}_1^2 \right] \left( \text{Tr} \left[ \tilde{M}_2 \right] \right)^2 + \left( \text{Tr} \left[ \tilde{M}_1 \right] \right)^2 \text{Tr} \left[ \tilde{M}_2^2 \right] - \frac{1}{N} \left( \text{Tr} \left[ \tilde{M}_1 \right] \right)^2 \left( \text{Tr} \left[ \tilde{M}_2 \right] \right)^2 \right]. \quad (1.16)$$

which is the correct answer. Thus, the master matrices $\tilde{M}_i, i = 1, 2$, given by (1.13), will yield all invariant Green’s functions, i.e., arbitrary words made out of $M_1$ and $M_2$. This construction can be generalized to the case of an arbitrary number of independent matrices, at the price of imbedding the diagonal matrices $m^{(i)}$ in larger and larger block matrices. This
is a very awkward construction. It indicates however the nature of the “$\infty \times \infty$” matrices that will be required to represent the master field.

Recently I. Singer [7] has presented an abstract existence proof for the master field for QCD$_2$ and pointed out the relationship to the work of Voiculescu on non-commutative probability theory [8]. Indeed, Voiculescu’s methods yield a much more satisfactory framework for representing the master field for independent matrix models [9]. More important we have been able to generalize these methods to deal with the most general matrix model, including QCD in any dimension, thus *yielding an explicit representation of the master field for any and all matrix models*. We do not mean that all matrix models are solvable, but rather that we can define a well defined Hilbert space and a well defined trace operation in which the master field of any matrix model can be explicitly constructed, if one possesses enough information about the solution of the theory. Although this construction can be viewed as repackaging it seems that the language that we shall review and develop is very appropriate to the $N = \infty$ theory and might lead to new methods for constructing the master field, or equivalently for solving the $N = \infty$ theory.

In Section 2 we discuss the general framework of non-commutative probability theory developed by Voiculescu, define the notion of *free random variables* and the construction of an appropriate Hilbert space in which the master fields of models of independent matrices can be constructed. We explore this construction for the most general such independent matrix model. We note that the generating functional introduced by Voiculescu in his construction of the representation of a free random variable has the interpretation of the generating functional of planar connected Green’s functions. We also show that the master field can be regarded as the solution of a certain *master field equation of motion*.

In Section 3 we consider the explicit construction of the master field for some particular solvable gauge theories. We first find an alternative, manifestly Hermitian form of the matrix field for independent Hermitian matrix models. We reformulate the master equations of motion in a form that is more useful. We then discuss the simplest gauge theory, the one-plaquette model, which undergoes a large-$N$ phase transition as a function of coupling. Here
we will find two master fields, one for each region of coupling.

In Section 4 we turn from theories of independent matrices to the general case of coupled matrices. Based on our interpretation of the generating functional introduced by Voiculescu in his construction of the representation of a free random variable we give a graphical proof of the construction of the master field for independent matrices. This argument can then be extended to deal with more general matrix models. We show that the master field for any number of coupled matrices can be formulated within the same Hilbert space as before and give its explicit construction. That is,

**If we can solve a matrix model then we can write an explicit expression for the master field as an operator in a well defined Hilbert space, whose structure only depends on the number of matrix variables.**

Section 5 is devoted to the construction of the master field for QCD$_2$. Here we shall give an explicit construction of the master field and show that we can choose a gauge in which it is spacetime independent.

In Section 6 we discuss an alternate description of QCD$_2$ in terms of loops. We construct master loop operators based on the observation that simple loops corresponded to free random variables and that any loop could be decomposed into words built out of simple loops. The simple structure of QCD$_2$ is then a consequence of the fact that these form a multiplicative free family. We use these master loop fields to recover the master gauge field.

Finally, in the last section we shall discuss some of the many directions of research that are suggested by this construction.

## 2 Non-Commutative Probability Theory

Voiculescu has introduced the concept of free random variables for non-commutative probability theory, which seems to be the appropriate mathematical framework for constructing the master field. We shall start by reviewing this framework, with no pretense at mathematical
rigor. For more details we refer the reader to [8].

2.1 Free Random Variables

For ordinary commuting random variables the notion of independence is simple, namely the probability measure of the random variables \( x_i \) factorizes,

\[
\mu(x_1, \ldots, x_n) = \prod_i \mu(x_i).
\]

Consequently the expectation value of products of functions of the \( x_i \)'s factorize

\[
\langle f_1(x_1)f_2(x_2) \ldots f_n(x_n) \rangle = \int \mu(x_1, \ldots, x_n)f_1(x_1)f_2(x_2) \ldots f_n(x_n)
= \prod_i \int \mu(x_i)f_i(x_i) = \langle f_1(x_1) \rangle \langle f_2(x_2) \rangle \ldots \langle f_n(x_n) \rangle.
\]

(2.1)

For non-commuting random variables this definition is much too strong. There is a weaker definition, that of free random variables which is conceptually analogous to independence, though completely non-commutative.

A non-commutative probability space is called free if the expectation value of products of functions of the non-commuting variables \( M_i \) vanish if the expectation value of all the individual functions vanish:

\[
\langle f_1(M_{i_1})f_2(M_{i_2}) \ldots f_n(M_{i_n}) \rangle = 0 \quad \text{if} \quad \begin{cases} 
\langle f_i(M_{i_k}) \rangle = 0 & \text{for all } k=1, \ldots, n-1 \\
\text{and } i_k \neq i_{k+1} & k=1, \ldots, n
\end{cases}.
\]

(2.2)

Note that in the above product the neighboring functions must be of different random variables.

This is a much weaker condition than the previous definition where, because of factorization, the expectation value vanishes if any of the individual expectation values vanish. Nonetheless, it is a powerful restriction on the non-commutative probability space, that is sufficient to express all expectation values of products of different variables in terms of the individual expectation values. This is shown by considering the product

\[
\langle [f_1(M_{i_1}) - \langle f_1(M_{i_1}) \rangle] [f_2(M_{i_2}) - \langle f_2(M_{i_2}) \rangle] \ldots [f_n(M_{i_n}) - \langle f_n(M_{i_n}) \rangle] \rangle = 0.
\]

(2.3)
Expanding this product one can express the expectation value of a product of \( n \) functions in terms of expectation values of \( n - 1, n - 2 \ldots \) functions. Iterating this procedure one can express the expectation value in terms of individual expectation values.

The expectation values of free random variables are obviously \textit{not} symmetric under the interchange of different non-commuting variables. However, it is a remarkable fact that if the variables are free, i.e., (2.2) is satisfied, then the expectation value is cyclically symmetric. This can be proved using the same strategy we just employed, namely (2.2) can be used to inductively show that if the expectation value of 2, 3, \ldots \( n \) variables is cyclic then it follows that the same is true for the expectation value of \( n + 1 \) variables. For details see [8].

The advantage of this definition is that independent matrix models in the limit of \( N = \infty \) are free non-commuting random variables. To see this we denote

\[
\text{tr} [f_1(M_{i_1})f_2(M_{i_2}) \ldots f_n(M_{i_n})] = \lim_{N \to \infty} \frac{1}{N} (\text{Tr} [f_1(M_{i_1})f_2(M_{i_2}) \ldots f_n(M_{i_n})]), \quad (2.4)
\]

where the expectation value is taken with the measure given by (1.8). Assume that the individual \( f_i \)'s have vanishing expectation value, \( \text{tr} [f_k(M_{i_k})] = 0, k = 1, \ldots, n \) and consider the Feynman diagrams that contribute to the product in perturbation theory. A given matrix \( M_i \) must be contracted, when we use Wick’s theorem, with the same \( M_i \) appearing in another, non-neighboring, term. Contracting two \( M_i \)'s will split the trace into a product of lower order traces that, when \( N = \infty \), factorize. Thus one can prove the above claim inductively. Of course in this case \( \text{tr} \) is manifestly cyclic, as it must be for free random variables.

We can use the fact that independent matrix models describe free random variables to disentangle the expectation values of arbitrary words. Thus, using the above method we see that

\[
\text{tr} [M_1M_2M_1M_2] = 2\text{tr} [M_1]\text{tr} [(M_2)^2M_1] + 2\text{tr} [M_2]\text{tr} [(M_1)^2M_2] - (\text{tr} [M_1])^2\text{tr} [(M_2)^2] - (\text{tr} [M_2])^2\text{tr} [(M_1)^2] - 4\text{tr} [M_1]\text{tr} [(M_2)]\text{tr} [M_1M_2] + 4(\text{tr} [M_1])^2(\text{tr} [M_2])^2 - (\text{tr} [M_1])^2(\text{tr} [M_2])^2 = \text{tr} [M_1]\text{tr} [(M_2)^2] + \text{tr} [M_2]\text{tr} [(M_1)^2] - (\text{tr} [M_1])^2(\text{tr} [M_2])^2, \quad (2.5)
\]

which agrees with (1.16). Therefore we see that the notion of free random variables auto-
matically captures the content of Haar measure for independent matrix variables in the limit of \( N = \infty \).

2.2 The Hilbert Space Representation of Free Random Variables

Given a collection of free random variables, \( \{M_i\}, i = 1, \ldots, n \), the correlation functions \( \langle M_{i_1}^{n_1} \cdots M_{i_k}^{n_k} \rangle \) are linear functionals on the free algebra generated by the \( M_i \)'s. There exists a very general mathematical construction that associates elements of a \( C^* \) algebra (with a positive linear functional \( \phi \) defined on it), with operators on a Hilbert space with a distinguished unit vector \( |\Omega\rangle \). In the case of matrix models of Hermitian or unitary matrices there is a natural involution operation—the adjoint, so that we wish to consider cases in which the above free algebra is actually a \( C^* \) algebra. States on this Hilbert space are generated by

\[
|M_i \ldots M_n\rangle \equiv \hat{M}_{i_1} \cdots \hat{M}_{i_n} |\Omega\rangle,
\]

where \( \hat{M}_i \)'s are the operators that represent the \( M_i \)'s. The inner product on this Hilbert space is defined via the linear functional \( \phi \)

\[
\langle A| B \rangle = \phi(A^\dagger B),
\]

where \( |A\rangle \) and \( |B\rangle \) are states of the form given in (2.6). In particular

\[
\langle \Omega| \hat{M}_{i_1} \cdots \hat{M}_{i_n} |\Omega\rangle = \langle \Omega| M_{i_1} \cdots M_{i_n} \rangle = \phi(M_{i_1} \cdots M_{i_n}).
\]

In the case of matrix models where our linear functionals are expectation values with respect to the measure \( \prod_i \mathcal{D}M_i \exp[-V_i(M_i)] \), together with the trace, we recognize that the above apparatus is the appropriate framework for constructing the master matrix operators. We see from the GNS construction that the required Hilbert space is huge—a Fock-like space consisting of states labeled by arbitrary words in the \( M_i \)'s. This is in agreement with our discussion of the master field above where we argued that the Hilbert space would have to be very large.

\[\uparrow\text{This is the Gelfand-Naimark-Segal (GNS) construction. See [10]}\]
For a one-matrix model-involving the matrix $M$ the space is actually quite simple and can be described by states labelled by $|\Omega\rangle$, $|M\rangle = \hat{M}|\Omega\rangle$, $|M^2\rangle = \hat{M}^2|\Omega\rangle$, $|M^n\rangle = \hat{M}^n|\Omega\rangle$. However, for a matrix model with $n$ independent matrices $M_i$ the Fock space of words is isomorphic to the an arbitrary ordered tensor product of one matrix Hilbert spaces. Note that the order is important since $\hat{M}_1\hat{M}_2\hat{M}_3|\Omega\rangle \neq \hat{M}_1\hat{M}_3\hat{M}_2|\Omega\rangle$.

An ordinary Fock space of totally symmetric or anti-symmetric states is generated by commuting or anti-commuting creation operators acting on the vacuum. We might try to construct the above Hilbert space in an analogous fashion, by creation operators $\hat{a}^\dagger_i$, for each $M_i$, acting on the vacuum $|\Omega\rangle$. However, since the words are all distinguishable we would have to use creation operators with no relations, i.e., there would be no relation between $\hat{a}^\dagger_i\hat{a}^\dagger_j$ and $\hat{a}^\dagger_j\hat{a}^\dagger_i$. This is indeed the case. As shown in [8] the above Hilbert space is identical to the Fock space constructed by acting on a vacuum state with creation operators $\hat{a}^\dagger_i$, one for each $M_i$, and that $\hat{M}_i$ can be represented in terms of $\hat{a}^\dagger_i$ and its adjoint $\hat{a}_i$. Specifically the Fock space is spanned by the states

$$(\hat{a}^\dagger_{i_1})^{n_{i_1}}(\hat{a}^\dagger_{i_2})^{n_{i_2}}\ldots(\hat{a}^\dagger_{i_k})^{n_{i_k}}|\Omega\rangle,$$

where

$$\hat{a}_i|\Omega\rangle = 0, \quad \hat{a}_i\hat{a}_j^\dagger = \delta_{ij}.$$ (2.10)

This is not an ordinary Fock space. There are no additional relations between different $\hat{a}_i$'s or different $\hat{a}_i^\dagger$'s, or even for $\hat{a}_j\hat{a}_i^\dagger$, except for the one that follows from completeness

$$\sum_i \hat{a}_i^\dagger\hat{a}_i = 1 - P_\Omega = 1 - |\Omega\rangle\langle\Omega|.$$ (2.11)

In the case of the one-matrix model this implies that $[\hat{a}, \hat{a}^\dagger] = P_\Omega$.

This algebra of the $\hat{a}_i$'s and the $\hat{a}_i^\dagger$'s is called the Cuntz algebra. It can also be regarded as a deformation of the ordinary algebra of creation and annihilation operators. Indeed it is the $q = 0$ case of the $q$-deformed algebra

$$\hat{a}_i\hat{a}_j^\dagger - q\hat{a}_j^\dagger\hat{a}_i = \delta_{ij},$$ (2.12)
an algebra that interpolates between bosons (for $q = 1$) and fermions ($q = -1$). The above space can be regarded as the Fock space we would use to describe the states of distinguishable particles, i.e., those satisfying Boltzmann statistics. Working in such a space is very different from working in ordinary bosonic Fock spaces. In some sense it is much more difficult, since we must remember the order in which the state was constructed. Thus simple operators in ordinary Fock space can become quite complicated here. For example the number operator in the case $n = 1$ is given by

$$\hat{N} =: \frac{\hat{a}^\dagger \hat{a}}{1 - \hat{a}^\dagger \hat{a}} := \sum_{k=1}^{\infty} (\hat{a}^\dagger)^{k} \hat{a}^{k}, \quad (2.13)$$

and obeys the usual commutation relations with $a$ and with $a^\dagger$. The reason that even such a simple operator is of infinite order in $\hat{a}$ and $\hat{a}^\dagger$ is that it must measure the presence of each particle in the state, thus it must be the sum of the operators $(\hat{a}^\dagger)^{k} \hat{a}^{k}$ that count whether a state has a particle in the $k$th position. In the general case, for any $n$, the corresponding number operator is given by

$$\hat{N} = \sum_{k=1}^{\infty} \sum_{i_{1},...,i_{k}} \hat{a}_{i_{1}}^\dagger \ldots \hat{a}_{i_{k}}^\dagger \hat{a}_{i_{k}} \ldots \hat{a}_{i_{1}}. \quad (2.14)$$

Clearly we need to develop methods for working in such strange spaces.

2.3 The Fock Space Representation of $\hat{M}_{i}$

It remains to show that we can construct an operator $M_{i}$, in terms of $\hat{a}_{i}$ and $\hat{a}_{i}^\dagger$ that reproduces the moments of the matrix $M_{i}$. Thus, suppressing the indices $i$, we wish to find an operator $M(\hat{a}, \hat{a}^\dagger)$ in the Fock space so that

$$\text{tr} [M^{p}] = \lim_{N \to \infty} \int \mathcal{D}M e^{-N \text{Tr}(M)} \frac{1}{N} \text{Tr} [M^{p}] = \langle \Omega | \hat{M}(\hat{a}, \hat{a}^\dagger) | \Omega \rangle. \quad (2.15)$$

Such an operator is clearly not unique, since we can always make a similarity transformation $M \to S^{-1}MS$, where $S$ leaves the vacuum unchanged $S | \Omega \rangle = | \Omega \rangle$ and $\langle \Omega | S^{-1} = \langle \Omega |.$

$\dagger$ Greenberg has discussed such particles with “infinite statistics” [12]
Voiculescu shows that we can always find such an operator in the form
\[ M(a, a^\dagger) = a + \sum_{i=0}^{\infty} M_n a^i, \]  
with an appropriate choice of the coefficients \( M_n \). To determine the coefficients we note that
\[
\text{tr}[M] = \langle \Omega | \hat{M} | \Omega \rangle = M_0; \quad \text{tr}[M^2] = \langle \Omega | \hat{M}^2 | \Omega \rangle = M_1 + M_0^2; \\
\text{tr}[M^p] = \langle \Omega | \hat{M}^p | \Omega \rangle = M_p + (\text{polynomial in } M_0, M_1, \ldots, M_{p-1}).
\]  
(2.17)

Therefore we can recursively construct \( M_0, M_1, \ldots, M_p \) in terms of \( \text{tr}[M], \text{tr}[M^2], \ldots, \text{tr}[M^p] \).

To construct the explicit form of these coefficients we establish the following lemma.

**Lemma** Given an operator of the form \( \hat{T} = \hat{a} + \sum_{i=0}^{\infty} t_n \hat{a}^i \) we associate the holomorphic function \( K = \frac{1}{z} + \sum_{i=0}^{\infty} t_n z^n \). Then
\[
\langle \Omega | F'(\hat{T}) | \Omega \rangle = \oint_C \frac{dz}{2\pi i} F[K(z)],
\]  
(2.18)

where \( C \) is a contour in the complex \( z \) plane around the origin.

To prove the lemma it is sufficient to prove it for monomial \( F' \)'s, namely to prove that
\[
n \langle \Omega | \hat{T}^{n-1} | \Omega \rangle = \oint_C \frac{dz}{2\pi i} K^n(z). \]

But \( n \langle \Omega | \hat{T}^{n-1} | \Omega \rangle = n \text{Tr}[\hat{T}^{n-1} P_\Omega] \). Then we use the fact that \( [\hat{T}, \hat{a}^\dagger] = [\hat{a}, \hat{a}^\dagger] = P_\Omega \) to write
\[
n \langle \Omega | \hat{T}^{n-1} | \Omega \rangle = n \text{Tr}[\hat{T}^{n-1}[\hat{T}, \hat{a}^\dagger]] = \text{Tr}[\hat{T}^n, \hat{a}^\dagger],
\]  
(2.19)

where the last equality follows from the fact that \( \text{Tr}[\hat{T}^n, \hat{a}^\dagger] = \sum_{i=0}^{n-1} \hat{T}^i [\hat{T}, \hat{a}^\dagger] \hat{T}^{n-i-1} \) and the fact that \( [\hat{T}^n, \hat{a}^\dagger] \) is a trace class operator. Finally we use the fact that if \( \hat{T}_f \) is the operator associated with the function \( f(z) \), that has the Laurent expansion \( f = \sum_{n=-\infty}^{\infty} f_n z^n \), i.e., \( \hat{T}_f = \sum_{n=1}^{\infty} f_n \hat{a}^n + f_0 + \sum_{n=1}^{\infty} f_n \hat{a}^i \), then
\[
\text{Tr}[\hat{T}_f, \hat{T}_g] = \oint_C \frac{dz}{2\pi i} f(z)g'(z).
\]  
(2.20)

It is sufficient to establish this formula for the case where \( f(z) \) and \( g(z) \) are monomials, then (2.20) follows by additivity. Consider \( f(z) = f_n z^n \) so that \( \hat{T}_f = f_n \hat{a}^n \). Clearly \( \text{Tr}[T_f, T_g] \)
will vanish unless $\hat{T}_g = g^{-n}\hat{a}^n$, i.e., $g(z) = g^{-n}z^{-n}$. Using $\hat{a}^\dagger n|m\rangle = \hat{a}^{(n+m)}|\Omega\rangle = |n + m\rangle$, 

$$\text{Tr}[\hat{T}_zn, \hat{T}_zn] = \sum_{m=0}^{\infty} \langle m|\hat{a}^\dagger n\hat{a}^n - \hat{a}^n\hat{a}^\dagger n|m\rangle = -\sum_{m=0}^{n-1} \langle m-n|m-n\rangle + \sum_{m=n}^{\infty} \langle m + n|m + n\rangle - \langle m - n|m - n\rangle = -n = \oint C \frac{dz}{2\pi i} z^n \frac{dz^{-n}}{dz}. \quad (2.21)$$

Using this formula to evaluate (2.19) we establish (2.18) for polynomial functions, namely

$$n \langle \Omega|\hat{T}^{-1}n\Omega\rangle = \oint C \frac{dz}{2\pi i} K^n(z). \quad (2.19)$$

We now apply this formula to determine the form of the operator $\hat{M}$ that reproduces the moments of the matrix $M$. Assuming that we have found such an operator, so that (2.15) holds. Then we can express the resolvent, $R(\zeta)$, the generating functional of the moments

$$R(\zeta) \equiv \sum_{n=0}^{\infty} \zeta^{-n-1} \text{tr}[M^n] = \text{tr}\left[\frac{1}{\zeta - M}\right] = \int dx \frac{\rho(x)}{\zeta - x}, \quad (2.22)$$

as

$$R(\zeta) = \sum_{n=0}^{\infty} \zeta^{-n-1} \langle \Omega|\hat{M}^n|\Omega\rangle = \sum_{n=0}^{\infty} \frac{1}{n+1} \zeta^{-n-1} \oint C \frac{dz}{2\pi i} M^{n+1}(z) = -\oint C \frac{dz}{2\pi i} \log[\zeta - M(z)], \quad (2.23)$$

where $M(z) = 1/z + \sum M_n z^n$. Now changing variables in the integral, $M(z) = \lambda$, $z = M^{-1}(\lambda) = H(\lambda)$ we have

$$R(\zeta) = -\oint C \frac{d\lambda}{2\pi i} H'(\lambda) \log[\zeta - \lambda] = \oint C \frac{d\lambda}{2\pi i} \frac{H(\lambda)}{\zeta - \lambda} = H(\zeta). \quad (2.24)$$

Therefore we find that $M(z)$ is the inverse, with respect to composition, of the resolvent, i.e., $R(M(z)) = M(R(z)) = z$.

This allows us to construct the master field for the one-matrix model explicitly, since the resolvent can be constructed algebraically in terms of the potential $V(M)$. In the simplest case of a Gaussian, $V(M) = \frac{1}{2\alpha} \text{Tr}[M^2]$, we have

$$G(z) = \frac{z - \sqrt{z^2 - 4\alpha}}{2\alpha} = \frac{2}{z + \sqrt{z^2 - 4\alpha}} \Rightarrow M(z) = \frac{1}{z} + \alpha z; \quad \hat{M} = \hat{a} + \alpha \hat{a}^\dagger. \quad (2.25)$$

This form for the Gaussian master field can be made explicitly Hermitian by a similarity transformation, using the number operator constructed above. Indeed if we take $S = \exp[-\frac{1}{2} \log \alpha \hat{N}]$, then

$$\hat{M} \rightarrow S\hat{M}S^{-1} = \sqrt{\alpha}[\hat{a} + \hat{a}^\dagger] \equiv \sqrt{\alpha} \hat{x}. \quad (2.26)$$
2.4 Connected Green’s Functions

For a non-Gaussian one-matrix model the master matrix $\hat{M} = \hat{a} + \sum_{n=0}^{\infty} M_n \hat{a}^{\dagger n}$ will have an infinite number of non-vanishing $M_n$’s. The function $M(z) = 1/z + \sum M_n z^n$ has, however, a simple interpretation. Let us recall the relation between the generating functional, $G(j)$, of Green’s functions and the generating functional of connected Green’s functions,

$$G(j) = \sum_{n=0}^{\infty} j^n \langle \text{tr} [M^n] \rangle = \frac{1}{j} R(1/j); \quad \psi(j) \equiv \sum_{n=0}^{\infty} j^n \langle \text{tr} [M^n] \rangle_{\text{conn.}} = \sum_{n=0}^{\infty} j^n \psi^n. \quad (2.27)$$

As shown by Brezin et.al. [12] the usual relation that $\psi = \log[G]$ does not hold for planar graphs. Rather the full Green’s functions can be obtained in terms of the connected ones by replacing the source $j$ in $\psi(j)$ by the solution of the implicit equation

$$z(j) = j \psi(z(j)). \quad (2.28)$$

Consequently, if one solves (2.28) for $z(j)$ then

$$G(j) = \psi(z(j)) = \frac{1}{j} R(1/j) \Rightarrow R(1/j) = z(j) \Rightarrow \frac{\psi(z(1/j))}{z(1/j)} = \frac{\psi(R(j))}{R(j)} = j. \quad (2.29)$$

Therefore the the function $\psi(z)/z$ is the inverse, with respect to convolution, of the resolvent $R(z)$. But we established above that $M(z)$ is the inverse of $R(z)$. Consequently

**The master field function $M(z)$ is such that $z M(z)$ is the generating functional of connected Green’s functions.**

This explains why in the Gaussian case $z M(z) = 1 + \alpha z^2$, since the only non-vanishing n-point function is the 2-point function, and why $M(z)$ will be an infinite series in $z$ for non-Gaussian distributions. Since the resolvent is a solution of an algebraic equation of finite order, for a polynomial potential, [12] it follows that $M(z)$ is a solution of an algebraic equation as well. This interpretation suggests a direct graphical derivation of the form of the master field that we shall present in Section 6 and that will prove to be the basis for generalizing this construction to the case of dependent matrices.
2.5 Equations of Motion

There are many ways in which independent matrix models can be solved. Saddle point equations, orthogonal polynomials or Schwinger-Dyson equations of motion. The later approach is particularly simple and leads to equations of motion for our master fields. The Schwinger Dyson equations of motion for the one-matrix model follow form the identity

$$\int \mathcal{D}M \sum_{ij} \frac{\partial}{\partial M_{ij}} \{ \exp[-N \text{Tr} V(M)] f(M)_{ij} \} = 0,$$

(2.30)

for an arbitrary function $f$ (a sum of polynomials) of $M$. Using the fact that

$$\frac{\partial}{\partial M_{ij}} (M^n)_{ab} = \sum_{j=0}^{n-1} (M^j)_{ai} (M^{n-j-1})_{jb},$$

(2.31)

and the factorization theorem for $N = \infty$, we derive for $f(M) = M^n$

$$\langle \frac{1}{N} \text{Tr} [V'(M) M^n] \rangle = \sum_{j=0}^{n-1} \langle \frac{1}{N} \text{Tr} [M^j] \rangle \langle \frac{1}{N} \text{Tr} [M^{n-j-1}] \rangle.$$

(2.32)

These equations yield recursion relations for the moments of $M$ that can be used to solve for the resolvent.

The $N = \infty$ equations can be reformulated in terms of the master field as

$$\langle \Omega | V'(\hat{M}) - \frac{\delta}{\delta \hat{M}} \rangle \hat{M} f(\hat{M}) | \Omega \rangle = 0,$$

(2.33)

for arbitrary $f(\hat{M})$. In this equation we must define what we mean by the derivative with respect to the master field. This is defined as

$$\frac{\delta}{\delta \hat{M}} \cdot f(\hat{M}) \equiv \lim_{\epsilon \to 0} \frac{f(\hat{M} + \epsilon P_\Omega) - f(\hat{M})}{\epsilon},$$

(2.34)

so that

$$\langle \Omega | \frac{\delta}{\delta \hat{M}} \cdot \hat{M}^n | \Omega \rangle = \sum_{j=0}^{n-1} \langle \Omega | \hat{M}^j | \Omega \rangle \langle \Omega | \hat{M}^{n-j-1} | \Omega \rangle.$$

(2.35)

With this definition (2.33) is equivalent to (2.32). Below we shall recast these equations in a form that might prove more useful.
2.6 The Hopf equation

The Hopf equation appears often in the treatment of large $N$ matrix models. It arises in the collective field theory description of $QCD_2$ [19, 18], where it determines the evolution of eigenvalue densities. It is also the equation of motion of the $c = 1$ matrix model [15] and governs the behavior of the Itzykson-Zuber integral [14]. We shall see that it arises very naturally in the context of non-commutative probability theory for families of free random variables.

Let us first introduce the concept of an additive free family. Given two free random variables $\hat{M}_1$ and $\hat{M}_2$, with distributions $\mu_1$ and $\mu_2$, their sum $\hat{M}_1 + \hat{M}_2$ has a distribution $\mu_3$ denoted by $\mu_1 \oplus \mu_2$. A one parameter family of free random variables, such that $\mu_{t_1} \oplus \mu_{t_2} = \mu_{t_1 + t_2}$, will be called an additive free family. In ordinary probability theory the distribution of the sum of two random variables is given by the convolution of the two individual distributions. However the Fourier transform is additive, i.e., we add the Fourier transforms of the individual distributions to get the Fourier transform of the sum. The non-commutative analog of the Fourier transform is the $R$-transform that we have already encountered above. In section 2.3 we represented the free random variable $M$ by the operator,

$$\hat{M} = \hat{a} + \sum_{n=0}^{\infty} M_n \hat{a}^\dagger_n,$$  (2.36)

with the associated series,

$$M(z) = \frac{1}{z} + \sum_{n=0}^{\infty} M_n z^n \equiv \frac{1}{z} + R(z)$$  (2.37)

Then it is shown in [8] that $R(z)$ is additive [8]. Namely, if $\hat{M}_1$ and $\hat{M}_2$ are two free random variables with $R$-transforms $R_1$ and $R_2$ respectively, then $\hat{M}_1 + \hat{M}_2$ has a distribution described by

$$\hat{M} = \hat{a} + R(\hat{a}^\dagger), \text{ with } R(z) = R_1(z) + R_2(z).$$  (2.38)

It immediately follows that for an additive free family, $R(z)$ must be linear in $t$. Thus,

**This also enables one to establish a central limit theorem for free random variables [8].**
for example, a free Gaussian additive family has

$$M(z, t) = \frac{1}{z} + tz,$$

(2.39)

corresponding to the family of distributions $\int \mathcal{D} M \exp[-\frac{1}{2t} \text{Tr} M^2]$. In general, for an additive free family

$$M(z, t) = \frac{1}{z} + t\varphi(z).$$

(2.40)

where $\varphi(z)$ need not be linear in $z$.

Consider the distribution for the free random variable $\hat{N}(t) = \hat{N}_0 + \hat{M}(t)$ where $\hat{N}_0$ is free with respect to the $\hat{M}$’s which are Gaussian, but otherwise has some arbitrary distribution. Due to the additivity of $\mathcal{R}(z)$,

$$\mathcal{R}_N(z) = \mathcal{R}_0(z) + tz.$$  

(2.41)

We shall show that the resolvent $R(\zeta, t)$, which is the inverse of $N(z, t) = \frac{1}{z} + \mathcal{R}_N(z)$ obeys the Hopf equation,

$$\frac{\partial R}{\partial t} + R \frac{\partial R}{\partial \zeta} = 0.$$  

(2.42)

To see this note that if

$$\zeta = \frac{1}{z} + \mathcal{R}_N(z) = \frac{1}{z} + \mathcal{R}_0(z) + tz$$

(2.43)

then,

$$R(\zeta, t) = R\left(\frac{1}{z} + \mathcal{R}_0(z) + tz, t\right) = z \Rightarrow \frac{dR}{dt}|_z = 0$$

$$\Rightarrow 0 = \frac{\partial R}{\partial t} |_{\zeta} + \frac{\partial R}{\partial \zeta} |_{\zeta} \frac{\partial \zeta}{\partial t} |_{z} = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial \zeta} z = \frac{\partial R}{\partial t} + R \frac{\partial R}{\partial \zeta}.$$  

(2.44)

This explains the ubiquitous appearance of the Hopf equation in large $N$ theories. In particular we can understand the origin of the Hopf equation in the $c = 1$ matrix model \[^3\]. It is easy to see from this argument that if instead of being Gaussian $\hat{M}(t)$ were some other additive free family, as described by (2.40), then the equation for the resolvent $R(\zeta, t)$ would be modified to

$$\frac{\partial R}{\partial t} + \varphi(R) \frac{\partial R}{\partial \zeta} = 0.$$  

(2.45)

These are the collective field theory equations for these general families.
We will show in section 6.4 that the Hopf equation also arises in the case of multiplicative free families. This will explain why it appears in QCD, where the Gaussian nature of the master field will be responsible for its occurrence (though it will not be the resolvent that will obey the equation.)

3 The One-Plaquette Model

The master field representation that we have constructed for independent Hermitian matrices is not manifestly Hermitian. However, as we remarked, there are many equivalent representations of the master field. In this section we shall derive a manifestly Hermitian representation of the master field for independent Hermitian matrices and then apply this construction of the simplest model of unitary matrices, the one-plaquette model that exhibits a large-N phase transition [16].

3.1 Hermitian Representation

We shall now give a prescription, again not unique, to construct a Hermitian master matrix \( \hat{M}(a, a^\dagger) = \hat{M}^\dagger(a, a^\dagger) \) that reproduces the moments of the one-matrix model of Hermitian matrices. The idea is to express \( \hat{M} \) as a function of the Hermitian operator \( \hat{x} \equiv a + a^\dagger \). But \( \hat{x} \) represents the master field for a Gaussian matrix model. Thus writing \( \hat{M} \) in terms of \( \hat{x} \) is equivalent to expressing a matrix with an arbitrary distribution in terms of one with a Gaussian distribution. This can be done directly by a change of variables in the probability measure of \( M \).

Write the moments of the matrix distribution, given in terms of the density of eigenvalues, as

\[
\text{tr} [M^n] = \int d\lambda \rho(\lambda) \lambda^n \equiv \int \frac{dx}{2\pi} \sqrt{4 - x^2} \lambda^n(x), \tag{3.46}
\]

where the function \( x(\lambda) \) is a solution of the differential equation \( dx/d\lambda = \rho(\lambda)/\sqrt{4 - x^2} \). Therefore if we are given the eigenvalue distribution \( \rho(\lambda) \) we can construct the master field
as
\[ \hat{M} = \lambda(a + a^\dagger) = \lambda(\hat{x}) \equiv M(\hat{x}); \quad \text{where } \lambda(x) \text{ is determined by } \frac{d\lambda(x)}{dx} = \frac{\sqrt{4 - x^2}}{\rho(\lambda)}. \] (3.47)

In the case of many independent matrices \( M_i \), we can find the master fields in Hermitian form as \( \hat{M}_i = \lambda_i(\hat{x}_i) = \lambda_i(\hat{a}_i + \hat{a}_i^\dagger) \), with each \( \lambda_i \) being determined separately from the distribution of eigenvalues of \( M_i \).

The master fields in this representation also obey the master equations of motion discussed above. It is amusing, and perhaps instructive for more complicated models, to reformulate these in a way that allows for the construction of the master field directly using the equations of motion. The equations of motion (2.33) can be rewritten as

\[
\langle \Omega | V'(M(\hat{x}))f(M(\hat{x})) - [\hat{\Pi}, f(M(\hat{x}))]|\Omega \rangle = 0,
\]
where \( \hat{\Pi} \) will be defined to be the conjugate operator to \( \hat{M} \) in the sense that

\[ [\hat{\Pi}, \hat{M}] = P_\Omega = |\Omega\rangle\langle\Omega|. \] (3.49)

Note that on the right hand side of the commutator we have the vacuum projection operator and not the identity. Since \( \hat{M} \) is Hermitian we can choose \( \Pi \) to be anti-Hermitian. Thus in the case of the Gaussian potential, where \( \hat{M} = \hat{x} \), we have

\[ \hat{M} = \hat{a} + \hat{a}^\dagger, \quad \hat{\Pi} = \hat{\rho} \equiv \frac{1}{2}(\hat{a} - \hat{a}^\dagger). \] (3.50)

With this definition we have that

\[ [\hat{\Pi}, f(\hat{M})] = \frac{\delta}{\delta \hat{M}} \cdot f(\hat{M}). \] (3.51)

Therefore the equations of motion are equivalent to

\[
\langle \Omega | V'(M(\hat{x}))f(M(\hat{x})) - 2\hat{\Pi}f(M(\hat{x})) + f(M(\hat{x}))\hat{\Pi}|\Omega \rangle = 0.
\]

But, since the states \( f(M(\hat{x}))|\Omega \rangle \) span the Fock space as we let \( f \) run over all functions of \( M(\hat{x}) \), these equations are equivalent to the condition that

\[ [V'(M(\hat{x})) - 2\hat{\Pi}] |\Omega \rangle = 0. \] (3.53)
This equation can be used to solve for the master field, i.e., given the potential \( V(M) \) solve (3.53) for a Hermitian operator \( \hat{M} \) in the Fock space where \( \hat{\Pi} \) is conjugate to \( \hat{M} \). The first step, given an ansatz for \( \hat{M} = M(\hat{x}) \) is to derive an explicit representation of \( \hat{\Pi} \). To do this we first note that

\[
[\hat{p}, M(\hat{x})] = \frac{M(\hat{x}_l) - M(\hat{x}_r)}{\hat{x}_l - \hat{x}_l},
\]

(3.54)

where the labels on the \( \hat{x} \) operators means that we are to expand the fraction in a power series in \( \hat{x}_l \) and \( \hat{x}_r \) and order the operators so that all the \( \hat{x}_l \)'s are to the left of all the \( \hat{x}_r \)'s. Using this notation we can then write

\[
\hat{\Pi} = \frac{\hat{x}_l - \hat{x}_l}{M(\hat{x}_l) - M(\hat{x}_r)} \hat{p}.
\]

(3.55)

In this expression, when the operators are ordered, \( \hat{p} \) appears to the right of all the \( \hat{x}_l \)'s and to the left of all the \( \hat{x}_r \)'s.

To illustrate how this goes consider the Gaussian case where \( V'(M) = M \). Take \( \hat{M} = g_1\hat{x} + g_2\hat{x}^2 + \cdots \). Then using (3.55) \( \hat{\Pi} = 1/g_1\hat{p} - g_2/g_1^2(\hat{x}\hat{p} + \hat{p}\hat{x}) + \cdots \). The equation of motion then reads

\[
[g_1\hat{x} + g_2\hat{x}^2 - 2/g_1\hat{p} + 2g_2/g_1^2(\hat{x}\hat{p} + \hat{p}\hat{x}) + \cdots]|\Omega\rangle = [\frac{(g_1 - 1/g_1)\hat{a}^\dagger + (\cdot)g_2 - 2g_2/g_1^2(\hat{a}^\dagger)^2 + g_2 + \cdots]}{\Omega}\rangle = 0.
\]

(3.56)

Consequently we deduce that \( g_1 = 1, g_2 = 0, \ldots \Rightarrow \hat{M} = \hat{x}, \hat{\Pi} = \hat{x} \).

3.2 The One-Plaquette Model

The one-plaquette model describes unitary matrices \( U \) with the distribution

\[
Z = \int \mathcal{D}U e^{-\frac{N}{2} \text{Tr}[U+U^\dagger]}.
\]

(3.57)

We shall derive a master field for \( U \) in the manifestly unitary form \( \hat{U} = \exp[iH(\hat{x})] \), where \( H(\hat{x}) \) will be the master field for the eigenvalues of \( U \),

\[
\langle \frac{1}{N} \text{Tr}[U^n] \rangle = \int d\theta \sigma(\theta) e^{in\theta} = \langle \Omega|e^{iNH(\hat{x})}|\Omega\rangle
\]

(3.58)
The $N = \infty$ eigenvalue distribution was determined in [16] to be

$$\sigma(\theta) = \begin{cases} \frac{2}{\pi \lambda} \cos \left( \frac{\theta}{2} \right) \sqrt{\frac{\lambda}{2} - \sin^2 \left( \frac{\theta}{2} \right)} & \lambda \leq 2 \\ \frac{1}{2 \pi} \left( 1 + \frac{1}{2 \lambda} \cos \theta \right) & \lambda \geq 2 \end{cases} \quad (3.59)$$

Following the strategy described above we can construct $H$ by the change of variables $2\pi \sigma(\theta) d\theta = \sqrt{4 - x^2} dx$ and $H(\hat{x}) = \theta(\hat{x})$. It immediately follows from (3.59) that for weak coupling the master unitary field is given by

$$\hat{U} = \exp \left[ 2i \sin^{-1} \sqrt{\lambda \hat{x}} / 8 \right], \text{ for } \lambda \leq 2. \quad (3.60)$$

The phase transition is visible in the master field, since $\lambda \hat{x} / 8$ is a Gaussian variable, whose means square value exceeds one for $\lambda \geq 2$, at which point $\hat{U}$ ceases to be unitary. In the strong coupling phase the master field is given by

$$\hat{U} = e^{iH(\hat{x})}, \text{ where } H(x) + \frac{1}{2\lambda} \sin H(x) = \frac{1}{2} x \sqrt{4 - x^2} + 2 \sin^{-1} \frac{x}{2} \quad (3.61)$$

This master field has the remarkable property that

$$\langle \Omega | \hat{U}^n | \Omega \rangle = \delta_{n,0} + \frac{1}{\lambda} (\delta_{n,1} + \delta_{n,-1}). \quad (3.62)$$

## 4 The General Matrix Model

So far we have discussed only independent matrix models where the action can be written as $S = \sum_i S_i(M_i)$ and there is no coupling between the various $M_i$'s. We found that the master fields can be constructed in a Fock space in terms of creation $\hat{a}_i$ and annihilation operators $\hat{a}_i^\dagger$, one for each degree of freedom, where the only relation satisfied by these operators is $\hat{a}_i \hat{a}_j^\dagger = \delta_{ij}$. Now let us consider the most general matrix model with coupled matrices, for example QCD in four dimensions. One might think that it would be necessary to enlarge the Hilbert space in which the matrices are represented, or to modify its structure. This is not the case. We show below that we can construct the master field in the same space as before, with no new degrees of freedom or relations between the $\hat{a}_i$'s and $\hat{a}_i^\dagger$'s. The only new
feature will be that $M_i$ will be constructed out of all the $\hat{a}_j$'s and $\hat{a}_j^\dagger$'s, not just those with $j = i$.

Let us go back to the construction of the master field for independent matrices and give a graphical proof that the master field defined by

$$\hat{M}_i = \hat{a}_i + \sum_n \psi_i^{n+1} \hat{a}_i^n,$$  

where $zM_i(z) = \psi_i(z) = 1 + \sum \psi_i^n z^n$ is the generating functional of connected Green’s functions of the matrix $M_i$, i.e., $\psi_i(z) = \sum_{n=0}^{\infty} \langle \text{tr} [M_i^n] \rangle z^n$, yields the correct Green’s functions.

Consider the most general Feynman graph that contributes to

$$\langle \frac{1}{N} \text{Tr} [M_{i_1} M_{i_2} M_{i_3} \ldots M_{i_n}] \rangle.$$

(4.64)

The most general contribution to such a Green’s function can be drawn, as in Fig. 1, in terms of connected Green’s functions. Fig.1 represents a contribution to the $N = \infty$ Green’s function $\langle \text{Tr} [M_1^2 M_2^2 M_3^2 M_4 M_5 M_6] \rangle$, where the solid circles represent the connected Green’s functions. We are using the standard double index line notation for the propagators of the matrices.

![Fig. 1 A contribution to $\langle \text{Tr} [M_1^2 M_2^2 M_3^2 M_4 M_5 M_6] \rangle$. The solid circles represent connected Green’s functions.](image)
What is special about these graphs is that none of the lines cross, i.e. the points around the circle corresponding to the matrices $M_i$, in the order determined by the above word, are joined by lines that do not intersect. In that case the double index graph can be drawn on the plane and contains the maximum number of powers of $N$.

Now let us note that these graphs are in one-to-one correspondence with the terms in the expansion of $\langle \Omega | \hat{M}_1^2 \hat{M}_2 \hat{M}_3 \hat{M}_4 \hat{M}_5 \hat{M}_4^* M_4^* | \Omega \rangle$, with $M_i$ given by (4.63). Writing out the expression for this vacuum expectation value we find a contribution that exactly corresponds to the above graph, namely

$$
\langle \Omega | (\hat{a}_1 + \ldots) \cdot (\hat{a}_1 + \ldots) \cdot (\hat{a}_2 + \ldots) \cdot (\ldots + \psi_2^2 \hat{a}_2^\dagger + \ldots) \cdot (\ldots + \psi_1^3 \hat{a}_1^2 + \ldots) \cdot (\hat{a}_3 + \ldots) \cdot (\ldots + \psi_3 \hat{a}_3^2 + \ldots) \cdot (\hat{a}_4 + \ldots) \cdot (\hat{a}_5 + \ldots) \cdot (\ldots + \psi_3^2 \hat{a}_5^2 + \ldots) \cdot (\ldots + \psi_4 \hat{a}_4^2 + \ldots) | \Omega \rangle = \psi_2^2 \psi_1^3 \psi_3^2 \psi_4^2.
$$

(4.65)

Conversely, every non-vanishing contribution to $\langle \Omega | \hat{M}_1^2 \hat{M}_2 \hat{M}_3 \hat{M}_4 \hat{M}_5 \hat{M}_4^* M_4^* | \Omega \rangle$ corresponds to a planar graph, such as that depicted in Fig.1. Start along the circle at a point denoted by $\langle \Omega |$, then $\hat{M}_1$ must create a line labeled by 1. The next operator is $\hat{M}_1$, so it can annihilate the line 1, with coefficient $\psi_1^2$, or create another 1 line as in Fig.1. Then the operator $\hat{M}_2$ cannot annihilate the line 1, since $\hat{a}_1 \hat{a}_2^\dagger = 0$. Then the next $\hat{M}_2$ must annihilate the line 2, with coefficient $\psi_2^2$, as in Fig.1. Clearly there is no way we can get graphs with lines 1 and 2 crossed, the Cuntz algebra ensures that this cannot happen. So on around the circle.

This argument is a graphical proof that the master fields is given by (4.63), with $\psi_i(z)$ being the generator of connected Green’s functions of the $M_i$’s. However, this argument does not really depend on the matrices being independent. Consider a general matrix model, with partition function

$$
Z = \int \prod_i \mathcal{D} M_i e^{-N \text{Tr} S(M_1, \ldots, M_i)},
$$

(4.66)

where $S$ will in general contain interactions between different matrices. The general graph that contributes to the Green’s function has a similar decomposition in terms of connected Green’s functions. An example of a contribution to $\langle \text{tr} [M_1 M_2 M_3 M_4 M_1 M_3 M_5 M_4] \rangle$ is depicted in Fig.2

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Fig. 2 A contribution to \( \langle \text{tr} [M_1 M_2 M_3 M_4 M_5 M_6 M_7 M_8] \rangle \). The solid circles represent connected Green’s functions.

The only difference is that the solid circles, representing the connected Green’s functions, now can involve matrices with different indices. We can construct a master field for each \( M_i \), in terms of the same creation and annihilation operators as before, as long as we let \( \hat{M}_i \) depend on \textit{all the creation operators}. Thus if

\[
\hat{M}_i = \hat{a}_i + \sum_{k=1}^{\infty} \psi_{i,j_1,j_2,\ldots,j_k} \hat{a}_{j_1} \hat{a}_{j_2} \cdots \hat{a}_{j_n},
\]

(4.67)

where \( \psi_{i,j_1,j_2,\ldots,j_k} \) is the connected Green’s function:

\[
\psi_{i,j_1,j_2,\ldots,j_k} \equiv \frac{1}{N} \langle \text{Tr} [M_i M_{j_1} M_{j_2} \cdots M_{j_n}] \rangle_{\text{conn}}.
\]

(4.68)

then the vacuum expectation values of products of \( \hat{M}_i \)'s will correctly reproduce the \( N = \infty \) limit of the general matrix model. Note that the coefficients \( \psi_{i,j_1,j_2,\ldots,j_k} \) are cyclically symmetric in the indices.

This argument shows that the master field exists as an operator in a Boltzmannian Fock space constructed with the use of a creation operator for each independent matrix field and gives an explicit expression for the master fields in terms of the solution of the theory. It
is not, of course, an explicit expression for the master fields—to do so would be to solve the theory.

One new approach to solving large $N$ theories might be to explore the equations of motion for the master field operators. In the case of coupled matrices the Schwinger-Dyson equations can also be formulated as

$$\left\langle \Omega \left[ \frac{\partial S[\hat{M}_1, \hat{M}_2, \ldots]}{\partial \hat{M}_i} - \frac{\delta}{\delta \hat{M}_i} \right] f[\hat{M}_1, \hat{M}_2, \ldots] \right\rangle = 0,$$

(4.69)

where $\frac{\delta}{\delta \hat{M}_i}$ is defined as before and $f$ is an arbitrary function of the $\hat{M}_i$. We can also define a conjugate operator, $\hat{\Pi}_i$, to $\hat{M}_i$ as before, that satisfies $[\hat{\Pi}_i, \hat{M}_j] = \delta_{ij}P\Omega$. If we can find a Hermitian representation of $\hat{M}_i$ then the above equations of motion are equivalent to

$$\left[ \frac{\partial S[\hat{M}_1, \hat{M}_2, \ldots]}{\partial \hat{M}_i} - 2\hat{\Pi}_i \right] |\Omega\rangle = 0.$$

(4.70)

It might be fruitful to explore this equation as a way of solving large $N$ theories.

5 Two Dimensional QCD

We now turn to discuss the master field for the simplest gauge theory, two-dimensional QCD. There are two approaches to the master field that one might pursue. One is to construct a master loop field, $\hat{U}_C$, that would be used directly to reproduce the expectation values of the Wilson loops for $N = \infty$. We shall discuss this approach in the following section. The other approach is to construct directly the master gauge field $\hat{A}_\mu(x)$. This is simple for QCD$_2$, since in an appropriate gauge the theory is Gaussian and corresponds to an independent matrix model, albeit one with continuum labels.

Consider two-dimensional Yang-Mills theory in flat space. We can work either in Euclidean space, $R_2$, or in Minkowski space $M_2$. If we choose an axial gauge $n^\mu A_\mu = 0$, where $n^\mu$ is a unit vector in any direction the theory becomes Gaussian. This is a legal gauge on $R_2$ or $M_2$. We choose for convenience the gauge $A_1(x) = 0$ and work in Euclidean space, in which case

$$F_{10}(x) = F(x) = \partial_1 A_0; \quad \mathcal{L} = \frac{1}{2} \text{Tr} F(x)^2 = \frac{1}{2} \text{Tr} (\partial_1 A_0(x))^2.$$  

(5.71)
The field strength $F$ is therefore an independent Gaussian matrix at each point in spacetime and we can easily construct a master field to represent it in the continuum Fock space defined by

$$\hat{a}(x)\hat{a}^\dagger(y) = \delta^2(x - y); \quad \hat{a}(x)|\Omega\rangle = 0; \quad |x_1, x_2, \ldots, x_n\rangle = \hat{a}^\dagger(x_1)\hat{a}^\dagger(x_2)\cdots\hat{a}^\dagger(x_n)|\Omega\rangle. \quad (5.72)$$

In this space we can write

$$\hat{F}(x) = \hat{a}(x) + \hat{a}^\dagger(x). \quad (5.73)$$

Alternatively we can work in momentum space, where $\hat{a}(p) = \int \frac{d^2 x}{2\pi} \exp(i p \cdot x)\hat{a}(x)$ satisfies, $\hat{a}(p)\hat{a}^\dagger(q) = \delta^2(p - q)$. We can then solve the equation $\partial_1 A_0(x) = F(x)$ to construct the master gauge field

$$\hat{A}_0(x) = \frac{1}{\partial_1} [\hat{a}(x) + \hat{a}^\dagger(x)] = \int \frac{d^2 p}{2\pi} \frac{i}{p_1} \left[ e^{-ip \cdot x}\hat{a}(p) - e^{ip \cdot x}\hat{a}^\dagger(p) \right]. \quad (5.74)$$

We can now make a gauge transformation to a gauge in which the master gauge field will be independent of $x^\mu$. To this end we define the momentum operator, $\hat{P}^\mu$, in Fock space so that

$$\hat{P}^\mu|p_1, p_2, \ldots, p_n\rangle = \left( \sum_{i=1}^n p_i^\mu \right)|p_1, p_2, \ldots, p_n\rangle. \quad (5.75)$$

An explicit representation of $\hat{P}^\mu$ in terms of creation and annihilation operators is

$$\hat{P}^\mu = \hat{P}^{\mu\dagger} = \sum_{k=1}^\infty \int d^2 p_1 \ldots d^2 p_k p_k^\mu \hat{a}(p_1)\cdots\hat{a}^\dagger(p_k)\hat{a}(p_k)\cdots\hat{a}(p_1). \quad (5.76)$$

Thus when the $k$th term in the sum acts on an $n$ particle state, for $k \leq n$, it removes $k$ particles from the state, measures the momentum of the $k$th particle and then puts the $k$ particles back in the original order. The momentum operator has the standard commutation relations with the creation and annihilation operators

$$[\hat{P}^\mu, \hat{a}(p)] = -p^\mu \hat{a}(p), \quad [\hat{P}^\mu, \hat{a}^\dagger(p)] = p^\mu \hat{a}^\dagger(p) \Rightarrow [\hat{P}^\mu, \hat{A}_0(x)] = -i\partial^\mu \hat{A}_0(x). \quad (5.77)$$

Therefore $A_0(x) = \exp(i\hat{P} \cdot x)A_0(x)\exp(-i\hat{P} \cdot x)$ and, as discussed in the introduction, we can make a gauge transformation on the above master field, with gauge function $\hat{U} = \exp(i\hat{P} \cdot x)$, to derive a spacetime independent master field:

$$\hat{A}_1 = \hat{P}_1, \quad \hat{A}_0 = \hat{P}_0 + \hat{A}(0) = \hat{P}_0 + \int \frac{d^2 p}{2\pi} \frac{i}{p_1} \left[ \hat{a}(p) - \hat{a}^\dagger(p) \right]. \quad (5.78)$$
In this gauge the field strength is given by

$$\hat{F}_{10} = i[\hat{A}_1, \hat{A}_0] = [\hat{P}_1, \hat{A}(0)] = \int \frac{d^2 p}{2\pi} \left[ \hat{a}(p) + \hat{a}^+(p) \right] = \hat{F}_{10}(0).$$

By means of a similarity transformation, we can rewrite $\hat{A}_0$ as

$$\hat{A}_0 = \hat{P}_0 + \int \frac{d^2 p}{2\pi} \left[ \hat{a}(p) + \frac{1}{p^2_1} \hat{a}^+(p) \right],$$

in which the second term we recognize as a sum of master fields for a continuum of Gaussian matrix variables, where the momentum space connected two-point function, the propagator, is $1/p^2_1$. One has to be careful in using this field to introduce an infrared regulator for small $p_1$. This can be done by cutting out a small hole in momentum space or by a principle value prescription for the propagator. As explained in [17, 18] gauge invariant observables do not depend on the regularization.

This master field satisfies the master equations $[D^\mu, \hat{F}_{\mu\nu}(x)] = \delta/\delta \hat{A}_\nu(x)$. In the original axial gauge this means that

$$\langle \Omega | [\partial_1^2 \hat{A}_0(x) - \frac{\delta}{\delta \hat{A}_0(x)}] f(A_0) | \Omega \rangle = 0,$$

where as before the operator derivative is defined as

$$\frac{\delta}{\delta \hat{A}_0(x)} f(A_0) = \lim_{\epsilon \to 0} \frac{f(A_0(y) + \epsilon \delta(y-x) P_\Omega) - f(A_0(y))}{\epsilon}, \quad P_\Omega = |\Omega \rangle \langle \Omega|.$$ (5.82)

These equations of motion can be used to show that the Wilson loop, which can be written in terms of the master field as,

$$\langle W_C \rangle = \langle \Omega | T\{ \exp[i \int_0^1 \hat{A}^\mu \dot{x}_\mu(t)] \} | \Omega \rangle = \langle \Omega | T\{ \exp[i g \int_0^1 \hat{A}^0(x(t)) \dot{x}_0(t)] \} | \Omega \rangle,$$ (5.83)

satisfies the Migdal-Makeenko equations [3]. Note that in the first integral, written in terms of the spacetime independent master field the path ordering is still necessary since, for non-straight paths, $\hat{A}^\mu \dot{x}_\mu(t)$ do not commute for different $t$’s.

## 6 Master Loop Fields
6.1 Wilson loops

One can alternatively describe the master field for $QCD$ in terms of Wilson loops. These are manifestly gauge invariant and contain, in principle, all information about gauge invariant quantities. They are also the natural variables for a string theory formulation of $QCD$. One would therefore like to have master loop operators to describe the large $N$ limit of these loops.

From the point of view of the master field of $QCD$, the loop approach is, in general, quite unwieldy. The space of loops is too large and overcomplete. There is a lot of redundancy in defining master loop operators for every possible loop. The space of loops is much bigger than the space of points. It is hard to see what a ‘basis’ might be in this space. Moreover, to extract information about, say, the spectrum of meson bound states seems extremely difficult in practice starting from these loops (even in $QCD_2$).

Nevertheless, in $QCD_2$, the loop space and loop variables have many nice and simplifying features (mainly due to the area preserving diffeomorphism symmetry of the theory). These features of the loop variables are not immediately apparent from the master connection that we constructed above. More importantly, they enable one to explicitly construct master loop operators that reproduce an arbitrary loop average fairly easily. Starting from these loops we can, by considering infinitesimal loops, recover the master field of Section 4. Alternatively, it will also be possible to start from the master field and derive the master loops without too much effort.

6.2 Free Random Variables in the Loop Space of $QCD_2$

The main tool in trying to solve for Wilson loop averages are the Makeenko-Migdal loop equations [6]. In 2 dimensions, they are especially tractable and Kazakov and Kostov have shown how the average of an arbitrary Wilson loop can be calculated for $N = \infty$ using these equations. We shall approach the calculation of loop averages in a somewhat different and suggestive manner. We shall start by decomposing an arbitrary loop into a word built of simple loops, all originating at some common base point but with otherwise non-overlapping
interiors. (By a simple loop we shall henceforth mean a non-self intersecting loop on the plane.) We shall argue that these simple loops form a family of free random variables. This will enable us to calculate an arbitrary loop average in terms of \( < \text{tr} [U_C^m] > \)'s where the \( U_C \)'s are the free random variables for simple loops \( C_i \). Simple loops will thus form a basis in loop space, though they still contain too much information and are overcomplete.

Let's first show that a set of simple loops, based at one point and non-overlapping, correspond to free random variables. For concreteness consider the loops \( C_1, C_2, C_3 \) based at the point \( P \) as in Fig.3.

We denote the holonomies along the loops \( C_i \) by \( U_i \), i.e

\[
U_i = P \exp \left( i \oint_{C_i} A_\mu dx^\mu \right)
\]

(6.84)

Note that the \( U_i \)'s are \( U(N) \) matrices. We claim that the \( U_i \) have independent distributions and hence are free random variables in the large \( N \) limit. One way to see this is to use the heat kernel action, which we know to be exact in the continuum limit,

\[
Z = \int \prod_L \mathcal{D}U_L \prod_{\text{plaquettes}} Z_P[U_P]; \quad Z_P[U_P] = \sum_R d_R \chi_R[U_P] e^{-C_2(R)A_P},
\]

(6.85)

where the sum runs over all representations of \( U(N) \), \( \chi_R \) is the character of the representation and \( C_2(R) \) its second Casimir operator. We can choose a triangulation of the plane such that the given contours, \( C_i \), are the borders of some of the triangles. The self similar nature of the heat kernel always allows us to choose such a triangulation. Then, when we come to use this measure to calculate averages of products of the \( C_i \)'s, we can integrate out all the other link variables, (this is only true on the plane), leaving us with an equivalent measure

\[
Z = \int \prod_i \mathcal{D}U_i \prod_i Z[U_i],
\]

where the product runs over all the simple loops, \( U_i \). Therefore the resulting integrals over the \( U_i \) are over independent distributions.

Naturally, this can also be seen directly from the loop equations. We shall give a rough sketch below. The loop equations are the \( N = \infty \) Schwinger-Dyson equations for Wilson
Here the L.H.S. refers to a variation of the area of the loop by $\delta \sigma_{\mu \nu}$ at $x = x(\tau)$. The R.H.S. vanishes unless $x(\tau)$ is a point of self intersection in which case $C_1$ and $C_2$ are the two loops into which $C$ breaks up at that point. (The $\int$ refers to the exclusion of $\sigma = \tau$ in the integral.) In the large $N$ limit, $W(C_1, C_2) = W(C_1)W(C_2)$ and, as shown by [21], because of the area preserving symmetry of QCD$_2$, the loop equations simplify to (See Fig.4)

\[
\frac{\partial}{\partial x_{\mu}} \frac{\delta}{\delta \sigma_{\mu \nu}} W(C) \bigg|_{x=x(\tau)} = \int d\sigma(\sigma) \delta^{(2)}(x(\sigma) - x(\tau)) W(C_1, C_2) 
\]

(6.86)

Fig. 4  The Loop Equations for QCD$_2$

\[
(\partial_k + \partial_i - \partial_l - \partial_j) W(C) = W(C_1)W(C_2), \quad (\partial_i \equiv \frac{\partial}{\partial A_i}).
\]

(6.87)

where the $A_i$ are the areas that meet at the point $P$ of self intersection, at which the loop splits up into $C_1$ and $C_2$. (the R.H.S. of Fig.4). The equations (6.87) form a closed set of equations that determine the loop average for a loop with $n$ self intersections in terms of ones with a lesser number of self intersections. They can be solved recursively in terms of the loop average for a simple loop. The latter can be computed either from perturbation theory or by imposing appropriate boundary conditions on loops with a large number of turns. It has the value $W(C) = e^{-A}$, where $A$ is the area of the loop.
If we now consider two simple loops, as before, based at some point and with holonomies $U$ and $V$, then it is possible to compute $<\text{tr} \left[ U^{n_1} V^{m_1} \ldots U^{n_k} V^{m_k} \right]>$ using the loop equations. They give a recursion relation for such a word of length $2k$ in terms of shorter words. We simply state the general expression.

$$<\text{tr} \left[ U^{n_1} V^{m_1} U^{n_2} V^{m_2} \ldots U^{n_{k-1}} V^{m_{k-1}} U^{n_k} V^{m_k} \right] >$$

$$= <\text{tr} [U^{n_1}]><\text{tr} [V^{m_1}]><\text{tr} [U^{n_2} V^{m_2} \ldots U^{n_{k-1}} V^{m_{k-1}} U^{n_k} V^{m_k}] >$$

$$- <\text{tr} [U^{n_1} V^{m_1}]><\text{tr} [U^{n_2} V^{m_2} \ldots U^{n_{k-1}} V^{m_{k-1}} U^{n_k} V^{m_k}] >$$

$$+ \ldots - <\text{tr} [U^{n_1} V^{m_1} U^{n_2} V^{m_2} \ldots U^{n_{k-1}} V^{m_{k-1}}]><\text{tr} [U^{n_k} V^{m_k}] >$$

$$+ <\text{tr} [U^{n_1} V^{m_1} U^{n_2} V^{m_2} \ldots U^{n_{k-1}} V^{m_{k-1}} U^{n_k}] > <\text{tr} [V^{m_k}] > . \quad (6.88)$$

This recursion relation can be obtained by combining the loop equations at the vertices where the loop on the L.H.S. breaks up into the loops represented by the various terms on the R.H.S. This relation is easily seen to be equivalent to a uniform (Haar) distribution (at large $N$) for the relative angular integrals between $U$ and $V$. Thus it implies that $U$ and $V$ are free random variables. Conversely, if we were to assume that the holonomies of simple loops are free random variables we could derive this relation from the defining property of such variables, as we discussed previously.

The above recursion relation is a very useful expression that will enable us to calculate arbitrary loop averages rather efficiently. For $k = 1$,

$$<\text{tr} [U^{n_1} V^{m_1}] > = <\text{tr} [U^{n_1}]><\text{tr} [V^{m_1}] > \quad (6.89)$$

and for $k = 2$,

$$<\text{tr} [UVUV] > = <\text{tr} U >^2 <\text{tr} [V^2] > - <\text{tr} U >^2 <\text{tr} V >^2 + <\text{tr} [U^2] ><\text{tr} V >^2 \quad (6.90)$$

which tallies with (2.3). If we have more than two such simple loops then these relations can be applied repeatedly to reduce the average to a product of averages of powers of the individual $U_i$'s.
The loop equations (6.87) can also be used to compute \( < \text{tr} [U^n] > \) for a simple loop with holonomy \( U \). Later we will obtain the same answer by other means as well. The answer can be expressed in terms of Laguerre polynomials \( L_n^1 \) [19, 20],

\[
< \text{tr} [U^n] > = \frac{1}{n} L_n^1 (nA) e^{-nA^2} = \frac{1}{n} \int \frac{dz}{2\pi i} (1 + \frac{1}{z})^n e^{-nA^2(1 + 2z)},
\]

where \( A \) is the area of the loop and we have exhibited the integral representation for the Laguerre polynomials. The first few terms are displayed below.

\[
< \text{tr} U > = e^{-\frac{A}{2}}, \quad < \text{tr} [U^2] > = (1 - A) e^{-A}, \quad < \text{tr} [U^3] > = (1 - 3A + \frac{3}{2} A^2) e^{-\frac{3A}{2}}. \quad (6.92)
\]

### 6.3 Decomposing a Loop Into a Word

Having seen that simple, non overlapping loops, based at a point are free random variables, we now proceed to show how an arbitrary loop can be written as a word built out of such simple loops. In fact, for a loop with \( n \) self intersections, there are \( (n + 1) \) windows (i.e. enclosed interiors) and the word will be built out of \( \{ U_i \} \)’s, \( i = 1, 2 \ldots n + 1 \), which will be associated with these windows. In the interests of clarity and to avoid notational clutter, we shall illustrate the decomposition in a few representative cases and its general nature will then be apparent.

The simplest self intersecting loop is the figure of eight in Fig.5. The loops \( C_1 \) and \( C_2 \) are simple loops and hence \( U \) and \( V \) are free random variables. Therfore the loop average is simply

\[
W(C) = < \text{tr} [UV] > = < \text{tr} U > < \text{tr} V > = e^{-\frac{(A_1 + A_2)}{2}}. \quad (6.93)
\]
The first, non-trivial example is the loop in Fig. 6a. We shall introduce a notation for loops in terms of line segments. Thus (13) denotes \( a \) and (31), \( b \).

The bar is to distinguish it from (31) which will denote \( a^{-1} \) – the oppositely directed line segment to (13) and similarly (13) for \( b^{-1} \).

Then, together with (12)=c and (21)=d, the loop itself can be written as

\[
C = (13)(31)(12)(21) = (13)(31)(21)(12)(21) = (13)(31)(12)(21) (6.94)
\]

Where we have inserted (21)(12)=1 and (21)(21) = 1 (two back tracking loops enclosing zero area). This is geometrically equivalent to the loop in Fig. 6b. As a word we see that it is \( UV^2 \) where \( U \) corresponds to the loop (13)(31)(21) and \( V \) to the loop (12)(21). These are simple loops and therefore, their holonomies \( U \) and \( V \) respectively, have independent distributions. Therefore,

\[
W(C) = \langle \text{tr}[UV^2] \rangle = \langle \text{tr} U \rangle \langle \text{tr} V^2 \rangle = e^{-\frac{A_1}{2}} e^{-A_2}(1 - A_2) (6.95)
\]

which is the standard answer.
Next consider the contour in Fig. 7. With the points labelled as shown,

\[ C = (12)(21)(13)(34)(45)(54)(41) \]

\[ = (12)(21)(14)(43)(31)(13)(34)(41)(13)(34)(45)(54)(41) \]

\[ = (12)(21)(14)(43)(31)(13)(34)(54)(41)(14)(45)(54)(41) \]

\[ = (12)(21)(14)(43)(31)(14)(45)(54)(41)(14)(45)(54)(41) \]

\[ = U_1U_2U_3U_2U_3^2 \]

(6.96)

where we have again introduced backtracking loops so as to peel off successively, the loops corresponding to the different windows. Note that all these loops are based at the common point 1, because of the introduction of backtracking or 'thin' loops. Therefore the \( U_i \)'s are free random variables and

\[ W(C) = < \text{tr} [U_1U_2U_3U_2U_3^2] > = < \text{tr} U_1 > < \text{tr} U_2 > < \text{tr} U_3 > < \text{tr} U_2^2 > < \text{tr} U_3^2 > \]

\[ + < \text{tr} U_2^2 > < \text{tr} U_3 > < \text{tr} U_3^2 > \]

(6.97)

This can be compared with the loop average computed by usual means, once we express the moments of \( U_i \) in terms of the appropriate polynomials, using (6.91).
Finally, consider a case which is 'non-planar' in the terminology of [21], i.e., the loop depicted in Fig. 8. With the segments denoted as shown,

\[
C = (12)(23)(31)U_2U_1(12)(23)(31)U_1U_2(12)(23)(31)U_1U_2(12)(23)(31)U_1U_2
\]

\[= U_3U_2U_4^{-1}U_3^{-1}U_4U_1 \quad (6.98)\]

Once again, the \(U_i\)'s are free random variables and therefore the loop average in this case is

\[
W(C) = <\text{tr}\left[U_3U_2U_4^{-1}U_3^{-1}U_4U_1\right]> = <\text{tr}\left[U_3U_2U_4^{-1}U_3^{-1}U_4\right]><\text{tr}\ U_1>
\]

\[= <\text{tr}\ U_1><\text{tr}\ U_2><\text{tr}\left[U_4^{-1}U_3^{-1}U_4U_3\right]><\text{tr}\ U_1><\text{tr}\ U_2>
\]

\[(<\text{tr}\ U_4^{-1}><\text{tr}\ U_4> - <\text{tr}\ U_4^{-1}><\text{tr}\ U_3^{-1}><\text{tr}\ U_4><\text{tr}\ U_3> + <\text{tr}\ U_3^{-1}><\text{tr}\ U_3>) = e^{-A_1/2} (e^{-A_1} - e^{-(A_1+A_2)} + e^{-A_3}) \quad (6.100)\]

which once again reproduces the usual answer.
By now the general procedure must be apparent – we decompose the loop starting with the segments bordering the outside and form loops from each of the windows giving a coherent orientation. It is possible to characterise these words algorithmically in terms of the graphical structure, but that is not pertinent to our present purpose. We also see that this process of associating a word with a loop and then using the recursion relations (6.88) makes the computation of complicated loop averages rather simple, in fact, mechanical.

6.4 The master loop operators

We have decomposed an arbitrary loop with holonomy $U_\Gamma$ into a word $\Gamma$ built of simple, non-overlapping loops $C_i$ and holonomies $U_{C_i}$ such that

$$U_\Gamma = \prod_{\Gamma} U_{C_i}^{n_i}$$  \hspace{1cm} (6.101)

Since the $U_{C_i}$ are free random variables, we can associate master loop operators to them, $\hat{U}_{C_i}$, by the general construction of Section 2. Then the master loop operator $\hat{U}_\Gamma$ that reproduces the loop average is

$$W(C) = \left\langle \Omega \left| \left( \prod_{\Gamma} \hat{U}_{C_i}^{n_i} \right) \right| \Omega \right\rangle.$$  \hspace{1cm} (6.102)

We shall now construct the loop operators $\hat{U}$ (supressing the contour labels) in the form

$$\hat{U} = \hat{a} + \sum_{k=0}^{\infty} \omega_k \hat{a}^{\dagger k}$$  \hspace{1cm} (6.103)

The $\omega_k$ can be determined from the $< \text{tr} [U^n] >$ which we saw, are given by (6.91). We claim that with

$$\omega_k = (-1)^{k-1} \frac{k^{-1} A^{k-1} e^{-kA/2}}{k!}$$  \hspace{1cm} (6.104)

we have

$$\left\langle \Omega | \hat{U}^n | \Omega \right\rangle = \frac{1}{n} L_{n-1}^1 (nA) e^{-nA^2} = < \text{tr} [U^n] >.$$  \hspace{1cm} (6.105)

We shall demonstrate this directly below. We can also represent $\hat{U}$ in an explicitly unitary manner, as in the one plaquette model. However, the manifestly unitary form is not particularly elegant and we shall not present it here.
Consider an infinitesimal rectangular loop (as in Fig.9) of area $\Delta A = \Delta x \Delta t$. The master loop operator associated with it is (6.103) for $A \to 0$. Thus to lowest order in $\Delta A$, we have

$$\hat{U}(\Delta A) = \hat{a} + \left(1 - \frac{\Delta A}{2}\right) - \Delta A \hat{a}^\dagger,$$

(6.106)

where $\hat{a}$ refers to the annihilation operator at the point $x$, $\hat{a}(x)$. We can equivalently represent this, by performing a similarity transformation, in the form

$$\hat{U}(\Delta A) = \left(1 - \frac{\Delta A}{2}\right) + i\sqrt{\Delta A}(\hat{a} + \hat{a}^\dagger) = \exp(i\hat{H}); \quad \hat{H} = \sqrt{\Delta A}(\hat{a} + \hat{a}^\dagger).$$

(6.107)

This is the explicitly unitary form for $\hat{U}$ for the infinitesimal loop. Note that if we naively drop the term linear in $\Delta A$, which arises from $-\frac{1}{2}H^2 = -\frac{1}{2}\Delta A(\hat{a} + \hat{a}^\dagger)^2 = -\frac{1}{2}\Delta A + \cdots$, as being of higher order than the $\sqrt{\Delta A}$, then $\hat{U}$ would not reproduce the correct leading behaviour in $\Delta A$ in $<\text{tr} [U^n]>$.

But we also know that the holonomy $U$ around such a loop is, in say, axial gauge

$$U = (1 - iA_0(x,t)\Delta t)(1 + iA_0(x + \Delta x, t)\Delta t) = (1 + i\partial_1 A_0 \Delta A) = \exp(i\partial_1 A_0 \Delta A).$$

(6.108)

Comparing (6.108) and (6.107) we have

$$\sqrt{\Delta A}\partial_1 A_0 = (\hat{a} + \hat{a}^\dagger).$$

(6.109)

This is equivalent to the master field of Section 4. Indeed, if we discretise the theory. i.e., smear the fields over plaquettes of size $\Delta A$, then the action reads as

$$S = \frac{1}{2} \sum_{\text{plaquettes}} (\Delta A)\text{Tr}(\partial_1 A_0)^2.$$

(6.110)

We see that $\sqrt{\Delta A}\partial_1 A_0$ are Gaussian free random variables, represented by $(\hat{a} + \hat{a}^\dagger)$, the result we obtained from the loop operator. Of course, this should come as no real surprise.

It is somewhat less trivial to start from the master gauge field and to calculate the master loop operators explicitly for finite loops. In $QCD_2$, we can do this rather easily since the
The product of two free random variables with distributions, \( \mu_1 \) and \( \mu_2 \) is again a free random variable with some distribution \( \mu_3 \) denoted by \( \mu_1 \otimes \mu_2 \). A one parameter family of free random variables, such that \( \mu_{t_1} \otimes \mu_{t_2} = \mu_{t_1 t_2} \), will be called a multiplicative free family. (Or equivalently \( \mu_{s_1} \otimes \mu_{s_2} = \mu_{s_1 + s_2} \), if we redefine the parameter \( t \rightarrow s = \log t \).)

We claim that \( \hat{U}(A) \) are a multiplicative free family with the area \( A \) playing the role of the parameter \( s \). In other words, given two simple loops \( C_1 \) and \( C_2 \), based at a point and non-overlapping, \( \hat{U}_{C_1}(A_1)\hat{U}_{C_2}(A_2) \) has the same distribution as \( \hat{U}_{C_1 \circ C_2}(A_1 + A_2) \). Again, there are many ways to see this. One is from the fact that the heat kernel action is self reproducing and exponentially dependent on the area of the plaquette.

A more explicit way is to note that \( \hat{U}_{C_1}(A_1)\hat{U}_{C_2}(A_2) \) has the same distribution as does \( \hat{U}_{C_1}(A_1)\hat{U}_{C_2}^\dagger(A_2) \). This is evident from Fig. 10, where we see that since \( \hat{W} \) has the same distribution as \( \hat{W}^\dagger(A_2) \) and both are independent of \( \hat{V} \). But \( \hat{V} \hat{W}^\dagger \) is a simple loop of area equal to the sum of the two areas.

This fact alone, actually enables us to construct the \( \hat{U} \) solely from the knowledge of the master gauge field, i.e. from the knowledge of an infinitesimal loop. To do so we need a non-commutative analog of the Mellin Transform in ordinary probability theory, which is multiplicative for the product of two random variables. It turns out that one can define such a transform—the S-transform, such that \( S_{\mu_1} S_{\mu_2} = S_{\mu_1 \otimes \mu_2} \). For a multiplicative free family \( S_{s_1} S_{s_2} = S_{s_1 + s_2} \) (dropping the \( \mu \)'s.) \( S(z) \) is therefore exponential in \( s \) in this case.
The function $S(z)$ for a non-commutative random variable $U$ is constructed as follows: If

$$\phi(j) = \sum_{n=1}^{\infty} \langle \Omega | \hat{U}^n | \Omega \rangle j^n,$$

then construct the inverse function $\chi(z)$, i.e. $\phi(\chi(z)) = z$. The S-transform is defined as:

$$S(z) = \frac{1 + z}{z} \chi(z).$$

Since the $\hat{U}$’s are multiplicative, we can use the S-transform of an infinitesimal loop to obtain the exact S-transform for one of finite area, knowing that it must necessarily exponentiate. For the infinitesimal rectangular loop in Fig.9. we saw, in axial gauge, that

$$U = (1 + i \partial_1 A_0 \Delta A) = \exp(i \partial_1 A_0 \Delta A).$$

Arguing backwards now, from the discretised action for $QCD_2$, for which $\sqrt{\Delta A} \partial_1 A_0 = (\hat{a} + \hat{a}^\dagger)$, we have

$$\hat{U} = \exp(i \hat{H}) \quad \hat{H} = \sqrt{\Delta A}(\hat{a} + \hat{a}^\dagger).$$

Now,

$$\phi(j) = \sum_{n=1}^{\infty} \langle \Omega | \hat{U}^n | \Omega \rangle j^n = \sum_{n=1}^{\infty} (1 - n^2 \Delta A / 2) j^n,$$

keeping only terms of order $\Delta A$. The sum can be performed giving

$$\phi(j) = \frac{j}{1 - j} - \frac{\Delta A}{2} j \frac{(3 - j)}{(1 - j)^3}.$$  

Equating this to $z$ and solving for $j \equiv \chi(z)$ (again to lowest order in $\Delta A$ only) gives

$$\chi_{\Delta A}(z) = \frac{z}{1 + z} (1 + \Delta A(1 + 2z)).$$

Therefore $S_{\Delta A}(z)$, defined as $\chi_{\Delta A}(z) \frac{1 + z}{z}$, is equal to $(1 + \Delta A(1 + 2z))$ for an infinitesimal loop. For finite area, this exponentiates as expected, to give

$$S_A(z) = e^{\frac{A}{2(1 + 2z)}} \Rightarrow \chi_A(z) = \frac{z}{1 + z} e^{\frac{A}{2(1 + 2z)}}.$$
We can now use the $S_A(z)$, which we obtained from the master gauge field, to give an alternative derivation of (6.91) for $\langle \mathrm{tr} [U^n] \rangle$. Since

$$\phi(e^{-i\theta}) = \sum_{n=1}^{\infty} < \mathrm{tr} [U^n] > e^{-i n \theta}$$

$$\Rightarrow < \mathrm{tr} [U^n] > = \frac{1}{2\pi} \int \phi(e^{-i\theta})e^{i n \theta} d\theta,$$

(6.119)

we have, with $\chi(z) = e^{-i\theta}$,

$$< \mathrm{tr} [U^n] > = \int \frac{dz}{2\pi i} z [\chi(z)]^{-(n+1)} \chi'(z),$$

(6.120)

or on integrating by parts

$$< \mathrm{tr} [U^n] > = \frac{1}{n} \int \frac{dz}{2\pi i} [\chi(z)]^{-n}.$$

(6.121)

In the case at hand $\chi_A(z) = \frac{z}{1+z} e^{\frac{A}{2}(1+2z)}$ and hence

$$< \mathrm{tr} [U(A)^n] > = \frac{1}{n} \int \frac{dz}{2\pi i} (1 + \frac{1}{z})^n e^{-n \frac{A}{2}(1+2z)}$$

(6.122)

which is (6.91).

We can now see why the Hopf equation arises in the QCD$_2$. In fact it, or its generalization (2.45), will appear for any multiplicatively free family of random variables. Define the resolvent of $U(A)$ to be

$$R(\zeta, A) = \sum_{n=0}^{\infty} < \mathrm{tr} [U(A)^n] > \zeta^{-(n+1)}.$$

(6.123)

By definition $\phi$ is related to the resolvent as

$$R(\zeta, A) = \frac{1}{\zeta} (\phi(\frac{1}{\zeta}) + 1) \Rightarrow R(\frac{1}{\chi(z)}, A) = \chi(z)(z + 1).$$

(6.124)

Now $\exp[i\theta] = 1/\chi(z) = (1 + z)/z \exp[-A/2(1+2z)]$, from which it follows that

$$\theta(z) = iA(z + \frac{1}{2}) - i \log \frac{1 + z}{z}.$$

(6.125)

Therefore, using (6.124) and the fact that $\phi(\chi(z)) = z$, we have

$$R(e^{i\theta} = \frac{1}{\chi(z)}, A) = e^{-i\theta}(\phi(e^{-i\theta} = \chi(z)) + 1) = e^{-i\theta}(z + 1) \Rightarrow e^{i\theta}R(e^{i\theta}, A) = 1 + z.$$

(6.126)
Redefining \( w = i(z + \frac{1}{2}) \) we have
\[
\theta(w) = Aw - i \log \frac{w - \frac{i}{2}}{w + \frac{i}{2}},
\]
(6.127)
Then \( F(\theta, A) \equiv i[e^{i\theta} R(e^{i\theta}, A) - \frac{1}{2}] = w \) satisfies
\[
\frac{\partial F}{\partial A} + F \frac{\partial F}{\partial \theta} = 0.
\]
(6.128)
We can also employ \( S_A(z) \) to explicitly compute \( \hat{U} \) in the form given by (6.103). Thus if
\[
U(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \omega_{k+1} z^k
\]
(6.129)
then \( U(z) \) is the inverse of the resolvent and
\[
R\left(\frac{1}{\chi(z)}\right) = \chi(z)(z + 1) \Rightarrow U((z + 1)\chi(z)) = \frac{1}{\chi(z)}.
\]
(6.130)
Therefore
\[
U(ze^{\frac{A}{2}(1+2z)}) = \frac{1+z}{z} e^{-\frac{A}{2}(1+2z)}.
\]
(6.131)
Consequently, if \( z(y) \) is determined from
\[
ze^{\frac{A}{2}(1+2z)} = y,
\]
(6.132)
then
\[
U(y) = \frac{1}{y}(1 + z(y)).
\]
(6.133)
This is essentially the result we need. We have obtained \( U(y) \), albeit thus far in an implicit form. To obtain the coefficients \( \omega_k \), we must examine the relation (6.132) more carefully. We have
\[
yU(y) = 1 + z(y) = 1 + \sum_{k=1}^{\infty} \omega_k y^k
\]
\[
= 1 + \sum_{k=1}^{\infty} \omega_k z^k e^{k \frac{A}{2}(1+2z)} \Rightarrow z = \sum_{k=1}^{\infty} \omega_k z^k e^{k \frac{A}{2}(1+2z)}.
\]
(6.134)
This is what determines the coefficients \( \omega_k \). In fact, if we redefine \( \omega_k = A^{k-1} e^{-k \frac{A}{2}} c_k \) and \( z \to Az \), then (6.134) becomes an equation for the \( c_k \)'s.
\[
z = \sum_{k=1}^{\infty} c_k z^k e^{kz}.
\]
(6.135)
All the area dependence is gone and the $c_k$ are just some numbers determined recursively by the above equation. In fact, the recursion relation is non trivial

$$c_k = - \sum_{r=1}^{k-1} c_{k-r} \frac{(k-r)^r}{r!}$$ \hspace{1cm} (6.136)

It can be checked that the $c_k$ are precisely $(-1)^{k-1} \frac{k^{k-1}}{k!}$ due to the non-trivial combinatorial relation

$$(-1)^{k-1} \frac{k^{k-1}}{k!} = (-1)^{k-r} \frac{(k-r)^{k-1}}{(k-r)!r!}$$ \hspace{1cm} (6.137)

But we can actually, rather simply argue that $\omega_k$ have to take this form. That is, it is simply the highest power of $A$ term that appears in $<\text{tr} U^k>$. This follows since $<\text{tr} U^k>$ is, a polynomial with highest power of $A$ being $A^{k-1}$ multiplying, of course, the $e^{-k \frac{A}{2}}$. Since $\omega_n \propto A^{n-1}$, the only term in $<0|\hat{U}^k|0>$ that can contribute an $A^{k-1}$ term is $\omega_k$ (other polynomial terms are of the form $\omega_{i_1} \omega_{i_2} \ldots \omega_{i_r}$ with $\sum_i i_r = k$ and thus are always of lower order). Therefore $\omega_k$ must be precisely the term appearing with $A^{k-1}$ in $<\text{tr} U^k>$ which is exactly what was given earlier (as can be checked from the expansion of the $L^{1}_{(n-1)}$).

7 Conclusions

In this paper we have reviewed the basic concepts of non-commutative probability theory and applied them to the large $N$ limit of matrix models. We discussed at length the work of Voiculescu on the properties and representation of free random variables. Since independent matrix models at $N = \infty$ are free random variables this appears to be the appropriate framework for constructing the master field. We discussed some of these models, including the one-plaquette model where we explicitly constructed the master field. We also discussed, at length, QCD$\,_{2}$. In an axial gauge this theory can be regarded as a theory of independent matrices and thus we could give an explicit construction of the master gauge field. We also showed that there exists a gauge in which the master gauge field is spacetime independent. We also constructed master loop operators based on the observation that simple loops corresponded to free random variables and that any loop could be decomposed into words built
out of simple loops. The simple structure of QCD$_2$ is then a consequence of fact that these form a multiplicative free family.

The most surprising and exciting of our results, however, is the extension of these techniques to deal with the general matrix model, in which the matrices do not have independent distributions and are coupled. Based on our observation that the generating function, introduced by Voiculescu to construct the representation of an independent random variable, can be identified as the generating function of connected planar Green’s functions, we were able to construct the master field for any and all matrix models. Remarkably the Hilbert space in which the master fields are represented is unchanged—it is the Fock space generated by a collection of creation and annihilation operators satisfying the Cuntz algebra—one for each matrix variable.

From some points of view our construction is somewhat disappointing. First, although we have an explicit construction of the master field for any matrix model in a well defined Hilbert space, to actually write the master field explicitly would require a knowledge of all the connected Green’s functions, which is tantamount to solving the theory. Thus from this point of view all we have done is to repackage the unknown solution. Second, we have almost as many degrees of freedom as before. The Hilbert space in which the master field is represented is almost as big as the full Hilbert space of the quantum field theory—i.e., there is an independent creation operator for each independent field variable. The only reduction is by a factor of $N^2$, since the large $N$ limit has been taken.

However, we believe that this reformulation is valuable. Clearly this is the appropriate framework for formulating the $N = \infty$ theory. It also suggests new approaches towards solving the theory by constructing the master field—now a well defined operator in a well defined space. For example, one approach might be to explore the operator equations of motion for the master field, as we have discussed above. Do these, for example, follow from some kind of variational principle that could be the basis for an approximation scheme? Can one develop similar techniques for the Hamiltonian formulation of large $N$ theories?
Acknowledgements

We would like to thank Mike Douglas, Andrei Matytsin, and Sasha Migdal for discussions.
D.G thanks I. Singer for discussions and for bringing the work of Voiculescu to his attention.

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$U_1 = A_1$

$U_2 = A_2$

$U_3 = A_3$

$A_4 = U_4$