ON THE SEMI-CONTINUITY PROBLEM OF NORMALIZED VOLUMES OF SINGULARITIES

YUCHEN LIU

ABSTRACT. We show that in any $\mathbb{Q}$-Gorenstein flat family of klt singularities, normalized volumes can only jump down at countably many subvarieties. A quick consequence is that smooth points have the largest normalized volume among all klt singularities. Using an alternative characterization of K-semistability developed by Li, Xu and the author, we show that K-semistability is a very generic or empty condition in any $\mathbb{Q}$-Gorenstein flat family of log Fano pairs.

1. Introduction

Given an $n$-dimensional complex klt singularity $(x \in (X, D))$, Chi Li [Li15a] introduced the normalized volume function on the space $\text{Val}_{x,X}$ of real valuations of $\mathbb{C}(X)$ centered at $x$. More precisely, for any such valuation $v$, its normalized volume is defined as $\hat{\text{vol}}_{x,(X,D)}(v) := A_{(X,D)}(v)^n \text{vol}(v)$, where $A_{(X,D)}(v)$ is the log discrepancy of $v$ with respect to $(X, D)$ according to [JM12, BdFFU15], and $\text{vol}(v)$ is the volume of $v$ according to [ELS03]. Then we can define the normalized volume of a klt singularity $(x \in (X, D))$ by

$$\hat{\text{vol}}(x, X, D) := \min_{v \in \text{Val}_{x,X}} \hat{\text{vol}}_{x,(X,D)}(v)$$

where the existence of minimizer of $\hat{\text{vol}}$ was shown recently by Blum [Blu16]. We also denote $\hat{\text{vol}}(x, X, 0) := \hat{\text{vol}}(x, X, 0)$.

The normalized volume of a klt singularity $x \in (X, D)$ carries some interesting information of its geometry and topology. It was shown by Xu and the author that $\hat{\text{vol}}(x, X, D) \leq n^n$ and equality holds if and only if $(x \in X \setminus \text{Supp}(D))$ is smooth (see [LiuX17, Theorem A.4] or Theorem 19). By [Xu14], the local algebraic fundamental group $\hat{\pi}_1^{\text{loc}}(X, x)$ of a klt singularity $x \in X$ is always finite. Moreover, assuming the conjectural finite degree formula of normalized volumes [LiuX17, Conjecture 4.1], then the size of $\hat{\pi}_1^{\text{loc}}(X, x)$ is bounded from above by $n^n/\hat{\text{vol}}(x, X)$ (see Remark 22). If $X$ is a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds, then Li and Xu [LX17] showed that $\hat{\text{vol}}(x, X) = n^n \cdot \Theta(x, X)$ where $\Theta(x, X)$ is the volume density of a closed point $x \in X$ (see [HS16, SS17] for background materials).

In this article, we investigate the behavior of normalized volumes of singularities under deformation. We first state the following natural conjecture on constructibility and lower semi-continuity of normalized volumes of klt singularities (see also [Xu17, Conjecture 4.11]).

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Conjecture 1. Let $\pi : (X, D) \to T$ together with a section $\sigma : T \to X$ be a $\mathbb{Q}$-Gorenstein flat family of complex klt singularities over a normal variety $T$. Then the function $t \mapsto \hat{\text{vol}}(\sigma(t), X_t, D_t)$ on $T(\mathbb{C})$ is constructible and lower semi-continuous with respect to the Zariski topology.

Our first main result partially confirms Conjecture 1 by showing that normalized volumes of klt singularities satisfy weak lower semi-continuity in the sense that they can only jump down at countably many subvarieties.

Theorem 2. Let $\pi : (X, D) \to T$ together with a section $\sigma : T \to X$ be a $\mathbb{Q}$-Gorenstein flat family of complex klt singularities. Then for any closed point $o \in T$, there exists an intersection $U$ of countably many Zariski open neighborhoods of $o$, such that $\hat{\text{vol}}(\sigma(t), X_t, D_t) \geq \hat{\text{vol}}(\sigma(o), X_o, D_o)$ for any closed point $t \in U$.

A quick consequence of Theorem 2 is that smooth points have the largest normalized volumes among all klt singularities (see Theorem 19 or [LiuX17, Theorem A.4]).

To verify the Zariski openness of K-semistability is an important step in the construction of algebraic moduli space of K-polystable $\mathbb{Q}$-Fano varieties. In a smooth family of Fano manifolds, Odaka [Oda13] and Donaldson [Don15] showed that the locus of fibers admitting Kähler-Einstein metrics (or equivalently, being K-polystable) with discrete automorphism groups is Zariski open. This was generalized by Li, Wang and Xu [LWX14] where they proved the Zariski openness of K-semistability in a $\mathbb{Q}$-Gorenstein flat families of smoothable $\mathbb{Q}$-Fano varieties in their construction of the proper moduli space of smoothable K-polystable $\mathbb{Q}$-Fano varieties (see [SSY16, Oda15] for related results). A common feature is that analytic methods were used essentially in proving these results.

Using the alternative characterization of K-semistability by the affine cone construction developed by Li, Xu and the author in [Li15b, LL16, LX16], we prove the following result on weak openness of K-semistability as an application of Theorem 2. Unlike the results described in the previous paragraph, our result is proved using purely algebraic method and hence can be applied to $\mathbb{Q}$-Fano families with non-smoothable fibers (or more generally, families of log Fano pairs).

Theorem 3. Let $\varphi : (Y, E) \to T$ be a $\mathbb{Q}$-Gorenstein flat family of complex log Fano pairs over a normal base $T$. Assume that $(Y_o, E_o)$ is log K-semistable for some closed point $o \in T$. Then

1. There exists an intersection $U$ of countably many Zariski open neighborhoods of $o$, such that $(Y_t, E_t)$ is log K-semistable for any closed point $t \in T$. In particular, $(Y_t, E_t)$ is log K-semistable for a very general closed point $t \in T$.

2. Denote by $\eta$ the generic point of $T$, then the geometric generic fiber $(Y_\eta, E_\eta)$ is log K-semistable.

3. Assume Conjecture 1 is true, then such $U$ from (1) can be chosen as a genuine Zariski open neighborhood of $o$.

The following corollary follows easily from Theorem 3.

Corollary 4. Suppose a complex log Fano pair $(Y, E)$ specially degenerates to a log K-semistable log Fano pair $(Y_o, E_o)$, then $(Y, E)$ is also log K-semistable.
Our strategy to prove Theorem 2 is to study various invariants of ideals instead of valuations. From the author’s characterization of normalized volume by normalized multiplicities of ideals (see [Lin16, Theorem 27] or Theorem 3, we know that \( \overline{\alpha}(\sigma(t), X_t, D_t) = \inf_{a} \text{lct}(X_t, D_t; a)^n \cdot e(a) \) where the infimum is taken over all ideals \( a \) cosupported at \( \sigma(t) \). These ideals are parametrized by some relative Hilbert scheme of \( X/T \) with countably many components. Clearly \( a \mapsto \text{lct}(X_t, D_t; a) \) is lower semi-continuous on the Hilbert scheme, but \( a \mapsto e(a) \) is usually upper semi-continuous which makes the desired lower semi-continuity of \( \text{lct}(a)^n \cdot e(a) \) obscure. To fix this issue, we introduce the \textit{normalized colength of singularities} by taking the infimum of \( \text{lct}(X_t, D_t; a)^n \cdot \ell(O_{\sigma(t)}, X_t/a) \) for certain classes of ideals \( a \). The normalized colength function behaves better in families since the colength function \( a \mapsto \ell(O_{\sigma(t)}, X_t/a) \) is always locally constant in the Hilbert scheme, so \( a \mapsto \text{lct}(X_t, D_t; a)^n \cdot \ell(O_{\sigma(t)}, X_t/a) \) is lower semi-continuous on the Hilbert scheme. Then we prove a key equality between the asymptotic normalized colength and the normalized volume (see Theorem 12) using local Newton-Okounkov bodies following [Cut13, KKL14] (see Lemma 13) and convex geometry (see Appendix A). Putting these ingredients together, we get the proof of Theorem 2.

This paper is organized as follows. In Section 2, we give the preliminaries including notations, normalized volumes of singularities, and \( \mathbb{Q} \)-Gorenstein flat families of klt pairs. In Section 3.1, we introduce the concept of normalized colengths of singularities. We show in Theorem 12 that the normalized volume of a klt singularity is the same as its asymptotic normalized colength. The proof of Theorem 12 uses a comparison of colengths and multiplicities established in Lemma 13. In Section 3.2, we study the normalized volumes and normalized colength after algebraically closed field extensions. The proofs of main theorems are presented in Section 4.1. We give applications of our main theorems in Section 4.2. Theorem 13 generalizes the inequality part of [LiuX17, Theorem A.4]. We give an effective upper bound on the degree of finite quasi-étale maps over klt singularities on Gromov-Hausdorff limits of Kähler-Einstein Fano manifolds (see Theorem 21). In Appendix A, we provide certain convex geometric results on lattice points counting that are needed in proving Lemma 13.

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2. Preliminaries

2.1. Notations. In the rest of this paper, all varieties are assumed to be irreducible, reduced, and defined over a (not necessarily algebraically closed) field \( k \) of characteristic 0. For a variety \( T \) over \( k \), we denote the residue field of any scheme-theoretic point \( t \in T \) by \( \kappa(t) \). Let \( \pi : X \to T \) between varieties over \( k \), we denote by \( X_t := X \times_T \text{Spec}(\kappa(t)) \) the scheme theoretic fiber over \( t \in T \). We also denote the geometric fiber of \( \pi \) over \( t \in T \) by \( X'_t := X \times_T \text{Spec}(\kappa(t)) \). Suppose \( X \) is a variety over \( k \) and \( x \in X \) is a \( k \)-rational point. Then for any field extension \( K/k \), we denote \( (x_K, X_K) := (x, X) \times_{\text{Spec}(k)} \text{Spec}(K) \).

Let \( X \) be a normal variety over \( k \). Let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \). We say that \( (X, D) \) is a \textit{Kawamata log terminal (klt) pair} if \( (K_X + D) \) is \( \mathbb{Q} \)-Cartier and \( \text{ord}_E(K_Y - ...)
$f^*(K_X+D)>-1$ (equivalently, $A_{(X,D)}(\text{ord}_E) > 0$) for any prime divisor $E$ on some log resolution $f: Y \to (X, D)$. A klt pair $(X, D)$ is called a log Fano pair if in addition $X$ is proper and $-(K_X+D)$ is ample. A klt pair $(X, D)$ together with a closed point $x \in X$ is called a klt singularity $(x \in (X, D))$.

Let $(X, D)$ be a klt pair. For an ideal sheaf $a$ on $X$, we define the log canonical threshold of $a$ with respect to $(X, D)$ by

$$\text{lct}(X, D; a) := \inf_E \frac{1 + \text{ord}_E(K_Y - f^*(K_X + D))}{\text{ord}_E(a)},$$

where the infimum is taken over all prime divisors $E$ on a log resolution $f: Y \to (X, D)$. We will also use the notation $\text{lct}(a)$ as abbreviation of $\text{lct}(X, D; a)$ once the klt pair $(X, D)$ is specified. If $a$ is co-supported at a single closed point $x \in X$, we define the Hilbert-Samuel multiplicity of $a$ as

$$e(a) := \lim_{m \to \infty} \frac{\ell(O_{x,X}/a^m)}{m^n/n!}$$

where $n := \dim(X)$ and $\ell(\cdot)$ is the length of an Artinian ring.

### 2.2. Normalized volumes of singularities.

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. For an $n$-dimensional klt singularity $x \in (X, D)$ over $\mathbb{k}$, C. Li [Li15a] introduced the normalized volume function $\hat{\text{vol}}_{x, (X,D)}: \text{Val}_{X,x} \to \mathbb{R}_{>0} \cup \{+\infty\}$ where $\text{Val}_{X,x}$ is the space of all real valuations of $\mathbb{k}(X)$ centered at $x$. Then we can define the normalized volume of the singularity $x \in (X, D)$ to be

$$\hat{\text{vol}}(x, X, D) := \inf_{v \in \text{Val}_{X,x}} \hat{\text{vol}}_{x, (X,D)}(v).$$

If in addition $\mathbb{k}$ is uncountable, then Blum [Blu16] proved the existence of a $\hat{\text{vol}}$-minimizing valuation. The following characterization of normalized volumes using log canonical thresholds and multiplicities of ideals is crucial in our study. Note that the right hand side of (2.1) was studied by de Fernex, Ein and Mustaţă [dFEM04] when $x \in X$ is smooth and $D = 0$.

**Theorem 5** ([Lin16, Theorem 27]). With the above notation, we have

(2.1) \[ \hat{\text{vol}}(x, X, D) = \inf_{a: m_x-\text{primary}} \text{lct}(X, D; a)^n \cdot e(a). \]

The following theorem provides an alternative characterization of K-semistability using the affine cone construction. Here we state the most general form, and special cases can be found in [Li15b, LL16].

**Theorem 6** ([LX16, Proposition 4.6]). Let $(Y, E)$ be a log Fano pair of dimension $(n-1)$ over an algebraically closed field $\mathbb{k}$ of characteristic 0. For $r \in \mathbb{N}$ satisfying $L := -r(K_Y + E)$ is Cartier, the affine cone $X = C(Y, L)$ is defined by $X := \text{Spec} \oplus_{m\geq 0} H^0(Y, L^m)$. Let $D$ be the $\mathbb{Q}$-divisor on $X$ corresponding to $E$. Denote by $x$ the cone vertex of $X$. Then

$$\hat{\text{vol}}(x, X, D) \leq r^{-1}(-K_Y - E)^{n-1},$$

and the equality holds if and only if $(Y, E)$ is log K-semistable.
2.3. \(\mathbb{Q}\)-Gorenstein flat families of klt pairs. In this section, the field \(\mathbb{K}\) is not assumed to be algebraically closed.

**Definition 7.**

(a) Given a normal variety \(T\), a \(\mathbb{Q}\)-Gorenstein flat family of klt pairs over \(T\) consists of a surjective flat morphism \(\pi : \mathcal{X} \to T\) from a variety \(\mathcal{X}\), and an effective \(\mathbb{Q}\)-divisor \(D\) on \(\mathcal{X}\) avoiding codimension 1 singular points of \(\mathcal{X}\), such that the following conditions hold:

- All fibers \(X_t\) are connected, normal and not contained in \(\text{Supp}(D)\);
- \(K_{X/T} + D\) is \(\mathbb{Q}\)-Cartier;
- \((X_t, D_t)\) is a klt pair for any \(t \in T\).

(b) A \(\mathbb{Q}\)-Gorenstein flat family of klt pairs \(\pi : (\mathcal{X}, D) \to T\) together with a section \(\sigma : T \to \mathcal{X}\) is called a \(\mathbb{Q}\)-Gorenstein flat family of klt singularities. We denote by \(\sigma(i)\) the unique closed point of \(X_i\) lying over \(\sigma(t) \in X_t\).

**Proposition 8.** Let \(\pi : (\mathcal{X}, D) \to T\) be a \(\mathbb{Q}\)-Gorenstein flat family of klt pairs over a normal variety \(T\). Then

1. There exists a closed subset \(Z\) of \(\mathcal{X}\) of codimension at least 2, such that \(Z_t\) has codimension at least 2 in \(X_t\) for every \(t \in T\), and \(\pi : \mathcal{X} \setminus Z \to T\) is a smooth morphism.

2. \(\mathcal{X}\) is normal.

3. For any morphism \(f : T' \to T\) from a normal variety \(T'\) to \(T\), the base change \(\pi_{T'} : (\mathcal{X}_{T'}, D_{T'}) = (\mathcal{X}, D) \times_T T' \to T'\) is a \(\mathbb{Q}\)-Gorenstein flat family of klt pairs over \(T'\), and \(K_{\mathcal{X}_{T'}/T'} + D_{T'} = g^*(K_{\mathcal{X}/T} + D)\) where \(g : \mathcal{X}_{T'} \to \mathcal{X}\) is the base change of \(f\).

**Proof.** (1) Assume \(\pi\) is of relative dimension \(n\). Let \(Z := \{x \in \mathcal{X} \mid \dim_{\kappa(x)} \Omega_{\mathcal{X}/T} \otimes \kappa(x) > n\}\). It is clear that \(Z\) is Zariski closed. Since \(\mathbb{K}\) is of characteristic 0, \(Z_t = Z \cap X_t\) is the singular locus of \(X_t\). Hence \(\text{codim}_{\mathcal{X}_t} Z_t \geq 2\) because \(X_t\) is normal.

(2) From (1) we know that \(Z\) is of codimension at least 2 in \(\mathcal{X}\), and \(\mathcal{X} \setminus Z\) is smooth over \(T\). Thus \(\mathcal{X} \setminus (Z \cup \pi^{-1}(T_{\text{sing}}))\) is regular, and \(Z \cup \pi^{-1}(T_{\text{sing}})\) has codimension at least 2 in \(\mathcal{X}\). So \(\mathcal{X}\) satisfies property (\(R_1\)). Since \(\pi\) is flat, for any point \(x \in X_t\) we have \(\text{depth}(O_{x, \mathcal{X}}) = \text{depth}(O_{x, X_t}) + \text{depth}(O_{t, T})\) by \([\text{MatSU}\, (21.C)\, \text{Corollary 1}]\). Hence it is easy to see that \(\mathcal{X}\) satisfies property (\(S_2\)) since both \(X_t\) and \(T\) are normal. Hence \(\mathcal{X}\) is normal.

(3) Let \(Z_{T'} := Z \times_T T'\), then \(X_{T'} \setminus Z_{T'}\) is smooth over \(T'\). Since the fibers of \(\pi_{T'}\) and \(T'\) are irreducible, we know that \(X_{T'}\) is also irreducible. Thus the same argument of (2) implies that \(X_{T'}\) satisfies both (\(R_1\)) and (\(S_2\)), which means \(X_{T'}\) is normal. Since \(\pi|_{\mathcal{X}\setminus Z}\) is smooth, we know that \(K_{\mathcal{X}_{T'}/T'} + D_{T'}\) and \(g^*(K_{\mathcal{X}/T} + D)\) are \(\mathbb{Q}\)-linearly equivalent after restricting to \(X_{T'} \setminus Z_{T'}\). Since \(Z_{T'}\) is of codimension at least 2 in \(X_{T'}\), the \(\mathbb{Q}\)-linear equivalence over \(X_{T'} \setminus Z_{T'}\) extends to \(X_{T'}\). Thus we finish the proof. \(\square\)

**Definition 9.**

(a) Let \(Y\) be a normal projective variety. Let \(E\) be an effective \(\mathbb{Q}\)-divisor on \(Y\). We say that \((Y, E)\) is a log Fano pair if \((Y, E)\) is a klt pair and \(-(K_Y + E)\) is \(\mathbb{Q}\)-Cartier and ample. We say \(Y\) is a \(\mathbb{Q}\)-Fano variety if \((Y, 0)\) is a log Fano pair.

(b) Let \(T\) be a normal variety. A \(\mathbb{Q}\)-Gorenstein family of klt pairs \(\varphi : (Y, E) \to T\) is called a \(\mathbb{Q}\)-Gorenstein flat family of log Fano pairs if \(\varphi\) is proper and \(-(K_{Y/T} + E)\) is \(\varphi\)-ample.
The following proposition is a natural generalization of [Blu16 Proposition A.2 and A.3]. It should be well-known to experts. The proof is omitted because it is essentially the same as [Blu16 Appendix A].

**Proposition 10.** Let $\pi : (X, D) \to T$ be a $\mathbb{Q}$-Gorenstein flat family of klt pairs over a normal variety $T$. Let $a$ be an ideal sheaf of $X$. Then

1. The function $t \mapsto \text{lct}(X_t, D_t; a_t)$ on $T$ is constructible;
2. If in addition $V(a)$ is proper over $T$, then the function $t \mapsto \text{lct}(X_t, D_t; a_t)$ is lower semi-continuous with respect to the Zariski topology on $T$.

3. Comparison of normalized volumes and normalized colengths

3.1. Normalized colengths of klt singularities.

**Definition 11.** Let $x \in (X, D)$ be a klt singularity over an algebraically closed field $\mathbb{k}$ of characteristic 0. Denote its local ring by $(R, m) := (O_{x, X}, m_x)$.

(a) Given constants $c \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$, we define the normalized colength of $x \in (X, D)$ with respect to $c, k$ as

$$\widehat{\ell}_{c,k}(x, X, D) := n! \cdot \inf_{m^c / a \subset m} \text{lct}(a)^n \cdot \ell(R/a).$$

Here $a$ is a $m$-primary ideal.

(b) Given a constant $c \in \mathbb{R}_{>0}$, we define the asymptotic normalized colength function of $x \in (X, D)$ with respect to $c$ as

$$\widehat{\ell}_{c,\infty}(x, X, D) := \liminf_{k \to \infty} \widehat{\ell}_{c,k}(x, X, D).$$

It is clear that $\widehat{\ell}_{c,k}$ is an increasing function in $c$. The main result in this section is the following theorem.

**Theorem 12.** For any klt singularity $x \in (X, D)$ over an algebraically closed field $\mathbb{k}$ of characteristic 0, there exists $c_0 = c_0(x, X, D) > 0$ such that

$$\widehat{\ell}_{c,\infty}(x, X, D) = \widehat{\text{vol}}(x, X, D) \quad \text{whenever } 0 < c \leq c_0.$$ 

**Proof.** We first show the "$\leq$" direction. Let us take a sequence of valuations $\{v_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} \text{vol}(v_i) = \text{vol}(x, X, D)$. We may rescale $v_i$ so that $v_i(m) = 1$ for any $i$. Since $\{\text{vol}(v_i)\}_{i \in \mathbb{N}}$ are bounded from above, by [Li15a Theorem 1.1] we know that there exists $C_1 > 0$ such that $A_{(X, D)}(v_i) \leq C_1$ for any $i \in \mathbb{N}$. Then by Li’s Izumi type inequality [Li15a Theorem 3.1], there exists $C_2 > 0$ such that $\text{ord}_m(f) \leq v_i(f) \leq C_2 \text{ord}_m(f)$ for any $i \in \mathbb{N}$ and any $f \in R$. As a result, we have $m^k \subset a_k(v_i) \subset m^{[k/C_2]}$ for any $i, k \in \mathbb{N}$. Thus $\ell(R/a_k(v_i)) \geq \ell(R/m^{[k/C_2]}) \sim \frac{e(m)}{2nk^2} k^n$. Let us take $c_0 = \frac{e(m)}{2nk^2}$, then for $k \gg 1$ we have $\ell(R/a_k(v_i)) \geq c_0 k^n$ for any $i \in \mathbb{N}$. Therefore, for any $i \in \mathbb{N}$ we have

$$\widehat{\ell}_{c_0,\infty}(x, X, D) \leq n! \liminf_{k \to \infty} \text{lct}(a_k(v_i))^n \ell(R/a_k(v_i)) = \text{lct}(a_\infty(v_i))^n \text{vol}(v_i) \leq \text{vol}(v_i).$$

In the last inequality we use $\text{lct}(a_\infty(v_i)) \leq A_{(X, D)}(v_i)$ as in the proof of [Liu16 Theorem 27]. Thus $\widehat{\ell}_{c_0,\infty}(x, X, D) \leq \lim_{i \to \infty} \text{vol}(v_i) = \text{vol}(x, X, D)$. This finishes the proof of the "$\leq$" direction.
For the “≥” direction, we will show that \( \ell_{c,\infty}(x, X, D) \geq \hat{\text{vol}}(x, X, D) \) for any \( c > 0 \). By a logarithmic version of the Izumi type estimate [[L15a, Theorem 3.1]], there exists a constant \( c_1 = c_1(x, X, D) > 0 \) such that \( v(f) \leq c_1 A_{X,D}(v) \text{ord}_m(f) \) for any valuation \( v \in \text{Val}_{X,x} \) and any function \( f \in R \). For any \( m \)-primary ideal \( a \), there exists a divisorial valuation \( v_0 \in \text{Val}_{X,x} \) computing \( \text{lev}(a) \) by [[L16, Lemma 26]]. Hence we have the following Skoda type estimate:

\[
\text{lev}(a) = \frac{A_{X,D}(v_0)}{v_0(a)} \geq \frac{A_{X,D}(v_0)}{c_1 A_{X,D}(v_0) \text{ord}_m(a)} = \frac{1}{c_1 \text{ord}_m(a)}.
\]

Let \( 0 < \delta < 1 \) be a positive number. If \( a \not\subset m^{[\delta k]} \) and \( \ell(R/a) \geq c k^n \), then

\[
\text{lev}(a)^n \cdot \ell(R/a) \geq \frac{c k^n}{c_1^n (|\delta k| - 1)^n} \geq \frac{c}{c_1^n \delta n}.
\]

If we choose \( \delta \) sufficiently small such that \( \delta^n \cdot c_1^n \hat{\text{vol}}(x, X, D) \leq n! c \), then for any \( m \)-primary ideal \( a \) satisfying \( m^n \subset a \not\subset m^{[\delta k]} \) and \( \ell(R/a) \geq c k^n \) we have

\[
n! \cdot \text{lev}(a) \cdot \ell(R/a) \geq \hat{\text{vol}}(x, X, D).
\]

Thus it suffices to show

\[
\hat{\text{vol}}(x, X, D) \leq n! \cdot \liminf_{k \to \infty} \inf_{m^n \subset a \subset m^{[\delta k]}} \inf_{\ell(R/a) \geq c k^n} \text{lev}(a) \ell(R/a).
\]

By Lemma 13, we know that for any \( \epsilon > 0 \) there exists \( k_0 = k_0(\delta, \epsilon, (R, m)) \) such that for any \( k \geq k_0 \) we have

\[
n! \cdot \inf_{m^n \subset a \subset m^{[\delta k]}} \text{lev}(a) \ell(R/a) \geq (1 - \epsilon) \inf_{m^n \subset a \subset m^{[\delta k]}} \text{lev}(a)^n e(a) \geq (1 - \epsilon) \hat{\text{vol}}(x, X, D).
\]

Hence the proof is finished. \( \square \)

The following result on comparison between colengths and multiplicities is crucial in the proof of Theorem 12. Note that Lemma 13 is a special case of Lech’s inequality [[Lec64, Theorem 3]] when \( R \) is a regular local ring.

**Lemma 13.** Let \((R, m)\) be an \( n \)-dimensional analytically irreducible Noetherian local domain. Assume that the residue field \( R/m \) is algebraically closed. Then for any positive numbers \( \delta, \epsilon \in (0, 1) \), there exists \( k_0 = k_0(\delta, \epsilon, (R, m)) \) such that for any \( k \geq k_0 \) and any ideal \( m^n \subset a \subset m^{[\delta k]} \), we have

\[
n! \cdot \ell(R/a) \geq (1 - \epsilon) e(a).
\]

**Proof.** By [[KK14, 7.8]] and [[Cut13, Section 4]], \( R \) admits a good valuation \( \nu : R \to \mathbb{Z}^n \) for some total order on \( \mathbb{Z}^n \). Let \( S := \nu(R \setminus \{0\}) \subset \mathbb{N}^n \) and \( C(S) \) be the closed convex hull of \( S \). Then we know that

- \( C(S) \) is a strongly convex cone;
- There exists a linear functional \( \xi : \mathbb{R}^n \to \mathbb{R} \) such that \( C(S) \setminus \{0\} \subset \xi > 0 \);
- There exists \( r_0 \geq 1 \) such that for any \( f \in R \setminus \{0\} \), we have

\[
\text{ord}_m(f) \leq \xi(\nu(f)) \leq r_0 \text{ord}_m(f).
\]

(3.2)
Suppose $a$ is an ideal satisfying $m^k \subset a \subset m^{[dk]}$. Then we have $\nu(m^k) \subset \nu(a) \subset \nu(m^{[dk]})$. By (3.2), we know that 
\[ S \cap \xi_{\geq \tau} \subset \nu(a) \subset S \cap \xi_{\geq \delta}. \]
Similarly, we have $S \cap \xi_{\geq \tau} \subset \nu(a') \subset S \cap \xi_{\geq \delta}$ for any positive integer $i$.

Let us define a semigroup $\Gamma \subset \mathbb{N}^{n+1}$ as follows:
\[ \Gamma := \{(\alpha, m) \in \mathbb{N}^n \times \mathbb{N} : x \in S \cap \xi_{\leq 2r_{\alpha}}\}. \]
For any $m \in \mathbb{N}$, denote by $\Gamma_m := \{\alpha \in \mathbb{N}^n : (\alpha, m) \in \Gamma\}$. It is easy to see $\Gamma$ satisfies [LM09] (2.3-5), thus [LM09] Proposition 2.1 implies 
\[ \lim_{m \to \infty} \frac{\# \Gamma_m}{m^n} = \text{vol}(\Delta), \]
where $\Delta := \Delta(\Gamma)$ is a convex body in $\mathbb{R}^n$ defined in [LM09] Section 2.1. It is easy to see that $\Delta = C(S) \cap \xi_{\leq \beta_0}$.

Let us define $\Gamma^{(k)} := \{(\alpha, i) \in \mathbb{N}^n \times \mathbb{N} : (\alpha, ik) \in \Gamma\}$. Then we know that $\Delta^{(k)} := \Delta(\Gamma^{(k)}) = k\Delta$. For an ideal $a$ and $k \in \mathbb{N}$ satisfying $m^k \subset a \subset m^{[dk]}$, we define 
\[ \Gamma_a^{(k)} := \{(\alpha, i) \in \Gamma^{(k)} : \alpha \in \nu(a')\}. \]
Then it is clear that $\Gamma_a^{(k)}$ also satisfies [LM09] (2.3-5). Since $\nu(a') = (S \cap \xi_{\geq 2r_{ik}}) \cup \Gamma_a^{(k)}$ and $R/m$ is algebraically closed, we have $\ell(R/a') = \#(\Gamma^{(k)}_i \setminus \Gamma_a^{(k)})$ because $\nu$ has one-dimensional leaves. Again by [LM09] Proposition 2.11, we have 
\[ n!e(a) = \lim_{i \to \infty} \frac{\ell(R/a')}{i^n} = \lim_{i \to \infty} \frac{\#(\Gamma^{(k)}_i \setminus \Gamma_a^{(k)})}{i^n} = \text{vol}(\Delta^{(k)}) - \text{vol}(\Delta^{(a,k)}), \]
where $\Delta^{(a,k)} := \Delta(\Gamma_a^{(k)})$. Since $\Gamma_a^{(k)} \subset \nu(a') \subset \xi_{\geq \delta}$, we know that $\Delta^{(a,k)} \subset \xi_{\geq \delta}$. Denote by $\Delta' := C(S) \cap \xi_{\leq \delta}$, then it is clear that $\Delta^{(a,k)} \subset k(\Delta \setminus \Delta')$.

On the other hand, 
\[ \ell(R/a) = \#(\Gamma^{(k)}_i \setminus \Gamma_a^{(k)}) \geq \#\Gamma_k - \#(\Delta^{(k)}_a \cap Z^n). \]
Denote by $\Delta_{a,k} := \frac{1}{k}\Delta^{(a,k)}$, then $\Delta_{a,k} \subset \Delta \setminus \Delta'$. Since $\text{vol}(\Delta_{a,k}) \leq \text{vol}(\Delta) - \text{vol}(\Delta')$, there exists positive numbers $\epsilon_1, \epsilon_2$ depending only on $\Delta$ and $\Delta'$ such that 
\[ (3.3) \quad \text{vol}(\Delta_{a,k}) \leq \text{vol}(\Delta) - \text{vol}(\Delta') \leq \left(1 - \frac{\epsilon_1}{\epsilon}\right) \text{vol}(\Delta) - \frac{\epsilon_2}{\epsilon}. \]
Let us pick $k_0$ such that for any $k \geq k_0$ and any $m^k \subset a \subset m^{[dk]}$, we have 
\[ \frac{\# \Gamma_k}{k^n} \geq (1 - \epsilon_1)\text{vol}(\Delta), \quad \frac{\#(\Delta^{(k)}_a \cap Z^n)}{k^n} \leq \text{vol}(\Delta_{a,k}) + \epsilon_2. \]
Here the second inequality is guaranteed by applying Proposition [24] to $\Delta_{a,k}$ as a sub convex body of a fixed convex body $\Delta$. Thus 
\[ \ell(R/a) - (1 - \epsilon)n!e(a) \geq \frac{\# \Gamma_k}{k^n} - \frac{\#(\Delta^{(k)}_a \cap Z^n)}{k^n} - (1 - \epsilon)(\text{vol}(\Delta) - \text{vol}(\Delta_{a,k})) \geq (1 - \epsilon_1)\text{vol}(\Delta) - \text{vol}(\Delta_{a,k}) - \epsilon_2 - (1 - \epsilon)(\text{vol}(\Delta) - \text{vol}(\Delta_{a,k})) = (\epsilon - \epsilon_1)\text{vol}(\Delta) - \epsilon(\Delta_{a,k}) - \epsilon_2 \geq 0. \]
Here the last inequality follows from \([3,3]\). Hence we finish the proof. \(\square\)

3.2. Normalized volumes under field extensions. In the rest of this section, we use Hilbert schemes to describe normalized volumes of singularities after a field extension \(\mathbb{K}/\mathbb{k}\). Let \((X,D)\) be a klt pair over \(\mathbb{k}\). Let \(x \in X\) be a \(\mathbb{k}\)-rational point. Let \(Z_k := \text{Spec}(\mathcal{O}_{x,X}/\mathfrak{m}_x^k)\) be the \(k\)-th thickening of \(x\). Consider the Hilbert scheme \(\text{Hilb}(Z_k/\mathbb{k})\) we know that \(H_{k,d}(\mathbb{k})\) parametrizes ideal sheaves \(c\) of \(X_k\) satisfying \(c \supseteq \mathfrak{m}_x^k\) and \(\ell(\mathcal{O}_{X_k}/c) = d\). In particular, any scheme-theoretic point \(h \in H_{k,d}\) corresponds to an ideal \(b\) of \(\mathcal{O}_{x,\kappa(h)}\) satisfying those two conditions, and we denote by \(h = [b]\).

**Proposition 14.** Let \(\mathbb{k}\) be a field of characteristic 0. Let \((X,D)\) be a klt pair over \(\mathbb{k}\). Let \(x \in X\) be a \(\mathbb{k}\)-rational point. Then

1. For any field extension \(\mathbb{K}/\mathbb{k}\) with \(\mathbb{K}\) algebraically closed, we have
   \[
   \ell_{c,k}(x, X, D_{\mathbb{K}}) = n! \cdot \inf_{d \geq ck^n, \ |b| \in H_{k,d}} d \cdot \text{lct}(\kappa([b]), D_{\kappa([b])}; b)^n.
   \]

2. With the assumption of (1), we have
   \[
   \mathbb{V}ol(x, X, D_{\mathbb{K}}) = \mathbb{V}ol(x, X, D_{\mathbb{K}}).
   \]

**Proof.** (1) We first prove the “\(\geq\)” direction. By definition, \(\ell_{c,k}(x, X, D_{\mathbb{K}})\) is the infimum of \(n! \cdot \text{lct}(X, D_{\mathbb{K}}; c)^n \ell(\mathcal{O}_{X_k}/c)\) where \(c\) is an ideal on \(X_k\) satisfying \(\mathfrak{m}_x^k \subseteq c \subseteq \mathfrak{m}_x\) and \(\ell(\mathcal{O}_{X_k}/c) = d \geq ck^n\). Hence \([c]\) represents a point in \(H_{k,d}(\mathbb{K})\). Suppose \([c]\) is lying over a scheme-theoretic point \([b] \in H_{k,d}\), then it is clear that \((X_k, D_{\mathbb{K}}, c) \cong (X_{\kappa([b])}, D_{\kappa([b])}, b) \times_{\text{Spec}(\kappa([b]))} \text{Spec}(\mathbb{K})\). Hence \(\text{lct}(X, D_{\mathbb{K}}; c) = \text{lct}(X_{\kappa([b])}, D_{\kappa([b])}; b)\), and the “\(\geq\)” direction is proved.

Next we prove the “\(\leq\)” direction. By Proposition 10 we know that the function \([b] \mapsto \text{lct}(X_{\kappa([b])}, D_{\kappa([b])}; b)\) on \(H_{k,d}\) is constructible and lower semi-continuous. Denote by \(H_{k,d}^1\) the set of closed points in \(H_{k,d}\). Since the set of closed points are dense in any stratum of \(H_{k,d}\) with respect to the lct function, we have the following equality:

\[
\begin{align*}
n! \cdot \inf_{d \geq ck^n, \ |b| \in H_{k,d}} d \cdot \text{lct}(X, D_{\mathbb{K}}; c)^n & = n! \cdot \inf_{d \geq ck^n, \ |b| \in H_{k,d}^1} d \cdot \text{lct}(X, D_{\mathbb{K}}; c)^n
\end{align*}
\]

Any \([b] \in H_{k,d}^1\) satisfies that \(\kappa([b])\) is an algebraic extension of \(\mathbb{k}\). Since \(\mathbb{K}\) is algebraically closed, \(\kappa([b])\) can be embedded into \(\mathbb{K}\) as a subfield. Hence there exists a point \([c] \in H_{k,d}(\mathbb{K})\) lying over \([b]\). Thus similar arguments implies that \(\text{lct}(X, D_{\mathbb{K}}; c) = \text{lct}(X_{\kappa([b])}, D_{\kappa([b])}; b)\), and the “\(\leq\)” direction is proved.

(2) From (1) we know that \(\ell_{c,k}(x, X, D_{\mathbb{K}}) = \ell_{c,k}(x, X, D_{\mathbb{K}})\) for any \(c, k\). Hence it follows from Theorem 12. \(\square\)

The following corollary is well-known to experts. We present a proof here using normalized volumes.

**Corollary 15.** Let \((Y, E)\) be a log Fano pair over a field \(\mathbb{k}\) of characteristic 0. The following are equivalent:

(i) \((Y, E)\) is log \(K\)-semistable;

(ii) \((Y, E)\) is log \(K\)-semistable for some field extension \(\mathbb{K}/\mathbb{k}\) with \(\mathbb{K} = \mathbb{K}\).


(iii) \((Y_{\mathbb{K}}, E_{\mathbb{K}})\) is log \(K\)-semistable for any field extension \(\mathbb{K}/\mathbb{K}\) with \(\mathbb{K} = \overline{\mathbb{K}}\).

We say that \((Y, E)\) is geometrically log \(K\)-semistable if one (or all) of these conditions holds.

**Proof.** Let us take the affine cone \(X = C(Y, L)\) with \(L = -r(K_Y + E)\) Cartier. Let \(D\) be the \(\mathbb{Q}\)-divisor on \(X\) corresponding to \(E\). Denote by \(x \in X\) the cone vertex of \(X\). Let \(\mathbb{K}/\mathbb{K}\) be a field extension with \(\mathbb{K} = \overline{\mathbb{K}}\). Then Theorem 6 implies that \((Y_{\mathbb{K}}, E_{\mathbb{K}})\) is log \(K\)-semistable if and only if \(\text{vol}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}}) = r^{-1}(-K_Y - E)^{n-1}\). Hence the corollary is a consequence of Proposition 14 (2).

We finish this section with a natural speculation. Suppose \(x \in (X, D)\) is a klt singularity over a field \(k\) of characteristic zero that is not necessarily algebraically closed. The definition of normalized volume of singularities extend verbatimly to \(x \in (X, D)\) which we also denote by \(\text{vol}(x, X, D)\). Then we expect \(\text{vol}(x, X, D) = \text{vol}(x_{\overline{k}}, X_{\overline{k}}, D_{\overline{k}})\), i.e. normalized volumes are stable under base change to algebraic closures. Such a speculation should be a consequence of the Stable Degeneration Conjecture (SDC) stated in [Li15a Conjecture 7.1] and [LX17 Conjecture 1.2] which roughly says that a \(\text{vol}\)-minimizing valuation \(v_{\min}\) over \(x_{\overline{k}} \in (X_{\overline{k}}, D_{\overline{k}})\) is unique and quasi-monomial, so \(v_{\min}\) is invariant under the action of \(\text{Gal}(\overline{k}/k)\) and hence has the same normalized volume as its restriction to \(x \in (X, D)\).

4. **Proofs and applications**

4.1. **Proofs.** The following theorem is a stronger result that implies Theorem 2.

**Theorem 16.** Let \(\pi : (\mathcal{X}, \mathcal{D}) \to T\) together with a section \(\sigma : T \to \mathcal{X}\) be a \(\mathbb{Q}\)-Gorenstein flat family of klt singularities over a field \(\mathbb{K}\) of characteristic 0. Then for any point \(o \in T\), there exists an intersection \(U\) of countably many Zariski open neighborhoods of \(o\), such that \(\text{vol}(\sigma(t), X_t, D_t) \geq \text{vol}(\sigma(o), X_o, D_o)\) for any point \(t \in U\). In particular, if \(t\) is a generalization of \(o\) then \(\text{vol}(\sigma(t), X_t, D_t) = \text{vol}(\sigma(o), X_o, D_o)\).

**Proof.** Let \(Z_k \to T\) be the \(k\)-th thickening of the section \(\sigma\), i.e. \(Z_{k,t} = \text{Spec}(\mathcal{O}_{X_t}/m_{(t),X_t}^k)\). Let \(d_k := \max_{t \in T} \ell(\mathcal{O}_{\sigma(t),X_t}/m_{(t),X_t}^k)\) for any \(d \in \mathbb{N}\), denote \(H_{k,d} := \text{Hilb}_d(Z_k/T)\). Since \(Z_k\) is proper over \(T\), we know that \(H_{k,d}\) is also proper over \(T\). Let \(H^n_{k,d}\) be the normalization of \(H_{k,d}\). Denote by \(\tau_{k,d} : H_{k,d} \to T\). After pulling back the universal ideal sheaf on \(\mathcal{X} \times_T H_{k,d}\) over \(H_{k,d}\) to \(H^n_{k,d}\), we obtain an ideal sheaf \(b_{k,d}\) on \(\mathcal{X} \times_T H^n_{k,d}\). Denote by \(\pi_{k,d} : (\mathcal{X} \times_T H^n_{k,d}, D \times_T H^n_{k,d}) \to H^n_{k,d}\), the projection, then \(\pi_{k,d}\) provides a \(\mathbb{Q}\)-Gorenstein flat family of klt pairs.

Following the notation of Proposition 14 assume \(h\) is scheme-theoretic point of \(H^n_{k,d}\), lying over \([b]\) \(\in H_{k,d}\). Denote by \(t = \tau_{k,d}([b]) \in T\). By construction, the ideal sheaf \(b_{k,d,h}\) on \(\mathcal{X} \times_T \text{Spec}(\kappa(h))\) is the pull back of \(b\) under the flat base change \(\text{Spec}(\kappa(h)) \to \text{Spec}(\kappa([b]))\).

Hence
\[
\ell_c((\mathcal{X}, D) \times_T \text{Spec}(\kappa(h)); b_{k,d,h}) = \ell_c((\mathcal{X}, D) \times_T \text{Spec}(\kappa([b])); b).
\]

For simplicity, we abbreviate the above equation to \(\ell_c(b_{k,d,h}) = \ell_c(b)\). Applying Proposition 10 to the family \(\pi_{k,d}\) and the ideal \(b_{k,d,h}\) implies that the function \(\Phi^n : H^n_{k,d} \to \mathbb{R}_{>0}\) defined as \(\Phi^n(h) := \ell_c(b_{k,d,h})\) is constructible and lower semi-continuous with respect to the Zariski topology on \(H^n_{k,d}\). Since \(\ell_c(b_{k,d,h}) = \ell_c(b)\), \(\Phi^n\) descend to a function \(\Phi\) on \(H_{k,d}\).
Theorem 17. Let $\phi : T \to \mathbb{R}_{>0}$ defined as
$$
\phi(t) := n! \cdot \min_{k \leq d \leq d_k} \Phi([b])^n
$$
is constructible and lower semi-continuous with respect to the Zariski topology on $T$. Then Proposition [14] implies $\phi(t) = \ell_{c,k}(\overline{\sigma(\bar{t})}, \mathcal{X}_t, D_t)$. Thus we conclude that $t \mapsto \ell_{c,k}(\sigma(\bar{t}), \mathcal{X}_t, D_t)$ is constructible and lower semi-continuous with respect to the Zariski topology on $T$. Hence for any $k, m \in \mathbb{N}$ there exists a Zariski open neighborhood $U_{k,m}$ of $o$, such that
$$
\ell_{1/m,k}(\sigma(\bar{t}), \mathcal{X}_t, D_t) \geq \ell_{1/m,k}(\sigma(\bar{t}), \mathcal{X}_t, D_t)
$$
whenever $t \in U_{k,m}$.

Let $U := \cap_{k,m} U_{k,m}$, then for any $m \in \mathbb{N}$ and any $t \in U$ we have $\ell_{1/m,\infty}(\sigma(\bar{t}), \mathcal{X}_t, D_t) \geq \ell_{1/m,\infty}(\sigma(\bar{t}), \mathcal{X}_t, D_t)$. By Theorem [12] for any $t \in U$ we have
$$
\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_t, D_t) = \lim_{m \to \infty} \ell_{1/m,\infty}(\sigma(\bar{t}), \mathcal{X}_t, D_t)
$$
$$
\geq \lim_{m \to \infty} \ell_{1/m,\infty}(\sigma(\bar{t}), \mathcal{X}_t, D_t)
$$
$$
= \text{vol}(\sigma(\bar{t}), \mathcal{X}_t, D_t).
$$
The proof is finished. □

The following theorem is a stronger result that implies Theorem 3.

Theorem 17. Let $\varphi : (\mathcal{Y}, \mathcal{E}) \to T$ be a $\mathbb{Q}$-Gorenstein flat family of log Fano pairs over a field $\mathbb{k}$ of characteristic 0. Assume that some geometric fiber $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K-semistable for a point $o \in T$. Then

1. There exists an intersection $U$ of countably many Zariski open neighborhoods of $o$, such that $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for any point $t \in T$. If, in addition, $\mathbb{k} = \overline{\mathbb{k}}$ is uncountable, then $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for a very general closed point $t \in T$.

2. The geometrically log K-semistable locus
$$
T^{K-ss} := \{ t \in T : (\mathcal{Y}_t, \mathcal{E}_t) \text{ is log K-semistable} \}
$$
is stable under generalization.

Proof. (1) For $r \in \mathbb{N}$ satisfying $\mathcal{L} = -r(K_{\mathcal{Y}/T} + \mathcal{E})$ is Cartier, we define the relative affine cone $\mathcal{X}$ of $(\mathcal{Y}, \mathcal{L})$ by
$$
\mathcal{X} := \text{Spec}_{T} \oplus_{m \geq 0} \varphi_*(\mathcal{L}^{\otimes m}).
$$
Assume $r$ is sufficiently large, then it is easy to see that $\varphi_*(\mathcal{L}^{\otimes m})$ is locally free on $T$ for all $m \in \mathbb{N}$. Thus we have $\mathcal{X} \cong \text{Spec} \oplus_{m \geq 0} H^0(\mathcal{Y}_t, \mathcal{L}_t^{\otimes m}) := O(\mathcal{Y}_t, \mathcal{L}_t)$. Let $\mathcal{D}$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ corresponding to $\mathcal{E}$. By [Kol13, Section 3.1], the projection $\pi : (\mathcal{X}, \mathcal{D}) \to T$ together with the section of cone vertices $\sigma : T \to \mathcal{X}$ is a $\mathbb{Q}$-Gorenstein flat family of klt singularities.

Since $(\mathcal{Y}_o, \mathcal{E}_o)$ is K-semistable, Theorem 3 implies
$$
\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_o, \mathcal{D}_o) = r^{-1}(-K_{\mathcal{Y}_o} - \mathcal{E}_o)^{n-1}.
$$
Then by Theorem 16 there exists an intersection $U$ of countably many Zariski open neighborhoods of $o$, such that $\text{vol}(\sigma(t), X_t, \mathcal{D}_t) \geq \text{vol}(\sigma(o), X_o, \mathcal{D}_o)$ for any $t \in U$. Since the global volumes of log Fano pairs are constant in $Q$-Gorenstein flat families, we have

$$\text{vol}(\sigma(t), X_t, \mathcal{D}_t) \geq \text{vol}(\sigma(o), X_o, \mathcal{D}_o) = r^{-1}(-K_{Y_o} - E_o)^{n-1} = r^{-1}(-K_{Y_t} - E_t)^{n-1}.$$ 

Then Theorem 6 implies that $(\tilde{Y}^i, \tilde{E}^i)$ is K-semistable for any $t \in U$.

(2) Let $o \in T^{K-\text{ss}}$ be a scheme-theoretic point. Then by Theorem 17 there exists countably many Zariski open neighborhoods $U_i$ of $o$ such that $\cap U_i \subset T^{K-\text{ss}}$. If $t$ is a generalization of $o$, then $t$ belongs to all Zariski open neighborhoods of $o$, so $t \in T^{K-\text{ss}}$. \hfill $\square$

**Proof of Theorem 4.** It is clear that (1) and (2) follows from Theorem 16. For (3), we only need to replace Theorem 16 by Conjecture 1, then the same argument in the proof of Theorem 17(1) works. \hfill $\square$

The following corollary is a stronger result that implies Corollary 4.

**Corollary 18.** Let $\pi: (\mathcal{Y}, \mathcal{E}) \to T$ be a $Q$-Gorenstein family of complex log Fano pairs. Assume that $\pi$ is isotrivial over a Zariski open subset $U \subset T$, and $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K-semistable for a closed point $o \in T \setminus U$. Then $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for any $t \in U$.

**Proof.** Since $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K-semistable, Theorem 17 implies that $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for very general closed point $t \in T$. Hence there exists (hence any) $t \in U$ such that $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable. \hfill $\square$

4.2. Applications. In this section we present applications of Theorem 2. The following theorem generalizes the inequality part of [LiuX17, Theorem A.4].

**Theorem 19.** Let $x \in (X, D)$ be a complex klt singularity of dimension $n$. Let $a$ be the largest coefficient of components of $D$ containing $x$. Then $\text{vol}(x, X, D) \leq (1 - a)n^n$.

**Proof.** Suppose $D_i$ is the component of $D$ containing $x$ with coefficient $D$. Let $D^n_i$ be the normalization of $D_i$. By applying Theorem 2 to $pr: (X \times D^n_i, X \times D^n_i) \to D^n_i$ together with the natural diagonal section $\sigma: D^n_i \to X \times D^n_i$, we have that $\text{vol}(x, X, D) \leq \text{vol}(y, X, D)$ for a very general closed point $y \in D_i$. We may pick $y$ to be a smooth point in both $X$ and $D$, then $\text{vol}(x, X, D) \leq \text{vol}(0, \mathbb{A}^n, a\mathbb{A}^{n-1})$ where $\mathbb{A}^{n-1}$ is a coordinate hyperplane of $\mathbb{A}^n$. Let us take local coordinates $(z_1, \cdots, z_n)$ of $\mathbb{A}^n$ such that $\mathbb{A}^{n-1} = V(z_1)$. Then the monomial valuation $v_a$ on $\mathbb{A}^n$ with weights $((1 - a)^{-1}, 1, \cdots, 1)$ satisfies $A_{\mathbb{A}^n}(v) = (1 - a) + (n - 1)$, $\text{ord}_{v_a}(\mathbb{A}^{n-1}) = 1 - a$ and $\text{vol}(v_a) = (1 - a)$. Hence

$$\text{vol}(x, X, D) \leq \text{vol}_{\mathbb{A}^n}(0, a\mathbb{A}^{n-1})(v_a) = (A_{\mathbb{A}^n}(v) - a\text{ord}_{v_a}(\mathbb{A}^{n-1}))^n \cdot \text{vol}(v_a) = (1 - a)n^n.$$ 

The proof is finished. \hfill $\square$

**Theorem 20.** Let $(X, D)$ be a klt pair over $\mathbb{C}$. Let $Z$ be an irreducible subvariety of $X$. Then for a very general closed point $z \in Z$ we have

$$\text{vol}(z, X, D) = \sup_{x \in Z} \text{vol}(x, X, D).$$

In particular, there exists a countable intersection $U$ of non-empty Zariski open subsets of $Z$ such that $\text{vol}(:, X, D)|_U$ is constant.
Proof. Denote by $Z^n$ the normalization of $Z$. Then the proof follows quickly by applying Theorem 2 to $\text{pr}_2 : (X \times Z^n, D \times Z^n) \to Z^n$ together with the natural diagonal section $\sigma : Z^n \to X \times Z^n$. □

Next we study the case when $X$ is a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds. Note that the function $x \mapsto \text{vol}(x, X) = n^n \cdot \Theta(x, X)$ is lower semi-continuous with respect to the Euclidean topology on $X$ by [SS17, LX17]. This result together with Theorem 2 provide strong evidence of the special case of Conjecture 1 on the constructibility and lower semi-continuity of the function $x \mapsto \text{vol}(x, X)$ for a klt variety $X$.

The following theorem partially generalizes [SS17, Lemma 3.3 and Proposition 3.10].

**Theorem 21.** Let $X$ be a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds. Let $x \in X$ be any closed point. Then for any finite quasi-étale morphism of singularities $\pi : (y \in Y) \to (x \in X)$, we have $\deg(\pi) \leq \Theta(x, X)^{-1}$. In particular, we have

1. $|\hat{\pi}_1 \text{loc}(X, x)| \leq \Theta(x, X)^{-1}$.
2. For any $\mathbb{Q}$-Cartier Weil divisor $L$ on $X$, we have $\text{ind}(x, L) \leq \Theta(x, X)^{-1}$ where $\text{ind}(x, L)$ denotes the Cartier index of $L$ at $x$.

Proof. By [LX17, Theorem 1.7], the finite degree formula holds for $\pi$, i.e. $\hat{\text{vol}}(y, Y) = \deg(\pi) \cdot \text{vol}(x, X)$. Since $\hat{\text{vol}}(y, Y) \leq n^n$ by [LX17, Theorem A.4] or Theorem 19 and $\text{vol}(x, X) = \Theta(x, X)$ by [LX17, Corollary 5.7], we have $\deg(\pi) \leq n^n / \text{vol}(x, X) = \Theta(x, X)^{-1}$. □

**Remark 22.** If the finite degree formula [LX17, Conjecture 4.1] were true for any klt singularity, then clearly $\deg(\pi) \leq n^n / \text{vol}(x, X)$ holds for any finite quasi-étale morphism $\pi : (y, Y) \to (x, X)$ between $n$-dimensional klt singularities. In particular, we would get an effective upper bound $|\hat{\pi}_1 \text{loc}(X, x)| \leq n^n / \text{vol}(x, X)$ where $\hat{\pi}_1 \text{loc}(X, x)$ is known to be finite by [Xu14, BGO17] (see [LiuX17, Theorem 1.5] for a partial result in dimension 3).

**Theorem 23.** Let $V$ be a $K$-semistable complex $\mathbb{Q}$-Fano variety of dimension $(n - 1)$. Let $q$ be the largest integer such that there exists a Weil divisor $L$ satisfying $-K_V \sim_\mathbb{Q} qL$. Then

$$q \cdot (-K_V)^{n-1} \leq n^n.$$

Proof. Consider the orbifold cone $X := C(V, L) = \text{Spec}(\oplus_{m \geq 0} H^0(V, \mathcal{O}_V([mL])))$ with the cone vertex $x \in X$. Let $\hat{X} := \text{Spec}_V \oplus_{m \geq 0} \mathcal{O}_V([mL])$ be the partial resolution of $X$ with exceptional divisor $V_0$. Then by [Kol04, 40-42], $x \in X$ is a klt singularity, and $(V_0, 0) \cong (V, 0)$ is a $K$-semistable Kollár component over $x \in X$. Hence [LX16, Theorem A] implies that $\text{ord}_{V_0}$ minimizes $\text{vol}_{x, V}$. By [Kol04, 40-42] we have $A_X(\text{ord}_{V_0}) = q$, $\text{vol}(\text{ord}_{V_0}) = (L^{n-1})$. Hence

$$\text{vol}(x, X) = A_X(\text{ord}_{V_0}) \cdot \text{vol}(\text{ord}_{V_0}) = q^n (L^{n-1}) = q(-K_V)^{n-1},$$

and the proof is finished since $\text{vol}(x, X) \leq n^n$ by [LX17, Theorem A.4] or Theorem 19. □
APPENDIX A. ASYMPTOTIC LATTICE POINTS COUNTING IN CONVEX BODIES

In this appendix, we will prove the following proposition.

**Proposition 24.** For any positive number \( \epsilon \), there exists \( k_0 = k_0(\epsilon, n) \) such that for any closed convex body \( \Delta \subset [0, 1]^n \) and any integer \( k \geq k_0 \), we have

\[
\left| \frac{\#(k\Delta \cap \mathbb{Z}^n)}{k^n} - \text{vol}(\Delta) \right| \leq \epsilon.
\]

**Proof.** We do induction on dimensions. If \( n = 1 \), then \( k\Delta \) is a closed interval of length \( k\text{vol}(\Delta) \), hence we know

\[ k\text{vol}(\Delta) - 1 \leq \#(k\Delta \cap \mathbb{Z}) \leq k\text{vol}(\Delta) + 1. \]

So (A.1) holds for \( k_0 = \lceil 1/\epsilon \rceil \).

Next, assume that the proposition is true for dimension \( n - 1 \). Denote by \( (x_1, \ldots, x_n) \) the coordinates of \( \mathbb{R}^n \). Let \( \Delta_t := \Delta \cap \{x_n = t\} \) be the sectional convex body in \([0, 1]^{n-1}\). Let \([t_-, t_+]\) be the image of \( \Delta \) under the projection onto the last coordinate. Then we know that \( \text{vol}(\Delta) = \int_{t_-}^{t_+} \text{vol}(\Delta_t) \, dt \).

By induction hypothesis, there exists \( k_1 \in \mathbb{N} \) such that

\[
\text{vol}(\Delta_t) - \frac{\epsilon}{3} \leq \frac{\#(k\Delta_t \cap \mathbb{Z}^{n-1})}{k^{n-1}} \leq \text{vol}(\Delta_t) + \frac{\epsilon}{3} \quad \text{for any } k \geq k_1.
\]

It is clear that

\[
\#(k\Delta \cap \mathbb{Z}^n) = \sum_{t \in [t_- \ldots t_+] \cap \frac{1}{k} \mathbb{Z}} \#(k\Delta_t \cap \mathbb{Z}^{n-1}),
\]

so for any \( k \geq k_1 \) we have

\[
(A.2) \quad \left| \#(k\Delta \cap \mathbb{Z}^n) - k^n \cdot \sum_{t \in [t_- \ldots t_+] \cap \frac{1}{k} \mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{\epsilon}{3} k^{n-1} \cdot \#([t_- \ldots t_+] \cap \frac{1}{k} \mathbb{Z}) \leq \frac{2\epsilon}{3} k^n.
\]

Next, we know that the function \( t \mapsto \text{vol}(\Delta_t)^{1/(n-1)} \) is concave on \([t_- \ldots t_+]\) by the Brunn-Minkowski theorem. In particular, we can find \( t_0 \in [t_- \ldots t_+] \) such that \( g(t) := \text{vol}(\Delta_t) \) reaches its maximum at \( t = t_0 \). Hence \( g \) is increasing on \([t_- \ldots t_0]\) and decreasing on \([t_0 \ldots t_+]\).

Then applying Proposition 25 to \( g|_{[t_- \ldots t_0]} \) and \( g|_{[t_0 \ldots t_+]}) \) respectively yields

\[
\left| \int_{t_-}^{t_0} \text{vol}(\Delta_t) \, dt - \frac{1}{k} \sum_{t \in [t_- \ldots t_0] \cap \frac{1}{k} \mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{2}{k},
\]

\[
\left| \int_{t_0}^{t_+} \text{vol}(\Delta_t) \, dt - \frac{1}{k} \sum_{t \in [t_0 \ldots t_+] \cap \frac{1}{k} \mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{2}{k}.
\]

Since \( 0 \leq \text{vol}(\Delta_{t_0}) \leq 1 \), we have

\[
(A.3) \quad \left| \int_{t_-}^{t_+} \text{vol}(\Delta_t) \, dt - \frac{1}{k} \sum_{t \in [t_- \ldots t_+] \cap \frac{1}{k} \mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{5}{k}.
\]

Therefore, by setting \( k_0 = \max(k_1, \lceil 15/\epsilon \rceil) \), the inequality (A.1) follows easily by combining (A.2) and (A.3). \( \square \)
Proposition 25. For any monotonic function $g : [a, b] \to [0, 1]$ and any $k \in \mathbb{N}$, we have

$$\left| \int_a^b g(s) ds - \frac{1}{k} \sum_{t \in [a,b] \cap \frac{1}{k}\mathbb{Z}} g(t) \right| \leq \frac{2}{k}.$$ 

Proof. We may assume that $g$ is an increasing function. Denote $a_k := \frac{ka}{k}$ and $b_k := \frac{kb}{k}$, so $[a, b] \cap \frac{1}{k}\mathbb{Z} = [a_k, b_k] \cap \frac{1}{k}\mathbb{Z}$. Since $\int_{t-1/k}^t g(s) ds \leq g(t)/k$ whenever $t \in [a_k + 1/k, b_k]$, we have

$$\int_{a_k}^{b_k} g(s) ds \leq \frac{1}{k} \sum_{t \in [a_k + 1/k, b_k] \cap \frac{1}{k}\mathbb{Z}} g(t) \leq \frac{1}{k} \sum_{t \in [a,b] \cap \frac{1}{k}\mathbb{Z}} g(t),$$

Similarly, $\int_{t}^{t+1/k} g(s) ds \geq g(t)/k$ for any $t \in [a_k, b_k - 1/k]$, we have

$$\int_{a_k}^{b_k} g(s) ds \geq \frac{1}{k} \sum_{t \in [a_k, b_k - 1/k] \cap \frac{1}{k}\mathbb{Z}} g(t) \geq \frac{1}{k} \sum_{t \in [a,b] \cap \frac{1}{k}\mathbb{Z}} g(t) - \frac{1}{k}.$$

It is clear that $a_k \in [a, a + 1/k]$ and $b_k \in [b - 1/k, b]$, so we have

$$\int_{a_k}^{b_k} g(s) ds \geq \int_a^b g(s) ds - \frac{2}{k}, \quad \int_{a_k}^{b_k} g(s) ds \leq \int_a^b g(s) ds.$$

As a result, we have

$$\frac{1}{k} \sum_{t \in [a,b] \cap \frac{1}{k}\mathbb{Z}} g(t) - \frac{1}{k} \leq \int_a^b g(s) ds \leq \frac{1}{k} \sum_{t \in [a,b] \cap \frac{1}{k}\mathbb{Z}} g(t) + \frac{2}{k}.$$

□

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