Energy Decay in $1+1$ Rindler Spacetime

Anne T. Franzen[1] and Yafet E. Sanchez Sanchez[2],†

[1] Center for Mathematical Analysis, Geometry and Dynamical Systems, Mathematics Department, Instituto Superior Técnico, Universidade de Lisboa, Portugal
[2] INFN Sezione di Genova Università di Genova, Italy

ABSTRACT

We consider solutions of the massless scalar wave equation $\Box_g \psi = 0$ on a fixed Rindler background and show polynomial decay of the energy flux related to Rindler observers near null infinity and to local observers near the Rindler horizon. The main estimates are obtained via the vector field method using suitable vector fields multipliers which are analogous to the ones used in Schwarzschild spacetime and Minkowski spacetime. Furthermore, we compare the Schwarzschild and Rindler scenarios and discuss the extent to which the principle of equivalence holds.

Contents

1 Introduction 2
1.1 Outline of the paper ............................................. 2

2 Preliminaries 3
2.1 Vector field method and energy currents ...................... 3
2.2 The Rindler solution ........................................... 3
2.2.1 The metric, ambient differential structure and Killing vector fields .............................................. 4
2.2.2 Proper coordinates of uniformly accelerated motion ......................................................... 4
2.2.3 Double-null coordinates ...................................... 5
2.2.4 $(r, \tau)$-coordinates ....................................... 5
2.3 Notation ......................................................... 6

3 The Vector Fields .............................................. 7
3.1 The Killing vector $\partial_{\tau}$ ................................ 7
3.2 The redshift vector field $Y$ and the local energy vector field $N$ ......................................................... 8
3.3 The vector field $X$ .............................................. 9
3.4 The vector field $r^p \partial_{\tau}$ .................................. 10

4 Energy Decay .................................................. 10
4.1 Boundedness of the $J^N$-Energy .............................. 10
4.2 The Intermediate Region ..................................... 12
4.3 Local Energy Decay near $RH^+$ .............................. 14
4.4 Far Away Estimates ............................................ 15
4.5 Energy decay for solutions with compactly supported initial data ......................................................... 16

1 e-mail address: anne.franzen@tecnico.ulisboa.pt
2 e-mail addresses:yafet.erasmo.sanchez.sanchez@edu.unige.it
1 Introduction

The Rindler spacetime \((\mathcal{M}, g)\) is a solution to the vacuum Einstein field equations. A brief introduction to the spacetime is for example given in \([32, 41, 40, 38, 36]\). The problem of analyzing the solution of the scalar wave equation

\[ \Box_g \psi = 0 \quad (1) \]

on Rindler backgrounds is intimately related to a better understanding of the equivalence principle \([1]\) and the Unruh effect \([39]\) which is often seen as a flat spacetime proxy for the Hawking radiation close to black hole event horizons. Therefore, solutions to the wave equation in Schwarzschild spacetime under suitable approximations must resemble solutions to the wave equation in Rindler spacetime.

On one hand, the analysis of solutions to the massless scalar wave equations in the exterior region of Schwarzschild spacetime was a first step towards understanding the non-linear behaviour of Einstein’s equations. Kay and Wald were the first to prove uniform boundedness of solutions up to and including the event horizon \([26]\). Later on, Dafermos and Rodnianski introduced robust physical space based methods in order to obtain non-degenerate energy and pointwise decay estimates in the exterior region including the event horizon and future null infinity \([10, 11, 12]\). In the last decades immense progress has been made in the analysis of linear and non-linear wave equations in the exterior of Schwarzschild \([28, 4, 5, 8, 22, 21, 9]\).

On the other hand, massless scalar wave equations in Rindler spacetime has been mainly analysed from the perspective of quantum field theory (see the review \([7, 3]\) and references therein). The analysis in these cases is done via mode decomposition. These decompositions has been useful to construct physical quantum states in the presence of Killing horizons \([15, 27]\) and has led to general constructions based on microlocal techniques \([34, 16, 25, 24]\). Furthermore, Higuchi, Iso, Ueda and Yamamoto have shown how different modes entangle and their role in the origin of the radiation \([20]\).

In this article, based on the Schwarzschild case, we define a future directed timelike vector field which captures the energy flux associated to local observers at the Rindler horizon and the energy flux associated to accelerated observers in the far away region. Furthermore, we show arbitrary polynomial energy decay of solutions with compactly supported initial data in \(1 + 1\) Rindler spacetime (Theorem 4.5).

1.1 Outline of the paper

In Section 2, we introduce the necessary preliminaries. We review the vector field method, introduce the coordinates and establish the general notation. In Section 3, we state the main vector fields used for the proof of energy decay. In Section 4, we show several estimates related to different region of the spacetime and show the main theorem (Theorem 4.5). In Section 5, we discuss the similarities and differences of our results compared to the Schwarzschild scenario.
2 Preliminaries

2.1 Vector field method and energy currents

In the following section we will briefly review the vector field method which we are going to use as an essential tool throughout this work. The solutions of the wave equation (1) have an associated symmetric stress-energy tensor given by

\[ T_{\mu\nu}(\psi) = \partial_{\mu}\psi\partial_{\nu}\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi. \] (2)

Since \( \psi \) is a solution to (1) it follows

\[ \nabla^\mu T_{\mu\nu} = 0. \] (3)

By contracting the energy-momentum tensor with a vector field \( V \), we define the current \( J_{V\mu}(\psi) \).

\[ J_{V\mu}(\psi) = T_{\mu\nu}(\psi) V_\nu. \] (4)

If the vector field \( V \) is timelike, then the one-form \( J_{V\mu}(\psi) \) can be interpreted as an energy flux.

In combination with the divergence theorem for a spacetime region \( D \) which is bound by two homologous hypersurfaces, \( \Sigma_\tau \) and \( \Sigma_0 \), we have

\[ \int_{\Sigma_\tau} J_{V\mu}(\psi) n_\mu d\text{Vol}_{\Sigma_\tau} + \int_B \nabla^\mu J_{V\mu}(\psi) d\text{Vol} = \int_{\Sigma_0} J_{V\mu}(\psi) n_\mu d\text{Vol}_{\Sigma_0}. \] (5)

The vector \( n_\mu \) denotes the normal to the subscript hypersurface \( \Sigma \) oriented according to Lorentzian geometry convention. Further, \( d\text{Vol} \) denotes the volume element over the entire spacetime region and \( d\text{Vol}_{\Sigma} \) the volume elements on \( \Sigma \), respectively. We have that the divergence of the current (4) is given by

\[ \nabla^\mu J_{\mu} = \nabla^\mu (T_{\mu\nu} V^\nu) = T_{\mu\nu}(\nabla^\mu V^\nu) + (\nabla^\mu T_{\mu\nu}) V^\nu. \] (6)

We decompose the divergence into two terms,

\[ K^V(\psi) \doteq T(\psi)(\nabla V) = (\pi^V)^{\mu\nu} T_{\mu\nu}(\psi), \] (7)

where \((\pi^V)^{\mu\nu} \doteq \frac{1}{2} (\mathcal{L}_V g)^{\mu\nu}\) is the so called deformation tensor along \( V \), and

\[ \mathcal{E}^V(\psi) \doteq (\nabla^\mu T_{\mu\nu}) V^\nu = (\Box g) V(\psi). \] (8)

Thus

\[ \nabla^\mu J_{\mu} = K^V + \mathcal{E}^V. \] (9)

From (7) we see that \( K^V \) is zero in case that our multiplier is a Killing vector field and \( \mathcal{E}^V \) is zero if \( \psi \) is a solution to the homogeneous wave equation.

2.2 The Rindler solution

In the following, we will briefly recall Rindler spacetime, a solution to the Einstein vacuum field equations represented as

\[ R_{\mu\nu} = 0, \] (10)

where \( R_{\mu\nu} \) is the Ricci tensor. This spacetime solution is crucial for understanding a family of uniformly accelerated observers in flat spacetime, and it is noteworthy that the Rindler spacetimes are isometric to a subset of the Minkowski spacetime. Uniform acceleration in relativity is defined as constant proper acceleration, which refers to acceleration as experienced from the observer’s frame of reference. For those unfamiliar with Rindler spacetime, references such as [32, 38] provide a concise overview.
2.2.1 The metric, ambient differential structure and Killing vector fields

To set the semantic convention, whenever we refer to the Rindler solution \((M, g)\) we mean the right wedge \(\mathcal{R} = J^{-}(\mathcal{H}^+_{\Lambda}) \cap J^{+}(\mathcal{H}^-_{\Lambda})\) of the spacetime, whose maximally analytic extension is the 1 + 1 Minkowski spacetime, see Figure 1. Since this paper deals exclusively with the right wedge, we will for simplicity in the following refer to \(\mathcal{H}_A\) just by using \(\mathcal{H}\).

The manifold \(M\) can be expressed by

\[
\mathcal{M} = \mathcal{Q} \cup \mathcal{RH}^+ \cup \mathcal{RH}^-, \quad \text{with} \quad \mathcal{Q} = (-\infty, \infty) \times (-\infty, \infty),
\]

with the retarded and advanced double-null coordinates \(u, v \in (-\infty, \infty)\), which we will introduce in Section 2.2.3. We can formally parametrize the future and past Rindler horizon by

\[
\mathcal{RH}^+ = \{\infty\} \times (-\infty, \infty), \quad \text{and} \quad \mathcal{RH}^- = (-\infty, \infty) \times \{-\infty\},
\]

as depicted in Figure 1. Notice that in Minkowski double null coordinates \(U, V\) (see Appendix A) we have

\[
\mathcal{RH}^+ = \{0\} \times [0, \infty), \quad \mathcal{RH}^- = (-\infty, 0] \times \{0\},
\]

2.2.2 Proper coordinates of uniformly accelerated motion

In the following we will derive Rindler coordinates simply by defining the trajectory of an uniformly accelerated observer in Minkowski spacetime with the metric

\[
ds^2 = -dt^2 + dx^2.
\]

We refer to the \((t, x)\)-coordinates as the laboratory frame. In the following we are interested in the observer’s proper frame which moves with the observer and is given in \((\tau, \xi)\)-coordinates. Without loss of generality we assume that our observer moves in \(x\)-direction. Using inverse Lorentz transformations and the comoving frame, which is the inertial frame for a given time at which the observer is instantaneously at rest, Mukhanov and Winitzki, [32] derive the following relations

\[
t(\tau, \xi) = \frac{1 + a\xi}{a} \sinh(a\tau), \quad -\infty < t < \infty,
\]

\[
x(\tau, \xi) = \frac{1 + a\xi}{a} \cosh(a\tau), \quad 0 < x < \infty,
\]
which cover only the subdomain $x > |t|$ of Minkowski space, with the horizon $\mathcal{R}H^+$ corresponding to $t = x$ and $\mathcal{R}H^-$ corresponding to $t = -x$, see Figure. Further, we have

$$\tau(t, x) = \frac{1}{2a} \ln \frac{x + t}{x - t}, \quad -\infty < \tau < \infty$$

$$\tilde{\xi}(t, x) = -\frac{1}{a} + \sqrt{x^2 - t^2}, \quad -\frac{1}{a} < \tilde{\xi} < \infty.$$ 

In these coordinates, the horizon $\mathcal{R}H$ corresponds to $\tilde{\xi} = -\frac{1}{a}$. The interpretation is that the accelerated observer cannot measure distances longer than $\frac{1}{a}$ in the opposite direction of their acceleration and can therefore not receive signals from beyond $\mathcal{R}H$. Relations (17) and (18) lead to the Rindler metric

$$ds^2 = -(1 + a\tilde{\xi})^2 d\tau^2 + d\tilde{\xi}^2.$$ 

Using the transformation

$$\xi = \frac{1 + a\tilde{\xi}}{a}, \quad 0 < \xi < \infty,$$

with $\xi = 0$ corresponding to $\mathcal{R}H$, we obtain the simplified metric

$$ds^2 = -a^2 \xi^2 d\tau^2 + d\xi^2.$$ 

### 2.2.3 Double-null coordinates

In double null coordinates $(u, v)$

$$u(\tau, \xi) = \tau - \frac{1}{a} \ln \xi, \quad -\infty < u < \infty,$$

$$v(\tau, \xi) = \tau + \frac{1}{a} \ln \xi, \quad -\infty < v < \infty,$$

where $u = \infty$ corresponds to $\mathcal{R}H^+$ and $v = -\infty$ corresponds to $\mathcal{R}H^-$. Further, we have

$$r(u, v) = \frac{v - u}{2} = \frac{1}{a} \ln \xi, \quad -\infty < r < \infty,$$

$$\tau(u, v) = \frac{v + u}{2}, \quad -\infty < \tau < \infty.$$ 

The metric (23) then transforms into

$$ds^2 = -e^{2a(r-u)} du dv.$$ 

To better understand the geometric meaning of the coordinates $r$ and $\tau$, the reader may refer for example to Figure.

### 2.2.4 $(r, \tau)$-coordinates

Note that using Section 2.2.3 we have

$$u(r, \tau) = \tau - r, \quad v(r, \tau) = r + \tau,$$

which leads to the metric in in $(r, \tau)$-coordinates

$$ds^2 = e^{2ar} [-dr^2 + d\tau^2].$$ 

\[^3\text{Note, that in Rindler spacetime, the } r \text{ coordinate is analogous to the familiar Tortoise coordinate, usually denoted by } r_*, \text{ of for example Schwarzschild spacetime and therefore also takes negative values.}\]
2.3 Notation

In the following we will use the notations given below, for the level sets of the coordinate functions $u, v, r$ and $\tau$:

\[
\begin{align*}
C_v &= \{ p \in \mathcal{M} \mid v = v_1 \}, \\
C_u &= \{ p \in \mathcal{M} \mid u = u_1 \}, \\
S_r &= \{ p \in \mathcal{M} \mid r = r_1 \}, \\
\Sigma_{\tau_1} &= \{ p \in \mathcal{M} \mid \tau = \tau_1 \}.
\end{align*}
\]

We will also write

\[
C_v(u_1, u_2) = \{ p \in \mathcal{M} \mid v(p) = v_1 \text{ and } u_1 \leq u(p) \leq u_2 \}.
\]

- The volume element associated to $g$ in $(u, v)$-coordinates is given by

\[
dVol = \frac{e^{a(v-u)}}{2} dv du,
\]

(31)

- On the null components $C_v$ and $C_u$, there is no natural choice of normal vector or volume form. So choosing for example $n_{C_v}$ to be any future directed vector orthogonal to $C_v$, then $dVol_{C_v}$ is completely determined by Stokes’ Theorem; for instance we have the following choices

\[
\begin{align*}
n_{C_v}^\mu &= 2e^{-a(v-u)} \partial_v, \quad \text{with} \quad dVol_{C_v} = \frac{e^{a(v-u)}}{2} dv, \\
n_{C_u}^\mu &= 2e^{-a(v-u)} \partial_u, \quad \text{with} \quad dVol_{C_u} = \frac{e^{a(v-u)}}{2} du,
\end{align*}
\]

(32, 33)

- Further, we have for the future timelike normal vector on a constant $\tau$ hypersurface and the outward pointing spacelikes normal vectors on a constant $r$ hypersurface

\[
\begin{align*}
n_{\tau=c}^\mu &= e^{-\frac{a(v-u)}{2}} (\partial_u + \partial_v), \quad \text{with} \quad dVol_\tau = \frac{e^{a(v-u)}}{2} (du + dv), \\
n_{r=c}^\mu &= \pm e^{-\frac{a(v-u)}{2}} \partial_r, \quad \text{with} \quad dVol_r = \frac{e^{\pm a(v-u)}}{2} (dv - du),
\end{align*}
\]

(34, 35)

- Using regular double null coordinates (Appendix A) we define the normal on $\mathcal{R}H^+$ and the volume element by

\[
n_{\mathcal{R}H^+}^\mu = \partial_V, \quad \text{with} \quad dVol_{\mathcal{R}H^+} = dV,
\]

(36)

and similarly by taking the usual Minkowski coordinates we define via a limiting process the normal vector and volume form in $\mathcal{I}^+$ by

\[
n_{\mathcal{I}^+}^\mu = \partial_U, \quad \text{with} \quad dVol_{\mathcal{I}^+} = dU,
\]

(37)

Now we define the foliations which we will use to bound certain regions where we will apply the divergence theorem. For arbitrary $r^* < \tilde{r} \in \mathbb{R}$ we define

\[
\Sigma_{r^*}^{\tilde{r}, \tau_0} := \begin{cases} \\
\Sigma_{\tau=\tau_0} & \ r^* < r < \tilde{r} \\
C_v = \tau_0 - r^* & \ r \leq r^* \\
C_u = \tau_0 + \tilde{r} & \ \tilde{r} \geq r
\end{cases}
\]

(38)

and

\[
\Sigma_{r_0}^{r^*} := \begin{cases} \\
\Sigma_{\tau=\tau_0} & \ r \geq r^* \\
C_v = \tau_0 - r_0 & \ r < r^*
\end{cases}
\]

(39)
In these hypersurfaces we have the normal vectors
\[
n_{\Sigma_r^* }^\mu \left\{ \begin{array}{ll}
n_{\Sigma = \tau_0}^\mu & r < \tau_0 \\
n_{\Sigma = \tau_0 - r^*}^\mu & r \leq \tau_0 \\
n_{\Sigma = \tau_0 + r^*}^\mu & \tau_0 \\
\end{array} \right. ,
\]
(40)
and
\[
n_{C_r^* }^\mu \left\{ \begin{array}{ll}
n_{\Sigma = \tau_0}^\mu & r \geq \tau_0 \\
n_{\Sigma = \tau_0 - r^*}^\mu & r < \tau_0 \\
\end{array} \right. .
\]
(41)

Notice that for fixed \( r^* \), \( \tilde{r} \) we have \( \mathcal{M} = \bigcup_{\tau \in \mathbb{R}} \Sigma_{\tau^*} \tilde{r} = \bigcup_{\tau \in \mathbb{R}} \Sigma_{\tau^*} \). Therefore, we have
\[
\int_{\Sigma_{\tau_0}^+} J^N_\mu n_\Sigma^\mu \, d\text{Vol}_{\Sigma_{\tau_0}^+} = \int_{C_\psi(u_0(r^*), \infty)} J^N_\mu n_\Sigma^\mu \, d\text{Vol}_{C_\psi} + \int_{\Sigma_{\tau_0} \cap \{r^* < r < \tilde{r}\}} J^N_\mu n_\Sigma^\mu \, d\text{Vol}_{\Sigma_{\tau_0}} + \int_{C_\psi(v_0(\tilde{r}), \infty)} J^N_\mu n_\Sigma^\mu \, d\text{Vol}_{C_\psi}
\]
where \( u_0(r^*) \) denotes the value of the \( u \) variable at the point of intersection of \( \Sigma_{\tau_0} \) and \( S_{r^*} \).

Similarly, \( v_0(\tilde{r}) \) denotes the value of the \( v \) variable at the point of intersection of \( \Sigma_{\tau_0} \) and \( S_{\tilde{r}} \).

### 3 The Vector Fields

#### 3.1 The Killing vector \( \partial_\tau \)

The future directed Killing vector
\[
\partial_\tau = a [\partial t + t \partial_x]
\]
(43)
for \( a > 0 \) is associated with the flow of accelerated observers through spacetime. It satisfies
\[
\nabla_\partial_\tau \partial_\tau |_{\mathcal{R}H^+} = a \partial_\tau,
\]
(44)
which defines the surface gravity \( a \) at \( \mathcal{R}H^+ \). Furthermore, the quantity \( J^\partial_\tau (\psi) \) can be interpreted as the energy flux related to the observer accelerating at a rate \( a \). Since \( \partial_\tau \) is Killing, it follows from (7) that \( K^\partial_\tau = 0 \). So, for solutions of the wave equation on any compact region \( \mathcal{D} \), the energy is conserved, i.e.
\[
E_{\partial \mathcal{D}} := \int_{\partial \mathcal{D}} J^\partial_\tau (\psi)n_{\partial \mathcal{D}}^\mu \, d\text{Vol}_{\partial \mathcal{D}} = 0,
\]
according to (9).

Further, we have for constant \( \tau \) hypersurfaces using (147)
\[
J^\partial_\tau n_\Sigma^\mu = e^{-ar} \left[ (\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 \right],
\]
(45)
and \( d\text{Vol}_{\Sigma_r} = e^{ar} \, dr \). For constant \( u \) hypersurfaces using (142), and (149) we have
\[
J^\partial_\tau n_{C_u}^\mu = e^{-a(u-v)} (\partial_u \psi)^2,
\]
(46)
and \( d\text{Vol}_{C_u} = e^{a(u-v)} \, du \). For constant \( v \) hypersurfaces
\[
J^\partial_\tau n_{C_v}^\mu = e^{-a(v-u)} (\partial_v \psi)^2
\]
(47)
and \( d\text{Vol}_{C_v} = e^{a(v-u)} \, du \). Moreover, we have the following proposition.
Proposition 3.1. Let \( r^*, \tilde{r}, t_0 \in \mathbb{R} \) with \( r^* < \tilde{r} \) and \( \psi \) be a solution to (4) with compactly supported data\(^4\) on \( \Sigma^{r_0}_{t} \) or in \( \Sigma^{r_0}_{t} \) not necessarily vanishing at \( \mathcal{R} \mathcal{H}^+ \). Then the following statements hold

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \leq B \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}}, \tag{48}
\]

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \leq B \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}}, \tag{49}
\]

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \leq B \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}}. \tag{50}
\]

and

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \leq B \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}}. \tag{51}
\]

Proof. We start by proving Eq. (48). Consider the region \( \mathcal{D} = \{ D^+(\Sigma^{r_0}_{t_0}) \cap D^-(\Sigma^{r_1}_{t_1}) \} \), where \( D^\pm \) are the future/past domain of dependence, respectively. Applying the divergence theorem\(^5\) gives

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \, d\text{Vol}_{\Sigma^{r^*}_{t}} + \int_{\mathcal{R} \mathcal{H}^+} J^\alpha_{\mu} n^\mu_{\mathcal{R} \mathcal{H}^+} \, d\text{Vol}_{\mathcal{R} \mathcal{H}^+} + \int_{\mathcal{I}^+} J^\alpha_{\mu} n^\mu_{\mathcal{I}^+} \, d\text{Vol}_{\mathcal{I}^+} = \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}} \, d\text{Vol}_{\Sigma^{r_0}_{t}}. \tag{52}
\]

Now because \( \mathcal{R} \mathcal{H}^+ \) is a null hypersurface \( n^-_{\mathcal{R} \mathcal{H}^+} \) is null and therefore

\[
\int_{\mathcal{R} \mathcal{H}^+} J^\alpha_{\mu} n^\mu_{\mathcal{R} \mathcal{H}^+} \, d\text{Vol}_{\mathcal{R} \mathcal{H}^+} \geq 0.
\]

Similarly, for \( \mathcal{I}^+ \) (which is defined in the limit sense \( v \to \infty \)), we have

\[
\int_{\mathcal{I}^+} J^\alpha_{\mu} n^\mu_{\mathcal{I}^+} \, d\text{Vol}_{\mathcal{I}^+} := \lim_{v \to \infty} \int_{\mathcal{C}_v} J^\alpha_{\mu} n^\mu_{\mathcal{C}_v} \, d\text{Vol}_{\mathcal{C}_v} \geq 0.
\]

Hence,

\[
\int_{\Sigma^{r^*}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r^*}_{t}} \, d\text{Vol}_{\Sigma^{r^*}_{t}} \leq \int_{\Sigma^{r_0}_{t}} J^\alpha_{\mu} n^\mu_{\Sigma^{r_0}_{t}} \, d\text{Vol}_{\Sigma^{r_0}_{t}}. \tag{53}
\]

The proof of Eq. (49), Eq. (50) and Eq. (51) is analogous. \( \square \)

3.2 The redshift vector field \( Y \) and the local energy vector field \( N \)

In [12, Corollary 3.1], Dafermos and Rodnianski construct a redshift vector field \( Y \) and vector field \( N \) in Schwarzschild spacetime that captures the local observers energy. An equivalent statement holds in Rindler spacetime which we state below.

First, we define the redshift vector field tailored for the Rindler Horizon.

**The vector field \( Y \)**

Let \( Y = (1 + \delta_1(e^{a(c-u)})\tilde{Y} + (\delta_2 e^{a(c-u)})\partial_\sigma \) where \( \tilde{Y} = e^{-a(v-u)}\partial_\sigma \). Then \( Y \) satisfies for suitable \( \delta_1, \sigma > 0 \)

1. \( Y \) is \( \phi_r \) invariant; i.e. \( [\partial_r, Y] = 0 \)

\( ^4\)The compactness of the data should be consider with respect to regular coordinates at the horizon. In particular, the compactness is not with respect the variable \( r \).

\( ^5\)The divergence theorem cannot be directly applied, since the domain is not compact. Therefore, we applied it by taking a sequence of compact regions and taking the limit.
2. \( g(Y,Y)|_{\mathcal{R}H^+} = 0 \), \( g(\partial_r,Y)|_{\mathcal{R}H^+} = -\delta_1 \)

3. \( \nabla_Y Y = -\sigma(Y + \partial_r) \)

Moreover, the following holds on \( \mathcal{R}H^+ \)

\[
\nabla_{\partial_r} Y = -aY \\
\nabla_Y Y = -\sigma(Y + \partial_r)
\]

The vector field \( N \)

**Proposition 3.2.** Let \( \psi \) be a solution of \( \square g \psi = 0 \). Given \( \epsilon > 0 \), there exist \( r_0, R \in (-\infty, \infty) \) and a \( \tau \)-invariant future directed timelike vector field \( N \) such that the following holds

\[
(a - \epsilon)J^N_{\mu}(\psi)n^\mu_{\Sigma_r} \leq K^N(\psi) \quad r \leq r_0
\]

\[
e J^N_{\mu}(\psi)n^\mu_{\Sigma_r} \leq C J^{\partial_r}_{\mu}(\psi)n^\mu_{\Sigma_r} \quad r_0 \leq r \leq R
\]

\[
|K^N| \leq C J^{\partial_r}_{\mu}(\psi)n^\mu_{\Sigma_r} \quad r_0 \leq r \leq R
\]

\[
J^N_{\mu}(\psi)n^\mu_{\Sigma_r} = J^{\partial_r}_{\mu}(\psi)n^\mu_{\Sigma_r} \quad R \leq r
\]

for constants \( c, C, \tilde{C} \).

**Proof.** Let \( N := Y + \partial_r \). As in the Schwarzschild case, one obtains \( (a - \epsilon)J^N_{\mu}(\psi)n^\mu_{\Sigma_r} \leq K^N(\psi) \) on \( \mathcal{R}H^+ \). By continuity we can extend this estimate to an open neighbourhood which determines \( r_0 \). Moreover, by compactness we obtain Eq.(55) and Eq.(56). Finally using suitable cutoffs one can extend the vector field smoothly to \( \partial_r \) for \( R \leq r \).

3.3 The vector field \( X \)

In this section we define a vector field \( X \) which will be useful to obtain Integrated Local Energy Decay estimates. The ansatz of the vector field goes back to Morawetz [30].

In the following analysis we use \((r, \tau)\)-coordinates. Now we choose a multiplier such that \( K^X \) is positive, with a general ansatz

\[
X_f = f(r)\partial_r.
\]

Using the above vector field multiplier in (138) with

\[
f(r) = \tanh(ar) + b
\]

\[
f'(r) = \frac{a}{\cosh^2(r)},
\]

with \( b \) a constant, we get

\[
K^X_f = \frac{a}{2\cosh^2(ar)}e^{-2ar} \left[(\partial_r \psi)^2 + (\partial_\tau \psi)^2\right]
\]

by using (138). Using the normal vector (34) and the multiplier (58) we get

\[
J^{X_f}_{\mu}(\psi)n^\mu_{\Sigma_r} = -((\tanh(ar) + b) e^{-ar}(\partial_r \psi \partial_\tau \psi)).
\]

Using the normal vector (35) and the multiplier (58) we get

\[
J^{X_f}_{\mu}(\psi)n^\mu_{\Sigma_r} = ((\tanh(ar) + b) e^{-ar}/2)[(\partial_r \psi)^2 + (\partial_\tau \psi)^2].
\]
3.4 The vector field $r^p \partial_v$

Analogous to the $r^p$ estimates of [11] (see also [31]) where this vector field is used to obtain decay in the far away region, we define the vector field multiplier

$$V = r^p \partial_v$$

and obtain

$$K^V = p r^{p-1} e^{-a(v-u)} (\partial_v \psi)^2$$

Using the normal vector (32) and the multiplier (64) we get

$$J^V_{\mu} n^\mu_{\Sigma_{r^*}} = 2r^p e^{-a(v-u)} (\partial_v \psi)^2$$

and with (148)

$$J^V_{\mu} n^\mu_{\Sigma_{r^*}} = 2r^p e^{-a(v-u)} (\partial_v \psi)^2,$$

(67)

4 Energy Decay

4.1 Boundedness of the $J^N$-Energy.

![Figure 2: Sketch of region $R_I = R_{IA} \cup R_{IB}$ depicted in darker shades](image)

**Proposition 4.1.** Let $r^*, \bar{r}, \tau_0 \in \mathbb{R}$ with $r^* < \bar{r}$ and $\psi$ be a solution to (1) with compactly supported data on $\Sigma^r_{\tau_0}$ or in $\Sigma_{\tau_0}$ not necessarily vanishing at $\mathcal{R}h^+$. Then the following statements hold

$$\int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma^r_{\tau_0}} \leq B \int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma^r_{\tau_0}},$$

(68)

$$\int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma_{\tau_0}} \leq B \int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma_{\tau_0}},$$

(69)

$$\int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma_{\tau_0}} \leq B \int_{\Sigma_{r^*}, \bar{r}} J^N_{\mu} n^\mu_{\Sigma_{\tau_0}},$$

(70)
\[
\int_{\Sigma_{r_0}^N} J^N_{\mu} n^\mu_{\Sigma_{r_0}^N} \leq B \int_{\Sigma_{r=r_0}} J^N_{\mu} n^\mu_{\Sigma_{r_0}^N},
\]

(71)

**Proof.** We start by proving Eq. (68). Now consider the region \( \mathcal{R}_I = \{ D^+ (\Sigma^N_{r_0}) \cap D^- (\Sigma^N_r) \} = \mathcal{R}_{I_A} \cup \mathcal{R}_{I_B} \), where \( D^\pm \) are the future/past domain of dependence of the underlying hypersurface, respectively. Further, \( \mathcal{R}_{I_A} = \{ p \in \mathcal{R}_I \text{ and } r \leq r_0 \} \) and \( \mathcal{R}_{I_B} = \{ p \in \mathcal{R}_I \text{ and } r_0 \leq r \} \), see Figure 2. Applying the divergence theorem gives

\[
\int_{\Sigma_{r_1}^N} J^N_{\mu} n^\mu_{\Sigma_{r_1}^N} d\text{Vol}_{\Sigma_{r_1}^N} + \int_{\mathcal{R}_I} K^N d\text{Vol}_{\mathcal{R}_I}
\]

(72)

Moreover, notice that \( \mathcal{R}_I \) can be chosen arbitrarily large. Hence, combining (75) and (80) in (73) we obtain

\[
\int_{\mathcal{R}_{IH}^+} J^N_{\mu} n^\mu_{\mathcal{R}_{IH}^+} d\text{Vol}_{\mathcal{R}_{IH}^+} \geq 0.
\]

Similarly, for \( I^+ \), we have

\[
\int_{I^+} J^N_{\mu} n^\mu_{I^+} d\text{Vol}_{I^+} \geq 0.
\]

Hence,

\[
\int_{\Sigma_{r_1}^N} J^N_{\mu} n^\mu_{\Sigma_{r_1}^N} d\text{Vol}_{\Sigma_{r_1}^N} + \int_{\mathcal{R}_I} K^N d\text{Vol}_{\mathcal{R}_I}
\]

(73)

Now consider \( \mathcal{R}_{I_A} \) and \( \mathcal{R}_{I_B} \), as shown in Figure 2, and write the bulk integral as

\[
\int_{\mathcal{R}_I} K^N d\text{Vol}_{\mathcal{R}_I} = \int_{\mathcal{R}_{I_A}} K^N d\text{Vol}_{\mathcal{R}_{I_A}} + \int_{\mathcal{R}_{I_B}} K^N d\text{Vol}_{\mathcal{R}_{I_B}}
\]

(74)

which using (54) of Proposition 3.2 and gives

\[
\int_{\mathcal{R}_I} K^N d\text{Vol}_{\mathcal{R}_I} = \int_{\mathcal{R}_{I_A}} K^N d\text{Vol}_{\mathcal{R}_{I_A}} + \int_{\mathcal{R}_{I_B}} K^N d\text{Vol}_{\mathcal{R}_{I_B}}
\]

(75)

Moreover, notice that \( |K^N| \) in \( \mathcal{R}_{I_B} \) is controlled by \( J^R_{\mu} n^\mu_{\Sigma_{r_0}} \) from (56) of Proposition 3.2.

Therefore using Proposition 3.1 and co-area formula,

\[
- \int_{\mathcal{R}_{I_B}} K^N d\text{Vol}_{\mathcal{R}_{I_B}} \leq \int_{\mathcal{R}_{I_B}} |K^N| d\text{Vol}_{\mathcal{R}_{I_B}}
\]

(76)

\[
\leq C \int_{\mathcal{R}_{I_B}} J^R_{\mu} n^\mu_{\Sigma_{r_0}} d\text{Vol}_{\mathcal{R}_{I_B}}
\]

(77)

\[
\leq C \int_{\mathcal{R}_I} J^R_{\mu} n^\mu_{\Sigma_{r_0}} d\text{Vol}_{\mathcal{R}_I}
\]

(78)

\[
\leq C \int_{\tau_0}^{\tau_1} \int_{\Sigma_{r_0}} J^R_{\mu} n^\mu_{\Sigma_{r_0}} d\text{Vol}_{\Sigma_{r_0}} d\tau
\]

(79)

\[
\leq C (\tau_1 - \tau_0) \int_{\Sigma_{r_0}} J^R_{\mu} n^\mu_{\Sigma_{r_0}} d\text{Vol}_{\Sigma_{r_0}}
\]

(80)

where \( C \) can be chosen arbitrarily large. Hence, combining (75) and (80) in (73) we obtain

\[
[(1 + C (\tau_1 - \tau_0)) \int_{\Sigma_{r_0}} J^N_{\mu} n^\mu_{\Sigma_{r_0}} d\text{Vol}_{\Sigma_{r_0}}
\]

(81)

\[
\geq (a - \epsilon) \int_{\mathcal{R}_I} J^N_{\mu} n^\mu_{\Sigma_{r_0}^N} d\text{Vol}_{\Sigma_{r_0}^N} + \int_{\Sigma_{r_1}^N} J^N_{\mu} n^\mu_{\Sigma_{r_1}^N} d\text{Vol}_{\Sigma_{r_1}^N}.
\]
Adding a term of the form \((a - \epsilon) \int_{R_{II}} J^N_{\mu \Sigma_{r_0}^\tau} d\text{Vol}_{R_{II}}\) to both sides and using (80) we obtain

\[
[(1 + C(\tau_1 - \tau_0))] \int_{\Sigma_{\tau_0}^\tau} J^N_{\mu \Sigma_{\tau_0}^\tau} d\text{Vol}_{\Sigma_{\tau_0}^\tau} \\
\geq (a - \epsilon) \int_{\Sigma_{\tau_0}^\tau} J^N_{\mu \Sigma_{\tau_0}^\tau} d\text{Vol}_{\Sigma_{\tau_0}^\tau} \\
+ \int_{\Sigma_{\tau_1}^\tau} J^N_{\mu \Sigma_{\tau_1}^\tau} d\text{Vol}_{\Sigma_{\tau_1}^\tau}.
\]

Let \(f(\tau) = \int_{\Sigma_{\tau_0}^\tau} J^N_{\mu \Sigma_{\tau_0}^\tau} d\text{Vol}_{\Sigma_{\tau_0}^\tau}\), then Eq. (82) is given by

\[
[(1 + C(\tau_1 - \tau_0))] f(\tau_0) \geq (a - \epsilon) \int_{\tau_0}^{\tau_1} f(\tau) + f(\tau_1). \quad (83)
\]

Rearranging terms, diving by \(\tau_1 - \tau_0\) we have

\[
Cf(\tau_0) \geq \frac{(a - \epsilon)}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f(\tau) + \frac{f(\tau_1) - f(\tau_0)}{\tau_1 - \tau_0}. \quad (84)
\]

Taking the limit \(\tau_1 \to \tau_0\) gives

\[
Cf(\tau_0) \geq (a - \epsilon)f(\tau_1) + \frac{d}{d\tau} f(\tau)|_{\tau_1}. \quad (85)
\]

This implies

\[
\frac{d}{d\tau} \left( f(\tau)|_{\tau_1} e^{(a-\epsilon)\tau_1} - \frac{C}{(a - \epsilon)} f(\tau_0) e^{(a-\epsilon)\tau_1} \right) \leq 0, \quad (86)
\]

which gives for some constant \(B\)

\[
B \int_{\Sigma_{\tau_0}^\tau} J^N_{\Sigma_{\tau_0}^\tau} d\text{Vol}_{\Sigma_{\tau_0}^\tau} \geq \int_{\Sigma_{\tau_1}^\tau} J^N_{\Sigma_{\tau_1}^\tau} d\text{Vol}_{\Sigma_{\tau_1}^\tau}. \quad (87)
\]

**4.2 The Intermediate Region**

![Figure 3: Sketch of region \(R_{II} = R_{II_A} \cup R_{II_B}\) depicted in darker shades](image)

**Proposition 4.2.** Let \(X_f\) be the vector field multiplier \(58\) given in Section 3.3 and \(\psi\) be a solution to \((1)\), the homogeneous wave equation on Rindler with compactly supported data. Then the following statements hold

\[
C(\tilde{r})K^X_f(\psi) > J^N_{\mu \Sigma_{\tau_0}^\tau}(\psi), \quad \text{for} \quad r \geq \tilde{r}. \quad (88)
\]
and

$$|J_{\mu}^{X_I} n_{\Sigma_{r_1}}^\mu(\psi)| \leq C J_{\mu}^{\partial_r} n_{\Sigma_{r_1}}^\mu(\psi).$$

(89)

Moreover, we have

$$\int_{\mathcal{R}_{II_B}} J_{\mu}^{\partial_r} n_{\Sigma_{r_1}}^\mu(\psi) d\text{Vol}_{\mathcal{R}_{II_B}} \leq C \int_{\Sigma_{r_0}} J_{\mu}^{\partial_r} n_{\Sigma_{r_0}}^\mu(\psi) d\text{Vol}_{\Sigma_{r_0}},$$

(90)

where $\mathcal{R}_{II} := \{D^+(\Sigma_{r_0}) \cap D^-(\Sigma_{r_1})\}$ and $\mathcal{R}_{II_B} : \{p \in \mathcal{R}_{II} \in s. t. r \geq \tilde{r}\}$ for $\tilde{r} \in \mathbb{R}$, see Figure 3.

**Proof.** We have

$$K_{X_I} = \frac{a}{2 \cosh^2(\alpha r)} e^{-2\alpha r} \left[ (\partial_r \psi)^2 + (\partial_r \psi)^2 \right]$$

(91)

and

$$J_{\mu}^{\partial_r} n_{\Sigma_{r_1}}^\mu = \frac{e^{-\alpha r}}{2} \left[ (\partial_r \psi)^2 + (\partial_r \psi)^2 \right],$$

(92)

Statement (88) can be shown by comparing the coefficients in (91) and (92). Since $\cosh^2(\alpha r)$ is a bounded function, then for $r \in [\tilde{r}, \infty)$ there exists, $c(\tilde{r})$, such that

$$\frac{a}{2 \cosh^2(\alpha r)} e^{-\alpha r} > c(\tilde{r}) > 0.$$  \hspace{1cm} (93)

Now using the normal vector (34) and the multiplier (58) we get

$$J_{\mu}^{X_I} n_{\Sigma_{r_1}}^\mu = - (\tanh(\alpha r) + b) e^{-\alpha r}(\partial_r \psi \partial_r \psi).$$

(94)

The proof of statement (89) follows from applying Cauchy–Schwarz inequality to (94) and then comparing with (92) to get

$$|J_{\mu}^{X_I} n_{\Sigma_{r_1}}^\mu| \leq - [\tanh(\alpha r) + b] \frac{e^{-\alpha r}}{2} \left[ (\partial_r \psi)^2 + (\partial_r \psi)^2 \right]$$

$$\leq C e^{-\alpha r} \left[ (\partial_r \psi)^2 + (\partial_r \psi)^2 \right]$$

$$\leq J_{\mu}^{\partial_r} n_{\Sigma_{r_1}}^\mu.$$  \hspace{1cm} (95)

In order to show statement (90), the following two steps are applied. First, we choose $b$ in (69) such that $\tanh(\alpha r) + b = 0$. Second, we apply the divergence theorem in $\mathcal{R}_{II_B}$ using the vector field $X_I$, which give the following inequality

$$\int_{\mathcal{R}_{II_B}} K_{X_I}^\mu(\psi) d\text{Vol}_{\mathcal{R}_{II_B}} \leq \int_{\Sigma_{r_1}} J_{\mu}^{X_I} (\psi) n_{\Sigma_{r_1}}^\mu d\text{Vol}_{\Sigma_{r_1}} + \int_{\Sigma_{r_0}} J_{\mu}^{X_I} (\psi) n_{\Sigma_{r_0}}^\mu d\text{Vol}_{\Sigma_{r_0}}.$$  \hspace{1cm} (96)

The timelike boundary term at $\tilde{r}$ vanishes due to our choice of $b$ and at infinity due to the compact support of $\psi$. By Eq. (88), Eq. (89) and Proposition (3.1), we have

$$\int_{\mathcal{R}_{II_B}} J_{\mu}^{\partial_r} (\psi) n_{\Sigma_{r_1}}^\mu d\text{Vol}_{\mathcal{R}_{II_B}} \leq C \int_{\mathcal{R}_{II_B}} K_{X_I}^\mu(\psi) d\text{Vol}_{\mathcal{R}_{II_B}} \leq$$

$$\leq \int_{\Sigma_{r_1 \cap (r > \tilde{r})}} J_{\mu}^{X_I} (\psi) n_{\Sigma_{r_1}}^\mu d\text{Vol}_{\Sigma_{r_1}} + \int_{\Sigma_{r_0 \cap (r > \tilde{r})}} J_{\mu}^{\partial_r} (\psi) n_{\Sigma_{r_0}}^\mu d\text{Vol}_{\Sigma_{r_0}}$$

$$\leq \int_{\Sigma_{r_1 \cap (r > \tilde{r})}} J_{\mu}^{X_I} (\psi) n_{\Sigma_{r_1}}^\mu d\text{Vol}_{\Sigma_{r_1}} + \int_{\Sigma_{r_0 \cap (r > \tilde{r})}} J_{\mu}^{\partial_r} (\psi) n_{\Sigma_{r_0}}^\mu d\text{Vol}_{\Sigma_{r_0}}$$

$$\leq \int_{\Sigma_{r_0}} J_{\mu}^{\partial_r} (\psi) n_{\Sigma_{r_0}}^\mu d\text{Vol}_{\Sigma_{r_0}}.$$  \hspace{1cm} (97-100)
4.3 Local Energy Decay near $\mathcal{R}H^+$

**Proposition 4.3.** Let $N$ be the vector field multiplier given by Proposition (3.2) and $\psi$ be a solution to (1) with compactly supported data. Then the following statement holds

$$
\int_{R_{III}} J_\mu^N n_\mu^\nu C_\nu \, dVol_{R_{III}} \leq C \int_{\Sigma_0} J_\mu^N n_\mu^\nu C_\nu \, dVol_{\Sigma_0},
$$

(101)

where $R_{III} := \left\{ D^+ (\Sigma_{r_0}^s) \cap D^- (\Sigma_{r_1}^t) \right\} = R_{III} \cup R_{III}^\prime$, with $R_{III} = \{ p \in R_{III} \in s. t. r \leq r_* \}$ and $R_{III}^\prime = \{ p \in R_{III} \in s. t. r_* \leq r \}$ for $r_*$ defined as below. See Figure 4.

**Proof.** Define $g(\bar{r}) = \int_{\tau_0}^{\tau_1} J_\mu^N n_\mu^\nu e^{at} \, dt$ and choose $\bar{r}_0, \bar{r}_1 < r_0$, then from the mean value theorem for integrals there is a $r_* \in (\bar{r}_0, \bar{r}_1)$ such that $g(r_*) = \frac{1}{\bar{r}_1 - \bar{r}_0} \int_{\bar{r}_0}^{\bar{r}_1} g(\bar{r}) \, d\bar{r}$. Take this $r_*$ to define $R_{III}$. We use the divergence theorem in the region $R_{III}$ with the vector field $N$ to obtain

$$
\int_{\Sigma_0} J_\mu^N n_\mu^\nu C_\nu \, dVol_{\Sigma_0} + \int_{R_{III}} K_\mu^N dVol_{R_{III}} \leq \int_{\Sigma_0} J_\mu^N n_\mu^\nu C_\nu \, dVol_{\Sigma_0} + \left| \int_{r_*} J_\mu^N n_\mu^\nu C_\nu \, dVol_{r_*} \right|
$$

(102)

Dropping the positive boundary term on the left hand side and using (54) we obtain

$$
(a - \epsilon) \int_{R_{III}} J_\mu^N n_\mu^\nu C_\nu \, dVol_{R_{III}} \leq \int_{\Sigma_0} J_\mu^N n_\mu^\nu C_\nu \, dVol_{\Sigma_0} + \left| \int_{r_*} J_\mu^N n_\mu^\nu C_\nu \, dVol_{r_*} \right|
$$

(103)

To estimate the integral given by a timelike boundary we recall that by the mean value
Theorem \( g(r_*) = \frac{1}{r_1 - r_0} \int_{r_0}^{r_1} g(\tilde{r})d\tilde{r} \) and we obtain the following inequalities

\[
g(r_*) \leq |\mathcal{g}(r_*)| \\
\leq \frac{1}{r_1 - r_0} \int_{r_0}^{r_1} g(\tilde{r})d\tilde{r} \\
\leq C \int_{r_0}^{r_1} \int_{r_0}^{r_1} ((\partial_r \psi)^2 + (\partial_r \tilde{\psi})^2) d\tau dr \\
\leq C \int_{r_0}^{r_1} \int_{r_0}^{r_1} J^N_\mu(\psi)n^{\mu}_{\Sigma_*} e^\alpha r d\tau dr \\
\leq C \int_{\mathcal{R}_{III}} J^N_\mu(\psi)n^{\mu}_{\Sigma_*} d\text{Vol}_{\mathcal{R}_{III}}
\]

Putting this together in Eq. \( 103 \), we get

\[
(a - \epsilon) \int_{\mathcal{R}_{IIIA}} J^N_\mu n^{\mu}_{\mathcal{C}_u} d\text{Vol}_{\mathcal{R}_{IIIA}} \\
\leq C \left( \int_{\mathcal{C}_{u}(u_0(r^*), \infty)} J^N_\mu n^{\mu}_{\mathcal{C}_u} d\text{Vol}_{\mathcal{C}_u} + \int_{\Sigma_{r_0}} J^N_\mu(\psi)n^{\mu}_{\Sigma_*} d\text{Vol}_{\Sigma_{r_0}} \right) \] (104)

which combined with Proposition \( 4.1 \) gives

\[
(a - \epsilon) \int_{\mathcal{R}_{IIIA}} J^N_\mu n^{\mu}_{\mathcal{C}_u} d\text{Vol}_{\mathcal{R}_{IIIA}} \leq C \int_{\Sigma_{r_0}} J^N_\mu(\psi)n^{\mu}_{\Sigma_*} d\text{Vol}_{\Sigma_{r_0}}. \] (105)

\[ \square \]

### 4.4 Far Away Estimates

In the following we will derive estimates in a far away region in close analogy to considerations in \[ 1 \] Section 4.

**Theorem 4.4.** Let \( \mathcal{N} \) be the vector field multiplier given by Proposition \( 3.2 \) and \( \psi \) be a solution to \( 1 \) with compactly supported data. Then we have in the region \( \mathcal{R}_{IV} := \{(u, r) \in \mathcal{R}_{IV} \text{ s.t.} r \geq w^* \}, u_0 \leq u \leq u_1 \} \) the following estimate

\[
\int_{\mathcal{C}_{u_1}(v_1(w^*), \infty)} r^p(\partial_v \phi)^2 dv + \int_{\mathcal{R}_{IV}} p^*(p-1)(\partial_v \phi)^2 dv du \\
= \int_{\mathcal{C}_{u_0}(v_0(w^*), \infty)} r^p(\partial_v \phi)^2 dv + \int_{\tau_0}^{\tau_1} 2w^{*p}(\partial_v \phi)^2 d\tau \] (106)

for all \( p > 0 \) where \( \mathcal{R}_{IV} := \{ D^+(\Sigma_{r_0}^{v_1}) \cap D^-((\Sigma_{r_1}^{v_1}) \} \) for \( w^* \) defined as below. See Figure 3.

Applying the divergence theorem using the multiplier Eq.64 in the entire region \( \mathcal{R}_{IV} \) and noting that the energy flux along \( I^+ \) does not contribute the given multiplier, leads to

\[
\int_{C_{u_1}(v_1(w^*), \infty)} J^V_\mu n^{\mu}_{u=u_1} d\text{Vol}_{C_{u_1}} + \int_{\mathcal{R}_{IV}} K^V d\text{Vol}_{\mathcal{R}_{IV}} \]

\[
\leq \int_{C_{u_0}(v_0(w^*), \infty)} J^V_\mu n^{\mu}_{v=v_0} d\text{Vol}_{C_{u_0}} + \int_{\tau_0}^{\tau_1} J^V_\mu n^{\mu}_{r=R} d\text{Vol}_{r}. \] (107)
Using (66), (65) and (67), we have

\[ Z_{C_u} \left( v_1(\tilde{r}), \infty \right) \left( v, \infty \right) \leq Z_{C_u} \left( v_0(\tilde{r}), \infty \right) \left( v, \infty \right) + \int \tau_1 \int_{\tau_0} \left( \frac{2R_p e^{-aR(\partial_v \psi)^2} dV} {R^2} \right) \left( \int_0 2R_p e^{-aR(\partial_v \psi)^2} dV \right) \]

for all \( n \in \mathbb{N} \).

\[ \text{which using (32), (35) and (31) leads to the desired result (106).} \]

4.5 Energy decay for solutions with compactly supported initial data

**Theorem 4.5.** Let \( \{ \Sigma^{r,r^*}_r \} \) be a foliation such that \( \tilde{r} < r_0 \) and \( r^* > r_0 \) as in Theorem 4.3 and Theorem 4.4, \( N \) be the vector field multiplier given by Proposition 4.2, and \( \psi \) be a solution to (1) with compactly supported data. Then the following statements hold:

\[ \int_{\Sigma^{r,r^*}_r} J^N_\mu (\psi) n^\mu_{r^*,r} dV_{\Sigma^{r,r^*}_r} \leq \frac{C}{r} \int_{\Sigma_0} J^N_\mu (\psi) n^\mu_r dV_{\Sigma_0} \]  

(109)

\[ \int_{\Sigma^{r,r^*}_r} J^N_\mu (\psi) n^\mu_r dV_{\Sigma^{r,r^*}_r} \leq C \left( \int_{\mathcal{L}_{w_0}(v_0(\tilde{r})))} r^n (\partial_v \phi)^2 dV + C \left( \int_{\Sigma_{r_0}} J^N_\mu (\psi) n^\mu_r dV_{\Sigma_{r_0}} \right) \right) , \]

(110)

for all \( n \in \mathbb{N} \).

**Proof.** Using Theorem 4.4 for \( p = 1 \) we have

\[ \int_{\mathcal{L}_{w_0}(v_0(\tilde{r})))} r (\partial_v \psi)^2 dV + \int_{\mathcal{R}_{IV}} (\partial_v \psi)^2 dV du \]

\[ = \int_{\mathcal{L}_{w_0}(v_0(\tilde{r})))} r (\partial_v \psi)^2 dV + \int_{\tau_0}^{\tau_1} 2R (\partial_v \psi)^2 d\tau , \]

(111)

and notice that in this region we have

\[ \int_{\mathcal{R}_{IV}} J^N_\mu (\psi) n^\mu_{C_u} dV_{C_u} = \int_{\mathcal{R}_{IV}} J^\beta_\mu (\psi) n^\mu_{C_u} dV_{C_u} = \frac{1}{2} \int_{\mathcal{R}_{IV}} (\partial_v \psi)^2 dV du . \]

(112)
Also,
\[ \left| \int_{\tau_0}^{\tau_1} 2r^* (\partial_r \psi)^2 dr \right| \leq C \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \] (113)

Putting this together in Eq. (111) and ignoring a positive contribution from the left side we get
\[ \int_{\mathcal{R}_{IVC}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\mathcal{R}_{IVC}} \leq C \left( \int_{\mathcal{C}_{u_0} (v_0 (\bar{r}))} r (\partial_r \psi)^2 dv + \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \right). \] (114)

Using the estimates (90), (101), we obtain
\[ \int_{\mathcal{R}_{IVC}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\mathcal{R}_{IVC}} + \int_{(\mathcal{R}_{IVC} \cup \mathcal{R}_{IVB})} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{(\mathcal{R}_{IVC} \cup \mathcal{R}_{IVB})} \]
\[ \leq \int_{\mathcal{R}_{IVC}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\mathcal{R}_{IVC}} + \int_{\mathcal{R}_{IVB}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\mathcal{R}_{IVB}} \]
\[ \leq C \left( \int_{\mathcal{C}_{u_0} (v_0 (\bar{r}))} r (\partial_r \psi)^2 dv + \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \right), \] (115)

Moreover, from Theorem 4.4 we have in the region $\mathcal{R}_{IVC}$ the following estimate
\[ \int_{\mathcal{C}_{u_1} (v_1 (\bar{r}))} r^p (\partial_r \phi)^2 dv + \int_{\mathcal{R}_{IVC}} pr^{(p-1)} (\partial_r \phi)^2 dv \]
\[ \leq \int_{\mathcal{C}_{u_0} (v_0 (\bar{r}))} r^p (\partial_r \phi)^2 dv + C \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \] (116)

This implies we can find a dyadic sequence $\tau_n \to \infty$ with the property that
\[ \int_{\mathcal{C}_{u_n} (v(w_{\tau_n} \cdot)^*)} pr^{(p-1)} (\partial_r \phi)^2 dv \]
\[ \leq C \tau_n^{-1} \left( \int_{\mathcal{C}_{u_0} (v_0 (\bar{r}))} r^p (\partial_r \phi)^2 dv + C \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \right) \] (117)

We now apply Eq. (106) with $p - 1$ in place of $p$ to obtain
\[ \int_{\mathcal{C}_{u_n} (v(w_{\tau_n} \cdot)^*)} r^{p-1} (\partial_r \phi)^2 dv + \int_{\mathcal{R}_{IVC} \tau_n} (p - 1) r^{(p-2)} (\partial_r \phi)^2 dv \]
\[ \leq \int_{\mathcal{C}_{u_{n-1}} (v(w_{\tau_n-1} \cdot)^*)} r^{p-1} (\partial_r \phi)^2 dv + C \int_{\Sigma_{\tau_n}^{\tau_n^*}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_{\tau_n-1}^{\tau_n}} \] (118)

in the region $\mathcal{R}_{IVC \tau_n}$, For $p = 2$ and adding a multiple of the ILED (90), (101) we get
\[ \int_{\mathcal{R}_{IVC} \tau_n} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\mathcal{R}_{IVC} \tau_n} \]
\[ + \int_{(\mathcal{R}_{IVC \tau_n} \cup \mathcal{R}_{IVB \tau_n})} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{(\mathcal{R}_{IVC \tau_n} \cup \mathcal{R}_{IVB \tau_n})} \]
\[ \leq C \tau_n^{-1} \left( \int_{\mathcal{C}_{u_0} (v_0 (\bar{r}))} r (\partial_r \phi)^2 dv + C \int_{\Sigma_{\tau_0}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_0} \right) \]
\[ + C \int_{\Sigma_{\tau_n-1}^{\tau_n^*}} J^N_{\mu} (\psi) \eta^* d\text{Vol}_{\Sigma_{\tau_n-1}^{\tau_n}} \] (119)
Using Eq. (109), the properties of the dyadic sequence and the energy boundness given by Proposition 4.1, we get

\[
\int_{\Sigma_{\tau_0}^\tau} J^N_\mu (\psi) n^\mu d\text{Vol}_{\Sigma_{\tau_0}^\tau} 
\leq C \frac{C}{\tau^2} \left( \int_{C_{v_0}(v_0, \infty)} r^2 (\partial_\tau \phi)^2 d\nu + C \int_{\Sigma_{\tau_0}} J^N_\mu (\psi) n^\mu d\text{Vol}_{\Sigma_{\tau_0}} \right) 
\]  

(120)

for any \( \tau \geq \tau_0 \). Using Theorem 4.4 with \( p = 3 \), another dyadic sequence, the mean value theorem, 4.1 and Eq. (117) we obtain the estimate

\[
\int_{\Sigma_{\tau_0}^\tau} J^0_\mu (\psi) n^\mu d\text{Vol} 
\leq C \frac{C}{\tau^2} \left( \int_{\Sigma_{\tau_0}^\tau} r^3 (\partial_\tau \phi)^2 d\nu + C \int_{\Sigma_{\tau_0}} J^0_\mu (\psi) n^\mu d\text{Vol}_{\Sigma_{\tau_0}} \right) 
\]  

for all \( \tau \geq \tau_0 \).

5 Discussion

Due to the equivalence principle, the energy decay of a family of uniformly accelerated observers must share similar properties with the energy decay of a family of stationary observers in a gravitational field. In this section, we focus on the comparison between Rindler spacetime and Schwarzschild spacetime. This comparison follows from the observation that in Schwarzschild the metric is given by

\[
ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2 
\]  

for \( \xi^2 := 1 - \frac{2m}{r} < 1 \) where \( t \) is proportional to the proper time of static observers and \( \xi \) small describes the near horizon geometry, while the Rindler metric is given by

\[
ds^2 = -\xi^2 dt^2 + d\xi^2 + dy^2 + dz^2 
\]  

for \( 0 < \xi < \infty \) where \( \tau \) is proportional to the proper time of uniformly accelerated observer and uniformly accelerated observers remain at fixed \( \xi \).

5.1 Comparison with Schwarzschild spacetime.

We discuss three separate regions as shown in the Figure below.
5.2 Region $B$

This region corresponds to compact regions away from the horizon. The analysis of trapping effects in Schwarzschild, due to the photon sphere located at $r = 3M$, are treated here. At the level of the estimates one needs to lose derivatives in order to obtain a Integrated Local Energy Decay estimate [12, Section 4.1]. Since in Rindler there is no such trapping behaviour, we obtain the Integrated Local Energy Decay estimate without loss of derivatives (Theorem 4.2 Eq.(90)). Notice that the situation also differs to what happens in Minkowski spacetime. There for compact regions there exists a finite $t$ such that the energy flux defined with respect to the Killing vector $\partial_t$ is zero. Although, Rindler spacetime is a subset of Minkowski spacetime, the energy flux of the solutions with respect to $\partial_\tau$ in any compact region does not necessarily vanish for any finite $\tau$.

5.3 Region $C$

The far away region is usually analysed using conformal compactifications. In the $1+1$ Rindler spacetime, by inspection one can show that part of the Minkowski null infinity region coincide with the null infinity region in Rindler [36, 38]. However, the conformal compactification of Rindler can be done in several ways e.g. by looking at the corresponding region in the conformal compactification of Minkowski spacetime or using a conformal transformation that sends the Rindler wedge into a casual diamond in Minkowski spacetime [23 Appendix E], [13 V.4.2]. It is in this region where we are able to obtain better estimates for the $r^p$ vector field compared to the Schwarzschild case. This may be a consequence of the $1+1$ case. In the higher dimensional case the transversal direction may constraint the range of possible values of $p$. Furthermore, in higher dimensional Rindler spacetimes there is also the ambiguity of what should be the suitable analogue to the usual spherical coordinates. In addition, the asymptotic symmetries at infinity in Rindler spacetime differ from the asymptotic symmetries in the asymptotically flat case [5]. Finally, unlike, Schwarzschild and Minkowski spacetime where timelike infinity (denoted by the white dot) is defined using timelike geodesics, in the Rindler case, the ‘timelike infinity’ (denoted by the black dot) is defined using the uniformly accelerated observers.

5.4 Region $A$

In the near horizon Rindler region, the existence of a bifurcated Killing horizon with positive surface gravity allows one to construct the redshift vector field $Y$ and the local observers vector field $N$ as in the Schwarzschild case. However, in Schwarzschild spacetime there is a compact bifurcation surface which is not the case in higher dimensional Rindler spacetimes where there is a bifurcation hyperplane. Nevertheless, the existence of the redshift vector

---

Figure 6: Left: Schwarzschild spacetime divided into three regions by the radial coordinate $r$. Right: Rindler spacetime divided in three regions by the coordinate $r$ described in [26].
in both spacetimes suggests further analogies. For example, in Schwarzschild, the redshift effect in scattering constructions from the future turns into a blueshift. This statement can be formulated as the non-invertibility of forward scattering maps. To be precise, one can define in Schwarzschild spacetime, the maps

\[ E^X_{\Sigma_{\tau_0}} \rightarrow E^X_{\mathcal{H}^+} \oplus E^X_{\mathcal{I}^+} \]

for \( X = \partial_t \) and \( X = N \) where

\[ E^X_M(\psi) := \int_{\mathcal{M}} T_{\mu \nu}(\psi)X^\mu n^\nu_M dV_M = \int_{\mathcal{M}} J^X_{\nu}(\psi)n^\nu_M dV_M \]

and \( \Sigma_{\tau_0} \) is a suitable initial Cauchy hypersurface. The existence of the forward scattering maps can be done using decay estimates combined either with the existence of radiation fields \([13, 31]\) or a suitable conformal compactification \([33]\). Moreover, it was shown in \([13, \text{Theorem 4.1}]\) that for \( X = \partial_t \) there is an inverse map

\[ E^\partial_t_{\mathcal{H}^+} \oplus E^\partial_t_{\mathcal{I}^+} \rightarrow E^\partial_t_{\Sigma_{\tau_0}} \]

while for \( X = N \) this is not possible. The non-invertibility is a consequence of the time reversed redshift at the horizon.

To further, give evidence that this is also the case in Rindler, we will use the characterisation of the energy of Gaussian Beams on Lorentzian manifolds. Notice that by \([35, \text{Theorem 5.1}]\) there exists a smooth solution \( v \) of the wave equation with \( E^X_{t=0}(v) = -g(X, \dot{\gamma}(\gamma(0))) \) such that

\[ |E^X_{t=0, N \cap T \cap \Sigma_{\tau_0}}(v) - (-g(X, \dot{\gamma}))|_{T = T_0} < \mu \]

for all \( 0 \leq t \leq T_1 \), where \( t \) is the Minkowski time coordinate map. Furthermore, \( \gamma : [0, S) \rightarrow M \) is the affinely parametrized generator of \( \mathcal{R}H^+ \), \( X \) a timelike, future-directed vector field and \( N \) a neighbourhood of \( \gamma \).

Explicitly, we have \( \gamma(s) = (s, s) \) and \( -g(N, \dot{\gamma}) = e^{-as} \) which means that the energy of the corresponding Gaussian beam decays exponentially. This is a direct manifestation of redshift effect at the Rindler Horizon. Moreover, since all hypothesis are satisfied, an analogous proposition to \([35, \text{Proposition 6.3}]\) holds. Explicitly,

**Proposition 5.1.** For every \( \mu > 0 \) and every \( t > 0 \), there exists a smooth solution \( v \) to the wave equation with \( E^N_0(v) = 1 \) and \( \int_{\mathcal{R}H^+} J^N_\nu n^\nu dV < \mu \) which satisfies

\[ E^N_0(v) \geq e^{at} - \mu, \]

where \( a \) is the surface gravity at the Rindler Horizon.

The proposition demonstrates that for every \( t_0 > 0 \), initial data can be specified for the mixed characteristic initial value problem on \( \mathcal{R}H^+ \cup \{(t, x) : t = t_0, x > t_0\} \) such that the total initial energy equals one. Meanwhile, by solving the equation backwards, the energy of the resulting solution approaches approximately \( e^{at_0} \) on \( t = 0 \).

### 5.5 The principle of equivalence.

The equivalence principle was initially stated by Einstein as the assumption that there is a “complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system” \([14]\). It is a geometric consequence of General Relativity and was a crucial guiding principle during the initial developing stages of the theory. Moreover, it has been experimentally verified with precision up to the atomic level, see for example \([42]\) or the contribution of Shapiro in \([19]\). It is worth noting that the analysis of the equivalence principle at the quantum level has been explored, as evidenced by references \([2], [17], \) and \([37]\). Precise definitions of the equivalence principle are crucial for this analysis. For a detailed
discussion on the various formulations of this principle, along with a historical overview, we refer the reader to [29].

In what follows, we aim to explore the extent to which classical wave equations render acceleration and gravity indistinguishable, and what distinct signatures might allow us to differentiate between them. As previously discussed, Rindler spacetime, which represents flat spacetime from the perspective of uniformly accelerated observers, exhibits similarities to, yet important differences from, the gravitational field produced by a spherically symmetric static mass, as described by the Schwarzschild metric. These differences and similarities provide key insights into the underlying physics of acceleration and gravity.

It is clear from the above comparison that in Region \( A \), which is close to the horizon, the behavior of wave equations is similar; for instance, there is a degeneration of energy with respect to static observers in the Schwarzschild case and with respect to accelerated observers in Rindler. Additionally, there is an exponential energy decay with respect to local observers, as indicated by Gaussian beams. This similarity is due to the presence of a Killing horizon with associated surface gravity. It is noteworthy that these effects occur regardless of whether one spacetime is flat and the other can be possibly strongly curved.

In the intermediate region \( B \), there are distinguishable effects between the two spacetimes. The behavior in Schwarzschild spacetime, particularly trapping phenomena, distinctively influences local energy decay estimates. From a physical perspective, the photon sphere acts as an obstruction to decay; however, the geodesics in this region are unstable. Nevertheless, in principle, an observer located near the photon sphere would not notice energy decay, in contrast to what is observed in Rindler spacetime, where such obstructions are absent. Therefore, one could distinguish between accelerated frames and gravitational fields by measuring the decay rates of energy in the associated field, provided one is sufficiently far away from the horizon and close to the photon sphere.

In the distant region \( C \), Schwarzschild and Rindler spacetimes exhibit a notable similarity concerning the interaction between the horizon, null infinity, and their associated scattering constructions. In Schwarzschild spacetime, an early-time energy blow-up occurs for decaying data observed at null infinity and vanishing at the event horizon, a phenomenon resulting from the redshift transforming into a blueshift. Above we have shown, that a similar blow-up is likely to occur also in Rindler. A parallel situation occurs in quantum physics. For instance, in Schwarzschild, the energy-momentum tensor of the Boulware vacuum diverges at the event horizon. Similarly, the energy-momentum tensor of the Rindler vacuum also exhibits a singularity at the horizon, indicating a similar kind of blow-up.

We have presented some results that shows the extent to which the principle of equivalence holds at the level of massless wave equations. This issue is subtle, in particular in relation with locality and the role of curvature. Our discussion highlights that certain global behaviors—specifically related to the energy of waves observed by accelerated observers in flat spacetime compared to that observed by static observers in a gravitational field—exhibit similarities. Moreover, close to the horizon, the redshift effect on the energy decay is similar, even in the presence of strong curvature in the gravitational case. Therefore, the validity of this principle depends critically on the specific statement one aims to prove.

Acknowledgements

The authors would like to thank the Max Planck Institute of Mathematics in Bonn for its hospitality during the writing of this paper. We also thank the Oberwolfach Institute, Leibniz University Hannover and CAMGSD, IST Lisboa for additional support. Special thanks to Pedro Girão for valuable comments on the manuscript. Further, we also benefited from discussions with José Natário and Michael Gruber. This work was partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020, by FCT/Portugal and through the FCT fellowship CEECIND/00936/2018 (A.F.).
We have no conflicts of interest to disclose.

A Regular double-null coordinates

In double null coordinates \((U, v)\) which are regular at \(\mathcal{R}\mathcal{H}^+\), we use the transformation

\[
u(U) = -\frac{1}{a} \ln |aU|, \quad U = -\frac{1}{a} e^{-av}, \tag{123}\]

with

\[
du = \frac{\partial u}{\partial U} dU = -\frac{1}{aU} dU = e^{au} dU, \tag{124}\]

and obtain the metric

\[
ds^2 = -e^{av} dU dv. \tag{125}\]

Defining the null coordinate \(V\) as

\[
v(V) = \frac{1}{a} \ln(aV), \quad V = \frac{1}{a} e^{av}, \tag{126}\]

with

\[
dv = \frac{\partial v}{\partial V} dV = \frac{1}{aV} dV = e^{-av} dV, \tag{127}\]

we obtain in the \((u, V)\) coordinate system the metric

\[
ds^2 = -e^{-au} du dV. \tag{128}\]

Hence, in the \((U, V)\) coordinate system we get the familiar Minkowski form of the metric

\[
ds^2 = -dU dV, \tag{129}\]

and the relationship between laboratory coordinates and regular double null coordinates is given by

\[U = t - x, \quad \text{and} \quad V = t + x, \tag{130}\]

restricted to the subdomain \(x > |t|\) yielding the range \(-\infty < U < 0\) and \(0 < V < \infty\).

B The \(K\)-current

In order to calculate the scalar-currents for the bulk term according to (7) we first derive the components of the deformation tensor

\[
(\pi^X)^{\mu \nu} = \frac{1}{2} (g^\mu \lambda \partial_\lambda X^\nu + g^\nu \lambda \partial_\lambda X^\mu + g^\mu \lambda g^{\nu \delta} g_{\lambda \rho} X^\rho) \tag{131}\]

in different coordinates. In \((r, \tau, y, z)\)-coordinates with the metric (30) and for the arbitrary vector field \(X = X^\tau (r, \tau) \partial_r + X^\tau (r, \tau) \partial_\tau\) we obtain

\[
(\pi^X)^{rr} = e^{-2ar} [\partial_r X^\tau + aX^\tau], \tag{132}\]

\[
(\pi^X)^{r\tau} = e^{-2ar} [\partial_r X^\tau + aX^\tau], \tag{133}\]

\[
(\pi^X)^{\tau \tau} = \frac{e^{-2ar}}{2} [-\partial_\tau X^\tau + \partial_r X^\tau], \tag{134}\]

Further, with (2) the energy-momentum tensor yields

\[
T_{\tau \tau} = \frac{1}{2} [(\partial_\tau \psi)^2 + (\partial_\tau \psi)^2], \tag{135}\]

\[
T_{rr} = \frac{1}{2} [(\partial_r \psi)^2 + (\partial_r \psi)^2], \tag{136}\]

\[
T_{r\tau} = (\partial_r \psi \partial_\tau \psi), \tag{137}\]
we obtain the scalar current

\[ K^X = (\partial_r \psi)^2 \frac{e^{-2ar}}{2} [\partial_r X^r - \partial_r X^\tau] + (\partial_r \psi)^2 \frac{e^{-2ar}}{2} [\partial_r X^\tau - \partial_r X^\tau] + (\partial_r \psi \partial_r \psi) e^{-2ar} [\partial_r X^\tau - \partial_r X^\tau] \] (138)

In \((u, v)\)-coordinates with the metric (28) and for the arbitrary vector field \(X = X^u(u, v)\partial_u + X^v(u, v)\partial_v\) we get

\[
\begin{align*}
(\pi^X)^{uu} & = -2e^{-a(v-u)} \partial_u X^u, \\
(\pi^X)^{vv} & = -2e^{-a(v-u)} \partial_v X^v, \\
(\pi^X)^{uv} & = -e^{-a(v-u)} [\partial_v X^v + \partial_u X^u] + ae^{-a(v-u)} [X^u - X^v].
\end{align*}
\] (139) (140) (141)

With (2) we get the following components for the energy-momentum tensor,

\[
\begin{align*}
T_{uu} & = (\partial_u \psi)^2, \\
T_{vv} & = (\partial_v \psi)^2, \\
T_{uv} & = 0.
\end{align*}
\] (142) (143) (144)

Multiplying the components then leads to

\[ K^X = -2e^{-a(v-u)} [(\partial_u \psi)^2 \partial_u X^u + (\partial_v \psi)^2 \partial_v X^v]. \] (145)

C The \(J\)-current

For the timelike normal vector on a constant \(\tau\) hypersurface we have

\[ n^\mu_{\tau=e} = e^{-ar} \partial_\tau, \text{ with } d\text{Vol}_\tau = e^{ar} dr, \] (146)

so that we can derive the currents

\[
\begin{align*}
J^X_{\mu} n^\mu_{\tau=e} & = \frac{X^r e^{-ar}}{2} \left[ [ (\partial_r \psi)^2 + (\partial_r \psi)^2 ] \right. + X^r e^{-ar} (\partial_r \psi \partial_r \psi), \\
J^X_{\mu} n^\mu_{\tau=e} & = \frac{X^r e^{-ar}}{2} \left[ [ (\partial_r \psi)^2 + (\partial_r \psi)^2 ] \right. + X^r e^{-ar} (\partial_r \psi \partial_r \psi),
\end{align*}
\] (147) (148)

with \(X = X^r(\tau, \tau) \partial_\tau + X^\tau(\tau, \tau) \partial_\tau\).

A vector field multiplier \(X^u(u, v)\partial_u + X^v(u, v)\partial_v\) leads to the currents

\[
\begin{align*}
J^X_{\mu} n^u_{\tau=e} & = X^u 2e^{-a(v-u)} (\partial_u \psi)^2, \\
J^X_{\mu} n^v_{\tau=e} & = X^v 2e^{-a(v-u)} (\partial_v \psi)^2, \\
J^X_{\mu} n^\mu_{\tau=e} & = e^{-a(v-u)} \left[ X^u (\partial_u \psi)^2 + X^v (\partial_v \psi)^2 \right].
\end{align*}
\] (149) (150)

References

[1] Peter C. Aichelburg. Is the equivalence principle useful for understanding general relativity? American Journal of Physics, 90(7):538–548, jul 2022.
[2] Brett Altschul, Quentin G. Bailey, Luc Blanchet, Kai Bongs, Philippe Bouyer, Luigi Cacciapuoti, Salvatore Capozziello, Naceur Gaaloul, Domenico Giulini, Jonas Hartwig, Luciano Less, Philippe Jetzer, Arnaud Landragin, Ernst Rasel, Serge Reynaud, Stephan Schiller, Christian Schubert, Fiodor Sorrentino, Uwe Sterr, Jay D. Tasson, Guglielmo M. Tino, Philip Tuckey, and Peter Wolf. Quantum tests of the einstein equivalence principle with the ste“quest space mission. Advances in Space Research, 55(1):501–524, 2015.

[3] J. S. Ben-Benjamin, M. O. Scully, S. A. Fulling, D. M. Lee, D. N. Page, A. A. Svidziński, M. S. Zubairy, R. M. Duff, G. Glauber, W. P. Schleich, and W. G. Unruh. Unruh acceleration radiation revisited. International Journal of Modern Physics A, 34(28):1941005, oct 2019.

[4] P. Blue and A. Soffer. Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates. Adv. Differential Equations, 8(5):595–614, 2003.

[5] P. Blue and A. Soffer. The wave equation on the Schwarzschild metric. II. Local decay for the spin-2 Regge-Wheeler equation. J. Math. Phys., 46(1):012502, 9, 2005.

[6] Hyeyoun Chung. Asymptotic symmetries of rindler space at the horizon and null infinity. Phys. Rev. D, 82:044019, Aug 2010.

[7] Luís C. B. Crispino, Atsushi Higuchi, and George E. A. Matsas. The Unruh effect and its applications. Rev. Modern Phys., 80(3):787–838, 2008.

[8] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. Acta Math., 222(1):1–214, 2019.

[9] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski, and Martin Taylor. The non-linear stability of the schwarzschild family of black holes, 2021.

[10] Mihalis Dafermos and Igor Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. Comm. Pure Appl. Math., 62(7):859–919, 2009.

[11] Mihalis Dafermos and Igor Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In XVIth International Congress on Mathematical Physics, pages 421–432. World Sci. Publ., Hackensack, NJ, 2010.

[12] Mihalis Dafermos and Igor Rodnianski. Lectures on black holes and linear waves. In Evolution equations, volume 17 of Clay Math. Proc., pages 97–205. Amer. Math. Soc., Providence, RI, 2013.

[13] Mihalis Dafermos and Yakov Shlapentokh-Rothman. Time-translation invariance of scattering maps and blue-shift instabilities on Kerr black hole spacetimes. Comm. Math. Phys., 350(3):985–1016, 2017.

[14] Albert Einstein. Über das Relativitätsprinzip und die aus demselben gezogene Folgerungen. (German) [On the Relativity Principle and the conclusions drawn from it]. Jahrbuch der Radioaktivität und Elektronik, 5:98–99, 1907.

[15] S A Fulling. Alternative vacuum states in static space-times with horizons. Journal of Physics A: Mathematical and General, 10(6):917–951, jun 1977.

[16] C. Gérard and M. Wrochna. Construction of Hadamard states by pseudo-differential calculus. Comm. Math. Phys., 325(2):713–755, 2014.

[17] Domenico Giulini. Equivalence principle, quantum mechanics, and atom-interferometric tests. In F. Finster et al., editors, Quantum Field Theory and Gravity, Birkhäuser Series, pages 345–370. Springer Verlag, Basel, 2012.
[18] Rudolf Haag. *Local quantum physics*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996. Fields, particles, algebras.

[19] A. Held, editor. *General Relativity and Gravitation. 100-years after the birth Albert Einstein Vol. 1*. Plenum, New York, 1980.

[20] Atsushi Higuchi, Satoshi Iso, Kazushige Ueda, and Kazuhiro Yamamoto. Entanglement of the vacuum between left, right, future, and past: The origin of entanglement-induced quantum radiation. *Phys. Rev. D*, 96:083531, Oct 2017.

[21] Peter Hintz. A sharp version of Price’s law for wave decay on asymptotically flat spacetimes. *Comm. Math. Phys.*, 389(1):491–542, 2022.

[22] Pei-Ken Hung, Jordan Keller, and Mu-Tao Wang. Linear Stability of Schwarzschild Spacetime: Decay of Metric Coefficients. *J. Diff. Geom.*, 116(3):481–541, 2020.

[23] Ted Jacobson and Manus R. Visser. Gravitational thermodynamics of causal diamonds in (A)dS. *SciPost Phys.*, 7:079, 2019.

[24] W. Junker and E. Schrohe. Adiabatic vacuum states on general spacetime manifolds: definition, construction, and physical properties. *Ann. Henri Poincaré*, 3(6):1113–1181, 2002.

[25] Wolfgang Junker. Hadamard states, adiabatic vacua and the construction of physical states for scalar quantum fields on curved spacetime. *Rev. Math. Phys.*, 8(8):1091–1159, 1996.

[26] B S Kay and R M Wald. Linear stability of schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Classical and Quantum Gravity*, 4(4):893–898, jul 1987.

[27] Bernard S. Kay and Robert M. Wald. Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon. *Phys. Rep.*, 207(2):49–136, 1991.

[28] Sergiu Klainerman and Jeremie Szeftel. Global nonlinear stability of schwarzschild spacetime under polarized perturbations, 2017.

[29] Dennis Lehmkuhl. The equivalence principle(s), 2019.

[30] Cathleen S. Morawetz. Time decay for the nonlinear klein-gordon equation. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 306(1486):291–296, 1968.

[31] Georgios Moschidis. The $r^p$-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications. *Ann. PDE*, 2(1):Art. 6, 194, 2016.

[32] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to quantum effects in gravity*. Cambridge University Press, 6 2007.

[33] Jean-Philippe Nicolas. Conformal scattering on the Schwarzschild metric. *Ann. Inst. Fourier (Grenoble)*, 66(3):1175–1216, 2016.

[34] Marek J. Radzikowski. Micro-local approach to the Hadamard condition in quantum field theory on curved space-time. *Comm. Math. Phys.*, 179(3):529–553, 1996.

[35] Jan Sbierski. Characterisation of the energy of gaussian beams on lorentzian manifolds: With applications to black hole spacetimes. *Analysis and PDE*, 8(6):1379–1420, 2015.
[36] Claude Semay. Penrose–carter diagram for a uniformly accelerated observer. *European Journal of Physics*, 28(5):877–887, jul 2007.

[37] Douglas Singleton and Steve Wilburn. Hawking radiation, unruh radiation, and the equivalence principle. *Phys. Rev. Lett.*, 107:081102, Aug 2011.

[38] M. Socolovsky. Rindler space, Unruh effect and Hawking temperature. *Ann. Fond. Louis de Broglie*, 39:1–49, 2014.

[39] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, Aug 1976.

[40] Robert M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.

[41] Robert M. Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. Chicago Lectures in Physics. University of Chicago Press, Chicago, IL, 1994.

[42] Clifford M. Will. *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 2 edition, 2018.