Out-of-equilibrium properties and non-linear effects for interacting quantum impurity systems in their strong coupling regime

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We provide an exact description of out-of-equilibrium fixed points in quantum impurity systems, that is able to treat time-dependent forcing. Building on this, we then show that analytical out-of-equilibrium results, that exactly treat interactions, can be obtained in interacting quantum impurity systems in their strong coupling regime, provided they are integrable at equilibrium and they are “super Fermi liquids”, i.e. they only allow for integer charge hopping. For such systems we build an out-of-equilibrium strong coupling expansion, akin to a Sommerfeld expansion in interacting systems. We apply our approach to the Interacting Resonant Level model, and obtain the exact expansion around the low energy fixed point of the universal scaling function for the charge current as a function of voltage, temperature, and frequency, up to order seven.

The study of nano structures forced out-of-equilibrium is a vivid field of research, driven amongst other things by long term efforts towards the miniaturization of electronics. Experimentally, rapid progresses in the realization of engineered micrometric structures coupled to macroscopic electrodes (e.g. quantum dots [1,2]), of hybrid devices consisting of atoms or molecules embedded in circuits 3-5 demonstrate the possibility of “single electron” electronics 6,7. A theoretical understanding of the mechanisms governing transport in those systems is thus of crucial importance. Whereas linear response – when the voltage across the nanostructure goes to zero – boils down to equilibrium properties and is fairly well understood, non-linear effects are significantly harder to predict: solving the out-of-equilibrium theory unfortunately turns out to be a considerable problem, mixing many-body aspects (interactions make it complicated) with the intrinsically open geometry of the out-of-equilibrium problem.

Those systems are modeled as quantum impurity systems (QIS), that consist in continua of electronic degrees of freedom representing the metallic electrodes (the baths) interacting with the nanostructure (the “impurity”). A generic feature of QIS at equilibrium is that the impurity/bath coupling, no matter how small, has drastic consequence on the groundstate of the system: at temperature \( T = 0 \), properties are described by a strong coupling (SC) fixed point (SC-FP) 8,10, that physically corresponds to the hybridization of the baths and governs the low voltage behavior; for example, for systems with a Fermi liquid SC-FP 11,12, hybridization results in a linear \( I(V) \) characteristic at small voltage, with the conductivity \( G_0 = \frac{\partial I}{\partial V} \mid_{V=0} \) being maximal at \( T = 0 \) and at the particle-hole symmetric point. A typical energy scale, called here the hybridization temperature \( T_b \) (and akin to a Kondo temperature, the scale below which the impurity spin hybridizes to conduction electrons in the Kondo model 13,14), marks the crossover to the SC regime: as soon as \( E/T_b \) acquires a finite value, \( E \) being any energy scale at which the system is probed, additional many-body scattering mechanisms must be taken into account to describe the physics – and, incidentally, non-linear effects. This SC regime, defined by all energy scales being smaller than \( T_b \), is the focus of this work (Fig 1); it is not captured by conventional perturbative expansions, performed around the weak coupling fixed point, that encounter convergence problems when the largest 52 of the ratios \( E/T_b \) becomes \( \lesssim 1 \).

Typically, QIS can be brought to effective one-dimensional systems homogeneous in space except at one point where the interaction with the impurity is concentrated. This one-dimensional character comes along with a realm of powerful methods – be they analytical or numerical – that gives the hope one could solve the out-of-equilibrium problem. Numerical techniques have made significant progresses recently, with the development of efficient time-dependent algorithms. Nevertheless, real-time numerical approaches performed on finite systems like time-dependent DMRG face a difficulty in extrapolating to large sizes or times 14, and methods formulated in infinite systems like diagrammatic Monte Carlo 15 or time dependent NRG 16 run into problems for accessing the stationary state in the SC regime.

On the analytical side, existing methods like e.g. real-time renormalization group (RG) 17,21 or functional RG 21,22 either work far from the SC-FP or rely on an approximation to produce explicit results in the SC regime. Exact results, beside their fundamental interest, would therefore be a precious tool to assess the validity of more general, approximate methods. Since many
QIS are integrable (and thus exactly solvable) at equilibrium, one could hope for a full exact solution out-of-equilibrium. While this is in fact the case for free QIS (by free we mean that the Hamiltonian is quadratic in fermions, a very special subset of integrable QIS) where the Landauer Buttiker formalism applies\cite{24,25}, and also for some interacting QIS that are amenable to free QIS by possibly complicated, non-local transformations\cite{27,28}, this might well be a property of free fermion systems. Indeed, there are to the best of our knowledge only two examples of interacting QIS that could be solved exactly out-of-equilibrium: the Boundary Sine Gordon model, solved by algebraic methods\cite{29} or Thermodynamical Bethe Ansatz (TBA)\cite{30,31}, and the Interacting Resonant Level Model (IRLM) at its self dual point (SD-IRLM)\cite{32}. It is likely that the integrable structure is not preserved when the system is coupled to reservoirs, except in a few exceptional cases. Moreover, the TBA approach is limited to static forcing, and cannot address e.g. the experimentally relevant AC regime.

In this Letter, we are going to use integrability in a weaker sense, by showing that it can provide an analytic address e.g. the experimentally relevant AC regime. The TBA approach is limited to static forcing, and cannot provide an analytic address e.g. the experimentally relevant AC regime.

We shall illustrate this method with the concrete example of the IRLM, for which we carried out this expansion at order $(\epsilon^{-7})$ summing up $\sim 1500$ diagrams using a Mathematica code. This model\cite{38} has recently earned the status of benchmark in out-of-equilibrium physics in QIS\cite{14,32,39,41} as one of the simplest QIS allowing for non equilibrium in the presence of interactions. The IRLM consists of two baths of free spinless fermions coupled via tunnel hopping to a single level, that interacts capacitively with the electrodes via a short range potential with strength $U$. After the standard steps of linearizing around the Fermi points and unfolding\cite{42}, the two semi-infinite wires are described by two right-moving free fermionic fields $\psi(x,t)$ coupled at $x = 0$ to the impurity level (with creation operator $d^\dagger$):

$$H = \sum_a H_a[\psi_a] + H_0, \quad H_0[\psi] = -i\nu \int_{-\infty}^{\infty} dx \psi \partial_x \psi \quad (2)$$

$$H_a = \gamma_a \psi_{a\dagger}(0)d + h.c. + U : \psi^\dagger_{a\dagger}\psi_{a\dagger}\psi^\dagger_\gamma\psi_{\gamma\dagger}(0)(d^\dagger d - \frac{1}{2}) + \epsilon_d d^\dagger d,$$

A sum over $a = 1, 2$ is implied in $H_a$, and in the following we set the Fermi velocity $\nu = 1$ as well as $e = h = k_B = 1$. For $U = 0$, one recovers a free theory, the Resonant Level Model (RLM). In the interacting model, standard manipulations (see e.g. Ref.\cite{41}) show that $U \neq 0$ gives the tunneling term an anomalous scaling dimension $D(U)$. In the half-filled lattice regularized IRLM, used e.g. in numerical simulations, one has\cite{52}

$$D(U) = \frac{1}{4} + \left(\frac{\tan^{-1}\epsilon_d - \epsilon_d}{2\epsilon_d}\right)^2.$$  

The free theory (RLM) corresponds to $D = \frac{3}{2}$, whereas $D = \frac{1}{4}$ corresponds to the self-dual point (SD-IRLM). In the following we restrict our attention to the interesting situation $D < 1$ where the tunneling term is relevant; in this case the tunneling amplitude $\gamma$ (one sets $\gamma_1 + i\gamma_2 = i\nu \epsilon_d^{\theta/2}$) flows to SC under the renormalization group and reaches the value 1 at an energy scale $T_h/W \sim (\gamma/\sqrt{W})^{1/(1-D)}$ (with $W$ the bandwidth) below this scale, the system enters the SC regime (Fig.1). $T_h$ is a non-universal quantity, that we fix via the charge susceptibility, $\chi = \frac{\partial \langle \hat{d}\hat{d}^\dagger \rangle}{\partial \epsilon_d} |_{\epsilon_d = 0} = \frac{2\pi D T_h}{\nu}$.

The physics in the SC regime is governed by the SC-FP,
that is reached at vanishing energy $E_T \to 0$, or by formally setting $T_b = \infty$ (in particular this implies $\frac{\epsilon}{T_b} = 0$: the SC-FP is particle-hole symmetric).

(i) Out-of-equilibrium SC fixed point. The SC-FP has conformal invariance implying it can be described by transparent fields $\Psi_a$, that are free fields of an homogeneous system (physically, homogeneity reflects the wires' perfect hybridization). The effect of the impurity is encoded in the relationship between densities of transparent $(Q_{ab} \equiv \langle \Psi_a \Psi_b \rangle)$ and of original $(\bar{Q}_{ab} \equiv \langle \bar{\psi}_a^\dagger \bar{\psi}_b \rangle)$ fields: in the incoming region $x < 0$, $Q_{ab}(x,t) = \bar{Q}_{ab}(x,t)$, whereas for outgoing fields $x > 0$, $Q_{ab}(x,t) = B_{sc}^e Q_{cd}(x,t)$.

We force the system out-of-equilibrium by considering a bath with thermodynamical variables $(\mu_a, T_a)$ that specifies the density matrix for incoming states in wire $a$. Since the system is homogeneous for transparent fields, we readily deduce the density matrix at the SC-FP of equilibrium $\rho_{n,eq}^{sc} = \rho_{n,sc}^{sc-1} \otimes \rho_{n,sc}^{sc-2}$. Since the system is homogeneous for transparent fields, for a generic operator one has maps the out-of-equilibrium theory onto an equilibrium $\langle \cdot \rangle$. A gauge transformation $\Psi = \exp \oint dx \mu_a(\bar{\psi}_a^\dagger \psi_a)$, with $\langle \cdot \rangle$.

In other words, the Hershfield's operator $[35]$ at the SC-FP is $\int dx \mu_a(\bar{\psi}_a^\dagger \psi_a)$. This seemingly simple operator becomes a complicated, non-local object that mixes the wires once reexpressed in terms of the physical fields $\psi_a$. Our Eq. $[33]$ generalizes an expression obtained earlier for $\hbar = \pi / 2$, $\mu_a = 0$ and $T_1 \neq T_2$ $[43]-[45]$. Besides, we generalize this approach for time-dependent forcing $\mu_a(t)$ $[55]$. As far as properties close to the impurity are concerned, one can equivalently couple the system to a space-varying chemical potential $[55]$ and replace $\mu_a(t)$ in $[33]$. Those “right-moving” potentials can be absorbed by a gauge transformation $\Psi_a(z) \to \Psi_{n,eq}^{-1}(z) \cdot \Psi_a(z)$ that maps the out-of-equilibrium theory onto an equilibrium one: for a generic operator one has $C_{\{\Psi_a\}}|_{n,eq} = \langle C_{\{\Psi_{n,eq} \} \Psi_a}\rangle_{eq}$, with $\langle \cdot \rangle_{eq}$ is the equilibrium average with density matrix $\rho_{n,eq}^{sc} = \bigotimes_a e^{-\mu_a(\bar{\psi}_a^\dagger \psi_a)}$. $\Psi_{n,eq}^{sc}$ can be written as:

$$\Psi_{n,eq}^{sc}(z) = \mathcal{R} e^{-\frac{\mu}{2} dz \Xi_a(z)} Q_{aa}(w)$$

where $\mathcal{R} = \int dz_0 \cdots dz_n X(z_0) \cdots X(z_n)$ is a radially ordered (around $z$) exponential with contours $\mathcal{C}_k \{z_k - z \} > |z_k - z| > 0$ and where $X_a(z,t) = \int_0^\infty dz' \mu_a(t')$ is continued to the complex plane $z = i(t-x)$.

Eq. $[33]$ proves that the forcing out of equilibrium at the SC-FP amounts to a deformation of the conformal field theory generated by the $U(1)$ charges $Q_{aa}$, just as finite temperature effects are generated by the energy-momentum tensor. Eqs. $[33]$ straightforwardly generalize to arbitrary SC-FP with charges $Q_{ab}$ conserved in the bulk – including non Fermi liquid fixed points. At the SC-FP, this gives a precise answer to the sometimes considered question whether ”voltage and temperature play similar roles” in QIS: they corresponds to different deformations of the same CFT describing the SC-FP.

On fermions, $\Psi_{n,eq}$ generates a phase $\mathcal{U}_{n,eq}^{-1}(z) \cdot \Psi_a(z)$. Starting from the current operator on wire a close to the impurity, $\hat{I}_n = Q_{aa}(z^-) - Q_{aa}(z^+)$, and acting on it with $\Psi_{n,eq}$ one recovers for the IRLM $[51]$ the linear regime $\mathcal{I}_{sc} = \langle \hat{I}_n \rangle_{n,eq} = G_0 V(t)$ with $G_0 = \exp^2 \rho^2 / \hbar$ and $V = \mu_1 - \mu_2$ the voltage.

(ii) Around the SC fixed point. We now consider finite values of $\rho_T$, $\frac{T_b}{T_1}$, $\frac{T_2}{T_1}$, that drive the IRLM away from the SC-FP. Inhomogeneity reappears: transparent fields $\Psi_a$ undergo scattering at $x = 0$, and we should consider all irrelevant processes allowed by symmetries. The IRLM is a Fermi liquid, so the lowest order process is an energy-momentum tensor, call it $\mathcal{O}_2$, with coupling $\propto \frac{1}{\hbar}$. It determines the back-scattered current $\mathcal{I}_{sc} = \langle \hat{I}_n \rangle \propto \sqrt{V / T_2}$ at lowest order: this is Fermi Liquid Theory. At higher orders $T_b^{-n}$ (e.g. to compute the first finite temperature correction $\propto T^3 V / T_3$ to $\mathcal{I}_{sc}$), $\mathcal{O}_2$ is not sufficient and new higher order processes must be considered.

In general one faces here several difficulties, that prevent any practical calculations: the number of independent processes (and corresponding couplings) grows very rapidly with $n$; moreover there is no way to fix the couplings $[55]$. On the contrary, the IRLM is integrable, and its infinity of conserved quantities $O_{2n}$ highly constrains the allowed processes. It has been shown $[40]$ that a dual description of the Hamiltonian can be derived: $H = \mathcal{H}_{0}^{sc} + \mathcal{H}_{\infty}^{sc}$ with $\mathcal{H}_{0}^{sc} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{T_n} O_{2n}(x = 0)$ that the couplings $g_{2n}$ are known. The operators $O_{2n}$ are the only independent allowed processes, and transfer integer charges across the impurity: we call such a system a “super Fermi liquid” which is a sufficient condition for our expansion.

We bring this dual description out-of-equilibrium and we obtain a systematic expansion for arbitrary correlators $G(t_1, ..., t_n)$ of local operators (expressible intern of the charges $Q_{ab}$). To do so, we evaluate $G$ within a Keldysh expansion in powers of $T_b^{-1}$, starting at time $t = -\infty$ at the out-of-equilibrium SC-FP $T_b = \infty$ and adiabatically turning on a finite $T_b^{-1}$ value. Using a super-operator formulation $[41]$, we can show $[51]$ a fact that in a super Fermi liquid the full Keldysh expansion can be implemented via the super-operator:

$$\mathcal{U}_0 = \mathcal{R} e^{-\frac{1}{2} dw H_{0}^{sc}(w)}$$

Short-distance divergences in the expansion are exactly canceled by the regularization (point splitting) inherited from integrability, even out-of-equilibrium. This proves formula $[41]$ $[58]$, an expression that is finite order by order, and that straightforwardly generalizes to arbitrary super Fermi liquids.

(iii) Predictions. We calculate the effective symmetrized current operator $J_{\text {eff}} = \frac{J_{\text {eff}} J_{\text {eff}}}{2}$ in the IRLM by evaluating at the operator level $[11]$ perturbatively in

$$\mathcal{U}_0 = \mathcal{R} e^{-\frac{1}{2} dw H_{0}^{sc}(w)}.$$
and SD-IRLM limits, as well as finite expressions match the exact results available in the RLM and SD-IRLM limits: it improves on the FL results (up to $T_b^{-2}$). The inset shows the temperature dependence of $\alpha$, the first non-linearity in $\frac{G_{\text{ex}}}{G_0} = \alpha(T) \frac{V^2}{T_B^2} + \mathcal{O}(V^4)$.

$T_b^{-1}$ at unprecedented large order – to do so, we turn the problem into a purely algebraic one \[\text{51}\] and solve it on a computer: this allows us to reach order $T_b^{-7}$ by resuming $\sim 1500$ diagrams.

We first consider a DC bias $\mu_{1/2} = \pm V/2$ and compute the back-scattered current $I_{bs}(V,T) = I_0^{(2)} - (I_0^{(3)} - H_0^{(3)}) = I_0^{(3)} \sum_{ij} a_{ij} V^{i (i+1)/2}$, where the coefficients $a_{ij}(D,\theta)$ are pure numbers which we obtain exactly. Our expressions match the exact results available in the RLM and SD-IRLM limits, as well as finite $T$ corrections in the linear regime \[\text{41}\]. The universal ratios $a_{44}/a_{62} = \frac{3-12D-16D^2+12(4D-1)(D-1)}{9(4D-1)}$ and $a_{40}/a_{20} = \frac{3(8D^2-4D+1)+92(2D^2-3D+1)}{5(4D-1)}$ match with results treating interactions at first order in $U$ \[\text{32}\] \[\text{40}\].

Interestingly, we can prove \[\text{51}\] at any order in $T_b^{-1}$ that the back scattered current vanishes at imaginary voltage $V = \pm 2i\pi T$, i.e. the quantity $I_{bs}(V,T) \Bigg|_{T^2+4\pi^2T^2} = 0$ is an analytical function.

The resulting back-scattered non-linear conductance $G_{bs}(V) = \frac{\partial I_{bs}}{\partial V}$ is shown in Fig. 2. We see the excellent agreement (up to $0.7 T_b$) with the exact expressions available in the RLM and SD-IRLM limits: it improves on the FL results (drastically when approaching the SD-IRLM).

We then consider an AC bias $\mu_{1/2}(t) = \pm V/2 \cos \omega t$. In the small voltage regime, the current can be written in terms of the admittance $Y^{(1)}(\omega) = (G + i\omega C)(\omega)$ ($C$ is the capacitance): $I(t) = V \Re \{Y^{(1)}(\omega) e^{i\omega t}\}$, and non-linear effects now lie in the $\omega$-dependence. We show in Fig. 3 the amplitude $|Y^{(1)}|$: it is maximal in the RLM limit. Beyond the $T, \omega \to 0$ limit $C(0,0) = \frac{\epsilon^2 \cos \theta}{4 D T_B h}$ predicted by FL theory, we find a rich dependence on $\omega$ and $T$, in particular $C(\omega=0)$ (Fig. 3a) is significantly affected by temperature in a way that depends on the interactions.

We are also able to compute higher harmonics of the current, defining higher order admittances $Y^{(n)}(\omega) = \lim_{V \to 0} \frac{1}{V} i \int_0^\infty \frac{d}{\epsilon} I(t) e^{-i\omega t} dt$. In the particle-hole symmetric case $\epsilon_d = 0$, the first non linear contribution is the third harmonic at $3\omega$. In Fig. 3b we show how frequency affects the amplitude $|Y^{(3)}|$. The FL predicts a constant $|Y^{(3)}| = \frac{(4D-1) \sin^2 \theta}{2 \pi D}$ whereas our approach reveals that beyond a critical value of the interactions $D < D_c = 0.3039\ldots$, increasing frequency enhances the third harmonic $|Y^{(3)}|$. We also predict a strong enhancement of $|Y^{(3)}|$ at finite $T$ (see Fig. 3d). Breaking particle-hole symmetry with a grid potential $\epsilon_d d^4/d$, $2\omega$ harmonics are generated, and the lowest contribution to the admittance reads $Y^{(2)} = -i g \frac{\cos \theta}{2} \sin \omega t / \omega^2$: it does not depend on the first process $O_2$, and is thus missed by the FL approach.

Conclusions. We have shown that out-of-equilibrium QIS at their SC fixed point can be described by a simple deformation of the underlying conformal field theory describing the equilibrium SC fixed point, in the same spirit as finite temperature can be obtained as a mapping from the plane to the cylinder geometry. Then, in integrable QIS that only allow for integer charge transfer (super Fermi liquids), we have shown that we can build a superoperator mapping the out-of-equilibrium interacting theory onto a free one at equilibrium, yielding controlled, exact analytical results in the whole SC regime for the expansion of the universal scaling function of all physical quantity. We have applied this method to the IRLM, revealing non-linear effects that are not captured by Fermi liquid theory. Our approach furnishes a mean to investigate in a controlled way many other out-of-equilibrium properties of super Fermi liquids, like (finite frequency)
noise, full counting statistics, or non-linear thermal transport. Our approach could also be married with numerical techniques like diagrammatic Monte Carlo, by e.g. appropriately sampling the set of Feynman diagrams contributing to physical quantities. It also raises the question whether it can be generalized to non (super) Fermi liquid integrable QIS.

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[51] Report to the on-line supplementary material.
[52] This is in agreement with the common wisdom that “the largest energy scale E cuts the RG flow”.
[53] t is the lattice hopping. D(U) is sometimes given in the literature[52] with an $U = U_0$ defined in a bosonic version of $\mathcal{H}_d$, the relation in the cut-off scheme associated with abelian bosonization is $U_{d} = \text{arg} (U_{\text{d},0}/4\nu_{d})$, see e.g. Ref. [43].
[54] During the redaction of this manuscript, we became aware of an independent construction of the non-equilibrium density matrix at the SC-FP [52] for time-
independent forcing.

[55] In general, the local chemical potential $\mu_a(t)$ close to the impurity differs from the one imposed far away on the electrodes[36].

[56] The spatial dependence we introduce by this trick is generically extremely weak in the SC regime $\omega \lesssim T_b$, with a wavelength $\gtrsim 1$ m for $T_b \lesssim 100$K.

[57] Only the coupling of the first operator, $\mathcal{O}_2$, can be fixed, since it effectively amounts to a definition of $T_b$.

[58] For incoming fields, we find that $\mathcal{H}_s \cdot \Psi_a(x < 0) = \Psi_a(x < 0)$ as expected from causality.
Supplementary material for:

Out-of-equilibrium properties and non-linear effects for interacting quantum impurity systems in their strong coupling regime

CURRENT AT THE SC FIXED POINT

Here we describe the steps needed to obtain the electrical current at the strong coupling fixed point (SC-FP), i.e. when all physical scales $E = V, T, \epsilon, \omega \ll T_B$, or equivalently when $T_B = \infty$. The electrical current operator $j_a(x)$ on wire $a$ at a point $x < 0$, is obtained through the continuity equation for the charge density $n_a(x) = \dot{\psi}^\dagger_a \psi_a(x) + \psi^\dagger_a \dot{\psi}_a(x)$, and reads $j_a(x) = (\psi^\dagger_a \psi_a(x) - \psi^\dagger_a \psi_a(-x))$. The current is then expressed in terms of the “transparent fields” $\Psi_a$ that do not feel the impurity (i.e. that have trivial boundary condition at the SC fixed point). The boundary conditions obeyed by the original fields at the SC FP are encoded in the relationship between original and transparent fields. They are conveniently expressed in terms of the densities $Q_{ab} : \equiv \psi^\dagger_a \psi^\dagger_b$ and $Q_{ab} : \equiv \Psi^\dagger_a \Psi^\dagger_b$. While incoming densities of transparent and original fields coincide, $Q_{ab}(x < 0) = Q_{ab}(x)$ (this traduces causality, i.e. the fact that at the left of the impurity, right moving fields are not influenced by the impurity), outgoing densities do obey:

$$Q_{ab}(x > 0) = B\theta_{ab} Q_{cd}(x),$$

$$[B(\theta)]_{ab}^{cd} = \cos^2 \theta \sigma^3 \otimes \tau^3 + \cos^2 \theta \sigma^1 \otimes \tau^1$$

$$\pm \sin \theta \cos \theta (\sigma^1 \otimes \tau^3 + \sigma^3 \otimes \tau^1)$$

The B matrix encoding the SC boundary conditions involves Pauli matrices $\sigma (\tau$ respectively) acting on the pair of indices $(a, c)$ (on the pair $(b, d)$ respectively). Taking the limit $x \to 0^-$ yields the current through the impurity in terms of transparent fields,

$$\dot{I}_a = j_a(0^-) = (\dot{\Psi}_a \dot{\Psi}_a - B_{\alpha\alpha}(0)) : \dot{\Psi}_a : (0, t).$$

Then, to force the system out-of-equilibrium at the SC fixed point, we apply $\mathcal{H}_{\text{eq}}$ to obtain the effective current operator $\dot{I}_a^{\text{eff}} = \mathcal{H}_{\text{eq}} \cdot \dot{I}_a$, whose equilibrium average value gives the non-equilibrium average value of the original current operator. The transformation $\mathcal{H}_{\text{eq}}$ mixes descendent fields amongst each other, and we find that the effective current acquires a part proportional to the identity which will give rise to the SC, non equilibrium expectation value

$$\dot{I}_a^{\text{eff}}(t) = \sum_{b,c} \Lambda^{bc}_{a}(t) : \dot{\Psi}_b \dot{\Psi}_c : (0, t) + \frac{\sin^2 \theta}{2\pi} (\mu_1(t) - \mu_2(t))$$

with explicitly

$$\Lambda^{bc}_{a}(t) = \left( \frac{-1}{2} \begin{array}{cc} 2 \sin^2 \theta & \sin 2\theta e^{i(\Xi_1 - \Xi_2)(\mu_1(t) - \mu_2(t))} \end{array} \right).$$

Taking the expectation value w.r.t the thermal density matrix $e^{-\beta H_0[\Psi]}$, we recover the linear regime:

$$I_{\text{sc}}^{\text{eff}} = \langle \dot{I}_a^{\text{eff}}(\tau) \rangle_{\text{sc}} = \frac{\sin^2 \theta}{2\pi} (\mu_1(t) - \mu_2(t)).$$

The current is linear in the bias, irrespective of the temperature (recall that $T_B = \infty$ here: the system has conformal symmetry and the finite temperature simply obtained via the usual mapping to the cylinder geometry, that does not affect the v.e.v. of the current).

AROUND THE SC FIXED POINT

Computation details

Departing from the SC fixed point amounts to setting $T_B < \infty$; the effective current operator now acquires contributions from back-scattering processes: it will become dressed by a cloud of particle-hole excitations. We write the effective current operator

$$\dot{I}_a^{\text{eff}} = \mathcal{H}_{\text{eq}} \cdot \dot{I}_a^{\text{eff,0}},$$

$$\dot{I}_a^{\text{eff,0}} = \mathcal{H}_{\text{a}}, \dot{I}_a,$$

clearly separating (i) on the one hand the effect of the interactions that dress the current operator by a cloud of particle-hole pairs yielding $\dot{I}_a^{\text{eff,0}}$ (this dressing is done at the fixed point), and (ii) on the other hand the boundary conditions imposed on incoming fields (at the far left of the impurity), i.e. incoming density matrices representing free Fermi seas biased by time dependent potentials encoded in an incoming Hershfield operator $Y^{\text{in}} = \int_{-\infty}^{\infty} \cdot \dot{\Psi}_a : \dot{\Psi}_a :$.

We expand the current

$$\dot{I}_a^{\text{eff,0}} = \sum_{n \geq 0} T^{-n} \dot{I}_a^{(n)}$$

by expanding Eq. (1) in the main text, i.e. by taking systematically into account the perturbation $H_B^{\text{eff}} =$
\[
\sum_{n=1}^{\infty} \frac{g_{2n}}{n^2} O_{2n}(x = 0) \quad \text{where the couplings read} 2, 3:
\]
\[
g_{2n} = \frac{(D)_{n-1}}{(n - \frac{1}{2}) n!} \Gamma \left( \frac{2n-1}{2(1-D)} \right) \Gamma \left( \frac{D}{2(1-D)} \right) \frac{2n-1}{2} \Gamma \left( \frac{2(n-1)D}{2(1-D)} \right)
\]
whereas the operators \(O_{2n}\) are the conserved quantities of the sine Gordon model and can be built out of an elementary field, a modified stress energy tensor \(O_2\). The first members of the series explicitly read \(2, 3\):
\[
O_2 = (Q^2) - \alpha_0 \partial Q_+ \quad \alpha_0 = \frac{1 - D}{\sqrt{D}}, \quad c = 1 - 6 \alpha_0^2
\]
\[
O_4 = (O_2 O_2)
\]
\[
O_6 = (O_2 O_4) + \frac{c + 2}{12} (O_2 \partial^2 O_2)
\]

The modified stress energy tensor is built out of the U(1) charge \(Q_+ = M_{ab} : \Psi^+_a \Psi^+_b :)\), with
\[
M = \frac{\pi}{\sqrt{4D}} \left( \sqrt{4D - 1 + \cos \theta} \frac{\sin \theta}{\sqrt{4D - 1 - \cos \theta}} \right).
\]

The perturbative expansion can be explicitly carried out at the level of operators. At each order \(\hat{I}^{(n)}\) bears contributions from all patterns \(\hat{P}^n\) of insertions of perturbing operators \(O_{2n_i}\), labelled by \(n = \{n_1, n_2, ..., n_k\}\) with the constraint \(\sum_i (2n_i - 1) = n\). Those operators are inserted along Keldysh contours. Using a super operator formulation \(3\) of the Keldysh expansion, we write the effective current operator as:
\[
\hat{I}^{(n)}(t) = \sum_{n'} \{C_n \prod_{i=1}^{n'} g_{2n_i} \sum_{bc} A_{bc}^{(n)}(t) \hat{P}_{bc}^{(n)}(t)
\]
\[
\hat{P}_{22}^{(n)}(t) = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 ... \int_{-\infty}^{t_{k-1}} dt_k \left[ O_{2n_1}(t_1), [O_{2n_2}(t_2), ... [O_{2n_k}(t_k), : \Psi^+_a \Psi^+_b :, (t)]...]\right]
\]

Here the symbol \(\sum'\) means that it is restricted to \(n\)'s satisfying the constraint \(\sum_i n_i = \frac{n+k}{2}\), and \(C_n\) is a combinatoric factor.

The binary operation "\(*\)" is in general a complicated, non-associative, non-commutative and non-local operation in the space of local operators of the CFT describing the SC-FP. In a super Fermi liquid, the perturbing operators \(O's\) are integer fields (specifically and technically, Kac Moody descendents of the identity operator \(1\), specifically and physically, fields built out of the fermion fields \(\Psi_a, \Psi^+_a\)). It results that nested commutators of \(H^{sc}_{ab}\) with \(\Psi^+_a \Psi^+_b\) can be represented as contour integrals (here we use the crucial property that the operator product expansion \(H^{sc}_{ab}(z) A(w) = \sum_k (H^{sc}_{ab} A)_k (z - w)^k\) only has integer exponents \(k\) (i.e. \(H^{sc}_{ab} \text{ and } A \text{ are mutually local operators}) if \(A\) is a fermionic bilinear operator : \(\Psi^+_a \Psi^+_b\);, and this property holds recursively, i.e. replacing \(A\) by \(H^{sc}_{ab} A\) etc...).

In a super Fermi liquid, the operation "\(*\)" on the space of local operators of the theory thus takes a simple and compact form:
\[
O \ast \Phi(z) = \oint dw \text{ O}(w) \Phi(z) = 2i\pi \left\{ O \Phi \right\}_1(z).
\]

so that the whole Keldysh expansion can be formally resumed (see Fig \(3\) by defining the effective operators:
\[
\hat{A}^{eff, 0}(z) = \Phi_0(z) \cdot \hat{A}(z) = \mathcal{R} e^{-\frac{i}{\hbar} \int dw H^{sc}_{cb} \ast \hat{A}(z)}
\]
\[
= \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \left( \begin{array}{c} H^{sc}_{ab} \ast H^{sc}_{cb} \ast \cdots \ast H^{sc}_{eb} \ast \hat{A}(z) \end{array} \right) \]

Being commuting conserved quantities, the operator product expansion (OPE) \(3\) between two \(O's\) has the property that the (operator-)coefficient of the term \((z - w)^{-1}\) is a total derivative, resulting in the operation \(\ast\) being associative and commutative; this simplifies the combinatorics of the expansion.

The determination of \(\hat{I}^{eff, 0}\) becomes a purely algebraic problem: at order \(T^{-n}\), \(\hat{I}^{(n)}\) lives in the space \(\mathcal{E}_{n+1}\) of Kac Moody descendents fields \(1\) of the identity operator with conformal weights \(\leq n + 1\), and all one needs to known is how to actually evaluate \(15\), or equivalently, using \(17\), the coefficient in \((z - w)^{-1}\) in the OPE
\[
\mathcal{O}(z) \Phi(w) = \sum_n \left( \begin{array}{c} \mathcal{O}(z) \Phi(w) \end{array} \right) \frac{1}{(z-w)^n} \text{ for an arbitrary field } \Phi \text{ in } \mathcal{E}_n.
\]

This can be done recursively by using the elementary OPE \(\Psi^+_a(z) \Psi^+_b(w) = \frac{\delta_{a,b}}{2\pi(z-w)} + \text{ regular}. \) The space \(\mathcal{E}_n\) is spanned by elementary operators \(\hat{F}_{\text{p}, \text{p'}, \text{q}, \text{q'}}\) that simply correspond to all physical processes transferring electrons from one wire to another at the impurity site, with the simultaneous emission of an arbitrary number of particle-
\[
\sum_{n=0}^{\infty} (-i)^n \int_{C_K} (\prod_{j=1}^{n} dt_j) \frac{\mathcal{A}_1 \mathcal{A}_2}{t - \mathcal{A}_1^\prime \mathcal{A}_2^\prime} = \sum_{n} \frac{(-i)^n}{n!} (\prod_{j=1}^{n} \int dt_j) \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}_1^{\text{eff,0}} \mathcal{A}_2^{\text{eff,0}}}
\]

FIG. 4: Effective operators in super Fermi liquids. The symbols \( t_i \) indicate insertions of the operator \( H_n^\text{sc}(t_i) \). The Keldysh expansion of a \( N \)-point correlator (here \( N = 2 \)) on top is formally resumed by introducing effective, dressed operators defined in (18), on the bottom. In the intermediate step, this correlator is written an exponential of nested (radiative operations defined in (18), on the bottom. In the intermediate step, this correlator is written an exponential of nested (radiatively ordered) contour integrals.

hole pairs:

\[
\hat{\mathcal{F}}_{\vec{p},\vec{p}^\prime,\vec{q},\vec{q}^\prime} = \hat{\mathcal{F}}_{\{p_1,\ldots,p_m,\bar{p}_1,\ldots,\bar{p}_m,q_1,\ldots,q_m,\bar{q}_1,\ldots,\bar{q}_m\}} = \left(\partial^{p_1} \Psi_1^\dagger \ldots \partial^{p_m} \Psi_1 \right) \left(\partial^{\bar{p}_1} \bar{\Psi}_1^\dagger \ldots \partial^{\bar{p}_m} \bar{\Psi}_1 \right) \left(\partial^{q_1} \Psi_2^\dagger \ldots \partial^{q_m} \Psi_2 \right) \left(\partial^{\bar{q}_1} \bar{\Psi}_2^\dagger \ldots \partial^{\bar{q}_m} \bar{\Psi}_2 \right)
\]

(19)

involving an arbitrary even number \( 2p = m_1 + \bar{m}_1 + m_2 + \bar{m}_2 \) of fermions, with \( p = m_1 + m_2 = \bar{m}_1 + \bar{m}_2 \). Fermi statistics allows one to choose ordered sets \( p_i > \bar{p}_{i+1} \), and similarly for the \( q_i \)’s, the \( \bar{p}_i \)’s, and the \( \bar{q}_i \)’s. The nested parenthesis indicate the normal order product of two local operators \( A(z) \) and \( B(w) \), \( (AB)(w) \equiv \frac{1}{2\pi} \oint_w \frac{dz}{z-w} A(z) B(w) \). Each process (19) is thus an elementary Feynman diagram, and one has to compute two things: (i) the weight of each elementary diagram (19) in the expansion of \( \hat{\mathcal{F}}^{\text{eff,0}} \) and (ii) the out-of-equilibrium thermal expectation value of each elementary diagram \( \mathcal{V}_{\text{eq}} \cdot \hat{\mathcal{F}}_{\vec{p},\vec{p}^\prime,\vec{q},\vec{q}^\prime} \).

The dimension of \( \mathcal{E}_n \) grows as \( \sim n^{-\frac{1}{2}} e^{2\pi \sqrt{n}} \), so actual calculations quickly become extremely tedious, since they e.g. require the computation of the full OPE for two arbitrary operators in \( \mathcal{E}_n \). We resort to a code, a calculator in \( \mathcal{E}_n \), to evaluate the \( \simeq 1500 \) diagrams necessary to reach order \( n = 6 \). Elementary calculations in \( \mathcal{E}_n \) are heavily recursive stemming from the fact that basic operations (like the (normal-ordered) product of two local operators, or more generally the OPE between two local operators) are neither commutative nor associative.

**Explicit dressed current operator**

Here we give explicitly, for illustration, the expression of the dressed (back-scattered) current operator, \( \hat{I}_{\text{ns}}^{\text{eff,0}} = (1 - \mathcal{V}) \cdot \hat{I} \) (we consider the symmetrized current \( \hat{I} = \frac{\hat{I} + \hat{I}^\dagger}{2} \)), on the basis of the set of diagrams (19). Since expressions quickly become cumbersome, we show only the expansion \( \hat{I}_{\text{ns}}^{\text{eff,0}} = \sum_{n \geq 0} T_n^{-n} \hat{I}_{\text{ns}}^{(n)} \) up to \( T_n^{-2} \) and we denote \( C = \cos \theta \)

\[
\hat{I}_{\text{ns}}^{(0)} = 0
\]

(20)

\[
\hat{I}_{\text{ns}}^{(1)} = \frac{i}{4D} \left( 1 - 2C^2 - C \sqrt{4D - 1} \right) \left[ (\partial \Psi_1 \Psi_1^\dagger + (\Psi_1 \partial \Psi_1^\dagger) \right] \\
- \frac{i}{4D} \left( 1 - 2C^2 + C \sqrt{4D - 1} \right) \left[ (\partial \Psi_2 \Psi_2^\dagger + (\Psi_2 \partial \Psi_2^\dagger) \right] \\
+ \frac{i}{4D} \sqrt{1 - C^2} \left( 2C + \sqrt{4D - 1} \right) \left[ (\partial \Psi_1 \Psi_2^\dagger - (\Psi_1 \partial \Psi_2^\dagger) \right] \\
+ \frac{i}{4D} \sqrt{1 - C^2} \left( 2C - \sqrt{4D - 1} \right) \left[ (\Psi_1 \partial \Psi_2^\dagger) - (\Psi_1 \partial \Psi_2^\dagger) \right]
\]

(21)
\[
\hat{I}_{\text{bs}}^{(2)} = \frac{1}{4D} \left( 1 - 2C^2 - C\sqrt{4D - 1} \right) \left[ (\partial^2 \Psi_1^\dagger \Psi_1) + (\Psi_1^\dagger \partial^2 \Psi_1) \right] \\
- \frac{1}{4D} \left( 1 - 2C^2 + C\sqrt{4D - 1} \right) \left[ (\partial^2 \Psi_2^\dagger \Psi_2) + (\Psi_2^\dagger \partial^2 \Psi_2) \right] \\
+ \frac{1}{4D^2} C\sqrt{1 - C^2} \left[ (\partial \Psi_1 \partial \Psi_2^\dagger) - (\partial \Psi_2 \partial \Psi_1^\dagger) \right] \\
+ \frac{1}{16D^2} \sqrt{1 - C^2} \left( 8CD + (2D + 1)\sqrt{4D - 1} \right) \left[ (\partial^2 \Psi_1 \Psi_2^\dagger) - (\partial^2 \Psi_2 \Psi_1^\dagger) \right] \\
+ \frac{1}{16D^2} \sqrt{1 - C^2} \left( 8CD - (2D + 1)\sqrt{4D - 1} \right) \left[ (\Psi_1 \partial^2 \Psi_2^\dagger) - (\Psi_2 \partial^2 \Psi_1^\dagger) \right] \\
+ \frac{1}{8D^2} \sqrt{1 - C^2}(1 - 2D)\sqrt{4D - 1} \left[ (\Psi_1^\dagger \left( \Psi_2^\dagger \left( \partial \Psi_2 \Psi_1 \right) \right)) - \left( \partial \Psi_1^\dagger \left( \Psi_1 \Psi_2 \right) \right) \right] \\
- \left( \partial \Psi_1^\dagger \left( \Psi_1 \Psi_2 \right) \right) + \left( \Psi_1^\dagger \left( \partial \Psi_1 \left( \Psi_1^\dagger \Psi_2 \right) \right) \right) - \left( \Psi_1 \left( \partial \Psi_2 \left( \Psi_2^\dagger \Psi_2 \right) \right) \right) \\
+ \frac{1}{4D^2} \left( C^2 - 2CD\sqrt{4D - 1} + 2D - 1 \right) \left( \partial \Psi_1^\dagger \partial \Psi_2 \right) \\
- \frac{1}{4D^2} \left( C^2 + 2CD\sqrt{4D - 1} + 2D - 1 \right) \left( \partial \Psi_1^\dagger \partial \Psi_1 \right)
\] 

(22)

**Current expansion**

To evaluate the expectation value of the current in the non-equilibrium theory, one needs to act with the super-operator \( \mathbb{H}_{N,\text{eq}} \) on the dressed current \( \hat{I}_{\text{bs}}^{(2)} \); in doing so one generates a component on the identity operator, that directly yields a finite expectation value. We give here the expression of the first terms of the backscattered current \( I_{\text{bs}} = \langle \mathbb{H}_{N,\text{eq}} \cdot \hat{I}_{\text{bs}}^{(2)} \rangle_{\text{tr}} \) for a static applied bias \( V \), at \( T \neq 0 \) and in the resonant case \( \epsilon_d = 0 \).

\[
\frac{I_{\text{bs}}}{I^\text{sc}} = \frac{X(V^2 + 4\pi^2T^2)}{48D^2 T^2} \left[ 1 + \frac{3g_4(8\pi^2T^2X(X - 15) + 3V^2(X^2 - 10X + 5))}{40DT^2} - \frac{8\pi^2T^2(1 + 5X + X^2) + 3V^2(X^2 + 1)}{40DT^2} \right] + \mathcal{O}(T^{-6})
\]

(23)

with \( X = 4D - 1 \), \( I^\text{sc} = (I_1^\text{sc} - I_2^\text{sc})/2 \) and the coupling \( g_4 \) is given by Eq.(13). One notices that the quantity \( V^2 + 4\pi^2T^2 \) factorizes in the expression of \( I_{\text{bs}}(V, T) \). This property can be shown to hold at arbitrary order in perturbation theory, and can be seen to be a direct consequence of the respective nature of finite voltage and finite temperature from the viewpoint of the deformations of the CFT (SC-FP) they are represented by. Indeed, to evaluate the contribution to the average value of a diagram \( \hat{I}_{\text{p, \bar{p}, a, \bar{a}}} \), where the average in taken in the finite temperature, non-equilibrium SC theory, one needs to first absorb the voltage (one writes \( z = i(t-x) \)) by the change \( \Psi_a(z) \rightarrow e^{i(-z)} V(t-x)/2 \Psi_a(z) \), and then the finite temperature effect by a mapping of the cylinder to the plane \( z \rightarrow \omega = e^{2i\pi T z} \). \( \Psi_a(z) \rightarrow \sqrt{2i\pi T} e^{i\pi T z} \Psi_a(\omega) \). All together, when choosing \( V = 2i\pi T \) (when \( V = -2i\pi T \), the role of \( \Psi_1 \) and \( \Psi_2 \) are exchanged) the fermions are changed according to:

\[
\Psi_1(z) \rightarrow \sqrt{2i\pi T} \Psi_1(\omega) \\
\Psi_1^\dagger(z) \rightarrow \sqrt{2i\pi T} \Psi_1^\dagger(\omega) e^{Vz} \\
\Psi_2(z) \rightarrow \sqrt{2i\pi T} \Psi_2(\omega) e^{Vz} \\
\Psi_2^\dagger(z) \rightarrow \sqrt{2i\pi T} \Psi_2^\dagger(\omega)
\]

Since \( \Psi_1 \) and \( \Psi_2^\dagger \) are untouched (apart from a constant prefactor), it is easy to convince oneself by looking at Eqs.(21,22), and further at arbitrary order in perturbation theory, that the identity operator cannot be in the back-scattered current under the action of the finite voltage and finite temperature deformations, so that \( I_{\text{bs}}(\pm 2i\pi T, T) = 0 \).

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