Robust large-margin learning in hyperbolic space

Melanie Weber
Manzil Zaheer
Ankit Singh Rawat
Aditya Menon
Sanjiv Kumar

1Princeton University
2Google Research

Abstract

Recently, there has been a surge of interest in representation learning in hyperbolic spaces, driven by their ability to represent hierarchical data with significantly fewer dimensions than standard Euclidean spaces. However, the viability and benefits of hyperbolic spaces for downstream machine learning tasks have received less attention.

In this paper, we present, to our knowledge, the first theoretical guarantees for learning a classifier in hyperbolic rather than Euclidean space. Specifically, we consider the problem of learning a large-margin classifier for data possessing a hierarchical structure. Our first contribution is a hyperbolic perceptron algorithm, which provably converges to a separating hyperplane. We then provide an algorithm to efficiently learn a large-margin hyperplane, relying on the careful injection of adversarial examples. Finally, we prove that for hierarchical data that embeds well into hyperbolic space, the low embedding dimension ensures superior guarantees when learning the classifier directly in hyperbolic space.

1 Introduction

Hyperbolic spaces have received sustained interest in recent years, owing to their ability to compactly represent data possessing hierarchical structure (e.g., trees and graphs). In terms of representation learning, hyperbolic spaces offer a provable advantage over Euclidean spaces for such data: objects requiring an exponential number of dimensions in Euclidean space can be represented in a polynomial number of dimensions in hyperbolic space [Sarkar, 2012]. This has motivated research into efficiently learning a suitable hyperbolic embedding for large-scale datasets [Nickel and Kiela, 2017, Chamberlain et al., 2017, Tifrea et al., 2019].

Despite this impressive representation power, little is known about the benefits of hyperbolic spaces for downstream tasks. For example, suppose we wish to perform classification on data that is intrinsically hierarchical. One may naively ignore this structure, and use a standard Euclidean embedding and corresponding classifier (e.g., SVM). However, can we design classification algorithms that exploit the structure of hyperbolic space, and offer provable benefits in terms of performance (e.g., predictive accuracy)? This fundamental question has received surprisingly limited attention. Indeed, while some prior work has proposed specific algorithms for learning classifiers in hyperbolic space [Cho et al., 2019, Monath et al., 2019], these have been primarily empirical in nature, and do not come equipped with theoretical guarantees on convergence and generalization.

In this paper, we take a first step towards addressing this question for the problem of learning a large-margin classifier. We provide a series of algorithms for provably learning such classifiers in hyperbolic space, and establish that these can be superior to classifiers learned in naive Euclidean space. This shows that by using a hyperbolic space that better reflects the intrinsic geometry of the data, one can see gains in both representation size and performance.

Specifically, our contributions are as follows:

(i) we provide a hyperbolic version of the classic perceptron algorithm and establish its convergence for data that is separable with a margin (Theorem 3.1).
(ii) we then establish how suitable injection of adversarial examples to gradient-based loss minimization yields an algorithm which can efficiently learn a large-margin classifier (Theorem 4.1). We further establish that simply performing gradient descent or using adversarial examples alone does not suffice to yield such a classifier efficiently.

(iii) we compare the Euclidean and hyperbolic approaches for hierarchical data and analyze the trade-off between low embedding dimensions and low distortion (dimension-distortion trade-off) when learning robust classifiers on embedded data. For hierarchical data that embeds well into hyperbolic space, the low embedding dimension is sufficient to ensure superior guarantees when learning the classifier in hyperbolic space.

Contribution (i) establishes that it is possible to design classification algorithms that exploit the structure of hyperbolic space, while provably converging to some admissible (not necessarily large-margin) separator. Contribution (ii) establishes that it is further possible to design classification algorithms that provably converge to a large-margin separator, by suitably injecting adversarial examples. Contribution (iii) shows that the adaptation of algorithms to the intrinsic geometry of the data can lead to significant performance gains.

1.1 Related work

Our results can be seen as hyperbolic analogue of classic results for Euclidean spaces. The large-margin learning problem is well studied in the Euclidean setting. Classic algorithms for learning classifiers include the perceptron [Rosenblatt, 1958, Novikoff, 1962, Freund and Schapire, 1999] and support vector machines [Cortes and Vapnik, 1995]. Robust margin-learning has been widely studied; notably by Lanckriet et al. [2003], El Ghaoui et al. [2003], Kim et al. [2006] and, recently, via adversarial approaches by Charles et al. [2019], Ji and Telgarsky [2018], Soudry et al. [2018]. Adversarial learning has recently gained interest through efforts to train more robust deep learning systems (see, e.g., [Madry et al., 2018, Fawzi et al., 2018]).

Recently, the representation of (hierarchical) data in hyperbolic space has gained a surge of interest. The literature focuses mostly on learning representations in the Poincare [Nickel and Kiela, 2017, Chamberlain et al., 2017, Tifrea et al., 2019] and Lorentz [Nickel and Kiela, 2018] models of hyperbolic space. Sala et al. [2018] analyze representation trade-offs in hyperbolic embeddings.

The body of work on performing downstream ML tasks in hyperbolic space is much smaller and mostly without theoretical guarantees. Monath et al. [2019] study hierarchical clustering in hyperbolic space. Cho et al. [2019] introduce a hyperbolic version of support vector machines for binary classification in hyperbolic space, albeit without theoretical guarantees.

2 Background and notation

We review some background material on hyperbolic spaces, as well as embedding into and learning in these spaces.

2.1 Hyperbolic space

Hyperbolic spaces are smooth Riemannian manifolds $\mathcal{M} = \mathbb{H}^d$ with constant negative curvature $\kappa$ and are as such locally Euclidean spaces. In the following, we introduce basic notation for two popular models of hyperbolic spaces; for a comprehensive overview, see Bridson and Haefliger [1999].

There are several equivalent models of hyperbolic space, each highlighting a different geometric aspect. In this work, we mostly consider the Lorentz model (aka hyperboloid model). For $x, x' \in \mathbb{R}^{d+1}$, let
\[ x \ast x' = x_0 x_0' - \sum_{i=1}^d x_i x_i' \] denote their \textit{Minkowski product}. The Lorentz model is defined as follows:

\[ \mathbb{L}^d = \{ x \in \mathbb{R}^{d+1} : x \ast x = 1 \} \]

\[ d_L(x, x') = \text{acosh}(x \ast x') \]

The distance \( d_L(x, y) \) corresponds to the length of the shortest line (\textit{geodesic}) along the manifold connecting \( x \) and \( x' \) (cf. Fig. 1a). Note that \( (\mathbb{L}, d_L) \) forms a metric space.

Another model of hyperbolic space that we use is the \textit{Poincare half-plane model}, which defines a hyperbolic metric space on the upper half of the complex plane as follows:

\[ \mathbb{P}^2 = \{ x = (x_0, x_1) \in \mathbb{R}^2 : x_1 > 0 \} \]

\[ d_P(x, x') = \text{acosh}\left( 1 + \frac{(x_0' - x_0)^2 + (x_1' - x_1)^2}{2x_1 x_1'} \right) . \]

The Poincare half-plane model can also be defined for higher dimensions. Furthermore, there exists an isometric mapping between both Lorentz model and Poincare half-plane model that preserves the Minkowski product (cf. Appendix A.2).

### 2.2 Embeddability of hierarchical data

A map \( \phi : X_1 \rightarrow X_2 \) between metric spaces \( (X_1, d_1) \) and \( (X_2, d_2) \) is called \textit{embedding}. We measure embeddability using the following \textit{multiplicative} notion of distortion: Define \( c_M \geq 1 \) such that for all \( x, x' \in X_1 \)

\[ d_2(\phi(x), \phi(x')) \leq d_1(x, x') \leq c_M \cdot d_2(\phi(x), \phi(x')) \]

Embeddings with multiplicative distortion \( c_M = 1 \) are termed \textit{isometric embeddings}.

Since hierarchical data is tree-like, we can use classic embeddability results for trees as a reference point. The embeddability of trees has been widely studied in the literature. \cite{bourgain1985} \cite{linial1995} showed that an \( N \)-point metric \( X \) (i.e., \(|X| = N\)) embeds into Euclidean space \( \mathbb{R}^{O(\log^2 N)} \) with the distortion \( c_M = O(\log N) \). This bound is tight for trees in the sense that embedding them in a Euclidean space (of any dimension) must incur the distortion \( c_m = \Omega(\log N) \) \cite{linial1995}. On the other hand, \cite{sarkar2012} showed that trees embed quasi-isometrically with \( c_M = O(1 + \epsilon) \) into hyperbolic space \( \mathbb{H}^d \). This holds even in the low-dimensional regime with the dimension as small as \( d = 2 \).

### 2.3 Classification in hyperbolic space

We consider classification problems of the following form: \( \mathcal{X} \subset \mathbb{L}^d \) denotes the feature space, \( \mathcal{Y} \) the label space and \( \mathcal{W} \subset \mathbb{R}^{d+1} \) the model space. For now, consider a binary classification task, i.e., \( Y = \{\pm 1\} \). In the following, we denote the training set as \( S \subset \mathcal{X} \times \mathcal{Y} \).
We begin by defining geodesic decision boundaries. Consider the Lorentz space \( \mathbb{L}^d \) with ambient space \( \mathbb{R}^{d+1} \). Then every geodesic decision boundary is a hyperplane in \( \mathbb{R}^d \) intersecting \( \mathbb{L}^d \) and \( \mathbb{R}^{d+1} \). Further, consider the set of linear separators or decision functions of the form
\[
\mathcal{H} = \{ h_w : w \in \mathbb{R}^{d+1}, w \cdot w < 0 \},
\]
where
\[
h_w(x) = \begin{cases} 
1, & w \cdot x > 0 \\
-1, & \text{otherwise.}
\end{cases}
\]
Note that the requirement \( w \cdot w < 0 \) in (2.1) ensures that the intersection of \( \mathbb{L}^d \) and the decision hyperplane \( h_w \) is not empty. The geodesic decision boundary corresponding to the decision function \( h_w \) is then given by
\[
\partial \mathcal{H}_w = \{ z \in \mathbb{L}^d : w \cdot z = 0 \}.
\]
The distance of a point \( x \in \mathbb{L}^d \) from the decision boundary \( \partial \mathcal{H}_w \) can be computed as [Cho et al., 2019]
\[
d(x, \partial \mathcal{H}_w) = \left| \text{asinh} \left( \frac{w \cdot x}{\sqrt{-w \cdot w}} \right) \right|.
\]

### 2.4 Large-margin classification in hyperbolic space

In this paper, we are interested in learning a large margin classifier in a hyperbolic space. Analogous to the Euclidean setting, the natural notion of margin is the minimal distance to the decision boundary over all training samples:
\[
\text{margin}_S(w) = \inf_{(x, y) \in S} y h_w(x) \cdot d(x, \partial \mathcal{H}_w)
\]
\[
= \inf_{(x, y) \in S} \text{asinh} \left( \frac{y (w \cdot x)}{\sqrt{-w \cdot w}} \right).
\]
The goal of large-margin classifier learning is then to find a linear separator \( h_{w^*} \) defined by
\[
w^* = \arg \max_w \text{margin}_S(w).
\]

### 3 Hyperbolic linear separator learning

The first step towards the goal of learning a large-margin classifier is to establish that we can provably learn some separator. To this end, we present a hyperbolic version of the classic perceptron algorithm and establish that it will converge on data that is separable with a margin.

#### 3.1 The hyperbolic perceptron algorithm

The hyperbolic perceptron (cf. Algorithm 1) learns a binary classifier \( w \) with respect to the Minkowski product. This is implemented in the update rule
\[
v_t \leftarrow w_t + y x.
\]
In contrast to the Euclidean case, the hyperbolic perceptron requires a normalization step
\[
w_{t+1} \leftarrow \frac{v_t}{\sqrt{-v_t \cdot v_t}}.
\]
Algorithm 1 Hyperbolic perceptron (h-Perceptron)

Initialize $w_0 \in \mathbb{R}^{d+1}$.

for $t = 0, 1, \ldots, T - 1$ do

for $j = 1, \ldots, n$ do

$y' \leftarrow \text{sgn}(x_j \ast w_t)$

if $y' \neq y_j$ then

$v_t \leftarrow w_t + y_j x_j$

if $\sqrt{-v_t \ast v_t} < 1$ then

$v_t \leftarrow \frac{v_t}{\sqrt{-v_t \ast v_t}}$

end if

break

end if

end for

end for

$w_{t+1} \leftarrow v_t$

end for

Output: $w_T$

This ensures that $w_{t+1} \ast w_{t+1} < 0$; as a result $w_{t+1}$ defines a meaningful classifier, i.e., $\mathbb{L}^d \cap \partial \mathcal{H}_{w_{t+1}} \neq \emptyset$. For more details, see Appendix B. While intuitive, it remains to establish that this algorithm converges, i.e., finds a solution which correctly classifies all the training samples. To this end, consider the following notion of hyperbolic linear separability with a margin: for $X, X' \subseteq \mathbb{L}^d$, we say that $X$ and $X'$ are linearly separable with margin $\gamma$, if there exists a $w \in \mathbb{R}^{d+1}$ with $\sqrt{-w \ast w} = 1$ such that

$w \ast x > \gamma' \quad \forall \ x \in X \quad \text{and} \quad w \ast x' < -\gamma' \quad \forall \ x' \in X'$,

where $\gamma' = \text{asinh}(\gamma)$. Assuming our training set is separable with a margin, the hyperbolic perceptron has the following convergence guarantee.

Theorem 3.1. Assume that there is some $\bar{w} \in \mathbb{R}^{d+1}$ with $\sqrt{-\bar{w} \ast \bar{w}} = 1$ and $w_0 \ast \bar{w} \geq 0$, and some $\gamma > 0$, such that $y_j(\bar{w} \ast x_j) \geq \sinh(\gamma)$ for $j = 1, \ldots, |S|$. Then, Algorithm 1 converges in $O\left(\frac{1}{\sinh(\gamma)}\right)$ steps and returns a solution with margin $\gamma$.

The proof of Theorem 3.1 follows the standard proof of the Euclidean perceptron and utilizes the Cauchy-Schwartz inequality for the Minkowski product. We defer the detailed proof of Theorem 3.1 to Appendix B.

Remark 3.2. Recall that the classic guarantee for the perceptron algorithm in Euclidean space establishes a $O(1/\gamma^2)$ convergence rate Novikoff [1962]. When $\gamma \sim 0$, the $\frac{1}{\sinh(\gamma)}$ convergence rate for hyperbolic spaces can be significantly faster than $\frac{1}{\gamma^2}$, indicating that exploiting the structure of hyperbolic space can be beneficial.

3.2 The challenge of large-margin learning

Theorem 3.1 establishes that the hyperbolic perceptron converges to some linear separator. However, for the purposes of generalization, one would ideally like to converge to a large-margin separator. As with the classic Euclidean perceptron, no such guarantee is possible for the hyperbolic perceptron; this motivates us to ask whether a suitable modification can rectify this.

Drawing inspiration from the Euclidean setting, a natural way to proceed is to consider the use of margin losses, such as the logistic or hinge loss. Formally, let $l: \mathcal{X} \times \{\pm 1\} \rightarrow \mathbb{R}_+$ be a loss function of the form

$l(x, y; w) = f(y \cdot (w \ast x))$, (3.1)
Algorithm 2 Adversarial Training

1: Initialize \( w_0 = 0, S' = \emptyset \).

2: for \( t = 0, 1, \ldots, T \) do

3: \( S_t \sim S \text{ iid with } |S_t| = m; S'_t \leftarrow \emptyset \).

4: for \( i = 0, 1, \ldots, m \) do

5: \( \tilde{x}_i \leftarrow \text{argmax}_{x \in X} d_L(x, z) \leq \alpha \)

6: end for

7: \( S'_t \leftarrow \{(\tilde{x}_i, y_i)\}_{i=1}^m \)

8: \( S' \leftarrow S' \cup S'_t \)

9: \( w_{t+1} \leftarrow A(w_t, S, S') \)

10: end for

where \( f : \mathbb{R} \rightarrow \mathbb{R}_+ \) is some convex, non-increasing function, e.g., the hinge loss. The empirical risk of the classifier parametrized by \( w \) on the training set \( S \subset X \times \{-1,1\} \) is

\[
L(w, S) = \frac{1}{|S|} \sum_{(x,y) \in S} l(x, y; w) .
\]

A common strategy to learn a suitable classifier is to minimize this empirical risk via gradient descent, which for learning rate \( \eta > 0 \) generates iterates

\[
w_{t+1} \leftarrow w_t - \frac{\eta}{|S|} \sum_{(x,y) \in S} \nabla l(x, y; w_t) .
\] (3.2)

Unfortunately, while this will yield a large-margin solution, the following result demonstrates that the number of iterations required may be prohibitively large.

Theorem 3.3. Let \( e_i \in \mathbb{R}^{d+1} \) be the \( i \)-th standard basis vector. Consider the training set \( S = \{(e_1, 1), (e_1, -1)\} \) and the initialization \( w_0 = e_2 \). Suppose \( \{w_t\}_{t \geq 0} \) is a sequence of classifiers generated by the gradient descent updates (cf. (3.2)). Then, the number of iterations needed to achieve margin \( \gamma \) is \( \Omega(\exp(\gamma)) \).

While this result appears disheartening, fortunately, we now present a simple resolution: by suitably adding adversarial examples, the solution found by the gradient descent converges to a large-margin solution in polynomial time.

4 Hyperbolic large-margin separator learning via adversarial examples

Theorem 3.3 reveals that gradient descent on a margin loss is insufficient to efficiently obtain a large-margin classifier. Adapting the approach proposed in Charles et al. [2019] for the Euclidean setting, we now show how to alleviate this problem by enriching the training set with adversarial examples before updating the classifier (cf. Algorithm 2). In particular, we minimize a robust loss of the following form:

\[
\min_{w \in \mathbb{R}^{d+1}} L_{\text{rob}}(w; S) := \frac{1}{|S|} \sum_{(x,y) \in S} l_{\text{rob}}(x, y; w) ,
\] (4.1)

\[
l_{\text{rob}}(x, y; w) := \max_{z \in \mathbb{R}^{d+1}} d_L(x, z) \leq a l(z, y; w) .
\] (4.2)

The inner optimization defined in (4.2) generates an adversarial example by perturbing a given input feature \( x \) on the hyperbolic manifold. Note that the magnitude of the perturbation added to the original example is bounded by \( a \), which we refer to as the adversarial budget. In particular, we want to construct

...
a perturbation that maximizes the loss, i.e., \( \hat{x} \leftarrow \arg \max_{d(x, z) \leq a} l(z, y; w) \). We will discuss how to efficiently generate the adversarial example \( \hat{x} \) in §4.3.

The outer optimization (cf. (4.1)) can be achieved by an iterative optimization procedure, which generates a sequence of classifiers \( \{w_t\} \). We update the classifier \( w_t \) according to an update rule \( A \), which accepts as input the current estimate of the weight vector, the original training set, and an adversarial perturbation of the training set. The update rule produces as output a weight vector which approximately minimizes the robust loss \( L_{\text{rob}} \) in (4.1).

We now establish that for a gradient based update rule, the above adversarial training procedure will efficiently converge to a large-margin solution. Table 1 summarizes the results of this section and compares with the corresponding results in the Euclidean setting [Charles et al., 2019].

### 4.1 Fast convergence via gradient-based update

Consider Algorithm 2 with \( A(w_t, S, S'_t) \) being a gradient-based update with learning rate \( \eta_t > 0 \):

\[
\begin{align*}
w_{t+1} &\leftarrow w_t - \frac{\eta_t}{|S'_t|} \sum_{(x, y) \in S'_t} \nabla_w l(\hat{x}, y; w_t), \\
\hat{w}_{t+1} &\leftarrow \frac{w_{t+1}}{\nabla - w_{t+1} \cdot w_{t+1}}.
\end{align*}
\]

(4.3a)

(4.3b)

Note that the normalization is performed to ensure that the update remains valid, i.e., \( L^d \cap \partial H_w \neq \emptyset \).

To compute the update, we need to compute gradients of the outer minimization problem, i.e., \( \nabla_w L_{\text{rob}} \) over \( S'_t \) (cf. (4.1)). However, this function is itself a maximization problem (cf. (4.2)). We therefore compute the gradient at the maximizer of this inner maximization problem. Danskin’s theorem [Danskin, 1966, Bertsekas, 2016] ensures that this gives a valid descent direction. We defer details to Appendix C.4.

As shall be detailed in §4.3, we can solve the inner maximization in closed form as:

\[
\begin{align*}
L_{\text{rob}}(x, y; w) &= \max_{d(x, z) \leq a} l(z, y; w) = l(\breve{x}, y; w) \\
&\text{with } \breve{x} = \left( \breve{x}_0, \sqrt{\breve{x}_0^2 - 1} \left( b \breve{x} + \sqrt{1 - b^2 \breve{x}^\perp} \right) \right),
\end{align*}
\]

where \( b, \breve{x}, \) and \( \breve{x}^\perp \) are defined in Theorem 4.4.

\[
\nabla_w l(\breve{x}, y; w) = f'(y(w \ast \breve{x})) \cdot \nabla_w y(w \ast \breve{x}) = f'(y(w \ast \breve{x})) \cdot \breve{y} \breve{x}^T,
\]

where \( \nabla_w y(w \ast \breve{x}) = \hat{y} \breve{x}^T = y(\breve{x}_0, -\breve{x}_1, \ldots, -\breve{x}_n)^T \). With Danskin’s theorem, \( \nabla l(\breve{x}, y; w) \in \partial L_{\text{rob}}(x, y; w) \), from which we can compute the descent direction and therefore the update step.

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| Space                        | Perceptron | adversarial margin | adversarial ERM | adversarial GD |
|------------------------------|------------|--------------------|-----------------|----------------|
| Euclidean (prior work)       | \( O\left( \frac{1}{\delta} \right) \) | \( \delta - a \) | \( \Omega(\exp(d)) \) | \( \Omega\left( \frac{\exp(\Omega)}{\sinh(\gamma)} \right) \) |
| Hyperbolic (this paper)      | \( O\left( \frac{1}{\sinh(\gamma)} \right) \) | \( \frac{1}{\cosh(\alpha)} \) | \( \Omega(\exp(d)) \) | \( \Omega\left( \frac{\cosh(\alpha)}{\sinh(\gamma)} \right) \) |

Table 1: Sample complexity for Perceptron and fundamental guarantees for adversarial training in both hyperbolic and Euclidean spaces. Recall that \( \gamma, \alpha, \) and \( d \) denote the margin of the training data, the adversarial perturbation budget, and the underlying dimension, respectively.
4.1.1 Convergence analysis

We now establish that the above gradient-based update converges to a large-margin solution in polynomial time. For this analysis, we need the following assumptions:

Assumption 1. 1. The training set $S$ is linearly separable, i.e., there exists a $\bar{w} \in \mathbb{R}^{d+1}$, such that $y(\bar{w} \cdot x) \geq 0$ for all $(x,y) \in S$.
2. There exists some constant $R_x \geq 0$, such that (i) $\|x\| \leq R_x$ and (ii) all possible adversarial perturbations remain within this constraint, i.e., $\|\tilde{x}\| \leq R_x$. Furthermore, we assume $\|w\| \leq R_w$ for some constant $R_w$. Let $R_a := R_x R_w$.
3. the function $f(s)$, underlying the loss (cf. (3.1)), has the following properties: (i) $f(s) > 0 \forall s$; (ii) $f'(s) < 0 \forall s$; (iii) $f$ is differentiable, and (iv) $f$ is $\beta$-smooth.

A loss function that fulfills Assumption 1 is the following hyperbolic equivalent of the logistic regression loss:

$$l(x,y;w) = \ln \left( 1 + e^{-\text{asinh} \left( \frac{y(w \cdot x)}{2R_a} \right)} \right),$$

where $R_a$ is as defined in Assumption 1. In the following, we focus on this choice of the loss function. Other loss functions as well as the derivation of the hyperbolic logistic regression loss are discussed in Appendix C.1.

We first show that Algorithm 2 with a gradient update is guaranteed to converge to a large-margin classifier.

**Theorem 4.1.** With constant step size and $A$ being the GD update with an initialization $w_0$ with $w_0 \cdot w_0 < 0$, $\lim_{t \to \infty} L(w_t; S \cup S'_t) = 0$.

The proof can be found in Appendix C.4. While this result guarantees convergence, it does not guarantee efficiency (e.g., by showing a polynomial convergence rate). Next, we quantify the convergence rate of Algorithm 2, showing that the algorithm with a gradient based update computes a max-margin classifier in polynomial time.

**Theorem 4.2 (Convergence rate GD update, Algorithm 2).** For a fixed constant $c \in (0,1)$, let the step size $\eta_t := \eta = c \cdot \frac{2 \sinh^2(\gamma)}{\beta \sigma_{\text{max}} \cosh^2(\alpha) R_a^2}$ with $\sigma_{\text{max}}$ denoting an upper bound on the maximum singular value of the data matrix, and $A$ be the GD update as defined in (4.3a) and (4.3b). Then, the iterates $\{w_t\}$ in Algorithm 2 satisfy the following:

$$L_{\text{rob}}(w_t; S) = O \left( \frac{\sinh^2(\ln(t))}{t} \cdot \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} \right).$$

The proof of Theorem 4.2 and the accompanying auxiliary results are presented in Appendix C.4.

4.2 On the necessity of combining gradient descent and adversarial training

We remark here that the enrichment of the training set with adversarial examples is critical for the polynomial-time convergence. Recall first that by Theorem 3.3, without adversarial training, we can construct a simple max-margin problem that cannot be solved in polynomial time. Interestingly, merely using adversarial examples by themselves does not suffice for fast convergence either.

Consider Algorithm 2 with an ERM as the update rule $A(w_t, S, S')$. In this case, the iterate $w_{t+1}$ corresponds an ERM solution for $S \cup S'$, i.e.,

$$w_{t+1} \leftarrow \arg \min_w \sum_{(x,y) \in S \cup S'} l(x,y;w).$$

(4.5)
Let \( S_t = S \), i.e., we utilize the full power of the adversarial training in each step. The following result reveals that even under this optimistic setting, Algorithm 2 may not converge to a solution with a non-trivial margin in polynomial time:

**Theorem 4.3.** Suppose Algorithm 2 (with an ERM update) outputs a linear separator of \( S \cup S' \). In the worst case, the number of iteration required to achieve a margin at least \( \epsilon \) is \( \Omega(\exp(d)) \).

We note that a similar result in the Euclidean setting appears in [Charles et al., 2019]. We establish Theorem 4.3 by extending the proof strategy of [Charles et al., 2019, Theorem 4] to hyperbolic spaces.

In particular, given a spherical code in \( \mathbb{R}^d \) with \( T \) codewords and \( \theta \sim \sinh(\epsilon) \cosh(\alpha) \) minimum separation, we construct a training set \( S \) and subsequently the adversarial examples \( \{S_i\} \) such that there exists an ERM solution \( w_t \) on \( S \cup S' \) (cf. (4.5)) that has margin less than \( \epsilon \). Now the result in Theorem 4.3 follows by utilizing a lower bound [Cohn and Zhao, 2014] on the size of the spherical code with \( T = \Omega(\exp(d)) \) codewords and \( \theta \sim \sinh(\epsilon) \cosh(\alpha) \) minimum separation. The detailed proof of Theorem 4.3 is presented in Appendix C.5.

### 4.3 Computing adversarial examples

We now return to the issue of efficiently computing the adversarial examples. We consider the \( \mathcal{T} \) with (Euclidean) linear program with a spherical constraint:

\[
\text{Theorem 4.4.} \quad \text{Given the input example } (x, y), \text{ let } x'_{\ell, 0} = (x_1, \ldots, x_d). \text{ We can efficiently compute a solution to } \text{CERT} \text{ or decide that no solution exists. If a solution exists, then based on a guess of } z_0 \text{ a maximizing adversarial example has the form } \tilde{x} = (z_0, \sqrt{z_0^2 - 1} \left( b \tilde{x} + \sqrt{1 - b^2} \tilde{x}_\perp \right)). \text{ Here, } b = \frac{\cosh(\alpha) - x_0 z_0}{\sqrt{z_0^2 - 1} \sqrt{z_0^2 - 1}} \text{ depends on the adversarial budget } \alpha, \text{ and } \tilde{x}_\perp \text{ is a unit vector orthogonal to } \tilde{x} = -x_{\ell, 0} / \|x_{\ell, 0}\|.
\]

The proof of Theorem 4.4 is presented in Appendix C.2.

**Remark 4.5.** Note that according to Theorem 4.4, it is possible that, for a particular guess of \( z_0 \), we may not be able to find an adversarial example \( \tilde{x} \) that leads to a prediction that is inconsistent with \( x \), i.e., \( h_w(x) \neq h_w(\tilde{x}) \). Thus, for some \( t \), we may have \( |S_t'| < m \) in Algorithm 2.
5 Dimension-distortion trade-off

So far we have focused on classifying data that is given in either Euclidean spaces $\mathbb{R}^d$ or Lorentz space $\mathbb{L}^d$. Now, consider data $(X, d_X)$ with similarity metric $d_X$ that was embedded into the respective spaces. We further assume access to maps $\phi_E : \mathcal{X} \to \mathbb{R}^d$ and $\phi_H : \mathcal{X} \to \mathbb{L}^d_+$ that embed $\mathcal{X}$ into the Euclidean space $\mathbb{R}^d$ and the upper sheet of the Lorentz space $\mathbb{L}^d_+$, respectively (see Remark A.1). Let $c_E$ and $c_H$ denote the multiplicative distortion induced by $\phi_E$ and $\phi_H$, respectively (cf. § 2.2). Upper bounds on $c_E$ and $c_H$ can be estimated based on the structure of $\mathcal{X}$ (e.g., tree-like) and the embedding dimensions $(d$ and $d'$, respectively).

How does the distortion $c_E, c_H$ impact our guarantees on the margin? In the previous sections, we noticed that some of the guarantees scale with the dimension of the embedding space. Therefore, we want to analyze the trade-off between the higher distortion resulting from working with smaller embedding dimensions and the higher cost of training robust models due to working with larger embedding dimensions.

We often encounter datasets in ML applications that is intrinsically hierarchical. Theoretical results on the embeddability of trees (cf. § 2.2) suggest that hyperbolic spaces are especially suitable to represent hierarchical data. We therefore restrict our analysis to such data. Further, we make the following assumptions on the underlying data $\mathcal{X}$ and the embedding maps, respectively.

**Assumption 2.** (1) Both $\phi_H(\mathcal{X})$ and $\phi_E(\mathcal{X})$ are linearly separable in the respective spaces, and (2) $\mathcal{X}$ is hierarchical, i.e., has a partial order relation.

**Assumption 3.** The maps $\phi_H, \phi_E$ preserve the partial order relation in $\mathcal{X}$ and the root is mapped onto the origin of the embedding space.

Our approach relates the distance between the support vectors to the size of the margin. The distortion of these distances via embedding then gives a bound on the error induced on the margin.

### 5.1 Euclidean case

In the Euclidean case, we relate the distance of the support vectors to the size of the margin via triangle relations. Let $x, y \in \mathbb{R}^d$ denote support vectors, such that $\langle x, w \rangle > 0$ and $\langle y, w \rangle < 0$ and margin($w$) = $\epsilon$. We can rotate the decision boundary, such that the support vectors are not unique. Without loss of generality, assume that $x_1, x_2$ are equidistant from the decision boundary and $\|w\| = 1$. In this setting, we show the following relation between the margin with ($\epsilon'$) and without ($\epsilon$) the influence of distortion:

**Theorem 5.1.** Let $\epsilon'$ and $\epsilon$ denote the margin with and without distortion. Then $\epsilon' \geq \frac{\epsilon}{c_E}$.

**Corollary 5.2.** If $\mathcal{X}$ is a tree embedded into $\mathbb{R}^{O(\log \log |\mathcal{X}|)}$, then $\epsilon' = O\left(\frac{\epsilon}{\log^3 |\mathcal{X}|}\right)$.
We now present empirical studies for hyperbolic linear separator learning in order to corroborate our theory. In particular, we evaluate the gradient based large-margin separator with adversarial examples (§4) on data that is linearly separable in hyperbolic space and dimension-distortion trade-off (§5). As in

Figure 3: Performance of Adversarial GD. Left: Loss $L(w)$ on the original data. Middle: $α$-robust loss $L_α(w)$. Right: Hyperbolic margin $γ$. We vary the adversarial budget $α$ over $\{0, 0.25, 0.5, 0.75, 1.0\}$.

The proof of Theorem 5.1 follows from a simple side length-altitude relations in the Euclidean triangle between support vectors (cf. Fig. 2(a)). Corollary 5.2 is then a simple application of Bourgain’s result on embedding trees into $\mathbb{R}^d$. For more details see Appendix D.2.

5.2 Hyperbolic case

As in the Euclidean case, we want to relate the margin to the pairwise distances of the support vectors. Such a relation can be constructed both in the original and in the embedding space, which allows us to study the influence of distortion on the margin in terms of $c_H$. In the following, we will work with the half-space model $\mathbb{P}^d$ (cf. §2.1). However, since the theoretical guarantees in the rest of the paper consider the Lorentz model $\mathbb{L}^d$, we have to map between the two spaces. We show in Appendix D.2 that such a mapping exists and preserves the Minkowski product, following prior work by Cho et al. [2019].

The hyperbolic embedding $φ_H$ has two sources of distortion: (1) the multiplicative distortion of pairwise distances, measured by the factor $\frac{1}{c_H}$; and (2) the distortion of order relations, in most embedding models captured by the alignment of ranks with the Euclidean norm. Under Assumption 3, order relationships are preserved and the root is mapped to the origin. Therefore, for $x \in \mathcal{X}$, the distortion on the Euclidean norms is given as follows:

$$||φ_H(x)|| = d_E(φ_H(x), φ_H(0)) = \frac{d_H(x, 0)}{c_H}$$

i.e., the distortion on both pairwise distances and norms is given by a factor $c_H^{-1}$.

In $\mathbb{P}^d$, the decision hyperplane corresponds to a hypercircle $K_w$. We express its radius $r_w$ in terms of the hyperbolic distance between a point on the decision boundary and one of the hypercircle’s ideal points [Cho et al., 2019]. The support vectors $x, y$ lie on hypercircles $K_x$ and $K_y$, which correspond to the set of points of hyperbolic distance $ε$ (i.e., the margin) from the decision boundary. We again assume, without loss of generality, that at least one support vector is not unique and let $x_1, x_2 \in K_x$ and $y \in K_y$ (cf. Fig. 2(b)). In this setting, we can show that the influence on the margin is negligible. More formally:

**Theorem 5.3.** Let again $ε', ε$ denote the margin with and without distortion. If $\mathcal{X}$ is a tree embedded into $\mathbb{L}^2_+$, $ε' \approx ε$.

The technical proof relies on a construction that reduces the problem to Euclidean geometry via circle inversion on the decision hypercircle. We defer all details to Appendix D.2.

6 Experiments

We now present empirical studies for hyperbolic linear separator learning in order to corroborate our theory. In particular, we evaluate the gradient based large-margin separator with adversarial examples (§4) on data that is linearly separable in hyperbolic space and dimension-distortion trade-off (§5). As in
our theory, we train hyperbolic linear classifiers $w$ whose prediction on $x$ is $y = \text{sgn}(w \cdot x)$. Additional experimental results can be found in Appendix F.

**Data.** We use the ImageNet ILSVRC 2012 dataset Russakovsky et al. [2015] along with its label hierarchy from Wordnet. Hyperbolic embeddings in Lorentz space is obtained for the internal label nodes as well as leaf image nodes using Sarkar’s construction [Sarkar, 2012]. For the first experiment to verify effectiveness of adversarial learning, we pick two classes from the dataset (n02464464 and n07831146) which are linearly separable in hyperbolic space. In this set, there were 1,648 positive and 1,287 negative examples. For the second experiment, to showcase better representational power of hyperbolic spaces for hierarchical data, we pick two disjoint subtrees (n00021939 and n00015388) from the hierarchy.

**Adversarial GD.** We utilize the hyperbolic hinge loss (C.2) for the experiment. To show the effect of adversarial examples, we compute three quantities: (i) loss on original data $L(w)$, (ii) $\alpha$-robust loss $L_\alpha(w)$, and (iii) the hyperbolic margin $\gamma$. We vary the budget $\alpha$ over $\{0, 0.25, 0.5, 0.75, 1.0\}$. For a given budget, to obtain adversarial examples we can see that the CERT problem (§4.3) will be feasible for $z_0 \in (x_0 \cosh(\alpha) - \Delta, x_0 \cosh(\alpha) + \Delta)$, where $\Delta = \sqrt{(x_0^2 - 1)(\cosh^2(\alpha) - 1)}$. We do a binary search in this range for $z_0$ to solve CERT and check if we can obtain an adversarial example. We utilize whatever adversarial examples we find, and ignore other data points. In all experiments, we use a constant step-size $\eta_t = 0.01 \forall t$. The results are plotted in Figure 3. We can see as $\alpha$ increases the problems become harder to solve (higher training robust loss) but we get better margin.

**Dimensional efficiency.** In this experiment, we illustrate the benefit of using hyperbolic space when underlying data is truly hierarchical. We subsample images from each subtree, so that in total we have 1000 vectors, to be more favourable to Euclidean setting. We obtain Euclidean embeddings following the setup and code from Nickel and Kiela [2017]. The Euclidean embeddings at 16 dim reached a mean rank (MR) $\leq 2$, which indicates reasonable quality in preserving distance to few-hop neighbors. We observe superior classification performance at much lower dimensions by leveraging hyperbolic space (see Table 2 in Apx. F.3). In particular, our hyperbolic classifier can perfectly classify with 8-dim embeddings, whereas Euclidean logistic regression struggles even with 16-dim embeddings. Clearly, due to high distortion, Euclidean embeddings struggle to capture the global structure among the data points, which make them easily separable.

7 Conclusion and future work

In this paper we studied the problem of learning robust classifiers with large margins in hyperbolic space. We introduced a hyperbolic perceptron algorithm and studied its sample complexity. Moreover, we introduce and analyze multiple adversarial approaches to robust large-margin learning. The second part of the paper analyzed the role of geometry in learning robust classifiers. We compared Euclidean and hyperbolic approaches with respect to the intrinsic geometry of the data. For hierarchical data that embeds well into hyperbolic space, the lower embedding dimension ensures superior guarantees when learning the classifier in hyperbolic space. This result suggests that it can be highly beneficial to perform downstream machine learning and optimization tasks in a space that naturally reflects the intrinsic geometry of the data.

Promising avenues for future research include (i) exploring the practicality of these results in machine learning and data science applications; and (ii) studying related methods in non-Euclidean spaces, together with an evaluation of dimension-distortion trade-offs.

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A Hyperbolic Space

Hyperbolic spaces are smooth Riemannian manifolds $\mathcal{M} = \mathbb{H}^d$ and as such locally Euclidean spaces. In the following we introduce basic notation for three popular models of hyperbolic spaces. For a comprehensive overview see Bridson and Haefliger [1999].

A.1 Models of hyperbolic spaces

Figure 4: Models of hyperbolic space: The Lorentz model $\mathbb{L}^d$, the Poincare ball $\mathbb{B}^d$, and the Poincare half-plane $\mathbb{P}^d$.

The Poincare ball defines a hyperbolic space within the Euclidean unit ball, i.e.

$$\mathbb{B}^d = \{ x \in \mathbb{R}^d : \|x\| < 1 \}$$

$$d_{\mathbb{B}}(x, x') = \text{acosh} \left( 1 + 2 \frac{\|x - x'\|^2}{(1 - \|x\|^2)(1 - \|x'\|^2)} \right).$$

Here, $\|\cdot\|$ is the usual Euclidean norm.

The closely related Poincare half-plane model is defined as

$$\mathbb{P}^2 = \{ x \in \mathbb{R}^2 : x_1 > 0 \}$$

$$d_{\mathbb{P}}(x, x') = \text{acosh} \left( 1 + \frac{(x_0' - x_0)^2 + (x_1' - x_1)^2}{2x_1 x_1'} \right).$$

Note that if $x_0 = x_0'$, the metric simplifies as

$$d_{\mathbb{P}}(x, x') = d_{\mathbb{P}}((x_0, x_1), (x_0, x_1')) = \left| \ln \frac{x_1'}{x_1} \right|.$$

The model can be generalized to higher dimensions with

$$\mathbb{P}^d = \{ (x_0, \ldots, x_{d-1}) \in \mathbb{R}^d : x_{d-1} > 0 \},$$

however, we will only use the two-dimensional model $\mathbb{P}^2$ here. We further define the hyperboloid as

$$\mathbb{L}^d = \{ x \in \mathbb{R}^{d+1} : x \ast x = 1 \}$$

$$d_{\mathbb{L}}(x, x') = \text{acosh}(x \ast x'),$$

where $\ast$ denotes the Minkowski product $x \ast x' = x_0 x_0' - \sum_{i=1}^{d} x_i x_i'$.

Remark A.1. The Lorentz model

$$\mathbb{L}^d = \{ x \in \mathbb{R}^{d+1} : x \ast x = 1 \}.$$
is also called double-sheet model. We will use this more general setting in sections 2-4. For simplicity, we restrict ourselves to the upper sheet

\[ \mathbb{L}^d_+ = \{ x \in \mathbb{R}^{d+1} : x \cdot x = 1, x_0 > 0 \} , \]

in section 5. All constructions of mappings between the different models of hyperbolic space can be extended to the double-sheet \( \mathbb{L}^d \).

Note that \( \mathbb{B}^d, \mathbb{L}^d \) and \( \mathbb{P}^d \) are embedded submanifolds of ambient real vector spaces.

### A.2 Equivalence of different models of hyperbolic spaces

The Poincare ball \( \mathbb{B}^d \) and the Lorentz model \( \mathbb{L}^d_+ \) are equivalent models of hyperbolic space. A mapping is given by

\[
\pi_{LB} : \mathbb{L}^d_+ \to \mathbb{B}^d \\
 x = (x_0, \ldots, x_d) \mapsto \left( \frac{x_1}{1 + x_0}, \ldots, \frac{x_d}{1 + x_0} \right). 
\]

We can further construct a mapping from \( \mathbb{B}^d \) to \( \mathbb{P}^d \) by inversion on a circle centered at \((-1, 0, \ldots, 0)\):

\[
\pi_{BP} : \mathbb{B}^d \to \mathbb{P}^d \\
 x = (x_0, \ldots, x_{d-1}) \mapsto \left( \frac{2x_1, \ldots, 2x_{d-1}, 1 - \|x\|^2}{1 + 2x_0 + \|x\|^2} \right). 
\]

### A.3 Embeddability

When analyzing the dimension-distortion trade-off, we make use of two key results on the embeddability (cf. §2.2) of trees into Euclidean and hyperbolic spaces. We state them below for reference.

**Theorem A.2** ([Bourgain, 1985]). An \( N \)-point metric \( \mathcal{X} \) (i.e., \( |\mathcal{X}| = N \)) embeds into Euclidean space \( \mathbb{R}^{O(\log^2 N)} \) with the distortion \( c_M = O(\log N) \).

This bound in Theorem A.2 is tight for trees in the sense that embedding them in a Euclidean space (of any dimension) must incur the distortion \( c_m = \Omega(\log N) \) [Linial et al., 1995].

**Theorem A.3** ([Sarkar, 2012]). Tree metrics embed quasi-isometrically with \( c_M = O(1 + \epsilon) \) into \( \mathbb{H}^d \).

### A.4 Spherical codes in hyperbolic space

Consider the unit sphere \( S^{d-1} \subseteq \mathbb{R}^d \). A spherical code is a subset of \( S^{d-1} \), such that any two distinct elements \( x, x' \) are separated by at least and angle \( \theta \), i.e. \( \langle x, x' \rangle \leq \cos \theta \). We denote the size of the largest code as \( A(d, \theta) \).

A similar construction of such “spherical caps” can be obtained in \( \mathbb{H}^d \). Note that the induced geometry of these caps is spherical, hence they inherit a spherical geometric structure. This allows in particular the transfer of bounds on \( A(d, \theta) \) to hyperbolic space [Cohn and Zhao, 2014]:

**Theorem A.4** (Chabauty, Shannon, Wyner (see, e.g., [Shannon, 1959])). \( A(d, \theta) \geq (1 + o(1)) \sqrt{2\pi d \frac{\cos \theta}{\sin \frac{\theta}{2}}} \).
B Hyperbolic Perceptron

In this section we analyze the convergence and generalization properties of the hyperbolic perceptron (cf. Algorithm 1). Note that the update \( v_t \leftarrow w_t + y_j x_j \) in Algorithm 1 always leads to a valid hyperplane, i.e., \( \mathbb{L}^d \cap \mathcal{H}_{v_t} \neq \emptyset \), which happens iff \( v_t \cdot v_t < 0 \). This can be verified as follows:

\[
v_t \cdot v_t = (w_t + y_j x_j) \cdot (w_t + y_j x_j) = w_t \cdot w_t + 2y_j(x_j \cdot w_t) + y_j^2(x_j \cdot x_j) < 0,
\]

where (i) is a consequence of the normalization step in Algorithm 1 and (iii) follows as \( x \cdot x = 1 \), since \( x \in \mathbb{L}^d \). As for (ii), note that we perform the update \( v_t \leftarrow w_t + y_j x_j \) only when \( y_j \neq \text{sign}(x_j \cdot w) \) (cf. Algorithm 1).

We now restate Theorem 3.1 and present a detailed proof of the result.

**Theorem B.1** (Convergence hyperbolic Perceptron in Algorithm 1 (Theorem 3.1)). Assume that there is some \( \bar{w} \in \mathbb{R}^{d+1} \) with \( \sqrt{-\bar{w} \cdot \bar{w}} = 1 \) and \( w_0 \cdot \bar{w} \geq 0 \), and some \( \gamma > 0 \), such that \( y_j(\bar{w} \cdot x_j) \geq \sinh(\gamma) \) for \( j = 1, \ldots, |\mathcal{S}| \). Then, Algorithm 1 converges in \( O\left(\frac{1}{\sinh(\gamma)}\right) \) steps and returns a solution with margin \( \gamma \).

**Proof.** Assume wlog \( w_0 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^{d+1} \). Then \( w_0 \cdot w_0 = -1 \), i.e., \( \mathbb{L}^d \cap \mathcal{H}_{w_0} \neq \emptyset \). Hence, \( w_0 \) is a valid initialization. Furthermore, assume that the \( t \)th error is made at the \( j \)th sample, i.e. update \( v_t \leftarrow w_t + y_j x_j \). For \( u \in \mathbb{R}^{d+1} \), let \( |u| = \sqrt{-u \cdot u} \).

Now let us consider two cases:

- **Case 1.** In this case, we assume that the normalization is not performed in \( t \)th step, i.e.,

\[
w_{t+1} = w_t + y_j x_j.
\]

Therefore,

\[
w_{t+1} \cdot \bar{w} = (w_t + y_j x_j) \cdot \bar{w} = w_t \cdot \bar{w} + y_j(x_j \cdot \bar{w}) \geq w_t \cdot \bar{w} + \gamma' \quad \text{. (B.1)}
\]

- **Case 2.** In this case, the normalization is performed in the \( t \)th step of Algorithm 1, i.e.,

\[
w_{t+1} = \frac{w_t + y_j x_j}{|w_t + y_j x_j|} \quad \text{. Thus,}
\]

\[
w_{t+1} \cdot \bar{w} = \frac{w_t + y_j x_j}{|w_t + y_j x_j|} \cdot \bar{w} \overset{(i)}{\geq} (w_t + y_j x_j) \cdot \bar{w} \geq w_t \cdot \bar{w} + y_j(x_j \cdot \bar{w}) \geq w_t \cdot \bar{w} + \gamma' \quad \text{. (B.2)}
\]

where (i) follows as the normalization is performed only if \( |w_t + y_j x_j| < 1 \) and numerator is positive by induction.

By utilizing (B.1) and (B.2), we obtain the following telescoping sum

\[
\sum_{k=0}^{T-1} (-w_{k+1} + w_k) \cdot \bar{w} \leq \sum_{k=0}^{T-1} -\gamma' \Rightarrow -w_T \cdot \bar{w} \leq -w_0 \cdot \bar{w} - T\gamma'.
\]
Recall that, for the Minkowski product, we have
\[
\cosh(\angle(u, u')) = -\frac{u \ast u'}{\sqrt{-u \ast u \sqrt{-u' \ast u'}}} = \frac{-u \ast u'}{|u| |u'|} .
\] (B.4)

By utilizing (B.4) with \((u, u') = (w_T, \bar{w})\) and \((u, u') = (w_0, \bar{w})\) in (B.3), we obtain that
\[
|w_T|\bar{w} \cosh(\angle(w_T, \bar{w})) \leq |w_0|\bar{w} \cosh(\angle(w_0, \bar{w})) - T \gamma'.
\] (B.5)

Since, we have \(|\bar{w}| = |w_0| = 1\), it follows from (B.5) that
\[
|w_T| \cosh(\angle(w_T, \bar{w})) \leq \cosh(\angle(w_0, \bar{w})) - T \gamma'.
\] (B.6)

Further, using the facts that, due to normalization in Algorithm 1, \(|w_T| \geq 1\) and \(\cosh(\cdot) \geq 1\), it follows from (B.6) that
\[
1 \leq \cosh(\angle(w_0, \bar{w})) - T \gamma' \leq C - T \gamma',
\] (B.7)

where \((ii)\) follows as \(\angle(w, \bar{w}) < \pi\), since the orientation is fixed by the requirement that \(L_d \cap H_{w_i} \neq \emptyset\); as a result, we can find an upper bound \(\cosh(\angle(w_0, \bar{w})) < \cosh(\pi) = C\). Now, it follows from (B.7) that
\[
T \leq \frac{C - 1}{\gamma'},
\] (B.8)

which completes the proof of the convergence guarantee. The margin is given by
\[
\text{margin}_S(w) = \inf_{(x, y) \in S} \text{asinh} \left( \frac{y(w \ast x)}{\sqrt{-w \ast w}} \right) = \text{asinh}(\gamma') = \text{asinh}(\sinh(\gamma)) = \gamma,
\]
which implies that a margin of \(\gamma\) is achieved in \(O\left(\frac{1}{\sinh(\gamma)}\right)\) steps. \(\square\)

## C Adversarial Learning

### C.1 Loss functions

For training the classifier, we consider the margin losses that have the following form
\[
l(x, y; w) = f(y \cdot (w \ast x)),
\] (C.1)

where \(f: \mathbb{R} \to \mathbb{R}_+\) is some convex, non-increasing function. Cho et al. [2019] introduce the hinge loss in the hyperbolic setting which is defined by the (hyperbolic) hinge function \(f(s) = \max\{0, \text{asinh}(1) - \text{asinh}(s)\}\), i.e.,
\[
l(x, y; w) = \max\{0, \text{asinh}(1) - \text{asinh}(y(w \ast x))\} .
\] (C.2)

A significant shortcoming of this notion is its non-smoothness and non-convexity. Therefore, we additionally consider a smoothed least squares loss:
\[
l(x_i, y_i; w) = \begin{cases} 
\frac{1}{2} (\text{asinh}(1) - \text{asinh}(y_i(w \ast x_i)))^2, & y_i(w \ast x_i) \leq 1, \\
0, & \text{else}
\end{cases}
\] (C.3)

We present experimental results for both losses.
The majority of the paper employs a hyperbolic version of the logistic loss to introduce the logistic regression problem in hyperbolic space. First, recall the logistic regression problem in the Euclidean setting. Given an input $x$ and a linear classifier defined by $w$, the prediction of the classifier is defined as

$$p(y|x; w) = 1/(1 + \exp(-y\langle x, w \rangle))$$

(C.4)

Thus the log-loss takes the following form

$$l(x, y; w) = -\log p(y|x; w) = \log (1 + \exp(-y\langle x, w \rangle))$$

$$= \log (1 + \exp(-\|w\|(x, \bar{w})))$$

$$= \log (1 + \exp(-y \text{sgn}(\langle x, \bar{w} \rangle)\|w\|d(x, \partial H_w)))$$

(C.5)

where $\bar{w} = w/\|w\|$ and $d(x, \partial H_w)$ is the distance of $x$ from the decision boundary $\partial H_w := \{z \in \mathbb{R}^{d+1} : \langle z, \bar{w} \rangle = 0\}$. Note that $y \text{sgn}(\langle x, \bar{w} \rangle)d(x, \partial H_w)$ denotes the Euclidean margin of the $(x, y)$ with respect to the decision boundary defined by $\bar{w}$.

We can define a hyperbolic version of the logistic regression problem, where we replace the Euclidean margin with the hyperbolic margin with respect to the linear classifier $w$. Recall that the hyperbolic margin has the following form (cf. (2.2)):

$$y \text{sgn}(x \ast w)d(x, \partial H_w) \bigg| \text{asinh} \left( \frac{w \ast x}{\sqrt{-w \ast w}} \right) \bigg| = \text{asinh} \left( \frac{y(w \ast x)}{\sqrt{-w \ast w}} \right)$$

(C.6)

Therefore, by combining (C.5) and (C.6), the hyperbolic logistic regression problem with a linear classifier corresponds to minimizing the following loss:

$$l(x, y; w) = \ln \left( 1 + \exp \left( -\text{asinh} \left( \frac{y(w \ast x)}{\sqrt{-w \ast w}} \right) \right) \right).$$

(C.7)

Note that the hyperbolic logistic loss and the Euclidean logistic loss differ in the scaling factor $\|w\|$. In order to ensure that the hyperbolic logistic loss satisfies Assumption 1, we introduce additional explicit scaling to obtain the following form of the loss.

$$l(x, y; w) = \ln \left( 1 + \exp \left( -\text{asinh} \left( \frac{y(w \ast x)}{2R} \right) \right) \right).$$

(C.8)

The following result verifies that the loss in (C.8) indeed satisfies Assumption 1.

**Lemma C.1.** For valid inputs $(x, y; w)$, the hyperbolic logistic loss in (C.8) fulfills Assumption 1.

**Proof.** The robust loss (Eq. 4.2) is evaluated over inputs $(x, y; w)$ only if $y(w \ast x) < 0$. A simple calculation shows, that Assumption 1.3 holds iff $\frac{|w \ast x|}{R_a} \leq 1$, where $R_a$ is as given in Assumption 1.2. As a result, we want to show $|w \ast x| \leq R_a$ for all allowable inputs $(x, y; w)$. Recall that

$$w \ast x = w_0x_0 - \sum_{i=1}^{d} w_i x_i$$

$$w \cdot x = w_0x_0 + \sum_{i=1}^{d} w_i x_i.$$

We consider the following cases:

1. $w_0x_0 > 0$ and $\sum_{i=1}^{d} w_i x_i < 0$: $|w \ast x| \geq |w \cdot x|$;

2. $w_0x_0 > 0$ and $\sum_{i=1}^{d} w_i x_i > 0$: $|w \ast x| \geq |w \cdot x|$;

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3. \( w_0 x_0 < 0 \) and \( \sum_{i=1}^{d} w_i x_i > 0 \): \(|w \ast x| \geq |w \cdot x|\);

4. \( w_0 x_0 < 0 \) and \( \sum_{i=1}^{d} w_i x_i < 0 \): \(|w \cdot x| \geq |w \ast x|\).

In case (2) and (4) we have

\[
|w \ast x| \leq |w \cdot x| \leq \|w\| \|x\| \leq R_x R_w = R_a, 
\]

where (i) follows from the Cauchy-Schwartz inequality and (ii) follows from Assumption 1.2. In case (1) and (3), we have

\[
|w \cdot x| = |w \cdot \hat{x}| \leq \|w\| \|\hat{x}\| \leq R_x R_w = R_a, 
\]

where \( \hat{x} = (x_0, -x_1, \ldots, -x_n) \) and (i) and (ii) again follow from the Cauchy-Schwartz inequality, respectively. This completes the proof. \( \square \)

**Remark C.2.** A conceptually similar loss is introduced in [Lebanon and Lafferty, 2004] for multinomial manifold. Max-margin learning with the above hyperbolic hinge loss was studied in [Cho et al., 2019].

### C.2 Generating adversarial examples (Certification problem)

Recall that to train a classifier with large margin, we enrich the training set with adversarial examples (cf. Algorithm 2). For a classifier \( w \), an adversarial example \( \hat{x} \) for a given \( (x, y) \) is generated by perturbing \( x \) in the hyperbolic space up to the maximum allowed perturbation budget \( \alpha \) such that

\[
\hat{x} \leftarrow \operatorname{argmax}_{z \in \mathbb{H}^d; \|x, z\| \leq \alpha} l(z, y; w) .
\]

For the underlying loss function (cf. Section C.1), due to the monotonicity of \( \text{asinh} \), the above problem can be equivalently expressed as

\[
\hat{x} \leftarrow \operatorname{argmin}_{z \in \mathbb{H}^d; \|x, z\| \leq \alpha} y \cdot (w \ast z) = \operatorname{argmax}_{w' \in \mathbb{H}^d; \|x, z\| \leq \alpha} w' \ast z
\]

\[
= \operatorname{argmax}_{z \in \mathbb{H}^d; \|x, z\| \leq \alpha} -w'_0 z_0 + \sum_{i} w'_i z_i .
\]

(C.10)

where \( w' = -yw \). Since \( w', z \in \mathbb{R}^{d+1} \), we can rewrite (C.10) as a constraint optimization task in the ambient Euclidean space:

\[
\max_{z \in \mathbb{R}^{d+1}} -w_0 z_0 + \sum_{i} w_i z_i \\
\text{s.t. } d_L(x, z) \leq \alpha \\
\quad z_0^2 - \sum_{i=1}^{d} z_i^2 = 1 .
\]

(C.11)

Assuming that we guess \( z_0 \) based on \( x_0 \), the constraint \( z_0^2 - \sum_{i=1}^{d} z_i^2 = 1 \) confines the solution space onto a \( d \)-dimensional sphere of radius \( r = \sqrt{z_0^2 - 1} \), which also implies that \( z_0 \geq 1 \). On the other hand the constraint \( d_L(x, z) \leq \alpha \) is equivalent to

\[
d_L(x, z) = \text{acosh}(x \ast z) = \text{acosh}(x_0 z_0 - \sum_{i=1}^{d} x_i z_i) < \alpha \quad \text{or} \quad \sum_{i} -x_i z_i \leq \cosh(\alpha) - x_0 z_0 .
\]
Thus, the problem in (C.11) reduces to the following linear program with a spherical constraint.

\[
(CERT) \quad \max_{z_0 \in \mathbb{R}^d} - w_0 z_0 + \sum_{i=1}^{d} w_i z_i \\
\text{s.t.} \quad \sum_{i=1}^{d} -x_i z_i \leq \cosh(\alpha) - x_0 z_0 \\
\|z_0\|^2 = z_0^2 - 1,
\]

where \(z_0 = (z_1, \ldots, z_d)\). We now present a proof of Theorem 4.4 which characterizes a solution of the program in (C.12). For the sake of readability, we first restate the result from the main text:

**Theorem C.3** (Theorem 4.4). Given the input example \((x, y)\), let \(x_0 = (x_1, \ldots, x_d)\). We can efficiently compute a solution to (CERT) or decide that no solution exists. If a solution exists, then based on a guess of \(z_0\) a maximizing adversarial example has the form \(\hat{x} = \left(z_0, \sqrt{\frac{z_0^2 - 1}{\|x_0\|^2}} \right)\). Here, \(b = \frac{(\cosh(\alpha) - x_0 z_0)}{(\|x_0\| \sqrt{z_0^2 - 1})}\) depends on the adversarial budget \(\alpha\), and \(\hat{x}_0\) is a unit vector orthogonal to \(\hat{x} = -x_0 / \|x_0\|\).

**Proof.** First, note that (CERT) can be rewritten as

\[
(CERT') \quad \max \langle \hat{w}, \hat{z} \rangle \\
\text{s.t.} \quad \langle \hat{x}, \hat{z} \rangle \leq b \\
\|\hat{z}\| = 1,
\]

where \(\hat{w} = w_0 / \|w_0\|\), \(\hat{x} = -x_0 / \|x_0\|\), and \(b = (\cosh(\alpha) - x_0 z_0) / (\|x_0\| \|z_0\|)\). We further set \(\hat{z} = z_0 / \|z_0\|\) so that the norm constraint confines the solution to the unit sphere to simplify the derivation. We can later rescale the solution to have the norm \(\sqrt{z_0^2 - 1}\).

The solution of \(CERT'\) lies on the cone \(\langle \hat{x}, \hat{z} \rangle = b\). We decompose \(\hat{w}\) along \(\hat{x}\) and its orthogonal complement \(\hat{x}^\perp\), i.e.

\[
\hat{w} = \zeta \hat{x} + \zeta \hat{x}^\perp.
\]

with \(\zeta \geq 0\) and \(\|\hat{x}^\perp\| = 1\). Without loss of generality, such a decomposition always exists. Note that

\[
\langle \hat{w}, \hat{z}^* \rangle = \zeta \langle \hat{x}, \hat{z}^* \rangle + \zeta \langle \hat{x}^\perp, \hat{z}^* \rangle = \zeta b + \zeta \langle \hat{x}^\perp, \hat{z}^* \rangle,
\]

where the second equality follows from \(\langle \hat{x}, \hat{z}^* \rangle = b\). This implies that for the objective \(\langle \hat{w}, \hat{z} \rangle\) to be maximized, \(\hat{z}^*\) has to have all of its remaining mass along \(\hat{x}^\perp\), i.e.,

\[
\hat{z}^* = b \hat{x} + \sqrt{1 - b^2} \hat{x}^\perp.
\]

After rescaling to satisfy the original norm constraint in CERT, the maximizing adversarial example (for a given \(z_0\)) is given as

\[
\hat{x} = \left(z_0, \sqrt{\frac{z_0^2 - 1}{\|z_0\|^2} \cdot \hat{z}^*} \right) = \left(z_0, \sqrt{\frac{z_0^2 - 1}{\|z_0\|^2}} \left(b \hat{x} + \sqrt{1 - b^2} \hat{x}^\perp\right) \right).
\]

\(\square\)

### C.3 Adversarial Perceptron

For the convergence analysis of the gradient-based update, we first need to analyze the convergence of the adversarial perceptron. We first state the following lemma that relates the adversarial margin to the max-margin classifier.
Lemma C.4. Let \( \bar{\omega} \) be the max-margin classifier of \( S \) with margin \( \gamma \). At each iteration of Algorithm 2, \( \bar{\omega} \) linearly separates \( S \cup S' \) with margin at least \( \frac{\gamma}{\cosh(a)} \).

Remark C.5. Note that this “adversarial Perceptron” corresponds to a gradient update of the form \( w_{t+1} \leftarrow w_t + y_t \bar{x}_t \), which resembles the adversarial SGD.

Proof. The proof reduces the problem to Euclidean geometry in the Poincare half plane. We defer the proof until Section E, since the respective geometric tools are introduced only in Section D.2.

With this result, we can show the following bound on the sample complexity of the adversarial perceptron:

Theorem C.6. Assume that there is some \( \bar{\omega} \in \mathbb{R}^{d+1} \) with \( \sqrt{\bar{\omega}^* \bar{\omega}} = 1 \) and \( w_0 \neq \bar{\omega} \geq 0 \), and some \( \gamma > 0 \), such that \( y_j (\bar{\omega}^* x_j) \geq \sinh(\gamma) \) for \( j = 1, \ldots, |S| \). Then, adversarial perceptron (with adversarial budget \( \alpha \)) converges after \( O \left( \frac{\cosh(a)}{\sinh(\gamma)} \right) \) steps, at which it has margin of at least \( \frac{\gamma}{\cosh(a)} \).

Proof. Without loss of generality, we initialize the classifier as \( w_0 = (0, 1, 0, \ldots, 0) \). Furthermore, assume that the \( t \)th error is made at the \( j \)th sample. For the ease of exposition, we assume that the normalization step is not performed at this update. (The case with normalization after the update can be handled as in the Proof of Theorem B.1.) Thus,

\[
    w_{t+1} \leftarrow w_t + y_t \bar{x}_t,
\]

which implies that

\[
    (w_{t+1} - w_t) \ast \bar{\omega} = (y_j \bar{x}_j) \ast \bar{\omega} = y_j (\bar{x}_j \ast \bar{\omega}) \geq \frac{\gamma'}{\cosh(a)},
\]

where \( \gamma' = \sinh(\gamma) \) and the last inequality follows from Lemma C.4. By summing and telescoping, we obtain that

\[
    \sum_{k=0}^{t} (w_{k+1} - w_k) \ast \bar{\omega} \geq \sum_{k=0}^{t} \frac{\gamma'}{\cosh(a)} \Rightarrow (w_{t+1} - w_0) \ast \bar{\omega} \geq \frac{t \gamma'}{\cosh(a)}.
\]

Now, by multiplying both sides by \(-1\) and rewriting the Minkowski product gives us that

\[
    -w_{t+1} \ast \bar{\omega} \leq -w_0 \ast \bar{\omega} - \frac{t \gamma'}{\cosh(a)} \leq \frac{|w_0| |\bar{\omega}|}{\cosh(\alpha)} \cosh(\angle(w_0, \bar{\omega})) \leq \frac{t \gamma'}{\cosh(a)} \leq \frac{t \gamma'}{\cosh(a)}. \tag{C.13}
\]

Now, note that

\[
    1 \leq \cosh(\angle(w_{t+1}, \bar{\omega})) \leq \frac{|w_{t+1}| |\bar{\omega}|}{|w_{t+1} \ast \bar{\omega}|} \leq \frac{-w_{t+1} \ast \bar{\omega}}{|w_{t+1}|} \leq C - \frac{t \gamma'}{\cosh(a)},
\]

where \((i)\) utilizes (C.13). Now, solving for \( t \) gives us that

\[
    t \leq (C - 1) \cdot \frac{\cosh(a)}{\gamma'}.
\]

Further, it follows from (2.2) that an adversarial hyperbolic margin of \( \frac{\gamma}{\cosh(a)} \) is then achieved after \( O \left( \frac{\cosh(a)}{\sinh(\gamma)} \right) \) steps. \( \square \)
C.4 Gradient-based update

Recall that, our objective in Algorithm 2 consists of an inner optimization (that computes the adversarial example) and an outer optimization (that updates the classifier). In particular, we consider

\[
\min_{w \in \mathbb{R}^{d+1}} L_{\text{rob}}(w; S) := \frac{1}{|S|} \sum_{(x,y) \in S} l_{\text{rob}}(x,y; w),
\]

where the robust loss is given by

\[
l_{\text{rob}}(x,y; w) := \max_{z \in \mathbb{R}^d, d_1(x,z) \leq \alpha} l(x,y; w) = l(\hat{x}, y; w),
\]

where \(\hat{x} \in \arg\max_{z \in \mathbb{R}^d, d_1(x,z) \leq \alpha} l(x,y; w)\).

Recall that, to compute the update, we need to compute gradients of the outer minimization problem, i.e., \(\nabla_w l_{\text{rob}}\) over \(S\). However, the function \(l_{\text{rob}}\) is itself a maximization problem (referred to as the inner maximization problem above). Therefore, we compute the gradient at the maximizer of the inner problem. Danskin’s theorem ensures that this gives a valid decent direction. For the sake of completeness, we recall the Danskin’s theorem here.

**Theorem C.7 (Danskin [1966], Bertsekas [2016]).** Suppose \(X\) is a non-empty compact topological space and \(g : \mathbb{R}^d \times X \to \mathbb{R}\) is a continuous function such that \(g(\cdot, \delta)\) is differentiable for every \(\delta \in X\). Let \(\delta^*_w = \arg\max_{\delta \in X} g(w, \delta)\). Then, the function \(\psi(w) = \max_{\delta \in X} g(w, \delta)\) is subdifferentiable and the subdifferential is given by

\[
\partial \psi(w) = \text{conv}\left(\{\nabla_w g(w, \delta) | \delta \in \delta^*_w\}\right).
\]

This approach has been previously used in Madry et al. [2018] and Charles et al. [2019]. Note that when we find an adversarial example in Algorithm 2, we can write it in a closed form (cf. Theorem C.3). In particular,

\[
l_{\text{rob}}(x,y; w) = \max_{d_1(x,z) \leq \alpha} l(z,y; w) = l(\hat{x}, y; w) \quad \text{with} \quad \hat{x} = \left(\hat{x}_0, \sqrt{\frac{\alpha^2}{\beta^2} - 1}\left(\beta \hat{x} + \sqrt{1 - \beta^2} \hat{x}^\perp\right)\right).
\]

Note that

\[
\nabla_w l(\hat{x}, y; w) = f'(y(w * \hat{x})) \cdot \nabla_w y(w * \hat{x}) = f'(y(w * \hat{x})) \cdot y(\hat{x})^T,
\]

where we have used the fact that \(\nabla_w y(w * \hat{x}) = y(\hat{x})^T = y(\hat{x}_0, -\hat{x}_1, \ldots, -\hat{x}_n)^T\). From Danskin’s theorem, we have \(\nabla_w l(\hat{x}, y; w) \in \partial l_{\text{rob}}(x,y; w)\). This enables us to compute the decent direction and perform the update step with

\[
\nabla L(w; S') = \frac{1}{|S'|} \sum_{(\hat{x}, y) \in S'} \nabla l(\hat{x}, y; w) \in \partial L_{\text{rob}}(w; S),
\]

Furthermore, we have

\[
\nabla^2_w l(\hat{x}, y; w) = f''(y(w * \hat{x})) \hat{x}^T \hat{x} \in \partial^2 l_{\text{rob}}(x,y; w),
\]

which enable the computation of the Hessian of \(L(w; S')\).

The convergence results in this section build on hyperbolic analogues of comparable Euclidean results in [Soudry et al., 2018, Ji and Telgarsky, 2018].

We first show a bound on the Hessian of the loss:
Lemma C.8.

\[ \nabla^2 L(w_t; S'_t) \preceq \beta \sigma_{\text{max}}^2 \cdot I, \]

where \( \sigma_{\text{max}} \) is an upper bound on the maximum singular value of the data matrix \( \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} x x^T \).

Proof.

\[
\nabla^2 L(w_t; S'_t) = \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} \nabla^2 l(\hat{x}, y; w_t)
\]

(i) \[
\preceq \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} f''(y(\hat{x} * w_t)) \hat{x} \hat{x}^T
\]

(ii) \[
\preceq \beta \cdot \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} \hat{x} \hat{x}^T
\]

\[
\leq \beta \sigma_{\text{max}}^2 \cdot I,
\]

where (i) and (ii) follow from (C.14) and the assumption that \( f \) is \( \beta \)-smooth. \( \square \)

With the help of Lemma C.8, we can show the following result (a restatement of Theorem 4.2), which establishes that the gradient updates are guaranteed to converge to a large-margin classifier:

**Theorem C.9 (Theorem 4.1).** Let \( \{w_t\} \) be the GD iterates

\[
w_{t+1} \leftarrow w_t - \eta \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} \nabla l(\hat{x}, y; w)
\]

\[
w_{t+1} \leftarrow \frac{w_{t+1}}{\sqrt{\nabla^2 l(w_{t+1}, S'_t)}}
\]

with constant step size \( \eta < \frac{2}{\beta \sigma_{\text{max}}^2} \) and an initialization \( w_0 \) with \( w_0 * w_0 < 0 \). Then, we have \( \lim_{t \to \infty} l(w_t; S \cup S'_t) = 0 \).

Proof. By Assumption 1.1 we can find a \( \bar{w} \) that linearly separates \( S \). Then, we have

\[
\langle \bar{w}, \nabla L(w; S'_t) \rangle = \langle \bar{w}, \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} f'(y(\hat{x} * w_t)) y \hat{x} \rangle
\]

\[
= \left( \frac{1}{|S'_t|} \sum_{(x,y) \in S'_t} f'(y(\hat{x} * w_t)) \right) \frac{y \langle \bar{w}, \hat{x} \rangle}{=y(\bar{w} * x) < 0}
\]

where the negativity of the first term follows from the assumptions on \( f \) (cf. Assumption 1.3) and the upper bound on the second term from the separability assumption. This implies that \( \langle \bar{w}, \nabla L(w; S'_t) \rangle \neq 0 \) for any finite \( w \). Therefore, there are no finite critical points \( w \) for which \( \nabla L(w; S'_t) = 0 \). However, GD is guaranteed to converge to a critical point for smooth objectives with an appropriate step size. Therefore, \( \|w_t\| \to \infty \) and \( y(w_t * x) > 0 \) \( \forall (x,y) \in S \cup S'_t \) and large enough \( t \). Then, we have \( l(x,y; w_t) \to 0 \), for all \( (x,y) \in S \cup S'_t \). This further implies that \( L(w_t; S \cup S'_t) = \frac{1}{|S \cup S'_t|} \sum_{(x,y) \in S \cup S'_t} l(x,y; w_t) \to 0 \). \( \square \)

We further show that the enrichment of the training set with adversarial examples is critical for polynomial-time convergence: Without adversarial training, we can construct a simple max-margin problem, that cannot be solved in polynomial time.
Theorem C.10 (Theorem 3.3). Consider $S = \{(e_1, 1), (-e_1, -1)\} \subset \mathbb{R}^{d+1} \times \{+1, -1\}$ and a typical initialization $w_0 = e_2 \in \mathbb{R}^{d+1}$ (with the standard basis vectors $e_1, e_2 \in \mathbb{R}^{d+1}$). Let $\{w_t\}_t$ is a sequence of classifiers generated by the GD updates (with fixed step size $\eta$)

$$w_{t+1} \leftarrow w_t - \frac{\eta}{|S|} \sum_{(x, y) \in S} \nabla l(x, y; w)$$

$$w_{t+1} \leftarrow \frac{w_{t+1}}{\sqrt{-w_{t+1}^1 * w_{t+1}}}. $$

Then, the number of iterations needed to achieve margin $\gamma$ where $\gamma$ is the derivative of the hyperbolic logistic regression loss (cf. (4.4)). Note that due to the structure of $S$ and $w_0$, the GD update will produce the following iteration sequence

$$a_{t+1} = a_t - f'(a_t)$$
$$w_t = (a_t, \sqrt{a_t^2 + 1}, 0, \ldots, 0),$$

where the first coordinate is determined through the GD update and the second through normalization to ensure the validity of the classifier, i.e., $w_t * w_t < 0$. In order to see this, note that $w_t * w_t = a_t^2 - (\sqrt{a_t^2 + 1})^2 = -1 < 0$. We now want to show that

$$a_t \leq \sinh((\ln(t + 1)) .$$

For the induction, note that $a_0 = 0 = \ln(1) = \sinh(\ln(1))$. Assume, that $a_t \leq \sinh(\ln(t + 1))$. We want to show

$$a_{t+1} \leq \sinh(\ln(t + 2)) .$$

Note, that

$$a_{t+1} = a_t + \frac{\exp(-\sinh(\frac{a_t}{2R}))}{2} \cdot \frac{\exp(-\sinh(\frac{a_t}{2R})) + 1}{\sqrt{\frac{a_t^2}{4R} + 1}}.$$ 

Since $\exp(-\sinh(\frac{a_t}{2R})) \leq \exp(-\sinh(\frac{a_t}{2R})) + 1$ and $R \sqrt{\frac{a_t^2}{4R} + 1} \geq 1$, clearly $\circ 2$ is bounded by 1. Inserting this above and replacing $\circ 1$ with the induction assumption, we have

$$a_{t+1} \leq \sinh(\ln(t + 1)) + 1 .$$

Note that, by definition, $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$. Thus,

$$\sinh(\ln(t + 1)) = \frac{1}{2} (t + 1 - (-(t + 1)) = t + 1 ,$$

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which further implies that
\[ a_{t+1} = \sinh(\ln(t+1)) + 1 \leq t + 2 = \sinh(\ln(t+2)) . \] (C.15)

This finishes the induction proof. Assuming a margin of at least \( \gamma \), we have
\[ \gamma \leq \text{margin}_S(w_i) = \sinh \left( \frac{y(x, w_i)}{\sqrt{-w_i \cdot w_i}} \right) \overset{(i)}{=} \sinh(a_{t+1}) \leq \text{asinh}(\sinh(\ln(t+2))) \leq \ln(t+2) , \]
where (i) follows \( w_i \cdot w_i = -1 \) after normalization and (ii) from the upper bound in (C.15). Now, by solving for \( t \), we obtain that \( t = \Omega(\exp(\gamma)) \). \( \square \)

Next, we quantify the convergence rate of adversarial training with GD updates (cf. (4.3a) and (4.3b)).
We start by presenting some auxiliary results.

**Lemma C.11 (Smoothness bound).** Let \( \eta_t =: \eta < \frac{2\sinh^2(\gamma)}{\beta \nu_{\text{max}} \cosh^2(\alpha) R^2} \) be the fixed step size and \( w_0 \) a valid initialization, i.e. \( w_0 \cdot w_0 < 0 \). Then, for the GD update (with fixed step size \( \eta_t =: \eta \))
\[
\begin{align*}
    w_{t+1} & = w_t - \eta \nabla L(w_t; S'_t) \\
    & = \frac{w_t}{\sqrt{-w_t \cdot w_t - \eta \nabla L(w_t; S'_t)}} \\
    & = \frac{w_t - \eta \nabla L(w_t; S'_t)}{\sqrt{-w_t \cdot w_t - \eta \nabla L(w_t; S'_t)}} .
\end{align*}
\]
we have
\[
1. \quad L_{\text{rob}}(w_{t+1}; S) \leq L_{\text{rob}}(w_t; S) - \eta \left( \frac{\sinh(\gamma)^2}{\cosh^2(\alpha) R^2} - \frac{\beta \nu_{\text{max}} \eta}{2} \right) \| \nabla L(w_t; S'_t) \|^2 ; \quad \text{and} \quad \| \nabla L(w, S'_t) \|_2 \leq \| \nabla L(w, S'_t) \|_2 .
\]

2. \( \sum_{t=0}^{\infty} \| \nabla L(w_t; S'_t) \|^2 < \infty; \) as a result, \( \lim_{t \to \infty} \| \nabla L(w_t; S'_t) \|^2 = 0. \)

**Proof.** In Algorithm 2 with gradient update rule, we have
\[
\begin{align*}
    w_{t+1} & = w_t - \eta \nabla L(w_t; S'_t) \\
    & = w_t - \frac{\eta}{|S'_t|} \sum_{(x,y) \in S'_t} l(x,y, w_t) \\
    & = w_t - \frac{\eta}{|S'_t|} \sum_{(x,y) \in S'_t} f'(y(x^* w_t)) y x .
\end{align*}
\]
Now, consider the inner product \( \langle w_{t+1}, \bar{w} \rangle \), where \( \bar{w} \) is the optimal classifier. With out loss of generality, we assume \( \| \bar{w} \| = 1. \)
\[
\langle w_{t+1}, \bar{w} \rangle = \langle w_t, \bar{w} \rangle - \frac{\eta}{|S'_t|} \sum_{(x,y) \in S'_t} f'(x^* w_t) y x ,
\]
\[
\overset{(i)}{=} \langle w_t, \bar{w} \rangle - \frac{\eta}{|S'_t|} \sum_{(x,y) \in S'_t} f'(x^* w_t) y x ,
\]
\[
\overset{(ii)}{=} \langle w_t, \bar{w} \rangle - \frac{\eta}{|S'_t|} \cosh(\alpha) \sum_{(x,y) \in S'_t} f'(y(x^* w_t)) ,
\]
where (i) and (ii) follow from \( \langle \bar{x}, \bar{w} \rangle = \bar{x} \cdot \bar{w} \) and \( y(x^* \bar{w}) \geq \frac{\gamma'}{\cosh(\alpha)} \) (cf. Lemma C.4), respectively. We use the shorthand \( \gamma' = \sinh(\gamma) \). With the linearity of the inner product, we get
\[
\langle w_{t+1} - w_t, \bar{w} \rangle \geq -\frac{\eta}{|S'_t|} \cosh(\alpha) \sum_{(x,y) \in S'_t} f'(y(x^* w_t)) .
\]
Since \( f' \) is negative (cf. Assumption 1.3), we can replace \( -f'(y(\bar{x} + w_l)) \) with \(|f'(y(\bar{x} + w_l))|\) to get

\[
\langle w_l - w_{l+1}, \bar{w} \rangle \geq \frac{\eta \gamma'}{\cosh(\alpha)} \sum_{(x,y) \in \mathcal{S}_l^t} |f'(y(\bar{x} + w_l))| \]

\[
= \frac{(i)}{\cosh(\alpha) R_a} \| \nabla L(w_l; \mathcal{S}_l^t) \| ,
\]

where \((i)\) holds as follows: Recall, that \( \| \nabla l(\bar{x}, y; w_l) \| \leq |f'(y(\bar{x} + w_l))| \| \bar{x} \| \). Thus,

\[
\| \nabla L(w_l; \mathcal{S}_l^t) \| = \| \frac{1}{|\mathcal{S}_l^t|} \sum_{(x,y) \in \mathcal{S}_l^t} \nabla l(\bar{x}, y; w_l) \| \leq \frac{1}{|\mathcal{S}_l^t|} \sum_{(x,y) \in \mathcal{S}_l^t} \| l(\bar{x}, y; w_l) \|
\]

\[
\leq \frac{1}{|\mathcal{S}_l^t|} \sum_{(x,y) \in \mathcal{S}_l^t} |f'(y(\bar{x} + w_l))| \| \bar{x} \| \leq \frac{R_a}{|\mathcal{S}_l^t|} \sum_{(x,y) \in \mathcal{S}_l^t} |f'(y(\bar{x} + w_l))| .
\]

This implies that

\[
\frac{1}{|\mathcal{S}_l^t|} \sum_{(x,y) \in \mathcal{S}_l^t} |f'(y(\bar{x} + w_l))| = \frac{1}{R_a} \| \nabla L(w_l; \mathcal{S}_l^t) \| .
\]

Applying Cauchy-Schwarz to the left hand side of (C.16) gives us that

\[
\| w_l - w_{l+1} \| \| \bar{w} \| \geq \langle w_l - w_{l+1}, \bar{w} \rangle \geq \frac{\eta \gamma'}{\cosh(\alpha) R_a} \| \nabla L(w_l; \mathcal{S}_l^t) \| .
\]

(C.17)

Now, using the fact that \( \| \bar{w} \| = 1 \) in (C.17), we get

\[
\| w_{l+1} - w_l \| \geq \frac{\eta \gamma'}{\cosh(\alpha) R_a} \| \nabla L(w_l; \mathcal{S}_l^t) \| .
\]

(C.18)

Now, consider the following Taylor approximation:

\[
L_{rob}(w_{l+1}; S) = L_{rob}(w_l; S) + \langle \nabla L(w_l; \mathcal{S}_l^t), w_{l+1} - w_l \rangle + \frac{(w_{l+1} - w_l)T}{\partial L_{rob}(v; S)} \frac{\nabla^2 L(v; \mathcal{S}_l^t)}{2} (w_{l+1} - w_l),
\]

where \( v \in \text{conv}(w_{l+1}, w_l) \). By utilizing Lemma 8 in (C.19), we get that

\[
L_{rob}(w_{l+1}; S) \leq L_{rob}(w_l; S) + \langle \nabla L(w_l; \mathcal{S}_l^t), w_{l+1} - w_l \rangle + \frac{\beta \sigma_{\text{max}}^2}{2} \| w_{l+1} - w_l \|^2 .
\]

(C.20)

Recall the update rule

\[
w_{l+1} = w_l - \eta \nabla L(w_l; \mathcal{S}_l^t)
\]

(C.21)

\[
\Rightarrow w_{l+1} - w_l = -\eta \nabla L(w_l; \mathcal{S}_l^t) .
\]

(C.22)

Inserting this in (C.20), we get

\[
L_{rob}(w_{l+1}; S) = L_{rob}(w_l; S) - \eta^{-1} \| w_{l+1} - w_l \|^2 + \frac{\beta \sigma_{\text{max}}^2}{2} \| w_{l+1} - w_l \|^2 .
\]

(C.23)
By combining (C.18) and (C.23), we obtain that
\[ L_{\text{rob}}(w_{t+1}; S) \leq L_{\text{rob}}(w_t; S) - \frac{\eta \gamma^2}{\cosh^2(\alpha) R_t^2} \| \nabla L(w_t; S') \|^2 + \frac{\beta c_{\max}^2}{2} \| w_{t+1} - w_t \|^2 . \] (C.24)

Again, utilizing (C.21), it follows from (C.24) that
\[ L_{\text{rob}}(w_{t+1}; S) \leq L_{\text{rob}}(w_t; S) - \frac{\eta \gamma^2}{\cosh^2(\alpha) R_t^2} \| \nabla L(w_t; S') \|^2 + \frac{\beta c_{\max}^2 \eta^2}{2} \| \nabla L(w_t; S') \|^2 \]
\[ = L_{\text{rob}}(w_t; S) - \eta \left( \frac{\gamma^2}{\cosh^2(\alpha) R_t^2} - \frac{\beta c_{\max}^2 \eta}{2} \right) \| \nabla L(w_t; S') \|^2 . \] (C.26)

This establishes the first claim of Lemma C.11. Now, we can rewrite (C.25) to obtain the following.
\[ \frac{L_{\text{rob}}(w_t; S) - L_{\text{rob}}(w_{t+1}; S)}{\eta \left( \frac{\gamma^2}{\cosh^2(\alpha) R_t^2} - \frac{\beta c_{\max}^2 \eta}{2} \right)} \geq \| \nabla L(w_t; S') \|^2 . \]

Note that our assumption on the step size \( \eta \) ensures that the denominator in (C.25) is \( \neq 0 \). Next, summing and telescoping gives us that
\[ \sum_{k=0}^{t} \| \nabla L(w_t; S') \|^2 \leq \sum_{k=0}^{t} \frac{L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_{k+1}; S)}{\eta \left( \frac{\gamma^2}{\cosh^2(\alpha) R_t^2} - \frac{\beta c_{\max}^2 \eta}{2} \right)} = \frac{L_{\text{rob}}(w_0; S) - L_{\text{rob}}(w_{t+1}; S)}{\eta \left( \frac{\gamma^2}{\cosh^2(\alpha) R_t^2} - \frac{\beta c_{\max}^2 \eta}{2} \right)} , \]

where the right term is bounded, since \( L_{\text{rob}}(w_0; S) < \infty \) and \( 0 \leq L_{\text{rob}}(w_{t+1}; S) \). This establishes the second claim of Lemma C.11 as
\[ \sum_{k=0}^{\infty} \| \nabla L(w_t; S') \|^2 < \infty \quad \Rightarrow \quad \lim_{t \to \infty} \| \nabla L(w_t; S') \|^2 = 0 . \]

\[ \square \]

**Lemma C.12.** With the assumptions of Lemma C.11, Lemma C.11.1 implies for all \( w \in \mathbb{R}^{d+1} \)
\[ 2 \sum_{k=0}^{l-1} \eta_k (L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w; S)) + \sum_{k=0}^{l-1} \eta_k^2 (L_{\text{rob}}(w_{k+1}; S) - L_{\text{rob}}(w_k; S)) \leq \| w_0 - w \|^2 - \| w_t - w \|^2 , \]
where \( \eta_k = \eta_k \left( \frac{\gamma^2}{\cosh(\alpha) R_t^2} - \frac{\beta c_{\max}^2 \eta_k}{4} \right) \).

**Proof.** First, note that the GD update
\[ w_{t+1} = w_t - \eta \nabla L(w_t; S') \in \partial L_{\text{rob}}(w_t; S) \]
implies that
\[ \| w_{t+1} - w \|^2 = \| w_t - w \|^2 - 2 \eta \langle \nabla L(w_t; S'), w_t - w \rangle + \eta^2 \| \nabla L(w_t; S') \|^2 \]
\[ = \| w_t - w \|^2 + 2 \eta \langle \nabla L(w_t; S'), w - w_t \rangle + \eta^2 \| \nabla L(w_t; S') \|^2 . \] (C.28)

Note that the hyperbolic logistic regression loss \( f(z) \) in (4.4) is convex for \( z < 0 \). As a consequence, \( l_{\text{rob}}(x, y; w) \) is convex for any adversarial example with \( \text{sgn}(w \ast x) \neq \text{sgn}(w \ast \tilde{x}) \). This implies that
\[ l_{\text{rob}}(x, y; w) \geq l_{\text{rob}}(x, y; w_t) + \langle \partial l_{\text{rob}}(x, y; w_t), w - w_t \rangle , \]

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for any \( w \in \mathbb{R}^{d+1} \) and any pair \((x, y)\) for which an adversarial example exists. Since the sum of convex function is convex, we further have
\[
L_{\text{rob}}(w; S) \geq L_{\text{rob}}(w_1; S) + \langle \nabla L(w_1; S)', w - w_1 \rangle.
\] (C.29)

By combining (C.27) and (C.29), we obtain that
\[
\|w_{k+1} - w\|^2 \leq \|w_k - w\|^2 + 2\eta_k (L_{\text{rob}}(w; S) - L_{\text{rob}}(w_k; S)) + \frac{\eta_k^2}{\tilde{\eta}_k} \|L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_{k+1}; S)\|
\] (i)
\[
\|w_0 - w\|^2 - \|w_0 - w\|^2 \leq 2 \sum_{k=0}^{t-1} \eta_k (L_{\text{rob}}(w; S) - L_{\text{rob}}(w_k; S)) + \frac{\eta_k^2}{\tilde{\eta}_k} \|L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_{k+1}; S)\|.
\]

Now, multiplying both sides by \(-1\) completes the proof as follows.
\[
2 \sum_{k=0}^{t-1} \eta_k (L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_0; S)) + \frac{\eta_k^2}{\tilde{\eta}_k} \|L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_{k+1}; S)\| \leq \|w_0 - w\|^2 - \|w_0 - w\|^2.
\]

We are now in a position to present the desired convergence result.

**Theorem C.13** (Convergence GD update, Algorithm 2). For a fixed constant \( c \in (0, 1) \), let the step size \( \eta_t := \eta \left( \frac{2\sinh^2(\gamma)}{\beta_\text{max} \cosh \alpha R_0^2} \right) \) and \( \Delta \) be the GD update as defined in (4.3a) and (4.3b). Then, the iterates \( \{w_t\} \) in Algorithm 2 satisfy
\[
L_{\text{rob}}(w_t; S) = 0 \left( \frac{\sinh^2(\ln(t))}{t} \cdot \frac{\sinh(\gamma)}{\cosh(\alpha)} \right).
\]

**Proof.** Without loss of generality, assume that \( w_0 = (0, e_j) \) where \( e_j \in \mathbb{R}^d \) is a standard basis vector whose \( i \)-th coordinate is 1. Note that this is a valid initialization, since \( w_0 \neq 0 \); furthermore, we have \( \|w_0\| = 1 \). Let \( w^* \in \mathbb{R}^{d+1} \) be a classifier that achieves the margin \( \gamma \) on \( S \), i.e., \( \forall (x, y) \in S \),
\[
y(x \ast w^*) \geq \sinh(\gamma) \iff \sinh \left( \frac{y \ast w^*}{\sqrt{-w^* \ast w^*}} \right) \geq \gamma.
\]

Without loss of generality, assume that \( \|w^*\| = 1 \). Let \( u_t := \frac{2R_a \sinh(\ln(t)) \cos(\alpha)}{\sinh(\gamma)} w^* \); then \( \|u_t\| = \frac{2R_a \sinh(\ln(t)) \cos(\alpha)}{\sinh(\gamma)} \). We have
\[
L_{\text{rob}}(u_t; S') = \frac{1}{|S'_{i1}|} \sum_{(x, y) \in S'_{i1}} l_{\text{rob}}(x, y; u_t) = \frac{1}{|S'_{i2}|} \sum_{(x, y) \in S'_{i2}} f(y(x \ast u_t))
\]
\[
\leq \frac{1}{|S'_{i2}|} \sum_{(x, y) \in S'_{i2}} (i) f(2R_a \sinh(\ln(t))) = f(2R_a \sinh(\ln(t))) \leq \ln(1 + \exp(-\ln(t))) \lesssim \frac{1}{t},
\]
(C.30)
where (i) follows from
\[
y(\hat{x} \ast u_t) = \frac{2R_u \sinh(\ln(t)) \cosh(\alpha)}{\sinh(\gamma)} \overset{\sinh(\gamma)}{\geq} 2R_u \sinh(\ln(t)),
\]
(ii) from \(\sqrt{-u_t \ast u_t} \geq 1\) and (iii) follows from the fact that \(\ln(1 + x) \leq x\).

Now, consider
\[
2\eta(t-1)(L_{\text{rob}}(w_t; S) - L_{\text{rob}}(u_i; S)) = 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_t; S) - L_{\text{rob}}(u_i; S))
\]
\[
= 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_t; S) - L_{\text{rob}}(u_i; S) + L_{\text{rob}}(w_k; S) - L_{\text{rob}}(w_k; S))
\]
\[
= 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_k; S) - L_{\text{rob}}(u_i; S) + 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_t; S) - L_{\text{rob}}(w_k; S)) \quad \text{(i)}
\]
\[
\leq 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_k; S) - L_{\text{rob}}(u_i; S)) + \sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_{k+1}; S) - L_{\text{rob}}(w_k; S)) \quad \text{(ii)}
\]
\[
\leq 2\sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_k; S) - L_{\text{rob}}(u_i; S)) + \sum_{k=0}^{t-1} \eta_k(L_{\text{rob}}(w_{k+1}; S) - L_{\text{rob}}(w_k; S)) \quad \text{(iii)}
\]
\[
\leq \|w_0 - u_i\|^2 - \|w_t - u_t\|^2,
\]
where (i) holds as we have a constant step-size, i.e., \(\eta_k = \eta\) and (ii) follows from the fact that
\[
L_{\text{rob}}(w_t; S) \leq L_{\text{rob}}(w_{k+1}; S) \quad \text{for } 0 \leq k \leq t - 1.
\]

The inequality in (iii) follows from Lemma C.12 with \(w = u_t\).

We can rewrite this as
\[
L_{\text{rob}}(w_t; S) \leq L_{\text{rob}}(u_i; S) + \frac{\|w_0 - u_i\|^2 - \|w_t - u_t\|^2}{2\sum_{k=0}^{t-1} \eta_k}
\]
\[
\overset{(i)}{\leq} \frac{1}{t} + \frac{\|w_0 - u_i\|^2}{2(t-1)\eta}
\]
\[
\overset{(ii)}{\leq} \frac{1}{t} + \frac{2\|w_0\|^2 + 2\|u_t\|^2}{2(t-1)\eta}
\]
\[
= \frac{1}{t} + \frac{\|w_0\|^2 + \|u_t\|^2}{(t-1)\eta}
\]
where (i) follows from (C.30) and (ii) follows from \((a + b)^2 \leq 2a^2 + 2b^2\). Now, using the fact that \(\|w_0\| = 1\) and \(\|u_t\| = \frac{2R_u \sinh(\ln(t)) \cosh(\alpha)}{\sinh(\gamma)}\), we obtain that
\[
L_{\text{rob}}(w_t; S) \leq \frac{1}{t} + \frac{1 + 4R_u^2 (\cosh(\alpha) / \sinh(\gamma))^2 \cdot \sinh^2(\ln(t))}{(t-1)\eta}
\]
By substituting \(\eta = c \cdot \frac{2\sinh^2(\gamma)}{R_u \cosh(\alpha) R_u^2}\), we get
\[
L_{\text{rob}}(w_t; S) = O\left(\frac{\sinh^2(\ln(t))}{t} \cdot \left(\frac{\sinh(\gamma)}{\cosh(\alpha)}\right)^4\right).
\]
Theorem C.14 (Sample complexity, GD update (Theorem C.13)). Consider Algorithm 2 with $\eta_t := \eta = c \cdot \frac{2 \sinh^2(\gamma)}{\beta \max \text{cosh}^2(a) R_\theta^2}$ and $A$ being the GD update. Then Algorithm 2 converges as $\Omega\left(\text{poly}\left(\frac{\sinh(\gamma)}{\text{cosh}(a)}\right)\right)$.

Proof. We want to show that the bound

$$L_{\text{rob}}(w_t) \leq \ln \left(1 + \exp \left(-\frac{\gamma}{\text{cosh}(a)}\right)\right)$$

implies that $w_t$ achieves margin $\gamma$ on $S \cup S'_t$ and then use the bound to derive the sample complexity of Adversarial GD.

First, note that

$$L_{\text{rob}}(w_t) = \frac{1}{|S \cup S'|} \sum_{(x,y) \in S \cup S'} l_{\text{rob}}(x,y;w_t),$$

and therefore, if $L_{\text{rob}}(w_t) \leq \ln \left(1 + \exp \left(-\frac{\gamma}{\text{cosh}(a)}\right)\right)$, we have

$$l_{\text{rob}}(x,y;w_t) \leq \ln \left(1 + \exp \left(-\frac{\gamma}{\text{cosh}(a)}\right)\right)$$

for any $(x,y) \in S \cup S'$ if we assume the contribution of each $l_{\text{rob}}$-term to be small. Then

$$l_{\text{rob}}(x,y;w_t) \leq \ln \left(1 + \exp \left(-\frac{\gamma}{\text{cosh}(a)}\right)\right)$$

and

$$\ln \left(1 + \exp \left(\text{asinh}(y(\bar{x} \ast w_t))\right)\right) \leq \ln \left(1 + \exp \left(-\frac{\gamma}{\text{cosh}(a)}\right)\right)$$

for any $(x,y) \in S \cup S'$, which implies that $w_t$ achieves margin $\gamma$ on $S \cup S'_t$.

Next, introduce the following constant:

$$C_q := \inf\{t \geq 2 : 2 + \ln(t)^2 \leq (t - 1) t^{-1/q}\}.$$
With this, we can rewrite the bound derived in Thm. C.13 as follows (with \( t \geq C_\eta \)):

\[
L_{\text{rob}}(\omega_t) \leq \frac{1}{t} + \frac{1 + \sinh(\ln(t))^2}{(t-1)\eta} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} 
\leq \frac{2 + \sinh(\ln(t))^2}{(t-1)\eta} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} 
\leq \frac{2 + \ln(t)^2}{(t-1)\eta} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} 
\leq \frac{(t-1)t^{-1/\eta}}{\eta(t-1)} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} 
\leq \frac{t^{-1/\eta}}{\eta} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right)^{-4} . 
\]

Solving for \( t \) and plugging in the above bound on \( L_{\text{rob}} \), for which \( \omega_t \) achieves the desired margin, we get

\[
t = \max\{C_\eta, \Omega \left( \left( \eta \left( \frac{\sinh(\gamma)^4}{\cosh(\alpha)^4} \right) \right)^{-\eta} \right) \}
\]

from which the claim follows directly. \( \square \)

### C.5 Algorithm 2 with an ERM update

Consider the unit sphere \( S^{d-1} \subseteq \mathbb{R}^d \). A spherical code with minimum separation \( \theta \) is a subset of \( S^{d-1} \), such that any two distinct elements \( u, u' \) in the subset are separated by at least an angle \( \theta \), i.e. \( \langle u, u' \rangle \leq \cos \theta \). We denote the size of the largest such code as \( A(d, \theta) \). A similar construction can be made in hyperbolic space, which allows the transfer of bounds on \( A(d, \theta) \) to hyperbolic space [Cohn and Zhao, 2014].

The following lemma shows that a spherical code with a suitable minimum separation \( \theta \) enables a simple pathological training set such that Algorithm 2 along with an ERM update rule cannot produce a classifier with a desired margin in a small number of iteration. In particular, the lemma shows that the number of iterations required to find the desired margin is lower-bounded by the size of the underlying spherical code.

**Lemma C.15.** Consider \( \mathcal{S} = \{(x_1, y_1) = ((1,0,\ldots,0),1), (x_2, y_2) = ((-1,0,\ldots,0),-1)\} \), where \( x_1, x_2 \in \mathbb{L}^d \) and \( y_1, y_2 \) the corresponding labels. For any \( \epsilon < \alpha \), there is an admissible sequence of classifiers \( \{\omega_t\}_{1 \leq t \leq T} \), with

\[
T = A \left( d, \arccos \left( \rho \cdot \frac{\sinh(\epsilon) \cosh(\alpha)}{\sqrt{\cosh^2(\alpha) - 1 + \sinh^2(\epsilon)}} \right) \right)
\]

**Proof.** First, note that \( x_1 \ast x_1 = x_2 \ast x_2 = 1 \), i.e., \( x_1, x_2 \in \mathbb{L}^d \) as desired. Let \( \epsilon' = \sinh(\epsilon) \) and \( \epsilon_t \in \mathbb{R}^{d+1} \) denotes the standard basis vector that has its \( i \)-th coordinate equal to 1. Now, consider classifiers of the form

\[
\omega_t = \left( \frac{\epsilon'}{\sqrt{1 + \epsilon_t^2}} v_t \right) \quad \text{where} \quad v_t \in \mathcal{C} \left( d, \arccos \left( \rho \cdot \frac{\sqrt{1 + \delta^2}}{\delta \sqrt{1 + \epsilon_t^2}} \right) \right) \quad \forall \ 1 \leq t \leq T ,
\]

\[ \text{C.36} \]
where $\rho < 1$; and $C \left( d, \arccos \left( \rho \cdot \frac{c' \sqrt{1 + \beta^2}}{\delta \sqrt{1 + \rho^2}} \right) \right)$ be the spherical code with the minimum separation

$\theta = \arccos \left( \rho \cdot \frac{c' \sqrt{1 + \beta^2}}{\delta \sqrt{1 + \rho^2}} \right)$ and size $A(\rho, \theta)$. Since $w_t \cdot w_t = (c')^2 - 1 - (c')^2 = -1$, we have $w_t \cdot w_t < 0$

for all $t$. This guarantees that the intersections of the decision boundaries defined by $\{w_t\}_t$ and $\mathbb{L}^d$ are not empty. Moreover, $\{w_t\}$ is an admissible sequence of classifiers with margin $\leq \epsilon$. To see this, note that, for $t = 1, \ldots, T$,

$$w_t \cdot x_1 = e' > 0$$

$$w_t \cdot x_2 = -e' < 0,$$

i.e., $\{w_t\}$ correctly classifies $\mathcal{S}$. Furthermore, with $-w_t \cdot w_t = 1$, we have

$$\text{asinh} \left( \frac{y_1(w_t \cdot x_1)}{\sqrt{-w_t \cdot w_t}} \right) = \text{asinh} \left( \frac{y_2(w_t \cdot x_2)}{\sqrt{-w_t \cdot w_t}} \right) = \epsilon,$$

which gives $\text{margin}_\mathcal{S}(w_t) = \epsilon$.

Now we perturb $x_1, x_2$ on $\mathbb{L}^d$ such that the magnitude of the perturbation is at most $\alpha$, i.e., we want to find $\bar{x}_1, \bar{x}_2 \in \mathbb{L}^d$ such that both $d_L(x_1, \bar{x}_1)$ and $d_L(x_2, \bar{x}_2)$ are at most $\alpha$. For $1 \leq t \leq T$, consider adversarial examples of the form

$$\bar{x}_{1t} = \left( \frac{\sqrt{1 + \delta^2}}{\delta v_t} \right)$$

and $\bar{x}_{2t} = -\left( \frac{\sqrt{1 + \delta^2}}{\delta v_t} \right)$.

Note that $\bar{x}_{1t}, \bar{x}_{2t} \in \mathbb{L}^d$ as $\bar{x}_{1t} \cdot \bar{x}_{1t} = \bar{x}_{2t} \cdot \bar{x}_{2t} = 1$. Let us verify the two conditions that we require the valid adversarial examples to satisfy:

- **Adversarial budget.** Note that we have

$$d_L(x_1, \bar{x}_{1t}) = d_L(x_2, \bar{x}_{2t}) = \text{acosh}(\sqrt{1 + \delta^2}).$$

Thus, by choosing $\delta = \sqrt{\cosh^2(\alpha) - 1}$, we achieve the maximal permitted perturbation $\alpha$.

- **Inconsistent prediction for the current classifier, i.e.,** $h_{w_t}(\bar{x}_{1t}/2t) \neq h_{w_t}(x_{1/2})$. Note that we have $\delta \geq \alpha > \epsilon$, which further implies that $\delta > \epsilon \geq e'$. In round $t$,

$$w_t \cdot \bar{x}_{1t} = e' \sqrt{1 + \delta^2} - \delta \sqrt{1 + e'^2} < 0$$

$$w_t \cdot \bar{x}_{2t} = -e' \sqrt{1 + \delta^2} + \delta \sqrt{1 + e'^2} > 0,$$

which is a consequence of the relation $\delta > e'$ as follows:

$$\delta^2 > e'^2 \Rightarrow \delta^2 + e'^2 \delta^2 > e'^2 + e'^2 \delta^2 \Rightarrow \delta^2 (1 + e'^2) > e'^2 (1 + e'^2) \Rightarrow \delta \sqrt{1 + e'^2} > e' \sqrt{1 + \delta^2}.$$

Recall that, in each round of Algorithm 2 with an ERM update, we create adversarial examples and add them to the training set, i.e., after round $t$ we have

$$\mathcal{S}_{<t} = \mathcal{S} \cup \bigcup_{i=0}^{t-1} \{(\bar{x}_{1i}, y_{1i}), (\bar{x}_{2i}, y_{2i})\}.$$

Now for each $t$ and any $i < t$, we have

$$w_t \cdot \bar{x}_{1i} = e' \sqrt{1 + \delta^2} - \delta \sqrt{1 + e'^2} \cdot \cos(\theta) > 0$$

$$w_t \cdot \bar{x}_{2i} = -e' \sqrt{1 + \delta^2} + \delta \sqrt{1 + e'^2} \cdot \cos(\theta) < 0,$$
i.e., \( w_t \) linearly separates \( S < t \).
Therefore, \( \{w_t\} \) in (C.36) form an admissible sequence of the classifiers, where \( w_t \) linearly separates \( S_t \) while achieving the margin of at most \( \epsilon \) on the original dataset \( S \). The length of the sequence is bounded by the size of the spherical code \( C(d, e' \cosh(\alpha)) \), which give us that

\[
T = A \left( d, \arccos \left( \frac{\epsilon \sqrt{1 + \frac{\sigma^2}{\delta \sqrt{1 + e'^2}}} \right) \right) = A \left( d, \arccos \left( \frac{\rho \cdot \sinh(\epsilon) \cosh(\alpha)}{\sqrt{\cosh^2(\alpha) - 1 \sqrt{1 + \sinh^2(\epsilon)}}} \right) \right).
\]

The following result (a restatement of Theorem 4.3 from the main text) then follows by applying a lower bound on the maximal size of spherical codes by Shannon.

**Theorem C.16 (Theorem 4.3).** Suppose Algorithm 2 (with an ERM update) outputs a linear separator of \( S \cup S' \). In the worst case, the number of iteration required to achieve the margin at least \( \epsilon \) is \( \Omega(\exp(d)) \).

**Proof.** The statement of the theorem follows from combining Lemma C.15 with Shannon’s lower bound (Theorem A.4) on the maximal size of spherical codes, namely

\[
T \geq (1 + o(1)) \sqrt{2\pi d} \frac{\cos(\theta)}{\sin^{d-1}(\theta)}.
\]

We introduce the shorthand \( \theta =: \arccos \left( \frac{4}{\rho} \right) \), where \( A = \rho \sinh(\epsilon) \cosh(\alpha) \) and \( B = \sqrt{\cosh^2(\alpha) - 1 \sqrt{1 + \sinh^2(\epsilon)}} \), as given by Lemma C.15. We then use two well-known trigonometric identities

\[
\cos(\arccos z) = z \quad \text{and} \quad \sin(\arccos z) = \sqrt{1 - z^2}
\]

to simplify the trigonometric fraction in Shannon’s bound:

\[
\frac{\cos \theta}{\sin^{d-1} \theta} = \frac{A}{B \left( 1 - \frac{A^2}{B^2} \right)^{\frac{d-1}{2}}} = \frac{AB^{d-2}}{(B^2 - A^2)^{\frac{d-1}{2}}}.
\]

For the denominator, note that

\[
B^2 - A^2 = (\cosh^2(\alpha) - 1)(1 + \sinh^2(\epsilon)) - \rho^2 \sinh^2(\epsilon) \cosh^2(\alpha)
\]
\[
= (1 - \rho^2) \sinh^2(\epsilon) \cosh^2(\alpha) + \cosh^2(\alpha) - 1 - \sinh^2(\epsilon)
\]
\[
\overset{(i)}{=} \cosh^2(\alpha) - 1 - \sinh^2(\epsilon)
\]

where (i) follows from the fact that we can choose \( \rho \) arbitrary close to \( 1 \). Putting everything together, we have the lower bound

\[
T \geq (1 + o(1)) \sqrt{2d} \frac{\rho \sinh(\epsilon) \cosh(\alpha) \left( \sqrt{\cosh^2(\alpha) - 1 \sqrt{1 + \sinh^2(\epsilon)}} \right)^{d-2}}{(\cosh^2(\alpha) - 1 - \sinh^2(\epsilon))^{\frac{d-1}{2}}}
\]

which is exponential in \( d \). \( \square \)


D Dimension-distortion trade-off

D.1 Euclidean case

In the Euclidean case, we relate the distance of the support vectors and the size of margin via side length - altitude relations. Let $x, y \in \mathbb{R}^d$ denote support vectors, such that $\langle x, w \rangle > 0$ and $\langle y, w \rangle < 0$ and margin$(w) = c$. We can rotate the decision boundary, such that the support vectors are not unique. Wlog, assume that $x_1, x_2$ are equidistant from the decision boundary and $\|w\| = 1$. In this setting, we show the following relation:

**Theorem D.1 (Thm. 5.1).** $\epsilon' \geq \frac{\epsilon}{c_E}$.

**Proof.** Let $d_1 = d_\mathcal{X}(\phi_E^{-1}(x_1), \phi_E^{-1}(y))$, $d_2 = d_\mathcal{X}(\phi_E^{-1}(x_2), \phi_E^{-1}(y))$ and $d_3 = d_\mathcal{X}(\phi_E^{-1}(x_1), \phi_E^{-1}(x_2))$ the distances between the support vectors in the original space. In the Euclidean embedding space we have

\[
\begin{align*}
    d_1' &= d_E(x_1, y) \geq \frac{d_1}{c_E}, \\
    d_2' &= d_E(x_2, y) \geq \frac{d_2}{c_E}, \\
    d_3' &= d_E(x_1, x_2) \geq \frac{d_3}{c_E}.
\end{align*}
\]

$d_1', d_2', d_3'$ are the side lengths of a triangle, whose altitude is given by the margin: $h = 2\epsilon'$. With Heron’s equation we get

\[
    h = 2\epsilon' = \frac{2}{d_3'} \sqrt{s'(s' - d_1')(s' - d_2')(s' - d_3')},
\]

where $s' = \frac{1}{2}(d_1' + d_2' + d_3')$. In $\mathcal{X}$ we have $s' = \frac{1}{2c_E}(d_1 + d_2 + d_3) = \frac{s}{c_E}$. Then we have with respect to the actual distance relations

\[
    h = 2\epsilon' \geq \frac{2}{c_E d_3'} \sqrt{c_E^4 s(s - d_1)(s - d_2)(s - d_3)} = 2\frac{\epsilon}{c_E},
\]

which gives the claim. \qed

D.2 Hyperbolic case

As in the Euclidean case, we want to relate the margin to the distance of the support vectors. Since the distortion can be expressed in terms of the distances of support vector in the original and the embedding space, this allows us to study the influence of distortion on the margin.

We will derive the relation in the half-space model ($\mathbb{P}^2$). However, since the theoretical guarantees above consider the upper sheet of the Lorentz model ($\mathbb{L}^d$), we have to map between the two spaces.

**Assumption 4.** We make the following assumptions on the underlying data $\mathcal{X}$ and the embedding $\phi_H$:

1. $\mathcal{X}$ is linearly separable;
2. $\mathcal{X}$ is hierarchical, i.e., has a partial order relation;
3. $\phi_H$ preserves the partial order relation and the root is mapped onto the origin of the embedding space.

Under these assumptions, the hyperbolic embedding $\phi_H$ has two sources of distortion:
1. the (multiplicative) distortion of pairwise distances, measured by the factor \( \frac{1}{c_H} \);

2. the distortion of order relations, in most embedding models captured by the alignment of ranks with the Euclidean norm.

Under Ass. 4, order relationships are preserved and the root is mapped to the origin. Therefore, the distortion on the Euclidean norms is given as follows:

\[
\|\phi_H(x)\| = d_E(\phi_H(x), \phi_H(0)) = \frac{d_X(x, 0)}{c_H},
\]

i.e., the distortion on both pairwise distances and norms is given by a factor \( \frac{1}{c_H} \).

**Note on notation:** In the following, a bar over any symbol indicates the Euclidean expression.

### D.2.1 Mapping from \( L^d_+ \) to \( P^2 \)

First, note that a transformation \( v \mapsto Bv \) with \( B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \) and an orthogonal matrix \( A \) is isometric, i.e., it preserves the Minkowski product [Cho et al., 2019]:

\[
(Bu) \ast (Bv) = u_0 v_0 - u_{1:d'}^T A^T v_{1:d'} = u_0 v_0 - u_{1:d'}^T v_{1:d'} = u \ast v.
\]

Setting the first column of \( A \) to \( \frac{w_1}{\|w_1\|} \) we can isometrically transform the decision hyperplane as \( \hat{w} = Bw = (\hat{w}_0, \|\hat{w}_{1:d'}\|, 0, \ldots, 0) \). Analogously, we can transform any point in \( L^d_+ \). In the following, we will use the shorthand \( \lambda = \frac{w_0}{w_1} \). We can then use the maps defined in section A.2 to map \( \hat{x} = Bx \in L^2_+ \) onto \( z \in P^2 \), i.e. applying \( (\pi_{BP} \circ (\pi_{LB} \circ B)) \) to any \( x \in L^2_+ \) gives \( z \in P^2 \).

**Remark D.2** (Effect of hyperbolic distortion on Euclidean distances in the Poincare half plane). Note that the hyperbolic distance in the Poincare half plane can be written as follows:

\[
d_P((x_0, x_1), (y_0, y_1)) = 2 \text{asinh} \left( \frac{1}{2} \sqrt{\frac{(x_0 - y_0)^2 + (x_1 - y_1)^2}{x_1 y_1}} \right)
\]

If \( c_H \) denotes the hyperbolic distortion, we get

\[
d'_P = \frac{d_P}{c_H} = 2 \text{asinh} \left( \frac{1}{2} \frac{d'_E}{\sqrt{x_1 y_1}} \right)
\]

\[
\Rightarrow \frac{1}{2} \frac{d_E}{\sqrt{x_1 y_1}} = \text{sinh} \left( \frac{2 \text{asinh} \left( \frac{1}{2} \frac{d_E}{\sqrt{x_1 y_1}} \right)}{2c_H} \right) \gtrsim \frac{1}{2} \frac{d_E}{c_H \sqrt{x_1 y_1}}.
\]

This suggests, that the effect of hyperbolic distortion on the Euclidean distances can be quantified by a comparable factor, i.e. \( d'_E \gtrsim \frac{d_E}{c_H} \).

**Lemma D.3** (Relation between h-margin and E-margin). Let \( \gamma \) be the margin of a hyperbolic classifier \( w \in \mathbb{R}^{d+1} \). Then the Euclidean margin \( \tilde{\gamma} \) of \( w \) is bounded as follows: \( \tilde{\gamma} \geq \text{sinh}(\gamma) \).
Proof. We again write the hyperbolic distance in the Poincare half plane in terms of the Euclidean distance of the ambient space:

\[
d_P((x_0, x_1), (y_0, y_1)) = 2 \text{asinh} \left( \frac{1}{2} \sqrt{\frac{(x_0 - y_0)^2 + (x_1 - y_1)^2}{x_1y_1}} \right) = 2 \text{asinh} \left( \frac{1}{2} \frac{d_E((x_0, x_1), (y_0, y_1))}{\sqrt{x_1y_1}} \right),
\]

where \( y \in \mathcal{H}_w \) is the point closest to the support vector \( x \in \mathcal{L}_{+1} \) on the decision boundary. Therefore, the hyperbolic margin is \( d_P(x, y) = \gamma \) and the Euclidean margin is \( d_E(x, y) = \bar{\gamma} \).

Since we mapped the feature space onto the Poincare half plane, \( y \) has the coordinates \( y = (\tilde{y}_0, \tilde{y}_1, 0, \ldots, 0) \) where \( \tilde{y}_0 = y_0 \) and \( \tilde{y}_1 = \frac{w^T y'}{\|w'\|} \). Similarly, \( x \) has the coordinates \( x = (\tilde{x}_0, \tilde{x}_1, 0, \ldots, 0) \). The transformation preserves the Minkowski product. Therefore we have

\[
y \cdot y = y_0^2 - y_1^2 = \tilde{y}_0^2 - \left( \frac{w^T y'}{\|w'\|} \right)^2 = 1
\]

and similarly \( x \cdot x = \tilde{x}_0^2 - \tilde{x}_1^2 = 1 \). This implies

(1) \( \tilde{y}_1 = \sqrt{\tilde{y}_0^2 - 1}, \quad \tilde{x}_1 = \sqrt{\tilde{x}_0^2 - 1} \),

and further

(2) \( \tilde{x}_0, \tilde{y}_0 \geq 1 \).

We want to show that \( \tilde{x}_1 \tilde{y}_1 \geq 1 \). For this, first, note that since \( y \in \mathcal{H}_w \) and the hyperbolic margin is \( \gamma \), we have

\[
0 = w \cdot y = w_0 y_0 - w^T y' \\
\Rightarrow w^T y' = w_0 y_0.
\]

This gives

\[
y \cdot y = y_0^2 - \frac{w_0^2 y_0^2}{\|w'\|^2} = 1
\]

\[
\Rightarrow 0 = y_0^2 - \frac{w_0^2 y_0^2}{\|w'\|^2} - 1,
\]

and therefore

(3) \( y_0 = \frac{1}{\sqrt{1 - \frac{w_0^2}{\|w'\|^2}}} \).

Since the hyperbolic margin is \( \gamma \), we further have

\[
d_P(x, y) = \text{acosh}(x \cdot y) \geq \gamma \Rightarrow x \cdot y \geq \cosh(\gamma) \geq 1,
\]

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and therefore

\[ x_0 y_0 - x_1 y_1 \geq 1 \]

\[ x_0 y_0 - \sqrt{x_0^2 - 1} \sqrt{y_0^2 - 1} \geq 1 \]

\[ (x_0 y_0 - 1)^2 \geq (x_0^2 - 1)(y_0^2 - 1) \]

\[ x_0^2 y_0^2 - 2x_0 y_0 + 1 \geq x_0^2 y_0^2 - x_0^2 - y_0^2 + 1 \]

\[ \Rightarrow 0 \leq (x_0 - y_0)^2 \]

which implies

\[ 4 \quad x_0 \geq y_0 \]

This gives for \( x_1 y_1 \) the following:

\[ x_1 y_1 = \frac{1}{1 - \frac{w_0^2}{\|w\|^2}} - 1 \]

\[ = \frac{w_0^2}{\|w\|^2 - w_0^2} \]

By assumption we have \( w \ast w = w_0^2 - \|w\|^2 = -1 \), which gives for the denominator \( -w_0^2 + \|w\|^2 = 1 \).

It remains to show that \( w_0^2 \geq 1 \).

For this last step, we want to show that mass concentrates on \( w_0 \) as the classifier is updated, ensuring \( w_0 \geq 1 \). By construction, we have initially \( w \ast w = -1 \). Wlog, assume that initially \( w_0 \geq 1 \). An initialization of this form can always be found, e.g., by setting \( w = (a, \sqrt{1 + a^2}, 0, \ldots, 0) \) for some \( a \geq 0 \). If the \( i \)th update is negative (\( y_i x_0 \leq 0 \)), then \( |w| \) will initially decrease, but the normalization step will scale away the effect on \( w_0 \). However, if the \( i \)th update is non-negative (\( y_i x_0 \geq 0 \)), it will increase \( w_0 \). Over time, the positive updates concentrate the mass on \( w_0 \). Since we initialized to \( w_0 \geq 1 \), the condition will always stay valid. With the arguments above, this implies \( x_1 y_1 \geq 1 \). Inserting the latter in the expression above, we get

\[ d_H = 2 \text{asinh} \left( \frac{1}{2} \left( \frac{d_E}{\sqrt{x_1 y_1}} \right) \right) \leq 2 \text{asinh} \left( \frac{d_E}{2} \right) \]

\[ \Rightarrow d_E \geq 2 \text{sinh} \left( \frac{d_H}{2} \right) \geq \text{sinh}(d_H) \].

\[ \square \]

**D.2.2 Characterizing the margin**

In \( \mathbb{P}^2 \) the decision hyperplane corresponding to \( \hat{w} = B w \) corresponds to a hypercircle \( K_w \). One can show, that its radius is given by \( r_w = \sqrt{\frac{1 - \lambda}{\lambda + 1}} \) [Cho et al., 2019], by computing the hyperbolic distance between a point on the decision boundary and one of the hypercircle’s ideal points. Further note, that the support vectors lie on hypercircles \( K_x \) and \( K_y \) which correspond to the set of points of hyperbolic distance \( \epsilon \) (i.e., the margin) from the decision boundary. We again assume wlog that at least one support vector is not unique and let \( x_1, x_2 \in K_x \) and \( y \in K_y \) (see Fig. 5).
Theorem D.4 (Thm. 5.3). $\epsilon' \approx \epsilon$.

Proof. Our proof consists of three steps:

**Step 1: Find Euclidean radii and centers of hypercircles.** The hypercircles $\mathcal{K}_x, \mathcal{K}_y$ correspond to arcs of Euclidean circles $\bar{\mathcal{K}}_x, \bar{\mathcal{K}}_y$ in the full plane that are related through circle inversion on the decision circle $\bar{\mathcal{K}}_w$ (i.e., the Euclidean circle corresponding to $\mathcal{K}_w$); see Fig. 5. We can construct a “mirror point” $y' \in \bar{\mathcal{K}}_x$ of $y$ by circle inversion on $\bar{\mathcal{K}}_w$. We have the following (Euclidean) distance relations: The circle inversion gives

$$\bar{d}(y', \bar{c}_w) \bar{d}(y, \bar{c}_w) = r_w^2 ,$$

where $\bar{c}_w$ denotes the center of $\bar{\mathcal{K}}_w$. Furthermore, we have (see Fig. 7)

$$\bar{d}(y, \bar{c}_w) = \bar{d}(y', \bar{c}_w) + \bar{d}(y, y') .$$

Putting both together, we get an expression for the Euclidean distance of $y$ and $y'$:

$$\bar{d}(y, y') = \bar{d}(\bar{c}_w, y) - \frac{r_w^2}{\bar{d}(ar{c}_w, y)} .$$
Here, we have by construction $\bar{c}_w = (0, a, 0, \ldots, 0)$ with a free parameter $a$. Wlog, assume $\bar{c}_w = (0, -1, 0, \ldots, 0)$. Next, consider the triangle $\Delta(x_1, x_2, y)$. We can express its altitude $h$ in terms of the side length $\bar{d}(x_1, x_2) =: d_1$, $\bar{d}(x_1, y) =: d_2$ and $\bar{d}(x_2, y) =: d_3$ via Heron’s formula:

\[
h = \frac{2}{d_1} \sqrt{s(s - d_1)(s - d_2)(s - d_3)},
\]

where $s = \frac{1}{2}(d_1 + d_2 + d_3)$. Now, consider the triangle $\Delta(x_1, x_2, y')$. Due to the relation between $y$ and $y'$, its altitude $h_x$ is related to $h$ as

\[
(2) \quad h_x = h - \bar{d}(y, y').
\]

With the side length - altitude relations given in $\Delta(x_1, x_2, y)$ and (2), we can compute the length of the other sides $\bar{d}(x_1, y')$ and $\bar{d}(x_2, y')$ as follows (with Pythagoras theorem):

\[
\bar{d}(x_1, y') = \left(\bar{d}_x^2 + \bar{d}(x_1, y)^2 - h^2\right)^{1/2},
\]

\[
\bar{d}(x_2, y') = \left(\bar{d}_x^2 + \bar{d}(x_2, y)^2 - h^2\right)^{1/2}.
\]

With that, we can compute the radius of $\bar{K}_x$ as follows: $\bar{K}_x$ circumscribes $\Delta(x_1, x_2, y')$, therefore its radius $\bar{r}_x$ can be computed via Heron’s formula as

\[
\bar{r}_x = \frac{\bar{d}(x_1, y') + \bar{d}(x_2, y') + \bar{d}(x_1, x_2)}{4A}
\]

\[
A = \sqrt{s(s - \bar{d}(x_1, y'))(s - \bar{d}(x_2, y'))(s - \bar{d}(x_1, x_2))}
\]

where $s = \frac{1}{2}(\bar{d}(x_1, y') + \bar{d}(x_2, y') + \bar{d}(x_1, x_2))$. With an analog construction, we can compute the radius $\bar{r}_y$ of $\bar{K}_y$ as function of $\bar{d}(x'_1, x'_2), \bar{d}(x'_1, y)$ and $\bar{d}(x'_2, y)$ via relations in the triangle $\Delta(x'_1, x'_2, y)$.

**Step 2: Express h-margin as distance between hypercircles.** As shown in Fig. 6, the margin is the hyperbolic distance from a point on $\bar{K}_x, \bar{K}_y$ to $\bar{K}_w$, corresponding to the length of a geodesic connecting the point with the closest point on $\bar{K}_w$. Let $v \in \bar{K}_x$ and $u \in \bar{K}_w$ the closest point on the decision
circle. From the geometry of the Poincaré half plane we know that there exists a Möbius transform \( \theta \in \text{Möb}(\mathbb{P}^2) \) such that the images \( \theta(u) = i\mu \) and \( \theta(v) = iv \) of \( u, v \) lie on the positive imaginary axis. Since the hyperbolic distance is invariant under Möbius transforms, we get

\[
d(u, v) = d(\theta(u), \theta(v)) = d(i\mu, iv) = \left| \log \frac{v}{\mu} \right|
\]

Similarly, we can express the distance between between support vectors \( x \in \mathcal{K}_x \) and \( y \in \mathcal{K}_y \), which is twice the hyperbolic margin: Let \( \theta(x) = i\mu_x \) and \( \theta(y) = i\mu_y \), where \( \mu_x, \mu_y \) are given by the intersection points of \( \mathcal{K}_x, \mathcal{K}_y \) with the imaginary axis. Then

\[
2\varepsilon = d(x, y) = \left| \log \frac{\mu_y}{\mu_x} \right|
\]

We can express \( \mu_x, \mu_y \) in terms of the centers and radii of \( \mathcal{K}_x, \mathcal{K}_y \) as follows (Fig. 6)

\[
\begin{align*}
\mu_x &= \bar{c}_x^{(2)} + \bar{r}_x \\
\mu_y &= \bar{c}_y^{(2)} + \bar{r}_y
\end{align*}
\]

where \( c^{(2)} \) denotes the second coordinate of the point \( c \in \mathbb{P}^m \). Putting everything together, we get the following expression for the margin:

\[
\varepsilon = \frac{1}{2} \left| \log \frac{\bar{c}_y^{(2)} + \bar{r}_y}{\bar{c}_x^{(2)} + \bar{r}_x} \right|
\]

**Step 3: Evaluate Distortion.** As discussed above (Prop. 5.1), the influence of distortion on the altitude \( h \) in the triangle \( \Delta(x_1, x_2, y) \) is given by the factor \( \frac{1}{c_H} \).

\[
\bar{h} = \frac{h}{c_H}
\]

\( \bar{r}_x \) depends on pairwise distances between support vectors and \( h \), which are distorted by a factor \( \frac{1}{c_H} \) (by assumption on \( \phi_H \) and 4). \( \bar{r}_x \) depends further on \( h \), which in turn depends on \( d(c_w, y) \). The latter depends on the Euclidean norm of the support vector \( y \), i.e., \( \| y \| \). With Ass. 4 the total multiplicative distortion is then at most of a factor \( \frac{1}{c_H} \). We can derive an analogue result for \( \bar{r}_y \). For the center \( \bar{c}_x \) note the following:

\[
\bar{c}_x^{(2)} = \frac{1}{2} \left[ (1 - \bar{r}_w^2)\bar{x}_0 - (1 + \bar{r}_w^2)\bar{x}_1 \right]
\]

where \( (\bar{x}_0, \bar{x}_1, 0, \ldots, 0) = \bar{x} = \left( \pi_{BP} \circ (\pi_{LB} \circ B) \right) \) and \( \bar{r}_w = \sqrt{\frac{1 - \lambda}{1 + \lambda}} \). Rewriting

\[
\begin{align*}
(1 - \bar{r}_w^2)\bar{x}_0 &= \frac{2\bar{w}_0\bar{x}_0}{\bar{w}_1 + \bar{w}_0} \\
(1 + \bar{r}_w^2)\bar{x}_1 &= \frac{2\bar{w}_1\bar{x}_1}{\bar{w}_1 + \bar{w}_0}
\end{align*}
\]

we get

\[
\bar{c}_x^{(2)} = \frac{\bar{w}^T\bar{x}}{\bar{w}_0 + \bar{w}_1}
\]
Similarly, one can derive

\[ \hat{c}_y^{(2)} = \frac{\hat{w}^T \hat{y}}{\hat{w}_0 + \hat{w}_1}, \]

for \((\bar{y}_0, \bar{y}_1, 0, \ldots, 0) = \bar{y} = (\pi_{BP} \circ (\pi_{LB} \circ B))\). Both are only affected by distortion of the form (2), i.e. the multiplicative distortion is given by a factor \(\frac{1}{c_H}\). Inserting this into the margin expression (4) gives

\[
\epsilon' = \frac{1}{2} \left| \log \frac{c_y + r_y}{c_x + r_x} \right| \geq \frac{1}{2} \left| \log \frac{c_y + r_y}{c_H c_x + c_H r_x} \right| = \frac{1}{2} \left| \log \left( 1 + \frac{c_y + r_y}{c_H c_x + c_H r_x} \right) \right|
\]

\[
= \frac{1}{2} \left| \log \frac{1}{c_H} + \log \frac{c_y + r_y}{c_x + r_x} \right| \approx \frac{1}{2} \left| \log \frac{c_y + r_y}{c_x + r_x} \right| = \epsilon,
\]

where (†) follows from \(c_H = O(1 + \epsilon)\) with \(\epsilon > 0\) small, by Thm. A.3.

\[ \square \]

### E Adversarial perceptron

With the geometric tools introduced in Appendix D, we can now also prove Lemma C.4. We restate the result from the main text:

**Lemma E.1.** *(Adversarial perceptron, Lem. C.4)* Let \(\bar{w}\) be the max-margin classifier of \(\mathcal{S}\) with margin \(\gamma\). At each iteration of Alg. 2, \(\bar{w}\) linearly separates \(\mathcal{S} \cup \mathcal{S}'\) with margin at least \(\gamma / \cosh(\alpha)\).

**Proof.** In the following, we again use the shorthand \(|u| = \sqrt{\pm u \cdot u}\), with "+", if \(u\) is space-like (i.e., \(u \cdot u > 0\)) and "-", if \(u\) is time-like (i.e., \(u \cdot u < 0\)). Since \(\bar{w}\) is "time-like" and \(x, \bar{x}\) space-like, we have

\[
\begin{align*}
\quad &\quad |\bar{w} \ast x| = |\bar{w}| \cdot |x| \cdot \cosh \angle (\bar{w}, x) \\
\quad &\quad |\bar{w} \ast \bar{x}| = |\bar{w}| \cdot |\bar{x}| \cdot \cosh \angle (\bar{w}, \bar{x}).
\end{align*}
\]

To prove the statement, we first transform the problem from the Lorentz model \(L^d\) to the Poincaré half plane \(\mathbb{H}^2\) using the map \((\pi_{BP} \circ (\pi_{LB} \circ B))\). Then the adversarial margin is given by the Euclidean distance of the hypercircle \(K_{\bar{x}}\) through \(\bar{x}\) and the decision hypercircle \(K_{\bar{w}}\). First, note that we can express this as the hyperbolic distance of the points \((0, \theta(\bar{x}))\) and \((0, r_w)\), where \(\theta \in \text{Möb}(\mathbb{H}^2)\) is a Möbius transform that maps \(\bar{x}\) to the imaginary axis. Importantly, any such \(\theta\) leaves the Minkowski product invariant. One can show [Cho et al., 2019] that

\[
\theta(\bar{x}) = c_{\bar{x}} + \sqrt{c_{\bar{x}}^2 + r_w}
\]

where \(c_{\bar{x}} = \frac{1}{2} \left((1 - r_w^2)\bar{x}_0 - (1 + r_w^2)\bar{x}_1\right)\) is the Euclidean center of \(K_{\bar{x}}\). The hyperbolic distance is then given by

\[
\quad \begin{align*}
\quad &\quad \left| \log \frac{\theta(\bar{x})}{r_w} \right| = \left| \log \frac{c_{\bar{x}}}{r_w} + \sqrt{c_{\bar{x}}^2 + 1} \right| = \sinh \left( \frac{c_{\bar{x}}}{r_w} \right).
\end{align*}
\]

Note, that

\[
\quad \frac{c_{\bar{x}}}{r_w} = \frac{1}{2} \left[ \left( \frac{1}{r_w} - r_w \right) \bar{x}_0 - \left( \frac{1}{r_w} + r_w \right) \bar{x}_1 \right],
\]

\[42\]
where
\[
\frac{1}{r_w} - r_w = \sqrt{\frac{1 + \lambda}{1 - \lambda}} - \sqrt{\frac{1 - \lambda}{1 + \lambda}} = \frac{2\lambda}{\sqrt{1 - \lambda^2}} = \frac{2w_0}{\sqrt{w_1^2 - w_0^2}}
\]
\[
\frac{1}{r_w} + r_w = \frac{2}{\sqrt{1 - \lambda^2}} = \frac{2w_1}{\sqrt{w_1^2 - w_0^2}}.
\]

This gives
\[
\begin{align*}
\text{(3)} \quad & \left| \text{asinh} \left( \frac{c_x}{r_w} \right) \right| = \left| \text{asinh} \left( \frac{w_0 \hat{x}_0 - w_1 \hat{x}_0}{\sqrt{w_1^2 - w_0^2}} \right) \right| = \left| \text{asinh} \left( \frac{w \ast \tilde{x}}{|w|} \right) \right|. \\
\end{align*}
\]

Using (2), we can express the adversarial margin in terms of the margin and the distance between features and adversarial samples as follows:
\[
\left| \text{asinh} \left( \frac{c_x}{w} \right) \right| = \left| \text{asinh} \left( \frac{c_x}{c_x} \frac{c_x}{r_w} \right) \right| \geq \dagger \left| \text{asinh} \left( \frac{c_x}{c_x} \sinh(\gamma) \right) \right|,
\]
where (\dagger) follows from the assumption that \( y(w \ast x) \geq \sinh(\gamma) \) (with margin \( \gamma \)). We further show above that we can express the Euclidean centers as
\[
c_x = \frac{w \ast x}{w_0 + w_1}, \quad c_{\tilde{x}} = \frac{w \ast \tilde{x}}{w_0 + w_1}.
\]

Wlog, assume that \( w \ast x > 0 \); then \( w \ast \tilde{x} < 0 \) and therefore
\[
c_x = \frac{|w \ast x|}{w_0 + w_1}, \quad c_{\tilde{x}} = -\frac{|w \ast \tilde{x}|}{w_0 + w_1}.
\]

Inserting (1) above in (3), we get
\[
\text{asinh} \left( -\frac{|w \ast \tilde{x}|}{|w \ast x|} \sinh(\gamma) \right) = \text{asinh} \left( -\frac{|w|}{|w|} \frac{|\tilde{x}|}{|x|} \cosh(\langle w, \tilde{x} \rangle) \sinh(\gamma) \right)
\]
\[
= \text{asinh} \left( -\frac{|\tilde{x}|}{|x|} \cosh(\langle w, \tilde{x} \rangle) \sinh(\gamma) \right)
\]
\[
\geq \dagger \text{asinh} \left( -\frac{|\tilde{x}|}{|x|} \sinh(\gamma) \right),
\]
where (\dagger) follows from \( w \) being a better classifier for \( x \) than for \( \tilde{x} \) by construction. Therefore, we have
\[
\left| \text{asinh} \left( -\frac{|w \ast \tilde{x}|}{|w \ast x|} \sinh(\gamma) \right) \right| \geq \left| \text{asinh} \left( \frac{|\tilde{x}|}{|x|} \sinh(\gamma) \right) \right|.
\]

Furthermore, note, that by construction we have \( d_L(x, \tilde{x}) \leq \alpha \) and therefore:
\[
\text{acosh}(x \ast \tilde{x}) \leq \alpha \Rightarrow x \ast \tilde{x} \leq \cosh(\alpha).
\]

Since \( x, \tilde{x} \) are both space-like, we further have \( |x| \mid \tilde{x} \mid \leq x \ast \tilde{x} \). In summary, this gives
\[
\text{(4)} \quad |x| \leq \frac{\cosh(\alpha)}{|x|}.
\]
Inserting (4) above, we get
\[
\text{asinh} \left( \frac{|\tilde{x}|}{|x|} \sinh(\gamma) \right) \geq \frac{4}{\gamma} \text{asinh} \left( \frac{|\tilde{x}|^2}{\cosh(\alpha)} \sinh(\gamma) \right)
\]
\[= \frac{4}{\gamma} \text{asinh} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right),
\]
where (‡) follows from $|\tilde{x}|^2 = \tilde{x} \ast \tilde{x} = 1$, since $\tilde{x} \in \mathbb{L}^m$. Finally, the claim follows from
\[
\text{asinh} \left( \frac{\sinh(\gamma)}{\cosh(\alpha)} \right) \geq \frac{\gamma}{\cosh(\alpha)}.
\]

\[\square\]

F Additional Experimental Results

F.1 Hyperbolic perceptron

To validate the hyperbolic perceptron algorithm, we performed two simple classification experiments. First, we picked two classes from the ImageNet dataset (n09246464 and n07831146) such that they are linearly separable in hyperbolic space. In the chosen set there were 1,648 positive and 1,287 negative examples. We observed that hyperbolic perceptron can successfully classify the points into the two groups. In a second experiment, we try hyperbolic perceptron on a linearly non-separable dataset. The algorithm was still able to classify reasonably well.

F.2 Adversarial Gradient decent

F.2.1 Choice of loss function

We want to discuss briefly our choice of loss function throughout the paper. Following the large body of work on large-margin learning in Euclidean space, we tested our approach with the classic hinge (Eq. C.2) and least squares losses (Eq. C.3). While both algorithms work well in practice (see sec. 6), they do not fulfill Ass. 1 on the whole domain. Therefore, our theoretical guarantees are not valid for those loss functions.

We derive theoretical results for the hyperbolic logistic loss (Eq. C.8) instead, which fulfills Ass. 1. Unfortunately, the hyperparameter $R_\alpha$ is difficult to determine in practice. We therefore decided to omit validation experiments with the hyperbolic logistic loss.

F.3 Dimension-distortion trade-off

Euclidean embeddings computed using implementation in Nickel and Kiela [2017] by Facebook Research\(^2\).

| $d$  | Euclidean | Hyperbolic |
|------|-----------|------------|
| 4    | 0.54      | 0.51       |
| 8    | 0.53      | 1.00       |
| 16   | 0.68      | 1.00       |

Table 2: Classification performance in hyperbolic vs. Euclidean space of dimension $d$.

\(^2\)https://github.com/facebookresearch/poincare-embeddings