Spin correlations in polarizations of P-wave charmonia $\chi_{cJ}$ and impact on $J/\psi$ polarization

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Based on a general form of the effective vertex functions for the decays of P-wave charmonia $\chi_{cJ}$, angular distribution formulas for the subsequent decays $\chi_{cJ} \rightarrow J/\psi \gamma$ decay and $J/\psi \rightarrow \mu^+\mu^-$ are derived. The formulas are the same as those obtained in a different approach in the literature. Our formulas are expressed in a more general form, including parity violation effects and the full angular dependence of $J/\psi$ and muon in the cascade decay $\chi_{cJ} \rightarrow J/\psi \gamma \rightarrow \mu^+\mu^-\gamma$. The $\chi_{cJ}$ polarization observables are expressed in terms of rational functions of the spin density matrix elements of $\chi_{cJ}$ production. Generalized rotation-invariant relations for arbitrary integer-spin particles are also derived and their expressions in terms of observable angular distribution parameters are given in the $\chi_{c1}$ and $\chi_{c2}$. To complement our previous direct-$J/\psi$ polarization result, we also discuss the impact on the observable prompt-$J/\psi$ polarization. As an illustrative application of our angular distribution formulas, we present the angular distributions in terms of the tree-level spin density matrix elements of $\chi_{cJ}$ and $\chi_{cJ}$ production in several different frames at the Large Hadron Collider. Moreover, a reweighting method is also proposed to determine the entire set of the production spin density matrix elements of the $\chi_{cJ}$, some of which disappear or are suppressed for vanishing higher-order multipole effects making the complete extraction difficult experimentally.

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I. INTRODUCTION

The polarization of heavy quarkonium in hadroproduction e.g. at the Tevatron and the LHC, is a longstanding issue in heavy quarkonium physics [1]. Nonrelativistic QCD (NRQCD) [2], a rigorous effective field theory founded on the nonrelativistic nature of heavy quarkonium, foresees that a $Q\bar{Q}$ pair may be formed in a color-octet (CO) state during the hard reaction at short-distances before it hadronizes into a color-singlet (CS) physical quarkonium by radiating soft gluons. In particular, the $J/\psi$ ($c\bar{c}$ bound state with quantum number $J^{PC}=1^{--}$), when produced at high transverse momentum ($p_T$), should predominantly originate from gluon fragmentation into $c\bar{c}[S_1^{[1]}]$, then evolve into the observed $c\bar{c}[S_1^{[1]}]$ [3]. The gluon fragmentation mechanism guarantees that the $J/\psi$ is produced transversely polarized in the helicity (HX) frame when its $p_T$ is sufficiently large. However, the data measured by the CDF [4, 5] Collaboration at the Tevatron indicate that the $J/\psi$ is mainly unpolarized and even slightly longitudinally polarized at large $p_T$, up to 20 GeV. This is the “polarization puzzle” of heavy quarkonium production.

The understanding of charmonium polarization is also important for the simulations in the experimental analyses: the detector acceptance for lepton pairs from the decay of $J/\psi$ (or other heavy quarkonia) strongly depends on the $J/\psi$ polarization [6]. The lack of a consistent description of the polarization in the simulation of quarkonium production results in one of the largest systematic uncertainties affecting the precision of cross section measurements.

Experimentally, measurements of direct-$J/\psi$ production at hadron colliders are incomplete. The measured prompt $J/\psi$ data include both direct production and feed-down contributions from $\chi_e$ and $\psi'$, through the decays $\chi_e \rightarrow J/\psi \gamma$ and $\psi' \rightarrow J/\psi \pi\pi$ (plus a small contribution of $\psi' \rightarrow \chi_{cJ} \rightarrow J/\psi \gamma \gamma$). Therefore, in order to compare the theoretical results with experimental data, the $\chi_e$ and $\psi'$ yield and polarizations must also be calculated. Moreover, the $\chi_e$ meson has its own phenomenological interest. The ratio of the differential cross sections for the $\chi_{c1}$ and $\chi_{c2}$ inclusive productions at the Tevatron has been measured by the CDF Collaboration [7]. Their results show that the ratio disagrees with the spin symmetry expectation from the leading-order (LO) computation. After including the next-to-leading (NLO) QCD radiative correction [8], the asymptotic behavior of $\frac{d\sigma}{dp_T}$ changes from $p_T^{-6}$ at LO to $p_T^{-4}$ at NLO for the $\frac{d\sigma}{dp_T}$ channel and, hence, becomes comparable to the contribution of $S_1^{[8]}$ at large $p_T$. The result provides an opportunity to solve the contradiction between the experiment and the theoretical prediction. The recent LHCb result [9] for $\frac{d\sigma}{dp_T}$ stays within the error bars of the NLO NRQCD prediction. Surely, as in the $J/\psi$ case, the investigation of polarization of $\chi_c$ will also be very helpful in understanding the charmonium production mechanisms in QCD.

We now briefly review the recent progress in the theory of heavy quarkonium hadroproduction. In Ref. [10], it was found that the NLO prediction for the direct-$J/\psi$
yield in the $^3S_1$ channel is 2 orders of magnitude larger than the LO one at large $p_T$, while NLO corrections for the CO S wave are small [11]. For the P wave, the NLO corrections for the $^3P_J$ channel [8] and $^3P_J$ channel [12–14] are found to be very large but negative. As for the polarization, the NLO QCD correction [15] for the direct-$J/\psi$ in the CS changes it from being transverse (LO) to longitudinal (NLO) in the HX frame. This can be understood in collinear factorization up to the NLO power in $m_c^2/\mu^2$ [16]. However, even after including NNLO* corrections [17], in which only tree-level diagrams at $\alpha_S^5$ are considered and infrared cutoffs are imposed to avoid soft and collinear divergences, theoretical predictions of CS contributions to the yields and polarizations are still in disagreement with the CDF data [4, 5]. Recently, two groups [18, 19] have presented their NLO results for the direct-$J/\psi$ production at hadron colliders, but drawn very different conclusions due to different treatments for the feed-down contributions from arbitrary integer-spin particles. In Sec. VII, we estimate rotation-invariant relations from the vector boson to arbitrary integer-spin particles. In the paper. In the next three sections, we derive the formulas of the decay angular distributions for spin-1 and spin-2 bosons taking into account higher-order radiative multipoles and allowing for parity-violating effects, are presented in the Appendixes A, B, C. A reweighting method is also proposed to extract the complete set of the SDMEs of the $\chi_{c2}$ in Appendix D.

II. KINEMATICS AND CONVENTIONS

In this section, we introduce the conventions and kinematics for our derivations performed in the following sections. Apart from $\chi_c \rightarrow J/\psi + \gamma$, we also consider the subsequent $J/\psi \rightarrow \mu^+ \mu^-$. The spin quantization axis $\vec{s}$ can be chosen arbitrarily in the rest frame of the decaying particle. Generally, the polarization vectors for a massive spin-1 particle are

$$
e_0 = (|\vec{k}|, E \sin \theta \cos \phi, E \sin \theta \sin \phi, E \cos \theta)/m,$$

$$
e_\pm = \frac{e^{\mp \gamma}}{\sqrt{2}}(0, \mp \cos \theta \cos \phi \pm i \sin \phi, 
\mp \cos \theta \sin \phi - i \cos \phi, \pm \sin \theta),$$

(1)

where $\theta$ and $\phi$ are the polar and azimuthal decay angles with respect to $\vec{s}$ and a chosen plane, 1 and the symbol $\gamma$ can be chosen as an arbitrary real number. $E$, $|\vec{k}|$, and $m$ are the particle’s energy, momentum, and mass. We set $\gamma = -\phi$ here. For a spin-2 tensor particle, its spin wave functions can be constructed from the spin-1 polarization four-vectors as

$$
e^\mu_\lambda = \sum_{\lambda_1, \lambda_2 = -1}^1 (1, \lambda_1; 1, \lambda_2|2, \lambda) \epsilon^\mu_\lambda \epsilon_\lambda^{\mu\nu},$$

(2)

where $(1, \lambda_1; 1, \lambda_2|2, \lambda)$ are the Clebsch-Gordan coefficients, and $\lambda, \lambda_1, \lambda_2$ denote the angular distribution components along the spin-quantization axis $\vec{s}$. Thus, we have the identities $p_\mu \epsilon^\mu_\lambda = p_\mu \epsilon^{\mu\nu}_\lambda = p_\nu \epsilon^\nu_\lambda = (\epsilon_\lambda)_\mu = 0$ and $\epsilon^{2\lambda}_\lambda = \epsilon^{\lambda\mu}_\mu$.

The $\chi_{cJ} \rightarrow J/\psi \gamma$ angular distribution can be written in terms of the $\chi_{cJ}$ production SDMEs $\rho_{\lambda\lambda'}$ and of the decay SDMEs $D_{\lambda\lambda'}$,

$$W(\theta, \phi) = \sum_{\lambda, \lambda' = -J}^J \rho_{\lambda\lambda'} D_{\lambda\lambda'}(\theta, \phi),$$

(3)

where $\theta$ and $\phi$ are the angles parameterizing the $J/\psi$ direction in the $\chi_{cJ}$ rest frame. Here, $\rho_{\lambda\lambda'}$ and $D_{\lambda\lambda'}$ represent the production and decay amplitudes of the $\chi_{cJ}$ with angular momentum projector component $\lambda$ along $\vec{s}$ multiplied by the corresponding complex conjugate amplitudes with component $\lambda'$.

1 The plane is an important component to define the polarization frames. At the end of this section, we will fix our chosen polarization axis and corresponding plane.
Several polarization frame definitions have been used in the literature to fully describe the polarization of heavy quarkonium [25], i.e. the HX (recoil or s-channel helicity) frame, the Collins-Soper frame, the Gottfried-Jackson frame, and the target frame. In the Collins-Soper frame, is chosen as the flight direction of the decaying quarkonium. In the Collins-Soper frame,

\[ \vec{s} = (\vec{p}_1^2/|\vec{p}_1|^2 - \vec{p}_2^2/|\vec{p}_2|^2)) / |\vec{p}_1|^2/|\vec{p}_2|^2/|\vec{p}_1| \],

(4)

where \( \vec{p}_1 \) and \( \vec{p}_2 \) denote the momenta of the two initial state colliding particles in the rest frame of the decaying quarkonium. In the Gottfried-Jackson frame, \( \vec{s} = \vec{p}_1^2/|\vec{p}_1|^2 \), and in the target frame, \( \vec{s} = -\vec{p}_2^2/|\vec{p}_2|^2 \). All the definitions of the X, Y, Z coordinates can be found in Ref. [25]. In particular, the Y coordinate points in the direction of \( \vec{p}_1 \times (-\vec{p}_2) \) in the \( \chi_c \) rest frame.

III. ANGULAR DISTRIBUTION OF \( \chi_{c1} \rightarrow J/\psi \gamma \)

The general vertex function for the decay of a vector or axial vector particle into two vector particles can be expressed as

\[
\mathcal{M}(V_0 \rightarrow V_1 V_2) = f_1 (\epsilon_{V_0} \cdot \epsilon_{V_1}^* [[\epsilon_{V_2} \cdot (-p_{V_0} - p_{V_1})] \]) + f_2 (\epsilon_{V_0} \cdot \epsilon_{V_2}^* [[\epsilon_{V_1} \cdot (p_{V_0} + p_{V_1})] \]) + f_3 (\epsilon_{V_1} \cdot \epsilon_{V_2}^* [[\epsilon_{V_0} \cdot (p_{V_1} - p_{V_2})] \]) + f_4 (\epsilon_{V_0} \cdot (p_{V_1} - p_{V_2})[[\epsilon_{V_1} \cdot (p_{V_0} + p_{V_1})] \]) + f_5 i\epsilon_{V_0} \epsilon_{V_1}^* \epsilon_{V_2}^* \epsilon_{V_3}^* p_{V_0} + f_6 i\epsilon_{V_0} \epsilon_{V_1}^* (p_{V_0} + p_{V_2})[[\epsilon_{V_1} \cdot (p_{V_0} + p_{V_2})] \]) + f_7 (\epsilon_{V_2}^* \cdot (p_{V_0} + p_{V_1})[[\epsilon_{V_1} \cdot (p_{V_0} + p_{V_1})] \])
\]

where

\[ p_{V_0} = p_{V_1} + p_{V_2}. \]

and \( \epsilon_{\mu \nu \rho \sigma} \) is the antisymmetric Levi-Civita tensor. A special notation about the vector contracting with the Levi-Civita tensor is used, for example, \( \epsilon_{\mu \nu \rho \kappa} \equiv \epsilon_{\mu \nu \rho \kappa} k^\kappa, \epsilon_{\mu \nu \rho \kappa} \equiv \epsilon_{\mu \nu \rho \kappa} g^{\kappa \kappa}. \) Specifically, in the case of \( \chi_{c1}(1^{++}) \rightarrow J/\psi(1^{+}^{-}) \gamma(1^{-}_{-}) \), the \( f_1, f_2, f_3, f_4 \) terms and the \( f_5 \) term can be dropped because of parity conservation in QED and the absence of a longitudinal polarization component for the photon. If we just consider the electric dipole (E1) transition, which is the dominant contribution according to the velocity scaling rule in NRQCD, the \( f_6 \) term can also be neglected. Therefore, we calculate the helicity amplitudes \( \mathcal{M}_{\chi_{c1} \rightarrow J/\psi \gamma} \) for \( \chi_{c1} \rightarrow J/\psi \gamma \) as

\[
\begin{align*}
\mathcal{M}_{+++} &= \mathcal{M}^*_{---} = \frac{m_{\chi_{c1}} e^{-i\phi} \sin \theta}{\sqrt{2}}, \\
\mathcal{M}_{+--} &= \mathcal{M}^*_{-+-} = -\frac{m_{\chi_{c1}} e^{2i\phi} \sin \theta}{\sqrt{2}}, \\
\mathcal{M}_{+-0} &= -\mathcal{M}_{-0+} = -\frac{(m_{\chi_{c1}} + m_{J/\psi}) \sin^2 \frac{\theta}{2}}{2m_{J/\psi}}, \\
\mathcal{M}_{0+-} &= -\mathcal{M}^*_{0-+} = -\frac{(m_{\chi_{c1}} + m_{J/\psi}) e^{2i\phi} \cos^2 \frac{\theta}{2}}{2m_{J/\psi}}, \\
\mathcal{M}_{00+} &= \mathcal{M}^*_{00-} = \frac{(m_{\chi_{c1}} + m_{J/\psi}) e^{-i\phi} \sin \theta}{2\sqrt{2}m_{J/\psi}},
\end{align*}
\]

where a factor \( f_5 (m_{\chi_{c1}} - m_{J/\psi}) \) common to all amplitudes has been omitted. The decay SDMEs are obtained as \( D_{\lambda \lambda'} = \sum \lambda_{J/\psi} \lambda_{\chi_{c1}} N_{\lambda \lambda', \chi_{c1}} \mathcal{M}_{\chi_{c1} \rightarrow J/\psi \gamma} \). Using these ingredients [i.e. Eqs. (3) and (6)] and assuming \( m_{\chi_{c1}} = m_{J/\psi} \), we can work out the general form of the angular distribution of \( \chi_{c1} \rightarrow J/\psi \gamma \):

\[
\begin{align*}
\mathcal{W}_{\chi_{c1} \rightarrow J/\psi \gamma}(\theta, \phi) &\propto \frac{N_{\chi_{c1} \rightarrow J/\psi \gamma}}{3 + \lambda_0} (1 + \lambda_0 \cos^2 \theta) \\
&\quad + \lambda_\phi \sin^2 \theta \cos 2\phi + \lambda_{\phi \phi} \sin 2\theta \cos \phi \\
&\quad + \lambda_{\phi \phi} \sin^2 \theta \sin 2\phi + \lambda_{\phi \phi} \sin 2\theta \sin \phi,
\end{align*}
\]

with

\[
\begin{align*}
\lambda_\theta &= \frac{3\rho_{1,0} - N_{\chi_{c1}}}{3N_{\chi_{c1}} - \rho_{0,0}}, \\
\lambda_\phi &= -\frac{2\sqrt{3}\rho_{1,0} - 3\rho_{1,-1}}{3N_{\chi_{c1}} - \rho_{0,0}}, \\
\lambda_{\phi \phi} &= \frac{2\sqrt{2}(3\rho_{1,0} + 3\rho_{1,-1})}{3N_{\chi_{c1}} - \rho_{0,0}},
\end{align*}
\]

where \( \rho_{i,j} \) are SDMEs for the \( \chi_{c1} \) yields and \( N_{\chi_{c1}} = \rho_{1,1} + \rho_{0,0} + \rho_{-1,-1} \).

IV. ANGULAR DISTRIBUTION OF \( \chi_{c2} \rightarrow J/\psi \gamma \)

Similarly, in the \( \chi_{c2} \) case, we can write down the general vertex function for a spin-2 tensor particle \( T \) decaying into two vector particles

\[
\begin{align*}
\mathcal{M}(T \rightarrow V_1 V_2) &= g_1 \epsilon_{V_2}^* \cdot \epsilon_T \cdot \epsilon_{V_1}^* \\
&\quad + g_2 ([p_{V_1} - p_{V_2}) \cdot \epsilon_T \cdot \epsilon_{V_1}^* ([\epsilon_{V_2} \cdot (-p_T - p_{V_1})]) \\
&\quad + g_3 ([p_{V_1} - p_{V_2}) \cdot \epsilon_T \cdot \epsilon_{V_1}^* (p_T + p_{V_2})) \\
&\quad + g_4 ([p_{V_1} - p_{V_2}) \cdot (p_T - p_{V_2})) ([\epsilon_{V_1} \cdot \epsilon_{V_2}^*]) \\
&\quad + g_5 ([p_{V_1} - p_{V_2}) \cdot (p_T - p_{V_2})]) \\
&\quad - \epsilon_{V_1} \cdot (p_T + p_{V_2}) \cdot \epsilon_{V_2} \cdot (p_T + p_{V_1}) + \text{Levi-Civita terms,}
\end{align*}
\]

2 Another useful frame is the “perpendicular helicity frame” [26, 27]. It has been used in the Υ polarization measurement [28] by the CMS Collaboration.
where $p_T = p_{V_1} + p_{V_2}$. Because of parity conservation, we drop the Levi-Civita terms in $\chi_{c2}(2^{++}) \rightarrow J/\psi(1^{--})\gamma(1^-)$. The $g_2, g_3, g_4, g_5$ terms can also be ignored in consideration of the fact that we only include the leading-order contribution, i.e., the E1 transition, and these terms are suppressed by $(m_{\chi_{c2}}^2 - m_{J/\psi}^2)^2$ as compared to the $g_1$ term. Moreover, some of these terms vanish exactly when the photon is transversely polarized. Thus, the helicity amplitudes $M_{\chi_{c2}\lambda_{J/\psi}}$ become

\[
M_{2^+} = M_{-2^-} = \frac{\sin^2 \theta}{4},
\]
\[
M_{2^-} = M_{-2^+} = e^{2i\phi} \cos^2 \frac{\theta}{2},
\]
\[
M_{-2^-} = M_{2^+} = e^{4i\phi} \sin^2 \theta.
\]
\[
M_{2^-} = M_{-2^+} = e^{2i\phi} \sin^4 \theta.
\]
\[
M_{2^0} = -M_{2^0} = -\frac{m_{\chi_{c2}}^2 + m_{J/\psi}^2}{\sqrt{2m_{\chi_{c2}} m_{J/\psi}}} e^{i\phi} \sin \frac{3}{2} \cos \frac{\theta}{2},
\]
\[
M_{2^-} = -M_{2^+} = -\frac{m_{\chi_{c2}}^2 + m_{J/\psi}^2}{\sqrt{2m_{\chi_{c2}} m_{J/\psi}}} e^{i\phi} \cos \frac{3}{2} \sin \frac{\theta}{2},
\]
\[
M_{2^0} = -M_{2^0} = -e^{-i\phi} \sin 2\theta.
\]
\[
M_{1^+} = -M_{-1^-} = -\frac{e^{i\phi} \sin \theta}{2}.
\]
\[
M_{1^-} = -M_{1^+} = -\frac{e^{i\phi} \sin \theta}{2},
\]
\[
M_{1^-} = -M_{-1^+} = e^{3i\phi} \sin 2\theta.
\]
\[
M_{10^+} = -M_{10^-} = \frac{m_{\chi_{c2}}^2 + m_{J/\psi}^2}{2\sqrt{2m_{\chi_{c2}} m_{J/\psi}}} \sin \frac{\theta}{2} (1 + 2 \cos \theta),
\]
\[
M_{10^-} = -M_{10^+} = \frac{m_{\chi_{c2}}^2 + m_{J/\psi}^2}{2\sqrt{2m_{\chi_{c2}} m_{J/\psi}}} e^{2i\phi} \cos^2 \frac{\theta}{2}(2 \cos \theta - 1),
\]
\[
M_{0^+} = -M_{0^-} = \frac{e^{-2i\phi}(1 + 3 \cos 2\theta)}{4\sqrt{6}},
\]
\[
M_{0^-} = -M_{0^+} = \frac{3 \sin^2 \theta}{2\sqrt{6}},
\]
\[
M_{00^+} = -M_{00^-} = -\frac{\sqrt{3}(m_{\chi_{c2}}^2 + m_{J/\psi}^2)}{8m_{\chi_{c2}} m_{J/\psi}} e^{-i\phi} \sin 2\theta.
\]

The angular distribution of the $\chi_{c2} \rightarrow J/\psi\gamma$ decay has, in the E1 approximation and assuming $m_{\chi_{c2}} = m_{J/\psi}$, the same general expression of the $\chi_{c1} \rightarrow J/\psi\gamma$ case:

\[
\mathcal{W}_{\chi_{c2} \rightarrow J/\psi\gamma}(\theta, \phi) \propto \frac{N_{\chi_{c2} \rightarrow J/\psi\gamma}}{3 + \lambda_\theta} (1 + \lambda_\theta \cos^2 \theta) + \lambda_\theta \sin^2 \theta \cos 2\phi + \lambda_\phi \sin 2\theta \cos \phi + \lambda_\theta \sin^2 \theta \sin 2\phi + \lambda_\phi \sin 2\theta \sin \phi.
\]

\[
\lambda_\theta = \frac{6N_{\chi_{c2}} - 9(\rho_{1,1} + \rho_{-1,-1}) - 12\rho_{0,0}}{6N_{\chi_{c2}} + 3(\rho_{1,1} + \rho_{-1,-1}) + 4\rho_{0,0}},
\]
\[
\lambda_\phi = \frac{2\sqrt{6}(\rho_{2,0} + \rho_{0,-2,0}) + 6\rho_{1,-1}}{6N_{\chi_{c2}} + 3(\rho_{1,1} + \rho_{-1,-1}) + 4\rho_{0,0}},
\]
\[
\lambda_{\phi\phi} = \frac{6(\rho_{2,1} + \rho_{0,-2,1}) + \sqrt{6}(\rho_{1,0} + 3\rho_{-1,0})}{6N_{\chi_{c2}} + 3(\rho_{1,1} + \rho_{-1,-1}) + 4\rho_{0,0}},
\]
\[
N_{\chi_{c2}} = \rho_{2,2} + \rho_{1,1} + \rho_{0,0} + \rho_{-1,-1} + \rho_{-2,-2}.
\]

If the magnetic quadrupole (M2) and electric octupole (E3) contributions are also taken into account by keeping the relevant terms in the vertex functions, the expression of the angular distribution acquires further terms in the $\chi_{c2}$ case while, both for $\chi_{c1}$ and $\chi_{c2}$, the existing terms are modified. The modifications depend on one additional coefficient (expressing the fractional M2 amplitude contribution) in the $\chi_{c1}$ case and on two additional coefficients (M2 and E3 contributions) in the $\chi_{c2}$ case. If these coefficients are not $< O(1\%)$, they can modify the angular distributions significantly [23]. However, inconsistencies in their current experimental determinations exist [29–32]. The complete formulas for the angular distributions including the higher-order multipole effects are presented in the appendix, while only the E1 transition is considered, for simplicity, throughout the body of the paper.

Our results (Eqs. (8) and (12)) are exactly the same as those given in Refs. [22, 23]. Actually, our derivations, which are based on the general effective decay vertex functions, are equivalent to those obtained there using the angular momentum conservation, since the effective amplitudes written by us are also originated from general considerations on the spins of the involved particles.

V. LEPTON DISTRIBUTION IN

\[ \chi_{cJ} \rightarrow J/\psi\gamma \rightarrow \mu^+ \mu^- \]

We are now in a position to investigate the $\mu^+$ angular distributions from the cascade decay $\chi_{cJ} \rightarrow J/\psi\gamma \rightarrow \mu^+ \mu^-$. We start from a general formalism to study it. We denote with $\vec{s}_1$ and $\vec{s}_2$ the quantization axes of the $\chi_{cJ}$ and of the $J/\psi$, respectively. The general form of the angular distribution of the $\mu^+$ is

\[
\mathcal{W}_{\chi_{cJ} \rightarrow J/\psi\gamma \rightarrow \mu^+ \mu^-}(\theta', \phi') = \int d^2 \Omega[\theta, \phi] \rho_{\chi_{cJ}} \rho_{J/\psi} \rho_{J/\psi} M_{\chi_{cJ} \rightarrow J/\psi\gamma}(\theta, \phi) M_{J/\psi \rightarrow \mu^+ \mu^-}(\theta', \phi')
\]

\[
\left( M_{J/\psi \rightarrow \mu^+ \mu^-}(\theta, \phi) M_{J/\psi \rightarrow \mu^+ \mu^-}(\theta', \phi') \right)^*.
\]
where the $\rho^{\chi_{cJ}}_{s_z, s_z'}$ coefficients are the production SDMEs of the $\chi_{cJ}, M^{\chi_{cJ} \rightarrow J/\psi \gamma}, M^{J/\psi \rightarrow \mu^+ \mu^-}$ are the amplitudes of the two successive decays, $J_z$ is the $\chi_{cJ}$ angular momentum projection with respect to $\vec{s}_1$, $s_z$ the $J/\psi$ angular momentum projection with respect to $\vec{s}_2$, $\lambda_\gamma, \lambda_{\mu^+}, \lambda_{\mu^-}$ are the photon and lepton helicities. The angles $\theta$ and $\phi$ define the $J/\psi$ direction in the $\chi_{cJ}$ rest frame with respect to $\vec{s}_1$. $\theta'$ and $\phi'$ determine the $\mu^+$ direction in the $J/\psi$ rest frame with respect to $\vec{s}_2$. Indices appearing twice imply a summation, with $J_z, J_z' = \pm J, \pm (J-1), \ldots, 0, s_z, s_z' = \pm 1, 0$, and $\lambda_\gamma, \lambda_{\mu^+}, \lambda_{\mu^-} = \pm 1$.

We will consider two different definitions of $\vec{s}_2$. In the first option, $\vec{s}_2$ is the flight direction of the $J/\psi$ in the rest frame of the $\chi_{cJ}$. The $J/\psi \rightarrow \mu^+ \mu^-$ angular distribution can be parametrized in the same form of Eqs. (7) and (11), with five observable coefficients depending on the $\chi_{cJ}$ SDMEs $\rho^{\chi_{cJ}}_{s_z, s_z'}$:

\begin{align}
\lambda^{\chi_{cJ}}_{\theta'} &= -\frac{1}{3}\lambda^{\chi_{cJ}}_{\phi'} = \lambda^{\chi_{cJ}}_{\theta' \phi'} = 0, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= -\frac{\sqrt{2} (\Re(\rho^{\chi_{cJ}}_{s_1, -1}) - \Re(\rho^{\chi_{cJ}}_{-1, 0}))}{12 N_{\chi_{cJ}}}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= -\frac{\sqrt{2} (3\rho^{\chi_{cJ}}_{-1, 0} + 3\rho^{\chi_{cJ}}_{-1, 0})}{12 N_{\chi_{cJ}}}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= \frac{1}{15}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= \frac{7\sqrt{6}(\Re(\rho^{\chi_{cJ}}_{s_1, -1}) + 2\Re(\rho^{\chi_{cJ}}_{-1, 0}) + 12\Re(\rho^{\chi_{cJ}}_{-1, 0})}{78 N_{\chi_{cJ}}}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= \frac{\sqrt{6}(\Re(\rho^{\chi_{cJ}}_{s_1, -1}) - \Re(\rho^{\chi_{cJ}}_{-1, 0}) + 24(\Re(\rho^{\chi_{cJ}}_{-1, 0}) - \Re(\rho^{\chi_{cJ}}_{-1, 0})}{156 N_{\chi_{cJ}}}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= \frac{7\sqrt{6}(3\rho^{\chi_{cJ}}_{s_1, -1} - 3\rho^{\chi_{cJ}}_{-1, 0}) - 123\Re(\rho^{\chi_{cJ}}_{-1, 0})}{78 N_{\chi_{cJ}}}, \\
\lambda^{\chi_{cJ}}_{\theta' \phi'} &= \frac{\sqrt{6}(3\rho^{\chi_{cJ}}_{-1, 0} + 3\rho^{\chi_{cJ}}_{-1, 0}) + 24(3\rho^{\chi_{cJ}}_{-1, 0} - 3\rho^{\chi_{cJ}}_{-1, 0})}{156 N_{\chi_{cJ}}}.
\end{align}

(14)

with $N_{\chi_{cJ}} = \sum_{j=-j}^{j} \rho^{\chi_{cJ}}_{j,j}$. We see that the same results can be obtained in another formalism using the language of angular momentum theory. The spin correlations between the $\chi_{cJ}$ production SDMEs $\rho^{\chi_{cJ}}_{j,j'}$, referred to the quantization axis $\vec{s}_1$, and the SDMEs of the $J/\psi$ coming from $\chi_{cJ}, \rho^{\chi_{cJ} \rightarrow J/\psi \gamma}$ are referred to $\vec{s}_2$, can be expressed as

\begin{equation}
\rho^{\chi_{cJ} \rightarrow J/\psi \gamma}_{s_z, s_z'} = \frac{3}{8\pi} \int d\Omega[\theta, \phi] \rho^{\chi_{cJ}}_{j,j'} D^{s_z}_{j,j} D^{s_z'}_{j,j'} (15)
\end{equation}

where implicit summations run over $J_z, J_z', I_z, I_z' = \pm J, \pm (J-1), \ldots, 0$ and over $\lambda_\gamma = \pm 1$, $\text{Br}(\chi_{cJ} \rightarrow J/\psi \gamma)$ is

the branching ratio of the radiative decay and $D^{j,j'}_{j_z, j_z'} (-\phi, \theta, \phi) = e^{i\phi(j',-j)}d^{j,j'}_{j_z, j_z'}(\theta)$, $d^{j,j'}_{j_z, j_z'}(\theta)$ being the well-known Wigner $d$ function

\begin{equation}
d^{j,j'}_{j_z, j_z'}(\theta) = \sum_{k=\text{max}(0,j'-j),j-1}^{\text{min}(j+j',j-j)} (-)^{k-j'+j} \sqrt{(j+j')!(j-j')!(j+j)! \times (j+j'-k)!(j-j-k)!(k-j'_z+j'_z)!(2j+2j')!}.
\end{equation}

(16)

One can easily verify that after substituting $\rho_{s_z, s_z'}^{\chi_{cJ} \rightarrow J/\psi \gamma}$ calculated from the above equation into the well-known expression of the angular distribution of the muon from $J/\psi \rightarrow \mu^+ \mu^-$,

\begin{align}
\lambda^{J/\psi}_{\theta'} &= \frac{N_{J/\psi} - 3\rho^{J/\psi}_{1,0}}{N_{J/\psi} + \rho^{J/\psi}_{1,0}}, \\
\lambda^{J/\psi}_{\theta'} &= \frac{2\Re(\rho^{J/\psi}_{1,1})}{N_{J/\psi} + \rho^{J/\psi}_{1,0}}, \\
\lambda^{J/\psi}_{\theta'} &= \frac{\sqrt{2}(\Re(\rho^{J/\psi}_{1,0}) - \Re(\rho^{J/\psi}_{1,0}))}{N_{J/\psi} + \rho^{J/\psi}_{1,0}}, \\
\lambda^{J/\psi}_{\theta'} &= -\frac{2\Im(\rho^{J/\psi}_{1,1})}{N_{J/\psi} + \rho^{J/\psi}_{1,0}}, \\
\lambda^{J/\psi}_{\theta'} &= \frac{\sqrt{2}(3\rho^{J/\psi}_{1,0} + 3\rho^{J/\psi}_{1,0})}{N_{J/\psi} + \rho^{J/\psi}_{1,0}},
\end{align}

(17)

the expressions in Eq.(14) are recovered.

As a second option, $\vec{s}_2$ is chosen as coinciding with $\vec{s}_1$. The $J/\psi$ spin state $|1, s_z\rangle$ with respect to $\vec{s}_2$ is no longer its helicity state, like in the first case. This option is actually a much “easier” choice for the experiment, at least at not very low $J/\psi$ momentum, because it does not require the use of the photon momentum, and it actually coincides with the usual set of reference frames adopted in the study of prompt $J/\psi$ [23]. With a direct calculation following Eq.(13), we obtain

\begin{align}
\lambda^{\chi_{cJ}}_{\theta'} &= \frac{N_{\chi_{cJ}} + 3\rho^{\chi_{cJ}}_{1,0}}{3N_{\chi_{cJ}} - \rho^{\chi_{cJ}}_{1,0}}, \\
\lambda^{\chi_{cJ}}_{\theta'} &= \frac{2\Re(\rho^{\chi_{cJ}}_{1,1})}{3N_{\chi_{cJ}} - \rho^{\chi_{cJ}}_{1,0}}, \\
\lambda^{\chi_{cJ}}_{\theta'} &= \frac{\sqrt{2}(\Re(\rho^{\chi_{cJ}}_{1,0}) - \Re(\rho^{\chi_{cJ}}_{1,0}))}{3N_{\chi_{cJ}} - \rho^{\chi_{cJ}}_{1,0}}, \\
\lambda^{\chi_{cJ}}_{\theta'} &= \frac{2\Im(\rho^{\chi_{cJ}}_{1,1})}{3N_{\chi_{cJ}} - \rho^{\chi_{cJ}}_{1,0}}, \\
\lambda^{\chi_{cJ}}_{\theta'} &= \frac{\sqrt{2}(3\rho^{\chi_{cJ}}_{1,0} + 3\rho^{\chi_{cJ}}_{1,0})}{3N_{\chi_{cJ}} - \rho^{\chi_{cJ}}_{1,0}}.
\end{align}

(18)
From Ref. [36], we know that the rotation-invariant property in the production plane is, indeed, fulfilled in the section, we want to generalize the relation to the arbitrary spin-$n$ ($n$ is an integer) particles. We straightforwardly write down the linear identities for the Wigner functions:

$$\sum_{m=-k}^{k} \langle k, m; k, m | 2k, 2m \rangle d_{2m,M}^{2k}(\theta)$$

$$= \langle k, M/2; k, M/2 | 2k, 2m \rangle \delta_{\text{mod}(M,2),0}, \quad n = 2k,$n = 2k + 1,$

where $k$ is a non-negative integer. The amplitudes with respect to a chosen polarization axis can be symbolically denoted as $|n\rangle = \sum_{m=-n}^{n} a_{m,n} |m,n\rangle$, where $|n,m\rangle$ is a $J$ eigenstate with the eigenvalues $m = -n,-n+1,...,n$, and $a_{m,n}$ is the production amplitude, i.e., $\langle a_{J_{z},J_{z}'} \rangle (\text{average over the events, assuming that for each event the particle can be produced in a different angular momentum state}).$ From Eq. (20), we can immediately draw a conclusion that the linear combinations of amplitudes

$$b_{2k} \equiv \sum_{m=-k}^{k} \langle k, m; k, m | 2k, 2m \rangle a_{2m,n}, \quad n = 2k,$n = 2k + 1,$

and

$$b_{2k+1} \equiv \sum_{m=0}^{k} \langle 2k+1 - m, 0; m, 0 | 2k+1, 0 \rangle (a_{2k+1-m,n} + a_{m-1-2k,n}), \quad n = 2k+1,$

are invariant under the rotation in the production plane. Therefore, the observables like $F_{n}$ defined as

$$F_{n} \equiv \frac{1}{B_{n}} \frac{\langle |b_{n}|^{2} \rangle}{N_{n}},$$

$$N_{n} \equiv \sum_{m=-n}^{n} \langle |a_{m}|^{2} \rangle = \sum_{m=-n}^{n} \rho_{m,m},$$

are rotation-invariant, where $B_{n}$ is a normalization factor to ensure $0 \leq F_{n} \leq 1$. In a more extended sense, any function of $F_{n}$ is rotation-invariant. $F_{n}$ can be expressed in terms of the coefficients of the decay angular distribution (e.g., $\lambda_{0}, \lambda_{2}, \phi_{2}$, etc). Specifically, for the spin-1 particles, the observable is

$$F_{1} \equiv \frac{1}{2} \frac{\langle |a_{1} + a_{-1}|^{2} \rangle}{\langle |a_{1}|^{2} + |a_{0}|^{2} + |a_{-1}|^{2} \rangle},$$

and
while for the spin-2 particles, its expression is

\[
F_2 \equiv \frac{1}{3} \frac{(|a_2 + \sqrt{3}a_0 + a_{-2}|^2)}{(|a_2|^2 + |a_1|^2 + |a_0|^2 + |a_{-1}|^2 + |a_{-2}|^2)}.
\]  

(23)

As examples of \(F_1\), we consider the \(J/\psi\) decay into two muons and the \(\chi_{c1}\) decay into a \(J/\psi\) and a photon. For the \(J/\psi\),\(^4\)

\[
F_{1J/\psi \rightarrow \mu^+\mu^-} = \frac{1 + \lambda_\rho + 2\lambda_{\rho'} - 2\lambda_\rho'}{3 + \lambda_\rho'},
\]  

(24)

which has been presented in Ref. [36], while for the \(\chi_{c1}\), one can derive

\[
F_{1\chi_{c1} \rightarrow J/\psi\gamma} = \frac{1 - \lambda_\rho - 4\lambda_{\rho'}}{3 + \lambda_\rho}
\]  

(25)

from Eqs.(8) and (22).\(^5\) The \(\chi_{c2}\) provides an example of the spin-2 particles. In fact, the complete angular distribution of the \(\chi_{c2}\)'s decay product \(J/\psi\) is Eq.(D3) in stead of Eq.(11). However, the terms absent in Eq.(11) are suppressed, as mentioned above. Hence, the spin information in Eq.(12) is not sufficient. In Appendix B, we have included the E1,M2, and E3 effects into the angular distribution of \(J/\psi\) in \(\chi_{c2} \rightarrow J/\psi\gamma\) and derived the \(F_{2\chi_{c2} \rightarrow J/\psi\gamma}\) there [see Eq.(B6)]. We suggest that the reader who is interested in this part to refer to Appendix B. These frame-invariant relations can be extended to the study of other bosons or mesons. The experimentalists can measure these observables to make a cross-check of their extractions of the angular distribution coefficients in different frames.

VII. UNCERTAINTY OF \(J/\psi\) POLARIZATION FROM FEED-DOWN

The CDF data for the prompt-\(J/\psi\) production include not only direct-\(J/\psi\) production but also the feed-down contributions from \(\chi_{cJ}\) and \(\psi'\). However, the recent NLO NRQCD calculations of \(J/\psi\) polarization in Refs. [18, 19] are devoid of the feed-down contributions. Though the LO NRQCD prediction of the feed down to the \(J/\psi\) polarization in Ref. [37] was found to have a minor impact on the final LO result, one may still doubt whether the NLO feed-down effect on \(\lambda_{\rho'}\)\(^6\) of the \(J/\psi\) can be neglected, because the NLO correction to the P wave is large [8]. In this section, we will estimate the possible uncertainty of the \(J/\psi\) polarization \(\lambda_{\rho'}\) arising from the feed down of the \(\chi_c\) and \(\psi'\) decays.

The calculation of the prompt-\(J/\psi\) polarization is complex. In general, prompt data are composed of four parts, i.e., the direct production of the \(J/\psi\), the single-cascade decays of the \(\chi_c\) and of the \(\psi'\), and the double-cascade decay \(\psi'\rightarrow \chi_c\gamma \rightarrow J/\psi\gamma\). The direct production of the \(J/\psi\) has been studied, e.g., in Refs. [18, 19]. The relation between the production SDMEs of the \(\chi_c\) and the SDMEs of the \(J/\psi\) from \(\chi_c\) decays is given by Eq.(19), whereas the relation between the production SDMEs of the \(\psi'\) and the SDMEs of the \(\chi_c\) coming from \(\psi'\) decay is

\[
\rho_{J/\psi, \chi_c} \propto \sum_{l_z, s_z, s_z'} \langle 1, l_z; 1, s_z | J, J_z / 1, l_z; 1, s_z' | J', J_z' \rangle \rho_{s_z, s_z'},
\]  

(26)

By combining Eqs.(27) and (19), the double-cascade decay component can also be calculated. However, in the following uncertainties estimation, we will neglect this contribution, because of the small branching ratio and small cross section ratio between \(\psi'\) and \(J/\psi\) [13]. Finally, the single-cascade decay \(\psi'\rightarrow J/\psi\pi\pi\) can be treated in analogy with the double chromoelectric dipole transition \(J/\psi \rightarrow J/\psi\pi\pi\) \([8]\). This part will also not be included in the uncertainties because of the small cross section ratio between \(\psi'\) and \(J/\psi\) [13] and the spin orientation conserved in \(\psi' \rightarrow J/\psi\pi\pi\) \([8]\).

In this way, the only contribution to be considered is the feed down from \(\chi_c\). We consider the total prompt-\(J/\psi\) yield \(\rho\) decomposed in the “direct” part, \(\rho^d\), already calculated in NRQCD at the NLO level, and the “feed-down” part \(\rho^f\), with their corresponding polarization observables \(\lambda_{\rho'}^d\) and \(\lambda_{\rho'}^f\), polar anisotropies of the dilepton decay distributions. The fraction of the \(J/\psi\) yield from feed down with respect to the total prompt yield is denoted as \(r\), i.e., \(r \equiv \frac{2\rho_{s_z, s_z'}^f + \rho_{s_z, s_z'}^d}{2\rho_{s_z, s_z'}^d + \rho_{s_z, s_z'}^f}\) with \(\rho_{s_z, s_z'}^f = \rho_{s_z, s_z'}^d + \rho_{s_z, s_z'}^d\) and \(\rho \equiv 2\rho_{s_z, s_z'}^d + \rho_{s_z, s_z'}^f\). Hence, the prompt-\(J/\psi\) decay polar anisotropy is

\[
\rho_{0,0}^d = \frac{r - \lambda_{\rho'}^d}{3 + \lambda_{\rho'}^d}, \quad \rho_{1,1}^d = \frac{1 + \lambda_{\rho'}^d}{3 + \lambda_{\rho'}^d},
\]  

(27)

\[
\rho_{0,0}^d = \frac{(1 - r) - \lambda_{\rho'}^d}{3 + \lambda_{\rho'}^d}, \quad \rho_{1,1}^d = \frac{(1 - r) + \lambda_{\rho'}^d}{3 + \lambda_{\rho'}^d}.
\]

\(^{6}\) Note that, to be consistent throughout the context, the polarization observables of the \(J/\psi\) or the angular distributions of the muon are all denoted by an extra prime.

\(^{7}\) Note that we use the symmetry property \(\rho_{\lambda,\lambda'}^H = (-)^{\lambda - \lambda'} \rho_{\lambda,\lambda'}^H\) which is guaranteed in hadroproduction by parity invariance [39].
All these considerations are valid for any polarization frame. From Eq. (18), we know that the $J/\psi$ from $\chi_{c1}$ and $\chi_{c2}$ can have $-1/3 \leq \lambda^{d}_{\psi} \leq 1$ and $-\frac{3}{2} \leq \lambda^{s}_{\psi} \leq 1$. Therefore, we take the $-1/3 \leq \lambda^{d}_{\psi} \leq 1$, which is weighted by the relative contributions of $\chi_{c1}$ and $\chi_{c2}$ to prompt $J/\psi$, i.e., $\sigma_{\chi_{c1}}B(\chi_{c1} \rightarrow J/\psi \gamma)/\sigma_{\chi_{c2}}B(\chi_{c2} \rightarrow J/\psi \gamma) \approx 5:2$, as measured by CDF [7]. For example, let us compare the cases $\lambda^{d}_{\psi} = 0$ and $\lambda^{d}_{\psi} = 1$, approximated values of the direct-$J/\psi$ polarization predictions in the HX frame for $p_T > 10\text{GeV}$ at the Tevatron according to Refs. [18, 19], respectively. The allowed prompt-$J/\psi$ polarization ranges in the two cases (with $\lambda^{d}_{\psi}$ varying from $-\frac{43}{105}$ to $1$) are $-\frac{228}{504}$ to $1$, and $0.08 - 0.12 \leq \lambda_{\rho} \leq 0.1$. If we fix $r = 0.3$ (the central value of the average $\chi_c$ feed-down fraction to prompt $J/\psi$ measured at the Tevatron [40]), the feed-down contribution may change $\lambda_{\rho}$ from 0.24 to $-0.14$ when $\lambda^{d}_{\psi} = 0$ and from 1 to 0.44 when the polarization of the direct $J/\psi$ is fully transverse. More generally, Fig. 1 shows curves for the prompt-$J/\psi$ polarization $\lambda^{d}_{\psi}$ as a function of the direct-$J/\psi$ polarization $\lambda^{d}_{\psi}$ for $\lambda^{d}_{\psi} = +1.0$, $-\frac{43}{105}$, and $r = 0.3$. The upper and lower curves represent physical bounds for $\lambda_{\rho}$. Figure 2 shows the maximum possible impact of the feed down from the $\chi_c$ decays on the prompt-$J/\psi$ polarization predicted in the HX frame for the Tevatron with $\sqrt{s} = 1.96\text{TeV}$ and $|y_{J/\psi}| < 0.6$. The LDMEs values (see Table I) used for the direct-$J/\psi$ prediction are those obtained in Ref. [19] by fitting the NLO NRQCD calculation to the Tevatron data. Only the central value of the direct-$J/\psi$ prediction is shown. In particular, the prediction of an almost unpolarized $J/\psi$ production obtained with the LDMEs determined in Ref. [19] is not drastically affected by the neglected impact of the $\chi_c$ feed-down contribution.

### TABLE I: CO LDMEs for $J/\psi$ from Ref. [19]

| LDME | $O^{J/\psi}(S^{[3]}_{1})$ | $O^{J/\psi}(S^{[3]}_{0})$ | $O^{J/\psi}(S^{[3]}_{0})$ | $O^{J/\psi}(P^{[3]}_{0})$/m_c |
|------|-----------------|-----------------|-----------------|-----------------|
| GeV^3 | 1.16 | 8.9 | 0.30 | 0.56 |

**VIII. AN EXAMPLE OF $\chi_{c1}$ AND $\chi_{c2}$ POLARIZATION**

In this section, we are in a position to give an example for the inclusive $\chi_{c1}$ and $\chi_{c2}$ production at the LHC with $\sqrt{s} = 8\text{ TeV}$. We also present the contributions of the individual Fock states to the SDMEs, for the convenience of the readers who want to use our results with different LDME values.

We use our automatic matrix element and event generator HELAC-Onia [42] to calculate all the SDMEs under...
The relevant conditions. The generator is built on rewritten versions of the published HELAC [43, 44], PHEGAS [45, 46], RAMBO [47], and VEGAS [48] codes. The program has been extensively tested.

The input parameters in our calculations are

1. \( m_c = 1.5 \text{ GeV}, m_{\chi_c} = 2m_c = 3 \text{ GeV}, \)

2. \( \sqrt{s} = 8 \text{ TeV}, |y_{\chi_c}| < 2.4, \)

3. the CTEQ6L1 [49] set of parton distribution functions,

4. renormalization and factorization scales \( \mu_R = \mu_F = \sqrt{(2m_c)^2 + p_T^2}, \)

5. CS LDMEs \( \langle O^{\chi-J}(p_J^{[1]}) \rangle = \frac{3(2j+1)2N_c}{4\pi} |R_p(0)|^2, \)

6. CO LDMEs \( \langle O^{\chi-J}(p_J^{[1]}) \rangle = \frac{2.2 \times 10^{-3} \text{GeV}^3}{S_1^{[8]}}, \)

The relations between the SDMEs of \( S_1^{[8]} \) and those of \( p_J^{[1]} \) are identical to those in Eq.(27) and are specifically

\[
S_1^{[8]} \rightarrow \chi-J \\
\rho_{J_z, J_z'} \propto \sum_{l_1,s_1,l_2,s_2} \langle l_1, s_1|J, J_z \rangle \langle l_2, s_2|J, J_z' \rangle \rho_{l_1,s_1,l_2,s_2}. \tag{29}
\]

The results of the calculations are organized in the following groups of figures, where in each case, from top to bottom, three different polarization frames are considered (HX, Collins-Soper, Gottfried-Jackson): Figs.(3,4,5) show the partial cross sections \( \langle d\sigma/dz_J \rangle \) and \( \langle d\sigma/dz_{J'} \rangle \) and the total cross sections \( \langle d\sigma_{\chi-J} \rangle \) as a function of \( p_T \) for each individual Fock state contributing to \( \chi_c \) production \( (F_1^{[1]}, F_2^{[1]}, \text{ and } S_1^{[8]}), \) with \( \frac{d\sigma_{\chi-J}}{dp_T} = \frac{d\sigma_{\chi-J}}{dp_T} + \frac{d\sigma_{\chi-J}}{dp_T} \) for \( F_1^{[1]} \) and \( S_1^{[8]} \) with \( \frac{d\sigma_{\chi-J}}{dp_T} = \frac{d\sigma_{\chi-J}}{dp_T} + \frac{d\sigma_{\chi-J}}{dp_T} \) for \( F_2^{[1]} \). The corresponding non-diagonal SDMEs are shown in Figs.(6,7,8). The frame-dependent parameters \( \lambda_0, \lambda_\phi, \text{ and } \lambda_{\theta_\phi} \) of the radiative \( \chi_c \) decays are shown in Figs.(9,10,11) for the \( \chi_{c1} \) and in Figs.(12,13,14) for the \( \chi_{c2} \), with distinct curves for the CS-only result and for the full CS+CO calculation. We remind that these parameters are also identical, in the E1-only approximation, to the parameters \( \lambda_0, \lambda_\phi, \text{ and } \lambda_{\theta_\phi} \) of the dilepton distribution of the \( J/\psi \) from \( \chi_{c1} \) decays in the second option, for each considered polarization frame. Finally, Figs.(15) and (16) show the results for the corresponding frame-independent polarization parameters \( F_1 \) and \( F_2 \).

Some features should be emphasized:

1. From the curves of the total cross sections in Figs.(3,4,5), we may conclude that the CS dominates in the low transverse momentum region, while the CO may dominate when \( p_T \) increases because gluon fragmentation processes [50] become important. The CS \( p_T \) distribution may receive significant contributions from the higher order radiative corrections [8]. The full NLO predictions for \( \chi_{c-J} \) polarizations are already presented in Ref. [24].

2. The polarization parameters in the Gottfried-Jackson frame tend to be very similar to those in the HX frame, especially at high \( p_T \), while significant differences exist between these two and the Collins-Soper frame. The HX and Collins-Soper frames are, therefore, sufficient for the characterization of the angular distributions. In particular, from Fig.5, we see that the longitudinal cross section of the \( S_1^{[8]} \) channel is largest in the Collins-Soper frame when \( p_T \gg 2m_c \), in agreement with the statement in Ref. [22].

3. We have verified that the results of \( \rho_{i,j} \) in the Gottfried-Jackson frame and the target frame coincide within error bars, aside from a factor \((-1)^{i-j}\)

4. In the HX and the Collins-Soper frames, the SDMEs \( \rho_{i,j} \) with \( |i-j| \) odd are almost zero. Therefore, they should be measured in the Gottfried-Jackson frame as well as \( \lambda_{\theta_\phi} \) and \( \lambda_{\theta_\phi} \) shown in Figs.(11) and (14). In fact, this is a consequence of the definition of the \( Y \) axis taken always as \( P_1 \times (-P_2) \) in the \( \chi_c \) rest frame both at positive and negative rapidity. Actually, the values of \( \lambda_{\theta_\phi} \) in the positive and negative rapidity are opposite. The existence of these relations between the choice of the \( Y \) axis and the sign \( \lambda_{\theta_\phi} \) was already pointed out in Ref. [6]. On the other hand, for \( \rho_{i,j} \) with \(|i-j|\) even, the measurements in the HX and the Collins-Soper frames are more significant than in the Gottfried-Jackson frame.

5. The frame-independent observables defined in Eqs.(22) and (23) for the \( \chi_{c1} \) and the \( \chi_{c2} \) are shown in Fig.15. The figures show the frame-independent property of \( F_1^{\chi_{c1}\rightarrow J/\psi} \) and \( F_2^{\chi_{c2}\rightarrow J/\psi} \) by direct numerical calculation. Hence, it would be interesting to measure these observables at the LHC. In
particular, it is remarkable that $F_2^{\chi_{c2}\rightarrow J/\psi\gamma}$ is practically independent of $p_T$, contrary to $\lambda_0$ and $\lambda_\phi$ in all frames considered (see Figs.(12) and (13)).

\section{Summary}

Finally, we draw our conclusion. The upgrade of the integrated luminosity at the LHC will not only allow us to measure the polarizations of $S$-wave quarkonium states like $J/\psi$ and $\psi'$ but also the angular distributions of decay products from the $P$-wave states $\chi_{cJ}$. This opens new opportunities to further test NRQCD factorization and the quarkonium production mechanisms, in general. We have presented a general calculation framework based on the shape of the decay vertex functions to investigate the polarizations of the $\chi_{cJ}$ states and their impact in the angular distributions have been calculated as a function of several polarization frames and also frame-independent quantities. Moreover, we have also estimated the impact of the $\chi_c$ feed down in the polarization of prompt $J/\psi$. We found that our previous direct $J/\psi$ polarization results \cite{19} will not change much after including this part. A more detailed phenomenological analysis of the yields and polarizations of $\chi_c$ in hadroproduction are performed in Ref. \cite{24}, based on our complete NLO NRQCD calculations.

Note added: While this paper was prepared, a new preprint \cite{51} for polarizations of the prompt-$J/\psi$ and $\psi'$ production at the LHC and Tevatron appeared. The authors extracted a set of CO LDMEs for the $J/\psi$ other than those in Refs. \cite{18, 19} by including the $\chi_{cJ}$ and $\psi'$ feed-down contributions. Their LDMEs result in two combinations of LDMEs that agree with those extracted in Refs. \cite{13, 19}, and their prompt polarization result is around the upper limit in Fig.2 of this paper.

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\addcontentsline{toc}{section}{Appendix A: Decay angular distribution of spin-1 bosons}

The most general expression for the decay angular distribution of a vector boson $V$ without assuming parity conservation is

$$W^V(\theta, \phi) \propto 1 + \lambda_0 \cos^2 \theta + \lambda_\phi \sin^2 \theta \cos \phi + \lambda_{\theta\phi} \sin 2\theta \cos \phi + \lambda_{\phi\phi} \sin \phi + 2\eta_{\theta\phi} \sin \theta \sin \phi \cos \phi,$$  

\begin{equation}
\lambda_0 = (1 - 3\delta) \frac{N_V - 3\rho_{0,0}^{V}}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\lambda_\phi = (1 - 3\delta) \frac{2\Re\rho_{0,-1}^{V}}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\lambda_{\theta\phi} = (1 - 3\delta) \frac{\sqrt{2}(\Re\rho_{1,0}^{V} - \Re\rho_{1,-1}^{V})}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\lambda_{\phi\phi} = (1 - 3\delta) \frac{2\Im\rho_{1,0}^{V} - 3\Im\rho_{1,-1}^{V}}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\eta_\theta = \alpha \frac{\rho_{1,1}^{V} - \rho_{1,-1}^{V}}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\eta_{\theta\phi} = \frac{\sqrt{2}(\Re\rho_{1,0}^{V} + \Re\rho_{1,-1}^{V})}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

\begin{equation}
\eta_{\phi\phi} = \frac{\sqrt{2}(3\Re\rho_{1,0}^{V} - \Re\rho_{1,-1}^{V})}{(1 + \delta)N_V + (1 - 3\delta)\rho_{0,0}^{V}},
\end{equation}

where $N_V = \rho_{1,1}^{V} + \rho_{0,0}^{V} + \rho_{1,-1}^{V}$, and the parameters $\alpha, \delta$ depend on the identity of $V$ and of its decay products. In particular, $\alpha$ is induced by the parity-violating interactions in the decay. In other words, it is nonzero only when the decay is not a parity conservative process. For the $J/\psi$ decays into dilepton, $\alpha = 0, \delta = 0$, whereas for the pure E1 radiative transition $\chi_{c1} \rightarrow J/\psi\gamma, \alpha = 0, \delta = \frac{1}{2}$. If one also wants to include the M2 transition in the $\chi_{c1}$ decay, $\delta$ should be changed to $\frac{1}{2} + a_{J/\psi\gamma}^{2} + a_{J/\psi\gamma,2}^{2} = 1$. They have been measured in Refs. \cite{29, 30, 32}. The numerical values measured are shown in Table II. Without losing generality, the rotation-invariant observable $F_1$ de-
fined in Eq.(22) can be written as:

$$F_1 = \frac{1 - 3\delta + (1 - \delta)\lambda_\theta + 2\lambda_\phi}{(1 - 3\delta)(3 + \lambda_\theta)}. \quad (A3)$$

| Experiment     | $a^{J=1}_2$ (10$^{-2}$) |
|----------------|-------------------------|
| CLEO [29]     | $-6.26 \pm 0.63 \pm 0.24$ |
| Crystal Ball [30] | $-0.2^{+0.8}_{-2.0}$ |
| E835 [32]     | $0.2 \pm 3.2 \pm 0.4$   |

In order to show the impact of the higher-order multipole M2 contribution to the $\chi_{c1}$ polarizations, we take the example illustrated in Sec. VII. As an illustrative case, only $\lambda_\phi$ in the HX frame is shown in Fig. 17, where E1 means pure E1 transition approximation and E1+M2 means that we have included the full E1 and M2 transitions using the CLEO [29] measured $a^{J=1}$ in Table II.

### Appendix B: Decay angular distribution of spin-2 bosons

The general decay angular distributions of spin-2 tensor particles $T$ can have up to 24 observable parameters:

$$W^T(\theta, \phi) \propto 1 + \lambda_\theta \cos^2 \theta + \lambda_\phi \cos^4 \theta$$

$$+ \lambda_{\theta \phi} \sin 2\theta \cos \phi + \lambda_{2\theta \phi} \sin 2\theta \sin^2 \theta \cos \phi$$

$$+ \lambda_{3\theta \phi} \sin 2\theta \sin^2 \theta \cos \phi$$

$$+ \lambda_{\phi \phi} \sin^4 \phi + \lambda_{\phi \phi} \sin^4 \phi$$

$$+ 2\eta_\theta \cos \theta + 2\eta_{\theta \phi} \cos^3 \theta$$

$$+ 2\eta_{\theta \phi} \sin \theta \cos \phi + 2\eta_{\phi \phi} \sin^3 \theta \cos \phi$$

$$+ 2\eta_{1\phi \phi} \sin \theta \sin \phi + 2\eta_{2\phi \phi} \sin^3 \theta \sin \phi$$

$$+ 2\eta_{3\phi \phi} \sin^2 \theta \cos \phi + 2\eta_{3\phi \phi} \sin^2 \theta \cos \phi$$

$$+ 2\eta_{3\phi \phi} \sin^3 \theta \cos \phi$$

$$+ 2\eta_{3\phi \phi} \sin^3 \theta \sin \phi, \quad (B1)$$

where

$$\lambda_\theta = 6[(1 - 3\delta_0 - \delta_1)N_T$$

$$- (1 - 7\delta_0 + \delta_1)(\rho_{T,1}^T + \rho_{T,-1}^T)]$$

$$-(3 - \delta_0 - 7\delta_1)p_{0,0}^T/R, \quad (B2)$$

$$\lambda_{2\theta} = (1 + 5\delta_0 - 5\delta_1)[N_T - 5(\rho_{T,1}^T + \rho_{T,-1}^T)]$$

$$+ 5p_{0,0}^T/R, \quad \lambda_{0\phi} = 4[2(1 - \delta_0 - 2\delta_1)(\Re \rho_{T,1}^T - \Re \rho_{T,-1}^T)]$$

$$- \sqrt{6}(2\delta_0 - \delta_1)(\Re \rho_{T,0}^T - \Re \rho_{T,-1,0})]/R, \quad \lambda_{2\phi} = -2(1 + 5\delta_0 - 5\delta_1)[(\Re \rho_{T,1}^T - \Re \rho_{T,-1,0})]$$

$$+ \sqrt{6}(2\delta_0 - \delta_1)(\Re \rho_{T,0}^T - \Re \rho_{T,-1,0})]/R, \quad \lambda_{1\phi} = 4[-2(1 - \delta_0 - 2\delta_1)(\Im \rho_{T,1}^T + \Im \rho_{T,-1,0})]$$

$$+ \sqrt{6}(2\delta_0 - \delta_1)(\Im \rho_{T,0}^T + \Im \rho_{T,-1,0})]/R, \quad \lambda_{2\phi} = (2(1 + 5\delta_0 - 5\delta_1)[(\Im \rho_{T,1}^T + \Im \rho_{T,-1,0})]$$

$$- \sqrt{6}(2\delta_0 - \delta_1)(\Im \rho_{T,0}^T + \Im \rho_{T,-1,0})]/R, \quad \lambda_{3\phi} = 4[6(1 + 5\delta_0 - 3\delta_1)(\Re \rho_{T,2,0}^T + \Re \rho_{T,-2,0}^T)$$

$$- 6(2\delta_0 - \delta_1)R \rho_{T,-1,0}^T]/R, \quad \lambda_{2\phi} = -2(1 + 5\delta_0 - 5\delta_1)[\sqrt{6}(\Re \rho_{T,2,0}^T + \Re \rho_{T,-2,0}^T)$$

$$- 4\Re \rho_{T,-1,0}^T]/R,$$
and

\[ \eta_\theta = 2[(2\alpha_1 + \alpha_2)(\rho_{T,2}^2 - \rho_{T,2,-2}^2) - 2(\alpha_1 - \alpha_2)(\rho_{T,1}^2 - \rho_{T,-1,-1}^2)]/R, \]
\[ \eta_{2\theta} = 2[(2\alpha_1 + \alpha_2)(\rho_{T,2}^2 - \rho_{T,2,-2}^2) - 2(\rho_{T,1}^2 - \rho_{T,-1,-1}^2)]/R, \]
\[ \eta_{\theta\phi} = 4(2(\alpha_1 - \alpha_2)(\rho_{T,1}^2 + \rho_{T,-1,0}^2) - \sqrt{6}(\rho_{T,0}^2 + \rho_{T,-2,0}^2))/R, \]
\[ \eta_{2\theta\phi} = 2(2\alpha_1 - \alpha_2)[3(\rho_{T,2}^2 + \rho_{T,-2,-1}^2) - \sqrt{6}(\rho_{T,1}^2 - \rho_{T,0}^2)]/R, \]
\[ \eta_{\phi} = -2\sqrt{6}(2\alpha_1 - \alpha_2)(\rho_{T,0}^2 - \rho_{T,-2,0}^2), \]
\[ \eta_{\phi} = 2\sqrt{6}(2\alpha_1 - \alpha_2)(\rho_{T,0}^2 + \rho_{T,-2,0}^2), \]
\[ \eta_{3\theta \phi} = -2(\alpha_1 - \alpha_2)(\rho_{T,0}^2 + \rho_{T,-2,1}^2), \]
\[ \eta_{3\theta \phi} = 2(2\alpha_1 - \alpha_2)(\rho_{T,2}^2 - \rho_{T,2,-2}^2), \]

with

\[ N_T = \rho_{T,2,2} + \rho_{T,-1,1}^2 + \rho_{T,0,0}^2 + \rho_{T,0,-1}^2 + \rho_{T,2,-2}^2, \]
\[ R = (1 + 5\delta_0 + 3\delta_1)^2, \]
\[ + 3(1 - 3\delta_0 - \delta_1)(\rho_{T,1}^2 + \rho_{T,-1,-1}^2), \]
\[ + (5 - 7\delta_0 - 9\delta_1)\rho_{T,0,0}^2. \]

The parameters \( \alpha_1 \) and \( \alpha_2 \) vanish when parity is conserved, as in the \( \chi_{c2} \) decay. Other two parameters \( \delta_0 \) and \( \delta_1 \) can be determined from the specific processes considered. For the \( \chi_{c2} \) decays into a \( J/\psi \) and a photon, through pure E1 transition, \( \delta_0 = \frac{1}{10} \) and \( \delta_1 = \frac{1}{10} \), while after including the higher-order multipole amplitudes in the radiative transitions, the coefficients \( \delta_0 \) and \( \delta_1 \) can be expressed as the following polynomials in the E1, M2, and E3 amplitudes \( a_1^{J=2}, a_2^{J=2}, a_3^{J=2} \):

\[ \delta_0 = [1 + 2a_1^{J=2}(\sqrt{5}a_2^{J=2} + 2a_3^{J=2}) + 4a_2^{J=2}(a_2^{J=2} + \sqrt{5}a_3^{J=2}) + 3(a_3^{J=2})^2]/10, \]
\[ \delta_1 = [9 + 6a_1^{J=2}(\sqrt{5}a_2^{J=2} - 2a_3^{J=2}) - 4a_2^{J=2}(a_2^{J=2} + 2\sqrt{5}a_3^{J=2}) + 7(a_3^{J=2})^2]/30. \]

Again, the normalization of \( (a_1^{J=2})^2 + (a_2^{J=2})^2 + (a_3^{J=2})^2 = 1 \) has been imposed. The measurements of the multipole amplitudes Refs. [29–32] are listed in Table III. Finally, the expression of the frame-independent parameter \( F_2 \) in terms of the coefficients in Eq. (B1):

\[ F_2 = \frac{n_1 + n_2\lambda_\theta + n_3\lambda_{2\theta} + n_4\phi + n_5\lambda_{2\phi} + n_6\lambda_{4\phi}}{d_1 + d_2\lambda_\theta + d_3\lambda_{2\theta}}, \]
\[ n_1 = \frac{1}{6}, \]
\[ n_2 = \frac{4 - 4\delta_0 - 3\delta_1}{18(2 - 4\delta_0 - 3\delta_1)}, \]
\[ n_3 = \frac{2 + 2\delta_0 - 7\delta_1 - 4\delta_0^2 + \delta_0\delta_1 + 3\delta_1^2}{6(1 + 5\delta_0 - 5\delta_1)(2 - 4\delta_0 - 3\delta_1)}, \]
\[ n_4 = \frac{1}{1}, \]
\[ n_5 = \frac{2\delta_0 - \delta_1}{(1 + 5\delta_0 - 5\delta_1)(2 - 4\delta_0 - 3\delta_1)}, \]
\[ n_6 = \frac{1}{1 + 5\delta_0 - 5\delta_1}, \]
\[ d_1 = \frac{15}{16}, d_2 = \frac{5}{16}, d_3 = 3 \frac{16}{16}. \]

In order to show the impact of the higher-order multipole M2 contribution\(^{11} \) to the \( \chi_{c2} \) polarizations, we take the example illustrated in Sec. VII as well. As an illustrative example, only \( \lambda_\phi \) in the \( \chi_\phi \) frame is illustrated in Fig. 18, where \( \chi \) is means pure E1 transition approximation and \( \chi_1 + \chi_2 \) means that we have included the full E1 and M2 transitions, with \( a_2^{J=2} \) as measured by the CLEO collaboration [29] (Table III).

\[ \text{(B6)} \]

\[ \text{TABLE III: The normalized M2 and E3 amplitudes} \]
\[ a_2^{J=2}, a_3^{J=2} \text{ for} \chi_{c2} \to J/\psi\gamma \text{ as measured by various experiments.} \]

| Experiment | \( a_2^{J=2}(10^{-2}) \) | \( a_3^{J=2}(10^{-2}) \) |
|------------|-----------------|-----------------|
| CLEO (Fit 1) [29] | \(-9.3 \pm 1.6 \pm 0.3 \) | 0 (fixed) |
| CLEO (Fit 2) [29] | \(-7.9 \pm 1.9 \pm 0.3 \) | 1.7 \pm 1.4 \pm 0.3 |
| Crystal Ball [30] | \(-33.3^{+11.6}_{-9.2} \) | 0 (fixed) |
| E760 (Fit 1) [31] | \(-14 \pm 6 \) | 0 (fixed) |
| E760 (Fit 2) [31] | \(-14^{+8}_{-7} \) | 0.5 \pm 0.5 |
| E835 (Fit 1) [32] | \(-9.3^{+3.9}_{-4.1} \pm 0.6 \) | 0 (fixed) |
| E835 (Fit 2) [32] | \(-7.6^{+5.4}_{-5.0} \pm 0.9 \) | 2.0^{+5.5}_{-4.4} \pm 0.9 |

\[ \text{Appendix C: Dilepton angular distribution in} \chi_c \text{ decay with multipole effects} \]

We consider here the general expression of the dilepton angular distribution in the decays \( \chi_c \to J/\psi\gamma \to \mu^+\mu^-\gamma \), including also the higher-order multipole transitions neglected in Sec. V.

In the first option discussed in Sec. V for the definition of the \( J/\psi \) quantization axis, the SDMEs for the

\[ \text{ref.}^{11} \] The E3 amplitude for the \( \chi_{c2} \) decay is zero from the consideration of the single quark radiation hypothesis.
$J/\psi$ from $\chi_c$ decays can be expressed in terms of the $\chi_c$ production matrix elements as\footnote{If one does not integrate the angles $\theta$ and $\phi$ in the following equation and put the SDMEs $\rho_{s_6,s_6'}^{\chi_c \rightarrow J/\psi \gamma}$ into Eq.(17), one obtains the full angular distribution of the decay chain $\chi_c \rightarrow J/\psi \gamma \rightarrow \mu^+\mu^-$, including the correlations between the $\chi_c$ decay angles $\theta, \phi$ and the $J/\psi$ decay angles $\theta', \phi'$, which might be useful in Monte Carlo simulations of experimental analyses.}

$$
\rho_{s_6,s_6'}^{\chi_c \rightarrow J/\psi \gamma} = \frac{1}{8\pi} \sum_{l=1}^{J+1} \sum_{l_z=-l}^{l_z} \int \! d\Omega[\theta, \phi] \rho_{l,s_6,l_z}^{\chi_c \rightarrow J/\psi \gamma} D_{l,s_6}^{l_z} D_{l_z,s_6'}^{l_z'} \langle l,l_z; 1,s_2|J,J_z|1,\lambda_c; 1,s_2' \rangle (2l+1)(a_l^{J=J})^2 \text{Br}(\chi_c \rightarrow J/\psi \gamma).
$$

(C1)

The coefficients are expressed as

$$
\chi_{\nu^c}^{\chi_c} = -\frac{1 - 3(a_j^{\nu^c})^2}{3 - (a_j^{\nu^c})^2}, \quad \chi_{\nu^c}^{\nu^c} = \chi_{\nu^c}^{\chi_c} = 0,
$$

$$
\chi_{\nu^c}^{\chi_c} = \frac{\sqrt{2}(a_j^{\nu^c})^2 (R(p_{\nu^c}^{\chi_c}) - R(p_{\nu^c}^{\chi_c}))}{4(3 - (a_j^{\nu^c})^2)^2 N_{\chi_c}},
$$

$$
\chi_{\nu^c}^{\chi_c} = \frac{\sqrt{2}(a_j^{\nu^c})^2 (3(p_{\nu^c}^{\chi_c}) + 3(p_{\nu^c}^{\chi_c}))}{4(3 - (a_j^{\nu^c})^2)^2 N_{\chi_c}},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{3(1 - 11(a_j^{\nu^c})^2)^2 + 9(a_j^{\nu^c})^2}{39 + 11(a_j^{\nu^c})^2 - 9(a_j^{\nu^c})^2},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{(a_j^{\nu^c})^2 (7\sqrt{6}(R(p_{\nu^c}^{\chi_c}) + R(p_{\nu^c}^{\chi_c})) + 12R(p_{\nu^c}^{\chi_c}))}{2(39 + 11(a_j^{\nu^c})^2 - 9(a_j^{\nu^c})^2)^2 N_{\chi_c}},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{[\sqrt{6}(1 - \frac{13}{3}(a_j^{\nu^c})^2)^2 (a_j^{\nu^c})^2 (3p_{\nu^c}^{\chi_c} + 3p_{\nu^c}^{\chi_c}) + 6(4 - 9(a_j^{\nu^c})^2)^2 (a_j^{\nu^c})^2 (3p_{\nu^c}^{\chi_c} + 3p_{\nu^c}^{\chi_c})]}{[4(39 + 11(a_j^{\nu^c})^2 - 9(a_j^{\nu^c})^2)^2 N_{\chi_c}}.
$$

As remarked previously, some of the polarization observables are trivial and devoid of spin information. However, these observables can, in principle, be measured to extract the multipole amplitudes of the $\chi_c$ decay, for example, in electron-positron collisions.

In the second option, in which the quantization axis for the $J/\psi \rightarrow \mu^+\mu^-$ decay coincides with the $\chi_c$ quantization axis, the general relation between the SDMEs of the $\chi_c J' \rho_{s_6,s_6'}^{\chi_c J' \rightarrow J/\psi \gamma}$ and those of the $J/\psi$ from the $\chi_c J$ decay

$$
\rho_{s_6,s_6'}^{\chi_c J \rightarrow J/\psi \gamma} \text{ in the full multipole expansion is}
$$

$$
\rho_{s_6,s_6'}^{\chi_c J \rightarrow J/\psi \gamma} \propto \sum_{l=1}^{J+1} \sum_{l_z=-l}^{l_z} \sum_{J_s,J_s'} (a_l^{J=J})^2 \langle l,l_z; 1,s_2|J,J_z|1,\lambda_c; 1,s_2' \rangle \langle J,J_z'|l,l_z; 1,s_2'|\rho_{s_6,s_6'}^{\chi_c J'} \text{Br}(\chi_c \rightarrow J/\psi \gamma).
$$

(C2)

The coefficients of the $\mu^+$ angular distribution are

$$
\chi_{\nu^c}^{\chi_c} = \frac{-N_{\chi_c} + 3p_{\nu^c}^{\chi_c}}{R_1},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{-2R_{p_{\nu^c}^{\chi_c}}}{R_1},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{-\sqrt{2}(R_{p_{\nu^c}^{\chi_c}} - R_{p_{\nu^c}^{\chi_c}})}{R_1},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{2\sqrt{2}(R_{p_{\nu^c}^{\chi_c}} + 3p_{\nu^c}^{\chi_c})}{R_1},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{6(N_{\chi_c} - 9(p_{\nu^c}^{\chi_c} + p_{\nu^c}^{\chi_c}) - 12R_{p_{\nu^c}^{\chi_c}}}{R_2},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{2\sqrt{6}(R_{p_{\nu^c}^{\chi_c}} + R_{p_{\nu^c}^{\chi_c}}) + 6R_{p_{\nu^c}^{\chi_c}}}{R_2},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{6(R_{p_{\nu^c}^{\chi_c}} - R_{p_{\nu^c}^{\chi_c}}) + \sqrt{6}(R_{p_{\nu^c}^{\chi_c}} - R_{p_{\nu^c}^{\chi_c}})}{R_2},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{6(3\rho_{\nu^c}^{\chi_c} - 3p_{\nu^c}^{\chi_c} - 63R_{p_{\nu^c}^{\chi_c}}}{R_2},
$$

$$
\chi_{\nu^c}^{\nu^c} = \frac{6(3\rho_{\nu^c}^{\chi_c} + 3\rho_{\nu^c}^{\chi_c} + 6\rho_{p_{\nu^c}^{\chi_c}})}{R_2},
$$

with

$$
R_1 = \left\{ (15 - 2(a_j^{\nu^c})^2)N_{\chi_c} \right. \\
- (5 - 6(a_j^{\nu^c})^2)^2 R_{p_{\nu^c}^{\chi_c}} \right\} / \left( (5 - 6(a_j^{\nu^c})^2)^2, \\
R_2 = \left( 2(21 + 14(a_j^{\nu^c})^2 + 5(a_j^{\nu^c})^2)N_{\chi_c} \right. \\
+ 3(7 - 14(a_j^{\nu^c})^2 - 5(a_j^{\nu^c})^2)(\rho_{\nu^c}^{\chi_c} + \rho_{\nu^c}^{\chi_c}) \right. \\
+ 4(7 - 14(a_j^{\nu^c})^2 - 5(a_j^{\nu^c})^2)\rho_{\nu^c}^{\chi_c} \right\} / \left( (7 - 14(a_j^{\nu^c})^2 - 5(a_j^{\nu^c})^2)^2. \\
$$

Appendix D: A possible way for determining $\rho_{J_s,J_s'}(|J_s - J_s'| > 2)$ of $\chi_c \rightarrow \chi_c \rightarrow \mu^+\mu^-$ with a reweighting method

From Eqs.(12), (14), (18), we see that the $\chi_c \rightarrow \chi_c$ SDMEs $\rho_{J_s,J_s'}(|J_s - J_s'| > 2)$ cannot be measured from the integrated angular distributions. The fact that the coefficients of these SDMEs are suppressed by $\nu^2$ or $(m_{\chi_c} - m_{J/\psi})$ makes the measurement of these polarization observables difficult. In this appendix, we propose a reweighting method to measure these SDMEs.

From Eq.(10), it can be recognized that this fact originates from the cancellation of the transverse and lon-
The explicit expressions for the coefficients are

\[ N_{\chi_{c2}} = \rho_{2.2} + \rho_{1.1} + \rho_{0.0} + \rho_{-1.1} + \rho_{-2.2}, \]

\[ R = 3(r_T + r_L)N_{\chi_{c2}} + 3r_T(\rho_{1.1} + \rho_{-1.1}) + (7r_T - 3r_L)\rho_{0.0}, \]

\[ \lambda_\phi = \frac{6r_TN_{\chi_{c2}} - 9r_L(\rho_{1.1} + \rho_{-1.1}) - 6(5r_T - 3r_L)\rho_{0.0}}{R}, \]

\[ \lambda_{2\phi} = \frac{(r_T - r_L)^3N_{\chi_{c2}} - 15(\rho_{1.1} + \rho_{-1.1}) + 15\rho_{0.0}}{R}. \]

From the above equations, we find that one can measure \( \lambda_{3\phi}, \lambda_{3\phi}^+, \lambda_{4\phi}, \lambda_{4\phi}^+ \) to determine the values of \( \rho_{J_z, J_z'}(|J_z - J_z'| > 2) \) if \( r_T \neq r_L \).
FIG. 3: (color online). Distributions of $d\sigma_{00}/dp_T$, $d\sigma_{11}/dp_T$, and $d\sigma_{tot}/dp_T = \gamma d\sigma_{11}/dp_T + d\sigma_{00}/dp_T$ for $p_T^{[1]}$ in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

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FIG. 4: (color online). Distributions of $\frac{d\sigma_{00}}{dp_T}$, $\frac{d\sigma_{11}}{dp_T}$, and $\frac{d\sigma_{02}}{dp_T}$ for $J/\psi(2S)$ in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

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FIG. 6: (color online). Real parts of the nondiagonal SDMEs (normalized by $\frac{d\sigma}{dp_T} = 2\frac{d\sigma}{dp_T} + \frac{d\sigma}{dp_T}$) for $\tilde{\mathcal{P}}_2^{[1]}$ in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

FIG. 7: (color online). Real parts of the nondiagonal SDMEs (normalized by $\frac{d\sigma}{dp_T} = 2\frac{d\sigma}{dp_T} + \frac{d\sigma}{dp_T} + \frac{d\sigma}{dp_T}$) for $\tilde{\mathcal{P}}_3^{[1]}$ in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames. In Fig.7(a), $\rho_{1,0,} \rho_{2,1,}$ and $\rho_{2,-1}$ are almost zero. In Fig.7(b), $\rho_{1,0,} \rho_{1,-1,} \rho_{2,1,}$ and $\rho_{2,-1}$ vanish.

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FIG. 8: (color online). Real parts of the nondiagonal SDMEs (normalized by $\frac{2\alpha_s}{\sqrt{p_T}}$) for $^3S_1^{[0]}$ in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

FIG. 9: (color online). Transverse momentum dependence of the parameter $\lambda_0$ of $\chi_{c1} \to J/\psi \gamma$ angular distribution, in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

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FIG. 10: (color online). Transverse momentum dependence of $\lambda_\phi$ of $\chi_{c1} \to J/\psi \gamma$ angular distribution, in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

FIG. 11: (color online). Transverse momentum dependence of $\lambda_{\theta\phi}$ of $\chi_{c1} \to J/\psi \gamma$ angular distribution, in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.
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FIG. 14: (color online). Transverse momentum dependence of $\lambda_{0\phi}$ of $\chi_{c2} \to J/\psi \gamma$ angular distribution, in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

FIG. 15: (color online). Transverse momentum dependence of the frame-independent parameter $F_{1}\chi_{c1} \to J/\psi \gamma$ [defined in Eq. (22)], in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

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FIG. 16: (color online). Transverse momentum dependence of the frame-independent parameter \( F_{\chi_{c2} \rightarrow J/\psi \gamma} \) [defined in Eq.(23)], in the (a) HX, (b) Collins-Soper, and (c) Gottfried-Jackson frames.

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