On the binomial edge ideals of block graphs

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Abstract

We find a class of block graphs whose binomial edge ideals have minimal regularity. As a consequence, we characterize the trees whose binomial edge ideals have minimal regularity. Also, we show that the binomial edge ideal of a block graph has the same depth as its initial ideal.

1 Introduction

In this paper we study homological properties of some classes of binomial edge ideals.

Let $G$ be a simple graph on the vertex set $[n]$ and let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables over a field $K$. For $1 \leq i < j \leq n$, we set $f_{ij} = x_i y_j - x_j y_i$. The binomial edge ideal of $G$ is defined as $J_G = \langle f_{ij} : \{i, j\} \in E(G) \rangle$. Binomial edge ideals were introduced in [8] and [12]. Algebraic and homological properties of binomial edge ideals have been studied in several papers. In [5], it was conjectured that $J_G$ and $\text{in}_<(J_G)$ have the same extremal Betti numbers. Here $<$ denotes the lexicographic order in $S$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. This conjecture was proved in [3] for cycles and complete bipartite graphs. In [6], it was shown that, for a closed graph $G$, $J_G$ and $\text{in}_<(J_G)$ have the same regularity which can be expressed in the combinatorial data of the graph. We recall that a graph $G$ is closed if and only if it has a quadratic Gröbner basis with respect to the lexicographic order.
In support of the conjecture given in [5], we show, in Section 3, that if \( G \) is a block graph, then \( \text{depth}(S/J_G) = \text{depth}(S/\text{in}_< (J_G)) \); see Theorem 3.2. By a block graph we mean a chordal graph \( G \) with the property that any two maximal cliques of \( G \) intersect in at most one vertex.

Also, in the same section, we show a similar equality for regularity. More precisely, in Theorem 3.4 we show that \( \text{reg}(S/J_G) = \text{reg}(S/\text{in}_< (J_G)) = \ell \) if \( G \) is a \( C_\ell \)-graph. \( C_\ell \)-graphs constitute a subclass of the block graphs; see Section 3 for definition and Figure 1 for an example.

In [10] it was shown that, for any connected graph \( G \) on the vertex set \([n]\), we have
\[
\ell \leq \text{reg}(S/J_G) \leq n - 1,
\]
where \( \ell \) is the length of the longest induced path of \( G \).

The main motivation of our work was to answer the following question. May we characterize the connected graphs \( G \) whose longest induced path has length \( \ell \) and \( \text{reg}(S/J_G) = \ell \)? We succeeded to answer this question for trees. In Theorem 4.1, we show that if \( T \) is a tree whose longest induced path has length \( \ell \), then \( \text{reg}(S/J_T) = \ell \) if and only if \( T \) is caterpillar. A caterpillar tree is a tree \( T \) with the property that it contains a path \( P \) such that any vertex of \( T \) is either a vertex of \( P \) or it is adjacent to a vertex of \( P \).

In [11], the so-called weakly closed graphs were introduced. This is a class of graphs which includes closed graphs. In the same paper, it was shown that a tree is caterpillar if and only if it is a weakly closed graph. Having in mind our Theorem 4.1 and Theorem 3.2 in [6] which states that \( \text{reg}(S/J_G) = \ell \) if \( G \) is a connected closed graph whose longest induced path has length \( \ell \), and by some computer experiments, we are tempted to formulate the following.

**Conjecture 1.1.** If \( G \) is a connected weakly closed graph whose longest induced path has length \( \ell \), then \( \text{reg}(S/J_G) = \ell \).

## 2 Preliminaries

In this section we introduce the notation used in this paper and summarize a few results on binomial edge ideals.

Let \( G \) be a simple graph on the vertex set \([n] = \{1, \ldots, n\} \), that is, \( G \) has no loops and no multiple edges. Furthermore, let \( K \) be a field and \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) be the polynomial ring in \( 2n \) variables. For \( 1 \leq i < j \leq n \), we set \( f_{ij} = x_i y_j - x_j y_i \). The binomial edge ideal \( J_G \subset S \) associated with \( G \) is generated by all the quadratic binomials \( f_{ij} = x_i y_j - x_j y_i \) such that \( \{i, j\} \in E(G) \). Binomial edge ideals were introduced in the papers [8] and [12].

We first recall some basic definitions from graph theory. A vertex \( i \) of \( G \) whose deletion from the graph gives a graph with more connected components
than G is called a cut point of G. A chordal graph is a graph without cycles of length greater than or equal to 4. A clique of a graph G is a complete subgraph of G. The cliques of a graph G form a simplicial complex, Δ(G), which is called the clique complex of G. Its facets are the maximal cliques of G. A graph G is a block graph if and only if it is chordal and every two maximal cliques have at most one vertex in common. This class was considered in [5, Theorem 1.1].

The clique complex Δ(G) of a chordal graph G has the property that there exists a leaf order on its facets. This means that the facets of Δ(G) may be ordered as F₁, ..., Fₙ such that, for every i > 1, Fᵢ is a leaf of the simplicial complex generated by F₁, ..., Fᵢ₋¹. A leaf F of a simplicial complex Δ is a facet of Δ with the property that there exists another facet of Δ, say G, such that, for every facet H ≠ F of Δ, H ∩ F ⊆ G ∩ F.

Let < be the lexicographic order on S induced by the natural order of the variables. As it was shown in [8, Theorem 2.1], the Gröbner basis of J₆ with respect to this order may be given in terms of the admissible paths of G. We recall the definition of admissible paths from [8].

Definition 2.1. [8] Let i < j be two vertices of G. A path i = i₀, i₁, ..., iᵣ = j from i to j is called admissible if the following conditions are fulfilled:

1. iₖ ≠ iᵢ for k ≠ l;
2. for each k = 1, ..., r - 1 on has either iₖ < i or iₖ > j;
3. for any proper subset {j₁, ..., jₛ} of {i₁, ..., iᵣ₋₁}, the sequence i, j₁, ..., jₛ, j is not a path in G.

Given an admissible path π in G from i to j, we set uₚ = (∏ᵢₖ>j xᵢₖ)(∏ᵢᵢ<i yᵢᵢ).

By [8, Theorem 2.1], it follows that

\[ \text{in}_<(J₆) = (uₚ xᵢ yⱼ : i < j, \text{π is an admissible path from } i \text{ to } j). \]

In particular, \( \text{in}_{<}(J₆) \) is a radical monomial ideal which implies that the binomial edge ideal \( J₆ \) is radical as well. Hence \( J₆ \) is equal to the intersection of all its minimal prime ideals. The minimal prime ideals were determined in [8, Section 3] in terms of the combinatorial data of the graph.

3 Initial ideals of binomial edge ideals of block graphs

In this section, we first show that, for a block graph G on \([n]\) with \(c\) connected components, we have depth(\(S/J₆\)) = depth(\(S/\text{in}_{<}(J₆)\)) = n + c, where <
denotes the lexicographic order induced by \( x_1 > \cdots > x_n > y_1 > \cdots > y_n \) in the ring \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \).

We begin with the following lemma.

**Lemma 3.1.** Let \( G \) be a graph on the vertex set \([n]\) and let \( i \in [n] \). Then
\[
\in_{\prec}(J_G, x_i, y_i) = (\in_{\prec}(J_G), x_i, y_i).
\]

**Proof.** We have \( \in_{\prec}(J_G, x_i, y_i) = \in_{\prec}(J_{G\setminus\{i\}}, x_i, y_i) = (\in_{\prec}(J_{G\setminus\{i\}}), x_i, y_i) \). Therefore, we have to show that \((\in_{\prec}(J_G), x_i, y_i) = (\in_{\prec}(J_{G\setminus\{i\}}), x_i, y_i)\). The inclusion \( \supseteq \) is obvious since \( J_{G\setminus\{i\}} \subset J_G \). For the other inclusion, let us take \( u \) to be a minimal generator of \( \in_{\prec}(J_G) \). If \( x_i \mid u \) or \( y_i \mid u \), obviously \( u \in (\in_{\prec}(J_{G\setminus\{i\}}), x_i, y_i) \). Let now \( x_i \notin u \) and \( y_i \notin u \). This means that \( u = u_x x_i y_i \) for some admissible path \( \pi \) from \( k \) to \( l \) which does not contain the vertex \( i \). Then it follows that \( \pi \) is a path from \( k \) to \( l \) in \( G\setminus\{i\} \), hence \( u \in \in_{\prec}(J_{G\setminus\{i\}}) \). \( \square \)

**Theorem 3.2.** Let \( G \) be a block graph. Then
\[
\text{depth}(S/J_G) = \text{depth}(S/\in_{\prec}(J_G)) = n + c,
\]
where \( c \) is the number of connected component of \( G \).

**Proof.** Let \( G_1, \ldots, G_c \) be the connected components of \( G \) and \( S_i = K[x_j, y_j]_{j \in G_i} \). Then \( S/J_G \cong S_1/J_{G_1} \otimes \cdots \otimes S_c/J_{G_c} \), so that
\[
\text{depth } S/J_G = \text{depth } S_1/J_{G_1} + \cdots + \text{depth } S_c/J_{G_c}.
\]

Moreover, we have \( S/\in_{\prec}(J_G) \cong S/\in_{\prec}(J_{G_1}) \otimes \cdots \otimes S/\in_{\prec}(J_{G_c}) \), thus
\[
\text{depth } S/\in_{\prec}(J_G) = \text{depth } S_1/\in_{\prec}(J_{G_1}) + \cdots + \text{depth } S_c/\in_{\prec}(J_{G_c}).
\]

Hence, without loss of generality, we may assume that \( G \) is connected. By [5, Theorem 1.1] we know that \( \text{depth}(S/J_G) = n + 1 \). In order to show that \( \text{depth}(S/\in_{\prec}(J_G)) = n + 1 \), we proceed by induction on the number of maximal cliques of \( G \). Let \( \Delta(G) \) be the clique complex of \( G \) and let \( F_1, \ldots, F_r \) be a leaf order on the facets of \( \Delta(G) \). If \( r = 1 \), then \( G \) is a simplex and the statement is well known. Let \( r > 1 \); since \( F_r \) is a leaf, there exists a unique vertex, say \( i \in F_r \), such that \( F_r \cap F_j = \{i\} \) for \( F_j \) is a branch of \( F_r \). Let \( F_{i_1}, \ldots, F_{i_q} \) be the facets of \( \Delta(G) \) which intersect the leaf \( F_r \) in the vertex \( \{i\} \). Following the proof of [5, Theorem 1.1] we may write \( J_G = J_1 \cap J_2 \) where \( J_1 = \bigcap_{s \in S} P_s(G) \) and \( J_2 = \bigcap_{l \in S} P_{F_l}(G) \). Then, as it was shown in the proof of [5, Theorem 1.1], it follows that \( J_1 = J_G' \) where \( G' \) is obtained from \( G \) by replacing the cliques \( F_1, \ldots, F_{i_q} \) and \( F_r \) by the clique on the vertex set \( F_r \cup (\bigcup_{j=1}^q F_{i_j}) \). Also, \( J_2 = (x_i, y_i) + J_{G''} \) where \( G'' \) is the restriction of \( G \) to the vertex set \([n] \setminus \{i\} \).
We have \( \text{in}_< (J_G) = \text{in}_< (J_1 \cap J_2) \). By [1, Lemma 1.3], we have \( \text{in}_< (J_1 \cap J_2) = \text{in}_< (J_1) \cup \text{in}_< (J_2) \) if and only if \( \text{in}_< (J_1 + J_2) = \text{in}_< (J_1) + \text{in}_< (J_2) \). But \( \text{in}_< (J_1 + J_2) = \text{in}_< (J_G') + (x_i, y_i) + \text{in}_< (J_G'') = \text{in}_< (J_G') + (x_i, y_i) \). Hence, by Lemma 3.1, we get \( \text{in}_< (J_1 + J_2) = \text{in}_< (J_G') + (x_i, y_i) = \text{in}_< (J_1) + \text{in}_< (J_2) \). Therefore, we get \( \text{in}_< (J_G) = \text{in}_< (J_1) \cap \text{in}_< (J_2) \) and, consequently, we have the following exact sequence of \( S \)-modules

\[
0 \rightarrow \frac{S}{\text{in}_< (J_G)} \rightarrow \frac{S}{\text{in}_< (J_1)} \oplus \frac{S}{\text{in}_< (J_2)} \rightarrow \frac{S}{\text{in}_< (J_1 + J_2)} \rightarrow 0.
\]

By using again Lemma 3.1, we have \( \text{in}_< (J_2) = \text{in}_< ((x_i, y_i), J_G'') = (x_i, y_i) + \text{in}_< (J_G'') \). Thus, we have actually the following exact sequence

\[
0 \rightarrow \frac{S}{\text{in}_< (J_G)} \rightarrow \frac{S}{\text{in}_< (J_G')} \oplus (x_i, y_i) + \text{in}_< (J_G'') \rightarrow \frac{S}{\text{in}_< (J_1 + J_2)} \rightarrow 0.
\]

(1)

Since \( G' \) inherits the properties of \( G \) and has a smaller number of maximal cliques than \( G \), it follows, by the inductive hypothesis, that

\[
\text{depth}(S/J_G') = \text{depth}(S/\text{in}_< (J_G')) = n + 1.
\]

Let \( S_i \) be the polynomial ring \( S/(x_i, y_i) \). Then \( S/((x_i, y_i) + \text{in}_< (J_G'')) \cong S_i/\text{in}_< (J_G'') \). Since \( G'' \) is a graph on \( n - 1 \) vertices with \( q + 1 \) connected components and satisfies our conditions, the inductive hypothesis implies that \( \text{depth} S/((x_i, y_i) + \text{in}_< (J_G'')) = n + q \geq n + 1 \). Hence,

\[
\text{depth}(S/\text{in}_< (J_G') \oplus S/((x_i, y_i) + \text{in}_< (J_G''))) = n + 1.
\]

Next, we observe that \( S/((x_i, y_i) + \text{in}_< (J_G'')) \cong S_i/\text{in}_< (J_H) \), where \( H \) is obtained from \( G' \) by replacing the clique on the vertex set \( F_r \cup (\bigcup_{j=1}^r F_r) \setminus \{i\} \) by the clique on the vertex set \( F_r \cup (\bigcup_{j=1}^r F_r) \setminus \{i\} \). Hence, by the inductive hypothesis, \( \text{depth} S/((x_i, y_i) + \text{in}_< (J_G'')) = n \) since \( H \) is connected and its vertex set has cardinality \( n - 1 \). Hence, by applying the Depth lemma to exact sequence (1), we get

\[
\text{depth} S/J_G = \text{depth} S/\text{in}_< (J_G) = n + 1.
\]

\[\Box\]

**Definition 3.3.** Let \( \ell \geq 2 \) be an integer. A \( C_\ell \)-graph is a connected graph \( G \) on the vertex set \([n]\) which consists of

(i) a sequence of maximal cliques \( F_1, \ldots, F_\ell \) with \( \dim F_i \geq 1 \) for all \( i \) such that \( |F_i \cap F_{i+1}| = 1 \) for \( 1 \leq i \leq \ell - 1 \) and \( F_i \cap F_j = \emptyset \) for any \( i < j \) such that \( j \neq i + 1 \), together with
(ii) some additional edges of the form $F = \{j, k\}$ where $j$ is an intersection point of two consecutive cliques $F_i, F_{i+1}$ for some $1 \leq i \leq \ell - 1$, and $k$ is a vertex of degree 1.

In other words, $G$ is obtained from a graph $H$ with $\Delta(H) = \langle F_1, \ldots, F_\ell \rangle$ whose binomial edge ideal is Cohen-Macaulay (see [5, Theorem 3.1]) by attaching edges in the intersection points of the facets of $\Delta(H)$. Obviously, such a graph has the property that its longest induced path has length equal to $\ell$. In the case that $\dim F_i = 1$ for $1 \leq i \leq \ell$, then $G$ is called a caterpillar graph. Figure 1 displays a $C\ell$-graph with $\ell = 5$.

![Figure 1: $C\ell$-graph](image)

We should also note that any $C\ell$-graph is chordal and has the property that any two distinct maximal cliques intersect in at most one vertex. So that any $C\ell$-graph is a connected block graph. But, obviously, there are block graphs which are not $C\ell$-graphs. Such an example is displayed in Figure 2.

![Figure 2: A block graph which is not a $C\ell$-graph](image)

**Theorem 3.4.** Let $G$ be a $C\ell$-graph on the vertex set $[n]$. Then

$$\text{reg}(S/J_G) = \text{reg}(S/\text{in}_{\Delta}(J_G)) = \ell.$$ 

**Proof.** Let $G$ consists of the sequence of maximal cliques $F_1, \ldots, F_\ell$ as in condition (i) in Definition 3.3 to which we add some edges as in condition (ii). So the maximal cliques of $G$ are $F_1, \ldots, F_\ell$ and all the additional whiskers. We proceed by induction on the number $r$ of maximal cliques of $G$. If $r = \ell$, then $G$ is a closed graph whose binomial edge ideal is Cohen-Macaulay, hence the statement holds by [6, Theorem 3.2]. Let $r > \ell$ and let $F'_1, \ldots, F'_r$ be a leaf order on the facets of $\Delta(G)$. Obviously, we may choose a leaf order on $\Delta(G)$
such that \( F'_r = F_r \). With the same arguments and notation as in the proof of Theorem 3.2, we get the sequence (1).

We now observe that \( G' \) is a \( C_{\ell-1} \)-graph, hence, by the inductive hypothesis,
\[
\text{reg} \left( \frac{S}{J_{G'}} \right) = \text{reg} \left( \frac{S}{\text{in}_<(J_{G'})} \right) = \ell - 1. \tag{2}
\]

The graph \( G'' \) has at most two non-trivial connected components. One of them, say \( H_1 \), is a \( C_{\ell'} \)-graph with \( \ell' \in \{\ell - 2, \ell - 1\} \). The other possible non-trivial component, say \( H_2 \), occurs if \( |F_{\ell}| \geq 3 \) and, in this case, \( H_2 \) is a clique of dimension \( |F_{\ell}| - 2 \geq 1 \). By the inductive hypothesis, we obtain
\[
\text{reg} \left( \frac{S}{J_{G''}} \right) = \text{reg} \left( \frac{S}{\text{in}_<(J_{G''})} \right) = \text{reg} \left( \frac{S}{J_{H_1}} \right) + \text{reg} \left( \frac{S}{J_{H_2}} \right) \leq \ell - 1 + 1 = \ell. \tag{3}
\]

Relations (2) and (3) yield \( \text{reg}(S/\text{in}_<(J_{G'}) \oplus S/(x_i, y_i) + \text{in}_<(J_{G''})) \leq \ell \).

From the exact sequence (1) we get
\[
\text{reg} \left( \frac{S}{\text{in}_<(J_G)} \right) \leq \max \{ \text{reg} \left( \frac{S}{\text{in}_<(J_{G'})} \oplus \frac{S}{(x_i, y_i) + \text{in}_<(J_{G''})} \right), \text{reg} \left( \frac{S}{\text{in}_<(J_G)} + 1 \right) \} \leq \ell. \tag{4}
\]

By [7, Theorem 3.3.4], we know that \( \text{reg}(S/J_G) \leq \text{reg}(S/\text{in}_<(J_G)) \), and by [10, Theorem 1.1], we have \( \text{reg}(S/J_G) \geq \ell \). By using all these inequalities, we get the desired conclusion.

\[ \blacksquare \]

4 Binomial edge ideals of caterpillar trees

Matsuda and Murai showed in [10] that, for any connected graph \( G \) on the vertex set \([n]\), we have \( \ell \leq \text{reg}(S/J_G) \leq n - 1 \), where \( \ell \) denotes the length of the longest induced path of \( G \), and conjectured that \( \text{reg}(S/J_G) = n - 1 \) if and only if \( T \) is a line graph. Several recent papers are concerned with this conjecture; see, for example, [6], [13], and [14]. One may ask as well to characterize connected graphs \( G \) whose longest induced path has length \( \ell \) and \( \text{reg}(S/J_G) = \ell \). In this section, we answer this question for trees.

A caterpillar tree is a tree \( T \) with the property that it contains a path \( P \) such that any vertex of \( T \) is either a vertex of \( P \) or it is adjacent to a vertex of \( P \). Clearly, any caterpillar tree is a \( C_{\ell} \)-graph for some positive integer \( \ell \).

Caterpillar trees were first studied by Harary and Schwenk [9]. These graphs have applications in chemistry and physics [4]. In Figure 3, an example of caterpillar tree is displayed. Note that any caterpillar tree is a narrow graph in the sense of Cox and Erskine [2]. Conversely, one may easily see that any narrow tree is a caterpillar tree. Moreover, as it was observed in [11], a tree is
a caterpillar graph if and only if it is weakly closed in the sense of definition given in [11].

In the next theorem we characterize the trees \( T \) with \( \text{reg}(S/J_T) = \ell \) where \( \ell \) is the length of the longest induced path of \( T \).

**Theorem 4.1.** Let \( T \) be a tree on the vertex set \([n]\) whose longest induced path \( P \) has length \( \ell \). Then \( \text{reg}(S/J_T) = \ell \) if and only if \( T \) is caterpillar.

**Proof.** Let \( T \) be a caterpillar tree whose longest induced path has length \( \ell \). Then, by the definition of a caterpillar tree, it follows that \( T \) is a \( C_\ell \)-graph. Hence, \( \text{reg}(S/J_T) = \ell \) by Theorem 3.4. Conversely, let \( \text{reg}(S/J_T) = \ell \) and assume that \( T \) is not caterpillar. Then \( T \) contains an induced subgraph \( H \) with \( \ell + 3 \) vertices as in Figure 4.

Then, by [15, Theorem 27], it follows that \( \text{reg}(S/J_H) = \ell + 1 \). Thus, since \( \text{reg}(S/J_H) \leq \text{reg}(S/J_G) \) (see [10, Corollary 2.2]), it follows that \( \text{reg}(S/J_G) \geq \ell + 1 \), contradiction to our hypothesis. \( \square \)

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