Domain Partition of the Hydro Production Function for Solving Efficiently the Short-Term Generation Scheduling Problem

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ABSTRACT: Short-term generation scheduling (STGS) is a fundamental task in the operational planning analysis of hydroelectric plants. For the multi-unit case, the STGS is represented as a large-scale nonconvex mixed-integer nonlinear optimization model. Then, considering the (usual) short time for providing a solution, it is vital to exploit all the structural properties of the problem at hand. The main issue for exploiting this problem is the hydro production function (HPF), which is a nonlinear nonconvex relationship between power, head, and turbined outflow of a generating unit (GU). Nevertheless, the HPF usually presents operating regions where the function is convex and regions in which it is concave due to physical reasons. Inspired by sequential convex mixed-integer nonlinear programming techniques, this paper proposes partitioning the domain of the HPF in regions in which it is convex and those in which it is concave. The HPF is approximated by a piecewise linear function using the logarithmic aggregation convex combination (LACC) model in convex regions. In turn, in the concave regions, the HPF is replaced by a convex hull approach, which, combined with symmetry strategies, reduces the number of binary variables in the resulting optimization problem. Using several computational instances of a 50-unit hydroelectric plant, we show that the partitioning-based proposed strategy significantly reduces the computational time compared to two other efficient MILP formulations.

INDEX TERMS Short-term hydro generation scheduling, hydroelectric power plants, domain partitioning, mixed-integer linear programming, logarithmic aggregate convex combination, symmetry-based techniques.

I. INTRODUCTION

Hydroelectric power plants play a significant role in the generation scheduling of power systems because their operation is cheap, renewable, and flexible. Nevertheless, the scarcity of water resources has prompted ever-increasing efforts to target a more efficient operation, especially in plants with many generating units (GUs). In this context, the short-term generation scheduling (STGS) problem, which aims to efficiently determine the status of GUs and their generation levels, usually in a day-head planning horizon, is one of the most important tasks for achieving an efficient operation because its solution is used in real-time to assist the operators in allocating the power production among the GUs. However, the formulation of the STGS problem depends heavily on the generation dispatch framework and plant operating characteristics [1]. In a centralized dispatch setting, hydro producers usually aim to minimize the turbined outflow necessary to meet the power target demand requested by the independent system operator (ISO); in contrast, in decentralized cases, a market-clearing process distributes power generation among producers by considering offers and demands from the market participants.
Consequently, hydro producers maximize their profits by trading electricity under the price-taker assumption. Independent of the dispatch framework, the STGS is usually modeled as a nonconvex mixed-integer nonlinear programming problem (MINLP), where the discrete decisions represent the status of the GUs. Nonlinearity and nonconvexity usually result from the hydro production function (HPF) and net head effects [1]. MINLPs are naturally challenging to solve, and the literature shows that searching for a solution includes exploring the STGS problem structure through strategies based on mathematical programming and heuristics [3]. Among the many strategies used to tackle this problem, the most prominent ones are based on mixed-integer linear programming (MILP) [4]–[14], MINLP [11], [15]–[17] and Lagrangian relaxation [11],[18]–[20].

In recent years, MILP-based approaches have become popular owing to the availability of efficient, friendly, commercial solvers capable of solving large-scale problems [3]. The central aspects in developing a strategy that seeks to solve the STGS via MILP efficiently can be summarized as follows: (i) how the HPF will be linearized and (ii) how the head effects will be incorporated in the strategy. The biggest challenge is to incorporate precision in the representation of (i) and (ii) in a strategy that is solved in a low computational time (preferably in less than an hour) because the STGS usually seeks to plan the generation for the next few hours the next day.

Some studies have used a concave approximation [4]–[7] regarding how the HPF is linearized to reduce the computational burden. In this strategy, only the binary variables present in the model correspond to the ON/OFF status of the GUs. However, a concave approximation of the HPF yields errors that are generally concentrated in the convex regions. On the other hand, with the new advances in MILP solvers, strategies have drawn attention for handling nonconvex functions precisely. Such strategies are employed in general (but not restricted) to univariate and bivariate functions. Although not all nonlinear models can be rewritten as piecewise linear (PWL) functions, such a reformulation is valid for HPF [8]–[14]. However, nonconvex MILP approaches introduce binary variables to identify each HPF polytope (e.g., straight line or triangle), requiring a trade-off between modeling precision and computational burden.

Three main approaches can be verified regarding head effects incorporated in MILP strategies for the STGS of hydro plants. First, we have strategies that incorporate the linearization of the gross or net head directly in the MILP formulation [5], [7], [10]–[13]. In this case, the HPF is approximated by a three-dimensional function, with power dependent on the turbined outflow and the head, which receives a linear approximation, usually through a straight segment to represent the forebay level and a straight segment to represent the tailrace level. Hydraulic losses can be approximated by a constant [10],[11] or incorporated in the determination of the HPF breakpoints used in the PWL approximation [13]. The second way to deal with head effects is to represent the three-dimensional HPF by several univariate functions associated with a net head [8][9]. In this case, binary variables were used to select the univariate function associated with the correct net head.

Third, to represent the head effects in the STGS problem, the head is dynamically obtained from the solution of problems that consider the univariate HPF. The commercial solution short-term hydro optimization program (SHOP) is an example of dynamically obtaining head effects [6]. The SHOP program is a robust STGS tool used by several hydropower plants in European countries and Chile. The basic idea is to update the net head in each iteration from the solutions obtained from a univariate concave HPF representation. This method of representing HPF in the STGS problem, concave and univariate, enables solutions in very short computational times [7]. However, when performing concave approximations in each iteration, the SHOP program may decrease the precision of the results. In this context, inspired by sequential convex mixed-integer nonlinear programming techniques [21], this paper proposes to associate precision and good computational performance with the STGS problem by partitioning the domain of the HPF in the regions in which it is convex and those in which it is concave. Considering a univariate HPF, domain partitioning is achieved to split the HPF into concave and convex regions. Subsequently, we employed PWL approximations for each partitioned region. A PLW based on a convex hull was used in the concave regions. Then, the symmetry concept determines the number of identical GUs (integer decisions) instead of determining which GU will be active (binary decisions). The symmetry, which considerably improves the computational efficiency, is based on the fact that, in the optimal solution, identical GUs operate at the same generation level in a concave region [22],[23]. Thus, it is possible to use a binary expansion to represent integer decisions and drastically reduce the number of variables associated with the HPF approximation.

On the other hand, in convex regions, the same approach is not suitable because linear inequalities do not effectively represent the HPF. Then, a PWL approximation based on the logarithmic aggregation convex combination (LACC) model is used. The study in [13] compared seven nonconvex models, finding that all reached the same results and that the LACC outperformed the other in terms of computational time. The superior performance of the LACC approach results from the small number of continuous and binary variables employed in the mixed-integer constraints used in the PWL approximation. Thus, the domain partitioning allows symmetry for the HPF in concave regions; in turn, the individual HPF representation is maintained for the GUs operating in convex regions. To the best of our knowledge,
This is the first study to develop a strategy based on HPF partitioning to solve the STGS problem in hydro plants.

The precision of the PWL approximation is also related to the number of selected points from the original HPF and their selection. The literature usually chooses equidistant points in the function domain [6], introducing errors in regions where nonlinearity is accentuated. Thus, to control and reduce the linearization errors, this paper proposes applying the RDP algorithm [24] to choose the HPF points to be used in the PWL approximation of both regions. Given a curve composed of line segments, the purpose of the RDP algorithm is to find a similar curve with fewer points. As a result, the simplified curve consists of a subset of points that define the original curve.

To validate and verify the computational performance of the proposed strategy in this study, we compared the partitioning-based approach with two STGS formulations: one that linearizes each HPF through the nonconvex model LACC, the most efficient among the nonconvex models [13], and one that performs a concave approximation in each FPH via CH [4].

In brief, the main contributions of this paper can be summarized as follows:

(i) Development of a solution strategy that separately explores the concave and convex regions of the HPF in the STGS problem.

(ii) Implementation of symmetry for identical GUs operating in concave regions of HPF.

(iii) Use the RDP algorithm to select the points used in the linearization of the HPF.

As it applies to a problem that considers a univariate HPF, the proposed strategy can be used directly in run-of-river hydro plants or with high regularization capacity, which has good gross drop predictability, or it can be used to solve a sub-problem associated with a strategy that updates the head iteratively, as is the case with SHOP. Thus, to facilitate the understanding of the proposed strategy, in this study, the formulation is developed for a run-of-river plant with many different types of GUs.

The remainder of this paper is organized as follows: Section II presents the mixed-integer nonlinear programming formulation of the STGS problem. Section III presents the partitioning-based approach, and Section IV presents the results of a 50-unit hydroelectric plant with different operating characteristics. Finally, the final remarks are presented in Section V.

II. THE STGS PROBLEM

This section presents the MINLP formulation associated with a typical STGS in a centralized dispatch framework. In this case, a hydroelectric power plant receives generation targets for the day ahead. Then, the plant operator must distribute the target generation among its GUs according to commercial and operational strategies. Although it is beyond the scope of this work, the proposed strategy can also be used in decentralized dispatch cases, where the agents seek to maximize their profits by generating bids for the day ahead.

To facilitate understanding the proposed strategy and link it to the case study analyzed in this study, the MINLP formulation with univariate HPF is related to a multi-unit run-of-river plant where the gross head can be known with a good level of precision. Nevertheless, the formulation can be adapted to solve a subproblem associated with an iteration of an algorithm similar to the SHOP program, which updates the head in each iteration to solve the STGS for hydro reservoir plants.

Next, constants are given in the upper case and boldface, whereas variables are given in the lower case. In addition, as this paper proposes to explore symmetry, it is important to include an index \( j \) associated with a group of identical GUs. Finally, we emphasize that the ISO considers the power flow constraints, including the flow in lines and load balance constraints, when determining the day-ahead power target of the power plant. Therefore, the power flow constraints were not considered in our formulation.

Finally, to understand how HPF can be represented as a univariate function in a run-of-river plant with an accurate day-ahead inflow forecast, consider the standard multidimensional representation of an HPF (1).

\[
g_{ijt} = A \cdot h_{ijt} \cdot (h_{ijt}, w_{ijt}) - w_{ijt} \cdot h_{ijt} - g_{ijt}(g_{ijt}),
\]

where:

- \( g_{ijt} \) power generation of GU \( i \), type \( j \), and time-step \( t \) (MW).
- \( A \) constant depending on gravity acceleration and water density (kg-m/s).
- \( h_{ijt} \) turbine hydraulic efficiency of GU \( i \), type \( j \), and time-step \( t \).
- \( w_{ijt} \) turbined outflow of GU \( i \), type \( j \), and time-step \( t \) (m³/s).
- \( h_{ijt} \) net head of GU \( i \) of type \( j \) in time-step \( t \) (m).
- \( g_{ijt} \) generator losses of GU \( i \), type \( j \), and time-step \( t \) (MW).

The net head is given by the difference between the gross head and hydraulic losses in the hydraulic circuit. For a run-of-river plant with accurate inflow forecasts, the gross head can be estimated a priori by subtracting the constant forebay level from the tailrace level calculated from the inflow, as in (2).

\[
G_{Ht} = FL - D_0 - D_1 \cdot I_1 - D_2 \cdot I_1^2 - D_3 \cdot I_1^3 - D_4 \cdot I_1^4
\]

where:

- \( G_{Ht} \) gross head in time-step \( t \) (m).
- \( FL \) forebay level (m).
- \( D_k \) constant \( k \) of the polynomial that represents the tailrace level function.
- \( I_1 \) forecast inflow in time-step \( t \) (m³/s).

Therefore, the net head, given by the difference between the \( G_{Ht} \) and hydraulic losses, is expressed in (3). Note that the net head is now a univariate function of \( w_{ijt} \).

\[
h_{ijt} = G_{Ht} - B \cdot w_{ijt}^2,
\]

where:

[1] A. G. Mallios and G. C. Alexopoulos, "Hydropower optimisation using a mixed-integer nonlinear programming formulation of the STGS problem", IEEE Access, 2021, 9, pp. 65069-65080.
The turbine hydraulic efficiency of each GU type \( j \) depends on the net head and turbined outflow. Because it is related to the operating characteristics of the turbine, at this point, we express the efficiency modeling as a generic nonlinear function, as shown in (4).

\[
he_{ij} = C_{ij} + C_{1j} \cdot w_{ij} + C_{2j} \cdot h_{ij} + C_{3j} \cdot w_{ij} \cdot h_{ij} + C_{4j} \cdot w_{ij}^2 + C_{5j} \cdot h_{ij}^2
\]

(4)

where:

- \( C_{ij} \): constant \( k \) of the function that represents the hydraulic efficiency of the type \( j \) GU.

The generator losses can be represented by (5) [25]:

\[
g_{ij} = H_{ij} + H_{ij} \cdot g_{ij}.
\]

(5)

where:

- \( H_{ij} \): constant \( k \) of the function that represents the generator losses of the type \( j \) GU.

Then, given that \( h_{ij} = G_{H} - B_{ij} \cdot w_{ij}^2 \), equation (4) becomes a univariate function of \( w_{ij} \). Thus, univariate HPF is described by (6).

\[
g_{ij} = \frac{A}{1 + H_{ij}} \cdot he_{ij}(w_{ij}) \cdot w_{ij} - h_{ij}(w_{ij}) - \frac{H_{ij}}{1 + H_{ij}}.
\]

(6)

According to the considerations and classical formulations of the STGS [1], the MINLP that represents the STGS related to this work is given by (7)–(17).

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{N} \sum_{j=1}^{J} \left( w_{ij} + \beta_{ij} \cdot st_{ij} + \gamma_{ij} \cdot sd_{ij} \right), \\
\text{s.t.:} & \quad \sum_{j=1}^{J} \sum_{i=1}^{N} g_{ij} = L_t, \\
& \quad \sum_{j=1}^{J} \sum_{i=1}^{N} w_{ij} + s_t = I_t, \\
& \quad g_{ij} = \frac{A}{1 + H_{ij}} \cdot he_{ij}(w_{ij}) \cdot w_{ij} - (G_{ij} - B_{ij} \cdot w_{ij}^2) - \frac{H_{ij}}{1 + H_{ij}}, \\
& \quad W_{ij}^\text{min} \cdot u_{ij} \leq w_{ij} \leq W_{ij}^\text{max} \cdot u_{ij}, \\
& \quad G_{ij}^\text{min} \cdot u_{ij} \leq g_{ij} \leq G_{ij}^\text{max} \cdot u_{ij}, \\
& \quad st_{ij} - u_{ij} + u_{ij-1} \geq 0, \\
& \quad sd_{ij} - u_{ij} + u_{ij+1} \geq 0, \\
& \quad 0 \leq s_t \leq S_t^\text{max}, \\
& \quad st_{ij} \geq 0, sd_{ij} \geq 0, \\
& \quad u_{ij} \in \{0, 1\}. \\
\end{align*}
\]

(7)–(17)

where:

- \( N_t \): number of time-steps.
- \( J \): number of groups with identical GUs.
- \( N_j \): number of GUs of type \( j \).
- \( \beta_j \): constant associated with the minimization of startups of GUs of type \( j \).
- \( \gamma_j \): constant associated with the minimization of shutdowns of GUs of type \( j \).
- \( st_{ij} \): variable that indicates if the GU \( i \) and type \( j \) is turned on in time-step \( t \).
- \( sd_{ij} \): variable that indicates if the GU \( i \) and type \( j \) is turned off in time-step \( t \).
- \( L_t \): power demand target in time-step \( t \) (MW).
- \( s_t \): spillage in time-step \( t \) (m\(^3\)/s).
- \( u_{ij} \): binary variable indicating the status (on/off) of GU \( i \) and type \( j \) in time-step \( t \).
- \( W_{ij}^\text{min} \): minimum turbined outflow of type \( j \) GU in time-step \( t \) (m\(^3\)/s).
- \( W_{ij}^\text{max} \): maximum turbined outflow of type \( j \) GU in time-step \( t \) (m\(^3\)/s).
- \( G_{ij}^\text{min} \): minimum power generation of type \( j \) GU in time-step \( t \) (MW).
- \( G_{ij}^\text{max} \): maximum power generation of type \( j \) GU in time-step \( t \) (MW).
- \( S_t^\text{max} \): maximum spillage (m\(^3\)/s).

The objective function (7) aims to minimize the total turbined outflow and the number of start-ups and shutdowns of the GUs. The concept is to operate the GUs at maximum efficiency and penalize start-ups and shutdowns with parameters \( \beta \) and \( \gamma \). Constraint (8) is the generation target supplied by the ISO. Constraint (9) represents the mass-balance requirements of the reservoir. The HPF of each GU is expressed in (10). Constraints (11) and (12) limit the turbined outflow and generation. Note that these limits are the same for identical \( j \)-type GUs. Constraints (13) and (14) represent the start-ups and shutdowns of the GUs. Inequalities (15) bound the spillage in time step \( t \). Finally, (16) and (17) set limits on the remaining variables.

### III. SOLUTION STRATEGY

The proposed solution strategy explores the characteristics of the concave and convex regions in the HPF separately. Furthermore, we applied the RDP algorithm to select the points used for approximating each region. In this context, this section first details the approach used to obtain the formulation (subsection D) by separating the nonconvexity of the HPF in the MINLP (subsection A) and the respective linear approximations of each concave region (Subsection B) and convex region (Subsection C). In subsection E, the procedure for point selection using the RDP algorithm is presented. Finally, section F summarizes the overall strategy of the flowchart.

### A. HPF DOMAIN PARTITION

The first step is partitioning the nonconvex and univariate HPFs into concave and convex regions according to their inflection points. FIGURE 1 illustrates a nonconvex HPF and inflection points where the function changes convexity/concavity. These inflection points were obtained by computing the zeros of the HPF second derivative [21]. As shown in FIGURE 1, the HPF possesses one concave region and two convex regions.
Usually, an HPF is quasi-concave, in which the convex region is associated with low turbined outflow values, as illustrated in [6]. More rarely, a convex region can also occur in the high-turbined outflow values, as shown in FIGURE 1. In this context, the convex MINLP derived from separating the HPF concave and convex regions is presented in (18)–(30). This formulation considers a single concave region and one or two convex regions (via index \(v\)). However, it is possible to generalize this formulation and the presented strategy to function with more concave and convex regions.

\[
\min \sum_{i=1}^{T} \sum_{j=1}^{N} \sum_{v=1}^{V} w_{vij} + wc_{ij} + \beta_j \cdot st_{ij} + \gamma_j \cdot sd_{ij},
\]

s.t.:

\[
\sum_{j=1}^{N} g_{vij} = L_{ij},
\]

\[
\sum_{i=1}^{T} \sum_{v=1}^{V} w_{vij} + \sum_{i=1}^{T} \sum_{v=1}^{V} wc_{ij} + s_i = I_{ij},
\]

\[
g_{c_{ij}} = \frac{A}{1 + H_{ij}} \cdot he_{ij}(wc_{ij}) \cdot wc_{ij} \cdot (GH_j - B_j \cdot wc_{ij}^2) - \frac{H_{ij}}{1 + H_{ij}},
\]

\[
g_{v_{ij}} = \frac{A}{1 + H_{ij}} \cdot he_{ij}(w_{vij}) \cdot w_{vij} \cdot (GH_j - B_j \cdot w_{vij}^2) - \frac{H_{ij}}{1 + H_{ij}},
\]

\[
WC_{ji}^{\min} \cdot uc_{ij} \leq wc_{ij} \leq WC_{ji}^{\max} \cdot uc_{ij},
\]

\[
WV_{vij}^{\min} \cdot uv_{vij} \leq w_{vij} \leq WV_{vij}^{\max} \cdot uv_{vij},
\]

\[
\sum_{i=1}^{T} u_{vij} + uc_{ij} = u_{ij},
\]

\[
st_{ij} - u_{ij} + u_{ij} \cdot st_{ij-1} \geq 0,
\]

\[
sd_{ij} - u_{ij} \cdot sd_{ij-1} + u_{ij} \geq 0,
\]

\[
0 \leq s_i \leq S_i^{\max},
\]

\[
st_{ij} \geq 0, \quad sd_{ij} \geq 0,
\]

\[
u_{vij}, uc_{ij}, u_{ij} \in [0,1],
\]

where:

- \(V\) number of convex regions in the HPF.
- \(w_{vij}\) turbined outflow in HPF convex region \(v\) of GU \(i\) and type-\(j\) in time-step \(t\) (m³/s).
- \(wc_{ij}\) turbined outflow in HPF concave region of GU \(i\) and type-\(j\) in time-step \(t\) (m³/s).
- \(g_{v_{ij}}\) power generation in HPF convex region \(v\) of GU \(i\) and type-\(j\) in time-step \(t\) (MW).
- \(gc_{ij}\) power generation in the concave region of GU \(i\) and type-\(j\) in time-step \(t\) (MW).
- \(uv_{vij}\) binary variable indicating if GU \(i\) and type-\(j\) is operating in the HPF convex region \(v\) in time-step \(t\).
- \(uc_{ij}\) binary variable indicating if GU \(i\) and type-\(j\) is operating in the HPF concave region in time-step \(t\).
- \(WC_{ji}^{\min}\) minimum turbined outflow in the HPF concave region of GUs of type \(j\) in time-step \(t\) (m³/s).
- \(WC_{ji}^{\max}\) maximum turbined outflow in the HPF concave region of GUs of type \(j\) in time-step \(t\) (m³/s).
- \(WV_{vij}^{\min}\) minimum turbined outflow in convex region \(v\) for GU of type \(j\) in time-step \(t\) (m³/s).
- \(WV_{vij}^{\max}\) maximum turbined outflow in convex region \(v\) for type \(j\) GU in time-step \(t\) (m³/s).

Compared to the previous MINLP problem, it is necessary to include new variables related to power generation, turbined outflow, and binary status variables for each region of the HPF. Furthermore, new constraints are required to represent partitioned HPF. Constraint (25) guarantees that the model chooses only one HPF region of GU \(i\) and type \(j\) in time step \(t\). The next step to reach the MILP proposed in this work is to linearize (21)–(22), as shown below.

### B. CONCAVE REGION LINEARIZATION

The logic behind a piecewise linear model is to approximate a nonlinear function using polytopes, \(P \in \mathcal{P}\), defined from predefined points, called vertices, \(v \in \chi(\mathcal{P})\). One of the advantages of partitioning the HPF is the use of a PWL approximation via the CH in the concave region. This approach is computationally appealing for avoiding binary variables to identify a straight line approximating the original nonlinear function. Thus, the univariate function (21), which represents the concave region of the HPF, can be represented precisely using a set of linear inequalities from the functions obtained through the convex hull (CH) technique [4]. Thus, (21) can be replaced by (31) as follows:

\[
gc_{ij} \leq E_{0_{ij}} \cdot uc_{ij} + E_{1_{ij}} \cdot wc_{ij}, \quad n = 1, \ldots, NH_{ji},
\]

where:

- \(NH_{ji}\) number of segments that create a convex hull over predefined breakpoints.
- \(E_{k_{ij}}\) is the coefficient \(k\) of segment \(n\) of the set \(NH_{ji}\) that creates the convex hull in the concave region of the GUs of type-\(j\) in time-step \(t\).
The coefficients $E_{kjt}$ are obtained using the points $[WC_{j}^n, GC_{j}^n]$, which represent turbined outflow ($WC_{j}^n$) and power generation ($GC_{j}^n$) of vertex $v$ in the HPF concave region for GUs of type $j$ in time step $t$.

It should be noted that the accuracy of the piecewise linear model in the concave region increases as a function of $t$.

On the other hand, there is also an increase in the number of binary and continuous variables. To that end, we consider that identical GUs operating in the concave region have identical continuous variables. To represent the symmetric representation of the HPF concave region consists of the following:

1. $k_j$ is an integer variable that represents the number of GUs of type $j$ operating in the concave region in time step $t$; and
2. $ws_{jt}$ and $gs_{jt}$ are, respectively, the turbined outflow and the power generation of the GU of type $j$ operating in the HPF concave region in time step $t$.

Then, the HPF (31) can be rewritten as (32):

$$k_j \cdot gs_{jt} \leq E_{0jt} \cdot k_j \cdot ws_{jt} + E_{1jt} \cdot k_j \cdot ws_{jt}, \quad n = 1, \ldots, NH_j,$$  (32)

Next, it is necessary to linearize the products $k_j \cdot ws_{jt}$ and $k_j \cdot gs_{jt}$. To that end, $k_j$ is first replaced by a binary expansion (33)–(35).

$$k_j = \sum_{m=1}^{M_j} 2^{m-1} \cdot y_{mj},$$  (33)

$$\sum_{m=1}^{M_j} 2^{m-1} \cdot y_{mj} \leq N_j,$$  (34)

$$y_{mj} \in \{0,1\},$$  (35)

where:

- $M_j$ is the number of binary variables necessary for representing the number of GUs of type $j$ operating in the concave region.
- $m$ is the position of the binary code with $M_j$ bits which indicates the number of GUs of type $j$ operating in the concave region.
- $y_{mj}$ is the binary variable associated with the position $m$ of the binary code that represents the integer variable $k_j$.

Note that the efficiency of the symmetry-based strategy depends on the number of identical units in the plant. For instance, for $N_j = 3$ and $M_j = 2$, whereas for $N_j = 26$, $M_j = 5$.

Subsequently, it is necessary to linearize the remaining products $ym_{jt} \cdot ws_{jt}$ and $ym_{jt} \cdot gs_{jt}$, with auxiliary variables $rw_{mj}$ and $rg_{mj}$ as shown in (36)–(40). Note that $rw_{mj} = y_{ mj} \cdot ws_{jt}$ is zero if $ym_{mj} = 0$ and $rw_{mj} = ws_{jt}$ if $ym_{jt} = 1$.

$$WC_{ j}^{min} \cdot y_{mj} \leq rw_{mj} \leq WC_{ j}^{max} \cdot y_{mj},$$  (36)

$$WC_{ j}^{max} \cdot (us_j - y_{mj}) \leq ws_{jt} - rw_{mj} \leq WC_{ j}^{max} \cdot (us_j - y_{mj}),$$  (37)

$$GC_{ j}^{min} \cdot y_{mj} \leq rg_{mj} \leq GC_{ j}^{max} \cdot y_{mj},$$  (38)

$$GC_{ j}^{max} \cdot (us_j - y_{mj}) \leq gs_{jt} - rg_{mj} \leq GC_{ j}^{max} \cdot (us_j - y_{mj}),$$  (39)

$$us_j \in \{0,1\},$$  (40)

where:

- $us_j$ is a binary variable that indicates if GU of type $j$ is dispatched in the concave region in time-step $t$.

Therefore, (32) can be replaced by (41).

$$\sum_{n=1}^{\frac{M_j}{2}} (2^{m-1} \cdot rg_{nj}^{n}) \leq E_{0jt} \cdot \sum_{n=1}^{\frac{M_j}{2}} (2^{m-1} \cdot y_{nj}^{n}) + E_{1jt} \cdot \sum_{n=1}^{\frac{M_j}{2}} (2^{m-1} \cdot rw_{nj}^{n}), \quad n = 1, \ldots, NH_{jt}.$$  (41)

Note that the symmetric representation of the HPF in the concave region is given by the set of constraints (35)–(41).

**TABLE I** shows the comparison between the individual representations, (23) and (31), and the symmetric representation, (35)–(41), of the HPF concave region for 5, 12, and 26 identical GUs. In this table, we use five segments to represent the piecewise linear model. The comparison is accomplished in terms of the number of constraints (NC), continuous variables (NCV), and binary variables (NBV).

| Model | Individual | Symmetric |
|-------|------------|-----------|
| GUs   | 5 | 12 | 26 | 5 | 12 | 26 |
| NC    | 35 | 84 | 182 | 17 | 21 | 25 |
| NCV   | 10 | 24 | 52  | 8  | 10 | 12 |
| NBV   | 5  | 12 | 26  | 4  | 5  | 6  |

### C. CONVEX REGION LINEARIZATION

The CH is not suitable in the convex region because the linear inequalities do not effectively represent the HPF. In this case, it is necessary to use a model with binary variables for PWL approximation in the convex region. Because LACC is one of the best models for our application [13][26], it is used for approximating (22) through the set of constraints (42)–(48), as described below.

$$wv_{vij} = \sum_{uv_{vij}^{(P)}} z^{uv}_{vij} \cdot Wv^{uv}_{vij},$$  (42)

$$gv_{vij} = \sum_{uv_{vij}^{(P)}} z^{uv}_{vij} \cdot Gv^{uv}_{vij},$$  (43)

$$z^{uv}_{vij} \geq 0,$$  (44)

$$\sum_{uv_{vij}^{(P)}} z^{uv}_{vij} = uv_{vij},$$  (45)

$$\sum_{uv_{vij}^{(P)}} z^{uv}_{vij} \leq x_{vij},$$  (46)

$$\sum_{uv_{vij}^{(P)}} z^{uv}_{vij} \leq 1 - x_{vij},$$  (47)

$$uv_{vij} \cdot x_{vij} \in \{0,1\}.$$  (48)

where:

- $Wv^{uv}_{vij}$ is the turbined outflow at vertex $v$ of convex region $v$ of GU of type $j$ in time-step $t$ (m³/s).
- $Gv^{uv}_{vij}$ is the power generation at vertex $v$ of convex region $v$ of GU of type $j$ in time-step $t$ (MW).
variable associated with point \([W^v_{ijr}, G^v_{ijr}]\) of vertex \(u\), convex region \(v\), GU \(i\) of type \(j\), in time-step \(t\).

\(B\) binary code that identifies each line segment of the HPF domain.

\(\mathcal{J}(B,l)\) set of vertices identifying the line segments with value \(* (0\ or\ 1)\ at\ position\ l \in \mathcal{L}\ of\ code\ B.\)

\(x^t_{ijr}\) binary variable \(l\ used\ in\ the\ binary\ code\ \(B\)\ to\ identify\ the\ line\ segment\ of\ convex\ region\ \(v\)\ of\ GU\ of\ type\ \(j\)\ in\ time-step\ \(t\).

Constraints (42)–(45) account for the convex combination of parameters in the vertices \([W^v_{ijr}, G^v_{ijr}]\) of the line segment selected in (46)–(48). Hence, as the number of points \([W^v_{ijr}, G^v_{ijr}]\) used in the linear approximation via LACC increases, the number of constraints, continuous variables, and binary variables in the model increases accordingly. Further details on the LACC model can be found in [26].

### D. PROPOSED FORMULATION

The proposed formulation to represent the STGS via MILP is given by:

\[
\min \sum_{t=1}^{T} \left\{ \sum_{j=1}^{J} \left[ (2^{n-1} \cdot r_{w_{mpj}}) + \beta_j \cdot ts_{jt} + \gamma_j \cdot ds_{jt} \right] \right\} + \sum_{j=1}^{J} \sum_{i=1}^{I} \sum_{v=1}^{V} w_{v_{ijr}}^t, \tag{49}
\]

s.t.:

\[
\sum_{j=1}^{J} \sum_{i=1}^{I} \sum_{v=1}^{V} g_{v_{ijr}} + \sum_{j=1}^{J} \sum_{i=1}^{I} \sum_{m=1}^{M} (2^{n-1} \cdot r_{g_{mpj}}) = L_t, \tag{50}
\]

\[
\sum_{j=1}^{J} \sum_{i=1}^{I} \sum_{v=1}^{V} w_{v_{ijr}} + \sum_{j=1}^{J} \sum_{i=1}^{I} \sum_{m=1}^{M} (2^{n-1} \cdot r_{w_{mpj}}) + s_j = I_t, \tag{51}
\]

\[
0 \leq s_j \leq S_{max}, \tag{52}
\]

\[
\sum_{i=1}^{I} \sum_{v=1}^{V} u_{v_{ijr}} + \sum_{m=1}^{M} (2^{n-1} \cdot y_{mpj}) = d_j, \tag{53}
\]

**FIGURE 2.** Piecewise linearization errors by using four and nine breakpoints.
Because STGS is a large-scale problem, increasing the number of points used in the linearization of the HPF negatively affects the computational performance. In this scenario, we propose using the RDP algorithm [24] to optimize the selection of breakpoints used in the linearization of the HPF.

The main idea of the RDP algorithm is, given a curve composed of line segments and \( X \) breakpoints, to achieve a similar curve with fewer points according to the maximum distance \( \epsilon \) allowed between the curves. A similar curve consists of a subset of the breakpoints that define the original curve. Therefore, the smaller the value of \( \epsilon \), the greater is the precision level of a similar curve concerning the curve with \( X \) breakpoints. In addition to the breakpoints selected by the RDP, points where changes in convexity/concavity occur are added to the selection.

FIGURE 3 shows the application of the RDP approach with \( X = 500 \) equidistant breakpoints and \( \epsilon = 0.005 \). In this example, the RDP chooses only 67 points to meet the precision level \( \epsilon \). By inserting the two points where the second derivative was zero, the number of breakpoints was 69, of which 44 were in the concave region, and the remaining were in the two convex regions. Note that the number of breakpoints can be even smaller if a higher value of \( \epsilon \) is accepted.

### F. SUMMARY OF THE PROPOSED STRATEGY

A step-by-step illustration of the procedure proposed in this study is shown in the flowchart in FIGURE 4. As shown in this figure, a pre-processing step is necessary to obtain the breakpoints used in the HPF piecewise linearization of each region.

![Flowchart of the proposed strategy](image)

#### IV. COMPUTATIONAL RESULTS

This section presents the computational performance of the proposed strategy. We used real-life data from Santo Antônio, a plant located in the northern region of Brazil, with 50 GUs and 3,568 MW. The plant has two types of GUs: 24 GUs with five-blade turbines and 26 GUs with four-blade turbines. Because of eco-friendly constraints, the plant operates with a forebay level (FL) of 71.3 meters when the inflow is less than 34,000 m³/s and 70.5 meters otherwise. The parameters associated with the tailrace-level functions are listed in TABLE II.

| W (m/s)      | W (m³/s) | W (MW) | W (m)            | B     | H0     | H1     |
|--------------|----------|--------|------------------|-------|--------|--------|
| Type 1       | 23.6     | 2.64   | 11.28            | 0.1641| 0.0182 |
| Type 2       | 22.4     | 65.6   | 25.15            | 0.0758| 0.0104 | 0.0182|

The specific parameters for each type of GU are listed in TABLE III. Because there are two GU types, the four-blade GUs are labeled as Type 1, whereas the five-blade GUs are denoted as Type 2. We cannot present the values of the hydraulic efficiency data owing to confidentiality issues.

![Table of GU parameters](image)
TABLE IV: NUMBER OF BREAKPOINTS OBTAINED BY RDP ALGORITHM

| Scenario | GU  | TNP | NPCV | NPCC |
|----------|-----|-----|------|------|
| 1        | Type 1 | 33  | 3    | 31   |
|          | Type 2 | 36  | 6/4  | 28   |
| 2        | Type 1 | 32  | 2    | 31   |
|          | Type 2 | 20  | 5    | 16   |
| 3        | Type 1 | 16  | 4    | 13   |
|          | Type 2 | 21  | 4    | 18   |

TABLE V shows the number of constraints (NC), continuous variables (NCV), and binary variables (NBV) of the LACC, CH, and PROP formulations for each scenario. The relative differences between PROP formulations LACC and CH are presented in DIF_LACC and DIF_CH, respectively.

TABLE V: NUMBER OF VARIABLES AND CONSTRAINTS IN LACC AND PROP APPROACHES

| Scenario | NC   | NCV  | NBV   |
|----------|------|------|-------|
| LACC     |      |      |       |
| CH       |      |      |       |
| PROP     |      |      |       |
| DIF_LACC |      |      |       |
| DIF_CH   |      |      |       |
| 1        |      |      |       |
| LACC     | 45,268 | 88,944 | 14,400 |
| CH       | 77,792 | 9,792  | 2,400  |
| PROP     | 50,739 | 29,808 | 11,616 |
| DIF_LACC | +12.1% | -66.5% | -19.3% |
| DIF_CH   | -34.8% | +204.4% | +384.0% |
| 2        |      |      |       |
| LACC     | 60,670 | 88,944 | 14,400 |
| CH       | 63,024 | 9,548  | 2,400  |
| PROP     | 24,096 | 14,928 | 5,472  |
| DIF_LACC | -60.3% | -83.2% | -62.0% |
| DIF_CH   | -61.8% | +56.3% | +128.0% |
| 3        |      |      |       |
| LACC     | 58,888 | 70,512 | 13,248 |
| CH       | 43,546 | 9,792  | 2,400  |
| PROP     | 30,770 | 18,384 | 7,776  |
| DIF_LACC | -47.7% | -73.9% | -41.3% |
| DIF_CH   | -29.3% | +87.7% | +224.0% |

Regarding the LACC formulation, note that PROP yields significant reductions in NCV and NBV for all scenarios. The NC increased only in scenario 1. There are two reasons for this increase: (i) the number of points \[ W_{ij}, G_{ij}^X \] resulting from the RDP algorithm for the concave region is proportionally larger for this scenario, which increases the number of constraints associated with the concave approximation (35)–(41); and (ii) this scenario is the only with two convex regions for the five-blade GUs. Concerning CH, PROP presents a significant reduction in NC, although it demands a greater number of binary and continuous variables.

The MILPs in this work were solved using GUROBI 9.0.0 via PYTHON 3.7. The computer had an Intel Core i7-8565U 4.60-GHz processor and 8 GB of RAM. Two stopping criteria are utilized: a 0.001% optimality gap and a 900-seconds time limit.

Tables VI, VII, and VIII present the main results for all scenarios and different values of \( \beta = \gamma \). For each scenario, the results presented are the optimal objective function (OBJ), number of GU switching (NSS), computation time (ST), in seconds, or the optimality gap (GAP), if ST > 900 s.

TABLE VI: MAIN RESULTS FOR SCENARIO 1

|          | \( \beta = \gamma \) | 10   | 50   | 100  | 500  |
|----------|----------------------|------|------|------|------|
| LACC     | 808,113              | 809,246 | 810,354 | 813,838 |
| CH       | 808,113              | 809,245 | 810,353 | 813,838 |
| PROP     | 808,115              | 809,245 | 810,360 | 813,838 |
| OBJ      | +0.0002%             | -0.0001% | +0.0007% | 0.0000% |
| DIF_LACC | +3.2%                | 0.0%    | 0.0%   | 0.0%  |
| DIF_CH   | +3.2%                | 0.0%    | 0.0%   | 0.0%  |
| NSS      | 31                   | 25      | 18     | 3    |
| PROP     | 32                   | 25      | 18     | 3    |
| DIF_LACC | +3.2%                | 0.0%    | 0.0%   | 0.0%  |
| DIF_CH   | +3.2%                | 0.0%    | 0.0%   | 0.0%  |
| LACC     | 0.0012%              | 30.66   | 28.88  | 63.28 |
| CH       | 0.0012%              | 18.67   | 19.56  | 26.24 |
| PROP     | -97.4%               | -15.1%  | -25.5% | -70.0% |
| DIF_LACC | -97.4%               | +39.5%  | +10.0% | -27.7% |
TABLE VII
MAIN RESULTS FOR SCENARIO 2

| β=γ  | LACC | CH | PROP | DIFF_LACC | DIFF_CH |
|-------|------|----|------|-----------|---------|
| 10    | 786,214 | 786,611 | 786,875 | 0.00000% | 0.0001% |
| 50    | 786,213 | 786,612 | 786,875 | 0.00000% | 0.00000% |
| 100   | 786,214 | 786,612 | 786,875 | 0.00000% | 0.00000% |
| 500   | 786,214 | 786,612 | 786,875 | 0.00000% | 0.00000% |

TABLE VIII
MAIN RESULTS FOR SCENARIO 3

| β=γ  | LACC | CH | PROP | DIFF_LACC | DIFF_CH |
|-------|------|----|------|-----------|---------|
| 10    | 561,812 | 563,675 | 565,345 | 0.00000% | 0.00000% |
| 50    | 561,812 | 563,675 | 565,345 | 0.00000% | 0.00000% |
| 100   | 561,812 | 563,675 | 565,345 | 0.00000% | 0.00000% |
| 500   | 561,812 | 563,675 | 565,345 | 0.00000% | 0.00000% |

TABLE IX
RESULTS OF PROP FOR THE FIRST 24 STAGES OF SCENARIO 3

| Stage | GH | D5_CV | G4_CV | D4_CV | G5_CV | W5_CC | W4_CC | W5_CV | W4_CV |
|-------|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 2449 | 19.08 | 26     | 9      | 0      | 68.82 | 73.29 | 0      | 0      |
| 2     | 2450 | 19.09 | 26     | 9      | 0      | 68.86 | 73.29 | 0      | 0      |
| 3     | 2452 | 19.10 | 26     | 9      | 0      | 68.94 | 73.29 | 0      | 0      |
| 4     | 2330 | 19.07 | 26     | 9      | 0      | 64.25 | 73.29 | 0      | 0      |
| 5     | 2201 | 19.05 | 26     | 8      | 0      | 62.10 | 73.29 | 0      | 0      |
| 6     | 2100 | 19.13 | 26     | 6      | 0      | 63.86 | 73.29 | 0      | 0      |
| 7     | 2060 | 19.19 | 26     | 6      | 0      | 62.32 | 73.29 | 0      | 0      |
| 8     | 1500 | 19.49 | 24     | 0      | 0      | 62.50 | 39.72 | 423.5  | 0      |
| 9     | 808  | 20.19 | 17     | 0      | 3      | 42.43 | 23.69 | 170.0  | 0      |
| 10    | 700  | 19.57 | 10     | 0      | 10     | 42.12 | 242.6 | 170.0  | 0      |
| 11    | 557  | 19.17 | 11     | 0      | 19     | 39.83 | 235.7 | 170.0  | 0      |
| 12    | 750  | 18.95 | 15     | 0      | 5      | 41.05 | 244.7 | 170.0  | 0      |
| 13    | 997  | 18.94 | 20     | 0      | 0      | 49.85 | 292.0 | 0      | 0      |
| 14    | 1050 | 18.73 | 20     | 0      | 0      | 52.50 | 309.8 | 0      | 0      |
| 15    | 1149 | 18.56 | 20     | 0      | 0      | 57.45 | 341.1 | 0      | 0      |
| 16    | 1300 | 18.95 | 21     | 0      | 0      | 61.90 | 359.2 | 0      | 0      |
| 17    | 1662 | 19.34 | 26     | 0      | 0      | 63.92 | 362.9 | 0      | 0      |
| 18    | 1750 | 19.24 | 26     | 1      | 0      | 64.49 | 368.3 | 420.1  | 0      |
| 19    | 1899 | 19.11 | 26     | 4      | 0      | 61.76 | 73.29 | 0      | 0      |
| 20    | 2000 | 19.24 | 26     | 5      | 0      | 62.83 | 355.1 | 422.8  | 0      |
| 21    | 2244 | 19.35 | 26     | 8      | 0      | 63.76 | 358.8 | 420.1  | 0      |
| 22    | 2250 | 19.32 | 26     | 8      | 0      | 63.99 | 361.9 | 417.9  | 0      |
| 23    | 2297 | 19.28 | 26     | 9      | 0      | 62.98 | 363.8 | 418.4  | 0      |
| 24    | 2297 | 19.28 | 26     | 9      | 0      | 62.98 | 358.8 | 419.3  | 0      |

Because LACC, CH, and PROP achieve essentially the same results, the advantage of PROP over LACC and CH lies in its computational performance. Note that in TABLES V, VI, and VII, reductions in computational time occur in all scenarios concerning LACC and in almost all scenarios with CH. The mean percentages of these reductions are listed in TABLE X. The PROP formulation provides an average speed-up of at least 75.7% and 60.3% for LACC and CH.
respectively, in which this reduction would be greater if a less restrictive time limit was set.

| Scenario 1 | Scenario 2 | Scenario 3 | General |
|-----------|-----------|-----------|---------|
| LACC      | -52.0%    | -78.0%    | -97.1%  | -75.7%  |
| CH        | -18.9%    | -65.1%    | -97.1%  | -60.3%  |

V. CONCLUSION

This paper proposes a MILP-based strategy for solving the STGS problem that efficiently explores the nonconvexity of the HPF. The strategy reduces the size of the problem approximating the HPF concave region via the convex hull techniques, exploring the symmetry of identical GUs in the concave region, using the LACC model for the convex region, and optimizing the breakpoint selection by using the RDP algorithm. The results show that the proposed strategy delivers the same optimal solutions obtained via LACC while still providing significant improvements in computational performance. In addition, the proposed strategy achieves computational times smaller, on average, than an approach that performs a concave approximation via the convex hull (CH) in each HPF. For the numerical tests analyzed, the proposed approach yielded time savings of up to 98.1%. Given the 12 computational instances performed with three distinct demand profiles, the overall time reduction is 75.7% compared to the formulation that performs a nonconvex approximation of the FPH via LACC, and 60.3%, when compared to a formulation that performs an approximation concave from FPH via CH. In hydroelectric power plants such as those analyzed in the case study (i.e., run-of-river operation and precise inflow forecast), the strategy can allow several modeling improvements, such as a longer planning horizon, a concave approximation of the FPH via LACC, and 60.3%, when compared to the formulation that performs a nonconvex approximation via the convex hull (CH) in each HPF. For the numerical tests analyzed, the proposed approach yielded time savings of up to 98.1%. Given the 12 computational instances performed with three distinct demand profiles, the overall time reduction is 75.7% compared to the formulation that performs a nonconvex approximation of the FPH via LACC, and 60.3%, when compared to a formulation that performs an approximation concave from FPH via CH. In hydroelectric power plants such as those analyzed in the case study (i.e., run-of-river operation and precise inflow forecast), the strategy can allow several modeling improvements, such as a longer planning horizon, a finer HPF discretization, or the inclusion of other operational constraints, such as power flow in the lines or ramping and power dumping. Furthermore, owing to the low computational time, the proposed strategy can improve the accuracy of programs that iteratively solve STGS problems in large power plants with reservoirs or cascade systems, as is the case with SHOP [6].

REFERENCES

[1] J. Kong, H. I. Skjelbred and O. B. Fosso, “An overview on formulations and optimization methods for the unit-based short-term hydro scheduling problem,” Electr. Power Syst. Res. 2020; 178. https://doi.org/10.1016/j.epsr.2019.106027.
[2] J. I. Perez-Diaz, J.R. Wilhelmi and L.A. Arevalo, “Optimal short-term operation schedule of a hydropower plant in a competitive electricity market,” Energy Convers. Manag., 2010. 51(12): 2955-2966. https://doi.org/10.1016/j.enconman.2010.06.038.
[3] R. Taktak and C. D’Ambrosio, “An overview on mathematical programming approaches for the deterministic unit commitment problem in hydro valleys,” Energy Systems. 2017; 8: 57-79.
[4] T. N. Santos and A. L. Diniz “A comparison of static and dynamic models for hydro production in generation scheduling problems,” In: IEEE PES General Meeting, Minnesota, USA; 2010. p. 1–5. https://doi.org/10.1109/PES.2010.5589895.
[5] A. Hamann and G. Hug, “Real-time optimization of a hydropower cascade using a linear modeling approach,” In: PSCC, Wroclaw, Poland; 2014. p. 1–7. doi: 10.1109 /PSCC.2014.7038354.
[6] H. I. Skjelbred, J. Kong and O. B. Fosso, “Dynamic incorporation of nonlinearity into MILP formulation for short-term hydro scheduling,” Int. J. Electr. Power Energy Syst. 2020; 116:105530. https://doi.org/10.1016/j.jpeps.2019.105530.
[7] L. Gualsânde and J. I. Perez-Diaz, “Mixed integer linear programming formulations for the hydro production function in a unit-based short-term scheduling problem,” Int. J. Electr. Power Energy Syst. 2021; 106747. https://doi.org/10.1016/j.ijepes.2020.106747.
[8] B. Tong, Q. Zhai and X. Guan, “An MILP based formulation for short-term hydro generation scheduling with analysis of the linearization effects on solution feasibility,” IEEE Trans. Power Syst. 2013; 28: 3588-3599.
[9] C. T. Cheng, J. Y. Wang and X. Y. Wu, “Hydro unit commitment with a head-sensitive reservoir and multiple vibration zones using MILP,” IEEE Trans. Power Syst. 2016; 31: 4842-4852, https://doi.org/10.1109/tpwrs.2016.2522469.
[10] X. Li, T. Li, J. Wei, G. Wang and W. G. Yeh, “Hydro unit commitment via mixed integer linear programming: a case study of the three gorges Project,” IEEE Trans. Power Syst. 2014; 29: 1232-1241.
[11] J. E. Finardi, F. Y. K. Takigawa and B. H. Brito, “Assessing solution quality and computational performance in the hydro unit commitment problem considering different mathematical programming approaches,” Electr. Power Syst. Res. 2016; 136: 212-222. doi: 10.1016/j.epsr.2016.02.018.
[12] S. Liao, Z. Liu, B. Liu and X. Wu, “Short-term hydro scheduling considering multiple units sharing a common tunnel and crossing vibration zones constraints,” Water Resour. Plann. Manage., 2021; 147(10), 04021063, 10.1061/(ASCE)WR.1943-5452.0001438.
[13] B. H. Brito, E. C. Finardi and F. Y. K. Takigawa, “Mixed-integer nonsolvable piecewise linear models for the hydropower production function in the Unit Commitment problem,” Electr. Power Syst. Res. 2020; 182; 106234. https://doi.org/10.1016/j.epsr.2020.106234.
[14] B. H. Brito, E. C. Finardi and F. Y. K. Takigawa, “Unit-commitment via logarithmic aggregated convex combination in multi-unit hydro plants,” Electr. Power Syst. Res. 2020; 189; 106784. https://doi.org/10.1016/j.epsr.2020.106784.
[15] T. D. Santo and A. S. Costa, “Hydroelectric unit commitment for power plants composed of distinct groups of generating units,” Electr. Power Syst. Res. 2016; 137: 16-25, https://doi.org/10.1016/j.epsr.2016.03.037.
[16] R. M. Lima, M. G. Marcovecchio, A. Q. Novais and I. E. Grossmann, “On the computational studies of deterministic global optimization of head dependent short-term hydro scheduling,” IEEE Trans. Power. Syst., 2013; 28(4), 4336-4347, https://doi.org/10.1109/TPWRS.2013.2274559.
[17] S. Seguin, P. Cote and C. Audet C, “Self-scheduling short-term unit commitment and loading problem,” IEEE Trans. Power Syst. 2016; 31; 133-142, https://doi.org/10.1109/tpwrs.2014.2383911.
[18] A. Marchand, M. Gendreau, M. Blais and G. Emiel, “Fast near-optimal heuristic for the short-term hydro-generation planning problem,” IEEE Trans. Power Syst. 2018; 33: 227-235, https://doi.org/10.1109/tpwrs.2017.2696438.
[19] E. C. Finardi and M. R. Scruzzato, “A Comparative analysis of different dual problems in the Lagrangian relaxation context for solving the hydro unit commitment problem,” Electr. Power Syst. Res. 2014; 107: 221-229.
[20] E. F. Finardi and M. R. Scruzzato, “Hydro unit commitment and loading problem for day-ahead operation planning problem,” Int. J. Electr. Power Energy Syst. 2013; 44: 7-16, https://doi.org/10.1016/j.ijpeps.2012.07.023.
[21] C. D’Ambrosio, J. Lee and A. Wächter, “An algorithmic framework for minlp with separable nonconvexity,” In Jon Lee and Sven Leyffer, editors, Mixed Integer Nonlinear Programming, pages 315-347, New York, NY, 2012. Springer New York.
[22] J. A. Almamia, F. Magnago, D. Moitre and H. Pintob, “Symmetry issues in mixed-integer programming based Unit Commitment,” Int. J. Electr. Power Energy Syst., 2014; 54, 86-90.
[23] J. Meus, K. Poncelet and E. Delarue, “Applicability of a Clustered unit commitment model in power system modeling,” IEEE Trans. Power Syst., 2017; 33, 2195-2204.
[24] D. Douglas and T. Peucker, “Algorithms for the reduction of the number of points required to represent a digitized line or its caricature,” The Canadian Cartographer 10(2), 112–122 (1973) doi:10.3138/FM57-6770-U75U-7727.
[25] M. M. Cordova, E. C. Finardi, F. A. C. Ribas, C. D. Pase, V. L. Matos, and M. R. Scuzziato, "Performance evaluation and energy production optimization in the real-time operation of hydropower plants," *Electr. Power Syst. Res.*, 2014; 116, 201–207. https://doi.org/10.1016/j.epsr.2014.06.012

[26] J. Vielma and G. Nemhauser, "Modeling disjunctive constraints with a logarithmic number of binary variables and constraints," *Math. Programming*, 2011; 128: 49-72.