FURTHER BOUNDS IN THE POLYNOMIAL SZEMERÉDI
THEOREM OVER FINITE FIELDS

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Abstract. We provide upper bounds for the size of subsets of finite fields lacking the polynomial progression
\[ x, x + y, \ldots, x + (m-1)y, x + y^m, \ldots, x + y^{m+k-1}. \]
These are the first known upper bounds in the polynomial Szemerédi theorem for the case when polynomials are neither linearly independent nor homogeneous of the same degree. We moreover improve known bounds for subsets of finite fields lacking arithmetic progressions with a difference coming from the set of \( k \)-th power residues, i.e. configurations of the form
\[ x, x + y^k, \ldots, x + (m-1)y^k. \]
Both results follow from an estimate of the number of such progressions in an arbitrary subset of a finite field.

1. Introduction

Generalizing Szemerédi’s theorem on arithmetic progressions in subsets of integers [Sze75], Bergelson and Leibman proved that each dense subset of \( \mathbb{Z} \) contains a configuration of the form \( x, x + P_1(y), \ldots, x + P_m(y) \), where \( y \in \mathbb{Z} \setminus \{0\} \) and \( P_1, \ldots, P_m \) are polynomials with integer coefficients and zero constant term [BL96]. Their proof, based on ergodic theory, does not give explicit quantitative bounds. Although no general bounds are known so far, they exist in certain special cases, for instance for \( x, x + y^k, \ldots, x + (m-1)y^k \) with \( m \geq 2 \) and \( k > 1 \) [Pre17] or for \( x, x + y, x + y^2 \) [PP19]. In the finite field analogue of the question, when we are looking for bounds on the size of \( A \subset \mathbb{F}_q \) lacking \( x, x + P_1(y), \ldots, x + P_m(y) \), bounds are known in the case of \( P_1, \ldots, P_m \) being linearly independent [Pel19].

In this paper, we give the first explicit upper bounds for the sizes of subsets of finite fields lacking certain polynomial progressions. Our main result is the following.

Theorem 1. Let \( m, k \in \mathbb{N}_+ \), and \( p \) be a prime. Suppose that \( A \subset \mathbb{F}_p \) lacks the progression
\[ x, x + y, \ldots, x + (m-1)y, x + y^m, \ldots, x + y^{m+k-1} \]
with \( y \neq 0 \). Then
\[
|A| < \begin{cases} 
    p^{1-c}, & m = 1, 2, \\
    p \frac{(\log \log p)^4}{\log p}, & m = 3, \\
    p(\log p)^{-c}, & m = 4, \\
    p(\log p)^{-c}, & m > 4 
\end{cases}
\]
where all constants are positive, and the implied constant depends on $k$ and $m$ while $c$ depends only on $m$. For $m > 4$, one can take the exponent $c$ to equal $c = 2^{-2^{m+9}}$.

It is worth noting that the exponent $c$ appearing in Theorem 1 for $c > 4$ is the same as the exponent that appeared in Gowers’ bounds in Szemerédi theorem [Gow01].

One can think of (1) as the union of an arithmetic progression and a shifted geometric progression. The cases $m = 1$ and $m = 2$ are in fact identical, and the bound in this case comes from the work of Peluse [Pel19]. Our contribution is the $m > 2$ case, for which there are no previous bounds in the literature. This is the first polynomial progression for which quantitative bounds are known where polynomials in $y$ are neither linearly independent nor homogeneous of the same degree. Theorem 1 is a special case of a more general result, which generalizes [Pel19] and uses it as a base case for induction.

**Theorem 2.** Let $m, k \in \mathbb{N}_+, m \geq 3$, and $P_m, ..., P_{m+k-1}$ be polynomials in $\mathbb{Z}[y]$ such that
\[
a_m P_m + ... + a_{m+k-1} P_{m+k-1}
\]
has degree at least $m$ unless $a_m = ... = a_{m+k-1} = 0$ (in particular, $P_m, ..., P_{m+k-1}$ are linearly independent and each of them has degree at least $m$). Let $r_m(p)$ be the size of the largest subset of $\mathbb{F}_p$ lacking $m$-term arithmetic progressions and $s_m : [p_0, \infty) \to [0, 1]$ be a decreasing function satisfying $r_m(p) \leq p \cdot s_m(p)$ for all primes $p \geq p_0 > 0$, with $s_m(n) \to 0$ as $n \to \infty$. If $A \subset \mathbb{F}_p$ lacks
\[
x, x + y, ..., x + (m - 1)y, x + P_m(y), ..., x + P_{m+k-1}(y)
\]
with $y \neq 0$, then
\[
|A| \ll p \cdot s_m(cp^C)
\]
where the constants $C$, $c$, and the implied constant depend on $m, k$, and $P_m, ..., P_{m+k-1}$ but not on the choice of $s_m$.

The best bounds for $r_m$ currently in the literature are of the form
\[
r_m(p) \ll \begin{cases} p \frac{\log p^4}{\log \log p}, & m = 3 \ [Blo16], \\ p \frac{\log p}{\log p}, & m = 4 \ [GT17], \\ p \frac{\log \log p}{\log p}, & m > 4 \ [Gow01] \end{cases}
\]
yielding the bounds given in Theorem 1. The content of Theorem 2 is that up to the values of constants, our bounds are of the same shape as the bounds in Szemerédi theorem. One cannot hope to do better, as each set containing (2) necessarily contains an $m$-term arithmetic progression. The function $s_m$ plays only an auxiliary role, allowing us to conveniently express known bounds in Szemerédi’s theorem as functions defined over positive real numbers.

We prove Theorem 2 by first proving an estimate for how many polynomial progressions a set $A \subset \mathbb{F}_p$ has. This counting result is the heart of this paper; once it is proved, deducing Theorem 2 is straightforward.

**Theorem 3** (Counting theorem). Let $m \in \mathbb{N}_+$ and $P_m, ..., P_{m+k-1}$ be polynomials in $\mathbb{Z}[y]$ such that
\[
a_m P_m + ... + a_{m+k-1} P_{m+k-1}
\]
has degree at least \( m \) unless \( a_m = \ldots = a_{m+k-1} = 0 \) (in particular, \( P_m, \ldots, P_{m+k-1} \) are linearly independent and each of them has degree at least \( m \)). Suppose that \( f_0, \ldots, f_{m+k-1} : \mathbb{F}_p \to \mathbb{C} \) satisfy \( |f_j(x)| \leq 1 \) for each \( 0 \leq j \leq m+k-1 \) and \( x \in \mathbb{F}_p \). Then

\[
\mathbb{E}_{x,y \in \mathbb{F}_p} \prod_{j=0}^{m-1} f_j(x + jy) \prod_{j=m}^{m+k-1} f_j(x + P_j(y))
\]

\[
= \mathbb{E}_{x,y \in \mathbb{F}_p} \prod_{j=0}^{m-1} f_j(x + jy) \left( \prod_{j=m}^{m+k-1} \mathbb{E} f_j \right) + O(p^{-c})
\]

where all the constants are positive and depend on \( m, k \) and polynomials \( P_m, \ldots, P_{m+k-1} \) but not on \( f_0, \ldots, f_{m+k-1} \).

Using the language of probability theory, we can interpret this result as “discorrelation”: up to an error \( O(p^{-c}) \), the polynomials \( P_m, \ldots, P_{m+k-1} \) occur independently from \( m \)-term arithmetic progressions.

The condition imposed on the polynomials \( P_m, \ldots, P_{m+k-1} \) may seem artificial, but Theorem 3 fails if this condition is not satisfied. As an example of failure, consider the configuration \( x, x + y, x + 2y, x + y^2 \). Because \( y^2 \) has degree 2, which is less than the length of the arithmetic progression, \( y^2 \) is contained in the span of \( x^2, (x + y)^2, (x + 2y)^2 \). Thus, there exist quadratic polynomials \( Q_0, Q_1, Q_2 \) and a nonzero linear polynomial \( Q_3 \) satisfying

\[
Q_0(x) + Q_1(x + y) + Q_2(x + 2y) + Q_3(x + y^2) = 0.
\]

As a consequence, if we take \( f_j(t) = e_p(aQ_i(t)) \) for \( a \neq 0 \), then

\[
\mathbb{E}_{x,y} f_0(x) f_1(x + y) f_2(x + 2y) f_3(x + y^2) = 1
\]

while the right-hand side of (3) in this case is \( O(p^{-c}) \), as \( \mathbb{E} f_3 = 0 \). More generally, if a linear combination of \( P_m, P_{m+1}, \ldots, P_{k+m-1} \) has degree \( d < m \), then there is a nontrivial polynomial relation connecting \( x, x + y, \ldots, x + (m-1)y \) with some of \( P_m, \ldots, P_{k+m-1} \), and this relation prevents discorrelation from happening.

A natural question that one could ask at this point is whether Theorems 1, 2 and 3 generalise to \( \mathbb{F}_q \) when \( q \) is a prime power and not just a prime number. Indeed, Theorem 3 remains true if we replace \( \mathbb{F}_p \) by \( \mathbb{F}_q \), with the error \( O(q^{-c}) \) instead of \( O(p^{-c}) \). However, Theorems 1 and 2 no longer need to hold. In the process of going from Theorem 3 to Theorems 1 and 2, one needs to apply known upper bounds for the largest subset of \( \mathbb{F}_q \) lacking \( m \)-term arithmetic progressions. These bounds differ in two extreme cases, one being \( \mathbb{F}_p \) and another being \( \mathbb{F}_q \) with \( q = p^m \) and \( p \) fixed. In the former case, the upper bounds for the largest subset lacking \( m \)-term arithmetic progressions vary from \( O(\frac{p}{\log p^{1-o(1)}}) \) to \( O(\frac{p}{(\log \log p)^c}) \) depending on the length \( m \) of the arithmetic progression, as indicated earlier. For the latter, Ellenberg and Gijswijt proved a bound of the form \( O(q^{1-c}) \) for 3-term arithmetic progressions \([EG16]\). The fact that polynomial bounds like that cannot be attained in \( \mathbb{F}_p \) comes from the celebrated construction of Behrend \([Beh46]\). Therefore, the bounds that we gave in Theorems 1 and 2 are given for \( \mathbb{F}_p \) and not for all \( \mathbb{F}_q \). If we wanted to work in the fixed characteristic case, then the largest subset of \( \mathbb{F}_q \) lacking the progression (2) would have size \( O(q^{1-c}) \) for \( m = 3 \), with constants depending on \( p, k, P_1, \ldots, P_{2+k} \). Polynomial bounds would also hold for \( m > 3 \) contingent on generalising polynomial bounds in Ellenberg and Gijswijt’s result to
longer arithmetic progressions. In the fixed characteristic case, one can thus do strictly better than in $\mathbb{F}_p$. However, between these two extremes there is a grey area of finite fields $\mathbb{F}_q$ which have both large characteristic $p$ and large dimension $n$ as a $\mathbb{F}_p$-vector space, for which our methods seem difficult to adapt.

To complement these results, we prove an upper bound for the size of subsets of $\mathbb{F}_p$ lacking progressions of the form

$$x, x + y^k, \ldots, x + (m-1)y^k$$

i.e. arithmetic progressions with $k$-th power common difference. An upper bound on subsets of $\mathbb{Z}$ lacking this configuration of the form $C \frac{n}{(\log \log n)^{\varepsilon}}$, with constants depending on $m$ and $k$, was proved by Prendiville [Pre17] using density increment, and it naturally carries over to subsets of finite fields. Our bound works only for finite fields, where it is of the same shape as Prendiville’s for $m > 4$, albeit with a better exponent, and strictly improves on it for $m = 3, 4$.

**Theorem 4** (Sets lacking arithmetic progressions with $k$-th power differences). Suppose $A \subset \mathbb{F}_p$ contains no arithmetic progression of length $m$ and common difference coming from the set of $k$-th power residues. Then

$$|A| \ll \begin{cases} \frac{p}{(\log \log p)^4}, & m = 3, \\ \frac{p}{p(\log p)^c}, & m = 4, \\ \frac{p(\log \log p)^c}{c}, & m > 4. \end{cases}$$

The constant $c$ depends only on $m$, and in fact for $m > 4$, we can take $c = 2^{-2^{m+9}}$. More generally,

$$|A| \ll p \cdot s_m(c' \cdot p^c)$$

where $s_m$ is defined as in Theorem 2. The constants $C, c'$ and the implied constants are positive and depend on $k$ and $m$.

Again, up to the values of constants involved, our bounds are optimal in the sense that they are of the same shape as the bounds in Szemerédi theorem.

We derive the bounds in Theorem 4 using a simple argument that heavily exploits the density and equidistribution of $k$-th power residues in the finite fields. With this argument, we prove the following more general counting theorem which implies Theorem 4.

**Theorem 5** (Counting theorem for linear forms with restricted variables). Let $L_1, \ldots, L_m$ be pairwise linearly independent linear forms in $x_1, \ldots, x_d$. Let $k_1, \ldots, k_d$ be positive integers. Moreover, if $k_j > 1$, assume that no linear form $L_i$ is of the form $L_i(x_1, \ldots, x_d) = cx_j$. If $f_1, \ldots, f_m$ satisfy $|f_i(x)| \leq 1$ for each $1 \leq i \leq m$ and each $x \in \mathbb{F}_p$, then

$$E_{x_1, \ldots, x_d \in \mathbb{F}_p} \prod_{j=1}^m f_j(L_j(x_1^{k_1}, \ldots, x_d^{k_d})) = E_{x_1, \ldots, x_d \in \mathbb{F}_p} \prod_{j=1}^m f_j(L_j(x_1, \ldots, x_d)) + O(p^{-c}).$$

In particular,

$$|\{(x_1, \ldots, x_d) \in \mathbb{F}_p^d : L_i(x_1^{k_1}, \ldots, x_d^{k_d}) \in A \text{ for } 1 \leq i \leq m\}|$$

$$= |\{(x_1, \ldots, x_d) \in \mathbb{F}_p^d : L_i(x_1, \ldots, x_d) \in A \text{ for } 1 \leq i \leq m\}| + O(p^{-c}).$$
Similarly to the discussion following Theorem 3, Theorem 5 remains true for $\mathbb{F}_q$, but going from Theorem 5 to Theorem 4 forces us to work in $\mathbb{F}_p$ instead of $\mathbb{F}_q$. An analogue of Theorem 4 for $\mathbb{F}_q$, $q = p^n$ with $p$ fixed and $n$ being the asymptotic parameter is that sets lacking $x, x + y^k, x + 2y^k$ have size $O(q^{1-c})$, with the implied constant dependent on $k$. A similar result would hold for longer progressions if Ellenberg and Gijswijt’s bound [EG16] could be generalised.

1.1. Known results. In this section we enumerate known bounds for subsets of $\mathbb{F}_q$ or $[N]$ lacking polynomial progressions

$$x, x + P_1(y), ..., x + P_m(y)$$

where not all of $P_1, ..., P_m$ are linear. There are some differences between the integral and finite field settings. Most importantly, finite fields contain significantly more polynomial progressions of a given form if at least one polynomial is nonlinear. That is because a nonlinear polynomial $P$ of degree $d > 1$ has only $\Theta(N^{1/d})$ images in $[N]$, but it is a dense subset of $\mathbb{F}_q$, in the sense that there are at least $q^d$ images of $P$ in $\mathbb{F}_q$.

The case $m = 1$ in natural numbers is often referred to as Furstenberg-Sárközy’s theorem, and it is equivalent to finding the largest subset $A$ of natural numbers whose difference set does not intersect the values of $P$ evaluated at integers. This problem has been studied, among others, by Sárközy [Sá78a, Sá78b], Balog, Pelikán, Pintz, and Szemeredi [BPPS94], Slijepčević [Sli03], Lucier [Luc06], and Rice [Ric19]. They showed that $A$ is sparse if and only if for each natural number $n$ there exists $m \in \mathbb{N}$ for which $n$ divides $P(m)$, getting explicit bounds on the way; such polynomials have been called intersective. When $P(y) = y^k$ for $k > 1$, a lower bound of the form $\Omega(N^c)$ for $0 < c < 1$ depending on $k$ can be obtained by trivial greedy algorithm, and the value of $c$ has been improved nontrivially by Ruzsa [Ruz84]. For finite fields $\mathbb{F}_q$, an elementary Fourier analytic argument gives upper bounds of the form $O(p^{2c})$ with the implied constant depending on $k$, while the best known lower bounds are of the form $\Omega(\log p \log \log p)$ for infinitely many primes $p$ [GR90].

In the case $m > 1$, bounds have only been known in two extremes. If $P_1, ..., P_m$ are all homogeneous of the same degree, i.e. we have a configuration of the form

$$x, x + c_1y^k, ..., x + c_my^k$$

then Prendiville [Pre17] proved that all subsets of $[N]$ lacking this configuration have size $O\left(\frac{N}{(\log \log N)^c}\right)$ for some $c > 0$ depending on $m$ and $k$. Theorem 4 improves this result over finite fields for configurations of length 3 and 4.

The other extreme is when $P_1, ..., P_m$ are all linearly independent. This case has recently been tackled over finite fields by Peluse [Pel19] who has showed that subsets of $\mathbb{F}_q$ lacking such progressions have size $O(q^{1-c})$ for $c > 0$ depending on $P_1, ..., P_m$. In the case $m = 2$, a specific exponent is known due to works of Bourgain and Chang [BC17], Peluse [Pel18], and Dong, Li and Sawin [DLS17]. Recently, the results on the case $x, x + y, x + y^2$ have been extended to the integers by Peluse and Prendiville [PP19].

1.2. Notation, terminology, and assumptions. Throughout the paper, $p$ always denotes the characteristic and cardinality of the finite field $\mathbb{F}_p$ in which we are currently working.
A function $f$ is 1-bounded if $\|f\|_\infty \leq 1$. We always assume that $f$ is a 1-bounded function from $\mathbb{F}_p$ to $\mathbb{C}$ unless explicitly stated otherwise. Sometimes, we use an expression $b(t_1, \ldots, t_n)$ to denote a 1-bounded function depending only on the variables $t_1, \ldots, t_n$ whose exact form is irrelevant and may differ from line to line.

We denote constants by $0 < c < 1 < C$. The exact values of these constants are generally unimportant, only their relative size, therefore we shall often use the same symbol $c, C$ to denote constants whose value changes from line to line or even in the same expression. If there are good reasons to distinguish between two constants in the same expression, we shall denote them as $c, c'$ or $C, C'$ respectively. If we need to fix a constant for the duration of an argument, we give it a numerical subscript, e.g. $c_0$. We also use asymptotic notation $f = O(g), g = \Omega(f), f \ll g$, or $g \gg f$ to denote that $|f(p)| \leq C|g(p)|$ for sufficiently large $p$. The constant may depend on parameters such as the length of the polynomial progression or the degrees and leading coefficients of polynomials $P_1, \ldots, P_m$ involved. However, if the asymptotic notation is used in an expression involving arbitrary functions $f_0, \ldots, f_m$, the constant never depends on the choice of $f_0, \ldots, f_m$. While it is quite common in additive combinatorics to denote the dependence of the constant on these parameters by e.g. writing $C_m$ when it depends on $m$, we refrain from doing so in order not to clutter the notation. Therefore the reader should always assume that constants depend on the shape and length of the polynomial progression, but never on the functions $f_0, \ldots, f_m$ weighting the progression. We shall reiterate this in the statements of our lemmas and theorems.

We often use expected values, which we denote by $\mathbb{E}_{x \in X} f(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$. If the set $X$ is omitted from the notation, it is assumed that $x$ is taken from $\mathbb{F}_p$ or from another specified set.

We denote the indicator function of the set $A$ by $1_A$. The map $C : x \mapsto \pi$ denotes the conjugation operator. Finally, we set $e_p(x) := e(x/p) = e^{2\pi i x/p}$.

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2. Basic concepts from additive combinatorics

The purpose of this section is to describe a few basic and standard concepts that are used extensively throughout this paper. We only introduce here ideas that are essential for all the arguments. There are tools which shall only be applied in specific proofs, and these will be discussed in relevant sections.

2.1. Fourier transform. Given a function $f : \mathbb{F}_p \to \mathbb{C}$ and $\alpha \in \mathbb{F}_p$, we define its Fourier transform by the formula

$$\hat{f}(\alpha) := \mathbb{E}_x f(x) e_p(\alpha x).$$

We also call $\hat{f}(\alpha)$ the Fourier coefficient of $f$ at $\alpha$. We define the inner product on $\mathbb{F}_p$ as well as $L^s$ and $\ell^s$ norms for functions from $\mathbb{F}_p$ to $\mathbb{C}$ to be

$$\langle f, g \rangle := \mathbb{E}_x f(x) \overline{g(x)}, \quad \|f\|_{L^s} = \left( \mathbb{E}_x |f(x)|^s \right)^{\frac{1}{s}}, \quad \|f\|_{\ell^s} = \left( \sum_x |f(x)|^s \right)^{\frac{1}{s}}.$$

for $1 \leq s < \infty$, and we set $\|f\|_\infty := \|f\|_{L^\infty} = \|f\|_{\ell^\infty} = \max \{ |f(x)| : x \in \mathbb{F}_p \}$. 
2.2. Gowers norms. Let $\Delta_h f(x) = f(x + h)\overline{f(x)}$ denote the multiplicative derivative of $f$ and $\nabla_h f(x) = f(x + h) - f(x)$ be its additive derivative. The $U^s$ norm of $f$ is defined as

$$||f||_{U^s} := \left( \mathbb{E}_{x,h_1,...,h_s} \prod_{w \in \{0,1\}^s} c|w| f(x + \overline{w} \cdot h) \right)^{1/s},$$

where $|w| = w_1 + ... + w_s$. If $f = 1_A$, then $||1_A||_{U^s}$ is the normalized count of $s$-dimensional parallelepipsed in $A$, i.e. configurations of the form

$$(x + w_1h_1 + ... + w_sh_s)_{w \in \{0,1\}^s}.$$  

It turns out that $||f||_{U^s}$ is a well-defined norm for $s > 1$ and a seminorm for $s = 1$ (for the proofs of these and other facts on Gowers norms described in this section, including Lemma 1, consult [Gre07] or [Tao12]). In fact, $||f||_{U^1} = ||\mathbb{E}_x f(x)|| = |\hat{f}(0)|$. Gowers norms enjoy several important properties that are used extensively in this paper. First, they are monotone:

$$||f||_{U^1} \leq ||f||_{U^2} \leq ||f||_{U^3} \leq ...$$

Second, one can express a $U^s$ norm of $f$ in terms of a lower-degree Gowers norm of its multiplicative derivatives:

$$||f||_{U^s}^2 = \mathbb{E}_{h_1,...,h_{s-k}} ||\Delta_{h_1,...,h_{s-k}} f||_{U^k}^2.$$  

In particular, taking $k = 2$ gives:

$$||f||_{U^2}^2 = \mathbb{E}_{h_1,...,h_{s-2}} ||\Delta_{h_1,...,h_{s-2}} f||_{U^2}^4.$$  

The utility of this formula for us is that $U^2$ norm is much easier to understand than the $U^s$ norms for $s > 2$. In particular, $||f||_{U^2} = ||\hat{f}||_{\ell^2}$, and from the fact that $\max_{\phi \in \mathbb{F}_p} |\hat{f}(\phi)| \leq ||\hat{f}||_{\ell^2} \leq \max_{\phi \in \mathbb{F}_p} |\hat{f}(\phi)|^{1/2}$ it follows that having a large $U^2$ norm is equivalent to having a large Fourier coefficient, which is the statement of $U^2$ inverse theorem. For $s > 2$, corresponding inverse theorems exist as well, but they are significantly more involved and we fortunately do not need them.

Gowers norms, introduced by Gowers in his celebrated proof of Szemerédi theorem, occur frequently in additive combinatorics because $||1_A||_{U^s}$ controls the number of $(s + 1)$-term arithmetic progressions in $A$ in the following way.

**Lemma 1** (Generalized von Neumann theorem). Let $f_0, ..., f_s$ be 1-bounded. Then

$$\mathbb{E}_{x,y} f_0(x)f_1(x + y)...f_s(x + sy) \leq \min_{0 \leq i \leq s} ||f_i||_{U^s}.$$  

2.3. Counting arithmetic progressions in subsets of finite fields. In Theorems 3 and 5, we show that a certain counting operator can be expressed in terms of

$$\Lambda_m(f_0, ..., f_{m-1}) := \mathbb{E}_{x,y} f_0(x)f_1(x + y)...f_{m-1}(x + (m-1)y)$$

which counts $m$-term arithmetic progressions weighted by $f_0, ..., f_{m-1}$. In particular, $\Lambda_m(1_A) = \Lambda_m(1_A, ..., 1_A)$ is a normalized count of $m$-term arithmetic progressions in $A$. Instead of giving the exact estimates for what this counting operator is, we want to bound it from below by an expression involving $N_m(\alpha)$, which is the smallest natural number such that $p > N_m(\alpha)$ implies that each subset of $\mathbb{F}_p$ of size at least $ap$ contains an $m$-term arithmetic progression. The reason why we want to have the estimate for $\Lambda_m$ in terms of $N_m$ is because the functions $N_m$ and
with an averaging argument of Varnavides, we obtain the
2
coupled with 3
using Theorem 2
2
3
it follows that

\[ r_m(n) := r_m(n)/n \]
are essentially inverses, where \( r_m(p) \) is the size of the largest subset of \( \mathbb{F}_p \) not containing an \( m \)-term arithmetic progression. What we mean by this is that if \( r_m' \) is bounded from above by a decreasing function \( s_m \), then subject to certain conditions \(- N_m \) is bounded from above by \( s_m^{-1} \). The following lemma makes this precise.

**Lemma 2.** Let \( r_m(p) \) be the size of the largest subset of \( \mathbb{F}_p \) lacking \( m \)-term arithmetic progressions. Let \( N_m(\alpha) \) be the smallest natural number such that \( p > N_m(\alpha) \) implies that each subset of \( \mathbb{F}_p \) of size at least \( \alpha p \) has an \( m \)-term arithmetic progression. Suppose that \( s_m : [p_0, \infty) \to (0,1] \) is a decreasing function with \( \lim_{n \to \infty} s_m(n) = 0 \). Let \( M_m \) be its inverse defined on \((0,\alpha_0]\), where \( \alpha_0 := s_m(p_0) \). Then \( r_m(p) \leq ps_m(p) \) for \( p \geq p_0 \) if and only if \( N_m(\alpha) \leq M_m(\alpha) \) for \( 0 < \alpha \leq \alpha_0 \).

Combining Lemma 2 with an averaging argument of Varnavides, we obtain the following lemma, the precise version of which has been borrowed from [RW19].

**Lemma 3** (Averaging over progressions). Suppose \( 0 < \alpha_0 \leq 1 \), and let \( M_m : (0,\alpha_0] \to \mathbb{R}_+ \) be a decreasing function satisfying \( N_m \leq M_m \). Suppose that \( A \subset \mathbb{F}_p \) has size \(|A| = \alpha p \) for some \( 0 < \alpha \leq \alpha_0 \). Then \(|A_m(1_A) \geq 1/M_m(\alpha/2)^2 \), where the implied constant depends on \( m \).

We conclude this section with the proof of Lemma 2.

**Proof of Lemma 2.** Assume that \( s_m \) is defined as in the statement of the lemma and that \( r_m(p) \leq s_m(p)p \) for all prime \( p \geq p_0 \). Fix a prime number \( p \geq p_0 \) and \( \alpha \in (0,\alpha_0] \). Suppose that \( A \subset \mathbb{F}_p \) of size \(|A| = \alpha p \) lacks an \( m \)-term arithmetic progression. The assumption of \( p \geq p_0 \) implies that \(|A| \leq r_m(p) \leq s_m(p)p \), or \( \alpha \leq s_m(p) \). From the monotonicity of \( s_m \) it follows that \( p \leq M_m(\alpha) \).

Thus, if a subset \( A \subset \mathbb{F}_p \) of size \(|A| = \alpha p \) lacks \( m \)-term arithmetic progression, it must be that either \( p \leq p_0 \) or \( p \leq M_m(\alpha) \), implying \( N_m(\alpha) \leq \max\{p_0,M_m(\alpha)\} \). The definition of \( p_0 \) and monotonicity of \( M_m \) imply that \( p_0 = M_m(\alpha_0) \leq M_m(\alpha) \), and so \( N_m(\alpha) \leq M_m(\alpha) \).

Conversely, suppose \( N_m(\alpha) \leq M_m(\alpha) \) for \( 0 < \alpha \leq \alpha_0 \). Suppose that a set \( A \subset \mathbb{F}_p \) of size \(|A| = \alpha p \) lacks an \( m \)-term arithmetic progression, and assume \( 0 < \alpha \leq \alpha_0 \), \( p \geq p_0 \). Then \( p \leq N_m(\alpha) \leq M_m(\alpha) \), and so \( \alpha \leq s_m(p) \).

It thus follows that if a subset \( A \subset \mathbb{F}_p \) of size \(|A| = \alpha p \) lacks an \( m \)-term arithmetic progression, then either \( \alpha \leq s_m(p) \) or \( \alpha > \alpha_0 \). If the latter holds, then \( \alpha > \alpha_0 \) implies \( M_m(\alpha) < M_m(\alpha_0) = p_0 \), and so this case is impossible whenever \( p \geq p_0 \). Thus we must have that \( \alpha \leq s_m(p) \) whenever \( p \geq p_0 \). \( \square \)

### 3. Deriving upper bounds in Theorem 2

This section is devoted to the proof of Theorem 2 using Theorem 3 coupled with the notation from Section 2.3.

**Proof of Theorem 2.** Throughout this proof, all the constants are allowed to depend on \( m, k \) and \( P_m, \ldots, P_{m+k-1} \). From Theorem 3 it follows that

\[
\mathbb{E}_{x,y} \prod_{j=0}^{m-1} 1_A(x+jy) \prod_{j=m}^{m+k-1} 1_A(x+P_j(y)) = \left( \mathbb{E}_{x,y} \prod_{j=0}^{m-1} 1_A(x+jy) \right) \alpha^k + O(p^{-c})
\]
If \( A \subset \mathbb{F}_p \) for \( p \geq p_0 \) has size \( |A| = \alpha p \) and lacks progressions (2), then the expression on the left-hand side is \( O(p^{-1}) \), and so

\[
(7) \quad \left( \mathbb{E}_{x,y} \prod_{j=0}^{m-1} 1_A(x + jy) \right) \alpha^k \ll p^{-c}.
\]

Let \( M_m \) be the inverse function for \( s_m \) on \((0, \alpha_0] \), where \( \alpha_0 = s_m(p_0) \), and set \( M = M_m(\alpha/2) \). The assumption \( p \geq p_0 \) and the fact that \( s_m \) is decreasing imply that \( 0 < \alpha \leq \alpha_0 \). Applying Lemma 3 to (7) gives \( \alpha^k M^{-2} \ll p^{-c} \). Behrend’s construction implies that \( M \) grows faster than polynomially in \( \alpha \): that is, for each \( C > 1 \) there exists \( c > 0 \) such that \( M \geq \alpha^{-C} \) [Beh46]. Consequently, we have \( M^{-3} \ll p^{-c} \) which implies that \( M \gg p^c \) for a different constant \( 0 < c < 1 \). From monotonicity of \( s_m \), it follows that \( \alpha \leq 2s_m(cp^c) \).

To illustrate the last bit of the above proof, we take Gowers’s [Gow01] estimate

\[
N_m(\alpha) \leq 2^{2^{\alpha-C}}
\]

for \( m > 4 \). Combined with \( N_m(\alpha/2) \gg p^c \), it gives the inequality \( 2^{2^{\alpha-C}} \gg p^c \). After rearranging, it yields

\[
\alpha \ll \frac{1}{(\log \log p)^c}.
\]

Note that the function \( s_m(p) = (\log_2 \log p)^{-c} \) is precisely the inverse function of \( M_m(\alpha) = 2^{\alpha-C} \) for an appropriate choice of constants.

4. Proof of Theorem 3

Finally, we come to the main part of the paper, which is the proof of the counting theorem for the progression (2). Like before, all the constants here are allowed to depend on \( m, k \) and \( P_m, \ldots, P_{m+k-1} \). First, we lexicographically order the set \( \mathbb{N}_+^2 \), i.e.

\[
(m, k) < (m', k') \iff m < m' \lor (m = m' \land k < k').
\]

We induct on \( (m, k) \) by following the lexicographic order on \( \mathbb{N}_+^2 \). Let \( S(m, k) \) denote the statement of Theorem 3 for \( (m, k) \); that is, \( S(m, k) \) holds iff for all linearly independent polynomials \( P_m, \ldots, P_{m+k-1} \) of degree at least \( m \) that do not span a polynomial of degree less than \( m \) there exists a constant \( c > 0 \) such that for all 1-bounded functions \( f_0, \ldots, f_{m+k-1} \), we have

\[
\mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy) \prod_{j=m}^{m+k-1} f_j(x + P_j(y))
\]

\[
= \left( \mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy) \right) \prod_{j=m}^{m+k-1} \mathbb{E} f_j + O(p^{-c}).
\]

\( S(1, k) \) and \( S(2, k) \) follow from the work of Peluse [Pel19], and they shall serve as our base cases. In the inductive step, we have to prove two cases:

1. \( S(m, 1) \), assuming the statement holds for all \( (m', k') < (m, 1) \) (although we shall only need to invoke \( S(m-1, 2) \)).
(2) $S(m,k)$ for $k > 1$, assuming it holds for $S(m,k')$ with $1 \leq k' < k$. The first case turns out to be the simpler of the two, and we shall carry it out promptly. The second case is much more involved, and it is where most of the difficulties lie.

Throughout this section, we denote the counting operator appearing in the statement of the Theorem 3 by $\Lambda$ with appropriate subscripts.

$$
\Lambda_{m,P_m,...,P_{m+k-1}}(f_0, ..., f_{m+k-1}) := \mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy) \prod_{j=m}^{m+k-1} f_j(x + P_j(y)).
$$

In particular, $\Lambda_m$ denotes the counting operator for $m$-term arithmetic progressions:

$$
\Lambda_m(f_0, ..., f_{m-1}) := \mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy).
$$

When $m$, $k$, and $P_m,...,P_{m+k-1}$ are clear out of the context, we shall suppress the subscripts and denote the operator just by $\Lambda$.

4.1. **Proof of $S(m,1)$ assuming $S(m-1,2)$**. As advertised earlier, we first prove the inductive step for $S(m,1)$. Let $P$ be a polynomial of degree at least $m$. Our goal is to show that the counting operator

$$
\Lambda_{m,P}(f_0, ..., f_m) = \mathbb{E}_{x,y} \left( \prod_{j=0}^{m-1} f_j(x + jy) \right) f_m(x + P(y))
$$

is in fact controlled by an operator involving an arithmetic progression of length $m-1$ of difference functions of $f_1, ..., f_{m-1}$. To accomplish this, we first rewrite (8) as

$$
\mathbb{E}_x f_0(x) \mathbb{E}_y \left( \prod_{j=1}^{m-1} f_j(x + jy) \right) f_m(x + P(y)).
$$

Applying the Cauchy-Schwarz inequality in $x$ together with 1-boundedness of $f_0$, changing variables, translating $x \mapsto x - y$, and finally using the triangle inequality, we obtain that

$$
|\Lambda_{m,P}(f_0, ..., f_m)|^2 \leq \mathbb{E}_x \left| \mathbb{E}_y \left( \prod_{j=1}^{m-1} f_j(x + jy) \right) f_m(x + P(y)) \right|^2
$$

$$
\leq \mathbb{E}_{x,y,h} \left( \prod_{j=1}^{m-1} \Delta_j f_j(x + jy) \right) \overline{f_m(x + P(y))f_m(x + P(y + h))}
$$

$$
\leq \mathbb{E}_h \mathbb{E}_{x,y} \left( \prod_{j=1}^{m-1} \Delta_j f_j(x + (j-1)y) \right) \overline{f_m(x + P(y) - y)f_m(x + P(y + h) - y)}.
$$

By the pigeonhole principle, there exists $h \neq 0$ such that

$$
|\Lambda_{m,P}(f_0, ..., f_m)|^2 \leq \mathbb{E}_{x,y} \left( \prod_{j=1}^{m-1} \Delta_j f_j(x + (j-1)y) \right) \overline{f_m(x + P(y) - y)f_m(x + P(y + h) - y)} + O(p^{-1})
$$

$$
= |\Lambda_{m-1,P_m,P_{m+1}}(g_0, ..., g_{m-2}, f_m)| + O(p^{-1})
$$
where we set
\[ P_m(y) = P(y) - y, \quad P_{m+1}(y) = P(y + h) - y \quad \text{and} \quad g_j(t) = \Delta_{(j+1)h} f_{j+1}(t). \]
From \( h \neq 0 \) it follows that \( P_m, P_{m+1} \) are linearly independent. Moreover, for any \((a, b) \neq (0, 0)\), the polynomial \( aP_m + bP_{m+1} \) has degree at least \( m - 1 \), attaining this degree precisely when \( a + b = 0 \). We have thus reduced the study of \( \Lambda_{m, P} \) to the analysis of \( \Lambda_{m-1, P_m, P_{m+1}} \), and so we are in the \( S(m - 1, 2) \) case. Applying Theorem 3 for this case, we see that
\[
|\Lambda_{m, P}(f_0, \ldots, f_m)|^2 \leq |\Lambda_{m-1, P_m, P_{m+1}}(g_0, \ldots, g_{m-2}, \overline{f_m, f_m})| + O(p^{-1})
\leq |\Lambda_{m-1}(g_0, \ldots, g_{m-2})| \cdot |\mathbb{E} f_m|^2 + O(p^{-c})
\leq |\mathbb{E} f_m|^2 + O(p^{-c})
\]
and hence
\[
|\Lambda_{m, P}(f_0, \ldots, f_m)| \leq |\mathbb{E} f_m| + O(p^{-c}).
\]
We have established so far that the \( U^1 \) norm of \( f_m \) controls \( \Lambda_{m, P}(f_0, \ldots, f_m) \) up to a power-saving error, i.e. \( \|f_m\|_{U^1} = 0 \) implies \( |\Lambda_{m, P}(f_0, \ldots, f_m)| = O(p^{-c}) \). To utilise this fact, we decompose \( f_m = \mathbb{E} f_m + (f_m - \mathbb{E} f_m) \) and split \( \Lambda_{m, P} \) accordingly. The term involving \( f_m - \mathbb{E} f_m \) has size at most \( O(p^{-c}) \) because \( \mathbb{E}(f_m - \mathbb{E} f_m) = 0 \), and so
\[
\Lambda_{m, P}(f_0, \ldots, f_m) = \Lambda_m(f_0, \ldots, f_{m-1}) \mathbb{E} f_m + O(p^{-c}),
\]
as required.

4.2. **Proof of \( S(m, k) \), \( k > 1 \).** Our next goal is to prove \( S(m, k) \) whenever \( k > 1 \). The natural thing to try would be to prove this case in a similar manner we proved \( S(m, 1) \); that is, to apply the Cauchy-Schwarz inequality to the counting operator
\[
\Lambda_{m, P_m, \ldots, P_{m+k-1}}(f_0, \ldots, f_{m+k-1})
\]
and bound it by the counting operator of
\[
\Lambda_{m-1, Q_m, R_m, \ldots, Q_{m+k-1}, R_{m+k-1}}(g_0, \ldots, g_{m-2}, \overline{f_m, f_m}, \ldots, \overline{f_{m+k-1}, f_{m+k-1}})
\]
where
\[
Q_j(y) = P_j(y) - y, \quad R_j(y) = P_j(y + h) - y \quad \text{and} \quad g_j(t) = \Delta_{(j+1)h} f_{j+1}(t).
\]
However, this simple extension of the method used to prove \( S(m, 1) \) does not work because there is no guarantee that \( Q_m, R_m, \ldots, Q_{m+k-1}, R_{m+k-1} \) are linearly independent (and in general, they may not be), nor that any nonzero linear combination of them has degree at least \( m - 1 \). To illustrate this problem, we look at
\[
x, \ x + y, \ x + 2y, \ x + y^3, \ x + y^4.
\]
Applying the Cauchy-Schwarz inequality and translating by \( x \mapsto x - y \), we control this configuration by the counting operator of the configuration
\[
x, \ x + y, \ x + y^3 - y, \ x + (y + h)^3 - y, \ x + y^4 - y, \ x + (y + h)^4 - y.
\]
Note that the polynomials \( y, y^3 - y, (y + h)^3 - y, y^4 - y, (y + h)^4 - y \) have degree at most 4, and there are 5 of them, hence there exist \( a_1, \ldots, a_5, b \) not all zero such that
\[
a_1 y + a_2(y^3 - y) + a_3((y + h)^3 - y) + a_4(y^4 - y) + a_5((y + h)^4 - y) = b.
\]
Consequently, one cannot apply induction hypothesis to this configuration. One therefore needs to come up with a different method.

In the remainder of this section, we outline the proof of $S(m, k)$, $k > 1$. We formulate consecutive steps of the proof as lemmas to be proved separately in the next section. Let

$$
\Lambda := \Lambda_{m, p^{1}, \ldots, p^{m+k-1}}.
$$

It shall become clear shortly that proving the general case of $S(m, k)$ for $k > 1$ can be reduced to the case of $f_{m+k-1}$ being a character. In this case, the following holds.

**Lemma 4.** Let $a \in \mathbb{F}_p^\times$ and $k \in \mathbb{N}_+$. Assume $S(m, k - 1)$. Then

$$
|\Lambda(f_0, \ldots, f_{m+k-2}, e_p(a \cdot))| \leq O(p^{-c})
$$

for a constant $c > 0$ depending on $m, k$, and the polynomials $P_m, \ldots, P_{m+k-1}$ but not on $a$ or $1$-bounded functions $f_0, \ldots, f_{m+k-2} : \mathbb{F}_p \to \mathbb{C}$.

Our first task before using this lemma is to show that $\Lambda$ is controlled by some Gowers norm of $f_{m+k-1}$. This follows from the so-called PET induction scheme, which originally appeared in Bergelson and Leibman’s ergodic-theoretic proof of polynomial Szemerédi theorem [BL96] and was subsequently applied in the works of Prendiville and Peluse [Pre17, Pel19, PP19].

**Lemma 5** (PET induction, Proposition 2.2 of [Pel19]). Let $P_1, \ldots, P_l$ be nonconstant polynomials in $\mathbb{Z}[y]$ such that $P_i - P_j$ is nonconstant whenever $i \neq j$. Then for any $1 \leq j \leq l$ there exist $s \in \mathbb{N}$ and $0 < \beta \leq 1$, depending only on the degrees and leading coefficients of $P_1, \ldots, P_l$, such that

$$
|\Lambda_x, x + P_i(y), \ldots, x + P_l(y)(f_0, \ldots, f_l)| \leq ||f_j||_{U^s} + O(p^{-\beta}).
$$

for all $1$-bounded functions $f_0, \ldots, f_l : \mathbb{F}_p \to \mathbb{C}$.

Our statement differs slightly from the statement of Proposition 2.2 in [Pel19] in that Peluse did not mention explicitly our condition that the difference between any two polynomials $P_i, P_j$ cannot be constant. However, she assumed throughout her paper that $P_1, \ldots, P_l$ were distinct polynomials with zero constant terms, which implies our condition. In our paper, the polynomials may have nonzero constant terms, in which case we replace $P_i(y)$ by $P'_i(y) := P_i(y) - P_i(0)$ and $f_i(t)$ by $f'_i(t) := f_i(t + P'_i(0))$, so that $f_i(x + P'_i(y)) = f'_i(x + P'_i(y))$. The facts that $f_i$ and $f'_i$ have the same Gowers norms and that $P'_1, \ldots, P'_l$ are all distinct polynomials with zero constant terms allows us to reduce to the case covered in Proposition 2.2 of [Pel19].

Our next step is to decompose $f_{m+k-1}$ into three terms using a decomposition based on the Hahn-Banach theorem.

**Lemma 6** (Hahn-Banach decomposition, Proposition 2.6 of [Pel19]). Let $f : \mathbb{F}_p \to \mathbb{C}$ and $|| \cdot ||$ be a norm on the space of $\mathbb{C}$-valued functions from $\mathbb{F}_p$. Suppose $||f||_{L^2} \leq 1$. Then there exists a decomposition

$$
f = f_a + f_b + f_c
$$

with $||f_a||^\ast \leq p^{\delta_1}$, $||f_b||_{L^1} \leq p^{-\delta_2}$, $||f_c||_{L^\infty} \leq p^{\delta_3}$, $||f_c|| \leq p^{-\delta_4}$ provided

$$
p^{\delta_4 - \delta_1} + p^{\delta_2 - \delta_3} \leq \frac{1}{2}.
$$

(9)
This decomposition was pioneered by Gowers and Wolf in their work on true complexity of linear forms [GW11c, GW11b, GW11a, Gow10]. The variant that we are using is due to Peluse and appeared in [Pel19, PP19]. The dual norm in the statement of Lemma 6 is defined by \( \|f\|_* = \sup\{ |\langle f, g \rangle| : \|g\|_\infty \leq 1\} \).

Using this decomposition, we can write \( f_{m+k-1} \) as a sum of three functions: the first has not too big \( U^s \)-dual norm, the second has small \( L^1 \) norm, and the third has a small \( U^s \) norm and not too big \( L^\infty \) norm. By taking appropriate values of \( \delta_1, \delta_2, \delta_3, \delta_4 \), we get rid of two error terms and only work with \( f_a \). This gives us control over \( \Lambda_{m,P_m,\ldots,P_{m+k-1}} \) by the \( U^s \) norm of a dual function

\[
F(x) := \mathbb{E}_y \prod_{j=0}^{m-1} f_j(x + jy - P_{m+k-1}(y)) \prod_{j=m}^{m+k-2} f_j(x + P_j(y) - P_{m+k-1}(y))
\]

and allows us to essentially replace \( f_{m+k-1} \) in the \( \Lambda_{m,P_m,\ldots,P_{m+k-1}} \) operator by a character. We call \( F \) a “dual function” because \( \Lambda \) has a small \( U^s \) norm and not too big \( L^\infty \) norm. For the special case of the dual function \( F \), we however show that \( \|F\|_{U^s} \) is indeed controlled by \( \|F\|_{U^2} \) for any \( s \in \mathbb{N} \). We achieve this in the following lemma.

**Lemma 7** (Degree lowering). Let \( F \) be defined as in (10). For each \( s \geq 2 \),

\[
\|F\|_{U^{s-1}} = \Omega(\|F\|_{U^s}^{2^{(s-2)(s+2)}}) - O(p^{-c})
\]

for \( c > 0 \) depending on \( m, k \), and \( P_m, \ldots, P_{m+k-1} \) but not on \( f_0, \ldots, f_{m+k-2} \). As a consequence,

\[
\|F\|_{U^2} = \Omega(\|F\|_{U^s}^{2^{(s-2)(s+2)}}) - O(p^{-c}).
\]

Having a control by the \( U^2 \) norm of the dual function \( F \) is important because this norm is in turn controlled by the \( U^1 \) norms of the component functions \( f_m, \ldots, f_{m+k-1} \), which follows from Lemma 4 coupled with \( S(m, k-1) \).

**Lemma 8** (\( U^1 \) control of the dual). Let \( F \) be defined as in (10). Then

\[
\|F\|_{U^2} \leq \min_{m \leq j \leq m+k-2} \|f_j\|_{U^1}^{1/2} + O(p^{-c})
\]

for some \( c > 0 \) depending on \( m, k \), and \( P_m, \ldots, P_{m+k-1} \) but not on \( f_0, \ldots, f_{m+k-2} \).

Combining the estimates of two previous lemmas with Hahn-Banach decomposition, we get a control of the \( \Lambda \) operator by \( U^1 \) norms of \( f_m, \ldots, f_{m+k-1} \).

**Lemma 9** (\( U^1 \) control of \( \Lambda \)). There exist constants \( c, c' > 0 \) and \( s \in \mathbb{N} \) depending only on \( m, k, P_m, \ldots, P_{m+k-1} \) but not on \( f_0, \ldots, f_{m+k-1} \) such that

\[
|\Lambda(f_0, \ldots, f_{m+k-1})| \ll p'^c \min_{m \leq j \leq m+k-2} \|f_j\|_{U^1}^{2^{-s}} + p^{-c}.
\]

Having established Lemma 9, it is straightforward to prove \( S(m, k) \). We split each of \( f_m, \ldots, f_{m+k-2} \) into \( f_j = \mathbb{E} f_j + (f_j - \mathbb{E} f_j) \), and decompose \( \Lambda \) accordingly. Then \( \Lambda(f_0, \ldots, f_{m+k-1}) \) splits into the main term

\[
\Lambda(f_0, \ldots, f_{m-1}, \mathbb{E} f_m, \ldots, \mathbb{E} f_{m+k-2}, f_{m+k-1})
\]
and $2^k - 1$ error terms, each of which involves at least one $f_j - Ef_j$ for $m \leq j \leq m + k - 2$. Using Lemma 9, each of the error terms has size $O(p^{-c})$; thus

$$
\Lambda(f_0, \ldots, f_{m+k-1}) = \Lambda(f_0, \ldots, f_{m-1}, Ef_m, \ldots, Ef_{m+k-2}, f_{m+k-1}) + O(p^{-c})
$$

$$= \Lambda_m, p_{m+k-1}(f_0, \ldots, f_{m-1}, f_{m+k-1}) \prod_{j=m}^{m+k-2} Ef_j + O(p^{-c}).
$$

Applying the $S(m, 1)$ case, we can split $\Lambda_m, p_{m+k-1}(f_0, \ldots, f_{m-1}, f_{m+k-1})$ into

$$
\Lambda_m, p_{m+k-1}(f_0, \ldots, f_{m-1}, f_{m+k-1}) = \Lambda_m(f_0, \ldots, f_{m-1}) Ef_{m+k-1} + O(p^{-c})
$$

and hence

$$
\Lambda(f_0, \ldots, f_{m+k-1}) = \Lambda_m(f_0, \ldots, f_{m-1}) \prod_{j=m}^{m+k-1} Ef_j + O(p^{-c}).
$$

This proves $S(m, k)$ for $k > 1$.

4.3. Proofs of Lemmas 7, 8 and 9. While in the previous section we outlined the proof of $S(m, k)$ for $k > 1$, here we derive the technical lemmas which are used in this proof.

**Proof of Lemma 7.** This proof follows the path of Proposition 6.6 in [PP19]. The main idea is to write the $U^s$ norm of the dual function $F$ as an average of the $U^2$ norms of derivatives of $F$, extract the maximum Fourier coefficients of $\Delta h_1, \ldots, h_{s-2} F$, and show that for a dense proportion of $(h_1, \ldots, h_{s-2})$ these coefficients satisfy certain linear relations provided $\| F \|_{U^s} \gg p^{-c}$. If $s = 3$ and $\phi(h)$ is the phase of the maximum Fourier coefficient of $\Delta h F$, then we show that $\phi$ is constant on a dense proportion of $h$. For $s > 3$, analogous relations are somewhat more complicated. These linear relations turn out to be sufficient to get a control of the $U^s$ norm of $F$ by its $U^{s-1}$ norm with polynomial bounds.

Using the definition of Gowers norms, we have

$$
\eta := \| F \|^2_{U^s} = \mathbb{E}_{h_1, \ldots, h_{s-2}} \| \Delta h_1, \ldots, h_{s-2} F \|^4_{U^2}.
$$

Let $H_1 = \{(h_1, \ldots, h_{s-2}) \in F_{p^{-2}}^s : \| \Delta h_1, \ldots, h_{s-2} F \|^4_{U^2} \geq \frac{1}{4} \eta \}$. To simplify the notation, let $\mathcal{Q}_{h} = (h_1, \ldots, h_{s-2})$ and $\mathbb{E}_{h} := \mathbb{E}_{h \in F_{p^{-2}}^s}$. From the popularity principle (see e.g. Exercise 1.1.4 in [TV06]) it follows that $|H_1| \geq \frac{1}{2} \eta p^{s-2}$, and so

$$
\eta^2 \leq \mathbb{E}_{h} || \Delta h F ||^4_{U^2} \cdot 1_{H_1}(h).
$$

The $U^2$ inverse theorem, stated in Section 2.2, implies that the square of the $U^2$ norm of a function is bounded by its maximum Fourier coefficient. Given $\Delta h F$, let $\Delta h F(\phi(h))$ denote its maximum Fourier coefficient. Then the right hand side of (11) is bounded by

$$
\mathbb{E}_{h} \mathbb{E}_{x, x'} \Delta h F(\phi(h)) |1_{H_1}(h)|
$$

$$\leq \mathbb{E}_{h} \mathbb{E}_{x} \Delta h F(x) e_p(\phi(h)x) |1_{H_1}(h)|
$$

$$\leq \mathbb{E}_{x, x'} \Delta h F(x) |1_{H_1}(h)|
$$

To simplify notation, we denote $Q_j = P_j - P_{m+k-1}$ for $0 \leq j \leq m + k - 2$, where we extend the definition of $P_j$ to $0 \leq j \leq m - 1$ by setting $P_j(y) = jy$ for these
values of \( j \). Unpacking the definition of the dual function \( F \), the expression (12) equals

\[
\mathbb{E}_{x,x',y} \Delta_h \left( \mathbb{E}_y \prod_{j=0}^{m+k-2} f_j(x + Q_j(y)) \right)
\]

After writing out the multiplicative derivatives, (13) is equal to

\[
\mathbb{E}_{x,x',y} \mathbb{E}_{y,y' \in \mathcal{E}_p^{(0,1)^{s-2}}} \prod_{j=0}^{m+k-2} f_j(x + w \cdot h + Q_j(y_{w}))
\]

The product in (14) contains \( 2^{s-2} \) copies of \( f_j \) for each \( j \) and each of \( x \) and \( x' \). In each of these copies the \( y \)-variable is different. We would like all the copies of \( f_j \) to be expressed in terms of the same \( y \)-variable. To achieve this, we modify (14) by applying the Cauchy-Schwarz inequality \( s-2 \) times. First, (14) can be rewritten as

\[
\mathbb{E}_{x,x',h_1,\ldots,h_{s-3},h_{s-2},h'_{s-2}} \prod_{j=0}^{m+k-2} \prod_{w \in \{0,1\}^{s-2}} f_j(x + w \cdot h + Q_j(y_{w})) e_p(\phi(h)(x - x')) 1_{H_1}(h).
\]

By the Cauchy-Schwarz inequality and change of variables, (15) is bounded by

\[
\left( \mathbb{E}_{x,x',h_1,\ldots,h_{s-3},h_{s-2},h'_{s-2}} \prod_{j=0}^{m+k-2} \prod_{w \in \{0,1\}^{s-2}} f_j(x + \sum_{i=1}^{s-3} w_i h_i + w_{s-2} h_{s-2} + Q_j(y_{w})) \right)^{s-2}
\]

The presence of so many terms in (16) comes from the fact that in the process of applying the Cauchy-Schwarz inequality and changing variables, each expression \( E(h_{s-2}) \) (depending possibly on other variables as well) is replaced by \( E(h_{s-2}) E(h'_{s-2}) \). Therefore the number of expressions in the product doubles, making (16) rather lengthy. Applying Cauchy-Schwarz another \( s-3 \) times to
and Combining all of this, we obtain the estimate (16) by

\[
\begin{aligned}
\mathbb{E}_{x,x',y,y',h,h'} \prod_{j=0}^{m+k-2} \prod_{w \in \{0,1\}^{s-2}} (1_{H_j} (\bar{h}(w)) \mathbb{C}^{[w]} f_j (x + w \cdot \bar{h}(w) + Q_j (y))) \\
\mathbb{C}^{[w]} f_j (x' + w \cdot \bar{h}(w) + Q_j (y')) e_p (\sum_{w \in \{0,1\}^{s-2}} (-1)^{|w|} \phi(\bar{h}(w))(x - x'))^{1/x}. 
\end{aligned}
\]

where

\[
\bar{h}(w) = \begin{cases} 
    h_i, & w_i = 0 \\
    h_i', & w_i = 1
\end{cases}
\]

The expression (17) can be simplified to

\[
\mathbb{E}_{h,h'} \mathbb{E}_{x,y} \left( \prod_{j=0}^{m+k-2} g_j (x + P_j (y)) e_p (\psi(h,h')(x + P_{m+k-1}(y))) \right)^2 \left( \square(H_1)(h,h') \right)^{1/x}
\]

where

\[
g_j (t) := \prod_{w \in \{0,1\}^{s-2}} \mathbb{C}^{[w]} f_j (t + w \cdot \bar{h}(w)),
\]

\[
\square(A) := \{(h,h') \in \mathbb{P}^{2(s-2)} : \forall w \in \{0,1\}^{s-2} h(w) \in A \}
\]

and

\[
\psi(h,h') := \sum_{w \in \{0,1\}^{s-2}} (-1)^{|w|} \phi(\bar{h}(w)).
\]

Combining all of this, we obtain the estimate

\[
\mathbb{E}_{h,h'} \mathbb{E}_{x,y} \left( \prod_{j=0}^{m+k-2} g_j (x + P_j (y)) e_p (\psi(h,h')(x + P_{m+k-1}(y))) \right)^2 \left( \square(H_1)(h,h') \right) \geq \left( \frac{\eta}{2} \right)^{2s-1}.
\]

We are now precisely in the situation of Lemma 4. By this lemma, the expression inside the absolute values equals \(O(p^{-c})\) unless \(\psi(h,h') = 0\). Therefore, the set

\[
H_2 := \{(h,h') \in \square(H_1) : \psi(h,h') = 0 \}
\]

has size at least

\[
\left( \left( \frac{\eta}{2} \right)^{2s-1} - O(p^{-c}) \right) p^{2(s-2)}.
\]

In particular, there exists \(h \in H_1\) such that the fiber

\[
H_3 := \{h' : (h,h') \in H_2 \}
\]

has size at least

\[
\left( \left( \frac{\eta}{2} \right)^{2s-1} - O(p^{-c}) \right) p^{s-2}.
\]
Fix this $h$. We now show that the phases $\phi$ possess some amount of low-rank structure which we subsequently use to complete the proof of the lemma. By the definitions of $H_2$ and $H_3$, for each $h' \in H_3$ we have $\psi(h, h') = 0$. Define

$$\psi_i(h, h') := (-1)^s \sum_{w \in (0, 1)^{s-2}} (-1)^{|w|} \phi(h, w).$$

Note that, $\psi(h, h') = \phi(h_1', ..., h_{s-2}') - \psi_1(h, h') - \cdots - \psi_{s-2}(h, h')$. Crucially, $\psi_i$ does not depend on $h_1',..., h_i'$. Thus, $\psi(h, h') = 0$ implies that

$$\phi(h_1', ..., h_{s-2}') = \sum_{i=1}^{s-2} \psi_i(h, h').$$

That is to say, $\phi(h_1', ..., h_{s-2}')$ can be decomposed into a sum of $s - 2$ functions, each of which does not depend on $h_i'$ for a different $i$.

To make the notation a bit more palatable, we illustrate the aforementioned for $s = 3$ and $4$. For $s = 3$,

$$\psi(h, h') = \phi(h) - \phi(h') = \psi_1(h) - \phi(h').$$

Hence $\psi(h, h') = 0$ implies that $\phi(h') = \phi(h)$. For $s = 4$,

$$\psi(h, h') = \phi(h_1, h_2) - \phi(h_1', h_2) - \phi(h_1, h_2') + \phi(h_1', h_2')$$

$$= \psi_1(h, h') - \psi_2(h, h') + \phi(h_1', h_2')$$

and so $\psi(h, h') = 0$ implies that

$$\phi(h_1', h_2') = \phi(h_1, h_2') + \phi(h_1', h_2) - \phi(h_1, h_2) = \psi_2(h, h') - \psi_1(h, h').$$

We now estimate the expression

(19)  \[ \mathbb{E}_{h'} \| \Delta_{h'} F \|_{U^3}^2 1_{H_3}(h') \]

from above and below. From below, it is bounded by

$$\frac{\eta}{2} \left( \frac{\eta}{2} \right)^{2^s - 1} - O(p^{-c}) \geq \left( \frac{\eta}{2} \right)^{2^s} - O(p^{-c}).$$

The upper bound is more complicated, and it relies on the fact that we can decompose $\phi(h')$ into a sum of $\psi_i$’s such that $\psi_i$ does not depend on $h_i'$. Using $U^2$-inverse theorem, (19) is bounded from above by:

(20)  \[ \mathbb{E}_{h'} \left| \Delta_{h'} F(\phi(h')) \right|^2 1_{H_3}(h') = \mathbb{E}_{h'} \left| \Delta_{h'} F \left( \sum_{i=1}^{s-2} \psi_i(h') \right) \right|^2 1_{H_3}(h'). \]

By positivity, we can extend (20) to the entire $\mathbb{F}_p^{s-2}$; that is, we have

(21)  \[ \mathbb{E}_{h'} \left| \Delta_{h'} F \left( \sum_{i=1}^{s-2} \psi_i(h') \right) \right|^2 1_{H_3}(h') \leq \mathbb{E}_{h'} \left| \Delta_{h'} F \left( \sum_{i=1}^{s-2} \psi_i(h') \right) \right|^2. \]
Rewriting, we obtain that
\[
\mathbb{E}_{h'} \left| \Delta_{h'} F \left( \sum_{i=1}^{s-2} \psi_i(h') \right) \right|^2 = \mathbb{E}_{x} \left| \Delta_{h'} F(x) e_p \left( \sum_{i=1}^{s-2} \psi_i(h') x \right) \right|^2
\]
(22)
\[
= \mathbb{E}_{x, h', h_{s-1}} \Delta_{h', h_{s-1}} F(x) e_p \left( \sum_{i=2}^{s-2} \psi_i(h') h_{s-1} \right).
\]

We apply Cauchy-Schwarz to (22) to get rid of the phases \(\psi_i(h')\). In the first application, we start by rewriting (22) as

(23)
\[
\mathbb{E}_{x, h_1', ..., h_s', h_{s-1}} (x, h_2', ..., h_s', h_{s-1}) \mathbb{E}_{h_1'} \Delta_{h_1', h_{s-2}} h_{s-2} \Delta_{h_1', h_{s-2}} h_{s-1} F(x + h_1') e_p \left( \sum_{i=2}^{s-2} \psi_i(h') h_{s-1} \right)
\]
and then we bound it by

(24)
\[
\left( \mathbb{E}_{x, h_1', h_2', ..., h_s', h_{s-2}, h_{s-1}} \Delta_{h_1', h_{s-2}} h_{s-1} \left( F(x + h_1') F(x + h_1') \right) \right. \\
\left. e_p \left( \sum_{i=2}^{s-2} (\psi_i(h_1', h_2', ..., h_s', h_{s-2}) - \psi_i(h_1', h_2', ..., h_s', h_{s-2})) h_{s-1} \right) \right)^{\frac{1}{2}}.
\]

After repeatedly applying Cauchy-Schwarz in this manner, we get rid of all the phases and bound (24) by \(\|F\|^2_{L^2} \). This proves the lemma.

The second proof is simpler.

Proof of Lemma 8. By \(U^2\)-inverse theorem, \(\|F\|^2_{U^2} \leq \max_{\alpha \in \mathbb{F}_p} |\hat{F}(\alpha)|\). By Lemma 4, this is \(O(p^{-\epsilon})\) unless \(\alpha = 0\), in which case \(\hat{F}(\alpha) = \Lambda_m (f_0, f_1, ..., f_{m+k-2})\). Thus,

\[
\|F\|^2_{U^2} \leq \lambda_{\alpha=0} \Lambda_m (f_0, f_1, ..., f_{m+k-2}) + O(p^{-\epsilon})
\]
\[
\leq |\Lambda_m (f_0, ..., f_{m-1})| \prod_{j=m}^{m+k-2} |\mathbb{E} f_j| + O(p^{-\epsilon}) \leq \min_{m \leq i \leq m+k-2} \|f_j\|_{U^1} + O(p^{-\epsilon})
\]
where the intermediate inequality follows from applying \(S(m, k-1)\). Taking square roots on both sides and applying Hölder’s inequality proves the lemma.

Next we prove Lemma 9 using the previous lemmas.

Proof of Lemma 9. Take \(s = s_0\) and \(\beta\) for which Lemma 5 holds. Using Lemma 6, we decompose \(f_{m+k-1}\) into

\[
f_{m+k-1} = f_a + f_b + f_c
\]
with \(\|f_a\|_{L^2} \leq p^{\delta_1}, \|f_b\|_{L^1} \leq p^{-\delta_2}, \|f_c\|_{L^2} \leq p^{\delta_3}, \|f_c\|_{L^1} \leq p^{-\delta_4}\), and split the \(\Lambda\) operator accordingly. The values of the parameters \(\delta_1, \delta_2, \delta_3, \delta_4\) have to satisfy (9) and will be determined later. The term involving \(f_b\) is easy to bound using Hölder inequality

\[
|\langle F, f_b \rangle| \leq \|F\|_{L^\infty} \|f_b\|_{L^1} \leq p^{-\delta_2}.
\]
The term involving $f_c$ can also be bounded from above provided $\delta_4$ is sufficiently large compared to $\delta_3$

$$
|\langle F, f_c \rangle| = \|f_c\|_{L^\infty} \left| \frac{F}{\|f_c\|_{L^\infty}} \right| \\
\leq p^{\delta_3} \left( \left( \frac{p^{-\delta_4}}{p^{\delta_3}} \right)^\beta + O(p^{-\beta}) \right) \\
\ll p^{\delta_3(1-\beta)-\beta\delta_4} + p^{\delta_3-\beta}
$$

where in the second inequality we are using Lemma 5. Finally, the term involving $f_a$ can be bounded using dual inequality

$$
|\langle F, f_a \rangle| \leq \|f_a\|_{U^{\alpha_0}} \|F\|_{U^{\alpha_0}} \leq p^\delta \|F\|_{U^{\alpha_0}}.
$$

Using the decomposition, we obtain the following bound on $\Lambda$ in terms of the $U^{\alpha_0}$ norm of the dual function $F$

$$
|\Lambda(f_0, \ldots, f_{m+k-1})| \leq |\langle F, f_a \rangle| + |\langle F, f_b \rangle| + |\langle F, f_c \rangle| \\
\leq p^{\delta_1} \|F\|_{U^{\alpha_0}} + p^{-\delta_2} + p^{\delta_3(1-\beta)-\beta\delta_4} + p^{\delta_3-\beta}.
$$

From Lemma 7 it follows that

$$
\|F\|_{U^2} = \Omega((\|F\|_{U^{\alpha_0}})^2) - O(p^{-c}).
$$

Let $s_1 = (s_0 - 2)(s_0 + 2)$. We thus have that

$$
|\Lambda(f_0, \ldots, f_{m+k-1})| \ll p^{\delta_1} \|F\|^{2^{-s_1}}_{U^{s_1}} + p^{\delta_1-2^{-s_1}c} + p^{-\delta_2} + p^{\delta_3(1-\beta)-\beta\delta_4} + p^{\delta_3-\beta}.
$$

Using Lemma 8, we are able to establish a $U^1$ control by $f_m, \ldots, f_{m+k-2}$

$$
|\Lambda(f_0, \ldots, f_{m+k-1})| \ll p^{\delta_1} \min_{m \leq i \leq m+k-2} \|f_i\|^{2^{-s_1}}_{U^{s_1}} + p^{\delta_1-2^{-s_1}c} + p^{-\delta_2} + p^{\delta_3(1-\beta)-\beta\delta_4} + p^{\delta_3-\beta}.
$$

Let $c_0$ be the value of $c$ appearing in (25). Setting the values of the parameters to be

$$
\delta_1 = 2^{-s_1} \frac{c_0}{2}, \quad \delta_2 = \beta 2^{-s_1} \frac{c_0}{8}, \quad \delta_3 = \beta 2^{-s_1} \frac{c_0}{4}, \quad \text{and} \quad \delta_4 = (1-\beta)2^{-s_1} \frac{c_0}{2}
$$

proves the lemma. \qed

4.4. **Proof of Lemma 4.** The last step is to prove Lemma 4. The proof of this lemma is quite lengthy, yet it requires few new ideas, as it closely resembles the proof of $S(m, k)$ for $k > 1$. Therefore we omit certain details which are identical to the details in the proof of $S(m, k)$ for $k > 1$.

Note also the order of induction: we use $S(m, k - 1)$ to derive Lemma 4 for $(m, k)$, and then we use the $(m, k)$ case of Lemma 4 to prove $S(m, k)$.

**Proof of Lemma 4.** We mimic the proof of $S(m, k)$ for $k > 1$ and split $f_{m+k-2}$ using the Hahn-Banach decomposition

$$
f_{m+k-2} = f_a + f_b + f_c,
$$

with $\|f_a\|_{U^{\alpha_0}} \leq p^{\delta_1}, \|f_b\|_{L^1} \leq p^{-\delta_2}, \|f_c\|_{L^\infty} \leq p^{\delta_3}, \|f_a\|_{U^{\alpha_0}} \leq p^{-\delta_4}$, where $s_0$ is the value of $s$ coming from Lemma 5. We extend the definition of $P_j$ to $0 \leq j \leq m-1$ by
setting \( P_j(y) = jy \) for these values, and let \( Q_j = P_j - P_{m+k-2} \) for \( 0 \leq j \leq m+k-1 \). Then we define the dual function by

\[
F(x) := \mathbb{E}_y \left( \prod_{j=0}^{m+k-2} f_j(x + Q_j(y)) \right) e_p(\alpha(x + Q_{m+k-1}(y))),
\]

so that

\[
\langle F, f_{m+k-2} \rangle = \Lambda(f_0, \ldots, f_{m+k-2}, e_p(\alpha)).
\]

The contributions coming from \( f_2 \) and \( f_3 \) are bounded by

\[
|\langle F, f_2 \rangle| \leq p^{-\delta_2}, \quad \text{and} \quad |\langle F, f_3 \rangle| \leq p^{\delta_3(1-\beta)} - \beta \delta_4 + p^{\delta_3 - \beta}.
\]

We bound the term involving \( f_a \) using the dual inequality

\[
|\langle F, f_a \rangle| \leq \|f_a\|_{L^0} \|F\|_{U^{s}} \leq p^{\delta_1} \|F\|_{U^{s}}.
\]

As in Lemma 7, we show that for \( s > 2 \), the \( U^s \) norm of \( F \) is controlled by its \( U^{s-1} \) norm, from which it follows that \( \|F\|_{U^{s}} \) is controlled by \( \|F\|_{L^2} \). The proof proceeds very much the same way. From the definition of \( U^s \) norm it follows that

\[
\eta := \|F\|^2_{U^s} = \mathbb{E}_H |\Delta_h F|_{L^2}^2 \leq \mathbb{E}_H |\Delta_h F(\phi(h))|^2
\]

where \( \Delta_h F(\phi(h)) \) is the largest Fourier coefficient of \( \Delta_h F \) and \( h = (h_1, \ldots, h_{s-2}) \). If

\[
H_1 := \{ h : |\Delta_h F|_{L^2}^2 \geq \frac{1}{2} \eta \}
\]

then the exact same analysis as applied in Lemma 7 shows that

\[
(26) \quad \mathbb{E}_{h, |h|} |\mathbb{E}_{x,y} \left( \prod_{j=0}^{m+k-3} g_j(x + P_j(y)) \right) e_p(\psi(h, h')(x + P_{m+k-1}(y)))
\]

\[
\left( \prod_{|w| \in \{0, 1\}^{s-2}} C^{|w|} e_p(aP_{m+k-1}(y)) \right) |^2 1_{\Box(H_1)}(h, h') \geq \left( \frac{\eta}{2} \right)^{2^{s-1}}
\]

where

\[
g_j(t) := \prod_{|w| \in \{0, 1\}^{s-2}} C^{|w|} f_j(t + w \cdot h(w)) \quad \text{and} \quad \psi(h, h') := \sum_{w \in \{0, 1\}^{s-2}} (-1)^{|w|} \phi(h, w).
\]

The only difference between (26) and (18) is that in (18), the exponential phase \( e_p(\psi(h, h')) \) was weighting \( x + P_{m+k-1} \). By contrast, in (26) the exponential phase \( e_p(\psi(h, h')) \) is weighting \( x + P_{m+k-2} \) whereas the polynomial \( x + P_{m+k-1} \) is weighted by \( e_p(a) \). Modulo that small difference, (26) and (18) are derived in an identical manner.

The crucial simplification comes from the fact that the product

\[
\prod_{|w| \in \{0, 1\}^{s-2}} C^{|w|} e_p(aP_{m+k-1}(y))
\]

is equal to 1 for \( s > 2 \), and so we have obtained that

\[
\left( \frac{\eta}{2} \right)^{2^{s-1}} \leq \mathbb{E}_{h, |h|} |\mathbb{E}_{x,y} \left( \prod_{j=0}^{m+k-3} g_j(x + P_j(y)) \right) e_p(\psi(h, h')(x + P_{m+k-2}(y))) |^2.
\]
To evaluate the inner sum, we apply $S(m, k - 1)$; this tells us that the inner sum is $O(p^{-c})$ for some $c > 0$ unless $\psi(h, h') = 0$ because the $U^1$ norm of $e_p(\psi(h, h'))$ vanishes for a nonzero $\psi(h, h')$. As in Lemma 7, the set

$$H_2 := \{(h, h') \in \square(H_1) : \psi(h, h') = 0\}$$

has size at least

$$\left(\left(\frac{n}{2}\right)^{2^s - 1} - O(p^{-c})\right) p^{2(s - 2)}$$

and there exists $h \in H_1$, which we fix, such that the fiber $H_3 := \{h' : (h, h') \in H_2\}$ has size at least

$$\left(\left(\frac{n}{2}\right)^{2^s - 1} - O(p^{-c})\right) p^{s - 2}.$$

Thus, the expression $\mathbb{E}_{h'}||\Delta_{h'} F||_{U^2}^4 1_{H_3}(h')$ is bounded from below by

$$\frac{n}{2} \cdot \left(\left(\frac{n}{2}\right)^{2^s - 1} - O(p^{-c})\right) \geq \left(\frac{n}{2}\right)^{2^s} - O(p^{-c}).$$

We want to bound it from above by $||F||_{U^{s+1}}^2$, which we do by mimicking again the argument from Lemma 4. Using the fact that $\psi(h, h') = 0$ for $h' \in H_3$, we can rewrite

$$\phi(h_1', ..., h_{s-2}') = \sum_{i=1}^{s-2} \psi_i(h, h')$$

for all $h' \in H_3$, where

$$\psi_i(h, h') := (-1)^s \sum_{w \in \{0,1\}^{s-2}, \sum_{w_i=0}^1} (-1)^{|w|} \phi(h(w))$$

does not depend on $h_1', ..., h_{s-2}'$.

Using positivity, expanding the definition of Fourier transform and changing variables, we obtain that

$$\mathbb{E}_{h'} \left|\Delta_{h'} F \left(\sum_{i=1}^{s-2} \psi_i(h')\right)\right|^2 1_{H_3}(h') \leq \mathbb{E}_{h'} \left|\Delta_{h'} F \left(\sum_{i=1}^{s-2} \psi_i(h')\right)\right|^2$$

(27)

$$= \mathbb{E}_{x, h', h_{s-1}} \Delta_{h', h_{s-1}} F(x) e_p \left(\sum_{i=1}^{s-2} \psi_i(h') h_{s-1}\right).$$

Applying the Cauchy-Schwarz inequality $s - 2$ times to (27), we get rid of all the phases $\psi_i(h')$ and bound (27) by $||F||_{U^{s+1}}^2$. Thus,

$$||F||_{U^s} \gg ||F||_{U^{s+1}}^2 - p^{-c}$$

for $s > 2$, and so

$$||F||_{U^2} \gg ||F||_{U^{s+1}}^2 - p^{-c}.$$
By the $U^2$ inverse theorem, $||F||_{U^2} \leq ||\hat{F}(b)||^{1/2}$ for some $b \in \mathbb{F}_p$. Expanding out, we see that

$$\hat{F}(b) = \mathbb{E}_{x,y} \left( \prod_{j=0}^{m+k-3} f_j(x + P_j(y)) e_p(aP_{m+k-1}(y) + bP_{m+k-2}(y)) \right).$$

Unless $a = b = 0$, the polynomial $aP_{m+k-1}(y) + bP_{m+k-2}(y)$ has degree at least $m$. Again, we are back in $S(m, k-1)$, as we are dealing with $k-1$ polynomials $P_m, P_{m+1}, \ldots, P_{m+k-3}, aP_{m+k-1} + bP_{m+k-2}$.

From $S(m, k-1)$ and $a \neq 0$ it follows that $||\hat{F}(b)|| = O(p^{-c})$. Thus,

$$||F||_{U^0} \ll p^{-c\beta^2 - (\rho_0 - 2)(\rho_0 + 2) - 1} + p^{-c \beta(1-\beta) - \beta^2 \delta_4} \ll p^{-c}.$$

Combining all the estimates, we have the bound

$$\langle F, f_{m+k-2} \rangle \ll p^{\delta_1 - c} + p^{-\delta_2} + p^{\delta_3(1-\beta) - \beta \delta_4} + p^\gamma \beta.$$

Set $c_0$ to be the value of $c$ in (28). Taking $\delta_1 = \frac{c_0}{2}$, $\delta_2 = \frac{\beta c_0}{8}$, $\delta_3 = \beta \frac{c_0}{4}$, and $\gamma = (1 - \beta) \frac{c_0}{2}$ gives a bound of the form

$$\langle F, f_{m+k-2} \rangle \ll p^{-c}$$

as desired.

5. Upper Bounds for Subsets of $\mathbb{F}_p$ Lacking Arithmetic Progressions with $k$-th Power Common Differences

We now switch gears, moving away from the progression (2) towards arithmetic progressions with common difference coming from the set of $k$-th powers. In this section, we prove Theorem 4 assuming Theorem 5. The argument goes much the same way as deriving Theorem 2 from Theorem 3.

First, we prove the following simple lemma which allows us to reduce to the case $k|p-1$.

**Lemma 10.** Let $k \in \mathbb{N}_+$ and $Q_k$ be the set of $k$-th power residues in $\mathbb{F}_p$. Then $Q_k = Q_{\gcd(k, p-1)}$.

**Proof.** Since $\mathbb{F}_p^\times$ is a cyclic group under multiplication, we can write it as $\mathbb{F}_p^\times = \langle a | a^{p-1} = 1 \rangle$. Note that for each $k \in \mathbb{N}$, $Q_k$ and $Q_{\gcd(k, p-1)}$ are subgroups of $\mathbb{F}_p^\times$ of cardinality $\frac{p-1}{\gcd(k, p-1)}$, generated respectively by $a^k$ and $a^{\gcd(k, p-1)}$. The property $\gcd(k, p-1)|k$ moreover implies that $Q_k$ is a subgroup of $Q_{\gcd(k, p-1)}$, and so they must be equal.

**Proof of Theorem 4.** The set of $k$-th powers in $\mathbb{F}_p$ is precisely $Q_k$, and by Lemma 10 it is the same as the set $Q_{\gcd(k, p-1)}$. Therefore we can assume that $k$ divides $p-1$, otherwise we replace $k$ with $\gcd(k, p-1)$. Suppose $A \subset \mathbb{F}_p$ for $p \geq p_0$ of size $|A| = \alpha p$ lacks $m$-term arithmetic progressions with difference coming from the set of $k$-th powers. From Theorem 5 it follows that

$$E_{x,y} 1_A(x) 1_A(x+y) \cdots 1_A(x+(m-1)y) 1_{Q_k}(y) = \frac{1}{k} E_{x,y} 1_A(x) 1_A(x+y) \cdots 1_A(x+(m-1)y) + O(p^{-c}).$$
Since $A$ lacks progressions with $k$-th power differences, the left-hand side of (29) is 0, and so we have
\[(30) \quad \mathbb{E}_{x,y} 1_A(x)1_A(x+y)\ldots 1_A(x+(m-1)y) = O(p^{-c}).\]

Applying Lemma 3 to (29) gives $M^{-2} \ll p^{-c}$ where $M = M_m(\frac{1}{2} \alpha)$ and $M_m$ is the inverse function to $s_m$ on $(0, \alpha_0]$, $\alpha_0 = s_m(p_m)$. Since $M$ grows faster than polynomially in $\alpha^{-1}$ by Behrend’s construction [Beh46], this gives $M_m \gg p^{c}$. Applying $s_m$ to both sides and noting that $s_m$ is decreasing, we obtain that $\alpha \leq 2s_m(Cp^c)$.

\[\square\]

6. Counting theorem for the number of linear configurations in subsets of $\mathbb{F}_p$ with variables restricted to the set of $k$-th powers

This section is devoted to the proof of Theorem 5. We will first show that without loss of generality, we can assume that $k_i$ divides $p - 1$ for each $1 \leq i \leq d$. This will simplify the notation in the rest of the argument.

**Lemma 11.** We have
\[
\mathbb{E}_{x_1,\ldots,x_d} \prod_{i=1}^{m} f_j(L_i(x_1^{k_i}, \ldots, x_d^{k_d})) = \mathbb{E}_{x_1,\ldots,x_d} \prod_{i=1}^{m} f_j(L_i(x_1^{k_i'}, \ldots, x_d^{k_d'}))
\]
\[
= k_1' \ldots k_d' \mathbb{E}_{x_1,\ldots,x_d} \prod_{i=1}^{m} f_j(L_i(x_1, \ldots, x_d)) \prod_{i=1}^{d} 1_{Q_{k_i'}}(x_i) + O(p^{-1})
\]

where $k_i' := \gcd(k_i, p - 1)$ for each $1 \leq i \leq d$.

**Proof.** By Lemma 10, $Q_k = Q_{\gcd(k, p-1)}$ for each $k \in \mathbb{N}_+$. Therefore the set of $k_i$-th power residues agrees with the set of $k_i'$-th power residues for each $1 \leq i \leq d$. Consequently, the set of tuples
\[
\{(x_1^{k_1'}, \ldots, x_d^{k_d'}) : (x_1, \ldots, x_d) \in \mathbb{F}_p^d\}
\]
equals the set of tuples
\[
\{(x_1^{k_1'}, \ldots, x_d^{k_d'}) : (x_1, \ldots, x_d) \in \mathbb{F}_p^d\},
\]
and moreover each tuple $(x_1^{k_1'}, \ldots, x_d^{k_d'})$ appears in $\mathbb{F}_p^d$ the same number of times as the tuple $(x_1^{k_i'}, \ldots, x_d^{k_i'})$. This implies the first equality, as the summations in both expressions are carried over the same sets of tuples the same number of times.

The second equality follows from the fact that each value of $y \in \mathbb{F}_p^\times$ equals $x_i^{k_i'}$ for precisely $k_i'$ different values of $x_i \in \mathbb{F}_p$. The error term $O(p^{-1})$ corresponds to the cases when at least one of the variables $x_1, \ldots, x_d$ is 0. Using union bound, there are at most $dp^{d-1}$ such cases, which together contribute at most $\frac{d}{p}$ to the expectation.

\[\square\]

We thus assume for the rest of this section that $k_1, \ldots, k_d$ are coprime to $p - 1$. With this assumption, we now describe a useful expression for $1_{Q_k}$ which is crucial in proving the error term in Theorem 5. Let $a$ be a generator for the multiplicative group $\mathbb{F}_p^\times$. Define the map
\[
\chi_k : \mathbb{F}_p^\times \to \mathbb{C}
\]
\[
a^l \mapsto e_k(l).
\]
The function $\chi_k$ is thus a *multiplicative character of order* $k$, i.e., a group homomorphism from $\mathbb{F}_p^\times$ to $\mathbb{C}$ satisfying $\chi_k(e) = 1$. We extend $\chi_k$ to $\mathbb{F}_p$ by setting $\chi_k(0) = 0$. Then $\chi_k$ picks out $Q_k$, in the sense that $\chi_k(x) = 1 \iff x \in Q_k$. Using the orthogonality of roots of unity, we can write

\[
1_{Q_k}(x) = \frac{1 + \chi_k(x) + \chi_k(x)^2 + \cdots + \chi_k(x)^{k-1}}{k} - \frac{1}{k}1_{\{0\}}(x).
\]

We now use (31) to replace each $1_{Q_k}$ by a sum of characters in (5). Using the multilinearity of the operator, we obtain a main term of the same form as in (5), which corresponds to the terms in (31) having $1_{Q_k}$ replaced by $\frac{1}{k}$. Terms where $1_{Q_k}$ is replaced by $\frac{1}{k}1_{\{0\}}(x)$ are of size $O(p^{-1})$, and there is a bounded number of them. It remains to deal with the terms that contain some $\frac{\chi_k(x)}{k}$ with $j > 0$ but have no $\frac{1}{k}1_{\{0\}}(x)$. Each such term is of the form

\[
E_{x_1, \ldots, x_d} \prod_{i=1}^m f_j(L_i(x_1, \ldots, x_d)) \prod_{i \in S} \frac{\chi_k(x_i)}{k_i}
\]

for a nonempty $S \subset \{1 \leq i \leq d : k_i > 1\}$ and $1 \leq j \leq k_i - 1$. From the fact that $k_i$ divides $d$ it follows that $\chi_k^{\prime}$ is also a character of order $k_i$, so without loss of generality we can take $j_i = 1$ for each $1 \leq i \leq d$.

Green and Tao proved that linear forms $L'_1(x_1, \ldots, x_d), \ldots, L'_m(x_1, \ldots, x_d)$ are controlled by a Gowers norm [GT10, Tao2]: specifically, they showed that

\[
\left| E_{x_1, \ldots, x_d} \prod_{j=1}^m g_j(L'_j(x_1, \ldots, x_d)) \right| \leq \min_{1 \leq j \leq m} ||f_j||_{U^s}
\]

whenever for each $1 \leq i \leq m$ one can partition $\{L'_j : j \neq i\}$ into $s + 1$ classes such that $L'_j$ does not lie in the span of each of them. The lowest $s - 1$ for which this is true is called *Cauchy-Schwarz complexity*, or *CS-complexity* of the system of linear forms $L'_1, \ldots, L'_m$. The only case when such $s$ may not exist is if two linear forms $L'_j$ and $L'_j$ are the same up to scaling. Otherwise we can partition linear forms into such classes: in the worst case, each of $\{L'_j : j \neq i\}$ forms a separate class, in which case the CS-complexity is $m - 2$. This extreme case occurs in arithmetic progressions, for instance: the operator

\[
E_{x,y} f_0(x + y) \cdots f_{m-1}(x + (m-1)y)
\]

is bounded by $||f||_{U^{m-1}}$ for each $0 \leq i \leq m - 1$, and the system of linear forms $\{x, x + y, \ldots, x + (m-1)y\}$ has CS-complexity $m - 2$.

We assumed specifically that no two linear forms $L_i, L_j$ are scalar multiples, and that $L_i$ is never a scalar multiple of $e_j$. From these assumptions we obtain the following lemma, which is essentially a restatement of Green and Tao’s result tailored to our context.

**Lemma 12.** For an arbitrary character $\chi_{k_i}$ of order $k_i$, we have the bound

\[
\left| E_{x_1, \ldots, x_d} \prod_{j=1}^m f_j(L_i(x_1, \ldots, x_d)) \prod_{i \in S} \frac{\chi_{k_i}(x_i)}{k_i} \right| \leq \left( \prod_{i \in S} \frac{1}{k_i} \right) \min_{i \in S} ||\chi_{k_i}||_{U^s},
\]
where $s - 1$ is the CS-complexity of the system

$$\{L_1, ..., L_m \} \cup \{x_j : j \in S\}$$

In particular, one can take $s = m + |S| - 1 \leq m + d - 1$.

**Proof.** By assumption, all forms in the system

$$\{L_1, ..., L_m \} \cup \{x_j : 1 \leq j \leq d, k_j > 1\}$$

are pairwise linearly independent. Since (35) is a subset of (36), all forms in (35) are also pairwise linearly independent. Therefore the CS-complexity of this system is finite, and is at most $m + |S| - 2$ because the system (35) consists of $m + |S| - 1$ linear forms.

It thus follows that the error term in (5) is controlled by Gowers norms of characters. The multiplicative property of characters makes it easy to bound their Gowers norms using tools such as Weil’s bound.

**Lemma 13** (Weil’s bound). Let $\chi$ be a nonprincipal multiplicative character of $\mathbb{F}_p$ of order $k$, and let $P \in \mathbb{F}_p[x]$ be a polynomial with $r$ distinct roots in the splitting field. If $P$ is not a $k$-th power, then

$$||E_x \chi(P(x))|| \leq (r - 1)q^{-\frac{k}{2}}.$$  

In particular, we use the following corollary, which is Corollary 11.24 in Iwaniec & Kowalski [IK04].

**Lemma 14** (Corollary to Weil’s bound). Let $\chi$ be a nonprincipal multiplicative character of $\mathbb{F}_p$, and let $b_1, ..., b_{2r} \in \mathbb{F}_p$. If one of them is different from others, then

$$||E_x \chi((x-b_1)...(x-b_r))\chi((x-b_{r+1})...(x-b_{2r}))|| \leq 2rp^{-\frac{k}{2}}.$$  

**Lemma 15** (Gowers norms of characters). If $\chi$ is a multiplicative character of $\mathbb{F}_p$ of order $k$ and $s$ is a natural number, then

$$||\chi||_{U^s} \leq 2p^{-2^{s(k+1)}}.$$  

**Proof.** By definition, the $U^s$ norm of $\chi$ is given by the following expression

$$||\chi||_{U^s}^2 = E_{h_1, ..., h_s} E_x \prod_{w \in \{0,1\}^s} |w||\chi(x + w \cdot \mathbf{h})|$$

$$= E_{h_1, ..., h_s} E_x \chi \left( \prod_{w \in \{0,1\}^s, |w| \text{ even}} (x + w \cdot \mathbf{h}) \right) \overline{\chi} \left( \prod_{w \in \{0,1\}^s, |w| \text{ odd}} (x + w \cdot \mathbf{h}) \right)$$

$$\leq E_{h_1, ..., h_s} E_x \chi \left( \prod_{w \in \{0,1\}^s, |w| \text{ even}} (x + w \cdot \mathbf{h}) \right) \overline{\chi} \left( \prod_{w \in \{0,1\}^s, |w| \text{ odd}} (x + w \cdot \mathbf{h}) \right).$$

If $w \cdot \mathbf{h}$ are not all equal, then by Lemma 14 we have

$$\left| E_x \chi \left( \prod_{w \in \{0,1\}^s, |w| \text{ even}} (x + w \cdot \mathbf{h}) \right) \overline{\chi} \left( \prod_{w \in \{0,1\}^s, |w| \text{ odd}} (x + w \cdot \mathbf{h}) \right) \right| \leq 2^s p^{-\frac{k}{2}}.$$


The only possibility for \( w \cdot h \) being equal for all \( w \in \{0, 1\}^s \) is when \( h_1 = \ldots = h_s = 0 \), which happens with probability \( p^{-s} \). Thus
\[
||\chi||_{U^s}^2 \leq 2^s p^{-\frac{s}{2}} + p^{-s}
\]
and so
\[
||\chi||_{U^s} \ll p^{-2(s+1)}.
\]

Applying the results of Lemma 15 to Lemma 12, we see that the error term in (5) is of the size \( O(p^{-c}) \), which proves Theorem 5.

7. Further discussion

There are many directions in which one could try to extend the results of this paper, in particular Theorem 3. One of the questions one might ask is whether there is a discorrelation result for progressions of the form
\[
x, x + Q(y), \ldots, x + (m - 1)Q(y), x + P_m(y), \ldots, x + P_{m+k-1}(y)
\]
where \( Q \) has degree greater than 1 while \( P_m, \ldots, P_{m+k-1} \) are linearly independent and presumably satisfy a further technical assumption of algebraic independence similar to one in Theorem 3. Combining methods used in the proofs of Theorems 3 and 5, one can easily derive a statement of the form:

**Theorem 6.** Let \( m, k, l \in \mathbb{N}_+ \) and \( P_m, \ldots, P_{m+k-1} \) be polynomials in \( \mathbb{Z}[y] \) such that
\[
a_mP_m + \ldots + a_{m+k-1}P_{m+k-1}
\]
has degree at least \( m \) unless \( a_m = \ldots = a_{m+k-1} = 0 \) (in particular, \( P_m, \ldots, P_{m+k-1} \) are linearly independent and each of them has degree at least \( m \)). Suppose \( f_0, \ldots, f_{m+k-1} \) are 1-bounded functions from \( \mathbb{F}_p \) to \( \mathbb{C} \). Then
\[
\mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy^l) \prod_{j=m}^{m+k-1} f_j(x + P_j(y^l)) = \left( \mathbb{E}_{x,y} \prod_{j=0}^{m-1} f_j(x + jy) \prod_{j=m}^{m+k-1} f_j \right) + O(p^{-c})
\]
where all the constants are positive and depend on \( m, k, l \) and polynomials \( P_m, \ldots, P_{m+k-1} \) but not on \( f_0, \ldots, f_{m+k-1} \).

This is a version of Theorem 3 where variable \( y \) is restricted to lie in the set of \( l \)-th powers. It essentially says that restricting the variables to the set of \( l \)-th powers does not matter. For instance, this theorem allows us to prove that a set \( A \subset \mathbb{F}_p \) lacking progressions of the form
\[
x, x + y^l, \ldots, x + (m - 1)y^l, x + y^{ml}, \ldots, x + y^{(m+k-1)l}
\]
has size at most
\[
|A| \ll \begin{cases} p^{-c}, & m = 1, 2, \\ p^\left(\frac{\log \log p}{\log p}\right)^c, & m = 3, \\ p^\left(\frac{\log p}{\log \log p}\right), & m = 4, \\ p^\left(\frac{\log \log p}{\log p}\right)^c, & m > 4. \end{cases}
\]
where the implied constant depends on \( k, m, l \) and \( c \) depends on \( m \) only. Note that the bounds here are of the same shape as the bounds in Theorem 1: this is because the proof of this corollary is identical to the proof of Theorem 2.

The drawback of this theorem is that it essentially only works for polynomials \( P_m, \ldots, P_{m+k-1} \) that can be expressed as polynomials in \( y \), i.e. \( P_i(y) = P_j(y') \) for some \( P_j \). For instance, it allows us to handle
\[
x, x + y^2, x + 2y^2, x + y^6
\]
but not
\[
x, x + y^2, x + 2y^2, x + y^5 \quad \text{or} \quad x, x + y^2, x + 2y^2, x + y^5.
\]
Replacing \( P_m(y^j), \ldots, P_{m+k-1}(y^j) \) in the statement of the theorem by \( P_m(y), \ldots, P_{m+k-1}(y) \) would require a completely different approach. We have an argument that would allow us to replace \( P_j(y') \) by \( P_j(y) \) for \( m = 3 \) and possibly \( m = 4 \), however it has two serious downsides. First, the argument only works if the minimal degree of \( P_j \)'s is unreasonably large depending on \( m \) and \( l \) - it in fact would have to be greater than the minimal value \( s \) obtained by applying Lemma 5 to \( x, x + y^j, \ldots, x + (m-1)y^j, x + P_m(y), \ldots, x + P_{m+k-1}(y) \), which has rather poor dependence on \( m \) and degrees of \( P_m, \ldots, P_{m+k-1} \). Second, the method does not generalize to higher \( m \) without resorting to higher order Fourier analysis. For this reason, we do not present this argument here, hoping to find a more robust version of it in the future.

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