The joint distribution of occupation times of skip-free Markov processes and a class of multivariate exponential distributions

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Abstract
For a skip-free Markov process on $\mathbb{Z}^+$ with generator matrix $Q$, we evaluate the joint Laplace transform of the occupation times before hitting the state $n$ (starting at 0). This Laplace transform has a very straightforward and familiar expression. We investigate the properties of this Laplace transform, especially the conditions under which the occupation times form a Markov chain.

1 Introduction

Consider a process $\{X_t\}_{t \geq 0}$ on $\{0, 1, 2, \ldots, n\}$ with generator $Q := ((q_{ij}))_{0 \leq i, j < \infty}$ satisfying

$$q_{i, i+1} > 0, q_{ij} = 0 \text{ if } j > i + 1 \text{ and } \sum_{j=0}^{i+1} q_{ij} = 0 \text{ for all } i \geq 0. \tag{1}$$

Such processes are called skip-free (to the left) processes. Note that birth and death processes are special cases of the process described above. Let $T_n$ be the first time when the process reaches $n$. As usual, $\tilde{l}_{i, T_n}$ denotes the amount of time $\{X_t\}_{t \geq 0}$ spends in state $i$ before it reaches $n$. Let $Q_n$ denote the upper $n \times n$ submatrix of $Q$. We are interested in the joint distribution of $\{l_{i, T_n}\}_{0 \leq i \leq n-1}$ starting at 0. We prove that

$$\mathbf{E}_0 \left[ e^{-\sum_{i=0}^{n-1} d_i l_{i, T_n}} \right] = \frac{| - Q_n |}{|D - Q_n|}, \tag{2}$$

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where \(d_i \geq 0, \ i = 0, 1, \ldots, n-1\) and \(D\) is a diagonal matrix with diagonal entries \(d_i, \ i = 0, 1, \ldots, n-1\). This formula generalizes the results of Kent [3] about birth and death processes. Using the identity (2) we prove that \(\{l_{T_n}\}_{0 \leq i \leq n-1}\) is a Markov chain iff \(Q_n\) is a tridiagonal matrix. For these purposes, we will derive an identity for the joint Laplace transform of occupation times for a general class of Markov processes and then specialize to skip-free processes.

2 An identity for a general class of Markov processes

Let \(\{X_t\}_{t \geq 0}\) be a Markov process with finite state space \(\mathcal{X}\). Let \(Q = ((q_{xy}))_{x,y \in \mathcal{X}}\) be the corresponding generator matrix.

Let \(S_i\) be the random time corresponding to the \(i^{th}\) transition for the Markov process \(\{X_t\}_{t \geq 0}\), \(i = 1, 2, 3, \ldots\). Set \(S_0 = 0\). Let \(\{Y_i\}_{i \geq 0} := \{X_{S_i}\}_{i \geq 0}\) be the embedded discrete-time Markov chain with one step transition probabilities

\[
p_{xy} = \frac{-q_{xy}}{q_{xx}} \quad \text{for} \quad x \neq y, \quad p_{xx} = 0.
\]

Let \(P := ((p_{xy}))_{x,y \in \mathcal{X}}\) and let \(Q^{diag}\) denote the diagonal matrix with diagonal entries same as \(Q\). Then,

\[
Q = Q^{diag}(I - P).
\] (3)

As usual, let

\[
l_t^x := \int_0^t 1_{\{X_s = x\}} ds, \ t \geq 0, x \in \mathcal{X}
\]

denote the occupation time of the Markov process \(\{X_t\}_{t \geq 0}\) in the state \(x\) till time \(t\).

We describe a typical path of \(\{X_t\}_{t \geq 0}\). The process starts at an initial state \(Y_0\). The process stays at \(Y_n\) during \([S_n, S_{n+1})\) for \(n = 0, 1, 2, \ldots\) and at a random time \(\xi = S_{\eta+1}\) jumps from the state \(Y_\eta\) to a “cemetery” \(\Delta\) (not included in \(\mathcal{X}\)), and stays there forever. (Here \(\xi\) takes non-negative real values and \(\eta\) takes non-negative integer values). The analysis presented below requires the assumptions that
(i) \( Q^{-1} \) exists.
(ii) \( P_x(\eta < \infty) = 1 \) \( \forall x \in \mathcal{X} \).

Let \( d = \{d_u\}_{u \in \mathcal{X}} \) be arbitrary with \( d_u \geq 0 \). Let \( D \) denote the diagonal matrix with diagonal entries \( \{d_u\}_{u \in \mathcal{X}} \). Note that \( -(D - Q) \) is the generator of a Markov process \( \{\tilde{X}_t\}_{t \geq 0} \) with the same structure as \( \{X_t\}_{t \geq 0} \) except that at every state \( x \in \mathcal{X} \), there is an additional killing rate of \( d_x \). Let \( \{\tilde{Y}_m\}_{m \geq 0} \) be the embedded discrete-time Markov chain corresponding to \( \{\tilde{X}_t\}_{t \geq 0} \). Let us establish the change of measure formula from \( \{\tilde{Y}_m\}_{m \geq 0} \) to \( \{Y_m\}_{m \geq 0} \).

\[
P_x\{\tilde{Y}_1 = y_1, \tilde{Y}_2 = y_2, ..., \tilde{Y}_n = y_n\}
= \prod_{i=1}^{n} \frac{q_{y_{i-1}y_i}}{-q_{y_{i-1}y_{i-1}} + d_{y_{i-1}}} \text{ (with } y_0 = x) 
= \left( \prod_{i=1}^{n} \frac{-q_{y_{i-1}y_{i-1}}}{-q_{y_{i-1}y_{i-1}} + d_{y_{i-1}}} \right) P_x\{Y_1 = y_1, Y_2 = y_2, ..., Y_n = y_n\}
\]

Hence,

\[
E_x[F(\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_n)] = E_x\left[\left( \prod_{i=1}^{n} \frac{-q_{Y_{i-1}Y_{i-1}}}{-q_{Y_{i-1}Y_{i-1}} + d_{Y_{i-1}}} \right) F(Y_1, Y_2, ..., Y_n) \right] \tag{4}
\]

for each bounded Borel measurable function \( F \).

We evaluate \( E_x \left[ e^{-\sum_{u \in \mathcal{X}} d_u l_u^\infty} \right] \) which is the joint Laplace transform of the occupation times \( \{l_u^\infty\}_{u \in \mathcal{X}} \) evaluated at \( d = \{d_u\}_{u \in \mathcal{X}} \). The argument is inspired by Dynkin [1].

Note that,

\[
\sum_{u \in \mathcal{X}} d_u l_u^\infty = \sum_{i=0}^{\eta} d_{Y_i}(S_{i+1} - S_i).
\]

Hence,

\[
E_x \left[ e^{-\sum_{u \in \mathcal{X}} d_u l_u^\infty} \right] 
= \sum_{n=0}^{\infty} E_x \left[ e^{-\sum_{i=0}^{n} d_{Y_i}(S_{i+1} - S_i)} 1_{\eta=n} \right] 
= \sum_{n=0}^{\infty} E_x \left[ e^{-\sum_{i=0}^{n} d_{Y_i}(S_{i+1} - S_i)} \sum_{y \in \mathcal{X}} 1\{Y_n = y, Y_{n+1} = \Delta\} \right] 
\]
\[
\sum_{y \in X} \sum_{n=0}^{\infty} E_x \left[ e^{-\sum_{i=0}^{n} d_i (S_{i+1} - S_i)} 1_{\{Y_n=y, Y_{n+1}=\Delta\}} \right]
\]

\[
\sum_{y \in X} \sum_{n=0}^{\infty} E_x \left[ e^{-\sum_{i=0}^{n} d_i (S_{i+1} - S_i)} | \{Y_m\}_{m \geq 0} \right] 1_{\{Y_n=y, Y_{n+1}=\Delta\}}
\]

\[
\sum_{y \in X} \sum_{n=0}^{\infty} E_x \left[ \prod_{i=0}^{n} \frac{-q_{Y_i} Y_i}{-q_{Y_i} Y_i + d_{Y_i}} \right] 1_{\{Y_n=y, Y_{n+1}=\Delta\}}
\]

The previous equality follows from the fact that conditioned on \{Y_m\}_{m \geq 0}, the intermediate transition times \{S_{i+1} - S_i\}_{i \geq 0} are independent and have \text{Exponential}(-q_{Y_i} Y_i) distribution for \(i = 0, 1, 2, \ldots\).

By Markov property and (4),

\[
E_x \left[ e^{-\sum_{u \in X} d_u l_u} \right]
\]

\[
= \sum_{y \in X} \sum_{n=0}^{\infty} \frac{-q_{yy}}{-q_{yy} + d_y} E_x \left[ \prod_{i=1}^{n} \frac{-q_{Y_{i-1}} Y_{i-1}}{-q_{Y_{i-1}} Y_{i-1} + d_{Y_{i-1}}} \right] 1_{\{Y_n=y\}} P(Y_1 = \Delta | Y_0 = y)
\]

\[
= \sum_{y \in X} \sum_{n=0}^{\infty} \frac{-q_{yy}}{-q_{yy} + d_y} E_x \left[ 1_{\{Y_n=y\}} \right] p(y, \Delta)
\]

\[
= \sum_{y \in X} -q_{yy} E_x \left[ \int_0^{\infty} 1_{(X_s=y)} ds \right] p(y, \Delta)
\]

\[
= \sum_{y \in X} -q_{yy} (D - Q)^{-1}(x, y) p(y, \Delta)
\]

\[
= \frac{|y-Q|}{|D-Q|} \left( \sum_{y \in X} -q_{yy} \frac{(D - Q)^{(x,y)}}{|y-Q|} p(y, \Delta) \right)
\]

Here, \(|(D - Q)^{(x,y)}|\) represents the determinant of the matrix obtained after removing the \(y^{th}\) row and the \(x^{th}\) column from \(D - Q\), multiplied by \(-1\) if the \(x^{th}\) row and the \(y^{th}\) row of \(D - Q\) differ by an odd number of rows. Thus, the joint Laplace transform of \{\(l_u\)\}_{u \in X} is given by

\[
E_x \left[ e^{-\sum_{u \in X} d_u l_u} \right] = \frac{|y-Q|}{|D-Q|} \left( \sum_{y \in X} \frac{|(D - Q)^{(x,y)}|}{|y-Q|} \left( -\sum_{u \in X} q_{yu} \right) \right). \quad (5)
\]

In [2], (5) is derived using Kac’s moment formula. Note that we did not require any reversibility assumption on \(Q\) in the above argument.
3 Application to skip-free processes

Let us investigate the case of skip-free processes with generator $Q$ given by (1). We are interested in the joint distribution of $\{l_{T_n}^i\}_{0 \leq i \leq n-1}$ starting at state 0. It follows by the structure of $Q$ that the joint distribution of $\{l_{T_n}^i\}_{0 \leq i \leq n-1}$ starting at 0 is the same as the joint distribution of the infinite occupation times of a Markov chain with generator matrix $Q_n$, starting at 0. Hence, by (5), for arbitrary $d_i \geq 0$, $i = 0, 1, ..., n-1$,

$$E_0 \left[ e^{-\sum_{i=0}^{n-1} d_i l_{T_n}^i} \right] = \frac{|-Q_n|}{|D - Q_n|} \left( \sum_{j=0}^{n-1} \frac{|(D - Q_n)(0,j)|}{|Q_n|} \left( - \sum_{i=0}^{n-1} q_{ji} \right) \right)$$

$$= \frac{|-Q_n|}{|D - Q_n|} \left( - \sum_{i=0}^{n-1} q_{n-1,i} \right)$$

The previous equality follows from the fact that

$$\sum_{i=0}^{n-1} q_{ji} = 0 \text{ for } j < n - 1.$$

Note that if we remove the $(n - 1)^{th}$ row and 0th column of $D - Q_n$, the resulting matrix is upper triangular with diagonal entries not depending on $d_i$, $i = 0, 1, ..., n - 1$. Hence,

$$E_0 \left[ e^{-\sum_{i=0}^{n-1} d_i l_{T_n}^i} \right] \propto \frac{1}{|D - Q_n|}.$$

By substituting $d_i = 0$ for $i = 0, 1, ..., n - 1$, we get,

$$E_0 \left[ e^{-\sum_{i=0}^{n-1} d_i l_{T_n}^i} \right] = \frac{|-Q_n|}{|D - Q_n|}.$$

Let us look more closely at the joint Laplace transform in (2). Note that, if $Q_n^{-1} := ((q_{ij}^n))_{0 \leq i, j \leq n-1}$, then

$$E_0 \left[ e^{-d_i l_{T_n}^i} \right] = \frac{-1}{d_i - \frac{1}{q_{ii}^n}} \text{ for } d_i \geq 0, \ 0 \leq i \leq n - 1.$$

Hence, the marginal distribution of $l_{T_n}^i$ is $Exponential \left( \frac{1}{q_{ii}^n} \right)$ for $0 \leq i \leq n - 1$, which means the joint distribution of $\{l_{T_n}^i\}_{0 \leq i \leq n-1}$ is a multivariate exponential distribution.
It is well known that if $\eta, \tilde{\eta} \sim_{i.i.d.} MVN_n(0, \Sigma)$, i.e. $\eta = (\eta_1, \eta_2, ..., \eta_n)$ and $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n)$ are multivariate normal with mean $0$ and covariance matrix $\Sigma$, then

$$E \left[ e^{-\sum_{i=1}^{n} \frac{d_i}{2} (\eta_i^2 + \tilde{\eta}_i^2)} \right] = \frac{|\Sigma^{-1}|}{|D + \Sigma^{-1}|} \text{ for } d_i \geq 0, \ i = 1, 2, ..., n.$$  

Note that if $M = \bar{D}\Sigma^{-1}\bar{D}^{-1}$, where $\bar{D}$ is a diagonal matrix, then

$$\frac{|\Sigma^{-1}|}{|D + \Sigma^{-1}|} = \frac{|M|}{|D + M|}.$$  

If $Q$ is reversible with respect to a probability measure $\pi$, it is possible to symmetrize $Q$ by diagonal conjugation, and hence we can identify the joint distribution of $\{l_{T_n}^i\}_{0 \leq i \leq n-1}$ with the sum of componentwise squares of two independent $MVN_n(0, (-Q_n^* )^{-1})$ random vectors multiplied by $\frac{1}{2}$. $Q_n^*$ is given by

$$Q_n^*(i, j) = q_{ij} \sqrt{\frac{\pi (i)}{\pi (j)}}, \ 0 \leq i, j \leq n - 1.$$  

But a generator for a skip-free process is diagonally conjugate to a symmetric matrix if and only if the process is a birth and death process. We refer to skip-free processes which are not birth and death processes as strictly skip-free processes. Hence, if $Q_n$ is not tridiagonal, then the joint distribution of $\{l_{T_n}^i\}_{0 \leq i \leq n-1}$ can not be identified with componentwise sums of squares of independent multivariate normal random vectors multiplied by $\frac{1}{2}$.

### 4 A complex Gaussian measure

If $A$ is real non-symmetric and positive definite, we still can interpret

$$\phi_A(d) = \frac{|A|}{|D + A|} \text{ for } d = (d_1, d_2, ..., d_n) \text{ with } d_i \geq 0, \ i = 1, 2, ..., n,$$  

as the Laplace transform of a signed measure. This will be proved using the methods in [4].

Let $C := \frac{A + A^T}{2}$ and $B := \frac{A - A^T}{2}$. Clearly, $C$ is symmetric, $B$ is skew-symmetric and $A = C + B$. Define the measure $\mu_A$ on $\mathbb{C}^n$ by

$$d\mu_A(z) = \left( e^{-\frac{z^T A z}{2}} + e^{-\frac{z^T A^* z}{2}} \right) dz,$$
where \(z = x + iy\) and \(dz = \prod_{j=1}^{n} dx_j \prod_{j=1}^{n} dy_j\). Note that \(z^T A \bar{z}\) and \(\bar{z}^T A z\) are complex conjugates, hence \(\mu_A\) is a real valued (although possibly signed) measure. Also, the transformation \(y \to -y\) (where \(z = x + iy\)) gives,

\[
\int_{\mathbb{C}^n} e^{-\frac{A z \overline{z}}{2}} \, dz = \int_{\mathbb{C}^n} e^{-\frac{\bar{z}^T A z}{2}} \, dz.
\]

To avoid confusion, we clarify that \(\int_{\mathbb{C}^n} f(z) \, dz\) stands for \(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z) \, dx \, dy\). Hence,

\[
\mu_A(\mathbb{C}^n) = 2 \int_{\mathbb{C}^n} e^{-\frac{A z \overline{z}}{2}} \, dz
\]

\[
= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{(x + iy)^T (C + B)(x - iy)}{2}} \, dx \, dy
\]

\[
= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{x^T C x + y^T C y - 2x^T B y}{2}} \, dx \, dy \quad (6)
\]

\[
= 2 \int_{\mathbb{R}^n} e^{-\frac{x^T C x + y^T B y}{2}} \left( \int_{\mathbb{R}^n} e^{-\frac{(y - (C^{-1} B^T)x)^T C (y - (C^{-1} B^T)x)}{2}} \, dy \right) \, dx
\]

\[
= 2 \int_{\mathbb{R}^n} e^{-\frac{x^T C x + y^T B y}{2}} \left( \sqrt{2\pi} \right)^n |C|^{-\frac{1}{2}} \, dx \quad (7)
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |C|^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\frac{x^T (C + B^{-1} B^T) x}{2}} \, dx
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |C|^{-\frac{1}{2}} |C + BC^{-1} B^T|^{-\frac{1}{2}}
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |C|^{-\frac{1}{2}} |C|^{-\frac{1}{2}} \left| I - C^{-\frac{1}{2}} B C^{-\frac{1}{2}} C^{-\frac{1}{2}} B C^{-\frac{1}{2}} \right|^{-\frac{1}{2}}
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |C|^{-\frac{1}{2}} |C|^{-\frac{1}{2}} \left| I - C^{-\frac{1}{2}} B C^{-\frac{1}{2}} \right|^{-\frac{1}{2}} \left| I + C^{-\frac{1}{2}} B C^{-\frac{1}{2}} \right|^{-\frac{1}{2}}
\]

Note that (6) follows from the fact that \(C\) is symmetric and \(B\) is skew-symmetric, and (7) follows by using properties of the multivariate normal distribution. Continuing further,

\[
\mu_A(\mathbb{C}^n) = 2 \left( \sqrt{2\pi} \right)^n |C - B|^{-\frac{1}{2}} |C + B|^{-\frac{1}{2}}
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |A^T|^ {-\frac{1}{2}} |A|^{-\frac{1}{2}}
\]

\[
= 2 \left( \sqrt{2\pi} \right)^n |A|^{-1}
\]
If we define the measure $\mu^*_A$ on $\mathbb{C}^n$ by

$$
\mu^*_A(.) = \frac{\mu_A(.)}{2(\sqrt{2\pi})^{2n}|A|^{-1}},
$$

then $\mu^*_A(\mathbb{C}^n) = 1$.

Let $D$ denote a diagonal matrix with non-negative entries $d_1, d_2, \ldots, d_n$. Then,

$$
\int_{\mathbb{C}^n} e^{-\frac{1}{2} \sum_{j=1}^{n} d_j |z_j|^2} d\mu^*_A(z) = \int_{\mathbb{C}^n} \frac{d\mu_{D+A}(z)}{2(\sqrt{2\pi})^{2n}|A|^{-1}} = \frac{\mu_{D+A}(\mathbb{C}^n)}{2(\sqrt{2\pi})^{2n}|A|^{-1}} = \frac{2(\sqrt{2\pi})^{2n}|D+A|^{-1}}{2(\sqrt{2\pi})^{2n}|A|^{-1}}
$$

Hence,

$$
\int_{\mathbb{C}^n} e^{-\frac{1}{2} \sum_{j=1}^{n} d_j |z_j|^2} d\mu^*_A(z) = \frac{|A|}{|D+A|}.
$$

Let $\mu_A^{abs}$ denote the measure induced on $\left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}, \ldots, \frac{|z_n|^2}{2}\right)$ by $\mu^*_A$. Since $\mu_A^{abs}$ is a signed measure, we can break it up as $\mu_A^{abs} = \mu_A^{abs,+} - \mu_A^{abs,-}$, where $\mu_A^{abs,+}$ and $\mu_A^{abs,-}$ are non-negative measures. One wonders for which $A$, $\mu_A^{abs,-} \equiv 0$, so that $\mu_A^{abs}$ is a probability measure. If $A$ is the upper $n \times n$ submatrix of the negative of the generator of a strictly skip-free Markov process, then it follows by (2) and the uniqueness of Laplace transform for a bounded measure, that $\mu_A^{abs}$ is a probability measure.

### 5 The Markov property

We now undertake the task of ascertaining the conditions under which $\{t_i^{T_n}\}_{0 \leq i \leq n-1}$ satisfies the Markov property for $n \geq 3$.

**Lemma 1** If $Q_n$ (with $Q$ as in (1)) is tridiagonal, then $\{t_i^{T_n}\}_{0 \leq i \leq n-1}$ satisfies the Markov property.
Proof Since $Q_n$ is tridiagonal, we can obtain a symmetric matrix $Q_n^*$ from $Q_n$ by diagonal conjugation. It follows that

$$\{\hat{\mu}_{T_n}\}_{0\leq i\leq n-1} = \frac{d}{2} (\eta^2 + \bar{\eta}^2),$$

where, $\eta, \bar{\eta} \sim MVN_n (\mathbf{0}, (-Q_n^*)^{-1})$ and $\eta, \bar{\eta}$ are independent. By the standard properties of Gaussian Markov random fields developed in [5], we get that

$$\eta_k \mid (\eta_{k-1}, \eta_{k-2}, \ldots, \eta_1) = \eta_k \forall k \leq n.$$  \hspace{1cm} (8)

Let $t > 0$ be fixed. Then,

$$E\left[e^{-t(\eta_k^2 + \bar{\eta}_k^2)} \mid \eta_{k-1}^2 + \bar{\eta}_{k-1}^2, \ldots, \eta_1^2 + \bar{\eta}_1^2\right] = E\left[E\left[e^{-t(\eta_k^2 + \bar{\eta}_k^2)} \mid \eta_{k-1}, \eta_{k-2}, \ldots, \eta_1\right] \mid \eta_{k-1}^2 + \bar{\eta}_{k-1}^2, \ldots, \eta_1^2 + \bar{\eta}_1^2\right]$$

The previous equality follows by the independence of $\eta$ and $\bar{\eta}$. By (8) we get,

$$E\left[e^{-t(\eta_k^2 + \bar{\eta}_k^2)} \mid \eta_{k-1}^2 + \bar{\eta}_{k-1}^2, \ldots, \eta_1^2 + \bar{\eta}_1^2\right] = E\left[e^{-t\eta_k^2} \mid \eta_{k-1}\right] E\left[e^{-t\bar{\eta}_k^2} \mid \eta_{k-1}, \eta_{k-2}, \ldots, \eta_1\right]$$

The previous equality follows by the fact that $\eta$ and $\bar{\eta}$ are i.i.d. Note that,

$$\eta_k \mid \eta_{k-1} \sim N(c_1\eta_{k-1}, c_2) \quad \text{and} \quad \bar{\eta}_k \mid \bar{\eta}_{k-1} \sim N(c_1\bar{\eta}_{k-1}, c_2)$$

for some constants $c_1$ and $c_2$. This gives,

$$E\left[e^{-t\eta_k^2} \mid \eta_{k-1}\right] = \frac{e^{-\frac{c_1^2\eta_{k-1}^2}{2c_2t+1}}}{\sqrt{2c_2t+1}} \quad \text{and} \quad E\left[e^{-t\bar{\eta}_k^2} \mid \bar{\eta}_{k-1}\right] = \frac{e^{-\frac{c_1^2\bar{\eta}_{k-1}^2}{2c_2t+1}}}{\sqrt{2c_2t+1}}.$$  \hspace{1cm} (9)

Combining everything,

$$E\left[e^{-t(\eta_k^2 + \bar{\eta}_k^2)} \mid \eta_{k-1}^2 + \bar{\eta}_{k-1}^2, \ldots, \eta_1^2 + \bar{\eta}_1^2\right] = E\left[\frac{e^{-\frac{c_1^2(\eta_{k-1}^2 + \bar{\eta}_{k-1}^2)}{2c_2t+1}}}{\sqrt{2c_2t+1}} \mid \eta_{k-1}^2 + \bar{\eta}_{k-1}^2, \ldots, \eta_1^2 + \bar{\eta}_1^2\right]$$

$$= \frac{e^{-\frac{c_1^2(\eta_{k-1}^2 + \bar{\eta}_{k-1}^2)}{2c_2t+1}}}{\sqrt{2c_2t+1}}$$

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Thus, the conditional Laplace transform given \( \eta_{k-1}^2 + \tilde{\eta}_{k-1}^2, ..., \eta_1^2 + \tilde{\eta}_1^2 \) is a function of \( \eta_{k-1}^2 + \tilde{\eta}_{k-1}^2 \). Since,

\[
\{ p_i T_n \}_{0 \leq i \leq n-1} = \frac{1}{2}(\eta^2 + \tilde{\eta}^2),
\]

it follows that

\[
l_T^n \mid (l_{T_n}^{k-1}, l_{T_n}^{k-2}, ..., l_{T_n}^0) \overset{d}{=} l_T^n \mid l_{T_n}^{k-1} \quad \forall 1 \leq k \leq n - 1.
\]

Hence proved.

We now prove the converse.

**Lemma 2** If \( Q_n \) (with \( Q \) as in (1)) is not tridiagonal, \( \{ p_i T_n \}_{0 \leq i \leq n-1} \) does not satisfy the Markov property.

**Proof** We first look at the scenario where \((X_1, X_2, X_3)\) are non-negative random variables with joint Laplace transform given by

\[
E \left[ e^{-\left(d_1 X_1 + d_2 X_2 + d_3 X_3\right)} \right] = \frac{| -A |}{| D - A |},
\]

where,

\[
A = \begin{pmatrix}
-a_{11} & a_{12} & 0 \\
a_{21} & -a_{22} & a_{23} \\
a_{31} & a_{32} & -a_{33}
\end{pmatrix},
\]

with \( a_{12}, a_{23}, a_{31}, a_{22}, a_{33} > 0, \ a_{21}, a_{32} \geq 0 \) and row sums less than zero or equal to zero. It follows that

\[
E \left[ e^{-\left(d_2 X_2 + d_3 X_3\right)} \right] = \frac{| -A |}{a_{11} \left| \begin{array}{cc}
d_2 + \tilde{a}_{22} & -\tilde{a}_{23} \\
-\tilde{a}_{32} & d_3 + a_{33}
\end{array} \right|},
\]

where \( \tilde{a}_{22} = a_{22} - \frac{a_{12} a_{23}}{a_{11}}, \tilde{a}_{23} = \tilde{a}_{32} = \sqrt{a_{23} a_{32} + \frac{a_{12} a_{23} a_{31}}{a_{11}}} \). It follows that

\[
(X_2, X_3) \overset{d}{=} \frac{1}{2}(\eta_1^2, \eta_2^2) + \frac{1}{2}(\tilde{\eta}_1^2, \tilde{\eta}_2^2),
\]

where \( \eta, \tilde{\eta} \sim MVN_2(0, (-A^*)^{-1}) \) are independent with

\[
A^* = \begin{pmatrix}
-\tilde{a}_{22} & \tilde{a}_{23} \\
\tilde{a}_{32} & -a_{33}
\end{pmatrix}.
\]
By (9),

\[ E\left[e^{-d_3 X_3} \mid X_2\right] = \frac{a_{33}}{d_3 + a_{33}} e^{-\frac{(\tilde{a}_{23})^2d_3 X_2}{a_{33}(d_3 + a_{33})}}. \]

Note that,

\[ X_3 \mid X_2, X_1 \overset{d}{=} X_3 \mid X_2 \]
\[ \Leftrightarrow E\left[E\left[e^{-d_3 X_3} \mid X_2, X_1\right] e^{-(d_1 X_1 + d_2 X_2)}\right] = E\left[E\left[e^{-d_3 X_3} \mid X_2\right] e^{-(d_1 X_1 + d_2 X_2)}\right] \quad \forall d_1, d_2, d_3 \geq 0 \]
\[ \Leftrightarrow \frac{|-A|}{|D - A|} = E\left[\frac{a_{33}}{d_3 + a_{33}} e^{-(d_1 X_1 + d_2 X_2)}\right], \text{ with } d_2^* = d_2 + \frac{(\tilde{a}_{23})^2d_3}{a_{33}(d_3 + a_{33})} \forall d_1, d_2, d_3 \geq 0 \]
\[ \Leftrightarrow \frac{|-A|}{|D - A|} = \frac{a_{33}}{d_3 + a_{33} (d_1 + a_{11}) (d_2^* + a_{22})a_{33} - (d_1 + a_{11})a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}} \quad \forall d_1, d_2, d_3 \geq 0 \]

After expanding the determinant and cancelling terms from both sides we obtain,

\[ X_3 \mid X_2, X_1 \overset{d}{=} X_3 \mid X_2 \]
\[ \Leftrightarrow \frac{(d_1 + a_{11})a_{12}a_{23}a_{31}d_3}{a_{33}a_{11}} - \frac{(d_3 + a_{33})a_{12}a_{23}a_{31}}{a_{33}} = 0 \quad \forall d_1, d_2, d_3 \geq 0 \]
\[ \Leftrightarrow a_{12}a_{23}a_{31}(d_1d_3 - a_{11}a_{33}) = 0 \quad \forall d_1, d_2, d_3 \geq 0, \]

which is not true, because \(a_{12}, a_{23}, a_{31} > 0\) by our assumptions. Hence, \((X_1, X_2, X_3)\) is not Markov.

We now return to \(\{l^i_T\}_{0 \leq i \leq n-1}^1\). The scenario considered above immediately gives the required result for \(n = 3\). But it also helps us in the general case. Since \(Q_n\) is not tridiagonal, we can find the smallest \(i\) such that \(q_{ij} > 0\) for some \(j \leq i - 2\). Let us call it as \(i_0\) and evaluate the joint Laplace transform of \((l^{i_0-2}_T, l^{i_0-1}_T, l^{i_0}_T)\). By (2),

\[ E\left[e^{-(d_1 l^{i_0-2}_T + d_2 l^{i_0-1}_T + d_3 l^{i_0}_T)}\right] = \left|\frac{|-Q_n|}{D^{i_0,3} - Q_n}\right|. \]

where \(D^{i_0,3}\) is a diagonal matrix with the \((i_0 - 2)\)\(^{th}\), \((i_0 - 1)\)\(^{th}\) and \(i_0\)\(^{th}\) entries given by \(d_1, d_2\) and \(d_3\) respectively and other diagonal entries being 0. We now reduce the determinant \(|D^{i_0,3} - Q_n|\) to a manageable form by performing appropriate row and column operations. Subtract a multiple of the first column from the second column to make the \((1, 2)\)\(^{th}\) entry zero. Successively
subtract a multiple of the \((j-1)^{th}\) column from the \(j^{th}\) column to make the \((j-1, j)^{th}\) entry 0, for \(j \leq i_0 - 2\). The upper \((i_0 - 3) \times (i_0 - 3)\) submatrix is now upper triangular. Similarly, subtract a multiple of the \(n^{th}\) row from the \((n-1)^{th}\) row to make the \((n-1, n)^{th}\) entry 0. Successively subtract a multiple of the \(j^{th}\) row from the \((j-1)^{th}\) row to make the \((j-1, j)^{th}\) entry 0, for \(j \geq i_0 + 1\). The lower \((n-i_0) \times (n-i_0)\) matrix is now upper triangular. The fact that \(Q\) is a generator matrix as in (1) ensures that all the diagonal entries are positive. Hence,

\[
|D^{i_0, 3} - Q_n| = c \begin{bmatrix}
    d_1 - \tilde{q}_{i_0 - 2, i_0 - 2} & -q_{i_0 - 2, i_0 - 1} & 0 \\
    -q_{i_0 - 1, i_0 - 2} & d_2 - q_{i_0 - 1, i_0 - 1} & -q_{i_0 - 1, i_0} \\
    -\tilde{q}_{i_0, i_0 - 2} & -\tilde{q}_{i_0, i_0 - 1} & d_3 - \tilde{q}_{i_0, i_0}
\end{bmatrix} \text{ with } c > 0.
\]

Again, \(Q\) being a generator matrix as in (1) along with \(q_{i_0, j} > 0\) for some \(j \leq i_0 - 2\) ensures that

\[
Q_n^{i, 3} = \begin{pmatrix}
    \tilde{q}_{i_0 - 2, i_0 - 2} & q_{i_0 - 2, i_0 - 1} & 0 \\
    q_{i_0 - 1, i_0 - 2} & q_{i_0 - 1, i_0 - 1} & q_{i_0 - 1, i_0} \\
    \tilde{q}_{i_0, i_0 - 2} & \tilde{q}_{i_0, i_0 - 1} & \tilde{q}_{i_0, i_0}
\end{pmatrix}
\]

is a generator matrix with \(q_{i_0 - 2, i_0 - 1}, q_{i_0 - 1, i_0}, \tilde{q}_{i_0, i_0 - 2} > 0\) and row sums less than zero or equal to zero. Hence,

\[
E \left[ e^{-(d_1 t_{i_0}^{i_0 - 2} + d_2 t_{i_0}^{i_0 - 1} + d_3 t_{i_0}^{i_0})} \right] = \frac{|-Q_n^{i_0, 3}|}{|D^3 - Q_n^{i_0, 3}|},
\]

where \(D^3\) is a diagonal matrix of dimension 3 with diagonal entries \(d_1, d_2, d_3\). The Laplace transform of \(\left(t_{i_0}^{i_0 - 2}, t_{i_0}^{i_0 - 1}, t_{i_0}^{i_0}\right)\) satisfies the conditions of the scenario considered at the beginning of this proof. Hence, \(\left(t_{i_0}^{i_0 - 2}, t_{i_0}^{i_0 - 1}, t_{i_0}^{i_0}\right)\) is not Markov, which is enough to conclude that \(\{\overline{T}_i\}_{0 \leq i \leq n-1}\) is not Markov. This completes the proof.

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