A density version for Häggström’s theorem

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Abstract

Given invariant percolation on a regular tree, where the probability of an edge to be open equals $p$, is it always possible to find an infinite self-avoiding path along which the density of open edges is bigger then $p$?

Let $S$ be an invariant percolation on the edges of the $d$-regular tree, where the probability of an edge being open equals $p$. We think of $S$ as an invariant process with values in $\{0, 1\}$ (1 corresponds to open edges). For $\overline{x} = (x_0, x_1, x_2, \ldots)$ an infinite self-avoiding path, let $D(\overline{x})$ be the density of the percolation along $\overline{x}$, that is,

$$D(\overline{x}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} S(x_{k-1}, x_k)$$

and let

$$D(S) = \sup_{\overline{x}} D(\overline{x}).$$

In general, this is a random variable that is $F_\infty$-measurable, where $F_\infty$ is the tail $\sigma$-algebra. We may look at the essential supremum of this random variable and define

$$D_d(p) = \inf_S \, \text{ess sup} \, D(S),$$

where the infimum is taken over all invariant percolation distributions on the $d$-regular tree. For background on invariant percolation, see e.g. [1].

Obviously, $D_d(p)$ is monotone in $p$ and $D_d(p) \geq p$.

**Question 1.** Is $D_d(p) > p$ for any $d \geq 3$ and $0 < p < 1$?

More generally we may ask

**Question 2.** What is $D_d(p)$?

In his seminal paper [2], Olle Häggström proved that any invariant percolation on the $d$-regular tree, with marginal at least $\frac{2}{d}$, has an infinite cluster. In particular, we get that $D_d\left(\frac{2}{d}\right) = 1$ and specifically $D_3\left(\frac{2}{3}\right) = 1$. 
Theorem 3. \( D_3 \left(1 - \frac{1}{\sqrt{3}}\right) \geq \frac{1}{2} \).

\( \text{Proof.} \) Take two iid samples from the percolation distribution and look at their maximum. If \( p \geq 1 - \frac{1}{\sqrt{3}} \) then this new percolation has marginal \( \geq \frac{2}{3} \), so by Häggström’s theorem there is an infinite cluster a.s. and in particular there is \( \mathcal{F} \) with all the edges open. At least one of the two original percolations must have \( D(\mathcal{F}) \geq \frac{1}{2} \), so \( D(S) \geq \frac{1}{2} \). □

More generally, define

\[ a(d, k) = 1 - \frac{k}{\sqrt{1 - \frac{2}{d}}} . \]

Theorem 4. \( D_d(a(d, k)) \geq \frac{1}{k} \)

\( \text{Proof.} \) The same proof as the previous theorem, except that you take \( k \) copies and work on the \( d \)-regular tree. □

Notice that for \( d = 3 \) and \( 2 \leq k \leq 5 \) we have \( a(d, k) < \frac{1}{k} \), so we get that \( D_3(p) > p \) for any \( p \in [a(3, k), \frac{1}{k}] \), but if \( k \geq 6 \) then \( a(3, k) > \frac{1}{k} \), so we obtain no new information.

However, for \( d \geq 4 \) we have \( a(d, k) < \frac{1}{k} \) for all \( k \), so we get some that \( D_d(p) > p \) for any \( p \in \cup_{k=1}^\infty [a(d, k), \frac{1}{k}] \).

In fact,

Theorem 5. For any \( d \geq 4 \) and any \( 0 < p < 1 \) we have \( D_d(p) > p \).

\( \text{Proof.} \) All we need to do is show that for \( d \geq 4 \) we have \( \cup_{k=1}^\infty [a(d, k), \frac{1}{k}] = (0, 1) \). We claim that for any \( d \geq 4 \) and any \( k \geq 1 \) we have \( a(d, k) \leq \frac{1}{k+1} \) which means that these intervals are overlapping.

Now

\[ 1 - \frac{k}{\sqrt{1 - \frac{2}{d}}} \leq \frac{1}{k+1} \]

is equivalent to

\[ \left(1 - \frac{1}{k+1}\right)^k \leq 1 - \frac{2}{d} \]

and the left hand side is decreasing (as a function of \( k \)) so the maximum is obtained for \( k = 1 \) and it is \( \frac{1}{2} \leq 1 - \frac{2}{d} \). □

Theorem 6. For any \( d \), the function \( D_d \) is uniformly continuous.

\( \text{Proof.} \) Fix \( d \). Let \( B \) be bernoulli percolation on the \( d \)-regular tree with marginal \( \varepsilon \). For a path of length \( n \) the probability of getting at least \( a \) 1’s is bounded by

\[ \binom{n}{an} \varepsilon^{an} \leq (2\varepsilon^a)^n \]
Since there are $d(d-1)^{n-1}$ paths of length $n$ we get that when $a > \frac{\log(2(d-1))}{\log(1/\varepsilon)}$ the probability of a path with $n$ 1’s decays exponentially. We conclude that

$$D(B) \leq f_d(\varepsilon) := \frac{\log(2(d-1))}{\log(1/\varepsilon)}.$$  

We now claim that if $0 \leq p < q \leq 1$ and $q - p \leq \varepsilon$ then $D_d(q) - D_d(p) \leq f_d(3\varepsilon)$. This implies uniform continuity since $f_d(\varepsilon) \to 0$ when $\varepsilon \to 0$.

To show the claim, let $S$ be an invariant percolation with marginal $p$ and $B$ bernoulli percolation with marginal $3\varepsilon$. Let $S'$ be their maximum. Then $S'$ has marginal $p + (1-p)3\varepsilon \geq q$ (since we may assume that $p \leq 2/3$, for $p > 2/3$ we have $D_d(p) = D_d(q) = 1$). Therefore, with positive probability, there is an infinite path $\pi$ such that the density of $s'$ along $\pi$ is at least $D_d(q)$. But the contribution of $B$ to the density of $\pi$ is at most $f_d(3\varepsilon)$, so the density of $S$ along $\pi$ is at least $D_d(q) - f_d(3\varepsilon)$. \[\square\]

In particular, $D_d(p) \to 1$ as $p \to \frac{2}{3}$ so for some $a < \frac{2}{3}$ we have $D_d(p) > p$ for all $p \in [a, 1)$. However, we still don’t know that $D_3(p) > p$ for all $0 < p < 1$ and specifically that $D_3(1/2) > (1/2)$.

When $d \to \infty$ we have that if $p = \frac{1}{d} + \frac{1}{2d}$ we have $1 - (1-p)^2 = 2p - p^2 > \frac{2}{3}$ so again we have $D(p) \geq \frac{1}{2}$. This works for any fixed $k$, so if we define the limit

$$D_\infty(x) = \lim_{d \to \infty} D\left(\frac{2}{d}x\right)$$

we know that $D_\infty(x) \geq \frac{1}{3}$ for any $x > \frac{1}{3}$.

**Question 7.** Is it true that $D_\infty(x) = x$?

Note that the same methods apply to site percolation on regular trees. However, in that case, as $d \to \infty$ the threshold in Häggström’s theorem tends to $\frac{1}{2}$ rather then 0. Indeed, the tree is a bipartite graph and the partition into two sides is invariant, hence we can define a percolation that choose one of the sides with equal probabilities and then put 1s on this side and 0s on the other. This gives a marginal of $\frac{1}{2}$ and also density of $\frac{1}{2}$ along any self-avoiding path. This percolation is ergodic, but have a nontrivial tail $\sigma$-algebra.

**Question 8.** What can be said about site percolation on regular trees if we require that the tail $\sigma$-algebra is trivial?

We may also consider more general processes, i.e. not $\{0, 1\}$-valued.

**Question 9.** Is it true that for any invariant, non-constant process $S$ on the edges of a regular tree, $D(S) > \mathbb{E}[S(e)]$, where $e$ is some/any edge of the tree?

An interesting side question is this:

**Question 10.** Is it true that when you replace the lim sup in the definition of $D(\pi)$ by lim inf you get the same function? If not, do our result still hold for the lim inf version?
Remark: Häggström’s theorem was extended to nonamenable Cayley graphs [1], all the discussion above adapts to this set up.

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References

[1] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, Group-invariant percolation on graphs. Geom. Funct. Anal. 9 (1999), no. 1, 29-66.

[2] O. Häggström, Infinite clusters in dependent automorphism invariant percolation on trees. Ann. Probab. 25 (1997), no. 3, 1423-1436.