Analysis of dynamical corrections to baryon magnetic moments

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Abstract

We present and analyze QCD corrections to the baryon magnetic moments in terms of the one-, two-, and three-body operators which appear in the effective field theory developed in our recent papers. The main corrections are extended Thomas-type corrections associated with the confining interactions in the baryon. We investigate the contributions of low-lying angular excitations to the moments quantitatively and show that they are completely negligible. When the QCD corrections are combined with the non-quark model contributions of the meson loops, we obtain a model which describes the moments within a mean deviation of 0.04 $\mu_N$. The nontrivial interplay of the two types of corrections to the quark-model moments is analyzed in detail, and explains why the quark model is so successful. In the course of these calculations, we parametrize the general spin structure of the $j = \frac{1}{2}^+$ baryon wave functions in a form which clearly displays the symmetry properties and the internal angular momentum content of the wave functions, and allows us to use spin-trace methods to calculate the many spin matrix elements which appear in the expressions for the moments. This representation may be useful elsewhere.

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I. INTRODUCTION

Baryon magnetic moments (baryon moments for short) have been studied intensively for many years using different models and approaches [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It initially appears somewhat surprising that the simple, nonrelativistic quark model (QM) [1, 2, 3, 4] is so good, with more complicated models giving only small improvements in the quantitative description of the baryon moments. This is now understood.

In our recent work on the connections between the QM and effective field theory [18, 19], we showed that the constituent QM for the baryon moments can be regarded as a rewriting of the relativistic chiral effective field theory (EFT) in which only the one-body moment operators are retained, a result also obtained by Morpurgo using his general parametrization method [7, 11, 17] derived exactly from QCD. Like Morpurgo, we found that the success of the additive QM is due to the numerical dominance of the one-body operators over the nonadditive two- and three-body operators, and suggested why the latter operators should, in fact, be small.

In our earlier dynamical model [15], an initial baryon moment operator was derived from quenched QCD using the Wilson-loop approach of Brambilla et al. [20]. This appears as a sum of single-particle moments $\mu_i \approx \langle eQ_i/2E_i \rangle$, where $E_i$ is the kinetic energy of quark $i$, and a set of Thomas precession terms connected to the binding interactions. In [16], we studied the further corrections to the baryon moments from from meson loops, which are absent in quenched QCD. This combined approach substantially improved the agreement between the theory and experiment, leaving an average difference between the theoretical and experimental octet moments of only $0.05\mu_N$. We did not, however, analyze the components of the model, or its description in EFT, in detail. The central idea in the context of our later work on EFT [18, 19] was to use a dynamical model, namely the QCD-based quark model, to obtain reasonable estimates of the input parameters connected to the one- and higher-body operators in the effective field theory, then to add the meson loop corrections using chiral perturbation theory.

Our goals here are (i) to present the general structure of the baryon moments in effective field theory, parametrized in a form which connects easily to dynamical ideas; (ii) to analyze the contributions of meson loops and the dynamical QCD corrections in terms of the EFT;
(iii) sketch the relevant parts of our dynamical calculations, including our analysis of the contributions of low-lying angular excitations; and (iv) to show how the loop and dynamical corrections combine to give a good description of the moments. Items (ii) and (iii) involve the structure of the wave functions used to estimate the dynamical corrections, so we give their form and the methods of calculation used. Some of this material is potentially useful in other contexts.

The paper is organized as follows: In Sec. II A, we briefly review the structure of the baryon moments in the effective field theory, where the QM moments arise from one-body operators. In Sec. II B we sketch a derivation of the QM moments and further two- and three-body dynamical corrections from QCD using a Wilson-loop approach [15, 16]. We determine the structure of the $j = \frac{1}{2}$ baryon wave functions needed to calculate these corrections and develop useful the trace methods for the calculation in Sec. III. We analyze the contributions of meson loops, the dynamical QCD corrections, and internal orbital angular momenta numerically in Sec. IV, and show why both the loop and dynamical corrections are needed to obtain a good description of the data. We present our conclusions and discussion in Sec. V, and give some additional information, such as proofs and sample calculations, in the appendices.

II. THEORY OF THE MOMENTS

A. Moments in chiral perturbation theory

In two earlier papers [18, 19], we analyzed the structure of the baryon magnetic moment operators to $O(m_s)$ in heavy-baryon chiral perturbations theory (HBChPT), connecting the general spin-flavor structure to the spin dependence of the underlying dynamical theory. The analysis was done using flavor-index labeling of the effective baryon fields $B_{ijk}^\gamma(x)$, where $i, j, k \in u, d, s$, and $\gamma$ is a Dirac spinor index. The transformation properties of the fields are the same as those of the operators

$$B_{ijk}^\gamma \simeq \frac{1}{6} \epsilon_{abc} q_i^{\alpha a} q_j^{\beta b} q_k^{\gamma c} (C\gamma^5)_{\alpha\beta}$$  \hspace{1cm} (1)

constructed in terms of free quark operators which carry the color, flavor, and spin structure of the baryon. This expression for the relativistic fields reduces in the rest frame of the
FIG. 1: One- and two-body contributions to the baryon magnetic moments at the quark level. Lines with small wiggles represent photons. There is a factor $\sigma \cdot B \leftrightarrow -\sigma^\mu F_{\mu \nu}$ in the Lagrangian at the quark-photon vertex, where $B$ is the external magnetic field. Crosses correspond to factors of the quark charge matrix $Q$, and dots to insertions of the flavor matrix $M$. Graphs (a) and (b) give the SU(6) structure for the moments.

As shown in [19], the octet moments $\mu$ can be written through first order in the chiral symmetry breaking parameter $m_s$ in terms of seven operators $m_r$, which correspond to the heavy baryon to the nonrelativistic structure familiar in the nonrelativistic quark model,

$$B^\gamma_{ijk} \rightarrow \frac{1}{6} \epsilon_{abc} \left( q_i^a T^i q_j^b q_k^c \right) q_k^c,$$

and provides a straightforward connection to semirelativistic dynamical models for the baryons. It also connects directly to Morpurgo’s general parametrization method for specifying the most general spin and flavor structure of matrix elements in QCD. In the following, we will refer to the structure in terms of “quarks” for convenience, but emphasize that the description is completely equivalent to a relativistic effective field theory.
FIG. 2: Three-body contributions to the baryon magnetic moments. The two graphs in (i) appear together with the same overall coefficient. The structures in graphs (h) and (i) are reducible to those in Figs. 1 and 2 (g).

The diagrams (a)-(g) in Figs. 1 and 2. The contribution of the octet moments to the chiral Lagrangian for is given to $O(m_s)$ as a matrix element of the form $(\vec{B}\mu B) \cdot \vec{B}$ where

$$\mu = \sum_r \mu_r m_r$$

(3)
is a combination of the quark-level operators

\[
\begin{align*}
\mathbf{m}_a &= \sum_i (Q\sigma)_i, \\
\mathbf{m}_b &= \sum_i (QM\sigma)_i, \\
\mathbf{m}_c &= \sum_{i\neq j} Q_i\sigma_j, \\
\mathbf{m}_d &= \sum_{i\neq j} M_i(Q\sigma)_j, \\
\mathbf{m}_e &= \sum_{i\neq j} (QM)_i\sigma_j, \\
\mathbf{m}_f &= \sum_{i\neq j} M_iQ_j\sigma_k, \\
\mathbf{m}_g &= \sum_{i\neq j\neq k} (QM)_i\sigma_j
\end{align*}
\]

\(Q\) is the diagonal quark charge matrix, \(Q = \text{diag}(2/3, -1/3, -1/3)\), and \(M\) is the diagonal mass matrix for the strange quark, \(M = \text{diag}(0, 0, 1)\). A flavor index attached to a \(Q\) or \(M\) means that the matrices are taken to act on that flavor quark so that, for example, \(\bar{B}(Q\sigma)_iB\) is the matrix element \(\bar{B}q_i^\dagger\sigma_iq_i\). These operators are shown diagrammatically in Figs. 1 and 2 (Figures 10 and 11 in [19]). The expressions for the baryon moments in terms of the \(\mu_r\) is given in Table I [26].

The coefficients \(\mu_r\) of the operators above are not specified in HBChPT, and the effective field theory is not predictive without further input. The additive quark model for the moments involves only the one-body operators \(\mathbf{m}_a\) and \(\mathbf{m}_b\), Figs. 1(a) and (b). These combine to give the effective QM moment operator \(\mu_aQ + \mu_bQM\). This describes the moments quite well, with a root-mean-square deviation of theory from experiment of 0.12 \(\mu_N\), about 11% of the average magnitudes of the moments. The underlying reasons for this striking success have been discussed in [13, 19] from the point of view of the weak spin dependence of the interactions in dynamical models, and in [7, 11, 17] in a QCD-based analysis. These related analyses conclude that the two-body contributions to the moments involving \(\mathbf{m}_c - \mathbf{m}_f\) and the three-body contributions proportional to \(\mathbf{m}_g - \mathbf{m}_i\) should all be small. Thus, graph 1(c) gives a Thomas-type contribution to the moments from the spin dependence of the quark interactions as discussed in [16] and [13]. Graph 1(d) describes the effect at O(\(m_s\)) of the strange-quark mass on the leading contribution to the moment from a different quark,
TABLE I: The coefficients for the expression of the octet baryon moments in terms of the operators corresponding to the diagrams in Figs. 1 and 2, labelled as in the figures, plus the operator $m_{MM} = \sum_{i \neq j} [Q_i M_j M_k - (QM)_i M_j]$ of second order in the strange-quark mass matrix $M$. The moment of the $\Sigma^0$ is determined by isospin invariance, $\mu_{\Sigma^0} = \frac{1}{2}(\mu_{\Sigma^+} + \mu_{\Sigma^-})$.

| Baryon | $\mu_a$ | $\mu_b$ | $\mu_c$ | $\mu_d$ | $\mu_e$ | $\mu_f$ | $\mu_g$ | $\mu_{MM}$ |
|--------|---------|---------|---------|---------|---------|---------|---------|-----------|
| $p$    | 1       | 0       | 0       | 0       | 0       | 0       | 0       | 0         |
| $n$    | -2/3    | 0       | 2/3     | 0       | 0       | 0       | 0       | 0         |
| $\Sigma^+$ | 1     | 1/9     | 0       | 8/9     | -4/9    | -4/9    | 8/9     | 0         |
| $\Sigma^-$ | -1/3   | 1/9     | -2/3    | -4/9    | 2/9     | -4/9    | -4/9    | 0         |
| $\Xi^0$ | -2/3    | -4/9    | 2/3     | -8/9    | 4/9     | -2/9    | 10/9    | 1         |
| $\Xi^-$ | -1/3    | -4/9    | -2/3    | -2/9    | -8/9    | -2/9    | -2/9    | 0         |
| $\Lambda$ | -1/3   | -1/3    | 1/3     | 0       | 1/3     | 0       | 0       | 0         |
| $\Sigma^0\Lambda$ | 1/$\sqrt{3}$ | 0       | -1/$\sqrt{3}$ | 1/$\sqrt{3}$ | 0       | 0       | -1/$\sqrt{3}$ | 0         |

Graph 1(a), through the mass dependence of the wave function, while 1(e) and 1(f) give the corresponding effects on the Thomas term. Graph (g) in Fig. 2 gives the three-body strange-mass correction to the Thomas term. The extra three-body operators (h) and (i) arise from the effect of spin-spin interactions on the leading and Thomas terms at $O(m_s)$. These appear in a combined treatment of the octet and decuplet moments, but are reducible to the preceding terms if we consider only the octet.

The dynamical problem in understanding the baryon moments is the actual calculation of the coefficients $\mu_r$, and of possible terms of $O(m_s^2)$ and $O(m_s^3)$ which could affect the $\Xi$, $\Xi^*$, and $\Omega^-$ moments. The origin of the leading one-body terms is understood at least semi-quantitatively, as discussed below. Two- and three-body contributions to the moments can be generated explicitly in HBChPT by meson loop corrections to the leading one-body contributions to the moments. The loop corrections have been considered by many authors [5, 6, 8, 9, 10, 12, 13, 14, 16]. Two- and three-body contributions also appear through the nonzero initial values of the coefficients $\mu_c-\mu_g$ which arise in dynamical models from Thomas-type terms associated with short-distance QCD interactions and the long-range confining interaction [13]. A model which combines these calculations [16] gives an excellent fit to the data, with a mean deviation of the calculated moments from experiment of only 0.05
\[ \mu_N. \] In the following sections, we discuss the origin of the dynamical corrections to the QM moments, show that possible orbital contributions to the moments are negligible, analyze how the loop and dynamical corrections combine to produce the good fit found in \[ 16 \], and point out where problems remain.

B. Baryon moments in a QCD-based quark model

In previous work \[ 15, 16 \], we derived the QM for the baryon moments, including dynamical corrections, in the context of QCD. Our approach was based on the work of Brambilla \textit{et al.} \[ 20 \], who derived the interaction potential and wave equation for the valence quarks in a baryon using a Wilson-line construction in quenched QCD. Their basic idea was to construct a Green’s function for the propagation of a gauge-invariant combination of quarks joined by path ordered Wilson-line factors

\[
U = P \exp \left( ig \int A_g \cdot dx \right),
\]

where \( A_g \) is the color gauge field. With internal quark loops omitted (the quenched approximation to QCD), the Wilson lines sweep out a three-sheeted world sheet of the form shown in Fig. \[ 3 \] as the quarks move from their initial to their final configurations. The approximation ignores the effect of meson loops.

By making an expansion in powers of \( 1/m_j \) using the Foldy-Wouthuysen approximation \[ 22 \], and considering only forward propagation of the quarks in time, Brambilla \textit{et al.} were able to derive a Hamiltonian and Schrödinger equation for the quarks, with an interaction which involves an average over the gauge field. That average was performed using the minimal surface approximation in which fluctuations in the world sheet are ignored, and the geometry is chosen to minimize the total area of the world sheet subject to the motion of the quarks. The short-distance QCD interactions were taken into account explicitly. Finally, the kinetic terms could be resummed. The result of this construction was an effective Hamiltonian \[ 20 \] to be used in a semirelativistic Schrödinger equation \( H \Psi = E \Psi, \)

\[
H = \sum_i \sqrt{p_i^2 + m_i^2} + \sigma (r_1 + r_2 + r_3) - \sum_{i<j} \frac{2 \alpha_s}{3 r_{ij}} + V_{SD},
\]

where \( V_{SD} \) is the spin-dependent part of the potential. Here \( r_{ij} = x_i - x_j \) is the separation of quarks \( i \) and \( j \), \( r_i \) is the distance of quark \( i \) from point at which the sum \( r_1 + r_2 + r_3 \)}
FIG. 3: World sheet picture for the structure of a baryon in the Wilson-loop approach.

is minimized, the parameter $\sigma$ is a “string tension” which specifies the strength of the long range confining interaction, and $\alpha_s$ is the strong coupling.

We will make one further approximation, known to be good [21], and will replace $(r_1 + r_2 + r_3)$ in the confining potential by $\frac{1}{2}(r_{12} + r_{23} + r_{31})$ and make corresponding changes in the associated spin-dependent terms. This approximation allows simple analytic calculation of a number of matrix elements we will need. With this change, the Hamiltonian becomes

$$H = H_0 + V_{SD},$$

$$H_0 = \sum_i \sqrt{p_i^2 + m_i^2} + \frac{\sigma}{2}(r_{12} + r_{23} + r_{31}) - \sum_{i<j} \frac{2\alpha_s}{3r_{ij}},$$

$$V_{SD} = -\frac{1}{4m_1^2 r_{12}} S_1 \cdot (r_{12} \times p_1) - \frac{1}{4m_1^2 r_{13}} S_1 \cdot (r_{13} \times p_1)$$

$$+ \frac{1}{3m_1^2 r_{12}} S_1 \cdot \left[(r_{12} \times p_1) \frac{\alpha_s}{r_{12}^3} + (r_{13} \times p_1) \frac{\alpha_s}{r_{13}^3}\right]$$

$$- \frac{2}{3m_1 m_2 r_{12}^3} S_1 \cdot r_{12} \times p_2 - \frac{2}{3m_1 m_3 r_{13}^3} S_1 \cdot r_{13} \times p_3$$

$$+ \frac{1}{m_1 m_2} \frac{2\alpha_s}{3r_{12}^3} \left[\frac{3}{r_{12}^2}(S_1 \cdot r_{12})(S_2 \cdot r_{12}) - S_1 \cdot S_2\right]$$

$$+ \frac{1}{m_1 m_2} \frac{16\pi}{9} \alpha_s \delta^3(r_{12}) S_1 \cdot S_2 + \text{permutations} + \cdots,$$
with $S_i = \sigma_i / 2$ the spin operator for quark $i$. The masses which appear in the spin-dependent terms are to be interpreted as effective masses, with $1/m_i \simeq 1/E_i$, and are not necessarily equal to the masses with the same labels that appear in $E_i = \sqrt{p_i^2 + m_i^2}$. The delta function spin-spin interactions will not play a role in the following, and will be dropped. The terms hidden in the ellipsis are momentum-dependent interactions of $O(p^2/m^2)$ relative to the terms given explicitly. These lead only to small corrections to the already-negligible orbital contributions to the moments, and will be ignored. This model has been used in detailed calculations in [21] and [23] to obtain good fits to the baryon spectrum up to $\sim 3$ GeV.

To derive an expression for the baryon moments in the same approximation to quenched QCD, we redo the calculation of Brambilla et al. with the gauge interaction extended to include the electromagnetic vector potential $A_{\text{em}}(x_i) = B \times x_q / 2$ associated with a constant external magnetic field $B$. We can then pick out the modified magnetic moment operator through the relation

$$\Delta H = -\mu \cdot B.$$  

The result is

$$\mu = \sum_j \left( \mu_j^{(QM)} + \Delta \mu_j^{QM} \right) + \mu_L.  \tag{17}$$

Here $\mu_j^{(QM)}$ is the QM moment operator $\mu_j \sigma_j$, while $\Delta \mu_j^{QM}$ and $\mu_L$ are the leading corrections to the baryon moments associated with the binding interactions in $V_{SD}$ and the nonzero orbital angular momentum in the baryon, respectively. The latter two can be obtained directly by making the minimal substitution $p_i \rightarrow p_i - e_i A(x_i)$ in Eq. (12) and isolating the $B$-dependent terms. This gives

$$\Delta \mu_1^{QM} = \frac{\mu_1}{6 m_1 r_{12}^3} \left[ (x_1 \cdot r_{12}) \sigma_1 - (x_1 \cdot \sigma_1) r_{12} \right] + (2 \leftrightarrow 3)$$

$$+ \frac{\mu_2}{2 m_1 r_{12}^3} \left[ (x_2 \cdot r_{21}) \sigma_1 - (x_2 \cdot \sigma_1) r_{21} \right] + (2 \leftrightarrow 3)$$

$$- \frac{\mu_1}{4 m_1 r_{12}} \left[ (x_1 \cdot r_{12}) \sigma_1 - (x_1 \cdot \sigma_1) r_{12} \right] + (2 \leftrightarrow 3),  \tag{18}$$

where $\mu_i = e_i / 2m_i$ with $m_i$ an effective mass. The results for $\Delta \mu_2^{QM}$ and $\Delta \mu_3^{QM}$ which can be obtained by cyclic permutation of the indices. Finally,

$$\mu_L = \sum_j \mu_j x_j \times p_j = L_j  \tag{19}$$

where $L_j$ is the orbital angular momentum of the $j$-th quark.
As discussed in [19], $\Delta \mu_{QM}$ arises from a set of Thomas precession terms. The diagonal terms in the first and third lines of Eq. (18) are associated with the spin-same-orbit interaction in Eq. (15), and give a largely multiplicative correction to $\mu_i$. The two-body spin-other-orbit terms introduce new, nonadditive structure, and are important in improving the simple quark-model fits to the moments [16]. These terms are proportional to the short-distance spin-dependent part of the potential of order $\alpha_s$ divided by $E_i E_j$, so are expected to be small.

To obtain an approximate operator form for the QM moment $\mu_j$ to use in later calculations, we act on the electromagnetic interaction term $-e \alpha_i \cdot A(x_i)$ with the Foldy-Wouthuysen transformation which reduces the operator Dirac Hamiltonian $(\alpha \cdot p + \beta m)$ for the free motion of a quark to the two-component form $H = \beta E$ [22], with $E = \sqrt{p^2 + m^2}$ the kinetic energy operator in Eq. (14). This leads to $O(e)$ to a two-component spin-dependent interaction

$$-\frac{e_j}{\sqrt{p_j^2 + m_j^2}} \sigma \cdot B - \frac{e_j}{4\sqrt{p_j^2 + m_j^2} (\sqrt{p_j^2 + m_j^2} + m_j)} \sigma \cdot (B \times p) \times p. \quad (20)$$

The first term can be identified with the QM moment interaction $-\mu_j \sigma \cdot B$ and gives

$$\mu_j = \langle e_j/(2E_j) \rangle \quad (21)$$

when averaged over the momentum distribution in a baryon. We will adopt this form in later calculations to obtain reasonable estimates of the effect of the different binding of the quarks in different baryons on the $\mu_j$, but will otherwise treat the $\mu_j$ as parameters. In this treatment, the second term in Eq. (20), which behaves similarly as far as the averages are concerned, gives a single-particle contribution to the baryon moments which is absorbed in the adjustment of the parameters $\mu_j \simeq e_j/2m_j$ in terms of the effective masses.

III. BARYON WAVE FUNCTIONS

A. Spin structure of the baryon wave functions

To estimate the dynamical corrections to the baryon moments, we need approximate wave functions for the Hamiltonian in Eq. (12). We have found it very useful to construct the wave functions for the $j = \frac{1}{2}^+$ baryons in a form that allows us to use trace methods to calculate
the many spin matrix elements which appear in the expressions for the magnetic moments. This construction also has the advantage of displaying clearly the symmetry properties and the internal angular momentum content of the wave functions. The basic idea is simple. We start with the \( j = \frac{1}{2} \) spin wave functions for three quarks in the quark-model ground state, and construct all further wave functions by the application of even-parity scalar operators with the correct symmetry to give allowed states. Because the total angular momentum operator \( J \) commutes with scalar operators, the new wave functions will retain the original eigenvalues \( j(j + 1) \), \( j_3 \) of \( J^2 \) and \( J_3 \). Our starting spin wave functions are of the familiar form

\[
\chi_{S_{12}=1}^{1,2} = \frac{1}{\sqrt{6}} \left[ (\uparrow\downarrow + \downarrow\uparrow) \uparrow - 2(\uparrow\uparrow) \downarrow \right], \\
\chi_{S_{12}=0}^{1,2} = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \uparrow, \\
\chi_{S_{12}=0}^{1,2} = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \downarrow,
\]

(22)

where we will label the quarks 1, 2, 3 in order. The spin functions for \( S_{12} = 0, 1 \) are connected by the operator \((\sigma_1 - \sigma_2) \cdot \sigma_3\) which changes the symmetry in spins 1 and 2,

\[
\chi_{S_{12}=0}^{1,2} = \frac{1}{2\sqrt{3}} (\sigma_1 - \sigma_2) \cdot \sigma_3 \chi_{S_{12}=1}^{1,2} \equiv \tilde{\phi} \chi_{S_{12}=1}^{1,2}, \\
\chi_{S_{12}=1}^{1,2} = \frac{1}{\sqrt{2}} (\sigma_1 - \sigma_2) \cdot \sigma_3 \chi_{S_{12}=0}^{1,2},
\]

(23)

as is easily checked by direct calculation.

The functions \( \chi_{S_{12}=0,1}^{1,2} \) give a complete basis for the \( j = \frac{1}{2} \) spin states for three quarks. A general \( j = \frac{1}{2} \) wave function can therefore be written in the form

\[
\psi_{1/2,j_3} = |1/2, j_3\rangle = \phi_1 \chi_{S_{12}=1}^{1,2}_{1/2,j_3} + \phi_0 \chi_{S_{12}=0}^{1,2}_{1/2,j_3} \\
= \left[ \phi_1 + \phi_0 \frac{1}{2\sqrt{3}} (\sigma_1 - \sigma_2) \cdot \sigma_3 \right] \chi_{S_{12}=1}^{1,2}_{1/2,j_3} \equiv \tilde{\phi} \chi_{S_{12}=1}^{1,2}_{1/2,j_3}
\]

(24)

where \( \phi_1 \) and \( \phi_2 \) are scalar operators symmetric in \( \sigma_1 \) and \( \sigma_2 \). We will construct the full operator \( \tilde{\phi} \) directly; the part antisymmetric in \( \sigma_1 \) and \( \sigma_2 \) will generate the \( S_{12} = 0 \) component of \( \psi \) when acting on the symmetrical \( S_{12} = 1 \) spin function.

We will concentrate initially on baryons containing two like quarks, taken as 1 and 2 in our labeling. The singlet color wave function of the baryon is antisymmetric in the interchange
of any two labels, so the space-spin wave function and, therefore, the operator $\tilde{\phi}$ must be symmetric under the interchange of 1 and 2. No symmetry requirement exists with respect to the third quark. We will work in the center-of-mass system of the three quarks, and use the Jacobi coordinates $\rho$ and $\lambda$, defined respectively as the separation of quarks 1 and 2, and the separation of quark 3 from the center-of-mass of 1 and 2, to describe the internal configuration of the baryon. It is also useful to introduce Jacobi coordinates in which the pairs 2,3 and 3,1 are singled out, with

$$
\rho = r_{12}, \quad \lambda = r_{12,3}, \\
\rho' = r_{23}, \quad \lambda' = r_{23,1}, \\
\rho'' = r_{31}, \quad \lambda'' = r_{31,2},
$$

where $r_{ij} = z_i - z_j$, $r_{ij,k} = R_{ij} - z_k$, and $R_{ij} = (m_i z_i + m_j z_j)/m_{ij}$. The relations among these coordinates and among the corresponding momenta are given in Appendix A. We note here that $\rho$ and $\lambda$ are, respectively, odd and even under the interchange of 1 and 2.

It will be useful before determining $\tilde{\phi}$ to determine the possible orbital angular momentum configurations for a spin $j = \frac{1}{2}^+$ baryon. We introduce two angular momenta, $L_\rho$ between quarks 1 and 2, and $L_\lambda$ between quark 3 and the center of mass of 1 and 2, and combine these to produce the total orbital angular momentum $L$. This must combine in turn with the total quark spin $S = \frac{1}{2}, \frac{3}{2}$ to give $j = \frac{1}{2}^+$. Clearly, $L \leq 2$, while $L_\rho$ and $L_\lambda$ must both be even or both odd to give a positive parity state. If we restrict our attention to states with $L_\rho$, $L_\lambda$, $L_\rho + L_\lambda \leq 2$ as well, the complete wave function can be constructed using the configurations

$$
(L_\rho, L_\lambda, L) = \begin{cases} 
(0, 0, 0) & (1, 1, 0) & (1, 1, 1) \\
(1, 1, 2) & (2, 0, 2) & (0, 2, 2)
\end{cases}
$$

Each unit of $L_\rho$ ($L_\lambda$) in the wave function corresponds to one factor of $\rho$ ($\lambda$) in the corresponding angular momentum tensor, and a symmetry factor -1 (+1) for an interchange of quarks 1 and 2. The required symmetry of the wave function under quark exchange then limits $S_{12}$ to the values $S_{12} = 1$ ($S_{12} = 0$) for $L_\rho$ even (odd) for the baryons with quarks 1 and 2 identical. This eliminates the configuration (1,1,2), $S_{12} = 0$ since this combination of angular momenta gives $j = 3/2, 5/2$ only.

It is now straightforward to construct the operator $\tilde{\phi}$ in Eq. (24) using the Pauli spin
matrices $\mathbf{\sigma}$, the coordinate vectors $\mathbf{\rho}$, $\mathbf{\lambda}$, and the tensor $t_{ij}$,

$$t_{i,j}(\mathbf{x}) = 3 \mathbf{\sigma}_i \cdot \mathbf{x} \mathbf{\sigma}_j \cdot \mathbf{x} - \mathbf{\sigma}_i \cdot \mathbf{\sigma}_j \mathbf{x}^2.$$  \hspace{1cm} (27)

The possible components are given in the labelling $(L_\rho, L_\lambda, L)_{S_{12}}$ by

\begin{align*}
(0,0,0)1 & : \quad 1 \\
(1,1,0) & : \quad (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \cdot \mathbf{\sigma}_3 \mathbf{\rho} \cdot \mathbf{\lambda} \\
(1,1,1)0 & : \quad (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \cdot \mathbf{\rho} \times \mathbf{\lambda}, \quad (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \times \mathbf{\sigma}_3 \cdot (\mathbf{\rho} \times \mathbf{\lambda}) \\
(2,0,2)1 & : \quad t_{12}(\mathbf{\rho}), \quad t_{23}(\mathbf{\rho}) + t_{31}(\mathbf{\rho}) \\
(0,2,2)1 & : \quad t_{12}(\mathbf{\lambda}), \quad t_{23}(\mathbf{\lambda}) + t_{31}(\mathbf{\lambda})
\end{align*}

(28) - (32)

Two operators are shown in the rows for $L = 1, 2$. However, it is clear that the two cannot be independent for the $j = \frac{1}{2}^+$ baryons because there is only one way to reach $j = \frac{1}{2}$ by combining the indicated values of $L$, $S_{12}$, and $s_3 = \frac{1}{2}$. We show directly in Appendix B that the second operators are not independent of the first for $j = \frac{1}{2}$ (this is not the case for $L = 2, j = \frac{3}{2}$). We will therefore write $\tilde{\phi}$ in terms of the first operator in each row, with

$$\tilde{\phi} = a + ib (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \cdot \mathbf{\rho} \times \mathbf{\lambda} + ct_{12}(\mathbf{\rho}) + dt_{12}(\mathbf{\lambda}) + e (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \cdot \mathbf{\sigma}_3 \mathbf{\rho} \cdot \mathbf{\lambda},$$  \hspace{1cm} (33)

where $a, \ldots, e$ are functions of $\rho^2$ and $\lambda^2$ only \[^{27}\] In the general case, the coefficient functions can also depend on $(\mathbf{\rho} \cdot \mathbf{\lambda})^2$, but this requires the introduction of values of $L_\rho = L_\lambda \geq 2$.

The analysis above holds for quarks 1 and 2 identical, that is, for the $p$, $n$, $\Sigma^+$, $\Xi^0$, and $\Xi^-$ baryons. The $\Sigma^0$ and $\Lambda$ are both $uds$ systems, and are distinguished by the interchange symmetry of the $ud$ pair. This is even for the $\Sigma^0$, so the $\Sigma^0$ wave function has the structure given in Eq. (33). The $\Lambda$ is different, with odd $ud$ interchange symmetry. An analysis similar to that above gives the spin dependence of the $\Lambda$ wave function as $\psi^\Lambda_{\frac{1}{2}, j_3} = \tilde{\phi}_\Lambda^{S_{12} = 1}_{\frac{1}{2}, j}$, with

$$\tilde{\phi}_\Lambda = a_\Lambda \mathbf{\sigma}_3 \cdot (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) + b_\Lambda \mathbf{\rho} \cdot \mathbf{\lambda} + c_\Lambda i \mathbf{\sigma}_3 \cdot \mathbf{\rho} \times \mathbf{\lambda} + d_\Lambda \mathbf{\rho} \times \mathbf{\lambda} \cdot \mathbf{\sigma}_3 \times (\mathbf{\sigma}_1 + \mathbf{\sigma}_2) + e_\Lambda t_{12}(\mathbf{\rho}, \mathbf{\lambda}),$$  \hspace{1cm} (34)

where

$$t_{12}(\mathbf{\rho}, \mathbf{\lambda}) = \frac{1}{2} [3 \mathbf{\sigma}_1 \cdot \mathbf{\rho} \mathbf{\sigma}_2 \cdot \mathbf{\lambda} - \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2 \mathbf{\rho} \cdot \mathbf{\lambda} + (1 \leftrightarrow 2)].$$  \hspace{1cm} (35)
B. Trace methods

Since the total angular momentum operator $J$ commutes with the Hamiltonian $H$, the matrix elements of $H$ satisfy the relations

\[
\langle j', j'_3 | H | j, j_3 \rangle = \langle j', j'_3 | H | j, j_3 \rangle \delta_{j'j} \delta_{j'_3j_3}, \tag{36}
\]

\[
\langle j, j'_3 | H | j, j_3 \rangle = \langle j, j_3 | H | j, j_3 \rangle, \tag{37}
\]

so, for $j = \frac{1}{2}$, $j_3 = \pm \frac{1}{2}$

\[
\langle \frac{1}{2}, j_3 | H | \frac{1}{2}, j_3 \rangle = \frac{1}{2} \left[ \langle \frac{1}{2}, \frac{1}{2} | H | \frac{1}{2}, \frac{1}{2} \rangle + \langle \frac{1}{2}, -\frac{1}{2} | H | \frac{1}{2}, -\frac{1}{2} \rangle \right]
= \frac{1}{2} \text{Tr} \frac{1}{2} H. \tag{38}
\]

We can use the projection operators to get rid of the restriction to $j = \frac{1}{2}$ in the trace. Formally,

\[
P_{S, S'_{12}} = \sum_{j_3} |S_{12}; s, j_3 \rangle \langle S_{12}; s, j_3|.
\tag{39}
\]

Now $P_{1/2, 1} \chi_{j_3=1}^{S_{12}=1} = \chi_{j_3=1}^{S_{12}=1}$, while $P_{1/2, 1}$ annihilates $\psi_{3/2, j_3}$, so by introducing the projection operator

\[
P_{1/2, 1} = \frac{1}{12} [3 + \sigma_1 \cdot \sigma_2 - 2 \sigma_1 \cdot \sigma_3 - 2 \sigma_2 \cdot \sigma_3] \tag{40}
\]

in matrix elements, we can use a basis of independent states $\uparrow, \downarrow$ for quarks 1, 2, and 3 and sum over the spin indices without restriction. This gives

\[
\langle \psi_{1/2, j_3} | H | \phi_{1/2, j_3} \rangle = \frac{1}{2} \sum_{j_3=\pm \frac{1}{2}} \chi_{1/2, j_3}^{S_{12}=1} \tilde{\psi} \bar{H} \phi \chi_{1/2, j_3}^{S_{12}=1}
= \frac{1}{2} \sum_{\lambda_1, \lambda_2, \lambda_3=\pm \frac{1}{2}} \tilde{\psi} \bar{H} \phi P_{1/2, 1}
= \frac{1}{2} \text{Tr} \tilde{\psi} \bar{H} \phi P_{1/2, 1}, \tag{41}
\]

where Tr $= \text{Tr}_1 \text{Tr}_2 \text{Tr}_3$ and $\lambda_i$ is the spin projection of the $i$-th quark ($i = 1, 2, 3$). Note that

\[
\text{Tr} \mathbb{1} = 8,
\frac{1}{2} \text{Tr} P_{1/2, 1} = 1, \tag{42}
\text{Tr} \sigma_1 = \text{Tr} \sigma_2 = \text{Tr} \sigma_3 = 0.
\]
C. Approximate wavefunctions

To get approximate wave functions to use in our calculations of the binding corrections to the moments, we have made variational calculations of the energies and approximate wave functions of the ground state baryons and their first excited states using the Hamiltonian in Eqs. (12) and (15). Our calculation and results are summarized below. Some details are given in Appendix A.

We emphasize that we are not trying to make detailed, accurate calculations, but rather to obtain crude wave functions which allow us to estimate the small binding corrections. More accurate calculations would probably only be useful in conjunction with further improvements in the underlying theory.

We rewrite the baryon wave function in Eq. (33) as

\[ \psi_{1/2,m} = (\psi_0^a + b_0 \hat{\psi}_b + c_0 \hat{\psi}_c + d_0 \hat{\psi}_d + e_0 \hat{\psi}_e) \chi^{S_{12}=1}_{1/2,m}, \]  

where \( \psi_0^a \) is a normalized wave function of the ground state and \( \hat{\psi}_b, \hat{\psi}_c, \hat{\psi}_d, \hat{\psi}_e \) are the normalized wave functions of the excited states. Now if we choose \( \psi_0^a, \cdots, \psi_0^e \) as the Gaussians [28],

\[ \psi_A^A = \left( \frac{\beta_A^A \beta_\lambda^A}{\pi} \right)^{3/2} \exp \left[ - (\beta_\rho^A \rho^2 + \beta_\lambda^A \lambda^2) \right], \quad A = a, \cdots, e, \]  

then the normalization coefficients \( N_b, \cdots, N_e \) are

\[ N_b = \frac{\beta_b^b \beta_\lambda^b}{\sqrt{2}}, \quad N_c = \frac{\beta_c^c}{\sqrt{30}}, \quad N_d = \frac{\beta_d^d}{\sqrt{30}}, \quad N_e = \frac{\beta_e^e \beta_\lambda^e}{3}. \]  

The coefficients \( b_0, \cdots, e_0 \) can be evaluated using perturbation theory in which the excited states are considered as the perturbation states. In such a way, we have

\[ A_0 = \frac{\langle \hat{\psi}_A | V_{SD} | \psi_0^A \rangle}{E_0 - E_A}, \quad A = b, \cdots, e, \]  

16
where $V_{SD}$ is the spin-dependent potential in Eq. (13) with the spin-spin and higher order terms omitted, and
\[ E_A = \langle \hat{\psi}_A | H_0 | \hat{\psi}_A \rangle, \quad E_0 = \langle \psi_0^0 | H_0 | \psi_0^0 \rangle. \] (48)

$H_0$ is given in Eq. (14).

To evaluate the coefficients $b_0, \ldots, e_0$ numerically, we first minimize the energies $E_A$ and $E_0$ by varying the parameters $\beta_\rho, \beta_\lambda$, and thus obtain the best variational wave functions $\psi_0^0, \hat{\psi}_b, \ldots, \hat{\psi}_e$ of Gaussian form. The calculation uses $\alpha_s = 0.39, \sigma = 0.18 \text{GeV}^2, m_u = m_d = 0.200 \text{GeV},$ and $m_s = 0.500 \text{GeV}$. The resulting values of $\beta_\rho$ and $\beta_\lambda$ are given in Table III. These values differ for different baryons, a point which will be important later. The values of excitation energies $E_A - E_0$ obtained this way and shown in Table II are generally consistent with those found in the more elaborate calculations [23] [29].

These results together with those from our calculation of the matrix elements $\langle \hat{\psi}_A | V_{SD} | \psi_0^0 \rangle$ between the ground state and lowest excited states $b, \ldots, e$ allow us to estimate the coefficients $b_0, \ldots, e_0$. We list the values found in Table IV.

**TABLE II:** The energy differences $\Delta E_A$ in MeV between the baryon ground states $\psi_0$ and the excited states $\hat{\psi}_A, A = b, \ldots, e$ in Eq. (45).

| Baryon | $\Delta E_b$ | $\Delta E_c$ | $\Delta E_d$ | $\Delta E_e$ |
|--------|-------------|-------------|-------------|-------------|
| $N$    | 870         | 821         | 820         | 789         |
| $\Sigma$ | 841        | 837         | 750         | 755         |
| $\Xi$  | 810         | 714         | 806         | 738         |
| $\Omega$ | 779        | 732         | 730         | 701         |

While the Hamiltonian does mix orbitally excited states into the $L_\mu = L_\lambda = L = 0$ quark-model ground state, the coefficients are very small, ranging from essentially zero to about 0.02 depending on the baryon. Because these coefficients only appear quadratically in the baryon moments, the orbital contributions to the moments will be completely negligible. This result is consistent with the Lichtenberg’s finding for the nonrelativistic QM [24].
TABLE III: Values of the parameters \(\beta_\rho\) and \(\beta_\lambda\) in the Gaussian approximation to the ground state wave functions of the octet baryons, Eq. (45), calculated for \(m_u = m_d = 0.200\) GeV, \(m_s = 0.500\) GeV, \(\alpha_s = 0.39\), and \(\sigma = 0.18\) GeV\(^2\).

| Baryon | \(N\) | \(\Sigma\) | \(\Xi\) | \(\Omega\) |
|--------|-------|----------|--------|--------|
| \(\beta_\rho\) | 0.340 | 0.347 | 0.387 | 0.394 |
| \(\beta_\lambda\) | 0.393 | 0.424 | 0.420 | 0.455 |

TABLE IV: The coefficients \(b_0, \ldots, e_0\) are evaluated using perturbation theory for \(\alpha_s = 0.39\), \(\sigma = 0.18\) GeV\(^2\), \(m_u = m_d = 0.343\) GeV, and \(m_s = 0.539\) GeV.

| Baryon | \(b_0\) | \(c_0\) | \(d_0\) | \(e_0\) |
|--------|-------|--------|--------|--------|
| \(N\) | 0 | -0.013 | 0.013 | -0.021 |
| \(\Sigma\) | 0.012 | -0.013 | 0.011 | -0.016 |
| \(\Xi\) | -0.006 | -0.009 | 0.011 | -0.020 |
| \(\Omega\) | 0 | -0.009 | 0.010 | -0.016 |

IV. ANALYSIS OF THE BARYON MOMENTS

A. The leading approximation to the moments

The leading approximation to the QM moments in Eq. (21) gives \(\mu_j = \langle e_j/E_j \rangle = \langle e_j/(2\sqrt{p^2 + m_j^2}) \rangle\). The matrix elements \(\langle 1/E_j \rangle\), customarily written in terms of an effective mass as \(e_j/m_j\), will actually differ for a given quark depending on the baryon in which it is confined. To estimate this effect, we have calculated the relevant matrix elements using the Gaussian variational wave functions and the methods sketched in Appendix C. The calculation was done using \(m_u = m_d = 0.200\) GeV and \(m_s = 0.500\) GeV in \(H_0\), values which give both a good ground-state baryon spectrum and reasonable values for the moments. We then estimate the corrections to \(\mu_u\) by taking the ratios of the calculated matrix elements in the \(\Sigma^+, \Lambda, \text{ and } \Xi^0\) to the matrix element in the proton, and multiplying by the actual value \(\mu_u = 2\mu_a/3\) determined from the data. The correction to \(\mu_s\) in the \(\Xi\) states is determined similarly using the \(\Sigma\) as the reference state. The corrections are small, with, for example, \(\mu_u\) decreasing by 0.046 \(\mu_N\) in the \(\Sigma\) and 0.088 \(\mu_N\) in the \(\Xi\) relative to its value in \(N\).
The overall corrections to the parameters $\mu_a, \ldots, \mu_{MM}$ are given in Table VI. The largest correction is to $\mu_d$. This arises, as suggested by Fig. 1(d), from the change in the moment of one quark, relative to the values for the $m_i$ given above, which arises from a change in the wave function when a second quark is strange.

The nonzero value of $\mu_{MM}$ found in this calculation corresponds to the introduction of the new structure

\[
m_{B,MM} = \sum_{i \neq j \neq k} [Q_i M_j M_k - \mathbb{1}_i (QM)_j M_k] \sigma
\]

at the baryon level, where $\sigma$ is twice the baryon spin operator. This is a mixed two- and three-body operator. The operator coefficient of $\sigma$ vanishes for all states which do not contain two strange quarks, and has the value 1 for the $\Xi^0$ and 0 for the $\Xi^-$. There are no further independent structures for the octet baryons [30].

B. Binding corrections

Without the corrections from the internal orbital angular moments, the moment of a baryon $B$ is given by

\[
\mu_B = \sum_j (\mu_j + \Delta \mu_j^B)(\sigma_{j,z})_B = \mu^M_B + \sum_j \Delta \mu_j^B(\sigma_{j,z})_B,
\]

where the sum is over the quarks in the baryon and we have quantized along $B$, taken along the $z$ axis. The spin expectation values are to be calculated in the baryon ground state. The correction $\Delta \mu_j^B$ to the moment of quark $j$ depends on the baryon $B$ in which it appears. The final baryon moments depart from the quark model pattern only when the ratios $\Delta \mu_j^B / \mu_j$ differ in different baryons.

The general result for the moment operator given above can be simplified considerably for the $L = 0$ ground state baryons. The absence of any Pauli matrices in the wave functions allows us to reduce the operators in $\Delta \mu_i^QM$ in Eq. (18) to the components proportional to $\sigma_i$ in calculating the ground-state matrix elements,

\[
\langle \Delta \mu_i^QM \rangle_B = \langle \Delta \mu_i^QM \sigma_i \rangle_B, \quad \Delta \mu_i^QM = \frac{1}{6} \text{Tr} \sigma_i \cdot \Delta \mu_i^QM,
\]

and the contribution to the quark moment $\mu_i$ in baryon $B$ is just $\Delta \mu_i^B = \langle \Delta \mu_i^QM \rangle_B$. We will write $\Delta \mu_i^B$ as the sum of terms arising from diagonal and Thomas-type corrections,

\[
\Delta \mu_i^B = \Delta \mu_i^D + \Delta \mu_i^T.
\]
These are given by
\[
\Delta \mu_i^D = \frac{\mu_i}{2m_i} \sum_{j \neq i} (\epsilon_{ij} - \Sigma_{ij}), \quad \Delta \mu_i^T = \frac{\mu_i}{e_i} \sum_{j \neq i} \frac{e_j - \epsilon_{ji}}{m_j},
\]
where the $\epsilon$’s and $\Sigma$’s are the ground state matrix elements
\[
\epsilon_{ij} = \frac{2\alpha_s}{9} \langle \frac{r_{ij} \cdot x_j}{r_{ij}^3} \rangle_B, \quad \Sigma_{ij} = \frac{\sigma}{6} \langle \frac{r_{ij} \cdot x_i}{r_{ij}^3} \rangle_B.
\]

Two quarks always have the same mass in each octet or decuplet baryon, so appear symmetrically in the spatial part of a flavor-independent wave function. We will label these quarks 1 and 2, let $m_1 = m_2 = m$, and take 3 as the odd-mass quark if there is one. With this labeling, the spatial matrix elements can be reduced to the small set
\[
\epsilon = \epsilon_{12} = \epsilon_{21}, \quad \epsilon' = \epsilon_{13} = \epsilon_{23}, \quad \bar{\epsilon} = \epsilon_{31} = \epsilon_{32},
\]
\[
\Sigma = \Sigma_{12} = \Sigma_{21}, \quad \Sigma' = \Sigma_{13} = \Sigma_{23}, \quad \bar{\Sigma} = \Sigma_{31} = \Sigma_{32},
\]
and the diagonal correction terms become
\[
\Delta \mu_i^D = \frac{\mu_i}{2m} (\epsilon + \epsilon' - \Sigma - \Sigma'),
\]
\[
\Delta \mu_i^T = \frac{\mu_i}{2m} (\epsilon + \epsilon' - \Sigma - \Sigma'),
\]
\[
\Delta \mu_i^D = \frac{\mu_i}{m_3} (\bar{\epsilon} - \bar{\Sigma}).
\]

The Thomas-type corrections are somewhat more complicated,
\[
\Delta \mu_1^T = \frac{\mu_1}{e_1} \left( \frac{\epsilon_2}{m_2} + \frac{\epsilon_3}{m_3} \bar{\epsilon} \right),
\]
\[
\Delta \mu_2^T = \frac{\mu_2}{e_2} \left( \frac{\epsilon_1}{m_1} + \frac{\epsilon_3}{m_3} \bar{\epsilon} \right),
\]
\[
\Delta \mu_3^T = \frac{\mu_3}{e_3} \frac{\epsilon_1 + \epsilon_2}{m} \epsilon'.
\]

Note that, for all masses equal, $\epsilon = \epsilon' = \bar{\epsilon}$, and $\Sigma = \Sigma' = \bar{\Sigma}$. The matrix elements differ only because of effects of the odd-quark mass on the wave functions.

We have evaluated the radial matrix elements above using the Gaussian wave functions obtained in our variational calculation of the ground state energies for the Hamiltonian in Eq. (12) and the methods sketched in Appendix C. The results are given in Table V for $\alpha_s = 0.39$ and $\sigma = 0.18$ GeV$^2$, values taken from fits to the baryon spectrum [21, 23] using the same Hamiltonian. As is clear from Table V the $\epsilon$’s and the $\Sigma$’s are all similar in
TABLE V: The values in GeV of the matrix elements $\epsilon$ and $\Sigma$ defined in Eqs. (55) and (56). The matrix elements were evaluated for $\alpha_s = 0.39$, $\sigma = 0.183$ GeV$^2$, $m_u = m_d = 0.200$ GeV, and $m_s = 0.500$ GeV for the quark moments $\mu_u = 1.943$, $\mu_s = -0.640$ obtained from a fit to the data.

| Baryon | $\epsilon$ | $\epsilon'$ | $\tilde{\epsilon}$ | $\Sigma$ | $\Sigma'$ | $\tilde{\Sigma}$ |
|--------|-------------|--------------|--------------------|-----------|-----------|-----------------|
| $N$    | 0.017       | 0.017        | 0.017              | 0.050     | 0.050     | 0.050           |
| $\Sigma$ | 0.017       | 0.024        | 0.011              | 0.049     | 0.064     | 0.031           |
| $\Xi$  | 0.019       | 0.013        | 0.023              | 0.044     | 0.033     | 0.060           |
| $\Omega$ | 0.019       | 0.019        | 0.019              | 0.044     | 0.044     | 0.044           |

magnitude even for $m_1 \neq m_3$, so the main effect of the diagonal corrections $\Delta \mu_i^D$ in Eq. (57) is a uniform shift in the input values of the $\mu_i$'s. This is absorbed when $\mu_u = -2\mu_d$ and $\mu_s$ are used as parameters in fitting the observed moments. The only nontrivial two- or three-body effect of these terms arises from the differences between the matrix elements in the different baryons. In contrast, the Thomas-type corrections $\Delta \mu_i^T$ in Eq. (58) are mainly two-body contributions and remain nontrivial even when the $\epsilon$'s are similar because of the different charges of the different quarks.

In Table VI we shown the values of the parameters $\mu_a - \mu_g$, $\mu_{MM}$ for the diagonal and Thomas-type corrections obtained using the input moments $\mu_u = 1.94 \mu_N$ and $\mu_s = -0.64 \mu_N$ found in a fit to the data which includes these and the other corrections. The effective masses used in Eqs. (57) and (58) were defined in terms of the input moments since the original coefficients in Eq. (58) are of the form $e_i/m_i^2$ or $e_i/m_i m_j$. The smallness of the parameters $\mu_c - \mu_g$ for the diagonal corrections compared to the values of $\mu_a$ and $\mu_b$ shows that the $\Delta \mu_i^D$ are indeed mostly contributions from the one-body operators. In contrast, the $\Delta \mu_i^T$ contribute mostly to the two-body coefficients $\mu_c - \mu_g$ with the largest contribution for the Thomas terms $\mu_c$ and $\mu_f$, Fig. 4. The value of $\mu_g$, which is associated with the three-body operators, is very small in both cases. Finally, the nonzero values of $\mu_{MM}$ obtained in both cases arise indirectly from the dependence of the matrix elements in Eqs. (55) and (56) on the strange-quark content of the state.
TABLE VI: The parameters $\mu_a - \mu_g$ and $\mu_{MM}$ for the estimated binding corrections to the moments, the diagonal and Thomas-type corrections defined in Eqs. (57) and (58), and for the meson loop corrections. The last lines give the total correction and the values of the parameters obtained in a least squares fit to the measured moments, and the differences with the experimental uncertainties.

| Type             | $\mu_a$ | $\mu_b$ | $\mu_c$ | $\mu_d$ | $\mu_e$ | $\mu_f$ | $\mu_g$ | $\mu_{MM}$ |
|------------------|---------|---------|---------|---------|---------|---------|---------|-------------|
| $\Delta \mu_0$   | 0       | 0.011   | 0       | -0.067  | 0.011   | 0.003   | 0.003   | -0.019      |
| $\Delta \mu_D$   | -0.306  | 0.231   | 0.000   | -0.023  | 0.001   | 0.000   | -0.008  | -0.012      |
| $\Delta \mu_T$   | 0       | 0.000   | 0.151   | 0.000   | -0.006  | -0.082  | 0.004   | -0.013      |
| meson loops      | 0.123   | -0.208  | -0.339  | -0.022  | 0.143   | 0.471   | 0.185   | 0.010       |
| $\Delta \mu_{total}$ | -0.183 | 0.034   | -0.188  | -0.112  | 0.149   | 0.392   | 0.184   | -0.034      |
| Fit to data      | 2.793   | -0.933  | -0.077  | -0.037  | 0.098   | 0.438   | 0.044   | 221         |
|                  | ±0.012  | ±0.070  | ±0.010  | ±0.036  | ±0.070  | ±0.140  |        |             |
| Fit – $\Delta \mu_{total}$ | 2.976   | -0.899  | 0.111   | -0.075  | -0.051  | 0.046   | -0.100  | 0.255       |
|                  | ±0.012  | ±0.070  | ±0.010  | ±0.036  | ±0.070  | ±0.140  |        |             |

C. Contributions of internal orbital angular momenta

As already remarked, the contributions of orbital angular momenta to the moments are negligible for the ground-state octet baryons. We will illustrate this in the special case in which $m_1 = m_2$ and $\mu_1 = \mu_2 = m$. $\mu_L$ is given in this case by

$$\mu_L = \mu_1 L_\rho + \frac{1}{M} (m_3 \mu_1 + 2m_3 \mu_3) L_\lambda,$$

(59)

where $L_\rho = \rho \times p_\rho$, $L_\lambda = \lambda \times p_\lambda$, and $M = 2m_1 + m_3$.

Using the wave function given in Eq. (43) and choosing the Gaussian spatial wave functions discussed above, we calculate $\langle \mu \rangle$ and obtain for the moment of a baryon $B$ of this type

$$\mu_B = \frac{4}{3} (\mu_1 + \Delta \mu_1^L) - \frac{1}{3} (\mu_3 + \Delta \mu_3^L),$$

(60)

$\Delta \mu_1^L$ and $\Delta \mu_3^L$ are the contributions from $L \neq 0$ configurations in the baryon wave function.
(Eq. (43)),
\[
\frac{\Delta \mu^L_3}{\mu_3} = \frac{m_3}{M} b_0^2 + c_0^2 + \frac{m_3 - 4m_1}{M} d_0^2 - 3e_0^2 - \frac{m_1}{M} b_0 e_0 R, \\
\frac{\Delta \mu^L_1}{\mu_1} = \frac{1}{4} (\frac{\Delta \mu^L_3}{\mu_3} + b_0^2 + 3e_0^2),
\]
with \( R \) an overlap integral of order unity. The results are quadratic in the small coefficients \( b_0, \ldots, e_0 \) in Table IV and are negligible on the scale of the other contributions.

D. Meson loop corrections

As noted earlier, the meson loop corrections to the baryon moments have been considered by many authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] in the context of chiral perturbation theory. We will use the loop corrections obtained in [16]. In that paper, the one-loop corrections were calculated using a meson-nucleon form factor to control the high-energy behavior of the loop integrals, which is not reliable in the context of a strict expansion in the chiral parameter \( m_s \). The results we obtained after correcting an error in [16] are given in Table VI as decomposed in terms of the parameters \( \mu_a, \ldots, \mu_g \) defined in Eq. (7). Their connection with the baryon moments is given in Table I. More detail on the calculations is given in [16].

The meson-loop corrections are distributed over all the parameters \( \mu_a - \mu_g \), corresponding to contributions from the one-, two-, and three- body operators. The largest parameters, \( \mu_c = -0.339 \) and \( \mu_f = 0.471 \), arise from the two-body Thomas-type contributions in Figs. 1c and 1f, the second a strange-mass correction to the first. There is also a significant three-body contribution to the moments through \( \mu_g \), Fig. 2g.

E. Analysis

The corrections to the moment parameters for the baryons discussed here are summarized in Table VII. The final rows in the table give the total calculated corrections to \( \mu_a, \ldots, \mu_g, \mu_{MM} \), the values of those parameters obtained by fitting the eight measured moments exactly, and their difference. The correspondence of the the individual baryon moments and the \( \Sigma^0 \Lambda \) transition moment to the parameters is given in Table I. The differences
TABLE VII: The fit to the baryon magnetic moments obtained using the calculated dynamical contributions to the chiral parameters $\mu_a, \ldots, \mu_{MM}$ given in Table VI, with $\mu_a$ and $\mu_s$ as free parameters.

| Baryon | $p$    | $n$    | $\Sigma^+$ | $\Sigma^-$ | $\Xi^0$  | $\Xi^-$  | $\Lambda$ | $\Sigma^0\Lambda$ |
|--------|--------|--------|-------------|------------|----------|----------|-----------|-------------------|
| Calculated | 2.744  | -1.955 | 2.461       | -1.069     | -1.278   | -0.598   | -0.607    | ±1.522           |
| Data   | 2.793  | -1.913 | 2.458       | -1.160     | -1.250   | -0.651   | -0.613    | ±1.610           |
|        | ±0.010 | ±0.025 | ±0.014      | ±0.003     | ±0.004   | ±0.080   |           |                   |

for $\mu_a$ and the combination $\mu_a + \mu_b$ give the values of $3\mu_u/2$ and $-3\mu_s$, the input quark moments given $\mu_d = -\mu_u/2$.

As was noted in [10], fits to the octet moments using $\mu_a$ and $\mu_s$, or equivalently, $\mu_a$ and $\mu_b$ as adjustable parameters give only slight improvements relative to the additive quark model when only the binding corrections, or only the meson loop corrections are included. Either set of corrections alone has an incorrect pattern of signs or magnitudes relative to the fitted parameters. However, when combined, there are significant cancellations, and the total result is much closer to experiment. This may be seen from Table VII where we give the best-fit results for the seven accurately known moments obtained using the calculated values of $\Delta \mu_{\text{total}}$, and allowing $\mu_a$ and $\mu_b$ to vary. The fit is quite good, with a mean deviation from experiment of 0.04 $\mu_N$ for $\mu_p, \ldots, \mu_\Lambda$. The poorly known $\Sigma^0\Lambda$ transition moment has very little weight in a complete fit, and is left as a prediction, $\mu_{\Sigma^0\Lambda} = 1.522 \mu_N$ compared to the measured value $1.61 \pm 0.08 \mu_N$.

The advantage of the decomposition in Table VI is the possible hints it provides for improving the theory. In particular, the most striking deviation of theory from experiment is in the value of the leading Thomas-type term $\mu_c$ corresponding to Fig. 1(c). This is well determined experimentally, with $\mu_n = -\frac{2}{3}(\mu_a - \mu_c)$. The QM ratio $\mu_n/\mu_p \approx -2/3$ is attained only for $\mu_c$ small, a result which is “accidental” in a general parametrization of the moments [17, 25]. Here $\mu_c$ is small because of a cancellation between the meson loop corrections at one loop and the Thomas-type correction $\Delta \mu^T$. Some further input will clearly be needed to obtain better agreement between theory and experiment.

The calculated value of the related coefficient $\mu_e$, is also significantly different from experiment. The uncertainties in $\mu_d$, $\mu_g$, and $\mu_{MM}$ are dominated by the relatively large
uncertainty in the $\Sigma^0\Lambda$ transition moment. The results nevertheless suggest that the three-body contribution $\mu_g$ may be overestimated, and the calculated two-strange quark term $\mu_{MM}$ is too small and of the wrong sign. We note that baryon mass insertions in the meson loop diagrams, so far not calculated, will affect these contributions [19].

V. SUMMARY

We have presented and analyzed the QCD corrections to the baryon magnetic moments in terms of the effective field theory description in terms of “quark” operators $m_a, \ldots, m_{MM}$ to be used in baryon matrix elements of the form $\mu_B = (B\mu B)$, with $B^{ij}_{\gamma}$ the effective field operator for the baryon in Eq. (1). We have then used the connection of this quark picture to dynamical models [7, 11, 18, 19] to estimate the input values of the unknown parameters in the chiral expansion using a semirelativistic dynamical model. This model can be derived in a quenched approximation to QCD [20] and gives a good description of the baryon spectrum up to $\sim 3$ GeV [21, 23]. This allows us to estimate the changes in the leading QM moments in different baryons and binding corrections to the moments identified earlier [15], and to study the contributions of orbital angular excitations.

We find that the binding corrections to the simple QM picture of the moments involve, as expected, mainly contributions from the one- and two-body operators, and are important. The changes in the QM moments in different baryons are significant but small. The orbital contributions to the moments are completely negligible since the terms in the wave functions which involve higher angular momenta are very small. The chiral meson loop contributions are large, but tend to cancel with the corresponding parameters for the binding corrections leading to a fairly successful overall fit to the moment data. Fits using the loop corrections alone are much less satisfactory. The deviations of the calculated parameters $m_a, \ldots, m_{MM}$ from those found in an exact fit to the baryons moments suggests where further input will be important.

We note finally that the procedure used in our construction of the $j = \frac{1}{2}^+$ baryon wave functions displays their symmetry properties and the internal angular momentum content very simply, and may be useful in other contexts as may the trace methods used to calculate the many spin matrix elements which appear in the expressions for the magnetic moments.
Acknowledgments

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APPENDIX A: JACOBI COORDINATES AND RELATIONS

With $i \neq j \neq k$ and $i$, $j$, and $k$ run from 1 to 3, we define the Jacobi-type coordinate systems as follows:

1. The space coordinates

$$r_{ij} = x_i - x_j, \quad R_{ij} = \frac{m_i x_i + m_j x_j}{m_{ij}},$$
$$r_{ij,k} = R_{ij} - x_k = \frac{m_i (x_i - x_k) + m_j (x_j - x_k)}{m_{ij}},$$
$$R_{ijk} = \frac{m_{ij} R_{ij} + m_k x_k}{M}. \quad (A1)$$

Here $m_{ij} = m_i + m_j$, $M = m_i + m_j + m_k$, and $R_{ijk}$ is the usual center-of-mass coordinate. With these definitions, the roles of $i$, $j$, and $k$ are completely symmetric.

Thus, if one defines coordinates $\rho = r_{12}$, $\lambda = r_{12,3}$, $\rho' = r_{23}$, $\lambda' = r_{23,1}$, $\rho'' = r_{31}$, and $\lambda'' = r_{31,2}$ then $\rho'$, $\lambda'$ and $\rho''$, $\lambda''$ can be expressed in terms of $\rho$ and $\lambda$ and conversely. For example, if $m_1 = m_2$, we find that

$$\rho' = \lambda - \frac{\rho}{2}, \quad \lambda' = -\frac{1}{m_{23}} \left( m_3 \lambda + \frac{M}{2} \rho \right), \quad (A2)$$
$$\rho'' = -\lambda - \frac{\rho}{2}, \quad \lambda'' = -\frac{1}{m_{31}} \left( m_3 \lambda - \frac{M}{2} \rho \right). \quad (A3)$$

One can therefore work with any of the pairs $\rho$, $\lambda$ or $\rho'$, $\lambda'$ or $\rho''$, $\lambda''$, and switch between them as necessary. The spatial volume element is simply $d^3R d^3\rho d^3\lambda$, or equivalently for the other pairs of internal coordinates.
2. The momentum coordinates

\[
p_{ij} = \frac{m_j p_i - m_i p_j}{m_{ij}}, \quad P_{ij} = p_i + p_j, \\
p_{ij,k} = \frac{m_k P_{ij} - m_{ij} p_k}{M}, \quad P_{ijk} = P_{ij} + p_k,
\]

where \( P_{ijk} \equiv P \) is the total momentum.

Now, if one denotes \( p_\rho = p_{12}, \quad p_\lambda = p_{123}, \quad p_{\rho'} = p_{23}, \quad p_{\lambda'} = p_{231}, \quad p_{\rho''} = p_{31}, \) and \( p_{\lambda''} = p_{312} \) then one can choose to work with either \( p_\rho, \ p_\lambda \) or \( p_{\rho'}, \ p_{\lambda'} \) or \( p_{\rho''}, \ p_{\lambda''} \) since there exist relations between them. For example, when \( m_1 = m_2 = m \) the coordinates \( p_{\rho'}, \ p_{\lambda'} \) and \( p_{\rho''}, \ p_{\lambda''} \) can be expresses in terms of \( p_\rho, \ p_\lambda \) as

\[
\frac{p_{\rho'}}{m_{\rho'}} = \frac{p_\lambda}{m_\lambda} - \frac{p_\rho}{2m_\rho}, \quad p_{\lambda'} = -p_\rho - \frac{p_\lambda}{2}, \\
\frac{p_{\rho''}}{m_{\rho''}} = -\frac{p_\lambda}{m_\lambda} - \frac{p_\rho}{2m_\rho}, \quad p_{\lambda''} = p_\rho - \frac{p_\lambda}{2},
\]

where \( m_\rho = m_1 m_2 / m_{12} = 2/m, \quad m_\lambda = m_3 m_{12} / m_{123}, \quad m_{\rho'} = m_2 m_3 / m_{23}, \) and \( m_{\rho''} = m_3 m_1 / m_{31} \). The volume element in momentum space is \( d^3 P d^3 p_\rho d^3 p_\lambda \), and equivalently for the other pairs of internal momenta.

**APPENDIX B: TRACE RELATIONS AND INDEPENDENT OPERATORS**

We present here some trace relations derived from the trace method given in subsection III B. These results are useful to an evaluation of the spin matrix elements. We also use them to prove the equivalence of some states shown in Eq.(28).
1. Results for spin traces

Let \( A, B, \) and \( C \) be any spin-independent vectors and \( x, y, \) and \( z \) be the coordinate-type vectors (such as \( \rho, \rho', \lambda, ... \)), we have

\[
\begin{align*}
\text{Tr} (\sigma_1 - \sigma_2) \cdot \sigma_3 \cdot \sigma_2 P_{\frac{1}{2},1} &= 0, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot \sigma_3 \cdot \sigma_2 \cdot \sigma_3 P_{\frac{1}{2},1} &= -12, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot A \cdot \sigma_j \cdot \sigma_2 P_{\frac{1}{2},1} &= 0, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot \sigma_3 \cdot \sigma_1 \cdot A P_{\frac{1}{2},1} &= 0, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot A (\sigma_1 - \sigma_2) \cdot B P_{\frac{1}{2},1} &= \frac{8}{3} A \cdot B, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot A (\sigma_1 - \sigma_2) \cdot B \cdot \sigma_3 \cdot C P_{\frac{1}{2},1} &= -\frac{8i}{3} (A \times B) \cdot C, \\
\text{Tr} (\sigma_1 - \sigma_2) \cdot A \cdot \sigma_j \cdot B (\sigma_1 - \sigma_2) \cdot A P_{\frac{1}{2},1} &= 0, (j = 1, 2, 3), \\
\text{Tr} t_{ij}(x) \sigma_k \cdot A P_{\frac{1}{2},1} &= 0, (i, j, k = 1, 2, 3), \\
\text{Tr} t_{12}^2(x) P_{\frac{1}{2},1} &= 16 x^4, \\
\text{Tr} t_{23}^2(x) P_{\frac{1}{2},1} &= 4 x^4, \\
\text{Tr} t_{12}(x) t_{23}(y) P_{\frac{1}{2},1} &= \text{Tr} t_{23}(y) t_{12}(x) P_{\frac{1}{2},1} = 4[x^2 y^2 - 3(x \cdot y)^2], \\
\text{Tr} t_{23}(x) t_{31}(y) P_{\frac{1}{2},1} &= \text{Tr} t_{31}(y) t_{23}(x) P_{\frac{1}{2},1} = -4[x^2 y^2 - 3(x \cdot y)^2].
\end{align*}
\]

2. Proof of the equivalence of redundant states

First consider states \(|\psi_1\rangle\) and \(|\psi_2\rangle\) defined in terms of the two operators in Eq. \((30)\) by

\[
\begin{align*}
|\psi_1\rangle &= |(\sigma_1 - \sigma_2) \times \sigma_3 \cdot \rho \times \lambda \chi_{j_3}^{S_{12}=1}\rangle \\
&= \left|[(\sigma_1 - \sigma_2) \cdot \rho \sigma_3 \cdot \lambda - (\sigma_1 - \sigma_2) \cdot \lambda \sigma_3 \cdot \rho] \chi_{j_3}^{S_{12}=1}\rightangle, \\
|\psi_2\rangle &= |i(\sigma_1 - \sigma_2) \cdot \rho \times \lambda \chi_{j_3}^{S_{12}=1}\rangle,
\end{align*}
\]

where the spatial wave functions are suppressed for simplicity. Using the trace identities shown above, it is easy to show that for \( j = \frac{1}{2} \)

\[
\langle \psi_1 | \psi_1 \rangle = 4 \langle \psi_1 | \psi_1 \rangle = 2 \langle \psi_1 | \psi_2 \rangle = 2 \langle \psi_2 | \psi_1 \rangle = \frac{16}{3} (\rho \times \lambda)^2.
\]

Thus, if we choose a state \(|\psi\rangle\) such that \(|\psi\rangle = |\psi_1 - 2 \psi_2\rangle\), then \(\langle \psi | \psi \rangle = 0\) and the states \(|\psi_1\rangle\) and \(2 |\psi_2\rangle\) are equivalent. We will therefore drop the second operator in Eq. \((30)\) in writing the general baryon wave function.
Next, we show that the states given by the action of the operators $t_{12}(x) + t_{31}(x)$ in Eq. (31) on $\chi_{j_3}^{S_{12}=1}$ are also not independent for $j = \frac{1}{2}$. Let us consider the following state

$$|\psi\rangle = |(t_{12}(x) + t_{23}(x) + t_{31}(x))\chi_{j_3}^{S_{12}=1}\rangle.$$  \hspace{1cm} (B16)

A calculation of $\langle \psi|\psi\rangle$ using the trace method yields

$$\langle \psi|\psi\rangle = \frac{1}{2}\text{Tr} [t_{12}(x) + 2t_{23}(x)]^2 P_{\frac{1}{2}, 1} = \frac{1}{2}\text{Tr} [t_{12}(x) + 4t_{23}(x) + 2t_{12}(x)t_{23}(x) + 2t_{23}(x)t_{12}(x)] P_{\frac{1}{2}, 1}$$

$$= \frac{1}{2}[16x^2 + 16x^2 + 2(-8x^2) + 2(-8x^2)] = 0.$$  \hspace{1cm} (B17)

Hence, the spin functions $t_{12}(x)\chi_{j_3}^{S_{12}=1}$ and $[t_{23}(x) + t_{31}(x)]\chi_{j_3}^{S_{12}=1}$ in Eq. (31) are equivalent.

The same result holds for the operators in Eq. (32).

**APPENDIX C: CALCULATION OF MATRIX ELEMENTS**

Let us consider the spatial matrix element $\langle \psi_0^A|V|\psi_0^B\rangle$, $A,B = a, b, c, d, e$.

Here $V = V_0(r_{ij})P(\rho, \lambda)$ with $V_0$ is a function of the absolute value of $r_{ij}$ and $P(\rho, \lambda)$ is a polynomial of $\rho$ and $\lambda$. The Gaussian wave functions $\psi(\beta_A)$ are defined by Eq. (45). Then, one has

$$\langle \psi_0^A|V|\psi_0^B\rangle = \left(\frac{\beta_{\rho A}\beta_{\lambda A}\beta_{\rho B}\beta_{\lambda B}}{\pi^2}\right)^{3/2}$$

$$\times \int d\rho d\lambda V_0(r_{ij})P(\rho, \lambda) \exp \left[-(\beta_{\rho A}^2 + \beta_{\lambda A}^2)\right],$$  \hspace{1cm} (C1)

where

$$\beta_{\rho}^2 = \frac{1}{2}(\beta_{\rho A}^2 + \beta_{\rho B}^2), \; \beta_{\lambda}^2 = \frac{1}{2}(\beta_{\lambda A}^2 + \beta_{\lambda B}^2).$$  \hspace{1cm} (C2)

If $r_{ij} = r_{12} = \rho$, then the integration is straightforward.

Now consider the case when

$$r_{ij} = a\lambda + b\rho,$$  \hspace{1cm} (C3)

where $a, b$ are constants. If one denotes $r_{ij} = s$, then $\lambda = (s - b\rho)/a$ and

$$\beta_{\rho}^2\rho^2 + \beta_{\lambda}^2\lambda^2 = \beta_{\rho}^2\left(1 + \frac{b^2}{a^2}\right)(\rho - ws)^2 + \frac{\beta_{\lambda}^2}{a^2 + \frac{b^2}{a^2}s^2},$$  \hspace{1cm} (C4)
where $\tilde{x} = \tilde{\beta}^2/\beta_\rho^2$ and $w = \tilde{b}/(a^2 + \tilde{x}b^2)$. A translation $\rho \to \rho + \omega s$ gives the result $\lambda = (\tilde{w}s + b\rho)/a$ with $\tilde{w} = 1 - bw$. Eq. (C1) now becomes

$$
\langle \psi_0^A | V | \psi_0^B \rangle = \left( \frac{\beta^A \beta^B \beta_\rho \beta_\lambda}{\pi^2} \right)^{3/2} \times \int \frac{d\rho \; ds}{|a|^3} V_0(|s|) P(\rho + ws, (\tilde{w}s + b\rho)/a) \exp \left[ - (\beta^2 \rho^2 + \beta_\lambda^2 s^2) \right],
$$

where

$$
\beta^2 = \beta_\rho^2 + \beta_\lambda^2 \left( 1 + \frac{\tilde{x}b^2}{a^2} \right), \quad \beta_\rho^2 = \frac{\tilde{\beta}^2}{a^2 + \tilde{b}^2}.
$$

At this step, the integration of the right hand side of Eq. (C5) becomes straightforward.

For illustration, let us consider a simple case when $\psi_0^A = \psi_0^B = \psi_0^a$ and $V = (\rho \times \lambda)^2/\rho'$. In this case, $V_0(r_{vd}) = 1/\rho'$ and $P(\rho, \lambda) = (\rho \times \lambda)^2$. Since $\rho' = \lambda - \rho/2$, one has $a = 1$, $b = -1/2$. Also in this case $\tilde{\beta}_\rho = \beta_\rho$, $\tilde{\beta}_\lambda = \beta_\lambda$, $\tilde{x} = \tilde{x} = \beta_\lambda^2/\beta_\rho^2$, and $w = -2x/(4 + x)$. Note that under the translation $\rho \to \rho + ws$, the expression $(\rho \times \lambda)^2$ transforms into $(\rho \times s)^2$ that can be replaced by $\tilde{\beta}^2/\beta^2$ after working out the angular dependent parts. One finds

$$
\langle \psi_0^a | (\rho \times \lambda)^2 | \psi_0^a \rangle = \left( \frac{\beta^2 \beta_\lambda^2}{\pi} \right)^{3/2} \int d\rho \; ds \frac{(\rho \times s)^2}{s} \exp \left[ - (\beta^2 \rho^2 + \beta_\lambda^2 s^2) \right]
$$

where

$$
\beta^2 = \beta_\rho^2 \left( 1 + \frac{x}{4} \right), \quad \beta_\lambda^2 = \beta_\lambda^2 \left( 1 + \frac{x}{4} \right), \quad \beta_\lambda \beta_\rho = \beta_\lambda \beta_\rho.
$$

We can use similar methods with the momentum-space wave functions

$$
\tilde{\psi}_0^a(\mathbf{p}_\rho, \mathbf{p}_\lambda) = \left( \frac{1}{\pi \beta_\rho \beta_\lambda} \right)^{3/2} \exp \left( - \frac{p_\rho^2}{2 \beta_\rho^2} - \frac{p_\lambda^2}{2 \beta_\lambda^2} \right)
$$

to evaluate integrals which involve the energies $E_i = \sqrt{p_i^2 + m_i^2}$. Thus,

$$
\begin{align*}
\langle \sqrt{p_1^2 + m_1^2} \rangle &= \beta_\rho J_0 \left( \frac{m_1^2}{\beta_\rho^2} \right), \\
\langle \sqrt{p_3^2 + m_3^2} \rangle &= \beta_\rho J_0 \left( \frac{m_3^2}{\beta_\rho^2} \right),
\end{align*}
$$

$$
\begin{align*}
\langle 1/\sqrt{p_1^2 + m_1^2} \rangle &= \frac{1}{\beta_\rho} J_0 \left( \frac{m_1^2}{\beta_\rho^2} \right), \\
\langle 1/\sqrt{p_3^2 + m_3^2} \rangle &= \frac{1}{\beta_\rho} J_0 \left( \frac{m_3^2}{\beta_\rho^2} \right),
\end{align*}
$$

where

$$
I_n(z^2) = \frac{4}{\sqrt{\pi}} \int_0^\infty dt \, t^{2n+2} \sqrt{t^2 + z^2} e^{-t^2}, \quad J_n(z^2) = \frac{4}{\sqrt{\pi}} \int_0^\infty dt \, \frac{t^{2n+2}}{\sqrt{t^2 + z^2}} e^{-t^2}.
$$
The integrals $I_n$ and $J_n$ must be evaluated numerically.

Working out the spin matrix elements and applying the technique of integration shown above, one can evaluate the matrix elements $\langle \psi_{1/2, m} | H | \psi_{1/2, m} \rangle$. For example, the matrix element $\langle \psi_0^a | H_0 | \psi_0^a \rangle$ is given by

$$\langle \psi_0^a | H_0 | \psi_0^a \rangle = 2 \beta' I_0 \left( \frac{m_1^2}{\beta^2} \right) + \beta\lambda I_0 \left( \frac{m_3^2}{\beta^2} \right) + \frac{\sigma}{\sqrt{\pi}} \left( \frac{3 \beta_\rho}{2 \beta_\lambda} \right) - \frac{4 \alpha s}{3 \sqrt{\pi}} \left( \beta_\rho + 2 \beta_\lambda' \right). \quad (C13)$$

The matrix elements of $H_0$ in the states in Eq. (33) with $L_\rho, L_\lambda \neq 0$ involve the $I_n$ with $n = 0, 1, 2$.

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[26] The connection of the coefficients $\mu_b$ and $\mu_d$ to the coefficients $\mu_1-\mu_7$ of the alternative operators defined in Eqs. (4.6)-(4.12) of [19], are unfortunately given incorrectly in Eq. (4.14) in that paper. The correct relations are $\mu_b = \mu_2 + \mu_4 + \mu_5 + \mu_6 + \mu_7$ and $\mu_d = \mu_4 + \mu_7$. The expressions for the baryon moments in terms of $\mu_1-\mu_7$ are simpler than those given here in Table I, but the simple connection with the underlying dynamics is lost.

[27] In the general case with one or both of the internal angular momenta $L_\rho, L_\lambda$ greater than two, there are additional tensors with the opposite 1,2 symmetry which appear multiplied by a factor $(\rho \cdot \lambda)$ to restore the overall symmetry of $\tilde{\phi}$, and the coefficient functions depend on $(\rho \cdot \lambda)^2$ as well as $\rho^2$ and $\lambda^2$. See the following discussion of the opposite symmetry tensors in connection with the $\Lambda$ hyperon.

[28] We actually considered somewhat more general functions with Gaussians multiplied by polynomials, but the basic results changed rather little. The simple Gaussians are sufficient to illustrate the main features of the corrections.

[29] Note that here we have not considered the first excited state of $L = 0$.

[30] As was noted in [19], any correction which affects only the $\Xi^-$ can be absorbed through an adjustment of the other parameters.