Calculation of radiation reaction effect on orbital parameters in Kerr spacetime

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We calculate the secular changes of the orbital parameters of a point particle orbiting a Kerr black hole, due to the gravitational radiation reaction. For this purpose, we use the post-Newtonian (PN) approximation in the first-order black hole perturbation theory, with the expansion with respect to the orbital eccentricity. In this work, the calculation is done up to the fourth post-Newtonian (4PN) order and to the sixth order of the eccentricity, including the effect of the absorption of gravitational waves by the black hole. We confirm that, in the Kerr case, the effect of the absorption appears at the 2.5PN order beyond the leading order in the secular change of the particle's energy and may induce a superradiance, as known previously for circular orbits. In addition, we find that the superradiance may be suppressed when the orbital plane inclines with respect to the equatorial plane of the central black hole. We also investigate the accuracy of the 4PN formulae by comparing to numerical results. If we require that the relative errors in the 4PN formulae are less than $10^{-5}$, the parameter region to satisfy the condition will be $p \gtrsim 50$ for $e = 0.1$, $p \gtrsim 80$ for $e = 0.4$, and $p \gtrsim 120$ for $e = 0.7$ almost irrespective of the inclination angle or the spin of the black hole, where $p$ and $e$ are the semi-latus rectum and the eccentricity of the orbit. The region can further be extended using an exponential resummation method to $p \gtrsim 40$ for $e = 0.1$, $p \gtrsim 60$ for $e = 0.4$, and $p \gtrsim 100$ for $e = 0.7$. Although we still need the higher-order calculations of the PN approximation and the expansion with respect to the orbital eccentricity to apply for data analysis of gravitational waves, the results in this paper would be an important improvement from the previous work at the 2.5PN order, especially for large-$p$ regions.

Subject Index E01, E02, E31, E36

1. Introduction

The gravitational two-body problem is a fundamental issue in general relativity. This also attracts great interest in gravitational wave physics because binary inspirals are promising sources of gravitational waves which are expected to be detected directly by ongoing gravitational wave experiments at observatories across the world. Understanding the dynamics of a binary system is required to predict the emitted gravitational waveforms accurately for efficient searches of the signal in observed data.

One of the major approaches for this purpose is the gravitational self-force (GSF) picture in black hole perturbation theory. In this picture, a binary is regarded as a point mass orbiting a black hole and the dynamics can be described by the equation of motion of the mass including the effect of the interaction with the self-field, that is, the GSF. After the formal expression of the GSF was presented by Mino, Sasaki, and Tanaka [1] and Quinn and Wald [2], a lot of effort has been devoted to...
developing practical formulations and methods to calculate the GSF (for example, refer to [3] for the formulation of GSF, and [4,5] for the recent progress in practical calculations of GSF).

Although a lot of progress has been made, however, it is still challenging to calculate the GSF directly for general orbits, especially in Kerr spacetime. Practical calculations of the GSF with high accuracy will require a huge amount of time and computer resources, mainly because of the regularization problem induced by the point mass limit. Therefore it is important to develop a way to reduce the cost of computing the GSF. The two-timescale expansion method [6] gives a hint: assuming that a point mass does not encounter any transient resonances (e.g., shown in [7]), the orbital phase, which is the most important information for predicting the waveform, can be expressed in the expansion with respect to the mass ratio, \( \eta \), as

\[
\Phi = \eta^{-1} \left[ \Phi^{(0)} + \eta \Phi^{(1)} + O(\eta^2) \right],
\]

where \( \Phi^{(0)} \) and \( \Phi^{(1)} \) are quantities of order unity. The leading term, \( \Phi^{(0)} \), can be calculated from the knowledge up to the time-averaged dissipative piece of the first-order GSF, corresponding to the secular growth. The calculation of this secular contribution can be simplified significantly by using the radiative field defined as half the retarded solution minus half the advanced solution for the equation of the gravitational perturbation [8–10], i.e. the adiabatic approximation method, because the radiative field is the homogeneous solution free from the divergence induced by the point mass limit. This method allows us to calculate the leading term accurately without spending huge computational resources. On the other hand, the calculation of \( \Phi^{(1)} \) requires the rest of the first-order GSF (the oscillatory part of the dissipative GSF and the conservative GSF) and the time-averaged dissipative piece of the second-order GSF. There is no simplification in calculating these post-1 adiabatic pieces at present. Since \( \Phi^{(1)} \) is subleading, however, the requirement for the accuracy is not so high compared to that of the leading term. This fact suggests that it is possible to reduce the computational cost by using a suitable method with an appropriate error tolerance to calculate each piece of the GSF (for example, a hybrid approach is proposed in [11]).

In this work, we focus on the time-averaged dissipative part of the first-order GSF, which has the dominant contribution to the evolution of inspirals, and present the analytic post-Newtonian (PN) formulae. So far, several works in this direction have been done for two restricted classes of orbits: circular orbits and equatorial orbits (see [12] and references therein for early works in the 1990s). Recently, thanks to the progress of computer technology, much higher order post-Newtonian calculations can be possible for circular equatorial orbits: the 22PN calculation of the energy flux is demonstrated in the Schwarzschild case [13], and the 11PN calculation in the Kerr case [14]. There is also the calculation of the secular GSF effects for slightly eccentric and slightly inclined (non-equatorial) orbits [10], which later was extended to orbits with arbitrary inclination [15], where the PN formulae of the secular GSF effects are presented in the expansion with respect to the orbital eccentricity. However, the calculation in [15] was done only up to the 2.5PN order with the second-order correction of the eccentricity. Also, the absorption to the black hole is ignored there. The main purpose of this work is to update the results in [15] up to the 4PN order and the sixth-order correction of the eccentricity, including the effect of the absorption to the black hole.

This paper is organized as follows. In Sect. 2, we give a brief review of the geodesic motion of a point particle in Kerr spacetime, the gravitational perturbations induced by the particle, and the adiabatic approximation method of calculating the secular effect of the GSF. In Sect. 3.1, we present the PN formulae of the secular changes of the the energy, azimuthal angular momentum, and Carter parameter of the particle due to the gravitational radiation reaction in the expansion with respect to
the orbital eccentricity. In Sect. 3.2, we investigate the accuracy of our PN formulae by comparing to numerical results given by the method in [16–18], which can give each modal flux at an accuracy of about 14 significant figures. In Sect. 3.3, we implement a resummation method to the PN formulae given in this work in order to improve the accuracy. In Sect. 3.4, we discuss the convergence of the analytic formulae with the PN expansion and the expansion with respect to the eccentricity. Finally, we summarize the paper in Sect. 4. For the readability of the main text, we present the PN formulae for the orbital parameters, the fundamental frequencies, and the orbital motion in Appendices A and B, which are used in calculating the secular changes of the orbital parameters. We also present the PN formulae for the secular changes of an alternative set of orbital parameters in Appendix C. Throughout this paper we use metric signature (−+++ ) and “geometrized” units with \( c = G = 1 \).

2. Review of formulation: Adiabatic radiation reaction

The orbital evolution of a point particle due to the time-averaged dissipative part of the GSF is often described in terms of the secular changes of the orbital parameters. In order to calculate the changes, we need the information on the first-order gravitational perturbations induced by the particle when it moves along the background geodesics. In this section, we review the geodesic dynamics of a point particle in Kerr spacetime, the gravitational perturbations induced by the particle, and the adiabatic evolution of the orbital parameters.

2.1. Geodesic motion

The Kerr metric in the Boyer–Lindquist coordinates, \( (t, r, \theta, \varphi) \), is given by

\[
g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{2Mr}{\Sigma}) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dtd\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2Ma^2r}{\Sigma} \sin^2 \theta \right) \sin^2 \theta d\varphi^2 .
\]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \), \( \Delta = r^2 - 2Mr + a^2 \), and \( M \) and \( aM \) are the mass and angular momentum of the black hole, respectively.

There are two Killing vectors related to the stationarity and axisymmetry of Kerr spacetime, which are expressed as \( \xi_{(t)}^\mu = (1, 0, 0, 0) \) and \( \xi_{(\varphi)}^\mu = (0, 0, 0, 1) \). In addition, it is known that Kerr spacetime possesses a Killing tensor, \( K_{\mu\nu} = 2\Sigma l_{(\mu\nu)} + r^2 g_{\mu\nu} \), where \( l^\mu \) and \( n^\mu \) are the Kinnersley null vectors given by

\[
l^\mu := \left( \frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \quad n^\mu := \left( \frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma} \right).
\]

For the geodesic motion of a particle in Kerr geometry, there are three constants of motion related to the symmetries:

\[
\hat{E} := -u^\alpha \xi_{(t)}^\alpha, \quad \hat{L} := u^\alpha \xi_{(\varphi)}^\alpha, \quad \hat{Q} := K_{\alpha\beta}u^\alpha u^\beta,
\]

where \( u^\alpha \) is the four-velocity of the particle. \( \hat{E} \) and \( \hat{L} \) correspond to the specific energy and azimuthal angular momentum of the particle, respectively. \( \hat{Q} \) is called the Carter constant, which corresponds to
the square of the specific total angular momentum in the Schwarzschild case. These specific variables are measured in units of the particle’s mass, $\mu$. One can recover the expressions in the standard units as

$$E := \mu \hat{E}, \quad L := \mu \hat{L}, \quad Q := \mu^2 \hat{Q}. \quad (5)$$

There is another definition of the Carter constant, $C \equiv Q - (aE - L)^2$, which vanishes for equatorial orbits. In this paper, we use $C$ as one of the orbital parameters, instead of $Q$.

By using these constants of motion, the geodesic equations can be expressed in the following form:

$$\left( \frac{dr}{d\lambda} \right)^2 = R(r), \quad \left( \frac{d\cos \theta}{d\lambda} \right)^2 = \Theta(\cos \theta), \quad (6)$$

$$\frac{dt}{d\lambda} = V_{tr}(r) + V_{t\theta}(\theta), \quad \frac{d\varphi}{d\lambda} = V_{\varphi r}(r) + V_{\varphi \theta}(\theta), \quad (7)$$

where we introduced a new parameter $\lambda$ through the relation $d\lambda = d\tau / \Sigma$, and the functions:

$$P(r) := \hat{E} \left( r^2 + a^2 \right) - a \hat{L}, \quad (8)$$

$$R(r) := \left[ P(r) \right]^2 - \Delta \left[ r^2 + \left( a \hat{E} - \hat{L} \right) \right]^2 + \hat{C}, \quad (9)$$

$$\Theta(\cos \theta) := \hat{C} - \left( \hat{C} + a^2 \left( 1 - \hat{E}^2 \right) + \hat{L}^2 \right) \cos^2 \theta + a^2 \left( 1 - \hat{E}^2 \right) \cos^4 \theta, \quad (10)$$

$$V_{tr}(r) := \frac{r^2 + a^2}{\Delta} P(r), \quad V_{t\theta}(\theta) := -a \left( a \hat{E} \sin^2 \theta - \hat{L} \right), \quad (11)$$

$$V_{\varphi r}(r) := \frac{a}{\Delta} P(r), \quad V_{\varphi \theta}(\theta) := - \left( a \hat{E} - \frac{\hat{L}}{\sin^2 \theta} \right). \quad (12)$$

A generic geodesic orbit in Kerr spacetime can be characterized by three parameters, \( \{ E, L, C \} \).\footnote{Strictly speaking, the orbit is also characterized by the initial position of the particle. However, the time-averaged dissipative part of the first-order GSF does not affect the initial position (the other parts of the first-order GSF and the higher-order GSF will do) \cite{6}. Also, the secular changes of \( \{ E, L, C \} \) do not depend on the initial position. Hence we do not need the information on the initial position to describe the secular evolution of the orbit at the order considered in this paper.}

In the case of a bound orbit, we can use an alternative set of parameters, \( \{ r_p, r_a, \theta_{\text{min}} \} \), instead of \( \{ E, L, C \} \), where \( r_p \) and \( r_a \) are the values of \( r \) at the periaxis and apoaipaxis and \( \theta_{\text{min}} \) is the minimal value of \( \theta \), respectively. Using this set of parameters, we can describe the range in which the motion takes place as \( r_p \leq r \leq r_a \) and \( \theta_{\text{min}} \leq \theta \leq \pi - \theta_{\text{min}} \). There is another useful choice of parameters used in \cite{19}, \( \{ p, e, \iota \} \), defined by

$$p := \frac{2 r_p r_a}{M (r_a + r_p)}, \quad e := \frac{r_a - r_p}{r_a + r_p}, \quad \cos \iota := \frac{L}{\sqrt{L^2 + C}}. \quad (13)$$

By analogy to the parameterization used in celestial mechanics, \( p, e, \iota \) are referred to as the semi-latus rectum, orbital eccentricity, and orbital inclination angle, respectively. For later convenience, we also introduce $Y = \cos \iota$ and $v = \sqrt{1/p}$. Since $v$ corresponds to the magnitude of the orbital velocity, it can be used as the post-Newtonian parameter. For example, we call the $O \left( v^8 \right)$ correction from the leading order the fourth-order post-Newtonian (4PN) correction.
It is worth noting that, by introducing \( \lambda \), the radial and longitudinal equations of motion in Eq. (6) are completely decoupled. For a bound orbit, therefore, the radial and longitudinal motions are periodic with the periods \( \{ \Lambda_r, \Lambda_\theta \} \) defined by

\[
\Lambda_r = 2 \int_{r_p}^{r_a} \frac{dr}{\sqrt{R(r)}}, \quad \Lambda_\theta = 4 \int_{\theta_{\min}}^{\pi/2} \frac{d\theta}{\sqrt{\Theta(\theta)}}. \tag{14}
\]

This means that these motions can be expressed in terms of Fourier series as

\[
r(\lambda) = p \sum_{n_r=0}^{\infty} \alpha_{n_r} \cos n_r \Omega_r \lambda, \tag{15}
\]

\[
\cos \theta(\lambda) = \sqrt{1 - Y^2} \sum_{n_\theta=0}^{\infty} \beta_{n_\theta} \sin n_\theta \Omega_\theta \lambda, \tag{16}
\]

where \( \Omega_r \) and \( \Omega_\theta \) are the radial and longitudinal frequencies given by

\[
\Omega_r := \frac{2\pi}{\Lambda_r}, \quad \Omega_\theta := \frac{2\pi}{\Lambda_\theta}, \tag{17}
\]

and we choose the initial values so that \( r(\lambda = 0) = r_p \) and \( \theta(\lambda = 0) = \pi/2 \).

Since the temporal and azimuthal equations of motion in Eq. (7) are divided into the \( r \)- and \( \theta \)-dependent parts, the solutions can be divided into three parts: the linear term with respect to \( \lambda \), the oscillatory part with period \( \Lambda_r \), and the oscillatory part with period \( \Lambda_\theta \). They can be expressed as

\[
t(\lambda) = \Omega_t \lambda + t^{(r)}(\lambda) + t^{(\theta)}(\lambda); \quad t^{(A)}(\lambda) := \sum_{n_A=1}^{\infty} \tilde{t}^{(A)}_{n_A} \sin n_A \Omega_A \lambda, \tag{18}
\]

\[
\varphi(\lambda) = \Omega_\varphi \lambda + \varphi^{(r)}(\lambda) + \varphi^{(\theta)}(\lambda); \quad \varphi^{(A)}(\lambda) := \sum_{n_A=1}^{\infty} \tilde{\varphi}^{(A)}_{n_A} \sin n_A \Omega_A \lambda, \tag{19}
\]

where the index \( A \) runs over \( \{r, \theta\} \), and

\[
\Omega_t := \left[ \frac{dt}{d\lambda} \right]_\lambda, \quad \Omega_\varphi := \left[ \frac{d\varphi}{d\lambda} \right]_\lambda \tag{20}
\]

with \( \langle \cdots \rangle_\lambda \equiv \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} d\lambda \cdots \), representing the time average along the geodesic. We choose the initial conditions as \( t(\lambda = 0) = \varphi(\lambda = 0) = 0 \). \( \Omega_\varphi \) corresponds to the frequency of the orbital rotation.

In Appendices A and B, we present the PN formulae of the orbital parameters, \( \{ E, L, C \} \), the fundamental frequencies, \( \{ \Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi \} \), and the Fourier coefficients of the motions in Eqs. (15), (16), (18), and (19).

\[\text{\textsuperscript{2}}\] If the ratio of the radial and longitudinal frequencies is irrational, we can adjust the origin of \( \lambda \) approximately so that the radial and longitudinal oscillations reach the minima simultaneously at \( \lambda = 0 \) [8]. On the other hand, this is not the case if the ratio is rational, i.e. the resonance case. This implies that the secular evolution of a resonant orbit cannot be described only by the PN formulae derived in this work [20].

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2.2. Secular evolution of orbital parameters

The gravitational perturbations in Kerr spacetime can be described by the Weyl scalar, \( \Psi_4 \), which satisfies the Teukolsky equation \( [21] \). To solve the Teukolsky equation, the method of separation of variables is often used, in which \( \Psi_4 \) is decomposed in the form

\[
\Psi_4 = \sum_{\ell m} \int d\omega R_\Lambda(r) S_\Lambda(\theta) e^{i m \phi - i\omega t},
\]

(21)

where \( S_\Lambda(\theta) \) is the spin-2 spheroidal harmonics and \( \Lambda \) represents a set of indices in the Fourier-harmonic expansion, \( \{\ell, m, \omega\} \). The separated equation for the radial function is given by

\[
\left[ \Delta^2 \frac{d}{dr} \left( \Delta^{-1} \frac{d}{dr} \right) + \left( \frac{K^2 + 4i(r - M)K}{\Delta} - 8i\omega r - \lambda \right) \right] R_\Lambda(r) = T_\Lambda,
\]

(22)

where

\[
K \equiv \left( r^2 + a^2 \right) \omega - ma,
\]

\( T_\Lambda \) is the source term constructed from the energy–momentum tensor of the point particle, and \( \lambda \) is the eigenvalue determined by the equation for \( S_\Lambda \) (to find the basic formulae for the Teukolsky formalism used in this paper, refer to Section 2 in \( [22] \), for example).

The amplitudes of the partial waves at the horizon and at infinity are defined by the asymptotic forms of the solution of the radial equation as

\[
R_\Lambda(r \to r_+) \equiv \mu Z_\Lambda^H \Delta^2 e^{-i k r^*}, \quad R_\Lambda(r \to \infty) \equiv \mu Z_\Lambda^\infty \Delta^2 e^{i \omega r^*},
\]

(23)

with \( r_+ \equiv M + \sqrt{M^2 - a^2} \) and \( k = \omega - ma/(2Mr_+) \). Since the spectrum with respect to \( \omega \) gets discrete in the case of a bound orbit, \( Z_\Lambda^H, \infty \) take the form

\[
Z_\Lambda^H,\infty = 2\pi \delta(\omega - \omega_{mn,n_0})\tilde{Z}_\Lambda^H,\infty,
\]

(24)

where \( \tilde{\Lambda} \) denotes the set of indices \( \{\ell, m, n_r, n_\theta\} \), and

\[
\omega_{mn,n_0} \equiv \Omega^{-1} \left( m \Omega_{\phi} + n_r \Omega_r + n_\theta \Omega_\theta \right).
\]

(25)

With these amplitudes, the secular changes of the orbital parameters, \( \{E, L, C\} \), can be expressed by

\[
\frac{dE}{dt} = -\mu^2 \sum_{\tilde{\Lambda}} \left[ \frac{1}{4\pi \omega_{mn,n_0}^2} \left( \left| \tilde{Z}_{\tilde{\Lambda}}^\infty \right|^2 + \alpha_{\ell m}(\omega_{mn,n_0}) \left| \tilde{Z}_{\tilde{\Lambda}}^H \right|^2 \right) \right],
\]

(26)

\[
\frac{dL}{dt} = -\mu^2 \sum_{\tilde{\Lambda}} \left[ \frac{m}{4\pi \omega_{mn,n_0}^3} \left( \left| \tilde{Z}_{\tilde{\Lambda}}^\infty \right|^2 + \alpha_{\ell m}(\omega_{mn,n_0}) \left| \tilde{Z}_{\tilde{\Lambda}}^H \right|^2 \right) \right],
\]

(27)

\[
\frac{dC}{dt} = -2 \left( a^2 E \cos^2 \theta \right) \frac{dE}{dt} + 2 \left( L \cot^2 \theta \right) \frac{dL}{dt} \left( \frac{dC}{dt} \right) - \mu^3 \sum_{\tilde{\Lambda}} \frac{n_\theta \Omega_\theta}{2\pi \omega_{mn,n_0}^3} \left( \left| \tilde{Z}_{\tilde{\Lambda}}^\infty \right|^2 + \alpha_{\ell m}(\omega_{mn,n_0}) \left| \tilde{Z}_{\tilde{\Lambda}}^H \right|^2 \right),
\]

(28)

where

\[
\alpha_{\ell m}(\omega) = \frac{256(2Mr_+)^5 k \left( k^2 + 4\tilde{C}^2 \right) \left( k^2 + 16\tilde{C}^2 \right) \omega^3}{|C_s|^2}, \quad \tilde{C} = \sqrt{M^2 - a^2/(4Mr_+)},
\]

(29)
and \( C_S \) is the Starobinsky constant given by [23]:

\[
|C_S|^2 = \left[ (\tilde{\lambda} + 2)^2 + 4awm - 4a^2w^2 \right] \left[ \tilde{\lambda}^2 + 36awm - 36a^2w^2 \right] + (2\tilde{\lambda} + 3) \left( 96a^2w^2 - 48awm \right) + 144w^2 \left( M^2 - a^2 \right).
\]  

(30)

It should be noted that, in these formulae, the averaged rates of change are expressed with respect to the Boyer–Lindquist time, which can be related to those with respect to \( \dot{\lambda} \) [24] as

\[
\left\langle \frac{dl}{dt} \right\rangle_t = \left\langle \frac{dl}{d\lambda} \right\rangle_{\dot{\lambda}}.
\]

(31)

for a function of time, \( I(t) \). It should also be noted that each formula in Eqs. (26)–(28) can be divided into the infinity part and the horizon part: the former consists of the terms including the amplitudes of the partial waves at infinity, \( \tilde{Z}_{\tilde{\lambda}}^\infty \); the latter consists of the terms including the amplitudes at the horizon, \( \tilde{Z}_{\tilde{\lambda}}^H \). As for the energy and azimuthal angular momentum, the infinity parts are balanced with the corresponding fluxes radiated to infinity and the horizon parts with the absorption of the gravitational waves into the central black hole [23,25].

The practical calculation of \( \tilde{Z}_{\tilde{\lambda}}^{H,\infty} \) involves solving the geodesic equations, calculating two independent homogeneous solutions of Eq. (22) and the spin-2 spheroidal harmonics, and the Fourier transformation of functions of them. In this work, we followed the same procedure proposed in [15] to perform these calculations analytically.

In performing the summation in Eqs. (26)–(28) practically, we need to truncate the summation to finite ranges of \( \tilde{\lambda} = (\ell, m, n_r, n_\theta) \). To obtain the accuracy of the 4PN and \( O(e^6) \), it is necessary to sum \( \ell \) in the range \( 2 \leq \ell \leq 6 \), \( n_r \) in the range \( -3 \leq n_r \leq 3 \), \( -2 \leq n_r \leq 3 \), and \( n_\theta \) in the range \( -8 \leq n_\theta \leq 12 \) for the infinity (horizon) part. The other modes out of these ranges are the higher PN corrections than the 4PN order or the higher-order corrections than \( O(e^6) \).

### 3. Results

#### 3.1. PN formulae of the secular changes of orbital parameters

In this work, we derived the analytic 4PN order formulae of Eqs.(26)–(28) in the expansion with respect to the orbital eccentricity, \( e \), up to \( O(e^6) \) \( \) (we simply call them the 4PN \( O(e^6) \) formulae).

Since the full expressions of the 4PN \( O(e^6) \) formulae are too lengthy to show in the text, we show the infinity parts up to the 3PN order and the horizon parts up to the 3.5PN order (while we keep the expansions with respect to \( e \) up to \( O(e^6) \)). The complete expressions of the 4PN \( O(e^6) \) formulae will be publicly available online [26].

The infinity parts of Eqs.(26)–(28) are given by

\[
\left\langle \frac{dE}{dt} \right\rangle^\infty_t = \left( \frac{dE}{dt} \right)_N \left[ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 + \left\{ -\frac{1247}{336} - \frac{9181}{672} e^2 + \frac{809}{128} e^4 + \frac{8609}{5376} e^6 \right\} v^2 
\right.
\]

\[
+ \left\{ 4\pi - \frac{73}{12} Yq + \left( \frac{1375}{48} \pi - \frac{823}{24} Yq \right) e^2 
\right.
\]

\[
+ \left. \left( \frac{3935}{192} \pi - \frac{949}{32} Yq \right) e^4 + \left( \frac{10007}{9216} \pi - \frac{491}{192} Yq \right) e^6 \right\} v^3
\]

\[
+ \left\{ -\frac{44711}{9072} + \frac{527}{96} \right. \]

\[
\left. Yq^2 - \frac{329}{96} q^2 + \left( -\frac{172157}{2592} - \frac{4379}{192} q^2 + \frac{6533}{192} Yq^2 \right) \right\} e^2
\]
\[
\left\{ \frac{dL}{dt} \right\}_t \bigg|_N = \frac{dL}{dt} \left[ \left\{ 1 + \frac{7}{8} e^2 \right\} Y + \left\{ -\frac{1247}{336} - \frac{425}{336} e^2 + \frac{10751}{2688} e^4 \right\} Y v^2 \right.
\]
\[
+ \left\{ \frac{61}{24} q - \frac{61}{8} Y^2 q + 4 \pi Y + \left( \frac{63}{8} q + \frac{97}{8} \pi Y - \frac{91}{4} Y^2 q \right) e^2 \right\} \]
\[
+ \left\{ \frac{95}{64} q + \frac{49}{32} \pi Y + \frac{461}{64} Y^2 q \right\} e^4 - \frac{49}{4608} \pi Y e^6 \right\} v^3
\]
\[
+ \left\{ -\frac{44711}{9072} q^2 + \frac{57}{16} q^2 + \frac{45}{8} Y^2 q^2 + \left( -\frac{302893}{6048} - \frac{201}{16} q^2 + \frac{37}{2} Y^2 q^2 \right) e^2 \right\} \]
\[
+ \left\{ -\frac{701675}{24192} q^2 + \frac{351}{128} q^2 + \frac{331}{64} Y^2 q^2 \right\} e^4 + \frac{162661}{16128} e^6 \right\} Y v^4
\]
\[
+ \left\{ \frac{4301}{224} Y^2 q - \frac{8191}{672} \pi Y - \frac{2633}{224} q + \left( -\frac{66139}{1344} q - \frac{48361}{1344} \pi Y + \frac{18419}{448} Y^2 q \right) e^2 \right\}
\]
\[
+ \left\{ \frac{3959}{1792} q + \frac{1657493}{43008} \pi Y - \frac{257605}{5376} Y^2 q \right\} e^4
\]
\[
+ \left\{ \frac{19161}{3584} q + \frac{5458969}{774144} \pi Y - \frac{52099}{1536} Y^2 q \right\} e^6 \right\} v^5
\]
\]
\[
\left\{ \begin{array}{l}
\frac{dC}{dt}_\infty = \left( \frac{dC}{dt} \right)_N \left[ 1 + \frac{7}{8} e^2 + \left( -\frac{743}{336} + \frac{23}{42} e^2 + \frac{11927}{2688} e^4 \right) v^2 \\
+ \left\{ 4 \pi - \frac{85}{8} Yq + \left( \frac{97}{8} \pi - \frac{211}{8} Yq \right) e^2 + \left( \frac{49}{32} \pi - \frac{517}{64} Yq \right) e^4 - \frac{49}{4608} \pi e^6 \right\} v^3 \\
+ \left\{ \frac{129193}{18144} - \frac{329}{96} q^2 + \frac{53}{8} Y^2 q^2 + \left( -\frac{84035}{1728} - \frac{929}{96} q^2 + \frac{163}{8} Y^2 q^2 \right) e^2 \\
+ \left( -\frac{1030273}{48384} - \frac{1051}{768} q^2 + \frac{387}{64} Y^2 q^2 \right) e^4 + \frac{1000131}{8064} e^6 \right\} v^4 \\
+ \left\{ -\frac{4159}{672} \pi + \frac{2553}{224} Yq + \left( -\frac{21229}{1344} \pi - \frac{553}{192} Yq \right) e^2 \\
+ \left( \frac{2017013}{43008} \pi - \frac{475541}{5376} Yq \right) e^4 + \left( \frac{6039325}{774144} \pi - \frac{153511}{3584} Yq \right) e^6 \right\} v^5 \\
+ \left\{ \frac{11683501663}{139708800} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma - \frac{3424}{105} \ln(2) + \frac{1277}{192} q^2 - \frac{193}{4} \pi Yq \\
+ \frac{2515}{48} Y^2 q^2 + \left( \frac{16319179321}{23284800} + \frac{229}{6} \pi^2 - \frac{24503}{210} \gamma + \frac{1391}{30} \ln(2) \\
- \frac{78003}{280} \ln(3) + \frac{16979}{1344} q^2 - \frac{2077}{8} \pi Yq + \frac{118341}{448} Y^2 q^2 \right) e^2 \\
+ \left( \frac{211889615389}{372556800} + \frac{109}{4} \pi^2 + \frac{3042117}{1120} \ln(3) - \frac{11663}{140} \gamma - \frac{418049}{84} \ln(2) \right\} v^6 \right\} \right],
\end{array} \right.
\]
\[
\begin{align*}
&- \frac{132193}{3584} q^2 - \frac{24543}{128} \pi Y q + \frac{91747}{330} Y^2 q^2 e^4 \\
&+ \left( \frac{33928992071}{186278400} - \frac{1044921875}{96768} \ln(5) + \frac{23}{16} \pi^2 - \frac{42667641}{3584} \ln(3) \right) e^4 \\
&+ \frac{94138279}{2160} \ln(2) - \frac{2461}{560} Y - \frac{24505}{5376} q^2 - \frac{4151}{288} \pi Y q + \frac{718799}{10752} Y^2 q^2 \right) e^6 \\
&- \left( \frac{1712}{105} + \frac{24503}{210} e^2 + \frac{11663}{140} e^4 + \frac{2461}{560} e^6 \right) \ln v \right] v^6 \right], \\
\end{align*}
\]

where the leading contributions are given by

\[
\begin{align*}
\left( \frac{dE}{dt} \right)_N &= - \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^3 (1 - e^2)^{3/2} \\
\left( \frac{dL}{dt} \right)_N &= - \frac{32}{5} \left( \frac{\mu^2}{M^2} \right) v^7 (1 - e^2)^{3/2} \\
\left( \frac{dC}{dt} \right)_N &= - \frac{64}{5} \mu^3 v^6 (1 - e^2)^{3/2} \left( 1 - Y^2 \right). \\
\end{align*}
\]

The horizon parts of Eqs.(26)–(28) are given by

\[
\begin{align*}
\left\langle \frac{dE}{dt} \right\rangle_H &= \left( \frac{dE}{dt} \right)_N \left[ - \frac{1}{512} \left( 16 + 120 e^2 + 90 e^4 + 5 e^6 \right) \right. \\
&\left. - \left( 1 + \frac{81}{32} q^2 - \frac{15}{32} Y^2 q^2 + \left( \frac{57}{4} + \frac{1143}{32} q^2 - \frac{195}{32} Y^2 q^2 \right) e^2 \right. \\
&\left. + \left( \frac{465}{16} + \frac{4455}{64} q^2 - \frac{225}{32} Y^2 q^2 \right) e^4 + \left( \frac{355}{32} + \frac{6345}{256} q^2 + \frac{75}{256} Y^2 q^2 \right) e^6 \right] q Y v^5 \right]. \\
\end{align*}
\]

\[
\begin{align*}
\left\langle \frac{dL}{dt} \right\rangle_H &= \left( \frac{dL}{dt} \right)_N \left[ - \frac{8 + 24 e^2 + 3 e^4}{1024} \right. \\
&\left. \left( 16 + 33 q^2 + 16 Y^2 q^2 + 45 Y^4 q^2 \right) q v^5 \right. \\
&\left. - \left( \frac{5}{4} + \frac{375}{128} q^2 - \frac{1}{4} Y^2 - \frac{63}{64} Y^2 q^2 + \frac{15}{128} Y^4 q^2 \right) \right. \\
&\left. + \left( \frac{10}{1} + \frac{5955}{256} q^2 - \frac{5}{16} Y^2 - \frac{855}{128} Y^2 q^2 + \frac{675}{256} Y^4 q^2 \right) e^2 \right. \\
&\left. + \left( \frac{255}{32} + \frac{18855}{1024} q^2 - \frac{15}{32} Y^2 - \frac{2295}{512} Y^2 q^2 + \frac{3375}{1024} Y^4 q^2 \right) e^4 \right. \\
&\left. + \left( \frac{15}{32} + \frac{2205}{2048} q^2 - \frac{225}{1024} Y^2 q^2 + \frac{525}{2048} Y^4 q^2 \right) e^6 \right] q Y v^7 \right]. \\
\end{align*}
\]
\[ \langle \frac{dC}{dt} \rangle_{\text{H}}^L = \left( \frac{dC}{dt} \right)_N \left[ -\frac{1}{1024} \left( 8 + 24 e^2 + 3 e^4 \right) \left( 16 + 3 q^2 + 45 Y^2 q^2 \right) q Y v^5 + \left\{ \begin{array}{l} \frac{1}{16} + \frac{93}{256} q^2 + \frac{165}{256} Y^2 q^2 + \left( \frac{5}{8} + \frac{705}{256} q^2 - \frac{1125}{256} Y^2 q^2 \right) e^2 \\
+ \left( \frac{27}{128} + \frac{4131}{2048} q^2 - \frac{8235}{2048} Y^2 q^2 \right) e^4 + \left( -\frac{3}{128} + \frac{27}{256} q^2 - \frac{165}{512} Y^2 q^2 \right) e^6 \end{array} \right\} q Y v^7 \right]. \tag{38} \]

\( \langle dE/dt \rangle_{\text{H}}^L, \langle dL/dt \rangle_{\text{H}}^L, \) and \( \langle dC/dt \rangle_{\text{H}}^L \) in Eqs. (36)–(38) are new PN formulae derived in this paper. \( \langle dE/dt \rangle_{\text{H}}^\infty, \langle dL/dt \rangle_{\text{H}}^\infty, \) and \( \langle dC/dt \rangle_{\text{H}}^\infty \) in Eqs. (32)–(34) are consistent with those in Ref. [15] up to 2.5PN and \( O(e^3) \).

From the leading-order expressions in Eq. (35), one will find that the Carter parameter, \( C \), does not change due to the radiation of the gravitational waves when \( Y = 1 \) (equatorial orbits) because \( (dC/dt)_N = 0 \).

In the Schwarzschild case, the Carter parameter corresponds to the square of the equatorial angular momentum (the normal component to the rotational axis of the central black hole). Then there is expected to exist the duality between \( L^2 \) and \( C \) due to the spherical symmetry. In fact, from Eqs. (33) and (34), [and also from (37) and (38)], one can find that \( \langle dL^2/dt \rangle \), vanishes in \( Y = 0 \) (polar orbits) while \( \langle dC/dt \rangle, \) for \( Y = 0 \) coincides with \( \langle dL^2/dt \rangle \) for \( Y = 1 \). This can be also realized by seeing that the secular change of the total angular momentum, \( \langle d(L^2 + C)/dt \rangle \), is independent of \( Y \). Then, it might be possible to understand that \( \langle dL/dt \rangle_{\text{H}}^\infty \) becomes 1.5PN from the leading order when \( q \neq 0 \) and \( Y = 0 \) (polar orbits) due to the spin–orbit coupling.

From the expressions of the horizon parts shown in Eqs. (36)–(38), we find that the absorption of the gravitational waves to the central black hole contributes at \( O(v^5) \) from the leading order in Eq. (35) for \( q \neq 0 \) and at \( O(v^3) \) for \( q = 0 \). The \( O(v^5) \) and \( O(v^7) \) corrections in \( \langle dE/dt \rangle_{\text{H}}^L \) can be positive for \( q > 0 \), which means that the particle can gain energy through a superradiance phenomenon. These observations are consistent with the results for circular, equatorial orbits shown in Refs. [27,28].

We also find that the superradiance terms in Eq. (36) vanish for \( Y = 0 \), and that \( \langle dE/dt \rangle_{\text{H}}^L \) has only the 4PN and higher-order corrections. The superradiance terms may come from the coupling between the black hole spin and the orbital angular momentum, like \( \propto L \cdot S \propto q \cos \iota \). Hence, when the orbital inclination increases (\( Y \) gets small correspondingly), the superradiance is suppressed [29].

### 3.2. Comparison to numerical results

To investigate the accuracy of the 4PN \( O(e^6) \) formulae derived in this work, we compare them to the corresponding numerical results given by the method established in Refs. [16–18], which enables one to compute the modal fluxes with a relative error of \( \sim 10^{-14} \) in double precision computations. In the practical computations, as well as in deriving the analytic expressions, we need to truncate the summation to finite ranges of \( \Lambda = \{ \ell, m, n_r, n_\theta \} \) in Eqs. (26)–(28). In order to save computational time in the numerical calculation, we sum \( \ell \) up to 7. We can check that the error due to neglecting terms for \( \ell \geq 8 \) is smaller than the relative error in the 4PN \( O(e^6) \) formulae from the corresponding numerical results up to \( \ell = 7 \). We also truncate \( n_r \) and \( n_\theta \) to achieve a relative error of \( \sim 10^{-7} \) in numerical results up to \( \ell = 7 \). For the parameters investigated in the comparison, the relative error of \( \sim 10^{-7} \) achieved by truncating \( n_r \) and \( n_\theta \) is again smaller than the relative error in the 4PN \( O(e^6) \)
fornumlae from the numerical results up to $\ell = 7$. Thus, we can regard the numerical results as benchmarks to investigate the accuracy in our analytic formulame.

Here we define the relative error in the analytic formulame of $\langle dE/dt \rangle_t$ by

$$\Delta_E \equiv \left| 1 - \frac{dE}{dt}^{\text{Ana}}_t \right| \big/ \left| \frac{dE}{dt}^{\text{Num}}_t \right|,$$

(39)

where $\langle dE/dt \rangle^{\text{Ana}}_t$ denotes the analytic formulame in order to distinguish it from the corresponding numerical result, $\langle dE/dt \rangle^{\text{Num}}_t$. We also define the relative errors in the analytic formulame of $\langle dL/dt \rangle_t$ and $\langle dC/dt \rangle_t$ in a similar manner and denote them as $\Delta_L$ and $\Delta_C$ respectively.

Figure 1 shows several plots of $\Delta_E$ for the 4PN $O(e^6)$ formulame as a function of $p$ for several sets of $(e, \iota)$ with $q = 0.9$. In the plots, we also show the relative errors in the 2.5PN $O(e^2)$ and 3PN $O(e^4)$ formulame for reference. From the plots for $e = 0.1$ (the three on the top), one can find that $\Delta_E$ falls off faster than $p^{-4}$ for $p \gtrsim 10$. Similarly, the relative errors in the 2.5PN $O(e^2)$ and 3PN $O(e^4)$ formulame fall off faster than $p^{-5/2}$ and $p^{-3}$. Noting that $v = \sqrt{1/p}$, this would be a good indication that our PN formulame has been derived correctly up to the required order.

$\Delta_E$ is expected to contain not only higher-order corrections than the 4PN order, but also the higher-order corrections of eccentricity than $O(e^6)$ in the lower PN terms, which will become dominant when $p$ and $e$ get larger. In fact, seeing the plots for $e = 0.7$ in Fig. 1, one can find that the relative error strays out of the expected power-law line for large $p$. This behavior is clearer in the plots of the relative error in the 2.5PN $O(e^2)$ formulame. From Eqs. (32) and (36), we know that the relative error in the 2.5PN $O(e^2)$ formulame contains the $O(e^4)$ correction in the $O(v^0)$ term. The effect of this correction appears as large-$p$ plateaus in the plots (see also Fig. 6). This may motivate us to perform the higher-order expansion with respect to the orbital eccentricity in the PN formulame or to derive the PN formulame without performing the expansion with respect to the orbital eccentricity [30–33]. In addition, it might be noted that the behavior of the relative error does not strongly depend on the inclination angle $\iota$ for fixed $q$ and $e$.

In Fig. 2, we show the relative errors in the 4PN $O(e^6)$ formulame for the secular changes of the three orbital parameters, $(E, L, C)$, for several sets of $(q, e)$ and $\iota = 50^\circ$. As in the case of $\Delta_E$ shown in Fig. 1, the relative errors, $\Delta_L$ and $\Delta_C$, fall off faster than $p^{-4}$ when $p \gtrsim 10$, except for the large-$p$ region ($p \gtrsim 100$) in the case of $e = 0.7$. Thus, the 4PN $O(e^6)$ formulame for the secular changes of the orbital parameters are expected to be valid up to $O(v^3)$. From Fig. 2, one might think that it is enough to investigate only $\Delta_E$ to discuss the accuracy of our formulame since there are not large differences in the relative errors, $\Delta_E$, $\Delta_L$, and $\Delta_C$.

Figure 3 shows contour plots for $\Delta_E$ as a function of $p$ and $e$ for several sets of $(\iota, q)$. From these plots, one may be able to comprehend the accuracy of our PN formulame more easily than using Figs. 1 and 2. One will find that the relative error becomes smaller (larger) for larger (smaller) $p$ and smaller (larger) $e$. Moreover, it might be noticed that the relative error does not strongly depend on the inclination angle $\iota$ for fixed $q$ as expected from Fig. 1. If one requires $\Delta_E < 10^{-5}$ as an error tolerance, one can use the contour line with the label $10^{-5}$ to find the region of validity in the figure. For example, one will find that $\Delta_E < 10^{-5}$ for $p \gtrsim 50$ and $e = 0.1$, $p \gtrsim 80$, and $e = 0.4$, and $p \gtrsim 120$ and $e = 0.7$ when $q = 0.9$.

3.3. Implementation of an exponential resummation method

In order to improve the accuracy in the analytic PN formulame, one may apply some resummation methods such as Padé approximation [34], the factorized resummation [35–37], and the exponential...
resummation [38]. Since the exponential resummation may be the simplest one to implement among them, we choose here to implement the exponential resummation. We apply it to our 4PN formulae and check how the accuracy is improved.

To introduce the exponential resummation, we make use of the following identity:

$$\left\langle \frac{dl}{dt} \right\rangle_t = \left( \frac{dl}{dt} \right)_N \exp \left\{ \ln \left[ \left( \frac{dl}{dt} \right)_t / \left( \frac{dl}{dt} \right)_N \right] \right\},$$

(40)

where $I = \{ E, L, C \}$. The exponential resummation can be obtained by replacing the exponent in (40) with the expansion with respect to $\nu$,

$$E_n^I := \ln \left[ \left( \frac{dl}{dt} \right)_t / \left( \frac{dl}{dt} \right)_N \right] \text{truncated after } n \text{th order of } \nu,$$

(41)
The relative errors in the analytic PN formulae for the secular changes of the three orbital parameters, \( \{E, L, C\} \), as functions of the semi-latus rectum \( p \) for \( \iota = 50^\circ \); \( q = 0.9, 0.5, 0.1, -0.9 \) (from top to bottom); and \( e = 0.1, 0.4, \) and 0.7 (from left to right). We truncated the plots at \( p = \max\{6, p_e(e, \iota)\} \), where \( p_e(e, \iota) \) is the value of \( p \) at the “separatrix” (the boundary between stable and unstable orbits), because the relative errors get too large in \( p < 6 \) to be meaningful and the orbit is not stable for \( p < p_e(e, \iota) \). As pointed out in Fig. 1, the relative errors in our 4PN \( O(e^8) \) formulae fall off faster than \( p^{-4} \) when the eccentricity is small, e.g. \( e \lesssim 0.4 \), while the fall-off gets slower when \( p \) is larger for \( e = 0.7 \). There are not large differences in the behaviors of \( \Delta_E, \Delta_L, \) and \( \Delta_C \), which suggests that it might be enough to focus only on \( \langle dE/dt \rangle \), to investigate the accuracy and convergence of our 4PN formulae.

where we do not perform the expansion with respect to \( e \). Since our PN formulae for \( \langle dI/dt \rangle \) are given at the 4PN order, we truncate \( F_{nI} \) after \( O(e^8) \). Finally, the exponential resummed form is expressed as

\[
\left\langle \frac{dI}{dt} \right\rangle^{\text{exp}} = \left( \frac{dI}{dt} \right)_0 \exp F_{8I}. \tag{42}
\]

Figure 4 shows the relative errors in the exponential resummed forms of the secular changes of \( E, L, \) and \( C \), estimated by using Eq. (39). We also show the relative errors in the Taylor-type formulae in the same graphs for comparison. One will find that the relative errors in the exponential resummed forms are less than those in the Taylor-type formulae in most cases, except for \( \langle dC/dt \rangle \).
3.4. Convergence with respect to $v$ and $e$ of the analytic formulae

Apart from comparisons to the numerical results, we may also discuss the convergence property in our PN formulae with respect to $v$ and $e$ by investigating the contribution of each order of $v$ and $e$ in the formulae, although this is a rough estimation.
First we assess the PN convergence of our formulae. For this purpose, we introduce $\Delta_n$ as

$$
\left( \frac{dE}{dt} \right)_i^{\text{PN}} = \left( \frac{dE}{dt} \right)_i^{\text{Exp}} + \sum_{n=0}^{8} \Delta_n,
$$

where $p = 1/v^2$ and $\Delta_n$ is the $O(v^n)$ term in the PN formula of $(dE/dt)_i$, e.g. $\Delta_0 = 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4$, $\Delta_1 = 0$, and $\Delta_2 = (-\frac{1247}{336} - \frac{9181}{672} e^2 + \frac{809}{128} e^4 + \frac{8609}{5376} e^6) v^2$. $\Delta_n$ depends on $(q, p, e, Y)$ in general, although we omit the argument for simplicity.

Since $\Delta_n$ shows the relative importance of the $O(v^n)$ term in the PN formulae, it can be used to investigate the convergence property with respect to $v$: it is expected that $|\Delta_{n+1}| < |\Delta_n|$ for moderately large $n$ if the PN formula converges. In Fig. 5, we plot the relative contribution of each order, $\Delta_n$, as a function of $p$ for several sets of $(e, \iota)$ and $q = 0.9$. From this figure, one may find that $\Delta_n$ does not strongly depend on the inclination angle, $\iota$, as shown in Sect. 3.2, while it strongly depends on $e$. The convergence gets worse when the orbital eccentricity becomes larger. This tendency is particularly evident in the small-$p$ region. Fixing the value of $p$, the orbit with larger $e$ passes by closer to the central black hole and will be affected by the stronger gravitational field. Hence the PN convergence is expected to be worse when the eccentricity becomes larger.

Fig. 4. The relative errors in the PN formulae and the exponential resummation formulae for the secular changes of the orbital parameters, $\{E, L, C\}$, as functions of the semi-latus rectum $p$ for $q = 0.9$, $\iota = 50^\circ$, and $e = 0.1, 0.4, 0.7$ (from top to bottom). We truncated the plots at $p = 6$ because the relative errors in the PN formulae get too large in $p < 6$ to be meaningful. Using the exponential resummation, the accuracy is improved in most cases. For example, the region to satisfy $\Delta_E < 10^{-3}$ is improved from $p \gtrsim 50$ to $p \gtrsim 40$ for $e = 0.1$, $p \gtrsim 80$ to $p \gtrsim 60$ for $e = 0.4$, and $p \gtrsim 120$ to $p \gtrsim 100$ for $e = 0.7$. This would suggest trying to apply resummation methods to the PN formulae even in the case of general orbits.
becomes larger. This tendency is clear for small $p$.

We have derived the secular changes of the orbital parameters, the energy, azimuthal angular momentum, and Carter parameter of a point particle orbiting a Kerr black hole, by using the post-Newtonian approximation in the first-order black hole perturbation theory. We have extended the previous approximation in the first-order black hole perturbation theory. We have extended the previous
determination of the semi-latus rectum $p$ for $e = 0.1, 0.4$, and 0.7 (from left to right), and $\iota = 20^\circ$, $50^\circ$ and $80^\circ$ (from top to bottom) when $q = 0.9$. We truncated the plots at $p = 6$ because the relative contributions get too large in $p < 6$ to be meaningful. It is expected that $|\Delta_{n+1}| < |\Delta_n|$ for moderately large $n$ if the PN formula converges. As shown in Sect. 3.2, $\Delta_n$ does not strongly depend on $\iota$ for a fixed $e$, although $\Delta_n$ strongly depends on $e$. In fact, the convergence seems worse as the orbital eccentricity becomes larger. This tendency is clear for small $p$, e.g. $p \lesssim 10$.

Next, in order to investigate the convergence of the expansion with respect to the orbital eccentricity in the PN formula, we introduce $A_n$ as

$$
\frac{dE}{dt}^{\text{PN}} = \left. \frac{dE}{dt} \right|_{n} \left[ A_0 e^0 + A_2 e^2 + A_4 e^4 + A_6 e^6 \right].
$$

(44)

where the term $A_0$ coincides with the energy flux for circular orbits and $A_n = 0$ when $n$ is odd.

One may ask whether the condition $|A_{2n+4} e^{2n+2}| < |A_{2n} e^{2n}|$ is satisfied for moderately large integer $n$ if the series converges. From Fig. 6, it is found that the condition is satisfied in most cases. As expected, the convergence becomes slower when the eccentricity is larger. In particular, the convergence gets worse when $p \lesssim 10$ in the $e = 0.7$ case. The calculation of the higher PN corrections will be necessary to improve the bad convergence for small $p$. We also note that $A_n$ does not strongly depend on $\iota$ for a fixed $q$, as in Sect. 3.2.

4. Summary

We have derived the secular changes of the orbital parameters, the energy, azimuthal angular momentum, and Carter parameter of a point particle orbiting a Kerr black hole, by using the post-Newtonian approximation in the first-order black hole perturbation theory. We have extended the previous
work [15], which derived formulae up to the 2.5PN order with the second-order correction with respect to the eccentricity, to the 4PN order with the sixth-order correction with respect to the eccentricity. We have also included the contribution due to the black hole absorption, which was not included in [15]. As shown in the case of equatorial, circular orbits [27,28], we have found that the secular changes of the three orbital parameters due to the absorption appear at the 2.5PN (4PN) from the leading order in the Kerr (Schwarzschild) case, and that the 2.5PN and 3.5PN contributions of the absorption to the secular change of the particle’s energy can be positive for \( q > 0 \), which implies that a superradiance can be realized in the Kerr case. We have also found that the superradiant contributions in the secular change of the energy get smaller when the inclination angle becomes larger and they vanish for polar (\( \nu = 0 \)) orbits. This means that the superradiant scattering may be suppressed for inclined orbits [29].

To investigate the accuracy in our 4PN formulae, we have compared the formulae to high-precision numerical results [18] in Sect. 3.2. We have found that the accuracy gets worse when the orbital velocity and the orbital eccentricity become larger, as expected. If the relative error in the 4PN \( O(e^6) \) formula for the secular change of the energy is required to be less than \( 10^{-5} \), the parameter region to satisfy it might be \( p \gtrsim 50 \) for \( e = 0.1 \), \( p \gtrsim 80 \) for \( e = 0.4 \), and \( p \gtrsim 120 \) for \( e = 0.7 \) when \( q = 0.9 \). The region does not strongly depend on the orbital inclination angle. From Fig. 1, one can clearly
find the improvement of the accuracy in our PN formulae from the previous work at the 2.5PN order and the second-order correction in the orbital eccentricity [15] whose relative error is larger than $10^{-2}$ for $p \gtrsim 100$ when $e \gtrsim 0.4$ since, in this region, the error due to the truncation of the expansion with respect to the orbital eccentricity is larger than that of the PN expansion.

One may improve the accuracy of our PN formulae by using resummation methods. In this paper, we have applied the exponential resummation [38] to our 4PN formulae and confirmed that the resummation method improves the accuracy in most cases investigated here. For example, we found that the region in which the relative errors are less than $10^{-5}$ can be extended from $p \gtrsim 50$ to $p \gtrsim 40$ for $e = 0.1$, $p \gtrsim 80$ to $p \gtrsim 60$ for $e = 0.4$, and $p \gtrsim 120$ to $p \gtrsim 100$ for $e = 0.7$.

We also investigated the convergence properties of the PN expansion and the expansion with respect to the orbital eccentricity, respectively. Both convergences get worse when the semi-latus rectum is smaller; in other words, the gravitational field becomes stronger. This tendency gets clearer in the case of large eccentricity, in which the particle passes by closer to the central black hole.

In order to improve the accuracy and convergence of the 4PN $O(e^6)$ formulae near the central black hole and to obtain the physical information of the source in the strong-field region, it is necessary to derive the higher-order corrections of the PN expansion and the expansion with respect to the eccentricity. It may be possible to avoid the expansion with respect to the eccentricity and to derive the PN formulae applicable to arbitrary eccentricity. So far, the PN formulae of the rate of the energy loss without performing the expansion with respect to the eccentricity had been derived for equatorial orbits in [30–33]. The extension of these results to the case of inclined orbits is challenging: we can obtain analytic expressions for general bound geodesic orbits in Kerr spacetime without performing the expansion with respect to the eccentricity or the inclination by using results in Ref. [39], while we need to reformulate the source term of the Teukolsky equation and the derivation of the partial waves constructed from the source term. We would like to leave this to future work.

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Appendix A. PN formulae for the orbital parameters and fundamental frequencies

In this section, we present the PN formulae of the orbital parameters, $\{\hat{E}, \hat{L}, \hat{C}\}$, and the fundamental frequencies, $\{\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi\}$. Here we show the formulae up to the 3PN $O(e^6)$ order to save space although it is possible to calculate them to the higher order. The higher order results will be publicly available online [26].

$$\hat{E} = 1 + \left\{ -\frac{1}{2} + \frac{1}{2} e^2 \right\} v^2 + \left\{ \frac{3}{8} - \frac{3}{4} e^2 + \frac{3}{8} e^4 \right\} v^4 + \left\{ -Yq + 2Yq e^2 - Yq e^4 \right\} v^5 + \left\{ \frac{27}{16} + \frac{1}{2} Y^2 q^2 + \left( -\frac{49}{16} - Y^2 q^2 \right) e^2 + \left( \frac{17}{16} + \frac{1}{2} Y^2 q^2 \right) e^4 + \frac{5}{16} e^6 \right\} v^6, \tag{A1}$$
\[ \hat{L} = p v \left[ Y + \left\{ \frac{3}{2} Y + \frac{1}{2} Y e^2 \right\} v^2 + \left\{ -3 Y^2 q - q Y^2 e^2 \right\} v^3 
+ \left\{ \frac{27}{8} Y + Y^3 + \left( \frac{9}{4} Y + Y^3 q^2 \right) e^2 + \frac{3}{8} Y e^4 \right\} v^4 + \left\{ -\frac{15}{2} Y^2 q - 7 q Y^2 e^2 - \frac{3}{2} Y^2 q e^4 \right\} v^5 
+ \left\{ \frac{135}{16} Y - 4 Y q^2 + 9 Y^3 q^2 + \left( \frac{135}{16} Y - 4 Y q^2 + 9 Y^3 q^2 \right) e^2 
+ \left( \frac{45}{16} Y + 2 Y^3 q^2 \right) e^4 + \frac{5}{16} Y e^6 \right\} v^6 \right] , \] (A2)

\[ \hat{C} = \left\{ \frac{1}{Y^2} - 1 \right\} L^2 , \]

\[ = p^2 v^2 \left( 1 - Y^2 \right) \left[ 1 + \left( 3 + e^2 \right) v^2 - 2 q Y \left( 3 + e^2 \right) v^3 
+ \left\{ 9 + 2 Y^2 q^2 + \left( 6 + 2 Y^2 q^2 \right) e^2 + e^4 \right\} v^4 - 4 q Y \left( 2 + e^2 \right) \left( 3 + e^2 \right) v^5 
+ \left\{ 27 - 8 q^2 + 30 Y^2 q^2 + \left( 27 + 28 Y^2 q^2 - 8 q^2 \right) e^2 + \left( 9 + 6 Y^2 q^2 \right) e^4 + e^6 \right\} v^6 \right] , \] (A3)

\[ \Omega_t = p^2 \left[ 1 + \frac{3}{2} e^2 + \frac{15}{8} e^4 + \frac{35}{16} e^6 + \left\{ \frac{3}{2} - \frac{1}{4} e^2 - \frac{15}{16} e^4 - \frac{45}{32} e^6 \right\} v^2 
+ \left\{ 2 q Y e^2 + 3 Y q e^4 + \frac{15}{4} Y q e^6 \right\} v^3 
+ \left\{ \frac{27}{8} - \frac{1}{2} Y^2 q^2 + \frac{1}{2} q^2 + \left( -\frac{99}{16} + q^2 - 2 Y^2 q^2 \right) e^2 
+ \left( -\frac{567}{64} + \frac{21}{16} q^2 - \frac{45}{16} Y^2 q^2 \right) e^4 + \left( -\frac{1371}{128} + \frac{25}{16} q^2 - \frac{55}{16} Y^2 q^2 \right) e^6 \right\} v^4 
+ \left\{ -3 Y q + \frac{43}{2} Y q e^2 + \frac{231}{8} Y q e^4 + \frac{555}{16} Y q e^6 \right\} v^5 
+ \left\{ \frac{135}{16} - \frac{1}{4} q^2 + \frac{3}{4} Y^2 q^2 + \left( -\frac{1233}{32} + \frac{47}{4} q^2 - \frac{75}{2} Y^2 q^2 \right) e^2 
+ \left( \frac{6567}{128} + \frac{499}{32} q^2 - \frac{1577}{32} Y^2 q^2 \right) e^4 + \left( -\frac{15565}{256} + \frac{75}{4} q^2 - \frac{1887}{32} Y^2 q^2 \right) e^6 \right\} v^6 \right] , \] (A4)

\[ \Omega_r = p v \left[ 1 + \left\{ -\frac{3}{2} + \frac{1}{2} e^2 \right\} v^2 + \left\{ 3 Y q - Y q e^2 \right\} v^3 
+ \left\{ -\frac{45}{8} + \frac{1}{2} q^2 - 2 Y^2 q^2 + \left( \frac{1}{4} q^2 + \frac{1}{4} Y^2 q^2 \right) e^2 + \frac{3}{8} e^4 \right\} v^4 \right] . \]
Here we show the PN formulae of the Fourier coefficients in Eqs. (15), (16), (18), and (19) up to the 3PN order. The 4PN results obtained in this work will be available online [26].

In this work, we follow the same procedure as in [15] to derive the amplitudes of the partial waves, $Z_{A}^{H,\infty}$ in (23). In the formal expression, the dependence of $\phi^{(n)}$ appears in the form of the combination $X \equiv \sin \theta e^{i\phi^{(n)}}$, which can be expressed in the Fourier series as

$$X = p \sum_{n_{0}=0}^{\infty} \left[ X_{n_{0}}^{\|} \cos n_{0} \Omega_{\theta} \lambda + i X_{n_{0}}^{\perp} \sin n_{0} \Omega_{\theta} \lambda \right]. \quad (B1)$$

We therefore show the Fourier coefficients of $X$ instead of $\phi^{(n)}$.

\begin{align}
\Omega_{\theta} &= p v \left[ 1 + \left\{ \frac{3}{2} + \frac{1}{2} e^{2} \right\} v^{2} - \left\{ 3 Yq + Yq e^{2} \right\} v^{3} \\
&\quad + \left\{ \frac{27}{8} + \frac{7}{4} Yq^{2} - \frac{1}{4} q^{2} + \frac{9}{4} q^{2} + \frac{1}{4} Yq^{2} \right\} e^{2} + \frac{3}{8} e^{4} \right\} v^{4} \\
&\quad - \left\{ \frac{15}{2} Yq + 7 Yq e^{2} + \frac{3}{2} Yq e^{4} \right\} v^{5} \\
&\quad + \left\{ \frac{135}{16} + \frac{57}{8} Yq^{2} - \frac{27}{8} q^{2} + \frac{135}{16} - \frac{19}{4} q^{2} + \frac{45}{4} Yq^{2} \right\} e^{2} \\
&\quad + \left\{ \frac{45}{16} + \frac{1}{8} q^{2} + \frac{13}{8} Yq^{2} \right\} e^{4} + \frac{5}{16} e^{6} \right\} v^{6}, \quad (A6) \\
\end{align}

\begin{align}
\Omega_{\phi} &= p v \left[ 1 + \left\{ \frac{3}{2} + \frac{1}{2} e^{2} \right\} v^{2} + \left\{ 2 q - 3 Yq - Yq e^{2} \right\} v^{3} \\
&\quad + \left\{ -\frac{3}{2} Yq^{2} + \frac{7}{4} Yq^{2} - \frac{1}{4} q^{2} - \frac{27}{8} + \frac{9}{4} q^{2} + \frac{1}{4} Yq^{2} \right\} e^{2} + \frac{3}{8} e^{4} \right\} v^{4} \\
&\quad + \left\{ 3 q - \frac{15}{2} Yq + (4 q - 7 Yq) e^{2} - \frac{3}{2} Yq e^{4} \right\} v^{5} \\
&\quad + \left\{ -\frac{9}{4} Yq^{2} + \frac{57}{8} Yq^{2} + \frac{135}{16} - \frac{27}{8} q^{2} + \frac{135}{16} - \frac{19}{4} q^{2} - \frac{35}{4} Yq^{2} + \frac{45}{4} Yq^{2} \right\} e^{2} \\
&\quad + \left\{ \frac{45}{16} + \frac{1}{8} q^{2} + \frac{13}{8} Yq^{2} \right\} e^{4} + \frac{5}{16} e^{6} \right\} v^{6}. \quad (A7) \\
\end{align}
B.1. Radial component

\[
\alpha_0 = 1 + e^2 \left\{ \frac{1}{2} - \frac{1}{2} v^2 + q Y v^3 + \left( -3 + \left( \frac{1}{2} - Y^2 \right) q^2 \right) v^4 \right. \\
+ 10 q Y v^5 + \left( -18 + \left( \frac{11}{2} - 18 Y^2 \right) q^2 \right) v^6 \left. \right\} \\
+ e^4 \left\{ \frac{3}{8} - \frac{3}{8} v^2 + \frac{3}{4} q Y v^3 + \left( -\frac{33}{16} + \left( \frac{5}{16} - \frac{11}{16} Y^2 \right) q^2 \right) v^4 \right. \\
+ \frac{27}{4} q Y v^5 + \left( -\frac{189}{16} + \left( -\frac{185}{16} Y^2 + \frac{61}{16} \right) q^2 \right) v^6 \left. \right\} \\
+ e^6 \left\{ \frac{5}{16} \frac{5}{16} - \frac{5}{16} v^2 + \frac{5}{8} q Y v^3 + \left( -\frac{27}{16} + \left( \frac{1}{4} - \frac{9}{16} Y^2 \right) q^2 \right) v^4 \right. \\
+ \frac{11}{2} q Y v^5 + \left( -\frac{19}{2} + \left( -\frac{75}{8} Y^2 + \frac{49}{16} \right) q^2 \right) v^6 \left. \right\}, \quad (B2) \\
\alpha_1 = e + e^3 \left\{ \frac{3}{4} - \frac{1}{2} v^2 + q Y v^3 + \left( -\frac{51}{16} + \left( \frac{7}{16} - \frac{15}{16} Y^2 \right) q^2 \right) v^4 \right. \\
+ \frac{43}{4} q Y v^5 + \left( -\frac{81}{4} + \left( -\frac{153}{8} Y^2 + \frac{11}{2} \right) q^2 \right) v^6 \left. \right\} \\
+ e^5 \left\{ \frac{5}{8} - \frac{1}{2} v^2 + q Y v^3 + \left( -\frac{93}{32} + \left( \frac{13}{32} - \frac{29}{32} Y^2 \right) q^2 \right) v^4 \right. \\
+ \frac{77}{8} q Y v^5 + \left( -\frac{277}{16} + \left( -\frac{133}{8} Y^2 + \frac{83}{16} \right) q^2 \right) v^6 \left. \right\}, \quad (B3) \\
\alpha_2 = e^2 \left\{ \frac{1}{2} + \frac{1}{2} v^2 - q Y v^3 + \left( 3 + \left( Y^2 - \frac{1}{2} \right) q^2 \right) v^4 \right. \\
- 10 q Y v^5 + \left( 18 + \left( 18 Y^2 - \frac{11}{2} \right) q^2 \right) v^6 \left. \right\} \\
+ e^4 \left\{ \frac{1}{2} + \frac{1}{2} v^4 + 2 q Y v^5 + \left( -\frac{11}{2} + \left( -4 Y^2 + \frac{1}{2} \right) q^2 \right) v^6 \left. \right\} \\
+ e^6 \left\{ \frac{15}{32} - \frac{5}{32} v^2 + \frac{5}{16} q Y v^3 + \left( -\frac{39}{32} + \left( \frac{1}{8} - \frac{9}{32} Y^2 \right) q^2 \right) v^4 \right. \\
+ \frac{17}{4} q Y v^5 + \left( -\frac{279}{32} + \left( -\frac{243}{32} Y^2 + 2 \right) q^2 \right) v^6 \left. \right\}, \quad (B4)
\[ \alpha_3 = e^3 \left\{ \frac{1}{4} + \frac{v^2}{2} - Yq v^3 + \left( \frac{51}{16} + \left( -\frac{7}{16} + \frac{15}{16} Y^2 \right) q^2 \right) v^4 \right. \\
- \frac{43}{4} Yq v^5 + \left( \frac{81}{4} + \left( \frac{153}{8} Y^2 - \frac{11}{2} \right) q^2 \right) v^6 \right\} \\
+ e^5 \left\{ \frac{5}{16} + \frac{v^2}{4} - \frac{Yq v^3}{2} + \left( \frac{69}{64} + \left( -\frac{13}{64} + \frac{29}{64} Y^2 \right) q^2 \right) v^4 \\
- \frac{53}{16} Yq v^5 + \left( \frac{135}{32} + \left( \frac{43}{8} Y^2 - \frac{69}{32} \right) q^2 \right) v^6 \right\}, \tag{B5} \]

\[ \alpha_4 = e^4 \left\{ \frac{1}{8} + \frac{3}{8} v^2 - \frac{3}{4} Yq v^3 + \left( \frac{41}{16} + \left( -\frac{5}{16} + \frac{11}{16} Y^2 \right) q^2 \right) v^4 \\
- \frac{35}{4} Yq v^5 + \left( \frac{277}{16} + \left( \frac{249}{16} Y^2 - \frac{69}{16} \right) q^2 \right) v^6 \right\} \\
+ e^6 \left\{ \frac{3}{16} + \frac{5}{16} v^2 - \frac{5}{8} Yq v^3 + \left( \frac{27}{16} + \left( -\frac{1}{4} + \frac{9}{16} Y^2 \right) q^2 \right) v^4 \\
- \frac{11}{2} Yq v^5 + \left( 9 + \left( \frac{75}{8} Y^2 - \frac{49}{16} \right) q^2 \right) v^6 \right\}, \tag{B6} \]

\[ \alpha_5 = e^5 \left\{ \frac{1}{16} + \frac{v^2}{4} - \frac{Yq v^3}{2} + \left( \frac{117}{64} + \left( -\frac{13}{64} + \frac{29}{64} Y^2 \right) q^2 \right) v^4 \\
- \frac{101}{16} Yq v^5 + \left( \frac{419}{32} + \left( \frac{45}{4} Y^2 - \frac{97}{32} \right) q^2 \right) v^6 \right\}, \tag{B7} \]

\[ \alpha_6 = e^6 \left\{ \frac{1}{32} + \frac{5}{32} v^2 - \frac{5}{16} Yq v^3 + \left( \frac{39}{32} + \left( -\frac{1}{8} + \frac{9}{32} Y^2 \right) q^2 \right) v^4 \\
- \frac{17}{4} Yq v^5 + \left( \frac{295}{32} + \left( \frac{243}{32} Y^2 - 2 \right) q^2 \right) v^6 \right\}, \tag{B8} \]

\[ \alpha_n = O \left( e^n \right) \text{ for } n \geq 7. \tag{B9} \]

\[ \beta_0 = 0, \tag{B10} \]

**B.2. Longitudinal component**
\[\beta_1 = 1 + \left(\frac{1}{16} - \frac{9}{16} Y^2\right) q^2 v^4 + \left(-\frac{1}{4} + \frac{9}{4} Y^2\right) q^2 v^6 + e^2 \left\{ \left(-\frac{1}{16} + \frac{9}{16} Y^2\right) q^2 v^4 + \left(-\frac{9}{4} Y^2 + \frac{1}{4}\right) q^2 v^6 \right\}, \quad (B11)\]
\[\beta_2 = 0, \quad (B12)\]
\[\beta_3 = \frac{1 - Y^2}{16} q^2 v^4 - \frac{1 - Y^2}{4} q^2 v^6 + e^2 \left\{ -\frac{1 - Y^2}{16} q^2 v^4 + \frac{1 - Y^2}{4} q^2 v^6 \right\}, \quad (B13)\]
\[\beta_n = \begin{cases} 0 & (n: \text{even}) \\ O(v^{2n-2}) & (n: \text{odd}) \end{cases} \quad (B14)\]

### B.3. \( r \)-part of the temporal component

\[
\frac{v}{p_1} \tau^{(r)}_1 = e \left\{ 2 + 4 v^2 - 6 Y q v^3 + \left(17 + \left(4 Y^2 - 1\right) q^2\right) v^4 - 54 Y q v^5 \right. \\
+ \left(88 + \left(84 Y^2 - 20\right) q^2\right) v^6 \left\} + e^3 \left\{ 3 + 3 v^2 - 4 Y q v^3 \\
+ \left(\frac{77}{8} + \left(\frac{21}{8} Y^2 - \frac{5}{8}\right) q^2\right) v^4 - \frac{57}{2} Y q v^5 + \left(\frac{173}{4} + \left(42 Y^2 - \frac{51}{4}\right) q^2\right) v^6 \right\} \\
+ e^5 \left\{ \frac{15}{4} + \frac{5}{2} v^2 - \frac{13}{4} Y q v^3 + \left(\frac{15}{2} + \left(\frac{17}{8} Y^2 - \frac{1}{2}\right) q^2\right) v^4 - \frac{45}{2} Y q v^5 \\
+ \left(\frac{67}{2} + \left(\frac{133}{4} Y^2 - 10\right) q^2\right) v^6 \right\}, \quad (B15)\]
\[
\frac{v}{p_2} \tau^{(r)}_2 = e^2 \left\{ \frac{3}{4} + \frac{7}{4} v^2 - \frac{13}{4} Y q v^3 + \left(\frac{81}{8} + \left(\frac{5}{2} Y^2 - \frac{7}{8}\right) q^2\right) v^4 - \frac{135}{4} Y q v^5 \\
+ \left(\frac{499}{8} + \left(\frac{55}{8} Y^2 - \frac{113}{8}\right) q^2\right) v^6 \right\} + e^4 \left\{ \frac{5}{4} + \frac{7}{4} v^2 - 3 Y q v^3 \\
+ \left(\frac{131}{16} + \left(\frac{37}{16} Y^2 - \frac{13}{16}\right) q^2\right) v^4 - \frac{103}{4} Y q v^5 + \left(\frac{691}{16} + \left(\frac{655}{16} Y^2 - \frac{197}{16}\right) q^2\right) v^6 \right\} \\
+ e^6 \left\{ \frac{105}{64} + \frac{105}{64} v^2 - \frac{175}{64} Y q v^3 + \left(\frac{905}{128} + \left(\frac{135}{64} Y^2 - \frac{95}{128}\right) q^2\right) v^4 - \frac{1413}{64} Y q v^5 \\
+ \left(\frac{4591}{128} + \left(\frac{2241}{64} Y^2 - \frac{1389}{128}\right) q^2\right) v^6 \right\}, \quad (B16)\]
\[
\frac{v}{p_3} \tau^{(r)}_3 = e^3 \left\{ \frac{1}{3} + v^2 - 2 Y q v^3 + \left(\frac{53}{8} + \left(\frac{13}{8} Y^2 - \frac{5}{8}\right) q^2\right) v^4 - \frac{45}{2} Y q v^5 \\
+ \left(\frac{523}{12} + \left(\frac{38}{4} Y^2 - \frac{39}{4}\right) q^2\right) v^6 \right\} + e^5 \left\{ \frac{5}{8} + \frac{5}{4} v^2 - \frac{19}{8} Y q v^3 \\
+ \left(\frac{7}{16} Y^2 - \frac{3}{4}\right) q^2\right) v^4 - \frac{91}{4} Y q v^5 + \left(\frac{647}{16} + \left(\frac{601}{16} Y^2 - \frac{175}{16}\right) q^2\right) v^6 \right\}, \quad (B17)\]
\[ \frac{v}{p} \hat{z}^{(r)}_4 = e^4 \left\{ \frac{5}{32} + \frac{19}{32} v^2 - \frac{39}{32} Yq v^3 + \left( \frac{137}{32} + \left( \frac{65}{64} Y^2 - \frac{13}{32} \right) q^2 \right) v^4 - \frac{473}{32} Yq v^5 \right. \\
+ \left( \frac{957}{32} + \left( \frac{1631}{64} Y^2 - \frac{207}{32} \right) q^2 \right) v^6 \right\} + e^6 \left\{ \frac{21}{64} v^2 - \frac{113}{64} Yq v^3 \right. \\
+ \left( \frac{89}{16} + \left( \frac{189}{128} Y^2 - \frac{19}{32} \right) q^2 \right) v^4 - \frac{1185}{64} Yq v^5 \right. \\
+ \left( \frac{553}{16} + \left( \frac{4005}{128} Y^2 - \frac{141}{16} \right) q^2 \right) v^6 \right\}, \]

(B18)

\[ \frac{v}{p} \hat{z}^{(r)}_5 = e^5 \left\{ \frac{3}{40} + \frac{7}{20} v^2 - \frac{29}{40} Yq v^3 + \left( \frac{27}{38} + \left( \frac{199}{20} Y^2 - \frac{1}{4} \right) q^2 \right) v^4 - \frac{189}{20} Yq v^5 \right. \\
+ \left( \frac{319}{60} + \left( \frac{1321}{80} Y^2 - \frac{331}{80} \right) q^2 \right) v^6 \right\}, \]

(B19)

\[ \frac{v}{p} \hat{z}^{(r)}_6 = e^6 \left\{ \frac{7}{20} v^2 - \frac{27}{64} Yq v^3 + \left( \frac{213}{128} + \left( \frac{23}{128} Y^2 - \frac{19}{128} \right) q^2 \right) v^4 - \frac{377}{64} Yq v^5 \right. \\
+ \left( \frac{4969}{384} + \left( \frac{665}{128} Y^2 - \frac{329}{128} \right) q^2 \right) v^6 \right\}, \]

(B20)

\[ \frac{v}{p} \hat{z}^{(r)}_n = O \left( e^n \right) \quad \text{(for } n \geq 7) \].

(B21)

### B.4. θ-part of the temporal component

\[ \frac{1}{p} \hat{z}^{(\theta)}_{t_4} = 0. \]

(B22)

\[ \frac{1}{p} \hat{z}^{(\theta)}_{t_2} = \frac{Y^2 - 1}{4} q^2 v^3 - \frac{Y^2 - 1}{2} q^2 v^5 + \frac{Y \left( Y^2 - 1 \right)}{4} \left( 3 + e^2 \right) q^3 v^6, \]

(B23)

\[ \frac{1}{p} \hat{z}^{(\theta)}_n = \begin{cases} 0 & \text{\( n \) : odd} \\ O \left( v^{2n-1} \right) & \text{\( n \) : even} \end{cases} \]

(B24)

### B.5. r-part of the azimuthal component

\[ \bar{\varphi}^{(r)}_1 = e \left\{ -2 q v^3 + 2 Y q^2 v^4 - 10 q v^5 + 18 Y q^2 v^6 \right\}, \]

(B25)

\[ \bar{\varphi}^{(r)}_2 = e^2 \left\{ -\frac{Y q^2 v^4}{4} + \frac{1}{2} q v^5 - \frac{3}{4} Y q^2 v^6 \right\}, \]

(B26)

\[ \bar{\varphi}^{(r)}_n = \begin{cases} O \left( v^{2n+1} \right) & \text{\( n \) : odd} \\ O \left( v^{2n} \right) & \text{\( n \) : even} \end{cases} \]

(B27)

### B.6. θ-part of the azimuthal component

\[ X_0^\theta = \frac{1 + Y}{2} - \frac{(9 Y - 1) \left( Y^2 - 1 \right)}{32} q^2 v^4 + \frac{(9 Y - 1) \left( Y^2 - 1 \right)}{8} q^2 v^6 \]

\[ + e^2 \left\{ \frac{(9 Y - 1) \left( Y^2 - 1 \right)}{32} q^2 v^4 - \frac{(9 Y - 1) \left( Y^2 - 1 \right)}{8} q^2 v^6 \right\}, \]

(B28)
\[ X_1^3 = 0, \]  
\[ X_2^3 = \frac{1 - Y}{2} + \frac{Y}{4} (Y^2 - 1) q^2 v^4 - Y (Y^2 - 1) q^2 v^6 \]  
\[ + e^2 \left\{ - \frac{Y}{4} (Y^2 - 1) q^2 v^4 + Y (Y^2 - 1) q^2 v^6 \right\}, \]  
\[ X_3^3 = 0, \]  
\[ X_4^3 = \frac{(Y + 1) (Y - 1)^2}{32} q^2 v^4 - \frac{(Y + 1) (Y - 1)^2}{8} q^2 v^6 \]  
\[ + e^2 \left\{ - \frac{(Y + 1) (Y - 1)^2}{32} q^2 v^4 + \frac{(Y + 1) (Y - 1)^2}{8} q^2 v^6 \right\}, \]  
\[ X_n^3 = \begin{cases} 0 & \text{(n: odd)} \\ O(v^{2n-4}) & \text{(n \geq 6: even)} \end{cases}, \]  
\[ X_0^3 = 0, \]  
\[ X_1^3 = 0, \]  
\[ X_2^3 = \frac{Y - 1}{2} - \frac{(5 Y + 1) (Y^2 - 1)}{16} q^2 v^4 + \frac{(5 Y + 1) (Y^2 - 1)}{4} q^2 v^6 \]  
\[ + e^2 \left\{ - \frac{(5 Y + 1) (Y^2 - 1)}{16} q^2 v^4 + \frac{(5 Y + 1) (Y^2 - 1)}{4} q^2 v^6 \right\}, \]  
\[ X_3^3 = 0, \]  
\[ X_4^3 = - \frac{(Y + 1) (Y - 1)^2}{32} q^2 v^4 + \frac{(Y + 1) (Y - 1)^2}{8} q^2 v^6 \]  
\[ + e^2 \left\{ - \frac{(Y + 1) (Y - 1)^2}{32} q^2 v^4 + \frac{(Y + 1) (Y - 1)^2}{8} q^2 v^6 \right\}, \]  
\[ X_n^3 = \begin{cases} 0 & \text{(n: odd)} \\ O(v^{2n-4}) & \text{(n \geq 6: even)} \end{cases}. \]  

**Appendix C. Secular evolution of the orbital parameters \( v, e, \) and \( Y \)**

An alternative set of the orbital parameters, \( J = \{ v, e, Y \} \), is also useful to specify the orbit. The secular changes of the parameters can be derived from those of \( I = \{ E, L, C \} \) as

\[ \left\langle \frac{dJ}{dt} \right\rangle_I = \sum_{I=I,E,L,C} (G^{-1})^J_I \left\langle \frac{dI}{dt} \right\rangle_I, \]  

where \( G^J_I = \frac{\partial(E, L, C)}{\partial(v, e, Y)} \) is the Jacobian matrix for the transformation from \( \{ E, L, C \} \) to \( \{ v, e, Y \} \).

---

\(^3\)It should be noted that, to calculate the Jacobian matrix up to \( O(v^6) \), one need to calculate \( \{ E, L, C \} \) up to \( O(v^8) \) since the leading terms do not depend on \( e \) and then the relative orders of accuracy of their first
Substituting the 3PN $O(e^6)$ formulae of $\langle dI/dt\rangle^\infty$ shown in Sect. 3 into the above relation, we obtain the secular changes of \{v, e, Y\} associated with the flux of gravitational waves to infinity as

$$
\left( \frac{dv}{dt} \right)_t^\infty = \left( \frac{dv}{dt} \right)_N \left[ 1 + \frac{7}{8} e^2 + \left\{ -\frac{743}{336} - \frac{55}{21} e^2 + \frac{8539}{2688} e^4 \right\} v^2 \\
+ \left\{ \frac{4 \pi - 133}{12} Y q + \left( \frac{97}{8} - \frac{379}{36} Y q \right) e^2 + \left( \frac{49}{32} \pi - \frac{475}{96} Y q \right) e^4 - \frac{49}{4608} \pi e^6 \right\} v^3 \\
+ \left\{ \frac{34103}{18144} - \frac{329}{96} q^2 + \frac{815}{96} Y^2 q^2 + \left( -\frac{526955}{12096} - \frac{929}{96} q^2 + \frac{477}{32} Y^2 q^2 \right) e^2 \\
+ \left( -\frac{1232809}{48348} - \frac{1051}{768} q^2 + \frac{999}{256} Y^2 q^2 \right) e^4 + \frac{105925}{16128} e^6 \right\} v^4 \\
+ \left\{ -\frac{4159}{672} \pi - \frac{1451}{56} Y q + \left( -\frac{48809}{1344} \pi - \frac{1043}{96} Y q \right) e^2 \\
+ \left( \frac{679957}{43008} \pi - \frac{15623}{336} Y q \right) e^4 + \left( \frac{4005097}{774144} \pi - \frac{35569}{1792} Y q \right) e^6 \right\} v^5 \\
+ \left\{ \frac{16447322263}{139708800} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma - \frac{3424}{105} \ln (2) - \frac{331}{192} q^2 \\
- \frac{289}{6} \pi Y q + \frac{145759}{1344} Y^2 q^2 + \left( \frac{8901670423}{11642400} + \frac{229}{6} \pi^2 - \frac{24503}{210} \gamma \right) \\
+ \frac{1391}{30} \ln (2) - \frac{78003}{280} \ln (3) + \frac{2129}{42} q^2 - \frac{4225}{24} \pi Y q + \frac{27191}{224} Y^2 q^2 \right\} e^2 \\
+ \left( \frac{269418340489}{372556800} + \frac{109}{4} \pi^2 + \frac{3042117}{1120} \ln (3) - \frac{11663}{140} \gamma \\
- \frac{418049}{84} \ln (2) - \frac{56239}{10752} q^2 - \frac{17113}{192} \pi Y q + \frac{414439}{3584} Y^2 q^2 \right\} e^4 \\
+ \left( \frac{174289281}{862400} - \frac{1044921875}{96768} \ln (5) + \frac{23}{16} \pi^2 - \frac{42667641}{3584} \ln (3) \\
+ \frac{94138279}{2160} \ln (2) - \frac{2461}{560} \gamma - \frac{3571}{3584} q^2 - \frac{108577}{13824} \pi Y q + \frac{41071}{1536} Y^2 q^2 \right\} e^6 \\
- \left( \frac{1712}{105} + \frac{24503}{210} e^2 + \frac{11663}{140} e^4 + \frac{2461}{560} e^6 \right) \ln v \right\} v^6 \right], \quad (C2)
$$

$$
\left( \frac{de}{dt} \right)_t^\infty = \left( \frac{de}{dt} \right)_N \left[ 1 + \frac{121}{304} e^2 + \left\{ -\frac{6849}{2128} - \frac{2325}{2128} e^2 + \frac{22579}{17024} e^4 \right\} v^2 \\
+ \left\{ \frac{985}{152} \pi - \frac{879}{76} Y q + \left( \frac{5969}{608} \pi - \frac{699}{76} Y q \right) e^2 + \frac{24217}{29184} \pi - \frac{1313}{608} Y q \right\} e^4 \right\} v^3 \right],
$$

derivatives with the eccentricity of the Jacobian matrix in Eq. (C1) are reduced by $O(e^2)$. For a similar reason, one also need to calculate $E$ up to 5PN order because the relative PN order of $\partial E/\partial v$ is reduced by $O(v^2)$ compared to $E$. 

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\[
\begin{align*}
&+ \left\{ \frac{286397}{38304} - \frac{3179}{608} q^2 + \frac{5869}{608} Y q^2 + \left( -\frac{2070667}{1216} + \frac{8925}{q^2} - \frac{633}{64} Y^2 q^2 \right) e^2 \\
&+ \left( -\frac{3506201}{306432} - \frac{3191}{4864} q^2 + \frac{9009}{4864} Y^2 q^2 \right) e^4 \right\} v^4 + \left\{ \frac{5678971}{817152} - \frac{442811}{17024} Y q \right\} e^4 \right\} v^5 \\
&+ \left\{ \frac{82283}{1995} - \frac{11021}{285} \ln 2 - \frac{234009}{5320} \ln 3 + \frac{11224646611}{46569600} \right\} v^6,
\end{align*}
\]

\[
\left\{ \frac{dY}{dt} \right\}_t = \left( \frac{dY}{dt} \right)_N \left[ 1 + \frac{189}{61} e^2 + \frac{285}{488} e^4 + \left\{ \frac{13}{244} Y q - \frac{277}{244} Y^2 q^2 - \frac{1055}{1952} Y q^3 \right\} e^2 + \left\{ -\frac{10461}{1708} - \frac{83723}{3416} e^2 - \frac{21261}{13664} e^4 + \frac{49503}{27328} e^6 \right\} v^2 + \left\{ \frac{290}{61} - \frac{12755}{3416} Y q + \left( \frac{1990}{61} - \frac{27331}{1708} Y q \right) e^2 + \left( \frac{21947}{976} - \frac{540161}{27328} Y q \right) e^4 + \left( \frac{38747}{35136} - \frac{140001}{27328} Y q \right) e^6 \right\} v^3 \right] ,
\]

where the leading contributions are given by

\[
\begin{align*}
\left( \frac{dv}{dt} \right)_N &= \frac{32}{5} \left( \frac{\mu}{M^2} \right) v^9 \left( 1 - e^2 \right)^{3/2} , \\
\left( \frac{de}{dt} \right)_N &= \frac{304}{15} \left( \frac{\mu}{M^2} \right) v^8 e \left( 1 - e^2 \right)^{3/2} , \\
\left( \frac{dY}{dt} \right)_N &= -\frac{244}{15} \left( \frac{\mu}{M^2} \right) v^{11} q \left( 1 - e^2 \right)^{3/2} \left( 1 - Y^2 \right) .
\end{align*}
\]
In the same way, substituting the 3.5PN $O(e^6)$ formulae of $\langle dI/dt \rangle^H_t$ shown in Sect. 3 into Eq. (C1), we obtain the secular changes of $\{v, e, Y\}$ associated with the flux of gravitational waves to the horizon as

$$
\left\langle \frac{dv}{dt} \right\rangle^H_t = \left( \frac{dv}{dt} \right)_N \left[ -\frac{1}{256} \left\{ 8 + 24 e^2 + 3 e^4 \right\} \left\{ 8 + 9 q^2 + 15 Y^2 q^2 \right\} Y q v^5 \\
\left\{ -\frac{11}{8} - \frac{189}{64} q^2 - \frac{15}{64} Y^2 q^2 + \left( -\frac{69}{8} - \frac{81}{4} q^2 + \frac{45}{32} Y^2 q^2 \right) e^2 \\
+ \left( -\frac{381}{64} - \frac{7479}{512} Y^2 q^2 + \frac{1035}{512} Y^2 q^2 \right) e^4 \\
+ \left( -\frac{11}{32} - \frac{423}{512} q^2 + \frac{45}{512} Y^2 q^2 \right) e^6 \right\} Y q v^7 \right],
$$

(C6)

$$
\left\langle \frac{de}{dt} \right\rangle^H_t = \left( \frac{de}{dt} \right)_N \left[ -\frac{33}{4864} \left\{ 8 + 12 e^2 + e^4 \right\} \left\{ 8 + 9 q^2 + 15 Y^2 q^2 \right\} Y q v^5 \\
\left\{ -\frac{453}{152} - \frac{8127}{1216} q^2 - \frac{45}{1216} Y^2 q^2 + \left( \frac{2979}{304} - \frac{28593}{1216} q^2 + \frac{1485}{608} Y^2 q^2 \right) e^2 \\
+ \left( -\frac{5649}{1216} - \frac{111591}{9728} q^2 + \frac{16515}{9728} Y^2 q^2 \right) e^4 \right\} Y q v^7 \right],
$$

(C7)

$$
\left\langle \frac{dY}{dt} \right\rangle^H_t = \left( \frac{dY}{dt} \right)_N \left[ -\frac{3}{7808} \left\{ 8 + 24 e^2 + 3 e^4 \right\} \left\{ 16 + 33 q^2 + 15 Y^2 q^2 \right\} v^2 \\
\left\{ -\frac{51}{122} - \frac{585}{1952} Y^2 q^2 - \frac{1953}{1952} q^2 + \left( -\frac{225}{61} - \frac{16875}{1952} q^2 + \frac{3375}{1952} Y^2 q^2 \right) e^2 \\
+ \left( -\frac{2961}{976} - \frac{109863}{15616} q^2 + \frac{16335}{15616} Y^2 q^2 \right) e^4 \\
+ \left( -\frac{171}{976} - \frac{3159}{7808} q^2 + \frac{405}{7808} Y^2 q^2 \right) e^6 \right\} v^4 \right].
$$

(C8)

Actually, we can obtain the higher PN results by using the 4PN $O(e^6)$ formulae for the secular changes of $\{E, L, C\}$, although we do not present them in the text. The full expressions of $\langle dJ/dt \rangle^\infty_t$ and $\langle dJ/dt \rangle^H_t$ for $J = \{v, e, Y\}$ will be available online [26].

Here we make a comment on the reliable order of the expansion with respect to $e$ in $\langle de/dt \rangle_t$. By using Eq. (C1), $\langle de/dt \rangle_t$ can be calculated from the linear combination of the secular changes of $\{E, L, C\}$. Since the leading order of the $(e, I)$ component of the inverse Jacobian matrix is $O(1/e)$, each term in the linear combination is apparently $O(1/e)$. However, the $O(1/e)$ contribution turns out to vanish due to a cancellation in taking the combination, and hence $\langle de/dt \rangle_t$ is $O(e)$, which corresponds to the well-known fact that circular orbits remain circular [40,41], i.e. $\langle de/dt \rangle_t = 0$ when $e = 0$. This cancellation reduces the reliable order in $\langle de/dt \rangle_t$ by $O(e^2)$, compared to the order of $\langle dI/dt \rangle_t$ for $I = \{E, L, C\}$. Since we calculate $\langle dI/dt \rangle_t$ up to $O(e^6)$ in this paper, we can obtain $\langle de/dt \rangle_t$ correctly up to $O(e^4)$ from the leading order.
The relative errors in the 4PN formulae for the secular changes of the orbital parameters

\[ \Delta_\varepsilon, \Delta_\iota, \Delta_\gamma \]

defined in a similar manner to Eq. (39), as a function of the semi-latus rectum \( p \) for \( q = 0.9, e = 0.1, 0.4, \) and 0.7 (from left to right), and \( \iota = 50^\circ \). We truncated the plots at \( p = 6 \) because the relative errors get too large in \( p < 6 \) to be meaningful. The relative error in the previous 2.5PN \( O(e^2) \) formula given in [15] strays off the \( p^{-3} \) line earlier than the 2.5PN \( O(e^2) \) formula in this paper. This trend is clearer for larger \( e \). The relative errors in the 3PN \( O(e^4) \) and 4PN \( O(e^6) \) formulae fall off faster than \( p^{-3} \) and \( p^{-4} \) for small \( e \) cases as expected, while this is not the case for \( e = 0.7 \) because of the higher-order correction of \( e \) than \( O(e^4) \).

\[ \langle dv/dt \rangle_\varepsilon^\infty \text{ and } \langle dY/dt \rangle_\varepsilon^\infty \text{ in Eqs. (C2) and (C4) are consistent up to the 2.5PN } O(e^2) \text{ order with the previous results in Ref. [15], while we find inconsistency in the } O(e^2) \text{ terms of the formula for } \langle de/dt \rangle_\varepsilon^\infty \text{ in [15]. This may be explained by the reduction in the reliable order mentioned above: the calculations of } \langle dI/dt \rangle_\varepsilon \text{ in [15] are done up to } O(e^2) \text{, and therefore the resultant formula of } \langle de/dt \rangle_\varepsilon \text{ is reliable only at the leading order. We can also confirm it numerically. In Fig. C1 we show the relative errors in the two analytic formulae by comparing to numerical results [18] in a similar manner to Eq. (39). It can be found that the relative error in the previous 2.5PN } O(e^2) \text{ formula strays out of the expected power law, } p^{-3} \text{, earlier than that in our 2.5PN } O(e^2) \text{ formula. This trend is clearer for larger eccentricity. We can also confirm the validity of our formula by seeing that the relative errors in our 4PN } O(e^4) \text{ formula fall off faster than } p^{-4}. \left( \text{The leading PN order of the difference in the } O(e^2) \text{ terms between the previous and our formulae is } \frac{39}{50} e^2 v^2 \text{. If our formula contains any error in the } e^2 v^2 \text{ term, the relative error will not fall off faster than } p^{-4} \text{ for large } p. \right) \]

A similar reduction in the PN order occurs in the calculation of \( \langle dY/dt \rangle_\varepsilon^\infty \): although each term in the linear combination of Eq. (C1) for \( J = Y \) is \( O(v^6) \), the terms at the first two orders, \( O(v^8) \) and \( O(v^{10}) \), vanish due to a cancellation in taking the combination. As a result, the leading order
of \( \langle dY/dt \rangle \), is \( O (v^{11}) \) and hence the reliable order relative to the leading term is reduced to \( O (v^5) \) (2.5PN order) when we have \( \langle dI/dt \rangle \) for \( I = \{ E, L, C \} \) up to \( O (v^8) \) (4PN order).

In Fig. C2, we show the relative errors in the analytic PN formulae for the secular changes of the orbital parameters, \( \{ v, e, Y \} \), derived from the 4PN \( O (e^0) \) formulae of \( \langle dI/dt \rangle \), for \( I = \{ E, L, C \} \). Similarly to Fig. 1, the relative errors in the analytic formulae for \( \langle dv/dt \rangle \) and \( \langle de/dt \rangle \) as functions of the semi-latus rectum \( p \) fall off faster than \( p^{-4} \) when the eccentricity is small. Observe, however, that the relative error in the analytic formula for \( \langle dY/dt \rangle \) falls off faster than \( p^{-5/2} \), but slower than \( p^{-4} \).

From the leading-order expressions in Eq. (C5), one will find the well-known fact that equatorial orbits stay in the equatorial plane \([10,15,42]\), i.e. \( \langle dY/dt \rangle = 0 \) when \( Y = 1 \). In the Schwarzschild case \( (q = 0) \), the secular changes of \( v \) and \( e \) do not depend on \( Y \) in addition to \( \langle dY/dt \rangle = 0 \). This implies that the orbital plane can be fixed on the equatorial plane \( (\theta = \pi/2) \) due to the spherical symmetry of Schwarzschild spacetime.

One will also find that the radiation reaction reduces the orbital eccentricity and increases the orbital velocity since \( \langle de/dt \rangle_N \leq 0 \) and \( \langle dv/dt \rangle_N \geq 0 \) \([40,41]\), while the radiation reaction increases (decreases) the inclination angle since \( \langle dY/dt \rangle_N \leq 0 \) \((\langle dY/dt \rangle_N \geq 0) \) when \( q \geq 0 \) \((q \leq 0) \) \([10,15,42]\). Moreover, the secular change of the inclination angle is smaller than those of the other orbital parameters since \( \langle d\ln e/d\ln v \rangle_t = O (v^0) \) and \( \langle d\ln Y/d\ln v \rangle_t = O (v^3) \).

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