Quasi-coherent structures and flows in turbulent transport

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Abstract. The nonlinear effects that appear in test particle (tracer) transport in two-dimensional incompressible velocity fields are discussed. Test particle trajectories show both random and quasi-coherent aspects. The coherent motion is associated with trapping or eddying in the structure of the stochastic field. It generates quasi-coherent trajectory structures. The random motion leads to diffusive transport while the structures determine a micro-confinement process. The strength of each of these aspects depends on the parameters of the turbulence. The transport coefficients and the average size of the quasi-coherent structures are determined as function of the characteristics of the turbulence. The transport process is completely different in the presence of structures in the sense that the dependence on the parameters is different. The quasi-coherence of the motion can also be represented by the generation of flows. We show that tracers flows yield in a stochastic potential with space-dependent amplitude.

1. Introduction
Tracer transport in stochastic velocity fields or the stochastic advection process is of fundamental importance in plasma physics, astrophysics, fluid mechanics and atmospheric and oceanic sciences. The trajectories are obtained from equations that are nonlinear due to the space-dependence of the velocity, and they can have rather complex structures in important special cases. One example is the turbulence in incompressible media that is dominantly two dimensional, which has a self-organizing character [1], [2]. It consists of the generation of quasi-coherent structure (vortices). This property appears at the basic level of particle (tracer) trajectories. They are random sequences of trapping or eddying events and long jumps. The trapping process [3]-[5] strongly modifies the diffusion coefficients and leads to non-Gaussian distribution of displacements. The direct numerical simulations largely dominate the analytical advance in this case.

The trapping events appear when the trajectory arrives around the maxima or the minima of the stream function (potential). The process is quasi-coherent because it affects all the particles situated in these regions in the same way. These trajectories form intermittent quasi-coherent structures similar to fluid vortices and represent eddying regions. The trapping events alternate with large displacements that correspond to large size contour lines of the potential. These large jumps are random. The transport is essentially determined by them, while the trapping events have negligible contributions.

The transport coefficients and the characteristics of the quasi-coherent structures are determined as function of the statistical parameters of the turbulence. The transport process is strongly modified in the presence of trajectory structures. We show that quasi-coherent flows can also be generated due to trapping. Our instrument, the decorrelation trajectory method [6], [7], is essentially analytical.

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2. Trajectory trapping or eddying in stochastic potential fields

The origin of trapping is the Hamiltonian structure of the equation of motion

\[
\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}, t), \quad \nu_j(\mathbf{x}, t) = \epsilon_{ij} \frac{\partial \phi(\mathbf{x})}{\partial x_j},
\]

where \(\mathbf{x}(t)\) is the trajectory in rectangular coordinates \(\mathbf{x} = (x_1, x_2)\), \(\epsilon_{12} = 1, \epsilon_{21} = -1, \epsilon_{ij} = 0\), and \(\phi(\mathbf{x}, t)\) is the stream function or the potential that defines the two-dimensional velocity field \(\mathbf{v}(\mathbf{x}, t)\).

The two components of the trajectory are couplet Hamiltonian variables.

The trajectories are periodic and they wind on the contour lines of the potential when the latter is time independent. This defines the state of permanent trapping. The Lagrangian potential is invariant in these conditions and the transport is subdiffusive because the particles are tied on the fixed contour lines of the potential.

If the motion is weakly perturbed (by time variation of the potential or by other components of the motion that can by introduced in Eq. (1)), the trapping is temporary and alternates with large jumps. The statistical importance of the trapping events depends on the strength of the perturbation that is represented by dimensionless factors, which are usually ratios of characteristic times. Two characteristic times are defined in (1): the correlation time of the potential \(\tau_c\) that gives the scale of the time variation of \(\phi(\mathbf{x}, t)\) and the time of flight of the particles \(\tau_{fl} = \lambda_c/V\), where \(\lambda_c\) is the correlation length of \(\phi(\mathbf{x}, t)\) and \(V\) is the amplitude of the velocity. These characteristic times define the Kubo number \(K = \tau_c/\tau_{fl}\), which is a measure of trapping in time dependent potentials. Namely, the permanent trapping corresponds to \(K \rightarrow \infty\), temporary trapping exists for \(K > 1\), and the statistical relevance of trapping is a decreasing function of \(K\). Any other perturbation of the basic Hamiltonian motion defines a characteristic time \(\tau_d\) for the decorrelation of the trajectory from the potential, and a dimensionless parameter similar to Kubo number, the decorrelation number \(K_d = \tau_d/\tau_{fl}\).

The trapping events appear when the trajectory arrives around the maxima and the minima of the potential. The process is quasi-coherent because it affects all the particles situated in these regions in the same way. The high degree of coherence of the trapped trajectories is evidenced in a study of the statistical properties of the distance \(\delta x\) between neighbor trajectories [7]. The time evolution of \(\langle \delta x^2(t) \rangle\) is very slow, which shows that neighbor particles have a coherent motion for a long time compared to \(\tau_c\). They are characterized by a strong clump effect. The increase of \(\langle \delta x^2(t) \rangle\) is slower than the Richardson law. These trajectories form intermittent quasi-coherent structures similar to fluid vortices and represent eddying regions. Their average size and life-time depend on the characteristic of the turbulence.

The particles that evolve on contour lines corresponding to small absolute values of the potential have much larger displacements. The large displacements are caused by the large size of these contour lines and also by the decorrelation, which allows transitions between neighbor potential cells. These large jumps are random. The transport is essentially determined by them, while the trapping events have negligible contributions.

Trapping was associated to a process of micro-confinement [8]. At a given moment, the micro-confinement affects a fraction of the particle, and, in a large time interval, every particle is subjected to micro-confinement events during a fraction of this time. The micro-confinement determines the existence of transport reservoirs [8] in the sense that diffusion can be strongly enhanced when these trapped particles are released by the increase of the perturbation strength.

Typical trajectories obtained from Eq. (1) for a potential with \(K=1\) and 10 are shown in Figure 1. One can see that the trapping events and the degree of coherence of the motion have significant increase at \(K=10\) compared to the example for \(K=1\).

The aim of the paper is to analyze the statistical characteristics of these complex trajectories and to identify the importance of the random and quasi-coherent aspects for given parameters of the potential. We show that the coherence of the trajectories can manifest also as quasi-coherent flows.
3. Tracer statistics in stochastic potential fields

The Eulerian and the Lagrangian correlations are the main concepts in test particle transport.

The two-point Eulerian correlation (EC) of the potential is

\[ E(x,t) = \langle \phi(x_1,t_1) \phi(x_2,t_2) \rangle \]

where \( x = x_1 - x_2 \), \( t = t_1 - t_2 \) and \( \langle \cdot \rangle \) is the statistical average over the realizations of the stochastic field or the space average over \( x_1 \). The main statistical parameters of the velocity field are evidenced by this function. The correlation length \( \lambda_c \) and time \( \tau_c \) correspond to the decay to zero of the function \( E(x,t) \) and the amplitude of the stochastic potential is \( \Phi^2 = E(0,0) \). The ECs for the components of the velocity are derivatives of \( E(x,t) \). The EC describes the space structure and the time variation of the stochastic velocity field.

The Lagrangian velocity correlation (LVC) is a time dependent function that describes statistical properties of particle motion in the stochastic velocity field. In most cases the LVC decays to zero since the Lagrangian velocity becomes statistically independent on the initial velocity (it decorrelates). The characteristic time \( \tau_L \) of the LVC decay is the measure of the memory of the stochastic Lagrangian velocity. The Lagrangian time \( \tau_L \) is the decorrelation time of the potential \( \tau_c \) in the case of the stochastic process described by Eq. (1).

Besides this important effect of memory, the LVC determines the main quantities related to test particle transport. As shown by Taylor [9], the mean square displacement (MSD) \( \langle x_i^2(t) \rangle \) and its derivative, the running diffusion coefficient \( D_i(t) \), are integrals of the LVC. The time dependent diffusion coefficients provide the "microscopic" characteristics of the transport process.

The diffusion at the transport space-time scales, which are much larger than \( \lambda_c \) and \( \tau_c \), is described by the asymptotic values \( D_i \). The decorrelation of the Lagrangian velocity after the memory time \( \tau_c \) leads to the possibility of decomposing the statistics of the displacements at large time \( t \gg \tau_c \) in a sequence of independent small-scale processes of time \( \tau_c \). The probability of the small scale displacements (the micro-probability), \( P^m(x,t) \), determines the elementary step \( \Delta \) of the transport process as the MSD. The statistics of the displacements at large times \( t \gg \tau_c \) is Gaussian, with the exception of the processes that have \( \Delta \rightarrow \infty \) or \( \tau_c \rightarrow \infty \) (see [10]).

The diffusion coefficients and the characteristics of the stochastic transport are determined by the micro-probability, which contains all statistical information on the trajectories at small time \( t \lesssim \tau_c \). The use of the ergodic theory is not necessary.

Figure 1. Typical trajectories obtained from numerical solutions of Eq. (1) for \( K=1 \) (left panel) and \( K=10 \) (right panel).
Most of the theoretical methods are essentially based on Corrsin hypotheses [11], which assume that the micro-probability $P^m(x,t)$ is Gaussian and that the displacements are statistically independent on the velocity field. They lead to diffusive transport in the limit of static velocity fields, and thus they cannot apply to the special case of zero divergence two-dimensional velocity fields. It can be shown that a diffusive transport is obtained even when the second assumption is eliminated, which suggest that the micro-probability is not Gaussian in the presence of trapping. The first theoretical approach that finds subdiffusive transport in this special case [12], [13] does not use the Gaussian assumption, but it is based on the percolation theory. It only determines the scaling of the asymptotic diffusion coefficients. The detailed statistical information that is contained in the LVC was first obtained by the decorrelation trajectory method (DTM) [6]. This semi-analytical method was developed and validated by the nested subensemble approach (NSA) [7]. Both methods are in agreement with the statistical consequences of the invariance of the potential.

The main idea of this approach [6-8] is to determine the Lagrangian averages not on the whole set of trajectories but to group together trajectories that are similar, to average on them and then to perform averages of these averages. Similar trajectories are obtained by imposing supplementary initial conditions besides the necessary one. Particularly important initial conditions are provided by the conserved quantities (the potential in this case). The value of the initial potential determines the average path of the trajectories. The supplementary initial conditions define a set of subensembles of the realizations of the potential. Conditional averages lead to space-time dependent average potential and velocity in each subensemble and to smooth trajectories that represent average particle motion: the decorrelation trajectories (DTs). The DTs are the main ingredient of DTM and NSA. They are smooth, simple trajectories determined from the EC of the stochastic fields. They are much different from particle trajectories as they saturate after the decorrelation time. The DTs represent the average evolution of the particles through the correlated zone of the potential. They describe the decorrelation process and provide characteristics of the transport.

The statistics of the trajectories is represented by weighted averages along the DTs. The weighting factors are the probabilities of the subensembles that correspond to the DTs. The LVC, $D_i(t)$, the probability of the small scale displacements $P^m(x,t)$ and other Lagrangian correlations are evaluated using the DTs. The detailed effective calculations are presented in [6-8]. Applications of the DTM for more complicated transport processes can be found in [14]-[19] for plasma turbulence and in [20]-[21] for space plasmas.

4. Quasi-coherent structures and transport

The parameters of the quasi-coherent structures are determined using the DTM as time dependent functions [14]. The fraction of trapped trajectories $n_{tr}(t)$ counts the number of trajectories that are closed at time $t$ in a frozen potential. The average sizes of the structures $S_i(t)$ are determined by the maxima of the displacements on the closed DTs.

The results presented in the figures are obtained for spectra with two maxima as in [15], which correspond to positive and negative cells with similar weight that yield zero space-integral of the EC. The time $t$ in Figure 2 and 3 is normalized with the time of flight.

Figure 2a shows that the sizes of the structures and the time dependent diffusion coefficients for a time-independent potential. The structures appear at times of the order of the time of flight ($t=\sim 1$) and they continuously increase. The fraction of trapped trajectories grows asymptotically towards one, and the sizes $S_i(t)$, $S_f(t)$ are increasing functions. Figure 2b demonstrates the time-correlation between the decrease of the fraction of free trajectories $n_{fr}(t)=1-n_{tr}(t)$ and the decay of the diffusion coefficients at $t>1$. The decrease of $D_i(t)$ at large times is produced by the quasi-coherent structures that trap a part of the trajectories. However, the diffusion coefficients are not proportional with $n_{tr}(t)$, which shows that the process is more complicated and depend on the size of the structures.

Figure 3 illustrates the effects of an average velocity $V/e_z$ superposed on a time independent stochastic potential. One can see in Figure 3a that the characteristics of the structures saturate. The fraction of trapped trajectories is limited, which shows that the average velocity destroys a part of the
structures. The released trajectories contribute to the increase of the diffusion (Figure 3b). It is interesting to underline that the transport along the average velocity strongly increases and it is superdiffusive of ballistic type $\langle x^2 \rangle_t \approx t^2$, while the perpendicular transport remains subdiffusive. The shape of the quasi-coherent structures is modified. They are elongated in the in the direction of the average velocity.

The existence of an average velocity in the equation of motion determines modifications of the DTs in the nonlinear regime [8], [15]. A new type of DTs appears. They are opened trajectories along the direction of the average velocity. Their number increases as $V_2$ increases and, when $V_2$ is much large than the stochastic velocity, all DTs are of this type. Thus, the average velocity eliminates progressively (as Vd increases) the quasi-coherent structures.

**Figure 2.** The time dependence of the parameters of the structure and of the diffusion coefficients obtained for an anisotropic potential. Left panel: the fraction of trapped trajectories $n_{tr}(t)$ and the sizes of the structures $S_x(t)$, $S_y(t)$. Right panel: $D_x(t)$ and $D_y(t)$ compared to the ratio of free trajectories $n_{fr}(t)$.

**Figure 3.** The effects of an average velocity $V_2 = 0.3 \Phi / \lambda_y$ added to the stochastic potential of Figure 2. Left panel: the fraction of trapped trajectories $n_{tr}(t)$ and the sizes of the structures $S_x(t)$, $S_y(t)$. Right panel: $D_x(t)$, $D_y(t)$ and the ratio of free trajectories $n_{fr}(t)$. 

5. Quasi-coherent flows

A stochastic velocity field generates random steps of the tracers that yield diffusive transport. They have zero average and are determined as the square root of the mean square displacement during the decorrelation time (obtained with the short-time probability $P_m(x,t)$). In special conditions, orientated steps of the tracers can appear even if the stochastic velocity field has zero average. They lead to quasi-coherent flows (also named pinches or direct transport). One mechanism for the generation of pinches in turbulent plasmas is related to the space variation of the confining magnetic field. It is a ratchet-type process that yields pinch velocities along the gradient of the magnetic field [16]. This flow is influenced by the presence of trajectory structures, which can reverse the direction of the flow from anti-parallel to parallel to the gradient.

A new mechanism of generation of quasi-coherent flows is presented here. The drive is the gradient of the amplitude of the stochastic potential. The potential is not a homogeneous stochastic field, but it can be described by

$$\phi(x,t) = \Phi(x_t) \bar{\phi}(x,t),$$

where $\bar{\phi}$ is a homogeneous stochastic field with the amplitude one and $\Phi(x_t)$ is a space-dependent function $\Phi(x_t) = \Phi^0 \exp\left(-\frac{x_t}{L}\right) \approx \Phi^0\left(1 - \frac{x_t}{L}\right)$. The characteristic length $L$ of the gradient of $\Phi(x_t)$ is large compared to the correlation length of the potential. The components of the velocity in the first order in $1/L$ are

$$v_1(x,t) = -\Phi(x_t) \frac{\partial}{\partial x_2} \bar{\phi}(x,t), \quad v_2(x,t) = \Phi(x_t) \frac{\partial}{\partial x_1} \bar{\phi}(x,t) - \frac{\Phi^0}{L} \bar{\phi}(x,t).$$

A supplementary term appears in the second component of the velocity at finite $L$. It determines a quasi-coherent flow along the gradient.

We have determined the diffusion coefficient and the velocity of the quasi-coherent flow using the DTM. The results show that the diffusion is weakly influenced by the gradient of $\Phi(x_t)$. The main effect consists of the average displacement $X$ that yields in this stochastic process, which determines the average velocity $V_f$. The flow is in the direction of the gradient of the amplitude $\Phi(x_t)$. As seen in Figure 4, $X$ and $V_f$ depend on the Kubo number of the stochastic potential. They are increasing functions of $K$ for $K<<1$ and decreasing functions at large $K$. The maximum of the velocity of the flow appears around $K=1$, in the range of the transition from quasilinear to nonlinear transport. This shows that the quasi-coherent structures that appear at $K>1$ determine the decrease of the velocity. They actually produce an average velocity of opposite direction, which reduces the pinch.

![Figure 4](image)

**Figure 4.** The average displacement $X$ and the velocity $V_f$ of the quasi-coherent flow as functions of the Kubo number $K$. The stochastic potential is as in Figure 2 and the gradient length $L=5$. 

6. Conclusions
The nonlinear effects that appear in test particle transport in two-dimensional incompressible turbulence are analyzed using the decorrelation trajectory method. Tracer trajectories show both random and quasi-coherent aspects. They consist of random sequences of trapping or eddying and large jumps. The strength of each of these aspects depends on the parameters of the turbulence.

The trapping generates quasi-coherent trajectory structures that have a micro-confinement effect represented by the decrease of the time dependent diffusion coefficients. A clear correlation between the decrease of the fraction of free trajectories and of the time dependent diffusion coefficients is evidenced.

An average velocity superposed on the stochastic potential has strong effects on the quasi-coherent structures. A part of them is destroyed and the released trajectories strongly increase the transport in the direction of the average velocity, which becomes superdiffusive. The remaining structures are have modified shapes elongated in the direction of the velocity.

The quasi-coherence of the trajectories appears in special conditions as flows of the tracers. We discuss here the effects of a large-scale gradient of the amplitude $\Phi$ of the stochastic potential. The tracers acquire an average velocity along the gradient of $\Phi$. This shows that in the space dependent stochastic potential, the tracers have the tendency to accumulate on the maxima of the amplitude.

7. References
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