Abstract
This is a short review of the statistical mechanical definition of entropy production for systems composed of a large number of interacting components. Emphasis is on open systems driven away from equilibrium where the entropy production can be identified with a logarithmic ratio of microstate multiplicities of the original macrostate with respect to the time-reversed state. A special role is taken by Gibbs measures for the stationary spatio-temporal distribution of trajectories. The mean entropy production is always non-negative and it is zero only when the system is in equilibrium. The fluctuations of the entropy production satisfy a symmetry first observed in [7] and then derived in [11] for the phase space contraction rate in a class of strongly chaotic dynamical systems. Aspects of the general framework are illustrated via a bulk driven diffusive lattice gas.

1 Introduction
The production of entropy in spatially extended systems was already discussed, both at length and in depth, by the founding fathers of thermodynamics and statistical mechanics. It is instructive to divide the discussion in two (very unequal) parts.
First there is the situation of a perfectly closed (or truly isolated) classical mechanical system undergoing a Hamiltonian dynamics and evolving towards equilibrium from nonequilibrium initial conditions. That is known as the approach or the convergence or the relaxation towards equilibrium. The issue here is to understand the usual second law of thermodynamics that associates an arrow of time to the macroscopic behavior leading to equilibrium and to derive
from the reversible microscopic laws the irreversible kinetic and hydrodynamic equations describing this evolution on the appropriate space-time scale. Secondly, there are the many phenomena of dissipation in open driven systems. Here one considers a stationary situation of a system or medium in contact with several reservoirs that interact with the system. Think of heat baths or particle reservoirs at different temperatures or concentrations through which a nonequilibrium state is maintained in the system or also of particle systems subject to an external driving field. As a result, a stationary current is installed and entropy is produced.

The plan is that I start by recalling some of the main ingredients in the qualitative understanding of macroscopic irreversibility for the first scenario, return to equilibrium. That has obtained most attention in the past and it is well understood. On the other hand, for the nonequilibrium steady state scenario the literature is mostly limited to the close to equilibrium situation and Section 3 must remind us of the phenomenology of entropy production. Then, I get a chance to say something new when in Section 6 I give a Boltzmann-like definition of entropy production in a simplified context. Formula (4.2) is most important. The general framework is presented in Sections 5 and 7. The application in the final Section 7 is on the asymmetric exclusion process.

The paper uses and studies aspects of thermodynamics, statistical mechanics and probability theory with a variety of language and tools. One application and source of inspiration is however not included, that of the theory of smooth dynamical systems. Yet, some of the work that I will present here was started in [19] (see also [24, 15]) from trying to understand the Gallavotti-Cohen results in [11] on the fluctuations of the phase space contraction in dynamical systems with very strong hyperbolicity assumptions and the ‘new theoretical ideas in nonequilibrium statistical mechanics’ à la Ruelle, see [27]. Some apology can be read in Section 6.

The present paper is, except for the Sections 4 and 7.2, a short review and, by lack of space, the many details and proofs are omitted. More can be found in [19]-[23] and I am especially grateful to Frank Redig for the joint work.

2 Approach to equilibrium

Here I only remind the reader about the problem of irreversibility in closed Hamiltonian systems and about its resolution. There is nothing new here (see e.g. [13, 17, 14, 8]) except perhaps for some measures of emphasis.

What is the problem? The microscopic dynamics is time-reversal invariant. Why then do we always see individual macroscopic systems taking a particular course in time, evolving towards equilibrium? What makes the direction of time in macroscopic behavior?

The resolution of this paradox has many sides and it can easily take you away for quite some time. Yet the beginning is easy and it says that there is absolutely no problem, it is a matter of counting and you could not expect otherwise: equilibrium is what you should expect. Here the basic ingredient is that there is a huge difference in scales between the microworld and the macroworld and that irreversibility belongs to the macroworld.
2.1 Boltzmann entropy

Take a system of \( N \) (large!) components \( x(i), i = 1, \ldots, N \), each having \( n \) possible values. A microstate \( x = (x(i), i = 1, \ldots, N) \) is one of the \( n^N \) possible configurations. The set of all these configurations (or, when additional constraints are given, some subset of this) is the microscopic phase space \( \Omega \). From a microscopic perspective each element of \( \Omega \) is equivalent and we remain indifferent in the estimates of plausibility: each \( x \) has the same probability for being realized. From a macroscopic point of view, some distinction can be made. For this we make a choice of macroscopic variables which are approximately additive (and, in presence of a ‘natural’ dynamics, are locally conserved). The macrovariable partitions \( \Omega \) in a number of subsets. Say for \( \Omega = \{+1, -1\}^N \) and macrovariable \( X_N(x) \equiv \sum_i x(i)/N \) each of these subsets will contain between 1 and, for \( N \) very large, almost \( 2^N \) elements. So we all expect that randomly picking an element from \( \Omega \) gives us a configuration that ‘typically’ has zero \( X_N \) (with standard error \( 1/\sqrt{N} \)). In other words, while all microstates are equivalent, some microstates are more typical from a macroscopic point of view.

To connect the microscopic world with macroscopic behavior Boltzmann introduced the entropy

\[
S(X) \equiv \log W(X)
\]

where \( W(X) \) is the number of microstates \( x \) for which \( X_N(x) = X \). The logarithm is very convenient and makes \( S(X) \) extensive. In the same way (and by some abuse of notation), we speak about the entropy \( S \) of a microstate \( x \):

\[
S(x) \equiv S(X_N(x)).
\]

Of course, the phase space for realistic systems is a bit more complicated. We should be speaking about a (classical) Hamiltonian dynamics for \( N \) point particles enclosed in a finite box with \( x \) corresponding to the positions and momenta of all the particles. The phase space \( \Omega \) is now the collection of all these microstates constrained to given values of certain macrovariables \( Y_N \). As we have a closed system with energy conservation, we can keep in mind that the total energy \( Y_N(x) = E \) is thus fixed. We are interested in the evolution of a second class of macrovariables \( X_N \). Let us think here about the macroscopic variable that corresponds to the spatial density profile, say the number of particles to the left of our box. Yet, for this more complicated phase space, more or less the same counting procedure can be followed. In this same language, microstates have a larger entropy when they correspond to a macroscopic value that can be microscopically realized in more ways. Now it is important to get a feeling of the huge differences that can arise when the system is large. For air, under normal circumstances, there are about \( 10^{25} \) molecules in a box of 1 m\(^3\). If we divide this box in pieces of 1 cm\(^3\) we get \( 10^{60000000000000000000000000000000} \) possible arrangements that would correspond to a homogeneous density profile (about the same number of particles in each piece) while only \( 10^6 \) possibilities of having all of them in just one such piece of the box. When considering these huge numbers and reasoning about plausibilities just as we do in everyday life, it appears that only a great conspiracy of initial conditions or dynamics can lead to an effective decrease in entropy: that is, we expect that \( S(x_t) \) grows as \( x_t \) follows the microscopic trajectory and equilibrium corresponds to maximal entropy. The law of increase of entropy is a statistical law, but one with a ‘moral certainty.’
It must be added that larger entropy does not necessarily mean less ordered or more spatially homogeneous. It depends on the type of interaction and/or the value of the energy that is fixed. It suffices to think of gravity (where clustering of mass is typical) or of molecular interactions at low energy leading to ordered structures.

### 2.2 Initial conditions

Note that the counting argument above is symmetric in time, valid as such for prediction as well as for retrodiction. So the real surprise is to see nonequilibrium initial conditions. When trying to see the origin of these special low entropy initial conditions we soon get in a chain of arguments leading to cosmological questions. Here input from the standard model of physical cosmology is necessary. The upshot is summarized by Richard Feynman: “it is necessary to add to the physical laws the hypothesis that in the past the universe was more ordered, in the technical sense, than it is today...to make an understanding of the irreversibility” or, much earlier, by Ludwig Boltzmann: “That in nature the transition from a probable to an improbable state does not take place as often as the converse, can be explained by assuming a very improbable initial state of the entire universe surrounding us. This is a reasonable assumption to make, since it enables us to explain the facts of experience, and one should not expect to be able to deduce it from anything more fundamental.” I refer to the pleasant Chapter 7 in [27] for some specific analysis.

### 3 Phenomenology of steady state entropy production

The definition of entropy production for nonequilibrium steady states cannot be completely arbitrary. Standard treatments are however largely restricted to close to equilibrium phenomenology. Entropy production appears in the close to equilibrium thermodynamics of steady state irreversible processes as the product of thermodynamic fluxes and thermodynamic forces. The forces are produced by gradients in intensive variables, the so called affinities that are maintained throughout. One thinks of gradients in chemical or electrostatic potential or in temperature. The fluxes are currents in the conjugate extensive variables. In linear response, they are linear combinations of the affinities. In this way, linear transport coefficients are defined that are functions of the intensive variables that locally characterize the state of the system.

As an example take a cylindric solid material through which a stationary heat current $J_Q$ is maintained by coupling the material at its right and left ends to reservoirs at temperature $T_r$ and $T_l$ respectively. Assuming a linear relation between the affinity $\nabla 1/T$ and the current $J_Q$ with a linear response coefficient $L_Q$, one arrives at Fourier’s law

$$J_Q = L_Q \nabla \left( \frac{1}{T} \right) = -h \nabla T$$

where $h \equiv L_Q/T^2$ is the heat conductivity. Here is expressed the empirical finding (close to equilibrium) of the proportionality of the heat current with the
temperature gradient. The entropy production in the material is then
\[ \dot{S} = J_Q \nabla \frac{1}{T} = hT^2[\nabla(\frac{1}{T})]^2 \]

Of course, the (close to equilibrium) entropy of the material remains constant (its macrostate is unchanged) and all produced entropy is carried away by the entropy current to be poured into the reservoirs. (The relation between entropy current and entropy production is established in the so-called entropy balance equation.) If we integrate the entropy production over the material, we see that the entropy of the reservoirs is changed by
\[ \frac{dS}{dt} = aJ_E(\frac{1}{T_r} - \frac{1}{T_{\ell}}) \]

where \( a \) is the cross-section area of the cylinder. To maintain the steady state we will have to carry away this extra entropy outside the coupled system. I refer to [1] for the standard treatment and to [3, 6] for more recent studies.

4 Multiplicity under constraints

An elementary mathematical clarification of the connection between Boltzmann entropy and thermodynamic equilibrium entropy goes by considering the multiplicity of microstates for a particular macroscopic observation, (see also [12]). Take \( N \) particles each of which can be in a certain phase space cell \( i \) (for example having energy \( E_i \), \( i = 1, \ldots, n \). In total, there are \( n^N \) microstates. The number of such microstates with \( m_1 \) particles in cell \( i = 1 \), \( \ldots \) and \( m_n \) particles in cell \( i = n \) (\( m_1 + \ldots + m_n = N \)) is given by the multinomial coefficient
\[ W_N(m_1, m_2, \ldots, m_n) = \frac{N!}{m_1! \ldots m_n!} \]

With proportions \( p_i \equiv m_i/N \) and via Stirling’s formula,
\[ \lim_{N \to \infty} \frac{1}{N} \ln W_N(p_1N, \ldots, p_nN) = -\sum_{i=1}^n p_i \ln p_i \]  
(4.1)

where the right-hand side is the Shannon entropy of the probability measure \( (p_i) \). Imagine now as further constraint that
\[ \sum_{i=1}^n p_i E_i = E \]

Then, the multiplicities or the Shannon entropy of (4.1) is maximal when
\[ p_i = \frac{e^{-\beta E_i}}{Z_\beta} \]

with \( \beta = \beta(E) \) found from the constraint. This maximal entropy is the equilibrium or Gibbs’ entropy
\[ S(E) = \ln Z_\beta + \beta E \]
We can now move $E$ to a new value $E + dE$ while supposing that the energy levels $E_i$ do not change. Obviously, the maximal entropy does change and it is doing so according to

$$dS = \beta dE$$

which is Clausius’ formula $dS = \delta Q/T$ for the change in equilibrium entropy from a heat transfer $\delta Q$ at reservoir temperature $T = 1/\beta$ for fixed volume.

I suggest a very similar scheme for entropy production. I sketch it here without explicit reference to dynamics just to emphasize, in this highly simplified setting, the structure of the definition that later, in fuller glory, will follow. (In brackets I will announce the analogue.)

Consider again $N$ variables $\sigma(i), i = 1\ldots, N$ with, for simplicity, $\sigma(i) = 0, 1$, i.e., $n = 2$ in the above (this should be thought of as the pathspace, the space of all microscopic space-time trajectories. Thus $N$ is the number of particles times the number of times they are observed; $\sigma(i) = 1$ indicates a right-moving particle at that time while $\sigma(i) = 0$ indicates left-moving). Let us select the first $M \leq N$ of these variables and let $(\eta(i), i = 1, \ldots, M)$ be a fixed configuration on them. (That is, we select a particular history for a given space-time volume.) That of course constrains the number $m_1$ of all variables that are in the first microstate (1):

$$N - M + \sum_{i=1}^{M} \eta(i) \geq m_1 \geq \sum_{i=1}^{M} \eta(i)$$

Let $W_M^{(N)}(\eta, m)$ be the number of microstates $\sigma \in \{0, 1\}^N$ so that $\sigma(i) = \eta(i)$ for $i = 1, \ldots, M$ and so that

$$\sum_{i=1}^{N} \sigma(i) = mN$$

(That will correspond to a given macroscopic value for some current.) Consider the involution $\eta(i) \rightarrow \bar{\eta}(i) \equiv 1 - \eta(i), i = 1, \ldots, M$ (time-reversal will take that role later). I suggest now to inspect the logarithmic ratio

$$\dot{S}_M^{(N)}(\eta, m) \equiv \ln \frac{W_M^{(N)}(\eta, m)}{W_M^{(N)}(\bar{\eta}, m)}$$

(4.2)

(That will be the entropy production.) Evaluating this for large $N$, while fixing all the rest, gives

$$\dot{S}_M(\eta, m) \equiv \lim_{N} \dot{S}_M^{(N)}(\eta, m) = \sum_{i=1}^{M} [\eta(i) - \bar{\eta}(i)] \ln \frac{m}{1 - m}$$

(4.3)

(Observe that for $\lambda \equiv \ln m/(1 - m)$ indeed $\exp \lambda/(1 + \exp \lambda) = m$ so that $\ln m/(1 - m)$ is the product of the field $\lambda \neq 0$ for $m \neq 1/2$ (= the equilibrium value) and the variable current or flux in the space-time volume $M$.) If we now also let $M$ grow very large, we will have that

$$\sum_{i=1}^{M} \eta(i) \approx mM, \sum_{i=1}^{M} [\eta(i) - \bar{\eta}(i)] \approx (2m - 1)M$$
and hence, \( \dot{S}_M(\eta, m)/M \) converges to the relative entropy

\[
\dot{S}(m) = (2m - 1) \ln \frac{m}{1 - m} = m \ln \frac{m}{1 - m} + (1 - m) \ln \frac{1 - m}{m}.
\]

(That is, the mean entropy production is positive and it equals the relative entropy between the original distribution and its time-reversal.)

5 **Gibbs measures with an involution**

The following must be considered as a program, the ideal context that we should keep in mind for more concrete realizations.

I write \( K \) for a finite set and take \( \Omega \equiv K \mathbb{Z}^{d+1} \). This \((d+1)\)-dimensional configuration space announces that \( \Omega \) plays the role of pathspace; its elements \( \sigma \equiv (\sigma_t(i), (t, i) \in \mathbb{Z}^{d+1}) \) are space-time trajectories with values \( \sigma_t(i) \in K \) at the space-time point \((t, i)\). Consider regular space-time cubes \( V_{T,L} \equiv \{(t, i) \in \mathbb{Z}^{d+1} : |t| \leq T, |i| \leq L \} \) centered around the origin, where \( T \) stipulates the temporal and \( L \) the spatial extension. Let \( \Theta_{T,L} \) be the involution on \( \Omega \) that reverses the time:

\[
(\Theta_{T,L}\sigma)_t(i) \equiv \sigma_{-t}(i) \text{ if } (t, i) \in V_{T,L},
\]

\[
\equiv \sigma_t(i) \text{ otherwise}
\]

(I could also have included a kinematical time-reversal \( \pi \) as involution on \( K \) and then write \( (\Theta_{T,L}\sigma)_t(i) \equiv \pi(\sigma_{-t}(i)) \) for \((t, i) \in V_{T,L} \) but, for simplicity, I will stick here to the choice \( \pi = \text{identity} \).) Given a local function \( f \) on \( \Omega \) it is possible to find large enough \( T_o, L_o \) so that for all \( T \geq T_o, L \geq L_o, f(\sigma) = f(\sigma_{-t}(i), (t, i) \in V_{T,L}) \). There is therefore no ambiguity in writing \( f\Theta \) for the new (time-reversed) function and similarly, for a probability measure \( \mu \) on \( \Omega \), to write \( \mu\Theta \) for the new probability measure with expectations \( \mu\Theta(f) \equiv \mu(f\Theta) \).

Next, consider a translation invariant space-time interaction potential \( U = (U_A) \) parametrized by the finite subsets \( A \) of \( \mathbb{Z}^{d+1} \). Each \( U_A \) is a function of the variables \( \sigma_t(i), (t, i) \in A \), and let me assume, just for convenience, that \( U_A \equiv 0 \) whenever the diameter of the set \( A \) is larger than a finite radius. I define the associated entropy production in a finite set \( \Lambda \) as

\[
\dot{S}_\Lambda \equiv \dot{S}_\Lambda^{(U)} \equiv \sum_{A \subset \Lambda} [U_A\Theta - U_A]
\]

Clearly, \( \dot{S}_\Lambda \) is asymmetric under time-reversal \( \Theta \) and it is measurable from the values of the variables in the set \( \Lambda \). Definition (5.5) is the analogue of (1.2).

In all realistic realizations of this definition, \( \dot{S}_\Lambda \) can be interpreted as a sum over products of fields and currents. The reader is probably waiting to see some dynamics and it may seem strange to speak about such an entropy production but I only deal with the general framework here and one illustration is contained in Section 7.

Let \( \mu \) be a translation-invariant probability measure on \( \Omega \). I define the mean entropy production (MEP) in \( \mu \) as the expectation

\[
\text{MEP}(U, \mu) \equiv \lim_{\Lambda} \frac{1}{|\Lambda|} \mu(\dot{S}_\Lambda)
\]
Limits are understood in the sense of increasing cubes \( \Lambda = V_{T,L} \). It is important to realize that you do not in fact need to take the \( \mu \)-average in the above but for a fixed (large enough) \( T, S/|\Lambda| \) will become \( \mu \)-almost surely equal to the MEP if \( \mu \) is ergodic with respect to spatial translations.

Let \( \mu \) be a translation invariant Gibbs measure for the potential \( (U_A) \). It has an entropy density \( s(\mu) \) that equals the entropy density \( s(\mu^\Theta) \) of the time-reversed Gibbs distribution. Yet, \( \mu \) and \( \mu^\Theta \) need not be equal; they can be discriminated via their relative entropy density \( s(\mu|\mu^\Theta) = s(\mu^\Theta|\mu) \) (see [12]).

**Theorem 1 (MEP)** Let \( \mu \) be a translation invariant Gibbs measure for the potential \( (U_A) \). Then

\[
\text{MEP}(U, \mu) = s(\mu|\mu^\Theta) \geq 0 \tag{5.7}
\]

with equality if and only if the potentials \( U \) and \( U^\Theta \) are physically equivalent.

**Remarks:**

- The identification of the MEP with the relative entropy density was announced at the very end of Section 4.
- There can be no spontaneous breaking of time-reversal symmetry: If the MEP is zero, then \( \mu \) and \( \mu^\Theta \) must be Gibbs measures for the same potential, which means that \( (U_A) \) and \( (U_A^\Theta) \) must be physically equivalent. This property of ‘no current without heat’ is established for infinite interacting particle systems in [24, 20, 21].

I now present the local (in space) fluctuation theorem (LFT) for Gibbs measures with an involution. Take \( T \) and \( L \) large and fix the volume \( V \equiv V_{T,L} \) and a subset \( \Lambda \equiv V_{T,L'} \) with \( L' << L \). For a function \( f \) of the variables in \( V \), define the expectation

\[
\mathbb{E}_V(f) \equiv \frac{1}{Z_V} \sum_{\sigma \in K^V} f(\sigma) e^{-\sum_{A \subset V} U_A(\sigma)}
\]

with \( Z_V \) the normalizing partition function. There is no need for translation invariance here and the potential \( (U_A) \) is allowed to contain time-reversal invariant hard core interactions. I write

\[
Z_{V\setminus \Lambda}(\sigma_\Lambda) \equiv \sum_{\sigma_{\Lambda^c} \in K^{V\setminus \Lambda}} e^{-\sum_{A \subset V, A \cap \Lambda \neq \emptyset} U_A(\sigma_{\Lambda^c}\sigma_\Lambda)}
\]

for the partition function in \( \Lambda^c \equiv V \setminus \Lambda \) with boundary condition \( \sigma_\Lambda \) in \( \Lambda \). Put

\[
F^\Lambda_V(\sigma_\Lambda) \equiv \ln \frac{Z_{V\setminus \Lambda}(\Theta \sigma_\Lambda)}{Z_{V\setminus \Lambda}(\sigma_\Lambda)}
\]

which, for local interactions, depends on the variables in (the interior boundary \( \partial \Lambda \) of) \( \Lambda \). Put \( R^\Lambda_V \equiv \dot{S}_\Lambda - F^\Lambda_V \).

**Theorem 2 (LFT)**

- For every function \( G \)

\[
\mathbb{E}_V[G(-R^\Lambda_V)] = \mathbb{E}_V[G(R^\Lambda_V)e^{-R^\Lambda_V}] \tag{5.8}
\]

\[
\mathbb{E}_V[G(-\dot{S}_\Lambda)] = \mathbb{E}_V[G(\dot{S}_\Lambda)e^{-\dot{S}_\Lambda + F^\Lambda_V}] \tag{5.9}
\]
• For every family \((\lambda_A)\) of complex numbers

\[
E_{V}[e^{-\sum_{\lambda \subset \Lambda} \lambda_A (U_\lambda \Theta - U_A)}] = E_{V}[e^{-\sum_{\lambda \subset \Lambda} (1 - \lambda_A)(U_\lambda \Theta - U_A)}e^{F_{\Lambda}}]
\]  

(5.10)

Remarks:

• I take it understood that, for sufficiently local interactions, the potential \(F_{\Lambda}\) is of the order of the boundary of \(\Lambda\). Therefore, asymptotically in logarithmic sense, \((5.4)\) leads directly to the symmetry \((5.3)\) for the distribution of the entropy production \(\tilde{S}_\Lambda\) itself, a symmetry first uncovered in \([7, 11]\) for the large deviation rate function of the phase space contraction in the SRB state of reversible dissipative mixing Anosov diffeomorphisms. One should however not take for granted that \(F_{\Lambda}\) is uniformly bounded by \(|\partial \Lambda|\). In many of the models to which one wishes to apply the above scheme, one really needs to prove that it concerns here just a ‘boundary term.’ One origin of problems can be that time is taken continuous and an unbounded number of changes can happen in any finite interval. This is less of a problem for jump processes where one uses that the Poisson process has all exponential moments but when working on non-compact phase spaces, the ‘boundary term’ \(F_{\Lambda}\) can possibly have non-existing exponential moments and thus really can change the large deviation rate function.

• The identity \((5.10)\) generates, by suitable differentiation, equalities between correlation functions. A discussion with an application to the Onsager reciprocity relations is contained in \([19]\). I like to compare \((5.10)\) with the Ward identities as we know them from quantum field theory; the mathematical origin is very similar and non-perturbative, liberating us from close to equilibrium assumptions.

6 Gibbsian hypothesis

The ambition in the above framework for the study of nonequilibrium driven systems is to use the standard Gibbs formalism and in particular its fluctuation theory in the ‘current’-ensemble. The mathematics will therefore not deviate significantly from what is e.g. found in the contributions \([5, 28, 12]\).

A frequently asked question is where these Gibbs measures come from or how they should be connected with a dynamics. Here two short answers.

6.1 Pathspace measure construction

The first answer is illustrated below in Section \([\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\] and many more examples can be found in \([19, 23]\) and \([24, 13, 8]\). One constructs the Gibbs measure explicitly as the pathspace measure of a stationary process. Most easy is the case of stochastic dynamics. It is not really necessary to realize the pathspace measure as a \textit{bona fide} Gibbs measure as we know it say for lattice spin systems; what is needed is to understand how the pathspace measure is governed by a space-time action which is approximately local and additive in space-time. Technically, this is a matter of setting up the appropriate Girsanov formula and I refer to the standard treatments in \([18, 4]\). Interestingly enough, this method of constructing
the process via a Gibbsian space-time distribution has also lead to new existence and uniqueness results for classes of diffusion processes, see e.g. [26].

Deterministic dynamics are here more of a problem but, for example, we have learnt from [8]-[11] how the Gibbsian structure can be exploited on the level of the symbolic dynamics for sufficiently strongly chaotic dynamics.

6.2 Maximum entropy principle

The first answer has the advantage of being explicit and directly connected to model dynamics that one may have in mind. I believe however that the second answer is more to the point. The reason to use Gibbs measures here is exactly the same as for using Gibbs measures in usual classical or quantum equilibrium statistical mechanics. All that changes is the type of ensemble because we must work with the currents as macro-observables. To do this on space-time is the only price that must be paid but it is a necessary one. Gibbs measures appear then as solutions of the maximum entropy principle but that need not be restricted to equilibrium conditions.

7 Asymmetric exclusion process

I illustrate the previous generalities using one concrete model, that of a bulk driven diffusive lattice gas. First about the mean entropy production, to check that it coincides with what one should expect. Secondly, about the local entropy production fluctuations.

7.1 MEP for ASEP

I start with the asymmetric exclusion process (ASEP) on a one-dimensional ring \(\{1, 2, \ldots, \ell\}\) (periodic boundary conditions). Each site \(i\) of the ring is either occupied by a particle (denoted by \(\eta(i) = 1\)) or is empty (\(\eta(i) = 0\)). The particle-configuration \(\eta\) is subject to a Markovian particle conserving asymmetric hopping dynamics with rates

\[
c(i, i + 1, \eta) = \eta(i)(1 - \eta(i + 1)) \frac{e^{E/2}}{2} + \eta(i + 1)(1 - \eta(i)) \frac{e^{-E/2}}{2}
\] (7.11)

for changing the configuration from \(\eta\) to \(\eta^{i,i+1}\) obtained by exchanging the occupations at sites \(i\) and \(i + 1\). In words, particles hop to nearest neighbor sites when there is a vacancy at a rate that depends on the direction. \(E\) is now an external driving field. We can easily obtain the space-time interaction from the standard Girsanov formula for Markov chains. The entropy production is then the relative action under time reversal and its expectation in a steady state is the MEP.

The product measure \(\rho_u\) with uniform density \(u \in [0, 1]\) is a stationary (non-reversible) measure for this dynamics. If we now consider a trajectory \((\eta_t, t \in [-T, T])\) of the stationary process in which at a certain time, when the configuration is \(\eta\), a particle hops from site \(i\) to \(i + 1\), then the time-reversed trajectory shows a particle jumping from \(i + 1\) to \(i\). The contribution of this event to the entropy production is therefore

\[
\ln c(i, i + 1, \eta) - \ln c(i, i + 1, \eta^{i,i+1}) = E[\eta(i)(1 - \eta(i + 1)) - \eta(i + 1)(1 - \eta(i))] (7.12)
\]
This jump in the trajectory itself happens with a rate $c(i, i+1, \eta)$ and therefore the mean entropy production equals

$$\text{MEP}(u, E) = \int c(0, 1, \eta) \ln \frac{c(0, 1, \eta)}{c(0, 1, \eta^0)} \rho_u(d\eta) = E \ u \ (1 - u) \ \sinh\left(\frac{E}{2}\right)$$  \hspace{1cm} (7.13)

That is correct: the MEP is the product of the field $E$ with the current $j(u, E)$ where the current $j(u, E)$ is the expected net number of particles passing through a given bond:

$$j(u, E) = \int \rho_u(d\eta) c(i, i+1, \eta) \left[ \eta(i)(1 - \eta(i+1)) - \eta(i+1)(1 - \eta(i)) \right]$$

In quadratic approximation (that is close to equilibrium) $j(u, E) \approx u(1 - u)\frac{E}{2}$ and,

$$\text{MEP}(u, E) \approx\frac{j(u, E)^2}{h_c}$$  \hspace{1cm} (7.14)

which is the dissipated heat through a conductor in an electric field $E$ with Ohmic conductivity $h_c \equiv u(1 - u)/2 = \rho_u((\xi(0)(1 - \xi(1)) - \xi(1)(1 - \xi(0)))^2)/4$ given in terms of the variance of the microscopic current (at $E = 0$).

### 7.2 LFT for ASEP

I take a $(2+1)$-dimensional set-up. There are (spatial) squares $V_0 \equiv [-L, L]^2 \cap \mathbb{Z}^2$ and $\Lambda_0 \equiv [-L', L']^2 \cap \mathbb{Z}^2$ with $L' < L$ large, and a continuous time interval $[-T, T]$ in which we observe the ASEP with the external field $E$ in the horizontal direction. That is the 2-dimensional analogue of 7.1 but now on $V_0$ with periodic boundary conditions and hopping rates

$$c(i, j, \eta) \equiv \frac{e^{E/2}}{2} \eta(i)(1 - \eta(j)) + \frac{e^{-E/2}}{2} \eta(j)(1 - \eta(i))$$

for a horizontal bond $\langle ij = i + e_1 \rangle$ with $e_1$ the unit vector in the positive horizontal direction, and

$$c(i, j, \eta) \equiv \frac{1}{2} \left[ \eta(i)(1 - \eta(j)) + \eta(j)(1 - \eta(i)) \right]$$

for a vertical bond $\langle ij = i \pm e_2 \rangle$. As stationary measure I take again $\rho_u$ (Bernoulli with density $u$) and $\eta_s(i), s \in [-T, T], i \in V_0$ denotes the stationary process. I show what becomes of relations (5.8) and (5.9). Let $\mathbb{E}_V^E[\cdot]$ denote the expectation with respect to the process in $V \equiv [-T, T] \times V_0$ with (pathspace) law $P_V^E$. For a function $f$ measurable from $\Lambda \equiv [-T, T] \times \Lambda_0$,

$$\mathbb{E}_V^E[f(\Theta_{T,L})] = \mathbb{E}_V^E[f(\Theta_{T,L})] = \mathbb{E}_V^E[f d\frac{d(P_V^E \Theta_{T,L})}{dP_V^E}]$$  \hspace{1cm} (7.15)

$$= \mathbb{E}_V^E[f d\frac{dP_V^{-E}}{dP_V^E}|_{\Lambda}]$$
so that, with
\[ R^V_\Lambda = \ln \frac{dP^E_\Lambda}{dP^\Lambda_\Lambda} \]
and \( f = G(R^V_\Lambda) \), the identity (7.15) is just (5.8). \( R^V_\Lambda \) is computed from a Girsanov formula for the non-Markovian point process \( dP^E_\Lambda \) and for this I first need to identify the intensities (see [18, 4]). In order to have a Gibbsian structure (allowing me to pass to (5.3)) these intensities must be the same in the bulk of \( \Lambda \) as they were in the bulk of \( V \). That is easy to verify from considering the conditional expectation of a function \( g \) in \( \Lambda_0 \) at time \( t+\delta \) given the past history in \( \Lambda_0 \):

\[
\begin{align*}
\mathbb{E}^E_\Lambda[g(\eta_{t+\delta}(i), i \in \Lambda_0)|\eta_s(i), s \in [-T, t], i \in \Lambda_0] = & \\
\mathbb{E}^E_\Lambda[\mathbb{E}^E_\Lambda[g(\eta_{t+\delta}(i), i \in \Lambda_0)|\eta_s(i), s \in [-T, t], i \in \Lambda_0]|\eta_s(i), s \in [-T, t], i \in \Lambda_0]
\end{align*}
\] (7.16)

The conditional expectation inside is explicit from the Markov process in \( V_0 \):

\[ \mathbb{E}^E_\Lambda[g(\eta_{t+\delta}(i), i \in \Lambda_0)|\eta_s(i), s \in [-T, t], i \in \Lambda_0] = g(\eta_t(i), i \in \Lambda_0) + \delta E^L_\Lambda g(\eta_t(i), i \in \Lambda_0) + O(\delta^2) \] (7.17)

with \( E^L_\Lambda \) the Markov generator of the process. Since \( E^L_\Lambda \) is a sum over all bonds with local rates, we see that the process restricted to \( \Lambda_0 \) has the same rates except for the boundary of \( \Lambda_0 \) where a birth and death process is added. As a result the Girsanov formula for \( R^V_\Lambda \) is indeed

\[ R^V_\Lambda = S_\Lambda - F^V_\Lambda \]

measurable in \( \Lambda \) with

\[
S_\Lambda(\eta_\Lambda) = E \int_{-T}^T \sum_{i, i+e_1 \in \Lambda_0} [\eta_t(i)(1-\eta_t(i+e_1)) - \eta_t(i+e_1)(1-\eta_t(i))] dN^1_i(t)
\] (7.18)

where \( N^1_i(t) \) is the number of jumps between \( i \) and \( i+e_1 \) up to time \( t \), and

\[
F^V_\Lambda(\eta_\Lambda) = \sum_{s \in \partial \Lambda} \sum_{b_i} \int_{-T}^T \ln \frac{\kappa^E_{b_i}(\eta, t)}{\kappa^E_{b_i}(\eta, t)}
\]
\[
+ \eta_t(i) \ln \frac{\lambda^E_{b_i}(\eta, t)}{\lambda^E_{b_i}(\eta, t)} dN_{b_i}(t)
\] (7.19)

where the second sum is over all bonds \( b_i \) starting at site \( i \in \Lambda_0 \) with the other end \( j \in \Lambda_0 \) and, with \( b_i = (ij) \),

\[ \lambda^E_{b_i}(\eta, t) \equiv E^{E_j/2} \mathbb{E}^E_\Lambda[1 - \eta_t(j)|\eta_s(k), k \in \Lambda_0, s \in [-T, t]] \]

and

\[ \kappa^E_{b_i}(\eta, t) \equiv E^{E_j/2} \mathbb{E}^E_\Lambda[\eta_t(j)|\eta_s(k), k \in \Lambda_0, s \in [-T, t]] \]

for \( E_j = \pm E \) if \( j = i \pm e_1 \) and \( E_j = 0 \) if \( j = i \pm e_2 \). The expression (7.18) is the entropy production, that is field times current in \( \Lambda \), and (7.19) is the boundary term. That establishes (5.3).
References

[1] R. Balian: du microscopique au macroscopique, TOME 2, Ecole Polytechnique (1982).

[2] J. Bricmont: Science of Chaos or Chaos in Science? In: The Flight from Science and Reason, Annals of the N.Y. Academy of Sciences 775, p. 131 (1996). Physicalia Magazine 17, 159, (1995).

[3] F. Bonetto, J.L. Lebowitz and L. Rey-Bellet: Fourier’s Law: a Challenge to Theorists. Preprint, mp-arc 00-89 (2000).

[4] P. Brémaud: Point Processes and Queues, Martingale Dynamics. Springer-Verlag (New-York), 1981.

[5] F. den Hollander: Large Deviations and Entropy. In this volume.

[6] J.-P. Eckmann, C.-A. Pillet and L. Rey-Bellet: Non-Equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. Commun. Math. Phys. 201, 657–697 (1999).

[7] D.J. Evans, E.G.D. Cohen and G.P. Morriss: Probability of second law violations in steady flows. Phys. Rev. Lett. 71, 2401–2404 (1993).

[8] G. Gallavotti: Chaotic hypothesis: Onsager reciprocity and fluctuation - dissipation theorem. J. Stat. Phys. 84, 899-926 (1996).

[9] G. Gallavotti: A local fluctuation theorem. Physica A 263, 39–50 (1999).

[10] G. Gallavotti: Chaotic dynamics, fluctuations, nonequilibrium ensembles. Chaos 8, 384–392 (1998).

[11] G. Gallavotti and E.G.D. Cohen: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Lett. 74, 2694–2697 (1995). Dynamical ensembles in stationary states. J. Stat. Phys. 80, 931–970 (1995).

[12] H.-O. Georgii: Aspects of Entropy. In this volume.

[13] S. Goldstein: Boltzmann’s Approach to Statistical Mechanics. To appear in Chance in Physics: Foundations and Perspectives, edited by D. Dürr (Springer-Verlag)

[14] E.T. Jaynes: The Gibbs Paradox. In: Maximum-Entropy and Bayesian Methods, G. Erickson, P. Neudorfer, and C. R. Smith (eds.), Kluwer, Dordrecht (1992).

[15] J. Kurchan: Fluctuation theorem for stochastic dynamics. J. Phys. A: Math. Gen. 31, 3719–3729 (1998).

[16] O.E. Lanford III: Entropy and equilibrium states in classical statistical mechanics. In Statistical Mechanics and Mathematical Problems (Batelle Seattle Rencontres 1971), Lecture Notes in Physics No. 20 (Springer-Verlag, Berlin), 1–113 (1973).

[17] J.L. Lebowitz: Microscopic Origins of Irreversible Macroscopic Behavior. Physica A. 263, 516–527 (1999).

[18] R.S. Lipster and A.N. Shiryaev: Statistics of Random Processes II Applications. Springer-Verlag (New-York), 1978.

[19] C. Maes: The Fluctuation Theorem as a Gibbs Property. J. Stat. Phys. 95, 367-392 (1999).
[20] C. Maes, F. Redig and M. Verschuere: Entropy Production for Interacting Particle Systems. Markov Proc. Rel. Fields, to appear.

[21] C. Maes, F. Redig and M. Verschuere: No Current Without Heat. Preprint (2000).

[22] C. Maes, F. Redig and A. Van Moffaert: On the definition of entropy production via examples. J. Math. Phys. 41, 1528–1554 (2000).

[23] C. Maes C. and F. Redig: Positivity of entropy production. J. Stat. Phys. 101, 3-16 (2000).

[24] J.L. Lebowitz and H. Spohn: A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics. J. Stat. Phys. 95, 333–365 (1999).

[25] R. Penrose: The Emperor’s New Mind. Oxford University Press (Oxford), 1989.

[26] R. Minlos, S. Roelly and H. Zessin: Gibbs states on space-time. Potential Analysis 13, issue 4 (2001).

[27] D. Ruelle: Smooth Dynamics and New Theoretical Ideas in Nonequilibrium Statistical Mechanics. J. Stat. Phys. 95, 393–468 (1999).

[28] S.R.S. Varadhan: Entropy, Large Deviations and Scaling Limits. In this volume.