Competing Compressible and Incompressible Phases in Rotating Atomic Bose Gases at Filling Factor $\nu = 2$

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We study the groundstates of weakly interacting atomic Bose gases under conditions of rapid rotation. We present the results of large-scale exact diagonalisation studies on a periodic geometry (a torus) which allows studies of compressible states with broken translational symmetry. Focusing on filling factor $\nu = 2$, we show a competition between the triangular vortex lattice, a quantum smectic state, and the incompressible $k = 4$ Read-Rezayi state. We discuss the corrections arising from finite size effects, and the likely behaviour for large system sizes. The Read-Rezayi state is stabilised by a moderate amount of additional dipolar interactions.

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I. INTRODUCTION

Under conditions of rapid rotation, ultra-cold gases of bosonic atoms are predicted to enter a very interesting regime of strongly-correlated many-particle physics. At sufficiently high vortex density, quantum fluctuations of the vortices can drive a quantum phase transition from the condensed vortex lattice phase into strongly-correlated liquid phases which are bosonic versions of conventional and unconventional fractional quantum Hall states.\textsuperscript{1,2,5,6} The condition for quantum melting of the vortex lattice was first discussed in Ref.\textsuperscript{2}, where the "filling factor", $\nu$, was shown to be the relevant control parameter, and exact diagonalisation studies were used to show evidence of a triangular vortex lattice for $\nu \gtrsim 6$. (For a uniform system, $\nu \equiv N/N_v$ where $N$ is the number of bosons and $N_v$ the number of vortices.)

While the limits of large\textsuperscript{2,3,4} and small\textsuperscript{1,2,5,6} filling factors are well understood, the nature of the groundstates in the regime of intermediate filling factors is less clear. This is a difficult regime to study theoretically, but is likely to show some of the most interesting new physics and will be the first to be accessed as experiments progress towards the strongly-correlated regime.\textsuperscript{10} In Ref.\textsuperscript{2} evidence was presented for the appearance of the sequence of Read-Rezayi (RR) states\textsuperscript{11} at $\nu = k/2$ with $k$ integer, which are incompressible liquid states whose quasiparticle excitations obey non-abelian exchange statistics.\textsuperscript{12} Subsequent exact diagonalisation studies have shown that good convergence to the RR states at $\nu \geq 3/2$ is not achieved with available system sizes.\textsuperscript{13} The role of the RR state at $\nu = 3/2$ was significantly clarified in Ref.\textsuperscript{13} where it was shown that this state provides an excellent description of the groundstate when a small additional non-local interaction is introduced; similar effects have been reported for $\nu = 2$ and $5/2$ in the spherical geometry.\textsuperscript{14}

In this paper we study the groundstates of rotating bosons in the regime of intermediate filling factor using exact diagonalisation studies in the torus geometry up to system sizes much larger than those previously reported in this regime. We show that the groundstates involve a competition between compressible states with broken translational symmetry and the incompressible RR states. Our results identify a competing "smectic" phase, and point out the important role that finite size effects can have in stabilising this state both in numerical studies and in experimental systems.

II. MODEL

We consider a system of bosonic atoms of mass $M$ confined to a harmonic trap with cylindrical symmetry, and with frequencies $\omega_\parallel$ and $\omega_\perp$ in the axial and transverse directions setting the lengthscales $a_{\parallel,\perp} \equiv \sqrt{\hbar/(M\omega_{\parallel,\perp})}$. The atoms interact through contact interactions $V(r) = g \delta^3(r)$, with $g = 4\pi\hbar^2 a_s/M$ chosen to reproduce the s-wave scattering length $a_s$. We study the regime of weak interactions, when the mean interaction energy is small compared to the trap energies, such that the particles are restricted to the quasi-2D lowest Landau level (LLL) regime.\textsuperscript{1,9,15} In this limit, the characteristic energy scale is set by the s-wave Haldane pseudopotential, $V_0 = g/(2\pi a_s^2 a_{\parallel,\perp}^2)$. We have investigated the nature of the bulk groundstates (i.e. of systems containing large numbers of vortices)\textsuperscript{10} using exact diagonalisation studies on a periodic rectangular geometry (a torus) of sides $a$ and $b$, which contains $N_v = ab/\pi a_s^2$ vortices. The torus lends itself naturally to the present study: it is commensurate with crystalline order so allows the study of states with broken translational invariance; the existence of a conserved momentum\textsuperscript{15} reduces the sizes of the matrices in the diagonalisations and permits the identification of translational-symmetry-broken states through the appearance of quasi-degeneracies at wavevectors equal to the reciprocal lattice vectors (RLVs) of the crystalline order.\textsuperscript{18} We express the momentum as a dimensionless wavevector $\mathbf{K} = (K_\parallel, K_\parallel)$ using units of $2\pi\hbar/a$ and $2\pi\hbar/b$ for the $x$ and $y$ components, and report only positive $K_\parallel, K_\parallel$ [states at $(\pm K_\parallel, \pm K_\parallel)$ are degenerate by symmetry].
The shaded symbols mark a band of excitations discussed in the text. [Note that the components $K_x, K_y$ are expressed in units of $2\pi/a$ and $2\pi/b$.]

III. RESULTS

We find that for filling factors $\nu \geq 2$ the spectrum is sensitive to the aspect ratio $a/b$ of the torus.\textsuperscript{15} The sensitivity to boundary conditions is a sign of a compressible groundstate, at least on the lengthscale defined by the finite system size in the calculation. Here, we shall focus on filling factor $\nu = 2$, which is convenient to study from a numerical point of view. (The qualitative features are reproduced at larger filling factors.)

A. Smectic Groundstate

In Fig. 1 we present the low-energy spectra at $\nu = 2$ for systems containing $N_v = 8$ and 12 vortices, at aspect ratios $(a/b = 2$ and $4/3$) which are very close to the values for which the groundstate energy has a local minimum.\textsuperscript{20} In both cases, the groundstate [at $K = (0, 0)$] is quasi-degenerate with a single state at non-zero wavevector $[a/b]$. This indicates a tendency to the formation of a smectic groundstate, \textit{i.e.}, a density wave state in which translational symmetry is broken in only one direction. In Figs.1(a) and (b) the smectic groundstates consist of four stripes parallel to the $y$-axis, with 4 and 6 particles per stripe respectively. Although the smectic period is close to the lattice constant of a square lattice, $a^{sq} = \sqrt{\pi}a_\perp$, there is no indication of ordering in a vortex lattice (there are no quasi-degenerate states at the requisite RLVs).

The low-energy excitation spectra above these quasi-degenerate states are consistent with the existence of a smectic groundstate. For $N_v = 8$, Fig. 1(a), the groundstate has four stripes of four particles each. There exist low-energy “particle-hole” excitations in which a particle is moved from one stripe to another (to give occupations of 4, 4, 3, 5). These excitations account for the band of states shown as shaded symbols in Fig. 1(a). This band is narrow showing that exchange interactions between the stripes are small. For larger systems (longer stripes) one expects the energy of this band to fall and exchange interactions to increase, so these particle-hole excitations will contribute increasingly to the groundstate. It is possible that this will lead to a phase locking of the stripes, causing the formation of a vortex lattice: a set of states at the RLVs of the lattice would then emerge from this band and become quasi-degenerate with the groundstate; the remaining states would contribute to the phonon mode of the lattice. The exact diagonalisation studies show that, at least up to $N_v = 12$, a vortex lattice does not form. Rather, the system shows behaviour expected of a smectic groundstate: the analogous low-energy band [shaded symbols in Fig. 1(b)] broadens to give low-energy states at $K = (0, 3)$ and $(0, 6)$ corresponding to the competing \textit{re-oriented} smectic state (with three stripes of eight particles parallel to the $x$-axis); the remaining low-energy states in this band form the expected gapless shear mode of this smectic state. Thus, the exact diagonalisation results up to $N_v = 12$ are consistent with a smectic groundstate at $\nu = 2$.\textsuperscript{22}

B. Triangular Vortex Lattice

Evidence for a competing vortex lattice state at $\nu = 2$ can be obtained by choosing an aspect ratio that is commensurate with the triangular vortex lattice (away from the local minimum in groundstate energy). There then develop approximate quasi-degeneracies of the groundstate with states at the RLVs of the triangular lattice, which emerge from the band of low-energy states discussed above. The spectra are shown in Fig.2 for systems with $N_v = 8, 12$. In all systems we have studied the states at the RLVs of the triangular lattice do not become well separated from other states in the low-energy band. Moreover, since this aspect ratio does not correspond to a local minimum of the energy, there is poor evidence that the groundstate is a triangular vortex lattice.
C. Finite-size Effects

As we now discuss, finite size corrections may be acting to favour the smectic state as compared to the triangular vortex lattice. We illustrate the relevance of finite size corrections by making simple mean-field ansatze for the triangular vortex lattice and smectic states.

The triangular vortex lattice is the fully condensed state in the 2D LLL that minimises the mean interaction energy. The mean energy of this state on a periodic lattice is

\[
\frac{E_{\text{tri}}}{N} = \beta_A \nu V_0 - \frac{\beta_A}{N_v} V_0
\]

where \(\beta_A = 1.1596\) and the correction at finite \(N_v\) arises from the lack of self-interaction of the particles [the mean interaction energy is proportional to \(N(N-1)\)].

As a simple description of the smectic state, we consider the Fock state, defined on a long cylinder with circumference \(L_y\), in which \(N_v\) bosons occupy each of a set of Landau-gauge orbitals (\"stripes\") that are extended around the circumference of the cylinder and that are equally spaced along the length of the cylinder by \(\Delta x\). (This state is similar to the Tao-Thouless state\textsuperscript{24}.) The smectic state describes a fragmented condensate, with \(N_v\) particles in each stripe and with no fixed phase relationship between the stripes. (In the LLL, the establishment of a fixed phase relationship between the stripes would establish fixed positions of the vortices, \textit{i.e.} form a vortex lattice.) For a large circumference, \(L_y \to \infty\), the mean interaction energy is minimised, at fixed filling factor \(\nu = N_v \pi a_0^2 / (L_y \Delta x)\), for a period \(\Delta x = 1.73a_\perp\). This is close to the period of the smectic state appearing in numerics, of approximately \(a_\perp = 1.77a_\perp\). At this period the energy per particle of the smectic state is

\[
\frac{E_{\text{smectic}}}{N} = \nu \beta_s V_0 - \frac{(\Delta x/a_\perp^2)}{N_{v,s}} V_0
\]

where \(\beta_s \equiv 1.17184\), and \(N_{v,s} \equiv N_v / \nu\) is the number of vortices associated with each stripe. The finite-size correction arises from the lack of self-interaction between the \(N_v\) particles in each stripe.

Since \(\beta_s > \beta_A\), the smectic state has higher energy than the triangular lattice in the thermodynamic limit \((N_v, N_{v,s} \to \infty)\). This is a consequence of the general result that repulsive interactions disfavour fragmentation of a condensate owing to the loss of exchange interactions.\textsuperscript{23} It is remarkable, however, just how small the energy difference between these states is \([(\beta_s - \beta_A)^2 / \nu \sim 1\%]\). For a system on a long cylinder with finite circumference \((N_v, N_{v,s} \to \infty)\) the smectic state has lower energy than the triangular vortex lattice for \(N_{v,s} \lesssim 80 / \nu\). The stabilisation of the smectic state can be viewed as a form of Mott transition: by fixing a definite particle number on each stripe, there is a reduction in Hartree energy at the expense of an increase in exchange energy. At \(\nu = 2\) these finite size effects cause the trial smectic state to have lower energy than the triangular vortex lattice up to systems of 40 vortices per stripe. This is far beyond the system sizes that can be studied in exact diagonalisations. It is also far beyond the maximum number of vortices crossing the clouds in existing experiments\textsuperscript{9,10} (on the order of ten), showing that finite size effects can play a significant role in determining the groundstates at intermediate filling factors in experimental geometries.

The simple mean-field analysis presented here is not intended to provide quantitative corrections to the exact diagonalisation results. This analysis neglects quantum fluctuations which reduce the mean energies of the smectic and vortex lattice states even in the thermodynamic limit. Nevertheless, there are indications that finite size corrections do play an important role in the energetics of the smectic state in the numerics: the smectic state has a lower energy the shorter is the length of the stripes. For example, for \(N_v = 12\), \(a/b = 4/3\) [Fig. 1(b)], the two high symmetry configurations correspond to configurations with either four short stripes with \(N_{v,s} = 3\) parallel to the \(x\)-axis, or three long stripes with \(N_{v,s} = 4\) parallel to the \(y\)-axis; they have energies obtained from the
states have periods that are larger than a more stable to quantum fluctuations states with stripe and "bubble crystal" ordering. The groundstate is well described by broken-symmetry on the spherical geometry for a model short-range interaction with the compressible states discussed above. An enhancement of the overlap for the groundstate at \( \nu = 2 \) is well described by the \( k = 4 \) RR state. For \( \nu = 2 \) and \( a/b = 1 \) by introducing interactions with non-zero range, we find that the spectrum becomes characteristic of the incompressible RR state (showing clear quasi-degeneracies at the expected wavevectors, and a gap to other excitations). The case of dipolar interaction is of particular experimental relevance, as these can be important for atomic species with large magnetic moments. Fig. 3 shows the overlap of the exact groundstate with the RR state as a function of the additional dipolar interaction. In a narrow range around \( V_2/V_0 = 0.3 \) the groundstate is well described by the \( k = 4 \) RR state. For \( V_2/V_0 \gtrsim 0.4 \) the groundstate is well described by broken-symmetry states with stripe and "bubble crystal" ordering. These states have periods that are larger than \( a^\text{m} \), and are much more stable to quantum fluctuations than are the compressible states discussed above. An enhancement of the overlap with the \( k = 4 \) RR state in related calculations on the spherical geometry for a model short-range interaction has also been reported.

**IV. SUMMARY**

We have shown that exact diagonalisation studies of rotating Bose gases with contact interactions indicate that the groundstate at \( \nu = 2 \) is a quantum smectic state. We argued that finite size corrections may lead to an enhanced stability of the smectic state in the system sizes amenable to exact diagonalisations, and the triangular vortex lattice may have lower energy in the thermodynamic limit. In the regime of intermediate filling factors, \( \nu \geq 2 \), the correlation length of the groundstate for contact interactions remains larger than the max-
mum system sizes that are available in exact diagonalisations. Comparisons with the RR states at $\nu = k/2$ quickly become severely limited, as these states involve clustering of groups of $k$ particles (at $\nu = 2$ exact diagonalisations currently allow studies of at most six clusters). The torus is likely to favour broken translational-symmetry states of short period, so it is possible that the true groundstates at intermediate filling factors are quantum states of more complex symmetry – including quantum nematic phases\(^{21}\) or incompressible liquids with small gaps. Clearly this is a regime in which there are strong quantum fluctuations of vortices, and where novel strongly correlated groundstates may form. It will be very interesting to explore this regime experimentally.

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