Writing $\pi$ as sum of arcotangents with linear recurrent sequences, Golden mean and Lucas numbers

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Abstract

In this paper, we study the representation of $\pi$ as sum of arcotangents. In particular, we obtain new identities by using linear recurrent sequences. Moreover, we provide a method in order to express $\pi$ as sum of arcotangents involving the Golden mean, the Lucas numbers, and more in general any quadratic irrationality.

1 Expressions of $\pi$ via arctangent function with linear recurrent sequences

The problem of expressing $\pi$ as the sum of arctangents has been deeply studied during the years. The first expressions are due to Newton (1676), Machin (1706), Euler (1755), who expressed $\pi$ using the following identities

$$\frac{\pi}{2} = 2 \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{4}{7}\right) + \arctan\left(\frac{1}{8}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{3}{79}\right),$$

respectively (see, e.g., [12] and [13]). Many other identities and methods to express and calculate $\pi$ involving the arctangent function have been developed. Some recent results are obtained in [6] and [2].

In this section, we find a method to generate new expressions of $\pi$ in terms of sum of arctangents, mainly using the properties of linear recurrent sequences. For the sake of simplicity, we will use the following notation:

$$A(x) = \arctan(x).$$
It is well-known that for $x, y \geq 0$, if $y \neq 1$

$$A(x) + A(y) = \begin{cases} A(x \odot y) & \text{if } xy < 1, \\ A(x \odot y) + \text{sign}(x) \pi & \text{if } xy > 1, \end{cases}$$

where

$$x \odot y = \frac{x + y}{1 - xy}$$

Let us denote by $x^{\odot n}$ the $n$-th power of $x$ with respect to the product $\odot$.

**Remark 1.** The product $\odot$ is associative, commutative and 0 is the identity.

**Definition 1.** We denote by $a = (a_n)_{n=0}^{+\infty} = W(\alpha, \beta, p, q)$ the linear recurrent sequence of order 2 with characteristic polynomial $t^2 - pt + q$ and initial conditions $\alpha$ and $\beta$, i.e.,

$$\begin{cases} a_0 = \alpha \\ a_1 = \beta \\ a_n = pa_{n-1} - qa_{n-2} & \forall n \geq 2 \ . \end{cases}$$

**Theorem 1.** Given $n \in \mathbb{N}$ and $x \in \mathbb{R}$, with $x \neq \pm 1$, we have

$$\left( \frac{1}{x} \right)^{\odot n} = \frac{v_n(x)}{u_n(x)}, \ \forall n \geq 1$$

where

$$(u_n(x))_{n=0}^{\infty} = W(1, x, 2x, 1 + x^2), \quad (v_n(x))_{n=0}^{\infty} = W(0, 1, 2x, 1 + x^2). \quad (1)$$

**Proof.** The matrix

$$M = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

has characteristic polynomial $t^2 - 2xt + x^2 + 1$. Consequently, it is immediate to see that

$$M^n = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix} .$$

Using the matrix $M$ we can observe that

$$\begin{pmatrix} u_{n-1}(x) & v_{n-1}(x) \\ -v_{n-1}(x) & u_{n-1}(x) \end{pmatrix} \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix} ,$$

i.e.,

$$\begin{cases} u_n(x) = xu_{n-1}(x) - v_{n-1}(x) \\ v_n(x) = u_{n-1}(x) + xv_{n-1}(x) \end{cases}, \ \forall n \geq 1.$$
Now, we prove the theorem by induction. It is straightforward to check that

\[
\frac{1}{x} = \frac{v_1(x)}{u_1(x)}, \quad \left(\frac{1}{x}\right)\odot^2 = \frac{\frac{1}{x} + \frac{1}{x}}{1 - \frac{1}{x^2}} = \frac{2x}{x^2 - 1} = \frac{v_2(x)}{u_2(x)}.
\]

Moreover, let us suppose

\[
\left(\frac{1}{x}\right)\odot^{(n-1)} = \frac{v_{n-1}(x)}{u_{n-1}(x)}
\]

for a given integer \(n \geq 1\), then

\[
\left(\frac{1}{x}\right)\odot^n = \frac{1}{x} \odot \left(\frac{1}{x}\right)\odot^{(n-1)} = \frac{1}{x} \odot \frac{v_{n-1}(x)}{u_{n-1}(x)} = \frac{u_{n-1}(x) + xv_{n-1}(x)}{xu_{n-1}(x) - v_{n-1}(x)}.
\]

\[
\frac{v_n(x)}{u_n(x)}.
\]

\[\Box\]

**Theorem 2.** Given \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\), with \(x \neq \pm 1\), we have

\[x\odot^n = (-1)^{n+1} \left(\frac{v_n(x)}{u_n(x)}\right)^{(-1)^n}, \quad \forall n \geq 1\]

where \(u_n(x)\) and \(v_n(x)\) are given by Eq. (1).

**Proof.** By using the same arguments of Theorem 1 we can write

\[x = \frac{u_1(x)}{v_1(x)} \quad \text{and} \quad x\odot^2 = \frac{2x}{1 - \frac{1}{x^2}} = \frac{v_2(x)}{u_2(x)}\]

Let us suppose by induction that \(x\odot^{(n-1)} = (-1)^n \left(\frac{v_{n-1}(x)}{u_{n-1}(x)}\right)^{(-1)^{n-1}}\)

if \(n\) is even

\[x\odot^n = \frac{x - v_{n-1}(x)}{u_{n-1}(x)} = \frac{xu_{n-1}(x) - v_{n-1}(x)}{u_{n-1}(x) + xv_{n-1}(x)} = \frac{u_n(x)}{v_n(x)},\]

if \(n\) is odd

\[x\odot^n = \frac{x + u_{n-1}(x)}{v_{n-1}(x)} = \frac{xv_{n-1}(x) + u_{n-1}(x)}{v_{n-1}(x) - xu_{n-1}(x)} = \frac{v_n(x)}{u_n(x)}.
\]

\[\Box\]
Let us highlight the matrix representation of the sequences \((u_n)_{n=0}^{\infty}\) and \((v_n)_{n=0}^{\infty}\) used in the previous theorem. Given the matrix

\[
M = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}
\]

we have

\[
M^n = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix}
\]

\[
M^n \begin{pmatrix} v_m(x) \\ u_m(x) \end{pmatrix} = \begin{pmatrix} v_{n+m}(x) \\ u_{n+m}(x) \end{pmatrix}
\]

The sequences \((u_n)_{n=0}^{\infty}\) and \((v_n)_{n=0}^{\infty}\) are particular cases of the Rédei polynomials \(N_n(d, z)\) and \(D_n(d, z)\), introduced by Rédei [10] from the expansion of \((z + \sqrt{d})^n = N_n(d, z) + D_n(d, z)\sqrt{d}\). The rational functions \(N_n(d, z)\) and \(D_n(d, z)\) have many interesting properties, e.g., they are permutations of finite fields, as described in the book of Lidl [7]. In [1], the authors showed that Rédei polynomials are linear recurrent sequences of degree 2:

\[
(N_n(d, z))_{n=0}^{\infty} = W(1, z, 2z, z^2 - d), \quad (D_n(d, z))_{n=0}^{\infty} = W(0, 1, 2z, z^2 - d).
\]

Thus, we can observe that

\[
u_n(x) = N_n(-1, x), \quad v_n(x) = D_n(-1, x), \quad \forall n \geq 0.
\]

Moreover, a closed expression of Rédei polynomials is well–known (see, e.g., [1]). In this way, we can derive a closed expression for the sequences \((u_n)_{n=0}^{\infty}\) and \((v_n)_{n=0}^{\infty}\):

\[
\begin{cases} 
\quad u_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} \\
\quad v_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} (-1)^k x^{n-2k-1}
\end{cases}
\]

Rational powers with respect to the product \(\odot\) can also be considered by defining the \(n\)–th root as usual by

\[
z = x^{\odot n} \quad \text{iff} \quad z^n = x.
\]

Moreover, by means of Theorem [2] we have that Eqs. [3] are equivalent to

\[
x = (-1)^{n+1} \left( \frac{v_n(z)}{u_n(z)} \right)^{(-1)^n},
\]
i.e., by Eqs. (2), the \( n \)-th root of \( x \) with respect to the product \( \odot \) is a root of the polynomial

\[
P_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} x^{1+\frac{k+1}{2}} z^k.
\]

Let us consider the equation

\[
nA \left( \frac{1}{x} \right) + A \left( \frac{1}{y} \right) = \frac{\pi}{4}, \quad (4)
\]

we want to solve it when \( n \) and \( x \) are integer values. We point out that Eq. (4) is equivalent to

\[
\left( \frac{1}{x} \right) \odot \frac{1}{y} = 1 \quad (5)
\]

By Theorem 1 we have

\[
\left( \frac{1}{x} \right) \odot \frac{1}{y} = \frac{v_n(x)}{u_n(x)} \odot \frac{1}{y} = \frac{u_n(x) + v_n(x)y}{-v_n(x) + u_n(x)y},
\]

Thus

\[
y = \frac{u_n(x) + v_n(x)}{u_n(x) - v_n(x)}
\]

solves Eq. (5), i.e.,

\[
\left( \frac{1}{x} \right) \odot \frac{u_n(x) + v_n(x)}{u_n(x) - v_n(x)} = 1, \quad \forall x \in \mathbb{Z}
\]

and consequently we can solve Eq. (4), i.e.,

\[
nA \left( \frac{1}{x} \right) + A \left( \frac{u_n(x) - v_n(x)}{u_n(x) + v_n(x)} \right) = \frac{\pi}{4} + k(n, x)\pi, \quad \forall x \in \mathbb{Z},
\]

where \( k \) is a certain integer number depending on \( n \) and \( x \). Precisely, we have

\[
k(n, x) = \text{sign} \left( nA \left( \frac{1}{x} \right) - \frac{\pi}{4} \right) \left( \left\lfloor T \right\rfloor + \chi_{\left( \frac{1}{2}, 1 \right)} \{T\} \right),
\]

where \( \chi_{\left( \frac{1}{2}, 1 \right)} \) is the characteristic function of the set \( \left( \frac{1}{2}, 1 \right) \) and

\[
T = \frac{\left| \frac{\pi}{4} - nA \left( \frac{1}{x} \right) \right|}{\pi}.
\]
In order to obtain Eq. (7), we can rewrite Eq. (6) as

\[ A \left( \frac{u_n(x) - v_n(x)}{u_n(x) + v_n(x)} \right) = \frac{\pi}{4} - nA \left( \frac{1}{x} \right) + k(n, x)\pi. \]

Let us consider the case in which the first member lies in the interval \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). If \( \frac{\pi}{4} - nA \left( \frac{1}{x} \right) \geq 0 \), then \( k(n, x) \) must be negative so that \( \frac{\pi}{4} - nA \left( \frac{1}{x} \right) + k(n, x)\pi \) lies in the correct interval. Since

\[ \frac{\pi}{4} - nA \left( \frac{1}{x} \right) = \pi \left( \lfloor T \rfloor + \{T\} \right), \]

it follows that if \( 0 \leq \{T\} \leq \frac{1}{2} \), then \( 0 \leq \pi \cdot \{T\} \leq \frac{\pi}{2} \) and consequently \( k = -\lfloor T \rfloor \). Conversely, if \( \frac{1}{2} < \{T\} < 1 \), then \( \frac{\pi}{2} < \pi \cdot \{T\} < \pi \) and, observing that

\[ \frac{\pi}{4} - nA \left( \frac{1}{x} \right) = \pi \left( \lfloor T \rfloor + 1 \right) + \pi \left( \{T\} - 1 \right), \]

we obtain \( -\frac{\pi}{2} < \pi (\{T\} - 1) < 0 \), that is \( k(n, x) = -(\lfloor T \rfloor + 1) \).

Similar considerations apply to \( \frac{\pi}{4} - nA \left( \frac{1}{x} \right) < 0 \), obtaining Eq. (7).

**Proposition 1.** The sequences \((u_n(x) + v_n(x))_{n=0}^{\infty}\) and \((u_n(x) - v_n(x))_{n=0}^{\infty}\) are linear recurrent sequences of order 2 and precisely

\[ (u_n(x) + v_n(x))_{n=0}^{\infty} = W(1, x+1, 2x, 1+x^2), \quad (u_n(x) - v_n(x))_{n=0}^{\infty} = W(1, x-1, 2x, 1+x^2) \]

**Proof.** It immediately follows from the definition of the sequences \((u_n)_{n=0}^{\infty}\) and \((v_n)_{n=0}^{\infty}\).

Eq. (6) provides infinitely many identities that express \( \pi \) as sum of arctangents.

**Example 1.** Taking \( n = 7 \) and \( x = 3 \) in Eq. (6) we have

\[ 7A \left( \frac{1}{3} \right) + A \left( \frac{u_7(3) - v_7(3)}{u_7(3) + v_7(3)} \right) = \frac{\pi}{4}, \]

i.e.,

\[ 7 \arctan \left( \frac{1}{3} \right) - \arctan \left( \frac{278}{29} \right) = \frac{\pi}{4}. \]
For $n = 8$ and $x = 3$, we have

$$8 \arctan \left( \frac{1}{3} \right) + \arctan \left( \frac{863}{191} \right) = \frac{\pi}{4} + \pi.$$ 

For $n = 5$ and $x = 2$, we have

$$5 \arctan \left( \frac{1}{2} \right) - \arctan \left( \frac{79}{3} \right) = \frac{\pi}{4}.$$ 

For $n = 2$ and $x = 7$, we have

$$2 \arctan \left( \frac{1}{7} \right) + \arctan \left( \frac{17}{31} \right) = \frac{\pi}{4}.$$ 

2 Golden mean and $\pi$

In Mathematics the most famous numbers are $\pi$ and the Golden mean. Thus, it is very interesting to find identities involving these special numbers. In particular, many expressions for $\pi$ in terms of the Golden mean have been found. For example, using the Machin formula of $\pi$ via arctangents, the following equalities arise

$$\frac{\pi}{4} = \arctan \left( \frac{1}{\phi} \right) + \arctan \left( \frac{1}{\phi^2} \right),$$

$$\frac{\pi}{4} = 2 \arctan \left( \frac{1}{\phi^3} \right) + \arctan \left( \frac{1}{\phi^5} \right),$$

$$\frac{\pi}{4} = 3 \arctan \left( \frac{1}{\phi^6} \right) + \arctan \left( \frac{1}{\phi^8} \right),$$

$$\pi = 12 \arctan \left( \frac{1}{\phi^9} \right) + 4 \arctan \left( \frac{1}{\phi^{10}} \right),$$

see [3], [4], [5]. Moreover, in [6], the authors found all possible relations of the form

$$\frac{\pi}{4} = a \arctan(\phi^k) + b \arctan(\phi^l),$$

where $a, b$ are rational numbers and $k, l$ integers.

In this section, we find new expressions of $\pi$ as sum of arctangents involving $\phi$. When $n = 2$, from Eq. (5) we find

$$y = \frac{x^2 + 2x - 1}{x^2 - 2x - 1}. \quad (8)$$

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It is well-known that the minimal polynomial of $\phi^m$ is
\[ f_m(t) = t^2 - L_m t + (-1)^m, \]
where $(L_m)_{m=0}^\infty = W(2, 1, 1, -1)$ is the sequence of Lucas numbers (A000032 in OEIS [11]). If we set $x = \phi^m$ in (3), then it is equivalent to replace $x^2 + 2x - 1$ and $x^2 - 2x - 1$ with
\[ x^2 + 2x - 1 \pmod{f_m(x)}, \quad x^2 - 2x - 1 \pmod{f_m(x)}, \]
respectively. When $m$ is odd, dividing by $x^2 - L_m x - 1$, we obtain
\[ y = \frac{(L_m + 2)x}{(L_m - 2)x} = \frac{L_m + 2}{L_m - 2} \]
and when $m$ is even, we have
\[ y = -\frac{2 + (2 + L_m)x}{-2 + (-2 + L_m)x} \]
and therefore
\[ y = \frac{-2 + (2 + L_m)\phi^m}{-2 + (-2 + L_m)\phi^m}. \]
We find the following identities
\[ \frac{\pi}{4} = 2 \arctan \left( \frac{1}{\phi^{2k+1}} \right) + \arctan \left( \frac{L_{2k+1} - 2}{L_{2k+1} + 2} \right) \quad (9) \]
\[ \frac{\pi}{4} = 2 \arctan \left( \frac{1}{\sqrt{2}^k} \right) + \arctan \left( \frac{1 - 2\sqrt{2}}{1 + 2\sqrt{2}} \right). \]
The above procedure can be reproduced for any root $\alpha$ of a polynomial $x^2 - hx + k$, finding expression of $\pi$ as the sum of arctangents involving quadratic irrationalities.

**Example 2.** Let us express $\pi$ in terms of $\sqrt{2}$. Its minimal polynomial is $x^2 - 2$ and
\[ x^2 + 2x - 1 \pmod{x^2 - 2} = 1 + 2x, \quad x^2 - 2x - 1 \pmod{x^2 - 2} = 1 - 2x. \]
We have
\[ \frac{\pi}{4} = 2 \arctan \left( \frac{1}{\sqrt{2}} \right) + \arctan \left( \frac{1 - 2\sqrt{2}}{1 + 2\sqrt{2}} \right). \]
In general, if $k$ is odd the minimal polynomial of $\sqrt{2^k}$ is $x^2 - 2^k$ and
\[ x^2 + 2x - 1 \pmod{x^2 - 2^k} = 2^k - 1 + 2x, \quad x^2 - 2x - 1 \pmod{x^2 - 2^k} = 2^k - 1 - 2x. \]
We have the following identity
\[ \frac{\pi}{4} = 2 \arctan \left( \frac{1}{\sqrt{2^k}} \right) + \arctan \left( \frac{2^k - 1 - 2^{k+1}}{2^k - 1 + 2^{k+1}} \right). \]
Example 3. Let us consider $\alpha = \frac{1}{2}(5 + \sqrt{29})$. The minimal polynomial of $\alpha^3$ is $x^2 - 140x - 1$ and

$$x^2 + 2x - 1 \pmod{x^2 - 140x - 1} = 142x, \quad x^2 - 2x - 1 \pmod{x^2 - 140x - 1} = 138x.$$  

Thus, we have

$$\frac{\pi}{4} = 2 \arctan \left( \frac{8}{(5 + \sqrt{29})^3} \right) + \arctan \left( \frac{69}{71} \right).$$

We can find different identities involving $\pi$ and the Golden mean considering the equation

$$x^{\frac{1}{4}} \odot y = 1. \quad (10)$$

**Proposition 2.** For any real number $x$, the following equalities hold

$$2A(-x \pm \sqrt{1 + x^2}) + A(x) = \pm \frac{\pi}{2}. \quad (11)$$

**Proof.** By Theorem 2 we know that the roots of the polynomial $P_2(z) = xz^2 + 2z - x$ are the values of $x^{\frac{1}{2}}$. Hence, from Eq. (10) we obtain

$$z_i \odot y = 1, \quad i = 1, 2, \quad (12)$$

where

$$z_1 = \frac{-1 + \sqrt{1 + x^2}}{x} \quad \text{and} \quad z_2 = \frac{-1 - \sqrt{1 + x^2}}{x}.$$  

Finally, solving Eq. (10) with respect to $y$ we get

$$y_1 = -x + \sqrt{1 + x^2} \quad \text{or} \quad y_2 = -x - \sqrt{1 + x^2}.$$  

It should be noted that if $x$ is positive then $y_2 < 0$ and $z_2 \cdot y_2 > 1$ so that

$$\frac{1}{2}A(x) + A(y_2) = A\left(x^{\frac{1}{2}} + y_2\right) - \frac{\pi}{2},$$

similar reasoning can be applied if $x$ is negative.

Now, substituting in Eqs. (12) we have

$$\frac{1}{2}A(x) + A(-x \pm \sqrt{1 + x^2}) = \pm \frac{\pi}{4},$$

or equivalently

$$2A(-x \pm \sqrt{1 + x^2}) + A(x) = \pm \frac{\pi}{2}.$$

\[\square\]
Eqs. (11) yield to other interesting formulas involving $\pi$, $\phi$ and Lucas numbers. To show this, we need some identities about Lucas numbers, Fibonacci numbers and the Golden mean:

$$\phi^m = \frac{L_m + F_m\sqrt{5}}{2}, \quad L_m^2 - 5F_m^2 = 4(-1)^m,$$

see, e.g., [9]. Considering $m$ odd, if we set

$$x = \frac{L_m}{2},$$

it follows

$$-x - \sqrt{1 + x^2} = -\frac{L_m - \sqrt{4 + L_m^2}}{2} = -\frac{L_m - F_m\sqrt{5}}{2} = -\phi^m. \quad (13)$$

Thus, substituting Eq. (13) into Eqs. (11) we find the formula

$$-\frac{\pi}{2} = \arctan \left(\frac{L_{2k+1}}{2}\right) - 2\arctan \left(\phi^{2k+1}\right). \quad (14)$$

On the other hand, if we consider $y = -x + \sqrt{1 + x^2}$ we have

$$-x + \sqrt{1 + x^2} = -\frac{L_m + \sqrt{4 + L_m^2}}{2} = -\frac{L_m + F_m\sqrt{5}}{2}. \quad (15)$$

Moreover,

$$\phi^m \cdot \frac{L_m + F_m\sqrt{5}}{2} = \frac{L_m^2 + 5F_m^2}{4} = 1,$$

and substituting in Eqs. (11) another interesting formula arises

$$\frac{\pi}{2} = \arctan \left(\frac{L_{2k+1}}{2}\right) + 2\arctan \left(\frac{1}{\phi^{2k+1}}\right). \quad (16)$$

Furthermore, by Eq. (9) we obtain an identity that only involves the Lucas numbers

$$\frac{\pi}{4} = \arctan \left(\frac{L_{2k+1}}{2}\right) - \arctan \left(\frac{L_{2k+1} - 2}{L_{2k+1} + 2}\right). \quad (17)$$

The previous identity corresponds to a special case of the following proposition.

**Proposition 3.** Let $f, g$ be real functions. If

$$g(x) = \frac{f(x) - 1}{f(x) + 1},$$

then

$$A(f(x)) - A(g(x)) = \frac{\pi}{4} + k\pi, \quad (18)$$

for some integer $k$. 

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Proof. We use the product $\odot$ for solving $A(f(x)) - A(g(x)) = \frac{\pi}{4}$. We have
\[
A \left( \frac{f(x) - g(x)}{1 + f(x)g(x)} \right) = \frac{\pi}{4}
\]
and
\[
\frac{f(x) - g(x)}{1 + f(x)g(x)} = 1
\]
from which
\[
g(x) = \frac{f(x) - 1}{f(x) + 1}.
\]

Remark 2. Eq. (18) has been found by means of only elementary algebraic considerations. The same result could be derived from analysis. Observe that given the functions $f$ and $g$ satisfying the hypothesis of the previous proposition, then $(\arctan f(x))' = (\arctan g(x))'$. When $f(x)$ and $g(x)$ are specified in Eq. (18), the value of $k$ can be retrieved as in Eq. (7) with analogous considerations.

The previous proposition allows to determine new beautiful identities. For example, the function $f(x) = \frac{ax}{b}$ determines the function $g(x) = \frac{ax - b}{ax + b}$ and
\[
A \left( \frac{ax}{b} \right) - A \left( \frac{ax - b}{ax + b} \right) = \frac{\pi}{4} + k\pi.
\]
For $a = 1$ and $b = 2$, we obtain the following interesting formulas
\[
\frac{\pi}{4} = \arctan \left( \frac{x}{2} \right) - \arctan \left( \frac{x - 2}{x + 2} \right),
\]
which holds for any real number $x > -2$ and
\[
-\frac{3\pi}{4} = \arctan \left( \frac{x}{2} \right) - \arctan \left( \frac{x - 2}{x + 2} \right),
\]
valid for any real number $x < -2$. Eqs. (19) and (20) provide infinitely many interesting identities, like Eq. (17) and, e.g., the following ones
\[
\frac{\pi}{4} = \arctan \left( \frac{\phi}{2} \right) - \arctan \left( \frac{\phi - 2}{\phi + 2} \right),
\]
\[
\frac{\pi}{4} = \arctan \left( \frac{F_m}{2} \right) - \arctan \left( \frac{F_m - 2}{F_m + 2} \right),
\]
\[
\frac{\pi}{4} = \arctan \left( \frac{\sqrt{2}}{2} \right) - \arctan \left( \frac{\sqrt{2} - 2}{\sqrt{2} + 2} \right).
\]
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