The Near-Critical Two-Point Function and the Torus Plateau for Weakly Self-avoiding Walk in High Dimensions

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Abstract
We use the lace expansion to study the long-distance decay of the two-point function of weakly self-avoiding walk on the integer lattice $\mathbb{Z}^d$ in dimensions $d > 4$, in the vicinity of the critical point, and prove an upper bound $|x|^{-(d-2)} \exp[-c|x|/\xi]$, where the correlation length $\xi$ has a square root divergence at the critical point. As an application, we prove that the two-point function for weakly self-avoiding walk on a discrete torus in dimensions $d > 4$ has a “plateau.” We also discuss the significance and consequences of the plateau for the analysis of critical behaviour on the torus.

Keywords  Self-avoiding walk · Lace expansion · Two-point function · Torus plateau

Mathematics Subject Classification 82B41 · 82B27 · 60K35

1 Introduction and Main Result

1.1 Introduction

A guiding but generally unproven principle in the scaling theory for critical phenomena in statistical mechanical models on $\mathbb{Z}^d$ is that the two-point function near a critical point generically has decay of the form

$$G_z(x) \approx \frac{1}{|x|^{d-2+\eta}} g(|x|/\xi(z))$$

in some reasonable meaning for “$\approx$”, when $|x|$ is comparable to the correlation length $\xi(z)$ and $z$ is close to its critical value $z_c$. The parameter $z$ depends on the model and represents, e.g., the fugacity for self-avoiding walk, the bond density for bond...
percolation, or the inverse temperature for the Ising model. The universal critical exponent \( \eta \) depends on dimension, the correlation length \( \xi(z) \approx (1 - z/z_c)^{-\nu} \) diverges as \( z \to z_c \) with a dimension-dependent universal critical exponent \( \nu \), and \( g \) is a function with rapid decay. The relation (1.1) is a basis for the derivation of the scaling relations between critical exponents, such as Fisher’s relation \( \gamma = (2 - \eta)\nu \), which are of primary importance in critical phenomena. Such derivations can be found for spin systems in [22, Sect. 4.1] (in a physics style), or (in a more mathematical style) in [14, Sect. 9.2] for percolation and in [26, Sect. 2.1] for self-avoiding walk. The relation (1.1) is fundamental in the physics literature but a mathematical justification is lacking in most examples.

In our main result, Theorem 1.1, we prove an upper bound of the form (1.1) for weakly self-avoiding walk in dimensions \( d > 4 \) (for which \( \eta = 0 \)), i.e.,

\[
G_z(x) \leq c_0 \frac{1}{1 \lor |x|^d z} e^{-c_1 m(z) |x|} \quad (x \in \mathbb{Z}^d, \ z \in [0, z_c]),
\]

with \( c_1 \in (0, 1) \) and with \( m(z) = \xi(z)^{-1} \) asymptotic to a multiple of \( (1 - z/z_c)^{1/2} \) as \( z \to z_c \). We generally write formulas in terms of the mass \( m(z) \) rather than the correlation length \( \xi(z) = m(z)^{-1} \). The norm \( |x| \) denotes the Euclidean norm \( \|x\|_2 \), and \( a \lor b = \max\{a, b\} \) is used in (1.2) to avoid division by zero.

The inequality (1.2) is a one-sided inequality so is not as strong as the assertion (1.1). Nevertheless, the upper bound is arguably more useful than a lower bound, and it has remarkable new consequences. One application of (1.2) is given in Theorem 1.2, where we prove that the decay of the two-point function for weakly self-avoiding walk on a discrete torus in dimensions \( d > 4 \) has a “plateau,” in the sense that it decays like the \( \mathbb{Z}^d \) two-point function for small \( x \) and then ceases to decay and levels off for larger \( x \) in the torus. As discussed in more detail in Sect. 1.6, the effect of periodic vs free boundary conditions on a two-point function above the upper critical dimension has been the subject of some debate in the physics literature. It is therefore useful to have theorems which provide definitive statements concerning the plateau. Furthermore, (1.2) and the existence of the plateau are used in [29] to provide a partial answer to the question: how long does a weakly self-avoiding walk on a discrete torus in dimensions \( d > 4 \) have to be before its behaviour differs significantly from that of a weakly self-avoiding walk on \( \mathbb{Z}^d \)? (The answer is at least length \( r^{d/2} \) on a torus of volume \( r^d \).) See also [27] for related work concerning the Brownian scaling limit on the torus.

For spread-out percolation in dimensions \( d > 6 \) an analogue of (1.2) is proved in [21] by methods very different from the ones we use here, and this allows the existence of the torus plateau to be proved via an adaptation of the proof of Theorem 1.2. Among other consequences, this leads to a proof of the percolation triangle condition for a high-dimensional torus (as defined in [4]) without the need for any torus lace expansion as in [5]—the torus critical behaviour can be inferred directly from consequences obtained via the percolation lace expansion on \( \mathbb{Z}^d \), which provides a new perspective on the much studied subject of torus percolation. These two applications of the torus plateau, to weakly self-avoiding walk in [29] and to percolation in [21], indicate that it is an invaluable tool for the analysis of torus critical phenomena.
As a byproduct and as an instructive comparison, we also give a short proof of the existence of a plateau for the torus two-point function for simple random walk in dimensions $d > 2$; this plateau was announced in [40] and proved in [13, 39] using methods different from ours.

1.2 The Model

Our results are for the nearest-neighbour weakly self-avoiding walk in dimensions $d > 4$. Background material can be found in [26, 34]. The model is defined as follows.

For $d \in \mathbb{N}$, let $D : \mathbb{Z}^d \to \mathbb{R}$ be the one-step transition probability for simple random walk on $\mathbb{Z}^d$, i.e., $D(x) = \frac{1}{2d}$ if $|x| = 1$ and otherwise $D(x) = 0$. For $n \in \mathbb{N}$, let $D^n$ denote the $n$-fold convolution of $D$ with itself; this is the $n$-step transition probability. We adopt the convention that $D^0(x) = \delta_{0,x}$. Let $W_n(x)$ denote the set of $n$-step walks from 0 to $x$, i.e., the set of $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ with each $\omega(i) \in \mathbb{Z}^d$, $\omega(0) = 0$, $\omega(n) = x$, and $|\omega(i) - \omega(i-1)| = 1$ for $1 \leq i \leq n$. The set $W_0(x)$ consists of the zero-step walk $\omega(0) = 0$ when $x = 0$, and otherwise it is the empty set. We write $\Omega_1 = 2d$ for the degree of the nearest-neighbour graph. The simple random walk two-point function (also called the lattice Green function) is defined, for $z \in [0, \frac{1}{\Omega_1})$, by

$$C_z(x) = \sum_{n=0}^{\infty} \sum_{\omega \in W_n(x)} z^n (z \Omega)^n D^n(x) \quad (x \in \mathbb{Z}^d). \tag{1.3}$$

For an $n$-step walk $\omega$, and for $0 \leq s < t \leq n$, we define

$$U_{st}(\omega) = \begin{cases} -1 & (\omega(s) = \omega(t)) \\ 0 & (\omega(s) \neq \omega(t)). \end{cases} \tag{1.4}$$

Given $\beta \in (0, 1)$, $z \geq 0$, and $x \in \mathbb{Z}^d$, the weakly self-avoiding walk two-point function is then defined by

$$G_z(x) = \sum_{n=0}^{\infty} \sum_{\omega \in W_n(x)} z^n \prod_{0 \leq s < t \leq n} (1 + \beta U_{st}(\omega)). \tag{1.5}$$

Compared to (1.3), the product in (1.5) discounts each $\omega$ by a factor $1 - \beta$ for each pair $s, t$ with an intersection for $\omega$, hence the name “weakly self-avoiding walk.” The choice $\beta = 1$ defines the strictly self-avoiding walk. The susceptibility is defined by

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x). \tag{1.6}$$

A standard subadditivity argument implies the existence of $z_c = z_c(\beta) \geq z_c(0) = \frac{1}{\Omega_1}$ such that $\chi(z)$ is finite if and only if $z \in [0, z_c)$; also $\chi(z) \geq 1/(1 - z/z_c)$ so
\( \chi(z_c) = \infty \) (see, e.g., [34, Theorem 2.3]). In particular, this implies that the series (1.5) converges at least for \( z \in [0, z_c) \).

### 1.3 Main Result

Our main result is the following theorem. Its proof, which uses the Brydges–Spencer lace expansion [7], is inspired by the methods of [37] and [26, Sect. 6.5.1] (the latter is based on [18]) but is more than the union of these methods. Its statement involves the mass (inverse correlation length) \( m(z) \); this is the exponential decay rate of the subcritical two-point function and is defined explicitly in (1.9).

We write \( f \sim g \) to mean \( \lim f / g = 1 \), and \( f \asymp g \) to mean that \( c_1 f \leq g \leq c_2 f \) with \( c_1, c_2 > 0 \). In general, constants are permitted to depend on the dimension \( d \), which is a fixed parameter in our analysis.

**Theorem 1.1** Let \( d > 4 \) and let \( \beta \) be sufficiently small. There are constants \( c_0 > 0 \) and \( c_1 \in (0, 1) \), which depend on \( d \) but not on \( \beta \), such that for all \( z \in (0, z_c) \) and \( x \in \mathbb{Z}^d \),

\[
G_z(x) \leq c_0 \frac{1}{1 \vee |x|^{d-2}} e^{-c_1 m(z) |x|}.
\]

(1.7)

The mass has the asymptotic form \( m(z) \sim c(1 - z/z_c)^{1/2} \) as \( z \to z_c \), with constant \( c = \Omega^{1/2} + O(\beta) \).

In the proof, the order of operations is:

1. Prove that (with constant independent of \( \beta \))

\[
G_{z_c}(x) \leq c_0 \frac{1}{1 \vee |x|^{d-2}}.
\]

(1.8)

This has been proved already in [37] and also in [3], in fact as an asymptotic relation rather than only as a bound, and the proof is not repeated here. Those proofs use the lace expansion with a bootstrap argument varying \( z \).

2. Prove that \( m(z) \sim \text{const}(1 - z/z_c)^{1/2} \) as \( z \to z_c \). An important element of the proof is the control of the lace expansion “tilted” by \( e^{m(z)x_1} \), where \( x_1 \) is the first component of \( x \in \mathbb{Z}^d \). The proof uses a bootstrap argument varying \( m \), as in [15, 18, 26].

3. Prove the inequality (1.7). The proof is based on the method of [37] but now applied to the exponentially tilted two-point function, which requires a significant extension of [37].

The proof has the potential to apply to spread-out strictly self-avoiding walk, to the Ising model, and to the \( \varphi^4 \) model, above the upper critical dimension 4. However this would need further development. It does not apply in its present form to percolation, nor to lattice trees and lattice animals; see Remark 5.3. A different approach is used to prove the analogue of (1.7) for percolation in high dimensions in [21], using methods special to percolation which do not apply to weakly self-avoiding walk.
1.4 Decay of the Two-Point Function

To place the estimate (1.7) in context, we summarise what was previously known about the decay of the two-point function. The theory for the two-point function is more developed for the more difficult case of strictly self-avoiding walk ($\beta = 1$) than for weakly self-avoiding walk, but typically the adaptation of proofs from the former to the latter is straightforward since the proofs are based on subadditivity arguments that apply equally well to both cases.

1. Let $z \in (0, z_c)$ and $d \geq 2$. There is a $z$-dependent norm $| \cdot |_z$ on $\mathbb{R}^d$, with $||x||_\infty \leq |x|_z \leq ||x||_1$ for all $x \in \mathbb{R}^d$, such that the mass $m(z)$ defined by the limit

\[
m(z) = \lim_{|x|_z \to \infty} \frac{-\log G_z(x)}{|x|_z}
\]

exists in $(0, \infty)$. The function $m$ is continuous and strictly decreasing in $z$, $m(z) \to \infty$ as $z \to 0$, and $m(z) \to 0$ as $z \to z_c$. These facts are proved in [26, Sect. 4.1] for strictly self-avoiding walk; the proofs directly adapt to the weakly self-avoiding walk.

2. Let $z \in (0, z_c)$ and $d \geq 2$. The bubble diagram is defined by $B(z) = \sum_{x \in \mathbb{Z}^d} G_z(x)^2$. The two-point function satisfies the inequality

\[
G_z(x) \leq B(z)^{1/2} e^{-m(z)|x|_z} \quad (x \in \mathbb{Z}^d).
\]

This is proved in [26, Theorem 4.1.18] for the strictly self-avoiding walk, and the proof adapts directly to the weakly self-avoiding walk.

3. Let $z \in (0, z_c)$ and $d \geq 2$. The two-point function obeys the Ornstein–Zernike decay

\[
G_z((x_1, 0, \ldots, 0)) \sim c_z \frac{1}{x_1^{(d-1)/2}} e^{-m(z) x_1} \quad (x_1 \to \infty),
\]

with $c_z > 0$. Off-axis behaviour is also known. This is proved in [10] and [26, Theorem 4.4.7] for the strictly self-avoiding walk. The bound (1.11) exhibits a power-law correction $x_1^{-(d-1)/2}$ to the exponential decay, which is different from the power $|x|^{-(d-2)}$ in (1.7) or (1.8) (unless $d = 3$).

4. Let $z \in (0, z_c)$ and $d > 4$. The asymptotic behaviour of the mass is

\[
m(z) \sim \text{const} \left(1 - z/z_c\right)^{1/2} \quad (z \to z_c).
\]

This is a statement that the critical exponent $\nu$ is equal to $\frac{1}{2}$. For the strictly self-avoiding walk, (1.12) is proved in [18] and [26, Theorem 6.1.2] using the lace expansion. In Sect. 4, we indicate the small changes need to prove (1.12) for weakly self-avoiding walk. The method of proof was first developed for percolation [15], and its elementary version for simple random walk is given in [26, Theorem A.2].
5. Let $d > 4$. The critical two-point function has asymptotic behaviour

$$G_{z_c}(x) \sim \text{const} \frac{1}{|x|^{d-2}} \quad (|x| \to \infty). \quad (1.13)$$

This is proved for strictly self-avoiding walk in [16, 17], for weakly self-avoiding walk in [3, 37], and for the continuous-time weakly self-avoiding walk (also known as the lattice Edwards model) in [6]. All these proofs use the lace expansion. The formula (1.13) is a statement that the critical exponent $\eta$ is equal to zero for $d > 4$.

A consequence of (1.13) is that the critical bubble diagram $B(z_c)$ is finite, which in turn implies that the susceptibility obeys

$$\chi(z) \sim A(1 - z/z_c)^{-1} \quad (z \to z_c)$$

with $A = 1 + O(\beta)$. The same is proved for the strictly self-avoiding walk in [18].

The importance of the estimate (1.7) resides in its uniformity as $z \to z_c$. Indeed, if we consider only $z \in (0, z_c - \delta]$ with fixed $\delta > 0$, then by (1.10) and the fact that $B(z_c - \delta) < \infty$, for any $c_1 \in (0, 1)$ we have

$$G_z(x) \leq \text{const} e^{-(1-c_1)m(z)\|x\|_{\infty}} e^{-c_1 m(z)\|x\|_{\infty}}. \quad (1.15)$$

Since $m(z) \geq m(z_c - \delta)$, and since $|x| \leq d^{1/2}\|x\|_{\infty}$,

$$e^{-(1-c_1)m(z)\|x\|_{\infty}} \leq e^{-(1-c_1)m(z_c - \delta)\|x\|_{\infty}} \leq \text{const} \frac{1}{|x|^{d-2}}. \quad (1.16)$$

Thus the bound (1.7) holds for $z \in (0, z_c - \delta]$, so our focus needs to be on $z$ close to $z_c$.

### 1.5 Conjectured Decay

The precise asymptotic behaviour of the lattice Green function $C_z(x)$ for fixed $z$ less than the critical value $z_0 = \frac{1}{\Omega_1}$ has recently been elucidated in [30]. It is proved there that there is an explicit norm $|\cdot|_z$ on $\mathbb{R}^d$ (not the same as the norm in (1.9)) that interpolates monotonically between the limiting values $\lim_{z \to 0} |x|_z = \|x\|_1$ and $\lim_{z \to z_0} |x|_z = \|x\|_2$, and an explicitly defined function $m_0(z)$ with $m_0(z) \sim (1 - z/z_0)^{1/2}$ as

For the more difficult dimension $d = 4$, $\eta = 0$ is proved for the continuous-time weakly self-avoiding walk in [2]. For dimensions $d = 2, 3$ it is predicted but not proved that the decay is of the form $|x|^{-(d-2+\eta)}$ with $\eta = \frac{5}{24}$ for $d = 2$ and $\eta \approx 0.03$ for $d = 3$. 

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\( z \to z_0 \) (see (2.1)–(2.2)), such that, as \( n \to \infty \),

\[
C_{z}(nx) \sim c_{z,\hat{x}} \frac{1}{(n|x|_z)^{d-2}} (m_0(z)n|x|_z)^{(d-3)/2} e^{-m_0(z)n|x|_z} \quad \text{(nonzero } x \in \mathbb{Z}^d). \tag{1.17}
\]

The constant \( c_{z,\hat{x}} \) has a limit as \( z \to z_0 \) which is positive and independent of the direction \( \hat{x} = \frac{x}{|x|_z} \). This is consistent with (1.1) with \( g(t) = t^{(d-3)/2} e^{-t} \). It is natural to conjecture that (1.17) will also apply to statistical mechanical models above their upper critical dimensions, including self-avoiding walk and the Ising model for \( d > 4 \), and percolation for \( d > 6 \). A potential factor such as \( (m(z)|x|_z)^{(d-3)/2} \) is compensated by giving up some exponential decay in the factor \( e^{-c_1 m|x|} \) in (1.7), with constant \( c_1 < 1 \).

### 1.6 The Plateau for the Torus Two-Point Function

Let \( \mathbb{T}_r^d = (\mathbb{Z}/r\mathbb{Z})^d \) denote the discrete \( d \)-dimensional torus of period \( r \geq 3 \). We are interested in large \( r \), and in obtaining estimates that remain valid uniformly in large \( r \). For notational convenience, we sometimes evaluate a \( \mathbb{Z}^d \) two-point function at a point \( x \in \mathbb{T}_r^d \), with the understanding that in this case we identify \( x \) with a point in \( [-r/2, r/2]^d \cap \mathbb{Z}^d \).

Let \( G_{z}^{\mathbb{T}}(x) \) denote the analogue of (1.5) for walks \( \omega \) on the torus rather than on \( \mathbb{Z}^d \), and let

\[
\chi^{\mathbb{T}}(z) = \sum_{x \in \mathbb{T}_r^d} G_{z}^{\mathbb{T}}(x)
\tag{1.18}
\]

denote the torus susceptibility. For any \( n \)-step torus walk \( \omega \), by the pigeon-hole principle there must be a vertex that is visited by \( \omega \) at least \( N_n = \lceil nr^{-d} \rceil \) times, so

\[
\prod_{0 \leq i \leq t \leq n} (1 + \beta U_{ii}(\omega)) \leq (1 - \beta)^{(N_n/2)}. \tag{1.19}
\]

For fixed \( r \), the binomial coefficient \( \binom{N_n}{2} \) grows quadratically in \( n \), so since the number of \( n \)-step walks is at most \( \Omega^n \), for any \( z \geq 0 \) we have

\[
\chi^{\mathbb{T}}(z) \leq \sum_{n=0}^{\infty} (z\Omega)^n (1 - \beta)^{(N_n/2)} < \infty. \tag{1.20}
\]

The susceptibility and two-point function on the torus are therefore entire functions of \( z \) when \( \beta > 0 \). This is in contrast to the situation for \( \mathbb{Z}^d \), where convergence is limited to the disk of radius \( z_c \).

For \( r \geq 3 \), walks on the torus are in one-to-one correspondence with walks on \( \mathbb{Z}^d \) via the canonical projection from \( \mathbb{Z}^d \) to \( \mathbb{T}_r^d \). We refer to the \( \mathbb{Z}^d \) walk corresponding to a torus walk as the “unfolding” of the torus walk.\(^2\) Since the unfolding of a walk cannot

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\(^2\) In detail, if \( \omega \) is a torus walk starting at 0 then its unfolding \( \tilde{\omega} \) is the walk on \( \mathbb{Z}^d \) with \( \tilde{\omega}(0) = 0 \) and \( \tilde{\omega}(i) = \tilde{\omega}(i - 1) + (\omega(i) - \omega(i - 1))_r \) where for \( x \in \mathbb{T}_r^d \) we write \( x_r \) for its unique representative in \( [-r/2, r/2]^d \cap \mathbb{Z}^d \).
have more self-intersections than the walk itself, for all dimensions, for all $\beta \in [0, 1]$, and for all $z \in [0, z_c]$ we have
\[
\chi^T(z) \leq \chi(z). \tag{1.21}
\]

The proofs of the theorems discussed in this section, Theorems 1.2–1.4, are given in Sect. 6.

1.6.1 The Plateau for Weakly Self-avoiding Walk for $d > 4$

The following theorem is one application of Theorem 1.1. It shows that the slightly subcritical weakly self-avoiding walk torus two-point function has asymptotic form given by two terms: an $x$-dependent term that is simply the $\mathbb{Z}^d$ two-point function, plus a constant term that can dominate for large $x$. The constant term is the torus “plateau.”

**Theorem 1.2** Let $d > 4$ and let $\beta$ be sufficiently small. There are constants $c_i > 0$, depending on $d$ but not on $\beta$, such that for all $x \in \mathbb{T}_r^d$,
\[
G_z(x) + c_1 \frac{\chi(z)}{r^d} \leq G^\tau_z(x) \leq G_z(x) + c_2 \frac{\chi(z)}{r^d}, \tag{1.22}
\]
where the upper bound holds for all $r \geq 3$ and all $z \in (0, z_c)$, whereas the lower bound holds provided that $z \in [z_c - c_3 r^{-2}, z_c - c_4 \beta^{1/2} r^{-d/2}]$.

Note that the interval of $z$ values for the lower bound is nonempty for sufficiently large $r$ since $d > 4$. Of course the upper bound of (1.22) also holds for $z = z_c$ since $\chi(z_c) = \infty$. The lower bound cannot hold when $z = z_c$ for the same reason, because, as noted above, $G^\tau_z(x)$ is an entire function of $z$ and hence is finite for all $z \geq 0$.

By universality we expect that (1.22) also holds for the strictly self-avoiding walk in dimensions $d > 4$; Monte Carlo verification of this has been carried out in [39, 40].

Let $\rho = z_c - z$. In the proof of Theorem 1.2 (see Remark 6.2), we show that (small) $c_3$ can be chosen such that
\[
G_z(x) \asymp \frac{1}{1 \vee |x|^{d-2}} \quad \text{when} \quad \rho \in [0, c_3 r^{-2}] \quad \text{and} \quad x \in \mathbb{T}_r^d. \tag{1.23}
\]

It therefore follows from (1.22) that
\[
G^\tau_z(x) \asymp \frac{1}{1 \vee |x|^{d-2}} + \frac{\chi(z)}{r^d} \quad \text{when} \quad \rho \in [c_4 \beta^{1/2} r^{-d/2}, c_3 r^{-2}] \quad \text{and} \quad x \in \mathbb{T}_r^d. \tag{1.24}
\]

By (1.14), the susceptibility diverges linearly at the critical point, so $\chi(z) \asymp r^p$ when $\rho \asymp r^{-p}$, and hence
\[
G^\tau_z(x) \asymp \frac{1}{1 \vee |x|^{d-2}} + \frac{1}{r^{d-p}} \quad \text{when} \quad \rho \asymp \frac{1}{r^p} \quad \text{with} \quad p \in [2, \frac{d}{2}] \quad \text{and} \quad x \in \mathbb{T}_r^d. \tag{1.25}
\]

For $p > 2$ the constant term dominates when $|x|^{d-2} \geq r^{d-p}$, i.e., when $|x| \geq r^{(d-p)/(d-2)}$. The latter domain of $x$ is the “plateau” where the torus two-point function no longer decays with distance.
1.6.2 The Scaling Window

In models defined on a finite graph such as a torus, rather than on an infinite graph such as $\mathbb{Z}^d$, there is not a precise notion of a critical point. Unlike the susceptibility $\chi$ which diverges at $z_c$, the torus susceptibility $\chi^T$ is an entire function with no singularity. Instead, there is a notion of a scaling window (or critical window), which is an interval of parameter values where the model exhibits critical behaviour.

Self-avoiding walk on the complete graph is exactly solvable [11, 36]. Consistent with the discussion in [36] for the complete graph, it is natural to define the scaling window for self-avoiding walk on a torus in dimensions $d > 4$ to consist of the values of $z$ with $|z - z_c| = O(r^{-d/2})$. Motivated by the complete graph, the susceptibility is conjectured to obey $\chi^T(z) \asymp r^{d/2}$ when $z$ is in the scaling window. For weakly self-avoiding walk, constants in these two relations naturally depend on $\beta$, since $\beta = 0$ is simple random walk for which there is no scaling window because even on the torus the susceptibility diverges at $z = \frac{1}{\Omega_1}$. The largest value $z = z_c - c_4 \beta^{1/2} r^{-d/2}$ that is permitted for the lower bound in Theorem 1.2 lies in this scaling window. For notational convenience, we write this value as $z^*$. Summation of the lower bound of (1.22) over $x \in T^d_r$, together with the general upper bound of (1.21), leads to the following immediate corollary of Theorem 1.2. In particular, with (1.14), the corollary shows that $\chi^T(z^*) \asymp r^{d/2}$, consistent with the above mentioned conjecture.

**Corollary 1.3** The two susceptibilities obey the inequality

$$c_1 \chi(z) \leq \chi^T(z) \leq \chi(z),$$

with the upper bound valid for general $d$, $\beta$, $z$, and the lower bound valid for $d > 4$, sufficiently small $\beta > 0$, and under the same restrictions on $z$ as in Theorem 1.2.

As a comparison, we note that the scaling window for high-dimensional percolation on a torus has been extensively studied; a summary can be found in [19, Chapter 13]. For percolation, the torus scaling window consists of points $p$ with $|p - p_c| = O(r^{-d/3})$. In particular, the $\mathbb{Z}^d$ critical point $p_c$ lies in the scaling window. In the scaling window, the torus susceptibility is of order $r^{d/3}$. Note the different exponent $r^{d/3}$ appearing here, compared with $r^{d/2}$ for self-avoiding walk. A new perspective on high-dimensional torus percolation is provided in [21], based on a percolation version of Theorem 1.1 (proved via different means than those used here) and of Theorem 1.2 (proved using similar means to those used here). An advantage of this new perspective is that it obviates the need for the torus lace expansion [5]. Instead, critical torus percolation can be analysed using only results derived from the $\mathbb{Z}^d$ lace expansion, particularly the $|x|^{-(d-2)}$ decay of the two-point function [16, 17], combined with percolation analogues of Theorems 1.1–1.2.

3 After this work was completed, it has been conjectured in [28] that for $d > 4$ and all $\beta \in (0, 1]$ the universal profile for the susceptibility in the scaling window is given by $\lim_{r \to \infty} r^{-d/2} \chi^T(z_c(1 + \lambda_1 sr^{-d/2})) = \lambda_2 \int_0^\infty t e^{-\frac{1}{4}t^2 + \frac{1}{2} st} dt$ for all $s \in \mathbb{R}$, for some constants $\lambda_1, \lambda_2$ depending on $d$ and $\beta$ (but not on $s, r$).
1.6.3 Boundary Conditions and the Plateau

There is a sizeable literature about the effect of boundary conditions on the decay of the two-point function of statistical mechanical models on a finite box above the upper critical dimension, including [25, 38, 40]. This literature has included some debate about what should be the correct behaviour. The emerging consensus is that with free boundary conditions there is no plateau and the susceptibility behaves as $r^2$ at the $\mathbb{Z}^d$ critical point (for Ising or self-avoiding walk) whereas with periodic boundary conditions (the torus) there is a plateau and the susceptibility behaves instead as $r^{d/2}$.

For the Ising model in dimensions $d > 4$, with free boundary conditions the lack of a plateau and the $r^2$ behaviour of the susceptibility has been proved in [8]. With periodic boundary conditions, a plateau lower bound is proved in [31], as is an $r^{d/2}$ lower bound for the susceptibility. The upper bounds remain unproved on the torus.

The results for percolation are consistent with this picture, again with the susceptibility of order $r^2$ for free boundary conditions but now of order $r^{d/3}$ on the torus. For free boundary conditions, the lack of a plateau at the $\mathbb{Z}^d$ critical point $p_c$ is proved in [9]. For periodic boundary conditions, a plateau upper bound at $p_c$ is proved in [20, Theorem 1.7], and plateau upper and lower bounds are proved throughout the entire scaling window in [21].

Theorem 1.2 provides a contribution to this general discussion by proving existence of a plateau for weakly self-avoiding walk in dimensions $d > 4$, up to the scaling window.

1.6.4 The Plateau for Simple Random Walk for $d > 2$

A more elementary version of Theorem 1.2 for simple random walk is given in the following theorem, which is a byproduct of our proof of Theorem 1.2. Let $C_T^\omega(x)$ denote the analogue of (1.3) for walks $\omega$ on the torus rather than on $\mathbb{Z}^d$. Let $z_0 = \frac{1}{\Omega} = \frac{1}{2^d}$.

The susceptibility $\chi(z)$ is the same for simple random walk on the torus or on $\mathbb{Z}^d$:

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} C^\omega(x) = \sum_{x \in T^d} C_T^\omega(x) = \frac{1}{1 - z\Omega} \quad (z \in [0, z_0)). \quad (1.27)$$

The isolation of $d = 4$ in the theorem is an unnatural artifact of our proof.

**Theorem 1.4** Let $d > 2$. For $d = 4$, there are constants $c'_i > 0$ such that for all $x \in T^d$, $r > 3$ and all $z \in (0, z_0)$, where $z_0 = \frac{1}{\Omega} = \frac{1}{2^d}$,

$$C^\omega(x) + c'_1 \frac{\chi(z)}{r^d} \leq C_T^\omega(x) \leq C^\omega(x) + c'_2 \frac{\chi(z)}{r^d}, \quad (1.28)$$

where the upper bound holds for all $r \geq 3$ and all $z \in (0, z_0)$, whereas the lower bound holds provided that $z \in [z_0 - c'_3 r^{-2}, z_0)$. For $d = 4$ the upper bound also holds as stated, but the constant term in the lower bound is weakened to $c'_4 \frac{\chi(z)}{r^d} \log \frac{1}{\chi(z)}$ with restriction that $\rho \log \rho |r^2$ is sufficiently small, where $\rho = z_0 - z$.

Note that both inequalities in (1.28) hold as equalities when $z = z_0$, since $\chi(z_0) = \infty$ by (1.27), and $C_T^\omega(x) = \infty$ because simple random walk on the torus is recurrent.

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We indicate in Remark 6.2 that \( C_z(x) \) obeys (1.23) (now with \( \rho = z_0 - z \)), and hence (1.25) also holds for \( C^T_z(x) \), now for all \( p \geq 2 \). Thus Theorem 1.4 implies that the “plateau” concept also applies to simple random walk (with an unnatural logarithmic caveat for \( d = 4 \)).

A different and earlier proof of Theorem 1.4 is based on the local central limit theorem [13, 39, 40]. In fact, the results of [13, 39, 40] are more general and for \( d = 4 \) are stronger since they do not have the logarithmic correction in the lower bound. Our proof instead uses the heat kernel estimate stated in (2.3) below (this has analogues for very general random walks), which affords a relatively simple proof of (1.28). As our principal interest is the weakly self-avoiding walk in dimensions \( d > 4 \), we do not aim for generality in Theorem 1.4, nor attempt to eliminate the logarithm for \( d = 4 \).

Theorem 1.4 has the following heuristic interpretation which provides some insight into the origin of the plateau. Consider the nearest-neighbour torus random walk \( S^T_n \) subjected to \( z \)-dependent killing, i.e., the walk has length \( N \) with geometric probability \( \mathbb{P}(N = n) = (z\Omega)^n(1-z\Omega) \) for \( n \geq 0 \), with the random variable \( N \) independent of the walk’s steps. The two-point function \( C^T_z(x) \) is the expected number of visits to \( x \) by the torus walk subjected to \( z \)-dependent killing:

\[
C^T_z(x) = \mathbb{E}\left( \sum_{n=0}^{N} \mathbb{I}_{S^T_n = x} \right) \quad (x \in \mathbb{T}^d), \tag{1.29}
\]

as can be verified by computing the right-hand side via conditioning on \( N \). The expected length is \( \mathbb{E}N = z\Omega \chi_0(z) \), so the susceptibility \( \chi_0(z) = (1-z\Omega)^{-1} \) is simply \( 1 + \mathbb{E}N \). As discussed above (1.21), for \( r \geq 3 \), for walks on the torus are in one-to-one correspondence with walks on \( \mathbb{Z}^d \) via unfolding. Thus, a torus walk to \( x \) unfolds to a walk on \( \mathbb{Z}^d \) ending at \( x \) or at a point \( x + ru \) with \( u \) a nonzero point in \( \mathbb{Z}^d \). The term \( C_z(x) \) in (1.28) is the expected number of visits to \( x \) by torus walks from 0 which unfold to walks on \( \mathbb{Z}^d \) which end at \( x \). In the proof of Theorem 1.4, the term \( r^{-d}\chi_0 \) arises from walks that wrap around the torus—these unfold to walks on \( \mathbb{Z}^d \) that end at \( x + ru \) for some nonzero \( u \in \mathbb{Z}^d \). For \( z \) with \( z_0 - z \leq r^{-2} \), the expected length is at least \( r^2 \), so the torus walk is well mixed (see [24, Theorem 5.5]) and its location is close to uniformly random on the torus. On average, it therefore spends time \( r^{-d}\chi_0 \) at each torus point, resulting in the constant term in (1.28).

An alternate heuristic interpretation is the following, which has been pointed out in [38]. The Fourier dual of the torus \( \mathbb{T}^d_r \) is \( \hat{\mathbb{T}}^d_r = \frac{2\pi}{r} \mathbb{T}^d_r \). Let \( k \cdot x = \sum_{j=1}^{d} k_j x_j \) denote the dot product of \( k \in \hat{\mathbb{T}}^d_r \) with \( x \in \mathbb{T}^d_r \). The Fourier transform of \( f : \mathbb{T}^d_r \to \mathbb{C} \) is defined by \( \hat{f}(k) = \sum_{x \in \mathbb{T}^d_r} f(x)e^{ik \cdot x} \) for \( k \in \hat{\mathbb{T}}^d_r \), and the inverse Fourier transform is \( f(x) = \frac{1}{r^d} \sum_{k \in \hat{\mathbb{T}}^d_r} \hat{f}(k)e^{-ik \cdot x} \) for \( x \in \mathbb{T}^d_r \). In particular,

\[
C^T_z(x) = \frac{1}{r^d} \sum_{k \in \hat{\mathbb{T}}^d_r} \hat{C}^T_z(k)e^{-ik \cdot x} \quad (x \in \mathbb{T}^d_r). \tag{1.30}
\]

The Fourier transform \( \hat{C}_z(k) \) on \( \mathbb{Z}^d \) (see (2.11)) and its torus counterpart \( \hat{C}^T_z(k) \) have the same functional form \((1 - z\Omega D(k))^{-1}\)—only the domains for \( k \) differ. Thus the
\[ k = 0 \text{ term in (1.30) is } r^{-d} \hat{C}_z(0) = r^{-d} \chi_0(z), \text{ so the constant term in (1.28) arises as the zero mode. The sum over nonzero } k \text{ is a Riemann sum approximation to the Fourier integral over the continuum torus } (\mathbb{R}/2\pi \mathbb{Z})^d \text{ that equals the two-point function } C_z(x) \text{ for } \mathbb{Z}^d \text{ via inverse Fourier transformation. If the Riemann sum has the same large-} x \text{ behaviour as the integral, then we would find that } C_z(x) - r^{-d} \chi_0(z) \text{ and } C_z(x) \text{ are comparable, as in (1.28). Verification of this last step is an oscillatory integral problem, which may be difficult. Nevertheless this heuristic discussion does support the idea that the plateau arises from the zero mode.} \\

\section{2 The Lattice Green Function}

Recall the definition of the lattice Green function \( C_\mu(x) \) in (1.3), where now we write \( \mu \) instead of \( z \). For the proof of Theorem 1.1, we need a version of (1.7) for \( C_\mu(x) \), as well as estimates on the Fourier transform of \( C_\mu(x) \). We develop these here.

\subsection{2.1 Massive Decay}

We now prove an inequality of the form (1.7) for \( C_\mu(x) \). Let \( \mu_c = \frac{1}{\Omega} = \frac{1}{2^d} \). For \( \mu \in (0, \mu_c] \), we define \( m_0(\mu) \geq 0 \) to be the unique solution to

\[ \cosh m_0(\mu) = 1 + \frac{1 - \mu \Omega}{2\mu}. \]  

(2.1)

In particular, \( m_0(\mu_c) = 0 \), \( m_0 \) is a strictly positive strictly decreasing function of \( \mu \in (0, \mu_c) \), and

\[ m_0(\mu)^2 \approx \frac{1}{\mu} - \frac{1}{\mu_c} (1 - \mu/\mu_c) \quad (\mu \to \mu_c). \]  

(2.2)

This \( m_0(\mu) \) is the exponential rate of decay of \( C_\mu(x) \) in (1.17), and \( C_\mu(x) \leq C_\mu(0)e^{-m_0(\mu)\|x\|_\infty} \) for all \( x \in \mathbb{Z}^d \) and \( \mu \in (0, \mu_c) \); both of these statements are proved in [26, Theorem A.2]. The following proposition is a variation of the above exponential estimate, with a power law correction as in (1.7). By (1.17), it is impossible for (2.4) to hold with \( a_1 = 1 \) (at least for \( d \neq 3 \)).

The proof of the proposition uses the fact there exist \( a, A > 0 \) such that the \( n \)-step transition probability obeys

\[ D^{*n}(x) \leq A \frac{1}{n^{d/2}} e^{-a\|x\|_\infty^2/n} \quad (n \geq \|x\|_\infty) \]  

(2.3)

(of course \( D^{*n}(x) = 0 \) when \( n < \|x\|_\infty \)). The heat kernel estimate (2.3) is proved in [1, Theorem 6.28]. Unlike local central limit theorems which give precise constants (e.g., [23]), (2.3) gives an exponential bound for \( x \) well beyond the diffusive scale, including for \( |x| \) and \( n \) of comparable size.
Proposition 2.1 For $d > 2$, there are constants $a_0 > 0$ and $a_1 \in (0, 1)$ such that for all $\mu \in (0, \mu_c)$,

$$C_\mu(x) \leq a_0 \frac{1}{1 + |x|^{d-2}} e^{-a_1 m_0(\mu) \|x\|_\infty} \quad (x \in \mathbb{Z}^d). \quad (2.4)$$

Proof There is nothing to prove for $x = 0$ since $C_{\mu_c}(0) < \infty$, so we assume $x \neq 0$. We write $\ell = \|x\|_\infty$ and $m_0 = m_0(\mu)$. As in the discussion around (1.15)–(1.16) (now with $C_{\mu}(0)$ in place of $B(z)$), it suffices to consider $\mu \in [\frac{1}{\Omega} - \delta, \frac{1}{\Omega}]$ for any small $\delta > 0$. For small enough $\delta$, there is a $c > 0$ such that $\mu/\Omega = 1 - (1 - \mu/\Omega) \leq e^{-(1 - \mu/\Omega)} \leq e^{-cm_0^2}$ by (2.2), so from (2.3) we obtain

$$C_\mu(x) \leq A \sum_{n=\ell}^{\infty} \frac{1}{n^{d/2}} e^{-at^2/2n}. \quad (2.5)$$

We apply the inequality $u^2 + v^2 \geq 2uv$ with $u^2 = cm_0^2n$ and $v^2 = a\ell^2/(2n)$, to obtain (with $a_1 = \sqrt{2ca}$)

$$C_\mu(x) \leq Ae^{-a_1m_0\ell} \sum_{n=\ell}^{\infty} \frac{1}{n^{d/2}} e^{-at^2/(2n)}. \quad (2.6)$$

The terms on the right-hand side increase while $n < a\ell^2/d$ and then decrease, so by crudely bounding as in the proof of the integral test for series convergence,

$$\sum_{n=\ell}^{\infty} \frac{1}{n^{d/2}} e^{-at^2/(2n)} \leq \sum_{\ell \leq n < a\ell^2/d} \frac{1}{n^{d/2}} e^{-at^2/(2n)} + \sum_{n \geq a\ell^2/d} \frac{1}{n^{d/2}} \leq 2 \int_\ell^{\infty} \frac{1}{t^{d/2}} e^{-at^2/(2t)} dt + O(e^{-(d-2)}). \quad (2.7)$$

Since the integral over $[0, \infty)$ converges, this gives an $O(e^{-(d-2)})$ upper bound and thus proves (2.4). \hfill \Box

2.2 Massive Infrared Bound

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ denote the continuous torus. The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x)e^{ik \cdot x} \quad (k \in \mathbb{T}^d). \quad (2.8)$$

\footnote{Not to be confused with the discrete torus $\mathbb{T}^d_r$ in Sect. 1.6, which will not reappear until Sect. 6.}
and the inverse Fourier transform is

\[ f(x) = \int_{\mathbb{T}^d} \hat{f}(k) e^{-ik \cdot x} \frac{dk}{(2\pi)^d} \quad (x \in \mathbb{Z}^d). \] (2.9)

In particular, the transform of the step distribution \( D \) is \( \hat{D}(k) = d^{-1} \sum_{j=1}^d \cos k_j \). We define \( A_\mu : \mathbb{Z}^d \to \mathbb{R} \) by \( A_\mu = \delta - \mu \Omega D \), with \( \delta \) the Kronecker delta \( \delta(x) = \delta_{0,x} \). It follows from the definition of \( C_\mu(x) \) in (1.3) that \( C_\mu(x) = \delta_{0,x} + \mu \Omega (D \ast C_\mu)(x) \), so

\[ C_\mu \ast A_\mu = \delta. \] (2.10)

Since the Fourier transform converts convolutions to products, this implies that

\[ \hat{C}_\mu(k) = \frac{1}{\hat{A}_\mu(k)} = \frac{1}{1 - \mu \Omega \hat{D}(k)}. \] (2.11)

Throughout the paper, for \( m \geq 0 \) and \( f : \mathbb{Z}^d \to \mathbb{C} \) we write \( f^{(m)} \) for the exponential tilt of \( f \):

\[ f^{(m)}(x) = f(x) e^{mx_1} \quad (x = (x_1, \ldots, x_d)). \] (2.12)

Also, for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with each \( \alpha_i \in \{0, 1, 2, \ldots\} \), we write \( |\alpha| = \sum_{j=1}^d \alpha_j \). We will use the fact that \( |k|^p + m_0^p \asymp (|k| + m_0)^p \) for any fixed \( p \in \mathbb{N} \).

The next lemma is a massive infrared bound. The purpose of its factor \( \sigma \) is to keep \( m \) bounded away from \( m_0(\mu) \), so that the tilt in \( C_\mu^{(m)}(x) \) does not remove all of the exponential decay from \( C_\mu(x) \). The decay remaining in \( C_\mu^{(m)}(x) \) has rate proportional to \( m_0(\mu) \) and is responsible for the \( m_0 \) term on the right-hand side of (2.13).

**Lemma 2.2** Let \( d > 2 \). Fix \( \sigma \in (0, 1) \). For any multi-index \( \alpha \) with \( |\alpha| \geq 0 \), there is a constant (depending on \( \alpha, \sigma \)) such that for all \( \mu \in [\frac{1}{\Omega T}, \frac{1}{\Omega}] \) and \( m \in [0, \sigma m_0(\mu)] \),

\[ |\nabla^\alpha \hat{C}_\mu^{(m)}(k)| \leq \text{const} \frac{1}{(|k| + m_0(\mu))^{2 + |\alpha|}} \quad (k \in \mathbb{T}^d). \] (2.13)

**Proof** It follows from (2.10) that \( C_\mu^{(m)} \ast A_\mu^{(m)} = \delta \). Since \( \hat{A}_\mu^{(m)}(k) = 1 - \mu \Omega \hat{D}^{(m)}(k) \), we obtain

\[ \hat{C}_\mu^{(m)}(k) = \frac{1}{\hat{A}_\mu^{(m)}(k)} = \frac{1}{\hat{A}_\mu^{(m)}(0) + \mu \Omega [\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)]}. \] (2.14)

By definition,

\[ \hat{D}^{(m)}(k) = \sum_{x \in \mathbb{Z}^d} D(x) e^{mx_1} e^{ik \cdot x} = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d : |x| = 1} e^{mx_1 + ik \cdot x} \]
\[ i \frac{1}{d} \sinh m \sin k_1 + \frac{1}{d} \cosh m \cos k_1 + \frac{1}{d} \sum_{j=2}^{d} \cos k_j, \]  
(2.15)

and hence

\[ \hat{D}^{(m)}(0) - \hat{D}^{(m)}(k) = -i \frac{1}{d} \sinh m \sin k_1 + \frac{1}{d} \cosh m (1 - \cos k_1) + \frac{1}{d} \sum_{j=2}^{d} (1 - \cos k_j). \]  
(2.16)

The origin of the formula for the mass \( m_0 = m_0(\mu) \) in (2.1) is that it satisfies

\[ \hat{C}^{(m_0)}(0) = \infty, \]

i.e.,

\[ 0 = \hat{A}^{(m_0)}(0) = 1 - 2\mu[\cosh m_0 + d - 1], \]  
(2.17)

an identity which can be verified from (2.1). Therefore, since \( m \in [0, \sigma m_0(\mu)] \) and \( \mu \geq \frac{1}{2\pi} \), there is a constant depending on \( \sigma \) such that

\[ \hat{A}^{(m)}(0) = \hat{A}^{(m)}(0) - \hat{A}^{(m_0)}(0) = 2\mu[\cosh m_0 - \cosh m] \geq \text{const} m_0^2. \]  
(2.18)

Also,

\[ |\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)| \geq \Re [\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)] \geq \frac{1}{d} \sum_{j=1}^{d} (1 - \cos k_j) \geq \text{const} |k|^2. \]  
(2.19)

Therefore, by using (2.18)–(2.19) together with (2.14) we obtain

\[ |\hat{A}^{(m)}(k)| \geq \text{const}(m_0^2 + |k|^2) \geq \text{const}(m_0 + |k|)^2. \]  
(2.20)

This proves the \( \alpha = 0 \) case of (2.13).

For \( |\alpha| \geq 1 \), explicit differentiation of (2.15) gives

\[ |\nabla^\alpha \hat{A}^{(m)}(\mu)(k)| \leq \text{const} \times \begin{cases} |k| + m & (|\alpha| \text{ odd}) \\ 1 & (|\alpha| \text{ even}). \end{cases} \]  
(2.21)

The significance of (2.21) is for small \( k, m \). Mixed partial derivatives of \( \hat{D}^{(m)}(k) \) and hence of \( \hat{A}^{(m)}(k) \) all vanish. As examples, we compute the first few derivatives (dropping the superscript \( (m) \)) using (2.20)–(2.21):

\[ \left| \frac{\nabla^2}{A^2} \right| \leq c_1 \frac{1}{(|k| + m_0)^3}, \quad \left| \frac{\nabla^2}{A^2} \right| \leq \left| \frac{\nabla^2}{A^2} \right| + \frac{2(\nabla^2 A)^2}{A^3} \leq c_2 \frac{1}{(|k| + m_0)^4}. \]  
(2.22)
\[ \left| \nabla_i \frac{1}{A} \right| \leq \left| \frac{\nabla^3 i A}{A^2} \right| + \frac{6(\nabla_i \dot{A})(\nabla^2_i \dot{A})}{\dot{A}^3} + \frac{6(\nabla_i \dot{A})^3}{\dot{A}^4} \leq c_3 \frac{1}{(|k| + m_0)^5}. \] (2.23)

In higher order derivatives, differentiation of the terms arising from the quotient rule at the previous order are of two types. When a denominator is differentiated, an additional factor \( \frac{(\nabla_i \dot{A})}{\dot{A}} \) is produced, and this worsens the upper bound by a factor \( (|k| + m_0)^{-1} \). When a numerator is differentiated, it either worsens the upper bound by a factor \( (|k| + m_0)^{-1} \) (if an odd order derivative in the numerator is differentiated) or it improves the upper bound by a factor \( (|k| + m_0)^{+1} \) (if an even order derivative in the numerator is differentiated). Thus, as claimed in (2.13), we obtain

\[ \left| \nabla^n \frac{1}{A} \right| \leq \text{const} \frac{1}{(|k| + m_0)^{n+2}}. \] (2.24)

This completes the proof. \( \square \)

### 3 Lace Expansion

The lace expansion was introduced by Brydges and Spencer [7] to prove that the weakly self-avoiding walk is diffusive in dimensions \( d > 4 \). In the decades since 1985, the lace expansion has been adapted and extended to a broad range of models and results [19, 34].

We restrict attention henceforth to dimensions \( d > 4 \) and sufficiently small \( \beta > 0 \). For the weakly self-avoiding walk, the lace expansion [7, 26, 34] produces an explicit formula for the \( \mathbb{Z}^d \)-symmetric function \( \Pi_z : \mathbb{Z}^d \rightarrow \mathbb{R} \) which for \( z \in [0, z_c) \) satisfies the convolution equation

\[ G_z(x) = \delta_{0,x} + z \Omega (D * G_z)(x) + (\Pi_z * G_z)(x) \quad (x \in \mathbb{Z}^d), \] (3.1)

or equivalently,

\[ \hat{G}_z(k) = \frac{1}{1 - z \Omega \hat{D}(k) - \hat{\Pi}_z(k)} \quad (k \in \mathbb{T}^d). \] (3.2)

We define

\[ F_z = \delta - z \Omega D - \Pi_z, \quad \hat{F}_z = 1 - z \Omega \hat{D} - \hat{\Pi}_z. \] (3.3)

Then (3.1)–(3.2) simplify to

\[ G_z * F_z = \delta, \quad \hat{G}_z(k) = \frac{1}{\hat{F}_z(k)}. \] (3.4)

In fact, \( \Pi_z \) is given by an alternating series \( \Pi_z(x) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(x) \), with each \( \Pi_z^{(N)}(x) \) nonnegative and monotone increasing in \( z \). It is proven, e.g., in [3] that...
there is a constant $K$ such that
\[
\Pi_z^{(N)}(x) \leq (K \beta)^N \frac{1}{1 + |x|^{3(d-2)}} \quad (N \geq 1, \ z \in [0, z_c], \ x \in \mathbb{Z}^d).
\] (3.5)

Consequently, for any $s < 2d - 6$, there is a constant $K_s$ such that
\[
\sum_{x \in \mathbb{Z}^d} |x|^s |\Pi_z(x)| \leq \sum_{x \in \mathbb{Z}^d} \sum_{N=1}^{\infty} |x|^s \Pi_z^{(N)}(x) \leq K_s \beta \quad (z \in [0, z_c]).
\] (3.6)

The inequality (3.6) is often referred to as a diagrammatic estimate since it is motivated by a diagrammatic representation of $\Pi_z$ (see [7] or [26, Sect. 5.4]). A similar diagrammatic estimate (as in [26, Theorem 5.4.4]) gives
\[
\sum_{x \in \mathbb{Z}^d} |\partial_{z} \Pi_z(x)| \leq \sum_{x \in \mathbb{Z}^d} \sum_{N=1}^{\infty} |\partial_{z} \Pi_z^{(N)}(x) \leq K' \beta \quad (z \in [0, z_c]).
\] (3.7)

Since $\chi(z) = \hat{G}_z(0)$ and $\chi(z_c) = \infty$, we have $0 = \hat{F}_{z_c}(0) = 1 - z_c \Omega - \hat{\Pi}_{z_c}(0)$. By (3.6) with $s = 0$, this implies that
\[
z_c - \frac{1}{\Omega} \leq K_0 \beta.
\] (3.8)

With $s = 1$, it also implies that there is a $z^* \in (z, z_c)$ such that
\[
\chi(z)^{-1} = \hat{F}_{z}(0) = \hat{F}_{z}(0) - \hat{F}_{z_c}(0)
\]
\[
= \Omega(z_c - z) + \hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z}(0)
\]
\[
= \Omega(z_c - z) + \partial_{z} \hat{\Pi}_{z}(0)|_{z=z^*} (z_c - z),
\] (3.9)

using the mean-value theorem for the last equality. It follows from (3.7) and the dominated convergence theorem that the coefficient in the last term approaches $\partial_{z} \hat{\Pi}_{z}(0)|_{z=z_c} = O(\beta)$ as $z \to z_c$. This proves that
\[
\chi(z) \sim A (1 - z/z_c)^{-1} \quad (z \to z_c)
\] (3.10)

with $A^{-1} = -z_c \partial_{z} \hat{F}_{z_c}(0) = z_c (\Omega + \partial_{z} \hat{\Pi}_{z_c}(0)) = 1 + O(\beta)$, using (3.7)–(3.8) in the last equality.

### 4 Asymptotic Formula for the Mass

In this section, we prove the statement from (1.12) that for $d > 4$ and sufficiently small $\beta > 0$,
\[
m(z) \sim \text{const} (1 - z/z_c)^{1/2} \quad (z \to z_c),
\] (4.1)
with constant equal to $\Omega^{1/2} + O(\beta)$. The essence of the proof is as in [26, Sect. 6.5] (originally in [18]), which itself is based on [15].

Let $z < z_c$, $m < m(z)$, and $\chi^{(m)}(z) = \sum_{x \in \mathbb{Z}^d} G_z^{(m)}(x)$. The tilted version of (3.4) is $G_z^{(m)} * F_z^{(m)} = \delta$ and hence

$$
\hat{G}_z^{(m)}(k) = \frac{1}{\hat{F}_z^{(m)}(k)} = \frac{1}{1 - z\Omega \hat{D}^{(m)}(k) - \hat{\Pi}_z^{(m)}(k)}.
$$

(4.2)

In particular (recall (2.15)),

$$
\frac{1}{\chi(z)} - \frac{1}{\chi^{(m)}(z)} = \hat{F}_z(0) - \hat{F}_z^{(m)}(0) = 2z[cosh m - 1] + [\hat{\Pi}_z^{(m)}(0) - \hat{\Pi}_z(0)].
$$

(4.3)

An argument\textsuperscript{5} based on the Lieb–Simon inequality gives $\chi^{(m)}(z) \to \infty$ as $m \to m(z)$, exactly as in the proof of [26, (6.5.7)]. Therefore, when we take the limit $m \to m(z)$ (from the left) in (4.3) we get

$$
\frac{1}{\chi(z)} = 2z[cosh m(z) - 1] + [\hat{\Pi}_z^{(m(z))}(0) - \hat{\Pi}_z(0)],
$$

(4.4)

provided we can justify that the limit $\lim_{m \to m(z)} \hat{\Pi}_z^{(m(z))}(0) = \hat{\Pi}_z^{(m(z))}(0)$ exists. The next proposition takes care of this last point, via dominated convergence. The proof of Proposition 4.1 is deferred to later in this section.

**Proposition 4.1** Let $d > 4$ and let $\beta$ be sufficiently small. Let\textsuperscript{6} $s \in [0, d - 2]$. There is a constant $K_s'$ such that, uniformly in $z \in [\frac{1}{\Omega}, z_c)$,

$$
\sum_{x \in \mathbb{Z}^d} |x|^s \sum_{N=1}^{\infty} |\Pi_z^{(N,m(z))}(x)| \leq K_s' \beta.
$$

(4.5)

By (3.10), the left-hand side of (4.4) is asymptotic to $A^{-1}(1 - z/z_c)$ as $z \to z_c$. Since $m(z) \to 0$, the first term on the right-hand side of (4.4) is asymptotic to $z_c m(z)^2$. Given $\epsilon \in (0, 2]$, it is an exercise with the Maclaurin series for $cosh t$ to prove that

$$
|cosh t - 1 - \frac{1}{2} t^2| \leq \text{const } t^{2+\epsilon} \cosh t.
$$

(4.6)

We apply (4.6) with $\epsilon = \min\{2, d - 4\}$, so $2 + \epsilon \leq d - 2$, and conclude from Proposition 4.1 and the fact that $\Pi_z(x) = \Pi_z(-x)$ that the last term on the right-hand

\textsuperscript{5} Briefly, if $\chi(m(z))$ were finite then $G_z^{(m(z))}$ would decay exponentially and this contradicts the fact that $m(z)$ is by (1.9) the exponential decay rate of $G_z$.

\textsuperscript{6} We expect that the proposition in fact remains true for all $s \in [0, 2d - 6)$ but its restriction to $s \in [0, d - 2]$ is sufficient for our needs.
side of (4.4) is
\[
\hat{\Pi}_z^{(m(z))}(0) - \hat{\Pi}_z(0) = \sum_{x \in \mathbb{Z}^d} (\cosh(m(z)x_1) - 1) \Pi_z(x)
\]
\[
= \frac{1}{2} m(z)^2 \sum_{x \in \mathbb{Z}^d} x_1^2 \Pi_z(x) + O(m(z)^{2+\epsilon}). \quad (4.7)
\]

Since \( m(z) \to 0 \) as \( z \to z_c \), we conclude from the above that
\[
A^{-1}(1 - z/z_c) \sim m(z)^2 \left[ z_c + \frac{1}{2} \sum_{x \in \mathbb{Z}^d} x_1^2 \Pi_{z_c}(x) \right]. \quad (4.8)
\]

This gives, as desired,
\[
m(z)^2 \sim c(1 - z/z_c) \quad (z \to z_c). \quad (4.9)
\]

From (4.8), we observe that the constant \( c \) is given by
\[
c = A^{-1}[z_c + \frac{1}{2} \sum_{x \in \mathbb{Z}^d} x_1^2 \Pi_{z_c}(x)]^{-1} = \Omega + O(\beta),
\]
\[
\text{since } A = 1 + O(\beta) \text{ (see below (3.10)) and } z_c = \Omega^{-1} + O(\beta) \text{ by (3.8).}
\]

It remains to prove Proposition 4.1, which we will do using the following lemma. Its hypothesis involves the tilted bubble diagram
\[
\Pi^{(m)}(z) = \sum_{x \in \mathbb{Z}^d} G^{(m)}_z(x)^2. \quad (4.10)
\]

**Lemma 4.2** Fix \( z \in \left[ \frac{1}{\Omega}, z_c \right) \) and \( m \geq 0 \). Let \( s \in [0, d - 2] \). Suppose that there is a constant \( \kappa \) such that \( B^{(m)}(z) \leq \kappa \). Then there is a constant \( K'_s \) depending on \( \kappa \) (and not on \( m, z \)) such that for sufficiently small \( \beta \) (depending on \( \kappa \)),
\[
\sum_{x \in \mathbb{Z}^d} |x|^s \sum_{N=1}^{\infty} \Pi^{(N,m)}_z(x) \leq K'_s \beta. \quad (4.11)
\]

**Proof** This is a standard diagrammatic estimate, just as in [26, Corollary 6.5.2]. We therefore only sketch the argument. The estimate is however better than [26, Corollary 6.5.2] by allowing a larger range of \( s \), as a consequence of (1.13) which was not available when [26, Corollary 6.5.2] was proved.

We illustrate the idea of the proof with the 4-loop term \( \Pi^{(4)} \), which obeys the estimate
\[
\Pi^{(4)}_z(x) \leq \beta^4 \sum_{u, v \in \mathbb{Z}^d} G_z(u)^2 G_z(v) G_z(u - v) G_z(x - u) G_z(x - v)^2. \quad (4.12)
\]

The proof of (4.12) is a small modification of the proof of [26, Theorem 5.4.2] (the small parameter \( \beta \) is more explicit here and this is a simplification). With an exponential
tilt, we obtain
\[ \Pi_z^{(4,m)}(x) \leq \beta^4 \sum_{u, v \in \mathbb{Z}^d} G_z(u)G_z^{(m)}(u) G_z(v) G_z(u-v) G_z^{(m)}(x-u) G_z(x-v)^2. \quad (4.13) \]

We multiply by $|x|^s$ and sum over $x \in \mathbb{Z}^d$, use the inequality $|x|^s \leq 2^s(|v|^s + |x-v|^s)$, and consider the effect of each of the two terms on the right-hand side for the resulting sum over $x$. For example, one of these two terms is
\[ T = \beta^4 \sum_{x, u, v \in \mathbb{Z}^d} G_z(u)G_z^{(m)}(u)|v|^s G_z(v) G_z(u-v) G_z^{(m)}(x-u) G_z(x-v)^2. \quad (4.14) \]

It follows from [26, Lemma 5.4.3] that the sum can be bounded by the product of $\sup_v |v|^s G_z(v)$ times the $\ell_2$ norm of each of the other six factors $G_z$ or $G_z^{(m)}$. By hypothesis, those $\ell_2$ norms are at most $\kappa^{1/2}$, and the supremum is bounded by a constant due to (1.8) and the restriction $s \leq d-2$. The result is a bound of order $\beta^4 \kappa^3$.

In more detail, by definition of $T$,
\[ T \leq \beta^4 \left( \sup_v |v|^s G_z(v) \right) \sup_u \left( \sum_{x, v} G_z(u-v) G_z^{(m)}(x-u) G_z(x-v)^2 \right) \times \left( \sum_u G_z(u)G_z^{(m)}(u) \right). \quad (4.15) \]

The first factor on the right-hand side is bounded via (1.8). The last factor is bounded by $\kappa$, by the Cauchy–Schwarz inequality and by hypothesis. The middle factor, after replacement of the summation indices by $x \mapsto x+u$ and $v \mapsto v+u$, followed by replacement of $x$ by $y = x - v$, becomes
\[ \sum_{x, v} G_z(v)G_z^{(m)}(x) G_z(x-v)^2 = \sum_{v, y} G_z(v)G_z^{(m)}(y+v) G_z(y)^2 \leq \left( \sup_y \sum_v G_z(v)G_z^{(m)}(y+v) \right) \sum_y G_z(y)^2 \leq \kappa^2, \quad (4.16) \]

where we used the Cauchy–Schwarz inequality for the supremum. Overall, this gives a bound on $T$ of order $\beta^4 \kappa^3$. This procedure can be mechanised for general $N$ as in [26, Sect. 5.4].

As a result, for general $N$, $\Pi^{(N)}$ can be bounded similarly by factoring the weight $e^{mx_1}$ along one side of the $N$-loop diagram and factoring $|x|^s$ along the other side of the diagram. The bound on the $N$th term in (4.11) is of order $N \beta^N N^{N-1}$, where the factor $N$ accounts for the use of the triangle inequality to distribute $|x|^s$ along fewer than $N$ diagram lines. Summation over $N$ gives (4.11), for $\beta$ small depending on $\kappa$.\[ \square \]
Proof of Proposition 4.1 Fix $z \in [\frac{1}{\Omega}, z_c)$. We define the critical simple random walk bubble diagram

$$B_0 = \sum_{x \in \mathbb{Z}^d} C_{1/\Omega}(x)^2 = \int_{\mathbb{T}^d} \frac{1}{(1 - \hat{D}(k))^2} \frac{dk}{(2\pi)^d} < \infty, \quad (4.17)$$

where the second inequality holds by the Parseval relation and (2.10). By Lemma 4.2, it suffices to prove that

$$B^{(m(z))}(z) \leq 2B_0. \quad (4.18)$$

We prove (4.18) with a bootstrap argument. In many applications of the lace expansion, the bootstrap argument is used to produce a forbidden interval for the parameter $z$. Here, we instead produce a forbidden interval for the mass parameter $m$; this strategy was first used in [15] for percolation and was subsequently applied also to self-avoiding walk in [18, 26].

To prove (4.18), we will prove that:

(i) $B^{(0)}(z) \leq \frac{3}{2}B_0$, and
(ii) if for $m \in (0, m(z))$ we assume $B^{(m)}(z) \leq 3B_0$ then in fact $B^{(m)}(z) \leq 2B_0$.

By (ii), the interval $(2B_0, 3B_0]$ is forbidden for values of $B^{(m)}(z)$ when $m \in (0, m(z))$. By (i), $B^{(0)}(z)$ lies below this forbidden interval. Since $B^{(m)}(z)$ is continuous in $m$ by monotone convergence, it follows from the Intermediate Value Theorem that $B^{(m)}(z) \leq 2B_0$ for all $m < m(z)$. By monotone convergence, this implies (4.18). So it remains to prove (i) and (ii).

The main work is to prove (ii); along the way we also prove (i). By (4.10), (4.2) and the Parseval relation, for $m \in [0, m(z))$ we have

$$B^{(m)}(z) = \int_{\mathbb{T}^d} \frac{1}{|\hat{F}^{(m)}_z(k)|^2} \frac{dk}{(2\pi)^d}. \quad (4.19)$$

The denominator of the integrand involves

$$\hat{F}_z^{(m)}(k) = \hat{F}_z^{(m)}(0) + [\hat{F}_z^{(m)}(k) - \hat{F}_z^{(m)}(0)]. \quad (4.20)$$

Thus, by the formula for $\hat{F}_z^{(m)}(k)$ in (4.2),

$$|\hat{F}_z^{(m)}(k)| \geq \text{Re} [\hat{F}_z^{(m)}(k) - \hat{F}_z^{(m)}(0)]$$

$$= z\Omega \text{Re} \{ \hat{D}^{(m)}(0) - \hat{D}^{(m)}(k) \} + \text{Re} [\hat{\Pi}_z^{(m)}(0) - \hat{\Pi}_z^{(m)}(k)], \quad (4.21)$$

where the inequality holds since $\hat{F}_z^{(m)}(0)$ is real and positive. Since $z\Omega \geq 1$ by assumption and because $\text{Re} [\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)] \geq 1 - \hat{D}(k)$ by (2.19),
\[ |\hat{F}^{(m)}_{z}(k)| \geq 1 - \hat{D}(k) + \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) \Pi^{(m)}_{z}(x) \]
\[ \geq 1 - \hat{D}(k) - \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) |\Pi^{(m)}_{z}(x)|. \]  
(4.22)

To bound the last sum in (4.22), we use the facts that \(1 - \cos t \leq \frac{1}{2} t^2\) for \(t \in \mathbb{R}\), and 
\[2\pi^{-2}t^2 \leq 1 - \cos t\) for \(t \in [-\pi, \pi]\). Since \(|k \cdot x| \leq |k||x|\),
\[\sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) |\Pi^{(m)}_{z}(x)| \leq \left[1 - \hat{D}(k)\right] \frac{d\pi^2}{4} \sum_{x \in \mathbb{Z}^d} |x|^2 |\Pi^{(m)}_{z}(x)|. \]  
(4.23)

By Lemma 4.2, if we assume that \(B^{(m)}(z) \leq 3B_0\) (as we do for \(m \in (0, m(z))\) in (ii)) then the right-hand side of (4.23) is \(O(\beta)[1 - \hat{D}(k)]\), so we obtain
\[|\hat{F}^{(m)}_{z}(k)| \geq \text{Re}[\hat{F}^{(m)}_{z}(k) - \hat{F}^{(m)}_{z}(0)] \geq [1 - O(\beta)](1 - \hat{D}(k)). \]  
(4.24)

But this implies the infrared bound
\[|\hat{G}^{(m)}_{z}(k)| \leq (1 + O(\beta)) \frac{1}{1 - \hat{D}(k)}, \]  
(4.25)

which by (4.19) and (4.17) implies that \(B^{(m)}(z) \leq 2B_0\), and completes the proof of item (ii).

Finally, we prove (i). For (i) we do not have an \textit{a priori} assumption that \(B^{(0)}(z) \leq 3B_0\) as was used the previous paragraph. Nevertheless, the proof of (ii) more than suffices for the proof of (i). Indeed, the above argument also implies (4.25) when \(m = 0\), using (3.6) to bound the right-hand side of (4.23) instead of making use of the \textit{a priori} assumption, and this implies that \(B^{(0)}(z) \leq \frac{3}{2}B_0\). This completes the proof of both items (i) and (ii) in the bootstrap argument, and concludes the proof. \(\Box\)

Note that (4.24), (4.25) and (4.11), although initially conditional on the bootstrap hypothesis, hold unconditionally now that the bootstrap argument has been completed. In particular, we have now proved that (4.18) does in fact hold, and hence for \(z \in [\frac{1}{\Omega}, z_c)\) we have
\[B^{(m(z))}(z) \leq 2B_0, \]  
(4.26)

which is a statement of a tilted bubble condition.

5 Proof of Theorem 1.1

We now complete the proof of Theorem 1.1 by proving the inequality (1.7) for \(G_{z}(x)\). The proof is based on an extension of the method of [37].
5.1 Isolation of Leading Term

We extend the method of [37] to include positive mass. Let \( \lambda > 0, \mu \in [0, \frac{1}{\Omega}] \), and \( A_\mu = \delta - \mu \Omega D \). Since \( C_\mu \ast A_\mu = \delta \) and \( F_z \ast G_z = \delta \) by (2.10) and (3.4), we have

\[
G_z = \lambda C_\mu + \delta \ast G_z - \lambda C_\mu \ast \delta \\
= \lambda C_\mu + C_\mu \ast E_{z, \lambda, \mu} \ast G_z \quad \text{with} \quad E_{z, \lambda, \mu} = A_\mu - \lambda F_z. \tag{5.1}
\]

As in [17, 37], we choose \( \lambda_z \) and \( \mu_z \) so that

\[
\sum_{x \in \mathbb{Z}^d} E_z(x) = \sum_{x \in \mathbb{Z}^d} |x|^2 E_z(x) = 0, \tag{5.2}
\]

where we set \( E_z = E_{z, \lambda_z, \mu_z} \). The solution to the two linear equations (5.2) in the two unknowns \( \lambda, \mu \) is

\[
\lambda_z = \frac{1}{1 - \hat{\Pi}_z(0) + \sum_x |x|^2 \Pi_z(x)}, \tag{5.3}
\]

\[
\mu_z \Omega = 1 - \lambda_z \hat{F}_z(0) = \frac{z \Omega + \sum_x |x|^2 \Pi_z(x)}{\hat{F}_z(0) + z \Omega + \sum_x |x|^2 \Pi_z(x)}. \tag{5.4}
\]

The term \( \hat{F}_z(0) = \chi(z)^{-1} \) is positive for \( z < z_c \). We are interested in the case \( z \in [\frac{1}{\Omega}, z_c) \), so \( z \Omega \geq 1 \). By (3.6), the \( \Pi \) terms in (5.3)–(5.4) are small, in particular \( \lambda_z = 1 + O(\beta) \). Also, the right-hand side of (5.4) lies in \((0, 1)\) and therefore \( \mu_z \in (0, \frac{1}{\Omega}) \) is subcritical. Explicit calculation using the definition of \( E_z \) from (5.1) and of \( \mu_z \) from (5.4) leads to

\[
E_z = (1 - \lambda_z)(\delta - D) - \lambda_z \hat{\Pi}_z(0) D + \lambda_z \Pi_z, \tag{5.5}
\]

\[
E_z^{(m)} = (1 - \lambda_z)(\delta - D^{(m)}) - \lambda_z \hat{\Pi}_z(0) D^{(m)} + \lambda_z \Pi_z^{(m)}. \tag{5.6}
\]

Multiplication of (5.1) by \( e^{mx_1} \) gives

\[
G_z^{(m)} = \lambda_z C_\mu^{(m)} + f_z^{(m)} \quad \text{with} \quad f_z^{(m)} = C_\mu^{(m)} \ast E_z^{(m)} \ast G_z^{(m)}. \tag{5.7}
\]

We will show that \( \lambda_z C_\mu^{(m)} \) gives the main contribution to \( G_z^{(m)} \), with \( f_z^{(m)} \) smaller by a factor \( \beta \).

5.2 The Key Ingredient

The key ingredient in the proof of the main result (1.7) is the following proposition.
Proposition 5.1 Let \( d > 4 \) and let \( \beta \) be sufficiently small. Let \( z \in [\frac{1}{\Omega}, z_c) \) and \( m \in [0, \frac{1}{2} m(z)] \). There is a constant \( A_1 > 0 \) (independent of \( m, z, \beta \)) such that

\[
\int_{\mathbb{R}^d} |\nabla^\alpha \hat{f}_z^{(m)}(k)| \frac{dk}{(2\pi)^d} \leq A_1 \beta \quad (|\alpha| \leq d - 2). \tag{5.8}
\]

Before proving Proposition 5.1, we show that it leads to a proof of (1.7) and thereby concludes the proof of Theorem 1.1. For this we need to anticipate a conclusion of Lemma 5.5, where it is proved that if \( z \in [\frac{1}{\Omega}, z_c) \) then

\[
\frac{1}{2} m(z) \leq \frac{2}{3} m_0(\mu_z). \tag{5.9}
\]

Here \( m_0(\mu) \) is the simple random walk mass defined in (2.1); according to (2.2) it vanishes at \( \frac{1}{\Omega} \) as \( m_0(\mu)^2 \sim \Omega(1 - \Omega \mu) \).

Proof of Theorem 1.1 Let \( z \in [\frac{1}{\Omega}, z_c) \) and set \( \tilde{m} = \frac{1}{2} a_1 m(z) \), with \( a_1 < 1 \) the constant of Proposition 2.1. Then \( \tilde{m} \leq \frac{1}{2} m(z) \). We use the fact that the smoothness of the Fourier transform \( \hat{f}_z^{(m)} \) controls the decay of \( f_z^{(m)} \), in the sense that Proposition 5.1 implies that

\[
|f_z^{(\tilde{m})}(x)| \leq A'_1 \beta (1 \vee |x|^{d-2})^{-1}. \tag{5.7}
\]

Therefore, by (5.7),

\[
G_z^{(\tilde{m})}(x) \leq \lambda_z C_{\mu_z}^{(\tilde{m})}(x) + A'_1 \beta \frac{1}{1 \vee |x|^{d-2}}. \tag{5.10}
\]

By Proposition 2.1,

\[
C_{\mu_z}^{(\tilde{m})}(x) = C_{\mu_z}(x) e^{\tilde{m} x_1} \leq a_0 \frac{1}{1 \vee |x|^{d-2}} e^{- (a_1 m_0(\mu_z) - \tilde{m}) \|x\|_\infty}. \tag{5.11}
\]

By (5.9), \( \tilde{m} = \frac{1}{2} a_1 m(z) \leq \frac{2}{3} a_1 m_0(\mu_z) \). As noted below (5.4), \( \lambda_z = 1 + O(\beta) \). With (5.10)–(5.11), this shows that by taking \( \beta \) sufficiently small we can obtain

\[
G_z^{(\tilde{m})}(x) \leq (1 + O(\beta)) a_0 \frac{1}{1 \vee |x|^{d-2}} + A'_1 \beta \frac{1}{1 \vee |x|^{d-2}} \leq 2 a_0 \frac{1}{1 \vee |x|^{d-2}}, \tag{5.12}
\]

and hence

\[
G_z(x) \leq 2 a_0 \frac{1}{1 \vee |x|^{d-2}} e^{- \frac{1}{2} a_1 m(z) x_1}. \tag{5.13}
\]

Without loss of generality, by the symmetry of \( G_z \) we may assume that \( x_1 = \|x\|_\infty \). Since \( \|x\|_\infty \geq d^{-1/2} |x| \), this proves that (1.7) holds with \( c_0 = 2 a_0 \) and \( c_1 = \frac{1}{2} a_1 d^{-1/2} \). This completes the proof. \( \square \)
5.3 Proof of Proposition 5.1

To complete the proof of (1.7), it remains to prove Proposition 5.1 and (5.9). By (5.7),

\[ \hat{f}_z^{(m)} = \hat{C}_{\mu_z}^{(m)} \hat{E}_z^{(m)} \hat{G}_z^{(m)}, \]  

(5.14)

so to prove Proposition 5.1 we need estimates on derivatives of each of the three factors on the right-hand side of (5.14). Lemmas 5.2 and 5.4 give the estimates we need for \( \hat{G}_z^{(m)} \) and \( \hat{E}_z^{(m)} \), respectively. Lemma 2.2 will give the required estimate for \( \hat{C}_{\mu_z}^{(m)} \), when combined with the relation between \( m_0(\mu_z) \) and \( m(z) \) claimed in (5.9) and established in Lemma 5.5.

**Lemma 5.2** Let \( d > 4 \) and let \( \beta \) be sufficiently small. Let \( z \in [\frac{1}{\Omega}, \zeta_c], m \in [0, \frac{1}{\Omega} m(z)], \) and \( 0 \leq |\alpha| \leq d - 2 \). There is a constant (independent of \( z, m, k, \beta, \alpha \)) such that

\[ |\nabla^\alpha \hat{G}_z^{(m)}(k)| \leq \text{const} \frac{1}{(|k| + m)^{2+|\alpha|}} \quad (k \in \mathbb{T}^d). \]  

(5.15)

**Proof** We have already proved a weaker version of the case \( \alpha = 0 \) of (5.15) in (4.25), with \( m = 0 \) on the right-hand side. To improve (4.25), we first observe that the inequality

\[ \text{Re} [\hat{F}_z^{(m)}(k) - \hat{F}_z^{(m)}(0)] \geq \text{const} |k|^2 \]  

(5.16)

follows directly from (4.24). To prove (5.15) for \( \alpha = 0 \), we will prove that \( \hat{F}_z^{(m)}(0) \geq \text{const} m^2 \). This indeed completes the proof, since with (5.16) it gives

\[ |\hat{F}_z^{(m)}(k)| \geq \text{Re} \hat{F}_z^{(m)}(k) = \hat{F}_z^{(m)}(0) + \text{Re} [\hat{F}_z^{(m)}(k) - \hat{F}_z^{(m)}(0)] \geq \text{const} (m^2 + |k|^2), \]  

(5.17)

which is the \( \alpha = 0 \) case of (5.15).

The proof that \( \hat{F}_z^{(m)}(0) \geq \text{const} m^2 \) is as follows. By (3.8), \( z_c - \frac{1}{\Omega} = O(\beta) \), so by (1.12) \( m(z) = O(\beta^{1/2}) \) is small. As noted below (4.3), \( \hat{F}_z^{(m(z))}(0) = 0 \), and hence by (2.15)

\[ \hat{F}_z^{(m)}(0) = \hat{F}_z^{(m)}(0) - \hat{F}_z^{(m(z))}(0) \]
\[ = z\Omega [\hat{D}^{(m(z))}(0) - \hat{D}^{(m)}(0)] + [\hat{\Pi}_z^{(m(z))}(0) - \hat{\Pi}_z^{(m)}(0)] \]
\[ = 2z [\cosh m(z) - \cosh m] + \sum_{x \in \mathbb{Z}^d} (\cosh(m(z)x_1) - \cosh(mx_1)) \Pi_z(x). \]  

(5.18)

For the first term, we use \( m \leq \frac{1}{2} m(z) \) and the fact that \( m(z) = O(\beta^{1/2}) \) to obtain

\[ 2z [\cosh m(z) - \cosh m] = z (m(z)^2 - m^2) + O(m(z)^4) = z (m(z)^2 - m^2) + O(\beta m(z)^2). \]  

(5.19)
For the $\Pi$ term we use the elementary inequality

$$0 \leq \cosh t - \cosh s \leq (t-s) \sinh t \leq (t-s) t \cosh t \quad (0 \leq s \leq t) \quad (5.20)$$

to conclude that

$$\sum_{x \in \mathbb{Z}^d} \left( \cosh(m(z)x_1) - \cosh(mx_1) \right) |\Pi_z(x)| \leq m(z)^2 \sum_{x \in \mathbb{Z}^d} x_1^2 \sum_{N=1}^{\infty} \Pi_z^{(N,m(z))}(x). \quad (5.21)$$

The right-hand side is $O(\beta m(z)^2)$ by Proposition 4.1, so the above leads to

$$\hat{F}^{(m)}(0) = z(m(z)^2 - m^2) + O(\beta m(z)^2). \quad (5.22)$$

Since $m \leq \frac{1}{2} m(z)$ by assumption, this gives $\hat{F}^{(m)}(0) \geq \text{const} \ m^2$ and the proof for $\alpha = 0$ is complete.

Examples of the first few derivatives of $\hat{G}^{(m)}_z$ are, with $\hat{F} = \hat{F}^{(m)}_z$,

$$\nabla_i \hat{G}^{(m)}_z = - \frac{\nabla_i \hat{F}}{\hat{F}^2}, \quad \nabla_i^2 \hat{G}^{(m)}_z = - \frac{\nabla_i^2 \hat{F}}{\hat{F}^2} + 2 \frac{(\nabla_i \hat{F})^2}{\hat{F}^3}, \quad (5.23)$$

$$\nabla_i^3 \hat{G}^{(m)}_z = - \frac{\nabla_i^3 \hat{F}}{\hat{F}^2} + \frac{6(\nabla_i \hat{F})(\nabla_i^2 \hat{F})}{\hat{F}^3} - \frac{6(\nabla_i \hat{F})^3}{\hat{F}^4}. \quad (5.24)$$

The denominators are bounded using the bound $\hat{F}^{(m)}_z(k) \geq \text{const} \ (|k| + m)^2$ from (5.17), exactly as in the proof of Lemma 2.2, and we again need to show that large powers in denominators are compensated by the numerators. This works essentially in the same way as in the proof of Lemma 2.2. For the numerators, as in (2.21) the $\hat{D}^{(m)}$ terms are bounded using

$$|\nabla^\alpha \hat{D}^{(m)}_\mu(k)| \leq \text{const} \times \begin{cases} (|k| + m) & (|\alpha| \text{ odd}) \\ 1 & (|\alpha| \text{ even}), \end{cases} \quad (5.25)$$

and it suffices to prove the same estimate for $\nabla^\alpha \hat{\Pi}^{(m)}_z$. In fact, it can be extrapolated from (5.23)–(5.24) that there is just one term in $\nabla^\alpha \hat{G}^{(m)}_z$ that requires an estimate on $\nabla^\alpha \hat{\Pi}^{(m)}_z$ with $|\alpha| = d-2$, namely a term proportional to $\hat{F}^{-2} \nabla_i \hat{F}$ with $|\alpha| = d-2$. Such a term is relatively small, since the numerator is bounded by a constant due to Proposition 4.1, so this term is $O((|k| + m)^{-4})$ which is better than the required bound $O((|k| + m)^{-2-|\alpha|})$ when $|\alpha| \geq 3$. It therefore suffices to prove the analogue of (5.25) for $\nabla^\alpha \hat{\Pi}^{(m)}_z$ only for $|\alpha| \leq d-3$. 

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By symmetry, $\hat{\Pi}_z^{(m)}$ can be written as

$$\hat{\Pi}_z^{(m)}(k) = \sum_{x \in \mathbb{Z}^d} \cos(k \cdot x) \cosh(mx_1)\Pi_z(x) + i \sum_{x \in \mathbb{Z}^d} \sin(k \cdot x) \sinh(mx_1)\Pi_z(x).$$

(5.26)

To estimates its $k$-derivatives, we use the inequalities

$$|\nabla^\alpha \cos(k \cdot x)| \leq \left\{ \begin{array}{ll} |x|^{|\alpha|} & (|\alpha| \text{ even}) \\ |x|^{|\alpha|+1}|k| & (|\alpha| \text{ odd}) \end{array} \right.,$$

(5.27)

$$|\nabla^\alpha \sin(k \cdot x)| \leq |x|^{|\alpha|},$$

(5.28)

$$\sinh(mx_1) \leq m|x_1| \cosh(mx_1).$$

(5.29)

The factor $|k|$ in (5.27) is bounded, so this gives

$$|\nabla^\alpha \hat{\Pi}_z^{(m)}(k)| \leq \text{const} \sum_{x \in \mathbb{Z}^d} |x|^{|\alpha|+1} |\Pi_z^{(m)}(x)| \times \left\{ \begin{array}{ll} (|k| + m) & (|\alpha| \text{ odd}) \\ 1 & (|\alpha| \text{ even}) \end{array} \right..$$

(5.30)

We have already observed that it is now sufficient to consider $|\alpha| \leq d - 3$, and by Proposition 4.1, $\sum_{x \in \mathbb{Z}^d} |x|^s |\Pi_z^{(m)}(x)| \leq O(\beta)$ for $s \leq d - 2$. This yields the required analogue of (5.25) for derivatives of $\hat{\Pi}_z^{(m)}$, and completes the proof.

$\square$

**Remark 5.3** In the proof of Lemma 5.2, we require $d - 2$ derivatives of $\hat{\Pi}_z^{(m)}$, which are controlled by the $(d - 2)^{nd}$ moment of $\Pi_z^{(m)}$. The latter ultimately goes back to (3.6) with $s = d - 2$, which itself requires $d + (d - 2) < 3(d - 2)$, i.e., $d > 4$. For the Ising and $\varphi^4$ models, $\Pi(x)$ also obeys an upper bound $|x|^{-3(d-2)}$ [6, 32, 33], which raises the possibility that our results could be extended to these spin models. However, for percolation the bound on $\Pi(x)$ is $|x|^{-2(d-2)}$ and for lattice trees and lattice animals it is $|x|^{-(2d-6)}$ [17], so for neither does $\Pi$ have finite $(d - 2)^{nd}$ moment in any dimension. Thus our approach cannot apply to percolation, nor to lattice trees and lattice animals, without a new idea. A completely different proof of Theorem 1.1 for high-dimensional percolation is given in [21]; that proof does not apply to weakly self-avoiding walk.

The following lemma illustrates the role of a key cancellation due to (5.2).

**Lemma 5.4** Let $d > 4$ and let $\beta$ be sufficiently small. Fix $z \in \{1, z_c\}$ and $m \in [0, \frac{1}{2} m(z)]$. There is a $c_0 > 0$ (independent of $z$, $m$, $k$) such that $|\nabla^\alpha \hat{E}_z^{(m)}(k)| \leq c_0 \beta$ for $|\alpha| \leq d - 2$, and moreover,

$$|\nabla^\alpha \hat{E}_z^{(m)}(k)| \leq c_0 \beta(|k| + m)^{3-|\alpha|} \quad (|\alpha| \leq 3).$$

(5.31)

**Proof** We first prove that $|\nabla^\alpha \hat{E}_z^{(m)}(k)| \leq c_0 \beta$ for $|\alpha| \leq d - 2$. To begin, we differentiate the formula for $\hat{E}_z^{(m)}(k)$ which arises from Fourier transformation of (5.6), and obtain

$$\nabla^\alpha \hat{E}_z^{(m)} = -(1 - \lambda_z) \nabla^\alpha (1 - \hat{D}^{(m)}) - \lambda_z \hat{\Pi}_z(0) \nabla^\alpha \hat{D}^{(m)} + \lambda_z \nabla^\alpha \hat{\Pi}_z^{(m)}.$$

(5.32)
We have seen previously that \( \lambda_z = 1 + O(\beta) \), and it follows from Proposition 4.1 that \( \hat{\Pi}_z(0) = O(\beta) \). Since \( z \in [\frac{1}{\Omega}, z_c] \) and since \( z_c - \frac{1}{\Omega} \leq O(\beta) \), the mass \( m \) is at most \( O(\beta^{1/2}) \) so certainly \( \nabla^a \hat{D}^{(m)}(k) = O(1) \). It also follows from Proposition 4.1 that \( \nabla^a \hat{\Pi}_z^{(m)}(k) = O(\beta) \) for \( |\alpha| \leq d - 2 \). This proves that, as claimed, \( |\nabla^a \hat{E}_z^{(m)}(k)| \leq c_0 \beta \)

for \( |\alpha| \leq d - 2 \).

It remains to prove (5.31). The case \( |\alpha| = 3 \) has already been proved in the previous paragraph (of course \( 3 \leq d - 2 \) for \( d \geq 5 \)), so we consider now \( 0 \leq |\alpha| \leq 2 \), beginning with \( |\alpha| = 0 \). We fix \( z \), define \( h : [0, \frac{1}{2} m(z)] \times \mathbb{T}^d \to \mathbb{C} \) by

\[
h(m, k) = \hat{E}_z^{(m)}(k) = \sum_{x \in \mathbb{Z}^d} E_z(x) e^{mx^1} e^{ik \cdot x}, \tag{5.33}
\]

and expand \( h \) to second order in \((m, k)\) with third order error term. The constant term \( h(0, 0) \) is zero by (5.2). First derivatives of \( h \) also vanish at \((0, 0)\) due to the reflection symmetry of \( E_z(x) \) in each component. In addition, it follows from (5.2) and symmetry that \( \sum_{x \in \mathbb{Z}^d} E_z(x) x_j^2 = 0 \) for each \( j = 1, \ldots, d \), and hence all second derivatives also vanish at \((0, 0)\) (second derivatives other than \( \partial_m^2, \partial_m \partial_{k_1} \) and \( \partial_{k_j}^2 \) all vanish by reflection symmetry). Therefore, the second order Taylor polynomial of \( h \) is identically zero and \( h \) is equal to its third order Taylor remainder, which we consider in its Lagrange form. All third derivatives of \( \hat{E}_z^{(m)}(k) \), with respect to \((m, k)\), are \( O(\beta) \) due to Proposition 4.1 (it is only derivatives of \( \hat{\Pi}_z^{(m)}(k) \) that require careful attention), and thus we see that by increasing the value of \( c_0 \) if necessary we obtain

\[
h(m, k) \leq c_0 \beta (|k| + m)^3, \tag{5.34}
\]

which is the \( |\alpha| = 0 \) case of (5.31).

The proof of (5.31) for \( |\alpha| = 1, 2 \) follows in exactly the same way, with the following minor change. Each \( k \)-derivative of \( h \) brings a factor \( x_j \) down from the exponent. This reduces the number of vanishing derivatives of \( \nabla^a h \) by \( |\alpha| \), with a corresponding reduction in the order of the error in Taylor expansion. This completes the proof. \( \square \)

The next lemma provides the relation between \( m_0(\mu_z) \) and \( m(z) \) claimed in (5.9).

**Lemma 5.5** Let \( d > 4 \) and let \( \beta \) be sufficiently small. If \( z \in [\frac{1}{\Omega}, z_c] \) then \( \mu_z \in [\frac{1}{2\Omega}, \frac{1}{\Omega}] \) and \( \frac{1}{z} m(z) \leq \frac{2}{3} m_0(\mu_z) \).

**Proof** Let \( z \in [\frac{1}{\Omega}, z_c] \). As in the proof of Lemma 5.2, \( m(z) = O(\beta^{1/2}) \). By (5.4) and (5.22) (with \( m = 0 \)),

\[
\mu_z \Omega = 1 - \lambda_z \hat{F}_z(0) = 1 - zm(z)^2 + O(\beta m(z)^2). \tag{5.35}
\]

Therefore, \( \mu_z \Omega = 1 - O(\beta) \) and hence \( \mu_z \in [\frac{1}{2\Omega}, \frac{1}{\Omega}] \) for small \( \beta \).
Since \( \cosh m_0(t) = 1 + \frac{1-t\Omega}{2t} \) by (2.1), we have
\[
m_0(t)^2 = \frac{1 - t\Omega}{t} + O\left(\frac{1 - t\Omega}{t}\right)^2 \tag{5.36}
\]
and hence, by (5.35) and the fact that \( \frac{1}{2\Omega} \leq \mu_z < \frac{1}{\Omega} \leq z \),
\[
m_0(\mu_z)^2 = \frac{z}{\mu_z} m(z)^2 (1 + O(\beta)) \geq m(z)^2 (1 + O(\beta)). \tag{5.37}
\]
For small \( \beta \), this gives \( \frac{2}{3} m_0(\mu_z) \geq \frac{1}{2} m(z) \), and the proof is complete. \( \Box \)

**Proof of Proposition 5.1** Let \( |\alpha| \leq d - 2 \), \( z \in \left[\frac{1}{\Omega}, z_c\right) \), and \( m \leq \frac{1}{2} m(z) \). It follows from the formula for \( f_z^{(m)} \) in (5.14), together with the product rule for differentiation, that \( \nabla \alpha f_z^{(m)} \) involves terms
\[
(\nabla^{\alpha_1} \hat{C}_{\mu_z}^{(m)})(\nabla^{\alpha_2} \hat{E}_{z}^{(m)})(\nabla^{\alpha_3} \hat{G}_{z}^{(m)}) \text{ with } |\alpha_1| + |\alpha_2| + |\alpha_3| = |\alpha|. \tag{5.38}
\]
The derivatives of \( \hat{E}_{z}^{(m)} \) are bounded using Lemma 5.4, and the derivatives of \( \hat{G}^{(m)}_z \) are bounded using Lemma 5.2. For the derivatives of \( \hat{C}^{(m)}_{\mu_z} \), we know from Lemma 5.5 that \( \mu_z \in \left[\frac{1}{2\Omega}, \frac{1}{\Omega}\right) \) and \( \frac{2}{3} m_0(\mu_z) \geq \frac{1}{2} m(z) \geq m \), so we can apply Lemma 2.2 (with \( \sigma = \frac{2}{3} \)) to obtain
\[
|\nabla^{\alpha_1} \hat{C}_{\mu_z}^{(m)}(k)| \leq \text{const} \frac{1}{(|k| + m_0(\mu_z))^{2 + |\alpha_1|}} \leq \text{const} \frac{1}{(|k| + m)^{2 + |\alpha_1|}}. \tag{5.39}
\]
Altogether, these facts lead to an upper bound for (5.38) of order
\[
\frac{\beta(|k| + m)^{3 - \min(|\alpha_2|, 3)}}{(|k| + m)^{2 + |\alpha_1|}(|k| + m)^{2 + |\alpha_3|}} \leq \frac{\beta}{(|k| + m)^{1 + |\alpha_1| + \min(|\alpha_2|, 3) + |\alpha_3|}} \leq \frac{\beta}{|k|^{d-1}}, \tag{5.40}
\]
The integral of the right-hand side over \( \mathbb{T}^d \) is \( O(\beta) \), so the proof is complete. \( \Box \)

### 6 The Plateau for the Torus Two-Point Functions

The proofs of Theorems 1.2 and 1.4 are largely the same so we present them together. We separate the proofs of the upper and lower bounds, beginning with the upper bound. We write \( \Gamma_z(x) \) to denote either of \( C_z(x) \) or \( G_z(x) \) in discussions that apply to both.

As a preliminary, we observe that if \( x \in \mathbb{T}^d \) is regarded as a point in \( [-\frac{r}{2}, \frac{r}{2})^d \cap \mathbb{Z}^d \), then \( \|x + ru\|_\infty \asymp r\|u\|_\infty \) uniformly in nonzero \( u \in \mathbb{Z}^d \), since
\[
\|x + ru\|_\infty \geq \|ru\|_\infty - \frac{r}{2} \geq \|ru\|_\infty - \frac{1}{2}\|ru\|_\infty = \frac{1}{2}\|ru\|_\infty, \tag{6.1}
\]
\[ \| x + ru \|_\infty \leq \frac{r}{2} + \| ru \|_\infty \leq \frac{1}{2} \| ru \|_\infty + \| ru \|_\infty = \frac{3}{2} \| ru \|_\infty. \] (6.2)

### 6.1 Upper Bound for the Torus Two-Point Functions

As discussed below (1.29), walks on the torus are in a one-to-one correspondence with walks on \( \mathbb{Z}^d \) via unfolding, and the simple random walk torus two-point function is given by

\[ C_z^T(x) = C_z(x) + \sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru). \] (6.3)

For weakly self-avoiding walk the equality does not hold, because the unfolding of a torus walk can have fewer intersections than the folded walk. Thus the above is replaced by an inequality

\[ G_z^T(x) \leq G_z(x) + \sum_{u \in \mathbb{Z}^d: u \neq 0} G_z(x + ru). \] (6.4)

By Theorem 1.1 and Proposition 2.1 (recall that \( \Gamma_z \) denotes either \( C_z \) or \( G_z \)),

\[ \Gamma_z(x) \leq c \frac{1}{1 \vee |x|^{d-2}} e^{-c' \nu(z)|x|} \quad (z \in (0, z_*)), \] (6.5)

with \( \nu(z) \) equal to \( m_0(z) \) or \( m(z) \), and with \( z_* \) equal to \( \frac{1}{12} \) or \( z_c \), for \( C_z \) and \( G_z \) respectively. The plateau upper bounds then follow immediately, as follows.

**Proof of upper bounds in Theorems 1.2 and 1.4** Let \( z \in (0, z_1) \), let \( d > 2 \) for simple random walk, and let \( d > 4 \) and \( \beta \) be small for weakly self-avoiding walk. By (6.5) and (6.1),

\[ \sum_{u \in \mathbb{Z}^d: u \neq 0} \Gamma_z(x + ru) \leq c \sum_{u \neq 0} \frac{1}{|x + ru|^{d-2}} e^{-c' \nu(z)|x + ru|} \leq c_0 \sum_{u \neq 0} \frac{1}{|ru|^{d-2}} e^{-c_0 v(z)|ru|}. \] (6.6)

We bound the sum on the right-hand side by an integral and make the change of variables \( y = vu \) to obtain

\[ \sum_{u \in \mathbb{Z}^d: u \neq 0} \Gamma_z(x + ru) \leq c_1 \frac{1}{r^d \nu(z)^2} \int_{\mathbb{R}^d} du \frac{1}{|y|^{d-2}} e^{-c_0 |y|} \leq c'_1 \frac{1}{r^d \nu(z)^2}. \] (6.7)

It remains to show that \( \nu(z)^{-2} \leq \text{const} \chi(z) \). Fix any \( z_1 \in (0, z_*) \). For \( z \leq z_1 \), since \( \nu \) is decreasing and since \( 1 = \chi(0) \leq \chi(z) \), we have \( \nu(z)^{-2} \leq \nu(z_1)^{-2} \leq \nu(z_1)^{-2} \chi(z) \) and the desired upper bound \( \text{const} \ r^{-d} \chi(z) \) follows for \( z \in (0, z_1] \). We can choose \( z_1 \) close enough to \( z_c \) that \( \nu(z)^{-2} \) and \( \chi(z) \) are comparable for \( z \in (z_1, z_*), \) since both
are asymptotic to \((1 - z/z_*)^{-1}\), and this gives the desired estimate for \(z \in (z_1, z_*)\) and thus completes the proof.7

6.2 Lower Bound for the Torus Two-Point Functions

We first consider dimensions \(d > 4\), which is the case relevant for the weakly self-avoiding walk. Let 
\[
\rho = z_* - z. \tag{6.8}
\]

Lemma 6.1 Let \(d > 4\), and for weakly self-avoiding walk let \(\beta\) be sufficiently small. There are \(a_i > 0\) such that, for all \(x \in \mathbb{Z}^d\) and all \(z \in [\frac{1}{2} z_*, z_*]\),
\[
a_4 \frac{1}{1 \lor |x|^{d-2}} \leq \Gamma_{z_*}(x) \leq a_5 \frac{1}{1 \lor |x|^{d-2}}, \tag{6.9}
\]
\[
\Gamma_{z_*}(x) - \Gamma_z(x) \leq a_3 \rho \frac{1}{1 \lor |x|^{d-4}}. \tag{6.10}
\]

Proof The fact that (6.9) holds for simple random walk is just the standard decay for the critical two-point function \([23]\). For weakly self-avoiding walk (6.9) follows from (1.13).

For (6.10), we first claim that 
\[
z \frac{d}{dz} \Gamma_z(x) \leq (\Gamma_z \ast \Gamma_z)(x). \tag{6.11}
\]

To prove this for simple random walk, we use the definition of \(C_z(x)\) in (1.3) to see that 
\[
z \frac{d}{dz} C_z(x) = \sum_{n=1}^{\infty} \sum_{\omega \in W_n(x)} n \rho^n \leq \sum_{n=0}^{\infty} \sum_{\omega \in W_n(x)} \sum_{m=0}^{n} z^{n-m} z^m, \tag{6.12}
\]
and then divide the walk \(\omega\) into subwalks of lengths \(m\) and \(n - m\) to factor the right-hand side as a convolution. The same technique applies to weakly self-avoiding walk, with the additional step of using the inequality
\[
\prod_{0 \leq s < t \leq n} (1 + \beta U_{st}) \leq \prod_{0 \leq s < t_1 \leq m} (1 + \beta U_{s_1 t_1}) \prod_{m \leq s_2 < t_2 \leq n} (1 + \beta U_{s_2 t_2}). \tag{6.13}
\]

7 The mechanism in this proof applies more generally—it is not necessary that the decay be exponential as in (6.5). For example, for a random walk on \(\mathbb{Z}^d\) whose step distribution is given by a fractional power \(-\Delta)^{\alpha/2}\) of the discrete Laplacian (a step from 0 to \(x\) has probability \(\frac{1}{c} |x|^{-(d+\alpha)}\)), the bound (6.5) is replaced, for \(d \geq 1, \alpha \in (0, \min\{2, d\})\), \(m^2 \in [0, 1]\), and \(x \neq 0\), by (see [35, Sect. 2.1])
\[
((\Delta)^{\alpha/2} + m^2)^{\alpha-1}_{0x} \leq \frac{1}{c |x|^{d-\alpha}}. \tag{6.14}
\]
The steps in (6.6)–(6.7) now give an upper bound \(r^{-d} m^{-2}\), which equals \(r^{-d} \chi\) since \(\sum_{x \in \mathbb{Z}^d} ((\Delta)^{\alpha/2} + m^2)^{\alpha-1}_{0x} = m^{-2}\).
to bound the interaction for $\omega$ by a product of interactions for the two subwalks. This proves (6.11), and then monotonicity in $z$, the upper bound of (6.9), and the elementary convolution estimate\(^8\) [17, Proposition 1.7(i)] give

$$z \frac{d}{dz} \Gamma_z(x) \leq (\Gamma_{z*} \ast \Gamma_{z*})(x) \leq c_0 \frac{1}{1 \vee |x|^{d-4}}.$$  \hfill (6.14)

Integration of (6.14) (using the assumed lower bound on the factor $z$) gives (6.10), and the proof is complete. \hfill \Box

**Remark 6.2** We can now prove the claim in (1.23). The upper bound follows by bounding $\Gamma_z(x)$ by $\Gamma_{z*}(x)$ and using the upper bound of (6.9). For the lower bound of (1.23), we take $\rho \leq \epsilon r^{-2}$ with $\epsilon = \frac{1}{2} a_3^{-1} a_4$ and apply Lemma 6.1 to see that

$$\Gamma_z(x) = \Gamma_{z*}(x) - (\Gamma_{z*}(x) - \Gamma_z(x)) \geq a_4 \frac{1}{1 \vee |x|^{d-2}} - a_3 \epsilon \frac{1}{r^2} \frac{1}{1 \vee |x|^{d-4}} \geq \frac{1}{2} a_4 \frac{1}{1 \vee |x|^{d-2}},$$ \hfill (6.15)

where we used $|x|^2 \leq dr^2/4$ in the last step. The constant $c_3$ in (1.22)–(1.23) can be taken to be at most $\epsilon$.

For the lower bound in dimensions $d > 4$, we first consider the easier case of simple random walk. Its proof is used also for weakly self-avoiding walk.

**Proof of lower bound in Theorem 1.4 for $d > 4$**. Let $d > 4$ and $x \in \mathbb{T}_r^d$, and suppose that $\rho \leq c_3' r^{-2}$, with the constant $c_3'$ to be chosen in the proof. Since $\chi_0(z) = (1 - z \Omega)^{-1} = (\Omega \rho)^{-1}$, by (6.3) it suffices to prove for some $c > 0$ that

$$\sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru) \geq \frac{c}{r^d \rho}.$$ \hfill (6.16)

For a lower bound, we only sum over $|u| \leq L$ with (large) $L$ depending on $r$, $\rho$ to be chosen later in the proof. By (6.9)–(6.10), for $y \neq 0$,

$$C_z(y) = C_{z*}(y) - (C_{z*}(y) - C_z(y)) \geq a_4 \frac{1}{|y|^{d-2}} - a_3 \rho \frac{1}{|y|^{d-4}}.$$ \hfill (6.17)

Therefore, by (6.1)–(6.2),

$$\sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru) \geq \sum_{0 < |r| \leq L} C_z(x + ru) \geq c_0 \sum_{0 < |u| \leq L} \frac{1}{|ru|^{d-2}} - c_0' \rho \sum_{0 < |u| \leq L} \frac{1}{|ru|^{d-4}}$$

---

\(^8\) The estimate implies that the convolution with itself of a function bounded by $|x|^{-(d-2)}$ is bounded by $|x|^{-(d-4)}$ if $d > 4$.  

\[ Springer\]
\begin{align}
\geq c_1 \frac{1}{r^{d-2}} L^2 - c'_1 \rho \frac{1}{r^{d-4}} L^4 \\
= c_1 \frac{1}{r^{d-2}} L^2 \left( 1 - c_2 \rho r^2 L^2 \right). \tag{6.18}
\end{align}

We choose $L^2 = (2c_2 \rho r^2)^{-1}$, and then $L^2 \geq (2c_2 c_3')^{-1}$ is large if we take $c_3'$ to be small. This choice gives
\begin{equation}
\sum_{u \in \mathbb{Z}^d : u \neq 0} C_z(x + ru) \geq \frac{1}{2} c_1 \frac{L^2}{r^{d-2}} = \frac{c_1}{4c_2 r^d \rho}, \tag{6.19}
\end{equation}
which proves (6.16) and completes the proof.

**Proof of lower bound of Theorem 1.2** We consider dimensions $d > 4$ and small $\beta$, and $z$ such that $\rho = z_c - z$ obeys $c_4 \beta^{1/2} r^{-d/2} \leq \rho \leq c_3 r^{-2}$ with $c_3$ equal to $c_3'$ from the previous proof and with $c_4$ to be chosen at the end of the proof. We seek a lower bound of the form $r^{-d} \chi$ for the difference
\begin{equation}
\psi_{r,z}(x) = G_{z}(x) - G_{z}(x), \quad (x \in \mathbb{T}_r^d). \tag{6.20}
\end{equation}

A torus walk to $x$ unfolds to a walk on $\mathbb{Z}^d$ ending at $x$ or at a point $x + ru$ with $u$ a nonzero point in $\mathbb{Z}^d$. The weight associated to the unfolded walk, as a weakly self-avoiding walk on $\mathbb{Z}^d$, can be larger than the weight of the original torus walk which is penalised by visits of its unfolding to distinct points in $\mathbb{Z}^d$ with the same projection to the torus. Without this penalty, the unfolded walks would have weight
\begin{equation}
\psi_{r,z}(x) = \sum_{u \in \mathbb{Z}^d : u \neq 0} G_{z}(x + ru) \quad (x \in \mathbb{T}_r^d). \tag{6.21}
\end{equation}

Exactly as in the proof of the lower bound of Theorem 1.4, $\psi_{r,z}(x)$ is bounded below by a multiple of $r^{-d} \chi$ if $\rho \geq c_3 r^{-2}$, since that proof only used (6.9)–(6.10). For later reference, note that in the proof of the upper bound of Theorem 1.2 we in fact obtained an upper bound on $\psi_{r,z}(x)$ without using this particular notation, and hence (provided $\rho \leq c_3 r^{-2}$ for the lower bound),
\begin{equation}
cc r^{-d} \chi(z) \leq \psi_{r,z}(x) \leq c'r^{-d} \chi(z), \quad (x \in \mathbb{T}_r^d). \tag{6.22}
\end{equation}

We make the decomposition
\begin{equation}
\psi_{r,z}^T(x) = \psi_{r,z}(x) - (\psi_{r,z}(x) - \psi_{r,z}^T(x)). \tag{6.23}
\end{equation}

By the lower bound of (6.22), it suffices to show that the subtracted term, which by (6.4) is nonnegative, obeys
\begin{equation}
\psi_{r,z}(x) - \psi_{r,z}^T(x) \leq \frac{1}{2} cc r^{-d} \chi(z), \quad (x \in \mathbb{T}_r^d), \tag{6.24}
\end{equation}
with $c$ the constant of (6.22).
Let $\pi_r : \mathbb{Z}^d \to \mathbb{T}_r^d$ be the canonical projection map onto the torus $\mathbb{T}_r^d = (\mathbb{Z}/r\mathbb{Z})^d$. In the following, all walks are on $\mathbb{Z}^d$. Given an $n$-step walk $\omega$ and $0 \leq s < t \leq n$, we define

$$U_{st}^\pi(\omega) = \begin{cases} -1 & (\pi_r\omega(s) = \pi_r\omega(t)) \\ 0 & \text{(otherwise)} \end{cases} \quad (6.25)$$

$$U_{st}(\omega) = \begin{cases} -1 & (\omega(s) = \omega(t)) \\ 0 & \text{(otherwise)} \end{cases} \quad (6.26)$$

$$U_{st}^+(\omega) = \begin{cases} -1 & (\pi_r\omega(s) = \pi_r\omega(t) \text{ and } \omega(s) \neq \omega(t)) \\ 0 & \text{(otherwise)} \end{cases} \quad (6.27)$$

as well as

$$K_T(\omega) = \prod_{0 \leq s < t \leq n} (1 + \beta U_{st}^\pi(\omega)), \quad (6.28)$$

$$K(\omega) = \prod_{0 \leq s < t \leq n} (1 + \beta U_{st}(\omega)), \quad (6.29)$$

$$K^+(\omega) = \prod_{0 \leq s < t \leq n} (1 + \beta U_{st}^+(\omega)). \quad (6.30)$$

Note that $K_T(\omega) = K(\omega)K^+(\omega)$. By definition,

$$\psi_{r,z}(x) = \sum_{u \neq 0} \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x+ru)} z^n K(\omega) \quad (x \in \mathbb{T}_r^d), \quad (6.31)$$

$$\psi_{r,z}^T(x) = \sum_{u \neq 0} \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x+ru)} z^n K_T(\omega) \quad (x \in \mathbb{T}_r^d), \quad (6.32)$$

and hence

$$\psi_{r,z}(x) - \psi_{r,z}^T(x) = \sum_{u \neq 0} \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x+ru)} z^n K(\omega)[1 - K^+(\omega)]. \quad (6.33)$$

The inequality

$$1 - \prod_{a \in A} (1 - u_a) \leq \sum_{a \in A} u_a \quad (u_a \in [0, 1]) \quad (6.34)$$
follows by induction on the cardinality of the set $A$. We apply it to $1 - K^+$ with $u_a = \beta |U_{st}^+(\omega)|$. The result is

$$
\psi_{r, z}(x) - \psi_{r, z}^T(x) \leq \beta \sum_{u \neq 0} \sum_{n=0}^{\infty} \sum_{0 \leq s < t \leq n} z^n \sum_{\omega \in \mathcal{W}_n(x+ru)} K(\omega)|U_{st}^+(\omega)|, \tag{6.35}
$$

The factor $U_{st}^+(\omega)$ is zero unless $\omega$ visits distinct points at times $s, t$ that project to the same torus point. We call two such points $y$ and $y + rv$ with $v \neq 0$, which entails that $\omega$ travel from 0 to $y$, from $y$ to $y + rv$, and from $y + rv$ to $x + ru$. We can therefore decompose $\omega$ as the concatenation of three subwalks, respectively $\omega_1 \in \mathcal{W}_{n_1}(y)$, $\omega_2 \in \mathcal{W}_{n_2}(rv)$, and $\omega_3 \in \mathcal{W}_{n_3}(x - y + r(u - v))$, of lengths $n_1 = s, n_2 = t - s$, and $n_3 = n - t$. Also $z^n = z^{n_1}z^{n_2}z^{n_3}$. The sums over $n, s, t$ in (6.35) are equivalent to summing over $n_1 \geq 0, n_2 \geq 1, n_3 \geq 0$, and $K(\omega)$ is bounded above by the product $K(\omega_1)K(\omega_2)K(\omega_3)$. After using this last bound (and relaxing the condition that $u$ be nonzero), (6.35) is bounded above by three independent sums over walks, which leads to an upper bound

$$
\beta \sum_{y \in \mathbb{Z}^d} G_z(y) \sum_{v \in \mathbb{Z}^d, v \neq 0} G_z(rv) \sum_{u \in \mathbb{Z}^d} G_z(x - y + r(u - v)) = \beta \psi_{r, z}(0) \sum_{y, w \in \mathbb{Z}^d} G_z(y)G_z(x - y + rw). \tag{6.36}
$$

To estimate the right-hand side of (6.36) for $x \in \mathbb{T}^d_r$, we proceed as follows. The constant $C$ in this paragraph can change value from one occurrence to the next. For fixed $w$, by (1.7) and by $|y - (x + rw)| \geq |x + rw| - |y|$ we have

$$
\sum_{y \in \mathbb{Z}^d} G_z(y)G_z(x - y + rw) \leq C \sum_{y \in \mathbb{Z}^d} \frac{1}{1 \vee |y|^{d-2}} e^{-c_1 m(z)|y|} \times \frac{1}{1 \vee |y - (x + rw)|^{d-2}} e^{-c_1 m(z)|y - (x + rw)|} \leq Ce^{-c_1 m(z)|x + rw|} \sum_{y \in \mathbb{Z}^d} \frac{1}{1 \vee |y|^{d-2}} \times \frac{1}{1 \vee |y - (x + rw)|^{d-2}} \leq C e^{-c_1 m(z)|x + rw|} \frac{1}{1 \vee |x + rw|^{d-4}}, \tag{6.37}
$$

with the last inequality a consequence of the convolution estimate [17, Proposition 1.7(i)]. The $w = 0$ term is bounded by a constant. For the rest of the sum over $w,$
As in the proof of the upper bound in Sect. 6.1, we use

$$
\sum_{w \neq 0} \frac{1}{1 \vee |x + rw|^{d-4}} e^{-c_1 m(z)|x + rw|} \leq C \sum_{w \neq 0} \frac{1}{|rw|^{d-4}} e^{-c_1 m(z)|rw|}
$$

$$
\leq C \int_{\mathbb{R}^d} \frac{1}{|ra|^{d-4}} e^{-c_1 m(z)|ra|} da
$$

$$
= C \frac{1}{m^4 r^d} \leq C \frac{\chi(z)^2}{r^d}.
$$

(6.38)

With the $w = 0$ term and the upper bound of (6.22), this leads to

$$
\psi_{r,z}(x) - \psi_{r,z}^T(x) \leq \text{const} \frac{\chi(z)}{r^d} \left( \beta + \beta \frac{\chi(z)^2}{r^d} \right).
$$

(6.39)

This proves (6.24) provided that $\beta$ and $\beta r^{-d} \chi(z)^2$ are sufficiently small. Since $\chi(z) \approx \rho^{-1}$, it is sufficient if $\beta r^{-d} \rho^2$ is sufficiently small, i.e., if $ho^2 \geq c^2_4 \beta r^{-d}$ for some large $c_4$. We have assumed this last inequality as a hypothesis for this reason. This concludes the proof of (6.24) and therefore completes the proof. □

Finally, we prove the lower bound in the remaining dimensions $d = 3, 4$ for simple random walk.

**Proof of lower bound of Theorem 1.4 for $d = 3, 4$.** As in the proof for $d > 4$, we seek a lower bound on $\sum_{u \neq 0} C_z(x + ru)$. We again write $z_* = \frac{1}{\Omega}$ and $\rho = z_* - z$. The lower bound is claimed to hold for $\rho \leq O(r^{-2})$, so we are considering small $\rho$ here.

We first prove that

$$
C_{z_\ast}(x) - C_z(x) \leq \text{const} \times \begin{cases} 
\rho^{1/2} & (d = 3) \\
|\rho| \log |\rho| & (d = 4).
\end{cases}
$$

(6.40)

By (2.10), the left-hand side of (6.40) can be written as the Fourier integral

$$
C_{z_\ast}(x) - C_z(x) = \int_{\mathbb{T}^d} e^{-ik \cdot x} \left( \frac{1}{1 - \hat{D}(k)} - \frac{1}{1 - z \Omega \hat{D}(k)} \right) \frac{dk}{(2\pi)^d}
$$

$$
= \Omega \rho \int_{\mathbb{T}^d} e^{-ik \cdot x} \frac{\hat{D}(k)}{[1 - \hat{D}(k)][1 - z \Omega \hat{D}(k)]} \frac{dk}{(2\pi)^d}.
$$

(6.41)

In the denominator, $1 - \hat{D}(k)$ is bounded below by a multiple of $|k|^2$, while $1 - z \Omega \hat{D}(k) = \Omega \rho + z \Omega [1 - \hat{D}(k)]$ is bounded below by a multiple of $\rho + |k|^2$ (since $z \Omega$ is bounded below because we are considering small $\rho$). The right-hand side of (6.41) is therefore bounded above by a multiple of

$$
\rho \int_{\mathbb{T}^d} \frac{1}{|k|^2(\rho + |k|^2)} dk = \rho^{(d-2)/2} \int_{\rho^{-1/2} \mathbb{T}^d} \frac{1}{|l|^2(1 + |l|^2)} dl.
$$

(6.42)

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For \( d = 3 \) this last integral is bounded as \( \rho \to 0 \), while for \( d = 4 \) it is \( O(|\log \rho|) \). This proves (6.40).

For simple random walk, (6.9) continues to hold in dimensions \( d = 3, 4 \) [23]. For \( d = 3 \) and \( y \neq 0 \), by (6.40) we therefore have

\[
C_z(y) = C_{z_0}(y) - (C_{z_0}(y) - C_z(y)) \geq a_4 \frac{1}{|y|} - c_1 \rho^{1/2},
\]  

so, for large \( L \), by (6.2),

\[
\sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru) \geq \sum_{0 <|u| \leq L} C_z(x + ru) \geq c_0' \sum_{0 <|ru| \leq L} \frac{1}{|ru|} - c_1 \rho^{1/2} L^3 \\
\geq c_0'' \frac{1}{r} L^2 - c_1 \rho^{1/2} L^3 = \frac{1}{2} c_0'' \frac{1}{r^3 \rho},
\]  

where we took \( L = c_0'' (2 c_1 \rho^{1/2} r)^{-1} \) in the last step. This \( L \) is large provided that our assumption \( \rho \leq c_3' r^{-2} \) holds with \( c_3' \) chosen sufficiently small. By (6.3), this proves the desired lower bound (1.28) for \( d = 3 \).

Finally, for \( d = 4 \), a similar computation gives

\[
C_z(y) \geq a_4 \frac{1}{|y|^2} - c_1 \rho |\log \rho|,
\]  

so

\[
\sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru) \geq c_0'' \frac{1}{r^2 \rho} L^2 - c_1 \rho |\log \rho| L^4 = c_0'' \frac{1}{r^2 \rho} L^2 (1 - c_1' \rho |\log \rho| r^2 L^2).
\]  

Now we choose \( L^2 = (2 c_1'' \rho |\log \rho| r^2)^{-1} \) to obtain

\[
\sum_{u \in \mathbb{Z}^d: u \neq 0} C_z(x + ru) \geq c_0'' \frac{1}{r^4 \rho |\log \rho|}.
\]  

The above assumes that \( L \) is large, which is true by our assumption that \( \rho |\log \rho| r^2 \) is sufficiently small. This completes the proof. □

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Declarations

Conflicts of interest  There are no financial and non-financial conflicts of interest.

References

1. Barlow, M.T.: Random Walks and Heat Kernels on Graphs. Cambridge University Press, Cambridge (2017)
2. Bauerschmidt, R., Brydges, D.C., Slade, G.: Critical two-point function of the 4-dimensional weakly self-avoiding walk. Commun. Math. Phys. 338, 169–193 (2015)
3. Bolthausen, E., van der Hofstad, R., Kozma, G.: Lace expansion for dummies. Ann. I. Henri Poincaré Probab. Statist. 54, 141–153 (2018)
4. Borgs, C., Chayes, J.T., van der Hofstad, R., Slade, G., Spencer, J.: Random subgraphs of finite graphs: I. The scaling window under the triangle condition. Random Struct. Alg. 27, 137–184 (2005)
5. Borgs, C., Chayes, J.T., van der Hofstad, R., Slade, G., Spencer, J.: Random subgraphs of finite graphs: II. The lace expansion and the triangle condition. Ann. Probab. 33, 1886–1944 (2005)
6. Brydges, D.C., Helmuth, T., Holmes, M.: The continuous-time lace expansion. Commun. Pure Appl. Math. 74, 2251–2309 (2021)
7. Brydges, D.C., Spencer, T.: Self-avoiding walk in 5 or more dimensions. Commun. Math. Phys. 97, 125–148 (1985)
8. Camia, F., Jiang, J., Newman, C.M.: The effect of free boundary conditions on the Ising model in high dimensions. Probab. Theory Related Fields 181, 311–328 (2021)
9. Chatterjee, S., Hanson, J.: Restricted percolation critical exponents in high dimensions. Commun. Pure Appl. Math. 73, 2370–2429 (2020)
10. Chayes, J.T., Chayes, L.: Ornstein–Zernike behavior for self-avoiding walks at all noncritical temperatures. Commun. Math. Phys. 105, 221–238 (1986)
11. Deng, Y., Garoni, T.M., Grimm, J., Nasrawi, A., Zhou, Z.: The length of self-avoiding walks on the complete graph. J. Stat. Mech: Theory Exp. 1, 103206 (2019)
12. Grafakos, L.: Classical Fourier Analysis, 3rd edn. Springer, New York (2014)
13. Grimm, J.: Finite-size effects in high dimensional physical systems. PhD thesis, Monash University (2018). https://doi.org/10.26180/5b9978f9dd0b6
14. Grimmett, G.: Percolation, 2nd edn. Springer, Berlin (1999)
15. Hara, T.: Mean field critical behaviour for correlation length for percolation in high dimensions. Probab. Theory Related Fields 86, 337–385 (1990)
16. Hara, T.: Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. Ann. Probab. 36, 530–593 (2008)
17. Hara, T., van der Hofstad, R., Slade, G.: Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. Ann. Probab. 31, 349–408 (2003)
18. Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. I. The critical behaviour. Commun. Math. Phys. 147, 101–136 (1992)
19. Heydenreich, M., van der Hofstad, R.: Progress in High-Dimensional Percolation and Random Graphs. Springer, Cham (2017)
20. van der Hofstad, R., Sapozhnikov, A.: Cycle structure of percolation on high-dimensional tori. Ann. Inst. H. Poincaré Probab. Statist. 50, 999–1027 (2014)
21. Hutchcroft, T., Michta, E., Slade, G.: High-dimensional near-critical percolation and the torus plateau. Ann. Probab. 51, 580–625 (2023)
22. Itzykson, C., Drouffe, J.-M.: Statistical Field Theory, vol. I. Cambridge University Press, Cambridge (1989)
23. Lawler, G.F., Limic, V.: Random Walk: A Modern Introduction. Cambridge University Press, Cambridge (2010)
24. Levin, D.A., Peres, Y.: Markov Chains and Mixing Times, 2nd edn. American Mathematical Society, Providence, RI (2017)
25. Lundow, P.H., Markström, K.: The scaling window of the 5D Ising model with free boundary conditions. Nucl. Phys. B 911, 163–172 (2016)
26. Madras, N., Slade, G.: The Self-Avoiding Walk. Birkhäuser, Boston (1993)
27. Michta, E.: The scaling limit of the weakly self-avoiding walk on a high-dimensional torus. Preprint https://arxiv.org/pdf/2203.07695 (2022)
28. Michta, E., Park, J., Slade, G.: Boundary conditions and universal finitesize scaling for the hierarchical $|\phi|^4$ model in dimensions 4 and higher.
29. Michta, E., Slade, G.: Weakly self-avoiding walk on a high-dimensional torus. Preprint https://arxiv.org/pdf/2107.14170 (2021). To appear in Probab. Math. Phys.
30. Michta, E., Slade, G.: Asymptotic behaviour of the lattice Green function. ALEA Lat. Am. J. Probab. Math. Stat. 19, 957–981 (2022)
31. Papathanakos, V.: Finite-Size Effects in High-Dimensional Statistical Mechanical Systems: The Ising Model with Periodic Boundary Conditions. PhD thesis, Princeton University (2006)
32. Sakai, A.: Lace expansion for the Ising model. Commun. Math. Phys. 272, 283–344 (2007). Correction: Sakai, A.: Correct bounds on the Ising lace-expansion coefficients. Commun. Math. Phys. 392, 783–823 (2022)
33. Sakai, A.: Application of the lace expansion to the $\phi^4$ model. Commun. Math. Phys. 336, 619–648 (2015)
34. Slade, G.: The Lace Expansion and its Applications. Springer, Berlin (2006). Lecture Notes in Mathematics Vol. 1879. Ecole d’Été de Probabilités de Saint–Flour XXXIV–2004
35. Slade, G.: Critical exponents for long-range $O(n)$ models below the upper critical dimension. Commun. Math. Phys. 358, 343–436 (2018)
36. Slade, G.: Self-avoiding walk on the complete graph. J. Math. Soc. Japan 72, 1189–1200 (2020)
37. Slade, G.: A simple convergence proof for the lace expansion. Ann. I. Henri Poincaré Probab. Statist. 58, 26–33 (2022)
38. Wittmann, M., Young, A.P.: Finite-size scaling above the upper critical dimension. Phys. Rev. E 90, 062137 (2014)
39. Zhou, Z., Grimm, J., Deng, Y., Garoni, T.M.: Random-length random walks and finite-size scaling on high-dimensional hypercubic lattices I: Periodic boundary conditions. Preprint https://arxiv.org/pdf/2008.00913 (2020)
40. Zhou, Z., Grimm, J., Fang, S., Deng, Y., Garoni, T.M.: Random-length random walks and finite-size scaling in high dimensions. Phys. Rev. Lett. 121, 185701 (2018)

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