Global weak solutions for a $2 \times 2$ balance nonsymmetric system of Keyfitz-Kranzer type

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Abstract

In this paper we consider the existence of global weak entropy solutions for a particular nonsymmetric Keyfitz-Kranzer type system, by using the compensated compactness method we get bounded entropy weak solutions.

1 introduction

In this chapter we consider the balanced nonsymmetric system

\begin{equation}
\begin{aligned}
\rho_t + \left( \rho \phi(\rho, w) \right)_t &= f(\rho, w), \\
(\rho w)_t + \left( \rho w \phi(\rho, w) \right)_x &= g(\rho, w)
\end{aligned}
\end{equation}

where $\phi(\rho, w) = \Phi(w) - P(\rho)$, $\Phi$ a convex function. This system was considered in \cite{2} where the author showed the existence of global weak solution for the homogeneous system \cite{1}. Another system of the type \cite{1} was considered in \cite{6} as a generalization to the scalar Buckley-Leverett equations describing two phase flow in porous media. The system \cite{1}, recently, has been object of constant studies, in \cite{1} the author considered the particular case in which $\Phi(w) = w$,
\( P(\rho) = \frac{1}{\rho}, \) in this case the two characteristics of the system are linear degenerate, solving the Riemann problem the existence and uniqueness of delta shock solution were established. In this line in the authors considered the case \( \Phi(w) = w \) and \( P(\rho) = \frac{B}{\rho^\alpha} \) with \( \alpha \in (0, 1) \), the existence and uniqueness of solutions to the the Riemann problem was got by solving the Generalize Rankine-Hugoniot condition. In both cases, when \( \Phi(s) = s \) the system models vehicular traffic flow in a highway without entry neither exit of cars, in this case the source term represents the entry or exit of cars see \( [4], [5] \) and reference therein for more detailed description of source term.

Noticing that when \( w \) is constant, the system reduces to the scalar balance laws
\[
\rho_t + (\rho \Phi(w) - \rho P(\rho))_x = f(\rho, w),
\]
and from the second equation in \( g \) should be of the form
\[wg = f.\]

Moreover, if we make \( h(\rho) = \rho \Phi(w) - P(\rho) \) the global weak solution of the Cauchy problem
\[
\begin{cases}
\rho_t + h(\rho)_x = f(\rho), \\
\rho(x, 0) = \rho_0(x)
\end{cases}
\]
there exists if \( h(\rho) \) is a convex function and the source term is dissipative, i.e
\[
h''(\rho) = -(2P'(\rho) + \rho P''(\rho)) > 0
\]
\[sf(s) \leq 0.
\]

We assume the following conditions,

\[C_1\] \( f, g \) are Lipschitz functions such that
\[wg(\rho, w) = f(\rho, w), \quad f(0, 0) = 0\]
\[C_2\] There exist a constant \( M > 0 \) such that
\[sf(s) \leq 0, \quad \text{for } |s| > M\]
\[C_3\] The function \( P(\rho) \) satisfies
\[P(0) = 0, \quad \lim_{\rho \to 0} \rho P'(\rho) = 0, \quad \lim_{\rho \to \infty} P(\rho) = \infty\]
\[\rho P''(\rho) + 2P'(\rho) < 0, \quad \text{for } \rho > 0\]

Remark 1.1. By example if \( f(\rho, m) = \rho \), then \( g(\rho, m) = \rho w \) in this case we have the nonsymmetric system with lineal damping.
Making $m = pw$, system (1) can be transformed in a symmetric system

$$
\begin{aligned}
\rho_t + (\rho \phi (\rho, m)) &= f(\rho, m), \\
m_t + (\rho w \phi (\rho, m)) &= g(\rho, m)
\end{aligned} \quad (10)
$$

for this system. Making $F(\rho, m) = (\rho \phi (\rho, m), m \phi (\rho, m))$, then

$$
dF = \begin{pmatrix}
\phi + \rho \phi & \rho \phi_m \\
m \phi & \phi + m \phi_m
\end{pmatrix},
$$

so the eigenvalues and eigenvector of $dF$ are given by

$$
\lambda_1(\rho, m) = \Phi \left( \frac{m}{\rho} \right) - P(\rho) \quad r_1 = (1, -\frac{\rho \phi_m}{\phi_m}) \quad (11)
$$

$$
\lambda_2(\rho, m) = \Phi \left( \frac{m}{\rho} \right) - \rho P' (\rho) \quad r_2 = (1, \frac{m}{\rho}) \quad (12)
$$

From (11), (12) the k-Riemann invariants are given by

$$
\begin{aligned}
W(\rho, m) &= \Phi \left( \frac{m}{\rho} \right) - P(\rho), \\
Z(\rho, m) &= \frac{m}{\rho}.
\end{aligned} \quad (13)
$$

Moreover

$$
\nabla \lambda_1 \cdot r_1 = 0, \quad (14)
\nabla \lambda_2 \cdot r_2 = 2 P'(\rho) + \rho P''(\rho). \quad (15)
$$

By $C_3$ condition, the system (18) is linear degenerate in the first characteristic field, non linear degenerate in the second characteristic field and non strictly hyperbolic. In this paper we obtain the main following theorem

**Theorem 1.2.** Let the initial data

$$
\rho(x, 0) = \rho_0(x), w(x, 0) = w_0(x) \in L^\infty(\Omega), \quad (16)
$$

whit $\rho_0(x) \geq 0$. The total variation of the Riemann invariants $W_0(x)$ be bounded, and the conditions $C_1, C_2, C_3$ holds, then the Cauchy problem (1), (16) has a global bounded weak entropy solution and $w_x(x, t)$ is bounded in $L^1(\mathbb{R})$. Moreover for $w$ costant $p$ is the global weak solution of the scalar balance laws

$$
\rho_t + h(\rho)x = f(\rho), \quad (17)
$$

where $h(\rho) = \Phi(w)\rho - \rho P(\rho), f(\rho) = f(\rho, w)$. 

3
2 A priori bounds and existence

In order to get weak solutions, in this section we investigate the problem of the existence of the solutions for the parabolic regularization to the system

\[
\begin{align*}
\rho_t + (\rho \phi(\rho, w)) &= \epsilon \rho_{xx} + f(\rho, w), \\
(\rho w)_t + (\rho w \phi(\rho, w)) &= \epsilon (\rho w)_{xx} + g(\rho, w)
\end{align*}
\]

with initial data

\[
\rho(x, 0) = \rho_0(x) + \epsilon, \quad w(x, 0) = w_0(x).
\]

We consider the transformation \( m = \rho w \), replacing in (18) we have

\[
\begin{align*}
\rho_t + (\rho \phi(\rho, m)) &= \epsilon \rho_{xx} + f(\rho, m), \\
m_t + (m \phi(\rho, m)) &= \epsilon m_{xx} + g(\rho, m),
\end{align*}
\]

with initial data

\[
\rho(x, 0) = \rho_0(x), \quad m(x, 0) = \frac{w_0(x)}{\rho_0(x)}.
\]

**Proposition 2.1.** Let \( \epsilon > 0 \) be, the Cauchy problem (18), (19), has a unique solution for any \((\rho_0, w_0)\). Moreover if \((\rho_0, w_0)\) is the solutions \((\rho, m)\) satisfies

\[
0 < c \leq \rho(x, t) \leq M, \quad \frac{|m(x, t)|}{\rho(x, t)} \leq M
\]

The proof of this theorem is postponed at the end of the section. We begin with some lemmas that will be useful afterward.

Let \( U = (\rho, m)^T \), \( H(U) = (f(U), g(U)) \) and \( M = DF \) where \( F(\rho, m) = (\rho \phi, m \phi) \). Then the system (20) can be written in the form

\[
U_t = \epsilon U_{xx} + MU_x + H.
\]

For any \( C_1, C_2 \) constants let

\[
\begin{align*}
G_1 &= C_1 - W, \\
G_2 &= Z - C_2,
\end{align*}
\]

where \( W, Z \) are the Riemann invariants given in (13). We proof that the region

\[
\Sigma = \{ (\rho, m) : G_1 \leq 0, G_2 \leq 0 \}
\]

is an invariant region.

**Lemma 2.2.** If \( \rho \in C^{1,2}([0, T] \times \mathbb{R}) \) satisfies

\[
\rho_t + (\rho \phi(w, \rho))_x = f(\rho, w),
\]

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with \( \rho(0, \cdot) \geq 0 \) and \( w \in C^1(\Omega) \) then \( \rho(t, \cdot) \geq 0 \), moreover if \( \rho(0, \cdot) \geq \delta > 0 \)

\[
\int_0^T \int_{-\infty}^{\infty} \rho |w - w_0| \, dx \, dt < K
\]  

with \( k \) constant, then \( \rho(x, t) \leq \delta(\epsilon, T) > 0 \) in \((0, T)\).

**Lemma 2.3.** The function \( G_1, G_1 \) defined in (24), (25), are quasi-convex.

**Proof.** Let \( r = (X, Y) \) be a vector. If \( r \cdot \nabla G_1 = 0 \) then \( Y = X (\frac{m}{\rho} + \frac{P'(\rho)}{\Phi'(w)}) \)
thus

\[
\nabla^2 G_1(r, r) = X^2 \left( -\frac{1}{\rho} (2P'(\rho + \rho P''(\rho)) + \Phi''(w)(\frac{P'(\rho)}{\Phi'(w)})^2 \right).
\]

If \( r \cdot \nabla G_2 = 0 \) then \( Y = \frac{m}{\rho} X \), thus we have

\[
\nabla^2 G_2(r, r) = 0.
\]

\[\square\]

**Lemma 2.4.** If the condition \( C_1 \) holds then \( G_1, G_2 \) satisty

\[
\nabla G_1 \cdot H \leq 0, \quad (29)
\]

\[
\nabla G_2 \cdot H \leq 0, \quad (30)
\]

**Proof.** From (24) and (25), we have that \( \nabla G_1 = (\frac{\Phi'}{\rho} m - \frac{P'}{\rho}, \Phi' 1) \rho \rho \rho \)
and \( \nabla G_2 = (\frac{\Phi'}{\rho} m \rho, \Phi' 1 - \rho) \rho \rho \rho \)
then

\[
\nabla G_1 \cdot H = \frac{\Phi'}{\rho} m (-\frac{m}{\rho} f + g) + P' f \leq 0, \quad (31)
\]

\[
\nabla G_2 \cdot H = \frac{\Phi'}{\rho} m (-\frac{m}{\rho} f + g) \leq 0. \quad (32)
\]

\[\square\]

From the Theorem 1.3.1, the region \( \Sigma \) defined in (26) is an invariant region for the system (20). It follows from (24), (25) that

\[
C_1 \leq \Phi(\frac{m}{\rho}) - P(\rho),
\]

\[
\frac{m}{\rho} \leq C_2.
\]
then

\[
C_1 - \Phi(\rho(C_2) \leq P(\rho), \quad (33)
\]

we appropriately choose \( C_1, C_2 \) such that

\[
0 < \delta \leq \rho, \quad P(\rho) \leq \Phi(C_2) - C_1. \quad (34)
\]

By (33) we have the proof of Proposition 4.1.1.
3 Weak convergence

In this section we show that the sequence $(\rho^\varepsilon, m^\varepsilon)$ has a subsequence that converges the weak solutions to the system (20). For this we consider the following entropy-entropy flux pairs construct in [2] by the author

\[ \eta(\rho, m) = \rho F\left( \frac{m}{\rho} \right), \]

(35)

\[ q(\rho, m) = \rho F\left( \frac{m}{\rho} \right) \phi(\rho, m) \]

(36)

The Hessian matrix of $\eta$ is given by

\[ \nabla^2 \eta \left( F'^{''} \frac{m^2}{\rho^2} - F'^{''} \frac{m}{\rho} \frac{1}{\rho} \right) \]

then we have that

\[ \nabla^2 \eta(X, X) = F'^{''} \rho \frac{m}{\rho} (\rho_x - m_x)^2, \]

(37)

where $X = (\rho_x, m_x)$. If $(\eta, q)$ is an entropy-entropy flux pair, multiplying in (20) by $\nabla \eta(\rho, m)$ we have

\[ n_t + q_x = \epsilon \eta_{xx} - \epsilon \nabla^2 \eta(X, X) + \nabla \eta \cdot G(\rho, m). \]

(38)

Replacing the equation (37) in (38) we have

\[ n_t + q_x = \epsilon \eta_{xx} - \epsilon F'^{''} \rho \frac{m}{\rho} (\rho_x - m_x)^2 + \nabla \eta \cdot G(\rho, m). \]

(39)

Chose a function $\varphi \in C_0^\infty(\mathbb{R}^2_+)$ satisfying $\varphi = 1$ on $[-L, L] \times [0, T]$. multiplying (39) by $\varphi$ and integrate the result in $(\mathbb{R}^2_+)$ we have

\[ \int_\mathbb{R} (\eta \varphi)(x, T) - \int_\mathbb{R} (\eta \varphi)(x, 0) - \int_0^T \int_\mathbb{R} (\eta \varphi_t + q \varphi_x) dx dt \]

\[ = -\epsilon \int_0^T \int_\mathbb{R} F'^{''} \rho \frac{m}{\rho} (\rho_x - m_x)^2 + \epsilon \int_0^T \int_\mathbb{R} \eta \varphi_{xx} dx dt - \int_0^T \int_\mathbb{R} \nabla \eta \cdot G(\rho, m) dx dt. \]

From the Proposition 2.1 we have

\[ \epsilon \int_0^T \int_{-L}^L \frac{F'^{''}}{\rho} \rho_x (\rho_x - m_x)^2 dx dt \leq C. \]

(40)

As a consequence of the inequality (40) we have the following Lemma.

Lemma 3.1. For any $\epsilon > 0$, if $(\rho, m)$ is a solutions to the Cauchy problem (20), (21), then $\sqrt{\epsilon} \rho_x, \sqrt{\epsilon} m_x$ are bounded in $L^2_{loc}(\mathbb{R}^2_+) \subset M_{loc}$. 

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For any bounded set $\Omega \subset \mathbb{R}_+^2$, we have
\[
\|\varepsilon \eta_{x} x\|_{W^{-1, 2}(\Omega)} = \sqrt{\varepsilon} \|\eta\|_{L^{\infty}(\Omega)} \|\sqrt{\varepsilon} \varphi_{x}\|_{L^2(\Omega)} \sup_{\varphi} \|\varphi_{x}\|_{L^2} \to 0, \ \varepsilon \to 0. \tag{41}
\]
and
\[
\nabla \eta \cdot G(\rho, m) \in L^\infty(\Omega) \subset L^1(\Omega) \subset \mathcal{M}_{loc}. \tag{42}
\]

Lemma 3.2.
\[
g(\rho)_t + \left(\int_{0}^{\rho} g'(s)f'(s)ds + g(\rho)\Phi(w)\right)_x, \tag{43}
\]
\[
(\rho \Phi(w))_t + \left(\rho \Phi^2(w) + f(\rho)\Phi(w)\right)_x \tag{44}
\]
are compact in $H_{loc}^{-1}(\mathbb{R}_+^2)$. Particularly
\[
\rho_t + (\rho \Phi(w) - \rho P(\rho))_x \tag{45}
\]
are compact in $H_{loc}^{-1}(\mathbb{R}_+^2)$.

Proof. The proof of (44) is a consequence of the Lemma 3.1 and the inequalities (41), (42) and the Murat’s Lemma. Multiplying in (43) by $g'$ we have
\[
g(\rho)_t + \left(\int_{0}^{\rho} g'(s)f'(s)ds + g(\rho)\Phi(w)\right)_x = g(\rho)_x x - \varepsilon g''(\rho)\rho^2 x + \rho g'(\rho)\Phi(\rho)_x + g' f(\rho, w). \tag{46}
\]
By a similar argument in the inequality (41) we have
\[
\|\varepsilon g(\rho)_x x\|_{W^{-1, 2}} \to 0, \text{ as } \varepsilon \to 0.
\]
From the Lemma 3.1 the last term in (46) is in $\mathcal{M}_{loc}$. By the Murat’s Lemma we conclude the proof of (44). \qed

According to the Young’measures Theorem 1.2.1, there exists a probability measure $v_{x,t}$ associated with the bounded sequence $(\rho^\varepsilon, w^\varepsilon)$ such that for almost $(x, t)$, $v_{x,t}$ satisfies the following Tartar equation,
\[
\langle v_{x,t}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle v_{x,t}, \eta_1 q_2 \rangle - \langle v_{x,t}, \eta_2 q_1 \rangle \tag{47}
\]
for any entropy-entropy flux pair. Here $\langle v_{x,t}, f(\lambda) \rangle = \int_{\mathbb{R}_+^2} f(\lambda)dv_{x,t}(\lambda)$. We consider the following entropy-entropy flux pairs
\[
\eta_1 = \rho^\varepsilon, \quad q_1 = \rho^\varepsilon (\Phi(\rho^\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon, \tag{48}
\]
\[
\eta_2 = \rho^\varepsilon w^\varepsilon, \quad q_2 = \rho^\varepsilon w^\varepsilon (\Phi(\rho^\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon)^2. \tag{49}
\]
Noticing that $\eta_1 q_2 - \eta_2 q_1 = 0$ we have that
\[
\eta_1 q_2 - \eta_2 q_1 = 0,
\]
then
\[
\rho^\varepsilon \rho^\varepsilon w^\varepsilon (\Phi(\rho^\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon)^2 - \rho^\varepsilon w^\varepsilon \rho^\varepsilon (\Phi(\rho^\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon
\]
References

[1] Hongjun Cheng, *Delta shock waves for a linearly degenerate hyperbolic systems of conservation laws of keyfitz-kranzer type*, Advances in Mathematical Physics 2013 (2013), 1–10.

[2] Yun guang Lu, *Existence of global entropy solutions to general systems of keyfitz-kranzer type*, Journal of Functional Analysis 264 (2013), 2457–2468.

[3] Hanchun Yang Hongjun Cheng, *On a nonsymmetric keyfitz-kranzer system of conservation laws with generalized and modified chaplygin gas pressure law*, Advances in Mathematical Physics 2013 (2013), 14.

[4] Andrea Corli Patricia Bagnerini, Rinaldo M. Colombo, *On the role of source terms in continuum traffic flow models*, Mathematical and computer modelling 44 (2006), 917–930.

[5] Andrea Corli Rinaldo M. Colombo, *Well posedness for multiline traffic models*, Ann. Univ. Ferrara, Sez. VII (2006), 291–301.

[6] Blake Temple, *Global solutions of the cauchy problem for a class of $2 \times 2$ non-strictly hyperbolic conservation laws*, Advances in Applied Mathematics 3 (1982), 335–375.