Wages and Utilities in a Closed Economy- A Strategic Analysis

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The broad objective of this paper is to initiate through a mathematical model, the study of causes of wage inequality and relate it to choices of consumption and the technologies of production in an economy. The paper constructs a simple Heterodox Model of a closed economy, in which the consumption and the production parts are clearly separated and yet coupled through a tatonnement process. The equilibria of this process correspond directly with those of a related Arrow-Debreu model. The formulation allows us to identify the combinatorial data which link parameters of the economic system with its equilibria, in particular, the impact of consumer preferences on wages. The Heterodox model also allows the formulation and explicit construction of the consumer choice game, where individual utilities serve as the strategies with total or relative wages as the pay-offs. We illustrate, through two examples, the mathematical details of the consumer choice game. We show that consumer preferences, expressed through modified utility functions, does indeed percolate through the economy, and influences not only prices, but also production and wages. Thus, consumer choice may serve as an effective tool for wage redistribution.
1 INTRODUCTION
As the present millenial edition of the global economy unfolds, many authors and agencies have pointed out several undesirable features which have emerged. These are the paucity of "good" jobs, rising inequality, excessive consolidation, and the possibility of linkages between modern production and consumption processes with climate change.

This paper deals with the first two problems, viz., the allocation of jobs and wages and rising inequality in wage incomes in the economy, and its structural determinants. For this purpose, we build a simple mathematical model, called the Heterodox Model, which illustrates some of the key features of the dependence of wages on the production and consumption parts of the economy.

Our model has two parts, the production as determined by a technology matrix $T$, which utilizes $m$ labour classes and produces $n$ goods and determines quantities of goods produced, labour utilization and wages. This part assumes prices as a given, i.e., which cannot be changed. The second part is the consumption part, which is modelled as a Fisher market and a utility matrix $U$. This part of the economy assumes the production part as a given, i.e., wages and quantities of goods produced, and allocates goods based on wages (i.e., endowments or disposable incomes) held by each labour class, and determines prices.

This paper shows the connection between $U$, the utility matrix and the wages obtained by various labour classes, as implemented by $T$. In other words, it traces the connection between personal consumption choice, with prices of goods, their production and finally wages received. Next, it shows that having a "private and real" utility $U_r$, and posting or posturing a different $U$ into the economy does indeed alter wages and has the potential to improve both the social welfare as well as the relative welfare for certain classes. This sets up the consumer choice game, where the manipulation of $U$ is the strategy, and the relative or total welfare, as measured by the allocation of goods and their utilities according to $U_r$, are the pay-offs.

The manipulation of $U$ has been studied before, e.g., in connection with the impact of advertising on the competition between firms and their profitability. It has been used by pressure groups to label certain products, e.g., coffee, as "compliant" with a desirable idiom, e.g., fair wages to the coffee-bean picker. Our analysis is largely that of a closed economy and the study of its modes as functions of the parameters of the system. In classical terms, it is to develop and study, as a strategic game, the dependence of market equilibria on various parameters which define the economy. It is useful to point out earlier work on the Fisher Market Game [1],[5][10], where it was shown that going to the market with postured utilities $U$ (instead of the real ones, $U_r$) may indeed bring rewards in terms of more favourable allocations. However, that model relates only to the consumption side of the economy.

Our aim is complement this by a simple production model to define a closed economy completely, so as to be able to find a consistent set of prices of goods, wages for each labour class, allocations of goods amongst classes so that an equilibrium exists in the economy. Then, building on the Fisher market concepts, we model the effects of strategizing on consumer preferences on wages, production and allocations. This work extends the Fisher Market Game conclusion that to show that, indeed consumer choice may be used to change wages as well.

The model borrows from many existing models, specially so from the Arrow-Debreu model and its earlier cousin, the Fisher model, and philosophically, from Sraffa’s accounting methods for calculating prices and wages [7] and labour inventory using the theory of value, the marginal production principle for calculating wages, and finally the use of utility functions to compute allocation of goods.

The rest of the paper is organized as follows. In Section 2 we describe the Heterodox Model $\mathcal{H}(C,P)$ as composed of two interconnected systems, the consumption model $C$ and the production...
model $\mathcal{P}$. The consumption model $\mathcal{C}$ consists of the key parameter $U$, the utility matrix, and the inputs as the endowments $w$ of agents, and $q$, the quantity of goods produced. The "output" variables are $p$, the prices, and $X$, the allocation of goods to labour classes (or agents). The production model $\mathcal{P}$ has the key parameter $T$, the technology matrix, and $Y$, the size of individual labour classes. The input variable is the price vector $p$. The outputs are the wages $w$, and the production vector $q$.

We also define two global optimization functions $f_C$ and $f_P$, which couple $\mathcal{C}$ and $\mathcal{P}$. They also set up the tatonnement as an iterative interaction between $\mathcal{C}$ and $\mathcal{P}$.

In Section 3, we further analyse the tatonnement process and exhibit certain non-convergent trajectories. Next, we associate combinatorial structures associated with equilibria and understand how they vary with the parameters of the economy, i.e., $T$, $Y$ and $U$. We use these results to define the CCG, the consumer choice game, where $T$, $Y$ are fixed, and $U$ is the strategy space. We do this over a collection of open sets and show that explicit description of the game is obtained over these open sets.

In Section 4, we use the above results to illustrate a particular market of three labour classes and three goods, and examine the vicinity of a particular fixed point $H_3$. We use the combinatorial data associated with the fixed point, viz., the Fisher solution forest, to explicitly construct the consumer choice game, i.e., the use of $U$ as strategies and the total utility as the pay-offs. We show that even within this open set, the strategic choice of $U$ makes eminent sense.

In Section 5, we cast the Heterodox market $\mathcal{H}$ as an Arrow-Debreu market. We show the equivalence of solutions of the Heterodox model, i.e., its fixed points, with the equilibria of the corresponding A-D market. Thus, this connects the two concepts and also gives an explicit description of the dependence of the A-D equilibria on the parameters of the economy.

In Section 6, we go back to the combinatorial data arising from a fixed point. First, we show through a 2-player example, the decomposition of the strategy space, i.e., the $U$-space into various regions indexed by Fisher forests. This also leads to a correspondence $\mathcal{N}$ between the strategy space and the space of possible pay-offs. In other words, $\mathcal{N} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$. We show that $\mathcal{N}$ is largely a 2-dimensional manifold.

Finally, in Section 7, we conclude by pointing out what was achieved, its economic significance, and possible future directions.

2 THE HETERODOX MODEL

We first list the basic parameters and internal variables of the Heterodox model.

2.1 The basic notations and assumptions

- A good can be both, a fixed unit of service or an output of a manufacturing plant, made available in a fixed time interval, called epoch, for example, a year. The set of goods is denoted by $G = \{g_1, \ldots, g_n\}$. Also, $q$ is a $(n \times 1)$ column vector with $q_j$ the amount of good $g_j$ manufactured and $p$ is an $1 \times n$ row vector with $p_j$ being the price of each good $g_j$.
- The whole population in the economy consists of agents divided into distinct classes, say according to their training. Thus, let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be these classes of labour, with each class $L_i$ willing to devote $Y_i$ units (e.g., person-years) of labour in every epoch. Thus, $Y$ forms a $k \times 1$ vector.
- Each (manufactured) good $g_j$ has exactly one production process or technology $T_j$ and each $T_j$ is a linear map $T_j : \mathcal{L} \rightarrow \mathbb{R}$. These tell us the amount of labour required from each labour-class to produce one unit of good $g_j$. The class of all such technologies is denoted by $\mathcal{T}$, which is represented as a $k \times m$-matrix $T$ with column $T_j$. Thus, in matrix $T$, $T_{ij}$ is the the amount of labour $L_i$ needed to produce $g_j$. 
We also note that when utilities are same for all persons in one labour class and are measured in happiness per person per kilo units. For example, \( u_{ij} \) is the happiness derived by a person from class \( L_i \) by consuming a unit of good \( G_j \). We denote the matrix formed by \( u_{ij} \) entries as \( U \).

Further, without loss of generality, we assume that for each labour class, there are technologies which utilize them. We also assume that the entries of \( T, Y \) and \( U \) are all in general position and satisfy no algebraic relation amongst themselves, with rational coefficients.

We assume that for each epoch in an economy, there is a non-negative tuple \((p, q, w, X)\), where

1. \( q \) is a \( n \times 1 \) column vector with \( q_j \) is the amount of good \( g_j \) manufactured,
2. \( p \) is an \( n \times 1 \) column vector and \( p_j \) is the price of each good \( g_j \),
3. \( w \) is an \( 1 \times k \) row vector where \( w_i \) are the wages received by each person in labour class \( L_i \) and finally,
4. \( X \) is a \( k \times m \)-matrix and \( x_{ij} \), the total amount of good \( g_j \) consumed by labour \( L_i \)

### 2.2 The consumption space \( C \)

Consumption in our economy is modelled as a Fisher market. Recall that, in a Fisher market, there are \( k \) buyers (\( L \), as in our case), and \( m \) goods \( G \). Each agent \( L_i \) is endowed with money \( m_i \), and each good \( G_j \) has quantity \( q_j \) for sale.

Solution of Fisher market is equilibrium prices \( p = [p_j]_{j \in G} \) and allocations \( X = [x_{ij}]_{i \in B, j \in G} \) such that they satisfy the following two constraints

**Market Clearing**: Allocations are such that all goods are completely sold and the money of all the buyers is exhausted, i.e.
\[
\forall j \in G, \sum_{i \in L} x_{ij} = q_j \quad \text{and} \quad \forall i \in L, \sum_{g \in G} p_j x_{ij} = m_i
\]

The consumption space is defined as: \( C(m, q) = \{X, p | X, p \text{ satisfy market clearance}\} \).

Next, on the set \( C(m, q) \) we define a solution to the Fisher market as one which satisfies:

**Optimal Goods**: Each buyer buys only those goods which give her the maximum utility per unit of money i.e if \( x_{ij} > 0 \), then \( \frac{u_{ij}}{p_j} = \max \frac{u_{ik}}{p_k} \)

While the above condition appears to be a multi-objective optimization problem, it is known that solutions to Fisher market are optimal points of the Eisenberg-Gale maximization function, a money weighted combination of the utilities of the buyers. Adsul etc. [2] too have given a convex program which captures the Fisher market solution as global optima. We denote such a global function as \( f_C \).

We now illustrate the Fisher market \( C \) through an example.

**Example - 1.** Consider a 2 buyers, 2 goods market with \( m_1 = 4, m_2 = 1, q_1 = 2, q_2 = 2, (u_{11}, u_{12}) = (1, 0.4) \) and \((u_{21}, u_{22}) = (0.5, 1)\). The equilibrium prices of this market are \((p_1, p_2) = (1.79, 0.71)\) and the unique equilibrium allocation is \((x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0.6, 0, 1.4)\).

We also note that when utilities \( U, m \) and \( q \) are generic, i.e. satisfy no algebraic relation amongst themselves with rational coefficients, then, the optimal solution to the Fisher market defines a unique weighted forest [1]: Let \( V(H) = L \cup G \) and \( E(H) = \{(i, j) | x_{ij} > 0\} \). For instance, the forest
corresponding to Example 1 given above has four nodes \((L_1, L_2, g_1, g_2)\) and three edges \((e_{11}, e_{12}, e_{22})\) with weights \((2, 0.6, 1.4)\) respectively i.e. the bipartite graph is a unique tree with three edges.

2.3 The production space \(P\)

The production space \(P(p)\) is the collection of all wages \(w\) and quantities \(q\) such that:

\[
Tq \leq Y \tag{1}
\]
\[
q_j \cdot (p_j - (wT)_j) \geq 0 \tag{2}
\]
\[
q \geq 0 \tag{3}
\]

Note that the first condition states that the quantities of goods produced are limited by labour constraints, while the second says that unprofitable goods are not produced.

The global maximization function \(f_P\) is defined as \(\sum_j p_j q_j\), i.e. revenue maximization. However, we must specify how wages \(w\) get decided. For this, we consider the following relaxation LP program:

\[
\max \quad p^T q \\
\text{s.t.} \quad Tq \leq Y \quad q_j \geq 0 \quad \forall j \in G
\]

To find the wages, we consider its dual program -

\[
\min \quad \lambda^T Y \\
\text{s.t.} \quad \lambda^T T \geq p^T
\]

Or

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} \cdot 
\begin{bmatrix}
T \\
-I
\end{bmatrix} = 
\begin{bmatrix}
p
\end{bmatrix}
\]

where \(\lambda_1\) and \(\lambda_2\) correspond to the dual variables associated with the first and second inequalities respectively. Here, \(\lambda_1\) and \(\lambda_2\) are \(1 \times m\) and \(1 \times n\) vectors respectively.

We take \(w = \lambda_1\) as the wages determined by \(f_P\). Karush-Kuhn-Tucker (KKT) conditions for the primal and dual program have the following implications-

1. If \(q_i > 0\), then \(\lambda_{2i} = 0\).

Therefore, it follows from (5) that \(\lambda_{1i} T_i = p_i\) i.e. \(wT_i = p_i\).

2. If \(\lambda_{1i} T_i \neq p_i\) i.e. \(wT_i \neq p_i\), then \(q_i = 0\) as \(\lambda_{2i} \neq 0\).

3. If \((Tq)_i < Y\), then \(\lambda_{3i} = 0\), i.e. corresponding \(w_i\) is zero.

4. Similarly, if \(w_i > 0\), then \((Tq)_i = Y\).

In particular, we see that the inequality in the dual program, viz. \(w^T T \geq p^T\), is opposite of the required constraint. However, in accordance with 3, complementary slackness implies \(q_j (p_j - (wT)_j) = 0\). The conditions reduce \(T\) to a square matrix \(\tilde{T}\) where all constraints are satisfied and tight for the active goods and classes. Let the corresponding prices, production and wages be \(p', q', w'\). We can find the dual variables \(\lambda\) i.e. wages as a product of \(\tilde{T}^{-1}\) and \(p'\). We see that the tight equation \(p' = w'T\) can also be derived through the marginal law of production. We also see that total money is conserved in the economy, i.e. \(pq = wTq = wY\) holds true. This sets up a remarkable result which connects wages and production amounts as the dual variables of each other.
2.4 A tatonnement process

We now set up the tatonnement. The basic objective of the tatonnement process is to arrive at an equilibrium \( \eta = (p, q, w, X) \) such that (i) \( p, X \) are the outputs of the consumption side Fisher market if input the money vector \( w \cdot Y \), and quantities \( q \), and (ii) \( q, w \) are the optimal solutions on the production side on input \( p \). The process begins with a candidate \( \eta \) and checks first if \( \eta \) is indeed an equilibrium. If not, it updates alternately, the consumption side and the production side.

The detailed description of the iterator function is given below.

1. Input \( \eta_0 = (p_0, q_0, w_0, X_0) \). Put \( n = 0 \).
2. We first check if the state \((p_n, q_n, w_n, X_n)\) is an equilibrium. This is done by first checking if \((q_n, w_n)\) is an optimal solution to \( f_P \) in the process \( P \) with input \( p_n \). Next, we check if \((p_n, X_n)\) satisfy the optimality conditions for the function \( f_C \) with inputs \((W_n, q_n)\) (where \( W_n = w_n \cdot Y \), the total wages. If it does indeed satisfy both conditions, we declare the point \( \eta_n \) as a Heterodox equilibrium.
3. If \( \eta_n \) is not in equilibrium, we follow the iterative steps below.
4. Using \( p_n \), we first compute \((q_{n+1}, w_{n+1})\) by optimizing \( f_P \). This is the \( n \)-th production-side update.
5. We next find \((p_{n+1}, X_{n+1})\) through the process \( f_C \) using the input \((q_{n+1}, W_{n+1})\).
6. Note that \( p_{n+1} \) does not set prices for goods not produced. These are set, assuming that a small \( \epsilon \) is indeed produced and predicting its price. Thus if \( b_1, \ldots, b_k \) are the maximum bang per buck values for the players, then

\[
    p_j = \max_i \frac{u_{ij}}{b_i}
\]

This tells us that when these Fisher-like prices are offered, for (at least) one player, the maximum bang per buck ratio equals the ratio these prices give, making the player buy the good. The computation of \( p \) as before and its modification is called \( C(n) \), i.e., the \( n \)-th consumption-side update.
7. This completes the definition of \( \eta_{n+1} \). We go back to Step 2.

We now illustrate two examples of equilibria obtained through the above iterative process.

Example 2 : Let us consider a 3 classes - 3 goods market with following specifications for technology, utility and labour availability.

\[
    T = \begin{bmatrix}
    1 & 0 & 2 \\
    3 & 4 & 0 \\
    0.5 & 2.5 & 2
    \end{bmatrix} \quad U = \begin{bmatrix}
    1.5 & 0.41 & 0 \\
    0.58 & 1.1 & 0.2 \\
    0.5 & 1.4 & 0.6
    \end{bmatrix} \quad Y = \begin{bmatrix}
    1
    \end{bmatrix}
\]

Starting with the price vector \([0.7379, 0.9379, 0.3617]\), the tatonnement process converges to an equilibrium point in 3 iterations, with the prices, production and wages in each iteration given by:

\[
    \begin{bmatrix}
    p_1 \\
    p_2 \\
    p_3 \\
    p_4
    \end{bmatrix} = \begin{bmatrix}
    2.0408 & 3.8706 & 0.7037 \\
    1.5961 & 3.027 & 1.2973 \\
    1.5099 & 2.8636 & 1.2273 \\
    1.5099 & 2.8636 & 1.2273
    \end{bmatrix} ;
\]

\[
    \begin{bmatrix}
    q_1 \\
    q_2 \\
    q_3 \\
    q_4
    \end{bmatrix} = \begin{bmatrix}
    0.2632 & 0.0526 & 0.3684 \\
    0 & 0.25 & 0.1875 \\
    0.2631 & 0.0526 & 0.3684 \\
    0.2631 & 0.0526 & 0.3684
    \end{bmatrix} ;
\]
Example 3: Let us now consider a $(2 \times 2)$ market with the following market specifications -

\[
T = \begin{bmatrix}
0.25 & 0 \\
0.25 & 1 \\
\end{bmatrix};
U = \begin{bmatrix}
1 & 0.81 \\
1.234 & 1 \\
\end{bmatrix};
Y = \begin{bmatrix}
2 \\
4 \\
\end{bmatrix}
\]

The \textit{tatonnement} process converges to the following output of prices, production and wages, when it starts with $p_0 = [0.14196, 0.75972]$

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 4 \\
8 & 2 \\
8 & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
\end{bmatrix} = \begin{bmatrix}
0.3085 & 0.25 \\
0.10394 & 0.084232 \\
0.10394 & 0.084232 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0.66307 & 0.33693 \\
0.66307 & 0.33693 \\
\end{bmatrix}
\]

The solution forest and the allocations are -
Example 4: Let us now consider a 3 classes - 3 goods market with following specifications.

\[ T = \begin{bmatrix} 0.05 & 1 & 0.9 \\ 0.5 & 0.8 & 0.15 \\ 0.4 & 0.5 & 0.4 \end{bmatrix}; U = \begin{bmatrix} 0.2 & 0.3 & 0.8 \\ 0.9 & 0.2 & 0.4 \\ 0.25 & 0.85 & 0.33 \end{bmatrix}; Y = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} \]

The production, prices and wages, as computed by the function described before, are-

\[ \text{Prod} = \begin{bmatrix} 4.35 & 9.78 & 0 \\ 14.7 & 0 & 10.3 \end{bmatrix}; \text{Prices} = \begin{bmatrix} 0.11 & 0.05 & 0.14 \\ 0.04 & 0.12 & 0.05 \end{bmatrix}; \text{Wages} = \begin{bmatrix} 0.52 & 0.48 & 0 \\ 0.12 & 0 & 0.88 \end{bmatrix} \]

The above two states toggle cyclically. In this case it is not possible for a state to exist where all goods and classes are simultaneously active. The solution involves two states, where class-1 and good-1 are active in both of them and other goods and classes alternate between the two. In general, we see the following necessary condition for an equilibrium to exist with a given set of active goods and classes. - For all goods \((j)\) that are not active in the economy,

\[ p_j = \max_i \frac{u_{ij}}{b_i} (\text{U-Level}) < \sum_i T_{ij} w_i (\text{T-Level}) \]

As we see earlier, if a good is not produced, it is allotted a Fisher like price, which is the \(U\)-level defined above. All produced goods have their prices greater than or equal to their \(U\) levels. It is clear that if \(U\)-Level for an unproduced good \(j\) is more that its \(T\)-Level i.e. \(p_j > (w \cdot T)_j\), then it being profitable, that good becomes active in the next iteration, by perhaps pushing a less efficient good out of production.

As seen from the examples, this method does work similar to the tatonnement process given in Walrus’ theory of general equilibrium [13]. It too starts with a price vector, computes production and wages and gives a next set of prices based on these market variables. It is clear that the process terminates if and only if it attains an equilibrium. From the above example, it can be observed that the process may not always converge, and there may be toggling states. Moreover, it can be shown that equilibria whose Fisher forests are disconnected are unlikely to arise from the above process, even though they are fixed points. We thus make a distinction between a heterodox equilibrium or a fixed point and a limit point of the tatonnement process.

However, as we show in Section 5, the Heterodox market has an equivalent Arrow-Debreau market. Whence, via the general theory of existence of equilibria, i.e., via Prop. 5.1 and 5.2 of the equivalence of the two, for any parameter set \((T, U, Y)\), satisfying certain broad conditions, a Heterodox equilibrium, i.e., a fixed point, always exists, but this need not be unique and it need not arise as a limit point.

3.2 Generic equilibrium and combinatorial data

We now associate a suitable combinatorial data with an equilibrium point \(\eta = (p, q, w, X)\) for the parameters \(T, Y, U\) of the economy. Define \(I(\eta) = \{i | w_i > 0\}\), \(J(\eta) = \{j | q_j > 0\}\) and \(F(\eta) = \{(i, j) | X_{ij} > 0\}\). The combinatorial data identify key features of the equilibrium, e.g., the labour classes with non-zero wages, the goods produced, and the Fisher forest, i.e., the price-determining consumptions. We now define the notion of ‘generic-ness’, which allows us to construct the equilibrium from its combinatorial data, and to extend such equilibria at a point to its vicinity.

Definition 3.1. We say that \(\eta\) is a generic equilibrium if (i) for \(j \notin J(\eta)\), we have \((wT)_j > p_j\), and (ii) for \((i, j) \notin F(\eta)\), we have \(u_{ij}/p_j < \max_k u_{ik}/p_k\).

Let us now fix \(T, Y\) and vary \(U\) over \(U = \mathbb{R}^{m \times n}\). Given a \(U \in U\), and an equilibrium point \(\eta\) with the parameters \(T, Y, U\) of the economy, we say that \(\eta\) sits over \(U\), since it is for this element of \(U\), that \(\eta\) was observed. Theorem 3.2 relates to the existence of generic equilibria.
**Theorem 3.2.** Let $T, Y, U$ be matrices in general position, i.e., there be no algebraic relationship between the entries, with rational coefficients. Given an equilibrium $\zeta$ over $U$, there are arbitrarily close $U'$ and equilibria $\eta$ sitting over $U'$ which are generic. Moreover, if $\eta$ has $m$ wage-earning labour classes, i.e., $|\eta| = m$ and $n$ goods produced, i.e., $|J(\eta)| = n$, then the number of connected components $(k)$ of the solution Fisher forest $F(\eta)$ is at least $n - m + 1$.

**Proof.** Appendix-B.

It is an important question if the data $(I, J, F)$ does indeed determine $\eta$, the equilibrium. This is summarized in the next theorem.

**Theorem 3.3.** Again, let $T, U, Y$ be in general position and $\eta$ be a generic equilibrium over $U$ with the combinatorial data $(I, J, F)$, then the parameters of $\eta$, viz., $p, w, q$ are solutions of a fixed set of algebraic equations in the coefficients of $U$. For an open set of the parameter space of $U$, the equilibria, as guaranteed by Prop. 5.1, 5.2, are generic and have the same combinatorial data as $\eta$.

**Proof.** The combinatorial data does give us the relationships $wIT = pJ$ and $T_{i,j}q_j = Y_i$. From this it follows that $|I| \leq |J|$ for otherwise there would be an algebraic relationship between $T$ and $Y$. However, if $|I| = |J|$, and $k = 1$, then $w$ is determined by $p$ and $q$ by $Y$. Since the forest $F$ is connected, $p$ is determined up to a scalar multiple and thus the whole system is solved. In summary, if $|I| = |J|$ and $k = 1$, there is a unique $\eta$ sitting above this combinatorial data. However, in the general case, we must first append to the variables $q_j$, a suitable subset $\{p_1, \ldots, p_k\}$, as in Appendix B.2. The $w$'s and the remaining $p$'s are expressible as homogeneous linear combinations of these $k$ prices. Next, to the linear set of equations $T_{i,j}q_j = Y_i$ we add the $k - 1$ independent money conservation equations to solve these simultaneously. Unfortunately, the conservation equations involve terms $p_iq_j$'s and are quadratic in the chosen variables with coefficients in the entries of $U$. Once these are solved, all other variables are known and the equilibrium point is reconstructed. Thus, over a given combinatorial data, we get an algebraic system with coefficients in $U$, but with finitely many solutions. By standard algebraic geometry results, other than a over a closed algebraic set, these solutions depend smoothly on the entries of $U$.

### 3.3 The Consumer Choice Game

We now define the consumer choice game $CCG(T, Y)$, which is parametrized by the technology matrix $T$ and the labour inventory $Y$, which are henceforth assumed to be fixed. The players are the labour classes, i.e., $L = \{L_1, \ldots, L_k\}$. The strategy space $S_i$ for player $i$ is the utility "row" vector $(u_{i,v}) \in \mathbb{R}^n$. These rows together constitute the matrix $U$. This strategy space is denoted by $U$. We also assume that there is a "real" utility matrix $U_r$, which is used to measure outcomes.

Given a play $U$, the outcome is given by an $\eta(U) = (q, w, y, p, X)$, an equilibrium over $U$ obtained in the Heterodox market. The payoffs, $U_j(X) = \sum_j (U_j)_{ij} x_j$, i.e., the equilibrium allocations evaluated by each player on their true utilities, define the preference relations for each player.

Let us now construct the pay-off functions in the vicinity of a generic equilibrium point $\eta(U)$ with the combinatorial data $(I, J, F)$. We first see that there is an open set $O_{I,J,F} \subseteq U$ containing $U$ which has the same combinatorial data $(I, J, F)$. The exact inequalities defining $O_{I,J,F}$ arise from the requirement that the Fisher forest $F$ have non-negative flows in all edges of $F$, that the edge $(i, j) \notin F$ has an inferior bang-per-buck, and that $(wT)_j - p_j > 0$ for $j \notin J$. As an example, consider an edge $(i, j) \in F$, and the requirement that the flow in this edge be positive. Now, the flow in this edge is a suitable linear combination of the wages $w$'s, prices $p$'s and quantities $q$'s. As we have argued before, these in turn, are smooth functions of the entries of $U$. Thus the condition that flow in the edge $(i, j)$ be positive is the requirement that $f(U) > 0$ for a suitable smooth function $f$ on $U$. 

Sanyukta Deshpande and Milind Sohoni
Thus, there is indeed such an open set $O_{I,J,F}$, and the pay-off functions are solutions of algebraic equations in the entries of $U$, the coefficients of which depend on the combinatorial data $(I,J,F)$. This gives us Theorem 3.4 below.

**Theorem 3.4.** For a generic equilibrium point $\eta(U)$ with the combinatorial data $(I,J,F)$, there is an open set $O_{I,J,F} \subseteq U$ containing $U$ and a smooth family $\eta'(U')$ of equilibria for each $U' \in O_{I,J,F}$ such that (i) $\eta'(U) = \eta(U)$ and (ii) the combinatorial data for $\eta'(U')$ is $(I,J,F)$.

The pay-off function in general is to be pieced together by such a collection of open sets, indexed by the combinatorics. On non-generic $U'$, the equilibrium $\eta(U')$ will have multiple feasible allocations and this determines a correspondence between the strategy space $U$ and $\mathbb{R}^k$, the pay-off space. Even for a generic $U$, there may be multiple equilibrium points, viz., $\eta_1, \ldots, \eta_k$, and each of these will define an analytic sheet of the correspondence over the generic open set.

We now demonstrate the theory described so far through a $3 \times 3$ market.

### 4 AN EXAMPLE

In this section, we describe an economy with three labour classes and three goods, viz., $\mathcal{H}_3$ and construct the consumer choice game where two of the labour classes engage in strategic behaviour.

Let $I = J = \{1, 2, 3\}$. Let $T, U, Y$ be as given below:

\[
T = \begin{bmatrix}
1 & 0 & 0 \\
1 & 2 & 0 \\
1 & 2 & 4
\end{bmatrix},
\quad
Y = \begin{bmatrix}
1 \\
10 \\
100
\end{bmatrix},
\quad
U_r = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix},
\quad
S = \begin{bmatrix}
1 & 0 & 0 \\
\alpha & 1 & 0 \\
0 & \beta & 1
\end{bmatrix}
\]

There are 3 labour types, with numbers 1, 10 and 100 respectively. $L_1$ only prefers good $G_1$, $L_2$ only $G_1, G_2$ and $L_3$ prefers $G_2$ and $G_3$ as shown in the true utilities $U_r$. The example can be understood as an instance of a market with three socio-economic classes and a good such as footwear which is produced in three different ways or qualities. In such cases, the given utility matrix catches the general preference towards the goods produced by different classes.

Let us consider labour class 2 and 3 as the players who exercise their strategies by choosing the variables $\alpha$ and $\beta$. This defines the strategy space $\mathcal{S}$ as shown above. Note that $U_r \in \mathcal{S}$. We compute (i) the dependence of the pay-offs on $\alpha$ and $\beta$, and (ii) the sub-domain of $\mathcal{S}$ over which the chosen forest $F$ below is the equilibrium forest.

We first solve for the production part. We see that $Tq = Y$ gives:

\[
q = T^{-1}Y = \begin{bmatrix}
1 \\
-0.5 \\
0
\end{bmatrix},
\quad
Y = \begin{bmatrix}
1 \\
4.5 \\
22.5
\end{bmatrix}
\]

thus, the production is determined. Next, we use $p = wT$,

\[
w = \begin{bmatrix}
w_1 & w_2 & w_3
\end{bmatrix},
\quad
p = \begin{bmatrix}
p_1 & p_2 & p_3
\end{bmatrix}
\]

\[
T^{-1} = \begin{bmatrix}
p_1 - p_2/2 & p_2/2 - p_3/4 & p_3/4
\end{bmatrix}
\]

This describes wages in terms of prices. All this does not need the equilibrium forest $F$.

For the consumption and allocation processes, let us assume that the solution forest $F$ is given by:
Note that this forest is motivated by the utility matrix given earlier. Using Fisher market constraints of optimum utility, we can write these equations:

\[ p_1 / \alpha = p_2 \quad \text{and} \quad p_2 / \beta = p_3 \]

Conserving the total money while allotting the goods, the equations result in the following money flows, with the conditions that \( 2p_1 - p_2 > 0 \), \( 9p_2 - 5p_3 \geq 0 \):

The flows mentioned on edges are the amounts spent by classes on the corresponding goods. This condition reduces to the requirements that \( \alpha > 0.5 \) and \( \beta > 5/9 \). Under these conditions, \( F \) will arise as the equilibrium forest.

Assuming total money in the economy as 1, we find class wages and allocations as functions of \( \alpha \) and \( \beta \)

\[
\begin{align*}
w_1 &= \frac{\beta(2\alpha - 1)}{2\alpha \beta + 9\beta + 45}, \quad x_{11} = 1 - \frac{0.5}{\alpha} \\
w_2 &= \frac{5(2\beta - 1)}{2\alpha \beta + 9\beta + 45}, \quad x_{21} = \frac{0.5}{\alpha}, \quad x_{22} = \frac{4.5 - 5}{2\beta} \\
w_3 &= \frac{50}{2\alpha \beta + 9\beta + 45}, \quad x_{32} = \frac{5}{9\beta}, \quad x_{33} = 22.5
\end{align*}
\]

where \( w_i \) are the class wages and dividing those by the number of people yields the values per person in each class.

We see that \( \alpha, \beta \) have significant impact on wages and allocations and thus, can be used as strategies. For example, if class-2 decides to keep the value of \( \alpha \) at 0.75 as opposed to 1, the wage share of class-1 decreases and thereby that of class-2 improves. Moreover, the allocations also increase.

For this forest, i.e., in the region \( \alpha > 0.5 \) and \( \beta > 5/9 \), the pay-offs based on the true utility \( U_r \) are given below as functions of \( \alpha, \beta \)

\[
\begin{align*}
u_1 &= 1 - \frac{0.5}{\alpha} \\
u_2 &= 4.5 + \frac{0.5}{\alpha} - \frac{5}{2\beta} \\
u_3 &= \frac{5}{9\beta} + 22.5
\end{align*}
\]

It is clear that decreasing \( \alpha \) and \( \beta \) are the strategies for class-2 and 3. In fact, the impact of \( \beta \) on \( L_2 \) is more significant than on \( L_3 \), and it would be in the interest of \( L_2 \) and \( L_3 \) to squeeze \( L_1 \) through the use of \( \alpha \).

Moreover, We see that multiple equilibrium forests are possible here, depending on \( \alpha \) and \( \beta \), including the one given above. For each of them, the number of active classes may be different and thus the utility functions will vary. In fact, for a sufficiently small value of \( \beta \), class-2 and thereby class-1 receive no wages. In this case, only good-3 is produced and its utility for class-3 is 25. This exceeds the utility of 23.5 which comes from the forest described above.
This illustrates that the local combinatorial data is sufficiently explicit to enable the computation of the pay-off functions. Moreover, significant benefits may accrue to players if they utilize the freedom of posturing their utility functions.

5 CONNECTING THE HETERODOX MODEL WITH THE A-D MARKET

The objective of this section is to construct an Arrow-Debreu (A-D) market $\hat{M}$ from the heterodox model $\hat{M}(T, U, Y)$, and show the equivalence of the equilibrium points in the Heterodox model and the market equilibria in the A-D sense. See Appendix C for a detailed description of the standard A-D market.

5.1 Heterodox Model as an A-D instance

We shall now build a suitable A-D market, given the data $T, U, Y$ for $\hat{M}$. We assume that $T$ and $U$ are $m \times n$, i.e., there are $m$ labour classes and $n$ processed goods, and $Y$ is the $m \times 1$ vector of labour class size. Recall that, $T_{ij}$ refers to the number of labour-units of type $i$ required to produce one unit of good $j$. The labour availability is given as a vector $Y$. We now construct $AD$ as follows.

- The set of firms in $AD$ is $F = \{f_1, \ldots, f_p\}$, where $n$ is the number of columns of $T$. The firm $f_i$ produces good $g_j$.
- The total number of goods are $m + n$, viz., $\{g_1, \ldots, g_m, r_1, \ldots, r_n\}$, where $r_i$ corresponds to the labour of class $i$. We call labour inputs as ‘raw’ goods.
- The number of agents is $m$, and each agent $A_i$ begins with an endowment $Y_i$ of good $r_i$ above.
- The production function of $f_j$ is $Y_j$ which arises from the column $j$ of $T$. Define $u_j$ as the $(m + n)$ vector $g_j - \sum_k T_{kj} r_k$ to represent that $T_{kj}$ units of labour type $k$ are used to make one unit of good $g_j$. Then $Y = \{\lambda \cdot u_j | \lambda \in [0, L]\}$ where $L$ is a large number. Thus firm $f_j$ produces some multiple of $u_j$.
- Agent $A_i$ owns a fraction $\alpha_{ij}$ of the firm $f_j$. The exact numbers will be irrelevant since we will see that in equilibrium, the firms make zero profits. Hence, income of each agent $i$, $M_i(\hat{p})$ is defined as (price i.e. wage) $\hat{p} \cdot r_i$ (initial endowment of labour units) $= w_i$ (wages) $\times 1$(labour input of one agent) $= w_i$.
- The utility matrix serves to define the continuous real valued utility function $u_i$ for each agent $i$. If $X_{ij} \geq 0$ is the amount of good $j$ allocated to agent $i$, then $u_i = \sum_j u_{ij} X_{ij}$. Utilities are zero for labour units hours, i.e., $U_{ij} = 0$ for $j > n$, as it is only the firms which have any use for labour. Since the utilities are linear, it is clear that the principle of non-satiation holds.

This completes the specification of the A-D market $AD$. An A-D equilibrium of the $AD$ are prices $p_1, \ldots, p_n, p_{n+1} = w_1, p_{n+m} = w_m$, production $q_j \cdot v_j$ in $Y_j$ and allocations $X_{ij}$ such that (i) each firm maximizes profits under the given global labour constraints, and (ii) each agent $A_i$ maximizes its utility under the expenditure constraint of the wages received from its endowments $Y_i$ priced at wages $w_i$ and (iii) demand meets supply for each good when the corresponding price is nonzero i.e. all produced goods are exhausted. If supply is more than demand, the price is zero.

5.2 Heterodox equilibrium as an equilibrium point in AD

For the market $\hat{M}(T, U, Y)$, we denote an Heterodox equilibrium as $(\hat{p}, \hat{q}, \hat{w}, \hat{X})$. Let us assume that in the $(m \times n)$ market, in the above equilibrium, it is the first $m'$ labour classes and the first $n'$ goods which are active. We let $(p, q, w)$ denote the prices, production and wages of active goods and labour i.e. $\hat{p} = (p, p_M)$, $\hat{q} = (q, 0)$ and $\hat{w} = (w, 0)$. As defined in the Heterodox model, $p_M$ refers to the modified prices of the unproduced goods.

By the feasibility of the equilibrium and the activity conditions, we have,

$$T \hat{q} \leq Y$$
where \( T' \) is the reduced technology matrix and \( Y' \) is the reduced \( Y \) vector in accordance with the active goods and classes. Let \( \tilde{Y} = (Y', 0) \) be the amount of labour used. The variables are such that the production is optimal given \( p \) and \( \tilde{p} \) are the amount of labour used. Also, by the price-setting mechanism of unproduced goods \( p_{n+1}, ..., p_n \), and the choice of \( \tilde{w} \) as the dual variables, we have

\[
p_j = \max_i \frac{u_{ij}}{b_i} \quad \text{(U-Level)} < \sum_i T_{ij} \tilde{w}_i \quad \text{(T-Level)}
\]

where \( b_i \) refers to the bang per buck ratio of \( i \)'s agent.

**Proposition 5.1.** Let \( M(T, U, Y) \) be the market with a Heterodox equilibrium point \((\tilde{p}, \tilde{q}, \tilde{w}, \tilde{X})\) as described above. Let \( P = (\tilde{p}, \tilde{w}) \) and \( Q = (\tilde{q}, \tilde{Y}) \). Then, \((P, Q, \tilde{X})\) is an equilibrium in the A-D market.

**Proof.** Along with the optimization constraints, we need to prove that total supply of all goods, producible and raw, is greater than or equal to the goods demanded or consumed. Moreover, if supply is more than the demand, the corresponding price/wage is zero. This translates to saying that all ‘produced’ and ‘raw’ goods satisfying \( T'q = Y' \) should be exhausted and all ‘raw’ goods satisfying \( T'q < Y' \) should receive zero wages. As described above, \( P = (\tilde{p}, \tilde{w}) = (p_1, p_2, ..., p_m, \tilde{0}, \tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n, \tilde{0}) \).

- Let us first consider the active goods and classes. We prove that each firm maximizes the profit, given the global constraints. For firm \( i \), let \( q_i^* = (q_i, -l_i) \) maximize the profit, given that it lies in the set of production possible technologies. Here, \( l_i \) is a vector of the amount of labour units consumed in making \( q_i \) amount of good \( i \). For all \( i \), \( q_i^* \) should satisfy

\[
\max \quad p_i q_i - w \cdot l_i = p_i q_i - (w \cdot T'_{s,i}) q_i
\]

\[
s.t. \quad q_i, l_i \geq 0,
\]

Using the Technology matrix, here we have \( l_i \) given by \( q_i(T'_{s,i}). \)

We note that since \( p = wT' \), the expression \( p_i q_i - w \cdot l_i \) equals \( p_i q_i - (w \cdot T'_{s,i}) q_i = 0 \). This means that whenever \( (q_k, -l_k) = (q_k, -q_l(T'_{s,i})) \), firm \( i \) gives an optimal production value, irrespective of \( q_i \).

- We now consider \( k > n' \). Since there is no production \( (q_k = 0) \) and consumption of labour, \( p_k - (w \cdot T_{s,k}) < 0 \) as given by the duality of \( q \) and \( w \) in the Heterodox model. Hence, the maximum occurs at \( q_k = 0, l_k = 0 \). Therefore, the optimization function is multivalued and \( (\tilde{q}, -T\tilde{q}) \) given \( T'q = Y' \) is an optimal point satisfying the global constraints. We also see that \( T'q = Y' \) implies that the raw ‘used’ goods are exhausted completely. On the other hand, there is a supply of raw ‘unused’ goods but no demand resulting in zero prices i.e. wages. This establishes that all firms maximize their profits and for raw goods: supply meets demand for ‘used’ goods and prices are zero for ‘unused’ goods.

- Next, we prove that each agent \( i \) finds an optimal consumption set \( X_i \) by maximizing her utility under the expenditure constraint. The optimization program given below exactly conveys this requirement. Given \( P \), we set up the equation for each agent \( i \):

\[
\max \quad \sum_j x_{ij} u_{ij}
\]

\[
s.t. \quad \tilde{p} \cdot x_i \leq M_i(\tilde{p}) = \tilde{w}_i,
\]

\[
0 < x_i \in X_i
\]
KKT conditions for this program imply that the optimal point satisfies 
\( (u_{i1}, u_{i2}, \ldots, u_{in}) = \mu(p_1 - \lambda_1, p_2 - \lambda_2, \ldots, p_n - \lambda_n) \), where \( \mu \) and \( \lambda_j \) are the Lagrange multipliers associated with constraints 1 and 2 respectively. This means that whenever \( x_{ij} \) is positive, \( \lambda_j = 0 \), and \( p_j = \mu u_{ij} \).

In other words, whenever agent \( i \) buys goods \( j1, j2 \), we have \( \frac{u_{ij1}}{u_{ij2}} = \frac{p_{i1}}{p_{i2}} \), which is a Fisher condition.

Since the utility function is convex, we observe that the Heterodox output for allocations, i.e., \( \hat{X} \) maximizes the above function, as given by the sufficiency of KKT. Moreover, the Heterodox output for consumption is such that all produced goods are completely exhausted. Since utility for raw goods is zero, we see that each agent maximizes her payoff by buying the right set of produced goods. Thus, we prove that all agents maximize their payoffs and demand equals supply of the reduced set of producible goods.

- This establishes that \((P, Q, \hat{X})\) so defined using the Heterodox model is an equilibrium point in the A-D market.

\[ \square \]

### 5.3 A-D equilibrium as an equilibrium in the Heterodox market

Now, let \((P, Q, \hat{X})\) be an equilibrium point in the A-D market. We assume that \(m'\) classes and \(n'\) goods are active. Let \( \hat{p}, \hat{w} \) be the corresponding prices and wages. Let the optimal production vector for each firm be \( \hat{y}_j = (q_j, l_j) \) so that the total output of firms is \( Q = (\sum_j q_j = \hat{q}, \sum_j l_j = Y, 0) \). Let \( \hat{X}_{ij} \) give the consumption.

**Proposition 5.2.** Let \( \hat{A}D \) be the market described above with a A-D equilibrium point \((P, Q, \hat{X})\). Then, \((\hat{p}, \hat{q}, \hat{w}, \hat{X})\) is an equilibrium point in the Heterodox model.

**Proof.** We let \( q, w \) be the ‘active’ vectors consisting of all positive entries from \( \hat{q}, \hat{w} \). Similarly, let \( p \) denote the prices of active goods.

- We first look at the conditions \( \hat{p}, \hat{q}, \hat{w} \) satisfy being a part of A-D equilibrium. For each firm \( j \) which is active, \( v_j = (q_j, -l_j) \) is its optimal solution where \( l_j \leq Y \). In other words, \( v_j \) has to maximize \( p_jq_j - (w \cdot T_{s,i})q_j \) subject to the non-negativity constraints. We can now consider these three cases -
  - \( p_i - (w \cdot T_{s,i}) < 0 \)
  - \( p_i - (w \cdot T_{s,i}) = 0 \) and
  - \( p_i - (w \cdot T_{s,i}) > 0 \).

Since \( p_j, w \) are given, we note that since firm \( j \) produces finite amount \( q_j \), we can only have \( p_i - (w \cdot T_{s,i}) \leq 0 \). If this is not true, then any finite \( q_j \) cannot maximize \( p_jq_j - (w \cdot T_{s,i})q_j \), which contradicts the definition. In other words, the function is strictly increasing as \( q_j \) increases, thus giving an unbounded solution. When we artificially put a bound on \( q_j \), the optimal solution is at an unattainable production plan. Moreover, the fact that firm \( j \) is active i.e. it is not making any losses, translates to the condition \( p_j - (w \cdot T_{s,j}) = 0 \). Therefore, we get that for all active firms/goods \( p_j = w \cdot T_{s,j} \), or \( p = w \cdot \hat{T}' \) where \( \hat{T}' \) represents the reduced \( T \) matrix corresponding to active goods. Continuing with the same concepts, for the inactive firms we must have \( p_j - (w \cdot T_{s,j}) \leq 0 \), for any feasible \( p_j \).

- The analysis for agents’ optimal consumption is exactly similar to that given in the earlier section, where the optimization program catches Fisher market conditions. Next, by the definition of an equilibrium point in AD, we know that total supply equals usage/consumption for all \( k \) such that \( w_k > 0 \). We have, zero utilities for raw goods i.e. labour hours. This forces that initial endowment of raw goods should equal the amount of raw goods consumed while producing other goods. This confirms that \( \hat{T}' q = Y' \). For the inactive classes, as \( p_i - (w \cdot T_{s,i}) < 0 \),
firms don’t utilize those. Therefore, there is supply but no demand for these goods. As the
equilibrium production plan is attainable, we have $T \bar{q} = \sum_j I_j \leq Y$. In all, we have that
$T \bar{q} \leq Y$ for all labour classes and $T' q = Y'$ for all classes that are active. Similarly, we have
that $\bar{p} \leq \bar{w} T$ for all goods, and $p = \bar{w} T'$ for active goods. Along with these two conditions,
we have that the allocations and prices follow Fisher market conditions i.e. all goods and
endowments are exhausted and every buyer maximises her utility and buys only those goods
which give her maximum bang per buck value.

- Building from the observations, we see that $\bar{q}$, $\bar{w}$ are dual variables of each other and optimal
for the following programs, as they satisfy the complementary slackness conditions.

$$\begin{align*}
\max_{\bar{q}} & \quad \bar{p} \cdot (T \bar{q}) \\
\text{s.t.} & \quad T \bar{q} \leq Y
\end{align*}$$

$$\begin{align*}
\min_{\bar{w}} & \quad \bar{w} \cdot Y \\
\text{s.t.} & \quad \bar{w} T \geq \bar{p}, \quad \bar{w}_i \geq 0
\end{align*}$$

(7)

Moreover, we see that $p_j$ for $j = n' + 1, \ldots n$ (unproduced goods) must satisfy

$$\max_i \frac{u_{ij}}{b_i} (U\text{- Level}) \leq p_j < \sum_i T_{ij} w_i (T\text{- Level})$$

If $p_j \geq T\text{- Level}$, then it violates the constraint of optimality of production for firm $j$. Similarly,
if $p_j < U\text{- Level}$, we have $p_j < u_{ij}/b_i$ for some player $i$. This means that $b_i$ is less than the bang
per buck that good $j$ offers, which contradicts the optimality condition of $D_i(p)$. Therefore,
we see that the A-D model allows for a band for each $p_j$ that corresponds to an unproduced
good. In the Heterodox model, we fix prices of such goods equal to their U-Levels, which
belong to this band for every $j$. In short, Heterodox model modifies the equilibrium prices of
inactive goods in A-D, while keeping all other variables and optimality conditions the same.

Thus we see that the A-D equilibrium point $(\bar{p}, \bar{q}, \bar{w}, X)$ satisfies all conditions for a fixed point in
the Heterodox settings. In other words, when this point is given as an input to the iterator defined
earlier, the production, prices, wages and allocations remain unchanged.

This proves that given any set of $(U, T, Y)$, an equilibrium $(p, q, w, X)$ exists in the Heterodox
model. As described before, Heterodox model can be considered an instance of the A-D model,
where the existence of equilibrium is proved. Using the proof given above, the equilibrium (with
little modifications) is a fixed point in the Heterodox model too.

In the next section, we analyse a 2-player scenario and look at the decomposition of strategy
space and also examine players’ strategic behaviour.

6 A 2 × 2 MARKET

Let us consider a two class economy with the following specifications - $T$ (Technology matrix), $Y$
(Labour availability), $U_r$ (True Utility matrix) and $U = \mathbb{R}^{2 \times 2}$ (Strategy matrix)

$$T = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad U_r = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

Since Fisher solutions do not change if the rows of $U$ are scaled independently, we see that effectively,$U$ is given by:

$$U_2 = \begin{bmatrix} \alpha & 1 \\ \beta & 1 \end{bmatrix}$$

We assume that $0 < \alpha, \beta < \infty$. Let us solve this case completely, i.e., decompose $U$ into various
zones by their combinatorial signatures. We also analyse the case when we transit from one zone
to another, and finally, when one of the labour classes is shut out of the market. Whenever both classes are active, \( q \) (production vector) is given by:

\[
q = T^{-1}Y = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} Y = \begin{bmatrix} 8 \\ 2 \end{bmatrix}
\]

and

\[
w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot T^{-1} = \begin{bmatrix} 4p_1 - p_2 \\ p_2 \end{bmatrix}
\]

Therefore, class-wise wages may be obtained as-

\[
m_1 = 2(4p_1 - p_2)
\]

\[
m_2 = 4p_2
\]

In a general 2 \times 2 economy, there are seven possible ways of allocating the produced goods are six forests and a cycle, where both classes participate, and two possible (1 \times 1) graphs, where only one class is active. Out of these 9 graphs, the given set of inputs \((T, Y)\) allows for six possible equilibrium solutions, each with a different allocation graph. Firstly, if both the classes are active in the equilibrium, we have these five possibilities-

![Forest-1](image1.png)

![Forest-2](image2.png)

![Forest-3](image3.png)

![Forest-4](image4.png)

![Cycle](image5.png)

It is easily observed that the conservation of money allows only these graphs. The combinatorial data \((I, J, F)\) and the actual equilibrium values \((p, q, w, X)\) depend on the choice of \(\alpha\) and \(\beta\). As described in Section 3, for each forest, there is an open set, a "zone", of the \(\alpha\)-\(\beta\) space over which the forest is the Fisher forest of the equilibrium. The conditions defining this zone arise from positivity of wages and allocations and the maximum bang per buck conditions. For the four forests, these are tabulated in the table given below.

The sixth possible equilibrium state is when \(w_1\) becomes zero. As we see later, it occurs when \(\beta < 1/4\). In this case, labour class 2 produces good 2 \((q = 4)\) and gets the whole share of economy. Note that for no set of prices, do we have \(w_2 = 0\) in equilibrium.

### 6.1 Market specifications

Here is a summary of the four forests given above. We analyse the condition \(\alpha = \beta\) i.e. the cycle in the next section. Recall that \(\alpha = \frac{u_{11}}{u_{12}}\) and \(\beta = \frac{u_{11}}{u_{22}}\). Let \(U_1, U_2\) denote the true utility functions of class-1 and 2 respectively. Also, \(w_1 + w_2 = 1\).

In addition to the conditions for these forests, if \(\beta < 1/4\), the equilibrium solution is \(w_1 = 0\). In that case, \(U_1 = 0\) and \(U_2 = 4\) as the production amount is \((0, 4)\). We denote this by forest-5 or zone-5.

We also classify the generic equilibrium points here. Recall that \(\eta\) is a generic equilibrium if (i) for \(j \notin J(\eta)\), we have \((wT)_j > p_j\), and (ii) for \((i, j) \notin F(\eta)\), we have \(u_{ij}/p_j < \max_k u_{ik}/p_k\). Therefore, we see that an equilibrium point belonging to, say zone-1, is generic if and only if \(\alpha > \beta\). As proved in section-3, we can define an open set for zone-1, viz., \(\alpha > \beta > 1/4\), comprising only of generic equilibrium points. Similarly, for zone-4, the open set would be \(\alpha < 1/2 < \beta\). As we see in the figure given below, the interior of each zone is an open set consisting of generic points and the boundaries correspond to the non generic equilibrium points.
We now look at the points where two or more forests are feasible, i.e. at the boundaries. It is clear from the figure that by changing $\alpha$ or $\beta$ it is possible to transit from one forest to another, by crossing the non-generic points where both forests are possible. It is shown earlier [6] that the set of allocations in Fisher market is hemicontinuous with respect to initial endowments and utility functions. Here, we show that though multiple allocations are possible at such points, utilities are bounded by the limits of utilities of forests at both sides. In other words, allocations and utilities at the transitions are convex combinations of the boundaries of those obtained in the adjoining zones. To make this precise, let us have the following definition.

**Definition 6.1.** Let $x \in \mathcal{U}$ be a point on the boundary of two zones, say Zone A and Zone B and let $\eta = (p, q, w, X)$ be a typical point above $x$, i.e., $\eta$ is an equilibrium for the parameter $x$. Let $X(x)$ be the collection of all allocations of equilibria above $x$ in the U-space. We say that $x$ is a manifold point if the set $U_i(X(x))$ is a bounded interval and its bounds are obtained as the limits $\lim_{q \to x} U_i(X(q))$ and $q \in Zone A$ and $q \in Zone B$.

Let us consider a point $x$ on the line $\alpha = \beta$, and let $\alpha(x) = \beta(x) = \mu > 0.5$. Thus $p$ sits on the transition between Zone 1 and Zone 2. Let $\eta(x) = (p, q, w, X)$ be a typical point above $x$. We see that (i) $I(x) = J(x) = \{1, 2\}$, and (ii) $F(p) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, i.e., the cycle, since $X$ has flows on all edges. In fact, there is a set $X$ of allocations possible at $\eta(p)$ and the bounds on $U_i(X)$ are precisely those achieved as $\lim_{q \to x} U_i(X(q))$ for $q$ in zone 1 and 2. Similarly, we see that for
$1/4 < \alpha = \beta < 1/2$, the bounds on the corresponding $U_i(X)$ are obtained through zone 1 and 3. For the points on the boundaries with forest 4, however, we see that the limits $\lim_{q \to p} U_i(X(q))$ are equal from both zones, thus giving a unique allocation $X$. On the other hand, we see that the points on the boundary $\beta = 1/4$ are not manifold points. This is because for each point on this line, there is a unique allocation $X$ leading to a unique $U_i(X)$ which does not equal $\lim_{q \to p} U_i(X(q))$ for $q$ in zone 1 or 3.

We now classify the zones into interior points and the boundaries where two or more forests are possible. As stated earlier, the interior points refer to generic points which form an open set. Corresponding to the interior points, we have $U_1, U_2$ defined uniquely, which are continuous functions of $\alpha, \beta$. Moreover, these are invertible functions on their restricted domain of $\alpha, \beta$, which makes the sets of possible payoffs $U_1^p, U_2^p$ open. It follows that the correspondences $N_i = (\alpha, \beta, U_1^p, U_2^p)$ are open for each forest $i$. Moving further, we see that for $\alpha = \beta$, which stands for a cycle, corresponding $U_1^p$ and $U_2^p$ belong to open sets. For example, on the boundary of forest 1 and 3, the open set $U_1^p$ is given by $1/4 < \alpha < 1/2$ and $(8\alpha - 2, 8 - 2/\alpha)$. The region looks like half of a parabola, bounded from all the sides. Thus, interior of every possible solution or zone is open and neighbourhood of each point is homeomorphic to open subsets of $\mathbb{R}^2$. For forests, the homeomorphism is given by the inverse of utility functions and for the cycle, the sets are open in $\mathbb{R}^2$.

We now claim that the boundaries of the forests and the cycle form 1 dimensional entities which serve as boundaries to the described 2 dimensional manifolds. Each boundary can be given by a unique equation in $\mathbb{R}^2$. On the line $\alpha = \beta$, there are two boundaries, one coming from $\alpha < \beta$ and another from $\alpha > \beta$. In between these two, a 2 dimensional plane is situated on each part of the segment $\alpha = \beta$, i.e. on the boundary of forest 1 and 3 and forest 1 and 2. When considered closures of the open sets, we see that correspondences intersect along these boundaries.

We therefore establish that in the region $\beta > 1/4$, payoff function $U_i$ is a 2 dimensional manifold with boundary, consisting of all 'manifold' points. Moreover, a correspondence $N = (\alpha, \beta, U_1^p, U_2^p)$ can be defined between the strategy space and the payoffs space. A general version is dealt with in the appendix D where we argue that the same results follow.

6.3 Strategic Analysis

Here, as we see from the figure, class-2 is dominant in the sense that it has a strategy to become the only active class in the economy, just by reducing $\beta$. However, it is not in its interest to completely drive out class-1. If the state is in any zone with 2 active classes, $\beta$ can be decreased to reach zone 5, where class-2 gets utility 4. But, due to a discontinuity in the utility function, we see that for any $\alpha$, class-2 achieves the highest payoff, which is arbitrarily close to 10, when $\beta$ approaches 1/4, but is, strictly more than 1/4. Thus, the best strategy for class-2 is to keep class-1 active and pose $\beta$ as close to, but greater than 1/4. Technically, we see that the discontinuity of the utility functions results in the non-existence of Nash equilibria.

Though this case rules out the possibility of a Nash equilibrium by making one class clearly dominant, the general scenario has a possibility of existence of Nash equilibria, which is discussed in the appendix.

7 CONCLUSION

This paper shows that consumer choice is indeed an important determinant of wage distribution in an economy. This connection provides an important tool for wage-earners to understand how they can adapt their consumption so as to support a more equitable distribution of wages. It does this by providing a modelling and analytic framework which allows us to explore concretely the thread between consumer choice, prices, production and wages.
The paper also helps us understand pricing of many everyday items, e.g., smartphones, where two similar devices may have very different prices, and also that these prices may dramatically change based on a fluid consumer choice. It also suggests that preferring goods and services provided by small-branded and local/regional players, rather than buying the "best" may be a better strategy to ensure better wages.

Next, the key data required of the economy, viz., $T$ and $Y$, is a labour inventory of the production processes of the economy and is part of some of the standard data sets of countries. Such an inventory could be used to develop a tool allowing each household to compute its labour footprint, i.e., an understanding of how their household consumption brings employment across the economy. One conjecture is that the consumption preferences of many wage-earners possibly do not support their own employment. Such an understanding may be useful to these very classes in modifying their personal consumption. Also note that the labour footprint, while very similar to the GDP calculation, does not need monetization. This is important in its own way.

The Heterodox model, may also be applied to standard "comparative advantage" arguments in Ricardian economics, see Appendix E. Here it actually helps illustrate how the social benefits of trade actually transmit as wages across various classes.

Technically, the computation of an equilibrium, given a $T, U, Y$ is an interesting problem. The A-D connection implies that already known efficient algorithms will shed some light. The tatonnement process of this paper needs to be strengthened and its "computing" power needs to be enhanced. A study of the pay-off correspondence and the existence of Nash equilibria needs to be undertaken.

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A 2 × 2 MARKET AND RESULTS

In this section, we give some numerical results for a 2 × 2 case. We look at the wages and total utilities of two classes as functions of their strategies i.e. utilities.

Let us assume that we have two labour classes with T (Technology matrix), U (Utility matrix) and Y (Labour availability) given by-

\[
T = \begin{bmatrix}
0.2501 & 0 \\
0.25 & 1
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
1 & \beta \\
\alpha & 1
\end{bmatrix}
\]

\[
Y = \begin{bmatrix}
2 \\
1 & 1
\end{bmatrix}
\]

Solving for q (Production vector), we get:

\[
q = T^{-1}Y = \begin{bmatrix}
4 & 0 \\
-1 & 1
\end{bmatrix} Y = \begin{bmatrix}
8 \\
2
\end{bmatrix}
\]

Using \( p = wT \),

\[
w = \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} = \begin{bmatrix}
p_1 \\
p_2
\end{bmatrix} \cdot T^{-1} = \begin{bmatrix}
4p_1 - p_2 \\
p_2
\end{bmatrix}
\]

Therefore, class-wise wages can be obtained as -

\[
m_1 = 2(4p_1 - p_2)
\]

\[
m_2 = 4p_2
\]

There are six possible solutions for a 2 × 2 market with no cycles and the solution is determined by the values of \( \alpha, \beta \) and the initial endowments. Note that we haven’t determined the complete market economy yet, where we find an equilibrium of prices, wages and allocations.

Adsul et al. [1] proved that a buyer may derive a better payoff by feigning a different utility and thus in the Fisher market game, buyers could strategize by posing different utility functions. In addition to that, here it is possible to see how buyers can improve their wages by changing their utilities and thereby, consumption.

Consider the above market where, for simplicity, there are two labour classes producing two similar goods, with different production functions as given. They derive their wages only by producing these two goods. Assuming that the players are perfectly rational, their actual utility functions are given by \( U_r \), comprising of all ones. For example, the footwear produced by a brand which employs both the classes and similar one produced by the lower class without any brand value should be valued equally by rational players. Now, depending on their consumption and utility matrix U, we compute their wages.

The following table (Table-1) shows the relationship between \( \beta \) i.e. the utility preference of class-1 towards good-2 and its wage fraction (wages of class-1 divided by the total money) and total utility. Note that these are equilibrium values i.e. given these \( \alpha, \beta \) and U, T, Y as above, the market attains an equilibrium with wage fraction and utilities as given in the table. For example, when \( \alpha = 0.75 \) and \( \beta = 0.81 \), utility of class-1 is 5.33 and wage fraction is 0.5. As the number of goods produced is [8, 2], the total utility of the society is 10 since we look at \( U_r \) while actually computing the utilities. Thus it follows that if utility of class-1 is \( x \), then that of class-2 is \( 10 - x \).

We consider four different values for \( \alpha \); 0.75, 1.001, 1.5, 1.7; and for each of them, we study the effect of increase in \( \beta \) on total utility and wages of class-1.

For each \( \alpha \), we see that the wage fraction is decreasing w.r.t \( \beta \). Also, U decreases when the utility for good-2 increases. This is because when \( \beta \) increases, total wages decrease, and thus purchasing power decreases, thereby lowering the total utility. The same results can be established for class-2. (Table 2)
We can see that as the wage fraction for class-1 increases, the same for class-2 decreases when $\alpha$ increases. Thus we conclude that a lower $\alpha$ value would help realize more payoff as well as wages. This clearly demonstrates the effects of consumption on wages and utilities or happiness derived.

**B. EXISTENCE OF EQUILIBRIUM IN A GENERAL TECHNOLOGY MATRIX CASE**

**Theorem B.1.** Let $T, Y, U$ be matrices in general position, i.e., there be no algebraic relationship between the entries, with rational coefficients. Given an equilibrium $\zeta$ over $U$, there are arbitrarily close $U'$ and equilibria $\eta$ sitting over $U'$ which are generic. Moreover, if $\eta$ has $m$ wage-earning labour classes, i.e., $|I(\eta)| = m$ and $n$ goods produced, i.e., $|J(\eta)| = n$, then the number of connected components ($k$) of the solution Fisher forest $F(\eta)$ is at least $n - m + 1$.

**Proof.** Let us consider an economy with $m$ labour classes and $n$ goods. WLOG, we examine the existence of an equilibrium so that all labour classes and goods are active. Let us $n - m$ as the deficit ‘$\text{def}$’. We first consider the case $k - 1 < \text{def}$ and prove that no equilibrium can exist in this case. We then consider the case $k - 1 \geq \text{def}$ and state the possibilities of an equilibrium by giving a system of solvable equations. With this, we prove that for any generic or non generic equilibrium, it is possible to find an arbitrarily close generic equilibrium by changing the control variables i.e $U'$. We conclude the proof by giving an example of a $2 \times 3$ case.

**B.1 $k - 1 < \text{def}$**

Let us assume that there exist $p, q, w$ such that $p = wT, Tq = Y$ and $p$ is the Fisher market solution for the market set up by $U, q$ and $w$ i.e. there exists a Heterodox equilibrium $(p, q, w, X)$. Using the
prices, it is possible to generate the Fisher forest. We assume that there are $k$ trees in the forest so that $k - 1 < \text{def}$.

Setting up the profitability conditions, we have,

$$
\begin{bmatrix}
T_{11} & T_{21} & \ldots & T_{m1} & | & p_1 \\
T_{12} & T_{22} & \ldots & T_{m2} & | & p_2 \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
T_{1n} & T_{2n} & \ldots & T_{mn} & | & p_n
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_m \\
-1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

We have $k \leq \text{def}$. Let us assume that the goods $g_1, \ldots, g_k$ are in distinct components and for the rows $R_1, \ldots, R_k \in \mathbb{R}^n$, where $R_i(i) = 1$, $R_i(j) = 0$ if $g_j$ is not in the component of $g_i$, and finally $R_i(j) = \alpha_{ij}$ a monomial in the entries of $U$ which relates the price of $g_i$ with $g_j$. If we are to assume that $p_1, \ldots, p_k$ as temporarily known, then the equation $wT = p$ leads us to the equation:

$$
\begin{bmatrix}
p_1 & \ldots & p_k & -w
\end{bmatrix}
\begin{bmatrix}
R_1 \\
\vdots \\
R_k \\
T
\end{bmatrix}
= 0 \quad (8)
$$

Thus, in case of $k = \text{def}$, we can rearrange the matrix so that the determinant of the following matrix is zero.

$$
\begin{bmatrix}
T_{11} & T_{21} & \ldots & T_{m1} & | & p_1 \\
T_{12} & T_{22} & \ldots & T_{m2} & | & \alpha p_1 \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
T_{1n} & T_{2n} & \ldots & T_{mn} & | & \gamma p_k
\end{bmatrix}
$$

Here, $\alpha, \beta, \ldots, \gamma$ represent the algebraic relationships between prices of goods belonging to the same components of the forest. Each column in the right corresponds to a tree. We have rearranged $p$ so that there are $k$ columns. It is clear that the $p$’s can be taken out from the determinant so that the condition translates to an algebraic relationship between $\alpha, \beta, \ldots, \gamma$ i.e. ratio of entries of $U$ and coefficients of $T$. This contradicts the assumption that $U$ and $T$ are independent. In case $k < \text{def}$, we can delete $\text{def} - k$ number of rows from the above matrix so that it becomes a $(m+k) \times (m+k)$ matrix. Determinant of this matrix is zero, which again contradicts the assumption. Thus, there does not exist any equilibrium when $k - 1 < \text{def}$.

**B.2 $k - 1 \geq \text{def}$**

We again consider (8), which is a system of $n$ homogeneous equations in $m + k$ unknowns. Thus, if the first $m + k - n = k'$ prices were fixed, then all prices and wages are defined using this $k'$ prices. Here, $k' \geq 1$.

Next, let us consider the conservation of money in each of the $k$ components $G_i$ and use these to solve for $q$. We of course have $wT = p$, whence $wY = wTq = pq$. Thus the overall conservation of money is already available and there are only $k - 1$ independent equations. Next, We have the $m$ equations $Tq = Y$. Now, for each component, the money available is given as $\sum_{L_j \in G_i} Y_j w_j$. On the goods side, we have $\sum_{g_j \in G_i} p_j q_j$. By the earlier argument, these translate into $k$ algebraic equations.
in the variables $p_1, \ldots, p_k$ and $q_1, \ldots, q_n$, which are homogeneous in $p$’s. By dividing throughout by $p_1$, we get $k - 1$ equations in $k' - 1$ variables. This gives us a total of $m + k - 1$ equations in $n + k' - 1$ variables. Substituting $k' = m + k - n$ gives us $m + k$ equations in $m + k - 1$ variables. These may be solved to obtain all quantities. Section-6 shows the existence of a disconnected $(2 \times 2$ market i.e. $k - 1 = 1 > def = 0$, which confirms that $k - 1 \geq def$.

Let us now consider the case where this equilibrium is generic. We recall the definition of generic equilibrium points- we say, $\eta$ is a generic equilibrium point if (i) for $j \notin J(\eta)$, we have $(wT)_j > p_j$, and (ii) for $(i, j) \notin F(\eta)$, we have $u_{ij}/p_j < \max_k u_{ik}/p_k$. In this case, the set of tight equations is the same as the set given by $(I, J, F)$ i.e. $Tq = Y, wT = p$ and the maximum bang per buck and money conservation constraints given by the Fisher market. Therefore, it is possible to change a $u_{ij}$ by $\delta$ and retain the equations- $(i, j) \notin F(\eta), u_{ij}/p_j < \max_k u_{ik}/p_k$, which result in a generic equilibrium point. It is clear that the equations $(wT)_j > p_j$ hold true. On the other hand, when we have a non-generic point, i.e. $(wT)_j = p_j$ for $j \notin J(\eta)$ or $u_{ij}/p_j = \max_k u_{ik}/p_k$ for $(i, j) \notin F(\eta)$, the number of tight equations is more than those given by its $I, J, F$. We can then relax an equation to reach a generic point.

In all, we have proved that given an equilibrium over $U$, there is an arbitrarily close $U'$ which corresponds to a generic equilibrium and no equilibrium exists if $k < def + 1$. Moreover, we can see that the condition $k - 1 \geq def$ is necessary but not sufficient, as there are more conditions like nonnegativity of variables and optimality of prices. We complete the existence proof by giving an example of a $2 \times 3$ case (next section).

### B.3 Example of a $2 \times 3$ case

Let us consider an economy with the following specifications.

$$ T = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0.75 & 0.8 \\ 0.4 & 0.9 & 0.7 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} $$

Here, two labour classes, with one labour unit each, can produce three goods using the above technology constraints. We show that an equilibrium exists in this economy so that all goods and classes are active, as opposed to the conventional square matrix solution where the number of active classes and goods is the same.

Let us consider $p, q, w$ as follows.

$$ p = \begin{bmatrix} 0.5235 \\ 0.8324 \\ 0.6470 \end{bmatrix}, \quad q^T = \begin{bmatrix} 0.5639 \\ 0.2426 \\ 0.3934 \end{bmatrix}, \quad w = \begin{bmatrix} 0.2952 \\ 0.4565 \end{bmatrix} $$

We see that these variables follow the equations-

$$ p = \begin{bmatrix} 0.5235 \\ 0.8324 \\ 0.6470 \end{bmatrix} = w \cdot T = \begin{bmatrix} 0.2952 \\ 0.2426 \\ 0.3934 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix} $$

And

$$ Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = T \cdot q = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} 0.5639 \\ 0.2426 \\ 0.3934 \end{bmatrix} $$

According to the definition of equilibrium in the Heterodox model, we see that $q$ and $w$ are dual variables of each other and follow the market constraints. On the consumption part, when $w$ and $q$ are given as inputs to the Fisher market, solution is the price vector $p$ with the following allocations-

$$ \text{Alloc} = \begin{bmatrix} 0.5639 & 0 & 0 \\ 0 & 0.2426 & 0.3934 \end{bmatrix} $$
That is, class 1 only consumes good 1 while class 2 buys goods 2 and 3. This confirms the existence of an equilibrium in a general \(m \times n\) economy with unequal number of active goods and classes. \(\square\)

C A-D MARKET NOTATION AND CONDITIONS

(1) Let us consider a finite set of firms \(F\), indexed \(j = 1, \ldots, l\). Let the possible production technologies be represented by a nonempty set \(\mathcal{Y}_j\) where an element \(y\) in \(\mathcal{Y}_j\) represents a technically possible combination of inputs (raw goods) and outputs (produced goods). We assume that \(\mathcal{Y}_j\) is convex, bounded, closed and includes 0 for every \(j\). The boundedness assumption is relaxed in the general model. We also assume that there can be no outputs without inputs and there does not exist any way to transform outputs to inputs.

(2) We define \(S_j\) as the optimal supply function of firm \(j\)

\[
S_j(p) = \{ y^* | y^* \in Y_j, \ p \cdot y^* \geq p \cdot y \ \forall y \in Y_j \}
\]

Under the above assumptions, \(S_j\) is nonempty, convex, and upper hemicontinuous for all \(p \in \mathbb{R}^n_+, p \neq 0\).

(3) Let us consider a finite set of agents/households \(H\), indexed \(i = 1, \ldots, m\). Let \(X_i \in \mathbb{R}^n_+\) denote the set of possible consumption plans of agent \(i\). We assume that \(X_i\) is closed, convex and unbounded above for each \(i\).

(4) The economy’s initial endowment of resources is denoted as \(r \in \mathbb{R}^n\). Each household has an initial endowment of goods given by \(r^i\) so that \(\sum r^i = r\). Let \(y \in \mathcal{Y} = \sum_j \mathcal{Y}_j\). The vector \(y\) is said to be attainable if \(y + r \geq 0\).

(5) Firm \(j\)’s profit function is

\[
\pi_j(p) = \max_{y \in \mathcal{Y}_j}(p \cdot y) = p \cdot S_j(p)
\]

Each household \(i\) has shares in firm \(j\)’s profit given by \(\alpha_{ij}\) so that \(\sum_i \alpha_{ij} = 1\).

(6) Each agent is endowed with a convex preference quasi ordering \(\succeq_i\) on \(X_i\). It is proved that the ordering can be represented by a continuous real-valued function \(u_i\) which is the utility function for each \(i\). We assume that there is always universal scarcity i.e. for \(x_i \in X_i\), there exists \(y_i\) so that agent \(i\) values it more than \(x_i\). We also assume that the sets \(A_i(x_0) = \{x | x \in X_i, x \succeq_i x_0\}\) and \(G_i(x_0) = \{x | x \in X_i, x_0 \succeq_i x\}\) are closed.

(7) Income of agent \(i\) is defined as

\[
M_i(p) = p \cdot r_i + \sum_{j \in F} \alpha_{ij} \pi_j(p)
\]

The artificially restricted convex budget set is defined as

\[
B_i(p) = \{x_i | x_i \in \mathbb{R}^n, p \cdot x_i \leq M_i(p), \ |x| \leq c\}
\]

\(c\) is defined so that \(|x| < c\) for all attainable \(X\). Existence of such \(c\) can be proved using assumptions in 1.

(8) Each agent has a convex demand set given by

\[
D_i(p) = \{x_i | x_i \in B_i(p) \cap X_i, x_i \succeq_i y \ \forall y \in B_i(p) \cap X_i\}
\]

\(B_i(p) \cap X_i\) is continuous (lower and upper hemicontinuous), compact valued, and nonempty for all \(p \in P\). Also, \(D_i(p)\) is upper hemicontinuous, convex, nonempty, and compact for all \(p \in P\).

(9) We assume that for all \(i\) in \(H\),

\[
M_i(p) > \inf_{x \in X_i \cap \{x | |x| \leq c\}} p \cdot x \ \forall p \in P
\]
For $x \in D_i(p)$, it can be proved that $p \cdot x \leq M_i(p)$. Moreover, if $p \cdot x < M_i(p)$, $|x| = c$.

(10) The excess demand correspondence at prices $p$ is defined as $Z(p) = D(p) - S(p) - r$. Under the assumptions given above, the weak Walrus’s law states that $(p \cdot z) \leq 0$. If $(p \cdot z) < 0$, then there exists $k = 1, 2, \ldots, N$ such that $z_k > 0$. Furthermore, if $Z(p)$ is well defined, then $(p \cdot z) = 0$.

Arrow and Debreu proved that under the assumptions and specifications mentioned above, there exists a competitive market equilibrium $P$. Precisely, the following four conditions should be satisfied for an equilibrium to exist. A set of vectors $(y_1, y_2, \ldots, y_l, P_1, P_2, \ldots, P_n, x_1, x_2, \ldots, x_m)$ is said to be an equilibrium if

1. $y_j^*$ maximizes $y_j \cdot P$ over the set $Y_j$, for each firm $j$. In other words, $y_j^*$ belongs to $S_j(P)$ for each $j$.
2. $x_i$ maximizes $u_i(x_i)$ over the set $B_i(P)$ i.e. $x_i \in D_i(P)$.
3. Prices are non-negative, bounded and not all are zero. Without loss of generality, we can assume that $\sum_j p_j = 1$.
4. $Z(P) \leq 0$. Moreover, $p_k = 0$ when $Z_k(P) < 0$.

The results can be extended to a general economy since the equilibrium of the restricted economy is an equilibrium in the unrestricted economy too, as shown using Walrus’s law.

### D. THE GENERAL 2 × 2 MARKETS

We now consider a general scenario involving 2 classes and 2 goods where $T, Y$ are given as constants. Here, given $T$ and $Y$, we rigorously solve for markets, and compute the conditions for a forest to be in equilibrium. We argue that wages are continuous functions of utilities and payoffs and allocations are continuous for each forest. Moreover, we generalize the results given in section 6 by proving the existence of a correspondence between the strategy space and utilities and also that utilities on the boundaries are linear combinations of those of the forests on both sides. We also look at the necessary conditions for Nash equilibria to exist.

#### D.1 Market and specifications

Let us consider a $2 \times 2$ market with inputs $T, Y$. Let $U_r$ be the true utility matrix. Let $U$ stand for the strategy matrix with $\frac{w_{ij}}{w_{ij}} = \alpha$, $\frac{w_{ij}}{w_{ij}} = \beta$.

Let $p_1/p_2 = \alpha$ where $\alpha$ is the appropriate ratio of utilities when the forest is connected or the ratio of some technology inputs when the forest is disconnected. For example, if class-1 buys good 1 and class-2 buys good-2, we have $p_2 T_{21}^{-1} = p_1 T_{12}^{-1}$. Assuming total money in the economy constant and equal to 1, we can solve for $q$ in terms of $T, Y$.

\[
q_1 = T_{11}^{-1} Y_1 + T_{12}^{-1} Y_2 \\
q_2 = T_{21}^{-1} Y_1 + T_{22}^{-1} Y_2
\]

We assume that $T$ and $Y$ are such that $q_1, q_2$ are positive. Solving for wages, we get

\[
w_1 = \frac{Y_1 (\alpha T_{11}^{-1} + T_{21}^{-1})}{Y_1 (\alpha T_{11}^{-1} + T_{11}^{-1}) + Y_2 (\alpha T_{12}^{-1} + T_{22}^{-1})} \\
w_2 = \frac{Y_2 (\alpha T_{11}^{-1} + T_{22}^{-1})}{Y_1 (\alpha T_{11}^{-1} + T_{21}^{-1}) + Y_2 (\alpha T_{12}^{-1} + T_{22}^{-1})}
\]

Like the case described before, we know that for a $2 \times 2$ market, there are six possible forests and one cycle. Also, there are two possible $(1 \times 1)$ markets. The possible $(2 \times 2)$ forests are-
where each is a solution under specific conditions, which place it in a specific zone. For both classes to be active, we first require \( \frac{p_1}{p_2} > \min \left( \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \right) = \min \left( \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \right) \). For connected forests such that class i1 buys only good j1 and class i2 buys j1, j2, we have \( w_{i1} < p_{j1}q_{j1} \) and \( w_{i2} > p_{j2}q_{j2} \). This ensures that all allocations are positive. Lastly, we require the bang per buck conditions to hold. For example, consider the first graph given above, where \( p_1/p_2 = u_{21}/u_{22} = \beta \). This graph constitutes a generic equilibrium if and only if,

1. \( \frac{p_1}{p_2} = \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) i.e. wages are positive.
2. \( \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) i.e. allocations are positive.

\[ w_{i1} = Y_1(p_1T_{11}^{-1} + p_2T_{21}^{-1}) < p_1q_1 = p_1(T_{11}^{-1}Y_1 + T_{21}^{-1}Y_2) \text{ or} \]
\[ w_{i2} = Y_2(p_1T_{12}^{-1} + p_2T_{22}^{-1}) > p_2q_2 = p_2(T_{21}^{-1}Y_1 + T_{22}^{-1}Y_2) \]

3. And, \( \beta < \alpha = \frac{w_{i1}}{w_{i2}} \) i.e. class-1 gets more bang per buck from good-1 than good-2. \( \left( \frac{u_{i1}}{p_1} > \frac{u_{i2}}{p_2} \right) \).

It can be seen that these conditions define an open set in the strategy space \( U \) which has a unique combinatorial data \((I, J, F)\). In the above example, it is given by then \( I = \{1, 2\}, J = \{1, 2\} \) and \( P \) as forest-1. We note that the third condition can be relaxed i.e. \( \beta \geq \alpha \), which results in a non generic equilibrium at \( \alpha = \beta \). This is because multiple allocations, including the forest-1 allocations are possible for this condition. We discuss the cycle i.e. the set \( \alpha = \beta \) in the next section.

In addition to the forests described before, we note that it is possible for wages of a particular class to become zero. When \( p_1/p_2 \leq \min \left( \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \right) \), only one class is active and it produces exactly one good. We refer to its corresponding domain in the \( U \)-space as Zone 7. In this case, if class \( i \) produces good \( j \), it is possible to compute its production and the utility \( U_i \) that it gives. We note that \( U_i \) is smaller than the total utility of both classes combined, as there is more net production when both classes are active.

### D.2 Continuity analysis

For the forests described above, the combinatorial data \((I, J, K)\) is defined by the following conditions

| Forest 1 | Forest 2 | Forest 3 | Forest 4 | Forest 5 | Forest 6 |
|----------|----------|----------|----------|----------|----------|
| \( \alpha \geq \beta = \frac{p_1}{p_2} \) | \( \frac{p_1}{p_2} = \alpha \geq \beta \) | \( \beta \geq \frac{p_1}{p_2} = \frac{T_{11}^{-1}Y_1}{T_{12}^{-1}Y_1} \geq \alpha \) | \( \beta \leq \frac{p_1}{p_2} = \frac{T_{11}^{-1}Y_1}{T_{12}^{-1}Y_2} \leq \alpha \) | \( \beta \geq \alpha = \frac{p_1}{p_2} \) | \( \frac{p_1}{p_2} = \beta \geq \alpha \) |
| \( \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \alpha > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \frac{T_{22}}{T_{12}} \) > \( \frac{T_{11}}{T_{12}} \) | \( \frac{T_{22}}{T_{12}} \) > \( \frac{T_{11}}{T_{12}} \) | \( \alpha > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) |
| \( \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \alpha > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \frac{T_{22}}{T_{12}} \) > \( \frac{T_{11}}{T_{12}} \) | \( \frac{T_{22}}{T_{12}} \) > \( \frac{T_{11}}{T_{12}} \) | \( \alpha > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) | \( \beta > \frac{T_{22}}{T_{12}}, \frac{T_{11}}{T_{12}} \) |

We also graphically represent these six forests by marking their zones. We number these forests from 1 to 6, ordered as given above. 7th zone refers to the case where only one class is active and in equilibrium. Here, the arrows indicate the direction of optimization for the classes, considering \( U_r \) as the strategy matrix comprising of all ones.
Here, \( c_1 = \frac{T_{11}^{-1} Y_1}{T_{12} Y_2} \) and \( c_2 = \frac{T_{21}^{-1} Y_1}{T_{22} Y_1} \) and we have assumed \( c_2 > c_1 \). The first graph refers to the condition \( \det(T) > 0 \) i.e. \( d_1 = \frac{T_{11}}{T_{12}} > d_2 = \frac{T_{21}}{T_{22}} \) and second refers to \( \det(T) < 0 \). There are four possible schemes corresponding to \( c_1, c_2 \) and \( \det(T) \), of which we have considered two with \( c_2 > c_1 \).

In the first scheme, class-1 always gets wages and vice versa. Note that the region for 7th zone can extend depending on \( d_1 \). If \( d_1 > c_1 \), as we will see, continuity of wages implies zone 4 becomes a part of zone 7.

We see that there are clear ranges of \( \alpha \) and \( \beta \) which define the solution forest. First, we observe that as wages are rational functions involving only \( p_1/p_2 \) i.e. the relevant \( \alpha, \beta \) in case the forest is connected, wages are always continuous within a forest. For disconnected forests, wages are constant. It is important to note that wages are bounded, \( w_1 + w_2 = 1 \). At the boundary of zone 4, \( p_1/p_2 \) i.e. \( \alpha \) or \( \beta \) equals \( c_1 \), which determines the wages. In other words, as \( \alpha \) or \( \beta \) continuously approach \( c_1 \), limit of \( w_1 \) equals \( w_1 \) for zone 4. Similarly, zone 3 allows for smooth transitions. It is clear that wages are continuous at the boundary \( \alpha = \beta \) also, as one relevant variable smoothly transitions into another. Similarly, in zone 7, wages continuously converge to zero.

Next, we observe that allocations and utilities are continuous within each forest, where they are functions. For disconnected forests, they are constants. For connected forests, where there are three edges, we see that 2 allocations are linearly dependent on \( p_1/p_2 \) and one is constant. For example, forest 1 has the following allocations:

\[
x_{11} = \frac{w_1}{p_1} = \frac{Y_1 T_{11}^{-1}}{Y_2} + \frac{Y_1 T_{21}^{-1}}{\beta}
\]

\[
x_{21} = q_1 - x_{11}, \quad x_{22} = q_2
\]

Since utilities (true and strategic) are linear functions of allocations, those are continuous too. Moreover, they are either linear or rational functions of \( \alpha, \beta \). Though we have considered a special strategy matrix with all ones here, the result is general.

**Transitions and the cycle**

Firstly, we note that except for the \( \alpha = \beta \) line, other boundaries allow for a continuous function where for each pair of \((\alpha, \beta)\), a unique allocation/utility exists. Let us see this by changing \( \alpha \) or \( \beta \) continuously to approach a forest/cycle from another. We see that when a tree is disconnected by making an edge weight zero, allocations and utilities continuously transform. For example, in zone -1, when \( \beta \) approaches \( c_1 \) from above, \( p_1/p_2 \) equals \( c_1 \) where zone-4 starts, and \( x_{11} \) becomes \( q_1 \). Therefore, forest 3,4 include their boundaries while defining utility functions i.e. boundaries have same utilities as the forests. In case of \( \alpha = \beta \), first an edge is added to get a cycle, where multiple allocations are possible and then another edge is removed.

The governing equations for a connected forest, say forest 1 are: \( w_1 = x < p_1 q_1, y < p_1 q_1 \) and \( z = p_2 q_2 \) along with \( x + y = p_1 q_1 \). When we increase \( \beta \) to approach the cycle, the equations become
\(x, y < p_1q_1\), and \(z, v < p_2q_2\) along with \(x + y = p_1q_1\) and \(z + v = p_2q_2\). Thus, the maximum value of \(z\) is \(p_2q_2\) where zone 1 is overlapping and minimum value is 0 where \(v = p_2q_2\) when zone 6 overlaps.

The utilities for forest 1 are-

\[
(U_1, U_2) = \left( \frac{w_1}{p_1}, q_1 + q_2 - \frac{w_1}{p_1} \right)
\]

For forest-6-

\[
(U_1, U_2) = \left( q_2 + q_1 - \frac{w_2}{p_1}, \frac{w_2}{p_1} \right)
\]

And for the cycle, these can be computed in terms of \(v, z\)-

\[
(U_1, U_2) = \left( \frac{w_1 - v}{p_1} + \frac{v}{p_2}, \frac{w_2 - x}{p_1} + \frac{z}{p_2} \right)
\]

Rearranging the terms, this means -

\[
(U_1, U_2) = \left( \frac{w_1}{p_1} + v \left( \frac{1}{p_2} - \frac{1}{p_1} \right), \frac{w_2}{p_1} + z \left( \frac{1}{p_2} - \frac{1}{p_1} \right) \right)
\]

Firstly, we note that the ratio \(1/p_2 - 1/p_1\) is a constant for any fixed \(\alpha, \beta\). Let this be a positive constant. Now, it is clear that when \(v = 0\), \(U_1\) for the cycle is minimum which coincides with that of forest-1. Similarly, when \(v = p_2q_2\), the maximum coincides with that of forest-6. This establishes that utilities at the cycle are linear combinations of those on the boundaries, so that for each \(\alpha\) on the \(\alpha = \beta\) line, utilities make a smooth transition. This makes us view the cycle as a linear combination of two forests depending on values of \(\alpha\) and \(\beta\). Though we have considered \(U_r\) made up of all ones, this result is valid for any matrix \(U_r\).

We see that the results generalize the example given in section 6. Using the same proof, we can now argue that there exists a correspondence \(N\) between the strategy space and utilities. Also, continuous payoffs for the forests and the range for the line \(\alpha = \beta\) allow for a 2 dimensional manifold whose boundaries are given by those of the forests. Next, we look at the conditions for a Nash equilibrium to exist.

### D.3 Existence of a Nash equilibrium

We observe that allocations are functions of these variables, either linear or inversely related (\(U_i = a + b\alpha\) or \(c + d/\alpha\)), when the forests are connected. In those cases, maximization can occur only at the boundaries i.e. at the points where transition of graphs occurs. When the forests are disconnected, allocations and utilities are constants for a range of \(\alpha, \beta\). Therefore, if any Nash equilibrium exists, it should be at the transition points or forest 3 or 4.

Let us now analyse the first scheme. Looking at the arrows, we know that forest-3 cannot be an equilibrium, which leaves \(\alpha = \beta\) and forest-4 as possibilities. Class-2 prefers forest-4 to forest-2, and similarly, class-1 prefers forest-4. For a point in forest-4 to be an equilibrium, \(\beta > d_1\) and \(\alpha > c_1\). In this region, given any \(\alpha\), class-2 prefers forest-4 to forest-1,3 and 5. We now have to find a region for \(\beta\) so that \(\alpha\) maximises \(U_1\) in that region. Note that when \(c_1 > \beta > d_1\), \(U_1\) is constant as a function of \(\alpha\) in zone 6. When it enters zone-2, \(U_1\) increases until it is in zone 4. Thus, we prove Nash equilibria exist in the region \(d_1 < \beta < c_1\) and \(c_1 < \alpha\). Similarly, it can be proved that Nash
equilibria exists in scheme-2 in zone-3 for $d2 < \alpha < c1$ and $c2 < \beta$. This equilibrium is, however, subject to the condition that the forest exists i.e. the region is feasible. For example, in scheme-1, forest 4 exists when $d1 < c1$ i.e. $\frac{T_{11}}{T_{12}} < \frac{T_{21}Y_{1}}{T_{22}Y_{2}}$ and $\beta < c1 < \alpha$.

This table also settles the question of existence and uniqueness of an equilibrium in a $2 \times 2$ case.

E  RICARDO’S THEORY OF COMPARATIVE ADVANTAGE AND FURTHER RESULTS

This classical theory was developed in 1817 by David Ricardo to explain why countries should engage in trade even if one country is more efficient at producing every single good as compared to other countries. In its basic form, it considers two goods produced by two countries and shows that it is profitable to trade in terms of the goods produced and societal happiness achieved in both countries. [10] A detailed version along with examples is given in the next section.

However, we note that this method does not allow us to compute the wages or allocation of money across the countries. In practical scenarios, though both countries are better off after trading, there’s reallocation of money due to technological differences and free market, which remains unexplained. Ricardian theory has the concept of trade and autarky wages which talks about real wages or purchasing power and how people gain from trade in terms of real wages. To explain the phenomenon, we divide the homogeneous labour classes from each country into two, each producing a good; for example, Rice producers and Computer makers. [11] Before trading, each labour division has same real wages in terms of rice and computers. After the trade begins, real wages of country 1 labour i.e. computer workers increase in terms of rice and remain the same in terms of computers and vice versa. However, the theory does not give a satisfactory and accurate account of wages and allocations.

In case of prices, those can be found by various ways, perhaps relating to various theories of relative prices. One way could be to feed the production values in the Fisher market and compute prices which assumes initial money allocations. Another focuses on the capital-labour ratio, which is not considered here. In this case, another way is to assume same wages in a class and thus derive autarky prices by finding the technology ratio. Lastly, one can find prices by computing the production ratio as per the Ricardian theory. However, none of these give a notion of equilibrium wages, prices and allocations when two countries trade given their technological differences and utility preferences. The theory developed here focuses exactly on these aspects.

We consider two cases here with different types of utilities- linear and piecewise linear concave (plc). We specify utility functions for countries so that we can enable them to compute their optimal production. Utility values perform the role of demand functions. For plc utilities, we consider an example to illustrate the Ricardian theory and then analyse it using the newly developed model. In case of linear utilities, we prove that both Ricardo’s model and the Heterodox model produce same results for productions. There are some remarkable properties exhibited by such structure which establish the relation between KKT conditions for optimality and the Fisher market.

E.1  Ricardo’s Theory of Comparative Advantage

Consider two countries A and B with 100 labour inputs each, that produce only two goods, computers and rice. Labour is homogeneous i.e. there’s no further division in terms of skills. Both countries have different production technologies and A is more efficient in producing both. The number of labour inputs needed by countries A and B to produce a unit of each good are given by-

|         | Computers | Rice |
|---------|-----------|------|
| Country A | 10        | 20   |
| Country B | 50        | 25   |
Thus, their production possibility frontiers (PPF) which give the possible tradeoff of producing goods in various combinations while keeping the labour and technology constraints are-

Now, for A

5 sacks of rice $\equiv$ 10 computers

Therefore, their relative prices in A are-

$$\frac{P_c}{P_r} = \frac{1}{2}$$

Similarly, for B

4 sacks of rice $\equiv$ 2 computers

And, their relative prices in B are-

$$\frac{P_c}{P_r} = \frac{2}{1}$$

If ratio equals any other value, then labour sources in each country would transfer entirely from the cheaper sector to the expensive one. For instance, if the ratio is 1 for each country, then a sack of rice is equivalent to a computer. So A will produce 10 computers and B will produce 4 sacks of rice in autarky, which doesn’t make sense because the demand of some goods is unmet. Hence, the ratio is derived from the PPF. Now, depending on the demand in each country, a point on its PPF is decided, which is the optimal production for that country. Thus relative prices are decided by labour requirements and production by optimization i.e. maximization of net utility.

When the countries start trading with each other other, it is claimed that the relative ratio of prices is now between 0.5 and 2. If it is not, then production of one good will completely stop and both countries will produce another good. For example, if the ratio is 2.5, then both countries will produce computers, which is absurd since there is no production of rice. though there is market demand for it. The exact ratio will depend on the total demand from both countries. Thus, with this new ratio of prices, PPFs will get modified-

Now, with this price ratio, since it is greater than 0.5, country A will produce only computers, something it is comparatively better at. Similarly, Country B will produce only rice. With this new
relative ratio, each country can buy more number of goods by exchanging the goods it is producing. For example, if the ratio is 1.5, country A will be able to sale one computer for 0.66 sacks of rice, as opposed to 0.5 before. Because of this outward shift in the PPFs, both countries are now better off. Hence, the theory of comparative advantage tells that trading makes countries produce goods that they are comparatively more efficient at, thereby increasing market’s efficiency.

E.2 Example

Here we consider a market with two producible goods computers and rice. By looking at the utility functions, we compute its most efficient production. We also look at how Fisher prices are derived in case of plc markets. Let us consider a market comprising of a labour class of 300 homogeneous units.

• Technology requirements are-

| Computers | Rice |
|-----------|------|
| 15        | 10   |

That is, 20 labour units are needed to make one computer and 10 for a rice sack.

• The utilities in two segments are as follows:

\[
U_1 = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 4 & 6 \end{bmatrix}, \\
U_2 = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 10 & 15 \end{bmatrix}
\]

Here, \( U_1 \) and \( U_2 \) denote the utilities in the first and second segment, respectively. Note that the utilities decrease from \( U_1 \) to \( U_2 \) after \( L_1 \) units of goods are consumed, to give a concave formulation. Also, instead of per person values, we consider per class values here i.e. utility of computers decreases when the society consumes 4 units as a whole.

Now, we consider the utility maximization problem to determine optimal production, where \( x_1 \) and \( x_2 \) are computers and rice produced in A and \( x_3 \) and \( x_4 \) are the same for B:

\[
\begin{align*}
\max \quad U(x) &= 0.5x_1 + 0.8x_2 + 0.4x_3 + 0.2x_4 \\
\text{s.t.} \quad 15x_1 + 15x_3 + 10x_2 + 10x_4 &= 300 \\
\end{align*}
\]

(9)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} \leq \begin{bmatrix} 4 \\
6 \\
10 \\
15 \\
\end{bmatrix}
\]

Where we get our first constraint from the PPF
We can now solve the optimization problem to get the following results:

| Computers | Rice |
|-----------|------|
| Production |      |
|   14      |  9   |

To find the prices, we can use KKT conditions and the Fisher market solution which uses the law of diminishing marginal utility.

Since 4 computers and 6 sacks of rice are used up in the first segment as required by L, the current allocation is in the second segment for both the goods. We know that utilities at the operating points are 0.4 and 0.2. Hence, the price ratio is equal to the utilities ratio. Thus we get the prices. This can also be obtained through the KKT conditions when we maximize total utility.

### E.3 Illustrations

Let us consider an example, in line with the Ricardian market. Here we have two countries, A with 200 homogeneous labour units and B with 50 homogeneous labour units. The countries produce two goods, Computers and Rice, with different technology inputs for each. We consider piece-wise linear concave utilities here, by expressing the values in terms of segments. Here are the specifications of the market.

- Let the technology and utilities be as follows:

  \[
  \begin{array}{ccc}
  C & R & C & R \\
  \hline
  A & 20 & 10 & U_1 & = & [0.5 & 0.8] \\
  B & 5 & 10 & U_2 & = & [0.4 & 0.2] \\
  \end{array}
  \]

  For example, country A produces one computer using 20 units of labour. The utilities are piece-wise linear, concave with \( U_1 \) and \( U_2 \) being the utility vectors in the first and second segments respectively.
The $L$ matrix ($2 \times 1 \times 2$) (maximum amounts of all segments) is where, values in these two segments are per person per good. For example, for a person in any of the countries, utility for Rice reduces after consuming 0.3 units of a Rice sack. Subsequently, for the combined economy of 250 people, the modified $L$ is

$$L_1 = \begin{bmatrix} 5 & 7.5 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 20 & 30 \end{bmatrix}$$

Note that the second segment values are superfluous in the sense that we consider only one decrement in the utilities and also that maximum possible production is $[20, 25]$.

Now, PPF(Production Possibility Frontier) of the resultant economy is

We maximize the total utility to find optimal production, which lies on this PPF. Let $z_1$ and $z_3$ be the total number of computers consumed in the first and second segments respectively. $z_2$ and $z_4$ are for rice. Also, $x_{ij}$ refers to computers produced by the $i$th country and consumed in the $j$th segment. Note that this does not talk about consumption by any country, it just says that out of the total computers produced in $i$, $x_{ij}$ computers were consumed with corresponding utility by both the countries together. Similarly, $y_{ij}$ is for rice. To find the production, the optimization program is

$$\text{max}_z U(z) = 0.5z_1 + 0.8z_2 + 0.4z_3 + 0.2z_4$$

$$\text{s.t. } \begin{cases} 20x_{11} + 20x_{12} + 10y_{11} + 10y_{12} = 200 \\ 5x_{21} + 5x_{22} + 10y_{21} + 10y_{22} = 50 \end{cases}$$

Here, inequalities mean that consumption of each good is bounded by the $L_1$ values. Also, the equalities are the resource constraints.

Optimization results for the above program are:

For example, $x_{11} + x_{12} = 1.15 + 3.57 = 4.72$ is the total number of computers produced by A. We see that at the optimal production, country A produces both the goods while country B produces only computers. Total production is $[14.72 \ 10.54]$ which lies on the PPF.
We note that because of plc utilities, it is possible for the optimal production to lie on points other than the corner ones on the PPF. On the other hand, in case linear utilities, the only possible optimal points are the cases where only rice or computers are produced or the point where one country produces rice and another produces computers.

As per the Ricardian theory, we can find the total production and relative prices in autarky for each country. So now the question is to determine prices and the wages in this combined economy. Also, it is important to see how these goods are allotted across these countries.

E.4 Ricardo’s Model: Parameters and Notation

Here we consider two cases—linear utilities and piecewise linear concave utilities and illustrate how various parameters are computed. Later, we look at the Heterodox results for the above example. Let there be two countries A and B with \( Y_1 \) and \( Y_2 \) labours in each, respectively.

- The Technology matrix and the Utility matrix:

\[
T = \begin{bmatrix}
T_{11} & T_{12} & 0 & 0 \\
0 & 0 & T_{21} & T_{22}
\end{bmatrix};
\]

\[
U = \begin{bmatrix}
u_{11} & u_{12} & u_{11} & u_{12} \\
u_{21} & u_{22} & u_{21} & u_{22}
\end{bmatrix}
\]

where the first row corresponds to country 1 and second to country 2. We assume that two types of goods are produced in each country—Computers and Rice. The four columns here correspond to C R C R respectively.

- Prices and Wages:

\[
w = \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix};
p = \begin{bmatrix}
p_1 & p_2 & p_3 & p_4
\end{bmatrix}
\]

Theorem E.1. Given a market comprising of two or more countries, corresponding technology matrix \( T \), utility matrix \( U \), and segments matrix \( L \), the optimal production found by the new method lies on its PPF. Moreover, in case the countries have same linear utilities, both the models—Ricardo and Heterodox produce the same output for production.

Proof. (Appendix E) \( \square \)

The Heterodox interpretation of Ricardo’s model

We first optimise the value of \( p^T q \) to find the optimal production given the technology constraints, where \( q \) : production and \( p \) : input prices. Wages are the dual variables of this program. Using KKT conditions, we know that for a general vector \( p \), only two entries of the \( q \) vector are nonzero. This is because we have the following type of feasibility conditions -

\[
p_1 = T_{11}w_1 - \lambda_1 \\
p_2 = T_{12}w_1 - \lambda_2 \\
p_3 = T_{21}w_2 - \lambda_3 \\
p_4 = T_{22}w_2 - \lambda_4
\]

where \( \lambda \) correspond to the multipliers associated with \( q_j \geq 0 \). It is evident that for positive \( q_j \)'s, \( \lambda_j \) should equal zero, so \( p_j \) should be equal to \( T_{ij}w_i \). Thus, it can be deduced that each country would specialise by producing exactly one good. However, toggling states can result in production of more than one good on average. We can find this good in each state by looking at the following
relation-

If \[ \frac{p_i}{p_j} > \frac{T_i}{T_j} \]

then, the country will produce good i, otherwise it will produce good j. Since \( T_i \) denotes the number of labour inputs required to produce good i, this equation makes perfect sense because it maximises the revenue the country can generate.

Once we know both the prices, wages can be computed as-

\[ \frac{w_1}{w_2} = \frac{T_2}{T_1} \cdot \frac{p_1}{p_2} = \frac{p_1}{T_1} / \frac{p_2}{T_2} \]

This can also be seen as a solution to \( T^{-1}p \), where \( T \) is diagonal. Here, the subscript 1 denotes the first country. Thus, \( T_1 \) is the selected technology from the first country and so on.

Now, when we feed this data into the Fisher market, we get a new set of prices. The ratio of these prices equals the ratio of appropriate utilities when the forest is connected. If the solution consists of only one state i.e. if no switching is observed, then the next set of wages is again computed by the above mentioned formula. Once the Fisher forest is stable, all parameters can be found easily.

- **Toggling states:** As a part of our iterator function, every time we run the Fisher market, we attribute some prices to the unselected goods given by the formula-

\[ p_{new} = u_{1new} \times \min_i \frac{p_i}{u_{2i}} \]

This makes sure that the unselected goods are made available at the prices which are low enough that at least one buyer class buys it when offered with other goods. Note that this doesn’t guarantee the production of that good and thereby its active status as only the most efficient goods are selected.

The states are stable unless the production changes. As stated above, it will change, if

\[ \frac{p_i}{p_{1new}} < \frac{T_i}{T_{new}} \quad \text{i.e.} \quad \frac{u_{2i}}{u_{1new}} < \frac{T_i}{T_{new}} \]

for any good \( new \) that was unselected in the earlier iteration and good \( i \) that was produced. Because we have two segments here, we often see this kind of phenomenon happening, which results in toggling states. For example, when Fisher market is such that the following holds

\[ \frac{p_1}{p_2} = \frac{u_{21}}{u_{22}} \quad (11) \]

and if,

\[ \frac{u_{2i}}{u_{1j}} < \frac{T_i}{T_j} < \frac{u_{1i}}{u_{2j}} \]

then goods \( i \) and \( j \) are produced alternatively.

If good \( i \) gets over in the first segment itself and second good \( j \) is allotted to both the classes in the next segment, we see -

\[ \frac{p_i}{p_j} = \frac{u_{1i}}{u_{2j}} \quad (12) \]

and the similar analysis holds for this case too.

- **When the forest is disconnected,** only one good is allocated to only one country in the second segment. When this happens, the utilities cannot be the same for both countries, which results in different bang per buck ratios. We can then find the maximum bang per buck ratio and effective utility value for the country which has lower ratio. It lies between the utility values of its first and
second segments which follows from the KKT conditions for Fisher market [?]. However, if this value doesn’t cause the system to toggle in the way described, the system is all determined. Using the price ratio, selected goods and wages can be computed. Even if it toggles, we are assured to get the same states since production solely decides and fixes other values. Now, the results for example-2 are-

| Wage Iterations- | Equilibrium Prices- |
|------------------|---------------------|
| A 0.55 0.45      | B C-A 0.0250 R-A 0.0625 C-B 0.0999 R-B 0.0500 |
| A 0.80 0.20      | B C-A 0.1005 R-A 0.0502 C-B 0.1005 R-B 0.0503 |

| Goods Iterations - | Total Allocations per person- |
|--------------------|-------------------------------|
| A C-A 10 0 10 0 B C-B 0 R-B 0 | A C-A 0.055 0 0.178 0 |
| A 0 20 10 0 B 0 R-B 0 0 | B C-A 0.033 R-A 0.0925 C-B 0.0646 R-B 0.03 |

Thus, the stable system has two states, one in which both countries produce computers and the second in which A produces Rice and B produces Computers. The average production of these two states is -

| Computers | Rice |
|-----------|------|
| A 5       | 10   |
| B 10      | 0    |

We note that the production is similar to Ricardo’s production with slight deviations. On the consumption part, Fisher market computes the prices. The unused goods are given Fisher-like prices in each iteration. Since the wages given here are per country wages, it can be computed that in first state, a person from A gets 0.0028 Rs and that from B gets 0.0089 Rs. While in the second, both get 0.0040 Rs. Because B is more efficient, on average, a person from B receives more than that from A.

### E.5 The basic difference

Ricardo’s theory is based on utility maximisation where, in a general sense, first the demand function is found using utilities and optimal production is decided. Thus, it maximises the total happiness a society can derive. On the other hand, as Eisenberg-Gale algorithm [12] suggests, an answer to Fisher’s market is the solution to the following optimization problem-

\[
\max_i \sum m_i \log(u_i) \tag{13}
\]

Thus, Eisenberg-Gale approach gives a more concrete notion as it weighs utility of each person/class with the money available. In other words, because of more purchasing power of the higher classes, their utility gets a higher weightage as compared to the lower ones. An interesting interpretation of this result is that the multi-agent model derives global optimal while all agents maximize their utilities. Also, it is interesting to see that prices and allocations emerge as a solution to this optimization problem. This is evident as while maximizing the utility, we have taken money considerations into account. Similarly, it is clear from the dual understanding of this problem,
which is the determination of wages and production, that wages are nothing but prices attributed to the skills labourers have and their demand from the society.

F THEOREM

Theorem F.1. Given a market comprising of two or more countries, corresponding technology matrix $T$, utility matrix $U$, and segments matrix $L$, the optimal production found by the new method lies on its PPF. Moreover, in case the countries have same linear utilities, both the models, Ricardo and Heterodox produce the same output for production.

Proof. To find the production, we use the following optimization program

$$\max_q \quad p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4$$

s.t.

$$q_j \geq 0 \quad \forall j$$

$$T_{11}q_1 + T_{12}q_2 \leq Y_1$$

$$T_{21}q_3 + T_{22}q_4 \leq Y_2$$

(14)

Whereas, the (PPF) is a curve consisting of all maximum output possibilities for two goods, given resource constraints. In other words, for a point on PPF, production of one good can only increase if that of other good decreases. Let us assume that the optimal production given by the above program does not lie on the PPF.

In case of a generic vector $p$, the KKT conditions require that -

$$p_1 = T_{11}w_1 - \lambda_1$$

$$p_2 = T_{12}w_1 - \lambda_2$$

$$p_3 = T_{21}w_2 - \lambda_3$$

$$p_4 = T_{22}w_2 - \lambda_4$$

where $\lambda$ correspond to the multipliers associated with $q_j \geq 0$.

It is evident that for positive $q_j$'s, $\lambda_j$ should equal zero, so $p_j$ should be equal to $T_{ij}w_i$. Therefore, for a given vector $p$, either $q_1$ or $q_2$ is positive. Similarly, one of $q_3$ and $q_4$ is positive. Thus, it can be deduced that each country would specialise by producing exactly one good.

Since the optimal point does not lie on the PPF, WLOG, let the optimum point $Q$ be such that $X = q_1 + q_3$ (Computers) can increase while $Y = q_2 + q_4$ (Rice) remains constant. This means that at least one of $q_1$ or $q_3$ increases while $T_{11}q_1 + T_{12}q_2 \leq Y_1$ and $T_{21}q_3 + T_{22}q_4 \leq Y_2$ hold. This implies that $U$ increases as $X$ increases, which contradicts the optimality of $Q$. Therefore, the production lies on the PPF of both countries.

We note that this also implies that inequality constraints are tight i.e. the amount that each country produces lies on that country’s PPF.

• In case of linear utilities, we have $U$ as follows

$$\begin{bmatrix}
C & R & C & R \\
u_1 & u_2 & u_1 & u_2 \\
u_1 & u_2 & u_1 & u_2
\end{bmatrix}$$

This has two implications. First, the Fisher Forest of allocations is a complete graph where both countries buy both the goods produced. Next, the iterative method gives the following output for prices, irrespective of the produced goods-

$$p = \begin{bmatrix}
p_1 & p_2 & p_1 & p_2
\end{bmatrix} = \frac{u_2}{u_1} \times \begin{bmatrix}
p_1 & p_1
\end{bmatrix}$$
Therefore, we have both optimization programs as-

\[
\begin{align*}
\max_q & \quad u_1 q_1 + u_2 q_2 + u_3 q_3 + u_4 q_4 \\
\text{s.t.} & \quad Tq \leq Y \\
q_j & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_q & \quad p_1 q_1 + p_2 q_2 + p_1 q_3 + p_2 q_4 \\
\text{s.t.} & \quad Tq \leq Y \\
q_j & \geq 0
\end{align*}
\]

(15)

It can be observed that both programs produce the same outputs for \( q \) since \( u_1 = p_1 \) \( u_2 = p_2 \)

We note that there exists only one stable state in case of linear utilities, which can be computed by solving the above program. \( \square \)

G MARKET CODE DOCUMENTATION

In this section, we illustrate the Fisher market code and the iterative process built using it, which is used to compute the Heterodox equilibrium.

G.1 Fisher market code - plc:

Fisher market is an economic model which determines prices and allocations of goods to the buyers, based on their utility preferences. The input to a typical Fisher market is a set of buyers \( B \) s.t. \( |B| = l \), a set of goods \( G \) s.t. \( |G| = n \), a utility function \( U_i(x) : \mathbb{R}^n \to \mathbb{R} \) for each buyer \( i \), a quantity vector \( q = (q_j)_{j \in G} \) and a money vector \( m = (m_i)_{i \in B} \), where \( x \) is a set of goods consumed, \( q_j \) is the quantity of good \( j \), and \( m_i \) is the money possessed by buyer \( i \). The market is solved in such a way that the money is completely spent and the market is cleared i.e. all goods are sold.

Here, instead of individual buyers, we deal with labour classes as the basic units. Also, We consider markets where we have piecewise linear concave (plc) utilities. We do so by giving utility inputs in the form of linear segments. For each good, slope of the utility decreases successively which follows from the law of diminishing marginal utilities . The construction of inputs is illustrated below.

G.1.1 Inputs.

(1) U: Utility matrix \((l * n * k)\)

- \( l \): Number of labour classes
- \( n \): Number of goods
- \( k \): Number of segments

Utility can be understood as happiness per person, per good i.e. \( u_{ijk} = \) utility/happiness that a person from labour class \( i \) derives by consuming one unit of good \( j \) in segment \( k \). Note that utilities are concave.

For example: When \( l = 3, g = 4, k = 2 \);

\[
U = \begin{bmatrix}
0.8 & 0.4 & 0.5 & 0.45 \\
0.3 & 0.75 & 0.2 & 0.5 \\
0.25 & 0.4 & 0.8 & 0.35 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.6 & 0.25 & 0.2 & 0.3 \\
0.2 & 0.4 & 0.1 & 0.4 \\
0.15 & 0.36 & 0.4 & 0.2 \\
\end{bmatrix}
\]

Here, we have three labour classes, four goods and two segments i.e. for each labour class, the utility of a good decreases to another value after consuming some fixed amount of it.

\( e.g. \, u_{141} = 0.45 = \) Utility of class 1 for good 4 in the first segment and, \( u_{142} = 0.3 = \) utility for the same in the second segment.

(2) L: Segments matrix \((l * n * k)\)

\( L \) denotes the maximum length of each segment. This is also per person, per good.

\( L_{ijk} = \) The maximum value of good \( j \) in the \( k \)th segment for which a person from class \( i \) has
utility $u_{ijk}$. The length of the last segment is assumed to be infinity i.e the final utility value lasts till the good gets over, and thus, the last segment values are ignored.

For example:

$$L = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 2.5 & 3.8 & 6 & 0.5 \\ 9 & 6 & 1.5 & 3 \end{bmatrix}; \begin{bmatrix} 6 & 4 & 3 & 3.5 \\ 2.5 & 8 & 4 \\ 8 & 5 & 4 & 8 \end{bmatrix}$$

e.g. If we consider U as above, a person from class 2 has utility 0.3 until she has consumed 2.5 units of good 1, and later, her utility reduces to 0.2 and remains 0.2 till the end, irrespective of the given corresponding second segment entry.

(3) $t$: The number of goods produced. ($1 \times n$)

For example,

$$t = \begin{bmatrix} 12 & 20 & 25 & 18 \end{bmatrix}$$

Here, four goods are produced in the above mentioned quantities.

(4) $m$: The money vector ($l \times 1$) For example,

$$m = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix}$$

Here, $m$ shows the money endowed to each class as a whole.

G.1.2 adplc To Fisher. We have two programs which together solve the Arrow Debreu market, namely adplc and FormAbeq. We make use of these to solve the Fisher market.

Inputs to adplc: $U$, $W$, $L$

where, $W$: Endowment matrix and $U$, $L$ are as above.

Outputs: Prices, Money allocations

We convert A-D to Fisher market by writing the endowment matrix as-

$$W = \begin{bmatrix} m_1/M & m_1/M & m_1/M \\ m_2/M & m_2/M & m_2/M \\ m_3/M & m_3/M & m_3/M \end{bmatrix}$$

where, $M=\text{sum}(m)$

The remaining inputs are kept same.

The adplc code assumes that quantity of each good is 1. It normalizes our inputs $W$ and $L$ accordingly. We appropriately scale the outputs by using $t$ and $m$ to derive corresponding outputs for the Fisher market. An example illustrating the solution to a Fisher market is given in the next section.

G.1.3 Example. Let us consider three labour classes, four goods and $U$, $L$, $t$ and $m$ as follows:

$$U = \begin{bmatrix} 0.8 & 0.4 & 0.5 & 0.45 \\ 0.3 & 0.75 & 0.2 & 0.5 \\ 0.25 & 0.4 & 0.8 & 0.35 \end{bmatrix}; \begin{bmatrix} 0.6 & 0.25 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.1 & 0.4 \\ 0.15 & 0.36 & 0.4 & 0.2 \end{bmatrix}$$

$$L = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 2.5 & 3.8 & 6 & 0.5 \\ 9 & 6 & 1.5 & 3 \end{bmatrix}; \begin{bmatrix} 6 & 4 & 3 & 3.5 \\ 2.5 & 8 & 4 \\ 8 & 5 & 4 & 8 \end{bmatrix}$$

$$m = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix}; t = \begin{bmatrix} 12 & 20 & 25 & 18 \end{bmatrix}$$
We solve the Fisher market to find prices of these four goods and their allocations.

Outputs are -

\[
\text{Money Allocations} = \begin{bmatrix}
2.205 & 1.47 & 2.859 & 1.47 \\
0 & 1.397 & 0 & 0.184 \\
0 & 2.205 & 0.613 & 0
\end{bmatrix};
\begin{bmatrix}
6.616 & 0 & 0 & 1.379 \\
0 & 1.836 & 0 & 3.583 \\
0 & 0.443 & 6.739 & 0
\end{bmatrix}
\]

This matrix tells how each class has spent money on these four goods. For Ex, labour class 2 (row 2) has bought only two goods; good 2 and good 4, and it has spent 1.397 + 0.184 + 1.836 + 3.583 = 7 bucks in total.

Here, sum of rows = [16 7 10] i.e. each labour class has completely spent its money.

\[
\text{Goods Allocations} = \begin{bmatrix}
3 & 4 & 7 & 4 \\
0 & 3.8 & 0 & 0.5 \\
0 & 6 & 1.5 & 0
\end{bmatrix};
\begin{bmatrix}
9 & 0 & 0 & 3.75 \\
0 & 5 & 0 & 9.75 \\
0 & 1.2 & 16.5 & 0
\end{bmatrix}
\]

This displays the allocation of goods. For ex, total allocation of good 3 (column 3) = 7 + 1.5 + 16.5 = 25 units. Note how first segment entries match with the corresponding L ones.

Here, sum of columns= [12 20 25 18] which equals the total production. Thus, all goods are sold completely.

And the prices are-

\[
\text{Prices} = [0.735 \ 0.368 \ 0.408 \ 0.368]
\]

Thus, \( \text{Prices} \times \text{Production}(t^T) = 33 = \text{Sum of m= 16 + 7 + 10} \)

### G.2 Closed loop code - Trade

#### G.2.1 Inputs

As explained below, this program takes T, U, L, Y, p as inputs and outputs production, next set of prices, wages, Allocations etc. U, L are as before. As explained in the theory, it uses convex optimization tools to find production (t) using the initial set of prices. Further, wages (m) are computed. Then, using 'plc', and U, L, t, m as inputs, we can derive other outputs like prices and allocations.

1. T is the technology matrix. \((l \times n)\)

   \(T_{ij}\) is the amount of labour \(L_j\) needed to produce one unit of \(G_i\). In simple terms, to produce one good, we have labour requirements as per its corresponding column.

   For example,

   \[
   T = \begin{bmatrix}
   1 & 0 & 0 \\
   1 & 1 & 0 \\
   1 & 1 & 1
   \end{bmatrix}
   \]

   Thus, to produce one unit of good 2, we need one person from labour class 2 and one person from labour class 3.

2. p is the initial price vector. Using p and T, 'Trade' finds the production by optimization.

3. Y: The number of labour units available in each class.

Thus, using 'Trade', we have set up an iterator which computes desired market variables. As per the proposed method, given a \(m \times n\) technology matrix, a sub matrix is chosen where the number of buyers and goods is same, and market variables are computed. In the next iteration, after modifying prices for the unused goods, the same procedure is repeated. All this is illustrated through the following examples.
G.2.2 \textit{Examples}. 

(1) Here we consider a market with three labour classes and three possible goods. We start with a price vector and see how various goods are produced and how classes derive wages from those. 

The following is the first iteration. 

Let $T, U, L, Y, p$ be as follows- 

\[
T = \begin{bmatrix}
1 & 0.4 & 0.5 \\
0.5 & 1.5 & 0.25 \\
0.2 & 0.35 & 0.6
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
0.85 & 0.5 & 0.4 \\
0.4 & 0.9 & 0.45 \\
0.55 & 0.4 & 0.8
\end{bmatrix} ; \begin{bmatrix}
0.6 & 0.2 & 0.17 \\
0.3 & 0.5 & 0.2 \\
0.2 & 0.35 & 0.6
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
0.33 & 0.40 & 0.1 \\
0.3 & 0.4 & 0.2 \\
0.93 & 0.6 & 0.7
\end{bmatrix} ; \begin{bmatrix}
0.58 & 0.4 & 0.2 \\
0.20 & 0.5 & 0.08 \\
0.84 & 0.5 & 0.45
\end{bmatrix}
\]

$Y = [10, 10, 10]$; $p = [1, 1.2, 1.3]$ 

And the outputs are (precise up to 3 decimal places )- 

\[\text{prices} = [2.182, 1.284, 1.15] \]

\[Wages = [5.384, 2.831, 16] \]

Normalized prices and wages - 

\[\text{prices} = [0.09, 0.053, 0.0475] \]

\[Wages = [0.222, 0.117, 0.66] \]

The production in units is (All the three goods are produced) - 

\[
q = \begin{bmatrix}
1.5 \\
3.85 \\
13.92
\end{bmatrix}
\]

Edited $L$ (according to $Y$)- 

\[
L = \begin{bmatrix}
3.3 & 4 & 1 \\
3 & 4 & 2 \\
9.3 & 6 & 7
\end{bmatrix} ; \begin{bmatrix}
5.8 & 4 & 2 \\
2 & 5 & 0.8 \\
8.4 & 5 & 4.5
\end{bmatrix}
\]

Goods allocation- 

\[
Alloc = \begin{bmatrix}
1.5 & 1.645 & 0 \\
0 & 2.205 & 0 \\
0 & 0 & 7
\end{bmatrix} ; \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 6.92
\end{bmatrix}
\]

Note that the total money is conserved- 

\[p1 \times q = 24.215 = \text{sum}(m) = 5.384 + 2.831 + 16 \]

(2) However, when above prices are used as input, we do not see the same market solution. In the third iteration, only good 1 and good 2 are produced and wages are allotted accordingly. The following tables summarize the results of 7 such iterations. It can be observed there is an underlying pattern in occurrence of these states. This is a toggling state result.
Active Goods =
(1 represents that the goods is produced and vice versa.)

| G-1 | G-2 | G-3 |
|-----|-----|-----|
| 1   | 1   | 1   |
| 1   | 1   | 1   |
| 1   | 1   | 0   |
| 1   | 1   | 1   |
| 1   | 1   | 0   |
| 1   | 1   | 1   |

Wage Iterations (up to two significant digits) =

class-1  class-2  class-3

|    |    |    |
|----|----|----|
| 0.22 | 0.12 | 0.66 |
| 0.83 | 0.12 | 0.05 |
| 0.86 | 0.14 |    0 |
| 0.70 | 0.06 | 0.24 |
| 0.86 | 0.14 |    0 |
| 0.70 | 0.06 | 0.24 |
| 0.86 | 0.14 |    0 |

(3) With T and U as given below and same L, p and Y as above, we get the following output. Here, all goods are produced and all classes receive wages.

\[
T = \begin{bmatrix}
1.00 & 0.10 & 0.50 \\
0.50 & 0.80 & 0.25 \\
0.20 & 0.35 & 0.60 \\
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
0.85 & 0.3 & 0.4 \\
0.4 & 0.9 & 0.35 \\
0.3 & 0.4 & 0.80 \\
\end{bmatrix}; \begin{bmatrix}
0.6 & 0.2 & 0.17 \\
0.3 & 0.5 & 0.2 \\
0.2 & 0.35 & 0.6 \\
\end{bmatrix}
\]

Outputs:

Active Goods =

| G-1 | G-2 | G-3 |
|-----|-----|-----|
| 1   | 1   | 1   |
| 1   | 1   | 1   |
| 1   | 1   | 1   |
| 1   | 1   | 1   |

Wage Iterations (up to two significant digits) =

| class-1 | class-2 | class-3 |
|---------|---------|---------|
| 0.11    | 0.29    | 0.6     |
| 0.092   | 0.207   | 0.7     |
| 0.086   | 0.096   | 0.82    |
| 0.067   | 0.091   | 0.84    |
|     | G-1 | G-2 | G-3 |
|-----|-----|-----|-----|
| 0.067 | 0.091 | 0.84 |

Price Iterations (in fractions) =

|     | G-1 | G-2 | G-3 |
|-----|-----|-----|-----|
| 1.00 | 1.20 | 1.30 |
| 0.03 | 0.04 | 0.05 |
| 0.03 | 0.04 | 0.06 |
| 0.03 | 0.04 | 0.06 |