Integration of algebroidal functions *

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Abstract

In this paper, we introduce the integration of algebroidal functions on Riemann surfaces for the first time. Some properties of integration are obtained. By giving the definition of residues and integral function element, we obtain the condition that the integral is independent of path. At last, we prove that the integral of an irreducible algebroidal function is also an irreducible algebroidal function if all the residues at critical points are zeros.

Keywords: algebroidal function, integral, integral function element, direct continuation.

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1 Introduction

Algebroidal functions are a kind of important multi-value functions. For example, in the field of complex differential equation, algebroidal solutions are more general than meromorphic solutions. But there are few results on algebroidal functions for lack of effective tools. Though integration is a basic definition, we have not seen any research on the integral of algebroidal functions, even in [1, 2]. Note that the integral of algebroidal functions we define is completely different from [3]. For holomorphic functions, the integrals can be defined as the limit of the same kind as that encountered in a usual integral. But for algebroidal functions, because of its multi-valuedness, we can not define the integral as what we usually do for holomorphic functions. In this paper, we see algebroidal functions as single-valued on the Riemann surfaces. The definite integral on the Riemann surface is defined and some of its properties are obtained in [4, 5]. Then by giving the definition of residues at critical points, we prove that the integral is independent of path if all the residues at critical points are zeros. Under the same assumption, we define the integral function element and prove that they are unchangeable for direct continuation. And last, we obtain the integral of an irreducible algebroidal function is also an irreducible algebroidal function if all the residues at critical points are zeros.

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First, we provide the definition of algebroidal functions. Let $A_1(z), A_2(z), \ldots, A_k(z)$ be a group of meromorphic functions in the complex plane $C$, then the equation in two variables

$$\Psi(W, z) = W^k + A_1(z)W^{k-1} + \ldots + A_k(z) = 0$$ (1)

defines a $k$-valued algebroidal function $W(z)$ in $C$.

The equation (1) is irreducible if it can not be expressed as the product of two non-mororphic functions. In this case, we say the algebroidal function is irreducible. In this paper, we confine our consideration on irreducible algebroidal functions. We use the standard definitions and notations of algebroidal functions; e.g., see [1, 6, 7, 8, 9].

The resultant of $\Psi(W, z)$ and its partial derivative $\Psi_W(W, z)$, which is said to be the discrimination of $W(z)$, is denoted by $R(\Psi, \Psi_W)(z)$. For an irreducible algebroidal function, we have $R(\Psi, \Psi_W)(z) \neq 0$. Hence, points in the complex plane can be divided into two kinds, say critical points and regular points. By critical points, we mean points in the set $S_W := \{z; R(\Psi, \Psi_W)(z) \neq 0\} \cup \{z; z$ is the pole of some $A_j(z), j = 0, 1, \ldots, k\}$. And points in the set $T_W := C - S_W$ are regular points. It can be deduced that critical points are isolated.

**Definition 1.** By a function element $(p(z), D_a)$, or $(p(z), a)$, we mean a simply-connected domain $D_a$ including $a \in C$ and a holomorphic function $p(z)$ in $D_a$. Two function elements $(p(z), a)$ and $(q(z), b)$ are equal, if $a = b$ and there is a neighborhood $U$ of $a$ such that $p(z) = q(z)$ in $U$. If for all $z \in D_a$, we have $\Psi(z, p(z)) = 0$, then $(p(z), D_a)$ is said to be a function element of the algebroidal function $W(z)$.

Suppose $W(z)$ is an irreducible algebroidal function defined by (1). If there is an ordered pair $(w_0, z_0)$ satisfying

(i) $\Psi(w_0, z_0) = 0$,

(ii) $\Psi_W(w_0, z_0) \neq 0$,

by the implicit function theorem, then there uniquely exists a function element $(w(z), D_{z_0})$ such that $w(z_0) = w_0$ and $\Psi(w(z), z) \equiv 0$ for all $z \in D_{z_0}$ (see [5]). Or, the ordered pair $(w(z), D_{z_0})$ is a function element of the algebroidal function $W(z)$. We denote the set of all the function element of the algebroidal function $W(z)$ by $\bar{T}_W$.

**Definition 2.** A function element $(q(z), b)$ is said to be the direct continuation of $(p(z), a) = (p(z), D_a)$, if two conditions hold:

(i) $b \in D_a$,

(ii) there is a neighborhood $U$ of $b$, such that $U \subset D_a$ and $p(z) = q(z)$ for all $z \in U$.

Hence, it can be denoted by $(p(z), b) = (q(z), b)$.

By this definition, we have for all $u \in D_a$, function element $(p(z), u)$ is a direct continuation of $(p(z), a)$.

**Remarks 1.** (i) If $(w(z), a)$ is a function element of the algebroidal function $W(z)$, by the uniqueness of direct continuation, all its direct continuations are belong to $W(z)$.

(ii) Function element $(w(z), D_a)$ and all its direct continuations $\{(q(z), b)\} (b \in D_a)$ form a neighborhood $U_w$ of $(w(z), D_a)$ on the Riemann surface. And the function $w[(q(z), b)] := w(b) \in C$ is an analytic function in $U_w$.  

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The critical points have been so far excluded from our considerations. Next we will consider the points in \( S_W \). For any point \( a \in S_W \), there is an element \( (q(z), U_a) = (q(z), a) \), where \( q(z) \) can be written as an Puiseux series

\[
q(z) = \sum_{n=0}^{\infty} B_n (z - a)^{n/m}, \quad B_u \neq 0.
\]  

(2)

Especially, when \( m = 1, u \geq 0 \), the element \( (q(z), U_a) \) is the function element we mentioned above. When \( u < 0 \), the element is said to be a pole element. When \( m > 1 \), we say that it is an algebraic element and \( a \) is a branch point of order \( m - 1 \). Both pole elements and algebraic elements are said to be singular elements.

From [4], we obtain that the domain of an irreducible algebroidal function is a Riemann surface composed by function elements and singular elements. We then define the integrals on these Riemann surfaces.

## 2 Definite integrals of algebroidal functions

**Definition 3.** Suppose that \( \tilde{C} \) is the Riemann surface defined by a \( k \)-valued algebroidal function \( W(z) \) and \( \tilde{L} \) is an arc on \( \tilde{C} \) satisfying

\[
\tilde{L} : (w_i(z), z(t)) \quad (\alpha \leq t \leq \beta) \subset \tilde{T}_W.
\]

The elements \((w_0(z), a = z(\alpha))\) and \((w(z), b = z(\beta))\) are the initial point and terminal point of \( \tilde{L} \), respectively. We choose \( n \) points along \( \tilde{L} \) arbitrarily:

\[
(w_0(z), z(t_0) = z(\alpha)), \quad (w_1(z), z(t_1)), \quad (w_2(z), z(t_2)), \quad \cdots,
\]

\[
(w_{n-1}(z), z(t_{n-1})), \quad (w(z), z(t_n) = z(\beta)).
\]

Since \( W(z) \) is continuous on \( \tilde{L} \), when \( \lambda = \max_{0 \leq j \leq n-1} |t_{j+1} - t_j| \) tends to zero, the limit

\[
J =: \lim_{\lambda \to 0} S_n = \sum_{j=0}^{n-1} w_j(z(t_j))(z(t_{j+1}) - z(t_j))
\]

exists. And we call it the definite integral of \( W(z) \) along \( \tilde{L} \).

It can be deduced from Definition 3 that the followings hold.

**Proposition 1.**

(i) \( \int_{\tilde{L}} \alpha W(z)dz = \alpha \int_{\tilde{L}} W(z)dz \), where \( \alpha \in C \);

(ii)

\[
\int_{\tilde{L}} W(z)dz = \int_{\tilde{L}_1} W(z)dz + \int_{\tilde{L}_2} W(z)dz + \cdots + \int_{\tilde{L}_n} W(z)dz,
\]

where \( \tilde{L} \) can be subdividing into \( n \) subarcs \( \tilde{L}_i \) \((i = 1, 2, \cdots, n)\);

(iii)

\[
\int_{\tilde{L}^-} W(z)dz = - \int_{\tilde{L}} W(z)dz,
\]

where \( \tilde{L}^- \) is the opposite path of \( \tilde{L} \);

(iv) If \( \tilde{L} \) is a close path on the Riemann surface and there is no singular element in it, then

\[
\int_{\tilde{L}} W(z)dz = 0.
\]

3
2.1 Independence of path

Definition 4. Suppose \((q(z), a)\) is a singular element of an algebroidal function \(W(z)\), where

\[
q(z) = \sum_{n=0}^{\infty} B_n(z-a)^{n/m}, \quad B_u \neq 0, m \geq 1 \text{ (or } u < 0). 
\]

The complex number \(m \cdot B_{-m}\) is called the residue of the singular element \((q(z), a)\).

Lemma 1. Suppose that \(\tilde{L}\) is a closed path on the Riemann surface with only one singular element \((q(z), a) = (q(z), D_a)\) inside of it. If the residue of the singular element \((q(z), a)\) is \(mB_{-m} = 0\), then

\[
\int_{\tilde{L}} W(z) dz = 0.
\]

Proof. Let \(\varepsilon \in (0, r)\) be sufficiently small and \(\tilde{\Gamma} \subset \tilde{T}_W\) be a closed curve on the Riemann surface satisfying

\[
\tilde{\Gamma}_m: (w_z(t)) (z(t) = a + \varepsilon e^{it}) \quad (0 \leq t \leq 2m\pi).
\]

By Proposition 1 and Definition 5, we have

\[
\int_{\tilde{L}} W(s) ds = \int_{0}^{2m\pi} \sum_{n=0}^{\infty} B_n r^{m/m} e^{int/m} \cdot i e^{it} dt
\]

\[
= i \sum_{n=0}^{\infty} B_n r^{1+\frac{m}{m}} \int_{0}^{2m\pi} e^{it(1+\frac{m}{m})} dt
\]

\[
= iB_{-m} \int_{0}^{2m\pi} dt = iB_{-m} 2m\pi = 0.
\]

\[\square\]

Theorem 2. (Independence of path) Suppose that \(\tilde{\mathbb{C}}\) is a Riemann surface defined by an irreducible algebroidal function \(W(z)\) and the residue of every singular element on it is zero. Then for any close path \(\tilde{L} \subset \tilde{T}_W\), we have

\[
\int_{\tilde{L}} W(z) dz = 0.
\]

Proof. Since \(\tilde{L}\) is a closed path, there are only finitely many singular elements in it, say \((q_j(z), B(a_j, r_j)) (j = 1, 2, \ldots, n)\). By adding finitely many curves \(\tilde{\gamma}_j \subset \tilde{T}_W\) \((j = 1, 2, \ldots, 2(n-1))\) in \(\tilde{L}\), then \(\tilde{L}\) and \(\tilde{\gamma}_j (j = 1, 2, \ldots, 2(n-1))\) form finitely many closed curves \(\tilde{L}_j (j = 1, 2, \ldots, n)\) and there is at most one singular element inside of each \(\tilde{L}_j\). Hence by Proposition 1, Lemma 4, and taking account of the cancellations along the curves \(\tilde{\gamma}_j (j = 1, 2, \ldots, 2(n-1))\), we have

\[
\int_{\tilde{L}} W(z) dz = \sum_{j=1}^{n} \int_{\tilde{L}_j} W(z) dz = 0.
\]

\[\square\]

Remarks 2. An algebroidal function satisfying the hypotheses in Theorem 4 does exist. For example, \(\sqrt{z}\) is an irreducible 2-valued algebroidal function in the complex plane. Element \((\sqrt{z}, 0)\) is the only singular element on its Riemann surface, whose residue is zero.
By Theorem 4, we have

**Corollary 1.** Suppose that $W(z)$ is an algebroidal function and all the residues of its singular elements are both zero. Then the define integration

$$\int_L W(z)dz$$

depends on the initial point $(w_0(z), a)$ and the terminal point $(w(z), b)$ only. Hence the define integration

$$\int_{(w_0(z), a)}^{(w(z), b)} W(z)dz$$

is well-defined.

### 2.2 Integral elements

In this part, we will define the integrals of all elements on the Riemann surface. First, we study the integral function elements:

**Definition 5.** Suppose that $W(z)$ is an algebroidal function and all the residues of its singular elements are both zero. Fix a function element $(w_a(z), a)$ on its corresponding Riemann surface. For any function element $(w_b(z), U_b)$, or $(w_b(z), b)$, we define its integral function element as

$$(\int_a w_b(z), b) := (c_{a,b} + \int_b^z w_b(s)ds, b),$$

where $\int_b^z w_b(s)ds$ is the complex integration and $c_{a,b}$ is the define integral

$$c_{a,b} = \int_{(w_a(z), a)}^{(w_b(z), b)} W(z)dz.$$

It can be deduced from Definition 5 that the integral function elements are also function elements. And we will see it is unchangeable for analytic continuation:

**Lemma 2.** Suppose that $W(z)$ is an algebroidal function and all the residues of its singular elements are both zero. Let function element $(w_b(z), u)$ be a direct continuation of function element $(w_b(z), b)$. Then the integral function element $(\int_a w_b(z), u)$ is the direct continuation of $(\int_a w_b(z), b)$.

**Proof.** It follows from Definition 5 that the integral function element of $(w_b(z), b)$ is

$$(\int_a w_b(z), b) \equiv (c_{a,b} + \int_b^z w_b(s)ds, b),$$

And the integral function element of $(w_b(z), u)$ is

$$\begin{align*} 
(\int_a w_b(z), u) & \equiv (\int_{(w_0(z), a)}^{(w_b(z), u)} W(z)dz + \int_u^z w_b(s)ds, u) \\
& = (\int_{(w_0(z), a)}^{(w_b(z), b)} W(z)dz + \int_{(w_0(z), b)}^{(w_b(z), u)} w_b(z)dz + \int_u^z w_b(s)ds, u) \\
& = (c_{a,b} + \int_b^u w_b(z)dz + \int_u^z w_b(s)ds, u)
\end{align*}$$
That is
\[
\left( \int_a w_b(z), u \right) = (c_{a,b} + \int_b^z w_b(s)ds, u)
\]  
(5)

By equality (4), (7) and Definition 2, we can obtain that integral function element \((\int_a w_b(z), u)\) is the direct continuation of integral function element \((\int_a w_b(z), b)\).

Hence, for all \(u \in U_b\), integral function element \((\int_a w_b(z), u)\) is the direct continuation of \((c_{a,b} + \int_b^z w_b(s)ds, b)\). The following lemma shows that the singular elements are weakly bounded.

This implies that every critical point is either analytic or a pole. For completeness, we give the proof of it. We should mention that the proof has been given in [5, 6].

**Lemma 3.** Let \(W(z)\) be an algebroidal function defined by (**1**). Then for any \(z_0 \in \mathbb{C}\), there exist \(r > 0, M > 0\) and \(n \in \mathbb{N}_+\), such that for function element \((p(z), a)\) satisfying \(\{0 < |z - a| < r\} \subset T_W\), we have

\[|(a - z_0)^n p(a)| < M.\]

**Proof.** Since the coefficients in equation (**1**) are meromorphic functions, they can be represented as Laurent series in \(\{0 < |z - z_0| < r\}\), respectively. That is

\[A_j(z) = \sum_{l=n_j}^{\infty} a^{(j)}_l (z - z_0)^l, \quad a^{(j)}_l \neq 0, j = 1, 2, \cdots, k\]

Let \(n_0 = \min \{n_j; j = 1, 2, \cdots, k\}\). Then there is a positive number \(M\) such that for any \(a \in \{0 < |z - z_0| < r\}\)

\[|(a - z_0)^{n_0} |(|A_1(a)| + |A_2(a)| + \cdots + |A_k(a)|) \leq M.\]

When \(|p(a)| \geq 1\), we have

\[|(a - z_0)^{n_0} p(a)| = \left| |(a - z_0)^{n_0} \left( A_1(a) + \frac{A_2(a)}{p(a)} + \cdots + \frac{A_k(a)}{p^{k-1}(a)} \right) \right| \leq |(a - z_0)^{n_0}| \left| \left| A_1(a) \right| + \left| \frac{A_2(a)}{p(a)} \right| + \cdots + \left| \frac{A_k(a)}{p^{k-1}(a)} \right| \right| \leq |(a - z_0)^{n_0}|(|A_1(a)| + |A_2(a)| + \cdots + |A_k(a)|) \leq M.\]

When \(|p(a)| < 1\), we have

\[|(a - z_0)^{n_0} p(a)| < r < M.\]

**Lemma 4.** (weakly bounded) Suppose that \(W(z)\) is an algebroidal function and all the residues of its singular elements are both zeros. Fix a function element \((w_a(z), a)\) on its corresponding Riemann surface. Then for any \(z_0 \in \mathbb{C}\), there exist two real positive numbers \(M, r\) and an integer \(n \geq 0\) such that for any integral function element \((\int_a w_a(z), u)\) satisfying \(u \in \{0 < |z - z_0| < r\}\), we have

\[|(u - z_0)^n \int_a w_a(u)| < M.\]

**Proof.** Let \(z_0 \in S^\circ_W\). Then by Lemma 3 there exist \(r > 0, M > 0\) and an integer \(n \geq 0\) such that for any function element \((w(z), u)\) satisfying \(u \in \{0 < |z - z_0| < r\}\), we have

\[|(u - z_0)^n w(u)| < M.\]
Cutting $B_0$ into simply connected domain $B_- := \{0 < |z - z_0| < r \cap |\arg(z - z_0)| < \pi \} \subset T_W$, we have $k$ single branches $w_1(z), w_2(z), \ldots, w_k(z)$ of $W(z)$.

For $b \in B_-$, there are $k$ distinct function elements $(w_1, b), (w_2, b), \ldots, (w_k, b)$. By Definition 6, the corresponding integral function elements are

$$(\int_a^{(w_j(z), b)} W(z)dz + \int_b^z w_j(s)ds, \quad j = 1, 2, \ldots, k).$$

Let

$$H := \max\{|\int_{w_0(z), a}^{(w_j(z), b)} W(z)dz|, \quad j = 1, 2, \ldots, k\}.$$  

For any $(u \in B_0)$, the integral function elements of $(w_j(z), u)$ $(j = 1, 2, \ldots, k)$ are $(\int_a^{w_j(z), u} W(z)dz, (w_1, b), (w_2, b), \ldots, (w_k, b))$. It follows from Lemma 2 that they are the direct continuations of $(\int_a^{w_j(z), b} W(z)dz, (w_1, b), (w_2, b), \ldots, (w_k, b))$. Hence, for any $u \in B_- \cap \{|u| > |b|\}$, we have

$$|\int_{w_0(z), a}^{(w_j(z), b)} W(z)dz| + |\int_b^u w_j(s)ds| \leq H + |\int_b^u w_j(s)ds| \leq H + 2\pi|b| \cdot \frac{M}{|u - z_0|^\alpha}.$$

\( \square \)

### 3 Integral of algebroidal function

Then we can define the integral of algebroidal function:

**Definition 6.** We say $M(z)$ is the integral of an algebroidal function $W(z)$ if $M'(z) = W(z)$.

**Theorem 3.** Suppose that $W(z)$ is an irreducible algebroidal function and all the residues of its singular elements are both zeros. Then the integral of $W(z)$ exists and it is also irreducible. Further, if $(w_a(z), a)$ is given, it is unique.

**Proof.**

**Step 1.** For all $b \in T_W$, there are $k$ distinct function elements $\{(w_j(z), b) = (w_j(z), U_b)\}_{j=1}^k$.

It follows from Definition 6 there are $k$ function elements

$$(\int_a^{w_j(z), U_b} W(z)dz + \int_b^z w_j(s)ds, \quad j = 1, 2, \ldots, k).$$

By Lemma 2, for $u \in U_b$, there are $k$ integral function elements

$$(\int_a^{w_j(z), U_b} W(z)dz + \int_b^z w_j(s)ds, \quad u), \quad j = 1, 2, \ldots, k; \quad u \in U_b.$$  

They are the direct continuations of $(\int_a^{w_j(z), b} W(z)dz, (w_1, b), (w_2, b), \ldots, (w_k, b))$. Then we can define a group of functions in $U_b$

$$B_1(z) := -\int_a^z w_1(z) - \int_a^z w_2(z) \ldots - \int_a^z w_k(z);$$

$$B_2(z) := \sum_{1 \leq i < j \leq k} \int_a^z w_i(z) \int_a^z w_j(z);$$

$$\ldots$$

$$B_k(z) := (-1)^k \int_a^z w_1(z) \int_a^z w_2(z) \cdots \int_a^z w_k(z).$$
It is trivial that \( \{B_j(z)\}_{j=1}^k \) are single-valued analytic functions in \( U_b \). Further, they are analytic in \( T_W \).

For any \( d \in S_W \), it is isolated. By Lemma 4, \( d \) is a pole of all \( \{B_j(z)\}_{j=1}^k \) at most. Hence, we obtain \( k \) meromorphic functions \( \{B_j(z)\}_{j=1}^k \) in \( C \).

We then turn to prove that \( q \) by \( a \)

its singular elements are both zeros. Then we can obtain a family of integral of \( \text{Theorem 4.} \)

Suppose that \( \) is also a meromorphic function. If the initial point is given, the integral is unique.

We obtain \( \)

\( \text{Step 2.} \) We then turn to prove that \( M(z) \) is irreducible. Otherwise, equation (7) can be composed by a \( q \)-valued \( (q < k) \) algebroidal function \( M_q(z) \). We suppose the derivative of \( M_q(z) \) is defined by

\[
(W - w_1(z))(W - w_2(z)) \cdots (W - w_q(z)) = W^q + d_1(z)W^{q-1} + \cdots + d_q(z) = 0.
\]

Then the above equation is a factor of the equation (7), which contradicts that \( W(z) \) is irreducible.

\[ \square \]

\textbf{Corollary 2.} Suppose \( f(z) \) is a meromorphic function in the simply-connected domain \( D \) and all the residues at poles are zeros. Then the integral of meromorphic function \( f(z) \) exists and it is also a meromorphic function. If the initial point is given, the integral is unique.

\textbf{Theorem 4.} Suppose that \( W(z) \) is an irreducible algebroidal function and all the residues of its singular elements are both zeros. Then we can obtain a family of integral of \( W(z) \) with an arbitrarily constant.

\textbf{Proof.} For a fixed function element \( (w_0(z), a) \), the unique equation which define the integral of algebroidal function \( W(z) \) is

\[
M^k + B_1(z)M^{k-1} + \cdots + B_k(z) = (M - \int_a w_1(z))(M - \int_a w_2(z)) \cdots (M - \int_a w_k(z)) = 0.
\]

Then the family of integral function with an arbitrarily constant \( c \) is

\[
M^k + B_1^c(z)M^{k-1} + \cdots + B_k^c(z) = \left[ M - (c + \int_a w_1(z)) \right] \left[ M - (c + \int_a w_2(z)) \right] \cdots \left[ M - (c + \int_a w_k(z)) \right]
\]

\[
= [(M - c) - \int_a w_1(z)][(M - c) - M - (c + \int_a w_2(z)) \cdots [(M - c) - \int_a w_k(z)]
\]

\[
= (M - c)^k + B_1(z)(M - c)^{k-1} + \cdots + B_k(z) = 0.
\]

Hence, we can obtain the coefficients are

\[
B_1^c = -cC_1^k + B_1;
\]

\[
B_2^c = c^2C_2^k - cC_{k-1}^1B_1 + B_2;
\]

\[
\cdots \cdots
\]

\[
B_j^c = (-1)^j[c^jC_{k-j}^j - c^{j-1}C_{k-j-1}^{j-1}B_1 + \cdots + (-1)^jB_j];
\]

\[
\cdots \cdots
\]

\[
B_k^c = (-1)^k[c^kC_k^k - c^{k-1}C_{k-1}^{k-1}B_1 + \cdots + (-1)^kB_k]
\]

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\[ \begin{align*}
B_1^c &= B_1 - cC_k^1; \\
b_2^c &= B_2 - cC_{k-1} B_1 + c^2 C_k^2; \\
\cdots \cdots \\
b_j^c &= B_j - cC_{k-j+1} B_{j-1} + \cdots + (-c)^j C_k^j; \\
\cdots \cdots \\
b_k^c &= B_k - cC_{k-k+1} B_{k-1} + \cdots + (-c)^k C_k^k.
\end{align*} \]

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