A survey on frame representations via dynamical sampling

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Abstract

Dynamical sampling deals with representations of a frame \( \{ f_k \}_{k=1}^\infty \) as an orbit \( \{ T^n \varphi \}_{n=0}^\infty \) of a linear and possibly bounded operator \( T \) acting on the underlying Hilbert space. It is known that the desire of boundedness of the operator \( T \) puts severe restrictions on the frame \( \{ f_k \}_{k=1}^\infty \). The purpose of the paper is to present an overview of the results in the literature and also discuss various alternative ways of representing a frame; in particular the class of considered frames can be enlarged drastically by allowing representations using only a subset \( \{ T^{\alpha(k)} \varphi \}_{k=1}^\infty \) of the operator orbit \( \{ T^n \varphi \}_{n=0}^\infty \). In general it is difficult to specify appropriate values for the scalars \( \alpha(k) \) and the vector \( \varphi \); however, by accepting an arbitrarily small and controllable deviation between the given frame \( \{ f_k \}_{k=1}^\infty \) and \( \{ T^{\alpha(k)} \varphi \}_{k=1}^\infty \) we will be able to do so.

1 Introduction

The purpose of this paper is to give an overview of the research topic dynamical sampling, seen from the frame perspective. We will also connect dynamical sampling with certain developments within linear dynamics, e.g., hypercyclic operators and operator orbits.

Dynamical sampling deals with representations of a frame \( \{ f_k \}_{k=1}^\infty \) as an orbit \( \{ T^n \varphi \}_{n=0}^\infty \) of a bounded linear operator \( T \) acting on the underlying Hilbert space. We will highlight a number of necessary conditions on the frame \( \{ f_k \}_{k=1}^\infty \) for such a representation to exist, restrictions that unfortunately exclude most classical frames. Motivated by this we discuss a number of alternative operator representations of frames, e.g., using unions of orbits. Most importantly, we will consider frame representations using suborbits \( \{ T^{\alpha(k)} \varphi \}_{k=1}^\infty \) for certain integers \( \alpha(k), k \in \mathbb{N} \), a step that enlarge the class of relevant frames considerably. As a final step we will consider approximate representations using
suborbits and show that by allowing a controllable and arbitrarily small deviation between \( \{ f_k \}_{k=1}^{\infty} \) and \( \{ T^{\alpha(k)} \varphi \}_{k=1}^{\infty} \) we will be able to specify appropriate choices of the scalars \( \alpha(k), k \in \mathbb{N} \).

The paper is presented in a problem-driven way, e.g., using Gabor frames and shift-invariant frames as motivation for the development. In the rest of the introduction we set the stage by providing the necessary background from operator theory and frame theory. The material about (exact) operator representations is collected in Section 2 while various methods for approximate representations are considered in Section 3. In the entire paper \( \mathcal{H} \) denotes a separable infinite-dimensional Hilbert space.

### 1.1 Operator representation of sequences

While our main interest is to analyse frames, several of our operator theoretical tools also occur in a more general context in the literature. For this reason we formulate the following definitions for general sequences in Hilbert spaces.

**Definition 1.1** Consider an ordered sequence \( \{ f_k \}_{k=1}^{\infty} = \{ f_1, f_2, f_3, \ldots \} \) of elements in \( \mathcal{H} \). If there exists a linear operator \( T : \text{span}\{ f_k \}_{k=1}^{\infty} \to \mathcal{H} \) such that \( f_{k+1} = T f_k \) for all \( k \in \mathbb{N} \), we say that the sequence \( \{ f_k \}_{k=1}^{\infty} \) is represented by the operator \( T \).

Note that the sequence \( \{ f_k \}_{k=1}^{\infty} \) being represented by \( T \) precisely means that

\[
\{ f_k \}_{k=1}^{\infty} = \{ f_1, f_2, f_3, \ldots \} = \{ f_1, T f_1, T^2 f_1, \ldots \} = \{ T^n f_1 \}_{n=0}^{\infty}.
\]

In operator theoretical terms, this is phrased by saying that the sequence \( \{ f_k \}_{k=1}^{\infty} \) is an orbit of the operator \( T \):

**Definition 1.2** Given a linear operator \( T : \mathcal{H} \to \mathcal{H} \) and some \( \varphi \in \mathcal{H} \), the sequence \( \{ T^n \varphi \}_{n=0}^{\infty} \) is called the orbit of \( \varphi \).

In general, the ordering of a given sequence \( \{ f_k \}_{k=1}^{\infty} \) is crucial for the question whether or not it can be represented by a linear operator. For example, if \( f \) is an arbitrary nonzero vector in \( \mathcal{H} \), the sequence \( \{ f, 0, 0, 0, \ldots \} \) is represented by the zero-operator, while the sequence \( \{ 0, f, 0, 0, \ldots \} \) can not be represented by a linear operator. We also observe that any linearly independent family \( \{ f_k \}_{k=1}^{\infty} \) can be represented by a linear operator \( T : \text{span}\{ f_k \}_{k=1}^{\infty} \to \mathcal{H} \), regardless of how the sequence is ordered; indeed, we simply define the operator \( T \) by \( T f_k := f_{k+1}, k \in \mathbb{N} \), and extend it by linearity. It was shown in [14] that for sequences \( \{ f_k \}_{k=1}^{\infty} \) spanning an infinite-dimensional space, linear independence is actually equivalent with the property of being representable by a linear operator:
Proposition 1.3 Consider any sequence \( \{f_k\}_{k=1}^{\infty} \) in \( \mathcal{H} \) for which \( \text{span}\{f_k\}_{k=1}^{\infty} \) is infinite-dimensional. Then the following are equivalent:

(i) \( \{f_k\}_{k=1}^{\infty} \) is linearly independent.

(ii) There exists a linear operator \( T : \text{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H} \) such that \( \{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty} \).

In the current paper we will exclusively be interested in representability via bounded operators. Even for linearly independent sequences, the possibility of such a representation depends on the ordering, as the next example demonstrates.

Example 1.4 Let \( \{e_k\}_{k=1}^{\infty} \) denote an orthonormal basis for \( \mathcal{H} \), and consider the sequence

\[
\{f_k\}_{k=1}^{\infty} = \{e_1, 2e_2, 3e_3, \ldots \} = \{ke_k\}_{k=1}^{\infty}.
\]

Then \( \{f_k\}_{k=1}^{\infty} \) is represented by the linear operator \( T \) defined by

\[
Te_k = \frac{k+1}{k} e_{k+1}, \quad k \in \mathbb{N};
\]

note that \( T \) extends to a bounded operator on \( \mathcal{H} \), with \( ||T|| = 2 \). On the other hand, the following reordering of the elements

\[
\{g_k\}_{k=1}^{\infty} = \{e_1, 10e_{10}, 2e_2, 10^2e_{10^2}, 3e_3, 10^3e_{10^3}, \ldots, 9e_9, 10^9e_{10^9}, 11e_{11}, 10^{10}e_{10^{10}}, \ldots \}
\]

clearly can not be represented by a bounded operator.

In the rest of the paper the focus will be on operator representations of frames. Readers who are interested in corresponding results for general sequences (in Banach spaces) are referred to [20].

1.2 Hypercyclic operators

The main focus of the paper is to consider exact and approximate representations of frames. Parts of the flow of the paper is motivated by the fact that there exist very special operators which have orbits that are dense in the underlying Hilbert space and thus allow to approximate arbitrary sequences.

Definition 1.5 A linear operator \( T : \mathcal{H} \to \mathcal{H} \) is hypercyclic if there exists \( \varphi \in \mathcal{H} \) such that the orbit \( \{T^n \varphi\}_{n=0}^{\infty} \) is dense in \( \mathcal{H} \). In this case the vector \( \varphi \) is called a hypercyclic vector.
Note that if \( T : \mathcal{H} \rightarrow \mathcal{H} \) is a hypercyclic operator with hypercyclic vector \( \varphi \), then the set \( \{ T^n \varphi \}_{n=N}^{\infty} = \{ T^n (T^N \varphi) \}_{n=0}^{\infty} \) is dense in \( \mathcal{H} \) for all \( N \in \mathbb{N} \); this implies that \( T^N \varphi \) is a hypercyclic vector for all \( N \in \mathbb{N} \), and thus that the set of hypercyclic vectors associated with \( T \) is dense in \( \mathcal{H} \).

The first example of a hypercyclic operator on a separable Hilbert space was constructed by Rolewicz [35] in 1969. It deals with the left-shift operator \( L(x_1, x_2, x_3, \ldots) := (x_2, x_3, \ldots) \) on \( \ell^2(\mathbb{N}) \):

**Lemma 1.6** Assume that \( a > 1 \) and let \( L \) denote the left-shift operator on \( \ell^2(\mathbb{N}) \). Then \( T := aL \) is a hypercyclic operator on \( \ell^2(\mathbb{N}) \).

More recently hypercyclic operators have also been constructed on \( L^2(\mathbb{R}) \):

**Example 1.7** In [11] it was proved that for any \( a \in \mathbb{R} \) there exists functions \( \omega : \mathbb{R} \rightarrow \mathbb{R} \) such that the operator \( T f(x) := \omega(x) f(x - a) \) is hypercyclic on \( L^2(\mathbb{R}) \). In particular, for the case \( a = 0 \) we can take any continuous decreasing function such that

\[
\omega(x) = \begin{cases} 
2 & \text{if } x \leq 0 \\
1/2 & \text{if } x \geq 1.
\end{cases}
\]

For a survey of other hypercyclic operators we refer to [26]. While the existence of hypercyclic operators is surprising in itself, the theory also contains several beautiful and intriguing results. We collect a few of them below, and refer to the literature [6, 25] for more results.

**Theorem 1.8** Consider a bounded linear operator \( T : \mathcal{H} \rightarrow \mathcal{H} \). Then the following hold:

(i) [4] If \( T \) is hypercyclic, then \( T^N \) is hypercyclic for every \( N \in \mathbb{N} \); moreover, \( T \) and \( T^N \) have the same set of hypercyclic vectors.

(ii) [23, 33] If there exist a finite collection of vectors \( \varphi_1, \ldots, \varphi_n \in \mathcal{H} \) such that the union of their orbits is dense in \( \mathcal{H} \), then one of the orbits is dense in \( \mathcal{H} \) by itself, i.e., \( T \) is hypercyclic.

(iii) [8] If an orbit \( \{ T^n \varphi \}_{n=0}^{\infty} \) is somewhere dense in \( \mathcal{H} \), it is automatically dense in \( \mathcal{H} \), i.e., \( T \) is hypercyclic with hypercyclic vector \( \varphi \).

Note in particular the result stated in Theorem 1.8 (i): phrased in words, it says that if \( T \) is hypercyclic with hypercyclic vector \( \varphi \), then the subset \( \{ \varphi, T^n \varphi, T^{2n} \varphi, \ldots \} \) is also dense in \( \mathcal{H} \), for every \( N \in \mathbb{N}! \)

Let us conclude this section with an interesting connection between hypercyclic operators and the celebrated **invariant subspace problem**, a classical open problem that can be phrased as follows:
Q: Does there exist a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ having no closed nontrivial invariant subspace?

Note that if $T : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator, the closed (sub)space $V := \overline{\text{span}}\{T^n \varphi\}_{n=0}^{\infty}$ is invariant for any $\varphi \in \mathcal{H}$. Thus, a hypercyclic operator for which every nonzero vector is hypercyclic would provide evidence for the existence of the type of operator asked for in the invariant subspace problem; indeed, such an operator is the only candidate. Taking Theorem 1.8 (iii) into account and following the formulation by Bourdon and Feldman in [8], we can phrase the invariant subspace problem as follows:

Q: Does there exists a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ having a nontrivial nowhere dense orbit?

1.3 Operator representations via suborbits

We already considered subsets of operator orbits in Section 1.2. This motivates the next definition.

Definition 1.9 Consider a linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ and some $\varphi \in \mathcal{H}$. Given an increasing sequence of nonnegative integers,

$$0 \leq \alpha(1) < \alpha(2) < \alpha(3) < \ldots,$$

the sequence $\{T^{\alpha(k)} \varphi\}_{k=1}^{\infty}$ is called a suborbit of $\{T^n \varphi\}_{n=0}^{\infty}$.

The following deep theorem by Halperin et al. [28] shows that we gain a significant freedom by asking for sequences to be represented by suborbits of bounded operators rather than the full orbit.

Theorem 1.10 Consider any linearly independent sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$. Then there exists a bounded operator $T : \mathcal{H} \rightarrow \mathcal{H}$ and an increasing sequence of nonnegative integers

$$0 \leq \alpha(1) < \alpha(2) < \alpha(3) < \cdots$$

such that

$$\{f_k\}_{k=1}^{\infty} = \{T^{\alpha(k)} f_1\}_{k=1}^{\infty}. \quad (1.1)$$

Theorem 1.10 is very appealing as a theoretical result, but it does not explain how one can choose the operator $T$ and the integers $\alpha(k)$ in practice. The proof in [28] involves the selection of a dense linearly independent sequence of elements in the underlying Hilbert space $\mathcal{H}$ and does not give a feasible way of identifying $T$ and $\alpha(k)$. For example, if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, it is easy to see directly that $\{e_k\}_{k=1}^{\infty} = \{T^n e_1\}_{n=0}^{\infty}$, where the operator $T$ is defined by $Te_k := e_{k+1}, k \in \mathbb{N}$; however, even in this simple case the
proof of Theorem 1.10 yields a very complicated procedure. One of the main
goals of the current paper is to provide more applicable and explicit methods
for representing sequences \( \{ f_k \}_{k=1}^{\infty} \) in an exact or approximate sense, with an
explicit given operator \( T \) and direct access to the integers \( \alpha(k) \).

1.4 Basic frame theory

In this section we will give a very short presentation of the key elements in
frame theory. Only aspects that are directly relevant for the current paper
will be addressed; for more information we refer to \cite{12} and \cite{29}.

A sequence \( \{ f_k \}_{k=1}^{\infty} \) in \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if there exist constants
\( A, B > 0 \) such that

\[
A \| f \|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H};
\]

it is tight if we can choose \( A = B \), and it is a Bessel sequence if at least the
upper frame condition holds. A Riesz basis is a frame which is at the same
time a basis; alternatively, a Riesz basis is a frame \( \{ f_k \}_{k=1}^{\infty} \) for which

\[
\sum_{k=1}^{\infty} c_k f_k = 0, \quad \{ c_k \}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \Rightarrow c_k = 0, \quad \forall k \in \mathbb{N}.
\]

If \( \{ f_k \}_{k=1}^{\infty} \) is a Bessel sequence, the synthesis operator is defined by

\[
U : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad U\{ c_k \}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k f_k;
\] (1.2)

it is well known that \( U \) is well-defined and bounded. An important role is
played by the kernel of the operator \( U \), i.e., the subset of \( \ell^2(\mathbb{N}) \) given by

\[
\mathcal{N}(U) = \left\{ \{ c_k \}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \left| \sum_{k=1}^{\infty} c_k f_k = 0 \right. \right\}. \quad (1.3)
\]

The frame operator is defined by

\[
S : \mathcal{H} \to \mathcal{H}, \quad Sf := UU^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.
\]

For a frame \( \{ f_k \}_{k=1}^{\infty} \), the frame operator is bounded, bijective, and self-adjoint; these
properties immediately lead to the important frame decomposition

\[
f = SS^{-1} f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}. \quad (1.4)
\]
The sequence \( \{S^{-1}f_k\}_{k=1}^\infty \) is also a frame; it is called the canonical dual frame. Frames that are not Riesz bases are overcomplete, and there exists \( \{g_k\}_{k=1}^\infty \neq \{S^{-1}f_k\}_{k=1}^\infty \) such that

\[
f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H}.
\]

Any frame \( \{g_k\}_{k=1}^\infty \) satisfying (1.5) for a given frame \( \{f_k\}_{k=1}^\infty \) is called a dual frame of \( \{f_k\}_{k=1}^\infty \).

The excess of a frame \( \{f_k\}_{k=1}^\infty \) is the maximal number of elements that can be removed yet leaving a frame. It is well-known that the excess equals \( \dim \mathcal{N}(U) \), the dimension of the kernel of the synthesis operator; see [5].

### 1.5 Structured function systems

In this section we will give a short introduction to a number of explicit frames in the Hilbert space \( L^2(\mathbb{R}) \). These frames will illustrate and motivate several results throughout the paper. As for Section 1.4 only results with direct relevance to the current paper are discussed.

In order to introduce the relevant frames we first need to consider a number of operators on \( L^2(\mathbb{R}) \). For \( a \in \mathbb{R} \), define the translation operator

\[
T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_a f(x) := f(x - a)
\]

and the modulation operator

\[
E_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad E_a f(x) := e^{2\pi i ax} f(x).
\]

The translation operators and the modulation operators are unitary.

Given a function \( \varphi \in L^2(\mathbb{R}) \) and some \( b > 0 \), the associated shift-invariant system is given by \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \). The frame properties of such systems are well understood, see, e.g., [7, 13, 5]. In particular, regardless of the choice of \( \varphi \in L^2(\mathbb{R}) \) and the parameter \( b > 0 \), the system can not be a frame for all of \( L^2(\mathbb{R}) \) but only for the subspace span\( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \). It is also known that \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \) is linearly independent for all \( \varphi \neq 0 \).

Classical examples of frames of the form \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \) are obtained by taking \( \varphi \) to be the sinc-function,

\[
sinc(x) := \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}
\]

Let us explain this in more detail. First, define the Fourier transform of \( f \in L^1(\mathbb{R}) \) by

\[
\hat{f}(\gamma) := \int_{-\infty}^\infty f(x) e^{-2\pi i \gamma x} dx;
\]
we extend the Fourier transform in the standard way to a unitary operator on $L^2(\mathbb{R})$. Then the functions $\{T_k \text{sinc}\}_{k \in \mathbb{Z}}$ form an orthonormal basis for the Paley-Wiener space,

$$PW := \left\{ f \in L^2(\mathbb{R}) \mid \text{supp}\hat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right] \right\}.$$ 

Also, the system $\{T_{k/2} \text{sinc}\}_{k \in \mathbb{Z}}$ can be considered as the union of two orthonormal bases for $PW$, and hence form a tight overcomplete frame for $PW$.

It is known that a finite union of shift-invariant systems at most can form a frame for a subspace for $L^2(\mathbb{R})$. However, by acting on a shift-invariant system with an infinite number of modulation operators, it is possible to construct frames for $L^2(\mathbb{R})$. This leads to the so-called Gabor frames. More formally, given some $a, b > 0$ and a function $g \in L^2(\mathbb{R})$, the associated Gabor system is the collection of functions given by

$$\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} = \{e^{2\pi i mbx} g(x - na)\}_{m,n \in \mathbb{Z}}.$$ 

The following result collects the information about Gabor systems that is needed in the current paper; much more information can be found, e.g., in [27, 12].

**Proposition 1.11** Let $g \in L^2(\mathbb{R}) \setminus \{0\}$. Then the following hold:

(i) $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is linearly independent.

(ii) If $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $ab \leq 1$.

(iii) If $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a Riesz basis if and only if $ab = 1$.

(iv) If $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is an overcomplete frame for $L^2(\mathbb{R})$, then it has infinite excess.

The result in (i) was proved in [32] (hereby confirming a conjecture stated in [30]); (ii) & (iii) are classical results [27, 12], and (iv) is proved in [5].

## 2 Frames and dynamical sampling

### 2.1 Frame properties of operator orbits

The analysis of the frame aspects related to dynamical sampling was initiated in the papers [1, 2] by Aldroubi et al. The paper [2] gives a complete treatment of the finite-dimensional case; we will not go into this topic as we only deal with the infinite-dimensional case in the current paper.

The approach in [1, 2] puts operator theory at the central spot. The central questions can be formulated as follows:
(i) Given a bounded operator \( T : \mathcal{H} \to \mathcal{H} \) belonging to a certain class, is it possible to choose \( \varphi \in \mathcal{H} \) such that the orbit \( \{ T^n \varphi \}_{n=0}^{\infty} \) is a basis?

(ii) Given a bounded operator \( T : \mathcal{H} \to \mathcal{H} \) belonging to a certain class, is it possible to choose \( \varphi \in \mathcal{H} \) such that the orbit \( \{ T^n \varphi \}_{n=0}^{\infty} \) is a frame?

The papers \([1, 2, 14]\) provide positive answers to these questions in certain cases, to be described in detail in Sections 2.2 & 2.3. However, except for these cases, most results in the literature are negative. We state a number of such results next.

**Proposition 2.1** Consider a bounded operator \( T : \mathcal{H} \to \mathcal{H} \). Then the following hold:

(i) \([2]\) If \( T \) is normal, then \( \{ T^n \varphi \}_{n=0}^{\infty} \) is not a basis.

(ii) \([3]\) If \( T \) is unitary, then \( \{ T^n \varphi \}_{n=0}^{\infty} \) is not a frame.

(iii) \([19]\) If \( T \) is compact, then \( \{ T^n \varphi \}_{n=0}^{\infty} \) is not a frame.

If \( T \) is a hypercyclic operator, clearly \( \{ T^n \varphi \}_{n=0}^{\infty} \) can not be a frame for any hypercyclic vector \( \varphi \in \mathcal{H} \). We will now prove that \( \{ T^n \eta \}_{n=0}^{\infty} \) and \( \{ (T^*)^n \eta \}_{n=0}^{\infty} \) can not be frames for any choice of \( \eta \in \mathcal{H} \).

**Lemma 2.2** Assume that \( \{ T^n \varphi \}_{n=0}^{\infty} \) is a frame for a bounded operator \( T : \mathcal{H} \to \mathcal{H} \) and some \( \varphi \in \mathcal{H} \). Then \( (T^n)^* f \to 0 \) as \( n \to \infty \), for all \( f \in \mathcal{H} \).

**Proof.** Fix any \( f \in \mathcal{H} \), and let \( k \in \mathbb{N} \). Then, denoting a lower frame bound for \( \{ T^n \varphi \}_{n=0}^{\infty} \) by \( A \), we have that

\[
A \| (T^k)^* f \|^2 \leq \sum_{n=0}^{\infty} | \langle (T^k)^* f, T^n \varphi \rangle |^2 = \sum_{n=k}^{\infty} | \langle f, T^n \varphi \rangle |^2 \to 0 \text{ as } k \to \infty.
\]

Thus also \( (T^k)^* f \to 0 \) as \( k \to \infty \). \( \square \)

**Proposition 2.3** Assume that \( T : \mathcal{H} \to \mathcal{H} \) is hypercyclic. Then \( \{ T^n \eta \}_{n=0}^{\infty} \) and \( \{ (T^*)^n \eta \}_{n=0}^{\infty} \) can not be frames for any choice of \( \eta \in \mathcal{H} \).

**Proof.** Let \( \varphi \in \mathcal{H} \) be a hypercyclic vector and consider any \( \eta \in \mathcal{H} \setminus \{0\} \). Then the scalar sequence \( \{ \langle \eta, T^n \varphi \rangle \}_{n=0}^{\infty} = \{ \langle (T^*)^n \eta, \varphi \rangle \}_{n=1}^{\infty} \) is unbounded, which implies that the sequence of norms \( \{ ||(T^*)^n \eta|| \}_{n=1}^{\infty} \) is unbounded. Using Lemma 2.2 it now follows that the sequence \( \{ T^n \eta \}_{n=0}^{\infty} \) is not a frame. For
the proof of the second claim, considering again any \( \eta \neq 0 \) and still letting \( \varphi \) denote a hypercyclic vector,

\[
\sum_{n=0}^{\infty} |\langle \varphi, (T^*)^n \eta \rangle|^2 = \sum_{n=0}^{\infty} |\langle T^n \varphi, \eta \rangle|^2 = \infty,
\]

which implies that \( \{(T^*)^n \eta \}_{n=0}^{\infty} \) is not a frame. \( \square \)

Yet we have not seen any examples of a frame with an operator representation \( \{T^n \varphi\}_{n=0}^{\infty} \). However, let us prove already now that if \( \{T^n \varphi\}_{n=0}^{\infty} \) is a frame for a surjective and bounded operator \( T \), the frame property is preserved if an arbitrary finite number of elements is removed from the frame:

**Proposition 2.4** Assume that \( \{T^n \varphi\}_{n=0}^{\infty} \) is a frame for \( \mathcal{H} \) for some bounded surjective operator \( T : \mathcal{H} \to \mathcal{H} \) and some \( \varphi \in \mathcal{H} \). Then \( \{T^n \varphi\}_{n \in \mathbb{N}_0 \setminus I}^{\infty} \) is a frame for \( \mathcal{H} \) for an arbitrary finite index set \( I \subset \mathbb{N}_0 \). In particular, \( \{T^n \varphi\}_{n=0}^{\infty} \) has infinite excess.

**Proof.** Consider any \( N \in \mathbb{N} \). By removing the first \( N \) elements of the frame \( \{T^n \varphi\}_{n=0}^{\infty} \) we are left with the family \( \{T^n \varphi\}_{n=N}^{\infty} \); since the operator \( T^N \) is bounded and surjective, this family is a frame. \( \square \)

### 2.2 The Carleson frame

The first construction of a frame of the form \( \{T^n \varphi\}_{n=0}^{\infty} \) was obtained by Al-droubi et al in [1] and further discussed in [2, 3]. We will formulate the result in our setting of an arbitrary separable infinite-dimensional Hilbert space.

**Theorem 2.5** Let \( \{e_k\}_{k=1}^{\infty} \) be an orthonormal basis for \( \mathcal{H} \) and consider a bounded operator \( T : \mathcal{H} \to \mathcal{H} \) such that \( Te_k = \lambda_k e_k \) for a bounded sequence of complex scalars \( \lambda_k \). Also, let \( \varphi \in \mathcal{H} \). Then \( \{T^n \varphi\}_{n=0}^{\infty} \) is a frame for \( \mathcal{H} \) if and only if the following conditions are satisfied:

(i) \( |\lambda_k| < 1 \) for all \( k \in \mathbb{N} \);

(ii) \( |\lambda_k| \to 1 \) as \( k \to \infty \);

(iii) The sequence \( \{\lambda_k\}_{k=1}^{\infty} \) satisfies the Carleson condition, i.e.,

\[
\inf_n \prod_{n \neq k} \frac{|\lambda_k - \lambda_n|}{|1 - \lambda_k \lambda_n|} > 0; \tag{2.1}
\]

(iv) \( \varphi = \sum_{k=1}^{\infty} m_k \sqrt{1 - |\lambda_k|^2} e_k \) for a scalar-sequence \( \{m_k\}_{k=1}^{\infty} \) that is bounded below away from zero and above.
For the sake of easy reference we will call the frames arising from the conditions in Theorem 2.5 for Carleson frames. The following result (see, e.g., [24, Thm. 9.2]) gives an easy verifiable criterion for the Carleson condition (2.1) to hold.

Proposition 2.6 Let \( \{\lambda_k\}_{k=1}^{\infty} \) be a sequence of distinct complex numbers such that \( |\lambda_k| < 1 \) for all \( k \in \mathbb{N} \). If there exists \( c \in ]0, 1[ \) such that

\[
\frac{1 - |\lambda_{k+1}|}{1 - |\lambda_k|} \leq c < 1, \quad \forall k \in \mathbb{N},
\]

then \( \{\lambda_k\}_{k=1}^{\infty} \) satisfies the Carleson condition. If \( \{\lambda_k\}_{k=1}^{\infty} \) is positive and increasing, the condition (2.2) is also necessary for \( \{\lambda_k\}_{k=1}^{\infty} \) to satisfy the Carleson condition.

Based on Proposition 2.6 it is easy to give an explicit example of a sequence satisfying the Carleson condition:

Corollary 2.7 For every \( \alpha > 1 \), the sequence \( \{\lambda_k\}_{k=1}^{\infty} = \{1 - \alpha^{-k}\}_{k=1}^{\infty} \) satisfies the Carleson condition.

Carleson frames have a number of very special features. For example, under a very weak condition, arbitrary finite subsets can be removed without destroying the frame property:

Corollary 2.8 Consider a Carleson frame \( \{T^n \varphi\}_{n=0}^{\infty} \) as in Theorem 2.5 and assume that \( \lambda_k \neq 0 \) for all \( k \in \mathbb{N} \). Then \( \{T^n \varphi\}_{n=N}^{\infty} \) is a frame for \( \mathcal{H} \) for any \( N \in \mathbb{N} \).

Proof. Since \( \lambda_k \neq 0 \) for all \( k \in \mathbb{N} \), the condition (ii) in Theorem 2.5 implies that there is a \( C > 0 \) such that \( |\lambda_k| \geq C \) for all \( k \in \mathbb{N} \); hence the operator \( T \) in Theorem 2.5 is surjective. The result now follows from Proposition 2.4. \( \Box \)

2.3 Frames and orbit representations

We will now continue the theme from Section 2.1, but with an important change of focus. While Section 2.1 was putting the operator \( T \) in the central spot and asking for frame properties of the associated orbits, we will now take a frame \( \{f_k\}_{k=1}^{\infty} \) as the starting point and analyze when and how it can be represented as an orbit of a bounded operator. Recall from Proposition 1.3 that a representation as the orbit of a linear operator is possible if \( \{f_k\}_{k=1}^{\infty} \) is linearly independent; thus, the key issue is to determine when the operator \( T \) can be chosen to be bounded.

We first state a result appearing in [19]. We will need the right-shift operator on \( \ell^2(\mathbb{N}) \), defined by

\[
T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), T\{c_k\}_{k=1}^{\infty} = \{0, c_1, c_2, \ldots\}.
\]
Theorem 2.9 Consider a frame $\{f_k\}_{k=1}^{\infty}$ with frame bounds $A, B$. Then the following are equivalent:

(i) The frame has a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$.

(ii) For some dual frame $\{g_k\}_{k=1}^{\infty}$ (and hence all),

$$f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1}, \forall j \in \mathbb{N}. \quad (2.4)$$

(iii) The kernel $\mathcal{N}(U)$ of the synthesis operator $U$ is invariant under the right-shift operator $T$.

In the affirmative case, letting $\{g_k\}_{k=1}^{\infty}$ denote an arbitrary dual frame of $\{f_k\}_{k=1}^{\infty}$, the operator $T$ has the form

$$T f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_{k+1}, \forall f \in \mathcal{H}, \quad (2.5)$$

and $1 \leq \|T\| \leq \sqrt{BA^{-1}}$.

Note that if $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, then $\{f_k\}_{k=1}^{\infty}$ and the (unique) dual frame $\{g_k\}_{k=1}^{\infty}$ form a biorthogonal system, and hence the condition (2.4) is trivially satisfied. Thus Theorem 2.9 immediately shows that every Riesz basis is the orbit of a bounded operator, a result that was first proved directly in [14]:

Corollary 2.10 If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, there exists a bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$.

Note that any reordering of a Riesz basis is a Riesz basis itself; thus, no matter how the elements in a Riesz basis are ordered, we can apply Corollary 2.10. On the other hand, we saw in Example 1.4 that for general linearly independent sequences in Hilbert spaces, the existence of a bounded operator representation depends on the ordering of the sequence. For general overcomplete frames, the question of representability via the orbit of a bounded operator is also extremely sensitive to reorderings of the elements, as demonstrated by results in [19].

The simplicity of the proof of Corollary 2.10 could make us optimistic with regard to construction of other classes of frames having an orbit representation via a bounded operator, but unfortunately this is misleading. Indeed, the Carleson frames and the Riesz bases are the only explicitly available examples in the literature of frames having such a representation. In particular, it was shown in [19] that a so-called near-Riesz basis (i.e., a frame with finite excess) never can be represented via a bounded operator:
Proposition 2.11 Let \( \{ f_k \}_{k=1}^{\infty} \) denote a frame which is a union of a Riesz basis and a finite non-empty collection of vectors. Then \( \{ f_k \}_{k=1}^{\infty} \) can not be represented as the orbit of a bounded operator.

Another serious restriction was proved in [18]. It excludes all overcomplete shift-invariant frames and Gabor frames from having a representation as the orbit of a bounded operator:

**Theorem 2.12** Assume that \( \{ T^n \varphi \}_{n=0}^{\infty} \) is an overcomplete frame for some \( \varphi \in \mathcal{H} \) and a bounded operator \( T : \mathcal{H} \rightarrow \mathcal{H} \). Then for each \( f \in \mathcal{H} \) we have that

\[
T^n f \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{2.6}
\]

The result in Theorem 2.12 is very disappointing. We have already seen that shift-invariant frames and Gabor frames automatically are linearly independent, so by Proposition 1.3 such systems can be represented as an orbit of a *linear* operator \( T \); however, in the overcomplete case \( T \) is forced to be unbounded. Since the problem is the boundedness and not the *existence* of a operator representation, it is natural to ask whether boundedness can be achieved by allowing a “scaling factor” in the operator representation. We formulate it as an open question:

**Q:** Do there exist overcomplete Gabor frames \( \{ E_{mb} T_{na} g \}_{m,n} \in \mathbb{Z} \) such that an appropriate ordering \( \{ f_k \}_{k=1}^{\infty} \) of the frame elements has a representation

\[
\{ f_k \}_{k=1}^{\infty} = \{ a_n T^n \varphi \}_{n=0}^{\infty}, \tag{2.7}
\]

for some scalars \( a_n \neq 0 \), a bounded operator \( T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), and some \( \varphi \in L^2(\mathbb{R}) \)?

Let us end the section with result connecting frame orbits and shift-invariant subspaces of the Hardy space. Recall that letting \( \mathbb{D} \) denote the open unit disc in the complex plane, the Hardy space \( H^2(\mathbb{D}) \) is defined as

\[
H^2(\mathbb{D}) = \{ f : \mathbb{D} \rightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n, \{ a_n \}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0) \}.
\]

The Hardy space is a Hilbert space, with the inner product defined by

\[
\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}, \quad \text{for} \quad f = \sum_{n=0}^{\infty} a_n z^n, \quad g = \sum_{n=0}^{\infty} b_n z^n.
\]

Furthermore the sequence of functions \( \{ z^n \}_{n=0}^{\infty} \) is an orthonormal basis for \( H^2(\mathbb{D}) \). The *shift operator* on the Hardy space is defined by

\[
S : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}), \quad Sf(z) = zf(z), \quad z \in \mathbb{D}
\]

Finally let \( 1_{\mathbb{D}} \) denote the constant function, i.e., \( 1_{\mathbb{D}}(z) = 1 \).
Theorem 2.13 Consider a bounded operator $T : \mathcal{H} \to \mathcal{H}$ and let $\varphi \in \mathcal{H}$. Then $\{T^n\varphi\}_{n=0}^\infty$ is a frame if and only if there exists a shift invariant subspace $\mathcal{F} \subset H^2(\mathbb{D})$ and a bounded bijective operator $W : \mathcal{F}^\perp \to \mathcal{H}$ such that

$$T = WP_{\mathcal{F}^\perp}SW^{-1}, \quad \varphi = WP_{\mathcal{F}^\perp}1_{\mathbb{D}}.$$ 

In the affirmative case, $\{T^n\varphi\}_{n=0}^\infty$ is a Riesz basis if and only if $\mathcal{F} = \{0\}$.

**Proof.** In the entire proof we let $\langle \cdot, \cdot \rangle$ denote the inner product on $H^2(\mathbb{D})$. First assume that $\{T^n\varphi\}_{n=0}^\infty$ is a frame. Consider the linear operator $V : H^2(\mathbb{D}) \to \mathcal{H}$ defined by

$$Vf = \sum_{n=0}^\infty \langle f, z^n \rangle T^n \varphi, \; f \in H^2(\mathbb{D}). \tag{2.8}$$

The operator $V$ is bounded; indeed, letting $B$ denote a Bessel bound for $\{T^n\varphi\}_{n=0}^\infty$, we have

$$\|Vf\|^2 = \|\sum_{n=0}^\infty \langle f, z^n \rangle T^n \varphi\|^2 \leq B \sum_{n=0}^\infty |\langle f, z^n \rangle|^2 = B\|f\|^2.$$ 

The frame assumption also implies that the operator $V$ is surjective. Moreover, for $f \in H^2(\mathbb{D})$ we have

$$TVf = T\sum_{n=0}^\infty \langle f, z^n \rangle T^n \varphi = \sum_{n=0}^\infty \langle f, z^n \rangle T^{n+1} \varphi = V\sum_{n=0}^\infty \langle f, z^n \rangle z^{n+1} = VSf. \tag{2.9}$$

This shows in particular that $\ker V$ is a closed shift-invariant subspace of $H^2(\mathbb{D})$. Now let $\mathcal{F} := \ker V$ and consider the restriction of the operator $V$ to the subspace $\mathcal{F}^\perp$, i.e., let $W := V_{|\mathcal{F}^\perp} : \mathcal{F}^\perp \to \mathcal{H}$. Also let $P_{\mathcal{F}^\perp} : H^2(\mathbb{D}) \to \mathcal{F}^\perp$ be the orthogonal projection. Then, using (2.9), on the subspace $\mathcal{F}^\perp$, we have

$$TW = TV = VS = VP_{\mathcal{F}^\perp}S = WP_{\mathcal{F}^\perp}S.$$

Therefore the invertibility of $W$ yields that $T = WP_{\mathcal{F}^\perp}SW^{-1}$; also $\varphi = V1_{\mathbb{D}} = VP_{\mathcal{F}^\perp}1_{\mathbb{D}} = WP_{\mathcal{F}^\perp}1_{\mathbb{D}}$.

Conversely, let $\mathcal{F} \subset H^2(\mathbb{D})$ be a closed shift-invariant subspace and $W : \mathcal{F}^\perp \to \mathcal{H}$ be an invertible bounded operator. Since $\{S^n1_{\mathbb{D}}\}_{n=0}^\infty$ is an orthonormal basis for $H^2(\mathbb{D})$, the sequence $\{P_{\mathcal{F}^\perp}S^n1_{\mathbb{D}}\}_{n=0}^\infty$ is a frame for $\mathcal{F}^\perp$. Now let $T := WP_{\mathcal{F}^\perp}SW^{-1}$ and $\varphi := WP_{\mathcal{F}^\perp}1_{\mathbb{D}}$. Since $\mathcal{F}$ is invariant under the action of $S$, we have

$$P_{\mathcal{F}^\perp}S = P_{\mathcal{F}^\perp}SP_{\mathcal{F}^\perp}.$$
which implies that \((P_{\mathcal{F}}^nS)^n = P_{\mathcal{F}}^nS^n\), for every \(n \in \mathbb{N}_0\). Thus
\[
T^n \varphi = W(P_{\mathcal{F}}^nS)^nW^{-1} \varphi = W(P_{\mathcal{F}}S)^nP_{\mathcal{F}}^n1_\mathcal{D} = WP_{\mathcal{F}}S^n1_\mathcal{D}
\]
Therefore the collection \(\{T^n \varphi\}_{n=0}^\infty = \{WP_{\mathcal{F}}S^n1_\mathcal{D}\}_{n=0}^\infty\) is a frame for \(\mathcal{H}\).

Theorem 2.13 is theoretically appealing, but it is not instrumental in terms of constructing more examples of explicitly given frames \(\{T^n \varphi\}_{n=0}^\infty\). The difficulty in constructing such frames is the motivation for the remaining sections in the paper.

### 2.4 Frames and bi-orbit representations

In this section we will consider a frame, indexed as \(\{f_k\}_{k \in \mathbb{Z}}\); clearly, we can bring any frame into this form by a reordering and reindexing.

A seemingly innocent modification of the topic from Section 2.3 would be to ask for representations of a given frame \(\{f_k\}_{k \in \mathbb{Z}}\) of the form \(\{T^k f_0\}_{k \in \mathbb{Z}}\) for an invertible operator \(T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \to \text{span}\{f_k\}_{k \in \mathbb{Z}}\); such a representation is called a bi-orbit representation. Interestingly, this leads to a theory that in certain aspects is very similar to the operator representations discussed in the previous sections, but also is completely different at some points. In this section we will highlight some of the similarities and differences for the two ways of representing a frame.

First, it was proved in [15] that Proposition 1.3 has a completely similar version for bi-orbit representations: any linearly independent frame \(\{f_k\}_{k \in \mathbb{Z}}\) has a representation \(\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}\) for some linear and invertible operator \(T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \to \text{span}\{f_k\}_{k \in \mathbb{Z}}\). It was also proved that Theorem 2.9 has a parallel version: indeed, if additionally the kernel of the synthesis operator is invariant under right and left-shifts, the operator \(T\) extends to a bijective and bounded operator \(T : \mathcal{H} \to \mathcal{H}\).

However, when it comes to concrete manifestations of frames of the form \(\{T^k f_0\}_{k \in \mathbb{Z}}\) as well as other properties for such frames, there are remarkable differences between orbit/bi-orbit representations. This becomes apparent already from the following example.

**Example 2.14**  As discussed in Section 1.5, the shift-invariant system \(\{T_k \text{sinc}\}_{k \in \mathbb{Z}}\) is an orthonormal basis for the Paley-Wiener space \(\text{PW}\) of functions in \(L^2(\mathbb{R})\), while \(\{T_k/2 \text{sinc}\}_{k \in \mathbb{Z}}\) is an overcomplete frame for the same space. Since \(\{T_k \varphi\}_{k \in \mathbb{Z}} = \{(T_{kb})^k \varphi\}_{k \in \mathbb{Z}}\), such frames are born to have a bi-orbit representation. However, since \(||T_{kb}\varphi|| = ||\varphi||\) for all \(k \in \mathbb{Z}\) it is clear from Theorem 2.12 that an overcomplete shift-invariant frame can not have a representation \(\{T^n \varphi\}_{n=0}^\infty\) for a bounded operator \(T\).

Recall that Lemma 2.2 and Theorem 2.12 provided necessary conditions for an orbit \(\{T^n \varphi\}_{n=0}^\infty\) to be a frame for a bounded operator \(T\). The next result,
proved in [18], shows that the corresponding conditions on a bi-orbit \( \{ T^n \phi \}_{n \in \mathbb{Z}} \) are fundamentally different:

**Theorem 2.15** Assume that \( \{ T^n \phi \}_{n \in \mathbb{Z}} \) is a frame with bounds \( A, B \) for some \( \phi \in H \) and a bounded invertible operator \( T : H \to H \). Then for each \( f \in H \) we have that

\[
\| T^n f \| \geq \sqrt{\frac{A}{B}} \| f \| \quad \text{and} \quad \| (T^*)^n f \| \geq \sqrt{\frac{A}{B}} \| f \|.
\]

Theorem 2.15 makes it natural to ask whether overcomplete Gabor frames have representations as bi-orbits of a bounded operator. We phrase the question as stated originally in [16]:

**Q:** Do there exist overcomplete Gabor frames \( \{ E_{mb} T_{na} g \}_{m,n \in \mathbb{Z}} \) such that an appropriate ordering \( \{ f_k \}_{k \in \mathbb{Z}} \) of the frame elements has a representation

\[
\{ f_k \}_{k \in \mathbb{Z}} = \{ T^n \phi \}_{n=-\infty}^{\infty},
\]

for some bounded operator \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) and some \( \phi \in L^2(\mathbb{R}) \)?

In [22] Corso provided a partial solution to the question, by showing that the answer is negative if \( ab \notin \mathbb{Q} \), as well as under certain support conditions on the function \( g \). However, the general question is still open.

### 2.5 Frame representations via multi-orbits

The difficulty in constructing explicitly given frames \( \{ T^n \phi \}_{n=0}^{\infty} \), discussed in detail in Section 2.3, makes it natural to explore various modifications of the basic idea. In this section we will discuss the additional flexibility that is obtained by allowing a frame representation using a finite number of operator orbits. Let us first connect the topics in Section 2.3 and Section 2.4 by showing that any frame with a bi-orbit representation \( \{ T^n \phi \}_{n \in \mathbb{Z}} \) can be considered as the union of two orbits:

**Example 2.16** Consider any frame that has a representation as a bi-infinite orbit \( \{ T^n \phi \}_{n \in \mathbb{Z}} \) for a bounded and bijective operator \( T : H \to H \). Then

\[
\{ T^n \phi \}_{n \in \mathbb{Z}} = \{ T^n \phi \}_{n=0}^{\infty} \cup \{ (T^{-1})^n T^{-1} \phi \}_{n=0}^{\infty},
\]

i.e., the frame \( \{ T^n \phi \}_{n \in \mathbb{Z}} \) can be represented as a union of two orbits, generated by the bounded operators \( T \) and \( T^{-1} \).

The following result, originally proved in [14], shows that the idea of multi-orbits significantly enlarges the class of frames that can be analyzed: for example, all the classical frames in harmonic analysis like Gabor frames and wavelet frames have such representations!
Theorem 2.17 Consider a frame \( \{ f_k \}_{k=1}^{\infty} \) which is norm-bounded below. Then there is a finite collection of vectors from \( \{ f_k \}_{k=1}^{\infty} \), to be called \( \varphi_1, \ldots, \varphi_J \), and corresponding bounded operators \( T_j : \mathcal{H} \to \mathcal{H} \) with closed range, such that

\[
\{ f_k \}_{k=1}^{\infty} = \bigcup_{j=1}^{J} \{ T_j^n \varphi_j \}_{n=0}^{\infty}.
\] (2.11)

While Theorem 2.17 is theoretically appealing, its practical applicability is limited by the fact that the proof is based on the Feichtinger Theorem and thus does not give direct access to the operators \( T_j \). More results on multi-orbit representations can be found in [10].

3 Approximate operator representations

In Section 2 the focus is on frames \( \{ f_k \}_{k=1}^{\infty} \) having an exact match with an operator orbit \( \{ T^n \varphi \}_{n=0}^{\infty} \); unfortunately, the results show that this is a very restrictive condition and that the standard frames in \( L^2(\mathbb{R}) \) are excluded. In the current section we will consider two relaxations on the orbit representations, which in combination yield a theory covering a much larger class of frames.

To be more precise, recall that Theorem 1.10 by Halperin et al. showed that suborbit representations \( \{ T^{\alpha(k)} \varphi \}_{k=1}^{\infty} \) via bounded operators are available for any linearly independent family \( \{ f_k \}_{k=1}^{\infty} \); however, in general we do not have a feasible procedure to identify the operator \( T \) and the appropriate integers \( \alpha(k) \). In this section we will show that for a large class of frames we can obtain approximate suborbit representations, where it is possible to specify as well the operator \( T \) as the integers \( \alpha(k) \). In the entire section we follow the presentation in [17]; we refer to that paper for proofs and more information.

We begin with a formal definition, explaining the exact meaning of “approximation.” In words, a frame \( \{ f_k \}_{k=1}^{\infty} \) is approximated by a sequence \( \{ \tilde{f}_k \}_{k=1}^{\infty} \) if \( \{ \tilde{f}_k \}_{k=1}^{\infty} \) satisfies the standard perturbation condition within frame theory:

**Definition 3.1** Let \( \{ f_k \}_{k=1}^{\infty} \) be a frame for \( \mathcal{H} \). Given any \( \epsilon > 0 \), a sequence \( \{ \tilde{f}_k \}_{k=1}^{\infty} \subset \mathcal{H} \) is called an \( \epsilon \)-approximation of \( \{ f_k \}_{k=1}^{\infty} \) if

\[
\left\| \sum c_k (f_k - \tilde{f}_k) \right\|^2 \leq \epsilon \sum |c_k|^2
\] (3.1)

for all finite sequences \( \{ c_k \} \).

For sufficiently small values of \( \epsilon \), an \( \epsilon \)-approximation \( \{ \tilde{f}_k \}_{k=1}^{\infty} \) of a frame \( \{ f_k \}_{k=1}^{\infty} \) is itself a frame and shares several key properties of the frame, e.g., its excess. Furthermore, the synthesis operator and the frame operator for \( \{ f_k \}_{k=1}^{\infty} \) are approximated “well” by the corresponding operators associated with \( \{ \tilde{f}_k \}_{k=1}^{\infty} \):
Theorem 3.2. Consider a frame \( \{ f_k \}_{k=1}^{\infty} \) for \( \mathcal{H} \) with frame bounds \( A, B \), and assume that \( \{ \tilde{f}_k \}_{k=1}^{\infty} \subset \mathcal{H} \) is an \( \epsilon \)-approximation of \( \{ f_k \}_{k=1}^{\infty} \) for some \( \epsilon \in ]0, A[ \). Then the following hold:

(i) \( \{ \tilde{f}_k \}_{k=1}^{\infty} \) is a frame with bounds \( A(1 - \sqrt{\frac{\epsilon}{A}})^2 \) and \( B(1 + \sqrt{\frac{\epsilon}{B}})^2 \), with the same excess as \( \{ f_k \}_{k=1}^{\infty} \).

(ii) Denoting the synthesis operators and frame operators of \( \{ f_k \}_{k=1}^{\infty} \) and \( \{ \tilde{f}_k \}_{k=1}^{\infty} \) by \( U, \tilde{U} \), respectively, \( S, \tilde{S} \), we have

\[
\| U - \tilde{U} \| \leq \sqrt{\epsilon}, \quad \| S - \tilde{S} \| \leq \sqrt{\epsilon B \left( 2 + \sqrt{\frac{\epsilon}{B}} \right)},
\]

and

\[
\| S^{-1} - \tilde{S}^{-1} \| \leq \frac{\sqrt{\epsilon B (2 + \sqrt{\frac{\epsilon}{B}})}}{A^2 (1 - \sqrt{\frac{\epsilon}{A}})^2}.
\]

The connection to approximate frame representations using suborbits of a bounded operator is explained in the next result, which provides a natural way to satisfy the condition (3.1):

Corollary 3.3 Let \( \{ f_k \}_{k=1}^{\infty} \) be a frame for \( \mathcal{H} \) with lower frame bound \( A \). Also, let \( \varphi \in \mathcal{H} \) and consider a bounded operator \( T : \mathcal{H} \rightarrow \mathcal{H} \). Assume that for a given \( \epsilon \in ]0, A[ \), and for any \( k \in \mathbb{N} \) there exists a nonnegative integer \( \alpha(k) \in \mathbb{N}_0 \) such that

\[
\| f_k - T^{\alpha(k)} \varphi \|^2 \leq \frac{\epsilon}{2^k}.
\]  (3.2)

Then \( \{ T^{\alpha(k)} \varphi \}_{k=1}^{\infty} \) is an \( \epsilon \)-approximation of \( \{ f_k \}_{k=1}^{\infty} \).

Note that Corollary 3.3 applies to any frame and any hypercyclic operator:

Example 3.4 Let \( T : \mathcal{H} \rightarrow \mathcal{H} \) be a hypercyclic operator with hypercyclic vector \( \varphi \). Then, for any frame \( \{ f_k \}_{k=1}^{\infty} \) for \( \mathcal{H} \) and any given \( \epsilon > 0 \) there exists nonnegative integers \( \alpha(k), k \in \mathbb{N} \), such that (3.2) holds. Thus, for sufficiently small values of \( \epsilon \), we obtain a frame \( \{ T^{\alpha(k)} \varphi \}_{k=1}^{\infty} \) with the same excess as the frame \( \{ f_k \}_{k=1}^{\infty} \) and approximating \( \{ f_k \}_{k=1}^{\infty} \) in the sense of Definition 3.1.

In general we do not have direct access to the numbers \( \alpha(k), k \in \mathbb{N} \), such that (3.2) holds, not even for hypercyclic operators. However, under natural conditions we can be very explicit about the choices of appropriate scalars \( \alpha(k) \) for certain choices of the operator \( T \). In the rest of the section we will use the following

General setup: Fix an orthonormal basis \( \{ e_k \}_{k=1}^{\infty} \) for the Hilbert space \( \mathcal{H} \), and let \( \{ f_k \}_{k=1}^{\infty} \) denote a frame for \( \mathcal{H} \). Fix some \( \lambda > 1 \), and define the scaled left/right-shift operators on \( \mathcal{H} \) by
$T \left( \sum_{k=1}^{\infty} c_k e_k \right) := \lambda \sum_{k=1}^{\infty} c_{k+1} e_k, \quad U \left( \sum_{k=1}^{\infty} c_k e_k \right) := \lambda^{-1} \sum_{k=1}^{\infty} c_{k+1}, \quad \{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}).$

For appropriately chosen nonnegative integers $\alpha(k), \ k \in \mathbb{N}$, let

$$\varphi := \infty \sum_{n=1}^{\infty} U^{\alpha(n)} f_n. \quad (3.3)$$

For the case of finitely supported vectors $\{f_k\}_{k=1}^{\infty}$ (meaning that each vector $f_k$ is a finite linear combination of elements from $\{e_k\}_{k=1}^{\infty}$) the following result specifies how to choose the powers $\alpha(k)$ such that the vector $\varphi$ in (3.3) is well-defined and (3.2) holds:

**Theorem 3.5** Under the conditions in the general setup, assume that $\{f_k\}_{k=1}^{\infty}$ consists of finitely supported vectors in $H_i$; for $k \in \mathbb{N}$, let $m(k)$ denote the largest index for a nonzero coordinate in $f_k$. Let $\{\alpha(k)\}_{k=1}^{\infty}$ be a strictly increasing sequence of nonnegative integers such that $\alpha(1) = 0$ and $\alpha(k+1) - \alpha(k) \geq m(k)$ for all $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N},$

$$||f_k - T^{\alpha(k)} \varphi||^2 \leq \frac{B \lambda^2}{\lambda^2 - 1} \lambda^{-2[\alpha(k+1) - \alpha(k)]}. \quad (3.4)$$

In particular, choosing the nonnegative integers $\alpha(k)$ such that $\alpha(1) = 0$ and for a given $\epsilon \in [0, A[,$

$$\alpha(k+1) - \alpha(k) \geq \max \left( m(k), \frac{k \ln(2) + \ln \left( \frac{2}{1} \right) + \ln \left( \frac{\lambda^2}{\lambda - 1} \right)}{2 \ln(\lambda)} \right), \quad (3.5)$$

the condition (3.2) is satisfied, i.e., the conclusions in Theorem 3.2 hold.

For certain values of the scaling parameter $\lambda$ we can be completely specific about how to choose the integers $\alpha(k)$:

**Corollary 3.6** In the setup of Theorem 3.5 let $\lambda = \sqrt{2},$ take an upper frame bound of the form $B = 2^N$ for some $N \in \mathbb{N}$ and a tolerance $\epsilon = 2^{-j}$ for some $j \in \mathbb{N}$. Then the following hold.

(i) Without any restriction on the support sizes $m(k)$ of the vectors $f_k$, the condition (3.3) is satisfied if $\alpha(1) = 0$ and

$$\alpha(k) = (k - 1) \left[ N + j + 1 + \frac{k}{2} \right] + \sum_{\ell=1}^{k-1} m(\ell), \ k \in \mathbb{N} \setminus \{1\}. \quad (3.6)$$
(ii) If \( m(k) \leq N + j + 1 + k \) for all \( k \in \mathbb{N} \), the condition (3.5) is satisfied if
\[
\alpha(k) = (k - 1) \left[ N + j + 1 + \frac{k}{2} \right], \quad k \in \mathbb{N}.
\] (3.7)

The main condition in Theorem 3.3, namely, that the frame \( \{f_k\}_{k=1}^\infty \) is finitely supported, can be relaxed: indeed, similar but slightly more technical results can be proved for localized frames. We refer to [17] for the details. Note also that generalizations to Banach spaces (for sequences \( \{f_k\}_{k=1}^\infty \) without assuming any frame property) have been obtained in [20].

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