An operator splitting method for solving dimensionless radiation hydrodynamics equations

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Abstract: By non-dimensionalizing the radiation hydrodynamics equations in the form of Boltzmann-Euler coupling, the two moments approximate model written with non-dimensional variables is proposed and the non-dimensionalization leads to two parameters. An operator splitting method is developed to solve the dimensionless radiation hydrodynamics equations. The advantage of this method is that instead of solving nonlinear equations, it solves a set of ordinary differential equations, and these ordinary differential equations are decoupled. Lastly, the paper describes the Su-Olson problem which has a semi-analytic solution. The analysis shows that the solutions of numerical simulation agree well with the semi-analytic solutions in equilibrium diffusion limit.

1. Introduction
Radiation hydrodynamics is widely used in astrophysics, inertial confinement fusion, nuclear explosion and other fields involving high temperature and high pressure. Because of the complexity of radiation transport model and its interaction with matter, the numerical method of radiation hydrodynamics has been the focus of research for a long time [1-2].

In the course of radiation transport, photons are absorbed and re-emitted by matter, and changes in radiation pressure and energy within the material cause changes in the state of motion of matter, which in turn affects the absorption and emission coefficient of matter. The interaction between radiation and matter shows a high degree of non-linearity and strong coupling. Physically, the radiation fluid equation has three important dynamic scales, which are related to material wave (local sound speed), radiation wave (velocity of light) and source term. The source term represents the slowest scale and the radiation wave represents the fastest scale in the region where the interaction between radiation and matter is weak. In the region where radiation and matter interact strongly, the source term represents the fastest scale and the material wave velocity is the slowest scale. The multi-scale characteristic of radiation hydrodynamics determines the difficulty of its numerical solution. In this paper, we deduce the dimensionless form of the radiation hydrodynamics equations, and the nondimensionalization leads to two parameters. An operator splitting method is developed to solve the dimensionless radiation hydrodynamics equations. Finally, the correctness of the method is verified by numerical tests.

2. Radiation Hydrodynamics Equations
Radiation hydrodynamic equations include hydrodynamic equations and radiative transfer equation. Without considering viscosity and heat conduction, the radiative transfer equation is:
\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + n \cdot \nabla \right) I(n, \nu) = S(n, \nu), \tag{1a}
\]
and the hydrodynamics equations are
\[
\rho \frac{\partial u}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{1b}
\]
\[
\left( \rho u \frac{\partial F_r}{\partial t} \right)_r + \nabla \cdot \left( \rho u \otimes u + \rho I + \frac{P_r}{c} \right) = 0, \tag{1c}
\]
\[
\left( \frac{1}{c^2} \rho u^2 + e + \frac{E_r}{c} \right)_r + \nabla \cdot \left( \left( \frac{1}{c^2} \rho u^2 + e + \frac{E_r}{c} \right) u + F_r \right) = 0, \tag{1d}
\]
where \( t \) is the time variable, \( c \) is the light speed, \( \nu \) is the photon frequency, \( I \) is the specific radiation intensity. \( \rho, \rho, u \) and \( T \) denote the gas pressure, density, velocity and temperature, respectively, \( e \) is the gas internal energy. Here, \( E_r, F_r \) and \( P_r \) are the radiation energy density, flux vector, and pressure tensor, respectively, which are defined as:
\[
E_r = \int_0^\infty n \, d\Omega \, d\nu, \quad F_r = \int_0^\infty n \otimes \, d\Omega \, d\nu, \quad P_r = \int_0^\infty n \, d\Omega \, d\nu. \tag{2c}
\]
the coupling term \( S(n, \nu) \) is also known as the radiation-matter interaction term, which is defined as:
\[
S(n, \nu) = \eta(n, \nu) - \chi(n, \nu) I(n, \nu), \tag{3}
\]
where \( \chi \) and \( \eta \) are the specific emission and absorption coefficients in the lab frame, respectively. For simplicity, we neglect scattering, and assume local thermodynamic equilibrium(LTE). To order \( \nu/c \), the coupling term \( S(n, \nu) \) can be written as:
\[
S(n, \nu) = \sigma_a(\nu) B_\nu(T) - \sigma_i(\nu) I(n, \nu) + \frac{1}{4\pi} \sigma_r(\nu) \int I(n, \nu) \, d\Omega + \frac{1}{4\pi c} \left[ C_0(\nu) + 3 n \cdot C_1(\nu) \right], \tag{4}
\]
where \( \sigma_a \) is the absorption opacity, \( \sigma_i \) is the scattering opacity, \( \sigma_r = \sigma_a + \sigma_s \) is the total opacity. \( B_\nu(T) \) is the Planck function, which is defined by
\[
B_\nu(T) = \frac{2\nu^3}{c^2} e^{-\nu h/kT}.
\]
\( C_0 \) and \( C_1 \) are defined as:
\[
C_0 = \left( \sigma_a - \sigma_s \right) u \cdot \left( \frac{1}{3} n \, d\Omega - \frac{4}{3} u \frac{1}{3} \, d\Omega \right), \quad C_1 = \frac{4}{3} \sigma_r u \frac{1}{3} n \, d\Omega.
\]
It is difficult to solve Eqs. (1) directly. It is noted that the influence of radiation on the material is mainly through the radiation energy density, the radiation flux and the radiation pressure. Then the radiation transport equation is considered to have an angle integral, so that the zero-order moment and first-order moment equation of the radiation transport equation are obtained:
\[
\frac{\partial E_r}{\partial t} + c \nabla \cdot F_r = S_E, \tag{5a}
\]
\[
\frac{\partial F_r}{\partial t} + c \nabla \cdot P_r = S_F, \tag{5b}
\]
where
\[
S_E = \int_0^\infty \int S \, d\Omega \, d\nu, \quad S_F = \int_0^\infty \int n S \, d\Omega \, d\nu.
\]
The dimensionless equation is helpful to analyze the relative important parameters in the equation. For the radiation fluid equation, consider the following dimensionless forms[3-4]:

\[
\hat{\dot{x}} = xl, \hat{\dot{t}} = t/l, \hat{\rho} = \rho \rho_a, \hat{u} = u a_v, \hat{p} = p \rho_a a_v^2, \hat{T} = T T_a, \hat{v} = v k T_a / h,
\]

\[
\hat{I} = \text{I}hca T_a^3 / k, \hat{\sigma}_a = \sigma_a l, \hat{\sigma}_s = \sigma_s l,
\]

where the superscript \( \hat{\cdot} \) represents the dimensional physical quantity, no superscript represents the dimensionless physical quantity.

Using Eqs. (6) into Eq. (1a) yields

\[
\frac{1}{c} \frac{\partial}{\partial t} \hat{I}(\hat{v}, n) + n \cdot \nabla \hat{I}(\hat{v}, n) = \hat{S}(\hat{v}, n).
\]

After the simplification, we get:

\[
\frac{1}{c/a_v} \frac{\partial}{\partial t} \hat{I}(\hat{v}, n) + n \cdot \nabla \hat{I} = \sigma_a \hat{B}_c - \sigma_s \hat{I} + \frac{1}{4\pi} \sigma \int \hat{d} \Omega + \frac{a_v}{4\pi c} \left[ (\sigma_a - \sigma_s) u \cdot \left( \int n \hat{d} \Omega - \frac{4}{3} a_v u \int \hat{d} \Omega \right) + 3n \cdot \hat{C} \right].
\]

Integrating Eq. (8) over angle and frequency, we get

\[
\frac{1}{C} \frac{\partial}{\partial t} E_r + \nabla \cdot F_r = \sigma_a \left( T^4 - E_r \right) + \left( \sigma_a - \sigma_s \right) \frac{u}{C} \cdot F_r,
\]

\[
\frac{1}{C} \frac{\partial}{\partial t} F_r + \nabla \cdot P_r = -\sigma_r F_r + \sigma_a \frac{u}{C} \left( T^4 - E_r \right).
\]

In the same way, we can get the hydrodynamics equations in dimensionless form:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + e \right) + \nabla \left[ \left( \frac{1}{2} \rho u^2 + e + P \right) u \right] = -\frac{a_v T_a^4}{\rho_a a_v^2} \left( \sigma_a \left( T^4 - E_r \right) + \left( \sigma_a - \sigma_s \right) \frac{u}{C} \cdot F_r \right).
\]

Two parameters are derived during the dimensionless process of radiation hydrodynamics equations:

\[
C = \frac{c}{a_v}, \quad R = \frac{a_v T_a^4}{\rho_a a_v^2},
\]

where \( C \) is the ratio of light speed to sound speed, which represents the effect of the relativistic effect. Similarly, \( R \) represents the effect of radiation on the dynamics of matter. Compared with the original equations, the meaning of each physical quantity is more clear. By adjusting the parameters of dimensionless equations, the numerical simulation results are compared and analyzed, so that we can understand the law of radiation wave and material wave transmission more clearly.

3. Operator-splitting scheme

Operator splitting is a common numerical method for solving multi-physical field[5-7] problems. We use operator-splitting to solve Eqs. (9) and (10) in three steps. The first step is to solve the hydrodynamics equations without the radiation source terms:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,
\]
\[ \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u + P) = 0, \]  
(11b)

\[ \frac{\partial \left( \frac{1}{2} \rho u^2 + e \right)}{\partial t} + \nabla \cdot \left( \frac{1}{2} \rho u^2 + e + P \right) \mathbf{u} = 0. \]  
(11c)

The second step is to solve the radiation transport equations without the radiation source terms:

\[
\frac{1}{C_t} \frac{\partial E_r}{\partial t} + \nabla \cdot F_r = 0, \quad (12a) \\
\frac{1}{C} \frac{\partial F_r}{\partial t} + \nabla \cdot P_r = 0. \quad (12b)
\]

The third step is to solve the functions with the radiation source terms:

\[
\frac{1}{C_t} \frac{\partial E_r}{\partial t} = \sigma_a \left( T^4 - E_r \right) + \left( \sigma_a - \sigma_s \right) \frac{u}{C} \cdot F_r, \quad (13a) \\
\frac{1}{C} \frac{\partial F_r}{\partial t} = -\sigma_a F_r + \sigma_a \frac{u}{C} \left( T^4 - E_r \right), \quad (13b) \\
\frac{\partial \left( \rho u \right)}{\partial t} = -\frac{a_s T^4}{\rho a_n^2} \left[ -\sigma_a F_r + \sigma_a \frac{u}{C} \left( T^4 - E_r \right) \right], \quad (13c) \\
\frac{\partial \left( \frac{1}{2} \rho u^2 + e \right)}{\partial t} = -\frac{a_s T^4}{\rho a_n^2} C \left[ \sigma_a \left( T^4 - E_r \right) + \left( \sigma_a - \sigma_s \right) \frac{u}{C} \cdot F_r \right]. \quad (13d)
\]

Eqs. (11) and (12) are hyperbolic, which can be solved by the standard methods. Eqs. (13) can be solved by an implicit method. Specific calculation details can be referred to [7].

### 4. Results

Su and Olson discovered a simple choice of material properties for the 1-D gray diffusion equations that removed the nonlinear interaction between the specific intensity and material temperature, which is used to test the accuracy and stability of numerical schemes in many literatures. The problem is commonly referred to as the Su-Olson problem[9].

The problem consists of a cold, infinite, homogeneous slab with a unit radiation source in \(|x| < 0.5\). Analytic solutions are provided at 10 points logarithmically spaced in \(0 < x < 10\) and for times \(t = 1, 10\) and 30. We present here only the results for \(\sigma_a = 0\).

In this problem, we set the cell number \(N_{cell} = 512\), \(x \in [0, 30]\), \(\sigma_a = 40\), \(C = 10^5\). Eqs. (5) based on M1 approximation is still used as the model equation. Figs. (1-3) plots the comparison of numerical simulation results with semi-analytical solutions at \(t = 1, 10\) and 30.
It can be seen that the numerical simulation results are in good agreement with those of semi-analytical solutions based on diffusion approximation when the time is longer (corresponding $t = 10$ and $30$), and the results are slightly different from those of short time (corresponding $t = 1$). The radiation temperature and the material temperature are basically in equilibrium state when $t \geq 10$, Then the radiation field is appropriate to be described by diffusion approximation. There are some differences between the radiation temperature and the material temperature at $t = 1$, and the numerical simulation results are also slightly different from the semi-analytical results, mainly because the radiation field is not balanced at this time, and the diffusion approximation is not suitable to describe the non-equilibrium radiation field. Therefore, the numerical simulation results of the first order moment approximation are slightly different.

5. Conclusions

In this paper, the M1 approximate radiation hydrodynamics equations based on local thermodynamic equilibrium and gray approximation are derived by dimensionless treatment of a series of parameters, and two parameters with certain physical significance are derived. An Operator-splitting scheme is employed to solve the dimensionless equations. The numerical simulation of Su-Olson problem proves the correctness of the dimensionless equations and methods.
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