Bayesian Approach to Foreground Removal

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Abstract. Our ability to extract the maximal amount of information from future observations at gigahertz frequencies depends on our ability to separate the underlying cosmic microwave background (CMB) from galactic and extragalactic foregrounds. We review the separation problem and its formulation within Bayesian inference, give examples of specific solutions with particular choices of prior density, and finally comment on the generalization of Bayesian methods to a multi-resolution framework. We propose a strategy for the regularization of solutions allowing a spatially varying spectral index, and discuss possible computational approaches such as multi-scale stochastic relaxation.

Key Words methods: data analysis - techniques: image analysis - cosmic microwave background

1. Introduction

Future observations from ground based, balloon borne, and satellite missions at gigahertz frequencies will contain a wealth of information. Our ability to extract the maximal amount of information from these experiments depends on our ability to separate the underlying cosmic microwave background (CMB) from galactic and extragalactic foregrounds. Anticipated components of the total foreground emission include synchrotron, free-free, and dust emission in our own galaxy, plus various extragalactic point sources. The interesting cosmological information, relevant for testing inflation and determining cosmological parameters, is found at sub-degree angular scales, but it is at these angular scales that we begin to resolve the microphysics of the galaxy, potentially complicating the separation of the underlying cosmic and foreground signals.

It has been demonstrated that, for foregrounds simulated by extrapolating existing observations to the relevant frequencies and spatial resolutions of the next generation of experiments, an accurate reconstruction of the CMB is

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possible (Brandt et al. 1994, Tegmark & Efstatiou 1996, Hobson et al. 1998). However, these foreground scenarios involve overly simple assumptions about the emissivity or spatial structure of the foregrounds. These methods assume the foregrounds have a power law emissivity, $I_\nu \propto \nu^\alpha$, and that the spectral index $\alpha$ is known or can be adequately inferred from the emission averaged over some spatial scale. The spatial structure of the variation of the spectral index, at the frequencies relevant for observations of the CMB, is in fact a priori unknown. Although further complexities in foreground models are a nuisance with respect to CMB extraction, they represent a “gold mine” of interesting physical quantities for ISM studies (A. Lazarian, 1998).

Because of unknown complexities in the physics of the foregrounds, we have a family of models, in which previous assumptions, such as a spatially constant spectral index, are successively relaxed. Inference within models with increasingly many degrees of freedom becomes impossible (completely degenerate), so the ability to constrain solutions is necessary for the extraction of useful information from the data. Bayesian inference provides a unifying framework within which the separation problem can be addressed in its full range of complexity. In addition, the Bayesian framework can provide computational solutions for more complicated, yet physically motivated, models through algorithms including stochastic relaxation.

In this article, we will review the separation problem and its formulation within Bayesian inference, give examples of specific solutions with particular choices of prior density, and finally comment on the generalization of Bayesian methods to a multi-resolution framework. The paper is meant to convey ideas of variations on current analysis strategies and present the range of possible approaches from linear filtering to multi-scale stochastic relaxation.

2. Review of the Separation Problem

The anticipated galactic foregrounds contributing to the total detected emission at frequencies of interest for CMB observations include free-free, synchrotron, and dust. The total specific intensity integrated along the line of sight from these sources is (following the convention in (Tegmark & Efstatiou 1996))

$$I(r, \nu) = \left[ \left( \frac{270.2 \, \text{MJy sr}^{-1}}{x_0} \right) \left( \frac{x_0^3}{x_0^3 - 1} \right) + \left( 24.8 \, \text{MJy sr}^{-1} \right) \left( \frac{x_0^3}{\sinh(x_0/2)} \right) \right]^2 \delta T \left( \text{CMB} \right)(r) + I_{\text{ff}}^0(r) \left( \frac{\nu}{\nu_0} \right)^{\alpha_{\text{ff}}} + I_{\text{s}}^0(r) \left( \frac{\nu}{\nu_0} \right)^{\alpha_{\text{s}}} + I_{\text{dust}}^0(r) \left( \frac{\nu}{\nu_0} \right)^{2+\beta(r)} \left( \frac{x_0^3}{x_d^3 - 1} \right)$$

(1)

where $I_0$ is a dimensionless map at a convenient reference frequency for the specific physical component, $x_d = h\nu/kT_d \sim \nu/417\text{GHz}$ for a dust temperature of $T_d = 20\text{K}$, $T_0$ is the temperature of the CMB blackbody, and $x_0 = \nu/56.8\text{GHz}$. Subtracting the isotropic blackbody term and converting to brightness temper-
ature (in micro-Kelvin) gives

\[ T(r, \nu) = \delta T^{(CMB)}(r) + \sum_i A^{(i)}(\nu) T_0^{(i)}(r) \left( \frac{\nu}{\nu_0} \right)^{\beta^{(i)}(r)} \] (2)

where \( A^{(i)}(\nu) \) is the spatially independent frequency dependence for the \( i \)th physical component, with spatial variations \( \beta^{(i)}(r) \). Note that the CMB is a frequency-independent, zero-mean field in these units. The frequency dependence of free-free emission is given by physics assuming typical electron densities and temperatures, while the synchrotron and dust spectral indices can be spatially varying due to spatial dependence of the galactic magnetic field and dust grain properties. The data returned from an experiment will be the integrated brightness temperature over the bandpass of the frequency channels of the instrument (with center frequency \( \nu_i \)) with additive noise \( \eta(r, \nu_i) \) giving

\[ T_{\text{obs}}(r, \nu_i) = \delta T^{(CMB)}(r) + \sum_j A^{(j)}(\nu_i) T_0^{(j)}(r) \left( \frac{\nu_i}{\nu_0} \right)^{\beta^{(j)}(r)} + \eta(r, \nu_i) \] (3)

(note that the maps at each frequency are the result of an initial processing stage from the time series returned from the experiment executing a particular scan strategy. We will not include details of this stage of analysis here.)

3. Bayesian Formulation

Given observations \( T_{\text{obs}} \) at the frequency channels of the instrument, we would like to recover the amplitude and spectral index for each of the assumed present physical components. The probability \( p[T_{\text{obs}}|T_0^{(i)}, \beta^{(i)}] \) of the data given the underlying amplitude \( T_0^{(i)}(r) \) and spectral indices \( \beta^{(i)}(r) \) is given by the statistics of the noise process and scan strategy. This probability is known as the likelihood, where \( \log p[T_{\text{obs}}|T_0^{(i)}, \beta^{(i)}] \sim -\chi^2 \) up to an additive constant (the normalization constant), and for pixel independent Gaussian noise

\[ \chi^2 = \left[ T_{\text{obs}} - \sum_j A^{(j)} T^{(j)} \left( \frac{\nu_i}{\nu_0} \right)^{\beta^{(j)}} \right]^T N^{-1} \left[ T_{\text{obs}} - \sum_j A^{(j)} T^{(j)} \left( \frac{\nu_i}{\nu_0} \right)^{\beta^{(j)}} \right] \] (4)

with \( N_{rs} = \langle \eta^2 \rangle \delta_{rs} \).

To find the underlying variables \( (T_0^{(i)}(r), \beta^{(i)}(r)) \), we could simply try to minimize \( \chi^2 \). In the limit that we have an overdetermined system with high signal-to-noise ratio, we can obtain good results with linear methods such as singular value decomposition (with the spectral index assumed known), or a non-linear \( \chi^2 \) method such as used by (Brandt et al. 1994). However, as noted in (Brandt et al. 1994), the recovery of the amplitude and spectral index for all the physical components in the presence of noise is numerically unstable. Some means of regularizing the solution is needed.

There are several strategies that can be employed to regularize solutions. (Brandt et al. 1994) discuss two reasonable simplifications - either assume the
contribution of a particular physical component is negligible at specific frequency channels, or find the spectral index from the average emission on patches of sky $10^7 \times 10^9$, and then find the amplitudes at all pixels holding the spectral index fixed within the given patch of sky.

Wiener filtering is another strategy that has been demonstrated successfully on simulated foregrounds, and is appropriate in cases where we have information about the spatial power spectrum of the foregrounds. The original method of (Tegmark & Efstathiou 1996) assumes the foregrounds are uncorrelated, with a power spectrum of the form $C_l \propto l^{-3}$. The Wiener matrix $W$ is constructed so that

$$T_0^{(i)} = WT_{\text{obs}}$$

and the expected value of the residual error is a minimum, giving the solution

$$T_0^{(i)} = \left[ A^T N^{-1} A + C^{-1} \right]^{-1} A^T N^{-1} T_{\text{obs}}$$

where $W = \left[ A^T N^{-1} A + C^{-1} \right]^{-1} A^T N^{-1}$, $A$ is the frequency response matrix as above, and $C_{ij} = \langle T_0^{(i)}(r_i) T_0^{(j)}(r_j) \rangle$ is the power spectrum of the $i$th foreground component.

As observed by (Hobson et al. 1998), solutions to the foreground separation problem can be generically formulated within Bayesian inference. In Bayesian inference, the posterior probability is interpreted as the figure of merit of a solution, quantifying our degree of confidence, and given by

$$p[T_0^{(i)}, \beta^{(i)} | T_{\text{obs}}] \propto p[T_{\text{obs}} | T_0^{(i)}, \beta^{(i)}] p[T_0^{(i)}, \beta^{(i)}]$$

The first term on the right-hand side is the likelihood as discussed above. The term $p[T_0^{(i)}, \beta^{(i)}]$ is the prior density which effectively regularizes solutions through a statistical characterization of the foregrounds known or assumed a priori. Choosing a Gaussian prior with the assumed power spectrum of the foregrounds gives the Wiener filter solution as the maximum posterior solution (as pointed out by (Hobson et al. 1998)). The maximum entropy (MAXENT) method of (Hobson et al. 1998)) finds the maximum posterior solution with the log-prior given by the entropy. The method of (Brandt et al. 1994) effectively uses a uniform prior over an allowed interval for the fluctuations within a simplified model.

We also note a slight variation on the Wiener filtering solution, in which we include previous observations as a spatial template for subtraction. For example, denoting the previous observations extrapolated to the frequencies and resolution of interest $T_{\text{other}}$, we can use the prior (in matrix notation)

$$- \log p[T_0^{(i)} | T_{\text{other}}] \sim \sum_i (T_0^{(i)})^T C^{-1} (T_0^{(i)}) + \sum_i (T_0^{(i)}) \Lambda_i (T_{\text{other}})$$

where $\Lambda$ quantifies the relative weight of the coupling of ($T_{\text{other}}$) to inferences made about ($T_0^{(i)}$). This prior gives the posterior

$$- \log p[T_0^{(i)} | \beta^{(i)}, T_{\text{obs}}, T_{\text{other}}] \sim \chi^2 + (T_0^{(i)})^T C^{-1} (T_0^{(i)}) + (T_0^{(i)}) \Lambda (T_{\text{other}})$$
The solution which maximizes the posterior is then

\[ T_0^{(i)}(r) = \left[ A^T N^{-1} A + C^{-1} \right]^{-1} \left[ A^T N^{-1} T_{\text{obs}} - \Lambda T_{\text{other}} \right] \]  

Coupling to previous observations as a spatial template, and choosing the maximum posterior solution, gives a subtraction of a particular foreground component through the filtered map \( \Lambda T_{\text{other}} \). This method has the obvious danger of the false addition of correlation in the various components, however we might want to include information from previous observations at large angular scales, where instrumental noise and systematic effects might be typically very low.

4. Separation Problem in a Multi-Resolution Setting

There is a natural way to define fluctuation maps at various scales by adopting a multi-resolution approach (which will not be discussed in great detail here - we follow the discussion in (Langer et al. 1993)). A multi-resolution approach can be implemented with a smoothing function to recursively generate low-pass, or Gaussian, images and band-pass, or Laplacian, images according to

\[ T_0^{(i)}(r, \sigma_j) = G * T_0^{(i)}(r, \sigma_{j-1}) \]

\[ LT_0^{(i)}(r, \sigma_{j-1}) = L * LT_0^{(i)}(r, \sigma_{j-1}) \]

where \( L = I - G \). An example for the smoothing filter is the discrete approximation to a Gaussian provided by a separable filter \( G(i,j) = g_i g_j \), where for example we can take \( g_i = [1/16, 1/4, 3/8, 1/4, 1/16] \). Note that this recursive scheme gives a non-orthogonal and overcomplete wavelet decomposition. The importance of a non-orthogonal and overcomplete basis for image analysis (as opposed to image compression) has been stressed elsewhere (see (Langer et al. 1993) and references therein for a discussion of this point). The basic motivation for a non-orthogonal, overcomplete basis is that the coefficients in an orthogonal wavelet bases can become diffuse upon translations of the original image.

In such a scale-space decomposition, we have fluctuation maps at various scales

\[ LT(r, \nu, \sigma) = L \delta T^{(CMB)}(r, \sigma) + \sum_i A^{(i)}(\nu) LT^{(i)}(r, \nu, \sigma) \]

where we have defined the effective scale-space spectral index as

\[ T^{(i)}(r, \nu, \sigma) = T_0^{(i)}(r, \sigma) \left( \frac{\nu}{\nu_0} \right)^{\beta^{(i)}(r, \sigma)} \]

Note that the above implicitly defines the spectral index in terms of the difference in the log brightness temperature at spatial scale \( \sigma \) of a physical component at two frequencies.

The motivation for the above conventions is simply that, given the spectral indices for the physical components at a given scale, we can iteratively update the fluctuation maps (the simple separation problem traditionally considered).
However, when adjusting the spectral index, we need to work with the total brightness temperature at the next scale, since

\[ c_\nu [\beta^{(i)}(r, \sigma_{j-1}) - \beta^{(i)}(r, \sigma_j)] = \log \left[ T^{(i)}(r, \nu, \sigma_j) + LT^{(i)}(r, \nu, \sigma_{j-1}) \right] \]

\[ - \log \left[ (T_0^{(i)}(r, \sigma_j) + LT_0^{(i)}(r, \sigma_{j-1})) \left( \frac{\nu}{\nu_0} \right) \beta^{(i)}(r, \sigma_j) \right] \]

where \( c_\nu = \log(\nu/\nu_0) \). The scale-space data can then be written as

\[ T_{\text{obs}}(r, \nu_i, \sigma) = \delta T^{(CMB)}(r, \sigma) + \sum_j A^{(j)}(\nu_i) T^{(j)}(r, \sigma) \left( \frac{\nu_i}{\nu_0} \right)^{\beta^{(j)}(r, \sigma)} + \eta(r, \nu_i, \sigma) \]

where the noise \( \eta(r, \nu_i, \sigma) \) has covariance matrix \( \langle \eta^T G \eta \rangle^{-1} \).

We explored a simple separation problem (standard galactic foregrounds with assumed known and spatially constant spectral index) within the above multi-resolution context as a test case in anticipation of more complex foreground scenarios. We considered only a single Gaussian and Laplacian component, so that

\[ T_0^{(i)}(r, 0) = LT_0^{(i)}(r, 0) + T_0^{(i)}(r, \sigma) \]

For pixel independent Gaussian noise, the majority of the noise power is contained in the Laplacian component of the total observed emission, and we expect the foregrounds to be locally smooth with sparsely distributed dominant features in the map. We are therefore motivated to assume a quadratic prior in the Laplacian component

\[ - \log p[LT_0^{(i)}] \sim \theta^j \sum_j (LT_0^{(i)}(r_j))^2 \]

where \( \theta^j \) is a parameter quantifying the relative weight between the prior and likelihood (see the review in (Geman 1990) for other applications of this type of prior). The maximum posterior solution is given by,

\[ T_0^{(i)} = [A^T N^{-1} A + B - H]^{-1} A^T N^{-1} T_{\text{obs}} \]

where the matrix elements \( B^j_{rs} = \theta^j \delta_{rs} \), and \( H^j_{rs} = \theta^j [2G_{rs} - (GG)_{rs}] \).

In order to implement the above however, an estimate for \( \theta^{(j)} \) was needed. We first found a preliminary solution with \( \chi^2 \) minimization, equivalent to the above solution with \( \theta^j = 0 \). After this solution was obtained, we estimate \( \theta^j \) from the initial solution as

\[ \theta^j = \frac{1}{2 \langle (LT_0^{(j)})^2 \rangle} \]

where the angle brackets denote the spatial average over the initial solution. After fixing \( \theta^j \), we continue with an iterative improvement on the Laplacian.
component according to

\[
\begin{align*}
LT_0^{(i)}(n+1) &= \alpha A^T N^{-1} [T_{\text{obs}} - AGT_0^{(i)}(n)] + MLT_0^{(i)}(n) \\
T_0^{(i)}(n+1) &= T_0^{(i)}(n) + LT_0^{(i)}(n+1) - LT_0^{(i)}(n) \\
GT_0^{(i)}(n+1) &= G * T_0^{(i)}(n+1)
\end{align*}
\]  

(19)

where the iteration matrix is \( M = I - \alpha (A^T N^{-1} A + B) \) and \( \alpha^k = 1/(A^T N^{-1} A + B)_{kk} \). The iteration relaxes to the desired maximum posterior solution. The motivation for this iteration is that, to a good approximation, the low-pass filtered \( \chi^2 \) solution is very close to the true low-pass filtered map (since the low-pass noise rms is drastically reduced). The needed improvement to the solution is in the Laplacian component, leading us to the iteration above.

5. Parameter Estimation

The approximation for \( \theta^{(j)} \) gave good results for our specific simple test case above, but in general we will want a more consistent approach to fixing parameters. Within Bayesian inference, the formal way to treat parameters is to include them in the posterior as another piece of information to be drawn from the data. In general, for a collection of weight parameters in the prior (such as the parameters \( \theta^{(j)} \) in our simple test case above), the full posterior becomes

\[
p[T_0^{(i)}, \beta^{(i)}, \theta | T_{\text{obs}}] \propto p[T_{\text{obs}} | T_0^{(i)}, \beta^{(i)}] p[T_0^{(i)}, \beta^{(i)} | \theta] p[\theta]
\]

(20)

where \( \theta \) is the vector of weight parameters. Note that the likelihood is not dependent on the parameters in the prior. The parameters can then be chosen from the density \( p[\theta, T_{\text{obs}}] \), obtained by marginalization over \( (T_0^{(i)}, \beta^{(i)}) \), so that

\[
p[\theta, T_{\text{obs}}] \propto p[\theta] \int d[T_0^{(i)}, \beta^{(i)}] p[T_{\text{obs}} | T_0^{(i)}, \beta^{(i)}] p[T_0^{(i)}, \beta^{(i)} | \theta]
\]

(21)

The density \( p[\theta, T_{\text{obs}}] \) is typically a very sharply peaked function. A reasonable choice of parameter values is then the one that maximizes the likelihood

\[
\theta = \text{max}_\theta p[T_{\text{obs}} | \theta]
\]

(22)

This is the strategy employed by (Hobson et al. 1998) to fix the relative weight between an entropic prior and the likelihood.

6. Generalization of Multi-Resolution Bayesian Inference

Returning to the multi-resolution setting described before, we want to discuss Bayesian methods proceeding from coarse to fine scales. For notational purposes, let \( T_0^{(i,j)}(r) = T_0^{(i)}(r, \sigma_j) \) be the emission at the reference frequency of the \( i^{th} \) physical component at the \( j^{th} \) scale. Let \( T_m^{(i,j)}(r) = T^{(i)}(r, \nu_m, \sigma_j) \) be the emission in frequency channel \( m \) of the \( i^{th} \) physical component at the \( j^{th} \)
scale, and finally let $\beta(i,j)(r) = \beta(i)(r, \sigma_j)$ be similarly defined. We can factor the probability density according to

$$p[T_0^{(i)}, \beta^{(i)} | T_{\text{obs}}] \propto \left( \prod_{\sigma} p[T_0^{(i,j)}, \beta^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}, T_{\text{obs}}] \right)$$  \hspace{1cm} (23)

Each term above has the form

$$p[T_0^{(i,j)}, \beta^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}, T_{\text{obs}}] \propto \frac{p[T_{\text{obs}} | T_0^{(i,j)}, \beta^{(i,j)}]}{p[T_0^{(i,j)}, \beta^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}]}$$  \hspace{1cm} (24)

where the first term is just the generalization of $\chi^2$ at scale $\sigma_j$, and the second term is the prior. We can factor the prior so that

$$p[T_0^{(i,j)}, \beta^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}] = \frac{p[\beta^{(i,j)} | T_0^{(i,j)}, T_0^{(i,j+1)}, \beta^{(i,j+1)}]}{p[T_0^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}]}$$  \hspace{1cm} (25)

The last term $p[T_0^{(i,j)} | T_0^{(i,j+1)}, \beta^{(i,j+1)}]$ is the prior on the amplitudes at the reference frequency with the constraint that the low-pass filtered map of the finer scale amplitude agree with the coarse scale above, $T_0^{(i,j+1)} = G * T_0^{(i,j)}$. We could use multi-resolution generalizations of the previous priors for this term, such as multi-resolution MAXENT methods (Starck & Pantin 1996, Starck et al. 1998). The term $p[\beta^{(i,j)} | T_0^{(i,j)}, T_0^{(i,j+1)}, \beta^{(i,j+1)}]$ is a prior on the spectral index given the amplitudes at the reference frequency, and can be formulated in terms of statistical characterizations of $T_m^{(i,j)}$, the emission of the physical components at each frequency.

One possible strategy in constructing a prior for $\beta^{(i,j)}$ is motivated by the observation that the foregrounds are non-Gaussian. Realizations of a non-Gaussian process can be more compressable in a suitable basis (depending on the type of non-Gaussianity), so that fewer wavelet coefficients are needed to reconstruct the image for a given rms error than in the Fourier basis. Therefore, we expect to be able to make good predictions about $T_m^{(i,j)}$ from $T_m^{(i,j+1)}$ simply by assuming that there are no new features at the finer scale. This approximation is a good one over large scale-space intervals for dominant features.

Dominant features can be associated with wavelet maxima, and tracking the location and amplitude of wavelet maxima across scale provide a way to constrain proposed fine scale maps $T_m^{(i,j)}$ given $T_m^{(i,j+1)}$, and therefore implicitly constrain variations in the spectral index. Wavelet maxima continuously merge in passing from fine to coarse scales, with no new maxima created at lower resolutions. This allows us to predict the location of maxima as we go to finer scales, with the creation of maxima only allowed if the data itself makes a strong case for it. In addition, wavelet maxima have been demonstrated to be a numerically stable representation of images, and a direct linear reconstruction solution given by the amplitude of the maxima (Mallat and Zhong 1991, Carmona et al. 1998). A generalization of the wavelet maxima representation is given by Basis Peruit (Chen et al. 1999), in which a decomposition is given by

$$T_m^{(i,0)}(r) = \sum_k \alpha_m^{(i)} \psi_k(r) \quad \text{such that } S[\alpha_m^{(i)}] \text{ is a minimum}$$  \hspace{1cm} (26)
where $\psi_k$ are the wavelet basis functions, and $S = \sum_k |\alpha_{m,k}^{(i)}|$ quantifies the sparseness of the wavelet coefficients. Basis Pursuit has been shown to generate solutions that are very close to the wavelet maxima representation, demonstrating that maxima can be understood as optimal representations in a specific sense for a wide variety of images.

We therefore propose that the relevant variations in the spectral index to consider are the variations in the spectral index which increase the likelihood and which are generated from a sparse representation according to

\[
C_{\nu}[\beta^{(i,j)} - \beta^{(i,j+1)}] = \log \left( G^j * \sum_k \alpha_{m,k}^{(i)} \psi_k(r) \right) - \log T_0^{(i,j)} \left( \frac{\nu_m}{\nu_0} \right)^{\beta^{(i,j+1)}}
\] (27)

where the wavelet coefficients should be consistent with the constraint

\[
T_{m}^{(i,j+1)} = T_0^{(i,j+1)} \left( \frac{\nu_m}{\nu_0} \right)^{\beta^{(i,j+1)}(r)} = G^j + \sum_k \alpha_{m,k}^{(i)} \psi_k(r)
\] (28)

Formally then, we can give a prior for the spectral index as a marginal process on the “hidden” wavelet coefficients according to (with the content on the right side of the condition bar implied in the above)

\[
p[\beta^{(i,j)}|T_0^{(i,j)}, T_0^{(i,j+1)}, \beta^{(i,j+1)}] = \int d[\alpha_{m,k}^{(i)}] p[\beta^{(i,j)}|\alpha_{m,k}^{(i)}], \ldots] p[\alpha_{m,k}^{(i)}] \ldots
\] (29)

where we can take

\[
-\log p[\beta^{(i,j)}|\alpha_{m,k}^{(i)}, \ldots] \sim \theta \left( C_{\nu}[\beta^{(i,j)} - \beta^{(i,j+1)}] - \log \left( G^j * \sum_k \alpha_{m,k}^{(i)} \psi_k(r) \right) + \log T_0^{(i,j)} \left( \frac{\nu_m}{\nu_0} \right)^{\beta^{(i,j+1)}} \right)^2
\] (30)

and

\[
-\log p[\alpha_{m,k}^{(i)}, \ldots] \sim S[\alpha_{m,k}^{(i)}]
\] (31)

To actually compute the marginalization will require the Metropolis algorithm or Gibbs sampler (Geman and Geman 1984). We could simply maximize the above, and always replace $\beta$ by the maximum - various approximations will have to be numerically experimented with to see what works.

By constraining the emission of a physical component at each frequency to be given according to a sparse representation, we have greatly reduced the effective degrees of freedom. This approach exploits non-Gaussianity, as non-Gaussian features will typically have long scale-space lifetimes and sparse representations in a wavelet basis. The numerical implementation of the above is, needless to say, much more complicated than the linear deterministic solutions discussed previously, driving us to stochastic relaxation. The local character of the posterior however, gives a conditional probability structure that enables various degrees of freedom to be adjusted in parallel.
7. Conclusions

Bayesian inference does not provide any single method for the separation of backgrounds and the CMB, but instead is a framework within which methods can be formulated, and in the process, explicitly reveal any assumptions made. The specific method to be used depends on what information is desired. Various analysis approaches can be understood within a Bayesian framework as the “clean sky” limit, or the “high SNR” limit, or the “only one suspected foreground component in these frequencies” limit, etc. It seems that a unified view of analysis would be important, especially when comparing data returned from different experiments. Probably the best way to proceed with actual data is to attack with any ‘reasonable’ Bayesian approach imaginable. With tools such as a multi-resolution approach and stochastic relaxation, we can attempt inference within the context of physically motivated models that address potential complexities in the foregrounds. Solutions that are consistent with approaches of varying complexity can be considered robust. However, discrepancies in inferences obtained with various methods provide an opportunity to learn about the data itself, and point the way to needed follow up studies and future experiments.

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