MacWilliams type identities on the Lee and Euclidean weights for linear codes over $\mathbb{Z}_\ell$

Yongsheng Tang\(^1\), Shixin Zhu\(^2\), Xiaoshan Kai\(^2,3\)

\(^1\)Department of Mathematics, Hefei Normal University, Hefei 230601, Anhui, P.R.China
\(^2\)School of Mathematics, Hefei University of Technology, Hefei 230009, Anhui, P.R.China
\(^3\)National Mobile Communications Research Laboratory, Southeast University, Nanjing 210096, P.R.China

Abstract Motivated by the works of Shiromoto [3] and Shi et al. [4], we study the existence of MacWilliams type identities with respect to Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$. Necessary and sufficient conditions for the existence of MacWilliams type identities with respect to Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$ are given. Some examples about such MacWilliams type identities are also presented.

keywords: Linear codes, Lee weight enumerator, Euclidean weight enumerator, MacWilliams type identity

1 Introduction

One of the most important results in coding theory is the MacWilliams identity [2] that describes how the weight enumerators of a linear code and its dual code relate to each other. The identity has been found widespread applications in coding theory and has been studied in a lot of ways. In the 1990s, Hammons et al. [1] found that the Lee weight of a codeword played an important role in studying a code over $\mathbb{Z}_4$. This urges that the Lee weight enumerators of linear codes over finite rings have been discussed by many authors. Shiromoto [3] gave the MacWilliams identities on Lee and Euclidean weights for linear codes over $\mathbb{Z}_4$. It is known that Shiromoto’s results [3] hold true for linear codes over $\mathbb{Z}_4$. Unfortunately, these results are not correct for more general rings. Shi et al. [4] presented two counterexamples to Shiromoto’s results [3] on the MacWilliams type identities with respect to Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$. However, the authors [4] did not give the MacWilliams type identities on the Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$. It is natural to ask whether the MacWilliams type identities with respect to the Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$ exist or not. In this paper, we solve this question and give necessary and sufficient conditions for the existence of MacWilliams type identities with respect to Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$.

2 Preliminaries

Let $\mathbb{Z}_\ell (\ell \geq 2)$ denote the ring of integers modulo $\ell$, and $\mathbb{Z}_\ell^n$ be the set of $n$-tuples over $\mathbb{Z}_\ell$. A linear code $C$ of length $n$ over $\mathbb{Z}_\ell$ is an additive subgroup of $\mathbb{Z}_\ell^n$. Hence, $C$ is a $\mathbb{Z}_\ell$-submodule of $\mathbb{Z}_\ell^n$. An element of $C$ is called a codeword of $C$. Any $\mathbb{Z}_\ell$-submodule of $C$ is called a subcode of $C$. Define the dual code $C^\perp$ of $C$ by

$$C^\perp = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{Z}_\ell^n \mid \sum_{i=1}^n x_i y_i = 0, \forall (y_1, y_2, \cdots, y_n) \in C \right\}.$$

Clearly, $C^\perp$ is also a linear code over $\mathbb{Z}_\ell$. The Lee weight for the elements of $\mathbb{Z}_\ell$ is defined as

\footnote{E-mail addresses: ysh\_tang@163.com(Y. Tang), sxinzhu@tom.com(S. Zhu), kxs6@sina.com(X. Kai).}
\[ \text{wt}_L(a) = \min\{a, \ell - a\} \] for all \( a \in \{0, 1, \cdots, \ell - 1\} \) and
\[ \text{wt}_L(c) = \sum_{i=1}^{n} \text{wt}_L(c_i), \]
for \( c = (c_1, c_2, \cdots, c_n) \in \mathbb{Z}_\ell^n \) (see [5]). It is obvious that \( \lfloor \ell/2 \rfloor = \max\{\text{wt}_L(a)\} \) for all \( a \in \{0, 1, \cdots, \ell - 1\} \), where \([a]\) denotes the integer part of \( a \). The Euclidean weight for the elements of \( \mathbb{Z}_\ell \) is defined as \( \text{wt}_E(a) = \text{wt}_L(a)^2 \) for all \( a \in \{0, 1, \cdots, \ell - 1\} \) and
\[ \text{wt}_E(c) = \sum_{i=1}^{n} \text{wt}_E(c_i)^2, \]
for \( c = (c_1, c_2, \cdots, c_n) \in \mathbb{Z}_\ell^n \). We easily find that \( \lfloor \ell/2 \rfloor^2 = \max\{\text{wt}_E(a)\} \) for all \( a \in \{0, 1, \cdots, \ell - 1\} \), where \([a]\) denotes the integer part of \( a \).

Throughout this paper, we denote by \( \ell_1 \) and \( \ell_2 \) the following integers, respectively, \( \ell_1 = \lfloor \ell/2 \rfloor \) and \( \ell_2 = \lfloor \ell/2 \rfloor^2 \). The Hamming weight enumerator of a linear code \( C \) of length \( n \) over \( \mathbb{Z}_\ell \) is defined as
\[ W(x, y) = \sum_{c \in C} x^{n - \text{wt}_H(c)} y^{\text{wt}_H(c)}. \]
Clearly, \( W(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i \), where \( A_i \) denote the number of codewords of Hamming weight \( i \) in \( C \).

The Lee weight enumerator of a linear code \( C \) of length \( n \) over \( \mathbb{Z}_\ell \) is defined as
\[ \text{Lee}(x, y) = \sum_{c \in C} x^{\ell_1 n - \text{wt}_L(c)} y^{\text{wt}_L(c)}. \]
Clearly, \( \text{Lee}(x, y) = \sum_{i=0}^{n} B_i x^{\ell_1 n-i} y^i \), where \( B_i \) denote the number of codewords of Lee weight \( i \) in \( C \).

The Euclidean weight enumerator of a linear code \( C \) of length \( n \) over \( \mathbb{Z}_\ell \) is defined as
\[ \text{Ew}(x, y) = \sum_{c \in C} x^{\ell_2 n - \text{wt}_E(c)} y^{\text{wt}_E(c)}. \]
Clearly, \( \text{Ew}(x, y) = \sum_{i=0}^{n} D_i x^{\ell_2 n-i} y^i \), where \( D_i \) denote the number of codewords of Euclidean weight \( i \) in \( C \).

The following MacWilliams identities on Lee and Euclidean weights for linear codes over \( \mathbb{Z}_\ell \) were obtained in [3].

**Theorem 2.1.** Let \( C \) be a linear code of length \( n \) over \( \mathbb{Z}_\ell \). Denote \( \ell_1 = \lfloor \ell/2 \rfloor \) and \( \ell_2 = \lfloor \ell/2 \rfloor^2 \). Then
\[ \text{Lee}_{C^\perp}(x, y) = \frac{1}{|C|} \text{Lee}_C(x + (\ell_1/\ell_1 - 1)y, x - y); \]
Let $\ell$ be a fixed integer. Recall that $\ell_1 = \lfloor \ell/2 \rfloor$. For any element $a \in \mathbb{Z}_\ell$, a Gray map $\varphi$ on $\mathbb{Z}_\ell$ is defined as

$$
\varphi : \mathbb{Z}_\ell \rightarrow \mathbb{F}_m^\ell,
$$

$$
a \mapsto (a_1, \ldots, a_i, \ldots, a_{\ell_1}),
$$

where $m(>1)$ is any divisor of $\ell$ and a prime power, and $\mathbb{F}_m$ is a finite field with $m$ elements. In detail,

- if $a = 0$, then $wt_\ell(0) = 0$ and $\varphi(a) = (0, \ldots, 0, 0, \ldots, 0)$;
- if $0 \neq a < \ell_1$ and $wt_\ell(a) = i$, then $\varphi(a) = (0, \ldots, 0, a_{\ell_i-i+1}, \ldots, a_{\ell_1})$, where $a_i \neq 0$ for $t = \ell_i - i + 1, \ldots, \ell_1$;
- if $a = \ell_1$ and $wt_\ell(a) = \ell_1$, then $\varphi(a) = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_{\ell_1})$, where $a_t \neq 0$ for $t = 1, \ldots, \ell_1$;
- if $a > \ell_1$ and $wt_\ell(a) = i$, then $\varphi(a) = (a_1, \ldots, a_i, 0, \ldots, 0)$, where $a_t \neq 0$ for $t = 1, \ldots, i$.

The Gray map $\varphi$ can be extended to $\mathbb{Z}_\ell^n$ in an obvious way.

**Example 3.1.** Let us consider a Gray map $\varphi$ on $\mathbb{Z}_6$. Since $\ell_1 = 6$, we have $\ell_1 = 3$. We can take $m = 2$ or 3. A Gray map $\varphi$ from $\mathbb{Z}_6$ to $\mathbb{F}_m^3$ can be defined as $\varphi(0) = (0, 0, 0)$, $\varphi(1) = (0, 0, a_1)$, $\varphi(2) = (0, b_2, b_1)$, $\varphi(3) = (c_3, c_2, c_1)$, $\varphi(4) = (d_2, d_1, 0)$, and $\varphi(5) = (c_1, 0, 0)$, where $a_1, b_1, c_1, d_1 \in \mathbb{F}_2 \setminus \{0\}$. In particular, the Gray map $\varphi$ from $\mathbb{Z}_6$ to $\mathbb{F}_3^3$ can be defined as $\varphi(0) = (0, 0, 0)$, $\varphi(1) = (0, 0, 1)$, $\varphi(2) = (0, 1, 1)$, $\varphi(3) = (1, 1, 1)$, $\varphi(4) = (1, 1, 0)$, $\varphi(5) = (1, 0, 0)$.

The following result about the Gray map is obvious from definition.

**Theorem 3.2.** Let the notation be as before. For any ring $\mathbb{Z}_\ell(\ell \geq 2)$, there exists a Gray map $\varphi$ from $\mathbb{Z}_\ell^n$ to $\mathbb{F}_m^n$, and the Gray map $\varphi$ is a weight preserving map from $(\mathbb{Z}_\ell^n$, Lee weight) to $(\mathbb{F}_m^n$, Hamming weight).

4 A MacWilliams type identity on Lee weight enumerator for linear codes over $\mathbb{Z}_\ell$

For our purpose, we introduce the Krawtchouk polynomials. Let $n$ be a fixed positive integers, $q$ a prime power, and $x$ an indeterminate. The polynomials

$$
K_k(x) = K_k(x, n) = \sum_{j=0}^{k} (-1)^j (q - 1)^{k-j} \binom{x}{j} \binom{n-x}{k-j}, k = 0, 1, 2, \ldots
$$

$$
Ew_{C, l}(x, y) = \frac{1}{|C|} Ew_{C}(x + (\ell^1/\ell^2 - 1)y, x - y).
$$

For linear codes over $\mathbb{Z}_4$, it is known that there exist the MacWilliams identities for Lee weight enumerators (see [1]). That is, Theorem 2.1 can be satisfied for linear codes over $\mathbb{Z}_4$. Unfortunately, it does not hold true for a general ring $\mathbb{Z}_l$. This was pointed out in [4] by giving two counterexamples. The purpose of this paper is to study the existence of the MacWilliams type identities with respect to the Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$. 

3 Gray map on $\mathbb{Z}_\ell$

The following result about the Gray map is obvious from definition.

**Theorem 3.2.** Let the notation be as before. For any ring $\mathbb{Z}_\ell(\ell \geq 2)$, there exists a Gray map $\varphi$ from $\mathbb{Z}_\ell^n$ to $\mathbb{F}_m^n$, and the Gray map $\varphi$ is a weight preserving map from $(\mathbb{Z}_\ell^n$, Lee weight) to $(\mathbb{F}_m^n$, Hamming weight).
are called the Krawtchouk polynomials. From definition of the Krawtchouk polynomials, we can obtain the following two lemmas (see [2] and [6]).

**Lemma 4.1.** For non-negative integers $k$ and $j$,
\[
\sum_{l=0}^{n} K_k(l) K_l(j) = q^n \delta_{k,j},
\]
where $\delta_{k,j} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$ is the Kronecker delta.

**Lemma 4.2.** Let $C$ and $C'$ be two codes of length $n$ over the finite field $\mathbb{F}_q$, and $A_i$ and $A_i'$ be the number of codewords of weight $i$ in $C$ and $C'$, respectively. Then
\[
W_{C'}(x, y) = \frac{1}{|C|} W_C(x + (q - 1)y, x - y).
\]
if and only if
\[
A_k' = \frac{1}{|C|} \sum_{j=0}^{n} A_j K_k(j), k = 0, 1, \ldots, n.
\]

Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$, and $m(> 1)$ be a positive divisor of $\ell$ and a prime power. Let the map $\varphi$ be a weight preserving map from $(\mathbb{Z}_\ell^2$, Lee weight) to $(\mathbb{F}_m^n$, Hamming weight). Then $\varphi(C)$ is a code of length $\ell_1 n$ over $\mathbb{F}_m$, which is not necessarily linear.

Let $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and $W_{\varphi(C)}(x, y)$ be the Hamming weight distribution and weight enumerator of the code $\varphi(C)$ of length $\ell_1 n$ over $\mathbb{F}_m$, respectively. Define their MacWilliams transforms to be $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and $W_{C'}(x, y)$ of a code $C'$ of length $\ell_1 n$ over $\mathbb{F}_m$, respectively. Furthermore, the MacWilliams transforms $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and $W_{\varphi(C)}(x, y)$ are the Hamming weight distribution $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and weight enumerator $W_{C'}(x, y)$ of the code $C'$, respectively, and the MacWilliams transforms of $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and $W_{C'}(x, y)$ are the weight distribution $\{A_0, A_1, \ldots, A_{\ell_1 n}\}$ and $W_{\varphi(C)}(x, y)$ of the code $\varphi(C)$, respectively. By Lemma 4.2, we have
\[
A_k' = \frac{1}{|\varphi(C)|} \sum_{j=0}^{\ell_1 n} A_j K_k(j), l = 0, 1, \ldots, \ell_1 n
\]
and
\[
W_{C'}(x, y) = \frac{1}{|\varphi(C)|} W_{\varphi(C)}(x + (m - 1)y, x - y).
\]
Moreover, for all $c \in \varphi(C)$, we have $A_0 = 1$. By the definition of Krawtchouk polynomials, we have $A_0' = 1$. We know that if $\varphi(C)$ is a linear codes of length $\ell_1 n$ over $\mathbb{F}_m$, then $C' = (\varphi(C))^\perp$.

**Theorem 4.3.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$, and let $m(> 1)$ be a positive divisor of $\ell$ and a prime power. Then the linear code $C$ has a MacWilliams type identity on the Lee weight over $\mathbb{Z}_\ell$ with the form
\[ \text{Lee}_C(x, y) = \frac{1}{|C|} \text{Lee}_C(x + (m-1)y, x - y) \]

if and only if the following conditions hold true

1) there exists a bijective map \( \varphi \) from \( \mathbb{Z}_\ell^n \) to \( \mathbb{F}_m^{\ell n} \) and the map \( \varphi \) is a weight preserving map from \( (\mathbb{Z}_\ell^n, \text{Lee weight}) \) to \( (\mathbb{F}_m^{\ell n}, \text{Hamming weight}) \);

2) there exists a code \( C' \) of length \( \ell_1 n \) over \( \mathbb{F}_m \) and the MacWilliams transform \( W_{C'}(x, y) \) of \( W_{\varphi(C)}(x, y) \) satisfying

\[ W_{\varphi(C)}(x, y) = W_{C'}(x, y). \]

**Proof.** First, suppose that a linear code \( C \) of length \( n \) has a MacWilliams type identity on the Lee weight over \( \mathbb{Z}_\ell \) with the form

\[ \text{Lee}_{C^\perp}(x, y) = \frac{1}{|C|} \text{Lee}_C(x + (m-1)y, x - y). \]

By Theorem 3.2, we attain that \( \varphi(C) \) is a code of length \( \ell_1 n \) over \( \mathbb{F}_m \) and

\[ \text{Lee}_C(x, y) = W_{\varphi(C)}(x, y). \]

For the code \( \varphi(C) \), there exists a code \( C' \) of length \( \ell_1 n \) over \( \mathbb{F}_m \) and the MacWilliams transform \( W_{C'}(x, y) \) of \( W_{\varphi(C)}(x, y) \) satisfying

\[ W_{C'}(x, y) = \frac{1}{|\varphi(C)|} W_{\varphi(C)}(x + (m-1)y, x - y). \]

Furthermore

\[ \text{Lee}_{C'}(x, y) = W_{\varphi(C')}(x, y) = \frac{1}{|C|} W_{\varphi(C)}(x + (m-1)y, x - y). \]

It follows that

\[ |C| W_{\varphi(C')}(x, y) = |\varphi(C)| W_{C'}(x, y). \]  \hspace{1cm} (1)

Note that \( A_0 = 1 \) and \( A'_0 = 1 \). By comparing the coefficient of \( x^{\ell_1 n} \) in the R.H.S. of Equation (1) with the L.H.S. of Equation (1), we obtain

\[ |C| = |\varphi(C)|. \]

Therefore

\[ W_{\varphi(C')}(x, y) = W_{C'}(x, y). \]

This shows that the conditions 1) and 2) hold true.

On the other hand, if there exists a bijective map \( \varphi \) from \( \mathbb{Z}_\ell^n \) to \( \mathbb{F}_m^{\ell n} \) and the map \( \varphi \) is a weight preserving map from \( (\mathbb{Z}_\ell^n, \text{Lee weight}) \) to \( (\mathbb{F}_m^{\ell n}, \text{Hamming weight}) \), then

\[ \text{Lee}_C(x, y) = W_{\varphi(C)}(x, y) \]

and

\[ |C| = |\varphi(C)|. \]

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Furthermore, for the code $\phi(C)$, there exists a code $C'$ of length $\ell_1n$ over $\mathbb{F}_m$ and the MacWilliams transform $W_{C'}(x, y)$ of $W_{\phi(C)}(x, y)$ satisfying

$$W_{C'}(x, y) = \frac{1}{|\phi(C)|} W_{\phi(C)}(x + (m - 1)y, x - y).$$

Since $W_{\phi(C^\perp)}(x, y) = W_{C'}(x, y)$, then

$$W_{\phi(C^\perp)}(x, y) = \frac{1}{|\phi(C)|} W_{\phi(C)}(x + (m - 1)y, x - y).$$

Therefore

$$\text{Lee}_{C^\perp}(x, y) = \frac{1}{|C|} \text{Lee}_{C}(x + (m - 1)y, x - y).$$

**Remark**  If the code $\phi(C)$ is a linear code of length $\ell_1n$ over $\mathbb{F}_m$, then $C' = (\phi(C))^\perp$ in the condition 2) of Theorem 4.3.

From Theorem 4.3, we easily get a necessary and sufficient condition for the existence of the MacWilliams type identities on the Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$.

**Corollary 4.4.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$, and let $m(>1)$ be a positive divisor of $\ell$ and a prime power. Then the linear code $C$ has a MacWilliams type identity on the Lee weight over $\mathbb{Z}_\ell$ with the form

$$\text{Lee}_{C^\perp}(x, y) = \frac{1}{|C|} \text{Lee}_{C}(x + (m - 1)y, x - y)$$

if and only if $\ell = m^{\ell_1}$ and there exists a code $C'$ of length $\ell_1n$ over $\mathbb{F}_m$ and the MacWilliams transform $W_{C'}(x, y)$ of $W_{\phi(C)}(x, y)$ satisfying $W_{\phi(C^\perp)}(x, y) = W_{C'}(x, y)$.

In fact, Corollary 4.4 gives a criterion for judging the existence of the MacWilliams type identities on the Lee weight enumerator for linear codes over $\mathbb{Z}_\ell$. Using this criterion we obtain the following result.

**Corollary 4.5.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell(\ell \geq 5)$, and let $m(>1)$ be a positive divisor of $\ell$ and a prime power. Then there is no MacWilliams type identity on the Lee weight for the linear code $C$ over $\mathbb{Z}_\ell$ with the form

$$\text{Lee}_{C^\perp}(x, y) = \frac{1}{|C|} \text{Lee}_{C}(x + (m - 1)y, x - y).$$

**Proof.** By Corollary 4.4, we have $m = \ell^{1/\ell_1}$. Now, we prove the result by considering three cases.

(i) $\ell = 5$. Then $\ell_1 = 2$. Since $m = \sqrt[4]{5}$ is not a positive integer, then there is no bijective map $\phi$ from $\mathbb{Z}_5^n$ to $\mathbb{F}_m^{2n}$. 6
(ii) \( \ell \geq 6 \) is even. Denote \( \ell = 2\kappa \). Then \( \kappa \geq 3 \) and \( \ell_1 = \kappa \). Let \( \sqrt{2\kappa} = t + 1 \). Then we have \( 2\kappa = (t + 1)^2 \) \( \kappa \geq \frac{(k-1)}{2} t^2 \). It follows that \( 2 > t + \ell^2 \), which means \( t < 1 \). Therefore \( 1 < m = \sqrt{2\kappa} < 2 \). This contradicts the fact that \( m(> 1) \) is a positive divisor of \( \ell \).

(iii) \( \ell > 6 \) is odd. Denote \( \ell = 2\kappa + 1 \). Then \( \kappa \geq 3 \) and \( \ell_1 = \kappa \). Let \( \sqrt{2\kappa + 1} = t + 1 \). Similar to Case 2, we can get a contradiction. \( \square \)

Let us use the above results to consider a linear code \( C \) of length \( n(\geq 1) \) over \( \mathbb{Z}_4 \) on the Lee weight. First, there exists a bijective map \( \varphi \) from \( \mathbb{Z}_4 \) to \( \mathbb{F}_2^n \). In fact, \( \varphi(0) = (0,0) \), \( \varphi(1) = (0,1) \), \( \varphi(2) = (1,1) \), and \( \varphi(3) = (1,0) \), and \( \varphi(C) \) is nonlinear (see [1] and [6]). The map \( \varphi \) can be extended to \( \mathbb{Z}_4^n \) in an obvious way and the extended \( \varphi \) is a bijection from \( \mathbb{Z}_4^n \) to \( \mathbb{F}_2^n \).

Second, there exists a code \( C' \) of length \( 2n \) over \( \mathbb{F}_2 \) and the MacWilliams transform \( W_{C'}(x,y) \) of \( W_{\varphi(C)}(x,y) \). By Lemma 4.2, we have \( W_{C'}(x,y) = \frac{1}{|\varphi(C)|} W_{\varphi(C)}(x+y,x-y) \).

Then, we get \( A'_l = \frac{1}{|\varphi(C)|} \sum_{j=0}^{2n} A_j K_j(j) \). By Lemma 4.1, we have

\[
\frac{1}{|C'|} \sum_{l=0}^{2n} A'_l K_k(l) = \frac{1}{|C'|} \frac{1}{|\varphi(C)|} \sum_{l=0}^{2n} A_j K_l(j) K_k(l) = \frac{1}{|\varphi(C)||C'|} \sum_{j=0}^{2n} A_j \sum_{l=0}^{n} K_l(j) K_k(l) = \frac{1}{|\varphi(C)||C'|} \sum_{j=0}^{2n} A_j 2^{2n} \delta_{j,k}.
\]

Therefore, \( \frac{1}{|C'|} \sum_{l=0}^{2n} A'_l K_k(l) = A_j \) if and only if \( \frac{1}{|\varphi(C)||C'|} \sum_{j=0}^{2n} A_j 2^{2n} \delta_{j,k} = A_j \), that is, \( |\varphi(C)||C'| = 2^{2n} \). On the other hand, for the code \( C' \), there exists a linear code \( C \) over \( \mathbb{Z}_4 \) such that \( C' = \varphi(C) \). Then, \( |\varphi(C)||C'| = |\varphi(C)||\varphi(C)| = 2^{2n} \). Since \( \varphi \) is a bijection, it follows that \( |\varphi(C)||\varphi(C)| = |C||C| = 2^{2n} \). Therefore, \( \varphi(C) = (\varphi(C))^\perp \) or \( C = C'^\perp \) (see [7]). Since \( \varphi(C) \) is nonlinear, it only follows that \( C = C'^\perp \). Then the MacWilliams transform \( W_{C'}(x,y) \) of \( W_{\varphi(C)}(x,y) \) satisfying \( W_{\varphi(C')} (x,y) = W_{C'} (x,y) \). Hence, \( W_{\varphi(C')} (x,y) = \frac{1}{|\varphi(C)|} W_{\varphi(C)} (x+y,x-y) \). Finally, the linear code \( C \) of length \( n \) over \( \mathbb{Z}_4 \) has MacWilliams type identity on the Lee weight with the form

\[
\text{Lee}_{C'}(x,y) = \frac{1}{|C|} \text{Lee}_C(x + y, x - y).
\]

The following two examples demonstrate the non-existence of MacWilliams type identities on the Lee weight for linear codes over \( \mathbb{Z}_6 \) and \( \mathbb{Z}_8 \), respectively.

**Example 4.6.** Consider any linear code \( C \) of length \( n(\geq 1) \) over \( \mathbb{Z}_6 \) equipped with the Lee weight. Since there does not exist a bijective map \( \varphi \) from \( \mathbb{Z}_6^n \) to \( \mathbb{F}_m^n \) \( (m = 2 \text{ or } 3) \), the linear code \( C \) of length \( n \) over \( \mathbb{Z}_6 \) does not have a MacWilliams type identity on the Lee weight with the form

\[
\text{Lee}_{C'}(x,y) = \frac{1}{|C|} \text{Lee}_C(x + (m - 1)y, x - y).
\]

**Example 4.7.** Consider any linear code \( C \) of length \( n(\geq 1) \) over \( \mathbb{Z}_8 \) equipped with the Lee weight. Since there does not exist a bijective map \( \varphi \) from \( \mathbb{Z}_8^n \) to \( \mathbb{F}_m^n \) \( (m = 2, 4 \text{ or } 8) \), then the
linear code $C$ of length $n$ over $\mathbb{Z}_8$ does not have a MacWilliams type identity on the Lee weight with the form

$$\text{Lee}_{C^\perp}(x,y) = \frac{1}{|C|}\text{Lee}_C(x + (m - 1)y, x - y).$$

5 A MacWilliams type identity on Euclidean weight enumerator for linear codes over $\mathbb{Z}_\ell$

In this section, we will use the similar methods in Section 4 to study the MacWilliams type identity on the Euclidean weight enumerator for linear codes over $\mathbb{Z}_\ell$. For every element $a \in \mathbb{Z}_\ell$, a map $\Phi$ on $\mathbb{Z}_\ell$ is defined as

$$\Phi : \mathbb{Z}_\ell \rightarrow \mathbb{F}_{q^n},$$

$$a \mapsto (a_1, \ldots, a_i, a_{i+1}, \ldots, a_\ell),$$

where $q > 1$ is a positive divisor of $\ell$ and a prime power, and $\mathbb{F}_q$ is a finite field with $q$ elements. Similar to Theorem 4.3, we can obtain the following results.

**Theorem 5.1.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$. Let $q > 1$ be a positive divisor of $\ell$, and a prime power. Then the linear code $C$ has a MacWilliams type identity on the Euclidean weight over $\mathbb{Z}_\ell$ with the form

$$\text{Ew}_{C^\perp}(x,y) = \frac{1}{|C|}\text{Ew}_C(x + (q - 1)y, x - y)$$

if and only if the following conditions hold true:

1) there exists a bijective map $\Phi$ from $\mathbb{Z}_\ell^n$ to $\mathbb{F}_{q^n}$, and the map $\Phi$ is a weight preserving map from $(\mathbb{Z}_\ell^n, \text{Euclidean weight})$ to $(\mathbb{F}_{q^n}, \text{Hamming weight});$

2) there exists a code $C''$ of length $\ell 2n$ over $\mathbb{F}_q$ and the MacWilliams transform $W_{C''}(x,y)$ of $W_{\Phi(C)}(x,y)$ satisfying $W_{\Phi(C^\perp)}(x,y) = W_{C''}(x,y)$.

**Corollary 5.2.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$. Let $q > 1$ be a positive divisor of $\ell$ and a prime power. Then the linear code $C$ has a MacWilliams type identity on the Euclidean weight over $\mathbb{Z}_\ell$ with the form

$$\text{Ew}_{C^\perp}(x,y) = \frac{1}{|C|}\text{Ew}_C(x + (q - 1)y, x - y)$$

if and only if $\ell = q^n$ and there exists a code $C''$ of length $\ell 2n$ over $\mathbb{F}_q$ and the MacWilliams transform $W_{C''}(x,y)$ of $W_{\Phi(C)}(x,y)$ satisfying $W_{\Phi(C^\perp)}(x,y) = W_{C''}(x,y)$.

By using the above corollary, we can obtain the nonexistence of a MacWilliams type identity on the Euclidean weight for linear codes over $\mathbb{Z}_\ell$ once the integer $\ell$ is more than 3.

**Corollary 5.3.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_\ell$ ($\ell \geq 4$). Let $q > 1$ be a positive divisor of $\ell$ and a prime power. There is no MacWilliams type identity on the Euclidean weight for the linear code $C$ over $\mathbb{Z}_\ell$ with the form

$$\text{Ew}_{C^\perp}(x,y) = \frac{1}{|C|}\text{Ew}_C(x + (q - 1)y, x - y).$$

**Proof.** By Corollary 5.2, we have $q = \ell^{1/\ell}$. Now, we divide into two cases to prove the result.
(i) $\ell \geq 4$ is even. Denote $\ell = 2\rho$. Then $\rho \geq 2$ and $\ell_2 = \rho^2$. Let $\sqrt{2\rho} = \lambda + 1$. Then $2\rho = (\lambda + 1)^2 > 2\lambda + \frac{2\sqrt{2(\rho - 1)}}{\rho - 1}\lambda^2$. It follows that $2 > \lambda + \lambda^2$, which gives $\lambda < 1$. Therefore $1 < q = \sqrt{2\rho} < 2$, which contradicts the fact that $q(> 1)$ is a positive divisor of $\ell$.

(ii) $\ell > 4$ is odd. Denote $\ell = 2\rho + 1$. Then $\rho \geq 2$ and $\ell_2 = \rho^2$. Let $\sqrt{2\rho + 1} = \lambda + 1$. Then we have $2\rho + 1 = (\lambda + 1)^2 > 1 + \rho^2\lambda + \frac{2\sqrt{2(\rho - 1)}}{\rho - 1}\lambda^2$. It follows that $2 > \lambda + \lambda^2$, which means $\lambda < 1$. Therefore $1 < q = \sqrt{2\rho + 1} < 2$, which contradicts the fact that $q(> 1)$ is a positive divisor of $\ell$.

Corollaries 4.5 and 5.2 give necessary and sufficient conditions for the existence of MacWilliams type identities on the Lee and Euclidean weight for linear codes over $\mathbb{Z}_\ell$, respectively. From them we can see that Theorem 2.1 does not always hold true for all positive integer $\ell$. The existence of MacWilliams type identities on the Lee and Euclidean weight enumerators for linear codes over $\mathbb{Z}_\ell$ depends on the value of $\ell$ and Gray map.

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