INITIAL VALUES PROBLEM FOR SURVIVAL PROBABILITIES IN HOMOGENEOUS DISCRETE TIME RISK MODEL WITH PREMIUM RATE TWO

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Abstract. We analyse $2 \times 2$ recurrent determinants $D_n$ that arise in initial values problem for ultimate time survival probability $\varphi(u)$ in homogeneous discrete time risk model $W(n) = u + \kappa n + \sum_{i=1}^{n} Z_i$, where $Z_i$ are positive integer valued i.i.d. random claims, the initial surplus $u \in \mathbb{N}_0$ and the income rate $\kappa = 2$. We prove the asymptotic version of a recent conjecture on the non–vanishing and monotonicity of $D_n$ and derive explicit formulas for the initial values $\varphi(0), \varphi(1)$ of a recurrence that yields survival probabilities. In cases when $Z_i$ are Bernoulli or Geometrically distributed, the conjecture on $D_n$ is shown to hold for all $n \in \mathbb{N}_0$.

1. Problem formulation and main results

Let $Z_i, 1 \leq i \leq n$ be i.i.d. copies of a discrete random variable (r.v.) $Z$ that takes only non-negative integer values. The sum $\sum_{i=1}^{n} Z_i$ is called a random walk (r.w.). Random walks appear on various occasions in many fields of mathematics, both pure and applied. For instance, in finance and insurance, the accumulated wealth at discrete moments of time $n \in \{0, 1, 2 \ldots \} := \mathbb{N}_0$, can be modeled by

$$W(0) := u, \quad W(n) = u + \kappa n - \sum_{i=1}^{n} Z_i, \quad n \geq 1,$$

where the parameter $u \in \mathbb{N}_0$ is called an initial surplus, $\kappa \in \mathbb{N}$ is the income rate and $Z_i$ represent randomly occurring claims. The model defined in (1) is a simplified discrete version of a more general Andersen model [2]. In such a form as in (1) it was introduced and studied in [13].

The main concern about the accumulated wealth in (1) is whether $W(n) > 0$ for every $n$. In other words, whether the initial savings and...
the subsequent income is always sufficient to cover incurred expenses. One desires to know the probability for the r.w. to hit the line \( u + \kappa n \) at least once up to some natural \( n \). To answer that for a finite \( n \) is just a simple probabilistic problem, see [13, Theorem 1]. However, the qualitative break appears as \( n \to \infty \). For this, let

\[
\varphi(u) := \mathbb{P}(\bigcap_{n=1}^{\infty} \{W(n) > 0\}).
\]

The function \( \varphi(u) \) is called the ultimate time survival probability. Using the law of total probability and elementary rearrangements (see [13, p.3]) it can be shown that

\[
\varphi(u) = \sum_{n=1}^{u+\kappa} h_{u+\kappa-n} \varphi(n),
\]

(2)

where \( h_k = \mathbb{P}(Z = k) \) for all \( k \in \mathbb{N}_0 \). We will assume that \( h_0 > 0 \), for otherwise one could replace every \( Z_i \) in (1) with \( Z_i - 1 \), and \( \kappa \) with \( \kappa - 1 \), which causes the reduction in order of the recurrence [13, Theorems 3 and 4].

In order to demonstrate the main futures of the recurrence (2) without getting entangled in too complicated expressions, we set \( \kappa = 2 \). For \( \kappa = 2 \), we call the process \( W(n) \) primitive if it satisfies \( h_{2n+1} \neq 0 \) for at least one \( n \in \mathbb{N}_0 \). If \( W(n) \) is not primitive, it means that \( \mathbb{P}(Z_i \in 2\mathbb{N}_0) = 1 \), for each \( i \geq 1 \). Then, one can consider the process \( W_1(n) \), obtained by replacing each \( Z_i \) with \( Z_i/2 \) and \( \kappa = 2 \) with \( \kappa/2 = 1 \) in Eq. (1). By denoting the ultimate time survival probability function of \( W_1(n) \) by \( \varphi_1(u) \), it is easy to show that, in the imprimitive case, \( \varphi(2u) = \varphi(2u - 1) = \varphi_1(u) \). So, it would be sufficient to consider the survival probabilities of the process \( W_1(n) \) with the reduced income rate \( \kappa = 1 \) instead of \( W(n) \) with \( \kappa = 2 \).

In order to find the values \( \varphi(u) \), for every \( u \in \mathbb{N}_0 \) with \( \kappa = 2 \), from the recurrence (2), we need to know initial values \( \varphi(0) \) and \( \varphi(1) \). Remaining values \( \varphi(u) \) are solved for by setting \( u = 0, 1, \ldots, n \) in (2):

\[
\varphi(0) = h_1 \varphi(1) + h_0 \varphi(2) \implies \varphi(2) = \frac{1}{h_0} \varphi(0) - \frac{h_1}{h_0} \varphi(1),
\]

\[
(1 - h_2) \varphi(1) - h_1 \varphi(2) - h_0 \varphi(3) = 0 \implies \varphi(3) = \frac{-h_1}{h_0} \varphi(0) + \frac{h_0 + h_1^2 + h_0 h_2}{h_0^2} \varphi(1),
\]

as so on. By mathematical induction [13, p. 17],

\[
\varphi(n) = x_n \varphi(0) + y_n \varphi(1),
\]

(3)
where the deterministic sequences $x_n$ and $y_n$ are given by

$$
x_0 := 1, \ x_1 := 0, \ x_n := \frac{1}{h_0} \left( x_{n-2} - \sum_{i=1}^{n-1} h_{n-i} x_i \right), \text{ for } n \geq 2 \quad (4)
$$

and

$$
y_0 := 0, \ y_1 := 1, \ y_n := \frac{1}{h_0} \left( y_{n-2} - \sum_{i=1}^{n-1} h_{n-i} y_i \right), \text{ for } n \geq 2. \quad (5)
$$

If we define a bi-valued r.v. $S$ according to the ultimate time survival or ruin at the surplus level $u$,

$$
S(u) := \begin{cases} 1, & \text{if } W(n) > 0 \text{ for all } n \in \mathbb{N}_0 \\ 0, & \text{if } W(n) \leq 0 \text{ for some } n \in \mathbb{N}_0, \end{cases}
$$

then one can deem $S(u)$ as a "long–memory" process with respect to $u$: in general, we must know the probabilities $\varphi(u) = P(S(u) = 1)$ for all the previous surplus levels up to $u$ to compute the next one for $u$.

For every $n \in \mathbb{N}_0$, (3) implies

$$
\begin{pmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix} = \begin{pmatrix} \varphi(n) \\ \varphi(n+1) \end{pmatrix}. \quad (6)
$$

Let

$$
D_n := \begin{vmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{vmatrix}, \quad n \in \mathbb{N}_0 \tag{7}
$$

be the principal determinant of (6). Namely $D_n$ is the main object of interest in the present paper. If $D_n \neq 0$, then $\varphi(0)$ and $\varphi(1)$ can be solved from the system (6):

$$
\varphi(0) := \frac{y_{n+1}}{D_n} \varphi(n) - \frac{y_n}{D_n} \varphi(n+1), \quad \varphi(1) := \frac{x_n}{D_n} \varphi(n+1) - \frac{x_{n+1}}{D_n} \varphi(n) \tag{8}
$$

Obviously, $\varphi(n) \leq \varphi(n+1) \leq 1$ for all $n \in \mathbb{N}_0$. This implies that $\varphi(\infty) := \lim_{n \to \infty} \varphi(n) \in [0, 1]$ exists. In addition, if the limits of the ratios

$$
\frac{x_n}{D_n}, \quad \frac{x_{n+1}}{D_n}, \quad \frac{y_n}{D_n}, \quad \frac{y_{n+1}}{D_n},
$$

exist, then

$$
\varphi(0) = \varphi(\infty) \lim_{n \to \infty} \frac{y_{n+1} - y_n}{D_n}, \quad \varphi(1) = \varphi(\infty) \lim_{n \to \infty} \frac{x_n - x_{n+1}}{D_n}. \quad (8)
$$

Using the law of large numbers it can be proved [13, Lemma 1] that $\varphi(\infty) = 1$ if the expectation $\mathbb{E}Z < 2$. If $\mathbb{E}Z \geq 2$, then $\varphi(n), \ n \in \mathbb{N}_0$ and $\varphi(\infty)$ all vanish [13, Theorem 9]. Intuitively, this means that the survival is possible if claims, represented by $Z$, on average are not too "aggressive".
Therefore, the non-vanishing of $D_n$ for each $n \in \mathbb{N}$, and, more generally, the asymptotic behavior of the $D_n$ for large $n$ is of great interest in the reconstruction of the initial values $\varphi(0), \varphi(1)$, and, subsequently, the remaining values $\varphi(u), u \geq 2$ via the recurrence in (2). In [13, p. 6], numerical computations with some selected distributions of $Z$ led to the following conjecture.

**Conjecture 1.** For every $n \in \mathbb{N}_0$,

$$1 \leq D_{2n} \leq D_{2n+2} \quad \text{and} \quad D_{2n+3} \leq D_{2n+1} \leq -1,$$

So far, numerical calculations did not reveal any counterexamples of Conjecture 1. We show that in the imprimitive case, Conjecture 1 admits an almost trivial proof which will be given in Section 2, Proposition 6. In the primitive case no proof is available yet. In Section 5, we verify this conjecture in cases when $Z$ is a Bernoulli or a Geometric r.v.

**Theorem 2.** Let $Z \sim B(p)$ or $Z \sim G(p)$, where $p \in (0, 1)$. For such r.v. $Z$, Conjecture 1 is true.

In the present paper we prove an asymptotic version of Conjecture 1 that requires the finiteness of the higher moments of $Z$ but does not depend on the specific distribution.

**Theorem 3.** Assume that $\mathbb{P}(Z \in 2\mathbb{N}_0 + 1) \neq 0$, and $\mathbb{E}Z < +\infty$. Furthermore, suppose that the higher moments of $Z$ satisfy:

$$\mathbb{E}Z^2 < +\infty, \quad \text{if } \mathbb{E}Z \neq 2 \quad \text{or} \quad \mathbb{E}Z^4 < +\infty, \quad \text{if } \mathbb{E}Z = 2.$$

Then,

$$1 < D_{2n} < D_{2n+2}, \quad D_{2n+3} < D_{2n+1} < -1,$$

for every $n > n_0$, where $n_0$ depends on $Z$ only, and

$$D_{2n} \to +\infty, \quad D_{2n+1} \to -\infty,$$

as $n \to +\infty$.

Theorem 3 is proved in Section 4. In the same section, we derive the asymptotics of $x_n, y_n$ and $D_n$ which, together with expressions in (8), allow to reconstruct explicitly the initial values $\varphi(0), \varphi(1)$ for recurrence relation in (2) with $\kappa = 2$.

**Theorem 4.** Let $h_0 = \mathbb{P}(Z = 0) > 0$ and $\mathbb{E}Z < 2$.

If $\mathbb{P}(Z \in 2\mathbb{N}_0 + 1) = 0$, then

$$\varphi(0) = \frac{2 - \mathbb{E}Z}{2}, \quad \varphi(1) = \frac{2 - \mathbb{E}Z}{2h_0}.$$
If $P(Z \in 2N_0 + 1) \neq 0$, and, in addition, $EZ^2 < +\infty$, then

$$
\varphi(0) = \frac{\alpha(2 - EZ)}{1 + \alpha}, \quad \varphi(1) = \frac{2 - EZ}{h_0(1 + \alpha)},
$$

where $-\alpha^{-1} \in (-1, 0)$ denotes the unique real solution of the equation $H(t) = t^2$, $t \in \mathbb{C}$ for the probability generating function (p.g.f.) $H(t)$ of $Z$.

It should be noted that proving monotonicity or even non-vanishing of $D_n$ for all $0 \leq n \leq n_0$ seems to be a really difficult problem in all but few trivial cases. For instance, when the p.g.f. of $Z$ is a rational function (the ratio of two polynomials with real coefficients), then to solve $D_n = 0$ for $n$ is equivalent to finding all zeros of a certain linear recurrence. This includes even the very basic case when the r.v. $Z$ has a finite support (its p.g.f. is a polynomial). There are no general explicit analytic formulas for finding such $n$. Typically, the solution involves obtaining a numerical upper bound $n_0$ for $n \in \mathbb{N}$ or the total number of such possible solutions using a sophisticated number-theoretical machinery, and then checking the range of possible $n \in \mathbb{N}$ with computers. Monograph [8] is an excellent source of references on this and related topics.

It is worth to mention that different generalizations, comparing to the model in (1), covering different r.w. setups, time, income rates and etc. are known: one can refer to [2], [6], [7], [9], [10], [11], [12], [21], and many other papers. Related versions of Conjecture 1 have been raised in [12, p. 8, 12], [13, p. 8, 9] by Šiaulys and the first named author of the present paper.

As the initial values of survival probabilities depend on the location of zeros of a p.g.f. of the r.v. $Z$, it is unlikely that there exists one simple formula that covers all possible cases for arbitrary $\kappa \in \mathbb{N}$. To avoid being buried by extensive technical details and the large number of cases to work through, in the present paper we deal with the most simple version of Conjecture 1 for $\kappa = 2$, and prepare mathematical tools to extend these results beyond $\kappa \geq 3$ in the future, including models with more complicated r.w. setups. Thus, some of our lemmas in Section 3 are stated in a form more general than needed for $\kappa = 2$ case.

2. Generating functions

Recall that the probabilities $h_n = P(Z = n) \geq 0$ for $n \in \mathbb{N}_0$ satisfy $\sum_{n \geq 0} h_n = 1$, with $h_0 > 0$. The probability generating function $H(t)$
of the r.v. \( Z \) is defined via the power series

\[
H(t) := \sum_{n \geq 0} h_n t^n, \quad t \in \mathbb{C}.
\]

We call an arbitrary power series \textit{imprimitive}, if there exists an integer \( d \geq 2 \), such that \( d \mid n \) whenever \( t^n \) is present in the series. In particular, \( H(t) \) is imprimitive when \( h_n \neq 0 \) only for \( n \) divisible by \( d \). In that case, one can re-write \( H(t) \) as \( H(t) = H_1(t^d) \) for a p.g.f. \( H_1(t) \) of the r.v. \( Z/d \). If no such \( d \geq 2 \) exists, then we call \( H(t) \) \textit{primitive}. Notice, that the process \( W(n) \) is primitive (resp. imprimitive), as described in Section 1, precisely when \( H(t) \) is primitive (resp. imprimitive).

We also use the notations

\[
\mathbb{D} = \{ t \in \mathbb{C} : |t| < 1 \}, \quad \partial \mathbb{D} = \{ t \in \mathbb{C} : |t| = 1 \}, \quad \overline{\mathbb{D}} = \{ t \in \mathbb{C} : |t| \leq 1 \}
\]

for the open unit disk and the unit circle in a complex plane \( \mathbb{C} \).

The generating functions of sequences \( x_n, y_n \ n \in \mathbb{N}_0 \) from (4), (5) for \( t \in \mathbb{C} \) are defined by

\[
X(t) := \sum_{n \geq 0} x_n t^n, \quad Y(t) := \sum_{n \geq 0} y_n t^n.
\]  

From (4) we have

\[
x_{n-2} = \sum_{i=0}^{n} x_i h_{n-i} - x_0 h_n, \quad n \geq 2.
\]

From this,

\[
\sum_{n \geq 2} x_{n-2} t^n = \sum_{n \geq 2} \left( \sum_{i=0}^{n} x_i h_{n-i} \right) t^n - x_0 \sum_{n \geq 2} h_n t^n
\]

or

\[
t^2 X(t) - X(t) H(t) - x_0 h_0 - (x_1 h_0 + x_0 h_1) t - x_0 (H(t) - h_0 - h_1 t),
\]

which simplifies to

\[
t^2 X(t) = X(t) H(t) - x_1 h_0 t - x_0 H(t).
\]  

From initial conditions \( x_0 = 1, x_1 = 0 \), one obtains

\[
X(t) = \frac{H(t)}{H(t) - t^2}.
\]  

By replacing \( x_n \) with \( y_n \) and using the appropriate initial conditions \( y_0 = 0, y_1 = 1 \) in (10), one obtains

\[
Y(t) = \frac{h_0 t}{H(t) - t^2}.
\]
Furthermore, from (11) and (12)
\[ h_0 X(t) - t Y(t) = h_0. \] (13)

Power series expansion of (13) yields
\[ h_0 x_n - y_{n-1} = 0, \quad n \geq 1 \quad \text{or} \quad y_n = h_0 x_{n+1}, \quad n \geq 0. \] (14)

Then \( D_n \) in (7) becomes
\[ D_n = \left| \begin{array}{cc} x_n & y_n \\ x_{n+1} & y_{n+1} \end{array} \right| = \left| \begin{array}{cc} x_n & h_0 x_{n+1} \\ x_{n+1} & h_0 x_{n+2} \end{array} \right| = h_0 \left( x_n x_{n+2} - x_{n+1}^2 \right). \] (15)

**Proposition 5.** For every \( n \in \mathbb{N}_0 \),
\[ 1 \leq x_{2n} \leq x_{2n+2} \quad \text{and} \quad x_{2n+3} \leq x_{2n+1} \leq 0. \]

*Proof of Proposition 5.* For \( n = 0 \), \( x_0 = 1 \leq 1/h_0 = x_2 \) and \( x_1 = 0 \geq -h_1/h_0^2 = x_3 \). By induction,
\[ x_{2n+2} = \frac{1}{h_0} \left( x_{2n} - \sum_{i=1}^{2n+1} h_{2n+2-i} x_i \right) = \]
\[ = \frac{1}{h_0} \left( x_{2n} - (h_{2n+1} x_1 + \ldots + h_1 x_{2n+1}) - (h_{2n} x_2 + \ldots + h_2 x_{2n}) \right) \]
\[ \geq \frac{1}{h_0} \left( x_{2n} - x_{2n}(h_1 + h_2 + \ldots) \right) = x_{2n} \]
and
\[ x_{2n+3} = \frac{1}{h_0} \left( x_{2n+1} - \sum_{i=1}^{2n+2} h_{2n+3-i} x_i \right) = \]
\[ = \frac{1}{h_0} \left( x_{2n+1} - (h_{2n+2} x_1 + \ldots + h_2 x_{2n+1}) - (h_{2n+1} x_2 + \ldots + h_1 x_{2n+2}) \right) \]
\[ \leq \frac{1}{h_0} \left( x_{2n+1} - x_{2n+1}(h_1 + h_2 + \ldots) \right) = x_{2n+1}. \]

Similar inequalities were obtained in [6] for a model that differs from given in (1).

It is curious that the monotonicity property from Proposition 5 is sufficient to establish Conjecture 1 when \( H(t) - t^2 \) is imprimitive.

**Proposition 6.** If \( \mathbb{P}(Z \in 2\mathbb{N}_0 + 1) = 0 \), then Conjecture 1 is true.

*Proof of Proposition 6.* The equality \( h_{2n+1} = 0 \) yields \( H(t) = H_1(t^2) \) for a p.g.f. \( H_1(t) \). By (11), we have \( X(t) = X_1(t^2) \) for \( X_1(t) = H_1(t)/(H_1(t) - t) \). It follows that \( X(t) \) is a power series of \( t^2 \), which implies \( x_{2n+1} = 0 \) for all \( n \in \mathbb{N}_0 \). Then, by (15),
\[ D_{2n} = h_0 x_{2n} x_{2n+2}, \quad D_{2n+1} = -h_0 x_{2n+2}^2. \]
As $x_0 = 1$ and $x_{2n}$ is non-decreasing by Proposition 5, $D_n$ have the required properties. □

However, such a simple trick is not sufficient to prove Conjecture 1 in the primitive case.

3. Location and properties of zeros

We prove a series of technical lemmas about the location of zeros and the vanishing multiplicity of the power series of the form $H(t) - t^\kappa$, $t \in \mathbb{C}$ and $\kappa \in \mathbb{N}$, where $H(t)$ is the p.g.f. of the r.v. $Z$. The power series of this form appear in the denominators of generating functions $X(t) = H(t)/(H(t) - t^\kappa)$ for the recurrence (2) with arbitrary natural $\kappa$. The choice $\kappa = 2$ corresponds generating functions derived in Section 2 and Conjecture 1. We allow arbitrary $\kappa \in \mathbb{N}$ in this section for the future references to the lemmas presented here.

We start with a basic properties of p.g.f. $H(t)$.

Lemma 7. The function $H(t)$ is holomorphic in $\mathbb{D}$ and continuous on its boundary $\partial \mathbb{D}$. In addition, if $H^{(k)}(1) < +\infty$, $k \in \mathbb{N}$, then the derivatives $H^{(j)}(t)$, $0 \leq j \leq k$ are continuous on $\partial \mathbb{D}$.

Proof of Lemma 7. For $|t| \leq 1$, $\sum_{n \geq 0} |h_n t^n| \leq \sum_{n \geq 0} h_n = 1$, so the convergence of the power series $H(t)$ is absolute and uniform. It follows that $H(t)$ is holomorphic inside the unit disk and continuous on its boundary.

Similarly, the sum of absolute values of the terms in the series $H^{(k)}(t)$ in $\overline{\mathbb{D}}$ is less or equal to $\sum_{n \geq 1} n! h_n/(n - k)! = H^{(k)}(1) < +\infty$. Thus, $H^{(k)}(t)$ converges uniformly to a continuous function for $|t| \leq 1$. This implies that the derivatives of $H(t)$ of order $\leq k$ are well defined and continuous on $\partial \mathbb{D}$. □

Lemma 8. The function $H(t) - t^\kappa$ has at most $\kappa$ zeros in $\mathbb{D}$, counted with their multiplicities.

Proof of Lemma 8. For every $t \in \partial \mathbb{D}$, and every real $\lambda > 1$, $|H(t)| \leq 1 < |\lambda t^\kappa|$. Hence, by Rouché’s theorem [18, Ch.10, Ex.24], $H(t) - \lambda t^\kappa$ has the same number of zeros in $\mathbb{D}$ as $t^\kappa$. By continuity of zeros inside $\mathbb{D}$ with respect to the parameter $\lambda$, as $\lambda \to 1^+$, the number of zeros cannot increase as $\lambda$ reaches 1 (it can only decrease, if some zeros from $\mathbb{D}$ reach the boundary $\partial \mathbb{D}$ at $\lambda = 1$). □

In subsequent lemmas, the positive integer $d$ is not restricted to $d \geq 2$ as before (it can also equal to 1).
Lemma 9. The equality $H(t) = \pm t^\kappa$ holds on $\partial \mathbb{D}$ only at points $t$ that satisfy $t^d = 1$ for $d \in \mathbb{N}$, such that $d \mid n$ whenever $h_n \neq 0$ and $d \mid \kappa$ in '+' case, $d \mid 2\kappa$ in '-' case. In particular, if $H(t) - t^\kappa$ is primitive, then $H(t) = t^\kappa$ holds on $\partial \mathbb{D}$ only at $t = 1$.

Proof of Lemma 9. In the triangle inequality,

$$1 = |H(t)t^{-\kappa}| \leq \sum_{n \geq 0} h_n |t^{n-\kappa}|,$$

"=" is attained only when all non-zero terms have the same complex argument. This implies that $t^{l-k}$ is real positive whenever $h_l h_k \neq 0$. Equality $|t| = 1$ implies $t^{l-k} = 1$ and all such $t \in \mathbb{D}$ must be roots of unity whose orders divide all the differences $l - k$ for $h_l h_k \neq 0$. Since $h_0 \neq 0$, for such a root of unity of the minimal order $d$, it follows that all $h_l \neq 0$ must lie in some arithmetic progression $l = dj$, $j \in \mathbb{N}$. Thus, one can write $H(t) = H_1(t^d)$. Then, for such a root of unity, $H_1(t^d) = 1$ and $H(t) = t^\kappa$ imply $t^\kappa = 1$, and $d \mid \kappa$, or $t^\kappa = -1$, which means $d \mid 2\kappa$. The primitive case now becomes obvious. □

Lemma 10. Let $r$ denote the order of vanishing of $H(t) - t^\kappa$ at $t = 1$. Then $r \leq 2$.

Proof of Lemma 10. For the real $t \in [0, 1]$, the repeated application of Cauchy Middle Value Theorem for $(H(t) - t^\kappa)/(t - 1)^k$, when $k \leq r$, implies that finite one sided limits

$$\lim_{t \to 1^-} \frac{H(t) - t^\kappa}{(t - 1)^k} = \begin{cases} \frac{H^{(k)}(1) - \kappa!}{k!}, & \text{for } k \leq \kappa \\ \frac{H^{(k)}(1)}{k!}, & \text{for } k > \kappa \end{cases}$$

exist for $k \leq r$. Limits in (16) must equal 0 for $k < r$ and, for $k = r$, limit must be finite and non–zero. It follows that $H^{(r)}(1)$ exist and are $< +\infty$. By Lemma 7, derivatives $H^{(j)}(t)$, $0 \leq j \leq r$ are continuous on $\overline{\mathbb{D}}$.

Assume that $r \geq 3$. Then, one has $H'(1) = \kappa$, $H''(1) = \kappa(\kappa - 1)$, $H'''(1) \notin \{0, +\infty\}$ by (16). Since $H(1) = \mathbb{E}Z$, $H'(1) = \mathbb{E}Z^2 - \mathbb{E}Z$, one obtains $\mathbb{E}Z = \kappa$, $\mathbb{E}Z^2 = \kappa^2$. Then, the variance $\text{var}(Z) = 0$ which means that r.v. $Z$ is degenerate $\mathbb{P}(Z = \kappa) = 1$. This means $H(t) = 1 - t^\kappa$ and $H(t)$ is a polynomial with the vanishing order at $t = 1$ at most one, which contradicts the assumption $r \geq 3$. □

Lemma 11. For the power series $H(t)$ it holds that $H(t) - t^\kappa = (1 - t)G(t)$, where $G(t) = \sum_{n \geq 0} g_n t^n$, and the coefficients

$$g_n = \begin{cases} \sum_{k \leq n} h_k, & \text{if } n < \kappa \\ -1 + \sum_{k \leq n} h_k, & \text{for } n \geq \kappa. \end{cases}$$
Moreover:

- If $H'(1) > \kappa$, then $G(t)$ has one simple real zero $t \in (0, 1)$, and $G(1) \neq 0$.
- If $H'(1) = \kappa$, then $G(t)$ vanishes for $t \in [0, 1]$ only at $t = 1$.
- If $H'(1) < \kappa$, then $G(t) > 0$ for all $t \in [0, 1]$.

Proof of Lemma 11. As $\sum_{n \geq 0} h_n = 1$, 

$$H(t) - t^\kappa = \sum_{n \geq 0} h_n (t^n - t^\kappa) = \sum_{n < \kappa} h_n (t^n - t^\kappa) - \sum_{n > \kappa} h_n (t^\kappa - t^n),$$

therefore

$$G(t) = \frac{H(t) - t^\kappa}{1 - t} = \sum_{n < \kappa} h_n t^n \frac{1 - t^{\kappa-n}}{1 - t} - \sum_{n > \kappa} h_n t^n \frac{1 - t^{n-\kappa}}{1 - t}$$

$$= \sum_{n < \kappa} h_n \left( \sum_{k=n}^{\kappa-1} t^k \right) - \sum_{n > \kappa} h_n \left( \sum_{k=\kappa}^{n-1} t^k \right)$$

$$= \sum_{k < \kappa} t^k \left( \sum_{n \leq k} h_n \right) - \sum_{k > \kappa} t^k \left( \sum_{n > k} h_n \right)$$

$$= \sum_{k < \kappa} t^k \left( \sum_{n \leq k} h_n \right) - \sum_{k \geq \kappa} t^k \left( 1 - \sum_{n \leq k} h_n \right).$$

Interchanging $k$ and $n$ yields the above claimed formulas.

As the coefficients $g_n \geq 0$, for $n < \kappa$, and $g_n \leq 0$, for $n \geq \kappa$, $G(t)$ and $G'(t)$ have at most one sign change each; by Descartes rule of signs for power series [5], it follows that each of $G(t)$ and $G'(t)$ can have at most 1 simple positive real zero in $(0, 1]$. As $G(0) = h_0 > 0$, $G(t)$ must have one simple zero in $(0, 1]$ if $G(1) < 0$. If $G(1) > 0$, then $G(t)$ does not vanish in $(0, 1]$, since in such case it must vanish twice, or have a zero of even multiplicity, which would contradict the aforementioned Descartes rule. It remains to consider the possibility that $G(1) = 0$. If $G(t)$ vanishes at some other point $s \in (0, 1)$, then $G'(t)$ must change its sign between 0 and $s$: indeed, as $G(0) = h_0 > 0$, $G'(0) = h_0 + h_1 > 0$ (for $\kappa \geq 2$), $G'(t)$ must become negative in $(0, s)$ for $G(t)$ to descend to 0 at $t = s$. Then, by Rolle's theorem, $G'(t)$ has at least two zeros: one in the interval $(0, s)$, as it was discussed above, and another in $(s, 1)$, contradicting the sign rule applied to $G'(t)$ [5]. The same is also true when $\kappa = 1$: $G'(t) < 0$ for $t \in (0, 1]$ or $G(t)$ is a constant (because $g_n \leq 0$, for $n \geq 1$, when $\kappa = 1$). Therefore $G(t)$ can have only one zero.
in $[0, 1]$, when $\kappa = 1$. One evaluates $G(1)$ by

$$G(1) = \lim_{t \to 1^-} \frac{H(t) - t^\kappa}{1 - t} = (t^\kappa - H(t))'|_{t=1} = \kappa - H'(1).$$

This proves all the properties of $G(t)$ claimed in Lemma 10.

**Lemma 12.** For any complex zero $\zeta \in \mathbb{D}$ of $H(t) - t^\kappa$, it’s absolute value $|\zeta|$ belongs to the interval $(0, s]$; here $s \in (0, 1]$ denotes the smallest positive zero of $H(t)$. Moreover, $|\zeta| = s$ is possible only when $\zeta = |s| e^{2\pi i j/d}$, for $0 \leq j \leq d - 1$, where the integer $d | \kappa$ and $d | n$ for each $n$ such that $h_n \neq 0$ and $i = \sqrt{-1}$.

**Proof.** By the previous Lemma 11, $H(t) - t^\kappa$ has unique positive simple real zero $s \in (0, 1]$, such that $H(t) > t^\kappa$ for $t \in (0, s)$, and $H(t) < t^\kappa$ for $t \in (s, 1)$ (if $s = 1$, then the later interval is empty). After taking absolute values on both sides of $H(\zeta) = \zeta^\kappa$, one obtains

$$\sum_{n \geq 0} h_n |\zeta^n| \geq |\zeta|^s, \quad (17)$$

which is equivalent to $H(|\zeta|)-|\zeta|^s \geq 0$. Thus, $|\zeta| \in (0, s]$. Furthermore, $|\zeta| = s$ is possible for $H(\zeta) = \zeta^\kappa$ only when equality is attained in the triangle inequality in (17). Reasoning the same way as in Lemma 9, all non-zero terms $h_n \zeta^n$ and $\zeta^\kappa$ must be real and positive, and there exists such smallest $d \in \mathbb{N}$, which $d | \kappa$, $d | n$ whenever $h_n \neq 0$ and $\zeta^d = s^d$.

The statement follows.

**Corollary 13.** Let $s \in (0, 1]$ denote the smallest positive zero of $H(t) - t^\kappa$. If $\kappa$ is even and $h_n \neq 0$ for at least one odd $n \in \mathbb{N}$, then $|H(-1)| < 1$ and $H(t) - t^\kappa$ has odd number of negative real zeros in $(-1, 0)$, each of them of odd order and located in $(-s, 0)$.

**Proof of Corollary 13.** As $0 < h_0 < 1$, one readily verifies that $H(-1) = \pm 1$ is impossible, if $h_n \neq 0$ for at least one odd $n$ (see also Lemma 9). Hence, $|H(-1)| < 1$. It follows that $H(t) - t^\kappa < 0$ at the point $t = -1$ and $H(t) - t^\kappa = h_0 > 0$ at $t = 0$, so the function $H(t) - t^\kappa$ must have an odd number of sign change points in $(-1, 0)$. Furthermore, by Lemma 12, $|t| \leq s$. As $t = -s$ is possible only for even integers $d$ in Lemma 9 (for $s = 1$) or Lemma 12 (for $s < 1$), we must have $|t| < s$ for each such sign change point $t \in (-s, 0)$.

**Corollary 14.** Assume that $H(t) - t^2$ is primitive. Then $H(t) - t^2$ has at most 2 simple, distinct zeros inside $\mathbb{D}$ and one zero on the boundary $\partial \mathbb{D}$ of multiplicity at most 3. More precisely, $H(t) - t^2$ has

a) a simple negative zero at $t = -\alpha^{-1} \in (-1, 0)$.

b) a simple positive zero at $t = \beta^{-1} \in (\alpha^{-1}, 1)$, when $H'(1) > 2$. 
c) a zero at $t = 1$. If $H'(t) \neq 2$, then this zero is simple. If $H'(1) = 2$, $H''(1) \notin \{2, +\infty\}$, then $t = 1$ is a double zero.

**Proof of Corollary 14.** By Lemma 8, $H(t) - t^2$ can have at most 2 distinct simple zeros inside $\mathbb{D}$ or 1 zero of order 2 in $\mathbb{D}$. By Corollary 13, there is precisely one real negative zero at $t = -\alpha - 1$ of order 1. Hence, another possible zero of $H(t) - t^2$ must be also of order $= 1$, so it must be real positive, because complex zeros of $H(t) - t^2$ with real coefficients should occur in conjugate pairs (see Lemma 11). If such zero exists, then denote it by $t = \beta - 1 \in (0, 1)$. One must have $\beta < \alpha$ by Lemma 12. It is obvious that $H(t) - t^2$ vanishes at $t = 1$. By Lemma 10, it must be of multiplicity $\leq 2$, and derivative calculations in (16) result in conditions for $H'(1)$ and $H''(1)$. □

4. **Asymptotic expansion**

In this section we decompose $X(t)$ in (11) into simple fractions and prove the main results of the article.

To deal with zeros on the boundary of the disk of convergence of $H(t)$ when decomposing $X(t)$, we need a lemma on the local behavior near the point of singularity.

**Lemma 15.** Let $k, n \in \mathbb{N}_0$, $\mathcal{D} \subset \mathbb{C}$ be non-empty convex open set and $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$. Suppose that the function $f : \overline{\mathcal{D}} \to \mathbb{C}$ is at least $n + k + 1$ times continuously complex-differentiable inside the intersection of $\overline{\mathcal{D}}$ and the open convex neighbourhood $\mathcal{U}$ of a point $\zeta \in \partial \mathcal{D}$; here, all the derivatives are taken in such a way, that the variable $t$ approaches $\zeta$ while staying in $\mathcal{U} \cap \overline{\mathcal{D}}$. If $\zeta$ is a removable singularity for $q(t) := f(t)/(t - \zeta)^{n+1}$ and its derivatives $q^{(j)}(t)$, $0 \leq j \leq k$. Hence, $q^{(j)}(t)$ may be deemed to be continuous at $\mathcal{U} \cap \overline{\mathcal{D}}$.

**Proof of Lemma 15.** Let $t \in \mathcal{U} \cap \overline{\mathcal{D}}$. For $\lambda \in [0, 1]$, let $s := s(\lambda) = \lambda(t - \zeta) + \zeta$. Define the function $g : [0, 1] \mapsto \mathbb{C}$ by $g(\lambda) := f(s) = f(\lambda(t - \zeta) + \zeta)$. By the convexity, $s \in \mathcal{U} \cap \overline{\mathcal{D}}$. Therefore, the complex-valued function $g(\lambda)$ of a real variable $\lambda$ is at least $n + k + 1$ times real-differentiable in the interval $[0, 1]$ with one-sided derivatives at endpoints. Let $T_n(\lambda) := \sum_{j=0}^{n} g^{(j)}(0)\lambda^j/j!$ be the Taylor polynomial of $g(\lambda)$ at $\lambda = 0$ of degree $n$, and let $R_n(\lambda) = g(\lambda) - T_n(\lambda)$ be the remainder term. As the Integral Remainder Theorem is applicable to
such function as \( g(\lambda) \) (see [20, Section 12.5.4 on p.94])

\[
R_n(\lambda) = 1/n! \int_0^\lambda g^{(n+1)}(\tau)(\lambda - \tau)^n d\tau,
\]

where \( \tau \in [0, \lambda] \). On the other hand, the Chain Rule differentiation yields

\[
g^{(j)}(\tau) = f^{(j)}(s(\tau))(t - \zeta)^j, \quad 0 \leq j \leq n + k.
\]

Since the first \( n \) derivatives of \( f(t) \) vanish at \( t = \zeta \), we have \( g^{(j)}(0) = 0 \), for \( 0 \leq j \leq n \). Therefore, \( T_n(\lambda) = 0 \), \( R_n(\lambda) = g(\lambda) \) and

\[
g(\lambda) = 1/n! \int_0^\lambda f^{(n+1)}(s(\tau))(\lambda - \tau)^n d\tau.
\]

or

\[
g(\lambda)/(t - \zeta)^{n+1} = 1/n! \int_0^\lambda f^{(n+1)}(s(\tau))(\lambda - \tau)^n d\tau.
\]

Setting \( \lambda = 1 \), one obtains

\[
f(t)/(t - \zeta)^{n+1} = 1/n! \int_0^1 f^{(n+1)}(\tau(t - \zeta) + \zeta)(1 - \tau)^n d\tau.
\]

Hence,

\[
\lim_{t \to \zeta} f(t)/(t - \zeta)^{n+1} = f^{(n+1)}(\zeta)/(n + 1)!,
\]

as long as \( t \in \overline{D} \). Thus, \( t = \zeta \) is a removable singularity. Moreover, the above integrand is \( k \) times continuously differentiable in \( U \cap \overline{D} \) with respect to the parameter \( t \). Therefore, by repeatedly differentiating under the integral with respect to \( t \) according to the Leibniz integral rule (an adaptation of [4, Ex.2, Ch.4] with an endpoint on \( \partial\overline{D} \)) and then taking the limit as \( t \to \zeta \) with \( t \in U \cap \overline{D} \), the function \( f(t)/(t - \zeta)^{n+1} \) is seen to be \( k \) times continuously differentiable in \( U \cap \overline{D} \). \( \square \)

**Theorem 16.** Assume that the \( H(t) - t^2 \) is primitive. As \( H(t) - t^2 \) vanishes at \( t = 1 \) with order \( r \leq 2 \), assume that \( H^{(2r+l)}(1) < +\infty \) for some \( l \in \mathbb{N} \). Then the generating function \( X(t) \) from (9) and (11) is represented in \( \overline{D} \) by

\[
X(t) = \frac{a}{1 + \alpha t} + \frac{b}{1 - \beta t} + \sum_{j=1}^r \frac{c_j}{(1 - t)^j} + F(t),
\]

where \( -\alpha^{-1} \) denotes the single negative real zero of \( H(t) - t^2 \) in \((-1, 0)\), \( \beta^{-1} \) (if present) denotes the smallest positive real zero of \( H(t) - t^2 \) in \((0, 1)\), \( \alpha > \beta > 1 \), and the \( l \)-th derivative \( F^{(l)}(t) \) of the remainder term function \( F : \overline{D} \to \mathbb{C} \) is holomorphic in \( D \) and continuous on \( \partial\overline{D} \). The coefficients \( a, b, c_j, 1 \leq j \leq r \) are real numbers.
Proof of Theorem 16. Since $H(0) = h_0 \neq 0$ and $t^m \neq 0$ for $t \neq 0$, the functions $H(t)$ and $H(t) - t^2$ have no common zero. By Lemma 7 and Corollary 14, $X(t)$ possesses simple poles at the zeros of $H(t) - t^2$: one always occurs at $t = -\alpha^{-1}$, another occurs at $\beta^{-1}$ when $H'(1) > 2$. Therefore, $X(t)$ is meromorphic inside $\mathbb{D}$. Since $H^{(2r+l)}(1) < +\infty$, the derivatives $H^{(j)}(t)$, $0 \leq j \leq 2r + l$ are continuous on $\overline{\mathbb{D}}$ by Lemma 7. As $X(t)$ is a ratio of $H(t)$ and $H(t) - t^2$, it follows that the derivatives $X^{(j)}(t)$, $0 \leq j \leq 2r + l$ are continuous on $\overline{\mathbb{D}} \setminus \{1\}$, since $t = 1$ is the single possible vanishing point of $H(t) - t^2$ on $\partial \mathbb{D}$ according to Corollary 14. In contrast to $-\alpha^{-1}$ and $\beta^{-1}$, the point $t = 1$ is not necessarily an isolated singularity of $X(t)$, as $H(t)$ and $X(t)$, in general, might not be continued holomorphically outside $\overline{\mathbb{D}}$.

To deal with this, rewrite $X(t)$ as $X(t) = H(t)/(t-1)^rG(t)$, where $G(t) := (H(t) - t^2)/(t-1)^r$. Since $t = 1$ is a zero of order $r$, $f(t) := H(t) - t^2$ satisfies $f^{(j)}(1) = 0$ for $0 \leq j \leq r - 1$, as it was shown in (16) of Lemma 10. By applying Lemma 15 to $f(t)$ with $n = r - 1$, $k = r + l$, $\mathcal{D} = \mathbb{D}$, $\zeta = 1$ one finds that, for $0 \leq j \leq r + l$, the $j$-th derivative of $G(t) = f(t)/(t-1)^r$ is continuous in $\mathbb{D}$. Then the function $Q(t) := H(t)/G(t)$ has continuous derivatives $Q^{(j)}(t)$ in $\mathbb{D}$ of the same order as $G(t)$ does, and it is meromorphic in $\mathbb{D}$.

Now, consider the Taylor polynomial of order $r - 1$ of the function $Q(t)$ at $t = 1$ and the corresponding remainder

$$T(t) := \sum_{j=0}^{r-1} \frac{Q^{(j)}(1)}{j!}(t-1)^j, \quad R(t) := Q(t) - T(t).$$

Then one can write

$$X(t) = \frac{Q(t)}{(t-1)^r} = \frac{R(t)}{(t-1)^r} + \frac{T(t)}{(t-1)^r}. \quad (19)$$

Setting $c_j = (-1)^j Q^{(r-j)}(1)/(r-j)!$, for $1 \leq j \leq r$ and $c_j = 0$ for $j > r$, $r \leq 2$ one obtains

$$T(t)/(t-1)^r = \sum_{j=1}^{2} c_j/(1-t)^j. \quad (20)$$

As $R(t)$ is a difference of $Q(t)$ and a polynomial, the derivatives $R^{(j)}(t)$, $0 \leq j \leq r + l$ are continuous in $\overline{\mathbb{D}} \setminus \{-\alpha^{-1}, \beta^{-1}\}$. Since $R^{(j)}(1) = 0$, for $j = 0, 1, \ldots, r - 1$, Lemma 15 can be applied to $f(t) = R(t)$, with $n = r - 1$, $k = l$. It follows that the $j$-th derivative of $R(t)/(t-1)^r$, $0 \leq j \leq l$ is continuous near $t = 1$ in $\mathbb{D}$. Thus, each of these derivatives of $R(t)/(t-1)^r$ are continuous on the whole $\mathbb{D} \setminus \{-\alpha^{-1}, \beta^{-1}\}$. However,
$R(t)/(t-1)^r$ in $\mathbb{D}$ still has a simple poles at $-\alpha^{-1}$ and at $\beta^{-1}$ if $H'(1) > 2$. Let

$$F(t) := \frac{R(t)}{(t-1)^r} - \frac{a}{1+\alpha t} - \frac{b}{1-\beta t},$$

(21)

where $a/\alpha$ and $b/\beta$ are equal to the residues of $R(t)/(t-1)^r$ at $t = -\alpha^{-1}$ and $t = \beta^{-1}$ respectively. Then, $F(t)$ and its derivatives up to the order $l$ are holomorphic inside $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ [18, Theorem 10.21]. Putting together (19), (20) and (21), we obtain the decomposition of $X(t)$ (18) in $\mathbb{D}$ with all the claimed properties. □

Corollary 17. The coefficients $a$, $b$, $c_1$, $c_2$ in Theorem 16 have the following expressions:

$$a = \frac{1}{2 + \alpha H'(-\alpha^{-1})}, \quad b = \begin{cases} 0, & \text{if } H'(1) \leq 2, \\ \frac{1}{2 - \beta H'(\beta^{-1})}, & \text{if } H'(1) > 2. \end{cases}$$

If $r = 1$, then $H'(1) \neq 2$, and $c_1 = 1/(2 - H'(1))$. If $r = 2$, then $H'(1) = 2 \neq H''(1)$, and

$$c_1 = \frac{2H''(1) - 12H''(1) + 24}{3(H''(1) - 2)^2}, \quad c_2 = \frac{2}{H''(1) - 2}.$$

Proof of Corollary 17. Since $-\alpha^{-1}$ is a solution of $H(t) = t^2$, one has $H(-\alpha^{-1}) = \alpha^{-2}$. From the decomposition (18) of $X(t)$ obtained in Theorem 16, one has

$$a = \lim_{t \to -\alpha^{-1}} (1 + \alpha t) X(t) = \lim_{t \to -\alpha^{-1}} \frac{\alpha H(t)}{(H(t) - t^2)/(t + \alpha^{-1})}$$

$$= \frac{\alpha H(-\alpha^{-1})}{(H(t) - t^2)/(t + \alpha^{-1})}'|_{t = -\alpha^{-1}}$$

$$= \frac{\alpha^{-1}}{H'(-\alpha^{-1}) + 2\alpha^{-1}} = \frac{1}{2 + \alpha H(-\alpha^{-1})}.$$

Replacing $-\alpha^{-1}$ with $\beta^{-1}$ in the above calculation, one finds $b$. For $r = 1$, using 1 in place of $-\alpha^{-1}$, one obtains $c_1$. For $r = 2$, the evaluation of $c_1$ and $c_2$ is slightly more complicated. One has

$$c_{2-j} = \lim_{t \to 1^-} \frac{(-1)^{2-j}}{j!} \left((t-1)^2 X(t)\right)^{(j)}, \quad \text{for } j = 0, 1.$$  

One evaluates this limit as follows. Write $(t-1)^2 X(t) = H(t)/G_2(t)$, where $G_2(t) := (H(t) - t^2)/(t-1)^2$. By Lemma 15, $G_2(t)$ is at least 2-times continuously differentiable in $\overline{\mathbb{D}}$ according to the assumptions.
of Theorem 16. This means the above limit evaluation can be replaced by
\[
c_{2-j} = \frac{(-1)^{2-j}}{j!} \left( \frac{H(t)}{G_2(t)} \right)^{(j)} \bigg|_{t=1}
\]
(22)

First, one evaluates \(H(t)\) and \(H'(t)\), using the appropriate assumptions of Theorem 16. Then one finds \(G_2(1), G_2'(1)\) by quotient rule, using higher derivatives and Cauchy Middle Value theorem on the real line (or, alternatively, Lemma 15) as \(t \to 1\) to resolve \(0/0\) ambiguities. Finally, one differentiates \(j\)-times the quotient \(H(t)/G_2(t)\) and substitutes the previously found values of \(H^{(j)}(1), G_2^{(j)}(1)\), in order to evaluate \(c_{2-j}\). For instance, \(j = 0\) in Eq. (22) yields
\[
c_2 = \frac{(-1)^{2-0}}{0!} \left( \frac{H(t)}{G_2(t)} \right)^{(0)} \bigg|_{t=1} = \frac{H(1)}{G_2(1)} = \left( \frac{((H(t) - t^2)^{(2)})_{t=1}}{((t - 1)^2)^{(2)}_{t=1}} \right)^{-1}
\]
\[
= \frac{2}{H''(1) - 2}.
\]
The evaluation of \(c_1\) for \(r = 2\) is similar, but more elaborate, so the technical details are omitted. \(\square\)

**Theorem 18.** Assume that \(H(t) - t^2\) is primitive, has vanishing order \(r \leq 2\) at \(t = 1\) and \(H^{(2r+l)}(1) < +\infty\) for some \(l \in \mathbb{N}_0\). Then the Taylor coefficients of \(X(t)\) have asymptotic expansion
\[
x_n = a(-1)^n\alpha^n + b\beta^n + p_{r-1}(n) + f_n,
\]
where \(a, b, \alpha, \beta, f_n\) are as in Theorem 16 and Corollary 17, \(p_{r-1}(t) \in \mathbb{R}[t]\) is a polynomial of degree \(r - 1\), and \(f_n = o(n^{-l})\), as \(n \to \infty\).

**Proof of Theorem 18.** The power series expansion at \(t = 0\) of the terms that appear in the decomposition equation (18) of Theorem 16 are
\[
\frac{1}{1 + \alpha t} = \sum_{n=0}^{\infty} (-1)^n \alpha^n t^n, \quad \frac{1}{1 - \beta t} = \sum_{n=0}^{\infty} \beta^n t^n,
\]
\[
\frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n, \quad \frac{1}{(1 - t)^2} = \sum_{n=0}^{\infty} (n + 1) t^n,
\]
\[
F(t) = \sum_{n=0}^{\infty} f_n t^n.
\]
Hence,
\[
X(t) = \sum_{n=0}^{\infty} (-1)^n a\alpha^n + b\beta^n + p_{r-1}(n) + f_n) t^n,
\]
where
\[ p_{r-1}(n) := c_1 + c_2(n + 1) = (c_1 + c_2) + c_2n. \] (24)

This proves (23). It remains to estimate the vanishing rate of the coefficients \( f_n \). By Theorem 16, the \( l \)-th derivative \( F^{(l)}(t) \) is holomorphic inside \( \mathbb{D} \) and continuous in \( \overline{\mathbb{D}} \). The coefficient of \( z^{n-l} \), \( n \geq l \), in the Taylor series of \( F^{(l)}(t) \) at \( z = 0 \) is \( n!(n-l)!f_n \). On the other hand, Cauchy’s integral formula [4, Ch.5, 1.11] yields
\[
\frac{n!}{(n-l)!} f_n = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} F^{(l)}(z) \frac{dz}{z^{n-l+1}} = \frac{1}{2\pi i} \int_0^1 F^{(l)}(e^{2\pi i\theta}) e^{-2\pi i(n-l)\theta} d\theta.
\] (25)

The last integral gives \( (n-l) \)-th coefficient of the Fourier series for \( F^{(l)}(e^{2\pi i\theta}) \). As \( F^{(l)}(t) \) is continuous on \( \partial \mathbb{D} \), it follows that \( F^{(l)}(e^{2\pi i\theta}) \in C[0,1] \), and, by Riemann-Lebesgue Lemma [14, Theorem 2.8, p.13], \( n!f_n/(n-l)! \to 0 \) as \( n \to +\infty \). Therefore, \( f_j = o(n^{-l}) \).

Remark 19. If the \( F(t) \) admits holomorphic continuation outside the circle of radius \( \rho > 1 \), centered at \( t = 0 \), then \( o(n^{-l}) \) in 18 can be strengthened to \( O(\rho^{-n}) \).

Remark 20. The weakest condition that ensures \( f_n = o(n^{-l}) \), as \( n \to +\infty \) is that of \( F^{(l-1)}(e^{2\pi i\theta}) \) being absolutely continuous on \( [0,1] \). However, there seems to be no easy ways to re-cast this condition in terms of the p.g.f. \( H(t) \).

Remark 21. If \( |F^{(l)}(t)| \leq M < +\infty \) on \( \partial \mathbb{D} \), where \( M \) can be estimated numerically, then a slightly weaker estimate \( |f_n| \leq M(n-l)!/n! = O(n^{-l}) \) that follows from (25) and (26) might be much more useful for numerical computations.

Theorem 22. Let \( H(t) - t^2 \) be primitive, with \( H^{(2r)}(1) < +\infty \) if its vanishing order at \( t = 1 \) is \( r \). Then, for \( n > n_0 \),
\[
1 < D_{2n} < D_{2n+2}, \quad D_{2n+3} < D_{2n+1} < -1,
\]
\[ D_{2n} \to +\infty, \quad D_{2n+1} \to -\infty, \]
as \( n \to +\infty \).

Proof of Theorem 22. If \( H'(1) \leq 2 \), then \( b = 0, r = 1 \) or 2. By Theorem 18, with \( l = 0 \),
\[
x_n = (-1)^n a\alpha^n + p_{r-1}(n) + o(1).
\]
Then,
\[
x_n x_{n+2} = a^2 \alpha^{2n+2} + (-1)^n a \alpha^n p_{r-1}(n+2) + (-1)^{n+2} a \alpha^{n+2} p_{r-1}(n) + o(\alpha^n),
\]
\[
x_{n+1}^2 = a^2 \alpha^{2n+2} + (-1)^{n+1} 2 a \alpha^{n+1} p_{r-1}(n+1) + o(\alpha^n),
\]
since \(\alpha > 1\). Therefore,
\[
D_n = h_0 \left( x_n x_{n+2} - x_{n+1}^2 \right)
\]
\[
= (-1)^n h_0 a \alpha^n \left( p_{r-1}(n+2) + 2 a p_{r-1}(n+1) + \alpha^2 p_{r-1}(n) \right) + o(\alpha^n).
\]
From (24), the leading term of \(p_{r-1}(n+2) + 2 a p_{r-1}(n+1) + \alpha^2 p_{r-1}(n)\) is equal to \(c_r(1 + \alpha)^2 n r^{-1}\). Therefore
\[
D_n \sim (-1)^n h_0 a c_r (1 + \alpha)^2 n r^{-1} \alpha^n, \text{ as } n \to \infty. \tag{27}
\]
Likewise, for \(H'(1) > 2, b \neq 0, r = 1,\) Theorem 18 (with \(l = 0\)) yields
\[
x_n = (-1)^n a \alpha^n + b \beta^n + c + o(1).
\]
Thus,
\[
x_n x_{n+2} = a^2 \alpha^{2n+2} + (-1)^n a \alpha^n \beta^n (\beta^2 + \alpha^2) + b^2 \beta^{2n+2} + o(\alpha^n),
\]
\[
x_{n+1}^2 = a^2 \alpha^{2n+2} + (-1)^{n+1} 2 a \alpha^{n+1} \beta^{n+1} + b^2 \beta^{2n+2} + o(\alpha^n),
\]
as \(\alpha > \beta > 1\). Hence,
\[
D_n = h_0 \left( x_n x_{n+2} - x_{n+1}^2 \right) = (-1)^n h_0 a \alpha^n \beta^n (\alpha + \beta)^2 + o(\alpha^n).
\]
As \(\alpha > 1\) and \(\beta > 1\), \(D_{2n} \to +\infty, D_{2n+1} \to -\infty,\) as \(n \to \infty,\) and,
\[
\lim_{n \to \infty} D_{n+2}/D_n = \begin{cases} 
\alpha^2, & \text{if } H'(1) \leq 2, \\
\alpha^2 \beta^2, & \text{if } H'(1) > 2.
\end{cases}
\]
Hence, there exists \(n_0 \in \mathbb{N}\), such that, for every \(n > n_0, D_{2n} \geq 1, D_{2n+1} \leq -1\) and \(|D_{n+2}| > |D_n|\). \(\Box\)

**Remark 23.** The expression \(x_n x_{n+2} - x_{n+1}^2\) from \(D_n\) takes part in Aitken’s \(\Delta^2\) convergence acceleration method [1, 17], while the ratio \(D_{n+1}/D_n\) is used as the numerical estimator for the radius of convergence of power series [16].

**Proof of Theorem 3.** As derivatives \(H^{(j)}(1)\) can be expressed via the moments \(\mathbb{E} Z^i, 0 \leq i \leq j \leq 2r\) (and vice versa), like \(H'(1) = \mathbb{E} Z, H''(1) = \mathbb{E} Z^2 - \mathbb{E} Z\), Theorem 3 is just a re-statement of Theorem 22. \(\Box\)
Proof of Theorem 4. Let us first consider the case \( \mathbb{P}(Z \in 2\mathbb{N}_0 + 1) = 0 \). This means that every r.v. \( Z_i \) in \( W(n) \) takes an even values only. Consequently, for integer \( u \geq 1 \), \( \varphi(0) = \varphi_1(0), \varphi(1) = \varphi(2 \cdot 1 - 1) = \varphi_1(1) \), where \( \varphi_1(u) \) denotes the ultimate survival probability of the process \( W_1(n) \) described in Section 1. By replacing \( Z \) with \( Z/2 \) in the well known (see, for instance [3]) ultimate survival probability formula for \( W_1(n) \) in \( \kappa = 1 \) case, one obtains
\[
\varphi(0) = 1 - \mathbb{E}Z/2, \text{ since } \mathbb{E}Z/2 < 1.
\]
Then, using recursion (2) for \( \varphi_1(n) \) with \( \kappa = 1 \), one obtains \( \varphi(1) \), as claimed.

Let us now consider the case \( \mathbb{P}(Z \in 2\mathbb{N}_0 + 1) \neq 0 \). Then \( H(t) - t^2 \) must be primitive. For the primitive case, \( H'(1) = \mathbb{E}Z < 2 \) results in the vanishing order \( r = 1 \) at \( t = 1 \) for \( H(t) - t^2 \), and \( \mathbb{E}Z^2 < +\infty \) yields \( H''(1) < +\infty \). Therefore, Theorem 18 and Theorem 22 are applicable (with \( r = 1 \) and \( l = 0 \)). Then
\[
x_n \sim (-1)^n a \alpha^n, \quad D_n \sim (-1)^n h_0 a c_1(1 + \alpha)^2 \alpha^n,
\]
as \( n \to \infty \) by Theorem 18 and by Eq. (27) in the proof of Theorem 22. Since \( y_n = h_0 x_{n+1} \) holds by Eq. (14), the limits of ratios \( x_n/D_n, x_{n+1}/D_n, y_n/D_n, y_{n+1}/D_n \) are
\[
\frac{1}{h_0 c_1(1 + \alpha)^2}, \quad \frac{\alpha}{h_0 c_1(1 + \alpha)^2}, \quad \frac{\alpha}{c_1(1 + \alpha)^2}, \quad \frac{\alpha^2}{c_1(1 + \alpha)^2},
\]
respectively. Substituting these limits into expressions in (8), and using \( c_1 = 1/(2 - H'(1)) = 1/(2 - \mathbb{E}Z) \) from Corollary 17, \( \varphi(\infty) = 1 \) for \( \mathbb{E}Z < 2 \), one obtains the claimed formulas for \( \varphi(0) \) and \( \varphi(1) \).

5. Some specific examples

5.1. Bernoulli’s distribution. Let \( Z \sim \mathcal{B}(p) \) denote the Bernoulli r.v. with success probability \( 0 < p < 1 \). Then its p.g.f. \( H(t) = q + pt \), where \( q = 1 - p \). In this case
\[
X(t) = \frac{q + pt}{q + pt - t^2} = \frac{1}{(1 + q)(1 - t)} + \frac{q}{(1 + q)(1 + t/q)}.
\]
Then
\[
x_n = \frac{1 + (-1)^n q^{1-n}}{1 + q}, \quad n \in \mathbb{N}_0.
\]
The determinant $D_n$ evaluates to

$$D_n = h_0(x_{n+2}x_n - x_{n+1}^2)$$

$$= \frac{q}{(1 + q)^2} \left( (1 + (-1)^n q^{-1-n}) (1 + (-1)^{n+2} q^{-1-n}) - (1 + (-1)^{n+1} q^{-n})^2 \right)$$

$$= \frac{q}{(1 + q)^2} \left( 1 + (-1)^n q^{-1-n} + (-1)^n q^{-1-n} + q^{-2n} - 1 - 2(-1)^{n+1} q^{-n} - q^{-2n} \right)$$

$$= \frac{(-1)^n q}{(1 + q)^2} \left( (-1)^n q^{-1-n} + (-1)^n q^{-1-n} - 2(-1)^{n+1} q^{-n} \right)$$

$$= \frac{(-1)^n q^{-1-n}}{(1 + q)^2} (q + q^{-1} + 2) = \frac{(-1)^n q^{1-n} (1 + q)^2}{q} = \frac{(-1)^n}{q^n}.$$ 

Therefore, for $Z \sim B(p)$, Conjecture 1 is true.

5.2. Geometric distribution. Let $Z \sim G(p)$ denote the geometric distribution $\mathbb{P}(Z = k) = p(1 - p)^k$, $k = 0, 1, \ldots$ with a p.g.f.

$$H(t) = \frac{p}{1 - qt}, \quad 0 < p, q < 1, \ p + q = 1$$

and

$$X(t) = \frac{1}{(1 - t)(1 + t - qp^{-1}t)} = \frac{1}{(1 - t)(1 + \alpha t)(1 - \beta t)}$$

where

$$\alpha = (\sqrt{4p^{-1} - 3} + 1)/2, \quad \beta = (\sqrt{4p^{-1} - 3} - 1)/2,$$

satisfy

$$\beta > 1, \text{ for } 0 < p < 1/3, \quad 0 < \beta < 1, \text{ for } 1/3 < p < 1, \quad (28)$$

$$\alpha > \max\{1, \beta\}, \text{ for } 0 < p < 1.$$

For $p \neq 1/3$, $X(t)$ decomposes into

$$X(t) = \frac{a}{1 + \alpha t} + \frac{b}{1 - \beta t} + \frac{c_1}{1 - t},$$

where

$$a = \frac{q + \alpha}{3q + 2\alpha}, \quad b = \frac{q - \beta}{3q - 2\beta}, \quad c_1 = \frac{p}{3p - 1},$$

(see Corollary 17) satisfy

$$1/2 < b < +\infty, \text{ if } p \in (0, 1/3), \quad -\infty < b < 0, \text{ if } p \in (1/3, 1), \quad (29)$$

$$0 > c_1 > -\infty, \text{ if } p \in (0, 1/3), \quad +\infty > c_1 > 1/2, \text{ if } p \in (1/3, 1),$$

$$4/9 < a < 1/2, \text{ for } 0 < p < 1.$$
It follows that
\[ x_n = (-1)^n \alpha^n + b \beta^n + c_1, \]
and
\[ \frac{(-1)^n}{h_0} D_n = (-1)^n (x_n x_{n+2} - x_{n+1}^2) \]
\[ = ab(\alpha + \beta)^2 \alpha^n \beta^n + (-1)^n bc_1 (\beta - 1)^2 \beta^n + c_1 a (1 + \alpha)^2 \alpha^n. \]
Consider
\[ (-1)^n / h_0 (D_{n+2} - \alpha^2 D_n) \]
\[ = ab(\alpha + \beta)^2 (\beta^2 - 1) \alpha^{n+2} \beta^n + (-1)^n bc_1 (\beta - 1)^2 (\beta^2 - \alpha^2) \beta^n \]
\[ = ab(\alpha + \beta)^2 (\beta^2 - 1) \beta^n \left( \alpha^{n+2} + (-1)^n c_1 (1 - \beta)(\alpha - \beta) / a(\alpha + \beta)(\beta + 1) \right) \]
\[ = ab(\alpha + \beta)^2 (\beta^2 - 1) \beta^n (\alpha^{n+2} + (-1)^n f(p)), \]
where
\[ f(p) := c_1 (1 - \beta)(\alpha - \beta) / a(1 + \beta)(\alpha + \beta) = p^2 (\alpha + p - 2) / (1 - p)^3. \]
The last expression for \( f(p) \) was found with Mathematica [15] and verified with Sage [19]. Since \( a > 0 \) and the signs of \( \beta^2 - 1 \) and \( b \) in (28), (29) coincide for \( p \in (0,1) \setminus \{1/3\} \), it follows that the sign of \( (-1)^n (D_{n+2} - \alpha^2 D_n) \) matches the sign of \( \alpha^{n+2} + (-1)^n f(p) \).

For \( f(p) \) it holds that \( \lim_{p \to 0^+} f(p) = 0 \), \( \lim_{p \to 1^-} f(p) = 1 \) and \( f'(p) > 0 \) for \( p \in (0,1) \), see Figure 1. Therefore, \( f(p) < 1 \) and \( \alpha^{n+2} + (-1)^n f(p) > 0 \) for \( p \in (0,1) \) due to \( \alpha > 1 \). Hence, for every \( n \in \mathbb{N}_0 \), it holds that \( D_{2n+3} < \alpha^2 D_{2n+1} < D_{2n+1} < -1 \) and \( D_{2n+2} > \alpha^2 D_{2n+1} > D_{2n+1} \geq 1 \).

**Figure 1.** \( \alpha = \alpha(p) \) (red) v.s. \( f(p) \) (blue), \( p \in (0,1) \)

It remains to consider the value \( p = 1/3 \). In this case,
\[ X(t) = \frac{1}{(1-t)(1+t-2t^2)} = \frac{1}{(1+2t)(1-t)^2} \]
\[
\frac{4/9}{1 + 2t} + \frac{2/9}{1 - t} + \frac{1/3}{(1 - t)^2}.
\]
This yields
\[
x_n = ((-2)^{n+2} + 5 + 3n)/9,
\]
\[
D_n = ((-2)^{n+2}(27n + 63) - 9)/81,
\]
\[
(-1)^n(D_{n+2} - D_n) = 2^{n+2}(n + 5) > 0.
\]
In conclusion, Conjecture 1 is true for \( Z \sim G(p) \).

References

[1] A. Aitken, *On Bernoulli’s numerical solution of algebraic equations*, Proceedings of the Royal Society of Edinburgh 46 (1926), 289–305.

[2] E. S. Andersen, *On the collective theory of risk in case of contagion between the claims*. Trans. XVth Int. Actuar. 1957, 2, 219–229.

[3] S. Asmussen, H. Albrecher, *Ruin Probabilities*, World Scientific: Singapore, 2010.

[4] J. B. Conway, *Functions of one complex variable*, Functions of one complex variable, 2nd ed., Springer–Verlag, New York, 1978.

[5] D. R. Curtiss, *Recent Extensions of Descartes’ Rule of Signs*, Annals of Mathematics 19 (4) (1918), 251–278.

[6] J. Damarackas, J. Šiaulys, *Bi-seasonal discrete time risk model*, Appl. Math. Comput. 2014, 247, 930–940.

[7] D. C. M. Dickson, R. Waters, *Recursive calculation of survival probabilities*, ASTIN Bull., 1991, 21, 199–221.

[8] E. Graham, A. van der Poorten, I. Shparlinsky, T. Ward, *Recurrence sequences*, Mathematical Surveys and Monographs, vol. 104, American Mathematical Society, Providence, RI, 2003. https://doi.org/10.1090/surv/104.

[9] H. U. Gerber, *Mathematical fun with the compound binomial process*, ASTIN Bull., 1988, 18, 161–168

[10] H. U. Gerber, *Mathematical fun with risk theory*, Insur. Math. Econ. 1988, 7, 15–23.

[11] A. Grigutis, A. Korvel, J. Šiaulys, *Ruin probability in the three-seasonal discrete-time risk model*, Mod. Stochastics: Theory Appl., 2015, 2, 421–441.

[12] A. Grigutis, J. Šiaulys, *Ultimate Time Survival Probability in Three-Risk Discrete Time Risk Model*, Mathematics 2020, 8, 147.

[13] A. Grigutis, J. Šiaulys, *Recurrent Sequences Play for Survival Probability of Discrete Time Risk Model*, Symmetry 2020, 12, 2111

[14] Y. Katznelson, *Introduction to Harmonic Analysis*, 3rd Edition, Stanford University, 2004.

[15] Mathematica (Version 9.0), Wolfram Research, Inc., Champaign, Illinois, 2012, https://www.wolfram.com/mathematica

[16] G. N. Mercer, A. J. Roberts, *A centre manifold description of contaminant dispersion in channels with varying flow properties*, SIAM Journal on Applied Mathematics 50 (6) (1990), 1547–565, http://www.jstor.org/stable/2101904

[17] S. B. Pomeranz, *Aitken’s \( \Delta^2 \) method extended*, Cogent Mathematics, 4 (1) (2017), http://dx.doi.org/10.1080/23311835.2017.1308622

[18] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987.
[19] SageMath, the Sage Mathematics Software System (Version 8.1), The Sage Developers, 2017, https://www.sagemath.org.
[20] G. E. Shilov, Mathematical Analysis. Part 3: Functions in One Variable (in Russian), Nauka, Moscow, 1973.
[21] E. S. W. Shiu, Calculation of the probability of eventual ruin by Beekman’s convolution series, Insur. Math. Econ., 1988, 7, 41–47.

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