A Unified Approach to Stein’s Method for Stable Distributions

Neelesh S Upadhye* and Kalyan Barman**

Department of Mathematics
Indian Institute of Technology Madras
Chennai-600036, India
*Email: neelesh@iitm.ac.in
**Email: kalyanbarman6991@gmail.com

Abstract

In this article, we first review the connection between Lévy processes and infinitely divisible random variables, and the classification of infinitely divisible distributions. Using this connection and the Lévy-Khinchine representation of the characteristic function, we establish a Stein identity for an infinitely divisible random variable. The classification and slight modification in approach give us a Stein identity for an \( \alpha \)-stable random variable with \( \alpha \in (0, 2) \). Using fine regularity estimates for the solution to Stein equation, we derive error bounds for \( \alpha \)-stable approximations. We then apply these results to obtain rates of convergence. Finally, we compare these rates with the results available in the literature.

Key words: Stein’s method, Stable distributions, Stable approximation, Semigroup approach.

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1 Introduction

Over the last five decades, Stein’s method has been an important tool for studying approximation problems. This method was first introduced for normal approximation by Charles Stein [35] in 1972. Thereafter, Chen [13] applied this method for Poisson approximation. Several extensions of this method for various well-known probability distributions are studied by several authors. For instance, Stein’s method for the classical distributions, such as the gamma distribution [27], Laplace distribution [29] and exponential distribution [16] is well-known. This technique is not limited to the classical distributions but also extends general families of distributions, such as the Pearson family [33], variance-gamma [18] and discrete Gibbs measure family [15, 26]. For an overview of Stein’s method, we refer the reader to [30], and for distribution specific development of Stein’s method, we refer to the web-page maintained by Yvik Swan: https://sites.google.com/site/steinsmethod/home and references therein.

In general, Stein’s method can be divided into three parts. In the first step, one finds an appropriate Stein operator \( \mathcal{A} \) for a given target random variable \( X \sim F_X \), a given probability distribution such that \( \mathbb{E}(\mathcal{A}f(X)) = 0 \), for \( f \in \mathcal{F} \) (an appropriate class of functions). In the second step, one solves a Stein equation

\[
\mathcal{A}(f)(x) = h(x) - \mathbb{E}h(X)
\]

with \( h \in \mathcal{H}_X \) (a class of smooth functions). In the last step, one derives regularity estimates for solution to the Stein equation \([1]\) with respect to \( h \), which are used to derive “Stein factors”, by bounding the quantity \( |\mathbb{E}(h(Y) - h(X))| \), where \( Y \sim G_Y \) is a random variable of interest. The problem of getting an upper bound
of $|E(h(Y) - h(X))|$ is equivalent to bounding $|E\mathcal{A}(f)(Y)|$ which gives an error bound in approximation of the distribution of a random variable $Y$ to the distribution of the target random variable $X$.

Application of Stein’s method for a random variable $X \sim F_X$ relies heavily on identifying a suitable operator $\mathcal{A}$ such that $E(\mathcal{A}f(X)) = 0$, as $\mathcal{A}$ characterizes the behavior of the distribution $F_X$. In recent years, several techniques have been developed for finding Stein operator such as the density approach [36], the generator approach [6], the probability generating function approach [38] and so on.

Recently, Arras and Houdré developed Stein’s method for infinitely divisible distributions (IDD) with finite first moment (see [3]), and for multivariate self-decomposable distributions (with finite first moment (see [1]), and without finite first moment (see [23])). An important subclass of these IDD and self-decomposable distributions is the $\alpha$-stable distributions. In this direction, Xu [40] developed Stein’s method for symmetric $\alpha$-stable distributions with $\alpha \in (1,2)$. Jin et. al. [22] extended Xu’s idea [40] and developed Stein’s method for asymmetric $\alpha$-stable distributions with $\alpha \in (1,2)$. Chen et. al. [11] also developed Stein’s method for asymmetric $\alpha$-stable distributions with $\alpha \in (1,2)$. Later, Chen et. al. [12] extended it for multivariate case and developed Stein’s method for multivariate $\alpha$-stable distributions with $\alpha \in (1,2)$. More recently, Chen et. al. [10] developed Stein’s method for $\alpha$-stable distributions with $\alpha \in (0,1]$. An overview of these articles is discussed in more detail in Section 2.

It is clear from the existing literature that the techniques for developing Stein’s method for $\alpha$-stable distributions depend on ranges of $\alpha$ ($\alpha \in (0,1]$ and $\alpha \in (1,2)$) and are different. It raises an interesting question:

(Question) For $\alpha \in (0,2)$, can one unify the Stein’s method for $\alpha$-stable distributions?

In this article, we obtain a Stein identity for an infinitely divisible random variable using the Lévy-Khinchine representation of the characteristic function. With slight modification in approach, we also obtain a Stein identity for an $\alpha$-stable random variable with $\alpha \in (0,2)$. We solve our Stein equation in a unified way via the semigroup approach. Using the fine regularity of the solution to our Stein equation, we demonstrate error bounds for $\alpha$-stable approximations. Finally, we apply these results to obtain the convergence rates for $\alpha$-stable approximations using two known examples, and we also compare our rates with the existing literature.

The organization of the article is as follows. In Section 2, we discuss some preliminaries and known results. In Section 3, we state our result on Stein identity for an infinitely divisible random variable, and in particular for an $\alpha$-stable random variable. Using regularity estimates of solution to our Stein equation, we compute bounds in appropriate probability metrics for $\alpha \in (0,1]$ and $\alpha \in (1,2)$ respectively. In Section 4, we discuss two applications of our results for $\alpha$-stable approximations and obtain the convergence rates. In Section 5, we provide the proofs of results presented in Section 3.

2 Preliminaries and known results

In this section, we review the relationship between IDD and Lévy processes. We also establish the classification of $\alpha$-stable distributions based on Lévy processes. Further, we discuss the results on convergence rates for $\alpha$-stable approximations.
2.1 Infinite divisibility and Lévy process

Let us first define the concept of infinite divisibility.

**Definition 2.1.** [25, p.3] The distribution of a random variable $X$ is said to be infinitely divisible, if, for every $n \in \mathbb{N}$,

$$X \overset{d}{=} X_{n,1} + \ldots + X_{n,n},$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and identically distributed (i.e., $X_{n,j} = X_n$, $j = 1, 2, \ldots, n$) and $X_n$ is called $n$-th factor of $X$.

In other words, a distribution function $F_X$ is infinitely divisible if, for each $n \in \mathbb{N}$, $F_X$ is $n$-fold convolution of $F_{n,X_n}$ with itself (i.e., $F_X = F_{n,X_n}^n$), where $F_{n,X_n}$ is $n$-th factor of $F_X$. This can also be summarized using a notion of characteristic exponent as follows: Define $\eta(z) := \log \phi_X(z) = \log \mathbb{E}(e^{izX})$, $z \in \mathbb{R}$ to be the characteristic exponent of a random variable $X$. Then, the distribution of random variable $X$ ($F_X$) is infinitely divisible, if, for each $n \in \mathbb{N}$, there exist a characteristic exponent $\eta_n(\cdot)$, such that $\eta(z) = n\eta_n(z)$, $z \in \mathbb{R}$.

Next, we use this property of characteristic exponents for some familiar distributions and show that these are in fact infinitely divisible.

**Example 2.2 (Normal distribution [39]).** Let $X \sim \mathcal{N}(\beta, \sigma^2)$, where $\beta \in \mathbb{R}$, and $\sigma > 0$. Then it is well-known that the characteristic exponent of $X$ is given by

$$\eta(z) = i\beta z - \frac{\sigma^2 z^2}{2} = n\left(i(\beta/n)z - \frac{\sigma^2/n}{2}z^2\right) = n\eta_n(z).$$

Observe now that, for every $n \in \mathbb{N}$, $\eta_n(z)$ is the characteristic exponent of the random variable $X_n \sim \mathcal{N}(\beta/n, \sigma^2/n)$. Hence the distribution of $X$ is infinite divisible.

**Example 2.3 (Poisson Distribution [14]).** Let $N \sim \text{Poisson}(\lambda)$, where $\lambda > 0$. Then the characteristic exponent of $N$ is given by

$$\eta(z) = \lambda(e^{iz} - 1) = n((\lambda/n)(e^{iz} - 1) = n\eta_n(z).$$

Note that, for each $n \in \mathbb{N}$, $N_n \sim \text{Poisson}(\lambda/n)$. Hence the distribution of $N$ is infinitely divisible.

Next, we recall the stochastic processes associated with these examples, namely, Brownian motion and Poisson process, and explore their connection with infinite divisibility.

**Definition 2.4 (Brownian Motion [39]).** A real-valued stochastic process $\{X_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion, if

(i) $X_0 = 0$ a.s.

(ii) For any fixed $\omega \in \Omega$, $t \mapsto X_t$ is continuous a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s \overset{d}{=} X_{t-s}$. 
(iv) For any partition of interval \([0, t]\), \(0 = t_0 < t_1 < \cdots < t_n = t\), the increments \(X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}\) are independent.

(v) For \(t > 0\), \(X_t \sim \mathcal{N}(0, t)\).

From (v), it is clear that the process is generated from a standard normal random variable \(X \sim \mathcal{N}(0, 1)\). Also, from (ii), we see that sample paths are continuous and not monotone.

**Definition 2.5** (Poisson Process \([14]\)). A non-negative integer-valued stochastic process \(\{N_t\}_{t \geq 0}\) on a probability space \((\Omega, \mathcal{F}, P)\) is said to be a Poisson process, if

(i) \(N_0 = 0\) a.s.

(ii) For any fixed \(\omega \in \Omega\), \(t \mapsto N_t\) is right continuous with left limits.

(iii) For \(0 \leq s \leq t\), \(N_t - N_s \overset{d}{=} N_{t-s}\).

(iv) For any partition of interval \([0, t]\), \(0 = t_0 < t_1 < \cdots < t_n = t\), the increments \(N_{t_1} - N_0, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}\) are independent.

(v) For \(t > 0\), \(N_t \sim \text{Poisson}(\lambda t)\).

From (v), it is clear that the process is generated from \(N \sim \text{Poisson}(\lambda)\). Also, from (ii), we can see that the sample paths are right continuous and non-decreasing.

Observe now that these two processes may appear to be quite different from each other, but the distributions that generate these processes are infinitely divisible. Let us look at the processes closely. We can see that these two processes have common properties, such as right-continuous sample paths, stationary and independent increments (from (iii) and (iv)), and generated from infinitely divisible random variables (from (v)). Let us use these common properties and define a new class of processes known as Lévy processes.

**Definition 2.6** (Lévy Process \([25]\)). A real-valued stochastic process \(\{X_t\}_{t \geq 0}\) on a probability space \((\Omega, \mathcal{F}, P)\) is said to be a Lévy process, if

(i) \(X_0 = 0\) a.s.

(ii) For any fixed \(\omega \in \Omega\), \(t \mapsto X_t\) is right continuous with left limits.

(iii) For \(0 \leq s \leq t\), \(X_t - X_s \overset{d}{=} X_{t-s}\).

(iv) For any partition of interval \([0, t]\), \(0 = t_0 < t_1 < \cdots < t_n = t\), the increments \(X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}\) are independent.

(v) For \(\varepsilon > 0\), \(\lim_{h \to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0\).

In short, a stochastic process can be characterized as Lévy process if its sample paths satisfy (ii) and have stationary and independent increments (from (iii) and (iv), respectively).
Next, we focus on the relation between infinite divisibility and Lévy process. From the definition of Lévy process, it is clear that the distribution of $X_t$ is infinitely divisible. To see this, observe that

$$X_t = (X_t - X_{(n-1)t}) + (X_{(n-1)t} - X_{(n-2)t}) + \cdots + (X_{ht} - X_0),$$

where $h = t/n$ and $n \in \mathbb{N}$, and these increments are independent and identically distributed with $X_0 = 0$. Hence, from Definition 2.1, the distribution of $X_t$ is infinitely divisible. We can also use characteristic exponent to show that $X_t$ has IDD, for any $t > 0$. To see this, let $\eta_t(z) = \log \mathbb{E}(e^{izX_t})$, $z \in \mathbb{R}$. Assume first that $t = m \in \mathbb{N}$ then, from (2), with $h = m/n$, $\eta_n(z) = m \eta_{m/n}(z)$. Similarly, for $t \in \mathbb{Q}_+$, set of positive rational numbers, say $t = m/n$, $\eta_t(z) = \eta_{m/n}(z) = (m/n)\eta_1(z)$ follows by choosing $h = n/m$ and (2). Now, for $t$ in positive irrationals, construct a decreasing sequence $\{t_n\}$ of positive rational numbers such that $t_n \to t$ as $n \to \infty$, then $\eta_t(z) = \lim_{n \to \infty} \eta_{t_n}(z) = \lim_{n \to \infty} t_n \eta_1(z) = t \eta_1(z)$. The last but one equality follows from continuity of sample paths (see, (ii) in the definition of Lévy process and dominated convergence theorem). Hence, using characteristic exponent, we have proved that $\eta_t(z) = t \eta_1(z)$, $z \in \mathbb{R}$ and $t > 0$. This shows that, for any $t > 0$, $X_t$ has IDD with characteristic exponent $\eta_1(z)$ and can be generated using the distribution of $X_1$ with characteristic exponent $\eta_1(z)$. The above discussion can now be summarized in the following theorem.

**Theorem 2.7.** [14, p.81] Let $\{X_t\}_{t \geq 0}$ be a real-valued Lévy process. Then there exists an IDD $F$ such that $X_1 \sim F$.

In particular, the representation of characteristic exponent of $F$ is given in following formulation.

**Theorem 2.8.** [24, p.5] Let $\{X_t\}_{t \geq 0}$ be a real-valued Lévy process. Then there exists a triplet $\beta, \sigma$, $\nu$ where $\beta \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge z^2) \nu(dz) < \infty$, such that

$$\mathbb{E}(e^{izX_t}) = e^{\eta_t(z)}, \text{ for } z \in \mathbb{R},$$

with $\eta(z) = i \beta z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{iu \rho} - 1 - iu \rho 1_{\{|\rho| \leq 1\}}) \nu(d\rho)$. Note that $\eta(\cdot)$ is the characteristic exponent of $F$ and the measure $\nu(\cdot)$ is known as Lévy measure (need not be a probability measure).

This brings us to the important question. Given an IDD $F$, can we construct a Lévy process $\{X_t\}_{t \geq 0}$ such that $X_1 \sim F$? To answer this question, in view of Theorem 2.8, we need assure the existence of the triplet $(\beta, \sigma, \nu)$ associated with $F$. This can be seen in following theorem.

**Theorem 2.9.** [24, p.3] A distribution $F$ with characteristic exponent $\eta(\cdot)$ is infinitely divisible if and only if there exists a triplet $(\beta, \sigma^2, \nu)$, where $\beta \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$, the Lévy measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge z^2) \nu(dz) < \infty$, with

$$\eta(z) = i \beta z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{iu \rho} - 1 - iu \rho 1_{\{|\rho| \leq 1\}}) \nu(d\rho).$$

The proofs of these theorems are quite lengthy and involved, we refer the interested readers to Sato [32, p.41] for more detailed discussion.
We have now established the fact that, for any IDD $F$ with triplet $(\beta, \sigma^2, \nu)$, there exist a unique Lévy process $\{X_t\}_{t \geq 0}$. Let us understand the concept through the following examples.

Ex. 1. Let $X \sim \mathcal{N}(0,1)$. Then $\eta(z) = -z^2/2$. On comparison with (3), we get the triplet $(\beta, \sigma^2, \nu) = (0, 1, 0)$ and associated Lévy process is Brownian motion as defined in Definition 2.4.

Ex. 2. Let $N \sim \text{Poisson}(\lambda)$, $\lambda > 0$. Then $\eta(z) = \lambda(e^{iz} - 1)$. On comparison with (3), we get the triplet $(\beta, \sigma^2, \nu) = (\lambda, 0, \lambda \delta_1)$, where $\delta_1$ is the Dirac measure at $\{1\}$, and associated Lévy process is Poisson process as defined in Definition 2.5.

Ex. 3. Let $X \sim \text{Gamma}(\lambda, \gamma)$, where $\lambda > 0, \gamma > 0$. Then $\eta(z) = -\gamma \log(1 - iz/\lambda)$. On comparison with (3), we get the triplet $(\beta, \sigma^2, \nu) = (\gamma(1 - e^\lambda), 0, \nu_1)$, where $\nu_1(du) = (\gamma e^{-\lambda u}/u) du$. For more details on computation of the triplet, we refer the readers to [25]. The associated Lévy process is known as gamma process.

Ex. 4. Let $X \sim \text{Cauchy}(x_0, c)$, $x_0 \in \mathbb{R}, c > 0$. Then $\eta(z) = ix_0 z - c|z|$. On comparison with (3), we get the triplet $(\beta, \sigma^2, \nu) = (x_0 - 2\Gamma/\pi, 0, \nu_2)$ where $\Gamma = \int_0^\infty \left( \frac{\sin u}{u^2} - \frac{1}{u} \right) du$, and $\nu_2(du) = (c/(\pi u^2)) du$. The associated Lévy process is known as 1-stable process.

In the examples discussed above, we see that the IDD and associated Lévy process are uniquely characterized by the triplet $(\beta, \sigma^2, \nu)$. Also, the behavior of $\nu$ is different in each of the examples. For Poisson distribution, $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(du) = \lambda < \infty$, for normal and gamma distribution, $\nu(\mathbb{R}) = 0$ and $\nu(\mathbb{R}) = \infty$, respectively, and for Cauchy distribution, $\int_{\{|u| \leq 1\}} \nu_2(du) = \infty$. Also, observe that the behavior $\sigma$ is also important for normal distribution. These two components of the triplet need to be further classified. Sato [32, Definition 11.9] has classified Lévy process $\{X_t\}_{t \geq 0}$ (infinitely divisible distribution $(X_1)$) into three different classes based on the triplet $(\beta, \sigma^2, \nu)$, as follows.

**Type A** $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$. (e.g. Poisson process).

**Type B** $\sigma = 0, \nu(\mathbb{R}) = \infty$, $\int_{\{|u| \leq 1\}} u \nu(du) < \infty$. (e.g. gamma process) 

**Type C** $\int_{\{|u| \leq 1\}} u \nu(du) = \infty$ or $\sigma > 0$. (e.g. 1-stable process or Brownian motion)

The examples studied here are by no means exhaustive. The class of IDD is very rich and include various well-known distribution like Student’s t-distribution, Pareto distribution, $F$-distribution among many others. Next, we focus on an important subclass of IDD, namely, non-Gaussian stable distributions. This class is characterized by the triplet $(\beta, \sigma^2, \nu) = (\beta, 0, \nu_\alpha)$, with $\beta \in \mathbb{R}$ and the Lévy measure $\nu_\alpha$ is given by

$$
\nu_\alpha(du) = \left( m_1 \frac{1}{u^{1+\alpha}} 1_{(0,\infty)}(u) + m_2 \frac{1}{|u|^{1+\alpha}} 1_{(-\infty,0)}(u) \right) du,
$$

where $\alpha \in (0, 2)$, $m_1, m_2 \in [0, \infty]$ and $m_1 + m_2 > 0$. (see [21, p.32]). Next, we give characteristic exponent representation for non-Gaussian stable distribution.

**Definition 2.10.** [21, p.168] A real-valued random variable $X$ is said to have non-Gaussian stable (also called $\alpha$-stable) distribution, if there exists a triplet $(\beta, 0, \nu_\alpha)$, such that for all $z \in \mathbb{R}$, the characteristic exponent is given by
\[ \eta_\alpha(z) = \log \phi_\alpha(z) = iz\beta + \int_\mathbb{R} (e^{izu} - 1 - izu1_{\{|u|\leq 1\}}(u)) \nu_\alpha(du), \quad (5) \]

where \( \beta \in \mathbb{R}, \alpha \in (0, 2), \) and \( \nu_\alpha \) is the Lévy measure defined in \([3]\), and is denoted by \( X \sim S(\alpha, \beta, m_1, m_2) \).

Note here that \( \beta \in \mathbb{R} \) is the location parameter and \( \alpha \in (0, 2) \) is stability parameter, useful in determining the decay of the tail of distribution of \( X \).

Observe next that, based on the classification of IDD summarized earlier in this section, we can classify \( \alpha \)-stable distributions as follows:

**Type B** \( \alpha \in (0, 1) \) (as \( \nu_\alpha(\mathbb{R}) = \infty \), but \( \int_{\{|u|\leq 1\}} u^\nu \nu_\alpha(du) < \infty \)).

**Type C** \( \alpha \in [1, 2) \). (As, \( \int_{\{|u|\leq 1\}} u^\nu \nu_\alpha(du) = \infty \)).

Observe now that, for stable distributions of Type B \( (\alpha \in (0, 1)) \), as \( \int_{\{|u|\leq 1\}} u^\nu \nu_\alpha(du) < \infty \). The characteristic exponent in \([5]\) can be rewritten as

\[ \eta_\alpha(z) = iz\beta_1 + \int_\mathbb{R} (e^{izu} - 1) \nu_\alpha(du), \quad (6) \]

where \( \beta_1 = \beta - \int_{\{|u|\leq 1\}} u^\nu \nu_\alpha(du) \).

Also, for stable distributions of Type C \( (\alpha \in (1, 2), \alpha \neq 1) \), as \( \int_{\{|u|\leq 1\}} u^\nu \nu_\alpha(du) = \infty \), but \( \int_{\{|u|>1\}} u^\nu \nu_\alpha(du) < \infty \). The characteristic exponent \([5]\) can be rewritten as

\[ \eta_\alpha(z) = iz\beta_2 + \int_\mathbb{R} (e^{izu} - 1 - izu) \nu_\alpha(du), \quad (7) \]

where \( \beta_2 = \beta + \int_{\{|u|>1\}} u^\nu \nu_\alpha(du) \).

Next, we show the connection between our representation and the various other representations of characteristic exponents available for \( \alpha \)-stable distributions. Based on the well-known four parameter representation of \( \alpha \)-stable distributions (see, \([31]\)), where the parameters \( \alpha \in (0, 2), \gamma_\alpha \in \mathbb{R}, \, d_\alpha \geq 0 \) and \( \theta \in [-1, 1] \) denote the stability, shift, scale and skewness parameters, respectively. Note here that, on careful adjustments of the integrals in \([5]\) with respect to \( \nu_\alpha \), one can obtain a well-known form of characteristic exponent (characteristic function) of \( \alpha \)-stable distributions of both types from the Lévy-Khinchine representation \([5]\) (see, \([31]\)). The explicit forms are given below:

**Type B** Let \( \theta = \frac{m_1 - m_2}{m_1 + m_2}, \gamma_\alpha = \beta - \frac{(m_1 - m_2)}{(1-\alpha)}, \) and \( d_\alpha = (m_1 + m_2) \cos \frac{\pi}{\alpha} \int_0^{\infty} (1 - e^{-u}) \frac{du}{u^{1+\alpha}} \). Then

\[ \eta_\alpha(z) = iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta \frac{z}{|z|} \tan \frac{\pi}{2\alpha} \right). \]

**Type C** Here we classify further into two cases \( \alpha = 1 \) and \( \alpha \in (1, 2) \)

(\( \alpha = 1 \)). Let \( \theta = \frac{m_1 - m_2}{m_1 + m_2}, \gamma_\alpha = \beta + (m_1 + m_2) \int_0^{\infty} \left( \frac{\sin u}{u^2} - \frac{1_{\{|u|\leq 1\}}(u)}{u} \right) du, \) and \( d_\alpha = (m_1 + m_2) \frac{\pi}{2} \). Then

\[ \eta_\alpha(z) = iz\gamma_\alpha - d_\alpha |z| \left( 1 + i\theta \frac{z}{|z|} \frac{2\log|z|}{\pi} \right). \]

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$\alpha \in (1, 2)$. Let $\theta = \frac{m_1 - m_2}{m_1 + m_2}$, $\gamma_{\alpha} = \beta - \frac{(m_1 - m_2)}{(1 - \alpha)}$, and $d_{\alpha} = (m_1 + m_2) \cos \frac{\pi \alpha}{2} \int_0^{\infty} (1 - e^{-u}) \frac{du}{u^{1+\alpha}}$. Then

$$\eta_{\alpha}(z) = iz\gamma_{\alpha} - d_{\alpha}|z|^\alpha \left(1 - i\theta \frac{z}{|z|} \tan \frac{\pi}{2} \alpha \right).$$

The derivation of these forms of characteristic exponents is given in the Appendix A. Observe also that, for $X \sim S(\alpha, 0, m, m)$, characteristic exponent of a symmetric $\alpha$-stable random variable is given by

$$\eta_{s,\alpha}^*(z) = \int_{\mathbb{R}} (e^{izu} - 1 - izu \mathbf{1}_{|u| \leq 1}(u)) \nu_{s,\alpha}(du) = -d_{\alpha}|z|^\alpha, \quad z \in \mathbb{R},$$

where $d_{\alpha} > 0$ is the scale parameter given by

$$d_{\alpha} = \begin{cases} 2m \cos \left(\frac{\pi \alpha}{2}\right) \int_0^{\infty} (1 - e^{-y}) \frac{dy}{y^{1+\alpha}}, & \alpha \in (0, 1), \\ 2m \cos \left(\frac{\pi \alpha}{2}\right) \int_0^{\infty} (1 - y - e^{-y}) \frac{dy}{y^{\alpha}}, & \alpha \in (1, 2), \\ \pi m, & \alpha = 1. \end{cases}$$

For brevity, if we set the scale parameter $d_{\alpha} = 1$. Then, the characteristic exponent $\eta_{s,\alpha}^*$ simplifies to

$$\eta_{s,\alpha}^*(z) = -|z|^\alpha, \quad z \in \mathbb{R}.$$

From the above discussed explicit form of $\eta_{\alpha}$, it is clear that characteristic function of an $\alpha$-stable random variable is differentiable for $\alpha \in (0, 1) \cup (1, 2)$, but fails at $\alpha = 1$. To fix this problem, we consider a tempered 1-stable random variable $Y_\gamma$ (see, [37]) with characteristic exponent $\eta_{1,\gamma}$, given by

$$\eta_{1,\gamma}(z) = iz\beta + \int_{\mathbb{R}} (e^{izu} - 1 - izu \mathbf{1}_{|u| \leq 1}(u)) \nu_{1,\gamma}(du), \quad z \in \mathbb{R}, \quad (8)$$

where tempering parameter $\gamma \in (0, \infty)$, and $\nu_{1,\gamma}$ is the Lévy measure defined as

$$\nu_{1,\gamma}(du) := \left(m_1 e^{-\gamma u} \mathbf{1}_{(0, \infty)}(u) + m_2 e^{-\gamma |u|} \mathbf{1}_{(-\infty, 0)}(u)\right) du.$$ 

Note that $Y_\gamma$ is infinitely divisible and its characteristic exponent $\eta_{1,\gamma}$ is differentiable on $\mathbb{R}$. Also, it can be easily shown that as $\gamma \to 0^+$, $\eta_{1,\gamma} \to \eta_1$, the characteristic exponent of 1-stable random variable $X$. This fact is used later to derive a Stein identity for 1-stable random variable $X$.

### 2.2 Probability metrics

Here, we review two well-known probability metrics used in this article.
1. The Wasserstein-\(\delta\) distance \([10]\). Let

\[
\mathcal{H}_\delta = \left\{ h : \mathbb{R} \to (\mathbb{R}, d_\delta) \bigg| |h^{(k)}(v) - h^{(k)}(x)| \leq d_\delta(v, x), k = 0, 1 \right\},
\]

where \(d_\delta(v, x) := |v - x| \wedge |v - x|^\delta\), \(h^{(1)}\) is the first derivative of \(h\), with \(h^{(0)} = h\) and the range of \(h^{(k)}\) is endowed with metric \(d_\delta\). Then, for any two random variables \(V\) and \(X\) the distance is given by

\[
d_{W_\delta}(V, X) := \sup_{h \in \mathcal{H}_\delta} |E[h(V)] - E[h(X)]|, \quad \delta < \alpha \leq 1,
\]

This distance is useful for studying stable approximations of Type B \((\alpha \in (0, 1))\) and Type C, Case 1 \((\alpha = 1)\). In \([10]\), authors use \(d_{W_\delta}\) distance with \(\delta \in (0, \alpha)\), for obtaining non-integrable stable approximations (Type B, and Type C, Case 1). We also see that Chen et. al. \([10\ Subsection 1.2]\) only consider the test function \(h\) to be bounded Lipschitz, and endowed with the metric \(d_\delta\). They also have discussed relation with other metrics in the existing literature. This is discussed in more detail in Section 3 and Section 4, respectively.

2. The Wasserstein-type distance \([3]\). Let

\[
\mathcal{H}_r = \left\{ h : \mathbb{R} \to \mathbb{R} \bigg| h \text{ is } r \text{ times continuously differentiable and, } \|h^{(k)}\| \leq 1, k = 0, 1, \ldots, r \right\},
\]

where \(h^{(k)}\), \(k = 1, \ldots, r\), is the \(k\)-th derivative of \(h\), with \(h^{(0)} = h\) and \(\|f\| = \sup_{x \in \mathbb{R}} |f(x)|\). Then, for any two random variables \(Y\) and \(Z\) the distance is given by

\[
d_{W_r}(Y, Z) := \sup_{h \in \mathcal{H}_r} |E[h(Y)] - E[h(Z)]|.
\]

This distance is useful for studying stable approximations of Type C, Case 2 \((\alpha \in (1, 2))\), see \([3]\). Johnson and Samworth \([23]\) use Mallows \(r\)-distance \(d_r\), \(r > 0\) for \(\alpha\)-stable approximations. Note that \(d_r\) is the classical Wasserstein \(r\)-distance for \(r \geq 1\). Moreover, these distances have the following order relationship.

\[
d_{W_r}(Y, Z) \leq d_{W_1}(Y, Z) \leq d_{1}(Y, Z) \leq d_{p}(Y, Z), \quad r, p \geq 1.
\]

In \([3\ Subsection 2.3]\), authors use \(d_{W_2}\) distance for obtaining approximations to infinitely divisible distributions with first finite moment and they also have discussed relation with other metrics in the existing literature. We also use these relationships and discuss the consequences in Section 3 and Section 4, respectively.

2.3 Literature review

Here, we review the known results on convergence rates for \(\alpha\)-stable approximations motivated by generalized CLT. The generalized CLT states that the sum of i.i.d. random variables when scaled and centered appropriately, converges to an \(\alpha\)-stable distribution. Mathematically, assume \((Y_n)_{n \geq 1}\) is a sequence of i.i.d.
random variables. For any $n \in \mathbb{N}$, define $S_n = a_n(Y_1 + Y_2 + \ldots + Y_n) - b_n$, where $a_n \in (0, \infty)$ and $b_n \in \mathbb{R}$ are constants. Then, the sum $S_n$ converges weakly to an $\alpha$-stable distribution with stability index $\alpha \in (0, 2]$, see [19]. For $\alpha = 2$, if we assume $\mathbb{E}Y_1 = 0$, $\mathbb{E}Y_1^2 = 1$, $\mathbb{E}|Y_1|^3 < \infty$ and set $a_n = \frac{1}{\sqrt{n}}$, $b_n = 0$, then the sum $S_n$ converges weakly to the standard normal distribution $F_{N(0,1)}$, and by Berry-Esseen theorem, it is shown that

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - F_{N(0,1)}(x)| \leq \frac{C}{\sqrt{n}} \mathbb{E}|Y_1|^3,$$

where $C > 0$ is some constant, see [28].

The problem of convergence rates for $\alpha$-stable approximations in the generalized CLT is studied by many authors using various approaches, see [8, [10, 21, 24] for more details. However, we consider an interesting article written by Johnson and Samworth [23] for comparing the results in this article, as it closely matches our framework. In [23], authors consider the generalized CLT, and develop the convergence rate for $\alpha$-stable approximation in the Mallows $r$-distance $d_r$, where $r > 0$, using the following framework. Let $(Y_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with distribution function $F$ such that $F(y) = \frac{c_1 + b(y)}{|y|^\alpha}$ for $y < 0$ and $1 - F(y) = \frac{c_2 + b(y)}{|y|^\alpha}$ for $y > 0$, where $c_1, c_2 > 0$ and $b(y) = O(|y|^{-d})$, $d > 0$. In [23, Theorem 1.2], it is shown that the partial sum $S_n = n^{-\frac{1}{\alpha}}(Y_1 + Y_2 + \ldots + Y_n)$ converges weakly to an $\alpha$-stable distribution with the rate $n^{\frac{1}{\alpha} - \frac{1}{\beta}}$ in the $d_\beta$ distance, where $\beta \in (0, 2]$. This result is proved using the coupling technique and Lindeberg method. More precisely, for $(Z_n)_{n \geq 1}$ an i.i.d. sequence of $\alpha$-stable distributed random variables, $n^{-\frac{1}{\alpha}}(Z_1 + Z_2 + \ldots + Z_n)$ has an $\alpha$-stable distribution, and hence

$$d_\beta\left(S_n, n^{-\frac{1}{\alpha}} \sum_{i=1}^{n} Z_i\right) = n^{-\frac{1}{\alpha}} d_\beta\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Z_i\right) = n^{-\frac{1}{\alpha}} d_\beta\left(\sum_{i=1}^{n} Y^*_i, \sum_{i=1}^{n} Z^*_i\right) \leq n^{-\frac{1}{\alpha} \mathbb{E}} \sum_{i=1}^{n} (Y^*_i - Z^*_i)^\beta,$$

where $(Y^*_i, Z^*_i)$ is a coupling of $Y_i$ and $Z_i$ with the property $\mathbb{E}|Y^*_i - Z^*_i| = d_\beta(Y_i, Z_i)$, and $\{(Y^*_i, Z^*_i)\}_{1 \leq i \leq n}$ are independent, see [23, Eq.(3)]. Finally, using Essen inequality [23, Eq.(11) and Eq.(12)] and by [23, Lemma 5.1], authors show that $d_\beta\left(S_n, n^{-\frac{1}{\alpha}} \sum_{i=1}^{n} Z_i\right) = O(n^{\frac{1}{\alpha} - \frac{1}{\beta}})$.

As mentioned in Section I, the goal of this article is to unify Stein’s method for $\alpha$-stable distributions. Let us now review the articles in the literature related to $\alpha$-stable distribution approximation via Stein’s method. Recently, Arras and Houdré [23] develop Stein’s method for IDD with finite first moment. Let us briefly discuss their ideas in [23]. In [23, Theorem 3.1], authors obtain a Stein characterization for IDD with finite first moment using covariance representation of functions of infinitely divisible random variables. We see that Stein characterizations of several distributions are followed by [23, Theorem 3.1]. Using the Fourier method, they provide general upper bounds for $d_{kol}(X_n, X)$, where both $X_n$ and $X$ are infinitely divisible random variables. In [23, Proposition 5.1], they also obtain generators for the self-decomposable distributions and prove a bound for self-decomposable distribution approximation in the Wasserstein-type distance $d_{W_2}$, see [23, Section 6]. Applying [23, Theorem 6.1 and Theorem 6.2], they demonstrate bounds for
α-stable approximations with α ∈ (1, 2) in the $d_{W_2}$ distance.

In this direction, Xu [40] develops Stein’s method for symmetric α-stable distributions with α ∈ (1, 2) and prove a rate of convergence $n^{-\frac{2}{\alpha}}$ for α-stable approximations in the Wasserstein distance $d_W$. Let us briefly discuss ideas in [40]. Let $S_n = Z_1 + Z_2 + \ldots + Z_n$ be a sum of $n$ centered i.i.d random variables. By [40] Theorem 1.4, a Stein operator for symmetric α-stable random variable is given by

$$Af(x) = \Delta^{\frac{\alpha}{2}} f(x) - \frac{1}{\alpha} x f'(x),$$

where $\Delta^{\frac{\alpha}{2}}$ is the fractional Laplacian and $f \in F$ (a class of functions $f$ with first and second derivatives bounded by a constant depending on α and that $\Delta^{\frac{\alpha}{2}} f$ is γ-Hölder continuous for any $0 < \gamma < 1$).

Using $K$-function approach [7], Xu shows that,

$$E(S_n f'(S_n)) = \sum_{i=1}^{n} \int_{-N}^{N} E(K_i(t, N) f''(S_n(i) + t)) \, dt + R,$$

where $N > 0$ is an arbitrary number, $S_n(i) = S_n - Z_i$, $K_i(t, N) = E(Z_i 1_{0 \leq t \leq Z_i \leq N}) - Z_i 1_{-N \leq Z_i \leq t \leq 0}$, and $R$ is a remainder.

Due to the heavy tail property of $Z_i$, Xu also shows that,

$$\Delta^{\frac{\alpha}{2}} f(S_n) = \int_{-N}^{N} K_\alpha(t, N) f''(S_n + t) \, dt + R',$$

where $K_\alpha(t, N) = \frac{m_\alpha}{\alpha(\alpha-1)}(|t|^{1-\alpha} - N^{1-\alpha})$ with $m_\alpha = \left( \int_{\mathbb{R}} \frac{1 - \cos y}{|y|^{1+\alpha}} \, dy \right)^{-1}$.

Using (9) and (10), it can be shown that

$$E(Af(S_n)) = \sum_{i=1}^{n} \int_{-N}^{N} E \left( \frac{K_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right) f''(S_n(i) + t) \, dt + R''$$

where $R''$ is another remainder. Hence,

$$|E Af(S_n)| \leq \left( \sum_{i=1}^{n} \int_{-N}^{N} E \left| \frac{K_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| \, dt \right) \|f''\| + \|R''\|,$$

where $\|f''\| = \sup_{x \in \mathbb{R}} |f''(x)|$. Therefore, to obtain a rate of convergence, it is sufficient to bound $\|f''\|$ and the remainder $\|R''\|$.

Jin et. al. [22] extend Xu’s idea [40] and develop Stein’s method for asymmetric α-stable distributions with α ∈ (1, 2). They also obtain kernel discrepancy type bound as (11) and demonstrate a rate of convergence for α-stable approximation, see [22] Theorem 1.1. Chen et. al. [11] also develop Stein’s method for asymmetric α-stable distributions with α ∈ (1, 2). Using leave-one-out approach developed by Stein [35], they derive a rate of convergence for α-stable approximations (see, [11] Theorem 1.4). Later, Chen et. al. [12] extend it for multivariate case and develop Stein’s method for multivariate α-stable distributions with α ∈ (1, 2).
this sequel, Arras and Houdré develop Stein’s method for multivariate self-decomposable distributions with finite first moment [4] and without finite first moment [5].

More recently, Chen et. al. [10] develop Stein’s method for α-stable distributions with α ∈ (0, 1]. In [10] Theorem 4 and Section 4, they compute the rate of convergence in the generalized CLT for the partial sum of i.i.d. random variables in the domain of normal attraction of an α-stable distribution. Due to lack of finite first moment, the strategy in deriving rate of convergence for α-stable approximation differs significantly from the case α ∈ (1, 2), obtained in [3, 22, 10]. Let us briefly discuss their ideas in [10]. Recall that \( d_{W_α}(V, X) := \sup_{h \in \mathcal{H}_α} |E h(V) - E h(X)|, \) \( \delta \in (0,\alpha) \), where \( \mathcal{H}_δ \) is the class of Lipschitz functions \( h : \mathbb{R} \to (\mathbb{R}, d_δ) \) such that \( |h(x) - h(y)| \leq d_δ(x, y) \) and the range \( \mathbb{R} \) of the function \( h \) is endowed with the metric \( d_δ(x, y) = |x - y| \wedge |x - y|^{\delta} \). Such a probability metric is suitable for α-stable approximation with \( \alpha \in (0, 1] \), see [10] Subsection 1.2. Let \( S_n = \frac{1}{\sigma_α} (Y_1 + Y_2 + \ldots + Y_n) \), \( \sigma_α > 0 \) be a partial sum of i.i.d. random variables in the domain of normal attraction of an α-stable distribution. For any α-stable random variable \( X \), Chen et al. [10] show that,

\[
d_{W_α}(S_n, X) \leq \sup_{h \in \mathcal{H}_α} \left| E \left( L^{\alpha,\theta} f(S_n) \right) - \frac{1}{\alpha} E \left( S_n f'(S_n) \right) \right|,
\]

where \( \theta \in [-1, 1] \) is a skewness parameter, \( L^{\alpha,\theta} \) is a generator of an α-stable Lévy process, and \( F_{\alpha,\theta} \) is a class of smooth functions. Due to the lack of first finite moment, one has to compute carefully contribution from the term \( E L^{\alpha,\theta} f(S_n) \) and that from the term \( E S_n f'(S_n) \) for obtaining an upper bound in \( d_{W_α} \) distance. In [10] Section 4, we see that the Taylor-like expansion and tail properties of \( Y_1 \) are the keys for obtaining the upper bound in (12). Also, the regularity estimates of the solution to Stein equation help them to obtain the rates of convergence for α-stable approximations.

3 Main results

In this section, we discuss the three important components of Stein’s method for IDD, as mentioned in the Introduction. First, we start with Stein identity for infinitely divisible random variables. Let \( \mathcal{S}(\mathbb{R}) \) be the Schwartz space defined by

\[
\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \lim_{|x| \to \infty} |x^{-m} \frac{d^n}{dx^n} f(x)| = 0, \text{ for all } m, n \in \mathbb{N} \right\},
\]

where \( C^\infty(\mathbb{R}) \) is the class of infinitely differentiable functions on \( \mathbb{R} \). It is important to note that the Fourier transform on \( \mathcal{S}(\mathbb{R}) \) is automorphism onto itself. This enables us to identify the elements of dual space \( \mathcal{S}^*(\mathbb{R}) \) with \( \mathcal{S}(\mathbb{R}) \). In particular, if \( f \in \mathcal{S}(\mathbb{R}) \), and \( \hat{f}(u) = \int_\mathbb{R} e^{-iux} f(x) dx, \ u \in \mathbb{R}, \) then \( \hat{f}(u) \in \mathcal{S}(\mathbb{R}) \). Similarly, if \( \hat{f}(u) \in \mathcal{S}(\mathbb{R}) \), and \( f(x) = \int_\mathbb{R} e^{iux} \hat{f}(u) du, \ x \in \mathbb{R}, \) then \( f(x) \in \mathcal{S}(\mathbb{R}) \), see [24].

Now, we state our first result on Stein identity for infinitely divisible random variables.

Theorem 3.1. Let \( X \sim IDD(\beta, \sigma^2, \nu) \) with characteristic exponent [4] and the characteristic function \( \phi_X \) be differentiable.
Then,

$$
E \left( (X - \beta)g(X) - \sigma^2 g'(X) - \int_{\mathbb{R}} u(g(X + u) - g(X))1_{|u| \leq 1}(u)\nu(du) \right) = 0, \quad g \in S(\mathbb{R}).
$$

(13)

**Remark 3.2.**

(i) Note that the differentiability of the characteristic function $\phi_X$ plays a crucial role in deriving the Stein identity (13). Indeed, our approach in deriving Stein identity for infinitely divisible random variables does not follow easily when $\phi_X$ is not differentiable. For example, the characteristic function of a Cauchy random variable is not differentiable (see, Section 2, Ex. 4.). We need to modify our approach to handle this problem (see Theorem 3.3).

(ii) Arras and Houdré [3, Theorem 3.1] provide a Stein identity of infinitely divisible random variables using covariance representation for functions of infinitely divisible random variables. However, they assume finite first moment and the function space as bounded Lipschitz. Our proof of Theorem 3.1 in Section 5 is without this assumption, and we consider the Schwartz space $S(\mathbb{R})$ as a suitable function space. It is important to note that the assumption of the first finite moment is an artefact of the technique of covariance representation. Also, our Stein identity (13) exactly matches with the Stein identity given in [3, Theorem 3.1].

Also, observe that several random variables such as Poisson, negative binomial, normal, Laplace, and gamma can be viewed as infinitely divisible by choosing appropriate triplet $(\beta, \sigma^2, \nu)$. Stein identities for these random variables can be easily derived using Theorem 3.1. In particular, $\alpha$-stable random variables are also infinitely divisible, but the derivation of Stein identity is not straightforward (see, Chen et. al. [10, 11]).

Next, we establish a Stein identity for $\alpha$-stable random variables. As noted in Section 1, the characteristic function of $\alpha$-stable variable is differentiable for $\alpha \in (0, 1) \cup (1, 2)$, but fails at $\alpha = 1$. Hence, we need to modify our approach for $\alpha = 1$, using tempered 1-stable random variable $Y$ in deriving the Stein identity, as mentioned in Section 2.

**Theorem 3.3.** Let $X \sim S(\alpha, \beta, m_1, m_2)$ with characteristic exponent (2). Then,

$$
E \left( (X - \beta)g(X) - \sigma^2 g'(X) - \int_{\mathbb{R}} (g(X + u) - g(X)1_{|u| \leq 1}(u))\nu_{\alpha}(du) \right) = 0, \quad g \in S(\mathbb{R}).
$$

(14)

Note here that $\sigma = 0$, as $X$ has a non-Gaussian stable distribution. The following corollary immediately follows for symmetric $\alpha$-stable random variables by setting $m_1 = m_2 = m$, $\beta = 0$ and adjusting $\nu_{\alpha}$.

**Corollary 3.4.** Let $X \sim S(\alpha, 0, m, m)$, for $\alpha \in (0, 2)$. Then a Stein identity for $X$ is given by

$$
E \left( Xg(X) - m \int_{0}^{\infty} g(X + u) - g(X - u)\frac{1}{u^\alpha}du \right) = 0, \quad g \in S(\mathbb{R}).
$$

(15)

In the following remark, we discuss and review the Stein identities available in the literature and in (iv), we compare them with our Stein identities.

**Remark 3.5.**

(i) Chen et. al. [10, Proposition 2.4] derive a Stein identity for an $\alpha$-stable random variable with $\alpha \in (0, 1]$, using Barbour’s generator approach [6], where the scale and the location parameters...
are chosen to be 1 and 0 respectively, for \( \alpha \in (0,1) \), and further, for \( \alpha = 1 \), the skewness parameter is also set to zero. Using truncation technique, Arras and Houdré [4, Theorem 3.1 and Theorem 3.2] also derive Stein identities for \( \alpha \)-stable random variables with \( \alpha \in (0,1) \) and \( \alpha = 1 \), respectively.

(ii) Chen et. al. [14, Theorem 1.2] derive a Stein identity for an \( \alpha \)-stable random variable with \( \alpha \in (1,2) \), using Barbour’s generator approach [6], where the scale and the location parameters are chosen to be 1 and 0 respectively. Chen et. al. [12] also extend their idea in deriving a Stein identity for multivariate \( \alpha \)-stable random vectors with \( \alpha \in (1,2) \), using Barbour’s generator approach [6], where the location parameter is chosen to be 0.

(iii) Xu [40, Theorem 4.1] derive a Stein identity for a symmetric \( \alpha \)-stable random variable with \( \alpha \in (0,2) \), using invariant measure property of Lévy-type operators [3]. Arras and Houdré [3, Examples 3.3, (viii)] also derive a Stein identity for symmetric \( \alpha \)-stable random variables with \( \alpha \in (1,2) \), using covariance representation of functions of infinitely divisible random variables [20].

(iv) Our Stein identity given in (14) is derived using the Lévy-Khinchine representation of the characteristic exponent given in [5] without any assumption on the scale, location and skewness parameter. Under the assumptions of Chen et. al. [10, 14], their Proposition 2.4 and Theorem 1.2 can be retrieved from Theorem 3.3. Using Proposition A.2, we see that Stein identities given in [5, Theorem 3.1 and Remark 5.3, (iv)] and [40, Theorem 4.1].

As noted in Section 1, the linchpin of Stein’s method is the Stein operator \( A \), and the properties of \( A \) play a crucial role in the success of this method. In this context, we adopt the following definition of Stein operator from [17].

**Definition 3.6.** [17, p.1] An operator \( A \) is said to be a Stein operator, if \( A \) acts on a class of test functions \( \mathcal{G} \) such that \( \mathbb{E}(Ag(X)) = 0 \) for all \( g \in \mathcal{G} \), where the random variable \( X \sim F \).

**Remark 3.7.** It is now clear from Theorem 3.1 that, for an infinitely divisible random variable \( X \), \( A_X(g)(x) := (-x + \beta)g(x) + \sigma^2 g'(x) + \int u(g(x + u) - g(x)1_{|u|\leq 1}(u))v(du) \) is an operator acting on \( \mathcal{S}(\mathbb{R}) \) such that \( \mathbb{E}(A_X(g)(X)) = 0 \) for all \( g \in \mathcal{S}(\mathbb{R}) \). Then, by the above definition, \( A_X \) is a Stein operator for an infinitely divisible random variable \( X \). Also, for any \( g \in \mathcal{S}(\mathbb{R}) \), \( A_X^\alpha(g)(x) := (-x + \beta)g(x) + \int u(g(x + u) - g(x)1_{|u|\leq 1}(u))\nu(du) \) is a Stein operator for an \( \alpha \)-stable random variable. Observe also that \( A_X^\alpha \) is an integral operator, where domain of the operator is \( \mathcal{F} = \mathcal{S}(\mathbb{R}) \), the closure of \( \mathcal{S}(\mathbb{R}) \) (see, [40] and references therein [22] for more details).

The existing literature on Stein’s method for \( \alpha \)-stable distributions (see, [3, 11, 11, 12, 22, 40]) suggests a variety of techniques for deriving a Stein operator depending on the stability parameter \( \alpha \in (0,1] \) or \( (1,2) \). As mentioned in Section 1, the purpose of this article is to unify Stein’s method for \( \alpha \)-stable distributions. To achieve this, let us use the Stein operator \( A_X^\alpha \) to set up Stein equation. For any \( h \in \mathcal{H}_X \) (a class of smooth functions), Stein equation of an \( \alpha \)-stable random variable \( X \) is

\[
A_X^\alpha(g)(x) = h(x) - \mathbb{E}(h(X)).
\]
To solve (16), we use well-known semigroup approach (see, [6]), and this can be motivated as follows. Recall first that, for $X \sim S(\alpha, 0, m, m)$ with $d_\alpha = 1$, characteristic function simplifies to

$$\phi_\alpha^\ast(z) = \exp\left(-|z|^\alpha\right), \quad z \in \mathbb{R}. $$

Also, observe that, for any $z \in \mathbb{R}$, $\phi_\alpha^\ast(z) = \phi_\alpha^\ast(1-e^{-t}z)$, $t \geq 0$, where $\phi_\alpha^\ast(e^{-t}z)$ and $\phi_\alpha^\ast((1-e^{-t})z)$ denote the characteristic functions of $e^{-t}X$ and $(1-e^{-t})X$ respectively. Note that $e^{-t}X$ and $(1-e^{-t})X$ are symmetric $\alpha$-stable random variables. Let us now generalize this idea for non-symmetric case. One can define a characteristic function, for all $z \in \mathbb{R}$, and $t \geq 0$, by

$$\phi_t(z) := \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} = \int_\mathbb{R} e^{izu} F_{X_{(t)}}(du),$$

where $F_{X_{(t)}}$ is the distribution function of $X_{(t)}$ and $\phi_\alpha$ is the characteristic function of $\alpha$-stable random variable given in [3]. The property given in (17) is also known as self-decomposability (see, [32]).

Following Barbour’s approach [6] and using (17), we choose a family of operators $(P_s^t)_{t \geq 0}$, for all $x \in \mathbb{R}$, as

$$P_s^t(g)(x) := \frac{1}{2\pi} \int_\mathbb{R} \hat{g}(z)e^{izx} e^{-tz} \phi_t(z) dz, \quad g \in F. $$

Using (17), we get

$$P_s^t(g)(x) = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \hat{g}(z)e^{izx} e^{izu} F_{X_{(t)}}(du) dz$$

$$= \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \hat{g}(z)e^{iz(u+xe^{-t})} F_{X_{(t)}}(du) dz$$

$$= \int_\mathbb{R} g(u + xe^{-t}) F_{X_{(t)}}(du),$$

where the last step follows by applying inverse Fourier transform.

**Proposition 3.8.** The family of operators $(P_s^t)_{t \geq 0}$ given in (18) is a $C_0$-semigroup on $F$.

**Proof.** For each $g \in F$, it is easy to show that $P_0^0 g(x) = g(x)$ and $\lim_{t \to \infty} P_s^t(g)(x) = \int_\mathbb{R} g(x) F_X(\alpha)(dx)$. Now, for any $s, t \geq 0$, we have

$$\phi_{t+s}(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-(t+s)}z)} = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-s}z)} \frac{\phi_\alpha(e^{-s}z)}{\phi_\alpha(e^{-(t+s)}z)} = \phi_s(z) \phi_t(e^{-s}z)$$

Using (20), we have

$$LHS = P_t^{s}(g)(x) = \frac{1}{2\pi} \int_\mathbb{R} \hat{g}(z)e^{izx} \phi_{t+s}(z) dz = \frac{1}{2\pi} \int_\mathbb{R} \hat{g}(z)e^{izx} \phi_s(z) \phi_t(e^{-s}z) dz.$$
\textbf{Proof.} The proof of this lemma is split into two parts.

(i) \( \alpha \in (0, 1) \): For all \( g \in \mathcal{S}(\mathbb{R}) \),

\[
\mathcal{T}_\alpha g(x) = \lim_{t \to 0^+} \frac{1}{t} (P^\alpha_t g(x) - g(x))
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z) e^{izx} \frac{1}{t} \left( e^{iz(x-t)} \phi_t(z) - 1 \right) dz
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z) e^{izx} \left( -x + \beta - \int_{\{|u| \leq 1\}} u \nu_\alpha (du) + \int_{\mathbb{R}} e^{izu} \nu_\alpha (du) \right) (iz) dz \quad \text{(using Prop. A.1)}
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z) e^{izx} \left( -x + \beta_1 + \int_{\mathbb{R}} e^{izu} \nu (du) \right) (iz) dz \quad \text{(where} \ \beta_1 = \beta - \int_{\{|u| \leq 1\}} u \nu_\alpha (du))
\]
\[
= (-x + \beta_1) g'(x) + \int_{\mathbb{R}} g'(x + u) \nu_\alpha (du)
\]
\[
= (-x + \beta) g'(x) + \int_{\mathbb{R}} (g'(x + u) - g'(x) \mathbf{1}_{\{|u| \leq 1\}}(u)) \nu_\alpha (du),
\]

and the desired conclusion follows.

Next, we find the generator of the semigroup defined in (18).

\textbf{Lemma 3.9.} Let \( (P^\alpha_t)_{t \geq 0} \) be a \( C_0 \)-semigroup as defined in (18). Then, its generator \( \mathcal{T}_\alpha \) is given by

\[
\mathcal{T}_\alpha g(x) = (-x + \beta) g'(x) + g'(x + u) - g'(x) \mathbf{1}_{\{|u| \leq 1\}}(u) \nu_\alpha (du), \quad g \in \mathcal{S}(\mathbb{R}),
\]

where \( \alpha \in (0, 1) \cup (1, 2). \)

\textbf{Proof.} The proof of this lemma is split into two parts.

(ii) \( \alpha \in (0, 1) \cup (1, 2) \): For all \( g \in \mathcal{S}(\mathbb{R}) \),

\[
\mathcal{T}_\alpha g(x) = (-x + \beta) g'(x) + \int_{\mathbb{R}} g'(x + u) \mathbf{1}_{\{|u| \leq 1\}}(u) \nu_\alpha (du),
\]
where the last equality follows by splitting $\beta$ (see (6)).

(ii) $\alpha \in (1, 2)$: For all $g \in \mathcal{S}(\mathbb{R})$,

$$
\mathcal{T}_\alpha(g)(x) = \lim_{t \to 0^+} \frac{1}{t} (P_t^\alpha(g)(x) - g(x))
$$

$$
= \frac{1}{2\pi} \lim_{t \to 0^+} \int_{\mathbb{R}} \hat{g}(z)e^{izx} \frac{1}{t} (e^{izx(e^{-t} - 1)}\phi_t(z) - 1) \,dz
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z)e^{izx} \left( -x + \beta + \int_{\{u > 1\}} u\nu_\alpha(du) + \int_{\mathbb{R}} (e^{izu} - 1)\nu(du) \right) i\beta \,dz 
$$

(see (5)).

Next, we handle the case $\alpha = 1$, using tempered 1-stable random variable $Y_\gamma$. Recall that the characteristic exponent of $Y_\gamma$ is given in (5). Let $\phi_{1,\gamma}(z) := e^{\eta_1(z)}$, $z \in \mathbb{R}$ be the characteristic function of $Y_\gamma$. Then, for all $z \in \mathbb{R}$, we define

$$
\phi_{1,1,\gamma}(z) := \frac{\phi_{1,\gamma}(z)}{\phi_{1,\gamma}(e^{-t}z)}, \quad t \geq 0. \tag{22}
$$

Using Corollary 15.11], it can be easily shown that $\phi_{1,1,\gamma}$ is a well-defined characteristic function. Now, using (22), define a family of operators $(P_t^{1,\gamma})_{t \geq 0}$, for all $x \in \mathbb{R}$, by

$$
P_t^{1,\gamma}(f)(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z)e^{izxe^{-t}}\phi_{1,1,\gamma}(z) \,dz = \int_{\mathbb{R}} g(u + xe^{-t})F_{Y_{\gamma,0}}(du), \quad g \in \mathcal{F}. \tag{23}
$$

Following similar steps as Proposition 3.8, one can show that $(P_t^{1,\gamma})_{t \geq 0}$, is a $\mathcal{C}_0$-semigroup on $\mathcal{F}$.

Next, we obtain a generator for the semigroup (23).

**Lemma 3.10.** Let $(P_t^{1,\gamma})_{t \geq 0}$ be a $\mathcal{C}_0$-semigroup as defined in (23). Then, its generator $\mathcal{T}_{1,\gamma}$ is given by

$$
\mathcal{T}_{1,\gamma}(g)(x) = -xg'(x) + \int_{\mathbb{R}} (g'(x + u) - g'(x)1_{\{|u| \leq 1\}}(u)) \nu_{1,\gamma}(du), \quad g \in \mathcal{S}(\mathbb{R}).
$$

**Proof.** For all $g \in \mathcal{S}(\mathbb{R})$,

we get,

$$
\mathcal{T}_{1,\gamma}(g)(x) = \lim_{t \to 0^+} \frac{1}{t} \left( P_t^{1,\gamma}(g)(x) - g(x) \right)
$$

where

$$
\frac{1}{2\pi} \lim_{t \to 0^+} \int_{\mathbb{R}} \hat{g}(z)e^{izx} \frac{1}{t} (e^{izx(e^{-t} - 1)}\phi_t(z) - 1) \,dz
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z)e^{izx} \left( -x + \beta + \int_{\{u > 1\}} u\nu_\alpha(du) + \int_{\mathbb{R}} (e^{izu} - 1)\nu(du) \right) i\beta \,dz
$$

(see (6)).
\[ T_{1,\gamma}(g)(x) = \frac{1}{2\pi} \lim_{t \to 0^+} \int_{\mathbb{R}} \hat{g}(z) e^{izx} \frac{1}{t} \left( e^{iz(x-1)} \frac{\phi_{1,\gamma}(z)}{\phi_{1,\gamma}(e^{-t}z)} - 1 \right) dz \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(z) e^{izx} \left( -x + \beta + \int_{\mathbb{R}} (e^{izu} - 1)_{\{|u| \leq 1\}} \nu_{1,\gamma}(du) \right) (iz) dz \]

\[ = (-x + \beta)g'(x) + \int_{\mathbb{R}} (g'(x + u) - g'(x)1_{\{|u| \leq 1\}}) \nu_{1,\gamma}(du), \]

where the last but one equality follows by doing computations similar to Proposition A.3. This completes the proof.

Now, observe that, \( \lim_{\gamma \to 0^+} P_{1}^{1,\gamma} g(x) = P_{1}^{1} g(x), g \in \mathcal{F}, \) as defined in (18). Hence \( \lim_{\gamma \to 0^+} T_{1,\gamma} = T_{1} \), where \( T_{1} \) is given by

\[ T_{1}(g)(x) = (-x + \beta)g'(x) + \int_{\mathbb{R}} (g'(x + u) - g'(x)1_{\{|u| \leq 1\}}) \nu_{1,\gamma}(du), \quad g \in \mathcal{S}(\mathbb{R}). \]

**Remark 3.11.** Note that, on careful adjustments of the integrals with respect to \( \nu_{\alpha} \) for \( \alpha \in (0, 2) \), we see that the operator \( T_{\alpha} \) is also a Stein operator for an \( \alpha \)-stable random variable, see [10, 11, 22, 40]. In these articles, authors use \( T_{\alpha} \) to set up their Stein equations. However, we consider \( A^{\alpha}_{X} \) to set up our Stein equation.

Next, we provide the solution to our Stein equation.

**Theorem 3.12.** (i) For \( \alpha \in (0, 1] \), let \( X \sim \mathcal{S}(\alpha, \beta, m_{1}, m_{2}) \) and \( h \in \mathcal{H}_{\delta}, 0 < \delta < \alpha \). Also, let

\[ A^{\alpha}_{X}(g)(x) = h(x) - \mathbb{E}h(X) \]  

be a Stein equation for \( X \). Then, the function \( g^{\alpha}_{h} : \mathbb{R} \to \mathbb{R} \) defined as

\[ g^{\alpha}_{h}(x) = -\int_{0}^{\infty} e^{-t} \int_{\mathbb{R}} h'(u + xe^{-t})F_{X(\alpha)}(du) dt, \]

solves (25).

(ii) For \( \alpha \in (1, 2) \), \( X \sim \mathcal{S}(\alpha, \beta, m_{1}, m_{2}) \) and \( h \in \mathcal{H}_{2} \). Also, let

\[ A^{\alpha}_{X}(g)(x) = h(x) - \mathbb{E}h(X) \]  

be a Stein equation for \( X \). Then, the function \( g^{\alpha}_{h} : \mathbb{R} \to \mathbb{R} \) defined as

\[ g^{\alpha}_{h}(x) = -\int_{0}^{\infty} e^{-t} \int_{\mathbb{R}} h'(u + xe^{-t})F_{X(\alpha)}(du) dt, \]

solves (27).

In the following remark, we review the techniques used by several authors to solve Stein equations under various constraints and justify our claim of unification.
Remark 3.13. (i) Chen et. al. [10, 11] derive the solution to Stein equation for an $\alpha$-stable random variable with $\alpha \in (0, 1]$ and $\alpha \in (1, 2)$ respectively using Barbour’s generator approach [6], and the transition density function of $\alpha$-stable processes. Xu [40] also uses the Barbour’s generator approach to solve Stein equation for a symmetric $\alpha$-stable random variable with $\alpha \in (1, 2)$. Arras and Houdré [3] provide the semigroup approach to solve Stein equation for an infinitely divisible random variable with the first finite moment. Recently, Arras and Houdré [5, Remark 4.3] show that semigroup approach for deriving the solution to Stein equation is also applicable for multivariate $\alpha$-stable random vectors with $\alpha \in (0, 1)$, and they also mention that the semigroup approach is also applicable for $\alpha = 1$, but requires different estimates.

(ii) Note that, for both parts of Theorem 3.12, we use only the semigroup approach to solve Stein equation for an $\alpha$-stable random variable with $\alpha \in (0, 2)$, and this unifies the method of solving the Stein equation for $\alpha$-stable random variables.

3.1 Properties of solution to Stein equation

Let us now study regularity estimates of the solution to our Stein equation. Recall that the Lévy measure $\nu_\alpha$ for $\alpha$-stable distributions is given by
$$
\nu_\alpha(du) = m_1 \frac{1}{|u|^{\alpha+1}} \mathbb{1}_{(0,\infty)}(u) + m_2 \frac{1}{|u|^{\alpha+1}} \mathbb{1}_{(-\infty,0)}(u) \, du,
$$
where $m_1, m_2 \in [0,\infty)$, $m_1 + m_2 > 0$ and $\alpha \in (0, 2)$. In the following theorem, we establish estimates of $g_\alpha^h$, which play a crucial role in the $\alpha$-stable approximation problem.

**Theorem 3.14.** (i) Let $\alpha \in (0, 1)$. For $h \in H_\delta$, $0 < \delta < \alpha$, let $g_\alpha^h$ be defined in (25). Then, for any $x, y \in \mathbb{R}$
$$
\|g_\alpha^h\| \leq \|h'\|, \quad (28)
$$
$$
\|g_\alpha^h(x) - g_\alpha^h(y)\| \leq \frac{1}{\delta + 1} |x - y|^{\delta}. \quad (29)
$$
Define $A_0 g_\alpha^h(x) := \int_{\mathbb{R}} u g_\alpha^h(x+u) \nu_\alpha(du)$. Then
$$
\|A_0 g_\alpha^h\| \leq C_{\alpha,\delta,m_1,m_2} := \frac{\alpha(m_1 + m_2)}{\delta(\alpha - \delta)} + \frac{\alpha(m_1 - m_2)}{1 - \alpha}, \quad (30)
$$
(ii) Let $\alpha = 1$. For $h \in H_\delta$, $0 < \delta < 1$, let $g_1^h$ be defined in (25). Then, for any $x, y \in \mathbb{R}$
$$
\|g_1^h\| \leq \|h'\|, \quad (31)
$$
$$
\|g_1^h(x) - g_1^h(y)\| \leq \frac{1}{\delta + 1} |x - y|^{\delta}. \quad (32)
$$
Define $A_1 g_1^h(x) := \int_{\mathbb{R}} u(g_1^h(x+u) - ug_1^1 1_{\{|u| \leq 1\}}) \nu_1(du)$. Then
$$
\|A_1 g_1^h\| \leq C_{1,\delta,m_1,m_2} := \frac{2(m_1 + m_2)}{\delta(1 - \delta^2)}. \quad (33)
$$
(iii) Let $\alpha \in (1, 2)$. For $h \in \mathcal{H}_2$, let $g_h^{\alpha}$ be defined in (27). Then, $g_h^{\alpha}$ is differentiable on $\mathbb{R}$,

$$
\|g_h^{\alpha}\| \leq \|h'\| \quad \text{and} \quad \|(g_h^{\alpha})'\| \leq \frac{1}{2} \|h''\|.
$$

(34)

Let $m_1 = m_2 = m$. Define $A_2 g_h^{\alpha}(x) := \int_{\mathbb{R}} (g_h^{\alpha}(x + u) - g_h^{\alpha}(x))w_{\alpha}(du)$. For any $x, y \in \mathbb{R}$

$$
\|A_2 g_h^{\alpha}(x) - A_2 g_h^{\alpha}(y)\| \leq C_{\alpha, m} \|h''\| |x - y|^{2 - \alpha},
$$

(35)

where $C_{\alpha, m}$ is a positive constant.

In the following remark, we review the estimates of solution to Stein equation available in the literature. In (iii) and (iv), we compare them with our results.

**Remark 3.15.**

(i) Chen et. al. \[10\] Proposition 1 and Proposition 2] provide estimates of solution to Stein equation for $\alpha \in (0, 1]$, using the properties of transition density of an $\alpha$-stable process. From the derivation of these estimates, we observe that the scaling property of transition density plays a crucial role in deriving the estimates. In particular, authors prove that for any $x, y \in \mathbb{R}$,

$$
|h'_x(x) - h'_y(y)| \leq C|x - y|^\alpha,
$$

where $f_h$ is the solution to Stein equation and $C$ is some positive constant, see \[10\] Proposition 1 (Eq.3.2)]. This fact helps them to determine a good rate of convergence and this is discussed in more detail in Section 4.

(ii) Chen et. al. \[11\] and Jin et. al. \[22\] derive estimates of solution to Stein equation for an $\alpha$-stable random variable with $\alpha \in (1, 2)$, using scaling property of transition density of an $\alpha$-stable process. Arras and Houdré \[3\] Section 5] also derive estimates of solution to Stein equation for an $\alpha$-stable random variable with $\alpha \in (1, 2)$, using self-decomposability and properties of the semigroup used to solve their Stein equation. In \[3\] Section 5], we see that for any $h \in \mathcal{H}_2$, upper bounds of first and second derivative of solution to Stein equation heavily rely on first and second derivative of $h$, respectively (due to the choice of Stein equation $\mathcal{T}_h g(x) = h(x) - \mathbb{E}h(X)$, where $X \sim S(\alpha, \beta, m_1, m_2)$).

(iii) Observe that, for any $h \in \mathcal{H}_5$ and $\alpha \in (0, 1]$, the upper bound of the solution to our Stein equation $g_h^{\alpha}$ relies on $h'$. Note that, for $\alpha \in (0, 1]$, other estimates of $g_h^{\alpha}$ are derived by suitably adjusting the integrals, and using the properties of the semigroup defined in \[18\]. These estimates are used to obtain the rate of convergence for $\alpha$-stable approximations with $\alpha \in (0, 1)$, and we obtain a flexible rate of convergence (faster rate when $\alpha \in (0, 0.5]$ and slower rate when $\alpha \in (0.5, 1)$). This is discussed in more detail in Section 4.

(iv) Observe also that, for $h \in \mathcal{H}_2$ and $\alpha \in (1, 2)$, the upper bound of the solution to Stein equation, $g_h^{\alpha}$ relies on first derivative of $h$, and upper bound for the first derivative of $g_h^{\alpha}$ relies on the second derivative of $h$. Other estimates of $g_h^{\alpha}$ are derived by suitably adjusting the integrals, and using the properties of the semigroup defined in \[18\]. These estimates provide a better rate of convergence for $\alpha$-stable approximations, whenever $\alpha \in (1, 2)$ and this is also discussed in more detail in Section 4.

Next, we provide bounds in the $d_{W_3}$ distance for $\alpha$-stable approximations of a partial sum of sequence of i.i.d random variables in the domain of normal attraction of an $\alpha$-stable distribution with $\alpha \in (0, 1]$. Let us define the domain of normal attraction of an $\alpha$-stable distribution as follows.

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Definition 3.16. [10, p.4] A real-valued random variable $Y$ is said to be in the domain of normal attraction of an $\alpha$-stable distribution with $\alpha \in (0, 1]$ if its CDF, $F_Y$, satisfies

$$1 - F_Y(y) = \frac{A + e(y)}{|y|^\alpha} (1 + \theta)$$ and $F_Y(-y) = \frac{A + e(-y)}{|y|^\alpha} (1 - \theta),$$

where $y > 1$, $\alpha \in (0, 1]$, $\theta = \frac{m_1 - m_2}{m_1 + m_2} \in [-1, 1]$, $A(>0)$ a constant and $e : \mathbb{R} \to \mathbb{R}$ is a bounded differentiable function vanishing at $\pm \infty$. Since $e$ is bounded, we denote $K = \sup_{y \in \mathbb{R}} e(y)$.

We denote $Y \in D_\alpha$, if $Y$ is in the domain of normal attraction of an $\alpha$-stable distribution, and for a positive constant $L$, the function $e$ defined in [36] is $C^2$ with the domain $\{ |y| > L \}$, and it satisfies $y e'(y) \to 0$ and $y^2 e''(y) \to 0$ as $|y| \to \infty$.

Theorem 3.17. Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of i.i.d random variables such that $Y_i \in D_\alpha$ and $X \sim S(\alpha, \beta, m_1, m_2)$ with $\alpha \in (0, 1]$. Define $S_n = n^{-1/\alpha}(Y_1 + Y_2 + \cdots + Y_n)$. Then,

(a) for $\alpha \in (0, 1)$

$$d_{W^s}(S_n, X) \leq C_{\alpha, \delta, m_1, m_2}^{-1} + C_{1, \delta, L} n^{-1-(1+\delta)} + C_{2, \delta} n^{-1-(1+\delta)} \sup_{L < |y| < n^\beta} (|\alpha e(y)| + |ye'(y)|) \int_{L < |y| < n^\beta} |y|^{\delta-\alpha} dy$$

$$+ C_{\alpha, \delta, m_1, m_2}^{-1} n^{-(1-\alpha)} + n^{-\alpha} \int_{|y| < n^\beta} |y| dF_Y(y) + R_{\alpha, n}.$$  

(b) For $\alpha = 1$

$$d_{W^s}(S_n, X) \leq C_{1, \delta, m_1, m_2}^{-1} + \frac{1}{\delta + 1} n^{-\delta} (L^2 + m_1 + m_2) + \frac{n^{-\delta}}{1 + \delta} \int_{L < |u| < n^\beta} \frac{|e(u) - u e'(u)|}{|u|^{1-\delta}} du$$

$$+ n^{-1} \int_{|u| > 1} \frac{|e(nu) - n u e'(nu)|}{|u|} du + R_{1,n}.$$  

Remark 3.18. Note that, in view of Theorem 3.17, we consider only real-valued random variables $Y_i \in D_\alpha$. Indeed, integer-valued random variables in general do not belong to $D_\alpha$, see [10]. The problem for developing an approach that allow to handle integer-valued sums is still open. Recently, Chen et. al. [10] also provide bounds in $d_{W^s}$ distance for $\alpha$-stable approximations of a partial sum of sequence of i.i.d random variables, belong to $D_\alpha$. Our bounds given in [37] and [38] are similar to the bounds given in [10, Theorem 4]. Note also that, our bounds include location parameter on $\alpha$-stable approximations with $\alpha \in (0, 1]$. Chen et. al.
In this section, we discuss the rates of convergence for \( \alpha \)-stable approximations using two examples and we compare them with existing literature.

Example 4.1 (Pareto distribution with \( \alpha \in (0,1) \)). Assume that \( Y_1, Y_2, \ldots, Y_n \) be i.i.d random variables having a Pareto distribution with \( \alpha \in (0,1) \), i.e.

\[
P(Y_1 > y) = \frac{1}{2y^\alpha}, \quad y \geq 1, \quad P(Y_1 \leq y) = \frac{1}{2y^\alpha}, \quad y \leq -1.
\]

In this case \( \theta = 0, A = \frac{1}{2}, e(y) = \frac{|y|^{\alpha - 1}}{2}I_{(-1,1)} \) and \( K = \frac{1}{2} \). Observe that \( L = 1, e(y) = 0 \) for \( |y| > 1 \). By Theorem 3.17 Case 1, one can easily verify that

\[
d_{W_2}(S_n, X) \leq Cn^{-\left(\frac{1}{\alpha} - 1\right)},
\]

where \( C \) is some positive constant. Moreover, \( d_{W_2}(S_n, X) = O(n^{-\left(\frac{1}{\alpha} - 1\right)}) \).
Let us compare our result with the known results in literature. The reference [27] shows a convergence rate \( d_{Kol}(S_n, X) \leq C_\alpha n^{-1} \), for \( \alpha \in (0, 1] \), where an exact value of \( C_\alpha \) was not given. Chen et. al. [10] proved that the rate \( n^{-1} \) is valid for the \( d_{W_2} \) distance, whenever \( \alpha \in (0, 1) \). For \( \alpha \in (0, 1) \), our rate is \( n^{-\frac{\alpha}{1-\alpha}} \), which is flexible. In comparison with the rates derived in [10, 24], we see that our rate is faster (\( \alpha \in (0, 0.5) \)), same (\( \alpha = 0.5 \)) and slower (\( \alpha \in (0.5, 1) \)).

**Example 4.2** (Pareto distribution with \( \alpha \in (1, 2) \)). Assume that \( Y_1, Y_2, \ldots, Y_n \) be i.i.d random variables having a Pareto distribution with \( \alpha \in (1, 2) \), i.e.

\[
P(Y_1 > y) = \frac{1}{2|y|^{\alpha}}, \quad y \geq 1, \quad P(Y_1 \leq y) = \frac{1}{2|y|^{\alpha}}, \quad y \leq -1.
\]

Assume also that \( \beta = 0 \) and \( m_1 = m_2 = m \). Denote \( Z_i = n^{-\frac{1}{\beta}}Y_i \) and \( S_n = Z_1 + Z_2 + \ldots + Z_n \). Now, using Theorem 3.19 we show \( d_{W_2}(S_n, X) = O(n^{-\frac{2-\alpha}{\alpha}}) \). Let us first compute the terms in Remainder \( R_{N,n} \).

We have,

\[
2\sum_{i=1}^{n} E \left( |Z_i|^{1/|Z_i|>N} \right) = 2n^{\frac{1}{\alpha}} \left( \int_{n^{\frac{1}{\alpha}}}^{\infty} xp(x)dx + \int_{-\infty}^{n^{\frac{1}{\alpha}}} xp(x)dx \right) = \frac{4}{\alpha - 1} N^{1-\alpha} = D_0 N^{1-\alpha}.
\]

We also have,

\[
\frac{C_{\alpha,m}}{n} \sum_{i=1}^{n} E|Z_i|^{2-\alpha} = \frac{C_{\alpha,m}}{n} \sum_{i=1}^{n} \int_{|x|>1} (n^{-\frac{1}{\beta}}|x|)^{2-\alpha} p(x)dx
\]

\[
= \frac{C_{\alpha,m}}{\alpha - 1} n^{\frac{2-\alpha}{\alpha}} = D_1 n^{\frac{2-\alpha}{\alpha}},
\]

hence

\[
R_{N,n} = D_0 N^{1-\alpha} + D_1 n^{\frac{2-\alpha}{\alpha}}.
\]

For any \( N > 0 \), we have

\[
K_{\nu_\alpha}(t, N) = 1_{[0,N]}(t) \int_{t}^{N} u\nu_\alpha(du) + 1_{[-N,0]}(t) \int_{-N}^{t} (-u)\nu_\alpha(du)
\]

\[
= \frac{m}{1-\alpha} (N^{1-\alpha} - t^{1-\alpha}) + \frac{m}{1-\alpha} (N^{1-\alpha} - (-t)^{1-\alpha})
\]

\[
= \frac{m}{\alpha - 1} (|t|^{1-\alpha} - N^{1-\alpha}).
\]

Using symmetry property of Pareto distribution, we have

\[
K_i(t, N) = \frac{\alpha}{2n(\alpha - 1)} \left( (|t| \wedge n^{-\frac{1}{\alpha}})^{1-\alpha} - N^{1-\alpha} \right)
\]

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Then by Theorem 3.19, we obtain
\[ d_{W_2}(S_n, X) \leq D_0 N^{1-\alpha} + D_1 n^{-\left(\frac{4-\alpha}{2}\right)} + D_2 n^{-\left(\frac{2-\alpha}{2}\right)}, \]
where \( D_0, D_1 \) and \( D_2 \) are positive constants. Since \( N \) is arbitrary, let \( N \to \infty \). Hence, \( d_{W_2}(S_n, X) = O(n^{-\left(\frac{4-\alpha}{2}\right)}) \). By [3, Lemma A.4], we have
\[ d_{W_1}(S_n, X) \leq D_3 \sqrt{n^{-\left(\frac{2-\alpha}{2}\right)}} = D_3 n^{-\left(\frac{1}{2} - \frac{1}{2}\right)}, \]
(40)
where \( D_3 \) is another positive constant. Moreover, \( d_{W_1}(S_n, X) = O(n^{-\left(\frac{1}{2} - \frac{1}{2}\right)}) \).

Let us compare our result with the known results in literature. Johnson and Samworth [23] show a convergence rate for \( \alpha \)-stable approximations in the Mallows distance \( d_r \) with some \( r > 0 \). They have shown that \( S_n = n^{-\frac{\alpha}{2}} \sum_{i=1}^{n} Y_i \) converges to an \( \alpha \)-stable distribution with a rate \( n^{\frac{\alpha}{2} - \frac{1}{2}} \) in the distance \( d_3 \) for some \( \beta \in (\alpha, 2] \). Hence, at \( \beta = 2 \), they show that the convergence rate is at most \( n^{1 - \frac{1}{2}} \). Xu [40] proved that \( S_n \) converges to an symmetric \( \alpha \)-stable distribution with a rate \( n^{\frac{1}{2} - \frac{1}{2}} \) in the Wasserstein-1 distance. From [40, Example 1], it is clear that the convergence rate \( n^{\frac{1}{2} - \frac{1}{2}} \) is not accessible. Note that, we obtain a rate \( n^{\frac{1}{2} - \frac{1}{2}} \) with \( \alpha \in (1, 2) \) in the \( d_{W_2} \) distance, which is faster rate than the rate obtained in [23], whenever \( \beta \in (\alpha, 2) \). Observe also that, at \( \beta = 2 \), the rate obtained in [23, Theorem 1.2] becomes \( n^{\frac{1}{2} - \frac{1}{2}} \). From (40), it immediately follows that the rate \( n^{\frac{1}{2} - \frac{1}{2}} \) is accessible in the \( d_{W_1} \) distance using our estimates.

5 Proofs

5.1 Proof of Theorem 3.1
Recall first that for \( X \sim IDD(\beta, \sigma^2, \nu) \), characteristic exponent \( \eta \) is given by
\[ \eta(z) = \log \phi_X(z) = iz\beta - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{izu} - 1 - izu \mathbb{1}_{|u| \leq 1}(u)) \nu(du), \ z \in \mathbb{R}. \]
(41)
Differentiating (41) with respect to \( z \), we have
\[ \phi_X'(z) = \left( i\beta - \sigma^2 z + i \int_{\mathbb{R}} u(e^{izu} - 1 \mathbb{1}_{|u| \leq 1}(u)) \nu(du) \right) \phi_X(z). \]
(42)
Recall from Section 2, \( F_X \) is the distribution function (cumulative distribution function) of \( X \) and if \( \phi_X' \) exists on \( \mathbb{R} \), then,
\[ \phi_X(z) = \int_{\mathbb{R}} e^{itz} F_X(dx) \text{ and } \phi_X'(z) = i \int_{\mathbb{R}} xe^{itz} F_X(dx), \ z \in \mathbb{R}. \]
(43)
Using (43) in (42) and rearranging the integrals, we have

\[
0 = i \int_R x e^{i \alpha x} F_X(dx) - \left( i (\beta + \int_R u(e^{iu} - 1_{|u| \leq 1}(u))\nu (du)) \phi_X(z) - \sigma^2 z \phi_X(z) \right)
\]

\[
= i \left( \int_R (x - \beta) e^{i \alpha x} F_X(dx) - \left( \int_R u e^{iu} \nu (du) \right) \phi_X(z) + \left( \int_R u 1_{|u| \leq 1}(u)\nu (du) \right) \phi_X(z) - i \sigma^2 z \phi_X(z) \right)
\]

\[
= \left( \int_R (x - \beta) e^{i \alpha x} F_X(dx) - \left( \int_R u e^{iu} \nu (du) \right) \phi_X(z) + \left( \int_R u 1_{|u| \leq 1}(u)\nu (du) \right) \phi_X(z) - i \sigma^2 z \phi_X(z) \right)
\]

The second integral of (44) can be written as

\[
\left( \int_R u e^{iu} \nu (du) \right) \phi_X(z) = \int_R \int_R u e^{iu} e^{i \alpha x} F_X(dx) \nu (du)
\]

\[
= \int_R \int_R u e^{i(u+x)} \nu (du) F_X(dx)
\]

\[
= \int_R \int_R u e^{i \alpha y} \nu (du) F_X(d(y - u))
\]

\[
= \int_R \int_R u e^{i \alpha x} \nu (du) F_X(d(x - u))
\]

\[
= e^{i \alpha x} \int_R u F_X(d(x - u))\nu (du).
\]

(45)

Substituting (44) in (44), we have

\[
0 = \left( \int_R (x - \beta) e^{i \alpha x} F_X(dx) - \int_R u F_X(d(x - u))\nu (du) + \left( \int_R u 1_{|u| \leq 1}(u)\nu (du) \right) \phi_X(z) - i \sigma^2 z \phi_X(z) \right)
\]

\[
= \int_R e^{i \alpha x} \left( (x - \beta) F_X(dx) - \int_R u F_X(d(x - u))\nu (du) + \left( \int_R u 1_{|u| \leq 1}(u)\nu (du) \right) F_X(dx) - i \sigma^2 F_X(dx) \right)
\]

(46)

On applying Fourier transform to (46), multiplying with g \in \mathcal{S}(\mathbb{R})$, and integrating over \mathbb{R}, we get

\[
\int_R g(x) \left( (x - \beta) + \int_R u 1_{|u| \leq 1}(u)\nu (du) \right) F_X(dx) - \int_R u g(x) F_X(d(x - u))\nu (du) - \sigma^2 \int_R g(x) F_X(dx) = 0.
\]

(47)

The third integral of (47) can be seen as

\[
\int_R \int_R u g(x) F_X(d(x - u))\nu (du) = \int_R \int_R u (g(y + u) F_X(dy)\nu (du)
\]

\[
= \int_R \int_R u (g(x + u) F_X(dx)\nu (du)
\]

\[
= \mathbb{E} \left( \int_R u (g(X + u)\nu (du) \right).
\]

(48)
Substituting (48) in (47), we have
\[
E \left( (X - \beta)g(X) - \int_{\mathbb{R}} u(g(X + u) - g(X)1_{\{|u|\leq 1\}}(u))\nu(du) - \sigma^2 g'(X) \right) = 0.
\]
This proves the theorem.

5.2 Proof of Theorem 3.3
Recall first that, for \( X \sim S(\alpha, \beta, m_1, m_2) \), characteristic exponent \( \eta_\alpha \) is given by
\[
\eta_\alpha(z) = \log \phi_\alpha(z) = iz\beta + \int_{\mathbb{R}} (e^{izu} - 1 - izu1_{\{|u|\leq 1\}}(u))\nu_\alpha(du), \quad z \in \mathbb{R}.
\]
Following similar steps to proof of Theorem 3.1, we get the result for \( \alpha \in (0, 1) \cup (1, 2) \).

For \( \alpha = 1 \), as the characteristic exponent \( \eta_1 \) is not differentiable over \( \mathbb{R} \). Let us consider tempered 1-stable random variable \( Y_\gamma \) with characteristic exponent (see, Section 2) given by
\[
\eta_{1,\gamma}(z) = iz\beta + \int_{\mathbb{R}} (e^{izu} - 1 - izu1_{\{|u|\leq 1\}}(u))\nu_{1,\gamma}(du), \quad z \in \mathbb{R},
\]
where \( \gamma \in (0, \infty) \), and \( \nu_{1,\gamma} \) is the Lévy measure defined as
\[
\nu_{1,\gamma}(du) := \left( m_1 e^{-\gamma u}1_{(0,\infty)}(u) + m_2 e^{-\gamma|u|}1_{(-\infty,0)}(u) \right) du.
\]
Observe that \( Y_\gamma \) is infinitely divisible and its characteristic exponent \( \eta_{1,\gamma} \) is differentiable on \( \mathbb{R} \). Also, it can be easily shown that as \( \gamma \to 0^+ \), \( \eta_{1,\gamma} \to \eta_1 \), the characteristic exponent of 1-stable random variable \( X \).

Now, applying Theorem 3.1, we get the Stein identity for \( Y_\gamma \) as follows.
\[
E \left[ (Y_\gamma - \beta)g(Y_\gamma) - \int_{\mathbb{R}} (g(Y_\gamma + u) - g(Y_\gamma)1_{\{|u|\leq 1\}}(u))\nu_{1,\gamma}(du) \right] = 0, \quad g \in S(\mathbb{R}).
\]
(49)

Now, taking limit as \( \gamma \to 0^+ \), (49) reduces to
\[
E \left[ (X - \beta)g(X) - \int_{\mathbb{R}} (g(X + u) - g(X)1_{\{|u|\leq 1\}}(u))\nu_{1}(du) \right] = 0, \quad g \in S(\mathbb{R}).
\]
This proves the theorem.

5.3 Proof of Theorem 3.12
For the proof of this theorem, we use the connection between the operators \( A^\alpha_X \) and \( T^\alpha \).
Proof of (i). The proof of this part is split into two parts.
where \( P \) is the solution to (24). Now, it remains to show that, \( \tilde{g}_h^\alpha \) is well-defined. Let us first consider \( \tilde{g}_h^\alpha : \mathbb{R} \to \mathbb{R} \) be defined as

\[
\tilde{g}_h^\alpha(x) = -\int_0^\infty (P_t^\alpha(h)(x) - \mathbb{E}h(X)) \, dt, \quad h \in \mathcal{H}_\delta, \delta \in (0, \alpha),
\]

where \( P_t^\alpha \) is the semigroup as defined in (18). We show that, for any \( h \in \mathcal{H}_\delta, 0 < \delta < \alpha, \tilde{g}_h^\alpha \) is well-defined.

Using (19), we have

\[
|P_t^\alpha(h)(x) - \mathbb{E}h(X)| = \left| \int_\mathbb{R} h(r + e^{-t}x)F_{X(t)}(dr) - \int_\mathbb{R} h(r)F_X(dr) \right|
\leq \min \left\{ e^{-t|x|}, (e^{-t|x|})^\delta \right\} + \left| \int_\mathbb{R} \tilde{h}(z) (\phi_t(z) - \phi_\alpha(z)) \, dz \right|
\leq \min \left\{ e^{-t|x|}, (e^{-t|x|})^\delta \right\} + \int_\mathbb{R} \tilde{h}(z) |\phi_t(z) - \phi_\alpha(z)| \, dz
\]

Now, let us calculate an upper bound between the difference of two characteristic functions \( \phi_t \) and \( \phi_\alpha \). For all \( t > 0 \) and \( z \in \mathbb{R} \),

\[
|\phi_t(z) - \phi_\alpha(z)| = \left| \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} - \phi_\alpha(z) \right| \leq |\phi_\alpha(e^{-t}z) - 1| = |e^{\omega_t(z)} - 1|,
\]

where \( \omega_t = iz\beta_1 e^{-t} + e^{-ta} \int_{\mathbb{R}} (e^{izu} - 1) \nu_\alpha(du), \) and \( \beta_1 = \beta - \int_{\{|u|\leq 1\}} u \nu_\alpha(du). \) Now, from (32) Lemma 7.9, the function \( z \mapsto e^{s\omega_t(z)} \) is a characteristic function for all \( s \in (0, \infty) \), thus, for all \( z \in \mathbb{R} \) and \( t > 0 \)

(a) \( \alpha \in (0, 1) \): We have

\[
A_X^h g_0^\alpha(x) = (-x + \beta)g_0^\alpha(x) + \int_\mathbb{R} (g_0^\alpha(x + u) - g_0^\alpha(x))1_{\{|u|\leq 1\}}(u) \, w_\alpha(du)
= \mathcal{T}_\alpha(\tilde{g}_h^\alpha)(x), \quad \text{(where} \quad \tilde{g}_h^\alpha(x) = -\int_0^\infty (P_t^\alpha(h)(x) - \mathbb{E}h(X)) \, dt, \ h \in \mathcal{H}_\delta, \delta \in (0, \alpha))
= -\int_0^\infty \mathcal{T}_\alpha P_t^\alpha(h)(x) \, dt
= -\int_0^\infty \frac{d}{dt} P_t^\alpha(h)(x) \, dt
= P_0 h(x) - P_\infty h(x)
= h(x) - \mathbb{E}h(X) \quad \text{(by Proposition 3.8).}
\]
\[|\phi_t(z) - \phi_0(z)| \leq \left| \int_0^1 \frac{ds}{ds} \left( \exp(s\omega_t(z)) \right) \right| \]

\[\leq |\omega_t(z)| \]

\[\leq |z||\beta_1|e^{-t} + e^{-\alpha t} \int_\mathbb{R} \left( e^{izu} - 1 \right) \nu_\alpha(du) \]

\[= |z||\beta_1|e^{-t} + e^{-\alpha t} \int_{\{\alpha>1\}} \left( e^{izu} - 1 \right) \nu_\alpha(du) + \int_{\{\alpha\leq1\}} \left( e^{izu} - 1 \right) \nu_\alpha(du) \]

\[\leq |z||\beta_1|e^{-t} + 2e^{-\alpha t} \int_{\{\alpha>1\}} \nu_\alpha(du) \]

\[+ e^{-\alpha t} \left( \int_{\{\alpha\leq1\}} (1 - e^{izu}) \nu_\alpha(du) \right) \]

\[= |z||\beta_1|e^{-t} + 2m_1 + m_2 e^{-\alpha t} + |z|^\alpha (M_1 + M_2) e^{-\alpha t}, \quad (51)\]

where \(M_1 = \int_{\{\alpha\leq1\}} (cos v - 1) \nu_\alpha(du)\) and \(M_2 = \int_{\{\alpha\leq1\}} \nu v \nu_\alpha(du)\).

Using (51) in (50), one can easily show that \(\int_0^\infty |P^{\alpha}_h(h)(x) - \mathcal{E}h(X)| dt < \infty\). Hence, \(\tilde{g}^{\alpha}_h\) is well-defined.

By dominated convergence theorem, we see that \(\tilde{g}^{\alpha}_h\) is differentiable and

\[ (\tilde{g}^{\alpha}_h)'(x) = - \lim_{\zeta \to \infty} \frac{d}{dx} \left( \int_0^\infty \left( P^{\alpha}_t(h)(x) - \mathcal{E}h(X) \right) dt \right) \]

\[= - \lim_{\zeta \to \infty} \int_0^\zeta \frac{d}{dx} \left( \int_\mathbb{R} h(xe^{-t} + u) F_{X_t}(du) \right) dt \]

\[= - \int_0^\infty e^{-t} \int_\mathbb{R} h'(u + xe^{-t}) F_{X_t}(du) dt = g^{\alpha}_h(x), \quad (52)\]

the desired conclusion follows.

(b) \(\alpha = 1\): To solve (24) for \(\alpha = 1\), consider a Stein equation (see, (49)) for tempered 1-stable random variable \(Y_\gamma\) is given by

\[(-x + \beta) g(x) + \int_\mathbb{R} \left( g(x + u) - g(x) \right) 1_{\{|u|\leq1\}}(u) \nu_{1,\gamma}(du) = h(x) - \mathcal{E}h(Y_\gamma), \quad h \in \mathcal{H}_\delta. \quad (53)\]

Following similar steps to proof as the Case 1 of (i), and using (24), we see that the function \(g^{1,\gamma}_h(x) = - \int_0^\infty e^{-t} \int_\mathbb{R} h'(u + xe^{-t}) F_{Y_{\gamma,t}}(du) dt\) solves (52), i.e.

\[(-x + \beta) g^{1,\gamma}_h(x) + \int_\mathbb{R} \left( g^{1,\gamma}_h(x + u) - g^{1,\gamma}_h(x) \right) 1_{\{|u|\leq1\}}(u) \nu_{1,\gamma}(du) = h(x) - \mathcal{E}h(Y_\gamma). \quad (53)\]
Observe that,

\[
\lim_{\gamma \to 0^+} g_h^{1,\gamma}(x) = - \lim_{\gamma \to 0^+} \int_0^\infty e^{-t} \int_\mathbb{R} h'(u + xe^{-t}) F_Y(v, t) \, du \, dt \\
= - \int_0^\infty e^{-t} \int_\mathbb{R} h'(u + xe^{-t}) F_X(v) \, du \, dt \\
= g_h^1(x).
\]

Also,

\[
\lim_{\gamma \to 0^+} Eh(Y_\gamma) = Eh(X), \quad \text{(since } Y_\gamma \xrightarrow{d} X).\]

Hence taking limit as \( \gamma \to 0^+ \) on \((53)\), we get

\[
(-x + \beta) g_h^1(x) + \int_\mathbb{R} (g_h^1(x + u) - g_h^1(x) 1_{\{|u| \leq 1\}}(u)) u \nu_1(du) = h(x) - Eh(X).
\]

Hence, \( g_h^1 \) is the solution to \((24)\) when \( \alpha = 1 \). Note here that, on careful adjustments of the integrals and suitably adjustments of parameters as previous case, one can verify that the function \( ˜g_h^1(x) = - \int_0^\infty (P_t^1(h)(x) - Eh(X)) \, dt, \quad h \in \mathcal{H}_\delta, \delta \in (0, 1) \) is well-defined and \( ( ˜g_h^1)'(x) = g_h^1(x) \) for all \( x \in \mathbb{R} \).

**Proof of (ii).** Following similar steps to proof of Case 1 of (i), it immediately shows that \( g_h^\alpha \) (where \( \alpha \in (1, 2) \) and \( h \in \mathcal{H}_2 \)) is the solution to \((20)\). So, it remains to show that \( g_h^\alpha \) is well-defined. Let us consider a function \( ˜g_h^\alpha : \mathbb{R} \to \mathbb{R} \) defined as

\[
( ˜g_h^\alpha)(x) = - \int_0^\infty (P_t^\alpha(h)(x) - Eh(X)) \, dt, \quad h \in \mathcal{H}_2,
\]

where \( P_t^\alpha \) is the semigroup as defined in \((18)\). Now, we show that for any \( h \in \mathcal{H}_2 \) and \( \alpha \in (1, 2) \), \( ˜g_h^\alpha \) is well-defined and \( ( ˜g_h^\alpha)'(x) = g_h^\alpha(x) \) for all \( x \in \mathbb{R} \).

Using \((19)\), we have

\[
|P_t^\alpha(h)(x) - Eh(X)| = \left| \int_\mathbb{R} h(r + e^{-t}x) F_X(v) \, dr - \int_\mathbb{R} h(r) F_X(dr) \right| \\
= \left| \int_\mathbb{R} (h(r + e^{-t}x) - h(r)) F_X(v) \, dr + \int_\mathbb{R} h(r) F_X(v) \, dr - \int_\mathbb{R} h(r) F_X(dr) \right| \\
\leq e^{-t} |x| |h'| + \left| \int_\mathbb{R} ˜h(z) (\phi_t(z) - \phi_\alpha(z)) \, dz \right| \\
\leq e^{-t} |x| |h'| + \int_\mathbb{R} | ˜h(z) (\phi_t(z) - \phi_\alpha(z)) | \, dz.
\]

Now, let us calculate an upper bound between the difference of two characteristic functions \( \phi_t \) and \( \phi_\alpha \). For all \( t > 0 \) and \( z \in \mathbb{R} \),

\[
|\phi_t(z) - \phi_\alpha(z)| = \left| \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-tz})} - \phi_1(z) \right| \leq |\phi_\alpha(e^{-tz}) - 1| = |e^{\alpha t} - 1|,
\]

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where \( \omega_t(z) = e^{-t\alpha}(iz\tilde{\beta} + \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{|u|\leq 1})\nu_\alpha(du)), \tilde{\beta} = \beta e^{(\alpha-1)t} + \int_{\mathbb{R}}(u\mathbf{1}_{|u|\leq 1} - u\mathbf{1}_{|u|\leq e^{-t}})\nu_\alpha(du). \) 

Note that the function \( z \to e^{s\omega_t(z)} \) is a characteristic function for all \( s \in (0, \infty) \). Indeed, \( e^{s\omega_t(z)} \) is a characteristic function of an \( \alpha \)-stable random variable with different parameters. Thus, for all \( z \in \mathbb{R} \) and \( t > 0, \)

\[
|\phi_t(z) - \phi_\alpha(z)| \leq \int_0^1 \frac{d}{ds}(\exp(s\omega_t(z)))ds \leq e^{-t\alpha} \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{|u|\leq 1})\nu_\alpha(du) \leq C_\alpha e^{-t\alpha}(1 + |z|^2), \quad C_\alpha > 0, \tag{55}
\]

where the last inequality is followed by \( [2] \) p.30, Ex. 1.2.16. Using (55) in (54), one can easily show that \( \int_0^\infty |P_t^n(h)(x) - Eh(X)|dt < \infty. \) Hence, \( \tilde{g}_h^n(x) \) is well-defined. The rest of this part follows from similar computations as Case 1 of (i).

### 5.4 Proof of Theorem 3.14

Recall the definition of \( (P_t^n)_{t\geq 0} \),

\[
P_t^n(g)(x) = \int_{\mathbb{R}} g(r + e^{-t}x)F_{X_{(t)}}(dr), \quad g \in \mathcal{F},
\]

where \( \alpha \in (0, 2) \) and \( F_{X_{(t)}} \) is the distribution function of \( X_{(t)} \) (see, (14)).

**Proof of (i).** Suppose \( \alpha \in (0, 1) \) and \( h \in \mathcal{H}_\delta, \delta \in (0, \alpha). \)

Let

\[
g_h^n(x) = -\int_0^\infty e^{-t} \int_{\mathbb{R}} h'(xe^{-t} + u)F_{X_{(t)}}(du)dt.
\]

It is clear that

\[
\|g_h^n\| \leq \int_0^\infty e^{-t}dt \int_{\mathbb{R}} \|h'\|F_{X_{(t)}}(du) \leq \|h'\|
\]

the desired conclusion follows.

Now observe that, for any \( x, y \in \mathbb{R} \) and \( h \in \mathcal{H}_\delta, \)

\[
|g_h^n(x) - g_h^n(y)| \leq \int_0^\infty e^{-t} \int_{\mathbb{R}} [h'(xe^{-t} + z) - h'(ye^{-t} + z)]F_{X_{(t)}}(dz)dt
\]

\[
\leq \int_0^\infty e^{-t} \int_{\mathbb{R}} |x - y|^\delta e^{-t\delta}F_{X_{(t)}}(dz)dt
\]

\[
= |x - y|^\delta \int_0^{\infty} e^{-t(1+\delta)}dt
\]

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Now observe that

\[ |g_h^\alpha(x) - g_h^\alpha(y)| \leq \frac{1}{1 + \delta} |x - y|^\delta, \]

the desired conclusion follows.

For \( \alpha \in (0, 1) \), we have

\[
\int_{\mathbb{R}} ug_h^\alpha(x + u) \nu_\alpha(du) = \alpha \int_{\mathbb{R}} \int_0^\infty (P_t^\alpha h(x + u) - P_t^\alpha h(x))dt \nu_\alpha(du) \quad \text{(using Proposition A.4)}
\]

\[
\leq \alpha \int_{|u|>1} \int_0^\infty (P_t^\alpha h(x + u) - P_t^\alpha h(x))dt \nu_\alpha(du) + \alpha \int_{|u|\leq1} \int_0^\infty (P_t^\alpha h(x + u) - P_t^\alpha h(x))dt \nu_\alpha(du)
\]

\[
:= I + II.
\]

Now observe that

\[
I = \alpha \int_{|u|>1} \int_0^\infty (P_t^\alpha h(x + u) - P_t^\alpha h(x))dt \nu_\alpha(du)
\]

\[
= \alpha \int_{|u|>1} \int_0^\infty (h((x + u)e^{-t} + y) - h(xe^{-t} + y)) F_{X(t)}(dy)dt \nu_\alpha(du)
\]

\[
\leq \alpha \int_{|u|>1} |u|^\delta \int_0^\infty e^{-t^\delta} dt \nu_\alpha(du)
\]

\[
= \frac{\alpha(m_1 + m_2)}{\delta(\alpha - \delta)}. \tag{56}
\]

Consider,

\[
II = \alpha \int_{|u|\leq1} \int_0^\infty (P_t^\alpha h(x + u) - P_t^\alpha h(x))dt \nu_\alpha(du)
\]

\[
= \alpha \int_{|u|\leq1} \int_0^\infty (h((x + u)e^{-t} + y) - h(xe^{-t} + y)) F_{X(t)}(dy)dt \nu_\alpha(du)
\]

\[
\leq \alpha \int_{|u|\leq1} |u| \int_0^\infty e^{-t} dt \nu_\alpha(du)
\]

\[
\leq \frac{\alpha(m_1 - m_2)}{1 - \alpha}. \tag{57}
\]

Hence, by (56) and (57), we get

\[
|A_0g_h^\alpha(x)| \leq \frac{\alpha(m_1 + m_2)}{\delta(\alpha - \delta)} + \frac{\alpha(m_1 - m_2)}{1 - \alpha} := C_{\alpha, \delta, m_1, m_2},
\]

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the desired conclusion follows.

**Proof of (ii).** The proofs of first two properties are similar to previous case. To prove the third property, we split $g^1_h$ in terms of the semigroup $P^1_t$ defined in (18). We write

$$A_1g^1_h := \int_{\mathbb{R}} u \left( g^1_h(x+u) - g^1_h(x) \right) \nu_1(du)$$

$$= \int_{\{|u| \leq 1\}} u \left( g^1_h(x+u) - g^1_h(x) \right) \nu_1(du) + \int_{\{|u| > 1\}} u g^1_h(x+u) \nu_1(du)$$

$$= \int_{\{|u| \leq 1\}} u \int_0^\infty \left( e^{-t} \left( \int_{\mathbb{R}} h'(x+u)e^{-t} + y) - h'(x e^{-t} + y) \right) F_{X(t)}(dy) \right) dt \nu_1(du)$$

$$+ \int_{\{|u| > 1\}} \int_0^\infty (P^1_t h(x+u) - P^1_t h(x)) dt \nu_1(du) \text{ (using Fubini's theorem)}$$

$$:= I + II.$$

By similar computation as Case 1 of (i), it is easy to show that

$$I = \int_{\{|u| \leq 1\}} u \int_0^\infty \left( e^{-t} \left( \int_{\mathbb{R}} h'(x+u)e^{-t} + y) - h'(x e^{-t} + y) \right) F_{X(t)}(dy) \right) dt \nu_1(du)$$

$$\leq \frac{1}{1+\delta} \int_{\{|u| \leq 1\}} |u|^{1+\delta} \nu_1(du) = \frac{m_1 + m_2}{\delta(1+\delta)}. \tag{58}$$

Consider,

$$II = \int_{\{|u| > 1\}} \int_0^\infty (P^1_t h(x+u) - P^1_t h(x)) dt \nu_1(du)$$

$$\leq \frac{1}{\delta} \int_{\{|u| > 1\}} |u|^\delta \nu_1(du) = \frac{m_1 + m_2}{\delta(1-\delta)}. \tag{59}$$

Hence, by (58) and (59), we get

$$\|A_1g^1_h\| \leq \frac{2(m_1 + m_2)}{\delta(1-\delta^2)} := C_{1,\delta,m_1,m_2},$$

the desired conclusion follows.

**Proof of (iii).** Suppose $\alpha \in (1,2)$ and $h \in H_2$.

Let

$$g^\alpha_h(x) = -\int_0^\infty e^{-t} \int_{\mathbb{R}} h'(x e^{-t} + u) F_{X(t)}(du) dt.$$

Then,

$$(g^\alpha_h)'(x) = -\int_0^\infty e^{-2t} \int_{\mathbb{R}} h'(x e^{-t} + u) F_{X(t)}(du) dt.$$
It is also easy to show that
\[ \| (g^\alpha_h)_1 \| \leq \| h' \|, \text{ and } \| (g^\alpha_h)_2 \| \leq \frac{1}{2} \| h'' \|. \]

Let \( m_1 = m_2 = m \). Let \( A_2 g^\alpha_h(x) = \int_{\mathbb{R}} (g^\alpha_h(x + u) - g^\alpha_h(x)) \nu_\alpha(du) \). Then, for any \( x, y \in \mathbb{R} \)
\[ |A_2 g^\alpha_h(x) - A_2 g^\alpha_h(y)| \leq \int_{\mathbb{R}} |(g^\alpha_h(x + u) - g^\alpha_h(y + u) - (g^\alpha_h(x) - g^\alpha_h(y))| |u| \nu_\alpha(du) \]
\[ = m \left( \int_{|u| > |x-y|} |(g^\alpha_h(x + u) - g^\alpha_h(y + u))| |u| \nu_\alpha(du) \right) \]
\[ =: I + II \]

Now observe that
\[ I = m \int_{|u| > |x-y|} |(g^\alpha_h(x + u) - g^\alpha_h(y + u)) - (g^\alpha_h(x) - g^\alpha_h(y))| \frac{du}{|u|^\alpha} \]
\[ \leq m \int_{|u| > |x-y|} |(g^\alpha_h(x + u) - g^\alpha_h(y + u) + |(g^\alpha_h(x) - g^\alpha_h(y))| |u| \nu_\alpha(du) \]
\[ \leq 4m \| (g^\alpha_h)' \| |x-y| \int_{|x-y|} u^{-\alpha} du \]
\[ \leq 2m \| h'' \| \frac{|x-y|^{2-\alpha}}{\alpha - 1}. \quad (60) \]

Consider,
\[ II = m \int_{|u| < |x-y|} |(g^\alpha_h(x + u) - g^\alpha_h(y + u)) - (g^\alpha_h(x) - g^\alpha_h(y))| \frac{du}{|u|^\alpha} \]
\[ \leq m \int_{|u| < |x-y|} |(g^\alpha_h(x + u) - g^\alpha_h(x)) + |(g^\alpha_h(y + u) - g^\alpha_h(y))| |u| \nu_\alpha(du) \]
\[ \leq 4m \| (g^\alpha_h)' \| \int_{0}^{\frac{|x-y|}{2}} u^{1-\alpha} du \]
\[ \leq 2m \| h'' \| \frac{|x-y|^{2-\alpha}}{2 - \alpha}. \quad (61) \]

Hence, by (60) and (61), we get
\[ \| A_2 g^\alpha_h(x) - A_2 g^\alpha_h(y) \| \leq C_{\alpha,m} \| h'' \| |x-y|^{2-\alpha}, \text{ where } C_{\alpha,m} = 2m \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right). \]
5.5 Proof of Theorem 3.17

Recall that $(Y_n)_{n \geq 1}$ is a sequence of i.i.d. random variables such that $Y_1 \in D_\alpha$. Let us denote

$$S_n = Y_1 + Y_2 + \ldots + Y_n,$$
$$S_{n,i} = S_n - n^{-\alpha}Y_i.$$

Note that, $S_{n,i}$ and $Y_i$ are independent. To prove this theorem, we first derive some lemmas. With the help of these lemmas, we obtain our bounds in the $d_{W_k}$ distance for $\alpha$-stable approximations with $\alpha \in (0, 1]$.

5.5.1 Proof of (a).

To prove this part of Theorem 3.17 we use the following lemmas. Recall the Lévy measure $\nu_\alpha$ for $\alpha$-stable distributions is given by $\nu_\alpha(du) = \left( m_1 \frac{1}{|u|^\alpha} 1_{(0,\infty)}(u) + m_2 \frac{1}{|u|^\alpha} 1_{(-\infty,0)}(u) \right) du$, where $m_1, m_2 \in [0, \infty)$, $m_1 + m_2 > 0$ and $\alpha \in (0, 2)$.

**Lemma 5.1.** Let $\alpha \in (0, 1)$. Let $g^\alpha_h$ is defined in (25). Then, for any $a > 0$,

$$\int_R u g^\alpha_h(x + u) \nu_\alpha(du) = a^{1-\alpha} \int_R u g^\alpha_h(x + au) \nu_\alpha(du).$$

**Proof.** We write

$$\int_R u g^\alpha_h(x + u) \nu_\alpha(du) = \int_R u g^\alpha_h(x + u) \frac{m_1 1_{(0,\infty)}(u) + m_2 1_{(-\infty,0)}(u)}{|u|^{\alpha+1}} du$$

$$= a^{1-\alpha} \int_R u g^\alpha_h(x + au) \frac{m_1 1_{(0,\infty)}(u) + m_2 1_{(-\infty,0)}(u)}{|u|^{\alpha+1}} du$$

$$= a^{1-\alpha} \int_R u g^\alpha_h(x + au) \nu_\alpha(du),$$

the desired conclusion follows.

**Lemma 5.2.** Let $\alpha \in (0, 1)$. Let $Y \in D_\alpha$ and $g^\alpha_h$ is defined in (25). Then, for $0 < a < 1$ and $z \in \mathbb{R}$,

$$\mathbb{E} \left( \left| \int_R u \left( g^\alpha_h(z + aY + u) - g^\alpha_h(z + u) \right) \nu_\alpha(du) \right| \right) \leq C_{A,K}^{\alpha,1} \alpha, m_1, m_2 a^\alpha.$$

**Proof.** We write

$$\mathbb{E} \left( \left| \int_R u \left( g^\alpha_h(z + aY + u) - g^\alpha_h(z + u) \right) \nu_\alpha(du) \right| \right) := I + II,$$

where

$$I := \mathbb{E} \left( \left| \int_R u \left( g^\alpha_h(z + aY + u) - g^\alpha_h(z + u) \right) \nu_\alpha(du) \right| 1_{|Y| > a^{-1}} \right),$$

$$II := \mathbb{E} \left( \left| \int_R u \left( g^\alpha_h(z + aY + u) - g^\alpha_h(z + u) \right) \nu_\alpha(du) \right| 1_{|Y| \leq a^{-1}} \right).$$
For $\alpha \in (0, 1)$, one can write by (30) and (36),
\[
I \leq 2C_{\alpha, \delta, m_1, m_2} \mathbb{P}(|Y| \geq a^{-1})
\leq 4C_{\alpha, \delta, m_1, m_2} \left( A + \sup_{|y| \geq a^{-1}} |e(y)| \right) a^\alpha
\leq 4C_{\alpha, \delta, m_1, m_2}(A + K)a^\alpha.  \tag{62}
\]

It is also easy to show that
\[
II \leq C_{\alpha}a^\alpha.  \tag{63}
\]

Hence, by (62) and (63), we have
\[
\mathbb{E} \left( \left| \int_{\mathbb{R}} u (g_h^\alpha(z + aY + u) - g_h^\alpha(z + u)) \nu_\alpha(du) \right| \right) \leq (4C_{\alpha, \delta, m_1, m_2} + C_{\alpha}) a^\alpha
\leq C_{\alpha, \delta, m_1, m_2, K} a^\alpha,
\]
the desired conclusion follows.

Recall the definition of $D_\alpha$ in Definition 3.16. We see that the function $e$ satisfies certain conditions with the domain $\{|y| > L\}$. These conditions play an important role for proving the following lemma.

**Lemma 5.3.** Let $\alpha \in (0, 1)$. Let $Y \in D_\alpha$ and $X$ be a random variable with finite $\delta$-th moment, which is independent of $Y$. For any $0 < a < \frac{1}{L}$ and $g_h^\alpha$ defined in (25), define
\[
J_1 := \left| \mathbb{E}(Y g_h^\alpha(X + aY)) - \mathbb{E}(Y 1_{(-1,1)}(aY)) \mathbb{E}(g_h^\alpha(X)) \right|.
\]

Then,
\[
J_1 \leq C_{1, \delta, L} a^\delta + C_{2, \delta} a^\delta \sup_{L < |y| < \frac{1}{a}} (\alpha |e(y)| + |y e'(y)|) \int_{L < |y| < \frac{1}{a}} |y|^{\delta - \alpha} dy
\]
\[
+ 2a^{\alpha - 1} \sup_{|y| > a^{-1}} (\alpha |e(y)| + |y e'(y)|) \int_{|y| > 1} \eta_{\alpha, \delta, m_1, m_2}(y)|y|^{-1 - \alpha} dy,
\]
where $C_{1, \delta, L} = \frac{2}{1+\delta} L^{1+\delta}$ and $C_{2, \delta} = \frac{2}{1+\delta}$.

**Proof.** We have by (36),
\[
J_1 = \left| \mathbb{E} \left( \int_{\mathbb{R}} (yg_h^\alpha(X + ay) - y 1_{(-1,1)}(ay) g_h^\alpha(X)) dF_Y(y) \right) \right|.
\]

Since $e$ is in $C^2$, for any $|y| > L$,
\[
dF_Y(y) = \frac{A \alpha + \alpha e(y) - ye'(y)}{|y|^{1+\alpha}} \kappa_\theta(y) dy,
\]
where $\kappa_\theta(y) = (1 + \theta)1_{(0,\infty)}(y) + (1 - \theta)1_{(-\infty,0)}(y)$.
Thus, we have

\[ \mathcal{J}_1 \leq E \int_{|y|<L} |y| |g_h^\alpha(X + ay) - g_h^\alpha(X)| dF_Y(y) + \frac{E}{\int_{|y|<\frac{L}{\alpha}}} |y| |g_h^\alpha(X + ay) - g_h^\alpha(X)| dF_Y(y) + \frac{E}{\int_{|y|>|y|<\frac{L}{\alpha}}} |yg_h^\alpha(X + ay)| dF_Y(y) := I + II + III. \]

It is easy to verify by (29),

\[ I \leq \frac{1}{1 + \delta} \frac{a^\delta}{\int_{|y|<L} |y| dF_Y(y)} \leq \frac{2}{1 + \delta} L^{1 + \delta} a^\delta, \]

and

\[ II \leq \frac{2a^\delta}{1 + \delta} \sup_{L<|y|<\frac{L}{\alpha}} \left( |\alpha e(y) - ye'(y)| \int_{|y|<L} |y|^{-\alpha - \delta} dy \right) \leq \frac{2a^\delta}{1 + \delta} \int_{|y|<\frac{L}{\alpha}} |y|^{\delta - \alpha} |ye'(y)| dy \leq \frac{2a^\delta}{1 + \delta} \int_{L<|y|<\frac{L}{\alpha}} (|\alpha e(y)| + |ye'(y)|) \int_{|y|<L} |y|^{\delta - \alpha} dy. \]

For the third term, we have,

\[ III \leq \frac{2a^\alpha}{1 + \delta} \sup_{|y|>\frac{L}{\alpha}} \left( |\alpha e(y)| + |ye'(y)| \right) \int_{|y|>\frac{L}{\alpha}} |yg_h^\alpha(X + ay)| |y|^{-\alpha - 1} dy \leq \frac{2a^\alpha}{1 + \delta} \sup_{|y|>\frac{L}{\alpha}} (|\alpha e(y)| + |ye'(y)|) \int_{|y|>1} \eta_{\alpha,\beta,\delta,m_1,m_2}(y)|y|^{-1 - \alpha} dy, \]

where the last inequality follows by Lemma 6.1 and Proposition A.5. Combining the estimates obtained in I, II and III, the desired conclusion follows.

**Proof of (a).** With the help of above lemmas, we now find bound in the \( d_{W^\alpha} \) distance for \( \alpha \)-stable approximation with \( \alpha \in (0,1) \).

By (16), we have

\[ |E[h(S_n) - h(X)| = \left| E\left[ -S_n g_h^\alpha(S_n) + \beta_1 g_h^\alpha(S_n) + \int_{\mathbb{R}} g_h^\alpha(S_n + u)w_\alpha(du) \right] \right| \leq I + II + III \]
where,

\[
I := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left| \int_{\mathbb{R}} u \left( g_h^n(S_{n,i} + n^{-\frac{1}{\alpha}} Y_i + u) - g_h^n(S_{n,i} + u) \right) \nu_{\alpha}(du) \right|
\]

\[
II := n^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \mathbb{E} \left| -Y_i g_h^n(S_n) + Y_i 1_{(-1,1)}(\lfloor n^{-\frac{1}{\alpha}} Y_i \rfloor) \mathbb{E} g_h^n(S_{n,i}) \right|
\]

\[
III := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left| \int_{\mathbb{R}} u g_h^n(S_{n,i} + u) \nu_{\alpha}(du) + \beta_1 g_h^n(S_n) - n^{1-\frac{1}{\alpha}} Y_i 1_{(-1,1)}(\lfloor n^{-\frac{1}{\alpha}} Y_i \rfloor) \mathbb{E} g_h^n(S_{n,i}) \right|
\]

For \( \alpha \in (0, 1) \), we have by Lemma 5.2 with \( a = n^{-\frac{1}{\alpha}} \),

\[
I \leq \frac{C_{\alpha, \delta, m_1, m_2}^{A}}{n}
\]

By Lemma 5.3 with \( a = n^{-\frac{1}{\alpha}} \), we have

\[
II \leq C_{1, \delta, L} n^{1-\frac{(1+\delta)}{\alpha}} + C_{2, \delta} n^{1-\frac{(1+\delta)}{\alpha}} \sup_{L<|y|<n^\frac{1}{\alpha}} \left( \alpha |e(y)| + |y e'(y)| \right) \int_{L<|y|<n^\frac{1}{\alpha}} |y|^{\delta-\alpha} dy
\]

\[
+ 2 \sup_{|y|>n^\frac{1}{\alpha}} \left( \alpha |e(y)| + |y e'(y)| \right) \int_{|y|>1} \eta_{\alpha, \beta, \delta, m_1, m_2}(y) |y|^{-1-\alpha} dy.
\]

Using Lemma 5.1 with \( a = n^{-\frac{1}{\alpha}} \), we have

\[
III \leq C_{\alpha, \delta, m_1, m_2} n^{-\frac{(1-a)}{\alpha}} + \beta_1 + n^{-\frac{(1-a)}{\alpha}} \int_{|y|<n^\frac{1}{\alpha}} |y| dF_{\gamma}(y).
\]

Combining the estimates obtained in I, II and III, the desired conclusion follows.

**5.5.2 Proof of (b).**

The following two lemmas play an important role for finding bound in the \( d_{W_\delta} \) distance for 1-stable approximation. Recall that \( A_1 g_h^1(x) := \int_{\mathbb{R}} u (g_h^1(x + u) - ug_h^1 1_{\{|u|\leq 1\}}) \nu_1(du) \), where \( g_h^1 \) is defined in (25) and \( \nu_1 \) is the Lévy measure (see [4]).

**Lemma 5.4.** Let \( \alpha = 1 \). Let \( Y \in D_\alpha \) and \( g_h^1 \) is defined in (25). Then, for any \( 0 < a < 1 \) and \( z \in \mathbb{R} \),

\[
\mathbb{E} \left( |A_1 g_h^1(z) - A_1 g_h^1(z + aY)| \right) \leq C_{1, \delta, m_1, m_2}^{A,K} a + 2C_{1, \delta, m_1, m_2} \int_0^\frac{1}{n} dF_{\gamma}(y),
\]

where \( C_{1, \delta, m_1, m_2}^{A,K} \) and \( C_{1, \delta, m_1, m_2} \) are constants.
Proof. We write

\[ \mathbb{E} \left( |A_1g_h^1(z) - A_1g_h^1(z + aY)| \right) := I + II, \]

where

\[ I := \mathbb{E} \left( |A_1g_h^1(z) - A_1g_h^1(z + aY)| 1_{|y| > \frac{1}{a}} \right), \]

\[ II := \mathbb{E} \left( |A_1g_h^1(z) - A_1g_h^1(z + aY)| 1_{|y| \leq \frac{1}{a}} \right). \]

When \( \alpha = 1 \), one can write by (33) and (36),

\[ I \leq 2C_{1, \delta, m_1, m_2} P(|Y| \geq a^{-1}) \]

\[ \leq 4C_{1, \delta, m_1, m_2} (A + \sup_{|y| \geq a^{-1}} |e(y)|) a \]

\[ \leq 4C_{1, \delta, m_1, m_2} (A + K)a, \]

and

\[ II \leq 2C_{1, \delta, m_1, m_2} P(|Y| < \frac{1}{a}) \]

\[ = C_{1, \delta, m_1, m_2} \int_0^\frac{1}{a} dF_Y(y), \]

Combining the estimates obtained in I and II, the desired conclusion follows.

Lemma 5.5. Let \( \alpha = 1 \). Let \( Y \in D_\alpha \) and \( X \) be a random variable with \( \delta \)-th finite moment such that \( X \) and \( Y \) are independent. For any \( 0 < a < \frac{1}{T} \), define

\[ J_2 := \left| \mathbb{E} \left( Yg_h^1(X + aY) \right) - \mathbb{E} \left( Y1_{(-1,1)}(aY) \right) \mathbb{E} \left( g_h^1(X) \right) - \mathbb{E} \left( A_0g_h^1(X) \right) \right|. \]

Then,

\[ J_2 \leq \frac{1}{\delta + 1} a^\delta (L^2 + m_1 + m_2) + \frac{a^\delta}{1 + \delta} \int_{L<|u|<\frac{1}{a}} \frac{|e(u) - u e'(u)|}{|u|^{1-\delta}} du + a \int_{|u|>1} \frac{|e(u/a) - u/a e'(u/a)|}{|u|} du. \]

Proof. We have

\[ \mathbb{E} A_1g_h^1(X) = \mathbb{E} \int_R u \left( g_h^1(X + u) - g_h^1(X) 1_{|u| \leq 1}(u) \right) \nu_1(du) \]

\[ = \mathbb{E} \int_R u \left( g_h^1(X + aY) - g_h^1(X) 1_{|ay| \leq 1}(ay) \right) \nu_1(du) \]

and
\[ \mathbb{E}(Yg_h(X+au)) - \mathbb{E}(Y\mathbf{1}_{(-1,1)}(au)) \mathbb{E}(g_h(X)) \]
\[ = \mathbb{E}\left( \int_{\mathbb{R}} (ug_h(X+au) - u\mathbf{1}_{(-1,1)}(au)g_h(X)) \ dF_Y(u) \right). \]

Since \( e \) is \( C^2 \), for any \(|y| > L\)
\[ dF_Y(y) = \frac{A\alpha + e(y) - ye'(y)|y|^2}{|y|^2} \kappa_\theta(y) dy, \]
where \( \kappa_\theta(y) = (1 + \theta)\mathbf{1}_{(0,\infty)}(y) + (1 - \theta)\mathbf{1}_{(-\infty,0)}(y). \)
Thus we have,
\[ J_2 \leq \mathbb{E}\int_{|u|<L} |ug_h(X+au) - ug_h(X)| (dF_Y(u) + \nu_1(du)) \]
\[ + \mathbb{E}\int_{L<|u|<\frac{1}{a}} |ug_h(X+au) - ug_h(X)| \frac{|\alpha e(u) - ue'(u)|}{|u|^2} du \]
\[ + \mathbb{E}\int_{|u|>\frac{1}{a}} |ug_h(X+au)| \frac{|\alpha e(u) - ue'(u)|}{|u|^2} du \]
\[ := I + II + III \]

Moreover, by (32), it is easy to verify
\[ I \leq \frac{1}{\delta + 1} a^\delta \int_{|u|<L} u^{1+\delta} (dF_Y(u) + \nu_1(du)) \]
\[ \leq \frac{1}{\delta + 1} a^\delta (L^2 + m_1 + m_2). \]

Using (32), we also have
\[ II \leq \frac{a^\delta}{1+\delta} \int_{L<|u|<\frac{1}{a}} \frac{|e(u) - ue'(u)|}{|u|^{1-\delta}} du. \]

For the third term, using (31), it can be immediately shown that
\[ III \leq a \int_{|u|>1} \frac{|e(u/a) - u/ae'(u/a)|}{|u|} du. \]

Combining the estimates obtained in I, II and III, the desired conclusion follows.
Proof of (b). With the help of above lemmas, we now find bound in the $d_{W_\delta}$ distance for 1-stable approximation. By (16), we have
\[
|\mathbb{E}[h(S_n) - h(X)]| = \left| \mathbb{E}\left[ (-S_n + \beta)g_h^1(S_n) + \int_{\mathbb{R}} (g_h^1(S_n + u) - g_h^1(S_n)) 1_{|u| \leq 1}(u) \nu_1(du) \right]\right| \leq I + II + III,
\]
where
\[
I := \frac{1}{n} \sum_{i=1}^{n} |\mathbb{E}A_1g_h^1(S_{n,i}) - \mathbb{E}A_1g_h^1(S_n)|
\]
\[
II := \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E}(Y_i 1_{(-1,1)}(\frac{1}{n} Y_i))g_h^1(S_{n,i}) - \mathbb{E}(A_1g_h^1(S_{n,i})) \right|
\]
\[
III := \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E}(Y_i 1_{(-1,1)}(\frac{1}{n} Y_i))g_h^1(S_{n,i}) - \beta \mathbb{E}g_h^1(S_n) \right|
\]
For $\alpha = 1$, we have by Lemma 5.4 with $a = \frac{1}{n}$,
\[
I \leq C_{\alpha,K} A_{\alpha,m_1,m_2} \frac{1}{n} + 2C_{1,\delta,m_1,m_2} \int_0^n d F_{|Y|}(y)
\]
By Lemma 5.5 with $a = \frac{1}{n}$, we have
\[
II \leq \frac{1}{\delta + 1} n^{-\delta} (L^2 + m_1 + m_2) + \frac{n^{-\delta}}{1 + \delta} \int_{L < |u| < \frac{1}{n}} \frac{|e(u) - ue'(u)|}{|u|^{1-\delta}} du + \frac{1}{n} \int_{|u| > 1} \frac{|e(nu) - nue'(nu)|}{|u|} du.
\]
Using (36) and (31), we have
\[
III \leq \left| \int_0^n \frac{e(y) - e(-y)}{y} dy \right| + 2K + \beta.
\]
Combining the estimates obtained in I, II and III, the desired conclusion follows.

5.6 Proof of Theorem 3.19
Recall that $(Y_n)_{n \geq 1}$ is a sequence of i.i.d. random variables with $\mathbb{E}Y_i = 0$ and $\mathbb{E}|Y_i| < \infty$ for $1 \leq i \leq n$. Let $Z_i = n^{-\frac{2}{3}} Y_i$ and define,
\[
S_n = Z_1 + Z_2 + \ldots + Z_n \quad \text{and} \quad S_n(i) = S_n - Z_i.
\]
Note that $S_n(i)$ and $S_n$ are independent. To derive an error bound in the $d_{W_2}$ distance for $\alpha$-stable approximations with $\alpha \in (1, 2)$, we need to go through three important lemmas.

**Lemma 5.6.** Let $\nu_\alpha$ be a Lévy measure for $\alpha$-stable distributions with $\alpha \in (1, 2)$. Let $g_\alpha^n$ is defined in [27]. Then for any $N > 0$,

\[
\int_{\mathbb{R}} (g_\alpha^n(S_n + u) - g_\alpha^n(S_n)) \nu_\alpha(du) = \int_{-N}^N K_{\nu_\alpha}(t, N)(g_\alpha^n)'(S_n + t)dt + R_N(S_n),
\]

where

\[
K_{\nu_\alpha}(t, N) = 1_{[0,N]}(t) \int_{-N}^N \nu_\alpha(du) \int_{-N}^t (-u)\nu_\alpha(du), \quad \text{and}
\]

\[
R_N(S_n) = \int_{|u|>N} (g_\alpha^n(S_n + u) - g_\alpha^n(S_n)) \nu_\alpha(du).
\]

The proof of this lemma follows by similar computations [3] Lemma 5.3].

**Lemma 5.7.** Let $g_\alpha^n$ is defined in [27]. Then for any $N > 0$, we have,

\[
\mathbb{E}[S_ng_\alpha^n(S_n)] = \sum_{i=1}^n \int_{-N}^N \mathbb{E}[(K_i(t, N)(g_\alpha^n)'(S_n(i) + t))]dt + R_1,
\]

where

\[
K_i(t, N) = \mathbb{E}[Z_i1_{\{0\leq t\leq Z_i\leq N\}} - Z_i1_{\{-N\leq Z_i\leq t\leq 0\}}], \quad \text{and}
\]

\[
R_1 = \sum_{i=1}^n \mathbb{E}[\xi_i\{g_\alpha^n(S_n) - g_\alpha^n(S_n(i))\}]1_{|\xi_i|\geq N}.
\]

The proof of this lemma follows by similar computations [40] Lemma 4.5].

Next, we derive a result using the above two lemmas which is as follows.

**Lemma 5.8.** Let $g_\alpha^n$ is defined in [27]. Then,

\[
\mathbb{E} \left[\int_{\mathbb{R}} (g_\alpha^n(S_n + u) - g_\alpha^n(S_n)) \nu_\alpha(du) - S_ng_\alpha^n(S_n)\right] = \sum_{i=1}^n \int_{-N}^N \mathbb{E} \left(\frac{K_{\nu_\alpha}(t, N)}{n} - K_i(t, N)\right)(g_\alpha^n)'(S_n(i) + t)dt + \frac{1}{n} \sum_{i=1}^n \mathbb{E}(R_N(S_n(i))) + R_1 + R_2,
\]

where $R_N(x)$ and $R_1$ are defined in Lemmas 5.6 and 5.7 respectively,

\[
R_2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\int_{\mathbb{R}} (g_\alpha^n(S_n + u) - g_\alpha^n(S_n)) \nu_\alpha(du) - \int_{\mathbb{R}} (g_\alpha^n(S_n(i) + u) - g_\alpha^n(S_n(i)) \nu_\alpha(du)\right].
\]
Proof. We have,
\[
\mathbb{E}\left[\int_{\mathbb{R}} (g_h^n(S_n + u) - g_h^n(S_n)) u \nu_\alpha(du) - S_n g_h^n(S_n)\right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\int_{\mathbb{R}} (g_h^n(S_n(i) + u) - g_h^n(S_n(i))) u \nu_\alpha(du) \right. \\
\left. - S_n g_h^n(S_n)\right] + R_1 + R_2 + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[R_N(S_n(i))\right] \\
= \frac{1}{n} \sum_{i=1}^{n} \int_{-N}^{N} \mathbb{E}\left(\frac{K_{\nu_\alpha}(t, N)}{n} - K_i(t, N)\right) (g_h^n)'(S_n(i) + t)dt \\
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(R_N(S_n(i))) + R_1 + R_2,
\]
the desired conclusion follows.

Proof of Theorem 3.19. By (10), we have
\[
\mathbb{E}[h(S_n) - h(X)] = \mathbb{E}\left(-S_n g_h^n(S_n) + \int_{\mathbb{R}} (g_h^n(S_n + u) - g_h^n(S_n)) u \nu_\alpha(du) \right) + \mathbb{E}\left(\beta + \int_{|u| > 1} u \nu_\alpha(du)\right) g_h^n(S_n).
\]
To bound \(\mathbb{E}[h(S_n) - h(X)]\), it is sufficient to find an upper bound of right hand side of the above result. By Lemma 5.8 and (34), we have
\[
\left| \frac{1}{n} \sum_{i=1}^{n} \int_{-N}^{N} \mathbb{E}\left(\frac{K_{\nu_\alpha}(t, N)}{n} - K_i(t, N)\right) (g_h^n)'(S_n(i) + t)dt \right| \leq \frac{1}{2} \|h''\| \sum_{i=1}^{n} \int_{-N}^{N} \left|\frac{K_{\nu_\alpha}(t, N)}{n} - K_i(t, N)\right| dt.
\]
Note that,
\[
\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(R_N(S_n(i))) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\int_{|u| > N} |f'(S_n(i) + u) - f'(S_n(i))| \nu_\alpha(du) \right] \leq 2 \|h''\| \int_{|u| > N} |u| \nu_\alpha(du), \text{ and}
\]
\[
|R_1| = \left| \sum_{i=1}^{n} \mathbb{E}\left[Z_i \{f'(S_n) - f'(S_n(i))\}\right] 1_{|Z_i| \geq N} \right| \leq 2 \|h''\| \sum_{i=1}^{n} \mathbb{E}[|Z_i| 1_{|Z_i| > N}].
\]
Using (33), we have
\[ |R_2| \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left| \int_{\mathbb{R}} (g^\alpha_h(S_n + u) - g^\alpha_h(S_n))u\nu_\alpha(du) - \int_{\mathbb{R}} (g^\alpha_h(S_n(i) + u) - g^\alpha_h(S_n(i)))u\nu_\alpha(du) \right| \right] \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left| \int_{\mathbb{R}} [(g^\alpha_h(S_n + u) - g^\alpha_h(S_n(i) + u)) - (g^\alpha_h(S_n) - g^\alpha_h(S_n(i)))] \omega_\alpha(du) \right| \]
\[ \leq C_{\alpha,m} \frac{1}{n} \sum_{i=1}^{n} |\mathbb{E}Z_i|^{2-\alpha}. \]

Also, for \( m_1 = m_2 = m \) and \( \beta = 0 \), we have
\[ \left| \mathbb{E} \left( \beta + \int_{|u|>1} u\nu_\alpha(du) \right) g^\alpha_h(S_n) \right| = 0. \]

Combining all the estimates above, we get the inequality of the theorem, as desired.

### A Appendix

In this section, we prove some technical results used in the previous sections.

**Proposition A.1.** Let \( X \sim S(\alpha, \beta, m_1, m_2) \). Then, its characteristic exponent \( \eta_\alpha \) given in (35) can be written in the following form.

\[ \eta_\alpha(z) = \begin{cases} 
iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta \frac{d_\alpha}{|z|^\alpha} \tan \frac{\pi}{2} \right), & \alpha \in (0, 2) \setminus \{1\}, \\
iz\gamma_1 - d_1 |z|^\alpha \{1 + i\theta \frac{d_1}{|z|^\alpha} \log |z|\}, & \alpha = 1,
\end{cases} \]

where \( \alpha \in (0, 2), \gamma_\alpha \in \mathbb{R}, d_\alpha \geq 0 \) and \( \theta \in [-1, 1] \).

**Proof.** Recall that for \( X \sim S(\alpha, \beta, m_1, m_2) \), the characteristic exponent is given by

\[ \eta_\alpha(z) = iz\beta + \int_{\mathbb{R}} (e^{izu} - 1 - izu 1_{\{|u|\leq 1\}}(u)) \nu_\alpha(du), \quad z \in \mathbb{R}, \]

where \( \nu_\alpha \) is the Lévy measure given by

\[ \nu_\alpha(du) = \left( m_1 \frac{1}{u^{1+\alpha}} 1_{(0, \infty)}(u) + m_2 \frac{1}{u^{1+\alpha}} 1_{(-\infty, 0)}(u) \right) du. \]

Now, we have to consider three different cases to proceed to the derivations of these expressions.

**i)** \( \alpha \in (0, 1) \)

As noted in Section 2, for \( \alpha \in (0, 1) \), the integral \( \int_{\{|u|\leq 1\}} u\nu_\alpha(du) < \infty \). Indeed \( \int_{\{|u|\leq 1\}} u\nu_\alpha(du) = \frac{m_1-m_2}{1-\alpha} \)

Denote \( \beta_1 = \beta - \frac{m_1-m_2}{1-\alpha} \). So, one can write \( \eta_\alpha \) as

\[ \eta_\alpha(z) = iz\beta_1 + \int_{\mathbb{R}} (e^{izu} - 1) \nu_\alpha(du), \quad z \in \mathbb{R} \quad (64) \]
Suppose $z > 0$, then from (64)

$$
\eta_\alpha(z) = iz\beta_1 + \int_0^\infty (e^{izu} - 1) \nu_\alpha(du) + \int_{-\infty}^0 (e^{izu} - 1) \nu_\alpha(du)
$$

$$
= iz\beta_1 + m_1 \int_0^\infty (e^{izu} - 1) \frac{du}{u^{1+\alpha}} + m_2 \int_{-\infty}^0 (e^{izu} - 1) \frac{du}{|u|^{1+\alpha}}
$$

$$
= iz\beta_1 + z^\alpha \left( m_1 \int_0^\infty (e^{-\alpha u} - 1) \frac{du}{|u|^{1+\alpha}} + m_2 \int_0^\infty (e^{-i\alpha u} - 1) \frac{du}{y^{1+\alpha}} \right)
$$

Applying Cauchy’s Theorem of contour integration on (65), we have

$$
\eta_\alpha(z) = iz\beta_1 + z^\alpha \left( m_1 e^{-i\pi\alpha} L(\alpha) + m_2 e^{i\pi\alpha} L(\alpha) \right),
$$

where $L(\alpha) = \int_0^\infty (e^{-y} - 1) \frac{dy}{y^{1+\alpha}} < 0$, see [19, p.164].

Thus,

$$
\eta_\alpha(z) = iz\beta_1 + z^\alpha L(\alpha) \left( (m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha \right) + i(m_2 - m_1) \sin\left(\frac{\pi}{2} \alpha \right) \right)
$$

$$
= iz\beta_1 + z^\alpha L(\alpha) (m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha \right) \left( 1 + \frac{m_2 - m_1}{m_1 + m_2} \tan\left(\frac{\pi}{2} \alpha \right) \right)
$$

For $z < 0$,

$$
\eta_\alpha(z) = \eta_\alpha(-z) = iz\beta_1 + (-z)^\alpha L(\alpha)(m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha \right) \left( 1 + \frac{m_2 - m_1}{m_1 + m_2} \frac{z}{|z|} \tan\left(\frac{\pi}{2} \alpha \right) \right).
$$

Therefore, for any $z \in \mathbb{R}$

$$
\eta_\alpha(z) = iz\beta_1 + |z|^\alpha L(\alpha)(m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha \right) \left( 1 + \frac{m_2 - m_1}{m_1 + m_2} \frac{z}{|z|} \tan\left(\frac{\pi}{2} \alpha \right) \right)
$$

$$
= iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta \frac{z}{|z|} \tan\left(\frac{\pi}{2} \alpha \right) \right),
$$

where $\gamma_\alpha = \beta_1 = \beta - \frac{m_1 - m_2}{1 - \alpha}$, $d_\alpha = (m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha \right) \int_0^\infty (1 - e^{-y}) \frac{dy}{y^{1+\alpha}}$ and $\theta = \frac{m_2 - m_1}{m_1 + m_2}$.

(ii) $\alpha \in (1, 2)$

As noted in Section 2, for $\alpha \in (1, 2)$, the integral $\int_{|u|>1} \nu_\alpha(du) < \infty$. Indeed $\int_{|u|>1} \nu_\alpha(du) = \frac{m_1 - m_2}{1 - \alpha}$

Denote $\beta_2 = \beta - \frac{m_1 - m_2}{1 - \alpha}$. So, one can write $\eta_\alpha$ as

$$
\eta_\alpha(z) = iz\beta_2 + \int_{\mathbb{R}} (e^{izu} - 1 - izu) \nu_\alpha(du), \quad z \in \mathbb{R}
$$

(66)
Suppose $z > 0$, then from (66)

$$\eta_\alpha(z) = iz\beta_2 + \int_0^\infty (e^{izu} - 1 - izu) \nu_\alpha(du) + \int_{-\infty}^0 (e^{izu} - 1 - izu) \nu_\alpha(du)$$

$$= iz\beta_2 + m_1 \int_0^\infty (e^{izu} - 1 - izu) \frac{du}{u^{1+\alpha}} + m_2 \int_{-\infty}^0 (e^{izu} - 1 - izu) \frac{du}{|u|^{1+\alpha}}$$

$$= iz\beta_2 + z^\alpha \left( m_1 \int_0^\infty (e^{iv} - 1 - iv) \frac{dv}{v^{1+\alpha}} + m_2 \int_0^\infty (e^{-iv} - 1 + iv) \frac{dv}{v^{1+\alpha}} \right)$$  \hspace{1cm} (67)

Applying Cauchy’s Theorem of contour integration on (67), we have

$$\eta_\alpha(z) = iz\beta_2 + z^\alpha \left( m_1 e^{-i\frac{\pi}{2} \alpha} M(\alpha) + m_2 e^{i\frac{\pi}{2} \alpha} M(\alpha) \right),$$

where $M(\alpha) = \int_0^\infty (e^{-y} - 1 + y) \frac{dy}{y^{1+\alpha}} > 0$, see [19, p.164].

Thus, for any $z > 0$

$$\eta_\alpha(z) = iz\beta_2 + z^\alpha M(\alpha) \left( (m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha\right) + i(m_2 - m_1) \sin\left(\frac{\pi}{2} \alpha\right) \right)$$

$$= iz\beta_2 + z^\alpha M(\alpha)(m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha\right) \left( 1 + i \frac{m_2 - m_1}{m_1 + m_2} \frac{z}{\|z\|} \tan\left(\frac{\pi}{2} \alpha\right) \right)$$

For $z < 0$,

$$\eta_\alpha(z) = \eta_\alpha(-z) = iz\beta_2 + (-z)^\alpha M(\alpha)(m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha\right) \left( 1 + i \frac{m_2 - m_1}{m_1 + m_2} \frac{z}{\|z\|} \tan\left(\frac{\pi}{2} \alpha\right) \right).$$

Therefore, for any $z \in \mathbb{R}$

$$\eta_\alpha(z) = iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta \frac{z}{\|z\|} \tan\left(\frac{\pi}{2} \alpha\right) \right),$$

where $\gamma_\alpha = \beta_1 = \beta - \frac{m_1 - m_2}{\alpha}$, $d_\alpha = (m_1 + m_2) \cos\left(\frac{\pi}{2} \alpha\right) \int_0^\infty (1 - e^{-y} - y) \frac{dy}{y^{1+\alpha}}$ and $\theta = \frac{m_1 - m_2}{m_1 + m_2}$.

(iii) $\alpha = 1$

For $z \in \mathbb{R}$, it is easy to show that

$$\int_0^\infty \frac{\cos zu - 1}{u^2} \, du = -\frac{\pi}{2} z$$

Now, suppose $z > 0$, then

$$\eta_1(z) = iz\beta + \int_0^\infty (e^{izu} - 1 - izu1_{|u| \leq 1}) \nu_1(du) + \int_{-\infty}^0 (e^{izu} - 1 - izu1_{|u| \leq 1}) \nu_1(du)$$  \hspace{1cm} (68)
Let us consider second integral of (68). Then, we have

\[ \int_0^\infty (e^{izu} - 1 - izu1_{[|u|\leq 1]})v_1(du) = m_1 \left( \int_0^\infty \frac{\cos zu - 1}{u^2} du + i \int_0^\infty (\sin zu - zu1_{[|u|\leq 1]}) \frac{du}{u^2} \right) = m_1 \left( -\frac{\pi}{2}z + i \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \left( \frac{\sin zu}{u^2} - \frac{zu1_{[|u|\leq 1]}}{u^2} \right) du \right) \]  

(69)

Using the transformation \( zu = v \) and changing suitably the limit of integration on (69), we have

\[ \int_0^\infty (e^{izu} - 1 - izu1_{[|u|\leq 1]})v_1(du) = m_1 \left( -\frac{\pi}{2}z + iz \log z + iz \int_0^\infty \left( \frac{\sin v}{v^2} - \frac{1_{[|v|\leq 1]}}{v} \right) dv \right) \]  

(70)

The last equality of (70) follows, since \( \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{\sin v}{v^2} dv = \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{1}{v} dv = \log z \). If we set \( \Gamma = \int_0^\infty \left( \frac{\sin v}{v^2} - \frac{1_{[|v|\leq 1]}}{v} \right) dv \), then (70) simplifies to

\[ \int_0^\infty (e^{izu} - 1 - izu1_{[|u|\leq 1]})v_1(du) = m_1 \left( -\frac{\pi}{2}z - iz \log z + iz \Gamma \right) \]

Similarly, the last integral of (68) leads to

\[ \int_{-\infty}^0 (e^{izu} - 1 - izu1_{[|u|\leq 1]})v_1(du) = m_2 \left( -\frac{\pi}{2}z + iz \log(-z) - iz \Gamma \right) \]

Thus, for any \( z > 0 \)

\[ \eta_1(z) = iz\beta - (m_1 + m_2)\frac{\pi}{2}z + i(m_2 - m_1)z \log z + iz(m_1 - m_2)\Gamma \]

\[ = iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2)\frac{\pi}{2}z \left( 1 - i\frac{(m_2 - m_1)}{m_1 + m_2} \frac{2}{\pi} \log z \right) \]

For any \( z < 0 \),

\[ \eta_1(z) = \eta_1(-z) = iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2)\frac{\pi}{2}(-z) \left( 1 - i\frac{(m_2 - m_1)}{m_1 + m_2} \frac{2}{\pi} \log(-z) \right) \]

Therefore, for any \( z \in \mathbb{R} \)

\[ \eta_1(z) = iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2)\frac{\pi}{2}|z| \left( 1 - i\frac{(m_2 - m_1)}{m_1 + m_2} \frac{2}{|z|\pi} \log |z| \right) \]

\[ = iz\gamma_1 - d_1|z| \left( 1 + i\frac{\beta}{|z|\pi} \frac{2}{|z|\pi} \log |z| \right) \]
where \( \gamma_1 = \beta + (m_1 - m_2)\Gamma, \) \( d_1 = (m_1 + m_2)\Gamma \) and \( \theta = \frac{(m_1 - m_2)}{m_1 + m_2} \).

This completes the proof.

**Proposition A.2.** Let \( x, z \in \mathbb{R} \) and \( \alpha \in (0, 1) \). Then, for all \( t \geq 0 \),

\[
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(e^{-t} - 1)} \phi_t(z) - 1 \right) = \left( -x + \beta_1 + \int_{\mathbb{R}} u e^{izu} \nu_\alpha(du) \right) (iz),
\]

where \( \beta_1 = \beta - \int_{\{ |u| \leq 1 \}} u \nu_\alpha(du) \).

**Proof.** Recall from Section 2, if \( X \) be a \( \alpha \)-stable random variable with \( \alpha \in (0, 1) \) one can write

\[
\phi_t(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-tz})} = \exp \left( iz \beta_1 (1 - e^{-t}) + \int_{\mathbb{R}} (e^{izu} - e^{iue^{-t}z}) \nu_\alpha(du) \right), \quad t \geq 0,
\]

where \( \beta_1 = \beta - \int_{\{ |u| \leq 1 \}} u \nu_\alpha(du) \) (see (6)).

Now, let us consider LHS of (71),

\[
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(e^{-t} - 1)} \phi_t(z) - 1 \right)
\]

= \lim_{t \to 0^+} \frac{1}{t} \left( \exp \left( izx(e^{-t} - 1) + iz \beta_1 (1 - e^{-t}) + \int_{\mathbb{R}} (e^{izu} - e^{iue^{-t}z}) \nu_\alpha(du) \right) - 1 \right)

= \lim_{t \to 0^+} \frac{1}{t} \left( \exp (A + iB) - 1 \right),
\]

where

\[
A = \int_{\mathbb{R}} (\cos(zu) - \cos(zue^{-t})) \nu_\alpha(du) \quad \text{and} \quad B = \left( zx(e^{-t} - 1) + z \beta_1 (1 - e^{-t}) + \int_{\mathbb{R}} (\sin(zu) - \sin(zue^{-t})) \nu_\alpha(du) \right).
\]

Applying Euler’s formula for complex exponential to (72), and rearranging the limits, we have

\[
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(e^{-t} - 1)} \phi_t(z) - 1 \right) = \lim_{t \to 0^+} \frac{e^{A} \cos(B)}{t} - 1 + i \lim_{t \to 0^+} \frac{e^{A} \sin(B)}{t}.
\]

It is easy to show that at \( t = 0 \), \( e^{A} \cos(B) = 1 = 0 \) and \( e^{A} \sin(B) = 0 \). Thus, on applying L’Hospital rule on (73), taking limit as \( t \) tend to \( 0^+ \), and using dominated convergence theorem, we have

\[
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(e^{-t} - 1)} \phi_t(z) - 1 \right) = \left( \int_{\mathbb{R}} iu \sin(zu) \nu_\alpha(du) - x + \beta_1 + \int_{\mathbb{R}} u \cos(zu) \nu_\alpha(du) \right) (iz)
\]

= \left( -x + \beta_1 + \int_{\mathbb{R}} u \cos(zu) + i \sin(zu) \nu_\alpha(du) \right) (iz)

= \left( -x + \beta_1 + \int_{\mathbb{R}} u e^{izu} \nu_\alpha(du) \right) (iz).
This completes the proof.

**Proposition A.3.** Let $x, z \in \mathbb{R}$ and $\alpha \in (1, 2)$. Then, for all $t \geq 0$,

$$
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(x-e^{-t})} \phi_t(z) - 1 \right) = \left( -x + \beta_2 + \int_{\mathbb{R}} u(e^{izu} - 1) \nu_\alpha(du) \right) (iz),
$$

where $\beta_2 = \beta + \int_{\{|u|>1\}} u \nu_\alpha(du)$ (see (7)).

**Proof.** Recall from Section 2, if $X$ be a $\alpha$-stable random variable with $\alpha \in (1, 2)$, one can write

$$
\phi_t(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} = \exp \left( iz \beta_2 (1 - e^{-t}) + \int_{\mathbb{R}} \left( e^{izu} - e^{iue^{-t}z} - iuz(1 - e^{-t}) \right) \nu_\alpha(du) \right), \quad t \geq 0,
$$

(74)

where $\beta_2 = \beta + \int_{\{|u|>1\}} u \nu_\alpha(du)$.

Now, let us consider LHS of (74),

$$
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(x-e^{-t})} \phi_t(z) - 1 \right)
= \lim_{t \to 0^+} \frac{1}{t} \left( \exp \left( ix(x-e^{-t}) + iz \beta_2 (1 - e^{-t}) + \int_{\mathbb{R}} \left( e^{izu} - e^{iue^{-t}z} - iuz(1 - e^{-t}) \right) \nu_\alpha(du) \right) - 1 \right)
= \lim_{t \to 0^+} \frac{1}{t} (\exp (C + iD) - 1),
$$

(75)

where

$$
C = \int_{\mathbb{R}} (\cos(zu) - \cos(zue^{-t})) \nu_\alpha(du) \quad \text{and}
$$

$$
D = \left( x(x-e^{-t}) + z \beta_2 (1 - e^{-t}) + \int_{\mathbb{R}} (\sin(zu) - \sin(zue^{-t}) - zu(1 - e^{-t})) \nu_\alpha(du) \right).
$$

Applying Euler’s formula for complex exponential to (75), and rearranging the limits, we have

$$
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(x-e^{-t})} \phi_t(z) - 1 \right) = \lim_{t \to 0^+} \frac{e^C \cos(D) - 1}{t} + i \lim_{t \to 0^+} \frac{e^C \sin(D)}{t}.
$$

(76)

It is easy to show that at $t = 0$, $e^C \cos(D) - 1 = 0$ and $e^C \sin(D) = 0$. Thus, on applying L’Hospital rule on (76), taking limit as $t$ tend to $0^+$, and using dominated convergence theorem, we have

$$
\lim_{t \to 0^+} \frac{1}{t} \left( e^{ix(x-e^{-t})} \phi_t(z) - 1 \right) = \left( \int_{\mathbb{R}} \left( iu \sin(zu) \nu_\alpha(du) - x + \beta_2 + \int_{\mathbb{R}} u(\cos(zu) - 1) \nu_\alpha(du) \right) (iz) \right)
= \left( -x + \beta_2 + \int_{\mathbb{R}} u(\cos(zu) + i \sin(zu) - 1) \nu_\alpha(du) \right) (iz)
= \left( -x + \beta_2 + \int_{\mathbb{R}} u(e^{izu} - 1) \nu_\alpha(du) \right) (iz)
$$

This completes the proof.
Proposition A.4. Let $\alpha \in (0, 2)$. Then,
\[
\frac{1}{\alpha} \int_{\mathbb{R}} u g'(x + u) \nu_\alpha(du) = \int_{\mathbb{R}} (g(x + u) - g(x)) \nu_\alpha(du), \quad g \in S(\mathbb{R}),
\]
where $\nu_\alpha$ is the Lévy measure defined in (4).

Proof. We use Fubini’s theorem and change in the order of integration in the following proof. For $\alpha \in (0, 2)$, we have
\[
\int_{\mathbb{R}} u g'(x + u) \nu_\alpha(du) = m_1 \int_{0}^{\infty} \frac{u g'(x + u)}{u^{1+\alpha}} du + m_2 \int_{-\infty}^{0} \frac{g'(x + u)}{(-u)^{1+\alpha}} du
\]
\[
= m_1 \int_{0}^{\infty} g'(x + u) \int_{u}^{\infty} \frac{1}{z^{1+\alpha}} dz du - m_2 \int_{-\infty}^{0} g'(x + u) \int_{u}^{0} \frac{1}{(-z)^{1+\alpha}} dz du
\]
\[
= m_1 \int_{0}^{\infty} \frac{1}{z^{1+\alpha}} \int_{0}^{\infty} g'(x + u) du dz - m_2 \int_{-\infty}^{0} \frac{1}{(-z)^{1+\alpha}} \int_{0}^{\infty} g'(x + u) du dz
\]
\[
= \alpha \int_{0}^{\infty} (g(x + z) - g(x)) \frac{m_1}{z^{1+\alpha}} + \alpha \int_{-\infty}^{0} (g(x + z) - g(x)) \frac{m_2}{(-z)^{1+\alpha}}
\]
\[
= \alpha \int_{\mathbb{R}} (g(x + u) - g(x)) \nu_\alpha(du)
\]

Proposition A.5. Let $\alpha \in (0, 1)$ and $h \in \mathcal{H}_\delta$ with $\delta \in (0, \alpha)$. Then,
\[
|x g_h^\alpha(x)| \leq \eta_{\alpha, \beta, \delta, m_1, m_2}(x) := |\beta_1||h'|| + C_{\alpha, \delta, m_1, m_2} + |x| \land |x|^{\delta} + \mathbb{E}|X|^{\delta}.
\]

Proof. For $\alpha \in (0, 1)$, we have by (16),
\[
|x g_h^\alpha(x)| = \left| -\beta_1 g_h^\alpha(x) - \int_{\mathbb{R}} u g_h^\alpha(x + u) \nu_\alpha(du) + (h(x) - h(0)) - (\mathbb{E}h(X) - \mathbb{E}h(0)) \right|
\]
Thus, by (28) and (30), we have
\[
|x g_h^\alpha(x)| \leq |\beta_1||h'|| + C_{\alpha, \delta, m_1, m_2} + |x| \land |x|^{\delta} + \mathbb{E}|X|^{\delta} := \eta_{\alpha, \beta, \delta, m_1, m_2}(x),
\]
the desired conclusion follows.
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References

[1] Albeverio, S., Rüdiger, B. and Wu, J.L. (2000). Invariant measures and symmetry property of Lévy-type operators. Potential Analysis, 13. pp. 147-168.

[2] Applebaum, D. (2009). Lévy processes and stochastic calculus, Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, xxx+460 pp.

[3] Arras, B. and Houdré, C. (2019). On Stein’s method for infinitely divisible laws with finite first moment. Springer Briefs in Probability and Mathematical Statistics.

[4] Arras, B. and Houdré, C. (2019). On Stein’s method for multivariate self-decomposable laws with finite first moment. Electron. J. Probab. 24(29). pp. 1-33.

[5] Arras, B. and Houdré, C. (2019). On Stein’s method for multivariate self-decomposable laws. Electron. J. Probab. 24(128). pp. 1-63.

[6] Barbour, A. D. (1990). Stein’s method for diffusion approximations. Probability Theory and Related Fields. 84. pp. 297-322.

[7] Chen, L.H.Y., Goldstein, L. and Shao, Q.M (2011). Normal approximation by Stein’s method. Springer, Heidelberg.

[8] Boonyasombut, V. and Shapiro, J. M. (1970). The accuracy of infinitely divisible approximations to sums of independent variables with application to stable laws. Ann. Math. Stat. 41. pp.237-250.

[9] Chen, P. and Xu, L. (2019). Approximation to stable law by the Lindeberg principle. Journal of Mathematical Analysis and Applications. 480. https://doi.org/10.1016/j.jmaa.2019.07.028.

[10] Chen, P., Nourdin, I., Xu, L., Yang, X., Zhang, R. (2021). Non-integrable stable approximation by Stein’s method. J Theor Probab. 114. https://doi.org/10.1007/s10959-021-01094-5

[11] Chen, P., Nourdin, I. and Xu, L. (2020). Stein’s method for asymmetric α-stable distributions, with applications to CLT. Journal of Theoretical Probability. 34. pp. 1382-1407.

[12] Chen, P., Nourdin, I., Xu, L. and Yang, X. (2019). Multivariate Stable Approximation in Wasserstein Distance By Stein’s Method. Preprint [http://arxiv.org/abs/1911.12917v1]

[13] Chen, L. H. Y. (1975). Poisson approximation for dependent trials. Annals of Probability 3, pp. 534-545.
[14] Cont, R. and Tankov, P. (2004). Financial Modelling with Jump Processes. Chapman and Hall/CRC Financial Mathematics Series.

[15] Eichelsbacher, P. and Reinert, G. (2008). Stein’s method for discrete Gibbs measures. The Annals of Applied Probability, 18, pp. 1588-1618.

[16] Fulman, J. and Ross, N. (2013). Exponential approximation and Stein’s method of exchangeable pairs. ALEA Latin American Journal of Probability and Mathematical Statistics 10(1), pp. 1-13.

[17] Gaunt, R.E., Mijoule, G. and Swan, Y. (2019). An algebra of Stein operators. Journal of Mathematical Analysis and Applications. Volume 469, Issue 1, pp. 260-279

[18] Gaunt, R. E. (2014). Variance-Gamma approximation via Stein’s method. Electronic Journal of Probability 19 no. 38, pp. 1-33.

[19] Gnedenko, B.V. and Kolmogorov, A.N. (1967). Limit distributions for sum of independent random variables. Addison-Wesley Publishing Company, Cambridge.

[20] Houdré, C., Pérez-Abreu, V. and Surgails (1997). Interpolation, correlation identities and inequalities for infinitely divisible random variables. J. Fourier Anal.Appl. 4(6), pp. 935-952

[21] Häusler, E. and Luschgy, H. (2015). Stable convergence and stable limit theorems. Probability Theory and Stochastic Modelling, 74. Springer, Cham. x+228 pp.

[22] Jin, X., Li, X. and Lu, X.(2020). A kernel bound for non-symmetric stable distribution and its applications. Journal of Mathematical Analysis and Applications. 488. 124063

[23] Johnson, O. and Samworth, R (2005). Central limit theorem and convergence to stable laws in Mallows distance. Bernoulli 11(5).pp.829-845

[24] Kuske, R. and Keller, J.B. (2001). Rate of convergence to a stable law. SIAM J.Appl.Math.61 pp.1308-1323.

[25] Kyprianou, A.E. (2014). Fluctuations of Lévy processes and applications. Introductory lectures (second edition). Springer

[26] Kumar, A.N. and Upadhye, N.S. (2020). On discrete Gibbs measure approximation to runs. Communications in Statistics - Theory and Methods. DOI: 10.1080/03610926.2020.1765256

[27] Luk, H. (1994). Stein’s method for the gamma distribution and related statistical applications. PhD thesis, University of Southern California.

[28] Nourdin, I. and Peccati, G. (2012). Normal approximations with Malliavin calculus. Cambridge University Press. Cambridge tracts in mathematics 192.

[29] Pike, J. and Ren, H. (2014). Stein’s method and the Laplace distribution. ALEA Latin American Journal of Probability and Mathematical Statistics 11, pp. 571-587.

[30] Ross, N. (2010). Fundamentals of Stein’s method. Probability Surveys 8, pp. 210-293.
[31] Samorodnitsky, G. and Taqqu, M.S. (1994). Lévy processes and infinitely divisible distributions, Cambridge University Press, Cambridge.

[32] Sato, K.I. (1999). Lévy processes and infinitely divisible distributions. Cambridge University Press, Cambridge.

[33] Schoutens, W. (2001). Orthogonal polynomials in Stein’s method. Journal of Mathematical Analysis and Applications 253, pp. 515-531.

[34] Stein, E. M. and Shakarchi, R. (2003). Fourier analysis. An introduction. Princeton Lectures in Analysis, 1. Princeton

[35] Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. 2, Univ. California Press, Berkeley, pp. 583-602.

[36] Stein, C. (1986). Approximate Computation of Expectations. IMS, Hayward, California.

[37] Küchler, U. and Tappe, S. (2013). Tempered stable distributions and processes. Stochastic processes and their applications 123, pp.4256-4293, University Press, Princeton.

[38] Upadhye, N. S., Čekanavičius, V. and Vellaisamy, P. (2017). On Stein operators for discrete approximations. Bernoulli 23, pp. 2828-2859.

[39] Walsh, J.B. (2011). Knowing the odds. An introduction to probability. Graduate Studies in Mathematics. 139. American Mathematical Society.

[40] Xu, L. (2019). Approximation of stable law in Wasserstein-1 distance by Stein’s method. The Annals of Applied Probability 29, No. 1, pp. 458-504.