On Rigidity of Unit-Bar Frameworks

József Solymosi · Ethan White

Received: 28 August 2018 / Revised: 4 April 2019 / Published online: 24 July 2019
© Springer Japan KK, part of Springer Nature 2019

Abstract
We construct infinitesimally rigid bipartite unit-bar frameworks in $\mathbb{R}^d$. Our constructions are variants of the knight’s graph. This answers problems proposed by Maehara.

Keywords Rigidity · Unit-bar graph · Knight graph

1 Introduction

A framework in $\mathbb{R}^d$ is a graph with vertices that are points in $\mathbb{R}^d$, and edges that are line segments between vertices. We refer to the vertices of a framework as joints and edges as bars. A framework is flexible if there is a continuous motion of its joints, keeping bar lengths constant, while changing the distance between two non-adjacent joints. If a framework is not flexible, it is rigid. For example, in the plane a square can be deformed into a family of rhombi, and so it is flexible. On the other hand, the shape of a triangle is uniquely determined by the lengths of its three sides, and so it is rigid.

An infinitesimal motion of $\mathbb{R}^d$ is a vector field $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for all pairs of points $x, y \in \mathbb{R}^d$:

$$(f(x) - f(y)) \cdot (x - y) = 0.$$  (1)

Let $F$ be a framework in $\mathbb{R}^d$ with joints $X$. An infinitesimal motion of the framework $F$ is a vector field $g: X \rightarrow \mathbb{R}^d$ that satisfies (1) for all bars $xy$ in $F$. If every infinitesimal motion of the framework $F$ is of the form $f|_X$ for some infinitesimal motion $f$ of $\mathbb{R}^d$, then we say $F$ is infinitesimally rigid, otherwise $F$ is infinitesimally flexible.

A framework possessing a continuous motion also admits a smooth motion, see [1]. The initial velocity of the joints in a framework undergoing a smooth motion is...
an infinitesimal motion. Hence flexible frameworks are infinitesimally flexible and infinitesimally rigid frameworks are rigid.

The infinitesimal motions of $\mathbb{R}^d$ arise from the initial velocities of smooth rigid motions, i.e. rotations and translations. As a result, the space of infinitesimal motions of $\mathbb{R}^d$ has dimension $\binom{d+1}{2}$. If $F$ is not contained in a $(d-2)$-dimensional hyperplane, then different motions of $\mathbb{R}^d$ induce different motions of $F$. On the other hand, if $F$ has at least $d$ joints and is contained in a $(d-2)$-dimensional hyperplane, $F$ will be infinitesimally flexible. As a result, a framework in $\mathbb{R}^d$ with at least $d$ joints is infinitesimally rigid if and only the space of infinitesimal motions of $F$ has dimension $\binom{d+1}{2}$. For more on infinitesimal rigidity, see [1,2].

A unit-bar framework has bars of only one length. Rigid unit-bar frameworks can be constructed by attaching equilateral triangles to each other, but finding rigid triangle-free unit-bar frameworks is harder. Maehara et al. [3–5] study triangle-free and bipartite rigid unit-bar frameworks. Maehara constructed a rigid bipartite unit-bar framework with 353 joints and 676 bars in [3]. His framework is not infinitesimally rigid. In [4] Maehara and Chinen construct an infinitesimally rigid triangle-free unit-bar framework with 22 joints and 41 bars. In [5], Maehara and Tokushige construct an infinitesimally rigid unit-bar triangle-free framework in $\mathbb{R}^3$ with 26 joints and 78 unit-bars. These two frameworks contain pentagons. In [4] and [6] the authors propose the following problems:

I. Find an infinitesimally rigid bipartite unit-bar framework in the plane
II. Find a general method to construct a triangle-free, infinitesimally rigid unit-bar framework in $\mathbb{R}^d$.

In the following section, we construct infinitesimally rigid bipartite unit-bar frameworks in $\mathbb{R}^d$, thereby solving the above problems.

## 2 Infinitesimally Rigid Unit-Bar Frameworks in $\mathbb{R}^d$

In our constructions we use variants of the knight’s graph. The knight’s graph has a vertex for each square on a chessboard and edges that represent legal moves of the knight (Fig. 1).

**Definition 1** The $m \times n$ knight’s framework in $\mathbb{R}^2$ has a joint at all integer coordinates $(x, y)$ where $0 \leq x \leq m - 1$ and $0 \leq y \leq n - 1$. Two joints $(x_1, y_1), (x_2, y_2)$ have a bar between them if $|x_1 - x_2| = 1$ and $|y_1 - y_2| = 2$, or if $|x_1 - x_2| = 2$ and $|y_1 - y_2| = 1$.

The knight’s framework is a unit-bar framework. Two joints $(x_1, y_1), (x_2, y_2)$ are adjacent only if $x_1 + y_1$ and $x_2 + y_2$ have different parity, and so the framework is bipartite. An infinitesimally rigid framework in the plane on $\nu$ joints must have at least $2\nu - 3$ bars [2]. The $m \times n$ knight’s framework has $2(m-1)(n-2)+2(m-2)(n-1)$ bars. It is easy to check that the smallest $m \times n$ knight’s framework with enough edges to be infinitesimally rigid is the $5 \times 5$ framework. The infinitesimal motions of a framework $F$ in $\mathbb{R}^d$ are described by the nullspace of the rigidity matrix of $F$. A framework in $\mathbb{R}^d$ with at least $d$ joints is infinitesimally rigid if and only if the
nullity of the rigidity matrix is \( \binom{d+1}{2} \). Equivalently, if the rank of the rigidity matrix is \( ud - \binom{d+1}{2} \), then the framework is infinitesimally rigid. For more on the rigidity matrix see [2].

**Theorem 2** The 5×5 knight’s framework is infinitesimally rigid.

For a proof of Theorem 2 not requiring computer aid, see [7]. Alternatively, the infinitesimal rigidity of the 5×5 knight’s framework can be verified by calculating the rank of the corresponding rigidity matrix, the computer program in Appendix 2 of [7] does this. The framework obtained by deleting the corner joints and one degree three joint from the 5×5 knight’s framework is also infinitesimally rigid. This framework has 20 joints and 37 edges. The infinitesimal rigidity of this framework can be verified by calculating the rank of its rigidity matrix. Every joint in the 5×6 knight’s framework that is not in the 5×5 framework has two bars in linearly independent directions connecting it to the 5×5 framework, and so the 5×6 framework is infinitesimally rigid. Inductively we see that the \( m \times n \) knight’s framework is infinitesimally rigid for all \( m, n \geq 5 \). The knight’s framework can be extended to higher dimensions.

**Definition 3** An \( n \)-lattice framework in \( \mathbb{R}^d \) has joints of the form \( (x_1, \ldots, x_d) \), where \( x_i \in \{0, 1, \ldots, n-1\} \). Let \( F \) be an \( n \)-lattice framework in \( \mathbb{R}^d \). For all integers \( 1 \leq i \leq d \) and \( 0 \leq c \leq n-1 \), define \( F_{i,c} \) to be the cross-section framework of \( F \) induced by all joints in \( F \) of the form \( (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_d) \). The framework \( F_{i,c} \) can be embedded in \( \mathbb{R}^{d-1} \) by deleting the \( i^{th} \) coordinate of all joints in \( F_{i,c} \). The resulting framework is an \( n \)-lattice framework in \( \mathbb{R}^{d-1} \), call it \( F'_{i,c} \).

In Fig. 2, the cross-section frameworks \( F_{1,0} \) and \( F_{2,1} \) are infinitesimally flexible in \( \mathbb{R}^3 \). In \( \mathbb{R}^2 \), \( F'_{1,0} \) remains infinitesimally flexible, but \( F'_{2,1} \) is infinitesimally rigid. If all 6 faces of the framework \( F \) in Fig. 2 had the two diagonal bars as \( F_{2,1} \) does, the framework would be infinitesimally rigid. We’ll prove that in general, if the cross-sections of a lattice framework have all possible bars, then the framework is infinitesimally rigid.

**Lemma 4** Let \( y, x_1, x_2, \ldots, x_n \) be joints of a framework \( F \) in \( \mathbb{R}^d \) such that \( yx_i \) is a bar for all \( i \). Let \( f \) be an infinitesimal motion of \( F \) such that \( f(x_i) = 0 \) for all \( i \). If \( z \) is in the span of \( \{y - x_1, \ldots, y - x_n\} \), then \( f(y) \cdot z = 0 \).
we have that e

Theorem 5 Let F be an n-lattice framework in \( \mathbb{R}^d \), \( d \geq 3 \), and \( n \geq 2 \). If for all \( 1 \leq i \leq d \) and \( 0 \leq c \leq n - 1 \), the framework \( F_{i,c} \) has bars between all pairs of joints, then F is infinitesimally rigid.

Proof Consider all infinitesimal motions \( f \) of \( F \) such that

\[
f(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad f_i(\mathbf{e}_k) = 0 \quad \text{for} \ 1 \leq k \leq d - 1 \ \text{and} \ 1 \leq k \leq d.
\]

The restrictions of (2) specify \( d + (d - 1) + \cdots + 1 = \binom{d+1}{2} \) degrees of freedom of \( f \). Hence the space of infinitesimal motions of \( F \) that satisfy (2) is at least the dimension of the space of all infinitesimal motions of \( F \) minus \( \binom{d+1}{2} \). It follows that if the only infinitesimal motions of \( F \) that satisfy (2) are identically zero, then \( F \) is infinitesimally rigid.

Let \( f \) be an infinitesimal motion of \( F \) satisfying (2). Note that \( e_0 \mathbf{0} \) is a bar of \( F \) and \( e_1 \) is in the span of \( \{e_1 - \mathbf{0}\} \). Since \( f(\mathbf{0}) = \mathbf{0} \), by Lemma 4 we see that \( f(e_1) \cdot e_1 = 0 \) and so \( f(e_1) = 0 \). Notice \( e_i \mathbf{0} \) is a bar for all \( 1 \leq i \leq d \). For all \( j \neq i \), since \( d \geq 3 \), we have that \( e_i e_j \) is also a bar. A simple induction and Lemma 4 gives the result \( f(e_i) = 0 \) for all \( 1 \leq i \leq d \). For any joint \( x \in F_{i,0} \) we have that \( x\mathbf{0} \) and \( xe_j \) are bars for all \( j \neq i \). Lemma 4 gives \( f_j(x) = 0 \) for all \( j \neq i \). Hence if a joint \( x \) has a zero in two or more coordinates, \( f(x) = \mathbf{0} \). Let \( y = (y_1, \ldots, y_d) \) be a joint of \( F \) such that \( y_i \neq 0 \) for all \( i \). Let \( y^{(i)} \) be the joint with \( y_i \) in the \( i^{th} \) coordinate and zeros in all other coordinates. Notice that \( y y^{(i)} \) is a bar for all \( 1 \leq i \leq d \). Furthermore, since \( d \geq 3 \), \( y^{(i)} \) has a zero in at least two coordinates, and so \( f(y^{(i)}) = \mathbf{0} \). It is easy to check that
the span of \( \{y - y^{(i)}\}_{1 \leq i \leq d} \) is all of \( \mathbb{R}^d \), and so by Lemma 4, \( f(y) = 0 \). Finally, let \( x = (x_1, \ldots, x_d) \) be a joint of \( F \) such that \( x_i = 0 \) and \( x_j \neq 0 \) for \( j \neq i \). Let \( z \) be the joint \((x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)\). The existence of \( z \) follows from \( n \geq 2 \). Since \( xz \) is a bar:

\[
(f(x) - f(z)) \cdot (x - z) = (f(x) - f(z)) \cdot e_i = 0.
\]

Since all coordinates of \( z \) are nonzero, \( f(z) = 0 \), and so \( f_i(x) = 0 \). It follows that \( f \equiv 0 \), and \( F \) is an infinitesimally rigid framework.

\[\Box\]

**Corollary 6** Let \( F \) be an \( n \)-lattice framework in \( \mathbb{R}^d \), \( d \geq 3 \) and \( n \geq 2 \). If for all \( 1 \leq i \leq d \), and \( 0 \leq c \leq n - 1 \), the framework \( F'_{i,c} \) is infinitesimally rigid in \( \mathbb{R}^{d-1} \), then \( F \) is infinitesimally rigid.

**Proof** Let \( 1 \leq i \leq d \) and \( 0 \leq c \leq n - 1 \) be arbitrary. Any infinitesimal motion \( f \) of \( F \) induces an infinitesimal motion \( f'_{i,c} \) of \( F'_{i,c} \) in the following way. For any joint \( x \in F'_{i,c} \) let \( \hat{x} \) denote the corresponding joint in \( F_{i,c} \), and put

\[
f'_{i,c}(x) = [f_1(\hat{x}), \ldots, f_{i-1}(\hat{x}), f_{i+1}(\hat{x}), \ldots, f_d(\hat{x})].
\]

It is clear that this defines \( f'_{i,c} \) as a vector field in \( \mathbb{R}^{d-1} \). Furthermore, for any bar \( xy \) of \( F'_{i,c} \), since \( f \) is an infinitesimal motion:

\[
(f'_{i,c}(x) - f'_{i,c}(y)) \cdot (x - y) = (f(\hat{x}) - f(\hat{y})) \cdot (\hat{x} - \hat{y}) = 0.
\]  \( \text{ (3) } \)

It follows that \( f'_{i,c} \) is an infinitesimal motion of \( F'_{i,c} \). Notice that the first equality in (3) holds for all \( x, y \in F'_{i,c} \), and not just bars. Since \( F'_{i,c} \) is infinitesimally rigid in \( \mathbb{R}^{d-1} \) we see that both equalities in (3) hold for all \( x, y \in F'_{i,c} \). Hence all infinitesimal motions of \( F \) are infinitesimal motions of the framework described in Theorem 5, and so \( F \) is infinitesimally rigid in \( \mathbb{R}^d \).

\[\Box\]

**Definition 7** The \( n \times \cdots \times n \) knight’s framework in \( \mathbb{R}^d \) is the \( n \)-lattice framework with bars between two joints \( x \) and \( y \) if the coordinates of \( x \) and \( y \) are equal except in two places where they differ by 1 and 2.

All bars in the knight’s framework have length \( \sqrt{5} \). The parity of the sum of the coordinates of two adjacent joints is different, the same as in the two dimensional case. Hence the knight’s framework in \( \mathbb{R}^d \) is bipartite, and in particular, triangle free. A consequence of Theorem 2 and Corollary 6 is the following.

**Theorem 8** The \( 5 \times \cdots \times 5 \) knight’s framework in \( \mathbb{R}^d \), for \( d \geq 2 \), is an infinitesimally rigid bipartite unit-bar framework.

Using a computer to calculate the rank of the rigidity matrix, we noticed that the \( 4 \times 4 \times 4 \) knight’s framework is infinitesimally rigid. The computer code of this program can be found in Appendix 1 of [7]. It follows that the \( 4 \times \cdots \times 4 \) knight’s framework in \( \mathbb{R}^d \) for \( d \geq 3 \) is also infinitesimally rigid by Corollary 6.
Acknowledgements  The research of the second author was supported in part by an NSERC CGS M. The research of the first author was supported in part by an NSERC Discovery Grant and OTKA NK Grant. The work of the first author was also supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant agreement nos. 741420, 617747, 648017).

References

1. Asimow, L., Roth, B.: The rigidity of graphs. Trans. Am. Math. Soc. 245, 279–289 (1978). https://doi.org/10.2307/1998867
2. Graver, J., Servatius, B., Servatius, H.: Combinatorial Rigidity, vol. 2. American Mathematical Society, Providence (1993)
3. Maehara, H.: A rigid unit-bar-framework without triangle. Math. Jpn. 36, 681–683 (1991)
4. Maehara, H., Chinen, K.: An infinitesimally rigid unit-bar-framework in the plane which contains no triangle. Ryuku Math. J. 8, 37–41 (1995)
5. Maehara, H., Tokushige, N.: A spatial Unit-bar-framework which is rigid and triangle-free. Graphs Comb. 12, 341–344 (1996)
6. Maehara, H.: Distance graphs and rigidity. Contemp. Math. 342, 149–168 (2004). https://doi.org/10.1090/conm/342/06139
7. Solymosi, J., White, E.: On rigidity of unit-bar frameworks. arXiv:1808.04005 (2018) (pre-print)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.