Abstract

Two types of intervention are commonly implemented in networks: characteristic intervention which influences individuals’ intrinsic incentives, and structural intervention which targets at the social links among individuals. In this paper we provide a general framework to evaluate the distinct equilibrium effects of both types of interventions. We identify a hidden equivalence between a structural intervention and an *endogenously determined* characteristic intervention. Compared with existing approaches in the literature, the perspective from such an equivalence provides several advantages in the analysis of targeting interventions of the network structure. We present a wide range of applications of our theory, including determining whether a structural intervention is beneficial, identifying the most wanted criminal(s) in delinquent networks, and targeting the key bridge nodes for disconnected communities.

*JEL Classification: D21; D29; D82.*

*Keywords: Network; Structural intervention; Katz-Bonacich centrality; Targeting;*
1 Introduction

Social ties shape economic agents’ decisions in a connected world, ranging from which product to adopt for consumers, how much time to spend on study for pupils, how much effort to exert for workers within a team, whether to conduct crime for teenagers, etc.\textsuperscript{1} These social ties, structurally represented as a network, govern individual incentives and therefore collectively determine equilibrium outcomes and welfare in the society. Thus, structural intervention on social ties provides an important policy instrument for the social planner. A natural research question arises: how to best intervene the social structure to maximize a certain performance objective subject to certain resource constraints. This research problem is inherently difficult as it is well-known that network operates in a complex way. Local changes of social links among a few nodes can influence actions of a large set of nodes, including those who are far away through ripple effects. Furthermore, the influence is not homogeneous: nodes that are closer to (further away) the origin of shocks tend to be more (less) responsive. These key features of shock propagation and heterogeneous responses make the analysis of structural interventions both intriguing and challenging.\textsuperscript{2}

In this paper, we propose a general yet tractable framework to quantitatively assess the consequences of an arbitrary structural intervention of social ties on the equilibrium actions. Overcoming the challenges mentioned above, we present a neat characterization result in Proposition 1 which evaluates the change of equilibrium behavior in response to changes in the network structure. Then we apply Proposition 1 to several economic settings, such as profitable structural intervention (in Section 3), key group removal in delinquent networks (in Section 4) and key connectors for isolated communities (in Sections 5).

More specifically, we consider linear quadratic games in which players are embedded in undirected weighted networks. The framework is based on a seminal paper by Ballester et al. (2006) (BCZ hereafter).\textsuperscript{3} They identify the equivalence between equilibrium actions

\textsuperscript{1}Numerous studies have highlighted the influence of networks in different contexts such as microfinance (Banerjee et al. (2013)), firm performance (Cai and Szeidl (2018)), productivity at work (Mas and Moretti (2009)), R&D (Goyal and Moraga-González (2001)), education (Sacerdote (2001); Calvó-Armengol et al. (2009)), crime (Ballester et al. (2006)), public goods provision (Bramoullé and Kranton (2007); Allouch (2017), brand choice (David and Dina (2004)), and policy intervention Galeotti et al. (2020)). For recent surveys, see, for instance, Bramoullé et al. (2016); Jackson et al. (2017); Elliott et al. (2019).

\textsuperscript{2}Admittedly, these two features are also true for other types of intervention, such as the characteristic intervention. As shown in Section 2.2, the problem of characteristic intervention in a fixed network is much simpler and has been extensively studied in the literature.

\textsuperscript{3}The model in BCZ has been applied, empirically tested, and generalized extensively in the network
in a network game and the Katz-Bonacich centralities in sociology (Bonacich (1987)). In unweighted networks, the Katz-Bonacich centrality of a node on a network simply counts the sum of geometrically discounted walks originated from this node to all the other nodes in the network, weighted by the characteristics of the ending nodes. Proposition 1 in our paper characterizes the impacts of structural intervention on the equilibrium by utilizing another equivalence result: any (local) intervention on the network structure is equivalent to a (local) endogenously determined interventions on characteristics. Specifically, we find that the equilibrium induced by a structural intervention coincides with that by an endogenously determined characteristic intervention without changing the network structure. Moreover, the endogenously determined characteristic intervention only changes the characteristics of the players whose social ties are altered by the structural intervention. The analysis of post-intervention equilibrium becomes much simpler after translating a structural intervention to the characteristic intervention since the later changes individual’s equilibrium behavior linearly and is well studied in the literature, while the former is non-linear.

Any structural intervention can be translated into some characteristic intervention which achieves the same equilibrium. However the inverse is not true since the network is undirected. Moreover, any action profile with finite and non-negative entries can be achieved by some plausible characteristic interventions. We then characterize the subset of action profiles which can be achieved by some plausible structural intervention. Proposition 2 completely pins down this set, which is shown to be neither convex nor a cone.

For applications, we first provide a sufficient condition under which a given structural intervention improves certain social objectives. Proposition 3 shows that, re-allocating weights from a node \( j \) with lower Katz-Bonacich centrality to a node \( i \) with higher centrality, such that \( i \) dominates \( j \) in all weights, improves a convex objective function over equilibrium behavior. The objective function includes both the aggregate effort and aggregate social welfare. Weight dominance is equivalent to that all \( j \)'s neighbours are also neighbours of \( i \) in unweighted networks. Based on this proposition, we then show that in weighted network setting any efficient network must belong to the class of generalized nested split graphs (Corollary 1). Proposition 3 can be further refined when the objective function is the aggregate

---

4This Katz-Bonacich centrality (and its variants and generalizations) plays important roles in shaping agents’ decisions in a wide range of network models, see, for instance, production networks (Acemoglu et al. (2012); Baqee (2018); Liu (2019)), pricing of social products (Candogan et al. (2012); Bloch and Quérou (2013); Chen et al. (2018a)).
gate equilibrium activity. Specifically, any structural intervention $C$ (not restricted to weight reallocation) strictly improves aggregate effort if $b^T C b \geq 0$, where $b$ is the Katz-Bonacich centrality profile prior intervention (Proposition 4).

In the second application, we adopt the outcome equivalence result to study the key group problem which aims to specify a group of players removing whom reduces the aggregate equilibrium activity the most. Specifically, the removal of a group of players is equivalent to a certain characteristic intervention restricted to this group. Such a characteristic intervention is chosen to decrease, within the original network, the induced equilibrium efforts of the group to zero. As a generalization of single node inter-centrality given by BCZ, we provide a form index of group inter-centrality explicitly based on indices in original network (Proposition 5). Such index takes into account both the activities nodes in this group and their influences on nodes outside of the group. Based on the this form, we show rigorously that the more the group members are connected to each other, the lower the group intercentrality (Corollary 2). The connectedness among group members is also the main reason that greedy algorithm fails in the search of the global optimum. In Proposition 6, we offer a toolkit to study information diffusion when re-transmission is not allowed for either sender or target node, a sensible assumption imposed by Bramoullé and Garance (2018). We define and derive intermediary centrality (Corollary 3). The node who maximizes this centrality is the key information intermediate and should be supervised most to check whether it has re-transmitted fake news or not.

In the third application, we introduce a bridge index to characterize the impact of building up a bridge between separated networks (Proposition 7). We use the bridge index to fully solve the key bridge problem. Furthermore, we show that the key bridge player must locate at the Pareto frontier of Katz-Bonacich centrality and self-loop in the network. In general, identity of key bridge player in one network depends on who is selected as his partner in the second network, therefore the selection of a bridge pair is not an independent decision across two networks. These findings are summarized in Corollary 4 and illustrated in Example 1. We further characterize the impact of building-up a new link in any network. We then use the characterization to simplify the proof of the argument that any node, who has the chance to build up a new link, prefers to connect with the node with higher degree, and any node who has to delete one link will remove the neighbour with lowest degree (Corollary 5). This argument serves as the key lemma to show that dynamic network formation process introduced by König et al. (2014) leads to nested split graph at each stage.
Our paper builds up on the vast literature on network games (see Ballester et al. (2006), Bramoullé and Kranton (2007) and Galeotti and Goyal (2010)). These papers typically characterize the effects of network structure on equilibrium behavior. Our paper instead focuses on how interventions of network structures affect equilibrium outcomes and sheds light upon the policy design targeting at network structure.

The literature on interventions in networks could be broadly divided into two categories: characteristic intervention and network structure intervention. In the first category, the characteristics of individuals can be changed by subsidy or taxation on the choices. For instance, Demange (2017) and Galeotti et al. (2020) study the optimal intervention on characteristics subject to a fixed budget constraint and a quadratic adjustment cost, respectively. Motivated by Ballester et al. (2006, 2010) and König et al. (2018), our paper mainly focuses on the second category, i.e. structural intervention. Past analysis on structure intervention is mainly built on path counting approach, i.e. counting the change in aggregate walks due to structure intervention, our paper proposes a novel approach, i.e. mapping the structure intervention into a characteristic intervention achieving the same equilibrium. We then adopt this approach to simplify proof of results in their papers, generalize their arguments and provide new insights. Ballester et al. (2006) consider the removal of a single node. Ballester et al. (2010) consider removal of a single link and that of a group. Our paper generalizes the analysis to any structural intervention. By applying our result to the analysis of group removal, we have proposed a way to calculate the group inter-centrality, which is only defined by Ballester et al. (2010) and a direct generalization of single node intercentrality in Ballester et al. (2006). Based on this closed form, we rigorously show that the intercentrality decreases with respect to intra-group connectedness, an observation drew by Ballester et al. (2010). By applying our result to link build-up, we have completely characterized the impact of a single link build-up. This result, in parallel to link removal in Ballester et al. (2010), can easily show that, in any nested split graph, any node who has a chance to add a link should build up a link to the node with highest degree, and any node who has to delete a link should always remove its neighbour with lowest degree. The argument is the key proposition to show that the dynamic network formation process, introduced by König et al. (2014), will yield nested split graph at each stage. In addition, we try to link these two strands of literature, i.e. characteristic intervention and structural intervention, by characterizing the set of characteristic interventions which achieve the same equilibrium with some structural intervention.
Our paper can also be linked to the literature of efficient network (e.g. Belhaj et al. (2016) on unweighted networks, Li (2020) and Hiller (2017) on weighted networks). Belhaj et al. (2016) show that the efficient network should be nested split graph by excluding profitable discrete link reallocation. Li (2020) has derived a similar result in weighted network through excluding profitable marginal weight reallocation. Our proposition 3 shows that any weight reallocation improves social planner’s objective when the objective function is convex and the post-intervention has weight dominance property. Such weight-reallocation includes both discrete and marginal one. Based on this argument, we can also show that the most efficient network is nested split without adopting first order approach. On one hand, our result generalizes the result of Belhaj et al. (2016) to weighted network; On the other hand, our result is complementary to Li (2020) since differentiability of either objective or cost function is not required. Moreover, we have proposed a sufficient condition under which any structural intervention (not restricted to weight-reallocation) will improve aggregate effort.

Our paper also contributes to the literature of information diffusion. Banerjee et al. (2013, 2019) and Cruz et al. (2017) have proposed a framework of information diffusion. They have defined target centrality and linked this centrality to Katz-Bonacich centrality. Bramoullé and Garance (2018) have imposed a sensible assumption, i.e. neither the sender nor the target can re-transmit the information, and proposed the targeting centrality. Given this sensible assumption, our paper provides a toolkit to study information diffusion by completely pin down the aggregate walks which start from one group, end at the second group, without passing the third group. We then define and derive intermediary centrality. This centrality helps to identify the key information intermediate whom should be supervised most by the social networking platforms to curtail fake news re-transmission.

This paper also speaks to the literature on the effect of bridge(s) between isolated communities. Verdier and Zenou (2015, 2018) study the communication between cultural leaders, who serve as bridge nodes connecting different race groups. Cai and Szeidl (2018) empirically demonstrate that business meetings facilitate interfirm communications and create enormous economic values by improving firm performance. To the best of our knowledge, Golub and Lever (2010) is the only network paper theoretically studying the impact of bridge nodes. In a DeGroot naive social learning model, they analyze the effects of adding bridge nodes on the eigenvalue centralities. Our paper, instead, focuses on Katz-Bonacich centralities and proposes an explicit index to characterize the key bridge nodes connecting two disconnected networks.
2 Interventions in networks: Theory

2.1 Setup

Baseline game played on a network Consider a network game played by a set of players $N = \{1, 2, \ldots, n\}$ embedded in a social network $G$, which is represented by an $n \times n$ matrix $G = (G_{ij})_{n\times n}$. Each player $i$ chooses an effort $a_i \in [0, \infty)$ simultaneously with payoff function given as follows:

$$u_i(a_i, a_{-i}) = \theta_i a_i - \frac{1}{2} a_i^2 + \delta \sum_{k=1}^{n} G_{ik} a_i a_k, \quad i \in N. \quad (1)$$

This specification of payoff is commonly adopted in the network literature, see Ballester, Calvó-Armengol, and Zenou (2006) and Galeotti, Golub, and Goyal (2020). Here $\theta_i > 0$ measures player $i$’s intrinsic marginal utility (hence $i$’s characteristic), $\frac{1}{2} a_i^2$ denotes player $i$’s cost of effort, and the last term, $\delta \sum_{k=1}^{n} G_{ik} a_i a_k$, captures the interaction term that represents local network effects among players. The scalar parameter $\delta$ controls the strength of network interaction. We assume $\delta > 0$ and $G_{ij} \geq 0$ for any $i, j \in N$, so the game exhibits strategic complementarity. We use $\Gamma (G, \theta, \delta)$ to denote the network game represented above, where $\theta = [\theta_1, \ldots, \theta_n]^T$ is the characteristics vector. Whenever there is no confusion, we use $(\theta_1, \ldots, \theta_n)$ to denote a vector $\theta$.

Throughout the paper, we consider weighted and undirected networks. That is, $G$ is symmetric with $G_{ij} = G_{ji} \in [0, \infty)$, and $G_{ii} = 0$ for all $i \in N$. Let $\lambda_{\text{max}} (G)$ denote the spectral radius of matrix $G$. By Perron-Frobenius theorem, $\lambda_{\text{max}} (G)$ also equals the largest eigenvalue of $G$. We first introduce the following well-known measure of centralities.

Definition 1. Given a network $G$, a scalar $\delta$, and an $n$-dimensional vector $\theta$, we define $\theta$-weighted Katz-Bonacich centralities as

$$b(G, \theta, \delta) = (b_1, b_2, \ldots, b_n) := (I - \delta G)^{-1} \theta, \quad (2)$$

provided that $\delta < \frac{1}{\lambda_{\text{max}}(G)}$. When $\theta = (1, 1, \ldots, 1) = 1$, we call $b(G, 1, \delta)$ the (unweighted) Katz-Bonacich centralities. Define the Leontief inverse matrix

$$M(G, \delta) = (m_{ij}(G))_{n \times n} := (I - \delta G)^{-1}, \quad (3)$$
so that $b_i(G, \theta, \delta) = \sum_{j=1}^{n} m_{ij}(G) \theta_j$. Whenever there is no confusion, we abbreviate $b(G, \theta, \delta)$, $b(G, 1, \delta)$ and $M(G, \delta)$ as $b(G, \theta)$, $b$ and $M$ respectively.

When the network is unweighted, i.e., $G_{ij} \in \{0, 1\}$, $m_{ij}$ counts the total number of walks from $i$ to $j$ in network $(N, G)$ with path of length $k$ discounted by $\delta^k$. So $i$’s Katz-Bonacich centrality $b_i(G, \theta, \delta)$ is the discounted sum of walks starting from $i$ ending at any node $j$ with weights $\theta_j$.\footnote{The Leontief inverse matrix has the following Neumann series representation
\[ M(G, \delta) = (I - \delta G)^{-1} = I + \delta G + \delta^2 G^2 + \cdots, \] provided that $0 \leq \delta < \frac{1}{\lambda_{\text{max}}(G)}$.}

Ballester et al. (2006) show that when $0 < \delta < \frac{1}{\lambda_{\text{max}}(G)}$, game $\Gamma(G, \theta, \delta)$ has a unique Nash equilibrium in which each player $i$’s equilibrium action $x_i$ is exactly equal to $i$’s Katz-Bonacich centrality, i.e.,

\[ x = b(G, \theta). \] (4)

More influential players, measured by Katz-Bonacich centralities, exert higher efforts in equilibrium. Such an elegant relationship between equilibrium outcomes and Katz-Bonacich centralities is the starting point of our analysis.

**Notation** Before proceeding, we introduce some notation. In the network $(N, G)$, for any subset $A \subseteq N$, let $|A|$ denote the cardinality of this set, and let $A^C = N \setminus A$ denote the complement of $A$. Let $G_{AA}$ denote the $|A| \times |A|$ adjacency matrix of the subnetwork formed by players in $A$. Moreover, the adjacency matrix $G$ can be written as a block matrix

\[ G = \begin{bmatrix} G_{A^C A^C} & G_{A^C A} \\ G_{A^C A} & G_{AA} \end{bmatrix}. \]

Similarly, we can rewrite a column vector $x$ of length $n$ as $\begin{bmatrix} x_{A^C} \\ x_A \end{bmatrix}$.

We use $x$ to denote the sum of all elements in vector $x = (x_1, x_2, \ldots, x_n)$, i.e., $x = \sum_{i=1}^{n} x_i$.

The transpose of a matrix $H$ is denoted by $H^T$. Consider two matrices $Q = (q_{ij})_{n \times m}$ and $P = (p_{ij})_{n \times m}$ of the same dimension. We write $Q \succeq (\preceq) P$ if and only if $q_{ij} \geq (\leq) p_{ij}$ for any $i, j$. Whenever there is no confusion, we denote $x$ and $\hat{x}$, the equilibrium action profile before and after intervention, respectively.
2.2 Effects of interventions

In this subsection, we first analyze the effects of characteristic intervention. Then we study the impact of structural intervention by translating it into a characteristic intervention which induces the same equilibrium outcome.

Characteristic interventions A characteristic intervention perturbs characteristic vector given a fixed network $G$. We restrict the set of plausible characteristic intervention $\Delta \Theta$ as

$$\Delta \Theta(G; \delta) := \{ \Delta \theta \in \mathbb{R}^n : (I - \delta G)^{-1}(\theta + \Delta \theta) \geq 0 \}.$$ 

The restriction is to ensure that the post-intervention equilibrium action profile is always linked to (weighted) Katz-Bonacich centrality. For any $\Delta \theta \in \Delta \Theta(G; \delta)$, denote

$$S \equiv \{ i \in N : \Delta \theta_i \neq 0 \}$$

the set of nodes involved in the characteristic intervention. The following lemma summarizes the impact of any plausible characteristic intervention $\Delta \theta$.

Lemma 1. Consider a plausible characteristic intervention $\Delta \theta = \begin{bmatrix} 0_{SC} \\ \Delta \theta_S \end{bmatrix} \in \Delta \Theta(G; \delta)$.

1. Agent $i$’s equilibrium effort increases by $\Delta x_i = M_{iS} \Delta \theta_S$;
2. The aggregate equilibrium effort level increases by $\Delta x = b_S^T \Delta \theta_S$;
3. The aggregate equilibrium welfare increases by $\Delta W = (\Delta \theta_S)^T M_{SN} [b(G, \theta) + \frac{1}{2} M_{NS} \Delta \theta_S]$.

This Lemma is straightforward. By Equation (4), the equilibrium action profile $x$ is linear in the characteristics vector $\theta$ with sensitivity matrix given by $M$ as the network structure is fixed during the intervention.

Consider a characteristic intervention at a single node $S = \{ j \}$ by $\Delta \theta_j$, then $\Delta x_i = m_{ij} \Delta \theta_j$ by Lemma 1 for any $i$. The marginal contribution of $j$’s characteristics on $i$’s equilibrium action is exactly $m_{ij}$. Summing over all $i$, the marginal contribution of $j$’s characteristics on the aggregate effort is just $\sum_{i \in N} m_{ij} = \sum_{i \in N} m_{ji} = b_j$. In other words, we have

$$\frac{\partial x}{\partial \theta_j} = \frac{\partial \{ \sum_{i \in N} x_i \}}{\partial \theta_j} = b_j.$$ 

(5)

When characteristics of multiple players are modified during the intervention (so $S$ contains multiple players), we employ the linearity (see Lemma 1): the change of player $i$’s equilibrium
action is simply the sum, over $j$ in $S$, of the effect caused by $j$, i.e., $\Delta x_i = \sum_{j \in S} m_{ij} \Delta \theta_j$. In other words,

$$\frac{\partial x_A}{\partial \theta_S} = M_{AS},$$

for any subset $A \subseteq N$. (6)

As we will see in next subsection, this desirable feature of linearity does not hold for structural intervention, the effects of which are nonlinear and hence more complex to analyze.

**Structural Interventions** A structural intervention perturbs the network $G$ while fixing the characteristic profile $\theta$. Define the plausible set of structural intervention as

$$C(G, \delta) := \left\{ C \in \mathbb{R}^{n \times n} : \delta < \frac{1}{\lambda_{\text{max}}(G + C)}, C_{ij} = C_{ji} \geq -G_{ij}, \forall i \neq j \in N \text{ and } C_{kk} = 0, \forall k \right\}.$$  

Specifically, we only consider the structural intervention after which the resulting network is still undirected with non-negative weights and zero self-loop. To analyze the effects of structural intervention $C$, we first translate it into a characteristic intervention which achieves the same equilibrium action profile. We call this relation as outcome equivalence and define it below.

**Definition 2.** Consider a network game $\Gamma(G, \theta, \delta)$, a plausible structural intervention $C \in C(G, \delta)$ and a plausible characteristic intervention $\Delta \theta \in \Delta \Theta(G, \delta)$. We call $C$ outcome equivalent to $\Delta \theta$ if

$$b(G + C, \theta) = b(G, \theta + \Delta \theta).$$

The next lemma characterizes the outcome equivalent characteristic intervention for any plausible structural intervention. Let $S = \{ i \in N : C_{ij} \neq 0 \text{ for some } j \in N \}$ denote the set of players involved in this intervention.

**Lemma 2.** Consider a network game $\Gamma(G, \theta, \delta)$.

1. For any plausible structural intervention $C = \begin{bmatrix} 0_{SC} & 0_{SC} \\ 0_{SC} & C_{SS} \end{bmatrix} \in C(G, \delta)$, if $\hat{x}$ is the equilibrium action profile after intervention, then the outcome equivalent characteristic intervention is $\Delta \theta = \begin{bmatrix} 0_{SC}^T \\ \Delta \theta_{SC} \end{bmatrix}^T \hat{x}_S C_{SS}$.

2. If the network game is perturbed by both characteristic and structural intervention at
the same time, i.e. a hybrid intervention \((C, \Delta \tilde{\theta}) = \left( \begin{bmatrix} 0_{SC}^SC & 0_{SC}^S \\ 0_{SS}^SC & C_{SS} \end{bmatrix}, \begin{bmatrix} 0_{SC}^S \\ \Delta \theta\end{bmatrix} \right)\).  

and let \(\tilde{x}\) be the equilibrium action profile post intervention, then the outcome equivalent characteristic intervention is \(\Delta \theta = \left( \begin{bmatrix} 0^T_{SC} \\ \delta \tilde{x}_SC \end{bmatrix}, \begin{bmatrix} \Delta \theta_S \\ C_{SS} + \Delta \tilde{\theta}^T_S \end{bmatrix} \right)\).

This lemma provides a new approach of studying structure (or hybrid) intervention by at first translating the structure intervention into its outcome-equivalent characteristic intervention and then studying the impact of the characteristic intervention. Many existing papers on structure intervention rely on path counting approaches, i.e. counting the change in aggregate walks.\(^8\) We will show in applications that our approach can be utilized to simplify the proof, generalize their arguments and provide new results. Note that the outcome equivalent characteristic intervention in Lemma 2 is determined by post-intervention action profile and therefore endogenously determined. However, we can then use Lemma 1 to pin down post-intervention action profile \(\tilde{x}_S\), which will be illustrated in the following proposition. Furthermore, when the post-intervention action profile \(\tilde{x}_S\) is partially known, like our applications on both weight-reallocation and group intercentrality, the analysis can be further simplified.

**Proposition 1** (Effects of structural interventions). Given a network game \(\Gamma (G, \theta, \delta)\) and a plausible structural intervention \(C \in C(G, \delta)\) with a set \(S\) of nodes being involved in the intervention, then we have

1. The outcome-equivalent characteristic intervention is \(\Delta \theta = \begin{bmatrix} 0_{SC}^S \\ \Delta \theta_S \end{bmatrix} \), where

   \[
   \Delta \theta_S = \delta C_{SS}(I - \delta M_{SS}C_{SS})^{-1}b_S(G, \theta).
   \]

2. Agent \(i \in N\)’s equilibrium effort increases by \(\Delta x_i = M_{iS}\Delta \theta_S\);

3. The aggregate equilibrium effort level increases by \(\Delta x = b^T_S\Delta \theta_S\);

\(^7\)A hybrid intervention \((C, \Delta \tilde{\theta})\) is plausible if \(C \in C(G, \delta)\) and \(\Delta \tilde{\theta} \in \Delta \Theta(G + C, \delta)\).

\(^8\)For instance, Ballester et al. (2006) focus on node removal, Ballester et al. (2010) study link removal and König et al. (2014) analyze link add-up and delete.
4. The aggregate equilibrium welfare increases by

\[ \Delta W = (\Delta \theta_S)^T M_{SN} \left[ b(G, \theta) + \frac{1}{2} M_{NS} \Delta \theta_S \right]. \]

By Lemma 1, the impact of the outcome equivalent characteristic intervention \( \Delta \theta = (0_{SC}, \delta C_{SS} \hat{x}_S) \) on the equilibrium action of nodes in \( S \) is given by

\[ \hat{x}_S - x_S = M_{SS} \delta C_{SS} \hat{x}_S. \]

We can use it to solve for \( \hat{x}_S \) and therefore \( \Delta \theta \) by substituting \( x = b(G, \theta) \) into the equation above. Finally, we adopt Lemma 1 to characterize the impact of \( \Delta \theta \).

Proposition 1 describes the impacts of an arbitrary structural intervention on the equilibrium outcome. There are two points worth noting. First, Proposition 1 only needs to pin down the Leontif inverse matrix of \( M_{SS} C_{SS} \) to derive the outcome-equivalent characteristic intervention. Whenever \( |S| \) is small compared with \( |N| \), Proposition 1 simplifies the calculation. Second, though local structural intervention generates global effects, the local information of Leontif inverse matrix, i.e. that of nodes involved in the structural intervention, suffices to characterize the global impact.

Lemma 2 implies that any structural intervention can be translated into some outcome-equivalent characteristic intervention. However, the inverse may not hold. Moreover, any action profile with finite and non-negative entries can be achieved by some plausible characteristic intervention. We study a related question: characterizing the subset of action profiles which can be achieved by some plausible structural intervention. Define

\[
X(\theta) := \left\{ (I - \delta(G + C))^{-1} \theta : C \in C(G, \delta) \right\}
\]

\[ := \left\{ (I - \hat{G})^{-1} \theta : \lambda_{\text{max}}(\hat{G}) < 1, \hat{G}_{kk} = 0, \forall k, \hat{G}_{ij} = \hat{G}_{ji} \geq 0, \forall i \neq j \in N \right\}. \]

These two ways of definition coincides and the second way implies that this set \( X(\theta) \) is independent of the original network \( G \) and network effect \( \delta \).

**Proposition 2.** \( x \in X(\theta) \) if and only if

\[
\begin{align*}
& x \geq \theta; \\
& \sum_{i=1}^{n} (x_i - \theta_i)x_i \geq 2 \max_{1 \leq j \leq n} (x_j - \theta_j)x_j.
\end{align*}
\]
We illustrate the “only if” direction of Proposition 2. Suppose an action profile $x$ can be achieved by some network $\hat{G}$ given characteristic vector $\theta$, then

$$x_i = \theta_i + \delta \sum_{j \neq i} \hat{G}_{ij} x_j.$$ 

Since each player exerts non-negative effort in equilibrium, $x_i \geq \theta_i$ by strategic complements. Fix a node $k$, we obtain

$$x_i - \theta_i = \delta \sum_{j \neq i} \hat{G}_{ij} x_j \geq \delta \hat{G}_{ik} x_k,$$

Suming over $j \neq k$ yields

$$\sum_{i \neq k} (x_i - \theta_i) x_i \geq \sum_{i \neq k} \delta \hat{G}_{ik} x_k x_i = \sum_{i \neq k} \delta \hat{G}_{ki} x_i x_k = (x_k - \theta_k) x_k,$$

which shows the second condition in (7). To prove the “if” direction, we adopt the Farkas’ Lemma. Note that there are $n(n-1)/2$ degrees of freedom in $G$ by symmetry, which is larger than $n$ when $n \geq 3$. The symmetry of $G$ puts certain restrictions on $X(\theta)$, which is neither a convex set nor a cone.

In the following sections, we will study three applications by applying the approach developed in Proposition 1, i.e. studying the impact of structural intervention through translating into outcome-equivalent characteristic intervention. These three applications include identifying profitable structural intervention, studying the key group problem and the key bridge problem.

### 3 Profitable structural intervention

Proposition 1 describes the effect of interventions on each player. Obviously, if weights are increased, i.e., $C \succeq 0$, then each player’s effort unambiguously increases. Likewise, when weights are reduced, i.e. $C \preceq 0$, the equilibrium effort decreases. However, the effect is often ambiguous if the signs of $C$ are mixed. In this section, we present a sufficient condition to guarantee that the weight reallocation leads to higher aggregate action and welfare. Such a condition is much simpler to check than that in Proposition 1.

To begin with, we introduce a lemma to show that if a node $i$ dominates $j$ in weight, then $i$ is more connected to the other nodes.

---

9The inequality holds if only if the underlying network is star. Besides, an action profile is at the boundary, i.e. $x \in \partial X(\theta)$ if and only if the post-intervention $G$ network is a star network or not connected.
Definition 3. Given a network \((N, G)\), node \(i\) dominates node \(j\) in weight if \(G_{ik} \geq G_{jk}\) for any \(k \notin \{i, j\}\).

Lemma 3. In network \((N, G)\), if node \(i\) dominates \(j\) in weight, then \(m_{ik} \geq m_{jk}\) for any \(k \notin \{i, j\}\) and \(m_{ii} \geq m_{jj}\).

In an unweighted network, \(i\) dominates \(j\) in weight implies that their neighbors are nested: each of \(j\)'s neighbor is always a neighbor of \(i\). Lemma 3 is intuitive from the path-counting perspective. Any walk from \(j\) to \(k\) is composed of two segments: in the first segment, the walk reaches one of \(j\)'s neighbors with one step; then it resumes from the neighbor and ends at \(k\). Since all \(j\)'s neighbors are \(i\)'s, the walk can be viewed as that from \(i\) to \(k\) by replacing the starting node \(j\) to \(i\). Therefore, there are more walks starting from \(i\) and ending at \(k\).

We investigate the impacts of weight reallocation between two nodes. A weight reallocation from \(j\) to \(i\) is a structural intervention \(C\) such that \(C_{ik} + C_{jk} = 0\) with \(C_{ik} \geq 0\) and \(C_{kl} = 0\) for any \(k, l \notin \{i, j\}\). That is, the structural intervention \(C\) switches some weights from \(j\) to \(i\). Consider an objective function \(Q(x) \equiv \sum_{l \in N} q(x_l)\), where \(q(\cdot)\) is assumed to be strictly increasing and weakly convex. Note that, when \(q(x) = x\), \(Q\) is the aggregate equilibrium effort, and when \(q(x) = \frac{x^2}{2}\), \(Q\) is the equilibrium welfare. The following proposition provides a sufficient condition on weight reallocation to ensure an improvement of the value of objective function.

Proposition 3. In network game \(\Gamma(G, 1, \delta)\), consider two players \(i, j\) with \(b_i > b_j\) and a weight reallocation \(C\) from \(j\) to \(i\). Then, \(Q(\hat{x}) > Q(x)\) if node \(i\) dominates node \(j\) in weight in (the post-intervention) network \(G + C\).

We prove this proposition by translating the structural intervention into characteristic intervention. Denote the set \(K = \{i, j\}\). In the first step, we consider a hybrid intervention \((C, \Delta \tilde{\theta})\) on \(\Gamma(G, 1)\), where \(C\) is the weight reallocation specified in Proposition 3, \(\Delta \tilde{\theta}_k = 0\) for any \(k \in K\) and \((\Delta \tilde{\theta}_i, \Delta \tilde{\theta}_j)\) is designed such that both node \(i\) and \(j\)'s effort does not change after the hybrid intervention. We then adopt Lemma 2 to translate the hybrid intervention into an outcome-equivalent characteristic intervention,

\[
\Delta \theta = (C_{ik} \hat{x}_i + C_{jk} \hat{x}_j, C_{ik} \hat{x}_K + \Delta \tilde{\theta}_i, C_{jk} \hat{x}_K + \Delta \tilde{\theta}_j),
\]
since $\hat{x}_l = x_l, \forall l \in \{i, j\}$, $\Delta \tilde{\theta}_k = 0$ and $C_{km} = 0, \forall k, m \in K$. The main reason that we design the hybrid intervention in this way is to ensure that the outcome-equivalent characteristic intervention $\Delta \theta_k \geq 0, \forall k \in K$, which then implies that all nodes in $K$ increase their efforts post intervention. We further show that $\Delta \tilde{\theta}_j < 0$ and $\Delta \tilde{\theta}_i + \Delta \tilde{\theta}_j < 0$ by node $i$ and $j$’s first order conditions. In the second step, we consider a characteristic intervention $-\Delta \tilde{\theta}$ on $\Gamma(G + C, 1 + \Delta \tilde{\theta})$. By Lemma 3, weight dominance implies that $m_{ij}(G + C) \leq m_{jk}(G + C)$. By Lemma 1, effort of node $k \notin \{i, j\}$ further increases by

$$\Delta x_k = -\Delta \tilde{\theta}_i m_{ki}(G + C) - \Delta \tilde{\theta}_j m_{kj}(G + C)$$

$$= -(\Delta \tilde{\theta}_i + \Delta \tilde{\theta}_j) m_{ki}(G + C) - \Delta \tilde{\theta}_j (m_{kj}(G + C) - m_{ki}(G + C)) > 0.$$  

We then show that node $j$’s effort increases, $i$’s effort decreases and increment in $j$’s effort outweighs decrement in $i$’s effort. The convexity of $q(\cdot)$ then implies the argument. “Weight-dominance” condition only plays a key role in the second step to ensure that nodes in $K$ increase their effort post the characteristic intervention $-\Delta \theta$.

In unweighted and symmetric networks, Belhaj et al. (2016)’s Lemma 1 states that re-allocating all neighbours from a node with lower Katz-Bonacich centrality to another node with higher centrality, will improve social welfare. Proposition 3 generalizes this result from unweighted networks to weighted networks. Note that it is not necessary to re-allocate all weights from $(i, k)$ to $(j, k)$ as long as the weight dominance condition holds.10 Lemma 1 in Belhaj et al. (2016) plays a key role in showing their main result that the most efficient network is always a nested split graph. Similarly, our Proposition 3 can be utilized to generalize their main finding in the context of weighted networks.

To formalize the idea, we define the optimal network design problem. Suppose a planner aims to design a network to maximize his objective function $Q(\cdot)$ under a total budget constraint $\bar{c}$. The cost of a weight $G_{ij}$ between node $i$ and $j$ is $c(G_{ij})$, where $c(\cdot)$ is a strictly increasing and strictly convex function. To form a network $G$, the total cost is given by

---

10This is not a trivial generalization, denote $C$ as the weight re-allocation between node $i$ and $j$. Let $C^+ := \max\{C_{ij}, 0\}_{i,j \in N}$ and $C^- = C - C^+$. Re-allocating all neighbours of $j$ (but not of $i$) to node $i$ in unweighted network implies that $(G + C^-)_{ik} \geq (G + C^-)_{jk}, \forall k \notin \{i, j\}$. But in weighted network, we only require that $(G + C^- + C^+)_{ik} \geq (G + C^- + C^+)_{jk}, \forall k \notin \{i, j\}$. Actually, in a working paper Sun et al. (2021), we show that both re-allocating all neighbours from node $j$ to node $i$ or the inverse, i.e. re-allocating all neighbours from node $i$ to node $j$, weakly increase social welfare.
\[ \sum_{i=1}^{n} \sum_{j>i}^{n} c(G_{ij}). \]

Assume that \( \theta = 1 \), the planner’s problem is described by

\[
\max_{G} \quad Q(x) \\
\text{s.t.} \begin{cases} 
\lambda_{\text{max}}(G) < \frac{1}{\delta}, \\
x = b(G), \\
\sum_{i=1}^{n} \sum_{j>i}^{n} c(G_{ij}) \leq \bar{c}.
\end{cases}
\]

The first constraint requires that the network game \( \Gamma(G, 1) \) has a unique equilibrium, i.e. the Katz-Bonacich centrality, which is given by the second constraint. To state our results, we introduce the definition of nested split graph in weighted networks (see Li (2020))

**Definition 4.** A symmetric weighted network \( (N, G) \) is a generalized nested split graph if for each \( i \neq j \), either \( i \) dominates \( j \) in weight or the inverse.

**Corollary 1.** Any solution to problem (8) must be a generalized nested split graph.

Corollary 1 is complementary to a recent work by Li (2020), which studies designing weighted and directed networks to maximize aggregate welfare. He assumes that each player’s action is an increasing and differentiable function of his neighbours’ aggregate effort. Besides, he adopts the first order approach and exploits Kuhn-Tucker conditions to exclude marginal profitable structural perturbation. Instead, Proposition 3, which analyzes the effect of any perturbation \( C \), enables us to derive similar result without turning to the first order approach. Therefore, Corollary 1 holds without the differentiability of both the objective function \( Q(\cdot) \) and cost function \( c(\cdot) \).

Recall that, in the proof of Proposition 3, “weight-dominance” condition is the foundation to show that all nodes in \( K = \{i, j\}^C \) increase their equilibrium effort. However, when the social planner’s objective is to maximize the aggregate effort, even without the “weight-dominance” condition, this characteristic intervention still improves aggregate effort since Lemma 1 implies\(^{11}\)

\[
\Delta x = -b_i(G + C)\Delta \tilde{\theta}_i - b_j(G + C)\Delta \tilde{\theta}_j \\
= -b_i(G + C)(\Delta \tilde{\theta}_i + \Delta \tilde{\theta}_j) - [b_j(G + C) - b_i(G + C)] \Delta \tilde{\theta}_j > 0
\]

\(^{11}\)We have implicitly relied on the argument that \( b_j(G + C) \geq b_i(G + C) \), which can be shown by Lemma 3.
We generalize this insight to any structural intervention in the following proposition.

Proposition 4. Consider a network game $\Gamma(G, 1, \delta)$, a plausible structural intervention $C$ such that $b^T_S C_{SS} b_S > 0$ will strictly increase aggregate effort.

Here is a sketch of the proof. For any $\alpha \in [0, 1]$, denote a function $J(\alpha)$ as

$$J(\alpha) := \delta \alpha b^T_S C_{SS} (I - \delta \alpha M_{SS} C_{SS})^{-1} b_S.$$ 

Proposition 1 implies that $\Delta x = J(1)$. Note that $J'(0) = \delta b^T_S C_{SS} b_S > 0$, and $J(\cdot)$ is shown to be convex in $\alpha$. The result follows from Jensen’s inequality.

Proposition 4 implies that any weight reallocation from a node with lower Katz-Bonacich centrality to a node with higher centrality will strictly improves aggregate effort, even without weight-dominance condition. To make it clear, we consider unweighted networks in which case $c_{ij} \in \{0, \pm 1\}$. Then, we can reformulate the expression in Proposition 4 as follows:

$$b'_S C_{SS} b_S = \sum_{i,j \in S} c_{ij} b_ib_j = \left( \sum_{(i,j): c_{ij} = 1} b_ib_j \right) - \left( \sum_{(i,j): c_{ij} = -1} b_ib_j \right). \quad (9)$$

To see some immediate implications of this proposition, we present two simple examples:

1. First, we consider a link reallocation which removes the link between $i, j$ and adds a new link between $k, l$.\(^{12}\) Then Proposition 4 implies that such a reallocation of links increases aggregate action if $b_ib_j < b_kb_l$. In particular, it holds when $b_i < b_k$ and $b_j \leq b_l$. Whenever the newly formed link contains nodes with higher Katz-Bonacich centralities than the removed link, this type of link reallocation increases aggregate action.

2. Second, we consider a link swap which removes the link between $i$ and $j$ and adds a new link between $i$ and $l$. Such a swap, by Proposition 4, increases aggregation action whenever $b_j < b_l$. Cutting an old link with a neighboring node $j$ of node $i$ with lower Katz-Bonacich centrality and creating a new link from $i$ to another unconnected node $l$ with higher Katz-Bonacich centrality makes the whole group overall more active.

\(^{12}\)To make such an intervention $C$ legitimate for unweighted network $G$, we assume $g_{ij} = 1$ and $g_{kl} = 0$. Moreover, we assume at least three elements of $\{i, j, k, l\}$ must be distinct.
4 Key players in network

4.1 Key group in delinquent network

Consider the following optimization problem:

$$\min_{A \subseteq N, |A| \leq k} b(G_{AC\bar{A}}, \theta_{AC}) ,$$

(10)

which is motivated by the application of criminal network (see Ballester et al. (2006, 2010) for detailed discussion): the government, facing a group of criminals in an (unweighted) network $G$, wants to identify a subset of criminals $A$ of $N$ (as known as the most wanted) so that the total action (criminal effort in this context) in the remaining network is minimized after removing $A$ from the original network $G$.\(^{13}\) Note that, before the intervention, the criminals play a game $\Gamma(G, \theta)$ with total criminal activities $b(G, \theta)$; after the removal of $A$ from $G$, the remaining criminals $A^C$ play the game $\Gamma(G_{AC\bar{A}}, \theta_{AC})$, which leads to the objective stated in program (10). To accommodate the constraint from the government side (for instance, limited police resource), the size of $A$ is bounded above by a positive integer $k$.

Problem (10) is called the key player problem for $k = 1$, and the key group problem in general for $k \geq 2$. Ballester et al. (2010) introduce the following definition.

**Definition 5.** The intercentrality index of group $A$ in network $(N, G)$ is defined as

$$d_A(G, \theta) := b(G, \theta) - b(G_{AC\bar{A}}, \theta_{AC}).$$

The intercentrality of group $A$, $d_A(G, \theta)$, is the precisely reduction of aggregate activity by removing group $A$ and it can be decomposed into two parts: a direct effect by the removed players in $A$, $\sum_{i \in A} b_i(G, \theta)$, and an indirect effect due to the decreasing of equilibrium actions of the remaining players $j \in A^C$, $\sum_{j \in A^C} \{b_j(G, \theta) - b_j(G_{AC\bar{A}}, \theta_{AC})\}$. The indirect effect induces a disparity between the group intercentrality and the sum of each group member’s Katz-Bonacich centralities. Ballester et al. (2006) derive a form of a single node’s intercentrality by original network index. However, a group’s intercentrality is only defined in Ballester et al. (2010), using post-intervention network index (Definition

\(^{13}\)It is well established that criminality is a social action with strong peer influences (see, for example, Jerzy (2001); Mark (2002); Patacchini and Zenou (2012)).
We apply our approach developed in Proposition 1 to derive a simple form for group intercentrality using original network index.

**Proposition 5.** For any $A \subseteq N$, we have

$$d_A(G, \theta) = b_A^T(G) (M_{AA}(G))^{-1} b_A(G, \theta).$$

(11)

In delinquent network, arresting a group of criminals not only makes them inactive in criminal activity but also eradicates their influence to the other criminals. Mathematically, removal of set $A$ is a hybrid intervention, which both cuts all links relating to $A$ and decreases their characteristics to 0. The key observation is that, this hybrid intervention is outcome equivalent to a characteristic intervention targeting only on group $A$, $\Delta\theta_A$, such that each group member will be exactly choosing zero in equilibrium after this characteristic intervention.\(^{14}\)

$$M_{AA} \Delta\theta_A = \hat{x}_A - x_A = -x_A.$$ (12)

Therefore, the removal of group $A$ has the same effects as changing the characteristics of players in $A$ by

$$\Delta\theta_A = M_{AA}^{-1} b_A(G, \theta).$$

According to Proposition 1 (3), this characteristic intervention leads to the reduction of aggregate action by $-b_A' \Delta\theta_A = d_A(G, \theta)$. Consequently, the key group problem (10) can be reformulated as

$$\max_{A \subseteq N, |A| \leq k} d_A(G, \theta) = b_A^T M_{AA}^{-1} b_A(G, \theta)$$

When $k = 1$, taking $S = \{i\}$, we obtain $d_i(G, \theta) = \frac{b_i b_i(G, \theta)}{m_i}$ by equation (11), which coincides with key player index in Ballester et al. (2006). To the best of our knowledge, there is no analogous simple expression for the key group index (or the intercentrality index) with $k \geq 2$, except for the definition. As a non-trivial generalization of the key player index, Proposition 5 uses self-loops and centralities of the removed players to construct the key group index. Thus, we can conveniently identify the key group from the information in the matrix $M(G)$ without re-computing the new equilibrium after the removal of nodes.

\(^{14}\)The hybrid intervention is given by $\left(\begin{bmatrix} 0_{A^c A^c} & -G_{A^c A} \\ -G_{A^c A} & -G_{AA} \end{bmatrix}, \begin{bmatrix} 0_{A^c} \\ -\theta_A \end{bmatrix}\right)$. By Lemma 2, we then translate into characteristic intervention $\Delta\theta = (-G_{A^c A} \hat{x}_A, \Delta\theta_A)$. Since $\hat{x}_A = 0$, $\Delta\theta_{A^c} = 0_{A^c}$. Besides, the critical value of $\Delta\theta_A$ is chosen such that Equation (12) holds.
Furthermore, the analytically simplicity of the expression in Proposition 5 enables us to make inference regarding the key group.

**Corollary 2.** In network game $\Gamma(G, 1, \delta)$, consider two subsets, $A$ and $A'$,

(i) if $A \subseteq (\subset) A'$, then $d_A(G, 1) \leq (\leq) d_{A'}(G, 1)$;

(ii) if $|A| = |A'|$, $b_A \preceq b_{A'}$, and $M_{AA} \succeq M_{A'A'}$, then $d_A(G, 1) \leq d_{A'}(G, 1)$;

(iii) if $A \subseteq A'$, then, for any $i \notin A'$,

$$d_{A \cup \{i\}}(G, 1) - d_A(G, 1) \geq d_{A' \cup \{i\}}(G, 1) - d_{A'}(G, 1)$$

Corollary 2 (i) is rather intuitive: Removing a larger group induces a more significant impact. In particular, to search for the optimal $S^*$ in equation (10), it is without loss of generality to consider $S$ with $|S| = k$.

Corollary 2 (ii) shows that when comparing groups of the same size, group $A'$ with greater Katz-Bonacich centralities, $b_{A'}$, and fewer walks within any pair of nodes in $A'$ (measured by $M_{A'A'}$) has a larger intercentrality. Corollary 2 (ii) generalizes the argument concerning key player in Ballester et al. (2006), who argue that, holding Katz-Bonacich centrality $b_i$ fixed, the single node’s intercentrality $d_i$ decreases with the proportion of $i$’s Bonachich centrality due to self-loops $m_{ii}/b_i$. When it comes to key group problem, holding the group Katz-Bonachich centrality $b_A$ fixed, the group’s intercentrality $d_A$ decreases with the proportion of $A$’s Bonachich centrality due to intra-group walks. That is, fixing the aggregate walks originating from the group, the higher the proportion of these walks ending at this group, the lower the group intercentrality. Ballester et al. (2010)’s Example 2 discusses this observation within a pair of nodes’ removal. Here, we extend and prove their observation rigorously. It worth noting that, Corollary 2 (ii) further implies $d_A(G, 1) \leq \sum_{i \in A} d_i(G, 1)$ for any $A \subseteq N$. In particular, the equality hold if and only if $m_{ij} = 0$ for any distinct nodes $i, j$ in $A$. That is, the value of group intercentrality is no larger than the sum of all single node’s intercentrality in this group.

Corollary 2 (iii) demonstrates that the intercentrality of group $d_A(G, 1)$ is a submodular function in $A$. As a result, the problem of finding a key group, which maximizes a submodular function, is NP-hard (Ballester et al. 2010).
4.2 Key players in information diffusion

In this subsection, we analyze the role of each node in an information diffusion model. Consider a standard model of information diffusion in network studied by Banerjee et al. (2013, 2019) and Cruz et al. (2017). A set of (identical) agents is embedded in a symmetric and unweighted social network $G$. Information is transmitted in discrete iterations. In period 0, one agent initially has the relevant piece of information. In period $t$, agents in a specific subset transmit the information independently to each of their neighbours in the network with some probability $\delta$. Information diffusion ends in infinite periods.

Bramoullé and Garance (2018) introduce three assumptions on the diffusion process:

1. Only agents who received the information at $t - 1$ can transmit at $t$;

2. If an agent $i$ receives the information from $S$ distinct resources at $t - 1$, she re-transmits the information independently $S$ times at $t$.

3. (no re-transmission) If one of the sender’s neighbours $i$ receives information at $t - 1$, then she re-transmits the information to all of $i$’s neighbours except the sender. If the targeting node receives information, then he will stop re-transmitting to other nodes.

Under these three assumptions, Bramoullé and Garance (2018) define the expected number of times the targeting node hears information (or political request) from any of nodes in the network as the targeting centrality.$^{15}$ Bramoullé and Garance (2018) argue that no re-transmission assumption is more appealing in the information diffusion and political favor transmission setting when the sender and target are clearly specified. To provide a toolkit to analyze information diffusion in this framework, we introduce a definition of walks that do not pass a given set of nodes.

**Definition 6.** Fixing a non-empty proper subset $A$ of $N$ in network $(N, G)$, for any $i, j \in N$, we define $w_{ij}(G, A)$ as total number of walks with length discount from $i$ to $j$ that do not pass any node in $A$ with the possible exception for the starting node $i$ and the ending node $j$. Let $W(G, A) = (w_{ij}(G, A))_{n \times n}$.

$^{15}$When the no re-transmission assumption is not imposed, Bramoullé and Garance (2018) define the expected number of times the targeting node hears information (or political request) from any of nodes in the network as diffusion centrality, which is closely related to Katz-Bonacich centrality.
Different from $m_{ij}$, $w_{ij}(G, A)$ excludes the walks from $i$ to $j$ that pass group $A$. In particular, $w_{ij}(G, \{i, j\})$ is the expected number of times $j$ hears about the information initiating from $i$ when both the sender $i$ and target $j$ can not re-transmit information. The target centrality of $i$ (proposed by Bramoullé and Garance (2018)) is given by

$$\bar{n}_i = \sum_{j \neq i} w_{ij}(G, \{i, j\}).$$

Moreover, Ballester et al. (2006) utilize the expression of $w_{ij}(G, \{k\})$ to derive the formation of $k$’s intercentrality. The following proposition generalizes these arguments by characterizing the matrix $W(G, A)$ using the indices in $G$.

**Proposition 6.** For any $A \subseteq N$, the following identities hold:

$$W_{AC, AC}(G, A) = M_{AC, AC} - M_{AC, A}(M_{AA})^{-1} M_{AC}$$

(13)

$$W_{AC, A}(G, A) = M_{AC, A}(M_{AA})^{-1}$$

(14)

$$W_{AA}(G, A) = 2I - (M_{AA})^{-1}$$

(15)

In particular, for any $A, B \subseteq N$ and $A \cap B = \emptyset$, then

$$W_{AB}(G, A \cup B) = (M_{AA})^{-1} M_{AB} (W_{BB}(G, A))^{-1}$$

$$= (W_{AA}(G, B))^{-1} M_{AB} (M_{BB})^{-1}$$

(16)

Proposition 6 characterizes the the total number of walks between each pair of nodes which do not pass group $A$. When $A = \{i\}$, for any pair $(j, k) \in A^C$, Equation (13) yields

$$m_{jk} - w_{jk}(G, \{i\}) = \frac{m_{ij}m_{ik}}{m_{ii}}.$$  

This equation is equivalent to Lemma 1 in Ballester et al. (2006), which characterizes the change in walks from $j$ to $k$ after removing a node $i$, summarizing which (among $(j, k)$) leads to the the intercentrality index. Equation (13) extends Lemma 1 in Ballester et al. (2006) to allow for removal of multiple nodes.

In the information diffusion scenario, Proposition 6 captures the expected times each agent receives the information, which is sent from one group and which can not be re-transmitted by another group. More specifically, Equation (16) implies that the expected
number of times $j$ hears from $i$ if both $i$’s and $j$’s re-transmission is excluded is

$$w_{ij}(G, \{i, j\}) = \frac{m_{ij}}{w_{ii}(G, \{j\}) m_{jj}} = \frac{m_{ij}}{m_{ii} w_{jj}(G, \{i\})}.$$ 

It generalizes Bramoullé and Garance (2018)’s targeting centrality in two directions: First, Equation (15) captures the expected times $j$ hears information from $i$ when re-transmission is excluded among a group including more nodes than just the $i$ and $j$. Second, Equation (14) covers the case where either the sender’s or the target’s re-transmission is excluded. Bramoullé and Garance (2018) has mentioned that we should only exclude re-transmission from the party (i.e. the sender or the target) which is clearly mentioned in the text. Finally, it is worth noting that Equation (16) demonstrates a symmetric decomposition of the walks between two disjoint sets of nodes in the network. This symmetric property is a group generalization of Bramoullé and Garance’s observation (cf. footnote 7 in Bramoullé and Garance (2018)).

In the framework of information diffusion, social network plays a key role as information or political intermediation. In the spirit of key player in delinquent network in Ballester et al. (2006), a natural question then arises, who is the key information or political intermediary? Nowadays, information transmission is always conducted through social networking application, like twitter, facebook etc. However, fake news are also spread and prevalent along network links, especially during the American presidential campaign and COVID-19. To deal with it, these platforms has to examine and verify the credibility of information transmitted by some internet celebrities. Whenever platforms find some fake news, they have to block the re-transmission from these celebrities to curtail the spreading of fake news. Therefore, we want to study which celebrity platforms should spend the most resources to supervise?

First, we define the intermediary centrality which characterizes the role of each node in the network in the information diffusion process. Recall that, the targeting centrality of agent $i$ in network $G$ is defined as $\bar{n}_i = \sum_{j \neq i} w_{ij}(G, \{i, j\})$.

---

$^{16}$Bramoullé and Garance (2018) state that (in our notation):

“For any $i$, $j$, $m_{ii} w_{jj}(G, \{i\}) = m_{jj}(G) w_{ii}(G, \{j\})$. To our knowledge, this provides a novel result in matrix analysis.”

The symmetric property holds since $w_{ii}(G, \{j\}) = m_{ii} - \frac{m_{ii}^2}{m_{jj}}$. When both $A = \{i\}$ and $B = \{j\}$ are singleton, one can cancel $m_{ij}$ on both sides of Equation (16), which yields the symmetric equation in Bramoullé and Garance (2018).
Definition 7. The intermediary centrality of node $i$ in network $(N, G)$ is defined as

$$r_i := \sum_{k \in N} \bar{n}_k - \sum_{k \neq i} \bar{n}_k(G_{(i)c}).$$

The node with the largest intermediary centrality is called the key intermediary.

When each node in the network has equal probability to be either the fake news sender or the target. Node $i$’s intermediary centrality is proportional to decrease in expected times the random target receives the fake news, due to removal of node $i$. Therefore, supervising the node with highest intermediary centrality, such that the node can neither initiate, receive nor re-transmit the fake news, results in the largest decrease in aggregate expected times the random target receives the fake news. Node $i$’s intermediary centrality can be decomposed into two parts, the reduction due to excluding $i$ from initiating the fake news, i.e. node $i$’s targeting centrality $\bar{n}_i$, and decrease in the rest nodes’ targeting centralities, i.e. $\sum_{k \neq i} \bar{n}_k(G_{(i)c})$. It is similar to intercentrality defined in Ballester et al. (2006), except that we have imposed the assumption that re-transmission is excluded from both the sender and target.

Definition 8. The key intermediary for sender-target pair $(i, j)$ is

$$k^*_1 \in \arg \max_{k \notin \{i,j\}} [w_{ij} - w_{ij}(G, \{i,j,k\})]$$

The key intermediary in network $(N, G)$ is $k^*_2 \in \arg \max_{k \in N} r_k$

Corollary 3. We have the following,

1. For any set of node $\{i, j, k\} \in N$, we have

$$w_{ij}(G, \{i, j, k\}) = \frac{m_{ij}m_{kk} - m_{ik}m_{jk}}{m_{ii}m_{jj}m_{kk} + 2m_{ij}m_{ik}m_{jk} - m_{ii}m_{jk}^2 - m_{ij}^2m_{kk} - m_{ik}^2m_{jj}}.$$

Moreover, for distinct nodes $k, k' \notin \{i, j\}$, if $m_{kk} \leq m_{k'k'}$, $m_{ik} \leq m_{ik'}$ and $m_{jk} \leq m_{j{k'}}$, then $\bar{n}_{ij}(G_{(k)c}) \geq \bar{n}_{ij}(G_{(k')c})$.

2. The key intermediary $k^* \in \arg \min_{k \in N} \sum_{i \neq k, j \notin \{i, k\}} w_{ij}(G, \{i, j, k\})$.

This corollary follows immediately from Equation (15) by setting $A = \{i, j, k\}$. If node
$k$ is more connected to both the sender $i$ and the target $j$, and node $k$ has larger self-loop, then node $k$ is more important as an intermediary for sender-target pair $(i, j)$.

5 The key bridge connecting separated networks

5.1 The bridge index and the key bridge

Consider two isolated networks, $(N_1, G^1)$ and $(N_2, G^2)$, where $N_i$ denotes the set of players and $G^i$ denotes the corresponding symmetric, zero-one matrix for $i \in \{1, 2\}$. Denote $N = N_1 \cup N_2$ and the adjacency matrix $G = \begin{bmatrix} G^1 & 0 \\ 0 & G^2 \end{bmatrix}$. Note that $m_{ij} = m_{ji} = 0$ for $i \in N_1, j \in N_2$. To simplify the notation, we set $\theta_k = 1$ for any $k \in N$ throughout this section.

The planner’s problem is to maximize aggregate equilibrium effort by adding a new link between some pair of nodes $(i, j) \in N_1 \times N_2$. Formally, the structural intervention connecting $(i, j)$ is represented by a matrix $E_{ij}$ where $E_{ij} = E_{ji} = 1$ and the rest entries are zero. The planner solves

$$\max_{(i,j) \in N_1 \times N_2} b(G + E_{ij}).$$

This problem is called key bridge problem (KBP). The pair of nodes $(i^*, j^*)$ that solves KBP is the key bridge pair. We call node $i^*$ (node $j^*$) as the key bridge player in network $G^1(G^2)$.

The key bridge problem naturally arise in many economic settings. For example, in the integration of new immigrants into a new country, the communication between culture leaders serves as a bond connecting two initially isolated communities (see Verdier and Zenou (2015, 2018)). For another example, a firm can be viewed as a network among workers with synergies as a worker’s productivity is influenced by his peers’ through knowledge sharing and skill complementarity. Building up interfirm social connections creates further economic values. For instance, Cai and Szeidl (2018) document the effects of interfirm meetings among young Chinese firms on their business performance.17 First, we define the bridge index as the difference in aggregate Katz-Bonacich centrality prior and post bridge build-up.

---

17In a large-scale experimental study of network formation, Choi et al. (2019) highlight the role of connectors and influencers.
Definition 9. For any pair \((i, j) \in N_1 \times N_2\), define the bridge index \(L_{ij}(G^1, G^2)\) as

\[
L_{ij}(G^1, G^2) = \frac{b(G + E_{ij}) - b}{\delta}.
\]  

(18)

By definition, the key bridge pair \((i^*, j^*)\) maximizes the bridge index. Our next proposition characterizes the bridge index based on the indices of \(G\).

Proposition 7. For any pair \((i, j) \in N_1 \times N_2\), a bridge index for this pair is given as

\[
L_{ij}(G^1, G^2) \equiv \frac{\delta m_{jj}b_i^2 + \delta m_{ii}b_j^2 + 2b_jb_i}{1 - \delta^2 m_{jj}m_{ii}}.
\]  

(19)

Only two players \(i, j\) are involved in this structural intervention. Therefore, by Lemma 2 (1), connecting the pair \((i, j)\) is outcome equivalent to increase bridge players’ characteristics \(\Delta \theta_i = \delta \hat{x}_j\) and \(\Delta \theta_j = \delta \hat{x}_i\) respectively. By Lemma 1, \(\hat{x}_i, \hat{x}_j\) and therefore \(\Delta \theta\) can be solved from the following linear equation system

\[
\begin{aligned}
\hat{x}_i &= b_i + m_{ii} \frac{\delta \hat{x}_j}{\Delta \theta_i} \\
\hat{x}_j &= b_j + m_{jj} \frac{\delta \hat{x}_i}{\Delta \theta_j}.
\end{aligned}
\]  

(20)

Finally, by Proposition 1 (3), Proposition 7 just follows

\[
\Delta x = b_i \frac{\delta \hat{x}_j}{\Delta \theta_i} + b_j \frac{\delta \hat{x}_i}{\Delta \theta_j} = \delta L_{ij}.
\]

The bridge index \(L_{ij}\) summarizes all of the walks passing new bridge \(i, j\) at least once. Equation (19) can be interpreted from path counting perspective. Specifically, the term \(b_jb_i\) in equation (19) counts the aggregate walks which pass the bridge once (see Figure 1). The term \(\delta m_{jj}b_i^2\) counts the aggregate walks starting from nodes in \(G^2\) and ending at nodes in \(G^2\), which passes the bridge for exactly two times (see Figure 2). Similarly applies for the term \(\delta m_{ii}b_j^2\). The term \(\frac{1}{1 - \delta^2 m_{jj}m_{ii}}\) counts the iterations.

To obtain further insights on who exactly is the key bridge player using the primitive information, we present the following Corollary. Let \(e_i = \sum_{j \in N} g_{ij}\) denote degree of player \(i\).
Figure 1: The walks start from and end at a different networks by passing the bridge once. These walks are composed of three segments: In the first segment, starting from some nodes in $N_1$, these walks transit to node $i$ in $G^1$, which are summed up as $b_i$; In the second segment, the walks pass the bridge with discount $\delta$; In the third segment, the walks resume from node $j$ and transit to any nodes in $G^2$ without passing the bridge, which are summed up as $b_j$.

**Corollary 4.** The following properties about the bridge index $L_{ij}(G^1, G^2)$ hold:

(i) For two nodes $i$ and $i'$ in $N_1$ with $b_i \geq b_{i'}$ and $m_{ii} \geq m_{i'i'}$, we have $L_{ij}(G^1, G^2) \geq L_{i'j}(G^1, G^2)$ for any $j \in N_2$.

(ii) For nodes $i, i'$ in $N_1$ and $j, j'$ in $N_2$ such that $b_i = b_{i'}, m_{jj} = m_{j'j'}$, $m_{ii} \geq m_{i'i'}$ and $b_j \geq b_{j'}$, we have

$$L_{ij}(G^1, G^2) - L_{i'j'}(G^1, G^2) \geq L_{i'j}(G^1, G^2) - L_{i'j'}(G^1, G^2).$$

(iii) For two nodes $i, i'$ in $N_1$ with $e_i > e_{i'}$, there exists $\tilde{\delta} > 0$ such that for any $0 < \delta < \tilde{\delta}$, $L_{ij}(G^1, G^2) > L_{i'j}(G^1, G^2)$ for any $j \in N_2$.

Corollary 4 (i) implies that to find key bridge player $i^*$ in first network $G^1$, it suffices to focus on the set of nodes in $N_1$ that lie on the Pareto frontier of Katz-Bonacich centrality $b_i$ and self-loops $m_{ii}$. Therefore, if the most active player (the one with highest $b_i$ also happens to the one with highest $m_{ii}$, it must be the key bridge player. The reason is that $L_{ij}(G^1, G^2)$ increases with $b_i$ and $m_{ii}$, fixing $j$. Meanwhile, the intercentrality index, $\frac{b_i^2}{m_{ii}}$, increases in $b_i$ but decreases in $m_{ii}$. Therefore, the key player may differ from the key bridge player.

Moreover, when the player with largest $b_i$ differs from the one with largest $m_{ii}$ in $N_1$, the selection of the key bridge player in $N_1$ crucially depends who is chosen as the bridge
Figure 2: The walks start from and end at a same network $G^1$ by passing the bridge twice. These walks are composed of five segments: In the first segment, these walks start from some node in $N_1$ and transit to node $i$ in network $G^1$, summed up as $b_i$; Then the walks pass the bridge first time with discount $δ$; In the third segment, the walks resume from node $j$ and transit back to node $j$ without passing the bridge, which is summed up by $m_{jj}$; Then, the walks passes the bridge for the second time; Finally, the walks resume from node $i$ and transit to any node in $G^1$, without passing the bridge. The last segment is summed up by $b_i$.

player $j$ in the other network $G^2$. In other words, the selection of the key bridge pair is not independent. As suggested by Corollary 4 (ii), the role of node $i$’s self-loops in determining the bridge index $L_{ij}(G^1, G^2)$ becomes more significant if a node $j$ with larger Katz-Bonacich centrality is selected. Correspondently, $i$’s Katz-Bonacich centrality plays a more important role than self-loops when a node $j$ with larger self-loops is selected. Consider a scenario in which these two networks are highly imbalanced, in the sense that the largest Katz-Bonacich centrality in network $G^2$ is much larger than the one in network $G^1$. Then the key bridge pair may consist of the central node in $G^2$ and the node with largest self-loop in network $G^1$ (even if such node may not be the most central one). It is because the self-loop of the player in $G^1$ contributes more than Katz-Bonacich centrality on the bridge index $L_{ij}(G^1, G^2)$ when Katz-Bonacich centrality of the bridge player in $G^2$ is significantly large. Furthermore, when $δ$ is below a threshold, the degree centrality plays the dominant role in the bridge index by item (iii). We illustrate these observations using the following example.

**Example 1.** Consider two isolated networks depicted in Figure 3.

Table 1 gives the Katz-Bonacich centrality $b_i$ and self-loops $m_{ii}$ measures for $δ = 0.25$. In the first network $(N_1, G^1)$, the hub player $h$ is more important than any of the periphery node both in terms of centrality and self-loops measures. In the second network $(N_2, G^2)$,

---

[18] This could be the case when network $G^2$ involves much more network members, who have denser connections among each other.
Table 1: Measures $\delta = 0.25$

| Players | $m_{ii}$   | $b_i$   |
|---------|------------|---------|
| $a_1$   | 1.5686     | 4.7059  |
| $a_2$   | 1.5980     | 4.6765  |
| $a_3$   | 1.2686     | 3.2059  |
| $a_4$   | 1.4255     | 4.1471  |
| $a_5$   | 1.0980     | 2.1765  |
| $h$     | 1.7778     | 4.8889  |
| $l$     | 1.1111     | 2.2222  |

Table 2: Bridges with $\delta = 0.25$

| bridge $i$-$j$ | $L_{ij}(G)$ |
|----------------|-------------|
| $h$-$a_1$      | 78.9970     |
| $h$-$a_2$      | 79.0258     |

$a_1$ dominates $a_2$ in terms of Katz-Bonacich centrality $b_i$, while $a_2$ dominates $a_1$ in terms of self-loops $m_{ii}$. All the other nodes in $N_2$ are dominated by $a_1$ and $a_2$ in both $m_{ii}$ and $b_i$. By Corollary 4 (i), the key bridge pair is either $(h,a_1)$ or $(h,a_2)$. Table 2 demonstrates that the bridge index $L_{h a_2}(G^1, G^2) > L_{h a_1}(G^1, G^2)$, thus $a_2$ is the key bridge player in $(N_2, G^2)$, yet $a_2$ is neither the most active player (in terms of $b_i$) nor the key player (in terms of the inter-centrality $b_i^2/m_{ii}$) in $(N_2, G^2)$.

Next, we consider $\delta = 0.23$ (see Tables 3 and 4). By the same logic, it suffices to consider connecting the hub $h$ in the first network to either $a_1$ or $a_2$ in the second network. However, for $\delta = 0.23$, the $b_i$ plays a more prominent role in the bridge index than $m_{ii}$, and indeed $a_1$ is now the key bridge player. This observation is consistent with Corollary 4 (iii): the key bridge player is the player with highest degree when $\delta$ is relatively small (the degree of $a_1$ is larger than that of $a_2$).

Keeping $\delta = 0.23$. Suppose we increase the periphery nodes in $(N_1, G^1)$ from 7 to 17. The Katz-Bonacich centrality of hub player $b_h = 48.76$ is significantly larger than that of players in $N_2$. Thus, the self-loops $m_{jj}$ of the bridge player $j$ in $N_2$ is more pronounced, compared with his Katz-Bonacich centrality $b_j$ in shaping the relative values of $L_{hj}$. As a
Table 3: Measures with $\delta = 0.23$

| Players | $m_{ii}$ | $b_i$ |
|---------|----------|-------|
| $a_1$   | 1.4213   | 3.8423 |
| $a_2$   | 1.4300   | 3.7545 |
| $a_3$   | 1.1969   | 2.6348 |
| $a_4$   | 1.3063   | 3.3533 |
| $a_5$   | 1.0752   | 1.8837 |
| $h$     | 1.5881   | 4.1448 |
| $l$     | 1.0840   | 1.9533 |

Table 4: Bridges with $\delta = 0.23$

| Bridge $i$-$j$ | $L_{ij}(G)$ |
|----------------|--------------|
| $h$-$a_1$      | 48.6711      |
| $h$-$a_2$      | 47.6461      |

result, $(h,a_2)$ is the key bridge (indeed, $L_{ha_2}(G^1,G^2) = 4744 > L_{ha_1}(G^1,G^2) = 4680$).

5.2 The link index

Instead of considering two isolated networks, in this subsection we consider a general network $(N,G)$ and examine the effects of link creation on the aggregate equilibrium efforts.

**Proposition 8.** For any nodes $i \neq j$ with $G_{ij} = 0$, then we have

$$b(G + E_{ij}) - b = \delta L_{ij},$$

where

$$L_{ij} = \frac{\delta m_{ii} b_j^2 + \delta m_{jj} b_i^2 + 2(1 - \delta m_{ij}) b_i b_j}{(1 - \delta m_{ij})^2 - \delta^2 m_{ii} m_{jj}}.$$  \hspace{1cm} (22)

Different from the derivation of key bridge index (Equation (20)), the post-intervened equilibrium efforts of newly connected players are given by

$$\begin{cases}
\hat{x}_i = b_i + m_{ii} \delta \hat{x}_j + m_{ij} \delta \hat{x}_i = \Delta \theta_i = \Delta \theta_j \\
\hat{x}_j = b_j + m_{jj} \delta \hat{x}_i + m_{ij} \delta \hat{x}_j = \Delta \theta_j = \Delta \theta_i
\end{cases}$$

(23)
which yields to

\[ \hat{x}_i = \frac{b_i + \delta_{m_{ii}}b_j}{(1 - \delta_{m_{ij}}) - \delta_{m_{ii}}m_{jj}}. \] (24)

Indeed, when \( G \) is the union of two isolated networks \( (N_1, G_1) \) and \( (N_2, G^2) \), we have \( m_{ij} = 0 \) for \( i \in N_1, j \in N_2 \), and the index \( L_{ij} \) in equation (22) reduces to \( L_{ij}(G_1, G^2) \) in equation (19).

The index \( L_{ij} \) measures the value of a potential new link \((i, j)\) in network \( G \). Therefore, this index is useful for determining the optimal location for a new link (for instance, the key bridge problem). In particular, König et al. (2014) introduce a network formation process and they show that the formed network is nested split graph at each stage. A key lemma for their result is that, in any nested split graph, any player who has the chance to build up a link will always connect the player with the highest degree, and any player who has to delete a link will always remove his neighbour with lowest degree. Equation (24) implies this observation directly.

**Corollary 5.** Consider two nodes \( j \) and \( k \) satisfying \( G_{jl} \geq G_{kl}, \forall l \notin \{i, j\} \), then

1. For any node \( i \) with \( G_{ij} = 0 \) and \( G_{ik} = 0 \), \( b_i(G + E_{ij}) \geq b_i(G + E_{ik}) \);

2. For any node \( i \) with \( G_{ij} = 1 \) and \( G_{ik} = 1 \), \( b_i(G - E_{ij}) \leq b_i(G - E_{ik}) \).

Equation (24), which characterizes the impacts of connecting \((i, j)\) on \( i \)'s Katz-Bonacich centrality, provides a simple proof of Corollary 5. Specifically, since \( j \) dominates \( k \) in degree (weight), Lemma 3 implies that \( m_{jj} \geq m_{kk} \), \( m_{ij} \geq m_{ik} \) and \( b_j \geq b_k \). The first part of Corollary 5 immediately holds since \( \hat{x}_i \) increases with these indexes. For the second part, consider an intervention that removes both link \((i, j)\) and \((i, k)\). Then, \( j \) also dominates \( k \) in the post intervened network \( G - E_{ij} - E_{ik} \). According to the first part, connecting \((i, j)\), which induces network \( G - E_{ik} \), generates a higher equilibrium outcome for \( i \) than that connecting \((i, k)\). That is, \( b_i(G - E_{ik}) \geq b_i(G - E_{ij}) \). Note that we can easily generalize this argument in weighted network.

The index \( L_{ij} \) is increasing in \( m_{ij} \). It implies that the social planner prefers to connect the node pair who has already been well connected in the original network \( G \). Therefore, given all else equal (i.e. same Katz-Bonacich centrality and Self-loop), the social planner prefers to build up a link within an already well-connected group instead of bridging separated groups.
Then should we encourage interaction among separated races, groups, ethics and so on? We argue that the bridge has exerted positive externality through connecting nodes in separated networks. This externality on connectedness lays the foundation for future links between these two networks. As a result, building up a bridge may be myopically sub-optimal but a necessary link at global optimum. We illustrate this insight through the following example.

Example 2. Consider a network $G$ composed of two separated cycles, each with size four. We search for the optimal way to add one or two links, where the links can be formed between or within two circles.

(a) Consider adding one link. By symmetry, it suffices to compare $\tilde{G}_1$ and $\tilde{G}_2$ in Figure 4. Given $m_{23} > m_{25} = 0$ and all other measures are equal, adding the intra-group link $(2,3)$ strictly dominates adding the inter-group link $(2,5)$, i.e., $\tilde{G}_2$ dominates $\tilde{G}_1$ (see Table 5).

![Figure 4: Adding a single link between two separated networks](image)

(b) What if we can add one additional link? It is easy to see that the optimal network is among the following three networks: $G_1$, $G_2$, and $G_3$ (see Figure 5). Among these three, see Table 5 shows that $G_3$ is the optimal. In other words, connecting two inter-group bridges $((2,5),(2,7))$ strictly dominates building up two intra-group links $((1,4),(2,3))$, even though building up one intra-group link is myopically optimal as in part (a).

---

19 All other ways of forming two links are dominated. Denote $((i,j),(k,l))$ as an intervention of adding two links. For instance, $((2,5),(4,7))$ is strictly dominated by $((2,5),(2,7))$ since $b_2 > b_4$, $m_{22} > m_{44}$ and $m_{27} > m_{47}$ once a bridge $(2,5)$ is added. $((2,5),(2,7))$ strictly dominates $((2,5),(2,8))$ since $m_{27} > m_{28}$ once a bridge $(2,5)$ is added. $((2,3),(6,7))$ is strictly dominated by $((1,4),(2,3))$ since $b_1 > b_6$, $m_{11} > m_{66}$ and $m_{14} > m_{67}$ once $(2,3)$ is added. $((2,5),(6,7))$ is strictly dominated by $((5,8),(6,7))$ since $b_8 > b_2$, $m_{88} > m_{22}$ and $m_{58} > m_{25}$ once $(6,7)$ is added.

20 Starting with $G_2$, the optimal network with one extra link by part (a), $(1,4)$ strictly dominates $(2,5)$ as the second link as $G_2$ has higher aggregate Katz-Bonacich centralities than $G_1$ in terms of (see Table 5).
6 Concluding remarks

In this paper, we present a theory of interventions in network. By showing an equivalence between a structural intervention and an endogenously determined characteristic intervention, we analyze how these two types of interventions affect the equilibrium actions and offer new insights regarding the optimal interventions in a range of applications.

We discuss several venues for future work. First, this paper mainly focuses on the benefit of structural interventions without explicitly modeling the cost of cutting/building links and nodes. It is interesting to study the optimal intervention policy with some budget on the cost of interventions (see Galeotti et al. (2020)). Second, we treat two instruments, i.e., characteristics and social links, independently in our analysis. In some contexts, intervention in one space (say the characteristics) may induce endogenous responses in the other space (the network links). For instance, Banerjee et al. (2018) show that new links are formed and existing links are removed after exposure to formal credit markets. Extending hybrid interventions to accommodate interdependence between two instruments is an intriguing subject. Finally, It is natural to extend our approach to network games with non-linear responses (see, for instance, Allouch (2017) and Elliott and Golub (2018)). These and other

However, neither $\bar{G}_1$ nor $\bar{G}_2$ is optimal. Nevertheless, starting with the dominated network, $\hat{G}_1$ in part (a), we can reach the optimal network $\bar{G}_3$ by adding the link $(2,7)$.
generalizations will enrich our understanding of optimal interventions in economic settings involving networks.
Appendix

A Proofs

Proof of Lemma 1: In the game $\Gamma (G, \theta)$, the equilibrium action profile is $x^* = M\theta$, and the aggregate equilibrium action is $x^* = 1'M\theta = b'(G)\theta$. Since the network $G$ is fixed for a characteristic intervention, the results directly follow. $\square$

Proofs of Lemma 2: The equilibrium actions of $\Gamma (G, \theta)$ satisfies $x = \theta + \delta Gx$. Under the structural intervention $C$, the new equilibrium action profile $\hat{x}$ satisfies

$$\hat{x} = \theta + \delta (G + C) \hat{x} = \left( \theta + \delta \frac{C \hat{x}}{\Delta \theta} \right) + \delta G \hat{x}. $$

That is, the structural intervention $C$ is outcome equivalent to a change of players’ intrinsic marginal utilities from $\theta$ to $\theta + \Delta \theta = \theta + \delta C \hat{x}$. Given $C = \begin{bmatrix} 0 & 0 \\ 0 & C_{SS} \end{bmatrix}$, we obtain

$$\Delta \theta = \delta C \hat{x} = \delta \begin{bmatrix} 0 & 0 \\ 0 & C_{SS} \end{bmatrix} \begin{bmatrix} \hat{x}_S^C \\ \hat{x}_S \end{bmatrix} = \begin{bmatrix} 0 \\ \delta C_{SS} \hat{x}_S \end{bmatrix}. $$

That is, the structural intervention $C$ is outcome equivalence to changing the characteristics of players in $S$ by $\Delta \theta_S = \delta C_{SS} \hat{x}_S$. Similarly, under hybrid intervention $(C, \Delta \tilde{\theta}) = \left( \begin{bmatrix} 0_{SC} & 0_{SC} \\ 0_{SS} & C_{SS} \end{bmatrix}, \begin{bmatrix} 0_{SC} \\ \Delta \theta_S \end{bmatrix} \right)$, the outcome-equivalent characteristic intervention is given by $\Delta \theta = C \hat{x} + \Delta \tilde{\theta} = \left[ 0_{SC}, \delta C_{SS} \hat{x}_S + \Delta \tilde{\theta}_S \right]^T.$ $\square$

Proofs of Proposition 1: The impact of the outcome equivalent characteristic intervention is $\hat{x}_S - x_S = M_{SS} \frac{\delta C_{SS} \hat{x}_S}{\Delta \theta}$. Since $x_S = b_S$, we have $\hat{x}_S = (I - \delta M_{SS} C_{SS})^{-1} b_S$, and thus, $\Delta \theta = \delta C_{SS} (I - \delta M_{SS} C_{SS})^{-1} b_S$. The rest of Proposition 1 directly follows Lemma 1. $\square$

Proof of Proposition 2: We prove this proposition by in Farkas’ Lemma, which is outlined below,
Lemma 4. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two assertions is true:

1. There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$;

2. There exists a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.

$x \in X(\theta; \delta)$ if and only if $(I - \delta \hat{G})x = \theta$. Denote the matrix $A$ as

$$A = \begin{bmatrix}
    x_2 & \ldots & x_{n-1} & x_n & 0 & 0 & 0 \\
    x_1 & 0 & 0 & 0 & x_3 & \ldots & x_n \\
    0 & x_1 & \ldots & x_2 & 0 & \ldots & \ldots & x_{n-1} & x_n & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & x_{n-2} & 0 & x_n \\
    0 & 0 & \ldots & x_1 & 0 & \ldots & x_2 & \ldots & \ldots & 0 & x_{n-2} & x_{n-1} \\
\end{bmatrix}. $$

Moreover, denote vector $g$ as

$$g^T = \begin{bmatrix}
    \hat{G}_{12} & \hat{G}_{13} & \ldots & \hat{G}_{1n} & \hat{G}_{23} & \ldots & \hat{G}_{2n} & \ldots & \hat{G}_{n-1,n} \\
\end{bmatrix}. $$

The equation $(I - \delta \hat{G})x = \theta$ can be rewritten as

$$x - \theta = \delta Ag. \quad (25)$$

By Farkas’ Lemma, there exists some $g \in (\mathbb{R}^+)^{\frac{n(n-1)}{2}}$ such that Equation (25) holds if and only if there does not exist $y \in \mathbb{R}^n$ such that

$$\begin{cases}
    A^T y \geq 0 \Leftrightarrow x_j y_i + x_i y_j \geq 0, \forall 1 \leq i < j \leq n \\
    (x - \theta)^T y < 0
\end{cases}. \quad (26)$$

Now we prove “if” part first. Suppose there does exist some $y \in \mathbb{R}^n$ such that Equation (26) holds. First, it implies that there is exactly a single entry $y_l < 0$ and the rest are weakly positive. Second, we have an upper bound for $-y_l$ as

$$-y_l \leq \min_{i \neq l} \frac{x_l}{x_i} y_i.$$ 

Therefore we have that
\[(x_l - \theta_l) \min_{i \neq l} \frac{x_l}{x_i} y_i \geq -(x_l - \theta_l) y_l = \sum_{i \neq l} (x_i - \theta_i) x_i^\top y_i = \sum_{i \neq l} (x_i - \theta_i) x_i^\top y_i \geq \min_{i \neq l} \frac{x_l}{x_i} \sum_{i \neq l} (x_i - \theta_i) x_i,\]

which therefore implies that

\[(x_l - \theta_l) x_l > \sum_{i \neq l} (x_i - \theta_i) x_i.\]

Since we allow for any \(l\), it must also be true that

\[\sum_{i=1}^{n} (x_i - \theta_i) x_i < 2 \max_{1 \leq j \leq n} (x_j - \theta_j) x_j,\]

which then leads to contradiction.

Then we prove the “only if” direction. Suppose that there exists some \(x \in X(\theta; \delta)\) such that

\[\sum_{i=1}^{n} (x_i - \theta_i) x_i < 2 \max_{1 \leq j \leq n} (x_j - \theta_j) x_j.\]

Without loss of generality, let me assume that \(n \in \arg \max_{1 \leq j \leq n} (x_j - \theta_j) x_j\). For any \(y > 0\), consider the vector \(y \in \mathbb{R}^n\) where

\[
\begin{cases}
  y_n = -y \\
y_i = \frac{x_i}{x_n} y \quad \text{if} \quad i \neq n.
\end{cases}
\]

One can easily verify that \(y\) is a solution to Equation (26), which therefore leads to contradiction. \(\Box\)

**Proof of Lemma 3:** We at first show that \(m_{ik} \geq m_{jk}\) for any \(k \notin \{i, j\}\). Suppose that \(m_{ik} < m_{jk}\), one then have that

\[m_{ik} = \delta \sum_{k' \notin \{i, j\}} G_{ik'} m_{k'k} + \delta G_{ij} m_{jk} \geq \delta \sum_{k' \notin \{i, j\}} G_{jk'} m_{k'k} + \delta G_{ji} m_{ik} = m_{jk},\]

which then leads to contradiction.
Second, $m_{ii} \geq m_{jj}$ since

$$m_{ii} = \delta \sum_{k \not \in \{i,j\}} G_{ik} m_{ki} + \delta G_{ij} m_{ji} \geq \delta \sum_{k \not \in \{i,j\}} G_{jk} m_{kj} + \delta G_{ji} m_{ij} = m_{jj}. \quad \square$$

**Proof of Proposition 3:**

Given a structural intervention $C$ such that $i$ dominates $j$ in $G + C$, consider a hybrid intervention $(C, \Delta \theta)$ such that $b_i(G) = b_i(G + C, \Delta \tilde{\theta})$ and $b_j(G) = b_j(G + C, \Delta \tilde{\theta})$. We use $\hat{x}$ and $\tilde{x}$ to denote the equilibrium profile of network game $\Gamma(G + C, 1)$ and $\Gamma(G + C, 1 + \Delta \tilde{\theta})$ respectively. Let $K = \{i, j\}^C$, for any $k \in K$, we have

$$\tilde{x}_k = 1 + C_{ki} (\tilde{x}_i - \tilde{x}_j) + \delta \sum_{l \in K} G_{kl} \tilde{x}_l + \delta G_{ki} \tilde{x}_i + \delta G_{kj} \tilde{x}_j$$

$$= [1 + C_{ki} (b_i - b_j) + \delta G_{ki} \tilde{x}_i + \delta G_{kj} \tilde{x}_j] + \delta \sum_{l \in K} G_{kl} \tilde{x}_l.$$

Rewriting in matrix form,

$$\tilde{x}_K = (I - \delta G_K)^{-1} [1 + C_K (b_i - b_j) + \delta G_{K,i} \tilde{x}_i + \delta G_{K,j} \tilde{x}_j]$$

$$\geq (I - \delta G_K)^{-1} [1 + \delta G_{K,i} \tilde{x}_i + \delta G_{K,j} \tilde{x}_j] = x_K.$$

That is, each player $k$ in $K$ increases his equilibrium effort under the hybrid intervention $(C, \Delta \tilde{\theta})$.

The first order conditions for node $l \in \{i, j\}$ is outlined as

$$\tilde{x}_l = 1 + \Delta \tilde{\theta}_l + \delta \sum_{m \in N} (G + C)_{lm} (x_m + \tilde{x}_m - x_m). \quad (27)$$

Combine this equation with the fact that

$$\tilde{x}_l = x_l = 1 + \delta \sum_m G_{lm} x_m,$n

we then have
\[ \Delta \tilde{\theta}_l = -\delta \sum_m G_{lm} (\tilde{x}_m - x_m) - \delta \sum_m C_{lm} \tilde{x}_m. \]

Since \( C_{im} \geq 0 \) for any \( m \in N \), \( \Delta \tilde{\theta}_i < 0 \). Moreover, since \( C_{im} + C_{jm} = 0 \), \( \forall m \in N \), we also have that

\[ \Delta \tilde{\theta}_i + \Delta \tilde{\theta}_j = -\delta \sum_{l \in \{i, j\}} \sum_m G_{lm} (\tilde{x}_m - \tilde{x}_m) < 0. \]

We then consider a characteristic intervention \(-\Delta \tilde{\theta}\) on the (post hybrid intervention) game \( \Gamma(G + C, 1 + \Delta \tilde{\theta}) \) to analyze the impact of weight reallocation \( C \) on the original game \( \Gamma(G, 1) \). Let \( \tilde{M} = (I - \delta(G + C))^{-1} \). By Lemma 1, the impact of characteristic intervention \(-\Delta \tilde{\theta}\) on the equilibrium effort of player \( k \in K \) is given by

\[ \Delta x_k = -\Delta \tilde{\theta}_i \tilde{m}_{ki} - \Delta \tilde{\theta}_j \tilde{m}_{kj} = - (\Delta \tilde{\theta}_i + \Delta \tilde{\theta}_j) \tilde{m}_{ki} - \Delta \tilde{\theta}_j (\tilde{m}_{kj} - \tilde{m}_{ki}) > 0. \]

According to Equation (27), for player \( l \in \{i, j\} \) we have

\[ \Delta x_l = -\Delta \tilde{\theta}_l + \delta C_{lK} \Delta x_K + \delta G_{lK} \Delta x_K. \]

Therefore,

\[ \Delta x_i + \Delta x_j = -\Delta \tilde{\theta}_i - \Delta \tilde{\theta}_j + \delta G_{iK} \Delta x_K + \delta G_{jK} \Delta x_K > 0. \]

Moreover, since \( \tilde{x}_i > \tilde{x}_j \) and \( q(\cdot) \) is weakly convex, the result trivially follows. \( \square \)

**Proof of Corollary 1:** It immediately follows from Proposition 3. \( \square \)

**Proof of Proposition 4:** Define \( \eta(t) = 1'(I - \delta(G + tC))^{-1} 1, t \in [0, 1] \). Since we have assumed that \( I - \delta G \) and \( I - \delta(G + C) \) are both symmetric positive definite, \( (I - \delta(G + tC)) \) is positive definite for any \( t \in [0, 1] \), hence \( \eta \) is well-defined. Given \( \theta = 1 \),

\[ \eta(0) = 1'(I - \delta(G))^{-1} 1 = b(G, 1) \]

is the equilibrium aggregate action before the intervention, and

\[ \eta(1) = 1'(I - \delta(G + C))^{-1} 1 = b(G + C, 1) \]

39
is the equilibrium aggregate action after the intervention. Direct computation shows that
\[
\eta'(0) = \eta'(t)_{t=0} = 1'(I - \delta(G + tC))^{-1}\delta C(I - \delta(G + tC))^{-1}1_{t=0} = 1'(I - \delta G)^{-1}\delta C(I - \delta G)^{-1}1 = \delta b'(G)Cb(G).
\]

Critically, \(\eta(\cdot)\) is convex in \(t\) by Lemma 5 below, therefore, \(\eta(1) - \eta(0) \geq \eta'(0)(1 - 0)\). In other words, \(b(G + C, 1) - b(G, 1) \geq \delta b'(G)Cb(G) \geq 0\). \(\square\)

**Lemma 5.** Let \(\mathcal{O}\) denote the set of \(n\) by \(n\) symmetric positive definite matrices. Then the function \(V(A) := 1'A^{-1}1\) is convex in \(A \in \mathcal{O}\).

**Proof of Lemma 5:** Define \(H(A, x) = 21'x - x'Ax\), where \(A \in \mathcal{O}, x \in \mathbb{R}^n\). Fixing a positive definite matrix \(A \in \mathcal{O}, H(A, \cdot)\) is strictly concave in \(x\) with the maximum value
\[
\max_{x \in \mathbb{R}^n} H(A, x) = 1'A^{-1}1 = V(A),
\]

obtained at \(x^* = A^{-1}1\). Moreover, \(H(A, x)\) is linear in \(A\) for fixed \(x\), so \(V(A) = \max_{x \in \mathbb{R}^n} H(A, x)\) is convex in \(A \in \mathcal{O}\), as the maximum of a family of linear functions is convex (see Boyd and Vandenberghe (2004)). \(\Box\)

**Proof of Proposition 5:** As demonstrated in the main text, \(d_s(G, \theta)\) exactly equals the effect of the characteristic intervention \(\Delta \theta_s = (M_{ss})^{-1} b_s(G, \theta)\) on the aggregate action. Therefore, by Lemma 1, \(d_s(G, \theta) = b_s \Delta \theta_s = b_s (M_{ss})^{-1} b_s(G, \theta)\). \(\Box\)

**Proof of Corollary 2:** The first part is straightforward from the definition of group intercentrality. We prove the last two parts of this corollary in the following. (ii) We first define \(v := (M_{ss})^{-1} b_s(G, 1)\). We first show that \(v\) is a positive vector, i.e, \(v \geq 0\):

\[
v = (M_{ss})^{-1} b_s(G, 1) = (M_{ss})^{-1} (M_{ssc}(G) 1_{|s_c|} + M_{ss}(G) 1_{|s|}) = (M_{ss})^{-1} M_{ssc} 1_{|s_c|} + 1_{|s|} = \delta G_{ssc} (I - \delta G_{scs})^{-1} 1_{|s_c|} + 1_{|s|} \geq 0,
\]

where in the last equality we use the identity \((M_{ss}(G))^{-1} M_{ssc} = \delta G_{ssc} (I - \delta G_{scs})^{-1}\). Moreover, given \(b_s \leq b_s'\), and \(M_{ss}(G) \geq M_{ss'}(G)\), and \(v \geq 0\), we have

\[
d_s(G, 1) = 2b_s(G) v - v'M_{ss}(G) v \leq 2b_s'(G) v - v'M_{ss'}v, \tag{28}
\]

\(^{21}\text{We use the fact that } d(A^{-1}) = -A^{-1}(dA)A^{-1}.\)
Solving the following concave programming yields

\[
\max_{x \in \mathbb{R}^{|S|}} \{2b_S'x - x'M_{S'S'}x\} = b'_S (G) (M_{S'S'})^{-1} b_S' = d_S (G, 1). \tag{29}
\]

By optimality,

\[
\max_{x \in \mathbb{R}^{|S|}} \{2b_S'x - x'M_{S'S'}x\} \geq 2b'_S (G) v - v'M_{S'S'}v. \tag{30}
\]

Combing equation (28), (29), and (30) yields

\[d_S (G, 1) \leq d_S' (G, 1). \tag{iii}
\]

Note that

\[d_{A \cup \{i\}} (G, 1) = d_{A} (G, 1) + d_i (G_{Ac}, 1) \text{ (c.f. Ballester et al. (2010)’s remark 1). Therefore, } d_{A \cup \{i\}} (G, 1) - d_{A} (G, 1) = d_i (G_{Ac}, 1). \text{ To prove Corollary 2 (iii), we only need to show } d_i (G_{Ac}, 1) \geq d_i (G_{Ac'}, 1) \text{ when } A \subseteq A'. \text{ It is straightforward from the definition of group intercentrality.} \]

\[\square\]

**Proof of Proposition 6:**

**Remark 6.** We can have alternative expressions for blocks of the matrix \(W\) as follows

\[
W_{ScSc} (G, S) = (I - \delta G_{ScSc})^{-1};
\]

\[
W_{ScS} (G, S) = (I - \delta G_{ScSc})^{-1} \delta G_{ScS};
\]

\[
W_{SS} (G, S) = \delta G_{SS} (I - \delta G_{ScSc})^{-1} \delta G_{ScS} + \delta G_{SS} + I.
\]

These expressions directly follow from the definition of \(W\). Unlike Proposition 6, these expressions use the centralities in the remaining network \(G_{ScSc}\).

The Leontief inverse matrix can be written in the block form

\[
(I - \delta G)^{-1} = \begin{bmatrix}
I - \delta G_{ScSc} & -\delta G_{ScS} \\
-\delta G_{SSc} & I - \delta G_{SS}
\end{bmatrix}^{-1} = \begin{bmatrix}
M_{ScSc} & M_{ScS} \\
M_{SSc} & M_{SS}
\end{bmatrix}.
\]

For easy notation let \(I - \delta G_{ScSc} = A, -\delta G_{ScS} = B, -\delta G_{SSc} = C\) and \(I - \delta G_{SS} = D\). Then by the remark above, we have \(W_{ScSc} (G, S) = A^{-1}, W_{ScS} (G, S) = -A^{-1} B\) and

\[22\text{As a principle submatrix of positive definite matrix } (I - \delta G)^{-1}, M_{S'S'} \text{ is also positive definite.}\]
\( W_{ss}(G, S) = CA^{-1} B - D + 2I. \) Using the block matrix inversion,

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} + A^{-1} B (D - CA^{-1} B)^{-1} CA^{-1} & -A^{-1} B (D - CA^{-1} B)^{-1} \\
-D (D - CA^{-1} B)^{-1} CA^{-1} & (D - CA^{-1} B)^{-1}
\end{bmatrix}
= \begin{bmatrix}
M_{scsc} & M_{scs} \\
M_{ssc} & M_{ss}
\end{bmatrix}.
\]

Thus, \((D - CA^{-1} B)^{-1} = M_{ss},\) which implies

\[
W_{ss}(G, S) = CA^{-1} B - D + 2I = 2 I + (M_{ss})^{-1}.
\]

We further have \(-A^{-1} B (D - CA^{-1} B)^{-1} = M_{scs}.\) Therefore,

\[
A^{-1} B = W_{scs}(G, S) = -M_{scs}(M_{ss})^{-1}.
\]

In addition, \(A^{-1} B (D - CA^{-1} B)^{-1} CA^{-1} = M_{scs}(M_{ss})^{-1} M_{ssc}.\) Substituting in the identity \(A^{-1} + A^{-1} B (D - CA^{-1} B)^{-1} CA^{-1} = M_{scsc},\) we can get

\[
A^{-1} = W_{scsc}(G, S) = M_{scsc} - M_{scs}(M_{ss})^{-1} M_{ssc}.
\]

Let \(S = A \cup B,\) then the inverse of matrix \(M_{ss}\) is given as

\[
(M_{ss})^{-1} = \begin{bmatrix}
M_{AA} & M_{AB} \\
M_{BA} & M_{BB}
\end{bmatrix}^{-1} = \begin{bmatrix}
(M_{AA})^{-1} \left( I + M_{AB}(W_{BB}(G, A))^{-1} M_{BA}(M_{AA})^{-1} \right) & -(M_{AA})^{-1} M_{AB}(W_{BB}(G, A))^{-1} \\
-(W_{BB}(G, A))^{-1} M_{BA}(M_{AA})^{-1} & (W_{BB}(G, A))^{-1}
\end{bmatrix}.
\]

\[
= \begin{bmatrix}
(W_{AA}(G, B))^{-1} & -(W_{AA}(G, B))^{-1} M_{AB}(M_{BB})^{-1} \\
-(M_{BB})^{-1} M_{BA}(W_{AA}(G, B))^{-1} & (M_{BB})^{-1} \left( I + M_{BA}(W_{AA}(G, B))^{-1} M_{AB}(M_{BB})^{-1} \right)
\end{bmatrix}.
\]
Using (15), we obtain that
\[ W_{AB}(G, A \cup B) = (2I - (M_{SS})^{-1})_{AB} = -(M_{SS})_{AB} \]
\[ = (M_{AA})^{-1}M_{AB}(W_{BB}(G, A))^{-1} \]
\[ = (W_{AA}(G, B))^{-1}M_{AB}(M_{BB})^{-1}. \]

\[ \square \]

**Proof of Corollary 3:** It follows from Equation (15) that
\[ w_{ij}(G, \{i, j, k\}) = (2I - (M_{\{i,j,k\}}^{-1}))_{ij} = \frac{m_{ij}m_{kk} - m_{ik}m_{jk}}{m_{ii}m_{jj} + 2m_{ij}m_{ik}m_{jk} - m_{ii}m_{jk}^2 - m_{ij}^2m_{kk} - m_{ik}^2m_{jj}}. \]

The proof of the second part is similar to the proof of Corollary 2 and therefore omitted. \[ \square \]

**Proof of Proposition 7:** The proof is clearly stated in the main text. \[ \square \]

**Proof of Corollary 4:** For item (i), we first note that by Equation (19), \( L_{ij} \) clearly increases with \( b_i \) and \( m_{ii} \) for each given \( j \). The claim just follows.

For item (ii), given \( m_{jj} = m_{j'j'} \) and \( b_j \geq b_{j'} \), we have
\[ L_{ij} - L_{ij'} = \frac{\delta m_{ii}(b_j^2 - b_{j'}^2)}{1 - \delta^2m_{ii}m_{jj}} + \frac{2b_i(m_j - m_{j'})}{1 - \delta^2m_{ii}m_{jj}} = \frac{\delta (b_j^2 - b_{j'}^2)}{m_{ii} - \delta^2m_{jj}} + \frac{2b_i(b_j - b_{j'})}{1 - \delta^2m_{ii}m_{jj}}, \]
which clearly increases in \( m_{ii} \). The result just follows by noting that \( b_i = b_{i'} \) and \( m_{ii} \geq m_{i'i'} \).

For item (iii), we apply the Taylor expansions to obtain that
\[ b_k(N^1) = 1 + \delta e_k(N^1) + O(\delta^2), \quad m_{kk}(N^1) = 1 + O(\delta^2), \quad k \in N_1, \]
where \( O(\delta^2) \) denotes a real-valued function such that \( \limsup_{\delta \to 0} |\frac{O(\delta^2)}{\delta^2}| < \infty \). Consequently,
\[ L_{ij}(N^1, N^2) = 2 + 2\delta \left(1 + e_i(N^1) + e_j(N^2)\right) + O(\delta^2). \]
Thus, when \( \delta \) is sufficiently small, only the degree centrality matters for the bridge index \( L_{ij} \). \[ \square \]

**Proof of Proposition 8:** The proof is clearly stated in the main text. \[ \square \]
Proof of Corollary 5: By Lemma 3, we have $m_{kl} > m_{jl}$ and $m_{kk} > m_{jj}$. Moreover, if we add a link $(i, j)$ to $G$, then node $i$’s equilibrium effort increases by

$$
\Delta x_i = \frac{\delta m_{ii} b_j^2}{1 - \delta m_{ij}} + 2b_i b_j \frac{m_{ii} m_{jj}}{1 - \delta m_{ij}}.
$$

One can easily find that $\Delta x_i$ is strictly increasing in $b_j$, $m_{ij}$ and $m_{jj}$. Therefore, we have $\Delta x_i(i, j) < \Delta x_i(i, k)$, which concludes the first part of the proposition.

To prove the second part, we just remove both link $(i, j)$ and $(i, k)$. Still, we have that $N_j \subset N_k$. Then we compare between adding a link $(i, j)$ and adding a link $(i, k)$. The first part then implies that adding a link $(i, k)$ increases agent $i$’s Katz-Bonacich centrality more, which then implies that removing a link $(i, j)$ in original graph decreases agent $i$’s centrality less.

References

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012). The network origins of aggregate fluctuations. *Econometrica* 80(5), 1977–2016.

Allouch, N. (2017). On the private provision of public goods on networks. *Journal of Economic Theory* 157, 527–552.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006). Who’s who in networks. wanted: The key player. *Econometrica* 74(5), 1403–1417.

Ballester, C., Y. Zenou, and A. Calvó-Armengol (2010). Delinquent networks. *Journal of the European Economic Association* 8(1), 34–61.

Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2013). The diffusion of microfinance. *Science* 341(6144).

Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2018). Changes in social network structure in response to exposure to formal credit markets. *Available at SSRN* 3245656.
Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2019). Using gossips to spread information: Theory and evidence from two randomized controlled trials. *The Review of Economic Studies* 86(6), 2453–2490.

Baqaee, D. R. (2018). Cascading failures in production networks. *Econometrica* 86(5), 1819–1838.

Belhaj, M., S. Bervoets, and F. Deroïan (2016). Efficient networks in games with local complementarities. *Theoretical Economics* 11(1), 357–380.

Bloch, F. and N. Quérou (2013). Pricing in social networks. *Games and Economic Behavior* 80, 243 – 261.

Bonacich, P. (1987). Power and centrality: A family of measures. *American Journal of Sociology* 92(5), 1170–1182.

Boyd, S. and L. Vandenberghe (2004). *Convex optimization*. Cambridge university press.

Bramoullé, Y., A. Galeotti, and B. Rogers (2016). *The Oxford handbook of the economics of networks*. Oxford University Press.

Bramoullé, Y. and G. Garance (2018). Diffusion centrality: Foundations and extensions. *Working paper*.

Bramoullé, Y. and R. Kranton (2007). Public goods in networks. *Journal of Economic Theory* 135(1), 478 – 494.

Cai, J. and A. Szeidl (2018). Interfirm relationships and business performance. *The Quarterly Journal of Economics* 133(3), 1229–1282.

Calvó-Armengol, A., E. Patacchini, and Y. Zenou (2009). Peer effects and social networks in education. *The Review of Economic Studies* 76(4), 1239–1267.

Candogan, O., K. Bimpikis, and A. Ozdaglar (2012). Optimal pricing in networks with externalities. *Operations Research* 60(4), 883–905.

Chen, Y.-J., Y. Zenou, and J. Zhou (2018a). Competitive pricing strategies in social networks. *The RAND Journal of Economics* 49(3), 672–705.

45
Chen, Y.-J., Y. Zenou, and J. Zhou (2018b). Multiple activities in networks. *American Economic Journal: Microeconomics* 10(3), 34–85.

Choi, S., S. Goyal, and F. Moisan (2019). Connectors and influencers.

Cruz, C., J. Labonne, and P. Querubin (2017). Politician family networks and electoral outcomes: Evidence from the philippines. *American Economic Review* 107(10), 3006–37.

David, G. and M. Dina (2004). Using online conversations to study word-of-mouth communication. *Marketing Science* 23(4), 545–560.

Demange, G. (2017). Optimal targeting strategies in a network under complementarities. *Games and Economic Behavior* 105, 84 – 103.

Elliott, M. and B. Golub (2018). A network approach to public goods. *Journal of Political Economy* forthcoming.

Elliott, M. L., S. Goyal, and A. Teytelboym (2019). Networks and economic policy. *Oxford Review of Economic Policy* 35(4), 565–585.

Galeotti, A., B. Golub, and S. Goyal (2020). Targeting interventions in networks. *Econometrica* 88(6), 2445–2471.

Galeotti, A. and S. Goyal (2010). The law of the few. *American Economic Review* 100(4), 1468–1492.

Golub, B. and C. Lever (2010). The leverage of weak ties how linking groups affects inequality. *Working paper*.

Goyal, S. and J. L. Moraga-González (2001). R&D networks. *The RAND Journal of Economics*, 686–707.

Hiller, T. (2017). Peer effects in endogenous networks. *Games and Economic Behavior* 105, 349 – 367.

Jackson, M. O., B. W. Rogers, and Y. Zenou (2017). The economic consequences of social-network structure. *Journal of Economic Literature* 55(1), 49–95.

Jerzy, S. (2001). *Delinquent Networks: Youth Co-Offending in Stockholm*. Cambridge Studies in Criminology. Cambridge University Press.
Konig, M. D., X. Liu, and Y. Zenou (2018). R& d networks: Theory, empirics and policy implications. *The Review of Economics and Statistics* forthcoming.

Konig, M. D., C. J. Tessone, and Y. Zenou (2014). Nestedness in networks: A theoretical model and some applications. *Theoretical Economics* 9(3), 695–752.

Li, X. (2020). Designing weighted and directed networks under complementarities. *Working paper at SSRN 3299331*.

Liu, E. (2019). Industrial Policies in Production Networks. *The Quarterly Journal of Economics* 134(4), 1883–1948.

Mark, W. (2002). *Companions in Crime: The Social Aspects of Criminal Conduct*. Cambridge Studies in Criminology. Cambridge University Press.

Mas, A. and E. Moretti (2009). Peers at work. *American Economic Review* 99(1), 112–45.

Patacchini, E. and Y. Zenou (2012). Juvenile delinquency and conformism. *Journal of Law, Economics, and Organization* 28(1), 1–31.

Sacerdote, B. (2001). Peer Effects with Random Assignment: Results for Dartmouth Roommates. *The Quarterly Journal of Economics* 116(2), 681–704.

Sun, Y., W. Zhao, and J. Zhou (2021). Building up efficient networks sequentially. *Working paper*.

Verdier, T. and Y. Zenou (2015). The role of cultural leaders in the transmission of preferences. *Economics Letters* 136, 158 – 161.

Verdier, T. and Y. Zenou (2018). Cultural leader and the dynamics of assimilation. *Journal of Economic Theory* 175, 374 – 414.