Control problems for the telegraph and wave equation networks

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Abstract. In this paper we consider control problems for the telegraph equation networks or, in other words, telegraph equations on metric graphs. If a network is homogeneous, i.e. its parameters are the same for all edges of the graph (and in some other cases), the problem can be reduced a control problem for the wave equation. For graphs without cycles, we obtain necessary and sufficient conditions of exact boundary controllability of the telegraph equation network and find the sharp time of controllability.

1. Introduction
The telegraph or telegrapher’s equations are a pair of coupled, linear first-order partial differential equations that describe the voltage and current on an electrical transmission line with distance and time. The telegraph equation first appeared in a paper by Kirchhoff [1] in 1857 and the next one by Heaviside [2] in 1876. It attracted close attention when it was treated by Poincaré [3] in 1893. The telegraph equation is widely used in the study of propagation of electric signals in a cable transmission line and also in wave phenomena. The transmission line is thought of as being composed of millions of tiny little circuit elements such as distributed resistance $R$ per unit length, distributed inductance $L$ per unit length, distributed capacitance $C$ per unit length and the leakage conductance of the dielectric material separating the two conductors is represented by a conductance $G$ per unit length. If the line voltage $V(x, t)$ and the current $I(x, t)$ where $x$ is the displacement and $t$ is the time, then the relationship between the voltage $V(x, t)$ and the current $I(x, t)$ along the transmission line can be described by the following coupled equations

$$L \partial_t I + \partial_x V + RI = 0 \quad (1.1)$$
$$C \partial_t V + \partial_x I + GV = 0. \quad (1.2)$$

In this paper we consider control problems for networks of the telegraph equations of the form (1.1), (1.2). Under differential equation networks (DENs) we understand differential operators on geometric graphs coupled by certain vertex matching conditions. Network-like structures play a fundamental role in many problems of science and engineering. The classical problem here that comes from applications is the problem of oscillations of the flexible structures
made of strings, beams, cables, and struts. These models describe bridges, space-structures, antennas, transmission-line posts, steel-grid reinforcements and other typical objects of civil engineering. More recently, the applications on a much smaller scale came into focus. In particular, hierarchical materials like ceramic or metallic foams, percolation networks and carbon and graphene nano-tubes, and graphene ribbons have attracted much attention. Papers discussing differential and difference equations on graphs have been appearing in various areas of science and mathematics since at least 1930s, but in the last two decades their number grew enormously. DENs arise as natural models of various phenomenon in chemistry (free-electron theory of conjugated molecules), biology (genetic networks, dendritic trees), geophysics, environmental science, and disease control. In physics, interest to quantum graphs arose, in particular, from applications to nano-electronics and quantum waveguides.

Networks of the telegrapher equations have been recently used for modeling electrical circuits, see, e.g. [4–6], and arterial blood flows [7–9]. Very little is known about controllability of such networks. Some results of this kind were obtained in [9] for star graphs of three edges. Exact controllability results for general trees of the telegrapher equations were obtained in [10] without an estimate of the controllability time. On the other hand, controllability of the wave equation on trees, i.e. graphs without cycles, is studied pretty well, see, e.g. monographs [11–13], surveys [14,15] and references therein. In the present paper we demonstrate that, in the case of homogeneous networks, control problem for the telegraph equation can be reduced to a problem for the wave equation for the voltage or current. For the voltage equation we obtain the Dirichlet control problem with the standard, Kirchhoff–Neumann matching conditions at the internal vertices. For the current equation we obtain the Neumann control problem with non-standard, so-called delta-prime matching conditions. We study these problems using and developing the method that was recently proposed in [16]. We prove that the systems of voltage, current and telegraph equations are exactly controllable if and only if the control is supported at all or at all but one boundary vertices of the tree. We also obtain the sharp estimate of the controllability time.

Eliminating \( I(x,t) \) in the system (1.1), (1.2) we obtain the second order equation for \( V \):

\[
(CL)\ddot{V} - V_{xx} + (CR + LG)V_t + (RG) V = 0.
\]

The same equation is valid for \( I \). If \( CL \neq 0 \), then using simple changes of variables this equation can be transformed to the wave equation with potential:

\[
\ddot{u} - u_{xx} + qu = 0
\]

with some constant \( q \).

If there is no inductance in the transmission line, i.e. \( L = 0 \) then equation (1.3) takes the form

\[
\ddot{u} - \frac{1}{RC} u_{xx} + \frac{G}{C} u = 0.
\]

This occurs in the case of transmission line of axons and dendrites of nerve cells. In that case (1.5) is called the cable equation; it describes the dynamics of trans-membrane potential \( u(x,t) \). Control and inverse problems for this equation on tree graphs were studied in [17] and [18].

In the case of no resistance and leakage, equation (1.3) takes the form

\[
\ddot{u} - c^2 u_{xx} = 0,
\]

which is the equation of wave motion with the phase speed of \( \frac{1}{\sqrt{CL}} \). On the other hand, when the inductance is negligible compared with the resistance and there is no leakage, equation (1.3) takes the form

\[
\ddot{u} - ku_{xx} = 0,
\]
which is the equation of diffusion with diffusivity \( k = \frac{1}{\rho c} \). The control problems for equations (1.6) and (1.7) on graphs were considered, e.g. in [19], for the case of variable coefficients.

2. Control problems on graphs

Let \( \Omega = \{V, E\} \) be a finite compact and connected metric graph, where \( V \) is a set of vertices and \( E \) is a set of edges. We recall that a graph is called a metric graph if every edge \( e_j \in E, j = 1, \ldots, N \), is identified with an interval \((a_{2j-1}, a_{2j})\) of the real line with a positive length \( l_j \). We denote the boundary vertices (i.e. vertices of degree one) by \( \Gamma = \{\gamma_1, \ldots, \gamma_m\} \), and interior vertices (whose degree is at least 2) by \( \{v_1, \ldots, v_M\} \). For each vertex \( v_k \), denote its degree by \( \gamma_k \). We write \( j \in J(v) \) if \( e_j \in E(v) \), where \( E(v) \) is the set of edges incident to \( v \).

The graph \( \Omega \) determines naturally the Hilbert space of square integrable functions \( \mathcal{H} := L^2(\Omega) = \oplus_j L^2(e_j) \). When convenient, we will denote the restriction of a function \( w \) on \( \Omega \) to \( e_j \) by \( w_j \). For any vertex \( v_k \) and function \( w(x) \) on the graph, we denote by \( \partial w_x(v_k) \) the derivative of \( w_j \) at \( v_k \) in the direction pointing away from the vertex. Along with \( \mathcal{H} \) we introduce two Sobolev-type spaces: \( \mathcal{H}^1 \) is the space of continuous functions \( \phi \) on \( \Omega \) such that \( \phi_j \in H^1(e_j) \forall j \) and \( \phi|_{\Gamma} = 0; \mathcal{H}^{-1} \) is the space dual to \( \mathcal{H}^1 \).

Let \( \Gamma \) be a union of two disjoint sets: \( \Gamma_1 = \{\gamma_1, \ldots, \gamma_m\} \), \( \Gamma_0 = \{\gamma_{m+1}, \ldots, \gamma_m\} \), and \( \Gamma_0 \) may be empty. We consider a system is described by the following initial boundary value problem (IBVP) on \( \Omega \) with Kirchhoff’s conditions at each internal vertex \( v_k \):

\[
\begin{align*}
L_j \partial_t I_j + \partial_x V_j + R_j I_j &= 0, \quad (x, t) \in (a_{2j-1}, a_{2j}) \times (0, T), \quad j = 1, \ldots, N, \quad (2.1) \\
C_j \partial_t V_j + \partial_x I_j + G_j V_j &= 0, \quad (x, t) \in (a_{2j-1}, a_{2j}) \times (0, T), \quad j = 1, \ldots, N, \quad (2.2) \\
V_j|_{t=0} &= I_j|_{t=0} = 0, \quad x \in (a_{2j-1}, a_{2j}), \quad j = 1, \ldots, N, \quad (2.3) \\
\sum_{j \in J(v_k)} \kappa_j I_j(v_k, t) &= 0, \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \quad (2.4) \\
V_j(\gamma_k, t) &= f_k(t), \quad j \in J(\gamma_k), \quad k = 1, \ldots, m_1, \quad t \in (0, T), \quad (2.6) \\
V_j(\gamma_k, t) &= 0, \quad j \in J(\gamma_k), \quad k = m_1 + 1, \ldots, m, \quad t \in (0, T). \quad (2.7)
\end{align*}
\]

Here \( T \) is arbitrary positive number, \( \kappa_j = 1 \) if \( v_k \) coincides with \( a_{2j-1} \) and \( \kappa_j = -1 \) if \( v_k \) coincides with \( a_{2j} \), and \( f_k \in L^2(0, T) \) for all \( j \) and \( k \). Coefficients \( C_j, L_j, R_j \) and \( G_j \) are arbitrary nonnegative numbers. In this paper we assume that \( C_j \) and \( L_j \) are strictly positive.

By \( V \) and \( I \) we denote the corresponding functions on the whole graph \( \Omega \); the vector function \( f = \{f_k\} \in L^2(0, T; \mathbb{R}^m) =: \mathcal{F}^T \) is referred to as boundary control. The well-posedness of this system was studied e.g in [10] (see also [20]. It was proved that for any \( f \in \mathcal{F}^T \), there exists a unique (generalized) solution of the IBVP (2.1)–(2.7) such that \( V, I \in C([0, T]; \mathcal{H}) \). It means that \( V(\cdot, t) \) and \( I(\cdot, t) \) belong to \( \mathcal{H} \) for any \( t \in [0, T] \) and continuously depend on \( t \) in \( \mathcal{H} \) norm.

The main question of our interest is the exact boundary controllability. (In this paper we consider only boundary controls.) The system (2.1)–(2.7) is called exactly \((V, I)\)-controllable in time \( T \) if, given arbitrary functions \( \varphi, \psi \in \mathcal{H} \), one can find \( f \in \mathcal{F}^T \) such that \( V(\cdot, T) = \varphi \) and \( I(\cdot, T) = \psi \). The system (2.1)–(2.7) is called exactly \((V, V_i)\)-controllable in time \( T \) if, given arbitrary functions \( \varphi \in \mathcal{H}, \psi \in \mathcal{H}^{-1} \) one can find \( f \in \mathcal{F}^T \), such that \( V(\cdot, T) = \varphi \) and \( V_i(\cdot, T) = \psi \). Using the equations (2.1) and (2.2) one can check that these two problems are equivalent: the system is exactly \((V, I)\)-controllable in time \( T \) if and only if it is exactly \((V, V_i)\)-controllable in the same time interval. This equivalence helps us to reduce the controllability problem for the telegraph equation on graphs to the problem for the wave equation.

If a graph \( \Omega \) has cycles, the wave or telegraph equation is not exactly (boundary) controllable in any time. For the wave equation it was proved in [12, Ch. 7]; the same argument works also
for the telegraph equation. Therefore, we will consider the exact controllability question for trees. We reduce the control problem for the system (2.1)–(2.7) to a similar problem for the wave equation. It is possible only in particular cases, and in this paper we consider two of them: (i) the coefficients $C_j, L_j, R_j, G_j$ are independent of $j$, and (ii) $R_j = G_j = 0$ for all $j$.

In the first case we can derive the control problem for $V$ or for $I$. We will start with a more straightforward problem for $V$. Eliminating $I_j(x, t)$ in the system (2.1)–(2.7) and introducing new functions $u_j(x, t) := V_j(x/\sqrt{CL}, t) \exp\{t(R/2L + G/2C)\}$, we obtain the IBVP

\[
\begin{align*}
\partial^2_{x} u_j - \partial^2_{t} u_j + q u_j &= 0, \quad (x, t) \in (b_{2j-1}, b_{2j}) \times (0, T), \quad j = 1, \ldots, N, \quad (2.8) \\
u_j|_{t=0} = \partial_t u_j|_{t=0} &= 0, \quad x \in (b_{2j-1}, b_{2j}), \quad j = 1, \ldots, N, \quad (2.9) \\
u_{v_k}(v_k, t) &= u_j(v_k, t), \quad i, j \in J(v_k), \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \quad (2.10) \\
\sum_{j \in J(v_k)} \partial u_j(v_k, t) &= 0, \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \quad (2.11) \\
u_j(\gamma_k, t) &= f_k(t), \quad j \in J(\gamma_k), \quad k = 1, \ldots, m_1, \quad t \in (0, T), \quad (2.12) \\
u_j(\gamma_k, t) &= 0, \quad j \in J(\gamma_k), \quad k = m_1 + 1, \ldots, m, \quad t \in (0, T). \quad (2.13)
\end{align*}
\]

Here $q$ is a constant and $b_j = \sqrt{CL} a_j$ for all $j$. For simplicity we keep the same notations for the control functions as in the IBVP (2.1)–(2.7) and for the vertices of a new graph $\tilde{\Omega} = (V, E)$, however, we notice that the lengths of the edges of the new graph are equal to $\sqrt{CL} l_j$, $j = 1, \ldots, N$. One can check that the system (2.1)–(2.7) is exactly $(V, V)$-controllable in time $T$ if and only if the system (2.8)–(2.13) is exactly $(u, u_t)$-controllable in the same time interval.

Controllability of the latter system, even with variable $q_j(x)$, was studied in several papers, see reviews [14, 15] and references therein. Here we follow the method proposed in [16], which splits a complicated problem of the $(u, u_t)$-controllability to several more simple problems. We need two more definitions.

The system (2.8)–(2.13) is called shape controllable in time $T$ if, for any $\varphi \in H$, there exists $f \in \mathcal{F}^T$ such that $u(\cdot, T) = \varphi$. The system (2.8)–(2.13) is called velocity controllable in time $T$ if, for any $\psi \in H^{-1}$, there exists $f \in \mathcal{F}^T$ such that $u_t(\cdot, T) = \psi$. It was demonstrated in [16] that the shape and velocity control problems are reduced to solving the systems of the second kind Volterra integral equations, and the system is shape controllable or velocity controllable if and only if control functions act at all or at all but one boundary vertices. The optimal (sharp) time $T_*$ of the shape or velocity controllability is determined by the following way.

Let $U$ be a union of disjoint (except for the end points) paths on $\tilde{\Omega}$. Each path $P$ begins at a controlled boundary vertex and ends anywhere in $\tilde{\Omega}$, and $\cup_{P \in U} = \tilde{\Omega}$. Then

\[
T_* = \min_{U} \max_{P \in U} \text{length } P. \quad (2.14)
\]

Figure 1: Controls act at all boundary vertices

Figure 2: Controls act at all but one boundary vertices
In figure 1 the union $U$ consists of three paths, and $T_s = (l_2 + l_3)/2$.
In figure 2 the union $U$ consists of two paths, and $T_s = l_2 + l_3$.

Now we will prove that the shape and velocity controllability imply the $(u, u_t)$-controllability. **Proposition 1.** Suppose that the system (2.8)–(2.13) is both shape and velocity controllable in time $T$. Then the system is $(u, u_t)$-controllable in time $2T$.

**Proof.** We consider the following eigenvalue problem on the graph $\Omega$:

$$-\varphi''(x) + q(x)\varphi(x) = \omega^2 \varphi(x),$$

$$\sum_{e_j \sim v} \partial_j \varphi(v) = 0 \quad \forall v \in V \setminus \Gamma;$$

$$\varphi \text{ is continuous on } \Omega.$$ 

$$\varphi|_s = 0.$$

It is known that the spectrum $\{\omega_n\}_{n \in \mathbb{N}}$ of this problem is purely discrete and the eigenfunctions $\{\phi_n\}_{n \in \mathbb{N}}$ form an orthonormal basis in $\mathcal{H}$. The solution $u(\cdot, t)$ of (2.8)–(2.13) can be represented in a form of a series with respect to $\{\phi_n\}$ (see, for example, [12,20]).

Control problems are reduced to moment problems using the Fourier method. The shape controllability is equivalent to the solvability of the moment problem [12, Ch. 3]

$$a_n = \langle f, s_n \rangle_{\mathcal{F}^2T}, \quad n \in \mathbb{N}, \quad s_n(t) := \frac{\varphi_n^t |_{\Gamma_1}}{\omega_n} \sin \omega_n(T - t).$$

(2.15)

Solvability means that for any $\{a_n\} \in \ell^2$, there exists $f \in \mathcal{F}T$ satisfying (2.15). For simplicity we assume here that $\omega_n \neq 0$. If $\omega_n = 0$ we use $t$ to replace $\frac{\sin \omega_n t}{\omega_n}$ in the expression of (2.15).

The velocity controllability in time $T$ is equivalent to the solvability of the moment problem

$$b_n = \langle g, c_n \rangle_{\mathcal{F}^2T}, \quad n \in \mathbb{N}, \quad c_n(t) := \frac{\varphi_n^t |_{\Gamma_1}}{\omega_n} \cos \omega_n(T - t).$$

(2.16)

Denote by $f_-(t)$ the odd extension with respect to $T$ of $f(t)$ from $[0, T]$ to $[0, 2T]$ and by $g_+(t)$ the even extension of $g(t)$. We observe that the function

$$h(t) = \frac{f_-(t) + g_+(t)}{2}$$

solves both moment problems

$$a_n = \langle h, s_n \rangle_{\mathcal{F}^2T}, \quad b_n = \langle h, c_n \rangle_{\mathcal{F}^2T},$$

(2.17)

where $\mathcal{F}^2T := L^2(0, 2T; \mathbb{R}^{m_1})$. It means that the moment problem (2.17) is solvable for any sequences $\{a_n\}, \{b_n\} \in \ell^2$. Therefore, the family $\{s_n, c_n\}$ forms a Riesz sequence in $\mathcal{F}^2T$ [12, Ch. 1]. It implies that both families

$$\left\{ \frac{\varphi_n^t |_{\Gamma_1}}{\omega_n} e^{\pm i \omega_n (T - t)} \right\} \quad \text{and} \quad \left\{ \frac{\varphi_n^t |_{\Gamma_1}}{\omega_n} e^{\pm i \omega_n t} \right\}$$

also form Riesz sequences in $\mathcal{F}^2T$, and by Theorem III.3.10 of [12] the system (2.8)–(2.13) is $(u, u_t)$-controllable in time $2T$. \hfill \Box

Taking into account equivalence between the $(u, u_t)$, $(V, V_t)$ and $(V, I)$ controllability, we can now formulate the first main result of the paper.
Theorem 1. If $\Omega$ is a tree, control functions act at all or at all but one boundary vertices, and the coefficients $C_j, L_j, R_j, G_j$ are independent of $j$, the system (2.1)–(2.7) is exactly $(V, I)$-controllable in time $T = 2T_*, \sqrt{C_L}$, where $T_*$ is defined for graph $\Omega$ by the relation similar to (2.14).

The same result can be obtained by eliminating $V_j(x, t)$ in the system (2.1)–(2.7) and deriving the IBVP for $w_j(x, t) := I_j(x/\sqrt{C_L}, t) \exp\{t(R/2L + G/2C)\}$:

$$
\begin{align*}
\partial_t^2 w_j - \partial_x^2 w_j + q_j w_j &= 0, \quad (x, t) \in (b_{2j-1}, b_{2j}) \times (0, T), \quad j = 1, \ldots, N, \\
w_j|_{t=0} &= \partial_t w_j|_{t=0} = 0, \quad x \in (b_{2j-1}, b_{2j}), \quad j = 1, \ldots, N, \\
\partial_{v_k} u_i(v_k, t) &= \partial_{w_j} w_j(v_k, t), \quad i, j \in J(v_k), \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \\
\sum_{j \in J(v_k)} \kappa_j w_j(v_k, t) &= 0, \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \\
\partial_\kappa w_j(\gamma_k, t) &= g_k(t), \quad j \in J(\gamma_k), \quad k = 1, \ldots, m_1, \quad t \in (0, T), \\
\partial_\kappa w_j(\gamma_k, t) &= 0, \quad j \in J(\gamma_k), \quad k = m_1 + 1, \ldots, m, \quad t \in (0, T).
\end{align*}
$$

Here the functions $g_k$ are connected with $f_k$ from (2.12) by the equalities $g_k(t) = -(C_j f'_k(t) + G f_k(t))$. The matching conditions (2.20), (2.21) are nonstandard matching conditions, and the authors have not found results on controllability of such systems in the literature. The methods developed for study of the system (2.8)–(2.13) can be extended to the system (2.18)–(2.23), and the following result is interesting in its own right in control theory for DENs.

Theorem 2. If $\Omega$ is a tree and control functions $g_k$ act at all or at all but one boundary vertices, the system (2.18)–(2.23) is $(w, w)$-controllable in time $T = 2T_*$. This controllability time estimate is sharp, i.e. it guarantees controllability of any tree and, generally, it cannot be improved.

Now we consider the system (2.1)–(2.7) in the case (ii), when $R_j = G_j = 0$ for all $j$. Eliminating $I_j(x, t)$ and introducing new functions $u_j(x, t) := V_j(x/\sqrt{C_j L_j}, t)$, we obtain the IBVP

$$
\begin{align*}
\partial_t^2 u_j - \partial_x^2 u_j + q_j u_j &= 0, \quad (x, t) \in (b_{2j-1}, b_{2j}) \times (0, T), \quad j = 1, \ldots, N, \\
u_j|_{t=0} &= \partial_t u_j|_{t=0} = 0, \quad x \in (b_{2j-1}, b_{2j}), \quad j = 1, \ldots, N, \\
u_i(v_k, t) &= u_j(v_k, t), \quad i, j \in J(v_k), \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \\
\sum_{j \in J(v_k)} \alpha_j \partial_{u_j} u_j(v_k, t) &= 0, \quad v_k \in V \setminus \Gamma, \quad t \in (0, T), \\
u_j(\gamma_k, t) &= f_k(t), \quad j \in J(\gamma_k), \quad k = 1, \ldots, m_1, \quad t \in (0, T), \\
u_j(\gamma_k, t) &= 0, \quad j \in J(\gamma_k), \quad k = m_1 + 1, \ldots, m, \quad t \in (0, T).
\end{align*}
$$

Here $q_j$ are some constants and $\alpha_j$ are positive constants. Controllability of this system, even with variable $q_j(x)$, can be studied by the same methods, and the results are the same as for the systems (2.8)–(2.13) and (2.18)–(2.23).

The corresponding result for the telegraph equation is formulated as follows. We introduce the optical length of the edge $e_j$ as $l_j \sqrt{C_j L_j}$, and the optical length of a path $P = \{e_{j_1}, \ldots, e_{j_\ell}\}$ as $l_{j_1} \sqrt{C_{j_1} L_{j_1}} + \ldots + l_{j_\ell} \sqrt{C_{j_\ell} L_{j_\ell}}$.

Theorem 3. Let $\Omega$ be a tree, control functions act at all or at all but one boundary vertices, and $R_j = G_j = 0$ for all $j$. Then the system (2.1)–(2.7) is exactly $(V, I)$-controllable in time $T = 2T^*$, where $T^*$ is defined by the relation similar to (2.14):

$$
T^* = \min_{U} \max_{P \in U} \text{optical length } P.
$$
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