On Parameter Selection of Nonsingular Predefined-time Terminal Sliding Mode With the Fixed-time Convergence Guarantee

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Abstract

This paper study the parameter selection of predefined-time sliding mode and try to design a general nonsingular predefined-time terminal sliding mode. 1). On parameter selection: Some existing predefined-time sliding modes are designed to focus on the reaching time and ignore the characterization of the equilibrium point i.e. $x_e = 0$. In actual engineer, the system can only converge to a small neighborhood near the equilibrium point because of the existence of uncertainty i.e. $x_e \to 0$. The actual equilibrium point should be deduced by taking the limit but not directly solve the equilibrium point. Hence, the actual selection of exponential term is suggested to be not 1 to make system convergence to the equilibrium point within predefined time. 2). On singularity-avoidance: Based on the exponential feature of predefined-time stability systems, a mathematic concept of switching sliding mode [1] is applied in the design of nonsingular predefined-time terminal sliding mode. However, this class of sliding mode switching method will cause a singular problem, which can be described as control input $u \to \infty$ if the system state $x \to 0$ and any-order time derivative of $x$ is not 0. Hence, a non-singular selection condition for each recursive sliding mode surface is explored in this paper to realize the global finiteness of control input. In addition, an unnecessary chattering of control input will also be caused by the sliding mode switching frequency, and a novel switching condition will be applied in the controller to reduce this chattering. Simulations will be carried out to validate the effect of the novel nonsingular predefined-time sliding mode control method to some extent.

Key words: exponential term; predefined-time stability; nonsingular terminal sliding mode; parameter selection.

1 Introduction

Sliding mode control is widely studied by researchers for its robustness, time convergence and high-precision [2,3,4]. An important research direction of sliding mode control is finite time control [5]. With the development of finite time sliding mode control, the characteristic of a terminal attractor is explored gradually to obtain faster convergence rate of system [6]. It is not satisfied that the convergence time depends on the initial state, a fixed-time attractivity is presented in [7] to obtain a fixed-time stability system. However, the upper bound of the settling time of traditional fixed-time is usually determined by two or more parameters, hence, a simplifying fixed-time stability system is presented in [8] to make the upper bound of settling time explicit in the system function. In addition, the improved fixed-time stability system can also obtain an upper bound of settling time without enlarging and reducing mathematically [9,10], but most of traditional fixed-time stability systems enlarged the upper bound of settling time. Especially, a note on predefined-time stability in [11] pointed that the predefined-time stability of system can be guaranteed without the exponential term. This result allows a construction of the predefined-time stabilizing controller with only polynomial term (instead of the exponential one) and its Lyapunov characterization analysis is perfect, but this class of predefined-time stability function [11,12,13,14] may not consider if system can reach and stay in the neighborhood of equilibrium point within predefined time. The related researchers study a reconstruction

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method of controller in Definition 3.1 of [8] to make system convergence to the equilibrium point theoretically, but it seems that the actual characterization of the equilibrium point will be different from the description in [8] by considering the analysis and simulation in this paper.

Furthermore, it is worth noting that almost all fixed-time or predefined-time sliding mode controllers have a singular problem. This singularity can be defined as a case where the control input may be infinite if the system state $x \to 0$ and any-order time derivative of $x$ is not 0 or $\infty$. For a second-order system, related researchers have presented many non-singular control methods, and these methods can be divided into two categories: one is to transfer the singularity caused by the state $x = 0$ to the time derivative of the state $\dot{x} = 0$ [15,12,16], another one is to deal with singular term of controller directly [17,18,19]. The first category seems to deal well with the singularity proposed in this paper, but the control input may be infinite if $\dot{x} = 0$. The value of this time derivative is possible in practice. Another one(except the method in [19]) can only guarantee finite-time stability of system, because its essence is to limit the max value of control input while ignoring the infinite control input required by the controller to ensure the fixed time convergence of the system when system is in singularity, then the actual convergence time may be longer than the predefined maximum convergence time. Fortunately, based on the exponential feature of predefined-time stability systems, a traditional sliding mode switching strategy [1] can be applied to design a nonsingular predefined-time terminal sliding mode controller in [19]. This controller can guarantee not only the strict fixed time stability of system but also the finiteness of the control input, however, it is difficult to find an appropriate switching sliding mode. For a higher-order system, there seems to be little research on non-singular fixed-time recursive sliding mode control. The above-mentioned methods may be too complex if apply them directly to higher-order system, hence, it is necessary to study a simplifying sliding mode switching scheme to overcome the singularity of fixed-time terminal sliding mode control method for second-order or high-order system.

The contribution of this paper lies on: 1). The relationship between the non-time parameter value of predefined-time stability system and the characterization of equilibrium point is studied to explore the better value range of parameters. 2). A simplifying sliding mode switching scheme is studied to overcome the singularity of fixed-time terminal sliding mode control. 3). A nonsingular predefined-time sliding mode controller for higher-order system is studied to explore the max range of non-time parameter value.

2 Preliminaries

Definition 1: (see [8]). Consider a system:

$$\dot{x}(t) = -\frac{1}{T_c} f(x(t); m), \; x(0) = x_0. \quad (1)$$

where $x = [x_1, x_2 \cdots x_n]^T \in \mathbb{R}^n$ is the vector of system state, $f(x): D \to \mathbb{R}^n$ is a continuous function and $f(x) = 0$. The system (1) is predefined-time stable if system is globally fixed-time stable and the minimum upper bound of settling time $T(x_0)$ can be deduced without enlarging and reducing in math.

Lemma 1: (see [11]). Consider the following system:

$$\dot{x} = -\frac{1}{T_c} \left\| \frac{\partial W(x)}{\partial x} \right\|^2 \left[ \frac{\partial W(x)}{\partial x} \right]^T. \quad (2)$$

where $T_c > 0$, and $W(x)$ satisfies the following conditions:

$$(i)0 < W(x) < 1 \text{ for all } x \in \mathbb{R}^n$$
$$(ii)W(x) = 0 \text{ if and only if } x = 0$$
$$(iii)W(x) \to 1 \text{ as } \|x\| \to \infty$$
$$(iv)W(x) \text{ is } C^1 \text{ in } \mathbb{R}^n \setminus \{0\}$$
$$(v) \left\| \frac{\partial W(x)}{\partial x} \right\| \neq 0 \text{ for all } x \in \mathbb{R}^n$$

Then, the system can be said predefined-time stable, and the minimum upper bound of settling time is $T_c$. 

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Lemma 2: Consider the following system:

\[
\dot{x} = -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-2} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^T.
\]

(4)

where \(W(\|x\|)\) satisfies the conditions in (3). The exponential term should be emphasized in here and \(0 < m < 1\) \((m \neq 1)\). The (3)-(v) should be modified as \(\frac{\partial W(\|x\|)}{\partial x} > 0\) for all \(x \in \mathbb{R}^n\). In addition, a supplementary condition should be: \(\frac{\partial W(\|x\|)}{\partial \|x\|} \neq 0\) for all \(x \in \mathbb{R}^n\).

Proof: The system (4) can be written as:

\[
\dot{x} = -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-2} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^T
\]

\[
= -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-1} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^T.
\]

(5)

1) By considering \(m = 1\), the \(W(\|x\|)\) still satisfies the condition in (3). However, a limit will be exemplified in the following to emphasize \(m \neq 1\):

\[
\lim_{x \to 0} \dot{x} = -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-1} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^T
\]

\[
= \begin{cases} 
-\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-1} \left[ \frac{\partial W(\|x\|)}{\partial x} \right] 
\end{cases} \begin{bmatrix} sign(x_1) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & sign(x_n) \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \quad x \to 0^+
\]

(6)

where \(E_{n1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^n\). It is clear that \(\dot{x}\) will be not zero when \(x \to 0\) by the theorem of equivalent infinitesimal. Hence, the system (5) may be not stable at 0 as \(t \to \infty\).

Example 1: Consider a simple on-order system: \(\dot{x} - x = 0\) in which \(u = -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^{-2} \left[ \frac{\partial W(\|x\|)}{\partial x} \right]^T\) and \(W(x) = 1 - exp(\|x^m\|)\). As shown in Figs. 1 and 2, the simulation is carried out to show the different convergence performance of predefined-time stable system under different parameters \(m\). It is clear that the system can converge to the equilibrium point more smoothly in \(0 < m \leq 0.5\) than \(0.5 < m \leq 1\) after \(T_c = 0.1\). In addition, there is a very interesting performance of system: by consider a traditional idea \((m=1)\) may be an appropriate value in Lemma 1, the chattering degree of controller should decrease as \(m\) decreases. However, according to the curves of simulation, the chattering degree of controller in \(m = 1\) is smaller than that in \(0.8 \leq m \leq 0.9\). Not only that, but this curve is a regular triangular curve, not as irregular as other curves. The mathematical principles of this phenomenon may be explained by (6).

Remark 1: Even by considering the following discontinuous predefined-time stable system which is given as the case 2 of Definition 3.1 in [8]:

\[
\dot{e} = -\frac{1}{T_c} exp(\|e\|) \text{sign}(e),
\]

(7)
Fig. 1. The curve of \( (x|\dot{x} = -\frac{1}{T_c} \exp(|x|^m) \frac{x}{|x|^m} ) \): 
\( T_c = 0.1, \ m = 1, 0.9, \ldots, 0.1 \)

Fig. 2. The curve of \( (\dot{x} | x = -\frac{1}{T_c} \exp(|x|^m) \frac{x}{|x|^m} ) \): 
\( T_c = 0.1, \ m = 1, 0.9, \ldots, 0.1 \)

However, the system is still difficult to converge to the equilibrium point within a predetermined time, because:

\[
\begin{cases}
\lim_{e \to 0^+} = -\frac{1}{T_c} \exp(\|e\|) \text{sign}(e) = -\frac{1}{T_c} \exp(\|e\|) \\
\lim_{e \to 0^-} = -\frac{1}{T_c} \exp(\|e\|) \text{sign}(e) = \frac{1}{T_c} \exp(\|e\|)
\end{cases}
\]  
(8)

2) By considering \( 0 < m < 1 \), \( W(\|x\|^m) \) still satisfies the condition in Lemma1, and a limit will be exemplified in the following to emphasize \( m \in (0,1) \) and \( m \neq 1 \):

\[
\lim_{x \to 0^+} \dot{x} = -\frac{1}{T_c} \left[ \frac{\partial W(\|x\|^m) m\|x\|^{m-1} x^T}{\|x\|^m} \right]^{-1} - \frac{1}{mT_c} \left[ \frac{\partial W(y)}{\partial y} \right]^{-1} \frac{x}{\|x\|^m} 
\]
\[
= 0
\]

(9)

It is clear that \( \dot{x} \) will be zero when \( x \to 0 \) according to the theorem of higher-order infinitesimal. Hence, the system (5) can be stable at 0 as \( t \to T_c \).

Consider a candidate Lyapunov function as the following:

\[
V_1 = \|x\|. 
\]

(10)
The time derivative of Lyapunov function (10) can be deduced as:

\[
\dot{V}_1 = \frac{x^T \dot{x}}{\|x\|} = -\frac{1}{T_c} \frac{x^T}{\|x\|} \left[ \frac{\partial W(\|x\|^m)}{\partial \|x\|^m} \right]_{\|x\|} -1 \|x\|^{-1-m}
\]

(11)

According to (3), the system (11) may not be stable. The (3)-(v) may only be a necessary but not sufficient condition, for example:

\[
\dot{x} = -\frac{1}{mT_c} \exp(\|x\|^m) \|x\|^{-1-m},
\]

(12)

Although all the conditions is satisfied, but (12) is not stable in \( x < 0 \), hence, (3)-(v) should be modified as a necessary and sufficient condition: \( \frac{\partial W(\|x\|^m)}{\partial x} > 0 \), then, (11) can be deduced as:

\[
\dot{V}_1 = -\frac{1}{mT_c} \left[ \frac{\partial W(V_1^m)}{\partial V_1^m} \right]_{V_1^m} \|x\|^{-1-m}
\]

\[
= -\frac{1}{mT_c} \left[ \frac{\partial W(V_1^m)}{\partial V_1^m} \right]_{V_1^m} V_1^{-m}
\]

(13)

Then, the settling time can be deduced as:

\[
T(x_0) = \frac{V_1(0)}{V_1(x_0)} \frac{1}{mT_c} \left[ \frac{\partial W(V_1^m)}{\partial V_1^m} \right]_{V_1^m} dV_1
\]

\[
= T_c \int_{V_1(x_0)^m}^{V_1(0)^m} \frac{\partial W(V_1^m)}{\partial V_1^m} dV_1^m
\]

\[
= T_c \left[ W(V_1(x_0)^m) - W(V_1(0)^m) \right]
\]

(14)

Hence, the max settling time can be direct defined as \( T_c \) without any math method of enlarging and reducing. In addition, in my opinion, it is also the essential difference between predefined time stability and fixed time one.

The proof is complete.

Definition 2: Until now, the definition of singularity for terminal sliding mode has been complex and confusing. Hence, we suggest that the singularity for fixed-time terminal sliding mode control can be defined: control input will not be infinite whenever system state is not to infinite. This may be the more proper description of singularity.

3 Main Results

3.1 A Second-order System

Consider a second-order MIMO system as the following:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x, t) + b(x, t)u \\
y_o &= x_1
\end{align*}
\]

(15)
where \( x_1 = [x_{1,1} \ x_{1,2} \cdots \ x_{1,n}]^T \), \( x = [x_1^T \ x_2^T]^T \). \( f(x, t) : D \rightarrow R^n \) is a continuous function. \( b(x, t) : D \rightarrow R^{n \times n} \) is a continuous function. \( u \in R^n \) is the control input, \( y_o \in R^n \) is system output.

Assumption 1 (for finite control input): \( \frac{\partial [\partial W(y)]^{-1}}{\partial y} > 0 \) and \( \frac{\partial^2 [\partial W(y)]^{-1}}{\partial y^2} > 0 \).

Corollary 1: According to Lemma 2, a predefined-time sliding mode can be designed as:

\[
s = \hat{e} + \frac{1}{T_c} \| \frac{\partial W(\|e\|^m)}{\partial e} \|^2 \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T.
\]

where \( e = y_o - y_d \) in which \( y_d \) is the desired system output. Then, a simple sliding mode switching method can be designed to make sliding mode control input nonsingular as:

\[
m = \begin{cases} 
1 + (m - 1) \text{sign}(\text{sat}(\frac{\|e\|}{\varepsilon})) , & \|e\| \neq 0 \\
1 , & \|e\| = 0
\end{cases}
\]

with

\[
\text{sat}(\frac{\|e\|}{\varepsilon}) = \begin{cases} 
1 , & \|e\| > \varepsilon \\
0 , & \|e\| \leq \varepsilon
\end{cases}
\]

where \( 0 < m \leq 0.5 \) and \( \varepsilon \) is a small parameter representing the risk of singularity. As a direct consequence, the singularity of sliding mode can be overcome by (17) and (18) if Assumption 1 is satisfied for \( y = \|e\|^m \).

Proof: This subsection mainly discusses the finiteness of control input, hence \( b(x, t) \) can set to a n-order unit matrix, \( f(x, t) \) can be set to 0.

The dynamic of sliding mode surface \( s \) can be expressed as:

\[
s' = u + \frac{1}{T_c} \frac{\partial}{\partial t} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e.
\]

1) By considering an arbitrary reaching law \( \hat{s} = f(s) \) which satisfies Lemma 2, then, the control input \( u \) can be deduced as:

\[
u = \frac{y = \|e\|^m - 1}{m T_c} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e - \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e
\]

Then, the singularity will come (generally, \( \hat{e} \to \infty \) will not be considered):

\[
\lim_{e \to 0, e \neq 0} u = \frac{y = \|e\|^m - 1}{m T_c} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e - \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e
\]

\[
gen f(s) - \frac{1}{T_c} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e - \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^{-1} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T e
\]
Let me break off the proof for a moment, an interesting phenomenon can be discussed to express my alternative understanding of singularity.

a). By considering $f(s)$ is finite, then,

$$
\lim_{e \to 0, \dot{e} \neq 0} \mathbf{u} = -\infty \text{sign}(\dot{e}). \tag{22}
$$

b). By considering $\|f(s)\|$ is the inequivalent infinity of $\left\| \frac{1}{T_c} \frac{\partial W(s)}{\partial s} \right\|^{-1} \dot{e} + \infty \text{sign}(\dot{e})$ in (21) and $\Psi_i = \text{sign}(f(s_i)) - \text{sign}(\dot{\epsilon}_i)$, then,

$$
\lim_{e \to 0, \dot{e} \neq 0} \mathbf{u} = \pm \infty \tag{23}
$$

c). By considering $\|f(s)\|$ is the equivalent infinity of $\left\| \frac{1}{T_c} \frac{\partial W(s)}{\partial s} \right\|^1 \dot{e} + \infty \text{sign}(\dot{e})$ in (21) and $\Psi_i = \text{sign}(f(s_i)) - \text{sign}(\dot{\epsilon}_i)$, then,

$$
\lim_{e \to 0, \dot{e} \neq 0} \mathbf{u} = \infty [\text{sign}(f(s)) - \text{sign}(\dot{e})]
= \left[ \infty \Psi_1 \infty \Psi_2 \cdots \infty \Psi_n \right]^T \tag{24}
$$

In case of (c), it is clear that the control input $\mathbf{u}$ may be zero if each $f(s_i)\dot{\epsilon}_i > 0$.

**Remark 2**: In the perspective of control engineering, the infinite initial condition may not be controlled, however, according to (21) and (24), the control input may be finite in a very strict condition as the equivalent infinity in (c). In addition, according to $\dot{s} = f(s)$ in (20), if each $\dot{s}, \dot{x}_i > 0$ and there is no unknown disturbance, the strict condition may be relaxed as: $\lim_{e \to \frac{1}{T_c}, |e| = c \rightarrow \infty} \|f(s)\| - \left\| \frac{1}{T_c} \frac{\partial W(s)}{\partial s} \right\|^1 \dot{e} + \frac{(1-m)}{m T_c} \left(\frac{\partial W(s)}{\partial s} \right)^{-1} \dot{\epsilon} \right\| \leq \text{a finite constant}$ in which $\|c\|$ is a big number, i.e. $\|c\| = 3.0 \times 10^8$. Hence, a special class of infinite initial condition may be controlled to be stable by a global finite control input if controller can make $\dot{s}_i \dot{\epsilon}_i > 0$ all the time. The Remark 2 is only a conjecture, but it also reveals some properties of singularity.

Go on the proof. Then, according to (21), (22), (23) and (24), we know that the possible reason of infinite control inputs. By considering actual engineer, $f(s)$ and $\dot{e}$ is finite, hence, we set $m = 1$ when $e \to 0$ and $s \neq 0$:

$$
\lim_{e \to 0, \dot{e} \neq 0} \|\mathbf{u}\|_{y = \|e\|^m}^{m = 1} \left( f(s) - \frac{1}{T_c} \frac{\partial W(s)}{\partial s} \right)^{-1} \dot{e} \tag{25}
= \text{a finite constant}
$$

Then, by considering a predefined-time reaching law $f(s) = -\frac{1}{T_c} \left\| \frac{\partial W(s)}{\partial s} \right\|^{-2} \left( \frac{\partial W(s)}{\partial s} \right)^T$, the candidate Lyapunov function is $V_s = \|s\|$ and the sliding mode existence condition can be expressed:

$$
\dot{V}_s = \frac{s^T \dot{s}}{\|s\|} \\
= -\frac{1}{T_c} \frac{s^T}{\|s\|} \left[ \frac{\partial W(|s|)}{\partial |s|} m |s|^{m-1} s^T \right]^{-1} \\
= -\frac{1}{m T_c} \left[ \frac{\partial W(V, 0)}{\partial V} \right]^{-1} V_s^{1-m} \tag{26}
< 0
$$
Fig. 3. The curve of $y$ in which $W(\|x\|^m) = 1 - \exp(\|x\|^m)$:

$T_c = 1$, $m = 1, 0.9, \cdots, 0.1$

The reaching time can be deduced by:

$$T = \frac{\int \frac{1}{V_s(e_0)} \left[ \frac{\partial W(V_s(e_0)^m)}{\partial e} \right]^{-1} V_s^{1-m} dV_s}{V_s(e_0)^m} \cdot (27)$$

Hence, the minimum upper bound of reaching time is $T_c$.

**Result 1:** the parameter $m$ satisfied the condition of (17) will not affect the minimum upper bound of reaching time.

2) When sliding mode surface reaches 0, then, (16) can be written as:

$$s = 0 = \dot{e} + \frac{1}{mT_c} \left\| \frac{\partial W(\|e\|^m)}{\partial e} \right\|^2 \left[ \frac{\partial W(\|e\|^m)}{\partial e} \right]^T. (28)$$

Then control input could be expressed by (20) and (28) as the following:

$$u = \frac{\partial W(\|e\|^m)}{\partial e} - \frac{1}{mT_c} \left( \frac{\partial W(\|e\|^m)}{\partial e} \right)^T \dot{e}.$$

According to higher-order infinitesimal theorem, it is clear that control input $u$ in (29) will not be infinite in any actual conditions when $0 < m \leq \frac{1}{2}$.

Then a candidate Lyapunov function can be shown as:

$$V_2 = \|e\|. (30)$$
The time derivative of Lyapunov function (30) can be deduced as:

$$
\dot{V}_2 = \frac{e^T e}{\| e \|^m} \\
= -\frac{1}{T_c} e^T \left[ \frac{\partial W(\| e \|^m)}{\partial \| e \|^m} \frac{m\| e \|^{m-1} e^T}{\| e \|^m} \right]^{-1}
$$

(31)

When $m = 1$:

$$
\dot{V}_2 = -\frac{1}{T_c} \left[ \frac{\partial W(V_2)}{\partial V_2} \right]^{-1} < 0.
$$

(32)

The settling time function of (32) can be deduced:

$$
T = \int_{V_2(e_0)}^{V_2(0)} \frac{1}{\frac{1}{T_c} \left[ \frac{\partial W(V_2)}{\partial V_2} \right]^{-1}} dV_2
$$

$$
= T_c \int_{V_2(e_0)}^{V_2(0)} \frac{\partial W(V_2)}{\partial V_2} dV_2
$$

$$
= T_c [W(V_2(e_0)) - W(V_2(0))].
$$

(33)

Note that $m \neq 1$ when $s = 0$, because $\dot{V} \neq 0$ if $V = 0$. In addition, by considering $f(x, t) = -x_2$ and $b(x, t) = E_{3 \times 3}$. The output of system with the controller designed by (20) can be showed in Fig. 3. According to Fig. 3, it is clear that the system errors will not converge to zero within $2T_c$ only if $m = 1$, and it converge wavy to zero at about 2.5s.

When $0 < m \leq \frac{1}{2}$,

$$
\dot{V}_2 = -\frac{1}{mT_c} \left[ \frac{\partial W(V_2^m)}{\partial V_2^m} \right]^{-1} V_2^{1-m} < 0.
$$

(34)

The settling time function of (34) can be deduced:

$$
T = \int_{V_2(e_0)}^{V_2(0)} \frac{1}{\frac{1}{mT_c} \left[ \frac{\partial W(V_2^m)}{\partial V_2^m} \right]^{-1}} dV_2
$$

$$
= T_c \int_{V_2(e_0)}^{V_2(0)} \frac{\partial W(V_2^m)}{\partial V_2^m} dV_2^m
$$

$$
= T_c [W(V_2(e_0)^m) - W(V_2(0)^m)].
$$

(35)

The upper bounds of reaching time in the case of (32) and (34) are same. However, these equilibrium states may be different: (32) is $(V_2, \dot{V}_2)_{V_2=0} = (V_2, \frac{1}{T_c} \left[ \frac{\partial W(V_2)}{\partial V_2} \right]^{-1} V_2 = 0 = (0, 1 \left[ \frac{\partial W(V_2^0)}{\partial V_2^0} \right]^{-1})$, (34) is $(V_2, \dot{V}_2)_{V_2=0} = (V_2, \frac{1}{T_c} \left[ \frac{\partial W(V_2^m)}{\partial V_2^m} \right]^{-1} V_2^{1-m} = (0, 0)$. In addition, none of the $y$ derivative of the known predefined-time stability functions is 0 if $V_2 \to 0$. Hence, the parameter should be $0 < m \leq \frac{1}{2}$ when $\| s \| = 0$.

In order to make control system nonsingular and predefined-time stable, the parameter $m$ should be changed as (17).
The proof is complete.

**Remark 3:** By considering a design method of semi-global predefined-time system in [9,10,20]. A class of candidate Lyapunov function can be set as:

\[ V = \|e\|^m. \]  

(36)

Then, the predefined-time system could be designed as:

\[ \dot{e} = -\frac{1}{T_c} \exp(\|e\|^{mp}) \frac{e}{\|e\|^{mp}}. \]  

(37)

In this case, the parameters \( m \) and \( p \) will be considered combined into a single parameter \( m = mp \).

**Remark 4:** In actual control engineering using the SMC control, \( s \) will not be zero in the presence of perturbations. Hence, the sliding mode switching strategy in (17) will work frequently, and then, an unavoidable chattering of control input will occur when sliding mode switches. We suggest that sliding mode switching strategy in (17) will work only if control input \( u \) is real large:

\[ \varepsilon = \begin{cases} 
\varepsilon , & \|u\| > u_{\max} \\
0 , & \text{others} 
\end{cases} \]  

(38)

where \( u_{\max} \) is the maximum allowable value of control input.

### 3.2 A N-order System

Consider a n-order MIMO system as the following:

\[
\begin{align*}
\dot{x}_i &= x_{i+1}; \quad i = 1, 2 \cdots , n-1 \\
\dot{x}_n &= f(x, t) + b(x, t)(u + u_r) + d(t) \\
y_o &= x_1
\end{align*}
\]

(39)

where \( x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_n^T \end{bmatrix}^T \), \( u_r \in \mathbb{R}^n \) is a robust controller, and an accurate estimation \( \hat{d}(t) \) of \( d(t) \) will be considered to design \( u_r \in \mathbb{R}^n = b^{-1}(x, t) \| \hat{d}(t) \| \text{sign}(s_{n-1}). \)

Consider a tracking problem \( y \rightarrow y_d \), and the error vector can be defined as:

\[ \mathbb{S} = x - x_d = \begin{bmatrix} e^T & \dot{e}^T & \cdots & e^{(n-1)T} \end{bmatrix}^T. \]  

(40)

**Corollary 2:** The singularity of n-order system can be described as: control input \( u \) will be infinite if \( e \rightarrow 0 \) and \( \|\mathbb{S}\| \neq 0 \). A nonsingular recursive predefined-time sliding mode can be designed by Lemma 2 as:

\[
\begin{align*}
s_1 &= \dot{s}_0 + \frac{1}{T_c} \left\| \frac{\partial W(\|s_0\|^m_0)}{\partial s_0} \right\|^{-2} \left[ \frac{\partial W(\|s_0\|^m_0)}{\partial s_0} \right]^T \\
s_2 &= \dot{s}_1 + \frac{1}{T_c} \left\| \frac{\partial W(\|s_1\|^m_1)}{\partial s_1} \right\|^{-2} \left[ \frac{\partial W(\|s_1\|^m_1)}{\partial s_1} \right]^T \\
&\vdots \\
s_{n-1} &= \dot{s}_{n-2} + \frac{1}{T_c} \left\| \frac{\partial W(\|s_{n-2}\|^m_{n-2})}{\partial s_{n-2}} \right\|^{-2} \left[ \frac{\partial W(\|s_{n-2}\|^m_{n-2})}{\partial s_{n-2}} \right]^T \\
0 &= \dot{s}_{n-1} + \frac{1}{T_c} \left\| \frac{\partial W(\|s_{n-1}\|^m_{n-1})}{\partial s_{n-1}} \right\|^{-2} \left[ \frac{\partial W(\|s_{n-1}\|^m_{n-1})}{\partial s_{n-1}} \right]^T,
\end{align*}
\]  

(41)
where \( s_0 = e \) and the parameters satisfy the following condition:

\[
m_k = \begin{cases} 
1 + (m_k - 1) \text{sign}(\text{sat}(\frac{\|s_k\|}{\varepsilon})) , & \|s_{k+1}\| \neq 0 \\
\frac{m_k}{\varepsilon} , & \|s_{k+1}\| = 0 \end{cases}
\] (42)

with

\[
\text{sat}(\frac{\|s_k\|}{\varepsilon}) = \begin{cases} 
1 , & \|s_k\| > \varepsilon \\
0 , & \|s_k\| \leq \varepsilon 
\end{cases}
\] (43)

where \( k = 0, \ldots, n-2 \). \( 0 < m_k \leq \frac{1}{n-k} \) and \( 0 < m_{n-1} < 1 \). \( \varepsilon \) is a small parameter representing the risk of singularity which satisfy the Remark 4.

Assumption 2: \( \frac{\partial^{(n)}[\frac{\partial W(\|s_{n-1}\|^{m})}{\partial|s_{n-1}|^{m}(\alpha)}]}{\partial \|s_{n-1}\|^{m-1}} \) should be bounded in finite area for \( 0 < m \leq 1, n \in R \) and \( s_{n-1} \in R^n \setminus \{\infty\} \). i.e. \( W(\|x\|^{m}) = 1 - e^{-\|x\|^m} \).

Corollary 2 is valid only if Assumption 2 is satisfied.

Then, according to Corollary 2, a sliding mode reaching law can be chose as:

\[
\dot{s}_{n-1} = -\frac{1}{T_c} \left[ \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial s_{n-1}} \right]^{-2} \left[ \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial s_{n-1}} \right]^T 
\] (44)

The nonsingular predefined-time sliding mode control input \( u \) for \( n \)-order MIMO system (39) with global finite control input can be designed by (29), (41) and (44) as:

\[
u = -b^{-1}(x, t) \left[ f(x, t) + y_d^{(n)} \right] + \sum_{k=0}^{n-2} \frac{1}{T_c} \left[ \frac{\partial W(\|s_k\|^{m_k})}{\partial s_k} \right]^{-1} \left[ \frac{\partial W(\|s_k\|^{m_k})}{\partial s_k} \right]^T (n-k-1) \\
+ \frac{1}{T_c} \left[ \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial s_{n-1}} \right]^{-2} \left[ \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial s_{n-1}} \right]^T 
\] (45)

where \( \cdot^{(n-k-1)} = \frac{\partial^{(n-k-1)}(\cdot)}{\partial (n-k-1)} \).

Proof of predefined-time stable control system:

Consider a candidate Lyapunov function as:

\[
V_1 = \|s_{n-1}\| 
\] (46)
The time derivative of Lyapunov function (46) can be deduced as:

\[ \dot{V}_1 = \frac{s_{n-1}^T}{\|s_{n-1}\|} \{ f(x, t) + b(x, t)(u + u_c) + d(t) \} + \sum_{k=0}^{n-2} \frac{1}{m_k T_c} \left( \frac{\partial W(\|s_k\|^{m_k})}{\partial s_k} \right)^{(n-k-1)} - \frac{s_k}{\|s_k\|^2} \]

Then, the reaching time of \( s_{n-1} \) can be deduced as:

\[ T_{n-1}(s_{n-1}(0)) = \int_{V_c(s_{n-1}(0))}^{V_c(0)} \frac{1}{m_{n-1} T_c} \left( \frac{\partial W(V_1^{m_{n-1}})}{\partial V_1^{m_{n-1}}} \right)^{m_{n-1}-1} \frac{1}{V_1^{1-m_{n-1}}} dV_1 = T_c(W(V_1(s_{n-1}(0))^{m_{n-1}} - W(V_1(0)^{m_{n-1}})) \]

It is clear that \( s_{n-1} = 0 \) after \( \sum_{k=1}^{n-1} T_{n-k}(s_{n-k}(0)) \). In a similar way of (47) and (48), each reaching time of \( s_{n-k} \) can be deduced as \( T_c(W(V_1(x_{s_{n-k}(0)})^{m_{n-1}} - W(V_1(x_{s_{n-k-1}(0)})^{m_{n-1}})) \), and \( k = 1, 2 \cdots n - 1 \). Then, the minimum upper bound of reaching time is \( (n-1)T_c \).

After \( (n-1)T_c \), the sliding mode surface \( s_1 = 0 \) has been reached, and (41) can be expressed as:

\[ \dot{e} = -\frac{1}{m_0 T_c} \left\| \frac{\partial W(\|e\|^{m_0})}{\partial e} \right\|^{-2} \left( \frac{\partial W(\|e\|^{m_0})}{\partial e} \right)^T \]

The time derivative of a candidate Lyapunov function: \( V_2 = \|e\| \) can be deduced as:

\[ \dot{V}_2 = \frac{e^T}{\|e\|} \dot{e} + \frac{1}{T_c} \|e\| \left( \frac{\partial W(\|e\|^{m_0})}{\partial e} \right)^T \left[ \frac{\partial W(\|e\|^{m_0})}{\partial e} \right] \]

\[ \dot{V}_2 = -\frac{1}{m_0 T_c} \left\| \frac{\partial W(\|e\|^{m_0})}{\partial e} \right\|^{-1} \|e\|^{1-m_0} \]

\[ = \frac{1}{m_0 T_c} \left\| \frac{\partial W(\|e\|^{m_0})}{\partial e} \right\|^{-1} \|e\|^{1-m_0} \]

\[ \dot{V}_2 = -\frac{1}{m_0 T_c} \left( \frac{\partial W(V_2^{m_0})}{\partial V_2^{m_0}} \right)^{m_0-1} V_2^{1-m_0} \]
Then the reaching time of \( e = 0 \) can be deduced as:

\[
T(x_0) = \int_{V_2(x_0)}^{V_2(0)} \frac{1}{m_{x_k} T_c} \left[ \frac{\alpha W(V_2(m_0))}{\partial V_2} \right] \frac{1}{V_2^{1-m_0}} dV_2
\]

\[
= T_c \int_{V_2(x_0)}^{V_2(0)} \frac{\partial W(V_2(m_0))}{\partial V_2} dV_2^{m_0}
\]

\[
= T_c (W(V_2(x_0))^{m_0}) - W(V_2(0)^{m_0})
\]

(51)

Then the minimum upper bound of settling time of system can be predefined as \( nT_c \).

The system will be said to be predefined-time stable if \( n \) is finite.

The proof is complete.

According to general Leibniz rule, (45) can be written as:

\[
u = -b^{-1}(x,t) f(x,t) + \frac{1}{m_{x_k} T_c} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-1} \binom{n-k-1}{j}
\]

\[
\cdot \left( \left( \frac{\partial W(\|s_k\|^{m_k})}{\partial \|s_k\|^{m_k}} \right)^{-1} (n-k-1-j) \left( \frac{s_k}{\|s_k\|^{m_k}} \right)^{(j)} + \frac{1}{m_{x_k} T_c} \left( \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial \|s_{n-1}\|^{|m_{n-1}}|} \right)^{-1} \left( \frac{s_{n-1}}{\|s_{n-1}\|^{m_{n-1}}} \right)^{(j)} \right)
\]

\[
\cdot \left( \left( \frac{\partial W(\|s_k\|^{m_k})}{\partial \|s_k\|^{m_k}} \right)^{-1} (n-k-1-j) \left( \sum_{l=0}^{j} \binom{j}{l} (s_k)^{(j-l)} (\|s_k\|^{-m_k})^{(l)} \right) + \frac{1}{m_{x_k} T_c} \left( \frac{\partial W(\|s_{n-1}\|^{m_{n-1}})}{\partial \|s_{n-1}\|^{|m_{n-1}}|} \right)^{-1} \left( \frac{s_{n-1}}{\|s_{n-1}\|^{m_{n-1}}} \right)^{(n-k-1)} \right)
\]

(52)

**Result 2:** According to Result 1, the singularity, which occurs at the reaching time of sliding mode \( s_k \): \( t \in (n-k-1)T_c, (n-k)T_c \), can also be avoided if \( m_k = 1 \) for any \( k \) and Assumption 2 is satisfied.

Proof of Result 2 (for nonsingular parameter selection): In this proof, we will study how to make control input finite when \( e \to 0 \) and \( \|\mathbf{s}\| \neq 0 \)

Consider \( m_k = 1 \) for all \( k \), then a term in the first equation of (52) can be expressed:

\[
\lim_{s_k \to 0} \frac{s_k}{\|s_k\|} = \frac{0}{\|0\|} = E_{n1}.
\]

(53)

If \( s_k \to 0 \), the control input \( u \) can be written as:

\[
u = -b^{-1}(x,t) f(x,t) + \frac{1}{m_{x_k} T_c} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-1} \binom{n-k-1}{j}
\]

\[
\cdot \left( \left( \frac{\partial W(\|s_k\|)}{\partial \|s_k\|} \right)^{-1} (n-k-1-j) \left( E_{n1} \right)^{(j)} + \frac{1}{T_c} \left( \frac{\partial W(\|s_{n-1}\|)}{\partial \|s_{n-1}\|} \right)^{-1} E_{n1} \right)
\]

(54)
It is clear that $E^{(j)}_{n_1}$ will not be infinite forever. Then, by considering a Faa di Bruno’s formula in [21], the singularity problem in controller (52) will be focused on the following:

$$
\left( \frac{\partial W(\|s_k\|)}{\partial \|s_k\|} \right)^{(n-k-1-j)} = \sum \frac{(n-k-1-j)!}{b_1 b_2 \cdots b_{n-k-1-j}!} \frac{\partial^h_p \left[ \frac{\partial W(\|s_k\|)}{\partial \|s_k\|} \right]^{-1}}{\partial (\|s_k\|)^{hp}} \cdot \left( \frac{\|s_k\|^2}{1!} \right)^{(1)} b_1 \left( \frac{\|s_k\|^2}{2!} \right)^{(2)} b_2 \cdots \left( \frac{\|s_k\|^2}{(n-k-1-j)!} \right)^{(n-k-1-j)} b_{n-k-1-j} \tag{55}
$$

where $b_1 + 2b_2 + \cdots + (n-k-1-j)b_{n-k-1-j} = n-k-1-j$ and $b_1 + b_2 + \cdots + b_{n-k-1-j} = h_p$ in which $b$ is non-negative integer.

**Remark 5:** There is a new definition in Faa di Bruno’s formula $\frac{\partial^\gamma \varphi(x)}{\partial x^\gamma} = \sum \frac{\gamma!}{b_1! \cdots b_n!} \frac{\partial^{\gamma \gamma} \varphi(x)}{\partial \varphi(x)^{\gamma \gamma}} (\frac{\varphi(x)}{1!})^{b_1} \cdots (\frac{\varphi(x)}{\gamma!})^{b_\gamma}$ for this paper; $\frac{\partial^\gamma \varphi(x)}{\partial x^\gamma}$ will be equal to $\varphi(\varphi(x))$ if $\gamma=0$.

Let $\|s_k\|^2 = s_k^T s_k$, then, $\|s_k\|^2$ in (55) can be expressed:

$$
(s_k^T s_k)^{(1)} = \sum_{i=1}^n \left( s_{1,k,v}^{2} \right)^{(1)} = \sum_{i=1}^n \sum_{v=1}^n \left\{ \frac{\partial^h \varphi_s^2}{\partial (s_{k,v})^{hv}} \left( \frac{s_{k,v}}{v} \right)^{b_1} \left( \frac{s_{k,v}}{2!} \right)^{b_2} \cdots \left( \frac{(s_{k,v})^{(l_v)}}{v} \right)^{b_l} \right\} \tag{56}
$$

where $s_k = \left[ s_{k,1} \ s_{k,2} \cdots \ s_{k,n} \right]^T$; $e = \left[ e_1 \ e_2 \cdots \ e_n \right]^T$; $l_v = 1, 2, \cdots, l_1$; $b_1 + 2b_2 + \cdots + l_v b_{l_v} = l_v$ and $b_1 + b_2 + \cdots + b_{l_v} = h_v$.

One of the possible singularity term in (56) can be focused on the following:

$$
\left. \frac{\partial^{h_v} s_{k,v}^2}{\partial (s_{k,v})^{h_v}} \right|_{s_{k,v}} = \sum_{v=1}^n \left[ 2s_{k,v} \right] |_{s_{k,v}} = \begin{cases} 
2 & h_v = 1 \\
0 & h_v \geq 3
\end{cases} \tag{57}
$$

and it is clear that the term (57) will not be infinite if $s_{k,v}$ is not infinite.

**Assumption 3:** There will be no infinite value in (57) if $s_{k,v}$ is bounded.

Then, other possible singularity terms: $s^{(i_k)}_k$ for $i_k = 1, 2, \cdots, n-k-1-j$ in (56) will be discussed in the following steps:

**Step 0:** The singularity definition of this subsection is that control input will be infinite if $e \to 0$ and $\|s\| \neq 0$. $s^{(i_1)}_{1,2,\ldots,n-1}$ will be assumed to be bounded first for $i_1 = 1, 2, \cdots, n-j$. Then, by substituting $s_0 = e$ in (55) and (56), it is clear that the singularity of controller about $s_0$ will not occur if $m_k = 1$ for any $k$.

**Step 1:** $s^{(i_1)}_{2,3,\ldots,n-1}$ will be assumed to be bounded first for $i_1 = 1, 2, \cdots, n-j$. If $m_0 = 1$ and $s_0 = e$, the
singularity of controller about $s_1$ can be expressed as:

$$
\begin{align*}
 s^{(i_1)}_1 &= e^{(i_1)} + \frac{1}{\tau_c} \sum_{j_1=0}^{i_1} \binom{i_1}{j_1} \\
 &\cdot \left[ \frac{\partial W(\|e\|)}{\partial \|e\|} \right]^{-1} (i_1-j_1) (E_{n1})(j_1) \\
 &= e^{(i_1)} + \frac{1}{\tau_c} \sum_{j_1=0}^{i_1} \binom{i_1}{j_1} \\
 &\left[ \sum_{i_1}^{(i_1-j_1)!} \frac{\partial^h e^{(i_1)}}{\partial (\|e\|)^{h}} \right] \left( \frac{\|e\|}{i_1} \right) h_1 \\
 &\cdot \left( \frac{(\|e\|)^{2}}{2!} \right) b_2 \cdots \left( \frac{(\|e\|)^{(i_1-j_1)}}{(i_1-j_1)} \right) b_{i_1-j_1} (E_{n1})(j_1)
\end{align*}
$$

where $b_1 + 2b_2 + \cdots + (i_1-j_1)b_{i_1-j_1} = i_1-j_1$ and $b_1 + b_2 + \cdots + b_{i_1-j_1} = h_1$.

Let $\|e\|^2 = e^T e$, then:

$$
(e^T e)^{(l_{v1})} = \sum_{v_1=1}^{n} \left( e_{v1}^2 \right)^{(l_{v1})}
$$

$$
= \sum_{v_1=1}^{n} \sum_{l_{v1}=1}^{(n)} \left\{ \frac{\partial^{h_{v1}} e_{v1}^2}{\partial (e_{v1})^{h_{v1}}} \right\} \left( \frac{\|e\|^2}{l_{v1}!} \right) b_{v1} l_{v1}
$$

$$
\text{where } e = \left[ e_{v1} \ e_{v2} \cdots e_{vn} \right]^T; \ l_{v1} = 1, 2, \cdots, l_1; \ b_1 + 2b_2 + \cdots + l_{v1}b_{v1} = l_{v1} \text{ and } b_1 + b_2 + \cdots + b_{v1} = h_{v1}.
$$

The singular problem will be focused on the following:

$$
\begin{align*}
\frac{\partial^{h_{v1}} e_{v1}^2}{\partial (e_{v1})^{h_{v1}}} &= \sum_{v_1=1}^{n} \left( 2e_{v1} \right)_h^{\geq 0} h_{v1} = 1 \\
&= \sum_{v_1=1}^{n} \left( 2e_{v1} \right)_h^{\geq 0} h_{v1} = 1 \\
0, &h_{v1} \geq 3
\end{align*}
$$

Then, by considering (58), (59) and (60), The singularity of controller about $s_1$ will not occur if $m_0 = 1$.

Steps 2, \cdots, k - 1:

Step k: \( s^{(i_2)}_{k+1, \cdots, n-1} \) will be be assumed to be bounded first for $i_k = 1, 2, \cdots, n - k - j$. The singularity of
controller about $s_k$ if $m_{k-1} = 1$ can be expressed as:

$$s_k^{(i_k)} = s_{k-1}(i_k) + \frac{1}{T_c} \sum_{j_k=0}^{i_k} \left( \frac{\partial W(\|s_{k-1}\|)}{\partial \|s_{k-1}\|} \right)^{-1} (i_k-j_k) (E_{n1})^{(j_k)}$$

$$= s_{k-1}(i_k) + \frac{1}{T_c} \sum_{j_k=0}^{i_k} \left( \frac{\partial W(\|s_{k-1}\|)}{\partial \|s_{k-1}\|} \right)^{-1} (i_k-j_k)$$

$$\left[ \sum \frac{(i_k-j_k)!}{b_1 b_2! \cdots b_{k-j_k}!} \left( \frac{\partial W(\|s_{k-1}\|)}{\partial \|s_{k-1}\|} \right)^{-1} \left( \frac{\|s_{k-1}\|}{s_{k-1}} \right)^{b_{k-j_k}} \right] (E_{n1})^{(j_k)}$$

where $b_1 + 2b_2 + \cdots + (i_k - j_k)b_{k-j_k} = i_k - j_k$ and $b_1 + b_2 + \cdots + b_{k-j_k} = h_k$.

Let $\|s_{k-1}\|^2 = s_{k-1}^T s_{k-1}$, then:

$$\left( s_{k-1}^T s_{k-1} \right)^{l_{vk}} = \sum_{v_k=1}^{n} \left( s_{k-1,v_k} \right)^2^{l_{vk}}$$

$$= \sum_{v_k=1}^{n} \sum_{l_{vk}=0}^{\infty} \left\{ \frac{l_{vk}!}{b_1 b_2! \cdots b_{l_{vk}}!} \left( \frac{\partial W(s_{k-1})}{\partial s_{k-1}} \right)^{b_{l_{vk}}} \right\}$$

$$\left( s_{k-1}^T s_{k-1} \right)^{l_{vk}}$$

where $s_{k-1} = \left[ s_{k-1,v1} \ s_{k-1,v2} \cdots s_{k-1,vn} \right]^T$; $l_{vk} = 1, 2, \cdots, l_k$; $b_1 + 2b_2 + \cdots + l_{vk} b_{l_{vk}} = l_{vk}$ and $b_1 + b_2 + \cdots + b_{l_{vk}} = h_{vk}$.

with

$$\left[ \frac{\partial h_{vk}^2 s_{k-1,vk}^2}{\partial (s_{k-1,vk})^h_{vk}} \right] = \sum_{v_k=1}^{n} \left[ \frac{2 s_{k-1,vk}}{h_{vk}} \right]$$

$$= \begin{cases} \sum_{v_k=1}^{n} \frac{2 s_{k-1,vk}}{h_{vk}}, & h_{vk} = 1 \\ \sum_{v_k=1}^{n} 2, & h_{vk} = 2 \\ 0, & h_{vk} \geq 3 \end{cases}$$

According to step 1 ((60) is substituted into (59), and (59) is substituted into (58)), it is clear that $s_{n,vk}^{(1),(2)\cdots,(vk)}$ will not be infinite if $m_{0,1,\cdots,k-2} = 1$. Then, similar to step 1, by considering (61), (62) and (63), The singularity of controller about $s_k$ will also not occur if $m_{k-1} = 1$.

Steps $k+1, \cdots, n-1$:

Step n: Make an identity: $s_n = 0$ in (41). According to the above-mentioned $n-1$ steps, $s_{n,1,2,\cdots,n-2}$ can be predefined to be bounded if $m_{0,1,\cdots,n-2} = 1$. The singularity of controller about $s_n = 0$ if $m_{n-1} = 1$ can be expressed as:

$$\left( s_{n-1}^T s_{n-1} \right)^{-1} = \frac{1}{T_c} \left[ \frac{\partial W(\|s_{n-1}\|)}{\partial \|s_{n-1}\|} \right]^{-1}$$

$$s_{n-1} \cdot E_{n1}$$

It is clear that the singularity of controller about $s_n$ will also not occur if $m_{n-1} = 1$.
According to the above steps, the Assumption 3 is actually satisfied everywhere if \( m_k = 1 \). Hence, the control input will not infinite if \( m_k = 1 \).

The proof is complete.

However, according to the simulation in Fig.3 and Result 1. It is clear that the predefined-time stability cannot be guaranteed if \( m_k = 1 \) for everywhere. Hence, a switching decision similar to (17) will be discussed in the following.

**Result 3:** In order to ensure the predefined-time stability of the system, parameter switching can only occur on the sliding mode surface \( s_k+1 \) relative to virtual system state \( s_k \). Hence, in reaching stage of virtual system state \( s_k \), the corresponding parameter switching cannot be made. In addition, the selection range of nonsingular parameters should be: \( 0 < m_k \le \frac{1}{n-k} \) and \( m_k = 1 \).

Proof of the selection range of nonsingular parameters Result 3 (\( 0 < m_k \le \frac{1}{n-k} \)):

According to the Faa di Bruno’s formula, (52) can be written as:

\[
\begin{align*}
\mathbf{u} &= -b^{-1}(x, t) \left\{ f(x, t) + \frac{1}{m_k T_c} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \right. \\
& \quad \left. \cdot \left[ \sum_{b_1, b_2, \cdots, b_{n-k-1-j}} \frac{\partial^p}{\partial ||s_k||^{m_k}} \left( \frac{\partial (||s_k||^{m_k})^{(j)}}{\partial ||s_k||^{m_k}} \right) \right] b_i \\
& \quad \left. \cdot \left( \frac{||s_k||^{m_k}}{2!} \right)^{j} b_2 \cdots \left( \frac{||s_k||^{m_k}}{n-k-1-j} \right) b_{n-k-1-j} \right) \right. \\
& \quad \left. \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \right\} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
& \quad \cdot \left( \frac{\partial W(||s_k||^{m_k})^{(m_k-1)}}{\partial ||s_k||^{m_k}} \right)^{-1} \frac{s_{n-1}}{s_{n-1}^{m_k-1}} \\
\end{align*}
\]

with

\[
||s_k||^{m_k} = \sum_{b_1, b_2, \cdots, b_{n-k-1-j}} \frac{(n-k-1-j)!}{b_1!b_2!\cdots b_{n-k-1-j}!} \left[ \prod_{z=0}^{n-k-1-j} \left( \frac{m_k}{2} - z \right) \right] ||s_k||^{m_k-2h_p} \\
\cdot \left( \frac{||s_k||^{m_k}}{2!} \right)^{b_2} \cdots \left( \frac{||s_k||^{m_k}}{n-k-1-j} \right)^{b_{n-k-1-j}} \\
\]

where \( b_1 + 2b_2 + \cdots + lb_l = l \) and \( b_1 + b_2 + \cdots + b_l = h; b_1 + 2b_2 + \cdots + (n-k-1-j)b_{n-k-1-j} = n-k-1-j \) and \( b_1 + b_2 + \cdots + b_{n-k-1-j} = h_p \).

Let \( ||s_k||^{2} = s_k^T s_k \), then:

\[
\begin{align*}
(s_k^T s_k)^{(l_v)} &= \sum_{v=1}^{n} (s_{k,v}^T)^{(l_v)} \\
&= \sum_{v=1}^{n} \sum_{s=1}^{n} \left\{ \frac{(l_v)!}{b_1!b_2!\cdots b_{l_v}!} \left[ \frac{\partial^{s+v} s_{k,v}^T}{\partial (s_{k,v})^{s+v}} \right] \\
& \quad \cdot \left( \frac{b_1}{s_1} \right)^{b_1} \left( \frac{b_2}{s_2} \right)^{b_2} \cdots \left( \frac{s_{k,v}}{s_{k,v}} \right)^{(s_{k,v})} \right\} b_{l_v} \\
&\end{align*}
\]

where \( s_k = \begin{bmatrix} s_{k,1} & s_{k,2} & \cdots & s_{k,n} \end{bmatrix}^T \). \( b_1 + 2b_2 + \cdots + l_v b_{l_v} = l_v \) and \( b_1 + b_2 + \cdots + b_{l_v} = h_v \).
Step 0: By assuming that $s_0 \neq 0$ and $s_1, s_2, \ldots, s_{n-1} = 0$, then, according to the results of Lemma 1 and (41), $(\dot{s}_1, \dot{s}_1, \ldots, (s_1)^{(n-1)})$ have to converge to zero in $t \geq (n-1)T_e$ if the system is predefined-time stable and $0 < m_0 \leq \frac{1}{n}$. Then, the control input $u$ can be written as:

$$u = -b^{-1}(x, t) \left\{ f(x, t) + \frac{1}{m_0 T_e} \sum_{j=0}^{n-1} \binom{n-1}{j} \right\}$$

$$\cdot \left[ \sum_{b_1, b_2, \ldots, b_{n-1}} \frac{\partial \rho_p}{\partial \rho_p} \left( \frac{\partial W(||e||^{m_0})}{\partial ||e||^{m_0}} \right)^{-1} \right.$$  

$$\left( ||e||^{m_0} \right)^{j} b_1 \left( \frac{||e||^{m_0}}{2} \right)^{b_2} \ldots \left( ||e||^{m_0} \right)^{(n-1)-j} b_{n-1} \right)$$

$$\left( \sum_{j=0}^{n-1} \binom{n-1}{j} \right) \left[ (e)^{-j} \sum_{b_1, b_2, \ldots, b_{n-1}} \frac{h-1}{2} \left( \sum_{z=0}^{m_0 \frac{2}{z}} \frac{b_1}{l!} \left( \frac{||e||^{2}}{l!} \right)^{b_2} \ldots \left( \frac{||e||^{2}}{l!} \right)^{b_{n-1}} \right) \right]$$

$$+ \frac{1}{m_0 T_e} \left[ \frac{\partial W(||s_0-n-1||)^{m_0-1}}{\partial ||s_0-n-1||} \right]^{-1} \left( \frac{||s_0-n-1||}{m_0} \right) \right)$$

(68)

According to (41), $s_0$ can be replaced by a function about $s_0$ when $s_1 = 0$:

$$\dot{s}_0 = -\frac{1}{T_e} \left[ \frac{\partial W(||s_0||^{m_0})}{\partial ||s_0||} \right] ||e||^{m_0} T_e$$

$$= \frac{1}{m_0 T_e} \left[ \frac{\partial W(||e||^{m_0})}{\partial ||e||^{m_0}} \right]^{-1} \frac{||e||^{m_0}}{||e||^{m_0}}$$

(69)

Then, (67) can be written by considering (57) and (69) as:

$$\left( e^T e \right)^{(l_v)} = \sum_{v=1}^{n} \left( \frac{e_v^2}{e_v} \right)^{(l_v)}$$

$$= \sum_{v=1}^{n} \left\{ \frac{1}{b_1 b_2 \ldots b_{n-1}} \left[ 2e_v \right] \left( \frac{\partial W(||e_v||^{m_0})}{\partial ||e_v||^{m_0}} \right)^{-1} e_v^{1-m_0} \right\}$$

$$\left[ \frac{e_v}{e_v} \right] \sum_{v=1}^{n} \left( \frac{e_v^2}{e_v} \right)^{(l_v)} b_v \right)$$

with

$$\left\{ b_1 + b_2 + \cdots + b_{n-1} = h_v \right\}$$

(70)

$$\left\{ b_1 + b_2 + \cdots + b_{n-1} = l_v \right\}$$

(71)

Consider a singularity expression: $e \to 0$ with $e^{(1,2, \ldots, n-1)} = a \text{ bounded vector}$. According to (57), (66) and (70), the main singularity term (the highest-order negative exponential) of controller in (68) about $s_0 = e$ can be simplified by considering $||0||_1 = ||0||_2$ as:

$$\sum_{v=1}^{n} \left( \frac{b_{n-1} e_v}{e_v} \right) \sum_{v=1}^{n} \left( \frac{b_{n-1} e_v}{e_v} \right) \sum_{v=1}^{n} \left( \frac{b_{n-1} e_v}{e_v} \right) \sum_{v=1}^{n} \left( \frac{b_{n-1} e_v}{e_v} \right)$$

(72)
with
\[
\begin{align*}
\sum_{v=0}^{n-1-j} b_v &= h_p \leq n - 1 - j \\
\sum_{v=0}^{l} b_v &= h \leq l \\
h_v - (b_2 + \cdots + b_v) &= b_1 \leq l
\end{align*}
\] (73)

where \( j = l \) because the final exponential term of \( e^{(j-l)} \) will be smaller than 1 if \( j \neq l \) by considering the decomposition term \( \sum_{k=0}^{n-2} \left( \frac{1}{T_k} \left\| \frac{\partial W(\|s_k\|^m_k)}{\partial s_k} \right\| \right)^{-2} \left[ \frac{\partial W(\|s_k\|^m_k)}{\partial s_k} \right]^T \right)^{(n-k-1)} \) in (45).

Then, (72) will not be infinite if the following inequality is satisfied:

\[
m_0(-h_p(1 - b_1 h_p) + 1 + h b_1) \leq (h_p^2 + h)(b_1 - 1) + (j - l)
\]

\[
\Rightarrow \begin{cases} 
  m_0 \leq (\frac{(h_p^2 + h)(j - l)}{(h_p(2h_p - 1) + 1 + 2h)}) \\
  m_0 \leq (\frac{(h_p(2h_p - 1) + 1 + 2h)}{(j - l)}) \\
  m_0 \leq (\frac{1}{(1(1 - 1) + 1 + n - 1)}) \leq \frac{1}{n}
\end{cases}
\] (74)

where \( m_0 \) will be minimum by considering the inequality relationship in (73).

Then the controller in condition of step 0 will not be singular if \( 0 < m \leq \frac{1}{n} \)

Step 1, \( \cdots, k - 1 \):

Step k: By assuming that \( s_{k+1,k+2,\ldots,n-1} = 0 \) and \( s_k \neq 0 \), then, according to the results of Lemma1 and (41), \((\dot{s}_{k+1}, \dot{s}_{k+1}, \ldots, (s_{k+1})^{(n-k-1)})\) have to converge to zero in \( t \geq (n - k - 1)T_c \) if the system is predefined-time stable and \( 0 < m_k \leq \frac{1}{n - k - 1} \). Then, according to (41), it is clear that it is that . The control input \( u \) can be written as:

\[
u = -b^{-1}(\mathbf{x}, t) \left\{ f(\mathbf{x}, t) + \frac{1}{m_0 T_c} \sum_{j=0}^{n-k-1} \left( \frac{n-k-1}{j} \right) b_j \left( \sum_{l=0}^{j} \binom{j}{l} \left( \frac{1}{\|e\|^m k} \right)^{(j-l)} b_l \left( \sum_{z=0}^{l} \binom{l}{z} \left( \frac{1}{\|e\|^m k} \right)^{z} \right) \right) \right.
\]

\[
\left. \left( \frac{1}{\|e\|^m k} \right)^{(j-l)} b_j \left( \sum_{l=0}^{j} \binom{j}{l} \left( \frac{1}{\|e\|^m k} \right)^{(j-l)} b_l \left( \sum_{z=0}^{l} \binom{l}{z} \left( \frac{1}{\|e\|^m k} \right)^{z} \right) \right) \right) \right)
\]

\[
\frac{1}{m_{n-1} T_c} \left[ \frac{\partial W(\|s_{n-1}\|^m_{n-1})}{\partial s_{n-1}} \right]^{-1} \left[ \frac{\partial W(\|s_{n-1}\|^m_{n-1})}{\partial s_{n-1}} \right]^T \right)^{(n-1)} \] (75)

According to (41), \( s_k \) can be replaced by a function about \( s_0 \) when \( s_{k+1} = 0 \):

\[
\dot{s}_k = -\frac{1}{T_c} \left[ \frac{\partial W(\|s_k\|^m_k)}{\partial s_k} \right]^{-2} \left[ \frac{\partial W(\|s_k\|^m_k)}{\partial s_k} \right]^T
\]

\[
= \frac{1}{m_k T_c} \left[ \frac{\partial W(\|s_k\|^m_k)}{\partial s_k} \right]^{-1} \frac{s_0}{\|s_k\|^m_k}
\] (76)
Then, (67) can be written by considering (57) and (76) as:

\[
(s_k^T s_k)^{(l_v)} = \sum_{v=1}^{n} (s_{k,v}^2)^{(l_v)}
\]

\[
= \sum_{v=1}^{n} \left\{ \frac{l_v}{2m_{k,v}^2} \left[ 2s_{k,v} \left| \frac{1}{m_{k,v}} \left[ \frac{\partial W(\|s_{k,v}\|_{m_k})}{\partial \|s_{k,v}\|_{m_k}} \right] \right|^{-1} s_{k,v} \right] \right\} (77)
\]

Then, (67) can be written by considering (57) and (76) as:

\[
(e \cdot \|s_k\|_{m_k})^{-1} s_{k,v} (h_v) \frac{(78)}{[\partial W(\|s_{k,v}\|_{m_k})]} \left[ \frac{\partial W(\|s_{k,v}\|_{m_k})}{\partial \|s_{k,v}\|_{m_k}} \right]^{-1} s_{k,v} \left| \frac{1}{m_{k,v}} \left[ \frac{\partial W(\|s_{k,v}\|_{m_k})}{\partial \|s_{k,v}\|_{m_k}} \right] \right|^{-1} s_{k,v} \right\}
\]

with

\[
\begin{align*}
& b_1 + b_2 + \cdots + b_{l_v} = h_v \quad (78) \\
& b_1 + 2b_2 + \cdots + l_v b_{l_v} = l_v
\end{align*}
\]

Consider a singularity expression: \( e \to 0 \) with \( e^{(1,2,\ldots,n-1)} = \text{a bounded constant} \). According to (57), (66) and (77), the main singular term (the highest-order negative exponential) of controller in (75) about \( s_0 = e \) can be simplified by considering \( \|0\|_1 = \|0\|_2 \) as:

\[
\|s_k\|^{m_k-2h_k} \|s_k\|^{m_k-(h_k-1)l_v} \left[ \frac{\partial W(\|s_k\|_{m_k})}{\partial \|s_k\|_{m_k}} \right] \left[ \frac{\partial W(\|s_k\|_{m_k})}{\partial \|s_k\|_{m_k}} \right]^{-1} s_k \left| \frac{1}{m_{k,v}} \left[ \frac{\partial W(\|s_{k,v}\|_{m_k})}{\partial \|s_{k,v}\|_{m_k}} \right] \right|^{-1} s_{k,v} \right\}
\]

with

\[
\left\{ \begin{align*}
& n-k-1-j \sum_{v=1}^{n} b_{v,k} = n - k - 1 - j \\
& \sum_{v=1}^{n} b_{v,k} = h \leq l
\end{align*} \right\}
\]

(80)

where \( j = l \) because the final exponential term of \( e^{(j-1)} \) will be smaller than 1 if \( j \neq l \) by considering the decomposition term \( \sum_{k=0}^{n-2} \left( \frac{\partial W(\|s_k\|_{m_k})}{\partial s_k} \right)^2 \left[ \frac{\partial W(\|s_k\|_{m_k})}{\partial \|s_k\|_{m_k}} \right]^{-1} \right\} \) in (45).

Then, (79) will not be infinite if the following inequality is satisfied:

\[
m_k(-h_p(1 - b_l h_p) + 1 + h b_1) \leq (h_p^2 + h)(b_1 - 1) + 1
\]

\[
\Rightarrow \left\{ \begin{align*}
& m_k \leq \frac{(h_p^2 + h)(b_1 - 1) + 1}{(h_p(1 - b_l h_p) + 1 + h b_1)} \\
& m_k \leq \frac{1}{(h_p(1 - b_l h_p) + 1 + h b_1)}
\end{align*} \right\}
\]

\[
(81)
\]

where \( m_k \) will be minimum by considering the inequality relationship in (80).

Then the controller in condition of step 0 will not be singular if \( 0 < m_k \leq \frac{1}{n-k} \)

Step \( k+1, \cdots, n-2: \)
Fig. 4. Some examples: (1) The curve of $y = u$ in the $W_1(x)$. $T_c = 1$, $m = 0.9$. (2) The curve of $y = u$ in the $W_1(x)$. $T_c = 1$, $m = 0.5$. $\cos(t)$ is the tracking signal.

Step $n - 1$: it is clear that the singular term of controller about $s_{n-1}$ is the final term of controller in (68), and the singularity will not occur if $0 < m_{n-1} < 1$ by considering Lemma 2.

According to the these steps, a selection scheme of nonsingular parameters $m_k$ can be designed as $0 < m_k \leq \frac{1}{n-k}$ if Assumption 1 is satisfied.

The proof is completed.

According to Result 2 and Result 3, a final selection scheme of nonsingular parameters $m_k$ which can guarantee the predefined-time can be designed as (42) and (43).

4 Simulation and Analysis

In order to show the effective of the proposed nonsingular sliding control method, two various-order control system will be shown in the following. In addition, the curve formed by singular parameters $m_k$ cannot be shown, because singularity is inevitable. Even if it is simulated by luck, it will not pass the above-mentioned mathematical argument.

Example 2: Fig. 4 shows a bad case: $m = 0.9$ in second-order system and $m = 0.5$ in third-order system. It is clear that the control input is infinite or very large in 1.143s in Fig. 4 (1), and the control input is also very large in 0.805s in Fig. 4 (2).

4.1 A Second-order MIMO System

$$\begin{aligned}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -x_2 + u \\
    y &= x_1
\end{aligned} \tag{82}$$

where $x_{1d} = \begin{bmatrix} \cos(t) & 1 & -5 & 5 \end{bmatrix}^T$, the tracking error is $e = x_1 - x_{1d}$

A predefined-time function can be chosen as: $W_1(x) = \sin(arctan(\|x\|^m))$. Then, a nonsingular predefined-time terminal sliding mode surface is designed by Corollary 2 as the following:

$$s = e + \frac{1}{T_c} \left[ \partial W_1(\|e\|^m) / \partial e \right]^{-1} \tag{83}$$
The curve of $y = x_1$ in the $W_1(x)$. $T_c = 1$, $m = 1, 0.5, \cdots 0.1$

$T_c = 1$, $\varepsilon = 0.0001$, $u_{\text{max}} = 15$.

According to Introduction, the case of Remark 4 was not considered in the method in [19], so the unnecessary buffeting will occur in control input caused by inappropriate singular risk parameter $\varepsilon$. As shown in Figs. 5 and 6, it is clear that both methods can make the tracking error converge to zero within predefined-time $2T_c = 2s$ if $m$ is satisfied the nonsingular parameter range deduced by this paper. As shown in Fig 7, it is also clear that the chattering of the proposed control input is lower than the controller in [19]. The tracking errors of second-order MIMO system can be shown in Figs. 8 and 9, and the tracking error $e$ cannot converge to zero within predefined-time if $m = 1$.
Fig. 7. (1) the method in [19] (2) the proposed method
The curve of $y = u$ in the $W_1(x)$. $T_c = 1$, $m = 1, 0.5, \cdots 0.1$

Fig. 8. (1) $x_{1d} = 1$ (2) $x_{1d} = -5$ (3) $x_{1d} = 5$
The curve of $y = x_1$ in the $W_1(x)$. $T_c = 1$, $m = 1, 0.5, \cdots 0.1$

Fig. 9. (1) $x_{1d} = 1$ (2) $x_{1d} = -5$ (3) $x_{1d} = 5$
The curve of $y = x_1$ in the $W_1(x)$. $T_c = 1$, $m = 1, 0.5, \cdots 0.1$

4.2 A Third-order MIMO System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -x_2 - 2x_3 + u \\
y &= x_1
\end{align*}
\]
Fig. 10. (1) the method in [19] (2) the proposed method
The curve of $y = x_1$ in $W_1(x)$ $T_c = 5$, $m = 1, 0.3, \cdots 0.1$

Fig. 11. (1) the method in [19] (2) the proposed method
The curve of $x_2$ in $W_1(x)$ $T_c = 5$, $m = 1, 0.3, \cdots 0.1$

where $x_{1d} = \left[ \cos(t) \ 1 \ -5 \ 5 \right]^T$, the tracking error is $e = x_1 - x_{1d}$

A predefined-time function can be chosen as: $W_1(x) = \sin(\arctan(||x||^{m_k}))$. Then, a nonsingular predefined-time terminal sliding mode surface is designed by Corollary 2 as the following:

$$s_{k+1} = s_k + \frac{1}{T_c} \left[ \frac{\partial W_1(||s_k||^{m_k})}{\partial s_k} \right]^{-1}$$

with

$$m_k = \begin{cases} 
1 + (m_k - 1) \text{sign}(\text{sat}(\|s_k\|/\varepsilon)) , & \|s_{k+1}\| \neq 0 \\
1, & \|s_{k+1}\| = 0
\end{cases}$$

where $k = 0, 1, 2$, $s_3 = 0$, $s_0 = e$, $T_c = 5$, $\varepsilon = 0.001$, $u_{max} = 50$. For convenience, $m_k = m = 1, 0.3, 0.2, 0.1$.

Then, according to (45), the control input $u$ can be designed as:

$$u = x_2 - 2x_3 - \frac{1}{m T_c} \sum_{k=0}^{n-2} \left( \frac{\partial W(||s_k||^{m})}{\partial ||s_k||^{m}} \right) \frac{s_k}{||s_k||^{m}} (n-k-1)$$

$$- \frac{1}{m T_c} \left( \frac{\partial W(||s_{n-1}||^{m})}{\partial ||s_{n-1}||^{m}} \right) \frac{s_{n-1}}{||s_{n-1}||^{m}}$$

where $n = 3$.

According to Introduction, the case of Remark 4 was not considered in the method in [19], so the unnecessary buffeting will also occur in control input caused by inappropriate singular risk parameter $\varepsilon$. As shown in Figs. 10, 11.
Fig. 12. (1) the method in [19] (2) the proposed method
The curve of $u$ in $W_1(x)$ $T_c = 5$, $m = 1, 0.3, \ldots, 0.1$

Fig. 13. (1) the method in [19] (2) the proposed method
The curve of $y = x_1$ in $W_1(x)$ $T_c = 5$, $m = 1, 0.3, \ldots, 0.1$

Fig. 14. (1) $x_{1d} = 1$ (2) $x_{1d} = -1$ (3) $x_{1d} = 2$
The curve of $x_2$ in $W_1(x)$ $T_c = 5$, $m = 1, 0.3, \ldots, 0.1$

and 12, it is clear that both methods can make the tracking error converge to zero within predefined-time $2T_c = 2s$ if $m$ is satisfied the nonsingular parameter range deduced by this paper. As shown in Fig 13, it is also clear that the chattering of the proposed control input is also lower than the controller in [19]. The tracking errors of third-order MIMO system can be shown in Figs. 14, 15 and 16, and the tracking error $e$ can not converge to zero within predefined-time if $m = 1$. 

25
5 Conclusion

Generally, the terminal sliding mode of fixed-time-class is often accompanied by the inevitable singularity problem and many researches have been done to overcome the singularity, but often at the larger cost of some controller performance. In this paper, we discuss a selection problem of nonsingular parameter in a predefined-time terminal sliding mode control strategy in as much detail as possible. Firstly, we discuss the parameter selection of traditional general predefined-time system expression equation and improve some conditions to make the system have a better stable performance. Secondly, we study the selection problem of nonsingular parameter for second-order system: a switching strategy is improved to overcome the singularity of reaching stage of sliding mode and chattering caused by switching sliding mode (Remark 4), and then, a selection range of nonsingular parameter is found to ensure the non-singularity of the sliding stage (s=0). Thirdly, we try to choose a recursive terminal sliding mode control method to deal with the predefined-time sliding mode control problem of high-order system, in this subsection, a selection method of nonsingular parameter is studied to try to achieve nonsingular predefined-time terminal sliding mode control for a n-order system. In the end, we use some simple system model to carry on the relevant numerical simulation, the final results, to some extent, confirm our inference.

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