Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations

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Abstract
In this paper, we study the local and global existence, and uniqueness of mild solution to initial value problems for fractional semilinear evolution equations with compact and noncompact semigroup in Banach spaces. In particular, we derive the form of fundamental solution in terms of semigroup induced by resolvent and ψ-function from Caputo fractional derivatives. These results generalize previous work where the classical Caputo fractional derivative is considered. Moreover, we prove the Mittag-Leffler–Ulam–Hyers stability result. Finally, we give examples of time-fractional heat equation to illustrate the result.

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1 Introduction
Fractional differential equations have been applied in many fields, such as economics, engineering, chemistry, physics, finance, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems (see [1–16]). The research on fractional calculus has become a focus area of study due to the fact that some dynamical models can be described more accurately with fractional derivatives than the ones with integer-order derivatives. In particular, it is shown that fractional calculus provides more realistic models demonstrating hidden aspects in a model of spring pendulum [13], the free motion of a particle in a circular cavity [11] and some epidemic models [15, 17].

Several researchers are interested in investigating various aspects of fractional differential equations such as existence and uniqueness of solutions, exact solutions, stability of solutions, and methods for explicit and numerical solutions [17–20]. The common techniques used to display the existence and uniqueness of solutions are fixed point theorem, upper-lower solutions, iterative method and numerical method. For stability of solutions, there is a concept of data dependence, which becomes one of significant topics in the analysis of fractional differential equations, called the Ulam–Hyers stability (see [21–23]).
One of the main research focuses on fractional calculus is the theory of fractional evolution equations since they are abstract formulation for many problems arising in engineering and physics. Evolution equations are commonly used to describe the systems that change or evolve over time. A number of studies have been conducted on the existence and unique of solutions for fractional evolution equations based on semigroup and fixed point theory (see [24–32]). On the other hand, there has been some studies about fundamental solution for homogeneous fractional evolution equations [33, 34]. Recently, [19] applied the homotopy analysis transform method (HATM) for solving time-fractional Cauchy reaction–diffusion equations. In addition, Wang and Zhou [35] presented four kind of stabilities of the mild solution of the fractional evolution equation in Banach space, namely Mittag-Leffler–Ulam–Hyers stability, generalized Mittag-Leffler–Ulam–Hyers stability, Mittag-Leffler–Ulam–Hyers–Rassias stability and generalized Mittag-Leffler–Ulam–Hyers–Rassias stability.

There is variation in the definition of fractional differential operators found in the literature, including Riemann–Liouville, Caputo, Hilfer, Riesz, Erdelyi–Kober, and Hadamard [2, 36] operators. The common definitions that triggered attention from many researchers are Riemann–Liouville and Caputo fractional calculus. In Riemann–Liouville fractional differential modeling, the initial condition involves limit values of fractional derivatives, which is difficult to interpret. The Caputo fractional derivative has the advantage of being suitable for physical models with initial condition because the physical interpretation of the prescribed data is clear and it is in general possible to provide these data by suitable measurements [37].

Almeida [38] generalized the definition of Caputo fractional derivative by considering the Caputo fractional derivative of a function with respect to another function \( \psi \) and studied some useful properties of the fractional calculus. The advantage of this new definition of the fractional derivative is that a higher accuracy of the model could be achieved by choosing a suitable function \( \psi \).

Recently, Jarad and Abdeljawad [39] introduced the generalized Laplace transform with respect to another function and the inverse version of the Laplace transform with respect to another function. This can be used to solve some fractional differential equations in the framework of generalized Caputo fractional derivative.

Motivated by the work of [25, 39], we consider the following fractional evolution equation in a Banach space \( E \):

\[
\begin{align*}
\begin{cases}
\mathcal{C}_\alpha \mathcal{D}_\psi^\alpha u(t) = Au(t) + f(t, u(t)), & t \in (0, T], \\
u(0) = u_0,
\end{cases}
\end{align*}
\]

where \( 0 < \alpha < 1, \ T < \infty, \ A \) is the infinitesimal generator of a \( C_0 \)-semigroup of uniformly bounded linear operators \( \{T(t)\}_{t \geq 0} \) on \( E \), \( u_0 \in E \) and \( f : [0, \infty) \times E \to E \) is given function. The fractional derivative \( \mathcal{C}_\alpha \mathcal{D}_\psi^\alpha \) considered in this work is in the sense of Caputo fractional derivative with respect to a function \( \psi \) which gives a more general framework to the results in the literature. Moreover, this problem is more general than the work in [39] where we consider the evolution operator \( A \) instead of a constant.

In this paper, we aim to establish a mild solution for the problem (1) in terms of semigroup depending on a function \( \psi \) from the generalized Caputo derivative. In addition, we prove the existence and uniqueness results of mild solution for the problem (1) in local and
global time under the condition that \( \{ T(t) \}_{t \geq 0} \) is both compact and noncompact operator. The results obtained in this work are in the abstract form which can be applied for further investigation such as the evolution equations with perturbation, delay and nonlocal term.

This paper will be organized as follows. In Sect. 2, we will briefly recall some basic definitions and some preliminary concepts about fractional calculus and auxiliary results used in the following sections. We then construct a mild solution by using semigroup for the problem in Sect. 3. We prove the existence and uniqueness of mild solutions of the problem (1) under compact and noncompact analytic semigroup by the Schauder fixed point theorem in Sects. 4 and 5, respectively. In Sect. 6 we present Mittag-Leffler–Ulam–Hyers stability result for the problem (1). Finally, we give some examples to illustrate the application of the results obtained in Sect. 7 and our conclusion in Sect. 8.

## 2 Preliminaries

In this section, we introduce preliminary background which is used throughout this paper.

Let \( E \) be a Banach space with the norm \( \| \cdot \| \) and let \( C(J, E) \) be the Banach space of continuous functions from \( J \) to \( E \) with the norm \( \| u \|_C = \sup_{t \in J} \| u(t) \| \).

### Definition 2.1 (\( \psi \)-Riemann–Liouville fractional integral [39])

Let \( \alpha > 0 \), \( f \) be an integrable function defined on \( [a, b] \) and \( \psi \in C^1([a, b]) \) be an increasing function with \( \psi'(t) \neq 0 \) for all \( t \in [a, b] \). The \( \psi \)-Riemann–Liouville fractional integral operator of order \( \alpha \) of a function \( f \) is defined by

\[
\left( aI^\alpha_\psi f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} f(s) \psi'(s) \, ds.
\]

(2)

It is obvious that when \( \psi(t) = t \), (2) is the classical Riemann–Liouville’s fractional integral.

### Definition 2.2 (\( \psi \)-Riemann–Liouville fractional derivative [39])

Let \( n - 1 < \alpha < n \), \( f \) be an integrable function defined on \( [a, b] \) and \( \psi \in C^1([a, b]) \) be an increasing function with \( \psi'(t) \neq 0 \) for all \( t \in [a, b] \). The \( \psi \)-Riemann–Liouville fractional derivative of order \( \alpha \) of a function \( f \) is defined by

\[
\left( aD^\alpha_\psi f \right)(t) = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \left( aI^{n-\alpha}_\psi f \right)(t)
\]

\[
= \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \left( \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f(s) \, ds \right),
\]

(3)

where \( n = [\alpha] + 1 \).

From the definition, when \( \alpha = n \in \mathbb{N} \), we have

\[
\left( aD^\alpha_\psi f \right)(t) = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t).
\]

### Definition 2.3

Let \( \psi \in C^n([a, b]) \) be such that \( \psi'(t) > 0 \) on \( [a, b] \). Then

\[
AC^\alpha_\psi ([a, b]) = \left\{ f : [a, b] \rightarrow \mathbb{C} \text{ and } f^{[n-1]} = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n-1} f \right\}.
\]
Lemma 2.4 ([39, 40]) Let $\alpha > 0$ and $\beta > 0$, then
\begin{enumerate}[(i)]  
  \item $aD^\alpha_\psi (f(x) - \psi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(a))^{\beta+\alpha-1}$;  
  \item $aD^\alpha_\psi (f(x) - \psi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(a))^{\beta-\alpha-1}$.
\end{enumerate}

Definition 2.5 ($\psi$-Caputo fractional derivative [38, 39]) Let $n - 1 < \alpha < n, f \in C^n([a, b])$ and $\psi \in C^\infty([a, b])$ be an increasing function with $\psi(t) \neq 0$ for all $t \in [a, b]$. The $\psi$-Caputo fractional derivative of order $\alpha$ of a function $f$ is defined by
\begin{align*}
\bigl( C_\psi^\alpha D_\psi^n f \bigr)(t) &= \bigl( aI^{n-\alpha}_\psi f^{[n]} \bigr)(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f^{[n]}(s) \psi'(s) \, du,
\end{align*}
where $n = [\alpha] + 1$ and $f^{[n]}(t) := \frac{1}{\psi'(t)} \psi f(t)$ on $[a, b]$.

From the definition, it is clear that, when $\alpha = n \in \mathbb{N}$,
\begin{align*}
\bigl( C_\psi^\alpha D_\psi^n f \bigr)(t) &= f^{[n]}(t).
\end{align*}

Remark 2.6 ([38, 39]) The relationship between the $\psi$-Caputo and the $\psi$-Riemann–Liouville derivatives can be written as follows:
\begin{align*}
\bigl( C_\psi^\alpha D_\psi^n f \bigr)(t) &= \bigl( aD_\psi^n f \bigr)(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(k-\alpha+1)} (\psi(t) - \psi(a))^{k-\alpha} \\
&= a\bigl( D_\psi^\alpha \bigl( f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (\psi(s) - \psi(a))^k \bigr) \bigr)(t),
\end{align*}
where $t > a$ and $n = [\alpha] + 1$.

Theorem 2.7 ([38]) Let $f \in C^n([a, b])$ and $\alpha > 0$. Then we have
\begin{align*}
aI^{\alpha}_\psi \bigl( aD_\psi^n f \bigr)(t) &= f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (\psi(t) - \psi(a))^k.
\end{align*}

In particular, given $\alpha \in (0, 1)$, we have
\begin{align*}
aI^{\alpha}_\psi \bigl( aD_\psi^n f \bigr)(t) &= f(t) - f(a).
\end{align*}

Definition 2.8 ([39]) Let $u, \psi : [a, \infty) \to \mathbb{R}$ be real valued functions such that $\psi(t)$ is continuous and $\psi'(t) > 0$ on $[0, \infty)$. The generalized Laplace transform of $f$ is denoted by
\begin{align*}
\mathcal{L}_\psi \{ u(t) \}(s) &= \int_a^\infty e^{-s(\psi(t)-\psi(a))} u(t) \psi'(t) \, dt
\end{align*}
for all $s$.

Definition 2.9 ([39]) Let $u$ and $v$ be two functions which are piecewise continuous at each interval $[0, T]$ and of exponential order. We define the generalized convolution of $u$ and $v$
by
\[(u \ast \psi \nu)(t) = \int_{a}^{t} u(\tau) \nu(\psi^{-1}(\psi(\tau) + \psi(a) - \psi(\tau))) \psi'(\tau) d\tau.\]

**Theorem 2.10** ([39]) Let \(\alpha > 0\) and \(f\) be a piecewise continuous function on each interval \([a, t]\) and \(\psi(t)\)-exponential order. Then
\[
\mathcal{L}_\psi \left\{ aI^\alpha_\psi u(t) \right\}(s) = \frac{aI^\alpha_\psi u(t)}{s^\alpha}.
\]

**Theorem 2.11** (Gronwall’s inequality [41, 42]) Let \(u, v\) be two integrable functions and \(h\) be a continuous function on \([a, b]\). Let \(\psi \in C^1([a, b])\) be an increasing function such that \(\psi'(t) \neq 0\) for all \(t \in [a, b]\). Assume that

1. \(u\) and \(v\) are nonnegative;
2. \(h\) is nonnegative and nondecreasing.

If
\[
u(t) \leq u(t) \leq v(t) + h(t) \int_{a}^{t} (\psi(t) - \psi(s))^{\alpha-1} u(s) \nu'(s) ds,
\]
then
\[
u(t) \leq v(t) + \int_{a}^{t} \sum_{k=1}^{\infty} \frac{[h(t) \Gamma(\alpha)]^k}{\Gamma(n\alpha)} (\psi(t) - \psi(s))^{k\alpha-1} \nu(s) \psi'(s) ds,
\]
for all \(t \in [a, b]\).

**Corollary 2.12** Under the hypotheses of Theorem 2.11, let \(v\) be a nondecreasing function on \([a, b]\). Then we have
\[
u(t) \leq u(t) E_\alpha(\psi(t) - \psi(a))
\]
for all \(t \in [a, b]\), where \(E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}\) is the Mittag-Leffler function with one parameter for \(z \in \mathbb{C}\) and \(\alpha > 0\).

**Definition 2.13** ([43, 44]) The Wright type function is given by
\[
\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(-\alpha k + 1 + \alpha)} = \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k + 1)) \sin(\pi(k + 1)\alpha)}{k!}
\]
for \(0 < \alpha < 1\) and \(z \in \mathbb{C}\).

**Proposition 2.14** ([43, 44]) The Wright function \(\phi_\alpha\) is an entire function and has the following properties:

1. \(\phi_\alpha(\theta) \geq 0\) for \(\theta \geq 0\) and \(\int_{0}^{\infty} \phi_\alpha(\theta) d\theta = 1;\)
(ii) \[ \int_0^\infty \phi_\alpha(\theta) \theta^r \, d\theta = \frac{\Gamma(1 + r)}{\Gamma(1 + \alpha r)} \quad \text{for } r > -1; \]

(iii) \[ \int_0^\infty \phi_\alpha(\theta) e^{-\alpha \theta} \, d\theta = E_\alpha(-z), \quad z \in \mathbb{C}; \]

(iv) \[ \alpha \int_0^\infty \theta \phi_\alpha(\theta) e^{-\alpha \theta} \, d\theta = E_{\alpha,\alpha}(-z), \quad z \in \mathbb{C}. \]

Next, we introduce the definition for Kuratowski measure of noncompactness, which will be used in the proof of our main results.

**Definition 2.15 ([45])** Let \( E \) be a Banach space and \( B(E) \) be the bounded subset of \( E \). The Kuratowski measure of noncompactness is the map \( \mu : B(E) \to [0, \infty) \) define by

\[ \mu(B) = \inf \left\{ \varepsilon > 0 : B \subset \bigcup_{j=1}^\infty B_j, \text{diam}(B_j) < \varepsilon \text{ for } i = 1, 2, \ldots, n \right\}, \]

where \( \text{diam}(B_j) = \sup\{|x - y| : x, y \in B_j\} \).

The following properties of the Kuratowski measure of noncompactness are well known.

**Lemma 2.16 ([45, 46])** Let \( E \) be Banach spaces and \( U, V \subset E \) be bounded. Then the noncompactness measure has the following properties:

(i) \( \mu(U) = 0 \) if and only if \( \overline{U} \) is compact, where \( \overline{U} \) means the closure hull of \( U \);

(ii) \( \mu(\lambda U) = |\lambda| \mu(U) \), where \( \lambda \in \mathbb{R} \);

(iii) \( \mu(U) = \mu(\overline{U}) = \mu(\text{conv } U) \), where \( \text{conv } U \) means the convex hull of \( U \);

(iv) \( \mu(U \cup V) = \max(\mu(U), \mu(V)) \);

(v) \( \mu(U) \leq \mu(V) \) if \( U \subset V \);

(vi) \( \mu(U + V) \leq \mu(U) + \mu(V) \); \( \text{where } U + V = \{x \mid x = y + z, y \in U, z \in V \} \);

(vii) \( \mu(U + x) = \mu(U) \), for any \( x \in E \);

(viii) If the map \( Q : \text{dom}(Q) \subset E \to X \) is Lipschitz continuous with constant \( k \), then \( \mu(Q(S)) \leq k \mu(S) \) for any bounded subset \( S \subset \text{dom}(Q) \), where \( X \) is another Banach space.

**Lemma 2.17 ([47])** Let \( E \) be a Banach space, and let \( D \subset E \) be bounded. Then there exists a countable set \( D_0 \subset D \) such that \( \mu(D) \leq 2 \mu(D_0) \).

**Lemma 2.18 ([45, 46])** Let \( E \) be a Banach space, and \( B \subset C(J, E) \), \( B(t) = \{u(t) : u \in B\} \) \((t \in J)\). If \( B \) is bounded and equicontinuous, then \( \mu(B(t)) \) is continuous on \( J \), and \( \mu(B) = \max_{t \in J} \mu(B(t)) = \mu(B(J)) \).

**Lemma 2.19 ([48])** Let \( E \) be a Banach space, and let \( B \subset C(J, E) \) be bounded and equicontinuous. Then \( \mu(B(t)) \) is continuous on \( J \), and

\[ \mu \left( \left\{ \int_j^b u(t) \, dt \mid u \in B \right\} \right) \leq \int_j^b \mu(B(t)) \, dt. \]
Lemma 2.20 ([49]) Let $E$ be a Banach space, and let $B = \{u_n\}_{n=1}^{\infty} \subset C(J, E)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integrable on $J$, and

$$\mu\left(\left\{ \int_J u_n(t) \, dt \right\}_{n=1}^{\infty}\right) \leq 2 \int_J \mu(B(t)) \, dt. \quad (7)$$

Lemma 2.21 ([50]) If $B \subset C(J, E)$ is bounded and equicontinuous, then $\overline{Co}B \subset C(J, E)$ is also bounded and equicontinuous.

Lemma 2.22 ([45]) Let $E$ be a Banach space. Assume that $D \subset E$ is a bounded closed and convex set on $E$, $Q : D \to D$ is condensing. Then $Q$ has at least one fixed point in $D$.

Lemma 2.23 (Schauder’s fixed point theorem) Let $E$ be a Banach space and $D \subset E$, a convex, closed and bounded set. If $T : D \to D$ is a continuous operator such that $T(D) \subset E$, $T(D)$ is relatively compact, then $T$ has at least one fixed point in $D$.

Next, we give some facts about the semigroups of linear operators. These results can be found in [51, 52].

For a strongly continuous semigroup (i.e., $C_0$-semigroup) $\{T(t)\}_{t \geq 0}$, the infinitesimal generator of $\{T(t)\}_{t \geq 0}$ is defined by

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad x \in E.$$  

We denote by $D(A)$ the domain of $A$, that is,

$$D(A) = \left\{ x \in E : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$  

Lemma 2.24 ([51, 52]) Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup, then there exist constants $C \geq 1$ and $a \geq 0$ such that $\|T(t)\| \leq Ce^{at}$ for all $t \geq 0$.

Lemma 2.25 ([51, 52]) A linear operator $A$ is the infinitesimal generator of a $C_0$-semigroup if and only if

(i) $A$ is closed and $D(A) = E$.

(ii) The resolvent set $\rho(A)$ of $A$ contains $\mathbb{R}^+$ and, for every $\lambda > 0$, we have

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda},$$  

where $R(\lambda, A) := (\lambda^2 I - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)x \, dt$.

Throughout this paper, let $A$ be the infinitesimal generator of a $C_0$-semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$ on $E$. Then there exists $M \geq 1$ such that $M = \sup_{t \in [0, \infty)} \|T(t)\|$.
3 Representation of mild solution using semigroup

According to Definition 2.5 and Theorem 2.7, it is suitable to rewrite the Cauchy problem in the equivalent integral equation

$$u(t) = u_0 + \frac{1}{T(\alpha)} \int_0^t (\psi(t) - \psi(t))^{a-1} (Au(t) + f(t, u(t))) \psi'(t) dt. \quad (8)$$

**Lemma 3.1** If (8) holds, then we have

$$u(t) = \int_0^\infty \phi_\alpha(\theta) T((\psi(t) - \psi(0))^{\alpha} \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty \phi_\alpha(\theta)(\psi(t) - \psi(s))^{a-1} T((\psi(t) - \psi(0))^{\alpha} \theta) f(s, u(s)) \psi'(s) d\theta ds. \quad (9)$$

**Proof** Let $\lambda > 0$. Applying the generalized Laplace transforms to (8), we have

$$U(\lambda) = \frac{u_0}{\lambda} + \frac{1}{\lambda^\alpha} (AU(\lambda) + F(\lambda)),$$

where

$$U(\lambda) = \int_0^\infty e^{-\lambda(\psi(t) - \psi(0))} u(t) \psi'(t) dt$$

and

$$F(\lambda) = \int_0^\infty e^{-\lambda(\psi(t) - \psi(0))} f(t, u(t)) \psi'(t) dt.$$

It follows that

$$U(\lambda) = \lambda^{\alpha-1} (\lambda^{\alpha} I - A)^{-1} u_0 + (\lambda^{\alpha} I - A)^{-1} F(\lambda)$$

$$= \lambda^{\alpha-1} \int_0^\infty e^{-\lambda^{\alpha} s} T(s) u_0 ds + \int_0^\infty e^{-\lambda^{\alpha} s} T(s) F(\lambda) ds$$

$$= \alpha \int_0^\infty (\lambda^2 s^{\alpha-1} e^{-\lambda^{\alpha} s}) T(\tilde{\tau}^\alpha) u_0 d\tau + \alpha \int_0^\infty \tilde{\tau}^{\alpha-1} e^{-\lambda^{\alpha} \tilde{\tau}^\alpha} T(\tilde{\tau}^\alpha) F(\lambda) d\tilde{\tau}$$

$$=: I_1 + I_2.$$

Taking $\tilde{\tau} = \psi(t) - \psi(0)$, we obtain

$$I_1 = \alpha \int_0^\infty \lambda^{\alpha-1} (\psi(t) - \psi(0))^{a-1} e^{-\lambda(\psi(t) - \psi(0))^{\alpha}} T((\psi(t) - \psi(0))^{\alpha}) \psi'(t) dt$$

$$= \int_0^\infty -\lambda \frac{d}{dt} e^{-\lambda(\psi(t) - \psi(0))^{\alpha}} T((\psi(t) - \psi(0))^{\alpha}) u_0 \psi'(t) dt$$

and

$$I_2 = \int_0^\infty \alpha(\psi(t) - \psi(0))^{a-1} e^{-\lambda(\psi(t) - \psi(0))^{\alpha}} T((\psi(t) - \psi(0))^{\alpha}) F(\lambda) \psi'(t) dt$$

$$= \int_0^\infty \int_0^\infty \alpha(\psi(t) - \psi(0))^{a-1} e^{-\lambda(\psi(t) - \psi(0))^{\alpha}} T((\psi(t) - \psi(0))^{\alpha})$$

$$e^{-\lambda(\psi(t) - \psi(0))^{\alpha}} f(s, u(s)) \psi'(s) \psi'(t) ds dt.$$
We consider the following one-sided stable probability density in [53]:

$$\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \theta^{-k-1} \frac{\Gamma(\alpha k + 1)}{k!} \sin(k\pi \alpha), \quad \theta \in (0, \infty)$$

whose integration is given by

$$\int_{0}^{\infty} e^{-\lambda \theta} \rho_\alpha(\theta) d\theta = e^{-\lambda^\alpha}, \quad \text{where } \alpha \in (0, 1). \quad (10)$$

Using (10), we get

$$\int_{0}^{\infty} \frac{1}{\lambda} \frac{d}{dt} \left((\psi(t) - \psi(0))^\alpha\right) f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \alpha(\psi(t) - \psi(0))^{\alpha-1} e^{-\lambda(\psi(t) - \psi(0))^\alpha} T\left((\psi(t) - \psi(0))^\alpha\right) T(\psi(t) - \psi(0))^\alpha e^{-\lambda(\psi(t) - \psi(0))^\alpha}\right)$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \alpha(\psi(t) - \psi(0))^{\alpha-1} \rho_\alpha(\theta) e^{-\lambda(\psi(t) - \psi(0))^\alpha} T\left((\psi(t) - \psi(0))^\alpha\right) u_0'\left(t\right) d\theta dt$$

$$= \int_{0}^{\infty} e^{-\lambda(\psi(t) - \psi(0))} \left( \int_{0}^{\infty} \rho_\alpha(\theta) T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right) u_0 d\theta\right) \psi'\left(t\right) dt$$

and

$$\int_{0}^{\infty} \int_{0}^{\infty} \alpha(\psi(t) - \psi(0))^{\alpha-1} e^{-\lambda(\psi(t) - \psi(0))^\alpha} T\left((\psi(t) - \psi(0))^\alpha\right) e^{-\lambda(\psi(t) - \psi(0))^\alpha}\right)$$

$$f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \alpha(\psi(t) - \psi(0))^{\alpha-1} \rho_\alpha(\theta) e^{-\lambda(\psi(t) - \psi(0))^\alpha} T\left((\psi(t) - \psi(0))^\alpha\right)$$

$$e^{-\lambda(\psi(t) - \psi(0))^\alpha}\right) f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) d\theta ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \alpha e^{-\lambda(\psi(t) - \psi(0))} \rho_\alpha(\theta) \left(\psi(t) - \psi(0)\right)^{\alpha-1} T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right)$$

$$f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) d\theta ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \alpha e^{-\lambda(\psi(t) - \psi(0))} \rho_\alpha(\theta) \left(\psi(t) - \psi(0)\right)^{\alpha-1}$$

$$T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right) f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) d\theta ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \alpha e^{-\lambda(\psi(t) - \psi(0))} \rho_\alpha(\theta) \left(\psi(t) - \psi(0)\right)^{\alpha-1}$$

$$T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right) f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) d\theta ds dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \alpha e^{-\lambda(\psi(t) - \psi(0))} \rho_\alpha(\theta) \left(\psi(t) - \psi(0)\right)^{\alpha-1}$$

$$T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right) f\left(s, u(s)\right) \psi'\left(s\right) \psi'\left(t\right) d\theta ds dt$$

$$= \int_{0}^{\infty} e^{-\lambda(\psi(t) - \psi(0))}$$

$$\times \left( \int_{0}^{\infty} \alpha \rho_\alpha(\theta) \left(\psi(t) - \psi(0)\right)^{\alpha-1} T\left(\frac{(\psi(t) - \psi(0))^\alpha}{\theta^\alpha}\right) f\left(s, u(s)\right) \psi'\left(s\right) d\theta ds\right)$$

$$\times \psi'\left(t\right) dt.$$
Then we get

\[ U(\lambda) = \int_0^\infty e^{-\lambda \psi(t)} \left( \int_0^\infty \rho_\alpha(\theta) T \left( \frac{(\psi(t) - \psi(0))}{\theta^\alpha} \right) u_0 d\theta \right) \psi'(t) dt \]

\[ + \int_0^\infty e^{-\lambda \psi(\tau)} \left( \int_0^\infty \rho_\alpha(\theta) \frac{(\psi(\tau) - \psi(s))}{\theta^\alpha} \right) T \left( \frac{(\psi(\tau) - \psi(s))}{\theta^\alpha} \right) f(s,u(s)) \psi'(s) d\theta ds \]

\[ \times \psi'(\tau) d\tau. \]

Now, we can invert the Laplace transform to get

\[ u(t) = \int_0^\infty \rho_\alpha(\theta) T \left( \frac{(\psi(t) - \psi(0))}{\theta^\alpha} \right) u_0 d\theta \]

\[ + \alpha \int_0^t \int_0^\infty \rho_\alpha(\theta) \frac{(\psi(t) - \psi(s))}{\theta^\alpha} \right) T \left( \frac{(\psi(t) - \psi(s))}{\theta^\alpha} \right) f(s,u(s)) \psi'(s) d\theta ds \]

\[ = \int_0^\infty \phi_\alpha(\theta) T ((\psi(t) - \psi(0))^{\alpha} \theta) u_0 d\theta \]

\[ + \alpha \int_0^t \int_0^\infty \theta \phi_\alpha(\theta) (\psi(t) - \psi(s)) \right) T ((\psi(t) - \psi(0))^{\alpha} \theta) f(s,u(s)) \psi'(s) d\theta ds , \]

where \( \phi_\alpha(\theta) = \frac{1}{\Gamma(1+\frac{\alpha}{\theta})} \rho_\alpha(\theta^{\frac{1}{\alpha}}) \) is the probability density function defined on \((0, \infty)\).

---

For any \( u \in E \), define operators \( S^\alpha_\psi(t,s) \) and \( T^\alpha_\psi(t,s) \) by

\[ S^\alpha_\psi(t,s)u = \int_0^\infty \phi_\alpha(\theta) T ((\psi(t) - \psi(s))^{\alpha} \theta) u d\theta \]

and

\[ T^\alpha_\psi(t,s)u = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T ((\psi(t) - \psi(s))^{\alpha} \theta) u d\theta \]

for \( 0 \leq s \leq t \leq T \).

**Lemma 3.2** The operators \( S^\alpha_\psi \) and \( T^\alpha_\psi \) have the following properties:

(i) For any fixed \( t \geq s \geq 0 \), \( S^\alpha_\psi(t,s) \) and \( T^\alpha_\psi(t,s) \) are bounded linear operators with

\[ \| S^\alpha_\psi(t,s)u \| \leq M \| u \| \quad \text{and} \quad \| T^\alpha_\psi(t,s)u \| \leq \frac{\alpha M}{\Gamma(1+\alpha)} \| u \| = \frac{M}{\Gamma(\alpha)} \| u \| \]

for all \( u \in E \).

(ii) The operators \( S^\alpha_\psi(t,s) \) and \( T^\alpha_\psi(t,s) \) are strongly continuous for all \( t \geq s \geq 0 \), that is, for every \( u \in E \) and \( 0 \leq s \leq t_1 < t_2 \leq T \) we have

\[ \| S^\alpha_\psi(t_2,s)u - S^\alpha_\psi(t_1,s)u \| \to 0 \quad \text{and} \quad \| T^\alpha_\psi(t_2,s)u - T^\alpha_\psi(t_1,s)u \| \to 0 \]

as \( t_1 \to t_2 \).
(iii) If $T(t)$ is compact operator for every $t > 0$, then $S_\alpha^\psi(t,s)$ and $T_\alpha^\psi(t,s)$ are compact for all $t,s > 0$.

(iv) If $S_\alpha^\psi(t,s)$ and $T_\alpha^\psi(t,s)$ are compact strongly continuous semigroup of bounded linear operators for $t,s > 0$, then $S_\alpha^\psi(t,s)$ and $T_\alpha^\psi(t,s)$ are continuous in the uniform operator topology.

Proof The proof follows the argument of [26].

Definition 3.3 A function $u \in C([0,T],E)$ is called a mild solution of (1) if it satisfies

$$u(t) = S_\alpha^\psi(t,0)u_0 + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T_\alpha^\psi(t,s)f(s,u(s))\psi'(s)ds, \quad t \in [0,T].$$

Before starting and proving the main results, we introduce the following hypotheses.

(H1) $T(t)$ is compact operator for every $t > 0$.

(H2) The function $f : [0,T] \times E \rightarrow E$ is Carathéodory function, that is:

(C1) For each $t \in [0,T]$ the function $f(t,\cdot) : E \rightarrow E$ is continuous.

(C2) For each $u \in E$ the function $f(\cdot,u) : [0,T] \rightarrow E$ is measurable.

(H3) For any $r > 0$, there exists a function $h_r \in L^\infty([0,T],E)$ such that

$$\sup_{\|u\| \leq r} \|f(t,u)\| \leq h_r(t), \quad \text{a.e. } t \in [0,T],$$

and there is a constant $L > 0$ such that

$$\limsup_{r \to \infty} \frac{\|h_r(t)\|_{L^\infty}}{r} = L.$$

(H4) For any $r > 0$, there exists $k(t) \in L^\infty([0,T],E)$ such that

$$\|f(t,u_1(t)) - f(t,u_2(t))\| \leq k(t)\|u_1 - u_2\|$$

for all $u_1, u_2 \in \Omega_r$.

(H5) There exist continuous functions $g_1, g_2$ on $[0,\infty)$ such that

$$\|f(t,u)\| \leq g_1(t) + g_2(t)\|u\|$$

for $t \geq 0$ and $u \in E$.

(H6) For any $r > 0$ and $T > 0$, there exists a positive constant $K$ such that, for any equicontinuous and countable set $D \subset \Omega_r = \{u \in E | \|u\| \leq r\}$,

$$\mu(f(t,D)) \leq K\mu(D), \quad t \in [0,T].$$

4 Existence and uniqueness of mild solution under compact analytic semigroup

In this section, we begin by proving a theorem concerning the existence and uniqueness of mild solution for the problem (1) under the condition of compact analytic semigroup. The discussions are based on fractional calculus and Schauder fixed point theorem. Our main results are as follows.
Theorem 4.1 Assume that conditions (H1)–(H3) hold. Then the problem (1) has at least mild solution provided that

\[ \frac{ML}{\Gamma(1+\alpha)}(\psi(T) - \psi(0))^\alpha < 1. \]  

(11)

Proof For any \( r > 0 \), let \( \Omega_r = \{ u \in C([0, T], E) : \| u \| \leq r \} \). Then \( \Omega_r \) is bounded closed convex subset of \( C([0, T], E) \). Define an operator \( K : \Omega_r \to C([0, T], E) \) by

\[ (Ku)(t) := S_p^\alpha (t, 0)u_0 + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T_p^\alpha (s, t)f(s, u(s))\psi'(s) \, ds, \]

for \( t \in [0, T] \).

Step 1: We will prove that \( K : \Omega_r \to \Omega_r \), that is, there exists \( r > 0 \) such that \( K(\Omega_r) \subset \Omega_r \), We assume that for each \( r > 0 \), there exists \( u_r \in \Omega_r \), and \( t \in [0, T] \), such that \( \| (Ku)(t) \| > r \). According to Lemma 3.2(i) and (H3), we have

\[
r < \| (Ku_r)(t) \| \leq \| S_p^\alpha (t, 0)u_0 \| + \int_0^t \| (\psi(t) - \psi(s))^{\alpha-1} T_p^\alpha (s, t)f(s, u_r(s))\psi'(s) \| \, ds \\
\leq M\| u_0 \| + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| T_p^\alpha (s, t)f(s, u_r(s))\| \psi'(s) \, ds \\
\leq M\| u_0 \| + \frac{M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| f(s, u_r(s)) \| \psi'(s) \, ds \\
\leq M\| u_0 \| + \frac{M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} h_r(s)\psi'(s) \, ds \\
\leq M\| u_0 \| + \frac{M\| h_r(t) \|_{L^\infty}}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) \, ds \\
= M\| u_0 \| + \frac{M\| h_r(t) \|_{L^\infty}}{\Gamma(1+\alpha)} (\psi(t) - \psi(0))^\alpha \\
\leq M\| u_0 \| + \frac{M\| h_r(t) \|_{L^\infty}}{\Gamma(1+\alpha)} (\psi(T) - \psi(0))^\alpha.
\]

Dividing to both side by \( r \) and taking the limit supremum as \( r \to \infty \), we obtain

\[
1 \leq \limsup_{r \to \infty} \frac{M\| u_0 \| + \frac{M\| h_r(t) \|_{L^\infty}}{r\Gamma(1+\alpha)} (\psi(T) - \psi(0))^\alpha}{r} \\
= \frac{ML}{\Gamma(1+\alpha)} (\psi(T) - \psi(0))^\alpha < 1,
\]

which is contradiction. Therefore \( K : \Omega_r \to \Omega_r \).

Step 2: We will prove that \( K : \Omega_r \to \Omega_r \), is continuous. Let \( \{ u_n \} \subset \Omega_r \) with \( u_n \to u \in \Omega_r \) as \( n \to \infty \).

From the assumptions (H2) and (H3), we have, for each \( t \in [0, T] \),

\[ f(t, u_n(t)) \to f(t, u(t)) \quad \text{as} \quad n \to \infty \]

and

\[ \| f(t, u_n(t)) - f(t, u(t)) \| \leq 2h_r(t) \quad \text{for all} \quad n \in \mathbb{N}. \]
By the Lebesgue dominated convergence theorem, for any $t \in [0, T]$ we have

$$
\| (K u_n)(t) - (K u)(t) \|
\leq \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| T_\psi^\alpha (t, s)[f(s, u_n(s)) - f(s, u(s))] \| \psi'(s) \, ds
\leq \frac{M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| f(s, u_n(s)) - f(s, u(s)) \| \psi'(s) \, ds \to 0
$$
as $n \to \infty$. Therefore, $\| (K u_n)(t) - (K u)(t) \|_C \to 0$ as $n \to \infty$. Hence $K : \Omega_r \to \Omega_r$ is continuous.

Step 3: We will prove that $K(\Omega_r)$ is equicontinuous. For any $u \in \Omega_r$ and $0 \leq t_1 < t_2 \leq T$, we have

$$
\| (K u)(t_2) - (K u)(t_1) \|
\leq \| S_\psi^\alpha (t_2, 0)u_0 - S_\psi^\alpha (t_1, 0)u_0 \|
+ \int_0^{t_2} (\psi(t_2) - \psi(s))^{\alpha-1} T_\psi^\alpha (t_2, s)f(s, u(s)) \psi'(s) \, ds
- \int_0^{t_1} (\psi(t_1) - \psi(s))^{\alpha-1} T_\psi^\alpha (t_1, s)f(s, u(s)) \psi'(s) \, ds
\leq \| S_\psi^\alpha (t_2, 0)u_0 - S_\psi^\alpha (t_1, 0)u_0 \|
+ \int_0^{t_2} (\psi(t_2) - \psi(s))^{\alpha-1} T_\psi^\alpha (t_2, s)f(s, u(s)) \psi'(s) \, ds
- \int_0^{t_1} (\psi(t_1) - \psi(s))^{\alpha-1} T_\psi^\alpha (t_1, s)f(s, u(s)) \psi'(s) \, ds
=: I_1 + I_2 + I_3 + I_4.
$$

By Lemma 3.2, it is clear that $I_1 \to 0$ as $t_1 \to t_2$ and we obtain

$$
I_2 \leq \frac{M \| h_r \|_{L^\infty}}{\Gamma(\alpha + 1)} \int (\psi(t_2) - \psi(t_1))^{\alpha-1} (\psi(t_2) - \psi(s))^{\alpha-1} \, ds
$$
and

\[ I_4 \leq \frac{M\|h_r\|_{L^\infty}}{\Gamma(\alpha + 1)} \left[ (\psi(t_2))^\alpha - (\psi(t_1))^\alpha - (\psi(t_2) - \psi(t_1))^\alpha \right] \]

and hence \( I_2 \to 0 \) and \( I_3 \to 0 \) as \( t_2 \to t_1 \). For \( t_1 = 0 \) and \( 0 < t_2 \leq T \), it is easy to see that \( I_4 = 0 \).

Then, for any \( \varepsilon \in (0, t_1) \), we have

\[
I_4 \leq \left\| \int_0^{t_1-\varepsilon} (\psi(t_1) - \psi(s))^\alpha - 1 \left[ T_\psi^\alpha(t_2, s) - T_\psi^\alpha(t_1, s) \right] f(s, u(s)) \psi'(s) \, ds \right\|
\]

\[
+ \left\| \int_{t_1-\varepsilon}^{t_1} (\psi(t_1) - \psi(s))^\alpha - 1 \left[ T_\psi^\alpha(t_2, s) - T_\psi^\alpha(t_1, s) \right] f(s, u(s)) \psi'(s) \, ds \right\|
\]

\[
\leq \frac{\|h_r\|_{L^\infty}}{\alpha} \left[ (\psi(t_1) - \psi(0))^\alpha - (\psi(t_1) - \psi(t_1 - \varepsilon))^\alpha \right] \sup_{0 \leq s \leq t_1-\varepsilon} \left\| T_\psi^\alpha(t_2, s) - T_\psi^\alpha(t_1, s) \right\|
\]

\[
+ \frac{2M\|h_r\|_{L^\infty}}{\Gamma(\alpha + 1)} \left[ (\psi(t_1) - \psi(t_1 - \varepsilon))^\alpha \right].
\]

It follows that \( I_4 \to 0 \) as \( t_2 \to t_1 \) and \( \varepsilon \to 0 \) by Lemma 3.2(iv) and (iii). Therefore,

\[
\| (\mathcal{K}u)(t_2) - (\mathcal{K}u)(t_1) \| \to 0 \quad \text{independent of } u \in \Omega_r \text{ as } t_2 \to t_1.
\]

which means that \( \mathcal{K}(\Omega_r) \) is equicontinuous.

**Step 4:** We will prove that, for any \( t \in [0, T] \), \( K(t) = \{ (\mathcal{K}u)(t) : u \in \Omega_r \} \) is relatively compact in \( E \).

Obviously, \( K(0) \) is relatively compact in \( E \). Let \( 0 \leq t \leq T \) be fixed. Then, for every \( \varepsilon > 0 \) and \( \delta > 0 \), let \( u \in \Omega_r \) and define an operator \( K_{t, \delta} \) on \( \Omega_r \) by

\[
(K_{t, \delta}u)(t) = \int_0^\infty \phi_u(\theta) T\left( (\psi(t) - \psi(0))^\theta \right) u_0 \, d\theta
\]

\[
+ \alpha \int_0^{t-\varepsilon} \int_0^\delta \theta \phi_u(\theta) (\psi(t) - \psi(s))^\alpha - 1 T\left( (\psi(t) - \psi(0))^\theta \right) f(s, u(s)) \psi'(s) \, d\theta \, ds
\]

\[
+ \alpha \int_0^{t-\varepsilon} \int_0^\delta \theta \phi_u(\theta) (\psi(t) - \psi(s))^\alpha - 1 T\left( (\psi(t) - \psi(0))^\theta + \varepsilon^\alpha \delta - \varepsilon^\alpha \delta \right)
\]

\[
f(s, u(s)) \psi'(s) \, d\theta \, ds
\]

\[
+ \int_0^\infty \phi_u(\theta) T\left( (\psi(t) - \psi(0))^\theta \right) u_0 \, d\theta
\]

\[
+ \alpha \int_0^{t-\varepsilon} \int_0^\infty \theta \phi_u(\theta) (\psi(t) - \psi(s))^\alpha - 1 \left[ T(\varepsilon^\alpha \delta) T\left( (\psi(t) - \psi(0))^\theta - \varepsilon^\alpha \delta \right) \right]
\]

\[
f(s, u(s)) \psi'(s) \, d\theta \, ds
\]

\[
+ \int_0^\infty \phi_u(\theta) T\left( (\psi(t) - \psi(0))^\theta \right) u_0 \, d\theta
\]
\[ + \alpha T(\varepsilon^\alpha \delta) \int_{0}^{t} \int_{0}^{\infty} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta - \varepsilon^\alpha \delta \right) f(s, u(s)) \psi'(s) d\theta \ ds. \]

Then, by the compactness of \( T(\varepsilon^\alpha \delta) \) for \( \varepsilon^\alpha \delta > 0 \), we see that the set \( K_{\varepsilon, \delta}(t) = \{ K_{\varepsilon, \delta}u(t) : u \in \Omega_r \} \) is relatively compact in \( E \) for all \( \varepsilon > 0 \) and \( \delta > 0 \). Furthermore, for any \( u \in \Omega_r \), we have

\[
\| (Ku)(t) - (K_{\varepsilon, \delta}u)(t) \| \\
= \alpha \left\| \int_{0}^{t} \int_{0}^{\infty} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \\
+ \int_{0}^{t} \int_{0}^{t-\delta} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \\
+ \int_{0}^{t-\delta} \int_{0}^{\delta} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \right\| \\
\leq \alpha \left\| \int_{0}^{t} \int_{0}^{\infty} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \\
+ \int_{0}^{t} \int_{t-\delta}^{\infty} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \\
+ \int_{t-\delta}^{\infty} \int_{0}^{\infty} \theta \phi_\alpha(\theta) \left( (\psi(t) - \psi(s))^\alpha \right) T \left( (\psi(t) - \psi(0))^\alpha \theta \right) f(s, u(s)) \psi'(s) d\theta \ ds \right\| \\
\leq \alpha M \| h_r \|_L \left( \int_{0}^{t} (\psi(t) - \psi(s))^\alpha \psi'(s) ds \right) \left( \int_{0}^{\delta} \theta \phi_\alpha(\theta) d\theta \right) \\
+ \alpha M \| h_r \|_L \left( \int_{0}^{t-\delta} (\psi(t) - \psi(s))^\alpha \psi'(s) ds \right) \left( \int_{\delta}^{\infty} \theta \phi_\alpha(\theta) d\theta \right) \\
\leq M \| h_r \|_L \left( \psi(t) - \psi(0) \right)^\alpha \left( \int_{0}^{\delta} \theta \phi_\alpha(\theta) d\theta \right) \\
+ M \| h_r \|_L \left( \psi(t) - \psi(t - \varepsilon) \right)^\alpha \left( \int_{\delta}^{\infty} \theta \phi_\alpha(\theta) d\theta \right) \\
\leq M \| h_r \|_L \left( \psi(t) - \psi(0) \right)^\alpha \left( \int_{0}^{\delta} \theta \phi_\alpha(\theta) d\theta \right) \\
+ M \| h_r \|_L \left( \psi(t) - \psi(t - \varepsilon) \right)^\alpha \left( \int_{0}^{\infty} \theta \phi_\alpha(\theta) d\theta \right) \\
= M \| h_r \|_L \left( \psi(t) - \psi(0) \right)^\alpha \left( \int_{0}^{\delta} \theta \phi_\alpha(\theta) d\theta \right) \\
+ M \| h_r \|_L \left( \frac{M \| h_r \|_L \left( \psi(T) - \psi(0) \right)^\alpha \theta - \varepsilon^\alpha \delta}{T^\alpha} + M \| h_r \|_L \left( \psi(t) - \psi(t - \varepsilon) \right)^\alpha \left( \int_{0}^{\delta} \theta \phi_\alpha(\theta) d\theta \right) \\
\to 0 \quad \text{as} \quad \varepsilon, \delta \to 0^+. \]

Therefore, there are relatively compact sets arbitrarily close to the set \( K(t) \) for \( t > 0 \). Hence, \( K(t) \) is relatively compact in \( E \).
Therefore, by the Arzelá–Ascoli theorem $\mathcal{K}(\Omega_r)$ is relatively compact in $C([0,T],E)$. Thus, the continuity of $\mathcal{K}$ and relatively compact of $\mathcal{K}(\Omega_r)$ imply that $\mathcal{K}$ is a completely continuous. By the Schauder fixed point theorem, we see that $\mathcal{K}$ has a fixed point $u^*$ in $\Omega_r$, which is a mild solution of (1). The proof is complete.

**Remark 4.2** From Theorem 4.1, we notice that if $\psi$ is a bijection function then the problem (1) has at least mild solution provided that

$$ T < \psi^{-1}\left[\left(\frac{\Gamma(1+\alpha)}{ML}\right)^{\frac{1}{\alpha}} + \psi(0)\right]. $$

**Theorem 4.3** Assume $(H_4)$ holds. Then the problem (1) has a unique mild solution.

**Proof** Let $u_1$ and $u_2$ be the solutions of the problem (1) in $\Omega_r$. Then, for each $i \in \{1,2\}$, the solution $u_i$ satisfies

$$ (\mathcal{K}u_i)(t) := S^\alpha_\psi(t,0)u_0 + \int_0^t \left(\psi(t) - \psi(s)\right)^{\alpha-1} T^\alpha_\psi(t,s)f(s,u_i(s))\psi'(s)\,ds. $$

Then, for any $t \in [0,T]$, we have

$$ \|u_1(t) - u_2(t)\| = \|(\mathcal{K}u_1)(t) - (\mathcal{K}u_2)(t)\| $$

$$ \leq \int_0^t \left(\psi(t) - \psi(s)\right)^{\alpha-1}\|T^\alpha_\psi(t,s)f(s,u_1(s)) - f(s,u_2(s))\|\psi'(s)\,ds $$

$$ \leq \frac{M}{\Gamma(\alpha)} \int_0^t \left(\psi(t) - \psi(s)\right)^{\alpha-1}\|f(s,u_1(s)) - f(s,u_2(s))\|\psi'(s)\,ds $$

$$ \leq \frac{Mk^+}{\Gamma(\alpha)} \int_0^t \left(\psi(t) - \psi(s)\right)^{\alpha-1}\|u_1(s) - u_2(s)\|\psi'(s)\,ds, $$

where $k^+ = \sup_{0 \leq t \leq T} |k(t)|$. By using the Gronwall inequality (Lemma 2.11), we obtain

$$ \|u_1(t) - u_2(t)\| = 0 \quad \text{for all } t \in [0,T] $$

which implies that $u_1 \equiv u_2$. Therefore, the problem (1) has a unique mild solution $u^* \in \Omega_r$. \hfill $\Box$

**Theorem 4.4** Suppose that conditions $(H_1)$–$(H_3)$ hold. Then, for any $u_0 \in E$, the problem (1) has a mild solution $u$ on a maximal interval of existence $[0,T_{\text{max}})$. If $T_{\text{max}} < \infty$, then $\lim_{t \to T_{\text{max}}} \|u(t)\| = \infty$.

**Proof** We notice that a mild solution $u$ of the problem (1) defined on $[0,T]$ can be extended to a larger interval $[0,T+\delta]$ with $\delta > 0$, by defining $v(t) = u(t+T)$, where $v(t)$ is a mild solution of

$$ \begin{cases} \frac{\partial}{\partial \psi} \psi(t) = A v(t) + f(t,v(t)), & t \in (T,T+\delta], \\ v(0) = u(T). \end{cases} $$

(12)
Therefore, repeating the procedure and using the methods of steps in Theorem 4.1, we can prove that there exists a maximal interval $[0, T_{\text{max}})$ such that the mild solution $u$ of the problem (1). We want to prove that if $T_{\text{max}} < \infty$ then $\lim_{t \to T_{\text{max}}} \|u(t)\| = \infty$.

First, we will prove that $\limsup_{t \to T_{\text{max}}} \|u(t)\| = \infty$. Assume by contradiction that $\limsup_{t \to T_{\text{max}}} \|u(t)\| < \infty$. Then there exists $K > 0$ such that $\|u(t)\| \leq K$ for $0 \leq t < T_{\text{max}}$. For $0 < t < t' < T_{\text{max}}$, we have

$$
\|u(t') - u(t)\|
\leq \|S^\alpha_{\psi}(t', 0)u_0 - S^\alpha_{\psi}(t, 0)u_0\|
+ \left| \int_t^{t'} \left( \psi(t') - \psi(t) \right) \left[ T^\alpha_{\psi}(t', s) - T^\alpha_{\psi}(t, s) \right] f(s, u(s)) \psi'(s) \, ds \right|
+ \left| \int_0^t \left[ (\psi(t') - \psi(s)) \left( T^\alpha_{\psi}(t', s) - T^\alpha_{\psi}(t, s) \right) f(s, u(s)) \psi'(s) \, ds \right] \right|
+ \left| \int_0^t (\psi(t) - \psi(s)) \left( T^\alpha_{\psi}(t, s) - T^\alpha_{\psi}(t, s) \right) f(s, u(s)) \psi'(s) \, ds \right|
=: I_1 + I_2 + I_3 + I_4.
$$

Similar to Step 3 of Theorem 4.1, we can prove that $\|u(t') - u(t)\| \to 0$ as $t', t \to T_{\text{max}}$. Therefore, by the Cauchy criteria we see that $\lim_{t \to T_{\text{max}}} u(t) = u_1$ exists. By the first part of the proof, there exists a $\delta > 0$ such that the solution can be extended to $[0, T_{\text{max}} + \delta]$ and we know that to the fractional evolution equation

$$
\begin{cases}
\frac{\partial}{\partial t} D^\alpha_{\psi} u(t) = Au(t) + f(t, u(t)), & 0 \leq t < \delta, \\
u(T_{\text{max}}) = u_1,
\end{cases}
(13)
$$

there exists a mild solution on $[T_{\text{max}}, T_{\text{max}} + \delta]$. This means that the mild solution of the problem (1) can be extended to $[0, T_{\text{max}} + \delta]$, which contradicts with the maximal interval $[0, T_{\text{max}})$. Hence, $\limsup_{t \to T_{\text{max}}} \|u(t)\| = \infty$.

Now, we will prove that if $T_{\text{max}} < \infty$, then $\lim_{t \to T_{\text{max}}} \|u(t)\| = \infty$. If this is not true, then there exist a constant $K > 0$ and a sequence $t_n \to T_{\text{max}}$ such that $\|u(t_n)\| \leq K$ for all $n$. Since $t \to \|u(t)\|$ is continuous and $\limsup_{t \to T_{\text{max}}} \|u(t)\| = \infty$, we can find a sequence $a_n$ such that $a_n \to 0$ as $n \to \infty$, $\|u(t)\| \leq M(K + 1)$ for $t_n \leq t \leq t_n + a_n$ and $\|u(t_n + a_n)\| = M(K + 1)$ for all $n$ sufficiently large. But we have

$$
M(K + 1) = \|u(t_n + a_n)\|
\leq \|S^\alpha_{\psi}(t_n + a_n, 0)u(t_n)\|
+ \int_{t_n}^{t_n + a_n} \psi(t) \psi'(s) ds
\leq M \frac{K}{\Gamma(\alpha + 1)} \|h_{\alpha}(t)\|_{L_\infty} \int_{t_n}^{t_n + a_n} \psi(t) \psi'(s) ds
\leq M \frac{K}{\Gamma(\alpha + 1)} \|h_{\alpha}(t)\|_{L_\infty} \psi(t_n) \psi'(t_n) a_n.
$$
which implies that $M(K + 1) \leq MK$ as $a_n \to 0$, a contradiction. Therefore, we find that if $T_{\text{max}} < \infty$, then $\lim_{t \to T_{\text{max}}} \|u(t)\| = \infty$. 

Next, we discuss the existence of a global mild solution for the problem (1). To this end, we need replace the assumption (H3) by (H5).

**Theorem 4.5** Assume that conditions (H1)–(H2) and (H5) hold, then for every $u_0 \in E$ the problem (1) has a global mild solution $u \in C([0, \infty), E)$.

**Proof** It is clearly that (H5) implies (H3). Therefore, by Theorem 4.4 we know that the problem (1) has a mild solution $u$ on a maximal interval of existence $[0, T_{\text{max}})$. By the proof process of Theorem 4.4, we can see that the problem (1) has a global mild solution if $u(t)$ is bounded for every $t$ in the interval of existence of $u$. If suffices to show that $u(t)$ is bounded for every $t \in [0, T_{\text{max}})$ with $T_{\text{max}} < \infty$.

Then for any $0 \leq t \leq T_{\text{max}}$ we have

$$
\|u(t)\| \leq \|S^\alpha(t, 0)u_0\| + \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \|T^\alpha_s(t, s)f(s, u(s))\| \psi(s) \, ds \\
\leq M\|u_0\| + \frac{\alpha M}{\Gamma(\alpha + 1)} \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \|f(s, u(s))\| \psi(s) \, ds \\
\leq M\|u_0\| + \frac{\alpha M}{\Gamma(\alpha + 1)} \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} (g_1(t) + g_2(t)\|u\|) \psi(s) \, ds \\
\leq M\|u_0\| + \frac{M}{\Gamma(\alpha + 1)} C_1(\psi(t) - \psi(0))^\alpha \\
+ \frac{M}{\Gamma(\alpha + 1)} C_2 \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \|u(s)\| \psi(s) \, ds \\
\leq M\|u_0\| + \frac{M}{\Gamma(\alpha + 1)} C_1(\psi(T_{\text{max}}) - \psi(0))^\alpha \\
+ \frac{M}{\Gamma(\alpha + 1)} C_2 \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \|u(s)\| \psi(s) \, ds \\
:= K_1 + K_2 \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \|u(s)\| \psi(s) \, ds,
$$

where

$$
C_1 = \sup_{0 \leq t \leq T_{\text{max}}} g_1(t), \quad C_2 = \sup_{0 \leq t \leq T_{\text{max}}} g_2(t),
$$

and

$$
K_1 = M\|u_0\| + \frac{M}{\Gamma(\alpha + 1)} C_1(\psi(T_{\text{max}}) - \psi(0))^\alpha, \quad K_2 = \frac{M}{\Gamma(\alpha)} C_2.
$$

By Corollary 2.12, we obtain

$$
\|u(t)\| \leq K_1 E_u(K_2 \Gamma(\alpha)[\psi(t) - \psi(0)]^\alpha) \\
\leq K_1 E_u(K_2 \Gamma(\alpha)[\psi(T_{\text{max}}) - \psi(0)]^\alpha),
$$

which means that $u(t)$ is bounded for every $t \in [0, T_{\text{max}})$. 

\qed
5 Existence and uniqueness of mild solution under noncompact analytic semigroup

In this section, we will prove the existence of mild solution for the problem (1) under the condition of a noncompact analytic semigroup.

**Theorem 5.1** Assume that conditions (H$_2$)–(H$_3$) and (H$_6$) hold. Then the problem (1) has at least one mild solution provided that

\[
\frac{ML}{\Gamma(1+\alpha)}(\psi(T) - \psi(0))^\alpha < 1 \quad \text{and} \quad \frac{4MK}{\Gamma(1+\alpha)}(\psi(T) - \psi(0))^\alpha < 1.
\]

**Proof** For any $r > 0$, let $\Omega_r = \{ u \in C([0, T], E) : \|u\| \leq r \}$.

Then, $\Omega_r$ is bounded closed convex subset of $C([0, T], E)$. Define an operator $K : \Omega_r \to C([0, T], E)$ by

\[
(Ku)(t) := S^\alpha_r(t, 0)u_0 + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_r(t, s)f(s, u(s))\psi'(s)ds, \quad t \in [0, T].
\]

Using the same argument in Theorem 4.1, we obtain $K : \Omega_r \to \Omega_r$ is continuous and $K(\Omega_r)$ is equicontinuous. Then it is sufficient to prove that $K : \Omega_r \to \Omega_r$ is condensing.

Let $D = \overline{C_0K(\Omega_r)}$, where $\overline{C_0}$ is the closure of convex hull. Then, by Lemma 2.21 we obtain $\overline{C_0K(\Omega_r)} \subset \Omega_r$ is bounded and equicontinuous. Now, we will prove that $K : D \to D$ is a condensing operator. For any $D \subset \overline{C_0K(\Omega_r)}$, by Lemma 2.17, we see that there exists a countable set $D_0 = \{ u_n \} \subset D$ such that

\[
\mu(K(D)) \leq 2\mu(K(D_0)).
\]

By the equicontinuity of $D$, we know that $D_0 \subset D$ is also equicontinuous. Therefore, by Lemma 2.20, we have

\[
\mu(K(D_0)(t)) = \mu\left( \left\{ S^\alpha_r(t, 0)u_0 + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_r(t, s)f(s, u_n(s))\psi'(s)ds \right\} \right)
\]

\[
\leq \mu\left( \left\{ S^\alpha_r(t, 0)u_0 \right\} \right)
\]

\[
+ \mu\left( \left\{ \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_r(t, s)f(s, u_n(s))\psi'(s)ds \right\} \right)
\]

\[
\leq 2\int_0^t \mu\left( \left\{ (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_r(t, s)f(s, u_n(s))\psi'(s) \right\} \right)ds
\]

\[
\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \mu\left( \left\{ f(s, u_n(s)) \right\} \right)\psi'(s)ds
\]

\[
\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} K\mu(D_0(s))\psi'(s)ds
\]

\[
= \frac{2MK}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\psi'(s)ds \cdot \mu(D)
\]

\[
= \frac{2MK}{\Gamma(\alpha + 1)} (\psi(t) - \psi(0))^{\alpha} \mu(D)
\]

\[
\leq \frac{2MK}{\Gamma(\alpha + 1)} (\psi(T) - \psi(0))^{\alpha} \mu(D).
\]
Since $K(D_0) \subset D$ is bounded and equicontinuous, we obtain

$$
\mu(K(D_0)) = \max_{t \in J} \mu(D_0(t))
$$

by Lemma (2.18). It follows that

$$
\mu(K(D)) \leq 2\mu(K(D_0)) \leq 4MK\Gamma(\alpha + 1)\left(\frac{M}{ML}\right)^{\frac{1}{2}} + \psi(0) \cdot \mu(D) < \mu(D).
$$

Thus, $K : D \to D$ is a condensing operator. Therefore, by Lemma 2.22, $K$ has at least one fixed point $u^*$ in $\Omega$, which is a mild solution of (1). The proof is complete. □

**Remark 5.2** From Theorem 5.1, we notice that if $\psi$ is bijection function then the problem (1) has at least one mild solution provided that

$$
T < \min \left\{ \psi^{-1}\left[ \psi(0) \right], \psi^{-1}\left[ \frac{M}{ML} \right] + \psi(0) \right\}.
$$

**Theorem 5.3** Assume that conditions (H2)–(H3) and (H6) hold. Then, for any $u_0 \in E$, the problem (1) has a mild solution $u$ on a maximal interval of existence $[0, T_{\text{max}})$. If $T_{\text{max}} < \infty$, then

$$
\lim_{t \to T_{\text{max}}^-} \| u(t) \| = \infty.
$$

**Proof** The proof uses the same argument as in Theorem 4.4. □

**Theorem 5.4** Assume that conditions (H2) and (H5) hold, then for every $u_0 \in E$ the problem (1) has a global mild solution $u \in C([0, \infty), E)$.

**Proof** The proof uses the same argument as in Theorem 4.5. □

**6 Mittag-Leffler–Ulam–Hyers stability**

For $f \in ([0, T] \times E, E), \psi \in C([0, T], \mathbb{R}^+)$ and $\epsilon > 0$ we consider the equation

$$
\frac{C}{\alpha}D_0^\alpha u(t) = Au(t) + f(t, u(t)), \quad t \in [0, T],
$$

(14)

and the inequalities

$$
\left| \frac{C}{\alpha}D_0^\alpha u(t) - Au(t) - f(t, u(t)) \right| \leq \epsilon, \quad t \in [0, T],
$$

(15)

$$
\left| \frac{C}{\alpha}D_0^\alpha u(t) - Au(t) - f(t, u(t)) \right| \leq \psi(t), \quad t \in [0, T],
$$

(16)

$$
\left| \frac{C}{\alpha}D_0^\alpha u(t) - Au(t) - f(t, u(t)) \right| \leq \epsilon \psi(t), \quad t \in [0, T].
$$

(17)

**Definition 6.1** Equation (14) is Mittag-Leffler–Ulam–Hyers stable, with respect to $E_\alpha$, if there exists a real number $C > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1([0, T], E)$ of inequality (15) there exists a mild solution $u \in C([0, T], E)$ of Eq. (14) with

$$
|v(t) - u(t)| \leq C\epsilon E_\alpha(t), \quad t \in [0, T].
$$
Definition 6.2 Equation (14) is generalized Mittag-Leffler–Ulam–Hyers stable, with respect to $E_\alpha$, if there exists a function $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0, T], E)$ of inequality (15) there exists a mild solution $u \in C([0, T], E)$ of Eq. (14) with

$$|v(t) - u(t)| \leq C\theta(\varepsilon)E_\alpha(t), \quad t \in [0, T].$$

Definition 6.3 Equation (14) is Mittag-Leffler–Ulam–Hyers–Rassias stable, with respect to $\varphi E_\alpha$, if there exists a real number $C_\varphi > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0, T], E)$ of inequality (17) there exists a mild solution $u \in C([0, T], E)$ of Eq. (14) with

$$|v(t) - u(t)| \leq C_\varphi \varphi(t)E_\alpha(t), \quad t \in [0, T].$$

Definition 6.4 Equation (14) is generalized Mittag-Leffler–Ulam–Hyers–Rassias stable, with respect to $\varphi E_\alpha$, if there exists a real number $C_\varphi > 0$ such that for each solution $v \in C^1([0, T], E)$ of inequality (16) there exists a mild solution $u \in C([0, T], E)$ of Eq. (14) with

$$|v(t) - u(t)| \leq C_\varphi \varphi(t)E_\alpha(t), \quad t \in [0, T].$$

Remark 6.5 It is clear that Definition 6.1 implies 6.2 and 6.3 implies 6.4.

Remark 6.6 A function $u \in C^1([0, T], E)$ is a solution of the inequality (15) if and only if there exists a function $g \in C^1([0, T], E)$ (which depend on $u$) such that

(i) $|g(t)| \leq \varepsilon$ for $t \in [0, T]$,

(ii) $\frac{D_\psi^\alpha}{\varphi} u(t) = Au(t) + f(t, u(t)) + g(t), \quad t \in [0, T].$

Remark 6.7 If $v \in C^1([0, T], E)$ is a solution of inequality (15), $v$ is a solution of the following integral inequality:

$$|v(t) - S_\psi^\alpha (t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T_\psi^\alpha (t, s)f(s, v(s)) \psi(s) ds|$$

$$\leq \varepsilon \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \|T_\psi^\alpha (t, s)\| \psi(s) ds.$$

Theorem 6.8 Assume that $f \in C([0, T] \times E, E)$ and there exists $L_f > 0$ such that

$$|f(t, u_1) - f(t, u_2)| \leq L_f |u_1 - u_2|$$

for all $t \in [0, T]$ and $u_1, u_2 \in E$. Then, Eq. (14) is Mittag-Leffler–Ulam–Hyers stable.

Proof Let $v \in C^1([0, T], E)$ be a solution of inequality (15). Let us denote by $u \in C([0, T], E)$ the unique mild solution of the Cauchy problem

$$\begin{cases}
\frac{D_\psi^\alpha}{\varphi} u(t) = Au(t) + f(t, u(t)), & t \in (0, T], \\
u(0) = v(0).
\end{cases} \quad (18)$$
We have
\[ u(t) = S^\alpha_{\psi}(t, 0)v(0) + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, u(s))\psi'(s) \, ds, \quad t \in [0, T]. \]

Then we get
\[
\begin{align*}
|v(t) - S^\alpha_{\psi}(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, v(s))\psi'(s) \, ds| & \\
& \leq \varepsilon \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \|T^\alpha_{\psi}(t, s)\|\psi'(s) \, ds \\
& \leq \frac{\alpha M \varepsilon}{\Gamma(1 + \alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\psi'(s) \, ds \\
& = \frac{M}{\Gamma(1 + \alpha)} (\psi(t) - \psi(0))^\alpha \varepsilon \\
& \leq \frac{M}{\Gamma(1 + \alpha)} (\psi(T) - \psi(0))^\alpha \varepsilon.
\end{align*}
\]

It follows that
\[
|v(t) - u(t)|
\]
\[
\leq |v(t) - S^\alpha_{\psi}(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, v(s))\psi'(s) \, ds| \\
\leq |v(t) - S^\alpha_{\psi}(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, v(s))\psi'(s) \, ds| \\
+ \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, v(s))\psi'(s) \, ds \\
- \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, u(s))\psi'(s) \, ds \\
\leq |v(t) - S^\alpha_{\psi}(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)f(s, v(s))\psi'(s) \, ds| \\
+ \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\alpha_{\psi}(t, s)\|f(s, v(s)) - f(s, u(s))\|\psi'(s) \, ds \\
\leq \frac{M}{\Gamma(1 + \alpha)} (\psi(T) - \psi(0))^\alpha \varepsilon \\
+ \frac{\alpha MLf}{\Gamma(1 + \alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\|v(s) - u(s)\|\psi'(s) \, ds.
\]

By Corollary 2.12, we obtain
\[
|v(t) - u(t)| \leq \frac{M}{\Gamma(1 + \alpha)} (\psi(T) - \psi(0))^\alpha E_u\left(MLf[\psi(t) - \psi(0)]^\alpha \varepsilon.\right)
\]

The proof is complete. \[\square\]

**Theorem 6.9** Assume that the following conditions hold:
\[
(i) \quad f \in C([0, \infty) \times E, E).
\]
(ii) \( k(t) \) is a nonnegative, nondecreasing continuous function defined on \( t \in [0, \infty) \) and
\[
|f(t, u_1) - f(t, u_2)| \leq k(t)|u_1 - u_2|
\]
for all \( t \in [0, \infty) \) and \( u_1, u_2 \in E \).

(iii) The function \( \varphi \in C([0, \infty), \mathbb{R}^n) \) is increasing and there exists \( \lambda > 0 \) such that
\[
\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| T^\varphi_\psi(t, s) \| \psi'(s) \, ds \leq \lambda \varphi(t)
\]
for all \( t \in [0, \infty) \). Then, Eq. (14) is generalized Mittag-Leffler–Ulam–Hyers–Rassias stable with respect to \( \varphi E_\alpha \).

**Proof** Let \( v \in C^1([0, T], \infty) \) be a solution of inequality (16). Then we get
\[
\left| v(t) - S^\varphi_\psi(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, v(s)) \psi'(s) \, ds \right|
\leq \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| T^\varphi_\psi(t, s) \| \psi'(s) \, ds \\
\leq \lambda \varphi(t),
\]
for all \( t \in [0, \infty) \). Let us denote by \( u \in C([0, T], \infty) \) the unique mild solution of the Cauchy problem
\[
\begin{cases}
C^\varphi_\psi D^\varphi u(t) = Au(t) + f(t, u(t)), & t \in (0, \infty), \\
u(0) = v(0).
\end{cases}
\]
We have
\[
u(t) - S^\varphi_\psi(t, 0)v(0) + \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, u(s)) \psi'(s) \, ds, \quad t \in [0, \infty).
\]
It follows that
\[
|\nu(t) - u(t)| \leq |\nu(t) - S^\varphi_\psi(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, u(s)) \psi'(s) \, ds |
\leq \left| \nu(t) - S^\varphi_\psi(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, v(s)) \psi'(s) \, ds \right|
+ \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, v(s)) \psi'(s) \, ds \\
- \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, u(s)) \psi'(s) \, ds \\
\leq \left| \nu(t) - S^\varphi_\psi(t, 0)v(0) - \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T^\varphi_\psi(t, s) f(s, v(s)) \psi'(s) \, ds \right|
+ \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \| T^\varphi_\psi(t, s) \| \left| f(s, v(s)) - f(s, u(s)) \right| \psi'(s) \, ds \\
\leq \lambda \varphi(t) + \frac{\alpha Mk(t)}{\Gamma(1 + \alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} |v(s) - u(s)| \psi'(s) \, ds.
\]
By Corollary 2.12, we obtain
\[
|v(t) - u(t)| \leq \lambda \psi(t) E_\alpha (Mk(t)[\psi(t) - \psi(0)])^\alpha.
\]

The proof is complete. □

7 Examples

In this section, we give examples of fractional differential equation of compact and non-compact semigroup cases. The main results can be applied for a larger class of Caputo fractional derivative with respect to \( \psi \). In particular, our results can be reduced to the examples in [25, 32] when \( \psi(t) = t \).

Example 7.1 Let \( E = L^2([0, \pi]) \) equipped with the norm and inner product defined, respectively, for all \( u, v \in L^2([0, \pi]) \) by
\[
\|u\| = \left( \int_0^\pi |u(x)|^2 \, dx \right)^{\frac{1}{2}} \quad \text{and} \quad \langle u, v \rangle = \int_0^\pi u(x)v(x) \, dx.
\]

Consider the following initial-boundary value problem of time-fractional parabolic partial differential equation with nonlinear source term:
\[
\begin{cases}
C_0 D_\alpha^\psi u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = f(t, u(x, t)), & t \in (0, 1], x \in [0, \pi], \\
u(0, t) = u(\pi, t) = 0, & t \in (0, 1], \\
u(x, 0) = u_0(x), & x \in [0, \pi],
\end{cases}
\tag{20}
\]
where \( \alpha = \frac{3}{2}, \psi(t) = \sqrt{t + 1} \) and \( f(t, u) = \frac{1}{3} e^{-t} u(x, t) + t^{2u} \sin(u(x, t)) \). We define an operator \( A : D(A) \subset E \to E \) by
\[
D(A) := \{ v \in E : v, v' \text{ are absolutely continuous and } v'' \in E, v(0) = v(\pi) = 0 \}
\]
and
\[
Au = \frac{\partial^2}{\partial x^2} u.
\]

It is well known that \( A \) has a discrete spectrum, the eigenvalue are \(-n^2, n \in \mathbb{N}\), with the corresponding normalized eigenvectors \( e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz) \). Then
\[
A x = \sum_{n=1}^{\infty} -n^2 \langle x, e_n \rangle e_n, \quad x \in D(A).
\]

Furthermore, \( A \) generates a uniformly bounded analytic semigroup \( \{T(t)\}_{t \geq 0} \) in \( E \) and is given by
\[
T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n, \quad x \in E.
\]
with \( \|T(t)\| \leq e^{-t} \) for all \( t \geq 0 \). Hence, we take \( M = 1 \) which implies that \( \sup_{t \in (0, \infty)} \|T(t)\| = 1 \) and \((H_1)\) are satisfies.

Then for \( t \in [0, 1] \) we have

\[
\|f(t, u)\| \leq \frac{1}{3} e^{-t} \|u\| + t^2 \pi,
\]

\[
\sup_{\|u\| \leq r} \|f(t, u)\| \leq \frac{1}{3} e^{-t} r + t^2 \pi =: \rho_r(t),
\]

\[
\limsup_{r \to \infty} \frac{\|\rho_r(t)\|_{L^\infty}}{r} = \frac{1}{3} =: L,
\]

and

\[
\|f(t, u_1) - f(t, u_2)\| \leq \frac{4}{3} \|u_1 - u_2\| \quad \text{for } u_1, u_2 \in \Omega_r.
\]

Therefore, \((H_2)\)–\((H_4)\) are satisfied. This yields

\[
\frac{M L}{\Gamma(1 + \alpha)} \left( \psi(T) - \psi(0) \right)^\alpha = \frac{1}{3 \Gamma(\frac{5}{2})} \left( \psi(1) - \psi(0) \right)^{\frac{3}{2}} \approx 0.2052 < 1.
\]

Hence, according to Theorems 4.1 and 4.3, the problem (20) has a unique mild solution on \([0, 1]\).

Moreover, \((H_5)\) is satisfied. Then, by Theorem 4.5, the problem (20) has a global mild solution \( u \in C([0, \infty), E) \).

**Example 7.2** Consider the following initial-boundary value problem of time-fractional parabolic partial differential equation with nonlinear source term:

\[
\begin{aligned}
&C_0^\alpha \psi u(x, t) - \Delta u(x, t) = f(t, u(x, t)), \quad t \in (0, 1], x \in [0, 1], \\
u(0, t) = u(1, t) = 0, \quad t \in [0, 1], \\
u(x, 0) = u_0(x), \quad x \in [0, 1],
\end{aligned}
\]

(21)

where \( \alpha = \frac{1}{2} \), \( \psi(t) = e^t \) and \( f(t, u) = \frac{1}{2} e^{-t} |u(x, t)| \). Let \( E = L^2([0, 1]) \) and \( A : D(A) \subset E \to E \) be an operator defined by

\[
D(A) := H^2(0, 1) \cap H^1_0(0, 1) = \left\{ v \in H^2(0, 1) : v(0) = v(1) = 0 \right\}
\]

and \( A^\mu u = \Delta u \),

where \( H^2(0, 1) \) is the completion of the space \( C^2(0, 1) \) with respect to the norm

\[
\|u\|_{H^2(0, 1)} = \left( \int_0^1 \sum_{|\mu| \leq 2} |D^\mu u(x)|^2 \, dx \right)^{\frac{1}{2}},
\]

and \( C^2 \) is the set of all the set of all continuous defined on \((0, 1)\) which have continuous partial derivatives of order less than or equal to 2, and \( H^2_0(0, 1) \) is the completion of \( C^1(0, 1) \) with respect to the norm \( \|u\|_{H^2(0, 1)} \).

Then for \( t \in [0, 1] \) we have

\[
\|f(t, u)\| \leq \frac{1}{5} e^{-t} \|u\|,
\]
\[ \sup_{\|u\| \leq r} \|f(t, u)\| \leq \frac{1}{5} e^{-t} r =: h_1(t), \]
\[ \limsup_{r \to \infty} \frac{\|h_1(t)\|_{L^\infty}}{r} = \frac{1}{2}, \]
\[ \|f(t, u_1) - f(t, u_2)\| \leq \frac{1}{5} \|u_1 - u_2\| \quad \text{for } u_1, u_2 \in \Omega_r, \]
and
\[ \mu(f(t, D)) \leq \frac{1}{5} \mu(D), \quad t \in [0, 1] \text{ and } D \in \Omega_r. \]

Therefore, (H2)–(H3) and (H6) are satisfied. We take \( M = 1, L = \frac{1}{5} \) and \( K = \frac{1}{5} \). This yields
\[ \frac{ML}{\Gamma(1 + \alpha)} \left( \psi(T) - \psi(0) \right)^\alpha = \frac{1}{25 \Gamma(\frac{3}{2})} \left( \psi(1) - \psi(0) \right)^\frac{1}{2} \approx 0.0592 < 1 \]
and
\[ \frac{4MK}{\Gamma(1 + \alpha)} \left( \psi(T) - \psi(0) \right)^\alpha = \frac{4}{25 \Gamma(\frac{3}{2})} \left( \psi(1) - \psi(0) \right)^\frac{1}{2} \approx 0.2367 < 1. \]

Hence, according to Theorem 5.1 and 4.3, the problem (21) has unique mild solution on [0, 1].

Moreover, (H3) is satisfied. Then Theorem 5.4 implies that the problem (21) has a global mild solution \( u \in C([0, \infty), E) \).

8 Conclusion

We construct a mild solution for fractional evolution equation based on Laplace transform with respect to \( \psi \)-function. We obtain the local and global existence and uniqueness of mild solution for the problem with \( \psi \)-Caputo fractional derivative, which can be reduced to the classical Caputo fractional derivative in previous work. Furthermore, the form of a fundamental solution obtained in this work is a foundation result for further investigation such as the problem with perturbation, delay and a nonlocal term.

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Availability of data and materials

The data that support the findings of this study are available from the authors, upon request.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

The main idea of this paper was proposed and mainly proved by PSN, while AS performed some proofs and provided some examples. All authors read and approved the final manuscript.

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