Online Bipartite Matching with Predicted Degrees

Justin Y. Chen, Piotr Indyk
MIT
justc@mit.edu, indyk@mit.edu

Abstract

We propose a model for online graph problems where algorithms are given access to an oracle that predicts the degrees of nodes in the graph (e.g., based on past data). Within this model, we study the classic problem of online bipartite matching. An extensive empirical evaluation shows that a greedy algorithm called MinPredictedDegree compares favorably to state-of-the-art online algorithms for this problem. We also initiate the theoretical study of MinPredictedDegree on a natural random graph model with power law degree distribution and show that it produces matchings almost as large as the maximum matching on such graphs.

1 Introduction

Online algorithms are algorithms that process their inputs “on the fly”, making irrevocable decisions based only on the data seen so far. Since they do not make any assumptions about the future, they are versatile and work even for adversarial inputs. Unfortunately, by focusing on the worst case, their performance in “typical” cases can be sub-optimal. As a result there has been a large body of research studying various relaxations of the worst-case model, where some extra information about the inputs, or the distribution they are selected from, is available (Unc 2016).

Motivated by the developments in machine learning, over the last few years, many papers have studied online algorithms with predictions [Mitzenmacher and Vassilvitski 2020]. Such algorithms are equipped with a predictor that, when invoked, provides an (imperfect) prediction of some features of the future part of the input, which is then used by the algorithm to improve its performance. The specific information provided by such predictors is problem-dependent. For graph problems studied in this paper, predictions could include: the list of edges incident to a given vertex [Kumar et al. 2019], the weight of an edge adjacent to a given node in an optimal solution [Antoniadis et al. 2020], or vertex weights that guide a proportional allocation scheme [Lavastida et al. 2020].

In this paper we focus on online graph problems, and propose a model where an algorithm is equipped with a “degree predictor”, i.e., an oracle that, given any vertex, predicts the degree of that vertex in the full graph (containing yet-unseen edges). This predictor has multiple appealing features. First it is simple, natural, and easy to interpret. Second, it is useful: vertex degree information is employed in many heuristic and approximation algorithms for graph optimization, for problems such as maximum independent set [Halldórsson and Radhakrishnan 1997] or maximum matching [Tinhofer 1984]. Third (as demonstrated in Section 4) such predictors can be easily obtained. Finally, degree prediction is closely related to the problem of estimating the frequencies of elements in a data set and frequency predictors have been already shown to improve the performance of algorithms for multiple data analysis problems [Hsu et al. 2019; Jiang et al. 2020; Eden et al. 2021].

The specific graph problem studied in this paper is online bipartite matching, where we are given a bipartite graph $G = (U \cup V, E)$, and the goal is to find a maximum matching in $G$. In the online setting, the set $U$ is known beforehand, while the vertices in $V$ arrive online one by one. When a new vertex $v$ arrives, the edges in $G$ adjacent to $v$ are provided as well. Online maximum bipartite matching is a classic question studied in the online algorithms literature, with many applications (Mehta 2013). It is known that a randomized online greedy algorithm, called Ranking, computes a matching of size at least $1 - 1/e$ times the optimum (Karp, Vazirani, and Vazirani 1990), and that this bound is tight in the worst-case. A large body of work studied various relaxations of the problem, obtained by assuming that vertex arrivals are random (Goel and Mehta 2008) or that the graph itself is randomly generated from a given “model” (Feldman et al. 2009). In this paper we extend the basic online model by assuming access to a predictor that, given any “offline” vertex $u \in U$, returns an estimate of its degree. (Note that the degree of any vertex in $V$ is known immediately upon its arrival.)

Our results We study the following simple greedy algorithm for bipartite matching: upon the arrival of a vertex $v$, if the set of neighbors $N(v)$ of $v$ in $G$ contains any yet-unmatched vertex, the algorithm selects $u \in N(v)$ of minimum predicted degree in $G$ and adds the edge $(u, v)$ to the matching. This algorithm, which we call MinPredicted-
Degree, is essentially identical to the algorithm proposed in [Borodin, Pankratov, and Salehi-Abari 2019] which in turn was inspired by the offline matching algorithm called MinGreedy [Thinofe 1984]. The intuition is that vertices with higher degree will have more chances to be matched in the future than vertices with lower degree. Our main contributions are as follows:

- We formulate a new model for data-driven online algorithms, equipped with degree predictors, and demonstrate its utility in the context of graph matching.
- We perform an extensive empirical evaluation of MinPredictedDegree for multiple random graph models and real graph benchmarks. Our experiments show that, on most benchmarks, MinPredictedDegree has the best performance among about a dozen state-of-the-art online algorithms. Furthermore, we observe that the performance of MinPredictedDegree remains strong even when the prediction quality degrades (up to a point).
- We initiate the theoretical study of MinPredictedDegree on a natural random graph model with power law degree distribution [Chung, Lu, and Vu 2004], henceforth referred to as CLV-B. Our theoretical predictions closely match the experimental results, and demonstrate that the competitive ratio achieved by our algorithm is very high. In particular, for several different power law distributions, it exceeds 0.99. We also point out that the ratio is always at least 0.5, even for worst case inputs and predictors.

2 Related Work

Online bipartite matching and its generalizations have been investigated extensively over the last few decades. The survey [Mehta 2013] as well as the recent paper [Borodin, Karavaslis, and Pankratov 2020] provide excellent overviews of this area.

There has been lots of interest in online algorithms with predictions over the last few years, for problems like caching [Lykouris and Vassilvitskii 2018; Rohatgi 2020; Wei 2020; Jiang, Panigrahi, and Sun 2020], ski-rental and its generalizations [Purohit, Svitkina, and Kumar 2018; Gollapudi and Panigrahi 2019; Anand, Ge, and Panigrahi 2020; Angelopoulos et al. 2020], scheduling [Mitzenmacher 2020; Lattanzi et al. 2020] and matching [Kumar et al. 2019; Antoniadis et al. 2020; Lavastida et al. 2020]. Other areas impacted by learning-based algorithms include combinatorial optimization [Dai et al. 2017; Balcan et al. 2018], similarity search [Salakhutdinov and Hinton 2009; Weiss, Torralba, and Fergus 2009; Jegou, Douze, and Schmid 2011; Wang et al. 2016; Dong et al. 2020], data structures [Kraska et al. 2018; Mitzenmacher 2018] and streaming/sampling algorithms [Hsu et al. 2019; Jiang et al. 2020; Eden et al. 2021]. See [Mitzenmacher and Vassilvitskii 2020] for an excellent survey of this area.

3 Algorithm

Online Bipartite Matching The online bipartite matching problem is defined as follows. Given a bipartite graph $G = (U \cup V, E)$, we call $U$ the “offline” side and $V$ the “online” side of the bipartition. Let $n = |U|$ and $m = |V|$. The nodes in $U$ are known beforehand and the nodes in $V$ arrive one at a time, along with their incident edges. An online bipartite matching algorithm maintains a matching throughout the process, with the goal of maximizing the size of the matching. As each node $v \in V$ arrives, the algorithm can pick one its neighboring edges to add to the matching.

MinPredictedDegree In addition to knowing the offline nodes $U$ beforehand, MinPredictedDegree is given a degree predictor $\sigma : U \to \mathbb{R}_{\geq 0}$. In practice, this predictor could be inferred from additional knowledge about the graph or from past data. When a node $v \in V$ arrives, MinPredictedDegree (see Algorithm 1) uses this predictor to greedily select the minimum predicted degree neighbor $u^*$ of $v$ that is not already covered in the matching and then adds the edge $\{u^*, v\}$ to the matching. If no such valid neighbor exists, MinPredictedDegree does nothing with $v$. Intuitively, low degree offline nodes should be matched as early as possible as they only appear a few times while we will have many chances to match high degree offline nodes.

The MinPredictedDegree algorithm has similar structure to the worst-case optimal Ranking algorithm [Karp, Vazirani, and Vazirani 1990] which assigns a random cost to each offline node and at each step greedily matches with the lowest cost offline neighbor. Specifically, if the degree predictor is random, MinPredictedDegree and Ranking are equivalent. As we show in the later sections, if the predictor is “good enough”, MinPredictedDegree often performs much better than Ranking. In particular, we find experimentally and through an analysis on random power law graphs that MinPredictedDegree competes favorably against baseline algorithms (including Ranking) and often outputs matchings almost as large as the maximum matching. In the worst case (even if the predictions are adversarial), MinPredictedDegree achieves a competitive ratio of $1/2$ as it returns a maximal matching (compared to $1 - 1/e$ achieved by Ranking). We also show that this bound is tight: see Appendix A for the details.

Algorithm 1: MinPredictedDegree

| Input: Offline nodes $U$ and degree predictor $\sigma : U \to \mathbb{R}_{\geq 0}$ |
| Output: Matching $M$ |
| Initialize $M \leftarrow \emptyset$. |
| while online node $v \in V$ arrives do |
| $N(v) \leftarrow$ unmatched neighbors of $v$ |
| if $|N(v)| > 0$ then |
| $u^* \leftarrow \arg\min_{u \in N(v)} \sigma(u)$ (ties broken arbitrarily) |
| $M \leftarrow M \cup \{u^*, v\}$ |
| end if |
| end while |

The main differences are syntactic: the algorithm of [Borodin, Pankratov, and Salehi-Abari 2019] computes the degrees based on the given “type graph” (see Section 4), while in this paper we allow arbitrary predictors.

\[99\]

\[80\]
4 Experiments

In this section, we evaluate the empirical performance of MinPredictedDegree on real and synthetic data. For each dataset, we report the empirical competitive ratio of MinPredictedDegree and a variety of baselines. In each case, the empirical competitive ratio is the average, over 100 trials, of the ratios of the sizes of the matchings outputted by a given algorithm and the sizes of the maximum matching. In addition to the average ratio, we report one standard deviation of the ratio across the trials.

Datasets We evaluate MinPredictedDegree on the following bipartite graph datasets.

- **Oregon:** 9 graphs1 sampled over 3 months representing a communication network of internet routers from the Stanford SNAP Repository (Leskovec and Krevl 2014). Each graph has ~ 10k nodes on each side of the bipartition and ~ 40k edges. For MinPredictedDegree, the offline degree predictor $\sigma : U \rightarrow \mathbb{R}$ is based on the first graph: if an offline node $u$ (i.e. a specific router) appeared in the first graph, $\sigma(u)$ is the degree of $u$ in that graph. If an offline node $u$ did not appear in the first graph, $\sigma(u) = 1$. For each trial, the order of arrival of the online nodes is randomized.

- **CAIDA:** 122 graphs sampled approximately weekly over 4 years representing a communication network of internet routers from the Stanford SNAP Repository (Leskovec and Krevl 2014). Each graph has ~ 20k nodes on each side of the bipartition and ~ 100k edges. The degree predictor is the same as for the Oregon dataset (for each year, the first graph of the year is used to form the predictor). As seen in Figure 9 (see Appendix H), the degree distribution of the graphs for both the Oregon and Caida datasets are long-tailed and the error of the first graph predictor increases over time as the underlying graph evolves. For each trial, the order of arrival of the online nodes is randomized.

- **CLV-B random graph:** Based on the Chung-Lu-Vu model used in prior work (Chung, Lu, and Vu 2004; Meka, Jain, and Dhillon 2009), given a vector of offline expected degrees $d = \{d_i\}_{i=1}^n$, the edge $\{u_i, v_i\}$ appears in the graph independently w.p. $d_i/m$. MinPredictedDegree uses the degree predictor which returns the expected degree for each offline node: $\sigma(u_i) = d_i$. This model corresponds to the case where consumers (online) pick their edges i.i.d. over producers (offline) and MinPredictedDegree has knowledge of the average preferences over producers. We consider the case where the expected offline degrees are distributed according to Zipf’s Law, a popular power law distribution where $d_i = C \cdot i^{-\alpha}$ (Mitzenmacher and Upfal 2005). In our experiments, we set size $n = m = 1000$, set $C = m/2$, and vary the exponent $\alpha$. The performance of MinPredictedDegree on CLV-B random graphs is further explored in Section 5.

---

1The graphs in the Oregon and CAIDA datasets are made bipartite following the bipartite double cover or duplicating method used in prior work (Borodin, Karavasilis, and Pankratov 2020). Given a graph $G = (V,E)$, the bipartite double cover of $G$ is the graph $G' = (U' \cup V', E')$ where $U'$ and $V'$ are copies of $V$ and there is an edge $\{u'_i, v'_j\} \in E'$ if and only if $\{v_i, v_j\} \in E$. 

---

Known i.i.d.: Following the methodology of Borodin et al. (Borodin, Karavasilis, and Pankratov 2020), we compare against competing methods in the known i.i.d. model for online bipartite matching. In the known i.i.d. model, algorithms are given access to a type graph $G = (U \cup V, E)$ and a distribution $P : V \rightarrow [0, 1]$. The nodes in $V$ and their incident edges represent “types” of online nodes. An input instance $\hat{G} = (U \cup \hat{V}, \hat{E})$ is formed by picking $m$ online nodes i.i.d. from the candidates types in $G$ according to the probabilities described by $P$. Similarly, to the CLV-B case, degree predictions are given by the expected degrees of the offline nodes (i.e. the degree of the offline nodes in the type graph rescaled by $|\hat{V}|/|V|$).

Within this model, we copy the methodology of Borodin et al. (Borodin, Karavasilis, and Pankratov 2020) and rerun the experiments from their work on synthetic power law graphs (Molloy Reed and Preferential Attachment) as well as on real world graphs. In the Molloy Reed experiments, the type graph is sample from a family of random graphs with degrees distributed according to a power law with exponential cutoff. In the Preferential Attachment experiments, the type graph is formed by the preferential attachment model in which edges are added sequentially with edges between high degree nodes being more likely. The Real World graphs are comprised of a variety of graphs from the Network Repository (Rossi and Ahmed 2015). See Appendix H for more results on Real World graphs.

Baselines We compare our algorithm to a variety of baseline algorithms.

- **Ranking** In all experiments, we compare to the classic, worst-case optimal Ranking algorithm (Karp, Vazirani, and Vazirani 1990).

- **MinDegree** The MinDegree algorithm is a version of MinPredictedDegree with a perfect oracle, i.e. $\sigma(u)$ returns the true degree of $u$. In comparison with MinPredictedDegree, MinDegree shows the effect of prediction error on the performance of MinPredictedDegree.

- **Known i.i.d. baselines** For the experiments in the known i.i.d. case, we also compare to the baselines in the extensive empirical study of Borodin et al. (2020)—see their paper for detailed descriptions of all algorithms. The code is distributed under the GPL license. Notably, the algorithms Category-Advice and 3-Pass are not strictly online algorithms: they take multiple passes over the data, using some limited information from previous passes to make better decisions in the next pass. It should also be noted that BKP-MinDegree is distinct from either the MinPredictedDegree or MinDegree algorithms we have described—it does not use the type graph but rather maintains and updates an estimate of the degree of the offline nodes throughout the runtime of the algorithm.

Most known i.i.d. baselines are not greedy—they do not always match an online node even if it has unmatched neighbors. Borodin et al. (2020) evaluate greedy augmentations of these algorithms (denoted by Algorithm(g)) which match to an arbitrary unmatched neighbor in these cases and generally show them to outperform their non-greedy...
counterparts. We additionally evaluate MinPredictedDegree augmented versions of these algorithms (denoted by Algorithm(MPD)) which applies the MinPredictedDegree rule in these cases using the expected degrees as predictions.

Figure 1: Comparison of empirical competitive ratios on the Oregon dataset. The first graph is used to form the degree predictions.

![Figure 1](image1.png)

Figure 2: Comparison of empirical competitive ratios on the CAIDA dataset. For each subfigure, the first graph of the year is used to form the degree predictions for the rest of the year.

![Figure 2](image2.png)

Results Across the various datasets, MinPredictedDegree performs favorably compared to the baseline algorithms. For the Oregon, CAIDA, and CLV-B random graph datasets, MinPredictedDegree significantly outperforms Ranking, and for Oregon and CAIDA, the performance of the algorithm mildly declines as the degree predictions degrade. For the known i.i.d. datasets, MinPredictedDegree often outperforms all online baselines, despite making only limited use of the known i.i.d. model. Additionally, augmenting the known i.i.d. algorithms with the (MPD) rule often improves their performance over both the base versions of the algorithms and the greedy versions.

- Oregon: On the Oregon dataset, MinPredictedDegree achieves a competitive ratio of $\sim 0.99$ across the graphs compared with competitive ratios ranging from 0.95 to 0.97 for Ranking. Compared with MinDegree, which uses knowledge of the true offline degrees, MinPredictedDegree’s performance slowly degrades over time as the graphs become less similar to Graph #1 (see Figure 9 in Appendix [H] for quantitative details).

- CAIDA: Similarly to the Oregon dataset, on the CAIDA dataset, MinPredictedDegree does significantly better than Ranking, achieving competitive ratios almost always greater than 0.98 compared to ratios around 0.95, respectively. As the performance of the degree predictor degrades over time, the performance of MinPredictedDegree gradually declines (though it still significantly outperforms Ranking for both datasets).

- CLV-B random graph: For CLV-B random graphs with offline expected degrees following Zipf’s Law with exponent $\alpha$. In subfigure (a), we vary $\alpha$ and MinPredictedDegree uses the expected degree as its predictor. In subfigure (b), the degree predictor is the offline degree in a random subgraph using a (varying) fraction of the online nodes.

Figure 3: Analysis of predictor noise (exponent $\alpha = 1$).

![Figure 3](image3.png)
In Figure 3b, we analyze the performance of MinPredictedDegree with a noisy degree predictor on Zipf’s Law CLV-B random graphs with exponent 1. To introduce noise, the degree predictor $\sigma(u)$ is given by the number of neighbors $u$ has with a random subset of the online nodes $V$. As we decrease the fraction of $V$ we subsample, thus increasing the variance of the predictor, the performance of MinPredictedDegree steadily declines. Even when the degree predictor only uses 10% or even 1% (the leftmost point on the graph) of the online nodes, it still outperforms the Ranking algorithm.

- **Known i.i.d.:** Across all of the experiments in the known i.i.d. model, MinPredictedDegree is among the top online algorithms, and is often the best performing online algorithm (note the algorithms in gray are not strictly online algorithms). Most of the algorithms (e.g., BahamiKapralov and ManshadiEtAl) rely heavily on the type graph, including precomputing an optimal matching on the type graph. By contrast, MinPredictedDegree only uses first-order information: it only looks at degrees and does not rely on any information about specific edges. Even so, in most cases, it outperforms all of the other online algorithms. Additionally, the (MPD) augmented versions of the known i.i.d. algorithms always beat the base algorithms and often beat the greedy (g) versions, indicating the potential of predicted degrees to be integrated with other algorithms. Note that while the standard deviations are quite wide (the known i.i.d. model is inherently stochastic), as the results are summarized over 100 trials, relatively small differences in the average performance of these algorithms are statistically significant as the standard error is small.

5 Analysis on CLV-B random graphs

Following in a long line of work in the average-case analysis of matching algorithms initiated by Karp and Sipser (1981), we analyze MinPredictedDegree under a natural random bipartite graph model we refer to as CLV-B, a bipartite version of the Chung-Yu-Vu random graph model (Chung, Lu, and Vu[2004]). A CLV-B random graph is parameterized by $n = |U|$, $m = |V|$, and a vector $d = \{d_i\}_{i=1}^n$ corresponding to the expected degrees of the offline nodes. Formally, for any $u_i \in U$ and $v_j \in V$, the edge $\{u_i, v_j\}$ appears in the graph with probability $d_i/m$. This model corresponds to the case where consumers pick their edges i.i.d. with $d$ describing the distribution over producers.

Within this model, we consider MinPredictedDegree where the degree predictions are given by the expected degrees $d$. Note that in this case, the degree predictor is noisy. In particular, for each offline node $u_i \in U$, the true degree of $u_i$ is distributed as a Binomial random variable with sample size $m$ and probability $d_i/m$. 

Figure 4: Comparison of empirical competitive ratios on Molloy-Reed model. Algorithms depicted in gray are not online algorithms (they use extra information or multiple passes). Algorithms in green make use of predicted degrees.

Figure 5: Comparison of empirical competitive ratios on Preferential Attachment graphs.

Figure 6: Comparison of empirical competitive ratios on Real World graphs. See Appendix for more results.
Table 1: Lower bound on the competitive ratio of MinPredictedDegree on CLV-B random graphs with offline expected degrees following a power law distribution as \( n, m \to \infty \). The fraction of offline nodes with expected degree \( d \) is proportional to \( d^{-\alpha}e^{-d/\lambda} \) for \( d = \{1, 2, \ldots \} \).

| CUTOFF \( \lambda \) | \( \alpha = 0.5 \) | \( \alpha = 1 \) | \( \alpha = 1.5 \) | \( \alpha = 2 \) |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| 10                  | 0.967           | 0.948           | 0.934           | 0.928           |
| 100                 | 0.998           | 0.986           | 0.958           | 0.937           |
| 1000                | 1.000           | 0.995           | 0.966           | 0.940           |
| 10000               | 1.000           | 0.997           | 0.969           | 0.940           |
| 100000              | 1.000           | 0.998           | 0.970           | 0.940           |

Our main results within this model are a set of equations that describe the size of the matching produced by MinPredictedDegree as well as the size of the maximum matching.

- Given a set of expected degrees \( d \), Equations 11 to 14 model the behavior of MinPredictedDegree on a CLV-B(\( d \)) graph. We extend these results to the asymptotic case in Appendix E, giving the expected matching size as \( n, m \to \infty \) for a given distribution of expected degrees.

- Given a set of expected degrees \( d \), Equations 12, 13, 14 give an upper bound on the expected size of the maximum matching on a CLV-B(\( d \)) graph, and in Appendix E we give the asymptotic equivalent. Empirically, we find this upper bound to be close to the maximum matching size when \( d \) follows a power law distribution.

- Using these equations, we show that in expectation MinPredictedDegree returns matchings almost as large as the maximum when the expected degrees of the offline nodes follow a power law distribution (see Figure 7 and Table 1). For both MinPredictedDegree and the maximum matching, we show that the matching sizes are concentrated about their expectations (Theorems 1, 2, 3), implying that on these graphs, MinPredicted achieves a large competitive ratio.

5.1 Analytic results

In Figure 7 and Table 1 we plot the ratio of the expected size of MinPredictedDegree’s matching to the expected size of the maximum matching, both modeled by the equations described in this section. Note that in the non-asymptotic case, our equations give an approximation to the true expectation of MinPredictedDegree’s matching size due to modeling the process with continuous differential equations. For the maximum matching analysis, our equations give an upper bound on the maximum matching size, which translates to a lower bound on our ratio (so we will only underestimate the true performance of MinPredictedDegree). For the types of graphs we consider, even for relatively small \( n \), the analytic ratios match the empirical ratios we found in Figure 3a, indicating the approximation/bound error is small for these graphs.

In Figure 7 we plot the ratio for CLV-B graphs with expected offline degrees following Zipf’s Law (at \( n = m = 1000 \), this setup corresponds to the experiment shown in Figure 3). Across all choices of the exponent other than \( \alpha = 1 \), MinPredictedDegree’s performance relative to the size of the maximum matching increases as the size of the graph grows. Further, for larger graphs, in many settings MinPredictedDegree achieves a ratio close to 1 and even for smaller graph achieves ratios above 0.9 for almost all settings.

Notably, at \( \alpha = 1 \), the ratio plateaus around 0.955, and at \( \alpha = 0.8 \), the ratio decreases from \( n = 10 \) to \( n = 100 \) before rising again as \( n \) increases. At \( \alpha = 1 \) across all values of \( n \) as well as at \( \alpha = 0.8 \) with \( n = 100 \), a large fraction of the offline nodes have expected degree close to one. Many of these nodes will have actual degree 1 and many will have actual degree \( \geq 2 \). MinPredictedDegree has no way of distinguishing between these two types of nodes as it only uses expected degrees and will mistakenly not match some offline nodes that only appear once. While there is always a discrepancy between actual and expected degrees, the issue of prioritizing a node with actual degree \( \geq 2 \) over a node with actual degree 1 is most detrimental, leading to worse performance when there are many offline nodes with expected degree close to one.

In Table 1, we plot the ratio for CLV-B graphs with expected offline degrees following a power law with exponential cutoff distribution (Borodin, Karavasilis, and Pankratov 2020; Mitzenmacher and Upfal 2005) and with \( n, m \to \infty \). For \( d = \{1, 2, \ldots \} \), the fraction of offline nodes with expected degree \( d \) is proportional to \( d^{-\alpha}e^{-d/\lambda} \) for exponent \( \alpha \) and cutoff \( \lambda \). Note that in the asymptotic case, as the sizes of MinPredictedDegree’s matching and the maximum matching are concentrated about their expectations (Theorems 1, 2, 3), the ratio of expectations is equivalent to the competitive ratio (expectation of ratio). When the exponent is small or the cutoff is large, MinPredictedDegree achieves a better competitive ratio, with the ratio exceeding 0.99 when both occur. When \( \alpha = 2 \), while MinPredictedDegree still achieves a competitive ratio of up to 0.94, the competitive ratio is not as affected by a larger cutoff as with smaller exponents (the power law factor is already significantly limiting the fraction of offline nodes with large expected degree).

In the rest of this section, we develop the equations that allowed us to produce these results. The analysis we present is general and can be used to evaluate MinPredictedDegree on CLV-B graphs with different parameters than those we have considered.
5.2 Analyzing MinPredictedDegree

In order to analyze MinPredictedDegree, we construct a family of random variables and corresponding differential equations that model the number of unmatched left nodes throughout the running of the algorithm. Let $Y_d^t$ be the number of offline nodes with expected degree $d$ who are unmatched by MinPredictedDegree after seeing the $t$th online node. Within this random graph model, $\{Y_d^t\}_{t=1}^m$ form a Markov chain with the following expected evolution:

$$E[Y_d^t+1 − Y_d^t] = − \left(1 − (1 − d/m)Y_d^t\right) \prod_{d′ < d} (1 − d′/m)Y_d^{t}. \quad (1)$$

The first term corresponds to the probability that at least one unmatched offline node with expected degree $d$ is incident on the $(t + 1)$st online node while the second term corresponds to the probability that this online node has no neighboring unmatched offline nodes with smaller expected degree (which would be prioritized).

Let $k_d = −\log(1−d/m)$. To simplify the analysis of MinPredictedDegree, it will be helpful to consider the random variables $Z_d^t = −k_d \ast Y_d^t$ where

$$E[Z_d^{t+1} − Z_d^t] = k_d \left(1 − e^{Z_d^t}\right) \prod_{d′ < d} e^{Z_d^{t}}. \quad (2)$$

Following the work of Kurtz and many subsequent researchers (Kurtz 1981; Wormald 1995; Mitzenmacher 1997; Luby et al. 2001), we show that the behavior of MinPredictedDegree as described by these Markov chains is well approximated by the trajectory of the following system of differential equations for all unique expected degrees $d$ in $d$:

$$\frac{dz_d(t)}{dt} = k_d \left(1 − e^{z_d(t)}\right) \prod_{d′ < d} e^{z_d(t)}. \quad (3)$$

These functions $z_d(t)$ represent continuous-time approximations of the Markov chains with their derivatives corresponding to expected change from Equation 2. In Appendix B we give the solution to these differential equations.

Relying on past work (Luby et al. 2001), we give the following theorem (see Appendix C for proof).

**Theorem 1.** Let $G$ be a CLV-B random graph with unique expected offline degrees $\{\delta_i\}_{i=1}^\ell$. Let $f_d = \lambda_d \cdot n$ be the number of offline nodes with expected degree $d$. Then, the expected (over the randomness in $G$) size of the matching formed by MinPredictedDegree approaches

$$\sum_{i=1}^\ell f_{\delta_i} + z_{\delta_i}(m)/k \quad (4)$$

as $n = m$ approach infinity, where $z_{\delta_i}(t)$ for $i \in \{1, \ldots, \ell\}$ form the solution to the system of differential equations in Equation 3. In addition, if $Y_d^t$ is the true number of unmatched offline nodes of expected degree $d$, then there exists a constant $c$ s.t.

$$\mathbb{P}(Y_d > −z_{\delta}(m)/k + cm^{5/6}) < \ell m^{2/3} \exp(−m^{1/3}/2). \quad (5)$$

The solution to the system of differential equations gives us a closed form continuous-time approximation for expected performance of MinPredictedDegree in terms of $d$. In particular, in the asymptotic case, the equations give the exact expected performance and in the non-asymptotic case give an approximation on the number of unmatched offline nodes (and thus the matching size).

In Appendix B we additionally show that MinPredictedDegree’s matching size is concentrated about its expectation.

**Theorem 2.** Let $G$ be a CLV-B random graph with expected offline degrees $d$ and let $X$ be the random variable corresponding to the size of the matching returned by MinPredictedDegree. Then,

$$\mathbb{P}(|X − \mathbb{E}[X]| ≥ 2\sqrt{m \log m}) ≤ \frac{2}{m}. \quad (6)$$

5.3 Analyzing the maximum matching

To analyze the maximum matching size within this model, we rely on an upper bound based on the matching version of Hall’s marriage theorem (Hall 1935). We first state the classic theorem.

**Theorem 3 (Hall’s Theorem).** Let $G = (U \cup V, E)$ be a bipartite graph. For any subset of nodes $S$, let $N(S)$ be the set of neighbors of the nodes in $X$. Then, $G$ has a perfect matching if and only if for all $S \subset U$ and $T \subset V$, $|S| ≤ |N(S)|$ and $|T| ≤ |N(T)|$.

Intuitively, if there is any subset $S$ with relatively few neighbors, then only $|N(S)|$ of the members of $S$ can possibly be matched. Let $\mu(G)$ correspond to the size of the maximum matching in $G$. For any bipartite graph $G = (U \cup V, E)$ and $S \subset U$,

$$\mu(G) ≤ n − (|S| − |N(S)|) \quad (7)$$

as out of all of the nodes in $S$, only $|N(S)|$ can be matched. Therefore, if we can calculate the expected size of $|S|$ and $|N(S)|$ for some subset of a random CLV-B graph $G$, we immediately get an upper bound on the expected size of the maximum matching in $G$.

Our upper bound, detailed in Appendix D involves constructing a specific subset $S^*$ of the offline nodes that makes use of our focus on power law graphs to provide a useful bound that is easy to evaluate. We empirically test how good of a bound Equation 7 gives using $S^*$ and find that for CLV-B random graphs with offline degrees following a power law distribution, the upper bound on $\mu(G)$ given by $n − (|S^*| − |N(S^*)|)$ is close to maximum matching size (often achieving the same value and in all trials was less than 2% greater than the true value). In addition to providing a good bound, we show that we can evaluate $\mathbb{E}[|S|]$ and $\mathbb{E}[|N(S)|]$ in Equations 12 and 13. By linearity of expectation, this directly gives us an upper bound on the expected size of the maximum matching.

In Appendix A we show that on CLV-B random graphs with power law distributed degrees, our bound on the maximum matching size is concentrated about its expectation. Combined with Theorem 2, this implies that our analytic results on the ratio of the expected sizes are closely related to the competitive ratio.
Acknowledgements

This research was supported in part by the NSF TRIPODS program (awards CCF-1740751 and DMS-2022448), NSF award CCF-2006798, Simons Investigator Award, NSF Graduate Research Fellowship under Grant No. 1745302, and MathWorks Engineering Fellowship.

References

2016. Special Semester on Algorithms and Uncertainty. [https://simons.berkeley.edu/programs/uncertainty2016]

Anand, K.; Ge, R.; and Panigrahi, D. 2020. Customizing ML Predictions for Online Algorithms. In International Conference on Machine Learning, 303–313. PMLR.

Angelopoulos, S.; Dürr, C.; Jin, S.; Kamali, S.; and Renault, M. 2020. Online Computation with Untrusted Advice. In 11th Innovations in Theoretical Computer Science Conference (ITCS 2020).

Antoniadis, A.; Gouleakis, T.; Kleer, P.; and Kolev, P. 2020. Secretary and Online Matching Problems with Machine Learned Advice. In 34th Conference on Neural Information Processing Systems.

Balcan, M.-F.; Dick, T.; Sandholm, T.; and Vitercik, E. 2018. Learning to Branch. In International Conference on Machine Learning.

Borodin, A.; Karavasilis, C.; and Pankratov, D. 2020. An Experimental Study of Algorithms for Online Bipartite Matching. Journal of Experimental Algorithmics (JEA), 25: 1–37.

Borodin, A.; Pankratov, D.; and Salehi-Abari, A. 2019. On conceptually simple algorithms for variants of online bipartite matching. Theory of Computing Systems, 63(8): 1781–1818.

Chung, F.; Lu, L.; and Vu, V. 2004. The spectra of random graphs with given expected degrees. Internet Mathematics, 1(3): 257–275.

Dai, H.; Khalil, E.; Zhang, Y.; Dilkina, B.; and Song, L. 2017. Learning combinatorial optimization algorithms over graphs. In Advances in Neural Information Processing Systems, 6351–6361.

Dong, Y.; Indyk, P.; Razenshteyn, I. P.; and Wagner, T. 2020. Learning Space Partitions for Nearest Neighbor Search. ICLR.

Eden, T.; Indyk, P.; Narayanan, S.; Rubinfeld, R.; Silwal, S.; and Wagner, T. 2021. Learning-based Support Estimation in Sublinear Time. In International Conference on Learning Representations.

Feldman, J.; Mehta, A.; Mirrokni, V.; and Muthukrishnan, S. 2009. Online stochastic matching: Beating 1-1/e. In 2009 50th Annual IEEE Symposium on Foundations of Computer Science, 117–126. IEEE.

Goel, G.; and Mehta, A. 2008. Online budgeted matching in random input models with applications to Adwords. In SODA, volume 8, 982–991.

Gollapudi, S.; and Panigrahi, D. 2019. Online algorithms for rent-or-buy with expert advice. In International Conference on Machine Learning, 2319–2327. PMLR.

Hall, P. 1935. On Representatives of Subsets. Journal of The London Mathematical Society-second Series, 26–30.

Halldórsson, M. M.; and Radhakrishnan, J. 1997. Greed is good: Approximating independent sets in sparse and bounded-degree graphs. Algorithmica, 18(1): 145–163.

Hsu, C.-Y.; Indyk, P.; Katabi, D.; and Vakilian, A. 2019. Learning-Based Frequency Estimation Algorithms. In International Conference on Learning Representations.

Jegou, H.; Douze, M.; and Schmid, C. 2011. Product quantization for nearest neighbor search. IEEE transactions on pattern analysis and machine intelligence, 33(1): 117–128.

Jiang, T.; Li, Y.; Lin, H.; Ruan, Y.; and Woodruff, D. P. 2020. Learning-augmented data stream algorithms. ICLR.

Jiang, Z.; Panigrahi, D.; and Sun, K. 2020. Online Algorithms for Weighted Paging with Predictions. In 47th International Colloquium on Automata, Languages, and Programming (ICALP 2020). Schloss Dagstuhl-Leibniz-Zentrum für Informatik.

Karp, R.; and Sipser, M. 1981. Maximum Matchings in Sparse Random Graphs. In FOCS 1981.

Karp, R. M.; Vazirani, U. V.; and Vazirani, V. V. 1990. An optimal algorithm for on-line bipartite matching. In Proceedings of the twenty-second annual ACM symposium on Theory of computing, 352–358.

Kraska, T.; Beutel, A.; Chi, E. H.; Dean, J.; and Polyzotis, N. 2018. The case for learned index structures. In Proceedings of the 2018 International Conference on Management of Data (SIGMOD), 489–504. ACM.

Kumar, R.; Purohit, M.; Schild, A.; Svitkina, Z.; and Vee, E. 2019. Semi-Online Bipartite Matching. 10th Innovations in Theoretical Computer Science.

Kurtz, T. 1981. 8. Approximations of Density Dependent Jump Markov Processes, 51–61.

Lattanzi, S.; Lavastida, T.; Moseley, B.; and Vassilvitskii, S. 2020. Online scheduling via learned weights. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 1859–1877. SIAM.

Lavastida, T.; Moseley, B.; Ravi, R.; and Xu, C. 2020. Learnable and Instance-Robust Predictions for Online Matching, Flows and Load Balancing. arXiv preprint arXiv:2011.11743.

Leskovec, J.; and Krevl, A. 2014. SNAP Datasets: Stanford Large Network Dataset Collection. [http://snap.stanford.edu/data].

Luby, M.; Mitzenmacher, M.; Shokrollahi, M. A.; and Spielman, D. 2001. Efficient erasure correcting codes. IEEE Transactions on Information Theory, 47(2): 569–584.

Lykouris, T.; and Vassilvitskii, S. 2018. Competitive caching with machine learned advice. In International Conference on Machine Learning, 3296–3305. PMLR.

Mehta, A. 2013. Online Matching and Ad Allocation. Foundations and Trends® in Theoretical Computer Science, 8(4): 265–368.

Meka, R.; Jain, P.; and Dhillon, I. 2009. Matrix Completion from Power-Law Distributed Samples. In Advances in Neural Information Processing Systems, volume 22, 1258–1266. Curran Associates, Inc.
Mitzenmacher, M. 1997. Studying Balanced Allocations with Differential Equations. *Combinatorics, Probability, and Computing*, 8: 473–482.

Mitzenmacher, M. 2018. A Model for Learned Bloom Filters and Optimizing by Sandwiching. In *Advances in Neural Information Processing Systems*.

Mitzenmacher, M. 2020. Scheduling with Predictions and the Price of Misprediction. In *ITCS*.

Mitzenmacher, M.; and Upfal, E. 2005. *Probability and Computing*.

Mitzenmacher, M.; and Vassilvitskii, S. 2020. Algorithms with predictions. arXiv preprint arXiv:2006.09123.

Purohit, M.; Svitkina, Z.; and Kumar, R. 2018. Improving online algorithms via ml predictions. In *Advances in Neural Information Processing Systems*, 9661–9670.

Rohatgi, D. 2020. Near-optimal bounds for online caching with machine learned advice. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1834–1845. SIAM.

Rossi, R. A.; and Ahmed, N. K. 2015. The Network Data Repository with Interactive Graph Analytics and Visualization. In *AAAI*.

Salakhutdinov, R.; and Hinton, G. 2009. Semantic hashing. *International Journal of Approximate Reasoning*, 50(7): 969–978.

Tinhofer, G. 1984. A probabilistic analysis of some greedy cardinality matching algorithms. *Annals of Operations Research*, 1(3): 239–254.

Wang, J.; Liu, W.; Kumar, S.; and Chang, S.-F. 2016. Learning to hash for indexing big data - a survey. *Proceedings of the IEEE*, 104(1): 34–57.

Wei, A. 2020. Better and Simpler Learning-Augmented Online Caching. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik.

Weiss, Y.; Torralba, A.; and Fergus, R. 2009. Spectral hashing. In *Advances in neural information processing systems*, 1753–1760.

Wormald, N. C. 1995. Differential Equations for Random Processes and Random Graphs. *Annals of Applied Probability*, 5(4): 1217–1235.
A Worst-Case Bound and Failure Modes

In the worst-case, MinPredictedDegree achieves a competitive ratio of 1/2. When $σ$ gives arbitrary predictions, MinPredictedDegree is equivalent to the simple greedy algorithm for online bipartite matching which achieves a competitive ratio of 1/2 [Karp, Vazirani, and Vazirani 1990; Mehta 2013]. Even if $σ$ is the perfect degree predictor where $σ(u) = deg(u)$ for all $u ∈ U$, MinPredictedDegree is still (1/2)-competitive in the worst-case.

As MinPredictedDegree forms a maximal matching, the matching it returns is always at least half the size of the maximum matching. For the matching upper bound on the competitive ratio in the perfect predictor case, consider the graph $G = (U ∪ V, E)$ where $n = m = 6$ with the following adjacency list:

- $\{u_1 : v_1, v_2, v_3\}$
- $\{u_2 : v_1, v_2, v_3\}$
- $\{u_3 : v_1, v_2, v_3\}$
- $\{u_4 : v_1, v_4\}$
- $\{u_5 : v_2, v_5\}$
- $\{u_6 : v_3, v_6\}$

The first half of the offline and online nodes form a complete bipartite subgraph while the second half of the offline nodes each connect to one online node in the first half and one online node in the second half. MinPredictedDegree with a perfect degree predictor will return the matching $M = \{\{u_4, v_1\}, \{u_5, v_2\}, \{u_6, v_3\}\}$, matching the first half of the online nodes with their lower degree neighbors in the second half of the offline nodes, therefore leaving the first half of the offline nodes unmatched.

More generally, MinPredictedDegree can perform poorly if the degree predictor is arbitrary or if high degree nodes have poor edge expansion compared to low degree nodes (making it disadvantageous to always prioritize low degree nodes as seen in the example above). However, often these adversarial structures do not appear in practice and matching low degree nodes first leads to better results. In fact, the hard instance given in the seminal paper by Karp, Vazirani, and Vazirani [Karp, Vazirani, and Vazirani 1990] that introduced the online bipartite matching problem and the Ranking algorithm relies on the fact that algorithms with no extra information on the graph (e.g. no degree predictions) must often mistakenly match high degree left nodes rather than low degree left nodes when given a choice.

B Solution to System of Differential Equations in Equation 3

Recall the system of differential equations from Equation 3:

$$\frac{dz_d(t)}{dt} = k_d \left(1 - e^{z_d(t)}\right) \prod_{d' < d} e^{z_{d'}(t)} \quad (8)$$

for all unique expected degrees $d$ in $d$. The solution to the system of differential equations is given by the following equations. Let $\{z_d\}_{i=1}^l$ be the ordered set of unique expected degrees and let $f_d$ be the number of offline nodes with expected degree $d$. We will define the auxiliary functions $α_δ(t)$ for $i ∈ \{2, \ldots, ℓ\}$ and variables $C_δ$ for $i = \{1, \ldots, ℓ\}$ as follows:

$$α_δ(t) = C_δ + e^{k_δ t}$$

$$α_δ(t) = (α_{δ-1}(t))^{k_{δ-1}/k_{δ-2}} + C_{δ-1} \quad (for i ≥ 3)$$

where

$$C_δ = e^{k_δ f_δ - 1}$$

$$C_δ = (α_δ(0))^{k_δ/k_{δ-1}}(e^{k_δ f_δ - 1}) \quad (for i ≥ 2).$$

Then,

$$z_δ(t) = -\log(C_δ e^{-k_δ t} + 1)$$

$$z_δ(t) = -\log(C_δ (α_δ(t))^{-k_δ/k_{δ-1}} + 1) \quad (for i ≥ 2).$$

(9)

In the rest of this section, we show that Equation 9 give the correct solutions to the differential equations in Equation 3.

Lemma 1. For $i ∈ \{2, 3, \ldots, ℓ\}$,

$$\frac{d}{dt}(α_δ(t)) = \prod_{j=1}^{i-1} e^{z_j(t)} \quad (10)$$

Proof. We will prove the lemma by induction. Consider the base case of $i = 2$:

$$\frac{d}{dt}(α_δ(t)) = \frac{k_δ e^{k_δ t}}{C_δ} = \frac{1}{C_δ} e^{k_δ t}$$

Now consider the inductive case of $i > 2$ under the assumption that

$$\prod_{i=1}^{i-1} e^{z_j(t)} = \prod_{j=1}^{i-2} e^{z_j(t)}$$

Then,

$$\frac{d}{dt}(α_δ(t)) = \frac{k_δ (α_δ(t))^{-k_δ/k_{δ-1}} - 1}{C_δ} \prod_{j=1}^{i-2} e^{z_j(t)}$$

This completes the proof.

Lemma 2. The expressions for $z_δ(t)$ in Equation 9 give a solution to system of differential equations in Equation 3 with initial conditions $z_δ(0) = -k_δ f_δ$. □
We give the proof of Theorem 1 which states that the solution of the differential equations gives the asymptotic expected behavior nonpositive as $z$ is the history up to time $t$ is bounded, (ii) that the proof of Theorem 1 follows a direct proof of Theorem 1.\hspace{1cm}

\textbf{Proof.} We will split the proof into two cases for $\delta_i$ and for $\delta_i$ with $i \geq 2$. Starting with $i = 1$, recall that

$$z_{\delta_i}(t) = -\log(C_{\delta_i} e^{-k\delta_i t} + 1).$$

First, we will show that this function has the correct derivative.

$$\frac{dz_{\delta_i}(t)}{dt} = -\frac{1}{C_{\delta_i} e^{-k\delta_i t} + 1} (C_{\delta_i}(-k\delta_i e^{-k\delta_i t})$$

$$= k\delta_i \frac{C_{\delta_i} e^{-k\delta_i t}}{C_{\delta_i} e^{-k\delta_i t} + 1} = k\delta_i (1 - e^{-z_{\delta_i}(t)})$$

It remains to be shown that $z_{\delta_i}(0) = -k\delta_i f_{\delta_i}$:

$$z_{\delta_i}(0) = -\log(C_{\delta_i} + 1) = -\log(e^{k\delta_i f_{\delta_i}}) = -k\delta_i f_{\delta_i}.$$ 

Now, consider the case where $i \geq 2$. Then

$$\frac{dz_{\delta_i}(t)}{dt} = \frac{d}{dt} \left( -\log(C_{\delta_i}(\alpha_{\delta_i}(t)^{-k\delta_i/k\delta_i - 1}) + 1) \right)$$

$$= -\frac{k\delta_i}{C_{\delta_i}(\alpha_{\delta_i}(t)^{-k\delta_i/k\delta_i - 1} + 1} \frac{d}{dt} (\alpha_{\delta_i}(t))$$

$$= k\delta_i (1 - e^{-z_{\delta_i}(t)}) \prod_{j=1}^{i-1} e^{z_{\delta_j}(t)}.$$ 

The last step makes use of Lemma 1. Finally, we must show that $z_{\delta_i}(0) = -k\delta_i f_{\delta_i}$:

$$z_{\delta_i}(0) = -\log(C_{\delta_i} \alpha_{\delta_i}(0)^{-k\delta_i/k\delta_i - 1} + 1)$$

$$= -\log(e^{k\delta_i f_{\delta_i} - 1 + 1}) = -k\delta_i f_{\delta_i}.$$ 

Thus, the given solution to the system of differential equations is correct. \hspace{1cm}

\textbf{C Proof of Theorem 1} \hspace{1cm}

We give the proof of Theorem 1 which states that the solution to the differential equations models the size of the matching returned by MinPredictedDegree.

\textbf{Proof of Theorem 1} The proof of Theorem 1 follows a direct application of Theorem 1 in Luby et al. [Luby et al., 2001]. We must show three conditions are satisfied: (i) if $|Z_d| - |Z_d'|$ is bounded, (ii) that $\frac{dz(t)}{dt} = \mathbb{E}[Z_d'^{t+1} - Z_d'|H_t]$ (where $H_t$ is the history up to time $t$, and (iii) that $\frac{dz(t)}{dt}$ satisfies a Lipschitz condition when $z_d(t) \leq 0$ (recall that $Z_d'$ is always nonpositive as $Y_d'$ is always nonnegative).

If these conditions hold, then the solution to the system of differential equations gives the asymptotic expected behavior of the variables $Z_d'$ and thus the asymptotic expected behavior of the variables $Y_d'$, which govern the size of the matching returned by MinPredictedDegree. In addition, we directly get the desired concentration bound which tells us in the non-asymptotic case that $Y_d'$ does not deviate too far from the predicted behavior from the differential equations:

$$\mathbb{P}(Y_d > -z_{\delta}(m)/k + cm^{5/6}) < \epsilon m^{2/3} \exp(-m^{1/3}/2).$$

(11)

It remains to show that these three conditions are met. Condition (i) is satisfied as the number of nodes of a given expected degree can change by at most one timestep, so $|Z_d^{t+1} - Z_d'| \leq k_d$. Condition (ii) is satisfied by construction in Equation 3. Finally, Condition (iii) is satisfied as $\frac{dz(t)}{dt}$ is comprised of a product of several terms resembling $C_1 e^{-C_2 x}$ for nonnegative constants $C_1, C_2$ and with $x$ nonnegative. Therefore, $\frac{dz(t)}{dt}$ has constant bounded first derivatives when $z_d(t) \leq 0$. \hspace{1cm}

\textbf{D Upper Bound on Expected Maximum Matching Size} \hspace{1cm}

As described in Section 5, our analysis of the maximum matching size on CLV-B random graphs works by constructing a subset $S$ of the nodes and evaluating $\mathbb{E}[|S|]$ and $\mathbb{E}[|N(S)|]$, which directly gives us an upper bound on the expected maximum matching size by Equation 7. Let $S^*$ be the subset of $U$ constructed as follows. Let $U_i$ be the set of degree 1 nodes in $U$ (here degree 1 referring to the actual degree of the node rather than the expected degree in the CLV-B model). Then, $S^*$ is the maximal set of nodes in $U$ s.t. $N(S^*) \subseteq N(U_i)$. In other words, $S^*$ is the maximal subset of nodes in $U$ whose neighbors completely overlap with the neighbors of the degree 1 nodes of $U$.

As $n - |S^*| + |N(S^*)|$ gives an upper bound on the maximum matching size, $\mathbb{E}[n - |S^*| + |N(S^*)|]$ gives an upper bound on the expected maximum matching size. The expected sizes of $S^*$ and $|N(S^*)|$ in a CLV-B random graph with expected degrees $d$ are given by the following equations. The expected size of $N(S^*)$ is simply the sum over all online nodes $v \in V$ of the probability that $v$ has at least one degree 1 neighbor. The expected size of $S^*$ is broken down as the sum over all offline nodes $u \in U$ of the probability that $u$ has actual degree $\Delta$ and then the probability that all $\Delta$ of $u$’s neighbors are members of $N(S^*)$. Let $S^*_\Delta$ be the subset of nodes in $S^*$ whose actual (as opposed to expected) degree are $\Delta$ and let $\beta_{\Delta}^0$ for $\Delta \in \{0, \ldots, m\}$ and $i \in \{1, \ldots, n\}$ be defined as

$$\beta_{\Delta}^0 = \beta_{\Delta}^1 = 1$$

$$\beta_{\Delta}^i = 1 + \sum_{r=1}^{\Delta} (-1)^r \left( \begin{array}{c} \Delta \\ r \end{array} \right) \prod_{\nu \neq i} \left[ 1 - r \left( \frac{d_{\nu}}{m} \right) \left( 1 - \frac{d_{\Delta}}{m} \right) \right] m^{-1}$$

(12)

(12)

$$\beta_{\Delta}^i$$ represents the probability of $u_i \in S$ conditioned on $u_i$ having actual degree $\Delta$. Then,

$$\mathbb{E}[|N(S^*)|] = m \left( 1 - \prod_{i=1}^{n} \left[ 1 - \frac{d_i}{m} \left( 1 - \frac{d_i}{m} \right) m^{-1} \right] \right)$$

This expression bounds the expected maximum matching size with high probability.
and
\[
\mathbb{E}[|S^*|] = \sum_{\Delta = 0}^{m} \mathbb{E}[|S^*_\Delta|] \tag{13}
\]
where
\[
\mathbb{E}[|S^*_\Delta|] = \sum_{i=1}^{n} \binom{m}{\Delta} \left( \frac{d_i}{m} \right)^\Delta \left( 1 - \frac{d_i}{m} \right)^{m-\Delta} \beta_i^\Delta. \tag{14}
\]

In the rest of this section, we show that the equations for the expected size of \( |S^*| \) and \(|N(S^*)|\) are correct by showing their derivations.

First, consider \( \mathbb{E}[|N(S^*)|] \). Recall that \( N(S^*) \) is the set of online nodes that have a neighbor with actual degree 1. Therefore, the expected size of \( N(S^*) \) is \( m \) minus the expected number of online nodes that have no degree one neighbors. For any online node \( u \in V \), the probability that \( u \) has no degree 1 neighbors is
\[
\prod_{u \in U} \mathbb{P}(u \text{ is not a deg 1 nbr of } v) = \prod_{u \in U} \left[ 1 - \mathbb{P}(u \text{ nbr of } v) \mathbb{P}(u \text{ has no other nbrs}) \right]
= \prod_{u \in U} \left[ 1 - \frac{d_u}{m} \left( 1 - \frac{d_u}{m} \right)^{m-1} \right].
\]

By linearity of expectation,
\[
\mathbb{E}[|N(S^*)|] = m \prod_{u \in U} \left[ 1 - \frac{d_u}{m} \left( 1 - \frac{d_u}{m} \right)^{m-1} \right].
\]

Now, we will deal with \( \mathbb{E}[|S^*_\Delta|] \). Recall that \( S^*_\Delta \) is the set of offline nodes with actual degree \( \Delta \) with all of their online neighbors having at least one offline neighbor with actual degree 1. For a given offline node \( u \in U \) with expected degree \( d_u \), the probability that \( u \) is in \( S^*_\Delta \) is the product of the probability of \( u \) having actual degree \( \Delta \) and the conditional probability of all of \( u \)'s neighbors having a degree 1 neighbor given \( u \) having actual degree \( \Delta \). We will call the first event \( A_{u,\Delta} \) and the second, conditional event \( B_{u,\Delta} \) \( |A_{u,\Delta}| \).

The probability of \( A_{u,\Delta} \) occurring corresponds to a Binomial random variable with size parameter \( m \) and probability parameter \( d_u/m \) taking on value \( \Delta \):
\[
\mathbb{P}(A_{u,\Delta}) = \binom{m}{\Delta} \left( \frac{d_u}{m} \right)^\Delta \left( 1 - \frac{d_u}{m} \right)^{m-\Delta}.
\]

The probability of \( B_{u,\Delta} \| A_{u,\Delta} \) equals 1 if \( \Delta = 0 \) or \( \Delta = 1 \) as either \( u \) has no neighbors or \( u \) is itself a degree 1 neighbor of its neighbors, respectively. If \( \Delta \geq 2 \), then \( \mathbb{P}(B_{u,\Delta} \| A_{u,\Delta}) \) is equal to the complement of the event that at least one of \( u \)'s neighbors has no degree 1 neighbor. Let \( C_{u,\Delta,r} \| A_{u,\Delta} \) be the event that any subset of \( r \) of \( u \)'s neighbors have no degree 1 neighbor given \( A_{u,\Delta} \). \( \mathbb{P}(C_{u,\Delta,r} \| A_{u,\Delta}) \) can be expressed as
\[
\Delta \prod_{u' \neq u} \left[ 1 - r \left( \frac{d_u}{m} \right) \left( 1 - \frac{d_u}{m} \right)^{m-1} \right].
\]

where the term within the product represents the probability of no offline nodes (excluding \( u \)) being degree one neighbors of a specific set of \( r \) online nodes (similarly to when expressing \( \mathbb{E}[|N(S^*)|] \) above). By the inclusion-exclusion rule,
\[
\mathbb{P}(B_{u,\Delta} \| A_{u,\Delta}) = 1 + \sum_{r=1}^{\Delta} (-1)^r \mathbb{P}(C_{u,\Delta,r} \| A_{u,\Delta}),
\]
thus completing the derivation of Equations [12] and [14].

### E Analysis on CLV-B random graphs in asymptotic case

In this section, we give slight modifications of the Equation [9] and Equations [12] [13] [14] in the case where \( n, m \to \infty \) to allow us to evaluate the equations to produce the results in Table [1]. The model will change slightly when considering the asymptotic case: we will describe the set of offline expected degrees \( d \) by a set of unique degrees \( \{\delta_i\} \) and corresponding fractions \( \{\lambda_i\} \) where a \( \lambda_i \) fraction of the offline nodes have expected degree \( \delta_i \).

Importantly for the asymptotic results in Table [1] while there are offline nodes with expected degree approaching infinity, a finite number of unique expected degrees account for all but an exponentially small fraction of the offline nodes, allowing us to evaluate the equations up to negligible error. In the following calculations, we will consider both \( \ell \) as well as all \( \delta_i \) for \( \ell = \{1, \ldots, \ell\} \) to be finite.

#### E.1 Asymptotic analysis of MinPredictedDegree

To start, we will replace \( f_d \) with \( m \cdot \lambda_d \) and we will replace \( t \) with \( \tau = t/m \). Recall \( k_d = -\log(1 - d/m) \). The Taylor expansion of \( \log(1 - x) \) at \( x = 0 \) is \(-\sum_{n=1}^{\infty} \frac{x^n}{n} \). Within Equation [9], \( k_d \) appears in terms \( k_d/k_d' \) and \( k_d \cdot f_d = k_d \cdot m \cdot \lambda_d \). In the asymptotic case, we will use the following substitutions for those terms:
\[
\lim_{m \to \infty} \frac{k_d}{k_d'} = d/d',
\]
and
\[
\lim_{m \to \infty} k_d \cdot m \cdot \lambda_d = -d \cdot \lambda_d.
\]

In both cases, we use the fact that as \( m \to \infty \), the first term in the Taylor series \( (d/m) \) dominates.

Using these substitutions, we can rewrite the equations for MinPredictedDegree as follows.
\[
\alpha \delta_i(\tau) = C_{\delta_i} + e^{\delta_i \tau}
\]
\[
\alpha \delta_i(\tau) = (\alpha \delta_{i-1}(\tau))^{\delta_{i-1}/\delta_{i-2}} + C_{\delta_{i-1}} \quad (\text{for } i \geq 3)
\]

where
\[
C_{\delta_i} = e^{\delta_i \lambda_{i-1}} - 1
\]
\[
C_{\delta_i} = (\alpha \delta_{i}(0))^{\delta_i/\delta_{i-1}} (e^{\delta_i \lambda_{i-1}} - 1) \quad (\text{for } i \geq 2).
\]

Then,
\[
\alpha \delta_i(\tau) = -\log(C_{\delta_i} e^{-\delta_i \tau} + 1)
\]
\[
\alpha \delta_i(\tau) = -\log(C_{\delta_i} (\alpha \delta_i(\tau))^{-\delta_i/\delta_{i-1}} + 1) \quad (\text{for } i \geq 2).
\]
Recall that the expected number of offline nodes with expected degree $d$ is given by $-z_d(\tau) / b_d$, evaluated when $t = m$. Then, in the asymptotic case, the expected fraction of offline nodes matched is

$$\sum_{i=1}^{\ell} \lambda_i + z_i(1)/\delta_i$$  \hspace{1cm} (16)

where $z_i(\tau)$ are given by Equation [15]

### E.2 Asymptotic analysis of maximum matching

For the equations for the upper bound on the expected maximum matching size, the key fact we will use is $\lim_{x \to 0} (1 + x) e^{-d} = 1$. Therefore, we can replace all terms of $(1 - \frac{d}{m})^{m-C}$ with $e^{-d}$. In addition, we can replace all terms of $(\frac{m}{C}) (\frac{d}{m})^C$ with $\frac{d^C}{C!}$. These substitutions give the following equations.

$$\beta_0 = \beta_1 = 1$$

$$\beta^\Delta = 1 - \sum_{r=1}^{\Delta} (-1)^r \left( \frac{\Delta}{r} \right) \prod_{i=1}^{t} \left[ 1 - p \left( \frac{\delta_i}{m} \right) e^{-\delta_i} \right] e^{m \lambda_i}$$

Note that the $\beta^\Delta$ terms are no longer indexed by $i$ as conditioning on the actual degree of a single node makes no difference on the probability in the asymptotic case. Then,

$$\mathbb{E} \left[ \frac{|N(S^*)|}{m} \right] = \left( 1 - \sum_{i=1}^{\ell} e^{-\delta_i} \lambda_i e^{-\delta_i} \right)$$  \hspace{1cm} (17)

and

$$\mathbb{E} \left[ \frac{|S^*|}{m} \right] \geq \sum_{\Delta=0}^{C} \mathbb{E} \left[ \frac{|S^\Delta|}{m} \right]$$  \hspace{1cm} (18)

where

$$\mathbb{E} \left[ \frac{|S^\Delta|}{m} \right] = \sum_{i=1}^{\ell} \lambda_i \left( \frac{\delta_i}{\Delta} \right) e^{-\delta_i} \beta^\Delta_i$$  \hspace{1cm} (19)

### F Concentration of MinPredictedDegree

In this section, we prove Theorem 2, showing that MinPredictedDegree’s performance on CLV-B random graphs is concentrated about its expectation.

**Proof.** Let $H_j$ represent the state of MinPredictedDegree after it has processed the $j$th online node, and let $Y_j = \mathbb{E}[X | H_j]$ be the expectation of the size of the returned matching conditioned on the history of the algorithm up to time $j$. Then $\{Y_j\}_{j=0}^{m}$ form a Doob martingale. We will proceed by bounding $|Y_j - Y_{j-1}|$.

If an offline node $i$ was matched with online node $j$, then the conditional expectation of the final matching size increases by $1 - \mathbb{P}(i \text{ matched } H_{j-1})$. For each unmatched offline node that is not matched with online node $j$, the conditional expectation decreases by the sum over all such nodes $i'$ of $\mathbb{P}(i' \text{ matched } j | H_{j-1})$. As both the increment and decrement are bounded in magnitude by 1, the martingale has bounded differences $|Y_j - Y_{j-1}| \leq 1$.

Applying the standard Azuma’s inequality bounds, we get the concentration result:

$$\mathbb{P}(|Y_m - Y_0| \geq 2\sqrt{m \log m}) \leq \frac{2}{m}.$$  \hspace{1cm} (20)

As $Y_0 = \mathbb{E}[X]$ and $Y_m = X$, this completes the proof. \qed

### G Concentration of the Upper Bound on Maximum Matching Size

In this section, we show that our upper bound on the size of a maximum matching in CLV-B random graphs with power law distributed degrees is concentrated about its expectation.

**Theorem 4.** Let $G$ be a CLV-B random graph with $n = m$ and with expected offline degrees following a power law distribution with exponent $\alpha > 3$, and let $X$ be the random variable corresponding to the difference $|S^*| - |N(S^*)|$ where $S^*$ and $N(S^*)$ are the subsets of the nodes in $G$ described in Section 2. Then, there exists some constant $C$ s.t.

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq C \sqrt{n \log n}) \leq \frac{1}{n}.$$  \hspace{1cm} (20)

**Proof of Theorem 2.** Let $H_t$ represent the $t$ offline nodes with the smallest degrees (ties broken arbitrarily) as well as their incident edges and let $Y_t = \mathbb{E}[|S^*| - |N(S^*)| | H_t]$ be the expected difference in the sizes of $S^*$ and $N(S^*)$ given knowledge of $H_t$. Note that here we are using true degrees and not expected degrees. Then, $\{Y_t\}_{t=0}^{n}$ form a Doob martingale.

Let $u_t$ be the offline node with the $t$th smallest degree and let $\text{deg}(u_t)$ be its degree. We will proceed by cases to show that the martingale has bounded differences.

1. Assume $\text{deg}(u_t) > 1$. From $H_{t-1}$ we know $N(S^*)$ as all degree one nodes have already been seen. Therefore, the contribution of $u_t$ to the difference $|S^*| - |N(S^*)|$ is independent of any subsequent offline nodes. Specifically, if $u_t \in S^*$, $Y_{t-1} = Y_t$; if $u_t \notin S^*$, $Y_{t-1} = Y_t$. If $u_t \notin S^*$, $Y_{t-1} = Y_t$.

2. Assume $\text{deg}(u_t) \leq 1$. In this case, we have to deal with the fact that $u_t$ can affect $N(S^*)$ as well as $S^*$. Part of the difference $Y_t - Y_{t-1}$ is due to the inclusion of $u_t$ in $S^*$ and the subsequent possibility that $u_t$ contributes a node to $N(S^*)$ if $\text{deg}(u_t) = 1$ and its neighbor is not already in $N(S^*)$. This part of the difference is bounded in magnitude by one as these events change the difference $S^* - N(S^*)$ by at most one. The other part of the difference $Y_t - Y_{t-1}$ is due to whether $u_t$ increments the size of $N(S^*)$ via its neighbor, affecting the probabilities $\mathbb{P}(u' \in S^*)$ for $u'$ where $\text{deg}(u_t) > 1$ as in Equation 21. As the expected size of $N(S^*)$ can

In any case, $|Y_t - Y_{t-1}| \leq 1$.

2. Assume $\text{deg}(u_t) \leq 1$. In this case, we have to deal with the fact that $u_t$ can affect $N(S^*)$ as well as $S^*$. Part of the difference $Y_t - Y_{t-1}$ is due to the inclusion of $u_t$ in $S^*$ and the subsequent possibility that $u_t$ contributes a node to $N(S^*)$ if $\text{deg}(u_t) = 1$ and its neighbor is not already in $N(S^*)$. This part of the difference is bounded in magnitude by one as these events change the difference $S^* - N(S^*)$ by at most one. The other part of the difference $Y_t - Y_{t-1}$ is due to whether $u_t$ increments the size of $N(S^*)$ via its neighbor, affecting the probabilities $\mathbb{P}(u' \in S^*)$ for $u'$ where $\text{deg}(u_t) > 1$ as in Equation 21. As the expected size of $N(S^*)$ can
change by at most one, the change in \( \mathbb{P}(u_t \in S^+) \) for each \( t' \) where \( \text{deg}(u_t) > 1 \) is at most

\[
\prod_{k=0}^{\text{deg}(u_t)-1} \left( \frac{t' - k}{m} \right) - \prod_{k=0}^{\text{deg}(u_t)-1} \left( \frac{t' - k - 1}{m} \right). \tag{22}
\]

The factors of \( t' \) in the numerators come from the fact that \( N(S^+) \leq t' \) if \( \text{deg}(u_t) > 1 \). As both parts of the difference contain many of the same terms, we can simplify Expression (22) as

\[
\frac{\text{deg}(u_t) - \text{deg}(u_t) - 1}{m} \prod_{k=1}^{\text{deg}(u_t) - 1} \left( \frac{t' - k}{m} \right) \leq \frac{\text{deg}(u_t)}{m}. \tag{23}
\]

As we assume that the expected degrees are distributed according to a power law distribution with exponent \( \alpha > 3 \), the expectation and variance of the degree of a given node \( u \) will be constant. Thus, with high probability, the sum over all offline degrees \( \sum u \text{deg}(u) = O(m) \). The contribution to \( Y_t - Y_{t-1} \) by Expression (23) is thus bounded by \( \sum u \text{deg}(u) = O(1) \). Overall, with high probability, \( |Y_t - Y_{t-1}| = O(1) \).

As in both cases, the martingale has constant bounded differences (with high probability), Azuma’s inequality directly gives us the theorem.

\[\Box\]

## H Additional Experiments

Figure 8 shows additional experiments on Real World graphs from the known i.i.d. model (based on the methodology of Borodin et al. (2020)). Overall, the results are very similar to those in Section 4. MinPredictedDegree does very well compared to the other online baselines (depicted in light blue) despite making relatively little use of the type graph information. Additionally, augmenting the known i.i.d. baselines with MinPredictedDegree (e.g. using the MinPredictedDegree rule when the base algorithm does not match the current online node even though it has unmatched neighbors) often improves the performance over the baseline algorithm and the greedy augmentation.

Figure 9 shows the degree distribution of the Oregon and Caida 2004 datasets as well as the \( \ell_2 \) prediction error (square root of the sum of the squared error of the degree prediction for each offline node in the current graph) over time of using the first days degrees as a prediction for future degrees. As the prediction quality degrades, the performance of MinPredictedDegree slowly declines.