Exact dynamics of quantum correlations of two qubits coupled to bosonic baths

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Abstract. Dynamics of the quantum entanglement and quantum discord of two qubits in two independent baths and a common bath with the Lorentzian spectrum are studied exactly in the numerical sense within the hierarchy approach. The effects of the counter-rotating-wave terms from the system–bath coherence on these quantum correlations are systematically discussed and comparisons with previous ones under the rotating-wave approximation are also performed. For two independent baths, beyond the weak system–bath coupling, the counter-rotating-wave terms essentially change evolutions of both the entanglement and quantum discord. With increase of the coupling, revival of the entanglement after a period of complete disentanglement is suppressed dramatically and finally disappears, and the quantum discord becomes smaller monotonically. For the common bath, the entanglement is also suppressed by the counter-rotating-wave terms, but the quantum discord shows quite different behaviors if initiated from spin-correlated states. In the non-Markovian regime, the quantum discord is almost not influenced by the counter-rotating-wave terms and is generally finite in the long-time evolution at arbitrary coupling while in the Markovian regime, it is significantly enhanced with the strong coupling.

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1. Introduction

Quantum correlations, originated from the coherent superposition in quantum mechanics [1, 2], have been attracting persistent attention ranging from quantum information, quantum optics, to many-body physics [3–5]. As the characteristic trait of the quantum correlation proposed by Schrödinger [6] without the classical counterpart, quantum entanglement played the crucial role in the highly efficient quantum computation and quantum information processing [7, 8]. It has also been applied to identify the phase transition in quantum many-particle systems [9]. However, there exist exceptions where the entanglement seems unnecessary, such as in the Grover search algorithm [10, 11] and the deterministic quantum computation with one pure qubit (DQC1) [12]. It is evident that other kind of the non-classical quantum correlation rather entanglement dominates the computational implementation of DQC1 [13]. Zurek and Vedral described one particular measure to quantify all non-classical correlations, termed as quantum discord (QD) [14–16]. Only when the QD disappears, the information can be safely obtained by locally measuring the distantly separate subsystem without disturbing the bipartite quantum system, where the corresponding state is fully classical. Recently, the QD has successfully been used to study the quantum phase transition of the critical systems [17–22], operational meanings for quantum processors [23] and state preparation [24].

It is known that the realistic system inevitably interacts with the environment, resulting in decoherence, which is a fundamental problem in the quantum information processing and measurement [25, 26]. In particular, Yu and Eberly [27] found that the dynamical behavior of the entanglement is significantly different from the single qubit in a pair of qubits separately coupling with Markovian baths, which can be modeled from the well-known spin–boson models [28]. They demonstrated a disentanglement in a finite amount of time, and termed it as entanglement sudden death (ESD) [27]. In these studies, they employed the quantum master equation under the Born and Markov approximations, assuming that system–bath interaction is weak and the relaxation of the bath is much faster than that of the system. Further works have been done in the non-Markovian regime in the framework of the rotating-wave approximation (RWA) [29–32]. Recently, the dynamics of the QD in the relevant spin–boson models were also studied in the non-Markovian regime with the RWA [33–36]. Besides, the non-Markovian dynamics of the entanglement in the spin–boson model beyond the RWA under

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Ohmic baths has been investigated based on an optimal polaronic transformation [37], where the higher-order terms are neglected. They found a monotonic decay of the entanglement as well as disentanglement. The entanglement oscillation or revival was not captured in their calculation, which is just a signature of the bath non-Markovian effect. It was pointed out that the corresponding accuracy in the wide parameter regime should be carefully analyzed [38]. More recently, a polaron trial state by the name of the Davydov $D_1$ ansatz has been utilized to probe the entanglement dynamics of two quits in Ohmic baths within the Dirac–Frenkel time-dependent variational procedure [39]. Therefore, it is highly desirable to solve the dynamics without performing any approximations.

Without the Born–Markov approximation and RWA, Tanimura et al [40–43] have developed an efficient hierarchy approach, which has been later extensively applied to some chemical and biophysical systems [44–49]. This method itself is exactly based on the Feynman–Vernon influence functional [50]. The numerical errors in the practical implementation are only from the order truncation of system–bath coupling and discrete time iteration, which can be safely controlled to a desired accuracy. Recently, this method has also been extended to study the dynamics of entanglement for two quits coupled to a common bath [51, 52] in the field of quantum information. The exact study of the entanglement dynamics for two quits coupled to their own baths are not available in the literature. This case is designed for two remote quits, each interacting locally with its own environment. Such a physical condition is also relevant to the quantum information science and quantum computation. On the other hand, dynamics of the QD, one kind of the quantum correlations recently mostly studied in two quits coupled to both independent and common baths beyond the RWA, have not been investigated. Due to the importance of the QD in fundamental concept and realistic applications, this gap should be necessarily filled.

In this paper, to give a comprehensive picture of two quits coupled to their own bath and common one, we will extend the hierarchical equation to these two kinds of bath structures to study dynamics of the pairwise entanglement and the QD in both Markovian and non-Markovian regimes. The effects of the counter-rotating-wave terms (CRTs) on these dynamics in a wide coupling regime will be systematically explored. This paper is organized as follows. In section 2, the definition of the QD is briefly reviewed. In section 3, the reduced density matrix of quits in two independent spin–boson models is exactly solved and the quantum correlations are evaluated. The discussions and comparisons with previous RWA results are also performed. In section 4, the quantum correlation for the common bath and corresponding analysis are presented. A summary is given in the last section.

2. Quantum discord

The QD is one main route to fully measure the non-classical correlation, which can be extended from the classical information theory [5, 14–16]. It is interpreted as the difference between two quantum mutual correlations of subsystems $A$ and $B$, before and after local measurement operated on subsystem $B$ [14, 15]. The total quantum correlation of two subsystems is determined by their joint density matrix $\rho$ as $I(\rho) = S(\rho_A) + S(\rho_B) - S(\rho)$, where $\rho_{A(B)} = \text{Tr}_{B(A)}(\rho)$. The von Neumann entropy $S(\rho_k) = -\text{Tr}(\rho_k \ln \rho_k)$. The other quantum mutual information $J(\rho)$, which is equivalent to $I(\rho)$ in classical information frame, is derived by locally measuring subsystem $B$ with a complete set of orthonormal projectors $\{\Pi_k^B\}$, where $k$ is one outcome state of $B$. After this measurement, the joint density is reduced to
conditional counterpart as \( \rho_k = \frac{[\mathbb{I} \otimes \Pi_k^B] \rho (\mathbb{I} \otimes \Pi_k^B)]}{p_k} \), where the corresponding probability is \( p_k = \text{Tr}[\mathbb{I} \otimes \Pi_k^B] \rho \). Then the mutual information based on specific performance is shown as \( Q = S(\rho_A) - \sum_k p_k S(\rho_k) \). Then, we should maximize it to capture all classical correlations by obtaining \( J(\rho) = \max \{ \Pi_k^B \} \{ Q \} \). Finally, the QD is defined as
\[
D(\rho) = I(\rho) - J(\rho).
\]
(1)
The quantum nature of the system can be explicitly observed from the QD. It is only zero in pure classical correlation condition, and has the same values as the entanglement in pure state. While it remains finite even in separable mixed states, where the corresponding entanglement may completely disappear \[14\].

Practically for an arbitrary density matrix, the QD can be obtained in the standard procedure. Take two qubits system for example, with the reduced density matrix as \( \rho_{AB} \). The total quantum correlation \( I(\rho_{AB}) \) can be straightforwardly obtained from the definition above. The complete set of orthonormal projectors \( \Pi_k^B \) can be chosen by
\[
|\Psi_1\rangle_B = \cos \theta |\downarrow\rangle_B + e^{i\phi} \sin \theta |\uparrow\rangle_B,
\]
\[
|\Psi_2\rangle_B = -e^{-i\phi} \sin \theta |\downarrow\rangle_B + \cos \theta |\uparrow\rangle_B.
\]

Then the conditional density \( \rho_k \) \( (k = 1, 2) \) of subsystem \( A \) is derived under the measurement parameters \( \theta \) and \( \phi \). Consequently, the conditional correlation \( Q \) is known. We maximize this conditional correlation by tuning \( \theta \) and \( \phi \) in full ranges to gain the classical correlation as \( J(\rho_{AB}) = \max_{\{ \theta, \phi \}} \{ Q \} \), which usually needs numerical search. Finally, the QD is obtained from equation (1).

3. Two independent spin–boson models

The Hamiltonian of two independent qubits coupled to two independent bosonic baths \( A \) and \( B \) is given by \( H = H_s + H_b + H_{sb} \), which reads
\[
H_s = \sum_{a=A,B} \frac{\omega_a}{2} \sigma_z^a,
\]
\[
H_b = \sum_{a=A,B; k} \omega_k^a b_{a,k}^\dagger b_{a,k},
\]
\[
H_{sb} = \sum_{a=A,B; k} \sigma_x^a (g_k^a b_{a,k} + g_k^{a*} b_{a,k}^\dagger),
\]
(2)

where \( a = A, B \), \( \sigma_\beta^a \) \( (\beta = x, y, z) \) is the Pauli operator with Zeeman energy \( \omega_a \) of spin \( \beta \), \( b_{a,k}^\dagger \) \( (b_{a,k}) \) creates (annihilates) one boson with frequency \( \omega_k^a \) in bath \( a \) and \( g_k^a \) is coupling strength between the system and the bath.

The spectral density of the bosonic bath is selected as the following Lorentzian distribution:
\[
J_a(\omega) = \frac{1}{2\pi} \frac{\lambda_a Y_a^2}{(\omega - \omega_a)^2 + Y_a^2},
\]
(3)
which can describe a bosonic field inside an imperfect cavity mode \( \omega_0^a \) with the system–bath coupling strength \( \lambda_a \). It was widely used in the recent studies of the quantum entanglement
The density matrix in the basis \{\text{system–bath interaction}\} is obtained numerically. In the fourth-order Runge–Kutta iteration of this hierarchical equation, we choose time step \(\Delta t\) in the following calculations, the two subsystems are identical as the corresponding initial time. It is very challenging to apply this approach to an arbitrary \(t\) we have \(n = (n_1, n_2)\), \(\bar{m} = (m_1, m_2)\). \(Q_a (a = A, B)\) in the system–bath interaction represents \(\sigma^a\). Since the initial state considered here is separable, shown as \(\rho = \rho_S \otimes \rho_B\), only \(\rho_{0(0,0),0(0)}(0)\) is not zero. For other states including system–bath correlation, such as steady state, we just pull back the initial time to \(t_0 = -\infty\). Then the density matrix already reaches the steady values when \(t = 0\). It should be noted that the hierarchical equation can only work properly based on special initial state mentioned above, starting from the corresponding initial time. It is very challenging to apply this approach to an arbitrary initial state including the system–bath correlation at the moment. For the simplest condition in the following calculations, the two subsystems are identical as \(\omega_\sigma = \omega_0\), \(\lambda_\sigma = \lambda\) and \(\gamma_\sigma = \gamma\). In the fourth-order Runge–Kutta iteration of this hierarchical equation, we choose time step \(\Delta t = 0.01\omega_0\) and order truncation \(N_r = \max\{n_1 + n_2, m_1 + m_2\} \leq 30\) so that convergent results can be obtained numerically.

For comparison, we also briefly review the main results derived in the RWA, where system–bath interaction is \(H_Q^B = \sum_{a,k} (g^a_{s,k} \sigma^a_{s,k} b^a_{s,k} + g^a_{s,k} \sigma^a_{s,k} b^a_{s,k})\). When the initial system reduced density matrix in the basis \{|1\} = |11\rangle, |2\} = |10\rangle, |3\} = |01\rangle, |4\} = |00\rangle\} is of \(X\) form

\[
\rho_{AB}(0) = \begin{pmatrix}
\rho_{11}(0) & 0 & 0 & \rho_{14}(0) \\
0 & \rho_{22}(0) & \rho_{23}(0) & 0 \\
0 & \rho_{32}(0) & \rho_{33}(0) & 0 \\
\rho_{41}(0) & 0 & 0 & \rho_{44}(0)
\end{pmatrix},
\]

we have

\[
\rho_{11}(t) = \rho_{11}(0) P_t^2, \quad \rho_{22}(t) = \rho_{22}(0) P_t + \rho_{11}(0) P_t (1 - P_t),
\]

\[
\rho_{33}(t) = \rho_{33}(0) P_t + \rho_{11}(0) P_t (1 - P_t), \quad \rho_{44}(t) = 1 - [\rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t)],
\]

\[
\rho_{14}(t) = \rho_{14}(0) P(t), \quad \rho_{23}(t) = \rho_{23}(0) P(t),
\]

where

\[
P_t = e^{-\gamma t} \left[\cos(Rt) + \frac{\gamma}{2R} \sin(Rt)\right]^2
\]

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\[ R = \sqrt{\frac{\lambda}{2}} - \frac{\sqrt{\lambda^2 - 4}}{4}. \]

It is known that the bath relaxation time is \( \tau_b \approx \frac{1}{\gamma} \), and the system–bath correlation time is \( \tau_r \approx \frac{1}{\gamma} \). From the time dependence of the function \( P_t \), we immediately know that the Markovian regime corresponds to \( \gamma / 2 > \lambda \) where \( P_t \) decays exponentially. While the non-Markovian regime locates in \( \gamma / 2 < \lambda \), where \( P_t \) shows oscillating behavior describing the coherent process between the system and the bath. This non-Markovian condition has been realized in cavity quantum electrodynamics having Rydberg atoms inside the Fabry–Perot cavities with \( \gamma / \lambda \approx 0.1 \) [53]. When the CRTs are included, numerical calculations should be performed, so the boundary cannot be given analytically. Therefore, we do not know whether it is in the non-Markovian regime or not \textit{a priori}. Fortunately, we do not need to distinguish the boundary between non-Markovian and Markovian regimes, the exact calculation procedures are the same in any case. The character of the baths will come out naturally with the final dynamical behaviors. For convenience, we still use the same notations defined under the RWA throughout this paper just for comparison.

As usual, the initial spin-anti-correlated state and spin-correlated one will be considered in this paper:

\[
\begin{align*}
|\Phi(\alpha)\rangle &= \alpha|00\rangle + \sqrt{1 - \alpha^2}|11\rangle, \\
|\Psi(\alpha)\rangle &= \alpha|00\rangle + \sqrt{1 - \alpha^2}|11\rangle,
\end{align*}
\]

which both obey \( X \) form in equation (6) during the evolution under RWA. The initial phonons are in their vacuum states. So it is straightforward to derive the concurrence, a pairwise entanglement [29]

\[
\begin{align*}
C_{\Phi}(t) &= \max\{0, 2\alpha\sqrt{1 - \alpha^2}P_t\}, \\
C_{\Psi}(t) &= \max\{0, 2\sqrt{1 - \alpha^2}P_t[\alpha - \sqrt{1 - \alpha^2}(1 - P_t)]\}.
\end{align*}
\]

The analytical expression for QD [54] in this case is more complicated and also only consists of \( \alpha \) and \( P_t \), which is implied in [34, 35].

It is shown analytically above that dynamics of both entanglement and QD under the RWA can be distinguished in the non-Markovian and Markovian regimes, characterized by the ratio \( \gamma / \lambda \). In each regime, the essential feature only depends on the weights \( \alpha^2 \) of the initial states, and independent of the system–bath coupling strength \( \lambda \). In other words, in the framework of the RWA, all essential properties for physical observables are independent of the system–bath coupling strength for given initial states, which should not be always true with the increasing coupling realized in many recent experiments [55]. Hence, it is necessary to go beyond the RWA.

3.1. Exact dynamics and comparisons

The dynamics of the pairwise concurrence of the two qubits in their own baths without the RWA can be obtained exactly by the hierarchy approach outlined above. Initiated from the anti-correlated Bell state (equation (9)) with \( \alpha = 1/\sqrt{2} \), the maximally entangled one, time evolutions of the concurrence from weak, intermediate, to strong coupling cases are presented in the upper panel of figure 1. Note that the concurrence under the RWA has been investigated exactly in [29] both in the Markovian and non-Markovian regimes. The relevant results for the same parameters are also reproduced for comparison.

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Figure 1. Dynamics of quantum correlation of two qubits coupled to two independent baths with initial anti-correlated Bell state $|\Phi^{(1)}_{\pm\frac{\sqrt{2}}{2}}\rangle$. $\omega_0 = 1$, $\gamma = f\lambda$.

In the weak system–bath coupling ($\lambda = 0.02$), the rotating-wave term dominates the entanglement evolution as indicated in figure 1(a), and the results without the RWA show negligible deviation from those in RWA. In the non-Markovian regime, the concurrence exhibits periodic oscillations with amplitude damping. The corresponding time of the zero points is at $t_n = [n\pi - \arctan(R/2\gamma)]/R$ ($n = 1, 2, \ldots$). While in the Markovian regime, the concurrence decays exponentially and vanishes asymptotically. The decay rate is larger than the non-Markovian one. Hence at such a weak coupling, we have not found any evident effect from the CRTs, which means that the previous RWA description is really valid.

What happens if we increase the system–bath coupling to the intermediate regime, such as $\lambda = 0.5$? It is interesting to observe in figure 1(b) that the dynamical behavior is apparently different from that in the RWA. From equation (11), one can see that the ESD never happens in the RWA if initiated from anti-correlated Bell state, which was first observed in [29] with the same system. But without the RWA, in both the Markovian ($f = 5$) and the non-Markovian ($f = 0.1$) regimes, the concurrence exhibits ESD obviously. Hence, we conclude that the RWA cannot describe the essential features of the entanglement dynamics beyond the weak coupling regime.

Practically, this physical condition can be realized in the recent strong coupling experiments [55]. In the non-Markovian limit ($\gamma/\lambda \ll 1$), the spectral function of baths,
normally defined as \( J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k) \). Therefore, the system can be approximately simplified to two independent Rabi models [56], and the corresponding effective system–bath coupling constant is given by \( g = \sqrt{\frac{\lambda}{\omega}} \). For \( \lambda = 0.5\omega_0 \) and \( f = 0.1 \), it is shown as \( g/\omega_0 = 0.11 \). This interaction strength was already carried out in the recent experiments of the superconducting qubits coupled to LC resonators made of an inductor L and a capacitor C, with the coupling constant \( g/\omega_0 \) reaching 10 percentages [55]. Moreover, clear evidences for physics beyond RWA have been shown in the strong coupling regime in experiments. Hence, by applying this setup based on qubit–resonator coupling system, entanglement sudden death and revival due to the CRTs may be directly observed, which is quite useful in quantum information processing.

In the Markovian regime, the entanglement vanishes quickly and permanently beyond the weak coupling as shown in figures 1(b) and (c). It mainly results from generation of extra phonons with increasing coupling, which disturbs the pairwise entanglement. No revival of the entanglement appears in this case. Interestingly, the evolution behavior is almost unchanged as \( \lambda \geq 0.5\omega_0 \), which follows that \( \lambda = 0.5\omega_0 \) is strong enough. However, in the non-Markovian regime, the entanglement dynamics show essential different behaviors with variation of the system–bath coupling. Revival of the entanglement after the ESD occurs at \( \lambda = 0.5\omega_0 \), but this revival phenomenon never happens in the strong coupling regime, as exhibited in figure 1(c) for \( \lambda = 2\omega_0 \). We propose that more extra phonons activated by the CRTs would suppress the feedback from the bath to the qubit [56] even in the non-Markovian regime, resulting in permanent disentanglement. The effective system–bath coupling strength for \( f = 0.1 \) in figure 1(c) can be estimated as \( g/\omega_0 = 0.45 \), which is hopefully realized in the near future [57, 58].

Then we investigate dynamics of the QD for two independent baths to explore the influences of the CRTs. The QD in the RWA has been discussed by Wang et al [34] and Fanchini et al [35]. In the lower panel of figure 1(d), we display evolutions of the QD for the same parameters as the upper panel by both the hierarchy approach and RWA. In the weak coupling regime, it behaves similar to the concurrence in figure 1(a) for both the Markovian and non-Markovian evolutions, where the CRTs can be ignored. Beyond the weak coupling regime, as shown in figures 1(e) and (f), the CRTs drives the QD to deviate from that in RWA obviously. Specifically for the non-Markovian case, the QD decays dramatically and shows weak revival signal, which mainly attributes to the excitation of the phonons in baths. For the Markovian case, the QD decreases faster than that in RWA and decays to zero asymptotically.

To see the comprehensive effects of the system–bath interaction, we plot dynamics of the quantum correlations as functions of both \( t \) and \( \lambda \) in figure 2 for initial anti-correlated Bell state. Both the entanglement and QD are suppressed monotonically with the increasing interaction. Revival signal for the entanglement is absent completely as the coupling strength exceeds a critical value, e.g. \( \lambda_c \approx 1.5\omega_0 \) for the parameters in figure 2(a). The region of ESD becomes wider with increase of the system–bath coupling. But figure 2(b) reveals that the QD never vanishes suddenly, consistent with the previous observation concluded from an arbitrary Markovian evolution [59]. Therefore, such a limitation is naturally removed in the present study, for the exact evolution definitely includes the non-Markovian effect. No sudden death of the QD is universal in this sense, in sharp contrast with the entanglement. Moreover, in our exact solution for the full model without the RWA, we actually have not found the real zero QD at any time, despite arbitrary small values. This non-zero nature for the QD may be intrinsic at least for the spin–boson model. Previous observation in the RWA that the QD can vanish at some discrete times [34, 35] may be an artificial result of ignoring the CRTs.
It should be noted that the entanglement dynamics only displays a monotonic decay in the non-Markovian study based on the optimal polaronic transformation [37]. Such behavior can also be induced by the Markovian effect. As a result, it is uncertain that the non-Markovian effect was included in their study where higher orders are neglected. Interestingly, the entanglement oscillation and revival are obviously found in our exact study, indicating that higher orders should be kept into account.

The essential feature for dynamics of the quantum entanglement and QD under the RWA is not altered qualitatively with the system–bath coupling in the same non-Markovian or Markovian regime. However, it is not that case in the real exact solution without the RWA. For the weak system–bath interaction, we have shown above that the RWA can basically give the right description of the quantum correlations of two separate spin–boson models for both Markovian and non-Markovian regimes. While in the strong coupling regime, the CRTs should be necessarily included for correctly depicting the phonon generation from the baths. And the quantum correlations show significantly different behaviors from those in RWA, especially in non-Markovian case. Hence, we focus on dynamics of the quantum correlations in the strong

Figure 2. Effect of system bath interaction on dynamics of quantum correlation of two qubits of two independent baths with initial anti-correlated Bell state $|\Phi(\frac{\sqrt{2}}{\sqrt{2}})\rangle$. $\omega_0 = 1$, $\gamma = 0.1\lambda$. 
coupling and non-Markovian regime in the following with various mostly used initial states in the literature.

3.2. Non-Markovian dynamics at strong coupling

First, we examine the dependence of dynamics of the quantum correlations on $\alpha$ for the initial state in equation (9), which characterizes weights of the two superposed anti-correlated spin states. Since the quantum correlation for $\alpha^2 \in (0, 1/2)$ is similar to that in $(1/2, 1)$, we only study the former case, and the results are exhibited in figures 3(a)–(c). In RWA, both the concurrence and QD show damping oscillations with stronger amplitude as $\alpha^2$ increases. The other evident feature is that they only vanish at some discrete moments. When the CRTs are included, the concurrence exhibits ESD after finite time evolutions in the whole regime. The starting time of ESD is much earlier than first vanishing moment in the RWA. The QD first decreases almost to zero and then revives with amplitude much weaker than that in the RWA, mainly due to the suppression from the phonons.

Figures 3(d)–(f) present the results for the spin-correlated state in equation (10). For the same reason, we only focus on the parameter regime $\alpha^2 \in (0, 1/2)$. Note that the corresponding
studies in the RWA have been given in [35], which are also listed for comparison. The present exact results show that the ESD occurs very quickly and irrelevant to $\alpha$. The QD at first decreases almost to zero, and then revives weakly. The revival amplitude becomes stronger with increasing $\alpha$. For the same $\alpha$, the revival amplitude is weaker than that in the RWA, resulting from the CRTs, as demonstrated in the inset.

For the two initial states studied above, it is generally observed that the larger initial quantum correlations are, the stronger they evolve in amplitudes at the same moments.

4. Two qubits coupled to one common bosonic bath

The Hamiltonian of two qubits coupled to the common bath $H = H_s + H_b + H_{sb}$ reads

$$H_s = \frac{\alpha}{2} \sum_{a=A,B} \sigma_z^a,$$

$$H_b = \sum_k \omega_k b_k^\dagger b_k,$$

$$H_{sb} = Q \sum_k (g_k b_k + g_k^* b_k^\dagger),$$

(13)

where the notations are the same as those in equation (2) and $Q = \sum_a=\{A,B\} \sigma_z^a$. The spectrum of the common bath is also Lorentzian distribution $J(\omega) = \frac{\gamma}{2\pi} \frac{\gamma_k^2}{(\omega-\omega_k)^2+\gamma_k^2}$, and the correlation function of the bath should be also $C(t) = \frac{\gamma t}{2} e^{-(\gamma t+i\omega_0)t}$. By choosing the proper auxiliary influence functional in the appendix, the hierarchical equation is given by (A.17) [52]

$$\frac{\partial \rho_{\tilde{n}}(t)}{\partial t} = -(iH_S^{\dagger} + \tilde{n} \cdot \tilde{v}) \rho_{\tilde{n}}(t) - i \sum_{k=1,2} Q^x \rho_{\tilde{n}+\tilde{e}_k}(t) - \frac{i\gamma \lambda}{2} \sum_{k=1,2} n_k [Q^x + (-1)^k Q^z] \rho_{\tilde{n}-\tilde{e}_k}(t),$$

(14)

where $\tilde{n} = (n_1, n_2)$, the unit vector $\tilde{e}_1 = (1,0)$ and $\tilde{e}_2 = (0,1)$. The reduced density matrix for qubits is $\rho_{AB}(t) = \rho_{\{0,0\}}(t)$, by which we can calculate the quantum correlations numerically.

In RWA, the system–bath interaction is $H_{sb}^{\text{RWA}} = (\sigma_+^A + \sigma_+^B) \sum_k g_k b_k + (\sigma_-^A + \sigma_-^B) \sum_k g_k^* b_k^\dagger$. If we represent the Hamiltonian in the basis $|0\rangle = |00\rangle$, $|+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$, $|\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$, $|2\rangle = |11\rangle$, $|\rangle$ is found to be isolated from the others. Hence, we can rewrite the Hamiltonian in three-level form as [31]

$$H_0 = 2\omega_0 |2\rangle \langle 2| + \omega_b |+\rangle \langle +| - \omega_b,$$

$$H_{int} = \sqrt{2} \sum_k g_k a_k |+\rangle \langle 0| + |2\rangle \langle +| + h.c.$$

(15)

For the single-particle excitation or double excitation case, the equation of motion for the system density matrix can be derived by the so-called pseudomode master equation in the interaction representation [31, 35, 60–63]

$$\frac{\partial \rho(t)}{\partial t} = -i[V, \rho(t)] - \gamma (a^\dagger a \rho(t) + \rho(t) a^\dagger a - 2a \rho(t) a^\dagger),$$

(16)

where $a^\dagger (a)$ is the creator (annihilator) of the bosonic pseudomode,

$$V = \sqrt{\frac{2}{2\gamma}} (a^\dagger |0\rangle \langle 0| + a^\dagger \langle 0| + a |2\rangle \langle +| + a^\dagger |2\rangle \langle 0|)$$
and \( \rho_0(t) = e^{iH_0t} \rho(t) e^{-iH_0t} \). By integrating the bosonic part of the density matrix, the reduced density of the system is given by \( \rho_{AB}(t) = \text{Tr}_a(\rho(t)) \). For single excitation, an alternatively exact solution has also been done in [32]. If the initial state is chosen by \( X \) form in equation (6), the concurrence can be expressed as [63]

\[
C(t) = 2 \max \{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\}.
\]

Figure 4. Dynamics of quantum correlation of two qubits coupled to the common bath with initial spin-correlated state \( |\Psi(\frac{1}{\sqrt{3}})\rangle \). \( \omega_0 = 1, \gamma = f \lambda \).

For the common bath, the entanglement evolution initiated from the spin-anti-correlated state without the RWA has been studied by Ma et al [52], where the system only evolves within the Hilbert space of odd number total excitations of the particles and bosons. On the other hand, in the previous studies of the dynamics of QD under the RWA, the spin-correlated state is usually selected as the initial state [31, 34, 35, 63]. Therefore, in the present exact study without the RWA, we will focus on the initial spin-correlated state. Note that for the Lorentzian bath, the parity symmetry is not broken with coupling. Hence in the present case, the system only evolves along the space of even number total excitations, with the parity quite different from that for the initial spin-anti-correlated state. By the way, for two qubits coupled to two independent baths initiated from Bell-like state, the parity symmetry of the whole system is also conserved.

First, we exhibit dynamics of the pairwise concurrence of two qubits coupled to a common bath with and without the RWA initiated from the spin-correlated state in equation (10), with \( \alpha = 1/\sqrt{3} \) in the upper panel of figure 4 for various coupling strengths in both Markovian and non-Markovian regimes. Similarly, for the weak coupling (\( \lambda = 0.02\omega_0 \)), the CRTs take little
effect on dynamics in different evolution regimes, demonstrating the same results as in the RWA [35]. In the intermediate coupling regime ($\lambda = 0.5\omega_0$), the ESD occurs more quickly than that in the RWA in the Markovian regime, and the evolution shows slight deviations from the RWA ones with damping oscillations in the non-Markovian regime. When the strong coupling regime is reached, the CRTs take remarkable effects in both evolution regimes, where concurrence vanishes completely and permanently without the RWA. Therefore, it is proposed that the CRTs also suppress the entanglement dramatically as the system–bath coupling enlarges for the common bath, similar to independent baths. This conclusion is also consistent with the results shown in [52] from different initial states.

Then we turn to the relevant QD, which is shown in the lower panel of figure 4. It is interesting to find that in the non-Markovian regime, the CRTs take very limited influences with the increasing coupling. Even at the strong coupling, only slightly deviations are observed. Qualitatively, in the non-Markovian regime, the QD in the common bath is dominated by the non-Markovian effect, which only weakly depends on the CRTs. More interestingly, the QD takes finite value at any time, and takes a moderate finite value asymptotically in the long-time limit, for arbitrary coupling strength.

In the Markovian regime, the QD tends to nearly the same remarkable value in the intermediate and strong coupling regimes. Actually, even at weak coupling like $\lambda = 0.05\omega_0$, the asymptotical value of QD already takes a considerable non-zero value. To explicitly show how the plateau reached by the QD at long times varies at different coupling strengths, we demonstrate dynamics of the QD at $\lambda = 0.05\omega_0$, $0.1\omega_0$ and $0.2\omega_0$ in figure 5(a). The QD approaches the plateau after several oscillations. This oscillated behavior is owing to the competitions between the effects of the common baths and CRTs at earlier dynamics. We also plot the curve of the steady-state value of the QD with the system–bath coupling in the Markovian regime in figure 5(b) for a wider coupling range. As the system–bath coupling increases, the QD rises quickly and saturates to a stable value around 0.23 beyond $\lambda \approx 0.5\omega_0$.

From the numerical iteration of the hierarchical equation in equation (14) with initial spin-correlated state in the Markovian regime (e.g. $f = 10$ in figure 4), it is found that the reduced

\[ |\Psi(t)\rangle = \left| \begin{array}{c} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{array} \right| \]
Figure 6. Dynamics of quantum correlation of two qubits coupled to the common bath with initial spin-correlated state $|\psi\rangle = |\frac{1}{\sqrt{2}}\rangle$ (a) concurrence and (b) QD. $\omega_0 = 1$, $\gamma = 0.1 \lambda$.

density matrix for two qubits under the same basis in equation (6) is always of the $X$ type

$$
\rho(t) = \begin{pmatrix}
    a(t) & 0 & 0 & w(t) \\
    0 & b(t) & b(t) & 0 \\
    0 & b(t) & b(t) & 0 \\
    w^*(t) & 0 & 0 & d(t)
\end{pmatrix}.
$$

(18)

If we express the Hamiltonian in equation (13) under the spin basis $\{|0\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle), |2\rangle = |11\rangle\}$, we find that $Q|-\rangle = 0$, whereas $Q|0\rangle = \sqrt{2}|+\rangle$, $Q|+\rangle = \sqrt{2}(|0\rangle + |2\rangle)$ and $Q|2\rangle = \sqrt{2}|+\rangle$. Meanwhile, $H_s$ will not generate transitions between these two-qubits states. As a result, the anti-symmetric state $|-\rangle$ is separated from the other ones, similar to the situation in RWA.

In the weak system–bath coupling (say $\lambda = 0.9 \omega_0$), $a(t)$ and $b(t)$ decrease gradually to 0 after the finite time evolution, whereas $d(t)$ increases and reaches 1 asymptotically. So both the concurrence and QD decay to 0 monotonically as indicated in figures 4(a) and (d), similar to that in the RWA. When the coupling strength becomes stronger, e.g. $\lambda = 0.5 \omega_0, 2 \omega_0$, the phonons are excited considerably due to the strong coupling [56]. From the CRTs, $\sum_k s_k^* b_k^\dagger (\sigma^A_k + \sigma^B_k)$, the qubits are also excited accompanying the extra excitations of the phonons, which stabilize $a(t)$ and $b(t)$ in long-time evolution. It is interesting to observe that they tend to 1/3 and 1/6 in the strong coupling limit at $t \to \infty$. The corresponding reduced density matrix is then described by $\rho(t \to \infty) = \frac{1}{3}(|00\rangle \langle 00| + |11\rangle \langle 11| + |+\rangle \langle +|)$, which is not altered by different $\alpha^2$. Under this
separable mixed state, the disentanglement occurs naturally from equation (17), and the QD is stabilized at 0.23, which is the same as that shown in figure 5.

To see the overall effects of the system–bath interaction on dynamics of the quantum correlations beyond the weak coupling regime, we draw the corresponding three-dimensional plots for \( \lambda \geq 0.1 \omega_0 \) in figure 6 in the non-Markovian regime. For the entanglement, two regimes are distinguished. One is damping oscillation regime at \( \lambda < \omega_0 \). The other is the regime where the entanglement revives following the ESD \( (\lambda > \omega_0) \), and the revival will be suppressed and finally disappears with further increase of the coupling. While for the QD, it is generally robust in the long-time evolution after weak oscillations, regardless of the coupling strength. Hence, it may be useful to act as one quantum information resource.

5. Conclusion

In summary, without performing any approximation, we exactly investigate the quantum correlations in two typical kinds of spin–boson models by applying the hierarchy approach. The corresponding RWA results are also reproduced for comparison. The Markovian and non-Markovian effects on dynamics of the quantum correlations are revealed in detail for two kinds bath structures in the strong system–bath interaction. It demonstrates that the CRTs should be necessarily included beyond the weak coupling regime to capture the essential features of the quantum correlations.

For two independent baths, the CRTs suppress both the entanglement and the QD dramatically, due to excitations of phonons in baths as the system–bath coupling increases. If initiated from the anti-correlated Bell state, the ESD is driven by extra activated phonons with increasing system–bath coupling, which is absent in the RWA study. It follows that the previous picture under the RWA is essentially modified within the strong coupling regime. The QD is found to be always higher than zero, despite extremely small, also different from that in the RWA. Dynamics with other initial conditions is also discussed in the strong coupling and non-Markovian regime. The general trend is that the larger the initial quantum correlations, the stronger they evolve in amplitudes at the same moments. For the model of two qubits coupled to the common bath, suppression of the entanglement by the CRTs is also observed. It is interesting to find that the non-classical correlation described by the QD can be enhanced as the coupling is strengthened, in contrast with the entanglement. It follows that while the entanglement becomes fragile with the increasing coupling to the surrounding environment, the QD is however robust in the long-time evolution. In recent experiments, the strong system–bath coupling has been reached in circuit QED systems [55], where \( g/\omega_0 \approx 0.1 \) for multi-modes and the RWA description broke down. The present study for the evolution of the quantum correlations beyond the RWA may motivate the corresponding experimental studies based on these strong-coupling solid-state devices.

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Appendix. Basic concept and application of the hierarchical equation

The Hamiltonian of the system–bath coupling system is modeled as $H = H_s + H_b + H_{sb}$. The time-dependent density matrix is given by $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$. In the interaction representation, the corresponding density matrix reads $\rho(t) = U_i(t)\rho(0)U_i^\dagger(t)$ where $U_i(t) = e^{-i\int_0^t dt H_{sb}(t)}$. On the specific basis $\{|x, \alpha\rangle\}$, we have

$$\langle x, \alpha| \rho(t) |x', \alpha'\rangle = \int dx_0 dx'_0 \int d\alpha_0 d\alpha'_0 \langle x, \alpha| U_i(t) \rho(0) U_i^\dagger(t) |x, \alpha\rangle$$

$$\times \langle x_0, \alpha_0| |x_0', \alpha_0'\rangle \langle x_0', \alpha_0'| U_i^\dagger(t) |x', \alpha'\rangle.$$  \hspace{1cm} \text{(A.1)}

It is known that $\rho_s'(t) = \text{Tr}_b[\rho(t)]$. Hence by integrating the bath degree of freedom, we can obtain the reduced density of the system

$$\langle \alpha| \rho_s'(t) |\alpha'\rangle = \int dx \langle x, \alpha| \rho(t) |x, \alpha\rangle$$

$$= \int dx dx_0 dx'_0 \int d\alpha_0 d\alpha'_0 \langle x, \alpha| U_i(t) |x_0, \alpha_0\rangle$$

$$\times \langle x_0, \alpha_0| \rho(0) |x_0', \alpha_0'\rangle \langle x_0', \alpha_0'| U_i^\dagger(t) |x', \alpha'\rangle.$$  \hspace{1cm} \text{(A.2)}

Considering the propagator in functional form

$$\langle x, \alpha| U_i(t) |x_0, \alpha_0\rangle = \int_{x_0}^{x_t} \mathcal{D}[x(\tau)] \int_{\alpha_0}^{\alpha_t} \mathcal{D}[\alpha(\tau)] e^{iS[x(\tau), \alpha(\tau)]}$$

and the initial factorized state $\rho(0) = \rho_s(0) \otimes \rho_b$ ($\rho_b = \exp(-\beta H_b)/Z$ is the equilibrium state), the reduced density matrix can be expressed as

$$\langle \alpha| \rho_s'(t) |\alpha'\rangle = \int d\alpha_0 d\alpha'_0 \langle \alpha| \rho_s(t) |\alpha'\rangle$$

$$\times \int_{\alpha_0}^{\alpha_t} \mathcal{D}[\alpha(\tau)] \int dx dx_0 dx'_0 \langle \alpha| \rho_0 |x_0, \alpha_0\rangle$$

$$\times \int_{x_0}^{x_t} \mathcal{D}[x(\tau)] e^{iS[x(\tau), \alpha(\tau)]}$$

$$\times \int_{x_0}^{x_t} \mathcal{D}^*[x'(\tau)] e^{iS[x'(\tau), \alpha'(\tau)]} \int dx_0 dx'_0 \langle \alpha| \rho_0 |x_0, \alpha_0\rangle$$

$$= \int_{\alpha_0}^{\alpha_t} \mathcal{D}[\alpha(\tau)] \int_{\alpha_0'}^{\alpha_t'} \mathcal{D}[\alpha'(\tau)] \mathcal{F}[\alpha(\tau), \alpha'(\tau)],$$  \hspace{1cm} \text{(A.3)}

where we introduce the functional field as

$$\mathcal{F}[\alpha(\tau), \alpha'(\tau)] = \int dx dx_0 dx'_0 \langle x_0| \rho_b |x_0'\rangle$$

$$\times \int_{x_0}^{x_t} \mathcal{D}[x(\tau)] \int_{x_0'}^{x_t'} \mathcal{D}^*[x'(\tau)] e^{iS[x(\tau), \alpha(\tau)]}$$

$$\times e^{iS[x'(\tau), \alpha'(\tau)]}$$

$$= \text{Tr}_b[\rho_b U_b^\dagger(\alpha') U_b(\alpha)]$$  \hspace{1cm} \text{(A.4)}

with $\langle x_0| U_b(\alpha)| x_0\rangle = \int_{x_0}^{x_t} D[x(\tau)] e^{iS[x(\tau), \alpha(\tau)]}$. The functional field generally satisfies $\mathcal{F}[\alpha, \alpha'] \leq 1$, where $\mathcal{F}[\alpha, \alpha'] = 1$ if there is no system–bath interaction.
Then, the harmonic bath is included, which is described by \( H_b = \sum_k \frac{\tilde{\beta}^2_k}{2m_k} + \frac{m_k \Omega_k^2}{2} x_k^2 \). Without loss of generality, we first only consider one mode case \( H_b = \frac{\tilde{\beta}^2}{2m} + \frac{m \Omega^2}{2} x^2 \). The system–bath coupling term is \( H_{sb} = \hat{Q}(\tau) \hat{x}(\tau) \). In the interaction representation, \( H_{sb} = \tilde{Q}(t) \hat{x}(t) \), and \( U_b^{[\alpha]}(\tau) = e^{-i \int_0^\tau d\tau' Q[\alpha(\tau')] \hat{x}(\tau')} \). Through a tedious but standard Feynman–Vernon influence functional calculation, we derive the influence functional as \( \mathcal{F}[\alpha, \alpha'] = e^{-\Phi[\alpha, \alpha']} \), where the influence phase is

\[
\Phi[\alpha, \alpha'] = \int_0^t dt' \int_0^t ds \{ Q[\alpha(t')] - Q[\alpha'(t')] \} [C(t' - s) Q[\alpha(s)] - C^*[t' - s) Q[\alpha'(s)]],
\]

where the corresponding bath correlation function is

\[
C(\tau) = \frac{1}{m \Omega^2} [\cot h(\beta \Omega/2) \cos \Omega \tau - i \sin \Omega \tau].
\]

Note here that the variable \( Q[\alpha] \) is a real number. For the present multi-mode case, the system–bath interaction is \( H_{sb} = \sum_a \hat{Q}_a \hat{F}_a \), and \( \hat{F}_a = \sum_k c_k^a \hat{x}_k \), then the influence functional is easily generalized to

\[
\mathcal{F}[\alpha(\tau), \alpha'(\tau)] = \exp \left( - \sum_a \int_0^t dt \{ \hat{Q}_a[\alpha(\tau)] - \hat{Q}_a[\alpha'(\tau)] \} [\tilde{Q}_a[\alpha(\tau)] - \tilde{Q}_a[\alpha'(\tau)] \right) \),
\]

where

\[
\tilde{Q}_a[\alpha(t)] = \sum_{\alpha'} \int_0^t d\tau C_{aa'}(t - \tau) \hat{Q}_a[\alpha(\tau)].
\]

The correlation function in independent baths is

\[
C_{aa'}(\tau) = \langle F_a(\tau) F_{a'}(0) \rangle
\]

\[
= \sum_k \delta_{aa'} \frac{\tilde{\beta}_a^2}{2m_k \Omega_k^2} \cot h\left( \frac{\beta_a \Omega_k^2}{2} \right) \cos(\Omega_k^2 \tau) - i \sin(\Omega_k^2 \tau). \]

By using the spectral function \( J_a(\omega) = \frac{1}{2} \sum_k \frac{\tilde{\beta}_k^2}{m_k \Omega_k^2} \delta(\omega - \Omega_k^2) \), we finally have

\[
C_{aa'}(\tau) = \delta_{aa'} \int d\omega J_a(\omega) [\cot h\left( \frac{\beta_a \omega}{2} \right) \cos(\omega \tau) - i \sin(\omega \tau)].
\]

In the Schrödinger representation, the equation of motion for \( \rho_s(t) \) is

\[
\frac{\partial \rho_s(t)}{\partial t} = -i[H_s, \rho_s(t)] + e^{-iH_s t} \frac{\partial \rho^1_s(t)}{\partial t} e^{iH_s t}.
\]

Before dealing with the derivative of \( \rho^1_s(t) \) under specific configurations \( \alpha_i^{(\tau)} \) as shown in equation (A.3), we should keep in mind that \( \alpha_i^{(\tau)} \) are the arbitrary configurations of the system in time \( t \), but not functions of \( t \) (\( \frac{\partial \alpha_i^{(\tau)}}{\partial \tau} = 0 \)). Moreover \( \int_{\alpha_i^{(\tau)}} \mathcal{D}[\alpha(\tau)] \) is a functional integral symbol commuting with \( \frac{\partial}{\partial t} \). Hence from equations (A.3) and (A.4), we have

\[
\frac{\partial \langle \alpha_i^{(\tau)} | \rho^1_s(t) | \alpha_i^{(\tau)} \rangle}{\partial t} = \int d\omega_0 d\omega_0' \langle \alpha_0 | \rho_s(0) | \alpha_0' \rangle \int_{\alpha_0}^{\alpha_i^{(\tau)}} \mathcal{D}[\alpha(\tau)] \int_{\alpha_0'}^{\alpha_i^{(\tau)}} \mathcal{D}^*[\alpha'(\tau)] \frac{\partial \mathcal{F}[\alpha(\tau), \alpha'(\tau)]}{\partial t}.
\]
The key step here is to connect time differentiation of the density matrix with that of the influence functional. The proper way is employing the hierarchical equation by introducing the auxiliary influence functional

$$\langle \alpha_i | \rho_s(n, t) | \alpha'_i \rangle = \int d\alpha_0 d\alpha'_0 \langle \alpha_i | \rho_s(0) | \alpha'_i \rangle \int_{\alpha_0}^{\alpha'_0} D[\alpha(\tau)] \int_{\alpha'_0}^{\alpha_0} D^*[\alpha'(\tau)] F_n[\alpha(\tau), \alpha'(\tau)]. \quad (A.12)$$

Such auxiliary choice is closely dependent on the specific form of the bath correlation function. The hierarchical equation used in this paper is described in the following.

For two independent baths with Lorentzian type spectral function $J_a(\omega) = \frac{\lambda_a \gamma_a}{2\pi (\omega - \omega_0^a)^2 + \gamma_a^2}$, the correlation function simply is

$$C_{aa}(\omega) = \frac{\lambda_a \gamma_a}{2} \exp[-(\gamma_a + i \omega_0^a)t]. \quad (A.13)$$

Based on the following formula

$$\frac{\partial}{\partial t} \tilde{Q}_a[\alpha(t)] = \frac{\lambda_a \gamma_a}{2} Q_a[\alpha(t)] - (\gamma_a + i \omega_0^a) \tilde{Q}_a[\alpha(t)],$$

and

$$\frac{\partial}{\partial t} F[\alpha(\tau), \alpha'(\tau)] = - \sum_a (Q_a[\alpha(t)] - Q_a[\alpha'(t)])[\tilde{Q}_a[\alpha(t)] - \tilde{Q}_a^*[\alpha'(t)] F[\alpha(\tau), \alpha'(\tau)],$$

we introduce the auxiliary influence functional of each subsystem as

$$F_{a,k}[\alpha(\tau), \alpha'(\tau)] = (\tilde{Q}_a[\alpha(t)])^{k_1} (\tilde{Q}_a^*[\alpha'(t)])^{k_2} F_a[\alpha(\tau), \alpha'(\tau)], \quad (A.14)$$

where $Q_a = \sigma_a$, $\alpha = A, B$ and index $k = (k_1, k_2)$. Then the auxiliary equations may be in proper hierarchy. The whole influence functional becomes

$$F_{\tilde{n}, \tilde{m}}[\alpha(\tau), \alpha'(\tau)] = F_{A, \tilde{n}}[\alpha(\tau), \alpha'(\tau)] F_{B, \tilde{m}}[\alpha(\tau), \alpha'(\tau)], \quad (A.15)$$

where $\tilde{n} = (n_1, n_2)$ and $\tilde{m} = (m_1, m_2)$. Finally, from equations (A.10), (A.12), (A.14) and (A.15) the evolution of the density matrix reads

$$\frac{\partial}{\partial t} \rho_{\tilde{n}, \tilde{m}}(t) = -(iH^R_A + \tilde{n} \cdot \tilde{v}_A + \tilde{m} \cdot \tilde{v}_B) \rho_{\tilde{n}, \tilde{m}}(t) + \sum_{k=1,2} (-1)^k Q^{x}_{A} \rho_{\tilde{n} - \tilde{e}_k, \tilde{m}}(t) + \sum_{k=1,2} (-1)^k Q^{x}_{B} \rho_{\tilde{n}, \tilde{m} + \tilde{e}_k}(t)$$

$$+ \frac{\lambda_A Y_A}{4} \sum_{k=1,2} n_k [Q^{\alpha}_{A} + (-1)^{k+1} Q^{\alpha^*}_{A}] \rho_{\tilde{n} - \tilde{e}_k, \tilde{m}}(t) + \sum_{k=1,2} (-1)^k Q^{x}_{B} \rho_{\tilde{n}, \tilde{m} + \tilde{e}_k}(t)$$

$$+ \frac{\lambda_B Y_B}{4} \sum_{k=1,2} m_k [Q^{\alpha}_{B} + (-1)^{k+1} Q^{\alpha^*}_{B}] \rho_{\tilde{n}, \tilde{m} - \tilde{e}_k}(t), \quad (A.16)$$

where unit vectors $\tilde{e}_1 = (1, 0)$ and $\tilde{e}_2 = (0, 1)$, $\tilde{v}_\alpha = (v^\alpha_+, v^\alpha_-)$ with $v^\alpha_+ = \gamma_a + i \omega_0^a$ and $v^\alpha_- = \gamma_a - i \omega_0^a$. We denote $A^X B = AB - BA$ and $A^Y B = AB + BA$.

For common bosonic bath with the similar Lorentzian distribution, if we choose the auxiliary influence functional

$$F_{\tilde{n}}[\alpha, \alpha'] = i^{n_1} (\tilde{Q}^*[\alpha'(t)])^{n_1} (-i)^{n_2} (\tilde{Q}[\alpha(t)])^{n_2} F[\alpha, \alpha'].$
the hierarchical equation can be expressed by

\[
\frac{\partial \rho_{\vec{n}}(t)}{\partial t} = - (iH_S^c + \vec{n} \cdot \vec{\nu}) \rho_{\vec{n}}(t) - i \sum_{k=1,2} Q^\times \rho_{\vec{n}+\vec{e}_k}(t)
- \frac{i\gamma\lambda}{4} \sum_{k=1,2} n_k [Q^\times + (-1)^k Q^o] \rho_{\vec{n}-\vec{e}_k}(t). \tag{A.17}
\]

It is to be noted that this equation is derived in an alternative way in [52].

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