Moment Matching Method for Pricing Spread Options with Mean-Variance Mixture Lévy Motions

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Abstract

The paper Borovkova et al. \cite{Borovkova2010} uses the moment matching method to obtain closed form formulas for spread and basket call option prices under log normal models. In this note, we also use the moment matching method to obtain semi-closed form formulas for the price of spread options under exponential Lévy models with mean-variance mixture. Unlike the semi-closed form formula in Caldana and Fusai \cite{Caldana2014}, where spread prices were expressed by using a Fourier inversion formula that involves all the log return processes, our formula gives spread prices in terms of the mixing distribution of the log returns. Numerical tests show that our formulas give accurate spread prices also.

Keywords: Spread Option · Moment Matching · Mean-Variance Mixture models · Lévy Price Dynamics

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1 Introduction

Spread options are popular financial derivatives in fixed income, currency, commodity, and equity markets. They are used to hedge portfolios of long and short positions in the underlying assets. A spread option is a European call option on the spread of two assets. It gives the holder the right, but not the obligation, to purchase the spread of two assets at a fixed strike price. Its price is given by the following risk-neutral valuation formula

$$
\Pi = e^{-rT} E^Q [S_1(T) - S_2(T) - K]^+,
$$

where $K$ is the strike price, $S_i(T)$ is the price of asset $i$ at maturity $T$ for $i = 1, 2$, $r > 0$ is the risk-free interest rate, and $Q$ is the risk-neutral measure, see Delbaen and Schachermayer \cite{Delbaen2000}.
Rásonyi [22], for the details of risk-neutral pricing and other related facts about risk-neutral measures.

Despite this simple European call option structure as in (1), spread option prices do not give closed form solutions even under the log normal models. In fact, many papers were devoted in obtaining closed form approximations to spread prices under log-normal models in the past, see Kirk [19], Bjerksund and Stensland [3], Carmonna and Durrleman [7], [8], Borovkova at al. [4] for example. Especially, the paper Borovkova at al. [4] used moment matching method to obtain a closed form formula for spread and in general for basket call option prices under the log normal models. Their main approach was to approximate a linear combination of log normal prices by shifted log normal random variables by moment matching. It was tested that, their closed-form formulas give highly precise spread option prices.

Moment matching techniques in pricing options were used in many other papers in the past also. For example, the seminal paper Jarrow and Rudd [17] proposed a method in pricing standard European options under distributions known only through their moments. They derived an option pricing formula through an Edgeworth series expansion of the distributions of the security prices. This Edgeworth expansion technique is in fact amounts to matching moments of random variables, see (A.5) in their paper for example. The series were truncated after the fourth term since for practical purposes the first four moments of the underlying distribution can capture most of the effect on option prices. Corrado and Su [10], following the approach in Jarrow and Rudd [17], found an approximate probability density function using a Gram-Charlier expansion of the normal density function, which amounts to matching higher order moments of underlying distributions.

Pricing spread options under non Gaussian models was also discussed in the past, see [5] for example. They express spread option price using a Fourier inversion formula which needs to be evaluated numerically, see Proposition 1 of their paper for the details. For other papers that use various numerical methods in pricing spread options see Hurd and Zhou [16], Cheang and Chiarella [9], Dempster and Hong [12].

The purpose of this note is to extend the results in Borovkova et al. [4] from log normal models to the case of mean-variance mixture Lévy motions. An m dimensional random variable $X$ has mean-variance normal mixture structure if

$$X \overset{d}{=} \alpha + \beta Y + \sqrt{Y} N^m,$$

(2)

where $\alpha, \beta \in \mathbb{R}^m$ are constant vectors, $Y$ is a non-negative scalar random variable, $N^m$ is an $m$–dimensional Normal random variable with correlation matrix $\Sigma$, i.e., $N^m \sim N(0, \Sigma)$, and $Y$ is a positive valued random variable independent from $N^m$.

We say that an $m$–dimensional Lévy process $L_t = (L_1^t, L_2^t, \ldots, L_m^t)$ has mean variance structure if $L_1 \overset{d}{=} X$ for some $X$ as in (1). These types of Lévy motions are quite popular in financial modelling, see Barndorff-Nielsen [1], Eberlein and Keller [13], Bingham and Kiesel [2], Schoutens [23], Prause [20], Raible [21].

In fact, it is well known that for any mean-variance mixture random variable $X$ as in (1), we have an associated Lévy process $L_t$ with $L_t \sim X$ as long as $Y$ is an infinitely divisible positive random variable, see Lemma 2.6 of EA Hammerstein [15].
Our goal in this note is to obtain a closed form formula for the price of the spread option \( (1) \) when the price dynamics is given by

\[
S_1(t) = S_1(0)e^{\omega_1 t + rt + L_1^t}, \quad S_2(t) = S_2(0)e^{\omega_2 t + rt + L_2^t},
\]

for \( L_t = (L_1^t, L_2^t) \) that has mean-variance mixture structure as in (2). In (3), \( S_1(0) \) and \( S_2(0) \) denote initial stock prices and \( \omega_1 \) and \( \omega_2 \) are determined by the martingale condition (see equation (4) of Jurczenko et al. [18]), i.e., the discounted prices \( e^{-rt} S_i(t), i = 1, 2 \), need to be martingales. Therefore we need to have \( E(e^{-rt} S_i(t)) = S_i(0), i = 1, 2 \), and these conditions give us

\[
\omega_1 = -\ln E e^{L_1^1}, \quad \omega_2 = -\ln E e^{L_2^2}.
\]

We can assume that the dynamics of \( L_i^t, i = 1, 2 \), satisfy

\[
L_i^t \overset{d}{=} \delta_i + \beta_i Y_t + \sigma_i \sqrt{Y_t} N(0, 1), \quad i = 1, 2,
\]

for some scalar constants \( \delta_i, \beta_i, \sigma_i, i = 1, 2 \), and some positive valued process \( Y_t \). Under these assumptions, the expressions for \( \omega_i, i = 1, 2 \), in (3) under the martingale condition are given by

\[
\omega_i = -\delta_i - \ln \left( \int_{-\infty}^{+\infty} e^{(\beta_i + \tau^2) y} f_Y(y) dy \right), \quad i = 1, 2,
\]

where \( f_Y \) is the density function of the mixing random variable \( Y \).

### 2 Approximations

As mentioned earlier, we would like to give a closed form expression for (1) by using the moment matching method. First we fix some notations. In (3), we denote \( \tilde{L}_1^t = \omega_1 t + rt + \ln S_1(0) + L_1^t \) and \( \tilde{L}_2^t = \omega_2 t + rt + \ln S_2(0) + L_2^t \). With these notations, we can write

\[
S_1(t) - S_2(t) = e^{\tilde{L}_1^t} - e^{\tilde{L}_2^t}.
\]

If we denote \( \alpha_1 = \delta_1 + r + \omega_1 + \ln S_1(0) \) and \( \alpha_2 = \delta_2 + r + \omega_2 + \ln S_2(0) \), we can write

\[
\tilde{L}_i^t \overset{d}{=} \alpha_i t + \beta_i Y_t + \sigma_i \sqrt{Y_t} N_i(0, 1), i = 1, 2.
\]

Here \( N_1 \) and \( N_2 \) are correlated standard normal random variables. We would like to approximate the spread price

\[
\Pi_T := e^{-rT} E(e^{\tilde{L}_1^T} - e^{\tilde{L}_2^T}^+).
\]

Without loss of any generality below we calculate (9) for \( T = 1 \). For the sake of notations we denote \( Y = Y_1 \) and \( X = (\tilde{L}_1^1, \tilde{L}_2^1) \). Then \( X \) has the form (3) in dimension two. We would like to approximate \( e^{X_1} - e^{X_2} \) by some appropriate random variable \( W \). Below we divide our
discussions into three cases; $\beta \neq 0$, $\beta = 0$, and when $\beta = 0$ and $N^2$ in (2) is replaced by the uniform random variable $U^2$ on the unit circle of the Euclidean space $R^2$.

In the following discussions we denote the moment generating function of the mixing distribution $Y$ in (2) by $\phi_Y(s)$. Since $Y$ is a non-negative random variable, $\phi_X(s)$ exists for all $s \in (-\infty, 0]$. We assume also that there exists some positive number $D > 0$ (can be equal to $+\infty$) such that $\phi_X(s)$ is finite for all $s \in (-\infty, D)$.

### 2.1 The case $\beta \neq 0$

This case corresponds to Lévy processes with mean-variance structure. We need to evaluate

$$\Pi_{mv} = e^{-r}E[(e^{X_1} - e^{X_2} - K)^+],$$

when $X = (X_1, X_2)^T$ has the mean-variance structure as in (2) with $\alpha = (\mu_1, \mu_2)^T, \beta = (\beta_1, \beta_2)^T$, and $N^2 = A(N_1, N_2)^T$ with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for some real numbers $a_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 2$. Here $\mu_1 = \delta_1 + r + \omega_1 + \ln S_1(0), \mu_2 = \delta_2 + r + \omega_2 + \ln S_2(0)$, and $N_1, N_2$ are two independent standard Normal random variables. The matrix $A$ is the square root of the covariance matrix $\Sigma$ of the two dimensional normal random variable $N^2$, i.e., $A = \Sigma^{\frac{1}{2}}$. In vector form, we can write $X$ as follows

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} Y + \sqrt{Y} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}. \quad (11)$$

Next, we introduce a random variable $W_{mv}$ as follows

$$W_{mv} \overset{d}{=} e^{a\sqrt{Y}N+bY+c} + d, \quad (12)$$

with four scalar parameters $a, b, c, d$. Our goal is to estimate these parameters by moment matching. Namely, we let

$$E(e^{X_1} - e^{X_2})^n = EW_{mv}^n, \quad n = 1, 2, 3, 4, \quad (13)$$

and obtain four equations related with these four parameters $a, b, c, d$. The calculations of the moments are presented in the Appendix A. The relations (13) give us the following four equations of four unknown scalars $a, b, c, d$,}
\[ M_1 = e^{\hat{c}} \phi_Y \left( \frac{a^2}{2} + b \right) + d, \]
\[ M_2 = e^{2\hat{c}} \phi_Y \left( 2a^2 + 2b \right) + 2de^{\hat{c}} \phi_Y \left( \frac{a^2}{2} + b \right) + d^2, \]
\[ M_3 = e^{3\hat{c}} \phi_Y \left( \frac{9a^2}{2} + 3b \right) + 3de^{2\hat{c}} \phi_Y \left( 2a^2 + 2b \right) + 3d^2e^{\hat{c}} \phi_Y \left( \frac{a^2}{2} + b \right) + d^3, \]
\[ M_4 = e^{4\hat{c}} \phi_Y \left( 8a^2 + 4b \right) + 4de^{3\hat{c}} \phi_Y \left( \frac{9a^2}{2} + 3b \right) + 6d^2e^{2\hat{c}} \phi_Y \left( 2a^2 + 2b \right) + 4d^3e^{\hat{c}} \phi_Y \left( \frac{a^2}{2} + b \right) + d^4. \]

(14)

**Remark 2.1.** We assume that the parameters \(a, b, c, d,\) in our above discussions are such that the arguments of the function \(\phi(\cdot)\) in (14) and in the expressions of \(M_1, M_2, M_3, M_4\) in the Appendix are less than \(D\) to guarantee the existence of the moment generating function.

From (14) we obtain the following simpler equations
\[
\hat{c} \phi_Y \left( \frac{\hat{a}}{2} + b \right) = M_1 - d,
\]
\[
e^{\hat{c}^2} \phi_Y \left( 2\hat{a} + 2b \right) = M_2 - 2dM_1 + d^2,
\]
\[
e^{\hat{c}^3} \phi_Y \left( \frac{9\hat{a}}{2} + 3b \right) = M_3 - 3dM_2 + 3d^2M_1 - d^3,
\]
\[
e^{\hat{c}^4} \phi_Y \left( 8\hat{a} + 4b \right) = M_4 - 4dM_3 + 6d^2M_2 - 4d^3M_1 + d^4,
\]

where \(\hat{c} = e^c, \hat{a} = a^2.\)

Then the spread price \(\Pi_{nv}\) in (10) can be approximated as
\[
\Pi_{nv} \approx e^{-r} E[(e^{a\sqrt{Y}N+bY+c} + d - K)^+] \quad (16)
\]

with \(a, b, c, d,\) satisfying (15).

Below we calculate the right-hand-side of (16). If \(d \geq K,\) the right-hand-side of (16) becomes
\[
E[(e^{a\sqrt{Y}N+bY+c} + d - K)^+] = E[e^{a\sqrt{Y}N+bY+c} + d - K] = e^{\hat{c}} \phi_Y \left( \frac{a^2}{2} + b \right) + d - K. \quad (17)
\]

If \(d < K,\) then we have
\[
E[(e^{a\sqrt{Y}N+bY+c} + d - K)^+] = E \left[ e^{a\sqrt{Y}N+bY+c} \cdot 1_{\{e^{a\sqrt{Y}N+bY+c} + d - K \geq 0\}} \right] + (d - K)P \left( e^{a\sqrt{Y}N+bY+c} + d - K \geq 0 \right). \quad (18)
\]
The first term in the right-hand-side of (18) is equal to

\[ E \left[ e^{a\sqrt{\gamma N} + bY + c} \cdot 1_{\{e^{a\sqrt{\gamma N} + bY + c + d - K} \geq 0\}} \right] \]

\[ = e^c E \left[ e^{(a^2/2 + b)Y} \cdot 1_{\{e^{a\sqrt{\gamma N} - a^2/2Y} \cdot 1\} \geq 0\}} \right] \]

\[ = e^c E \left[ e^{(a^2/2 + b)Y} \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right] \]

Here we can assume \( a \) is a positive number as \( aN \) has the same distribution as \( |a|N \). The second term in the right-hand-side of (18) is equal to

\[ P \left( e^{a\sqrt{\gamma N} + bY + c + d - K} \geq 0 \right) \]

\[ = E \left[ \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right] \]

(19)

If we plug (19) and (20) into (18) we obtain

\[ E \left[ e^{a\sqrt{\gamma N} + bY + c} \cdot 1_{\{e^{a\sqrt{\gamma N} + bY + c + d - K} \geq 0\}} \right] \]

\[ = e^c E \left[ e^{(a^2/2 + b)Y} \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right] \]

+ \( (d - K)E \left[ \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right] \]

(21)

Finally, we obtain the following approximate formula for the spread price

\[ \Pi_{mv} \approx \begin{cases} 
\begin{align*}
e^{-r} \phi_Y \left( \frac{a^2}{2} + b \right) + d - K, \\
e^{-r} E \left[ e^{(a^2/2 + b)Y} \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right] \\
+ (d - K)e^{-r} E \left[ \Phi \left( -\frac{\ln(K - d) - (a^2 + b)Y - c}{a\sqrt{Y}} \right) \right], 
\end{align*} \end{cases} \quad d \geq K. \]

(22)

Remark 2.2. The formula (22) is obtained by matching the first four moments of the spread price and of \( W_{mv} \) in (12). The main difficulty in using the formula (22) is to calculate the parameters \( a, b, c, d \) from (15). Especially, since these parameters show up in the argument of the moment generating function \( \phi_Y(s) \) that is finite only for \( s < D \), one can only apply this formula to models with model parameters that satisfy the stated conditions in Remark 2.1 above. If no \( a, b, c, d \) exist that satisfy the stated conditions in remark 2.1 above for a given model, then our moment matching method is not applicable for such models.
Remark 2.3. To solve (15) for $a, b, c, d$ in our numerical results, we approximated the moment generating function $\phi_Y(\cdot)$ by a fourth order polynomial, i.e., we approximated $\phi_Y(s)$ by $\bar{\phi}_Y(s) = 1 + sEY + \frac{s^2}{2} EY^2 + \frac{s^3}{3!} EY^3 + \frac{s^4}{4!} EY^4$ in (16). Still our numerical results gave accurate spread prices.

Remark 2.4. We remark that the formula in the right-hand-side of (22) has the potential to be used in pricing spread options based on empirical data without relying on (15) to obtain $a, b, c, d$. To see this let $\bar{L}(a, b, c, d)$ denote the right-hand-side of (22) with $\phi_Y$ replaced by $\bar{\phi}_Y$ in remark 2.3 above. Then one can obtain $a, b, c, d$, by minimizing the distance of $\bar{L}(a, b, c, d)$ with the real world spread prices.

2.2 The case $\beta = 0$

Now we assume $\beta_1 = \beta_2 = 0$. In this case, $X$ in (2) has normal variance mixture structure. We single out this case as in this case we can have a simpler formula than (22) for the price of spread options as will be seen below. In this case, the vector form of $X$ takes the following form

$$(X_1, X_2) \sim \begin{pmatrix} N(\mu_1, \sigma^2_1) \\ N(\mu_2, \sigma^2_2) \end{pmatrix}.$$

We denote the corresponding spread price by

$$\Pi_v =: e^{-r} E[(e^{X_1} - e^{X_2} - K)^+].$$

(24)

To approximate $e^{X_1} - e^{X_2}$ in (24), we introduce the following random variable

$$W_v \sim e^{a \sqrt{Y} N + bY + c}$$

with three scalar parameters $a, b, c$. Similar to the case of $\beta \neq 0$, we match the first three moments, i.e.,

$$E(e^{X_1} - e^{X_2})^n = E[W_v]^n, \quad n = 1, 2, 3.$$

(26)

The relation (26) leads us to the following equations (for the details see the Appendix B)

$$\begin{align*}
M_1 &= e^{b \phi_Y(\frac{a^2}{2})} + c, \\
M_2 &= e^{2b \phi_Y(2a^2)} + 2ce^{b \phi_Y(\frac{a^2}{2})} + c^2, \\
M_3 &= e^{3b \phi_Y(3a^2)} + 3ce^{2b \phi_Y(2a^2)} + 3c^2 e^{b \phi_Y(\frac{a^2}{2})} + c^3,
\end{align*}$$

(27)

where $M_1, M_2, M_3$ denote the first three moments of $e^{X_1} - e^{X_2}$.

Following the same steps as the normal mean-variance mixture case, we get the following closed form approximate spread price

$$\Pi_v \approx \begin{cases} 
\begin{align*}
e^{b-r} \phi_Y(\frac{a^2}{2}) + c - K, & c \geq K, \\
e^{b-r} E \left[ e^{\frac{a^2}{2} Y} \Phi \left( -\frac{\ln(K-c) - a^2 Y - b}{a \sqrt{Y}} \right) \right] & c < K \\
+ (c - K) e^{-r} E \left[ \Phi \left( -\frac{\ln(K-c) - b}{a \sqrt{Y}} \right) \right]. &
\end{align*}
\end{cases}$$

(28)
Remark 2.5. The formula (28) involves three parameters $a, b, c$, only. Our numerical tests show that this approximation is more accurate than (22).

2.3 Elliptical Case

In this section, we assume that the log returns \( \left( \ln[S_1(t)/S_1(0)],\ln[S_2(t)/S_2(0)] \right) \) of price processes have the following elliptical structure under the risk neutral measure

\[
\begin{pmatrix}
\ln[S_1(1)/S_1(0)] \\
\ln[S_2(1)/S_2(0)]
\end{pmatrix}
\overset{d}{=}
\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix}
+ \sqrt{R} \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
U_1^{(2)} \\
U_2^{(2)}
\end{pmatrix},
\] (29)

for some real numbers $\mu_i, i = 1, 2$, $a_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 2$, and a positive valued random variable $R$ which is independent from the uniform random variable $U^{(2)} = (U_1^{(2)}, U_2^{(2)})^T$ on the unit circle in $\mathbb{R}^2$. It is well known that the density functions of $U_1^{(2)}$ and $U_2^{(2)}$ take the following form

\[
f_{U_1^{(2)}}(x) = f_{U_2^{(2)}}(x) = \frac{1}{\pi} \sqrt{1 - x^2}, \quad -1 < x < 1.
\] (30)

We denote $X_1 =: \ln[S_1(t)/S_1(0)], X_2 =: \ln[S_2(t)/S_2(0)]$ and let

\[
\Pi_e =: e^{-r}E[(S_1(0)e^{X_1} - S_2(0)e^{X_2} - K)^+].
\] (31)

be the price of the corresponding spread option. Our goal in this section is to approximate $S_1(0)e^{X_1} - S_2(0)e^{X_2}$ by an appropriate random variable $W_e$. For the random variable $W_e$, we take

\[
W_e = e^{aU_1^{(2)}+b} + c.
\]

This is a random variable with three parameters $a, b, c$, and we determine them by the following moment conditions

\[
E(S_1(T) - S_2(T))^n = EW_e^n, \quad n = 1, 2, 3.
\] (32)

To find the right-hand-side of (32), we need to recall the following two properties of the spherical random variable $U^{(2)}$. The first property, we need to recall is

\[
v'U^{(2)} = ||v||U_1^{(2)}
\] (33)

for any vector $v \in \mathbb{R}^2$. The second property is that, for any integers $m_1, m_2$, with $m = \sum_{i=1}^2 m_i$, the following equation on mixed moments holds

\[
E \left[ \Pi_{i=1}^2 U_i^{m_i} \right] = \begin{cases}
\Pi_{i=1}^2 \frac{(m_i)!}{2^{m_i}(m_i/2)!}, & \text{if } m_i \text{ are even for both } i = 1, 2, \\
0, & \text{if at least one of the } m_i \text{ is odd}.
\end{cases}
\] (34)

For the details of these results see Fang et al. [14].

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By using the relations (33) and (34), we can calculate the three moments of \( W_c \). Let \( M_1, M_2, M_3 \) be the moments up to order three of \( S_1(T) - S_2(T) \) (for the calculation of these three moments see Appendix C). Then by (32), we obtain the following equations

\[
\begin{align*}
M_1 &= e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + c, \\
M_2 &= e^{2b} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + 2c e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + c^2, \\
M_3 &= e^{3b} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + 3c e^{2b} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + 3c^2 e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + c^3.
\end{align*}
\]

\( (35) \)

After we determine \( a, b, \) and \( c \) from (35), we need to calculate \( E[(e^a \sqrt{RU_1^{(2)}} + b + c - K)^+] \).

We divide into cases: if \( c \geq K \) we have

\[
E[(e^a \sqrt{RU_1^{(2)}} + b + c - K)^+] = E[e^a \sqrt{RU_1^{(2)}} + b + c - K] = e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + c - K.
\]

(36)

If \( c < K \), we have

\[
E[(e^a \sqrt{RU_1^{(2)}} + b + c - K)^+] = E \left[ e^a \sqrt{RU_1^{(2)}} + b + \mathbf{1}_{e^a \sqrt{RU_1^{(2)}} + b + c - K \geq 0} \right] + (c - K) P(e^a \sqrt{RU_1^{(2)}} + b + c - K \geq 0).
\]

(37)

The first term of (37) can be written as

\[
E \left[ e^a \sqrt{RU_1^{(2)}} + b + \mathbf{1}_{e^a \sqrt{RU_1^{(2)}} + b + c - K \geq 0} \right] = e^b \int_{\ln(K-c)-b}^{+\infty} e^{az+b} h(z) dz.
\]

(38)

where \( h(z) \) is the probability density function of \( \sqrt{RU_1^{(2)}} \). By using this probability density function \( h(z) \), we can calculate the probability in the second term on (37) as follows

\[
P(e^a \sqrt{RU_1^{(2)}} + b + c - K \geq 0) = P \left( \sqrt{RU_1^{(2)}} \geq \frac{\ln(K-c) - b}{a} \right) = \int_{\ln(K-c)-b}^{+\infty} h(z) dz.
\]

(39)

We plug (38) and (39) into (37) and obtain the following relation

\[
E[(e^a \sqrt{RU_1^{(2)}} + b + c - K)^+] = e^b \int_{\ln(K-c)-b}^{+\infty} e^{az+b} h(z) dz + (c - K) \int_{\ln(K-c)-b}^{+\infty} h(z) dz.
\]

(40)

Finally, from (40) and (36) we obtain the following approximate closed form formula for the spread price

\[
\Pi_c \approx \begin{cases} 
 e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + c - K, & c \geq K, \\
 e^b \int_{(\ln(K-c)-b)}^{+\infty} e^{az+b} h(z) dz + (c - K) e^{-r} \int_{(\ln(K-c)-b)}^{+\infty} h(z) dz, & c < K.
\end{cases}
\]

(41)
2.4 Numerical Results

In this part we check the performance of the approximations (22) and (28). We test these formulas when the mixing distribution $Y$ is equal to the exponential, gamma, and inverse Gaussian random variables separately. We calculate the spread price (24) by the Monte Carlo method and also by our approximate formulas (22) and (28).

The Tables give the comparison of the performance of our formulas (22) and (28) against the spread prices obtained by the Monte Carlo methods. The results obtained by Monte Carlo method (with one million trials) is listed in the bracket next to the result obtained by the formulas (22) and (28). In Table 1, the mixing random variable is $Y \sim \text{Exp}(1)$, and the other parameters are as follows $r = 0$, $\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$. From this table it can be seen that when $Y \sim \text{Exp}(1)$, the formula (22) approximates the true spread price (obtained by the Monte Carlo method) pretty well.

In Table 2 below we take $Y \sim \text{Gamma}(2, 1)$, i.e., $Y$ is a gamma random variable with shape parameter 2 and scale parameter 1. For other parameters we pick $r = 0$, $\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$. From the table it can be seen that the formula (22) gives very precise results for the spread prices also in this case.

In Table 3 below we take $Y \sim IG(\frac{1}{\sqrt{2}}, 1)$. The other parameters are $r = 0$, $\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$. We can see that, in this case the approximated spread prices are very close to Monte Carlo results also.

Table 4 compares the performance of our formula (22) with the spread price obtained by Monte Carlo methods. For the mixing distribution we used $Y \sim \text{Exp}(1)$. The other parameters of the model are listed in and below the table. Similarly, Table 5 compares the performance of (22) with Monte Carlo method when $Y \sim \text{Gamma}(2, 1)$, and Table 6 compares the performance of (22) with the Monte Carlo method when $Y \sim IG(\frac{1}{\sqrt{2}}, 1)$. We observe that the approximation works better for variance mixture models than mean-variance mixture models as can be seen in these tables.
### Table 1: Exponential Mixture

| $S_1 = 5, S_2 = 1$ | $K = 1.9$ | $K = 2.0$ | $K = 2.1$ | $K = 2.2$ | $K = 2.3$ |
|-------------------|-----------|-----------|-----------|-----------|-----------|
|                   | 1.9842    | 1.9502    | 1.8696    | 1.7798    | 1.6865    |
|                   | (2.0994)  | (1.9994)  | (1.8995)  | (1.7996)  | (1.6998)  |
| $S_1 = 4, S_2 = 1$ | $K = 1.4$ | $K = 1.5$ | $K = 1.6$ | $K = 1.7$ | $K = 1.8$ |
|                   | 1.5306    | 1.4708    | 1.3833    | 1.2901    | 1.1950    |
|                   | (1.5997)  | (1.4997)  | (1.3999)  | (1.3001)  | (1.2006)  |
| $S_1 = 3, S_2 = 1$ | $K = 0.9$ | $K = 1.0$ | $K = 1.1$ | $K = 1.2$ | $K = 1.3$ |
|                   | 1.0691    | 0.9863    | 0.8941    | 0.7989    | 0.7043    |
|                   | (1.1000)  | (1.0002)  | (0.9006)  | (0.8014)  | (0.7029)  |
| $S_1 = 2, S_2 = 1$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ | $K = 0.6$ | $K = 0.7$ |
|                   | 0.6615    | 0.5932    | 0.4990    | 0.4051    | 0.3158    |
|                   | (0.7006)  | (0.6011)  | (0.5023)  | (0.4048)  | (0.3105)  |
| $S_1 = S_2 = 1$   | $K = 0.1$ | $K = 0.2$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ |
|                   | 0.0209    | 0.0087    | 0.0043    | 0.0024    | 0.0015    |
|                   | (0.0221)  | (0.0103)  | (0.0054)  | (0.0031)  | (0.0018)  |

$r = 0$, $\beta_1 = \beta_2 = 0.1$, $a_{11} = 0.15$, $a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$

### Table 2: Gamma Mixture

| $S_1 = 5, S_2 = 1$ | $K = 1.8$ | $K = 1.9$ | $K = 2.0$ | $K = 2.1$ | $K = 2.2$ |
|-------------------|-----------|-----------|-----------|-----------|-----------|
|                   | 2.1840    | 2.0977    | 1.9994    | 1.9008    | 1.8028    |
|                   | (2.1998)  | (2.1000)  | (2.0004)  | (1.9010)  | (1.8020)  |
| $S_1 = 4, S_2 = 1$ | $K = 1.4$ | $K = 1.5$ | $K = 1.6$ | $K = 1.7$ | $K = 1.8$ |
|                   | 1.5986    | 1.5000    | 1.4018    | 1.3050    | 1.2103    |
|                   | (1.6002)  | (1.5008)  | (1.4018)  | (1.3034)  | (1.2060)  |
| $S_1 = 3, S_2 = 1$ | $K = 0.9$ | $K = 1.0$ | $K = 1.1$ | $K = 1.2$ | $K = 1.3$ |
|                   | 1.0994    | 1.0011    | 0.9045    | 0.8112    | 0.7221    |
|                   | (1.1008)  | (1.0019)  | (0.9041)  | (0.8078)  | (0.7143)  |
| $S_1 = 2, S_2 = 1$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ | $K = 0.6$ | $K = 0.7$ |
|                   | 0.6985    | 0.6009    | 0.5062    | 0.4184    | 0.3395    |
|                   | (0.7019)  | (0.6040)  | (0.5080)  | (0.4160)  | (0.3314)  |
| $S_1 = 1, S_2 = 1$ | $K = 0.1$ | $K = 0.2$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ |
|                   | 0.0340    | 0.0194    | 0.0116    | 0.0074    | 0.0049    |
|                   | (0.0420)  | (0.0231)  | (0.0135)  | (0.0083)  | (0.0054)  |

$r = 0$, $\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$
Table 3: Inverse Gaussian Mixture

| $S_1 = 3, S_2 = 2$ | $K = 0.4$ | $K = 0.5$ | $K = 0.6$ | $K = 0.7$ | $K = 0.8$ |
|---------------------|-----------|-----------|-----------|-----------|-----------|
|                     | 0.5981    | 0.5005    | 0.4077    | 0.3224    | 0.2468    |
|                     | (0.6049)  | (0.5084)  | (0.4146)  | (0.3257)  | (0.2452)  |
| $S_1 = 3, S_2 = 1$ | $K = 1.0$ | $K = 1.1$ | $K = 1.2$ | $K = 1.3$ | $K = 1.4$ |
|                     | 0.9990    | 0.9000    | 0.8002    | 0.7008    | 0.6025    |
|                     | (1.0003)  | (0.9004)  | (0.8007)  | (0.7014)  | (0.6028)  |
| $S_1 = 2, S_2 = 2$ | $K = 0.1$ | $K = 0.2$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ |
|                     | 0.0479    | 0.0279    | 0.0165    | 0.0101    | 0.0064    |
|                     | (0.0512)  | (0.0285)  | (0.0163)  | (0.0097)  | (0.0060)  |
| $S_1 = 2, S_2 = 1$ | $K = 0.4$ | $K = 0.5$ | $K = 0.6$ | $K = 0.7$ | $K = 0.8$ |
|                     | 0.6000    | 0.5001    | 0.4012    | 0.3054    | 0.2170    |
|                     | (0.6004)  | (0.5008)  | (0.4021)  | (0.3056)  | (0.2155)  |
| $S_1 = 1, S_2 = 1$ | $K = 0.1$ | $K = 0.2$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ |
|                     | 0.0139    | 0.0050    | 0.0021    | 0.0010    | 0.0005    |
|                     | (0.0142)  | (0.0049)  | (0.0019)  | (0.0009)  | (0.0005)  |

$\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$

Table 4: Exponential Mixture (Normal Variance Mixture)

| $S_1 = 5, S_2 = 1$ | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ |
|---------------------|--------|--------|--------|--------|--------|
|                     | 2.9196 | 1.9859 | 1.0189 | 0.2516 | 0.0509 |
|                     | (3.0001) | (2.0008) | (1.0204) | (0.2514) | (0.0507) |
| $S_1 = 4, S_2 = 1$ | 1.9692 | 1.0045 | 0.1966 | 0.0268 | 0.0055 |
|                     | (2.0002) | (1.0071) | (0.1964) | (0.0266) | (0.0055) |
| $S_1 = 3, S_2 = 1$ | 0.9958 | 0.1426 | 0.0101 | 0.0014 | 0.0003 |
|                     | (1.0013) | (0.1423) | (0.0101) | (0.0014) | (0.0003) |
| $S_1 = 2, S_2 = 1$ | 0.0914 | 0.0018 | 0.0001 | 0.0000 | 0.0000 |
|                     | (0.0908) | (0.0019) | (0.0001) | (0.0000) | (0.0000) |
| $S_1 = 1, S_2 = 1$ | $K = 0.1$ | $K = 0.2$ | $K = 0.3$ | $K = 0.4$ | $K = 0.5$ |
|                     | 0.0203 | 0.0082 | 0.0034 | 0.0014 | 0.0006 |
|                     | (0.0189) | (0.0075) | (0.0032) | (0.0014) | (0.0007) |

$\beta_1 = \beta_2 = 0.1$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$
Table 5: Gamma Mixture (Normal Variance Mixture)

|       | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ |
|-------|---------|---------|---------|---------|---------|
| $S_1 = 5, S_2 = 1$ | 2.9980 (2.9997) | 2.0028 (2.0027) | 1.0554 (1.0556) | 0.3766 (0.3762) | 0.1174 (0.1168) |
| $S_1 = 4, S_2 = 1$ | 1.9998 (2.0000) | 1.0215 (1.0216) | 0.2943 (0.2938) | 0.0673 (0.0667) | 0.0183 (0.0182) |
| $S_1 = 3, S_2 = 1$ | 1.0038 (1.0039) | 0.2136 (0.2129) | 0.0289 (0.0286) | 0.0055 (0.0055) | 0.0014 (0.0014) |
| $S_1 = 2, S_2 = 1$ | 0.1372 (0.1358) | 0.0064 (0.0064) | 0.0006 (0.0007) | 0.0001 (0.0001) | 0.0000 (0.0000) |
| $S_1 = 1, S_2 = 1$ | 0.0413 (0.0373) | 0.0212 (0.0181) | 0.0107 (0.0090) | 0.0055 (0.0046) | 0.0028 (0.0024) |

$r = 0$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$

Table 6: Inverse Gaussian Mixture (Normal Variance Mixture)

|       | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ |
|-------|---------|---------|---------|---------|---------|
| $S_1 = 5, S_2 = 1$ | 2.9999 (2.9999) | 2.0002 (2.0000) | 1.0093 (1.0093) | 0.2217 (0.2218) | 0.0266 (0.0268) |
| $S_1 = 4, S_2 = 1$ | 2.0000 (1.9999) | 1.0025 (1.0025) | 0.1733 (0.1732) | 0.0122 (0.0124) | 0.0016 (0.0016) |
| $S_1 = 3, S_2 = 1$ | 1.0003 (1.0002) | 0.1256 (0.1255) | 0.0037 (0.0039) | 0.0003 (0.0003) | 0.0000 (0.0001) |
| $S_1 = 2, S_2 = 1$ | 0.0804 (0.0800) | 0.0005 (0.0005) | 0.0000 (0.0000) | 0.0000 (0.0000) | 0.0000 (0.0000) |
| $S_1 = 1, S_2 = 1$ | 0.0138 (0.0130) | 0.0042 (0.0038) | 0.0014 (0.0012) | 0.0005 (0.0004) | 0.0002 (0.0002) |

$r = 0$, $a_{11} = a_{22} = 0.15$, $a_{12} = a_{21} = 0.05$

3 Conclusion

The moment matching method is employed in option pricing in many papers in the past. In this note, we give semi-closed form formulas for spread prices under exponential Lévy models with mean-variance structure by using the moment matching method. The main step in this procedure is to choose the form of the random variable to be used to approximate the spread price as accurately as possible. For this, we make use of the structure of the mean-variance mixture random variables and introduce appropriate approximating random variables with a
couple of scalar parameters. These scalar parameters are then determined by the moment matching method resulting in several equations that needs to be solved.

Unlike the simple and easy to implement closed form formulas for log normal models in the paper Borovkova et al. [4], where a linear combination of log normal models were approximated by shifted log normal models in pricing basket call options, our semi-closed form formulas can be implemented only after some associated parameters are calculated from a system of equations. This difficulty is due to the mean-variance structure of log prices that we imposed on the price dynamics.

Although, the above stated difficulties exist in our approach, in this note we demonstrated the application of the moment matching method to pricing spread options under popular log price dynamics like multi-dimensional variance gamma model, hyperbolic Lévy motions etc. Our approach can be seen as an alternative to the Fourier inversion approach employed in Caldana and Fusai [5] for these types of popular models.

Appendix A: Mean-Variance Mixture Case

Below we calculate the moments of $e^{X_1} - e^{X_2}$. Let $\phi_Y(s)$ denote the moment generating function of $Y$. The moments of $e^{X_1} - e^{X_2}$ up to fourth order is given as follows

\[
M_1 = : E[e^{X_1} - e^{X_2}] \\
= E \left[ e^{\mu_1 + \beta_1 Y + \sqrt{Y} Z_1} - e^{\mu_2 + \beta_2 Y + \sqrt{Y} Z_2} \right] \\
= e^{\mu_1} E[E[e^{\beta_1 Y + \sqrt{Y} Z_1}|Y]] - e^{\mu_2} E[E[e^{\beta_2 Y + \sqrt{Y} Z_2}|Y]] \\
= e^{\mu_1} E\left[ \exp \left\{ (\beta_1 + \frac{a_{11}^2 + a_{12}^2}{2}) \cdot Y \right\} \right] - e^{\mu_2} E\left[ \exp \left\{ (\beta_2 + \frac{a_{21}^2 + a_{22}^2}{2}) \cdot Y \right\} \right] \\
= e^{\mu_1} \phi_Y \left( \beta_1 + \frac{a_{11}^2 + a_{12}^2}{2} \right) - e^{\mu_2} \phi_Y \left( \beta_2 + \frac{a_{21}^2 + a_{22}^2}{2} \right).
\]

We have

\[
M_2 =: E[(e^{X_1} - e^{X_2})^2] = E[e^{2X_1}] - 2E[e^{X_1+X_2}] + E[e^{2X_2}],
\]

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where

\[
E[e^{2X_1}] = E[e^{2\mu_1 + 2\beta_1 Y + 2\sqrt{Y} Z_1}]
\]
\[
= e^{2\mu_1} E[e^{2\beta_1 Y + 2\sqrt{Y} Z_1} | Y]
\]
\[
= e^{2\mu_1} E \left\{ \exp \left\{ 2(\beta_1 + a_{11}^2 + a_{12}^2) \cdot Y \right\} \right\}
\]
\[
= e^{2\mu_1} \varphi_Y (2(\beta_1 + a_{11}^2 + a_{12}^2))
\]
\[
E[e^{X_1 + X_2}] = e^{\mu_1 + \mu_2} E[e^{(\beta_1 + \beta_2) Y + \sqrt{Y} (Z_1 + Z_2)}]
\]
\[
= e^{\mu_1 + \mu_2} E \left\{ \exp \left\{ (\beta_1 + \beta_2 + (a_{11} + a_{21})^2 + (a_{12} + a_{22})^2) / 2 \right\} \right\}
\]
\[
E[e^{2X_2}] = e^{2\mu_2} \varphi_Y (2(\beta_2 + a_{21}^2 + a_{22}^2)).
\]

After plugging in these expressions in \( M_2 \) we obtain

\[
M_2 =: E[(e^{X_1} - e^{X_2})^2]
\]
\[
= e^{2\mu_1} \varphi_Y (2(\beta_1 + a_{11}^2 + a_{12}^2)) - 2e^{\mu_1 + \mu_2} \varphi_Y (\beta_1 + \beta_2 + (a_{11} + a_{21})^2 + (a_{12} + a_{22})^2) / 2
\]
\[+ e^{2\mu_2} \varphi_Y (2(\beta_2 + a_{21}^2 + a_{22}^2)).
\]

We have

\[
M_3 =: E[(e^{X_1} - e^{X_2})^3] = E[e^{3X_1}] - 3E[e^{2X_1 + X_2}] + 3E[e^{X_1 + 2X_2}] - E[e^{3X_2}],
\]

where

\[
E[e^{3X_1}] = e^{3\mu_1} \varphi_Y \left( 3\beta_1 + \frac{9(a_{11}^2 + a_{12}^2)}{2} \right)
\]
\[
E[e^{2X_1 + X_2}] = e^{2\mu_1 + \mu_2} \varphi_Y \left( 2\beta_1 + \beta_2 + \frac{(2a_{11} + a_{21})^2 + (a_{12} + 2a_{22})^2}{2} \right)
\]
\[
E[e^{X_1 + 2X_2}] = e^{\mu_1 + 2\mu_2} \varphi_Y \left( \beta_1 + 2\beta_2 + \frac{(a_{11} + 2a_{21})^2 + (a_{12} + 2a_{22})^2}{2} \right)
\]
\[
E[e^{3X_2}] = e^{3\mu_2} \varphi_Y \left( 3\beta_2 + \frac{9(a_{21}^2 + a_{22}^2)}{2} \right)
\]
we plug these five formulas back into $M_3$ and obtain

$$M_3 = E[(e^{X_1} - e^{X_2})^3]$$

$$= e^{3\mu_1} \phi_Y \left( 3\beta_1 + \frac{9(a_{11}^2 + a_{12}^2)}{2} \right) - 3e^{2\mu_1 + \mu_2} \phi_Y \left( 2\beta_1 + \beta_2 + \frac{(2a_{11} + a_{21})^2 + (2a_{12} + a_{22})^2}{2} \right)$$

$$+ 3e^{\mu_1 + 2\mu_2} \phi_Y \left( \beta_1 + 2\beta_2 + \frac{(a_{11} + 2a_{21})^2 + (a_{12} + 2a_{22})^2}{2} \right) - e^{3\mu_2} \phi_Y \left( 3\beta_2 + \frac{9(a_{21}^2 + a_{22}^2)}{2} \right).$$

We have

$$M_4 := E[(e^{X_1} - e^{X_2})^4] = E[e^{4X_1}] - 4E[e^{3X_1 + X_2}] + 6E[e^{2X_1 + 2X_2}] - 4E[e^{X_1 + 3X_2}] + E[e^{4X_2}],$$

where

$$E[e^{4X_1}] = e^{4\mu_1} \phi_Y \left( 4\beta_1 + 8(a_{11}^2 + a_{12}^2) \right)$$

$$E[e^{3X_1 + X_2}] = e^{3\mu_1 + \mu_2} \phi_Y \left( 3\beta_1 + \beta_2 + \frac{(3a_{11} + a_{21})^2 + (3a_{12} + a_{22})^2}{2} \right)$$

$$E[e^{2X_1 + 2X_2}] = e^{2\mu_1 + 2\mu_2} \phi_Y \left( 2(\beta_1 + \beta_2 + (a_{11} + a_{21})^2 + (a_{12} + a_{22})^2) \right)$$

$$E[e^{X_1 + 3X_2}] = e^{\mu_1 + 3\mu_2} \phi_Y \left( \beta_1 + 3\beta_2 + \frac{(a_{11} + 3a_{21})^2 + (a_{12} + 3a_{22})^2}{2} \right)$$

$$E[e^{4X_2}] = e^{4\mu_2} \phi_Y \left( 4\beta_2 + 8(a_{21}^2 + a_{22}^2) \right)$$

We plug these five formulas back into $M_4$ and obtain

$$M_4 = E[(e^{X_1} - e^{X_2})^4]$$

$$= e^{4\mu_1} \phi_Y \left( 4\beta_1 + 8(a_{11}^2 + a_{12}^2) \right)$$

$$- 4e^{3\mu_1 + \mu_2} \phi_Y \left( 3\beta_1 + \beta_2 + \frac{(3a_{11} + a_{21})^2 + (3a_{12} + a_{22})^2}{2} \right)$$

$$+ 6e^{2\mu_1 + 2\mu_2} \phi_Y \left( 2(\beta_1 + \beta_2 + (a_{11} + a_{21})^2 + (a_{12} + a_{22})^2) \right)$$

$$- 4e^{\mu_1 + 3\mu_2} \phi_Y \left( \beta_1 + 3\beta_2 + \frac{(a_{11} + 3a_{21})^2 + (a_{12} + 3a_{22})^2}{2} \right)$$

$$+ e^{4\mu_2} \phi_Y \left( 4\beta_2 + 8(a_{21}^2 + a_{22}^2) \right).$$
Below we calculate the moments of $e^{a\sqrt{YN}+bY+c} + d$.

\[ A_1 = : E[e^{a\sqrt{YN}+bY+c} + d] = e^d E[e^{a\sqrt{YN}+bY}] + d = e^d E[e^{a\sqrt{YN}+bY} | Y] + d. \]
\[ = e^d E[e^{(a^2/2)+bY}] + d = e^d \phi_Y\left(\frac{a^2}{2} + b\right) + d \]

\[ A_2 = : E[(e^{a\sqrt{YN}+bY+c} + d)^2] = E[e^{2a\sqrt{YN}+2bY+2c}] + E[2de^{a\sqrt{YN}+bY+c}] + E[d^2] \]
\[ = e^{2d} \phi_Y(2a^2 + 2b) + 2de^c \phi_Y\left(\frac{a^2}{2} + b\right) + d^2. \]

\[ A_3 = : E[(e^{a\sqrt{YN}+bY+c} + d)^3] = E[e^{3a\sqrt{YN}+3bY+3c}] + E[3de^{2a\sqrt{YN}+2bY+2c}] + E[3d^2 e^{a\sqrt{YN}+bY+c}] \]
\[ + E[d^3] = e^{3d} \phi_Y\left(\frac{9a^2}{2} + 3b\right) + 3de^c \phi_Y(2a^2 + 2b) + 3d^2 e^c \phi_Y\left(\frac{a^2}{2} + b\right) + d^3. \]

\[ A_4 = : E[(e^{a\sqrt{YN}+bY+c} + d)^4] \]
\[ = E[e^{4a\sqrt{YN}+4bY+4c}] + E[4de^{3a\sqrt{YN}+3bY+3c}] + E[6d^2 e^{2a\sqrt{YN}+2bY+2c}] + E[4d^3 e^{a\sqrt{YN}+bY+c}] + E[d^4] \]
\[ = e^{4d} \phi_Y(8a^2 + 4b) + 4de^3c \phi_Y\left(\frac{9a^2}{2} + 3b\right) + 6d^2 e^{3c} \phi_Y(2a^2 + 2b) + 4d^3 e^c \phi_Y\left(\frac{a^2}{2} + b\right) + d^4 \] (49)

**Appendix B: Variance Mixture Case**

The calculations of this part are similar to the calculations in Appendix A. If we let $\beta = (\beta_1, \beta_2)^T = 0$ in the formulas in Appendix A, we obtain the corresponding formulas for the variance mixture case.

**Appendix C: General Elliptical Case**

First we write $S_1(0)e^{X_1} - S_2(0)e^{X_2} = e^{\ln S_1(0)+X_1} - e^{\ln S_2(0)+X_2}$. In the following we denote $\mu_1 := \mu_1 + \ln S_1(0)$ and $\mu_2 := \mu_2 + \ln S_2(0)$ for the simplicity of notations. We calculate the moments as follows

\[ M_k = : E[e^{X_1} - e^{X_2}] \]
\[ = e^{\mu_1} E\left[ e^{\sqrt{R}(a_{11}U_1^{(2)} + a_{12}U_2^{(2)})} \right] - e^{\mu_2} E\left[ e^{\sqrt{R}(a_{21}U_1^{(2)} + a_{22}U_2^{(2)})} \right] \]
\[ = e^{\mu_1} E\left[ e^{\sqrt{R}\frac{a_{12} + a_{11}^2}{m!} \sqrt{R}U_1^{(2)}} \right] - e^{\mu_2} E\left[ e^{\sqrt{R}\frac{a_{22} + a_{21}^2}{m!} \sqrt{R}U_1^{(2)}} \right] \]
\[ = e^{\mu_1} \sum_{m=0}^{\infty} E\left( \frac{\sqrt{R}(a_{11}^2 + a_{12}^2) R U_1^{(2)}}{m!} \right)^m - e^{\mu_2} \sum_{m=0}^{\infty} E\left( \frac{\sqrt{R}(a_{22}^2 + a_{21}^2) R U_1^{(2)}}{m!} \right)^m \] (50)

\[ = e^{\mu_1} \sum_{k=0}^{\infty} \frac{(\frac{a_{11}^2 + a_{12}^2}{4k(k+1)!})^{k}}{E[R^k]} - e^{\mu_2} \sum_{k=0}^{\infty} \frac{(\frac{a_{22}^2 + a_{21}^2}{4k(k+1)!})^{k}}{E[R^k]} \]

where $k = 2m$. The second equation above is because of (33), the forth equation follows from
we have
\[ E \left[ \left( \sqrt{RU_1^{(2)}} \right)^m \right] = E \left[ \left( \sqrt{R} \right)^m \right] \frac{1}{m!} \frac{m!}{2^m \left( \frac{m}{2} \right)!} \] when \( m \) is even, else 0
\[ k = 2m \] \[ E[R^k] \frac{(2k)!}{4^k (k!)^2} \] (51)

hence
\[ E \left[ \frac{\left( \sqrt{a_{11}^2 + a_{12}^2 \sqrt{RU_1^{(2)}}} \right)^m}{m!} \right] = \frac{\left( \sqrt{a_{11}^2 + a_{12}^2} \right)^m}{m!} E \left[ \left( \sqrt{RU_1^{(2)}} \right)^m \right] \]
\[ = \frac{\left( a_{11}^2 + a_{12}^2 \right)^k}{4^k (k!)^2} E[R^k] \] (52)

and
\[ E \left[ \frac{\left( \sqrt{a_{21}^2 + a_{22}^2 \sqrt{RU_1^{(2)}}} \right)^m}{m!} \right] = \frac{\left( \sqrt{a_{21}^2 + a_{22}^2} \right)^m}{m!} E \left[ \left( \sqrt{RU_1^{(2)}} \right)^m \right] \]
\[ = \frac{\left( a_{21}^2 + a_{22}^2 \right)^k}{4^k (k!)^2} E[R^k] \] (53)

The second and third moments can be calculated as follows
\[ M_2^* = : E[(e^{X_1} - e^{X_2})^2] \]
\[ = e^{2\mu_1} \sum_{k=0}^{\infty} \frac{(a_{11}^2 + a_{12}^2)^k}{(k!)^2} E[R^k] - 2e^{\mu_1 + \mu_2} \sum_{k=0}^{\infty} \frac{((a_{11} + a_{21})^2 + (a_{12} + a_{22})^2)^k}{4^k (k!)^2} E[R^k] \]
\[ + e^{2\mu_2} \sum_{k=0}^{\infty} \frac{(a_{21}^2 + a_{22}^2)^k}{(k!)^2} E[R^k] \] (54)

\[ M_3^* = : E[(e^{X_1} - e^{X_2})^3] \]
\[ = e^{3\mu_1} \sum_{k=0}^{\infty} \frac{9(a_{11}^2 + a_{12}^2)^k}{4^k (k!)^2} E[R^k] - 3e^{2\mu_1 + \mu_2} \sum_{k=0}^{\infty} \frac{((2a_{11} + a_{21})^2 + (2a_{12} + a_{22})^2)^k}{4^k (k!)^2} E[R^k] \]
\[ + 3e^{\mu_1 + 2\mu_2} \sum_{k=0}^{\infty} \frac{((a_{11} + 2a_{21})^2 + (a_{12} + 2a_{22})^2)^k}{4^k (k!)^2} E[R^k] - e^{3\mu_2} \sum_{k=0}^{\infty} \frac{9(a_{21}^2 + a_{22}^2)^k}{4^k (k!)^2} E[R^k] \]
Below we calculate the moments of \( W^e = e^{a\sqrt{R U^{(2)}}} + b + c \). We have

\[
A_1^e = : E[e^{a\sqrt{R U^{(2)}}} + b + c] = e^b E[e^{a\sqrt{R U^{(2)}}}] + c = e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)} E[R^k] + c.
\]

\[
A_2^e = : E[(e^{a\sqrt{R U^{(2)}}} + b + c)^2] = e^{2b} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} E[R^k] + 2ce^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)} E[R^k] + c^2.
\]

\[
A_3^e = : E[(e^{a\sqrt{R U^{(2)}}} + b + c)^3] = e^{3b} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)^2} E[R^k] + 3ce^{2b} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)} E[R^k] + 3c^2 e^b \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k(k!)} E[R^k] + c^3.
\]

(55)

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