Holographic renormalization for $z = 2$ Lifshitz spacetimes from AdS

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Abstract

Lifshitz spacetimes with the critical exponent $z = 2$ can be obtained by the dimensional reduction of Schrödinger spacetimes with the critical exponent $z = 0$. The latter spacetimes are asymptotically AdS solutions of AdS gravity coupled to an axion–dilaton system and can be uplifted to solutions of type IIB supergravity. This basic observation is used to perform holographic renormalization for four-dimensional asymptotically $z = 2$ locally Lifshitz spacetimes by the Scherk–Schwarz dimensional reduction of the corresponding problem of holographic renormalization for five-dimensional asymptotically locally AdS spacetimes coupled to an axion–dilaton system. We can thus define and characterize a four-dimensional asymptotically locally $z = 2$ Lifshitz spacetime in terms of five-dimensional AdS boundary data. In this setup the four-dimensional structure of the Fefferman–Graham expansion and the structure of the counterterm action, including the scale anomaly, will be discussed. We find that for asymptotically locally $z = 2$ Lifshitz spacetimes obtained in this way, there are two anomalies each with their own associated nonzero central charge. Both anomalies follow from the Scherk–Schwarz dimensional reduction of the five-dimensional conformal anomaly of AdS gravity coupled to an axion–dilaton system. Together, they make up an action that is of the Horava–Lifshitz type with a nonzero potential term for $z = 2$ conformal gravity.

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1. Introduction

In recent years we have witnessed a development in which it was realized that certain asymptotically AdS gravitational systems have features in common with systems encountered in the study of quantum phase transitions that occur in condensed matter physics when a system reaches a quantum critical point. See [1–3] for some review papers. From the condensed matter point of view, one is interested in the effective IR description of a system that in the UV consists of strongly coupled electrons. There exist cases where the effective field theory that is valid near the quantum critical point is described by a strongly coupled CFT [3]. The idea is to study such systems holographically by identifying sectors in holographically dual theories (consistent truncations of the complete theory) that via the concept of universality have the same universal properties as the condensed matter system one is interested in. On the gravity side this maps to a particular choice of matter fields on a background that becomes asymptotically AdS.

However not all quantum critical points are described by CFTs. In general, theories at a critical point are scale invariant with a scaling that is of the non-relativistic type:

\[ t \rightarrow \lambda^z t \quad \& \quad \vec{x} \rightarrow \lambda \vec{x}, \]

where \( t \) and \( \vec{x} \) are the respective time and space coordinates describing the system. The parameter \( z \) is called the critical exponent. When \( z \neq 1 \) the theory can be either Lifshitz or Schrödinger invariant. Again such systems can occur at strong coupling. To study Lifshitz- or Schrödinger-invariant systems holographically, we need to consider a spacetime whose isometry group is the Lifshitz or the Schrödinger symmetry group. Such spacetimes are called Lifshitz [4, 5] and Schrödinger [6, 7] spacetimes, respectively.

Another interesting motivation to study (asymptotically) Lifshitz or Schrödinger spacetimes comes from the question: How general is holography? Since Lifshitz and Schrödinger spacetimes are no longer asymptotically AdS, they form interesting examples to extend holographic techniques to asymptotically non-AdS spacetimes. In this work we will focus on spacetimes that are asymptotically locally \( z = 2 \) Lifshitz\(^4\) in a sense to be made precise below (and agreeing with the definition given in [10]). For earlier work on asymptotically Lifshitz spacetimes and holographic renormalization, see [10–15]. These studies have so far focused on Lagrangians with no known string theory origin that contain gravity coupled to a massive vector field described by a Proca Lagrangian, but that do not contain dilatonic scalars. On the other hand we do know how to embed Lifshitz spacetimes into string theory [16–22]. In particular when \( z = 2 \) the embedding of Lifshitz into string theory is quite straightforward. Here we will use the explicit model of [22] (based on [17, 19]). This case is interesting for a number of reasons: (1) it is within the context of string theory, (2) there is an explicit relation with AdS via the dimensional reduction (see below) and (3) it is explicitly \( z = 2 \) which is a special value having properties that are different from generic \( z \) values, so that it would be good to have an explicit detailed study of this case.

The basic idea of this paper is as follows. Lifshitz spacetimes with the critical exponent \( z = 2 \) can be obtained by the dimensional reduction of Schrödinger spacetimes with the critical exponent \( z = 0 \). The latter spacetimes are asymptotically AdS solutions of AdS gravity coupled to an axion–dilaton system. This basic observation is used to perform holographic renormalization for four-dimensional asymptotically locally \( z = 2 \) Lifshitz spacetimes by the dimensional reduction of the corresponding problem of holographic renormalization for pure Lifshitz spacetimes suffer from IR singularities (divergent tidal forces in the bulk) [5, 8, 9]. In this work we will be primarily interested in the UV properties, i.e. close to the boundary, where there are no singularities.

\(^4\) We briefly mention here that pure Lifshitz spacetimes suffer from IR singularities (divergent tidal forces in the bulk) [5, 8, 9]. In this work we will be primarily interested in the UV properties, i.e. close to the boundary, where there are no singularities.
five-dimensional asymptotically locally AdS (AlAdS) spacetimes coupled to an axion–dilaton system.

Recently, interesting work appeared in relation to the Lifshitz scale anomaly [23, 24, 14, 15] generalizing the conformal anomaly for the AdS gravity of [25] to other values of $z$. In our setup we can make an explicit relation between the five-dimensional AdS conformal anomaly (in the presence of an axion–dilaton system) and the four-dimensional Lifshitz scale anomaly for $z = 2$. We find that in the model we have studied, there are two nonzero central charges and thus two associated anomalies for asymptotically locally $z = 2$ Lifshitz spacetimes. In the remainder of this paper we will simply refer to AlLif spacetimes without explicitly writing $z = 2$.

This paper is organized as follows. In section 2 we will review holographic renormalization for the five-dimensional AdS gravity coupled to an axion–dilaton system [26]. In section 3 we will work out the form of the four-dimensional Fefferman–Graham (FG) expansions by Scherk–Schwarz reducing the FG expansions of section 2. Finally, in section 4 we use these results to obtain the counterterm action of AlLif spacetimes by the dimensional reduction of the counterterms of section 2 and evaluate the anomaly counterterms on-shell using the results of section 3.

2. Holographic renormalization for the AdS gravity coupled to an axion–dilaton field

In this section we discuss the five-dimensional model of AdS gravity coupled to an axion–dilaton system and review the holographic renormalization carried out in [26]. We will however not use the Hamiltonian formalism of [26], but instead work within a Lagrangian framework. We will explicitly solve the equations of motion up to NNLO and discuss the local and anomaly counterterms as well as the one-point functions for AlAdS boundary conditions [27, 28].

2.1. FG expansions and counterterms

The bulk action is

$$ S_{\text{bulk}} = \frac{1}{2\kappa_5^2} \int_M d^5x L_{\text{bulk}}, $$

where

$$ L_{\text{bulk}} = \sqrt{-g} \left( R + 12 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \partial^\mu \chi \right), $$

and where $\kappa_5^2 = 8\pi G_5$ with $G_5$ being the five-dimensional Newton’s constant. The Gibbons–Hawking boundary action is given by

$$ S_{\text{GH}} = \frac{1}{\kappa_5^2} \int_{\partial M} d^4x \sqrt{-h}K, $$

where $h$ denotes the boundary metric. We have set the AdS$_5$ length equal to 1.

The equations of motion that we would like to obtain by varying $S_{\text{bulk}} + S_{\text{GH}}$ (supplied with additional boundary terms for AlAdS boundary conditions) are

$$ E_{\mu\nu} = G_{\mu\nu} - 6g_{\mu\nu} - T_{\mu\nu}^{\text{bulk}} = 0, $$

$$ E_\phi = \Box \phi - e^{2\phi} (\partial \chi)^2 = 0, $$

$$ E_\chi = \Box \chi + 2\partial_\mu \phi \partial^\mu \chi = 0, $$

where

$$ T_{\mu\nu}^{\text{bulk}} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{2\phi} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} ((\partial \phi)^2 + e^{2\phi} (\partial \chi)^2). $$
The solution expressed as an asymptotic series in a radial gauge, i.e. as an FG expansion [29, 30], reads

\[ g_{\mu\nu} \, dx^\mu \, dx^\nu = \frac{dr^2}{r^2} + h_{ab} \, dx^a \, dx^b, \]

\[ h_{ab} = \frac{1}{r^4} [ \chi(0)_{(3)} + r^4 \log r \chi(4,1)_{(3)} + r^4 \chi(4)_{(3)} + O(r^6 \log r)], \]

\[ \phi = \phi_0 + \frac{r^4}{2} \phi_1 + \frac{r^4}{4} \log r \phi_2 + \frac{r^4}{3} \phi_3 + O(r^6 \log r), \]

\[ \chi = \chi_0 + \frac{r^4}{2} \chi_1 + \frac{r^4}{3} \chi_2 + \frac{r^4}{4} \log r \chi(4,1) + \frac{r^4}{3} \chi(4) + O(r^6 \log r), \]

where the coefficients are given by

\[ h_{(2)ab} = -\frac{1}{2} \left( R(0)_{(ab)} - \frac{1}{2} \partial_a \phi_0 \partial_b \phi_0 - \frac{1}{2} e^{2\phi_0} \partial_a \chi_0 \partial_b \chi_0 \right) + \frac{1}{r^2} h_{(0)ab}(R(0) - \frac{1}{2} (\partial \phi_0)^2 - \frac{1}{2} e^{2\phi_0} (\partial \chi_0)^2), \]

\[ \phi_2 = \frac{1}{2} (\nabla^0(0) \phi_0 - e^{2\phi_0} (\partial \chi_0)^2), \]

\[ \chi_2 = \frac{1}{2} (\nabla^0(0) \chi_0 + 2 \partial a \phi_0 \partial^a \chi_0), \]

at second order and by

\[ h_{(3)ab} = h_{(3)ab}(h_{(2)ab} + \frac{1}{2} \nabla^0(0) \nabla_{(2)ab} + \frac{1}{2} \nabla^0(0) \nabla_{(2)ab} - \nabla^0(0) h_{(2)ab}) - \frac{1}{2} \nabla^0(0) \nabla^0(0) h_{(2)ab} - \frac{1}{2} \nabla^0(0) \nabla^0(0) h_{(2)ab} \]

\[ \phi_1 = -\frac{1}{2} \nabla^0(0) \phi_2 + 2 \nabla^0(0) \h_{(2)ab} + 4 e^{\phi_0} \chi_2(0) + \frac{1}{2} \nabla^0(0) h_{(2)ab} - 2 \nabla^0(0) h_{(2)ab} \]

at order \( r^4 \log r \). We note that \( h_{(4)ab} \) is traceless. Indices of the expansion coefficients are raised and lowered with the AdS boundary metric \( h_{(0)ab} \). At order \( r^4 \) we have that \( h_{(4)ab} \) is constrained by

\[ h_{(4)ab} = h_{(2)ab} h_{(2)ab} - \frac{1}{2} \nabla^0(0) \nabla^0(0) h_{(2)ab} - \frac{1}{2} \nabla^0(0) \nabla^0(0) h_{(2)ab} \]

Following [27] we will write \( h_{(4)ab} \) as

\[ h_{(4)ab} = \chi_{ab} + t_{ab}, \]

where \( t_{ab} \) is the boundary energy–momentum tensor whose trace and divergence will be given below together with the explicit form of \( \chi_{ab} \). In the expansion for the scalars we have that \( \phi_4 \) and \( \chi_4 \) are fully arbitrary functions of the boundary coordinates.

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5 We will denote here and further below by \( a_{(n,0)} \) the coefficient at order \( r^n (\log r)^m \) of the field \( r^a \) where \( r^{-\Delta} \) is the leading term in the expansion of \( a \) with the exception of \( a_{(0,0)} \) which we will simply denote as \( a_0 \).
A counterterm action that kills all divergences of the on-shell action $S_{\text{bulk}} + S_{\text{GH}}$ is given by

$$S_{ct} = \frac{1}{\kappa^2} \int_{\partial M} d^4 x \sqrt{-h} \left( -3 - \frac{1}{4} Q + A (\lambda + \log r) \right),$$

(21)

where $\lambda$ is some scheme-dependent parameter (minimal subtraction corresponds to $\lambda = 0$) and

$$Q = h^{ab} Q_{ab}, \quad Q_{ab} = R_{(h)ab} - \frac{1}{2} \partial_a \phi \partial_b \phi - \frac{1}{2} e^{2 \phi} \partial_a \chi \partial_b \chi,$$

(22)

$$A = \frac{1}{8} (Q^{ab} Q_{ab} - \frac{1}{2} Q^2 + \frac{1}{2} \left( \Box^{(h)} \phi - e^{2 \phi} (\partial \chi)^2 + \frac{1}{2} e^{2 \phi} (\Box^{(h)} \chi + 2 \partial_a \phi \partial^a \chi)^2 \right).$$

(23)

This expression for the conformal anomaly $A$ differs slightly (by one term) from the expression given in [26, appendix B].

### 2.2. One-point functions

We write the total variation of $S_{\text{ren}}$ as

$$\delta S_{\text{ren}} = \frac{1}{2 \kappa^2} \int_{\partial M} d^4 x \sqrt{-h} \left( E_{\mu \nu} \delta g^{\mu \nu} + E_{\phi} \delta \phi + E_{\chi} \delta \chi \right)
- \frac{1}{2 \kappa^2} \int_{\partial M} d^4 x \sqrt{-h} \left( T_{ab} \delta h^{ab} + 2 T_{\phi} \delta \phi + 2 T_{\chi} \delta \chi \right),$$

(24)

where $E_{\mu \nu}, E_{\phi}, E_{\chi}$ are the equations of motion (4)–(6) and

$$T_{ab} = (K - 3) h_{ab} - K_{ab} + \frac{1}{2} Q_{ab} - \frac{1}{2} h_{ab} Q + (\lambda + \log r) T_{ab}^{(A)},$$

(25)

$$T_{\phi} = \frac{1}{2} h^{ab} \partial_a \phi + \frac{1}{2} \left( \Box^{(h)} \phi - e^{2 \phi} (\partial \chi)^2 + (\lambda + \log r) T_{\phi}^{(A)},$$

(26)

$$T_{\chi} = \frac{1}{2} e^{2 \phi} h^{ab} \partial_a \chi + \frac{1}{2} e^{2 \phi} \left( \Box^{(h)} \chi + 2 \partial_a \phi \partial^a \chi \right) + (\lambda + \log r) T_{\chi}^{(A)},$$

(27)

in which we defined

$$T_{ab}^{(A)} = - \frac{2 \kappa^2}{\sqrt{-h}} \frac{\delta A}{\delta h^{ab}}, \quad T_{\phi}^{(A)} = - \frac{\kappa^2}{\sqrt{-h}} \frac{\delta A}{\delta \phi}, \quad T_{\chi}^{(A)} = - \frac{\kappa^2}{\sqrt{-h}} \frac{\delta A}{\delta \chi},$$

(28)

with

$$A = \frac{1}{\kappa^2} \int_{\partial M} d^4 x \sqrt{-h} A.$$  

(29)

Using the fact that from the expansions it follows that $\sqrt{-h} = r^{-4} \sqrt{-h_0} + O(r^{-2})$, $\delta h^{ab} = r^2 \delta h^{ab}_0 + O(r^0)$, $\delta \phi = \delta \phi_0 + O(r^2)$ and $\delta \chi = \delta \chi_0 + O(r^2)$, we obtain the following one-point functions (we take the cut-off boundary at $r = \epsilon$)

$$\langle T_{(0)ab} \rangle = - \frac{2 \kappa^2}{\sqrt{-h_0}} \frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta h^{ab}_0} = \lim_{\epsilon \to 0} \epsilon^{-2} T_{ab} = 2 h_{(4)ab} - 2 X_{ab} = T_{ab},$$

(30)

$$\langle O_{\phi} \rangle = - \frac{\kappa^2}{\sqrt{-h_0}} \frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta \phi_0} = \lim_{\epsilon \to 0} \epsilon^{-4} T_{\phi} = - 2 \phi_0 - \frac{1}{2} \phi_2 h^{(2)ab} + e^{2 \phi_0} \chi_2^2 - \frac{1}{2} (3 - 4 \lambda) \phi_4(1),$$

(31)

$$\langle O_{\chi} \rangle = - \frac{\kappa^2}{\sqrt{-h_0}} \frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta \chi_0} = \lim_{\epsilon \to 0} \epsilon^{-4} T_{\chi} = - 2 e^{2 \phi_0} \chi_4(1) - \frac{1}{2} e^{2 \phi_0} \left( \chi_2 h^{(2)ab}_0 + 4 \chi_2 \phi_2 + (3 - 4 \chi) \chi_4(1) \right),$$

(32)
where

\[
X_{ab} = \frac{1}{4} h_{(2)ab} h_{(2)ab} - \frac{1}{4} h_{(2)ab} h_{(2)ab} A_{(0)} - \frac{1}{4} (3 - 4 \lambda) h_{(4,1)ab},
\]

with

\[
A_{(0)} = \lim_{\epsilon \to 0} \epsilon^{-4} A = \frac{1}{4} \left( h_{(2)ab} h_{(2)ab} - (h_{(2)ab})^2 \right) + \phi_2^2 + e^{2\phi_0} \phi_2^2.
\]

The contribution to the one-point functions from the \(r^4 \log r\) terms in the FG expansions can all be removed by choosing \(\lambda = \frac{1}{4}\). The boundary energy–momentum tensor is identified with \(t_{ab}\) in (20). For any choice of \(\lambda\) we compute its trace and divergence (by using equations (18) and (19)) and we find

\[
t_{ab} = A_{(0)},
\]

\[
\nabla_{(0)} t_{ab} = - \langle \mathcal{O}_\phi \rangle \delta_{(0)\phi} - \langle \mathcal{O}_\phi \rangle \delta_{(0)\phi}.
\]

2.3. Manifest SL(2, \(\mathbb{R}\)) invariance of the counterterm action

To make the SL(2, \(\mathbb{R}\)) invariance of the counterterm action \(S_c\) manifest, define the matrix of Noether currents (transforming in the adjoint of SL(2, \(\mathbb{R}\)))

\[
\mathcal{J}_\mu = \left( \partial_\mu \mathcal{M} \right) \mathcal{M}^{-1} = \begin{pmatrix} -J_{(1)}(1) & J_{(3)}(1) \\ J_{(2)}(1) & J_{(1)}(1) \end{pmatrix},
\]

where \(\mathcal{M}\) is given by

\[
\mathcal{M} = e^\phi \begin{pmatrix} X^2 + e^{-2\phi} & X \\ X & 1 \end{pmatrix}.
\]

We have the three on-shell conserved SL(2, \(\mathbb{R}\)) Noether currents

\[
J_{(1)\mu} = \partial_\mu \phi - \phi \partial_\mu X,
\]

\[
J_{(2)\mu} = e^{2\phi} \partial_\mu X,
\]

\[
J_{(3)\mu} = 2 \phi \partial_\mu \phi - \phi^2 e^{2\phi} \partial_\mu X + \partial_\mu X.
\]

The matrix \(\mathcal{J}_\mu\) of Noether currents satisfies the properties

\[
\nabla^a \mathcal{J}_b - \nabla^b \mathcal{J}_a = \mathcal{J}_\mu \mathcal{J}_\nu - \mathcal{J}_\nu \mathcal{J}_\mu,
\]

\[
\mathcal{J}_\mu \mathcal{J}_\nu + \mathcal{J}_\nu \mathcal{J}_\mu = \text{Tr}(\mathcal{J}_\mu \mathcal{J}_\nu) \mathcal{1}.
\]

The counterterm action (21) can be rewritten as

\[
S_c = \frac{1}{k^2_2} \int d^4 x \sqrt{-h} \left[ -3 + \frac{1}{4} Q - \frac{1}{8} \log r \left( Q^{ab} Q_{ab} - \frac{1}{3} Q^2 + \frac{1}{4} \text{Tr}(\nabla_a \mathcal{J}^a \nabla_b \mathcal{J}^b) \right) \right],
\]

where

\[
Q_{ab} = R_{(2)ab} - \frac{1}{4} \text{Tr}(\mathcal{J}_a \mathcal{J}_b), \quad Q = h^{ab} Q_{ab},
\]

making manifest its SL(2, \(\mathbb{R}\)) invariance.

3. FG expansions for asymptotically locally \(z = 2\) Lifshitz spacetimes

A pure \(z = 2\) Lifshitz spacetime can be obtained by writing a pure \(z = 0\) Schrödinger spacetime in the form of a Kaluza–Klein Ansatz. In order to support the geometry of a \(z = 0\) Schrödinger spacetime, we need an axionic scalar field. The massive vector field supporting the Lifshitz geometry [31] can be obtained by the Scherk–Schwarz reduction in which the axion shift symmetry is gauged by the Kaluza–Klein vector. Hence we can obtain Lagrangians supporting

\footnote{The 2-form and 3-form matter supporting the Lifshitz geometry that was introduced in [5] can be obtained by first dualizing the axion in five dimensions to a 3-form potential and then performing an ordinary Kaluza–Klein reduction.}
$z = 2$ Lifshitz spacetimes by the Scherk–Schwarz reduction of Lagrangians supporting $z = 0$ Schrödinger spacetimes [16, 17, 19, 32, 22]. We are now in a position to use these observations to perform holographic renormalization for this class of Lagrangians supporting $z = 2$ Lifshitz spacetimes by the Scherk–Schwarz reduction using the results of the previous section.

3.1. Scherk–Schwarz circle reduction

We will from now on distinguish between five- and four-dimensional objects by putting a hat on all five-dimensional quantities of the previous section. We split the five-dimensional coordinates as $x^\hat{\mu} = (x^\mu, u)$. Consider the following reduction Ansatz:

$$\hat{d}s^2 = \hat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} = \frac{dr^2}{r^2} + \hat{h}_{ab} dx^a dx^b = g_{\mu\nu} dx^\mu dx^\nu + e^{2\phi} (du + A_\mu dx^\mu)^2$$

$$\hat{\chi} = \chi + ku,$n

$$\hat{\phi} = \phi,$$

where the four-dimensional unhatted fields are all independent of the fifth coordinate $u$ which is periodically identified as $u \sim u + 2\pi L$. The reduced theory expressed in terms of the four-dimensional metric $g_{\mu\nu}$ will not be in the Einstein frame. The frame in (46) is such that we preserve the five-dimensional radial gauge (8) in four dimensions. We will perform the holographic renormalization of the reduced four-dimensional theory in this frame. This construction is very reminiscent of the methods used in [33, 34] in the case of dimensional reduction from AlAdS spacetimes to spacetimes that are (in the Einstein frame) asymptotically locally AdS.

For the dimensional reduction of (1)–(3) the following relations are useful:

$$\sqrt{-\hat{g}} = e^{\phi} \sqrt{-\hat{g}},$$

$$\hat{n}^\mu = n^\mu,$n

$$\sqrt{-\hat{h}} = e^{\phi} \sqrt{-\hat{h}},$$

$$\hat{K} = K + n^\mu \partial_\mu \Phi,$n

$$\hat{R} = R - 2 \Box \Phi - 2 (\partial \phi)^2 - \frac{1}{2} e^{2\phi} F^2.$$

Using these relations we find

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-\hat{g}} \left( \hat{R} + 12 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \hat{\chi})^2 \right) + \frac{1}{\kappa_5^2} \int d^4x \sqrt{-\hat{h}} \hat{K} + S_{S}$$

$$= \frac{2\pi L}{2\kappa_5^2} \int d^4x \sqrt{-g} \left( e^{\phi} R - \frac{1}{4} e^{2\phi} F^2 - \frac{1}{2} e^{\phi} (\partial \phi)^2 - \frac{1}{2} e^{\phi} (D \chi)^2 - e^{2\phi} \phi V \right)$$

$$+ \frac{2\pi L}{\kappa_5^2} \int d^3x \sqrt{-h} e^\phi K + S_{S},$$

where

$$D_\mu \chi = \partial_\mu \chi - k A_\mu \equiv -kB_\mu,$$
\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}, \quad (56) \]

\[ V = \frac{k^2}{2} e^{-3\Phi + 2\phi} - 12 e^{-\phi}, \quad (57) \]

in which \( B_{\mu} \) is the massive vector field (that only exists for \( k \neq 0 \)) and where \( S_{ct} \) is a counterterm. From now on we will take \( k \neq 0 \) and replace \( D_{\mu} X \) by \( -kB_{\mu} \) giving

\[ S = \frac{2\pi L}{k^2} \int d^4 x \sqrt{-g} \left( e^\Phi R - \frac{1}{4} e^\Phi F^2 - \frac{1}{2} e^\phi (\partial\phi)^2 - \frac{k^2}{2} e^{\Phi + 2\phi} B^2 - e^{2\phi} V \right) + \frac{2\pi L}{k^2} \int d^3 x \sqrt{-h} e^\Phi K + S_{ct}. \quad (58) \]

The four-dimensional equations of motion associated with the action (58) are

\[ R_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \Phi + \frac{1}{2} g_{\mu\nu} (\Box \Phi + \partial_{\alpha} \Phi \partial_{\nu} \Phi + \frac{1}{2} g_{\mu\nu} (\partial\Phi)^2 + \frac{k^2}{2} B_{\mu} B_{\nu} + \frac{1}{2} k^2 e^{2\phi} \left( F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F^2 \right) \). \quad (59) \]

\[ 0 = \Box \Phi + (\partial\Phi)^2 - \frac{1}{4} e^{2\phi} F^2 + \frac{k^2}{2} e^{-2\Phi + 2\phi} - 4, \quad (60) \]

\[ 0 = \nabla_{\mu} (e^\Phi \partial_{\nu} \phi) - k^2 e^{\Phi + 2\phi} B^2 - k^2 e^{-\Phi + 2\phi}, \quad (61) \]

\[ 0 = \nabla_{\mu} (e^{3\Phi} F_{\mu\nu}) - k^2 e^{2\phi + 2\phi} B^\nu. \quad (62) \]

### 3.2. The \( z = 2 \) Lifshitz spacetime

Equations (59)–(62) admit the pure \( z = 2 \) Lifshitz spacetime as a solution

\[ ds^2 = \frac{dr^2}{r^2} - e^{-2\phi \Phi} \left( \frac{du}{r} \right)^2 + \frac{1}{r^2} (dx^2 + dy^2), \quad (63) \]

\[ B = - e^{-2\Phi \Phi} \frac{dr}{r^2}, \quad (64) \]

\[ \Phi = \Phi(0) = \phi(0) + \log \frac{k}{2}, \quad (65) \]

\[ \phi = \phi(0) = \text{cst.} \quad (66) \]

From a five-dimensional perspective this solution is a \( z = 0 \) Schrödinger spacetime and reads

\[ ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} (-2 dt du + dx^2 + dy^2) + \frac{k^2}{4} e^{2\Phi \Phi} du^2, \quad (67) \]

\[ \hat{\Phi} = \hat{\phi}(0) = \phi(0) = \text{cst.} \quad (68) \]

\[ \hat{\chi} = ku + \text{cst.} \quad (69) \]

Before studying more general solutions that asymptote to (63) in a certain sense, we will first study the pure Lifshitz spacetime from a five-dimensional point of view using an arbitrary FG coordinate system where we only keep manifest the \( u \) coordinate for the purpose of performing the Scherk–Schwarz reduction. In the language of the previous section, this means that we should take

\[ \hat{\phi}(0) = \text{cst.} \quad (70) \]
\[ \hat{\phi}(4) = 0, \]  
(71)  
\[ \hat{x}(0) = ku + \text{cst}, \]  
(72)  
\[ \hat{x}(4) = 0, \]  
(73)  
\[ \hat{h}_{(0)ab} \text{ is conformally flat and admits a hypersurface orthogonal null Killing vector } \partial_u, \]  
(74)  
\[ \hat{t}_{ab} = 0. \]  
(75)  

The properties of \( \hat{h}_{(0)ab} \) can be easily understood. From the reduction Ansatz (46) we learn that

\[ e^{2\Phi}/\Phi_1 = \hat{h}_{uu}. \]  
(76)  

In order that \( \Phi_1 \) is a constant, it is necessary that

\[ \hat{h}_{(0)uu} = 0. \]  
(77)  

Since in order to do the reduction we need that \( \partial_u \) is a Killing vector of the five-dimensional metric and because \( u \) is a boundary coordinate we find that \( \partial_u \) is a null Killing vector of the boundary metric. Furthermore we also need that the boundary value of \( \Phi \) is fixed by (65). This requires that

\[ \hat{h}_{(2)uu} = k^2/4 \ e^{2\Phi_0}. \]  
(78)  

This in turn is only possible provided we impose (as follows from (12))

\[ \hat{R}_{(0)uu} = 0. \]  
(79)  

Using that \( \partial_u \) is a null Killing vector and thus tangent to a null geodesic congruence, it will be shown below, with the help of the Raychaudhuri equation, that provided (79) holds, the null Killing vector \( \partial_u \) is hypersurface orthogonal. Finally since the metric (67) is asymptotically AdS it follows that the AdS boundary metric is conformally flat. This explains the condition imposed on \( \hat{h}_{(0)ab} \) which together with (75) imply that \( \Phi \) is a constant satisfying (65). Furthermore these conditions for \( \hat{h}_{(0)ab} \) combined with (75) are necessary and sufficient in order that the five-dimensional metric is of the form

\[ ds^2 = ds^2_{\text{AdS}} + k^2/4 \ e^{2\Phi_0} \ du^2, \]  
(80)  

where \( ds^2_{\text{AdS}} \) is the metric of a pure five-dimensional AdS spacetime. This completes the five-dimensional uplift of the pure Lifshitz metric in FG coordinates.

### 3.3. Boundary parametrizations

In subsection 3.6 we will generalize the results of the previous subsection to the case of AILif spacetimes defined from a five-dimensional point of view. In order to prepare for that we will now discuss one of the conditions that goes into the definition of AILif spacetimes defined from a five-dimensional point of view as this will provide guidance for how to proceed. For the pure Lifshitz solution the value of \( \Phi - \phi \) was not a free parameter but equal to \( \log k^2. \) Since it is not a free parameter we will require that this is still true for AILif spacetimes. This is enough to deduce that once again \( \hat{h}_{(0)ab} \) must admit a hypersurface orthogonal null Killing vector. To see this one just observes that we again need equations (77) and (79) and of course that \( \partial_u \) is a Killing vector of the complete five-dimensional metric. We do not require that \( \hat{h}_{(0)ab} \) is conformally flat so that an AILif spacetime uplifts to an AdS spacetime as defined in [28].
Furthermore, the only condition imposed on \( \hat{h}_{(0)\hat{a}\hat{b}} \) admits a hypersurface orthogonal null Killing vector, it will be useful to consider the following double null split of the boundary metric:

\[
\hat{h}_{(0)\hat{a}\hat{b}} = -\hat{N}_{(0)\hat{a}}\hat{H}_{(0)\hat{b}} - \hat{N}_{(0)\hat{b}}\hat{H}_{(0)\hat{a}} + \hat{\Pi}_{(0)\hat{a}\hat{b}},
\]

where, say, \( \hat{H}_{(0)} \) is identified with the null Killing direction \( \partial_\alpha \) and \( \hat{N}_{(0)} \) is a second null vector satisfying

\[
\hat{N}_{(0)\hat{a}}\hat{H}_{(0)\hat{a}} = -1.
\]

Furthermore, we impose that \( \hat{\Pi}_{(0)\hat{a}\hat{b}} \) is a projector onto a two-dimensional Euclidean subspace orthogonal to both \( \hat{N}_{(0)} \) and \( \hat{H}_{(0)} \). The indices on \( \hat{H}_{(0)\hat{a}}, \hat{N}_{(0)\hat{a}} \) and \( \hat{\Pi}_{(0)\hat{a}\hat{b}} \) are raised and lowered using \( \hat{h}_{(0)\hat{a}\hat{b}} \):

We will now use the above-introduced notation to show that a null Killing vector field \( \hat{H}_{(0)} = \partial_\alpha \) satisfying (79) is indeed hypersurface orthogonal. The vector \( \hat{H}_{(0)} \), being a null Killing vector, is tangent to a null geodesic congruence. Define

\[
\hat{B}_{(0)\hat{a}\hat{b}} = \hat{\nabla}_{(0)\hat{a}}\hat{H}_{(0)\hat{b}},
\]

as well as

\[
\hat{S}_{(0)\hat{a}\hat{b}} = \hat{\Pi}_{(0)\hat{a}\hat{c}}\hat{\Pi}_{(0)\hat{b}\hat{d}}\hat{B}_{(0)\hat{c}\hat{d}},
\]

which is the projected version of \( \hat{B}_{(0)\hat{a}\hat{b}} \) with the projection onto the co-dimension-2 subspace orthogonal to both \( \hat{N}_{(0)} \) and \( \hat{H}_{(0)} \). This space is not uniquely defined as \( \hat{N}_{(0)} \), being only constrained by (82), is not uniquely defined. Anyway, the results will not depend on the specific choice for \( \hat{N}_{(0)} \). Because \( \hat{H}_{(0)} \) is Killing, the shear and expansion of the null geodesic congruence are zero and the Raychaudhuri equation reads

\[
\hat{H}_{(0)}\hat{H}_{(0)}\hat{R}_{(0)\hat{a}\hat{b}} = \hat{\omega}_{(0)\hat{a}\hat{b}}\hat{\omega}_{(0)\hat{a}\hat{b}},
\]

where

\[
\hat{\omega}_{(0)\hat{a}\hat{b}} = \hat{S}_{(0)\hat{a}\hat{b}}.
\]

Hence, whenever (79) holds we have for a null Killing vector that

\[
\hat{\omega}_{(0)\hat{a}\hat{b}}\hat{\omega}_{(0)\hat{a}\hat{b}} = 0 \rightarrow \hat{\omega}_{(0)\hat{a}\hat{b}} = 0.
\]

We now show that this implies that \( \hat{H}_{(0)} \) is hypersurface orthogonal. We have from the definition of \( \hat{\omega}_{(0)\hat{a}\hat{b}} \) and the properties of \( \hat{H}_{(0)} \) that

\[
d\hat{H}_{(0)} = \hat{\nabla}_{(0)}\hat{V}_{(0)},
\]

where \( \hat{V}_{(0)\hat{a}} = 2\hat{N}_{(0)}\hat{V}_{(0)\hat{a}} \hat{V}_{(0)\hat{b}} \hat{H}_{(0)\hat{b}} \). It follows that for arbitrary \( \hat{V}_{(0)\hat{a}} \) and hence for any choice of \( \hat{N}_{(0)} \) that

\[
\hat{H}_{(0)} \bigwedge d\hat{H}_{(0)} = 0,
\]

which is the Frobenius integrability condition for \( \hat{H}_{(0)} \) to be hypersurface orthogonal.

Because \( \hat{H}_{(0)} \) is hypersurface orthogonal we can always (locally) choose coordinates such that

\[
\hat{H}_{(0)\hat{a}} = H_{(0)}\partial_\alpha.
\]

Furthermore the only condition imposed on \( \hat{N}_{(0)} \) is that it satisfies (82). Hence we can assume without loss of generality that also \( \hat{N}_{(0)} \) is hypersurface orthogonal and given by

\[
\hat{N}_{(0)\hat{a}} = N_{(0)}\partial_\alpha.
\]
It follows that $N_{(0)} = -1$ and $\dot{H}_{(0)} = 0$. Hence we thus have
\[ \dot{h}_{(0)ab} \, dx^a \, dx^b = 2H_{(0)} \, dt \, dr + \Pi_{(0)ij}(dx^i + H_{(0)}N^i_{(0)} \, dt)(dx^j + H_{(0)}N^j_{(0)} \, dt), \] (92)
where we dropped the hat on $\dot{\Pi}_{(0)ij}$ and $\dot{N}^i_{(0)}$ and where all metric components are arbitrary functions of $t$ and $x^i$ but do not depend on $u$. In appendix A we provide some explicit formulas for the geometric quantities of interest depending on $\dot{h}_{(0)ab}$ expressed in the coordinate system (92).

### 3.4. Dimensional reduction of the FG expansions

From the reduction Ansatz (46) and (47) together with (55) it follows that in the radial gauge for $\tilde{g}_{\tilde{\rho}\tilde{\sigma}}$, we have the following relation between the four- and five-dimensional fields:

\[
\begin{align*}
g_{rr} &= \frac{1}{r^2}, \\
g_{rt} &= 0, \\
h_{ab} &= \dot{h}_{ab} - \frac{\dot{\hat{h}}_{au} \dot{\hat{h}}_{bu}}{\hat{h}_{uu}}, \\
B_t &= -\frac{1}{k} \hat{\partial}_r \hat{x}, \\
B_u &= \frac{\dot{\hat{h}}_{uu}}{\hat{h}_{uu}} - \frac{1}{k} \hat{\partial}_u \hat{x}, \\
\Phi &= \frac{1}{2} \log \hat{h}_{uu},
\end{align*}
\]

where the hatted fields satisfy the reduction Ansatz. The double null split of the five-dimensional boundary metric $\ddot{h}_{(0)ab}$ puts the three-dimensional metric $\hat{h}_{ab}$, defined in (95), in the ADM form. We will work out the expansions of the four-dimensional fields assuming only that $\ddot{h}_{(0)ab}$ is parametrized as in (92).

Using equations (95)–(98) together with (48) as well as the five-dimensional expansions (9)–(11) and the boundary parametrization (92), we obtain the following expansions for the four-dimensional fields:\n
\[
\begin{align*}
h_{tt} &= \frac{1}{r^2} \left( h_{(0)tt} + r^2 \log r h_{(2,1)tt} + r^2 h_{(2,2)tt} + r^4 (\log r)^2 h_{(4,2)tt} \\
&\quad + r^4 \log r h_{(4,4)tt} + r^4 h_{(4,4)tt} + O\left(r^6 (\log r)^3\right)\right), \\
h_{tu} &= \frac{1}{r^2} \left( h_{(0)tu} + r^2 \log r h_{(2,1)tu} + r^2 h_{(2,2)tu} + O\left(r^4 (\log r)^2\right)\right), \\
h_{ij} &= \frac{1}{r^2} \left( h_{(0)ij} + r^2 h_{(2,ij)} + r^4 \log r h_{(4,1)ij} + r^4 h_{(4,ij)} + O\left(r^6 (\log r)^2\right)\right), \\
B_t &= r(B_{(0)t} + r^2 \log r B_{(2,1)t} + r^2 B_{(2,2)t} + O\left(r^4 \log r\right)) , \\
B_u &= \frac{1}{r^2} \left( B_{(0)u} + r^2 \log r B_{(2,1)u} + r^2 B_{(2,2)u} + r^4 (\log r)^2 B_{(4,2)u} + O\left(r^6 (\log r)^3\right)\right),
\end{align*}
\]

We note that the $r$ component of the massive vector field goes to zero as we approach the boundary. This boundary condition is very similar to what has been proposed for asymptotically Schrödinger spacetimes in [35].
\[
B_i = B_{(0)i} + r^2 \log r B_{(2,1)i} + r^2 B_{(2)i} + \mathcal{O}(r^4 (\log r)^2),
\]

\[
\Phi = \Phi_{(0)} + r^2 \log r \Phi_{(2,1)} + r^2 \Phi_{(2)} + r^4 (\log r)^2 \Phi_{(4,1)} + r^4 \log r \Phi_{(4)} + \mathcal{O}(r^6 (\log r)^3),
\]

\[
\phi = \phi_{(0)} + r^2 \phi_{(2)} + r^4 \log r \phi_{(4,1)} + r^4 \phi_{(4)} + \mathcal{O}(r^6 \log r).
\]

For the determinant and inverse metric we obtain the following expansions:

\[
\sqrt{-h} = e^{-\Phi} \sqrt{\hat{h}} = r^{-4} \sqrt{\hat{h}} = \hat{h} + \mathcal{O}(r^4 (\log r)^2),
\]

\[
\hat{h}^\mu = \hat{h}^\mu = r^4 (s_{(0)}^\mu + r^2 \log r s_{(2,1)}^\mu + r^2 s_{(2)}^\mu + \mathcal{O}(r^4 (\log r)^2)),
\]

\[
\hat{h}^i = \hat{h}^i = r^4 (s_{(0)i} + r^2 \log r s_{(2,1)i} + r^2 s_{(2)i} + \mathcal{O}(r^4 (\log r)^2)),
\]

\[
\hat{h}^{ij} = \hat{h}^{ij} = r^2 (s_{(0)}^{ij} + r^2 s_{(2,1)}^{ij} + r^2 s_{(2)}^{ij} + \mathcal{O}(r^4 \log r)),
\]

where

\[
h_{(2,1)} = \frac{1}{2} h_{(0)0},
\]

\[
h_{(2)} = \frac{1}{2} \left( h_{(2)} - \frac{\Pi_{(0)i} h_{(0)0} h_{(0)i}}{h_{(0)0}^2} + \frac{\Pi_{(0)}^i h_{(2)i}^0}{h_{(0)0}} \right),
\]

\[
s_{(0)}^\mu = \frac{1}{h_{(0)0}},
\]

\[
s_{(0)}^{ij} = -\frac{\Pi_{(0)}^{ij}}{h_{(0)0}},
\]

\[
s_{(0)}^{(2,1)} = \frac{1}{h_{(0)0}^2},
\]

\[
s_{(0)}^{(2,1)} = \frac{\Pi_{(0)}^{ij}}{h_{(0)0}^2} \left[ \frac{-h_{(2,1)ij} h_{(0)0} h_{(0)0}^{ij}}{h_{(0)0}^2} \right],
\]

\[
s_{(2)}^\mu = \frac{1}{h_{(0)0}^2} \left[ \frac{-h_{(2)} h_{(0)0} h_{(0)0}}{h_{(0)0}^2} \right] + \frac{\Pi_{(0)}^{ij} h_{(0)0j} h_{(0)0} h_{(0)0i}}{h_{(0)0}^2},
\]

\[
s_{(1)}^{ij} = \frac{\Pi_{(0)}^{ij}}{h_{(0)0}^2} \left[ \frac{-h_{(2)} h_{(0)0} h_{(0)0}}{h_{(0)0}^2} + \frac{\Pi_{(0)}^{ij} h_{(0)0j} h_{(0)0i}}{h_{(0)0}^2} + \frac{\Pi_{(0)}^{ij} h_{(2)} h_{(0)0} h_{(0)0i}}{h_{(0)0}^2} \right],
\]

\[
s_{(0)}^{ij} = \frac{\Pi_{(0)}^{ij} h_{(0)0} h_{(0)0}}{h_{(0)0}^2}.
\]

We next work out the coefficients appearing in the above expansions (99)–(106) up to second order. From plugging in (9)–(11) into (95)–(98) we find at leading order the following relations:

\[
h_{(0)0} = - (B_{(0)i})^2 e^{2\Phi_{(0)}},
\]

\[
\Phi_{(0)} = \phi_{(0)} + \log \frac{k}{\pi}.
\]
The constraint (121) is a consequence of the fact that the Killing vector \( \partial_{\mu} \) is null on the boundary of the five-dimensional AlAdS spacetime. We will parametrize the constraint equation (121) by writing

\[
B_{(0)\mu} = H_{(0)} e^{-2\phi_{(0)}},
\]

\[
h_{(0)\mu} = - H_{(0)}^2 e^{-2\phi_{(0)}},
\]

where \( H_{(0)} \) originates from the reduction \((\hat{h}_{(0)\mu} = H_{(0)})\) and we eliminate \( \Phi_{(0)} \) in favor of \( \Phi_{(0)} \). Furthermore since we have \( h_{(0)ij} = \Pi_{(0)ij} \) we will use \( \Pi_{(0)ij} \) in the expansions.

In order to express the four-dimensional expansions that are obtained by the reduction procedure in terms of the coefficients appearing in (99)–(106) we first work out the expansion expressed in terms of the Kaluza–Klein vector field \( A_{\mu} \) and the axion \( \chi \). The result of this analysis is given in appendix B. In order to re-express the result of appendix B in terms of the coefficients of the massive vector field \( B_{\mu} \), we go about as follows. Replace \( \partial_{\mu} \chi_{(0)} \) using that \( h_{(0)\mu} = H_{(0)}(N_{(0)\mu} - \frac{1}{k} \partial_{\mu} \chi_{(0)}) \). This leads to an explicit dependence on \( N_{(0)\mu} \) which we do not want. However, it turns out that \( N_{(0)\mu} \) only appears as \( N_{(0)\mu} \Pi_{(0)ij} \) and this can be removed using equation (B.22) which can be written as

\[
H_{(0)}^2 N_{(0)\mu} N_{(0)} = h_{(2)\mu} + 2H_{(0)} A_{(2)\mu} + 2H_{(0)}^2 e^{-2\phi_{(0)}} \Phi_{(2)}.
\]

Once we have removed \( \partial_{\mu} \chi_{(0)} \) and \( N_{(0)\mu} \) from the expansions of appendix B in this way, the result can be easily expressed in terms of the coefficients of the massive vector field \( B_{\mu} \) using that

\[
B_{(0)\mu} = A_{(0)\mu},
\]

\[
B_{(2,1)\mu} = A_{(2,1)\mu},
\]

\[
B_{(2)\mu} = A_{(2)\mu} - \frac{1}{k} \partial_{\mu} \chi_{(0)},
\]

\[
B_{(0)\mu} = A_{(0)\mu} - \frac{1}{k} \partial_{\mu} \chi_{(0)},
\]

\[
B_{(2)\mu} = A_{(2)\mu} - \frac{1}{k} \partial_{\mu} \chi_{(2)},
\]

\[
B_{(0)\mu} = - \frac{2}{k} \chi_{(2)}.
\]

We end up with the following expressions:

\[
h_{(2,1)\mu} = 2 e^{-2\phi_{(0)}} H_{(0)}^2 \Phi_{(2,1)},
\]

\[
h_{(2,1)\mu} = - H_{(0)} B_{(2,1)\mu},
\]

\[
h_{(2)\mu} = e^{2\phi_{(0)}} H_{(0)}^4 h_{(0)\mu} B_{(2)\mu} + 2h_{(0)\mu} \Phi_{(2)} - H_{(0)} B_{(2)\mu} + \frac{1}{2} H_{(0)} \partial_{\mu} B_{(0)\mu}
\]

\[
+ \frac{1}{2} \partial_{\mu} \Phi_{(0)} \partial_{\mu} \Phi_{(0)} + \frac{1}{2} \Pi_{(0)}^2 (D_{(0)}^0 \partial_{(0)}) \Pi_{(0)} + \frac{1}{2} H_{(0)}^{-1} \partial_{(0)} H_{(0)}
\]

\[
- \frac{1}{2} H_{(0)}^{-2} \partial_{(0)} H_{(0)} \partial_{(0)} H_{(0)} - \frac{1}{8} H_{(0)}^{-1} \partial_{(0)} H_{(0)} \Pi_{(0)}^2 \partial_{(0)} \Pi_{(0)} + \frac{1}{2} D_{(0)}^0 D_{(0)}^0 h_{(0)\mu}
\]

\[
- \frac{1}{4} D_{(0)}^0 D_{(0)}^0 h_{(0)\mu} + \frac{1}{2} H_{(0)}^{-1} h_{(0)\mu} D_{(0)}^0 \partial_{(0)} H_{(0)} - \frac{1}{2} h_{(0)\mu} H_{(0)}^{-1} D_{(0)}^0 \partial_{(0)} H_{(0)}
\]

\[
+ \frac{1}{2} H_{(0)}^3 \partial_{(0)} H_{(0)} D_{(0)}^0 h_{(0)\mu} - \frac{1}{2} H_{(0)}^{-1} (\partial_{(0)} H_{(0)}) D_{(0)}^0 h_{(0)\mu},
\]

\[
h_{(2)\mu} = - \frac{1}{2} R_{(0)\mu} + \frac{1}{2} H_{(0)}^4 D_{(0)}^0 \partial_{(0)} H_{(0)} - \frac{1}{2} H_{(0)}^2 \partial_{(0)} H_{(0)} \partial_{(0)} H_{(0)} + \frac{1}{2} \partial_{(0)} \Phi_{(0)} \partial_{(0)} \Phi_{(0)}
\]

\[
+ \Pi_{(0)}^2 (e^{2\phi_{(0)}} H_{(0)}^2 B_{(2)\mu} + 2 \Phi_{(2)} - \frac{1}{2} H_{(0)} D_{(0)}^0 \partial_{(0)} H_{(0)}),
\]
\[ B_{(0)\mu} = \frac{1}{2} H_{(0)}^2 \Pi^{ij}_{(0)} B_{(0)ij} = \frac{1}{2} H_{(0)}^{-1} \Pi^{ij}_{(0)} \partial_i \Pi_{(0)ij} + H_{(0)}^{-1} h_{(0)ij} \Phi_{(0)} - H_{(0)}^{-1} \Phi_{(0)}, \]  
(136)

\[ B_{(2,1)\mu} = -2 e^{2\Phi_{(2)}} H_{(0)} \Phi_{(2,1)}, \]  
(137)

\[ B_{(2)\mu} = \frac{1}{8} e^{-2\Phi_{(2)}} H_{(0)} \left( \mathcal{R}_{(0)} + H_{(0)}^{-1} D^0\mu \partial_\mu H_{(0)} + \frac{1}{2} H_{(0)}^{-2} \partial_\mu H_{(0)} \right) - \frac{1}{2} \partial_\mu \Phi_{(0)} + 20 \Phi_{(2)} + \frac{1}{2} H_{(0)}^{-1} h_{(2)\mu} - \frac{1}{2} H_{(0)}^{-1} h_{(0)\mu} h_{(0)i} \right), \]  
(138)

\[ B_{(0)\mu} = 0, \]  
(139)

\[ B_{(2,1)\mu} = \frac{1}{4} H_{(0)}^{-1} \mathcal{R}_{(0)\mu} h_{(0)} + \frac{1}{2} H_{(0)}^{-1} D^0\mu h_{(0)i} + \frac{1}{2} H_{(0)}^{-2} \partial_\mu h_{(0)i} - \frac{1}{2} H_{(0)}^{-1} D^0_j h_{(0)i} \partial_\mu \Phi_{(0)} + \frac{1}{2} H_{(0)}^{-1} \partial^i \Phi_{(0)} h_{(0)i} \partial_\mu \Phi_{(0)} - \frac{1}{2} H_{(0)}^{-1} \partial^i \Phi_{(0)} h_{(0)i} \partial_\mu \Phi_{(0)} - \frac{1}{2} H_{(0)}^{-1} \partial^i \Phi_{(0)} h_{(0)i} \partial_\mu \Phi_{(0)} \]  
(140)

\[ \Phi_{(2,1)} = -\frac{8}{3} \Phi_{(2)} - \frac{8}{3} e^{2\Phi_{(2)}} H_{(0)}^2 B_{(2)\mu} - \frac{4}{3} e^{2\Phi_{(2)}} H_{(0)}^{-2} h_{(2)\mu} - \frac{1}{2} D^0 i \partial_\mu \Phi_{(0)} - \frac{12}{27} \partial_\mu \Phi_{(0)} \partial^i \Phi_{(0)} + \frac{4}{3} e^{2\Phi_{(2)}} H_{(0)}^{-2} h_{(0)i} h_{(0)i} \]  
(141)

\[ \Phi_{(2)} = 2 e^{2\Phi_{(2)}} h_{(0)}^2 h_{(0)i} h_{(0)i} + 2 e^{2\Phi_{(0)}} H_{(0)}^2 h_{(2)\mu} + 2 \Phi_{(2)} + \frac{1}{2} D^0 i \partial_\mu \Phi_{(0)} \]  
(142)

Some of the notations used in these expressions are explained in appendix A. The way in which we write these coefficients is slightly ambiguous because of the various relations among the coefficients, e.g. we could replace \( B_{(2)\mu} \) by \( h_{(2)\mu} \) using (138). In the way we write the coefficients, we consider the set of fields: \( H_{(0)}, h_{(0)i}, h_{(2)\mu}, \Pi_{(0)i\mu}, H_{(0)}, \Phi_{(0)}, \Phi_{(2)}, B_{(2)i}, B_{(2)\mu}, h_{(0)i}, h_{(4)i}, h_{(4)i}, \Phi_{(4)} \) and \( B_{(2)i} \) as arbitrary boundary functions whose specification fixes the asymptotic expansion. The corresponding five-dimensional data are given by the set: \( \chi_{(0)}, \chi_{(4)}, \Phi_{(0)}, \Phi_{(4)}, H_{(0)}, N_{(0)}, \Pi_{(0)i\mu}, \bar{t}_{(0)i} \). The data in \( \bar{t}_{(0)i} \) are constrained by (35) and (36). Since the reduction distributes the components of \( \bar{t}_{(0)i} \) over the functions \( \Phi_{(2)} (\bar{t}_{(0)i}), B_{(2)i} (\bar{t}_{(0)i}), B_{(2)i} (\bar{t}_{(0)i}), h_{(0)i} (\bar{t}_{(0)i}), h_{(4)i} (\bar{t}_{(0)i}), h_{(4)i} (\bar{t}_{(0)i}) \) these functions must satisfy the constraints that result from reducing (35) and (36).

The transition from the variables \( (A_{\mu}, \chi) \) to \( B_{\mu} \) via (55) at the level of the expansions is much more straightforward when \( \chi_{(0)} \) is a constant. This can be seen from (126) to (131) or from the fact that in that case (125) becomes a relation among the coefficients with the left-hand side equal to \( h_{(0)i} h_{(0)i} \).

### 3.5. Radial gauge in the Einstein frame

In the frame (46) the expansions (99)–(106) form an asymptotically locally Lifshitz spacetime according to the definition of [10]. We will now discuss the expansion from the point of view of the four-dimensional Einstein frame to see in which sense also in that case we are dealing...
with an AlLif spacetime. In the Einstein frame the metric, using the above expansions, takes
the following form:

\[ g^E_{rr} = \frac{1}{r^2} e^{\Phi_0} + O(\log r), \]

(143)

\[ g^E_{\mu\nu} = \frac{1}{r^2} e^{\Phi_0} h_{(0)\mu\nu} + O\left(\frac{\log r}{r^2}\right), \]

(144)

\[ g^E_{ij} = \frac{1}{r^2} e^{\Phi_0} h_{(0)ij} + O(\log r), \]

(145)

\[ g^E_{ij} = \frac{1}{r^2} e^{\Phi_0} h_{(0)ij} + O(\log r). \]

(146)

To write down a FG-type expansion in the Einstein frame requires that we write \( g^E_{\mu\nu} \) and consider coordinate transforming it,
\[ g^E_{\mu\nu}(x) = g^E_{\mu\nu}(x) + \delta g^E_{\mu\nu}(x) = \epsilon^E_{\mu\nu}(x) + \mathcal{L}_\xi g^E_{\mu\nu}(x), \]
where the coordinates transform as \( x^a = x^\mu - \xi^\mu \). We require that
\[ \delta g^E_{\mu\nu} = \mathcal{L}_\xi g^E_{\mu\nu} = \epsilon^E_{\mu\nu} + \delta g^E_{\mu\nu} = 0, \]
\[ \delta g^E_{rr} = \mathcal{L}_\xi g^E_{rr} = 2 \left( \partial_r \xi^a - \frac{1}{r} \xi^r + \frac{1}{2} \xi^{ab} \partial_a \Phi \right) g^E_{rr} = - (\xi^\mu \partial_\mu \Phi) g^E_{rr}, \]
where \( \delta \Phi = \xi^\mu \partial_\mu \Phi \). This is the infinitesimal version of a coordinate transformation that
brings us to the radial gauge in the Einstein frame. The general solution to condition (148) is
given by
\[ \xi^a = \xi^a_{(0)}(x) - \int \frac{dr}{r^2} h^{ab} \partial_b \xi^r. \]
Equation (149) can be written as
\[ \partial_r \xi^r - \frac{1}{r} \xi^r + \xi^a \partial_a \Phi - \xi^a_{(0)} \partial_a \Phi - \partial_r \right[ \frac{dr}{r^2} h^{ab} \partial_b \xi^r = 0. \]
Using the expansions for \( \Phi \) and \( h^{ab} \) given in (106) and (108)–(110) we can deduce that at
leading order the equation for \( \xi^r \) simplifies to
\[ \partial_r \xi^r - \frac{1}{r} \xi^r + \xi^a_{(0)} \partial_a \Phi = 0. \]
(152)

The solution to this equation is given by
\[ \xi^r = \xi^r_{(0)}(x) - \xi^a_{(0)} \partial_a \Phi = 0. \]
(153)

where \( \xi^r_{(0,1)} = -\xi^a_{(0)} \partial_a \Phi \). As a solution to (151) the error in (153) is \( O(r^3(\log r)^2) \).
We note that when \( \Phi_{(0)} \) is constant then at leading order the metric \( g^E_{\mu\nu} = e^{\Phi_0} g_{\mu\nu} \) with
\( g_{\mu\nu} \) as given via (46) agrees with a radial gauge coordinate system with a Lifshitz length scale
that is given by \( e^{\Phi_0} \) measured in units of the AdS length scale which we set equal to 1.
Knowing the coordinate transformation at leading order in \( \xi^r \) is good enough to decide whether we obtain an AlLif spacetime for general boundary dependence of \( \Phi_{(0)} \) and thus \( \Phi_{(0)} \)
by looking at the leading terms in the FG expansion in the radial gauge in the Einstein frame. Solving for \( \xi^r \) beyond leading order would not modify the leading behavior of the metric but
only affect it at subleading orders.

We now use equations (150) and (153) to work out the effect of the coordinate transformation on \( h_{ab} \) to the radial gauge at leading order using
\[ \delta h_{ab} = \xi^a \partial_a h_{ab} + h_{ac} \partial_b \xi^c + h_{bc} \partial_a \xi^c + \xi^c \partial_a h_{ab}. \]
Using (150) and (153) we see that due to the $r \log r$ term in (153) we obtain via the last term in (154) logarithmic violations of the leading Lifshitz behavior. For example the leading term in $\delta h_{tt}$ is of order $r^{-4} \log r$, whereas for pure Lifshitz we only have $r^{-4}$. The $r^{-4} \log r$ term disappears if and only if we take $\Phi(0)$ constant.

It is nonetheless useful to perform the analysis of holographic renormalization for arbitrary $\Phi(0)$ because it allows us to treat $\Phi(0)$ as a source and compute the vev for the dual operator. Further turning on $\Phi(0)$ as a non-constant boundary field may be an interesting class of deformations in their own right. It would be interesting to study this more precisely from a renormalization group point of view.

### 3.6. Asymptotically locally $z = 2$ Lifshitz spacetimes

We are now in a position to define (from a five-dimensional perspective) the notion of an ALLif spacetime. We will call a solution to the equations of motion (59)–(62) ALLif if and only if the five-dimensional uplift of this solution (which always exists as the reduction is consistent) satisfies the following properties:

\begin{align}
\phi(0) &= \text{cst}, \quad (155) \\
\hat{\chi}(0) &= k u + \chi(0)(x), \quad (156) \\
\hat{h}(0)_{\hat{a}\hat{b}} &= \text{such that it admits a hypersurface orthogonal null Killing vector } \partial_{u}. \quad (157)
\end{align}

We will show that this agrees nicely with the definition of an ALLif spacetime as given in [10]. When the metric $\hat{h}(0)_{\hat{a}\hat{b}}$ is conformally flat and $\chi(0)$ constant, we call the reduced spacetime asymptotically Lifshitz.

The reduced solution was already ALLif in the frame defined by (46). Now that we require $\Phi(0)$ to be constant, it is also guaranteed to be ALLif in the Einstein frame. To compare with the vielbein-based definition of ALLif spacetimes as given in [10] we can simply decompose the metric $g_{\mu\nu}$ into vielbeins. Doing so we obtain

\begin{align}
e^t &= r^{-2} \tilde{e}^t_0 \, dt + \tilde{e}^t_i \, dx^i, \\
e^i &= r^{-1} \tilde{e}^i_0 \, dt + r^{-1} \tilde{e}^i_j \, dx^j,
\end{align}

where the tangent space metric $\eta_{\hat{a}\hat{b}}$ is

\begin{align}
\eta_{tt} &= -1, & \eta_{ti} &= 0, & \eta_{ij} &= \delta_{ij}. \quad (160)
\end{align}

The boundary conditions are such that

\begin{align}
\tilde{e}^t_i |_{r=0}, & \quad \tilde{e}^i_j |_{r=0}, \quad (161)
\end{align}

are nonzero functions of the boundary coordinates, whereas

\begin{align}
\tilde{e}^t_i |_{r=0}, & \quad \tilde{e}^i_j |_{r=0}, \quad (162)
\end{align}

can be chosen freely (zero or nonzero functions of the boundary coordinates). These boundary conditions nicely agree with those of [10] including the condition that $r^2 \tilde{e}^t$ is hypersurface orthogonal as $r$ goes to zero. This is tied to the fact that we have chosen coordinates such that $h(0)_{iiu} = 0$ which in turn is related to choosing adapted coordinates for $\hat{h}(0)_{\hat{a}\hat{b}}$ to make the hypersurface orthogonality of the null Killing vector $\partial_u$ manifest.
4. Lifshitz counterterms and scale anomalies

With the results of the previous two sections we are now in a position to discuss the counterterms for the AIlf spacetimes, i.e. to work out the form of $S_{ct}$ in (58) and to work out the anomaly counterterms on-shell. From the five-dimensional point of view the on-shell anomaly counterterm is related to the trace anomaly (35). Upon the Scherk–Schwarz dimensional reduction we will see that from the four-dimensional perspective we are dealing with anisotropic rescalings and two associated anomaly terms, one second order and one fourth order in derivatives.

4.1. Anisotropic conformal rescalings

Conformal rescalings of the boundary metric $\hat{h}_{(0)ab}$ can be generated by Penrose–Brown–Henneaux (PBH) transformations [36, 37], i.e. diffeomorphisms that preserve the radial gauge choice. Infinitesimally, these transformations act on the five-dimensional fields as

$$\delta \hat{g}_{\mu \nu} = L_{\xi} \hat{g}_{\mu \nu},$$

$$\delta \hat{\phi} = L_{\xi} \hat{\phi},$$

$$\delta \hat{\chi} = L_{\xi} \hat{\chi},$$

such that $L_{\xi} \hat{g}_{rr} = L_{\xi} \hat{g}_{0a} = 0$ so that the radial gauge of the five-dimensional metric (8) is preserved. The solution to these equations gives

$$\hat{\xi}_r = r \hat{\xi}_r(0),$$

$$\hat{\xi}_a = \hat{\xi}_a(0) - \int dr \hat{h}_{0b} \partial_b \hat{\xi}_r(0),$$

where $\hat{\xi}_r(0)$ and $\hat{\xi}_a(0)$ are independent of $r$. Acting with such diffeomorphisms assuming $\hat{\xi}_r(0) \neq 0$ on the five-dimensional solution leads to conformal rescalings and reparametrizations of the boundary metric $\hat{h}_{(0)ab}$ via

$$\delta \hat{h}_{ab} = \hat{\xi}_r \partial_r \hat{h}_{ab} + \hat{\xi}_a \partial_a \hat{h}_{rb} + \hat{\xi}_b \partial_b \hat{h}_{ra} + \hat{\xi}_r \partial_r \hat{h}_{ab}.$$  

(168)

If we further demand that the transformed metric still satisfies the reduction Ansatz, then we must also require that $\hat{\xi}_r(0)$ and $\hat{\xi}_a(0)$ are independent of $u^8$. This means that the boundary rescalings and diffeomorphisms preserve the existence of a hypersurface orthogonal null Killing vector given by $\partial_x^u$.  

The finite version of these transformations (with $\hat{\xi}_r(0) = 0$) transforms the leading terms in the FG expansion as follows:

$$\hat{h}_{(0)ab} \rightarrow \Omega^2 \hat{h}_{(0)ab},$$

$$\hat{\chi}_{(0)} \rightarrow \hat{\chi}_{(0)},$$

$$\hat{\phi}_{(0)} \rightarrow \hat{\phi}_{(0)},$$

(169)  

(170)  

(171)

with $\partial_u \Omega = 0$. In the parametrization (92) the conformal rescalings act as

$$H_{(0)} \rightarrow \Omega^2 H_{(0)},$$

(172)

8 A similar restriction for the AdS PBS transformations has also been observed in FG expansions for asymptotically $z = 2$ Schrödinger spacetimes that can be obtained from asymptotically AdS spacetimes via the so-called TsT transformation [38].

9 These restrictions are not strong enough to preserve the form of the parametrization (92). That will only be the case if we furthermore demand that $\hat{\xi}_r(0)$ is independent of $x'$ and is thus a function of $r$ only.
\[ \Pi_{(0)ij} \rightarrow \Omega^2 \Pi_{(0)ij}, \quad \text{(173)} \]

\[ N_{(0)}^i \rightarrow \Omega^{-2} N_{(0)}^i. \quad \text{(174)} \]

Furthermore the scalars \( \Phi_{(0)}, \phi_{(0)} \) and \( \chi_{(0)} \) transform with weight 0. This implies that \( h_{(0)\mu} \) scales with weight 4, while \( h_{(0)r} \) and \( h_{(0)ij} \) scale with weight 2. These are precisely the anisotropic conformal rescalings of [39]. We will now study the associated anisotropic conformal anomalies by the dimensional reduction of the five-dimensional counterterm action.

### 4.2. Dimensional reduction of the counterterm action

Performing a dimensional reduction of the counterterm action (21) we obtain

\[
S_{\text{ct}} = \frac{2\pi L}{\kappa_5^2} \int d^3x \sqrt{-h} e^{\Phi} \left[ -3 - \frac{1}{4} \left( R_{(h)} - \frac{1}{2} e^{2\Phi} F^2 - \frac{1}{4} \left( \Phi_{(h)} \right)^2 - \frac{k^2}{2} e^{2\Phi} B^2 - \frac{k^2}{2} e^{2\Phi} \Phi \right) \right]
\]

\[ + (\log r) \frac{2\pi L}{\kappa_5^2} \int d^3x \sqrt{-h} \left( A^{(0)} + A^{(2)} + A^{(4)} \right), \quad \text{(175)} \]

where

\[
A^{(0)} = \frac{k^4}{12} e^{2\Phi} \left( B^2 + e^{-2\Phi} \right),
\]

\[
A^{(2)} = -\frac{k^2}{8} e^{2\Phi} B^2 B^\Phi \left( R_{(h)ab} - \nabla_a^{(h)} \nabla_b \Phi - \partial_a \Phi \partial_b \Phi - \frac{1}{2} e^{2\Phi} F_a^c F_{cb} - \frac{1}{2} \partial_a \phi \partial_b \phi \right)
\]

\[ + \frac{k^2}{8} e^{2\Phi} \left( \nabla_a^{(h)} F_{ab} + 6 F_{ab} \partial^a \Phi \right) + \frac{k^2}{24} e^{2\Phi} \left( \Phi_{(h)} + \Phi \right) - \frac{1}{8} e^{2\Phi} F^2
\]

\[ - 3 \left( \Phi_{(h)} + \Phi \right) \left( \nabla_a^{(h)} B + B^2 \partial_a \Phi + 2 B^a \partial_a \Phi \right)^2 \],

\[
A^{(4)} = \frac{1}{8} \left( R_{(h)ab} - \nabla_a^{(h)} \nabla_b \Phi - \partial_a \Phi \partial_b \Phi - \frac{1}{2} e^{2\Phi} F_a^c F_{cb} - \frac{1}{2} \partial_a \phi \partial_b \phi \right)^2
\]

\[ + \frac{1}{16} e^{2\Phi} \left( \nabla_a^{(h)} F_{ab} + 6 F_{ab} \partial^a \Phi \right)^2 + \frac{1}{8} \left( \Phi_{(h)} + \Phi \right)^2 - \frac{1}{8} e^{2\Phi} F^2
\]

\[ - \frac{1}{24} \left( R_{(h)} - 2 \Phi_{(h)} - 2 \Phi \right)^2 - \frac{1}{8} e^{2\Phi} F^2 - \frac{1}{2} \left( \partial \Phi \right)^2
\]

\[ + \frac{1}{16} \left( \Phi_{(h)} + \Phi \right)^2. \quad \text{(178)} \]

The superscript on \( A \) refers to the number of derivatives.

The local counterterms given in the first line of (175) agree exactly with the counterterms given in [13] for what they call the minimal action provided we set \( \Phi \) and \( \phi \) equal to constants such that \( \Phi - \phi = \log \frac{r}{L} \). To compare with the expression given [13], one must perform some mild field redefinitions. Even though setting \( \Phi \) and \( \phi \) equal to constants is not a consistent truncation from the model discussed here to the massive vector model without any scalars, it is interesting that we nonetheless obtain the same answer. This is because it is consistent and in fact necessary in order to obtain AILif solutions to set \( \Phi_{(0)} \) and \( \phi_{(0)} \) equal to constants, i.e. the scalars become asymptotically constant. Since in this case the counterterms do not depend on the \( r \) dependent part of \( \Phi \) and \( \phi \) (from a five-dimensional point of view, this is to say that
the divergent parts of the on-shell action do not depend on $\phi_{(4)}$ and $i_{ab}$, we should obtain the same answer as for the massive vector model without scalars. Since for asymptotically constant scalars the term in $S_{ct}$ proportional to $B^2 + e^{-2\phi}$ in the first line of (175) is at least of order $r^0$, i.e. at best a finite counterterm (we did not check if the coefficient is nonzero), the result also agrees with the local counterterm used in [10, 14] for the no scalar massive vector model.

Even though the on-shell four-dimensional action, (54) and (175), is finite by construction, we have checked that it is finite when we evaluate it for the reduced expansions of appendix B. For this purpose one needs to expand the fields up to the orders indicated in (B.7)–(B.14). We have not listed all coefficients for reasons as explained just below (B.32). The check of the finiteness has been performed with the software package Cadabra [40, 41]. We consider this an important check on our algebra. The form of the counterterms is not unique (and this has nothing to do with the freedom to add finite counterterms). There are many ways of rewriting them that would equally lead to a finite renormalized on-shell action with the same on-shell expression. We have simply chosen a form that one obtains from the reduction (modulo a few total derivatives in the boundary Lagrangian that have been removed).

We will now use this result to study the Lifshitz scale anomaly by evaluating the term proportional to $\log r$ in $S_{ct}$ on-shell. When doing so we will make one simplifying assumption which is to take the four-dimensional $\chi_{(0)}$ constant. For $\chi_{(0)}$ constant we find that $A^{(0)}$ is zero. Otherwise it would have been second order in derivatives of $\chi_{(0)}$. Furthermore we have, using that $\Phi_{(0)} - \phi_{(0)} = \log \frac{4}{z}$ and that $\phi_{(0)}$ is constant,

$$\frac{2\pi L}{\kappa^2} \int_{\partial M} d^3 x \sqrt{-h} e^{\Phi} A^{(2)} \frac{2\pi L}{8\kappa^2} e^{2\phi_{(0)}} \int_{\partial M} dt d^2 x H'_{(0)} \sqrt{\Pi_{(0)}} \left(4K_{\mu\nu}K^\mu_{(0)} - 2K_{(0)}^2 \right),$$

(179)

$$\frac{2\pi L}{\kappa^2} \int_{\partial M} d^3 x \sqrt{-h} e^{\Phi} A^{(4)} \frac{2\pi L}{48\kappa^2} e^{2\phi_{(0)}} \int_{\partial M} dt d^2 x H'_{(0)} \sqrt{\Pi_{(0)}} \left(\mathcal{R}_{(0)} + D^{0\mu} \partial_\mu \log H'_{(0)} \right)^2,$$

(180)

where we defined

$$H'_{(0)} = (-h_{(0)\mu\nu})^{1/2} = H_{(0)} e^{-\phi_{(0)}},$$

(181)

and where

$$K_{(0)\mu\nu} = \frac{1}{2H'_{(0)}} \left(\partial_\mu \Pi_{(0)\nu} - D^{0\mu} h_{(0)\nu} - D^{0\nu} h_{(0)\mu} \right), \quad K_{(0)} = \Pi_{(0)\mu} K_{(0)\mu},$$

(182)

is the extrinsic curvature and its trace. The integrands in (179) and (180) are invariant under the anisotropic Weyl rescalings (172)–(174). The on-shell expression for the anomaly counterterm of (175) forms an action that is of the Horava–Lifshitz type with a nonzero potential term for $z = 2$ conformal gravity [42, 14].

The expression for the anomaly at second order in derivatives (179) agrees with what has been found in [14, 15]. The anomaly at fourth order in derivatives (180) has been shown in [14, 15] to exist on general grounds but was not observed in the no scalar massive vector model for generic $z$ values. Its presence here does not seem to rely on the presence of scalars in the analysis. It would be interesting to understand this difference better, e.g. by performing an explicit $z = 2$ calculation for the no scalar massive vector model as $z = 2$ is a special case.

The term in parenthesis in (179) and (180) vanishes for asymptotically Lifshitz spacetimes, i.e. for a conformally flat boundary metric $h_{(0)ab}$. In the notation of [15] we have the following values for the central charges $C_1, C_2^{10}$:

$$C_1 = \frac{2\pi L}{64\pi G_5} l^2 e^{2\phi_{(0)}},$$

(183)

10 We thank the authors of [15] for useful discussions.
\[ C_2 = \frac{2\pi L}{384\pi G_5} I_2^2 e^{2\Phi_0} = \frac{1}{6} C_1, \]  

(184)

where we have inserted the AdS length parameter \( l \) of the five-dimensional AlAdS spacetimes. This can also be written as

\[ C_1 = \frac{l_{\text{Lif}}^2}{64\pi G_4} = 6C_2, \]  

(185)

where

\[ l_{\text{Lif}}^2 = l^2 e^{\Phi_0} \]  

(186)

is the Lifshitz length parameter\(^{11}\) and \( G_4 \) is the four-dimensional Newton’s constant given by

\[ \frac{1}{G_4} = \frac{2\pi L e^{\Phi_0}}{G_5}; \]  

(187)

in which \( L e^{\Phi_0} \) is the asymptotic value of the radius of the compactification circle.

5. Discussion

We have performed holographic renormalization for ALif spacetimes with \( z = 2 \) in the context of solutions of type IIB supergravity. The approach was based on the observation that a four-dimensional \( z = 2 \) Lifshitz spacetime in IIB string theory can be obtained by combining a stack of extremal D3 branes with an axion plane wave. From a five-dimensional point of view the intersection of the D3 brane and the axion wave leads to a \( z = 0 \) Schrödinger spacetime which is an asymptotically AdS spacetime. The relation to a \( z = 2 \) Lifshitz spacetime is then via the Scherk–Schwarz reduction. This situation has been observed, in various forms, in [16, 17, 19, 32, 22].

As mentioned in [32] the reduction from the point of view of the boundary theory is along a lightlike circle and should therefore be viewed as some DLCQ of \( N = 4 \) SYM in the background of a theta angle that depends linearly on the null circle coordinate leading to Lifshitz–Chern–Simons gauge theory [43]. This fact however does not prevent us from performing holographic renormalization in the bulk as the reduction in the bulk is everywhere along a spacelike circle.

From a five-dimensional point of view the boundary of the AlAdS spacetime must admit a hypersurface orthogonal null Killing vector \( \partial_u \). This vector \( \partial_u \) generates the compact null circle on the boundary. We have used boundary coordinates (92) that are suitably adapted to the existence of such a vector field and this has played a central role in the construction of ALif spacetimes. It would be interesting to define the boundary conditions for having an ALif spacetime in a coordinate independent manner. Once this parametrization has been chosen, the structure of the four-dimensional FG expansions agrees with the boundary condition for ALif spacetimes given in [10] provided we choose the five-dimensional dilaton to asymptote to a constant. Regarding the five-dimensional axion there is the restriction that it asymptotes to \( \tilde{\chi}(0) = ku + \chi(0)(\chi) \). From a four-dimensional perspective one then has the possibility to describe the boundary data from the point of view of either the \((A_\mu, \chi)\) (with a gauge symmetry) or the \( B_\mu \) variables. When the four-dimensional \( \chi(0) \) is non-constant, the relation between these two sets of variables from the boundary point of view, i.e. the free functions appearing in the FG expansions, is not so simple. It would be interesting to understand this better and to see if there is a preferred set of variables.

Upon the dimensional reduction of the local counterterms of the AlAdS spacetime, we obtain the local counterterms of the ALif spacetimes and these nicely agree with what has been

\(^{11}\) In the Einstein frame the coefficient of the \( \frac{\partial^2}{\partial r^2} \) term in metric (63) is given by \( l^2 e^{\Phi_0} \).
found in the literature so far [13, 10, 14, 15]. Furthermore, upon the dimensional reduction of the anomaly counterterms we find that the four-dimensional anomaly counterterm evaluated on-shell (for AlLif spacetimes with \( \chi(0) \) constant) consists of two pieces that together form the action of \( z = 2 \) conformal gravity in \( 2+1 \) dimensions with the nonzero potential term [42, 14]. The presence of the potential term has so far not been seen in studies of the no scalar massive vector model. At the same time in the setting in which we computed the on-shell anomaly, all scalars were asymptotically constant and our on-shell anomalies do not depend on the scalars. It would be interesting to understand this situation better. What is noteworthy about the reduced on-shell anomaly is that there now appear two central charges (in the notation of [15] these are \( C_1 \) and \( C_2 \)) that are proportional to each other. From the reduction we can see that both originate from the single central charge in five dimensions. It would be interesting to understand this from the dual field theory point of view, i.e. the DLCQ of \( N = 4 \) SYM in the background of a theta angle that depends linearly on the null circle coordinate.

Furthermore it would also be of interest to see if this setup can be used to understand better the asymptotic symmetry group for AlLif spacetimes (see [13]) and to see if it is possible to understand the presence of the two central charges from that point of view. In the five-dimensional case the central charge shows up in the transformation of the boundary stress tensor under a PBH transformation. It would be interesting to investigate if there are similar statements possible for AlLif spacetimes and what the role of the stress tensor complex of [11, 10] is in this respect.

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Appendix A. Double null split

In this appendix we collect expressions for the Christoffel connections and the Ricci tensor components using the following double null split of the boundary metric of the five-dimensional AlAdS spacetimes:

\[
\hat{h}_{(0)\hat{a}\hat{b}} \, dx^\hat{a} \, dx^\hat{b} = 2H(0) \, du \, dt + \Pi_{(0)ij}(dx^i + H(0)N^i_0 \, dt)(dx^j + H(0)N^j_0 \, dt),
\]

where all metric components are arbitrary functions of \( t \) and \( x^i \) but do not depend on \( u \). The nonzero inverse metric components are given by

\[
\hat{h}^{\mu
}_{(0)} = H^{\mu
}_{(0)}^{-1}, \quad \hat{h}^{\mu}_{(0)} = -N^\mu_0, \quad \hat{h}^{ij}_{(0)} = \Pi^{ij}_{(0)}.
\]

For the determinant we have

\[
\sqrt{-\hat{h}(0)} = H(0)\sqrt{\Pi(0)},
\]

where all metric components are arbitrary functions of \( t \) and \( x^i \) but do not depend on \( u \). The nonzero inverse metric components are given by

\[
\hat{h}^{\mu
}_{(0)} = H^{\mu
}_{(0)}^{-1}, \quad \hat{h}^{\mu}_{(0)} = -N^\mu_0, \quad \hat{h}^{ij}_{(0)} = \Pi^{ij}_{(0)}.
\]

For the determinant we have

\[
\sqrt{-\hat{h}(0)} = H(0)\sqrt{\Pi(0)},
\]
where $\Pi_{(0)}$ is the determinant of the $2 \times 2$ metric $\Pi_{(0)ij}$. The nonzero Christoffel connections are

1. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{N}_\mu^{\nu} \partial_\nu \mathcal{H}_{(0)}.
\]
2. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)}^{-1}.
\]
3. \[
\hat{\Gamma}_\mu^{(0)u} = -\frac{1}{2} \mathcal{H}_{(0)} \mathcal{N}_{\mu}^{\nu} \partial_\nu \mathcal{H}_{(0)ij} + \mathcal{H}_{(0)} \left( \mathcal{N}_{\mu}^{\nu} \partial_\nu \mathcal{H}_{(0)} \right) \mathcal{N}_{(0)ij}^{\nu} + \mathcal{H}_{(0)}^{2} \mathcal{N}_{\mu}^{\nu} \mathcal{N}_{(0)ij}^{\nu} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu}.
\]
4. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{N}_{ij}^{\nu} \mathcal{N}_{(0)ij}^{\nu} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{H}_{(0)} \mathcal{N}_{ij}^{\nu} \left( \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} \right)
\]
   \[+ \frac{1}{2} \mathcal{N}_{ij}^{\nu} \partial_\nu \mathcal{H}_{(0)} - \frac{1}{2} \mathcal{N}_{ij}^{\nu} \partial_\nu \mathcal{H}_{(0)ij}.
\]
5. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)}^{-1} \partial_\mu \mathcal{H}_{(0)},
\]
6. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)}^{-1} \partial_\mu \mathcal{H}_{(0)},
\]
7. \[
\hat{\Gamma}_\mu^{(0)u} = -\frac{1}{2} \partial_\nu \mathcal{H}_{(0)},
\]
8. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} (\partial_\mu \mathcal{H}_{(0)} + \mathcal{H}_{(0)ij} \partial_\mu \mathcal{H}_{(0)ij} - \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} ),
\]
9. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} (\partial_\mu \mathcal{H}_{(0)} + \mathcal{H}_{(0)ij} \partial_\mu \mathcal{H}_{(0)ij} - \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} ),
\]
10. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} (\partial_\mu \mathcal{H}_{(0)} + \mathcal{H}_{(0)ij} \partial_\mu \mathcal{H}_{(0)ij} - \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} ),
\]
11. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} (\partial_\mu \mathcal{H}_{(0)} + \mathcal{H}_{(0)ij} \partial_\mu \mathcal{H}_{(0)ij} - \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} ),
\]
1. \[
\hat{\Gamma}_\mu^{(0)u} = \frac{1}{2} \mathcal{H}_{(0)} (\partial_\mu \mathcal{H}_{(0)} + \mathcal{H}_{(0)ij} \partial_\mu \mathcal{H}_{(0)ij} - \frac{1}{2} \mathcal{H}_{(0)} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} + \mathcal{D}_{(0)ij}^{(0)0} \mathcal{N}_{(0)ij}^{\nu} ),
\]

where indices are raised and lowered with $\Pi_{(0)ij}$, so $N_{(0)ij}^{\nu} = \Pi_{(0)ij}^{\nu}$ not to be confused with the $i$th component of $\tilde{N}_{(0)3}^{\nu}$ in (90), and $D_{(0)ij}^{(0)0}$ is the covariant derivative with respect to the metric $\Pi_{(0)ij}$ and where finally $C_{ij}^{\alpha}$ are the associated Christoffel connections. For the nonzero components of the Ricci tensor we obtain

where $\mathcal{R}_{(0)ij}$ is the Ricci scalar associated with the metric $\Pi_{(0)ij}$. One can also define the shift vector $K_{(0)}^{\nu} = \mathcal{H}_{(0)} N_{(0)}^{\nu}$, but it does not make the expressions shorter. The Ricci scalar is given by

\[
\hat{\mathcal{R}}_{(0)} = \mathcal{R}_{(0)} - 2 \mathcal{H}_{(0)}^{-1} \partial_\mu \mathcal{H}_{(0)} + \frac{1}{2} \mathcal{H}_{(0)}^{-2} \partial_\mu \mathcal{H}_{(0)}.
\]
Appendix B. The four-dimensional Fefferman–Graham expansions in terms of $A_\mu$ and $\chi$

Here we will present the four-dimensional Fefferman–Graham expansions for the on-shell configurations of the theory described by the action (54) in terms of the vector field $A_\mu$ and the axion $\chi$. The reduction Ansatz reads

$$g_{rr} = \frac{1}{r^2}, \quad (B.1)$$
$$g_{ra} = 0, \quad (B.2)$$
$$h_{ab} = \hat{h}_{ab} - \frac{\hat{h}_{am}\hat{h}_{bm}}{\hat{h}_{mm}}, \quad (B.3)$$
$$A_r = 0, \quad (B.4)$$
$$A_a = \frac{\hat{h}_{aa}}{\hat{h}_{rr}}, \quad (B.5)$$
$$\Phi = \frac{1}{2} \log \hat{h}_{rr}. \quad (B.6)$$

Using these equations together with the reduction Ansatz (47) and (48) as well as the five-dimensional expansions (9)–(11) and the boundary parametrization (92), we obtain expansions for the four-dimensional fields in terms of $H_0$, $N_0^i$, $\Pi_{0ij}$, $\phi_0$, $\chi_0$ as well as the free parameters appearing at higher order in the expansions (9)–(11), i.e. the reduced versions of the coefficients $\hat{h}_{ab}$, $\hat{\phi}_{(4)}$ and $\hat{\chi}_{(4)}$. These expansions take the following form:

$$h_{tr} = \frac{1}{r^2}(h_{(0)tr} + r^2 \log rh_{(2,1)tr} + r^2 h_{(2,2)tr} + r^4 (\log r)^2 h_{(4,2)tr} + r^4 \log rh_{(4,1)tr} + r^4 h_{(4,4)tr} + O(r^6 (\log r)^3)), \quad (B.7)$$

$$h_{tt} = \frac{1}{r^2}(h_{(0)tt} + r^2 \log rh_{(2,1)tt} + r^2 h_{(2,2)tt} + O(r^4 (\log r)^2)), \quad (B.8)$$

$$h_{ij} = \frac{1}{r^2}(h_{(0)ij} + r^2 h_{(2)ij} + r^4 \log rh_{(4,1)ij} + r^4 h_{(4,4)ij} + O(r^6 (\log r)^2)), \quad (B.9)$$

$$A_r = \frac{1}{r^2}(A_{(0)r} + r^2 \log rA_{(2,1)r} + r^2 A_{(2,2)r} + r^4 (\log r)^2 A_{(4,2)r} + r^4 \log rA_{(4,4)r} + r^4 A_{(4,4)r} + O(r^6 (\log r)^3)), \quad (B.10)$$

$$A_\mu = A_{(0)\mu} + r^2 \log rA_{(2,1)\mu} + r^2 A_{(2,2)\mu} + O(r^4 (\log r)^2), \quad (B.11)$$

$$\Phi = \Phi_{(0)} + r^2 \log r\Phi_{(2,1)} + r^2 \Phi_{(2,2)} + r^4 (\log r)^2 \Phi_{(4,2)} + r^4 \log r\Phi_{(4,1)} + r^4 \Phi_{(4,4)} + O(r^6 (\log r)^3)), \quad (B.12)$$

$$\phi = \phi_{(0)} + r^2 \phi_{(2)} + r^4 \log r\phi_{(4,1)} + r^4 \phi_{(4,4)} + O(r^6 \log r), \quad (B.13)$$

$$\chi = \chi_{(0)} + r^2 \chi_{(2)} + r^4 \log r\chi_{(4,1)} + r^4 \chi_{(4,4)} + O(r^6 \log r). \quad (B.14)$$

Just as in the massive vector case discussed in section 3 we have the constraints at leading order

$$h_{(0)\mu} = -(A_{(0)\mu})^2 e^{2\Phi_{(0)}}, \quad (B.15)$$
Again we will deal with this by writing
\[ A_{(0)\nu} = H_{(0)} e^{-2\Phi_{(0)}}, \]
and eliminating \( \phi_{(0)} \) in favor of \( \Phi_{(0)} \). Since we have the following relations:
\[ h_{(0)\mu\nu} = H_{(0)} \mathcal{N}_{(0)\mu\nu} - \frac{1}{k} \partial_{\mu} \chi_{(0)}, \]
\[ h_{(0)\mu\nu} = \Pi_{(0)\mu\nu}, \]
we will keep writing \( \Pi_{(0)\mu\nu} \) and replace \( \mathcal{N}_{(0)\mu\nu} \) by \( H_{(0)}^{-1} h_{(0)\mu\nu} + \frac{1}{k} \partial_{\mu} \chi_{(0)} \) in order to express the four-dimensional expansions in terms of the coefficients appearing in \((B.7)-(B.14)\). Translating in this manner the reduced expansions we obtain the following coefficients:
\[ h_{(2,1)\mu} = 2H_{(0)}^2 e^{-2\Phi_{(0)}} \Phi_{(2,1)}, \]
\[ h_{(2)\mu} = h_{(0)\nu} h_{(0)\mu} - \frac{1}{k} \mathcal{R}_{(0)\mu\nu} \partial_{\nu} \chi_{(0)} + \frac{1}{k^2} H_{(0)}^2 \partial_{\mu} \chi_{(0)} \partial_{\nu} \chi_{(0)} - 2H_{(0)} A_{(2)\nu} - 2H_{(0)}^2 e^{-2\Phi_{(0)}} \Phi_{(2)}, \]
\[ h_{(2,1)\mu} = -H_{(0)} A_{(2,1)\mu}, \]
\[ h_{(2)\mu} = \frac{1}{4} H_{(0)}^{-1} h_{(0)\nu} D_{(0)} \partial_{\nu} \chi_{(0)} + \frac{1}{4} D_{(0)} \partial_{\nu} H_{(0)} - H_{(0)}^{-1} \partial_{\nu} H_{(0)} + \frac{1}{4} D_{(0)} \partial_{\nu} H_{(0)} - \frac{1}{4} \partial_{\nu} H_{(0)} h_{(0)\mu\nu} - \frac{1}{4} \partial_{\nu} H_{(0)} h_{(0)\mu\nu} - \frac{1}{4} \partial_{\nu} H_{(0)} h_{(0)\mu\nu} + \frac{1}{4} \Pi_{(0)\mu\nu} D_{(0)} \partial_{\nu} \chi_{(0)} h_{(0)\mu\nu} + H_{(0)} A_{(2)\nu} e^{2\Phi_{(0)}} h_{(0)\mu\nu} - H_{(0)} A_{(2)\nu} + 2\Phi_{(2)} h_{(0)\mu\nu}, \]
\[ h_{(2)\mu\nu} = -\frac{1}{2} (R_{(0)\mu\nu} + \frac{1}{2} H_{(0)}^{-1} D_{(0)} \partial_{\mu} \chi_{(0)} + \frac{1}{2} \partial_{\nu} \chi_{(0)} \partial_{\mu} \chi_{(0)} + \frac{1}{4} \partial_{\nu} \chi_{(0)} \partial_{\mu} \chi_{(0)} + \frac{1}{4} \partial_{\nu} \chi_{(0)} \partial_{\mu} \chi_{(0)}) \]
\[ + H_{(0)}^{-1} \Pi_{(0)\mu\nu} (A_{(2)\nu} e^{2\Phi_{(0)}} + 2H_{(0)} \Phi_{(2)} - \frac{1}{4} D_{(0)} \partial_{\nu} H_{(0)} - \frac{1}{k} e^{2\Phi_{(0)}} \partial_{\nu} \chi_{(0)}), \]
\[ A_{(2)\nu} = -2H_{(0)} e^{-2\Phi_{(0)}} \Phi_{(2)}, \]
\[ A_{(2)\mu} = -2\Phi_{(2)} H_{(0)} e^{-2\Phi_{(0)}} + \frac{2}{3k} \partial_{\mu} \chi_{(0)} + \frac{1}{3k} h_{(0)\nu} \partial_{\nu} \chi_{(0)} + \frac{1}{6k^2} H_{(0)} \partial_{\nu} \chi_{(0)} \partial_{\nu} \chi_{(0)} + \frac{1}{12} H_{(0)} e^{-2\Phi_{(0)}} (R_{(0)\nu} + H_{(0)}^{-1} D_{(0)} \partial_{\nu} H_{(0)} + \frac{1}{2} H_{(0)}^{-1} \partial_{\nu} \chi_{(0)} \partial_{\nu} \chi_{(0)} + \frac{1}{2} \partial_{\nu} \chi_{(0)} \partial_{\mu} \chi_{(0)}), \]
\[ A_{(0)\mu} = \frac{1}{k} \partial_{\mu} \chi_{(0)}, \]
\[
A_{(2,1)i} = \frac{1}{4} H_{(0)}^{-1} R_{(0)ij} h_{(0)i}^j + \frac{1}{4} H_{(0)}^{-1} D_{(0)}^{(ij)} h_{(0)i}^j + \frac{1}{4} H_{(0)}^{-2} \partial_i H_{(0)} D_{(0)}^{(i)} h_{(0)i}^j \\
- \frac{1}{4} H_{(0)}^{-2} \partial^i H_{(0)} (D_{(0)}^{(ij)} h_{(0)ij} + D_{(0)}^{(ij)} h_{(0)ji}) - \frac{1}{2} H_{(0)}^{-1} D_{(0)}^{(ij)} h_{(0)i}^j \partial_i \Phi_{(0)} \\
+ \frac{1}{2} H_{(0)}^{-1} \partial^i \Phi_{(0)} (D_{(0)}^{(ij)} h_{(0)ij} + D_{(0)}^{(ij)} h_{(0)ji}) - \frac{5}{4} H_{(0)}^{-1} h_{(0)i}^j \partial_i \Phi_{(0)} \partial_j \Phi_{(0)} \\
- \frac{1}{4} H_{(0)}^{-1} D_{(0)}^{(ij)} \partial_i \Phi_{(0)} \partial_j \Phi_{(0)} + \frac{1}{8} H_{(0)}^{-1} \partial_i \Phi_{(0)} \partial_j \Phi_{(0)} + \frac{1}{4} H_{(0)}^{-1} \partial^i H_{(0)} \partial_j \Pi_{(0)ij} \\
- \frac{1}{8} H_{(0)}^{-1} \partial_i H_{(0)} \Pi_{(0)ik} \partial_j \Pi_{(0)kj} + \frac{1}{4} H_{(0)}^{-1} \partial^i \Phi_{(0)} \Pi_{(0)ik} \partial_j \Pi_{(0)kj} \\
- \frac{1}{2} H_{(0)}^{-1} \partial^i \Phi_{(0)} \partial_i \Pi_{(0)ij} + \frac{5}{4} H_{(0)}^{-1} \partial_i \Phi_{(0)} \partial_j \Phi_{(0)}, \quad (B.29)
\]

\[
\Phi_{(2,1)} = \frac{1}{12} R_{(0)} + \frac{1}{12} D_{(0)}^{(ij)} \partial_i \log H_{(0)} - \frac{1}{2} D_{(0)}^{(ij)} \partial_i \Phi_{(0)} - \frac{13}{24} \partial^i \Phi_{(0)} \partial_i \Phi_{(0)} \\
+ \frac{4}{3k} \varepsilon^{2\Phi_{(0)}} ( - \partial_i \chi_{(0)} \partial^i \chi_{(0)} + 2k H_{(0)}^{-1} \partial_i \chi_{(0)} - 2k H_{(0)}^{-1} \partial_j \chi_{(0)} ), \quad (B.30)
\]

\[
\phi_{(2)} = \frac{1}{4} D_{(0)}^{(ij)} \partial_i \Phi_{(0)} + \frac{1}{4} H_{(0)}^{-1} \partial_i H_{(0)} \partial^i \Phi_{(0)} \\
+ \frac{1}{k} H_{(0)}^{-1} \varepsilon^{2\Phi_{(0)}} \left( \frac{1}{k} H_{(0)} \partial_i \chi_{(0)} \partial^i \chi_{(0)} + 2 \partial_i \chi_{(0)} \partial_j \chi_{(0)} + 2 \partial_j \chi_{(0)} \partial_i \chi_{(0)} - 2 \partial_i \chi_{(0)} \partial_j \chi_{(0)} \right), \quad (B.31)
\]

\[
\chi_{(2)} = \frac{k}{2} H_{(0)}^{-1} \left( \frac{1}{4} \Pi_{(0)ij} \partial_i \Pi_{(0)ij} - \frac{1}{2} D_{(0)}^{(ij)} h_{(0)ij} - \partial_i \chi_{(0)} \partial_j \chi_{(0)} + \partial_i \chi_{(0)}, \partial_j \chi_{(0)} \right), \quad (B.32)
\]

We have not listed coefficients of the form \( a_{(4,m)} \) for some field \( a \) even though they can be computed from the reduction; the expressions are typically half a page and so we will not write them. Furthermore we did not write coefficients that depend explicitly on the reduction of \( \hat{L}_{\partial a} \) as these can be considered ‘arbitrary’ from a four-dimensional point of view. We put arbitrary in quotation marks because these coefficients are constrained by the reduced version of equations (35) and (36).

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12 The coefficients that depend explicitly on \( \hat{L}_{\partial a} \) are \( \Phi_{(2)}(\hat{\ell}_{a}), A_{(2)}(\hat{\ell}_{a}), A_{(4)}(\hat{\ell}_{a}), h_{(4i)j} (\hat{\ell}_{i}), h_{(4ij)} (\hat{\ell}_{ij}) \) and \( h_{(4ij)} (\hat{\ell}_{ij}) \).
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