Spectral estimates for degenerate critical levels.

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Abstract

We establish spectral estimates at a critical energy level for $h$-pseudors. Via a trace formula, we compute the contribution of isolated (non-extremum) critical points under a condition of "real principal type". The main result holds for all dimensions, for a singularity of any finite order and can be invariantly expressed in term of the geometry of the singularity. When the singularities are not integrable on the energy surface the results are significative since the order w.r.t. $h$ of the spectral distributions are bigger than in the regular setting.

Keywords : Semi-classical analysis; Trace formula; Oscillatory integrals.

1 Introduction.

If $P_h$ is a self-adjoint $h$-pseudo-differential operator, or more generally $h$-admissible (see [18]), acting on a dense subset of $L^2$, a classical and accessible problem is to study the asymptotic behavior, as $h$ tends to 0, of the spectral distributions :

$$\gamma(E, \varphi, h) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \varphi\left(\frac{\lambda_j(h) - E}{h}\right),$$

where the $\lambda_j(h)$ are the eigenvalues of $P_h$. Here we suppose that the spectrum is discrete in $[E - \varepsilon, E + \varepsilon]$, a sufficient condition for this is given below. One motivation is that it is in general not possible to compute the spectrum and one has to use statistical methods to gain spectral information. A second motivation is the existence of a duality between spectrum of quantum operators and the classical mechanic attached to their symbols.

If $p_0$ is the principal symbol of $P_h$, an energy $E$ is regular when $\nabla p_0(x, \xi) \neq 0$ on the energy surface $\Sigma_E = \{(x, \xi) \in T^*\mathbb{R}^n / p_0(x, \xi) = E\}$ and critical when it is not regular. It is well known that asymptotics of (1), as $h$ tends to 0, are closely
related to the closed trajectories of the classical flow of $p_0$ on the surface $\Sigma_E$.
Hence, there is a duality between the following objects:
\[
\lim_{h \to 0} \gamma(E, \varphi, h) = \{(t, x, \xi) \in \text{supp}(\hat{\varphi}) \times \Sigma_E / \Phi_t(x, \xi) = (x, \xi)\},
\]
where $\Phi_t$ is the flow of the Hamiltonian vector field $H_{p_0} = \partial_\xi p_0 \partial_x - \partial_x p_0 \partial_\xi$.
This duality has a universal character and does not systematically require the presence of an asymptotic parameter, as can show the trace formulae of Selberg and Duistermaat-Guillemin [9]. In the semi-classical setting, this relation was initially pointed out in the physics literature [11] & [1]. From a mathematical point of view, and $E$ a regular energy, a non-exhaustive list of references concerning this subject is [4], [16], [17] and more recently with a different approach [7]. See also [12] for the case of elliptic operators.

If $E$ is no more a regular value, the behavior of (1) depends on the singularities of $p$ on $\Sigma_E$ which leads to technical complications. The case of non-degenerate critical energies, that is such that the critical-set $C(p_0) = \{(x, \xi) \in T^*\mathbb{R}^n / dp_0(x, \xi) = 0\}$ is a compact $C^\infty$ manifold with a Hessian $\frac{\partial^2}{\partial \xi \partial x} p_0$ transversely non-degenerate along this manifold has been studied first by Brummelhuis et al. [3]. The problem was solved for quite general operators but for some "small times", i.e. for $\text{supp}(\hat{\varphi})$ contained in a neighborhood of the origin so that the only period of the linearized flow in $\text{supp}(\hat{\varphi})$ was 0. Later, Khuat-Duy [15] has obtained the contributions of the non-zero periods of the linearized flow with $\text{supp}(\hat{\varphi})$ compact, but for Schrödinger operators with symbol $\xi^2 + V(x)$ and a non-degenerate potential $V$. Our contribution was to generalize the result of [15] for more general operators but under extra assumptions on the flow (see [5]). Finally, in [6] the case of totally degenerate extremum was treated and the objective of this work is to study degenerate singularities which are not an extremum of the symbol.

After a reformulation, based on the theory of Fourier integral operators, the asymptotics of (1) can be expressed in terms of oscillatory integrals whose phases are related to the flow of $p_0$ on $\Sigma_E$. When $(x_0, \xi_0)$ is a critical point of $p_0$, it is well known that the relation $\text{Ker}(d_{x, \xi} \Phi_t(x_0, \xi_0) - \text{Id}) \neq \{0\}$ leads to the study of degenerate oscillatory integrals. In this work we consider the case of a totally degenerate energy, that is such that the Hessian matrix at our critical point is zero. Hence, the linearized flow for such a critical point satisfies $d_{x, \xi} \Phi_t(x_0, \xi_0) = \text{Id}$, for all $t \in \mathbb{R}$. A fortiori:
\[
\text{Ker}(d_{x, \xi} \Phi_t(x_0, \xi_0) - \text{Id}) = T_{x_0, \xi_0} T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}, \forall t \in \mathbb{R}, \tag{2}
\]
and the oscillatory integrals we have to consider are totally degenerate. In particular, it is impossible to use the stationary phase method to determine the asymptotic behavior of Eq. (1).

To solve this problem we will establish suitable normal forms, for the phase functions of our oscillatory integrals, for whom it is possible to generalize the stationary phase formula. The construction is geometric and is independent of the dimension but the asymptotic expansion of the related oscillatory integrals
depends on the dimension and on the order of the singularity at the critical point. Finally, since the normal forms have a geometrical meaning, it is possible to express invariently the top order coefficients of the asymptotic expansions in term of the geometry of the singularity on the energy surface.

2 Hypotheses and main result.

Let $P_h = Op_h^w(p(x, \xi, h))$ an $h$-pseudodifferential operator, obtained by Weyl quantization, in the class of $h$-admissible operators with symbol $p(x, \xi, h) \sim \sum h^j p_j(x, \xi)$, i.e. there exists $p_j \in \Sigma_0^n(T^*\mathbb{R}^n)$ and $R_N(h)$ such that:

$$ P_h = \sum_{j<N} h^j p_j^w(x, hD_x) + h^N R_N(h), \ \forall N \in \mathbb{N}. $$

Here $R_N(h)$ is a bounded family of operators on $L^2(\mathbb{R}^n)$ for $h \leq h_0$ and:

$$ \Sigma_0^n(T^*\mathbb{R}^n) = \{ a : T^*\mathbb{R}^n \to \mathbb{C}, \sup |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < C_{\alpha, \beta} m(x, \xi), \ \forall \alpha, \beta \in \mathbb{N}^n \}, $$

where $m$ is a tempered weight on $T^*\mathbb{R}^n$. For a detailed exposition on $h$-admissible operators we refer to the book of Robert [18]. In particular, $p_0(x, \xi)$ is the principal symbol of $P_h$ and $p_1(x, \xi)$ the sub-principal symbol. We note $\Phi_t := \exp(t\mathcal{H}_{p_0})$, the Hamiltonian flow of $\mathcal{H}_{p_0} = \partial_t p_0, \partial_x - \partial_x p_0, \partial_\xi$ and $\Sigma_E = p_0^{-1}(E) \subset T^*\mathbb{R}^n$ the energy surfaces of $p_0$.

We study here asymptotics of the spectral distributions:

$$ \gamma(E_c, \varphi, h) = \sum_{\lambda_j(h) \in [E_c - \varepsilon, E_c + \varepsilon]} \varphi(\frac{\lambda_j(h) - E_c}{h}), \ h \to 0^+, \ (3) $$

under the hypotheses $(H_1)$ to $(H_4)$ given below. We use here the notation $E_c$ to recall that this energy will be chosen critical.

$(H_1)$ The symbol of $P_h$ is real and there exists $\varepsilon_0 > 0$ such that the set $p_0^{-1}([E_c - \varepsilon_0, E_c + \varepsilon_0])$ is compact in $T^*\mathbb{R}^n$.

Remark 1 By Theorem 3.13 of [18] the spectrum $\sigma(P_h) \cap [E_c - \varepsilon, E_c + \varepsilon]$ is discrete and consists in a sequence $\lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h)$ of eigenvalues of finite multiplicities, if $\varepsilon$ and $h$ are small enough. A fortiori, $(H_1)$ insures that $\Sigma_{E_c}$ is compact.

To simplify notations we write $z = (x, \xi)$ for any point of the phase space.

$(H_2)$ On $\Sigma_{E_c}$, $p_0$ has a unique critical point $z_0 = (x_0, \xi_0)$ and near $z_0$:

$$ p_0(z) = E_c + \sum_{j=k}^N p_j(z) + \mathcal{O}(|(z - z_0)|^{N+1}), \ k > 2, $$

where the functions $p_j$ are homogeneous of degree $j$ in $z - z_0$. 

Remark 2 Strictly speaking, one could consider $k = 2$ with $\text{supp}(\hat{\varphi})$ small. But there is nothing new here since this case is precisely treated in [3].

$(H_3)$ We have $\hat{\varphi} \in C_0^\infty(\mathbb{R})$.

Since we are interested in the contribution of the fixed point $z_0$, to understand the new phenomenon it suffices to study a local problem:

$$
\gamma_{z_0}(E_c, \varphi, h) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{itE_c} \hat{\varphi}(t) \psi^w(x, hD_x) \exp\left(-\frac{i}{\hbar} t P_h \right) \Theta(P_h) dt.
$$

(4)

Here $\Theta$ is a function of localization near the critical energy surface $\Sigma_{E_c}$ and $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ has an appropriate support near $z_0$. Rigorous justifications are given in section 3 for the introduction of $\Theta(P_h)$ and in section 4 for $\psi^w(x, hD_x)$. The case of a critical point which is not an extremum is quite difficult, in particular because the singularity is transferred on the blow up of the critical point. To obtain a reasonable problem we consider the following hypothesis inspired by H"ormander’s real principal condition for distributions:

$(H_4)$ We have $\nabla p_k \neq 0$ on the set $C(p_k) = \{ \theta \in S^{2n-1} / p_k(\theta) = 0 \}$.

Remark 3 Contrary to the case of a local extremum [3], $z_0$ is not isolated on $\Sigma_{E_c}$. This imposes to study the classical dynamic in a neighborhood of $z_0$.

By homogeneity we have $\nabla p_k \neq 0$ on the cone $\{ z \in T^*\mathbb{R}^n \setminus \{0\} / p_k(z) = 0 \}$, but mainly $(H_4)$ will be used on the unit sphere. $C_{p_k}$ is a smooth manifold of codimension 1 which can be equipped with an invariant (Liouville) measure.

Construction of a geometrical measure. To state the main result clearly, we explain how to construct a relevant measure on the line. By $(H_4)$ we can locally construct on $C(p_k)$ the $2n - 2$ dimensional Liouville form $dL$ via:

$$
dL(\theta) \wedge dp_k(\theta) = d\theta, \forall \theta \in C(p_k).
$$

(5)

Here the differential are w.r.t. coordinates on $S^{2n-1}$, $d\theta$ is the standard surface density and we note again $p_k(\theta)$ for the restriction of $p_k$ on $S^{2n-1}$. The form $dL$ induces a density on $C(p_k)$ and by continuity we can extend the construction to close surfaces $p_k(\theta) = \varepsilon$ for $\varepsilon$ small enough. Accordingly we define the integrated density as:

$$
\text{Lvol}(u) = \int_{\{p_k(\theta) = u\}} |dL(\theta)|.
$$

(6)

An alternative definition is to compute the volume in $S^{2n-1}$ of the pullback:

$$
p_k^{-1}(\{0, x\}) = \{ \theta \in S^{n-1} / p_k(\theta) \in [0, x] \},
$$

(7)

4
and to interpret the result as a measure:

\[ V(p_k^{-1}([0,x])) = \int_0^x \text{Lvol}(s)ds. \]  

(8)

This relation is known in geometry as the co-area formula. The most important point, that will be exploited in this work is that \((H_4)\) insures that \(\text{Lvol}(u)\) is smooth near the origin. In other words, viewing \(\text{Lvol}(u)\) as a distribution we obtain that \(0 \notin \text{sing}\text{supp}(\text{Lvol}(u))\).

With these objects the new contributions to the trace formula are given by:

**Theorem 4** Under hypotheses \((H_1)\) to \((H_4)\) we obtain the existence of a full asymptotic expansion:

\[ \gamma_{z_0}(E_c, \varphi, h) \sim h^{\frac{2n}{k} - n} \sum_{m=0,1} \sum_{j=0}^{\infty} \int_0^L \text{vol}(s) ds. \]  

(8)

where the logarithms only occur when \((2n + j)/k \in \mathbb{N}^*\) and \(\Lambda_{j,m} \in \mathcal{S}'(\mathbb{R})\).

As concerns the leading term we obtain:

(1) If \(k > 2n\) (non-integrable singularity on \(\Sigma_{E_c}\)) we have:

\[ \gamma_{z_0}(E_c, \varphi, h) \sim h^{\frac{2n}{k} - n} \Lambda_{0,0}(\varphi) + O(h^{\frac{2n}{k} - n} \log(h)), \quad \text{as } h \to 0, \]

where the distributional coefficient \(\Lambda_{0,0}(\varphi)\) is given by:

\[ \frac{1}{(2\pi)^n k} \langle t^{k-1}, \varphi(t) \rangle \int_{\{p_k \geq 0\}} \|p_k(\theta)\|^{-\frac{2n}{k}} d\theta + \langle t^{k-1}, \varphi(t) \rangle \int_{\{p_k \leq 0\}} \|p_k(\theta)\|^{-\frac{2n}{k}} d\theta. \]  

(10)

(2) If the ratio \(2n/k = q\) is an integer we obtain logarithmic contributions:

\[ \gamma_{z_0}(E_c, \varphi, h) \sim h^{\frac{2n}{k} - n} \Lambda_{0,1}(\varphi) + O(h^{\frac{2n}{k} - n}), \quad \text{as } h \to 0, \]

where:

\[ \Lambda_{0,1}(\varphi) = \frac{1}{(2\pi)^n} \frac{d^{q-1}\text{Vol}_{q-1}(0)}{dq-1(0)} \int_\mathbb{R} |t|^{q-1} \varphi(t). \]  

(11)

(3) For \(2n > k\) and \(2n/k \notin \mathbb{N}\) the asymptotic is as in 1) with the modified distributions:

\[ \langle t^{k-1}_+, \varphi(t) \rangle \left\langle \frac{d^{2n}}{du^{2n}} u^{\frac{2n}{k} - \frac{2n}{k}}, \text{Vol} \right\rangle + \langle t^{k-1}_-, \varphi(t) \rangle \left\langle \frac{d^{2n}}{du^{2n}} u^{\frac{2n}{k} - \frac{2n}{k}}, \text{Vol} \right\rangle, \]

where the derivatives w.r.t. \(w\) are normalized distributional derivatives.
The meaning of normalized derivative it that one choose the normalization:

\[ \frac{d^{2n}}{dw^{2n}} = \prod_{j=1}^{2n} \frac{1}{j - \frac{2n}{k}} \frac{d^{2n}}{dw^{2n}}, \] (12)

so that the distributional derivatives satisfy:

\[ \left\langle \frac{d^{2n}}{dw^{2n}} w^{2n-\frac{2n}{k}}, f(w) \right\rangle = \left\langle w^{\frac{-2n}{k}}, f(w) \right\rangle, \] (13)

for all \( f \in C^\infty_0 \) with \( f = 0 \) in a neighborhood of the origin. With the method we employ here this normalization appears naturally in the expansion. The distributional bracket involving \( L_{\text{vol}} \) is detailed in section 6.

**Remark 5** In cases (1) & (3) the remainders \( O(h^{\frac{2n+1}{k} - n \log(h)}) \) can be replaced by \( O(h^{\frac{2n+1}{k} - n}) \) under the only condition that \( (2n+1)/k \notin \mathbb{N} \). An interesting point is that the degree \( k \) of the singularity on \( T^*\mathbb{R}^n \) has the inverse scaling property on the spherical blow-up since the new degree is \( -\frac{2n}{k} \).

Results (3)&(2) for \( q \geq 2 \) are not intuitive and are certainly difficult to be reached without geometry. In particular one has to work in the dual since both Fourier transforms w.r.t. \( t \) are distributional. In (3), the order \( 2n \) is arbitrary and the result is the same for any derivative of order greater than \( E(2n/k) \). Viewing the top order coefficients of the trace as distributions, i.e.:

\[ \gamma_{z_0}(E_c, \varphi, h) \sim f(h) \langle \gamma, \varphi \rangle, \ h \to 0, \] (14)

in all cases at hand we obtain:

**Corollary 6** Under the previous assumptions, \( \text{singsupp}(\gamma) = \{0\} \).

A similar result presumably holds for all terms of the expansion since the asymptotic involves distributions \(|t|^n \log(|t|)\). Note that Corollary 6 is not obvious in view of Eq. (2). Also, it must be pointed out that results (1)&(2) for \( q = 1 \) are bigger than the standard estimate for non-critical energies for which one obtain:

\[ \gamma(E, \varphi, h) \sim \frac{h^{1-n}}{(2\pi)^n \hat{\varphi}(0)\text{Vol}(\Sigma_E)}, \] (15)

where \( \text{Vol}(\Sigma_E) \) is the usual Liouville volume of the regular (compact) surface of energy \( E \). Hence the presence of non-integrable singularities on the energy surface has a significative spectral effect which can perhaps be measured by eigenfunctions estimates as in [3]. On the other side, for an integrable singularity, we obtain the global result:

**Corollary 7** Under the conditions of Theorem 4, if \( k < 2n \) we have:

\[ \gamma(E_c, \varphi, h) \sim \frac{h^{1-n}}{(2\pi)^n \hat{\varphi}(0)\text{Vol}(\Sigma_{E_c})}, \text{ as } h \to 0. \] (16)
This result is a consequence of Theorem 4 and of the results of section 4 to which we refer for a detailed proof. Finally, an interesting problem that we have not investigated here is the problematic of small $h$ and $|E - E_c|$ estimates for a non-integrable singularity on $\Sigma_{E_c}$: in this setting $\text{Vol}(\Sigma_{E_c})$ of Eq. (15) diverges as $E \to E_c$. See [3] for more details.

3 Oscillatory representation.

The construction below is more or less classical. We recall here important facts and results concerning the approximation of the propagator by Fourier integral operators, or FIO, depending on a parameter $h$. We recall that:

$$\gamma(E_c, \varphi, h) = \sum_{\lambda_j(h) \in I_\varepsilon} \varphi(\frac{\lambda_j(h) - E_c}{h}), \quad I_\varepsilon = [E_c - \varepsilon, E_c + \varepsilon],$$

with $\hat{\varphi} \in C_0^\infty(\mathbb{R})$ and $p_0^{-1}(I_{c_0})$ compact in $T^*\mathbb{R}^n$. In this setting the spectrum of $P_h$ is discrete in $I_\varepsilon$ for $h > 0$ small enough and $\varepsilon < \varepsilon_0$ (see Remark 1) and the sum is well defined. We localize near $E_c$ with a cut-off $\Theta \in C_0^\infty([E_c - \varepsilon, E_c + \varepsilon])$, $\Theta = 1$ near $E_c$ and $0 \leq \Theta \leq 1$ on $\mathbb{R}$. The associated decomposition is:

$$\gamma(E_c, \varphi, h) = \gamma_1(E_c, \varphi, h) + \gamma_2(E_c, \varphi, h),$$

with:

$$\gamma_1(E_c, \varphi, h) = \sum_{\lambda_j(h) \in I_\varepsilon} (1 - \Theta)(\lambda_j(h))\varphi(\frac{\lambda_j(h) - E_c}{h}), \quad \gamma_2(E_c, \varphi, h) = \sum_{\lambda_j(h) \in I_\varepsilon} \Theta(\lambda_j(h))\varphi(\frac{\lambda_j(h) - E_c}{h}).$$

A classical result, see e.g. [3], is that the sum of Eq. (17) satisfies:

**Lemma 8** $\gamma_1(E_c, \varphi, h) = O(h^\infty)$, as $h \to 0$.

Consequently, asymptotic behaviors of $\gamma(E_c, \varphi, h)$ and $\gamma_2(E_c, \varphi, h)$ are equivalent modulo $O(h^\infty)$. By inversion of the Fourier transform we obtain:

$$\Theta(P_h)\varphi(\frac{P_h - E_c}{h}) = \frac{1}{2\pi} \int e^{\frac{i\hat{\varphi} t}{h}} \varphi(t)e^{-\frac{i}{h}tP_h}\Theta(P_h)dt.$$ 

Since the trace of the left hand-side is exactly $\gamma_2(E_c, \varphi, h)$, we have:

$$\gamma_2(E_c, \varphi, h) = \frac{1}{2\pi} \text{Tr} \int e^{\frac{i\hat{\varphi} t}{h}} \varphi(t)e^{-\frac{i}{h}tP_h}\Theta(P_h)dt. \quad (19)$$
Observe that Eq. (19) generalizes the Poisson summation formula.

Let be \( U_h(t) = \exp(-\frac{it}{h}P_h) \), the evolution operator. For each integer \( N \) we can approximate \( U_h(t)\Theta(P_h) \), modulo \( O(h^N) \), by a Fourier integral-operator, or FIO, depending on a parameter \( h \). Let \( \Lambda \) be the Lagrangian manifold associated to the flow of \( p_0 \), i.e.:

\[
\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \tau = p_0(x, \xi), (x, \xi) = \Phi_t(y, \eta)\}.
\]

**Theorem 9** \( U_h(t)\Theta(P_h) \) is an \( h \)-FIO associated to \( \Lambda \). There exist \( U^{(N)}_{\Theta, h}(t) \) with integral kernel in Hörmander’s class \( I(\mathbb{R}^{2n+1}, \Lambda) \) and \( R^{(N)}_{h}(t) \) bounded, with a \( L^2 \)-norm uniformly bounded for \( 0 < h \leq 1 \) and \( t \) in a compact subset of \( \mathbb{R} \), such that \( U_h(t)\Theta(P_h) = U^{(N)}_{\Theta, h}(t) + h^N R^{(N)}_{h}(t) \).

We refer to Duistermaat [8] for a proof of this theorem.

**Remark 10** By a theorem of Helffer&Robert (Theorem 3.11 and Remark 3.14 of [13]), \( \Theta(P_h) \) is an \( h \)-admissible operator with a symbol supported in \( p_0^{-1}(I_c) \). This allows us to consider only oscillatory-integrals with compact support.

For the control of the remainder, associated to \( R^{(N)}_{h}(t) \), we use :

**Corollary 11** Let \( \Theta_1 \in C_0^\infty(\mathbb{R}) \) such that \( \Theta_1 = 1 \) on \( \text{supp}(\Theta) \) and \( \text{supp}(\Theta_1) \subset I_c \), then \( \forall N \in \mathbb{N} \):

\[
\text{Tr}(\Theta(P_h)\varphi(\frac{P_h - E_c}{h})) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} \hat{\varphi}(t)e^{\frac{i}{h}tE_c}U^{(N)}_{\Theta, h}(t)\Theta_1(P_h)dt + O(h^N).
\]

For a proof of this result, based on the cyclicity of the trace, see [9] or [13].

If \( (x_0, \xi_0) \in \Lambda \) and \( \varphi(x, \theta) \in C^\infty(\mathbb{R}^k \times \mathbb{R}^N) \) parameterizes \( \Lambda \) in a sufficiently small neighborhood \( U \) of \( (x_0, \xi_0) \), then for each \( u_h \in I(\mathbb{R}^k, \Lambda) \) and \( \chi \in C_0^\infty(T^*\mathbb{R}^k) \), \( \text{supp}(\chi) \subset U \), there exists a sequence of amplitudes \( a_j = a_j(x, \theta) \in C_0^\infty(\mathbb{R}^k \times \mathbb{R}^N) \) such that for all \( L \in \mathbb{N} \):

\[
\chi^w(x, hD_x)u_h = \sum_{-d \leq j < L} h^j I(a_j e^{\frac{i}{h}x}) + O(h^L).
\]

We will use this remark with the following result of Hörmander (see [13], tome 4, proposition 25.3.3). Let be \( (T, \tau, x_0, \xi_0, y_0, -\eta_0) \in \Lambda_{\text{flow}}, \eta_0 \neq 0 \), then near this point there exists, after perhaps a change of local coordinates in \( y \) near \( y_0 \), a function \( S(t, x, \eta) \) such that :

\[
\phi(t, x, y, \eta) = S(t, x, \eta) - \langle y, \eta \rangle,
\]

8
parameterizes $\Lambda_{\text{flow}}$. In particular this implies that:
\[
\{(t, \partial_t S(t, x, \eta), x, \partial_x S(t, x, \eta), \partial_\eta S(t, x, \eta), -\eta)\} \subset \Lambda_{\text{flow}},
\]
and that the function $S$ is a generating function of the flow, i.e.:
\[
\Phi_t(\partial_\eta S(t, x, \eta), \eta) = (x, \partial_x S(t, x, \eta)).
\]
Moreover, $S$ satisfies the Hamilton-Jacobi equation:
\[
\begin{cases}
\partial_t S(t, x, \xi) + p_0(x, \partial_x S(t, x, \eta)) = 0, \\
S(0, x, \xi) = \langle x, \xi \rangle.
\end{cases}
\]
Mainly, we will apply this result locally near $(x_0, \xi_0) = (y_0, \eta_0)$, our unique fixed point of the flow on the energy surface $\Sigma_{E_0}$. If $\xi_0 = 0$ we can replace the operator $P_h$ by $e^{i(x, \xi_0)} P_h e^{-i(x, \xi_0)}$ with $\xi_1 \neq 0$. The spectrum is the same since the new operator has symbol $p(x, \xi - \xi_1, h)$ with critical point $(x_0, \xi_1)$, $\xi_1 \neq 0$.

Hence, for each $N \in \mathbb{N}^*$ and modulo an error $O(h^{N-d})$, the localized trace $\gamma_2(E, \varphi, h)$ of Eq. (14) can be written as:
\[
\gamma_2(E, \varphi, h) = \sum_{j < N} (2\pi h)^{-d+j} \int_{\mathbb{R} \times \mathbb{S}^2} e^{iS(t, x, \xi) - i\varphi(t, x, \xi)} a_j(t, x, \xi) \hat{\varphi}(t) dtd\xi.
\]
To obtain the right power $-d$ of $h$ we apply results of Duistermaat [8] (following here Hörmander for the FIO, see [14] tome 4) concerning the order. An $h$-pseudo-differential operator obtained by Weyl quantization:
\[
(2\pi h)^{-\frac{d}{2}} \int_{\mathbb{R}^N} a(x, y, \xi) e^{i(x, y, \xi)} d\xi,
\]
is of order 0 w.r.t. $1/h$. Since the order of $U_h(t)\Theta(P_h)$ is $-\frac{1}{h}$, we have:
\[
\psi^w(x, hD_x)U_h(t)\Theta(P_h) \sim \sum_{j < N} (2\pi h)^{-n+j} \int_{\mathbb{R}^n} a_j(t, x, y, \eta) e^{iS(t, x, \eta) - i\varphi(t, x, \eta)} dy.
\]
Multiplying Eq. (21) by $\hat{\varphi}(t) e^{i\varphi(Et)}$ and passing to the trace we find Eq. (23) with $d = n$ and where we write again $a_j(t, x, \eta)$ for $a_j(t, x, x, \eta)$.

To each element $u_h$ of $I(\mathbb{R}^k, \Lambda)$ we can associate a principal symbol $e^{\frac{i}{h}S} \sigma_{\text{princ}}(u_h)$, where $S$ is a function on $\Lambda$ such that $\xi dx = dS$ on $\Lambda$. In fact, if $u_h = I(ac\hat{\varphi}^\prime)$ then we have $S = \varphi \circ i_{\varphi}^{-1}$ and $\sigma_{\text{princ}}(u_h)$ is a section of $|\Lambda|\hat{\pi} \otimes M(\Lambda)$, where $M(\Lambda)$ is the Maslov vector-bundle of $\Lambda$ and $|\Lambda|\hat{\pi}$ the bundle of half-densities on $\Lambda$. If $p_1$ is the sub-principal symbol of $P_h$, in the global coordinates $(t, y, \eta)$ on $\Lambda_{\text{flow}}$ the half-density of the propagator $U_h(t)$ is given by:
\[
\exp(i \int_0^t p_1(\Phi_s(y, -\eta)) ds |dtdy\eta|^\frac{d}{2}).
\]
This expression is related to the resolution of the first transport equation for the propagator, for a proof we refer to Duistermaat and Hörmander [10].

4 Classical dynamic near the equilibrium.

The function $S$ of Eq. (23) is related to the classical mechanic and we obtain a link with the dynamic generated by $p_0$. Precisely, a critical point of the oscillatory integral of Eq. (23) satisfies the equations:

$$
\begin{align*}
E_c &= -\partial_t S(t, x, \xi), \\
x &= \partial_\xi S(t, x, \xi), \\
\xi &= \partial_x S(t, x, \xi),
\end{align*}
$$

where the right hand side defines a closed trajectory of the flow inside $\Sigma_{E_c}$. Accordingly, the non-stationary phase method shows that $\gamma_2(E_c, \varphi, h)$ is asymptotically determined by the closed orbits of the flow on $\Sigma_{E_c}$. We recall that we are mainly interested in the contribution of the equilibrium $z_0$. With $\psi \in C_0^\infty(T^*\mathbb{R}^n)$, $\psi = 1$ near $z_0$, we write:

$$
\gamma_2(E_c, \varphi, h) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{iE_c}{h}} \tilde{\varphi}(t) \psi^w(x, hD_x) \exp(-\frac{i}{h} t P_h) \Theta(P_h) dt
$$

$$
+ \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{iE_c}{h}} \tilde{\varphi}(t)(1 - \psi^w(x, hD_x)) \exp(-\frac{i}{h} t P_h) \Theta(P_h) dt.
$$

With the additional hypothesis of having a clean flow, the second term can be fully treated by the semi-classical trace formula on a regular level. We also observe that the contribution of the first term is micro-local and allows to introduce local coordinates near $z_0$. To distinguish the contribution of $z_0$ from eventual closed trajectories we use the following result on the dynamic.

**Lemma 12** For all $T > 0$ there exists a neighborhood $U_T$ of the critical point such that $\Phi_t(z) \neq z$ for all $z \in U_T \setminus \{z_0\}$ and for all $t \in [-T, 0] \cup [0, T]$.

For a proof of this Lemma, based mainly on a result of Yorke [20], see [10].

With $\text{supp}(\tilde{\varphi})$ compact we can choose $\psi$ such that Lemma 12 holds on $\text{supp}(\psi)$ for all $t \in \text{supp}(\tilde{\varphi})$. Hence, on the support of $\psi$ there are two contributions:

1) Points $(t, x, \xi) = (0, x, \xi)$ for $(x, \xi) \in \Sigma_{E_c}$.
2) Points $(t, x, \xi) = (t, z_0)$ for $t \in \text{supp}(\tilde{\varphi})$.

Now, we restrict our attention to the second contribution. In the following, until further notice, the derivatives $d$ are derivatives w.r.t. initial conditions $z$ (resp. $x, \xi$). If $z_0$ is an equilibrium of the vector field $X$ then the linearized flow $d\exp(tX)(z_0)$ is the flow of the linearized vector field $dX(z_0)$. In our setting, the linearized operator is zero and we have:

$$
d\Phi_t(z_0) = \exp(0) = \text{Id}, \ \forall t.
$$
Moreover, with \((H_2)\) we clearly obtain that:

\[ d^j \Phi_t(z_0) = 0, \forall t, \forall j \in \{2, ..., k-2\}. \]  

(27)

The non-zero terms of the Taylor expansion of the flow are computed by:

**Lemma 13** Let be \(z_0\) an equilibrium of the \(C^\infty\) vector field \(X\) and \(\Phi_t\) the flow of \(X\). Then for all \(m \in \mathbb{N}^*\), there exists a polynomial map \(P_m\), vector valued and of degree at most \(m\), such that:

\[ d^m \Phi_t(z_0)(z^m) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0)P_m(d\Phi_{s}(z_0)(z), ..., d^{m-1}\Phi_{s}(z_0)(z^{m-1}))ds. \]

See [5] or [6] for a proof. In our setting, the \((k-1)\)-jet of \(p_0\) is flat in \(z_0\) and we have \(P_{k-1}(y_1, ..., y_{k-2}) = d^{k-1}H_{p_0}(z_0)(y_1^{k-1})\). Here, for any vector \(u\) the notation \(u^l\) stands for \((u, ..., u)\) repeated \(l\)-times, the same convention is used below. In view of Eq. (26) by integration from 0 to \(t\) we obtain:

\[ d^{k-1} \Phi_t(z_0)(z^{k-1}) = \int_0^t d^{k-1}H_{p_0}(z_0)(z^{k-1})ds = td^{k-1}H_{p_0}(z_0)(z^{k-1}). \]  

(28)

This provides an explicit formula for the germ of \(\Phi_t\) in \(z_0\):

\[ \Phi_t(z) = z + \frac{1}{(k-1)!}d^{k-1}\Phi_t(z_0)(z^{k-1}) + O(\|z\|^k). \]  

(29)

We describe now more precisely the singularities of our phase function. Without loss of generality, we can assume that \(z_0 = 0\)

**Lemma 14** Near the origin we have:

\[ S(t, x, \xi) - \langle x, \xi \rangle + tE_x = -t(\mathfrak{p}_k(x, \xi) + R_{k+1}(x, \xi) + tG_{k+1}(t, x, \xi)), \]  

(30)

where \(R_{k+1}(x, \xi) = O(\|(x, \xi)\|^k)\) and \(G_{k+1}(t, x, \xi) = O(\|(x, \xi)\|^k)\), uniformly for \(t\) in a compact subset of \(\mathbb{R}\).

**Proof.** In view of Eq. (29), we search our local generating function as:

\[ S(t, x, \xi) = -tE_x + \langle x, \xi \rangle + S_k(t, x, \xi) + O(\|(x, \xi)\|^k), \]

where \(S_k\) is homogeneous of degree \(k\) w.r.t. \((x, \xi)\). Let \(J\) be the matrix of the usual symplectic form. Comparing terms of degree \(k-1\) in the implicit relation \(\Phi_t(\partial_\xi S(t, x, \xi)) = (x, \partial_\xi S(t, x, \xi))\) provides:

\[ J\nabla S_k(t, x, \xi) = \frac{1}{(k-1)!}d^{k-1}\Phi_t(0)(\langle x, \xi \rangle^{k-1}). \]
By homogeneity and with Eq. (28) we obtain:

\[ S_k(t, x, \xi) = \frac{1}{k!} \langle (x, \xi), t Jd^{k-1} H_p_k(x, \xi)^{k-1} \rangle = -tp_k(x, \xi). \]

As concerns the remainders, we have \( S(0, x, \xi) = \langle x, \xi \rangle \), so that:

\[ S(t, x, \xi) - \langle x, \xi \rangle = tF(t, x, \xi), \]

where \( F \) is smooth in a neighborhood of \( (x, \xi) = 0 \). Now, the Hamilton-Jacobi equation imposes that \( F(0, x, \xi) = -p_0(x, \xi) \) and we have:

\[ R_{k+1}(x, \xi) = p_0(x, \xi) - E_c - p_k(x, \xi) = \mathcal{O}(||(x, \xi)||^{k+1}). \]

Finally, the time dependant remainder can be written:

\[ S(t, x, \xi) - S(0, x, \xi) - t\partial_t S(0, x, \xi) = \mathcal{O}(t^2), \]

since by construction this term is of order \( \mathcal{O}(||(x, \xi)||^{k+1}) \) we get the desired result when \( t \) is in a compact subset of \( \mathbb{R} \).

\[ \blacksquare \]

\textbf{Remark 15} With Lemma 13 one can compute explicitly terms of higher degree for \( S \) and \( \Phi_t \). But we do not need them for the present contribution because of some considerations of homogeneity below.

\section{Normal forms and oscillatory integrals.}

Retaining only the coefficient of highest degree w.r.t. \( h \) in Eq. (28), we have to study the asymptotic behavior of oscillatory integrals:

\[ I_h^1 = \int_{\mathbb{R} \times T^*\mathbb{R}^n} e^{\frac{1}{h}(S(t, x, \xi) - \langle x, \xi \rangle - tE_c)} a(t, x, \xi) dt dx d\xi, \quad h \to 0^+. \quad (31) \]

Here, we have temporary discarded the factor \((2\pi h)^{-n}\) to avoid it’s constant repetition in the calculations. Since the contribution we study is local, we can work with local coordinates and we identify locally our neighborhood of the critical point in \( T^*\mathbb{R}^n \) with an open of \( \mathbb{R}^{2n} \). With \( z = (x, \xi) \in \mathbb{R}^{2n} \), we define:

\[ \Psi(t, z) = \Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi \rangle + tE_c. \quad (32) \]

The next Lemma provides a resolution of singularities for \( \Psi \) w.r.t. \( C(p_k) \).

\textbf{Lemma 16} Assume \( P_h \) satisfies conditions \((H_2)\) and \((H_4)\). For all \( t \) in a compact, after a blow-up w.r.t. \( z \) in a neighborhood of \( z_0 \), there exists local coordinates \( \eta \) such that:

\[ \Psi(t, z) \simeq -\eta_0 \eta_k^k, \quad \text{in all directions where } p_k > 0, \]

\[ \Psi(t, z) \simeq +\eta_0 \eta_k^k, \quad \text{in all directions where } p_k < 0, \]

\[ \Psi(t, z) \simeq -\eta_0 \eta_k^1 \eta_2, \quad \text{locally near } C(p_k). \]
Proof. We can assume that \( z_0 = 0 \). To perform the blow-up, we use polar coordinates \( z = (r, \theta) \), \( \theta \in S^{2n-1}(R) \). By Lemma \ref{lem:blow-up} near \( z_0 \) we have:

\[
\Psi(t, z) \simeq -tr^k(p_k(\theta)) + rR_{k+1}(r, \theta) + trG_{k+1}(t, r, \theta),
\]

where \( p_k(\theta) \) is the restriction of \( p_k \) on \( S^{2n-1} \) and \( Q(t, 0, \theta) = 0 \). If \( p_k(\theta_0) \neq 0 \) and \( t \) in a compact, then for \( r < r_0 \) we have \( p_k(\theta) + Q(t, r, \theta) \neq 0 \). We define:

\[
\eta_0(t, r, \theta) = \frac{1}{k} p_k(\theta), \quad \eta_1(t, r, \theta) = r|p_k(\theta) + Q(t, r, \theta)|^{1/k}.
\]

In these coordinates the phase becomes \(-\eta_0 \eta_1^k\) if \( p_k(\theta_0) \) is positive (resp. \( \eta_0 \eta_1^k \) for a negative value). Near \( \theta_0 \), we have:

\[
\frac{\partial \eta_1}{\partial r}(t, 0, \theta) = |p_k(\theta)|^{1/k} \neq 0, \forall t,
\]

hence, the corresponding Jacobian satisfies \(|J\eta|(t, 0, \theta) = |p_k(\theta)|^{1/k} \neq 0\).

Now, let \( \theta_0 \in C(p_k) \). Up to a permutation, we can suppose that \( \partial_{\theta_1} p_k(\theta_0) \neq 0 \). We accordingly choose the new coordinates:

\[
(\eta_0, \eta_1, \eta_3, ..., \eta_{2n})(t, r, \theta) = (t, r, \theta_2, ..., \theta_{2n-1}),
\]

which are locally admissible since \(|J\eta|(t, 0, \theta_0) = |\partial_{\theta_1} p_k(\theta_0)| \neq 0\). Finally, lemma follows by compactness of \( C(p_k) \).

If necessary, we can shrink the support of \( \psi \) to obtain the existence of the normal forms inside \( \text{supp}(\hat{\varphi}) \times \text{supp}(\psi) \). We define now a partition of unity associated to \( p_k \). We pick a family of functions \( \phi_j \in C^\infty(S^{2n-1}) \) such that:

\[
C(p_k) \subset \bigcup_j \text{supp}(\phi_j), \quad \sum_j \phi_j = 1 \text{ near } C(p_k).
\]

We can also choose each \( \text{supp}(\phi_j) \) small enough so that normal forms of Lemma \ref{lem:normal-form} exist in \([0, r_0] \times \text{supp}(\phi_j) \). Clearly, this family can be chosen finite and we obtain a partition of unity with \( \phi_0 = 1 - \sum \phi_j \). The support of \( \phi_0 \) is not connected and we define \( \phi_0^+ \), with \( p_k(\theta) > 0 \) on \( \text{supp}(\phi_0^+) \), similarly we define \( \phi_0^- \) where \( p_k < 0 \), so that \( \phi_0 = \phi_0^+ + \phi_0^- \).

Remark 17 The family \( \{\phi_j\} \) depends only on \( p_k \), e.g. we can impose \( \sum \phi_j = 0 \) for \(|p_k(\theta)| \geq \varepsilon > 0\). This point is useful for the globalization in section 6.
Let be $\lambda = h^{-1}$. We accordingly split up the integral of Eq. (31) to obtain:

$$I_{\pm}(\lambda) = \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{2n-1}} e^{i\lambda \Psi(t, r, \theta)} \phi_{0, \pm}^\pm (\theta) a(t, r\theta) r^{2n-1} dt dr d\theta$$

$$= \int_{\mathbb{R} \times \mathbb{R}_+} e^{-i\lambda (\pm \eta_0 \eta_1^k)} A_{0, \pm}^\pm (\eta_0, \eta_1) d\eta_0 d\eta_1 = \int_{\mathbb{R}_+} \hat{A}_{0, \pm}^\pm (\pm \lambda \eta_1^k, \eta_1),$$

respectively for the directions where $p_k(\theta) > 0$ and $p_k(\theta) < 0$. Here the notation $\hat{A}$ stands for the Fourier transform w.r.t. the first argument and is also used below.

Similarly, the covering of $C(p_k)$ gives:

$$I_j(\lambda) = \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{2n-1}} e^{i\lambda \Psi(t, r, \theta)} \phi_j (\theta) a(t, r\theta) r^{2n-1} dt dr d\theta$$

$$= \int_{\mathbb{R} \times \mathbb{R}_+} e^{-i\lambda \eta_0 \eta_1^k \eta_2} A_j (\eta_0, \eta_1, \eta_2) d\eta_0 d\eta_1 d\eta_2 = \int_{\mathbb{R}_+ \times \mathbb{R}} \hat{A}_j (\eta_1^k \eta_2, \eta_1, \eta_2) d\eta_1 d\eta_2.$$

These new amplitudes are obtained by pullback and integration, i.e.:

$$A_{0, \pm}^\pm (\eta_0, \eta_1) = \int \eta^* (\phi_{0, \pm}^\pm (\theta) a(t, r\theta) r^{2n-1} |J\eta|) d\eta_2 ... d\eta_{2n},$$

(33)

$$A_j (\eta_0, \eta_1, \eta_2) = \int \eta^* (\phi_j (\theta) a(t, r\theta) r^{2n-1} |J\eta|) d\eta_3 ... d\eta_{2n}.$$  

(34)

With $C(p_k)$ compact, our oscillatory integral can be written as a finite sum:

$$I(\lambda) = I_+(\lambda) + I_-(\lambda) + \sum_{j=0}^{L} I_j(\lambda), \quad \lambda = h^{-1},$$

(35)

where each term of the r.h.s. will be treated by elementary methods.

**Remark 18** By pullback of the measure $r^{2n-1} dr$, we have $A_{0, \pm}^\pm = O(\eta_1^{2n-1})$ near $\eta_1 = 0$. Same remark for $A_j = O(\eta_1^{2n-1})$ near $\eta_1 = 0$. This point is important, since Lemmas [19, 21] below involve Dirac-distributions w.r.t. $\eta_1$.

**Expansion of the related oscillatory integrals.**

We end this section with two results on asymptotics. In fact, to save a lot of computations we will take benefit of the linear term $\eta_0$ in our normals forms. This approach is more economic than the strategy proposed in [19]. But the reader must keep in mind that the method of [19] can be applied in a more general setting. The next elementary Lemma can be found in [6] and allows to expand both integrals $I_{\pm}(\lambda)$.
Lemma 19  For any $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$ we have :

$$
\int_0^\infty \hat{a}(\lambda \eta_1^k, \eta_1) d\eta_1 \sim \sum_{j=0}^\infty \lambda^{-\frac{j+1}{k}} c_j(a), \; \lambda \to +\infty,
$$

(36)

where the distributional coefficients are :

$$
c_j = \frac{1}{k!} \left( \mathcal{F}(x_+^{\frac{j+1-k}{k}})(\eta_0) \otimes \delta_0^{(j)}(\eta_1) \right), \; x_+ = \max(x,0).
$$

Remark 20 The same result holds for a phase $-\eta_0 \eta_1^k$ if we replace terms $x_+$ by $x_-$ in Lemma 19. For oscillatory integrals on $\mathbb{R}^2$ one can conclude by splitting the domain of integration and 2 applications of Lemma 19.

For $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$, we define the family of elementary fiber integrals :

$$
I_{n,k}(\lambda) = \int_0^\infty \int_\mathbb{R} a(\lambda y_1^k y_2, y_1, y_2) dy_2 y_1^{2n-1} dy_1.
$$

(37)

Lemma 21 There exists a sequence of distributions $(D_{j,p})$ such that :

$$
I_{n,k}(\lambda) \sim \sum_{p=0,1} \sum_{j \in \mathbb{N}, j \geq 2n} D_{j,p}(a) \lambda^{-\frac{j}{k}} \log(\lambda)^p, \; \text{as} \; \lambda \to \infty,
$$

(38)

where the logarithms only occur when $(j/k)$ is an integer.

As concerns the leading term, if $(2n/k) \notin \mathbb{N}^*$ we obtain :

$$
I_{n,k}(\lambda) = \lambda^{-\frac{2n}{k}} d(a) + \mathcal{O}(\lambda^{-\frac{2n+1}{k}} \log(\lambda)),
$$

(39)

with :

$$
d(a) = C_{n,k} \int_0^\infty \int_0^\infty t^{\frac{2n-1}{k}} y_2^{2n-\frac{2n}{k}} \left( \partial_{y_2}^{2n} a(t, 0, y_2) + \partial_{y_2}^{2n} a(-t, 0, -y_2) \right) dy_2 dt.
$$

But when $2n/k = q \in \mathbb{N}^*$, we have :

$$
I_{n,k}(\lambda) = \lambda^{-\frac{2n}{k}} \log(\lambda) \int_\mathbb{R} |t|^{q-1} \partial_{y_2}^{q-1} a(t, 0, 0) dt + \mathcal{O}(\lambda^{-\frac{2n}{k}}).
$$

(40)

Remark 22 The remainder of Eq. (39) can be optimized to $\mathcal{O}(\lambda^{-\frac{2n+1}{k}})$ if $(2n + 1)/k$ is not an integer and is optimal otherwise.
Proof. By a standard density argument we can assume that the amplitude is of the form \( a(s, y_1, y_2) = f(s)b(y_1, y_2) \). The justification is that our coefficients below are computed by continuous linear functionals, i.e. distributions. We define the Melin transforms of \( f \) as:

\[
M_\pm(\lambda) = \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-\lambda} b(y_1, y_2) y_1^{2n-1} dy_1 dy_2 dz, (41)
\]

We split-up \( I_{n,k} \) as \( J_+ \) and \( J_- \) by separating integrations \( y_2 > 0 \) and \( y_2 < 0 \). By Melin inversion formula, we accordingly obtain:

\[
J_+(\lambda) = \frac{1}{2i\pi} \int_{\gamma} M_+(z) \lambda^{-z} \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-z} b(y_1, y_2) y_1^{2n-1} dy_1 dy_2 dz, (42)
\]

where \( \gamma = c + i\mathbb{R} \) and \( 0 < c < k^{-1} \). Similarly we have:

\[
J_-(\lambda) = \frac{1}{2i\pi} \int_{\gamma} M_-(z) \lambda^{-z} \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-z} b(y_1, -y_2) y_1^{2n-1} dy_1 dy_2 dz. (43)
\]

The existence of a full asymptotic expansion is a direct consequence of:

Lemma 23 The family of distributions \( z \mapsto (y_1 y_2^k)^{-z} \) on \( C^\infty_0(\mathbb{R}^2_+) \) initially defined in the domain \( \Re(z) < k^{-1} \) is meromorphic on \( \mathbb{C} \) with poles \( z_{j,k} = j/k, j \in \mathbb{N}^* \). These poles are of order 2 when \( z_{j,k} \in \mathbb{N}^* \) and of order 1 otherwise.

Proof. We form the Bernstein-Sato polynomial \( b_k \) attached to our problem:

\[
T((y_2 y_1^k)^{1-z}) := \frac{\partial}{\partial y_2} \frac{\partial^k}{\partial y_1^k} (y_2 y_1^k)^{1-z} = b_k(z)(y_2 y_1^k)^{-z},
\]

\[
b_k(z) = (1 - z) \prod_{j=1}^{k} (j - kz).
\]

If \( \Re(z) < k^{-1} \), \((k+1)\)-integrations by parts yield:

\[
\int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-z} f(y_1, y_2) dy_1 dy_2 = \frac{(-1)^{k+1}}{b_k(z)} \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{1-z} (T f)(y_1, y_2) dy_1 dy_2.
\]

Now the integral in the r.h.s. is analytic in \( \Re(z) < 1 + k^{-1} \). After \( p \) iterations the poles, with their orders, can be read off the rational functions:

\[
\Re_p(z) = \prod_{l=1}^{p} \frac{1}{b_k(z - l)}.
\]

This gives the result since \( p \) can be chosen arbitrary large. ■
Hence, the following functions are meromorphic on $\mathbb{C}$:

$$g^\pm(z) = \int_{\mathbb{R}^2} (y_1 y_2)^{-z} b(y_1, \pm y_2) dy_1 dy_2. \quad (45)$$

A classical result, see e.g. [2], is that $M^\pm(c + ix) \in \mathcal{S}(\mathbb{R}^x)$ when $c \notin -\mathbb{N}$. If we shift the path of integration $\gamma$ to the right in our integral representation, Cauchy’s residue method provides the asymptotic expansion. For any $d > c$, outside of the poles, we obtain that:

$$\int_{c+i\mathbb{R}} \lambda^{-z} M^+_z(z) g^\pm(z) dz - \int_{d+i\mathbb{R}} \lambda^{-z} M^+_z(z) g^\pm(z) dz = \sum_{c<z,j,k<d} \text{res}(\lambda^{-z} M^+_z g^\pm)(z_{j,k}).$$

Since $d$ is not a pole the second integral can be estimated via:

$$| \int_{d+i\mathbb{R}} \lambda^{-z} M^+_z(z) g^\pm(z) dz | \leq C(f, b) \lambda^{-d} = \mathcal{O}(\lambda^{-d}), \quad (46)$$

where, for each $d$, the constant $C$ involves the $L^1$-norm of a finite number derivatives of $b$. This will indeed lead to an asymptotic expansion with precise remainders. Applying this method to $J^+_\lambda$ and $J^-_\lambda$ we obtain the existence of a full asymptotic expansion of the form:

$$I_{n,k}(\lambda) \sim \sum_{p=0,1, j \in \mathbb{N}^*} \sum_{C_{j,p} \lambda^{-\frac{j}{k}} \log(\lambda)^p}. \quad (47)$$

Moreover, by Lemma [23] these logarithms only occur when $j/k$ is integer.

**Computation of the leading term.**

To avoid unnecessary discussions and calculations below, we remark that we can commute the polynomial weight of Eqs. (42,43) via:

$$T((y_1 y_2)^{k} y_1^{2n-1}) = b(z)(y_2 y_1^k)^{-z} y_1^{2n-1}, \quad (48)$$

$$b(z) = (1 - z) \prod_{j=1}^{k} (j - k z + 2n - 1). \quad (49)$$

By iteration, we obtain that the poles are the rational numbers:

$$z_{p,j,k,n} = p + \frac{j + 2n - 1}{k}, \quad j \in [1, \ldots, k], \quad p \in \mathbb{N}.$$

By examination of the integrals w.r.t. $y_1$, no residue contributes before:

$$z_0 = \frac{2n}{k}. \quad (50)$$
This result explains the effect of the dimension \( n \) and justifies fully Eq. (38). To reach \( z_0 \) we need \( E(2n/k) + 1 \) iterations. But, by analytic continuation, any bigger integer is acceptable. For the computation of residuum below we use:

\[
\lambda - z M_+(z) \mathcal{B}_n(z) \int_{\mathbb{R}^2} (y_1^k y_2)^{2n-z} y_1^{2n-1} T^{2n} b(y_1, y_2) dy_1 dy_2, \tag{51}
\]

\[
\mathcal{B}_n(z) = \prod_{l=0}^{2n-1} \frac{1}{b(z-l)}. \tag{52}
\]

This choice of \( 2n \) iterations is arbitrary but avoids a lot of calculations.

a) Case of \( z_0 \) simple pole.

Here the residue can be computed by the limit \( z \to z_0 \). We find:

\[
\lambda - \frac{2n}{k} \lim_{z \to \frac{2n}{k}} (z - \frac{2n}{k}) \mathcal{B}_n(z) M_+ \left( \frac{2n}{k} \right) \int_{\mathbb{R}^2} (y_1^k y_2)^{2n-\frac{2n}{k}} y_1^{2n-1} T^{2n} b(y_1, y_2) dy_1 dy_2.
\]

In particular, we can compute the integral w.r.t. \( y_1 \) via:

\[
\int_0^\infty y_1^{2kn-1} \partial_1^{2n-2}(\partial_2^{2n} b(y_1, y_2)) dy_1 = (2kn - 1)! \partial_2^{2n} b(0, y_2).
\]

A similar result holds for \( J_-(\lambda) \) and we obtain:

\[
J_+(\lambda) = \lambda - \frac{2n}{k} C_{n,k} M_+ \left( \frac{2n}{k} \right) \int_0^\infty y_2^{2n-\frac{2n}{k}} \partial_1^{2n} b(0, y_2) dy_2 + R_1(\lambda), \tag{53}
\]

\[
J_-(\lambda) = \lambda - \frac{2n}{k} C_{n,k} M_- \left( \frac{2n}{k} \right) \int_0^\infty y_2^{2n-\frac{2n}{k}} \partial_1^{2n} b(0, -y_2) dy_2 + R_2(\lambda). \tag{54}
\]

Here \( C_{n,k} \) is the canonical constant:

\[
C_{n,k} = \frac{1}{k} \prod_{j=1}^{2n} \frac{1}{j - \frac{2n}{k}}. \tag{55}
\]

Finally, according to Lemma \( \ref{lem:remainder} \) each remainder \( R_j \) is of order \( \mathcal{O}(\lambda^{-\frac{2n+1}{k}}) \) if \( (2n+1)/k \notin \mathbb{N} \) and \( \mathcal{O}(\lambda^{-\frac{2n+1}{k}} \log(\lambda)) \) otherwise. This justifies Remark \( \ref{rem:remainder} \).

b) Case of \( z_0 \) double pole.

If \( h \) is meromorphic with a pole of order 2 in \( \xi_0 \) we have:

\[
\text{res}(h)(z_0) = \frac{1}{2} \lim_{z \to z_0} \frac{\partial}{\partial z}(z-z_0)^2 h(z).
\]
Applying Leibnitz’s rule to \( \partial_z (\lambda^{-z} M_+ (z) g^+ (z)) \), we obtain:

\[
J_+ (\lambda) = B \lambda^{-2n} \log (\lambda) + O (\lambda^{-2n}),
\]

where the distribution \( B \) is computed almost as before. We find:

\[
B = - \frac{1}{2} D_{n,k} M_+ (\frac{2n}{k}) \int \limits_0^\infty y_2^{2n - \frac{2n}{k}} (\partial_{y_2}^n b)(0, y_2) dy_2.
\]

\[
D_{n,k} = \lim_{z \to 2n/k} (z - \frac{2n}{k})^2 B_n (z).
\]

But \( q = 2n/k \) is an integer and by integration by parts we obtain:

\[
\int \limits_0^\infty y_2^{2n - q} (\partial_{y_2}^{2n} b)(0, y_2) dy_2 = (-1)^{q+1} (2n - q)! \partial_{y_2}^{-q} b(0, 0).
\]

Since a similar result holds for \( J_- (\lambda) \), we obtain the desired result by gathering all the constants and summation. Finally, we can extend our formulas since all coefficients in the expansion are of the form:

\[
\langle D^1, f \otimes b \rangle = \langle D^1_1, f \rangle \langle D^1_2, b \rangle, \quad D^1_{1,2} \in \mathcal{D}'(\mathbb{R}).
\]

By linearity and continuity, the result holds for a general amplitude \( a \).

Remark 24 The previous method allows to compute all coefficients of the expansion. In particular the expansion also involves logarithmic distributions (associated to double poles). We do not detail all these terms since they have, a priori, no invariant formulation in the trace formula.

6 Proof of the main result.

We have now the desired results concerning the asymptotic behavior of the trace. Hence, to prove the main result, it remains to express the top order coefficients of the expansion invariantly. Taking Remark 18 into account, to avoid unnecessary calculations we define:

\[
A^+_0 (t, y_1) = y_1^{2n-1} \tilde{A}^+_0 (t, y_1),
\]

\[
A_j (t, y_1, y_2) = y_1^{2n-1} \tilde{A}_j (t, y_1, y_2).
\]

Note that these definitions have no effect on the Fourier transform w.r.t. \( t \).

Directions where \( p_k (\theta) \neq 0 \).
By Lemma 21 we obtain that the first non-zero coefficient is obtained for \( l = 2n - 1 \) (see Remark 18) and is given by

\[
\frac{1}{k} \frac{1}{(2n - 1)!} \left( |\eta_0|^{\frac{2n-k}{k}} \otimes \delta_0^{(2n-1)} \right) = \frac{1}{k} \left( |\eta_0|^{\frac{2n-k}{k}} \otimes \delta_0, \hat{A}_0^+ (\eta_0, \eta_1) \right).
\]

Since by construction:

\[
\hat{A}_0^+ (\eta_0, 0) = \int_{\mathbb{S}^{2n-1}} a(\eta_0, 0) \phi_0^+ (\theta)|p_k (\theta)|^{-\frac{2n}{k}} d\theta,
\]

we obtain that the local contribution, associated to \( \text{supp}(\phi_0^+) \), is:

\[
\frac{1}{k} \left( \mathcal{F} (x_0^{\frac{2n-k}{k}}) (\eta_0), a(\eta_0, 0) \right) \int_{\mathbb{S}^{2n-1}} \phi_0^+ (\theta)|p_k (\theta)|^{-\frac{2n}{k}} d\theta,
\]

and a similar result holds on \( \text{supp}(\phi_0^-) \). Now, since \( a(t, 0) = \hat{\varphi}(t) \), cf. Eq. (25), the directions where \( p_k (\theta) \neq 0 \) contribute as:

\[
I_+ (\lambda) \sim \frac{1}{k} \lambda^{-\frac{2n}{k}} \left( \int |t|^{\frac{2n-k}{k}}, \varphi(t) \right) \int_{\mathbb{S}^{2n-1}} \phi_0^+ (\theta)|p_k (\theta)|^{-\frac{2n}{k}} d\theta,
\]

\[
I_- (\lambda) \sim \frac{1}{k} \lambda^{-\frac{2n}{k}} \left( \int |t|^{\frac{2n-k}{k}}, \varphi(t) \right) \int_{\mathbb{S}^{2n-1}} \phi_0^- (\theta)|p_k (\theta)|^{-\frac{2n}{k}} d\theta.
\]

**Microlocal contribution of \( C(p_k) \).**

According to the analysis above, we will distinguish out the case \( k \) divides \( 2n \).

(1) Case of \( k > 2n \), integrable singularity on the blow-up.

Here \( 2n/k \in [0, 1] \). According to Lemma 21 the contribution of \( I_j (\lambda) \) is:

\[
\frac{1}{k} \lambda^{-\frac{2n}{k}} \int_{\mathbb{R}^2_+} |t|^{\frac{2n-k}{k}} |y_2|^{-\frac{2n}{k}} \left( \hat{A}_j (t, 0, y_2) + \hat{A}_j (-t, 0, -y_2) \right) dtdy_2 + R(\lambda).
\]

Reminding that \( y_2(t, 0, \theta) = p_k (\theta) \), we obtain:

\[
\int_{\mathbb{R}^+} |y_2|^{\frac{2n}{k}} \hat{A}_j (t, 0, y_2)dy_2 = a(t, 0) \int_{\{p_k (\theta) \geq 0\}} |p_k (\theta)|^{-\frac{2n}{k}} \phi_j(\theta) d\theta.
\]

Via Eq. (25), by summation \( I(\lambda) \) is asymptotically equivalent to:

\[
\lambda^{-\frac{2n}{k}} \left( \int |t|^{\frac{2n-k}{k}}, \varphi(t) \right) \int_{\{p_k \geq 0\}} |p_k (\theta)|^{-\frac{2n}{k}} d\theta + \int_{\{p_k \leq 0\}} |p_k (\theta)|^{-\frac{2n}{k}} d\theta.
\]
Note that none of the coefficients above are equal unless \( \varphi \) or \( p_k \) are symmetric.

(2) Case of \( q = 2n/k \) integer.

Here the contribution of each \( I_j(\lambda) \) is dominant since we obtain:

\[
I_j(\lambda) = \frac{1}{k} \lambda^{-q} \log(\lambda) \int_{\mathbb{R}} |t|^q \partial_{y_2}^{-1} \hat{A}_j(t,0,0) dt + \mathcal{O}(z^{-q}).
\]

Unless \( q = 1 \), there is no way to take the limit directly, and the geometric properties are still hidden in the Jacobian. To reach the result we will use the Schwartz kernel technic. Clearly, it is enough to evaluate our derivative and to integrate w.r.t. \( t \).

With \( s = (s_1, s_2) \in \mathbb{R}^2 \), we write the evaluation as:

\[
\partial_{y_2}^{-1} \hat{A}_j(t,0,0) = \frac{1}{(2\pi)^2} \int e^{i(s, (y_1, y_2))} (is_2)^{q-1} \hat{A}_j(t, y_1, y_2) dy_1 dy_2 ds.
\]

Here we have used an oscillatory Schwartz kernel for \( \delta_{y_1} \otimes \delta_{y_2}^{-1} \). This integral representation allows to inverse our diffeomorphism to obtain:

\[
\partial_{y_2}^{-1} \hat{A}_j(t,0,0) = \frac{1}{(2\pi)^2} \int e^{i(s, (r, y_2(r, \theta)))} (is_2)^{p-1} a(t, r\theta) \theta_j(\theta) d\theta dr ds.
\]

If we extend the integrand by 0 for \( r < 0 \), the normalized integral w.r.t. \((r, s_1)\) provides \( \delta_r \). By construction \( y_2(0, \theta) = p_k(\theta) \) and we accordingly have:

\[
\partial_{y_2}^{-1} \hat{A}_j(t,0,0) = a(t,0) \frac{1}{(2\pi)} \int_{\mathbb{R} \times S^{n-1}} e^{iup_k(\theta)(iu)^{q-1} \theta_j(\theta)} d\theta du.
\]

This Fourier integral makes sense with \( S^{n-1} \) compact. We define here a local version of the integrated density:

\[
J_j(w) = \int_{\{p_k(\theta) = w\}} \phi_j(\theta) dL_w(\theta),
\]

where \( dL_w \) is the density induced by the Leray-form \( dL_{p_k} : dp_k \wedge dL_{p_k}(\theta) = d\theta \).

Note that all these objects can be constructed by mean of local coordinates under the only condition that \( \text{supp}(\phi_j) \) is small enough near \( C(p_k) \). Moreover, since \( p_k \) is continuous on \( S^n \), each \( J_j(w) \) defines a compactly supported distribution, smooth near the origin according to \((H_4)\). The sum over all the \( \phi_j \) gives the geometric contribution:

\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{iuw}(iu)^{p-1} \sum_j J_j(w) du = \frac{d^{p-1} \text{Vol}}{dw^{p-1}}(0).
\]

By integration w.r.t. \( t \) we obtain the result in general position.
Remark 25 The case $2n = k$ is directly accessible with:

$$I(\lambda) = \frac{1}{k} \log(\lambda) \text{LVol}(0) \int \varphi(t) dt + O(\lambda^{-1}),$$

where $\text{LVol}(0)$ is usual the Liouville volume of $C(p_k)$.

(2) $k < 2n$ and simple pole, non-integrable singularity. The distributional coefficients for the positive part of each $I_j(\lambda)$ are:

$$\langle \nu_{j,+}, a \rangle = C_{n,k} \int_0^\infty \int_0^\infty t^{\frac{2n}{k}-1} y_2^{2n-\frac{2n}{k}} (\partial_{y_2}^2 \hat{A}_j)(t, 0, y_2) dy_2 dt.$$

With the same oscillatory technic as above, for $p_k(\theta) \geq 0$ we obtain globally:

$$\langle T_+, a \rangle = C_{n,k} \langle \left| w \right|_+^{2n-\frac{2n}{k}} \partial_{u}^{2n} \text{LVol}(w) \rangle \int_0^\infty \left| t \right|^{\frac{2n}{k}-1} \hat{\varphi}(t) dt.$$

The duality bracket is well defined since $\text{LVol}$ is compactly supported. A similar result holds for the negative part of each $I_j(\lambda)$. The value of $C_{n,k}$, see Eq. (55), gives the result stated in Theorem 4 by normalizing the distributional derivatives. Note that our choice is convenient since $\partial_{u}^{2n}$ is symmetric. ■

Now we detail the construction of the distributional bracket of the part (3) of Theorem 4. Clearly, $\text{LVol}$ is supported in $[\inf p_k, \sup p_k]$. Let be $\chi \in C_0^\infty$, $0 \leq \chi \leq 1$ on $\mathbb{R}$ chosen such that $\chi = 1$ near the origin and $\chi(u) = 0$ for $|u| \geq \varepsilon$, with $\varepsilon > 0$ small enough. We write the geometric contribution as:

$$\langle U, \text{LVol} \rangle = \langle U, \chi \text{LVol} \rangle + \langle U, (1 - \chi)\text{LVol} \rangle.$$

Away from the origin, e.g. for $u > 0$, $U$ is smooth and we obtain directly:

$$\begin{aligned}
C_{n,k} \left\langle \frac{d^n}{du^{2n}} u_+^{2n-\frac{2n}{k}}, \chi(u)\text{LVol}(u) \right\rangle &= \frac{1}{k} \int_{\{f_k(\theta) > 0\}} \chi(f_k(\theta))|f_k(\theta)|^{-\frac{2n}{k}} d\theta.
\end{aligned}$$

Here the value of $C_{n,k}$ from Eq. (35) justifies the normalization of Eq. (13). For the singular part of $u_+^{2n-\frac{2n}{k}}$, we use the local regularity of $\text{LVol}(u)$ near $u = 0$ and integrations by parts to conclude. Finally, using the previous trick we obtain a formulation which does not depend on the partition of unity by a double covering of the sphere and $\varepsilon$ small enough.

Remark 26 The key point here is that we can put in duality the distributions $\partial_{u}^{2n}|u|^{2n-\alpha}$ and $\text{LVol}(u)$ since their singular supports are disjoints.
Effect of the sub-principal symbol.
Until now we have only considered the case of an operator given by quantization of a symbol $p_0$. But in presence of a sub-principal symbol $p_1$ the construction is the same. The only important change, see Eq. (25), is that the amplitude has to be modified by:

$$a(t, z_0) = \hat{\phi}(t) \exp(i \int_0^t p_1(\Phi_s(z_0))ds).$$

But $z_0$ in a fixed point of $\Phi_t$ and a fortiori:

$$a(t, z_0) = \hat{\phi}(t) \exp(itp_1(z_0)).$$

Hence if $p_1(z_0) = 0$, which is the case in many practical situations, the trace formula remains the same. Otherwise, by Fourier inversion formula, the effect is a shift on $\phi$ by $p_1(z_0)$ in all integral formulae.

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