Blow-up sets for a complex valued semilinear heat equation

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Abstract
This paper is concerned with finite blow-up solutions of a one dimensional complex-valued semilinear heat equation. We provide locations and the number of blow-up points from the viewpoint of zeros of the solution.

Keyword system of semilinear parabolic equation; blow-up point

1 Introduction

We study blow-up solutions of a one dimensional complex-valued semilinear heat equation:
\[
z_t = z_{xx} + z^2, \tag{1}
\]
where \(z(x, t)\) is a complex valued function and \(x \in \mathbb{R}\). If \(z(x, t)\) is written by \(z = a + ib\), where \(a, b \in \mathbb{R}\), (1) is rewritten as
\[
a_t = a_{xx} + a^2 - b^2, \quad b_t = b_{xx} + 2ab. \tag{2}
\]
This equation is a special case of Constantin-Lax-Majda equation with a viscosity term, which is a one dimensional model for the 3D Navier-Stokes equations (see [3, 4, 13, 16, 17, 3]). When \(z\) is real-valued (i.e. \(b = 0\)), (2) coincides with the so-called Fujita equation [5]:
\[
a_t = a_{xx} + a^p. \tag{3}
\]

In a recent paper [6], they clarify the difference the dynamics of solutions between (1) and (2). A goal of paper is to extend their results and to provide new properties of solutions of (1) based on results in [4]. The Cauchy problem of (1) admits an unique local solution in \(R \in C(R)\). We call a solution \(z\) blow-up in a finite time, if there exists \(T > 0\) such that
\[
\limsup_{t \to T} \|z(t)\|_{L^\infty(R)} = \limsup_{t \to T} \sqrt{\|a(t)\|_{L^\infty(R)} + \|b(t)\|_{L^\infty(R)}} = \infty.
\]

Moreover we call a point \(x_0 \in R\) a blow-up point, if there exists a sequence \(\{(x_j, t_j)\}_{j \in \mathbb{N}} \subset R \times (0, T)\) such that \(x_j \to x_0\), \(t_j \to T\) and \(|z(x_j, t_j)| \to \infty\) as \(j \to \infty\). The set of blow-up points is called a blow-up set.

We first consider an ODE solution \((a(x, t), b(x, t)) = (a(t), b(t))\) of (1). Then equation (1) is reduced to
\[
a_t = a^2 - b^2, \quad b_t = 2ab.
\]
This ODE system has an unique solution given by
\[
a(t) = \frac{T_1 - t}{(T_1 - t)^2 + T_2^2}, \quad b(t) = \frac{T_2}{(T_1 - t)^2 + T_2^2},
\]
where
\[
T_1 = \frac{a(0)}{a(0)^2 + b(0)^2}, \quad T_2 = \frac{b(0)}{a(0)^2 + b(0)^2}.
\]
Therefore this ODE solution exists globally in time, if \(b(0) \neq 0\). From this observation, we expect that the component \(b\) prevents a blow-up phenomenon in (1). In fact, the following result is given in [6].

Theorem 1.1 (Theorem 1.1 [6]). Suppose that the initial data \((a_0, b_0) \in L^\infty(R) \cap C(R)\) satisfy
\[
a_0(x) < A b_0(x) \quad \text{for all } x \in R
\]
with some constant \(A \in R\). Then the solution of (1) exists globally in time and \(\lim_{t \to \infty} (a(t), b(t)) = (0, 0)\) in \(L^\infty(R)\).

Furthermore for the case \(b_0(x) > 0\) with asymptotically positive constants, they prove that the condition \(a_0(x) < A b_0(x)\) in Theorem 1.1 is not needed to assure the same conclusion.

Theorem 1.2 (Theorem 1.4 [6]). Suppose that the initial data \((a_0, b_0) \in L^\infty(R) \cap C(R)\) satisfy
\[
0 \leq a_0 \leq M, \quad 0 \leq b_0 \leq L, \quad a_0 \neq M, \quad \lim_{|x| \to \infty} a_0(x) = M, \quad \lim_{|x| \to \infty} b_0(x) = N.
\]
for some \(L > 0\) and \(M > N > 0\). Then the solution of (1) exists globally in time and \(\lim_{t \to \infty} (a(t), b(t)) = (0, 0)\) in \(L^\infty(R)\).

Our first result is a local version of Theorem 1.2. To state our results, we assume
\[
\sup_{0 < t < T} (T - t)(\|a(t)\|_{L^\infty(R)} + \|b(t)\|_{L^\infty(R)}) < \infty. \tag{3}
\]
Theorem 1.3. Let \((a, b)\) be a solution of \((1)\) and \(T > 0\) be its blow-up time. Assume that \((3)\) holds and there exists a neighborhood \(O\) of \((x_0, T)\) in \(\mathbb{R} \times (0, T)\) such that \(b(x, t) > 0\) or \(b(x, t) < 0\) for \((x, t) \in O\). Then \(x_0\) is not a blow-up point of \((a, b)\).

Theorem 1.3 implies that if a solution \((a, b)\) blows up in a finite time, the component \(b\) must be sign changing near blow-up points. A main goal of this paper is to characterize the location and the number of blow-up points by using zeros of the component \(b\).

Theorem 1.4. Let \((a, b)\) and \(T > 0\) be as in Theorem 1.3 and \(\gamma(t)\) be a zero of \(b(t)\) (i.e. \(b(\gamma(t), t) = 0\)). Assume that \((3)\) holds and \(b_0(x)\) has exact one zero. Then \(\gamma(t)\) is continuous on \([0, T]\) and its blow-up point \(x_0 \in \mathbb{R}\) is given by \(x_0 = \gamma(T)\).

The existence of blow-up solutions are proved in \([6]\) and \([18]\). In \([6]\), they provide sufficient conditions on the initial data for a finite time blow-up by using a comparison argument in the Fourier space based on \([3]\). In particular, the exact initial data satisfying their blow-up conditions is given by (see Remark 3.3 in \([5]\))

\[
a_0(x) = (3 - 4x^2)e^{-x^2}, \quad b_0(x) = 2xe^{-x^2}.
\]

For this case, Theorem 1.4 suggests that the solution blows up only on the origin. On the other hand, they \([18]\) construct blow-up solutions with exact blow-up profiles \((a^*(x), b^*(x)) = \lim_{t \to T}(a(x, t), b(x, t))\). Two blow-up solutions constructed in \([6]\) and \([18]\) have different type of asymptotic forms.

2 Preliminary

2.1 Functional setting

To study the asymptotic behavior of blow-up solutions, we introduce a self-similar variable around \(\xi \in \mathbb{R}\) and a new unknown function \((u_\xi, v_\xi)\), which is defined by

\[
\begin{align*}
\quad u_\xi(y, s) &= (T - t)a(\xi + e^{-s/2}y, t), \quad v_\xi(y, s) = (T - t)b(\xi + e^{-s/2}y, t), \quad t = T - e^{-s}.
\end{align*}
\]

Let \(s_T = -\log(T - t)\). Then \((u, v) = (u_\xi, v_\xi)\) satisfies

\[
\begin{align*}
\frac{\partial u_\xi}{\partial s} + \frac{\partial u_\xi}{\partial y} &= \frac{y^2}{2}u_\xi - u + u^2 - v^2, \quad y \in \mathbb{R}, \quad s > s_T, \\
\frac{\partial v_\xi}{\partial s} + \frac{\partial v_\xi}{\partial y} &= \frac{y^2}{2}v_\xi - v + 2uv, \quad y \in \mathbb{R}, \quad s > s_T.
\end{align*}
\]

We here introduce functional spaces which are related to \([5]\). Put \(\rho(y) = e^{-y^2/4}\) and

\[
L^2_\rho(\mathbb{R}) = \{ f \in L^2_{\text{loc}}(\mathbb{R}); \| f \|_\rho < \infty \}, \quad H^1_\rho(\mathbb{R}) = \{ f \in L^2_\rho(\mathbb{R}); \| f \|_{H^1_\rho} = \sqrt{\| f \|_\rho^2 + \| f_x \|_\rho^2} < \infty \},
\]

where the norm of \(L^2_\rho(\mathbb{R})\) is defined by

\[
\| f \|_\rho^2 = \int_{-\infty}^{\infty} f(y)^2 \rho(y)dy.
\]

Here we recall the following inequality (see Lemma 2.1 in \([11]\) p. 430).

\[
\int_{-\infty}^{\infty} y^2e^{2\rho}dy < c\| f \|_{H^1_\rho}^2.
\]

(6)

For the convenience of the reader, we provide the proof of this inequality. Let \(g(y) = f(y)e^{-y^2/8}\). Then a direct computation shows that

\[
g^2_y = \left( f^2_y + \frac{y^2}{16}f^2 - \frac{y}{4}(f^2)_{yy} \right)e^{-y^2/4}
\]

The integration of the last term is calculated as

\[
-\int_{-\infty}^{\infty} \frac{y}{4}(f^2)_{yy}e^{-y^2/4}dy = \int_{-\infty}^{\infty} \left( \frac{y^2}{16}e^{-y^2/4} \right) f^2dy.
\]

Therefore we obtain

\[
\int_{-\infty}^{\infty} f^2_ydy + \frac{1}{4} \int_{-\infty}^{\infty} f^2_{yy}dy - \frac{1}{16} \int_{-\infty}^{\infty} y^2f^2dy > 0,
\]

which proves the desired inequality.

2.2 Boundedness of solutions in self-similar variables

We here provide some conditions for a boundedness of solutions. These conditions are useful to apply a scaling argument, which is often used in the proof of Theorem 1.3 and Theorem 1.4

Lemma 2.1. Let \((a, b)\) be a solution of \((1)\) satisfying \((3)\) and \((u_\xi, v_\xi)\) be given in \([5]\). Then there exist \(R > 0\) and \(c_0 > 0\) such that if \(\| u_\xi(s_1) \|_{L^\infty(-R, R)} + \| v_\xi(s_1) \|_{L^\infty(-R, R)} < c_0\) for some \(\xi \in \mathbb{R}\) and \(s_1 > s_T\), then \(\xi\) is not a blow-up point of \((a, b)\).
Proof. For simplicity of notations, we omit the subscript $\xi \in \mathbb{R}$. Let $M = \sup_{s \geq 0}(\|u(s)\|_{L^\infty(\mathbb{R})} + \|v(s)\|_{L^\infty(\mathbb{R})}) < \infty$ and set $F(s) = \|u(s)\|_p^2 + \|v(s)\|_p^2$. $G(s) = \|u_y(s)\|_p^2 + \|v_y(s)\|_p^2$. Multiplying (4) by $u$ and $v$, we get
\[
\frac{1}{2} F_s < -G + F + c \int_{-\infty}^{\infty} (|u|^3 + |u|^2) \rho dy.
\]
We assume $\|u(s)\|_{L^\infty(-R,R)} + \|v(s)\|_{L^\infty(-R,R)} < \epsilon$. Then from (1), the last term is estimated by
\[
\int_{-\infty}^{\infty} (|u|^3 + |v|^3) \rho dy < \epsilon \int_{|y| < R} (u^2 + v^2) \rho dy + MR^{-2} \int_{|y| > R} g^2 (u^2 + v^2) \rho dy < \epsilon F + cMR^{-2}G.
\]
We now choose $R_0 > 0$ and $\epsilon_0 > 0$ such that $\epsilon_0 < 1/2$ and $cMR_0^{-2} < 1/2$, which implies $F_s(s) < 0$ if $\|u(s)\|_{L^\infty(-R_0,R_0)} + \|v(s)\|_{L^\infty(-R_0,R_0)} < \epsilon_0$. To construct a comparison function for $v$, we first consider
\[
w_s = w_{yy} - \frac{y}{2} w_y + (-1 + 2M) w \tau > s, \quad w(s) = |v(s)|.
\]
We easily see that
\[
\|w(\tau)\|_p^2 < e^{(-2+4M)(\tau-s)}\|v(s)\|_p^2.
\]
Next we construct a comparison function for $u$.
\[
z_s = z_{yy} - \frac{y}{2} z_y + (-1 + M) z + w(\tau)^2 \tau > s, \quad z(s) = |u(s)|,
\]
where $w(\tau)$ is defined above. Then we get
\[
\|z(\tau)\|_p^2 < e^{(-1+2M)(\tau-s)}\|u(s)\|_p^2 + M^2 \int_s^\tau e^{-(1+2M)(\tau-\mu)}\|w(\mu)\|_p^2 d\mu
\]
\[
< e^{(-1+2M)(\tau-s)}\|u(s)\|_p^2 + \left(\frac{M^2}{2 + 4M}\right) e^{-(3+6M)(\tau-s)}\|v(s)\|_p^2.
\]
Combining above estimates, we obtain
\[
F(\tau) < c_1 e^{cz(\tau-s)} F(s) \quad \text{for } \tau > s
\]
for some $c_1, c_2 > 0$. Furthermore by a regularity theory for parabolic equations, it holds that
\[
\|u(s)\|_{L^\infty(-R_0,R_0)} + \|v(s)\|_{L^\infty(-R_0,R_0)} < \epsilon_0 \int_{s-1}^s F(\mu) d\mu.
\]
Let $\epsilon_1 = \min\{c_1 e^{cz/2}, c_2/2\} \epsilon_0$ and $\epsilon_2 = \epsilon_1/2$. We now claim that if $F(s) < \epsilon_2$ for some $s > s_T$, then it holds that $F(\tau) < \epsilon_1$ for $\tau > s_T$. In fact, we assume that there exists $s_1 > s$ such that $F(\tau) < \epsilon_1$ for $s < \tau < s_1$ and $F(s_1) = \epsilon_1$. By definition of $\epsilon_1$ and (7), we find that $s_1 > s + 1$. Therefore we get from definition of $\tau_1$ and (3) that
\[
\|u(\tau)\|_{L^\infty(-R_0,R_0)} + \|v(\tau)\|_{L^\infty(-R_0,R_0)} < c_1 \epsilon_1 < \frac{\epsilon_0}{2}
\]
for $\tau \in (s+1, s_1)$. As a consequence, from definition of $R_0$ and $\epsilon_0$, it follows that $F_s(s) < 0$ for $s \in (s+1, s_1)$. However this contradicts definition of $\tau_1$, which completes the proof.

Lemma 2.2. Let $(a,b)$ and $(\xi, \nu) \in \mathbb{R}$, $(\nu, \xi) \in \mathbb{R}$ and put $u_s(y, s) = u_s(y, s_1, s)$, $v_s(y, s) = v_s(y, s_1, s)$. Then if $(u_s, v_s) \rightarrow (U, V)$ in $L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R})$ as $s \rightarrow \infty$ and $(U(s), V(s)) \rightarrow (0, 0)$ in $L^\infty_{\text{loc}}(\mathbb{R})$, then $\xi, \nu \in \mathbb{R}$ is not a blow-up point of $(a, b)$ for large $i \in \mathbb{N}$.

Proof. Let $R > 0$ and $\epsilon_0 > 0$ be given in Lemma 2.1. Since $(U(s), V(s)) \rightarrow (0, 0)$, there exists $s_0 > 0$ such that $\|U(s_0)\|_{L^\infty(-2R,2R)} + \|V(s_0)\|_{L^\infty(-2R,2R)} < \epsilon_0/2$. Furthermore since $(u_s, v_s) \rightarrow (U, V)$ as $s \rightarrow \infty$, we see that
\[
\|u_s(s_0)\|_{L^\infty(-R,R)} + \|v_s(s_0)\|_{L^\infty(-R,R)} < \epsilon_0
\]
for large $i \in \mathbb{N}$. Therefore from Lemma 2.1, $\xi$ is not a blow-up point of $(a, b)$, which completes the proof.

Lemma 2.3. Let $(a_i, b_i)$ be a solution of (11) and satisfies $\sup_{0 \leq t \leq 1}(a_i(x_t, t) + b_i(x_t, t)) < c/(1-t)$ for $t \in (0, 1)$. If $(a_i, b_i) \rightarrow (A, B)$ and $(A, B)$ does not blow up on $x = x_0$ at $t = 1$, then $x_0$ is not a blow-up point of $(a_i, b_i)$ at $t = 1$ for large $i \in \mathbb{N}$.

Proof. Set $1 - t = e^{-s}$, $u_i(y, s) = (1-t) a_i(x_0 + e^{-s/2}y, t)$ and $v_i(y, s) = (1-t) b_i(x_0 + e^{-s/2}y, t)$. From the assumption, we see that $(u_i, v_i)$ is uniformly bounded on $[0, \infty)$. Since $(a_i, b_i) \rightarrow (A, B)$ and $u_i(y, 0) = a_i(x_0 + y, 0)$, $v_i(y, 0) = b_i(x_0 + y, 0)$, we see that $(u_i, v_i) \rightarrow (U, V)$ and $U(y, s) = (1-t) A(x_0 + e^{-s/2}y, t)$, $V(y, s) = (1-t) B(x_0 + e^{-s/2}y, t)$ for $s > 0$. Since $\sup_{0 \leq t \leq 1}(A(x_t, t) + B(x_t, t)) < c/(1-t)$, $(A, B)$ does not blow up for $t \in (0, 1)$. If $(A, B)$ does not blow up on $x = x_0$ at $t = 1$, it holds that $(U, V) \rightarrow (0, 0)$ as $s \rightarrow \infty$. Therefore by the same way as in the proof of Lemma 2.2 we conclude that $x_0$ is not a blow-up point of $(a_i, b_i)$ at $t = 1$ for large $i \in \mathbb{N}$. The proof is completed.
3 Local conditions for boundedness of solutions

In this section, we provide the proof of Theorem 3.3. Let \( x_0 \in \mathbb{R} \) be a blow-up point, \( T > 0 \) be a blow-up time and \( \mathcal{O} \) be the neighborhood of \((x_0, T)\) stated in Theorem 1.2. Since the proof for the case \( b(x, t) < 0 \) for \((x, t) \in \mathcal{O}\) is the same as for the case \( b(x, t) > 0 \) for \((x, t) \in \mathcal{O}\), we here only consider the later case. For such a case, by shifting the initial time, we can assume

\[
b(x, t) > 0 \quad \text{for } x \in (L_1, L_2), \ t \in (0, T)
\]

for some \( L_1 < x_0 < L_2 \). Furthermore throughout this section, we assume \( b(x, t) > 0 \).

**Lemma 3.1.** Either one of the intervals \((L_1, x_0)\) and \((x_0, L_2)\) is included in the blow-up set.

**Proof.** Assume that their exist \( l_1 \in (L_1, x_0) \) and \( l_2 \in (x_0, L_2) \) such that \( x = l_1 \) and \( x = l_2 \) are not blow-up points. From this assumption, \( a(x, t) \) and \( b(x, t) \) are uniformly bounded on \((l_1 - \epsilon, l_1 + \epsilon)\) and \((l_2 - \epsilon, l_2 + \epsilon)\) for some \( \epsilon > 0 \). Therefore since \( b(x, t) > 0 \) in \((L_1, L_2)\), by a comparison argument, we easily see that

\[
b(l_1, t) > \delta \quad \text{for } t \in (0, T), \quad b(l_2, t) > \delta \quad \text{for } t \in (0, T), \quad b_0(x) > \delta \quad \text{for } x \in (l_1, l_2)
\]

with some \( \delta > 0 \). Set \( \gamma = a/b \). Then \( \gamma \) satisfies

\[
\gamma = \gamma_{xx} + 2\nu \gamma_x - \left( \frac{a^2 + b^2}{b} \right),
\]

where \( \nu = b_x/b \). Since \( x = l_1 \) and \( x = l_2 \) are not blow-up points, it is clear that \( M = \sup_{0 < t < T} (|a(l_1, t)| + |a(l_2, t)| + |a_0(x)|) < \infty \). Combining this fact and \( 4 \), we get

\[
\gamma(l_1, t) < M/\delta \quad \text{for } t \in (0, T), \quad \gamma(l_2, t) < M/\delta \quad \text{for } t \in (0, T), \quad \gamma_0(x) < M/\delta \quad \text{for } x \in (l_1, l_2).
\]

Therefore we obtain from a maximum principle that

\[
\gamma(x, t) > M/\delta \quad \text{for } x \in (l_1, l_2), \ t \in (0, T).
\]

Let \( \lambda_i = 1/i \) and set \( a_i(x, \tau) = \lambda_i a(x_0 + \sqrt{\lambda_i x}, T - 1/i + \lambda_i \tau) \) and \( b_i(x, \tau) = \lambda_i b(x_0 + \sqrt{\lambda_i x}, T - 1/i + \lambda_i \tau) \). Then we easily see that \( 3 \) is equivalent to

\[
\sup_{x \in \mathbb{R}} (|a_i(x, \tau)| + |b_i(x, \tau)|) < \frac{c_0}{1 - \tau}.
\]

Therefore by taking a subsequence, we get \((a_i, b_i) \to (A, B)\) and

\[
\sup_{x \in \mathbb{R}} (|A(x, \tau)| + |B(x, \tau)|) < \frac{c_0}{1 - \tau} \quad \text{for } \tau \in (-1, 1).
\]

Furthermore we get from \( 10 \) that

\[
A(x, \tau)/B(x, \tau) < M/\delta \quad \text{for } x \in \mathbb{R}, \ \tau \in (-1, 1).
\]

Since \((A, B)\) is a solution of \( 1 \), Theorem 1.1 \( 2 \) stated in Introduction implies that \((A, B)\) exists globally in time. Therefore from Lemma 2.3, the origin is not a blow-up point of \((a_i, b_i)\) for large \( i \in \mathbb{N} \), which implies that \( x_0 \) is not a blow-up point of \((a, b)\). This contradicts the assumption, which completes the proof.

From Lemma 3.1, we can assume that the interval \((-L, L)\) is included in the blow-up set and \( b \) satisfies

\[
b(x, t) > 0 \quad \text{for } x \in (-L, L), \ t \in (0, T).
\]

We now introduce self-similar variables and define a new unknown function \((u, v)\) as in Section 2.1. Let \( \xi \in \mathbb{R} \) and set

\[
T - t = e^{-s}, \quad s_T = -\log(T - t),
\]

\[
u_{\xi}(y, s) = (T - t)a(\xi + e^{-s/2}y, t), \quad v_{\xi}(y, s) = (T - t)b(\xi + e^{-s/2}y, t).
\]

Then \((u, v) = (u_{\xi}, v_{\xi})\) satisfies \((5)\).

**Lemma 3.2.** Let \( \{\xi_i\}_{i \in N} \subset (-L/2, L/2) \) and \( \{s_i\}_{i \in N} \) be sequences. Put

\[
a_i(x, \tau) = \lambda_i a(\xi_i + \sqrt{\lambda_i x}, t_i + \lambda_i \tau), \quad b_i(x, \tau) = \lambda_i b(\xi_i + \sqrt{\lambda_i x}, t_i + \lambda_i \tau).
\]

If \((a_i, b_i) \to (A, B)\) as \( i \to \infty \) and \((A, B)\) blows up on the origin at \( \tau = 1 \), then the origin is not an isolated blow-up point of \((A, B)\).

**Proof.** We prove by contradiction. Assume that the origin is an isolated blow-up point of \((A, B)\). Then there exist \( \theta_1, \theta_2 \) \((0 < \theta_1 < \theta_2 < 1)\) such that

\[
\sup_{0 < \tau < 1, \theta_1 < x < \theta_2} (|A(x, \tau)| + |B(x, \tau)|) < \infty.
\]

Therefore from Lemma 2.3 there exists \( c > 0 \) such that

\[
\sup_{0 < \tau < 1, \theta_1 < x < \theta_2} (|a_i(x, \tau)| + |b_i(x, \tau)|) < c \quad \text{for } i \gg 1.
\]

Let \( \theta = (\theta_1 + \theta_2)/2 \) and

\[
\tilde{a}_i(y, s) = (1 - \tau)a_i(\theta + e^{-s/2}y, \tau), \quad \tilde{b}_i(y, s) = (1 - \tau)b_i(\theta + e^{-s/2}y, \tau),
\]

\( 1 - \tau = e^{-s} \).
Then we see that

\[ \tilde{u}_i(y, s) = e^{-(s+s_i)}a(\tilde{\xi}_i + e^{-(s+s_i)/2}y, T - e^{-(s+s_i)}) = u_{\xi_i}(y, s_i + s), \]

\[ \tilde{v}_i(y, s) = v_{\xi_i}(y, s_i + s), \]

where \( \tilde{\xi}_i = \xi_i + \sqrt{\lambda_i} \). Put \( \Delta = (\theta_2 - \theta_1)/2 \). Then we get from (12) that

\[ \sup_{|y| < c e^{s/2\Delta}} (|u_{\xi_i}(y, s_i + s)| + |v_{\xi_i}(y, s_i + s)|) = \sup_{|y| < c e^{s/2\Delta}} (|\tilde{u}_i(y, s)| + |\tilde{v}_i(y, s)|) \]

\[ = \sup_{\theta_1 < x < \theta_2} e^{-s}(|a_i(x, \tau)| + |b_i(x, \tau)|) < c e^{-s} \quad \text{for } s > 0, \ i \gg 1. \]

This implies

\[ \sup_{|y| < c e^{s/2\Delta}} (|u_{\xi_i}(y, s)| + |v_{\xi_i}(y, s)|) < c e^{-(s-s_i)} \quad \text{for } s > s_i, \ i \gg 1. \]

Therefore from Lemma 2.1 \( \tilde{\xi}_i \) is not a blow-up point of \((a, b)\), which contradicts that \( \tilde{\xi}_i \) is a blow-up point of \((a, b)\). \( \square \)

Lemma 3.3. For any \( R > 0 \), there exists \( \epsilon_1 > 0 \) such that if \( \inf_{-R < y < R} v_\xi(y, s) < \epsilon_1 \), then it holds that

\[ \sup_{-R < y < R} |u_\xi(y, s)| = 1 \quad \text{for } s > s_T, \ \xi \in (-L/2, L/2). \]

Proof. We prove by contradiction. Assume that there exist \( R > 0, \ \{s_i\}_{i \in \mathbb{N}} \ (s_i \to \infty) \) and \( \{\xi_i\}_{i \in \mathbb{N}} \subset (-L/2, L/2) \) such that

\[ \inf_{-R < y < R} v_{\xi_i}(y, s_i) < 1/i, \quad \sup_{-R < y < R} |u_{\xi_i}(y, s_i)| - 1 > 1/8. \]

Put \( \lambda_i = e^{-s_i}, \ t_i = T - \lambda_i \) and

\[ a_i(x, \tau) = \lambda_i a(\xi_i + \sqrt{\lambda_i}x, t_i + \lambda_i \tau), \quad b_i(x, \tau) = \lambda_i b(\xi_i + \sqrt{\lambda_i}x, t_i + \lambda_i \tau). \]

Then we easily see from (13) that

\[ |a_i(x, \tau)| + |b_i(x, \tau)| < c(1 - \tau)^{-1} \quad (14) \]

for some \( c > 0 \). Therefore by taking a subsequence, we get \( (a_i, b_i) \to (A, B) \). Since \( b_i(x, 0) = v_{\xi_i}(x, s_i) \), by a strong maximum principle, \( B \) must be zero on \( \mathbb{R} \times (0, 1) \). If \( A \equiv 0, \) Lemma 2.3 implies that \( (a_i, b_i) \) does not blow up on the origin. Therefore it is sufficient to consider the case \( A \neq 0 \) on \( \mathbb{R} \times (0, 1) \). We note from (3) that \( A \) exists at least until \( \tau = 1 \). Since the origin is a blow-up point of \((a_i, b_i)\) at \( \tau = 1 \), \( A \) must blow up at the origin at \( \tau = 1 \) from Lemma 2.3. Since \( B \equiv 0, \) \( A \) satisfies \( A_{t} = A_{xx} + A^2 \). From Theorem 7 p.209, there are two possibilities: (I) \( A \equiv 1 \) or (II) the origin is the isolated blow-up point. Since (II) is excluded from Lemma 3.2 (I) occurs. Therefore this contradicts (3), which completes the proof.

Lemma 3.4. Let \( v_\pm = v_\xi \) with \( \xi = \pm L/4 \). Then it holds that

\[ \lim_{s \to \infty} \inf_{0 < s} v_\pm(0, s) > 0. \]

Proof. Let \( A = \partial^2_x - \frac{\xi}{2} \partial_y. \) Since the first eigenvalue of \( A \) is \( H^2_x(\mathbb{R}) \) is zero, we can choose \( R_0 > 0 \) such that the first eigenvalue of \( A |_{\text{Dirichlet}} \) in \( H^2_x((-R_0, R_0)) \) is \( \{ f \in H^2_x(\mathbb{R}); f(y) = 0 \text{ for } |y| > R_0 \} \) is less than 1/8. Put \( v_\pm = v_\xi \) with \( \xi = \pm L/4 \). From (11), we see that \( v_\pm \) is positive on \((-R_0, R_0)\) for large \( s > s_T \). Let \( \phi(y) > 0 \) be the first eigfunction of \( A |_{\text{Dirichlet}} \). Then from Lemma 3.3, if we choose \( \epsilon > 0 \) sufficiently small, \( \psi = \epsilon \phi \) becomes a subsolution of \( v_\pm \) in \((-R_0, R_0)\), which completes the proof.

Proof of Theorem 1.3. Combining Lemma 3.4 and (3), we obtain \( a(\pm L/4, t)/b(\pm L/4, t) < c' \) for some \( c' > 0 \). Therefore by the same argument as in the proof of Lemma 3.2, we see that the origin is not a blow-up point, which contradicts the assumption. The proof of Theorem 1.3 is completed.

4 Location of blow-up points

This section is devoted to the proof of Theorem 1.4. From Theorem 1.3 if a solution of (11) blows up in a finite time, \( b \) must be sign changing near the blow-up point. Here we discuss a relation between blow-up points and zeros of \( b \). Since \( b \) satisfies \( b_t = b_{xx} + 2ab \), the number of zeros of \( b(t) \) is nonincreasing in \( t \) (see e.g. (10)). Therefore from assumption of Theorem 1.3, the number of zeros of \( b(t) \) is one or zero for \( t \in (0, T) \). However since \((a,b) \) blows up at \( t = T \), \( b(t) \) has one zero for \( t < T \) from Theorem 1.3. Throughout this section, we assume that \( b(t) \) has one zero for \( t \in (0, T) \) and denote a zero of \( b(t) \) by \( \gamma(t) \). Furthermore we assume

\[ b(x, t) = \begin{cases} \text{negative} & \text{if } x < \gamma(t), \\ \text{positive} & \text{if } x > \gamma(t). \end{cases} \]

Proposition 4.1. Let \( x_0 \in \mathbb{R} \) be an isolated blow-up point. Then the blow-up set on \( \mathbb{R} \) consists of \( x_0 \).
Proof. To derive contradiction, we assume that \( x_1 > x_0 \) is another blow-up point. Since \( x_0 \) and \( x_1 \) are blow-up points, we see from Theorem 1.3 that
\[
\liminf_{t \to T} \gamma(t) \leq x_0, \quad \limsup_{t \to T} \gamma(t) \geq x_1.
\]
(15)

Let \( x_2 = (x_0 + x_1)/2, \delta = (x_1 - x_0)/2 \) and set
\[
u(y, s) = e^{-s}a(x_2 + e^{-s/2}y, T - e^{-s}), \quad \nu(y, s) = e^{-s}b(x_2 + e^{-s/2}y, T - e^{-s}).
\]
Since \((a, b)\) is uniformly bounded on \((x_2 - \delta, x_2 + \delta)\), \((a, b)\) satisfies
\[
\sup_{|y| < \delta e^{-s/2}} (|u(y, s)| + |v(y, s)|) < c_1 e^{-s}
\]
for some \( c_1 > 0 \). Therefore we get from (16) and (3)
\[
\int_{-\infty}^{\infty} |u|^2 \rho dy < c_1 e^{-s} \int_{|y| < \delta e^{-s/2}} |v|^2 \rho dy + \delta^2 e^{-s} \|u\|_{L^\infty(\mathbb{R})}^2 \int_{|y| > \delta e^{-s/2}} |y|^2 |v|^2 \rho dy
\]
\[
< (c_1^2 + \delta^2) e^{-s} \|v\|_{H^2_\rho(\mathbb{R})}^2.
\]

Let \( A = \partial_y^2 - \frac{\gamma}{2} \partial_y \). Then we see that
\[
\|u_s - (A - 1)v\|_{\rho} < 2\|u\|_{\rho} < 2\sqrt{c_1^2 + \delta^2} e^{-s/2} \|v\|_{H^2_\rho(\mathbb{R})}.
\]
Therefore from Lemma A.16 [1] (see also 2 [2]), we obtain \( \|v(s)\|_{\rho} \geq c e^{-\mu s} \) for some \( \mu > 0 \). As a consequence, there exists \( k \in \mathbb{N} \) such that
\[
v(s) = \alpha_k (1 + o(1)) e^{-\lambda_k s} \phi_k \quad \text{in } L^2_\rho(\mathbb{R}).
\]
However this contradicts (15), which completes the proof. \( \square \)

Let \( x_0 \in \mathbb{R} \) be a blow-up point of \((a, b)\). If \( x_0 \) is an isolated blow-up point, Proposition 1.1 implies that no other blow-up points exist on \( \mathbb{R} \). Then we see that \( \gamma(t) \) is continuous on \([0, T]\). In fact, if \( \gamma \) is not continuous at \( t = T \), it satisfies
\[
\liminf_{t \to T} \gamma(t) < \liminf_{t \to T} \gamma(t).
\]
However by the same argument as in the proof of Lemma 1.1 we derive contradiction. Therefore if \( x_0 \) is an isolated blow-up point of \((a, b)\), the proof is completed. We here consider the case where there are no isolated blow-up points. Let \( x_1 > x_0 \) be another blow-up point. Then the interval \((x_0, x_1)\) is included in the blow-up set. By shifting the origin, we can assume that
\[
\text{the interval } (-L, L) \text{ is included in the blow-up set.}
\]
(16)

We put \( e^{-s} = T - t \) and
\[
u(y, s) = (T - t)a(e^{-s/2}y, t), \quad v(y, s) = (T - t)b(e^{-s/2}y, t).
\]
We denote a zero of \( v(s) \) by \( \Gamma(s) \), which satisfies \( \Gamma(s) = e^{s/2} \gamma(t) \).

Lemma 4.1. For any \( \epsilon_0 > 0 \) there exists \( K > 0 \) such that if \( |v(y, s)| > \epsilon_0 \) for some \( |y| < s \) and \( s \gg 1 \), then it holds that \( |y_1 - \Gamma(s)| < K \).

Proof. We prove by contradiction. Assume that there exist \( \epsilon_0 > 0 \), \( \{y_1\}_{i \in \mathbb{N}} \) and \( \{s_i\}_{i \in \mathbb{N}} \) satisfying \( |y_1| < s_i \) and \( s_i \to \infty \) such that
\[
|v(y_i, s_i)| > \epsilon_0, \quad |y_i - \Gamma(s_i)| > i.
\]
(17)

We put \( \lambda_i = e^{-s_i}, t_i = T - e^{-s_i} \) and
\[
ai(x, \tau) = \lambda_i a(\sqrt{\lambda_i} y_i + \sqrt{\lambda_i} x, t_i + \lambda_i \tau), \quad bi(x, \tau) = \lambda_i b(\sqrt{\lambda_i} y_i + \sqrt{\lambda_i} x, t_i + \lambda_i \tau).
\]
Then (3) implies
\[
\sup_{x \in \mathbb{R}} (|ai(x, \tau)| + |bi(x, \tau)|) < \frac{c_1}{1 - \tau} \quad \text{for } \tau \in (0, 1)
\]
(18)
with some \( c_1 > 0 \). Furthermore we easily see that \( ai(x, 0) = u(y_i + x, s_i) \) and \( bi(x, 0) = v(y_i + x, s_i) \). Therefore it follows from (17) that
\[
|b_i(x, 0)| > 0 \quad \text{for } |x| < i.
\]
(19)

By taking a subsequence, we get
\[
(ai, bi) \to (A, B).
\]
Then we get from (18) and (19)
\[
|B(x, 0)| > 0 \quad \text{for } x \in \mathbb{R}, \quad \sup_{x \in \mathbb{R}} (|A(x, \tau)| + |B(x, \tau)|) < \frac{c_1}{1 - \tau} \quad \text{for } \tau \in (0, 1).
\]
From Theorem 1.3 we find that \((A, B)\) does not blow up on the origin at \( \tau = 1 \). As a consequence, from Lemma 2.2 the origin is not a blow-up point of \((ai, bi)\) at \( \tau = 1 \) for large \( i \in \mathbb{N} \), which implies that \( \sqrt{\lambda_i} y_i \) is not a blow-up point of \((a, b)\) for large \( i \in \mathbb{N} \). However since \( \sqrt{\lambda_i} y_i \to 0 \) as \( i \to \infty \), this contradicts (15). \( \square \)
Lemma 4.2. For any $\delta > 0$ and $r > 0$ there exists $m_0 > 0$ such that if $\|v(s)\|_{L^\infty((-1,1))} < m_0$ for some $s \gg 1$, then it holds that
\[ \sup_{-r < y < r} (|u(y, s) - 1| + |u_y(y, s)|) < \delta. \]

Proof. Since the proof of this lemma is the same as that of Lemma 3.3, we omit the detail. \qed

Lemma 4.3. $\liminf_{s \to \infty} \|v(s)\|_{L^\infty((-1,1))} = 0$.

Proof. Since the interval $(-L, L)$ is included in the blow-up set, we get from Theorem 1.3 that
\[ \liminf_{t \to T} \gamma(t) \leq -L, \quad \limsup_{t \to T} \gamma(t) \geq L. \]
Therefore since $\Gamma(s) = e^s \gamma(t)$, Lemma 4.2 proves this lemma. \qed

Proposition 4.2. $\lim_{s \to \infty} \|v(s)\|_{L^\infty((-1,1))} = 0$.

The proof of this Proposition is given in Section 4.1, which is a crucial step in this paper.

4.1 Proof of Proposition 4.2

This proof is based on the argument in [1]. We carefully investigate the behavior of solutions through a dynamical system approach in $L^2_p(R)$. Since $v(s)$ has exactly one zero for $s > s_T$, we focus on the behavior of the corresponding eigenmode of $v(s)$.

4.1.1 Choice of $\eta, \zeta, \delta, R$

Let $A = \partial_{yy} - \frac{1}{y^2} \partial_y$. It is known that $H^1_p(R)$ is spanned by eigenfunctions $\{\phi_i\}_{i \in \mathbb{N}}$ of $A$. A function $v$ in $H^1_p(R)$ is decomposed to
\[ v = \alpha \phi_0 + \beta \phi_1 + \gamma \phi_2 + w. \]
Since $\phi_2(y) = c_1(y^2 - 1)$ for some $c_1 > 0$, it follows that $\phi_2(0) = -c_1$ and $\phi_2(2) = 3c_1$. Here we recall the inequality:
\[ \|w\|_{L^\infty((-2,2))} < c \|w\|_{H^1_p(R)}. \]
Therefore there exists $c_1 > 0$ such that if $v \in H^1_p(R)$ satisfies $\alpha^2 + \beta^2 + \|w\|_{H^1_p(R)} < \epsilon_1 \gamma^2$, then $v$ has at least two zeros in $(-2, 2)$. Here we fix $\eta > 0$, $\zeta > 0$ and $\bar{\epsilon} \in (0, 1/4)$ such that
\[ 2 \left( \frac{1}{\eta} + \frac{\zeta}{\bar{\epsilon}} + \frac{1}{\zeta} < \epsilon_1, \quad \bar{\epsilon} \left( \frac{1}{\eta} \left( \frac{1}{1 + \epsilon} + \frac{1}{\epsilon} \right) + \frac{1}{\zeta} \right) < \frac{1}{8}, \quad \left( \frac{1}{4} - 2 \bar{\epsilon} \right) \eta - (2 + \eta^2) \bar{\epsilon} > 0, \quad \bar{\epsilon} \eta < \frac{1}{8} \right) \]
(20)
Furthermore we put
\[ M = \sup_{y \in \mathbb{R}, s > s_T} (|u(y, s) - 1| + |u_y(y, s)|). \]
By using [6], we can fix $\delta > 0$ and $\bar{R} > 0$ such that if $|P(y)| < \delta$ for $|y| < \bar{R}$ and $\|P\|_{L^\infty(R)} < M$, then it holds that
\[ \int_{-\infty}^{\infty} P(y)^2 \left( \sum_{k=0}^{2} (|\phi_k|^2 + |\phi_k'|^2) \right) dy < \left( \frac{\epsilon}{24} \right)^2, \quad \int_{-\infty}^{\infty} |P(y)|^2 dy < \frac{\epsilon}{8} \|v\|^2_{H^1_p(R)}. \]

4.1.2 Assumptions and setting

To prove Proposition 4.2 we assume
\[ m_* = \limsup_{s \to \infty} \|v(s)\|_{L^\infty((-1,1))} > 0 \]
throughout this section. Since $v$ satisfies $v_s = A v + K(y, s)v$ with $K(y, s) = -1 + 2u$, this assumption is equivalent to $\limsup_{s \to \infty} \|v(s)\|_{\rho} > 0$. We apply Lemma 4.2 with $\delta = \delta$ and $r = \bar{R}$. Then there exists $\bar{m} \in (0, m_*)$ such that if $\|v(s)\|_{L^\infty((-1,1))} < \bar{m}$, then it holds that
\[ \sup_{-\bar{R} < y < \bar{R}} (|u(y, s) - 1| + |u_y(y, s)|) < \delta. \]
From Lemma 4.3 there exists $\{s_i\}_{i \in \mathbb{N}}$ such that $\|v(s_i)\|_{L^\infty((-1,1))} \to 0$ as $i \to \infty$. By definition of $m_*$ ($m < m_*$), we can choose $s_i^{-}$ and $s_i^{+}$ such that $s_i^{-} < s_i < s_i^{+}$ by
\[ \|v(s_i)\|_{L^\infty((-1,1))} < \bar{m} \quad \text{for } s_i \in (s_i^{-}, s_i^{+}), \quad \|v(s_i^{+})\|_{L^\infty((-1,1))} = \bar{m}. \]
Since $\|v(s_i)\|_{L^\infty((-1,1))} \to 0$ as $i \to \infty$, we easily see that $\|v(s_i)\|_{\rho} + \|v(s_i)\|_{\rho} \to 0$ as $i \to \infty$. Therefore it follows that $s_i^{+} - s_i \to \infty$ as $i \to \infty$. Put $\Delta_i = s_i^{+} - s_i^{-}$, $\Delta_i \to \infty$ and
\[ u_i(y, s) = u(y, s_i^{-} + s), \quad v_i(y, s) = v(y, s_i^{-} + s). \]
To analyze the dynamics of $v_i(s)$ in $L^2_p(R)$, we decompose a function $v_i$ by using eigenfunctions of $A$.
\[ v_i = \alpha_i \phi_0 + \beta_i \phi_1 + \gamma_i \phi_2 + w_i, \quad \partial_y v_i = \mu_i \phi_0 + \nu_i \phi_1 + q_i. \]
Lemma 4.4. For any $d > 0$, it holds that
\[
\liminf_{i \to \infty} \inf_{0 < s < d} \|v_i(s)\|_\rho > 0, \quad \liminf_{i \to \infty} \inf_{0 < s < d} (|\alpha_i(s)| + |\beta_i(s)|) > 0.
\]

Proof. First we assume
\[
\liminf_{i \to \infty} \inf_{0 < s < d} \|v_i(s)\|_\rho = 0.
\]
Then there exists \( \{d_i\}_{i \in \mathbb{N}} \subset (0, d) \) such that \( \|v_i(d_i)\|_\rho \to 0 \) as \( s \to \infty \). By taking a subsequence, we get \( d_i \to d_* \in (0, d] \) and \( (u_i, v_i) \to (U, V) \) as \( i \to \infty \). Then by definition of \( s_i \) and \( d_i \), it follows that \( V(0) \neq 0 \) and \( V(d_*) = 0 \) holds. However since \( V \) satisfies \( V_2 = AV + (1 - 2U)V \), \( V(d_*) = 0 \) contradicts the backward uniquenes for parabolic equations, which proves the first statement. To prove the second statement, we repeat the same argument above. Assume that there exists \( \{d_i\}_{i \in \mathbb{N}} \subset (0, d) \) such that
\[
\liminf_{i \to \infty} \inf_{0 < s < d} (|\alpha_i(d_i)| + |\beta_i(d_i)|) = 0.
\]
From the first statement of this lemma and Lemma 4.1, we see that \( |\Gamma(s_i)| < K \) for some \( K > 0 \). By taking a subsequence, we get \( d_i \to d_* \), \( (u_i, v_i) \to (U, V) \) and \( \Gamma(s_i) \to \Gamma_* \in (-K, K) \). Then from definition of \( \Gamma(s) \), we see that \( V(y, 0) \leq 0 \) for \( y < \Gamma_* \), \( V(y, 0) \geq 0 \) for \( y > \Gamma_* \).

Since \( V \neq 0 \) on \( \mathbb{R} \times (0, \infty) \), the number of zeros of \( V(s) \) is decreasing in \( s \). Therefore the number of zeros of \( V(d_i) \) is one or zero. On the other hand, we see from (23) that \( (V(d_*), \phi_0)_\rho = 0, (V(d_*), \phi_1)_\rho = 0 \). Therefore from Corollary 6.17 \([9]\), we find that the number of \( V(d_*') \) has more than one zeros, which is contradiction. The proof is completed. \( \Box \)

4.1.3 Dynamics of \( v_i(s) \) on \( L^2_\rho(\mathbb{R}) \)
In the following argument, we always assume \( s \in (0, \Delta_i) \). Therefore it follows from definition of \( \tilde{m} \) that
\[
\sup_{-R < y < R} (|u_i(y, s) - 1| + |\partial_y u_i(y, s)|) < \tilde{\delta} \quad \text{for } s \in (0, \Delta_i).
\]
Then \( v_i \) satisfies
\[
\partial_s v_i = \partial_{yy} v_i - \frac{y}{2} \partial_y v_i + v_i + 2(u_i - 1)v_i.
\]
Multiplying equation by \( \phi_k \) \( (k = 0, 1, 2) \), we get
\[
\alpha_i = \alpha_i + 2h_{0i}, \quad \beta_i = \frac{1}{2}\beta_i + 2h_{1i}, \quad \gamma_i = 2h_{2i},
\]
where \( h_{ki} \) \( (k = 0, 1, 2) \) is given by
\[
h_{ki} = \int_{-\infty}^{\infty} (u_i - 1)v_i \phi_k \rho dy.
\]
Furthermore since \( w_i \) satisfies
\[
\partial_s w_i = Aw_i + w_i + 2(u_i - 1)w_i + 2(u_i - 1)(\alpha_i \phi_0 + \beta_i \phi_1 + \gamma_i \phi_2) - 2 \sum_{k=0}^2 h_{ki} \phi_k,
\]
we get
\[
\frac{1}{2} \partial_s \|w_i\|^2_\rho = -\|\partial_s w_i\|^2_\rho + \|w_i\|^2_\rho + 2 \int_{-\infty}^{\infty} (u_i - 1)w_i^2 \rho dy + 2H_i,
\]
where \( H_i \) is given by
\[
H_i = \int_{-\infty}^{\infty} (u_i - 1)(\alpha_i \phi_0 + \beta_i \phi_1 + \gamma_i \phi_2) w_i \rho dy - \sum_{k=0}^2 \int_{-\infty}^{\infty} h_{ki} \phi_k w_i \rho dy.
\]
By choice of \( R \) and \( \delta \), we see that
\[
\int_{-\infty}^{\infty} |u_i - 1|w_i^2 \rho dy < \frac{\epsilon}{8} \|w_i\|^2_{H^1_\rho(\mathbb{R})}, \quad |h_{ki}| < \left( \int_{-\infty}^{\infty} (u_i - 1)^2 \phi_k^2 \rho dy \right)^{1/2} \|w_i\|_\rho < \frac{\epsilon}{24} \|w_i\|_\rho,
\]
\[
|H_i| < \left( \int_{-\infty}^{\infty} (u_i - 1)^2 (\phi_0 + |\phi_1| + |\phi_2|)^2 \rho dy \right)^{1/2} \|w_i\|_\rho \|w_i\|_\rho + \|w_i\|_\rho \sum_{k=0}^2 |h_{ki}| < \frac{\epsilon}{24} \|w_i\|_\rho \|w_i\|_\rho + \frac{\epsilon}{8} \|w_i\|_\rho \|w_i\|_\rho = \frac{\epsilon}{6} \|w_i\|_\rho \|w_i\|_\rho.
\]
Applying these estimates in (24) and (25), we get
\[
\begin{align*}
\partial_s \left( \alpha_i^2 + \beta_i^2 \right) &> \frac{1}{2} \left( \alpha_i^2 + \beta_i^2 \right) - \varepsilon^2 \left( \gamma_i^2 + \|w_i\|_\rho^2 \right), \\
\partial_s \gamma_i^2 &< \frac{\varepsilon}{2} \left( \left( \alpha_i^2 + \beta_i^2 \right) + \gamma_i^2 + \|w_i\|_\rho^2 \right), \\
\partial_s \|w_i\|_\rho^2 &< -\frac{1}{2} \|w_i\|_\rho^2 + \varepsilon^2 \left( \left( \alpha_i^2 + \beta_i^2 \right) + \gamma_i^2 \right).
\end{align*}
\]  
(26)

Next we provide estimates for $\partial_y v_i$. Let $z_i = \partial_y v_i$. Then $z_i$ satisfies
\[
\partial_s z_i = Az_i + \frac{z_i}{2} + 2(u_i - 1)z_i + 2(\partial_y u_i)v_i.
\]

Since $z_i = \mu_i \phi_0 + \nu_i \phi_1 + q_i$, $\mu_i$ and $\nu_i$ satisfy
\[
\mu_i = \frac{1}{2} \mu_i + 2\hat{h}_{0i} - 2\hat{h}_{0i}, \quad \nu_i = 2\hat{h}_{1i} - 2\hat{h}_{1i},
\]
where $\hat{h}_i$ and $\hat{h}_k (k = 0, 1)$ are given by
\[
\hat{h}_i = \int_{-\infty}^\infty (1 - u_i)z_i \phi_k \rho dy, \quad \hat{h}_k = \int_{-\infty}^\infty (\partial_y u_i)v_i \phi_k \rho dy.
\]
Furthermore $q_i$ satisfies
\[
\partial_s q_i = Aq_i + \frac{1}{2} q_i + 2(u_i - 1)q_i - 2(\partial_y u_i)v_i + 2(u_i - 1)(\mu_i \phi_0 + \nu_i \phi_1) - 2(\hat{h}_{0i} - \hat{h}_{0i})\phi_0 + 2(\hat{h}_{1i} - \hat{h}_{1i})\phi_1.
\]

By the same calculation as $v_i$, we obtain
\[
\begin{align*}
\partial_s \mu_i^2 &> \frac{\mu_i^2}{2} - \varepsilon^2 \left( \nu_i^2 + \|q_i\|_\rho^2 + \|v_i\|_\rho^2 \right), \\
\partial_s \nu_i^2 &< \frac{\varepsilon}{2} \left( \nu_i^2 + \mu_i^2 + \|q_i\|_\rho^2 + \|v_i\|_\rho^2 \right), \\
\partial_s \|w_i\|_\rho^2 &< -\frac{1}{2} \|w_i\|_\rho^2 + \varepsilon^2 \left( \mu_i^2 + \nu_i^2 + \|v_i\|_\rho^2 \right).
\end{align*}
\]  
(27)

We here put
\[
X_i = \alpha_i^2 + \beta_i^2 + \gamma_i^2, \quad Y_i = \mu_i^2 + \nu_i^2, \quad Z_i = \|w_i\|_\rho^2 + \|q_i\|_\rho^2.
\]  
(28)

Since $\varepsilon < 1/2$, combining (20) and (27), we obtain
\[
\begin{align*}
\left\{ 
X_i > \frac{1}{4} X_i - \varepsilon(Y_i + Z_i), \\
|Y_i| < \varepsilon(X_i + Y_i + Z_i), \\
|Z_i| < -\frac{1}{4} Z_i + \varepsilon(X_i + Y_i).
\right.
\]
\]  
(29)

Let $\eta > 0$ be given in (20). We define $\kappa_i$ by
\[
\kappa_i = \eta X_i - Y_i - Z_i.
\]

We investigate the behavior of $\kappa_i$.
\[
\kappa_i' = \frac{\eta}{4} X_i - \eta \varepsilon(Y_i + Z_i) - \varepsilon(X_i + Y_i + Z_i) + \frac{1}{4} Z_i - \varepsilon(X_i + Y_i)
\]
\[
= \left( \frac{\eta}{4} - 2\varepsilon \right) X_i - (2 + \eta)\varepsilon Y_i + \left( \frac{1}{4} - (1 + \eta)\varepsilon \right) Z_i.
\]

Since $\kappa_i \geq 0$ is equivalent to $Y_i + Z_i \leq \eta X_i$, it holds that
\[
\kappa_i' > \left( \frac{\eta}{4} - 2\varepsilon - (2 + \eta)\varepsilon \eta \right) X_i + \left( \frac{1}{4} - (1 + \eta)\varepsilon \right) Z_i
\]
\[
= \left( \frac{1}{4} - 2\varepsilon \right) \eta - (2 + \eta^2) \varepsilon \right) X_i + \left( \frac{1}{4} - (1 + \eta)\varepsilon \right) Z_i \quad \text{if } \kappa_i > 0.
\]

Therefore from (20), we conclude
\[
\kappa_i' > 0 \quad \text{if } \kappa_i \geq 0.
\]

Since $Y_i = \gamma_i^2 + \nu_i^2 = 2\gamma_i^2$ and $Z_i = \|w_i\|_\rho^2 + \|z_i\|_\rho^2 = \|w_i\|_\rho^2$ (see Lemma 6.2 [1]), if $\kappa_i < 0$ $(\Leftrightarrow \eta X_i < Y_i + Z_i)$ and $Z_i < \eta Y_i$, it holds that
\[
\alpha_i^2 + \beta_i^2 + \|w_i\|_\rho^2 < X_i + Z_i < \left( \frac{1 + \tilde{\gamma}}{\eta} + \frac{1 + \tilde{\gamma}}{\eta} \right) Y_i = 2 \left( \frac{1 + \tilde{\gamma}}{\eta} + \frac{2}{\eta} \right) \eta^2 < \varepsilon_1 \gamma_i^2,
\]
where we use (20) in the last inequality. Therefore by definition of $\epsilon_1$, $v_i$ has more than one zeros if $\kappa_i < 0$ and $Z_i < \eta Y_i$. Summarizing the above estimates, we obtain the following lemma.
Lemma 4.5. If \( \kappa_i(s') \geq 0 \) for some \( s' \in (0, \Delta_i) \), then it holds that \( \kappa_i(s) > 0 \) for \( s \in (s', \Delta_i) \). Furthermore if \( \kappa_i(s) < 0 \) for some \( s \in (0, \Delta_i) \), then it holds that \( \zeta_i(s) < Z(s) \).

Lemma 4.6. Let \( \Delta_i^- = \{ s \in (0, \Delta_i); \kappa_i(s') < 0 \text{ for some } s' < (0, s) \} \). Then it holds that \( \lim_{i \to \infty} \Delta_i^- = \infty \) and \( \zeta_i(s) < Z_i(s) \) for \( s \in (0, \Delta_i^-) \).

Proof. Since the second statement is trivial from Lemma 4.5, it is enough to show the first statement. We prove by contradiction. Assume that there exists a subsequence \( \{ j \} \in \Lambda \subset \{ i \} \in \mathbb{N} \) such that \( \{ \Delta_j^- \} \in \Lambda \) is bounded. Then from Lemma 4.4 there exists \( \theta > 0 \) such that

\[
\inf_{0 < s < \Delta_i^-} (|a_j(s)| + |b_j(s)|) > \theta \quad \text{for } j \in \Lambda. \tag{30}
\]

From definition of \( \Delta_j^- \) and Lemma 4.5, we see that \( \kappa_j(s) > 0 \) for \( s \in (\Delta_j^-, \Delta_i) \). Therefore since \( X_i, Y_i \) and \( Z_i \) satisfy (29) for \( s \in (0, \Delta_i) \), we get from (29) that

\[ X_i > \frac{1}{4} X_i - \bar{e} \eta X_i > \frac{1}{8} X_i \quad \text{for } s \in (\Delta_j^-, \Delta_i). \]

Since we note from (30) that \( X_j(\Delta_j^-) > \theta \) for \( j \in \Lambda \), we obtain

\[ X_j(s) > \theta e^{s - \Delta_j^-} / \theta \quad \text{for } s \in (\Delta_j^-, \Delta_j). \]

However since \( \Delta_j \to \infty \) as \( j \to \infty \) and \( \Delta_j^- \) is bounded, \( X_j(s) \) becomes arbitrary large for large \( j \in \Lambda \), which contradicts a boundedness of \( X_i(s) \). \( \square \)

Proof of Proposition 4.2. From Lemma 4.3, there exists a subsequence \( \{ (u_i, v_i) \} \in \mathbb{N} \) such that

\[ \eta X_i \leq Y_i + Z_i, \quad \zeta Y_i < Z_i \quad \text{for } s \in (0, \Delta_j^-), \quad \lim_{s \to \infty} \Delta_j^- = \infty. \tag{31} \]

Therefore we get from (29) that

\[ \bar{Z}_i < \left( -\frac{1}{4} + \bar{e} \left( \frac{1}{\bar{\eta}} \left( 1 + \frac{1}{\bar{\zeta}} \right) \right) \right) \bar{Z}_i < \frac{1}{8} \bar{Z}_i \quad \text{for } s \in (0, \Delta_j^-), \]

which implies \( Z_i < Z_i(0)e^{-s/8} \). Combining this estimate and (31), we obtain

\[ \| t_i(s) \| \rho < ce^{-s/8} \quad \text{for } s \in (0, \Delta_i^-) \]

for some \( c > 0 \). As a consequence, from Lemma 4.2, there exists a positive continuous function \( F(s) \) on \( s > 0 \) such that \( F(s) \to 0 \) as \( s \to \infty \) and

\[ \| u_i(s) - 1 \| \rho < F(s) \quad \text{for } s \in (0, \Delta_i^-). \]

Then by taking a subsequence, we get \( (u_i, v_i) \to (U, V) \) as \( i \to \infty \). From above estimates, we see that

\[ \lim_{s \to \infty} \| U(s) - 1 \| \rho = 0, \quad \| V(s) \| \rho = O(e^{-s/8}). \]

Then Lemma 4.7 implies that \( \| U(s) - 1 \| \rho = O(e^{-s/8}) \) for some \( \gamma > 0 \). Therefore we get form Lemma 4.8 that

\[ |U(y, s) - 1| + |V(y, s)| < ce^{-\gamma s/2} \quad \text{for } |y| < e^{\delta s} \tag{32} \]

for some \( \theta > 0 \) and \( c > 0 \). Since \( V \) satisfies (5), it holds that

\[ \| V_s - (A - 1)V \| \rho < 2\| (U - 1)V \| \rho. \]

Since \( U \) is uniformly bounded, by using (3) and (32), we get

\[ \| (U - 1)V \| \rho^2 = \int_{|y|<e^{\delta s}} (U - 1)^2 V^2 \rho dy + \int_{|y|>e^{\delta s}} (U - 1)^2 V^2 \rho dy < ce^{-2\gamma s} \int_{R} V^2 \rho dy + ce^{-2\delta s} \int_{|y|>e^{\delta s}} |y|^2 V^2 \rho dy < c \left( e^{-2\gamma s} + e^{-2\delta s} \right) \| V \|_{H_i^j(R)}^2. \]

Therefore we obtain

\[ \| V_s - (A - 1)V \| \rho < ce^{-\mu s} \| V \|_{H_i^j(R)}^2 \]

for some \( \mu > 0 \). Repeating the argument as in the proof of Proposition 4.1, which derives contradiction. Therefore the assumption (21) is false. \( \square \)

Lemma 4.7. If \((u_i, v_i)\) converges to some function \((U, V)\) in \( L^\infty_{loc}(\mathbb{R} \times (0, \infty)) \) satisfying \( \lim_{s \to \infty} \| U(s) - 1 \| \rho = 0 \) and \( \| V(s) \| \rho \) decays exponentially, then \( \| U(s) - 1 \| \rho \) decays exponentially.

Proof. Put \( \lambda_i = e^{-\delta s}, \quad t_i = T - \lambda_i \) and

\[ a_i(x, \tau) = \lambda_i a(\sqrt{\lambda_i} x, t_i + \lambda_i \tau), \quad b_i(x, \tau) = \lambda_i b(\sqrt{\lambda_i} x, t_i + \lambda_i \tau). \]
Then we see that
\[ u_i(y, s) = (1 - \tau) a_i(x, \tau), \quad v_i(y, s) = (1 - \tau) b_i(x, \tau), \]
with \( x = e^{-s/2} y \) and \( 1 - \tau = e^{-\theta s} \). Therefore since \((a_i(0), b_i(0)) = (u_i(0), v_i(0))\), we get \((a_i, b_i) \to (A, B)\) and
\[ U(y, s) = (1 - \tau) A(x, \tau), \quad V(y, s) = (1 - \tau) B(x, \tau). \]
Since \( |V(s)|_\rho = O(e^{-\gamma s}) \), we see from Lemma 4.5 that \( |V(y, s)| = O(e^{-\gamma s}) \) for \( |y| < e^{\theta s} \). Therefore applying the same argument as [4] with a slight modification, we find that there are two possibilities: (I) there exists \( \gamma_1 > 0 \) such that \( |u(s) - 1|_\rho = O(e^{-\gamma_1 s}) \) or (II) there exists \( \Lambda \neq 0 \) such that \( U(s) - 1 = \Lambda(1 + o(1))s^{-1}\phi_0 \) in \( L^2_\rho(\mathbb{R}) \). Assume that (II) holds. Since \(|V(y, s)| = O(e^{-\gamma_1 s})\) for \(|y| < e^{\theta s}\), the argument in the proof of Proposition 2.3 in [5] shows
\[
\lim_{s \to 0} \sup_{|y| < \sqrt{s}} \left| \int_0^s \left( U(y, s) - \left( 1 + \frac{y^2}{s} \right)^{1/2} \right) d\xi = 0 \right. \quad \text{for any } l > 0
\]
with some \( c > 0 \). Furthermore applying the argument in [7], we can verify that the origin is an isolated blow-up point of \((A, B)\). Therefore (II) is excluded from Lemma 4.2 which completes the proof.

**Lemma 4.8.** Let \((U, V)\) be a bounded solution of (5) satisfying \( |V(s)|_\rho = O(e^{-\gamma s}) \). Then there exist \( \theta > 0 \) and \( c > 0 \) such that
\[
|V(y, s)| < e^{-\gamma s/2} \quad \text{for } |y| < e^{\theta s}.
\]
Furthermore if \( |U(s) - 1|_\rho + |V(s)|_\rho = O(e^{-\gamma s}) \). Then there exist \( \theta > 0 \) and \( c > 0 \) such that
\[
|U(y, s) - 1| + |V(y, s)| < e^{-\gamma s/2} \quad \text{for } |y| < e^{\theta s}.
\]

**Proof.** We apply the method given in [7]. Let \( K = 2 \sup_{s > 0} |U(s)|_{L^\infty(\mathbb{R})} \). To construct a comparison function for \( V \), we consider
\[
W = AW + KW \quad \tau > s, \quad W_0 = |V(s)|.
\]
Then this solution \( W \) is given by
\[
W(\tau) = \frac{e^{K(\tau-s)}}{2\sqrt{\pi}\sqrt{1 - e^{-(\tau-s)}}} \int_{-\infty}^\infty \exp \left( \frac{-(ye^{-(\tau-s)/2} - \xi)^2}{4(1 - e^{-(\tau-s)})} \right) W_0(\xi) d\xi.
\]
Then it holds that
\[
\int_{-\infty}^\infty \exp \left( -\frac{(ye^{-(\tau-s)/2} - \xi)^2}{4(1 - e^{-(\tau-s)})} \right) W_0(\xi) d\xi < \left( \int_{-\infty}^\infty \exp \left( -\frac{(ye^{-(\tau-s)/2} - \xi)^2}{2(1 - e^{-(\tau-s)})} \right) \right)^{1/2} \||W_0||_\rho.
\]
Since
\[
\frac{(ye^{-(\tau-s)/2} - \xi)^2}{2(1 - e^{-(\tau-s)})} + \frac{\xi^2}{4} = -\frac{1 + e^{-(\tau-s)}}{4(1 - e^{-(\tau-s)})} \left( \xi - \frac{2e^{-(\tau-s)}}{1 + e^{-(\tau-s)}} \right)^2 + \frac{e^{-(\tau-s)}}{4(1 - e^{-(\tau-s)})^2} \left( 2 - e^{-(\tau-s)} \right) y^2,
\]
we obtain
\[
W(\tau) < ce^{-\gamma s/2} \exp \left( \frac{2e^{-\gamma s/2} y^2}{2(1 - e^{-(\tau-s)})^2} \right) e^{-\gamma s}.
\]
We choose \( \tau = (1 + \frac{2s}{2\sigma})s \). Since \( \tau - s = \frac{2s}{2\sigma} > \log 2 \) for large \( s \), it follows that
\[
W(\tau) < ce^{-\gamma s/2} \exp \left( \frac{2e^{-\gamma s/2} y^2}{4\pi(1 - e^{-(\tau-s)})^2} \right) e^{-\gamma s}.
\]
for \(|y| < e^{\gamma s/4K} \) and \( s \gg 1 \). Therefore the first estimate is proved. Next we provide estimates for \( U - 1 \). Let \( C = U - 1 \). Then it satisfies
\[
C_\sigma = AC + C + C^2 - V^2.
\]
By the same way as above, we consider
\[
W = AW + KW + V^2 \quad \tau > s, \quad |W_0| = |V(s)|,
\]
where \( K = 1 + \sup_{s > 0} |U(s)|_{L^\infty(\mathbb{R})} \). Then \( W \) is given by
\[
W(\tau) = \frac{e^{K(\tau-s)}}{2\sqrt{\pi}\sqrt{1 - e^{-(\tau-s)}}} \int_{-\infty}^\infty \exp \left( -\frac{(ye^{-(\tau-s)/2} - \xi)^2}{4(1 - e^{-(\tau-s)})} \right) W_0(\xi) d\xi
\]
\[
+ \int_0^\tau \frac{e^{K(\tau-\mu)}}{2\sqrt{\pi}\sqrt{1 - e^{-(\tau-\mu)}}} \int_{-\infty}^\infty \exp \left( -\frac{(ye^{-(\tau-\mu)/2} - \xi)^2}{4(1 - e^{-(\tau-\mu)})} \right) V(\xi, \mu)^2 d\xi d\mu.
\]
By the same way as above, we choose \( \tau = (1 + \frac{2s}{2\sigma})s \). Then it is enough to estimate the second term on the right-hand side. Since \(|V(\xi, s)| < ce^{-\gamma s/2} \) for \(|\xi| < e^{\theta s}\) and \( s \gg 1 \), we get
\[
\int_{-\infty}^\infty \exp \left( -\frac{(ye^{-(\tau-\mu)/2} - \xi)^2}{4(1 - e^{-(\tau-\mu)})} \right) V(\xi, \mu)^2 d\xi < \int_{|\xi| < e^{\theta s}} d\xi + \int_{|\xi| > e^{\theta s}} d\xi < ce^{-\gamma \mu} \sqrt{4\pi(1 - e^{-(\tau-\mu)})} + c \int_{|\xi| > e^{\theta s}} \exp \left( -\frac{(ye^{-(\tau-\mu)/2} - \xi)^2}{4(1 - e^{-(\tau-\mu)})} \right) d\xi.
\]
If $|y| < e^{\theta s}/2$ and $|\xi| > e^{\theta s}$, it holds that $|ge^{-(\tau-\mu)/2} - \xi| > \xi/2$. Therefore we get
\[
\int_{|\xi| > e^{\theta s}} \exp \left( -\frac{(y e^{-(\tau-\mu)/2} - \xi)^2}{4(1 - e^{-(\tau-\mu)})} \right) d\xi < e^{-\gamma s} \int_{|\xi| > e^{\theta s}} |\xi|^\theta \exp \left( -\frac{\xi^2}{16(1 - e^{-(\tau-\mu)})} \right) < c(1 - e^{-(\tau-\mu)/(\gamma^2 + \theta/28)} e^{-\gamma s}
\]
for $|y| < e^{\theta s}/2$. As a consequence, we obtain
\[
\int_s^\tau \frac{e^{K(\tau-\mu)}}{2\sqrt{\pi}} \frac{1}{1 - e^{-(\tau-\mu)}} d\mu \int_{-\infty}^{\infty} \exp \left( -\frac{(y e^{-(\tau-\mu)/2} - \xi)^2}{4(1 - e^{-(\tau-\mu)})} \right) V(\xi, \mu)^2 d\xi < c e^{K(\tau-s)} \int_s^\tau e^{-\gamma \mu} d\mu < c e^{K(\tau-s)} e^{-\gamma s} = ce^{-\gamma s/2}
\]
for $|y| < e^{\theta s}/2$, which completes the proof. 

\[\square\]

4.2 Proof of Theorem 1.4

The proof of Theorem 1.4 is almost the same as that of Proposition 1.2.

**Proof of Theorem 1.4.** Assume that (16) holds true. Then from Proposition 1.2, $v(s)$ converges to zero in $L_\rho^2(\mathbb{R})$ as $s \to \infty$. Then we see from Lemma 1.2 that $u(s) \to 1$ in $L_\rho^2(\mathbb{R})$ as $s \to \infty$. Once $\|v(s)\|_\rho = O(e^{-\gamma s})$ for some $\gamma > 0$ is derived, by the same argument as in the proof of Proposition 1.2, we obtain contradiction. Therefore it is enough to show that $v(s)$ decays exponentially in $L_\rho^2(\mathbb{R})$. In fact, we decompose $v(s)$ as (22) and define $X$, $Y$ and $Z$ as (28). Since $(u(s), v(s)) \to (0, 1)$, repeating arguments in Section 4.3, we obtain (29). Therefore we obtain
\[\kappa(s) = \bar{\eta}_X(s) - Y(s) - Z(s) < 0, \quad \bar{\zeta}Y(s) < Z(s).
\]
This implies that $v(s)$ decays exponentially in $L_\rho^2(\mathbb{R})$, which completes the proof. 

\[\square\]

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