Local Maxima of White Noise Spectrograms and Gaussian Entire Functions

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Abstract
We confirm Flandrin’s prediction for the expected average of local maxima of spectrograms of complex white noise with Gaussian windows (Gaussian spectrograms or, equivalently, modulus of weighted Gaussian Entire Functions), a consequence of the conjectured double honeycomb mean model for the network of zeros and local maxima, where the area of local maxima centered hexagons is three times larger than the area of zero centered hexagons. More precisely, we show that Gaussian spectrograms, normalized such that their expected density of zeros is 1, have an expected density of 5/3 critical points, among those 1/3 are local maxima, and 4/3 saddle points, and compute the distributions of ordinate values (heights) for spectrogram local extrema. This is done by first writing the spectrograms in terms of Gaussian Entire Functions (GEFs). The extrema are considered under the translation invariant derivative of the Fock space (which in this case coincides with the Chern connection from complex differential geometry). We also observe that the critical points of a GEF are precisely the zeros of a Gaussian random function in the first higher Landau level. We discuss natural extensions of these Gaussian random functions: Gaussian Weyl–Heisenberg functions and Gaussian bi-entire functions. The paper also reviews recent results on the applications of white noise spectrograms, connections between several developments, and is partially intended as a pedestrian introduction to the topic.

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1 Introduction

The present paper is concerned with the critical points (divided in local maxima, local minima and saddle points) of spectrogram transformed white noise. More precisely, we are interested in statistical averages of critical points and local extrema of white noise spectrograms with Gaussian windows. As first observed in [14], such white noise spectrograms can be written in terms of a Gaussian Entire Function (GEF), an observation which had the virtue of connecting the traditionally applications-oriented field of time-frequency analysis, to the traditionally more fundamental topic of Gaussian Analytic Functions.

We will both deal with the Gaussian Entire Function

\[ F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \]

and its translation invariant derivative

\[ F_1(z) = F_1(z, \overline{z}) = (\partial_z - \overline{z}) F(z) = \sum_{k=0}^{\infty} a_k \frac{(k - |z|^2)z^{k-1}}{\sqrt{k!}}, \]

where \( a_k \) are i. i. d. Gaussian random variables. The zeros of \( F_1(z) \) describe the critical points of \( H(z) = |F(z)|^2 e^{-|z|^2} \), which can be divided in saddle points and local maxima/minima, while, due to the maximum modulus principle, the equation \( F'(z) = 0 \) can only yield saddle points. As we will see below, \( |e^{-|z|^2/2} F(z)|^2 \) and \( |e^{-|z|^2/2} F_1(z)|^2 \) can be identified with white noise spectrograms windowed by the Gaussian and the first Hermite function, respectively. This leads to a Gaussian functions companion to the determinantal point processes in higher Landau levels [3, 5, 31, 43, 45, 46, 56], also defined using time-frequency transforms. Observe that, while \( F(z) \) is analytic, \( F_1(z) \) is just polyanalytic [13], since \( \partial_z^2 F_1(z, \overline{z}) = 0 \) (see [2] for an introduction to polyanalytic functions from the time-frequency analysis perspective).

We now make an intermezzo in this introduction to point out two curious properties.

- **Silence** points (zeros) repel each other strongly (2-point intensity vanishes at zero [17], Sect. 3).
- **Loud** points (critical points) display weaker repulsion (positive 2-point intensity at zero [11]).

Back to our work, we will first compute the expected number of critical, saddle and local maxima coordinates, defined under constraints on the critical points equation,

\[ F_1(z_c) = 0, \]
and then the corresponding average distribution of \textit{ordinate values} (the expected average values of $F(z)$ at the critical, saddle and local maxima coordinates), where the ordinate value $x_c$ associated to the point $z_c$ is
\[
x_c = \left| e^{-|z|^2/2} F(z_c) \right| = \text{Spec} \mathcal{W}(z_c)^{1/2} \in \mathbb{R}^+.
\]

The description of the local extrema and corresponding ordinates of spectrogram transformed white noise is likely to be useful in signal analysis applications involving local maxima of spectrograms as in, for instance, [61], but also in reassignment [32] and syn-crosqueezing [21] techniques. This is particularly evident in differential reassignment [20], where local maxima are attractors of the corresponding vector field.

Our results also complement the information used in the recent algorithms aimed at recovering a signal embedded in white noise from its spectrogram zeros [12, 14–16, 25, 33, 34, 38] (these ‘zero-based’ methods have also been recently extended to wavelets, see [6, 15, 47], to the sphere [52] and to the Stockwell transform [49]) and level sets [41]. Such algorithms have found notable applications, for instance, in methods for anonymize motion sensor data and enhance privacy in the Internet of Things (IoT) [54].

The analysis of local extrema of white noise spectrograms relies on the eigenvalues of the Hessians of the Gaussian vectors, from which one can derive the Kac-Rice type formulas as in [23]. The resulting expected number of local maxima is $1/3$ times the expected number of zeros. One can provide heuristics for the $1/3$ proportion using a random honeycomb model, suggested by Flandrin to describe zeros and local maxima of spectrograms of white noise (see Fig. 1 on p. 6). By extrapolating such heuristics to the setting of holomorphic sections of a Hermitian holomorphic line bundle over a complex manifold [23], the model also suggests a hand-waving explanation for the occurrence of the factor $1/3$ in the first term of the statistics of supersymmetric vacua (modelled as fixed Morse index critical points of random holomorphic sections). Moving to a different physical context, a correspondence between the critical points of Gaussian entire functions and the zeros of a Gaussian bi-entire function will be described. The Gaussian bi-entire function lives on the first higher Landau level eigenspace, a space of paramount importance in condensed matter physics, used to model electron dynamics on a energy level above the first layer of charged-like particles. The Landau levels are the classical explanation for the step-like changes in conductivity under the action of a constant magnetic field associated with the integer Quantum–Hall effect [60].

The results in this paper are novel to a certain extent, since their statement (both in the GEF and time-frequency language) has not appeared before in the literature and given a direct proof. But one should not go as far as calling them ‘genuinely new’, since general results have been obtained for compact Kähler manifolds and the particular calculations done for $SU(2)$ random polynomials in [23] and [29]. It may come as a surprise that the more simple (and also the most well-studied) case of the GEF has not been studied before, since the cases treated in [23] and [29] are much more complicated. At least formally, our main results could have been indirectly derived (one may say ‘guessed’, since actually some definitions are a bit different for the non-compact case, most notably in the section on critical values) by properly considering...
limit cases of the results for $SU(2)$ polynomials in [23] and [29]. Even if this argument could be made rigorous, it would be a long journey in comparison to the direct and relatively elementary proofs we offer in this paper.

An outline of the paper follows. Since we will alternate between the spectrogram and GEF formulations, to avoid confusing readers not familiar with both topics, we try to always present both perspectives, sometimes at the cost of a little redundancy. In the next section we present a short review of the required fundamental concepts about random Gaussian functions and time-frequency analysis. In the third section we present the main results, together with some remarks concerning the motivations, implications and connections to the work of other authors. The proofs of the main results are gathered in Sect. 4. In Sect. 5, we will consider Gaussian non-analytic functions motivated by the results of the paper. First we observe that the nonzero critical points of $H(z)$ are precisely the zeros of a bi-analytic Gaussian function with correlation kernel given by the reproducing kernel of the second Landau level. This suggests a more general class of Gaussian random functions with correlation kernels given by the kernel of the Weyl–Heisenberg ensemble [3, 5]. We will also consider Gaussian bi-analytic functions, with correlation kernel given by the sum of the reproducing kernels of the first two Landau levels. We close the presentation with a conclusion section, discussing the connections between the topics and results involved in the paper, and a few considerations about their symbiotic nature.

2 Background

2.1 Gaussian Entire Functions and Spectrograms of White Noise

Given a window function $g \in L^2(\mathbb{R})$, the short-time Fourier transform of $f \in L^2(\mathbb{R})$ is

$$V_g f(x, \xi) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i \xi t} dt, \quad (x, \xi) \in \mathbb{R}^2. \quad (2.1)$$

when $\|g\|_2 = 1$, the map $V_g$ is an isometry between $L^2(\mathbb{R})$ and a closed subspace of $L^2(\mathbb{R}^2)$:

$$\|V_g f\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})}, \quad f \in L^2(\mathbb{R}).$$

If we choose the Gaussian function $h_0(t) = 2^{1/4} e^{-\pi t^2}$, $t \in \mathbb{R}$, as a window in (2.1), then a simple calculation shows that, identifying $z = (x, \xi) \in \mathbb{R}^2$ with $z := x + i\xi \in \mathbb{C}$,

$$e^{-i\pi x\xi + \frac{\pi}{2}|z|^2} V_{h_0} f(x, -\xi) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi i z - \pi t^2 - \frac{\pi}{2} z^2} dt = B f(z), \quad (2.2)$$

where $B f(z)$ is the Bargmann transform of $f$ [41]. The Bargmann transform $B$ is a unitary isomorphism from $L^2(\mathbb{R})$ onto the Bargmann–Fock space $\mathcal{F}(\mathbb{C})$ consisting of all entire functions satisfying

$$\|F\|_{\mathcal{F}(\mathbb{C})}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} dz < \infty. \quad (2.3)$$
We will consider Hermite functions normalized as follows:

\[ h_r(t) = \frac{2^{1/4}}{\sqrt{r!}} \left( \frac{-1}{2\sqrt{\pi}} \right)^r e^{\pi t^2} \frac{dr}{dt} \left( e^{-2\pi t^2} \right), \quad r \geq 0, \]

Let \( g(t) := \frac{2^{1/4}}{\sqrt{\pi}} e^{-\pi t^2}, t \in \mathbb{R} \), be the one-dimensional, \( L^2 \)-normalized Gaussian. The short-time Fourier transform of the Hermite functions with respect to \( g \) is

\[ V_{g} h_k \left( \frac{z}{\sqrt{\pi}} \right) = e^{ix \xi} \frac{z^k}{\sqrt{k!}} e^{-|z|^2/2}, \quad k \geq 0. \quad (2.4) \]

One can formally define complex white noise \( \mathcal{W} \) by the series expansion

\[ \mathcal{W}(t) = \sum_{k=0}^{\infty} a_k h_k(t) \quad (2.5) \]

where \( a_k \) are i.i.d. Gaussian random variables (the probability of such sequences being square summable is zero, thus, with probability one, the series is not convergent in \( L^2 \), but with a little extra effort the argument can be made rigorous, by defining the sum \( (2.5) \) in a larger Banach space, see ([14, 15], Sect. 3) for a detailed approach). Then, by linearity of the Short-time Fourier transform \( V_g \), we have formally from \( (2.5) \) and \( (2.4) \)

\[ V_g \mathcal{W} \left( \frac{z}{\sqrt{\pi}} \right) = \sum_{k=0}^{\infty} a_k V_g h_k \left( \frac{z}{\sqrt{\pi}} \right) = e^{ix \xi} e^{-|z|^2/2} \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \quad (2.6) \]

### 2.2 Gaussian Entire Functions and Chern Critical Points

Gaussian entire functions are defined, for \( z \in \mathbb{C} \), as

\[ F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}}, \quad (2.7) \]

where \( a_k \) are i.i.d. Gaussian random variables, and can be related to the Short-time Fourier transform of white noise (see the formal manipulation \( (2.6) \) and [14, 15] for details) as follows:

\[ V_g \mathcal{W} \left( \frac{z}{\sqrt{\pi}} \right) = e^{ix \xi} e^{-|z|^2/2} \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \]

Since \( F(z) \) is an entire function, all local minima are zeros (see Proposition 1 in Sect. 4 for a proof). Moreover, the maximum principle rules out the possibility of finding local maxima for \( |F(z)| \). However, this is possible to do if we look at the extrema of white noise spectrograms. More precisely, at the critical, saddle and local maxima.
points and corresponding ordinate values, of the following random function:

\[ Spec_g \mathcal{W}(z) = \left| V_g \mathcal{W}(\frac{\bar{z}}{\sqrt{\pi}}) \right|^2 = \left| e^{-|z|^2/2} \sum_{k=0}^{\infty} a_k \frac{\bar{z}^k}{\sqrt{k!}} \right|^2 = \left| e^{-|z|^2/2} F(z) \right|^2 = H(z) \]

(2.8)

where \( V_g \) stands for the Short-Time Fourier Transform with a Gaussian window, or, equivalently, of

\[ \log \left( Spec_g \mathcal{W}(z) \right)^{\frac{1}{2}} = \log \left| F(z) e^{-\frac{|z|^2}{2}} \right| = \log |F(z)| - \frac{|z|^2}{2} = U(z). \]

The non-zero critical points of \( H(z) \) and \( U(z) \) are given by the solutions of the equation

\[ \nabla'_z F(z) = 0, \quad (2.9) \]

where \( \nabla'_z \) is the translation-invariant derivative (the symmetric part of the Chern connection, see Sect. 4 for more details):

\[ \nabla'_z = \partial_z - \bar{z} \]

leading us to consider the following Gaussian non-Entire Function,

\[ F_1(z) = \nabla'_z F(z) = \sum_{k=0}^{\infty} a_k \frac{\nabla'_z \bar{z}^k}{\sqrt{k!}} = \sum_{k=0}^{\infty} a_k \frac{(\partial_z - \bar{z}) \bar{z}^k}{\sqrt{k!}} = \sum_{k=0}^{\infty} a_k (k - |z|^2) \bar{z}^{k-1} \]

(2.10)

and the associated correlation kernel

\[ K(z, w) = \mathbb{E}(\nabla'_z F(z) \nabla'_z F(w)) = L_1^0(|z - w|^2) e^{z \bar{w}}, \]

where \( L_j^\alpha \) denotes the Laguerre polynomial

\[ L_j^\alpha(x) = \sum_{i=0}^{j} (-1)^i \binom{j + \alpha}{j - i} \frac{x^i}{i!}, \quad x \in \mathbb{R}, \quad j \geq 0, \quad j + \alpha \geq 0. \]

\( K(z, w) \) turns out to be the reproducing kernel of the eigenspace linked to the second eigenvalue of the Landau Laplacian with a magnetic field (the second Landau Level—see the last section of the paper for more details). The solutions of (2.9) are the Chern critical points for holomorphic functions in \( \mathbb{C} \), following the terminology of [23], where general Riemann surfaces have been considered.

In what follows, we will use the concept of average number of a point process \( \mathcal{X} \) on \( \mathbb{C} \), \( \mathcal{N}^{\mathcal{X}}(z) \), well known in point processes theory as the 1-point intensity (the most popular notation is \( \rho_1 \)) because it does not measure correlations with other points of the process. It simply gives the probability that \( z \in \mathcal{X} \). For a precise definition (for
details, we suggest consulting the introductory material in [17]), one considers the expectation of the random distribution
\[ \delta = \sum_{w \in X} \delta_w, \]
which can be defined weakly on test functions as
\[ \mathbb{E}(\{\delta, \varphi\}) = \int_{\mathbb{C}} \varphi(z) \mathcal{N}^{\mathcal{X}}(z) d\nu(z), \]
where \( d\nu(z) = \frac{1}{\pi} dz \), where \( dz \) stands for Lebesgue measure. For instance, given \( \Omega \subset \mathbb{C} \), denote by \( \mathcal{X}(\Omega) \) the number of points to fall in \( \Omega \) on a realization of \( \mathcal{X} \). To compute the expected number of points to fall in \( \Omega \subset \mathbb{C} \), one chooses \( \varphi(z) = 1_{\Omega}(z) \), leading to
\[ \mathbb{E}(\mathcal{X}(\Omega)) = \int_{\Omega} \mathcal{N}^{\mathcal{X}}(z) d\nu(z). \quad (2.11) \]
For the point process defined by the zeros of \( F(z) \), it is well known (from a trivial computation using Edelman–Kostlan formula [24], see [17]) that \( \mathcal{N}^{zer} = 1 \), which means that one zero is expected per unit area of the plane. It may be instructive to observe that Formula (2.11) allows to compute the density of the expected number of points. For instance, for the density of expected number of zeros, we choose \( \Omega \) as a disc of center \( z \) and radius \( R \) and obtain from (2.11) that \( \mathbb{E}(\mathcal{X}(D(z, R))) = \int_{D(z, R)} 1 d\nu(z) = R^2 \). Then,
\[ \lim_{R \to \infty} \sup_{z \in \mathbb{C}} \frac{\#\text{points expected in } D(z, R)}{|D(z, R)|} = \lim_{R \to \infty} \frac{\mathbb{E}(\mathcal{X}(D(0, R)))}{R^2} = 1. \]

3 Results and Comments

3.1 Critical, Saddle and Local Maxima Points

In the next result we compute the corresponding averages for the critical, saddle and local maxima points of \( H(z) \). We denote such averages by \( \mathcal{N}^{\text{crit}}, \mathcal{N}^{\text{sadd}} \) and \( \mathcal{N}^{\text{max}} \), respectively.

**Theorem 1** All local minima of \( H(z) \) are zeros of \( F(z) \) and none of them is a zero of \( \nabla_{z} F(z) \). The remaining critical points of \( H(z) \) are zeros of \( \nabla_{z} F(z) \). Their average number is \( \mathcal{N}^{\text{crit}} = 5/3 \), which can be divided in an expected average number of \( \mathcal{N}^{\text{max}} = 1/3 \) local maxima of \( H(z) \) and \( \mathcal{N}^{\text{sadd}} = 4/3 \) saddle points.

From (2.11), the expected number of local maxima in a set \( \Omega \subset \mathbb{C} \) with Lebesgue measure \( |\Omega| < \infty \) will then be given by
\[ \mathbb{E}(\mathcal{X}_{\text{max}}(\Omega)) = \frac{1}{3} |\Omega|. \]
Taking into account the identification between spectrograms of complex white noise with Gaussian windows and Gaussian Entire Functions provided by (2.8), we have

$$\text{Spec}_g \mathcal{W}(z) = H(z)$$

and

$$\log \left( \text{Spec}_g \mathcal{W}(z)^{\frac{1}{2}} \right) = U(z),$$

where $V_g$ stands for the Short-Time Fourier Transform with a Gaussian window $g(t) := 2^{1/4}e^{-\pi t^2}$, $t \in \mathbb{R}$. Under this identification, Theorem 1 can be reformulated as:

**Corollary 1** All local minima of $\text{Spec}_g \mathcal{W}(z)$ are zeros of $V_g \mathcal{W}$. The remaining critical points of $\text{Spec}_g \mathcal{W}(z)$ are zeros of $\nabla \text{Spec}_g \mathcal{W}(z)$. Their expected average number is $5/3$, which can be divided in an expected average number of $1/3$ local maxima and $4/3$ saddle points of $\text{Spec}_g \mathcal{W}(z)$.

This confirms a heuristic prediction of Patrick Flandrin, based on a conjectured double mean honeycomb model for the distribution of zeros and local maxima of Gaussian spectrograms, illustrated by the scheme of Fig. 1 below. According to such model, zeros define a mean honeycomb structure made of hexagons, which themselves define a larger honeycomb structure attached to local maxima, resulting in a distribution where the area of the local maxima centered hexagons is three times larger than the area of zero centered hexagons [besides Fig. 1, see also ([36], p. 148; Fig. 15.9) ([35], Fig. 1) ([37], Formula (22))]. Such a distribution would have a density of zeros three times bigger than the density of local maxima, as the above result demonstrates to be true. We note in passing that Sodin and Tsirelson [57], have shown that the random set with a triangular structure (slightly different from the perturbed hexagonal lattice)

$$\Lambda = \left\{ \sqrt{3\pi} (k + li) + ce^{2\pi im/3} \eta_{k,l} : k, l \in \mathbb{Z}; \ m = 0, 1, 2 \right\},$$

where $\eta_{k,l}$ are i.i.d Gaussian, achieves asymptotic similarity with the zeros of the GEF. Gathering the discussion of this paragraph, as a further step towards Flandrin’s conjecture, we conjecture that $\Lambda$ achieves asymptotic similarity with the local maxima of the GEF.

Hexagonal lattices are ubiquitous in Euclidean point configuration problems. In time-frequency analysis this is discussed in Feichtinger’s survey [27]. An important conjecture by Strohmer and Beaver [58] expects the condition number of (deterministic) Gabor frames with Gaussian windows to be optimized by a hexagonal lattice. Significant progress towards a proof has been made by Faulhuber and collaborators [26], and a preprint with a full solution of the problem has recently been posted in [18].

**Remark 1** In [15] the expected number of holomorphic critical points of $F(z)$ (given by the equation $F'(z) = 0$) has been computed using the Edelman–Kostlan formula, since $F'(z)$ is also an entire function. It coincides with the large $N$ limit of the number of saddle points of the chaotic analytic polynomial [22]. While sharing as common

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Fig. 1 Flandrin’s honeycomb: the area of the hexagons centered at local maxima (big dots) is three times the area of the hexagons centered at zeros (small dots).

aspect an increase in the number of saddles when compared to the number of zeros, the result is quite different from the above. The average, explicitly given as

\[ N_{\text{hol}}^{\text{crit}}(z) = \left(1 + (1 + |z|^2)^{-2}\right). \]

depends on \(|z|\), since the zero set of \(F'(z) = 0\) is rotation invariant (whence the radial average) but not translation invariant like \(\nabla_z F(z)\). In this case all solutions of the equation \(F'(z) = 0\) yield saddle points, due to the restriction imposed by the maximum modulus principle of an analytic function, which prevent the existence of local extrema. By considering \(\nabla_z F(z) = 0\) instead (which is equivalent to the zeros of the gradient of the full spectrogram, as we will see in the next section), we were able to provide a full description of the local extrema and saddle points of \(H(z)\).

Remark 2 Due to the factor \(|z|^2\) in (2.10), \(\nabla_z F(z)\) is not an entire function. Thus, one cannot take advantage of the simplifications leading to the Edelman-Kostlan formula and the direct use of Kac-Rice type formulas is necessary.

Remark 3 The fact that all local minima of \(H(z)\) are zeros of \(F(z)\) confirms what has been observed numerically in [14].
Remark 4 In [30], it has been shown that zeros of $F(z)$ and $\nabla'_z F(z)$ exhibit repulsion at small scales (when their distance tends to zero). The impossibility of $F(z)$ and $\nabla'_z F(z)$ having common zeros (see Proposition 1 for a proof) is another manifestation of such repulsion.

Remark 5 A striking result of Ghosh and Nishry [39] shows that the zero set of $F(z)$, conditioned on the rare event of a disc with no zeros with radius $r$, converges (scaled) to an explicit limiting Radon measure with a large “forbidden region” between a singular part supported on the boundary of the (scaled) hole and the equilibrium measure far from the hole. Given the structure of zeros and local maxima suggested by Flandrin’s honeycomb network, partially supported by our results, it would be interesting to know what happens to the critical/local maxima points under the same rare event conditioning. At the moment we have no hint on what to expect. Technically, the main obstruction in adapting the methods used in [39] seems to be the absence of Jensen’s formula for $\nabla'_z F(z)$ (a bi-analytic function). The same obstruction shows up if one attempts to adapt the known proofs for upper bounding the probability of existence of a disc without local maxima (the so-called hole probability).

Also related to the above results are two questions of Nazarov, Sodin and Volberg in [50, 12.3] and [51, 2.2.3], where it is also suggested that the techniques used in [23] should also work here. Indeed, the proof of Theorem 1 consists of a simplification (possible in this case) of the methods used in [23], where the full calculations have been performed in one dimension for an elliptic model of Gaussian polynomials. In our case, results from analysis in one complex variable can be applied to rule out the possibility of local minima, simplifying the discussion on Morse indexes (see [23] and [29]) to the sign of the Hessian determinant at critical points. We give now a glimpse of the questions stated in [50, 12.3] and [51, 2.2.3] (see also the comments starting on page 146 of [36] for a time-frequency perspective combined with other ideas, such as differential reassignment and Voronoi tessellation).

The origin of the problem lies on the idea of gravitational allocation. Letting the Lebesgue measure fall by gravity and descend along gradient lines to the zeros of $F(z)$, one is lead to an evolution described by a stochastic gradient flow involving the log-spectrogram $U(z)$ as a potential:

$$\frac{dZ(t)}{dt} = -\nabla U(Z(t)).$$

Denoting by $\Gamma_z$ the corresponding gradient curves that pass through $z$ (assumed to be neither a zero of $F(z)$ nor a critical point of $U(z)$), a random partition of the plane is obtained:

$$\mathbb{C} = \bigcup_{a:F(a)=0} B(a),$$

where $B(a)$ is the basin of attraction of $a$: the set of points $z$ such that $\Gamma_z$ terminates at $a$. Now, associate to each $z \in \mathbb{C}$, the basin such that $z \in B(a_z)$ and define two basins as neighbors if they have a common gradient curve on their boundary. Then $N_z$ is the number of basins neighbors of $B(a_z)$, which equals the number of saddle points of
U(z) connected by gradient curves to a_z. Since almost surely each saddle is connected with two zeros, at least heuristically, we have:

\[ \mathbb{E}N_z = (1) \frac{2}{\text{expected average number of saddle points}} \frac{2}{\text{expected number of zeros}} = R_{sadd/\text{zeros}}. \]

By Theorem 1, \( R_{sadd/\text{zeros}} = \frac{8}{3} \). Likewise, the number of basins that meet at the same local maximum is

\[ \mathbb{E}N_z^{\text{max}} = (2) \frac{2}{\text{expected average number of saddle points}} \frac{2}{\text{expected number of local maxima}} = R_{sadd/\text{local max}}. \]

By Theorem 1, \( R_{sadd/\text{local max}} = 8 \).

**Remark 6** To provide an answer for the questions in [50] it is necessary to show the identities (1) and (2).

In another direction, the connections explored in this note may shed some light regarding the ubiquitous presence of the factor 1/3 in the first terms of the expansion of the number of critical points under the so-called Chern connection which, for entire functions in the Fock space, boils down to the operator \( \nabla'_z \). Indeed, in the comments after formulas (15) and (16) of [23], Douglas, Shiffman and Zelditch mention that ‘It would be interesting to find a heuristic reason for the factor 1/3’. Flandrin’s mean honeycomb model provides such an heuristic reason in the planar case, and suggests that a geometric analogue of such model may exist for the expected distribution of zeros and maxima of random functions in general complex Riemann surfaces.

### 3.2 Critical, Saddle and Local Maxima Ordinate Values

The observations leading to the proof of Theorem 1 allow, with a slight extra effort, to understand the average distribution of the ordinate values of the critical, saddle and local maxima. By an ordinate critical value we mean the value of \( (\text{Spec}_g \mathcal{W}(z))^{1/2} \), at the critical point \( z_c \)

\[ x_c = \left( \text{Spec}_g \mathcal{W}(z_c) \right)^{1/2} = |F(z_c)| e^{-\frac{|z_c|^2}{2}} \in \mathbb{R}^+ \]

Formally, the expected density of the average ordinates of critical values of local maxima of \( (\text{Spec}_g \mathcal{W}(z))^{1/2} \), \( D(x) \), is defined weakly on a compact set \( \Omega \subset \mathbb{C} \) for a test function \( \psi \) by the identity:

\[
\mathbb{E} \left( \frac{1}{|\Omega|} \sum_{\xi \in \Omega : \nabla'_{\xi} F(\xi) = 0} \delta \left| F(\xi) e^{-\frac{|\xi|^2}{2}} \right|, \psi \right) = \mathbb{E} \left( \frac{1}{|\Omega|} \sum_{\xi \in \Omega : \nabla'_{\xi} F(\xi) = 0} \psi \left| F(\xi) e^{-\frac{|\xi|^2}{2}} \right| \right) = \int_{\mathbb{R}^+} \psi(x) D(x) dx
\]
and similarly for saddles and local maxima. The following theorem, which is our second main result, describes the expected density distribution of ordinate values of critical, local maxima and saddle points (which are not probability distributions). The proof is based on methods of Feng and Zelditch [28, 29].

**Theorem 2** The expected density of the average ordinate critical values of local maxima of \( (\text{Spec}_g W(z))^{1/2} \) is given by the function \( D^{\text{max}} \), defined on \([0, \infty)\) by

\[
D^{\text{max}}(x) = 2x \left( x^2 - 2 + 2e^{-\frac{x^2}{2}} \right) e^{-x^2},
\]

of the ordinate values of saddle points, by

\[
D^{\text{sadd}}(x) = 4xe^{-\frac{3}{2}x^2},
\]

and of the ordinates of critical values by

\[
D^{\text{crit}}(x) = D^{\text{sadd}}(x) + D^{\text{max}}(x) = 2x \left( x^2 - 2 + 4e^{-\frac{x^2}{2}} \right) e^{-x^2}.
\]

**4 Proof of Theorems 1 and 2**

**4.1 Critical Points Under the Chern Connection**

The so called Chern connection under the Bargmann–Fock metric is given as

\[
\nabla_z = \nabla'_z + \nabla''_z
\]

with

\[
\nabla'_z = \partial_z - \bar{z}
\]

and

\[
\nabla''_z = \partial_{\bar{z}}.
\]

Following [23], we define a critical point of \( F(z) \) under the Chern connection as a solution of

\[
\nabla_z F(z) = 0.
\]

If \( F(z) \) is entire as in the case of the GEF, then \( \nabla''_z F(z) = 0 \) and \( \nabla_z F(z) = \nabla'_z F(z) \), simplifying the critical points equation:

\[
\nabla'_z F(z) = \partial_z F(z) - \bar{z}F(z) = 0.
\]

**Proposition 1** The nonzero critical points of \( H(z) \) and \( U(z) \) are given by the equation \( \nabla'_z F(z) = 0 \). All nonzero critical points which are local extrema points of \( H(z) \) and \( U(z) \) are necessarily local maxima. Thus, all local minima are zeros of \( F(z) \) and cannot be critical points given by the equation \( \nabla'_z F(z) = 0 \).
**Proof** The critical points equation

\[
0 = \left| \nabla \log \left| F(z)e^{-|z|^2} \right| \right| = \left| \frac{\partial_z F(z)}{F(z)} - \frac{\bar{z}}{2} \right|,
\]

where

\[
\nabla f = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}},
\]

is equivalent, for \( F(z) \neq 0 \), to

\[
\partial_z F(z) - \bar{z} F(z) = 0 \iff \nabla'_z F(z) = 0
\]

Likewise, the critical point equation

\[
0 = \left| \nabla \left| F(z) \right|^2 e^{-|z|^2} \right| = 2e^{-|z|^2} \overline{F}(z)(\nabla_z F(z))
\]

is equivalent, for \( F(z) \neq 0 \), to \( \nabla'_z F(z) = 0 \). Thus, the critical points of

\[
H(z) = \left| F(z) \right|^2 e^{-|z|^2}
\]

are the same as those of

\[
U(z) = \log \left| F(z)e^{-\frac{|z|^2}{2}} \right| = \log \left| F(z) \right| - \frac{|z|^2}{2}
\]

Now observe that, if \( F(z) \neq 0 \), the Laplacian of \( \log \left| F(z)e^{-\frac{|z|^2}{2}} \right| \) is negative:

\[
\Delta \log \left| F(z)e^{-\frac{|z|^2}{2}} \right| = \Delta \log \left| F(z) \right| - 2 = -2.
\]

This implies that \( \log \left| F(z)e^{-\frac{|z|^2}{2}} \right| \) is super harmonic outside the zero set of \( F \). Thus, all local minima of \( \log \left| F(z)e^{-\frac{|z|^2}{2}} \right| \) are attained when \( F(z) = 0 \), and the same happens with the local minima of \( \left| F(z) \right|^2 e^{-|z|^2} \). As a result, all nonzero critical points which are local extrema points of \( H(z) \) and \( U(z) \) are local maxima. Finally, by [17, Lemma 2.4.1] all the zeros of \( F(z) \) are simple. It follows that if \( F(z) = 0 \) then \( \nabla'_z F(z) \neq 0 \).

The proofs of Theorems 1 and 2 in the next sections will use Kac-Rice formulas [8] and Hessians [7]. They are inspired by the scheme of [23, 28, 29], but in our much simple setting, we managed to obtain a direct, more elementary proof.
4.2 Proof of Theorem 1

The average of critical points is given by the expected number of zeros of $\nabla_z' F(z) = 0$.

Since $\nabla_z' F(z)$ is not analytic, to compute the expected number of its zeros one cannot use the Edelman-Kostlan formula, as usually done for Gaussian analytic functions.

The key of the proof is to start with an appropriate Hessian matrix whose determinant sign allows to separate maxima, minima and saddles, and show that, at the critical points $\nabla_z' F(z_c) = 0$, this Hessian determinant is proportional to $\text{det} DF(z_c)$, which, under the critical point $\nabla_z' F(z_c) = 0$ constraint, equals the determinant of the complex differential matrix which appears on the Kac-Rice formula for the average of critical points.

The nonzero critical points of $U(z) = \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|$ are given by $\nabla_z' F(z) = 0$.

The same is true for $2U(z) = \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|^2$. The corresponding Hessian matrix is

$$D^{Hess}(2U)(z) = \begin{bmatrix}
\partial_z \partial_z \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|^2 & \partial_z \partial_z \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|^2 \\
\partial_z \partial_z \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|^2 & \partial_z \partial_z \log \left| F(z) e^{-\frac{|z|^2}{2}} \right|^2 
\end{bmatrix}$$

$$= \begin{bmatrix}
\partial_z \left( \frac{1}{F(z)} \nabla_z' F(z) \right) & \partial_z \left( \frac{1}{F(z)} \nabla_z' F(z) \right) \\
\partial_z \left( \frac{1}{F(z)} \nabla_z' F(z) \right) & \partial_z \left( \frac{1}{F(z)} \nabla_z' F(z) \right)
\end{bmatrix}.$$

Observing that

$$\partial_z \left( \frac{1}{F(z)} \nabla_z' F(z) \right) = -\frac{\partial_z F(z)}{F(z)^2} \nabla_z' F(z) + \frac{1}{F(z)} \partial_z \left( \nabla_z' F(z) \right),$$

one can evaluate the entries of the above matrix at the critical points $\nabla_z' F(z_c) = 0$, leading to

$$\text{det} D^{Hess}(2U)(z_c) = \text{det} \begin{bmatrix}
\frac{1}{F(z_c)} \partial_z \nabla_z' F(z_c) & \frac{1}{F(z_c)} \partial_z \nabla_z' F(z_c) \\
\partial_z \nabla_z' F(z_c) & \partial_z \nabla_z' F(z_c)
\end{bmatrix}$$

$$= \frac{1}{|F(z_c)|^2} \text{det} \begin{bmatrix}
\nabla_{z'}' \nabla_z' F(z_c) & \nabla_{z'}' \nabla_z' F(z_c) \\
\nabla_z' \nabla_z' F(z_c) & \nabla_z' \nabla_z' F(z_c)
\end{bmatrix}.$$

We have concluded that

$$|F(z_c)|^2 \text{det} D^{Hess}(2U)(z_c) = |\nabla_{z'}' \nabla_z' F(z_c)|^2 - |\nabla_z' \nabla_z' F(z_c)|^2 \quad (4.1)$$
coincides with the determinant of $D(F)(z_c)$ at the critical points $\nabla'_z F(z_c) = 0$, where

$$D(F)(z) = \begin{bmatrix} \nabla''_z \nabla'_z F(z) & \nabla''_z \nabla''_z F(z) \\ \nabla'_z \nabla'_z F(z) & \nabla''_z \nabla'_z F(z) \end{bmatrix}. \quad (4.2)$$

Thus, $\det D^{Hess}(2U)(z_c)$ and $\det DF(z_c)$ have the same sign. This allows to separate maxima, minima and saddles as follows. If $\det DF(z_c) > 0$, then both eigenvalues of $DF(z)$ (and consequently, of $\det D^{Hess}(2U)(z_c)$) are negative (if they were both positive then $z_c$ would be a local minima, and this is not possible by Proposition 2.1). Thus, one has a non-zero local extrema that cannot be a local minimum and $z_c$ is a local maximum. Thus, $z$ is a local maximum if

$$\det DF(z) > 0,$$

or

$$|\nabla''_z \nabla'_z F(z)|^2 > |\nabla'_z \nabla''_z F(z)|^2. \quad (4.3)$$

Likewise, if

$$\det DF(z) < 0,$$

or, equivalently, if

$$|\nabla''_z \nabla'_z F(z)|^2 < |\nabla'_z \nabla''_z F(z)|^2, \quad (4.4)$$

then $DF(z)$ has both positive and negative eigenvalues and $z$ is a saddle point. Thus, saddle points are characterized by the condition $(4.4)$. Furthermore, at a critical point $\nabla'_z F(z_c) = 0$,

$$\det DF(z_c) = \det \begin{bmatrix} \nabla''_z \nabla'_z F(z_c) & \nabla''_z \nabla''_z F(z_c) \\ \nabla'_z \nabla'_z F(z_c) & \nabla''_z \nabla'_z F(z_c) \end{bmatrix} = \det \begin{bmatrix} \partial_z \nabla'_z F(z_c) & \partial_z \nabla''_z F(z_c) \\ \partial_z \nabla'_z F(z_c) & \partial_z \nabla''_z F(z_c) \end{bmatrix},$$

We will now consider the 3-dimensional Gaussian process

$$(\nabla'_z F(z), \nabla'_z \nabla'_z F(z), \nabla''_z \nabla'_z F(z)) \quad (4.5)$$

with correlation kernel

$$K(z, w) = \mathbb{E}(\nabla'_z F(z) \nabla'_z F(w)) = \nabla'_z \nabla'_z e^{z \nabla'_z w} = L_1^0(|z - w|^2) e^{z \nabla'_z w},$$

Using the vector $(4.5)$, the Kac-Rice formula for the average of critical points can be given as the conditional expectation of $\det D(\nabla'_z F)$ at a critical point as in [23, (90)] or using [8, Theorem 6.2] as in [42, Lemma 2.3] (the $\frac{1}{\pi}$ factor is included in the measure $d\nu(z)$ of (2.11)). Thus,

$$\mathcal{N}^{crit} = \mathbb{E}(|\det DF| \delta_0(\nabla'_z F)). \quad (4.6)$$

Gathering the contents of this discussion, once we compute the Gaussian density $p(0, v, u)$, by $(4.3)$, the expected number of local maxima can be obtained using the
following Kac-Rice formula:

\[
\mathcal{N}_{\max} = \mathbb{E}(|\det D(F) : \det D(F) > 0| \delta_0(\nabla_z' F)) \\
= \mathbb{E}\left[|\nabla_z'' \nabla_z' F|^2 \right. \\
\left. - \left|\nabla_z' \nabla_z F(z)\right|^2 > \left|\nabla_z' \nabla_z F(z)\right|^2 \delta_0(\nabla_z' F)\right] \\
= \int_{\mathbb{C}^2 \cap \{|u|^2 > |v|^2\}} (|u|^2 - |v|^2) p(0, v, u) dv du.
\] (4.7)

and from (4.4), the expected number of saddle points is

\[
\mathcal{N}_{\text{sadd}} = \mathbb{E}(|\det D(F) : \det D(F) < 0| \delta_0(\nabla_z' F)) \\
= \mathbb{E}\left[|\nabla_z'' \nabla_z' F|^2 \right. \\
\left. - \left|\nabla_z' \nabla_z F(z)\right|^2 < \left|\nabla_z' \nabla_z F(z)\right|^2 \delta_0(\nabla_z' F)\right] \\
= \int_{\mathbb{C}^2 \cap \{|u|^2 < |v|^2\}} (|v|^2 - |u|^2) p(0, v, u) dv du.
\] (4.8)

The Gaussian measure \( p(0, v, u) \) is given in terms of the covariance matrix

\[
\Lambda = \begin{pmatrix}
1 & \nabla_z' \\
\nabla_z'' & (1 \nabla_w' \nabla_w'T) K(z, w)
\end{pmatrix}
\]

and its inverse by

\[
p(0, v, u) = e^{-\frac{\left\langle \begin{pmatrix} 0 \\ v \\ u \end{pmatrix}, \Lambda^{-1} \begin{pmatrix} 0 \\ v \\ u \end{pmatrix} \right\rangle}{\det \Lambda}}.
\]

To compute \( p(0, v, u) \) we observe that

\[
K(z, z) = e^{|z|^2}
\]

\[
\nabla_z' K(z, w)|_{z=u} = 0
\]

\[
\nabla_z'' K(z, w)|_{z=u} = 0
\]

\[
\nabla_z'' \nabla_w'' K(z, w)|_{z=u} = e^{|z|^2}
\]

\[
\nabla_z' \nabla_w' K(z, w)|_{z=u} = 2e^{|z|^2}
\]

\[
\nabla_z' \nabla_w' K(z, w)|_{z=u} = 0,
\]

and we multiply the matrix \( \Lambda \) by \( e^{-|z|^2} \), since this keeps the measure \( p(0, v, u) dv du \) invariant, leading to the following covariance matrix and respective inverse:

\[
\Lambda = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
We arrive thus at
\[ p(0, v, u) = \frac{1}{2} e^{-\frac{|v|^2}{2} - |u|^2}. \]

For the local maxima, we obtain from (4.7):
\[
N_{\text{max}}(z) = \frac{1}{2} \int_{\mathbb{C}^2 \cap \{|u|^2 > |v|^2\}} (|u|^2 - |v|^2) e^{-\frac{|u|^2}{2} - |u|^2} dvdu
= \int_{x>0} \int_{t>2x} (t - 2x) e^{-t-x} dt dx
= \int_{\mathbb{R}^+} e^{-3x} dx = \frac{1}{3}.
\]

Likewise, (4.8) gives the average number of saddle points:
\[
N_{\text{sadd}}(z) = \frac{1}{2} \int_{\mathbb{C}^2 \cap \{|u|^2 < |v|^2\}} (|v|^2 - |u|^2) e^{-\frac{|u|^2}{2} - |u|^2} dvdu
= \int_{x>0} \int_{t<2x} (2x - t) e^{-t-x} dt dx
= 2 \int_{\mathbb{R}^+} e^{-3/2x} dx = \frac{4}{3}.
\]

Now, from Proposition 1 all nonzero critical points which are local extrema points of \( H(z) \) and \( U(z) \) are necessarily local maxima. Thus, critical points can only be saddles or local maxima and the expected number of critical points is:
\[ N_{\text{crit}} = N_{\text{sadd}} + N_{\text{max}} = 1/3 + 4/3 = 5/3. \]

Thus, we have an average of 5/3 critical points. Among those, 1/3 are local maxima and 4/3 saddle points.

### 4.3 Proof of Theorem 2

We consider the empirical point measure defined by the values of \( |F(\bar{\xi})| e^{-\frac{|\xi|^2}{2}} \) of critical points for \( \bar{\xi} \) in a compact set \( \Omega \subset \mathbb{C} \), as the finite sum normalized by the Lesbegue measure of \( \Omega \), \( |\Omega| \):
\[
\frac{1}{|\Omega|} \sum_{\bar{\xi} \in \Omega : \nabla_{\bar{\xi}} F(\bar{\xi}) = 0} \delta_{|F(\xi)e^{-\frac{|\xi|^2}{2}}|}.
\]

The density of the ordinates of critical values, \( D(x) \), is defined weakly by the identity:
\[
\mathbb{E}\left( \frac{1}{|\Omega|} \sum_{\bar{\xi} \in \Omega : \nabla_{\bar{\xi}} F(\bar{\xi}) = 0} \delta_{|F(\bar{\xi})e^{-\frac{|\bar{\xi}|^2}{2}}|} , \psi \right) = \mathbb{E}\left( \frac{1}{|\Omega|} \sum_{\bar{\xi} \in \Omega : \nabla_{\bar{\xi}} F(\bar{\xi}) = 0} \psi \left| F(\bar{\xi})e^{-\frac{|\bar{\xi}|^2}{2}} \right| \right).
\]
\[ = \int_{\mathbb{R}^+} \psi(x) D(x) dx \]

We first consider the density \( D^\wedge(u) \), defined by

\[
\mathbb{E} \left( \frac{1}{|\Omega|} \sum_{\xi \in \Omega : \nabla'_\xi F(\xi) = 0} \delta_{F(\xi)} e^{-|\xi|^2/2}, \psi \right) = \int_{\mathbb{C}} \psi(u) D^\wedge(u) du
\]

and then we will integrate out the angle variable by \( D(x) = \int_{0}^{2\pi} D^\wedge(x, \theta) d\theta \), after observing that, for radial \( \psi \),

\[
\int_{\mathbb{R}^+} \psi(x) D(x) dx = \mathbb{E} \left( \frac{1}{|\Omega|} \sum_{\xi \in \Omega : \nabla'_\xi F(\xi) = 0} \psi \left| F(\xi) e^{-|\xi|^2/2} \right| \right) = \int_{\mathbb{C}} \psi(u) D^\wedge(u) du.
\]

When \( \nabla'_z F(z) = 0 \),

\[
\nabla'_{z} (e^{-|z|^2/2} \nabla'_{z} F(z)) = e^{-|z|^2/2} \nabla'_{z} (\nabla'_{z} F(z)) - \frac{z}{2} e^{-|z|^2/2} \nabla'_{z} F(z) = e^{-|z|^2/2} \nabla'_{z} \nabla'_{z} F(z)
\]

and

\[
\nabla''_{z} e^{-|z|^2/2} \nabla'_{z} F(z) = e^{-|z|^2/2} \nabla''_{z} (\partial_z F(z) - \tilde{z} F(z)) - \frac{z}{2} e^{-|z|^2/2} \nabla'_{z} F(z) = e^{-|z|^2/2} \nabla'_{z} \nabla'_{z} F(z)
\]

\[
= -e^{-|z|^2/2} F(z),
\]

so that we can write the 3-dimensional Gaussian process

\[
(e^{-|z|^2/2} \nabla'_{z} F(z), \nabla'_{z} (e^{-|z|^2/2} \nabla'_{z} F(z)), \nabla''_{z} (e^{-|z|^2/2} \nabla'_{z} F(z))) = (e^{-|z|^2/2} \nabla'_{z} F(z), e^{-|z|^2/2} \nabla'_{z} F(z), -e^{-|z|^2/2} F(z)),
\]

The Kac-Rice formula is not changed by the resulting multiplication of the correlation kernel by \( e^{-|z|^2/2 - |w|^2/2} \). To obtain the density of the ordinates of critical values, using [8, Theorem 6.4] gives

\[
\mathbb{E} \left( \frac{1}{|\Omega|} \sum_{\xi \in \Omega : \nabla'_\xi F(\xi) = 0} \psi \left( F(\xi) e^{-|\xi|^2/2} \right) \right)
\]

\[
= \frac{1}{\pi |\Omega|} \int_{\Omega} \mathbb{E} \left[ \left| e^{-|z|^2/2} \nabla'_{\xi} \nabla'_{\xi} F(\xi) \right|^2 - \left| e^{-|z|^2/2} F(\xi) \right|^2 \right] \psi(e^{-|z|^2/2} F(\xi)) \delta_0(\nabla'_\xi F(\xi)) d\xi
\]

\[
= \frac{1}{\pi |\Omega|} \int_{\mathbb{C}^2} \psi(u) \left| u \right|^2 - \left| u \right|^2 \left| p(0, u) \right| du d\xi.
\]
\[
= \int_{\mathcal{C}} \psi(u) \left[ \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathcal{C}} \left| |v|^2 - |u|^2 \right| p(0, v, u) d\xi d\nu \right] du
\]

Thus, from (4.9),
\[
D^\wedge(u) = \frac{1}{\pi |\Omega|} \int_{\Omega} \int_{\mathcal{C}} \left| |v|^2 - |u|^2 \right| p(0, v, u) d\xi d\nu
\]

and, from the previous section,
\[
p(0, v, u) = \frac{1}{2} e^{-|v|^2/2 - |u|^2}.
\]

Writing \( u \) in polar coordinates \((x, \theta)\) as \( u = e^{i\theta} x, x \geq 0, \)
we obtain the expected density as a function of the ordinates \( x \in \mathbb{R}^+ \):
\[
D(x) = \int_0^{2\pi} D^\wedge(x, \theta) x d\theta.
\]

Now,
\[
D(x) = \frac{1}{2\pi |\Omega|} \int_0^{2\pi} \int_{|\Omega|} \int_{\mathcal{C}} \left| |v|^2 - |x|^2 \right| x e^{-|v|^2/2 - |x|^2} d\xi d\eta
\]
\[
= 2xe^{-|x|^2/2} \int_{\mathcal{C}} \left| 2|v|^2 - |x|^2 \right| e^{-|v|^2} d\nu
\]
\[
= 2xe^{-|x|^2} \int_{t>0} \left| 2t - |x|^2 \right| e^{-t} dt
\]
\[
= 2xe^{-|x|^2} \left( \int_{t>|x|^2/2} \left( 2t - |x|^2 \right) e^{-t} dt + \int_{t<|x|^2/2} \left( |x|^2 - 2t \right) e^{-t} dt \right).
\]

For the first integral in brackets,
\[
\int_{t>|x|^2/2} \left( 2t - |x|^2 \right) e^{-t} dt = \int_{t>|x|^2/2} 2te^{-t} dt - |x|^2 \int_{t>|x|^2/2} e^{-t} dt
\]
\[
= 2(1 + \frac{|x|^2}{2})e^{-|x|^2/2} - |x|^2 e^{-|x|^2/2}
\]
\[
= 2e^{-|x|^2/2}
\]

while the second gives
\[
\int_{t<|x|^2/2} \left( |x|^2 - 2t \right) e^{-t} dt = |x|^2 + 2e^{-|x|^2/2} - 2.
\]
Thus, using the arguments of the previous section to separate saddle points from local maxima, we have

\[ D^{sadd}(x) = 4xe^{-\frac{3}{2}x^2} \]

and

\[ D^{\text{max}}(x) = 2x \left( x^2 - 2 + 2e^{-\frac{x^2}{2}} \right) e^{-x^2}. \]

The total density of critical values is then given by

\[ D^{\text{crit}}(x) = D^{sadd}(x) + D^{\text{max}}(x) = 2x \left( x^2 - 2 + 4e^{-\frac{x^2}{2}} \right) e^{-x^2}. \]

5 Gaussian Non-Analytic Functions

5.1 Gaussian Functions in the Second Landau Level

In this section we will rephrase the results on zeros and critical points of a GEF within the realm of the euclidean Landau levels [10], a popular mathematical physics model used to describe the quantum dynamics of charged-like particles, which, under the presence of a constant perpendicular magnetic field (which will be assumed constant \( B = 1 \)), organize themselves in layers with unit increase of energy measured by the corresponding eigenvalues (the terminology Landau levels is mostly used to refer to such eigenvalues). Each such Landau level eigenvalue \( r \) has an associated Hilbert space with reproducing kernel, which can be used to define planar wave functions for repulsive interacting particles distributed in a particular energy level. This leads to an integer indexed sequence of determinantal point processes in the plane [56], an important class of Weyl–Heisenberg ensembles [3], which boil down to the Ginibre ensemble in the case \( r = 0 \). The increase in energy required for the allocation of particles in higher Landau levels was for a long time the mainstream physical explanation for the plateau-like pattern of perfect integer quantizations observed in the Integer Quantum-Hall Effect [60], when the electron filling factor of the observed sample saturates its bounded region degeneracy (corresponding to the dimension of the sample as a vector space conditioned by the Pauli exclusive allocation of an electron per unit cell). The integer jumps are thus explained by a simple electron density model, requiring no understanding of the higher order correlations. Thus, they can be described by the behavior of the single particle electron density (1-point intensity), which has been shown to have universal circular-law type behavior in all Landau levels, as a special case of the limit law of finite Weyl–Heisenberg ensembles [5], a geometrical flexible indicator function, initially studied with the aim of approximating localization regions (or, in the language of anti-Wick operators, of quantization symbols) by sums of spectrograms with windows orthogonal in a particular strong sense [4].

Back to the recurring theme of the paper, the critical points of (2.7) coincide with the zeros of the random analytic function

\[ F_1(z) = \nabla' z F(z), \quad (5.1) \]
with correlation kernel
\[ K_1(z, w) = \mathbb{E}(F_1(z)\overline{F_1(w)}) = \nabla'_z \overline{\nabla'_w} e^{z\overline{w}} = L_1(|z - w|^2)e^{z\overline{w}}. \] (5.2)

The Landau operator acting on the Hilbert space \( L^2\left(\mathbb{C}, e^{-|z|^2}\right) \) can be defined as minus the composition of the symmetric and non-symmetric part of the Chern connection:
\[ L_z := -\partial_z \partial_{\overline{z}} + \overline{z} \partial_{\overline{z}} = -\nabla'_z \nabla''_z. \] (5.3)

The spectrum of \( L_z \) is given by \( \sigma(L_z) = \{ r : r = 0, 1, 2, \ldots \} \).

We now observe that the symmetric part of the Chern connection, \( \nabla'_z = \partial_z - \overline{z} \) is precisely the inter Landau levels raising operator (maps eigenfunctions of (5.3) with eigenvalue \( r \) to eigenfunctions with eigenvalue \( r + 1 \)), while the anti-symmetric part \( \nabla''_z = \partial_{\overline{z}} \) is the corresponding lowering operator. Denote by \( F^r_2(\mathbb{C}) \) the space of functions which can be written as
\[ f_r(z) = (\nabla'_z)^r f(z), \]
for some \( f \in F^r_2(\mathbb{C}) \). Then \( F^r_2(\mathbb{C}) \) is the eigenspace of \( L_z \) associated with the eigenvalue \( r \) and the corresponding reproducing kernel is
\[ K_r(z, w) = (\nabla'_z)^r (\overline{\nabla'_w})^r e^{z\overline{w}} = r! L_r(|z - w|^2)e^{z\overline{w}}. \]

Setting \( r = 0 \) gives the correlation kernel of the GEF (2.7), while \( r = 1 \) leads to the correlation kernel (5.2) of the Chern connection critical points for the random Gaussian function (5.1). As a result, the distribution of critical points of a GEF coincides with the distribution of zeros of a Gaussian random function in the second Landau level \((r = 1)\).

### 5.2 Spectrograms of White Noise with the First Hermite Window

The random function \( F_1(z) \) can also be understood as the STFT of white noise, since
\[ Spec_{g}\mathcal{W}(z_c) = \left| V_{h_1}\mathcal{W}(\frac{z_c}{\pi}) \right|^2 \leq \left| e^{-|z|^2/2} \sum_{k=0}^{\infty} a_k \frac{(k - |z|^2)_{z-1}}{\sqrt{k!}} \right|^2 = \left| e^{-|z|^2/2} F_1(z) \right|^2. \]

Thus, Theorem 1 implies that \( Spec_{h_1}\mathcal{W}(z) \) has an average number of 5/3 zeros. Heuristically, one can think that the zero-crossings of the Hermite window \( h_1 \) will be reflected in the spectrogram, leading to an increase of the number of zeros. Following this reasoning, it is likely to expect a monotone increasing in the average number of zeros of the spectrogram with the order (which is fine tuned with the number of real zero-crossings) of the Hermite window. These heuristics have been confirmed in [42, Corollary 1.10], in work done simultaneously and independently from this, where the result \( r + 1/2 + \frac{1}{4r+2} \) is obtained for the the average number of zeros of \( V_h, \mathcal{W}(\frac{z}{\pi}) \).
Remark 7 By Proposition 1, all local maxima of \(|V_{h_0} W(\bar{z})|\) are zeros of \(|V_{h_1} W(\bar{z})|\). Thus, the local maxima of \(|V_{h_0} W(\bar{z})|\) and of \(|V_{h_1} W(\bar{z})|\) are intertwined, confirming the observations in [36, Fig. 10.4] for \(r = 1\). However, it still remains an open question how to proof the suggested intertwining property of local maxima for general \(r\).

5.3 Gaussian Weyl–Heisenberg Functions

In this section we will consider the usual assumption \(a_k \sim N_C(0, 1)\) i.i.d. The previous section suggests considering the STFT of white noise with a general normalized window \(g \in L^2(\mathbb{R})\):

\[ V_g W\left(\frac{\bar{z}}{\pi}\right) = \sum_{k=0}^{\infty} a_k V_g h_k\left(\frac{\bar{z}}{\sqrt{\pi}}\right). \]

This leads us to a family of Gaussian functions with correlation kernel

\[ K_g(z, z') = V_g g\left(\frac{\bar{z} - \bar{z}'}{\pi}\right) = \int_{\mathbb{R}} e^{-2i(y'-y)t} (t - x'/\pi) g(t - x/\pi) dt. \]

The first two intensities of Gaussian Weyl–Heisenberg functions have been computed in [42]. Setting \(g = h_0\) we obtain, up to a phase factor, the correlation kernel of the GEF (corresponding to the Fock kernel, or the lowest Landau level):

\[ K_{h_0}(z, z') = e^{i(x'y' - xy)} e^{-\frac{1}{2}(|z|^2 - |z'|^2)} e^{zz'}. \]

The choice \(g = h_r\) leads to a similar relation with the higher Landau level kernels:

\[ K_{h_r}(z, z') = e^{i(x'y' - xy)} e^{-\frac{1}{2}(|z|^2 - |z'|^2)} e^{zz'} L_r(|z - z'|^2). \]

Determinantal point processes with the kernel \(K_g(z, z')\) have been studied in [3, 5], under the name of Weyl–Heisenberg ensembles.

In general, the non-analiticity of the basis functions creates an obstruction in several of the arguments used in the fundamental results about Gaussian Entire Functions, as it should be clear for a reader familiar with the engrossing presentation of the topic in [17]. For instance, the arguments used in Proposition 1 break down if we try to look at the Chern connection critical points of \(V_g W\) when \(g\) is not a Gaussian (the only window leading to entire functions, see [9]). Thus, while it is clear how to obtain the zero-intensities, the analysis leading to the statistics of local extrema may pose some obstructions when trying to separate local minima from local maxima for more general windows.

5.4 Short-Time Fourier Transforms with White Noise Windows

In some signal analysis applications it may be useful to consider a random window, as done in [53, 55, Sect. 8], where a ‘white noise window’ has been used in the context
of compressed sensing in the finite Gabor setting. In the continuous infinite setting considered in this paper, writing
\[ f(t) = \sum_{k=0}^{\infty} a_k h_k(t), \]
and observing that
\[ V_g f(x, \xi) = \int_{\mathbb{R}} f(t) g(t-x) e^{-2\pi i \xi t} dt = \int_{\mathbb{R}} f(t+x) g(t) e^{-2\pi i \xi (t+\bar{\xi} x)} dt = e^{-2\pi i \xi} V_f g(x, \xi) \]
we can write the STFT modulus of \( f \) with a random white noise window \( W \) as a Gaussian Weyl–Heisenberg function with a window \( f \):
\[ \left| V_W f \left( \frac{\bar{\xi}}{\pi} \right) \right| = \left| V_f W \left( \frac{\bar{\xi}}{\pi} \right) \right|, \]
leading to the set-up of the previous section.

A related problem of interest in signal analysis is the study of the STFT transform of non-white random noise. Some heuristics regarding the invariance of the zeros of Gaussian spectrogram noise under ‘coloring’ can be found in [36, p. 156].

### 5.5 Gaussian Bi-entire Functions

A further observation will lead us to the consideration of a related Gaussian non-analytic function. A complex valued function is said to be polyanalytic of order \( n \) if it satisfies the higher order Cauchy–Riemann equations,
\[ \partial_z^n f = 0. \quad (5.4) \]
The Fock space of polyanalytic functions, \( \mathbf{F}^n(\mathbb{C}) \) is the space of polyanalytic functions of order \( n \), with finite Fock norm (2.3). Vasilevski [59] obtained the following decomposition of \( \mathbf{F}^n(\mathbb{C}) \), in terms of the Landau levels eigenspaces \( \mathcal{F}_2^r(\mathbb{C}) \):
\[ \mathbf{F}^n(\mathbb{C}) = \mathcal{F}_2^0(\mathbb{C}) \oplus \cdots \oplus \mathcal{F}_2^{n-1}(\mathbb{C}). \quad (5.5) \]

Despite some recent research activity related to determinantal point processes in higher Landau levels and polyanalytic Fock spaces [3, 5, 31, 43, 45, 46, 56], we were unable to find any literature about the corresponding Gaussian random (non-analytic) functions with exception of [42, Theorem 1.8] (though they occurred implicitly in some form in [23, 51], as we have made clear in the previous section).

We will finish this note by indicating how to compute the expected number of zeros of a Gaussian bi-entire function \( F(z) \) (5.6). By bi-entire we mean polyanalytic of order 2 in \( \mathbb{C} \), according to (5.4). The decomposition (5.5) suggests considering the Gaussian random function
\[ F(z) = F(z) + F_1(z) = \sum_{k=0}^{\infty} \left( a_k^{(0)} \frac{z^k}{\sqrt{k!}} + a_k^{(1)} \frac{(k-|z|^2)z^{k-1}}{\sqrt{k!}} \right) \quad (5.6) \]
which, according to the models in previous sections, represents the sum of two STFT’s of independent realizations of white noise,

\[ \mathcal{W}^{(0)} = \sum_{k=0}^{\infty} a_k^{(0)} h_k(t); \quad \mathcal{W}^{(1)} = \sum_{k=0}^{\infty} a_k^{(1)} h_k(t), \]

transformed with orthogonal windows \( h_0 \) and \( h_1 \):

\[ V_{h_0} \mathcal{W}^{(0)} \left( \frac{z}{\sqrt{\pi}} \right) + V_{h_1} \mathcal{W}^{(1)} \left( \frac{z}{\sqrt{\pi}} \right) = e^{\pi i x} e^{-|z|^2/2} \mathbf{F}(z). \]

The associated correlation kernel is the sum of the correlation kernel of the zeros of \( F(z) \), with the correlation kernel of the zeros of \( F_1(z) \):

\[ K(z, w) = L_1^{(1)}(|z - w|^2)e^{z\overline{w}} = e^{z\overline{w}} + L_1(|z - w|^2)e^{z\overline{w}}. \]

This is the reproducing kernel of the Fock space of bi-analytic functions \( F^2(\mathbb{C}) \), given by the sum of the kernels of the first two Landau levels eigenspaces. The average number of zeros of \( F(z) \) can be computed with a procedure similar to the one used in the proof of Theorem 1, by considering the 3-dimensional Gaussian process

\[ (F(z), \nabla'_z F(z), \nabla''_z F(z)) \]

with correlation kernel \( K(z, w) \). One can also use the correspondences between polyanalytic functions and time-frequency analysis \([1, 5]\) and deduce the result from the theory of Gaussian Weyl–Heisenberg functions (see \([42, Theorem 1.8]\)). A related problem is the study of bi-analytic functions with more general weights. For this purpose, the asymptotic expansion of the bi-analytic Bergman kernel obtained in \([44]\) may come in handy.

### 6 Conclusion

Based on the correspondence between Gaussian entire functions and spectrograms of white noise explored in \([14, 15]\), we have provided a rigorous proof that spectrograms of white noise with Gaussian windows have an average of \( \frac{5}{3} \) critical points, among those \( \frac{1}{3} \) are local maxima, and \( \frac{4}{3} \) saddle points, providing, simultaneously, information about the number of neighbors of basins of zeros and local extrema, in the model of Nazarov, Sodin, and Volberg \([50]\). This confirms what is predicted by Flandrin’s double honeycomb mean mode for the average of local maxima and zeros of white noise spectrograms, where the area of local maxima centered hexagons is three times larger than the area of zero centered hexagons. We suggest that such mean model may provide a heuristic reason for the factor \( 1/3 \) in the topological terms of similar problems in complex Riemann surfaces, a question asked by Douglas, Shiffman and Zelditch in \([23]\). Using methods of Feng and Zelditch \([28, 29]\), we have also provided the distribution of the ordinates of such points. The information may be useful in
signal recovering thresholding-based methods and in algorithms based on spectrogram maxima in the style of the publicly released Shazam Entertainment’s recognition algorithm [61]. We also made a connection to Euclidean Landau levels [10], a model used in condensed matter physics related to the Quantum-Hall Effect: by noting that the translation invariant part of the Chern connection coincided with the inter-Landau levels raising operator, we pointed out that the critical points of the GEF are the zeros of a Gaussian random function with correlations given by the reproducing kernel of the first higher Landau level eigenspace, an instance of a space of bi-entire functions. This suggested the introduction of a bi-entire Gaussian random function which represents the sum of two independent realizations of white noise, STFT transformed with the first two Hermite windows. Further directions with applied potential in physics and signal analysis involve the setting of compact Riemann surfaces covered in [23]. In particular, $SU(2)$ polynomials provide finite dimensional models with treatable formulas [23, 28, 29], which may find applications in the context of [52]. The 2-point intensities of critical points seem to be untreatable without computer algebra support [11, 42].

Finally, a few comments which, before risking contempt by more scientifically orthodox thinkers, should be considered within the umbrella of speculative assertions. The analogies between the statistics of supersymmetric vacua, spectrogram extrema and the structure of zeros and critical points could suggest using the combination of acoustic-related numerical and thought experiments, of the kind presented in the recent monograph [36], as a source of intuition to large scale macroscopic and microscopic phenomena. The combination of ideas used in this article intertwine basic principles of time-frequency representations, Gaussian analytic functions, and condensed matter physics. One may accept the possibility of a cross-fertilization between such topics, which, traditionally, tend to be treated in different subdisciplines of mathematics, physics and acoustics. Moving a step further in these speculations, one could point to a more abstract picture, since all these topics may be brought together by the formalism of noncommutative geometry [19]. For an overview of some of the links between time-frequency analysis and noncommutative geometry, see for instance the survey of Luef and Manin [48].

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