AUTOMORPHISM GROUPS OF GENERIC STRUCTURES:
EXTREME AMENABILITY AND AMENABILITY

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Abstract. We investigate correspondences between extreme amenability and amenability of automorphism groups of Fraïssé-Hrushovski generic structures that are obtained from smooth classes, and their Ramsey type properties of their smooth classes, similar to [7, 13]. In particular, we focus on some Fraïssé-Hrushovski generic structures that are obtained from pre-dimension functions. Using these correspondences, we prove that automorphism groups of ordered Hrushovski generic graphs are not extremely amenable in both cases of collapsed and uncollapsed. Moreover, we prove that automorphism groups of Fraïssé-Hrushovski generic structures that are obtained from pre-dimension functions with rational coefficients are not amenable.

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1. Introduction

An extensive research has been devoted to studying dynamical properties of automorphism groups of the Fraïssé-limit of a class of finite structures satisfying joint embedding and amalgamation properties. Suppose \( \mathcal{K} \) is a class of finite structures in a relational language \( \mathcal{L} \) with the joint embedding (JEP), the amalgamation (AP) and the hereditary (HP) properties. It is well-known that there exits a unique ultra-homogeneous countable structure \( M \) whose class of finite substructures up to isomorphism is \( \mathcal{K} \) (see [4, 7] for more information). The structure \( M \) is called the Fraïssé-limit of \( \mathcal{K} \).
A rich model theoretic studies has been developed for understanding the first-order theory of \( M \). In particular, the automorphism groups of Fraïssé-limit structures have been recently of central attention. It is well-known that the automorphism group of \( M \), denoted by \( \text{Aut}(M) \), is a Polish closed subgroup of the permutation group of its underlying set i.e. it can be seen as closed subgroup of \( S_\omega \). A good survey for various kind of questions and results in the topic can be found in [10].

In the seminal paper [7] of Kechris, Pestov and Todorcevic a close correspondence between extreme amenability of \( \text{Aut}(M) \) and certain combinatorial property of the class \( K \), called the Ramsey property, has been discovered. Let \( G \) be a topological group. A continuous action \( \Gamma \) of \( G \) on a compact Hausdorff space \( X \) is called a \( G \)-flow. Group \( G \) is called extremely amenable if every \( G \)-flow \((G, \Gamma, X)\) has a fix point in \( X \).

In [7], they have shown that the automorphism group of an ordered Fraïssé-limit structure \( M \) is extremely amenable if and only if its ordered Fraïssé-class has the Ramsey property. Later in [13], a connection has been found between amenability of \( \text{Aut}(M) \) and another combinatorial property called the convex Ramsey property. A Hausdorff topological group \( G \) is amenable if every \( G \)-flow \((G, \Gamma, X)\) supports an \( G \)-invariant Borel probability measure on \( X \). It has been proved in [13] that \( \text{Aut}(M) \) is amenable if and only if \( K \) has the convex Ramsey property.

Our paper follows similar paths for adopting their line of research for Fraïssé-Hrushovskii limits of smooth classes. The Fraïssé-Hrushovskii limits of smooth classes includes the original construction of Hrushovskii of CM-trivial strongly minimal sets [5], which are very important structures model-theoretically, as well as the original Fraïssé-limits of finite structures. A class of (finite) structures \( K \) together with a partial ordering \( \preceq \) is called a smooth class if for every \( A, A_1, A_2 \in K \) with \( A_1, A_2 \subseteq A \) whenever \( A_1 \preceq A \), it follows that \( A_1 \cap A_2 \preceq A_2 \) (see Definition 1). It is worth noting that the notion of substructure satisfies this condition. Similar to the Fraïssé-limit case, one can show that for a smooth class \((K, \preceq)\) with HP and the adopted JEP and AP, there is a \((K, \preceq)\)-generic structure (see Proposition 4). A natural question is to verify (extreme) amenability of the automorphism group of a \((K, \preceq)\)-generic structure.

In this paper in Section 2, firstly we show that indeed a similar correspondence of [7] between extreme amenability of the automorphism group of a \((K, \preceq)\)-generic structure, and a modified definition of Ramsey property for \((K, \preceq)\) is valid. Later, this correspondence enables us to show that for each \( \alpha \geq 1 \) the automorphism groups of ordered ab-initio generic graphs \( M_\alpha \), ordered collapsed generic graphs \( M_\alpha^c \) and ordered \( \omega \)-categorical generics \( M_\alpha^f \), are not extremely amenable.

In Section 3, we prove a similar correspondence of [13] between amenability of the automorphism group of a \((K, \preceq)\)-generic structure, and again a modified version of convex Ramsey property for \((K, \preceq)\). This helps us, in Section 4, to rule out the amenability of the automorphism group of the ab-initio generic structures that are obtained from pre-dimension functions with rational coefficients. However, the amenability question of the ab-initio generic structures that are obtained from pre-dimension functions with
irrational coefficients and \( \omega \)-categorical Hrushovski generic structures remain unanswered in this manuscript.

Acknowledgement: The authors would like to thank the anonymous referee for the encouraging comments and the thoughtful suggestions.

2. Extreme amenability of automorphism groups of generic structures

In [7], the general correspondence between the extreme amenability of the automorphism group of an ordered Fraïssé structure and the Ramsey property of its finite substructures has been discovered. In this section, in Theorem 1, we prove that indeed a similar correspondence for the automorphism group of a generic structure and the Ramsey property of its \( \leq \)-closed finite substructures holds. Below some backgrounds about the smooth classes and Fraïssé-Hrushovski limits is presented.

2.1. Background.

2.1.1. Smooth class.

Definition 1. Let \( \mathcal{L} \) be a finite relational language and \( \mathcal{K} \) be a class of \( \mathcal{L} \)-structures which is closed under isomorphism and substructure. Let \( \leq \) be a reflexive and transitive relation on elements of \( A \subseteq B \) of \( \mathcal{K} \) and moreover, invariant under \( \mathcal{L} \)-embeddings such that it has the following properties:

1. \( \emptyset \in \mathcal{K} \), and \( \emptyset \leq A \) for all \( A \in \mathcal{K} \);
2. If \( A \subseteq B' \subseteq B \), then \( A \leq B \) implies that \( A \leq B' \);
3. If \( A, A_1, A_2 \in \mathcal{K} \) and \( A_1, A_2 \subseteq A \), then \( A_1 \leq A \) implies \( A_1 \cap A_2 \leq A_2 \).

The class \( \mathcal{K} \) together with the relation \( \leq \) is called a smooth class. For \( A, B \in \mathcal{K} \) if \( A \leq B \), then we say that \( A \) is \( \leq \)-closed substructure of \( B \), or simply \( A \) is \( \leq \)-closed in \( B \). Moreover, if \( N \) is an infinite \( \mathcal{L} \)-structure such that \( A \subseteq N \), we denote \( A \leq N \) whenever \( A \leq B \) for every finite substructure \( B \) of \( N \) that contains \( A \). We say an embedding \( \Gamma \) of \( A \) into \( N \) is \( \leq \)-embedding if \( \Gamma [A] \leq N \).

Notation. Suppose \( A, B, C \) are \( \mathcal{L} \)-structures with \( A, B \subseteq C \). We denote \( AB \) for the \( \mathcal{L} \)-substructure of \( C \) with domain \( A \cup B \). For an \( \mathcal{L} \)-structure \( N \), denote \( \text{Age} (N) \) for the set of all finite substructures of \( N \); up to isomorphism.

Definition 2. Let \( (\mathcal{K}, \leq) \) be a smooth class.

1. We say \( (\mathcal{K}, \leq) \) has the hereditary property (HP) if \( A \in \mathcal{K} \) and \( B \subseteq A \), then \( B \in \mathcal{K} \).
2. Suppose \( A, B \) and \( C \) are elements of \( \mathcal{K} \) such that \( A \leq B, C \). The free-amalgam of \( B \) and \( C \) over \( A \) is a structure with domain \( BC \) whose only relations are those from \( B \) and \( C \) such that \( B \cap C = A \). We denote it by \( B \otimes_A C \).

\[\text{After the earlier version of this paper was uploaded in ArXiv, David M. Evans in an email correspondence informed us that he can show, using a different method, the automorphism groups of generic structures that are obtained from pre-dimension functions with irrational coefficients and the \( \omega \)-categorical generic structures are not amenable.}\]
(3) We say \((\mathcal{K}, \leq)\) has the \(\leq\)-amalgamation property (AP) if for every \(A, B, C \in \mathcal{K}\) and \(\leq\)-embeddings \(\gamma_1 : A \to B\) and \(\gamma_2 : A \to C\), there are \(D\) and \(\leq\)-embeddings \(\lambda_1 : B \to D\) and \(\lambda_2 : C \to D\) such that \(\lambda_1 \circ \gamma_1 = \lambda_2 \circ \gamma_2\) (equivalently; as we assume \(\mathcal{K}\) is closed under isomorphism, for every \(B, C \in \mathcal{K}\) that have a common substructure \(A\) with \(A \leq B, C\), there is \(D \in \mathcal{K}\) such that \(B \leq D\) and \(C \leq D\)).

(4) We say \((\mathcal{K}, \leq)\) has the free-amalgamation property if for \(B, C \in \mathcal{K}\) that have a common substructure \(A\) with \(A \leq B, C\), then \(B \otimes_A C \in \mathcal{K}\).

**Remark 3.** We included \(\emptyset\) in the class \(\mathcal{K}\) in order to consider the joint embedding property (JEP) as a special case of the \(\leq\)-amalgamation property.

**Proposition 4.** If \((\mathcal{K}, \leq)\) is a smooth class with the \(\leq\)-amalgamation property, then there is a unique countable structure \(\mathbf{M}\), up to isomorphism, satisfying:

1. \(\text{Age}(\mathbf{M}) = \mathcal{K}\);
2. \(\mathbf{M} = \bigcup_{i \in \omega} \mathbf{A}_i\) where \(A_i \in \mathcal{K}\) and \(A_i \leq A_{i+1}\) for every \(i \in \omega\);
3. If \(A \leq M\) and \(A \leq B \in \mathcal{K}\), then there is an embedding \(\Gamma : B \to \mathbf{M}\) with \(\Gamma |_A = \text{id}_A\) and \(\Gamma[B] \leq \mathbf{M}\).

**Proof.** See [9]. \(\square\)

**Definition 5.** The structure \(\mathbf{M}\), that is obtained in the above proposition, is called the Fraïssé-Hrushovski \((\mathcal{K}, \leq)\)-generic structure or simply \((\mathcal{K}, \leq)\)-generic structure.

**Fact 6.** (See [9]) Suppose \(A \subseteq_{\text{fin}} \mathbf{M}\). Then, there is a unique smallest finite closed set that contains \(A\) in \(\mathbf{M}\). It is called \(-\text{closure of } A\) in \(\mathbf{M}\) that is denoted by \(\text{cl}(A)\).

2.1.2. Ab-initio classes of graphs. Let \(\mathcal{L} = \{R\}\) consist of a binary relation \(R\) and let \(\mathcal{K}\) be the class of all finite graphs. For \(\alpha \geq 1\), define \(\delta_\alpha : \mathcal{K} \to \mathbb{R}\) as \(\delta_\alpha(A) = \alpha \cdot |V(A)| - |E(A)|\) where \(V(A)\) is the set of vertices of \(A\) and \(E(A)\) the set of edges of \(A\). For every \(A \subseteq B \in \mathcal{K}\), define \(A \leq_\alpha B\) if

\[\delta_\alpha(C/A) := \delta_\alpha(C) - \delta_\alpha(A) \geq 0,\]

for every \(C\) with \(A \subseteq C \subseteq B\). Finally put \(\mathcal{K}_\alpha^+ := \{A \in \mathcal{K} : \delta_\alpha(B) \geq 0, \text{ for every } B \subseteq A\}\).

**Fact 7.** \((\mathcal{K}_\alpha^+, \leq_\alpha)\) is a smooth class with the free-amalgamation property and HP.

Hence, there is the unique countable \((\mathcal{K}_\alpha^+, \leq_\alpha)\)-generic structure \(\mathbf{M}_\alpha\). When the coefficient \(\alpha\) is rational, using a finite-to-one function \(\mu\) over the 0-minimally algebraic elements (see Definition [1]), one can restrict the ab-initio class \(\mathcal{K}_\alpha^+\) to \(\mathcal{K}_\alpha^\mu\) such that \((\mathcal{K}_\alpha^\mu, \leq_\alpha)\) has AP (see [1]).

**Definition 8.** Suppose \(\alpha \geq 1\) is a rational number and \(\mathbf{M}_\alpha\) is the \((\mathcal{K}_\alpha^+, \leq_\alpha)\)-generic structure:

1. Suppose \(A\) and \(B\) are two disjoint finite sets in \(\mathbf{M}_\alpha\). \(B\) is called 0-algebraic over \(A\) if \(\delta(B/A) = 0\) and \(\delta(B_0/A) > 0\) for all proper subset \(\emptyset \neq B_0 \subseteq A\). \(B\) is called 0-minimally algebraic over \(A\) if there is no proper subset \(A_0\) of \(A\) such that \(B\) is 0-algebraic over \(A_0\).
(2) Let $E \in K^+_{\alpha} \times K^+_{\beta}$ be the set of all $(A, B)$ such that $B$ is 0-minimally algebraic over $A$ and $A \neq \emptyset$. Define a function $\mu : E \to \mathbb{N}$ such that $\mu$ is finite-to-one, and $\mu(A, B) \geq \delta(A)$ for every $(A, B) \in E$.

(3) Let $K^+_{\alpha} \subseteq K^+_{\beta}$ be such that $A \in K^+_{\alpha}$ if for every $A' \subseteq A$ and $B'$, a 0-minimally algebraic set over $A'$, the number of pairwise disjoint isomorphic copies of $B'$ over $A'$ in $A$ is bounded by $\mu(A', B')$.

**Fact 9.** (cf. [5]) The class $(K^+_{\alpha}, \leq_\alpha)$ is a smooth class with AP and HP.

Let $M^\alpha$ for the $(K^+_{\alpha}, \leq_\alpha)$-generic structure.

To obtain an $\omega$-categorical generic structure one needs further restrictions: Suppose $f : \mathbb{R}^\geq \to \mathbb{R}^\geq$ is an increasing unbounded function. Then let

$$K^f_{\alpha} := \{ A \in K^+_{\alpha} : \delta_\alpha(A') \geq f(|A'|) \forall A' \subseteq A \} .$$

**Fact 10.** For suitable choice of $f$ (called good) the class $(K^f_{\alpha}, <_\alpha)$ a smooth class with the free-amalgamation property and HP, where $A <_\alpha B$ iff $\delta_\alpha(A) < \delta_\alpha(B')$ for every $A \subseteq B' \subseteq B$.

Hence, for a good $f$, there is the countable $(K^f_{\alpha}, <_\alpha)$-generic structure $M^\alpha$. Moreover, the generic structure $M^\alpha$ is an $\omega$-categorical structure (see [2] for more details).

### 2.2. $\leq_\alpha$-Ramsey property and its correspondence with extreme amenability.

Denote $S_\omega$ for the set of all permutations of $\mathbb{N}$. It is a well-known fact that $S_\omega$ with the point-wise convergence topology forms a Polish group. From now on, we consider the point-wise convergence topology on $S_\omega$.

**Definition 11.** For a topological group $G$ and a subgroup $H$ of $G$ by a $k$-coloring $c$ of $G/H$ with $k \in \mathbb{N}\setminus\{0\}$, we mean a map $c : \{hH : h \in G\} \to \{0, 1, 2, \ldots, k-1\}$, from the set of left cosets of $H$ into $\{0, 1, \ldots, k-1\}$.

**Fact 12.** (See [7] Proposition 4.2) Let $G$ be a closed subgroup of $S_\omega$. Then, the followings are equivalent:

1. $G$ is extremely amenable;
2. For any open subgroup $H$ of $G$, any $k$-coloring $c : G/H \to \{0, 1, \ldots, k-1\}$ and any finite $A \subseteq_{\text{fin}} G/H$, there are $g \in G$ and $i \in \{0, 1, \ldots, k-1\}$ such that $c(ga) = i$, for all $a \in A$.

We work with a fixed smooth class $(K, \leq)$ with AP and HP. Let $M$ be the countable $(K, \leq)$-generic structure with $\mathbb{N}$ as the underlying universe. Put $G := \text{Aut}(M)$. It is also well-known that $G$ is a closed subgroup of $S_\omega$. Let $A \subseteq M$ be a finite subset of $M$. Write

$$G_{(\alpha)} := \{ g \in G : g(a) = a, \text{ for all } a \in A \},$$

for the point-wise stabilizer of $A$ in $G$, and write $G_A := \{ g \in G : g[A] = A \}$ for the set-wise stabilizer of $A$ in $G$ where $\emptyset \neq A \subseteq_{\text{fin}} M$.

**Remark 13.** Note that $\{G_{(\alpha)} : \emptyset \neq A \subseteq M\}$ forms a basis of neighborhood of $1_G$. 
Definition 14. Suppose \( A \in \mathcal{K} \) and let \( N \) is any \( \mathcal{L} \)-structure. We denote \( \binom{N}{A} \) for the set of all \( \leq \)-embeddings of \( A \) into \( N \). For \( k \in \mathbb{N} \setminus \{0\} \), we call a function \( c : \binom{N}{A} \to \{0, 1, \cdots, k-1\} \) a \( k \)-coloring function.

Suppose \( A \in \mathcal{K} \). The group \( G \) acts naturally on \( \binom{M}{A} \) in the following way: When \( \Gamma \in \binom{M}{A} \)
\[
g \cdot \Gamma := \Gamma'
\]
if \( \Gamma' (A) = g [\Gamma (A)] \). It is worth noting, since elements of \( G \) sends \( \leq \)-closed sets to \( \leq \)-closed sets, this action is well-defined.

Definition 15. We say that \( G \) preserves a linear ordering \( \preceq \) on \( M \) if \( a \preceq b \) implies \( g(a) \preceq g(b) \), for every \( a, b \in M \) and \( g \in G \).

Proposition 16. The following conditions are equivalent:

1. \( G \) is extremely amenable;
2. \( G_{(A)} = G_A \), for any finite \( \emptyset \neq A \subseteq M \);
   \( a \) \( G \)
   \( b \) \( G \)
   \( M \)
   \( k \) \( G \)
   \( G \)
   \( G \)
   \( a \)
   \( M \)
   \( a \)
   \( M \)
   \( a \)

3. \( G \) preserves a linear ordering;
   \( G \)
   \( G \)
   \( G \)

Proof. (Similar to the proof of Proposition 4.3. in [7]) An easy argument shows that (2) and (3) are equivalent.

1 \( \implies \) 3. Assume that \( G \) is extremely amenable. Since \( LO \), the space of invariant linear orderings defined on \( M \), forms a \( G \)-flow, it follows that the action of \( G \) on \( LO \) has a fixed point. This is exactly our expected ordering.

To show 3-(b), fix \( A \leq B \in \mathcal{K} \) and suppose \( c : \binom{M}{A} \to \{0, 1, \cdots, k-1\} \) is a \( k \)-coloring. Fix \( \Lambda \) a \( \leq \)-embedding of \( B \) in \( M \) and let \( B_0 := \Lambda (B) \). Then, \( B_0 \leq M \). Take \( A_0 \leq B_0 \) to be the corresponding \( \leq \)-closed copy of \( A \) inside \( B_0 \) under \( \Lambda \). Put \( H = G_{A_0} = G_{(A_0)} \) which is an open subgroup of \( G \). We can identify the set \( G/H \) of left cosets of \( H \) in \( G \) with \( \binom{M}{A_0} \). Now by applying Fact [12] to \( H \), \( c \) and \( \binom{M}{A_0} \), one can find \( i \) with \( 0 \leq i \leq k-1 \) and \( g \in G \) such that \( c(g \cdot (\Lambda \circ \gamma)) = i \), for all \( \gamma \in \binom{B_0}{A_0} \).

Let \( B' = g [B_0] = g [\Lambda (B)] \leq M \). Pick \( \Lambda' \in \binom{M}{B} \) such that \( \Lambda'(B) = g [\Lambda (B)] \). Then, \( c(\Lambda' \circ \gamma) = i \), for any \( \gamma \in \binom{B'}{A} \).

2 \( \implies \) 1. A similar argument as above shows (2) implies Fact [12]. Hence, \( G \) is extremely amenable.

\( \square \)

Definition 17. Assume \( A \leq B \leq C \in \mathcal{K} \) and \( k \geq 1 \). We write
\[
C \to (B)^A_k,
\]
if for every \( k \)-coloring \( c : \binom{C}{A} \to \{0, 1, \cdots, k-1\} \), there exists \( \lambda \in \binom{C}{B} \) such that \( c(\lambda \circ \gamma) \) is constant for all \( \gamma \in \binom{\Lambda (B)}{A} \). In this case \( \lambda (B) \) is called a \( c \)-monochromatic
copy of \( B \) in \( C \). We say that the class \(( K, \leq \rangle \) has the \( \leq \)-Ramsey property if for every \( A \leq B \in K \) and \( k \geq 2 \), there exists \( C \in K \) with \( B \leq C \) such that \( C \rightarrow (B)_k^A \).

**Remark 18.** A similar argument as in classical Ramsey theory shows that if for every \( A \leq B \in K \) we have \( C \rightarrow (B)_2^A \) for some \( C \in K \), then the smooth class \(( K, \leq \rangle \) has the \( \leq \)-Ramsey property (See [12], page 81-82).

The following theorem gives the main correspondence that we have mentioned in the introduction. The proof is similar to the proof of Proposition 4.5. in [7], and we give the analogues modification of the proof in order to highlight the role of \( \leq \) relation in the \(( K, \leq \rangle \)-generic structure.

**Theorem 19.** The followings are equivalent:

1. \( G \) is extremely amenable;
2. (a) \( G \) preserves a linear ordering;
   (b) \(( K, \leq \rangle \) has the \( \leq \)-Ramsey property.

**Proof.** 1 \( \Longrightarrow \) 2. We have already presented a proof for 2-(a). Now we are going to show the \( \leq \)-Ramsey property, assuming the extremely amenability of \( G \). By Remark 18 we only need to check the \( \leq \)-Ramsey property for \( k = 2 \). Suppose, on the contrary that, there are \( A \leq B \in K \) such that \( C \rightarrow (B)_2^A \) for all \( C \in K \). Pick \( \Lambda \in \binom{\mathbf{M}}{B} \) and let \( B_0 := \Lambda \circ B \). Then, for every finite \( E \leq \mathbf{M} \) with \( B_0 \leq E \), there exists a 2-coloring \( c_E : \binom{E}{A} \rightarrow \{0, 1\} \) such that for any \( \lambda \in \binom{E}{B} \) the value of \( c_E (\lambda' \circ \gamma) \) is not constant when \( \gamma \in \binom{X(B)}{A} \).

Take \( \mathcal{I} := \{ F \leq M : F \subseteq_{\text{fin}} \mathbf{M} \} \) as an index set and for \( D \in \mathcal{I} \), let \( \mathcal{X}_D := \{ F \in \mathcal{I} : D \leq F \} \). From Fact 6 it follows that \( \mathcal{E} := \{ \mathcal{X}_A : A \in \mathcal{I} \} \) has the finite intersection property. Hence, there exists an ultra-filter \( \mathcal{U} \) on the index set \( \mathcal{I} \) such that for every finite \( D \in \mathcal{I} \) the set \( \mathcal{X}_D \in \mathcal{U} \). For each \( \Gamma \in \binom{\mathbf{M}}{A} \) exactly one of the followings cases holds:

1. \( \{ E \in \mathcal{I} : \text{cl} (\Gamma (A) B_0) \leq E \text{ and } c_E (\Gamma) = 0 \} \in \mathcal{U} \), or
2. \( \{ E \in \mathcal{I} : \text{cl} (\Gamma (A) B_0) \leq E \text{ and } c_E (\Gamma) = 1 \} \in \mathcal{U} \).

Define a 2-coloring \( c : \binom{\mathbf{M}}{A} \rightarrow \{0, 1\} \) as follows: for \( \Gamma \in \binom{\mathbf{M}}{A} \)

\[
c (\Gamma) := i \iff \{ E \in \mathcal{I} : \text{cl} (\Gamma (A) B_0) \leq E \text{ and } c_E (\Gamma) = i \} \in \mathcal{U}.
\]

Now by Proposition 16.2-(b), there are \( \Lambda' \in \binom{\mathbf{M}}{B} \) and \( i \in \{0, 1\} \) such that \( c (\Lambda' \circ \gamma) = i \), for all \( \gamma \in \binom{X(B)}{A} \). For each \( \gamma \in \binom{X(B)}{A} \) put

\[
\mathcal{A}_\gamma = \{ E \in \mathcal{I} : \text{cl} (\Lambda' (B) B_0) \leq E \text{ and } c (\Lambda' \circ \gamma) = c_E (\Lambda' \circ \gamma) = i \}.
\]

Note that both sets

\[
\mathcal{I}_{B_0, \gamma} := \{ E \in \mathcal{I} : \text{cl} (\Lambda' \circ \gamma (A) B_0) \leq E \text{ and } c_E (\Lambda' \circ \gamma) = i \}
\]

and

\[
\mathcal{X}_{\text{cl}(\Lambda' (B) B_0)} = \{ E \in \mathcal{I} : \text{cl} (\Lambda' (B) B_0) \leq E \}
\]

are in \( \mathcal{U} \). Furthermore, \( \mathcal{A}_\gamma \supseteq \mathcal{I}_{B_0, \gamma} \cap \mathcal{X}_{\text{cl}(\Lambda' (B) B_0)} \). Hence, \( \mathcal{A}_\gamma \in \mathcal{U} \). Let \( E \in \bigcap_{\gamma \in \binom{X(B)}{A}} \mathcal{A}_\gamma \neq \emptyset \). Note that if \( B_0 \leq E \) then for each \( \gamma \in \binom{X(B)}{A} \), \( c_E (\gamma) = i \).
Therefore, $\lambda(B)$ is a monochromatic subset with respect to $c_E$ where $\lambda \in (E_B)$ which is a contradiction.

2 $\implies$ 1 Part (b) of 2 in Proposition 16 trivially follows from the $\leq$-Ramsey property and part (a) of 2 in Proposition 16 follows from 2-(a). Hence, $G$ is extremely amenable. \hfill \Box

2.3. $\leq$-Ramsey property for some ab-initio classes. In this subsection, we show certain ab-initio classes obtained from pre-dimension functions does not have the $\leq$-Ramsey property.

Recall the followings from [12].

**Definition 20.** Suppose $A = (V, E)$ is a graph where $V$ is the set of vertices and $E$ is set of edges of $A$.

1. Let $e_A := |E(A)|$ and $v_A := |V(A)|$.
2. For a vertex $a \in A$, $\deg(a)$ denotes the degree of $a$ in $A$.
3. The maximum density of $A$, denoted by $m(A)$, is defined as $m(A) := \max \left\{ \frac{e_B}{v_B} : B \subseteq A \right\}$.
4. Let $\eta(A) := \min \left\{ \deg(a) : a \in V(A) \right\}$ and define $\eta^*(A) := \max \left\{ \eta(B) : B \subseteq A \right\}$.
5. For two graphs $B \subseteq C$, we abbreviate $C \overset{r}{\rightarrow} (B)_\gamma$ to indicate any vertex $r$-coloring of $C$ has a subgraph $B'$ isomorphic to $B$, whose all vertices are monochromatic.

**Definition 21.** The smooth class $(\mathcal{K}, \leq)$ with HP has the one-point $\leq$-Ramsey property if for every one-point structure $A$ and every structure $B$ with $A \leq B$ and $k \geq 2$, there exists $C \in \mathcal{K}$ with $B \leq C$ such that $C \overset{A}{\rightarrow} (B)^A_k$.

The following lemma provides the key idea for proving Theorem 23. For its proof, the reader is referred to [12] (Lemma 12.2, page 130).

**Lemma 22.** Suppose $B$ and $C$ are two graphs such that $m(C) < \frac{1}{2} \cdot r \cdot \eta^*(B)$, for some $r \geq 2$. Then, $C \overset{r}{\rightarrow} (B)_\gamma$.

**Theorem 23.** Suppose $(\mathcal{K}, \subseteq) \in \{ (\mathcal{K}_\alpha^+, \leq_\alpha), (\mathcal{K}_\alpha, \leq_\alpha), (\mathcal{K}_\alpha^-, \leq_\alpha) \}$ for $\alpha \geq 1$. Then the class $(\mathcal{K}, \subseteq)$ does not have the one-point $\subseteq$-Ramsey property.

**Proof.** Let $A$ be the singleton graph and let $L$ be a loop with $n$ vertices, $n \geq 3$ that contains $A$. It is easy to see that $L \in \mathcal{K}$, for every $\alpha \geq 1$. Furthermore, any embedding of $A$ in $L$ is $\subseteq$-closed. In particular, $A \subseteq L$. An easy calculation shows for any $C \in \mathcal{K}$, we have $m(C) \leq \alpha$. Moreover, $\eta^*(L) = 2$.

Now by choosing $r > \alpha$ we have $m(C) < \frac{1}{2} \cdot r \cdot \eta^*(L)$, for every $C \in \mathcal{K}$. Hence, in the light of Lemma 23 $C \overset{r}{\rightarrow} (L)^A_r$ for every $C \in \mathcal{K}$. On the other hand, since all embeddings of $A$ inside $L$ are $\subseteq$-closed, it follows that $C \overset{A}{\rightarrow} (L)^A_r$, for every $C \in \mathcal{K}$. Hence, the class $(\mathcal{K}, \subset)$ does not have the one-point $\subset$-Ramsey property. \hfill \Box

In fact, the above proof shows something stronger. Suppose $\mathcal{L}^*$ is a finite expansion of $\mathcal{L}$ that contains a binary relation $\prec$. Let $\mathcal{K}_{\mathcal{L}^*}$ be the all $\mathcal{L}^*$-expansions of structures...
C ∈ K, in which the relation < is interpreted as a linear-ordering on the universe of C. Subsequently, for A², B² ∈ K₊, we define A² ⊆ B² if and only if A²⁺ ⊆ B²⁺ and A ⊆ B, where A²⁺ and B²⁺ are Λ⁺-expansions of graphs A and B; respectively.

**Corollary 24.** The class (K₊, ⊆⁺) does not have the one-point ⊆⁺-Ramsey property.

In this case, the class (K₊, ⊆⁺) has the JEP, HP and AP. Therefore, if we take the (K₊, ⊆⁺)-generic structure M₊, then it is easy to see that this structure is formed by adding a generic linear ordering to the (K, ⊆)-generic structure M. Note that by our notation M ∈ {Mₓ, Mₓ⁺, Mₓ⁺₁}. In the light of Theorem [19] and the corollary above, the following theorem where R is binary is established. It has to be noted that it seems in the case of hypergraphs (relations with arity > 2) a similar result to Lemma [22] is true.

**Theorem 25.** Let G be Aut(Mₓ⁺) where Λ⁺ is a finite expansion Λ that contains a linear-ordering relation as explained above, and M ∈ {Mₓ, Mₓ⁺, Mₓ⁺₁} for α ≥ 1. Then G is not extremely amenable.

3. **Amenability of automorphism groups of generic structures**

Here, we further continue our project, this time similar to [13]. We give the correspondence between amenability of the automorphism groups of Fraïssé-Hrushovski generic structures, and the convex ⊆-Ramsey property of the automorphism group of the structure with respect to its smooth class, that has been defined later.

In the following subsection, we first adapt the notion of convex Ramsey property and then study the convex ⊆-Ramsey property with a slightly different approach. We present the expected correspondence in Theorem [32]. Later, we prove our main result Theorem [40] that shows automorphism groups of generic structures of certain class of smooth classes does not have the convex ⊆-Ramsey property and hence they are not amenable. Theorem [40] provides the ingredient for the next section to investigate the convex ⊆⁻α-Ramsey property of the (Kₓ, ≤⁻α)-generic structure when the coefficient α is rational.

Throughout this section, (K, ≤) is a smooth class of finite relational Λ-structures with the ≤-amalgamation property and HP, and M is the (K, ≤)-generic structure. Suppose A ∈ K and N is a substructure of M. Denote ⟨N⟩ for the set of all finitely supported probability measures on (N). Suppose X ⊆ Y ⊆ Z are ≤-closed substructures of M and let r ∈ ⟨Z⟩. We define ⟨X⟩ to be the set

\[ \left\{ q ∈ \langle Z \rangle : ∃ p ∈ \langle Y \rangle : ∀ Γ ∈ \langle Y \rangle : ∀ Λ ∈ \langle Z \rangle : q (Λ ∩ Γ) = r (Λ) · p (Γ) \right\}. \]

3.1. **The convex ⊆-Ramsey property.**

**Definition 26.** We say Aut (M) has the convex ⊆-Ramsey property with respect to (K, ≤) if for every A, B ∈ K with A ≤ B and every 2-coloring function f : (M) → {0, 1}, there exists p ∈ ⟨M⟩ such that for every q₁, q₂ ∈ ⟨M⟩, |f (q₁) − f (q₂)| ≤ 1/2.

If the condition above holds for a coloring function f, we say f satisfies the convex ⊆-Ramsey condition. Note that the convex ⊆-Ramsey property demands that all coloring functions satisfy the convex ⊆-Ramsey condition.
Let $A, B \in \mathcal{K}$ such that $A \leq B$. Fix an enumeration $\tilde{\eta} = (\eta_1, \ldots, \eta_m)$ of $(B_A)$ where $(B_A) = \{\eta_1, \ldots, \eta_m\}$; hence $m = |(B_A)|$. For each $\Lambda \in (M_B)$ define $\Lambda \cdot \tilde{\eta} := (\Lambda \circ \eta_1, \ldots, \Lambda \circ \eta_m)$, and note that $\Lambda \circ \eta_i \in (M_A)$, for each $i \in \{1, \ldots, m\}$. Let $f : (M_A) \to \{0, 1\}$ be a 2-coloring function. For each $\Lambda \in (M_B)$ define $f(\Lambda \cdot \tilde{\eta}) := (f(\Lambda \circ \eta_1), \ldots, f(\Lambda \circ \eta_m))$ which is a finite sequence of 0 and 1 and hence $f(\Lambda \cdot \tilde{\eta}) \in \{0, 1\}^m$.

Fix $\Lambda \in (M_B)$ and let $\tilde{k} := f(\Lambda \cdot \tilde{\eta}) \in \{0, 1\}^m$. There are two possibilities for elements of $\tilde{k}$:

Case 1: $k_i = k_j$ for all $i, j \in \{1, \ldots, m\}$.

Then, we can assign a finitely supported probability measure $p$ on $(M_B)$ which concentrates on $\Lambda$. In which the following corollary holds:

**Corollary 27.** Suppose $A, B \in \mathcal{K}$ and $A \leq B$. Suppose $f : (M_A) \to \{0, 1\}$ is a coloring function such that there exists $\Lambda \in (M_B)$ where $f(\Lambda \cdot \tilde{\eta})$ is constant (or monochromatic). Then, the convex $\leq$-Ramsey condition holds for $f$.

Case 2. There are $i, j \in \{1, \cdots, m\}$ such that $k_i = 0$ and $k_j = 1$.

Then, we can show the following lemma which is needed for Theorem 38.

**Lemma 28.** Suppose $(\mathcal{K}, \leq)$ has the free-amalgamation property. Then, for every $v \in \{1, \ldots, m\}$ and $w \in \{0, 1\}$, there is $\bar{k}^{v, w} := (k_1^{v, w}, \ldots, k_m^{v, w}) \in \{0, 1\}^m$ with $k_v^{v, w} = w$ such that there exist infinitely many distinct $\Lambda_s^{v, w} \in (M_B)$, $s < \omega$ with $f(\Lambda_s^{v, w} \cdot \tilde{\eta}) = \bar{k}^{v, w}$.

**Proof.** Let $v \in \{1, \ldots, m\}$ and $w \in \{0, 1\}$. Note that $\eta_v$ corresponds to a $\leq$-embedding of $A$ into $B$. By our assumption $k_j = w$ or $k_j = w$. Without loss of generality, let $k_j = w$. By the $\leq$-genericity of $\mathcal{M}$ and the free-amalgamation property, there are infinitely many distinct embeddings $\Lambda_s$ for $s < \omega$ such that $\Lambda_s \circ \eta_v = \Lambda \circ \eta_v$. Since $\Lambda \circ \eta_v(A)$ and $\Lambda \circ \eta_v(A)$ are $\leq$-closed and isomorphic, there exists an automorphism $g$ of $\mathcal{M}$ such that $g[\Lambda \circ \eta_v(A)] = \Lambda \circ \eta_v(A)$. It is easy to check $g \cdot (\Lambda_s \circ \eta_v) = \Lambda_s \circ \eta_v$ and hence $f(g \cdot (\Lambda_s \circ \eta_v)) = w$. Let $\Sigma_v := \{g \cdot \Lambda_s : s < \omega\}$. Now consider $\{f(\Lambda \cdot \tilde{\eta}) \in \{0, 1\}^m : \Lambda \in \Sigma_v\}$. Since $|\Sigma_v|$ is infinite, there is $\bar{k}^{v, w} \in \{0, 1\}^m$ such that $f(\Lambda_s^{v, w} \cdot \tilde{\eta}) = \bar{k}^{v, w}$ for infinitely many distinct $\Lambda_s^{v, w}$'s in $\Sigma_v$. Note that $k_v^{v, w} = k_j = w$, and hence we are done. \hfill \square

**Remark 29.** In general for any smooth class $(\mathcal{K}, \leq)$, one can easily modify the argument above to guarantee that under the assumption of AP, there exists at least one embedding with the desired property.

Suppose $q_1, q_2$ are two elements of $(M_A)$. Let $\{\Gamma_i : i \in I\}$ to be an enumeration of all elements $(M_A)$; without repetition. Note that $|I| = \omega$ when $(\mathcal{K}, \leq)$ has the free-amalgamation property. Now, we define $w_{q_1, q_2}^i := q_1(\Gamma_i) - q_2(\Gamma_i)$. Since by our definition $q_1, q_2$ are finitely supported probability measures, it follows $I_0 := \{i \in I : q_1(\Gamma_i) = q_2(\Gamma_i) = 0\}$ is cofinite. Moreover, $\sum_{i \in I} w_{q_1, q_2}^i = \sum_{i \in I} q_1(\Gamma_i) - \sum_{i \in I} q_2(\Gamma_i) = 1 - 1 = 0$. Let $r \in (M_B)$. Then, for $q_1, q_2 \in \langle r \rangle$:

$$f(q_1) - f(q_2) = \sum_{i \in I} f(\Gamma_i) \cdot (q_1(\Gamma_i) - q_2(\Gamma_i)) = \sum_{i \in I} w_{q_1, q_2}^i \cdot f(\Gamma_i).$$
Now fix \( \{ \Lambda_j : j \in J \} \) to be an enumeration of all elements \( (M_B) \); without repetition. Since \( r \) is finitely supported probability measure, there is a finite \( J_r \subseteq J \) such that \( r (\Lambda_j) \neq 0 \) if and only if \( j \in J_r \). For \( q_1, q_2 \in \langle r \rangle \) it is clear that \( w_{q_1, q_2} = q_1 (\Gamma_i) = q_2 (\Gamma_i) = 0 \) when \( \Gamma_i \notin \Lambda_j (B) \), for all \( j \in J_r \). Let \( q_1, q_2' \in \langle R_A \rangle \) such that \( q_1 = r \cdot q_1' \) and \( q_2 = r \cdot q_2' \). Let \( I_r \) be the finite subset of \( I \) such that \( i \in I_r \) if and only if \( \Gamma_i (A) \subseteq \Lambda_j (B) \), for some \( j \in J_r \). Therefore,

\[
f (q_1) - f (q_2) = \sum_{i \in I_r} w_i \cdot f (\Gamma_i) = \sum_{j \in J_r} r_j \cdot \left( \sum_{i \in I_r : \Gamma_i (A) \subseteq \Lambda_j (B)} w_{i, q_1, q_2} \cdot f (\Gamma_i) \right); \]

where \( r_j \)’s are the coefficients calculated from \( r \): namely \( r_j = r (\Lambda_j) \) for \( j \in J_r \). Note that \( \sum_{j \in J_r} r_j = 1 \). We have already fixed an enumeration of \( (B_A) \). Then, it follows that

\[
f (q_1) - f (q_2) = \sum_{j \in J_r} r_j \cdot \left( \sum_{1 \leq i \leq m} w_i \cdot f (\Lambda_j \circ \eta_i) \right). \]

Denote \( \bar{f}_j := \left( f (\Lambda_j \circ \eta_1) \quad \cdots \quad f (\Lambda_j \circ \eta_m) \right)_{1 \times m} \) for \( j \in J_r \) which we call it the coloring matrix of \( \Lambda_j \). We can demonstrate all the calculations above in the following matrix presentation:

\[
f (q_1) - f (q_2) = \left( r_{j_1} \quad \cdots \quad r_{j_r} \right)_{1 \times r} \times \begin{pmatrix} \bar{f}_1 \\ \vdots \\ \bar{f}_r \end{pmatrix}_{r \times m} \times \begin{pmatrix} w^1_{q_1, q_2} \\ \vdots \\ w^m_{q_1, q_2} \end{pmatrix}_{m \times 1}, \]

where \( r := |J_r| \) and \( J_r = \{ j_1, \ldots, j_r \} \). We call the matrix \( \begin{pmatrix} w^1_{q_1, q_2} \\ \vdots \\ w^m_{q_1, q_2} \end{pmatrix}_{m \times 1} \) a weight matrix. Denote \( R^r \) for the matrix \( \left( r_{j_1} \quad \cdots \quad r_{j_r} \right)_{1 \times r} \) of coefficients of \( r \), which we call it a probability matrix. Denote \( F^f \) for the matrix

\[
\begin{pmatrix} \bar{f}_1 \\ \vdots \\ \bar{f}_r \end{pmatrix}_{r \times m}. \]

\textbf{Definition 30.} A \( m \times 1 \)-matrix \( W \) is called a Dirac-weight matrix if there are exactly one entry in \( W \) with value 1, exactly one entry value \(-1\) and all the other entries of \( W \) are 0.

It is clear that there are at most \( 2 \cdot \left( \begin{pmatrix} m \\ 2 \end{pmatrix} \right) \) many different Dirac-weight matrices.
Lemma 31. A coloring function \( f \) satisfies the convex \( \preceq \)-Ramsey condition if and only if there is a positive real valued probability \( 1 \times r \)-matrix \( R \) such that \( R \times F^j \times W \leq \frac{1}{2} \), for every Dirac-weight matrix \( W \).

Proof. \((\Rightarrow)\) Obvious.
\((\Leftarrow)\) Suppose \( V \) is a weight \( m \times 1 \)-matrix which is not 0 everywhere. Let \( V^+ := \{ v \in V : v > 0 \} \) and \( V^* := \{ v \in V : v \notin V^+ \} \). In the light of Corollary 27, we only need to check the cases that there are no monochromatic coloring matrices of \( \preceq \)-closed copies of \( B \). One can show that \( \sum_{v_i \in V^+} f_{j,i} \cdot v_i \leq \sum_{v_i \in V^+} v_i \leq 1 \), for all \( 1 \leq j \leq r \). For each \( 1 \leq j \leq r \) let \( W_j \) be a Dirac-weight matrix such that

1. \( w_{l,1} = -1 \) for some \( 1 \leq l \leq m \) whenever \( f_{j,l} = 0 \);
2. \( w_{i,1} = 1 \) for some \( 1 \leq i \leq m \) with \( v_i \in V^+ \) and \( f_{j,i} = 1 \); In case there no such \( v_i \in V^+ \) with \( f_{j,i} = 1 \), choose any \( i \neq l \) and let \( w_{i,1} = 1 \).

By the assumption, there is a real valued probability \( 1 \times r \)-matrix \( R \) such that \( R \times F^j \times W_j \leq \frac{1}{2} \), for each \( 1 \leq j \leq r \). A straightforward calculation shows that \( R \times F^j \times V \leq \frac{1}{2} \).

\[ \Box \]

3.2. The convex \( \preceq \)-Ramsey property and its correspondence with amenability. Similar to Section 22, we investigate the correspondence between the amenability of the automorphism group of the generic structure of a smooth class, and its convex \( \preceq \)-Ramsey property. In [13], similar correspondence has been given for the case of automorphism groups of Fraïssé-limit structures.

Theorem 32. Suppose \( M \) is the \((K, \preceq)\)-generic structure of a smooth class \((K, \preceq)\) with HP and AP. Then, the followings are equivalent:

1. \( \text{Aut}(M) \) has the convex \( \preceq \)-Ramsey property with respect to \((K, \preceq)\).
2. For every \( A, B \in K \) with \( A \preceq B \), there is \( C \in K \) such that \( B \preceq C \) and for every \( f : \langle C \rangle_A \rightarrow \{0,1\} \) there is \( p \in \langle C \rangle_B \) such that for every \( q_1, q_2 \in \langle C \rangle_A \),
   \[ |f(q_1) - f(q_2)| \leq \frac{1}{2}. \]
3. For every \( A, B \in K \) with \( A \preceq B \) and every \( \varepsilon > 0 \), there is \( C \in K \) such that \( B \preceq C \) and for every \( f : \langle C \rangle_A \rightarrow [0,1] \) there is \( p \in \langle C \rangle_B \) such that for every \( q_1, q_2 \in \langle C \rangle_A \),
   \[ |f(q_1) - f(q_2)| \leq \varepsilon. \]
4. For every \( A, B \in K \) with \( A \preceq B \) and every \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), there is \( C \in K \) such that \( B \preceq C \) for every sequence of functions \( f_i : \langle C \rangle_A \rightarrow [0,1] \) with \( i < n \), there is \( p \in \langle C \rangle_B \) such that for every \( q_1, q_2 \in \langle p \rangle_A \) and \( i < n \),
   \[ |f_i(q_1) - f_i(q_2)| \leq \varepsilon. \]
5. \( \text{Aut}(M) \) is amenable.

Proof. Proof of Theorem 6.1. in [13] can easily be modified for this case. \( \Box \)

Remark 33. We say a smooth have \((K, \preceq)\) have the convex \( \preceq \)-Ramsey property if condition (2) of the theorem above holds for \((K, \preceq)\). Then, on the bases of Theorem 32.
Aut \((M)\) has the convex \(\leq\)-Ramsey property with respect to \((K, \leq)\) if and only if \((K, \leq)\) have the convex \(\leq\)-Ramsey property.

3.3. Main results.

**Definition 34.** Suppose \(A, B \in K\) and \(A \leq B\). Let \(f : (M)_A \to \{0, 1\}\) be a coloring function and \(m := |(B)_A|\). For \(n > 0\), a \(n \times m\)-matrix \(Y\) is called a full-coloring matrix of \(f\) if every row of the matrix \(Y\) corresponds to a coloring matrix of a \(\leq\)-closed copy of \(B\) in \(M\), and conversely every coloring matrix of a \(\leq\)-closed copy of \(B\) corresponds to a row of \(Y\).

**Convection.** We always assume there is no repetition of similar rows in \(Y\).

**Remark 35.** Let \(Y\) be the full-coloring matrix of a coloring function \(f\). Then

1. It is an easy observation that when we interchange rows of \(Y\) we obtain a full-coloring matrix of the same coloring function.
2. It follows from Lemma 28 if the smooth class \((K, \leq)\) has the free-amalgamation property then, if \(f\) is not constant, \(Y\) should contain for every \(w \in \{0, 1\}\) and every \(1 \leq i \leq m\) at least a row whose \(i\)-th entry is \(w\), and moreover, there are infinitely many distinct \(\leq\)-closed copies of \(B\) such that the coloring matrix of them are exactly of that given row.
3. Suppose \(Y\) satisfies the convex \(\leq\)-Ramsey condition. This means there are a finite set \(\{\Lambda_i : i \in I_0\}\) of \(\leq\)-closed embeddings of \(B\) in \(M\), and a finitely supported probability measure \(p \in \langle M_B \rangle\) such that \(\text{supp}(p) = \{\Lambda_i : i \in I_0\}\) and \(|f(q_1) - f(q_2)| \leq \frac{1}{2}\), for every \(q_1, q_2 \in \langle p_A \rangle\). Suppose \(f'\) is another coloring function that the set of rows of its full-coloring matrix contains the set of the coloring of the rows of \(\{\Lambda_i(B) : i \in I_0\}\) under \(f\). Then clearly the convex \(\leq\)-Ramsey condition also holds for \(f'\).

So far, the matrices that we have considered are obtained from the full-coloring matrices of coloring functions for some \(A, B\) with \(A \leq B\) in a smooth class \((K, \leq)\). The matrix presentation suggests the following definition, without specifically referring to the class \((K, \leq)\) and any coloring function.

**Definition 36.** Let \(Y = \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix}\) be a \(n \times m\)-matrix whose entries are 0 or 1. Moreover, assume there is no repetition of similar rows in \(Y\). We say \(Y\) satisfies the convex Ramsey condition if there exist a \(r \times m\)-matrix \(X = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_r \end{bmatrix}\) such that \(\{\bar{x}_1, \ldots, \bar{x}_r\} \subseteq \{\bar{y}_1, \ldots, \bar{y}_n\}\) and a probability \(1 \times r\)-matrix \(R\) such that for every Dirac-weight \(m \times 1\)-matrices \(W\)

\[
R \times X \times W \leq \frac{1}{2}.
\]
**Question.** It is an interesting question to fully understand or classify, for fixed \( n, m \in \mathbb{N} \), all \( n \times m \)-matrices with the convex Ramsey condition. Then, the question of the convex \( \leq \)-Ramsey property for a \((K, \leq)\) is reduced to investigate whether in a smooth class \((K, \leq)\) any of such matrices can be full-coloring matrix of a coloring function.

**Lemma 37.** Suppose \( X \) is a \( r \times m \)-matrix with \( r > 0 \) such that \( R \) contains a column whose entries are 1, and a column whose entries are 0. Then, there is no probability \( 1 \times r \)-matrix \( R \) such that

\[
R \times X \leq \frac{1}{2}
\]

holds for all Dirac-weight matrices \( W \).

**Proof.** Suppose \( R \) is any probability matrix. Let \( Q := R \times X \). Suppose \( X_i^c, X_j^c \) are two columns of \( X \) whose all its entries are 1, and 0; respectively. Let \( W \) be a Dirac-weight matrix such that \( w_{i,1} := 1, w_{j,1} := -1 \) and \( w_{k,1} = 0 \) for all \( k \in \{1, \cdots, m\} \setminus \{i, j\} \). Then, \( q_{1,i} = R \times X_i^c = \sum_{1 \leq i \leq r} r_i = 1 \) and \( q_{1,j} = R \times X_j^c = 0 \). Now \( Q \times W = q_{1,i} \cdot w_{i,1} + q_{1,j} \cdot w_{j,1} = 1 > \frac{1}{2} \) which is a failure for above condition. \( \square \)

**Lemma 38.** For the following matrix

\[
Y := \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}_{6 \times 6}
\]

the convex Ramsey condition fails.

**Proof.** We need to prove for any \( r > 0 \) and \( r \times 6 \)-matrix \( X \), whose rows are chosen from rows of \( Y \), there is no probability \( 1 \times r \)-matrix \( R \) such that \( R \times X \times W \leq \frac{1}{2} \), for all Dirac-weight matrices \( W \).

From Lemma 37, it follows that for \( r = 1, 2 \) there is no probability \( 1 \times r \)-matrix \( R \) such that the convex Ramsey condition holds for \( X \), since a column of with constant 1 and a column of constant 0 appear in \( X \). For \( r = 3 \), the only cases that remain to be checked, again using Lemma 37, are

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]
Let $R := (r_1 \ r_2 \ r_3)_{1 \times 3}$ be any probability matrix i.e. $\sum_{1 \leq i \leq 3} r_i = 1$. Then $Q := R \times X = (1 \ 1 \ r_1 \ r_2 \ r_3)_{1 \times 6}$ and $(r_2 + r_3 \ r_1 + r_3 \ r_1 + r_2 \ 0 \ 0 \ 0)_{1 \times 6}$; respectively. Now consider

$$W_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad W_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad W_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix};$$

to be Dirac-weight matrices. It is easy to see that $Q \times W_1 = 1 - r_1$, $Q \times W_2 = 1 - r_2$, and $Q \times W_3 = 1 - r_3$. The convex Ramsey condition requires that $1 - r_1 \leq \frac{1}{2}$, $1 - r_2 \leq \frac{1}{2}$ and $1 - r_3 \leq \frac{1}{2}$. Hence $r_i \geq \frac{1}{2}$ for all $i \in \{1, 2, 3\}$, which is contradictory with the fact that $r_1 + r_2 + r_3 \leq 1$.

Now we assume $r > 3$. Note that for any $r \times m$-matrix $X$, we have the following easy properties which will help us for the subsequent calculations:

1. The number of entries $1$ in the first three columns is at most $3 \cdot r$ and at least $2 \cdot r$;
2. The number of entries $0$ in the first three columns is at most $r$. Moreover, entry $0$ might not occurs in these columns;
3. Dually, the same statement as (2) also holds for the number of entries $1$ in the last three columns.
4. In the first three columns never two entries $0$ appear in the same row;
5. Dually, in the last three columns never two entries $1$ appear in the same row.

Suppose now $R$ is a probability $1 \times r$-matrix. Then, the convex Ramsey condition requires the following matrix inequality to be true:

$$Q \times \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}_{6 \times 9} \leq \begin{pmatrix} \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}_{1 \times 9},$$

where $Q = R \times X$. Define the following set of indices: For $1 \leq l \leq 3$, let $I'_0 := \{i : r_{i,l} = 0\}$ and for $3 < l' \leq 6$, let $I''_1 := \{i : r_{i,l'} = 1\}$. By the above observation all
Then, it follows that 3·

Definition 39. Let \((\mathcal{K}, \leq)\) be a smooth class and \(\mathbf{M}\) the \((\mathcal{K}, \leq)\)-generic model. Let \(A, B \in \mathcal{K}\) with \(A \leq B\) and assume \(\Lambda_i, \Lambda_j \in \binom{\mathbf{M}}{B}\).

1. We say \(\Lambda_i\) and \(\Lambda_j\) are in the same connected competent with respect to \(A\) if there are \(n \geq 1\) and \(\Lambda_{i_1}, \ldots, \Lambda_{i_n} \in \binom{\mathbf{M}}{B}\) such that \(\Lambda_{i_1} = \Lambda_i, \Lambda_{i_n} = \Lambda_j\) and \(\Lambda_{i_j}(B) \cap \Lambda_{i_{j+1}}(B)\) contains at least one \(\leq\)-closed copy of \(A\) for \(1 \leq j \leq n\). We say \(\Lambda_i\) and \(\Lambda_j\) have distance \(n-1\) if \(n\) is the minimum number that satisfies the condition.

2. Let \(m \geq 2\). We say \(\Lambda_i\) and \(\Lambda_j\) lay on an \(m\)-cycle of embeddings over \(A\) if there exists distinct \(\Lambda_{i_1}, \ldots, \Lambda_{i_m} \in \binom{\mathbf{M}}{B}\) where \(\Lambda_{i_1} = \Lambda_i, \Lambda_{i_m} = \Lambda_j\) such that:
   a. \(\Lambda_{i_j}(B) \cap \Lambda_{i_{j+1}}(B)\) contain at least one \(\leq\)-closed copy of \(A\), for \(1 \leq j \leq m\);
   b. No pairs of elements of the set \(\{\Lambda_{i_j}(B) \cap \Lambda_{i_{j+1}}(B) : 1 \leq j \leq m\}\) contain a common \(\leq\)-closed copy of \(A\).

3. We say \((A; B)\) is a tree-pair if the following conditions hold:
   a. \(\Lambda(B) \cap \Lambda'(B)\) contains at most one \(\leq\)-closed copy of \(A\) for any two distinct \(\Lambda, \Lambda' \in \binom{\mathbf{M}}{B}\);
   b. Any two distinct \(\Lambda, \Lambda' \in \binom{\mathbf{M}}{B}\) never lay in an \(l\)-cycle for \(l \geq 2\).

Theorem 40. Suppose \((\mathcal{K}, \leq)\) is a smooth class with \(\text{AP}\) and \(\text{HP}\), and \(\mathbf{M}\) the \((\mathcal{K}, \leq)\)-generic structure. Suppose there are \(A, B \in \mathcal{K}\) and \(A \leq B\) such that \((A; B)\) is a tree-pair with \(|\binom{B}{A}| = 6\). Then, \(\text{Aut}(\mathbf{M})\) does not have the convex \(\leq\)-Ramsey property with respect to \((\mathcal{K}, \leq)\).
Proof. Our strategy is to present a coloring function $f : (M_A) \to \{0, 1\}$ such that the full-coloring matrix of $f$ for copies of $B$ is

$$Y := \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}_{6 \times 6}.$$

Then, Lemma 38 implies that $\text{Aut}(M)$ does not have the convex $\leq$-Ramsey property with respect to $(K, \leq)$. Take the connected components of elements of $(M_B)$ with respect to $A$. It is enough to present a suitable coloring function for each connected component, denoted by $\Lambda$ where $\Lambda \in (M_B)$. Since $(A; B)$ is a tree-pair, each connected component $[\Lambda]$ does not contain a cycle. We can define a coloring function $f$ on each $[\Lambda]$ by induction on the distance of elements of $[\Lambda]$ from a fixed element in the class.

Choose a row $\bar{y}$ in $Y$, and let $f_0$ be a coloring of $\leq$-closed copies of $A$ in $\Lambda(B)$ such that $f_0(\Lambda) = \bar{y}$. For each $i \geq 1$ and embeddings of distance $i$ from $\Lambda$, we inductively extend $f_i$ to $f_{i+1}$ and then define $f := \bigcup_{i<\omega} f_i$. Here, we only explain how to define $f_i$ on elements of $(M_B)$ which have distance one to $\Lambda$. Let $\Lambda' \in [\Lambda]$ be such an element. Then $\Lambda'(B) \cap \Lambda(B) = \Lambda' \circ \eta_i(A)$ for exactly one $i \in \{1, \ldots, 6\}$ (follows from the definition of a tree-pair).

Choose a row $\bar{y}$ of $Y$ such that $y_{1.1} = f_0(\Lambda' \circ \eta_i(A))$. Let $f_1$ be such that $f_1(\Lambda') = \bar{y}$. Note that any two embeddings of distance one from $\Lambda$ intersect at most in one copy of $A$ in $\Lambda(B)$. This follows from the fact that there are no 2-cycles in a connected competent of a tree-pair. Hence, one can assign a coloring, consistently, to each embedding of distance one and by Remark 29, each row of $Y$, will eventually appear in $f_1$. A similar argument works for embeddings of distance $i \geq 1$ inductively, as there are no $l$-cycles in a connected component in a tree-pair for $2 \leq l \leq 2i$. Therefore, we have $f_i$’s defined and finally the desired $f$ is obtained for each connected component. □

4. Amenability of automorphism groups of Ab-initio generic structures

Using the correspondence in Subsection 3.2 and Theorem 32, we show that the automorphism groups of Hrushovski-Fraïssé structures which are obtained from pre-dimensions with rational coefficients are not amenable groups. Moreover, for Hrushovski structures in these cases, we strengthen Theorem 25, by showing that the automorphism groups of their ordered generic structures are not extremely amenable (Theorem 25 only solves the binary case).

4.1. Pre-dimension functions with rational coefficients.

4.1.1. Setting. Suppose $(\mathcal{K}_{\alpha}^+, \leq)$ is an ab-initio class of finite relational $\mathfrak{L}$-structures, where $\mathfrak{L} = \{R\}$, that has been defined in Subsection 2.1.2. In this section, we assume $\alpha$ is a rational number. For simplicity, we denote $\delta, \mathcal{K}_0, \leq$ and $M_0$ for $\delta, \mathcal{K}_{\alpha}^+, \leq_\alpha$ and
M_0; respectively. We fix α = 2; the arguments can be easily modified for any rational α ≥ 1. Recall that we denote AB for the L-substructure A ∪ B in C and δ(A/B) = δ(AB) − δ(A) where A, B, C ∈ K_0 and A, B ⊆ C.

**Fact 41.** (See [14]) For each finite A, B, C ∈ M_0, the following holds

\[ \delta(ABC) = \delta(AB/C) + \delta(C) = \delta(A/BC) + \delta(B/C) + \delta(C). \]

Recall the definition of 0-minimally algebraic sets from Subsection 2.1.2 (i.e. Definition 3). The following remark is used later in the proof of Theorem 43.

**Remark 42.** Suppose A ∈ K_0, and let m ≥ max{3, |A|} be an integer. Consider C_m = \{c_1, \ldots, c_m\} such that \( \bigwedge_{1 ≤ i < m} R^{C_m}(c_i, c_{i+1}) \land R^{C_m}(c_1, c_m) \); i.e. C_m is a single m-cycle. Note that C_m ∈ K_0. Suppose (a_i : 1 ≤ i ≤ n) is an enumeration of elements of A. Now let D_m := A∪C_m such that \( \bigwedge_{1 ≤ i ≤ n} R^{D_m}(c_i, a_i) \land \bigwedge_{n < i ≤ m} R^{D_m}(c_i, a_n) \). It is easy to check that \( \delta(D_m/A) = 0 \), D_m ∈ K_0 and C_m is 0-minimally algebraic over A. Hence, there are infinitely many non-isomorphic 0-minimally algebraic sets over each A in K_0.

**Theorem 43.** There are A, B in K_0 with A ≤ B and \(|(B)| = 6\) such that (A; B) is a tree-pair. Therefore, Aut(M_0) does not have the convex ≤-Ramsey property with respect to (K_0, ≤).

**Proof.** Note that the number 6 (i.e. the number of copies A in B) is only needed to obtain the tree-pair that is required in Theorem 40 and proving the existence of tree-pairs that have more that 6 copies of A are similar. Define P_2(6) := \{u ⊆ \{1, \ldots, 6\} : |u| = 2\}.

**Claim (A).** There are A, B ∈ K_0 such that following conditions hold

1. A ∈ K_0 such that |A| = 2, δ(A) = 3 and there is no A′ ⊆ A with δ(A′) = 0.
2. B contains exactly six disjoint ≤-closed isomorphic copies A_1, · · ·, A_6 of A.
3. B = \bigcup_{1 ≤ i ≤ 6} A_i \cup \bigcup_{u ∈ P_2(6)} X_u such that the followings hold:
   a. \( \delta(X_u/(A_u A_{u,2})) = -1 \) for each \( u := \{u_1, u_2\} ∈ P_2(6) \);
   b. \( δ(X_u A_{u,2} \not\subseteq X_{v} A_{v,2} A_{v,2} \) for all \( v, u ∈ P_2(6) \) where \( u \neq v \);
   c. \( δ(X'_u/(A_u A_{u,2})) ≥ 0 \) for all \( \emptyset ≠ X'_u ⊆ X_u \) where \( u ∈ P_2(6) \).
4. \( δ(B) = δ(A) = 3. \)

**Proof of Claim A.** For each 1 ≤ i ≤ 6 let A_i := \{a_{1i}, a_{2i}, a_{3i}\} be an L-structure with \( R^{A_i}(a_{1i}, a_{2i}) \land R^{A_i}(a_{2i}, a_{3i}) \land R^{A_i}(a_{3i}, a_{1i}) \) such that \( A_i ∩ A_j = \emptyset \) for 1 ≤ i ≠ j ≤ 6. Let A denote the isomorphic type of A_i’s. It is clear that \( δ(A) = 3 \) and A ∈ K_0. Fix \( ζ : P_2(6) → \{1, \cdot \cdot \cdot, 15\} \) to be an enumeration of elements of P_2(6), without repetition. For each \( u ∈ P_2(6) \) put \( m_u := 6 · ζ(u) \). Now let X_u be isomorphic to C_m_u := \{c_1, · · ·, c_{m_u}\} (a cycle of length m_u; see Remark 42). It is clear that \( δ(C_{m_u}) = m_u \). As \( m_u ≥ 6 \) then X_u does not contain any substructure isomorphic to A, for each \( u ∈ P_2(6) \). Now let \( B = \bigcup_{1 ≤ i ≤ 6} A_i \cup \bigcup_{u ∈ P_2(6)} X_u \) be an L-structure such that A_i’s and X_u’s are L-structures as above, for 1 ≤ i ≤ 6 and \( u ∈ P_2(6) \); respectively, with the following additional relations: For each \( u ∈ \{u_1, u_2\} ∈ P_2(6) \) we have \( R^B(c_{1, u_2}, a_{2p, u_2}) \land R^B(c_{2p−1, u_2}, a_{3p}^u) \land R^B(c_{2p}, a_{3p}^u) \) where \( p^u ∈ \{1, 2, 3\} \) such that
Suppose $B \equiv p \pmod{3}$. One can check that in the $\mathcal{L}$-structure $B$ we have $\delta(X_u/A_{u_1}A_{u_2}) = m_u - (m_u + 1) = -1$ where $u = \{u_1, u_2\} \in P_2(6)$. Moreover, the only isomorphic copies of $A$ are $A_i$'s for $1 \leq i \leq 6$ in $B$. Hence $|\{(B)\}| = 6$. Furthermore as the $X_u$ contain cycles of different length, hence 3-(b) follows. One can also check that 3-(c) also follows. Finally, one can see $\delta(B) = 6 \cdot 3 - \binom{6}{2} = 18 - 15 = 3$, and $B \in K_0$. 

Let $A, B \in K_0$ be $\mathcal{L}$-structures that are obtained from Claim (A). Then

**Claim (B).**

1. $\delta(B') \geq 3$ for every $\bigcup_{i \leq 6} A_i \subseteq B' \subseteq B$.
2. Suppose $B' \not\subseteq B$ such that it contains at least two $\preceq$-closed copy of $A$. Then, $\delta(B') > 3$ and $cl(B') = B$.
3. $A_i \leq B$, for each $i \in \{1, \cdots, 6\}$.

**Proof of Claim B.** (1) By Condition 3-(c) of Claim (A), we have $\delta(X'/ (A_{u_1}A_{u_2})) \geq 0$, for all $\emptyset \neq X' \subsetneq X_u$. Therefore, $\delta(B') \geq \delta(B) = 3$.

(2) If $B'$ contains all $A_i$'s for $1 \leq i \leq 6$, then at least one $X_u$ does not fully contain $B'$ where $u \in P_2(6)$. So it is enough to show if $X' \subsetneq X_u$, then $\delta((B \setminus X_u) \cup X') > 3$. We have the following

\[
\delta(B) = \delta(B \setminus X_u) + \delta(X_u/(B \setminus X_u)) = \delta(B \setminus X_u) + \delta(X_u/A_{u_1}A_{u_2}) = \delta(B \setminus X_u) - 1.
\]

Therefore, $\delta(B \setminus X_u) = \delta(B) + 1 > 3$. Again by the assumption $\delta(X'/A_{u_1}A_{u_2}) \geq 0$ and then $\delta((B \setminus X_u) \cup X') = \delta(B \setminus X_u) + \delta(X'/ (B \setminus X_u)) = \delta(B \setminus X_u) + \delta(X'/ (A_{u_1}A_{u_2})) > 3$.

Suppose $B'$ contains only $n$-many $\preceq$-closed copies of $A$ for $1 < n < 6$. Without loss of generality assume $A_1, \cdots, A_n \subseteq B'$. With abuse of notation by $i \in n$, we mean $1 \leq i \leq n$. Then, $B' = \bigcup_{i \in n} A_i \cup X^* \cup A^*$ where $X^* \subsetneq \bigcup_{u \in P_2(6)} X_u$ and $A^* \subsetneq \bigcup_{j \notin n} A_j$. Note that by our assumption $A^*$ is a disjoint union of proper subsets of $A_j$. Let $X^* := X^*_1 \cup X^*_2$ where $X^*_1$ is the union of all $X_u$'s such that $X^* \cap X_u = X_u$ and let $X^*_2 := X^* \setminus X^*_1$. Note that $\delta(X'/ \bigcup_{1 \leq i \leq 6} A_i) = \delta(X'/A_{u_1}A_{u_2})$ for $X' \subsetneq X_u$. Then, by putting $X := \bigcup_{u \in P_2(6)} X_u$, from Condition 3-(c) of Claim (A), it follows

\[
\delta\left(X^*/ \bigcup_{i \in n} A_i \cup A^*\right) = \delta\left(X_1^*/ \bigcup_{i \in n} A_i \cup A^*\right) + \delta\left(X_2^*/ \bigcup_{i \in n} A_i \cup A^*\right) \geq \delta\left(X/ \bigcup_{i \in n} A_i \cup A^*\right).
\]

Thus

\[
\delta(B') = \delta\left(\bigcup_{i \in n} A_i \cup X^* \cup A^*\right) = \delta\left(\bigcup_{i \in n} A_i \cup A^*\right) + \delta\left(X^*/ \bigcup_{i \in n} A_i \cup A^*\right) \geq \delta\left(\bigcup_{i \in n} A_i \cup A^*\right)
\] \[\geq \delta(B'X).
\]

Now we only need to calculate $\delta(B'X)$. It is easy to see that $\delta\left(\bigcup_{i \in n} A_i \cup \bigcup_{u \in P_2(6)} X_u\right) = n \cdot 3 - \binom{n}{2} > 3$. Let $X_1^+ := \bigcup_{u \in P_2(6)} X_u$ and $X_2^+ := X \setminus X_1^+$. 

\[
\delta\left(\bigcup_{i \in n} A_i \cup \bigcup_{u \in P_2(6)} X_u\right) = n \cdot 3 - \binom{n}{2} > 3.
\]
We have $\delta \left( \frac{X_u}{\bigcup_{i \in n} A_i \cup A^*} \right) > \delta \left( \frac{X_u}{A_u, A_{u_1}} \right)$ when $X_u \subseteq X_2^+$. Therefore, it follows that $\delta \left( \frac{X_2^+}{\bigcup_{i \in n} A_i \cup A^*} \right) \geq 0$. Now

\[
\delta \left( B'X \right) = \delta \left( \bigcup_{i \in n} A_i \cup X \cup A^* \right) = \delta \left( \bigcup_{i \in n} A_i \cup A^* \right) + \delta \left( X/ \bigcup_{i \in n} A_i \cup A^* \right) = \delta \left( \bigcup_{i \in n} A_i \cup A^* \right) + \delta \left( X/ \bigcup_{i \in n} A_i \cup A^* \right) = \delta \left( \bigcup_{i \in n} A_i \right) + \delta \left( A^* \right) + \delta \left( X_1^+ / \bigcup_{i \in n} A_i \cup A^* \right) + \delta \left( X_2^+ / \bigcup_{i \in n} A_i \cup A^* \right) \geq \delta \left( \bigcup_{i \in n} A_i \cup X_1^+ \right) + \delta \left( A^* \right) \geq \delta \left( \bigcup_{i \in n} A_i \cup X_1^+ \right) \geq 3.
\]

Hence, $\text{cl}(B') = B$.

(3) Follows from (2) and (1). \qed

**Claim (C).** Let $\Lambda_1, \Lambda_2, \Lambda_3 \in \binom{M_0}{B}$. Then, the followings hold

1. If $\Lambda_1(B) \cap \Lambda_2(B)$ contains at least two $\ll$-closed copy of $A$, then $\Lambda_1 = \Lambda_2$.
2. Suppose $\Lambda_1(B) \cap \Lambda_2(B)$ contains exactly one $\ll$-closed copy of $A$. Then, $\Lambda_1(B) \Lambda_2(B) = \Lambda_1(B) \otimes_{\Lambda_3(B)} \Lambda_2(B)$ and $\delta \left( \Lambda_1(B) \Lambda_2(B) \right) = 3$. Furthermore, $\delta \left( \Lambda_1(B) \Lambda_2(B) \right) = 3$.

**Proof of Claim C.** (1) Follows from Claim B.

(2) For simplicity let $B_1 := \Lambda_1(B)$ and $B_2 := \Lambda_2(B)$.

It is easy to verify that

\[
\delta \left( B_1 B_2 \right) \leq \delta \left( B_1 \right) + \delta \left( B_2 \right) - \delta \left( B_1 \cap B_2 \right).
\]

The equality holds if and only if $B_1$ and $B_2$ are in the free-amalgamation over $B_1 \cap B_2$. Now

\[
3 \leq \delta \left( B_1 B_2 \right) \leq 3 + 3 - 3.
\]

Hence, $B_1$ and $B_2$ are in the free-amalgamation over $B_1 \cap B_2$ and $B_1 B_2$ is $\ll$-closed. Moreover, $\delta \left( \Lambda_1(B) \Lambda_2(B) \right) = 3$ follows from the fact that $\Lambda_1(B) \Lambda_2(B)$ is a $\ll$-closed set that it contains a $\ll$-closed copy of $A$, and it contained in $\Lambda_1(B)$. \qed

Now, we want to show $(A; B)$ is a tree-pair. Suppose, on the contrary, that a connected component of $\binom{M_0}{B}$ does contain an $m$-cycle for some $m > 1$ (i.e. there are distinct $\Lambda_1, \ldots, \Lambda_m \in \binom{M_0}{B}$ such that $\Lambda_j, \Lambda_{j+1}$ have distance one for $1 \leq j < m$, and $\Lambda_1(B) \Lambda_m(B)$ contains at least one $\ll$-closed copy of $A$ such that $\Lambda_1(B) \Lambda_m(B) \neq \Lambda_1(B) \Lambda_2(B)$). For simplicity, let $B_j := \Lambda_j(B)$, for $1 \leq j \leq m$. We further assume that the $m$-cycle is minimal which implies that the intersection of each successor pair of elements of $B_j$’s are distinct. By Claim C, $B_i \cap B_{i+1}$ contains exactly one $\ll$-closed
copy of $A$. Now

$$
\begin{align*}
\delta(B_1 \cdots B_m) & \leq \quad \delta(B_1 \cdots B_{m-1}) + \delta(B_m) - \delta((B_1 \cdots B_{m-1}) \cap B_m) \\
& \leq \quad \delta(B_1 \cdots B_{m-1}) + 3 - 2\cdot \delta(A) \\
& \leq \quad \delta(B_1 \cdots B_{m-1}) - 3 \\
& \leq \quad \delta(B_1 \cdots B_{m-2}) + \delta(B_{m-1}) - \delta((B_1 \cdots B_{m-2}) \cap B_{m-1}) - 3 \\
& \vdots \\
& \leq \quad \delta(B_1) - 3 = 0.
\end{align*}
$$

This contradicts with the fact that for each $B_i$ we have $B_i \subseteq B_1 \cdots B_n$ and $B_i$ is $\leq$-closed in $M_0$. Therefore, each connected component of $\binom{M_0}{B}$ with respect to $A$ does not contain a cycle and hence $(A; B)$ is a tree-pair.

$\square$

Epand the language $L$ by adding a binary relation $\prec$ and let $\mathcal{L}^+ := \mathcal{L} \cup \{\prec\}$. Let $\mathcal{K}_{\alpha}^+$ be the set all $\mathcal{L}^+$-expansions of structures $C \in \mathcal{K}_0$, in which the relation $\prec$ is interpreted as a linear-ordering on the universe of $C$. For $E^{\mathcal{L}^+}, F^{\mathcal{L}^+} \in \mathcal{K}_{\alpha}^+$ we define $E^{\mathcal{L}^+} \preceq F^{\mathcal{L}^+}$ if and only if $E^\mathcal{L} \subseteq F^\mathcal{L}$ and $E \preceq F$ where $A^\mathcal{L}$ and $B^\mathcal{L}$ are $\mathcal{L}$-expansions of graphs $A$ and $B$; respectively. Similar to the proof above, we can show $\left(A^{\mathcal{L}^+}; B^{\mathcal{L}^+}\right)$ in again a tree-pair where the $A^{\mathcal{L}^+} \subseteq B^{\mathcal{L}^+}$, $A^{\mathcal{L}^+} \upharpoonright \mathcal{L} = A$ and $B^{\mathcal{L}^+} \upharpoonright \mathcal{L} = B$. Therefore, the following theorem is established. Indeed, in the case of pre-dimension functions with rational coefficients, Theorem 25 extends to a wider class of generic structures.

**Theorem 44.** The automorphism groups of ordered Hrushovski generic structures that are obtained from pre-dimension functions with rational coefficients are not extremely amenable.

**Remark 45.** In [10], it was asked whether there are any links between the extension property (known also as Hrushovski property in [8, 6]) of the Fraïssé class and the extremely amenability of the automorphism group of the Fraïssé-limit. By a result of the first author in [3], the class $\mathcal{K}_0$ does not have the extension property. Existence of certain kind of tree-pairs and the extension property of the class $\mathcal{K}_0$ seems to be related.

### 4.2. Collapsed ab-initio generic structures.

In [5], in order to obtain a strongly minimal structure, Hrushovski restricts the uncollapsed ab-initio class $\mathcal{K}_{\alpha}^+$ to a smaller class $\mathcal{K}_{\alpha}^\mu$, using a finite-to-one function $\mu$ over the 0-minimally algebraic elements. We have already mentioned the class $\mathcal{K}_{\alpha}^\mu$ and the $(\mathcal{K}_{\alpha}^\mu, \preceq_\alpha)$-generic structure $M_{\alpha}^\mu$ in Subsection 2.1.2. The structure $M_{\alpha}^\mu$ is called the collapsed ab-initio generic structure. It has to be indicated that the relation $\mathfrak{R}$ in the language $\mathcal{L}$ that Hrushovski considers is a ternary symmetric relation and $\alpha = 1$. Note that “collapsing” with the $\mu$ function is applicable just for ab-initio generic structures which are obtained from pre-dimension functions with rational coefficients. Similar to Theorem 43, one can show there are $A, B \in \mathcal{K}_{\alpha}^\mu$ such that $|\binom{B}{A}| = 6$ and $(A; B)$ form a tree-pair. Hence,
Theorem 46. The automorphism groups of collapsed ab-initio generic structures and specially the automorphism group of Hrushovski’s strongly minimal set are not amenable.

4.3. Remaining cases. In the previous subsection, we only dealt with ab-initio generic structures that are obtained from pre-dimension functions with rational coefficients. For ab-initio generic structures that are obtained from pre-dimension functions with irrational coefficients, the amenability of their automorphism group is left unanswered in this manuscript. This includes the \( \omega \)-categorical pseudo-plane constructed by Hrushovski (see [14]) and \( M'_\alpha \) that we have mentioned in Subsection 2.1.2. There are also other interesting variants of Hrushovski’s construction that are simple (see [11]). In all these cases tree-pairs do not exist. However, it seems plausible to modify the techniques of the present paper to assign a coloring function, to a pair \((A;B)\) whose graph might have cycles, that corresponds to a matrix which does not satisfy the convex Ramsey condition. As we mentioned before, David M. Evans shows, using a different method, the automorphism groups of generic structures that are obtained from pre-dimension functions with irrational coefficients and the \( \omega \)-categorical generic structures are not amenable.

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