Solution to the gauge-Higgs analyticity paradox

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Abstract

The Fradkin-Shenker theorem proves analyticity in a region that connects Higgs to confinement regimes, precluding a phase transition. This conflicts with a simpler analyticity argument applicable to any symmetry-breaking phase transition that requires the phase diagram to be bifurcated. A flaw in the Fradkin-Shenker and related Osterwalder-Seiler proofs is found which removes this paradox. Higgs and Confinement regions are everywhere separated by a phase boundary. A new order parameter allowing this transition to be traced with Monte-Carlo simulations without gauge fixing is introduced.

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One of the few rigorous results in lattice gauge theory is the Fradkin Shenker (FS) theorem[1]. It states that when both gauge and Higgs fields are in the fundamental representation a finite-width region exists in the gauge-Higgs coupling plane where expectation values of local operators are analytic functions of the couplings. Consequently, the Higgs and confinement phases are continuously connected, not separated by a phase transition, despite differing qualitatively. That both phases are massive makes this conceivable. The FS theorem applies to the fixed-magnitude Higgs field. It specializes an earlier theorem by Osterwalder and Seiler[2] (OS) which proved a region of analyticity in the Higgs phase for a variable-magnitude Higgs, and for pure gauge theory at strong coupling. The proof relies on a convergent cluster expansion, a technique developed earlier for $p(\phi)_2$ theories by Glimm, Jaffee, and Spencer(GLS)[3]. Although the series converges, it is shown below that individual terms being summed can themselves be non-analytic, which invalidates the proof. In fact a simpler analyticity argument based on the exactness of the broken symmetry leads to the opposite conclusion,
that Higgs and confinement regions are everywhere separated by a phase transition.

The system has action

\[ S = -\beta \sum_p \frac{1}{D} (\text{tr}(U_p) - D) - \lambda \sum_{\vec{r}, \mu} \frac{1}{D} \text{tr}(\phi^\dagger(\vec{r}) U_\mu(\vec{r}) \phi(\vec{r} + \hat{\mu})) \]  

(1)

where \( U_\mu \) are gauge fields on links belonging to the fundamental representation of a gauge group, \( U_p \) is the elementary plaquette made from the product of four links \( UU^\dagger U^\dagger \), and \( \phi \) are site-based matrix-valued Higgs fields also in the fundamental representation. \( \beta \approx 1/g^2 \) is the inverse gauge coupling and \( \lambda \) is the Higgs coupling. \( D \) is the representation dimension. For complex representations, real parts are taken. The cases of 3-d \( \mathbb{Z}_2 \), which is self-dual, and 4-d \( SU(2) \) are emphasized here. Fig. 1 shows the phase diagram for \( \mathbb{Z}_2 \). The analyticity region (AR) is the striped area and is representative - not drawn to scale. All theories in the above category have similar analyticity regions and a Higgs transition that presumably ends before reaching it. Monte Carlo simulations show a generally strong transition at high \( \beta \) becoming weaker as \( \beta \) is lowered and eventually looking like a crossover. The exact terminus of the transition (critical point) in the different theories is somewhat disputed, as is whether it is first or second order at the endpoint (orders are group dependent)[4, 5]. Another transition joining the Higgs transition originates on the pure-gauge axis in some theories, separating Coulomb and confinement phases.

At \( \beta = \infty \) is a spin model, 3-d Ising in the 3-d \( \mathbb{Z}_2 \) case and O(4) Heisenberg for 4-d \( SU(2) \). These have well known symmetry-breaking magnetic phase transitions. The phase transition survives entering the phase diagram, in some cases changing order. The symmetry-breaking nature of the phase transition becomes hidden by the prohibition of local symmetry breaking(Elitzur’s theorem[6]). The same issue confronts the Higgs mechanism in the standard model, which no one doubts is driven by spontaneous symmetry breaking. Symmetry breaking becomes visible in Landau gauge, which leaves a remnant global symmetry unfixed. The remnant symmetry is the symmetry broken in the Higgs phase, driving the Higgs mechanism. The symmetry-breaking nature of the Higgs transition conflicts with the FS theorem. A phase transition line separating symmetry-realized and symmetry-broken regions cannot end in a critical point, as in Fig. 1, because an analytic function which vanishes in a finite region vanishes everywhere. For an exact symmetry this applies to the order parameter in the symmetry-realized phase. Everywhere it touches the symmetry-broken region is non-analytic. Thus the transition must continue until it hits an edge of the phase diagram. There is only one option for the endpoint (explained in detail later), the upper left corner. Thus a paradox exists. The FS theorem claims analyticity in the same region that the simple argument above, also based on analytic
Figure 1: Gauge-Higgs phase diagram for the 3-d Z2 theory. Phase transitions (on right) are solid, based on ref. [5]. Self-dual line is dotted. A representative Fradkin-Shenker analyticity region(AR) is striped. Dashed lines represent a possible alternative AR based on roughening transitions, with on-axis locations from ref. [10].

function theory, predicts a phase transition. Numerical observation of apparently continuous crossovers in energy quantities in this region cannot be trusted to disprove the existence of a phase transition, because some phase transitions are only weakly displayed in energy quantities. Generally, using numerical methods applied to energy quantities alone, one can detect only phase transitions for which the infinite-lattice specific heat becomes infinite. A specific heat peak is observed growing with lattice size if the critical exponent $\alpha > 0$. However, if $\alpha < 0$ the specific heat remains finite at the critical point and shows little finite-size effect. In this case it is a derivative of the specific heat which diverges. It may develop a cusp or a curvature cusp, for instance, which are difficult to distinguish numerically from continuous crossovers.

If one has an order parameter to study, however, then weak phase transitions are detectable from finite-size crossings of Binder cumulant or normalized correlation length. Both have opposite scaling behavior on opposite sides of a phase transition and are lattice-size invariant at the phase transition. Caudy and Greensite(CG) have studied the Higgs transition in 4-d SU(2) using Landau gauge[7]. A regular gauge-invariant Monte Carlo sim-
ulation is run and gauge configurations are transformed to lattice Landau
gauge which seeks to maximize $\text{tr}(U)$ for all links. Since a global gauge trans-
formation leaves $\text{tr}(U)$ invariant, that remains an exact symmetry subject
to spontaneous symmetry breaking. To the extent that gauge fixing can
produce many links close to the identity, the system approaches the spin
model (where links are unity). It resembles a spin glass where some of the
interactions are not fully ferromagnetic. As $\beta$ decreases interaction disorder
increases until eventually ferromagnetism is lost. It can be regained by in-
creasing $\lambda$ which favors order. This explains the slope of the phase transition
line. The Higgs field expectation value $\langle \phi \rangle$ is the order parameter. CG find
a definite symmetry breaking for all $\beta$ tested. To rationalize this result with
the FS theorem, they have suggested the phase transition might at some
point become non-thermal, and turn into something like a Kertész line. However a Kertész line occurs when explicit symmetry breaking smooths a
singularity, such as an Ising model in a weak magnetic field. The percola-
tion transition persists, but is not associated with an energy singularity. But
neither is there an order parameter singularity in this case. What is needed
for the gauge-Higgs model is for the energy to be non-singular even though
the order parameter is singular, in other words for the energy to simply not
care about the order parameter anymore. Such a critical point would be
unique in the annals of statistical mechanics. If the fields carry energy then
how could a singularity in the field not affect the energy in some way? If the
energy no longer cared about the field, then it would randomize, preventing
magnetization. A system magnetizes due to the dependence of the free en-
ergy on magnetization. This requires the free energy and magnetization to
be correlated, which will transmit singularities in the magnetization to the
free energy, because these are both sharp quantities in the thermodynamic
limit.

CG give another argument to discount the observed phase transition. A
similar study was done in the lattice Coulomb gauge, which maximizes the
traces of links in only three of four directions, leaving a much larger remnant
gauge symmetry, global in three directions and local in one. Expectation
values of gauge links pointing in the fourth direction $\langle \sum U_4 \rangle$ summed over
each hyperlayer are the order parameters. They find symmetry breaking in
this quantity also, however in one region it is along a line above the Landau
gauge result. They argue this apparent “gauge ambiguity” could signify an
artifact. However, there is a good reason why the Coulomb gauge result
may differ from Landau gauge. These are different order parameters testing
different symmetries (see Appendix). There may well be two parallel phase transitions in part of the phase diagram, with an interesting phase between
them, in which the Higgs symmetry is spontaneously broken but the gauge
fields are still confined. Without a long range force to drive the Higgs
mechanism a massless Goldstone boson may exist here. Another possible
explanation for the Coulomb gauge result differing are potential problems with Gribov copies, local gauge condition minima. So this result should probably be checked for possible systematic errors from gauge fixing, and a finite size scaling study be undertaken to precisely fix the location, before a separate transition is definitively concluded. To demonstrate that gauge fixing isn’t somehow creating a false signal, below I introduce a new order parameter for the Higgs transition that doesn’t require gauge fixing. The phase transition it sees corresponds to that seen using Landau gauge, which makes sense since both are sensitive to the Higgs symmetry breaking. So at least this transition is not gauge-dependent.

Consider the shape of the analyticity region (AR). FS have computed this explicitly for the Z2 case shown. Based on inequalities, it is a lower limit of the true AR. The strong coupling expansion in the pure gauge theory appears to converge up to the roughening transition which for pure-gauge SU(2) is around $\beta = 1.9$ [9]. The first actual singularity appears to occur here. Roughening also exists for the 3-d Ising model and its dual, the 3-d Ising gauge theory [10]. These transitions presumably enter the phase diagram. Since increasing $\lambda$ increases order, one would expect that to stay on this transition one would have to compensate by lowering $\beta$. This results in the “true” AR (roughening transition line) sloping oppositely to the FS AR. For 3-d Z2, a dual roughening transition exists in the Higgs phase. Opposite slope of the FS AR from the likely behavior of the physical AR is not necessarily inconsistent, because the roughening transitions could conceivably join the main transition at some point. However, the opposite slope of the FS AR from an order-disorder contour still seems odd.

The upper left corner of the phase diagram has a demonstrable singularity [11]. The $\beta = 0$ theory is solvable. In axial gauge, no interactions perpendicular to the gauge-fixed direction remain, resulting in a set of 1-d Ising or Heisenberg models. These are disordered for all finite $\lambda$ but are ordered at $\lambda = \infty$. The 1-d Ising model has a phase transition at $T = 0$, accompanied by an essential singularity, in our notation at $1/\lambda = 0$. This result can also be obtained by a single integration in unitary gauge. The Heisenberg case is somewhat less singular but still the correlation length diverges [12]. Thus a singularity exists deep within the FS AR. This makes a natural endpoint for the phase transition and is the self-dual line endpoint. However, it is possible this is an isolated singular point. The FS theorem is evaded simply because the first term in the cluster expansion is itself singular here.

Summarizing, two different analyticity arguments, the FS theorem and the exact symmetry-breaking argument, have opposite conclusions. One forbids a phase transition whereas the other requires one. Accepting FS means accepting the concept of an unprecedented exact-symmetry non-thermal transition. The other possibility is a flaw in the FS argument, with the
AR region sloping in the physically sensible direction, pinching down at the upper left corner, allowing the phase transition through (alternative AR in Fig. 1). The FS theorem relies on proving the convergence of the cluster expansion. It uses unitary gauge which removes the Higgs field entirely. The Higgs action becomes

$$S_H = -\lambda \sum_{\vec{r}, \mu} \frac{1}{D} \text{tr} U_{\mu}(\vec{r}).$$  \hspace{1cm} (2)

Consider the expectation value of a local operator $F$ such as the average plaquette. The cluster expansion rewrites the individual plaquette Boltzmann factor

$$\exp(-\beta S_p) = 1 + p_p$$ \hspace{1cm} (3)

For small $\beta$, $p_p$ are small. The Higgs action, now local on links, is absorbed into the link measure $dU$. The cluster expansion for $\langle F \rangle$ is given by

$$\langle F \rangle = \sum_{Q_1(Q_0)} \int dU F \prod_{p \in Q_1} p_p \frac{Z(Q_1 \cup Q_0)}{Z}. \hspace{1cm} (4)$$

Here $Q_0$ is the set of plaquettes connected to $F$. The expansion is over all connected sets of plaquettes $Q_1$ connected with $Q_0$. $Z$ is the partition function and $Z(Q_1 \cup Q_0)$ is the partition function missing all plaquettes in $Q_1 \cup Q_0$ and any touching its boundary. For details see FS\textsuperscript{[1]} and OS\textsuperscript{[2]}. As $Q_1$ grows terms have a larger number of the small factors $p_p$ which for small $\beta$ form a convergent series. The ratio of the two partition functions sums the disconnected diagrams and does not spoil the series convergence for small enough $\beta$ or large enough $\lambda$. Small $\beta$ and large $\lambda$ both aid convergence which explains the slope of the FS AR. Quoting FS, “Analyticity of $\langle F \rangle$ in (the couplings) (follows), because the series converges uniformly and the terms are each analytic.” There seems little doubt the series converges in the region claimed. However, the second condition that the individual terms are analytic is not addressed in either FS or OS. Presumably it was thought too obvious to require proof. Recall that non-analyticity of an individual term is the loophole that allows a singularity at the upper corner. Could this problem be more widespread? The suspicious factor is the ratio of the two partition functions, one missing some of its plaquettes. Since a partition function is the sum of an infinite number of terms in the thermodynamic limit, here we have a finite ratio of two infinite quantities. It is precisely such a ratio that gives rise to thermodynamic singularities. It is not at all apparent that such a factor is singularity free. Consider the simplest case of $Z(\text{missing a single plaquette})/Z$ which can be written $\langle \exp(\beta S_p) \rangle$ where $S_p$ is the single plaquette action. If we expand the exponential this contains a term of $\langle S_p \rangle$ the expectation value of the average plaquette itself. So
in the cluster expansion for the average plaquette, some of the terms on the RHS also include factors of the average plaquette and other more complex expectation values. Thus if the average plaquette has a singularity, there are singularities on both sides of the equation which seems perfectly consistent regardless of convergence. Therefore a convergent cluster expansion is not sufficient to prove analyticity. It is consistent with expectation values either being singular or not. One must also prove that each ratio of partition functions on the RHS is itself analytic. This flaw is common to both FS and the second part of OS which concerns the Higgs mechanism. It may be possible to save the first part of OS which covers the pure gauge theory, by first proving exponential clustering from a symmetry argument. Singularities in expectation values arise from massless excitations and infinite-range forces. Say the plaquette-plaquette correlation function follows a power law. Then the \( n \)th derivative of the average plaquette with respect to \( \beta \) is an integrated \( n + 1 \)-point function which diverges for large enough \( n \). This will not happen for a massive theory where correlation functions fall exponentially (exponential clustering). So if one can prove exponential clustering independently of the cluster expansion, that may serve as input to an analyticity proof. In GJS, exponential clustering is proven early on from the \( \phi \to -\phi \) symmetry of even-power \( p(\phi) \) theories. A similar symmetry is present in the pure gauge theory, \( U \to -U \), but not for \( \lambda \neq 0 \).

The above doubt cast on the FS proof removes the paradox. The fact that the symmetry being broken is exact yields a powerful argument in favor of a phase transition bifurcating the diagram. It is not necessary to argue away the transitions observed in Monte Carlo simulations. Higgs and confinement may both be massive phases but apparently they are different massive phases.

**Appendix**

Here the symmetries of the gauge-Higgs system are examined more carefully and a new gauge-invariant order parameter for the Higgs transition is introduced. This helps ally any concern that the observed phase transitions could be gauge-fixing artifacts. The original action in eqn. (1) is invariant under the local gauge transformation, \( V(\vec{r}) \)

\[
U_\mu(\vec{r}) \to V(\vec{r})U_\mu(\vec{r})V^\dagger(\vec{r} + \hat{\mu}), \quad \phi(\vec{r}) \to V(\vec{r})\phi(\vec{r})
\]

(5)

There is also a separate global symmetry transformation, \( W \), that affects only the \( \phi \) fields,

\[
\phi(\vec{r}) \to \phi(\vec{r})W.
\]

(6)

In the Landau gauge, a single global \( V \) remains and the \( \phi \) transforms according to \( \phi \to V\phi W \). If \( \phi \) takes an expectation value, both the \( W \) and
\( V \) symmetries are spontaneously broken. For the SU(2) case this gives an SU(2) \( \times \) SU(2) = O(4) space of broken vacua. To further separate these and probe the W symmetry alone, one can borrow a trick from the study of spin glasses, the “two real replica” technique\[13\]. For a given gauge configuration (which has an associated Higgs configuration), one equilibrates another instance of the Higgs field \( \phi_r \). Because this is not a genuine second Higgs field that the gauge field “knows” about through detailed balance, a full Monte-Carlo equilibration is required. Measured quantities approach asymptotic values exponentially with equilibration time, which can be adjusted to drive systematic errors as low as desired. Generally \( \epsilon \)-folding times are in the hundreds of sweeps and practical equilibration times in the low thousands. Now there are two independently generated \( \phi \) fields, each of which is from either a spontaneously broken ensemble of the W symmetry or from an unbroken one. If broken, the \( \phi_r \) and \( \phi \) will choose different symmetry-breaking directions. Indeed \( \phi_r \) has a separate global symmetry \( W_r \). The order parameter is \(< m >\), where
\[
\begin{align*}
m &= |\sum_{\vec{r}} \phi_r^\dagger(\vec{r})\phi(\vec{r})|.
\end{align*}
\]
(7)
The sum is over 4-space. The norm is O(4). Because both \( \phi \)’s are equilibrated to the same gauge background they similarly adjust to the gauge topography. The order parameter is gauge invariant, but sensitive to the W-symmetry. For unbroken W-symmetry both \( \phi \) and \( \phi_r \) will have tunnelings within each configuration which will destroy any overall correlation when spatially summed. However, for broken W symmetry an O(4) set of spontaneous broken vacua arise, with nonzero \(< m >\). Such an order parameter is usually used to find spin-glass order but it will also detect ferromagnetic order, which is the case here because the transition observed is consistent with the location of the Landau-gauge transition seen by CG. Fig. 2 shows the Binder cumulant, \( U_B = 1 - \frac{< m^4 >}{3 < m^2 >^2} \) for \( \beta = 1.2 \) and various \( \lambda \) on 16\(^4\) and 24\(^4\) lattices. An equilibration study was first performed to ensure systematic errors were less than 10% of statistical errors. This required 1800 sweeps for the 24\(^4\) case. One sees a crossing at \( \lambda = 1.36 \) (verified in upper region at 15 standard deviations and also in the normalized correlation length). The susceptibility shows a growing peak. Preliminary fits to finite size scaling give a critical exponent \( \nu = 0.8 \). The current uncertainty is roughly ±0.15. This study is being extended to include more lattice sizes and better statistics to further narrow this estimate and establish a connection with energy quantities. The \( \nu \) estimate predicts \( \alpha = 2 - d\nu = -1.2 \) with about a 50% uncertainty, the negative value consistent with a finite specific heat. Crossings are also observed at \( \beta = 0.5 \). The correlation between this order parameter and internal energy is non-zero in the broken phase, which demonstrates that the energy “cares” about this symmetry breaking. Details will be reported elsewhere.
Figure 2: The Binder cumulant for the replica-field correlation order parameter (s-shape, left scale) and susceptibility (right scale), for $\beta = 1.2$ on $16^4$ (triangles) and $24^4$ (boxes) lattices. Error bars computed from binned fluctuations.

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