On gravitational lensing by deflectors in motion

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ABSTRACT
Gravitational lensing by a spinning deflector in translational motion relative to the observer is discussed in the weak field, slow motion approximation. The effect of rotation, which generates an intrinsic gravito-magnetic field, separates from that due to radial motion. Corrections to the lens equation, deflection angle and time delay are derived.

Key words: gravitation – gravitational lensing – relativity

1 INTRODUCTION
Gravitational lensing is a well recognized practical tool in modern astrophysics. It is one of the most deeply investigated phenomena of gravitation and progressive development of technological capabilities demands to lead a full analysis on the basis of higher-order effects (Sereno 2003a). Corrections due to the motion of the deflector deserve particular attention on both theoretical and phenomenological sides. The motion of the lens might play a role in various astrophysical systems (Frittelli 2003b, Sereno 2003a, Sereno & Cardone 2002, Sereno 2004). The measurement of an effect on time delay of luminous signals, due to a translational motion of the deflector, was recently claimed in Fomalont & Kopeikin (2003) but a debate about correct theory and analysis behind the experiment is still under way (Samuel 2004, Kopeikin 2005). Besides translational motion on a static background, mass-energy currents relative to other masses generate space-time curvature. This phenomenon, known as intrinsic gravito-magnetism, is a new feature of gravity and is still waiting for an experimental confirmation (Ciufolini & Wheeler 1995).

Various authors have investigated the problem of light deflection by lenses in motion. The effect of a translational motion of the deflector on a static background has been widely discussed (Pyne & Birkinshaw 1993, Kopeikin & Schäfer 1999, Frittelli 2003b, Heyrovsky 2004, Wucknitz & Sperhake 2004) and an early disagreement on the correction factor of first order in velocity has been solved (see Wucknitz & Sperhake 2004 and references therein). Gravitational lensing by spinning deflectors has been also addressed with very different approaches (Epstein & Shapiro 1980, Ibáñez & Martin 1983, Ibáñez 1983, Dymnikova 1986, Glicenstein 1999, Ciufolini et al. 2003, Sereno 2004, 2002, 2003b, Sereno & Cardone 2002). The two phenomena are really different since the intrinsic angular momentum of a body cannot be generated or eliminated by a Lorentz transformation. Whereas the effect due to a translational motion is a consequence of the existence of the standard gravito-electric field plus local invariance, the problem of light deflection by a deflector with angular momentum is related to the existence of the dragging of inertial frames and of the related gravito-magnetic field. In order to show that intrinsic gravito-magnetism and the local Lorentz invariance on a static background are two fundamentally different phenomena, space-time curvature by mass-energy currents relative to other mass can be precisely characterized by a frame- and coordinate-independent method, based on space-time curvature invariants (see Ciufolini & Wheeler 1995, Section 6.11).

In this letter, we address the problem of light deflection by an extended lens with angular momentum in translational motion on a static background. To this aim, we adopt the usual framework of gravitational lensing theory, i.e. \( i \) weak field and slow motion approximation for the lens and \( ii \) thin lens hypothesis (Schneider et al. 1993, Petters et al. 2001). The paper is as follows. In Sec. 2 gravitational lensing in a stationary space-time is reviewed. In Sec. 3 we discuss the combined effect of a roto-translational motion of the deflector on time delay. Section 4 is devoted to the discussion of the bending angle and to the writing of the lens equation. Section 5 contains some final considerations.

2 STATIONARY SPINNING LENSES
We are interested in the gravitational field by a weak deflector in slow motion. Matter velocities are much less than \( c \), the speed of light in the vacuum, and matter stresses are also small. The metric is asymptotically flat and deviates only slightly from the Minkowski one. Up to leading order in \( c^{-3} \), such a metric can be written in Cartesian coordinates \( x^i \equiv \{ t, \mathbf{x} \} \), \( i \) as
\[
\begin{align*}
\frac{ds^2}{c^2} & \simeq \left( 1 + 2 \frac{\phi}{c^2} \right) c^2 dt^2 - 8 c dt \frac{\mathbf{V} \cdot \mathbf{dx}}{c^3} - \left( 1 - 2 \frac{\phi}{c^2} \right) d\mathbf{x} \cdot d\mathbf{x},
\end{align*}
\]
where a dot represents the Euclidean scalar product between two three-dimensional vectors. Let us consider a stationary space-time. Here, \( \phi \) reduces to the Newtonian potential,
\[
\phi_s(\mathbf{x}) \simeq -G \int_{\mathbb{R}^3} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} \, d^3x',
\]
where \( \rho \) is the mass density and \( G \) is the gravitational constant, and \( V \) is a vector potential taking into account the gravito-magnetic field produced by mass currents,
\[
V_s(\mathbf{x}) \simeq -G \int_{\mathbb{R}^3} \frac{(\mathbf{v})(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} \, d^3x',
\]
where \( \mathbf{v} \) is the velocity field of the mass elements of the deflector. In terms of an angular velocity \( \omega \), \( \mathbf{v} = \omega \times \mathbf{x} \).

In a conformally stationary space-time, actual light rays can be determined through the Fermat’s principle (Schneider et al. 1992), which states that light travels a path whose optical length is stationary compared with neighbouring paths (Rossi 1957). A curved space-time embedded with a stationary metric can be interpreted as a flat one with an effective index of refraction, \( n_s \) (Schneider et al. 1992; Sereno 2003b). A light signal emitted at the source will arrive after
\[
\Delta T = \frac{1}{c} \int_{\gamma} n_s \, dl,
\]
where \( \gamma \) is the spatial projection of the light curve. The Fermat’s principle can be expressed as
\[
\delta \int_{\gamma} n_s \, dl = 0.
\]

For a stationary, slowly rotating, weak lens, the effective refraction index, \( n_s \), is given by (Schneider et al. 1992; Sereno 2003b)
\[
n_s \simeq 1 - \frac{2}{c^2} \phi_x + \frac{4}{c^2} V_s \cdot \mathbf{e},
\]
where \( \mathbf{e} \equiv \frac{dx}{dt} \) is the unit tangent vector of a ray and \( dl \equiv \sqrt{n_s (dx^2 + dy^2 + dz^2)} \) is the Euclidean arc length. The total travel time is (Sereno 2003b)
\[
\Delta T_s \simeq \Delta T_s^{\text{geo}} + \Delta T_s^{\text{Sh}} + \Delta T_s^{\text{GRM}}
\]
where \( \Delta T_s^{\text{geo}} \) is the geometrical time delay, \( \Delta T_s^{\text{Sh}} \) is the Shapiro time delay, i.e. the gravitational delay at the post-Newtonian order,
\[
\Delta T_s^{\text{Sh}} \equiv \frac{2}{c^3} \int_{\gamma} \phi_s \, dl,
\]
and \( \Delta T_s^{\text{GRM}} \) is the gravito-magnetic time delay
\[
\Delta T_s^{\text{GRM}} \equiv \frac{4}{c^4} \int_{\gamma} V_s \cdot d\mathbf{e} \cdot dl.
\]

As usual, the lens is assumed to be thin and weak. The actual path of the photon deviates negligibly from the undeflected path. It is useful to employ the spatial orthogonal coordinates \( (\xi, \eta, l) \), centred on the lens and such that the \( l \)-axis is along the incoming light ray direction \( \mathbf{e}_l \) and the vector \( \xi \) spans the lens plane. So, in the integrand functions, we can use \( \mathbf{x} = \mathbf{e}_l + \xi \mathbf{e}_\xi + \eta \mathbf{e}_\eta \). The unperturbed photon impacts the lens plane in \( \xi_\text{los} \). The main contribution to the potential time delay is (Schneider et al. 1992)
\[
\Delta T_s^{\text{Sh}} \simeq -\frac{4G}{c^3} \int_{\mathbb{R}^2} d^2 \xi' \Sigma(\xi') \ln \frac{\langle |\xi' - \xi| \rangle}{\xi_0},
\]
where \( \xi_0 \) is a length scale in the lens plane and \( \Sigma \) is the projected surface mass density of the deflector,
\[
\Sigma(\xi) \equiv \int_{\xi_\text{los}} \rho(\xi, l) \, dl;
\]
the gravito-magnetic correction to the potential time delay, up to the order \( v/c \), can be expressed as (Sereno 2002)
\[
\Delta T_s^{\text{GRM}} \simeq \frac{8G}{c^3} \int_{\mathbb{R}^2} d^2 \xi' \Sigma(\xi') \langle v \cdot e_{\text{los}}(\xi') \rangle \ln \frac{|\xi - \xi'|}{\xi_0},
\]
where \( \langle v \cdot e_{\text{los}}(\xi) \rangle \) is the weighted average, along the line of sight \( e_{\text{los}} \), of the component of the velocity \( v \) along \( e_{\text{los}} \).

In the thin lens approximation, the only components of the velocities parallel to the line of sight enter the equations of gravitational lensing. We remind that the time delay function is not an observable, but the time delay between two actual rays can be measured. In the above expressions for the time delays, we have neglected some not relevant additive constants.

### 3 TIME DELAY BY SHIFTING AND SPINNING LENSES

We want now to consider an additional translational motion of the deflector. The corresponding metric can be obtained by applying a coordinate transformation to the metric of the same deflector with the center of mass at rest, which is expressed by Eq. (1) with the potentials in Eqs. (2, 3). We limit to a motion with slow velocity, \( u \sim \mathcal{O}(v) \), and negligible acceleration. Hereafter, in this section, the primed coordinates will refer to the rest frame. A rigid motion along a path \( \gamma \) can be accounted for by a change of coordinates \( x \to x = x + \gamma(t) \) (Frittelli 2003a). Limiting to negligible acceleration of the path, the change in the time coordinate is such that \( dt = dt' + \gamma(t) dx / c^2 \) (Frittelli 2003a). The metric takes the same form in Eq. (1) with
\[
\phi(x, t) = \phi_0(x - \gamma(t)),
\]
\[
V(x, t) = V_s(x - \gamma(t)) + \frac{u}{c} \phi_0(x - \gamma(t)),
\]
where \( u \equiv \dot{\gamma} \).

The metric in Eq. (1) with the potentials given in Eq. (12, 13) is no more stationary and Fermat’s principle cannot be applied as done in Sec. 2. However, under suitable assumptions, the computation of the gravitational lensing quantities can proceed nearly in the same way. In fact, in usual astrophysical systems, the transit time through the deflector can be considered small with respect to the total travel time (Schneider et al. 1992). We can assume that interaction happens instantaneously at the moment when the photon passes the lens, \( t_d \). The position of the center of mass of the lens at \( t_d = 0 \) locates the origin of the coordinate system. During the transit time, the velocities are approximately constant. So, we can consider the components of the metric tensor fixed at \( t = t_d \) and proceed as in the stationary case. The corresponding ad hoc refraction index is written solving for \( dt \) the equation \( ds^2 = 0 \), and properly considering the difference between the proper arc-length in a curved space-time and the Euclidean arc length (Sereno 2003b). We obtain
\[
n \simeq 1 - \left( 1 - 2 \frac{u_{\text{los}}(t_d)}{c} \right) \frac{2 \phi_0(x - \gamma(t_d))}{c^2}
\]
On the motion of a gravitational lens

4 LENS EQUATION

Just as for a stationary space-time, where the Fermat’s principle exactly holds, even if the lens is in slow motion, the bending angle $\alpha$ and the gravitational time delay, $\Delta T_{\text{pot}} = \Delta T - \Delta T_{\text{geo}}$, can be related by a gradient (Frittelli 2003a).

$$\alpha = -\nabla_\perp(c\Delta T_{\text{pot}}),$$

where $\nabla_\perp \equiv \nabla - \mathbf{e}_l (\mathbf{e}_l \cdot \nabla)$. The bending angle turns out to be

$$\alpha(\xi) \simeq \frac{4G}{c^2} \int_{R^2} d^2\xi' \left\{ \frac{\xi - \xi'}{|\xi - \xi'|^2} - \mathbf{U}(\xi') \cdot \mathbf{e}_l \right\} \left( 1 - \frac{u_{\text{los}}}{c} + \frac{2}{c} (\mathbf{v} \cdot \mathbf{e}_l)_{\text{los}} (\xi - \mathbf{u}_l t_d) \right).$$

Once again, three terms contribute to $\alpha$. The main term is due to the static, gravito-electric field of the corresponding lens at rest. The second term derives from the spin and take the same form as for a stationary lens with angular momentum. The third effect on the bending angle, due to a translational motion of the deflector, can be interpreted in terms of standard aberration of light in an optically active medium with an effective index of refraction induced by the gravitational field of the lens (Frittelli 2003a). The bending angle by a point-like moving lens carries a pre-factor $(1 - u_{\text{los}}/c)$ with respect to the bending angle in the static case (Frittelli 2003a). For a stationary spinning deflector, the path of the light-ray does not stay on a plane (Sereno 2004), as it is for a spherically symmetric lens. However, since in our approximations, deviations induced by the angular momentum of the lens are small, the same arguments based on aberration can be applied as in Frittelli (2003a), providing an independent confirmation to Eq. (25) as far as a radial motion is concerned.

Let us finally write the lens equation. We introduce the angular position of the source in the plane of the sky, $\beta$, and the angular position of the image in the lens plane $\theta = \xi/D_s$, where $D_s$ is the angular diameter distance from the observer to the deflector. For a system of point-like lenses, velocity effects do not alter the form of the lens equation with respect to the static case (Kopeikin & Schäfer 1999; Heyrovsky 2004). For the general case of a spinning and shifting deflector, the lens equation takes the form,

$$\beta = \theta - \frac{D_{\text{los}}}{D_s} \alpha(\theta),$$

where and $D_s$ and $D_{\text{los}}$ are the angular diameter distances from the observer to the source and from the deflector to the source, respectively. With respect to the static case, the position of the lens is time dependent and the bending angle is corrected for factors due to the radial motion and the angular momentum of the deflector.

In the limit of slow velocities, translational and rotational motions are separated and add together to correct lensing quantities. In general, this is not true. To first order in deflection, the deflection angle for a lens with a relativistic translational motion can be directly obtained from the stationary deflection angle caused by a lens at rest by applying a Lorentz transformation (Wucknitz & Sperhake 2004). The stationary deflection angle can be written as a sum of the gravito-electric and the gravito-magnetic terms, $\alpha^{\text{GRE}} + \alpha^{\text{GRM}}$. For a relativistic translational velocity, $\alpha(\mathbf{u}) = \sqrt{\frac{1 - u_{\text{los}}/c}{1 + u_{\text{los}}/c}} (\alpha^{\text{GRE}} + \alpha^{\text{GRM}})$.
The scaling factor in Eq. (27), which reduces to $1 - u_{los}/c$ for a slow motion, applies in the same way to the main term and to the gravito-magnetic correction.

5 CONCLUDING REMARKS

We have considered the effects on gravitational lensing of both the translational motion on a static background and of an intrinsic angular momentum of the deflector. In the weak-field, slow motion approximation the effects on bending and time delay of light rays are separated and added together. Only the radial motion of the centre of mass enters the corrections, whereas the transverse motion has to be considered only when determining the lens position and the relative separation between lens and source. The radial and the angular velocity enter the corrections with a different scaling. The internal rotational velocity must be weighted along the line of sight and carries an extra factor 2 with respect to the radial velocity, that must be evaluated at the time the photon passes the lens. This shows that results in Sereno (2002), Sereno & Cardone (2002), which refer to spinning lenses, are correct, differently from what claimed in Wucknitz & Sperhake (2004). However, results in Sereno (2002) can not be applied to a radial motion.

Thanks to usual approximations, our analysis does not limit to point-like lenses and extended deflector have been considered. The internal velocity field enters the equations. This allows to solve the question whether the gravito-magnetic correction to the deflection of light can probe the inner structure of a lens (Wucknitz & Sperhake 2004). Even if the main contribution for a light ray propagating outside the lens is due to the total angular momentum, light rays crossing the inner structure of the deflector probe only a “partial” angular momentum.

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