ROUNDING THE ARITHMETIC MEAN VALUE OF THE SQUARE ROOTS OF THE FIRST \( n \) INTEGERS

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Abstract. In this article we study the arithmetic mean value \( \Sigma(n) \) of the square roots of the first \( n \) integers. For this quantity, we develop an asymptotic expression, and derive a formula for its integer part which has been conjectured recently in the work of M. Merca. Furthermore, we address the numerical evaluation of \( \Sigma(n) \) for large \( n \gg 1 \).

The aim of this article is to derive an explicit formula for the integer part of the arithmetic mean value of the square roots of the first \( n \) integers. More precisely, we consider the sequence

\[
\Sigma(n) = \frac{1}{n} \sum_{k=1}^{n} \sqrt{k}, \quad n \in \mathbb{N},
\]

and show the following identity.

**Theorem 1.** For any \( n \in \mathbb{N} \), there holds that

\[
\lfloor \Sigma(n) \rfloor = \lfloor A(n) \rfloor,
\]

where we define the function

\[
A(x) = \frac{2}{3} \sqrt{x+1} \left(1 + \frac{1}{4x}\right),
\]

for \( x \geq 1 \). Here, \( \lfloor \cdot \rfloor \) signifies the integer part of a positive real number.

This result is motivated by the recent work [Mer17, see Conjecture 2], where Theorem 1 has been conjectured.

1. An asymptotic result

In order to prove Theorem 1, we begin by deriving an asymptotic result for the sum of the square roots of the first \( n \) integers. Here, we employ an idea presented in [Mer17], which is based on using the trapezium rule for the numerical approximation of integrals. In this context, we also point to the related work [She13], where upper and lower Riemann sums have been applied. In comparison to the analysis pursued in [Mer17], in the current paper, we use a different approach to control the error in the trapezium rule. Thereby, we arrive at a slightly sharper asymptotic representation for large \( n \). Incidentally, an asymptotic representation has been derived already in the early work [Ram00].

**Theorem 2.** For any \( \nu, n \in \mathbb{N} \), with \( \nu < n \), there holds

\[
\sum_{k=\nu}^{n} \sqrt{k} = nA(n) - \frac{2}{3} \sqrt{\sigma} \left(\nu - \frac{3}{4}\right) - \frac{\delta_{\nu,n}}{24},
\]

with

\[
\sigma(\nu+2, n+2) < \delta_{\nu,n} < \sigma(\nu, n),
\]

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where

\[ \sigma(\nu, n) = \begin{cases} 
\frac{3}{2} - n^{-1/2} & \text{if } \nu = 1, \\
(\nu - 1)^{-1/2} - n^{-1/2} & \text{if } \nu \geq 2. 
\end{cases} \]

Proof. Let us consider the function \( f(x) = \sqrt{x} \), for \( x \geq 1 \). We interpolate it by a piecewise linear function \( \ell \) in the points \( x = 1, 2, \ldots \), i.e., for any \( k \in \mathbb{N} \) there holds \( \ell(k) = \sqrt{k} \), and \( \ell \) is a linear polynomial on the interval \([k, k+1]\). We define the remainder term

\[ \tilde{\delta}_{\nu,n} := \int_{\nu}^{n+1} (\sqrt{x} - \ell(x)) \, dx = \frac{2}{3} \left( (n+1)^{3/2} - \nu^{3/2} \right) - \int_{\nu}^{n+1} \ell(x) \, dx. \]

Then, we note that

\[ \int_{\nu}^{n+1} \ell(x) \, dx = \frac{1}{2} \sqrt{\nu} + \sum_{k=\nu+1}^{n} \sqrt{k} + \frac{1}{2} \sqrt{n+1}; \]

this is the trapezium rule for the numerical integration of \( f \). Therefore,

\[ \sum_{k=\nu+1}^{n} \sqrt{k} = \frac{2}{3} \left( (n+1)^{3/2} - \nu^{3/2} \right) - \frac{1}{2} \sqrt{\nu} - \frac{1}{2} \sqrt{n+1} - \tilde{\delta}_{\nu,n} \]

\[ = \frac{2}{3} \sqrt{n+1} \left( n + \frac{1}{4} \right) - \frac{2}{3} \sqrt{\nu} \left( \nu + \frac{3}{4} \right) - \tilde{\delta}_{\nu,n}. \]

Hence,

\[ \sum_{k=\nu}^{n} \sqrt{k} = \frac{2}{3} \sqrt{n+1} \left( n + \frac{1}{4} \right) - \frac{2}{3} \sqrt{\nu} \left( \nu + \frac{3}{4} \right) - \tilde{\delta}_{\nu,n} \]

\[ = nA(n) - \frac{2}{3} \sqrt{\nu} \left( \nu + \frac{3}{4} \right) - \tilde{\delta}_{\nu,n}. \]

It remains to study the error term \( \tilde{\delta}_{\nu,n} \). For this purpose, applying twofold integration by parts, for \( k \in \mathbb{N} \), we note that

\[ \int_{k}^{k+1} (\sqrt{x} - \ell(x)) \, dx = \frac{1}{8} \int_{k}^{k+1} (\sqrt{x} - \ell(x)) \frac{d^2}{dx^2} \left( 1 - 4(x - k - 1/2)^2 \right) \, dx \]

\[ = \frac{1}{32} \int_{k}^{k+1} x^{-3/2} \left( 1 - 4(x - k - 1/2)^2 \right) \, dx \]

\[ < \frac{1}{32} k^{-3/2} \int_{k}^{k+1} \left( 1 - 4(x - k - 1/2)^2 \right) \, dx \]

\[ = \frac{1}{48} k^{-3/2}. \]

Thus, if \( \nu \geq 2 \), we obtain

\[ \tilde{\delta}_{\nu,n} = \sum_{k=\nu}^{n} \int_{k}^{k+1} (\sqrt{x} - \ell(x)) \, dx \]

\[ < \frac{1}{48} \sum_{k=\nu}^{n} k^{-3/2} \]

\[ < \frac{1}{48} \int_{\nu-1}^{n} x^{-3/2} \, dx \]

\[ = \frac{1}{24} \left( (\nu - 1)^{-1/2} - n^{-1/2} \right). \]
Otherwise, if $\nu = 1$, the above bound implies
\[ \hat{\delta}_{1,n} < \frac{1}{48} + \frac{1}{24} \left( \frac{3}{2} - n^{-1/2} \right). \]
Similarly, we have
\[
\hat{\delta}_{\nu,n} > \frac{1}{48} \sum_{k=\nu}^{n} (k + 1)^{-3/2} \\
> \frac{1}{48} \int_{\nu+1}^{n+2} x^{-3/2} \, dx \\
= \frac{1}{24} \left( (\nu + 1)^{-1/2} - (n + 2)^{-1/2} \right).
\]
This completes the proof. \(\square\)

For $\nu = 1$ the above result implies the identity
\[
(4) \quad \Sigma(n) = A(n) - \frac{1}{6n} - \frac{\hat{\delta}_{1,n}}{24n^2},
\]
which will be crucial in the analysis below.

**Remark 1.** Proceeding in the same way as in the proof of Theorem 2, a formula for the more general case of the arithmetic mean value of the $r$-th roots of the first $n$ integers, with $r \geq 1$, can be derived: More precisely, for any $\nu, n \in \mathbb{N}$, with $\nu < n$, there holds
\[
\sum_{k=\nu}^{n} k^{1/r} = \frac{r}{r+1} (n+1)^{1/r} \left( n + \frac{1-1/r}{2} \right) - \frac{r}{r+1} \nu^{1/r} \left( \nu - \frac{1+1/r}{2} \right) - \hat{\delta}_{\nu,n,r},
\]
with $\hat{\delta}_{\nu,1} = 0$ (i.e., for $r = 1$), and $\sigma_r(\nu + 2, n + 2) < \hat{\delta}_{\nu,n,r} < \sigma_r(\nu, n)$ for $r > 1$, where
\[
\sigma_r(\nu, n) = \begin{cases} 
2^{1/r} - n^{-1+1/r} & \text{if } \nu = 1, \\
(\nu - 1)^{-1+1/r} - n^{-1+1/r} & \text{if } \nu \geq 2.
\end{cases}
\]

### 2. Proof of Theorem 1

The proof of Theorem 1 is based on the ensuing two auxiliary results. The first lemma provides tight upper and lower bounds on $A$ from (2). The purpose of the second lemma is to identify any points where the integer part of $A$ changes.

**Lemma 1.** The function $A$ from (2) is strictly monotone increasing for $x \geq 1$. Furthermore, there hold the bounds
\[
(5) \quad A(x) < \frac{2}{3} \sqrt{x + 2} \quad \text{for } x \geq 2,
\]
and
\[
(6) \quad A(x) > \frac{2}{3} \sqrt{x + \frac{5}{4} + \frac{1}{4x}} \quad \text{for } x \geq 6.
\]

**Proof.** The strict monotonicity of $A$ follows directly from the fact that
\[ A'(x) = \frac{4x^2 - x - 2}{12x^2 \sqrt{x + 1}} > 0, \]
for any $x \geq 1$. Furthermore, we notice that the graph of $A$ and of the function
\[ h(x) = \frac{2}{3} \sqrt{x + 2}, \]
which is the upper bound in (5), have exactly one positive intersection point at $x^* = (9 + \sqrt{113})/16 < 2$. Moreover, choosing $x = 2 > x^*$, for instance, we have

$$A(2) = \frac{3\sqrt{3}}{4} < \frac{4}{3} = h(2).$$

Thus, we conclude that $A(x) < h(x)$ for any $x > x^*$; this yields (5). The lower bound (6) follows from an analogous argument.

\textbf{Lemma 2.} For any $m \in \mathbb{N}_0$, let $\alpha(m) = \frac{9}{4}(m + 1)^2 - 2$. Then, for $n, m \in \mathbb{N}$, with $n \leq \alpha(m)$, there holds that $A(n) < m + 1$. Conversely, we have $A(n) - 1/4n > m + 1$, for any integer $n > \alpha(m)$.

\textbf{Proof.} Consider $m, n \in \mathbb{N}$ such that $1 \leq n \leq \alpha(m)$. Applying the monotonicity of $A$, cf. Lemma 1, together with (5), we infer

$$A(n) \leq A(\alpha(m)) < \frac{2}{3} \sqrt{\alpha(m) + 2} = m + 1.$$ 

Furthermore, if $m = 2s$, with $s \in \mathbb{N}$, is even, then we have $\alpha(2s) = 9s^2 + 9s + 1/4$; moreover, if $m = 2s - 1$, with $s \in \mathbb{N}$, is odd, then there holds $\alpha(2s - 1) = 9s^2 - 2$. In particular, we conclude that $n \geq \alpha(m) + 1$ for any $n \in \mathbb{N}$ with $n > \alpha(m)$. Then, involving (6), it follows that

$$A(n) - \frac{1}{4n} > 2 \sqrt{n + \frac{5}{4}} \geq \frac{2}{3} \sqrt{\alpha(m) + 2} \geq m + 1,$$

which yields the lemma. \hfill \Box

\textbf{Proof of Theorem 1.} We are now ready to prove the identity (1). To this end, given $n \in \mathbb{N}$, we define

$$m := \min\{k \in \mathbb{N} : \alpha(k) \geq n\} \in \mathbb{N}.$$ 

Evidently, there holds $n \leq \alpha(m)$ as well as $n > \alpha(m - 1)$. By virtue of (1) and due to Lemma 2 there holds

$$1 \leq \Sigma(n) < A(n) < m + 1.$$ 

If $m = 1$, the proof of the theorem is complete. Otherwise, if $m \geq 2$, then by means of Theorem 2 we notice that $\delta_{1,n} < 3/2$. Then, recalling (4), this leads to

$$\Sigma(n) > A(n) - \frac{1}{6n} - \frac{1}{16n} = A(n) - \frac{11}{48n}.$$ 

Hence, upon employing Lemma 2 (with $n > \alpha(m - 1)$), it follows that

$$\Sigma(n) > A(n) - \frac{1}{4n} > m.$$ 

Combining the above estimates, we deduce the bounds

$$m < \Sigma(n) < A(n) < m + 1.$$ 

This shows (1).

\section{3. Numerical evaluation of $\Sigma(n)$}

For large values of $n$ the straightforward computation of $\Sigma(n)$, i.e., simply adding the numbers $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}$, and dividing by $n$, is computationally slow and prone to roundoff errors. For this reason, we propose an alternative approach: if $n \gg 1$, we choose $\nu \in \mathbb{N}$, with $\nu + 1 < n$, of moderate size (so that the numerical evaluation of $\Sigma(\nu)$ is well-conditioned and accurate). Then, we write

$$\Sigma(n) = \frac{1}{n} \left( \nu \Sigma(\nu) + \sum_{k=\nu+1}^{n} \sqrt{k} \right).$$
Here, employing Theorem 2 we notice that

\[ \sum_{k=\nu+1}^{n} \sqrt{k} = nA(n) - \nu A(\nu) - \frac{\delta_{\nu+1,n}}{24}. \]

Hence, upon defining the approximation

\[ \tilde{\Sigma}(\nu, n) := \frac{1}{n} \left( nA(n) + \nu \Sigma(\nu) - \nu A(\nu) \right), \]

it follows that

\[ \left| \Sigma(n) - \tilde{\Sigma}(\nu, n) \right| \leq \frac{\delta_{\nu+1,n}}{24n}, \]

with

\[ \delta_{\nu+1,n} < \sigma(\nu + 1, n) = \nu^{-1/2} - n^{-1/2} = n^{-1/2} \left( \left( \frac{\nu}{n} \right)^{-1/2} - 1 \right); \]

cf. (3). In this way, we infer the error estimate

(7) \[ \left| \Sigma(n) - \tilde{\Sigma}(\nu, n) \right| < \frac{n^{-3/2}}{24} \left( \left( \frac{\nu}{n} \right)^{-1/2} - 1 \right). \]

In particular, given a prescribed tolerance \( \epsilon > 0 \), we require

\[ \nu = \left\lceil \frac{n (24\epsilon n^{-1/2} + 1)^{2}} {n - 2} \right\rceil, \]

with \( \nu \leq n - 2 \), in order to deduce the guaranteed bound

\[ \left| \Sigma(n) - \tilde{\Sigma}(\nu, n) \right| < \epsilon. \]

To give an example, we consider \( n = 10^7 \). Then, choosing \( \nu = 100 \) leads to the numerical value

\[ \tilde{\Sigma}(n) = 2108.1852648724285 \ldots \]

In this particular case, the error bound (7) gives

\[ \left| \Sigma(n) - \tilde{\Sigma}(\nu, n) \right| \leq 4.1535 \times 10^{-10}. \]

This estimate is fairly sharp; indeed, the true error is approximately \( 4.1328 \times 10^{-10} \).

**References**

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