Ideals in Toeplitz Algebras*

Ronald G. Douglas

Abstract

We determine the ideal structure of the Toeplitz $C^*$-algebra on the bidisk

0 Introduction

A large part of doing research in mathematics is asking the right question. Posing a timely question can trigger thought and provoke insights, even when the matter has no good resolution. That is what happened in the case at hand. After devoting much time to the study of Hilbert modules and quotient Hilbert modules in particular, I had occasion to wonder if a $C^*$-algebra perspective on this topic might not be useful. More precisely, I asked myself whether there was any relationship between the $C^*$-algebras for a Hilbert module and its quotient module generated by the operators obtained from module multiplication. The answer turned out to be negative but the techniques and approach used to reach that conclusion, namely an analysis of the ideal structure of the Toeplitz $C^*$-algebras, seemed worth reporting on. Although this general topic has been considered earlier (cf. [6]), these studies focused almost exclusively on index theory which will not concern us here.

Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $A(\Omega)$ be the function algebra obtained from the closure in the supremum norm on $\Omega$ of all functions holomorphic on some neighborhood of the closure of $\Omega$. In this paper we will focus on the case of the unit ball $\mathbb{B}^m$ and the polydisk $\mathbb{D}^m$ for $m = 2$.

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A Hilbert module $\mathcal{M}$ over $A(\Omega)$ is a Hilbert space together with a unital module map $A(\Omega \times \mathcal{M} \to \mathcal{M}$ which is continuous. Here we confine attention to contractive Hilbert modules; that is, a module $\mathcal{M}$ for which the inequality $\|\varphi f\|_{\mathcal{M}} \leq \|\varphi\|_{A(\Omega)} \|f\|_{\mathcal{M}}$ holds for $\varphi$ in $A(\Omega)$ and $f$ in $\mathcal{M}$. Actually, we will be concerned only with the Hardy modules $H^2(D^m)$ and $H^2(B^m)$ on $D^m$ and $B^m$, respectively, and their quotient modules. Recall that the Hilbert space for this module can be obtained as the closure of $A(D^m)$ in the space $L^2((\partial D)^m)$ and $L^2(B^m)$ relative to Lebesgue measure on the distinguished boundary $(\partial D)^m$ of $D^m$ and the boundary $\partial D^m$, respectively.

In [4] quotient modules obtained from a hypersurface $\mathcal{J}$ are studied and a characterization is obtained of them based on the intrinsic geometry of how $\mathcal{J}$ sets in $\Omega$ and the spectral sheaf for the ambient Hilbert module. We will not provide any details on those results here except to point out that their precision might suggest that an intimate relationship might also exist for the corresponding $C^*$-algebras. In particular, let $\mathcal{C}^*(\mathcal{M})$ denote the $C^*$-algebra generated by the operators on $\mathcal{M}$ defined by module multiplication. The question which spawned this study is whether there is a relationship between $\mathcal{C}^*(R)$ and $\mathcal{C}^*(R/R_0)$ for $R$ the Hardy module and $R_0$ the submodule of functions in $R$ that vanish on $\mathcal{J}$. Another way to phrase the question is whether the map defined by $M_\psi \to N_\psi$ for $\psi$ in $A(\Omega)$ extends to the $C^*$-algebras $\mathcal{C}^*(R)$ and $\mathcal{C}^*(R/R_0)$, where $M_\psi$ and $N_\psi$ are the operators defined by module multiplication on $R$ and $R/R_0$, respectively.

1 Quotient Modules Determined by Hypersurfaces

One way to approach this question is by first determining the ideal $I$ in $\mathcal{C}^*(\mathcal{R})$ generated by $\{M_\varphi \mid \varphi |_{\mathcal{J}} \equiv 0\}$, which leads to the topic of this paper: What are the ideals in $\mathcal{C}^*(\mathcal{R})$? More particularly, determine the ideal generated by a collection of operators and calculate the quotient $C^*$-algebra. For the case of the ball, $\mathcal{C}^*(H^2(B^m))$ contains the ideal $\mathfrak{K}(H^2(B^m))$ of compact operators on $H^2(B^m)$ and the quotient is commutative and isomorphic to $C(\partial B^m)$. Thus the ideal structure of $\mathcal{C}^*(H^2(B^m))$ is especially simple. The ideals in $\mathcal{C}^*(H^2(B^m))$ are $\mathfrak{K}(H^2(B^m))$ and those corresponding to closed subsets of $\partial B^m$. As a result, there can be no map from $\mathcal{C}^*(H^2(B^m))$ to the $C^*$-algebra determined by a quotient module since the latter algebras are not commutative. The situation for $\mathcal{C}^*(H^2(D^m))$, however, is more interesting. Rather than tackling this problem in full generality at the outset, however, we begin by analyzing some examples.

Let us consider three hypersurfaces in the closure of $D^2$; the quotient modules they determine;
and the $\mathcal{C}^*$-algebra maps:

$$Z_1 = \{(z_1, z_2) \mid z_1 = 0, |z_2| \leq 1\},$$

$$Z_2 = \{(z_1, z_2) \mid z_1 = 1, |z_2| \leq 1\}$$

and

$$Z_3 = \{(z_1, z_2) \mid z_1 = z_2, |z_1| \leq 1, |z_2| \leq 1\}.$$

Using the results in [4] or by explicit calculation, one can show the quotient $C^*$-algebras are:

$$Q_1 = H^2(D^2)/\{f \in H^2(D^2) \mid f|_{Z_1} \equiv 0\} \cong H^2(D),$$

$$Q_2 = H^2(D^2)/\{f \in H^2(D^2) \mid f|_{Z_2} \equiv 0\} = (0),$$

and

$$Q_3 = H^2(D^2)/\{f \in H^2(D^2) \mid f|_{Z_3} \equiv 0\} \cong B^2(D).$$

(Here, $B^2(D)$ is the Hilbert module over the disk algebra $A(D)$ determined by the Bergman space on $D$.)

To make sense of these equivalences, we must provide a few more details. In the first case, the associated module multiplication on $H^2(D)$ has $z_1$ acting as the zero operator while $z_2$ is just multiplication by $z$ on $H^2(D)$. In the second case, we interpret the condition $f|_{Z_2} \equiv 0$ to select the set of functions in $A(D^2)$ that vanish on $Z_2$. Hence, the closure in $H^2(D^2)$ of such functions is all of $H^2(D^2)$ and thus the quotient module is $(0)$. Finally, the quotient $Q_3$ is isometrically isomorphic to the Bergman space with both multiplication by $z_1$ and $z_2$ corresponding to multiplication by $z$. (This equivalence was first observed by Rudin [5].) To examine whether the maps from $M_\varphi$ to $N_\varphi$ for $\varphi$ in $A(D^2)$ extend to the $C^*$-algebras, we want to consider the ideals $I_1$, $I_2$ and $I_3$ in $\mathcal{C}^*(H^2(D^2))$ generated by the sets $\{M_\varphi \mid \varphi|_{Z_i} \equiv 0\}$ for $i = 1, 2, 3$. We state and prove our general theorem characterizing ideals after working out these examples.

## 2 Ideals in Toeplitz $C^*$-Algebras—Three Examples

We begin by recalling something about the nature of the various Toeplitz $C^*$-algebras. For $H^2(D^2)$ we now denote it by $\mathfrak{T}(H^2(D^2))$. This bidisk Toeplitz $C^*$-algebra was introduced by R. Howe and the author in [3] where it was used to study the invertibility and Fredholmness of Toeplitz operators in the quarter-plane. We’ll recall the techniques developed in that paper a little later. Next, the standard Toeplitz $C^*$-algebra $\mathfrak{T}(H^2(D))$ for the disk is obtained by considering module multiplication on the quotient module $H^2(D^2)/[z_1]$, where $[z_1]$ denotes the closed submodule generated by $z_1$. In the second case, asking about the algebra for the quotient module is vacuous.
since $H^2(D^2)/[z_1 - 1] = (0)$. Finally, the $C^*$-algebra defined by module multiplication on the Bergman module is known to be $\mathfrak{T}(H^2(D))$, the same as the Toeplitz $C^*$-algebra for $H^2(D)$. This result follows from an explicit calculation but also from the BDF characterization of essentially normal operators [2].

To study the maps on $\mathfrak{T}(H^2(D^2))$ defined by $M_\varphi \to N_\varphi|_{\mathfrak{I}_i}$ for $\varphi$ in $A(D^2)$, we begin by considering the ideal $\mathfrak{I}_i$ in $\mathfrak{T}(H^2(D^2))$ generated by the Toeplitz operators whose symbols vanish on $\mathfrak{z}_i$ for $i = 1, 2, 3$. Recall that corresponding to the identification of $H^2(D^2)$ with the Hilbert space tensor product $H^2(D) \otimes H^2(D)$, we have $\mathfrak{T}(H^2(D^2)) \cong \mathfrak{T}(H^2(D)) \otimes \mathfrak{T}(H^2(D))$, where the latter is the spatial tensor product. Moreover, we use the standard notation of $T_\varphi$ for the Toeplitz operator defined by module multiplication by $\varphi$. Finally, we let $z$ denote the variable on $D$ and $z_1$ and $z_2$ denote the variables on $D^2$.

2.1 Example 1

For $\mathfrak{I}_1$, we note that $T_{z_1} = T_z \otimes I$ and $T^*_{z_1}T_{z_1} = (T_z \otimes I)^*(T_z \otimes I) = I \otimes I = I$ and hence the identity operator is in $\mathfrak{I}_1$ which means that $\mathfrak{I}_1 = \mathfrak{T}(H^2(D^2))$. Therefore, the map from module multipliers on $H^2(D^2)$ to their restrictions acting on the quotient $H^2(D^2)/[z_1]$ does not extend.

2.2 Example 2

Before tackling the next example we need to recall the basic results from [3] in which the short exact sequence for the Toeplitz $C^*$-algebra on the disk

$$0 \to \mathfrak{K}(H^2(D)) \longrightarrow \mathfrak{T}(H^2(D)) \overset{\sigma}{\longrightarrow} C(\partial D) \longrightarrow 0$$

is tensored with itself. (In [1] it was pointed out that while the preservation of exactness for tensor products is not automatic, it is valid in this case and, moreover, all the tensor products are unique.) We will use $\mathfrak{K}$ and $\mathfrak{T}$ to denote the $C^*$-algebras $\mathfrak{K}(H^2(D))$ and $\mathfrak{T}(H^2(D))$, respectively.

**Proposition 2.1.** The following diagram is exact:
We observe that spatially we have $K \otimes K \cong K(H^2(D^2))$ and $T \otimes T \cong T(H^2(D^2))$. Moreover, one has $C(\partial D) \otimes C(\partial D) \cong C((\partial D)^2)$ and $K \otimes C(\partial D) \cong C(\partial D) \otimes K \cong C(\partial D, K)$ and $T \otimes C(\partial D) \cong C(\partial D) \otimes T \cong C(\partial D, T)$, where $C(\partial D, K)$ and $C(\partial D, T)$ are the $C^*$-algebras of continuous functions on $\partial D$ taking values in $K$ and $T$, respectively. Finally, we record for later use:

**Proposition 2.2.** If $i$ and $\Sigma$ are defined by the diagram, then

$$\text{Range } i = \text{Kernel}(T \otimes T \to (T \otimes C(\partial D) \oplus C(\partial D) \otimes K)$$

and

$$\text{Kernel } \Sigma = \text{Range}((T \otimes T \oplus K \otimes T) \to T \otimes T).$$

The proof involves a standard diagram chase.

Now $\mathcal{I}_2$ is the ideal in $\mathfrak{K}(H^2(D^2))$ generated by the operator $T_{z_1-1}$, since the set of functions of the form $(z_1 - 1)\varphi(z_1, z_2)$ is uniformly dense in the ideal of functions in $A(D^2)$ that vanish on $\mathfrak{I}_2$. Calculating, we have

$$\Sigma(T_{z_1-1} \otimes 1) = (z - 1) \otimes 1 = z_1 - 1$$

and therefore

$$\Sigma(\mathcal{I}_2) = \{ f \in C((\partial D)^2) \mid f|_{\mathfrak{I}_2} \equiv 0 \}.$$ 

Further, the commutator $[T_{z_1-1}^* \otimes I, T_{z_1-1} \otimes I] = P_0 \otimes I$ is in $(K \otimes T) \cap \mathcal{I}_2$, where $P_0$ is the projection onto the constant functions in $H^2(D)$. Since $\mathcal{I}_2$ is an ideal, we see that $P_0 \otimes T$ is contained in $\mathcal{I}_2$ and hence $K \otimes T \subset \mathcal{I}_2$. Now appealing to the exact sequence $0 \to K \otimes T \to T \otimes T \xrightarrow{\sigma \otimes 1} C(\partial D, T) \to 0$, we see
since $\mathcal{R} \otimes \mathfrak{T} \subset \mathcal{I}_2$ that $\mathcal{I}_2$ is determined by $(\sigma \otimes 1)(\mathcal{I}_2)$. Therefore, $\mathcal{I}_2 = \{ X \in \mathfrak{T} \otimes \mathfrak{T} | (\sigma \otimes 1)(X)|_{z=1} \equiv 0 \}$.

Now consider the following diagram obtained by taking quotients:

\[
\begin{array}{ccc}
\mathfrak{T} \otimes \mathfrak{T} & \rightarrow & C(\partial \mathbb{D}, \mathfrak{T}) \\
\downarrow & & \downarrow \\
\mathfrak{T} \otimes \mathfrak{T} / [T_{z_1-1}] & \rightarrow & C(\partial \mathbb{D}, \mathfrak{T}) / [z_1 - 1] \cong \mathfrak{T}(H^2(\mathbb{D})).
\end{array}
\]

Using the bottom line, we see that the map $T_\varphi \rightarrow T_{\varphi |_{\mathfrak{T}_3}}$ defined for $\varphi$ in $A(\mathbb{D}^2)$ extends to a $C^*$-homomorphism from $\mathfrak{T}(H^2(\mathbb{D}^2))$ to $\mathfrak{T}(H^2(\mathbb{D}))$. Thus, although the underlying Hilbert modules don’t match up in this case, the $C^*$-algebras generated by the module multipliers do.

### 2.3 Example 3

Now let us consider $\mathcal{I}_3$ which is the ideal in $\mathfrak{T}(H^2(\mathbb{D}^2))$ generated by $T_{z_1 - z_2}$ or $T_z \otimes I - I \otimes T_z$ in $\mathfrak{T} \otimes \mathfrak{T}$. We proceed as in the last case by calculating the commutator

\[
[(T_z \otimes I - I \otimes T_z)^*, T_z \otimes I - I \otimes T_z] = P_0 \otimes I - I \otimes P_0,
\]

which is in $\mathcal{I}_3$. Again, since $\mathcal{I}_3$ is an ideal in $\mathfrak{T} \otimes \mathfrak{T}$, we see that both $P_0 \otimes I$ and $I \otimes P_0$ are in $\mathcal{I}_3$ and hence $\mathcal{I}_3$ contains both $P_0 \otimes \mathfrak{T}$ and $\mathfrak{T} \otimes P_0$. Therefore, $\mathcal{R} \otimes \mathfrak{T} + \mathfrak{T} \otimes \mathcal{R}$ is contained in $\mathcal{I}_3$. Now we consider the exact sequence

\[
\mathcal{R} \otimes \mathfrak{T} \oplus \mathfrak{T} \otimes \mathcal{R} \rightarrow \mathfrak{T} \otimes \mathfrak{T} \xrightarrow{\Sigma} C((\partial \mathbb{D})^2) \rightarrow 0
\]

and note that $\mathcal{I}_3$ is determined by $\Sigma(\mathcal{I}_3)$. Hence, we have $\mathcal{I}_3 = \{ X \in \mathfrak{T} \otimes \mathfrak{T} | \Sigma(X)|_{\mathfrak{T}_3} \equiv 0 \}$ which implies that

\[
\mathfrak{T}(H^2(\mathbb{D}^2)) / \mathcal{I}_3 \cong C(\{ (z_1, z_2) \in (\partial \mathbb{D})^2 | z_1 = z_2 \}) \cong C(\partial \mathbb{D}).
\]

Therefore, there is no $C^*$-map from $\mathfrak{T}(H^2(\mathbb{D}^2))$ to $\mathfrak{T}(H^2(\mathbb{D}))$ that extends the correspondence $T_\varphi \rightarrow T_{\varphi |_{\mathfrak{T}_3}}$ defined for $\varphi$ in $A(\mathbb{D}^2)$.

### 3 Ideals in Toeplitz $C^*$-Algebras—General Case

The foregoing results might seem, at first, to be a little peculiar. For both of the cases in which the quotient Hilbert modules are well-behaved, namely examples one and three, there is no $C^*$-map, while in the second example there is a $C^*$-map but no quotient module map. As we said
at the beginning of this note, at this point in our study we saw that we were attempting to answer the wrong question. However, there is a characterization of ideals in $\mathfrak{T}(H^2(D))$ which makes sense of the results about the foregoing examples.

We begin by analyzing the ideals in $\mathfrak{T} \otimes \mathfrak{K}$, one of the $C^*$-algebras in the diagram.

**Lemma 3.1.** The non-zero ideals $\mathfrak{I}$ in $\mathfrak{T} \otimes \mathfrak{K}$ are parametrized by the closed subsets $M$ of $\partial \mathbb{D}$ such that $\mathfrak{I}_M = \{ X \in \mathfrak{T} \otimes \mathfrak{K} \mid (\sigma \otimes 1)(X)(z) = 0 \text{ for } z \in M \}$, where $\sigma \otimes 1 : \mathfrak{T} \otimes \mathfrak{K} \rightarrow C(\partial \mathbb{D}) \otimes \mathfrak{K} \cong C(\partial \mathbb{D}, \mathfrak{K})$.

**Proof.** If $\mathfrak{I} \neq (0)$, then $\mathfrak{K}(H^2(D)) \cong \mathfrak{K} \otimes \mathfrak{K} \subset \mathfrak{I} \subset \mathfrak{T} \otimes \mathfrak{K}$ and the result follows from the exactness of the sequence

$$0 \rightarrow \mathfrak{K} \otimes \mathfrak{K} \rightarrow \mathfrak{T} \otimes \mathfrak{K} \rightarrow \mathfrak{T} \otimes \mathfrak{K} \sigma \otimes 1 \rightarrow C(\partial \mathbb{D}) \otimes \mathfrak{K} \rightarrow 0$$

and the structure of the ideals in $C(\partial \mathbb{D}) \otimes \mathfrak{K} \cong C(\partial \mathbb{D}, \mathfrak{K})$.

Next we consider the ideals in $C(\partial \mathbb{D}) \otimes \mathfrak{T}$.

**Lemma 3.2.** A non-zero ideal $\mathfrak{J}$ in $C(\partial \mathbb{D}) \otimes \mathfrak{T} \cong C(\partial \mathbb{D}, \mathfrak{T})$ is characterized by two closed sets $Z' \subseteq \partial \mathbb{D}$ and $Z \subseteq (\partial \mathbb{D})^2$ with the properties that

1. $Z' \times \partial \mathbb{D} \subset Z$,
2. $\mathfrak{J} = \{ X \in C(\partial \mathbb{D}, \mathfrak{T}) \mid X(z_2) = 0 \text{ for } z_2 \in Z' \text{ and } \sigma|X(z_2)|(z_1) = 0 \text{ for } (z_1, z_2) \in Z \}$.

**Proof.** The set of values at $z_1$ in $\partial \mathbb{D}$ of the elements $X$ in $\mathfrak{J}$ forms an ideal in $\mathfrak{T}$. Since the ideals in $\mathfrak{T}$ are $(0)$, $\mathfrak{K}$, and the operators $\{ X \in \mathfrak{T} \mid \sigma(X)|S \equiv 0 \}$ for some closed subset $S$ of $\partial \mathbb{D}$, the result follows. The set $Z'$ corresponds to the subset of $\partial \mathbb{D}$ on which evaluation yields the zero ideal.

An ideal in $\mathfrak{T} \otimes \mathfrak{T}$ gives rise to two ideals to which the preceding lemma applies.

**Lemma 3.3.** If $\mathfrak{J}$ is an ideal in $\mathfrak{T} \otimes \mathfrak{T}$, then the pairs of sets $(Z'_1, Z_1)$ and $(Z'_2, Z_2)$ that characterize $(\sigma \otimes 1)(\mathfrak{J})$ and $(1 \otimes \sigma)(\mathfrak{J})$, respectively, satisfy $Z_1 = Z_2$.

**Proof.** The result follows by identifying $Z_1$ and $Z_2$ with the zero set of the ideal $\Sigma(\mathfrak{J}) \subset C((\partial \mathbb{D})^2)$.
Lemma 3.4. If $\mathcal{I}$ is a non-zero ideal in $\mathfrak{T} \otimes \mathfrak{T}$, $\mathcal{I}_1$ is the ideal $\mathcal{I} \cap (\mathfrak{T} \otimes \mathfrak{R})$ in $\mathfrak{T} \otimes \mathfrak{R}$ characterized by $M \subset \partial \mathbb{D}$, and $\mathcal{I}'$ is the ideal $(\sigma \otimes 1)(\mathcal{I})$ in $C(\partial \mathbb{D}) \otimes \mathfrak{R}$ characterized by the pair $(Z', Z)$ in $\partial \mathbb{D}$ and $(\partial \mathbb{D})^2$, respectively, then $Z' = M$.

Proof. If we let $\mathcal{I}'_1 = (\sigma \otimes 1)(\mathcal{I}_1)$, then $X$ is in $\mathcal{I}_1$ if and only if $(\sigma \otimes 1)(X)(z) = 0$ for $z$ in $M$. Thus, $\mathcal{I}'_1$ consists of the functions in $C(\partial \mathbb{D}) \otimes \mathfrak{R} \cong C(\partial \mathbb{D}, \mathfrak{R})$ that vanish on $M$. Moreover, the set of values of the functions in $\mathcal{I}'_1$ at $z$ in $\partial \mathbb{D}$ forms an ideal $\mathcal{I}'_1(z)$ in the subalgebra $\mathcal{I}'(z)$ of values at $z$ of the functions in $\mathcal{I}' \subset C(\partial \mathbb{D}, \mathfrak{T})$. We claim that $\mathcal{I}'_1(z_0) = (0)$ for $z_0$ in $\partial \mathbb{D}$ implies $\mathcal{I}'(z_0) = (0)$ from which the result follows.

Suppose $X$ is an element of $\mathcal{I}$ for which $(\sigma \otimes 1)(X)(z_0) \neq 0$. Since $\mathcal{I}$ is an ideal, by multiplying it by an appropriate element in $\mathfrak{T} \otimes \mathfrak{T}$ we can assume $(\sigma \otimes 1)(X)(z_0)$ is compact and nonzero. In particular, there exists a compact operator $K$ such that $(\sigma \otimes 1)(X)(z_0)K \neq 0$. Then $X' = X(I \otimes K)$ is in $\mathcal{I}$ as well as in $\mathfrak{T} \otimes \mathfrak{R}$ and thus in $\mathcal{I}'$. But $(\sigma \otimes 1)(X')(z_0) = (\sigma \otimes 1)(X)(z_0)K$ is a non-zero element of $\mathcal{I}'_1(z_0)$ which is a contradiction. \hfill \Box

We can now state the main result describing the ideals in $\mathfrak{T}(H^2(\mathbb{D}^2))$.

Theorem. A non-zero ideal $\mathcal{I}$ in $\mathfrak{T}(H^2(\mathbb{D}^2))$ is characterized by a triple $(Z', Z'' , Z)$ with $Z'$ and $Z''$ closed subsets of $\partial \mathbb{D}$ and $Z$ a closed subset of $(\partial \mathbb{D})^2$ which contains both $Z' \times \partial \mathbb{D}$ and $\partial \mathbb{D} \times Z''$.

Moreover,

$$\mathcal{I} = \{ X \in \mathfrak{T}(H^2(\mathbb{D}^2)) \mid \Sigma(X)|_{Z'} \equiv 0, (\sigma \otimes 1)(X)(z) = 0 \text{ for } z \text{ in } Z'' \}.$$ 

Further, there is an isometry $\alpha = (\oplus \alpha_z \oplus (\oplus \beta_z) \oplus \mu$ from $\mathfrak{T} \otimes \mathfrak{T}/\mathcal{I}$ to $\left( \bigoplus_{z \in Z'} \mathfrak{T} \right) \oplus \left( \bigoplus_{z \in Z''} \mathfrak{T} \right) \oplus C(Z)$, where $\alpha_z(X) = (\sigma \otimes 1)(X)(z)$, $\beta_z(X) = (1 \otimes \sigma)(X)(z)$ and $\mu(X) = \Sigma(X)$. Finally, the range of $\alpha$ consists of the $\left( \bigoplus_{z \in Z'} X_z \right) \oplus \left( \bigoplus_{z \in Z''} Y_z \right) \oplus f$, where $\{ X_z \}, \{ Y_z \} \subset \mathfrak{T}$ and $f$ in $C((\partial \mathbb{D})^2$ and 1) the maps $z \to X_z$ and $z \to Y_z$ are continuous from $Z'$ and $Z''$ to $\mathfrak{T}$, respectively, and 2) $(1 \otimes \sigma)X_z(w) = (\sigma \otimes 1)Y_w(z) = f(z, w)$ for $(z, w)$ in $Z' \times Z''$.

Proof. Using the preceding lemmas, it is possible to define the sets $Z', Z''$ and $Z$ as indicated. Namely, $Z$ is the zero set of the ideal $\Sigma(\mathcal{I})$ in $C((\partial \mathbb{D})^2)$, and $Z'$ and $Z''$ are the subsets of those $z$ in $\partial \mathbb{D}$ for which $(\sigma \otimes 1)(\mathcal{I})(z) = (0)$ and $(1 \otimes \sigma)(\mathcal{I})(z) = (0)$, respectively. Clearly then $\mathcal{I}$ is contained in the ideal defined in the statement of the Theorem and our task is to show that they are equal. Hence, let $X$ be an operator in $\mathfrak{T} \otimes \mathfrak{T}$ that satisfies (1) $\Sigma(X)|_{Z'} \equiv 0$, (2) $(\sigma \otimes 1)(X)(z) = 0$ for $z$ in $Z'$ and (3) $(1 \otimes \sigma)(X)(w) = 0$ for $w$ in $Z''$. We must show that $X$ is in $\mathcal{I}$. 8
Since $\Sigma(\mathcal{I}) = \{ f \in C((\partial \mathbb{D})^2) \mid f|_3 \equiv 0 \}$, there exists $X_0$ in $\mathcal{I}$ such that $\Sigma(X_0) = \Sigma(X)$ and hence $\Sigma(X - X_0) = 0$. Therefore, by Proposition 2.2, there exist $Y_1$ and $Y_2$ in $\mathcal{I} \otimes \mathfrak{K}$ and $\mathfrak{K} \otimes \mathcal{I}$, respectively, such that $X - X_0 = Y_1 + Y_2$. If we evaluate the equation $(\sigma \otimes 1)(X - X_0) = (\sigma \otimes 1)(Y_1) + (\sigma \otimes 1)(Y_2)$ at $z$ on $Z'$, then we obtain $0 = (\sigma \otimes 1)(Y_1)(z) + (\sigma \otimes 1)(Y_2)(z)$ by the assumptions on $X$ and the definition of $Z'$. Since $(\sigma \otimes 1)(Y_2) = 0$, it follows that $(\sigma \otimes 1)(Y_1)(z) = 0$ for $z$ in $Z'$. Similarly we have $(1 \otimes \sigma)(Y_1) = 0$ and $(1 \otimes \sigma)(Y_2)(z) = 0$ for $z$ in $Z''$. If we can show that $Y_1$ and $Y_2$ are in $\mathcal{I}$, then it will follow that $X$ is in $\mathcal{I}$ thereby completing the proof. By symmetry, it is enough to show that $Y_1$ is in $\mathcal{I}$.

By Lemma 3.4, it follows that $\mathcal{I} \cap (\mathcal{I} \otimes \mathfrak{K})$ is characterized by a closed subset $Z'_0$ of $\partial \mathbb{D}$ consisting of those $z$ for which $(\sigma \otimes 1)(\mathcal{I})(z) = (0)$. Clearly, $Z' \subseteq Z'_0$ and it is enough to show equality of these sets. For $z_0$ in $Z'_0 \setminus Z'$, there exists an operator $W$ in $\mathcal{I}$ such that $(\sigma \otimes 1)(W)(z_0) \neq 0$. As in the proof of Lemma 3.4, one can show that one can obtain $W'$ in $\mathcal{I} \otimes \mathfrak{K}$ for which $(\sigma \otimes 1)(W')(z_0) \neq 0$ which completes the proof of the characterization of the ideals in $\mathcal{I}(H^2(\mathbb{D})^2))$.

To establish that $\alpha$ is an isometry we first we define $\overline{\alpha}$ as the direct sum of the maps $\overline{\alpha}_z(X) = (\sigma \otimes 1)(X)(z)$ for $z$ in $\partial \mathbb{D}$, $\overline{\beta}_w(X) = (1 \otimes \sigma)(X)(w)$ for $w$ in $\partial \mathbb{D}$, and $\overline{\mu}(X) = \Sigma(X)$ for $X$ in $\mathcal{I} \otimes \mathcal{I}$. The range of both $\overline{\alpha}_z$ and $\overline{\beta}_w$ is $\mathcal{I}$, while the range of $\overline{\mu}$ is $C((\partial \mathbb{D})^2)$. It remains to observe that $\bigcap_{z \in \partial \mathbb{D}} \ker \overline{\alpha}_z = \ker(\sigma \otimes 1) = \mathfrak{K} \otimes \mathcal{I}$ and $\bigcap_{w \in \partial \mathbb{D}} \ker \overline{\beta}_w = \ker(1 \otimes \sigma) = \mathcal{I} \otimes \mathfrak{K}$. Thus $\ker \overline{\alpha} = (\mathcal{I} \otimes \mathfrak{K}) \cap (\mathfrak{K} \otimes \mathcal{I}) = \mathfrak{K} \otimes \mathfrak{K}$. If $\mathcal{I}$ is a non-zero ideal in $\mathcal{I}(H^2(\mathbb{D})^2))$, it contains $\mathfrak{K} \otimes \mathfrak{K}$.

Now consider $\overline{\alpha}: \mathcal{I} \otimes \mathcal{I} \rightarrow \left( \bigoplus_{z \in Z'} \mathcal{I} \right) \oplus \left( \bigoplus_{w \in Z''} \mathcal{I} \right) \oplus C(Z)$, where $\mu(X) = \Sigma(X)|_Z$. Clearly the kernel of $\overline{\alpha}$ is $\mathcal{I}$ by the first statement of the Theorem and hence we can define $\alpha = \overline{\alpha} \circ \pi^{-1}$ on $\mathcal{I} \otimes \mathcal{I}/\mathcal{I}$, where $\pi$ is the quotient map from $\mathcal{I} \otimes \mathcal{I}$ to $\mathcal{I} \otimes \mathcal{I}/\mathcal{I}$. Since $\ker \overline{\alpha} = \mathcal{I}$, $\alpha$ is one-to-one and hence isometric.

Finally, we must show that the range of $\alpha$ consists of all elements $x = \left( \bigoplus_{z \in Z'} X_z \right) \oplus \left( \bigoplus_{w \in Z''} Y_w \right) \oplus f$ in $\left( \bigoplus_{z \in Z'} \mathcal{I} \right) \oplus \left( \bigoplus_{w \in Z''} \mathcal{I} \right) \oplus C(Z)$ that satisfying conditions (1) and (2). That all elements $x$ in the range of $\alpha$ satisfy conditions (1) and (2) follows from properties of the maps in the tensor product diagram in Proposition 2.1. Hence, the remainder of the proof requires showing that for an element $x = \left( \bigoplus_{z \in Z'} X_z \right) \oplus \left( \bigoplus_{w \in Z''} Y_w \right) \oplus f$ satisfying (1) and (2), there exists an $X$ in $\mathcal{I} \otimes \mathcal{I}$ such that $\alpha_z(X) = X_z$ for $z$ in $Z'$, $\beta_w(X) = Y_w$ for $w$ in $Z''$ and $\Sigma(X) = f$.

Taking $\mathcal{X}$ in $\mathcal{I} \otimes \mathcal{I}$ such that $\Sigma(\mathcal{X}) = f$, the element

$$x' = \left( \bigoplus_{z \in Z'} X_z - \alpha_z(\mathcal{X}) \right) \oplus \left( \bigoplus_{w \in Z''} Y_w - \beta_w(\mathcal{X}) \right) \oplus (f - \Sigma(\mathcal{X}))$$
also satisfies (1) and (2). Moreover, showing that \( x' \) is in the range of \( \alpha \) implies that \( x \) is also. Hence we can assume for the given \( x \) that \( f \equiv 0 \).

Assume we have such an element \( x \). Extend the function \( z \to X_z \) on \( Z' \) to a function \( \Phi \) defined from \( \partial \mathbb{D} \) to \( \mathcal{T} \) by extending it linearly on the open intervals in \( \partial \mathbb{D} \setminus Z' \). Observe that \( \Phi \) is continuous by (1) and hence in \( C(\partial \mathbb{D}, \mathbf{T}) \cong C(\partial \mathbb{D}) \otimes \mathbf{r} \). Moreover, for \( z' \) in \( Z' \) and \( w \) in \( \partial \mathbb{D} \), we have

\[
((1 \otimes \sigma)\Phi)(z, w) = ((1 \otimes \sigma)X_z)(w) = f(z, w) = 0.
\]

Hence, \((1 \otimes \sigma)X_z = 0 \) for \( z \) in \( Z' \) and since \( \Phi \) was defined by linear extension we have \((1 \otimes \sigma)\Phi(z) = 0 \) or \((1 \otimes \sigma)\Phi = 0 \). By the exactness of the sequence

\[
0 \to C(\partial \mathbb{D}, \mathcal{K}) \to C(\partial \mathbb{D}, \mathbf{T}) \to C((\partial \mathbb{D})^2) \to 0,
\]

we see that \( \Phi \) is in \( C(\partial \mathbb{D}, \mathcal{K}) \) or \( \Phi(z) \) is compact for \( z \) in \( \partial \mathbb{D} \). Since the sequence

\[
0 \to \mathcal{K} \otimes \mathcal{K} \to \mathcal{T} \otimes \mathcal{K} \to C(\partial \mathbb{D}) \otimes \mathcal{K} \to 0
\]

is exact, there exists \( \exists \mathcal{X} \) in \( \mathcal{T} \otimes \mathcal{K} \), and hence also in \( \mathcal{T} \otimes \mathcal{T} \), such that \((\sigma \otimes 1)(\exists \mathcal{X}) = \Phi \). Replace \( x \) by \( x - (\exists \mathcal{X}) \) to obtain an element \( x'' = \left( \bigoplus_{z \in Z'} X'_z \right) \oplus \left( \bigoplus_{w \in Z''} Y'_w \right) \oplus f' \) in \( \left( \bigoplus_{z \in Z'} \mathcal{T} \right) \oplus \left( \bigoplus_{w \in Z''} \mathcal{T} \right) + C(Z) \) that satisfies (1) and (2) and such that \( f' \equiv 0 \) and \( X'_z = X_z - (\sigma \otimes 1)(\exists \mathcal{X})(z) = X_z - \Phi(z) = 0 \). Repeating the previous construction but reversing the roles of the first and second factors, we can replace \( x'' \) by an element \( x''' = \left( \bigoplus_{z \in Z'} 0 \right) \oplus \left( \bigoplus_{w \in Z''} 0 \right) \oplus 0 \) such that \( x''' = x'' - \alpha(\exists \mathcal{X}) \). Hence, \( x'' \) is in the range of \( \alpha \) if and only if the zero element is, which concludes the proof.

Examining the proof, one sees that the key is the fact that the intersection of \( \ker(\sigma \otimes 1) \) and \( \ker(1 \otimes \sigma) \) in \( \mathcal{T} \otimes \mathcal{T} \) is the intersection of \( \mathcal{K} \otimes \mathcal{T} \) and \( \mathcal{T} \otimes \mathcal{K} \) or \( \mathcal{K} \otimes \mathcal{K} \), and that the latter ideal is contained in every nontrivial ideal in \( \mathcal{T} \otimes \mathcal{T} \). If we consider the Toeplitz \( C^* \)-algebra over \( H^2(\mathbb{D}^3) \), then one has a symbol map \( \Sigma \) from \( \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \) to \( C((\partial \mathbb{D})^3) \) and three symbol maps \( \Sigma_1, \Sigma_2, \) and \( \Sigma_3 \) from \( \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \) to \( C((\partial \mathbb{D})^2, \mathcal{T}) \) with the intersection of their kernels again being \( \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K} \). Thus one can formulate a description of the ideals in \( \mathcal{T}(H^2((\partial \mathbb{D})^3)) \) analogous to the theorem in terms of \( \Sigma, \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \). A similar approach will work for Toeplitz \( C^* \)-algebras defined for domains of rank greater than one. To some extent, this idea structure also shows up in index theory (cf. [6]).

References

[1] C.A. Berger, L.A. Coburn, and A. Lebow, Representation and index theory for \( C^* \)-algebras generated by commuting isometries, J. Funct. Anal. 27 (1978), 51–99.
[2] L.G. Brown, R.G. Douglas, and P.A. Fillmore, Unitary equivalence modulo the compact operators and extensions of $C^*$-algebras, *Proceedings of a Conference on Operator Theory (Dalhousie Univ. Halifax, N.S., 1973)*, pp. 58–128. Lecture Notes in Math., Vol. 346, Springer, Berlin, 1973.

[3] R.G. Douglas and Roger Howe, On the $C^*$-algebra of Toeplitz operators on the quarter plane, *Trans. Amer. Math. Soc.* 158 (1971), 203–217.

[4] R.G. Douglas, Gadadhar Misra and Cherian Varaghere, On quotient modules—the case of arbitrary multiplicity, *J. Funct. Anal.* 174 (2000), 364–398.

[5] Walter Rudin, *Function Theory in Polydiscs*, W.A. Benjamin Inc., New York-Amsterdam, 1969.

[6] Haral Upmeier, *Toeplitz Operators and Index Theory in Several Complex Variables*, Operator Theory: Advances and Applications, 81, Birkhäuser Verlag, Basel, 1996.