The general solutions of some nonlinear second and third order PDEs with constant and nonconstant parameters

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Abstract
Selection of 25 examples from extensive nontrivial families for different types of nonlinear PDEs and their formal general solutions are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like.

1 Introduction
Nonlinear partial differential equations (PDEs) play very important role in many fields of mathematics, physics, chemistry, and biology, and numerous applications. Despite the fact that various methods for solving nonlinear PDEs have been developed in 19-20 centuries [1]-[5], there exists a very disadvantageous opinion that only a very small minority of nonlinear second- and higher-order PDEs admit general solutions in closed form (see, e.g., percentage of PDEs with general solutions in fundamental handbook [6]).

Nevertheless there exist some extensive nontrivial families for different types of nonlinear PDEs which general solutions can be expressed in closed form and which seemingly are not described in literature.

In present paper, as a preliminary result, some relatively simple examples of such PDEs of second and third order and their formal general solutions are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like.

2 Notations remarks
The expression of the following type

\[ \text{RootOf}[F(Z)] \]

means any root of the algebraic equation \( F(Z) = 0 \) with respect to indeterminate \( Z \), and

\[ X = \text{RootOf}[Z^p - X] . \]
For shortness
\[ \int_{t}^{x} f(t,x) dt = \{ \int f(t,x) dt \}_{x} \]

3 Second order PDEs with two independent variables and constant parameters

3.1 \[ \frac{\partial^2 w}{\partial t \partial x} - \left( \frac{1}{w} \frac{\partial w}{\partial t} + b \right) \frac{\partial w}{\partial x} - \frac{c}{w} \frac{\partial w}{\partial t} - cb = 0 , \]

where \( b, c \) are constants.

General solution
\[ w(t,x) = \{-c \int \exp[-e^{bt}G(x)] dx + F(t)\} \exp[e^{bt}G(x)] , \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.2 \[ \frac{\partial^2 w}{\partial t \partial x} - \frac{1}{w} \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} - \frac{c}{w} \frac{\partial w}{\partial t} - kw = 0 , \]

where \( c, k \) are constants.

General solution
\[ w(t,x) = \{-c \int \exp[x(1-kt)]G(x) dx + F(t)\} \exp[-x(1-kt)] , \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.3 \[ \frac{\partial^2 w}{\partial t \partial x} = \frac{a}{w} \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{1}{w} \frac{\partial w}{\partial t} + b + \frac{c}{w} \right) \frac{\partial w}{\partial x} + \frac{2c \frac{\partial w}{\partial x} + (bw+c)^2}{4aw} , \]

where \( a, b, c \) are constants, and \( a \neq 0. \)

General solution
\[ w(t,x) = \{-c \int \exp[1/2a \int \frac{2dx}{t+G(x)} + bx] dx + F(t)\} \exp[-1/2a \int \frac{2dx}{t+G(x)} + bx] , \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.
3.4 \[ \frac{\partial^2 w}{\partial t \partial x} - \left( \frac{1}{w} \frac{\partial w}{\partial t} + b \right) \frac{\partial w}{\partial x} - \frac{c}{w} \frac{\partial w}{\partial t} - kw - cb = 0 , \]

where \( b, c, k \) are constants, and \( b \neq 0, k \neq 0 \).

General solution

\[ w(t, x) = \left[ -c \int \exp \left( \frac{k}{b^2} \{ e^{bt} G(x) + bx \} \right) dx + F(t) \right] \exp \left( -\frac{k}{b^2} \{ e^{bt} G(x) + bx \} \right) , \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.5 \[ \frac{\partial^2 w}{\partial t \partial x} - \frac{a}{w} \left( \frac{\partial w}{\partial x} \right)^2 - \left( \frac{1}{w} \frac{\partial w}{\partial t} + b + \frac{c}{w} \right) \frac{\partial w}{\partial x} - \frac{c}{2aw} \frac{\partial w}{\partial t} - \frac{bc}{2a} - \frac{c^2}{4aw} = 0 , \]

where \( a, b, c, k \) are constants, and \( a \neq 0, b^2 - 4ak \neq 0 \).

General solution

\[ w(t, x) = \]

\[-\frac{c}{2a} \{ \int \exp \left( \frac{1}{2a} \int \exp \left( t \sqrt{b^2 - 4ak} G(x) \right) \frac{(b + \sqrt{b^2 - 4ak}) - \sqrt{b^2 - 4ak} + b}{1 + \exp(t \sqrt{b^2 - 4ak} G(x))} dx \right) dx + F(t) \} \exp \left( -\frac{1}{2a} \int \frac{\exp \left( t \sqrt{b^2 - 4ak} G(x) \right) \left( b + \sqrt{b^2 - 4ak} \right) - \sqrt{b^2 - 4ak} + b}{1 + \exp(t \sqrt{b^2 - 4ak} G(x))} dx \right) , \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.6 \[ \frac{ac}{w} + b + \frac{a}{w} \frac{\partial w}{\partial x} - \frac{a^2}{w^4} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} - w \frac{\partial^2 w}{\partial t \partial x} \right)^2 = 0 , \]

where \( a, b, c \) are constants, and \( a \neq 0 \).

General solution
\[ w(t, x) = \{-c \int \exp[-\frac{1}{4a}(-4bx + \int (t + G(x))^2 dx)] dx + F(t) \} \]

\[ \times \exp[\frac{1}{4a}(-4bx + \int (t + G(x))^2 dx)], \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.7  \[ \frac{\partial^2 w}{\partial t \partial x} - \left( \frac{1}{w} \frac{\partial w}{\partial t} + b \right) \frac{\partial w}{\partial x} - \frac{g}{w} \frac{\partial w}{\partial t} \]

\[ - \frac{aw^2}{g+hw+\frac{\partial w}{\partial x}} - kw - bg = 0, \]

where \( a, b, g, h, k \) are constants.

General solution

\[ w(t, x) = \]

\[ \{-g \int e^{hx} \exp\{- \int RootOf[t - \int^Z \frac{\xi d\xi}{(k-hb)\xi + b\xi^2 + a} + G(x)] dx] dx + F(t) \} \]

\[ \times e^{-hx} \exp\{ \int RootOf[t - \int^Z \frac{\xi d\xi}{(k-hb)\xi + b\xi^2 + a} + G(x)] dx \}, \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

3.8  \[ \frac{\partial^2 w}{\partial t \partial x} - \left( \frac{1}{w} \frac{\partial w}{\partial t} - m \right) \frac{\partial w}{\partial x} \]

\[ + \frac{kw}{a} \exp\left( \frac{ac}{w} + b + \frac{a}{w} \frac{\partial w}{\partial x} \right) \]

\[ - \frac{c}{w} \frac{\partial w}{\partial t} + \frac{gw}{a} + cm = 0, \]

where \( a, b, c, k, g, m \) are constants, and \( a \neq 0 \).

General solution
\[ w(t, x) = \{-c \int \exp\left( \frac{bx}{a} \right) \]

\[ \times \exp\left( -\frac{1}{a} \int \text{RootOf}[t + \int^Z \frac{d\xi}{g - bm + k \exp(\xi) + m\xi} + G(x)]dx dx + F(t) \right) \]

\[ \times \exp\left( \frac{-bx}{a} \right) \exp\left( \frac{1}{a} \int \text{RootOf}[t + \int^Z \frac{d\xi}{g - bm + k \exp(\xi) + m\xi} + G(x)]dx, \right) \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

### 3.9 \( w \left( \frac{\partial w}{\partial x} + cw + b \right) \frac{\partial^2 w}{\partial t \partial x} \)

\[ = \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial x} + cw + 2b \right) + b(cw + b) \frac{\partial w}{\partial t} + mw^3, \]

where \( b, c, m \) are constants.

**General solution**

\[ w(t, x) = \{-b \int \exp[-\int (\sqrt{G(x)} + 2mt - c)dx + F(t)] \} \exp[\int (\sqrt{G(x)} + 2mt - c)dx], \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

### 3.10 \( w \left( \frac{\partial w}{\partial x} + cw + b \right)^2 \frac{\partial^2 w}{\partial t \partial x} \)

\[ = \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial x} \right)^3 + (2cw + 3b) \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial x} \right)^2 \]

\[ + (cw + b)(cw + 3b) \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} \]

\[ + b(cw + b)^2 \frac{\partial w}{\partial t} + mw^4, \]

where \( b, c, m \) are constants.

**General solution**
\[ w = -\left\{ \int \exp \left[ - \int ((G(x) + 3mt)^{\frac{1}{3}} - c) \, dx \right] \, dx + F(t) \right\} \exp \left[ \int ((G(x) + 3mt)^{\frac{1}{3}} - c) \, dx \right], \]

where \( F(t) \) and \( G(x) \) are arbitrary functions.

### 3.11

\[
\begin{align*}
\frac{w}{\partial x} + cw + b \frac{\partial^2 w}{\partial t \partial x} &= a \left( \frac{\partial w}{\partial x} \right)^3 + \left( \frac{\partial w}{\partial t} + (3ac + k) w + 3ab \right) \left( \frac{\partial w}{\partial x} \right)^2 \\
&\quad + ((cw + 2b) \frac{\partial w}{\partial t} + (\frac{k^2}{3a} + 2kc + 3ac^2) w^2 \\
&\quad + 2b(k + 3ac) w + 3ab^2) \frac{\partial w}{\partial x} + b(cw + b) \frac{\partial w}{\partial t} \\
&\quad + \left( \frac{k^3}{27a^2} + \frac{k^2c}{3a} + ac^3 + kc^2 \right) w^3 \\
&\quad + b(\frac{k^2}{3a} + 2ck + 3ac^2) w^2 + b^2(3ac + k) w + ab^3, \\
\end{align*}
\]

where \( a, b, c, k \) are constants, and \( a \neq 0, k \neq 0 \).

General solution

\[
\begin{align*}
w(t, x) &= -\left\{ \int \exp \left[ \frac{1}{3} \int \frac{(k + 3ac) \sqrt{6kt + G(x)} - 9ac}{\sqrt{6kt + G(x)} - 3} \, dx \right] \, dx + F(t) \right\} \\
&\quad \times \exp \left[ -\frac{1}{3a} \int \frac{(k + 3ac) \sqrt{6kt + G(x)} - 9ac}{\sqrt{6kt + G(x)} - 3} \, dx \right], \\
\end{align*}
\]

where \( F(t) \) and \( G(x) \) are arbitrary functions.
4 Second order PDEs with two independent variables and non-constant parameters

4.1 \[
\frac{\partial^2 w}{\partial t \partial x} = a \left( \frac{\partial w}{\partial x} \right)^2 - \left( \frac{2 ahw}{hx+b} - cx - g \right) \frac{\partial w}{\partial x} + \frac{h}{hx+b} \frac{\partial w}{\partial t} \\
+ \frac{ah^2w^2}{(hx+b)^2} - \frac{h(cx+g)w}{hx+b} + \frac{(hx+b)(2hg-cb+chx)c}{4h^2a},
\]
where \( a, b, c, g, h \) are constants, and \( a \neq 0, h \neq 0, hg - cb \neq 0 \).

General solution
\[
w = \frac{(hx+b)c}{2ah} \left\{ F(t) - x + 2(hg-cb) \int \left[ 1 / (-c(hx+b) + \exp \left( \frac{(cb-hg)t}{h} \right) ) \right] G(x) dx \right\},
\]
where \( F(t) \) and \( G(x) \) are arbitrary functions.

4.2 \[
w[b(t, x) + c(t, x)]w + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial t \partial x} = \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial x} \right)^2 \\
+ \left[ (c(t, x)w + 2b(t, x)) \frac{\partial w}{\partial t} - \frac{\partial c(t,x)}{\partial t}w^2 - w \frac{\partial b(t,x)}{\partial t} \right] \frac{\partial w}{\partial x} \\
+ b(t, x) \left[ c(t, x)w + b(t, x) \right] \frac{\partial w}{\partial t} - a(t, x)w^3 \\
- \left[ \frac{\partial b(t,x)}{\partial t}c(t, x) + \frac{\partial c(t,x)}{\partial t}b(t, x) \right] w^2 - w \frac{\partial b(t,x)}{\partial t}b(t, x),
\]
General solution
\[
w = - \int b(t, x) \exp \left[ - \int (-c(t, x) + \sqrt{c^2(t, x) - 2 \int a(t,x) dt + G(x) dx}) dx \right] + F(t) \\
\times \exp \left[ \int (-c(t, x) + \sqrt{c^2(t, x) - 2 \int a(t,x) dt + G(x) dx}) dx \right],
\]
where \( F(t) \) and \( G(x) \) are arbitrary functions.
\[ \frac{\partial^2 w}{\partial t \partial x} = \frac{a(t,x)}{w} \left( \frac{\partial w}{\partial x} \right)^2 + \left[ \frac{1}{w} \frac{\partial w}{\partial t} + b(t,x) + \frac{c(t,x)}{w} \right] \frac{\partial w}{\partial x} \\
+ \frac{c(t,x)}{2a(t,x)w} \frac{\partial w}{\partial t} + \frac{c^2(t,x)}{4a(t,x)w} \\
+ \frac{1}{2a^2(t,x)} \left[ a(t,x)b(t,x)c(t,x) \\
+ c(t,x) \frac{\partial a(t,x)}{\partial t} - a(t,x) \frac{\partial c(t,x)}{\partial t} \right], \]

where \( a(t,x) \neq 0, c(t,x) \neq 0. \)

General solution

\[ w = - \frac{1}{2W(t,x)} \left[ \int \frac{c(t,x)W(t,x)}{a(t,x)} dx + F(t) \right], \]

where

\[ W(t,x) = \exp \left\{ \int c(t,x) \exp \left( \int b(t,x) dt \right) / [-c(t,x)] \times \int \left\{ \exp \left( \int b(t,x) dt \right) a(t,x) c(t,x) (-c(t,x) + 2b(t,x)) \right. \right. \\
+ 2c(t,x) \frac{\partial a(t,x)}{\partial t} - 2 \frac{\partial c(t,x)}{\partial t} a(t,x) \left[ c^2(t,x) \right] \left\} dt \right. \]

\[ - 2c(t,x) G(x) + 2a(t,x) \exp \left( \int b(t,x) dt \right) dx \}, \]

and \( F(t) \) and \( G(x) \) are arbitrary functions.
\[ 4.4 \quad \frac{\partial^2 w}{\partial t \partial x} = \left[ \frac{1}{w} \frac{\partial w}{\partial t} + b(t, x) \right] \frac{\partial w}{\partial x} \]
\[ + \frac{g(t, x)}{w} \frac{\partial w}{\partial t} + k(t, x)w \]
\[ + g(t, x)b(t, x) - \frac{\partial g(t, x)}{\partial t} , \]

General solution
\[
 w = -\frac{1}{W(t, x)} \int g(t, x)W(t, x)dx + F(t) ,
\]

where
\[
 W(t, x) = \exp\left\{ - \int \frac{\exp(\int b(t, x)dt)}{G(x)(g(t, x) + 1)} \right\} 
\times \left\{ ([G(x)(g(t, x) + 1) + 1] 
\times \int k(t, x) \exp[- \int b(t, x)dt]dt + (g(t, x) + 1) \right) 
\times \left\{ \frac{1}{(g(t, x) + 1)^2} \frac{\partial g(t, x)}{\partial t} \int k(t, x) \exp[- \int b(t, x)dt]dt \right) 
- \left( \int \frac{k(t, x)}{g(t, x) + 1} \exp[- \int b(t, x)dt]dt \right) \right\} dx ,
\]

and \( F(t) \) and \( G(x) \) are arbitrary functions.
Second order PDEs with four independent variables and constant parameters

5.1 \[ A_1 \frac{\partial^2 w}{\partial x_1 \partial x_4} + A_2 \frac{\partial^2 w}{\partial x_2 \partial x_4} + A_3 \frac{\partial^2 w}{\partial x_3 \partial x_4} + C_0 + B_1 \frac{\partial w}{\partial x_4} + \]
\[ + C_1 (A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3} + B_1 w + B_0) \]
\[ + C_2 (A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3} + B_1 w + B_0)^2 = 0, \]

where \( w = w(x_1, x_2, x_3, x_4) \) and \( A_i, B_i, C_i \) are constants, and \( A_1 \neq 0, C_2 \neq 0, 4C_0C_2 - C_1^2 \neq 0 \).

General solution

\[ w(x_1, x_2, x_3, x_4) = -\frac{1}{2A_1 C_2} \exp(-B_1 x_1) \int^{x_1} \exp(B_1 \xi \frac{A_1}{A_1}) (2B_0 C_2 + C_1 + \tan\frac{1}{2} x_4 \sqrt{4C_0 C_2 - C_1^2}) d\xi \]
\[ + G(\xi, (A_2 \xi + A_1 x_2 - A_2 x_1), (A_3 \xi + A_1 x_3 - A_3 x_1)) |\sqrt{4C_0 C_2 - C_1^2}) d\xi \]
\[ + \exp(-B_1 \xi \frac{A_1}{A_1}) F((A_1 x_2 - A_2 x_1), (A_1 x_3 - A_3 x_1), x_4), \]

where \( F(t_1, t_2, t_3) \) and \( G(t_1, t_2, t_3) \) are arbitrary functions.

5.2 \[ w(A_1 \frac{\partial^2 w}{\partial x_1 \partial x_4} + A_2 \frac{\partial^2 w}{\partial x_2 \partial x_4} + A_3 \frac{\partial^2 w}{\partial x_3 \partial x_4}) + C_0 \]
\[ + C_1 (w(A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3}) + B_0) \]
\[ + C_2 (w(A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3}) + B_0)^2 \]
\[ + (A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3}) \frac{\partial w}{\partial x_4} = 0, \]

where \( w = w(x_1, x_2, x_3, x_4) \) and \( A_i, B_i, C_i \) are constants, and \( A_1 \neq 0, C_2 \neq 0, 4C_0 C_2 - C_1^2 \neq 0 \).
5.3 General solution

\begin{align*}
w(x_1, x_2, x_3, x_4) &= \frac{1}{A_1 C_2} \left\{ A_1 C_2 \left(-\tan\left(\frac{x_4}{2}\sqrt{4C_0 C_2 - C_1^2}\right)(2B_0 C_2 + C_1) + \sqrt{4C_0 C_2 - C_1^2}\right) \right. \\
&\quad \times \int_{x_1}^{x_4} G(\xi, (\xi A_2 + x_2 A_1 - x_1 A_2), (\xi A_3 + x_3 A_1 - x_1 A_3)) \\
&\quad \left./\left[-1 + \tan\left(\frac{x_4}{2}\sqrt{4C_0 C_2 - C_1^2}\right)G(\xi, (\xi A_2 + x_2 A_1 - x_1 A_2), (\xi A_3 + x_3 A_1 - x_1 A_3))\right]\right\} d\xi \\
&\quad + A_1 C_2 (2B_0 C_2 + C_1) + \sqrt{4C_0 C_2 - C_1^2} \tan\left(\frac{x_4}{2}\sqrt{4C_0 C_2 - C_1^2}\right) \\
&\quad \times \int_{x_1}^{x_4} 1/\left[-1 + \tan\left(\frac{x_4}{2}\sqrt{4C_0 C_2 - C_1^2}\right)G(\xi, (\xi A_2 + x_2 A_1 - x_1 A_2), (\xi A_3 + x_3 A_1 - x_1 A_3))\right]\right\} d\xi \\
&\quad + F((x_2 A_1 - x_1 A_2), (x_3 A_1 - x_1 A_3), (x_4)) \frac{1}{x_4},
\end{align*}

where \(F(t_1, t_2, t_3)\) and \(G(t_1, t_2, t_3)\) are arbitrary functions.

\begin{align*}
5.3 \quad w(A_1 &\frac{\partial^2 w}{\partial x_1 \partial x_4} + A_2 \frac{\partial^2 w}{\partial x_2 \partial x_4} + A_3 \frac{\partial^2 w}{\partial x_3 \partial x_4}) + C_0 \\
&\quad + C_1 \exp[C_2 (w(A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3}) + B_0)] \\
&\quad + (A_1 \frac{\partial w}{\partial x_1} + A_2 \frac{\partial w}{\partial x_2} + A_3 \frac{\partial w}{\partial x_3}) \frac{\partial w}{\partial x_4} = 0,
\end{align*}

where \(w = w(x_1, x_2, x_3, x_4)\) and \(A_i, B_i, C_i\) are constants, and \(A_1 \neq 0, C_0 \neq 0, C_2 \neq 0\).

General solution

\begin{align*}
w(x_1, x_2, x_3, x_4) &= \left\{ -\frac{2}{A_1 C_2} (B_0 C_2 x_1 - x_1 \ln C_0 + \int_{x_1}^{x_4} \ln(\exp[C_2 C_0(x_4 \\
&\quad + G(\xi, (\xi A_2 + x_2 A_1 - x_1 A_2), (\xi A_3 + x_3 A_1 - x_1 A_3))] - C_1) d\xi \\
&\quad - F((x_2 A_1 - x_1 A_2), (x_3 A_1 - x_1 A_3), (x_4)) \frac{1}{x_4},
\end{align*}

where \(F(t_1, t_2, t_3)\) and \(G(t_1, t_2, t_3)\) are arbitrary functions.
6 Third order PDEs with two independent variables and constant parameters

6.1 $\frac{\partial^3 w}{\partial t^2 \partial x} = \left( \frac{\partial^2 w}{\partial t \partial x} \right)^2 + \frac{3}{2w} \frac{\partial w}{\partial t} \frac{\partial^2 w}{\partial t \partial x} - \frac{3}{2w^2} \left( \frac{\partial w}{\partial t} \right)^2 \frac{\partial w}{\partial x} ,$

General solution

$w(t, x) = \frac{G(x)}{[F(t) + H(x)]^2} ,$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.2 $\frac{\partial^3 w}{\partial t^2 \partial x} = \left( \frac{\partial^2 w}{\partial t \partial x} \right)^2 + \frac{2}{w} \frac{\partial w}{\partial t} \frac{\partial^2 w}{\partial t \partial x} - \frac{aw^2 \frac{\partial^2 w}{\partial t^2}}{\frac{\partial w}{\partial t}} - \frac{2}{w^2} \left( \frac{\partial w}{\partial t} \right)^2 \frac{\partial w}{\partial x} + 2aw \frac{\partial w}{\partial t} ,$

General solution

$w(t, x) = -\frac{\frac{dG(x)}{dx}}{F(t) + H(x) + atG(x)} ,$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.3 $\frac{\partial^3 w}{\partial t \partial x^2} = \frac{2}{w^2} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial t \partial x} + \left( b - \frac{1}{w} \frac{\partial w}{\partial t} \right) \frac{\partial^2 w}{\partial x^2} + \frac{g}{w} \left( \frac{\partial w}{\partial x} \right)^2 ,$

where $b, g$ are constants, and $b \neq 0$.

General solution

$w(t, x) = H(t) \exp\left[ b \int \frac{dx}{(b + g)x - e^{bt}G(x) + F(t)} \right] ,$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.
6.4 \[ [aw(aw{\partial^2 w}{\partial x^2}) - g({\partial w}{\partial x})^2)] [{\partial w}{\partial t}{\partial^2 w}{\partial x^2} - 2{\partial^2 w}{\partial^2 w}{\partial x^2}{\partial t}{\partial x}] \]
\[ + a{\partial w}{\partial x}{\partial w}({\partial^2 w}{\partial x^2})^2w - g{\partial w}{\partial t}({\partial w}{\partial x})^3{\partial^2 w}{\partial x^2} + h({\partial w}{\partial x})^5 = 0, \]
where \(a, g, h\) are constants, and \(a \neq 0\).

General solution

\[ w(t, x) = H(t) \exp[a \int \frac{dx}{(a - g)x + F(t) + \int \sqrt{G(x) - 2ah^t} dx}], \]
where \(F(t), G(x)\) and \(H(x)\) are arbitrary functions.

6.5 \[ [aw(aw{\partial^2 w}{\partial x^2}) - (a + b)({\partial w}{\partial x})^2)] \]
\[ \times (\frac{\partial w}{\partial x} \frac{\partial^3 w}{\partial t \partial x^2} - 2\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial t \partial x}) \]
\[ - a^2 w(mw - \frac{\partial w}{\partial t}) \frac{\partial w}{\partial x}({\partial^2 w}{\partial x^2})^2 \]
\[ + a(a + b)(2mw - \frac{\partial w}{\partial t}) \frac{\partial w}{\partial x}(\frac{\partial^2 w}{\partial x})^2 \]
\[ + k({\partial w}{\partial x})^5 = 0, \]
where \(a, b, m, k\) are constants, and \(a \neq 0, m \neq 0\).

General solution

\[ w(t, x) = H(t) \exp\{- \int \frac{am dx}{bm(t) + \int \sqrt{m[a + b]^2 + k]} + e^{2mG(x)} dx\}, \]
where \(F(t), G(x)\) and \(H(x)\) are arbitrary functions.
7 Third order PDEs with two independent variables and non-constant parameters

7.1 $w^2 (w + x \frac{\partial w}{\partial x}) \frac{\partial^3 w}{\partial t \partial x^2} = w [x w \frac{\partial^2 w}{\partial x^2} + 2 x (\frac{\partial w}{\partial x})^2 + 4 w \frac{\partial w}{\partial x}] \frac{\partial^2 w}{\partial t \partial x} + w^2 (w^2 + \frac{\partial w}{\partial t}) \frac{\partial^2 w}{\partial x^2} - 2 x (\frac{\partial w}{\partial x})^3 \frac{\partial w}{\partial t} - 2 w (w^2 + 2 \frac{\partial w}{\partial t}) (\frac{\partial w}{\partial x})^2,$

General solution

$$w(t, x) = \frac{dF(t)}{dt} \left( \frac{dF(t)}{dx} - F(t) + xH(t) \right),$$

where $F(t), G(x)$ and $H(t)$ are arbitrary functions.

7.2 $\frac{\partial^3 w}{\partial t^2 \partial x} = \frac{a}{t} \frac{\partial^2 w}{\partial t \partial x} + \frac{k}{x} \frac{\partial^2 w}{\partial t^2} - \frac{ak}{tx} \frac{\partial w}{\partial t} - \frac{b}{t^2} \frac{\partial w}{\partial x} + \frac{bkw}{t^2 x},$

General solution

$$w(t, x) = F(t)x^k + H(x)t^\frac{3}{4} + \frac{\frac{3}{4} + \frac{1}{2} \sqrt{1 + 2a + a^2} - 4b}{4},$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

8 Conclusion

The details of the method and more extensive lists of solvable PDEs are preparing for publication. The method in principle fit for implementation in CAS, e.g., in Maple.
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