Folded Strings*

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Abstract

Recent progress on the complete set of solutions of two dimensional classical string theory in any curved spacetime is reviewed. When the curvature is smooth, the string solutions are deformed folded string solutions as compared to flat spacetime folded strings that were known for 19 years. However, surprizing new stringy behavior becomes evident at singularities such as black holes. The global properties of these solutions require that the “bare singularity region” of the black hole be included along with the usual black hole spacetime. The mathematical structure needed to describe the solutions include a recursion relation that is analogous to the transfer matrix of lattice theories. This encodes lattice properties on the worldsheet on the one hand and the geometry of spacetime on the other hand. A case is made for the presence of folded strings in the quantum theory of non-critical strings for $d \geq 2$.

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1 Introduction

Feza Gürsey was a great master and an artist in finding connections between Mathematics and Physics. In this conference we have the pleasure to hear from many of his friends that have admired his leadership in several areas of Physics. I am very appreciative for having been given the opportunity to express my gratitude to Feza for the inspiration he has provided to me as my teacher as well as my colleague and friend.

Among his first discoveries was the non-linear sigma models, which he applied to pion physics. Nowadays, sigma models are at the basis of string theory in the form of conformal field theories. In recent years, through the use of gauged Wess-Zumino-Witten models based on non-compact groups, it has been possible to construct exact conformal field theories that describe (super)strings propagating in curved space-time in 2D to 4D. These models combine several fields that deeply interested Feza: non-linear sigma models, conformal invariance, classical and quantum gravity, non-compact groups, unification of forces, string theory. In honoring Feza today, I would like to highlight recent progress made in this field.

2 Motivation

The original physical motivations for studying string theory were: (1) understanding unification of forces including quantum gravity, and (2) understanding the Standard Model. In recent years it has become more and more evident that these goals should be examined in the presence of curved 4D space-time string backgrounds. The construction of 4D curved space-time string theories that correspond to exact conformal theories have provided models in which various questions can be investigated.

The usual scenario of flat 4D plus extra curved dimensions may not be the right approach for making predictions about the Standard Model. The gauge symmetries and spectrum of quark + lepton families, which are the main ingredients of the Standard Model, were probably fixed during the early times in the evolution of the Universe. At such times 4D space-time was curved.

\footnote{Feza was the first to introduce the idea of the sigma models. Many people think of the paper by Gell-Mann and Lévy in connection with sigma models, but it is important to recall that Gell-Mann and Lévy refer to Feza’s paper.}
Since curvature contributes to the central charge and other topological aspects of String Theory, it is likely that the predictions of String Theory under such conditions may be quite different than the flat 4D approach. Therefore, String Theory in curved space-time must be better understood before attempting to make connections to low energy physics. One should consider all kinds of curved backgrounds, not only the traditional cosmological backgrounds, since the passage from curved space-time to flat space-time may involve various phase transitions, including inflation of a small region of the original curved universe to today’s universe that is essentially homogeneous and flat. The gauge bosons, and chiral families of quarks and leptons in a small region of the early curved universe would become the ones observed in today’s inflated flat universe. The possibility of such a scenario suggests that curved space-time string theory deserves intensive study. In addition, the issues surrounding gravitational singularities should be answered in the context of curved space-time string theory, as it is the only known theory of quantum gravity.

3 Some Results in 1-time G/H

With these questions in mind, we have been pursuing a program of building and analyzing exactly solvable models of string theory in curved spacetime based on conformal field theory. The main tool is the G/H gauged WZW model based on non-compact groups, such that the coset contains a single time coordinate. A lot of progress was made on the construction of exact conformal field theory models for bosonic, supersymmetric and heterotic strings in curved spacetime, and some exact results were derived. These include:

- Classification of G/H models with 1-time + (d - 1)-space coordinates [1]. G is non-compact and H can be non-compact or compact. After identifying the simple cases for G/H that yield a single time coordinate, the classification is easily extended to semi-simple, with Abelian factors, and their contractions to solvable groups [3][4]. There may also exist other exact conformal models which may not be G/H models.

- The non-linear sigma model geometries for these models have been derived [3][7][1][2], giving the metric $G_{\mu\nu}(x)$, the anti-symmetric tensor $B_{\mu\nu}(x)$, and the dilaton $\Phi(x)$. These automatically solve Einstein’s
equations for dilaton gravity. The global spaces for these manifolds have been constructed, and rich duality symmetries have been identified. Furthermore, the exact point particle geodesics in the global space have been obtained through group theoretical methods [6][7][1].

- Quantum corrections to these geometries have been computed to all orders in the sigma model interactions, thanks to the group theoretical construction, mainly by using algebraic methods[8]. This led to an exact quantum effective action [8] with the quantum corrected metric, antisymmetric tensor and dilaton. The results show that for type-II superstrings, thanks to the supersymmetry, there are no corrections to the classical expressions. In bosonic or heterotic cases the corrections show that certain singularities of the metric get shielded by quantum corrections in parts of spacetime.

- In these models the exact spectrum of the Laplacian can be obtained through unitary representation theory of non-compact groups. This provides the method for extracting the spectrum of quarks and leptons, but more work is needed along these lines.

4 Classical solutions

More recently, it became apparent that a physical interpretation of the models as well as further progress will be accomplished through the study of the classical equations of such models. Therefore, we have turned to the classical theory. This is relevant to fundamental questions of singularities in gravitational physics, as well as stringy questions about the early universe and its influence on the low energy spectrum of quarks and leptons. The classical string solution for any gauged WZW model was obtained in general terms in [9], and its specialization to particle solutions was given explicitly in [8].

A more detailed exploration of the general 2D classical string theory in any curved spacetime (i.e. not only WZW models) was done in [5]. In 2D the only non-trivial stringy solutions turn out to be necessarily folded strings, and therefore they are the only path toward analyzing stringy questions in a toy model. In addition to the interest in singular gravitational behavior (such as black holes) there has also been a long-standing interest in exploring consistent generalizations of non-critical strings with the hope that they may
be relevant for some branch of physics. Folded strings fall into this category, especially in the area of string-QCD relations. Therefore two aspects of string theory were investigated: (i) strings in curved space-time and (ii) folded strings.

In papers [5][10][11] the complete set of solutions of two dimensional classical string theory were constructed for any 2D curved spacetime. The classical action is given by $\int d^2\sigma G_{\mu\nu}(x)\partial_+ x^\mu \partial_- x^\nu$. In 2D $B_{\mu\nu}(x)$ can be eliminated since it produces a total derivative in the action, and in the classical theory the dilaton is absent. The most general metric can always be transformed into the conformal form $G_{\mu\nu} = \eta_{\mu\nu}G(x)$. Then the most general 2D classical string equations of motion and conformal (Virasoro) constraints take the form

$$\begin{align*}
\partial_+(G \partial_- u) + \partial_-(G \partial_+ u) &= \frac{\partial G}{\partial u}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+(G \partial_- v) + \partial_-(G \partial_+ v) &= \frac{\partial G}{\partial v}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+ u \partial_+ v &= 0 = \partial_- u \partial_- v,
\end{align*}$$

where we have used the target space lightcone coordinates $u(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 + x^1)$, $v(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 - x^1)$, and the world sheet lightcone coordinates $\sigma^\pm = (\tau \pm \sigma)/\sqrt{2}$, $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/\sqrt{2}$.

In flat space-time the solutions are given in terms of arbitrary left-moving and right-moving functions $x^\mu_L(\sigma^+), x^\mu_R(\sigma^-)$

$$x^\mu(\tau, \sigma) = x^\mu_L(\sigma^+) + x^\mu_R(\sigma^-).$$

As shown by BBHP [12][13], the constraints are also satisfied provided\footnote{Although the original BBHP solutions were for open strings, the same solutions also apply to closed strings by simply taking independent functions $f, g$ for left movers and right movers.}

$$\begin{align*}
u(\sigma^+, \sigma^-) &= v_0 + \frac{\nu^-}{\sqrt{2}} \left[ (\sigma^+ + f(\sigma^+)) + (\sigma^- - g(\sigma^-)) \right] \\
v(\sigma^+, \sigma^-) &= v_0 + \frac{\nu^-}{\sqrt{2}} \left[ (\sigma^+ - f(\sigma^+)) + (\sigma^- + g(\sigma^-)) \right]
\end{align*}$$

where $f(\sigma^+)$ and $g(\sigma^-)$ are any two periodic functions, $f(\sigma^+) = f(\sigma^+ + \sqrt{2})$, $g(\sigma^-) = g(\sigma^- + \sqrt{2})$, with slopes $f'(\sigma^+) = \pm 1$ and $g'(\sigma^-) = \pm 1$. The slope can change discontinuously any number of times at arbitrary locations $\sigma^+_i, \sigma^-_j$ within the basic intervals $-1/\sqrt{2} \leq \sigma^\pm \leq 1/\sqrt{2}$ (and then repeated periodically), but the functions $f, g$ are continuous at these points. The discontinuities in the slopes are allowed since the equations of motion are first order in
either $\partial_+$ or $\partial_-$. The number of times the slope changes in the basic interval corresponds to the number of folds for left movers and right movers respectively. The simplest BBHP solution is the so called yo-yo solution given by $f = |\sigma^+|_{\text{per}}$ and $g = |\sigma^-|_{\text{per}}$ which are the periodically repeated absolute value. These solutions describe folded strings, with the folds oscillating against each other, and moving at the speed of light. Examples are plotted in Figures 1,2. In Fig.1 one sees the yo-yo solution with equal periods for $|\sigma^+|_{\text{per}}$, and $|\sigma^-|_{\text{per}}$. Fig. 2 is generated by taking the period of $|\sigma^-|_{\text{per}}$ to be half of that of $|\sigma^+|_{\text{per}}$.

As discovered in [5][10], the complete set of classical solutions in curved spacetime are classified by their behavior in the asymptotically flat region of spacetime $G(u, v) \to 1$, where they tend to the folded string solutions of BBHP given in (2) as boundary conditions. The curved space solutions are given in the form of a map from the world sheet to target spacetime, where (as a mathematical convenience) the world sheet is divided into lattice-like patches corresponding to different maps. The world-sheet lattice structure is determined by the sign patterns of $(f', g') = (\pm, \pm)$ inherent in the BBHP solutions, thus the lattice is dictated by the boundary conditions in the asymptotically flat region of spacetime $G(u, v) \to 1$. The lattice is on the world-sheet, not in curved spacetime, it is only a mathematical tool to keep track of patches, and the world sheet is not at all discretized. In each patch of the lattice one set of signs holds, hence there are 4 types of patches called $A, B, C, D$. For each such patch there is a solution of the equations of motion that is valid within the patch. The forms of the solutions in patches labelled by an integer $k$ are (see eq.(4) for an example of a pattern of patches)

$$
\begin{align*}
A : & \quad u = U_k(\sigma^+), \quad v = V_k(\sigma^-) \\
B : & \quad u = U_k(\sigma^-), \quad v = V_k(\sigma^+) \\
C : & \quad u = u_k, \quad v = W[\alpha_k(\sigma^+) + \beta_k(\sigma^-), u_k] \\
D : & \quad u = W[\alpha_k(\sigma^-) + \beta_k(\sigma^+), v_k], \quad v = v_k,
\end{align*}
$$

(3)

where the constants $u_k, v_k$ and the functions $U_k(\sigma^\pm), V_k(\sigma^\pm), \alpha_k(\sigma^\pm), \beta_k(\sigma^\pm)$ are given by a recursion relation whose form depends on the metric $G$. It is easy to verify that, independently of the recursion relation, the forms listed in (3) solve the differential equations for any $U_k(\sigma^\pm), V_k(\sigma^\pm), \alpha_k(\sigma^\pm), \beta_k(\sigma^\pm)$
provided the functions $W, \bar{W}$ are defined by inverting the following functions

$$
\int^W du' G(u_k, v') = \alpha + \beta, \quad \int^\bar{W} du' G(u', v_k) = \bar{\alpha} + \beta.
$$

By construction, in flat spacetime the recursion reproduces the BBHP solutions given above.

The recursion relation, which is analogous to a “transfer matrix”, connects the maps in different patches into a single continuous map. It is derived by demanding continuity across the boundaries of each patch (see below for an example). Thus, the functions in the various patches get related to each other. This “transfer matrix” encodes the properties of the world sheet lattice on the one hand and the geometry of spacetime on the other hand. Thus, lattices on the world-sheet plus geometry in space-time lead to “transfer matrices”. Recall that the lattice is dictated by the nature of the solution (2) in the asymptotically flat region of target spacetime. This seems to be a rich area of mathematical physics to explore in more detail in the future.

As an example we consider the simplest yo-yo solution as a boundary condition. This defines the sign patterns according to the slopes of the periodic functions $|\sigma^+|_{\text{per}}$ and $|\sigma^-|_{\text{per}}$, and the following lattice emerges from the periodic behavior of these functions. The world sheet is labelled by $\sigma$ horizontally and by $\tau$ vertically. Periodicity in $\sigma$ is imposed, hence the world sheet is a cylinder. It is sliced by equally spaced 45° lines that form a light-cone lattice in $\sigma^\pm$. The crosses in the diagram represent the corners of the cells on the world sheet.
The transfer matrix for this “yo-yo lattice” was derived in [3][11][10] for any metric \( G \). Here we give only the results for the \( SL(2, R) / R \) black hole space-time \( ds^2 = du \, dv (1 - uv)^{-1} \) and for the cosmological deSitter space-time \( ds^2 = dt^2 - R^2(t) \frac{dr^2}{1 - kr^2} = \frac{4}{H^2} (u + v)^{-2} \, du \, dv \), for \( |R(t)| = e^{Ht}, \, k = 0 \).

For the black hole metric the “transfer matrix” is

\[
\bar{W}_k = \frac{1}{v_k} \left( 1 - \frac{(1 - U_k(\sigma^+)) v_k}{1 - u_{k-1} v_k} \right),
\]

\[
U_{k+1}(z) = \frac{1 - u_{k+1} v_{k+1}}{1 - u_{k-1} v_k} \left[ U_k(z) + \frac{u_k - u_{k-1}}{1 - u_k v_k} \right],
\]

\[
u_{k+1} = \frac{2 u_k - u_{k-1} - u_k^2 v_k}{1 - u_{k-1} v_k}, \tag{5}
\]

and similarly \( W_k, V_k, v_k \) are obtained from the above by interchanging \( U \leftrightarrow V \) and \( u \leftrightarrow v \). The constants \( u_k, v_k \) are the values of the functions \( U_k(z), V_k(z) \) at the boundaries of the cell labelled by \( k \):

\[
u_{k-1} = U_k(-1/\sqrt{2}), \quad u_k = U_k(1/\sqrt{2}), \quad \text{etc.}
\]

These constants describe the motion of folds that move at the speed of light. Note that for \( u, v \to 0 \) or \( \infty \) the metric approaches the flat metric.

Remarkably, there is an invariant of this “transfer matrix” that corresponds to the “lattice area” swept by the string [for comparison, recall the form of the action density whose meaning is area \( dA = (1 - uv)^{-1} (\partial_u u \partial_v v + \partial_v v \partial_u u) \)]

\[
dA_k = \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{1 - \frac{4}{3} (u_k + u_{k-1})(v_k + v_{k-1})}. \tag{6}
\]

It can be easily verified that \( dA_{k+1} = dA_k \), implying that this quantity remains a constant even in the vicinity of singularities \( uv \approx 1 \). This observation leads to new interesting phenomena as discussed below.

The “transfer matrix” for the deSitter spacetime is

\[
\bar{W}_k(\sigma^+, \sigma^-) = \left[ \frac{1}{U_k(\sigma^+)} + \frac{1}{U_k(\sigma^-)} - \frac{1}{u_{k-1} v_k} \right]^{-1} - v_k
\]

\[
U_{k+1}(z) = \left[ \frac{1}{U_k(z)} + \frac{1}{u_k + v_k - u_{k-1} v_k} \right]^{-1} - v_k
\]

\[
u_{k+1} = \frac{2 u_k + v_k - u_{k-1} v_k}{1 - u_{k-1} v_k} - v_k \tag{7}
\]

Similar formulas hold for \( W_k, V_k, v_k \) respectively. In this case too there is an invariant area

\[
dA_k = \frac{4}{H^2} \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{(u_k + v_{k-1})(u_{k-1} + v_k)}
\]
The 2D deSitter space can be embedded in 3D as the surface of a hyperboloid described by

$$x_0^2 - x_1^2 - x_2^2 = -H^{-2}, \quad x_1 = \frac{uv-H^{-2}}{u+v}, \quad x_2 = \frac{1}{H} \frac{u-v}{u+v}$$  (8)

Then the deSitter metric takes the flat form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2.$$  (9)

The motion is more easily visualized in this parametrization.

The recursion relations are solved in terms of two functions $U_0(z), V_0(z)$ that are associated with the initial cell $k = 0$. The remaining conformal invariance may be used to fix the form of the functions $U_0(z), V_0(z)$ in the initial cell (although this is not necessary). For the yo-yo solution the initial functions $U_0(z), V_0(z)$ need not contain more than 4 constants that are related to the initial positions and velocities of the two folds. However, there is a physical requirement: the time coordinate $x^0(\tau, \sigma)$ constructed from $U_0(\sigma^\pm), V_0(\sigma^\pm)$ must be an increasing function of the proper time $\tau$ for any value of $\sigma$, so that physically every point on the string moves forward in time (no bits of anti-strings). Therefore, the simplest physical gauge fixed form is

$$U_0(z) = \frac{1}{2}(u_0 + u_{-1}) + \frac{1}{\sqrt{2}} (u_0 - u_{-1}) z_{\text{per}},$$
$$V_0(z) = \frac{1}{2}(v_0 + v_{-1}) + \frac{1}{\sqrt{2}} (v_0 - v_{-1}) z_{\text{per}},$$  (10)

where $z_{\text{per}}$ is the linear function $z_{\text{per}} = z$ in the interval $-1/\sqrt{2} \leq z \leq 1/\sqrt{2}$, and then repeated periodically. This form reproduces the BBHP yo-yo solution in flat spacetime from the recursion relations, provided one uses $G(u, v) = 1$. However, any other function with the same 4 boundary constants and general increasing character will produce the same, gauge independent, physical motion for the folds in flat or curved spacetime, since their motion is given by gauge independent equations involving only the gauge independent constants $u_k, v_k$. Evidently, the motion of the intermediate points of the string is gauge dependent, as expected.

The constants $(u_k, v_k)$ are sufficient to describe the physical motion of the folds (or end points), as well as the whole string. The trajectories of the folds are plotted in Figs.4,5,6 by feeding the recursion relations to a computer. As in Fig.3, the string performs oscillations that are similar to those of flat
spacetime around a center of mass that follows on the average the geodesic of a massive point particle. This is expected intuitively. A detailed discussion of the black hole case was given in [5]. The main surprise is the tunnelling of the string into the forbidden region in Fig.5 (the bare singularity region of the black hole), where particles cannot go. This behavior cannot be avoided since it follows from minimal area conservation laws that were given above [5][10][11]. It could be compared to diffraction, or light illuminating the wall around a corner, that can happen with waves, but not with particles. In addition, the massive point particle geodesic (as well as the string geodesic) does not stop at the black hole, rather it reaches the black hole in a finite amount of proper time, and then it continues into a second sheet of spacetime that is glued to the first sheet at the black hole singularity. The observers on the second sheet see it as if the massive particle or the string is coming out of a white hole. The motion may continue from white hole to black hole singularities, each time moving into a new sheet, interpreted as a new world, like in the Reissner-Nordstrom spacetime. For more details see [5][10][11].

5 Quantum Folded String

Given the fact that the string in 2D is quite non-trivial classically, we expect that there is a consistent quantization procedure that includes the non-trivial folded states. Therefore we should try to make a case for folded strings in the quantum theory.

As pointed out many times in our past work, folded 2D-string states are present in the $d = 2$ and $c \leq 25$ sector of the quantum theory in flat as well as curved spacetime. In simple string models, when it has been possible to compute the spectrum, their norm is positive and is proportional to $(c - 26)$. Only if $d = 2$ and $c = 26$ simultaneously (e.g. $d = 2$ flat space-time with linear dilaton such that $c = 26$) the folded string states become zero norm states and then the special discrete momentum states survive as the only stringy states. A simple model in which these properties may be easily seen is the covariant quantization of the 2D string theory, in which the physical states are identified as the subset that satisfies the Virasoro constraints, i.e. $L_0 - \frac{d-2}{24} = L_{\alpha \geq 1} = 0$ applied on states. For example, it has been known for a long time that the $d \leq 25$ sector of the flat string theory has non-trivial
positive norm states (including for \( d = 2 \)) that satisfy the Virasoro constraints and that there are no ghosts [14]. A similar covariant quantization can be carried out for the 2D black hole string by using the Kac-Moody current algebra formulation, and relaxing the \( c = 26 \) condition (i.e. \( k < 9/4 \)) to include the folded strings.

Why \( c = 26 \)? There are several approaches to the quantization of strings that converge on the requirement of \( c = 26 \). These include the light-cone gauge, the Polyakov path integral and the BRST quantization. However, they each involve certain steps that seem to inadvertently exclude the \( c < 26 \) string. We can point out that

(i) The usual light-cone approach throws away the folded states from the beginning by assuming a uniform momentum density \( P^+(\tau, \sigma) = p^+ \), a statement that is not true for the BBHP solutions even in flat spacetime.

(ii) The Polyakov approach assumes a certain measure for the path integral, thus locking into a definition of a quantum theory. A different measure that takes into account folds can be considered as in [15] mentioned below.

(iii) the BRST approach requires \( Q^2_{\text{BRST}} = 0 \) as an operator. This is a stronger requirement than the Virasoro constraints satisfied only in the physical subspace \( \langle \text{phys}|L_n - \alpha_0 \delta_{n0}|\text{phys} \rangle = 0 \). An analogous statement would be \( \langle \text{phys}|Q_{\text{BRST}}|\text{phys} \rangle = 0 \), which does not lead to \( c = 26 \). Actually, the fact that there exists a consistent covariant quantization of the flat free string in \( d < 26 \) is already proof that the \( Q^2_{\text{BRST}} = 0 \) approach is too strong.

Therefore, it appears that a more general quantization of string theory for \( c < 26 \), that would permit folded string states, seems possible. What would also be interesting is to find the correct formulation for interacting folded strings. The path integral approach discussed in [15] seems to be promising, and it may be possible to make faster progress by reformulating it in the conformal gauge and relating it to our classical solutions. Note that the definition of fold in ref. [15] does not take into account that the map from the world sheet to spacetime may be many to one (i.e. a region maped to a segment, as is the case for our solutions). This feature may be important in
the formulation of folds and their interactions in the path integral approach. In particular, the description of folds in the conformal gauge, as in our papers, may eventually prove to be a more convenient mathematical formulation than the one used in [15].

6 Higher dimensions

Folded strings exist in higher dimensions as well. One can display the general solution in flat space-time in the temporal gauge

\[ x^0 = p^0 \tau, \quad \vec{x}(\tau, \sigma) = \vec{x}_L(\sigma^+) + \vec{x}_R(\sigma^-), \quad (\partial_+ \vec{x}_L)^2 = p_0^2 = (\partial_- \vec{x}_R)^2 \]

\[ \partial_+ \vec{x}_L = p^0 \left( \frac{2f}{1+f^2}, \frac{1-f^2}{1+f^2} \varepsilon_L \right), \quad \partial_- \vec{x}_R = p^0 \left( \frac{2g}{1+g^2}, \frac{1-g^2}{1+g^2} \varepsilon_R \right) \] (11)

where \( f(\sigma^+), \, g(\sigma^-) \) are arbitrary periodic vectors in \( d-2 \) dimensions, \textit{which could be discontinuous}, and \( \varepsilon_L(\sigma^+), \, \varepsilon_R(\sigma^-) \) take the values \( \pm 1 \) in patches of the corresponding variables such that the sign patterns repeat periodically (as in the 2D string). When \( f, g \) are both zero the solution reduces to the 2 dimensional BBHP case. In general, the presence of discontinuous \( \varepsilon_L, \varepsilon_R \), and the discontinuities in \( f(\sigma^+), \, g(\sigma^-) \) gives a larger set of solutions, which include strings that are partially or fully folded. Discontinuities are allowed since the differential equations are first order in the derivatives \( \partial_+ \) and \( \partial_- \). Such solutions are usually missed in the lightcone gauge even in the flat classical theory (therefore, the lightcone “gauge” is not really a gauge).

The curved space-time analogs of such solutions in higher dimensions are presently under investigation.

7 Comments and Conclusions

We have solved generally the classical 2D string theory in any curved space-time. All stringy solutions correspond to folded strings. All solutions tend to the BBHP solutions (as boundary conditions) in the asymptotically flat region of the curved space-time. Therefore, the BBHP solutions of eq. (3) serve to classify all the solutions for any curved space-time. In fact, the sign
patterns of the BBHP solutions provide the method for dividing the world-sheet into patches, thus defining the lattices associated with the $A, B, C, D$ solutions. The matching of boundaries for these functions gives the general solution in curved space-time in the form of a “transfer matrix”. Thus, lattices on the world-sheet plus geometry in space-time lead to transfer matrices. This seems to be a rich area to explore in more detail.

The general physical motion of the string is: oscillations around a center of mass that follows on the average a geodesic of a massive particle, consistent with intuition. The oscillations are deformed by curvature as compared to the BBHP solutions in flat spacetime, but they maintain the same general character as long as the curvature is smooth. However, new stringy behavior becomes evident in the vicinity of singularities where new phenomena, such as tunneling (similar to diffraction), take place. There is also the continuation of the motion into new worlds, in a finite amount of proper time, that the string as well as the massive particle geodesics do (but not the massless particle! - see [10][11]). Because of the tunelling and the new worlds, the global space of the $SL(2, R)/R$ black hole is not just the usual black hole space, $uv < 1$. Rather, it must include also the $uv > 1$ “bare singularity” region even for the classical description of strings (actually this region is not really singular, as argued in [10][11]). We conjecture that the inclusion of the bare singularity region is a more general requirement than the $SL(2, R)/R$ case for the correct description of string motion. Of course, by duality, the quantum theory must include all the regions.

Folded string are also of interest in a string-QCD relation. Gluons are expected to behave just like the folds, since only at the location of a gluon the color flux tube can fold. Some recent discussion if this point can be found in [16] and [10][11].

We suspect that the inclusion of the quantum states corresponding to folded strings may lead to a consistent quantum theory in less than 26 dimensions. As already emphasized earlier in the paper, the free string is perfectly consistent as a quantum theory for $c < 26$, including the folded states. The interacting quantum string with folds remains as an open possibility.
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Figures

Fig. 1.— Minimal surface of flat string with 2 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.

Fig. 2.— Minimal surface of flat string with 3 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.
Fig. 3 – Minimal area in curved spacetime. The sizes of the rectangles change depending on the curvature.

Fig. 4. Ingoing string on 1st sheet meets black hole, moves out to 2nd sheet.
Fig. 5. String minimal area tunnels to forbidden region beyond black hole. Arrows along trajectories of midpoint.