The hierarchy recurrences in local relaxation

Sheng-Wen Li\textsuperscript{1} and C. P. Sun\textsuperscript{2,3}

\textsuperscript{1}Center for quantum technology research, School of Physics, Beijing Institute of Technology, Beijing 100081, China
\textsuperscript{2}Graduate School of China Academy of Engineering Physics, Beijing 100193, China
\textsuperscript{3}Beijing Computational Science Research Center, Beijing 100193, China

Inside a closed many-body system undergoing the unitary evolution, a small partition of the whole system exhibits a local relaxation. If the total degrees of freedom of the whole system is a large but finite number, such a local relaxation would come across a recurrence after a certain time, namely, the dynamics of the local system suddenly appear random after a well-ordered oscillatory decay process. It is found in this paper, for a collection of $N$ two-level systems (TLSs), the local relaxation of one TLS within has a hierarchy structure hiding in the randomness after such a recurrence: similar recurrences appear in a periodical way, and the later recurrence brings in stronger randomness than the previous one. Both analytical and numerical results that we obtained well explains such hierarchy recurrences: the population of the local TLS (as an open system) diffuses out and regathers back periodically due the finite-size effect of the bath \cite{theoretical-framework}. We also find that the total correlation entropy, which sums up the entropy of all the $N$ TLSs, approximately exhibit a monotonic increase; in contrast, the entropy of each single TLS increases and decreases from time to time, and the entropy of the whole $N$-body system keeps constant during the unitary evolution.

\textbf{Introduction:} When an open system is contacted with a bath infinitely large, the open system would approach a certain steady state after a long time relaxation. However, such an irreversible behavior cannot be seen in the dynamics of one or few-body systems. Thus the macroscopic irreversibility seems contradicted with the microscopic reversibility \cite{Stueckelberg,Chernov,Mah Exact,Reco,Reco 1}.

One useful way to look through this problem is to study the system relaxation contacted with a finite bath, namely, the bath contains a finite number of degrees of freedom (DoF), and then consider its transition to the thermodynamics limit \cite{Nordahl,Dauxois,Bonilla}. The open system and the bath as a whole isolate system always follows the unitary evolution and keeps a constant entropy as the initial state, while the open system itself seems relaxing towards a certain steady state, thus such a relaxation behavior of the open system itself is called the \textit{local relaxation} \cite{Nordahl,Dauxois,Bonilla}.

Due to the finite-size effect of the bath, the local relaxation of the open system would come across a recurrence after a certain time: at first the system dynamics shows a well-ordered oscillatory decay behavior, but then suddenly becomes random \cite{Nordahl,Dauxois,Bonilla,Reco,Reco 1}. With the increase of the DoF number in the bath, such a recurrence time appears much later, thus it does not show up in practice.

In this paper, we find that, in the region after such a recurrence, indeed there exists a hierarchy structure hiding in the randomness: similar recurrences appear in a periodical way, and the later recurrence brings in stronger randomness than the previous one, therefore, we call them \textit{hierarchy recurrences}.

Here we study the dynamics of a chain of $N$ two-level systems (TLSs). One of the TLSs is treated as the open system, and all the other ($N-1$) TLSs make up a finite bath. We obtain a Bessel function expansion for the system dynamics, which well explains the appearance of such hierarchy recurrences. Further, we also find the physical reason for the appearance of such hierarchy recurrences: with the time increases, the population of the open system diffuses out and propagates in the finite bath (the periodic TLS chain); once the population regathers back to the open system, the system dynamics exhibits such a recurrence, and this process happens again and again, which gives rise to the hierarchy recurrences.

We also study the dynamics of the \textit{total correlation entropy} of the $N$-body system, which sums up the entropy of all the $N$ TLSs \cite{Calderaro,Calderaro 2,Calderaro 3}. It turns out the total correlation approximately exhibits a monotonic increasing behavior, and the increasing curve becomes more and more “smooth” with the increase of the bath size. Thus, the total correlation exhibits a quite similar behavior as the irreversible entropy increase in the standard thermodynamics \cite{Melloni,Tracy,Chisnell}. In contrast, the whole $N$-body system always keeps a constant due to the unitary evolution, and the entropy of each single TLS increases and decreases from time to time.

\textbf{Local relaxation:} We consider a chain of $N$ TLSs. They have equal on-site energies ($\omega \geq 0$), and exchange energy with the nearest neighbors (interaction strengths $g$):

$$\hat{H} = \sum_{n=0}^{N-1} \frac{1}{2} \omega \hat{\sigma}_{n}^{z} + g(\hat{\sigma}_{n}^{+}\hat{\sigma}_{n+1}^{-} + \hat{\sigma}_{n}^{-}\hat{\sigma}_{n+1}^{+}).$$ \hspace{1cm} (1)

Here $\hat{\sigma}_{n}^{\pm} := (\hat{\sigma}_{n}^{-})^{{}\dagger} = |\psi\rangle_{n}(\pi/2), \hat{\sigma}_{n}^{z} := |\psi\rangle_{n}(\pi) - |\psi\rangle_{n}(\pi/2)$, and $|\psi\rangle_{n}, |\pi\rangle_{n}$ are the excited and ground states of the $n$-th TLS.

Here site-0 is regarded as an open “\textit{SYSTEM}”, while all the other ($N-1$) TLSs build up a finite “\textit{BATH}”. Initially, the “\textit{SYSTEM}” (site-0) starts from the excited state as its initial state, and all the TLSs in the “\textit{BATH}” start from the ground state. Thus effectively the “\textit{BATH}” has a temperature $T \rightarrow 0^{+}$. And now we study the dynamics of the open “\textit{SYSTEM}”.

The Hamiltonian (1) is a quantum XX model \cite{Haldane}, and the dynamics of the whole chain is exactly solvable. Applying the Jordan-Wigner transform, the Hamiltonian (1) becomes a
fermionic one,
\[ \sigma^z_n = 2\hat{c}^\dagger_n \hat{c}_n - 1, \quad \hat{\sigma}_n^+ = \hat{c}^\dagger_n \prod_{i=0}^{n-1} (-\hat{\sigma}_i^z), \]
\[ \hat{H} = \sum_{n=0}^{N-1} \omega \hat{c}^\dagger_n \hat{c}_n + g(\hat{c}^\dagger_{n+1} \hat{c}_n + \hat{c}^\dagger_n \hat{c}_{n+1}). \] (2)

Under the periodic boundary condition, it can be further diagonalized by the Fourier transform \( \hat{c}_n = \sum_{k=0}^{N-1} \exp(i\frac{2\pi nk}{N}) \hat{b}_k \sqrt{N} \), which reads \( \hat{H} = \sum_n \varepsilon_n \hat{b}_k^\dagger \hat{b}_k \), with the eigen mode energy \( \varepsilon_k = \omega + 2g \cos \frac{2\pi k}{N} \).

The \( N \)-body chain as a whole isolated system follows the unitary evolution. From the above transformations, the above initial condition gives \( \langle \hat{c}^\dagger_n \hat{c}_n \rangle_{t=0} = 1 \), and \( \langle \hat{c}^\dagger_n \hat{c}_n \rangle_{t=0} = 0 \) for the other \( m, n \), and that gives the following dynamics
\[ \langle \hat{c}^\dagger_m \hat{c}_n \rangle(t) = \sum_{k, q=0}^{N-1} \frac{1}{N} e^{i \frac{2\pi nq - i \frac{2\pi}{N} nk \hat{b}_k^\dagger(0) e^{i \omega t} \hat{b}_q(0)}{N} e^{-i \varepsilon_k t}} \]
\[ = \sum_{k, q, x, y} \langle \hat{c}^\dagger_m \hat{c}_n \rangle(0) e^{i \frac{2\pi}{N} (n-y) - i \frac{2\pi}{N} k(m-x) + i(\varepsilon_k - \varepsilon_q) t} \]
\[ := \lbrack \Phi_m^{(N)}(2gt) \rbrack^* \Phi_n^{(N)}(2gt), \] (3)

where we call \( \Phi_m^{(N)}(2gt) \) as the coherence function, and
\[ \Phi_n^{(N)}(\tau) := \frac{1}{N} \sum_{k=0}^{N-1} \exp \left[ -i \tau \cos \frac{2\pi k}{N} + \frac{2\pi k}{N} n \right] \]
\[ \overset{N \rightarrow \infty}{\longrightarrow} \int_0^{2\pi} \frac{dx}{2\pi} e^{-i \tau \cos x + imx} = (-i)^n J_n(\tau). \] (4a)

In the thermodynamics limit \( N \rightarrow \infty \), \( \Phi_n^{(N)}(\tau) \) becomes the Bessel function \( J_n(\tau) \), which approaches zero when \( \tau \rightarrow \infty \) [6–8, 10, 11].

It can be seen from Eqs. (2, 3) that, each site always keeps a diagonal density state \( \rho_n(t) = p_{n,e}(t) |e\rangle_n \langle e| + p_{n,g}(t) |g\rangle_n \langle g| \), where \( p_{n,e}(t) := \langle \hat{\sigma}_n^+ \hat{\sigma}_n^- \rangle_{t=0} = |\hat{c}^\dagger_n \hat{c}_n \rangle_{t=0} \) is the excited population of site-\( n \). Therefore, if the “BATH” is infinitely large (\( N \rightarrow \infty \)), the “SYSTEM” would reach and stay at the ground state after long relaxation, namely, \( p_{n,e}(t) = |J_0(2gt)|^2 \rightarrow 0 \) (here the limit \( N \rightarrow \infty \) is taken before \( t \rightarrow \infty \)).

**Scaling behavior of recurrences:** If the “BATH” is a finite one composed of \((N-1)\) TLSs, due to the finite-size effect, the above coherence functions \( \Phi_n^{(N)}(\tau) \) exhibit a recurrence behavior \(2\); within the time \( 0 \leq t \leq t_{\text{rec}} := N/2g \), \( \Phi_n^{(N)}(\tau) \) fits the above Bessel function (4b) quite closely and decays towards zero, but then it shows a “sudden bump” and starts to look random after \( t \geq t_{\text{rec}} \) [see the solid blue line in Fig. 1(b), \( t_{\text{rec}} \equiv N/2g \) is the recurrence time] [6–8, 10, 11].

Therefore, based on the local observation within a finite time smaller than \( t_{\text{rec}} \), we may conclude the open “SYSTEM” itself is relaxing towards a certain steady state, but indeed the full \( N \)-body state always keeps a pure state during the unitary evolution. With the increase of the size \( N \), the recurrence time \( t_{\text{rec}} \equiv N/2g \) becomes larger and larger, thus such a recurrence behavior does not show up in practice.

In Fig. 1(c), the scaling behavior of \( \text{Re} \left[ \Phi_0^{(N)}(\tau) \right] \) for different sizes \( N \) is shown. Besides the above recurrence appearing around \( t \approx t_{\text{rec}} \equiv N/2g \), it is worth noting that some well-organized recurrence patterns also appear in the region \( t \geq t_{\text{rec}} \). It can be seen similar recurrences also appear periodically around \( t \approx q t_{\text{rec}} \) for \( q = 2, 3, 4, \ldots \) [see the arrows in Fig. 1(c)]. Moreover, each recurrence seems bringing in stronger randomness to \( \Phi_0^{(N)}(\tau) \) than the previous one, which forms a hierarchy structure, thus we call them hierarchy recurrences.

We find that the appearance of such hierarchy recurrences can be explained by the following expansion of \( \Phi_0^{(N)}(\tau) \) [Eq.
Here we used the relation \( \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}(n-m)} = N\delta_{n-m, qN} \), with \( q \) as an arbitrary integer.

For example, site-0 \((n = 0)\) gives a simple Bessel function series [using \( J_{-n}(\tau) = (-1)^n J_n(\tau) \)]

\[
\Phi_0^{(n)}(\tau) = J_0(\tau) + (-i)^n[1 + (-1)^n]J_n(\tau) + (-i)^{2n}[1 + (-1)^{2n}]J_{2n}(\tau) + \ldots
\]

For a large \( N \), the Bessel function \( J_n(\tau) \approx 0 \) in the area \( 0 \leq \tau \leq N \), and starts to exhibit significant oscillations after \( \tau \approx N \) [see \( J_{100}(\tau) \) in Fig. 1(b)]. Therefore, in the above expansion of \( \Phi_0^{(n)}(\tau) \), each term \( J_{qn}(\tau) \) contributes a “sudden bump” around \( \tau \approx qN \), and this is just why the above recurrences appear around \( t \approx q\tau_{\text{rec}} \) (\( q = 1, 2, 3, \ldots \)).

**Population propagation:** The population dynamics of all the \( N \) TLSs is shown in Fig. 2(b), i.e., \( p_{n, e}(t) = \langle \hat{\sigma}_n^+ \hat{\sigma}_n^- \rangle(t) = |\Phi_n^{(n)}(2gt) \rangle^2 \), and a propagation pattern is clearly seen. Initially, the population distribution of the \( N \) TLSs forms a “cusp” around site-0 \([p_{0, e}(t) = 1, \text{ and } p_{n, e}(t) = 0 \text{ for } n \neq 0]\). Within the time \( t \leq \tau_{\text{rec}} \), the initial population “cusp” on site-0 propagates towards the two directions of the periodic chain, and the propagation “speed” is almost a constant [21, 22]. This constant speed also can be seen from the leading terms of \( p_{n, e}(t) = |\Phi_n^{(n)}(2gt) \rangle^2 \approx |J_n(2gt)|^2 + \ldots \) for \(-N/2 < n < N/2\), see Eq. (5); the leading Bessel function indicates the first “sudden bump” of site-\( n \) appears around \( t \approx |n|/2g \), which linearly depends on the distance \(|n|\) to site-0 [here site-(\( n \)) and site-(\( N-n \)) are the same one due to the periodic boundary condition].

The two-side propagations would meet each other at the periodic boundaries at \( n \sim \pm N/2 \), and then regathers back to site-0 again. Notice that this is just the moment that \( \Phi_0^{(n)}(2gt) \) exhibits its first recurrence \( t \approx \tau_{\text{rec}} \approx N/2g \), see the dashed vertical lines in Fig. 2. The propagation regathered back would be superposed with the original one, and that makes the system dynamics appear more random. Clearly, since such propagation and regathering happens again and again, the “SYSTEM” (site-0) experiences the above hierarchy recurrences periodically around \( t \approx q\tau_{\text{rec}} \).

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[3] When \( N \rightarrow \infty \), the function series \( \{ \Phi_0^{(n)}(\tau) \} \) converges pointwise to \( J_0(\tau) \) but not uniformly.
With the increase of the size $N$ between populations $N$, the maximum of $C_p$ under the constraints (1) might achieve [30]. With the help of Lagrangian multipliers, scaled time $2^g t/N$ for $N = 20, 100, 1000$. The blue dashed lines are the correlation maximum $C_{(\text{max})}^{(N)}$. (d) The relative error $\eta_{\text{rel}}$ between max $\{C_r(t)\}$ and the maximum $C_{(\text{max})}^{(N)}$ decreases with the site number $N$.

In this sense, the above correlation maximization effectively gives a pseudo-equilibrium state $\hat{\rho}_{\text{eq}} \equiv \bigotimes_n \hat{\rho}_n$, where $\hat{\rho}_n := \frac{1}{N} |\psi_n\rangle\langle \psi_n| (|\omega\rangle + (1 - \frac{1}{N}) |\rho\rangle)_n (|\rho\rangle)$, and the whole N-body state $\hat{\rho}(t)$ “looks” like approaching this pseudo-equilibrium state during the unitary evolution [6]. But we emphasize indeed $\hat{\rho}(t)$ and $\hat{\rho}_n(t)$ never have any steady states when $t \to \infty$, and $\hat{\rho}(t)$ always keeps a pure state.

The increasing rate of the above total correlation (7) also can be rewritten in the form of relative entropy [26, 31–33]

$$\partial_t C_r(t) = \partial_t D[\hat{\rho}(t) \mid \bigotimes_n \hat{\rho}_n(t)],$$

where $D[\rho||\varrho] = \text{tr}[\rho (\ln \rho - \ln \varrho)]$ is the relative entropy. Approximately, the reference state $\bigotimes_n \hat{\rho}_n(t)$ here can be replaced by the pseudo-equilibrium state $\hat{\rho}_{\text{eq}} \equiv \bigotimes_n \hat{\rho}_n$.

We emphasize that the pseudo-equilibrium state $\hat{\rho}_{\text{eq}}$ here is determined by the above correlation maximization, but irrelevant with the on-site energy $\omega$ and the interaction strength $g$, thus it is different from the canonical state like $\hat{\rho}_n \sim \exp[-\hat{H}/T]$. If the on-site energy $\omega \leq 0$, all the above results for the system dynamics still remains the same.

Moreover, when $\omega < 0$, the status of $|\rho\rangle_n$ and $|\omega\rangle_n$ are indeed reversed: initially, the open “SYSTEM” starts from the ground state while the TLSs in the “BATH” start from the excited state. Therefore, effectively the “BATH” has a negative temperature $T \to 0^-$ [34, 35]. Notice that all the above results of the total correlation entropy still applies in this situation.

Summary: In this paper, we study the local relaxation process of an open system contacted with a finite bath. We find that, due to the finite-size effect of the bath, the local relaxation of the open system exhibits hierarchy recurrences periodically, which makes the system dynamics appear more and more random. Essentially, that is because the energy diffuses out of the open system regathers back from the finite bath again and again. During the unitary evolution, the open system and the bath as a whole isolated system keeps a constant entropy, and the entropy of each single TLS increases and decreases from time to time, while the total correlation entropy approximately exhibits a monotonic increasing behavior, which is similar as the irreversible entropy increase in the standard thermodynamics [16–19]. We emphasize throughout the above discussions there is no average on time or any random configurations. The quantum XX model here could be realized in many physical systems, such as optical lattices [36], superconducting circuits [37, 38], and ion trap arrays [39, 40].

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