Supersymmetry on the lattice and the Leibniz rule

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Abstract

The major obstacle to a supersymmetric theory on the lattice is the failure of the Leibniz rule. We analyze this issue by using the Wess-Zumino model and a general Ginsparg-Wilson operator, which is local and free of species doublers. We point out that the Leibniz rule could be maintained on the lattice if the generic momentum \( k_\mu \) carried by any field variable satisfies \( |ak_\mu| < \delta \) in the limit \( a \to 0 \) for arbitrarily small but finite \( \delta \). This condition is expected to be satisfied generally if the theory is finite perturbatively, provided that discretization does not induce further symmetry breaking. We thus first render the continuum Wess-Zumino model finite by applying the higher derivative regularization which preserves supersymmetry. We then put this theory on the lattice, which preserves supersymmetry except for a breaking in interaction terms by the failure of the Leibniz rule. By this way, we define a lattice Wess-Zumino model which maintains the basic properties such as \( U(1) \times U(1)_R \) symmetry and holomorphicity. We show that this model reproduces continuum theory in the limit \( a \to 0 \) up to any finite order in perturbation theory; in this sense all the supersymmetry breaking terms induced by the failure of the Leibniz rule are irrelevant. We then suggest that this discretization may work to define a low energy effective theory in a non-perturbative way.

1 Introduction

There are basically two different motivations for defining a field theory on the lattice. The first is to regularize a divergent theory and simultaneously define the theory in a non-perturbative sense. The second is to define a discretized version of a theory, which is finite in continuum perturbation theory, so that one can apply the numerical and other techniques in a non-perturbative way. Though this second aspect is not commonly discussed in the context of lattice theory, we want to show that this second aspect may be essential in putting supersymmetric theories on the lattice.

There are several difficulties to define supersymmetry on the lattice. The most notable and difficult issue is the failure of the Leibniz rule. To be explicit, we have on the lattice

\[
\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x))
\]

If one applies a discretization to superstring theory, for example, it also corresponds to a discretization of a perturbatively finite theory.
\[ \frac{1}{a}(f(x + a) - f(x))g(x) + f(x)\frac{1}{a}(g(x + a) - g(x)) \]

\[ + a\frac{1}{a}(f(x + a) - f(x))\frac{1}{a}(g(x + a) - g(x)) \]

(1.1)

namely the “lattice version of the Leibniz rule” is given by

\[ (\nabla(fg))(x) = (\nabla f)(x)g(x) + f(x)(\nabla g)(x) + a(\nabla f)(x)(\nabla g)(x). \]

(1.2)

This shows that the breaking of supersymmetry by the lattice artifact is formally of order \( O(a) \), but actually the breaking is of order \( O(1) \) if the momentum carried by the field variables is of order \( O(1/a) \). To recover the conventional Leibniz rule, a necessary condition for the momentum variable is

\[ |ak_\mu| < \delta \]

(1.3)

for \( a \to 0 \) with arbitrarily small but finite \( \delta \). Here \( k_\mu \) is a generic momentum carried by any field variable in the Feynman diagrams so that the last term in the lattice Leibniz rule (1.2) is neglected to give

\[ (\nabla(fg))(x) = (\nabla f)(x)g(x) + f(x)(\nabla g)(x). \]

(1.4)

This requirement is expected to be satisfied if the theory in continuum is finite in a perturbative sense so that all the momentum variables in Feynman diagrams are finite and thus infinitesimally small measured by the lattice unit \( 1/a \) in the limit \( a \to 0 \), provided that the lattice discretization does not introduce further symmetry breaking terms.

If the above condition is satisfied, all the supersymmetry breaking terms for finite lattice spacing, which are induced by the failure of the Leibniz rule, are expected to be irrelevant in the sense that those supersymmetry breaking terms vanish in the limit \( a \to 0 \). In the context of supersymmetric Yang-Mills theory on the lattice, an argument to the effect that all the supersymmetry breaking terms are irrelevant was given in the past\(^2\), though the \( N = 1 \) supersymmetric Yang-Mills theory is not finite and thus the basic reasoning is completely different. In the context of the Wess-Zumino model\(^4\), we would like to show that a sensible lattice discretization, which is based on the presently available technique, may be to first render the continuum Wess-Zumino model finite by applying the higher derivative regularization. This higher derivative regularization is known to preserve supersymmetry in a perturbative sense\(^5\). We then apply the lattice discretization to this regularized continuum theory.

In this paper, we present a detailed analysis of the above procedure for the Wess-Zumino model in the framework of perturbation theory, and then suggest that this scheme may work in a non-perturbative sense also. We utilize the Ginsparg-Wilson fermion

\[ (\nabla(fg))(x) = (f(x + a)g(x) - f(x - a)g(x - a))/(2a) \]

\[ = (\nabla f)(x)g(x) + f(x)(\nabla g)(x) + a(\nabla f)(x)((g(x + a) - g(x))/a) + a(\nabla f)(x - a))/a)((\nabla g)(x) \]

and still the limit \( a \to 0 \) is not smoothly defined in general.

\(^2\)If one uses the symmetric difference,

\[ (\nabla(fg))(x) = (f(x + a)g(x + a) - f(x - a)g(x - a))/a) \]
operators which are local and free of species doublers\cite{6-10}. In the course of this analysis, we clarify some of the subtleties appearing in the Ginsparg-Wilson operators when utilized in the present context.

2 Wess-Zumino model on the lattice

The Wess-Zumino model is the simplest supersymmetric model in 4-dimensional space time, and the non-renormalization theorem was first discovered in this model: If renormalized at vanishing momenta, all the potential terms including mass terms do not receive any (even finite) renormalization, except for a uniform wave function renormalization, up to all orders in perturbation theory\cite{11}.

As for the previous studies of the Wess-Zumino model on the lattice, see, for example, \cite{12-14}.

2.1 Lattice Lagrangian

We define the Wess-Zumino model on the lattice by\cite{15}

$$
\mathcal{L} = \frac{1}{2} \chi^T C \Gamma_5 H \chi + \frac{1}{2} m \chi^T C \chi + g \chi^T C (P_+ \phi P_+ + P_- \phi^\dagger P_-) \chi - \phi^\dagger D^\dagger D \phi + F^\dagger \frac{1}{\Gamma_5} F + m [F \phi + (F \phi)^\dagger] + g [F \phi^2 + (F \phi^2)^\dagger]
$$

$$
= \frac{1}{2} \chi^T C \frac{H}{\Gamma_5} a \chi + \frac{1}{2} m \chi^T C \chi + g \chi^T C (P_+ \phi P_+ + P_- \phi^\dagger P_-) \chi - \phi^\dagger \frac{H^2}{a^2} \phi + F^\dagger \frac{1}{\Gamma_5} F + m [F \phi + (F \phi)^\dagger] + g [F \phi^2 + (F \phi^2)^\dagger].
$$

(2.1)

Here the hermitian operator

$$
H = a \gamma_5 D = H^\dagger
$$

(2.2)

satisfies the general Ginsparg-Wilson relation\cite{9}

$$
\gamma_5 H + H \gamma_5 = 2 H^{2k+2}
$$

(2.3)

with a non-negative integer $k$, which implies

$$
\gamma_5 H^2 = (\gamma_5 H + H \gamma_5) H - H (\gamma_5 H + H \gamma_5) + H^2 \gamma_5 = H^2 \gamma_5,
$$

(2.4)

and

$$
\Gamma_5 = \gamma_5 - H^{2k+1}
$$

(2.5)

which satisfies $H \Gamma_5 + \Gamma_5 H = 0$. An explicit form of $H$ is given in Appendix. When we have $H^2$ and $\Gamma_5$ in the bosonic terms, we adopt the convention to discard the unit Dirac matrix in $H^2$ (see Appendix). Note that $\Gamma_5^2 = 1 - H^{4k+2}$, and $H^2 = 1$ just on top of the would-be species doublers.
The projection operators are defined by

\[ P_\pm = \frac{1}{2} (1 \pm \gamma_5), \quad \hat{P}_\pm = \frac{1}{2} (1 \pm \hat{\gamma}_5) \]  

(2.6)

with \( \hat{\gamma}_5 = \gamma_5 - \frac{2H^{2k+1}}{a} \) which satisfies \( \hat{\gamma}_5^2 = 1 \).

A salient feature of our Lagrangian is that it is invariant under the continuum chiral transformation, except for the mass term, if one performs simultaneously a suitable phase rotation of the fields \( \phi \) and \( \phi^\dagger \). We here note the relation which follows from the defining relation of \( H \) (see also Appendix)

\[ \Gamma_5 H/a \gamma_5 H = \frac{1}{a} (\gamma_5 H - H^2 + 2k) = \frac{1}{2a} [\gamma_5, H] \propto \gamma^\mu \sin \frac{ap_\mu}{a}. \]  

(2.7)

We then have

\[ \{ \gamma_5, \Gamma_5 H \} = \frac{1}{2} \{ \gamma_5, [\gamma_5, H] \} = 0 \]  

(2.8)

which suggests that the fermion kinetic operator satisfies

\[ \{ \gamma_5, \Gamma_5 H \} = \{ \gamma_5, \Gamma_5 H \} = 0 \]  

(2.9)

by using \( [\gamma_5, \Gamma_5] = 0 \). The factor \( \Gamma_5 \) in the fermion kinetic term \( (1/\Gamma_5)H \) vanishes at the momentum corresponding to the would-be species doublets \([15, 17]\), but this non-locality is compensated for in the present supersymmetric theory by the corresponding singularity in the term \( F^\dagger F \). Note that variables \( \{ \chi, \phi, F \} \) are treated as components of a single superfield in supersymmetric theory. In fact one can confirm that the partition function of the free part of the Lagrangian gives unity and thus the factor \( \Gamma_5 \) is cancelled among the component fields:

\[
\int D\chi D\phi D\phi^\dagger D F D F^\dagger \exp \left\{ \int \left[ \frac{1}{2} \chi^T C \frac{1}{\Gamma_5} H a + \frac{1}{2} m \chi^T C \chi - \phi^\dagger \frac{H^2}{a^2} \phi + F^\dagger \frac{1}{\Gamma_5} F + m [F \phi + (F \phi)^\dagger] \right] \right\}
\]

\[
= \int D\chi D\phi D\phi D F^\dagger D (F')^\dagger \exp \left\{ \int \left[ \frac{1}{2} \chi^T C \frac{1}{\Gamma_5} H a + \frac{1}{2} m \chi^T C \chi - \phi^\dagger \frac{H^2}{a^2} \phi - (F')^\dagger \frac{1}{\Gamma_5} F' \right] \right\}
\]

\[
= \sqrt{\frac{\det \left[ \frac{H^2}{a^2} + (m \Gamma_5)^2 \right]}{\det \left[ \frac{H^2}{a^2} + (m \Gamma_5)^2 + \frac{1}{\Gamma_5^2} \right]}} = \sqrt{\frac{\det [\frac{1}{\Gamma_5} + m]}{\det \left[ \frac{H^2}{a^2} + (m \Gamma_5)^2 \right]}}
\]

(2.10)

\[ = \frac{\{ \det \left[ \frac{H^2}{a^2} + (m \Gamma_5)^2 \right] \}^{1/4}}{\det \left[ \frac{H^2}{a^2} + (m \Gamma_5)^2 \right] \det \left[ \frac{1}{\Gamma_5} \right]} = 1 \]
if one recalls that both of $\Gamma^2_5$ and $(\frac{H}{a})^2$ are proportional to a $4 \times 4$ unit matrix and that this unit matrix is neglected in the bosonic sector appearing in the denominator.

We shall also confirm that perturbation theory is well defined without any singularity, though the fermion propagator vanishes at the momenta corresponding to the would-be species doublers. In the non-perturbative formulation, the factor $1/\Gamma_5$ may be compensated for by rescaling the variables $F$ and $F^\dagger$, as will be shown later.

Our convention of the charge conjugation matrix is

\begin{align}
C\gamma^\mu C^{-1} &= - (\gamma^\mu)^T, \\
C\gamma_5 C^{-1} &= \gamma_5^T, \\
C^\dagger C &= 1, \quad C^T = -C
\end{align}

and the Ginsparg-Wilson operator satisfies

\begin{align}
C\gamma_5 \Gamma_5 C^{-1} &= (\gamma_5 \Gamma_5)^T, \\
CDC^{-1} &= D^T.
\end{align}

### 2.2 Supersymmetry

If one defines the real components by

\[ \phi \rightarrow \frac{1}{\sqrt{2}}(A + iB), \quad F \rightarrow \frac{1}{\sqrt{2}}(F - iG) \]

the above Lagrangian is written as

\[
\mathcal{L} = \frac{1}{2} \chi^T C \frac{1}{\Gamma_5} \frac{1}{a} H \chi - \frac{1}{2a^2} [AH^2 A + BH^2 B] + \frac{1}{2} [F \frac{1}{\Gamma_5} F + G \frac{1}{\Gamma_5} G] + \frac{1}{2} m \chi^T C \chi + m [FA + GB] + \frac{1}{\sqrt{2}} g \chi^T C (A + i\gamma_5 B) \chi + \frac{1}{\sqrt{2}} g [F(A^2 - B^2) + 2G(AB)].
\]

The free part of the action formed from this Lagrangian is confirmed to be invariant under the “lattice supersymmetry” transformation\footnote{We assume that our kinetic operator $\Gamma_5 H$ satisfies the proper charge conjugation symmetry $CT_5 HC^{-1} = (\Gamma_5 H)^T$, which includes an operation corresponding to partial integration.}

\[
\begin{align*}
\delta \chi &= - \frac{1}{a} \Gamma_5 H (A - i\gamma_5 B) \epsilon - (F - i\gamma_5 G) \epsilon, \\
\delta A &= \epsilon^T C \chi = \chi^T C \epsilon, \\
\delta B &= -i \epsilon^T C \gamma_5 \chi = -i \chi^T C \gamma_5 \epsilon \\
\delta F &= \epsilon^T C \Gamma_5 \frac{1}{a} H \chi \sim \chi^T C \Gamma_5 \frac{1}{a} H \epsilon \\
\delta G &= i \epsilon^T C \Gamma_5 \frac{1}{a} H \gamma_5 \chi \sim -i \chi^T C \Gamma_5 \frac{1}{a} H \gamma_5 \epsilon
\end{align*}
\]
with a constant Majorana-type Grassmann parameter $\epsilon$. Note that the order of the operators is important in these expressions. The second expressions of $\delta F$ and $\delta G$ need to be treated carefully when these variations are multiplied with other field variables. If one recalls the correspondence to continuum theory in the naive limit $a \to 0$

$$\frac{1}{a} H \leftrightarrow \gamma_5 \partial$$
$$\Gamma_5 \leftrightarrow \gamma_5,$$  \hspace{1cm} (2.18)

the above transformation defines a lattice generalization of the continuum supersymmetry transformation\[4\][5].

For example, the variation of the kinetic terms under the above transformation is given by

$$\delta \int \mathcal{L}_{\text{kin}} = \int \{ \chi^T C \frac{1}{\Gamma_5 a} \frac{1}{\Gamma_5 a} H( - \Gamma_5 \frac{1}{\Gamma_5 a} H( A - i \gamma_5 B) \epsilon - ( F - i \gamma_5 G) \epsilon ] - \frac{1}{a^2} A \Gamma_5 [ \epsilon^T C \chi ] - \frac{1}{a^2} B H^2 [ - i \epsilon^T C \gamma_5 \chi ] + F \frac{1}{\Gamma_5 a} [ \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \chi ] + G \frac{1}{\Gamma_5 a} [ i \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \gamma_5 \chi ] \}$$

$$= \int \{ \chi^T C \frac{1}{a^2} H^2 ( A - i \gamma_5 B) \epsilon - \chi^T C \frac{1}{\Gamma_5 a} \frac{1}{\Gamma_5 a} H( F - i \gamma_5 G) \epsilon ] - \frac{1}{a^2} \chi^T C H^2 ( A - i \gamma_5 B) \epsilon + \chi^T C \frac{1}{\Gamma_5 a} \frac{1}{\Gamma_5 a} H( F - i \gamma_5 G) \epsilon ] = 0$$  \hspace{1cm} (2.19)

which is consistent with (2.10). Here we used (2.8).

The lattice supersymmetry variation of the interaction terms is given by

$$\delta \int \mathcal{L}_{\text{int}} = \int \{ \frac{2}{\sqrt{2}} g \chi^T C ( A + i \gamma_5 B ) [ - \Gamma_5 \frac{1}{\Gamma_5 a} H( A - i \gamma_5 B) \epsilon - ( F - i \gamma_5 G) \epsilon ] + \frac{2}{\sqrt{2}} g \{ F A [ \epsilon^T C \chi ] - \frac{1}{a^2} A H^2 [ - i \epsilon^T C \gamma_5 \chi ] + 2 G A [ - i \epsilon^T C \gamma_5 \chi ] + 2 B [ \epsilon^T C \chi ] \} + \frac{1}{\sqrt{2}} g \{ ( A^2 - B^2 ) [ \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \chi ] + 2 ( A B ) [ i \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \gamma_5 \chi ] \} \}$$

$$= \int \{ \frac{2}{\sqrt{2}} g \chi^T C ( A + i \gamma_5 B ) [ - \Gamma_5 \frac{1}{\Gamma_5 a} H( A - i \gamma_5 B) \epsilon ] + \frac{1}{\sqrt{2}} g \{ ( A^2 - B^2 ) [ \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \chi ] + 2 ( A B ) [ i \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \gamma_5 \chi ] \} \}$$

$$= \int \{ \frac{2}{\sqrt{2}} g \chi^T C ( A + i \gamma_5 B ) [ - \Gamma_5 \frac{1}{\Gamma_5 a} H( A - i \gamma_5 B) \epsilon ] \}$$

\[5\]For example, in the presence of a scalar field $A(x)$, we have $\int A \epsilon^T C \Gamma_5 \frac{1}{\Gamma_5 a} H \chi = \int \chi^T C \Gamma_5 \frac{1}{\Gamma_5 a} H( A \epsilon )$. 


\[ + \frac{1}{\sqrt{2}} g \{ [\chi^T \Gamma_5 \frac{1}{a} H (A^2 - B^2) \epsilon] - 2i [\chi^T \Gamma_5 \frac{1}{a} H \gamma_5 (AB) \epsilon] \} \]
\[ = - \int \left\{ \frac{2}{\sqrt{2}} g \chi^T C (A \Gamma_5 \frac{1}{a} H A - B \Gamma_5 \frac{1}{a} H B) \epsilon \right. \\
\left. + \frac{2i}{\sqrt{2}} g \chi^T C [A \Gamma_5 \frac{1}{a} H B + B \Gamma_5 \frac{1}{a} H A] \gamma_5 \epsilon \right. \\
\left. + \frac{1}{\sqrt{2}} g \{ [\chi^T \Gamma_5 \frac{1}{a} H (A^2 - B^2) \epsilon] - 2i [\chi^T \Gamma_5 \frac{1}{a} H (AB) \epsilon] \} \} \right. \] (2.20)

Here we used the relation \( \gamma_5 \Gamma_5 H = -\Gamma_5 H \gamma_5 \) (2.8). If the operator \( \Gamma_5 H/a \) (2.7) satisfies the Leibniz rule, the above variation of the interaction terms vanishes. We thus encounter the notorious issue related to the Leibniz rule, which is basically the lattice artifact.

The propagators for perturbative calculations are given by

\[ \langle \phi \phi \rangle = \frac{a^2}{H^2 + (am\Gamma_5)^2} \]
\[ \langle FF \rangle = (-) \frac{H^2 \Gamma_5^2}{H^2 + (am\Gamma_5)^2} \]
\[ \langle F \phi \rangle = \langle F^\dagger \phi \rangle = (-) \frac{a^2 m\Gamma_5^2}{H^2 + (am\Gamma_5)^2} \]
\[ \langle \chi(y)\chi^T(x)C \rangle = (-) \frac{a}{H + am\Gamma_5}\Gamma_5 = (-) \gamma_5 \Gamma_5 \frac{a}{H + am\Gamma_5}\gamma_5 \] (2.21)

and other propagators vanish. When we have \( H^2 \) and \( \Gamma_5^2 \) in the bosonic propagators, we adopt the convention to discard the unit Dirac matrix in \( H^2 \). Note that \( \Gamma_5^2 = 1 - H^{4k+2} \).

If one uses the identities

\[ P_+ \phi(x) \hat{P}_+ = P_+ \phi(x) P_+ \gamma_5 \Gamma_5, \]
\[ P_- \phi(x) \hat{P}_- = P_- \phi(x) P_- \gamma_5 \Gamma_5, \] (2.22)

the Feynman rules in the present scheme are essentially identical to those in the previous calculation\[18\]. The one-loop level non-renormalization theorem when renormalized at vanishing momenta is thus satisfied, though the kinetic terms receive non-uniform finite renormalization in addition to uniform logarithmic renormalization\[18\].

The holomorphic properties in a naïve sense are preserved in our Lagrangian\[6\]. As for the \( U(1) \times U_R(1) \) charges, where \( U_R(1) \) stands for the R-symmetry, we first write the potential part of the Lagrangian as

\[ \mathcal{L}_{pot} = \frac{1}{2} m (P_+ \chi)^T C P_+ \chi + m F \phi + \chi^T C P_+ g \phi P_+ \chi + g F \phi^2 \]
\[ +\frac{1}{2} m^\dagger (P_- \chi)^T C P_- \chi + (m F \phi)^\dagger + \chi^T C P_- (g \phi)^\dagger P_- \chi + (g F \phi^2)^\dagger \] (2.23)

\[ ^6\text{The decisive factor to ensure the non-renormalization theorem is supersymmetry, while } U(1) \times U_R(1) \text{ symmetry and holomorphicity provide additional constraints.} \]
and we assign:

\[
\begin{align*}
\phi &= (1, 1), \\
F &= (1, -1), \\
P_+ \chi &= \xi = (1, 0), \\
m &= (-2, 0), \\
g &= (-3, -1)
\end{align*}
\]  

by regarding \(m\) and \(g\) as complex parameters. Here \(\xi\) is a two component spinor in the representation where \(\gamma_5\) is diagonal. For \(m = g = 0\), our Lagrangian preserves these charges if one recalls

\[
\frac{1}{\Gamma_5} H = P_+ \frac{1}{\Gamma_5} HP_+ + P_+ \frac{1}{\Gamma_5} HP_-
\]  

and \(P_- \chi = \xi^*\) in the representation where \(\gamma_5\) is diagonal.

### 3 Higher derivative regularization on the lattice

In the perturbative treatment of the above lattice Lagrangian (2.1), the higher order diagrams generally break supersymmetry because all the momentum regions contribute to the loop diagrams; it is not easy to preserve supersymmetry in this strict sense (i.e., for all the momentum regions) on the lattice because of the failure of the Leibniz rule.

A way to resolve this difficulty may be to apply a higher derivative regularization on the lattice. By this way, one can transfer all the divergences to the infrared divergences measured by the lattice unit \(1/a\). In the infrared region, the momenta in loop diagrams are constrained to the momentum region in continuum theory, for which the lattice artifact such as the failure of the Leibniz rule could be negligible in the limit \(a \rightarrow 0\).

The higher derivative regularization in the present lattice Lagrangian is implemented by

\[
\mathcal{L} = \frac{1}{2} \chi^T C \left( \frac{1}{\Gamma_5} \frac{\gamma_5 D (D^\dagger D + M^2)}{M^2} \right) \chi + \frac{1}{2} m \chi^T C \left( \frac{H (H^2 + (aM)^2)}{a^2 (aM)^2} \right) \chi
\]

\[
- \phi^\dagger D^\dagger D \left( \frac{D^\dagger D + M^2}{M^2} \right) \phi + F^\dagger \frac{1}{\Gamma_5} \left( \frac{D^\dagger D + M^2}{M^2} \right) F
\]

\[
+ m \left[ F \left( \frac{D^\dagger D + M^2}{M^2} \right) \phi + (F \phi^2)^\dagger \right]
\]

\[
+ g \chi^T C (P_+ \phi P_+ + P_- \phi^3 P_-) \chi + g [F \phi^2 + (F \phi^2)^\dagger]
\]

\[
= \frac{1}{2} \chi^T C \left( \frac{1}{\Gamma_5} \frac{H (H^2 + (aM)^2)}{a^2 (aM)^2} \right) \chi + \frac{1}{2} m \chi^T C \left( \frac{H^2 + (aM)^2}{aM} \right) \chi
\]

\[
- \phi^\dagger \frac{H^2 (H^2 + (aM)^2)}{a^2 (aM)^2} \phi + F^\dagger \frac{1}{\Gamma_5} \left( \frac{H^2 + (aM)^2}{aM} \right) F
\]

\[
+ m \left[ F \left( \frac{H^2 + (aM)^2}{aM} \right) \phi + (F \phi^2)^\dagger \right]
\]

\[
+ g \chi^T C (P_+ \phi P_+ + P_- \phi^3 P_-) \chi + g [F \phi^2 + (F \phi^2)^\dagger].
\]
The $U(1) \times U_R(1)$ symmetry and holomorphicity are preserved in this regularization.

The propagators for perturbative calculations are given by

\[
\langle \phi \phi^\dagger \rangle = \frac{a^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]

\[
\langle FF^\dagger \rangle = (-) \frac{H^2 \Gamma_5}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]

\[
\langle F \phi \rangle = \langle F^\dagger \phi^\dagger \rangle = (-) \frac{a^2 m \Gamma_5^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]

\[
\langle \chi(y) \chi^T(x) C \rangle = (-) \gamma_5 \Gamma_5 \frac{a}{H + am\Gamma_5} \frac{(aM)^2}{H^2 + (aM)^2} \gamma_5
\]

\[
= \frac{a \Gamma_5 H - a^2 m (\Gamma_5)^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2} \tag{3.2}
\]

and other propagators vanish. When we have $H^2$ and $\Gamma_5^2$ in the bosonic sector, we adopt the convention to discard the unit Dirac matrix in $H^2$. Note that $\Gamma_5 H + H \Gamma_5 = 0$ and $\Gamma_5^2 = 1 - H^4 k^2$. Here $M$ is a new mass scale which may be chosen to be

\[
1 \gg (aM)^2 \gg (am)^2 \tag{3.3}
\]

One can confirm that the free part of this Lagrangian with higher derivative regularization (3.1) is still invariant under the lattice supersymmetry transformation (2.17) by noting $[\Gamma_5, H^2] = 0$, while the interaction terms are not modified by the higher derivative regularization.

### 3.1 One-loop tadpole and self-energy corrections

It has been shown previously that the superpotential is not renormalized in the one-loop level even for a finite $a$ when renormalized at vanishing momenta\[18\]. This conclusion still holds in the present model with higher derivative regularization. One can also confirm that the cancellation of tadpole diagrams is still maintained even for a finite $a$ in the one-loop level in the present model\[13\].

It is instructive to analyze the tadpole diagrams in some detail. The scalar tadpole contribution to the fields $\phi$ and $\phi^\dagger$ is given by

\[
2g[\langle F \phi \rangle + \langle F^\dagger \phi^\dagger \rangle] = -2g \int \phi \frac{a^2 m \Gamma_5^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2} + \phi^\dagger \frac{a^2 m \Gamma_5^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]

\[
= -2 m g a^2 (\phi + \phi^\dagger) \int \frac{\Gamma_5^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]

\[
= -2 m g a^2 (\phi + \phi^\dagger) \int_\pi (-2\pi)^4 H^2(k) \frac{\Gamma_5^2(k)}{H^2(k) + (am\Gamma_5(k))^2} M^2 \tag{3.4}
\]

Intuitively, this one-loop cancellation arises from the fact that the interaction terms, when one of $\phi$ or $\phi^\dagger$ is set to a constant, are reduced to the effective mass terms which are invariant under lattice supersymmetry.
where we chose the basic Brillouin zone at
\[-\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a}\] (3.5)
and rescaled the integration variable as \(a k_\mu \to k_\mu\). If one considers the limit \(a \to 0\) in the above integral, one obtains a logarithmic divergence from the region \(k_\mu \sim 0\). But it is important to recognize that the entire region of the basic Brillouin zone gives a finite contribution in the above integral even at the limit \(a \to 0\). This is the peculiar feature of the one-loop tadpole diagrams in the present minimal higher derivative regularization\[8\], and all other diagrams receive non-vanishing contributions only from the infrared region \(k_\mu \sim 0\) in the limit \(a \to 0\). In any case, it is confirmed that the above scalar tadpole contribution is precisely cancelled by a fermion tadpole contribution even for a finite \(a\).

Since the renormalization of kinetic terms was not uniform \(\text{without}\) the higher derivative regularization\[18\], we here analyze the kinetic terms and associated quadratic and logarithmic divergences in the one-loop level in more detail. The one-loop correction to the “kinetic term” \(F F^\dagger\) is given by

\[
g^2 \frac{2!}{2!} [F(\phi)^2 + (F(\phi)^2)^\dagger]^2
\]
\[
\to g^2 \frac{2!}{2!} [4F\langle\phi\phi^\dagger\rangle\langle\phi\phi^\dagger\rangle F^\dagger]
\]
\[
= 2g^2 a^4 F\int \frac{1}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}
\]
\[
\times \frac{1}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2} F^\dagger
\]
which is logarithmically divergent.

The integral for \(F F^\dagger\) is written in more detail as

\[
2g^2 \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} \frac{1}{H^2(k + ap) + (am\Gamma_5)^2(k + ap)} \frac{(aM)^2}{H^2(k + ap) + (aM)^2}
\]
\[
\times \frac{1}{H^2(k) + (am\Gamma_5)^2(k)} \frac{(aM)^2}{H^2(k) + (aM)^2}
\]
(3.7)
where we rescaled the integration variable as \(a k_\mu \to k_\mu\) by choosing the basic Brillouin zone as in (3.5). In this integral, if one chooses the integration domain \(\text{outside}\) the infrared region
\[
\delta > k_\mu > -\delta \quad \text{for all} \quad \mu
\]
(3.8)
for arbitrarily small but finite \(\delta\), the integral vanishes for \(a \to 0\) since we have no infrared divergences\[9\]. In this analysis, the absence of species doubling in the Ginsparg-Wilson

---

\[8\] If one considers the higher derivative regularization with the factor \(((H^2 + (aM)^2))/((aM)^2)^2\) instead of \((H^2 + (aM)^2))/((aM)^2)\) in (3.1), even the one-loop tadpole diagrams receive non-vanishing contributions only from the infrared region in the limit \(a \to 0\). This stronger regularization may be necessary for a numerical simulation.

\[9\] This and following analyses are extended to \(\delta = (aM)^\epsilon\) with sufficiently small positive \(\epsilon\).
operator is essential: Namely, $H^2 \sim 1$ for the momentum domain of the would-be species doublers.

The non-vanishing contribution to the above integral in the limit $a \to 0$ is thus given by

$$2g^2 \left[ \int_{-\delta}^{\delta} \frac{d^4k}{(2\pi)^4} \frac{1}{H^2(k+ap) + (am \Gamma_5)^2(k+ap) + (aM)^2} \right] \frac{1}{H^2(k) + (am \Gamma_5)^2(k) H^2(k) + (aM)^2}$$

$$= 2g^2 \left[ \int_{-\delta/a}^{\delta/a} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + (am)^2 k^2 + (aM)^2} \right] \frac{M^2}{k^2 + m^2 k^2 + M^2}$$

$$\to 2g^2 \left[ \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 k^2 + M^2} \right] \frac{M^2}{k^2 + m^2 k^2 + M^2}$$

$$\times \frac{1}{k^2 + m^2 k^2 + M^2}$$

in the limit $a \to 0$. Here we used $H^2(k) \simeq k^2$ for $|k| < \delta$ by recalling the rescaling $ak^\mu \to k^\mu$. See Appendix. This last expression is identical to the continuum result in the higher derivative regularization.

The crucial aspect of this analysis is that the momentum variables are constrained to the infrared region in the limit $a \to 0$. Namely, the typical momentum variable is constrained to be

$$|ak^\mu| < \delta$$

for arbitrarily small but finite $\delta$, which is a necessary condition for the validity of the Leibniz rule on the lattice.

The correction to the kinetic term of the scalar particle $\phi$ by the fermion loop diagram is given by

$$g^2 a^2 Tr \phi^4 \frac{H \Gamma_5}{H^2 + (ma \Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2} \frac{\Gamma_5 H}{H^2 + (ma \Gamma_5)^2} \frac{(aM)^2}{H^2 + (aM)^2}$$

where the symbol $Tr$ includes the integral over the loop momentum as well as the trace over Dirac matrices. The quadratic divergence and the logarithmic divergence associated with the mass term in this expression, when evaluated at vanishing external momentum, are cancelled by the scalar loop diagram (3.14) given below even for a finite $a$. 
By analyzing the infrared structure in the limit \( a \to 0 \), one can confirm that the expression (3.12) is reduced to the continuum expression in the higher derivative regularization

\[
g^2 \text{tr} \int \frac{d^4 k}{(2\pi)^4} \frac{k^+ \phi^+(p)}{(k+p)^2 + m^2 (k+p)^2 + M^2} \frac{M^2}{k^2 + m^2 k^2 + M^2}. \tag{3.13}
\]

The one-loop self-energy of the scalar particle \( \phi \) produced by the scalar particle loop is given by

\[
g^2 \frac{2!}{2!} [F\phi^2 + (F\phi^2)\Gamma^+] [F\phi^2 + (F\phi^2)\Gamma^-] \\
\rightarrow g^2 \frac{2!}{2!} [8\phi\phi^\dagger (FF\dagger) \langle \phi \phi^\dagger \rangle] \\
= g^2 \frac{2!}{2!} \int [8\phi\phi^\dagger \frac{-H^2 \Gamma_5^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \frac{a^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2}] \\
= -4g^2 a^2 \phi \phi^\dagger \int \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \frac{1}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \\
+ \frac{(mg)^2 a^4}{2!} \int [8\phi\phi^\dagger \frac{\Gamma_5^4}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \frac{a^2}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2}]. \tag{3.14}
\]

The first term is quadratically divergent, and the remaining term is logarithmically divergent. These terms, when evaluated at vanishing external momentum, precisely cancel the corresponding fermion contributions (3.12) even for a finite \( a \).

The fermion self-energy correction is given by

\[
4g^2 a^3 \bar{\psi} P_+ \int \frac{\Gamma_5 H}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \frac{1}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} P_+ \psi \\
+ 4g^2 a^3 \bar{\psi} P_- \int \frac{\Gamma_5 H}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} \frac{1}{H^2 + (am\Gamma_5)^2} \frac{(aM)^2}{H^2 + (am\Gamma_5)^2} P_- \psi \tag{3.15}
\]

where we used the relation

\[
P_\pm \Gamma_5 H = \Gamma_5 H P_\mp. \tag{3.16}
\]

By analyzing the infrared structure in the limit \( a \to 0 \), we again have

\[
4g^2 \bar{\psi} P_+ \int \frac{d^4 k}{(2\pi)^4} \frac{k^+ \phi^+(p)}{(k+p)^2 + m^2 (k+p)^2 + M^2} \frac{M^2}{k^2 + m^2 k^2 + M^2} \frac{1}{M^2} \frac{M^2}{k^2 + m^2 k^2 + M^2} P_+ \psi \\
+ 4g^2 \bar{\psi} P_- \int \frac{d^4 k}{(2\pi)^4} \frac{k^+ \phi^+(p)}{(k+p)^2 + m^2 (k+p)^2 + M^2} \frac{M^2}{k^2 + m^2 k^2 + M^2} \frac{1}{M^2} \frac{M^2}{k^2 + m^2 k^2 + M^2} P_- \psi. \tag{3.17}
\]

We have to examine if a universal wave function renormalization is sufficient to remove the divergences from these 3 contributions (3.10), (3.13) and (3.17) in the limit \( a \to 0 \).
One can in fact show that a uniform subtraction of logarithmic infinity (for $M \to \text{large}$) renders all these expressions finite when renormalized at vanishing momentum. For the fermion contribution to the scalar kinetic term (3.13), we first rewrite it as

$$4g^2 \int \frac{d^4k}{(2\pi)^4} \phi^\dagger(p)\phi(p) 3! \int d\alpha d\beta d\gamma d\delta (1 - \alpha - \beta - \gamma - \delta) \times$$

$$\frac{-p^2(1 - \alpha - \beta)M^4}{[p^2(\alpha + \beta)(1 - \alpha - \beta) + \alpha(k^2 + m^2) + \beta(k^2 + M^2) + \gamma(k^2 + m^2) + \delta(k^2 + M^2)]^4}$$

and similarly for the fermion self-energy correction (3.17). We then renormalize all the kinetic terms at $p = 0$ by using the relation

$$3! \int d\alpha d\beta d\gamma d\delta (1 - \alpha - \beta - \gamma - \delta) \times$$

$$\frac{1 - \alpha - \beta}{[\alpha(k^2 + m^2) + \beta(k^2 + M^2) + \gamma(k^2 + m^2) + \delta(k^2 + M^2)]^4}$$

$$= 3! \int d\alpha d\beta d\gamma d\delta (1 - \alpha - \beta - \gamma - \delta) \times$$

$$\frac{1/2}{[\alpha(k^2 + m^2) + \beta(k^2 + M^2) + \gamma(k^2 + m^2) + \delta(k^2 + M^2)]^4}. \quad (3.19)$$

In the one-loop level, we can thus maintain supersymmetry including renormalization factors$^{10}$ in the limit $a \to 0$. In other words, all the supersymmetry breaking terms for finite lattice spacing should vanish for $a \to 0$.

### 3.2 Two and higher-loop diagrams

#### 3.2.1 Tadpole diagrams

We start with the analysis of tadpole diagrams. One can confirm that the tadpole diagrams for the auxiliary field $F$ in the two-loop level precisely cancel even for a finite lattice spacing $a$, and similarly for the field $F^\dagger$.

For the scalar field $\phi$, we have 4 tadpole diagrams in the two-loop level. These diagrams do not quite cancel for a finite $a$ due to the failure of the Leibniz rule for general momenta. But for non-vanishing contributions in the limit $a \to 0$, one can reduce the Feynman amplitudes to those of the continuum theory with higher derivative regularization, which are then shown to cancel precisely. We can thus maintain the vanishing tadpole diagrams in the two-loop level.

For example, we have a two-loop tadpole contribution arising from a fermion loop diagram with a scalar exchange correction, which contains a quadratic divergence in a naive sense,

$$4\phi g^3 \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} tr \left\{ \frac{a\Gamma_5 H(p)}{H^2(p) + (am\Gamma_5(p))^2} H^2(p) + (aM)^2 \right\} M^2$$

$^{10}$Here we are repeating the known analysis in continuum theory$^{[3]}$. 

13
\[
\times \frac{a^2 m(\Gamma_5(p))^2}{H^2(p) + (am\Gamma_5(p))^2} \frac{M^2}{H^2(k + p)} \frac{H^2(k + p) + (am\Gamma_5(k + p))^2}{a\Gamma_5H(k + p) + (aM)^2} \frac{M^2}{H^2(k) + (am\Gamma_5(k))^2}
\]

where we used the rescaled variables \( ap_{\mu} \rightarrow p_{\mu} \) and \( ak_{\mu} \rightarrow k_{\mu} \). By analyzing the limit \( a \rightarrow 0 \), one can confirm that only the infrared regions

\[
|p_{\mu}| < \delta, \quad |k_{\mu}| < \delta \quad \text{for all} \quad \mu
\]

with arbitrarily small but finite \( \delta \) give a non-vanishing contribution

\[
4\phi^3 \int_{-\delta}^{\delta} \frac{d^4 p}{(2\pi)^4} \int_{-\delta}^{\delta} \frac{d^4 k}{(2\pi)^4} tr \left\{ \frac{a\Gamma_5H(p)}{H^2(p) + (am\Gamma_5(p))^2} \frac{M^2}{H^2(k + p) + (am\Gamma_5(k + p))^2} \frac{M^2}{H^2(k) + (am\Gamma_5(k))^2} \right\}
\]

By rescaling the momentum variables to original ones and considering the limit \( a \rightarrow 0 \), one recovers the continuum result in the higher derivative regularization. Other tadpole amplitudes in the two-loop level are analyzed similarly.

### 3.2.2 General diagrams in two and higher-loop level

Similarly, one can confirm that the non-vanishing parts of all the Feynman amplitudes in the two-loop level in the limit \( a \rightarrow 0 \) are reduced to those of the continuum theory with higher derivative regularization. Namely, all the loop momenta in the Feynman amplitudes are constrained to be in the infrared region, where the Leibniz rule is satisfied.

To the extent that the non-renormalization and other good properties are maintained in the continuum theory with higher derivative regularization\(^3\), our lattice regularization thus reproduces all the good properties of the supersymmetric Wess-Zumino model in the limit \( a \rightarrow 0 \).

One can extend this analysis up to any finite order in perturbation theory, since the power counting in this regularized theory is effectively superconvergent; the higher order diagrams are thus more ultra-violet convergent and thus less sensitive to the lattice cut-off for \( a \rightarrow 0 \). All the supersymmetry breaking terms induced by the failure of the Leibniz rule are thus shown to vanish in the limit \( a \rightarrow 0 \).
3.3 Non-perturbative treatment

As for the non-perturbative treatment of our model, one may first perform the path integral over the Majorana fermion which produces the Pfaffian

\[
\int \mathcal{D}\phi \mathcal{D}\phi^\dagger \mathcal{D}F \mathcal{D}F^\dagger \times \sqrt{\det \left[ \frac{1}{\Gamma_5} \frac{H (H^2 + (aM)^2)}{a} + m \frac{(H^2 + (aM)^2)}{(aM)^2} + 2g (P_+ \phi P_+ + P_- \phi^\dagger P_-) \right]}
\]

\[
\times \exp \left\{ \int \left[ -\phi^\dagger \frac{H^2 (H^2 + (aM)^2)}{a^2} \phi + F^\dagger \frac{1}{\Gamma_5^2} \frac{H^2 + (aM)^2}{(aM)^2} F + m \left[ \frac{(H^2 + (aM)^2)}{(aM)^2} \phi + \left( \frac{H^2 + (aM)^2}{(aM)^2} \phi \right)^\dagger \right] + g \left[ F \phi^2 + (F \phi^2)^\dagger \right] \right\}
\]

\[
= \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \mathcal{D}F \mathcal{D}F^\dagger \times \sqrt{\det [- \frac{H (H^2 + (aM)^2)}{a} + \frac{m (H^2 + (aM)^2)}{(aM)^2} \Gamma_5 + 2g (P_+ \phi P_+ + P_- \phi^\dagger P_-) \Gamma_5]}
\]

\[
\times \exp \left\{ \int \left[ -\phi^\dagger \frac{H^2 (H^2 + (aM)^2)}{a^2} \phi + F^\dagger \frac{H^2 + (aM)^2}{(aM)^2} F + m \left[ \frac{H^2 + (aM)^2}{(aM)^2} \phi + \left( \frac{H^2 + (aM)^2}{(aM)^2} \phi \right)^\dagger \right] + g \left[ F \phi^2 + (F \phi^2)^\dagger \right] \right\}
\]

where in the second expression we rescaled the field variables as

\[
F \rightarrow \Gamma F, \quad F^\dagger \rightarrow F^\dagger \Gamma
\]

with \( \Gamma \equiv \sqrt{\Gamma_5^2} = \sqrt{1 - H^4 k^2 + 2} \). In the last expression in (3.23) we have no singularity associated with \( 1/\Gamma_5 \). This path integral (or after performing the path integral over \( F \) and \( F^\dagger \) ) may be evaluated non-perturbatively. Since the Wess-Zumino model is not asymptotically free, the non-perturbative result in the continuum limit \( a \rightarrow 0 \) may be defined with a finite \( M \), which provides a finite mass scale to specify the renormalized parameters; one can thus prevent the coupling constant from increasing indefinitely. In this sense our possible non-perturbative formulation, which is inferred from perturbative considerations, is consistent.

If the above path integral (3.23), when evaluated non-perturbatively, gives a well-defined result in the limit \( a \rightarrow 0 \), it is expected that the path integral defines a theory which incorporates the quantum effects up to the energy scale \( M \). The path integral may then be effective in defining a non-perturbative low energy effective theory for

\[
|p_\mu| \sim m \ll M
\]

where \( p_\mu \) is the typical external momentum carried by field variables.

\[11\] If one should be able to evaluate the continuum theory with higher derivative regularization in a non-perturbative way, one would obtain the same result.
4 Discussion

We have discussed a way to ensure the Leibniz rule for the supersymmetric Wess-Zumino model on the lattice. The basic observation is that the lattice Leibniz rule is reduced to that of continuum theory if the generic momentum $k_\mu$ carried by any field variable is constrained in the infrared region $|ak_\mu| < \delta$ for arbitrarily small but finite $\delta$ in the limit $a \to 0$. A way to ensure this momentum condition is to apply the higher derivative regularization to the Wess-Zumino model so that the theory becomes finite up to any finite order in perturbation theory. On the basis of the analysis of Feynman amplitudes, we have shown that this is in fact realized in our lattice formulation which incorporates a lattice version of higher derivative regularization. All the supersymmetry breaking terms induced by the failure of the Leibniz rule thus become irrelevant in the sense that they all vanish in the limit $a \to 0$. We suggested that this mechanism may work even in a non-perturbative sense.

In this analysis, it is crucial that the Ginsparg-Wilson operator is free of species doublers so that the lattice operator $H$ is of order $O(1)$ in the momentum regions of the would-be species doublers\(^{12}\). Besides, the Ginsparg-Wilson operator maintains some of the basic symmetries such as chiral symmetry, $U(1) \times U_R(1)$ symmetry and holomorphicity.

In conclusion, we have presented a possible way to maintain the Leibniz rule on the lattice. It is yet to be seen if this analysis is extended to supersymmetric Yang-Mills theories on the lattice\(^{20-21}\), though we naively expect that supersymmetric theories which are perturbatively finite (such as $N = 4$ theory) may be put on the lattice consistently. One may also want to find a more drastic way to overcome the difficulty associated with the Leibniz rule, which might lead to a completely new understanding of lattice regularization.

A Ginsparg-Wilson operators

An explicit form of the general Ginsparg-Wilson operator $H$\(^{9}\), which satisfies the algebra (2.3), is given in momentum space by

$$H(ap_\mu) = \gamma_5 \left( \frac{1}{2} \right)^{\frac{k+1}{2k+1}} \left( \frac{1}{\sqrt{F(k)}} \right)^{\frac{k+1}{2k+1}} \left\{ \left( \sqrt{H_W^2 + M_k} \right)^{\frac{k+1}{2k+1}} - \left( \sqrt{H_W^2 - M_k} \right)^{\frac{k}{2k+1}} i \frac{s}{a} \right\}$$

$$= \gamma_5 \left( \frac{1}{2} \right)^{\frac{k+1}{2k+1}} \left( \frac{1}{\sqrt{F(k)}} \right)^{\frac{k+1}{2k+1}} \left\{ \left( \sqrt{F(k) + \tilde{M}_k} \right)^{\frac{k+1}{2k+1}} - \left( \sqrt{F(k) - \tilde{M}_k} \right)^{\frac{k}{2k+1}} i \frac{s}{a} \right\}$$

(A.1)

where $k$ is a non-negative integer and

$$F(k) = (s^2)^{2k+1} + \tilde{M}_k^2,$$

\(^{12}\)It is possible to implement the present mechanism for the model based on the Wilson fermion\(^{12}\), if one defines the higher derivative regulator suitably by including the Wilson term. The non-perturbative analysis would, however, become more involved since the basic symmetries such as chiral symmetry and holomorphicity are spoiled.
\[ \tilde{M}_k = \left( \sum_{\mu}(1 - c_\mu) \right)^{2k+1} - m_0^{2k+1} \]  
(A.2)

with

\[
\begin{align*}
    s_\mu &= \sin ap_\mu \\
    c_\mu &= \cos ap_\mu \\
    s' &= \gamma^\mu \sin ap_\mu.
\end{align*}
\]  
(A.3)

This operator is known to be local and free of species doublers\cite{22}, and this operator for \( k = 0 \) is reduced to Neuberger’s overlap operator\cite{8}. Our Euclidean Dirac matrices are hermitian, \((\gamma^\mu)^\dagger = \gamma^\mu\), and the inner product is defined to be \( s^2 \geq 0 \). Note that \( H^2 \) (and consequently \( \Gamma_5^2 = 1 - H^{4k+2} \)) is independent of Dirac matrices. The parameter \( m_0 \) is constrained by \( 0 < m_0 < 2 \) to avoid species doublers, and \( 2m_0^{2k+1} = 1 \) gives a proper normalization of \( H \), namely, for an infinitesimal \( p_\mu \), i.e., for \( \left| ap_\mu \right| \ll 1 \),

\[ H \approx -\gamma_5 ai \ p(1 + O(ap)^2) + \gamma_5(\gamma_5 ai \ p)^{2k+2} \]  
(A.4)

to be consistent with \( H = \gamma_5 aD \); the last term in the right-hand side is the leading term of chiral symmetry breaking terms.

We thus have

\[ H^2 = 1 \]  
(A.5)

just on top of the would-be species doublers, for example, \( ap_\mu = (\pi, 0, 0, 0) \), and

\[ H^2 \approx (ap_\mu)^2 \]  
(A.6)

for \( \left| ap_\mu \right| \ll 1 \) independently of the parameter \( k \).

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