ON SHORT PRODUCTS OF PRIMES IN ARITHMETIC PROGRESSIONS

IGOR E. SHPARLINSKI

Abstract. We give several families of reasonably small integers $k, \ell \geq 1$ and real positive $\alpha, \beta \leq 1$, such that the products $p_1 \cdots p_k s$, where $p_1, \ldots, p_k \leq m^\alpha$ are primes and $s \leq m^\beta$ is a product of at most $\ell$ primes, represent all reduced residue classes modulo $m$. This is a relaxed version of the still open question of P. Erdős, A. M. Odlyzko and A. Sárközy (1987), that corresponds to $k = \ell = 1$ (that is, to products of two primes). In particular, we improve recent results of A. Walker (2016).

1. Introduction

Since our knowledge of distribution of primes in short arithmetic progressions is rather limited, it is certainly interesting to consider various modifications and relaxations of this question. In particular, as one of such relaxations, Erdős, Odlyzko and Sárközy [3] have introduced a question about the distribution of products of two small primes in residue classes. Namely, given an integer $m \geq 1$, Erdős, Odlyzko and Sárközy [3] ask whether all reduced classes $a$ modulo $m$ can be represented as the product

\begin{equation}
\label{eq:1.1}
p_1 p_2 \equiv a \pmod{m}
\end{equation}

of two primes $p_1, p_2 \leq m$ and prove a series of conditional results towards this under various assumptions about the zero-free regions for Dirichlet $L$-functions. However, it appears that even the Extended Riemann Hypothesis is not powerful enough to answer the original question.

Some more accessible relaxations of this problem have been introduced and studied by Friedlander, Kurlberg and Shparlinski [6], where, in particular, the congruence (1.1) is considered on average over $a$ and

\begin{itemize}
  \item Date: May 13, 2018.
  \item 2010 Mathematics Subject Classification. Primary 11N25; Secondary 11B25, 11L07, 11N36.
  \item Key words and phrases. residue classes, primes, sieve method, exponential sums.
\end{itemize}
Furthermore, one can find in [6] some results on several ternary modifications of (1.1) such as the congruences
\[(1.2) \quad p_1(p_2 + p_3) \equiv a \pmod{m} \quad \text{and} \quad p_1p_2(p_3 + h) \equiv a \pmod{m},\]
where \(p_1, p_2, p_3 \leq x\) are primes and \(h\) is a fixed integer. Recently the results of [6] about the congruences (1.2) have been improved by Garaev [7, 8]. Furthermore, the congruence
\[p_1p_2 + p_2p_3 + p_1p_3 \equiv a \pmod{m}\]
with primes \(p_1, p_2, p_3 \leq x\) has been studied in [1, 4], with some applications to the size of largest prime divisor of the bilinear quadratic form \(p_1p_2 + p_2p_3 + p_1p_3\).

Yet another relaxation of the original question of [3] have been introduced in [14], where one of the components of the product on the left hand side is prime and the other one is almost prime (that is, a product of a small number of primes.

More precisely, or an integer \(\ell \geq 1\) we use \(\mathcal{P}_\ell\) to denote the set of integers that are products of at most \(\ell\) primes. Thus \(\mathcal{P} = \mathcal{P}_1\) is the set of primes.

Now, for some real positive \(x, y \leq m\) and integer \(\ell \geq 1\), we consider the congruence
\[(1.3) \quad ps \equiv a \pmod{m}, \quad p \in \mathcal{P} \cap [1, x], \quad s \in \mathcal{P}_\ell \cap [1, y],\]
that is, with variables \(p \leq x\) which is prime and \(1 \leq s \leq y\) which is a product of at most \(\ell\) primes.

**Definition 1.** We say that a triple \((\ell; \alpha, \beta) \in \mathbb{N} \times \mathbb{R}^2\) is admissible, if for any fixed \(\varepsilon > 0\) and
\[x = m^{\alpha + \varepsilon} \quad \text{and} \quad y = m^{\beta + \varepsilon}\]
the congruence (1.3) has a solution for any reduced residue class a modulo \(m\), provided that \(m\) is large enough, and we denote by \(\mathcal{A}_3\) the set of admissible triples.

Thus the question of [3] is equivalent, apart from the presence of \(\varepsilon\), to proving that \((1; 1, 1)\) is an admissible triple, which seems to be out of reach nowadays. However, some families of admissible triples, have been given in [14, Theorem 3]. In particular, it is observed in [14, Section 4] that
\[(1.4) \quad (17; 0.997, 0.997) \in \mathcal{A}_3.\]

Walker [16] has recently considered a different variant of this question and asked about the solvability of the congruence
\[(1.5) \quad p_1 \ldots p_k \equiv a \pmod{m}, \quad p_1, \ldots, p_k \in \mathcal{P} \cap [1, x].\]
that is, where \( p_1, \ldots, p_k \leq x \) are primes.

**Definition 2.** We say that a pair \((k; \alpha) \in \mathbb{N} \times \mathbb{R}\) is admissible, if for any fixed \( \varepsilon > 0 \) and

\[ x = m^{\alpha + \varepsilon} \]

the congruence (1.5) has a solution for any reduced residue class a modulo \( m \), provided that \( m \) is large enough, and we denote by \( \mathcal{A}_2 \) the set of admissible pairs.

Thus in these settings, the question of [3] is equivalent to proving that \((2; 1)\) is admissible (again, apart from the presence of \( \varepsilon \)).

We also introduce a similar definition with respect to prime moduli

**Definition 3.** We say that a pair \((k; \alpha) \in \mathbb{N} \times \mathbb{R}\) is admissible for primes, if for any fixed \( \varepsilon > 0 \) and

\[ x = m^{\alpha + \varepsilon} \]

the congruence (1.5) has a solution for any reduced residue class a modulo \( m \), provided that \( m \) is prime and large enough, and we denote by \( \mathcal{A}_2^p \) the set of admissible for primes pairs.

Walker [16, Theorem 2] has shown that

(1.6) \((6; 15/16) \in \mathcal{A}_2^p,\)

(note that \( 15/16 = 0.9375 \ldots \)), as well as that for any \( \eta > 0 \) there is an integer \( k_\eta \), for which

(1.7) \((k_\eta; 3/4 + \eta) \in \mathcal{A}_2^p,\)

However, we note that the claim made in [16] that (1.6) is an improvement over (1.4) does not seem to be justified. Even ignoring the difference between arbitrary and prime moduli \( m \), which distinguishes (1.4) and (1.6), we note that while

(1.8) \((\ell; \alpha, \alpha) \in \mathcal{A}_3 \implies (\ell + 1; \alpha) \in \mathcal{A}_2,\)

the opposite implication is not clear and most likely to be false.

The main goal of this work is to show that there is an alternative and more efficient approach to producing pairs \((k; \alpha) \in \mathcal{A}_2\) with reasonably small \( k \) and \( \alpha \). In particular, we obtain a series of improvements of (1.6) and (1.7), see Section 3 for the numerical values.

In fact, given some real positive \( x, y \leq m \) and an integer \( \ell \geq 1 \), we consider a more general congruence, which includes (1.3) and (1.5) as special cases:

(1.9) \( p_1 \ldots p_k s \equiv a \pmod{m}, \quad p_1, \ldots, p_k \in \mathcal{P} \cap [1, x], \ s \in \mathcal{P}_\ell \cap [1, y], \)
that is, with variables \( p_1, \ldots, p_k \leq x \) which are and \( 1 \leq s \leq y \) which is a product of at most \( \ell \) primes.

**Definition 4.** We say that a quadruple \((k, \ell; \alpha, \beta) \in \mathbb{N}^2 \times \mathbb{R}^2\) is admissible, if for any fixed \( \varepsilon > 0 \) and

\[
x = m^{\alpha + \varepsilon} \quad \text{and} \quad y = m^{\beta + \varepsilon}
\]

the congruence (1.3) has a solution for any reduced residue class a modulo \( m \), provided that \( m \) is large enough, and we denote by \( \mathfrak{A}_4 \) the set of admissible quadruples.

Then aforementioned improvements of (1.6) follow as special cases from the obvious analogue of (1.8) that

\[
(1.10) \quad (k, \ell; \alpha, \alpha) \in \mathfrak{A}_4 \implies (k + \ell; \alpha) \in \mathfrak{A}_2.
\]

Our method is based on bounds of some exponential sums with reciprocals of primes. These bounds are then coupled with the sieve method in the form given by Greaves [11, Section 5].

In particular, we use this opportunity to improve slightly the result of [14] about admissible triples via a more careful choice of parameters and then we introduce a new argument which allows us to produce a large family of admissible quadruples. In turn, using (1.10) we significantly improve the results of Walker [16]. For example, we replace 1/4 with 1/2 in (1.7), and extend it to composite moduli, see (3.1) below.

Finally, we remark that here all elements of the product are less than the modulus, that is, we always have \( \alpha, \beta \leq 1 \). For products of large primes one can achieve more, and for example by a result of Ramaré and Walker [13] every reduced class modulo \( m \) can be represented by a product of three primes \( p_1, p_2, p_3 \leq m^4 \) (provided that \( m \) is large enough).

2. **Main result and its implications**

Following the results of Greaves [10, Equation (1.4)], see also [11, pp. 174–175], we also define the constants

\[
\delta_2 = 0.044560, \quad \delta_3 = 0.074267, \quad \delta_4 = 0.103974,
\]

and, after rounding up,

\[
\delta_\ell = 0.124821, \quad \ell \geq 5.
\]

We also define

\[
\vartheta_\ell = \ell - \delta_\ell, \quad \ell = 2, 3, \ldots.
\]

First we prove the following general statement.
Theorem 2.1. For any fixed real $\alpha \geq 1/2$ and $\beta \geq 0$ and any integer $\ell \geq 2$:

(i) if $\alpha/16 + \beta > 1$ and $\alpha/3 + \beta > 5/4$ and
$$\max \left\{ \frac{\beta}{\alpha/16 + \beta - 1}, \frac{\beta}{\alpha/3 + \beta - 5/4} \right\} \leq \vartheta_{\ell}$$
then we have $(\ell; \alpha, \beta) \in \mathcal{A}_3$;

(ii) if $\alpha + \beta > 3/2$ and
$$\frac{\beta}{\alpha + \beta - 3/2} \leq \vartheta_{\ell}$$
then we have $(2, \ell; \alpha, \beta) \in \mathcal{A}_4$;

(iii) if $\alpha/2 + \beta > 1$ and
$$\frac{\beta}{\alpha/2 + \beta - 1} \leq \vartheta_{\ell}$$
then we have $(3, \ell; \alpha, \beta) \in \mathcal{A}_4$;

(iv) if $\beta \geq 1/2$ and
$$\frac{\beta}{\beta - 1/2} \leq \vartheta_{\ell}$$
then we have $(4, \ell; \alpha, \beta) \in \mathcal{A}_4$.

We now consider the special case of $\alpha = \beta$.

Corollary 2.2. For any integer $\ell \geq 3$:

(i) For $\alpha = 16\vartheta_{\ell}/(17\vartheta_{\ell} - 16)$ we have $(\ell; \alpha, \alpha) \in \mathcal{A}_3$;

(ii) for $\alpha = 3\vartheta_{\ell}/(4\vartheta_{\ell} - 2)$ we have $(2, \ell; \alpha, \alpha) \in \mathcal{A}_4$;

(iii) for $\alpha = 2\vartheta_{\ell}/(3\vartheta_{\ell} - 2)$ we have $(3, \ell; \alpha, \alpha) \in \mathcal{A}_4$;

(iv) for $\alpha = \vartheta_{\ell}/(2\vartheta_{\ell} - 2)$ we have $(4, \ell; \alpha, \alpha) \in \mathcal{A}_4$.

3. Numerical Examples

First, we note that Corollary 2.2 (i), taken with $\ell = 17$ improves (with respect to $\alpha$) the result from [14, Section 4], which we have presented in (1.4). However this reduction in the value of $\alpha$ is rather minor and is “invisible” at the level of numerical precision with which we present our results.

On the other hand, for $k = 2, 3, 4$ our improvements are more significant. For example, for $\ell = 3$, we derive

$$(2, 3; 0.905, 0.905), (3, 3; 0.864, 0.864), (4, 3; 0.760, 0.760). \in \mathcal{A}_4,$$

In particular, recalling (1.10), we obtain

$$(5; 0.905), (6; 0.864) \in \mathcal{A}_2$$
each of which improves (1.6) and we also have
\[(7; 0.760, 0.760) \in A_2\]
which maybe compared with (1.7).

Furthermore, with \(\ell = 4\) we obtain
\[(4, 4; 0.673, 0.673) \in A_4,\]
and hence
\[(8; 0.673) \in A_2\]
which improves (1.7).

We also see from Corollary 2.2 (iv) and (1.10) that for any \(\eta > 0\)
there is some \(k_\eta\) such that
\[(3.1) \quad (k_\eta; 1/2 + \eta) \in A_2,\]
which is yet another improvement of (1.7).

We remark that here we have used the value of \(\delta_3\) and \(\delta_4\) given
by (2.1), that has been announced by Greaves [10, Equation (1.4)], see
also [11, pp. 174–175], however full details of calculation have never
been supplied (although there seems to be no reason to doubt the
validity of these values). However, even with slightly larger values, as
those reported in [9] our approach still leads to improvements of (1.6)
and (1.7).

4. Notation

Throughout the paper, \(p\) and \(q\) always denote prime numbers, while
\(k, \ell, m\) and \(n\) (in both the upper and lower cases) denote positive
integer numbers.

We use \(\mathbb{Z}_m\) to denote the residue ring modulo \(m\).

As we have mentioned, for an integer \(\ell \geq 1\), we use \(\mathcal{P}_\ell\) to denote the
set of integers that are products of at most \(\ell\) primes.

As usual, we use \(\pi(x)\) to denote the number of primes \(p \leq x\) and
\(P(n)\) to denote the largest prime divisor of \(n \geq 2\) (we also set \(P(1) = 0\)).

We fix a sufficiently large integer \(m\) and for any integer \(n\) with
\(\gcd(m, n) = 1\) we denote by \(\overline{n}\) the multiplicative inverse of \(n\) mod-
ulo \(m\), that is, the unique integer \(u\) defined by the conditions

\[nu \equiv 1 \pmod{m} \quad \text{and} \quad 1 \leq u < m.\]

We remark that once we write \(\overline{n}\) we automatically assume that
\(\gcd(m, n) = 1\).

The implied constants in the symbols ‘\(O\)’ and ‘\(\ll\)’ may occasionally,
where obvious, depend on the small positive parameter \(\varepsilon\). We recall
that the notations $U = O(V)$ and $U \ll V$ are all equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$.

Finally, the notation $z \sim Z$ means that $z$ must satisfy the inequality $Z < z \leq 2Z$.

5. Exponential sums with reciprocals of primes

For an integer $m \geq 2$, we define the exponential function $e_m = \exp(2\pi k/n)$ consider the exponential sums

$$S_k(a; x) = \sum_{p_1, \ldots, p_k \leq x} \sum_{\gcd(p_1, \ldots, p_k, m) = 1} e_m(a\overline{p}_1 \ldots \overline{p}_k), \quad k = 1, 2, \ldots,$$

where $x \geq 2$ is a real number and $a$ is an integer.

Note that in [14] only the sums $S_1(a; x)$ have been employed together with the following bound of Fouvry and Shparlinski [4, Theorem 3.1],

$$|S_m(a; x)| \leq (x^{15/16} + m^{1/4} x^{2/3}) m^{o(1)}$$

uniformly for $x \leq m^{4/3}$ and integers $a$ with $\gcd(a, m) = 1$.

We note that the bound (5.1) extends a similar bound of Garaev [7, Theorem 1.1] from prime to composite moduli. For convenience, we have dropped the condition $x \geq m^{3/4}$ from [4, Theorem 3.1] as for smaller values of $x$ the bound is trivial.

Here, since we study a modified question, we also make use of the sums $S_k(a; x)$ with $k = 2, 3, 4$. First we need the following simple statement:

**Lemma 5.1.** For any real $x \geq 2$ the number of solutions to the congruence

$$\overline{p}_1 \overline{p}_2 \equiv \overline{q}_1 \overline{q}_2 \pmod{m}, \quad p_1, p_2, q_1, q_2 \leq x,$

is at most $x^{2+o(1)} (x^2/m + 1)$.

**Proof.** Clearly we can rewrite this congruence as

$$p_1 p_2 \equiv q_1 q_2 \pmod{m}.$$

When a pair $(q_1, q_2)$ is chosen (trivially, in at most $x^2$ ways), this puts the product $p_1 p_2 \leq x^2$ in an arithmetic progression of the fork $a + km$ with $k = 0, \ldots, K$, where $K = \left\lfloor x^2/m \right\rfloor$ (and $a \neq 0$). Hence, each out of the $K + 1 \leq x^2/m + 1$ elements of this progression gives rise to at most 2 pairs $(p_1, p_2)$. Therefore, the number of such solutions is $2x^2 (x^2/m + 1)$. This concludes the proof.

Clearly we ignored some possible logarithmic savings in the proof of Lemma 5.1 which do not affect our main results.
Our bounds rely on the following classical bound on bilinear sums, which dates back to Vinogradov [15, Chapter 6, Problem 14.a] and has reappeared in many forms since then.

**Lemma 5.2.** For arbitrary sets $U, V \subseteq \mathbb{Z}_m$, complex numbers $\varphi_u$ and $\psi_v$ with
$$
\sum_{u \in U} |\varphi_u|^2 \leq \Phi \quad \text{and} \quad \sum_{v \in V} |\psi_v|^2 \leq \Psi,
$$
and an integer $a$ with $\gcd(a, m) = 1$, we have
$$
\left| \sum_{u \in U} \sum_{v \in V} \varphi_u \psi_v e_m(auv) \right| \leq \sqrt{\Phi \Psi m}.
$$

**Lemma 5.3.** For any real $x \geq m^{1/2}$, uniformly over integers $a$ with $\gcd(a, m) = f$, we have

$$
S_2(a; x) \ll x \left( \frac{fx}{m} + 1 \right) (m/f)^{1/2},
$$

$$
S_3(a; x) \ll x^{5/2} \left( \frac{fx}{m} + 1 \right)^{1/2},
$$

$$
S_4(a; x) \ll x^4 (m/f)^{-1/2}.
$$

**Proof.** The bound on $S_2(a; x)$ is instant from Lemma 5.2, if one uses the trivial bound
$$
\Phi = \Psi \leq \# \{ \overline{p}_1 \equiv \overline{p}_2 \pmod{m/f}, \ p_1, p_2 \leq x \} \leq x (fx/m + 1)
$$
(note that hereafter $\overline{p}_1$ and $\overline{p}_2$ are computed modulo $m$ rather than modulo $m/f$ but this does not affect the argument).

To estimate $S_3(a; x)$, we group $p_1$ and $p_2$ together, and again use Lemma 5.2 with
$$
\Phi = \# \{ \overline{p}_1 \overline{p}_2 \equiv \overline{q}_1 \overline{q}_2 \pmod{m/f}, \ p_1, p_2, q_1, q_2 \leq x \}
$$
$$
\ll x^2 \left( \frac{fx^2}{m} + 1 \right) \ll fm^{-1} x^4
$$
by Lemma 5.1 (where we have also used that $x \geq m^{1/2}$), and also, as before, with
$$
\Psi \leq x (fx/m + 1).
$$

Finally for $S_4(a; x)$, we group $p_1$ and $p_2$ as well as $p_3$ and $p_4$ together, and again Lemma 5.2 with
$$
\Phi = \Psi \ll fm^{-1} x^4
$$
which concludes the proof. \qed
Note that the assumption $x \geq m^{1/2}$ of Lemma 5.3 is used only for the purpose of typographical simplicity of the bounds; one can obtain more general statements which apply to any $x$.

Let
\begin{equation}
T_k(a; x, y) = \# \{(p_1, \ldots, p_k, v) : p_1, \ldots, p_k \leq x, \ v \leq y,
\ a\overline{p}_1 \ldots \overline{p}_k \equiv v \pmod{m}\}\).
\end{equation}

We recall our convention that in the definition of $T_k(a; x, y)$ we automatically assume that $\gcd(p_j, m) = 1, j = 1, \ldots, k$.

We also use
\begin{equation}
\Delta_k(a; x, y) = T_k(a; x, y) - \pi(x)^k y/m
\end{equation}
to denote the deviation between $T_k(a; x, y)$ and its expected value.

**Lemma 5.4.** For any real $x$ and $y$ and with $m \geq x \geq m^{1/2}, m \geq y \geq 1$ and integer $a$ with $\gcd(a, m) = 1$, we have
\begin{align*}
\Delta_1(a; x, y) &\leq (x^{15/16} + m^{1/4} x^{2/3} m^{o(1)}) m^{o(1)}, \\
\Delta_2(a; x, y) &\leq x m^{1/2+o(1)}, \\
\Delta_3(a; x, y) &\leq x^{5/2+o(1)}, \\
\Delta_4(a; x, y) &\leq x^{4+o(1)} m^{-1/2}.
\end{align*}

**Proof.** The bound on $\Delta_1(a; x, y)$ is given by [14, Lemma 2] (the fact that in [14] the prime $p$ is from a dyadic interval $[x/2, x]$ is inconsequential).

Using the same standard technique as in the proof of [14, Lemma 2], in particular, the Erdős–Turán inequality, see [2, 12], we easily derive the other bounds from Lemma 5.3.

We remark, that this method gives $(\pi(x) - \omega(m))^k y/m$ for the main term, where $\omega(m)$ is the number of distinct prime divisors of $m$. Since we trivially have $\omega(m) \ll \log m$ this difference gets absorbed in the error term. \qed

### 6. Sieving

Here we collect some results of Greaves [10, 11] which underly our approach.

Let $\mathcal{A} = (a_1, \ldots, a_N)$ be a sequence of $N$ integers in the interval $[1, Y]$ for some real $Y > 1$. For an integer $d \geq 1$ we define $\mathcal{A}(d)$ as a subsequence of $\mathcal{A}$ consisting of elements $a_n$ with $d \mid a_n$.

We say that $\mathcal{A}$ has a level of distribution $D$ if there is a multiplicative function $\rho(d)$ and some constant $\kappa > 0$ such that for
\[
|\mathcal{A}(d) - \rho(d) N| \leq \kappa \mathcal{A}(d) / d.
\]
we have
\[ \sum_{d \leq D} r(d) \ll N^{1-\kappa}. \]

We remark the condition on the sum of error terms can be relaxed a little bit and generally instead of a power saving and logarithmic saving is sufficient.

Given a level of distribution, we also define the degree
\[ g = \frac{\log Y}{\log D}. \]

We also recall the definition of the constants \( \delta_\ell \) from Section 2.

Then in the above notation by [11, Proposition 1, Chapter 5] we have:

**Lemma 6.1.** If for some integer \( \ell \geq 2 \) we have
\[ g < \ell - \delta_\ell \]
then for some element \( a_n \) of \( \mathcal{A} \) we have \( a_n \in \mathcal{P}_\ell \).

### 7. Proof of Theorem 2.1

We fix \( a \) with \( \gcd(a, m) = 1 \) and for an integer \( k \geq 1 \), consider the sequence \( \mathcal{A}_k \) consisting of the smallest nonnegative residues
\[ v \equiv a\overline{p}_1 \ldots \overline{p}_k \pmod{m} \]
for \( p_1, \ldots, p_k \leq x \) and such that these residues satisfy \( v \leq y \). In particular \( \#\mathcal{A}_k = T_k(a; x, y) \) as defined by (5.2).

As usual, for an integer \( d \geq 1 \) we denote by \( \mathcal{A}_k(d) \) the number of \( v \in \mathcal{A}_k \) with \( d \mid v \). Clearly \( \#\mathcal{A}_k(d) \) is number of solutions to the congruence
\[ a\overline{p}_1 \ldots \overline{p}_k \equiv du \pmod{m}, \quad p_1, \ldots, p_k \leq x, \ u \leq y/d. \]

Thus \( \#\mathcal{A}_k(d) = 0 \) if \( \gcd(d, m) > 1 \). Otherwise, that is, for \( \gcd(d, m) = 1 \), we have
\[ \left| \#\mathcal{A}_k(d) - \frac{\pi(x)ky}{dm} \right| \leq \Delta_k(a; x, y), \]
where \( \Delta_k(a; x, y) \) is defined by (5.3).

We now fix some sufficiently small \( \varepsilon > 0 \).
Using Lemma 5.4 we see that the levels of distribution $D_k$ of $A_k$, $k = 1, 2, 3, 4$, satisfy
\[
D_1 \geq \min\{x^{1/16}ym^{-1-\varepsilon}, x^{1/3}ym^{-5/4-\varepsilon}\}
\]
\[
D_2 \geq xym^{-3/2-\varepsilon},
\]
\[
D_3 \geq x^{1/2}ym^{-1-\varepsilon},
\]
\[
D_4 \geq ym^{-1/2-\varepsilon},
\]
provided that $m$ is large enough.

Since all elements of the sequences $A_k$ are in the interval $[1, y]$, their degree satisfies
\[
g_k \leq \frac{\log y}{\log D_k}.
\]
Hence, for $x = m^{\alpha+\varepsilon}$ and $y = m^{\beta+\varepsilon}$ with $1 \geq \alpha \geq 1/2$ and $1 \geq \beta \geq 0$ we have
\[
g_1 \leq \max\left\{\frac{\beta + \varepsilon}{\alpha/16 + \beta - 1}, \frac{\beta + \varepsilon}{\alpha/3 + \beta - 5/4}\right\},
\]
\[
g_2 \leq \frac{\beta + \varepsilon}{\alpha + \beta - 3/2},
\]
\[
g_3 \leq \frac{\beta + \varepsilon}{\alpha/2 + \beta - 1},
\]
\[
g_4 \leq \frac{\beta + \varepsilon}{\beta - 1/2},
\]
provided that the denominators are positive. Recalling Lemma 6.1, we conclude the proof.

8. Comments

We remark that Theorem 2.1 resembles results about the distribution of elements of $\mathcal{P}_k$ in arithmetics progressions, see, for example, [5, Theorem 25.8]. However, these results seem to be completely independent and do not imply each other. For example, the elements of $\mathcal{P}_2$ produced by [5, Theorem 25.8] cannot be ruled out to be prime, and they are also relatively large compared to the modulus).

Acknowledgement

The author is grateful to John Friedlander for enlightening discussions on sieves and comments on an earlier version of this paper.

Part of this work was also done when the authors was visiting the Max Planck Institute for Mathematics, Bonn, and Fields Institute,
Toronto, whose generous support and hospitality are gratefully acknowledged.

This work was also partially supported by by the Australian Research Council Grant DP140100118.

References

[1] R. C. Baker, ‘Kloosterman sums with prime variable’, Acta Arith., 156 (2012), 351–372. (p. 2)
[2] M. Drmota and R. Tichy, Sequences, discrepancies and applications, Springer-Verlag, Berlin, 1997. (p. 9)
[3] P. Erdős, A. M. Odlyzko and A. Sárközy, ‘On the residues of products of prime numbers’, Period. Math. Hung., 18 (1987), 229–239. (pp. 1, 2, and 3)
[4] É. Fouvry and I. E. Shparlinski, ‘On a ternary quadratic form over primes’, Acta Arith., 150 (2011), 285–314. (pp. 2 and 7)
[5] J. B. Friedlander and H. Iwaniec, Opera de Cribro, Amer. Math. Soc., Providence, RI, 2010. (p. 11)
[6] J. B. Friedlander, P. Kurlberg and I. E. Shparlinski, ‘Products in residue classes’, Math. Res. Letters, 15 (2008), 1133–1147. (pp. 1 and 2)
[7] M. Z. Garaev, ‘An estimate of Kloosterman sums with prime numbers and an application’, Matem. Zametki, 88 (2010), 365–373, (in Russian). (pp. 2 and 7)
[8] M. Z. Garaev, ‘On multiplicative congruences’, Math. Zeitschrift, 272 (2012), 473–482. (p. 2)
[9] G. Greaves, ‘A weighted sieve of Brun’s type’, Acta Arith., 40 (1982) 297–332. (p. 6)
[10] G. Greaves, ‘The weighted linear sieve and Selberg’s λ2-method’, Acta Arith., 47 (1986) 71–96. (pp. 4, 6, and 9)
[11] G. Greaves, Sieves in number theory, Springer, Berlin, 2001. (pp. 4, 6, 9, and 10)
[12] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, Wiley-Interscience, New York-London-Sydney, 1974. (p. 9)
[13] O. Ramaré and A. Walker, ‘Products of primes in arithmetic progressions: a footnote in parity breaking’, Preprint, 2016 (available from http://arxiv.org/abs/1610.05304). (p. 4)
[14] I. E. Shparlinski, ‘On products of primes and almost primes in arithmetic progressions’, Period. Math. Hungarica, 67 (2013), 55–61. (pp. 2, 4, 5, 7, and 9)
[15] I. M. Vinogradov, Elements of number theory, Dover Publ., NY, 1954. (p. 8)
[16] A. Walker, ‘A multiplicative analogue of Schnirelmann’s theorem’, Bull. London Math. Soc., 48 (2016), 1018–1028. (pp. 2, 3, and 4)