Inflation and Braneworlds: Degeneracies and Consistencies

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(March 19, 2022)

Scalar and tensor perturbations arising in an inflationary braneworld scenario driven by a single scalar field are considered, where the bulk on either side of the brane corresponds to Anti–de Sitter spaces with different cosmological constants. A consistency relation between the two spectra is derived and found to have an identical form to that arising in standard single–field inflation based on conventional Einstein gravity. The dS/CFT correspondence may provide further insight into the origin of this degeneracy. Possible ways of lifting such a degeneracy are discussed.

PACS numbers: 98.80.Cq, 04.50.+h

I. INTRODUCTION

The observational evidence for acoustic peak structure in the power spectrum of the cosmic microwave background (CMB) radiation continues to increase [1]. The data is consistent with an initial spectrum of adiabatic scalar (density) perturbations generated in a spatially flat universe [2–4]. This provides strong support for the simplest class of inflationary models, where the accelerated expansion of the universe is driven by the potential energy arising from the self–interactions of a scalar ‘inflaton’ field. (For recent reviews, see, e.g., Ref. [5]). Density perturbations in the post–inflationary universe are generated due to quantum fluctuations in this field as it rolls slowly down its potential [6].

In general, the spectrum of density perturbations is given by

\[ A_S^2 \propto k^{n_S - 1}, \]

where the dependence of the amplitude, \( A_S \), on comoving wavenumber, \( k \), is parametrized in terms of the spectral index, \( n_S = n_S(k) \). During inflation, a primordial spectrum of tensor (gravitational wave) perturbations is also generated quantum mechanically with amplitude \( A_T^2 \propto k^{n_T} \). To lowest–order in the slow–roll approximation, the spectral indices, \( \{ n_S, n_T \} \), have constant, but different, values [7]. However, the amplitudes of the two spectra on a given scale are not independent and are related through the so–called ‘consistency’ relation:

\[ \frac{A_T^2}{A_S^2} = -\frac{1}{2} n_T. \] \hspace{1cm} (1.1)

The primary importance of the consistency equation (1.1) is that it reduces the number of independent inflationary parameters to three. More optimistically, since it does not depend on the specific functional form of the inflaton potential, it may be regarded as a model–independent test of the single–field inflationary scenario, although the practical difficulties in measuring the relevant parameters are considerable. To date, the contribution from tensor perturbations to the CMB power spectrum is constrained to be no more than 30 per cent [2], but interest is growing in the possibility that forthcoming measurements of the CMB polarization may result in a direct detection of the gravitational waves through their contribution to the curl (B–mode) component of the polarization [9]. (For a recent survey of the observational status see Ref. [10]).

From the theoretical perspective, there has been considerable interest in recent years in the possibility that our observable (four–dimensional) universe may be represented as a domain wall or membrane that is embedded in a higher–dimensional ‘bulk’ spacetime [11–18]. In particular, in the type II Randall–Sundrum model (RSII) [14], a domain wall is embedded in five–dimensional Anti–de Sitter (AdS) space. Corrections to Newtonian gravity are suppressed even though the fifth dimension is uncompactified because the higher–dimensional geometry is non–factorizable [4].

1In this paper, we adopt the normalization conventions of Ref. [8] for the amplitudes of the perturbation spectra. For the tensor spectrum, this implies that \( A_T^2 = \langle \varphi^2 \rangle /50 \), where \( \langle \varphi^2 \rangle^{1/2} = 2\kappa_4 (H/2\pi) \) is the amplitude of the quantum fluctuation in each of the polarization states, \( \varphi_0 \), and \( \kappa_4^2 = 8\pi/m_4^2 \), where \( m_4 \) is the four–dimensional Planck mass. Units are chosen such that \( h = c = 1 \).
The motion of the brane through the bulk space is interpreted as cosmic expansion or contraction by an observer confined to the brane [13,14]. The warping of the bulk on the brane leads to corrections to the standard Hubble law and the Friedmann equation acquires a term that is quadratic in the energy density [15]. These corrections enhance the friction acting on the inflaton field at high energies, thereby extending the region of parameter space where inflation may proceed [16,24].

Such effects also significantly modify the amplitudes and scale dependences of the scalar and tensor perturbations [19,23]. Remarkably, however, it was recently shown that the corresponding consistency equation in the RSII scenario has precisely the form given in Eq. (1.1) [21]. This results in a degeneracy between the predictions of the different scenarios. An important question that arises, therefore, is whether this degeneracy between the consistency equations of the standard and braneworld inflationary scenarios is merely a coincidence, or whether it is indicative of a more universal relationship.

In this paper we explore such a degeneracy further. One of the primary features of the RSII model is the $Z_2$ reflection symmetry that is imposed on the fifth dimension of the bulk metric. We relax this assumption by considering an asymmetric braneworld scenario, where the bulk space on either side of the brane is represented by different AdS spaces with different, negative cosmological constants. A natural mechanism for breaking the reflection symmetry in this way is to couple the brane to a four-form gauge field through a Wess–Zumino action [24]. The cosmology of asymmetric braneworlds has been considered previously by a number of authors [15,16,24–27]. In this paper, we consider a scenario where the inflationary expansion of the braneworld is driven by a single, self–interacting scalar field that is confined to the brane. The effect of the $Z_2$ symmetry on the form of the consistency equation can then be determined.

We begin in Section II by presenting the Friedmann equation for the scenario under consideration and proceed to derive the scalar perturbation amplitude generated during an epoch of inflation. We derive the corresponding expression for the tensor perturbations in Section III. In Section IV, it is shown that the consistency equation (1.1) holds even when the reflection symmetry is not imposed. In Section V, the degeneracy is discussed within the context of the recently conjectured dS/CFT correspondence [28] and possible ways to lift the degeneracy are highlighted.

II. COSMOLOGICAL FIELD EQUATIONS AND SCALAR PERTURBATIONS

When deriving the Friedmann equation, it is convenient to choose coordinates in the vicinity of the brane such that the brane is stationary and the metric dependence on the fifth coordinate is separable, i.e., $ds^2 = γ_{μν}(x^σ,y)dx^μdx^ν + dy^2$, where the brane is located at $y = 0$. The effective Friedmann equation is then derived from the junction conditions [29]:

$$[K_{μν}] = -\kappa_5^2 \left( T_{μν} - \frac{1}{3} T \gamma_{μν} \right), \tag{2.1}$$

where $[K_{μν}] = K_{μν}(0^+) - K_{μν}(0^-)$, $K^{AB} = (4)\nabla^A n^B$ is the extrinsic curvature of the boundary with unit normal vector, $n^A$, $T_{μν}$ is the energy–momentum of the matter confined to the brane, $T = T^μ_μ$ is its trace, $\kappa_5^2 ≡ 8πG_5$ and $G_5$ is the five–dimensional Newton constant.

Since we focus on inflation of the braneworld, we consider the case where the bulk metric corresponds to pure AdS space and the induced metric on the brane is the spatially flat, Friedmann–Robertson–Walker (FRW) metric. It is assumed that the brane energy–momentum tensor is given by the perfect fluid form $T^{(B)}_{μν}|_{brane} = δ(y)\text{diag}(−ρ, p, p, p, 0)$, where $ρ$ and $p$ represent the energy density and pressure, respectively. We focus on the case where the energy–momentum is sourced by the tension of the brane, $λ$, and a single, self–interacting scalar field, $φ$, with potential, $V(φ)$, such that $ρ = ρ_φ + λ$ and $p = p_φ − λ$, where $ρ_φ = ˙φ^2/2 + V(φ)$, $p_φ = ˙φ^2/2 − V(φ)$ and a dot denotes differentiation with respect to proper time on the brane. The spatial components of the junction conditions (2.1) then reduce to [13,14,23,29]

$$(α_+ + H^2)^{1/2} + (α_- + H^2)^{1/2} = \frac{\kappa_5^2ρ}{3}, \tag{2.2}$$

$^2$Upper case, Latin indices denote five–dimensional variables, $A = (0,1,\ldots,4)$, lower case, Greek indices represent world–volume indices, $μ = (0,\ldots,3)$, and spatial indices on the domain wall are denoted by lower case, Latin indices, $i = (1,2,3)$. 
where $\alpha_{\pm} \equiv -\kappa_5^2 \Lambda_{\pm}/6$, $\Lambda_{\pm}$ represent the bulk cosmological constants on either side of the brane, $H = \dot{a}/a$ is the Hubble parameter and $a(t)$ is the scale factor of the world–volume metric. Eq. (2.1) can be solved to yield the effective Friedmann equation on the brane [15,16,25,26]:

$$H^2 = \frac{\kappa_5^2 \rho}{36} - \frac{1}{2} (\alpha_- + \alpha_+) + \frac{9}{4 \kappa_5^2 \rho^2} (\alpha_- - \alpha_+)^2. \quad (2.3)$$

Relaxing the $Z_2$ symmetry results in the non–trivial third term on the right hand side of Eq. (2.2). It is interesting to note that when this term is present, Eq. (2.2) is invariant under an infra–red/ultra–violet duality transformation on the total energy density, $\kappa_5^2 \rho \leftrightarrow 3(\alpha_+ - \alpha_-)/\kappa_5^2 \rho$. As in the symmetric scenario, the standard, linear dependence on the energy density may be recovered at low energy scales by tuning the brane tension to cancel the effects of the negative cosmological constants. Dimensional reduction implies that the four– and five–dimensional Newton constants are related by [25]

$$\kappa_5^2 \rightarrow \kappa_4^2 = \frac{1}{2} \left( \frac{1}{\sqrt{\alpha_+}} + \frac{1}{\sqrt{\alpha_-}} \right). \quad (2.4)$$

In the symmetric limit, $\Lambda_{\pm} = \Lambda$, Eq. (2.3) reduces to [18]

$$H^2 = \frac{\kappa_4^2 \rho \phi}{3} \left[ 1 + \frac{\rho \phi}{2 \lambda} \right], \quad (2.5)$$

where the brane tension is tuned to satisfy $\kappa_4^2 \lambda = \sqrt{-6 \Lambda}$.

When Eq. (2.2) is valid, the time–component of the junction condition (2.1) enforces covariant conservation of energy–momentum on the brane [17]. This implies that

$$\ddot{\phi} + 3H \dot{\phi} + V' = 0 \quad (2.6)$$

for a self–interacting scalar field, where a prime denotes $d/d\phi$, and Eqs. (2.2) and (2.6) then determine the classical dynamics of the asymmetric braneworld.

We now proceed to determine the amplitude of the scalar perturbations that are generated during inflation of the asymmetric braneworld. We consider the era when the slow–roll approximation, $|\dot{H}|/H^2 \ll 1$ and $|\dot{\phi}| \ll H|\phi|$, is valid. In general, the curvature perturbation on uniform density hypersurfaces is given by $\zeta = H \delta \phi/\phi$ and this in turn is determined by the scalar field fluctuation, $\delta \phi$, on spatially flat hypersurfaces [30]. The perturbations generated from a single, self–interacting scalar field are adiabatic and conservation of energy–momentum then implies that $\zeta$ is conserved on large scales [31]. Consequently, the amplitude of a mode when it re–enters the Hubble radius after inflation is related to the curvature perturbation by $A_S^2 = 4\langle \zeta^2 \rangle/25$. The right–hand side of this expression is evaluated when the mode goes beyond the Hubble radius during inflation, i.e., when the comoving wavenumber is given by

$$k(\phi) = a_e H(\phi) \exp[-N(\phi)], \quad (2.7)$$

where $a_e$ denotes the value of the scale factor at the end of inflation and $N = \ln a = \int dt H(t)$ represents the number of e–foldings between a scalar field value, $\phi$, and the end of inflation, $\phi_e$. Since the field fluctuation at this epoch is determined by the Gibbons–Hawking temperature of de Sitter space, $\langle \delta \phi^2 \rangle = H^2/(4\pi^2)$, it follows that the scalar perturbation amplitude has the form

$$A_S^2 = \frac{1}{25 \pi^2} \left. \frac{H^4}{\phi^2} \right|_{k = aH} \quad (2.8)$$

Substitution of the scalar field equation (2.6) then relates the amplitude directly to the inflaton potential:

$$A_S^2 = \frac{9}{25 \pi^2 \sqrt{2}} \left[ \frac{\kappa_5^2 (V + \lambda)}{36} - \frac{1}{2} (\alpha_- + \alpha_+) + \frac{9}{4 \kappa_5^2 (V + \lambda)^2} (\alpha_- - \alpha_+)^2 \right]^{3/2} \left|_{k = aH} \quad (2.9)$$
III. TENSOR PERTURBATIONS

The calculation of the gravitational wave spectrum is more involved, because the tensor perturbations extend into the bulk. Langlois, Maartens and Wands [23] have considered the generation of gravitational waves in the symmetric RSII scenario in the limit where the world–volume of the brane corresponds to pure de Sitter space. This case arises when the scalar field is constant and represents a good approximation to a field that is rolling slowly down its potential. In this limit, the evolution equation for the tensor perturbations admits a separable solution and this allows an analytical expression for the amplitude on large scales to be derived.

We briefly review the method of Ref. [23] and then extend the analysis to the asymmetric scenario. The unperturbed bulk metric in the Z_2 symmetric model is written as [32]

\[ ds_5^2 = A^2(y) [-dt^2 + a^2(t)dx^2] + dy^2, \]

where \( A = (H/\alpha) \sinh[\alpha(y_b - |y|)] \), the Cauchy horizons, \( g_{00}(\pm y_b) = 0 \), are located at \( y = \pm y_b \), and the constant \( \alpha = \kappa_4/\kappa_5 = (-\Lambda/6)^{1/2} \) is determined by the bulk cosmological constant, \( \Lambda \). The perturbed metric is given by \( ds_5^2 = A^2[-dt^2 + a^2(\delta_{ij} + E_{ij})dx^i dx^j] + dy^2 \) and the metric perturbations, \( E_{ij} \), are decomposed into Fourier modes with amplitude, \( E(t, y; \vec{k}) \). The linearly perturbed junction conditions (2.1) then imply that

\[ \frac{dE}{dy} \bigg|_{y=0} = 0 \]  

in the absence of anisotropic stresses. The equation of motion for the gravitational waves separates into ‘on–brane’ and ‘off–brane’ components and the amplitude is expanded into eigenmodes, \( E(t, y; \vec{k}) = \int dm \varphi_m(t; \vec{k}) \xi_m(y) \), where \( m \) represents the separation constant. The general solution for the zero–mode is given by

\[ \xi_0 = C_1 + C_2 \int_0^y \frac{1}{A^2(y')} dy' \]  

where \( C_{1,2} \) are constants. Although Eq. (3.3) diverges at the Cauchy horizons, the boundary condition (3.2) removes the divergent part of the zero–mode, since it requires \( C_2 = 0 \). The general solution to the off–brane equation for the ‘light’ modes (\( m < 3H/2 \)) remains divergent at the Cauchy horizon even when Eq. (3.2) is satisfied. Consequently, these modes do not contribute to the spectrum of orthonormal modes that forms the basis of the Hilbert space for the quantum field. ‘Heavy’ modes with \( m > 3H/2 \) remain in the vacuum state during inflation.

The solution to the on–brane equation for the zero–mode asymptotes to \( \varphi_0 \rightarrow \) constant at late times. Thus, the amplitude of the zero–mode remains constant on super–Hubble radius scales as in the standard inflationary scenario. The amplitude of the quantum fluctuations in this mode is determined by deriving an effective, five–dimensional action for the tensor perturbations and integrating over the fifth dimension. Normalizing the action relative to the standard, four–dimensional result imposes the condition [23]

\[ 2 \int_0^{y_b} dy C_1^2 A^2 = 1 \]  

and this implies that \( C_1 = \sqrt{\alpha} F(x) \), where

\[ \frac{1}{F^2} = \sqrt{1 + x^2 - x^2 \sinh^{-1}\left(\frac{1}{x}\right)} \]  

and \( x \equiv H/\alpha \). This normalization results in a four–dimensional action for the zero–mode that is formally equivalent to that of a massless scalar field in a FRW universe, but with an overall factor of \((8\kappa_5^2)^{-1}\) instead of the conventional factor of \((8\kappa_5^2)^{-1}\). Consequently, the standard four–dimensional results can be employed by viewing each polarization, \( \varphi_0 \), as a quantum field evolving in a time–dependent potential. This implies that \( \langle \varphi_0^2 \rangle^{1/2} = 2\kappa_5 (H/2\pi) \) and the tensor perturbation amplitude is therefore given by

\[ A_T^2 = \frac{C_1^2 \langle \varphi_0^2 \rangle}{50} = \frac{\kappa_5^2}{50\pi^2} H^2 F^2. \]

We now determine the gravitational wave amplitude when the Z_2 symmetry is not imposed in the fifth dimension. The method of Ref. [23] may be employed directly, and we therefore omit many of the details. The first question to address is the appropriate form of the bulk metric. In the asymmetric scenario, regions of two AdS spaces with different
cosmological constants are effectively separated by the brane. Consequently, when the bulk metric is expressed in Gaussian normal coordinates, it is possible to consider cases where a Cauchy horizon exists on only one side of the brane. However, in the derivation of Eq. (2.2), it was assumed implicitly that the unit normal vector, \( n^A \), pointed towards a Cauchy horizon. (In the cases where this is not so, the signs of one or both of the terms on the left hand side of Eq. (2.2) would change, thus requiring unphysical forms of matter).

For consistency, therefore, we should consider the scenario where Cauchy horizons exist on both sides of the brane. The appropriate form for the unperturbed bulk metric is then given by Eq. (3.1), where

\[
A(y) = \begin{cases} 
A_+(y) \equiv \text{cosech}(\sqrt{\alpha_+} y_{h_+}) \sinh(\sqrt{\alpha_+}(y_{h_+} - y)), & \text{if } y > 0 \\
A_-(y) \equiv \text{cosech}(\sqrt{\alpha_-} y_{h_-}) \sinh(\sqrt{\alpha_-}(y - y_{h_-})), & \text{if } y < 0 ,
\end{cases} 
\]

(3.7)

where the Cauchy horizons are located at

\[
y_{h_\pm} = \pm \frac{1}{\sqrt{\alpha_\pm}} \sinh^{-1} \left( \frac{\sqrt{\alpha_\pm}}{H} \right) 
\]

(3.8)

and \( H \) is the four–dimensional Hubble parameter. The bulk metric is continuous at \( y = 0 \), as required. The locations of the Cauchy horizons are determined by the values of the bulk cosmological constants and the two horizons are equidistant from the brane when the cosmological constants have the same value. This implies that the \( Z_2 \)–symmetric scenario can be recovered in a smooth fashion in the limit that \( \Lambda_+ \rightarrow \Lambda_- \).

In general, the linearly perturbed junction conditions imply that

\[
\left. \frac{dE}{dy} \right|_{y=0^+} = \left. \frac{dE}{dy} \right|_{y=0^-} (3.9)
\]

in the absence of a reflection symmetry. Thus, a given mode and its first derivative must be continuous across the brane, unless there exist anisotropic stresses. (We assume such effects to be negligible in the present work). The general solution for the zero–mode is now given in terms of the quadrature (3.3), where Eq. (3.7) is satisfied, and this diverges at the Cauchy horizon unless \( C_2 = 0 \), as in the symmetric model. Consequently, the \( C_2 \) part of the solution does not contribute to the quantum vacuum. This leaves the constant solution \( E_0 = J = \text{constant} \) as the physically relevant solution to the off–brane equation of motion. Specifying this solution is equivalent to imposing the boundary condition \( \left( \frac{dE}{dy} \right)_{y=0^\pm} = 0 \) and results in a normalizable zero–mode. The asymptotic form of the light modes \( E_m \) may also be deduced and the non–trivial solutions, as well as the general solution, diverge at the Cauchy horizon.

It follows from Eq. (3.4) that the normalization of the zero–mode is sensitive to the location of the Cauchy horizon. In the asymmetric case, this condition generalizes to

\[
\frac{1}{J^2} = \int_{y_{h_-}}^{y_{h_+}} dy A_+^2(y) + \int_{0}^{y_{h_+}} dy A_-^2(y) (3.10)
\]

and substituting the bulk metric (3.7) and evaluating the integrals then implies that

\[
\frac{1}{J^2} = \frac{1}{2 \sqrt{\alpha_- F^2(x_-)}} + \frac{1}{2 \sqrt{\alpha_+ F^2(x_+)}} , (3.11)
\]

where \( F = F(x_\pm) \) is given in Eq. (3.5) and \( x_\pm \equiv H/\sqrt{\alpha_\pm} \).

Finally, integrating over the fifth dimension in the effective action for the tensor perturbations results in a four–dimensional action for the zero–mode, \( \varphi_0 \), that is identical in form to that of the symmetric model. We conclude, therefore, that the gravitational wave spectrum generated in the asymmetric inflationary braneworld scenario is given by

\[
A_T^2 = \frac{\kappa_5^2}{50\pi^2} H^2 J^2 \bigg|_{k=\alpha H} (3.12)
\]

We refer to Eq. (3.11) as the correction function for the gravitational waves. At low energies, \( x_\pm \ll 1 \) and \( J \rightarrow \kappa_4/\kappa_5 \), implying that the standard expression is recovered in this limit.
IV. CONSISTENCY EQUATION

Eqs. (2.9) and (3.12) represent the scalar and tensor perturbation spectra for a given inflationary potential. Relaxing the $Z_2$ reflection symmetry has resulted in both spectra receiving significant modifications. Given the complicated functional form of the two spectra, it might be expected that the form of consistency equation (1.1) would be altered by these additional terms.

In the standard approach to deriving the consistency equation, we differentiate the tensor spectrum with respect to comoving wavenumber, $k$, and employ the condition $k = aH$ to relate a given scale to a particular value of the inflaton field $\phi$. To lowest–order in the slow–roll approximation, this implies that $d\ln k = Hdt \equiv dN$. The consistency equation then follows by substituting in the expressions for the amplitudes.

On the other hand, it is more illuminating to first consider the nature of the consistency relation between the two spectra in the high energy limit, where $\kappa^2 \rho^2 \gg 1$ ($x \gg 1$). In this limit, the first term on the right hand side of Eq. (2.9) dominates. Thus, at high energies, both the Friedmann equation (2.3) and the scalar perturbation amplitude (2.8) reduce to the corresponding equation arising in the symmetric model. Moreover, it follows from Eq. (3.11) that $J^2 \approx 3H/2$ in this limit, implying that the tensor amplitude also reduces to the symmetric limit. Thus, the degeneracy of the consistency equation is not lifted in the high–energy limit and the tilt of the tensor spectrum is related to the ratio of the scalar and tensor amplitudes by Eq. (1.1).

Motivated by the above observation, we now consider the possibility that Eq. (1.1) is valid for all energy scales. In this case, substituting Eqs. (2.8) and (3.12) into Eq. (1.1) implies that

$$\frac{dH}{dV} \frac{d(HJ)}{dH} = -\frac{\kappa^2 J^2 V'^2}{18 H^3}$$

and Eq. (4.1) can be simplified by expressing the scalar field equation (2.3) in the form

$$\frac{dH}{dN} = \frac{dH}{dV} \frac{V'^2}{3H^2}$$

Substituting Eq. (4.2) into (4.1) then yields

$$H^4 \frac{dH}{dV} \frac{d(HJ)^{-2}}{dH} = -\frac{\kappa^2}{3}.$$ (4.3)

Eq. (4.3) represents a necessary condition that the effective Friedmann equation, $H = H(V)$, and correction function, $J = J(H)$, must satisfy for the consistency equation (1.1) to remain valid.

To proceed, it proves convenient to define the new variables

$$u_\pm \equiv \sinh^{-1} \left( \frac{\sqrt{\alpha_\pm}}{H} \right).$$

It then follows from the definition (3.3) that the correction function (3.11) may be expressed in the compact form

$$\frac{1}{H^2 J^2} = \frac{1}{4} \sum_{i=\pm} \frac{1}{\alpha_{i}^{3/2}} [\sinh (2u_{i}) - 2u_{i}]$$

and differentiating Eq. (4.5) with respect to the Hubble parameter implies that

$$\frac{d(HJ)^{-2}}{dH} = -\frac{1}{H^2} \sum_{i=\pm} \frac{1}{\alpha_{i}} (\sinh u_{i}) (\tanh u_{i}).$$ (4.6)

Eq. (4.3) may now be simplified by substituting in Eq. (4.6):

$$\frac{dH}{dV} [\text{sech} u_+ + \text{sech} u_-] = \frac{\kappa^2}{3}.$$ (4.7)

3In the following discussion, equality denotes equality at this level of the slow–roll approximation.
and the identity
\[
\frac{d}{dV} \left( \sqrt{\alpha} \coth u \right) = \frac{dH}{dV} \sech u
\] (4.8)
then results in a further simplification:
\[
\frac{d}{dV} \left[ \sqrt{\alpha} \coth u_+ + \sqrt{\alpha} \coth u_- \right] = \frac{\kappa^2}{3}
\]
(4.9)
Thus, the consistency equation for the asymmetric braneworld inflationary scenario is given by Eq. (1.1) if Eq. (4.9) is satisfied. We may establish that this is indeed the case by rewriting the junction condition (2.2) in terms of the new variables (4.4):
\[
\sqrt{\alpha} \coth u_+ + \sqrt{\alpha} \coth u_- = \frac{\kappa^2}{3}(V + \lambda),
\]
(4.10)
where we have employed the slow–roll approximation. Since the tension of the brane, \(\lambda\), is constant, we deduce immediately that Eq. (4.9) follows as a direct consequence of the junction condition (2.2), or equivalently, the effective Friedmann equation (2.3).

V. DISCUSSION

The consistency equation (1.1) has long been regarded as a strong observational signature of the standard, single–field inflationary scenario formulated within the context of conventional Einstein gravity [8]. We have considered the corrections that arise to the scalar and tensor perturbation spectra in a Randall–Sundrum braneworld scenario, where the bulk cosmological constants on either side of the brane are different. We have found that the corresponding consistency equation for such an asymmetric braneworld model is also given by Eq. (1.1). This identifies a third class of inflationary models where such a relationship between observable parameters arises and the result is surprising given that the gravitational physics and perturbation spectra are radically different in all three scenarios. In other models where the spectra differ from the standard expressions, such as in the warm inflationary scenario [33] and models driven by higher–order terms in the curvature invariants [34], the consistency equation is modified.

In effect, the consistency equation is given by Eq. (1.1) because the Hubble parameter, \(H = H(V)\), and correction function, \(J = J(H)\), satisfy the first–order differential equation (4.3). The corrections to the scalar and tensor spectra in both the symmetric and asymmetric braneworld scenarios have precisely the necessary form for this equation to remain valid. It is not obvious \textit{a priori} from Eqs. (2.9), (3.11) and (3.12) that this should be the case.

It is of interest to explore the degeneracy of the consistency equation from a theoretic viewpoint. Further insight may be gained by considering the symmetric Randall–Sundrum model. By defining the new variables
\[
b = \frac{1}{2} \sinh^{-1} x
\]
(5.1)
\[
\beta = \kappa \frac{d\phi}{dN},
\]
(5.2)
where \(x\) is defined after Eq. (3.5), it is possible to rewrite the cosmological field equations (2.3) and (2.4) as a first–order system [11]
\[
\dot{b} = -\left( \frac{3\kappa^2}{8\lambda} \right)^{1/2} \dot{\phi}^2
\]
(5.3)
and
\[
\dot{\beta} = -\left( \frac{8\lambda}{3} \right)^{1/2} \frac{b'}{H'}
\]
(5.4)
The correction function (2.3) to the gravitational wave spectrum can then be expressed directly in terms of a quadrature with respect to the \(b\)-function:
\[
\frac{1}{F^2} = -4 \sinh^2 2b \int \frac{db}{\sinh^3 2b}
\]
(5.5)
and the scalar perturbations (2.8) take the form

\[ A_S^2 = \frac{\kappa_4^2}{25\pi^2} \frac{H^2}{\beta^2}. \]  

(5.6)

By employing the definitions (2.7), (5.1) and (5.2) and substituting Eqs. (3.6), (5.4) and (5.6) into Eq. (5.3), we find that

\[ \frac{1}{A_T^2} = 2 \int \frac{d\ln k}{A_S^2} \]  

(5.7)

when the slow–roll approximation, \(|\dot{H}|/H^2 \ll 1\) and \(|\ddot{\phi}| \ll H|\dot{\phi}|\), is obeyed. Thus, differentiation with respect to comoving wavenumber recovers the consistency equation (1.1).

The degeneracy between the consistency equations of the standard and symmetric braneworld scenarios arises because the observable quantities, \(\{A_T^2/A_S^2, n_T\}\), in the latter case are independent of the brane tension. (The standard expressions for the perturbation spectra are formally recovered in the limit \(\lambda \to \infty\)). The variables \(\{b, \beta\}\) play a central role in establishing this independence. Moreover, introducing these variables allows the second–order field equations (2.3) and (2.4) to be expressed as a coupled, first–order system (5.3) and (5.4), where \(b\) is a monotonically decreasing function of cosmic time when the null energy condition is satisfied [41]. This is analogous to the flow equations recently considered within the context of the conjectured dS/CFT correspondence [28,35,36] and motivates us to explore a physical interpretation of these parameters within such a context.

The dS/CFT correspondence states that pure quantum gravity in de Sitter (dS) space has a dual description in terms of a conformal field theory (CFT), where the latter is located on the Euclidean boundary of de Sitter space at future infinity [28]. Although still at the conjectural level, this correspondence establishes a ‘dictionary’ relating variables in the bulk to those of the boundary theory.

In a field theory with an operator, \(\mathcal{O}\), and coupling parameter, \(g\), the coupling is constant at the classical level, but renormalization introduces a scale–dependence. The breaking of scale invariance is parametrized by the \(\beta\)-function, \(\beta \equiv \partial g/\partial \ln \mu\), where \(\mu\) is the renormalization group (RG) scale. In the dS/CFT correspondence, a scalar field is dual to the operator, \(\mathcal{O}\). The symmetries of de Sitter space, and consequently the scale invariance of the CFT, are broken when the field rolls slowly down its potential [33]. By analogy with the AdS/CFT correspondence [37], the value of the scalar field is identified with the field theory coupling and the RG scale with the scale factor of the universe, i.e., \(g = \kappa_4 \phi\) and \(\mu \propto a\) [35]. Thus, we may identify Eq. (5.2) as the \(\beta\)-function of the quantum theory expressed in terms of the bulk variables. This first–order equation then describes the RG flow between the fixed points, \(\beta = 0\).

Another quantity of key importance is the central charge (\(c\)-function). This parametrizes the number of degrees of freedom of the CFT and decreases along the RG flow to the infra–red fixed point. (In the dS/CFT correspondence, this limit corresponds to the past). A natural candidate for the \(c\)-function has been proposed in standard inflationary cosmology, \(c \equiv (\kappa_4^2 H^2)^{-1}\) [28]. Adopting such an expression in the case of the braneworld implies that the \(c\)-function is related to the \(b\)-function (5.3) such that \(c = (3/4\pi\kappa_4^2) \cosh^2(2b)\). This decreases monotonically as we go back in time by virtue of Eq. (5.3). Thus, the functions (5.1) and (5.2) admit a physical interpretation in terms of the \(\beta\)– and \(c\)-functions of the dual three–dimensional Euclidean field theory and the relative amplitudes of the perturbation spectra can be expressed directly in terms of these variables:

\[ \frac{A_T^2}{A_S^2} = \frac{1}{2} \beta^2 F^2[c]. \]  

(5.8)

Since the dS/CFT correspondence introduces a new perspective on inflationary cosmology, it would be interesting to extend this preliminary analysis further along the lines outlined in Ref. [35] for standard inflationary cosmology.

The degeneracy of the consistency equation (1.1) has a number of significant implications. In particular, it implies that observations of the CMB may not be able to discriminate between conventional and braneworld inflation [39] and this leads us to consider how the degeneracy might be lifted.

It should be emphasized that the degeneracy arises in the predictions of the primordial perturbations, but the bulk space may influence the evolution of these perturbations. In particular, the derivation of Eq. (1.1) neglected the backreaction of the metric fluctuations in the fifth dimension. This is a consistent approximation when the scalar perturbations are perturbations of a homogeneous distribution of matter on the brane, as is the case considered above [19]. More generally, however, the backreaction induces a non–trivial Weyl curvature in the bulk and this manifests itself as a non–local source of energy–momentum in four dimensions [33]. The background dynamics is altered by such a source and it would be interesting to incorporate these effects into the analysis. Furthermore, we have focused on a specific realization of the braneworld scenario, where the bulk space is conformally flat and the degeneracy may be broken by relaxing this condition. This could be achieved, for example, by introducing a scalar field into the bulk.
The results discussed above hold only to lowest–order in the slow–roll approximation. To next–to–leading order, the consistency equation of the standard scenario receives corrections \[1\]:

\[
n_{T} = -2 \frac{A_{T}^{2}}{A_{S}^{2}} \left[ 1 - \frac{A_{T}^{2}}{A_{S}^{2}} + (1 - n_{S}) \right].
\] (5.9)

Significantly, these corrections do not depend on the tilt of the tensor spectrum and it is therefore important to establish whether similar terms arise in the braneworld scenario. Eq. (5.9) was derived by performing a general expansion about the exact perturbation spectra that are generated during power law inflation. This particular model is exactly solvable because the kinetic and potential energies of the scalar field redshift at the same rate. The corresponding model in the braneworld scenario was recently found \[11\] and, in principle, could serve as a basis for addressing higher–order effects in this class of models.

It is known that the consistency equation (1.1) relaxes to an inequality, \(A_{T}^{2}/A_{S}^{2} \leq - n_{T}/2\), in multiple–field inflationary models \[2\]. Isocurvature (entropy) perturbations can also be generated when more than one scalar field is present. Recently, it was shown that for a general two–field model, the cross correlation between the adiabatic and isocurvature modes modifies the consistency equation such that \(A_{T}^{2}/A_{S}^{2} = - n_{T}(1 - r_{C}^{2})/2\), where \(r_{C}\) is determined by the cross–correlation power spectrum \[12\]. It is possible, therefore, that the degeneracy between the single field consistency equations might be lifted by considering isocurvature perturbations in a two–field braneworld scenario. This would require a detailed analysis and derivation of the evolution equations for the perturbations.

In conclusion, there exists a surprising degeneracy between the observational predictions of different inflationary scenarios based on conventional Einstein gravity and the braneworl picture. This provides us with potentially crucial observational insight into the gravitational physics of extra dimensions.

ACKNOWLEDGMENTS

GH is supported by the Particle Physics and Astronomy Research Council. JEL is supported by the Royal Society. We thank D. Wands for helpful discussions.

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