On covering paths with 3 dimensional random walk

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Abstract

In this paper we find an upper bound for the probability that a 3 dimensional simple random walk covers each point in a nearest neighbor path connecting 0 and the boundary of an $L_1$ ball of radius $N$ in $\mathbb{Z}^d$. For $d \geq 4$, it has been shown in [5] that such probability decays exponentially with respect to $N$. For $d = 3$, however, the same technique does not apply, and in this paper we obtain a slightly weaker upper bound: $orall \varepsilon > 0, \exists c_\varepsilon > 0,$

$$P\left(\text{Trace}(P) \subseteq \text{Trace}\left(\{X_n\}_{n=0}^\infty\right)\right) \leq \exp\left(-c_\varepsilon N \log^{-\left(1+\varepsilon\right)}(N)\right).$$

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1 Introduction

In this paper, we study the probability that the trace of a nearest neighbor path in $\mathbb{Z}^3$ connecting 0 and the boundary of an $L_1$ ball of radius $N$ is completely covered by the trace of a 3 dimensional simple random walk.

First, we review some results we proved in a recent paper for general $d$’s. For any integer $N \geq 1$, let $\partial B_1(0, N)$ be the boundary of the $L_1$ ball in $\mathbb{Z}^d$ with radius $N$. We say that a nearest neighbor path $P = (P_0, P_1, \cdots, P_K)$ is connecting 0 and $\partial B_1(0, N)$ if $P_0 = 0$ and $\inf\{n : \|P_n\|_1 = N\} = K$. And we say that a path $P$ is covered by a $d$ dimensional random walk $\{X_{d,n}\}_{n=0}^\infty$ if

$$\text{Trace}(P) \subseteq \text{Trace}(X_{d,0}, X_{d,1}, \cdots) := \{x \in \mathbb{Z}^d, \exists n X_{d,n} = x\}.$$

In [5], we have shown that for any $d \geq 2$ such covering probability is maximized over all nearest neighbor paths connecting 0 and $\partial B_1(0, N)$ by the monotonic path that stays within distance one above/below the diagonal $x_1 = x_2 = \cdots = x_d$.

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**Theorem 1.1.** *(Theorem 1.4 in [5]) For each integers \( L \geq N \geq 1 \), let \( \mathcal{P} \) be any nearest neighbor path in \( \mathbb{Z}^d \) connecting 0 and \( \partial B_1(0,N) \). Then

\[
P(\text{Trace}(\mathcal{P}) \in \text{Trace}(X_{d,0}, \cdots, X_{d,L})) \leq P(\hat{\mathcal{P}} \in \text{Trace}(X_{d,0}, \cdots, X_{d,L}))
\]

where

\[
\hat{\mathcal{P}} = (\arcc[0 : d-1, \arcc[0 : d-1, \cdots, \arcc[N/d][0 : d-1, \arcc[N/d+1][0 : N-d[N/d]])
\]

\[
\text{arc}_i[0 : d-1] = \left( 0, e_1, e_2, \cdots, \sum_{i=1}^{d-1} e_i \right)
\]

and \( \arcc = (k-1) \sum_{i=1}^{d} e_i + \arcc \).

Then noting that the probability of covering \( \hat{\mathcal{P}} \) is bounded above by the probability that a simple random walk returns to the exact diagonal line for \( [N/d] \) times, one can introduce the Markov process

\[
\hat{X}_{d-1,n} = \left( X^1_{d,n} - X^2_{d,n}, X^2_{d,n} - X^3_{d,n}, \cdots, X^{d-1}_{d,n} - X^d_{d,n} \right)
\]

where \( X^i_{d,n} \) is the \( i \)th coordinate of \( X_{d,n} \) and see that \( \{\hat{X}_{d-1,n}\}_{n=0}^{\infty} \) is another \( d-1 \) dimensional non simple random walk, which is transient when \( d \geq 4 \). In particular, starting from any point \((x_1,x_2,\cdots,x_{d-1}) \in \mathbb{Z}^{d-1}\), the transition probability of \( \hat{X}_{d-1,n} \) is given as follows:

- \((x_1,x_2,\cdots,x_{d-1}) \rightarrow (x_1 \pm 1, x_2,\cdots,x_{d-1}) \), both with probability \( 1/(2d) \).
- For any \( 2 \leq i \leq d-1 \), \((x_1,\cdots,x_{i-1},x_i+1,\cdots,x_{d-1}) \rightarrow (x_1,\cdots,x_{i-1} \mp 1,x_i \pm 1,x_{i+1},\cdots,x_{d-1}) \) each with probability \( 1/(2d) \).
- \((x_1,x_2,\cdots,x_{d-1}) \rightarrow (x_1,x_2,\cdots,x_{d-1} \pm 1) \), both with probability \( 1/(2d) \).

Thus, we immediately have the following upper bound:

**Theorem 1.2.** *(Theorem 1.5 in [5]) There is a \( P_d \in (0,1) \) such that for any nearest neighbor path \( \mathcal{P} = (P_0, P_1, \cdots, P_K) \) connecting 0 and \( \partial B_1(0,N) \) and \( \{X_{d,n}\}_{n=0}^{\infty} \) which is a \( d \)-dimensional simple random walk starting at 0 with \( d \geq 4 \), we always have

\[
P(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{d,n}\}_{n=0}^{\infty})) \leq P_d^{[N/d]}.
\]

Here \( P_d \) equals to the probability that \( \{X_{d,n}\}_{n=0}^{\infty} \) ever returns to the \( d \) dimensional diagonal line.

Theorem 1.2 implies that for each fixed \( d \geq 4 \), the covering probability decays exponentially with respect to \( N \).

For \( d = 3 \), the same technique may not apply since now \( \{\hat{X}_{2,n}\}_{n=0}^{\infty} \) is a recurrent 2 dimensional random walk, which means that \( P_3 = 1 \) and that the original 3 dimensional random walk will return to the diagonal line infinitely often. To overcome this issue, we note that although the diagonal line

\[
D_\infty = \{(0,0,0), (1,1,1), \cdots \}
\]

is recurrent, it is possible to find an infinite subset \( \hat{D}_\infty \subset D_\infty \) that is transient. And if we can further show for this specific transient subset that the return probability is uniformly bounded away from 1 (which is not generally true for all transient subsets, as is shown in Counterexample 1 in Section 3), then we are able to show

\[
P(\hat{\mathcal{P}} \in \text{Trace}(X_{3,0}, X_{3,1}, \cdots)) \leq \exp \left( -c |\hat{D}_\infty \cap \hat{\mathcal{P}}| \right).
\]

With this approach, we have the following theorem:
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**Theorem 1.3.** For each \( \varepsilon > 0 \), there is a \( c_\varepsilon \in (0, \infty) \) such that for any \( N \geq 2 \) and any nearest neighbor path \( \mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3 \) connecting 0 and \( \partial B_1(0,N) \), we have
\[
P \left( \text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty) \right) \leq \exp \left( -c_\varepsilon N \log^{1-\varepsilon}(N) \right).
\]

Note that the upper bound in Theorem 1.3 seems to be non-sharp. The reason is that we did not fully use the geometric structure of path \( \hat{\mathcal{P}} \) to minimize the covering probability. I.e., although we require our simple random walk to visit the transient subset for \( O(N \log^{1-\varepsilon}(N)) \) times, those returns may be not enough to cover every point in \( \tilde{D}_\infty \cap \hat{\mathcal{P}} \). In fact, the following conjecture seems to be supported by numerical simulations, which is shown in Section 4.

**Conjecture 1.4.** There is a \( c \in (0, \infty) \) such that for any \( N \geq 2 \) and any nearest neighbor path \( \mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3 \) connecting 0 and \( \partial B_1(0,N) \), we always have
\[
P \left( \text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty) \right) \leq \exp (-cN).
\]

The structure of this paper is as follows: In Section 2, we construct the infinite subset \( \tilde{D}_\infty \) of the diagonal line, calculate its density, and show it is transient. In Section 3, we show the return probability of \( \tilde{D}_\infty \) is uniformly (in the starting point) bounded away from 1, and with these techniques, finish the proof of Theorem 1.3. In Section 4, we present a numerical simulation which seems to support Conjecture 1.4.

## 2 Infinite transient subset of the diagonal

Without loss of generality we can concentrate on the proof of Theorem 1.3 for sufficiently large \( N \). Recall that
\[
\hat{\mathcal{P}} = \left( \text{arc}_1[0 : d-1], \text{arc}_2[0 : d-1], \cdots, \text{arc}_{\lfloor N/d \rfloor}[0 : d-1], \text{arc}_{\lfloor N/d \rfloor+1}[0 : N - d \lfloor N/d \rfloor] \right)
\]
is the path connecting 0 and \( B_1(0,N) \) that maximizes the covering probability. When \( d = 3 \), let
\[
\mathcal{D}_{\lfloor N/d \rfloor} = \{(0,0,0),(1,1,1), \cdots, ([N/3],[N/3],[N/3])\}
\]
be the points in \( \mathcal{P} \) that lie exactly on the diagonal. Although it is clear that for simple random walk \( \{X_{3,n}\}_{n=0}^\infty \) starting at 0, \( \mathcal{D}_\infty \) is a recurrent set, following a similar construction to Spitzer [6, Chapter 6.26], we find a transient infinite subset of \( \mathcal{D}_\infty \) as follows: for \( n_1 = 0 \), \( n_2 = \lfloor \log^{1+\varepsilon}(2) \rfloor = 1 \), and for all \( k \geq 3 \)
\[
n_k = \left\lceil \sum_{i=1}^{k} \log^{1+\varepsilon}(i) \right\rceil \in \mathbb{Z}, \quad \text{(2.1)}
\]
define
\[
\tilde{D}_\infty = \{(n_k, n_k, n_k)\}_{k=1}^\infty \subset \mathcal{D}_\infty.
\]
Since \( \log^{1+\varepsilon}(k) > 1 \) for all \( k \geq 3 \), it is easy to see that \( \{n_k\}_{k=1}^\infty \) is a monotonically increasing sequence. Moreover, for each \( 1 \leq k_1 < k_2 < \infty \),
\[
n_{k_2} - n_{k_1} = \left\lceil \sum_{i=1}^{k_2} \log^{1+\varepsilon}(i) \right\rceil - \left\lceil \sum_{i=1}^{k_1} \log^{1+\varepsilon}(i) \right\rceil \geq \sum_{i=k_1+1}^{k_2} \log^{1+\varepsilon}(i) - 1.
\]
This implies that for all $k_2 \geq 8$ and $1 \leq k_1 < k_2$,

$$n_{k_2} - n_{k_1} \geq \frac{1}{2} \int_{k_1}^{k_2} \log^{1+\varepsilon}(x)dx. \quad (2.2)$$

For any $N \in \mathbb{Z}$, define

$$\tilde{D}_N = \tilde{D}_\infty \cap D_N$$

and

$$C_N = \left| \tilde{D}_N \right| = \sup \{ k : n_k \leq N \}.$$

Recalling the definition of $n_k$ in (2.1), we also equivalently have

$$C_N = \sup \left\{ k : \sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq N \right\} = \inf \left\{ k : \sum_{i=1}^{k} \log^{1+\varepsilon}(i) > N \right\} - 1.$$

**Lemma 2.1.** For any $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that

$$C_N \in \left( 2^{-1-\varepsilon} N \log^{-1-\varepsilon}(N), C_\varepsilon N \log^{-1-\varepsilon}(N) \right)$$

for all $N \geq 2$.

**Proof.** Note that for any $k$ such that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) > N$$

we must have that $k > C_N$, and that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \geq \int_{1}^{k} \log^{1+\varepsilon}(x)dx \geq \frac{1}{2^{1+\varepsilon}} (k - k^{1/2}) \log^{1+\varepsilon}(k). \quad (2.3)$$

For $K_N = \left\lfloor \frac{2^\varepsilon N}{\log^{1+\varepsilon}(N)} \right\rfloor$, we have by (2.3)

$$\sum_{i=1}^{K_N} \log^{1+\varepsilon}(i) \geq \frac{1}{2^{1+\varepsilon}} (K_N - K_N^{1/2}) \log^{1+\varepsilon}(K_N) \geq \frac{1}{2^{1+\varepsilon}} \cdot K_N \cdot \frac{K_N - K_N^{1/2}}{K_N} \cdot \log^{1+\varepsilon} \left( \frac{2^\varepsilon N}{\log^{1+\varepsilon}(N)} \right) \quad (2.4)$$

Noting that $K_N \to \infty$ as $N \to \infty$ and that

$$\lim_{N \to \infty} \frac{\log^{1+\varepsilon} \left( \log^{1+\varepsilon}(N) \right)}{\log^{1+\varepsilon}(N)} = \lim_{N \to \infty} (1 + \varepsilon)^{1+\varepsilon} \left[ \frac{\log(\log(N))}{\log(N)} \right]^{1+\varepsilon} = 0,$$

for sufficiently large $N$

$$\sum_{i=1}^{K_N} \log^{1+\varepsilon}(i) \geq 2N \cdot \frac{K_N - K_N^{1/2}}{K_N} \cdot \frac{\log^{1+\varepsilon} \left( \frac{2^\varepsilon N}{\log^{1+\varepsilon}(N)} \right)}{\log^{1+\varepsilon}(N)} > N \quad (2.5)$$

which implies $C_N < K_N$ and finishes the proof of the upper bound. On the other hand, note that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq \int_{1}^{k+1} \log^{1+\varepsilon}(x)dx \leq k \log^{1+\varepsilon}(k+1).$$
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So for any \( k \leq 2^{-1-\varepsilon}N \log^{-1-\varepsilon}(N) \),

\[
\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq k \log^{1+\varepsilon}(k + 1) \leq 2^{-1-\varepsilon}N \frac{\log^{1+\varepsilon}(2^{-1-\varepsilon}N \log^{-1-\varepsilon}(N) + 1)}{\log^{1+\varepsilon}(N)} < N.
\]

Thus we have shown the lower bound and the proof of Lemma 2.1 is complete. \( \square \)

Next using Lemma 2.1 we can show that \( \tilde{D}_\infty \) is transient for 3 dimensional simple random walk:

**Lemma 2.2.** For 3 dimensional simple random walk \( \{X_{3,n}\}_{n=0}^\infty \), \( \tilde{D}_\infty \) is a transient subset.

**Proof.** According to Wiener’s test (see Corollary 6.5.9 of [3]), it is sufficient to show that

\[
\sum_{k=1}^\infty 2^{-k} \text{cap}(A_k) < \infty \quad (2.6)
\]

where \( A_k = \tilde{D}_\infty \cap [B_2(0, 2^k) \setminus B_2(0, 2^{k-1})] \). Then according to the definition of capacity (see Section 6.5 of [3]), we have for all \( k \geq 1 \)

\[
\text{cap}(A_k) \leq |A_k| \leq |\tilde{D}_\infty \cap B_2(0, 2^k)| \leq |\tilde{D}_\infty| = C_{2^k}. \quad (2.7)
\]

By Lemma 2.1,

\[
\text{cap}(A_k) \leq C_{2^k} \leq \frac{C\varepsilon}{\log^{1+\varepsilon}(2)} \frac{2^k}{k^{1+\varepsilon}}. \quad (2.8)
\]

Thus we have

\[
\sum_{k=1}^\infty 2^{-k} \text{cap}(A_k) \leq \frac{C\varepsilon}{\log^{1+\varepsilon}(2)} \sum_{k=1}^\infty \frac{1}{k^{1+\varepsilon}} < \infty
\]

which implies that \( \tilde{D}_\infty \) is transient. \( \square \)

### 3 Uniform upper bound on returning probability

Now we have \( \tilde{D}_\infty \) is transient, i.e.,

\[
P \left( X_n \notin \tilde{D}_\infty \text{ i.o.} \right) = 0,
\]

which immediately implies that there must be some \( \bar{x} \in Z^3 \setminus \tilde{D}_\infty \) such that

\[
P_{\bar{x}}(T_{\tilde{D}_\infty} < \infty) < 1,
\]

where \( T_{\tilde{D}_\infty} \) is the first time a simple random walk visits \( \tilde{D}_\infty \), and \( P_{\bar{x}}(\cdot) \) is the distribution of the simple random walk conditioned on starting at \( \bar{x} \). Then note that \( \tilde{D}_\infty \) is a subset of the diagonal line, which implies \( \tilde{D}_\infty \) has no interior point while \( Z^3 \setminus \tilde{D}_\infty \) is connected. Thus for any \( x_k \in \tilde{D}_\infty \), there exists a nearest neighbor path

\[
Y = \{y_0, y_1, \ldots, y_m\}
\]

with \( y_0 = x_k, y_m = \bar{x} \) while \( y_i \in Z^3 \setminus \tilde{D}_\infty \), for all \( i = 1, 2, \ldots, m - 1 \). Combining this with the fact that

\[
P_{\bar{x}}(T_{\tilde{D}_\infty} < \infty) = \frac{1}{6} \sum_{i=1}^{3} \left[ P_{\bar{x} + e_i}(T_{\tilde{D}_\infty} < \infty) + P_{\bar{x} - e_i}(T_{\tilde{D}_\infty} < \infty) \right]
\]

ECP 23 (2018), paper 57. http://www.imstat.org/ecp/
for all $x \in \mathbb{Z}^3 \setminus \tilde{D}_\infty$, we have
$$P_{x_0}(T_{\tilde{D}_\infty} < \infty) < 1,$$
for all $i \geq 1$, which in turns implies that
$$P_{x_k}(\bar{T}_{\tilde{D}_\infty} < \infty) < 1$$
for all $k$, where $\bar{T}_{\tilde{D}_\infty}$ is the first returning time, i.e. the stopping time a simple random walk first visits $\tilde{D}_\infty$ after its first step.

However, in order to use the transient set $\tilde{D}_\infty$ in our proof, (3.2) is not enough. We need to show that starting from each point $x_k = (n_k, n_k, n_k) \in \tilde{D}_\infty$, the probability $P_{x_k}(\bar{T}_{\tilde{D}_\infty} < \infty)$ is uniformly bounded away from 1. And this is not generally true for all transient subsets $A$. First of all, when $A$ has interior points, the return probability of those points are certainly one. And even if $A$ has no interior point and $\mathbb{Z}^3 \setminus A$ is connected, we have the following counter example:

**Counterexample 1:** Consider subsets
$$A_k = \{(2^k, 1, n), (2^k, -1, n), (2^k + 1, 0, n), (2^k - 1, 0, n)\}_{n=-k}^{k} \cup \{(2^k, 0, 0)\}$$
and
$$A = \bigcup_{k=1}^{\infty} A_k$$
where the 2 dimensional projection of $A$ is illustrated in Figure 1 (the distances between $A_k$’s are not exact in the figure):

![Figure 1: A counter example to uniform upper bound on returning probability](image)

Using Wiener’s test, it is easy to see $A$ is a transient subset. However, for points $a_k = (2^k, 0, 0) \in A$, $k \geq 1$, in order to have a simple random walk starting at $a_k$ never returns to $A$, we must have the first $k$ steps of the random walk be along the $z$–coordinate. Thus
$$P_{a_k}(T_A = \infty) < \frac{1}{3^k},$$
which implies that
$$\lim_{k \to \infty} P_{a_k}(T_A < \infty) \geq \lim_{k \to \infty} \left(1 - \frac{1}{3^k}\right) = 1.$$
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Proof. With (3.2) showing all returning probabilities are strictly less than 1, it is sufficient for us to show that

\[ \lim_{k \to \infty} \sup_{x \in \mathbb{R}} P_{x_k}(\bar{T}_{\bar{D}} < \infty) < 1. \]  

(3.4)

Actually, here we prove a stronger statement

\[ \lim_{k \to \infty} P_{x_k}(\bar{T}_{\bar{D}} < \infty) = P_0(T_0 < \infty) < 1. \]  

(3.5)

Note that for each \( k \)

\[ P_{x_k}(\bar{T}_{\bar{D}} < \infty) = P_0(T_0 < \infty), \]

and that

\[ P_{x_k}(\bar{T}_{\bar{D}} < \infty) \leq P_{x_k}(\bar{T}_{\bar{x}_k} < \infty) + P_{x_k}(T_{\bar{D}} - \bar{x}_k) < \infty), \]

and that

\[ P_{x_k}(\bar{T}_{\bar{D}} - \bar{x}_k) < \infty) \leq \sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) + \sum_{i=k+1}^{\infty} P_{x_k}(T_{x_i} < \infty). \]

It suffices for us to show that

\[ \lim_{k \to \infty} \sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) = 0, \]  

(3.6)

and that

\[ \lim_{k \to \infty} \sum_{i=k+1}^{\infty} P_{x_k}(T_{x_i} < \infty) = 0. \]  

(3.7)

To show (3.6) and (3.7), we first note the well known result that there is a \( C < \infty \) such that for any \( x \neq y \in \mathbb{Z}^3 \),

\[ P_x(T_y < \infty) \leq \frac{C}{|x - y|}. \]

First, to show (3.6) recall that \( x_k = (n_k, n_k, n_k) \), which implies that for any \( i \) and \( k \), \( |x_{k,i} - x_{i-1}| \geq |n_k - n_{i-1}| \). We have according to (2.2), for any \( k \geq 8 \)

\[ \sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) \leq \sum_{i=1}^{k-1} \frac{C}{|x_{k,i} - x_{i-1}|} \leq 2C \sum_{i=1}^{k-1} \int_{i_{i-1}}^{i} \frac{1}{\log^{1+\varepsilon}(x)} dx. \]  

(3.8)

Thus it is again sufficient to show that

\[ \lim_{k \to \infty} \sum_{i=1}^{k-1} \frac{1}{\int_{i_{i-1}}^{i} \log^{1+\varepsilon}(x)} dx = 0. \]  

(3.9)

Note that

\[ \sum_{i=1}^{k} \frac{1}{\int_{i_{i-1}}^{i} \log^{1+\varepsilon}(x)} dx = \sum_{i=1}^{[k/2]} \frac{1}{\int_{i_{i-1}}^{i} \log^{1+\varepsilon}(x)} dx + \sum_{i= [k/2]}^{k-1} \frac{1}{\int_{i_{i-1}}^{i} \log^{1+\varepsilon}(x)} dx. \]  

(3.10)

For each \( k \geq 8 \) and \( i \leq [k/2] \), we have

\[ \int_{i}^{k_{i/2}} \log^{1+\varepsilon}(x) dx \geq \int_{k/2}^{k_{i/2}} \log^{1+\varepsilon}(x) dx \geq \int_{k/2}^{k} 1 dx = k/2. \]

Thus

\[ \sum_{i=1}^{[k/2]} \frac{2}{k} \leq \frac{k}{k} = o(1). \]  

(3.11)
Then for each \( k \geq 8 \) and \( i \in \left[ \left\lceil k^{1/2} \right\rceil, k - 1 \right] \),
\[
\int_i^k \log^{1+\varepsilon}(x)dx \geq \int_i^k \log^{1+\varepsilon}(k^{1/2})dx = \frac{1}{2(k-1)} \log^{1+\varepsilon}(k).
\]

Thus
\[
\sum_{i=\left\lceil k^{1/2} \right\rceil}^{k-1} \int_i^k \log^{1+\varepsilon}(x)dx \leq \frac{2^{1+\varepsilon}}{\log^{1+\varepsilon}(k)} \sum_{i=1}^k \frac{1}{i}.
\]
(3.12)

Noting that
\[
\sum_{i=1}^k \frac{1}{i} \leq 1 + \int_1^k \frac{1}{x}dx = 1 + \log(k)
\]

one can immediately have
\[
\sum_{i=\left\lceil k^{1/2} \right\rceil}^{k-1} \int_i^k \log^{1+\varepsilon}(x)dx \leq \frac{2^{1+\varepsilon} \log(k) \log^{1+\varepsilon}(k)}{1+\log(k)} = o(1).
\]
(3.13)

Combining (3.9), (3.11) and (3.13), we obtain (3.6).

Then, to show (3.7) we have according to (2.2), for any \( k \geq 8 \)
\[
\sum_{i=k+1}^\infty P_{k+1}(T_{x_i} < \infty) \leq \sum_{i=k+1}^\infty \frac{C}{|x_i - x_k|} \leq 2C \sum_{i=k+1}^\infty \int_i^k \log^{1+\varepsilon}(x)dx.
\]
(3.14)

Thus it is again sufficient to show that
\[
\lim_{k \to \infty} \sum_{i=k+1}^\infty \int_i^k \log^{1+\varepsilon}(x)dx = 0.
\]
(3.15)

Now for each \( k \) we separate the infinite summation in (3.15) as
\[
\sum_{i=k+1}^\infty \int_i^k \log^{1+\varepsilon}(x)dx = \sum_{i=k+1}^{k^2} \int_k^i \log^{1+\varepsilon}(x)dx + \sum_{i=k^2+1}^\infty \int_k^i \log^{1+\varepsilon}(x)dx.
\]
(3.16)

For its first term we use similar calculation as in (3.12) and have
\[
\sum_{i=k+1}^{k^2} \int_k^i \log^{1+\varepsilon}(x)dx \leq \frac{1}{\log^{1+\varepsilon}(k)} \sum_{i=1}^{k^2} \frac{1}{i-k} \leq \frac{1}{\log^{1+\varepsilon}(k)} \sum_{i=1}^{k^2} \frac{1}{i}.
\]
(3.17)

And since
\[
\sum_{i=1}^{k^2} \frac{1}{i} \leq 1 + \int_1^{k^2} \frac{1}{x}dx = 1 + 2 \log(k)
\]
we have
\[
\sum_{i=k+1}^{k^2} \int_k^i \log^{1+\varepsilon}(x)dx \leq \frac{1 + 2 \log(k)}{\log^{1+\varepsilon}(k)} = o(1).
\]
(3.18)

At last for the second term in (3.16), we have for each \( k \geq 8 \) and \( i \geq k^2 + 1 \),
\[
\int_k^i \log^{1+\varepsilon}(x)dx \geq \int_k^{i/2} \log^{1+\varepsilon}(x)dx \geq (i - i^{1/2}) \log^{1+\varepsilon}(i^{1/2}) \geq \frac{1}{2^{2+\varepsilon}} \log^{1+\varepsilon}(i).
\]

Thus
\[
\sum_{i=k^2+1}^\infty \int_k^i \log^{1+\varepsilon}(x)dx \leq 2^{2+\varepsilon} \sum_{i=k^2+1}^\infty \frac{1}{i} \log^{1+\varepsilon}(i).
\]
(3.19)
Finally, noting that
\[ \sum_{i=3}^{\infty} \frac{1}{i \log^{1+\varepsilon}(i)} \leq \int_{2}^{\infty} \frac{1}{x \log^{1+\varepsilon}(x)} \, dx = \frac{1}{\varepsilon \log(2)} < \infty, \]
we have the tail term
\[ \sum_{i=k^2+1}^{\infty} \frac{1}{i \log^{1+\varepsilon}(i)} = o(1) \quad (3.20) \]
as \( k \to \infty \). Thus combining (3.15)- (3.20), we have shown (3.7) and thus finished the proof of this lemma.

Proof of Theorem 1.3. With Lemma 3.2, and recalling that
\[ \tilde{D}_N = \tilde{D}_\infty \cap D_N \]
and
\[ C_N = |\tilde{D}_N| = \sup \{ k : n_k \leq N \}, \]
we can define the stopping times \( \tilde{T}_{\tilde{D}_{[N/3]};0} = 0 \),
\[ \tilde{T}_{\tilde{D}_{[N/3];1}} = \inf \{ n > 0, X_{3,n} \in \tilde{D}_{[N/3]} \} \]
and for all \( k \geq 2 \)
\[ \tilde{T}_{\tilde{D}_{[N/3];k}} = \inf \{ n > \tilde{T}_{\tilde{D}_{[N/3];k-1}}, X_{3,n} \in \tilde{D}_{[N/3]} \} \].

Then by Lemma 3.2, one can immediately see that for any \( k \geq 0 \)
\[ P\left( T_{\tilde{D}_{[N/3];k+1}} < \infty \big| \tilde{T}_{\tilde{D}_{[N/3];k}} < \infty \right) \leq P_{X_{3,\tilde{T}_{\tilde{D}_{[N/3];k}}}} (\tilde{D}_{\tilde{D}_\infty} < \infty) \leq 1 - c_{\varepsilon,1}, \]
and thus
\[ P\left( T_{\tilde{D}_{[N/3];k+1}} < \infty \big| \tilde{T}_{\tilde{D}_{[N/3];k}} < \infty \right) \leq (1 - c_{\varepsilon,1})^{C_{[N/3]}}. \]

By Lemma 2.1 we have
\[ C_{[N/3]} \geq 2^{-\varepsilon-1} [N/3] \log^{-1-\varepsilon}([N/3]) \geq \frac{2^{-\varepsilon-2}}{3} N \log^{-1-\varepsilon}(N) \quad (3.22) \]
for all \( N \geq 4 \). Thus combining (3.21) and (3.22)
\[ P\left( \tilde{P} \subseteq \text{Trace}\left( \{ X_{3,n} \}_{n=0}^{\infty} \right) \right) \leq P\left( D_{[N/3]} \subseteq \text{Trace}\left( \{ X_{3,n} \}_{n=0}^{\infty} \right) \right) \leq P\left( D_{[N/3]} \subseteq \text{Trace}\left( \{ X_{3,n} \}_{n=0}^{\infty} \right) \right) \leq P\left( T_{\tilde{D}_{[N/3];C_{[N/3]}}} < \infty \right) \leq \exp\left( -c_{\varepsilon,N} \log^{-1-\varepsilon}(N) \right) \]
where \( c_{\varepsilon} = -\frac{2^{-\varepsilon-2}}{3} \log(1 - c_{\varepsilon,1}) \). And the proof of Theorem 1.3 is complete. \( \square \)
4 Discussions

In Conjecture 1.4, we conjecture that the cover probability should have exponential decay just as the $d \geq 4$ case. This conjecture seems to be supported by the following preliminary simulation which shows the log-plot of probabilities that the first 5000 steps of a 3 dimensional simple random walk starting at 0 cover $D_i = \{(0,0,0), (1,1,1), \cdots, (i,i,i)\}$ for $i = 1, 2, \cdots, 9$.

The simulation result above seems to indicate that after taking logarithm, the covering probability decays almost exactly as a linear function, which implies the exponential decay we predicted, indicating that the upper bound we found in Theorem 1.3 is not sharp.

Another possible approach towards a sharp asymptotic is noting that although $\{\hat{X}_{2,n}\}_{n=0}^\infty$ is recurrent and will return to 0 with probability 1, the expected time between each two successive returns is $\infty$. Moreover, in order to cover $\mathcal{P}$, only those returns to diagonal before that $\{X_{3,n}\}_{n=0}^\infty$ has left $B_2(0,N) \supset B_1(0,N)$ forever could possibly help. This observation, together with the tail probability asymptotic estimations using local central limit theorem and techniques in [1] and [2] applied on the non simple random walk $\{\hat{X}_{2,n}\}_{n=0}^\infty$, and some large deviation argument, enable us to find a proper value of $T$ such that

- with high probability $\{X_{3,n}\}_{n=T}^\infty \cap B_2(0,N) = \emptyset$,
- with high probability $\{\hat{X}_{2,n}\}_{n=0}^T$ will not return to 0 for $\lceil N/3 \rceil$ times or more.

Right now this approach can only give us the following weaker upper bound (a detailed proof can be found in technical report [4]):

**Proposition 4.1.** There are $c, C \in (0, \infty)$ such that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3$ connecting 0 and $\partial B_1(0,N)$,

$$P(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty)) \leq C \exp\left(-cN^{1/3}\right).$$

However, this seemingly worse approach might have the potential to fully use the geometric structure of path $\mathcal{P}$ to minimize the covering probability. Note that in order to cover $D_{\lfloor N/3 \rfloor}$ we not only need $\{\hat{X}_{2,n}\}_{n=0}^\infty$ to return to 0 for at least $\lfloor N/3 \rfloor$ times before leaving $B_2(0,N)$, but also must have that the locations of $X_{3,n}$ at such visits cover each
point on the diagonal. I.e., define the stopping times \( \tau_{3,0} = 0 \)
\[
\tau_{3,1} = \inf\{n \geq 1 : \hat{X}_{2,n} = 0\}
\]
and for all \( i \geq 2 \)
\[
\tau_{3,i} = \inf\{n > \tau_{3,i-1} : \hat{X}_{2,n} = 0\}.
\]
Define
\[
\{Z_{3,n}\}_{n=0}^{\infty} = \left\{X_{3,\tau_{3,n}}^{1} + X_{3,\tau_{3,n}}^{2} + X_{3,\tau_{3,n}}^{3}\right\}_{n=0}^{\infty}.
\]
Noting that \( \tau_{3,i} < \infty \) for any \( i \), and that \( \{X_{3,n}\}_{n=0}^{\infty} \) is translation invariant, \( \{Z_{3,n}\}_{n=0}^{\infty} \) is a well defined one dimensional random walk with infinite range. And we have
\[
P\left(\text{Trace}(P) \subseteq \text{Trace}\left(\{X_{3,n}\}_{n=0}^{\infty}\right)\right) \leq P\left(\left(0, 1, \cdots, \lfloor N/3 \rfloor\right) \subseteq \text{Trace}\left(\{Z_{3,n}\}_{n=0}^{\infty}\right)\right).
\]
Thus Conjecture 1.4 would follow from the techniques described above for Proposition 4.1 if the following conjecture is proved.

**Conjecture 4.2.** There is a \( c \in (0, \infty) \) such that for any \( N \geq 2 \)
\[
P\left(\left(0, 1, \cdots, \lfloor N/3 \rfloor\right) \subseteq \text{Trace}\left(\{Z_{3,n}\}_{n=0}^{\lfloor N/3 \rfloor}\right)\right) \leq \exp(-cN).
\]

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