Opial’s inequality in $q$-Calculus revisited

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Abstract
We have fundamentally corrected the proofs of the theorems from our paper [9] by giving an entirely different approach, using quite a simple method based on applications of some elementary inequalities, well-known Hölder’s inequality and the Gauchman $q$-restricted integral.

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1 Introduction and preliminaries
In the recent paper [9] a generalization of the Opial integral inequality
\[ \int_0^h |f(x)f'(x)|\,dx \leq \frac{h}{4} \int_0^h (f'(x))^2\,dx \tag{1} \]
in $q$-calculus was given. Here we eliminate some inaccuracies by simplifying and modifying the proofs of the theorems.

First of all, we present necessary definitions and facts from the $q$-calculus, where $q$ is a real number satisfying $0 < q < 1$, and $q$-natural number is defined by
\[ [n]_q = \frac{1 - q^n}{1 - q} = q^{n-1} + \cdots + q + 1, \ldots, n \in \mathbb{N}. \]

Definition 1.1. Let $f$ be a function defined on an interval $(a, b) \subset \mathbb{R}$, so that $qx \in (a, b)$ for all $x \in (a, b)$. For $0 < q < 1$, we define the $q$-derivative as
\[ (D_qf)(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0; \quad D_qf(0) = \lim_{x \to 0} D_qf(x). \tag{2} \]

In the paper [8], Jackson defined $q$-integral, which in the $q$-calculus bears his name.

Definition 1.2. The $q$-integral on $[0, a]$ is
\[ \int_0^a f(x)d_qx = a(1 - q) \sum_{j=0}^{\infty} q^j f(aq^j). \]
On this basis, in the same paper, Jackson defined an integral on \([a, b]\)

\[
\int_a^b f(x) \, dq \, x = \int_0^b f(x) \, dq \, x - \int_0^a f(x) \, dq \, x,
\]  

(3)

For a positive integer \(n\) and \(a = bq^n\), using the left-hand side integral of \(3\), in the paper \[7\], Gauchman introduced the \(q\)-restricted integral

\[
\int_a^b f(x) \, dq \, x = \int_{bq^n}^b f(x) \, dq \, x = b(1 - q) \sum_{j=0}^{n-1} q^j f(q^j b).
\]  

(4)

**Definition 1.3.** The real function \(f\) defined on \([a, b]\) is called \(q\)-increasing (\(q\)-decreasing) on \([a, b]\) iff \(qx \leq f(x)\) \((f(qx) \geq f(x))\) for \(x, qx \in [a, b]\).

It is easy to see that if the function \(f\) is increasing (decreasing), then it is \(q\)-increasing (\(q\)-decreasing) too.

## 2 Results and discussions

Our main results are contained in three theorems.

**Theorem 2.1.** Let \(f(x)\) be \(q\)-decreasing function on \([a, b]\) with \(f(bq^n) = 0\). Then, for any \(p \geq 0\), there holds

\[
\int_a^b |D_q f(x)| |f(x)|^p \, dq \, x \leq (b - a)^p \int_a^b |D_q f(x)|^{p+1} \, dq \, x.
\]  

(5)

**Proof.** Using Definition 1.1 and 4, we have

\[
\int_a^b |D_q f(x)||f(x)|^p \, dq \, x = \int_{bq^n}^b \left| \frac{f(x) - f(qx)}{x - qx} \right| |f(x)|^p \, dq \, x
\]

\[
= b(1 - q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| |f(bq^j)|^p,
\]

whence, taking into account that \(f(x)\) is \(q\)-decreasing, we have

\[
\sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})||f(bq^j)|^p \leq |f(bq^n)|^p \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})|.
\]

In view of \(f(bq^n) = \sum_{j=0}^{n-1} f(bq^{j+1}) - f(bq^j)\), we obtain

\[
|f(bq^n)|^p = \left( \sum_{j=0}^{n-1} |f(bq^{j+1}) - f(bq^j)| \right)^p \leq \left( \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})| \right)^p,
\]
so that
\[ |f(bq^n)|^p \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})| \leq \left( \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})| \right)^{p+1}. \]

Thus
\[ \int_a^b |D_q f(x)||f(x)|^p d_q x \leq \left( \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})| \right)^{p+1}. \] (6)

The right-hand side of this inequality we can write in the form of
\[ \left( \sum_{j=0}^{n-1} |f(bq^j) - f(bq^{j+1})| \right)^{p+1} = \left( \sum_{j=0}^{n-1} |bq^j - bq^{j+1}| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right)^{p+1} \]

After rewriting \( q^j = (q^j)^{\frac{p}{p+1}} (q^j)^{\frac{1}{p+1}} \), and applying Hölder’s inequality to the last sum, we have
\[ \sum_{j=0}^{n-1} (q^j)^{\frac{p}{p+1}} (q^j)^{\frac{1}{p+1}} \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| \leq \left( \sum_{j=0}^{n-1} (q^j)^{\frac{p}{p+1}} \right)^{\frac{p}{p+1}} \times \left( \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{p+1} \right)^{\frac{1}{p+1}}. \]

After raising both sides to the power \( p + 1 \), we find
\[ \left( \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| \right)^{p+1} \leq \left( \sum_{j=0}^{n-1} q^j \right)^p \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{p+1}. \]

Multiplying this inequality by \( b^{p+1}(1-q)^{p+1} \), and relying on the formula for the sum of the first \( n \) terms of the geometric series, we arrive at the inequality
\[ \left( b(1-q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| \right)^{p+1} \leq b^p(1-q)^p \times \]
\[ b(1-q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{p+1}. \] (7)

Considering that \( b^p(1-q)^p = (b - bq^n)^p = (b-a)^p \), taking into account (6), we have proved the inequality
\[ \int_a^b |D_q f(x)||f(x)|^p d_q x \leq (b-a)^p \left( b(1-q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{p+1} \right). \]
Referring to (4), there holds
\[ b(1 - q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{p+1} = \int_a^b |D_q f(x)|^{p+1} d_q x, \]
whereby we prove the theorem.

**Remark 2.2.** In particular, by taking \( p = 1 \), the inequality (5) in Theorem 2.1 reduces to the following Opial’s inequality in \( q \)-Calculus.
\[ \int_a^b |D_q f(x)||f(x)| d_q x \leq (b - a) \int_a^b |D_q f(x)|^2 d_q x. \]

The following theorems are concerned with \( q \)-monotonic functions.

**Theorem 2.3.** If \( f(x) \) and \( g(x) \) are \( q \)-decreasing functions on \([a, b]\) satisfying \( f(bq^0) = 0 \) and \( g(bq^0) = 0 \), then there holds the inequality
\[ \int_a^b \left( f(x)D_q g(x) + g(qx)D_q f(x) \right) d_q x \leq \frac{b - a}{2} \int_a^b (|D_q f(x)| + |D_q g(x)|)^2 d_q x. \quad (8) \]

**Proof.** Replacing (2) in the integral
\[ \int_a^b \left( f(x)D_q g(x) + g(qx)D_q f(x) \right) d_q x, \]
we obtain
\[ \int_{bq^0}^b \left( f(x) \frac{g(x) - g(qx)}{x - qx} + g(qx) \frac{f(x) - f(qx)}{x - qx} \right) d_q x, \]
whence, using Gauchman \( q \)-restricted integral, we have
\[ b(1 - q) \left( \sum_{j=0}^{n-1} q^j f(bq^j) \frac{g(bq^j) - g(bq^{j+1})}{bq^j - bq^{j+1}} + \sum_{j=0}^{n-1} q^j g(bq^{j+1}) \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right) \]
\[ = \sum_{j=0}^{n-1} \left( f(bq^j)g(bq^j) - g(bq^{j+1})f(bq^j) \right) + g(bq^{j+1})f(bq^j) - f(bq^{j+1})g(bq^j)) \]
\[ = \sum_{j=0}^{n-1} \left( f(bq^j)g(bq^j) - g(bq^{j+1})f(bq^j) \right) = -f(bq^n)g(bq^n). \]

Using the elementary inequality \(-xy \leq \frac{1}{2}(x^2 + y^2), \quad x, y \in \mathbb{R}\), and considering that
\[ f(bq^n) = \sum_{j=0}^{n-1} \left( f(bq^{j+1}) - f(bq^j) \right), \quad g(bq^n) = \sum_{j=0}^{n-1} \left( g(bq^{j+1}) - g(bq^j) \right), \]

\[ \frac{b - a}{2} \int_a^b (|D_q f(x)| + |D_q g(x)|)^2 d_q x. \]

\[ \int_a^b (|D_q f(x)| + |D_q g(x)|)^2 d_q x. \]

\[ \frac{b - a}{2} \int_a^b (|D_q f(x)| + |D_q g(x)|)^2 d_q x. \]
we find
\[-f(bq^n)g(bq^n) \leq \frac{1}{2} \left( \left( \sum_{j=0}^{n-1} (f(bq^{j+1}) - f(bq^j)) \right)^2 + \left( \sum_{j=0}^{n-1} (g(bq^{j+1}) - g(bq^j)) \right)^2 \right)\]

Applying (7) for \( p = 1 \), knowing that \( f(x) \) and \( g(x) \) are \( q \)-decreasing, we obtain
\[
\left( b(1-q) \sum_{j=0}^{n-1} q^j \frac{f(bq^{j+1}) - f(bq^j)}{bq^j - bq^{j+1}} \right)^2 \leq b(1-q^n)b(1-q) \sum_{j=0}^{n-1} q^j \left( \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right)^2
\]
as well as
\[
\left( b(1-q) \sum_{j=0}^{n-1} q^j \frac{g(bq^{j+1}) - g(bq^j)}{bq^j - bq^{j+1}} \right)^2 \leq b(1-q^n)b(1-q) \sum_{j=0}^{n-1} q^j \left( \frac{g(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right)^2.
\]
Since \( b(1-q^n) = b - a \), making use of (4), we have
\[
\int_a^b (f(x)D_q g(x) + g(qx)D_q f(x)) \, dq \leq \frac{b-a}{2} \int_a^b ((D_q f(x))^2 + (D_q g(x))^2) \, dq,
\]
whereby (4) is proved. \( \square \)

**Theorem 2.4.** If \( f(x) \) and \( g(x) \) are \( q \)-decreasing functions on \([a, b] \) satisfying \( f(bq^0) = g(bq^0) = 0 \), then there holds the inequality
\[
\int_a^b |f(x)|^s |g(x)|^t \, dq \leq (b-a)^{s+t} \int_a^b \left( \frac{s}{s+t} |D_q f(x)|^{s+t} \, dq + \frac{t}{s+t} |D_q g(x)|^{s+t} \, dq \right). \tag{9}
\]

**Proof.** First, we apply (4) to the left-hand side of (4), and have
\[
\int_a^b |f(x)|^s |g(x)|^t \, dq = b(1-q) \sum_{i=0}^{n-1} q^i |f(bq^i)|^s |g(bq^i)|^t.
\]
For real numbers \( z, w \geq 0 \) and \( s, t > 0 \), we rely on the elementary inequality
\[
z^s w^t \leq \frac{s}{s+t} z^{s+t} + \frac{t}{s+t} w^{s+t}.
\]
After setting \( z = (q^j)^{\frac{s}{s+t}} |f(bq^j)|, w = (q^j)^{\frac{t}{s+t}} |g(bq^j)| \), we find
\[
\sum_{i=0}^{n-1} q^i |f(bq^i)|^s |g(bq^i)|^t \leq \sum_{i=0}^{n-1} \left( (q^i)^{\frac{s}{s+t}} |f(bq^i)| \right)^s \left( (q^i)^{\frac{t}{s+t}} |g(bq^i)| \right)^t \leq \frac{s}{s+t} \sum_{i=0}^{n-1} q^i |f(bq^i)|^s + \frac{t}{s+t} \sum_{i=0}^{n-1} q^i |g(bq^i)|^s t.
\]

5
Considering that \( f \) and \( g \) are \( q \)-decreasing functions, so \( |f(bq^i)|^{s+t} \leq |f(bq^n)|^{s+t} \) and \( |g(bq^i)|^{s+t} \leq |g(bq^n)|^{s+t} \), the last inequality becomes

\[
\sum_{i=0}^{n-1} q^i |f(bq^i)|^s |g(bq^i)|^t \leq \frac{1-q^n}{1-q} \left( \frac{s}{s+t} |f(bq^n)|^{s+t} + \frac{t}{s+t} |g(bq^n)|^{s+t} \right).
\]

However, there holds

\[
|f(bq^n)|^{s+t} = \left\| \sum_{i=0}^{n-1} f(bq^{i+1}) - f(bq^i) \right\|^{s+t} \leq \left( \sum_{i=0}^{n-1} |f(bq^{i+1}) - f(bq^i)| \right)^{s+t},
\]

\[
|g(bq^n)|^{s+t} = \left\| \sum_{i=0}^{n-1} g(bq^{i+1}) - g(bq^i) \right\|^{s+t} \leq \left( \sum_{i=0}^{n-1} |g(bq^{i+1}) - g(bq^i)| \right)^{s+t},
\]

so that we have

\[
\int_a^b |f(x)|^s |g(x)|^t \, dx = b(1-q) \sum_{i=0}^{n-1} q^i |f(bq^i)|^s |g(bq^i)|^t
\]

\[
\leq b(1-q^n) \left( \frac{s}{s+t} \left( \sum_{i=0}^{n-1} |f(bq^{i+1}) - f(bq^i)| \right)^{s+t} + \frac{t}{s+t} \left( \sum_{i=0}^{n-1} |g(bq^{i+1}) - g(bq^i)| \right)^{s+t} \right) \leq b(1-q^n) \left( b(1-q) \sum_{i=0}^{n-1} q^i \left| \frac{f(bq^i) - f(bq^{i+1})}{bq^i - bq^{i+1}} \right|^{s+t} + t \left( b(1-q) \sum_{i=0}^{n-1} q^i \left| \frac{g(bq^i) - g(bq^{i+1})}{bq^i - bq^{i+1}} \right|^{s+t} \right) \right),
\]

(10)

Here we follow the same procedure as in the proof of Theorem 2.1. So, after rewriting \( q^i = (q^i)^{\frac{1}{s+t}} (q^i)^{\frac{s}{s+t}} \), and applying Hölder’s inequality to both sums on the right side of the last inequality, for the first sum we have

\[
\sum_{j=0}^{n-1} (q^j)^{\frac{s}{s+t}} (q^j)^{\frac{s}{s+t}} \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| \leq \left( \sum_{j=0}^{n-1} \left( (q^j)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \left( \sum_{j=0}^{n-1} \left( (q^j)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \times
\]

\[
\left( \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{s+t} \right)^{\frac{s}{s+t}},
\]

and for the second as well

\[
\sum_{j=0}^{n-1} (q^j)^{\frac{s}{s+t}} (q^j)^{\frac{s}{s+t}} \left| \frac{g(bq^j) - g(bq^{j+1})}{bq^j - bq^{j+1}} \right| \leq \left( \sum_{j=0}^{n-1} \left( (q^j)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \left( \sum_{j=0}^{n-1} \left( (q^j)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \right)^{\frac{s}{s+t}} \times
\]

\[
\left( \sum_{j=0}^{n-1} q^j \left| \frac{g(bq^j) - g(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{s+t} \right)^{\frac{s}{s+t}},
\]
We multiply both inequalities by \( b(1 - q) \), then raise them to the power \( s + t \). Thus, we obtain

\[
\left( b(1 - q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right| \right)^{s+t} \leq b^{s+t-1}(1 - q^n)^{s+t-1} \times
\]

\[
b(1-q) \sum_{j=0}^{n-1} q^j \left| \frac{f(bq^j) - f(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{s+t} = b^{s+t-1}(1 - q^n)^{s+t-1} \int_a^b |D_qf(x)|^{s+t} dq_x,
\]

and similarly

\[
\left( b(1 - q) \sum_{j=0}^{n-1} q^j \left| \frac{g(bq^j) - g(bq^{j+1})}{bq^j - bq^{j+1}} \right| \right)^{s+t} \leq b^{s+t-1}(1 - q^n)^{s+t-1} \times
\]

\[
b(1-q) \sum_{j=0}^{n-1} q^j \left| \frac{g(bq^j) - g(bq^{j+1})}{bq^j - bq^{j+1}} \right|^{s+t} = b^{s+t-1}(1 - q^n)^{s+t-1} \int_a^b |D_qg(x)|^{s+t} dq_x,
\]

so that, because \( b(1 - q^n) = b - a \), from (10) there follows (9), whereby we complete the proof.

\[\square\]

**Remark 2.5.** In the special case when \( s = t = r \) and \( f(x) = g(x) = h(x) \), the inequality established in (9) reduces to the \( q \)-Wirtinger-type inequality

\[
\int_a^b |h(x)|^{2r} dq_x \leq (b-a)^{2r} \int_a^b (D_qh(x))^{2r} dq_x.
\]

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