The complex pre-potential and the Aharonov–Bohm effect

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Abstract

The Aharonov–Bohm (AB) effect is traditionally attributed to the affect of the electromagnetic 4-potential $A$, even in regions where both the electric field $E$ and the magnetic field $B$ are zero. We argue that the quantity measured by AB experiments may be the difference in values of a multiple-valued complex function, which we call a pre-potential. The pre-potential is a combination of the two scalar potential functions introduced by Whittaker. We show that any electromagnetic field can be described by such a pre-potential, and give an explicit expression for the electromagnetic field tensor through this potential. The covariance of the pre-potential is explained. A program for deriving the AB effect from the pre-potential is proposed.

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1. Introduction

In their seminal paper [1], Aharonov and Bohm (AB) claimed that, contrary to the conclusions of classical mechanics, the electromagnetic 4-potential $A$ affects the motion of an electron beam, even in regions where the electromagnetic (EM) field vanishes. They proposed two kinds of experiments which were successfully performed later (see, for instance,[2, 3]).

In [1], a scalar function $S$ such that $\nabla S = (e\hbar/c)A$ was introduced. The function $S$ may be called a pre-potential. It has been shown that if $\psi_0$ is the solution of the Schrödinger equation in the absence of an EM field, then the function $\psi = \psi_0 e^{-iS/\hbar}$ is the solution of the equation in the presence of the field, at least in a simply connected region in which the EM field vanishes. In the magnetic AB experiment, however, the region outside the solenoid is not simply connected. Thus, they consider separately the wavefunction $\psi_i$ of each of the two beams. Since the domain of each $\psi_i$ is a simply connected region with the EM field being zero, we have $\psi_i = \psi_0^i e^{-iS_i/\hbar}$. Now, adding the two wavefunctions together, we obtain a single-valued wavefunction $\psi = \psi_1 + \psi_2$. 

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The pre-potential $S$ is multi-valued because the value of $S(F)$ (where $F$ is the point of interference) depends on the path from $A$ to $F$. This is exactly the manifestation of the AB effect. Nevertheless, by combining the single-valued wavefunction of each beam (using the superposition principle), we obtain a single-valued wavefunction $\psi = \psi_1 + \psi_2$.

Silverman [4] showed that the use of multi-valued wavefunctions in some AB-type experiments is not only permissible, but also necessary. He even provided a physical interpretation to multi-valued wavefunctions. Our approach does not need multi-valued wavefunctions, only a multi-valued pre-potential.

Attributing the AB effect to the 4-potential $A$ is possible only where the EM field vanishes. In this paper, we propose an alternative explanation of the AB effect. We shift the focus from the 4-potential $A$ to a complex-valued multi-valued function $S(x)$, where $x$ is a point in spacetime. Obviously, a real-valued function $S$ satisfying $\nabla S = (e\hbar/c)A$ cannot produce a non-trivial field, since in this case the expression $F = \nabla \times A$ vanishes. Nevertheless, following [5], we describe an EM field by a complex-valued function $S(x)$, which we call the complex pre-potential of the EM field. The real and imaginary parts of $S(x)$ are, in fact, the two known ‘scalar potentials’ introduced by Whittaker in 1904 [6]. We will show that the components of the EM field at any spacetime point $x$ are combinations of second derivatives of the pre-potential $S(x)$, similarly to the function $S(x)$ of [1].

Note that a complex scalar pre-potential was already introduced by Green and Wolf in 1953 [7]. They described the similarity of the expressions for energy and momentum densities between their pre-potential and the wavefunction. The relation of the Green–Wolf pre-potential to the Whittaker scalar potentials is still unclear.

2. Definition of the complex pre-potential of a point charge

Denote by $P = (t, x, y, z)$ a point in spacetime at which we want to calculate the pre-potential. We call $P$ the observer. Denote by $I$ the world-line of the point charge $q$ generating our EM field. Let $Q \in I$ be the unique point of intersection of the past light-cone at $P$ with $I$. We denote the time of the event $Q$ by $\tilde{\tau}$ and refer to this time as the retarded time of the potential. Note that radiation emitted by the charge at $Q$ at retarded time will reach $P$ at time $t$. Thus, the potential at $P$ will depend only on the position described by the vector $a = \vec{QP}$ of the charge at the proper time $\tilde{\tau}$, see figure 1.

Let $K$ be an inertial reference frame in spacetime with coordinates $(ct, x, y, z) = x^\mu$. For the rest of the paper, we will use units in which the speed of light $c = 1$ and omit $c$ from equations. The inner product of two 4-vectors is defined as

$$ a \cdot b = \eta_{\mu\nu}a^\mu b^\nu, \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (1) $$

The space of 4-vectors with this inner product is the Minkowski spacetime $M$. Let $x^\mu$ denote the coordinates of $P$, and let $\bar{x}^\mu$ be the coordinates of $Q$, the charge at the retarded time. Introduce a 4-vector $a(x) = \vec{QP}$. Then

$$ a^\mu(x) = x^\mu - \bar{x}^\mu \quad \text{and} \quad a^2 = a \cdot a = 0. \quad (2) $$

The vector $a(x)$ is a null (light-like) vector in spacetime.

Since $a \in M$ is a null vector, we have

$$ (a^0 + a^3)(a^0 - a^3) = (a^1 + i a^2)(a^1 - i a^2). \quad (3) $$

We may therefore define a dimensionless complex number

$$ \zeta(x) = \frac{a^1 - i a^2}{a^0 + a^3} = \frac{a^0 - a^3}{a^1 + i a^2}. \quad (4) $$

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This constant coincides with the ‘single complex parameter’ occurring during the stereographic projection of the celestial sphere to the Agrand plane (see [8] vol1 p 15).

We want the scalar potential to be a function of the dimensionless scalar $\zeta$. To identify the ‘right’ function, note that the electric force depends on the distance from the charge as $\frac{1}{r^2}$ and, as explained in the introduction, the force is a second derivative of the potential. Hence, the natural candidate for the scalar potential is a multiple of the logarithm function.

**Definition.** The complex pre-potential $S(x)$ at the observer point $x$ of a moving charge $q$ is defined by

$$S(x) = q \ln \zeta = q \ln \frac{a^1(x) - ia^2(x)}{a^0(x) + a^3(x)},$$

where $a(x)$ is defined by (2).

To make clear the meaning of the complex pre-potential $S(x)$, we introduce new coordinates $(r_0, r_1, \theta, \varphi)$ on $M$:

$$x^0 = r_0 \cosh \theta, \quad x^1 = r_1 \cos \varphi, \quad x^2 = r_1 \sin \varphi, \quad x^3 = r_0 \sinh \theta,$$

which we call relativistic bi-polar coordinates. In these coordinates the equation of the light-cone takes the form $r_0 = r_1$, and any ray on the light-cone is defined by a pair of angle $\varphi$ and pseudo-angle $\theta$, which can be described by a single complex number $\theta - i\varphi$. Since $a(x)$ belongs to the light-cone,

$$\zeta = \frac{a^0 - a^3}{a^1 + ia^2} = \frac{r_0 e^\theta}{r_1 e^{i\varphi}} = e^{\theta - i\varphi},$$

so, the complex pre-potential

$$S(x) = q(\theta - i\varphi)$$
is proportional to a complex pseudo angle which describes the position of the charge on the light-cone.

In order to define a covariant expression of the EM field through the pre-potential, we will introduce a complex EM field tensor and a representation of the Lorentz group associated with this tensor.

3. The Faraday vector and complex EM field tensor

An EM field can be defined by an electric field intensity \( E(t, x) \) and a magnetic field intensity \( B(t, x) \). Equivalently, at any spacetime point \((t, x)\), one can represent the EM field by a complex 3D-vector \( F \), called the Faraday vector, as

\[
F = E + iB \quad \text{with} \quad F^2 = E^2 - B^2 + 2iE \cdot B = z^2.
\]

The two Lorentz invariants of the field are the real and imaginary parts of \( z^2 \).

In [9] a complex electromagnetic tensor \( F^\beta_\alpha \) was introduced for the description of an EM field, similar to the one introduced by Silberstein [10]. If \( F^\beta_\alpha \) is the usual electromagnetic tensor, then

\[
F^\beta_\alpha = F^\beta_\alpha (F, -iF) \quad \text{or} \quad F^\beta_\alpha = \sum_{j=1}^{3} (\rho_j)^\beta_\alpha F_j.
\]

where the matrices \( (\rho_j)^\mu_\nu \), called the Majorana–Oppenheimer matrices, (see [11]) are

\[
(\rho_1)^\mu_\nu = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (\rho_2)^\mu_\nu = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},
\]

\[
(\rho_3)^\mu_\nu = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

Direct calculation shows that

\[
F^2 = (\frac{z^2}{2})^2 I,
\]

with \( z^2 \) as in (6). Since \( T^2 = ((2/z)\mathcal{F})^2 = I \), the operator \( T \) is a symmetry operator, and \( \mathcal{F} = (z/2)T \) is a multiple of a symmetry. We denote the complex conjugate of the complex EM field tensor by \( \bar{F}^\beta_\alpha = \sum_{j=1}^{3} (\bar{\rho}_j)^\beta_\alpha F_j \). With this notation, the usual electromagnetic tensor \( F^\beta_\alpha \) can be decomposed as

\[
F^\beta_\alpha = F^\beta_\alpha + \bar{F}^\beta_\alpha.
\]

Direct computation shows that the matrices \( \sigma_j = i\rho_j \) obey the commutator relations for the generators of the rotation group \( SO(3) \), while the \( \rho_j \) matrices obey the commutator relations of boosts in the Lorentz group, i.e.

\[
[\sigma_j, \sigma_k] = -\epsilon_{jkl} \sigma_l, \quad [\rho_j, \rho_k] = \epsilon_{jkl} \rho_l \quad [\sigma_j, \rho_k] = \epsilon_{jkl} \rho_l.
\]
As a result, we can use the six matrices $\rho_j, \sigma_j$ as the generators of the Lorentz group. In addition, the complex conjugates $\bar{\rho}_j$ of these matrices satisfy the same commutation relations. Thus, $\bar{\rho}_j$ and $\bar{\sigma}_j = -i\bar{\rho}_j$ are also the generators of the Lorentz group. Moreover,

$$[\bar{\rho}_j, \rho_l] = 0,$$

for all $j, l = 1, 2, 3$.

In addition to the above commutation relations, these matrices also satisfy the following anti-commutation relations:

$$\{\rho_j, \rho_l\} = \frac{\eta_{jl}}{2} I,$$

$$\{\bar{\rho}_j, \bar{\rho}_l\} = \frac{\bar{\eta}_{jl}}{2} I,$$

where the anti-commutator of two operators is defined as $\{A, B\} = AB + BA$.

### 4. Lorentz group representations in $\mathcal{M}^4$

Our complex pre-potential $S(x)$ is a function $M \to \mathbb{C}$ on the Minkowski space $M$. Its gradient at any spacetime point belongs to the complexified cotangent space at that point, which we identify with $\mathbb{C}^4$. Denote by $\mathcal{M}^4$ the complex space $\mathbb{C}^4$ endowed with the bilinear $\mathbb{C}$-valued form $x \cdot y = \eta_{\mu\nu} x^\mu y^\nu$, which can be considered as a complexification of the Minkowski space. The bilinear form on $\mathcal{M}^4$ is an extension of the inner product (1).

Denote by $\pi$ the lift to $\mathcal{M}^4$ of the fundamental representation of the Lorentz group $L$, see figure 2.

We denote by $\tilde{\pi}$ the representation in $\mathcal{M}^4$ generated by matrices $\rho_j$ and $i\rho_j$ (so, the boost in direction $j$ is given by $\Upsilon^j = \exp(\rho_j)$), and by $\tilde{\pi}^*$ the representation in $\mathcal{M}^4$ generated by matrices $\bar{\rho}_j$ and $-i\bar{\rho}_j$. In the Newman–Penrose basis of $\mathcal{M}^4$ (also known as Bondi tetrad, see [5]), the matrices $\rho_j$ become $\left(\begin{array}{cc} 0 & \sigma_j \\ \sigma_j & 0 \end{array}\right)$, where $\sigma_j$ are the Pauli matrices. This shows that the representation $\tilde{\pi}$ is a direct sum of two $SL(2, \mathbb{C})$ representations of the Lorentz group. The representation $\tilde{\pi}$ is a representation of pairs of spinors, while the representation $\tilde{\pi}^*$ is a representation of pairs of dotted spinors.

Note that the matrices $\rho_j + \bar{\rho}_j$ are the generators of boosts in direction $j$. Thus, using (12), the representation $\pi$ of the Lorentz group $L$ can be decomposed as

$$\Lambda^j = \exp(\rho_j + \bar{\rho}_j) = \exp(\rho_j) \exp(\bar{\rho}_j) = \Upsilon^j \bar{\Upsilon}^j.$$  

More generally, for any $g \in L$, since $\Lambda = \pi(g) = \exp\left(F^g_\theta\right)$ for some tensor $F^g_\theta$, from (10) we have

$$\Lambda = \pi(g) = \Upsilon \bar{\Upsilon} = \tilde{\pi}(g) \tilde{\pi}^*(g),$$

with $\Upsilon = \exp\left(F^g_\theta\right)$.
5. Covariance and invariance under the representations in $\mathcal{M}^4$

We will now prove two claims.

**Claim 1.** (a) The covariance of the tensor $F^\beta_\alpha$ under the representation $\pi$ is equivalent to the covariance of $\tilde{F}^\beta_\alpha$ under $\tilde{\pi}$ (or, equivalently, to the covariance of $\bar{F}^\beta_\alpha$ under $\tilde{\pi}^*$).

(b) For any $j$ the $j$-component of the Faraday vector $F_j$ is invariant under $\tilde{\pi}$.

**Proof.** Let $\Lambda = \pi(g)$ for an arbitrary element $g$ of the Lorentz group. From (15) we have

$$F^\beta_\alpha = \Lambda^{-1} F^\beta_\alpha \Lambda = (\Upsilon \bar{\Upsilon})^{-1} (\bar{F} + F) \Upsilon \bar{\Upsilon}.$$  

Using (12), we get $[\Upsilon, \bar{\rho}_l] = [\Upsilon, \bar{\Upsilon} l] = [\Upsilon, \bar{F}] = [\bar{\Upsilon}, \rho_l] = [\bar{\Upsilon}, F] = 0$. Hence, the above equation can be rewritten as

$$\Lambda^{-1} F^\beta_\alpha \Lambda = \Upsilon^{-1} F \Upsilon + \bar{\Upsilon}^{-1} \bar{F} \bar{\Upsilon}.$$  

This proves that covariance of the tensor $F$ under the representation $\pi$ is equivalent to the covariance of $\tilde{F}$ under $\tilde{\pi}$.

For any $j = 1, 2, 3$, from (7) and (13) we get

$$F_j I = 2(\rho_j, F).$$

Applying the transformation $\tilde{\pi}(g)$, we get after the transformation

$$F'_j I = 2[\tilde{\pi}(g) \rho_j, \tilde{\pi}(g) F].$$

But

$$F'_j I = 2[\tilde{\pi}(g) \rho_j, \tilde{\pi}(g) F] = 2\tilde{\pi}(g)\{(g), (g) F\} = \tilde{\pi}(g)(F_j I) = F_j I,$$

proving (b) of the claim. □

**Claim 2.** The dimensionless constant $\zeta$ defined by (4) and the complex potential $S(x)$ defined by (5) are invariant under the representation $\tilde{\pi}$.

**Proof.** Note that from (13) it follows that $\Upsilon^i = \exp(\rho_j \psi) = \cosh(\psi/2)I + \sinh(\psi/2)2\rho_j$. Thus, if we apply, for example, $\Upsilon^1$ on the vector $a$, we get

$$\Upsilon^1 a = \cosh(\psi/2)(a^0, a^1, a^2, a^3) + \sinh(\psi/2)(a^1, a^0, -ia^3, ia^2).$$

So, applying this transformation to $\zeta$ and using the identity in (4), we get

$$\Upsilon^1(\zeta) = \frac{\cosh(\psi/2)(a^1 - ia^2) + \sinh(\psi/2)(a^0 - a^3)}{\cosh(\psi/2)(a^0 + a^3) + \sinh(\psi/2)(a^1 + ia^2)} = \frac{a^1 - ia^2}{a^0 + a^3} \cdot \frac{1 + \tanh(\psi/2)(a^0 - a^3)/(a^1 - ia^2)}{1 + \tanh(\psi/2)(a^1 + ia^2)/(a^0 + a^3)} = \zeta.$$  

The same argument is valid for any basic boost or rotation. This proves claim 2. □

The existence of a complex Lorentz invariant associated with a null vector has been indicated in [12].
6. The Faraday vector and the complex pre-potential of a uniformly moving charge

To define the connection between the complex pre-potential $S$ and the Faraday vector $F$, we need a new operation on $M^4$. This operation acts by multiplication with the matrix $C = 2\hat{p}_3$, namely, $a^\mu \mapsto C_\mu^a a^\mu$. Since the square of this operation is the identity, we call it the conjugation. From (12) it follows that this conjugation is covariant under the representation $\tilde{\pi}$.

**Proposition.** Let $S$ be the complex pre-potential of a moving charge defined by (5). Then the components of the Faraday vector $F$ can be derived from the complex potential by

$$F_j = \partial^\mu (\rho_j) a^\mu C_\mu^a \partial_a S.$$  

(16)

**Proof.** Formula (16) defines an explicit expression for each component of the Faraday vector $F$:

$$F_1 = S_{13} + iS_{02}, \quad F_2 = S_{23} - iS_{01}$$  

(17)

and

$$F_3 = \frac{1}{2}(S_{00} - S_{11} - S_{22} + S_{33}).$$  

(18)

By the above two claims, equation (16) is covariant under the representation $\tilde{\pi}$. Hence, it is enough to check this formula only for the field of a charge resting at the origin.

Consider a charge resting at the origin. The world-line of the charge is $l = (t, 0, 0, 0)$. From definition (2), we get $a = (|x|, x^1, x^2, x^3)$, where $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Thus,

$$S(x) = q \ln \frac{x^1 - |x|^2}{|x|^2}.$$  

Let $\varrho = (x^1)^2 + (x^2)^2 = |x|^2 + x^3$. Then, since $\frac{\varrho}{|x|^2} |x| = \frac{x^1}{|x|^2}$, we obtain $S_{00} = 0$,

$$S_1 = q \left( \frac{1}{x^1 - |x|^2} - \frac{x^1/|x|}{|x|^2} \right) = \frac{q}{\varrho} \left( x^1 + i\varrho^2 - \frac{(|x|^2 - x^3)x^1}{|x|^2} \right) = \frac{q}{\varrho} \left( \frac{x^1x^3}{|x|^2} + i\varrho \right),$$

$$S_2 = \frac{q}{\varrho} \left( \frac{x^2x^3}{|x|^2} - i\varrho \right), \quad S_3 = -\frac{q}{|x|^2}.$$  

Then, from (17), we obtain

$$F_1 = S_{13} + iS_{02} = S_{13} = -\partial_{x^1} q \frac{|x|}{|x|^2} = \frac{q x^1}{|x|^2},$$

and

$$F_2 = S_{23} - iS_{01} = S_{23} = -\partial_{x^2} q \frac{|x|}{|x|^2} = \frac{q x^2}{|x|^2}.$$  

To calculate $F_3$ using (18), we first calculate

$$S_{11} = \frac{\partial}{\partial x^1} q \left( \frac{x^1x^3}{|x|^2} + i\varrho \right) = -2q x^1 \frac{x^1x^3}{|x|^2} + \frac{q x^3}{|x|^2} \frac{x^1}{|x|^2},$$

$$S_{22} = \frac{\partial}{\partial x^2} q \left( \frac{x^2x^3}{|x|^2} - i\varrho \right) = -2q x^2 \frac{x^2x^3}{|x|^2} + \frac{q x^3}{|x|^2} \frac{x^2}{|x|^2}.$$  

This implies that

$$S_{11} + S_{22} = -\frac{q x^3}{|x|^2}.$$
Since $S_{00} = 0$ and $S_{33} = \frac{q x^3}{|x|^3}$, equation (18) yields
\[ F_3 = \frac{q x^3}{|x|^3} \]
This coincides with the usual formula for the electric force of a charge resting at the origin. □

7. The complex pre-potential for an EM field

Any EM field is generated by a collection of moving charges. We may assume that charges close to each other move with velocities that do not vary significantly. The sources of the EM field may be represented by the charge density $\sigma(y)$ on the spacetime 4-vector $y$. We assume that the potential depends additively on the charges generating the field. Thus, the complex pre-potential of the EM field is given by
\[ S(x) = \int_{K^{-}(x)} \ln \left( \frac{a^{1} + ia^{2}}{a^{0} + a^{3}} \right) \sigma(x + a) \, da, \tag{19} \]
where $K^{-}(x)$ denotes the backward light-cone at $x$. Since the complex logarithm is a multi-valued function, the complex pre-potential is in general a multi-valued function.

Note that the operators $\alpha_j := \rho_j C$ occurring in (16) satisfy the canonical anti-commutation relations (13). In the Newman–Penrose basis of $\mathcal{M}^4$ (also known as Bondi tetrad, see [5]), the matrices $\alpha_j$ take the usual form of the Dirac’s $\alpha$-matrices $(\sigma_j^0 0)$, where $\sigma_j$ are the Pauli matrices (see, for example [13]). Note that the matrices $\rho_j$, which define the representation $\tilde{\pi}$, also satisfy the canonical anti-commutation relations (13). However, the set $\rho_1, \rho_2, \rho_3$ cannot be completed by a forth anti-commuting matrix, needed for the Dirac equation.

By using this notation and the superposition principle, equation (16) can be rewritten as
\[ F_j = \hat{\partial}^\nu(\alpha_j)^{\nu}_\lambda \partial_\lambda S, \tag{20} \]
for a pre-potential $S$ of an arbitrary EM field. From this we get explicit formulas for the components of the EM field and the EM field tensor:
\[ E_j = F_j^0 = F_0^j = 2 \Re (\hat{\partial}^\nu(\alpha_j)^{\nu}_\lambda \partial_\lambda S) \tag{21} \]
and
\[ B_j = \epsilon_{jk}^l = F_l^j = 2 \Im (\hat{\partial}^\nu(\alpha_j)^{\nu}_\lambda \partial_\lambda S). \tag{22} \]
Note that the components of the field are expressed through second derivatives of the pre-potential $S(x)$.

8. Discussion

We introduced a new description of an EM field by a complex-valued function $S(x)$ (pre-potential) on the Minkowski spacetime. The advantages of this approach are as follows.

- Due to its multiple-valued nature, the complex pre-potential becomes a natural candidate for analytical description of the AB effect.
- Based on Whittaker scalar potentials, the description of an EM field is reduced to two degrees of freedom.
- It reveals a new connection between the Dirac equation and classical electrodynamics.
We obtain a complex Lorentz invariant $\zeta$ associated with position of a charge with respect to an observer.

Our approach reveals a new connection between the fundamental and spinor representations of the Lorentz group.

The following problems are still open.

- Incorporate the pre-potential into the Dirac and Schrödinger equations.
- Understand the effect of the EM field on the solutions of these equations through the pre-potential.
- Derive the formulas for the pre-potential for standard sources of an EM field.

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