Abstract
Starting from integrable cellular automata we present a novel form of Painlevé equations. These equations are discrete in both the independent variable and the dependent one. We show that they capture the essence of the behavior of the Painlevé equations organizing themselves into a coalescence cascade and possessing special solutions. A necessary condition for the integrability of cellular automata is also presented.
A novel extension of the Painlevé equations, well-known for their numerous applications in mathematics and physics, will be presented in this paper. Recent progress in the domain of integrable discrete systems has led to the derivation of discrete analogs of the Painlevé equations [1]. They were first identified in a 2D model of quantum gravity where they appeared as an integrable recursion relation for the calculation of the partition function [2]. With hindsight, their first occurrence can be traced back to the work of Jimbo and Miwa on the spin-spin correlation function of a 2D Ising lattice [3]. To date, the full list of discrete Painlevé equations has been established [4] and their properties (in perfect parallel with those of their continuous counterparts) are being actively investigated. Moreover the study of discrete integrable systems has made clear the fact that they, rather than the continuous ones, are the fundamental entities. In fact, the latter are contained, through appropriate limits, in the discrete ones and thus we can establish the hierarchical diagram:

Discrete systems $\rightarrow$ Semi-continuous systems $\rightarrow$ Continuous systems

A recent discovery has made possible to extend this structure in another direction. In [5] one of the present authors has proposed a systematic way for the introduction of integrable ultra-discrete systems. In previous works on discrete systems, while the independent variables were discrete, the dependent variables were assumed to vary continuously. The ultra-discrete limit provides a systematic way to discretize the dependent variable. One can, starting from a given evolution equation, obtain the cellular automaton (CA) equivalent. The aim of this paper is two-fold. First, introduce the ultra-discrete analogs of the Painlevé equations and investigate their properties and, second, provide integrability conditions for cellular-automaton like equations.

Cellular automata have been the object of an impressive number of studies and their behavior is known to be of the utmost richness. The integrability of such systems has not been thoroughly studied, since it represents considerable difficulties. An occurrence of an integrable automaton has been noted in [6] by Pomeau who obtained explicitly its constant of motion. Cellular automata representing evolution equations have been studied from the point of view of the existence of localized, soliton-like, solutions. The notion of soliton for CA’s was first introduced by Park et al. in [7]. Further examples of such CA’s with soliton-like structures were given by the Clarkson group [8]. Integrable CA were introduced by Bruschi and collaborators [9] who derived Lax pairs for their cellular-automaton equations. Bobenko et al. [10] have proposed an interesting approach to integrable CA’s by considering them as the restriction of an integrable discrete equation over a finite field. However in many cases the relation to the well-known integrable evolution equations was based on circumstantial evidence rather than a systematic derivation. The situation has changed recently due to the introduction of a method [5] that allows one to convert a given discrete evolution equation to one where the dependent variable also takes discrete values. The starting point was the CA proposed by one of the authors in collaboration with Satsuma [11]. This simple model (essentially a “box and ball” system) was shown later to be the ultra-discrete
Again, the dynamics are very similar: the two particles converge towards each other, rebound once and have, in perfect analogy to the continuous case, 
\[ X^{t+1} - 2X^t + X^{t-1} = 2(Y^t)_+ - 2(X^t)_+ \text{ and } Y^{t+1} - 2Y^t + Y^{t-1} = 2(X^t)_+ - 2(Y^t)_+. \] 
Again, \( \Delta^2(X^t + Y^t) = 0 \) and we can take \( X^t + Y^t = mt + p \) (where \( m, t, p \) take integer values). We find thus that \( X \) obeys the ultra-discrete equation:
\[ X^{t+1} - 2X^t + X^{t-1} = 2(mt + p - X^t)_+ - 2(X^t)_+ \] 
This is the ultra-discrete analog of the special form of the Painlevé P_{III} equation (5).
Figure 1 gives a comparison of the evolution under (4) and (6). We remark that the dynamics are very similar: the two particles converge towards each other, rebound once
or twice, get captured and go on oscillating around some equilibrium point. Thus, starting from a well-known physical problem we have introduced the corresponding cellular automaton equation and, restricting it to the simplest periodic lattice, we obtained the ultra-discrete form of a Painlevé equation.

In order to construct the ultra-discrete analogs of the Painlevé equations (u-P) we must start with the discrete form that allows the ultra-discrete limit to be taken. The general procedure is to start with an equation for \( x \), introduce \( X \) through \( x = e^{X/\epsilon} \) and then take appropriately the limit \( \epsilon \to 0 \). Clearly the substitution \( x = e^{X/\epsilon} \) requires \( x \) to be positive. This is a stringent requirement that limits the exploitable form of the d-P’s to multiplicative ones. Fortunately many such forms are known for the discrete Painlevé transcendents [13]:

\[
\text{d-P}_1
\]

\[
x_{n+1}x_{n-1} = \frac{\alpha \lambda^n}{x_n} + \frac{1}{x_n^2}
\]  
(7a)

\[
x_{n+1}x_{n-1} = \alpha \lambda^n + \frac{1}{x_n}
\]  
(7b)

\[
x_{n+1}x_{n-1} = \alpha \lambda^n x_n + 1
\]  
(7c)

\[
\text{d-P}_II
\]

\[
x_{n+1}x_{n-1} = \frac{\lambda^n(x_n + \alpha \lambda^n)}{x_n(1 + x_n)}
\]  
(7d)

\[
x_{n+1}x_{n-1} = \frac{x_n + \alpha \lambda^n}{1 + \beta x_n \lambda^n}
\]  
(7e)

\[
\text{d-P}_III
\]

\[
x_{n+1}x_{n-1} = \frac{(x_n + \alpha \lambda^n)(x_n + \beta \lambda^n)}{(1 + \gamma x_n \lambda^n)(1 + \delta x_n \lambda^n)}
\]  
(7f)

We remark that in some cases, more than one form exists for a given d-P. The derivation of the equivalent ultra-discrete forms is elementary: we take the logarithm of both sides of the equation and whenever a sum appears we apply the limit leading to the truncated power function. We find thus:

\[
\text{u-P}_1
\]

\[
X_{n+1} + X_{n-1} + 2X_n = (X_n + n + a)_{+}
\]  
(8a)

\[
X_{n+1} + X_{n-1} + X_n = (X_n + n + a)_{+}
\]  
(8b)

\[
X_{n+1} + X_{n-1} = (X_n + n + a)_{+}
\]  
(8c)

\[
\text{u-P}_II
\]

\[
X_{n+1} + X_{n-1} = n + (n + a - X_n)_{+} - (X_n)_{+}
\]  
(8d)

\[
X_{n+1} + X_{n-1} - X_n = (n + a - X_n)_{+} - (X_n + n + b)_{+}
\]  
(8e)

\[
\text{u-P}_III
\]

\[
X_{n+1} + X_{n-1} - 2X_n = (n + a - X_n)_{+} + (n + b - X_n)_{+} - (X_n + c + n)_{+} - (X_n + d + n)_{+}
\]  
(8f)
These equations describe cellular automata provided we restrict the values of the parameters as well as the initial values of the dependent variable to integers. Figure 2 shows a comparison of the discrete P_I (7c) with the ultra-discrete P_I (8c). It is remarkable that the behavior of the two equations is very similar (provided that we plot the logarithm of the variable of the discrete equation, as expected). The forms (8) are not canonical in the sense that they can be simplified by a translation of $X$ and a linear transformation in $n$. We will not enter into these details but merely list the canonical forms obtained:

canonical u-P_I

$$X_{n+1} + X_{n-1} + \sigma X_n = (X_n + n)_+ \quad \text{with} \quad \sigma = 0, 1, 2 \quad (9a)$$

canonical u-P_{II}

$$X_{n+1} + X_{n-1} - \sigma X_n = a + (n - X_n)_+ - (n + X_n)_+ \quad \text{with} \quad \sigma = 0, 1 \quad (9b)$$

canonical u-P_{III}

$$X_{n+1} + X_{n-1} - 2X_n = (n+a-X_n)_+ + (n-a-X_n)_+ - (X_n+b+n)_+ - (X_n-b+n)_+ \quad (9c)$$

Note that equation (6), for $m = 2, p = 0$ is the subcase $a = b = 0$ of (9c) after the change of variable $X^t = X_n + n$.

At this point two questions appear unavoidable. First, is it justified to call these equations ultra-discrete Painlevé equations? What do they have in common with the familiar Painlevé equations? One first remark is that the u-P’s form a coalescence cascade just like their continuous and discrete counterparts [13]. Indeed, starting from u-P_{II} (8d) (respectively (8e)) we can take $b \to -\infty$ in which case the nonlinear term containing it is always zero. We transform the resulting equation using the identity $(x)_+ = x + (-x)_+$, then translate $X$ and obtain equation u-P_I in the form (8b), (resp. (8c)). We can also recover (8a) from (8d) (and (8b) also from (8c)). Similarly starting from u-P_{III} (8f) we can obtain u-P_{II} (8d) by taking $b \to +\infty$ and $c \to +\infty$ such that $b-c$ is finite. Next we translate $X$ and through a linear transformation of $n$ we find u-P_{II} (8d). This is not the only property the u-P’s share with the continuous and discrete Painlevé equations as we shall see below. The second question is whether it is possible to guess the forms of the u-P’s. In other words, what is the (integrability) criterion that would single them out among all possible equations?

In the case of discrete systems the criterion for integrability (equivalent to the Painlevé property) is the property known as singularity confinement [14]. For cellular automata no singularity can exist and thus this criterion is inoperable. The situation is analogous to polynomial mappings where no singularity can appear. There, the criterion for integrability is based on arguments of growth of the degree of the iterate (or the similar notion of complexity introduced by Arnold [15]). Veselov [16] has shown that among mappings of the form $x_{n+1} - 2x_n + x_{n-1} = P(x_n)$ with polynomial $P(x)$, only the linear one has non-exponential growth of the degree of the polynomial that results
from the iteration of the initial conditions. Let us apply such a low-growth criterion to a family of ultra-discrete $P_I$ equations of the form:

$$X_{n+1} + \sigma X_n + X_{n-1} = (X_n + \phi(n))_+$$

The three $u\text{-}P_I$ obtained from (7) correspond to $\sigma = 0, 1, 2$. What is the condition for $X$ not to grow exponentially towards $\pm \infty$? We ask simply that the polynomials $r^2 + \sigma r + 1$ and $r^2 + (\sigma - 1)r + 1$ have complex roots (otherwise exponential growth ensues). The only integer values of $\sigma$ satisfying this criterion are $\sigma = -1, 0, 1, 2$. We remark that the three values mentioned above are exactly retrieved plus the value $\sigma = -1$. A close inspection of this mapping shows that it is also integrable: it is just a form of an ultra-discrete $P_{III}$, obtained from the discrete system $x_{n+1}x_{n-1} = x_n(x_n + \lambda^n)$ which leads to (10) with $\phi(n) = 0$.

We have applied the low-growth criterion to other cases like $u\text{-}P_{II}$ and $u\text{-}P_{III}$ and in every case the results of the growth analysis correspond to the already obtained integrable cases. However low-growth is not a sufficiently powerful integrability criterion. In particular the inhomogeneous terms ($\phi$ in equation (10)) cannot be fixed by this argument. Any $\phi(n)$ would satisfy this requirement. So another criterion must complement this first one.

In the case of (continuous) evolution equations two integrability criteria are often used in conjunction: the Painlevé property and the existence of multisoliton solutions. In the case of Painlevé equations the latter are the special solutions that exist for particular values of the parameters [17] (except for $P_I$ which is parameter-free). A particular class of these solutions (existing also in the discrete case) are the rational ones. We shall investigate this property in the case of $u\text{-}P$ equations. This will strengthen the argument that (9) are indeed Painlevé equations and will allow us to fix completely their form. For $d\text{-}P_{II}$ the simplest rational solution is a constant. Thus a constant solution should exist for $u\text{-}P_{II}$ and indeed for (9b) with $\sigma = 0$ we find that $X = 0$ is a solution for $a = 0$. However this solution exists if we replace $n$ in (9b) by any function of $n$. The next solution for $u\text{-}P_{II}$ is a step-function one. Indeed when $n$ is large constant positive solution exists with $X$ equal to $a/4$ while a constant solution with $X = a/2$ exists when $n$ is large negative. Thus, for instance for $a = 4$ a solution for $n << 0$ is $X = 2$, while for $n >> 0$ a solution is $X = 1$. The remarkable fact is that that these constant “half” solutions do really join to form a solution of (9b) with a unique jump at $n = -1$. It is straightforward to check that this will not be the case in general if the non-autonomous part is not linear in $n$. The general solution of this type becomes now clear. For $a = 4m$ we have a solution with $m$ jumps from the value $X = 2m$ to $X = m$. The first jump occurs at $n_0 = 1 - 2|m|$ and we have successive jumps of $-|m|/m$ at $n = n_0 + 3k$, $k = 0, 1, 2, \ldots, |m| - 1$. Analogous results can be obtained for the other $u\text{-}P_{II}$ corresponding to $\sigma = 0$. Thus $u\text{-}P_{II}$ has a rich structure of particular solutions.

On the basis of these results we can conclude that the integrability criterion for automata-like Painlevé equations appears to be based on low-growth arguments together
with the existence of a rich class of explicit, globally described, solutions. The need for such a two-step process is the price we have to pay in the CA case because of the absence of singularities. Can these ideas be applied to other CA evolution equations? Let us present the example of ultra-discrete Burgers equation:

\[ X_{n+1}^t = X_n^t + (X_{n+1}^t)_+ - (X_n^t)_+ \]  

(11)

Using the identity \((X)_+ = X + (-X)_+\) we can rewrite (11) as:

\[ X_{n+1}^t = a(X_{n+1}^t)_+ - b(-X_n^t)_+ \]  

(12)

where, from (11), \(a = b = 1\). How can we obtain these values of the parameters based on a low-growth argument? If either \(|a| > 1\) or \(|b| > 1\) we can find a special direction where either \(X_{n+k}^t\) or \(X_{n-k}^t\) grows exponentially. Thus the only integer values that \(a\) and \(b\) can take are \(\pm 1\) and 0. The value 0 leads to uninteresting, one dimensional evolution and the case \(a = b = -1\) is equivalent to the case \(a = b = 1\). Thus the only other case that cannot be excluded on low-growth arguments is the case \(a = 1\) and \(b = -1\). Although the discrete equation from which we could have obtained it is not integrable the behavior of the automaton-like equation is regular and there is as yet no indication as to its nonintegrability.

It is clear from the results presented in [5] and the ones above that we can produce cellular-automaton-like equations starting from integrable nonlinear evolution equations. The present work has focused on the ultra-discrete forms of Painlevé equations of which the first three were given. The forms of the remaining u-P equations will also be investigated: probably they will require a two-component description. The properties of their solutions as well as the relations between the various u-P’s must also be studied. These open questions will be addressed in future works. What is important at this stage is that we have shown that this new domain of integrable systems is particularly rich. While the discrete systems are the fundamental entities and contain all the structure, the cellular automata are their bare-bones version capturing the essence of the dynamics. This explains the interest that these ultra-discrete systems present both from the mathematical and the physical points of view.
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Figure captions.

Figure 1. Distance between the two particles as a function of time in the case (a) of the continuous Toda potential and (b) its ultra-discrete analog.

Figure 2. Solution of the discrete Painlevé I equation (a) and of its ultra-discrete analog (b).