New elliptic solutions of the Yang-Baxter equation

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Abstract

We consider finite-dimensional reductions of the most general known solution of the Yang-Baxter equation with a rank 1 symmetry algebra, which is described by an integral operator with an elliptic hypergeometric kernel. The reduced R-operators reproduce at their bottom the standard Baxter’s R-matrix for the 8-vertex model and Sklyanin’s L-operator. The general formula has a remarkably compact form and yields new elliptic solutions of the Yang-Baxter equation based on the finite-dimensional representations of the elliptic modular double. The same result is reproduced using the fusion formalism.
1 Introduction

Exactly solvable models play an important role in the investigation of critical phenomena in statistical mechanics [3]. One of the key structural elements leading to exact solvability is the Yang-Baxter equation (YBE) [22, 46], which, in a sense, admits reduction of the many-body problems to the investigation of dynamics of pairwise two-body interacting systems. In the context of discrete spin lattice systems, solutions of the YBE are given by ordinary finite rank matrices with trihotomic matrix elements expressed either in terms of the rational, trigonometric (or hyperbolic) or elliptic functions. In the context of quantum field theories (the continuous spin systems), the YBE is solved in terms of the integral operators associated with the plain hypergeometric functions, their various $q$-analogues (trigonometric or hyperbolic) or the elliptic hypergeometric integrals [40].

The first elliptic solution of YBE given by a $4 \times 4$ matrix was derived by Baxter and it played a crucial role in solving the 8-vertex model [2], which is related also to the XYZ spin chain [3, 46]. A different $4 \times 4$ R-matrix with elliptic entries was found by Felderhof [18]. As shown by Krichever [27], the Baxter and Felderhof R-matrices exhaust all $4 \times 4$ matrix solutions of YBE. Elliptic R-matrices with higher rank symmetry algebras were constructed by Belavin [5].

The concept of the quantum group $U_q(s\ell_2)$ naturally arises from degeneration of Baxter’s R-matrix to the trigonometric level [22]. However, as shown by Faddeev [16], in general it is necessary to deal with a more complicated algebra called the modular double. At the elliptic level corresponding algebraic structures are determined by the Sklyanin algebra [33, 34] and the elliptic modular double [39]. Various connections between the Sklyanin algebra and elliptic hypergeometric functions were considered in [29, 30, 31, 39].

One can construct higher dimensional matrix solutions of YBE out of the fundamental ones using the fusion procedure [25], which was applied to many situations. For instance, fusion of elliptic R-matrices for the SOS type models has been constructed by the Kyoto group [11, 12, 13]. This procedure yielded rather complicated forms of the Boltzmann weights which appeared to be
nicely combined to an elliptic hypergeometric series \cite{19}. The fusion was applied also to the Sklyanin algebra representations, see e.g. \cite{15, 20, 21, 23, 24, 44, 45} and references therein.

Continuous spin integrable systems (noncompact spin chains) are important in quantum field theory. A systematic approach to solving YBEs for such systems based on a twisted representation of the permutation group generators by integral operators was developed in some of our previous papers. Construction of an elliptic hypergeometric solution of YBE following this line and using some formal series was given in \cite{8}. A similar construction based on the technique of intertwining vectors was described later in \cite{49}. The most complicated known solution of YBE (with the rank 1 symmetry algebra) as an integral operator with elliptic hypergeometric kernel has been constructed in \cite{9} following the logical line of \cite{8} boosted by a powerful machinery of elliptic hypergeometric integrals \cite{35, 37, 40}. The key ingredients in this construction are the elliptic beta integral evaluation formula discovered in \cite{35} and an integral operator introduced in \cite{38}, the Bailey lemma for which yields the star-triangle relation in the operator form. Some further development of the results of \cite{9} is described in \cite{6, 10}. In particular, in \cite{9, 10} some finite-dimensional representations of the elliptic modular double have been constructed.

Various finite-dimensional R-matrices and L-operators were obtained in \cite{7} from a reduction of general integral operator solutions of YBE. In the rational case this procedure yields perhaps all such finite-dimensional R-matrices. In the \( q \)-deformed cases the situation is more complicated. Namely, such finite-dimensional solutions of YBE were built for the plain \( U_q(\mathfrak{sl}_2) \)-algebra and the Faddeev modular double cases only under the restriction that \( q \) is not a root of unity. Here we extend the analysis of \cite{7} to the elliptic level and find new elliptic solutions of the YBE as reductions of the general integral R-operator. They are characterized by the presence of some two-dimensional discrete lattices for each spin variable leading to “a doubling” of the dimension of the finite-dimensional representation space. Existence of such “two-index” solutions of the YBE was conjectured in \cite{36} on the basis of some properties of the elliptic hypergeometric integrals leading to a new class of biorthogonal functions of hypergeometric type \cite{37, 40}. For completeness we derive the same elliptic solutions of YBE using the fusion procedure as well. In this respect, our approach is close to the one developed by Takebe \cite{44, 45} and Konno \cite{24}, which is based on Rosengren’s results \cite{30}.

The plan of the paper is as follows. We start with a short review of well-know facts about elliptic solutions of YBE. In Sect. 2 we present Baxter’s R-matrix of the 8-vertex model and some basic properties of the Jacobi theta functions. In Sect. 3 we review the Sklyanin algebra, its finite-dimensional and infinite-dimensional representations, and corresponding Lax operator. In particular, we show that the reduction of the Lax operator to two-dimensional representation in the quantum space coincides with Baxter’s R-matrix. In Sect. 4 we proceed to the elliptic modular double. Our considerations are heavily based on the intertwining operator of equivalent representations of this algebra. It serves as a basic tool enabling us to describe finite-dimensional representations. Moreover, it suggests a natural way to construction of the generating function of finite-dimensional representations embracing all basis vectors of the representation space. In the beginning of Sect. 5 we outline the construction of the general R-operator which acts in a tensor product of two infinite-dimensional representations of the elliptic double. Then in Sect. 5.1 we proceed to the main topic of the paper. There we calculate reductions of the general R-operator to finite-dimensional representations in one of the spaces as well as in both spaces. The results are described by the remarkably compact formulae \cite{73} and \cite{74}. They comprise an enormous number of elliptic YBE solutions, both plain R-matrices and L-operators with the elliptic function entries. In order to demonstrate the power of the reduction formulae we recover Sklyanin’s Lax operator out of the general R-operator in Sect. 5.2. Then in Sect. 6 we demonstrate how to reproduce this results by means of the fusion method. In Sect. 6.1 we perform fusion of arbitrary number of Baxter’s R-matrices and explicitly reconstruct Sklyanin’s Lax operator for finite-dimensional representations. In Sect. 6.2 we proceed to a higher level of complexity. There we accomplish the
fusion of arbitrary number of Lax operators for infinite-dimensional representations. The result is a higher-spin R-operator – a generalization of Sklyanin’s Lax operator – which is defined on a tensor product of an infinite-dimensional representation and arbitrary finite-dimensional representation. The derived formula is in a nice agreement with the reduction result from Sect. 5.1. We conclude in Sect. 7 where we discuss possible applications of the obtained solutions of YBE.

2 Baxter’s R-matrix

The Yang-Baxter equation with the spectral parameter which we investigate in this work has the form

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).$$

(1)

Its solutions depend on the complex spectral parameter $$u \in \mathbb{C}$$ and are called the R-matrices (or R-operators). The operator $$R_{ik}(u)$$ act in the tensor product of three (in general different) spaces $$V_1 \otimes V_2 \otimes V_3$$. The indices $$i$$ and $$k$$ indicate that $$R_{ik}(u)$$ acts nontrivially in the subspace $$V_i \otimes V_k$$ and it is the unity operator in the remaining part of $$V_1 \otimes V_2 \otimes V_3$$. In the most general situation all three spaces $$V_i$$ are infinite-dimensional and $$R_{ik}(u)$$ are described by integral operators with some singular kernels.

The equation (1) is inherently universal since it does not involve information neither on the symmetry algebra underlying an integrable model nor on its particular representations in the spaces $$V_i$$. In this paper we deal with solutions of (1) associated with an elliptic deformation of a rank 1 Lie algebra to be specified below. Further we specify spaces $$V_i$$ in (1) and corresponding representations of the symmetry algebra, starting from two-dimensional representations and heading towards infinite-dimensional representations. Thus we go through the hierarchy of elliptic solutions from the simplest to the most intricate elliptic R-matrices.

In the simplest case all $$V_i$$-spaces in (1) are two-dimensional, $$V_i = \mathbb{C}^2$$. The corresponding elliptic R-matrix has been found by Baxter in solving the eight-vertex model [2] [3] [19]

$$R_{12}(u) = \sum_{\alpha=0}^{3} w_{\alpha}(u) \sigma_\alpha \otimes \sigma_\alpha = \begin{pmatrix} w_0(u) \sigma_0 + w_3(u) \sigma_3 & w_1(u) \sigma_1 - iw_2(u) \sigma_2 \\ w_1(u) \sigma_1 + iw_2(u) \sigma_2 & w_0(u) \sigma_0 - w_3(u) \sigma_3 \end{pmatrix},$$

(2)

where $$\sigma_0 = 1$$ and $$\sigma_\alpha, \alpha = 1, 2, 3,$$ are the standard Pauli matrices and the coefficient functions $$w_{\alpha}(u) = \frac{\theta_{\alpha + 1}(u + \eta)}{\theta_{\alpha + 1}(\eta)}$$. We use the shorthand notation $$\theta_{\alpha}(u) \equiv \theta_{\alpha}(u|\tau)$$ for Jacobi theta-functions with modular parameter $$\tau$$

$$\theta_1(z|\tau) = -\sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2} \tau \cdot e^{2\pi i (n + \frac{1}{2}) (z + \frac{1}{2})} = e^{-\pi iz \theta(e^{2\pi iz}; p)} R(\tau) = \frac{p^{-\frac{1}{2}}}{R(p; \tau)}$$

(3)

where in the multiplicative notation

$$p = e^{2\pi i \tau}, \quad \theta(t; p) = (t; p)_\infty (pt^{-1}; p)_\infty, \quad (t; p)_\infty = \prod_{k=0}^{\infty} (1 - tp^k).$$

(4)

The rest three theta-functions are obtained by shifts of the argument of $$\theta_1$$ by quasi-period halves

$$\theta_2(z|\tau) = \theta_1(z + \frac{1}{2}|\tau), \quad \theta_3(z|\tau) = e^{\pi iz + \pi iz} \theta_2(z + \frac{1}{2}|\tau), \quad \theta_4(z|\tau) = \theta_3(z + \frac{1}{2}|\tau).$$

Quasi-periodic properties of these functions follow from the relations

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -e^{-2\pi iz - 2\pi i} \theta_1(z|\tau).$$

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -e^{-2\pi iz - 2\pi i} \theta_1(z|\tau).$$

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -e^{-2\pi iz - 2\pi i} \theta_1(z|\tau).$$

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -e^{-2\pi iz - 2\pi i} \theta_1(z|\tau).$$

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -e^{-2\pi iz - 2\pi i} \theta_1(z|\tau).$$
Other indispensable elliptic special functions and some necessary identities for theta-functions will be indicated in proper places below.

The Baxter R-matrix \( R \) depends on the spectral parameter \( u \in \mathbb{C} \) and two additional free variables \( \eta, \tau \in \mathbb{C} \), such that \( \theta_{\alpha}(\eta) \neq 0, \alpha = 1, \ldots, 4 \), and \( \text{Im}(\tau) > 0 \). Its connection to a symmetry algebra and representation theory meaning are explained in the next section.

The Baxter’s R-matrix \( R \) is not the only \( 4 \times 4 \) matrix solution of YBE. As mentioned already, there exist a different solution of YBE for \( V_i = \mathbb{C}^2 \), which has been found by Felderhof [18]. In this paper we concentrate on the solutions of YBE associated with the Baxter’s case postponing the study of Felderhof’s solution to a separate work.

3 L-operator and Sklyanin algebra

3.1 Algebraic relations

At the next level of complexity of YBE [11] one of the spaces, say \( V_3 \), is arbitrary, and the rest two spaces are 2-dimensional \( V_1 = \mathbb{C}^2 \), \( V_2 = \mathbb{C}^2 \). In this case the R-matrix \( \mathbb{R}_{13}(u) = L_{13}(u) \) (and \( \mathbb{R}_{23}(u) = L_{23}(u) \)) is known as the quantum L-operator or the Lax matrix. It acts as a \( 2 \times 2 \) matrix constructed out of the Pauli matrices in the space \( V_1 \)

\[
L_{13}(u) = L(u) \equiv \sum_{\alpha=0}^{3} w_{\alpha}(u) \sigma_{\alpha} \otimes S^{\alpha} = \left( \begin{array}{cc}
w_0(u) S^0 + w_3(u) S^3 & w_1(u) S^1 - iw_2(u) S^2 \\
w_1(u) S^1 + iw_2(u) S^2 & w_0(u) S^0 - w_3(u) S^3 \end{array} \right),
\]

whose entries \( S^{\alpha} \) are some operators acting in \( V_3 \) which do not depend on the spectral parameter \( u \). The functions \( w_{\alpha}(u) \) are the same as in the Baxter R-matrix \( R \). The space \( V_1 = \mathbb{C}^2 \) in [5] is usually referred to as an auxiliary space, and \( V_3 \) is called the quantum space of the Lax operator. The same operators \( S^{\alpha} \) enter the second copy of the Lax operator \( L_{23}(u) \), which acts as a \( 2 \times 2 \) matrix in the space \( V_2 = \mathbb{C}^2 \). In this setting the equation (11) takes the form of a RLL-relation [16]

\[
\mathbb{R}_{12}(u-v) L_{13}(u) L_{23}(v) = L_{23}(v) L_{13}(u) \mathbb{R}_{12}(u-v),
\]

where \( \mathbb{R}_{12}(u) \) is Baxter’s R-matrix \( R \). Since \( \mathbb{R}_{12} \) in (6) has been already specified, this relation can be considered as a nontrivial restriction for the operators \( S^{\alpha} \) and the space \( V_3 \) where the operators \( S^{\alpha} \) are acting. After some laborious work one can see that the equation (6) is equivalent to the following set of commutation relations for four operators \( S^0, S^1, S^2, S^3 \) forming the Sklyanin algebra [33 [34]:

\[
S^{\alpha} S^{\beta} - S^{\beta} S^{\alpha} = i \cdot \left( S^0 S^\gamma + S^\gamma S^0 \right),
\]

\[
S^0 S^{\alpha} - S^{\alpha} S^0 = i \mathbf{J}_{\beta \gamma} \cdot \left( S^\beta S^\gamma + S^\gamma S^\beta \right),
\]

where the triplet \((\alpha, \beta, \gamma)\) is an arbitrary cyclic permutation of \((1,2,3)\). The structure constants \( \mathbf{J}_{\beta \gamma} \) are parametrized in terms of theta functions as

\[
\mathbf{J}_{12} = \frac{\theta^2_2(\eta)\theta^2_4(\eta)}{\theta^2_3(\eta)\theta^2_1(\eta)}; \quad \mathbf{J}_{23} = \frac{\theta^2_1(\eta)\theta^2_3(\eta)}{\theta^2_2(\eta)\theta^2_4(\eta)}; \quad \mathbf{J}_{31} = -\frac{\theta^2_2(\eta)\theta^2_3(\eta)}{\theta^2_1(\eta)\theta^2_4(\eta)}
\]

and satisfy the constraint \( \mathbf{J}_{12} + \mathbf{J}_{23} + \mathbf{J}_{31} + \mathbf{J}_{12}\mathbf{J}_{23}\mathbf{J}_{31} = 0 \). One can write \( \mathbf{J}_{\alpha \beta} = \frac{\mathbf{J}_{2 \gamma} - \mathbf{J}_{\gamma}}{\gamma - \eta} \), \( \gamma \neq \alpha, \beta \), where

\[
\mathbf{J}_1 = \frac{\theta_2(2\eta)\theta_2(0)}{\theta^2_2(\eta)}; \quad \mathbf{J}_2 = \frac{\theta_3(2\eta)\theta_3(0)}{\theta^2_3(\eta)}; \quad \mathbf{J}_3 = \frac{\theta_4(2\eta)\theta_4(0)}{\theta^2_4(\eta)}.
\]
Casimir operators are indispensable for classification of irreducible representations of any algebra. There are two Casimir operators in the Sklyanin algebra commuting with all generators: 
\[ [K_0, S^\alpha] = [K_2, S^\alpha] = 0, \]
\[ K_0 = \sum_{\alpha=0}^{3} S^\alpha S^\alpha; \quad K_2 = \sum_{\alpha=1}^{3} J_\alpha S^\alpha S^\alpha. \] (10)

A remarkable feature of the algebraic structure (7) is that it admits a highly nontrivial explicit realization of generators as finite-difference operators with elliptic coefficients found by Sklyanin in his pioneering paper \[34\]
\[ S^\alpha = i^{\delta_{\alpha,2}}\theta_{\alpha+1}(\eta) \left[ \theta_{\alpha+1}(2z - 2\eta\ell) e^{\eta\partial} - \theta_{\alpha+1}(-2z - 2\eta\ell) e^{-\eta\partial} \right], \] (11)
where \( e^{\eta\partial} \) is a shift operator, \( e^{\eta\partial} f(z) = f(z + \eta) \). The variable \( \ell \in \mathbb{C} \) is called the spin. In this realization the Casimir operators reduce to the following scalar expressions
\[ K_0 = 4\theta_1^2(2\ell\eta + \eta); \quad K_2 = 4\theta_1(2\ell\eta + 2\eta) \theta_1(2\ell\eta). \]
The spin \( \ell \) labels the Sklyanin algebra representations since it fixes (together with \( \eta \) and \( \tau \)) the Casimir operator values.

There exists a useful factorized representation for the Lax operator \( L(u) \) \[5\] when the operators \( S^\alpha \) are given by the explicit expression (11) \[8, 28, 48, 49\],
\[ L(u_1, u_2) = \frac{1}{\theta_1(2z)} \begin{pmatrix} \theta_3(z - u_1) & -\theta_3(z + u_1) \\ -\theta_4(z - u_1) & \theta_4(z + u_1) \end{pmatrix} \begin{pmatrix} e^{\eta\partial} & 0 \\ 0 & e^{-\eta\partial} \end{pmatrix} \begin{pmatrix} \theta_4(z + u_2) & \theta_3(z + u_2) \\ \theta_4(z - u_2) & \theta_3(z - u_2) \end{pmatrix}. \] (12)
Here we use shorthand notations \( \theta_{\alpha}(z) \equiv \theta_{\alpha}(z; \tau) \) for theta-functions with the modular parameter \( \tau \). Instead of working with the spectral parameter \( u \) and the spin \( \ell \) separately we prefer to combine them in a pair of new “light-cone” parameters which are linear combinations of the latter
\[ u_1 = \frac{u}{2} + \frac{\eta\ell + \eta}{2}; \quad u_2 = \frac{u}{2} - \frac{\eta\ell - \eta}{2}. \] (13)
The factorization formula for the Lax operator plays an essential role in the construction of the general R-operator (see \[9\] and Sect. \[3\]) and it is crucial in the “fusion” of Baxter’s R-matrices (see Sect. \[6.1\]).

### 3.2 Finite-dimensional representations of the Sklyanin algebra

For generic values of the spin \( \ell \) the Sklyanin algebra representation determined by the generators (11) is irreducible and infinite-dimensional. It can be realized in the space of holomorphic functions of one complex variable \( z \). However, for a discrete set of spin values, say, \( 2\ell = n, \ n \in \mathbb{Z}_{\geq 0} \), finite-dimensional representations arise and the action of generators (11) leaves invariant the corresponding finite-dimensional subspace. Indeed, the irreducible representation of the Sklyanin algebra at (half)-integer spin \( \ell = \frac{n}{2} \) is \((n + 1)\)-dimensional and it can be realized in the space \( \Theta_{2n}^+ \) consisting of even theta functions of order \( 2n \) \[34\]. The space \( \Theta_{2n}^+ \) is formed by the holomorphic functions of complex variable \( z \) which are even \( f(z) = f(-z) \) and have simple quasiperiodicity properties under the shifts of \( z \) by 1 and \( \tau \):
\[ f(z + 1) = f(z); \quad f(z + \tau) = e^{-2n\pi i\tau - 4n\pi iz} f(z). \]

Further, it will be useful for us to choose a particular basis in the space \( \Theta_{2n}^+ \). Let us consider monomials in a pair of theta functions \( e_1 = \theta_4(z), e_2 = \theta_3(z) \) of the quasi-periods 1 and \( \tau/2 \). Taking
into account their quasiperiodicity properties, we see that the following set of \( n + 1 \) functions belongs to the space \( \Theta^+_{2n} \) and, due to the linear independence, forms a basis

\[
\bar{\theta}^j_3(z) \bar{\theta}^{n-1-j}_4(z) \quad ; \quad j = 0, 1, \cdots, n. \tag{14}
\]

In order to clarify how Baxter’s R-matrix \(^{(2)}\) is related to the Sklyanin algebra let us consider in more detail its two-dimensional representation that corresponds to the spin \( \ell = 1/2 \). In that case the Sklyanin algebra generators \(^{(11)}\) take the following form

\[
[S^\alpha \Phi](z) = \left(\frac{i}{\theta_1(2z)}\right)^{\delta_{\alpha,2}} \theta_{\alpha+1}(2z - \eta) \cdot \Phi(z + \eta) - \theta_{\alpha+1}(-2z - \eta) \cdot \Phi(z - \eta). \tag{15}
\]

Let us restrict them to the two-dimensional space \( \Theta^+_{2} \) using the basis \( e_1 = \bar{\theta}_3(z) \) and \( e_2 = \bar{\theta}_4(z) \) \(^{(14)}\). To do this, we compute the action of generators \(^{(15)}\) on this basis vectors and expand the result in the same basis. Using the following identities for theta functions

\[
\begin{align*}
2 \theta_1(x + y) \theta_1(x - y) &= \bar{\theta}_3(x) \bar{\theta}_3(y) - \bar{\theta}_4(y) \bar{\theta}_3(x), \\
2 \theta_2(x + y) \theta_2(x - y) &= \bar{\theta}_3(x) \bar{\theta}_3(y) - \bar{\theta}_4(y) \bar{\theta}_4(x), \\
2 \theta_3(x + y) \theta_3(x - y) &= \bar{\theta}_3(x) \bar{\theta}_3(y) + \bar{\theta}_4(y) \bar{\theta}_4(x), \\
2 \theta_4(x + y) \theta_4(x - y) &= \bar{\theta}_4(x) \bar{\theta}_3(y) + \bar{\theta}_4(y) \bar{\theta}_3(x),
\end{align*} \tag{16}
\]

one obtains in a straightforward fashion

\[
\begin{align*}
S^0 \bar{\theta}_3(z) &= \bar{\theta}_3(z) \theta_1(2\eta), & S^0 \bar{\theta}_4(z) &= \bar{\theta}_4(z) \theta_1(2\eta), \\
S^1 \bar{\theta}_3(z) &= \bar{\theta}_4(z) \theta_1(2\eta), & S^1 \bar{\theta}_4(z) &= \bar{\theta}_3(z) \theta_1(2\eta), \\
S^2 \bar{\theta}_3(z) &= -i \bar{\theta}_4(z) \theta_1(2\eta), & S^2 \bar{\theta}_4(z) &= i \bar{\theta}_3(z) \theta_1(2\eta), \\
S^3 \bar{\theta}_3(z) &= -\bar{\theta}_3(z) \theta_1(2\eta), & S^3 \bar{\theta}_4(z) &= \bar{\theta}_4(z) \theta_1(2\eta). \tag{17}
\end{align*}
\]

These relations \(^{(17)}\) can be arranged in a matrix form. Consequently, we find that the Sklyanin algebra generators \(^{(15)}\) are reduced to the Pauli sigma-matrices in the basis \(^{(14)}\),

\[
S^\alpha (e_1, e_2) = (S^\alpha e_1, S^\alpha e_2) = (e_1, e_2) \sigma^\alpha \theta_1(2\eta). \tag{18}
\]

Finally, we perform this reduction \(^{(18)}\) in the quantum space of the Lax operator \(^{(5)}\) and find that it is proportional\(^{(1)}\) to Baxter’s R-matrix \( R(u) \) \(^{(2)}\),

\[
L(u) (e_1, e_2) = \sum_{\alpha=0}^3 w_\alpha(u) \sigma_\alpha \otimes (S^\alpha e_1, S^\alpha e_2) = \theta_1(2\eta) \sum_{\alpha=0}^3 w_\alpha(u) \sigma_\alpha \otimes (e_1, e_2) \sigma^\alpha = (e_1, e_2) \theta_1(2\eta) R_{12}(u). \tag{19}
\]

Thus the Baxter R-matrix is incorporated to the set of RLL-relation solutions \(^{(5)}\) and its entries provide two-dimensional (the fundamental) representation of the Sklyanin algebra \(^{(7)}\). In Sect. \( \S 6.1 \) we follow an opposite strategy showing explicitly how Lax operators with arbitrary finite-dimensional representations in the quantum space can be “fused” out of Baxter’s R-matrix \(^{(2)}\).

\(^{1}\)We warn the reader that due to an unconventional choice of the spectral parameter shift in \(^{(5)}\) (cf. equations \((2.5)\) and \((2.6)\) in \(^{3}\)) there is no discrepancy in dependence on spectral parameter between the Baxter R-matrix and the Lax operator reduction.
4 Elliptic modular double

4.1 Algebraic relations

In this section we consider an algebra which is called the elliptic modular double and which, roughly speaking, consists of two Sklyanin algebras. First, we outline the structure of finite-dimensional representations of this algebra postponing corresponding solutions of YBE (1) to the next section. The necessity of the doubling of Sklyanin algebras will be clarified there. We will see that the symmetry requirements with respect to the extended algebra pose additional restrictions as compared to the Sklyanin algebra that enable us to fix uniquely the YBE solutions.

The elliptic modular double was introduced in [39]. A particular version of this algebra degenerates in a special limit to Faddeev’s modular double of the quantum algebra $U_q(sl_2)$ [16]. Let us take two sets of generators $S^\alpha$ and $\tilde{S}^\alpha$ both respecting Sklyanin algebra commutation relations. The generators $S^\alpha$ fulfill commutation relations (7), and the operators $\tilde{S}^\alpha$, $\tilde{S}^\beta$ commute with all generators in this sector: \[
\tilde{S}^\alpha \tilde{S}^\beta - \tilde{S}^\beta \tilde{S}^\alpha = i \left( \tilde{S}^0 \tilde{S}^\gamma + \tilde{S}^\gamma \tilde{S}^0 \right), \\
\tilde{S}^0 \tilde{S}^\alpha - \tilde{S}^\alpha \tilde{S}^0 = i \tilde{J}_{\beta\gamma} \left( \tilde{S}^\beta \tilde{S}^\gamma + \tilde{S}^\gamma \tilde{S}^\beta \right),
\]
where the triplet $(\alpha, \beta, \gamma)$ is an arbitrary cyclic permutation of $(1, 2, 3)$. The modular double which we shall be using below corresponds to the $\tilde{S}^\alpha$-generators obtained from (9) after the permutation $2\eta \equiv \tau$. This yields the following parametrization of the tilded structure constants $\tilde{J}_{\alpha\beta} = \frac{\tilde{J}_{\beta\gamma}}{J_{\gamma}}$, $\gamma \neq \alpha, \beta$, where (cf. (9))

\[
\tilde{J}_1 = \frac{\theta_2(\tau|2\eta)\theta_2(0|2\eta)}{\theta_2^2(\tau/2|2\eta)}, \quad \tilde{J}_2 = \frac{\theta_3(\tau|2\eta)\theta_3(0|2\eta)}{\theta_3^2(\tau/2|2\eta)}, \quad \tilde{J}_3 = \frac{\theta_4(\tau|2\eta)\theta_4(0|2\eta)}{\theta_4^2(\tau/2|2\eta)}.
\]

The cross-commutation relations between generators of two Sklyanin algebras, $S^\alpha$ and $\tilde{S}^\alpha$, have the form

\[
S^\alpha \tilde{S}^\beta = \tilde{S}^\beta S^\alpha, \quad \alpha, \beta \in \{0, 3\} \quad \text{or} \quad \alpha, \beta \in \{1, 2\}, \\
S^\alpha \tilde{S}^\beta = -\tilde{S}^\beta S^\alpha, \quad \alpha \in \{0, 3\}, \beta \in \{1, 2\} \quad \text{or} \quad \alpha \in \{1, 2\}, \beta \in \{0, 3\}.
\]

Because of the non-commutativity of $S^\alpha$ and $\tilde{S}^\beta$ it is not a direct product of two Sklyanin algebras, though it is not difficult to trace the difference of actions of the subalgebra generators on modules in different orders.

The Casimir operators of the elliptic double’s second half (cf. (10))

\[
\tilde{K}_0 = \sum_{\alpha=0}^{3} \tilde{S}^\alpha \tilde{S}^\alpha, \quad \tilde{K}_2 = \sum_{\alpha=1}^{3} \tilde{J}_\alpha \tilde{S}^\alpha \tilde{S}^\alpha,
\]

commute with all generators in this sector: $[\tilde{K}_0, \tilde{S}^\alpha] = [\tilde{K}_2, \tilde{S}^\alpha] = 0$. Then from the cross-commutation relations (22) it follows that $K_0, K_2$ commute with $S^\alpha$ and, vice versa, $\tilde{K}_0, \tilde{K}_2$ commute with $\tilde{S}^\alpha$, i.e. there are in total four Casimir operators:

\[
[K_0, S^\alpha] = [K_2, \tilde{S}^\alpha] = [\tilde{K}_0, S^\alpha] = [\tilde{K}_2, \tilde{S}^\alpha] = 0.
\]

For building representations of the described elliptic modular double the Sklyanin realization of the generators (11) plays a crucial role. However, it is convenient in what follows to modify slightly the generators (11) by means of a similarity transformation:

\[
S^\alpha_{\text{mod}} \equiv e^{\pi iz^2/\eta} S^\alpha e^{-\pi iz^2/\eta}.
\]
The reason for such a conjugation is explained in [9]—it simplifies the form of the intertwining operator to be described below. Obviously, this modification does not alter the Sklyanin’s algebra commutation relations (11). Since in the rest of the paper we deal only with the new set of generators we will omit the mark ‘mod’ for the sake of brevity and denote by $S_\alpha$ the new set of generators,

$$S_\alpha = e^{\pi iz^2/\eta} i^{\delta_\alpha} \frac{\theta_{\alpha+1}(\eta|\tau)}{\theta_1(2z|\tau)} \left[ \theta_{\alpha+1} (2z - g + \eta|\tau) e^{\eta \partial_z} - \theta_{\alpha+1} (-2z - g + \eta|\tau) e^{-\eta \partial_z} \right] e^{-\pi iz^2/\eta}. \tag{24}$$

The parameter $g \in \mathbb{C}$ is related to the spin $\ell$ from (11) as $g = \eta (2\ell + 1)$ and we call it as the spin as well. In the following we will use $g$-spin as an independent parameter instead of $\ell$. The operators (24) provide a realization for the first Sklyanin algebra of the elliptic double. Permutation of the quasi-periods $2\eta \approx \tau$ yields a finite-difference operator realization of generators forming the second Sklyanin algebra,

$$\bar{S}_\alpha = e^{2\pi iz^2/\tau} i^{\delta_\alpha} \frac{\theta_{\alpha+1}(\tau/2|2\eta)}{\theta_1(2z|2\eta)} \left[ \theta_{\alpha+1} (2z - g + \tau/2|2\eta) e^{\frac{1}{2}\tau \partial_z} - \theta_{\alpha+1} (-2z - g + \tau/2|2\eta) e^{-\frac{1}{2}\tau \partial_z} \right] e^{-2\pi iz^2/\tau}, \tag{25}$$

where $g$-spin is the same arbitrary parameter as in (24). The cross-commutation relations (22) can be verified for the set of generators (24) and (25).

In the considered realization of the elliptic double four Casimir operators (10) and (23) reduce to the following scalar expressions

$$K_0 = 4 \theta_1^2 (g|\tau), \quad K_2 = 4 \theta_1 (g - \eta|\tau) \theta_1 (g + \eta|\tau),$$

$$\bar{K}_0 = 4 \theta_1^2 (g|2\eta), \quad \bar{K}_2 = 4 \theta_1 (g - \frac{\tau}{2}|2\eta) \theta_1 (g + \frac{\tau}{2}|2\eta),$$

which are invariant under the reflections $g \rightarrow -g$. The variables $\eta$ and $\tau$ are fixed by the structure constants (9) (or (21)). Therefore the $g$-spin parameter fixes the values of all Casimirs and specifies representations of the elliptic modular double.

In the previous section we considered Lax operators for the Sklyanin algebra. Let us extend corresponding formulae to the elliptic modular double. Since the latter contains two halves there are two different Lax operators $L^{doub}$ and $\bar{L}^{doub}$. The Lax operator related to the first algebra is constructed out of the generators $S_\alpha$, i.e. $L^{doub}$ is given by the expression (5) with $S_\alpha$ fixed in (24),

$$L^{doub}(u) = \sum_{\alpha=0}^3 w_\alpha(u) \sigma_\alpha \otimes S_\alpha; \quad w_\alpha(u) = \frac{\theta_{\alpha+1}(u + \eta|\tau)}{\theta_{\alpha+1}(\eta|\tau)}. \tag{26}$$

Analogously to (12) it can be factorized as follows

$$L^{doub}(u_1, u_2) = e^{\pi iz^2/\eta} \frac{1}{\theta_1(2z|\tau)} \left( \begin{array}{cc} \theta_3 (z - u_1|\frac{\tau}{2}) & -\theta_3 (z + u_1|\frac{\tau}{2}) \\ -\theta_4 (z - u_1|\frac{\tau}{2}) & \theta_4 (z + u_1|\frac{\tau}{2}) \end{array} \right) \left( \begin{array}{cc} e^{\eta \partial_z} & 0 \\ 0 & e^{-\eta \partial_z} \end{array} \right) \cdot$$

$$\left( \begin{array}{cc} \theta_4 (z + u_2|\frac{\tau}{2}) & \theta_3 (z + u_2|\frac{\tau}{2}) \\ \theta_4 (z - u_2|\frac{\tau}{2}) & \theta_3 (z - u_2|\frac{\tau}{2}) \end{array} \right) e^{-\pi iz^2/\eta}, \tag{27}$$

where the “light-cone” combinations of the spectral parameter and $g$-spin are (cf. (13))

$$u_1 = \frac{u + g}{2}, \quad u_2 = \frac{u - g}{2}. \tag{28}$$

Similarly, the Lax operator for the second Sklyanin algebra is constructed out of the generators $\bar{S}_\alpha$ (25),

$$\bar{L}^{doub}(u) = \sum_{\alpha=0}^3 \bar{w}_\alpha(u) \sigma_\alpha \otimes \bar{S}_\alpha; \quad \bar{w}_\alpha(u) = \frac{\theta_{\alpha+1}(u + \tau/2|2\eta)}{\theta_{\alpha+1}(\tau/2|2\eta)}. \tag{29}$$
It takes the factorized form as well which is obtained from (27) by interchange of the quasi-periods \( \tau \leftrightarrow 2 \eta \). Both Lax operators (26) and (29) respect YBE (6) with the Baxter’s R-matrices. For \( L^{doub} \) it is R-matrix (2), and for \( \tilde{L}^{doub} \) the R-matrix is given by (2) with \( 2 \eta \leftrightarrow \tau \) (i.e. by (2) with \( w_\alpha(u) \) substituted for \( \tilde{w}_\alpha(u) \)).

4.2 An intertwining operator

Since we are going to construct finite-dimensional solutions of YBE (1) with the symmetry of the elliptic modular double we need to get insight into the structure of its finite-dimensional representations. It is well known that intertwining operators of equivalent representations provide an extremely useful tool enabling one to uncover finite-dimensional representations. They give an evidence on the decoupling of finite-dimensional representations from infinite-dimensional ones through their null-spaces.

Let us consider an integral operator \( M(g) \) depending on a parameter \( g \in \mathbb{C} \) acting on holomorphic functions of one complex variable \( \Phi(z) \) as follows

\[
[M(g) \Phi](z) = \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g, \pm 2x)} \Phi(x) \, dx,
\]

(30)

where we assume the constraints \( \text{Im}(-g \pm z) > 0 \). An extension of the action of this operator to other domains of values of \( g \) and \( z \) is obtained by analytical continuation (a deformation of the integration contour to admissible limits). Here we denote

\[
\kappa = \frac{1}{2} \left( q; q \right)_\infty \left( p; p \right)_\infty, \quad p = e^{2\pi i \tau}, \quad q = e^{4\pi i \eta}.
\]

Parameters \( p \) and \( q \) are quasi-periods in the multiplicative notation (cf. (4)). We extensively use the short-hand notation \( \Gamma(a, b) := \Gamma(a)\Gamma(b), \quad \Gamma(\pm x) := \Gamma(x)\Gamma(-x), \quad \Gamma(\pm z \pm x) := \Gamma(z + x)\Gamma(z - x)\Gamma(-z + x)\Gamma(-z - x) \), etc. in order to avoid bulky expressions. The kernel of integral operator (30) is expressed through the elliptic gamma function \([17, 32, 37]\)

\[
\Gamma(z) \equiv \Gamma(z|\tau, 2\eta) \equiv \prod_{n,m=0}^{\infty} \frac{1 - e^{-2\pi i z}p^{n+1}q^{m+1}}{1 - e^{2\pi i z}p^n q^m}
\]

(31)

defined for \( |p|, |q| < 1 \). This special function is omnipresent in our considerations and it possesses a number of important identities indicate below. The \( \Gamma(z) \)-function is quasi-periodic with respect to the shifts by \( 2\eta \) and \( \tau \) that justifies its name,

\[
\Gamma(z + 2\eta) = R(\tau) e^{i\pi z} \theta_1(z|\tau) \Gamma(z), \quad \Gamma(z + \tau) = R(2\eta) e^{i\pi z} \theta_1(z|2\eta) \Gamma(z),
\]

(32)

where on the right-hand sides one actually has the coefficients \( \theta(e^{2\pi i z}; p) \) and \( \theta(e^{2\pi i z}; q) \), respectively (cf. (4) and (14)). One has the reflection identity

\[
\Gamma(z) \Gamma(-z + 2\eta + \tau) = 1.
\]

(33)

Zeros of \( \Gamma(z) \) are located on the two-dimensional lattice \( z = \mathbb{Z} + \tau \mathbb{Z}_{>0} + 2\eta \mathbb{Z}_{>0} \) and poles on the lattice \( z = \mathbb{Z} + \tau \mathbb{Z}_{\leq0} + 2\eta \mathbb{Z}_{\leq0} \).

The integral operator \( M(g) \) was introduced in \([38]\) in order to define a universal integral transform of hypergeometric type yielding an integral analogue of the Bailey chain techniques \([1]\). It is relevant to YBE since the general R-operator factors as a product of either \( M(g) \) or its kernel (see \([9]\) and Sect.
This operator satisfies a very simple inversion relation resembling the key Fourier transformation property \[ M(g) M(-g) = 1, \] which is true for an appropriate space of test functions at least for the values of parameter \( g \) away from two discrete two-dimensional quarter-infinite lattices \( g = n\eta + m\frac{\tau}{2} \) and \( g = \frac{1}{2} + n\eta + m\frac{\tau}{2}, \) \( n, m \in \mathbb{Z}. \)

As shown in [9] the operator (30), being symmetric in \( 2\eta \) and \( \tau, \) satisfies the following intertwining relations with the generators of elliptic modular double,

\[
M(g) S^\alpha(g) = S^\alpha(-g) M(g), \quad M(g) \tilde{S}^\alpha(g) = \tilde{S}^\alpha(-g) M(g). \tag{35}
\]

Here we explicitly indicate the \( g \)-spin dependence of the algebra generators (see [24] and [25]) in order to show that \( g \) changes the sign under the action of \( M. \) Thus the integral operator \( M(g) \) (30) is an intertwining operator of equivalent representations of the elliptic modular double [9]. Evidently, it is an intertwining operator of equivalent representations of the Sklyanin algebra itself. However, in the single Sklyanin algebra case there is a family of intertwining operators depending on a periodic function of two complex variables, i.e. there is a functional freedom in their choice. The elliptic modular double enables one to fix uniquely the representative (30) out of this family. In the conventional Sklyanin algebra setting usage of the spin variable \( \ell \) is preferable. From the relation \( g = \eta(2\ell + 1) \) one can see that the equivalent representations are obtained by the transformation \( \ell \rightarrow -1 - \ell. \)

Now we consider the intertwining operator (30) for two-index discrete lattice of the \( g \)-spin parameter \( g = n\eta + m\tau/2 \) with \( n, m \in \mathbb{Z}_{\geq 0}. \) In this case \( M(g) \) drastically simplifies. This can be seen by means of the contiguous (or recurrence) relations for the intertwining operator [6] [10]

\[
A_k(g) M(g) = M(g + \eta) \theta_k \left( z\frac{\tau}{2} \right); \quad B_k(g) M(g) = M \left( g + \frac{\tau}{2} \right) \theta_k \left( z\eta \right), \tag{36}
\]

where \( A_k(g) \) and \( B_k(g), \) \( k = 3, 4, \) are the following difference operators

\[
A_k(g) = e^{\frac{\pi i z^2}{\eta}} \cdot \frac{c_A}{\theta_1(2z|\tau)} \left[ \theta_k \left( z + g + \eta\frac{\tau}{2} \right) e^{\eta\partial_z} - \theta_k \left( z - g - \eta\frac{\tau}{2} \right) e^{-\eta\partial_z} \right] e^{-\pi i z^2},
\]

\[
B_k(g) = e^{2\pi i z^2} \cdot \frac{c_B}{\theta_1(2z|2\eta)} \left[ \theta_k \left( z + g + \frac{\tau}{2}|\eta \right) e^{\frac{\tau}{2}\partial_z} - \theta_k \left( z - g - \frac{\tau}{2}|\eta \right) e^{-\frac{\tau}{2}\partial_z} \right] e^{-2\pi i z^2},
\]

with the normalization constants

\[
c_A = \frac{e^{\pi i \eta}}{R(\tau)}, \quad c_B = \frac{e^{\pi i \frac{\tau}{2}}}{R(2\eta)}. \tag{37}
\]

The contiguous relations (36) describe transformations of \( M(g) \) under shifts along the two-dimensional \( g \)-spin lattice \( g = n\eta + m\tau/2, \) \( n, m \in \mathbb{Z}_{\geq 0}. \) As shown in [10] (see also [6]) they lead to a factorized representation of \( M(g) \) on this lattice. Indeed, using the initial condition \( M(0) = 1, \) which is proved by the residue calculus [9], one solves straightforwardly the contiguous relations (36) in \( \eta \)-direction or \( \frac{\tau}{2} \)-direction on the lattice and obtains

\[
M(n\eta) = A_k(n\eta - \eta) \cdots A_k(\eta) A_k(0) \cdot \theta_k^{-n} \left( z\frac{\tau}{2} \right), \tag{38}
\]

\[
M \left( m\frac{\tau}{2} \right) = B_k \left( m\frac{\tau}{2} - \frac{\tau}{2} \right) \cdots B_k \left( \frac{\tau}{2} \right) B_k(0) \cdot \theta_k^{-m} \left( z\eta \right) \theta_k^{-n} \left( z\frac{\tau}{2} \right).
\]

Note, that the form of the intertwiner does not depend on the values of index \( k \) for theta functions involved. In the general case of \( M \left( n\eta + m\frac{\tau}{2} \right) \) we have [10]

\[
M \left( n\eta + m\frac{\tau}{2} \right) = A_k(n\eta - \eta + m\frac{\tau}{2}) \cdots A_k(\eta + m\frac{\tau}{2}) A_k(m\frac{\tau}{2}) \cdot
\]

\[
\cdot B_k \left( m\frac{\tau}{2} - \frac{\tau}{2} \right) \cdots B_k \left( \frac{\tau}{2} \right) B_k(0) \cdot \theta_k^{-m} \left( z\eta \right) \theta_k^{-n} \left( z\frac{\tau}{2} \right). \tag{38}
\]
Thus we see that \( M(n \eta + m \tau^2) \) is a finite-difference operator with elliptic coefficients. Of course there are many equivalent ways to represent \( M(n \eta + m \tau^2) \) as a product of \( A_k \) and \( B_k \)-operators corresponding to all possible zigzags on the lattice from the point \( g = 0 \) to \( g = n \eta + m \tau/2 \). In \( \langle 35 \rangle \) we choose the trajectory starting from \( g = 0 \), going along the \( \tau \)-direction to \( g = m \tau/2 \), then turning towards the \( \eta \)-direction and proceeding to \( g = n \eta + m \tau/2 \).

Expanding \( \langle 35 \rangle \) one obtains the intertwiner in the form of a sum over shift operators \( e^{(k \eta + l \tau^2)} \partial \), \( k, l \in \mathbb{Z} \) with elliptic coefficients. Explicit expressions can be found in \([10]\). In this form the discrete intertwining operator \( M(n \eta + m \tau^2) \) has been obtained first in \([47]\).

There is the second lattice \( g = \frac{1}{2} + n \eta + m \tau^2 \), \( n, m \in \mathbb{Z}_{\geq 0} \), leading to a crucial simplification of the intertwiner. The residue calculation from \([9]\) shows that the intertwiner \( \langle 30 \rangle \) simplifies at \( g = \frac{1}{2} \) to \( M(\frac{1}{2}) = e^{\frac{1}{2} \partial} \). As shown in \([10]\) solving the contiguous relations \( \langle 36 \rangle \) the intertwiners on both lattices are related to each other by the half-period shift operator

\[
M(\frac{1}{2} + n \eta + m \tau^2) = M(n \eta + m \tau^2) e^{\frac{1}{2} \partial}. \tag{39}
\]

### 4.3 Finite-dimensional representations of the elliptic double

The intertwining operator gives us an insight to finite-dimensional representations of the elliptic modular double. Indeed, equalities \( \langle 35 \rangle \) show that the null-space of the operator \( M(g) \) and the image of the operator \( M(-g) \) form invariant spaces for the elliptic modular double, i.e. they both are invariant under the action of two constituent Sklyanin algebra generators \( S^\alpha(g) \) \( \langle 24 \rangle \) and \( \tilde{S}^\alpha(g) \) \langle 25 \rangle,

\[
S^\alpha, \tilde{S}^\alpha : \text{Ker} M(g) \to \text{Ker} M(g); \quad S^\alpha, \tilde{S}^\alpha : \text{Im} M(-g) \to \text{Im} M(-g). \tag{40}
\]

Consequently, if the intertwining operator has a nontrivial null-space then a sub-representation of the algebra decouples and a corresponding invariant subspace naturally emerges.

The finite-dimensional null-space of the intertwiner \( \langle 30 \rangle \) was partially characterized in \([9]\) and extensively investigated in \([10]\). There it was proven in two ways – by means of the factorized representation \( \langle 38 \rangle \) and studying limits of \( g \)-spin in the integral representation \( \langle 30 \rangle \) – that the nontrivial null-space of the intertwiner arises on two distinct \( g \)-spin lattices:

\[
g = n \eta + m \frac{\tau}{2} \quad \text{and} \quad g = \frac{1}{2} + n \eta + m \frac{\tau}{2}, \quad n, m \in \mathbb{Z}_{\geq 0}, \quad (n, m) \neq (0, 0). \tag{41}
\]

Here we outline the second proof since it provides a natural basis of finite-dimensional representations and the corresponding generating function of the representation.

The whole null-space of the intertwiner is too big, so we need imposing some additional constraints. The intersection of two invariant subspaces \( \langle 40 \rangle \) is a finite-dimensional representation of the elliptic double \( \langle 10 \rangle \)

\[
\text{Ker} M(g) \cap \text{Im} M_{\text{ren}}(-g) \tag{41}
\]

where the \( g \)-spins are on the lattices

\[
g = (n + 1) \eta + (m + 1) \frac{\tau}{2} \quad \text{or} \quad g = \frac{1}{2} + (n + 1) \eta + (m + 1) \frac{\tau}{2}, \quad n, m \in \mathbb{Z}_{\geq 0}, \tag{42}
\]

and \( M_{\text{ren}}(g) \) is the renormalized intertwining operator which, obviously, fulfills the intertwining relations \( \langle 35 \rangle \) with the elliptic modular double generators (cf. \( \langle 30 \rangle \)),

\[
[M_{\text{ren}}(g) \Phi](z) = \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(\pm 2x)} \Phi(x) \, dx. \tag{43}
\]
To demonstrate this fact we rewrite the inversion relation \((34)\) multiplying it by the numerical factor \(\Gamma(2g)\) and take the limit \(g \to (n + 1)\eta + (m + 1)\frac{\tau}{2}\) at \(n, m \in \mathbb{Z}_{\geq 0}\). Since \(\Gamma(2g)\) vanishes in this limit (recall \((31)\)) we come to the operator identity

\[
M \left( (n + 1)\eta + (m + 1)\frac{\tau}{2} \right) M_{ren} \left( -(n + 1)\eta - (m + 1)\frac{\tau}{2} \right) = 0, \tag{44}
\]

which implies that the intertwiner annihilates the kernel of integral operator \(M_{ren} \) considered as a function of \(z\) and containing an auxiliary parameter \(x\),

\[
M_z \left( (n + 1)\eta + (m + 1)\frac{\tau}{2} \right) \cdot \Gamma \left( \mp z \mp x + (n + 1)\eta + (m + 1)\frac{\tau}{2} \right) = 0.
\]

Here the subindex \(z\) indicates that the \(M_z\)-operator acts by integration over the \(z\)-variable. Thus we conclude that the function

\[
\Gamma \left( \mp z \mp x + (n + 1)\eta + (m + 1)\frac{\tau}{2} \right) \tag{45}
\]

is a generating function of \((n + 1)(m + 1)\)-dimensional representation for the spin lattice spin \(g = (n + 1)\eta + (m + 1)\frac{\tau}{2}\), \(n, m \in \mathbb{Z}_{\geq 0}\). Expanding it with respect to theta-functions of the auxiliary parameter \(x\) we recover all basis vectors of the finite-dimensional representation. Since the generating function \((45)\) is a kernel of \(M_{ren}\), all basis vectors belong not only to the null-space of \(M(g)\), but they also lie in the intersection with \(\text{Im} M_{ren}(-g) \) \((41)\). Moreover, this representation is irreducible.

Now we explicitly extract basis vectors from \((45)\). We simplify the generating function to a finite product of theta functions using the equations \((32)\) and the inversion relation \((33)\),

\[
\Gamma \left( \mp z \mp x + (n + 1)\eta + (m + 1)\frac{\tau}{2} \right) = \prod_{k=0}^{n-1} \theta(e^{2\pi i(z \pm x + (2k + 1)\eta + (1 + m)\frac{\tau}{2})}; p) \prod_{l=0}^{m-1} \theta(e^{2\pi i(z \pm x + (1 - n)\eta + (2l + 1 - m)\frac{\tau}{2})}; q).
\]

Passing to \(\theta_{3,4}(z|\frac{\tau}{2})\) and \(\theta_{3,4}(z|\eta)\) functions, one can rewrite the latter products as

\[
c \cdot \prod_{r=0}^{n-1} \left[ \theta_3(z|\frac{\tau}{2}) \theta_4(x + (n - 1 - 2r)\eta|\frac{\tau}{2}) + (-1)^m \theta_4(z|\frac{\tau}{2}) \theta_3(x + (n - 1 - 2r)\eta|\frac{\tau}{2}) \right] \cdot \prod_{s=0}^{m-1} \left[ \theta_3(z|\eta) \theta_4(x + (m - 1 - 2s)\frac{\tau}{2}|\eta) + (-1)^n \theta_4(z|\eta) \theta_3(x + (m - 1 - 2s)\frac{\tau}{2}|\eta) \right], \tag{46}
\]

where \(c\) is a normalization constant,

\[
c = (-2)^{-m-n} R^{2n}(\tau) R^{2m}(2\eta) e^{-\pi i m (m^2 - 1) - 2 i \pi n (n^2 - 1)}.
\]

Thus we conclude that the set of \((n + 1)(m + 1)\) homogeneous polynomials in theta functions of degree \(n\) with respect to \(\theta_3(z|\frac{\tau}{2})\) and \(\theta_4(z|\frac{\tau}{2})\) and of degree \(m\) with respect to \(\theta_3(z|\eta)\) and \(\theta_4(z|\eta)\), i.e.

\[
\varphi_{j,l}^{(n,m)}(z) = \left[ \theta_3(z|\frac{\tau}{2}) \right]^j \left[ \theta_4(z|\frac{\tau}{2}) \right]^{m-1-j} \cdot \left[ \theta_3(z|\eta) \right]^l \left[ \theta_4(z|\eta) \right]^{m-1-l} \tag{47}
\]

with \(j = 0, 1, 2, \ldots, n\) and \(l = 0, 1, 2, \ldots, m\), is a basis of the finite-dimensional representation which arises at the spin values \(g = (n + 1)\eta + (m + 1)\frac{\tau}{2}\), with \(n, m \in \mathbb{Z}_{\geq 0}\). This result is in line with the basis \((14)\) in the space of finite-dimensional representations of the Sklyanin algebra.

Note that the variable \(z\) and the auxiliary parameter \(x\) appear on equal footing in the arguments of \(\Gamma\)-functions in \((45)\), however the symmetry \(z \leftrightarrow x\) is not obvious in the form \((46)\). Thus implementing the change \(z \leftrightarrow x\) we obtain another expansion of the generating function in theta functions. Consequently the generating function \((46)\) produces two natural bases: the basis \((47)\)
\{ \varphi_{j,l}^{(n,m)}(z) \}_{l=0,1,\ldots,m} \text{ and the basis } \{ \psi_{j,l}^{(n,m)}(z) \}_{j=0,1,\ldots,n} \text{ formed by sums of theta function products with the arguments shifted by a multiple of } \eta \text{ or } \frac{\eta}{2}.

\psi_{j,l}^{(n,m)}(z) = \text{Sym} \prod_{r=0}^{n-1} \theta_{a_r} (z + (n - 1 - 2r)\eta | \frac{\eta}{2} ) \cdot \text{Sym} \prod_{s=0}^{m-1} \theta_{b_s} (z + (m - 1 - 2s)\frac{\eta}{2} | \eta ) \quad (48)

where \( a_r, b_s \in \{3, 4\} \), \( \theta_3 \) appears \( j \) times in the first product and \( \theta_3 \) appears \( l \) times in the second product, and Sym stands for symmetrization with respect to indices 3, 4 specifying theta functions.

Analogous results take place for the second \( g \)-spin lattice. The generating function of finite-dimensional representations at spin \( g = \frac{1}{2} + (n + 1)\eta + (m + 1)\frac{\eta}{2} \), with \( n, m \in \mathbb{Z}_{\geq 0} \), is

\[ \Gamma \left( \mp z \mp x + \frac{1}{2} + (n + 1)\eta + (m + 1)\frac{\eta}{2} \right) . \quad (49) \]

Since the shift of \( z \to z + \frac{1}{2} \) results in the permutation of theta functions \( \theta_3(z) \) and \( \theta_4(z) \),

\[ e^{\frac{1}{2} \theta} : (\theta_3(z), \theta_4(z)) \to (\theta_4(z), \theta_3(z)) \]

we conclude that \( \varphi_{j,l}^{(n,m)}(z) \) (47) and \( \psi_{j,l}^{(n,m)}(z) \) (48) still form appropriate bases.

\section{The general elliptic R-operator}

\subsection{Intertwining operators and factorization of the R-operator}

We proceed to increasing the complexity level in YBE (1) and consider its solutions with the symmetry of the elliptic modular double (see Sect. 4), which imposes more severe restrictions as compared to the plain Sklyanin algebra case. The reason for this choice will become evident shortly. Let the spaces \( \mathbb{V}_1 \) and \( \mathbb{V}_2 \) in (1) are arbitrary and the third space is two-dimensional, \( \mathbb{V}_3 = \mathbb{C}^2 \).

Like in Sect. 3 two out of three R-matrices from (1) reduce to the Lax operators \( \mathbb{R}_{13}(u) \equiv L_{13}(u) \) and \( \mathbb{R}_{23}(u) \equiv L_{23}(u) \), but this time they share a common auxiliary two-dimensional space \( \mathbb{V}_3 = \mathbb{C}^2 \), matrix entries of \( L_{13} \) are operators acting in \( \mathbb{V}_1 \), and entries of \( L_{23} \) act in \( \mathbb{V}_2 \). Further, we omit the index 3 in \( L_{13}, i = 1, 2, \) of the common auxiliary space. Since there are two different Lax operators associated with the elliptic double, \( L_{\text{doub}}^{(\alpha \beta)} (26) \) and \( \tilde{L}_{\text{doub}}^{(\alpha \beta)} (29) \), equation (11) yields two different relations, known as RLL-relations [25, 33]:

\begin{align*}
\mathbb{R}_{12}(u - v) L_{12}^{\text{doub}}(u) L_{22}^{\text{doub}}(v) &= L_{22}^{\text{doub}}(v) L_{12}^{\text{doub}}(u) \mathbb{R}_{12}(u - v), \quad (50) \\
\mathbb{R}_{12}(u - v) \tilde{L}_{12}^{\text{doub}}(u) \tilde{L}_{22}^{\text{doub}}(v) &= \tilde{L}_{22}^{\text{doub}}(v) \tilde{L}_{12}^{\text{doub}}(u) \mathbb{R}_{12}(u - v). \quad (51)
\end{align*}

The operator \( \mathbb{R}_{12}(u) : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V}_1 \otimes \mathbb{V}_2 \) is called the general R-operator for the elliptic modular double. Let us stress that it has to respect both relations (50) and (51) simultaneously. They can be interpreted as operator equations that the general R-operator has to satisfy. Indeed, the general elliptic R-operator has been constructed in [9] solving (50) and (51) which jointly fix it uniquely up to a normalization constant.

Some more comments are in order. We assume that \( \mathbb{V}_1 \) is a space of functions of a complex variable \( z_1 \) and \( \mathbb{V}_2 \) is a space of functions of a complex variable \( z_2 \). Therefore the operator \( \mathbb{R}_{12} \) acts in the space \( \mathbb{V}_1 \otimes \mathbb{V}_2 \) of functions \( \Phi(z_1, z_2) \) of two independent complex variables \( z_1 \) and \( z_2 \). We use shorthand notations: the index \( k \) of \( L_k^{\text{doub}} \) and \( \tilde{L}_k^{\text{doub}} \) indicates that the elliptic double generators \( S_k^\alpha \) and \( \tilde{S}_k^\alpha \) forming these matrices are finite-difference operators [24] and [25] with a spin \( g_k \), and they act in the space \( \mathbb{V}_k \), i.e. \( S_k^\alpha \), \( \tilde{S}_k^\alpha \) : \( \mathbb{V}_k \to \mathbb{V}_k \). The operators \( S_1^\alpha, \tilde{S}_1^\alpha \) and \( S_2^\beta, \tilde{S}_2^\beta \) act in different spaces and, obviously, they commute with each other, \([ S_1^\alpha, S_2^\beta ] = [ \tilde{S}_1^\alpha, \tilde{S}_2^\beta ] = [ S_1^\alpha, \tilde{S}_2^\beta ] = [ \tilde{S}_1^\alpha, S_2^\beta ] = 0 \). Matrices \( L_k^{\text{doub}} \) in equation (50) are multiplied as usual \( 2 \times 2 \) matrices acting in the space \( \mathbb{V}_3 = \mathbb{C}^2 \),
and the same is valid for $\tilde{L}_{k}^{doub}$ in (51). In the relations (50), (51) we omit dependence on the spins $g_k$ in the Lax operators and in the general R-operator.

Let us concentrate our attention to the first RLL-relation (50), i.e. to a one half of the elliptic modular double. The Lax operator for the Sklyanin algebra is not unique. The transformations $L^{doub} \rightarrow \sigma_\alpha L^{doub}$, where $\sigma_\alpha$ is any Pauli matrix, are automorphisms of the Sklyanin algebra [33], i.e. they are consistent with the algebra commutation relations (7). Consequently, there are several possible forms of the equation (50) with different general R-operators labeled by a pair of supplementary indices $\alpha$ and $\beta$ specifying a choice of the Lax operators

$$R_{12}^{\alpha\beta}(u - v) \sigma_\alpha L_1^{doub}(u) \sigma_\beta L_2^{doub}(v) = \sigma_\beta L_2^{doub}(v) \sigma_\alpha L_1^{doub}(u) R_{12}^{\alpha\beta}(u - v).$$

(52)

For a technical reason (see [9]) we fix $\alpha = \beta = 3$ from the very beginning and denote $R_{jk}(u) \equiv R_{jk}^{33}(u)$ that corresponds to the $c$-series of the Sklyanin algebra representations [33]. In this case it is possible to cancel one of the $\sigma_3$-matrices such that the defining RLL-relation takes the form

$$R_{12}(u - v) L_1^{doub}(u_1, v_2) = L_2^{doub}(v_1, v_2) L_1^{doub}(u_1, u_2) R_{12}(u - v),$$

(54)

where we restored dependence on the spins $g_1$ and $g_2$ in the Lax operators by means of the parameters $u_1, u_2, v_1, v_2$ (cf. (28))

$$u_1 = \frac{u + g_1}{2}, \quad u_2 = \frac{u - g_1}{2}; \quad v_1 = \frac{v + g_2}{2}, \quad v_2 = \frac{v - g_2}{2}.$$

(55)

In (52) we still omit dependence of the general R-operator on the spins $g_1, g_2$. The full-fledged notation would be $R_{12}(u_1, u_2|v_1, v_2)$ or $R_{12}(u - v|g_1, g_2)$.

The general R-operator, i.e. an integral operator solution of the equation (54), has been constructed in [9] following the general ideological scheme of [8] powered by the techniques of elliptic hypergeometric integrals [35, 37, 40]. The construction naturally implies factorization of the R-operator to a product of several simple operators. In [8] a number of factorized forms of the R-operator has been derived which were related to an operator representation of the symmetric group $\mathfrak{S}_4$. Here we do not go into details of the construction and just present the factorized form of R-operator needed for our current purposes. The R-operator is a composite operator given by the following product [9]

$$R_{12}(u - v) = S(u_1 - v_2) M_2(u_2 - v_2) M_1(u_1 - v_1) S(u_2 - v_1),$$

(56)

where the constituting blocks - elementary intertwining operators - are the following operators

$$[S(a) \Psi](z_1, z_2) = \Gamma(\pm z_1 \pm z_2 + a + \eta + \frac{t}{2}) \Psi(z_1, z_2),$$

(57)

$$[M_1(b) \Psi](z_1, z_2) = \kappa \int_0^1 \frac{\Gamma(\pm z_1 \pm x - b)}{\Gamma(-2b, \pm 2x)} \Psi(x, z_2) dx,$$

(58)

$$[M_2(c) \Psi](z_1, z_2) = \kappa \int_0^1 \frac{\Gamma(\pm z_2 \pm x - c)}{\Gamma(-2c, \pm 2x)} \Psi(z_1, x) dx,$$

(59)

and the normalization constant is $\kappa = \frac{1}{2}(q; q)_\infty (p; p)_\infty$. Note that operators $M_1$ and $M_2$ are copies of the intertwining operator of the equivalent representations $M$ (50). They act in the spaces of
functions of variables $z_1$ and $z_2$, respectively. One can straightforwardly recast the factorized form \( [56] \) of the R-operator in an integral operator form whose kernel contains products of 16 nontrivial elliptic gamma functions.

It was shown in \( [9] \) that the elementary intertwining operators solve the following defining relations, which justify the chosen terminology,

\[
M_1(u_1 - u_2) L_1^{doub}(u_1, u_2) = L_1^{doub}(u_2, u_1) M_1(u_1 - u_2), \tag{60}
\]

\[
M_2(v_1 - v_2) L_2^{doub}(v_1, v_2) = L_2^{doub}(v_2, v_1) M_2(v_1 - v_2), \tag{61}
\]

\[
S(u_2 - v_1) L_1^{doub}(u_1, u_2) \sigma_3 L_2^{doub}(v_1, v_2) = L_1^{doub}(u_1, v_1) \sigma_3 L_2^{doub}(u_2, v_2) S(u_2 - v_1). \tag{62}
\]

Using these relations one can check straightforwardly that the general R-operator \( [56] \) does satisfy the RLL-relation \( [53] \). The indicated operators \( [57] \), \( [58] \), \( [59] \) are not the most general solutions of defining relations \( [60] \), \( [61] \), \( [62] \). We are free to multiply the kernels of integral operators $M_1$, $M_2$, and multiplication by a function operator $S$ by periodic functions of $z_{1,2}$ with the period $\eta$ (see \( [9] \) for details). Resorting to the second half of the elliptic double we are able to completely fix this residual freedom. Indeed, the elementary intertwining operators $S$, $M_1$, $M_2$ are invariant with respect to the permutation of moduli $\tau \rightarrow 2\eta$. The same is true for their product \( [56] \) – the general R-operator. Consequently the general R-operator \( [56] \) satisfies as well the accompanying RLL-relation, which contains $L^{doub}$ and is responsible for the second half of the elliptic double

\[
R_{12}(u - v) \tilde{L}_1^{doub}(u_1, u_2) \sigma_3 \tilde{L}_2^{doub}(v_1, v_2) = \tilde{L}_1^{doub}(v_1, v_2) \sigma_3 \tilde{L}_2^{doub}(u_1, u_2) R_{12}(u - v). \tag{63}
\]

As a result, the ambiguity periodic functions should be periodic with respect to the shifts of $z_{1,2}$ by $\pi/2$. Joining to this analyticity structures the demand that we are working in the space of periodic functions with the period 1 (which is equivalent to the analyticity requirements in the multiplicative variables $e^{2\pi i z_{1,2}}$), we fix these functions to be constants by the Jacobi theorem on the absence of nontrivial triply periodic functions (the variables $1, \eta, \pi/2$ should be incommensurate for its validity).

The same ambiguity phenomenon in the R-operator construction takes place for Lie algebra $\mathfrak{sl}_2$ and the quantum algebra $U_q(\mathfrak{sl}_2)$ which is cured automatically for the group $\text{SL}(2, \mathbb{C})$ and the modular double $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ (see for example \( [7] \) and references therein).

Finally, one needs checking that the outlined construction of the general R-operator \( [56] \) is selfconsistent. For that it is necessary to go one step up in the complexity hierarchy of YBE \( [1] \) and reaching the highest level. More precisely, it is necessary to show that derived R-operator solves this equation when all three spaces $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$ are infinite-dimensional representations of the elliptic modular double. Let us keep the corresponding spins $g_1$, $g_2$, $g_3$ in the generic position. Stripping off the permutation operators $P_{ij}$ from \( [1] \) we find the Yang-Baxter relation for the general R-operator

\[
R_{23}(u - v) R_{12}(u) R_{23}(v) = R_{12}(v) R_{23}(u) R_{12}(u - v), \tag{64}
\]

which is indeed satisfied by \( [56] \). The proof is straightforward and is based on the following relations for the elementary intertwining operators

\[
M_k(a) M_k(-a) = 1, \quad M_k(a) S(a + b) M_k(b) = S(b) M_k(a + b) S(a), \quad k = 1, 2, \tag{65}
\]

\[
M_1(a) M_2(b) = M_2(b) M_1(a), \quad S(a) S(-a) = 1. \tag{66}
\]

The quadratic relations in \( [65] \) are two copies of \( [33] \), and the second quadratic relation in \( [66] \) is a consequence of the reflection identity \( [33] \). The cubic relations in \( [65] \) coincide with an operator form of the star-triangle relation

\[
\Gamma(\pm x \pm z + \alpha + \eta + \frac{\pi}{2}) M_2(\alpha + \beta) \Gamma(\pm x \pm z + \beta + \eta + \frac{\pi}{2}) = M_2(\beta) \Gamma(\pm x \pm z + \alpha + \beta + \eta + \frac{\pi}{2}) M_2(\alpha), \tag{67}
\]
and it is a consequence of the Bailey lemma associated with an elliptic Fourier transformation \( [35] \).

The latter lemma is a direct consequence of the following elliptic beta integral evaluation formula discovered in \( [35] \)

\[
\kappa \int_0^1 dz \frac{\Gamma(\pm z + z_1 - a)}{\Gamma(-2a, \pm 2z)} \cdot \Gamma(\pm z + z_2 + a + b + \eta + \frac{\tau}{2}) \cdot \frac{\Gamma(\pm x + z - b)}{\Gamma(-2b, \pm 2x)} =
\]

\[
= \Gamma(\pm z_1 + z_2 + b + \eta + \frac{\tau}{2}) \cdot \frac{\Gamma(\pm x + z_1 - a - b)}{\Gamma(-2a - 2b, \pm 2x)} \cdot \Gamma(\pm x + z_2 + a + \eta + \frac{\tau}{2}),
\]

where \( \kappa = \frac{1}{2} (q; q)_\infty (p; p)_\infty \) and imaginary parts of the parameters \(-a \pm z_1\), \(-a \pm x\), \((a + b \pm z_2 + \eta + \tau/2)\) are positive (the latter constraints on the parameters can be relaxed by an analytical continuation). A functional form of the star-triangle relation \((67)\) has been considered in \([4]\).

### 5.2 Reductions of the general R-operator to finite-dimensional representations

Now we proceed to the main subject of our work and reduce the general R-operator \((56)\) to finite-dimensional representations in the first space that yields a series of new YBE solutions. The principle possibility for implementing this reduction is given by the following intertwining relation with the general R-operator \((56)\)

\[
M_1(u_1 - u_2) \mathbb{R}_{12}(u_1, u_2 | v_1, v_2) = \mathbb{R}_{12}(u_2, u_1 | v_1, v_2) M_1(u_1 - u_2),
\]

which can be proved using the identities \((65), (66)\). This relation shows that both, the null-space of the intertwining operator \(M_1(g_1)\) and the image of \(M_1(-g_1)\) (recall \((55)\)), are mapped onto themselves by the R-operator \(\mathbb{R}_{12}\). Therefore, if we find invariant finite-dimensional subspaces of the latter spaces they will be invariant with respect to the action of the R-operator itself.

Let us stress that the reduction problem is quite nontrivial. Indeed, the general R-operator \((56)\) is an integral operator acting in a pair of infinite-dimensional spaces, but the reduced R-operator (in one the spaces) becomes a matrix whose entries are finite-difference operators. The reduction in both spaces of \(R_{12}\) converts an integral operator into a matrix with \(u\)-dependent numeric coefficients.

In order to single out a finite-dimensional subspace out of the infinite-dimensional representation in the first space we consider the generating function \(\Gamma(\mp z_2 + u_1 - u_2)\) (recall \((45)\) and \((49)\)), where \(z_2\) is an auxiliary parameter, and act upon it by the R-operator. We break down the calculation in several steps according to the factorized form of the general R-operator \((56)\). In the end we choose the spin parameter at the first space to sit on the two-dimensional discrete lattice (recall \((55)\))

\[
g_1 = u_1 - u_2 = g_{n,m} = (n + 1) \eta + (m + 1) \tau/2 \quad \text{with} \quad n, m \in \mathbb{Z}_{\geq 0},
\]

such that the generating function produces a basis of the finite-dimensional representation in the first space \((66)\). However, for a while we assume \(g_1 = u_1 - u_2\) to be generic.

We act by the first two factors \(M_1(u_1 - v_1) S(u_2 - v_1)\) of the R-operator \((56)\) on the function \(\Gamma(\mp z_1 \mp z_3 + u_1 - u_2) \Phi(z_2)\), where \(\Phi(z_2)\) is an arbitrary holomorphic test function,

\[
M_1(u_1 - v_1) S(u_2 - v_1) \cdot \Gamma(\mp z_1 \mp z_3 + u_1 - u_2) \Phi(z_2) = \frac{\Gamma(2u_1 - 2u_2)}{\Gamma(2v_1 - 2u_2)}
\]

\[
\cdot \frac{\Gamma(\mp z_2 \pm z_1 \mp z_3 + u_1 - u_1 + \eta + \frac{\tau}{2})}{\Gamma(\mp z_2 \mp z_1 + u_1 - v_1 + \eta + \frac{\tau}{2})} \Phi(z_2).
\]

The right-hand side expression was obtained using the elliptic beta integral evaluation formula \((68)\).

In order to apply the third factor \(M_2(u_2 - v_2)\) \((58)\) of the R-operator \((56)\), we resort to the relation

\[
M_2(u_2 - v_2) \Gamma(\mp z_1 \mp z_2 + u_2 - u_1 + \eta + \frac{\tau}{2}) \Gamma(\mp z_2 \mp z_1 + u_1 - v_1 + \eta + \frac{\tau}{2}) \Phi(z_2) =
\]

\[
= \frac{\Gamma(2u_2 - 2u_1 + 2\eta + \tau)}{\Gamma(2v_2 - 2u_2)} M_1(u_1 - u_2 - \eta - \frac{\tau}{2}) \Gamma(\mp z_1 \mp z_2 - v_2 - u_2) \Gamma(\mp z_2 \mp z_1 + u_1 - v_1 + \eta + \frac{\tau}{2}) \Phi(z_2),
\]
which follows immediately from the integral representation (30) for $M(g)$. A merit of the previous formula is that we traded a complicated integral operator $M_2(u_2 - v_2)$ for $M_1(u_1 - u_2 - \eta - \frac{\tau}{2})$, which turns to the finite-difference operator $M_1(n\eta + m\frac{\tau}{2})$ (38) on the lattice (70) for the spin $g_1$ values. Incorporating into the latter formula the inert factors from (71) and the last factor $S(u_1 - v_2)$ (see (57)) of the R-operator (56), we find

$$R_{12}(u_1, u_2|v_1, v_2) \Gamma(\mp z_1 + z_3 + u_1 - u_2) \Phi(z_2) = \frac{1}{\Gamma(2v_1 - 2u_2)} \cdot \frac{1}{\Gamma(2v_2 - 2u_2)} \cdot M_1(u_1 - u_2 - \eta - \frac{\tau}{2}) \Gamma(\mp z_2 + v_2 - u_2) \Gamma(\mp z_1 + z_3 + v_1 - u_2) \cdot S(u_1 - v_2) \Gamma(\mp z_1 + z_3 + v_1 - u_2) \cdot \frac{1}{\Gamma(2v_1 - 2u_2)} \cdot M_1(u_1 - u_2 - \eta - \frac{\tau}{2}) \Gamma(\mp z_1 + z_3 + u_1 - u_2) \cdot \frac{1}{\Gamma(2v_2 - 2u_2)} \cdot M_2(n\eta + m\frac{\tau}{2}) \Phi(z_1).$$

Finally, we apply the reflection relation $\Gamma(\mp x + \eta + \frac{\tau}{2}) = 1$ for elliptic gamma function (83), go back to the spin variables (55), supplement $R_{12}$ with the permutation operator $\mathbb{P}_{12}$, and put the spin $g_1$ onto the two-dimensional discrete lattice (70): $g_1 = g_{n,m} = (n + 1)\eta + (m + 1)\tau/2$, at $n, m \in \mathbb{Z}_{\geq 0}$. Then the previous formula takes the form

$$R_{12}(u|g_1 = g_{n,m}, g_2) \Gamma(\mp z_1 + z_3 + g_{n,m}) \Phi(z_2) =$$

$$= c \cdot \frac{\Gamma(\mp z_2 + z_3 - \frac{\nu}{2} - \frac{g_{n,m} + g_2}{2} + \frac{\eta + \frac{\tau}{2}}{2}) \Gamma(\mp z_1 + z_2 - \frac{\nu}{2} - \frac{g_{n,m} + g_2}{2} + \frac{\eta + \frac{\tau}{2}}{2})}{\Gamma(\mp z_2 + z_3 - \frac{\nu}{2} - \frac{g_{n,m} - g_2}{2} + \frac{\eta + \frac{\tau}{2}}{2}) \Gamma(\mp z_1 + z_2 - \frac{\nu}{2} - \frac{g_{n,m} - g_2}{2} + \frac{\eta + \frac{\tau}{2}}{2})} \cdot M_2(n\eta + m\frac{\tau}{2}) \Phi(z_2).$$

(72)

where the finite-difference operator $M_2(n\eta + m\frac{\tau}{2})$ acting in the second space is given explicitly in (38) and the normalization constant $c$ is

$$c^{-1} = \Gamma(-u + g_{n,m} + g_2).$$

The formula (72) constitutes the central result of this paper. It gives a rich class of solutions of the YBE (1), which are endomorphisms on a tensor product of finite-dimensional and infinite-dimensional representations of elliptic modular double belonging to the series specified in Sect. 4.

Some comments how to apply formula (72) are in order. Expressions on both sides of equality (72) depend on the auxiliary parameter $z_3$. We expand them with respect to the auxiliary basis $\varphi^{(n,m)}(z_3)$ (17) that induces an expansion of the left-hand side with respect to the basis $\psi^{(n,m)}(z_1)$ (48) of the first space (see (16)). The finite-difference operator $M_2(n\eta + m\frac{\tau}{2})$ (38) acts from the left-hand side and shifts the arguments in the elliptic gamma functions as well as in the test function $\Phi(z_2)$ by multiples of $\eta$ and $\tau/2$. After the shifts are performed we can trade all gamma functions for theta functions by means of the recurrence (32) and the reflection identity (33). Expanding the product of theta functions on the right-hand side of (73) with respect to the auxiliary basis $\varphi^{(n,m)}(z_3)$, we find explicitly

$$R_{12}(u|g_{n,m}, g_2) \psi_{j,l}^{(n,m)}(z_1) \Phi(z_2).$$

Expanding this expression in the basis $\varphi^{(n,m)}(z_1)$ by means of the identities (16), we obtain a matrix $[R_{12}]_{j,l}^{r,s}$ of the operator $R_{12}$ reduced with respect to the first space written in a pair of bases $\psi^{(n,m)}(z_1)$ and $\varphi^{(n,m)}(z_1)$, i.e.

$$R_{12}(u|g_{n,m}, g_2) \psi_{j,l}^{(n,m)}(z_1) = \varphi^{(n,m)}(z_1) \left[ R_{12}(u|g_{n,m}, g_2) \right]_{j,l}^{r,s},$$

where summation over the repeated indices $r, s$ is assumed. Let us stress that entries of the derived matrices are finite-difference operators acting in the second space.
After the reduction to a finite-dimensional representation in the first space is implemented, we can easily perform reduction to a finite-dimensional representation in the second space as well. In order to achieve this goal we just substitute the test function \( \Phi(z_2) \) for the generating of finite-dimensional representations in the second space \((15)\). Indeed, we choose spins in the first space as \( g_1 = g_{n,m} = (n + 1) \eta + (m + 1) \tau / 2 \) and in the second space as \( g_2 = g_{k,l} = (k + 1) \eta + (l + 1) \tau / 2 \), for \( n, m, k, l \in \mathbb{Z}_{\geq 0} \). Then, the formula \((73)\) yields

\[
\mathbb{R}_{12}(u|g_{n,m}, g_{k,l}) \Gamma(\mp z_1 \mp z_3 + g_{n,m}) \Gamma(\mp z_2 \mp z_4 + g_{k,l}) = c \cdot \frac{\Gamma(\mp z_2 \mp z_3 - \eta + \frac{g_{n,m} - g_{k,l}}{2})}{\Gamma(\mp z_1 \mp z_2 - \frac{u}{2} - \frac{g_{n,m} + g_{k,l}}{2} + \eta + \frac{\tau}{2})} \cdot M_2(n \eta + m \frac{\tau}{2}) \Gamma(\mp z_1 \mp z_2 - \frac{u}{2} + \frac{g_{n,m} - g_{k,l}}{2} + \eta + \frac{\tau}{2}) \Gamma(\mp z_2 \mp z_4 + g_{k,l}) \cdot (74)
\]

Here \( z_3 \) and \( z_4 \) are auxiliary parameters. Applying this formula one can straightforwardly extract the finite-dimensional \( \mathbb{R} \)-matrices with respect to both spaces in a pair of bases \( \psi^{(n,m)}(z_1) \psi^{(k,l)}(z_2) \) and \( \varphi^{(n,m)}(z_1) \varphi^{(k,l)}(z_2) \), i.e.

\[
\mathbb{R}_{12}(u|g_{n,m}, g_{k,l}) \psi^{(n,m)}_{i_1,j_1}(z_1) \psi^{(k,l)}_{i_2,j_2}(z_2) = \varphi^{(n,m)}_{r_1,s_1}(z_1) \varphi^{(k,l)}_{r_2,s_2}(z_2) \bigg[ \mathbb{R}_{12}(u|g_{n,m}, g_{k,l}) \bigg]^{r_1,s_1}_{i_1,j_1} \bigg[ \mathbb{R}_{12}(u|g_{n,m}, g_{k,l}) \bigg]^{r_2,s_2}_{i_2,j_2} \cdot (75)
\]

This relation defines new solutions of YBE given as plain square matrices with \( nmk \) rows and columns with the elliptic function entries.

The derived reduction formulae have analogous in the case of rational solutions of YBE as well as in the case of trigonometric deformation which were obtained in \[7\]. In the case of the Lie group \( \text{SL}(2, \mathbb{C}) \) the analogues of \((73)\) and \((74)\) are given by formulae \((32)\) and \((36)\) in \[7\], respectively. In the case of the modular double \( U_q(sl_2) \otimes U_q(sl_2) \) the analogues of \((73)\) and \((74)\) are given by formulae \((101)\) and \((102)\) in \[7\], respectively.

In Sect. 4.3 we considered two discrete lattices of \( g \)-spin corresponding to finite-dimensional representations \((12)\). The reduction of the general \( R \)-operator to the second lattice

\[
g_1 = g_{n,m} = \frac{1}{2} + (n + 1) \eta + (m + 1) \tau / 2 , \quad n, m \in \mathbb{Z}_{\geq 0}
\]

is also possible. All previous calculations are valid in this case as well since we fix the spin \( g_1 \) at the last step. Since the resulting reduction formulae are almost the same we will only indicate the needed modifications. In the reduction formula \((73)\) we just need substituting the finite-difference operator on the right-hand side for another one \( M_2(\frac{u}{2} + n \eta + m \frac{\tau}{2}) \) (see \[(39)\]) . The generating function of finite-dimensional representations on the left-hand side of \((73)\) automatically takes the correct form \[(9)\]. The same remark is true for the reduction formula \((74)\). As we have already observed in Sect. 4.3 both bases \( \varphi^{(n,m)} \) and \( \psi^{(n,m)} \) are valid for both lattices.

### 5.3 Two-dimensional reduction and the Lax operator

In order to illustrate explicitly how the reduction formula \((73)\) works, consider reduction of the general \( R \)-operator to Sklyanin’s \( L \)-operator \((26)\) going through all details of the calculation. We pick the spin \( g_1 = g_{1,0} = 2 \eta + \tau / 2 \) (see \((70)\)) that corresponds to the two-dimensional representation in the first space of \( \mathbb{R}_{12} \) in \((73)\),

\[
\mathbb{R}_{12}(u|g = 2 \eta + \frac{\tau}{2} , g) \Gamma(\mp z_1 \mp z_3 + 2 \eta + \frac{\tau}{2}) \Phi(z_2) = \\
c \cdot \frac{\Gamma(\mp z_2 \mp z_3 - \frac{u+g}{2} + \eta + \frac{\tau}{4})}{\Gamma(\mp z_1 \mp z_2 - \frac{u+g}{2} - \frac{\tau}{4})} \cdot M_2(\eta) \frac{\Gamma(\mp z_1 \mp z_2 - \frac{u+g}{2} + \eta + \frac{\tau}{4})}{\Gamma(\mp z_2 \mp z_3 - \frac{u+g}{2} + \frac{\tau}{4})} \Phi(z_2) . \quad (75)
\]
The generating function \( \Gamma(z_1 \mp z_3 + 2\eta + \frac{\tau}{4}) = -\frac{1}{2} R^2(\tau) e^{\frac{3\pi i}{2}} (\bar{\theta}_4(z_1) \bar{\theta}_3(z_3) + \bar{\theta}_3(z_1) \bar{\theta}_4(z_3)) \),

yields the basis \( \varphi^{1,0}(z_1) = \{ \bar{\theta}_4(z_1), \bar{\theta}_3(z_1) \} \) (see (77)) which coincides with the second basis \( \psi^{1,0}(z_1) \) (see (45)). Thus in this particular situation we do not need to deal with two different bases. Consequently, the reduction formula (73) produces automatically a matrix of the \( R \)-operator written in the single basis \( \varphi^{1,0}(z_1) \), not a pair of bases.

The next ingredient of (73) — the finite-difference operator \( M_2(\eta) \) — has the form (see (57))

\[
M_2(\eta) = e^{\pi i z_2^2/\eta} \frac{\theta_1(2z_2)}{\theta_1(z_2)} \left[ e^{\eta \delta z_2} - e^{-\eta \delta z_2} \right] e^{-\pi i z_2^2/\eta}.
\]

Using equations for the elliptic gamma function (32), we simplify the right-hand side expression in (75) and obtain

\[
\Re_{12}(u|2\eta + \frac{\tau}{4}, g) \left( \bar{\theta}_4(z_1) \bar{\theta}_3(z_3) + \bar{\theta}_3(z_1) \bar{\theta}_4(z_3) \right) \Phi(z_2) = \\
= \lambda \frac{\bar{\theta}_3(z_3)}{\theta_1(2z_2)} e^{\pi i z_2^2/\eta} \left[ A(z_2, z_1) e^{-\pi i (z_2^2 + \eta^2)} \Phi(z_2 + \eta) + A(-z_2, z_1) e^{-\pi i (z_2^2 + \eta^2)} \Phi(z_2 - \eta) \right] + \\
+ \lambda \frac{\bar{\theta}_4(z_3)}{\theta_1(2z_2)} e^{\pi i z_2^2/\eta} \left[ B(z_2, z_1) e^{-\pi i (z_2^2 + \eta^2)} \Phi(z_2 + \eta) + B(-z_2, z_1) e^{-\pi i (z_2^2 + \eta^2)} \Phi(z_2 - \eta) \right],
\]

(76)

where the functions \( A(z_2, z_1) \) and \( B(z_2, z_1) \) are given by products of three theta functions:

\[
\begin{align*}
A(z_2, z_1) &= \bar{\theta}_4(-z_2 - \frac{u+g}{2} - \eta + \frac{\tau}{4}) \theta_1(z_2 \pm z_1 - \frac{u+g}{2} + \frac{\tau}{4}) \\
B(z_2, z_1) &= -\bar{\theta}_3(-z_2 - \frac{u-g}{2} - \eta + \frac{\tau}{4}) \theta_1(z_2 \pm z_1 - \frac{u+g}{2} + \frac{\tau}{4})
\end{align*}
\]

(77)

and the normalization constant \( \lambda \) is

\[
\lambda = -R^2(\tau) c c_A e^{-2\pi i(u+\eta)}.
\]

Due to linear independence of the functions \( \bar{\theta}_4(z_3) \) and \( \bar{\theta}_3(z_3) \) forming the basis \( \varphi^{1,0}(z_3) \) the expression (76) can be split to a pair of operator relations

\[
\Re_{12}(u|2\eta + \frac{\tau}{4}, g) \bar{\theta}_4(z_1) = \frac{\lambda}{\theta_1(2z_2)} e^{\pi i z_2^2/\eta} \left[ A(z_2, z_1) e^{\eta \delta z_2} + A(-z_2, z_1) e^{-\eta \delta z_2} \right] e^{-\pi i z_2^2/\eta},
\]

\[
\Re_{12}(u|2\eta + \frac{\tau}{4}, g) \bar{\theta}_3(z_1) = -\frac{\lambda}{\theta_1(2z_2)} e^{\pi i z_2^2/\eta} \left[ B(z_2, z_1) e^{\eta \delta z_2} + B(-z_2, z_1) e^{-\eta \delta z_2} \right] e^{-\pi i z_2^2/\eta},
\]

where we imply that both side expressions are applied to a test function \( \Phi(z_2) \).

Next we expand the product of \( \theta_1 \)-functions in (77) over the basis functions \( \bar{\theta}_4(z_1) \) and \( \bar{\theta}_3(z_1) \) by means of identities (17)

\[
2 \theta_1(z_2 \pm z_1 - \frac{u+g}{2} + \frac{\tau}{4}) = \bar{\theta}_3(z_1) \bar{\theta}_4(z_2 - \frac{u+g}{2} + \frac{\tau}{4}) - \bar{\theta}_4(z_1) \bar{\theta}_3(z_2 - \frac{u+g}{2} + \frac{\tau}{4})
\]

that yields an expansion of the functions \( A(z_2, z_1) \) and \( B(z_2, z_1) \) (77) in this basis

\[
A(z_2, z_1) = a(z_2) \bar{\theta}_4(z_1) + c(z_2) \bar{\theta}_3(z_1) \;
B(z_2, z_1) = b(z_2) \bar{\theta}_4(z_1) + d(z_2) \bar{\theta}_3(z_1).
\]

\footnote{We use the shorthand notation \( \theta_1(x \mp y) \equiv \theta_1(x + y) \theta_1(x - y) \), etc to avoid bulky formulæ.
The coefficients in this expansion are given by the following functions of $z_2$,

\[
\begin{align*}
    a(z_2) &= -\tilde{\theta}_4(-z_2 - \frac{u - z_2}{2} - \eta + \frac{\tau}{4}) \tilde{\theta}_3(z_2 - \frac{u + z_2}{2} + \frac{\tau}{4}), \\
    b(z_2) &= \tilde{\theta}_3(-z_2 - \frac{u - z_2}{2} - \eta + \frac{\tau}{4}) \tilde{\theta}_3(z_2 - \frac{u + z_2}{2} + \frac{\tau}{4}), \\
    c(z_2) &= \tilde{\theta}_4(-z_2 - \frac{u - z_2}{2} - \eta + \frac{\tau}{4}) \tilde{\theta}_4(z_2 - \frac{u + z_2}{2} + \frac{\tau}{4}), \\
    d(z_2) &= -\tilde{\theta}_3(-z_2 - \frac{u - z_2}{2} - \eta + \frac{\tau}{4}) \tilde{\theta}_4(z_2 - \frac{u + z_2}{2} + \frac{\tau}{4}).
\end{align*}
\]

Now we are able to write the action of the R-operator on the basis \( \{ \tilde{\theta}_4(z_1), \tilde{\theta}_3(z_1) \} \) in a matrix form with operator entries acting in the second space

\[
\mathbb{R}_{12}(u|2\eta + \frac{\tau}{2}, g) \left( \tilde{\theta}_4(z_1), \tilde{\theta}_3(z_1) \right) = \lambda \cdot \left( \tilde{\theta}_4(z_1), \tilde{\theta}_3(z_1) \right) \cdot e^{\pi iz_2^2 / \eta}.
\]

Finally, the previous formula can be rewritten up to a normalization constant in the form of Sklyanin’s Lax operator with the shifted spectral parameter \( \Gamma_{\text{doub}}(u - \tau/2) \) [26],

\[
\mathbb{R}_{12}(u|2\eta + \frac{\tau}{2}, g) \left( \tilde{\theta}_4(z_1), \tilde{\theta}_3(z_1) \right) = -\lambda \cdot \left( \tilde{\theta}_4(z_1), \tilde{\theta}_3(z_1) \right) \cdot e^{\pi iz_2^2 / \eta} \left( \begin{array}{c}
    w_0(u - \frac{\tau}{2}) S^0 + w_3(u - \frac{\tau}{2}) S^3 \\
    w_1(u - \frac{\tau}{2}) S^1 + iw_2(u - \frac{\tau}{2}) S^2 \\
    w_1(u - \frac{\tau}{2}) S^1 + iw_2(u - \frac{\tau}{2}) S^2 \\
    w_0(u - \frac{\tau}{2}) S^0 - w_3(u - \frac{\tau}{2}) S^3
\end{array} \right) ; \quad w_a(u) = \frac{\theta_{a+1}(u + \eta)}{\theta_{a+1}(\eta)}
\]

and with generators \( S^a \) in the spin \( g \) representation which are specified in [24].

In a similar way the reduction of the R-operator with the spin \( g_1 = g_{0,1} = \eta + \tau \) gives rise to the Lax operator for the second half of the elliptic modular double \( \Gamma_{\text{doub}} \) [29].

### 6 Fusion construction

Fusion is a generally accepted method of constructing higher-spin finite-dimensional solutions of YBE out of the fundamental ones. Initially it has been developed for the Lie algebra \( sl_2 \) (as well as for \( sl_N \)) [25] and then generalized to deformed trigonometric and elliptic rank 1 symmetry algebras [26].

The elliptic higher-spin finite-dimensional solutions found numerous applications in condensed matter models [22]. Another motivation to consider the fusion in the elliptic case is purely mathematical. On the one hand, the representation theory of \( sl_2 \) and \( U_q(sl_2) \) is well understood. Indeed, in these cases the Cartan subalgebra exists and Verma modules are formed by raising and lowering generators acting on the lowest weight states. On the other hand, the representation theory of the Sklyanin algebra is not elaborated well enough and requires further comprehensive studies. All four generators of the Sklyanin algebra appear on equal footing [7], which is the main obstacle preventing the Verma module construction in the elliptic setting. The fusion method is tightly related to the structure of representations of the underlying symmetry algebra [25]. Thus the fusion construction for elliptic deformations is of special interest since it enables one to shed some light on the Sklyanin algebra representations. All steps of the general R-operator construction are essentially the same for deformed and non-deformed algebras [8,9]. In this section we will demonstrate that curiously enough the fusion recipe for the Sklyanin algebra can be formulated literally in the same way as for the \( sl_2 \) algebra.

The fusion for integrable models with underlying Sklyanin algebra and its higher rank generalization specified by RLL-relation with Belavin’s R-matrix has been extensively studied in a number of papers, e.g., [11,12,13,21,25]. In spite of the progress reached in these works, the corresponding elliptic fusion still looks rather complicated and not much explicit formulae are available. The
aim of this section is to fill this gap in the rank 1 case and our approach is most close to the one considered in \cite{24, 14}.

Additionally to the simplification of the previous considerations, in this section we develop in a novel fashion the fusion for the elliptic modular double generalizing the results of our previous work \cite{7}. We start from fusing Baxter’s R-matrices \cite{2} and construct out of them the Lax operator for spins $\frac{3}{2}, n \in \mathbb{Z}_{\geq 0}$ \cite{5}. In order to simplify calculations we resort to symbols of finite-dimensional operators and auxiliary spinors notation. In \cite{15} a laborious calculation enabled to implement explicitly the fusion of two, three, and four Baxter’s R-matrices reproducing the Lax operators \cite{5} at spin $1, \frac{3}{2},$ and 2, respectively. We will explicitly perform fusion of arbitrary number of Baxter’s R-matrices yielding all finite-dimensional Lax operators \cite{5} and corresponding all finite-dimensional representations of the Sklyanin algebra from the series \cite{11}. At the same time our calculation is fairly simple. It is based on the factorization of the symbol of Baxter’s R-matrix which is similar to the Lax factorization \cite{12}.

After that we proceed to the fusion of Lax operators with infinite-dimensional representations in the quantum space. In this way we produce higher-spin R-operators acting on the tensor product of an infinite-dimensional and arbitrary finite-dimensional representations of the Sklyanin algebra. They do coincide with solutions given by formula (73) obtained by reductions of the general elliptic R-operator. In this way we find a complete agreement between two approaches to constructing finite-dimensional (in one or both tensor space factors) YBE solutions.

### 6.1 Fusion of the Baxter’s R-matrices and finite-dimensional representations of the Sklyanin algebra

For the rank 1 symmetry algebras underlying an integrable system the fusion recipe of \cite{25, 26} looks as follows. One forms an inhomogeneous monodromy matrix $T_{i_1...i_n}^{j_1...j_n}$ out of L-operators \cite{5} $L_i^j$ (or out of Baxter’s R-matrices \cite{2} which are L-operators for the spin $\frac{1}{2}$ \cite{19}) multiplying them as operators in the quantum space and taking tensor products of the auxiliary spaces $\mathbb{C}^2$,

$$T_{i_1...i_n}^{j_1...j_n}(u) = L_{i_1}^{j_1}(u) L_{i_2}^{j_2}(u-2\eta) \cdots L_{i_n}^{j_n}(u-2(n-1)\eta),$$

and then symmetrizes the monodromy matrix over the spinor indices. The result $T_{(i_1...i_n)}^{(j_1...j_n)}$ is a R-operator which has a higher-spin auxiliary space and solves the YBE because the parameters of inhomogeneity are adjusted in a special way.

Thus constructing higher-spin R-operators one has to deal with $\text{Sym}(\mathbb{C}^2)^{\otimes n}$, which is a space of symmetric tensors with a number of spinor indices $\Psi_{(i_1...i_n)}$. The usual matrix-like action of the operators $\hat{T}$ has the form

$$\left[\hat{T} \Psi\right]_{(i_1...i_n)} = T_{(i_1...i_n)}^{(j_1...j_n)} \Psi_{(j_1...j_n)},$$

where the summation over repeated indices is assumed. We prefer not to deal with a multitude of spinor indices. Instead we introduce auxiliary spinors $\lambda, \mu$ and contract them with the tensors

$$\lambda_{i_1} \cdots \lambda_{i_n} \Psi_{i_1...i_n} = \Psi(\lambda), \quad \lambda_{i_1} \cdots \lambda_{i_n} T_{i_1...i_n}^{j_1...j_n} \mu_{j_1} \cdots \mu_{j_n} = T(\lambda|\mu).$$

Thus the symmetrization over spinor indices is taken into account automatically. Henceforth, in place of tensors we work with corresponding generating functions which are homogeneous polynomials of degree $n$ of two variables

$$\Psi(\lambda) = \Psi(\lambda_1, \lambda_2); \quad \Psi(\alpha \lambda_1, \alpha \lambda_2) = \alpha^n \Psi(\lambda_1, \lambda_2).$$

$T(\lambda|\mu)$ is usually called the symbol of the operator. In this way the formula \cite{80} acquires a rather concise form

$$\left[\hat{T} \Psi\right](\lambda) = \frac{1}{n!} T(\lambda|\partial_\mu) \Psi(\mu)|_{\mu=0}.$$
Let us note that we do not need in fact to take $\mu = 0$ in (83). The $\mu$ variable disappears automatically on the right-hand side of (83) since $T(\lambda|\mu)$ and $\Psi(\mu)$ have the same homogeneity degree.

There is a dual description of the finite-dimensional matrix-like action on $\text{Sym}(\mathbb{C}^2)^\otimes n$ which enables as well to get rid off spinor indices. Let us use a spinor $\lambda$ to form the rank $n$ symmetric tensor $\Psi(j_1 \cdots j_n) = \lambda_{j_1} \cdots \lambda_{j_n}$. Then we interpret $T(j_1 \cdots j_n)_{(i_1 \cdots i_n)}$ as a matrix of the operator $\hat{T}$ relating them in a standard way

$\hat{T} \cdot \lambda_{j_1} \cdots \lambda_{j_n} = \lambda_{i_1} \cdots \lambda_{i_n} T(j_1 \cdots j_n)_{(i_1 \cdots i_n)}$.

Contraction of the latter relation with a symmetric tensor formed by another auxiliary spinor $\mu_{j_1} \cdots \mu_{j_n}$ yields the relation between two formalisms

$\hat{T} \cdot \langle \lambda|\mu \rangle^n = T(\lambda|\mu)$.

(84)

According to (81), $\langle \lambda|\mu \rangle^n$ is a symbol of the identity operator $\delta_{j_1}^{i_1} \cdots \delta_{j_n}^{i_n}$ in the space of rank $n$ symmetric tensors. It is also a generating function of the basis vectors since expanding $\langle \lambda|\mu \rangle^n$ in $\mu$ we recover all of them. Thus (84) implies that the operator $\hat{T}$ acting on the generating function produces the symbol of $\hat{T}$.

We construct the Lax operator (5) with a finite-dimensional local quantum space starting from the Baxter R-matrix. The latter acts on the tensor product of two spin $\frac{1}{2}$ representations and it is given by (2). Following the recipe from (26) we form the product of the Baxter R-matrices

$R(j_1 \cdots j_n)_{(i_1 \cdots i_n)}(u) = \text{Sym} R^{j_1}_{i_1}(u) R^{j_2}_{i_2}(u - 2\eta) \cdots R^{j_n}_{i_n}(u - 2(n - 1)\eta)$,

(85)

where $\text{Sym}$ implies symmetrization with respect to $(i_1 \cdots i_n)$ and $(j_1 \cdots j_n)$, and the indices refer to the first space of the R-matrix (2),

$R^j_i(u) = \sum_{\alpha=0}^3 w_\alpha(u) \sigma_\alpha^j \sigma_\alpha^i$; \hspace{1cm} w_\alpha(u) = \frac{\theta_{\alpha+1}(u + \eta)}{\theta_{\alpha+1}(\eta)}$.

In such a way one obtains an operator acting in the space of symmetric rank $n$ tensors, i.e. the space of spin $\frac{n}{2}$ representation, and in two-dimensional auxiliary space where the Pauli $\sigma$-matrices are acting. According to (26) it respects the Yang-Baxter relations. We are going to find an explicit expression for $R(j_1 \cdots j_n)_{(i_1 \cdots i_n)}(u)$ for all integer $n$.

With that end in view we calculate the symbol of (85) with respect to the quantum space contracting (83) with a number of auxiliary spinors $\lambda$ and $\mu$,

$R(u|\lambda,\mu) = \lambda_{i_1} \cdots \lambda_{i_n} R^{j_1 \cdots j_n}_{i_1 \cdots i_n}(u) \mu_{j_1} \cdots \mu_{j_n} =

= \langle \lambda|R(u)|\mu \rangle \langle \lambda|R(u - 2\eta)|\mu \rangle \cdots \langle \lambda|R(u - 2(n - 1)\eta)|\mu \rangle$.

(86)

Let us stress that it is still an operator in the auxiliary space, but for the sake of brevity we refer to it as a symbol of the R-matrix. The symbol $R(u|\lambda,\mu)$ factorizes in a product of Baxter’s R-matrix symbols $\langle \lambda|R(u)|\mu \rangle = \lambda_i R^i_j(u) \mu_j$. The homogeneity (82) implies that an auxiliary spinor have one redundant degree of freedom that can be eliminated by fixing a particular gauge by scale transformations $\lambda \rightarrow a \cdot \lambda$. Further we constrain $\lambda$ by $\lambda_1 = \theta_1(z), \lambda_2 = \theta_3(z)$ with a new parameter $z$, but we keep components $\mu_1$ and $\mu_2$ independent. Let us mention that an isomorphism between the representation space of the Sklyanin algebra $\Theta_{2n}^+$ and the space of symmetric tensors has been constructed in (20) (41) (45).

Then taking into account the matrix form of the Sklyanin algebra generators $S^a$ in spin $\frac{1}{2}$ representation (13) we obtain the symbol $\langle \lambda|R(u)|\mu \rangle$,

$\langle \lambda|R(u)|\mu \rangle = \begin{pmatrix}
\langle \tilde{\theta}_1, \tilde{\theta}_3 | (w_0(u) \sigma_0 + w_3(u) \sigma_3) | \mu \rangle \\
\langle \tilde{\theta}_1, \tilde{\theta}_3 | (w_1(u) \sigma_1 - i w_2(u) \sigma_2) | \mu \rangle \\
\langle \tilde{\theta}_1, \tilde{\theta}_3 | (w_0(u) \sigma_0 - w_3(u) \sigma_3) | \mu \rangle
\end{pmatrix}$.

22
\[
\frac{1}{\theta_1(2\eta)} \left( \begin{array}{ccc}
 w_0(u) S^0 + w_3(u) S^3 & w_1(u) S^1 - iw_2(u) S^2 & w_0(u) S^0 - w_3(u) S^3 \\
w_1(u) S^1 + iw_2(u) S^2 & w_0(u) S^0 - w_3(u) S^3 & \end{array} \right) (\bar{\theta}_4, \bar{\theta}_3|\mu). \quad (87)
\]

Let us note that \( \langle \lambda | \mu \rangle = \langle \bar{\theta}_4, \bar{\theta}_3 | \mu \rangle = \mu_1 \bar{\theta}_4(z) + \mu_2 \bar{\theta}_3(z) \) is a symbol of the identity operator. Thus we represented \( \langle \lambda | R(u)|\mu \rangle \) as a matrix with elliptic finite-difference operator entries acting on the symbol of the identity operator.

Then we need to calculate the product (86) of symbols (87). It seems at first sight to be an extremely complicated task that demands an extensive use of numerous identities for Jacobi theta-functions. Fortunately we will be able to avoid difficult calculations due to a peculiar property of the symbol (87). It can be factorized in a product of three matrices respecting the normal ordering of \( z \) and \( \partial \),

\[
\langle \lambda | R(u)|\mu \rangle = \frac{1}{\theta_1(2\eta)\theta_1(2z)} \left( \begin{array}{ccc}
 \bar{\theta}_3 (z - \frac{\eta}{2} - \eta) & -\theta_3 (z + \frac{\eta}{2} + \eta) & 0 \\
 -\theta_4 (z - \frac{\eta}{2} - \eta) & \theta_4 (z + \frac{\eta}{2} + \eta) & 0 \\
 0 & 0 & e^{-\eta \partial_z} \\
\end{array} \right) \cdot \left( \begin{array}{ccc}
 \bar{\theta}_4 (z + \frac{\eta}{2}) & \bar{\theta}_3 (z + \frac{\eta}{2}) & \langle \bar{\theta}_4(z_1), \bar{\theta}_3(z_1)|\mu \rangle |_{z_1 = z} \\
 \theta_4 (z - \frac{\eta}{2}) & \theta_3 (z - \frac{\eta}{2}) & \langle \theta_4(z_1), \theta_3(z_1)|\mu \rangle |_{z_1 = z} \\
\end{array} \right).
\quad (88)
\]

The latter factorization is similar to the factorization of the L-operator (12). However the normal ordering in (88) results in a shift of the spectral parameter in the rightmost matrix factor compared to (12). The factorized elliptic Lax operator with normal ordering appeared before in (28) in a different context.

We appreciate the merits of the factorization (88) if we calculate the product of two consecutive symbols from (86), i.e. \( \langle \lambda | R(u)|\mu \rangle \langle \lambda | R(u-2\eta)|\mu \rangle \). Indeed, a pair of adjacent lateral matrices cancels in the product owing to

\[
\left( \begin{array}{ccc}
 \bar{\theta}_4 (z + \frac{\eta}{2}) & \bar{\theta}_3 (z + \frac{\eta}{2}) & 0 \\
 \theta_4 (z - \frac{\eta}{2}) & \theta_3 (z - \frac{\eta}{2}) & 0 \\
 0 & 0 & e^{-\eta \partial_z} \\
\end{array} \right) \left( \begin{array}{ccc}
 \bar{\theta}_3 (z - \frac{\eta}{2} - \eta) & -\theta_3 (z + \frac{\eta}{2} + \eta) & 0 \\
 -\theta_4 (z - \frac{\eta}{2} - \eta) & \theta_4 (z + \frac{\eta}{2} + \eta) & 0 \\
 0 & 0 & e^{-\eta \partial_z} \\
\end{array} \right) = 2 \theta_1(2z) \theta_1(u) \left( \begin{array}{ccc}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
\end{array} \right)
\]
which is equivalent to identities (16). Thus we obtain once again a product of three matrices such that all shift operators are gathered in the middle matrix,

\[
\langle \lambda | R(u)|\mu \rangle \langle \lambda | R(u-2\eta)|\mu \rangle = \frac{2 \theta_1(u)}{\theta_1(2\eta)\theta_1(2z)} \left( \begin{array}{ccc}
 \bar{\theta}_3 (z - \frac{\eta}{2} - \eta) & -\theta_3 (z + \frac{\eta}{2} + \eta) \\
 -\theta_4 (z - \frac{\eta}{2} - \eta) & \theta_4 (z + \frac{\eta}{2} + \eta) \\
 0 & 0 \\
\end{array} \right) \cdot \left( \begin{array}{ccc}
 e^{\eta \partial_z} & 0 & 0 \\
 0 & e^{-\eta \partial_z} & 0 \\
\end{array} \right) \cdot \left( \begin{array}{ccc}
 \bar{\theta}_4 (z + \frac{\eta}{2}) & \bar{\theta}_3 (z + \frac{\eta}{2}) & \langle \bar{\theta}_4(z_1), \bar{\theta}_3(z_1)|\mu \rangle |_{z_1 = z} \\
 \theta_4 (z - \frac{\eta}{2}) & \theta_3 (z - \frac{\eta}{2}) & \langle \theta_4(z_1), \theta_3(z_1)|\mu \rangle |_{z_1 = z} \\
\end{array} \right).
\]

Since the product of two symbols has the same structure as a single symbol (88), the generalization of the previous result for the product of \( n \) symbols (86) is evident

\[
R(u|\lambda, \mu) = \frac{r_n(u)}{\theta_1(2z)} \left( \begin{array}{ccc}
 \bar{\theta}_3 (z - \frac{\eta}{2} - \eta) & -\theta_3 (z + \frac{\eta}{2} + \eta) \\
 -\theta_4 (z - \frac{\eta}{2} - \eta) & \theta_4 (z + \frac{\eta}{2} + \eta) \\
 0 & 0 \\
\end{array} \right) \cdot \left( \begin{array}{ccc}
 e^{\eta \partial_z} & 0 & 0 \\
 0 & e^{-\eta \partial_z} & 0 \\
\end{array} \right) \cdot \left( \begin{array}{ccc}
 \bar{\theta}_4 (z + \frac{\eta}{2} - (n-1)\eta) & \bar{\theta}_3 (z + \frac{\eta}{2} - (n-1)\eta) & \langle \bar{\theta}_4(z_1), \bar{\theta}_3(z_1)|\mu \rangle |_{z_1 = \ldots = z = z} \\
 \theta_4 (z - \frac{\eta}{2} + (n-1)\eta) & \theta_3 (z - \frac{\eta}{2} + (n-1)\eta) & \langle \theta_4(z_1), \theta_3(z_1)|\mu \rangle |_{z_1 = \ldots = z = z} \\
\end{array} \right),
\quad (89)
\]

where the normalization factor \( r_n(u) = 2^{n-1} \theta_1(u) \cdots \theta_1(u-2(n-1)\eta) \theta_1^{-n}(2\eta) \). Further we multiply three matrices in the latter formula which have mutually commuting entries, then act by the shift operators from the left by means of the obvious formula

\[
e^{\pm \eta \partial_z} \langle \bar{\theta}_4(z_1), \bar{\theta}_3(z_1)|\mu \rangle \cdots \langle \bar{\theta}_4(z_n), \bar{\theta}_3(z_n)|\mu \rangle |_{z_1 = \ldots = z = z} = e^{\pm \eta \partial_z} \langle \bar{\theta}_4(z), \bar{\theta}_3(z)|\mu \rangle^n,
\]

23
and, finally, implement matrix factorization according to (12) this time taking into account non-commutativity of the matrices' entries

\[
R(u|\lambda, \mu) = \frac{r_n(u)}{\theta_1(2\zeta)} \begin{pmatrix} \bar{\theta}_3(z - \frac{u}{2} - \eta) & -\bar{\theta}_3(z + \frac{u}{2} + \eta) \\ -\bar{\theta}_4(z - \frac{u}{2} - \eta) & \bar{\theta}_4(z + \frac{u}{2} + \eta) \end{pmatrix} \begin{pmatrix} e^{\eta \partial} & 0 \\ 0 & e^{-\eta \partial} \end{pmatrix} \cdot \begin{pmatrix} \bar{\theta}_1(z + \frac{u}{2} - n\eta) & \bar{\theta}_3(z + \frac{u}{2} - n\eta) \\ \bar{\theta}_1(z - \frac{u}{2} + n\eta) & \bar{\theta}_3(z - \frac{u}{2} + n\eta) \end{pmatrix} (\bar{\theta}_4(z), \bar{\theta}_3(z)|\mu)^n. \tag{90}
\]

In view of the factorized form of the Lax-operator (12) the previous formula is equivalent to

\[
R(u|\lambda, \mu) = r_n(u) L \left( \frac{u}{2} + \eta, \frac{u}{2} - n\eta \right) (\bar{\theta}_4(z), \bar{\theta}_3(z)|\mu)^n. \tag{91}
\]

Once again the two factorized forms of the symbol, normal ordered (89), and without ordering (90), are related to each other by a shift of the spectral parameter in the rightmost matrix factor. In order to recover the Lax operator from the symbol (91) we recall that \((\bar{\theta}_4(z), \bar{\theta}_3(z)|\mu)^n = (\lambda|\mu)^n\) and apply the formula (54). Thus the fusion of \(n\) Baxter’s R-matrices resulted in the Lax operator \(L(u + (1 - n)\eta)\) with the Sklyanin algebra generators in spin \(\ell = \frac{n}{2}\) representation.

It is a well known fact that the representation of the algebra (7) by elliptic finite-difference operators (11) found by Sklyanin [34] is highly intricate. Indeed the straightforward calculation demonstrating that the operators (11) do respect the commutation relations (7) is extremely laborious. The explicit fusion formulae derived in this section do provide an independent check of the algebra commutation relations. Let us emphasize one more time the crucial role played by the factorization formula (88) in our considerations.

### 6.2 Fusion for the elliptic modular double

In the previous section we “fused” Baxter’s R-matrices. Now we proceed to the fusion of Lax operators in the auxiliary space which have infinite-dimensional representation in the quantum space. We will concentrate on the elliptic modular double. The results for the Sklyanin algebra are particular cases of the latter.

We form the inhomogeneous monodromy matrix (recall (79)) of out the Lax operators of two species: \(L_{doub}^{\text{doub}}\) (26) and \(L_{doub}^{\text{doub}}\) (29), and symmetrize it over the indices of auxiliary \(\mathbb{C}^2\) spaces (recall (79)),

\[
\begin{pmatrix} R_{\text{fus}}(u) \end{pmatrix}^{(j_1, \ldots, j_{n+m})}_{(i_1, \ldots, i_{n+m})} = \text{Sym} \left( L_{doub}^{\text{doub}} \right)^{j_1}_{i_1}(u) \left( L_{doub}^{\text{doub}} \right)^{j_2}_{i_2}(u - 2\eta) \cdots \left( L_{doub}^{\text{doub}} \right)^{j_n}_{i_n}(u - 2(n - 1)\eta) \cdot \left( L_{doub}^{\text{doub}} \right)^{j_{n+1}}_{i_{n+1}}(u - 2n\eta) \cdots \left( L_{doub}^{\text{doub}} \right)^{j_{n+m}}_{i_{n+m}}(u - 2m\eta - (m - 1)\tau). \tag{92}
\]

In the quantum spaces of Lax operators an infinite-dimensional representation with spin \(g\) is realized. \(R_{\text{fus}}(u)\) is a higher-spin R-operator which solves YBE and acts in the tensor product of an infinite-dimensional spin \(g\) representation and finite-dimensional \((n + 1)(m + 1)\)-dimensional spin \(g_{n,m} = (n + 1)\eta + (m + 1)\frac{\tau}{2}, n, m \in \mathbb{Z}_{\geq 0}\), representation.

In the previous section we benefited a lot from auxiliary spinors and the symbol notation that considerably simplified the calculation. Working with the elliptic modular double we will need two times more auxiliary spinors: \(\lambda, \bar{\lambda}, \mu, \bar{\mu}\). In view of homogeneity (82) the spinors contain redundant degrees of freedom. In this calculation we will retain only independent variables parametrizing the components of spinors \(\lambda, \mu\) by \(a, b\),

\[
\lambda_1 = \lambda_1(a) = \theta_4(a|\frac{\tau}{2}) , \quad \lambda_2 = \lambda_2(a) = \theta_3(a|\frac{\tau}{2}) , \quad \mu_1 = \mu_1(b) = \theta_3(b|\frac{\tau}{2}) , \quad \mu_2 = \mu_2(b) = -\theta_4(b|\frac{\tau}{2}). \tag{93}
\]
The same formulae hold for \( \tilde{\lambda} \), \( \tilde{\mu} \) with the permutation \( 2\eta \leftrightarrow \tau \). The spinors \( \lambda \) and \( \tilde{\lambda} \) are algebraically independent at generic \( a \), as well as \( \mu \) and \( \tilde{\mu} \) at generic \( b \). At this point we bear in mind an isomorphism \( [45, 20] \) between the space of even theta functions that the Sklyanin algebra generators act upon and the space of symmetric tensors.

In order to find the symbol of \( R_{\text{fus}} \) with respect to the finite-dimensional space we firstly need to calculate symbols of Lax operators. According to \( [81] \) we contract Lax operators \( [20, 29] \) with a pair of auxiliary spinors in \( \mathbb{C}^2 \) space yielding scalar operators:\[^3\]

\[
\Lambda(u, \lambda, \mu) = \lambda_i \left( L_{\text{doub}} \right)_i^j \mu_j \quad \text{and} \quad \tilde{\Lambda}(u, \tilde{\lambda}, \tilde{\mu}) = \tilde{\lambda}_i \left( \tilde{L}_{\text{doub}} \right)_i^j \mu_j ,
\]

which are linear combinations of the generators \( S^a \) \( [21] \) and \( \tilde{S}^a \) \( [25] \), respectively, in spin \( g \) representation. The operator \( \Lambda \) arose previously in \( [28] \) in the study of vacuum curves of elliptic Lax operators.

By means of identities \( [16] \) one can gather Jacobi theta functions and rewrite the symbol as follows

\[
\Lambda(u, \lambda, \mu) = c e^{\pi i z^2/\eta} \frac{1}{\theta_1(2z|\tau)} \left[ \theta_1(z - u_1 \pm a|\tau) \theta_1(z + u_2 + \eta + \tau \pm b|\tau) e^{4\pi i z} e^{\eta \theta} \right. \\
- \theta_1(-z - u_1 \pm a|\tau) \theta_1(-z + u_2 + \eta + \tau \pm b|\tau) e^{-4\pi i z} e^{-\eta \theta} \left] e^{-\pi i z^2/\eta} ,
\]

where the normalization constant \( c = -4e^{-2\pi i(u-g+2\eta-\tau)} \). The linear combinations \( u_1, u_2 \) of the spectral parameter \( u \) and spin \( g \) are defined in \( [28] \). Let us remind shorthand notations adopted here: \( \theta_1(x \pm y) \equiv \theta_1(x + y) \theta_1(x - y) \). Then we take into account that the intertwining operator of equivalent representations \( M \) \( [30] \) simplifies to a finite-difference operator \( [37] \) on a discrete lattice of spin parameters, in particular

\[
M_z(\eta) = e^{\pi i z^2/\eta} \frac{1}{\theta_1(2z|\tau)} \left( e^{\eta \theta_2} - e^{-\eta \theta_1} \right) e^{-\pi i z^2/\eta} .
\]

Applying the identity

\[
\Gamma(\pm(z + \eta) \pm a + \lambda) = R^2(\tau) e^{2\pi i(z + \lambda - \eta)} \theta_1(z + \lambda - \eta \pm a|\tau) \Gamma(\pm z \pm a + \lambda - \eta) ,
\]

which is a consequence of the recurrence relations \( [32] \) and the reflection formula \( [33] \), we rewrite \( [95] \) in terms of \( M_z(\eta) \) and the elliptic gamma functions

\[
\Lambda(u, \lambda, \mu) = c \cdot \Gamma(\pm a \pm z \pm u_1 + 2\eta + \tau) \Gamma(\pm b \pm z + \eta - u_2) M_z(\eta) \cdot \\
\cdot \Gamma(\pm a \pm z + \eta - u_1) \Gamma(\pm b \pm z + 2\eta + \tau + u_2) ,
\]

where the constant \( c = -4R^{-4}(\tau) e^{-2\pi i(u+3\eta)} \).

In fact, the chain of transformations leading from \( [94] \) to \( [97] \) follows in the counter direction as compared to the calculation in Sect. 5.3. Indeed, there we recovered the Lax operator from the general R-operator by means of the reduction formula \( [33] \). The right-hand side of the latter does coincide with \( [97] \) after appropriate identification of parameters. The factorized representation \( [97] \) of the symbol \( \Lambda \) where the finite-difference operator \( [96] \) is sandwiched in between two multiplication by function operators is a shadow of the Lax factorization \( [27] \). Further we profit from the star-triangle relation \( [67] \) which we rewrite here in an equivalent form

\[
S_b(\alpha) M_z(\alpha + \beta) S_b(\alpha) = M_z(\beta) S_b(\alpha + \beta) M_z(\alpha) ,
\]

[^3]: We are grateful to D. Karakhanyan and R. Kirschner for a discussion on this point.
where the following shorthand notation is adopted
\[ S_b(\alpha) = \Gamma(\pm b \pm z + \alpha + \eta + \frac{\tau}{2}). \] (99)

Thus due to (98) the operator (97) is equal to
\[ \Lambda(u, \lambda, \mu) = c \cdot S_a(u_1 + \eta + \frac{\tau}{2}) M_z(u_2 + \eta + \frac{\tau}{2}) S_b(\eta) M_z(-u_2 - \frac{\tau}{2}) S_a(-u_1 - \frac{\tau}{2}). \] (100)

Obviously, the analogues of equalities (97) and (100) obtained after the permutation \(2\eta \leftrightarrow \tau\) take place for the second symbol \(\Lambda\) (94). Now we are ready to calculate the symbol of the higher-spin R-operator (92) which factorizes in a product of Lax operators’ symbols.

\[ R_{\text{fus}}(u) = \Lambda(u) \Lambda(u - 2\eta) \cdots \Lambda(u - 2(n - 1)\eta) \cdot \tilde{\Lambda}(u - 2n\eta - (m - 1)\tau), \] (101)

where we omit dependence on auxiliary spinors for brevity. The second factorized representation of the symbols (100) enables us to calculate straightforwardly the product (101). Indeed, in the product of two consecutive symbols from (101), two adjacent pairs of operator factors cancel out in view of the reflection formula for elliptic gamma functions (89). Thus we obtain an explicit expression for the symbol
\[ R_{\text{fus}}(u) = c^n c^m \cdot S_a(u_1 + \eta + \frac{\tau}{2}) M_z(u_2 + \eta + \frac{\tau}{2}) S_b(\eta) S_b^m(\frac{\tau}{2}) \cdot M_z(-u_2 + (n - 1)\eta + (m - 1)\frac{\tau}{2}) S_a(-u_1 + (n - 1)\eta + (m - 1)\frac{\tau}{2}). \] (102)

Let us note that previous multiple cancellations in the product (101) are similar to cancellations which we encountered in calculating the symbol \(R(\lambda, \mu)\) (88) in the previous section. There the factorized form (88) of the building blocks, which constitute the string (86), played a crucial role as well.

Since we know the symbol (102), we reconstruct immediately the corresponding operator by means of (88)
\[ [R_{\text{fus}}(u) \Phi](\lambda, \tilde{\lambda}; z) = R_{\text{fus}}(u|\lambda, \tilde{\lambda}, \partial_\mu, \partial_{\tilde{\mu}}) \Phi(\mu, \tilde{\mu}| z)|_{\mu=\tilde{\mu}=0}. \] (103)

The test function \(\Phi(\lambda, \tilde{\lambda}| z)\) is homogeneous in spinor variables \(\lambda, \tilde{\lambda}\) of degree \(n\) and \(m\), respectively. Several comments concerning (103) are in order. The formula (103) contains differentiation with respect to auxiliary spinors that is meaningful since according to (101) the symbol is polynomial in \(\mu, \tilde{\mu}\). However this property of the symbol is no more obvious for (102) where a sole parameter \(b\) is responsible for dependence on both spinors (93). In order to recover polynomiality of (102) we resort to the identities
\[ S_b(\eta) = \Gamma(\pm z \pm 2\eta + \frac{\tau}{2}) = -\frac{1}{2} R^2(\tau) e^{i\eta \tau} \left[ \mu_1 \theta_4(z|\frac{\tau}{2}) + \mu_2 \theta_3(z|\frac{\tau}{2}) \right], \] (104)
\[ S_b(\frac{\tau}{2}) = \Gamma(\pm z \pm 2\eta + \eta + \tau) = -\frac{1}{2} R^2(2\eta) e^{i\eta \tau} \left[ \tilde{\mu}_1 \theta_4(z|\eta) + \tilde{\mu}_2 \theta_3(z|\eta) \right] \] (105)

following from (16), (32), (33). Thus (102) is homogeneous in \(\mu\) and \(\tilde{\mu}\) of degree \(n\) and \(m\), respectively. The polynomiality of (102) on \(\lambda\) and \(\tilde{\lambda}\) is less trivial although it is guaranteed by (101). We will verify it explicitly in a moment.

Moreover, we need to check that the “fused” higher-spin R-operator (103) is identical with the R-operator resulted from the reduction formula (73). With this in mind we take the generating

\[ \text{Let us note that strange looking shifts of the spectral parameter in (101) are owing to an unconventional definition of the Lax operators (25) and (29) which include a supplementary shift of the spectral parameter as compared to the usual definitions.} \]
function of finite-dimensional spin $g_{n,m} = (n+1)\eta + (m+1)\frac{z}{2}$, $n, m \in \mathbb{Z}_{\geq 0}$, representation (see (106)) and rewrite it in terms of the auxiliary spinors’ components (recall (93))

$$
\lambda_1 = \theta_4(a|\frac{\tau}{2}) \), \ \lambda_2 = \theta_3(a|\frac{\tau}{2}) \), \ \bar{\lambda}_1 = \theta_4(a|\eta) \), \ \bar{\lambda}_2 = \theta_3(a|\eta) \),
$$

that results in a finite product of linear combinations of theta functions depending on an auxiliary parameter $x$,

$$
\Gamma(\mp a \mp x + (n+1)\eta + (m+1)\frac{z}{2}) =
\begin{align*}
&= c \cdot \prod_{r=0}^{n-1} \left[ \lambda_2 \theta_4(x + (n - 1 - 2r)\eta | \frac{\tau}{2}) + (-1)^m \lambda_1 \theta_3(x + (n - 1 - 2r)\eta | \frac{\tau}{2}) \right] \\
& \cdot \prod_{s=0}^{m-1} \left[ \bar{\lambda}_2 \theta_4(x + (m - 1 - 2s)\frac{z}{2} | \eta) + (-1)^n \bar{\lambda}_1 \theta_3(x + (m - 1 - 2s)\frac{z}{2} | \eta) \right].
\end{align*}
$$

The generating function has the homogeneity degrees $n$ and $m$ in $\lambda$ and $\bar{\lambda}$, respectively. Then we act on the generating function by $R_{n,m}$ in the first space according to (103). Let us stress that the second space of $R_{n,m}$ is untouched. The derivatives with respect to spinors in (103) are easily calculated by means of the formula

$$
S_b^n(\eta) S_m^{n}(\frac{z}{2}) |_{\mu \to \partial_{a,\bar{\mu}} \to \partial_{\eta}} \Gamma(\pm x \pm b + (n+1)\eta + (m+1)\frac{z}{2}) =
\begin{align*}
&= n!m!(-2)^{-n-m} R^{2n}(\tau) R^{2m}(2\eta) e^{\frac{\pi i}{2} (n+\pi)\eta} \Gamma(\pm z \pm x + n\eta + m\frac{z}{2} + \eta + \frac{z}{2}),
\end{align*}
$$

which follows from (104), (105), (106). Effectively, the previous formula implies the substitution (recall (99))

$$
S_b^n(\eta) S_m^{n}(\frac{z}{2}) \to \Gamma(\pm z \pm x + (n+1)\eta + (m+1)\frac{z}{2}) = S_x(n\eta + m\frac{z}{2})
$$

in (102) that yields

$$
R_{n,m}(u) \Gamma(\mp a \mp x + (n+1)\eta + (m+1)\frac{z}{2}) = S_a(u_1 + \eta + \frac{z}{2}) M_x(u_2 + \eta + \frac{z}{2}) S_x(n\eta + m\frac{z}{2}) \cdot
\begin{align*}
& \cdot M_x(-u_2 + (n-1)\eta + (m-1)\frac{z}{2}) S_a(-u_1 + (n-1)\eta + (m-1)\frac{z}{2}) = S_a(u_1 + \eta + \frac{z}{2}) \cdot
\end{align*}
\begin{align*}
& \cdot S_x(-u_2 + (n-1)\eta + (m-1)\frac{z}{2}) M_x(n\eta + m\frac{z}{2}) S_x(u_2 + \eta + \frac{z}{2}) S_a(-u_1 + (n-1)\eta + (m-1)\frac{z}{2}).
\end{align*}

At the last step we rearranged the operator factors by means of the star-triangle relation (98). Finally, renaming the variables $a = z_1$, $z = z_2$, $x = z_3$, recalling definitions of $S_x(u)$ (99) and $u_1$, $u_2$ (28), and applying the reflection formula (33), we transform (108) in the reduction formula (73). The precise relation is

$$
R_{n,m}(u + g_{n,m} - \eta - \frac{z}{2}) = R_{12}(u|g_{n,m}, g)
$$

where both sides are assumed to be restricted to a finite-dimensional spin $g_{n,m}$ representation in the first space. Thus we have found a nice agreement between two approaches to constructing finite-dimensional (in one of the spaces) solutions of YBE.

### 7 Conclusion

Our previous paper [7] and the present one are devoted to construction of finite-dimensional matrix solutions of the YBE with plain function entries as well as to building of quantum L-operators given by finite-dimensional matrices with differential or finite-difference operator entries. We derived concise formulae for these objects from reductions of general integral operator YBE solutions with a rank 1 symmetry algebra. At the rational level this procedure probably yields all finite-dimensional solutions of YBE. At the $q$-deformed level we see at least two missing classes of solutions. Namely,
we did not consider the notorious root of unity cases for the basic parameter $q$, which form its own separate representation theory world. Also we did not consider the R-operator and corresponding reductions for the model suggested in [41].

At the elliptic level the situation is even more complicated. Our consideration misses the cases when the base parameters $p$ and $q$ are commensurable, which yields the situation reacher than the plain root of unity cases $q^N = 1$ and/or $p^M = 1$. Moreover, there exists the Felderhof model [18], which should be properly understood from our point of view. Namely, it is necessary to understand from which integral operator it can be derived as a reduction and what is the whole hierarchy of finite-dimensional solutions associated to it? It may happen that they are obtained from the general R-operator of [9] as a result of a more intricate reduction procedure than we have considered here. We hope to address these questions in the future.

Integral operator solutions of YBE play an important role in various quantum field theories. As mentioned in [7] (see the references given there), the rational level case is important for the investigation of high-energy behaviour of the quantum chromodynamics. However, it is not completely clear where our finite-dimensional reductions appear within such applications. We mention also a different important role played by the elliptic hypergeometric integrals [35, 40] in field theories. Namely, they emerge as superconformal indices of four-dimensional supersymmetric gauge theories [14, 42] and encode an enormous amount of information on the corresponding models and corresponding dualities. Again, it is not clear what particular kind of useful information on these models is hidden in the finite-dimensional representations of the elliptic modular double [9, 10] and R-matrices we have constructed in this work.

The most evident application of our results for future investigations consists in the detailed consideration of solvable models analogous to the 8-vertex model using the most general elliptic R-matrix we have derived. There should exist also other new many-body integrable models in classical and quantum mechanics related to our results. In particular, it would be interesting to investigate the classical Poisson algebra systems associated with them.

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