Abstract. In a recent paper by McCormick and Cohen (2021), the new vertical profiles were discussed of electron density in the D/E region of ionosphere that accounts for the observations of VLF atmospherics (sferics). In the present paper, we use a typical profile from this paper and apply the classical profile of electron effective collision frequency for obtaining the conductivity of middle atmosphere. This air conductivity profile is compared then with those matching the Schumann resonance observations. By using this novel conductivity model, we compute the propagation parameters of ELF-VLF modes in the Earth–ionosphere cavity with the help of full-wave solution. An emphasis is made on the comparison of Schumann resonance spectra found for the existing and the novel profiles. The multi-mode full-wave solutions allowed us to compute the wide-band complex spectra of vertical electric and horizontal magnetic field components by using the classical modal expansions. Finally, the pulsed waveforms were calculated using the Fourier transform of complex ELF/VLF spectra, which belong to typical slow tail atmospherics (sferics). The conclusion is made that the profile by McCormick and Cohen underestimates the radio wave attenuation only at ELF, so that relevant data predicts somewhat higher peak frequencies and Q-factors of Schumann resonance.

Key words: ELF/VLF atmospherics (sferics), conductivity of middle atmosphere, modal expansion, Schumann resonance, model slow tail atmospherics (sferics)

1. Introduction

In their latest work, McCormick and Cohen (2021) introduced a new four-parameter model of the D-region ionospheric electron density (60–90 km) relevant to the wide-band very low frequency (VLF, 3–30 kHz) pulsed radio signals (atmospherics or sferics). This type of models containing a few layers is well known. For example, a two-layer model with four free parameters was used in the paper by Jones (1964). He introduced the step-like profile of electron content in the lower ionosphere. The merit of the approach used in McCormick and Cohen (2021) is in the application of exponential height profile of electron density, which contains a “vertical jump” by ~10 km around the altitude of 60 km. Parameters of this model, being defined in the altitude range from 45 to 90 km, are linked to the classical exponential profile introduced by Wait and Spies (1964).

McCormick and Cohen (2021) discuss the applicability of their model to the wide-band observations of atmospherics in the ambient day and night conditions. In the present paper, we focus on their typical profile from the other point of view. We compare the model results relevant to their profile with those pertinent to the conductivity profiles of the middle atmosphere that were successfully used for interpretations of natural electromagnetic signals from ULF (ultra low frequency) to VLF bands (Kudintseva et al., 2016; Nickolaenko et al., 2016, 2017, 2021a,b). These
bands cover, in particular, the currents of the global electric circuit (Kudintseva et al., 2016), Schumann resonances (Jones and Kemp, 1971; Kudintseva et al., 2016; Nickolaenko et al., 2016, 2017, 2021a) and the slow tail atmospherics in the day and night conditions (Nickolaenko et al., 2021b). These Schumann resonance and slow tails belong to ELF (extremely low frequency, below 3 kHz) range.

As the first step, we obtain the vertical profile of atmosphere conductivity using the typical profile of electron density from the paper by McCormick and Cohen (2021). After obtaining the relevant conductivity profile (we regard it as a 'new' profile in what follows), we compare it with our previous profiles defined in Kudintseva et al. (2016) and Nickolaenko et al. (2016, 2017, 2021a,b).

As the second step, we find the full-wave solution for the particular Earth–ionosphere waveguide models using the Riccati differential equation. This provides the modal complex propagation constants $\nu_p(\beta)$ and the modal norms of fields relevant to different waveguide modes of order $p = 0, 1, 2, \ldots$

In the third step, we compute the Schumann resonance spectra corresponding to the ‘old’ (Kudintseva et al., 2016; Nickolaenko et al., 2016, 2017, 2021a,b) and the ‘new’ (McCormick and Cohen, 2021) models relevant to the globally uniform distribution of lightning strokes. The power spectra of vertical electric field component are compared revealing the distinctions conditioned by different model profiles.

In the fourth step, we compute the complex spectra of the vertical electric and horizontal magnetic fields that arrive at an observer with a given distance from the Bruce and Golde vertical stroke of lightning [see the book of Nickolaenko and Hayakawa (2002)]. Finally, we obtain the pulsed waveforms at the source-observer distances of 4 and 10 Mm. The waveforms demonstrate that the new model profile (McCormick and Cohen, 2021) predicts similarly to the old model (Nickolaenko et al., 2021b) the slow tail atmospherics, which might be detected using a wide-band ELF–VLF receiver.

2. Model of atmosphere conductivity

Let us list the parameters of electron density profiles and display the relevant formulas. The step-like altitude dependence of electron density profile $N_e(h)$ is described by the following equation (McCormick and Cohen, 2021):

$$N_e(h) = \begin{cases} N_1 & \text{when } h < s_L \\ N_L & \text{when } s_L \leq h < U \\ N_1 \cdot N_L \cdot \frac{Z}{h} & \text{when } h > U \end{cases} \quad (1)$$

Here, the profile by Wait and Spice (1964) is used as a reference, which is equal to,

$$N_1 = 3.93771 \cdot 10^8 \cdot \exp [0.35 \cdot (h - h_0)] \quad [\text{m}^{-3}] \quad (2)$$

The elements of the step-like profiles are correspondingly equal to:

$$N_L = 3.93771 \cdot 10^8 \cdot \exp [0.35 \cdot (s_L - h_0)] \quad [\text{m}^{-3}] \quad (3)$$

$$Z = 3.93771 \cdot 10^8 \cdot \exp [0.35 \cdot (s_L + \Delta h - h_0)] \quad [\text{m}^{-3}] \quad (4)$$

Parameters of the profile are denoted in the following way: $h$ is the height above the ground surface in km; $s_L = 60$ km denotes the lower height of split; $\Delta h = 8$ km indicates the size of the split or the width of the gap in the stepped increase in the electron density where it is held at a constant $N_L$ value; $U = s_L + \Delta h$ is the upper height of the split. $h_0$ is effective reflection height.

The model by Eq. (1) is identical to that by Wait and Spies (1954) below the split altitude $s_L$, and it is equivalent to the reference profile elevated by $\Delta h$ above the split. The height scale is
identical on both sides of the split. The relevant conductivity of middle atmosphere is found from the following equation:

\[ \sigma = \frac{N_e e^2}{m_e \nu_{en}} \quad \text{[S/m]} \]  

(5)

where \( e = 1.602 \times 10^{-19} \text{C} \) is the electron’s charge; \( m_e = 9.109 \times 10^{-31} \text{kg} \) is the electron’s mass, and the effective collision frequency of electrons in Eq. (5) is equal to:

\[ \nu_{en} = 5.0 \times 10^6 \exp \left[-0.15(h - 70)\right] \quad \text{[s}^{-1}] \]  

(6)

It was taken from Wait and Spies (1964).

Figure 1. Comparison of conductivity profiles derived for VLF and ELF/ULF signals. Schumann resonance (SR) night, day, and average, and McCormick and Cohen (as Cohen) model. The above parameters correspond to the typical model introduced in McCormick and Cohen (2021). Figure 1 demonstrates a set of height profiles of air conductivity. There are profiles developed for explaining Schumann resonance (SR) data and the data of global current circuit (Kudintseva et al., 2016; Nickolaenko et al., 2016, 2017, 2021a,b). The dashed line shows the averaged SR profile used in the work by Kudintseva et al. (2016). The lines with dots and asterisks depict the SR ambient day and night profiles used in the paper by Nickolaenko et al. (2021b). The straight broken line in Figure 1 depicts the ‘new’ conductivity profile (as Cohen) described by Eq. (5). One may observe that all the profiles are rather close to each other above the 50 km altitude, but they are seen to seriously deviate below this height. It is obvious that such a departure of the new profile (McCormick and Cohen, 2021) must reduce the electromagnetic losses in the Earth–ionosphere cavity, raising peak frequencies and Q-factors of Schumann resonance. We must also remark that the vertical discontinuity in the electron density at 60 km altitude had turned into a rather steep, but a gradual increase. This is the result of the height variations in the electron collision frequency.

Figure 2 compares the characteristic (magnetic and electric) heights of the lower ionosphere (upper panel) \((h_L \text{ and } h_C)\) and the complex propagation constants \((\psi)\) of zeroth order mode \((p=0)\) (lower panel). These characteristic heights were computed for particular profiles using the full-wave solution in the form of Riccati equation (Kudintseva et al., 2016; Nickolaenko et al., 2021).
These heights were initially introduced in the paper by Greifinger and Greifinger (1978). One may observe substantial departures in the upper (magnetic) and the lower (electric) heights \((h_L \text{ and } h_C)\) relevant to different models. Nevertheless, the propagation constants, which depend on the ratio of these heights, do not show any dramatic deviations.

Figure 2. Characteristic heights \((h_L \text{ and } h_C)\) and propagation constants \(\nu(f)\) relevant to the average Schumann resonance (SR) profile and to the McCormick and Cohen (2021) profile (as Coh).

3. Power spectra of Schumann resonance

After obtaining the propagation constant of ELF radio waves \(\nu(f)\) \((f, \text{frequency})\), we can compute the typical power spectra of Schumann resonance observed in the vertical electric field component. To avoid the dependence of resonance spectra on the source–observer distance, we assume that mutually independent lightning sources have the ‘white’ spectra and are uniformly distributed over the whole globe. In this case, the power spectrum is described by the following equation (Nickolaenko and Hayakawa, 2002):

\[
\langle |E|^2 \rangle = \left| \frac{\nu(f+1)}{f} \right|^2 \sum_{n=0}^{\infty} \frac{(2n+1)}{[\nu(f+1)-n(n+1)]^2}
\]

Here, \(\nu(f) = \nu_0(f)\) is the zeroth order mode propagation constant depending on the signal frequency \(f\) and \(n\) is the Schumann resonance mode number.

In our computations, we have summed 100 terms in Eq. (7). The resulting model spectra are shown in Figure 3, and we present three curves there. The black line shows the reference Schumann resonance (SR) spectrum. The relevant heuristic \(\nu(f)\) dependence was published in Ishaq and Jones (1977):

\[
\nu(f) = 1.67-0.1759 \ln(f)+0.0179 [\ln(f)]^2-i0.063 f^{0.64}
\]

Here, \(i\) denotes the square root of \(-1\).
Formula (8) summarizes the long-term global observations performed in interest of the “Sanguine – Seafarer” project (Nickolaenko and Hayakawa, 2002). The smooth line and the line with diamonds in Figure 3 depict correspondingly the power spectra $\langle |E|^2 \rangle$ for the heuristic $\nu(f)$ dependence (8) and to the average conductivity profile developed from Schumann resonance (SR) observations. The latter was shown in Figure 1 by dashed line. The line with dots in Figure 3 is the power spectrum (divided by 1.5) computed for the ‘new’ stepped conductivity profile.

Obviously, the atmosphere conductivity profile by McCormick and Cohen (2021) developed for modeling mainly the VLF atmospherics does not account for the ions content in the lower part of mesosphere. This is the reason why it overestimates the Schumann resonance amplitudes, its peak frequencies, and the Q-factors. These features were expected when introducing data of Figure 1.

![Figure 3](image.png)

Figure 3. Model power spectra of Schumann resonance (SR) in the vertical electric field component for a uniform source distribution.

Deviations among the Schumann resonance spectra are evident, but not so crucible. They become obvious only when the direct comparison of data is performed plotted in the same figure. Deviations in the signal amplitude become a rather poor argument when comparing models, since we do not know the actual level of global thunderstorm activity during ‘observations’. In this context, the ‘new’ profile based exclusively on the VLF data seems to be not bad at all. To catch deviations, one has to turn to the observed peak frequencies, as the experimental estimates of the Q-factors (the peak widths) of Schumann resonances cannot be measured with sufficient accuracy.

We arrive at the following conclusion. The Schumann resonance is described in a better way when the ionic conductivity is accounted for in the atmosphere. The stepped profile in Figure 1 should be modified accordingly below the altitude of 50 km. After such a correction, the further modifications might be applied within the other height intervals. Clearly, the endless improvements might be suggested for finding the upgraded model profile.

4. Amplitude spectra of atmospherics

After postulating the exponential altitude dependence of effective electron collision frequency, we obtain the conductivity profile corresponding to the model (McCormick and Cohen,
This allows us to use the full-wave solution in computations of the parameters of waveguide modes used in expansion of electromagnetic field (Nickolaenko et al., 2021b): the propagation constants and the norms of three waveguide modes ($p = 0, 1, \text{ and } 2$). The most informative characteristic of the waveguide modes is the frequency variations of their attenuation rate shown in Figure 4.

One may observe that the wave attenuation (as imaginary part of $\nu$) of waveguide modes in Figure 4 of the ‘new’ profile resembles the classical plots demonstrated in the books by Wait (1962) and Galejs (1972). In particular, the zeroth order mode attenuation (line with open circles) grows monotonically with signal frequency. The attenuation of the higher-order modes decreases with frequency and has a kind of plateau in the vicinity of the cut-off frequency of relevant mode. Fortunately, the ‘new’ conductivity profile causes no mode degeneracy, as seen in e.g. Nickolaenko et al. (2021b). Thus, the ‘new’ profile provides the conventional VLF propagation characteristics.

We can compute now the complex spectra of the vertical electric and the horizontal magnetic fields by using the classical modal expansions (Wait, 1962; Galejs, 1972). The relevant amplitude spectra are shown in Figure 5 for the vertical electric field component. The Bruce and Golde model was used for the current moment of the ‘parent’ lightning stroke (Nickolaenko and Hayakawa, 2002). The source–observer distance was chosen to be 4 Mm.

![Figure 4. Wave attenuation of three waveguide modes (as imaginary part of $\nu$) as the functions of frequency.](image-url)
Figure 5. Wide-band amplitude spectra of first three waveguide modes in the Earth–ionosphere spherical cavity for the source–observer distance of 4 Mm.

The abscissa in Figure 5 shows the signal frequency ranging from 1 to 10,000 Hz on the logarithmic scale. The ordinate is the logarithmic scale of the signal amplitude in arbitrary units. The bold black line depicts the sum $E_s$ of three waveguide modes ($p=0,1,2$). The line with open circles shows the spectrum of the zeroth order mode (the TEM wave). This curve outlines the distinct Schumann resonance peaks with the general pattern corresponding to the particular 4 Mm distance from the vertical stroke of lightning. The zeroth order mode amplitude severely attenuates when the signal frequency exceeds the value of 1 kHz.

The line with asterisks shows the first mode amplitude. This wave appears above the frequency of 2 kHz, and its amplitude becomes comparable with that of the zeroth order mode at around 3 kHz. Here, one may notice the modal interference in the $E_s$ amplitude. The line with wedges shows the amplitude spectrum of the second mode. It emerges around the frequency of 6 kHz. However, its amplitude is so small, that a contribution from the second mode is negligible at all frequencies below 10 kHz.

5. Time domain waveforms of atmospherics

By applying the Fourier transform to the complex spectra of field components, we obtain the pulsed waveforms shown in Figure 6. The left panel in this figure shows the pulses at the source–observer distance of 4 Mm, and the right panel depicts the pulses at the distance of 10 Mm.

The abscissas in Figure 6 show the time in seconds passed from the lightning stroke initiation. The upper plots in the both panels depict the ELF pulse arriving at the observer as the zeroth order mode. These pulses are detected by usual Schumann resonance receivers of the band-pass below 100 Hz, and they are regarded in the literature as Q-bursts (Ogawa et al., 1966) or ELF-transients (Jones and Kemp, 1971). In the wide-band observations, the zero-order mode signal forms the slow tail part of atmospherics. The second plots in each panel show the signal arriving as the first mode wave. It contains rather fast oscillations at a frequency around the first cut-off frequency of the waveguide (~2 kHz). Such pulses are detected by VLF receivers with the band-pass above 1.5 kHz, which are called atmospherics.

The lower plots show the assembly of the zeroth order and the first order modes driven by the wide-band radiation of the Bruce and Golde discharge (Nickolaenko and Hayakawa, 2002). We observe the presence of the high frequency precursor ‘head’ followed by a slow tail. Figure 6
demonstrates that the attenuating VLF (the first-order mode) component disappears rapidly, and only the ‘rear’ part of the Q-burst (the zeroth order mode) remains in the waveform. This part of ELF pulse is recognized as the ‘tail’ dragged by the ‘head’. The slow tail atmospherics might be observed by the wide-band receivers covering the interval of 1 – 10,000 Hz, provided that the parent lightning stroke radiates in this entire band. The time delay between the VLF precursor and the ELF slow tail monotonously increases with the source–observer distance (Nickolaenko et al., 2021b).

Figure 6. Pulsed waveforms in the Earth–ionosphere spherical cavity for the source–observer distances of 4 and 10 Mm.

6. Conclusion
The presented model data allows us to conclude that the general outline of atmospherics (sferics) modeled using the profile by McCormick and Cohen (2021) is a reasonable one especially at VLF. The distinctions among the computational data are observed in the Schumann resonance band below the frequency of 100 Hz. The ‘new’ profile by McCormick and Cohen (2021) predicts the unrealistically high parameters of Schumann resonance spectra such as the too high mode amplitudes, peak frequencies, and the quality factors. These features might be observed as increased max to min ratio of the spectral peaks.

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