The scaling behavior over four decades of the ratio of temperature $T$ to magnetic field $B$ observed in the magnetization in $\beta$-YbAlB$_4$ is theoretically examined. By developing a theoretical framework that exhibits the quantum critical phenomena of Yb-valence fluctuations under a magnetic field, we show that the $T/B$-scaling behavior can appear near the quantum critical point of the valence transition. The emergence of the $T/B$ scaling indicates the presence of the small characteristic energy scale of critical Yb-valence fluctuations. It is argued that the quantum valence criticality offers a unified explanation for the unconventional quantum criticality as well as the $T/B$ scaling in $\beta$-YbAlB$_4$.

Quantum critical phenomena in itinerant electron systems that do not follow the conventional spin-fluctuation theory$^{1-4}$ have attracted attention in condensed matter physics.$^5$ The heavy-electron metal $\beta$-YbAlB$_4$ has recently attracted great interest since the unconventional quantum criticality, such as the magnetic susceptibility $\chi \sim T^{-0.5}$, the electronic specific-heat coefficient $C_e/T \sim -\log T$, and approximately $T$-linear resistivity, has been observed at low temperatures at least below 3 K down to a few hundred mK.$^6,7$

Interestingly, from the magnetization data for $T \lesssim 3$ K and the magnetic field $B \lesssim 2$ T, in $\beta$-YbAlB$_4$ it has been discovered that the magnetic susceptibility $\chi$ shows the following $T/B$ scaling behavior over four decades of $T/B$:

$$\chi^{-1} = (\mu_B B)^{1/2} \varphi \left( \frac{k_B T}{\mu_B B} \right),$$

where $\mu_B$ and $k_B$ are the Bohr magneton and Boltzmann constant, respectively, and $\varphi$ is the function $\varphi(x) = \Lambda (\Gamma^2 + x^2)^{1/4}$ with $\Lambda$ and $\Gamma$ being constants.$^7$ Namely, $\chi^{-1}/(\mu_B B)^{1/2}$ is expressed as a single scaling function of the ratio $T/B$.

This striking behavior of Eq. (1) calls for theoretical explanation, and it has so far been proposed that anisotropic hybridization between f and conduction electrons is the key origin...
of the emergence of $T/B$ scaling.\(^8\) However, this theory requires an assumption that the renormalized f level is pinned at the hybridized band edge, and it also seems unclear whether the unconventional criticality observed in $C_c/T$ and the resistivity can be explained by the anisotropic hybridization.

Recently, it has been shown theoretically that a new type of quantum criticality emerges near the quantum critical point (QCP) of the first-order valence transition in Yb- and Ce-based heavy-electron systems.\(^9\) Critical valence fluctuations of Yb or Ce cause the quantum criticality in physical quantities such as $\chi$, $C_c/T$, resistivity, and the NMR/NQR relaxation rate $(T_1T)^{-1}$, which give a unified explanation for the measured unconventional criticality in $\beta$-YbAlB$_4$.\(^9,10\) Hence, it is interesting to examine whether the critical Yb-valence fluctuation can account for the $T/B$ scaling observed in $\beta$-YbAlB$_4$.

In this Letter, we show that the $T/B$ scaling can be understood from the viewpoint of the quantum valence criticality. By developing a theoretical framework for the quantum critical phenomena of Yb-valence fluctuations under a magnetic field, we show that the $T/B$ scaling emerges near the QCP of the valence transition. We demonstrate that the emergence of the $T/B$ scaling is a hallmark of the presence of the small characteristic energy scale of the critical Yb-valence fluctuations.

We employ the theoretical framework developed in Ref. 9, whose formulation is extended so as to describe the effect of a magnetic field. Hereafter, we take the energy units of $k_B = 1$, $\hbar = 1$, and $\mu_B = 1$ unless otherwise noted. We consider the simplest minimal model

$$H = H_{\text{PAM}} + H_{Ufc} + H_{\text{Zeeman}}$$

as the starting Hamiltonian, where $H_{\text{PAM}} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + E_i \sum_{i\sigma} n_{i\sigma}^f + \sum_{\mathbf{k}\sigma} \left(V_{i\mathbf{k}} f_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \text{h.c.} \right) + U \sum_{ij} n_{i\sigma}^f n_{j\sigma}^f$, $H_{Ufc} = \sum_{\mathbf{i}\mathbf{r}\sigma} n_{i\sigma}^f c_{i\sigma}^\dagger$, and the Zeeman term $H_{\text{Zeeman}} = -h \sum_i S_i^z$ with $n_{i\sigma}^f \equiv \sigma_i^a \sigma_i^{a\prime}$ for $a = f$ or $c$ and $S_i^z \equiv \frac{1}{2}(n_{i1}^f - n_{i1}^c)$ in the standard notation.

To discuss the quantum critical phenomena of Yb- (and Ce-) valence fluctuations, first we take into account the local correlation effect by the $U$ term, which is the strongest interaction in Eq. (2) responsible for the realization of the heavy-electron state. Then perturbative expansion with respect to the $U_{fc}$ term is performed. To perform the procedure, we employ the slave-boson large-$N$ expansion method.\(^11\) Here we set the orbital degeneracy $N = 2$ to discuss $\beta$-YbAlB$_4$, where the Kramers-doublet ground state is realized. Hence, $\sigma = \uparrow, \downarrow$ in Eq. (2) should be regarded as the effective “spin” index that specifies the Kramers doublet. The slave-boson operator $b_i$ is introduced to eliminate the doubly occupied state for $U \to \infty$ under the constraint $\sum_{\sigma} n_{i\sigma}^f + 2b_i^\dagger b_i = 1$. The Lagrangian is written as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, where $\mathcal{L}_0$...
is the Lagrangian for $H_{PAM} + H_{Zeeeman}$ with the term $-\sum_i \Lambda_i \left( \sum_r n_{i\sigma}^f + 2b^\dagger_i b_i - 1 \right)$ with $\Lambda_i$ being the Lagrange multiplier and $L'$ is the Lagrangian for $H_{U'_s}$ (see Ref. 9 for detail).

For $\exp(-S_0)$ with the action $S_0 = \int_0^\beta d\tau L_0(\tau)$, the saddle-point solution is obtained via the stationary condition $\delta S_0 = 0$ by approximating spatially uniform and time-independent solutions, i.e., $\lambda_q = \lambda_0\delta q$ and $b_q = b_0\delta q$. The solution is obtained by solving the mean-field equations $\delta S_0 / \delta \lambda = 0$ and $\delta S_0 / \delta b = 0$ self-consistently.

For $\exp(-S')$ with the action $S' = \int_0^\beta d\tau L'(\tau)$, we introduce the identity applied by a Stratonovich-Hubbard transformation $e^{-S'} = \int D\varphi \exp \left[ \sum_i \int_0^\beta d\tau \left\{ -\frac{U_0}{2} \varphi_{i\sigma}(\tau)^2 + i\frac{U_0}{2}(c_{i\sigma} f_{i\sigma}^\dagger - f_{i\sigma} c_{i\sigma}^\dagger)\varphi_{i\sigma}(\tau) \right\} \right]$. The partition function is expressed as $Z = \int D\varphi \exp(-S[\varphi])$ with $S = S_0 + S'$. By performing Grassmann number integrations for $cc^\dagger$ and $ff^\dagger$, we obtain $Z = \int D\varphi \exp(-S[\varphi])$ with

$$S[\varphi] = \sum_\sigma \left[ \frac{1}{2} \sum \bar{q} \Omega_{2\sigma}(\bar{q})\varphi_{\sigma}(\bar{q})\varphi_{\sigma}(-\bar{q}) + \sum_{\bar{q}_1, \bar{q}_2, \bar{q}_3} \Omega_{3\sigma}(\bar{q}_1, \bar{q}_2, \bar{q}_3) \times \varphi_{\sigma}(\bar{q}_1)\varphi_{\sigma}(\bar{q}_2)\varphi_{\sigma}(\bar{q}_3) \delta \left( \sum_{i=1}^3 \bar{q}_i \right) + \sum_{\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4} \Omega_{4\sigma}(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4) \times \varphi_{\sigma}(\bar{q}_1)\varphi_{\sigma}(\bar{q}_2)\varphi_{\sigma}(\bar{q}_3)\varphi_{\sigma}(\bar{q}_4) \delta \left( \sum_{i=1}^4 \bar{q}_i \right) + \cdots \right],$$

(3)

where the abbreviation $\bar{q} \equiv (q, i\omega_l)$ with $\omega_l = 2\hbar T$ is used. Since the long wavelength around $q = 0$ and the low-frequency regions play dominant roles in the critical phenomena, $\Omega_{i\sigma}$ for $i = 2, 3,$ and 4 are expanded for $q$ and $\omega_l$ around $(0, 0)$: $\Omega_{2\sigma}(q, i\omega_l) \approx \eta_{0\sigma} + A_\sigma q^2 + C_\sigma |q|$, where $\eta_{0\sigma} = U_{fc} \left[ 1 - U_{fc} \chi^{\text{fct}}_0(0, 0, 0) \right]$. Here $\chi^{\text{phys}}_0(q, i\omega_l) = -\frac{T}{N_s} \sum_{k,n} G^{bb}_k(\epsilon_{k\sigma})$, where $G^{\text{fct}}_{k\sigma}(\epsilon_{k\sigma}) = 1/[(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{fct}} - V_{k\sigma}^2)/(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{fct}})]$, $G_{k\sigma}^c(\epsilon_{k\sigma}) = 1/[(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{c}} - V_{k\sigma}^2)/(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{c}})]$, and $G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma}) = \bar{V}_{k\sigma}/[(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{cct}} - V_{k\sigma}^2)/(\epsilon_{k\sigma} - \bar{\epsilon}_{\text{cct}})]$ with $\epsilon_{k\sigma} = (2n + 1)\pi T$. Here, $\bar{\epsilon}_{\text{fct}}$, $\bar{\epsilon}_{\text{cct}}$, and $\bar{V}_k$ are defined as $\bar{\epsilon}_{\text{fct}} \equiv \epsilon_k + \frac{U_{0\sigma}}{2}$, $\bar{\epsilon}_{\text{cct}} \equiv \epsilon_l + \frac{U_{2\sigma}}{2} + \frac{\lambda}{N_\sigma} - P(\sigma)\frac{b}{2}$, and $\bar{V}_k \equiv \frac{V_{ab}}{\sqrt{N_s}}$, respectively, with $P(\uparrow) \equiv +1$ and $P(\downarrow) \equiv -1$. Since $\chi^{\text{fct}}_0(0, 0) \gg \chi^{\text{fct}}_0(0, 0, 0)$ in simplicity of calculation. For $\Omega_{3\sigma}$ and $\Omega_{4\sigma}$, expansion up to the zeroth order is performed as $\Omega_{3\sigma}(\bar{q}_1, \bar{q}_2, \bar{q}_3) \approx v_{3\sigma}/\sqrt{N_s}$ and $\Omega_{4\sigma}(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4) \approx v_{4\sigma}/(\beta N_s)$, respectively. The mode-coupling constant $v_{4\sigma}$ is derived as

$$v_{4\sigma} = \frac{U_{fc}^4}{4} \left[ \frac{T}{N_s} \sum_n \sum_k G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma})^2 G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma}) G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma}) \right]$$

$$+ \frac{T}{2N_s} \sum_n \sum_k G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma})^2 G_{k\sigma}^{\text{fct}}(\epsilon_{k\sigma})^2,$$

(4)
Green function by Eq. (4). The solid and dashed lines with an arrow represent $f$ and conduction-electron Green functions $G^f_{k\sigma}(i\varepsilon_n)$ and $G^c_{k\sigma}(i\varepsilon_n)$, respectively. The half-dashed and solid line with an arrow represents the off-diagonal Green function $G^f_{k\sigma}(i\varepsilon_n)$. The wiggly line represents critical valence fluctuations.

where the first and second terms are expressed by a Feynman diagram in Figs. 1(a) and 1(b), respectively.

Since renormalization-group analysis has shown that higher order terms $v_i$ ($i \geq 3$) are irrelevant for the $d = 3$ spatial dimension, we construct the action for the Gaussian fixed point. Taking account of the mode-coupling effects up to the 4th order in $S[\varphi]$ in Eq. (3), we employ Feynman’s inequality for the free energy: $F \leq F_{\text{eff}} + T(S - S_{\text{eff}})_{\text{eff}} \equiv \tilde{F}(\eta)$, where $S_{\text{eff}}$ is the effective action for the best Gaussian, $S_{\text{eff}}[\varphi] = \frac{1}{2} \sum_{\sigma} \sum_{q} \chi_{\varphi\sigma}(q, i\omega_n)^{-1} |\varphi_{\sigma}(q, i\omega_n)|^2$. Here, $\chi_{\varphi\sigma}(q, i\omega_n)$ is the valence susceptibility defined as

$$\chi_{\varphi\sigma}(q, i\omega_n)^{-1} \approx \eta + A_{\sigma} q^2 + C_{\sigma} \frac{|\omega_n|}{q},$$

where the notation follows in Ref. 9. Under the optimal condition $\frac{d\tilde{F}(\eta)}{d\eta} = 0$, the self-consistent renormalization (SCR) equation under a magnetic field in the $A_{\sigma} q_B^2 \leq \eta$ regime is obtained:

$$\sum_{\sigma} A_{\sigma} q_{\text{Bar}}^4 \frac{T_{0\sigma}}{T_{A\sigma}^2} \left( 1 + \frac{v_{4\sigma} q_{\text{Bar}}^3 T_{0\sigma}}{\pi^2} \frac{T_{0\sigma}}{T_{A\sigma}} \right)$$

$$\times \left[ y_{0\sigma} - \tilde{y}_{\sigma} + \frac{3}{2} y_{1\sigma} t_{\sigma} \left\{ \frac{x_c^3}{6\tilde{y}_{\sigma}} - \frac{1}{2\tilde{y}_{\sigma}} \int_0^{x_c} \frac{x^3}{x + \frac{t_{\sigma}}{\tilde{y}_{\sigma}}} dx \right\} \right]$$

$$\times \left[ C_{2\sigma} + \frac{x_c^2}{3} \frac{t_{\sigma}}{\tilde{y}_{\sigma}} \int_0^{x_c} \frac{x^4}{\left(x + \frac{t_{\sigma}}{\tilde{y}_{\sigma}}\right)^2} \right] = 0,$$

where $\tilde{y}_{\sigma} = y A_{\sigma} \left( \frac{q_{\text{Bar}}}{q_{\text{Bar}}} \right)^2$, $t_{\sigma} = T_{0\sigma}$, $T_{0\sigma} = \frac{\lambda_{\sigma} q_{\text{Bar}}}{2\pi e_{\sigma}}$, and $T_{A\sigma} = \frac{\lambda_{\sigma} q_{\text{Bar}}}{2}$ with $q_{\text{Bar}}$ being the Brillouin zone for “spin” $\sigma$. Note that $A$, $C$, and $q_{\text{B}}$ are the zero-field values of $A_{\sigma}$, $C_{\sigma}$, and $q_{\text{Bar}}$, respectively. Here, $y$ is defined as $y \equiv \frac{q}{A q_{\text{B}}}$, and the dimensionless integral variable and its cutoff are defined as $x \equiv q/q_{\text{B}}$ and $x_c \equiv q_c/q_{\text{B}}$, respectively. The parameters $y_{0\sigma}$ and $y_{1\sigma}$ are
given by

\[ y_{0\sigma} = \frac{\eta_{0\sigma}}{A_{\sigma} q_{0\sigma}} + v_{4\sigma} \frac{N_{\sigma} q_{0\sigma}^3}{T_{\sigma}^2}, \]
\[ y_{1\sigma} = \frac{v_{4\sigma} \frac{N_{\sigma} q_{0\sigma}^3}{T_{\sigma}^2}}{1 + v_{4\sigma} \frac{N_{\sigma} q_{0\sigma}^3}{T_{\sigma}^2}}, \]

respectively, where \( C_{1\sigma} \) and \( C_{2\sigma} \) are constants given by \( C_{1\sigma} = \int_0^L d\chi \ln \left( \frac{N_{\sigma} q_{0\sigma}^3 C_{\chi}^2}{(A_{\sigma} q_{0\sigma}^3)^2 + (C_{\sigma} q_{0\sigma})^2} \right) \)
and \( C_{2\sigma} = 2(C_{\sigma} q_{0\sigma})^2 \int_0^L d\chi \frac{\chi^2}{(A_{\sigma} q_{0\sigma}^3)^2 + (C_{\sigma} q_{0\sigma})^2} \), respectively.

Note that in the zero-field case, \( h = 0 \), Eq. (6) is reduced to the simple form

\[ y = y_0 + \frac{3}{2} y_1 I \left( \frac{x_c^3}{6y} - \frac{1}{2y} \int_0^{x_c} \frac{x^3}{x + \frac{1}{6y}} dx \right) \]

with \( y_0 = y_{0\sigma}, y_1 = y_{1\sigma}, \) and \( I = T/T_0 \), where \( T_0 = \frac{A_{\sigma} q_{0\sigma}^3}{2\pi C} \), which reproduces Eq. (6) in Ref. 9.

It is noted that at the QCP of the valence transition, the magnetic susceptibility diverges, whose singularity is the same as the valence susceptibility \( \chi \propto \chi_v(0, 0) \propto y^{-1} \) since the main contribution to \( \chi \) and \( \chi_v \) comes from the common many-body effects caused by \( U_{\text{fc}} \), which can be expressed by the common Feynman diagrams near the QCP. 9

In this Letter, we demonstrate that the \( T/B \) scaling behavior appears when the characteristic temperature of critical valence fluctuations \( T_0 \) is smaller than (or at least comparable to) the measured lowest temperature. Hence, we here set the coefficient \( A_{\sigma} \) in Eq. (5) as a small input parameter to discuss the effect of a small \( T_0 \) on physical quantities. The procedure of our calculation is summarized as follows. First, we solve the saddle-point solution for \( \exp(-S_0) \) at \( T = 0 \) for given parameters of \( \varepsilon_f, V_k, U = \infty, \) and \( h \) at the filling \( n = \frac{1}{2N_c} \sum_{i \sigma} (n_{i\sigma} + n_{i\sigma}^c) \) by using the slave-boson mean-field theory. Second, we calculate \( \chi_{\text{free}}(0, 0) \) and the [..] part in Eq. (4) by using the saddle-point solution. Then we obtain \( \eta_{0\sigma} \) and \( v_{4\sigma} \) for a given \( U_{\text{fc}} \). Third, by using \( y_{0\sigma} \) and \( y_{1\sigma} \) obtained from Eqs. (7) and (8), respectively, we solve the valence SCR equation [Eq. (6)] and finally obtain \( y(t) \).

We note that the crystalline electronic field (CEF) ground state of \( \beta\)-YbAlB\(_4\) has been suggested to be the Kramers doublet, which is well separated from the excited CEF levels. 6,13 Since the analysis of the CEF-level scheme, which well reproduces the anisotropy of the magnetic susceptibility, deduces that a hybridization node exists along the \( c \)-axis in \( \beta\)-YbAlB\(_4\), 13–15 we employ the anisotropic hybridization in the form of \( V_k = V(1 - k_c^2) \) with \( \hat{k} = k/|k| \) to simulate \( \beta\)-YbAlB\(_4\) most simply.
For evaluation of the saddle-point solution, we employ the typical parameter set for heavy-electron systems: $D = 1$, $V = 0.65$, and $U = \infty$ at the filling $n = 0.8$. Here, $D$ is the half bandwidth of conduction electrons given by $\varepsilon_k = k^2/(2m_0) - D$, which is taken as the energy unit. The mass $m_0$ is set such that the integration from $-D$ to $D$ of the density of states of conduction electrons per “spin” and site is equal to 1.

Following the argument in Ref. 9, we discuss the general property at the QCP of the valence transition by defining it as the point with the solution of Eq. (6) $y$ being zero at $T = 0$, which is identified to be $(\varepsilon_f, U_{fc}) = (-0.7, 0.700328652)$ for $A = 5 \times 10^{-6}$ at $h = 0$. This $U_{fc}$ is larger than $U_{fc}^{RPA} = 1/\chi_{00}^{RPA}(0, 0) = 0.62404$ for $\varepsilon_f = -0.7$, which reflects the mode-coupling effect of critical valence fluctuations. Namely, a positive $\nu_{A\sigma}$ overcomes a negative $\eta_{0\sigma}$ for $U_{fc} > U_{fc}^{RPA}$ [see Eq. (7)], giving rise to $0.700328652 > U_{fc}^{RPA}$.

It is noted that here we set a rather large c-f hybridization strength $V$ to simulate $\beta$-YbAlB$_4$ with a large fundamental characteristic energy scale $\approx 200$ K. Actually, the characteristic energy for heavy electrons, which is defined as the Kondo temperature $T_K \equiv \tilde{\varepsilon}_f - \mu$ within the saddle-point solution for $\exp(-S_0)$, is estimated to be $T_K \approx 0.02437$.

To examine the magnetic-field dependence of $y(t)$ at the QCP, we solve the valence SCR equation [Eq. (6)] for $U_{fc} = 0.700328652$. To make a comparison with experiments where a magnetic field from the order of $B = 10^{-4}$ T to $B = 2$ T is applied, we apply a magnetic field ranging from $h = 10^{-8}$ to $h = 10^{-4}$. Here we note that the energy unit of our theory is the conduction bandwidth $D = 1$, which is of the order of $10^4$ T ($\approx 10^4$ K). To compare with experiments measured in the temperature range from the order of $T = 10^{-2}$ K to $T = 3$ K, we solve the valence SCR equation [Eq. (6)] for $6 \times 10^{-6} \leq T \leq 3 \times 10^{-4}$. As noted above, $A$ is set as $A = 5 \times 10^{-6}$, which gives $T_0 = 3 \times 10^{-6}$, slightly smaller than the lowest temperature but of the same order. Owing to the smallness of $A_{\sigma}$, hereafter we neglect its field dependence and set $A = A_{\sigma}$ for $h \neq 0$.

The results are shown in Fig. 2. Intriguingly, we find that all the data over four decades of the magnetic field fall down to a single scaling function of the ratio $T/h$:

$$y = h^{1/2} \varphi \left( \frac{T}{h} \right).$$

The least-squares fit of the scaling function $\varphi(x) = ax^{1/2}$ to the data for $10^1 \leq T/h \leq 10^4$ shows that the data are well fitted by the dashed line in Fig. 2. Namely, $y/h^{1/2} \approx a(T/h)^{1/2}$, i.e., $y \approx T^{1/2}$. This implies that the quantum criticality of Yb-valence fluctuations is dominant, giving rise to the non-Fermi liquid regime. This behavior coincides with the measured scaling function Eq. (1) for $x = T/h \gg \Gamma$. It is noted that as $x$ decreases the data tend to
deviate from $ax^{1/2}$, i.e., there is a tendency of upward deviation from the dashed line toward $x = T/h \ll 1$ in the smaller $T/h$ region than that shown in Fig. 2. This reflects the suppression of the valence susceptibility by applying the magnetic field. Namely, as $x = T/h$ decreases, the crossover from the non-Fermi-liquid regime with the quantum valence criticality to the Fermi liquid regime with suppressed valence fluctuation occurs. As noted above, the uniform magnetic susceptibility $\chi$ has the same temperature dependence as the valence susceptibility $\chi \propto \chi_{\nu}(0,0) \propto y^{-1.9}$. This indicates that general tendency of the $T/B$ scaling observed in the magnetization data of $\beta$-YbAlB$_4$ can be reproduced by the solutions of the valence SCR equation [Eq. (6)] under a magnetic field.

To analyze how the $T/h$ scaling behavior appears in the present theoretical framework, let us rewrite the valence SCR equation [Eq. (6)] with the scaled form of $y/t^{1/2}$ and $t/h$:

$$\sum_{\sigma} A_{\sigma} q_{B\sigma}^4 \frac{T_{0\nu}}{T_{A\sigma}^2} \left( 1 + \frac{v_{4\nu} g_{B\sigma}^2 T_{0\nu}}{\pi^2 T_{A\sigma}^2} \right) \left[ y_{0\nu} - \frac{y_{\nu}}{h^{1/2}} \right]$$

$$+ \frac{3}{2} y_{1\nu} \left( \frac{t_{\nu}}{h} \right) \left\{ \frac{\chi}{6 \left( \frac{y_{\nu}}{h^{1/2}} \right)} - \frac{1}{2} \frac{\chi}{\left( \frac{y_{\nu}}{h^{1/2}} \right)} \frac{\chi}{x + h^{1/2}} \left[ \frac{y_{\nu}}{6 \left( \frac{y_{\nu}}{h^{1/2}} \right)} \right] dx \right\}$$

$$\times \left[ C_{2\nu} + \frac{\chi}{3 \left( \frac{y_{\nu}}{h^{1/2}} \right)^2} \int_0^\chi \frac{x^4}{\left( x + h^{1/2} \left[ \frac{y_{\nu}}{6 \left( \frac{y_{\nu}}{h^{1/2}} \right)} \right] \right)^2} dx \right] = 0. \tag{11}$$

We see that most terms can be expressed in the form of $y/h^{1/2}$ and $t/h$, except for the term...
with the $x$-integration in each square bracket [ ]. Namely, extra $h^{1/2}$ factors appear in the denominators of the integrands. This implies that \textit{the $T/h$ scaling does not hold exactly}. From Eq. (11), however, it turns out that in the case of $t_{\sigma}/\bar{\gamma}_{\sigma} \gg 1$, the denominators of the integrands become large, which make the $x$-integrations negligibly small. We confirmed that this is the case when $T_0$ is below (or at least comparable to) the measured lowest temperature. In the present calculation, we set $T_0 = 3.0 \times 10^{-6}$ and the lowest temperature for the data in Fig. 2 is $T = 6.0 \times 10^{-6}$, i.e., $T_0$ is a few times smaller than the lowest temperature. Note that $T_0$ is the same order as the lowest temperature.

From these results, the $T/B$ scaling observed in $\beta$-YbAlB$_4$ suggests that a small characteristic temperature of critical valence fluctuations $T_0$ exists. Since the measured lowest temperature is on the order of $10^{-2}$ K in $\beta$-YbAlB$_4$, $T_0$ is considered to be of the same order or smaller.

As shown in Ref. 9, because of the strong local-correlation effect by $U \gg D$, an almost dispersionless critical valence-fluctuation mode appears, giving rise to the extremely small $q^2$-coefficient $A$ in the momentum space. This almost flat mode is reflected in the emergence of the extremely small characteristic temperature $T_0$. Owing to the extremely small $T_0$, the temperature at the low-$T$ measurement can be regarded as a “high” temperature in the scaled temperature $t = T/T_0 \gtrsim 1$, where unconventional quantum criticality emerges in physical quantities such as $\chi$, ($T_1 T$)$^{-1}$, $C_e/T$, and resistivity, which well account for the behavior of $\beta$-YbAlB$_4$.\textsuperscript{9} Our results show that observation of the $T/B$ scaling indicates the presence of the small characteristic temperature $T_0$. In other words, quantum valence criticality gives a unified explanation for the unconventional criticality in physical quantities as well as the $T/B$ scaling in $\beta$-YbAlB$_4$.

To verify the existence of such a small $T_0$ experimentally, the measurement of the dynamical valence susceptibility $\chi'_{\nu}(q, \omega)$ is desirable as a direct observation. Mössbauer measurement and ESR measurement also seem to be possible probes to detect $T_0$,\textsuperscript{19,20} which are interesting future studies.

Although Eq. (1) shows that $\chi \approx (\mu_B B)^{-1/2}$ for $x = \frac{h_0 T}{\mu_B B} \ll \Gamma$, it should be noted that a very narrow range of experimental data is used to derive this limiting behavior: A large magnetic field of $B = 2$ T and intermediate temperatures of $0.2 \text{ K} \leq T \leq 0.5 \text{ K}$ (but \textit{not} the lowest temperature) are used.\textsuperscript{7} Namely, the scaling form in the $x \ll \Gamma$ regime is an outcome of the transient behavior of the magnetization, where $\chi$ is greatly suppressed to be almost constant around $B = 1 - 2$ T.\textsuperscript{7} Furthermore, we should be careful about the fact that the whole scaling range of $10^{-1} \leq T/B \leq 10^3$ is not covered by a series of experimental data as a function
of $T$ for a fixed $B$. From these circumstances, it seems to be appropriate to consider that
\[
\chi^{-1}/(\mu_B B)^{1/2} \approx (k_B T/\mu_B)^{1/2}
\]
for the $x = k_B T/\mu_B \gg \Gamma$ regime in Eq. (1), derived from the experimental
data for the wide $T$ and $B$ range is a substantial scaling function.

Theoretically, as shown in Ref. 21, the location of the QCP in the ground-state phase
diagram in the $\epsilon_f-U_{fc}$ plane is moved by applying $h$. If the system is located in the vicinity of
the QCP at $h = 0$, applying $h$ moves the system away from the QCP, which causes the marked
suppression of $\chi$ at large $h$. In this Letter, we discussed the $h$-dependence of $\chi$ through the
$h$-dependence of $\eta_{0r}$ and $\nu_{4r}$ with the QCP being unmoved for simplicity of analysis. Taking
account of this effect is expected to make the crossover $T/h$ between the Fermi liquid and
non-Fermi liquid regimes shift to the larger-$T/h$ direction in Fig. 2, which is an interesting
future study for quantitative analysis.

We note that in the present theory the key origin of the emergence of the $T/B$ scaling is not
the anisotropic hybridization but the quantum valence criticality. In the present calculation,
the renormalized f level is not located at the band edge as expected in the general (and natural)
situation for heavy-electron state. Namely, in our framework, even without the pinning of the
f-level, i.e., the fine tuning of the f-level position, the $T/B$ scaling behavior can emerge, which
is in sharp contrast to Ref. 8.

We also note that the $T/B$ scaling does not hold exactly as discussed below Eq. (11). When
$T_0$ is comparable to the middle-$T$ range applied to the scaling plot of the data, the deviation
from the single scaling function shown in Fig. 2 becomes visible. As shown in Ref. 9, the
valence susceptibility, i.e., the magnetic susceptibility behaves as $y^{-1} \sim t^{-1/2}$ for $t \geq 1$
and $y^{-1} \sim t^{-2/3}$ for $T_K/T_0 > t \gg 1$. At sufficiently high temperatures, $T \gg T_K$, the Curie-Weiss
behavior $y^{-1} \sim t^{-1}$ appears. Hence, we stress that the emergence of the $T/B$ scaling is an
approximate outcome for the intermediate-temperature region which satisfies $t_{cr}/\tilde{y}_{cr} \gg 1$ as
explained above.

In summary, we have shown that the $T/B$ scaling together with the unconventional quantum
criticality observed in $\beta$-YbAlB$_4$ can be understood from the viewpoint of the quantum
valence criticality in a unified way.

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