Cut-eliminability in second order logic calculi

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Abstract

In this paper we propose a semantics in which the truth value of a formula is a pair of elements in a complete Boolean algebra. Through the semantics we can unify largely two proofs of cut-eliminability (Hauptsatz) in classical second order logic calculus, one is due to Takahashi-Prawitz and the other by Maehara.

1 Takeuti’s fundamental conjecture

\(G^1_{LC}\) defined in subsection 1.1 below is an impredicative sequent calculus with the (cut) rule for the second order logic. \(G^1_{LC}\)\(^\text{cf}\) denotes the cut-free fragment of \(G^1_{LC}\), and \(LK = G^0_{LC}\) the first order fragment.

(Takeuti’s fundamental conjecture for the second order calculus \(G^1_{LC}\))

(cut) inferences are eliminable from proofs in \(G^1_{LC}\): if \(G^1_{LC}\) proves a sequent, then it is provable without (cut).

It seems to me that G. Takeuti’s intention in the conjecture is to reduce or paraphrase the consistency problem of the second order arithmetic \(Z_2 = (\Pi^1_{\infty}\text{-CA})\) to a mathematical problem of cut-eliminability in the second order calculus \(G^1_{LC}\), and the consistency of higher order arithmetic to the cut-eliminability in the higher order calculus \(GLC\).

Some partial results are obtained on the conjecture. Takeuti [16] shows a cut-elimination theorem for a fragment of \(G^1_{LC}\), and one for a fragment of the higher order calculus \(GLC\) in [17], both of which implies the 1-consistency of the subsystem \((\Pi^1_{\infty}\text{-CA})_0\) of the second order arithmetic, the strongest one in the big five. In [1] a cut-elimination theorem for a fragment of \(G^1_{LC}\) is shown, which implies the 1-consistency of the subsystem \((\Delta^1_2\text{-CA+BI})\) of the second order arithmetic. All of these proofs in [16] [17] [1] are based on transfinite induction on computable notation systems of ordinals, and hence are ordinal-theoretically informative ones.

Although no proof of the full conjecture has been obtained as Takeuti had expected, the cut-eliminability holds for second order calculus.
Theorem 1.1 [12]

\[ G^1 \text{LC} \vdash \Gamma \Rightarrow \Delta \Rightarrow G^1 \text{LC}^{cf} \vdash \Gamma \Rightarrow \Delta \]

Moreover the cut-eliminability holds for higher order calculus GLC.

Theorem 1.2 [13, 9]

\[ \text{GLC} \vdash \Gamma \Rightarrow \Delta \Rightarrow \text{GLC}^{cf} \vdash \Gamma \Rightarrow \Delta \]

In this paper let us focus on the second order calculus for simplicity, and we propose a semantics in which the truth value of a formula is a pair of elements in a complete Boolean algebra. Through the semantics we can unify largely two proofs of cut-eliminability (Hauptsatz) in classical second order logic calculus, one is due to Takahashi-Prawitz and the other by Maehara.

In Section 2 a soundness theorem 2.9 of \( G^1 \text{LC} \) is shown for semi valuations based on the semantics with pairs of elements in a complete Boolean algebra. Our proof of the theorem is essentially the same as in Takahashi[13], Prawitz[9] and Maehara[6]. In Section 3 Theorem 1.1 is concluded.

In Section 4 a \( \text{cBa} = X \subset P(X) \) is introduced from a relation \( M \) on an arbitrary set \( X \neq \emptyset \). The construction of the \( \text{cBa} \) is implicit in [6]. Theorem 1.1 is proved using a semi valuation defined from cut-free provability.

In Section 5 the proof theoretic strength of cut-eliminability is calibrated. It is well known that Theorem 1.1 is equivalent to the 1-consistency of \( \text{Z}_2 \) over a weak arithmetic. We sharpen it with respect to end sequents of proofs and fragments. Finally some open problems are mentioned.

1.1 Logic calculi

Let us recall second order sequent calculi briefly. Details are found in [18].

Logical connectives are \( \neg, \lor, \land, \exists, \forall \). A second order language is obtained from a first order language by adding countably infinite \( n \)-ary variables \( X^n_i \) for each \( n = 1, 2, \ldots \). For simplicity let us assume that our language contains no relation (predicate) symbol nor function symbol. Formulas are quantified by second order quantifiers \( \exists X^n, \forall X^n \) as well as first order quantifiers \( \exists x, \forall x \).

For a formula \( G \) and a list \( \bar{x} = (x_1, \ldots, x_n) \) of distinct variables, the expression \( \lambda \bar{x}.G \) is an \( n \)-ary abstract or a term of second order, and denoted by \( T, \ldots \).

A finite set of formulas are said to be a cedent, denoted \( \Gamma, \Delta, \ldots \) \( \Gamma, \Delta := \Gamma \cup \Delta \), \( \Gamma, A := \Gamma \cup \{A\} \). A pair of cedents \( (\Gamma, \Delta) \) is denoted \( \Gamma \Rightarrow \Delta \), and called a sequent. \( \Gamma \) is said to be the antecedent, \( \Delta \) succedent of the sequent \( \Gamma \Rightarrow \Delta \).
A sequent calculus \( \mathcal{G}^1 \mathcal{LC} \) is a logic calculus for the second order logic. Its initial sequents are

\[
A, \Gamma \Rightarrow \Delta, A \quad (A: \text{atomic})
\]

Inference rules are first order ones (\( L^{-} \), \( (R^{-}) \), \( (L \lor) \), \( (R \lor) \), \( (L \land) \), \( (R \land) \), \( (L \exists^0) \), \( (R \exists^0) \), \( (L \forall^0) \), \( (R \forall^0) \))

\[
\frac{\neg F, \Gamma \Rightarrow \Delta, F}{\Gamma \Rightarrow \Delta, \neg F} \quad (L^{-})
\]

\[
\frac{F, \Gamma \Rightarrow \Delta, \neg F}{\Gamma \Rightarrow \Delta, \neg F} \quad (R^{-})
\]

where \( F \) is the \textit{minor formula}, and \( \neg F \) the \textit{major formula} of the inference rules (\( L^{-} \)), (\( R^{-} \)).

\[
\frac{F_0, F_0 \lor F_1, \Gamma \Rightarrow \Delta}{F_0 \lor F_1, \Gamma \Rightarrow \Delta} \quad \text{(L\lor)}
\]

\[
\frac{F_1, F_0 \land F_1, \Gamma \Rightarrow \Delta}{F_0 \land F_1, \Gamma \Rightarrow \Delta} \quad \text{(R\land)}
\]

where \( i \in \{0, 1\} \), \( F_0, F_1 \) are the \textit{minor formula}, and \( F_0 \lor F_1 \) the \textit{major formula} of the inference rules (\( L \lor \)), (\( R \lor \)).

\[
\frac{F(a), \exists x F(x), \Gamma \Rightarrow \Delta}{\exists x F(x), \Gamma \Rightarrow \Delta} \quad \text{(L\exists^0)}
\]

\[
\frac{\exists x F(x), F(t), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \exists x F(x)} \quad \text{(R\exists^0)}
\]

\[
\frac{F(t), \forall x F(x), \Gamma \Rightarrow \Delta}{\forall x F(x), \Gamma \Rightarrow \Delta} \quad \text{(L\forall^0)}
\]

\[
\frac{\forall x F(x), F(a), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \forall x F(x)} \quad \text{(R\forall^0)}
\]

where in (\( L \exists^0 \)), (\( R \forall^0 \)), \( a \) is an eigenvariable which does not occur in the lower sequent, and \( F(a) \) is the \textit{minor formula}. In (\( R \exists^0 \)), (\( L \forall^0 \)), \( t \) is a first order term, and \( F(t) \) is the \textit{minor formula}. \( \exists x F(x) \) is the \textit{major formula} of the inference rules (\( L \exists^0 \)), (\( R \exists^0 \)), and \( \forall x F(x) \) the \textit{major formula} of the inference rules (\( L \forall^0 \)), (\( R \forall^0 \)).

The (\textit{cut}) inference

\[
\frac{\Gamma \Rightarrow A, C, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Delta, \Theta} \quad \text{(cut)}
\]

There is no minor nor major formula of \textit{(cut)} inference.

Rules for second order quantifications (\( L \exists^1 \)), (\( R \exists^1 \)), (\( L \forall^1 \)), (\( R \forall^1 \)).

\[
\frac{F(Y), \exists X F(X), \Gamma \Rightarrow \Delta}{\exists X F(X), \Gamma \Rightarrow \Delta} \quad \text{(L\exists^1)}
\]

\[
\frac{\Gamma \Rightarrow \Delta, \exists X F(X), F(T)}{\Gamma \Rightarrow \Delta, \exists X^n F(X)} \quad \text{(R\exists^1)}
\]

\[
\frac{F(T), \forall X F(X), \Gamma \Rightarrow \Delta}{\forall X^n F(X), \Gamma \Rightarrow \Delta} \quad \text{(L\forall^1)}
\]

\[
\frac{\Gamma \Rightarrow \Delta, \forall X F(X), F(Y)}{\Gamma \Rightarrow \Delta, \forall X F(X)} \quad \text{(R\forall^1)}
\]
where in \((L^\exists), (R^\forall)\), \(Y\) is an eigenvariable which does not occur in the lower sequent, and \(F(Y)\) is the minor formula. In \((R^\exists), (L^\forall)\), \(T\) is an \(n\)-ary second order term, and \(F(T)\) is the minor formula. \(\exists XF(X)\) is the major formula of the inference rules \((L^\exists), (R^\exists)\), and \(\forall XF(X)\) the major formula of the inference rules \((L^\forall), (R^\forall)\).

Since cedents here are finite sets of formulas, there are no explicit structural rules, weakening (or thinning), contraction nor exchange in our sequent calculi.

2 Valuations

In this section let us propose a semantics in which the truth value of a formula is a pair of elements in a complete Boolean algebra, and a soundness theorem \ref{thm:Soundness} of \(G^1\text{LC}\) is shown for semi valuations based on the semantics.

For a cBa (complete Boolean algebra) \(B\) let \(\mathbb{D}_B\) denote the set of pairs \((a, b)\) of elements \(a, b \in B\) such that \(a \leq b\). Here \(\mathcal{D}\) stands for the axiom \(\Box A \rightarrow \Diamond A\) in the modal logic. \(\neg a\) denotes the complement of \(a \in B\).

\begin{definition}
For a cBa \(B\) let

\[ \mathbb{D}_B := \{(a, b) \in B \times B : a \leq b\}. \]

Each \(a \in \mathbb{D}_B\) is written \(a = (\Box a, \Diamond a)\), where \(\Box a \leq \Diamond a\). For \(a, b \in \mathbb{D}_B\) let

\[ a \leq b :\Leftrightarrow \Box a \leq \Box b \land \Diamond a \leq \Diamond b \quad \neg a := (\neg \Diamond a, \neg \Box a) \]

\[ a \leq b :\Leftrightarrow \Box a \leq \Box b \land \Diamond a \geq \Diamond b \]

Then for \(\{a_\lambda\}_\lambda \subset \mathbb{D}_B\), the following hold.

\[ \begin{align*}
\sup_\prec \{a_\lambda\}_\lambda &= (\sup_\lambda \Box a_\lambda, \sup_\lambda \Diamond a_\lambda) \\
\inf_\prec \{a_\lambda\}_\lambda &= (\inf_\lambda \Box a_\lambda, \inf_\lambda \Diamond a_\lambda)
\end{align*} \]

\[ \begin{align*}
\sup_\prec \{a_\lambda\}_\lambda &= (\sup_\lambda \Box a_\lambda, \inf_\lambda \Diamond a_\lambda) \\
\inf_\prec \{a_\lambda\}_\lambda &= (\inf_\lambda \Box a_\lambda, \sup_\lambda \Diamond a_\lambda)
\end{align*} \]

Obviously \(\mathbb{D}_B\) is a complete lattice under the order \(\leq\) as well as under the order \(\prec\). Note that \(a \leq b \Leftrightarrow (\Box a \rightarrow \Box b) = (\Diamond b \rightarrow \Diamond a) = 1\), where \((a \rightarrow b) := \sup\{\neg a, b\}\) for \(a, b \in B\).

For example for \(B = 2 = \{0, 1\}\), \(\mathbb{D}2\) is the set of three truth values \(3 := \mathbb{D}2 = \{f, u, t\} = \{(0, 0), (0, 1), (1, 1)\}\), where \(f < u < t\) and \(u < f, t\).

\begin{proposition}[Monotonicity]
For \(a, b, a_\lambda, b_\pi \in \mathbb{D}_B\)

\[ a \leq b \Rightarrow \neg a \leq \neg b. \]

\[ \forall a \exists \lambda (a_\lambda \leq b_\pi) \land \forall a \exists \pi (a_\pi \leq b_\lambda) \Rightarrow \sup\{a_\lambda\}_\lambda \leq \sup\{b_\pi\}_\pi \land \inf\{a_\lambda\}_\lambda \leq \inf\{b_\pi\}_\pi. \]

\end{proposition}

\begin{definition}
A \(\mathbb{D}_B\)-valued model \(M\) is a pair \((D_0, D_1)\), where \(D_0 \neq \emptyset\) is a non-empty set, \(D_1 = \bigcup_{n \geq 1} D_1^{(n)}\) and \(D_1^{(n)}\) a non-empty set of functions \(\alpha : D_0^n \rightarrow \mathbb{D}_B\) for each \(n = 1, 2, \ldots\)

\end{definition}

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For each $\alpha \in D_1^{(n)}$ introduce an $n$-ary relation constant $\bar{\alpha}$, and each $t \in D_0$ is identified with the individual constant for $t$. For formulas $A$ and $n$-ary abstracts $\lambda \vec{x}.G(x_1, \ldots, x_n)$ with $t \in D_0$ and $\bar{\alpha}$, let us define recursively $M(A) \in \mathbb{D}_B$, $M(\lambda \vec{x}.G(x_1, \ldots, x_n)) : D_0^n \rightarrow \mathbb{D}_B$ as follows.

1. $M(\bar{\alpha}(t_1, \ldots, t_n)) = \alpha(t_1, \ldots, t_n)$ for $t_1, \ldots, t_n \in D_0$. $M(\neg F) = \neg M(F)$.
2. $M(F_0 \lor F_1) = \sup\{M(F_0), M(F_1)\}$. $M(F_0 \land F_1) = \inf\{M(F_0), M(F_1)\}$.
3. $M(\exists x F(x)) = \sup\{M(F(t)) : t \in D_0\}$. $M(\forall x F(x)) = \inf\{M(F(t)) : t \in D_0\}$.
4. $M(\exists X^n F(X)) = \sup\{M(F(\bar{\alpha})) : \alpha \in D_1^{(n)}\}$. $M(\forall X^n F(X)) = \inf\{M(F(\bar{\alpha})) : \alpha \in D_1^{(n)}\}$.
5. $M(\lambda \vec{x}.G(x_1, \ldots, x_n))(t_1, \ldots, t_n) = M(G(t_1, \ldots, t_n))$.

Intuitively $M(A) = (a, b)$ means that the degree of truth of $A$ is $a$, and one of non-falsity of $A$ is $b$. When $\mathbb{B} = 3$, $\square M(A) = 1$ [$\Diamond M(A) = 1$] is related to the fact that $\neg A$ is valid [?A is valid] in a three-valued structure for Girard’s three-valued logic with modal operators $!, ?$ in $[5]$, resp.

For $\alpha, \beta : D_0^n \rightarrow \mathbb{D}_B$ let

$$\alpha \preceq \beta : \Leftrightarrow \forall \bar{t} \in D_0^n (\alpha(\bar{t}) \preceq \beta(\bar{t}))$$

and

$$M \models 3CA : \Leftrightarrow \text{for each formula } G(x_1, \ldots, x_n, X^k) \text{ and each } \beta \in D_1^{(k)},$$

there exists an $\alpha \in D_1^{(n)}$ such that $\alpha \preceq M(\lambda \vec{x}.G(x_1, \ldots, x_n, \bar{\beta}))$

Recall that $Tm_0$ denotes the set of first order terms, and $Tm_1^{(n)}$ the set of $n$-ary abstracts $\lambda \vec{x}.G(x_1, \ldots, x_n)$.

**Definition 2.4** Let $V$ be a map from the set of formulas $A$ to $\mathbb{D}_B$, $A \mapsto V(A) \in \mathbb{D}_B$. $V$ is said to be a semi $\mathbb{D}_B$-valuation if it enjoys the following conditions:

1. $V(\neg F) \preceq \neg V(F)$.
2. $V(F_0 \lor F_1) \preceq \sup\{V(F_0), V(F_1)\}$. $V(F_0 \land F_1) \preceq \inf\{V(F_0), V(F_1)\}$.
3. $V(\exists x F(x)) \preceq \sup\{V(F(t)) : t \in Tm_0\}$. $V(\forall x F(x)) \preceq \inf\{V(F(t)) : t \in Tm_0\}$.
4. $V(\exists X^n F(X)) \preceq \sup\{V(F(T)) : T \in Tm_1^{(n)}\}$. $V(\forall X^n F(X)) \preceq \inf\{V(F(T)) : T \in Tm_1^{(n)}\}$. 

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Definition 2.5 Let 
\[ \mathbb{B}_\Delta := \{(a, a) : a \in \mathbb{B}\}. \]

A \( \mathbb{B} \)-valued model \( \mathcal{N} \) is a pair \((D_0, I)\) such that \( D_0 \) is a non-empty set, \( I = \bigcup I^{(n)} \) and \( I^{(n)} \) is a non-empty set of functions \( X : D_0^n \rightarrow \mathbb{B}_\Delta \).

Let \( \mathcal{N}(A) := \Box \mathcal{N}(A) = \Diamond \mathcal{N}(A) \) for any formula \( A \), and 
\[ \mathcal{N} \models 2CA : \iff \]
for each \( n \geq 1 \) and each formula \( G(x_1, \ldots, x_n, X) \), 
\[ \mathcal{N}(\forall X \exists Y^n x_1, \ldots, x_n(Y(x_1, \ldots, x_n) \leftrightarrow G(x_1, \ldots, x_n, X))) = 1 \]
where 1 denotes the largest element in \( \mathbb{B} \).

Proposition 2.6 Let \( \mathbb{B} \) be a cBa.

1. Suppose \( G^1 \text{LC}^f \vdash \Gamma \models \Delta \). Then \( \inf \{ \square V(A) : A \in \Gamma \} \leq \sup \{ \Diamond V(B) : B \in \Delta \} \), i.e., \( \Diamond V(\bigwedge \Gamma \supset \bigvee \Delta) = 1 \) for any semi \( \mathbb{B} \)-valuation \( V \).

2. Suppose \( G^1 \text{LC} \vdash \Gamma \models \Delta \). Then \( \inf \{ \mathcal{N}(A) : A \in \Gamma \} \leq \sup \{ \mathcal{N}(B) : B \in \Delta \} \), i.e., \( \mathcal{N}(\bigwedge \Gamma \supset \bigvee \Delta) = 1 \) for any \( \mathbb{B} \)-valued model \( \mathcal{N} = (D_0, I) \) with \( \mathcal{N} \models 2CA \).

Lemma 2.7 (Cf. [6].) Let \( V \) be a semi \( \mathbb{B} \)-valuation. Define a \( \mathbb{B} \)-model \( \mathcal{M} = (D_0, D_1) \) by \( D_0 = Tm_0 \) and \( D_1^{(n)} = \{ V(T) \in D_0^\mathbb{B} : T \in Tm_1^{(n)} \} \) with \( V(\lambda \vec{x}. G(x_1, \ldots, x_n))(t_1, \ldots, t_n) := V(G(t_1, \ldots, t_n)) \) for \( t_1, \ldots, t_n \in Tm_0 \). Then for formula \( F(X^n), T \in Tm_1^{(n)}, \) and \( \alpha = V(T) \)
\[ V(F(T)) \leq \mathcal{M}(F(\vec{a})) \] (1)
and
\[ \mathcal{M} \models 3CA \] (2)

Proof. 1: This is seen by induction on formulas \( F(X) \) using Proposition 2.2.

For example consider the case \( F(X) \equiv (\exists Y^k G(Y, X)) \). By the induction hypothesis we have \( V(G(S, T)) \leq \mathcal{M}(G(\vec{\beta}, \vec{\alpha})) \) for any \( S \in Tm_1^{(k)} \) and \( \beta = V(S) \). Then \( V(F(T)) \leq \sup \{ V(G(S, T)) : S \in Tm_1^{(k)} \} \leq \sup \{ \mathcal{M}(G(\vec{\beta}, \vec{\alpha})) : \beta = V(S), S \in Tm_1^{(k)} \} = \mathcal{M}(F(\vec{a})) \).

2: Let \( G(x_1, \ldots, x_n, Y^k) \) be a formula. For a \( k \)-ary abstract \( T \in Tm_1^{(k)} \), let \( \alpha = V(T), \beta = V(\lambda \vec{x}. G(x_1, \ldots, x_n, T)) \in D_1^{(n)} \). From 1 we see for any \( t_1, \ldots, t_n \in Tm_0 \) that \( \beta(t_1, \ldots, t_n) = V(\lambda \vec{x}. G(x_1, \ldots, x_n, T))(t_1, \ldots, t_n) = V(G(t_1, \ldots, t_n, \alpha)) \leq \mathcal{M}(G(t_1, \ldots, t_n, \vec{a})) = \mathcal{M}(\lambda \vec{x}. G(x_1, \ldots, x_n, \vec{a}))(t_1, \ldots, t_n) \).

Lemma 2.8 (Cf. [13] [9] [6].) Let \( \mathcal{M} = (D_0, D_1) \) be a \( \mathbb{B} \)-valued model such that \( \mathcal{M} \models 3CA \). Let for \( \alpha \in D_1^{(n)} \)
\[ I(\alpha) := \{ \chi \in D_0^\mathbb{B} : \alpha \leq \chi \} \quad I^{(n)} := \bigcup \{ I(\alpha) : \alpha \in D_1^{(n)} \} \]
Then for the $\mathbb{B}$-valued model $\mathcal{N} = (D_0, I)$, $\alpha \in D_1^{(n)}$, $\mathcal{X} \in I^{(n)}$ and formulas $F(X^n)$, the following hold:
\[
\alpha \trianglelefteq \mathcal{X} \Rightarrow M(F(\bar{\alpha})) \trianglelefteq \mathcal{N}(F(\bar{\mathcal{X}}))
\]  
(3)

and
\[
\mathcal{N} \models 2CA
\]  
(4)

**Proof.** Note that $\alpha \trianglelefteq \mathcal{X} \iff \forall t_1, \ldots, t_n \in D_0 [\square \alpha(t_1, \ldots, t_n) \leq \mathcal{X}(t_1, \ldots, t_n) \leq \Diamond \alpha(t_1, \ldots, t_n)]$.

(3): This is seen by induction on formulas $F(X)$. The case when $F(X)$ is an atomic formula $X(t_1, \ldots, t_n)$ is seen from the assumption $\alpha \trianglelefteq \mathcal{X}$. Other cases follow from Proposition 2.2. For example consider the case $F(X) = (\exists Y^k G(Y, X))$. By the induction hypothesis we have $M(G(\beta, \bar{\alpha})) \trianglelefteq \mathcal{N}(G(\bar{\mathcal{Y}}, \bar{\mathcal{X}}))$ for any $\beta \in D_1^{(k)}$ and $\mathcal{Y} \in I^{(k)}$ with $\beta \trianglelefteq \mathcal{Y}$. On the other hand we have $\forall \mathcal{Y} \in I^{(k)} \exists \beta \in D_1^{(k)} (\beta \trianglelefteq \mathcal{Y})$ and $\forall \beta \in D_1^{(k)} \exists \mathcal{Y} \in I^{(k)} (\beta \trianglelefteq \mathcal{Y})$ by the definition of $I^{(k)}$. Hence Proposition 2.2 yields $M(F(\bar{\alpha})) = \sup \{ M(G(\beta, \bar{\alpha})) : \beta \in D_1^{(k)} \} \leq \sup \{ \mathcal{N}(G(\bar{\mathcal{Y}}, \bar{\mathcal{X}})) : \mathcal{Y} \in I^{(k)} \} = \mathcal{N}(F(\bar{\mathcal{X}}))$.

(4): For formulas $G(x_1, \ldots, x_n, X^k)$ we need to show that $\mathcal{N}(\forall \mathcal{X} \exists \bar{\mathcal{Y}}(Y(x_1, \ldots, x_n) \leftrightarrow G(x_1, \ldots, x_n, X))) = 1$. Let $\mathcal{X} \in I^{(k)}$, and $D_1^{(k)} \ni \alpha \trianglelefteq \mathcal{X}$. From $\mathcal{N} \models 3CA$ pick a $\beta \in D_1^{(n)}$ such that $\beta \trianglelefteq M(\lambda \bar{x}.G(x_1, \ldots, x_n, \bar{\alpha}))$. On the other hand we have $M(\lambda \bar{x}.G(x_1, \ldots, x_n, \bar{\alpha})) \trianglelefteq \mathcal{N}(\lambda \bar{x}.G(x_1, \ldots, x_n, \bar{\mathcal{X}}))$ by (3). Now let $\mathcal{Y}(t_1, \ldots, t_n) = \mathcal{N}(G(t_1, \ldots, t_n, \mathcal{X}))$. Then $\beta \trianglelefteq \mathcal{Y}$ and $\mathcal{Y} \in I^{(n)}$. Therefore $\mathcal{N}(\forall \mathcal{X} \exists \bar{\mathcal{Y}}(Y(x_1, \ldots, x_n) \leftrightarrow G(x_1, \ldots, x_n, \bar{\mathcal{X}}))) = 1$. 

**Theorem 2.9** Suppose $G^{1}\mathcal{L}\mathcal{C} \vdash \Gamma \Rightarrow \Delta$. Then for any cBa $\mathcal{B}$, $\inf\{ \Box V(A) : A \in \Gamma \} \leq \sup\{ \Diamond V(B) : B \in \Delta \}$, i.e., $\Diamond V(\bigwedge \Gamma \supset \bigvee \Delta) = 1$ for any semi $\mathcal{B}$-valuation $V$.

**Proof.** For a given semi $\mathcal{B}$-valuation $V$, let $\mathcal{M}$ be the $\mathcal{B}$-model in Lemma 2.7. By (1) we see that $V(C) \leq M(C)$ for formulas $C$. Also $\mathcal{M} \models 3CA$ by $\mathcal{N} \models 2CA$ by (4). Now assume $G^{1}\mathcal{L}\mathcal{C} \vdash \Gamma \Rightarrow \Delta$. We obtain $\inf\{ \Box V(A) : A \in \Gamma \} \leq \inf\{ \Box M(A) : A \in \Gamma \} \leq \inf\{ \mathcal{N}(A) : A \in \Gamma \} \leq \sup\{ \Diamond V(B) : B \in \Delta \} \leq \sup\{ \Diamond M(B) : B \in \Delta \}$ by Proposition 2.6.2. 

Although the intermediate step with $\mathcal{B}$-models in Lemma 2.7 due to J. Y. Girard is intuitively appealing, it is dispensable. The following Lemma 2.10 is seen as in Lemmas 2.7 and 2.8.

**Lemma 2.10** Let $V$ be a semi $\mathcal{B}$-valuation. Define a 2-valued model $\mathcal{N} = (D_0, I)$ with $D_0 = Tm_0$ as follows.

Let $T \equiv (\lambda \bar{x}.G(x_1, \ldots, x_n)) \in Tm_1^{(n)}$. Then $v(T)(t_1, \ldots, t_n) := v(G(t_1, \ldots, t_n))$ for $t_1, \ldots, t_n \in Tm_0$, and $I(T) = \{ \mathcal{X} \in D_0 2 : v(T) \trianglelefteq \mathcal{X} \}$. Let $I^{(n)} = \bigcup\{ I(T) : T \in Tm_1^{(n)} \}$. 

Then for $T \in \text{TM}_1^{(n)}$, $X \in I^{(n)}$ and formulas $F(X^n)$,
\[ v(T) \leq X \Rightarrow v(F(T)) \leq N(F(X)) \] (5)
and
\[ N \models 2CA \] (6)

3 Semi valuation through proof search

It is easy to conclude Theorem 1.1 from Theorem 2.9 and the following Lemma 3.1. This is the proof by Takahashi[13] and Prawitz[9].

Lemma 3.1 (Cf. [11].)
Suppose $G^1 \text{LC} \not\vdash \Gamma \Rightarrow \Delta$. Then there exists a semi 3-valuation $V$ such that $V(A) = t$ for $A \in \Gamma$ and $V(B) = f$ for $B \in \Delta$.

Proof. By a canonical proof search, we get an infinite binary tree of sequents supposing $G^1 \text{LC} \not\vdash \Gamma \Rightarrow \Delta$. Pick an infinite path through the tree. Let us define formulas occurring in antecedents of the path to be $t$, formulas occurring in succedents to be $f$. This results in a semi valuation $V(A) = 3 = B2$ such that $\forall A \in \Gamma(V(A) = t)$, $\forall B \in \Delta(V(B) = f)$. \qed

(Proof of Theorem 1.1, ver.1)
Suppose $G^1 \text{LC} \not\vdash \Gamma \Rightarrow \Delta$. By Lemma 3.1 pick a semi 3-valuation $V$ such that $\forall A \in \Gamma[V(A) = t]$, $\forall B \in \Delta[V(B) = f]$. Namely $1 = \inf\{\Box V(A) : A \in \Gamma\} \leq \sup\{\Diamond V(B) : B \in \Delta\} = 0$. Theorem 2.9 yields $G^1 \text{LC} \not\vdash \Gamma \Rightarrow \Delta$. \qed

4 Semi valuation defined from cut-free provability

In this section following Maehara[4], a cBa $\mathbb{B}_X \subset \mathcal{P}(X)$ is first introduced from a relation $M$ on an arbitrary set $X \neq \emptyset$. $M$ is a symmetric relation such that if $(x, x) \in M$, then $(x, y) \in M$ for any $y \in X$. The construction of the cBa $\mathbb{B}_X$ is implicit in [6]. Second the Hauptsatz for $G^1 \text{LC}$ is concluded using a semi valuation defined from cut-free provability as in [4].

It seems to me that Maehara’s proof compares more straightforward with the proof in Section 3 due to Takahashi-Prawitz in the sense that the latter proves the contraposition of the Hauptsatz. The cost we have to pay is to elaborate a cBa from relations in Subsection 4.1 which gives an inspiration to researches in non-classical logics, e.g., cf. [3].

4.1 complete Boolean algebras induced from relations

Let $X \neq \emptyset$ be a non-empty set, and $M : X \ni x \mapsto M(x) \subset X$ a map. Assume $M$ enjoys the following two conditions for any $x, y \in X$:
\[ x \in M(x) \iff M(x) = X \] (7)
Then let
\[ \mathbb{B}_X := \{ \alpha \subset X : \alpha = \bigcap \{ M(x) : \alpha \subset M(x), x \in X \} \}. \]

In the following we consider only subsets of the set \( X \). Let \( \bigcap \emptyset := X \).

**Lemma 4.1** \( \forall \alpha \subset X \bigcap \{ \gamma \in \mathbb{B}_X : \alpha \subset \gamma \} \cap \bigcap \{ M(x) : \alpha \subset M(x) \} \in \mathbb{B}_X \).

**Proof.** Let \( \beta = \bigcap \{ \gamma \in \mathbb{B}_X : \alpha \subset \gamma \} \) and \( \delta = \bigcap \{ M(x) : \alpha \subset M(x) \} \). First it is clear that \( \alpha \subset M(x) \Rightarrow \delta \subset M(x) \), and hence \( \delta \in \mathbb{B}_X \).

We show \( \delta \subset \beta \). Assume \( \alpha \subset \gamma \in \mathbb{B}_X \) and \( \gamma \subset M(x) \). Then \( \alpha \subset M(x) \). Hence \( \delta \subset M(x) \), and \( \delta \subset \bigcap \{ M(x) : \gamma \subset M(x) \} = \gamma \). Thus \( \delta \subset \beta \). \( \square \)

**Theorem 4.2** \( \mathbb{B}_X \) is a cBa with the following operations for \( \alpha, \beta \in \mathbb{B}_X \), and \( \{ M(x) : x \in X \} \subset \mathbb{B}_X \).

1. \( 1 = X \). \( 0 = \bigcap_{y \in X} M(y) = \{ x \in X : x \in M(x) \} \).
2. \( \inf \lambda \alpha \lambda = \bigcap \lambda \alpha \lambda \) and \( \alpha \leq \beta \Leftrightarrow \alpha \subset \beta \).
3. \( \sup \lambda \alpha \lambda = \bigcap \{ \gamma \in \mathbb{B}_X : \bigcup \lambda \alpha \lambda \subset \gamma \} \).
4. \( \complement \alpha = \bigcap \{ M(x) : x \in \alpha \} \).

**Proof.** It is clear that \( \{ M(x) : x \in X \} \subset \mathbb{B}_X \). \( \square \)

We show \( \bigcap_{y \in X} M(y) \subset \mathbb{B}_X \). \( \bigcap_{y \in X} M(y) \subset \mathbb{B}_X \) is obvious. If \( \exists x \in X \) \( (x \in M(x)) \), then \( X \subset \mathbb{B}_X \) follows from \( \square \). Otherwise \( \bigcap \{ M(x) : X \subset M(x) \} = \bigcap \emptyset = X \).

Next we show \( \{ x \in X : x \in M(x) \} \Rightarrow x \in \bigcap_{y \in X} M(y) \). Assume \( x \in M(x) \). Then by \( \exists y \in X = M(x) \). \( \square \) yields \( x \in M(y) \).

Next we show \( \{ \alpha \lambda \alpha \lambda \} \subset \mathbb{B}_X \). Let \( \beta = \bigcap \{ M(y) : \bigcap \lambda \alpha \lambda \subset \gamma \} \). We show \( \beta \subset \alpha \lambda \alpha \lambda \). \( \bigcap \lambda \alpha \lambda \subset \gamma \) for any \( \lambda \). Let \( \alpha \lambda \alpha \alpha \subset M(x) \). Then \( \bigcap \lambda \alpha \lambda \subset M(x) \), and \( \beta \subset M(x) \). Hence \( \beta \subset \alpha \lambda \alpha \alpha \). We obtain \( \beta \subset \bigcap \lambda \alpha \lambda \), and hence \( \beta \in \mathbb{B}_X \). Therefore \( \inf \lambda \alpha \lambda = \bigcap \lambda \alpha \lambda \). On the other side we see \( \sup \lambda \alpha \lambda = \bigcap \{ \gamma \in \mathbb{B}_X : \bigcup \lambda \alpha \lambda \subset \gamma \} \). Then \( \beta \subset \bigcap \lambda \alpha \lambda \) if \( \gamma \subset \beta \).

Next we show \( \alpha \subset M(x) \Rightarrow x \in -\alpha \) \( \square \)

Assume \( \alpha \subset M(x) \) and \( y \in \alpha \). Then \( y \in M(x) \), and \( x \in M(y) \) by \( \square \). Thus \( x \in -\alpha = \bigcap \{ M(y) : y \in \alpha \} \).

Third we show \( \alpha \cap (-\alpha) = 0 \) if \( \alpha \subset \bigcap \lambda \alpha \lambda \alpha \). Then \( x \in \alpha \cap (-\alpha) \). \( \square \)

Finally we show \( \sup \{ \alpha, -\alpha \} = X \). Let \( \alpha, -\alpha \subset \beta \in \mathbb{B}_X \). If \( \beta \subset M(x) \), then by \( \square \) we have \( x \in -\alpha \subset M(x) \). \( \square \) yields \( M(x) = X \). Therefore \( \beta = \bigcap \{ M(x) : \beta \subset M(x) \} = X \). \( \square \)

The complement \( -M(y) \) of \( M(y) \) is given in the following Proposition \( \square \).
Proposition 4.3  For \( y \in X \), let \( m(y) := \bigcap \{ M(x) : y \in M(x) \} \). Then

\[
m(y) = -M(y)
\]  

(10)

**Proof.**  (I): By Theorem 4.2 and (8) we have \(-M(y) = \bigcap \{ M(x) : x \in M(y) \} = \bigcap \{ M(x) : y \in M(x) \} = m(y)\). □

4.2 semi valuation induced from relation

In what follows let \( X = S \) be the set of all sequents.

**Definition 4.4**  For sequents \( \Gamma \Rightarrow \Delta \)

\[
M(\Gamma \Rightarrow \Delta) := \{(\Lambda \Rightarrow \Theta) \in S : G^{1}LC^{f} \vdash \Gamma, \Lambda \Rightarrow \Delta, \Theta\}.
\]

It is clear that the map \( S \ni x \mapsto M(x) \subseteq S \) enjoys (7) and (8). (7) follows from the contraction and weakening (thinning) rules, while (8) is seen from the exchange rule. All of these rules are implicit in our calculus \( G^{1}LC^{f} \).

Let \( B_{S} \subseteq P(S) \) be the cBa induced by the map, cf. Theorem 4.2. We have for sequents \( x \in S, x \in M(x) \Leftrightarrow G^{1}LC^{f} \vdash x, 0 = \{ x \in S : x \in M(x) \} = M(\Rightarrow) \) for the empty sequent \( \Rightarrow \).

**Definition 4.5**  For formulas \( A \)

\[
\Diamond V(A) := M(\Rightarrow A) \\
\Box V(A) := m(A \Rightarrow) = \bigcap \{ M(\Gamma \Rightarrow \Delta) : (A \Rightarrow) \in M(\Gamma \Rightarrow \Delta) \}
\]

By Theorem 4.2 and (10) in Proposition 4.3 we have \( \Diamond V(A), \Box V(A) \in B_{S} \).

**Lemma 4.6**  \( V \) is a semi \( \Box B_{S} \)-valuation.

**Proof.**  \( \Box V(A) \subset \Diamond V(A) \) is seen from \( G^{1}LC^{f} \vdash A \Rightarrow A \).

The conditions of the \( \Diamond \) are seen from the right rules.

\( \Diamond V(\exists X^{n}F(X)) \supset \sup_{<} \{ \Diamond V(F(T)) : T \in T_{m^{1}}(n) \} \): From the rule \( (R3) \) we see that \( \Diamond V(\exists X^{n}F(X)) = M(\Rightarrow \exists X^{n}F(X)) \supset \bigcup_{T \in T_{m^{1}}(n)} M(\Rightarrow F(T)) \). Hence \( B_{S} \ni \Diamond V(\exists X^{n}F(X)) \supset \bigcap \{ \alpha : \bigcup_{T \in T_{m^{1}}(n)} M(\Rightarrow F(T)) \subset \alpha \} = \sup_{<} \{ \Diamond V(F(T)) : T \in T_{m^{1}}(n) \} \).

\( \Diamond V(\forall X^{n}F(X)) \supset \inf_{<} \{ \Diamond V(F(T)) : T \in T_{m^{1}}(n) \} : \Diamond V(\forall X^{n}F(X)) = M(\Rightarrow \forall X^{n}F(X)) \supset \bigcap_{T \in T_{m^{1}}(n)} M(\Rightarrow F(T)) = \inf_{<} \{ \Diamond V(F(T)) : T \in T_{m^{1}}(n) \} \) is seen from the rule \( (R3) \).

\( \Diamond V(\neg A) \supset \Diamond (\neg V(A)) \): By (10) and the rule \( (R\neg) \), we obtain \( \Diamond (\neg V(A)) = -\Box V(A) = -m(A \Rightarrow) = M(\Rightarrow \neg A) = \Diamond V(\neg A) \).

The conditions for \( \Box \) are seen from the left rules using (10) in Proposition 4.3.

\( \Box V(\exists X^{n}F(X)) \subset \sup_{<} \{ \Box V(F(T)) : T \in T_{m^{1}}(n) \} : \) By (10) it suffices to show that \( M(\exists X^{n}F(X) \Rightarrow) \supset \bigcap \{ M(F(T) \Rightarrow) : T \in T_{m^{1}}(n) \} \), which follows from the
rule \((L\exists^3)\).
\[ \square V(\forall^n X^n F(X)) \subset \inf \{ \square V(F(T)) : T \in Tm_{1}^{(n)} \} : \]
Again by \((10)\) it suffices to show that \(M(\forall^n X^n F(X) \Rightarrow) \supset \sup \{ M(F(T) \Rightarrow) : T \in Tm_{1}^{(n)} \} \), which follows from the rule \((L\forall^2)\).
\[ \square V(\neg A) \subset \neg \bigcirc V(A) : \]
The rule \((L\neg)\) yields \(M(\neg A \Rightarrow) \supset M(\Rightarrow A)\), from which and \((10)\) we obtain \(\square V(\neg A) = m(\neg A \Rightarrow) = -M(\neg A \Rightarrow) \subset \neg \bigcirc V(A)\). \(\square\)

(Proof of Theorem 5.1, ver.2)
Suppose \(G^{1}\LC \vdash \Gamma \Rightarrow \Delta\). From Theorem 2.9 and Lemma 1.6 we see that \(\bigcap \{ \square V(A) : A \in \Gamma \} \subset \sup \{ \bigcirc V(B) : B \in \Delta \} \) for the semi \(\mathbb{R}_S\)-valuation \(V\) defined in Definition 4.5. Now we have \((\Gamma \Rightarrow) \in \square V(A) = m(A \Rightarrow)\) for any \(A \in \Gamma\) by weakening. Hence \((\Gamma \Rightarrow) \in \sup \{ \bigcirc V(B) : B \in \Delta \} = \bigcap \{ M(x) : \bigcup_{B \in \Delta} M(\Rightarrow B) \subset M(\Rightarrow \Delta) \}\). On the other hand we have \(\bigcup_{B \in \Delta} M(\Rightarrow B) \subset M(\Rightarrow \Delta)\) by weakening. Therefore \((\Gamma \Rightarrow) \in M(\Rightarrow \Delta)\), i.e., \(G^{1}\LC \vdash \Gamma \Rightarrow \Delta\). \(\square\)

5 Proof-theoretic strengths

In the final section let us calibrate proof theoretic strengths of cut-eliminability. For a class \(\Phi\) of sequents \(CE\Phi(G^{1}\LC)\) denotes the statement that any \(G^{1}\LC\)-provably sequent in \(\Phi\) is provable without the \((\text{cut})\) rule. When \(\Phi\) is the set of all sequents, let \(CE(G^{1}\LC) \equiv CE\Phi(G^{1}\LC)\). \(I\Sigma_1\) denotes the fragment of the first-order arithmetic in which the complete induction schema is restricted to \(\Sigma^0_1\) formulas in the language of first-order arithmetic. Let \(\Sigma^0_1\) denote the set of \(\Sigma^0_1\) sequents in which no second-order quantifier occurs, and first-order existential quantifier \([\text{first-order universal quantifier}]\) occurs only positively \([\text{occurs only negatively}]\), resp. Then \(1-\text{CON}(\mathbb{Z}_2)\) denotes the 1-consistency of the second order arithmetic \(\mathbb{Z}_2 = (\Pi^1_2,\text{-CA})\), which says that every \(\mathbb{Z}_2\)-provable \(\Sigma^0_1\)-sequent is true.

Theorem 5.1
\[ I\Sigma_1 \vdash CE(G^{1}\LC) \leftrightarrow CE\Sigma^0_1(G^{1}\LC) \leftrightarrow 1-\text{CON}(\mathbb{Z}_2). \]

Proof: \((CE\Sigma^0_1(G^{1}\LC) \rightarrow 1-\text{CON}(\mathbb{Z}_2))\). This is shown in [15] as follows. Argue in \(I\Sigma_1\).

Let \(L^2\) denote the class of lower elementary recursive functions. The class of functions contains the zero, successor, projection and modified subtraction functions and is closed under composition and summation of functions. \(L^2\) denotes the class of lower elementary recursive relations. Then it is easy, cf. [10] to see that the class \(L^2\) is closed under boolean operations and bounded quantifications, each function in \(L^2\) is bounded by a polynomial, and the truth definition of atomic formulas \(R(x_1, \ldots, x_n)\) for \(R \in L^2\) is elementary recursive.

Suppose that \(\mathbb{Z}_2 \vdash \exists x R\) for a \(\Sigma^0_1\)-sentence \(\exists x R\) with an \(R \in L^2\). In the \(\mathbb{Z}_2\)-proof, restrict each first-order quantifier \(\forall x, \exists x\) to \(\forall x \in \mathbb{N}, \exists x \in \mathbb{N}\), where \(\mathbb{N}(a) := \forall X (X(0) \land \forall y(X(y) \supset X(Sy)) \supset X(a))\) with the successor function \(S\), and \(\exists x \in \mathbb{N} B \equiv (\exists x (\mathbb{N}(x) \land B))\), etc. Let us denote the restriction of a formula
A by $A^3$. The comprehension axiom (CA) $\exists X\forall y(X(y) \leftrightarrow G(y))$ follows from
$(R^3\exists)$. Complete induction schema follows $\forall a \in \mathbb{N} \forall X(X(0) \land \forall y \in N(X(y) \supset X(Sy))) \supset X(a))$. We obtain a $G^1\text{LC}$ proof of a sequent $Eq^3$, $A_0^3 \Rightarrow \exists x \in \mathbb{N} R$ for an axiom $A_0$ of finitely many constants for functions in $\mathcal{L}^2$ and the equality axiom $Eq :\Leftrightarrow (\forall x y, y(x = y \rightarrow (X(x) \leftrightarrow X(y))))$. $A_0$ is a universal formula $\forall x_1, \ldots, x_n Q$ with a $Q \in \mathcal{L}^2_\forall$. Thus we obtain a $G^1\text{LC}$-proof of the sequent $E\varphi, A_0 \Rightarrow \exists x R$.

Next let $\mathcal{E}(a) :\Leftrightarrow (\forall X\forall y(a = y \rightarrow (X(a) \leftrightarrow X(y))))$, and restrict each first-order quantifier $\forall x, \exists x$ occurring in the $G^1\text{LC}$-proof to $\forall x \in \mathcal{E}, \exists x \in \mathcal{E}$. Then we obtain a $G^1\text{LC}$-proof of the sequent $Eq^x, A_0 \Rightarrow \exists x R$, where $Eq^x \Leftrightarrow (\forall x y, y \in \mathcal{E}(x = y \rightarrow (X(x) \leftrightarrow X(y))))$, which is provable. Hence we obtain a $G^1\text{LC}$-proof of the sequent $A_0 \Rightarrow \exists x R$. Now by $CE_{\Sigma_\forall^0}(G^1\text{LC})$, i.e., the cut-eliminability from the proof with $\Sigma_\forall^0$-end sequents, we get $G^1\text{LC}^{\text{cf}} \vdash A_0 \Rightarrow \exists x R$, i.e., $L\varphi, A_0 \Rightarrow \exists x R$. Then we see that $\exists x R$ is true.

$(1\text{-CON}(\mathbb{Z}_2) \rightarrow CE(G^1\text{LC}))$. Although this is a folklore, cf. [5], let us show it briefly.

It suffices to show in $\mathbb{Z}_2$, the cut-eliminability from each proof $P$ of a sequent $\Gamma \Rightarrow \Delta$ since the statement $CE(G^1\text{LC})$ is a $\Pi^0_2$. In what follows argue in $\mathbb{Z}_2$, and consider the Takahashi-Prawitz' proof in Section 3 for simplicity. First observe that Lemma 2.10 of the existence of a semi 3-valuation $V$ is provable[4] in WKL, a fortiori in $\mathbb{Z}_2$, assuming that $G^1\text{LC}^{\text{cf}} \not\vdash \Gamma \Rightarrow \Delta$.

In Lemma 2.10 the satisfaction relation $\mathcal{N} \models F$ in the 2-model $\mathcal{N} = (D_0, I)$ is second-order definable for each formula $F$. Then for each formulas $F(X^n)$ and $G(x_1, \ldots, x_n, X)$, we have $v(T) \leq X \Rightarrow v(F(T)) \leq \mathcal{N}(F(X^n))$ and $\mathcal{N}(\forall X\exists Y^n\forall x_1, \ldots, x_n (Y(x_1, \ldots, x_n) \leftrightarrow G(x_1, \ldots, x_n, X))) = 1$. This suffices to evaluate the truth values of formulas occurring in the proof $P$, and $\mathcal{N}(\Gamma \Rightarrow \Delta) = 0$. Hence $P$ is not a $G^1\text{LC}$-proof of the sequent $\Gamma \Rightarrow \Delta$. A contradiction.

Proposition 5.2 $\Pi_1 \vdash CE_{\Sigma_\forall^0}(G^1\text{LC}) \rightarrow CE(G^1\text{LC})$.

Proof: This follows from Theorem 4.1 indirectly. Here is a direct proof.

P. Päppinghaus[8] shows that $\Pi_1 \vdash CE_{\Pi^0_1}(G^1\text{LC}) \rightarrow CE(G^1\text{LC})$ by using cut-absorption and the jocker translation, where $\Pi^1$ denotes the set of sequents in which second-order universal quantifier [second-order existential quantifier] occurs only positively [occurs only negatively], resp. In what follows argue in $\Sigma_1$.

Let $\Gamma \Rightarrow \Delta$ be a $\Pi^1$-sequent. Erase each second-order quantifier $\forall X, \exists Y$ in the sequent to get a first-order sequent $\Gamma_0 \Rightarrow \Delta_0$. It is easy to see that if $G^1\text{LC} \vdash \Gamma \Rightarrow \Delta$, then $G^1\text{LC} \vdash \Gamma_0 \Rightarrow \Delta_0$, and if $G^1\text{LC}^{\text{cf}} \vdash \Gamma_0 \Rightarrow \Delta_0$, then $G^1\text{LC}^{\text{cf}} \vdash \Gamma \Rightarrow \Delta$. Hence we obtain $CE_{\Pi^0_1}(G^1\text{LC}) \rightarrow CE_{\Pi^0_1}(G^1\text{LC})$ for the set $\Pi^0$ of first-order sequents.

1Maehara’s proof in Section 3 is formalizable in ACA$_0$. $\alpha \in \mathbb{B}S$ is definable by an arithmetical formula.
Next let \( H \in \Sigma_1^0 \) be an Herbrand normal form of the first-order formula \( \forall \Gamma_0 \supset \exists \Delta_0 \). Then again it is easy to see that if \( G^1 \text{LC} \vdash \Gamma_0 \supset \Delta_0 \), then \( G^1 \text{LC} \vdash \exists \Rightarrow H \), and if \( \text{LK} \vdash \exists \Rightarrow H \), then \( \text{LK}^{\exists/\forall} \vdash \Gamma_0 \supset \Delta_0 \). Therefore \( CE_{\Sigma_1^0}(G^1 \text{LC}) \rightarrow CE_{\Pi_1^0}(G^1 \text{LC}) \).

Let us mention a refinement for fragments. \( \Pi_1^1 \) denotes the class of formulas \( G \equiv (\forall X_1 \exists X_2 \cdots Q X_n A) \) with a first-order matrix \( A \), and \( Q = \forall \) when \( n \) is odd, \( Q = \exists \) else. An abstract \( T \equiv (\lambda \vec{x} . G(x_1, \ldots, x_k)) \) is in \( \Pi_1^1 \) iff \( G \in \Pi_1^1 \). Then \( G^1 \text{LC}(\Pi_1^1) \) denotes a fragment of the calculus \( G^1 \text{LC} \) in which inference rules \((R^\exists 1), (L^\forall 1)\) are restricted to \( T \in \Pi_1^1 \):

\[
\begin{array}{c}
\Gamma \Rightarrow \Delta, \exists X F(X), F(T) \\
\Gamma \Rightarrow \Delta, \exists X F(X)
\end{array}
\quad
\begin{array}{c}
F(T), \forall X F(X), \Gamma \Rightarrow \Delta \\
\forall X F(X), \Gamma \Rightarrow \Delta
\end{array}
\quad
(R^\exists 1)
\quad
(L^\forall 1)
\]

An inspection to the proof of Theorem 5.1 shows the following. Note that \( N(a) \) as well as \( E(a) \) is a \( \Pi_1^1 \)-formula without second-order free variable.

**Corollary 5.3** For each \( n > 0 \)

\[ \Sigma_1^1 \vdash CE(G^1 \text{LC}(\Pi_1^1)) \leftrightarrow CE_{\Sigma_1^1}(G^1 \text{LC}(\Pi_1^1)) \leftrightarrow 1\text{-CON}((\Pi_1^1\text{-CA})_0). \]

Finally let us mention some open problems.

**Problem 1.** What is the proof theoretic strength of the statement \( CE_{\Pi_1^1}(G^1 \text{LC})? \)

\( CE_{\Pi_1^1}(G^1 \text{LC}) \) says that any \( G^1 \text{LC}\)-provable \( \Pi_1^1 \)-sequent is provable without the \((\text{cut})\) rule, where \( \Pi_1^1 \) denotes the dual class for \( \Sigma_1^1 \). Specifically does \( \Sigma_1 \) prove \( CE_{\Pi_1^1}(G^1 \text{LC})? \)

To state the next problem we need first some definitions.

**Definition 5.4**

1. An inference rule is said to be reducible if there is a minor formula \( A \) of the inference rule such that either the formula \( A \) is in the antecedent and the sequent \( \Rightarrow A \) is provable, or \( A \) is in the succedent and the sequent \( A \Rightarrow \) is provable.

2. A proof \( P \) enjoys the pure variable condition if in \( P \), a free variable occurs in a sequent other than the end-sequent, then it is an eigenvariable of an inference rule \( J \) and the variable occurs only in the upper part of the inference rule \( J \).

3. A proof is said to be in irreducible or in Mints’ normal form if it is cut-free, enjoys the pure variable condition and contains no reducible inference rules.

Mints’ normal form theorem for a sequent calculus \( C \) states that every \( C \)-provable sequent has an irreducible proof (with respect to \( C \)).
Theorem 5.5 \(\left[7, 2\right]\)

Over \(\Sigma_1\), Mints’ normal form theorem for the first-order calculus \(\text{LK}\) is equivalent to the 2-consistency of the first-order arithmetic \(\text{PA}\).

Problem 2. Does Mints’ normal form theorem hold for \(G^1\text{LC}\)?

It is easy to see that Mints’ normal form theorem for \(G^1\text{LC}\) implies the 2-consistency of the second-order arithmetic \(Z_2\) as follows. Assume that \(Z_2 \vdash \exists x \forall y R(x)\) for a false \(\Sigma_0^1\)-sentence \(\exists x \forall y R(x)\). Let \(\text{Ind} \iff (\forall a \mathbb{N}(a))\). Then \(G^1\text{LC} \vdash \exists x (\text{Eq} \land \text{Ind} \land A_0 \supset \forall y R(x))\) for a true \(\Pi_0^1\)-sentence \(A_0\). Pick an irreducible proof \(P\) of the sequent \(\Rightarrow \exists x (\text{Eq} \land \text{Ind} \land A_0 \supset \forall y R(x))\) in \(G^1\text{LC}\). Then for a closed term \(t\) the last inference must be a right rule \((R\exists^0)\):

\[
\Rightarrow \exists x (\text{Eq} \land \text{Ind} \land A_0 \supset \forall y R(y)) \quad (R\exists^0)
\]

From the \(\Sigma_0^1\)-completeness, we see for the false \(\Pi_0^1\)-sentence \(\forall y R(t)\), that there exists a proof of the sequent \(\text{Eq} \land \text{Ind} \land A_0 \supset \forall y R(t)\) even in the weak fragment \(BC\) of \(G^1\text{LC}\) defined in p.166, \(\left[8\right]\), in which the abstracts \(T\) in the inference rules \((R\exists^1), (L\forall^1)\) are restricted to variables and predicate constants.

This means that \(P\) is reducible. A contradiction.

Cut-elimination by absorption in \(\left[8\right]\) is useless to prove the Mints’ normal form theorem since in

\[
\begin{align*}
\Gamma &\Rightarrow \Delta, A
\end{align*}
\]

\[
A \supset A, \Gamma \Rightarrow \Delta
\]

\[
\Rightarrow \forall X (X \supset X), \Gamma \Rightarrow \Delta
\]

\[
(L \supset)
\]

\[
(L\forall^1)
\]

\[
\Rightarrow \forall X (X \supset X) \text{ as well as } \Rightarrow A \supset A \text{ is provable, and both inferences } (L\forall^1) \text{ and } (L \supset) \text{ are reducible.}
\]

A proof of Mints’ normal form theorem hold for \(\text{LK}\) in \(\left[2\right]\) runs as follows. Assume that a sequent \(\Gamma_0 \Rightarrow \Delta_0\) has no irreducible proof. By a proof search, we get an infinite binary tree of sequents, where we don’t analyze, e.g., a succedent formula \(\exists x A(x)\) for a term \(t\) when its instance \(A(t)\) can be refuted, i.e., \(A(t) \Rightarrow\) is provable.

\[
\Rightarrow \exists x A(x), \Gamma \Rightarrow \Delta
\]

\[
\Rightarrow \forall x A(x)
\]

Pick an infinite path \(P\) through the tree. Let \(\mathcal{P}_a [\mathcal{P}_s]\) denote the set of formulas occurring in an antecedent [occurring in a succedent] of a sequent on the path \(P\), resp. Let atomic formulas in \(\mathcal{P}_a\) to be true, and atomic formulas in \(\mathcal{P}_s\) to be false. From the truth values of atomic formulas define a first-order structure \(\mathcal{M}\).

\(\left[8\right]\) Then we see by induction on formulas \(A\) that if \(A \in \mathcal{P}_a\), then \(\mathcal{M} \models A\), and if

\(\text{Here we need } \omega\text{-times iterated jump operations. In a canonical proof search for cut-free provability in } \text{LK}, \text{ we obtain a valuation from an infinite path, which enjoys the Tarski’s conditions without appealing iterated jump operations. This is known as the Kreisel’s trick.}\)
A ∈ Pr, then M ⊭ A. In the case of unanalyzed formula as above, M ⊭ A(t) follows from the soundness of the calculus LK for any first-order structures M.

An obstacle in extending this proof to GLC lies in the fact that we need first prove (6), and then (5) follows from (6) in the proof of Lemma 2.10. However in proving (6) for an infinite path obtained from a search tree with respect to the non-existence of irreducible proof, we need the soundness of GLC for 2-models N, but the soundness holds only if the model N enjoys the Comprehension axiom. In other words we need (6) before we prove (5), and we are in a circle.

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