Empirical Chaos Processes and Blind Deconvolution

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Abstract

This paper investigates conditions under which certain kinds of systems of bilinear equations have a unique structured solution. In particular, we look at when we can recover vectors $w, q$ from observations of the form

$$y_\ell = \langle w, b_\ell \rangle \langle c_\ell, q \rangle,$$

where $b_\ell, c_\ell$ are known. We show that if $w \in \mathbb{C}^{M_1}$ and $q \in \mathbb{C}^{M_2}$ are sparse, with no more than $K$ and $N$ nonzero entries, respectively, and the $b_\ell, c_\ell$ are generic, selected as independent Gaussian random vectors, then $w, q$ are uniquely determined from

$$L \geq \text{Const} \cdot (K + N) \log^5 (M_1 M_2)$$

such equations with high probability.

The key ingredient in our analysis is a uniform probabilistic bound on how far a random process of the form

$$Z(X) = \sum_{\ell=1}^L |b_\ell^* X c_\ell|^2$$

deviates from its mean over a set of structured matrices $X \in \mathcal{X}$. As both $b_\ell$ and $c_\ell$ are random, this is a specialized type of 4th order chaos; we refer to $Z(X)$ as an empirical chaos process. Bounding this process yields a set of general conditions for when the map $X \rightarrow \{b_\ell^* X c_\ell\}_{\ell=1}^L$ is a restricted isometry over the set of matrices $\mathcal{X}$. The conditions are stated in terms of general geometric properties of the set $\mathcal{X}$, and are explicitly computed for the case where $\mathcal{X}$ is the set of matrices that are simultaneously sparse and low rank.

Our theoretical study is motivated by practical problems in signal processing and communications. Specifically, the blind deconvolution problem, where we observe $y = s * h$ and want to recover both $s$ and $h$, involves solving bilinear equations with the form above. Our results show that deconvolution is possible when $s$ and $h$ are sparse in generic bases, with their total number of nonzero terms within a polylogarithmic factor of their length, and they obey an incoherence condition.

1 Introduction

We study the problem of recovering, from linear measurements, a matrix $X$ that is low rank and whose factors are sparse. The linear measurements are themselves structured, and have the form of taking a linear
combination of the columns of $X$ and then a different linear combination of the result: $y_\ell = b_\ell^* X c_\ell$. We are particularly interested in the case where the matrix has rank 1 — this problem then amounts to solving a system of bilinear equations in two unknown vectors, both of which are sparse.

A rich theory for recovering structured signals from limited measurements has been developed over the last decade, a body of work now known as “compressed sensing”. While the literature has explored a diverse set of problems of this general type (see [1,2] for overviews of this topic), most of the attention has been directed towards theory and methods for recovering sparse vectors and low rank matrices from underdetermined linear measurements (cf. Section 1.1 below for a more detailed overview and more references).

Combining these structures, that is, recovering matrices that simultaneously sparse and low rank (SSLR), is not as straightforward as it first appears. For example, while provably effective convex relaxations are available for each type of structure by itself, combing these relaxations into a single penalty known to be suboptimal [3], and when the matrix is rank 1, there is no convex relaxation that will be effective [4].

We will start by looking at a special but important case of particular interest. The problem is to recover vectors $w, q \in \mathbb{C}^{M_1}, q \in \mathbb{C}^{M_2}$ from observations that are products of linear functionals:

$$y_\ell = \langle w, b_\ell \rangle \langle c_\ell, q \rangle + \text{noise},$$

$$= b_\ell^* (w q^*) c_\ell + \text{noise}$$

$$= \langle w q^*, b_\ell c_\ell^* \rangle + \text{noise}, \quad \text{for } \ell = 1, \ldots, L.$$  \hfill (1)

where $b_\ell \in \mathbb{C}^{M_1}, c_\ell \in \mathbb{C}^{M_2}$ are known. As the manipulations above suggest, the measurements can be modeled as a Frobenius inner product between a rank 1 matrix formed from the outer products of the unknown vectors and a rank 1 matrices formed from the outer product of the measurement vectors $b_\ell, c_\ell$.

When $w = q$ and $c_\ell = b_\ell$, we are observing the squared magnitude $|\langle w, b_\ell \rangle|^2$ of linear measurements — this problem has been studied in the recent literature under the name of phase retrieval [5,6]. For different $w, q$ and $b_\ell, c_\ell$, it is a kind of blind deconvolution problem analyzed in [9] (see also Section 1.1 below). In both of these cases, recovering $w$ and $q$ is possible from $L \sim \max(M_1, M_2)$, within logarithmic factors, with no structural assumptions on the unknown vectors.

Our structural model is that both $w$ and $q$ are sparse — $w$ has at most $K \leq M_1$ nonzero entries, and $q$ has at most $N \leq M_2$ nonzero entries. We show below that when the $b_\ell, c_\ell$ are random vectors, then there is an algorithm exhibiting local convergence to the right solution $(w, q)$ (up to a constant factor) provided

$$L \gtrsim (K + N) \log^5(M_1 M_2).$$

This result is optimal, up to logarithmic factors, as it would take on the order of $K \log M_1 + N \log M_2$ measurements to recover $w$ and $q$ if we had the advantage of observing the $\langle w, b_\ell \rangle$ and $\langle c_\ell, q \rangle$ independently.

Our approach uses the lifting technique popularized by the recent work on phase retrieval and blind deconvolution. We explicitly treat the measurements in (1) above as linear measurements of a structured matrix, and then develop general conditions on when matrices with this structure can be recovered from inner products against known matrices $A_\ell = b_\ell c_\ell^*$. This formulation also allows us to naturally extend the results to the more general setting of recovering a matrix $X$ that is simultaneously sparse and low rank. For $M_1 \times M_2$ matrices that can be written as $X = Y Q^*$, where both $Y$ and $Q$ have $R < \min(M_1, M_2)$ columns and have at most $K$ and $N$ non-zero rows, respectively, Theorem 2 below implies that one needs

$$L \gtrsim R^{3/2} (K + N) \log^5(M_1 M_2),$$

measurements of the form $b_\ell^* X c_\ell$, where again $b_\ell$ and $c_\ell$ are random. Note that such $X$ have in general $K N$ non-zero terms while the bound above scales with $K + N$. This indicates that taking advantage of the low rank structure has a tangible advantage over exploiting sparsity by itself. We expect that the dependence on the rank can even be improved from $R^{3/2}$ to $R$.

As with many structured recovery problems, a core proof ingredient is to establish that map from the set of structured matrices to the measurements preserves distances. To make this precise, let $A : \mathbb{C}^{M_1 \times M_2} \to \mathbb{C}^L$
denote a linear map from the space of $M_1 \times M_2$ matrices to $\mathbb{C}^L$. The operator $A$ acts by taking inner products against a series of fixed matrices $A_1, \ldots, A_L$:

$$A(X) = \begin{bmatrix} \langle X, A_1 \rangle \\ \vdots \\ \langle X, A_L \rangle \end{bmatrix}.$$  \hspace{1cm} (2)

Given a set of matrices, $X \subset \mathbb{C}^{M_1 \times M_2}$, we say that $A$ has the restricted isometry property (RIP) of level $\delta$ for $X$ if

$$(1 - \delta_a)\|X\|_F^2 \leq \|A(X)\|_2^2 \leq (1 + \delta_a)\|X\|_F^2,$$  \hspace{1cm} (3)

for all $X \in X$ for some $\delta_a \leq \delta$. Notice that if $X$ is formed by taking linear combinations of pairs from another set $X'$, $X = X' \oplus X'$, then (3) says that $A$ preserves distances between matrices in $X'$, thus providing a stable embedding from $X'$ to $\mathbb{C}^L$.

Theorem 1 below gives general conditions under which (3) holds when $A$ has the form

$$A_\ell = b_\ell c'_\ell, \quad \Rightarrow \quad \langle X, A_\ell \rangle = b_\ell^* X c_\ell,$$  \hspace{1cm} (4)

where $b_\ell$ and $c_\ell$ are Gaussian random vectors. The result is stated in terms of abstract geometric parameters for the set $X$. In Theorem 2, we estimate these geometric parameters for the particular case where $X = X_{R,K,N}$ is the set of matrices that have at most rank $R$ and have right and left factors with $K$ and $N$ non-zero rows. We will refer to $X_{R,K,N}$ as the set of *simultaneously sparse and low rank* (SSLR) matrices.

The restricted isometry property is sufficient to ensure local convergence, for signals with a certain peakiness condition even global convergence, of a simple and stable algorithm for recovering SSLR $X = YQ^*$ from $A(X)$. In [10], the *sparse power factorization* (SPF) method was introduced for expressly this purpose. The results in that paper were stated specifically for the rank 1 case, but it is clear that they extend to general $R$. The SPF algorithm alternates between two steps. In the first, it holds the left factor $Y$ constant and updates the right factor $Q$ using a standard sparse recovery algorithm. The second step uses the same algorithm to estimate $Y$ with $Q$ held constant. For $X \in X_{R,K,N}$, this process is guaranteed to provide a stable recovery when (3) holds for $X_{R',K',N'}$, where $R' = 2R$, $K' = 3K$, and $N' = 3N$. More details on SPF are given in Section 2 below.

1.1 Relation to previous works and applications

Bilinear Inverse Problems

Bilinear inverse problems of the form (1) have been extensively studied in the signal processing and applied mathematics literature for decades. However, the techniques used to solve these problems, and the mathematics used to analyze the methods, tended to be very specialized to the application at hand. Using lifting to recast the bilinear problem into the recovery of a rank-1 from linear constraints allows us to treat many types of problem in a unified manner, and draw on the recently developed low-rank recovery literature. The problem of phase retrieval, or recovering a vector from observations of the magnitude of a series of inner products, was the first subject to be rigorously analyzed in this way [6, 8]. In this problem, the measurements have the form $|a^*_\ell w|^2 = a^*_\ell X a_\ell$, with $X = w w^*$. Recovery guarantees for this problem were first derived for generic $a_\ell$ and later for more structured $a_\ell$ that reflected the constraints present in certain imaging applications [11, 12]. While the first works on this topic operated explicitly in the lifted domain, were the goal is to recover a matrix, there has been recent work on efficient algorithms that work directly in the signal space [13, 15]. An alternative line of research focuses on conditions on the number of measurements, which are necessary and sufficient to establish injectivity [16, 17]. While these results do not yield any tractable methods for signal recovery, they do provide important benchmarks regarding the number of measurements that are necessary for any method to work.
The bilinear problem we study here, with measurements of the form \( b_\ell^* X c_\ell, X = wq^* \), have two important differences from the phase retrieval problem. First, the lifted matrix is non-symmetric (and not necessarily square), giving the target slightly less structure. Second, the measurement vectors that we apply to the left and right of \( X \) are different, making the randomness in the linear operators we analyze of a different nature. Whether or not a rank-1 matrix \( X \) is symmetric, it can be recovered through a completely random linear map (i.e. the \( A_\ell \) in (2) have independent subgaussian entries) with \( L \) on the order of maximum of the number of rows and columns in \( X \) using a number of different algorithms \cite{18,20}; conditions for injectivity have been established in \cite{21}. In contrast, the (lifted) linear map considered in this paper has a more structured randomness, with the \( A_\ell \) taken as the outer product of two random vectors.

We also use sparsity as an additional constraint, a model which is of general interest across many applications. It can be shown that while minimizing a linear combination of sparsity and rank is a reasonable strategy, replacing both of them individually by their convex relaxation will necessarily require a number of measurements comparable to using just one of the regularizers \cite{4}, so basically ignoring one of the two structural properties. To date, no approach has been proposed to overcome this difficulty in a general context. For the phase retrieval problem and a carefully designed measurement matrix of reduced rank, one can employ a two-stage procedure based on a combination of compressed sensing and phase retrieval \cite{22,23}. For the more general case of bilinear measurements, the sparse power factorization algorithm \cite{10} guarantees local convergence and allows for recovery guarantees for very peaky signals (see Section 2 for details) and Gaussian measurement maps.

As in the case of phase retrieval, however, one can typically not expect bilinear maps, which are in this sense generic, rather some structure is imposed by the applications at hand. A first scenario that has been studied is the calibration of a sensor system, here one seeks to estimate both the calibration parameters and the signal. This problem has been analyzed from an algorithmic viewpoint in \cite{24}, with injectivity conditions studied in \cite{25}. A second sequence of works, discussed in the following paragraph, studies blind deconvolution problems, which are also the focus of this paper. A third application area of bilinear inverse problems, channel estimation with a coded aperture, is also covered by our results, and is discussed in more detail below.

**Blind Deconvolution**

In blind deconvolution problems, the bilinear map is a (circular) convolution, that is,

\[
B(w, q) = w * q,
\]

where, with \( \odot \) denoting subtraction mod \( L \), the entries of the convolution are given by

\[
(w * q)_j = \sum_{\ell=1}^{L} w_\ell q_{j \odot \ell}.
\]

The blind deconvolution then consists of recovering \( w \) and \( q \) from these observations using some structural model assumptions. This problem has been of continuous interest for many years (see \cite{26} for an overview of classical work with many further references).

Especially in communication applications, it is natural to transform the problem to the Fourier domain. Then the convolution becomes a pointwise multiplication. The interpretation is the following. A source encodes a (generally complex-valued) message as a series of amplitude and phase shifts on tones at frequencies \( \omega_1, \ldots, \omega_L \); these values are collected in the vector

\[
s = \begin{bmatrix}
s(\omega_1) \\
s(\omega_2) \\
\vdots \\
s(\omega_L)
\end{bmatrix}.
\]
As the message is transmitted through a linear time-invariant channel, this vector gets weighted frequency-by-frequency by a channel response \( h(\omega_\ell) \). The receiver observes

\[
y_\ell = s(\omega_\ell) \cdot h(\omega_\ell), \quad \ell = 1, \ldots, L.
\]

The blind deconvolution task is to recover both the \( s(\omega_\ell) \) and \( y(\omega_\ell) \) from the \( y_\ell \). Note that in contrast to other scenarios discussed before, the number of measurements is the same as the dimension of each of the input signals, so without structural assumptions the map is not injective.

Again, a structural assumption that is often empirically satisfied is sparsity in a suitable known basis. If the support of the signals is known, this assumption boils down to requiring that the signals lie in given known low-dimensional subspaces. More precisely, \( s \) lies in the span of the columns of an \( L \times M_1 \) matrix \( B \) and \( h \) lies in span of the columns of an \( L \times M_2 \) matrix \( C \): \( s = Bx \), and \( h = Cq \). Then \( y_\ell = (b_\ell, x)(c_\ell, q) = (\bar{x}, \bar{b}_\ell)(c_\ell, q) \), similar to (1), where the \( b_\ell \) and \( c_\ell \) are the rows of \( B \) and \( C \), respectively. Injectivity conditions of the resulting measurement operation were studied in \cite{27} and later improved in \cite{28,29}. One finds that the map is injective for generic choices of \( B \) and \( C \) as long as \( L \geq 2(M_1 + M_2) - 4 \), which can also be shown to be optimal \cite{29}. A compressed sensing viewpoint was assumed in \cite{4}, which shows recovery guarantees for this setup for nuclear norm minimization. The assumptions on the basis matrices are that \( B \) is a fixed matrix whose rows is dispersed in the Fourier domain and \( C \) is chosen at random. The relation between signal and subspace dimensions required in this approach agrees with the optimal one from \cite{29} up to logarithmic factors. A more efficient algorithm reconstruction algorithm has been studied in \cite{30}, under similar assumptions on the subspace dimensions. If one of the signals can be modulated in a controlled way before being convolved with the other signal, these structural assumptions can be relaxed \cite{31,32}.

As in the other cases discussed above, the case of general sparsity, i.e., the case that the support of the signals is unknown, involves two simultaneous objective and is consequently considerably more difficult as it involves the two competing objectives sparsity and rank. Again, one can prove optimal injectivity conditions: the map is injective for generic choices of \( B \) and \( C \) and sparsity levels \( K \) and \( N \) provided \( L \geq 2(N + K) - 2 \). \cite{27,29}. Without substantial additional assumptions on the signals, no recovery guarantees are known for measurement numbers, which get close to this limit. A variant of the sparse power factorization algorithm has been designed specifically for this setup \cite{32}. Recovery guarantees are proved for \( B \) and \( C \) both random; they require not only the peakiness assumption, but also an additional spectral flatness assumption with respect to one of these bases. The latter aspect is also related to the fact that the results on the original variant of Sparse Power Factorization do not apply: The proofs are based on restricted isometry properties (RIP, cf. the last paragraph in this section), and the RIP conditions derived in the companion paper \cite{34} only cover spectrally flat signals. Consequently, the spectral flatness needs to be artificially imposed in each step. The RIP conditions derived in this paper (see Section 1.3 below) are also for \( B \) and \( C \) both random, but do not require spectral flatness, so the results on the original version of the sparse power factorization directly apply.

### Channel estimation with a coded aperture

Multiple-input multiple-output (MIMO) channel estimation can also be abstracted as a matrix recovery problem from linear measurement of the form \cite{4}. The general scenario is that \( M_1 \) sources are propagating information to \( M_2 \) receivers. The joint channel between the sources and receivers is characterized by a collection of \( M_1 \times M_2 \) matrices \( H(\omega) \) of complex-valued fading coefficients that describe how the amplitude and phase of a sinusoid at frequency \( \omega \) is altered as it travels from each source to each receiver. Estimating this matrix at one or more frequencies \( \omega \) is typically done by having the sources periodically emit sinusoids at known amplitude and phases, then measuring the response at the receiver.

In a typical application, the \( M_1 \) sources might be different users of a channel at unknown locations whose activity can be coordinated, while the receiver consists of a fixed array of \( M_2 \) sensors. If the sources emit pilot sequences simultaneously, and the pilot sequences at a particular time are captured in the vector \( b_\ell \), then the complex amplitudes incident on the sensor array are \( b_\ell^* H(\omega) \). At the receiver side, instead of
making a reading at each element independently, we might instead observe a linear combination of the array elements; if these weights are tabulated in the vector $c_\ell$, the final reading is $b_\ell^* H(\omega) c_\ell$. The inner product against $c_\ell$ can be thought of as a kind of generalized beamforming. If the $c_\ell$ are random vectors, then the measurements are being made with something akin to coded aperture; instead of using the weights to focus the return from a particular direction, they are being used to reduce the throughput off of the sensor.

Our recovery results below tell us that the length of the probe signals and the number of generalized beams we need to get an accurate estimate of $H(\omega)$ could be far less than $M_1 M_2$ if the channels are jointly structured, exhibiting strong correlations across source/receiver pairs, or sparsity/subspace constraints along rows or columns.

Restricted Isometry Properties

A key tool in this paper is a restricted isometry property (RIP). In the last years, such conditions played an important role in the mathematical theory of signal processing. First introduced in the context of compressed sensing [35], such conditions have subsequently also been used for low-rank recovery problems [18] and in arbitrary Hilbert space settings provided an efficient projector exists [36]. For sparsity and low rank as a joint prior (which is not covered by the setting of [36], as a projector would solve the sparse PCA problem), the RIP has been established for Gaussian measurements as a tool to show local convergence of the Sparse Power Factorization in [10]. An RIP for the blind deconvolution problem has first been conjectured in [37]. In [34], an RIP was established for two random bases and signals that satisfy an incoherence assumption. While such assumptions will typically be satisfied, they nevertheless significantly restrict the applicability, as one needs to ensure that it continues to hold in each iteration. In this paper, we remove the incoherence assumption and prove the corresponding RIP for random bases and all signals.

1.2 Empirical chaos processes

We will establish (3) by relating $\delta_A$ to the supremum of a particular empirical process. This notion refers to quantities of the form

$$\sup_{f \in \mathcal{F}} \frac{1}{L} \sum_{\ell=1}^{L} f(Y_\ell) - \mathbb{E}[f(Y_\ell)], \quad (5)$$

where $Y_\ell$ are independent random variables (which can be random scalars, vectors, matrices, etc.) and $\mathcal{F}$ is a pre-defined class of functions. Empirical processes have been extensively studied in the statistical learning theory literature (see [38, 39] for a survey). Their use in establishing restricted isometries was pioneered in [40, 41]. Here, the random variables forming the process are the measurement vector pairs $b_\ell, c_\ell$, and the function class $\mathcal{F}$ is parameterized by the set $\mathcal{X}$ of matrices of interest. With the quick calculation that $\mathbb{E}[|b_\ell^* X c_\ell|^2] = \|X\|^2_F$, we know that $\delta_A$ is bounded by

$$\sup_{X \in \mathcal{X}} \frac{1}{L} \left( \|A(X)\|^2_F - \|X\|^2_F \right) = \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^{L} |b_\ell^* X c_\ell|^2 - \mathbb{E}[|b_\ell^* X c_\ell|^2], \quad (6)$$

which is indeed a special case of (5). The additional structure of this process is related to scenarios considered in various recent works, e.g., by Mendelson in [42]. There, the set $\mathcal{F}$ in (5) consists of positive valued functions (each of which is representable as a square of a function as above), and a specialized bound is derived that is much tighter than in the general case. Our functions have even more structure, which we use below to get yet a tighter bound than would arise from simply applying the results from [42].

The individual summands in (6), the “ingredients” of the empirical process, are themselves known as decoupled Gaussian chaos process, a random variable of the form $\langle g_1, X g_2 \rangle$, where the $g_1, g_2$ have independent Gaussian entries. For this reason, we refer to (6) as an empirical chaos process.
Tail bounds for standard chaos process have been established by Hanson and Wright in [43]. They showed that the tail decay is bounded by a combination of a subexponential decay parametrized by the spectral norm of $\mathbf{X}$ and a subgaussian decay parametrized by the Frobenius norm of $\mathbf{X}$. A chaining argument then leads to a bound for suprema of chaos processes which involves the Talagrand $\gamma_1$ functional of $\chi$ with respect to the operator norm and a $\gamma_2$ functional of $\chi$ with respect to the Frobenius norm [44] (see Definition 1 in the next section for the precise definition of these functionals). This bound is known to yield suboptimal estimates in certain applications. For example, in compressed sensing, the restricted isometry property for subsampled random convolutions can also be expressed as the supremum of a chaos process. When applying the general results from [44], one only obtains the RIP for suboptimal embedding dimension [45]. Only incorporating the fact that in this application, the set $\mathcal{X}$ only consists of positive definite matrices led to embedding dimensions which are optimal up to logarithmic factors [46].

Similar difficulties arise when estimating the tail decay of the empirical chaos process in (6). Firstly, as just argued, Gaussian chaos processes are subexponential random variables, while the bounds in [42] require that the $f(Y_t)$ are subgaussian. For the lower bound in (5), the Mendelson’s small ball method (see, e.g., [47]) may apply for more general distributions, but for the upper bound, one will need to use more precise properties of the chaos processes, which is what we pursue in this paper.

### 1.3 Main results

The main mathematical contribution of this paper is a concentration inequality for empirical chaos processes (as introduced in the previous section) over general subsets $\mathcal{X}$ of $M_1 \times M_2$ matrices that have rank at most $R$. This first result, Theorem 1 below, is written in terms of abstract functionals that quantify the geometry of the set $\mathcal{X}$. Our second contribution, which gives us the embedding result (3) for the set of SSLR matrices when $\mathcal{A}(\cdot)$ consists of a series of inner products against random rank-1 matrices as in (4), comes from bounding these geometrical functionals for the particular case of matrices that are both low rank and have sparse factors.

We are ultimately interested in $M_1 \times M_2$ matrices that are simultaneously sparse and low rank. The concentration inequality that we present in Theorem 1, however, can be applied to any set of low rank matrices that have structured factors. Let $\mathcal{L}$ and $\mathcal{R}$ be the sets of $M_1 \times R$ and $M_2 \times R$ matrices with orthogonal columns:

$$\mathcal{L} = \{ \mathbf{Y} \in \mathbb{C}^{M_1 \times R} : \mathbf{Y}^* \mathbf{Y} = \text{diag}(\{\omega_r\}), \ \omega_r \geq 0 \},$$

$$\mathcal{R} = \{ \mathbf{Q} \in \mathbb{C}^{M_2 \times R} : \mathbf{Q}^* \mathbf{Q} = \text{diag}(\{\tau_r\}), \ \tau_r \geq 0 \}.$$

Through the singular value decomposition (SVD), any rank-$R$ matrix can be written as the product of a matrix from $\mathcal{L}$ and (the Hermitian transpose of) a matrix from $\mathcal{R}$. We will consider subsets of rank-$R$ matrices of the form

$$\mathcal{X} = \{ \mathbf{X} \in \mathbb{C}^{M_1 \times M_2} : \mathbf{X} = \mathbf{Y} \mathbf{Q}^*, \ \mathbf{Y} \in \mathcal{Y}, \ \mathbf{Q} \in \mathcal{Q} \}, \text{ for some } \mathcal{Y} \subset \mathcal{L}, \ \mathcal{Q} \subset \mathcal{R}. \quad (7)$$

For a matrix in this set, the SVD $\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^*$ can be formed simply by normalizing the columns of $\mathbf{Y}$ and $\mathbf{Q}$ to get $\mathbf{U}$ and $\mathbf{V}$, then taking $\Sigma = \text{diag}(\{\omega_r \tau_r\})$. Our main results follow from an analysis of an empirical chaos process on this set:

$$Z(\mathbf{X}) = \frac{1}{L} \sum_{\ell=1}^{L} |b_{\ell}^* \mathbf{X} c_{\ell}|^2, \quad b_{\ell}, c_{\ell} \sim \text{Normal}(0, 1), \quad \mathbf{X} \in \mathcal{X}. \quad (8)$$

Our main task is to bound the deviation of $Z(\mathbf{X})$ from its mean uniformly over all $\mathbf{X} \in \mathcal{X}$. This bound depends on a notion of geometrical complexity of both the factor sets $\mathcal{Y}$ and $\mathcal{Q}$ and their combination into

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1 Throughout the paper, we are working in the complex field. When we say $b_{\ell} \sim \text{Normal}(0, 1)$, this means that the entries of $b_{\ell}$ have independent real and imaginary parts that are Gaussian, each with variance 1/2, so $\mathbb{E}[|b_{\ell}[n]|^2] = 1$. We will use the same convention for complex Gaussian scalars.
\( \mathcal{X} \). The definition of this complexity is subtle, and is measured in terms of the Talagrand-\( \gamma_2 \) functional \(^{[48]}\) for these sets relative to two different distance metrics. Given a set \( \mathcal{S} \) and a distance defined by a norm \( \| \cdot \| \), the \( \gamma_2 \) functional quantifies how well \( \mathcal{S} \) can be approximated at different scales. In general, the \( \gamma_\beta \) functionals are defined as follows (see \(^{[48]}\) for detailed discussion).

**Definition 1.** For a metric space \(( \mathcal{S}, \| \cdot \| )\), an admissible sequence of \( \mathcal{S} \) is a collection of subsets \( \{ \mathcal{S}_n : n \geq 0 \} \), such that for every \( n \geq 1 \), \( |\mathcal{S}_n| \leq 2^n \) and \( |\mathcal{S}_0| = 1 \). The \( \gamma_\beta \) functional is

\[
\gamma_\beta(\mathcal{S}, \| \cdot \| ) = \inf \sup_{Z \in \mathcal{S}} \sum_{n=0}^{\infty} 2^{n/\beta} \min_{W \in \mathcal{S}_n} \| Z - W \|,
\]

where the infimum is taken with respect to all admissible sequences of \( \mathcal{S} \).

As we will discuss further in Section \(^{[3]}\) below, the \( \gamma_2 \)-functional can be directly related to the rate at which the size of the best \( \epsilon \)-cover of the set \( \mathcal{S} \) grows as \( \epsilon \) decreases. Although this is a purely geometric characteristic of \( \mathcal{S} \), the \( \gamma_2 \) functional gives a tight bound on the supremum of a Gaussian process. For example, if \( \mathbf{G} \) is an \( M_1 \times M_2 \) random matrix whose entries are independent and distributed Normal(0, 1), then \( \sup_{S \in \mathcal{S}} \langle \mathbf{S}, \mathbf{G} \rangle \sim \gamma_2(\mathcal{S}, \| \cdot \|_F) \).

Along with the \( \gamma_2 \), the other geometrical quantity that appears in our results is the diameter of the set \( \mathcal{S} \) with respect to a specified norm, \( D(\mathcal{S}, \| \cdot \| ) = \sup_{Z \in \mathcal{S}} \| Z \| \).

Along with the Frobenius (sum-of-squares) norm \( \| \cdot \|_F \) and the spectral (largest singular value) norm \( \| \cdot \|_2 \), we will measure distances using a “random norm” on \( \mathcal{Y} \) and \( \mathcal{Q} \). This norm is defined relative to a matrix \( \mathbf{G} \) with \( L \) columns, whose entries we will eventually choose independently from a standard complex Gaussian distribution. For an \( M \times R \) matrix \( \mathbf{Y} \), we define

\[
\| \mathbf{Y} \|_{2,G} = \max_{1 \leq \ell \leq L} \left( \sum_{r=1}^{R} |(g_{\ell r}, y_{r})|^2 \right)^{1/2} = \max_{1 \leq \ell \leq L} \| \mathbf{Y}^* g_\ell \|_2, \tag{9}
\]

where the \( y_r \) are the columns of \( \mathbf{Y} \), and \( g_\ell \in \mathbb{C}^M \) is the \( \ell \)th column of \( \mathbf{G} \).

With \( \mathbf{G} \) drawn at random, these norms are random variables, and so in turn the quantity \( \gamma_2(\mathcal{Y}, \| \cdot \|_{2,G}) \) is also a random variable. We will quantify their size using the Orlicz \( \psi_2 \) norm:

\[
\| X \|_{\psi_2} = \inf \left\{ c > 0 \mid \mathbb{E}[e^{-|X|^2/c^2} - 1] \leq 1 \right\},
\]

Note that a random variable \( X \) is subgaussian if and only if \( \| X \|_{\psi_2} < \infty \).

The machinery is now in place to state our main results. We start with a statement about bounding the maximum deviation of the empirical chaos process in \(^{[5]}\) from its mean over a subset of rank-\( R \) matrices having the form \(^{[7]}\).

**Theorem 1 (Tail bounds for Empirical Chaos Processes).** Let \( \mathcal{X} = \mathcal{Y} \cap \mathcal{Q} \) as in \(^{[7]}\), and let \( Z(\mathbf{X}) \) be the random process defined in \(^{[8]}\). Then there exist universal constants \( C_1, C_2 \) such that for every \( t > 0 \),

\[
P \left( \sup_{\mathbf{X} \in \mathcal{X}} |Z(\mathbf{X}) - \mathbb{E}[Z(\mathbf{X})]| > C_1 E + t \right) \leq 2 \exp \left( -C_2 \min \left\{ \frac{t^2}{V^2}, \frac{t}{U / W^{1/2}} \right\} \right), \tag{10}
\]

where

\[
E = \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2)}{\sqrt{L}} \left( \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2)}{\sqrt{L}} + D(\mathcal{X}, \| \cdot \|_F) \right),
\]

\[
V = \frac{1}{L} D(\mathcal{X}, \| \cdot \|_F) \left( \|\gamma_2(\mathcal{Y}, \| \cdot \|_{2,G})\|_{\psi_2} + \|\gamma_2(\mathcal{Q}, \| \cdot \|_{2,G})\|_{\psi_2} + D(\mathcal{X}, \| \cdot \|_2) \right),
\]

\[
U = \frac{1}{L} \left( \|\gamma_2(\mathcal{Y}, \| \cdot \|_{2,G})\|_{\psi_2}^2 + \|\gamma_2(\mathcal{Q}, \| \cdot \|_{2,G})\|_{\psi_2}^2 + D(\mathcal{X}, \| \cdot \|_2)^2 \right) + \sqrt{W} D(\mathcal{X}, \| \cdot \|_F),
\]

\[\]
Our embedding result for simultaneously sparse and low rank matrices come by applying Theorem 1 to
\[
\mathcal{X}_{R,K,N} = \mathcal{X}_{K,N} = \{ X \in \mathbb{C}^{M_1 \times M_2} : X = YQ^*, \ Y \in \mathcal{Y}_K, \ Q \in \mathcal{Q}_N \},
\]
with
\[
\mathcal{Y}_K = \left\{ Y \in \mathbb{C}^{M_1 \times R} : \|Y\|_{0,2} \leq K, \ Y^*Y = \text{diag}([\omega_r]), \ \omega_r \geq 0, \ \sum_{r=1}^R \omega_r^4 \leq 1 \right\},
\]
\[
\mathcal{Q}_N = \left\{ Q \in \mathbb{C}^{M_2 \times R} : \|Q\|_{0,2} \leq N, \ Q^*Q = \text{diag}([\tau_r]), \ \tau_r \geq 0, \ \sum_{r=1}^R \tau_r^4 \leq 1 \right\},
\]
where \( \| \cdot \|_{0,2} \) is the number of rows with at least one non-zero entry. A given \( X = YQ^* \) in this set can be refactored into its SVD, \( X = U\Sigma V^* \), where \( U \) has at most \( K \) active rows, \( V \) has at most \( N \) active rows, and \( \|X\|_F^2 = \|\Sigma\|_F^2 = \sum_r \omega_r^4 \leq 1 \). \( X \) itself has rank at most \( R \) and has at most \( KN \) non-zero entries. Ideally, we should be able to recover \( X \) from \( \sim R(K + N) \) measurements, perhaps within a logarithmic factor. For measurements \( \langle X, A_l \rangle \) with the entries of \( A_l \) chosen independently from a standard Gaussian distribution, this is indeed the case \cite{10}. The result we present here for structured random measurements of the form \eqref{11} has slightly sub-optimal scaling in the rank \( R \): we show that we can recover \( X \) from \( \sim R^{3/2}(K + N) \) measurements, to within logarithmic factors.

To apply Theorem 1 to the set of SSLR matrices, we need to bound the associated geometrical quantities. It is clear that \( D(\mathcal{X}_{K,N}, \| \cdot \|_F) = 1 \) and \( D(\mathcal{X}_{K,N}, \| \cdot \|_2) = 1 \). In Section 4 we prove the following:
\[
\gamma_2(\mathcal{X}_{K,N}, \| \cdot \|_2) \lesssim \sqrt{R(K + N) + K \log(M_1/K) + N \log(M_2/N)}, \sup_{X \in \mathcal{X}_{K,N}} \|X\|_{2,G}^2 \lesssim \log(LR),
\]
\[
\|\gamma_2(\mathcal{Y}_K, \| \cdot \|_2,G)\|_{\psi_2} \lesssim \frac{R^{3/4}}{\sqrt{L}} \left[ \sqrt{K\alpha_1} + N\alpha_2 \right], \|\gamma_2(\mathcal{Q}_N, \| \cdot \|_2,G)\|_{\psi_2} \lesssim \frac{R^{3/4}}{\sqrt{L}} \left[ \sqrt{K\alpha_1} + N\alpha_2 \right],
\]
where we have made the following substitutions to ease notation:
\[
\alpha_1 = \log(M_1 R) \log^2 K \log \log(LM_1), \ \alpha_2 = \log(M_2 R) \log^2 N \log L \log(LM_2).
\]

Bounds on the quantities in Theorem 1 follow immediately:
\[
E \lesssim \frac{R}{L} (K \log(M_1/K) + N \log(M_2/N)),
\]
\[
V \lesssim \frac{R^{3/4}}{\sqrt{L}} \left( \sqrt{K\alpha_1} + N\alpha_2 \right),
\]
\[
U \lesssim \frac{R^{3/2}}{L} (K\alpha_1 + N\alpha_2) + \frac{\log(LR)}{L},
\]
\[
W \lesssim \frac{\log(LR)}{L}.
\]

Combining these bounds with Theorem 1 gives us conditions under which the linear map in \eqref{11} is a restricted isometry (with high probability) for matrices that are simultaneously sparse and low rank. For the probability on the right-hand side of \eqref{11} to be at most \( \epsilon \), we need to take
\[
t \geq \max \left\{ V \frac{\log(2/\epsilon)}{C_2}, \ U \frac{\log(2/\epsilon)}{C_2}, \ W \left( \frac{\log(2/\epsilon)}{C_2} \right)^2 \right\}.
\]
As we also want \( t < 1 \) (so that \( \delta = C_1 t < 1 \) in \eqref{6}), the term involving \( V \) above will dominate the expression once we put aside the constants.

\footnote{We are dropping the \( R \) from the descriptors of the sets at this point to ease the notation.}
Theorem 2 (SSLR $\ell_2$-RIP). Let $A : \mathbb{R}^{M_1 \times M_2} \to \mathbb{R}^L$ be the linear map defined in (1) with $b_\ell, c_\ell$ as in (8), and let $\alpha_1, \alpha_2$ be defined as in (14). For a fixed $\delta \in (0,1)$, if
\[
L \geq C \delta^{-2} R^{3/2} \max(K\alpha_1, N\alpha_2) \log(1/\epsilon),
\]
then with probability at least $1 - \epsilon$, the linear map $A$ satisfies the restricted isometry property (3) for $X_{K,N}$ with $\delta_A \leq \delta$. The constant $C$ is universal.

In [10], a restricted isometry for $(R,K,N)$ SSLR matrices is derived in the case where $A(\cdot)$ is a random projection. In this case, the number of measurements of $X$ scaled as $L \gtrsim R(K+N)$ to within logarithmic factors. Theorem [4] shows that for the more structured measurements of type (4), we have the same scaling in the sparsity parameters $K, N$, but worse scaling with the rank $R$. The result, however, is interesting even when the rank $R = 1$, as this corresponds to solving a system of bilinear equations.

Theorem 2 coupled with the sparse power factorization algorithm, gives us recovery guarantees for recovering SSLR matrices from measurements against random rank-1 measurement matrices. As discussed above, this is immediately related to our ability to solve certain structured bilinear inverse problems. We state the implications for blind deconvolution as a corollary. It states that if we observe the convolution of two vectors $L$ which are sparse in generic bases of $\mathbb{C}^L$, then we can recover the individual components when the sparsity is well-conditioned in the ambient dimension $L$.

Corollary 1 (Blind Deconvolution of Generic Vectors). Let $B \in \mathbb{C}^{L \times L}$, and $C \in \mathbb{C}^{L \times L}$ be Gaussian random matrices and set $\hat{s} = Bx$, and $\hat{h} = Cq$ for coefficients $x \in \mathbb{C}^L$, $\|x\|_0 \leq K$, and $q \in \mathbb{C}^L$, $\|q\|_0 \leq N$. If $x$ and $q$ satisfy the incoherence condition
\[
\|x\|_\infty \cdot \|q\|_\infty \geq 3.97\delta + (\sqrt{2} - 0.97),
\]
for some $\delta \in (0,1)$ and
\[
K + N \leq C_\epsilon \delta^2 \cdot \frac{L}{\log^5(L) \log(1/\epsilon)},
\]
for some $0 < \epsilon < 1$, then the sparse-power-factorization algorithm recovers $\hat{s}$, and $\hat{h}$ exactly (to within a global scalar multiple) from $y = \hat{s} \circ \hat{h}$ with probability at least $\epsilon$. The constant $C_\epsilon$ above depends only on $\epsilon$.

Proof of Corollary 1 is a straightforward combination of Theorem 2 and the guarantees from [10] for the SPF algorithm which we will review in the next section.

2 Recovery Framework

Theorem 2 proves that the ensembles of random rank-1 measurement matrices form an invertible and a well-conditioned embedding for low-rank and sparse matrices. However, the recovery of the true solution is related to our ability to effectively extract it from the measurements. As discussed in the introduction, the naïve convexification approach of recovering simultaneous structures using mixed norms has been shown in [3] to have fundamental limitations, and one of the few non-convex approaches with preliminary recovery guarantees is the so-called sparse power factorization algorithm, which is adapted from [49] and was studied in detail in [10]. In this section, we present a brief review of the algorithm for the case of sparse and rank-1 matrices. However, the algorithm can be generalized to sparse matrices of larger rank, see [10] for details.

That is, this section will consider measurements of the form
\[
y = A(hm^*) + n,
\]
where $A$ is a known linear operator acting on matrix space and $n$ is a noise vector. From such measurements, SPF aims to estimate $h$ and $m$ via alternating minimization, that is, one alternates between updating $h$ and $m$ while keeping the respective other input fixed.
More precisely, given some initialization \( \mathbf{m}_0 \), one employs a combination of a least squares program and hard thresholding pursuit (HTP) \(^{50}\) for an update of \( \mathbf{h} \), that is, one minimizes the cost function \( \| \mathbf{y} - \mathbf{A}(\mathbf{ht}_0) \|_2^2 \) with respect to \( \mathbf{h} \) while at the same time promoting sparsity. The output \( \mathbf{h}_0 \) is then employed to compute an updated vector \( \mathbf{m}_1 \) by minimizing the objective \( \| \mathbf{y} - \mathbf{A}(\mathbf{h}_0 \mathbf{m}^*) \|_2^2 \) with respect to \( \mathbf{y} \), again hard thresholding pursuit is employed to enforce sparsity. From \( \mathbf{m}_1 \), update \( \mathbf{h}_0 \) as before and iterate.

A main result of \(^{10}\) is that provided \( \mathcal{A} \) has a restricted isometry property in the sense of \(^{3}\) for the set \( \mathcal{X}_{2,3K,3N} \) of simultaneously sparse and rank 2 matrices (cf. Section \(^1\)), one obtains local convergence of SPF. Thus the problem boils down to finding good initializations \( \mathbf{m}_0 \). In \(^{10}\) the so-called thresholding initialization is proposed, which we will discuss in the following paragraph. For the remainder of this section after that, we will present a global recovery guarantee for thresholding initialization and very peaky signals also derived in \(^{10}\).

**Thresholding Initialization**

Denote by \( \| \cdot \|_{0,2} \), \( \| \cdot \|_{\infty,0} \) the number of non-zero rows in a matrix and the maximum row support size of an \( M \times M \) matrix, respectively. That is,

\[
\| \mathbf{Z} \|_{0,2} := \| (\| \mathbf{Z}^{(j)} \|_2)_{j=1}^L \|_0 \quad \text{and} \quad \| \mathbf{Z} \|_{\infty,0} := \max_{j \in \{1, \ldots, M\}} \| \mathbf{Z}^{(j)} \|_0,
\]

where \( \mathbf{Z}^{(j)} \) denotes the \( j \)th row of \( \mathbf{Z} \). Now define a set

\[
\mathcal{Z} := \{ \mathbf{Z} \in \mathbb{C}^M \times M : \| \mathbf{Z} \|_{0,2} \leq K, \| \mathbf{Z} \|_{\infty,0} \leq N \},
\]

and let \( \mathcal{P}_\mathcal{Z} \) denote the orthogonal projection onto the set \( \mathcal{Z} \). The thresholding initialization as presented in \(^{10}\) is now obtained by first computing the projection \( \mathcal{P}_\mathcal{Z}(\mathcal{A}^*(\mathbf{y})) \). To this end, define the \( S \)-support norm of \( \mathbf{x} \in \mathbb{C}^L \) as

\[
\| \mathbf{x} \|_{(S)}^2 = \sum_{s=1}^S (|x|^s)^2,
\]

where \( |x|^s \) denotes the \( s \)th largest element of the \( |x| \). That is, the \( S \)-support norm is determined by the largest (in absolute value) entries in \( \mathbf{x} \). To compute the projection \( \mathcal{P}_\mathcal{Z}(\mathcal{A}^*(\mathbf{y})) \), find \( K \) rows with largest \( N \)-support norms and set the rest of the rows to zero. Then replace each of the selected rows by its best \( N \)-support approximation to obtain \( \mathcal{P}_\mathcal{Z}(\mathcal{A}^*(\mathbf{y})) \). The projection can be computed as the solution of the following least-squares program

\[
\mathcal{P}_\mathcal{Z}(\mathcal{A}^*(\mathbf{y})) := \arg\inf_{\mathbf{Z} \in \mathcal{Z}} \| \mathcal{A}^*(\mathbf{y}) - \mathbf{Z} \|_F.
\]

Now determine the \( N \) columns of \( \mathcal{P}_\mathcal{Z}(\mathcal{A}^*(\mathbf{y})) \) with largest \( \ell_2 \) norms and set the rest of the columns to zero. As the initialization \( \mathbf{m}_0 \) for the input vector \( \mathbf{m} \) choose the leading right singular vector of the resulting matrix.

**Global recovery guarantees**

Let us denote the limits of the iterations by \( \mathbf{h} \) and \( \hat{\mathbf{m}} \) (for details on the superlinear rate of convergence, see \(^{10}\)) , and \( \nu := \| \mathbf{n} \|_2 \). Then under the assumption that the linear map \( \mathcal{A} \) obeys an RIP with constant \( \delta \leq 0.04 \) for \( (3K,3N) \) sparse, rank-2 matrices in the sense of \(^{3}\) above, it is shown in \(^{10}\) that if the signal-to-noise ratio (SNR) \( \| \mathbf{y} \|_2 / \| \mathbf{n} \|_2 \) is bounded by 0.04 and the vectors \( \mathbf{h} \) and \( \mathbf{m} \) are very peaky, i.e.,

\[
\| \mathbf{h} \|_\infty \geq 0.78 \| \mathbf{h} \|_2 \text{ and } \| \mathbf{m} \|_\infty \geq 0.78 \| \mathbf{m} \|_2,
\]

then the error scales inversely with the SNR, that is,

\[
\| \hat{\mathbf{h}} \mathbf{m}^* - \mathbf{h}_m^* \|_F \leq C \| \mathbf{n} \|_2 / \| \mathbf{y} \|_2,
\]

where \( C \) is an absolute constant.

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The peakiness condition in (17) may seem quite strict but somewhat peaky vectors occur naturally in certain contexts; for example, in the context of blind estimation, the channel vector $\mathbf{h}$ may be considered peaky as the power of fixed (line of sight) path is comparable or even bigger than the power of scatterers in certain applications. Some of the studies on the peakiness of channel responses can be found in [51], and the references therein. Furthermore, this peakiness condition can be relaxed if the support of the non-zeros of one of the vectors is known — this case is of interest in blind channel estimation with randomly coded messages; for more details, see [9]. Algorithm initialization in this case reduces to finding the leading right singular vector of $\Pi_J \mathbf{A}^*(\mathbf{y})$, where $\Pi_J$ is the projection operator that sets those rows in the matrix $\mathbf{A}^*(\mathbf{y})$ to zero that are not included in the index set $J$. The set $J$ can be found in a computationally tractable way as the solution of the maximization problem

$$
\hat{J} = \arg\max_{|J|=K} \|\Pi_J \mathbf{A}^*(\mathbf{y})\|_F,
$$

where $K$ is the sparsity of the vector with unknown support. Instead of condition (17) one then needs $\|\mathbf{h}\|_\infty > 0.4 \|\mathbf{h}\|_2$.

3 Proof of Theorem 1

Noting that,

$$
\mathbb{E} \frac{1}{L} \sum_{\ell=1}^L |b^*_\ell \mathbf{X} c_\ell|^2 = \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_b (b^*_\ell \mathbf{X} (\mathbf{c}_\ell \mathbf{C}_\ell^*) \mathbf{X}^* b_\ell) = \text{Tr} (\mathbf{X} \mathbf{X}^*) = \|\mathbf{X}\|_F^2,
$$

the process under consideration can be rewritten as

$$
\Gamma := \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L (|b^*_\ell \mathbf{X} c_\ell|^2 - \|\mathbf{X}\|_F^2).
$$

(18)

Our strategy will be to proceed via the moments. As a first step, we apply the following decoupling results for Gaussian chaos processes.

**Proposition 1** ([52]). There exists an absolute constant $C$ such that the following holds. Let $\mathbf{g} = (g_1, \ldots, g_L)$ be a sequence of independent standard normal random variables. If $\mathcal{Q}$ is a collection of square matrices $\mathbf{Q}$, and $\mathbf{g}^\prime$ is an independent copy of $\mathbf{g}$, then

$$
\mathbb{E} \sup_{\mathbf{Q} \in \mathcal{Q}} |\langle \mathbf{g}, \mathbf{Q}\mathbf{g} \rangle - \text{Tr} (\mathbf{Q})|^p \leq C^p \mathbb{E} \sup_{\mathbf{Q} \in \mathcal{Q}} |\langle \mathbf{g}^\prime, \mathbf{Q}\mathbf{g} \rangle|^p,
$$

where $\text{Tr} (\mathbf{Q})$ denotes the trace of matrix $\mathbf{Q}$.

We estimate

$$
\|\Gamma\|_{L_p} = \left\| \sup_{\mathbf{X} \in \mathcal{X}} \left( \frac{1}{L} \sum_{\ell=1}^L |b^*_\ell \mathbf{X} c_\ell|^2 - \|\mathbf{X}\|_F^2 \right) \right\|_{L_p} \leq \left( \mathbb{E} \sup_{\mathbf{X} \in \mathcal{X}} \left( \frac{1}{L} \sum_{\ell=1}^L |b^*_\ell \mathbf{X} c_\ell|^2 - \|\mathbf{X}\|_F^2 \right)^{p/2} \right)^{1/p} + \left( \mathbb{E} \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L \|\mathbf{X} c_\ell\|_2^2 - \|\mathbf{X}\|_F^2 \right)^{1/p}
$$

$$
\leq \left( \mathbb{E} \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L |b^*_\ell \mathbf{X} c_\ell|^2 - \|\mathbf{X}\|_F^2 \right)^{1/p} + \left( \mathbb{E} \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L \|\mathbf{X} c_\ell\|_2^2 - \|\mathbf{X}\|_F^2 \right)^{1/p}
$$

$$
\leq \left( \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L |b^*_\ell \mathbf{X} c_\ell c_\ell^* \mathbf{X}^* b_\ell| \right)_{L_p} + \left( \sup_{\mathbf{X} \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^L c_\ell^* \mathbf{X} \mathbf{X}^* c_\ell^* \right)_{L_p}
$$

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where the first inequality is the result of triangle inequality, the second inequality follows from an application of Proposition 1 on Gaussian vectors \( \{b_t\} \), and \( \{c_t\} \), respectively, in the first term, and on \( \{c_t\} \) in the second term. The vectors \( \{b_t\}_{t=1}^L \) and \( \{c_t\}_{t=1}^L \) are independent copies of the vectors \( \{b_t\}_{t=1}^L \) and \( \{c_t\}_{t=1}^L \), respectively. Now the first term on the right hand side above can be split as follows

\[
\left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{t=1}^L b_t^* X c_t^* b_t^* b_t \right\|_{L^p} \leq \left( \mathbb{E}b \mathbb{E}c \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L (c_t^* X b_t^* b_t^* X c_t - b_t^* X b_t^* b_t^* X c_t) \right|^p \right)^{1/p} + \left( \mathbb{E} \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L b_t^* X b_t^* b_t \right|^p \right)^{1/p}
\]

\[
\leq \mathbb{E} \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L c_t^* X b_t^* b_t^* X c_t \right| + \mathbb{E} \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L b_t^* X b_t^* b_t \right|
\]

where the last inequality again follows by an application of Proposition 1. Putting it all together and noting that \( b_t \) and \( c_t \) are identically distributed, we obtain

\[
\|\Gamma\|_{L^p} \lesssim \mathbb{E} \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L c_t^* X b_t^* b_t^* X c_t \right| + 2 \mathbb{E} \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L b_t^* X b_t^* b_t \right|
\]

(19)

A bound for the second summand follows from the following lemma, which is a consequence of (6):

**Lemma 1.** Let \( b_t, 1 \leq \ell \leq L \) be distributed as in (5), and \( b_t^o \) be independent copy of \( b_t \). Let \( \mathcal{X} \subset \mathbb{R}^{M \times M} \) be a set matrices as defined in (7) equipped with distance metric \( \| \cdot \|_F \). Then

\[
\left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{t=1}^L b_t^o X b_t^o \right| \right\|_{L^p} \lesssim \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2)}{\sqrt{L}} \left( \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2) + d_F(\mathcal{X})}{\sqrt{L}} + \sqrt{\frac{p}{L} d_2(\mathcal{X})} + \frac{p}{L} d_2^2(\mathcal{X}) \right).
\]

**Proof.** Let \( \xi \) and \( \xi^o \) be length \( LM \) vectors formed by stacking \( g_t \in \mathbb{C}^M, \ell \in [L] \), and \( g_t^o \in \mathbb{C}^M, \ell \in [L] \), respectively. Denote \( H_X = X \otimes I_L \) to be a \( LM \times LM \) block-diagonal matrix formed by stacking \( M \times M \) matrices \( X \) along the diagonal. Define \( \mathcal{H} := \{H_X : X \in \mathcal{X}\} \) to be the set of such matrices. Then we want to control the quantity

\[
\sup_{H_X \in \mathcal{H}} \langle H_X \xi, H_X \xi^o \rangle,
\]

and it follows by the direct application of Theorem 3.5 in (6) (noting that \( \sqrt{pd_2(\mathcal{H})} \gamma_2(\mathcal{H}, \| \cdot \|_2) \lesssim \gamma_2(\mathcal{H}, \| \cdot \|_2)^2 + pd_2^2(\mathcal{H}) \)) that

\[
\sup_{H_X \in \mathcal{H}} \langle H_X \xi, H_X \xi^o \rangle \lesssim \gamma_2(\mathcal{H}, \| \cdot \|_2) (\gamma_2(\mathcal{H}, \| \cdot \|_2) + d_F(\mathcal{H}) + \sqrt{pd_2(\mathcal{H})} + \frac{p}{L} d_2(\mathcal{X}) + \frac{p}{L} d_2^2(\mathcal{X}) \),
\]

(20)

where \( d_F(\mathcal{H}) \) and \( d_2(\mathcal{H}) \) denote the diameter of the space \( \mathcal{H} \) in the Frobenius and the operator norms, respectively. Since the matrix \( H_X \) is a formed by stacking \( L \) copies of the matrix \( X \) along the diagonal, this implies that for all \( H_X \in \mathcal{H}, \|H_X\|_F = \sqrt{L} \|X\|_F \) and \( \|H_X\|_2 = \|X\|_2 \), and hence

\[
d_F(\mathcal{H}) = \sqrt{L} d_F(\mathcal{X}) \text{ as well as } d_2(\mathcal{H}) = d_2(\mathcal{X}).
\]

Given \( H_X, H_Y \in \mathcal{H} \), the distance metric becomes \( \|H_X - H_Y\|_2 = \|X - Y\|_2 \), which implies that \( \gamma_2(\mathcal{H}, \| \cdot \|_2) = \gamma_2(\mathcal{X}, \| \cdot \|_2) \). Plugging these in (20) completes the proof of the lemma. \( \square \)

A bound for the first summand in (19) is established by the following lemma, which is our main technical contribution.
Lemma 2. Let \( b_\ell \) and \( c_\ell \) be distributed as in \([3]\), and \( b_\ell^* \) and \( c_\ell^* \) be independent copies of \( b_\ell \) and \( c_\ell \), respectively. Let \( \mathcal{X} \subset \mathbb{R}^{M \times M}, \mathcal{Y} \subset \mathbb{R}^{M \times R}, \mathcal{Q} \subset \mathbb{R}^{R \times M} \) such that \( \mathcal{X} = \mathcal{Y} \odot \mathcal{Q} \) is the factorization according to the singular value decomposition. Then

\[
\left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \left| \sum_{\ell=1}^{L} c_\ell^* X^* b_\ell^* b_\ell^* X c_\ell \right| \right\|_{L_p} \\
\leq \frac{1}{\sqrt{L}} \left( \left\| \gamma_2(\mathcal{Y}, \cdot, \cdot, 2, \mathcal{G}) \right\|_{L_p} + \left\| \gamma_2(\mathcal{Q}, \cdot, \cdot, 2, \mathcal{G}) \right\|_{L_p} \right) + \sqrt{p} \sup_{X \in \mathcal{X}} \left\| X \right\|_{2, \mathcal{G}} \left\| \left( \left\| X \right\|_{2, \mathcal{G}} \right) \right\|_{L_p} \sup_{X \in \mathcal{X}} \left\| \sum_{\ell=1}^{L} b_\ell^* X c_\ell \right\|_{L_p}^{1/2}
\]

Proof. Our strategy will be to apply a chaining argument, but to only consider admissible sequences \( X_n \) of \( \mathcal{X} = \mathcal{Y} \odot \mathcal{Q} \), which are products of admissible sequences \( \mathcal{Y}_n \) of \( \mathcal{Y} \) and \( \mathcal{Q}_n \) of \( \mathcal{Q} \). Thus we relate to each \( X = YQ^* \in \mathcal{Y} \odot \mathcal{Q} \), a “closest point” in \( X_n \). which is given by

\[
\pi_n(X) := \pi_n(Y)\pi_n(Q)^*,
\]

where, as common in chaining arguments,

\[
\pi_n(Y) := \operatorname{arg\, min}_{Y \in \mathcal{Y}_n} \| Y - \bar{Y} \|_{2,G}; \quad \pi_n(Q) := \operatorname{arg\, min}_{Q \in \mathcal{Q}_n} \| Q - \bar{Q} \|_{2,G};
\]

the random \( 2, \mathcal{G} \)-norm was defined earlier in \([3]\). Note that because of the product construction, we have that, for \( n \geq 1 \), \( |X_n| = 2^{2^{n+1}} \) in contrast to \( 2^n \) in the standard setup.

Furthermore, we use the notation \( \Delta_n(\cdot) = \pi_{n+1}(\cdot) - \pi_n(\cdot) \) and \( \rho_n = \pi_{n+1}(\cdot) + \pi_n(\cdot) \) to indicate difference and sum of the approximations at adjacent scales \( n + 1 \) and \( n \). For example, for some \( Y \in \mathcal{Y} \)

\[
\Delta_n(Y) = \pi_{n+1}(Y) - \pi_n(Y), \quad \rho_n(Y) = \pi_{n+1}(Y) + \pi_n(Y);
\]

similarly for \( Q \in \mathcal{Q} \).

As the involved matrix spaces are finite dimensional, we may assume w.l.o.g. that \( \mathcal{Y} \) and \( \mathcal{Q} \) are both finite (for example via a covering argument). That is, for some \( n_0 \), \( \mathcal{Y} \) or \( \mathcal{Q} \) \( \leq 2^{2^{n_0+1}} \). Fix \( p \geq 1 \), let \( k \) be the largest integer such that \( 2^k \leq p \) and \( k < n_0 \). Thus for every \( X = YQ^* \in \mathcal{Y} \odot \mathcal{Q} \), one has \( \pi_{n_0}(X) = X \) and a telescoping sum argument together with the triangle inequality yields

\[
\left| \sum_{\ell=1}^{L} b_\ell^* X c_\ell c_\ell^* X^* b_\ell \right| \leq \sum_{n=k}^{n_0-1} \sum_{\ell=1}^{L} \left| (b_\ell^* \pi_{n+1}(X)c_\ell c_\ell^* \pi_{n+1}(X)^* b_\ell^* - b_\ell^* \pi_n(X)c_\ell c_\ell^* \pi_n(X)^* b_\ell^*) \right| + \sum_{\ell=1}^{L} b_\ell^* \pi_k(X)c_\ell c_\ell^* \pi_k(X)^* b_\ell^* \right|.
\]

Using binomial formulas and the fact that \( \pi_n(X) = \pi_n(Y)(\pi_n(Q))^* \), it follows that the increments in the first sum can be rewritten as

\[
b_\ell^* \pi_{n+1}(X)c_\ell c_\ell^* \pi_{n+1}(X)^* b_\ell^* - b_\ell^* \pi_n(X)c_\ell c_\ell^* \pi_n(X)^* b_\ell^*
\]

\[
= \frac{1}{2} \left( b_\ell^* X_{1,n} c_\ell c_\ell^* X_{1,n}^* b_\ell^* + b_\ell^* X_{2,n} c_\ell c_\ell^* X_{2,n}^* b_\ell^* + b_\ell^* X_{3,n} c_\ell c_\ell^* X_{3,n}^* b_\ell^* + b_\ell^* X_{4,n} c_\ell c_\ell^* X_{4,n}^* b_\ell^* \right),
\]

where \( X_{1,n} = (\rho_n Y)(\pi_{n+1} Q^*) \), \( X_{2,n} = (\Delta_n Y)(\pi_{n+1} Q^*) \), \( X_{3,n} = (\pi_n Y)(\rho_n Q^*) \), and \( X_{4,n} = (\pi_n Y)(\Delta_n Q^*) \).

We obtain, again using the triangle inequality,

\[
\left| \sum_{\ell=1}^{L} (b_\ell^* X c_\ell c_\ell^* X^* b_\ell^* - b_\ell^* \pi_k(X)c_\ell c_\ell^* \pi_k(X)^* b_\ell^*) \right| \leq
\]

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As $b_\ell, c_\ell$ and $b'_\ell, c'_\ell$ are independent and identically distributed, one has
\[
\left\| \sum_{n=k}^{n_0-1} \sum_{\ell=1}^{L} b_\ell X_{1,n} c_\ell c'_\ell X_{2,n} b'_\ell \right\|_{L_p} = \left\| \sum_{n=k}^{n_0-1} \sum_{\ell=1}^{L} b'_\ell X_{2,n} c_\ell c'_\ell X_{1,n} b_\ell \right\|_{L_p},
\]
and it follows that $\|\Lambda_1\|_{L_p} \lesssim \left| \sum_{n=k}^{n_0-1} \sum_{\ell=1}^{L} b_\ell X_{1,n} c_\ell c'_\ell X_{2,n} b'_\ell \right|_{L_p}$.

Thus, to estimate $\Lambda_1$, it suffices to bound
\[
\sum_{n=k}^{n_0-1} \sum_{\ell=1}^{L} b_\ell X_{1,n} c_\ell c'_\ell X_{2,n} b'_\ell.
\]
Since $c'_\ell$ is a Gaussian vector with independent entries each of variance 1, rotation invariance implies that
\[
\left| \sum_{\ell} c'_\ell X_{2,n} b'_\ell b_\ell X_{1,n} c_\ell \right| \leq u \left( \sum_{\ell=1}^{L} \|X_{2,n} b'_\ell b_\ell X_{1,n} c_\ell\|^2 \right)^{1/2}
\]
holds, conditionally on $b_\ell, b'_\ell$ and $c_\ell$, with probability at least $1 - 2 \exp(-u^2/2)$ for $u \geq c_1$, where $c_1$ is an absolute constant. Further note that $\max_{r} \left( \sum_{\ell=1}^{R} \|b'_\ell, Y^{(r)}\|^2 \right)^{1/2}$ is identically distributed as the random norm $\|Y\|_{2,\mathcal{G}}$ introduced above. In the spirit of this observation, we write $\|Y\|_{2,B'} := \max_{r} \left( \sum_{\ell=1}^{R} \|b'_\ell, Y^{(r)}\|^2 \right)^{1/2}$.

Hence, observing that, as all $Q \in \mathcal{Q}$, and in particular $\|\pi_{n+1}(Q)\| \leq 1$, are isometries,
\[
\sum_{\ell=1}^{L} \|X_{2,n} b'_\ell b_\ell X_{1,n} c_\ell\|^2 \leq \|\Delta_n(Y)\|_{2,B'} \cdot \sum_{\ell=1}^{L} |b'_\ell| \sum_{r=1}^{R} (\Delta_n(Y))_{r} |b'_\ell, Y^{(r)}| c_\ell |^2
\]
The last inequality uses the fact that
\[
\|b'_\ell, \Delta_n(Y)\|_{2} = \left( \sum_{r=1}^{R} (\Delta_n(Y))_{r} \right)^{1/2} \leq \|\Delta_n(Y)\|_{2,B'},
\]
where $(\Delta_n(Y))_{r}$ specifies the $r$th column of $\Delta_n(Y)$. In summary, one has with probability at least $1 - 2 \exp(-u^2/2)$, conditional on $B, B'$, and $C$,
\[
\left| \sum_{\ell} c'_\ell X_{2,n} b'_\ell b_\ell X_{1,n} c_\ell \right| \leq u \|\Delta_n(Y)\|_{2,B'} \left( \sum_{\ell=1}^{L} |b'_\ell, \pi_{n+1}(Q) c_\ell|^2 \right)^{1/2}
\]
We now consider for $t > 0$, conditional on $B, B'$ and $C$, the event $\delta'$ on which the above conclusion holds for each level $k \leq n \leq n_0$ with $u = u(n) = t 2^{n/2}$ uniformly for all choices of $(\pi_{n+1}(Q))^*$. To estimate the number of choices for a fixed level $n$, note that $|\{\rho_{n}(Y) : Y \in Y\}| \leq |Y_n \times Y_n| = 2^{2^n}$. Consequently, the number of choices is bounded by $|Y_n \times Y_{n+1} \times \mathcal{Q}_{n+1}| \leq 2^{3 \cdot 2^{n}} \cdot 2^{2^{n+1}} \cdot 2^{2^{n+2}}$. One obtains
\[
\left| \sum_{\ell} c'_\ell X_{2,n} b'_\ell b_\ell X_{1,n} c_\ell \right| \lesssim t^{2^n/2} \|\Delta_n(Y)\|_{2,B'} \left( \sum_{\ell=1}^{L} |b'_\ell, \pi_{n+1}(Q) c_\ell|^2 \right)^{1/2}.
\]
This produces the following conclusion on the event $\mathcal{E}$

$$
\sum_{n=k}^{n_0-1} \sum_{t} \left| \frac{c'_t X_{2,n}^* b'_t b'_t X_{1,n} c_t}{t} \right| \leq t \sum_{n=k}^{n_0-1} 2^{n/2} \| \Delta_n(Y) \|_{2,B'} \left( \sum_{\ell=1}^{L} |b'_t(\rho_n Y)(\pi_{n+1} Q^*) c_t|^2 \right)^{1/2} .
$$

(24)

The probability of $\mathcal{E}^c$ can be estimated using a union bound over all levels and, for each level, all (at most) $2^{2^{n+3}}$ choices of $(\rho_n Y)(\pi_{n+1} Q^*)$. One obtains

$$
P(\mathcal{E}^c | B, B', C) \leq \sum_{n=k}^{n_0-1} 2 \exp(-t^2 2^{n-1} + 2^{n+3}).
$$

Then for $t > 4$, we have

$$
P(\mathcal{E}^c | B, B', C) \leq 2 \sum_{n=k}^{n_0-1} \exp(-2^n t^2/4) \leq 4 \exp(-2^t t^2/4).
$$

(25)

The idea of tightening the chaining bound by first using a union bound over a level that depends on the moment one is calculating is due to [33]. We now continue estimating the last factor in (22), maximizing over $Y$ and $Q$:

$$
\sup_{Y \in Y'} \sup_{Q \in Q} \left| \sum_{n=1}^{n_0-1} |b'_t(\rho_n Y)(\pi_{n+1} Q^*) c_t|^2 \right| \leq \sup_{Y \in Y'} \sup_{Q \in Q} \left| \sum_{n=1}^{n_0-1} |b'_t \pi_{n+1}(Y)(\pi_{n+1} Q^*) c_t|^2 \right| + \sup_{Y \in Y'} \sup_{Q \in Q} \left| \sum_{n=1}^{n_0-1} |b'_t \pi_n(Y)(\pi_{n+1} Q^*) c_t|^2 \right|
$$

$$
\leq 2 \sup_{Y \in Y'} \sup_{Q \in Q} \left| \sum_{n=1}^{n_0-1} |b'_t \pi_n(Y)(\pi_{n+1} Q^*) c_t|^2 \right| = 2 \sup_{Y \in Y'} \sup_{Q \in Q} \left| \sum_{n=1}^{n_0-1} |b'_t \pi_n(Y)(\pi_{n+1} Q^*) c_t|^2 \right|
$$

This gives

$$
\sum_{n=k}^{n_0-1} \left| \frac{c'_t X_{2,n}^* b'_t b'_t X_{1,n} c_t}{t} \right| \leq t \sup_{Y \in Y'} \sum_{n=k}^{n_0-1} 2^{n/2} \| \Delta_n(Y) \|_{2,B'} \sup_{X \in \mathcal{X}} \left( \sum_{\ell=1}^{L} |c'_t X^* b'_t|^2 \right)^{1/2}
$$

$$
\leq t \sup_{Y \in Y'} \sum_{n=k}^{n_0-1} 2^{n/2} \| \Delta_n(Y) \|_{2,B'} \sup_{X \in \mathcal{X}} \left( \sum_{\ell=1}^{L} |b'_t X c_t|^2 \right)^{1/2},
$$

To compute the $L_p$ norm of $A_1$, we start with evaluating the $p$-th moment with respect to $c'_t$, that is

$$
\mathbb{E}_{c'} A_1^p = \int_0^\infty p t^{p-1} P_{c'}(A_1 > t|b, b', c) dt.
$$

For that, denote

$$
\mathcal{F}(B, B', C) = \sup_{Y \in Y'} \sum_{n=k}^{n_0-1} 2^{n/2} \| \Delta_n(Y) \|_{2,B'} \sup_{X \in \mathcal{X}} \left( \sum_{\ell=1}^{L} |b'_t X c_t|^2 \right)^{1/2},
$$

and observe that for an absolute constant $c_1 > 0$

$$
\int_0^\infty p t^{p-1} P_{c'}(A_1 > t|B, B', C) dt \leq c_1^p \mathcal{F}(B, B', C)^p + \int_{c_1 \mathcal{F}(B, B', C)}^\infty p t^{p-1} P_{c'}(A_1 > t|\{B, B', C\}) dt
$$

$$
\leq 4^p \mathcal{F}(B, B', C)^p + \mathcal{F}(B, B', C)^p \int_4^\infty p u^{p-1} P_{c'}(A_1 > u \mathcal{F}(B, B', C)|\{b_t, b'_t, c_t\}) du
$$

$$
\leq c_1^p \mathcal{F}(B, B', C)^p.
$$

(26)
The last equality follows as follows: First, one has $2^k \leq p$ and hence by (25), for all $t \geq 4$,

$$P\{A_1 \geq tF(B, B', C)|B, B', C\} \leq P\{\delta^n(B, B', C)|B, B', C\} \leq 2 \exp(-c_2 2^k t^2 / 4) \leq 2 \exp(-pt^2 / 4).$$

Moreover, it follows from the standard identity $\int_0^\infty pt^{p-1}e^{-t^2}dt \leq \frac{p}{2} \cdot (\frac{\pi}{2})^{p/2}$ by change of variables that $\int_0^\infty pt^{p-1}e^{-pt^2}dt \leq p$, which yields (26). Hence,

$$\left(\mathbb{E}(\Lambda_n^p|B, B', C)\right)^{1/p} \leq \mathcal{F}(B, B', C) = \sup_{X \in \mathcal{X}} \sum_{n=1}^{n_0} 2^{n/2} \|\Delta_n Y\|_{2, B'} \cdot \sup_{X \in \mathcal{X}} \left(\sum_{\ell=1}^{L} |b^*_\ell X c_\ell|^2\right)^{1/2}$$

and thus, taking the infimum over all admissible sequences as well as invoking $\|\Delta_n Y\|_{2, B'} \leq \|\pi_n(Y) - Y\|_{2, B'} + \|\pi_{n+1}(Y) - Y\|_{2, B'}$,

$$\left(\mathbb{E}(\Lambda_n^p|B, B', C)\right)^{1/p} \leq \gamma_2(\mathcal{Y}, \|\cdot\|_{2, B'}) \cdot \sup_{X \in \mathcal{X}} \left(\sum_{\ell=1}^{L} |b^*_\ell X c_\ell|^2\right)^{1/2}$$

Using independence and that $B'$ is a Gaussian random matrix, we obtain

$$\|\Lambda_1\|_{L_p} = \mathbb{E}||\mathbb{E}(\Lambda_n^p|B, B', C)|\|_{L_p} \cdot \sup_{X \in \mathcal{X}} \left(\sum_{\ell=1}^{L} |b^*_\ell X c_\ell|^2\right)^{1/2} \leq \|\gamma_2(\mathcal{Y}, \|\cdot\|_{2, B'})\|_{L_p} \cdot \sup_{X \in \mathcal{X}} \left(\sum_{\ell=1}^{L} |b^*_\ell X c_\ell|^2\right)^{1/2}$$

where the last inequality uses that in probability spaces, the $L_p$ norm of a random variable is monotonic in $p$.

To bound $\|\Lambda_2\|_{L_p}$, we first establish exactly analogously to the previous case that conditionally on $B, C$, and $C'$

$$\left|\sum_{\ell=1}^{L} b^*_\ell X_{k,n} c'_\ell X'_{k,n} b'_\ell\right| \leq u \|c'_\ell (\Delta_n Q)(\pi_n Y)^*\|_{2} \cdot \left(\sum_{\ell=1}^{L} |b^*_\ell (\pi_n Y)(\rho_n Q)^* c_\ell|^2\right)^{1/2}$$

holds with probability at least $1 - 2 \exp(-u^2 / 2)$ for $u \geq c_1$. By definition of $\mathcal{Y}$, $||\pi_n(Y)|| \leq 1$. To bound the right hand side, observe that

$$\|c'_\ell (\Delta_n Q)(\pi_n Y)^*\|_{2}^2 \leq \|\Delta_n Q\|_{2, C'}^2$$

where the norm $\|\cdot\|_{2, C'}$ is as defined earlier. The remainder of the estimate proceeds exactly as for $\Lambda_1$, and we obtain

$$\|\Lambda_2\|_{L_p} \leq \|\gamma_2(\mathcal{Q}, \|\cdot\|_{2, B'})\|_{L_p} \cdot \sup_{X \in \mathcal{X}} \left(\sum_{\ell=1}^{L} |b^*_\ell X c_\ell|^2\right)^{1/2}$$

(31)

It remains to bound the last term in (21). For that, note that $2^{2^k} \leq \exp(p)$,

$$\mathbb{E} \sup_{X \in \mathcal{X}} \left|\sum_{\ell} c'_\ell \pi_k(X^*) b'_\ell b^*_\ell \pi_k(X) c_\ell\right|^p \leq \sum_{X \in \mathcal{A}_k} \mathbb{E} \left|\sum_{\ell} c'_\ell \pi_k(X^*) b'_\ell b^*_\ell \pi_k(X) c_\ell\right|^p \leq \exp(p) \sup_{X \in \mathcal{X}} \mathbb{E} \left|\sum_{\ell} c'_\ell \pi_k(X^*) b'_\ell b^*_\ell \pi_k(X) c_\ell\right|^p$$
This concludes the proof.

Putting the ingredients in (29), (31), and (32) together; normalizing by Corollary 2.

\[
\begin{align*}
&\leq \exp(p) \sup_{X \in \mathcal{X}} \mathbb{E} \left( \mathbb{E} \left[ \left| \sum_{\ell} c^*_\ell \pi_k(X^*) b^*_\ell b^*_\ell \pi_k(X) c_\ell \right|^p \right] |B, B', C, \right) \\
&\leq C^p \tau^p/2 \sup_{X \in \mathcal{X}} \mathbb{E} \left( \left| \sum_{\ell} \pi_k(X^*) b^*_\ell \pi_k(X) c_\ell \right|^2 \right)^{p/2} \\
&\leq C^p \tau^p/2 \sup_{X \in \mathcal{X}} \mathbb{E} \left( \left| X^* b^*_\ell \pi_k(X) c_\ell \right|^2 \right)^{p/2} \\
&\leq C^p \tau^p/2 \sup_{X \in \mathcal{X}} \mathbb{E} \left( \left| X \right|^p_{2,G} \sup_{X \in \mathcal{X}} \mathbb{E} \left( \left| b^*_\ell X c_\ell \right|^2 \right)^{p/2} \right)
\end{align*}
\]

where the fourth inequality uses rotation invariance as well as the moments of a Gaussian random variables, and \(C_1\) is an absolute constant. One obtains

\[
\left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{\ell} b^*_\ell X c_\ell c^*_\ell X^* b^*_\ell \right\|_{L_p} \lesssim \sqrt{\tau} \left( \left\| \gamma_2(\mathcal{Y}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} + \left\| \gamma_2(\mathcal{Q}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} \right) \cdot \left\| \sup_{X \in \mathcal{X} \cap \mathcal{Q}} \frac{1}{L} \sum_{\ell=1}^{L} \left| b^*_\ell X c_\ell \right|^2 \right\|_{L_p}^{1/2}.
\]

Putting the ingredients in (29), (31), and (32) together; normalizing by \(L\), and using (21), we obtain

\[
\left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{\ell} b^*_\ell X c_\ell c^*_\ell X^* b^*_\ell \right\|_{L_p} \lesssim \sqrt{\tau} \left( \left\| \gamma_2(\mathcal{Y}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} + \left\| \gamma_2(\mathcal{Q}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} \right) \cdot \left\| \sup_{X \in \mathcal{X} \cap \mathcal{Q}} \frac{1}{L} \sum_{\ell=1}^{L} \left| b^*_\ell X c_\ell \right|^2 \right\|_{L_p}^{1/2}.
\]

This concludes the proof. \(\square\)

Lemmas 1 and 2 have the following direct corollary, which will in turn imply Theorem 1.

**Corollary 2.** Consider the exact same setup as in Lemma 3 and denote

\[
\begin{align*}
\alpha &:= \frac{1}{\sqrt{L}} \left( \left\| \gamma_2(\mathcal{Y}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} + \left\| \gamma_2(\mathcal{Q}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} \right) \\
\beta &:= \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2)}{\sqrt{L}} \left( \frac{\gamma_2(\mathcal{X}, \| \cdot \|_2)}{\sqrt{L}} + d_F(\mathcal{X}) \right) + \sqrt{\tau} \left( \left\| \gamma_2(\mathcal{Y}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} + \left\| \gamma_2(\mathcal{Q}, \| \cdot \|_2, \mathcal{G}) \right\|_{L_p} \right).
\end{align*}
\]

Then

\[
\left\| \Gamma \right\|_{L_p} \lesssim \alpha^2 + \beta + \alpha d_F(\mathcal{X})
\]

**Proof.** Note that \(\sup_{X \in \mathcal{X}} \sum_{\ell=1}^{L} \left| b^*_\ell X c_\ell \right|^2 \leq \Gamma + \sup_{X \in \mathcal{X}} \left\| X \right\|^2_{p,2}\) and hence \(\sup_{X \in \mathcal{X}} \sum_{\ell=1}^{L} \left| b^*_\ell X c_\ell \right|^2 \leq \left\| \Gamma \right\|_{L_p} + d_F^2(\mathcal{X})\). Combining this observation with (13) as well as Lemmas 1 and 2, we obtain

\[
\left\| \Gamma \right\|_{L_p} \lesssim \left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^{L} c^*_\ell X^* b^*_\ell b^*_\ell X c_\ell \right\|_{L_p} + \left\| \sup_{X \in \mathcal{X}} \frac{1}{L} \sum_{\ell=1}^{L} b^*_\ell X^* b^*_\ell \right\|_{L_p}
\]
\[
\lesssim \alpha \left( \left\| \Gamma \right\|_{L_p} + d_F^2(\mathcal{X}) \right)^{1/2} + \beta.
\]
Inequalities of similar form appear in many related problems and are typically solved in the following way. Note that the inequality is of the form \( G \lesssim \alpha \sqrt{G + \gamma} + \beta \), which can be rewritten as \((G - \beta)^2 \lesssim \alpha^2(G + \gamma)\), which in turn is equivalent to \((G - (\beta + \frac{\alpha^2}{2}))^2 \lesssim \alpha^2 \gamma + 2\beta \frac{\alpha^2}{4} + \frac{\alpha^4}{4} \) and hence implies

\[
G \lesssim \beta + \frac{\alpha^2}{2} + \sqrt{\alpha^2 \gamma + 2\beta \frac{\alpha^2}{4} + \frac{\alpha^4}{4}} \lesssim \beta + \alpha \sqrt{\gamma},
\]

which proves the result.

This corollary will be combined with the following which is a variant of Proposition 7.15 in [1]. The proof proceeds exactly in the same way and is hence omitted.

**Proposition 2.** Suppose \( Z \) is a random variable satisfying

\[
(\mathbb{E}|Z|^p)^{1/p} \leq \alpha + \beta \sqrt{p} + \gamma p^2, \quad \forall p \geq p_0
\]

for some \( \alpha, \beta, \gamma, \delta, \eta \) \( > 0 \). Then, for \( u \geq p_0 \)

\[
P(|Z| \geq e(\alpha + \beta \sqrt{u} + \gamma u^2)) \leq e^{-u}.
\]

We now have all the ingredients to complete the proof of Theorem 1. Noting that both the two random \( \gamma_2 \)-functionals in the statement of Corollary 2 and \( \|X\|_{2,G} \) are subgaussian random variables, we can bound their moments via their \( \psi_2 \) norm, obtaining the bound

\[
\alpha \leq \frac{\|Y\|_{2,G}}{L} \left( \frac{\gamma_2(X, \| \cdot \|_2)}{\sqrt{L}} + d_F(X) \right) + \frac{\|X\|_{2,G}}{L} \sup_{X \in \mathcal{X}} \|X\|_{2,G} \psi_2
\]

and

\[
\alpha^2 \lesssim \frac{\|Y\|_{2,G}}{L} \left( \frac{\gamma_2(X, \| \cdot \|_2)}{\sqrt{L}} + d_F(X) \right) + \frac{\|X\|_{2,G}}{L} \sup_{X \in \mathcal{X}} \|X\|_{2,G} \psi_2^2.
\]

So Corollary 2 yields

\[
\|\Gamma\|_{L_\rho, \psi_2} \lesssim \frac{\|\gamma_2(X, \| \cdot \|_2)\|_2}{\sqrt{L}} \left( \frac{\gamma_2(X, \| \cdot \|_2)}{\sqrt{L}} + d_F(X) \right)
\]

\[+ \frac{\|Y\|_{2,G}}{L} \left( \frac{\gamma_2(X, \| \cdot \|_2)}{\sqrt{L}} + \frac{\gamma_2(Y, \| \cdot \|_2)\psi_2 + \gamma_2(Q, \| \cdot \|_2)\psi_2 + d_2(X)}{\sqrt{L}} \right) \]

\[+ \frac{\|Y\|_{2,G}}{L} \left( \frac{\gamma_2(Y, \| \cdot \|_2)\psi_2 + \gamma_2(Q, \| \cdot \|_2)\psi_2 + d_2(X)}{\sqrt{L}} \right) \]

\[+ \frac{\|X\|_{2,G}}{L} \sup_{X \in \mathcal{X}} \|X\|_{2,G} \psi_2^2.
\]

Now Theorem 1 follows from the direct application of Proposition 2.

\[\square\]

### 4 Proof of Theorem 2

In this section, we prove a series of lemmas that bound the geometric quantities appearing in Theorem 1 for the particular case where \( \mathcal{X} = \mathcal{X}_{K,N} \) is the set of simultaneously sparse and low rank matrices defined in equations (11), (12), and (13).

The \( \gamma_2 \)-functional, which appears for several different sets and norms in the statement of Theorem 1, can be directly related to the covering number of the space under consideration. To make this precise, we need to
Lemma 3.

The covering number $N(S, \| \cdot \|, \epsilon)$ is the size of the smallest $\epsilon$-cover of $S$. We can bound the $\gamma_2$ functional defined above in terms of the covering numbers using Dudley’s integral \[48, 54\]:

\[
\gamma_2(S, \| \cdot \|) \leq c \int_0^{D(S, \| \cdot \|)} \sqrt{\log N(S, \| \cdot \|, \epsilon)} \, d\epsilon,
\]

where $c$ is a known constant, and $D(S, \| \cdot \|)$ is the diameter of $S$: $D(S, \| \cdot \|) = \sup_{Z \in S} \|Z\|$. We will use this approximation in all of our bounds involving $\gamma_2$ functionals below.

**Lemma 4.**

The first inequality follows from the definition of $\gamma_2(S, \| \cdot \|)$ and the fact that $\|X - W\|_2 \leq \|X - W\|_F$ for all matrices $X, W$.

For the second relation, first note that $\|X\|_F \leq 1$ for all $X \in X_{K,N}$, so we can take $D(X_{K,N}, \| \cdot \|_F) = 1$ as the upper limit in the integral \[43\]. To bound the covering number, we employ a volumetric estimate very similar to Lemma 3.1 in \[55\]. In that reference, it is shown that for set of matrices of the form

\[X = \{ X : X = U\Sigma V^*, \quad U \in \mathcal{U}, \quad V \in \mathcal{V}, \quad \Sigma \text{ diagonal}, \quad \|\Sigma\|_F \leq 1\},\]

where all of the matrices in $\mathcal{U} \subset \mathbb{C}^{M_1 \times R}$ and $\mathcal{V} \subset \mathbb{C}^{M_2 \times R}$ have unit-norm columns, we have

\[
N(X, \| \cdot \|_F, \epsilon) \leq \left(1 + \frac{6}{\epsilon}\right)^{KR} \cdot N(U, \| \cdot \|_1, \epsilon/3) \cdot N(V, \| \cdot \|_1, \epsilon/3),
\]

where $\|X\|_{1,2}$ is the maximum $\ell_2$ norm of the columns of $X$. If we take $\mathcal{U}$ and $\mathcal{V}$ to be the sets of matrices with orthonormal columns and no more than $K$ and $N$ non-zero rows, respectively, then $X_{K,N}$ is a subset of the set in \[35\]. Restricted to a fixed support, this $\mathcal{U}$ can be covered to precision $\epsilon/3$ with a size of $(1 + 6/\epsilon)^{KR}$, and so,

\[
N(U, \| \cdot \|_1, \epsilon/3) \leq \left(\frac{M_1}{K}\right)^{(1 + 6/\epsilon)^{KR}}, \quad N(V, \| \cdot \|_1, \epsilon/3) \leq \left(\frac{M_2}{N}\right)^{(1 + 6/\epsilon)^{NR}},
\]

meaning

\[
\log N(X, \| \cdot \|_F, \epsilon) \leq R(N + K + 1) \log \left(1 + \frac{6}{\epsilon}\right) + K \log \left(\frac{M_1\epsilon}{K}\right) + N \log \left(\frac{M_2\epsilon}{N}\right).
\]

Integrating the square-root of the expression above from 0 to 1 yields the result.

**Lemma 4.**

\[
\sup_{X \in X_{K,N}} \|\|X\|_{2,G}\|_{\psi_2} \leq \sqrt{\log(LR)}.
\]

**Proof.** Any $X \in X_{K,N}$ can be written $X = U\Sigma V^*$ with $U$ and $V$ having orthonormal columns and $\Sigma$ being an $R \times R$ diagonal matrix with $\|\Sigma\|_F \leq 1$. We have

\[
\|X\|_{2,G} = \max_{1 \leq \ell \leq L} \|U\Sigma V^* g_\ell\|_2 = \max_{1 \leq \ell \leq L} \|\Sigma V^* g_\ell\|_2 \leq \max_{1 \leq \ell \leq L} \|g_\ell\|_\infty.
\]

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where the $z_t = V^* g_t$ are again independent standard Gaussian random vectors of length $R$. The lemma follows from a well-known bound (see [50, Lemma 2.2.2], for example) on the $\psi_2$ norm of a supremum of a set of independent Gaussian random variables:

$$\left\| \max_{1 \leq t \leq L} \| z_t \|_2 \right\|_{\psi_2} \lesssim \sqrt{\log(LR)}.$$

The most involved part in establishing Theorem 2 is estimating the covering numbers for sets of matrices that are row-sparse using the random norm $\| \cdot \|_{2,G}$. The following lemma, from [57], gives us a method for bounding the covering numbers of a convex set $U$ in an arbitrary norm in cases where $U$ is naturally written as the hull of a small set of vectors.

**Lemma 5** ([57]). There exists an absolute constant $C$ for which the following holds. Let $H$ be a normed space, consider a finite set $U \subset H$, and assume that for every $J \in \mathbb{N}$ and $(U_1, \ldots, U_J) \in U^J$, $\mathbb{E} \| \sum_{j=1}^J \epsilon_j U_j \|_H \leq A\sqrt{J}$, where $(\epsilon_j)_{j=1}^L$ denotes a Rademacher vector. Then for every $u > 0$,

$$\log N(\text{conv}(U), \| \cdot \|_H, u) \leq C(A/u)^2 \log |U|.$$

Our final two estimates are established by applying Lemma 5 with a careful choice of $U$.

**Lemma 6.** Let $\mathcal{Y}_K$ be the set of $M_1 \times R$ matrices in [12]. Then with $G$ fixed,

$$\gamma_2(\mathcal{Y}_K, \| \cdot \|_{2,G}) \lesssim \| G \|_\infty \sqrt{KR^{3/2} \log(M_1 R) \log^2(K) \log(L)}.$$

Thus with the entries of $G$ drawn independently from a standard Gaussian distribution,

$$\| \gamma_2(\mathcal{Y}_K, \| \cdot \|_{2,G}) \|_{\psi_2} \lesssim \sqrt{KR^{3/2} \log(M_1 R) \log^2(K) \log(L) \log(M_1 L)}.$$

**Proof.** We will bound the $\gamma_2$ functional using the Dudley integral in the standard way:

$$\gamma_2(\mathcal{Y}_K, \| \cdot \|_{2,G}) \lesssim \int_0^{D(\mathcal{Y}_K, \| \cdot \|_{2,G})} \log N(\mathcal{Y}_K, \| \cdot \|_{2,G}, u) \, du. \tag{36}$$

The diameter of $\mathcal{Y}_K$, appearing in the upper limit of the integral above, can be bounded as follows. For every $Y \in \mathcal{Y}_K$, with columns $y_1, \ldots, y_R$, we have

$$\| Y \|_{2,G}^2 = \max_{1 \leq t \leq L} \| Y^* g_t \|_2^2 \leq \left( \max_{1 \leq t \leq L} \| g_t \|_\infty \right) \left( \sum_{r=1}^R \| y_r \|_1^2 \right)^{1/2} \leq \| G \|_\infty \sqrt{K} \| Y \|_F,$$

and since $\| Y \|_F \leq R^{1/4}$,

$$D(\mathcal{Y}_K, \| \cdot \|_{2,G}) \leq \| G \|_\infty \sqrt{KR^{1/4}}. \tag{37}$$

We develop two estimates on the covering numbers $N(\mathcal{Y}_K, \| \cdot \|_{2,G}, u)$, one which we will apply for small values of $u$ in the integral above, and one for larger values of $u$. For small $u$, we have the simple volumetric estimate

$$N(\mathcal{Y}_K, \| \cdot \|_{2,G}, u) \leq \left( \frac{M_1}{K} \right) \left( 1 + \frac{2\| G \|_\infty \sqrt{KR^{1/4}}}{u} \right)^{RK}. \tag{38}$$

This follows from the facts that: a) subsets of $\mathcal{Y}_K$ that have all $Y$ with the same support (i.e. locations of non-zero rows) are contained in a subspace of dimension $RK$ with $\| Y \|_{2,G}$ bounded as in [57], and b) there are $\binom{M_1}{K}$ ways to choose the support.
For larger \( u \), we apply Lemma 5. Every \( Y \in \mathcal{Y}_K \) has \( \|Y\|_1 \leq \sqrt{K} R^{3/2} \), and so \( \mathcal{Y}_K \subseteq K^{1/2} R^{3/4} \mathcal{Y}_1' \), where

\[
\mathcal{Y}_1' = \left\{ Y \in \mathbb{R}^{M_1 \times R} : \sum_{m,r} |Y_{m,r}| \leq 1 \right\},
\]

and so

\[
N(\mathcal{Y}_K, \| \cdot \|_2, G, u) \leq N(\mathcal{Y}_1', \| \cdot \|_2, G, K^{-1/2} R^{-3/4} u).
\]

We estimate the covering numbers of \( \mathcal{Y}_1' \) by taking

\[
U = \{ \pm e_{i,k} \}_{i,k=1}^{M_1,R}
\]

in Lemma 5 and so \( |U| = 2M_1 R \) and conv(\( U \)) = \( \mathcal{Y}_1' \). Let \( U_1, \ldots, U_J \) be arbitrary elements of \( U \). With \( G \) fixed, for the Rademacher sequence \( \epsilon_1, \ldots, \epsilon_J \) we have

\[
\mathbb{E}_\epsilon \left[ \sum_{j=1}^J \epsilon_j U_j^* g \right]_{2,G} = \mathbb{E}_\epsilon \left[ \max_{1 \leq \ell \leq L} \left( \sum_{j=1}^J \epsilon_j U_j^* g \right)_{2,G} \right].
\]

The vectors \( U_j^* g \) each have exactly one non-zero entry containing an entry from \( g \); we will denote these vectors as \( g^{(j)} \). The sum \( \sum_j \epsilon_j U_j^* g \) is a random vector whose Euclidean norm is subgaussian:

\[
\left\| \sum_{j=1}^J \epsilon_j U_j^* g \right\|_{\psi_2} \lesssim \left( \sum_{j=1}^J \|g^{(j)}\|_2^2 \right)^{1/2} \lesssim \sqrt{J} \|g\|_\infty.
\]

Thus a standard result on the expected maximum of a sequence of subgaussian random variables yields

\[
\mathbb{E}_\epsilon \left[ \max_{1 \leq \ell \leq L} \left\| \sum_{j=1}^J \epsilon_j U_j^* g \right\|_{2,G} \right] \lesssim \sqrt{J} \|G\|_\infty \sqrt{\log L},
\]

and so

\[
\log N(\mathcal{Y}_1', \| \cdot \|_2, G, K^{-1/2} R^{-3/4} u) \lesssim \frac{\|G\|_\infty^2 K R^{3/2} \log L}{u^2} \log(M_1 R).
\]  \hfill (39)

For a \( \kappa \) to be chosen below, we break the integral in (38) into two parts, and apply the two bounds (35) and (39),

\[
\gamma_2(\mathcal{Y}_K, \| \cdot \|_2, G) \lesssim \kappa \left( \log \left( \frac{M_1}{K} \right) + K R \log \left( 1 + \frac{2 \|G\|_\infty \sqrt{K} R^{1/2}}{\kappa} \right) \right)^{1/2} + \|G\|_\infty \sqrt{K} R^{3/2} \log(L) \log(M_1 R) \int_\kappa \|G\|_\infty \sqrt{K} R^{1/2} u^{-1} du.
\]

Taking \( \kappa = \|G\|_\infty R^{1/2} \), and using the fact that \( R \leq K \), we see that the second term above dominates and we have

\[
\gamma_2(\mathcal{Y}_K, \| \cdot \|_2, G) \lesssim \|G\|_\infty \sqrt{K} R^{3/2} \log(L) \log(M_1 R) \log K.
\]

\[\square\]

A similar bound for the set \( \mathcal{Q}_N \) comes from repeating the argument above.

**Lemma 7.** Let \( \mathcal{Q}_N \) be the set of \( M_2 \times R \) matrices in (13). Then with \( G \) fixed,

\[
\gamma_2(\mathcal{Q}_N, \| \cdot \|_2, G) \lesssim \|G\|_\infty \sqrt{N} R^{3/2} \log(M_2 R) \log^2(N) \log(L),
\]

and so with the entries of \( G \) drawn independently from a standard Gaussian distribution,

\[
\|\gamma_2(\mathcal{Q}_N, \| \cdot \|_2, G)\|_{\psi_2} \lesssim \sqrt{N} R^{3/2} \log(M_2 R) \log^2(N) \log(L) \log(M_2 L).
\]
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