Decoherence in quantum walks and quantum computers

Andrew P. Hines$^1$ and P.C.E. Stamp$^1$

$^1$Pacific Institute of Theoretical Physics, Department of Physics and Astronomy, University of British Columbia, 6224 Agricultural Rd, Vancouver BC, Canada V6T 1Z1

Decoherence is the major stumbling block in the realization of a large-scale quantum computer. Ingenious methods have been devised to overcome decoherence, but their success has been proven only for over-simplified models of system-environment interaction. Whether such methods will be reliable in the face of more realistic models is a fundamental open question. In this partly pedagogical article, we study two toy models of quantum information processing, using the language of quantum walks. Decoherence is incorporated in 3 ways - by coupling to a noisy ‘projective measurement’ system, and by coupling to oscillator and spin baths.

I. INTRODUCTION

Any realistic quantum computer must be implemented fault-tolerantly – the ideal quantum computation must be reliably performed using non-ideal components [1]. The great difficulty, when compared with classical information processing is that the “errors” that can affect qubits are much more complicated. Classical bits can only be in one of two states (‘0’ or ‘1’), so the sole possible error is an unwanted change of the bit-value – a bit-flip. In the case of quantum bits, there is a continuum of different errors that may occur. Qubits can be placed in arbitrary superposition $\alpha|0\rangle + \beta|1\rangle$ and with multiple qubits, superpositions give entangled states. Superpositions can decohere into statistical mixtures of states, destroying the phase correlations necessary for a quantum computation.

Decoherence is caused by interactions between a quantum system and its environment [2, 3]. A system starting in some superposition of states becomes entangled with its environment, and the resulting state cannot be decomposed into a simple product state. Averaging over the environmental variables gives a full, or partial, statistical mixture of states, described by the reduced density matrix of the system, rather than a coherent superposition.

To be able to perform quantum computation in the face of decoherence, some form of quantum error correction is necessary. One of the major breakthroughs in the field of quantum information processing was the discovery of quantum error correcting protocols [4, 5, 6]. The continuum of potential single-qubit errors can be overcome by correcting for just a discrete subset [1], by encoding a single logical qubit in a number of entangled qubits. It was then claimed that if the total strength of the decoherence is below a certain, finite threshold value, any quantum computation could be performed to some arbitrary close accuracy – this is a fault-tolerant threshold. Estimates of threshold values depend greatly upon the quantum error-correcting code used, assumptions about the architecture of the quantum computer, and most importantly, how the interaction with the environment is described.

The simplest approach, used in almost all papers until recently, was to treat the environment essentially as a stochastic noise source – decoherence was modelled as a sequence of independent errors, each affecting a single qubit, occurring randomly in time. Each ideal unitary component of a quantum computation can then be replaced by a non-unitary map describing evolution with some probability of an error. Such noise models were used to derive fault-tolerant thresholds bounding the probability of an error occurring at each location in the circuit [7, 8]. Two key assumptions about the interactions with the environment are implicit in these models: they are local – meaning no correlations between different qubits, except through gates – and Markovian – the environment is memory-less, so there are no correlations between different times.

Neither of these assumptions is realistic; they miss all the non-local effects (in space and time) which result when a set of quantum systems are coupled to a real environment. Quantum environments are usually described by either oscillator baths [2, 9], or spin-baths [3, 10] – representing delocalized or localized environmental modes, respectively – or a combination of both. Applying such models to a fault-tolerance calculation requires a Hamiltonian formulation, starting with a description of the system and environment dynamics, and their interaction.

Recently, several attempts were made to introduce more realistic assumptions. These include analysis of a local, but non-Markovian, environment [11], then extended to include long-range spatial correlations between qubit errors [12, 13]. In these analyses the threshold value was derived in terms of an operator norm on the interaction term in the full system-environment Hamiltonian - this operator norm depended explicitly on an arbitrary high-energy cut-off in the Hamiltonian. This makes no physical sense - no physical result should depend on the value chosen for these cut-offs. Clearly one needs a formulation of the problem which is based on the physical mechanisms operating in the system. Certainly this is possible – indeed, several quantitative analyses of correlated errors for real systems exist already in the literature [14, 15, 16], including one which incorporated long-range dipolar interactions [16].

This paper discusses one way of setting up such a formulation (although there is no space here for a complete discussion). We use two ‘toy models’ discussed below –
our goal is frankly to give some intuition for the problem. Our approach is formulated in the language of quantum walks. A quantum walk describes the dynamics of a particle on some arbitrary mathematical graph, in general coupled to an external environment. Hamiltonians describing a ‘quantum walker’ can be mapped to a very large class of Hamiltonians describing quantum information processing systems. Again, we shall see that a realistic formulation of environmental couplings in these Hamiltonians is necessary to get sensible results.

We begin with a brief discussion of quantum walks without environments, showing how to reformulate these in a Hamiltonian framework, and to map between quantum walk Hamiltonians, spin networks and quantum gates. We then extend these ‘free walker’ models to general models of walks coupled to quantum environments. We then look at our two toy models of quantum walks (with the walker on a hypercube and hyperlattice respectively). We compare how the walker dynamics is modified using 3 models of decoherence – first a diagonal coupling to a spin-bath. No attempt is made to give a complete treatment - our aim is to show, in these simple models, the kind of results one gets by using physically realistic environments to analyze decoherence (and how radically different these are from the models usually used in error-correction theory).

II. ‘BARE’ QUANTUM WALKS

Quantum walks were introduced as the quantized version of classical random walks \([17]\), which are one of the cornerstones of computer science \([18, 19]\). They display fascinating behaviour quite different from their classical counterparts \([13, 20, 21, 22, 23]\). One motivation for introducing them was the hope that one could thereby generate new kinds of quantum algorithm, which have so far proved very hard to find. The quantum walk algorithms proposed so far fall into one of two classes \([24]\). The first is based on exponentially faster hitting times \([20, 21, 22, 23]\): generally the hitting time is defined as the mean ‘first passage’ time taken to reach a given target node from some initial state. The second class provides a quadratic speed-up using a quantum walk search \([24, 27]\). In the case of a spatial search, the quantum walk algorithms can outperform the usual quantum searches based on Grover’s algorithm.

‘Bare’ quantum walks (ie., with no environment) are defined over an undirected graph with \(N\) nodes, each labelled by an integer \(n \in [0, N-1]\). Suppose we now want to describe them in a Hamiltonian framework. We can then define ‘simple quantum walks’ \([28]\) by the Hamiltonian

\[
\hat{H}^S = - \sum_{ij} \left( \Delta_{ij}(t) \hat{c}_i^\dagger \hat{c}_j + H.c. \right) + \sum_j \epsilon_j(t) \hat{c}_j^\dagger \hat{c}_j. \tag{1}
\]

Here a walker at node \(n\) corresponds to the quantum state \(|n\rangle = \hat{c}_n^\dagger |0\rangle\). The first term in (1) is a ‘hopping’ term with amplitude \(\Delta_{ij}(t)\) along the link \(|ij\rangle\) between nodes \(i\) and \(j\); the second describes ‘on-site’ node energies \(\epsilon_j(t)\). This is a generalization of the continuous-time quantum walks originally introduced by Farhi and Gutmann \([20]\). One can also imagine a walker coupled to some other degrees of freedom, which partially control the walker dynamics (an example of this is where the walk is controlled by a discrete ‘coin’ variable). We therefore define a ‘composite quantum walk’ Hamiltonian, in which the simple walker couples at each node \(j\) to a mode with Hilbert space dimension \(l_j\), and on each link \(|ij\rangle\) to a mode with Hilbert space dimension \(m_{ij}\), giving:

\[
\hat{H}_o^C = - \sum_{ij} \left( F_{ij}(\mathcal{M}_{ij}; t) \hat{c}_i^\dagger \hat{c}_j + H.c. \right) + \sum_j G_j(\mathcal{L}_j; t) \hat{c}_j^\dagger \hat{c}_j + \hat{H}_o(\{\mathcal{M}_{ij}, \mathcal{L}_j\}). \tag{2}
\]

While there have been proposals for implementing quantum walks in real space \([24, 30]\), it is more likely that the walk will take place in the ‘information’ space of some other physical system, like a register of qubits.

Within this general formulation, one can map between qubit Hamiltonians, spin Hamiltonians, quantum gate systems, and quantum walk Hamiltonians \([28]\). In Ref. \([28]\) we construct such mappings, exhibiting different ways of encoding quantum walks in multi-spin/qubit systems. These mappings are useful to explore quantum algorithms and quantum information processing hierarchies. On the more practical side, they offer a general way of implementing quantum walks over various graphs. In this paper we will use them to study the dynamics of quantum information processing, particularly the dynamics of decoherence in the presence of realistic quantum environments.

As examples, consider the two toy models we will analyze. The first is a symmetric hypercube – in the Hamiltonian (1), one assumes constant nearest-neighbour hopping between nodal sites (ie., \(\Delta_{ij} = \Delta_o, \forall i, j\)), and no on-site terms (ie., \(\epsilon_j = 0, \forall j\)), in a network restricted to a single \(D\)-dimensional hypercube. This is equivalent...
to the simple qubit Hamiltonian

$$H^\text{HC}_o = -\Delta_o \sum_{n=1}^{D} \hat{x}_n^2,$$

(3)

describing $D$ independent qubits, with the Pauli matrix $\hat{x}_j$ describing the $j$-th qubit. The node states for the walk are encoded in terms of multi-qubit states, using the binary representation of the node number, as shown in figure 1.

Our second toy model extends this $D$-dimensional hypercube to an infinite $d$-dimensional cubic hyperlattice, so the bare Hamiltonian is now

$$\hat{H}^\text{HL}_o = -\Delta_o \sum_{ij} (\hat{c}_i^\dagger \hat{c}_j + H.c.) \equiv \sum_{p} \epsilon_o(p) \hat{c}_{p}^\dagger \hat{c}_{p}$$

(4)

where $p$ is a ‘quasi-momentum’ (called the ‘crystal momentum’ in the solid-state literature); and the ‘band energy’ is

$$\epsilon_o(p) = 2\Delta_o \sum_{\mu=1}^{d} \cos(p_{\mu} a_o)$$

(5)

where we assumed a cubic lattice spacing $a_o$; henceforth we put $a_o = 1$.

### III. DECOHERENCE IN QUANTUM WALKS

The effects of decoherence on quantum walks have been considered by several authors, mostly for discrete-time quantum walks (for a review see [31]). Most investigations to date have either used simple, Markovian ‘noise’ sources [32, 33, 34, 35, 36, 37], or imperfect (non-unitary) evolution [38], to produce decoherence. In the discrete-time walk, the effect of the environment is represented as a weak measurement on the system: at each time step, there is some probability $p$ that a measurement is made, and the outcome lost to the environment. This could affect the ‘coin’ and/or walker degree of freedom. The measurement monitors the position of the walker and state of the ‘coin’, in the basis coupling to the walker transition, though other mechanisms on the coin have been considered. Extensions to continuous-time walks have similarly described the environment as a position-monitoring measurement device [39].

The focus of such models has primarily been to demonstrate the transition of the quantum walk to a classical random walk [42], even for quite exotic models [52]. The analysis has used random measurements to simulate decoherence. Using such analyses it is argued that decoherence may even be useful in some quantum walks [22], and that one gets a quantum speedup of classical mixing processes [39]. However it has also been argued that quantum walks must approach a classical random walk in the long-time limit when decohered [31].

In this paper we use a Hamiltonian description which allows us to incorporate realistic couplings to the kinds of environment which exist in Nature. It then becomes possible to solve for the dynamics of quantum walk systems, without using ad hoc decoherence models. This approach gives very different results from those noted above (for example, one sees in some cases quite dramatic departures from classical random walks in the long-time limit). It also has the great advantage of bringing work on decoherence in both quantum walks and qubit networks into contact with experiment.

Although there is a large variety of different coupling mechanisms in Nature, one can classify them in a simple way. First, the environments can be divided into two classes. Environments of extended or delocalised modes (like phonons, photons, conduction electrons, spin waves, etc) can be mapped to an oscillator bath [2], whereas environments of localised modes (local spins, including nuclear spins, two-level systems, defects, dislocations, etc.) must be mapped to a spin bath [10]; the two environments have very different effects on some central quantum system coupled to them [3]. The second classification one can make is between diagonal and non-diagonal environmental couplings to the central quantum system. The former couple to individual states of the central system, the latter to transitions between these states (thus the distinction depends on which basis we choose). In the case of quantum walks a diagonal coupling has the form $H_{\text{int}} = \sum_{j,\alpha} U_j(X_\alpha)n_j$, where $n_j = \hat{c}_j^\dagger \hat{c}_j$, and the $\{X_\alpha\}$ are the environmental modes, which couple here to states localised on a node. A non-diagonal coupling couples the $\{X_\alpha\}$ to inter-node transitions, and has the form $H_{\text{int}} = \sum_{ij,\alpha} V_{ij}(X_\alpha)[\hat{c}_i^\dagger \hat{c}_j + H.c.]$. We will see examples of these various classes of coupling in what follows.

We now turn to the dynamics of decoherence in our two toy models.

#### A. Quantum walk on the hypercube

For a simple hypercube the free walker dynamics is trivial. Thus, for a walker initialized at the $\vec{z} = 0 \equiv \downarrow \ldots \downarrow \downarrow$ corner, the probability of being at an arbitrary site $\vec{z}$, as a function of time, is

$$P_{\vec{z}_n}(t) = \cos^{2n_i}(2\Delta_o t) \sin^{2n_{\uparrow}}(2\Delta_o t),$$

(6)

where $n_{\downarrow}$ is the number of $\downarrow$‘s, and $n_{\uparrow}$ the number of $\uparrow$‘s appearing in $\vec{z}$.

1. Master equation approach

We begin by simply coupling to a memory-free system which randomly monitors the walker ‘position’ on the hypercube. One then describes the dynamics by the master equation

$$\frac{d\rho(t)}{dt} = -i[\hat{H}, \rho(t)] + \gamma \sum_{k=1}^{N} \mathcal{D}[\hat{M}_k] \rho(t),$$

(7)
\[ D \left[ \hat{M}_k \right] \rho = \hat{M}_k \rho \hat{M}^\dagger_k - \frac{1}{2} \left( \hat{M}^\dagger_k \hat{M}_k \rho + \rho \hat{M}^\dagger_k \hat{M}_k \right), \quad (8) \]
such that the \{ \hat{M}_k \} describe the interaction with the environment; this is the continuous-time generalisation of a discrete-time description \cite{31}. There are two obvious ways to couple to the walker position. The usual approach \cite{31} has been to have \{ \hat{M}_k \} project onto individual hypercube nodes, ie., \( M_k \rightarrow |j\rangle \langle j| \). In a qubit representation, this site-based decoherence actually corresponds to a rather unphysical multi-qubit interaction with the environment. A recent paper \cite{37} claims to consider this standard case, however, they actually choose projectors onto the single qubit basis states for the \( M_k \)'s. This is clearly a more physical representation of the decoherence when the quantum walk is implement on a multi-qubit register. In the hypercube representation, it projects the state onto one or other of the 2D faces of the hypercube, rather than individual sites.

Let us consider the results for these 2 cases. For the case where decoherence dephases individual qubits, the master equation is

\[ \frac{d\rho(t)}{dt} = -i [H, \rho(t)] + \gamma \sum_{k=1}^{D} D [\sigma_k^z] \rho(t), \quad (9) \]

Since the qubits do not couple, the total density matrix is a product of individual qubit density matrices, which we write in the form

\[ \rho(t) = \frac{1}{2} \begin{pmatrix} 1 + z(t) & x(t) - iy(t) \\ x(t) + iy(t) & 1 - z(t) \end{pmatrix}, \quad (10) \]

where \( x(t) = \langle \hat{x}(t) \rangle \), etc.

Then the solution is as follows. For the \( x \)-coordinate, we have

\[ x(t) = x(0)e^{-\gamma t}. \quad (11) \]

When \( \gamma < 4\Delta_o \), the system is underdamped, resulting in decaying oscillations around the origin in the \( y-z \)-plane), given by

\[ y(t) = e^{-\gamma t/2} \left[ \begin{array}{c} \frac{1}{\sqrt{1 - r^2}} (z(0) - ry(0)) \sin(\omega t) \\ + y(0) \cos(\omega t) \end{array} \right], \quad (12) \]

\[ z(t) = e^{-\gamma t/2} \left[ \begin{array}{c} z(0) \cos(\omega t) \\ + \frac{1}{\sqrt{1 - r^2}} (rz(0) - y(0)) \sin(\omega t) \end{array} \right], \quad (13) \]

where \( r = \gamma/4\Delta_o \) and \( \omega = \sqrt{4\Delta_o^2 - \gamma^2/2} \). In the over-damped regime, \( \gamma > 4\Delta \), we see exponential decay,

\[ y(t) = -\frac{1}{2\sqrt{r^2 - 1}} \left[ A_+ e^{-\lambda_+ t} + A_- e^{-\lambda_- t} \right], \quad (14) \]

\[ z(t) = -\frac{1}{2\sqrt{r^2 - 1}} \left[ B_+ e^{-\lambda_+ t} + B_- e^{-\lambda_- t} \right], \quad (15) \]

where \( \lambda_\pm = 1/2[\gamma \pm \sqrt{\gamma^2 - 16\Delta_o^2}] \), and

\[ A_\pm = \pm \left( (r \mp \sqrt{r^2 - 1}) y(0) - z(0) \right), \]

\[ B_\pm = \pm \left( y(0) - (r \mp \sqrt{r^2 - 1}) z(0) \right). \]

Suppose the walker is initialized at the origin (‘0’), so that \( z(0) = 1 \) and \( x(0) = y(0) = 1 \) for each qubit. The probability of being found at this same site after a time \( t \) is \( P_{00}(t) = \left( \frac{1 + z(t)}{2} \right)^D \), while the probability of being at the far corner is \( P_{DD}(t) = \left( \frac{1 - z(t)}{2} \right)^D \).

Consider now the results for the case where the master equation projects onto each site. Figure 2 compare these ‘site’-based results (calculated numerically) with

FIG. 2: Comparison of the two decoherence mechanisms modelled by the master equation for the quantum walk on the hypercube. The walker in initially localized on a single corner. Plotted above in (a) and (b) is the probability that the walker remains at this location (blue and green), and the probability of being at the far corner (red and orange), over time: lines for qubit-based, circles for site-based decoherence. (a) shows results for \( D = 3 \) with various \( \gamma \), while (b) is for \( \gamma = 0.5 \) and differing \( D \). In each plot, the dashed black line represents the classical behaviour.
the single-qubit based decoherence. The difference in the resulting dynamics is easy to understand. In site-based decoherence, all off-diagonal elements of the total density operator, which describe quantum interference, decay at the same rate. However because single-qubit dephasing monitors which face of the hypercube the walker is located on, different off-diagonal elements decay at different rates; the element $\rho_{nm}$ decays at rate $-(C/D)\gamma$, where $D$ is the number of qubits, and $1 \leq C \leq D$ is a constant, given by the Hamming weight of the difference between the binary representations of the two indices $m$ and $n$.

An interesting question arising from these results is to what extent the useful properties of a decoupling quantum system, with a site-based environmental interaction, can be simulated in a multi-qubit representation. These issues will be addressed in a forthcoming article where a detailed analysis of different decoherence channels acting on the quantum walk on the hypercube, with a focus on those characteristics important for potential algorithms, is presented.

2. Hypercube coupled to an oscillator bath

Let us now see what one gets by coupling the hypercube to a bath of oscillators. We assume a Hamiltonian $\mathcal{H} = H^\text{HC}_0 + V + H_{osc}$, with $H^\text{HC}_0$ given in above, and with oscillator Hamiltonian given by

$$H_{osc} = \sum_{\bf{q}} \left( \frac{P^2_q}{m_q} + m_q \omega_q^2 x^2_q \right)$$

in terms of oscillators $\{x_q\}$, and a diagonal interaction $V = \sum_{\bf{q}=1}^{N_z} v_q^2 \sum_{\bf{q}=1}^{N_z} x_q$, between oscillators and the qubits (note in line with the remarks above, this interaction is not diagonal in the hypercube node basis).

Implicit but not mentioned in these Hamiltonians is a UV cutoff $\omega_c$; it is assumed that modes at energy scales $> \omega_c$ do not couple to the walker. Suppose we now choose a lower UV cutoff $\omega_c$. Then the result is (i) a renormalisation of $\Delta_0$, and (ii) an extra inter-qubit interaction term $\sum_{\bf{nm}} V_{nm}^{zz} \rho_{nm}$ in the effective Hamiltonian, with

$$V_{nm}^{zz}(\Omega_c) = \int_{\Omega_c}^{\Omega_0} \frac{d\omega}{\pi} J_{nm}^{zz}(\omega)$$

where the function $J_{nm}^{zz}(\omega)$ is defined in terms of the propagator $D(\mathbf{R}, \omega) = \sum_{\mathbf{q}} D(\mathbf{q}, \omega) \exp[i \mathbf{q} \cdot \mathbf{R}]$ for the oscillator modes (the zero-temperature propagator has the form $D(\mathbf{q}, \omega) = 2\omega_q / (\omega^2 - \omega_q^2)$, with the usual Matsubara generalisation to finite $T$). One has

$$J_{nm}^{zz}(\omega) = \frac{\pi}{2} \sum_{\mathbf{q}} \frac{|c_n^m(q)c_m^l(q)|}{\omega_q} D(\mathbf{R}_{nm}, \omega_q) \delta(\omega - \omega_q)$$

where we assume qubits $n$ and $m$ separated by a radius vector $\mathbf{R}_{nm}$. The crucial point is that consistent calculations of the qubit dynamics must give the same results for either the ‘bare’ form without interactions and with the bare $\Delta_0$, or the lower-energy form with the renormalisation and the interaction.

To say anything more about this dynamics we need the form of the bath spectrum and couplings - the details are very lengthy and will be given elsewhere. But note that the dynamics is now utterly different from what was found using the master equation approach - the induced couplings between qubits generate entanglement between them (in hypercube language they generate couplings across the diagonals), as well as long-time tails in the single-qubit dynamics. This is seen already in solutions for the single qubit, ie., in the well-known spin-boson model, and in the solution for a pair of qubits interacting with oscillators.

B. Hyperlattice Walker

The quantum walk on the $d$-dimensional hyperlattice, with Hamiltonian, is also trivially solvable. For a walker initialized at the origin (centre) of the hyperlattice, $\vec{n} = 0$, the probability of being at some other site, with the position described by the $d$-dimensional vector $\vec{n}$, after some time $t$ is given by

$$P_{\vec{n}0}(t) = \prod_{\mu=1}^{d} J_{\mu\mu}^2(z),$$

where $J_{\mu}(z)$ is the $m^\text{th}$ order Bessel function, of dimensionless time $z = 2\Delta t$. The purely quantum evolution is characterized (at long times) by $P_{00}(t) \propto 1/t^d$ and by mean-square displacement - given by $\langle n^2(t) \rangle = \sum_{\vec{n}\neq 0} n^2 P_{\vec{n}0}(t)$ - behaving as $\langle n^2(t) \rangle \propto t^2$.

This result is often compared to classical diffusion on a hyperlattice, where $\langle n^2(t) \rangle \propto 1/t^{d/2}$ and $\langle n^2(t) \rangle \propto t$.

1. Master equation approach

We first consider decoherence modelled by Eq. (7) with

$$\{M_{\vec{n}}\} = \{|\vec{n}\rangle \langle \vec{n}|\}$$

where $\vec{n}$ is the lattice vector defining a given site on the hyperlattice, and we initialize the particle at the origin of the lattice. This corresponds to an environment which monitors the position of the walker – an assumption consistent with a diagonal (on-site) coupling to the environment. We consider both the probability of the particle remaining at the origin, $P_{00}(t)$, and the mean square displacement. Results from numerical calculations, for $d = 1, 2$ are presented in figure. As $\gamma$ increases from zero, the quantum walk begins to behave more classically, as exemplified in the behaviour of $\langle n^2(t) \rangle$. However, as
Consider a spin-bath of two-level systems which couples *non-diagonally* to the walker; the environment then monitors the transitions between nodes. Suppose that each time the quantum walker hops between nodes it can flip the \( k^{th} \) bath spin, \( \sigma_k \), with complex flipping amplitude \( \alpha_k \). This gives an effective Hamiltonian,

\[
H = \Delta_o \sum_{<ij>} \left\{ c_i^\dagger c_j \cos \left( \sum_k \alpha_k \sigma_k^z \right) + H.c. \right\}.
\]

Hamiltonians of this form have previously been derived for a number of physical systems \[10\]. The strength of the decoherence is parametrized by \( \lambda = \sum_k \alpha_k^2 \), which measures the number of bath spins flipped during a transition between 2 nodes. We let \( \lambda \gg 1 \), thereby considering the limit of strong decoherence.

The dynamics of this model are exactly solvable \[41\], with the reduced density operator of the walker given by a phase average over the propagator of the ‘free’ quantum walk. Taking the strong decoherence limit, with the walker initially at the origin, we have

\[
P_{00}(t) = \frac{1}{\Delta_o} \int_0^{2\pi} d\varphi \prod_{\mu=1}^d J_{2\varphi}(z \cos \varphi).
\]

While for strong decoherence one might expect long-time diffusive behaviour, what we actually see is quite different. For long times (\( z \gg 1 \)), one has the limiting short-distance behaviour (return probability) given by

\[
P_{00}(z \rightarrow \infty) \approx \frac{1}{\pi} \int_{-\infty}^{\infty} d\varphi J_o^2(\varphi) = \frac{A_d}{\Delta_o^d},
\]

where \( A_d = (2\pi)^{-1} \int_{-\infty}^{\infty} dx J_o^2(x) \) is a constant (in \( d = 1 \) there is an additional \( \ln(2\Delta_o t) \) factor). This gives a divergence in the total time spent at the origin; the interaction with the bath causes ‘sub-diffusive’ behaviour near the origin and a form of quasi-localization. However quite remarkably, the mean-square displacement calculated from Eq. \[22\] is

\[
\langle n^2(t) \rangle = \frac{d}{2} (\Delta t)^2,
\]

which is but a factor of two smaller than the coherent quantum evolution! The exact solution\[41\] shows that the density matrix has one component showing quasi-localization at the origin, coexisting with another showing coherent ballistic dynamics far from the origin. This is quite a departure from the classical diffusion expected from a strong coupling to the environment! This behaviour arises because the environmental coupling does not distinguish different walker positions on the hyperlattice, or the direction of transitions between nodes. It only records when a transition between nodes has occurred, allowing for constructive interference over

\[\gamma\] becomes large, we see a quantum Zeno effect, with the particle trapped at the origin.

This transition to classical diffusion is very commonly found when we couple the position of the walker to an environment - it is not just found in memory-free noise models like this one, but also in more realistic oscillator bath models\[9\]. However it is by no means generally valid. To see this, we consider what happens with a spin bath environment.
many differing paths on the graph. For more details see Ref. [41].

We emphasize here that the graph over which the quantum walk takes place can be represented using different sets of basis states; there is no general principle forcing environmental couplings to distinguish different nodes or transition directions in these different encodings. In the design of quantum computers and certain search algorithms, the above result shows the importance of investigating quantum walks for which environmental couplings do not distinguish different ‘position’ (or ‘nodal’) states in the information space in which the quantum walk is encoded [42].

IV. CONCLUDING REMARKS

The extent to which quantum information processing and entanglement are robust to decoherence is a central question in quantum computation, and by implication in quantum walk theory. The two examples we have discussed here highlight the importance of the choice of ‘information space’ in which the structure of the walk is encoded. To fully appreciate the potential impact for quantum walk algorithms, one must focus on the relevant algorithmic properties of the quantum walk, and the effect of different decoherence processes.

Quantum walks were originally devised as a new way to develop quantum algorithms. However we have found that they may actually be more useful in understanding the effect of the environment on quantum information processing, using mappings between arbitrary quantum information systems and quantum walk systems. Most of the promising architectures for quantum computing, especially solid-state proposals, have qubits which couple to quantum environments which cannot be treated simply as ‘noise’ sources. In reality the coupling to the environment generates complex correlated errors, as well as highly non-local (in space and time) correlations in the decoherence dynamics. By representing this complex dynamics in terms of quantum walks coupled to general quantum environments, one makes it possible to arrive at very useful results.

All of these investigations may usefully be understood as a new direction in research on the quantum dynamics of various systems (moving particles, spins, etc.) on different kinds of lattice and graphs. This field has an interesting history, with many applications in statistical mechanics [43], and the theory of disordered systems [44]. It seems likely that methods imported from these fields will be very useful in understanding quantum information processing.

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