Non-Bayesian Estimation Framework for Signal Recovery on Graphs

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Abstract—Graph signals arise from physical networks, such as power and communication systems, or as a result of a convenient representation of data with complex structure, such as social networks. We consider the problem of general graph signal recovery from noisy, corrupted, or incomplete measurements and under structural parametric constraints, such as smoothness in the graph frequency domain. In this paper, we formulate the graph signal recovery as a non-Bayesian estimation problem under a weighted mean-squared-error (WMSE) criterion, which is based on a quadratic form of the Laplacian matrix of the graph and its trace WMSE is the Dirichlet energy of the estimation error w.r.t. the graph. The Laplacian-based WMSE penalizes estimation errors according to their graph spectral content and is a difference-based cost function which accounts for the fact that in many cases signal recovery on graphs can only be achieved up to a constant addend. We develop a new Cramér-Rao bound (CRB) on the Laplacian-based WMSE and present the associated Lehmann unbiasedness condition w.r.t. the graph. We discuss the graph CRB and estimation methods for the fundamental problems of 1) A linear Gaussian model with relative measurements; and 2) Bandlimited graph signal recovery. We develop sampling allocation policies that optimize sensor locations in a network for these problems based on the proposed graph CRB. Numerical simulations on random graphs and on electrical network data are used to validate the performance of the graph CRB and sampling policies.

Index Terms—Non-Bayesian parameter estimation, graph Cramér-Rao bound, Dirichlet energy, Graph Signal Processing, Laplacian matrix, sensor placement, graph signal recovery

I. INTRODUCTION

Graphs are fundamental mathematical structures that are widely used in various fields for network data analysis to model complex relationships within and between data, signals, and processes. Many complex systems in engineering, physics, biology, and sociology constitute networks of interacting units that result in signals that are supported on irregular structures and, thus, can be modeled as signals over the vertices (nodes) of a graph, i.e. graph signals. Thus, graph signals arise in many modern applications, leading to the emergence of the area of graph signal processing (GSP) in the last decade (see, e.g. [2]–[5]). GSP theory extends concepts and techniques from traditional digital signal processing (DSP) to data indexed by generic graphs. However, most of the research effort in this field has been devoted to the purely deterministic setting, while methods that exploit statistical information generally lead to better average performance compared to deterministic methods and are better suited to describe practical scenarios that involve uncertainty and randomness [6], [7]. In particular, the development of performance bounds, such as the well-known Cramér-Rao bound (CRB), on various estimation problems that are related to the graph structure is a fundamental step towards having statistical GSP tools.

Graph signal recovery aims to recover graph signals from noisy, corrupted, or incomplete measurements. Applications include registration of data across a sensor network, time synchronization across distributed networks [8], [9], and state estimation in power systems [10]–[12]. The conventional CRB does not provide an appropriate tool for the recovery of graph signals since it does not display consistency with the geometry of the data. For example, the CRB does not take into account the graph smoothness, bandlimitedness w.r.t. the graph, nor the connectivity of the graph and the degrees of the different nodes, and treats the information in connected and separated nodes equally. In addition, the classical CRB is applied on the mean-squared-error (MSE), where this criterion may be inappropriate for characterizing the performance when parameters are defined on a manifold [13], [14]. Moreover, in many cases, graph signals are only a function of relative values, i.e. the differences between vertex values. Such signals arise, for example, in community detection [1], motion consensus [15], [16], time synchronization in networks [8], [9], and power system state estimation (PSSE) [17], [18]. In these cases, signal recovery on graphs can only be achieved up to a constant addend, which is a situation which the CRB does not tolerate. Therefore, new evaluation measures and performance bounds are required for statistical GSP. New performance bounds can be useful also for designing the sensing network topology, which is a crucial task in both data-based and physical networks.

Sampling and recovery of graph signals are fundamental tasks in GSP that have received considerable attention recently. In particular, recovery with a regularization using the Dirichlet energy, i.e. the Laplacian quadratic form, has been used in various applications, such as image processing [19], [20], non-negative matrix factorization [21], principal component analysis (PCA) [22], data classification [23], [24], and semisupervised learning on graphs [25]. Characterization of graph signals, dimensionality reduction, and recovery using the graph Laplacian as a regularization have been used in various fields [20]. In the case of isotropic Gaussian noise with relative measurements, the CRB for synchronization of rotations has been developed in the seminal work in [27], [28], and it is shown to be proportional to the pseudo-inverse of the Laplacian matrix. In addition, the effect of an incomplete
measurement graph on the CRB has been shown to be related to the Laplacian of the graph \([27]\). Thus, the graph Laplacian represents the information on the structure of the underlying graph and can be used for the development of performance assessment, analysis, and practical inference tools \([29]\).

In this paper, we study the problem of graph signal recovery in the context of non-Bayesian estimation theory. First, we introduce the Laplacian-based weighted MSE (WMSE) criterion as an estimation performance measure for graph signal recovery. This measure is used to quantify changes w.r.t. the variability that is encoded by the weights of the graph \([30]\) and is a difference-based criterion whose trace is the Dirichlet energy. We show that the WMSE can be interpreted as the MSE in the graph frequency domain, in the GSP sense. We present the concept of graph-unbiasedness in the sense of the Lehmann-unbiasedness definition \([31]\). We develop a new bound. The performance of the proposed graph CRB for linear Gaussian model with relative measurements and for bandlimited graph signal recovery. We show that the new bound provides analysis and design tools, where we optimize the sensor locations by using the graph CRB. In simulations, we demonstrate the use of the graph CRB and associated estimators for recovery of graph signals in random graphs and for the problem of PSSE in electrical networks. In addition, we show that the proposed sampling policies lead to better estimation performance in terms of Dirichlet energy.

The rest of the paper is organized as follows: Section II presents the mathematical model for the graph signal recovery. In Section III we present the proposed performance measure and discuss its properties. In Section IV and Section V we develop the graph CRB for linear Gaussian model with relative measurements and of bandlimited graph signal recovery, respectively, and discuss the design of sample allocation policies based on the new bound. The performance of the proposed graph CRB is evaluated in simulations in Section VII. Finally, our conclusions can be found in Section VIII.

II. MODEL AND PROBLEM FORMULATION

In this section, we present the considered model and formulate the graph signal recovery task as a non-Bayesian estimation problem. Subsection II-A includes the notations used in this paper. Subsection II-B introduces the background and relevant concepts of GSP. In Subsection II-C and in Subsection II-D we present the estimation problem and the influence of linear parametric constraints, respectively.

A. Notations

In the rest of this paper, we denote vectors by boldface lowercase letters and matrices by boldface uppercase letters. The operators \((\cdot)^T\), \((\cdot)^{-1}\), \((\cdot)^\dagger\), and \(\text{Tr}(\cdot)\) denote the transpose, inverse, Moore-Penrose pseudo-inverse, and trace operators, respectively. For a matrix \(A \in \mathbb{R}^{M \times K}\) with a full column rank, \(P_A = I_M - A(A^T A)^{-1} A^T\), where \(I_M\) is the identity matrix of order \(M\). The \(m, q\)th element of the matrix \(A\), and the \((m_1 : m_2 \times q_1 : q_2)\) submatrix of \(A\) are denoted by \(a_{m,q}\), \(A_{m,q}\), and \(A_{m_1:m_2,q_1:q_2}\), respectively. The notation \(A \geq B\) implies that \(A - B\) is a positive-semidefinite matrix, where \(A\) and \(B\) are positive-semidefinite matrices of the same size. The gradient of a vector function \(g(\theta)\), \(\nabla g(\theta)\), is a matrix in \(\mathbb{R}^{K \times M}\), with the \((k,m)\)th element equal to \(\frac{\partial g_k(\theta)}{\partial \theta_m}\), where \(g = [g_1, \ldots, g_K]^T\) and \(\theta = [\theta_1, \ldots, \theta_M]^T\). For any index set, \(S \subset \{1, \ldots, M\}\), \(\theta_S\) is a subvector of \(\theta\) containing the elements indexed by \(S\), where \(|S|\) and \(S^\perp \triangleq \{1, \ldots, M\} \setminus S\) denote the set’s cardinality and the complement set, respectively. The vectors \(1\) and \(0\) are column vectors of ones and zeros, respectively, \(e_n\) is the \(n\)th column of the identity matrix, all with appropriate dimension, \(1_A\) denotes the indicator function of an event \(A\), and the number of non-zero entries in \(\theta\) is denoted by \(||\theta||_0\). Finally, the notation \(E_{\theta}[\cdot]\) represents the expected value parameterized by a deterministic parameter \(\theta\).

B. Background: Graph signal processing (GSP)

In this section, we briefly review relevant concepts related to GSP \([2]\, [3]\) that will be used in this paper. Consider an undirected weighted graph, \(G(M, \xi, W)\), where \(M = \{1, \ldots, M\}\) denotes the set of \(M\) nodes or vertices and \(\xi\) denotes the set of edges with cardinality \(|\xi| = K\). We only consider simple graphs, with no self-loops and multi-edges. The symmetric matrix \(W\) is the weighted adjacency matrix with entry \(W_{m,k}\) denoting the weight of the edge \((m, k) \in \xi\), reflecting the strength of the connection between the nodes \(m\) and \(k\). This weight may be a physical measure or conceptual, such as a similarity measure. We assume for simplicity that the edge weights in \(W\) are non-negative (\(W_{m,k} \geq 0\)). When no edge exists between \(m\) and \(k\), the weight is set to 0, i.e. \(W_{m,k} = 0\).

Definition 1. Given \(G(M, \xi, W)\), the neighborhood of a node \(m \in M\) is defined as \(N_m = \{k \in M : (m, k) \in \xi\}\).

The Laplacian matrix, which contains the information on the graph structure, is defined by \(L \triangleq D - W\), where \(D\) is a diagonal matrix with \(D_{m,m} = \sum_{k=1}^{M} W_{m,k}\). The Laplacian matrix, \(L\), is a real, symmetric, and positive semidefinite matrix, which satisfies the null-space property, \(LI_M = 0\), and with nonpositive off-diagonal elements. Thus, its associated singular value decomposition (SVD) is given by

\[
L = VAV^T,
\]

where the columns of \(V\) are the eigenvectors of \(L\), \(V^T = V^{-1}\), and \(A \in \mathbb{R}^{M \times M}\) is a diagonal matrix consisting of the distinct eigenvalues of \(L\), \(0 = \lambda_1 < \lambda_2 < \ldots < \lambda_M\). Throughout this paper we will focus on the case where the observed graph is connected and, thus, \(\lambda_2 \neq 0\). If the graph is not connected, the proposed approach can be applied to each connected component separately. The eigenvalues \(\lambda_1, \ldots, \lambda_M\) can be interpreted as graph frequencies, and eigenvectors, i.e. the columns of the matrix \(V\), can be interpreted as corresponding graph frequency components. Together they define the graph spectrum for graph \(G(M, \xi, W)\).

In this framework, a graph signal is defined as a function \(\theta : M \rightarrow \mathbb{R}^M\), assigning a scalar value to each vertex, where
entry \( \theta_m \) denotes the signal value at node \( m \in \mathcal{M} \). The graph Fourier transform (GFT) of a graph signal \( \theta \) w.r.t. the graph \( \mathcal{G}(\mathcal{M}, \xi, W) \) is defined as the projection onto the orthogonal set of the eigenvectors of \( L \) \cite{2, 3}:

\[
\hat{\theta} \triangleq V^T \theta.
\] (2)

Similarly, the inverse GFT is obtained by left multiplication of a vector by \( V \), i.e. by \( V \hat{\theta} \). The GFT plays a central role in GSP since it is a natural extension of filtering operations and the notion of the spectrum of the graph signals \cite{2, 3}.

C. Estimation problem

We consider the problem of estimating the graph signal, \( \theta \in \mathbb{R}^M \), which is considered in this paper to be a deterministic parameter vector. The estimation is based on a random observation vector, \( x \in \Omega_x \), where \( \Omega_x \) is a general observation space. We assume that \( x \) is distributed according to a known probability distribution function (pdf), \( f(x; \theta) \). Our goal is to integrate the side information in the form of graph structure, encoded by the Laplacian matrix, in the estimation approach.

Let \( \hat{\theta} : \Omega_x \rightarrow \mathbb{R}^M \) be an estimator of \( \theta \), based on a random observation vector, \( x \in \Omega_x \). We consider estimators in the Hilbert space of bounded energy on \( \mathcal{G}(\mathcal{M}, \xi, W) \) \cite{2}:

\[
\mathcal{H}_E \triangleq \{ g(x) \in \mathbb{R}^M | E_\theta [g^T(x) L g(x)] < \infty \}.
\] (3)

Our goal in this paper is to investigate estimation methods and bounds that are based on defining an appropriate estimation performance measure that reflects the graph topology and graph signal properties.

D. Linear constraints in graph recovery

In various GSP problems there is side information on the graph signals in the form of linear parametric constraints. These linear constraints describe properties of the graph signals, such as bandlimitedness (see the example in Subsection \[24\]), or the existence of reference nodes (anchors) \cite{15, 16}, or can serve to obtain a well-posed estimation problem \cite{27}. Formally, in these cases it is known a-priori that \( \theta \) satisfies the following linear constraint:

\[
G \theta + a = 0, \ G \in \mathbb{R}^{K \times M},
\] (4)

where \( G \in \mathbb{R}^{K \times M} \) and \( a \in \mathbb{R}^K \) are known, and \( 0 \leq K \leq M \). We assume that the matrix \( G \) has full row rank, i.e. the constraints are not redundant. The constrained set is:

\[
\Omega_\theta \triangleq \{ \theta \in \mathbb{R}^M | G \theta + a = 0 \}.
\] (5)

Thus, we are interested in the problem of recovering graph signals that belong to \( \Omega_\theta \). We define the orthonormal null space matrix, \( U \in \mathbb{R}^{M \times (M-K)} \), such that

\[
GU = 0 \quad \text{and} \quad U^T U = I_{M-K}.
\] (6)

The case \( K = 0 \) implies an unconstrained estimation problem in which \( U = I_M \). The matrix \( U \) can be found based on the eigenvector matrix of the orthogonal projection matrix \( P_G \triangleq I_M - G^T G G^T \)^{-1}G.

III. LAPLACIAN-BASED WMSE ESTIMATION

In this paper, we suggest the use of the following matrix cost function for graph signal recovery:

\[
C(\hat{\theta}, \theta) \triangleq \Lambda^\frac{1}{2} V^T (\hat{\theta} - \theta) (\hat{\theta} - \theta)^T V \Lambda^\frac{1}{2},
\] (7)

where \( V \) and \( \Lambda \) are defined in \cite{1}. The corresponding risk, which is the expected cost from (7), is the following WMSE:

\[
E_\theta [C(\hat{\theta}, \theta)] = \Lambda^\frac{1}{2} V^T \text{MSE}(\hat{\theta}, \theta) V \Lambda^\frac{1}{2},
\] (8)

where the MSE matrix is defined as:

\[
\text{MSE}(\hat{\theta}, \theta) \triangleq E_\theta [(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T].
\] (9)

The proposed risk in (8) can be interpreted as a WMSE criterion \cite{23, 24} with a positive semidefinite weighting matrix, \( \Lambda^\frac{1}{2} V^T \). Different weights are assigned to the individual errors, where these weights are based on the graph topology. The Laplacian matrix, \( L \) which is used to determine these weights through \( \Lambda \) and \( V^T \), can be based on a physical network or on statistical dependency, such as a probabilistic graphical model that is obtained from historic, offline data.

The rationale behind this cost function, as well as some of its properties, are as follows:

1) Difference-based criterion: It is proved in Appendix \[25\] that the \((m,n)\)th element of the cost \( C(\theta, \theta) \) is

\[
[C(\theta, \theta)]_{m,n} = \Lambda^\frac{1}{2}_{m,m} \Lambda^\frac{1}{2}_{n,n} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m} V_{l,n} \left( \epsilon_k - \epsilon_m \right) \left( \epsilon_l - \epsilon_n \right) - \frac{1}{2} (\epsilon_m - \epsilon_n)^2,
\] (10)

where the estimation error at node \( m \) is defined by:

\[
\epsilon_m \triangleq \hat{\theta}_m - \theta_m, \ m = 1, \ldots, M.
\] (11)

The elements of the cost matrix in (10) imply that the proposed cost function only takes into account the relative estimation errors of the differences between the error signals at different nodes. Thus, any translation of all errors by a vector \( c_1 \), would not change the cost. This property reflects the fact that in various applications (see, e.g. \cite{8, 9, 15–17}) graph signals are only a function of the differences between vertex values, and estimation can only be achieved up to a constant addend \cite{15, 16, 18}.

In addition, since for the Laplacian matrix \( \lambda_1 = 0 \), then \( [\Lambda]_{1,1} = 0 = 1 \). By substituting this value in (10), we obtain:

\[
[C(\theta, \theta)]_{m,n} = 0, \ \text{if} \ m = 1 \ \text{and/or} \ n = 1.
\] (12)

Therefore, minimization of the expected matrix cost, \( \mathbb{E}[C(\theta, \theta)] \), in the sense of positive semidefinite matrices, is equivalent to minimization of the submatrix \( \mathbb{E}[[C(\theta, \theta)]_{2:M,2:M}] \).

2) Dirichlet energy interpretation: By substituting (1) in (7) and using the trace operator properties, it can be verified that the trace of \( C(\theta, \theta) \) is given by

\[
\text{Tr}(C(\theta, \theta)) = (\hat{\theta} - \theta)^T V A V^T (\hat{\theta} - \theta) = (\hat{\theta} - \theta)^T L (\hat{\theta} - \theta).
\] (13)
By substituting $\mathbf{L} = \mathbf{D} - \mathbf{W}$ in (13) it can be shown that

$$\text{Tr}(\mathbf{C}(\hat{\theta}, \theta)) = \frac{1}{2} \sum_{m=1}^{M} \sum_{k \in N_m} W_{m,k} (\epsilon_k - \epsilon_m)^2,$$

(14)

where $N_m$ is the set of connected neighbors of node $m$, as defined in Definition 1.

The cost in (13) is the Dirichlet energy of the estimation error signal, $\epsilon = \hat{\theta} - \theta$, w.r.t. the Laplacian matrix $\mathbf{L}$ (26), (25). This smoothness measure, which is well known in spectral graph theory, quantifies how much the signal changes w.r.t. the variability encoded by the graph weights. Intuitively, since the weights are nonnegative, the graph Dirichlet energy quantifies how much the signal change $s$ in networks, is derived in Section IV-B. The proposed graph CRB is a lower bound on the WMSE from (3) of any unbiased estimator, where the unbiasedness w.r.t. the graph is defined in Section IV-A by using the concept of Lehmann unbiasedness.

A. Graph unbiasedness

Lehmann (31) proposed a generalization of the unbiasedness concept based on the chosen cost function for each scenario, which can be used for various cost functions (see, e.g. (14), (32), (36), (37)). The following proposition defines the graph unbiasedness property of estimators w.r.t. the Laplacian-based WMSE and under the linear parametric constraints.

Proposition 1. An estimator, $\hat{\theta}$, is an unbiased estimator of the graph signal, $\theta$, in the Lehmann sense w.r.t. the weighted squared-error cost function from (7) and under the constrained set in (4) if it satisfies

$$U^T \mathbf{E}_0 [\hat{\theta} - \theta] = 0, \forall \theta \in \Omega_\theta,$$

(17)

where $\Omega_\theta$ is defined in (5).

Proof: The proof is given in Appendix B.

It can be seen that if an estimator has a zero mean bias, i.e. $E_0[\hat{\theta} - \theta] = 0, \forall \theta \in \Omega_\theta$, then it satisfies (17), but not vice versa. Thus, the uniform graph unbiasedness condition is a weaker condition than requiring the mean-unbiasedness property. For example, since $\mathbf{L} = 0$, the condition in (17) is oblivious to the estimation error of a constant bias over the graph, $c_1$, for any constant $c \in \mathbb{R}$, in contrast with mean unbiasedness. It can be seen that the graph unbiasedness in (17) is a function of the specific graph topology. In addition, the unbiasedness definition can be reformulated by using any matrix that spans the null space of $\mathbf{G}$, $N(\mathbf{G})$, instead of $\mathbf{U}$.

Special case 1. For the case where no constraint is imposed, we have $U = I_M$ and the condition in (17) is reduced to

$$V \mathbf{E}_0 [\hat{\theta} - \theta] = 0, \forall \theta \in \mathbb{R}^M,$$

(18)

where we used the GFT operator from (3). Since $\lambda_1 = 0$ and $\lambda_m > 0, m = 2, \ldots, M$, the condition in (18) is equivalent to the requirement that for each nonzero eigenvalue, the bias in the frequency domain should be zero.

Another special case of the graph-unbiasedness for a bandlimited graph signal is discussed in Subsection VI.

In non-Bayesian estimation theory, two types of unbiasedness are usually considered: uniform unbiasedness at any point in the parameter space, and local unbiasedness, in which the estimator is assumed to be unbiased only in the vicinity of the parameter $\theta_0$. By using the feasible directions of the constraint set, similar to the derivations in (34), (35), it can be shown that the local graph-unbiasedness conditions are as follows:

Definition 2. Necessary conditions for an estimator $\hat{\theta} : \Omega_\theta \rightarrow \mathbb{R}^M$ to be a locally Lehmann-unbiased estimator in the vicinity of $\theta_0 \in \Omega_\theta$ w.r.t. the WMSE are

$$U^T L \mathbf{E}_0 [\hat{\theta} - \theta] |_{\theta = \theta_0} = 0$$

(19)

and

$$U^T L \nabla_\theta \mathbf{E}_0 [\hat{\theta} - \theta] |_{\theta = \theta_0} = 0.$$

(20)
B. Graph CRB

In the following, a novel graph CRB for the estimation of graph parameters is derived. Various low-complexity, distributive algorithms exist for graph signal recovery. The new bound is a useful tool for assessing their performance.

In the following theorem, we present the proposed graph CRB on the WMSE of local graph-unbiased estimators in the vicinity of \( \theta \), as defined in Definition 3.

**Theorem 1.** Let \( \hat{\theta} \) be a locally graph-unbiased estimator of \( \theta \), and assume the following regularity conditions:

1. The operations of integration w.r.t. \( x \) and differentiation w.r.t. \( \theta \) can be interchanged, as follows:

\[
\nabla_\theta \int_{\Omega_\theta} g(x; \theta) \, dx = \int_{\Omega_\theta} \nabla_\theta g(x; \theta) \, dx.
\]

2. The Fisher information matrix (FIM),

\[
J(\theta) \triangleq \mathbb{E}_\theta [\nabla^T_\theta \log f(x; \theta) \nabla_\theta \log f(x; \theta)],
\]

is well defined and finite.

Then,

\[
\mathbb{E}_\theta [C(\hat{\theta}, \theta)] \geq B(\theta), \quad \forall \theta \in \Omega_\theta,
\]

where

\[
B(\theta) \triangleq \Lambda^2 \nabla^T \nabla U(\mathbb{U}^T J(\theta) \mathbb{U})^{\dagger} \mathbb{U}^T \Lambda^2.
\]

Equality in (23) is obtained iff the estimation error in the graph-frequency domain satisfies

\[
\Lambda^2 (\hat{\theta} - \theta) = \Lambda^2 \nabla^T \nabla U (\mathbb{U}^T J(\theta) \mathbb{U})^{\dagger} \mathbb{U}^T \nabla^T \log f(x; \theta),
\]

\[\forall \theta \in \Omega_\theta.\]

**Proof:** The proof is given in Appendix C.

Similar to the explanations in Subsection II-C, it can be seen that the first row and column of the proposed matrix bound in (24) is zero. Thus, in practice, we use the submatrix bound, \( B(\theta) \), and its properties in the general case can be found, for example, in [39].

**1) Relation with the constrained CRB (CCRB) on the MSE:** It can be verified that the proposed bound in (24) is a weighted version of the CCRB, i.e.

\[
B(\theta) = \Lambda^2 \mathbb{V}^T \mathbb{B}_{\text{CCRB}}(\theta) \mathbb{V} \Lambda^2,
\]

where the CCRB is given by (40).

\[
\mathbb{B}_{\text{CCRB}}(\theta) = U (\mathbb{U}^T J(\theta) \mathbb{U})^{\dagger} \mathbb{U}^T.
\]

This result stems from the fact that the performance criterion in this paper is a weighted version of the MSE with the structure described in (6). The equality condition of the CCRB holds iff (34), (38). (40).

\[
\hat{\theta} - \theta = U (\mathbb{U}^T J(\theta) \mathbb{U})^{\dagger} \mathbb{U}^T \nabla^T \log f(x; \theta).
\]

It can be verified that if \( \theta \) is a constrained-efficient estimator which satisfies (28), then it is also an estimator which satisfies (25), but not vice versa. The equality conditions of the graph signal recovery are less restrictive, since the error w.r.t. the zero-graph frequency can be neglected.

2) **Graph CRB on the Dirichlet energy:** The WMSE bound in Theorem 1 is a matrix bound. As such, it implies bounds on the marginal WMSE of each node and on submatrices that are related to subgraphs. In particular, by applying the trace operator on the bound from (23)-(24) and using the trace operator properties, (1), and (13), we obtain the associated graph CRB on the expected Dirichlet energy:

\[
\mathbb{E}[(\hat{\theta} - \theta)^T \mathbb{L}(\hat{\theta} - \theta)] \geq \text{Tr} (B(\theta)) = \text{Tr} \left( \mathbb{L} (\mathbb{U}^T J(\theta) \mathbb{U})^{\dagger} \mathbb{U}^T \right).
\]

For an unconstrained estimation problem, in which \( \mathbb{U} = I_M \), the bound in (29) is reduced to

\[
\mathbb{E}[(\hat{\theta} - \theta)^T \mathbb{L}(\hat{\theta} - \theta)] \geq \text{Tr} (B(\theta)) = \text{Tr} (\mathbb{L}^{\dagger}(\theta)).
\]

3) Efficiency:

**Definition 3.** A graph-unbiased estimator, in the sense of Proposition 7 that achieves the graph CRB in Theorem 7 is said to be an efficient estimator on the graph \( G(\mathcal{M}, \xi, W) \).

Similar to the conventional theory on estimators’ efficiency, it can be shown that if there is an estimator which satisfies (25) and it is not a function of \( \theta \), then this is an efficient estimator. Moreover, similar to the uniformly minimum variance unbiased estimator [41, p. 20], the graph-efficient estimator from Definition 3 is also the uniformly minimum risk unbiased (MRU) estimator, i.e., an estimator that is uniformly graph-unbiased (i.e., it satisfies (17)), and achieves minimum WMSE, defined in (5). However, while in classical estimation theory efficiency is associated with uniqueness, the efficient estimator defined in Definition 3 is not unique. For example, if \( \hat{\theta}_e \) is an efficient estimator, then any shifted estimator \( \hat{\theta}_e + c \mathbb{I} \), where \( c \in \mathbb{R} \) is a constant, is also an efficient estimator, since it also satisfies (25) and is not a function of \( \theta \).

In the following sections, we discuss the problem of estimating vector-valued node variables from noisy relative measurements. This problem arises in many network applications, such as localization in sensor networks and motion consensus [13], [16], synchronization of translations [27], [42], and state estimation in power systems [11], [12], where the vector \( \theta \) represents parameters such as positions, states, opinions, and voltages. The measurement sensor network in this model may be different from the physical network used in the WMSE cost in Subsection II-C. This model represents the fact that many cyber-physical systems consist of two interacting networks: an underlying physical system with topological structure and a sensor that may be with a different topology. Similarly, in other real-life applications such as in genetics, there exist dual networks, with a physical-world network and a second network that represents the conceptual or statistical world [43].
A. Model

We consider a noisy measurement of the weighted relative state of each edge, as follows:

\[ x_{m,k} = \bar{w}_{m,k}(\theta_m - \theta_k) + \nu_{m,k}, \quad \forall (m,k) \in \xi, \]  

where \( \{\nu_{m,k}\}, \forall (m,k) \in \xi, m > k, \) is an i.i.d. Gaussian noise sequence with variance \( \sigma^2 \). The sequence \( \{\bar{w}_{m,k}\}, \forall (m,k) \in \xi, \) contains positive weights that are given by the system parameters. We assume that the edge weights satisfy \( \bar{w}_{m,k} = \bar{w}_{k,m} \). Thus, from symmetry, the measurement and the noise sequences satisfy \( x_{k,m} = -x_{m,k} \) and \( \nu_{k,m} = -\nu_{m,k} \), respectively, \( \forall (m,k) \in \xi \). In general, \( \xi \) and \( \{\bar{w}_{m,k}\} \) that are associated with the measurements and determined by the sensing approach, are different from \( \xi \) and \( \{\bar{w}_{m,k}\} \) that are based on the physical graph or on graph that was generated from \( \text{a-priori} \) data. The goal here is to estimate the state vector, \( \theta = [\theta_1, \ldots, \theta_M]^T \), from the observation vector, \( \mathbf{x} \). For example, this problem with the model in (31) where \( \bar{w}_{m,k} = 1, \forall (m,k) \in \xi, \) is the problem of synchronization of translations from (24). Another example is PSSE in electrical networks, (10)–(12), as described in Section VII.

Let \( G(M, \bar{\xi}, \mathbf{W}) \) be defined as the measurement graph associated with the model in (31) over the same vertex set as in the “physical” graph, \( G(M, \xi, \mathbf{W}) \), which is used in (8). We assume that \( G(M, \bar{\xi}, \mathbf{W}) \) is a simple graph and define its associated Laplacian matrix, \( \mathbf{L} \), with the elements

\[ \mathbf{L}_{m,k} = \begin{cases} \sum_{k \in \mathcal{N}_m} \bar{w}_{m,k} & \text{if } m = k \\ -\bar{w}_{m,k} & \text{if } (m,k) \in \bar{\xi} \\ 0 & \text{otherwise} \end{cases}, \]  

where, similar to Definition 1, \( \mathcal{N}_m = \{k \in \mathcal{M} : (m,k) \in \bar{\xi}\} \) be the oriented incidence matrix of the graph \( G(M, \bar{\xi}, \mathbf{W}) \), where each of its columns has two nonzero elements, 1 at the \( m \)th row and \(-1\) at the \( k \)th row, representing an edge connecting nodes \( m \) and \( k \), where the sign is chosen arbitrarily. Then, the model in (31) can be rewritten in a matrix form as follows:

\[ \mathbf{E}\mathbf{x} = \mathbf{L}\theta + \mathbf{E}\nu, \]  

where \( \mathbf{x}, \mathbf{w}, \) and \( \mathbf{\nu} \) include the elements \( \{x_{m,k}\}, \{\bar{w}_{m,k}\}, \{\nu_{m,k}\}, \) respectively, \( \forall (m,k) \in \bar{\xi}, m > k, \) in the same order, and we use the fact that \( \mathbf{L} = \mathbf{E}\text{diag}(\mathbf{w})\mathbf{E}^T \). Since \( \{\nu_{m,k}\}, m > k, \) is an i.i.d. Gaussian noise sequence with variance \( \sigma^2 \), then \( \mathbf{E}\nu \) is a zero-mean Gaussian vector with a covariance matrix \( \sigma^2\mathbf{E}\mathbf{E}^T \).

B. Graph CRB and an efficient estimator

The log-likelihood function for the modified model in (33), after removing constant terms and by using \( \mathbf{L}^T = \mathbf{L} \), is

\[ \log f(\mathbf{x}; \theta) = -\frac{1}{2\sigma^2}(\mathbf{E}\mathbf{x} - \mathbf{L}\theta)^T(\mathbf{E}\mathbf{E}^T)^\dagger(\mathbf{E}\mathbf{x} - \mathbf{L}\theta). \]  

Thus, the gradient of (34) satisfies

\[ \nabla_\theta \log f(\mathbf{x}; \theta) = \frac{1}{\sigma^2}\mathbf{E}(\mathbf{E}\mathbf{E}^T)^\dagger(\mathbf{E}\mathbf{x} - \mathbf{L}\theta). \]  

The matrix \( \mathbf{E}\mathbf{E}^T \) is a Laplacian matrix with \( M \) nodes and equal unit weights for all edges. Thus, its pseudo-inverse is given by (43)

\[ (\mathbf{E}\mathbf{E}^T)^\dagger = \left( \mathbf{E}\mathbf{E}^T - \frac{1}{M}\mathbf{1}\mathbf{1}^T \right)^{-1} + \frac{1}{M}\mathbf{1}\mathbf{1}^T. \]  

By substituting (36) in (35) and using the null-space property, \( \mathbf{L}\mathbf{1} = \mathbf{0} \), one obtains

\[ \nabla_\theta^T \log f(\mathbf{x}; \theta) = \frac{1}{\sigma^2} \mathbf{L}(\mathbf{E}\mathbf{E}^T - \frac{1}{M}\mathbf{1}\mathbf{1}^T)^{-1}(\mathbf{E}\mathbf{x} - \mathbf{L}\theta). \]  

Thus, the FIM from (22) for this case is given by

\[ \mathbf{J}(\theta) = \frac{1}{\sigma^2} \mathbf{L}(\mathbf{E}\mathbf{E}^T - \frac{1}{M}\mathbf{1}\mathbf{1}^T)^{-1}\mathbf{L}. \]  

The FIM in (38) is a function of the graph topology via the Laplacian and the oriented incidence matrices. For the special case of unit weights, i.e. for \( \mathbf{L} = \mathbf{E}\mathbf{E}^T \), the FIM satisfies \( \mathbf{J}(\theta) = \frac{1}{\sigma^2} \mathbf{L} \), which coincides with the result in (27).

It is shown in Appendix D that the pseudo-inverse of the FIM in (38) is given by

\[ \mathbf{J}^\dagger(\theta) = \sigma^2\mathbf{L}^\dagger\mathbf{E}\mathbf{E}^T\mathbf{L}^\dagger. \]  

By substituting (39) and \( \mathbf{U} = \mathbf{I}_M \) (since there are no constraints in the considered model) in (23)–(24), we obtain that the graph CRB for this case is

\[ \mathbf{E}[\mathbf{C}(\theta, \theta)] = \mathbf{B}(\theta) = \sigma^2\mathbf{A}^\dagger\mathbf{V}^T\mathbf{L}^\dagger\mathbf{E}\mathbf{E}^T\mathbf{L}^\dagger\mathbf{V}\mathbf{A}^\dagger, \]  

which is not a function of the specific values of \( \theta \). By applying the trace operator on (40), we obtain the bound on the expected Dirichlet energy, which is

\[ \mathbf{E}[(\hat{\theta} - \theta)^T(\hat{\theta} - \theta)] \geq \text{Tr}(\mathbf{B}(\theta)) = \sigma^2\text{Tr}(\mathbf{E}\mathbf{E}^T\mathbf{L}^\dagger\mathbf{L}^\dagger\mathbf{E}). \]  

The graph CRBs are lower bounds on the WMSE and on the Dirichlet energy of any graph unbiased estimators as defined for this unconstrained setting in (18).

For the estimator, by substituting (35), (39), and \( \mathbf{U} = \mathbf{I}_M \) in (25), equality in (40) is obtained iff

\[ \mathbf{A}^\dagger\mathbf{V}^T(\hat{\theta} - \theta) = \mathbf{A}^\dagger\mathbf{V}^T\mathbf{L}^\dagger\mathbf{E}\mathbf{E}^T\mathbf{L}^\dagger\mathbf{V}\mathbf{A}^\dagger(\mathbf{E}\mathbf{x} - \mathbf{L}\theta). \]  

By using the properties of the Laplacian and incidence matrices, as well as pseudo-inverse properties, it can be shown that \( \mathbf{E}\mathbf{E}^T\mathbf{L}^\dagger\mathbf{L}(\mathbf{E}\mathbf{E}^T)^\dagger = \mathbf{E} = \mathbf{E}^T\mathbf{E}^T\mathbf{L}^\dagger\mathbf{L} \). Thus, (42) implies the following condition for achievability of the bound:

\[ \mathbf{A}^\dagger\mathbf{V}^T\hat{\theta} = \mathbf{A}^\dagger\mathbf{V}^T\mathbf{L}^\dagger\mathbf{E}\mathbf{x} + \mathbf{A}^\dagger\mathbf{V}^T(\mathbf{I}_M - \mathbf{L}^\dagger\mathbf{L})\theta \]

\[ = \mathbf{A}^\dagger\mathbf{V}^T\mathbf{L}^\dagger\mathbf{E}\mathbf{x} + \frac{1}{M}\mathbf{A}^\dagger\mathbf{V}^T\mathbf{1}\mathbf{1}^T\theta \]

\[ = \mathbf{A}^\dagger\mathbf{V}^T\mathbf{L}^\dagger\mathbf{E}\mathbf{x}, \]  

where the second equality is obtained by using the fact that for connected graphs, there is only one zero eigenvalue of the Laplacian matrix which is associated with the eigenvector \( \mathbf{v}_1 = \frac{1}{\sqrt{M}}\mathbf{1}, \) and the last equality is obtained by using the Laplacian null-space property, \( \mathbf{V}^T\mathbf{1} = \mathbf{0} \). Then the estimator on the r.h.s. of (43) is not a function of \( \theta \), then it is also an efficient estimator, in the sense of Definition 3. By using
the null-space property $V^T 1 = 0$, it can be verified that any estimator of the form
\[ \hat{\theta} = \tilde{L}^T \text{Ex} + c1, \] (44)
with an arbitrary scalar $c \in \mathbb{R}$, satisfies the equality condition in (43). Thus, the efficient estimator is not unique and the true signals, $\theta$, can be recovered up to a constant vector, $c1$. This result is reasonable, since, with relative measurements alone, as given in the model in (31), determining the signal is possible only up to an additive constant [15], [16], [18].

C. Sample allocation

In this subsection, we design a sample allocation rule for the sensing model from Subsection V.A based on solving an optimization problem that aims to minimize the graph CRB in (41). In this case, the graph CRB is achievable by the efficient estimator in (44) and is not a function of the unknown graph signal, $\theta$. Thus, minimizing the graph CRB in (41) will result in the minimum WMSE over graph unbiased estimators.

In many cases, the physical topology is known and is the one that is both used in the cost function in (5) and governs the measurement model in (31) for any edge that is measured. Thus, in the following we assume that the edge weights in (31) satisfy $\bar{w}_{m,k} = w_{m,k}, \forall (m,k) \in \xi$, and the sensors can be located only in existing edges of the physical network, where if $(m,k) \in \xi$, then $(k,m) \in \xi$. We assume a constrained amount of sensing resources, e.g., due to limited energy and communication budget. We thus state the sensor placement problem as follows:

Problem 1. Given a graph $G(M, \xi, W)$, find the smallest subset $\xi \subset \xi$ such that the graph signal, $\theta$, can be correctly recovered (up-to-a-constant) by the measurements on $\xi$ and the graph CRB from (41), $\sigma^2 \text{Tr}(\tilde{L}^T \tilde{L} \tilde{L}^T \tilde{E} \tilde{E}^T)$, is minimized, where $\bar{w}_{m,k} = w_{m,k}, \forall (m,k) \in \xi \cap \xi$.

The requirement of correct graph signal recovery in Problem 1 is defined as recovery up to a constant vector, $c1$, which is an inherent ambiguity with relative measurements. In order to correctly reconstruct the full graph signal by the estimator from (44) up-to-a-constant, we need to have $\text{rank}(\tilde{L}) = M - 1$. That is, for each node, $m \in M$, we need to have at least one edge in $\xi$ which is associated with this node. It can be shown that this condition for unique up-to-a-constant recovery is that $\xi$ is a spanning tree of $G(M, \xi, W)$ [46]. Thus, Problem 1 is equivalent to the following optimization problem.

Problem 2. Given a graph $G(M, \xi, W)$, find the Laplacian matrix $\tilde{L}^*$ such that
\[ \tilde{L}^* = \arg \min_{\tilde{L}} \text{Tr}(\tilde{L}^T \tilde{E} \tilde{E}^T \tilde{L}^T \tilde{L}), \text{ such that } \tilde{L} \in S_T(G), \] (45)
where $S_T(G)$ is the set of spanning trees of $G(M, \xi, W)$.

If $\tilde{L}$ is a spanning tree of a connected (loopless) graph, then it can be seen that its oriented incidence matrix, $E$, has $M - 1$ linearly independent columns. Thus, pseudo-inverse properties $E^T (EE^T)^{-1} = (EE^T)^{-1} E^T = I_{M-1}$. Thus, by substituting $\tilde{L} = \text{Ediag}(\bar{w}) \tilde{E}^T$, we obtain that for spanning trees
\[ \tilde{L} = \tilde{L}(E \tilde{E}^T)^{-1} \tilde{L} = \text{Ediag}(\bar{w}) \tilde{E}^T (E \tilde{E}^T)^{-1} \text{Ediag}(\bar{w}) \tilde{E}^T = \text{Ediag}(\bar{w}) \text{diag}(\bar{w}) \tilde{E}^T. \] (46)
That is, $\tilde{L}$ is also a Laplacian matrix of a spanning tree of $L$ with the weights $\{w_{m,k}^2\}, \forall (m,k) \in \xi$, and the optimization from (45) is reduced to:

Problem 3. Given a graph $G(M, \xi, W)$, find the Laplacian matrix $\tilde{L}^*$ such that
\[ \min_{\tilde{L}} \text{Tr}(\tilde{L}^T \tilde{L}), \text{ such that } \tilde{L} \in S_T(G), \] (47)
where $\tilde{L} = \text{Ediag}(\bar{w}) \text{diag}(\bar{w}) \tilde{E}^T$ and $\tilde{L} = \text{Ediag}(\bar{w}) \tilde{E}^T$.

It is shown in [27] that
\[ \text{Tr}(\tilde{L}^T \tilde{L}) = st_{\tilde{L}}(\tilde{L}), \] (48)
where $st_{\tilde{L}}(\tilde{L})$ is the total stretch of the graph $G(M, \xi, W)$ w.r.t. the spanning tree represented by $\tilde{L}$. Total stretch is a parameter used to measure the quality of a spanning tree in terms of distance preservation and can represent the average effective resistance [47], [48]. The objective function in Problem 3 is the $\ell _{-\frac{1}{2}}$ total stretch of the graph $G(M, \xi, W)$ [49]. Thus, the problem of finding the minimum graph CRB is equivalent to the problem of finding a spanning tree with minimum average $\ell _{\frac{1}{2}}$ stretch, which is a classical problem in network design, graph theory, and discrete mathematics. Although this problem in general is an NP-hard problem, it was shown that a standard maximum weight spanning tree algorithm [50] yields good results in practice [51]. Thus, in this paper we solve the sensor allocation problem in Problem 3 by finding the minimum weight spanning tree of $G(M, \xi, W)^2$, which is one of the fundamental easy problems of algorithmic graph theory and can be solved by existing algorithms [50].

VI. EXAMPLE 2: BANDLIMITED GRAPH SIGNAL RECOVERY

In this section, we consider the problem of signal recovery from noisy, corrupted, and incomplete measurements, under the constraint of bandlimited graph signals [52], [53]. We assume the following measurement model:
\[ x[n] = M(\theta + w[n]), n = 0, \ldots, N - 1, \] (49)
where $n$ represents a time index, $\theta$ is an unknown signal, $w[n]$, $n = 0, \ldots, N - 1$ are time-independent noise vectors with zero mean and a known distribution, and $M \in \mathbb{R}^{D \times M}$ is a mask matrix with $D \leq M$. The task is to recover the true signal, $\theta$, based on the accessible measurement vector, $x = [x[0], \ldots, x[N - 1]]^T$. Signal recovery from inaccessible and corrupted measurements requires additional knowledge of signal properties. In GSP, a widely-used assumption is that the signal of interest, $\theta$, is a graph-bandlimited signal [2], [4], i.e. its GFT, $\tilde{\theta}$, defined in (2), is a $R$-sparse vector. Here we assume a graph low-frequency signal defined as follows:

Definition 4. A graph signal $\theta \in \mathbb{R}^M$ is bandlimited w.r.t. a GFT basis $\mathbb{V}$, as defined in (2), with bandwidth $R$ when
\[ \tilde{\theta}_m = 0, \quad \forall m = R + 1, \ldots, M. \] (50)
The condition in (50) can be represented as the linear constraint from (4) with
\[ G = Q \mathbf{V}^T \text{ and } \mathbf{a} = 0_{(M-R) \times 1}, \quad (51) \]
where \( Q \) is an \((M - R) \times M\) matrix of the form \( Q = \begin{bmatrix} 0_{1:(M-R),1:R} & \mathbf{I}_{M-R} \end{bmatrix} \). Thus, a null-space matrix \( \mathbf{U} \), defined in (6), can be chosen to be \( \mathbf{U} = \mathbf{V}Q \), where \( Q = \begin{bmatrix} \mathbf{I}_R \ 0_{1:(M-R),1:R} \end{bmatrix} \). By substituting these values in (51) it can be verified that in this case an estimator, \( \hat{\theta} \), is a Lehmann-unbiased estimator of \( \theta \) under the constrained set in (50) iff
\[ \hat{Q}^T \mathbf{V}^T \Lambda \hat{\mathbf{q}} \mathbf{V} e\theta \hat{\theta} = 0, \quad \forall \theta \in \Omega_R^R, \quad (52) \]
where \( \Omega_R^R \) is the subspace of \( R \)-bandlimited graph signals that satisfy Definition (4). By using the GFT operator from (2), as well as the fact that \( \lambda_1 = 0 \), it can be verified that the unbiasedness condition in (52) is equivalent to requiring that
\[ e\theta \hat{\theta}_m = 0, \quad \forall m \leq R. \quad (53) \]
The condition in (53) reflects the fact that the high graph frequencies are known to be zero for a \( R \)-bandlimited graph signal, and, thus, there is no need for an unbiasedness condition on frequencies higher than \( R \). In addition, since the proposed cost is oblivious to the estimation error of a constant bias over the graph, \( c_1 \), then the estimator of the first graph frequency, \( \hat{\theta}_1 \), can also have an arbitrary bias.

By substituting \( \mathbf{U} = \mathbf{V}Q \), \( \mathbf{V}^T \mathbf{V} = \mathbf{I} \), and the model from (29) in (24), we obtain that the proposed graph CRB for graph bandlimited signals is given by
\[ B(\theta) = \frac{1}{N} \Lambda^{\frac{1}{2}} \hat{Q} \left( \mathbf{Q}^T \mathbf{V}^T \mathbf{M}^T \mathbf{J}(\theta) \mathbf{M} \mathbf{V} \hat{Q} \right)^\dagger \hat{Q}^T \Lambda^{\frac{1}{2}}, \quad (54) \]
where \( \mathbf{J}(\theta) \) is the FIM for a single measurement \( \theta + \mathbf{w}[n] \). Since \( |A|_{1,1} = 0 \), and due to the structure of \( Q \), the first row and the last \( M - R \) rows of \( \Lambda^{\frac{1}{2}} \) are zero rows. Thus, the relevant bound in (24) is the submatrix \( B(\theta)|_{2:R,2:R} \), which deploys a bound in the estimation error of the low graph frequencies. As a result, by substituting (1), (4), and (54) in (29), by using the trace operator properties and the definition of \( \hat{Q} \), and under the assumption that \( R \leq D \), we obtain that the graph CRB on the expected Dirichlet energy is
\[ \sum_{m=2}^{R} \lambda_m E\theta[(\hat{\theta}_m - \hat{\theta}_m)^2] \geq \frac{1}{N} \sum_{m=2}^{R} \lambda_m \left[ (\mathbf{V}^T_{1:M,1:R} \mathbf{M} \mathbf{J}(\theta) \mathbf{M} \mathbf{V}^T_{1:M,1:R})^{-1} \right]_{m,m}. \quad (55) \]
It can be seen that a necessary and sufficient condition for a unique up-to-a-constant reconstruction of a graph bandlimited signal as defined in Definition (50), i.e. the reconstruction of \( \theta_m, m = 2, \ldots, R \), is that \( \text{rank}(\mathbf{V}^T_{1:M,1:R} \mathbf{M} \mathbf{J}(\theta) \mathbf{M} \mathbf{V}^T_{1:M,1:R}) = R \). For the special case where \( \mathbf{M} \) is a sampling matrix associated with the subset of nodes \( S \), i.e. it satisfies \( \mathbf{M}_{:,S} = \mathbf{I}_S \) and \( \mathbf{M}_{:,S^c} = 0 \), then \( \mathbf{M} \mathbf{V}^T_{1:M,1:R} = \mathbf{V} S,1:R \) and (55) is reduced to
\[ \sum_{m=2}^{R} \lambda_m E\theta[(\hat{\theta}_m - \hat{\theta}_m)^2] \geq \frac{1}{N} \sum_{m=2}^{R} \lambda_m \left[ (\mathbf{V}^T_{S,1:R} \mathbf{M} \mathbf{J}(\theta) \mathbf{V} S,1:R)^{-1} \right]_{m,m}. \quad (56) \]
The graph CRB in (56) allows us to define a criterion for sampling set selection. We thus state the sensor placement problem as follows:

**Problem 4.** Given a graph \( G(M, \mathcal{E}, W) \), a cutoff frequency, \( \lambda_R \), and a number of sensors \( L \geq R \), find the subset of nodes (sensor placements) \( S^* \) such that all \( R \)-bandlimited signals can be uniquely recovered from their samples on this subset and the worst-case graph CRB from (56) is minimized, i.e.
\[ S^* = \arg \min_{S:|S|=L} \max_{\theta \in \Omega_R^R} \frac{1}{N} \sum_{m=2}^{R} \lambda_m \left[ (\mathbf{V}^T_{S,1:R} \mathbf{M} \mathbf{J}(\theta) \mathbf{V} S,1:R)^{-1} \right]_{m,m}. \]

Since finding the optimal set in Problem 4 is an NP-hard problem, a greedy algorithm is described in Algorithm 1. At each iteration of this algorithm we remove the node that maximally increases the graph CRB in (56) (maximized w.r.t. \( \theta \)), until we obtain the subset \( S \subset M \) with cardinality \( L \).

**Algorithm 1 Sensor allocation for bandlimited graph signals**

**Input:** 1) Laplacian matrix, \( \mathbf{L} = \mathbf{V} \Lambda \mathbf{V}^T \), 2) number of sensors \( L \geq R \); and 3) cutoff frequency, \( \lambda_R \)

**Output:** Subset of sensors, \( S \)

1: Initialize the set of available locations, \( L^{(0)} = M \)
2: Initialize iteration index, \( i = 0 \)
3: while \( |L^{(i)}| > L \) do
4: Find the optimal node to remove:
5: \( w = \arg \min_{w \in L^{(i)}} \max_{\theta \in \Omega_R^R} \frac{1}{N} \sum_{m=2}^{R} \lambda_m \left[ (\mathbf{V}^T_{L^{(i)} \setminus w,1:R} \mathbf{M} \mathbf{J}(\theta) \mathbf{V} L^{(i)} \setminus w,1:R)^{-1} \right]_{m,m}. \)
6: Update the available locations, \( L^{(i)} \leftarrow L^{(i-1)} \setminus w \), and the iteration index, \( i \leftarrow i + 1 \)
7: end while
8: Update the set of removed locations: \( S = M \setminus L^{(i)} \)

In the following we develop an efficient estimator in the graph frequency domain for the special case where \( \mathbf{w}[n] \) is zero-mean Gaussian noise with a known covariance matrix, \( \Sigma \). By substituting \( \mathbf{U} = \mathbf{V}Q \) and the model from (29) in the equality condition of Theorem 1 in (25), we obtain
\[ \Lambda^{\frac{1}{2}}(\hat{\theta} - \theta) = \Lambda^{\frac{1}{2}} \hat{Q} \left( \mathbf{Q}^T \mathbf{M}^T \Sigma^{-1} \mathbf{Q} \right)^\dagger \hat{Q}^T \Sigma^{-1} \left( \frac{1}{N} \sum_{n=1}^{N-1} \mathbf{x}[n] - \mathbf{M} \hat{\theta} \right), \quad (57) \]
\[ \forall \theta \in \Omega_R^R, \text{ where } \hat{\theta} = \mathbf{Q} \left( \mathbf{Q}^T \mathbf{M}^T \Sigma^{-1} \mathbf{Q} \right)^\dagger \hat{Q}^T \mathbf{M}^T \left( \frac{1}{N} \sum_{n=1}^{N-1} \mathbf{x}[n] \right). \quad (58) \]
satisfies the efficiency condition in (57). By using the definition of $Q$, the estimator from (58) can be written as

$$
\hat{\theta}_{1:N} = \left( Q^T M^T \Sigma^{-1} M Q \right)^{\dagger} \left( Q^T M^T \Sigma^{-1} \sum_{n=0}^{N-1} x[n] \right)
$$

(59)

Since $\hat{\theta}$ is also the constrained ML (CML) estimator of $\theta$, then, by using the invariance property of the ML estimator [41] and [59], the CML estimator of $\theta$ is given $\hat{\theta}_{\text{CML}} = V \hat{\theta}$.

VII. Simulations

In this section, we evaluate the performance of the proposed graph CRB, estimators, and sensor allocation methods. In Subsection VII-A we present simulations of the linear Gaussian model with relative measurements, as described in Section V. In Subsection VII-B we present simulations of the bandlimited graph signal recovery, as described in Section VI.

The performance of all estimators is evaluated using at least 1,000 Monte-Carlo simulations.

We consider the following two test cases:

1) State estimation in electrical networks: A power system can be represented as an undirected weighted graph, $G(M, \xi, W)$, where the set of vertices, $M$, is the set of buses (generators or loads) and the edge set, $\xi$, is the set of transmission lines between these buses. The weight matrix, $W$, is determined by the branch susceptances [12], [18]. The goal of PSSE, which is at the core of energy management systems for various monitoring and analysis purposes, is to estimate $\theta$ based on system measurements [17]. Since $\theta$ is measured over the buses of the electrical network, PSSE can be interpreted as a graph signal recovery problem. The simulations of this test case are implemented on the IEEE 118-bus system from [54] that represents a portion of the American Electric Power System. Sensor allocation in power systems is of great importance (see, e.g. [53], [56] and references therein).

2) Graph signal recovery in random graphs: In this test case we simulate synthetic graphs from the Watts-Strogatz small-world graph model [52] with varying numbers of nodes, $M$, and an average nodal degree of 4.

A. Linear Gaussian model with relative measurements

In this subsection we consider the linear Gaussian model with relative measurements from Section VI.C. We compare the estimation performance of the estimator from (44) and the proposed graph CRB from (41) for the following sensor allocation policies:

1) Max. ST - maximum spanning tree of $G(M, \xi, W^2)$, which is the approximated solution of Problem 3.

2) Min. ST - minimum spanning tree of $G(M, \xi, W^2)$, which can be considered as the worst-case solution of Problem 3.

3) Rand ST - an arbitrary spanning tree of $G(M, \xi, W^2)$, choosing randomly over the network.

To find the minimum and maximum spanning trees in 1 and 2 we use the MATLAB function $\text{graphminspantree}$, which has a computational complexity of $O(M^2 \log(M))$.

First, we consider PSSE with supervisory control and data acquisition (SCADA) sensors that measure the power flow at the edges. The linear approximations of the power flow model, named DC model [17], which represents the measurement vector of the active powers at the buses, $x$, can be written as the model in (31), where $\theta_m$ is the voltage phase at bus $m$, $\forall m = 1, \ldots, M$, and $w_{m,k} = w_{m,k}, \forall (m,k) \in \xi$, are the susceptances of the transmission lines. Since the DC model is an up-to-a-constant model, conventional PSSE considers a reference bus with $\hat{\theta}_1 = 0$ and then uses the ML estimator of the other parameters, $\hat{\theta}_{2:M} = (L_{2:M})^\dagger x$. This procedure is equivalent to the simulated estimator from (44), where $c$ is chosen such that $\bar{\theta}_1 = 0$.

The root WMSE (square of the Dirichlet energy) of the estimator from (44) and the root of the graph CRB are presented for PSSE in electrical networks under the different sensor allocation policies is presented in Fig. 1.a versus $\frac{\bar{\theta}_1}{\sigma}$, where $\sigma^2$ is the variance of the Gaussian noise, $\nu_{m,k}$, from [51]. Similarly, in Fig. 1b the estimation performance and the root of the graph CRB are presented for a random graph with the three sample allocation policies versus the number of nodes in the system for $\sigma^2 = 1$. It can be seen that in both figures the maximum spanning tree policy has significantly lower Dirichlet energy than that of the random and the minimum spanning tree schemes, where the minimum spanning tree is worse than random sampling by an arbitrary spanning tree. The results indicate that by applying knowledge of the physical nature of the grid, we can achieve significant performance gain by using limited numbers of well-placed sensors. Finally, it can be verified that since the estimator from (44) is an efficient estimator on the graph $G(M, \xi, W)$ in the sense of Definition 3 then it coincides with the associated graph CRB for all the considered scenarios.

B. Bandlimited graph signal recovery

In this subsection we consider the problem of bandlimited graph signal recovery from Section VII for electrical networks, where $w[n] = \text{zero-mean Gaussian noise with a known co-variance matrix, } \Sigma = \sigma^2 I$. Thus, in this case $J(\theta) = \frac{1}{\sigma^2} I$.

In this case, we assume a PSSE with direct access to a limited number of state (voltage) measurements. Thus, we assume the model in (49) for the special case where $M$ is a sampling matrix associated with the subset of nodes $S$, i.e. it satisfies $M_{S,S} = I_S$ and $M_{S,S^c} = 0$. This can be obtained in practical electrical networks by using Phasor Measurement Units (PMUs) [58]. However, installing PMUs onto all possible buses is impossible due to a budget constraint and limitations on power and communication resources. Thus, there is a need to establish a method to determine which information should be observed in the course of designing electrical networks. In power systems, the voltage signal, $\theta$, is shown to be smooth [11], [59]. Similarly, it can be shown to be a graph bandlimited signal. However, the existing PSSE methods do not incorporate the smoothness and bandlimitedness constraints.

In Figs. 2a and 2b the root WMSE of the estimator from (59) and the root of the graph CRB are presented for
Fig. 1: Linear Gaussian model with relative measurements: The graph CRB and the root WMSE for the three sensor allocation policies (Max. ST, Min. ST, and rand ST) for a) state estimation in power systems versus \(1/\sigma^2\), in IEEE 118-bus system; and b) signals in random graphs versus the number of nodes in the network, for \(\sigma^2 = 1\).

Fig. 2: Bandlimited graph signal recovery: The graph CRB and the root WMSE for a random sensor allocation policy and for Algorithm 1 for state estimation in power systems in IEEE 118-bus system for a) real-data values and assumed \(R = 45\) cutoff frequency versus \(1/\sigma^2\); and b) \(R = 10\) bandlimited graph signal versus the number of selected sensors.

VIII. CONCLUSION

We consider the problem of graph signal recovery as a non-Bayesian parameter estimation under the Laplacian-based WMSE performance evaluation measure. We develop the graph CRB, which is a lower bound on the WMSE of any graph-unbiased estimator in the Lehmann’s definition sense. We present new sensor allocation policies that aim to reduce the graph CRB under a constrained amount of sensing nodes. The relation between the problem of finding the optimal sensor locations of relative measurements in the graph-CRB sense and the problem of finding the maximum weight spanning tree of a graph is demonstrated. The proposed graph CRB is evaluated and compared with the Laplacian-based WMSE of the ML estimator, for signal recovery over random graphs and for PSSE in electrical networks. Significant performance gains are observed from these simulations for using the optimal sensor locations. Thus, the proposed graph CRB can be used as a system design tool for sensing networks. Future work includes extensions to graph signal recovery in complex-value systems, as well as methods that include the cost of communication and computation, in order to assess the performance of distributive algorithms for graph signal recovery.

APPENDIX A: DERIVATION OF (10)

By using the facts that \(\left( V \Lambda^{\frac{1}{2}} \right)^T = \Lambda^{\frac{1}{2}} V^T\) and that \(\Lambda^{\frac{1}{2}}\) is a diagonal matrix, the \((m,n)\)th element of \(C(\hat{\theta}, \theta)\), defined
in (7), satisfies
\[
[C(\hat{\theta}, \theta)]_{m,n} = \sum_{k=1}^{M} \sum_{l=1}^{M} [A_k^T V^T]_{m,k} [A_l^T V^T]_{n,l} \epsilon_k \epsilon_l
\]
\[
= \Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m} V_{l,n} \epsilon_k \epsilon_l
\]
\[
= \frac{1}{2} \Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m} V_{l,n} (\epsilon_k - \epsilon_l)^2
\]
\+
\Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m}^2 V_{l,n}^2.
\]
where \( \epsilon_m \triangleq \hat{\theta}_m - \theta_m, m = 1, \ldots, M \) is defined in (11). The smallest eigenvalue of the Laplacian matrix is always \( \lambda_1 = 0 \) and is associated with the eigenvector \( v_1 = \frac{1}{\sqrt{M}} \). It can be seen that \( (\lambda_1, v_1) \) is also a pair of eigenvalue-eigenvector of \( L^+ \), and thus,
\[
\Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} V_{l,n} = 0, \forall n, l = 1, \ldots, M.
\]
By substituting (61) in (60), we obtain
\[
[C(\hat{\theta}, \theta)]_{m,n} = \frac{1}{2} \Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m} V_{l,n} (\epsilon_k - \epsilon_l)^2.
\]
By substituting
\[
(\epsilon_k - \epsilon_l)^2 = (\epsilon_k - \epsilon_m + \epsilon_m - \epsilon_l + \epsilon_l - \epsilon_n)^2
\]
\[
= (\epsilon_k - \epsilon_m)^2 + (\epsilon_l - \epsilon_n)^2 + (\epsilon_m - \epsilon_l)^2
\]
\[
- 2[(\epsilon_k - \epsilon_m)(\epsilon_l + \epsilon_n) + (\epsilon_k - \epsilon_m)(\epsilon_m - \epsilon_l)]
\]
\[
= (\epsilon_k - \epsilon_m)(\epsilon_l + \epsilon_n) + (\epsilon_k - \epsilon_m)(\epsilon_m - \epsilon_l)
\]
\[
+ (\epsilon_l - \epsilon_n)(\epsilon_m - \epsilon_l) + (\epsilon_l + \epsilon_n)(\epsilon_m - \epsilon_l),
\]
into (62), we obtain
\[
[C(\hat{\theta}, \theta)]_{m,n} = \Lambda_{m,m}^{1/2} \Lambda_{n,n}^{1/2} \sum_{k=1}^{M} \sum_{l=1}^{M} V_{k,m} V_{l,n}
\]
\[
\times \left( \frac{1}{2} [(\epsilon_k - \epsilon_m)^2 + (\epsilon_l + \epsilon_n)^2 + (\epsilon_m - \epsilon_l)^2]
\]
\[
+ [(\epsilon_k - \epsilon_m)(\epsilon_l + \epsilon_n) + (\epsilon_k - \epsilon_m)(\epsilon_m - \epsilon_l)]
\]
\[
+ (\epsilon_l - \epsilon_n)(\epsilon_m - \epsilon_l) + (\epsilon_l + \epsilon_n)(\epsilon_m - \epsilon_l) \right).
\]
By substituting (61) in (63), we obtain the term in (10).

APPENDIX B: PROOF OF PROPOSITION 1

The scalar Lehmann unbiasedness definition is extended in p. 13 in (60) to matrix cost functions, as follows:

**Definition 5.** The estimator, \( \hat{\theta} \), is said to be a uniformly unbiased estimator of \( \theta \) in the Lehmann sense w.r.t. a general positive semidefinite matrix cost function, \( C(\hat{\theta}, \theta) \), if
\[
E_{\theta}[C(\hat{\theta}, \theta)] \succeq E_{\theta}[C(\hat{\theta}, \theta)], \forall \eta, \theta \in \Omega_{\theta},
\]
where \( \Omega_{\theta} \) is the parameter space.

In this appendix, we prove that the graph-unbiasedness is obtained from the Lehmann unbiasedness with the cost function from (7) and under the constrained set in (4). By substituting (7) in (65), we obtain that the Lehmann unbiasedness in this case requires that
\[
\Lambda^{+} V^{T} E_{\theta} \left[ (\hat{\theta} - \eta)(\hat{\theta} - \eta)^{T} \right] V \Lambda^{+}
\]
\[
\succeq \Lambda^{+} V^{T} E_{\theta} \left[ (\hat{\theta} - \theta)(\hat{\theta} - \theta)^{T} \right] V \Lambda^{+},
\]
(66)
\[\forall \theta, \eta \in \Omega_{\theta}.\]

Similar to the derivation in p. 14 of (31), on adding and subtracting \( E_{\theta}[\theta] \) inside the four round brackets in (66), this condition is reduced to
\[
\Lambda^{+} V^{T} E_{\theta} \left[ (E_{\theta}[\theta] - \eta)(E_{\theta}[\theta] - \eta)^{T} \right] V \Lambda^{+}
\]
\[
\succeq \Lambda^{+} V^{T} E_{\theta} \left[ (E_{\theta}[\theta] - \theta)(E_{\theta}[\theta] - \theta)^{T} \right] V \Lambda^{+},
\]
(67)
\[\forall \theta, \eta \in \Omega_{\theta}.\]

By using the definition of the constrained set in (5), and since the range of \( U \) is the null-space of \( G \), it can be seen that for a given \( \theta \in \Omega_{\theta} \), any \( \eta \in \Omega_{\theta} \) can be written as (see, e.g. Section 4.2.4 in (61))
\[
\eta = \theta + U w,
\]
(68)
for some vector \( w \in \mathbb{R}^{M-K} \). By substituting (68) in (67), we obtain
\[
\Lambda^{+} V^{T} E_{\theta} \left[ (E_{\theta}[\theta] - \theta - U w)(E_{\theta}[\theta] - \theta - U w)^{T} \right] V \Lambda^{+}
\]
\[
\succeq \Lambda^{+} V^{T} E_{\theta} \left[ (E_{\theta}[\theta] - \theta)(E_{\theta}[\theta] - \theta)^{T} \right] V \Lambda^{+},
\]
(69)
\[\forall \theta, \eta \in \Omega_{\theta}, w \in \mathbb{R}^{M-K} \]. The condition in (69) can be rewritten as
\[
\Lambda^{+} V^{T} U w w^{T} U^{T} V \Lambda^{+}
\]
\[
\succeq \Lambda^{+} V^{T} \left( (E_{\theta}[\theta] - \theta) w^{T} U^{T} + U w (E_{\theta}[\theta] - \theta)^{T} \right) V \Lambda^{+},
\]
(70)
A necessary condition for (70) to be satisfied is that
\[
\text{Tr}(U^{T} L U w w^{T}) \geq 2 \text{Tr}(U^{T} L (E_{\theta}[\theta] - \theta) w^{T}),
\]
(71)
where we used the trace operator properties. Since the condition in (71) should be satisfied for any \( w \in \mathbb{R}^{M-K} \), and, in particular, for both \( \pm \epsilon w_{0} \in \mathbb{R}^{M-K} \) with small positive \( \epsilon \) and for the unit vectors \( e_{k} \in \mathbb{R}^{M-K}, k = 1, \ldots, M-K \), then a necessary condition is that (71) is satisfied.

APPENDIX C: PROOF OF THEOREM 1

The following proof for the development of the graph CRB is along the path of the development of the CCRB on the MSE in a conventional estimation problem in (49). Let \( W \in \mathbb{R}^{M \times M} \) be an arbitrary matrix. Then, the Cauchy-Schwartz inequality implies that
\[
\Lambda^{+} V^{T} E \left[ (\hat{\theta} - \theta - W U U^{T} \theta_{o}^{T}) \log f(\mathbf{x}; \theta) \right]
\]
\[
\times \left( \hat{\theta} - \theta - W U U^{T} \theta_{o}^{T} \right) V \Lambda^{+} \succeq 0.
\]
(72)
Under Condition (2), the matrix inequality in (72) implies that
\[
\Lambda^{+} V^{T} E[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^{T}] V \Lambda^{+}
\]
\[
\succeq \Lambda^{+} V^{T} E[(\theta - \theta) \log f(\mathbf{x}; \theta)] U U^{T} W^{T} V \Lambda^{+}
\]
\+
\Lambda^{+} V^{T} W U U^{T} E[\theta_{o}^{T} \log f(\mathbf{x}; \theta)(\hat{\theta} - \theta)^{T}] V \Lambda^{+}
\]
\[
- \Lambda^{+} V^{T} W U U^{T} J(\theta) U^{T} W^{T} V \Lambda^{+}.
\]
(73)
By using regularity condition \( \zeta \) it can be verified that (see, e.g. Appendix 3B in [41])

\[
E \left[ (\theta - \theta) \nabla_\theta \log f(x; \theta) \right] = I_M + \nabla_\theta E[\theta - \theta].
\] (74)

Thus, by multiplying (74) by \( L \) and \( U \) from left and right, respectively, and substituting the local graph-unbiasedness from (20), we obtain

\[
LE \left[ (\theta - \theta) \nabla_\theta \log f(x; \theta) \right] U = LU.
\] (75)

Or, equivalently, by using \( L = V \Lambda V^T \) and the fact that \( \Lambda \) is a nonnegative diagonal matrix,

\[
\Lambda^{\frac{1}{2}} V^T E \left[ (\theta - \theta) \nabla_\theta \log f(x; \theta) \right] U = \Lambda^{\frac{1}{2}} V^T U. \tag{76}
\]

By substituting (8) and (76) in (73), we obtain

\[
E[\theta] \mathbb{C}(\theta, \theta)] \geq \Lambda^{\frac{1}{2}} V^T U U^T V^T \Lambda^{\frac{1}{2}} + \Lambda^{\frac{1}{2}} V^T W U U^T V^T \Lambda^{\frac{1}{2}} - \Lambda^{\frac{1}{2}} V^T W U U^T J(\theta) U U^T W^T V^T \Lambda^{\frac{1}{2}}. \tag{77}
\]

Now, let \( W \) be such that

\[
W^T U = U(\nabla_\theta J(\theta) U)^\dagger. \tag{78}
\]

By substituting (78) in (77) and using \( A^\dagger A A^\dagger = A^\dagger \) for any matrix \( A \), we obtain the bound in (23), (44).

According to Cauchy-Schwartz conditions, for any given matrix \( W \), the equality holds in (73) iff

\[
\Lambda^{\frac{1}{2}} V^T (\theta - \theta) = \zeta(\theta) \Lambda^{\frac{1}{2}} V^T W U U^T V^T \lambda \nabla_\theta \log f(x; \theta), \tag{79}
\]

where \( \zeta(\theta) \) is a scalar function of \( \theta \). By substituting (78) in (79), we obtain the following condition for the proposed graph CRB:

\[
\Lambda^{\frac{1}{2}} V^T (\theta - \theta) = \zeta(\theta) \Lambda^{\frac{1}{2}} V^T U (U^T J(\theta) U)^\dagger U^T V^T \lambda \nabla_\theta \log f(x; \theta). \tag{80}
\]

Computing the expected quadratic term \( \langle a a^T \rangle \) of each side of (80), substituting (8) and (22), we obtain

\[
E[\theta] \mathbb{C}(\theta, \theta)] = \zeta^2(\theta) B(\theta), \tag{81}
\]

where \( B(\theta) \) is defined in (24). Thus, for obtaining equality in (22), we require \( \zeta(\theta) = \pm 1 \). It can be verified that if ordering for \( \theta \) from (81) to satisfy (20), we must take the positive value \( \zeta(\theta) = 1 \). By substituting \( \zeta(\theta) = 1 \) in (80), we obtain (25).

**APPENDIX D: DERIVATION OF (39)**

Since \( E E^T \) is a Laplacian matrix, it satisfies

\[
1^T \left( E E^T - \frac{1}{M}11^T \right)^{-1} = -1^T \text{ and } (E E^T - \frac{1}{M}11^T)^{-1} = \frac{1}{M}11^T.
\]

Under these results, the null-space property of \( L \), and \( L = L - \frac{1}{M}11^T + \frac{1}{M}11^T \), we obtain that (38) can be rewritten as

\[
J(\theta) = \frac{1}{\sigma^2} \left( L - \frac{1}{M}11^T \right) \times \left( EE^T - \frac{1}{M}11^T \right)^{-1} \left( L - \frac{1}{M}11^T \right). \tag{82}
\]

By using Lemma 3 from [62], we obtain

\[
J(\theta) = \sigma^2 \left( I - \frac{y^T \psi \psi^T}{\psi^T \psi} \left( L - \frac{1}{M}11^T \right)^{-1} \left( E E^T - \frac{1}{M}11^T \right) \right)
\times \left( \psi^T \psi \right)^{-1} \left( \frac{1}{M}11^T \right)^{-1} \left( L - \frac{1}{M}11^T \right), \tag{83}
\]

where \( \psi = \frac{1}{M}1 \), \( y^T = 1^T \), and we used the facts that \( 1^T \left( E E^T - \frac{1}{M}11^T \right)^{-1} = -1^T \) and \( (E E^T - \frac{1}{M}11^T)^{-1} = 1^T \). By substituting \( \psi \) and \( y \) in (83), one obtains

\[
J^i(\theta) = \sigma^2 \left( I - \frac{1}{M}11^T \right) \left( L - \frac{1}{M}11^T \right)^{-1} \times \left( E E^T - \frac{1}{M}11^T \right) \left( L - \frac{1}{M}11^T \right)^{-1} \left( I - \frac{1}{M}11^T \right). \tag{84}
\]

It is well known that the pseudo-inverse of a Laplacian matrix with \( M \) nodes is given by

\[
\tilde{L} = \left( L - \frac{1}{M}11^T \right)^{-1} + \frac{1}{M}11^T. \tag{85}
\]

By substituting (85) in (84) and using the null-space property of \( EE^T \) and \( \tilde{L}^i \), we obtain (39).

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