ON DENSE SUBSETS IN SPACES OF METRICS

YOSHITO ISHIKI

Abstract. In spaces of metrics, we investigate topological distributions of the doubling property, the uniform disconnectedness, and the uniform perfectness, which are the quasi-symmetrically invariant properties appearing in the David–Semmes theorem. We show that the set of all doubling metrics and the set of all uniformly disconnected metrics are dense in spaces of metrics on finite-dimensional and zero-dimensional compact metrizable spaces, respectively. Conversely, this denseness of the sets implies the finite-dimensionality, zero-dimensionality, and the compactness of metrizable spaces. We also determine the topological distribution of the set of all uniformly perfect metrics in the space of metrics on the Cantor set.

1. Introduction

For a metrizable space $X$, we denote by $M(X)$ the set of all metrics on $X$ that generate the same topology on $X$. Define a metric $D_X : M(X)^2 \rightarrow [0, \infty]$ by $D_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|$. In [10], the author introduced the notion of a transmissible property which unifies geometric properties defined by finite subsets of metric spaces, and proved that for every non-discrete metrizable space $X$, the set of all metrics in $M(X)$ not satisfying a transmissible property with a singular transmissible parameter is dense $G_\delta$ in $M(X)$. Since the doubling property and the uniform disconnectedness are transmissible properties with singular parameters, the set of all non-doubling metrics and the set of all non-uniformly disconnected metrics are dense $G_\delta$ in spaces of metrics (see [10]). In contrast to [10], in this paper, we investigate topological distributions of the doubling property, the uniform disconnectedness, and the uniform perfectness in spaces of metrics.

Niemytzki and Tychonoff [12] proved that a metrizable space $X$ is compact if and only if all metrics in $M(X)$ are complete, and the author [9] proved an ultrametric analogue of their result (see [9, Corollary 1.3], see also [5, Proposition 4.10]). Nomizu and Ozeki [13] proved that a second countable differentiable manifold is compact if and only if all Riemannian metrics on the manifold are complete. As a development of these theorems, in the present paper, we characterize the compactness and the finite-dimensionality or the 0-dimensionality of a metrizable

2010 Mathematics Subject Classification. Primary 54E45, Secondary 30L05.

Key words and phrases. Space of metrics, Ultrametric, Topological distribution.

1
space by the denseness of the set of all doubling metrics or all uniform disconnected metrics in the space of metrics.

In this paper, a topological space is said to be finite-dimensional (resp. 0-dimensional) if its covering dimension is finite (resp. 0). The definition and basic properties of the covering dimension can be seen in [1].

For a metric space \((X,d)\) and for a subset \(A\) of \(X\), we denote by \(\delta_d(A)\) the diameter of \(A\), and we define \(\alpha_d(A) = \inf\{d(x,y) \mid x \neq y, \ x, y \in A\}\). A metric space \((X,d)\) is said to be doubling if there exist \(\beta \in (0,\infty)\) and \(C \in [1,\infty)\) such that for every finite subset \(A\) of \(X\) we have \(\text{Card}(A) \leq C \cdot (\delta_d(A)/\alpha_d(A))^{\beta}\), where the symbol “Card” stands for the cardinality. Let \(X\) be a topological space. A subset \(S\) is said to be \(F_\sigma\) (resp. \(G_\delta\)) if \(S\) is the union of countably many closed subsets of \(X\) (resp. the intersection of countably many open subsets of \(X\)).

**Theorem 1.1.** Let \(X\) be a metrizable space. Then the space \(X\) is finite-dimensional and compact if and only if the set of all doubling metrics in \(M(X)\) is dense \(F_\sigma\) in \((M(X), \mathcal{D}_X)\).

A metric space is said to be uniformly disconnected if there exists \(\delta \in (0,1)\) such that for every non-constant finite sequence \(\{z_i\}_{i=1}^N\) in \(X\) we have \(\delta d(z_1, z_N) \leq \max_{1 \leq i \leq N} d(z_i, z_{i+1})\). This notion was introduced in [2] in a different but equivalent way. Note that a metric space is uniformly disconnected if and only if the metric space can be bi-Lipschitz embeddable into an ultrametric space (see [2, Proposition 15.7]). Similarly to Theorem 1.1, we obtain:

**Theorem 1.2.** Let \(X\) be a metrizable space. Then \(X\) is 0-dimensional and compact if and only if the set of all uniformly disconnected metrics in \(M(X)\) is dense \(F_\sigma\) in \((M(X), \mathcal{D}_X)\).

Let \(X\) be a set. A metric \(d\) on \(X\) is said to be an ultrametric if for all \(x,y,z \in X\) the metric \(d\) satisfies the so-called strong triangle inequality: \(d(x,y) \leq d(x,z) \vee d(z,y)\), where the symbol \(\vee\) stands for the maximum operator on \(\mathbb{R}\). We say that a set \(S\) is a range set if \(S\) is a subset of \([0,\infty)\) and \(0 \in S\). For a range set \(S\), we say that a metric \(d\) on \(X\) is \(S\)-valued if \(d(X)^2\) is contained in \(S\). For a range set \(S\), and for a topological space \(X\), we denote by \(\text{UM}(X,S)\) the set of all \(S\)-valued ultrametrics on \(X\) that generate the same topology on \(X\).

For a topological space \(X\), and for a range set \(S\), we define a function \(\mathcal{UD}_X^S : \text{UM}(X,S)^2 \rightarrow [0,\infty]\) by assigning \(\mathcal{UD}_X^S(d,\epsilon)\) to the infimum of \(\epsilon \in S \cup \{\infty\}\) such that for all \(x,y \in X\) we have \(d(x,y) \leq e(x,y) \vee \epsilon\), and \(e(x,y) \leq d(x,y) \vee \epsilon\). The function \(\mathcal{UD}_X^S\) is an ultrametric on \(\text{UM}(X,S)\) valued in \(\text{CL}(S) \cup \{\infty\}\), where \(\text{CL}(S)\) is the closure of \(S\) in \([0,\infty)\).

In [2], the author investigate the topological distributions of metrics not satisfying a transmissible property in \((\text{UM}(X,S), \mathcal{UD}_X^S)\).
We say that a range set \( S \) has \textit{countable coinitiality} if there exists a strictly decreasing sequence \( \{r_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( S \) convergent to 0 as \( i \to \infty \).

As an ultrametric analogue of Theorem 1.1 we obtain:

\[ \text{Theorem 1.3. Let } S \text{ be a range set with the countable coinitiality. Let } X \text{ be an ultrametrizable space. Then } X \text{ is compact if and only if the set of all doubling metrics in } \text{UM}(X, S) \text{ is dense } F_{\sigma} \text{ in } \text{UM}(X, S), \text{UD}_{X, S}. \]

\[ \text{Remark 1.1. There are some results on relations between topological properties of metrizable spaces and properties of spaces of ultrametrics. Let } X \text{ be an ultrametrizable space. Dovgoshey–Shcherbak } [5] \text{ proved that } X \text{ is compact if and only if all } d \in \text{UM}(X, [0, \infty)) \text{ are totally bounded, and proved that } X \text{ is separable if and only if for all } d \in \text{UM}(X, [0, \infty)) \text{ we have Card}(\{d(x, y) \mid x, y \in X\}) \leq \aleph_0. \]

Let \( c \in (0, 1) \). A metric space \((X, d)\) is said to be \( c \)-\textit{uniformly perfect} if for every \( x \in X \), and for every \( r \in (0, \delta_d(X)) \), there exists \( y \in X \) with \( c \cdot r \leq d(x, y) \leq r \). A metric space is said to be \textit{uniformly perfect} if it is \( c \)-uniformly perfect for some \( c \in (0, 1) \). We next investigate topological distributions of the set of all uniformly perfect metrics and its complement in the space of metrics on the Cantor set. In this paper, let \( \Gamma \) denote the Cantor set (\( \Gamma \) is also called the middle-third Cantor set).

\[ \text{Theorem 1.4. The following statements hold true:} \]

\begin{enumerate}
    \item The set of all uniformly perfect metrics in \( M(\Gamma) \) is dense \( F_{\sigma} \) in \( (M(\Gamma), \mathcal{D}_\Gamma) \).
    \item The set of all non-uniformly perfect metrics in \( M(\Gamma) \) is a dense \( G_{\delta} \) set in \( (M(\Gamma), \mathcal{D}_\Gamma) \).
\end{enumerate}

We say that a range set \( S \) is \textit{exponential} if there exist \( a \in (0, 1) \) and \( M \in [1, \infty) \) such that for every \( n \in \mathbb{Z}_{\geq 0} \) we have \([M^{-1}a^n, Ma^n] \cap S \neq \emptyset \).

As an ultrametric analogue of Theorem 1.1 we obtain:

\[ \text{Theorem 1.5. Let } S \text{ be a range set. Then the following hold true:} \]

\begin{enumerate}
    \item The set \( S \) is exponential if and only if the set of all uniformly perfect metrics in \( \text{UM}(\Gamma, S) \) is dense \( F_{\sigma} \) in \( (\text{UM}(\Gamma, S), \text{UD}_{\Gamma}) \).
    \item The set of all non-uniformly perfect metrics in \( \text{UM}(\Gamma, S) \) is dense \( G_{\delta} \) in \( (\text{UM}(\Gamma, S), \text{UD}_{\Gamma}) \).
\end{enumerate}

Let \((X, d)\) and \((Y, e)\) be metric spaces. A homeomorphism \( f : X \to Y \) is said to be \textit{quasi-symmetric} if there exists a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that for all \( x, y, z \in X \) and for every \( t \in [0, \infty) \) the inequality \( d(x, y) \leq td(x, z) \) implies the inequality \( e(f(x), f(y)) \leq \eta(t)e(f(x), f(z)) \). For example, all bi-Lipschitz homeomorphisms are quasi-symmetric. Note that the doubling property, the uniform disconnectedness, and the uniform perfectness are invariant under quasi-symmetric maps. David and Semmes [2] proved that if
a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set $\Gamma$ equipped with the Euclidean metric ([2, Proposition 15.11]). To simplify our description, the symbols $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ stand for the doubling property, the uniform disconnectedness, and the uniform perfectness, respectively. Before stating our results, for the sake of simplicity, we introduce the following notions:

**Definition 1.1.** If a metric space $(X, d)$ satisfies a property $\mathcal{P}$, then we write $T_\mathcal{P}(X, d) = 1$; otherwise, $T_\mathcal{P}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space $(X, d)$ is of type $(u_1, u_2, u_3)$ if we have $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

A topological space is said to be a Cantor space if it is homeomorphic to the Cantor set $\Gamma$. For a metric space $(X, d)$, we denote by $G(X, d)$ the conformal gauge of $(X, d)$ defined as the quasi-symmetric equivalent class of $(X, d)$, which is a basic concept in the conformal dimension theory (see e.g., [11]). For each $(u, v, w) \in \{0, 1\}^3$, we define $M(u, v, w) = \{ G(X, d) \mid (X, d) \text{ is a Cantor space of type } (u, v, w) \}$.

The David–Semmes theorem mentioned above states that $M(1, 1, 1)$ is a singleton. In contrast to this, the author [8] proved that for every $(u, v, w) \in \{0, 1\}^3$ except $(1, 1, 1)$, we have $\text{Card}(M(u, v, w)) = 2^{\aleph_0}$ (see [8, Theorem 2]). As a development of this result, we investigate topological distributions of metrics of type $(u, v, w)$ for all $(u, v, w) \in \{0, 1\}^3$.

For every $(u, v, w) \in \{0, 1\}^3$, we put $E(u, v, w) = \{ d \in M(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w) \}$. Let $X$ be a topological space. A subset $M$ of $X$ is said to be $F_{\sigma\delta}$ (resp. $G_{\delta\sigma}$) if $M$ is the intersection of countably many $F_{\sigma}$ subsets of $X$ (resp. the union of countably many $G_{\delta}$ subsets of $X$).

**Theorem 1.6.** The following three statements hold true:

1. The set $E(1, 1, 1)$ is a dense $F_{\sigma}$ set of $(M(\Gamma), D_\Gamma)$.
2. The set $E(0, 0, 0)$ is a dense $G_{\delta}$ subset of $(M(\Gamma), D_\Gamma)$.
3. For every $(u, v, w) \in \{0, 1\}^3$ except $(1, 1, 1)$ and $(0, 0, 0)$, the set $E(u, v, w)$ is a dense $G_{\delta\sigma}$ and $F_{\sigma\delta}$ subset of $(M(\Gamma), D_\Gamma)$.

**Acknowledgements.** The author would like to thank Professor Koichi Nagano for his advice and constant encouragement.

## 2. The Doubling Property

The following is known as the McShane–Whitney extension theorem.

**Theorem 2.1.** Let $l \in (0, \infty)$. Let $(X, d)$ be a metric space, and let $A$ be a subset of $X$. Then, for every $l$-Lipschitz map $f : A \to \mathbb{R}$, there exists an $l$-Lipschitz map $F : X \to \mathbb{R}$ such that $F |_A = f$. Moreover,
every $l$-Lipschitz map $f$ from $(A, d|_{A^2})$ into $\mathbb{R}^n$ with $\ell^\infty$-norm can be extended to an $l$-Lipschitz map from $(X, d)$ into $\mathbb{R}^n$ with $\ell^\infty$-norm.

**Theorem 2.2.** Let $X$ be a metrizable compact finite-dimensional space. Let $Q$ be the set of all metrics $d \in M(X)$ for which $(X, d)$ is isometrically embeddable into a Euclidean space equipped with $\ell^\infty$-norm. Then the set $Q$ is dense in $(M(X), D_X)$.

**Proof.** In this proof, let $L_N$ denote the metric induced from the $\ell^\infty$-norm on $\mathbb{R}^N$ for all $N \in \mathbb{Z}_{\geq 1}$. Put $n = \dim(X)$. Take arbitrary $d \in M(X)$ and $\epsilon \in (0, \infty)$. Since $X$ is compact, there exists a finite sequence $P = \{p_i\}_{i=1}^m$ in $X$ such that $d(P) \geq \epsilon$ and $X = \bigcup_{i=1}^m U(p_i, \epsilon)$. By the Kuratowski embedding theorem (see [7]), there exists an isometric embedding from $(P, d|_{P^2})$ into $(\mathbb{R}^m, L_m)$. Thus, by Theorem 2.1 there exists a $1$-Lipschitz map $F : X \to \mathbb{R}^m$ such that $F|_P$ is isometry. By [11 Theorem 9.6], there exists a topological embedding $I : X \to \mathbb{R}^{2n+1}$. Since $\mathbb{R}^{2n+1}$ is homeomorphic to its open ball with radius $\epsilon/2$, we may assume that $\delta_{L_{2n+1}}(I(X)) < \epsilon$. Define $D \in M(X)$ by $D(x, y) = L_m(F(x), F(y)) \lor L_{2n+1}(I(x), I(y))$. Then the map $E : (X, D) \to (\mathbb{R}^{m+2n+1}, L_{m+2n+1})$ defined by $E(x) = (F(x), I(x))$ is an isometric embedding. Since $F|_P$ is an isometric embedding, and since $\alpha_d(P) \geq \epsilon$, we have $D(p_i, p_j) = d(p_i, p_j)$. Since $F$ is $1$-Lipschitz, for every $x \in X$, the inequality $d(x, p_i) < \epsilon$ implies $D(x, p_i) < \epsilon$. For all $x, y \in X$, take $p_i, p_j$ with $d(x, p_i) < \epsilon$ and $d(y, p_j) < \epsilon$. Then,

$$|D(x, y) - d(x, y)| \leq d(x, p_i) + d(y, p_j) + D(x, p_i) + D(y, p_j) < 4\epsilon.$$  

Thus, we conclude that $Q$ is dense in $(M(X), D_X)$. \hfill \Box

The following is the Hausdorff metric extension theorem [6]:

**Theorem 2.3.** Let $X$ be a metrizable space, and $A$ a closed subset of $X$. Then for every $d \in M(A)$, there exists $D \in M(X)$ with $D|_{A^2} = d$.

By Corollary 4.4 and Proposition 4.9 in [10], we obtain:

**Lemma 2.4.** For a metrizable spaces $X$, the set of all doubling metrics in $M(X)$ is $F_\sigma$ in $(M(X), D_X)$.

**Proof of Theorem 2.4.** Let $T$ be the the set of all doubling metrics in $M(X)$. Assume first that $X$ is a metrizable compact finite-dimensional space. Since all metric subspaces of the Euclidean spaces are doubling, $T$ contains the set $Q$ stated in Theorem 2.2. Then, by Theorem 2.2 and Lemma 2.4, the set $T$ is dense $F_\sigma$ in $M(X)$. We next prove the opposite. If there exists a doubling metric in $M(X)$, then, by the Assouad embedding theorem (see [7 Theorem 12.1]), $X$ is finite-dimensional. For the sake of contradiction, suppose that $X$ is not compact. Then, there exists a countable closed discrete subspace $F$ of $X$. Let $\epsilon$ be the metric on $F$ such that $\epsilon(x, y) = 1$ whenever $x \neq y$. By Theorem 2.3 there exists $D \in M(X)$ with $D|_{F^2} = \epsilon$. Let $U$ be the open ball
centered at $D$ with radius $1/2$ in $(M(X), D_X)$. Take $d \in U$. Then, for every finite subset $A$ of $F$, we have $\delta_D(A) - 1/2 \leq \delta_d(A)$ and $\alpha_d(A) \leq \alpha_D(A) + 1/2$. Since $1 \leq \alpha_D(A)$ and $1 \leq \delta_D(A)$, we have $\delta_D(A)/2 \leq \delta_d(A)$ and $\alpha_d(A) \leq 2\alpha_D(A)$. Since $D$ is not doubling, for every $C \in [1, \infty)$, and for every $\beta \in (0, \infty)$, there exists a finite subset $B$ of $F$ with $4C \cdot (\delta_D(B)/\alpha_D(B))^{\beta} \leq \text{Card}(B)$, and hence we obtain $C \cdot (\delta_d(B)/\alpha_d(B))^{\beta} \leq \text{Card}(B)$. Then $d$ is not doubling. Thus, the open set $U$ consists of non-doubling metrics, and hence the set $T$ is not dense in $M(X)$. This finishes the proof. \qed

3. Amalgamation Lemmas

We provide new amalgamation lemmas of metrics and ultrametrics.

**Proposition 3.1.** Let $I$ be a set. Let $(X, d)$ be a metric space. Let $\{B_i\}_{i \in I}$ be a covering of $X$ consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i \in I}$ be points with $p_i \in B_i$. Let $\{e_i\}_{i \in I}$ be a set of metrics such that $e_i \in M(B_i)$. Define a function $D : X^2 \rightarrow [0, \infty)$ by

$$D(x, y) = \begin{cases} e_i(x, y) & \text{if } x, y \in B_i; \\ e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, y) & \text{if } x \in B_i \text{ and } y \in B_j. \end{cases}$$

Then $D \in M(X)$ and $D|_{B_i^2} = e_i$ for all $i \in I$. Moreover, if for every $i \in I$ we have $\delta_d(B_i) \leq \epsilon$ and $\delta_e(B_i) \leq \epsilon$, then $D_X(D, d) \leq 4\epsilon$.

**Proof.** We first show that $D$ satisfies the triangle inequality. Take distinct $i, j, k \in I$, and take $x, y, z \in X$. In the case of $x, y \in B_i$ and $z \in B_j$, we have $D(x, y) = e_i(x, y) \leq e_i(x, p_i) + e_i(p_i, y) \leq D(x, z) + D(z, y)$. In the case of $x \in B_i$ and $y, z \in B_j$, we have

$$D(x, y) = e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, y) \leq e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, z) + e_i(z, y) = D(x, z) + D(z, y).$$

In the case of $x \in B_i$, $y \in B_j$ and $z \in B_k$, we have

$$D(x, y) = e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, y) \leq e_i(x, p_i) + d(p_i, p_k) + d(p_k, p_j) + e_j(p_j, z) + e_i(z, y) \leq D(x, z) + D(z, y).$$

Since $i, j, k$ and $x, y, z$ are arbitrary, we conclude that $D$ satisfies the triangle inequality. Since $\{B_i\}_{i \in I}$ is a disjoint family of clopen subsets, $X$ is homeomorphic to the disjoint union space induced from $\{B_i\}_{i \in I}$. Thus, $D \in M(X)$. We next prove the latter part. Take $x, y \in X$. If $x, y \in B_i$, then, by the assumption, we have $|D(x, y) - d(x, y)| \leq 2\epsilon$. If $x \in B_i$ and $y \in B_j$ for some $i, j \in I$ with $i \neq j$, then we have

$$|D(x, y) - d(x, y)| \leq D(x, p_i) + D(p_j, y) + d(x, p_i) + d(y, p_j) \leq \delta_d(B_i) + \delta_e(B_i) + \delta_d(B_j) + \delta_e(B_j) \leq 4\epsilon.$$

This completes the proof. \qed
Proposition 3.2. Let $I$ be a set. Let $S$ be a range set. Let $(X, d)$ be an $S$-valued ultrametric space. Let $\{B_i\}_{i \in I}$ be a covering of $X$ consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i \in I}$ be points with $p_i \in B_i$. Let $\{e_i\}_{i \in I}$ be a set of ultrametrics with $e_i \in \text{UM}(B_i, S)$. Define a function $D : X^2 \to [0, \infty)$ by

$$D(x, y) = \begin{cases} 
  e_i(x, y) & \text{if } x, y \in B_i; \\
  e_i(x, p_i) \lor d(p_i, p_j) \lor e_j(p_j, y) & \text{if } x \in B_i \text{ and } y \in B_j.
\end{cases}$$

Then $D \in \text{UM}(X, S)$ and $D|_{B_i^2} = e_i$ for all $i \in I$. Moreover, if for every $i \in I$ we have $\delta_d(B_i) \leq \epsilon$ and $\delta_{e_i}(B_i) \leq \epsilon$, then $\text{UD}_{\epsilon}^X(D, d) \leq \epsilon$.

Remark 3.1. Let $X$ be a metrizable space, let $A$ be a closed subset of $X$, and let $\{B_i\}_{i \in I}$ be a covering of mutually disjoint clopen subsets of $X$. The Hausdorff metric extension theorem (Theorem 2.3) states that a metric defined on the squared set $A^2$ can be extended to a metric defined on $X^2$ (see also Theorem 3.7). On the other hand, Proposition 3.1 states that a metric defined on the set $\bigcup_{i \in I} B_i^2$ can be extended to a metric defined on $X^2$. Note that the set $\bigcup_{i \in I} B_i^2$ is not a squared subset of $X^2$ in general. Dovgoshey–Martio–Vuorinen [4] found the necessary and sufficient condition under which a weight of a weighted graph can be extended to a pseudometric on the vertex set of the graph. Dovgoshey–Petrov [3] prove an ultrametric version of it. Propositions 3.1 and 3.2 can be considered as a generalization of Dovgoshey–Martio–Vuorinen and Dovgoshey–Petrov’s results. We also remark that Proposition 3.2 is a generalization of Dovgoshey–Shcherbak’s construction of ultrametrics (see the definition (4.11) in the proof of Theorem 4.7 in [5]).

By the definition of the uniform perfectness, we obtain:

Lemma 3.3. A metric space $(X, d)$ is uniformly perfect if and only if there exist $c \in (0, 1)$ and $\delta \in (0, \infty)$ such that for every $x \in X$ and for every $r \in (0, \delta)$, there exists $y \in X$ with $cr \leq d(x, y) \leq r$.

Recall that the symbols $P_1$, $P_2$, and $P_3$ stand for the doubling property, the uniform disconnectedness, and the uniform perfectness, respectively.

Lemma 3.4. Let $(X, d)$ be a metric space. Let $\{B_i\}_{i=1}^n$ be a covering of $X$ consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i=1}^n$ be points with $p_i \in B_i$. Let $\{e_i\}_{i=1}^n$ be a set of metrics such that $e_i \in \text{M}(B_i)$. Let $D$ be the metric constructed in Proposition 3.1 from $d$ and $\{e_i\}_{i=1}^n$. Then for every $k \in \{1, 2, 3\}$, the following hold true:

1. If each $e_i$ satisfies $P_k$, then so does $(X, D)$.
2. If some $e_i$ does not satisfy $P_k$, then neither does $(X, D)$. 

Proof. We first prove the statement (1). In the case of $k = 1$, take a finite subset $A$ of $X$, and put $A_i = A \cap B_i$ for all $i \in \{1, \ldots, n\}$. Since each $S_i$ is doubling, for each $i \in \{1, \ldots, n\}$ there exist $C_i \in (0, \infty)$ and $\beta_i$ such that $\text{Card}(A_i) \leq C_i(\delta_{\alpha_i}(A_i)/\alpha_{\epsilon_i}(A_i))^{\beta_i}$. Put $C = \max_{1 \leq i \leq n} C_i$ and $\beta = \max_{1 \leq i \leq n} \beta_i$, then we have $\text{Card}(A) \leq nC(\delta_{\alpha}(A)/\alpha_{\epsilon}(A))^{\beta}$. This proves the statement (1) for $k = 1$. We next deal with the case of $k = 2$. By the assumption and by [2 Proposition 15.7], for each $i \in \{1, \ldots, n\}$ there exist an ultrametric $w_i \in M(B_i)$ and a bi-Lipschitz map $f_i : (B_i, \epsilon_i) \rightarrow (B_i, w_i)$. By applying Proposition 3.2 to $\{w_i\}_{i=1}^n$, we obtain an ultrametric $R \in M(X)$. Define $f : (X, D) \rightarrow (X, R)$ by $f(x) = f_i(x)$ if $x \in B_i$. Then, $f$ is bi-Lipschitz, and hence $(X, D)$ is uniformly disconnected. We next prove the case of $k = 3$. Assume that all $B_i$ are $c$-uniformly perfect for some $c \in (0, 1)$. Put $m = \min_{1 \leq i \leq n} \delta_{\epsilon_i}(B_i)$. Since each $B_i$ possesses at least two elements, we have $m > 0$. Put $C = (1/2) \min\{c, m/\delta_D(X)\}$. Then, $(X, D)$ is $C$-uniformly perfect.

We now prove the statement (2) in the lemma. Since the doubling property and uniform disconnectedness are hereditary to all metrics subspaces, the statements (2) for $k = 1, 2$ are true. We now treat the case of $k = 3$. Take $c \in (0, 1)$. Put $\delta = \min\{d(p_i, p_j) \mid i \neq j\}$. We may assume that $\epsilon_1$ is not uniformly perfect. Then, by Lemma 3.3 there exist $x \in B_1$ and $r \in (0, \delta)$ such that for all $y \in B_1$ we have $e_1(x, y) < cr$ or $r < e_1(x, y)$. By the definition of $D$, for all $y \in X$ we have $D(x, y) < cr$ or $r < D(x, y)$. Thus, $D$ is not uniformly perfect. □

By Corollary 4.4 and Proposition 4.11 in [10], we obtain:

Lemma 3.5. For a metrizable spaces $X$, the set of all uniformly disconnected metrics in $M(X)$ is $F_\sigma$ in $(M(X), D_X)$.

Proof of Theorem 7.4. Assume that $X$ is compact and 0-dimensional. Take arbitrary $d \in M(X)$ and $\epsilon \in (0, \infty)$. Since $X$ is compact and 0-dimensional, there exists a covering $\{B_i\}_{i=1}^n$ of $X$ of mutually disjoint clopen subsets with $\delta_d(B_i) \leq \epsilon$. Since each $B_i$ is 0-dimensional, there exists a uniformly disconnected metric $e_i \in M(B_i)$ with $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.1 to $d$ and $\{e_i\}_{i=1}^n$, we obtain $D \in M(X)$ with $D(X, D, d) \leq 4\epsilon$. By Lemma 3.4 the metric $D$ is uniformly disconnected. Thus, by Lemma 3.5 the set of all uniformly disconnected metrics is dense $F_\sigma$ in $M(X)$. We prove the opposite. If there exists a uniformly disconnected metric in $M(X)$, then $X$ is 0-dimensional. For the sake of contradiction, suppose that $X$ is not compact. Then, there exists a countable closed discrete subset $F$ of $X$. Identify $F$ with $\mathbb{Z}$, and let $e$ be the relative Euclidean metric on $\mathbb{Z}(= F)$. By Theorem 2.3 there exists $D \in M(X)$ such that $D|_{F^2} = e$. Let $U$ be the open ball centered at $D$ with radius $1/2$ in $(M(X), D_X)$. Take $d \in U$. Since $e$ is not uniformly disconnected, for every $\delta \in (0, 1)$ there exists a non-constant finite sequence $\{z_i\}_{i=1}^N$ in
$F$ with $\max_{1 \leq i \leq N} D(z_i, z_{i+1}) < 4\delta D(z_1, z_N)$. By $D_X(d, D) < 1/2$, and by $1 \leq D(z_1, z_2)$ and $1 \leq \max_{1 \leq i \leq N} D(z_i, z_{i+1})$, we have $D(z_1, z_N) \leq 2d(z_1, z_N)$ and $\max_{1 \leq i \leq N} d(z_i, z_{i+1}) \leq 2 \max_{1 \leq i \leq N} D(z_i, z_{i+1})$. Then we obtain $\max_{1 \leq i \leq N} d(z_i, z_{i+1}) < \delta d(z_1, z_N)$. This implies that $d$ is not uniformly disconnected. Thus, the open subset $U$ consists of non-uniformly disconnected metrics, and hence the set of all uniformly disconnected metrics is not dense in $M(X)$. This finishes the proof. \hfill \Box

Similarly to Lemma 3.4, we obtain:

**Lemma 3.6.** Let $S$ be a range set. Let $(X, d)$ be an $S$-valued ultrametric space. Let $\{B_i\}_{i=1}^n$ be a covering of $X$ consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i=1}^n$ be points with $p_i \in B_i$. Let $\{e_i\}_{i=1}^n$ be metrics such that $e_i \in UM(B_i, S)$. Let $D$ be the metric constructed in Proposition 3.2 from $d$ and $\{e_i\}_{i=1}^n$. Then for all $k \in \{1, 3\}$, the following hold true:

1. If each $e_i$ satisfies $\mathcal{P}_k$, then so does $(X, D)$.
2. If some $e_i$ does not satisfy $\mathcal{P}_k$, then neither does $(X, D)$.

The author [9] proved the extension theorem on ultrametrics, which is an ultrametric analogue of Theorem 2.3 (see [9, Theorem 1.2]).

**Theorem 3.7.** Let $S$ be a range set. Let $X$ be a topological space with $UM(X, S) \neq \emptyset$, and let $A$ be a closed subset of $X$. Then for every $d \in UM(A, S)$ there exists $D \in UM(X, S)$ with $D|_{A^2} = d$.

By Corollary 6.4 and Proposition 6.8 in [9], we obtain:

**Lemma 3.8.** Let $S$ be a range set. For a topological space $X$, the set of all doubling metrics in $UM(X, S)$ is $F_\sigma$ in $UM(X, S)$.

**Proof of Theorem 4.1.** By Lemma 3.8, the set of all doubling metrics in $UM(X, S)$ is $F_\sigma$ in $UM(X, S)$. Assume first that $X$ is compact. By [9, Proposition 2.12], we have $UM(X, S) \neq \emptyset$. Take arbitrary $d \in UM(X, S)$ and $\epsilon \in (0, \infty)$. Since $X$ is compact and 0-dimensional, there exists a disjoint covering $\{B_i\}_{i=1}^n$ of clopen subsets with $\delta_d(B_i) \leq \epsilon$. Then, there exists a doubling metric $e_i \in UM(B_i, S)$ with $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.2 to $d$ and $\{e_i\}_{i=1}^n$, we obtain $D \in UM(X, S)$ with $UD_X^S(D, d) \leq \epsilon$. By Lemma 3.6, the metric $D$ is doubling. Thus, the set of all doubling metrics in $UM(X, S)$ is dense in $UM(X, S)$. We next prove the opposite. Similarly to the proof of Theorem 4.1 by using Theorem 3.7, we conclude that if the set of all doubling metrics in $UM(X, S)$ is dense in $UM(X, S)$, then $X$ is compact. \hfill \Box

4. **The Uniform Perfectness**

Fix a countable dense subset $P$ of $\Gamma$. For $c \in (0, 1)$, let $K(c)$ denote the set of all $d \in M(\Gamma)$ such that for every $r \in (0, 1) \cap \mathbb{Q}$, and for every $x \in P$, there exists $y \in P$ satisfying that $c \cdot r \leq d(x, y) \leq r$. Let $K$ denote the set of all uniformly perfect metrics in $M(\Gamma)$.
Lemma 4.1. The set $K$ is an $F_σ$ subset of $M(Γ)$.

Proof. By the definitions, we have $K = \bigcup_{c \in (0,1) \cap \mathbb{Q}} K(c)$. For every $c \in (0,1) \cap \mathbb{Q}$, we prove that $CL(K(c))$ is contained in $K(c/4)$, where $CL$ is the closure operator of $M(Γ)$. Take $d \in CL(K(c))$, and take a sequence $(d_n)_{n \in \mathbb{Z}_{\geq 0}}$ in $K(c)$ such that $d_n \to d$ as $n \to \infty$ in $(M(Γ), D_Γ)$. Then for every $n \in \mathbb{Z}_{\geq 0}$, for every $r \in (0,1) \cap \mathbb{Q}$, and for every $x \in P$, there exists $y(n, r, x) \in P$ such that $c \cdot (r/2) \leq d_n(x, y(n, r, x)) \leq r/2$. Since $(0,1) \cap \mathbb{Q}$ and $P$ are countable, and since $Γ$ is compact, we can apply Cantor’s diagonal argument to $y(n, r, x)$, and hence we obtain a strictly increasing map $φ : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that for every $r \in (0,1) \cap \mathbb{Q}$, and for every $x \in P$, the sequence $\{y(φ(n), r, x)\}_{n \in \mathbb{Z}_{\geq 0}}$ converges to a point in $X$, say $z(r, x)$. Take $p(r, x) \in P$ with $d(p(r, x), z(r, x)) \leq (c/4)r$. Letting $n \to \infty$, we have $c \cdot (r/2) \leq d(x, z(r, x)) \leq r/2$, and hence we have $(c/4) \cdot r \leq d(x, p(r, x)) \leq r$. Thus, $d \in K(c/4)$, and hence $CL(K(c))$ is contained in $K(c/4)$. By this observation, we conclude that $K = \bigcup_{c \in (0,1) \cap \mathbb{Q}} CL(K(c))$. Thus $K$ is $F_σ$ in $M(Γ)$.

Similarly to Lemma 4.1 we obtain:

Lemma 4.2. Let $S$ be a range set. The set $K \cap UM(Γ, S)$ is $F_σ$ in $UM(Γ, S)$.

Proof of Theorem 1.4. By Lemma 4.1, it suffices to show that $K$ is dense in $M(Γ)$. Take arbitrary $d \in M(Γ)$ and $ε \in (0,∞)$. Since $Γ$ is 0-dimensional and compact, there exists a covering $\{B_i\}_{i=1}^n$ of mutually disjoint clopen non-empty subsets with $δ_d(B_i) ≤ ε$. Note that each $B_i$ is a Cantor space (see [14] Corollary 30.4). Identify $B_i$ and $Γ$, and let $ε_i \in M(B_i)$ be the identified metric with $ε \cdot E$, where $E$ is the relative Euclidean metric on $Γ$. Then each $ε_i$ is uniformly perfect and satisfies $δ_{ε_i}(B_i) ≤ ε$. Applying Proposition 3.1 to $d$ and $\{ε_i\}_{i=1}^n$, we obtain $D \in M(Γ)$ with $D_Γ(d, D) ≤ 4ε$. Lemma 3.2 implies that $D$ is uniformly perfect. Thus $K$ is dense in $M(Γ)$. This finishes the proof.

Lemma 4.3. Let $S$ be an exponential range set. Then there exists a strictly decreasing sequence $\{s(n)\}_{n \in \mathbb{Z}_{\geq 0}}$ in $S$ satisfying that for every $n \in \mathbb{Z}_{\geq 0}$ we have $M^{-1}a^n ≤ s(n) ≤ Ma^n$.

Proof. By the assumption on $S$, there exist $b \in (0,1)$ and $M \in [1,∞)$ such that for every $n \in \mathbb{Z}_{\geq 0}$, we have $[M^{-1}b^n, Mb^n] \cap S \neq \emptyset$. Put $p = −\log M/\log b$. Put $a = b^{2p+1}$. Then, $[M^{-1}a^n, Ma^n] \cap S \neq \emptyset$ and $Ma^{n+1} < M^{-1}a^n$ for all $n \in \mathbb{Z}_{\geq 0}$. This leads to the lemma.

A sequence $s : \mathbb{Z}_{\geq 0} \to (0,∞)$ said to be shrinking if it is a strictly decreasing sequence $(0,∞)$ convergent to 0. For a shrinking sequence $s : \mathbb{Z}_{\geq 0} \to (0,∞)$, and for every $m \in \mathbb{Z}_{\geq 0}$, define $s^{(m)} : \mathbb{Z}_{\geq 0} \to (0,∞)$ by $s^{(m)}(n) = s(m+n)$. Then $s^{(m)}$ is shrinking.
Let $2^\omega$ be the set of all maps from $\mathbb{Z}_{\geq 0}$ into $\{0, 1\}$. Define a valuation $\nu : 2^\omega \times 2^\omega \to [0, \infty]$ by $\nu(x, y) = \min\{n \in \mathbb{Z}_{\geq 0} \mid x(n) \neq y(n)\}$ if $x \neq y$; otherwise $\nu(x, y) = \infty$. Let $s$ be a shrinking sequence. Define a metric $d_s$ on $2^\omega$ by $d_s(x, y) = s(\nu(x, y))$. Then for every shrinking sequence $s : \mathbb{Z}_{\geq 0} \to (0, \infty)$, and for every $m \in \mathbb{Z}_{\geq 0}$, the metric space $(2^\omega, d_s(m))$ is a Cantor space. Note that for every $x \in 2^\omega$, and for every $n \in \mathbb{Z}_{\geq 0}$, there exists $d_s(x, y) = s(n)$. The metric space $(2^\omega, d_s)$ is called a sequentially metrized Cantor space in the author’s paper [8], and the author investigated the doubling property, the uniform disconnectedness, and the uniform perfectness of sequentially metrized Cantor spaces. The following lemma is essentially contained in [8] Lemma 6.4.

**Lemma 4.4.** Let $S$ be a range set. Let $s : \mathbb{Z}_{\geq 0} \to S$ be a shrinking sequence in $S$. If there exist $a \in (0, 1)$ and $M \in [1, \infty)$ such that $M^{-1}a^n \leq s(n) \leq Ma^n$, then for every $m \in \mathbb{Z}_{\geq 0}$, the metric space $(2^\omega, d_s(m))$ is uniformly perfect.

**Proof.** It suffices to show the case of $m = 0$. Put $c = M^{-2}a$. Take arbitrary $x \in 2^\omega$. For every $r \in (0, \infty)$, take $n \in \mathbb{Z}_{\geq 0}$ such that $s(n + 1) < r \leq s(n)$. Take $y \in 2^\omega$ with $d(x, y) = s(n + 1)$. Then, $cr \leq d(x, y) \leq r$. This implies that $(2^\omega, d_s)$ is uniformly perfect. □

**Proof of Theorem 1.3** Similarly to Theorem 1.3, the statement (2) follows from Proposition 3.2 and Lemmas 3.6 and 4.2.

We now prove the statement (1). Assume first that $S$ is exponential. Then by Lemma 4.3 there exist $a \in (0, 1)$ and $M \in [1, \infty)$ and a strictly decreasing sequence $\{s(n)\}_{n \in \mathbb{Z}_{\geq 0}}$ in $S$ such that $M^{-1}a^n \leq s(n) \leq Ma^n$. Take arbitrary $d \in UM(\Gamma, S)$ and $\epsilon \in (0, \infty)$. Then, there exists a covering $\{B_i\}_{i=1}^N$ of $\Gamma$ consisting of mutually disjoint non-empty clopen subsets of with $\delta_d(B_i) \leq \epsilon$. Note that each $B_i$ is a Cantor space. For a sufficiently large $m \in \mathbb{Z}_{\geq 0}$, the space $(2^\omega, d_{s(m)})$ satisfies $\delta_{d_{s(m)}}(2^\omega) \leq \epsilon$. Identify $B_i$ and $2^\omega$, and let $e_i \in UM(B_i, S)$ be the identified metric with $d_{s(m)}$. By applying Proposition 3.2 to $\{e_i\}_{i=1}^N$ and $d$, we obtain an $S$-valued ultrametric $D$ in $UM(\Gamma, S)$ with $UD^D_X(D, D) \leq \epsilon$. By Lemmas 4.1 and 4.2, the metric $D$ is uniformly perfect. This leads to the former part of the statement (1). Assume next that $S$ is not exponential. We now prove that every $d \in UM(\Gamma, S)$ is not uniformly perfect. Take arbitrary $c \in (0, 1)$. Since $S$ is not exponential, there exists $n \in \mathbb{Z}_{\geq 0}$ with $[c^{n+1}, c^n] \cap S = \emptyset$. Take any $x \in \Gamma$. Put $r = c^{n+1}$. Then every $y \in \Gamma$ satisfies $d(x, y) \leq cr$ or $r \leq d(x, y)$. Thus, every $d \in UM(\Gamma, S)$ is not uniformly perfect. This finishes the proof. □

Before proving Theorem 1.6, remark that each $E(u, v, w)$ contains a metric with arbitrary small diameter. This follows from the facts that $E(u, v, w) \neq \emptyset$ (see [8] Theorem 1.2), and that for every $(u, v, w) \in \{0, 1\}^3$ and for every $\epsilon \in (0, \infty)$, if $d \in E(u, v, w)$, then $\epsilon d \in E(u, v, w)$.
Proof of Theorem 1.6. Fix \((u, v, w) \in \{0, 1\}^3\). We now prove that \(E(u, v, w)\) is dense in \(M(\Gamma)\). Take arbitrary \(d \in M(\Gamma)\) and \(\epsilon \in (0, \infty)\). Since \(\Gamma\) is compact and 0-dimensional, there exists a covering \(\{B_i\}_{i=1}^n\) of \(\Gamma\) consisting of mutually disjoint non-empty clopen subsets with \(\delta_d(B_i) \leq \epsilon\). Since the set \(E(u, v, w)\) contains a metric with arbitrary small diameter, there exists \(e_i \in M(B_i)\) of type \((u, v, w)\) with \(\delta_{e_i}(B_i) \leq \epsilon\). Applying Proposition 3.1 to \(d\) and \(\{e_i\}_{i=1}^n\), we obtain \(D \in M(\Gamma)\) with \(D, d \leq 4\epsilon\). By Lemma 3.4, the metric \(D\) is of type \((u, v, w)\). Thus, \(E(u, v, w)\) is dense in \(M(\Gamma)\). For each \(k \in \{1, 2, 3\}\), let \(W(k, 1)\) denote the set of all metrics satisfying the property \(P_k\) in \(M(\Gamma)\), and put \(W(k, 0) = M(\Gamma) \setminus W(k, 1)\). By Lemmas 2.4, 3.5, and 4.1, for all \(k \in \{1, 2, 3\}\), the sets \(W(k, 1)\) and \(W(k, 0)\) are \(F_\sigma\) and \(G_\delta\) in \(M(\Gamma)\), respectively. Thus, for all \((u, v, w) \in \{0, 1\}^3\), we have

\[
E(u, v, w) = W(1, u) \cap W(2, v) \cap W(3, w).
\]

By the equality (4.1), the sets \(E(1, 1, 1)\) and \(E(0, 0, 0)\) are \(F_\sigma\) and \(G_\delta\), respectively. If \((u, v, w) \in \{0, 1\}^3\) coincides with neither \((0, 0, 0)\) nor \((1, 1, 1)\), then by the equality (4.1), the set \(E(u, v, w)\) is the intersection of an \(F_\sigma\) set and a \(G_\delta\) set in \(M(\Gamma)\). Since \(M(\Gamma)\) is a metrizable space, the set \(E(u, v, w)\) is \(F_\sigma\delta\) and \(G_\delta\sigma\) in \(M(\Gamma)\). This finishes the proof. □

References

1. M. G. Charalambous, Dimension theory: A selection of theorems and counterexamples, Atlantis Studies in Mathematics, vol. 7, Springer, Cham, 2019.
2. G. David and S. Semmes, Fractured fractals and broken dreams: Self-similar geometry through metric and measure, Oxford Lecture Ser. Math. Appl., vol. 7, Oxford Univ. Press, 1997.
3. A. A. Dovgoshey and E. A. Petrov, Subdominant pseudoultrametric on graphs, Sb. Math 204 (2013), 1131–1151.
4. O. Dovgoshey, O. Martio, and M. Vuorinen, Metrization of weighted graphs, Ann. Comb. 17 (2013), no. 3, 455–476.
5. O. Dovgoshey and V. Shcherbak, The range set of ultrametrics, compactness, and separability, preprint, arXiv:2102.10901, 2021.
6. F. Hausdorff, Erweiterung einer homöorphie, Fund. Math. 16 (1930), 353–360.
7. J. Heinonen, Lectures on analysis on metric spaces, Springer-Verlag, New York, 2001.
8. Y. Ishiki, Quasi-symmetric invariant properties of Cantor metric spaces, Ann. Inst. Fourier (Grenoble) 69 (2019), no. 6, 2681–2721.
9. □, An embedding, an extension, and an interpolation of ultrametrics, preprint, arXiv:2008.10209, to appear in p-Adic Numb. Ultr. Anal. Appl., 2020.
10. □, An interpolation of metrics and spaces of metrics, preprint, arXiv:2003.13277., 2020.
11. J. M. Mackay and J. T. Tyson, Conformal dimension: Theory and application, Univ. Lecture Ser., vol. 54, Amer. Math. Soc., 2010.
12. V. Niemytzki and A. Tychonoff, Beweis des satzes, dass ein metrisierbarer raum dann und nur dann kompakt ist, wenn er in jeder metrik vollständig ist, Fund. Math. 12 (1928), 118–120.
13. K. Nomizu and H. Ozeki, The existence of complete Riemannian metrics, Proc. Amer. Math. Soc. 12 (1961), 889–891.
14. S. Willard, *General topology*, Dover Publications, 2004; originally published by the Addison-Wesley Publishing Company in 1970.

(Yoshito Ishiki)

Graduate School of Pure and Applied Sciences
University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan

Email address: ishiki@math.tsukuba.ac.jp