Pair-Tunneling Induced Localized Waves in a Vector Nonlinear Schrödinger Equation

Li-Chen Zhao\textsuperscript{1}, Liming Ling\textsuperscript{2}, Zhan-Ying Yang\textsuperscript{1}, Jie Liu\textsuperscript{3,4}\textsuperscript{1}Department of Physics, Northwest University, 710069, Xi’an, China

\textsuperscript{2}Department of Mathematics, South China University of Technology, Guangzhou 510640, China

\textsuperscript{3}Science and Technology Computation Physics Laboratory, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China and

\textsuperscript{4}Center for Applied Physics and Technology, Peking University, 100084, Beijing, China

(Dated: August 14, 2013)

We investigate the localized waves of the coupled two-mode nonlinear Schrödinger equations with a pair-tunneling term representing strongly interacting particles can tunnel between the modes as a fragmented pair. Facilitated by Darboux transformation, we have derived exact solution of nonlinear vector waves such as bright solitons, Kuznetsov–Ma soliton, Akhmediev breathers and rogue waves and demonstrated their interesting temporal-spatial structures. The phase diagram that demarcates the parameter ranges of the nonlinear waves is obtained. Our results have implications in such diverse fields as Bose-Einstein condensate, nonlinear fibers and super fluids.

PACS numbers: 05.45.Yv, 02.30.Ik, 67.85.Hj, 03.70.+k

Introduction—Vector nonlinear waves such as vector bright soliton (BS), vector Akhmediev breather (AB) and vector rogue waves (RW), have recently become a topic of intense research in theory because of their widespread applications \[1\]–\[3\]. These kinds of waves appear in oceans, atmosphere, optics, plasmas, as well as in the quanta world of super fluids and Bose-Einstein condensates (BEC) \[4\]–\[8\]. Most studies are based on the coupled nonlinear Schrödinger equations (NLS), in which coupling between different modes (or components) are described by cross-phase modulation (XPM) term. One mode can then influence others through imposing a phase that is dependent of its instantaneous density distribution. Thus, for the XPM-type coupling, the population or particle number in each component is conserved. However, in practical physical systems, the mode population in each component is not necessarily conserved. For instance, in microscopic particle transport or light propagation, the particle (or light) in one well (or mode) can transfer to another well (mode) through quantum tunneling \[9\]–\[13\]. More interestingly, for the BEC systems where the ultra-cold atoms behave coherently, pair-tunneling (PT) that the strongly correlated atoms can tunnel back and forth as a fragmented pair in a double-well, were observed in recent experiments \[14\]–\[16\]. Behind dynamics can be described by a two-mode vector NLS that contain both XPM and PT coupling terms. In such systems, particle population in each mode is not conserved and nonlinear waves are expected to be more marvelous.

In this letter, we show that the two-mode vector NLS with both XPM and PT couplings can be integrable and present exact solutions of vector BS, vector Kuznetsov–Ma soliton (K-MS), vector AB and vector RW using Darboux transformation method. These nonlinear waves demonstrate marvelous temporal-spatial structures. We observe Josephson-like oscillation for vector K-MS solution. For the vector RW, the typical eye-shape structure of total density distribution is found to decompose into a hump density distribution of one component and a two-valley density distribution of the other component. We obtain the phase diagram that demarcates the parameter ranges of the distinctive nonlinear waves and further demonstrate more exotic second-order nonlinear wave.

Integrable Model—With including tunneling effects, dynamics of two-component systems can be described by following vector nonlinear Schrödinger Equation in general \[10\],

\[
i \left( \frac{\Phi_{1\ell}}{\Phi_{2\ell}} \right) = H \left( \frac{\Phi_1}{\Phi_2} \right),
\]

where \(\delta\) denotes the energy difference between the two components. The star means complex conjugation, \(H^0 = -\partial_x^2\) are free evolution. \(H^{MF} = g_{i,i} |\Phi_i|^2 + g_{3-i,i} |\Phi_{3-i}|^2\) \((i = 1, 2)\) are the interactions accounting for self-phase modulation and XPM (incoherent coupling effect), represented by the first and second term respectively in the righthand. \(\Omega\) represents the tunneling term, denoting coherent coupling between the components. In most studies, \(\Omega\) is set to be zero because it was usually believed that the presence of tunneling makes the systems become non-integrable \[3\]–\[5\]. While in a more recent work \[4\], it was found that, when the tunneling term takes a specific form of delta function of \(\Omega = \delta(x)\), a static vector BS solution can be derived with dropping the XPM term. In the present work, we consider \(\Omega = \beta \Phi_1^* \Phi_2\), denoting PT effects observed in recent BEC experiments \[14\]–\[16\], and the coherent energy coupling effects in a nonlinear birefringent fiber \[16\]. Moreover, the nonlinear XPM term is present.

*Electronic address: liu.jie@iapcm.ac.cn
With setting $\delta = 0$, $g_{1,1} = g_{2,2} = \beta = -2$ and $g_{1,2} = g_{1,1} = -4$, the equation (1) can be rewritten explicitly as the following integrable coupled NLS,

$$i\Phi_{1t} + \Phi_{1xx} + 2|\Phi_1|^2\Phi_1 + 4|\Phi_2|^2\Phi_1 + 2\Phi_2^2\Phi_1^* = 0, (2)$$

$$i\Phi_{2t} + \Phi_{2xx} + 2|\Phi_2|^2\Phi_2 + 4|\Phi_1|^2\Phi_2 + 2\Phi_1^2\Phi_2^* = 0. (3)$$

It should be pointed that the integrable coupled NLS is distinctive from the well-known Manakov system [17, 18], since the pair-tunneling effects are considered in the coupled model. In following, we will present the exact solutions of nonlinear waves such as vector BS, vector K-MS, vector AB, vector RW, and even high-order ones, and demarcate the parameter boundary of these nonlinear waves.

**Pair-tunneling induced localized waves**—With solving the corresponding Lax-pair from the seed solution $\Phi_{10} = s \exp[2s^2t]$ ($s > 0$) and $\Phi_{20} = 0$, the generalized vector nonlinear wave solution can be deduced as follows using Dauboux transformation method [19].

$$\Phi_1 = s \exp[i2s^2t] + \frac{2a}{|P_1|^2 + |P_2|^2}P_1P_2^*,$$

$$\Phi_2 = \frac{2a}{|P_1|^2 + |P_2|^2}P_1P_2^*,$$

where

$$P_1 = \frac{\sqrt{\tau + a}}{\tau}\Psi_1 - \frac{\sqrt{\tau - a}}{\tau}\Psi_2,$$

$$P_2 = -\left(\frac{\sqrt{\tau + a}}{\tau}\Psi_1 - \frac{\sqrt{\tau - a}}{\tau}\Psi_2\right)\exp[-i2s^2t],$$

and

$$\Psi_1 = \exp[t\tau + i(\tau^2 + 2a\tau - a^2 + 2s^2)t],$$

$$\Psi_2 = \exp[-t\tau + i(\tau^2 - 2a\tau - a^2 + 2s^2)t],$$

with $\tau = \sqrt{a^2 - s^2}$ ($a$ is an arbitrary real number). $a$ determines the initial nonlinear localized wave’s shape, and $s$ is related with the amplitude of each background for localized waves. In the following, we discuss properties of the nonlinear wave solution according to the two parameters.

Case 1: When $s > 0$ and $|a| > s$, the temporal spatial structures of the solution is presented in Fig. 1. The densities of both components show breathing structure. The sum over them, i.e., total density, is found to be analogous to the K-MS of scalar NLS [20, 21]. As an extension, we term it as vector K-MS. Interestingly, we observe periodic conversion between two components. By calculating the whole particle number $N_1[t] = \int_{-\infty}^\infty |\Phi_1|^2dx$, we find that, the oscillation is not cosine or sine type as in standard Josephson oscillation [22], shown in Fig. 1 (c). Instead, it has been modified by the nonlinear interactions, similar to the nonlinear Josephson effects observed in BEC experiments [23]. The oscillation period is obtained as follows,

$$T = \frac{\pi}{2a\sqrt{a^2 - s^2}}.$$
FIG. 3: (color online) The evolution of vector RW: (a) the density evolution of component Φ₁, and (b) the density evolution of component Φ₂. The coefficients are s = a = 1.

Case 2: When s > 0 and |a| < s, the vector wave is periodic in space and the behavior of sum density distribution is analogous to the AB of scalar NLS [24]. Therefore, it is marked by vector AB as shown in Fig. 2. It is seen that a hump train appears in the density distribution of the first component, while a double-valley train emerges for the second component. Each double-valley structure looks like a butterfly. The temporal periodic oscillation is greatly suppressed so that the particle conversion between the two components only emerges in certain time range. The density period can be derived analytically,

$$S = \frac{\pi}{\sqrt{s^2 - a^2}}.$$  (8)

Case 3: When |a| = s > 0, vector RW solution can be derived from the general solution with taking a limit τ → 0. The periodicity in both time and space is absent, as shown in Fig. 3. A double-valley structure appears in the density distribution of the second component Φ₂, while, a hump structure appears in the density distribution of the first component Φ₁. In the double-valley structure, the highest density emerges at the center between the two valleys and its amplitude is equal to the background density. We certify that the double-valley structure in component of Φ₂ is quite distinctive from usual RW structure for scalar NLS. However, the distribution of the sum density of both components is analogous to the well-known Peregrine solution of scalar NLS [23], which has been considered as RW [20]. The above observation suggests that, in the presence of PT, the typical eye-shape RW of total density is decomposed into a hump density distribution of Φ₁ component and a double-valley density distribution of Φ₂ component.

Case 4: When s = 0 and a ≠ 0, the vector bright soliton(BS) can be given directly from the general expression of the exact solution. In this case, no tunneling emerges between two components. The vector solution is a trivial combination of the well known scalar solitons of NLS, that is, the solitons’ shape in the two components are identical.

To summarize the above cases in parameter a and s space, we plot an interesting phase diagram in Fig. 4.

FIG. 4: (color online) The phase diagram of nonlinear waves with parameters a and s. The BS denotes vector bright solitons, RW denotes vector rogue waves, K-MS denotes vector Kuznetsov-Ma solitons which are periodic in time, and AB denotes vector Akhmediev breathers which are periodic in space.

FIG. 5: (color online) The evolution plot of the second-order RW in the coupled system. (a) for one kind RW in Φ₁ component, (b) for the RW in Φ₂ component correspondingly. (c) for another kind RW in Φ₁ component, (d) for the RW in Φ₂ component correspondingly. It is seen that the valleys in Φ₁ correspond to the humps in Φ₂ for pair-tunneling effects.

We just plot the first quadrant with s ≥ 0, a ≥ 0, it can be extended to other quadrants according to centric symmetry. On the s axis, all particles keeps staying in Φ₁ component, no population in Φ₂, corresponding to a trivial plane solution.

In addition to the above marvelous localized waves, high-order nonlinear solutions can also be obtained through Darboux transformation method. As an example, we show the second-order RW with the PT effects. The actual expression of the solution is quite complicated.
initial conditions are given by Eq. (4) and (5) with vector K-MS wave: (a) the density evolution of component \( \Phi_1 \), and (b) the density evolution of component \( \Phi_2 \). The initial conditions are given by Eq. (4) and (5) with \( s = 1 \), \( a = 1.2 \), and \( t = -1 \). It is seen that the numeric evolution of localized waves is similar to the exact ones in Fig. 1.

and will be presented elsewhere. In Fig. 5, we plot the second-order RWs for different cases, which shows more interesting patterns emerged. We emphasize that the PT effects are essential in generating the above exotic structures.

**Application into two-component condensate system**—Experimentally, vector soliton has been realized in multi-component BEC systems \([23, 25]\), and in a nonlinear birefringent fiber \([23]\). On the other hand, scalar AB, RW and K-MS have also been observed very recently in single-mode nonlinear fibers \([24, 30, 31]\). The PT induced localized waves could be observed in the birefringent fiber or two-component BEC systems through combining these experimental techniques.

Let us consider a cigar-shaped condensate with two hyperfine states, \( \Phi_1 \) and \( \Phi_2 \). For simplicity, we assume the initial condensation occurring in the trapped state \( \Phi_2 \). State \( \Phi_1 \) is coupled to \( \Phi_2 \) by an RF or microwave field tuned near the \( \Phi_2 \rightarrow \Phi_1 \) transition. The PT effects can be realized by the RF field in the strong interaction regimes \([14, 15, 32]\). The total number of \( ^{87} \text{Rb} \) atoms in the condensate is \( N = 5 \times 10^4 \). \( a_{i,j}(i,j = 1,2) \) are s-wave scattering lengths which can be adjusted by Feshbach resonance technique. Setting \( a_{1,2} = a_{2,1} = 1.6 \text{ nm} \) and \( a_{1,1} = 0.8 \text{ nm} \), under mean-field approximation, the s-wave scattering effective interaction strengths between atoms in the same hyperfine state are \( U_{j,j} = 4\hbar^2 a_{j,j}/m \) (\( m \) is the atom mass), and the scattering effective interaction strengths between atoms in different hyperfine state are \( U_{j,i,j} = 4\hbar^2 a_{j,i,j}/m \). When the interaction between atoms is attractive and the PT coefficient is \( N \cdot U_{1,1} \), the units in axial direction and time are scaled to be \( 0.2 \mu \text{m} \) and \( 1.0 \text{ ms} \) respectively, the dynamics of the condensate with PT effects can be described well by the Eq. (2) and (3). The exact solution can be used to design proper initial density and phase conditions in the two components for each localized wave presented here. For example, from the K-MS initial condition, we would observe the nonlinear Josephson oscillation between the two hyperfine states. From the above results, we can know that the oscillation period of the K-MS waves in Fig. 1 will be \( 1.97 \text{ ms} \). Since the density and phase can be manipulated well in BEC systems \([1]\), the localized waves could be observed in the two-component condensate system.

In summary, the PT induced localized waves, such as BS, AB, K-MS, and RW, are investigated exactly in the two-component system. We stimulate the localized waves by numeric calculation from the initial condition given by the exact solution. It is found that the localized waves can be excited in the system. As an example, we show the stimulation results for vector K-MS localized wave in Fig. 6, which correspond to the exact ones in Fig. 1. Our obtained solutions contribute to better control and understanding of localized wave phenomena in a variety of complex dynamics, ranging from optical communications, to Bose-Einstein condensates, and four-wave mixing systems.

L.C. Zhao is grateful to Dr. Hui Cao for helpful discussions. This work is supported by the National Fundamental Research Program of China (Contact No. 2011CB921503, 2013CB834100), the National Science Foundation of China (Contact Nos. 11274051, 91021021).

---

[1] P. G. Kevrekidis, D. Frantzeskakis, and R. Carretero-Gonzalez, Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment (Springer, Berlin Heidelberg, 2009).

[2] C. Becker, S. Stellmer, P.S. Panahi, S. Dorcher, M. Baumert, Eva-Marie Richter, J. Kronjager, K. Bongs, K. Sengstock, Nature phys., 4 (2008) 496-501.

[3] Y.V. Bludov, V.V. Konotop, and N. Akhmediev, Eur. Phys. J. Special Topics 185 (2010) 169.

[4] T. Kanna and M. Lakshmanan, Phys. Rev. Lett. 86 (2001) 5043-5046; M. Vijayajayanthi, T. Kanna, and M. Lakshmanan, Phys. Rev. A 77 (2008) 013820.

[5] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, Phys. Rev. Lett. 109 (2012) 044102.

[6] L.C. Zhao, J. Liu, J. Opt. Soc. Am. B 29 (2012) 3119-3127.

[7] N. Akhmediev and E. Pelinovsky, Eur. Phys. J. Special Topics 185 (2010) 1; E. Pelinovsky and C. Kharif, Extreme Ocean Waves (Springer, Berlin, 2008).

[8] A.R. Osborne, Nonlinear Ocean Waves and the Inverse Scattering Transform (Elsevier, New York, 2010).

[9] Y.Y. Li, W. Pang, S.H. Fu, and B. A. Malomed, Phys. Rev. A 85 (2012) 053821.

[10] J. Williams, R. Walser, J. Cooper, et al., Phys. Rev. A 59 (1999) R31; J. Williams, R. Walser, J. Cooper, E. A. Cornell, and M. Holland, Phys. Rev. A 61 (2000) 033612.

[11] J. Ieda, T. Miyakawa, and M. Wadati, Phys. Rev. Lett.
[12] Z.J. Qin, G. Mu, Phys. Rev. E 86 (2012) 036601.
[13] Y. Lahini, F. Pozzi, M. Sorel, R. Morandotti, D. N. Christodoulides, and Y. Silberberg, Phys. Rev. Lett. 101 (2008) 193901.
[14] S. Fölling, S. Trotzky, P. Cheinet, et al., Nature 448 (2007) 06112.
[15] S. Zöllner, H.D. Meyer, and P. Schmelcher, Phys. Rev. Lett. 100 (2008) 040401.
[16] G.P. Agrawal, Nonlinear Fiber Optics (4th Edition, Academic Press, Boston, 2007).
[17] M.G. Forest, S.P. Sheu, O.C. Wright, Phys. Lett. A 266 (2000) 24-33.
[18] O. C. Wright, Applied Mathematics Letters 16 (2003) 647-652; M.G. Forest, O.C. Wright, Physica D: Nonlinear Phenomena 178 (2003) 173-189.
[19] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons (Springer, Berlin Heidelberg, 1991).
[20] E. Kuznetsov, Sov. Phys. Dokl. 22 (1977) 507-508.
[21] Y.C. Ma, Stud. Appl. Math. 60 (1979) 43-58.
[22] B.D. Josephson, Phys. Lett. 1 (1962) 251.
[23] M. Albiez, R. Gati, J. Fölling, et al., Phys. Rev. Lett. 95 (2005) 010402.
[24] N. Akhmediev, V. I. Korneev, Theor. Math. Phys. 69 (1986) 1089-1093.
[25] D. H. Peregrine, J. Aust. Math. Soc. Ser. B 25 (1983) 16-43.
[26] B. Kibler, J. Fatome, C. Finot, G. Millot, et al., Nature Phys. 6 (2010) 790.
[27] P. Das, T.S. Raju, U. Roy, and Prasanta K. Panigrahi, Phys. Rev. A 79 (2009) 015601.
[28] C. Hamner, J. J. Chang, and P. Engels, Phys. Rev. Lett. 106 (2011) 065302; M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, Phys. Rev. A 84 (2011) 041605(R).
[29] D.Y. Tang, H. Zhang, L.M. Zhao, and X. Wu, Phys. Rev. Lett. 101 (2008) 153904.
[30] K. Hammani, B. Kibler, et al., Opt. Lett. 36 (2011) 112.
[31] B. Kibler, J. Fatome, C. Finot, G. Millot, G. Genty, et al., Sci. Rep. 2 (2012) 463.
[32] R.J. Ballagh, K. Burnett, and T.F. Scott, Phys. Rev. Lett. 78 (1997) 1607-1611.