On Treves’ Algebraic Characterization of the KdV Hierarchy.

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Abstract

We have found a possibility to streamline the proof of the Treves’ theorem [1] on an algebraic characterization of the KdV hierarchy which makes it significantly shorter, following essentially the logic of the original proof.

1. One of possible methods to construct the equations of the KdV hierarchy is based on the recursion formula

\[ R''_{n-1} + 4uR'_{n-1} + 2u'R_{n-1} = -4R'_n, \quad R_0 = 1. \]  

(1.1)

Quantities \( R_n \) are differential polynomials in \( u \), i.e., polynomials in \( u \) and its derivatives. There is a grading in the algebra of differential polynomials of \( u \):

\[ w(u) = 2, \quad w(\partial) = 1 \]

where \( \partial = d/dx \). Eq. (1.1) determines \( R_n \) up to a constant of integration. Two sequences of \( R_n \) obtained with different choice of constants are related by a linear triangular transformation, i.e., terms of one sequence are linear combinations of terms of the other sequence with less or equal numbers. If the requirement is imposed that \( \{ R_n[u] \} \) are homogeneous in weight differential polynomials, \( w(R_n) = 2n \), then the recursion formula (1) uniquely determines all \( R_n \)’s. The equations of the KdV hierarchy are

\[ \partial_{t_n} u = R'_n[u], \]  

(1.2)

it is supposed that \( u \) depends on parameters \( t_n \). All \( R_m \)'s are first integrals of each Eq. (1.2) in the sense that

\[ \partial_{t_m} R_n = \partial Q_{nm} \]

where \( Q_{nm} \) is a differential polynomial (see, e.g., [2], [3]). Any linear combination of \( R_n \) is also a first integral. Two first integrals are equivalent if they differ by a differential polynomial which is the derivative of another polynomial.

Treves [1] gave the following criterion of the fact that a given differential polynomial \( P[u] \) is equivalent (differs by an exact derivative) to a linear combination of \( R_n \):

**Theorem** (Treves). A differential polynomial \( P[u] \) is \( \sum c_n R_n[u] + \partial Q[u] \) where \( Q \) is a differential polynomial if and only if the following criterion is satisfied: let the formal series

\[ -\frac{2}{x^2} + a_0 + \sum_{2}^{\infty} a_k x^k, \quad a_k = \text{const} \]

be substituted for \( u \) in \( P[u] \). Then

\[ \text{res}_x P \left[ -\frac{2}{x^2} + a_0 + \sum_{2}^{\infty} a_k x^k \right] = 0. \]  

(1.3)

The residue \( \text{res}_x \) symbolizes, as usual, the coefficient of \( x^{-1} \).

The theorem is remarkable for the following two reasons. (1) This criterion has a touch of that enigmatical universality which distinguishes the celebrated Sato bilinear identity: there is no hint on the KdV equations in Eq. (1.3), it has a very general character, and nevertheless the hierarchy is invisibly present there. (2) It suggests the use of the “trial functions” in the form of formal

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Laurent series in $x$ in the study of differential polynomials. This can work for other problems, too. For example, it is possible to prove the following necessary and sufficient criterion for a differential polynomial of $u$ to be a derivative of another differential polynomial: if an arbitrary semi-infinite Laurent series $\sum_{n=N}^{\infty} a_n x^n$ is substituted for $u$, the residue of the obtained Laurent series is zero. The Treves theorem states that if the class of trial series is restricted in an appropriate way then the Hamiltonians of the KdV hierarchy should be added to the exact derivatives.

2. Necessity of the criterion (1.3).

Proof. With the second term, $\partial Q[u]$, it is clear: when any Laurent series of $x$ is substituted for $u$, then $Q[u(x)]$ is a Laurent series and its derivative, $\partial Q[u(x)]$, has the zero residue. Now we have to prove that the formal series in powers of $x$

$$R_n \left[ -\frac{2}{x^2} + a_0 + \sum_{k=1}^{\infty} a_k x^k \right] = \hat{R}_n(x)$$

satisfy (1.3). The sequence of formal series in powers of $x$: $\{\hat{R}_n(x)\}$ satisfies the recursion formula

$$\hat{R}_{n-1}(x) + 4 \left( -\frac{2}{x^2} + a_0 + \sum_{k=1}^{\infty} a_k x^k \right) \hat{R}'_{n-1}(x) + 2 \left( -\frac{2}{x^2} + a_0 + \sum_{k=1}^{\infty} a_k x^k \right)' \hat{R}_{n-1}(x) = -4 \hat{R}'_{n}(x).$$

Taking $\hat{R}_0 = 1$, the other $\hat{R}_n$ can be recovered from the sequence of the recursion relations uniquely up to arbitrary additive constants. The choice of the constants is irrelevant since if the theorem is proven for one choice of constants it is true for all the others since the residue is a linear functional. For simplicity, the free of $x$ terms can be taken as zero.

Let us find $\hat{R}_1$ and $\hat{R}_2$. We have $2(4x^{-3} + 2a_2 x + 3a_3 x^2 + ...) = -4 \hat{R}'_1$ whence

$$\hat{R}_1 = \frac{1}{x^2} - \frac{1}{2}(a_2 x^2 + a_3 x^3 + ...).$$

Denote $a_0 + a_2 x^2 + a_3 x^3 + ... = \phi$ and $T_1 = \sum_{k=1}^{\infty} T_{1,k} x^k = -(1/2)(a_2 x^2 + a_3 x^3 + ...)$, so, $\hat{R}_1 = x^{-2} + T_1$. The next recursion formula is

$$((-2)(-3)(-4)x^{-5} + T_1''') + 4(-2x^{-2} + \phi)(-2x^{-3} + T'_1) + 2(4x^{-3} + \phi')(x^{-2} + T_1) = -4 \hat{R}'_2.$$

The terms with $x^{-5}$ go. The terms with $x^{-3}$ are $-8a_0 + 8T_{1,0} = -8a_0$. The terms with $x^{-2}$ are $-8T_{1,1} + 8T_{1,1} = 0$, and the terms with $x^{-1}$ are $-16T_{1,2} + 8a_2 + 8T_{1,2} + 4a_2 = -8T_{1,2} - 4a_2 = 4a_2 - 4a_2 = 0$. Now, $-4 \hat{R}'_2 = (-8a_0)x^{-3} +$ (terms with nonnegative powers of $x$). Therefore, $\hat{R}_2 = A_2 x^{-2} + T_2$ where $A_2$ is a constant and $T_2$ is a series in positive powers of $x$.

Now we can make a hypothesis that all $\hat{R}_m$ have a form $\hat{R}_m = A_m x^{-2} + T_m$, where $A_m$ is a constant and $T_m$ a series in positive powers of $x$ ($T_{m-1,0} = 0$), and prove it by induction. The recursion formula has the form:

$$(-2)(-3)(-4)A_{m-1} x^{-5} + T''_{m-1} + 4(-2x^{-2} + \phi)(-2A_{m-1} x^{-3} + T'_{m-1}) + 2(4x^{-3} + \phi')(A_{m-1} x^{-2} + T_{m-1}) = -4 \hat{R}'_m.$$

Collect the terms with the same power of $x$:

$x^{-5}$: $-24A_{m-1} + 16A_{m-1} + 8A_{m-1} = 0$,
$x^{-3}$: $-A_{m-1} a_0 = A_m$,
$x^{-2}$: $-8T_{m-1,1} + 8T_{m-1,1} = 0$,
$x^{-1}$: $-16T_{m-1,2} - 8A_{m-1} a_2 + 8T_{m-1,2} + 4A_{m-1} a_2 = -8T_{m-1,2} - 4A_{m-1} a_2$.

In the cases of $\hat{R}_1$ and $\hat{R}_2$ this term was zero. Therefore, we make another hypothesis that it is always zero, and also prove it by induction. Thus, we suppose

$$2T_{m-1,2} + A_{m-1} a_2 = 0. \quad (2.1)$$
We continue:

\[ x^0 : \quad 6T_{m-1,3} - 8 \cdot 3T_{m-1,3} - 8A_{m-1}a_3 + 4a_6T_{m-1,1} + 2A_{m-1} \cdot 3a_3 + 8T_{m-1,3} = -4T_{m,1} \]

or

\[ x^0 : \quad -10T_{m-1,3} + 4a_6T_{m-1,1} - 14A_{m-1}a_3 = -4T_{m,1} \]

\[ x^1 : \quad 4 \cdot 3 \cdot 2T_{m-1,4} - 8 \cdot 4T_{m-1,4} - 8A_{m-1}a_4 + 4 \cdot 2a_0T_{m-1,2} + 8T_{m-1,4} + 2A_{m-1} \cdot 4a_4 = -8T_{m,2} \]

or, taking into account (2.1), \( a_0a_2A_{m-1} = 2T_{m,2} \). We had the equation \(-A_{m-1}a_0 = A_m \). Therefore,

\[ x^1 : \quad A_m a_2 + 2T_{m,2} = 0 \]

which is nothing but our hypothesis (2.1) for the next number \( m \). The rest of equations determine \( T_{m,3}, \ldots \) Now both the hypotheses are proven and we have

\[ \hat{R}_m = A_m x^{-2} + T_m, \quad (T_m = T_{m,1}x + T_{m,2}x^2 + \ldots). \]

Take the residue:

\[ \text{res}_x \hat{R}_m = 0 \]

q.e.d.

3. Sufficiency of the criterion (1.3). Beginning of the proof.

A differential polynomial \( P[u] \) is a polynomial \( P(\xi_0, \xi_1, \xi_2, \ldots) \) where \( u, u', u'', \ldots \) are substituted for \( \xi_0, \xi_1, \xi_2, \ldots \). As it was said before, there is a grading: \( w(u^{(n)}) = n + 2, \ w(\partial) = 1 \). If all terms of a polynomial \( P \) have the same weight \( \kappa \) then \( P(\lambda^2u, \lambda^3u', \lambda^4u''\ldots) = \lambda^\kappa P(u, u', u'', \ldots) \).

We must prove the second part of the Treves theorem:

A differential polynomial \( P[u] \) satisfying the Treves condition

\[ \text{res}_x P \left[ -2/x^2 + a_0 + \sum_2 a_n x^n/n! \right] = 0 \]  

(3.1)

for any \( \{a_n\} \) is a sum of an exact derivative \( \partial S[u] \) and a linear combination of the KdV polynomial \( R_k[u] \).

**Proof.** First we are proving a lemma:

**Lemma.** If a differential polynomial satisfies the Treves condition (3.1) then so does each homogeneous in weight component of this polynomial.

**Proof of the lemma.** Let \( P = \sum P_\kappa \) where \( P_\kappa \) a homogeneous polynomial of weight \( \kappa \). Since \( \{a_n\} \) are arbitrary, we can replace them by \( a_n \lambda^{n+2} \). Now,

\[
\text{res}_x \sum P_\kappa \left( -2/x^2 + \lambda^2a_0 + \sum_2 \lambda^{n+2}a_n x^n/n! + 4/x^3 + \sum_2 \lambda^{n+2}a_n x^{n-1}/(n-1)! - 12/x^4 + \sum_2 \lambda^{n+2}a_n x^{n-2}/(n-2)! \ldots \right)
\]

\[ = \text{res}_x \sum \lambda^\kappa P_\kappa \left( -2/(lx)^2 + a_0 + \sum_2 \alpha_n (lx)^n/n! + 4/(lx)^3 + \sum_2 \alpha_n (lx)^{n-1}/n! - 12/(lx)^4 + \sum_2 \alpha_n (lx)^{n-2}/(n-2)! \ldots \right) \]

\[ = \sum \lambda^{\kappa-1} \text{res}_x P_\kappa \left[ -2/x^2 + a_0 + \sum_2 a_n x^n/n! \right]. \]
If this is zero, then each term is zero since \( \lambda \) is arbitrary, q.e.d.

Therefore, we can consider each component of weight \( \kappa \) separately. It is easy to prove by induction, using the recursion relation for \( R_k \) (1.1), that all \( R_k \) (recall that \( w(R_k) = 2k \)) contain the term \( u^k \) with a non-zero coefficient. Thus, if the given polynomial \( P_\kappa \), \( \kappa = 2k \) contains this term, then there is a constant \( c \) such that \( P - cR_k \) is without this term and satisfies the Treves condition (3.1) since both \( P_\kappa \) and \( R_k \) do. If \( \kappa \) is odd, \( P_\kappa \) cannot have terms \( cu^l \). Further we are showing that a polynomial of the weight \( \kappa \) satisfying (3.1) and without terms \( cu^l \) is an exact derivative. Let \( P \) be such a polynomial. Any non-zero polynomial can always by transformed by adding an exact derivative to a “reduced form” which means that all its monomials have a form

\[
c(u^{j_1})^{\eta_1} \cdots (u^{j_\mu})^{\eta_\mu}, \ j_1 < \cdots < j_\mu, \ \eta_\mu \geq 2.
\]

The number \( \mu \) is the order of the term. After this reduction, the polynomial \( P \) still satisfies (1.1) since an exact derivative always does, and it does not have terms \( cu^l \). A reduced polynomial cannot be an exact derivative. Therefore, what we need to prove is that \( P \) is identically zero. Making an obvious change of variables, we will write the Treves condition as

\[
\text{res}_x P \left[ \frac{1}{x^2} + a_0 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \right] = 0
\]
or

\[
\text{res}_x P \left( \frac{1}{x^2} + a_0 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}, -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, \ldots \right) = 0. \tag{3.2}
\]

We can differentiate this equation with respect to \( a_0 \). Since \( a_0 \) enters only the first argument of \( P(u, u', u'', \ldots) \), this will be

\[
\text{res}_x \frac{\partial}{\partial a_0} P \left( \frac{1}{x^2} + a_0 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}, -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, \ldots \right) = 0.
\]

We can repeat this operation again and again until the polynomial does not contain \( u \) and still is not zero if it was not initially, since there is no term \( u^k \) and the others contain besides \( u \) other variables, \( u', u'' \ldots \). Moreover, it preserves its reduced form. Thus, we can assume that \( P \) does not contain \( u \) and is \( P(u', u'', \ldots) \). The Treves condition is then

\[
\text{res}_x P \left( -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, 6/x^4 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots \right) = 0.
\]

Now take the derivative with respect to \( a_j \):

\[
\text{res}_x \sum_{i=1}^{j} \frac{\partial^i}{\partial x^i} \left( \frac{x^j}{j!} \right) \frac{\partial}{\partial a_0} P \left( -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, 6/x^4 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots \right) = 0.
\]

Adding an exact derivative, “integrating by parts”, we do not change the residue:

\[
\text{res}_x \frac{x^j}{j!} \left( \frac{\partial}{\partial u} \right) P \left( -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, 6/x^4 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots \right) = 0, \ j = 2, 3, \ldots
\]

where \( \delta/\delta u = \sum_{i=0}^{\infty} (-1)^i \partial^i (\partial/\partial u^{(i)}) \) (\( \partial = \partial/\partial x \)) is the variational derivative. Since \( \partial P/\partial u = 0 \),

\[
\frac{\delta P}{\delta u} = -\frac{\partial P}{\delta u} = \partial \sum_{i=0}^{\infty} (-1)^{i+1} \delta^i \frac{\partial P}{\delta u^{(i+1)}} = \partial Q, \ w(Q) = \kappa - 3.
\]

Then the last equation, being integrated by parts, takes the form

\[
\text{res}_x x^j Q \left( -2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, 6/x^4 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots \right) = 0, \ j = 1, 2, \ldots
\]
which means that the negative (principal) part of the Laurent expansion of the function

\[
xQ \left(-2/x^3 + \sum_{n=2}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}, 6/x^4 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots\right)
\]

vanishes.

If we prove that \( Q = 0 \), it will follow that \( \delta P/\delta u = 0 \), i.e., \( P \) is an exact derivative which is impossible for a non-zero reduced differential polynomial. Thus, it will be proven that reduced \( P \) is zero, or that \( P \) is an exact derivative.

The homogeneity means \( Q(\lambda^4 \xi_1, \lambda^4 \xi_2, \ldots) = \lambda^{k-3} Q(\xi_1, \xi_2, \ldots) \). Taking the derivative with respect to \( \lambda \) and letting \( \lambda = 1 \), we get the Euler identity

\[
3 \xi_1 \partial Q/\partial \xi_1 + 4 \xi_2 \partial Q/\partial \xi_2 + \ldots = (2k - 3)Q.
\]

Using the fact that \( Q \) is homogeneous, one can rewrite (3.3) as

\[
\frac{1}{x^k-4} Q \left(-2 + \sum_{n=0}^{\infty} a_j n^2 \frac{x^n}{n!}, 6 + \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}, \ldots\right).
\]

4. The proof continued.

Now, we must expand (3.5) in powers of \( x \). If doing this directly, the expression is too involved. The following trick (Treves) simplifies the task. Let

\[
\eta_j(x) = \sum_{n=0}^{\infty} a_j n^2 \frac{x^n}{n!},
\]

The conversion of this formula is an exact derivative which is

\[
\sum_{n=0}^{\infty} (-1)^n \eta_j n^2 x^n n! \]

\[
= \sum_{n=0}^{\infty} (-1)^n \sum_{m=0}^{\infty} a_j n^2 m^2 \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_j \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^p}{n!} = a_j.
\]

The expression (3.5) takes the form

\[
\frac{1}{x^k-4} Q \left(-2 + \eta_1 x^3, 6 + \eta_2 x^4, \ldots, (-1)^j (j+1)! + \eta_j x^{j+2}, \ldots\right).
\]

Quantities \( \eta_j \) are not independent since \( \alpha_1 = 0 \). This gives a relation \( \sum_{n=0}^{\infty} (-1)^n \eta_j n^2 x^n n! = 0 \) or

\[
\eta_1 = \sum_{n=1}^{\infty} (-1)^{n+1} \eta_1 n^2 \frac{x^n}{n!}
\]

whence the expression (3.5) is

\[
\frac{1}{x^k-4} Q \left(-2 + \sum_{n=1}^{\infty} (-1)^{n+1} \eta_1 n^2 \frac{x^{n+3}}{n!}, 6 + \eta_2 x^4, \ldots\right).
\]

Expand \( Q(\cdot) \) in powers of \( \eta_j \)'s. We have

\[
\frac{\partial}{\partial \eta_j} Q(\cdot) = x^{j+2} D_j Q(\cdot) \quad \text{where} \quad D_j = \frac{\partial}{\partial \xi_j} + \frac{(-1)^j}{(j+1)!} \frac{\partial}{\partial \xi_1}.
\]

Therefore, the expression (4.1) is

\[
\sum_{q_2, \ldots, q_r} \frac{D_j^{q_2} \cdots D_r^{q_r} Q(\theta)}{q_2! \cdots q_r! x^{k-4+\lambda(\theta)} \eta_2^{q_2} \cdots \eta_r^{q_r}}
\]
where
\[ (\theta) = (-2!, 3!, ..., (-1)^\mu(\mu + 1)!), \quad \lambda(q) = \sum_{j=2}^{\mu} (j + 2)q_j. \]

We had independent variables \(a_2, a_3, ..., a \) Now we changed them to new independent variables \(\eta_2, \eta_3, ..., \eta \). The condition that the expression (4.2) does not contain negative powers of \(x \) becomes
\[ D_2^0 \cdots D_r^0 Q(\theta) = 0 \text{ when } \lambda(q) < \kappa - 4. \quad (4.3) \]

It is convenient to replace the operators \(D_j \) by \(L_j\):
\[ L_2 = D_2 \text{ and } L_j = D_j + (j - 1)^{-1}D_{j-1}, \quad j > 2. \]

We have, \(L_j = \partial/\partial \xi_j + \partial/\partial \xi_{j-1} \). This is a triangular change of operators: \(L_j \) is a linear combination of \(D_i \) with \(i \leq j \) and vice versa. Therefore, we can replace (4.3) by an equivalent equation
\[ L_2^0 \cdots L_r^0 Q(\xi_1, \xi_2, ...)|_{\xi_2=-2, \xi_2=6,...} = 0, \text{ when } \lambda(q) < \kappa - 4. \quad (4.4) \]

5. End of the proof.

**Lemma.** The condition (4.3) (or, equivalently, (4.4)), implies \(Q \equiv 0\).

**Proof.** We use induction on \(\mu \), the number of arguments of the polynomial \(Q(\xi_1, ..., \xi_\mu)\). For \(\mu = 1\) the theorem is trivial. Indeed, since \(Q \) is a homogeneous polynomial of weight \(\kappa - 3 \), and \(\xi_1 \) has the weight 3, \(Q = c_\xi^{(\kappa-3)/3} \) if \(\kappa - 3 \) is divisible by 3 and zero otherwise. If \(c \neq 0 \), then \(Q(-2) \neq 0 \) which contradicts Eq. (4.3). Thus, \(c = 0 \) and \(Q = 0 \). Let the lemma be proven for \(\mu - 1 \).

We have \(Q = \sum_{r_1, ..., r_\mu} a_{r_1, ..., r_\mu} \xi_{1}^{r_1} \cdots \xi_{\mu}^{r_\mu} \) where
\[ 3r_1 + 4r_2 + ... + (\mu + 2)r_\mu = \kappa - 3. \quad (5.1) \]

This homogeneity implies the Euler equation
\[ (3\xi_1 \partial_{\xi_1} + 4\xi_2 \partial_{\xi_2} + ... + (\mu + 2)\xi_\mu \partial_{\xi_\mu})Q(\xi) = (\kappa - 3)Q(\xi). \quad (5.2) \]

Besides,
\[ (\partial_{\xi_1} + \partial_{\xi_2})^{q_1}(\partial_{\xi_2} + \partial_{\xi_3})^{q_2} \cdots (\partial_{\xi_{\mu-1}} + \partial_{\xi_\mu})^{q_\mu} Q((\theta)) = 0 \quad (5.3) \]
when \(\lambda(q) = 4q_2 + 5q_3 + ... + (\mu + 2)q_\mu < \kappa - 4 \).

One has to prove that under these conditions \(Q \equiv 0 \).

Let us apply the operator \(\partial_{\xi_1}^{q_1} \cdots \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} \) to (5.2) and put \((\xi) = (\theta)\):
\[ (-3!)^q \partial_{\xi_1}^{q_1} + 4! \partial_{\xi_2}^{q_2} + ... + (\mu - 1)! \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} \frac{\partial^{\mu-1-q_\mu-1}}{\partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu}} Q((\theta)) \]
\[ + (-1)^{q_1} \partial_{\xi_1}^{q_1} \cdots \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} Q((\theta)) - A \]

where
\[ A = c \partial_{\xi_1}^{q_1} \cdots \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} Q((\theta)) \]

Suppose it is already proven that all partial derivatives of order \(q - 1 \) vanish. Then we have \(A = 0 \) and
\[ \partial_{\xi_1}^{q_1} \cdots \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} \frac{\partial^{\mu-1-q_\mu-1}}{\partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu}} Q((\theta)) \]
\[ = \left( (-1)^{q_1} \frac{3!}{(\mu + 2)!} \partial_{\xi_1}^{q_1} + (-1)^{q_1+1} \frac{4!}{(\mu + 2)!} \partial_{\xi_2}^{q_2} + ... + \frac{(\mu + 1)!}{(\mu + 2)!} \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \right) \times \]
\[ \times \partial_{\xi_1}^{q_1} \cdots \partial_{\xi_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^{q_\mu} Q((\theta)) \]
We have diminished the order of the derivative with respect to $\xi_\mu$ by 1. We can proceed until the derivative disappears at all:

$$\partial^{i_1}_1 \cdots \partial^{i_{\mu-1}}_{\xi_{\mu-1}} \partial^{j-i_1-\cdots-j_{\mu-1}}_\mu Q((\theta))$$

$$= \left( (-1)^\mu \frac{3!}{(\mu+2)!} \partial_{\xi_1} + (-1)^{\mu+1} \frac{4!}{(\mu+2)!} \partial_{\xi_2} + \cdots + \frac{(\mu+1)!}{(\mu+2)!} \partial_{\xi_{\mu-1}} \right)^{q-i_1-\cdots-j_{\mu-1}} \times$$

$$\times \partial^{i_1}_1 \cdots \partial^{i_{\mu-1}}_{\xi_{\mu-1}} Q((\theta)).$$

Thus, computing a derivative of $q$th order one can always replace $\partial_{\xi_\mu}$ by the expression in parentheses in the last two formulas. Denote this expression as $\partial^*_\xi$. In particular, (5.3) implies

$$(\partial_{\xi_1} + \partial_{\xi_2})^{q_2} (\partial_{\xi_2} + \partial_{\xi_3})^{q_3} \cdots (\partial_{\xi_{\mu-1}} + \partial^*_\xi)^{q_{\mu-1}} = 0$$

(5.4)

if $4q_2 + 5q_3 + \cdots + (\mu + 2)q_{\mu} < \kappa - 4$. Derivations $\partial_{\xi_1}, \ldots, \partial_{\xi_{\mu-1}}, \partial^*_\xi$ can be expressed as linear combinations of $(\partial_{\xi_1} + \partial_{\xi_2}), \ldots, (\partial_{\xi_{\mu-1}} + \partial^*_\xi)$, therefore any partial derivative of order $q < (\kappa-4)/(\mu+2)$ of $Q$ at the point $(\theta)$ is a linear combination of the expressions in the left-hand side of (5.4), and, therefore, vanishes. The induction is proven.

This is still not the whole story. We have proven only that the derivatives of order $< (\kappa-4)/(\mu+2)$ vanish at the point $(\theta)$. In particular $\partial^*_\xi Q(\theta) = 0$ when $j < (\kappa-4)/(\mu+2)$. On the other hand, this derivative vanishes when $j > (\kappa-3)/(\mu+2)$ since $r_{\mu} \leq (\kappa-3)/(\mu+2)$ (see (5.1)). If $\kappa-3$ is not divisible by $\mu+2$, this means that the derivatives of all orders with respect to $\xi_\mu$ vanish. $Q$ does not depend on $\xi_\mu$. Since we use the induction on $\mu$, we have assumed that for $\mu-1$ the theorem is proven, so $Q \equiv 0$. If $(\kappa-3)/(\mu+2)$ is an integer, say $n$, then the only non-zero derivative with respect to $\xi_\mu$ at $(\theta)$ can be the $n$th one. Then, $Q$ has a form $Q^* \cdot (\xi_\mu - (-1)^\mu(\mu+1)!)^n$. However, this is homogeneous only if $Q^* \equiv Q \equiv 0$ q.e.d. The theorem is also proven.

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