LOCATION OF ZEROS OF NON-TRIVIAL POSITIVE SUPER SOLUTIONS TO SCHRÖDINGER EQUATIONS

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Abstract. We study Schrödinger operators on $L^2(E;m)$ of the form $-A+V$ with singular potentials $V$. We address the question posed by H. Brezis about the structure of the set $\{u=0\}$ for non-negative supersolutions to $-Au+Vu=0$. The class of operators $A$ we study in the paper includes, in particular, symmetric Lévy type operators and symmetric diffusions in divergence form, with strictly positive Green functions. The class of potentials $V$ consists of positive smooth measures, which contains, in particular, Coulomb potentials and harmonic potentials, as well as generalized potentials, i.e. positive Borel measures concentrated on $m$-negligible sets.

1. Introduction

Let $(E,\mathfrak{g})$ be a locally compact separable metric space, $m$ be a Radon measure on $E$ with full support and $A$ be a self-adjoint operator on $L^2(E;m)$ that generates a Markov semigroup of contractions $(T_t)_{t\geq 0}$ on $L^2(E;m)$. We assume that the Green function $G$ for the operator $A$ exists and is strictly positive.

Formulation of the problem. In the present paper, we answer the question posed by H. Brezis (see [5, 8]): what is the structure of the set $\{u=0\}$, where $u$ satisfies

$$-Au+Vu \geq 0, \quad u \geq 0, \quad (1.1)$$

for a locally integrable potential $V : E \to [0,\infty]$. In fact in the paper we consider a wider class of potentials, allowing $V$ in (1.1) to be a positive smooth measure (see Section 2.1), i.e. a Borel measure absolutely continuous with respect to the capacity $\text{Cap}_A$ naturally generated by the operator $A$, and such that $\int_E \eta dV < \infty$ for a strictly positive quasi-continuous function $\eta : E \to \mathbb{R}$; so $V$ as a function need not be locally integrable and as a measure need not be Radonian. Thus, we encompass the class of so called generalized Schrödinger operators.

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The classical strong maximum principle (Hopf [29], 1927) stands if \( A = \Delta \) (or \( A \) is a diffusion with coefficients satisfying some suitable assumptions) on a bounded domain \( D \subset \mathbb{R}^d \) and \( V \in L^p(D;m) \) for some \( p > d/2 \), then there are only two possibilities for \( u \in C^2_0(D) \) satisfying (1.1) pointwisely in \( D \): either \( \{u = 0\} = \emptyset \) or \( \{u = 0\} = D \) (see e.g. Littman [33], Stampacchia [44], Gilbarg and Trudinger [27]). In general, however, for merely locally integrable potentials, we cannot expect the strong maximum principle to hold. For instance, \( \{u = 0\} = \{0\} \) for the function \( u(x) = |x|^2 \), which satisfies (1.1) with \( A = \Delta_{|B(0,1)} \) and \( V(x) = 2d|x|^{-2} \). We see that \( V \in L^1(D;m) \) for \( d \geq 3 \) (\( V \cdot m \) is a smooth measure for \( d = 2 \)).

The problem of the structure of the set \( \{u = 0\} \) for \( u \) satisfying (1.1), was studied for the Dirichlet Laplacian or uniformly elliptic diffusion operator by Ancona [5], Bénilan and Brezis [8], Brezis and Ponce [15], and recently by Orsina and Ponce [37]. The results obtained in [5, 8, 15, 37] can be briefly summarized as follows. In the paper by Brezis and Bénilan [8, Appendix C] it is assumed that \( A = \Delta \) and \( V \in L^1_{loc}(\mathbb{R}^d;m) \). It is shown there that if \( u \in L^1_{loc}(\mathbb{R}^d;m) \) satisfies (1.1) a.e., then

\[
\text{boundedness of the set } \{u > 0\} \text{ implies that } \{u = 0\} = \mathbb{R}^d \text{ a.e.}
\]

Ancona [5] (see also Brezis and Ponce [15]) have considered a uniformly elliptic divergence form operator \( Au = \sum_{i,j=1}^d (a_{ij} u_{x_j})_{x_j} \) on a bounded domain \( D \subset \mathbb{R}^d \). In [5] it is proved that if a quasi-continuous \( u \in H^1(D) \) (or \( u \in L^1(D;m) \) in case \( a \) is smooth) satisfies (1.1) in the sense of measures, and is non-trivial \((m\text{-a.e.)}, \) then (for quasi-continuous version of \( u \))

\[
\text{Cap}_2(\{u = 0\}) = 0.
\] (1.2)

Here Cap2 is the Newtonian capacity. In the paper by Orsina and Ponce [37], \( A = \Delta \) on a bounded domain \( D \subset \mathbb{R}^d \) and \( V \in L^p(D;m) \) for some \( p > 1 \). It is proved there that if \( u \in L^1(D;m) \cap L^1(D;V \cdot m) \) satisfies (1.1), and is non-trivial \((m\text{-a.e.)}, then

\[
\text{Cap}_{W^{1,p}}(x \in D : \limsup_{r \to 0^+} \int_{B(x,r)} u(y) \, dy = 0) = 0.
\] (1.3)

Three conclusions can be drawn from the above results. They all concern a "size" of the set \( \{u = 0\} \) - recall that if \( \text{Cap}_A(B) = 0 \), then \( \mathcal{H}_G(B) < \infty \), where \( \mathcal{H}_G \) is the Hausdorff measure related to the Green function \( G \) (see [28]), and if \( \text{Cap}_{W^{1,p}}(B) = 0 \), then \( \mathcal{H}_{d-2p}(B) < \infty \), where \( \mathcal{H}_{d-2p} \) is \((d - 2p)\)-dimensional Hausdorff measure. For their formulation, if we are talking about a subtler measure than the Lebesgue measure, a precise version of \( u \) is needed, thus the existence of a regular representative of \( u \) must be a part of the assertion for the results describing the set \( \{u = 0\} \). Finally, the "size" of \( \{u = 0\} \) depends on the regularity of potential \( V \). A companion problem is the rigorous meaning of the inequality (1.1). Besides the classical pointwise formulation, applicable only to specific operators and regular \( u \), one may consider weak formulation for (1.1):

\[
- \int_E u A\eta \, dm + \int_E V u \eta \, dm \geq 0, \quad \eta \in C,
\] (1.4)

for a suitable class \( C \) of test functions, and one may understood (1.1) in the sense of measures, with the assumption that \( Au \) is a Borel measure. At this point it is worth mentioning that first results on strong maximum principle (for diffusion operators) with some non-classical formulations of (1.1) are due to Calabi [16] and Littman [33].
Main results of the paper. In the present paper we study the structure of the set \( \{u = 0\} \) for a wide class of operators. A complete novelty of the present paper is the fact that we treat the question by H. Brezis for non-local operators which are now of great interest both in pure and in applied mathematics (see e.g. [14] and references therein). As we mentioned before, in the literature the attention has been focused on the problem of a 'size' of the set \( \{u = 0\} \). We go much further in this research, and this is the second novelty of the present paper, namely we indicate a set \( N_V \) depending only on \( A \) and \( V \) - where all possible zeros of any non-trivial solution to (1.1) are located. As corollaries, we get results on a 'size' of the set \( \{u = 0\} \). It appears that the said set \( N_V \) admits the following form:

\[
N_V := \{ x \in E : \exists \text{ finely-open } U_x, \text{ with } x \in U_x, \text{ such that } \int_{U_x} G(x,y)V(y)\,m(dy) < \infty \}.
\]

This is an interesting object, which naturally appears in the context of Schrödinger equations with measure data (see [31]), and is well known in the probabilistic potential theory as it is the complement of the set of permanent points for \( V \) (see e.g. [13]). It appears that independently of the potential \( V \) and operator \( A \) we always have

\[
\text{Cap}_A(N_V) = 0. \tag{1.5}
\]

The main result of the paper (see Theorem 5.2) stands as follows.

**Theorem 1.** Let \( u \in L^1(E;m) \cap L^1(E;V \cdot m) \) be a positive function satisfying

\[
- \int_E u \eta \, dm + \int_E V u \eta \, dm \geq 0, \quad \eta \in C, \tag{1.6}
\]

where \( C = \{ \eta \in \mathcal{D}(A) : \eta \in B^+_E(E), A \eta \text{ is bounded} \} \). Then there exists an m-version \( \tilde{u} \) of \( u \) which is finely-continuous on \( E \setminus N_V \). Moreover, if \( \tilde{u}(x) = 0 \) for some \( x \in E \setminus N_V \), then \( \tilde{u} \equiv 0 \).

Note that the finely-continuous m-version \( \tilde{u} \) of \( u \) is given by the following formula:

\[
\tilde{u}(x) = \lim_{t \to 0} \int_E u(y)p_t(x,y)\,m(dy) = \lim_{\alpha \to \infty} \int_E u(y)G_\alpha(x,y)\,m(dy), \quad x \in E \setminus N_V, \tag{1.7}
\]

where \( p_t(\cdot,\cdot) \) is the transition kernel of the semigroup \( (T_t) \), and \( G_\alpha(\cdot,\cdot) \) is its \( \alpha \)-Green’s function. The above result combined with (1.5) yields a generalization of the result by Ancona (1.2).

From the above theorem, we conclude the second main result of the paper, which stands that \( N_V \) is exactly the set of zeros of all non-trivial positive functions satisfying (1.6).

**Theorem 2.** Let \( \mathcal{A} := \{ u \in L^1(E;m) \cap L^1(E;V \cdot m) : u \geq 0, u \text{ satisfies } (1.6), u \text{ is finely-continuous and } u \not\equiv 0 \} \). Then

\[
N_V = \bigcup_{u \in \mathcal{A}} \{ u = 0 \}.
\]

Although the above theorems are the main results of the paper, we put the major work into proving the following, interesting in its own, result which provides Feynman-Kac type representation for solutions to (1.6).
Theorem 3. Assume that \( u \in L^1(E; m) \cap L^1(E; V \cdot m) \) is a positive function that satisfies (1.6). Then there exists an \( m \)-version \( \tilde{u} \) of \( u \) which is finely-continuous on \( E \setminus N_V \). Moreover, for any \( k \geq 0 \) there exists a positive smooth measure \( \beta_k \) such that
\[
\tilde{u}_k(x) = \int_{\Omega} \left[ e^{-\int_0^{\tau_D(\omega)} V(\omega(r)) dr} \tilde{u}_k(\omega(t \wedge \tau_D(\omega))) \right] dP_x(\omega)
\]
\[
+ \int_{\Omega} \left[ \int_0^{\tau_D(\omega)} e^{-\int_0^s V(\omega(r)) ds} dA^{\beta_k}_r(\omega) \right] dP_x(\omega),
\]
for any open relatively compact set \( D \subset E \), \( x \in D \), and \( t \geq 0 \), where \( u = \tilde{u} \wedge k \).

Here \( (P_x)_{x \in E} \) is a family of Borel measures on the Skrochod path space \( \Omega \subset E^{[0,\infty)} \) consisting of càdlàg functions, so called Markov family associated with the semigroup \( \{\mathbf{V} \} \). \( A^{\beta_k} \) is a positive continuous additive functional of \( (P_x) \) associated with \( \beta_k \) - in case \( \beta_k \) is a function \( A^{\beta_k}_t(\omega) = \int_0^t \beta_k(\omega(r)) dr \) - and
\[
\tau_D : \Omega \to [0, \infty], \quad \tau_D(\omega) = \inf \{ t > 0 : \omega(t) \notin D \}
\]
(see introductory Section 2.2).

In Section 6, we prove our last main result of the paper concerning the strong maximum principle for operators \(-A+V\) - we say that SMP holds for \(-A+V\) if for any finely-continuous positive function \( u \in L^1_{\text{loc}}(E; m) \cap L^1(E; V \cdot m) \) that satisfies (1.6) we have the following implication: if \( u(x_0) = 0 \) for some \( x_0 \in E \), then \( u(x) = 0, x \in E \). Section 6 is the only one where we dispense with the assumption that \( V \cdot m \) is a smooth measure. In this section the only requirement from \( V : E \to [0, \infty] \) is being Borel measurable. By Theorem 1 if \( V \cdot m \) is a smooth measure and \( N_V = \emptyset \), then SMP holds. However, \( N_V = \emptyset \) already implies that \( V \cdot m \) is smooth. Thus implication \( N_V = \emptyset \Rightarrow \text{SMP} \) is true for arbitrary Borel measurable \( V : E \to [0, \infty] \). We prove that if \( N_V \neq \emptyset \), then there is only one possibility when SMP still holds, namely if there is no non-trivial positive solution to (1.6). The class of such potentials is reach. For example, in [36] it has been proven that for \( A = \Delta \) and \( V(x) := |x_1|^{-\gamma} \), with \( \gamma \in [1, 2) \) there is no non-trivial positive solution to (1.6).

Theorem 4. Let \( V : E \to [0, \infty] \) be a Borel measurable function. Then the SMP holds for \(-A+V\) if and only if
1. either \( N_V = \emptyset \)
2. or \( N_V \neq \emptyset \) and there is no non-trivial positive finely-continuous solution to (1.6).

The conclusion of the above theorem, in case \( A = \Delta \) and \( d = 1 \), follows from the recent paper by Bertsch, Smarrazzo and Tesei [7], where the authors went even a step further and gave a necessary and sufficient condition on \( V \) guaranteeing that under \( N_V = \emptyset \) there is no non-trivial positive solution to (1.6) (see Remark 6.2).

In Section 7, we generalize the result by Orsina and Ponce (1.3) (see also Section 8) by proving that for \( V \in L^p(E; m) \), with \( p > 1 \), and non-trivial positive \( u \in L^1(E; m) \cap L^1(E; V \cdot m) \) that satisfies (1.6) we have
\[
C_p(\{\tilde{u} = 0\}) = 0,
\]
where \( \tilde{u} \) is the finely-continuous \( m \)-version of \( u \) on \( N_V \), and \( C_p \) is Riesz’s capacity. Recall that for \( A = \Delta \) we have \( C_p \sim \text{Cap}_{W^{2,p}} \).
In Section 8 (see Theorem 8.4) we provide a result, especially important from the practical point of view, saying that in case $A$ is a Lévy operator, i.e.

$$-A = \psi(-\Delta)$$  \hspace{1cm} (1.8)

for a Bernstein function $\psi$, then Theorems 1–4 holds true with $C$ replaced by $C^\infty_c(\mathbb{R}^d)$ in (1.6). We also make some comments and provide some results concerning finely-continuous versions of Borel functions, especially for the Laplacian and the fractional Laplacian.

**Smooth measures.** In order to make the exposition of the main results of the present paper more clear, we formulated them in the Introduction for potentials $V : E \to [0, \infty]$ such that measure $V \cdot m$ is smooth; equivalently for potentials $V$ being locally quasi-integrable: for any compact $K \subset E$ and $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subset K$ such that $Cap_A(K \setminus F_\varepsilon) \leq \varepsilon$ and

$$\int_{F_\varepsilon} V(y) m(dy) < \infty$$

(we denote the class of such functions by $L_{q,loc}(E;m)$). Clearly, $L_{loc}^1(E;m) \subset L_{q,loc}^1(E;m)$. There are many interesting and very important in applications subclasses of $L_{q,loc}^1(E;m)$ which go beyond the space $L_{loc}^1(E;m)$, such as, for example, the class of repulsive potentials

$$V(x) = \sum_{i=1}^l \frac{1}{\delta_{K_i}(x)^p}, \quad x \in E.$$  

Here $K_i$ are compact sets satisfying $Cap_A(K_i) = 0$, $i = 1, \ldots, l$ ($\delta_{K_i}(x) := \text{dist}(x, K_i)$), and $p_i > 0$, $i = 1, \ldots, l$. We see that $V$ explodes when approaches $K_i$. For the Dirichlet fractional Laplacian $(\Delta^\alpha)_{|D}$, with a bounded open $D$ in $\mathbb{R}^d$ ($d \geq 2$, $\alpha \in (0, 1)$), we may take for $K_i$ any set of Hausdorff dimension less than $d - 2\alpha$ (see [1, Theorem 5.1.9]).

However, most of the results of the paper (Theorems 1–3) holds true with $V \cdot m$ replaced by a positive smooth measure $\nu$. The class of smooth measures, denoted by $S_A$, depends on the operator $A$, but, as we mentioned before, we always have the inclusion $L_{loc}^1(E;m) \subset S_A$. An interesting subclass of $S_A$, of the great importance in applications, consists of generalized potentials, i.e. positive measures which charge $m$-negligible sets (see e.g. [2]). For example, for $A = (\Delta^\alpha)_{|D}$ any $\sigma$-finite positive Borel measure $\nu$ satisfying

$$\nu(dx) \ll \mathcal{H}_\lambda(dx),$$

for some $\lambda \in (d - 2\alpha, d)$, is a generalized potential (see [1, Theorem 5.1.13]).

It is known that $N_\nu = \emptyset$ if and only if $\nu$ is strictly smooth, and this is so e.g. when $\nu$ is Green finite or $\nu$ is from Kato’s class, see Section 2.3. As a particular application of the main results of the paper, we have that the classical strong maximum principle holds for the operator $-\Delta_D + \nu$ (with connected $D$) and $\nu(dy) = \gamma_M(dy)$ (where $M$ is a compact $d - 1$ dimensional manifold and $\gamma_M(dy)$ is the surface measure on $M$), since in this case $\nu$ is of Kato’s class (see [3]).

**Dirichlet operators.** The class of operators considered in the present paper includes the operators studied in [5, 8, 15, 37] as well as other fundamental operators, for instance Laplacian with mixed boundary condition on a connected open set (see [20]), fractional Laplacian, Dirichlet fractional Laplacian, regional fractional Laplacian, fractional Laplacian with mixed boundary condition (see [6]) on arbitrary open set, Lévy type operators of the form $\psi(-\Delta)$, where $\psi$ is a Bernstein function (see [21, Example 1.4.1], [12]), uniformly elliptic operators on manifolds (see [21, Example 5.7.2]), degenerate diffusion operators
and (see [21, Exercise 3.1.1.]), Laplacian on Sierpiński’s gasket (see [22]). The general structure of the operator $A$ is known in case $E \subset \mathbb{R}^d$. Due to Beurling-Deny decomposition and the transformation rule for additive functionals (see [21, Theorem 5.6.2])

$(-Au, v)_{L^2(E;m)} = \int_E u_xv dx = \int_{E \times E \setminus \delta} (u(x) - u(y))(v(x) - v(y))dJ(dx, dy)$

for any $u \in \mathcal{D}(A), v \in \mathcal{D}(\mathcal{E})$, where $(\nu_{ij})_{i,j=1,\ldots,d}$ is a positive definite matrix of smooth measures and $J$ is a positive symmetric Borel measure on $E \times E \setminus \delta$ ($\delta$ denotes the diagonal in $E \times E$).

**Final comments.** After finishing this manuscript, we have learned about the results of [38]. In this paper Orsina and Ponce studied the set of zeros of solutions to the equation

$$-\Delta u + Vu = f \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

with $f \in L^\infty(D;m)$, where $V : E \to [0, \infty]$ is Borel measurable. The authors introduced in [38] a set $Z \subset D$, called universal zero-set,

$$Z := \bigcap_{f \in L^\infty(D;m), f \equiv 0} \{ x \in D : \hat{\omega}_f(x) := \limsup_{r \to 0^+} \int_{B(x,r)} w_f(y) m(\text{dy}) = 0 \},$$

where $\omega_f \in W^{1,2}_0(D) \cap L^\infty(D)m \cap L^1(D;V \cdot m)$ is a unique solution to (1.9), and they proved that if $\hat{\omega}_f(x) = 0$ for some $x \in D \setminus Z$, then $\hat{\omega}_f \equiv 0$ on a finely-connected component of $D \setminus Z$ containing $x$. From our results it follows easily that $Z = N_V$ provided $V$ is locally quasi-integrable. Thus, as a corollary, we obtain a simple characterization of the set $Z$ by means of the Green function of $A$. Observe that the result by Orsina and Ponce agrees with our results since $D \setminus N_V$ is finely-connected due to (1.5) (see Remark 8.11 and [38, Corollary 1.2]).

### 2. Notation and standing assumptions

We denote by $\mathcal{B}(E)$ (resp. $\mathcal{B}^+(E)$) the set of all Borel (resp. positive Borel) measurable functions on $E$. We say that a measure $\mu$ on $E$ is not trivial if $\mu(B) \neq 0$ for some Borel set $B \subset E$. For $x \in E$ and $r > 0$, $B(x, r) := \{ y \in E : \varrho(x, y) < r \}$.  

#### 2.1. Dirichlet forms and potential theory

In the paper, we assume that $(A, \mathcal{D}(A))$ is a negative definite self-adjoint operator on $L^2(E;m)$ generating a strongly continuous Markov semigroup of contractions $(T_t)_{t \geq 0}$ on $L^2(E;m)$. It is well known (see [21, Section 1]) that there exists a unique symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ such that

$$\mathcal{D}(A) \subset \mathcal{D}(\mathcal{E}) \quad \mathcal{E}(u, v) = (-Au, v), \quad u \in \mathcal{D}(A), \quad v \in \mathcal{D}(\mathcal{E}).$$

We assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient and regular, i.e. there exists a strictly positive bounded function $g$ on $E$ such that

$$\int_E |u|g dm \leq \sqrt{\mathcal{E}(u,u)}, \quad u \in \mathcal{D}(\mathcal{E}),$$

and $\mathcal{D}(\mathcal{E}) \cap C_c(E)$ is dense in $C_c(E)$ in the uniform convergence topology, and in $\mathcal{D}(\mathcal{E})$ equipped with the norm generated by the inner product $\mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2}$.

**Remark 2.1.** There is no loss of generality in assuming that $\mathcal{E}$ is transient. Indeed, if $-A$ generates a Dirichlet form $\mathcal{E}$ which is not transient, then for any $\alpha > 0$ the operator $-\alpha A$ generates the form $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}))$, with $\mathcal{E}_\alpha(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \alpha \langle \cdot, \cdot \rangle_{L^2(E;m)}$, which is transient. It is clear that if a positive $u$ satisfies $-Au + u \cdot \nu \geq 0$, then $-Au + \alpha u + u \cdot \nu \geq 0$. Moreover, if $\nu$ is
smooth with respect to $E$, then it is smooth with respect to $E_{\alpha}$. Therefore, we may apply the results of the paper to the operator $-A + \alpha$ perturbed by $\nu$.

For an open set $U \subset E$, we put
\[ \text{Cap}_A(U) = \inf \{ \mathcal{E}(u, u) : u \geq 1_U, u \text{ m-a.e.}, u \in \mathcal{D}(E) \}, \]
and then, for arbitrary $B \subset E$, we set $\text{Cap}_A(B) = \inf \text{Cap}_A(U)$, where the infimum is taken over all open subsets $U$ of $E$ such that $B \subset U$. We say that a property holds q.e. if it holds except for a set of capacity $\text{Cap}_A$ zero.

By $\mathcal{M}(E)$ (resp. $\mathcal{M}^+(E)$, $\mathcal{M}_b(E)$) we denote the set of Borel (resp. positive Borel, bounded Borel) measures on $E$. In the paper we adopt the following notation: for a $\mu \in \mathcal{M}^+(E)$, and $f \in \mathcal{B}^+(E)$ we set
\[ \langle \mu, f \rangle = \int_E f \, d\mu. \]
For $f$ and $\mu$ as above, we denote by $f \cdot \mu$ the Borel measure on $E$ such that
\[ \langle f \cdot \mu, \eta \rangle = \langle \mu, f \eta \rangle, \quad \eta \in \mathcal{B}^+(E). \]

We say that a function $u$ on $E$ is quasi-continuous if for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subset E$ such that $\text{Cap}_A(E \setminus F_\varepsilon) \leq \varepsilon$ and $u|_{F_\varepsilon}$ is continuous. By [21, Theorem 2.1.3], each function $u \in \mathcal{D}(E)$ admits a quasi-continuous $m$-version. In the sequel, for $u \in \mathcal{D}(E)$, we denote by $\tilde{u}$ its quasi-continuous $m$-version.

We say that $\mu \in \mathcal{M}(E)$ is smooth if
\begin{enumerate}[(a)]
  \item $\mu \ll \text{Cap}_A$,
  \item $\langle |\mu|, \eta \rangle < \infty$ for some strictly positive quasi-continuous function $\eta$ on $E$.
\end{enumerate}
It is well known that every $\mu \in \mathcal{M}(E)$ admits the following unique decomposition
\[ \mu = \mu_d + \mu_c, \]
where $\mu_d \ll \text{Cap}_A$ and $\mu_c \perp \text{Cap}_A$. In the literature, $\mu_d$ is called the diffuse part of $\mu$, and $\mu_c$ is called the concentrated part of $\mu$.

We set
\[ \mathcal{D}(E^\nu) = \mathcal{D}(E) \cap L^2(E; \nu), \quad E^\nu(u, v) = \mathcal{E}(u, v) + \langle \tilde{u} \cdot \nu, \tilde{v} \rangle, \quad u, v \in \mathcal{D}(E^\nu). \]
By [34, Theorem 4.6], $(E^\nu, \mathcal{D}(E^\nu))$ is a quasi-regular symmetric Dirichlet form on $L^2(E; \nu)$.

By [34, Corollary 2.10], there exists a unique negative definite self-adjoint operator $(A^\nu, \mathcal{D}(A^\nu))$ such that $\mathcal{D}(A^\nu) \subset \mathcal{D}(E^\nu)$ and
\[ E^\nu(u, v) = -A^\nu u, v, \quad u \in \mathcal{D}(A^\nu), v \in \mathcal{D}(E^\nu). \]

We put $-A + \nu := -A^\nu$ and we denote by $(T_t^\nu)_{t \geq 0}$ the strongly continuous Markov semigroup of contractions on $L^2(E; m)$ generated by $-A^\nu$.

For an open set $D \subset E$, we denote by $(E^D, \mathcal{D}(E^D))$ the part of $(E, \mathcal{D}(E))$ on $D$, that is a symmetric form defined as
\[ \mathcal{D}(E^D) = \{ u \in \mathcal{D}(E) : \tilde{u} = 0 \text{ q.e. on } E \setminus D \}, \quad E^D(u, v) = \mathcal{E}(u, v), \quad u, v \in \mathcal{D}(E^D). \]

By [21, Theorem 4.4.3], $(E^D, \mathcal{D}(E^D))$ is again a regular symmetric transient Dirichlet form on $L^2(D; m)$. The operator generated by $(E^D, \mathcal{D}(E^D))$ shall be denoted by $A_{|D}$. 
We denote by \( \Delta^\alpha \), \( \alpha \in (0,1) \), the operator associated with the form
\[
\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \hat{u}(x) \hat{v}(x) |x|^{2\alpha} \, dx, \quad \mathcal{D}(\mathcal{E}) = \{ u \in L^2(\mathbb{R}^d; dx) : \int_{\mathbb{R}^d} |\hat{u}|^2(x) |x|^{2\alpha} \, dx < \infty \},
\]
where \( \hat{u} \) stands for the Fourier transform of \( u \), and by \( \Delta \) the usual Laplace operator, which can be viewed as the operator associated with the above form with \( \alpha = 1 \). It is well known that if \( \alpha \in (0,1) \), then for \( u \in C_c^\infty (\mathbb{R}^d) \),
\[
\Delta^\alpha u(x) = c_{d,\alpha} \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|x-y|^{d+2\alpha}} \, dy = \frac{c_{d,\alpha}}{2} \int_{\mathbb{R}^d} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2\alpha}} \, dy
\]
for some constant \( c_{d,\alpha} > 0 \) (see, e.g., [32]). The operator corresponding to the part of the above form on \( D \) shall be denoted by \( (\Delta^\alpha)|_D \) if \( \alpha \in (0,1) \), and by \( \Delta|_D \) if \( \alpha = 1 \).

2.2. Probabilistic potential theory. Let \( \mathcal{D} \) be the set of functions \( \omega : [0,\infty) \rightarrow E \cup \{ \partial \} \) (where \( \partial \) is a one-point compactification of \( E \) in case \( E \) is not compact, and an isolated point in case \( E \) is compact) such that

(1) \( \omega \) is càdlàg, i.e. it is right continuous on \([0,\infty)\) and has left limits on \((0,\infty)\),
(2) if \( \omega(t) = \partial \) for some \( t > 0 \), then \( \omega(s) = \partial \), \( s \geq t \).

Let \( d_S \) be the Skorohod metric on \( \mathcal{D} \) (see e.g. Section 12 of [11]). With this metric \( \mathcal{D} \) is a separable complete metric space. Let \( X := \text{Id}_\mathcal{D} \) and
\[
X_t(\omega) := (X(\omega))(t), \quad t \geq 0, \omega \in \mathcal{D}.
\]

We see that \( X_t \) is a projection onto "t-coordinate". From now on any function \( u : E \rightarrow \mathbb{R} \cup \{ \pm \infty \} \) extends to \( E \cup \{ \partial \} \) by letting \( u(\partial) = 0 \). By [21, Theorem 4.2.4], there exists a family \( \{ P_x, x \in E \cup \{ \partial \} \} \) of (Borel) probability measures on \( \mathcal{D} \) such that for any \( f \in B_b(E) \cap L^2(E;m) \) and any \( t > 0 \),

(1) \( T_t f(x) = \int_{\mathcal{D}} f(X_t(\omega)) P_x(\omega)(dw) \) \( m \)-a.e.,
(2) \( x \mapsto \int_{\mathcal{D}} f(X_t(\omega)) P_x(\omega)(dw) \) is quasi-continuous.

As is customary, we denote \( \mathbb{E}_x Y := \int_\mathcal{D} Y(\omega) P_x(\omega)(dw) \) for any \( Y \in \mathcal{B}(\mathcal{D}) \). The operator \( \mathbb{E}_x \) is called the expectation (with respect to \( P_x \)). Let \( \mathcal{P} \) denote the set of all probability measures on \( \mathcal{B}(\mathbb{R}) \). For a given \( \mu \in \mathcal{P} \), we let
\[
P_\mu(A) := \int_E P_x(A) \mu(dx), \quad A \in \mathcal{B}(\mathcal{D}).
\]

Denote \( \mathcal{F}_t := \sigma(X_s : s \leq t) \), \( \mathcal{F}_\infty^0 := \sigma(X_s : s \geq 0) \). Next, let \( \mathcal{F}_\infty^\mu \) be a completion of \( \mathcal{F}_\infty^0 \) with respect to \( P_\mu \), and \( \mathcal{F}_t^\mu \) be a completion of \( \mathcal{F}_t^0 \) in \( \mathcal{F}_\infty^\mu \) with respect to \( P_\mu \). We set
\[
\mathcal{F}_t := \bigcap_{\mu \in \mathcal{P}} \mathcal{F}_t^\mu, \quad \mathcal{F}_\infty := \bigcap_{\mu \in \mathcal{P}} \mathcal{F}_\infty^\mu.
\]

By [21, Theorem 7.2.1]
\[
\mathcal{X} = (X_t)_{t \geq 0}, \quad (P_x)_{x \in E \cup \{ \partial \}}, \quad \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \quad \zeta, \quad (\theta_t)_{t \geq 0}
\]
is a Hunt process, i.e. admits some extra properties described in [21, Section A.2], where
\[
\zeta(\omega) = \inf \{ t > 0 : X_t(\omega) = \partial \}, \quad \theta_t : \mathcal{D} \rightarrow \mathcal{D}, \quad \theta_t(\omega)(s) := \omega(t+s).
\]
For \( f \in \mathcal{B}^+(E) \), we put
\[
\mathbb{P}_t f(x) = \mathbb{E}_x f(X_t), \quad t \geq 0, \quad R_\alpha f(x) = \mathbb{E}_x \int_0^\zeta e^{-\alpha t} f(X_t) \, dt, \quad \alpha \geq 0, \quad x \in E,
\]
and \( R := R_0 \). We say that a property holds q.a.s. if it holds \( P_x \)-a.s. for q.e. \( x \in E \), and we say that it holds a.s. if it holds \( P_x \)-a.s. for every \( x \in E \).

Any family \( (Y_t)_{t \geq 0} \) of mappings \( Y_t : \mathcal{D} \to \mathbb{R} \) that are \( \mathcal{F}_\infty/\mathcal{B}(E) \) measurable for any \( t \geq 0 \) is called a stochastic process. We say that a stochastic process \( (Y_t) \) is \( \mathbb{F} \)-adapted if for each \( t \geq 0 \), \( Y_t \) is \( \mathcal{F}_t/\mathcal{B}(E) \) measurable. We say that a stochastic process \( (Y_t) \) is càdlàg if \( P_x(Y \in \mathcal{D}) = 1, \ x \in E \).

Throughout the paper, we assume that \( X \) satisfies absolute continuity condition, which means that \( R_1 f(x) = 0, \ x \in E \) whenever \( f \in L^2(E; m) \cap \mathcal{B}^+(E) \) and \( \int_E f \, dm = 0 \). Recall, see [21, Theorem 4.2.4], that due to symmetry of \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) absolute continuity of \( X \) implies that
\[
\mathbb{P}_t f(x) = 0, \ x \in E, \ t > 0 \quad \text{whenever} \quad f \in \mathcal{B}^+(E) \quad \text{and} \quad \int_E f \, dm = 0. \quad (2.1)
\]
We say that \( f \in \mathcal{B}^+(E) \) is an \( \alpha \)-excessive function if
\[
\sup_{t > 0} e^{-\alpha t} \mathbb{P}_t f(x) = f(x), \quad x \in E.
\]
It follows from the above definition and the absolute continuity of \( X \), that
\[
f, g \text{ are \( \alpha \)-excessive and } f \leq g \text{ m-a.e imply } f(x) \leq g(x), \ x \in E. \quad (2.2)
\]
We will use frequently this property without special mentioning. In the sequel 0-excessive functions will be called simply excessive.

By [21, Lemma 4.2.4], for any \( \alpha \geq 0 \), there exists a unique Borel function \( G_\alpha : E \times E \to \mathbb{R}^+ \cup \{+\infty\} \) (called the \( \alpha \)-Green function) such that for all \( f \in \mathcal{B}^+(E) \)
\[
R_\alpha f(x) = \int_E G_\alpha(x,y) f(y) \, dy, \quad x \in E, \quad (2.3)
\]
and \( G_\alpha(x,\cdot), G_\alpha(\cdot, y) \) are \( \alpha \)-excessive for any \( x, y \in E \). We let \( G := G_0 \). For a given \( \mu \in \mathcal{M}^+(E) \), we set
\[
R_\alpha \mu(x) = \int_E G_\alpha(x,y) \mu(dy), \quad x \in E.
\]
We also let \( R = R_0 \). Observe that from the very definition of the Green function one readily concludes that
\[
R_\alpha \mu \text{ is \( \alpha \)-excessive for any } \mu \in \mathcal{M}^+(E). \quad (2.4)
\]
By [39, Corollary 1.3.6], due to the assumption that \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is transient, there exists a strictly positive function \( \eta \in \mathcal{B}(E) \) such that
\[
R\eta(x) \leq 1, \quad x \in E. \quad (2.5)
\]
From this, we conclude that
\[
\mu \in \mathcal{M}^+_1(E) \implies R\mu < \infty \text{ q.e.} \quad (2.6)
\]
Indeed, it is enough to apply [24, Proposition II.3.5] and the fact that \( \{R\mu, \eta \} = \{\mu, R\eta \} \leq \mu(E) < \infty \) for any strictly positive function \( \eta \in \mathcal{B}(E) \) satisfying (2.5).
We say that $A \subset E$ is *nearly Borel* if there exist $B_1, B_2 \in \mathcal{B}(E)$ such that $B_1 \subset A \subset B_2$ and $\text{Cap}_A(B_2 \setminus B_1) = 0$. The class of all nearly Borel subsets of $E$ shall be denoted by $\mathcal{B}^n(E)$. It is clear that $\mathcal{B}(E) \subset \mathcal{B}^n(E)$. For $B \in \mathcal{B}^n(E)$, we set

$$\sigma_B = \inf \{ t > 0 : X_t \in B \}, \quad \tau_B = \sigma_{E \setminus B}.$$  

By [21, (A.2.7)] for any $B \in \mathcal{B}^n(E)$,

$$\lim_{t \to 0^+} \tau_B \circ \theta_t + t = \tau_B. \tag{2.7}$$

We say that a set $N \subset E$ is *polar* if there exists $B \in \mathcal{B}^n(E)$ such that $N \subset B$ and $P_x(\sigma_B < \infty) = 0, \quad x \in E$.

By [21, Theorems 4.1.2, 4.2.1], $\text{Cap}_A(N) = 0$ if and only if $N$ is polar.

Finally, observe that for any relatively compact open $D \subset E$

$$P_x(\tau_D < \infty) = 1, \quad x \in E. \tag{2.8}$$

Indeed, since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is assumed to be regular, $\text{Cap}_A(D) < \infty$. Thus, by [21, Lemma 2.1.1, Lemma 2.2.6], there exists an excessive function $e_D$ that satisfies:

$$e_D(x) = 1, \quad x \in D, \quad e_D = R\mu \text{ for a positive smooth measure } \mu \text{ on } E.$$  

Therefore, applying [23, proof (e), page 403] but with $U\mu$, appearing there, replaced by $R\mu$ yields (2.8).

Let $\mathcal{T}$ be the topology generated by $\varrho$ (the metric on $E$). We denote by $\mathcal{T}_f$ the *fine topology* on $E$, that is the smallest topology on $E$ for which all excessive functions are continuous. By [13, Section II.4], $\mathcal{T} \subset \mathcal{T}_f$ and $A$ is a finely-open set if and only if for every $x \in A$ there exists $D \in \mathcal{B}^n(E)$ such that $D \subset A$ and

$$P_x(\tau_D > 0) = 1.$$  

In other words, starting from $x \in A$, the process $X$ spends some nonzero time in $A$ until it exits $A$. Observe that each polar set is finely-closed. Therefore, every non-empty finely-open set $V$ satisfies $\text{Cap}_A(V) > 0$. By [13, Theorem II.4.8],

$$f \in \mathcal{B}^n(E) \text{ is finely-continuous} \iff f(X) \text{ is right-continuous a.s.} \tag{2.9}$$

Therefore, using absolute continuity condition, one concludes that

$$f, g \text{ are finely-continuous, } f \leq g \text{ m-a.e imply } f(x) \leq g(x), \quad x \in E. \tag{2.10}$$

By [21, Theorem 4.6.1],

$$f \text{ is finely-continuous and finite q.e. } \Rightarrow f \text{ is quasi-continuous.} \tag{2.11}$$

If $w$ is an excessive function, then by [13, Theorem III.5.7], $w(X)$ is a càdlàg $\mathbb{F}$-supermartingale under measure $P_x$ for every $x \in E$. In particular, $w$ is finely-continuous and hence quasi-continuous.

By [21, Theorem A.2.10], a Hunt process

$$X^D = \{(X_t)_{t \geq 0}, (P^D_x)_{x \in D \cup \{\partial\}}, \mathbb{P}^D = (\mathcal{F}^D_t)_{t \geq 0}, \, \zeta, (\theta_t)_{t \geq 0}\}$$

associated with the form $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$ satisfies

$$P^D_x(\zeta = \tau_D), \quad x \in D. \tag{2.12}$$

Moreover,

$$P^D_t f(x) := \mathbb{E}^D_x f(X_t) = \mathbb{E}_x[f(X_t)1_{\{t < \tau_D\}}], \quad x \in D. \tag{2.13}$$
Remark 2.2. The notions of excessive functions, harmonic functions, nearly Borel sets, polar sets, fine topology, quasi-continuous functions, smooth measures, Cap\(_A\) introduced above depend on the process \(X\) or the associated Dirichlet form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\). We omit this dependence in our notation since in most of the present paper we use them for a fixed process \(X\) and form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) associated with the operator \(A\) in (1.1). In the case where the process or the form under consideration will change, we will write this explicitly.

2.3. Additive functionals and perturbation of self-adjoint operators by smooth measures.

Definition 2.3. We say that an \(\mathbb{F}\)-adapted process \(A = (A_t)_{t \geq 0}\) is an additive functional (AF) of \(X\) if there exists a polar set \(N\) and \(\Lambda \in \mathcal{F}_\infty\) such that

(a) \(P_x(\Lambda) = 1, \ x \in E \setminus N\),
(b) \(P_x(A_0 = 0), \ x \in E \setminus N\),
(c) \(\theta_t(\Lambda) \subset \Lambda, \ t \geq 0,\) and for every \(\omega \in \Lambda, \ A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_s \omega), \ s, t \geq 0,\)
(d) \(A_t(\omega) < \infty, \ t < \zeta(\omega), \ \omega \in \Lambda,\) and \(A_t(\omega) = A_{\zeta(\omega)}, \ t \geq \zeta(\omega), \ \omega \in \Lambda,\)
(e) \([0, \infty) \ni t \mapsto A_t(\omega)\) is càdlàg for every \(\omega \in \Lambda.\)

The set \(N\) in the above definition is called an exceptional set for \(A\), and \(\Lambda\) is called a defining set for \(A\).

Definition 2.4. We say that an \(\mathbb{F}\)-adapted process \(A = (A_t)_{t \geq 0}\) is a positive additive functional (PAF) of \(X\) if it is an additive functional with defining set \(\Lambda \in \mathcal{F}_\infty\) and exceptional set \(N\), and

(a) \(A_t(\omega) \geq 0, \ t \geq 0, \ \omega \in \Lambda.\)

Definition 2.5. We say that an \(\mathbb{F}\)-adapted process \(A = (A_t)_{t \geq 0}\) is a positive continuous additive functional (PCAF) of \(X\) if it is a positive additive functional with defining set \(\Lambda \in \mathcal{F}_\infty\) and exceptional set \(N\), and

(a) \([0, \infty) \ni t \mapsto A_t(\omega)\) is continuous for every \(\omega \in \Lambda.\)

If \(N = \emptyset\), then \(A\) is called a strict PCAF of \(X\).

A Borel measure \(\mu\) on \(E\) is called strictly smooth if it is smooth and there exists an increasing sequence \(\{B_n\}\) of Borel finely-open subsets of \(E\) such that \(\bigcup_{n \geq 1} B_n = E\), and \(R(1_{B_n}, \mu)\) is bounded for every \(n \geq 1\). By [21, Theorem 5.1.7], there is a one-to-one correspondence between strict PCAFs and positive strictly smooth measures. By [21, Theorem 2.2.4], for any positive smooth measure \(\nu\), there exists an increasing sequence \(\{\nu_n\}\) of strictly smooth measures such that \(\nu_n \prec \nu\), i.e. \(\nu_n(B) \prec \nu(B)\) for any \(B \in \mathcal{B}(E)\).

For given \(\alpha \geq 0, f \in \mathcal{B}^+(E)\), and non-negative \(\mathbb{F}\)-adapted càdlàg process \(Y\), we let

\[
\phi_Y^{\alpha, f}(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) e^{-Y_t} dt, \quad x \in E.
\]

The following result has been proven in [31]. Its close formulations can be found in many publications, however it should be emphasized that the assertion of the result below provides a point-by-point analysis of the exceptional sets (e.g. (iv) is well known but with "q.e." in place of "\(x \in E_\nu\)"); properties (i) and (v) together, identify \(N_\nu\) as a minimal exceptional set for a PCAF \(A\) associated with \(\nu\).
Proposition 2.6. Let $\nu$ be a positive smooth measure on $E$ and let

$$N_\nu = E \setminus E_\nu, \quad E_\nu = \{x \in E : \exists V_x \text{ - finely-open neighborhood of } x \text{ such that } \int_{V_x} G(x, y) \nu(dy) < \infty\}.$$ 

Let $\{\nu_n\}$ be a sequence of positive strictly smooth measures such that $\nu_n \nearrow \nu$, and for $n \geq 1$ let $A^n$ be a strict PCAF of $\mathbb{X}$ in the Revuz correspondence with $\nu_n$. Then,

(i) The process $A_t := \sup_{n \geq 1} A^n_t$, $t \geq 0$, is a PCAF of $\mathbb{X}$ with the exceptional set $N_\nu$.
(ii) $\phi^n_A$ is finely-continuous for any $\alpha > 0$ and $f \in B^+_\alpha(E)$.
(iii) If $f \in B^+(E)$ and $R_f$ is finite, then $\phi^0_A$ is finely-continuous.
(iv) For all $x \in E_\nu$, and $f \in B^+(E)$,

$$\mathbb{E}_x \int_0^\zeta f(X_t) \, dA_t = \int_{E} G(x, y) f(y) \nu(dy), \quad (2.15)$$

(v) For every $x \in E_\nu$, $P_x(A_t = \infty, t > 0) = 1$.

From now on, for a given smooth measure $\nu$, we denote by $A^\nu$ the PCAF of $\mathbb{X}$ with exceptional set $N_\nu$ constructed in Proposition 2.6. The one-to-one correspondence between PCAFs of $\mathbb{X}$ and positive smooth measures, expressed in our case by (2.15), is called the Revuz duality.

In what follows we adopt the convention that for any $\mathbb{F}$-adapted positive process $Y$ and positive smooth measure $\nu$,

$$\int_0^\zeta Y_r \, dA^\nu_r := \lim_{n \to \infty} \int_0^\zeta Y_r \, dA^{\nu_n}, \quad \text{a.s.}, \quad (2.16)$$

where $\{\nu_n\}$ is as in Proposition 2.6.

In the sequel, to emphasize the dependence of the set $E_\nu$ on the operator $A$, we sometimes write $E_\nu(A)$, $N_\nu(A)$ instead of $E_\nu, N_\nu$. Observe that for any open set $D \subset E$ and positive smooth measure $\nu$,

$$N_\nu(A|_D) = N_\nu(A) \cap D. \quad (2.17)$$

Indeed, from (2.12) and (2.13) it follows that $A^\nu_{\mathbb{X} \cap D}$ is a PCAF of $\mathbb{X}^D$ in Revuz duality with $\nu|_D$. Since, by Proposition 2.6, $A^\nu < \zeta$, $t < \zeta$, or $A^\nu = \infty$, $t > 0$, we easily get (2.17).

Thanks to the notion of PCAF of $\mathbb{X}$ one can give a beautiful probabilistic interpretation of the semigroup $(T^\nu_t)_{t \geq 0}$ generated by the operator $-A + \nu$ (see Section 2.1). It can be viewed as a generalization of the famous Feynman-Kac formula. By [21, Theorem A.2.11], there exists a Hunt process

$$\mathbb{X}^\nu = ((X_t)_{t \geq 0}, (P^\nu_x)_{x \in E \cup \{\partial\}}, \mathbb{F}^\nu = \{\mathcal{F}_t^\nu, t \geq 0\}, \zeta, \{(\theta_t)_{t \geq 0}\})$$

associated with the form $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ in the sense that for every $f \in B(E) \cap L^2(E; m)$,

$$T^\nu_tf(x) = \mathbb{E}^\nu_x f(X_t) \quad \text{m-a.e. } x \in E. \quad (2.18)$$

We set

$$P^\nu_x f(x) = \mathbb{E}^\nu_x f(X_t), \quad R^\alpha_x f(x) = \mathbb{E}^\nu_x e^{-\alpha t} f(X_t) \, dt, \quad x \in E, \quad \alpha \geq 0, \quad (2.19)$$
where $\mathbb{E}_{x}'$ stands for the expectation with respect to $P_x'$. We put $R^\nu := R^\nu_0$. By [21, Section 6.1],
\[ P_t^\nu f(x) = \mathbb{E}_x e^{-At^\nu} f(X_t), \quad R^\nu_0 f(x) = \mathbb{E}_x \int_0^\zeta e^{-at} e^{-At^\nu} f(X_t) dt, \quad x \in E^\nu. \tag{2.19} \]
By [21, Exercise 6.1.1], $-A + \nu$ possesses the Green function $G^\nu$ on $E^\nu \times E^\nu$ and
\[ G^\nu(x,y) + \int_{E^\nu} G(x,z) G^\nu(z,y) \nu(dz) = G(x,y), \quad (x,y) \in E^\nu \times E^\nu. \tag{2.20} \]
From this identity and symmetry of $G$ and $G^\nu$, we infer that for any $\mu \in \mathcal{M}^+(E)$,
\[ R^\nu \mu + R((R^\nu \mu) \cdot \nu) = R^\nu \mu + R(\mu_{E^\nu} \cdot \nu) = R(\mu_{E^\nu}) \text{ in } E^\nu. \tag{2.21} \]

3. Irreducibility and Feynman-Kac formula

In this section, we recall the notion of irreducibility of Markov semigroups $(T_t)_{t \geq 0}$, which in some sense (see Section 5) is equivalent to obeying by $-A$ the strong maximum principle.

We close the section with a simple proposition which suggests how by the Feynman-Kac representation for a function $u : E \to \mathbb{R}$ one can deduce the structure of the set $\{u = 0\}$.

We say that an $m$-measurable set $C \subset E$ is $(T_t)_{t \geq 0}$-invariant if for every $f \in \mathcal{B}^+(E)$, $T_t(1_C) f = 1_C T_t f$, $t \geq 0$, $m$-a.e. A semigroup $(T_t)_{t \geq 0}$ is called irreducible if any invariant set $C$ satisfies $m(C) = 0$ or $m(E \setminus C) = 0$. It is known (see, e.g., [10, Proposition 2.10]) that an $m$-measurable set $C \subset E$ is $(T_t)_{t \geq 0}$-invariant if and only if there exists an excessive function $u_1$ (or $u_2$) such that $C = \{u_1 = 0\}$ (or $C = \{u_2 < \infty\}$) $m$-a.e. By [21, Theorem 1.6.1], an $m$-measurable set $C \subset E$ is $(T_t)_{t \geq 0}$-invariant if and only if for every $u \in \mathcal{D}(E)$, $1_C u \in \mathcal{D}(E)$.

Let us also note here, that for any $\alpha$-excessive function $w$ the set $F := \{w = 0\}$ is finely-open (it is obviously finely-closed as $w$ is finely-continuous). Indeed, let $V_n := \{w > 1/n\}$. Then, since $(e^{-at} w(X_t))_{t \geq 0}$ is a càdlàg supermartingale, we have
\[ \mathbb{E}_x e^{-\alpha \sigma_V_n} w(x_{\sigma_V_n}) \leq w(x), \quad x \in E. \]
Note that for $x \in F$, the right-hand side of the above inequality equals zero. On the other hand, since $w(X)$ is càdlàg, we have that $e^{-\alpha \sigma_V_n} w(x_{\sigma_V_n}) \geq e^{-\alpha \sigma_V_n} 1/n > 0$ provided $\sigma_V_n < \infty$. Therefore, for any $x \in F$, $P_x (\sigma_V_n < \infty) = 0$. Letting $n \to \infty$, we get $P_x (\tau_F < \infty) = 0$, $x \in F$. In other words, $F$ is finely-open.

**Lemma 3.1.** A symmetric Markov $C_0$-semigroup $(T_t)_{t \geq 0}$ on $L^2(E;m)$ satisfying the absolute continuity condition is irreducible if and only if its Green function $G$ is strictly positive on $E \times E$.

**Proof.** Sufficiency is clear. By [21, Exercise 4.7.1], for every finely-open $V \subset E$ such that $\text{Cap}_A(V) > 0$, we have
\[ \int_V G_\alpha(x,y) m(dy) = \mathbb{E}_x \int_0^\zeta e^{-at} 1_V(X_t) dt > 0, \quad x \in E, \alpha \geq 0. \]
Since $F_\alpha(x) := \{y \in E : G_\alpha(x,y) = 0\}$ is finely-open (see the comment preceding the lemma), we get by the above equation that $\text{Cap}_A(F_\alpha(x)) = 0$ for any $\alpha \geq 0$, $x \in E$. Consequently, for every $x \in E$, $G_\alpha(x,\cdot) > 0$ $m$-a.e., and so, by symmetry of $G_\alpha$, $G_\alpha(\cdot,y) > 0$ $m$-a.e. for every $y \in E$. From this and (2.20) applied to $\nu = m$, we infer that
\[ G(x,y) \geq \int_E G(x,z) G_1(z,y) m(dz) > 0, \quad x,y \in E. \]
Remark 3.2. Since it is known that the Green function for \((\Delta^\alpha)_{D}\) is strictly positive (see e.g. [26]), it follows from the above lemma that for any open set \(D \subset \mathbb{R}^d \) and \(\alpha \in (0,1)\), the semigroup generated by \((\Delta^\alpha)_{D}\) is irreducible. Notice that this is not true for the classical Dirichlet Laplacian, because by [42], \(\Delta_{D}\) is irreducible if and only if \(D\) is finely-connected.

Remark 3.3. Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be a regular symmetric Dirichlet form on \(L^2(E; m)\). If \(E\) is finely-connected, then \((T_t)_{t \geq 0}\) is irreducible (see [42]). If \(\mathcal{E}\) is local (i.e. \(\mathcal{E}(u,v) = 0\) for any \(u,v \in \mathcal{D}(\mathcal{E})\) such \(\text{supp}[u] \cap \text{supp}[v] = \emptyset\)), then \((T_t)_{t \geq 0}\) is irreducible if and only if \(E\) is finely-connected (see [42]).

Lemma 3.4. Let \(\nu\) be a non-trivial positive smooth measure on \(E\) and \((T_t)_{t \geq 0}\) be irreducible. Then for every \(x \in E\),

\[
P_x(\exists_{t \geq 0} A_\nu^x > 0) > 0.
\]

Proof. By Proposition 2.6(iv), the assertion of the lemma holds true for \(x \in N_\nu\). If \(x \in E_\nu\), then by (2.15),

\[
\mathbb{E}_x \int_0^t dA^\nu_x = \int_E G(x,y) \nu(dy), \quad x \in E_\nu.
\]

By Lemma 3.1, \(G(x,y) > 0\), \(x, y \in E\). From this, the fact that \(\nu\) is non-trivial and the above equality, we conclude that the result also holds true for \(x \in E_\nu\).

Proposition 3.5. Let \((T_t)_{t \geq 0}\) be irreducible and \(u\) be a positive function on \(E\). If there exist positive smooth measures \(\mu, \nu\) such that \(\mu\) is non-trivial and

\[
u(x) \geq \mathbb{E}_x \int_0^t e^{-A^\nu_t} dA^\mu_x, \quad x \in E,
\]

then \(\{u = 0\} \subset N_\nu\).

Proof. Suppose that \(u(x) = 0\) for some \(x \in E_\nu\). Then \(\mathbb{E}_x \int_0^t e^{-A^\nu_t} dA^\mu_x = 0\), which implies, in particular, that \(x \in E_\mu\) (cf. (2.16)). On the other hand, by Lemma 3.4, \(P_x(\exists_{t \leq \zeta} A_\nu^x > 0) > 0\). Therefore \(P_x(\exists_{t \leq \zeta} A_\nu^x = \infty) > 0\). So, \(x \not\in N_\nu\), a contradiction.

4. Feynman-Kac representation for supersolutions

In this section, we show that any function \(u\) satisfying (1.1) has a finely-continuous version on \(E_\nu\), which admits a Feynman-Kac representation. This result plays a pivotal role in the proof of the main result of the paper.

We start with providing rigorous meaning to (1.1). Set

\[
U_b := \{\eta \in \mathcal{D}(A) : \eta \in \mathcal{B}_b^{+}(E), A\eta \text{ is bounded}\}.
\]

Lemma 4.1. The set \(U_b\) is dense in \(L^{2,+}(E;m)\) equipped with the standard norm.

Proof. Let \(f \in L^{2,+}(E;m)\). By [40, Theorem 2.4] and the fact that \((T_t)_{t \geq 0}\) is Markov, \(\frac{1}{t} \int_0^t T_s(f \wedge k) ds \in U_b, t > 0, k > 0\). By the contractivity and the strong continuity of \((T_t)_{t \geq 0}\) we get the result.

For positive \(u \in L^1(E;m) \cap L^1(E;\nu)\) we let

\[
I_u[\eta] := \langle u, -A\eta \rangle + \langle u \cdot \nu, \eta \rangle, \quad \eta \in U_b.
\]
Definition 4.2. We say that $\mathcal{C} \subseteq \mathcal{U}_0$ is a set of test functions for $A$ (we occasionally write $\mathcal{C}(A)$ to underline the operator) if there exists $\mathcal{C}_1 \subseteq \mathcal{U}_0$ that satisfies

(a) $\mathcal{C} \subseteq \mathcal{C}_1$, 
(b) $\mathcal{C}_1$ is dense in $L_{2,+}^2(E;m)$, and satisfies $R_\alpha \mathcal{C}_1 \subseteq \mathcal{C}_1$, $\alpha > 0$, 
(c) for any positive $u \in L^1(E;m) \cap L^1(E;\nu)$, that is assumed additionally to be quasi-continuous in case $\nu$ is not absolutely continuous with respect to $m$, we have

$$I_u[\nu] \geq 0, \quad \nu \in \mathcal{C} \quad \Rightarrow \quad I_u[\eta] \geq 0, \quad \eta \in \mathcal{C}_1.$$ 

Definition 4.3. We say that an $\mathcal{F}$-adapted process $A = (A_t)_{t \geq 0}$ is a (local) martingale additive functional (MAF) of $X$ if it is an additive functional with defining set $\Lambda \in \mathcal{F}_\infty$ and exceptional set $\Lambda$, and $A$ is a (local) martingale under measure $P_x$ for any $x \in E \setminus \Lambda$.

Theorem 4.4. Let $\mathcal{C}$ be a set of test functions for $A$. Assume that $u \in L^1(E;m) \cap L^1(E;\nu)$ is a positive function such that

$$\langle u, -\nu \rangle + \langle u \cdot \nu, \eta \rangle \geq 0, \quad \eta \in \mathcal{C}.$$ 

(4.1)

Suppose that either $\nu \ll m$ or $u$ is quasi-continuous. Then there exists an $m$-version $\tilde{u}$ of $u$ which is finely-continuous on $E_\nu$. Moreover, for every $k \geq 0$ there exists a positive smooth measure $\beta_k$ such that

$$\tilde{u}_k(x) = E_x e^{-\lambda \nu \cdot \beta_k} \tilde{u}_k(x) + E_x \int_0^{t \wedge \tau_D} e^{-\lambda \nu \cdot \beta_k} dA \beta_k, \quad t \geq 0,$$ 

for every open relatively compact set $D \subseteq E$ and every $x \in D$, where $u_k = u \wedge k$.

Proof. The proof shall be divided into five steps.

Step 1. We shall show that $\alpha R_\alpha (u + R(u \cdot \nu)) \leq u + R(u \cdot \nu)$ m-a.e. for every $\alpha > 0$. Let $\eta \in \mathcal{C}_1$ ($\mathcal{C}_1$ is an extension of $\mathcal{C}$ according to Definition 4.2). By [40, Theorem 2.4]

$$\alpha R_\alpha \eta - \eta = AR_\alpha \eta.$$ 

It follows from this and (4.1) that

$$\langle u, \alpha R_\alpha \eta - \eta \rangle + \langle R(u \cdot \nu), \alpha R_\alpha \eta - \eta \rangle = \langle u, AR_\alpha \eta \rangle + \langle R(u \cdot \nu), \alpha R_\alpha \eta - \eta \rangle$$

$$\leq \langle u \cdot \nu, R_\alpha \eta \rangle + \langle R(u \cdot \nu), \alpha R_\alpha \eta - \eta \rangle = 0,$$

where the last equality being a consequence of the resolvent identity. Since $\eta$ was an arbitrary function from $\mathcal{C}_1$, and $\mathcal{C}_1$ is dense in $L_{2,+}^2(E;m)$, we get the desired property.

Step 2. We show that $u$ has an $m$-version $\tilde{u}$ which is finely-continuous on $E_\nu$. By [10, Proposition 2.4], $w := u + R(u \cdot \nu)$ possesses an $m$-version that is excessive, and we let $\tilde{w}$ denote this version. By the construction, $\tilde{w} = \lim_{t \downarrow 0} P_t w$. Set $\tilde{u}(x) := \limsup_{t \downarrow 0} P_t u(x), \quad x \in E$. Observe that

$$\tilde{w} = \tilde{u} + R(u \cdot \nu).$$ 

From this (cf. (2.4), (2.11)), in particular, we deduce that $\tilde{u}$ is quasi-continuous. Consequently, if $u$ is assumed to be quasi-continuous, then $u = \tilde{u}$ q.e. (see [21, Theorem 2.1.2]) and, as a result, $\tilde{u} + R(u \cdot \nu) = \tilde{u} + R(u \cdot \nu)$, so $\tilde{u} + R(u \cdot \nu)$ is excessive. On the other hand, if $\nu$ is assumed to be absolutely continuous with respect to $m$, then clearly $\tilde{u} + R(u \cdot \nu) = \tilde{u} + R(u \cdot \nu)$, as $\tilde{u}$ is an $m$-version of $u$, thus $\tilde{u} + R(u \cdot \nu)$ is excessive again. By [13, Theorem III.5.7]
\[\bar{u} + R(\bar{u} \cdot \nu)](X)\] is a càdlàg supermartingale under measure \(P_x\) for q.e. \(x \in E\). By [18, Theorem 3.18] there exists a PCAF \(A\) of \(X\) and a local MAF of \(X\) such that for q.e. \(x \in E\)

\[
\bar{u}(X_t) = \bar{u}(X_0) + \int_0^t \bar{u}(X_r) dA_r^x - A_t + M_t, \quad t \geq 0, \quad P_x\text{-a.s.}
\] (4.3)

Applying integration by parts formula to the product \(e^{-At} \bar{u}(X_t)\) yields

\[
e^{-At} \bar{u}(X_t) = \bar{u}(X_0) - \int_0^t e^{-A(r-t)} dA_r + \int_0^t e^{-At} dM_r, \quad t \geq 0, \quad P_x\text{-a.s.}
\] (4.4)

Let \((\tau_k)\) be a non-decreasing sequence of stopping times such that \(\tau_k \to \infty\), \(E \int_0^{\tau_k} e^{-At} dA_r \leq k\) q.e., and \(\left(\int_0^{\tau \land \tau_k} e^{-At} dM_r\right)_{t \geq 0}\) is a martingale under \(P_x\) q.e. Then, by (4.4)

\[
E_x e^{-At} \bar{u}(X_{t \land \tau_k}) \leq \bar{u}(x) \quad \text{q.e.}
\] (4.5)

From this and Fatou’s lemma, we deduce that \(P_t \bar{u} \leq \bar{u} \text{ q.e. for any } t \geq 0\) (see also (2.19)). Thus, by [10, Proposition 2.4], \(\bar{u} = \lim_{\tau_n \to 0} P_{\tau_n}^x \bar{u}\) is excessive on \(E_\nu\) with respect to \((P_t^x)\). We put \(\bar{u}(x) = 0\) for \(x \in N_\nu\). Since \(\bar{u}\) is a function with respect to \((P_t^x)_{t \geq 0}\) on \(E_\nu\), it is finely-continuous on \(E_\nu\) with respect to \((P_t^x)_{t \geq 0}\). This is equivalent to the fact that the process \(t \mapsto e^{-At} \bar{u}(X_t)\) is right-continuous under the measure \(P_x\) for \(x \in E_\nu\) (see (2.9) and (2.19)). Since for any \(x \in E_\nu\) we have \(e^{-At} > 0\), \(t \geq 0\), \(P_x\text{-a.s.},\) we see that \(\bar{u}(X)\) is right-continuous under the measure \(P_x\) for \(x \in E_\nu\). In other words, \(\bar{u}\) is finely-continuous on \(E_\nu\) (see (2.9)). In particular, \(\bar{u}\) is quasi-continuous (see (2.11)). Therefore, with the aid of [21, Theorem 2.1.2], \(\bar{u} = \bar{u} \text{ q.e. as } \bar{u}, \bar{u}\text{ are } m\text{-versions of } u\), and both \(\bar{u}\) and \(\bar{u}\) are quasi-continuous.

As a result, we get, by (4.3), that for q.e. \(x \in E\)

\[
\bar{u}(X_t) = \bar{u}(X_0) + \int_0^t \bar{u}(X_r) dA_r^x - A_t + M_t, \quad t \geq 0, \quad P_x\text{-a.s.}
\] (4.6)

**Step 3.** We show that (4.2) holds q.e. Write \(\bar{u}_k = \bar{u} \wedge k\). By the Tanaka-Meyer formula (see, e.g., [41, IV. Theorem 70]) applied to (4.6) we have

\[
\bar{u}_k(X_t) = \bar{u}_k(X_0) + \int_0^t 1_{\{\bar{u}_k \leq k\}}(X_r) \bar{u}(X_r) dA_r^x
\]

\[
- \int_0^t 1_{\{\bar{u}_k \leq k\}}(X_r) dA_r - C_t^k + \int_0^t 1_{\{\bar{u}_k \leq k\}}(X_r-) dM_r, \quad t \geq 0 \text{ q.a.s.,}
\) (4.7)

where \(C^k\) is an increasing càdlàg process with \(C^k_0 = 0\). Let \(\{\tau_n\}\) be a fundamental sequence (for the definition see, e.g., [41, Section I.6]) for the local martingale \(\int_0^{\tau_n} 1_{\{\bar{u}_k \leq k\}}(X_r-) dM_r\). By (4.7)

\[
\bar{u}_k(x) + E_x \int_0^{\tau_n} 1_{\{\bar{u}_k \leq k\}}(X_r) \bar{u}(X_r) dA_r^x = E_x C^k + E_x \bar{u}_k(X_{\tau_n}) + E_x \int_0^{\tau_n} 1_{\{\bar{u}_k \leq k\}}(X_r) dA_r \quad \text{q.e.}
\]

Letting \(n \to \infty\) we get \(E_x C^k \leq k + R(u \cdot \nu)(x)\) q.e. Thus \(E_x C^k < \infty\) q.e. Let \(C^{k,p}\) be the dual predictable projection of \(C^k\) (see [21, Section A.3]). It exists q.e. since \(E_x C^k < \infty\) q.e. By (4.7), \(C^k\) is a positive additive functional, so by [21, Theorem A.3.16], \(C^{k,p}\) is also a positive additive functional. By the definition of a Hunt process, \(X\) is quasi-left continuous. Therefore, by [17, Proposition 2, Proposition 4] every local \(\mathbb{F}\)-martingale has only totally inaccessible jumps. So, by (4.6), \(\bar{u}(X)\) has only totally inaccessible jumps. Consequently, by (4.7), \(C^{k,p}\) has only totally inaccessible jumps too. However, \(C^{k,p}\) is predictable. Therefore \(C^{k,p}\) is continuous. By the Revuz duality, there exists a unique positive smooth measure.
\( \gamma_k \) such that \( C^{k,p} = A^{\gamma_k} \). By the definition of the dual predictable projection, there exists a uniformly integrable martingale \( N \) such that \( C^k = C^{k,p} + N \). We let
\[
L^k_t = N_t + \int_0^t 1_{\{\tilde{u} \leq k\}}(X_r) \, dM_r, \quad t \geq 0 \text{ q.a.s.}
\]
Furthermore, by the Revuz duality there exists a positive smooth measure \( \mu \) such that \( A = A^\mu \). Consequently, by (4.7)
\[
\tilde{u}_k(X_t) = \tilde{u}_k(X_0) + \int_0^t 1_{\{\tilde{u} \leq k\}}(X_r) \tilde{u}(X_r) \, dA_r^\mu - \int_0^t 1_{\{\tilde{u} > k\}}(X_r) \, dA_r^k - L^k_t, \quad t \geq 0 \text{ q.a.s.}
\]
Now, put \( \beta_k = (1_{\{\tilde{u} \leq k\}} \tilde{u}_k) \cdot \nu + 1_{\{\tilde{u} > k\}} \cdot \mu + \gamma_k \). From (4.8) we conclude that
\[
\tilde{u}_k(X_t) = \tilde{u}_k(X_0) + \int_0^t \tilde{u}_k(X_r) \, dA_r^\mu - A_r^k + L^k_t, \quad t \geq 0 \text{ q.a.s.}
\]
Applying the integration by parts formula to the product \( e^{-A_t^\nu} \tilde{u}_k(X_t) \) yields
\[
e^{-A_t^\nu} \tilde{u}_k(X_t) = \tilde{u}_k(X_0) - \int_0^t e^{-A_r^\nu} \, dA_r^{\beta_k} + \int_0^t e^{-A_r^\nu} \, dL^k_r, \quad t \geq 0 \text{ q.a.s.}
\]
Let \( \{\tau_n\} \) be a fundamental sequence for the local martingale \( L^k \). By the above equation for any stopping time \( \alpha \) we have
\[
\tilde{u}_k(x) = \mathbb{E}_x e^{-A_{\alpha \wedge \tau_n \wedge \tau}^\nu} \tilde{u}_k(X_{\alpha \wedge \tau_n \wedge \tau}) + \mathbb{E}_x \int_0^{\alpha \wedge \tau_n \wedge \tau} e^{-A_r^\nu} \, dA_r^{\beta_k}.
\]
Letting \( n \to \infty \), we get that for q.e. \( x \in D \) and any stopping time \( \alpha \),
\[
\tilde{u}_k(x) = \mathbb{E}_x e^{-A_{\alpha \wedge \tau}^\nu} \tilde{u}_k(X_{\alpha \wedge \tau}) + \mathbb{E}_x \int_0^{\alpha \wedge \tau} e^{-A_r^\nu} \, dA_r^{\beta_k}.
\]
We used here (2.8).

**Step 4.** We show that \( N_{\beta_k} \subset N_\nu \). First observe that by (4.11), \( \mathbb{E}_x \int_0^\zeta e^{-A_r^\nu} \, dA_r^{\beta_k} \leq k \) for q.e. \( x \in E \). Let \( \{\beta_n^{\beta_k}\} \) be a sequence of smooth measures with bounded potentials such that \( \beta_n^{\beta_k} \to \beta_k \) (see [21, Theorem 2.2.4]). The function \( w_n = \mathbb{E}_x \int_0^\zeta e^{-A_r^\nu} \, dA_r^{\beta_n^{\beta_k}} \) is finely-continuous. Indeed, by [21, Lemma 5.1.5],
\[
w_n(x) = \mathbb{E}_x \int_0^\zeta dA_r^{\beta_n^{\beta_k}} - \mathbb{E}_x \int_0^\zeta w_n(X_r) \, dA_r^\nu, \quad x \in E.
\]
By (2.4) both functions on the right-hand side of the above equation are finely-continuous. Moreover, \( \mathbb{E}_x \int_0^\zeta dA_r^{\beta_n^{\beta_k}} \) is bounded, so \( w_n \) is finely-continuous. Therefore \( w_n(x) \leq k \) for every \( x \in E \) (see (2.10)). From this inequality, definition of \( w_n \), and Proposition 2.6(iv), we easily deduce that \( N_{\beta_k} \subset N_\nu \).

**Step 5.** Conclusion. Fix \( t > 0 \), and put
\[
v(x) = \mathbb{E}_x \int_0^{\tau \wedge \tau_D} e^{-A_r^\nu} \, dA_r^{\beta_k}, \quad w(x) = \mathbb{E}_x e^{-A_{\tau \wedge \tau_D}^\nu} \tilde{u}_k(X_{\tau \wedge \tau_D}), \quad x \in D.
\]
It is an elementary check that if \( s \succ 0 \) then \( t \wedge (\tau_D \circ \theta_s) + s \succ t \wedge \tau_D \) a.s. (cf. (2.7)). For \( x \in E_\nu \), if \( s \succ 0 \), then
\[
P_s(w)(x) = P_s(\mathbb{E}_x e^{-A_{\tau_D \circ \theta_s}^\nu} \tilde{u}_k(X_{\tau_D \circ \theta_s}))(x)
= \mathbb{E}_x e^{-A_{\tau_D \circ \theta_s}^\nu} \tilde{u}_k(X_{\tau_D \circ \theta_s} + x) \rightarrow w(x).
\]
We have used here Markov property of \( \mathbb{X} \), fine continuity of \( \tilde{u}_k \) on \( E_\nu \) and continuity of \( A^\nu \) under the measure \( P_x \) for \( x \in E_\nu \). Next, observe that by the strong Markov property of \( \mathbb{X} \),
\[
v(x) = R^\nu \beta_k(x) - \mathbb{E}_s^\nu((R^\nu \beta_k(X_{t\wedge T_D})), \quad x \in E_\nu.
\]
(4.13)
Since \( N_{\beta_k} \subset N_\nu \), \( A^\beta_k \) is a strict PCAF of \( \mathbb{X}^\nu \). Hence, by [13, Proposition II.4.2], \( R^\nu \beta_k \) is finely-continuous with respect to \((P^\nu_t)_{t \geq 0}\). In particular, by (2.9) and (2.19),
\[
\lim_{s \searrow 0} P_s(R^\nu \beta_k)(x) = \lim_{s \searrow 0} P^\nu_s(R^\nu \beta_k)(x) = R^\nu \beta_k(x), \quad x \in E_\nu.
\]
Since \( R^\nu \beta_k \) is bounded one shows, as in the case of \( w \), that
\[
P_s((E^\nu R^\nu \beta_k(X_{t\wedge T_D}))(x) \to E^\nu R^\nu \beta_k(X_{t\wedge T_D}), \quad x \in E_\nu, \quad s \searrow 0.
\]
Thus \( P_s \nu(x) \to \nu(x) \) for \( x \in E_\nu \) as \( s \searrow 0 \). By (4.11)
\[
\tilde{u}_k = w + v, \quad \text{q.e.}
\]
Therefore, by the absolute continuity condition for \( \mathbb{X} \) (see (2.1)),
\[
P_s(\tilde{u}_k)(x) = P_s(w)(x) + P_s(v)(x), \quad x \in E_\nu.
\]
Letting \( s \searrow 0 \) we get the desired result. \( \square \)

**Corollary 4.5.** Let \( \mathcal{C} \) be a set of test functions for \( A \). Let \( u \in L^1(E; \nu) \cap L^1(E; \nu) \) be a quasi-continuous positive function such that (4.1) is satisfied. Then
\[
\{ u \wedge k, -A\eta \} + \{(1_{\{u \leq k\}} u) \cdot \nu, \eta \} \geq 0, \quad \eta \in \mathcal{C}.
\]

**Proof.** We adopt the notation of Theorem 4.4 and its proof. Let \( \{D_n\} \) be an increasing sequence of relatively compact open subsets of \( E \) satisfying \( \bigcup_{n \geq 1} D_n = E \). Substituting \( \tau_{D_n} \) in place of \( t \) in (4.9) (cf. (2.8)), putting the expectation to both sides of (4.9), and then letting \( n \to \infty \), we find that
\[
u_k = h^{(k)} - R(u_k \cdot \nu) + R_{\beta_k} \quad \text{a.e.},
\]
where \( h^{(k)}(x) := \lim_{n \to \infty} \mathbb{E}_x \tilde{u}_k(X_{t\wedge T_{D_n}}), \quad x \in E \). It is an elementary check that \( h^{(k)} \) is an excessive function up to \( m \)-equivalence. Let \( \eta \in \mathcal{C} \). Then, by the above equation,
\[
\langle u_k, -A\eta \rangle + \langle R((1_{\{u \leq k\}} u) \cdot \nu), -A\eta \rangle = \langle h^{(k)}, -A\eta \rangle + \langle R_{\beta_k}, -A\eta \rangle \geq 0.
\]
In the last inequality, we used the facts that \( h^{(k)}, R_{\beta_k} \) are excessive functions, and \( \eta \) is positive. Now, one easily concludes the result. \( \square \)

5. **Location of zeros of non-trivial supersolutions**

It appears that without additional assumption on the Green function \( G^\nu \), the set of positive function \( u \in L^1(E; m) \cap L^1(E; \nu) \) satisfying (4.1) may be trivial.

**Proposition 5.1.** There exists a non-trivial quasi-continuous positive function \( u \in L^1(E; m) \cap L^1(E; \nu) \) satisfying (4.1) if and only if \( G^\nu(x_0, \cdot) \in L^1(E; m) \) for some \( x_0 \in E_\nu \).

**Proof.** Suppose that \( u \in L^1(E; m) \cap L^1(E; \nu) \) is the asserted function. By the reasoning following (4.5), there exists a function \( \tilde{u} \) that is excessive with respect to \((P^\nu_t)_{t \geq 0}\) and equals \( u \) q.e. Let \( \mu \) be a bounded positive non-trivial measure on \( E_\nu \). Then \( R^\nu \mu \) is an excessive function with respect to \((P^\nu_t)_{t \geq 0}\) which is finite a.e. (see (2.4), (2.6)). Therefore \( u \wedge R^\nu \mu \) shares the same properties. By [24, Proposition 3.9], there exists a positive Borel
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measure \( \gamma \) on \( E_\nu \), such that \( u \wedge R^\nu \mu = R^\nu \gamma \). Of course, \( \gamma \) is non-trivial, and \( R^\nu \gamma \in L^1(E;m) \).

Hence

\[
\langle \gamma, R^\nu 1 \rangle = \langle R^\nu \gamma, 1 \rangle = \| R^\nu \gamma \|_{L^1(E;m)} < \infty .
\]

Thus, \( R^\nu 1 < \infty \) \( \gamma \)-a.e. Since \( \gamma \) is non-trivial, there exists \( x_0 \in E_\nu \) such that

\[
\langle R^\nu 1 \rangle(x_0) = \int_E G^\nu(x_0,y)m(dy) < \infty .
\]

Now suppose that there exists \( x_0 \in E_\nu \) such that \( G^\nu(x_0,\cdot) \in L^1(E;m) \). Then \( u := G^\nu(x_0,\cdot) \in L^1(E;m) \cap L^1(E;\nu) \) and \( u \) satisfies (4.1). Indeed, it is clear that \( u \) is an excessive function with respect to \( (P_t^\nu)_{t \geq 0} \), so (4.1) is satisfied. Furthermore, by (2.21), \( R^\nu \nu \leq 1 \), so \( (u, \nu) = \langle R^\nu \delta_{x_0}, \nu \rangle = \langle \delta_{x_0}, R^\nu \nu \rangle \leq 1 \).

In the reminder of this section, we assume that \( G^\nu(x_0,\cdot) \in L^1(E;m) \) for some \( x_0 \in E_\nu \).

This assumption is satisfied for instance if \( m(E) < \infty \) or \( R1 \) is bounded.

**Theorem 5.2.** Let \( \mathcal{C} \) be a set of test functions for \( A \). Assume that \( (T_t)_{t \geq 0} \) is irreducible. Let \( u \in L^1(E;m) \cap L^1(E;\nu) \) be a positive function satisfying

\[
\{u, -A\eta\} + \{u \cdot \nu, \tilde{\eta}\} \geq 0, \quad \eta \in \mathcal{C}.
\]

Suppose that either \( \nu \ll m \) or \( u \) is quasi-continuous. Then there exists an \( m \)-version \( \tilde{u} \) of \( u \) which is finely-continuous on \( E_\nu \). Moreover, if \( \tilde{u}(x) = 0 \) for some \( x \in E \setminus N_\nu \), then \( \tilde{u} \equiv 0 \).

**Proof.** Let \( \tilde{u} \) be the function constructed in the proof of Theorem 4.4. It is finely-continuous on \( E_\nu \) and \( \tilde{u}(x) = 0, x \in N_\nu \). By Theorem 4.4, for any \( k \geq 0 \) and relatively compact open set \( D \subset E \),

\[
\tilde{u}_k(x) = \mathbb{E}_x e^{-A\nu t_D} \tilde{u}_k(x_{t_D}), \quad t \geq 0,
\]

for every \( x \in D \), where \( \beta_k \) is a positive smooth measure and \( \tilde{u}_k = \tilde{u} \wedge k \). From (5.2) it follows that

\[
\tilde{u}_k(x) \geq \mathbb{E}_x \int_0^\zeta e^{-A\nu} d\gamma_k, \quad x \in E_\nu.
\]

Assume that \( \tilde{u}(x) = 0 \) for some \( x \in E_\nu \). By Proposition 3.5 and (5.3), \( \beta_k = 0 \). Therefore, letting \( t \to \infty \) in (5.2), we get

\[
\tilde{u}(x) = \mathbb{E}_x e^{-A\nu t_D} \tilde{u}_k(x_{t_D}), \quad t \geq 0, \quad x \in D.
\]

Let \( \{D_n\} \) be an increasing sequence of compact subsets of \( E \) such that \( \bigcup_{n \geq 1} D_n = E \). Applying strong Markov property to (5.4) we find that for any \( n \geq 1 \) and any stopping time \( \alpha \),

\[
e^{-A\nu} \tilde{u}_k(X_\alpha) = \mathbb{E}_x (e^{-A\nu} \tilde{u}_k(X_{\tau_D})) 1_{\{\alpha < \tau_D\}}(\mathcal{F}_\alpha) \quad \text{a.s.}
\]

From this and the choice of sequence \( \{D_n\} \), we infer that for any bounded stopping time \( \alpha \), \( \mathbb{E}_x e^{-A\nu} \tilde{u}_k(X_\alpha) = \tilde{u}(x), x \in E \). As a result, \( (\tilde{u}_k(X_t))_{t \geq 0} \) is a martingale under the measure \( P_x \) for any \( x \in E \). Therefore, if \( \tilde{u}(x) = 0 \) for some \( x \in E_\nu \), then \( e^{-A\nu} \tilde{u}_k(X_\alpha) = 0, t \geq 0, P_x\)-a.s. Since \( x \in E_\nu \), we have that \( e^{-A\nu} \tilde{u}_k(X_\alpha) = 0, t \geq 0, P_x\)-a.s. Therefore, \( \tilde{u}_k(X_\alpha) = 0, t \geq 0, P_x\)-a.s. Consequently,

\[
0 = \mathbb{E}_x \int_0^\zeta \tilde{u}_k(X_r) dr = \int_E G(x,y)\tilde{u}(y) dy.
\]

Since \( (T_t)_{t \geq 0} \) is irreducible, \( G(x,\cdot) \) is strictly positive (see Lemma 3.1). Thus, \( \tilde{u} = 0 \) \( m \)-a.e. Since \( \tilde{u} \) is finely-continuous on \( E_\nu \), and \( X \) satisfies absolute continuity condition, we have \( \tilde{u} = 0 \) on \( E_\nu \) (see (2.10)). By the very definition of \( \tilde{u} = 0 \), we have \( \tilde{u}(x) = 0, x \in N_\nu \).
Example 5.3. Observe that if $\nu \ll m$ does not hold, then, in general, the conclusion of Theorem 5.2 does not hold without assumption that $u$ be quasi-continuous. Indeed, let

$$w(x) := 1 - (2 - |x|)^+ \wedge 1, \quad x \in E := (-2, 2), \quad A := \Delta, \quad \nu := \delta_{\{-1\}} + \delta_{\{1\}}.$$ 

One easily finds that $-\Delta w = -\nu$, i.e. $\langle w, -\Delta \eta \rangle = -\int_E \eta \, d\nu, \eta \in C^\infty_c(E)$. Set

$$u(x) := w(x), \quad x \in E \setminus \{-1, 1\}, \quad u(-1) = u(1) := 1.$$ 

Then, $-\Delta u + u \cdot \nu \geq 0$, i.e. $\langle u, -\Delta \eta \rangle + \int_E u \eta \, d\nu \geq 0, \eta \in C^\infty_c(E)$. Since the Green function for $-\Delta$ on $(-2, 2)$ is bounded, we have $N_\nu = \emptyset$. We see that $u(x) = 0, x \in [-1, 1]$ but $u \not\equiv 0$ in $E_\nu = E$. This is so because $u$ is not continuous on $E$, which in our case, i.e. one dimension and $A = \Delta$, is equivalent to quasi-continuity of $u$.

Corollary 5.4. Let the assumptions of Theorem 5.2 hold with quasi-continuous $u$ (if $u$ is not assumed to be quasi-continuous in Theorem 5.2, then by its assertion we know that such version exists). If $\text{Cap}_A\{\{u = 0\}\} > 0$, then $u = 0$ q.e.

Proof. Since $u$ is quasi-continuous, $\tilde{u} = u$ q.e. (see [21, Theorem 2.1.2]). Hence $\text{Cap}_A\{\{\tilde{u} = 0\}\} > 0$. From this and the fact that $\text{Cap}_A(N_\nu) = 0$ we conclude that there exists $x \in E_\nu$ such that $\tilde{u}(x) = 0$. Therefore, by Theorem 5.2, $\tilde{u} = 0$, so $u = 0$ q.e. \hfill \Box

Remark 5.5. The above corollary, in case $A = \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j})$ is a symmetric uniformly elliptic operator on a bounded domain, was proved by Ancona [5, Theorem 8]. In fact, Ancona needed some additional regularity assumptions on $u$ or coefficients $a_{ij}$ (either $u \in H^1_{\text{loc}}(D)$ or $a_{i,j}$ are smooth and $Au$ is a measure). Note, however, that thanks to these additional conditions the author dispensed with the assumption that $u \in L^1(E; \nu)$.

We close this section with the result saying that $N_\nu$ is the set of all possible zeros of positive non-trivial solutions to (5.1).

Theorem 5.6. Assume that $(T_1)_{t \geq 0}$ is irreducible and $\nu$ is a positive smooth measure. Then

$$N_\nu = \bigcup_{u \in \mathcal{A}} \{u = 0\},$$

where $\mathcal{A} = \{u \in L^1(E; m) \cap L^1(E; \nu) : u \geq 0, u \text{satisfies (5.1), } u \text{ is finely-continuous and } u \not\equiv 0\}$.

Proof. Let $u \in \mathcal{A}$. Then $u(x) > 0$ for some $x \in E$. Since $u$ is finely-continuous, there exists a finely-open neighborhood $V_x$ of $x$ such that $u(y) > 0$ for $y \in V_x$. Since $V_x$ is finely-open, $\text{Cap}_A(V_x) > 0$ (see the comments on polar sets in Section 2.2), so there exists $y_0 \in E_\nu$ such that $u(y_0) > 0$. Therefore, by Theorem 5.2, $\{u = 0\} \subset N_\nu$. Consequently, $\bigcup_{u \in \mathcal{A}} \{u = 0\} \subset N_\nu$. To prove the opposite inclusion, we use the assumption that $G^\nu(x_0, \cdot) \in L^1(E; m)$ for some $x_0 \in E_\nu$. In the proof of Proposition 5.1 we have shown that $G^\nu(x_0, \cdot) \in L^1(E; \nu)$. By the very definition of the Green function, $G^\nu(x_0, \cdot)$ is excessive with respect to $(P^\nu_t)_{t \geq 0}$. Let $v := G^\nu(x_0, \cdot) \wedge 1$. Of course $v \in L^1(E; m) \cap L^1(E; \nu)$ and $v$ is excessive with respect to $(P^\nu_t)_{t \geq 0}$. By [24, Proposition 3.9], there exists a positive Borel measure $\gamma$ such that $v = R^\nu \gamma$. Let $\gamma_1 := R^\nu \gamma$. Then $R^\nu \gamma_1 \leq v$. Let $\eta$ be a strictly positive bounded function on $E$ such that $R\eta$ is bounded (see (2.5)), and let

$$u(x) := \mathbb{E}_{x} \int_0^\infty (\gamma_1 \wedge \eta)(X_r)e^{-Ar} \, dr, \quad x \in E.$$
Observe that $u = R^\nu (\gamma_1 \land \eta) \wedge R^\nu \gamma_1 \leq v \in L^1(E; m) \cap L^1(E; \nu)$ (see (2.19)). Since $u$ is excessive with respect to $(P_t^\nu)_{t \geq 0}$ (see (2.4)), it satisfies (5.1). By Proposition 2.6(iii), $u$ is finely-continuous. Thus $u \in \mathcal{A}$. It is clear that $\{u = 0\} = N_\nu$, which proves that $N_\nu \subset \bigcup_{u \in \mathcal{A}} \{u = 0\}$. \hfill \Box

6. Strong maximum principle

In this section we are concerned with the strong maximum principle (SMP) for operators of the form $-A + V$, where $V : E \to [0, +\infty]$ is assumed to be merely a Borel measurable function (no additional assumptions). In consequence $\nu := V \cdot m$ is not, in general, a smooth measure, and so the results of the previous sections cannot be directly applied. Nevertheless, $N_{V \cdot m}$ is still well defined. Moreover,

$$N_{V \cdot m} = \{x \in E : P_x \left( \int_0^t V(X_r) \, dr = \infty, \, t > 0 \right) = 1 \}. \quad (6.1)$$

The above equation one easily gets repeating the proof of [31, Proposition 3.2] with $\nu_n := (V \land n) \cdot m$.

Fix a set $C$ of test functions for $A$. We say that SMP holds for an operator $-A + V$ if for any finely-continuous positive function $u \in L^1_{\text{loc}}(E; m) \cap L^1(E; V \cdot m)$ that satisfies (4.1) with $\nu = V \cdot m$, we have the following implication: if $u(x_0) = 0$ for some $x_0 \in E$, then $u(x) = 0, \, x \in E$.

**Theorem 6.1.** Let $V : E \to [0, +\infty]$ be a Borel measurable function. Assume that $(T_t)_{t \geq 0}$ is irreducible. Then the strong maximum principle holds for $-A + V$ if and only if

1. either $N_{V \cdot m} = \emptyset$
2. or $N_{V \cdot m} \neq \emptyset$ and there is no non-trivial positive finely-continuous solution to (4.1) with $\nu = V \cdot m$.

**Proof.** Set $\nu := V \cdot m$. That (1) implies SMP follows from Theorem 5.2. The implication (2)⇒SMP is trivial. Now assume that SMP holds and $x_0 \in N_\nu \neq \emptyset$. Let $D$ be an open relatively compact subset of $E$ such that $x_0 \in D$. Let $f$ be a positive finely-continuous solution to (4.1). Repeating the argument of the proof of Theorem 4.4 that led to (4.3), we conclude (4.3) with $\tilde{u}$ replaced by $f$ and $\int_0^t \tilde{u}(X_r) \, dA^\nu_r$ replaced by $A^f_\nu$. Observe that although $u$ is not smooth, we know that $f \cdot \nu$ is smooth, as it is a bounded measure, thus, $A^f_\nu$ is well defined. Now, repeating the argument of Step 3. of the proof of Theorem 4.4 that led to (4.9), we find that

$$f_k(X_t) = f_k(X_0) + A^f_\nu t - A^\beta_k t + L^k_t, \quad t \geq 0 \text{ q.a.s.,} \quad (6.2)$$

for a local martingale $L^k$ and a positive smooth Green bounded measure $\beta_k$, where $f_k := f \land k$. Let $\nu_l := (V \land l) \cdot m$ (clearly it is a smooth measure), and let $\{\tau_n\}$ be a fundamental sequence for the local martingale $L^k$. Applying integration by parts formula to $e^{-A^\nu_l} f_k(X_t)$ we find that for any stopping time $\alpha$ the following equation holds in $D$,

$$f_k(x) = \mathbb{E}_x \left[ e^{-A^\nu_{\alpha \land T^\nu}} f_k(X_{\alpha \land T^\nu}) + \int_0^{\alpha \land T^\nu} e^{-A^\nu r} dA^\beta_r + A^{f \land \alpha}_{\alpha \land T^\nu} - A^{f \land \nu}_{\alpha \land T^\nu} \right] \text{ q.e.}$$

Letting $l \to \infty$ and then $n \to \infty$ yields

$$f_k(x) = \mathbb{E}_x \left[ e^{-A^\nu_{\alpha \land T^\nu}} f_k(X_{\alpha \land T^\nu}) + \int_0^{\alpha \land T^\nu} e^{-A^\nu} dA^\beta_r \right] \text{ q.e. in } D, \quad (6.3)$$
where \( A_t := \int_0^t V(X_r) \, dr, \, t \geq 0 \). Let \( w(x) \) denote the right-hand side of (6.3) and let \( w_l(x) \) denote the right-hand side of (6.3) but with \( A \) replaced by \( A^\alpha \). By Step 5. of the proof of Theorem 4.4 \( w_l \) is finely-continuous. Clearly, \( w = \inf_{l \geq 1} w_l \). Therefore, since \( f_k \) is assumed to by finely-continuous, we conclude from (6.3) that

\[
f_k(x) \leq \mathbb{E}_x \left[ e^{-A_{\alpha \wedge T}^D} f_k(X_{\alpha \wedge T}) + \int_0^{\alpha \wedge T} e^{-A_r^D} \, dA_r^\beta \right], \quad x \in D.
\]

(6.4)

From this and (6.1), we deduce that \( u(x_0) = 0 \). Thus, by SMP, \( u(x) = 0, \, x \in E \). □

Remark 6.2. The conclusion of Theorem 6.1, in case \( A = \Delta \) and \( d = 1 \), follows from [7], where the authors went a step further and give a necessary and sufficient condition on \( V \) guaranteeing (2) of Theorem 6.1. (recall here that in case \( A = \Delta \) and \( d = 1 \), fine topology and Euclidean topology agree).

7. Schrödinger equations with \( L^p \) potentials

For \( p > 1 \) and \( A \in C \), we set

\[
C_p(A) = \inf\{ \| f \|^p_{L^p} : f \geq 0, \, f \in L^p(E;m), \, Rf \geq 1_A \}.
\]

We also let

\[
C_1(A) = \inf\{ \| \mu \|_{TV} : \mu \in \mathcal{M}^+(E), \, R\mu \geq 1_A \}
\]

for any \( A \in E \). The above set functions are called Riesz capacities. For any \( p \geq 1 \) and \( A \in B^p(E) \) we also set

\[
c_p(A) = \sup\{ \| \mu \|_{TV} : \mu \in \mathcal{M}_0^+(E), \, \mu(A^c) = 0, \, R\mu \leq 1 \}.
\]

Observe that \( c_1(A) \leq C_1(A) \) for every \( A \in B^p(E) \). Indeed, let \( \mu \in \mathcal{M}^+(E) \) be such that \( \mu(A^c) = 0 \) and \( R\mu \leq 1 \), and let \( \nu \in \mathcal{M}_0^+(E) \) be such that \( R\nu \geq 1_A \). Then

\[
\| \mu \|_{TV} = \mu(A) \leq \langle R\nu, \mu \rangle = \langle \nu, R\mu \rangle \leq \| \nu \|_{TV},
\]

from which the desired result easily follows. By [21, Exercise 2.2.2], for every compact \( K \subset E \),

\[
\text{Cap}_A(K) = c_1(K).
\]

Since \( \text{Cap}_A \) is a Choquet capacity, we conclude from the above equality that

\[
\text{Cap}_A \ll c_1 \leq C_1.
\]

Lemma 7.1. If \( f \in L^p(E;m) \) for some \( p \geq 1 \), then \( f \cdot m \) is a smooth measure.

Proof. Let \( g \) be a strictly positive function in \( L^p(E;m) \). Then \( R_1g \in L^p(E;m) \). Of course, \( R_1g \) is finely-continuous (see (2.4)) and \( (R_1g, f) < \infty \). □

Theorem 7.2. Let \( \mathcal{C} \) be a set of test functions for \( A \). Let \( (T_t)_{t \geq 0} \) be irreducible and \( V \) be a positive Borel function such that \( V \in L^p(E;m) \) for some \( p \geq 1 \). Assume that \( u \in L^1(E;m) \cap L^1(E;V \cdot m) \) is a positive function such that

\[
\langle u, -A\eta \rangle + \langle u \cdot V, \eta \rangle \geq 0, \quad \eta \in \mathcal{C}.
\]

(7.2)

Then there exists an \( m \)-version \( \bar{u} \) of \( u \) which is finely-continuous on \( E_\nu \). Moreover, if \( C_p(\{ \bar{u} = 0 \}) > 0 \), then \( \bar{u} \equiv 0 \).
Proof. Let \( \tilde{u} \) be the function constructed in the proof of Theorem 4.4. By Lemma 7.1, \( \nu := V \cdot m \) is a positive smooth measure. Since \( N_\nu \subseteq \{ RV = \infty \} \), it follows from [35, Theorem 3] (see also the comments at the beginning of [35, Section 3]) that \( C_p(N_\nu) = 0 \). Hence, since \( C_p(\{\tilde{u} = 0\}) > 0 \), there is \( x \in E_\nu \) such that \( \tilde{u}(x) = 0 \). By Theorem 5.2, \( \tilde{u} \equiv 0 \).

\[ \square \]

Corollary 7.3. Let \( (T_t)_{t \geq 0} \) be irreducible, \( V \) be a positive function such that \( V \in L^1(E;m) \) and \( u \) be a positive function in \( L^1(E;m) \cap L^1(E;V \cdot m) \) such that (7.2) is satisfied. If \( \text{Cap}_A(\{\tilde{u} = 0\}) > 0 \), then \( \tilde{u} \equiv 0 \).

Proof. Assume that \( \text{Cap}_A(\{\tilde{u} = 0\}) > 0 \). Then, by (7.1), \( C_1(\{\tilde{u} = 0\}) > 0 \), so by Theorem 7.2, \( \tilde{u} \equiv 0 \).

\[ \square \]

8. The set of test functions and finely-continuous versions of supersolutions

In the present section we are concerned with the structure of the set of test functions \( C \) in (4.1). Usually, in applications, we are interested in cases when \( C = C_c^\infty(\mathbb{R}^d) \) or \( C = C_0^\infty(\mathbb{R}^d) \) are allowed. In the second part of the section we will provide several remarks and simple results on finely-continuous versions of functions. Especially, we focus on the Laplacian and the fractional Laplacian. In these two particular cases one may provide some special formulas for finely-continuous \( m \)-versions \( \tilde{u} \). We shall close this section with a result on equivalence between Riesz’ capacity for the fractional Laplacian and some Sobolev capacity.

8.1. The set of test functions. We will start with the following useful result.

Proposition 8.1. Assume that \( u \in \mathcal{D}_e(\mathcal{E}) \cap L^1(E;\nu) \) and \( \mathcal{C}(\mathcal{E}) \) is a standard core for \( \mathcal{E} : \mathcal{C}(\mathcal{E}) \subset \mathcal{D}_e(\mathcal{E}) \cap C_c(\mathcal{E}) \) is dense in \( C_c(\mathcal{E}) \) and \( \mathcal{D}_e(\mathcal{E}) \), with standard norms, and \( u \in \mathcal{C}(\mathcal{E}) \) implies \( \phi(u) \in \mathcal{C}(\mathcal{E}) \) for any 1-Lipschitz smooth function on \( \mathbb{R} \) with \( \phi(0) = 0 \) (see [21, page 6]). If

\[ \mathcal{E}(u, \eta) + \int_E \tilde{u} \eta \, d\nu \geq 0, \quad \eta \in \mathcal{C}(\mathcal{E}), \]

then the conclusion of Theorem 5.2 holds true, namely, there exists an \( m \)-version \( \tilde{u} \) of \( u \) which is finely-continuous on \( E_\nu \), and if \( \tilde{u}(x) = 0 \) for some \( x \in E \setminus N_\nu \), then \( \tilde{u} \equiv 0 \).

Proof. First note, that \( u \) being in \( \mathcal{D}(\mathcal{E}) \) implies the existence of \( \tilde{u} \) (quasi-continuous \( m \)-version of \( u \)). Since \( \mathcal{C}(\mathcal{E}) \) is a standard core, we get at once that (8.1) holds for any \( \eta \in \mathcal{U}_b \). Clearly, for \( \eta \in \mathcal{U}_b \), \( \mathcal{E}(u, \eta) = (\langle u, -A\eta \rangle) \). Therefore, the assumptions of Theorem 5.2 are satisfied. \[ \square \]

In many interesting cases a standard core of \( \mathcal{E} \) may be taken to be equal \( C_c^\infty(\mathbb{R}^d) \) (see e.g. [4], [21, Example 1.2.4, Exercise 3.1.1], [43]).

Now let us return to the case we have no information about the regularity of \( u \).

For positive functions \( u \in \mathcal{B}_c^0(E) \), we let

\[ H_D(u)(x) = \mathbb{E}_x u(X_{\tau_D}), \quad x \in E. \]

By [21, Theorem 4.3.2], \( \Pi_D(u) := u - H_D(u) \), \( \Pi_D : (\mathcal{D}_e(\mathcal{E}), \mathcal{E}) \rightarrow (\mathcal{D}_e(\mathcal{E}|_D), \mathcal{E}) \) is the orthogonal projection onto \( \mathcal{D}_e(\mathcal{E}|_D) \). Observe also that for any positive \( u \in \mathcal{D}_e(\mathcal{E}|_D) \),

\[ P_t^D(H_D(u))(x) = \mathbb{E}[u(X_{t}1_{\{t < \tau_D\}})] = \mathbb{E}_x[u(X_{\tau_D})1_{\{t < \tau_D\}}] \leq H_D(u)(x), \quad x \in D. \]

(8.2)
Lemma 8.2. Let $D$ be a relatively compact open subset of $E$. Let $u \in \mathcal{D}(A)$ and $\eta \in \mathcal{D}(A|D)$ be positive. Then

$$(-Au, \eta)_{L^2(D;m)} \leq (u, -A|D\eta)_{L^2(D;m)}. \quad (8.3)$$

Moreover, there exists $c > 0$ depending only on $\|H_D(u)\|_{L^1(D;m)}$ such that

$$|(u, -A|D\eta)_{L^2(D;m)} - (-Au, \eta)_{L^2(D;m)}| \leq c\|\eta\|_\infty. \quad (8.4)$$

Proof. Let $(T^D_t)_{t \geq 0}$ be the semigroup generated by $A|D$. By Dynkin’s formula (see [21, (4.4.2)]), $u - H_D(u) \in \mathcal{D}(A|D)$. Therefore,

$$(-Au, \eta)_{L^2(D;m)} = \mathcal{E}(u, \eta) = \mathcal{E}(u - H_D(u), \eta) = \mathcal{E}_D(u - H_D(u), \eta) = (u - H_D(u), -A|D\eta)_{L^2(D;m)}.$$  

Consequently,

$$(-Au, \eta)_{L^2(D;m)} - (u, -A|D\eta)_{L^2(D;m)} = -(H_D(u), -A|D\eta)_{L^2(D;m)}.$$  

Next,

$$(H_D(u), -A|D\eta)_{L^2(D;m)} = \lim_{t \to 0^+} \frac{1}{t} (H_D(u), \eta - T^D_t \eta)_{L^2(D;m)}$$

$$= \lim_{t \to 0^+} \frac{1}{t} (H_D(u) - T^D_t H_D(u), \eta)_{L^2(D;m)}.$$  

Inequality (8.4) follows from (8.5), (8.6) and [9, Proposition 4.4]. Combining (8.2) with (8.5) and (8.6) yields (8.3). \qed

Let $j_\varepsilon$ be a standard mollifier on $\mathbb{R}^d$. In the remainder of this subsection we assume that $E = \mathbb{R}^d$, $C_c^\infty(E) \subset \mathcal{D}(A)$ and

$$A(u * j_\varepsilon) = j_\varepsilon * (Au), \quad u \in C_c^\infty(E). \quad (8.7)$$

One easily shows that the above condition fulfill Lévy operators, i.e. operators of the form

$$Au(x) := a\Delta u(x) + \int_{\mathbb{R}^d} \left( u(x + y) - u(x) - \nabla u(x) \cdot y 1_{B(0,1)}(y) \right) \mu(dy),$$

where $a \geq 0$ and $\mu$ is a positive Borel measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \min\{1, |y|^2\} \mu(dy) < \infty.$$

Lemma 8.3. Let $D$ be a bounded open subset of $\mathbb{R}^d$, and $\nu$ be a positive smooth measure on $D$. Let $u \in L^1(D;m) \cap L^1(D;\nu)$ be a positive quasi-continuous function such that

$$\langle u, -A\xi \rangle + \langle u \cdot \nu, \xi \rangle \geq 0, \quad \xi \in C_c^\infty(D), \; \xi \geq 0. \quad (8.9)$$

Then

$$\langle u, -A|D\eta \rangle + \langle u \cdot \nu, \eta \rangle \geq 0, \quad \eta \in \mathcal{U}_0(A|D). \quad (8.10)$$

Proof. We extend $u$ (resp. $\nu$) to $\mathbb{R}^d$ by letting $u = 0$ (resp. $\nu = 0$) on $\mathbb{R}^d \setminus D$, and then set $u_\varepsilon := j_\varepsilon * u$, $(u \cdot \nu)_\varepsilon := j_\varepsilon * (u \cdot \nu)$. Let $U$ be an open set such that $\overline{U} \subset D$. Let $\eta \in \mathcal{U}_0(A|U)$. By (8.9), (8.7), for $\varepsilon, \delta > 0$ small enough, we have

$$0 \leq \langle u, -A(j_\varepsilon * (j_\delta \star \eta)) \rangle + \langle u \cdot \nu, (j_\varepsilon * (j_\delta \star \eta)) \rangle$$

$$\quad = \langle u_\varepsilon, -A(j_\delta \star \eta) \rangle + \langle (u \cdot \nu)_\varepsilon, j_\delta \star \eta \rangle$$

$$\quad = \langle -Au_\varepsilon, j_\delta \star \eta \rangle + \langle (u \cdot \nu)_\varepsilon, j_\delta \star \eta \rangle.$$
Then there exists an \( m \) such that
\[
0 \leq \langle -Au, \eta \rangle + \langle (u \cdot \nu)_\varepsilon, \eta \rangle \leq \langle u_\varepsilon - A[U] \eta, \eta \rangle + \langle (u \cdot \nu)_\varepsilon, \eta \rangle.
\]
Letting \( \varepsilon \to 0 \) we obtain
\[
\langle u, -A[U] \eta \rangle + \langle u \cdot \nu, \eta \rangle \geq 0, \quad \eta \in \mathcal{U}_0(A[U]).
\]
By Step 1 of the proof of Theorem 4.4, \( u + R^U(u \cdot \nu) \) is an excessive function with respect to \( (P^U_t) \). By Riesz’s decomposition theorem, there exists a positive Borel measure \( \mu_U \) such that
\[
u^U = H^U(u) + R^U \mu_U.
\]
By the uniqueness argument, \( \mu_U = (\mu_W)_U \) for \( U \subset W \) and \( W \subset D \). Therefore there exists a positive Borel measure \( \mu \) on \( D \) such that \( \mu|_U = \mu_U \) for every open \( U \subset D \) with \( \bar{U} \subset D \). Thus
\[
u^U = H^U(u) + R^U \mu \quad \text{q.e.}
\]
Let \( D_n \) be an increasing sequence of open sets such that \( \bar{D}_n \subset D \) and \( \bigcup_{n \geq 1} D_n = D \). We have
\[
u^U = H_{D_n}(u) + R^{D_n} \mu \quad \text{q.e.}
\]
Letting \( n \to \infty \) we get
\[
u^U = h + R^D \mu \quad \text{q.e.}
\] with \( h = \lim_{n \to \infty} H_{D_n}(u) \). One easily shows that \( h \) is an excessive function with respect to \( (P^D_t) \). Let \( \eta \in \mathcal{U}_0(A[D]) \). Then there exists \( \rho \in \mathcal{B}_0(E) \) such that \( \eta = R^D \rho \). By (8.11),
\[
\langle u, -A[D] \eta \rangle + \langle u \cdot \nu, \eta \rangle = \langle u, \rho \rangle + \langle u \cdot \nu, R^D \rho \rangle = \langle u, \rho \rangle + \langle R^D (u \cdot \nu), \rho \rangle = \langle h, \rho \rangle + \langle R^D \mu, \rho \rangle = \langle h, \rho \rangle + \langle \mu, R^D \rho \rangle = \langle h, -A[D] R^D \rho \rangle + \langle \mu, \eta \rangle \geq \langle h, -A[D] R^D \rho \rangle.
\]
This implies (8.10) since \( \langle h, -A[D] R^D \rho \rangle = \lim_{n \to 0^+} \frac{1}{t} (h - T^D_t h, R^D \rho) \geq 0 \).

**Theorem 8.4.** Assume that \( (T^D_t)_{t \geq 0} \) is irreducible. Let \( \nu \) be a positive smooth measure on \( D \) and \( u \in L^1(D; m) \cap L^1(D; \nu) \) be a positive quasi-continuous function such that
\[
\langle u, -A \xi \rangle + \langle u \cdot \nu, \xi \rangle \geq 0, \quad \xi \in C_c^\infty(D), \xi \geq 0.
\] Then there exists an \( m \)-version \( \bar{u} \) of \( u \), which is finely-continuous on \( E_\nu \cap D \). Moreover, if \( \bar{u}(x) = 0 \) for some \( x \in E_\nu \cap D \), then \( \bar{u} \equiv 0 \) in \( D \).

**Proof.** It follows from Theorem 5.2 and Lemma 8.3.

**Remark 8.5.** If \( A \) is given by (8.8), then, by [26, Proposition 2.2], \( (T^D_t) \) is irreducible provided one of the following conditions holds:
(1) \( a > 0 \) and \( D \) is connected,
(2) \( \overline{\mathcal{D}} \) is a subset of the support of \( \mu \).

**Corollary 8.6.** Under the assumptions of Theorem 8.4, if \( \text{Cap}_A(\{u = 0\} \cap D) > 0 \), then \( u = 0 \) q.e. in \( D \).
Proof. By Theorem 8.4, there exists an m-version \( \hat{u} \) of \( u \), which is finely-continuous on \( E_\nu \cap D \). Since \( u \) is quasi-continuous, \( \hat{u} = u \) q.e. in \( D \), so by the assumptions of the corollary, \( \text{Cap}_A(\{\hat{u} = 0\} \cap D) > 0 \). Since \( \text{Cap}_A(N_\nu) = 0 \), there exists \( x \in E_\nu \cap D \) such that \( \hat{u}(x) = 0 \). Hence, by Theorem 8.4, \( \hat{u} = 0 \) in \( D \), which implies that \( u = 0 \) q.e. in \( D \).

8.2. Finely-continuous versions of supersolutions. In general, if \( u \) is a finely-continuous positive function on a finely-open set \( U \subset E \), then by (2.9)

\[
\lim_{r \searrow 0} \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = \lim_{t \searrow 0} \frac{1}{\alpha^{d/2}} \int_0^t e^{-\alpha t} u(X_t) \, dt, \quad x \in U. \tag{8.13}
\]

The last two equalities combined with (2.3) imply that if a function \( u : E \to \mathbb{R} \) has an m-version \( \hat{u} \) which is finely-continuous on \( U \), then (1.7) holds. The first equation in (8.13) can be recast as follows

\[
u \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = \lim_{t \to 0^+} \frac{1}{\alpha^{d/2}} \int_0^t e^{-\alpha t} u(X_t) \, dt, \quad x \in U.
\]

where for fixed \( x \in U \), \( P_{B(x,r)}(x,dy) \) is a Borel measure on \( E \setminus B(x,r) \), so called harmonic measure. The kernel \( P_{B(x,r)}(x,dy) \) is called in the literature the Poisson kernel. It is given by the following formula

\[
P_{B(x,r)}(x,dy) := P_x(X_{\tau_{B(x,r)}} \in dy).
\]

In case of the Laplacian and the fractional Laplacian, Poisson’s kernels may be computed explicitly. In many cases, although no explicit formula is known, asymptotic behavior of the Poisson kernel is well studied (see e.g. [30]).

Lemma 8.7. Let \( A = \Delta_D \). Let \( u \) be a finely-continuous positive bounded function on \( D \). Then

\[
u \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = \lim_{r \to 0} \int_{B(x,r)} u(y) \, dy, \quad x \in D.
\]

Proof. Let \( a_{d,r} = m(B(x,r)) = r^d \cdot a_d \) where \( a_d = \pi^{d/2}/\Gamma(\frac{d}{2} + 1) \), and let \( b_{d,r} = S(\partial B(x,r)) = r^{d-1} \cdot b_d \) where \( b_d = 2\pi^{d/2}/\Gamma(\frac{d}{2}) \). Note that \( b_d/a_d = d \). By using [19, Proposition 1.21] we find that

\[
u \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = \frac{1}{a_{d,r}} \int_0^r \int_{\partial B(x,s)} u(y) \, dS(y) \, ds = \frac{1}{a_{d,r}} \int_0^r b_{d,s} E_x u(X_{\tau_{B(x,s)}}) \, ds.
\]

Hence

\[
u \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = \frac{b_d}{a_d} \frac{1}{r^d} \int_0^r E_x u(X_{\tau_{B(x,s)}}) \, ds = \frac{1}{r^d} \int_0^r E_x u(X_{\tau_{B(x,s)}}) \, ds.
\]

Since \( u \) is finely-continuous, \( E_x u(X_{\tau_{B(x,s)}}) \to u(x) \) as \( s \searrow 0 \), which combined with the above equation gives the desired result.

Now Theorem 8.4 can be restated as follows.

Theorem 8.8. Let \( D \) be an open subset of \( \mathbb{R}^d \) such that \( (T^D) \) is irreducible. Let \( \nu \) be a positive smooth measure on \( D \) and \( u \in L^1(D; \mu) \cap L^1(D; \nu) \) be a positive quasi-continuous function such that (8.12) is satisfied. If

\[
u \limsup_{r \to 0} \frac{1}{r^d} \int_{B(x,r)} u(y) \, dy = 0
\]

for some \( x \in E_\nu \cap D \), then \( u = 0 \) q.e. in \( D \).
Proof. Follows from Corollary 4.5, Theorem 8.4 and Lemma 8.7. □

We shall provide one another corollary, which was the main result of the recent paper by Orsina and Ponce [37].

For any compact $K \subset D$ and $p > 1$, we define

$$\text{Cap}_{W^{2,p}}(K) := \inf \{ \| u \|_{W^{2,p}} : u \in C_c^\infty(\mathbb{R}^d), u \geq 1_K \}.$$ 

In the standard way $\text{Cap}_{W^{2,p}}$ can be extended to arbitrary set $A \subset \mathbb{R}^d$ (see [1, Definition 2.2.4]).

Remark 8.9. By the Calderón-Zygmund $L^p$-theory, $\text{Cap}_{W^{2,p}}$ is equivalent to $C_p$ for the operator $A = \Delta_D$ (see Section 6 for the definition of $C_p$).

Theorem 8.10. Let $D$ be an open subset of $\mathbb{R}^d$ such that $(T^D)$ is irreducible. Let $p > 1$ and $V \in L^p(D; m)$ be positive. Let $u \in L^1(D; m) \cap L^1(D; V; m)$ be a positive function that satisfies

$$(u, -\Delta \xi) + (Vu, \xi) \geq 0, \quad \xi \in C_c^\infty(D), \xi \geq 0.$$ 

Write

$$Z := \left\{ x \in D : \limsup_{r \to 0^+} \int_{B(x, r)} u(y) \, dy = 0 \right\}.$$ 

If $\text{Cap}_{W^{2,p}}(Z) > 0$, then $u = 0$ m-a.e. in $D$.

Proof. By Remark 8.9, $C_p(Z) > 0$. Write $\nu = V \cdot m$. Since $C_p(N_\nu) = 0$ (see the proof of Theorem 7.2), $\nu \cap Z \neq \emptyset$, so by Theorem 8.8, $u = 0$ m-a.e. □

Remark 8.11. Let $D$ be a bounded domain in $\mathbb{R}^d$. Let $Z$ be as in (1.10), and, for given positive $f \in L^\infty(D; m)$, let $\omega_f \in W^{1,2}_0(D) \cap L^\infty(D; m) \cap L^1(D; V; m)$ be a unique solution to (1.9). By [31, Proposition 3.2(ii)], Theorem 6.4] $\omega_f$ has a finely-continuous $m$-version $\tilde{\omega}_f$ and

$$\tilde{\omega}_f(x) = E_x \int_0^{\tau_D} e^{-\int_0^t V(X_r) \, dr} f(X_r) \, dt, \quad x \in D.$$ 

We see that $\omega_f(x) = 0$, whenever $x \in N_V$. On the other hand $\{\omega_1 = 0\} = N_V$. Thereby, applying Lemma 8.7 yields that $N_V = Z$.

In what follows, $c_{d,\alpha} = \pi^{1+d/2} \Gamma(d/2) \sin \pi \alpha$ and

$$I_r^{(\alpha)} u(x) = c_{d,\alpha} \int_{B_d(x, r)} \frac{r^\alpha}{|y-x|^d (|y-x|^2 - r^2)^\alpha} u(y) \, dy, \quad x \in D.$$ 

Lemma 8.12. Let $A = \Delta^\alpha$ for an $\alpha \in (0, 1)$. Let $u$ be a finely-continuous positive bounded function on $D$. Then

$$u(x) = \lim_{r \to 0^+} I_r^{(\alpha)} u(x), \quad x \in D.$$ 

Proof. It is well known that $E_x u(X_{\tau_{B(x, r)}}) = I_r^{(\alpha)} u(x), x \in D$ (see, e.g., [32, Section 4]). On the other hand, since $u$ is finely-continuous, $E_x u(X_{\tau_{B(x, r)}}) \to u(x)$ as $r \to 0^+$. □

Similarly to the case of the Laplacian, we can now restate Theorem 8.4 as follows.

Theorem 8.13. Let $\nu$ be a positive smooth measure on $D$ and $u \in L^1(D; m) \cap L^1(D; \nu)$ be a positive quasi-continuous function such that (8.12) is satisfied. If $\limsup_{r \to 0^+} I_r^{(\alpha)} u(x) = 0$ for some $x \in E_\nu \cap D$, then $u = 0$ q.e. in $D$. 


Proof. Follows from Corollary 4.5, Theorem 8.4 and Lemma 8.12. □

For $0 < s < 1$ and $p > 1$, we define

$$|u|_{W^{s,p}(D)} = \left( \int_D \int_D \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p},$$

and for an arbitrary $s > 0$ such that $s \notin \mathbb{N}$ we define

$$W^{s,p}(D) = \left\{ u \in W^{[s],p}(D) : \left| \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \right|_{W^{s-[s],p}(D)} < \infty, \ |k| = [s], \ k \in \mathbb{N}^d \right\},$$

where $|k| = k_1 + \ldots + k_d$. We adopt the convention that $W^{0,p}(D) = L^p(D)$. Let $K$ be a compact subset of $D$. We set

$$\text{Cap}_{W^{2s,p}}(K) := \inf \{ \|u\|_{W^{2s,p}(D)} : u \in C_c^\infty(D), \ u \geq 1_K \}.$$

In the standard way $\text{Cap}_{W^{2s,p}}$ can be extended to arbitrary set $A \subset D$ (see [1, Definition 2.2.4]).

Proposition 8.14. Let $K$ be a compact subset of $D$ and $p \geq 2$. Let $C_p$ be Riesz’s capacity defined for the operator $A = (\Delta^\alpha)$. Then $C_p(K) = 0$ if and only if $\text{Cap}_{W^{2s,p}}(K) = 0$.

Proof. Since $\{ u \in C_c^\infty(D) : u \geq 1_K \} \subset \{ R^D f : f \in L^p(D), \ R^D f \geq 1_K \}$, we have $C_p(K) \leq \text{Cap}_{W^{2s,p}}(K)$. To show the necessity part, suppose that $C_p(K) = 0$. By the definition of $C_p$, for every $\varepsilon > 0$ there exists a positive $f_\varepsilon \in L^p(D)$ such that $R^D f_\varepsilon \geq 21_K$ and $\|f_\varepsilon\|_{L^p(D)} \leq \varepsilon$. We set $u = R^D f_\varepsilon$ and extend to $\mathbb{R}^d$ by putting $f_\varepsilon = 0$ on $\mathbb{R}^d \setminus D$. By Dynkin’s formula,

$$u(x) + E_x R f_\varepsilon(X_{\tau_D}) = R f_\varepsilon(x), \quad x \in D.$$ 

Set $v(x) = R f_\varepsilon(x)$, $h(x) = E_x R f_\varepsilon(X_{\tau_D})$, $x \in \mathbb{R}^d$. By the Calderón-Zygmund $L^p$-theory,

$$\varepsilon \geq \|f_\varepsilon\|_{L^p(D)} = \|f_\varepsilon\|_{L^p(\mathbb{R}^d)} \geq \|v\|_{W^{2s,p}(\mathbb{R}^d)}.$$

Let $v_\delta = j_\delta * v$, where $j_\delta$ is the standard mollifier. A straightforward computation shows that $\|v_\delta\|_{W^{2s,p}(\mathbb{R}^d)} \leq \|v\|_{W^{2s,p}(\mathbb{R}^d)}$. Hence $\varepsilon \geq \|v_\delta\|_{W^{2s,p}(\mathbb{R}^d)}$. Let $\xi \in C_c^\infty(D)$ be such that $\xi \geq 1_K$. By [25, Theorem 1.4.1.4], there exists $c_\xi$ such that

$$\|\xi v_\delta\|_{W^{2s,p}(\mathbb{R}^d)} \leq c_\xi \|v_\delta\|_{W^{2s,p}(\mathbb{R}^d)}.$$ 

Since $R$ is strongly Feller, $v$ is l.s.c. Therefore for a sufficiently small $\delta > 0$, $v_\delta \geq 1_K$. Of course $\xi v_\delta \geq 1_K$, $\xi v_\delta \in C_c^\infty(D)$ and

$$\varepsilon \geq c_\xi \|\xi v_\delta\|_{W^{2s,p}(\mathbb{R}^d)} \geq c_\xi \|\xi v_\delta\|_{W^{2s,p}(D)}.$$ 

Since $\varepsilon > 0$ was arbitrary, this implies that $\text{Cap}_{W^{2s,p}}(K) = 0$. □

Remark 8.15. The assertion of Proposition 8.14 holds true for $p \in (1,2)$ and $\alpha = 1/2$ (the proof is analogous to the proof given above). In case $p \in (1,2)$ and $\alpha \neq 1/2$ it is not true that $\|f\|_{L^p(\mathbb{R}^d)} \sim \|R f\|_{W^{2s,p}(\mathbb{R}^d)}$ for $f \in L^p(\mathbb{R}^d)$. As a consequence, in that case Proposition 8.14 does not hold as stated. However, it holds true if in its formulation we replace the space $W^{2s,p}(\mathbb{R}^d)$ by the Besov space $B^\alpha_{p,s}(\mathbb{R}^d)$ (see [45, Theorem 5, page 155]).

Theorem 8.16. Let $V$ be a positive function in $L^p(D; m)$ for some $p \geq 2$ and $u \in L^1(D; m) \cap L^1(E; V \cdot m)$ be a positive function such that

$$\langle u, -\Delta^\alpha \xi \rangle + \langle u \cdot V, \xi \rangle \geq 0, \quad \xi \in C_c^\infty(D), \ \xi \geq 0.$$
Let $Z = \{ x \in D : \limsup_{r \to 0^+} r^{\alpha} I_r^{(\alpha)}(u)(x) = 0 \}$. If $\text{Cap}_{W^{2,\alpha}}(Z) > 0$, then $u = 0$ m.a.e.

Proof. By Proposition 8.14, $C_p(Z) > 0$. Let $\nu = V \cdot m$. Since $C_p(N_\nu) = 0$ (see the proof of Theorem 7.2), $E_\nu \cap Z \neq \emptyset$. Hence, by Theorem 8.13, $u = 0$ m-a.e. □

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