The Kerr theorem, Kerr-Schild formalism and multi-particle Kerr-Schild solutions

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We consider an extended version of the Kerr theorem incorporated in the Kerr-Schild formalism. It allows one to construct the series of exact solutions of the Einstein-Maxwell field equations from a holomorphic generating function $F$ of twistor variables. The exact multiparticle Kerr-Schild solutions are obtained from generating function of the form

$$F = \prod_i F_i,$$

where $F_i$ are partial generating functions for the spinning particles $i = 1..k$. Gravitational and electromagnetic interaction of the spinning particles occurs via the light-like singular twistor lines. As a result, each spinning particle turns out to be ‘dressed’ by singular pp-strings connecting it to other particles. Physical interpretation of this solution is discussed.

I. INTRODUCTION

In the fundamental work by Debney, Kerr and Schild [1], the Einstein-Maxwell field equations were integrated out for the Kerr-Schild form of metric

$$g_{\mu\nu} = \eta_{\mu\nu} + 2he^{3}_{\mu}e^{3}_{\nu},$$

where $\eta_{\mu\nu}$ is metric of an auxiliary Minkowski spacetime $M^4$, and vector field $e^3_\mu(x)$ is null ($e^3_\mu e^{\mu\nu} = 0$) and tangent to a principal null congruence (PNC) which is geodesic and shear-free (GSF) [1]. PNC is determined by a complex function $Y(x)$ via the one-form

$$e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv$$

written in the null Cartesian coordinates

$$2\bar{\zeta} = x + iy, \quad 2\zeta = x - iy, \quad 2\bar{u} = z + t, \quad 2u = z - t.$$  \hspace{1cm} (3)

One of the most important solution of this class is the Kerr-Newman solution for the rotating and charged black hole. Along with astrophysical applications, it finds also application as a model of spinning particle in general relativity, displaying some relationships to the quantum world. In particular, it has the anomalous gyromagnetic ratio $g = 2$, as that of the Dirac electron [2], stringy structures [3–6] and other features, allowing one to construct a semiclassical model of the extended electron [6–9] which has the Compton size and possesses the wave properties [6,8,10].

The principal null congruence (PNC) of the Kerr-Newman solution represents a vortex of the light-like rays (see Fig.1) which are twistors indeed. So, the Kerr geometry is supplied by a twistorial structure which is described in twistor terms by the Kerr theorem. In addition to the important role of the Kerr theorem in twistor theory [12–14], the Kerr theorem represents in the Kerr-Schild approach [1] a very useful technical instrument allowing one to obtain the Kerr-Newman solution and its generalizations.

In accordance with the Kerr theorem, the general geodesic and shear-free congruence on $M^4$ is generated by the simple algebraic equation

$$F = 0,$$  \hspace{1cm} (4)

where $F(Y, \lambda_1, \lambda_2)$ is any holomorphic function of the projective twistor coordinates

$$Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}.$$  \hspace{1cm} (5)

Since the twistor coordinates $\lambda_1, \lambda_2$ are itself the functions of $Y$, one can consider $F$ as a function of $Y$ and $x \in M^4$, so the solution of (4) is a function $Y(x)$ which allows one to restore PNC by using the relation (2). We shall call the function $F$ as generating function of the Kerr theorem.

It should be noted that the Kerr theorem has never been published by R. Kerr as a theorem. First, it has been published without a proof by Penrose [12]. However, in a very restricted form, for a special type of generating function $F$, it has been used in [1] by derivation of the Kerr-Newman solution. The text of paper [1] contains some technical details which allows one to reconstruct the proof of the Kerr theorem in a general form which is valid for the Kerr-Schild class of metrics [15,16].

The basic results of the fundamental work [1] were obtained for the quadratic in $Y$ generating function $F$, which corresponds to the Kerr PNC. In particular, for the Kerr-Newman solution the equation $F(Y) = 0$ has two roots $Y^\pm(x)$ [3,17,15], and the space-time is double sheeted, which is one of the mysteries of the Kerr geometry, since the (+) and (−) sheets are imbedded in the same Minkowski background having dissimilar gravitation (and electromagnetic) fields, and the fields living on
the (+)-sheet do not feel the existence of different fields on the (−)-sheet.

It has been mentioned in [18], that for quadratic in \( Y \) functions \( F \) the Kerr theorem determines not only congruence, but also allows one to determine the metric and electromagnetic field (up to an arbitrary function \( \psi(Y) \)).

In this paper we consider an extended version of the Kerr theorem which allows one to determine the corresponding geodesic and shear-free PNC for a very broad class of holomorphic generating functions \( F \), and also to reconstruct the metric and electromagnetic field, i.e. to describe fully corresponding class of the exact solutions of the Einstein-Maxwell field equations.

In particular, we consider polynomial generating functions \( F \) of higher degrees in \( Y \) which lead to the multiparticle Kerr-Schild solutions. These solutions have a new peculiarity: the space-time and corresponding twistorial structures turn out to be multi-sheeted.

The wonderful twosheetedness of the usual Kerr space-time is generalized in these solutions to multi-sheeted space-times which are determined by multi-sheeted Riemann holomorphic surfaces and induce the corresponding multi-sheeted twistorial structures.

Twistorial structures of the \( i \)-th and \( j \)-th particles do not feel each other, forming a type of its internal space. However, the corresponding exact solutions of the Einstein-Maxwell field equations show that particles interact via the common singular twistor lines – the lightlike pp-strings.

We find out that the mystery of the known two-sheetedness of the Kerr geometry is generalized to some more mystical multi-sheetedness of the multiparticle solutions.

As a result, besides the usual Kerr-Newman solution for an isolated spinning particle, we obtain a series of the exact solutions, in which the selected Kerr-Newman particle is surrounded by external particles and interacts with them by singular pp-strings. It is reminiscent of the known from quantum theory difference between the “naked” one-particle electron of the Dirac theory and a multi-particle structure of a “dressed” electron which is surrounded by virtual photons in accordance with QED. The multiparticle space-time turns out to be penetrated by a multi-sheeted web of twisters.

In the Appendix A we give a brief description of the basic relations of the Kerr-Schild formalism. In the Appendix B we give some Corollaries from the proof of Kerr Theorem following to the papers [15,16]. In the Appendix C we give the basic Kerr-Schild equations obtained in [1] for the general geodesic and shearfree congruences.

II. THE KERR THEOREM AND ONE-PARTICLE KERR-SCHILDM SOLUTIONS

The Kerr-Schild form of metric (1) has the remarkable properties allowing one to apply rigorously the Kerr Theorem to the curved spaces. It is related to the fact that the PNC field \( e^{3i}_{\mu} \), being null and GSF with respect to the Kerr-Schild metric \( g \), \( e^{3i}_{\mu}e^{3i}_{\nu}g = 0 \), will also be null and GSF with respect to the auxiliary Minkowski metric, and this relation remains valid by an analytic extension to the complex region. In the Appendix A we show that the geodesic and shear-free conditions on PNC coincide in the Minkowski space and in the Kerr-Schild background. Therefore, obtaining a geodesic and shear-free PNC in Minkowski space in accordance with the Kerr theorem, and using the corresponding null vector field \( e^{3i}_{\mu}(x) \) in the Kerr-Schild form of metric, one obtains a curved Kerr-Schild space-time where PNC will also be null, geodesic and shearfree.

![FIG. 1. The Kerr singular ring and 3-D section of the Kerr principal null congruence.](image)

It was shown in [15,18] that the quadratic in \( Y \) generating function of the Kerr theorem can be expressed via a set of parameters \( q \) which determine the position, motion and orientation of the Kerr spinning particle.

For some selected particle \( i \), function \( F_{i}(Y) \), may be represented in general form

\[
F_{i}(Y|q_{i}) = A(x|q_{i})Y^{2} + B(x|q_{i})Y + C(x|q_{i}).
\]

The equation \( F_{i}(Y|q_{i}) = 0 \) can be resolved explicitly, leading to two roots \( Y(x) = Y_{i}^{\pm}(x|q_{i}) \) which correspond to two sheets of the Kerr space-time. The root \( Y^{+}(x) \) determines via (2) the out-going congruence on the (+)-sheet, while the root \( Y^{-}(x) \) gives the in-going congruence on the (−)-sheet. By using these root solutions, one can represent function \( F_{i}(Y) \) in the form

\[
F_{i}(Y) = A_{i}(x)(Y - Y_{i}^{+}(x))(Y - Y_{i}^{-}(x)).
\]

The relation (2) determines the vector field \( e^{3i}_{\mu} \) of the Kerr-Schild ansatz (1), and metric acquires the form

\[
g^{(i)}_{\mu\nu} = \eta_{\mu\nu} + 2h^{(i)}_{\mu\nu}e^{3i}_{\nu}e^{3i}_{\nu}. \tag{8}
\]

Based on this ansatz, after rather long calculations and integration of the Einstein-Maxwell field equations performed in the work [1] under the conditions that PNC is geodesic and shearfree (which means \( Y_{2} = 0 \) and \( Y_{4} = 0 \), see Appendix A), one can represent the function \( h^{(i)} \) in the form
\[ h^{(i)} = \frac{1}{2} M^{(i)} (Z^{(i)} + \bar{Z}^{(i)}) - \frac{1}{2} A^{(i)} \bar{A}^{(i)} Z^{(i)} \bar{Z}^{(i)}, \]

where
\[ M^{(i)} = m^{(i)} (P^{(i)})^{-3} \]

and
\[ A^{(i)} = \psi^{(i)}(Y)(P^{(i)})^{-2}. \]

Here \( m^{(i)} \) is mass and \( \psi^{(i)}(Y) \) is arbitrary holomorphic function.

Electromagnetic field is determined by two complex self-dual components of the Kerr-Schild tetrad form \( \mathcal{F} = \mathcal{F}_{ab} e^a \wedge e^b \),
\[ \mathcal{F}_{12}^{(i)} = A^{(i)} (Z^{(i)})^2 \]

and
\[ \mathcal{F}_{31}^{(i)} = -(A^{(i)} Z^{(i)})_{1}, \]

see Appendix C.

We added here the indices \( i \) to underline that these functions depend on the parameters \( q_i \) of \( i \)-th particle.

Setting \( \psi^{(i)}(Y) = e = \text{const.} \), we have the charged Kerr-Newman solution for \( i \)-th particle, vector potential of which may be represented in the form
\[ A_{\mu}^{(i)} = \text{Re}(e Z^{(i)}) (P^{(i)})^{-2}. \]

It should be emphasized (1), that integration of the field equations has been performed in [1] in a general form, before concretization of the form of congruence, only under the general conditions that PNC is geodesic and shear free.

On the other hand, it was shown in [15,18] (see also Appendix B), that the unknown so far functions \( P^{(i)} \) and \( Z^{(i)} \) can also be determined from the generating function of the Kerr theorem \( F^{(i)} \). Namely,
\[ P^{(i)} = \partial_{\lambda_i} F_1 - \dot{Y} \partial_{\lambda_2} F_1, \quad P^{(i)}/Z^{(i)} = - \frac{dF_1}{dY}. \]

Therefore, for the quadratic in \( Y \) functions (6), we arrive at the first extended version of the Kerr theorem

1/ For a given quadratic in \( Y \) generating function \( F_1 \), solution of the equation \( F_1 = 0 \) determines the geodesic and shear free PNC in the Minkowski space \( M^4 \) and in the associated Kerr-Schild background (8).

2/ The given function \( F_1 \) determines the exact stationary solution of the Einstein-Maxwell field equations with metric given by (8), (9), (10) and electromagnetic field given by (14), where functions \( P^{(i)} \) and \( Z^{(i)} \) are given by (15).

For practical calculations the Kerr-Schild form \( g_{\mu \nu} = \eta_{\mu \nu} - H k_\mu k_\nu \) is more useful, where functions \( H = H^{(i)} = h^{(i)} (P^{(i)})^2 \), and the normalized null vector fields are \( k_\mu = e^{3(i)}/P^{(i)} \).

In this form function \( H^{(i)} \) will be
\[ H^{(i)} = \frac{m_i}{2} \frac{1}{\dot{r}_i} + \frac{1}{\bar{r}_i} e^{2/|\bar{r}_i|^2}, \]

and the Kerr-Newman electromagnetic field is determined by the vector potential
\[ A_{\mu}^{(i)} = \text{Re}(e/\bar{r}_i) k_\mu^{(i)}, \]

where \( \bar{r}_i = P^{(i)}/Z^{(i)} = \frac{dF_1}{dY} \)

is the so called complex radial distance which is related to a complex representation of the Kerr geometry [15,16,18,20].

For a standard oriented Kerr solution in the rest, \( \tilde{r} = \sqrt{x^2 + y^2 + (z - ia)^2} = r + ia \cos \theta \), which corresponds to the distance from a complex point source positioned at the complex point \( \tilde{x} = (0, 0, ia) \). One sees, that the Kerr singular ring is determined by \( \tilde{r} = 0 \Rightarrow r = \cos \theta = 0 \). For the Kerr geometry this representation was initiated by Newman, however this scheme works rigorously in the Kerr-Schild approach, where the complex source represents a complex world line \( x(\tau) \) in the complexified auxiliary Minkowski space-time \( CM^4 \).

In accordance with Corollary 4 (Appendix B), position of the Kerr singular ring is determined by the system of equations
\[ F_1 = 0, \quad \bar{r}_i = \frac{dF_1}{dY} = 0. \]

Extended version of the Kerr theorem allows one to get exact solution for an arbitrary oriented and boosted charged spinning particle [18].

### III. MULTI-SHEETED TWISTOR SPACE

Following [11], we consider the case of a system of \( k \) spinning particles having the arbitrary displacement, orientations and boosts. One can form the function \( F \) as a product of the corresponding blocks \( F_i(Y) \),
\[ (Y) \equiv \prod_{i=1}^{k} F_i(Y). \]

The solution of the equation \( F = 0 \) acquires \( 2k \) roots \( Y^\pm_i(x) \), forming a multi-sheeted covering space over the Riemann sphere \( S^2 = CP^1 \equiv Y \).

Indeed, \( Y = e^{i\phi} \tan \frac{\theta}{2} \) is a complex projective angular coordinate on the Minkowski space-time and on the corresponding Kerr-Schild space-time.\(^2\)

\(^2\)Two other coordinates in the Kerr-Schild space-time may be chosen as \( \tilde{r} = PZ^{-1} \) and \( \rho = x^\mu e^\mu_{\mu} \), where \( x^\mu \) are the four Cartesian coordinates in \( M^4 \).
The twistorial structure on the i-th (+) or (−) sheet is determined by the equation \( F_i = 0 \) and does not depend on the other functions \( F_j, \; j \neq i \). Therefore, the particle \( i \) does not feel the twistorial structures of other particles.

The equations for singular lines

\[
F = 0, \; dF/dY = 0 \tag{21}
\]

acquires the form

\[
\prod_{i=1}^{k} F_i = 0, \quad \sum_{i=1}^{k} \prod_{l \neq i} F_l dF_i/dY = 0 \tag{22}
\]

which splits into \( k \) independent relations

\[
F_i = 0, \quad \prod_{l \neq i} F_l dF_i/dY = 0. \tag{23}
\]

One sees, that the Kerr singular ring on the sheet \( i \) is determined by the usual relations \( F_i = 0, \; dF_i/dY = 0 \), and \( i \)-th particle does not feel also the singular rings of the other particles. The space-time splits on the independent twistorial sheets, and the twistorial structure related to the \( i \)-th particle plays the role of its “internal space”.

One should mention that it is a direct generalization of the well known two-sheetedness of the usual Kerr space-time.

Since the twistorial structures of different particles are independent, it seems that the \( k \)-particle solutions \( \{ Y_i^\pm(x) \}, \; i = 1, 2, \ldots, k \) form a trivial covering space \( K \) over the sphere \( S^2 \), i.e. \( K \) is a trivial sum of \( k \) disconnected two-sheeted subspaces \( K = \bigcup_{i=1}^{k} S_2^2 \).

However, there is one more source of singularities on \( K \) which corresponds to the multiple roots: the cases when some of twistor lines of one particle \( i \) coincides with a twistor line of another particle \( j \), forming a common \((ij)\)-twistor line. Indeed, for each pair of particles \( i \) and \( j \), there are two such common twistor lines: one of them \((ij)\) is going from the positive sheet of particle \( i \), \( Y_i^+(x) \) to negative sheet of particle \( j \), \( Y_j^-(x) \) and corresponds to the solution of the equation \( Y_i^+(x) = Y_j^-(x) \), another one \((ji)\) is going from the positive sheet of particle \( j \), \( Y_j^+(x) \) to negative sheet of particle \( i \) and corresponds to the equation \( Y_j^+(x) = Y_i^-(x) \).

The common twistor lines are also described by the solutions of the equations (21) and correspond to the multiple roots which give a set of “points” \( A_j \), where the complex analyticity of the map \( Y_i^\pm(x) \to S^2 \) is broken.\(^3\)

\(^3\)The given in [22] analysis of the equations (21) shows that for the holomorphic functions \( F(Y) \) the covering space \( K \) turns out to be connected and forms a multisheeted Riemann surface over the sphere with the removed branch points \( S^2 \setminus \cup_j A_j \).

The solutions \( Y_i(x) \), which determine PNC on the \( i \)-th sheet of the covering space, induce multisheeted twistor fields over the corresponding Kerr-Schild manifold \( K^4 \).

\[\text{IV. MULTIPARTICLE KERR-SCHILD SOLUTIONS.}\]

As we have seen, the quadratic in \( Y \) functions \( F \) generate exact solutions of the Einstein-Maxwell field equations. In the same time, the considered above generating functions \( \prod_{i=1}^{k} F_i(Y) = 0 \), leads to a multisheeted covering space over \( S^2 \) and to the induced multisheeted twistor structures over the Kerr-Schild background which look like independent ones. Following to the initiate naive assumption that twistorial sheets are fully independent, one could expect that the corresponding multisheeted solutions of the Einstein-Maxwell field equations will be independent on the different sheets, and the solution on \( i \)-th sheet will reproduce the result for an isolated \( i \)-th particle. However, it is obtained that the result is different.

Formally, we have to replace \( F_i \) by

\[
F = \prod_{i=1}^{k} F_i(Y) = \mu_i F_i(Y), \tag{24}
\]

where

\[
\mu_i = \prod_{j \neq i} F_j(Y) \tag{25}
\]

is a normalizing factor which takes into account the external particles. In accordance with (15) this factor will also appear in the new expression for \( P/Z \) which we mark now by capital letter \( \hat{R} \)

\[
\hat{R}_i = P/Z = -d_Y F = \mu_i P^{(i)}/Z^{(i)}, \tag{26}
\]

and in the new function \( P_i \) which we will mark by hat

\[
\hat{P}_i = \mu_i P_i. \tag{27}
\]

Functions \( Z \) and \( \hat{Z} \) will not be changed.

By substitution of the new functions \( P_i \) in the relations (9), (10) and (11), we obtain the new relations

\[
M^{(i)} = m^{(i)}(\mu_i(Y)P^{(i)})^{-3}, \tag{28}
\]

\[
A^{(i)} = \psi^{(i)}(Y)(\mu_i(Y)P^{(i)})^{-2} \tag{29}
\]

and

\[
h_i = \frac{m}{2(\mu_i(Y)P_i)^3}(Z^{(i)} + \hat{Z}^{(i)}) = \frac{|\psi|^2}{2|\mu_i(Y)P_i|^2}Z^{(i)}\hat{Z}^{(i)}. \tag{30}
\]
For the new components of electromagnetic field we obtain
\[ F_{12}^{(i)} = \psi^{(i)}(Y)(\mu_i P^{(i)})^{-2}(Z^{(i)})^2 \]  
and
\[ F_{31}^{(i)} = -\left(\frac{\psi^{(i)}(Y)}{(\mu_i Y)^2}\right)Z^{(i)} \]  
In terms of \( \tilde{r}_i \) and \( H_i \) the Kerr-Newman metric takes the form:
\[ H_i = \frac{m_0}{2} \left( \frac{1}{\mu_i \tilde{r}_i} + \frac{1}{\mu_i^2 \tilde{r}_i^2} \right) + \frac{e^2}{2|\mu_i \tilde{r}_i|^2}, \]
where we have set \( \psi^{(i)}(Y) = e \) for the Kerr-Newman solution.

The simple expression for vector potential (14) is not valid more.\(^4\)

One sees, that in general case metric turns out to be complex for the complex mass factor \( m(\mu_i(Y))^{-2} \), and one has to try to reduce it to the real one.

This problem of reality \( M \) was also considered in [1]. The function \( M \) satisfies the equation
\[ (\ln M + 3 \ln P), Y = 0 \]  
which has the general solution
\[ M = m(Y)/P^3(Y, \bar{Y}), \]
where \( m(Y) \) is an arbitrary holomorphic function. The simplest real solution is given by a real constant \( m \) and a real function \( P(Y, \bar{Y}) \). As was shown in [1], it results in one-particle solutions.

In our case, functions \( P_i \) have also to be real, since they relate the real one-forms \( e^3 \) and \( k \),
\[ e^{3(i)} = P_i k^{(i)} dx. \]
Functions \( \mu_i(Y) \) are the holomorphic functions given by (25), and functions \( m_i = m_i(Y) \) are arbitrary holomorphic functions which may be taken in the form
\[ m_i(Y) = m_0(\mu_i(Y))^3 \]
to provide reality of the mass terms \( M_i = m_i(Y)/P_i^3 \) on the each i-th sheet of the solution.

Therefore, we have achieved the reality of the multi-sheeted Kerr-Schild solutions, and the extended version of the Kerr theorem is now applicable for the general multiplicative form of the functions \( F_i \), given by (20).

One can specify the form of functions \( \mu_i \) by using the known structure of blocks \( F_i \)

\[^{4}\text{Besides the related with } \mu_i(Y) \text{ singular string factor, it acquires an extra vortex term.}\]

\[\mu_i(Y_i) = \prod_{j \neq i} A_j(x)(Y_i - Y_j^+)(Y_i - Y_j^-).\]  
If the roots \( Y_i^\pm \) and \( Y_j^\pm \) coincide for some values of \( Y_i^\pm \), it selects a common twistor for the sheets \( i \) and \( j \). Assuming that we are on the i-th (\( + \)-)sheet, where congruence is out-going, this twistor line will also belong to the in-going (\( - \)-)sheet of the particle \( j \). The metric and electromagnetic field will be singular along this twistor line, because of the pole \( \mu_i \sim A(x)(Y_i^+ - Y_j^-) \). This singular line is extended to a semi-infinite line which is common for the \( i \)-th and \( j \)-th particle. However, the considered in [?] simple example shows that there exists also a second singular line related to interaction of two particles. It is out-going on the \( Y_j^+ \)-sheet and belongs to the in-going (\( - \)-)sheet of the particle \( i \), \( Y_i^- \).

Therefore, each pair of the particles \( (ij) \) creates two opposite oriented in the space (future directed) singular twistor lines, pp-strings. The field structure of this string is described by singular pp-wave solutions (the Schild strings) [6,10].

If we have \( k \) particles, then in general, for the each Kerr’s particle \( 2k \) twistor lines belonging to its PNC will turn into singular null strings.

As a result, one sees, that in addition to the well known Kerr-Newman solution for an isolated particle, there are series of the corresponding solutions which take into account presence of the surrounding particles, being singular along the twistor lines which are common with them.

By analogue with QED, we call these solutions as ‘dressed’ ones to differ them from the original ‘naked’ Kerr-Newman solution. The ‘dressed’ solutions have the same position and orientation as the ‘naked’ ones, and differ only by the appearance of singular string along some of the twistor lines of the Kerr PNC.

V. CONCLUSION

We considered the extended version of the Kerr theorem which, being incorporated in the Kerr-Schild formalism, allows one to get exact multiparticle Kerr-Schild solutions.

Recall, that for the parameters of spinning particles \( a \gg m \), and horizons of the Kerr geometry disappear, obtaining the naked ring-like singularity which is branch line of space. As a result, the space-time acquires a two-sheeted topology which is exhibited at the Compton distances \( a = h/2m \) and has to play important role in the structure of spinning particle. In fact, the space-time in the Compton region turns out to be strongly “polarized” by the Kerr twistorial structure [20]. The obtained multi-sheeted solutions represent a natural generalization of the Kerr’s two-sheetedness and are related with multi-sheetedness of the corresponding twistorial spaces of the geodesic and shear-free principal null congruences.

The ‘naked’ Kerr-Newman solution is the usual Kerr-Newman solution for an isolated particle, while the
‘dressed’ Kerr-Newman particle is linked by twistorial (photon and/or graviton) lines to other external particles.

The following from this picture gravitational and electromagnetic interaction of the particles via the null singular lines (pp-strings) is surprising and gives a hint that the these lines may be related to virtual photons and gravitons. The resulting structure of the multiparticle Kerr-Newman solution acquires the features of the multiparticle structure of electron in QED [20]. It shows that a twistorial web of pp-string may cover the spacetime forming a twistorial structure of vacuum. It looks not too wonder, since the multiplicative generating function of the Kerr theorem (20) has been taken in analogue with the structure of higher spin gauge theory [23] and is reminiscent of a twistorial version of the Fock space.

In the opposite case, \( a \leq m \), which is important for astrophysical applications, the spinning body acquires the event horizon, however the singular axial twistor lines form the holes in horizon [24], which may have classical and quantum consequences.

For the both applications, \( a \gg m \) and \( a \leq m \), the obtaining of the exact rotating Kerr-Schild solutions with a wave electromagnetic field (case \( \gamma \neq 0 \)) represents the extremely important and extremely hard problem which is unsolved up to now.

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**APPENDIX A. BASIC RELATIONS OF THE KERR-SCHILD FORMALISM**

The Kerr-Schild null tetrad \( e^a = e^a_\mu dx^\mu \) is determined by relations:

\[
\begin{align*}
e^1 &= d\zeta - Y dv; \quad e^2 = \bar{d}\bar{\zeta} - \bar{Y} d\bar{v}; \\
e^3 &= du + \bar{Y} d\bar{\zeta} + Y d\zeta - YY \bar{d}\bar{v}; \\
e^4 &= dv + he^3,
\end{align*}
\]

(39)

The inverse (dual) tetrad has the form

\[
\begin{align*}
\partial_1 &= \partial_\zeta - \bar{Y} \partial_\bar{v}; \quad \partial_2 = \partial_\bar{\zeta} - Y \partial_u; \quad \partial_3 = \partial_u - h \partial_4; \\
\partial_4 &= \partial_v + Y \partial_\zeta + \bar{Y} \partial_{\bar{\zeta}} - Y \bar{Y} \partial_u.
\end{align*}
\]

(40)

The congruence \( e^3 \) is geodesic if \( Y,1 = 0 \), and is shear free if \( Y,2 = 0 \).

**APPENDIX B. THE KERR THEOREM.**

Proof of the Kerr Theorem on the Kerr-Schild background is given in [15] and has the following Corollaries:

1: For arbitrary holomorphic function of the projective twistor variables \( F(Y,\lambda_1, \lambda_2) \), the equation \( F = 0 \) determines function \( Y(x) \) which gives the congruence of null directions \( e^3 \), (39) satisfying the geodesic and shearfree conditions \( Y,2 = Y,4 = 0 \).

2: Explicit form of the geodesic and shearfree conditions is \( (\partial_\zeta - Y \partial_u)Y = 0 \) and \( (\partial_v + Y \partial_\zeta) \partial_u)Y = 0 \). It does not depend on function \( h \) and coincides with these conditions in Minkowski space.

3: Function \( F \) determines two important functions

\[
P = \partial_\lambda_1 F - \bar{Y} \partial_\lambda_2 F,
\]

and

\[
PZ^{-1} = - dF/dY,
\]

4: Singular points of the congruence are defined by the system of equations

\[
F = 0, \quad dF/dY = 0,
\]

5: The following relations are useful

\[
\bar{Z}Z^{-1}Y,3 = -(\log P),,2, \quad P,,4 = 0.
\]

**APPENDIX C. BASIC FIELD EQUATIONS FOR ARBITRARY GSF PNC**

The geodesic and shearfree conditions \( Y,2 = Y,4 = 0 \) reduce strongly the list of gravitational and Maxwell equations. As a result, one obtains for the tetrad components

\[
R_{24} = R_{22} = R_{44} = R_{14} = R_{11} = R_{41} = R_{42} = 0.
\]

(45)

It simplifies also e.m. field \( \mathcal{F}_{ab} \), up to two nonzero complex components \( \mathcal{F}_{12} = \mathcal{F}_{34} = F_{12} + F_{34} \) and \( \mathcal{F}_{31} = 2F_{31} \). The general form of function \( h \) for any geodesic and shearfree PNC is

\[
h = \frac{1}{2} M(Z + \bar{Z}) + AA\bar{Z}Z.
\]

(46)

Solutions of the Maxwell equations lead to the equations

\[
\mathcal{F}_{31} = \gamma Z - (AZ),,1, \quad \gamma,4 = 0,
\]

(47)

\[
A_{,2} - 2Z^{-1}Z\bar{Y},3 A = 0,
\]

(48)

\[
A_{,3} - Z^{-1}Y,3 A_{,1} - \bar{Z}^{-1}\bar{Y},3 A_{,2} + \bar{Z}^{-1}\bar{Y},2 - Z^{-1}Y,3 \gamma = 0.
\]

(49)

Two extra gravitational equations are

\[
M_{,2} - 3Z^{-1}Z\bar{Y},3 M - A\bar{Z} = 0,
\]

(50)

and
Corollary 5 of the Kerr theorem yields \( \bar{Z} Z^{-1} Y, 3 = -(\log P), 2 \), \( P, 4 = 0 \).

By the used in [1] restriction \( \gamma = 0 \), corresponding to electromagnetic field without wave excitations, the field equations are simplified and may be reduced to the form \( (\log A P^2), 2 = 0 \), \( (\log M P^3), 2 = 0 \), \( A, 4 = M, 4 = 0 \), leading to the general solution
\[
A = \psi(Y)/P^2, \quad M = m(Y)/P^3.
\] (52)