The Szegö–Asymptotics for Doubly–Dispersive Gaussian Channels

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Abstract

We consider the time–continuous doubly–dispersive channel with additive Gaussian noise and establish a capacity formula for the case where the channel operator is represented by a symbol which is periodic in time and fulfills some further integrability, smoothness and oscillation conditions. More precisely, we apply the well–known Holsinger–Gallager model for translating a time–continuous channel for a sequence of time–intervals of increasing length \( \alpha \to \infty \) to a series of equivalent sets of discrete, parallel channels, known at the transmitter. We quantify conditions when this procedure converges. Finally, under periodicity assumptions this result can indeed be justified as the channel capacity in the sense Shannon. The key to this is result is a new Szegö formula for certain pseudo–differential operators with real–valued symbol. The Szegö limit holds if the symbol belongs to the homogeneous Besov space \( \dot{B}^1_{\infty,1} \) with respect to its time–dependency, characterizing the oscillatory behavior in time. Finally, the formula justifies the water–filling principle in time and frequency as general technique independent of a sampling scheme.

I. INTRODUCTION

The information–theoretic treatment of the time–continuous channel dispersive in time and frequency (doubly–dispersive) with additive Gaussian noise has been a problem of long interest. A well known result for the time–invariant and power–limited case has been achieved by Gallager and Holsinger [2] and [3] in discretizing the time–continuous problem into an increasing sequence of parallel memoryless channels with known information capacity \( I_n \). Coding theorems for the time–discrete Gaussian channel can be used for the time–continuous channel whenever such a discretization is realizable. A direct coding theorem without discretization has been established by Kadota and Wyner [4] for the causal, stationary and asymptotically memoryless channel.

The discretization in [3] was achieved by representing a single use of the time–continuous channel as the restriction of the channel operator to time intervals \( \alpha \Omega \) of length \( \alpha \). The quantity \( I_n \) is then determined by spectral properties of the restricted operator. A major step in the calculation for the time–invariant case was the exact determination of the limit:

\[
I(S) := \lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \lim_{n \to \infty} I_n(\alpha S) \right)
\]

* This work has been partially presented on the IEEE ISIT conference, 2011 [1].
which relies on the Kac–Murdock–Szegö result [5] on the asymptotic spectral behaviour of convolution operators. As the classical result of Shannon for the time–continuous band–limited channel and the discussion in [6] shows, $I(S)$ has only a meaning of coding capacity for a power budget $S$ whenever there exists a sequence $\alpha_k$ of realizable discretization approaching this limit as $k \to \infty$. Some remaining problems in this direction, like for example the robustness of this limit against interference between different blocks, have been resolved for Gallager–Holsinger model in [7]. The limit has the advantage of nice interpretation as ”water–filling” along the frequencies:

$$I(S) = \int_{B \cdot \sigma(\omega) \geq 1} \log(B \cdot \sigma(\omega)) d\omega$$

where $\sigma$ denotes Fourier transform of the correlation operator $L_\sigma$ (required to be absolute integrable and bounded), i.e. the positive symbol of a convolution operator. The constant $B$ is implicitly determined for a given power budget by a relation similar to (2).

Since the time–invariant case represents the commutative setting a joint signaling scheme (like for example orthogonal frequency division multiplexing) is permitted and the determination of the capacity is essentially reduced to a power allocation problem. Although the coherent setting is considered so far only the channel gains have to be given to the transmitter in this case.

However, doubly–dispersive channels represent the non–commutative generalization and do not admit a joint diagonalization such that there still remains the problem of proper signal design. Here, the channel operator can be characterized for example by the time–varying transfer function, i.e. the symbol $\sigma(x,\omega)$ of a so called pseudo-differential operator which depends on the frequency $\omega$ and the time instant $x$. Obviously, by uncertainty an exact characterization of frequencies at time instants is meaningless and the symbol can reflect spectral properties only in an averaged sense. Thus, it is important to know whether the limit in (1) for a real symbol is asymptotically given by the average:

$$\frac{1}{\alpha} \int_{\alpha \times \mathbb{R}} r(B \cdot \sigma(x,\omega)) dx d\omega$$

for $\alpha \to \infty$ and $r(x) = \log(x) \cdot \chi_{[1,\infty]}(x)$. Then, (3) with a similar integral for the function $(x-1)/x \cdot \chi_{[1,\infty]}(x)$ represents the water–filling principle in time and frequency. Obviously, this strategy is used already in practice when optimizing rate functions in some long–term meaning. But, in fast–fading scenarios for example it not clear whether this procedure on a short time scale is indeed related to (1).

Averages closely related to the one in (3) have been studied for a long time in the context of asymptotic symbol calculus of pseudo-differential operators and semi–classical analysis in quantum physics [8], [9], [10]. Unfortunately, the results therein are not directly applicable in the information and communication theoretic setting because here 1.) the symbols of the restricted operators are (in general) discontinuous and usually not decaying in time 2.) the functions $r$ to be considered are neither analytic nor have the required smoothness 3.) the path of approaching the limit has to be explicitly in terms of an increasing sequence of interval restrictions (infinite–dimensional subspaces) in order to establish realisability. For operators with semigroup property as the ”heat channel” [11] it is possible approaching the limit via projections onto the (finite–dimensional) span of an increasing sequence of Hermite functions as established in [12] for Schrödinger operators. However, in the problem considered here this approach
does not guarantees the existence of signaling schemes of finite length \( \alpha \) to practically achieve the limit and a semi–group property of this particular type is not present.

The idea of approximate eigenfunctions of so called underspread channels [13], [14] has been used to obtain information–theoretical statements for the non–coherent setting [15]. Signal design has then to be considered with respect to statistical properties [16]. The method presented in this paper suggests that in the coherent setting the approximation in terms of trace norms is relevant.

A. Main Results

We establish a procedure for estimating the deviation of formula (3) to desired quantity (1). It will be shown that both terms asymptotically agree for \( \alpha \to \infty \) if the difference of symbol products \( L_{\sigma^r} \) and operator composition \( L_{\sigma} L_{\tau} \) can be controlled in trace norm on \( \alpha \Omega \) with a sub-linear scaling in \( \alpha \). We will further discuss the information–theoretical impacts:

**Theorem 1.** Let be \( \sigma \in C^3 \) be the symbol of the channel’s correlation operator \( L_{\sigma} \) and \( \sigma(x, \cdot) \in L_2 \) uniformly in \( x \). If \( \| \sigma(x, \cdot) - \sigma(x, \cdot + h) \|_{L_2}^2 \leq c|h|^{\beta} \) for \( \beta \leq 1 \) and \( \sigma(x, \omega) \) is 1–periodic in \( x \) the time–continuous capacity under an average power constraint \( S \) is given as:

\[
I(S) = \int \int_{(\Omega \times \mathbb{R}) \cap \{ B_{\sigma} \geq 1 \}} \log(B \cdot \sigma(x, \omega)) dx d\omega
\]  

with the constant \( B = B(S) \) implicitly given by the equation:

\[
S = \int \int_{(\Omega \times \mathbb{R}) \cap \{ B_{\sigma} \geq 1 \}} \frac{B \cdot \sigma(x, \omega) - \frac{1}{B \cdot \sigma(x, \omega)}}{dx d\omega}
\]  

if the (inverse) Fourier transform of \( \sigma(x, \omega) \) in \( \omega \) (the impulse response of \( L_{\sigma} \)) is supported in a fixed interval.

The paper is organized as follows: In Section II we introduce the channel model and establish the problem as a Szegö statement on the asymptotic symbol calculus for pseudo–differential operators. The asymptotic is investigated in Section III as series of four sub–problems: an increasing family of interval sections, the asymptotic symbol calculus, an approximation method and finally a result on “products” of symbols. Following this line of four arguments we able establish (4).

II. System Model and Problem Statement

We use \( L_p(\Omega) \) for usual Lebesgue spaces \( (1 \leq p \leq \infty) \) of complex–valued functions on \( \Omega \subseteq \mathbb{R}^n \) and abbreviate \( L_p = L_p(\mathbb{R}^n) \) with corresponding norms \( \| \cdot \|_{L_p} \). For \( p = 2 \) the Hilbert space has inner product \( \langle u, v \rangle := \int uv \). Classes of smooth functions up to order \( k \) are denoted with \( C^k \) and \( \hat{f} = \mathcal{F} f \) is the Fourier transform of \( f \). Partial derivatives of a function \( \sigma(x, \omega) \) are written as \( \sigma_x \) and \( \sigma_\omega \), respectively. \( \mathcal{I}_2 \) and \( \mathcal{I}_1 \) are Hilbert–Schmidt and trace class operators with square–summable and absolute summable singular values and the symbol \( \text{tr}(X) \) denotes the trace of an operator \( X \) (more details will be given later on) on \( L_2 \).
A. System Model

We consider the common model of transmitting a finite energy signal \( s \) with support in an interval \( \Omega \) of length \( \alpha \) through a channel represented by a fixed linear operator \( H \) and additive distortion \( n_k \), i.e. quantities simultaneously measured at the receiver within the interval are expressed as noisy correlation responses:

\[
\langle r_k, Hs \rangle + n_k
\]

where \( \{ \langle r_k, \cdot \rangle \} \) are suitable normalized linear functionals implemented at the receiver. We assume Gaussian noise with \( E(\bar{n}_k n_l) = \langle r_k, r_l \rangle \).

Let us denote with \((Pu)(x) = \chi(x/\alpha)u(x)\) the restriction of a function \( u \) onto the interval \( \alpha \Omega \). Note that in what follows: \( P \) always depends on \( \alpha \). We will make in the following the assumption that the restriction \( HP \) of the channel operator \( H \) to input signals of length \( \alpha \) with finite energy is compact, i.e. the restriction \( PL_\sigma P \) of the correlation operator \( L_\sigma := H^*H \) is compact as well (\( H^* \) denotes the adjoint operator on \( L_2 \)). This excludes certain channel operators - like the identity - which usually referred to as "dimension-unlimited", i.e. the wideband cases.

Assume that the kernel \( k(x, y) \) of \( L_\sigma \) fulfills:

\[
|k(x, x - z)|^2 \leq \psi(z)
\]

for some \( \sqrt{\psi} \in L_1 \cap L_2 \). Then its (Kohn–Nirenberg) symbol or time-varying transfer function is given by Fourier transformation:

\[
\sigma(x, \omega) = \int e^{i2\pi \omega(x-y)}k(x, x-y)dy
\]

The symbol is continuous and from the Riemann–Lebesgue Lemma it follows that \( |\sigma(x, \omega)| \to 0 \) as \( |\omega| \to \infty \).

Throughout the paper we assume that \( \sigma \) is real–valued (this can be circumvented when passing to the Weyl symbol since \( L_\sigma \) is positive–definite). It follows that \( \||\sigma(x, \cdot)||^2_{L_2} = ||k(x, x - \cdot)||^2_{L_2} \leq ||\psi||_{L_1} \) uniformly in \( x \) and that \( L_\sigma \) is bounded on \( L_2 \):

\[
|\langle u, L_\sigma v \rangle| = |\langle u \otimes \tilde{v}, k \rangle| \leq \langle |u \otimes v|, \sqrt{\psi} \rangle
\]

\[
= \langle |u|, \sqrt{\psi} * |v| \rangle \leq \| \sqrt{\psi} \|_{L_1} \||u||_2 \||v||_2 \|
\]

We will from now on use \( \| \cdot \|_{op} := \| \cdot \|_{L_2 \to L_2} \) to denote the operator norm on \( L_2 \). A compact operator \( HP \) can be written via the Schmidt representation (singular value decomposition) as a limit of a sum of rank–one operators \( HP = \sum_k s_k \langle u_k, \cdot \rangle v_k \) with singular values \( s_k = \sqrt{\lambda_k(PL_\sigma P)} \) and orthonormal bases \( \{ u_k \} \) and \( \{ v_k \} \) – all depending on \( \alpha \). For the coherent setting we assume that finite subsets of these bases are known and implementable at the transmitter and the receiver, respectively. Obviously, this is an idealized and seriously strong assumption which can certainly not be fulfilled without error in practise. The investigations in [17] suggest that underspreadness of \( H \) is necessary prerequisite for reliable error control. When representing the signal \( s \) as a finite linear combination of

\[
\sup_{x \in \mathbb{R}} k(x, x - \cdot) \in L_1 \cap L_2
\]
a single use of the time–continuous channel $H$ over the time interval $\alpha \Omega$ with power budget $S$ is decomposed into a single use of a finite set of parallel Gaussian channels jointly constrained to $\alpha S$.

We will consider in the following independent uses of the channel in (6) as our preliminary model and restrict to $r_k = u_k$, i.e. $E(i_k u_k) = \delta_{kl}$. Then, the capacity and the power budget of the equivalent memoryless Gaussian channel are related through the water–filling level $B$ as (see for example [3]):

$$
\frac{1}{\alpha} \sum_{B\lambda_k \geq 1} \log(B\lambda_k) = \frac{1}{\alpha} \text{tr}_\alpha r(BPL_\alpha P) \\
\frac{B}{\alpha} \sum_{B\lambda_k \geq 1} \frac{B\lambda_k - 1}{B\lambda_k} = \frac{B}{\alpha} \text{tr}_\alpha p(BPL_\alpha P)
$$

with $r(x) = \log(x) \cdot \chi_{[1,\infty)}(x)$ and $p(x) = \frac{x-1}{x} \cdot \chi_{[1,\infty)}(x)$. The symbol $\text{tr}_\alpha Y := \text{tr}(PYP)$ denotes the trace of the operator $Y$ on the range of $P$ and the operators $r(PXP)$ and $p(PXP)$ for $X$ being self–adjoint are meant by spectral mapping theorem.

If the time–varying impulse response of $L_\sigma$ (or $H$) has finite delay ($k(x, x-z)$ is zero for $z$ outside a fixed interval) and is periodic in the time instants $x$ (the symbol $\sigma(x, \omega)$ is periodic in $x$) multiple channel uses in the preliminary model can be taken as consecutive uses of the same time–continuous channel. Inserting guard periods of appropriate fixed (independent of $\alpha$) size will not affect the asymptotic for $\alpha \to \infty$. Thus, any further results will then indeed refer to the information (and coding) capacity. The assumptions on finite delay might be relaxed using direct methods like in [7] or [18] whereby extensions to almost–periodic channels seems to lie at the heart of information theory.

### B. Problem Statement

The interval restriction $P$ has the symbol $\chi_{(-1,\infty)}$. The symbol of operator products is given as the twisted multiplication of the symbol of the factors. Under the trace this is reduced to ordinary multiplication (see for example [19] in the case of Weyl correspondence). Thus, the term in (3) can be written as the following trace:

$$
\frac{1}{\alpha} \text{tr}_\alpha L_f(\sigma) = \frac{1}{\alpha} \int_{\alpha \Omega \times \mathbb{R}} f(\sigma(x, \omega)) dx d\omega
$$

when taking $f(x) = r(Bx)$. Comparing (10) with (11) means to estimate the asymptotic behavior of:

$$
\frac{1}{\alpha} \text{tr}_\alpha (f(PL_\alpha P) - L_f(\sigma))
$$

for $\alpha \to \infty$ (we abbreviate $f(\sigma) := f \circ \sigma$). As seen from $r$ and $p$ in (10) the functions $f$ of interest are continuous but not differentiable at $x = 1$.

### III. Asymptotic Trace Formulas

The procedure for estimating the difference in (12) essentially consists in the following arguments: A functional calculus will be used to represent the function $f$ in the operator context. For $L_f(\sigma)$ this can be done independently...
of $\alpha$ but for $f(PL_\sigma P)$ such an approach is much more complicated because of the remaining projections $P$. Hence, the first step is to estimate its deviation to $f(L_\sigma)$ by inserting the zero term $\frac{1}{\alpha} \left( \frac{1}{\alpha} \right)$ into (12):

$$
\frac{1}{\alpha} \left( \frac{1}{\alpha} \right) \left( f(PL_\sigma P) - f(L_\sigma) \right)
$$

and use $|\text{tr}(a+b)| \leq |\text{tr}a| + |\text{tr}b|$ to estimate both terms separately. The first contribution refers to the stability of interval sections (in Section III-A). For second term a Fourier–based functional calculus reduces the problem to the characterization of the approximate product rule for symbols (in Section III-B) which can then be estimated independently of the particular function $f$ (in Section III-D). Unfortunately, the last steps require certain smoothness of $f$. Therefore we will approach the limit via smooth approximations $f_\varepsilon$ as discussed in Section III-C.

A. Stability of Interval Sections

The following stability result was inspired by the analysis on the Widom conjecture in [20]. Let $\text{spec}(L_\sigma)$ denote the spectrum of $L_\sigma$. Then the interval $I := \bigcup_{t \in [0,1]} t \cdot \text{spec}(L_\sigma)$ contains the spectra of the family $PL_\sigma P$ for each $\alpha$.

**Theorem 2.** Let $L_\sigma$ be an operator with a kernel which fulfills $|k(x, x-z)|^2 \leq \psi(z)$ with $\psi \in L_1$. If $\|\psi(1 - \chi_{[-s,s]})\|_{L_1} \leq c/s$ then:

$$
\frac{1}{\alpha} \int |\text{tr} \alpha \left( f(PL_\sigma P) - f(L_\sigma) \right)| \leq \|f''\|_{L_\infty} \frac{\log(\alpha)}{\alpha}
$$

for $f'' \in L_\infty(I)$.

Further details, see here [21]. Recall that the functions $f$ to be considered here are continuous and differentiable a.e. on $I$ (except at point $x = 1$). We will shortly discuss the proof of this theorem since it is only a minor variation of [20].

**Proof:** Laptev and Safarov [22] have obtained from Berezin inequality the following estimate. For functions $f'' \in L_\infty(I)$ the operator $P[f(L_\sigma) - f(PL_\sigma P)]P$ is trace class if $PL_\sigma$ and $PL_\sigma(1-P)$ are Hilbert–Schmidt with the trace estimate:

$$
|\text{tr} \alpha \left( f(L_\sigma) - f(PL_\sigma P) \right)| \leq \frac{1}{2} \|f''\|_{L_\infty} \|PL_\sigma(1-P)\|_{L_2}^2
$$

Recall that the interval projection $P$ is multiplication with the scaled characteristic function $\chi(\cdot/\alpha)$. Thus, change of variables $x = y' + x'$ and $y = y' - x'$ gives:

$$
\|PL_\sigma\|_{L_2}^2 = \int \chi(x/\alpha)|k(x, y)|^2 dxdy
$$

$$
\leq 2\alpha^2 \int \psi(2\alpha x')dx' \int \chi(y' + x')dy' \leq \alpha \|\psi\|_1
$$

In the same manner we get:

$$
\|PL_\sigma(1-P)\|_{L_2}^2 = \int \chi(x/\alpha)(1 - \chi(\frac{x}{\alpha}))|k(x, y)|^2 dxdy
$$

$$
\leq \alpha^2 \int \chi(x)(1 - \chi(y))\psi(\alpha(x - y))dxdy
$$

$$
= \alpha^2 \int \psi(2\alpha x) \cdot \omega(2x)dx
$$

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with \( \omega(x) := 4|x| \leq 2 \) for \( |x| \leq 1/2 \) and \( \omega(x) := 2 \) outside this interval. With \( u = 2\alpha x \) and \( \phi(u) = \psi(u) + \psi(-u) \) we split and estimate the integral as follows:

\[
\|PL_\sigma(1 - P)\|_{L_2}^2 = \frac{\alpha}{2} \int_0^\infty \phi(u)\omega(\frac{u}{\alpha})du \\
\leq \frac{\alpha}{2} \left( \frac{8}{\alpha} \int_0^2 + \frac{4u}{\alpha} + 2 \int_{2\alpha}^\infty \right) \phi(u)du
\]

(18)

With the assumptions of the theorem the terms are bounded separately:

\[
\|PL_\sigma(1 - P)\|_{L_2}^2 = 4\|\psi\|_1 + 2 \int_2^{2\alpha} \phi(u)udu + \frac{c^2}{2}
\]

(19)

Finally we use \( \phi(u) = -\frac{d}{du} \int_u^\infty \phi(s)ds \) and integrate by parts to obtain \( \int_2^{2\alpha} \phi(u)udu = c(1 + \log \alpha) \).

**Discussion:** Consider again the impulse response \( h(x, z) = k(x, x - z) \). The condition in the theorem is then:

\[
\|h(x, \cdot)(1 - \chi_{[-s, s]}(\cdot))\|_{L_2}^2 = \int_{-\infty}^{-s} (|h(x, z)|^2 + |h(x, -z)|^2) dz \leq c/s
\]

(20)

for \( s > 0 \). Differentiating both sides \( (d/d(-s)) \) gives the sufficient condition:

\[
|h(x, s)|^2 + |h(x, -s)|^2 \leq c/s^2
\]

(21)

Thus, if the kernel has the decay \( |k(x, x - s)| \leq c/|s| \) for any \( s \neq 0 \) the condition is fulfilled, i.e. from integration by parts for symbols \( \sigma(x, \cdot) \in C^1 \) and \( \sigma_\omega(x, \cdot) \in L_1 \) uniformly in \( x \) and vanishing for \( \omega \) at infinity. More generally this holds for \( \sigma(x, \cdot) \in L_2 \) having \( L_2 \)-modulus of continuity uniformly in \( x \) \( \|\sigma(x, \cdot) - \sigma(x, \cdot + h)\|_{L_2}^2 \leq c|h|^\beta \) for \( \beta \leq 1 \) (see [23, Lemma 2.10] for \( \beta = 1 \) and its generalization [20, Lemma 3.4.1] for \( 0 < \beta \leq 1 \)).

### B. Asymptotic Symbol Calculus

Here we shall use Fourier techniques to estimate the right term in (13). We abbreviate in the following \( e(x) = \exp(i2\pi x) \).

**Lemma 3.** Let \( f \) be a \( L_1 \)-function with \( \hat{f}(\omega) = \mathcal{O}(\omega^{-4-\delta}) \) for some \( \delta > 0 \) and \( f(0) = 0 \). For \( L_\sigma \) being bounded and self-adjoint on \( L_2 \) with real-valued symbol \( \sigma \in C^3 \) it follows that:

\[
\frac{1}{\alpha} |\rho_\alpha(f(L_\sigma) - L_\sigma f)| \leq 2\pi \int d\omega |\hat{f}(\omega)| \int_0^\omega Q_\alpha(\nu)\frac{d\nu}{\alpha}
\]

(22)

with \( Q_\alpha(\nu) := \|P(L_\sigma L_{e(\nu)} - L_{\sigma e(\nu)})\|_{L_2} \).

The lemma shows that whenever the rhs in (22) is finite the asymptotics for \( \alpha \to \infty \) is determined only by \( Q_\alpha/\alpha \). The function \( Q_\alpha \) essentially compares the twisted product of \( \sigma \) and \( e(\nu\sigma) \) with the ordinary product \( \sigma \cdot e(\nu\sigma) \) in trace norm reduced to intervals of length \( \alpha \).

**Proof:** The operator \( e(\omega L_\sigma) \) depends continuously on \( \omega \) and could be defined as the usual power series converging in norm since \( L_\sigma \) is bounded. In particular \( e(\omega L_\sigma) \) is unitary on \( L_2 \) \( (L_\sigma \) is self-adjoint) and \( \|e(\omega L_\sigma)\|_{\sigma} = 1 \). Thus, for \( \hat{f} \in L_1 \) the operator-valued integral:

\[
f(L_\sigma) = \int e(\omega L_\sigma)\hat{f}(\omega)d\omega
\]

(23)

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is a Bochner integral and \( \|f(L_\sigma)\|_\var{op} \leq \|\hat{f}\|_1 \).

Next, we consider the operator \( L_{f(\sigma)} \) with symbol \( f(\sigma) = f \circ \sigma \). The value of \( f \circ \sigma \) at each point can be expressed in terms of \( \hat{f} \). This suggests an integral formula similar to (23):

\[
L_{f(\sigma)} = \int L_{e(\nu \sigma)} \hat{f}(\omega) d\omega
\]

where its convergence has to be discussed. Ensuring convergence in the sense of Bochner requires further control of \( \|L_{e(\nu \sigma)}\|_\var{op} \). From Calderon Vaillancourt Theorem [24, Ch.5] we have:

\[
\|L_{e(\nu \sigma)}\|_\var{op} \leq \|e(\nu \sigma)\|_{C^3} := \sum_{\alpha + b \leq 3} |2\pi \nu|^{\alpha + b} \|\partial_\nu^\alpha \partial_\omega^b \sigma\|_\infty
\]

Thus, for \( \hat{f}(\omega) = O(\omega^{-4-\delta}) \) and \( \delta > 0 \) also the integral (24) converge in the sense of Bochner. From the considerations above we get therefore:

\[
|\text{tr}_\alpha (L_{f(\sigma)} - f(L_\sigma))| \leq \int |\hat{f}(\omega)| \cdot |\text{tr}_\alpha u(\omega)| d\omega
\]

with \( u(\omega) = e(\omega L_\sigma) - L_{e(\nu \sigma)} \). As suggested in [10] the operator \( u(\omega) \) fulfills the following identity:

\[
u' = i2\pi \left( L_\sigma u(\omega) + L_\sigma L_{e(\nu \sigma)} - L_{\sigma e(\nu \sigma)} \right)
\]

i.e. an inhomogenous Cauchy problem with initial condition \( u(0) = 0 \). By the Stones theorem, \( \{u(\omega)\}_{\omega \in \var{R}} \) is a strongly (norm-) continuous one–parameter family of operators on \( L_2(\alpha \var{O}) \). For a function \( g_0 \in L_2(\alpha \var{O}) \) the solution of the homogeneous equation is its unitary evolution:

\[
g_\omega = e(i2\pi \omega L_\sigma) g_0 = [e(i2\pi \omega L_\sigma) P] g_0
\]

By Duhamel’s principle (see for example [26, p.50] for the Banach–space valued case) the solution of (27) are the operators:

\[
u = \frac{2\pi}{i} \int_0^\omega e((\omega - \nu) L_\sigma P) (L_\sigma L_{e(\nu \sigma)} - L_{\sigma e(\nu \sigma)}) d\nu
\]

considered on \( L_2(\alpha \var{O}) \). With \( e((\omega - \nu) L_\sigma P) = e((\omega - \nu) L_\sigma) P \) this gives the estimate:

\[
|\text{tr}_\alpha u(\omega)| \leq 2\pi \int_0^\omega \|P (L_\sigma L_{e(\nu \sigma)} - L_{\sigma e(\nu \sigma)}) P\|_{\var{L}_1} d\nu = 2\pi \int_0^\omega Q_\alpha(\nu) d\nu
\]

since \( \|Pe((\omega - \nu) L_\sigma)\|_{\var{op}} \leq 1 \).

The smoothness assumptions in the theorem can be weakened to \( \sigma \in C^{2+\delta} \) and \( \hat{f}(\omega) = O(\omega^{-3-\delta}) \) when using Hőlder-Zygmund spaces. Furthermore, there is variant in terms of the modulation space \( M^{\infty,1} \) (see [27, pp.320]). We expect that these conditions can be further reduced when using in (24) some weaker convergence in \( \text{tr}_\alpha \) instead of requiring a Bochner integral. The proof of the theorem can also be based on the Paley–Wiener theorem, i.e. \( f \to f(L_\sigma) \) and \( f \to L_{f(\sigma)} \) are operator–valued distributions of compact support with order at most 3 and have therefore \( C^3 \) as natural domain (hier nochmal auf die decay condition im theorem eingehen).

in the case of operators we use \( \partial_\omega e(\omega \sigma) = i2\pi \sigma e(\omega \sigma) \) [25, Lemma 5.1].
C. An Approximation Procedure

Since $L_\sigma$ is bounded (see (9)) the functions $f$ will be evaluated only on a finite interval contained in $I$. We consider functions $f$ of the form $f(x) = h(x) \cdot \chi_{[1,\infty)}(x)$ with a critical point at $x = 1$ (the function $h$ is “sufficiently nice” on $I$, e.g., in $C^\infty(I)$). In general therefore, the first derivative of $f$ will be not continuous at $x = 1$ but of bounded variation. By smooth extension outside the interval $I$ its Fourier transforms $\hat{f}(\omega)$ can decay only as $O(\omega^{-2})$, see here for example [28, Theorem 2.4], i.e. $f \in L_1 \cap \mathcal{F}L_1$. Unfortunately, this is not sufficient for Lemma 3. Therefore, we replace the Heaviside function $\chi_{[1,\infty)}$ in $f$ by a series of smooth approximations $\phi_\varepsilon$ as done for example in [9]. Let be $\phi \in C^\infty$ strictly increasing with $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$. Define $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$ and consider $f_\varepsilon = h \phi_\varepsilon \in C^\infty_c$ instead of $f$ (again by smooth extension outside the interval $I$):

$$|\hat{f}_\varepsilon(\omega)| \leq \frac{c_n |I|}{2\pi \omega^n} \varepsilon^{-n}$$  \hspace{1cm} (31)

We abbreviate $d_\varepsilon = f - f_\varepsilon$ and obtain from triangle inequality and linearity in $f$ that:

$$|\text{tr}_\alpha (f(L_\sigma) - L_f(\sigma))| \leq |\text{tr}_\alpha (f_\varepsilon(L_\sigma) - L_{f_\varepsilon}(\sigma))| + |\text{tr}_\alpha d_\varepsilon(L_\sigma)| + |\text{tr}_\alpha L_{d_\varepsilon(\sigma)}|$$  \hspace{1cm} (32)

We assume wlog that $\max_{t \in I} h(t) = 1$. The function $d_\varepsilon$ is non–negative and of the following form:

$$0 \leq d_\varepsilon = (\chi_{[1,\infty)} - \phi_\varepsilon)h = (\chi_{[1,\infty)} - \phi((\cdot - 1)/\varepsilon)) \cdot h \leq 1$$  \hspace{1cm} (33)

Furthermore, $d_\varepsilon$ is smooth except $x = 1$ where it has no continuous derivative. Therefore $\hat{d}_\varepsilon(\omega)$ decays as $O(\omega^{-2})$ implying that $d_\varepsilon \in L_1 \cap \mathcal{F}L_1$. On its support $\text{supp}(d_\varepsilon) \subseteq [1, 1 + \varepsilon]$ the function $d_\varepsilon$ is upper–bounded by the strictly
decreasing function $g_\varepsilon$:
\[ d_\varepsilon \leq g_\varepsilon := (1 - \phi((\cdot - 1)/\varepsilon)) \quad (34) \]

\( a) \) Scaling of $\text{tr}_\alpha L_{d_\varepsilon(\sigma)}$: For the last approximation term in (32) we compute the trace using (11). Let $\mu$ be Lebesgue–measure in $\alpha \Omega \times \mathbb{R}$ and $z = (x, \omega) \in \alpha \Omega \times \mathbb{R}$. Since we assume $\|\sigma\|_2^2(\alpha \Omega \times \mathbb{R}) = \mathcal{O}(\alpha)$ we have the following standard estimate:
\[ \mu\{|\sigma| \geq t\} \leq \frac{1}{t^d}\|\sigma\|_2^2(\alpha \Omega \times \mathbb{R}) = \mathcal{O}(\alpha) \quad (35) \]
and $\mu\{|\sigma| \geq t\}$ is therefore decreasing in $t$. We also use the layer cake representation of integrals (see [29]). Let $\{\nu : F(z) \geq t\}$ be super–level set of the non–negative measurable function $F$. Then:
\[ F(z) = \int_0^\infty \chi_{\{\nu : F(\nu) \geq t\}}(z)dt \quad (36) \]
We start from (11) with the property that $d_\varepsilon \leq g_\varepsilon \leq 1$ on its support $[1, 1+\varepsilon]$ and therefore also $g_\varepsilon(\sigma) \leq \sigma = |\sigma|$ on the set where $1 \leq \sigma$. Thus, using Fubini:
\[ \text{tr}_\alpha L_{d_\varepsilon(\sigma)} \leq \int_{\alpha \Omega \times \mathbb{R}} g_{\varepsilon}(\sigma(z))d\mu(z) \leq \int_{\{1 \leq \sigma \leq 1+\varepsilon\}} |\sigma|d\mu = \int_{\{1 \leq \sigma \leq 1+\varepsilon\}} \left( \int_0^\infty \chi_{\{|\sigma| \geq t\}}(z)dt \right) d\mu \]
\[ = \int_{1}^{1+\varepsilon} \left( \int_{\{1 \leq \sigma \leq 1+\varepsilon\}} \chi_{\{|\sigma| \geq t\}}(z)dt \right) d\mu \leq \int_{1}^{1+\varepsilon} \mu\{|\sigma| \geq t\}dt \leq \varepsilon \cdot \mu\{|\sigma| \geq 1\} = \mathcal{O}(\varepsilon \cdot \alpha) \quad (37) \]
The last inequality follows since $\mu\{|\sigma| \geq t\}$ is decreasing in $t$.

\( b) \) Scaling of $\text{tr}_\alpha d_\varepsilon(L_{\sigma})$: Let $S = \text{spec}(PL_{\sigma}P)$ be the spectrum of $PL_{\sigma}P$. We have the relation (see here (11)):
\[ \sum_{1 \leq \lambda \in S} \lambda \leq \sum_{\lambda \in S} \lambda = \text{tr}_\alpha L_{\sigma} = \int_{\alpha \Omega \times \mathbb{R}} \sigma(x, \omega)dx d\omega = \mathcal{O}(\alpha) \quad \text{to check} \quad (38) \]
We perform now similar steps as above for a counting measure instead of $\mu$ and we abbreviate $S_\varepsilon := \{\lambda \in S : 1 \leq \lambda \leq 1+\varepsilon\}$. From spectral theorem it follows that:
\[ \text{tr}_\alpha d_\varepsilon(L_{\sigma}) = \sum_{\lambda \in S} d_\varepsilon(\lambda) \leq \sum_{\lambda \in S} g_\varepsilon(\lambda) \leq \sum_{\lambda \in S} \lambda \leq \sum_{\lambda \in S} \int_0^\infty \chi_{\{\nu \in S_\varepsilon : \nu \geq t\}}(\lambda)dt \]
\[ = \int_0^\infty \left( \sum_{\lambda \in S_\varepsilon} \chi_{\{\nu \in S_\varepsilon : \nu \geq t\}}(\lambda) \right) dt = \int_0^\infty \{\lambda \in S_\varepsilon : \lambda \geq t\}dt \]
\[ \leq \int_{1}^{1+\varepsilon} (\sum_{1 \leq \lambda \in S_\varepsilon} 1)dt \leq \varepsilon (\sum_{1 \leq \lambda \in S} \lambda) \leq \mathcal{O}(\varepsilon \alpha) \quad (39) \]

In essence: polynomial grow of $Q_{\alpha}(\nu)$ in $\nu$ can always be compensated by taking $n$ large enough such that at the rhs in (22) remains a finite quantity $R_\alpha(\varepsilon)$. If for example $R_\alpha(\varepsilon) = \mathcal{O}(\alpha^{-\gamma})$, we choose $\varepsilon = \alpha^{-\delta}$ with $\delta < \gamma/n$. Then $R_\alpha(\varepsilon) \to 0$ and $\varepsilon \to 0$ for $\alpha \to \infty$ which is obviously sufficient for the limit.
D. Approximate Symbol Products

Polynomial orders of $Q_\alpha(\nu)$ in $\nu$ which will occur in the following will be compensated by the approximation method in Section III-C. The role of $\tau$ and $\sigma$ can also be interchanged since according (25) $L_\tau$ is bounded polynomially in $s$.

Let us abbreviate $\tau = e(\nu\sigma) = \exp(i2\pi\nu\sigma)$. Then the operator in the term $Q_\alpha(\nu)/\alpha$ of Lemma 3 is the deviation between operator and symbol product $L_\sigma L_\nu - L_{\sigma\tau}$. As in [8] we insert $L_\sigma L_\nu^* - L_\sigma L_\nu^* = 0$. Define the operators $T = L_\nu^* - L_\tau$ and $T' = L_\sigma L_\nu^* - L_{\sigma\tau}$ and apply triangle inequality to obtain:

$$Q_\alpha(\nu) \leq \|PL_\sigma TP\|_{L_1} + \|PT'P\|_{L_1}$$

$$\leq \|PL_\sigma PT\|_{L_1} + \|PL_\sigma(1 - P)TP\|_{L_1} + \|PT'P\|_{L_1}$$

$$\leq \|PL_\sigma PT\|_{L_1} + \|PT'\|_{L_1} + \|PL_\sigma(1 - P)\|_{L_2} \cdot \|TP\|_{L_2}$$

(40)

where (17) from the proof of Theorem 2 has been used. The operators $T$ and $T'$ have the kernels $t(x, y)$ and $t'(x, y)$ defined formally as:

$$t(x, y) = \int e^{i2\pi(x-y)\omega}(\tau(x, \omega) - \tau(y, \omega))d\omega$$

$$t'(x, y) = \int e^{i2\pi(x-y)\omega}\sigma(x, \omega)(\tau(x, \omega) - \tau(y, \omega))d\omega$$

(41)

The meaning of these integrals has to be discussed. Note again that $\tau(x, \omega) = \exp(i2\pi
\nu\sigma(x, \omega))$ is a pure phase symbol and $\sigma \in C^3$, i.e. from $\sigma(x, \cdot) \in L_2$ follows that $\sigma(x, \cdot)$ vanishes at infinity for each $x$. We have therefore for all $x$ and $y$:

$$2 \geq |\tau(x, \omega) - \tau(y, \omega)| \to 0 \quad \text{as} \quad |\omega| \to \infty$$

(42)

From integration by parts ($n$ times) we have:

$$t(x, y) = \int e^{i2\pi(x-y)\omega}\frac{\partial^n}{\partial\omega^n}(\tau(x, \omega) - \tau(y, \omega))d\omega =: \int e^{i2\pi(x-y)\omega}t_{\omega}(x, y)d\omega$$

$$t'(x, y) = \int e^{i2\pi(x-y)\omega}\frac{\partial^n}{\partial\omega^n}(\sigma(x, \omega)(\tau(x, \omega) - \tau(y, \omega)))d\omega =: \int e^{i2\pi(x-y)\omega}t'_{\omega}(x, y)d\omega$$

(43)

Both kernels $t_{\omega}$ and $t'_{\omega}$ are finite linear combinations of the form:

$$\left(\sigma^{(k)}(x, \omega)\right)^{\frac{\partial^n}{\partial\omega^n}(\tau(x, \omega) - \tau(y, \omega))}$$

(44)

for $j = 0 \ldots$ and $m \geq n$ where $n \geq 1$ is fixed. We shall argue later that it will be sufficient to consider the case $n = 1$. Let then $T_{\omega}$ and $T'_{\omega}$ be the corresponding operators with the kernels $t_{\omega}$ and $t'_{\omega}$. Once the trace norms of the restricted operators $PT_{\omega}P$ and $PT'_{\omega}P$ decay sufficiently the estimate:

$$\|PTP\|_{L_1} \leq \int \|PT_{\omega}P\|_{L_1}d\omega \quad \text{(same for $T'$)}$$

(45)

is possible. This will be related to the oscillatory behavior of the symbol and we will investigate this in the next section. Before continuing on this point, we consider the Hilbert–Schmidt norm $\|TP\|_{L_2}$ in (40). From (43) for
Thus, it remains to evaluate trace norms of kernels $t(x, y)$ and $t'(x, y)$ restricted to $\alpha \Omega \times \alpha \Omega$ (recall that the restriction operators $P = P_{\alpha}$ depend on $\alpha$ and the operators $T, T'$ depend on $\nu$ through their kernels $t$ and $t'$).

**E. Paracommutators and Schur Multipliers**

Under mild assumptions on the $\omega$–dependency of $\sigma(x, \omega)$ the overall trace norm $\|PTP\|_{L_1}$ and $\|PT'P\|_{L_1}$ in (47) can be estimated following the decomposition in (45) and the oscillatory character of $\sigma(x, \omega)$ in $x$ will play the major role. We will express the contributions $PT_\omega P$ and $PT'_\omega P$ in (45) in the form of paracommutators which have been studied for example by Janson and Peetre in [30]. A paracommutator $T_b$ with symbol $b$ is a “para–multiplication” of the following form:

$$T_b f(\xi) = \int b(\xi - \eta) A(\xi, \eta) f(\eta) d\eta$$

(48)

where the function $A(\xi, \eta)$ acts as a Schur multiplier (see below). This definition includes Toeplitz and Hankel operators, i.e., for example for $A(\xi, \eta) = 1$ it's a pointwise multiplication $f \cdot b$.

**a) Schur Multipliers:** An important tool therein are bounded Schur–multipliers (see here also [31]). For exposition, let $m(x, y)$ be a function represented by:

$$m(x, y) = \int m_x(x, t)m_y(y, t) d\mu(t)$$

(49)

with some sigma–finite measure $\mu$ and denote with $M$ the corresponding mapping which operates on kernels $k$ of operators $K$ of a particular Schatten ideal $L_p$ for $1 \leq p \leq \infty$. Then:

$$\|MK\|_{L_p} \leq \|K\|_{L_p} \cdot \|m_x(\cdot, t)\|_{L_\infty} \cdot \|m_y(\cdot, t)\|_{L_\infty} d\mu(t) \leq \|K\|_{L_p} \cdot \|M\|_M$$

(50)

and $\|M\|_M$ is essentially defined here as the infimum over all representations (49). For $\mu$ being a simple point–measure at $t_0$ the kernel of $MK$ takes the form $m_x(x) k(x, y) m_y(y)$ with $m_x = m_x(\cdot, t_0)$ and $m_y = m_y(\cdot, t_0)$ such that $\|MK\|_{L_p} \leq \|K\|_{L_p} \cdot \|m_x\|_{L_\infty} \|m_y\|_{L_\infty}$.

A direct consequence of these tools is that for investigation of scaling

the term $\sigma^{(k)}(x, \omega)^d$ in (44) can be dropped if $\sigma(x, \omega)$ and its derivatives like $\partial_\omega \sigma(x, \omega)$ are uniformly bounded in $x$ and $\omega$. They are Schur–multipliers in $x$ for fixed $\omega$ with $\|\sigma(\cdot, \omega)\|_M = \|\sigma(\cdot, \omega)\|_{L_\infty} \leq C$ where the constant $C$ does not depend on $\omega$ and the size $\alpha$ of interval $\alpha \Omega$. 

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b) The Paracommutator: From Schur–multiplier techniques it follows also that we can replace the outer support restriction \( \chi_\alpha(x)\chi_\alpha(y) \) with certain (inner) smooth functions \( \phi_\alpha(x) := \phi(x/\alpha), \phi_\alpha(y) \) and \( \phi_\alpha((x - y)/2) \) where \( \phi \in C^\infty \) with \( \phi(x) = 1 \) for \( |x| \leq 1 \) and \( \phi(x) = 0 \) for \( |x| \geq 2 \). More precisely, since

\[
\chi_\alpha(x)\chi_\alpha(y) = \chi_\alpha(x)\chi_\alpha(y)\phi_\alpha(x)\phi_\alpha(y)((x - y)/2)
\]

we can write the kernels (we do not write here \( \omega \)-dependency explicitly) as:

\[
k(x, y) = [\tau(x)\phi_\alpha(x) - \tau(y)\phi_\alpha(y)] \frac{\phi_\alpha((x - y)/2)}{x - y} \cdot \chi_\alpha(x)\chi_\alpha(y)
\]

where either \( \tau(x) = \exp(i2\pi s\sigma(x, \omega)) \) or \( \tau(x) = \partial_v\exp(i2\pi s\sigma(x, \omega)) \). Since the restriction \( \chi_\alpha(x)\chi_\alpha(y) \) is a Schur–multiplier of norm one, the trace norm is upper–bounded by the trace of the smooth kernel:

\[
k(x, y) = [\tau(x)\phi_\alpha(x) - \tau(y)\phi_\alpha(y)] \frac{\phi_\alpha((x - y)/2)}{x - y} := [b(x) - b(y)] a(x - y)
\]

where \( a \) is a distribution defined for \( z \neq 0 \) by \( a(z) = \phi(z/(2\alpha))/z \) and \( b \) is a smooth function defined by:

\[
b(x) = \begin{cases} 
\phi_\alpha(x)e^{i2\pi s\sigma(x,\omega)} & \text{for case } (T) \\
\phi_\alpha(x)\partial_v e^{i2\pi s\sigma(x,\omega)} & \text{for case } (T')
\end{cases}
\]

Next, we path to the Fourier kernel and we use here abbreviations from [30], i.e. we write:

\[
k(x, y) = \int_{u + v = 1} b(u x + v y) a(x - y) d\mu(u, v)
\]

with the point measure \( \mu(u, v) = \delta(u - 1, v) - \delta(u, v - 1) \). The Fourier kernel \( \hat{k} = (\mathcal{F} \otimes \mathcal{F}^*) k \) of \( T \) is the kernel of the operator \( \mathcal{F}T\mathcal{F}^* \) defined by \( \langle f, Tg \rangle = \langle \hat{f}, \mathcal{F}T\mathcal{F}^* \hat{g} \rangle \) since

\[
\langle f \otimes \tilde{g}, k \rangle = \langle (\mathcal{F}^* \hat{f} \otimes \mathcal{F}^* \hat{g}) k \rangle = \langle (\mathcal{F}^* \otimes \mathcal{F}) \hat{f} \otimes \hat{g}, k \rangle = \langle \hat{f} \otimes \hat{g}, (\mathcal{F} \otimes \mathcal{F}^*) k \rangle
\]

We get therefore (repeating the steps in [30, p.499]):

\[
\hat{k}(\xi, \eta) = \iint e^{-i2\pi(x\xi - y\eta)} k(x, y) dx dy
\]

\[
= \int_{u + v = 1} d\mu(u, v) \int e^{-i2\pi(x\xi - y\eta)} b(u x + v y) a(x - y) dx dy \quad \text{with } z = x - y
\]

\[
= \int_{u + v = 1} d\mu(u, v) \int e^{-i2\pi(z\xi + y(\xi - \eta))} b(u z + y) a(z) dz dy \quad \text{with } w = uz + y
\]

\[
= \int_{u + v = 1} d\mu(u, v) \int e^{-i2\pi(z\xi + (w - uz)(\xi - \eta))} b(w) a(z) dz dw
\]

\[
= \int_{u + v = 1} \hat{b}(\xi - \eta)\hat{a}(v\xi + w\eta) d\mu(u, v) = \hat{b}(\xi - \eta) [\hat{a}(\eta) - \hat{a}(\xi)]
\]

\[
=: \hat{b}(\xi - \eta) A(\xi, \eta)
\]

According to (48) this is the Fourier kernel of a paracommutator with \( A(\xi, \eta) = \hat{a}(\eta) - \hat{a}(\xi) \). Except of the singularity of \( a(z) \) at point \( z = 0 \) the definition of the Fourier transforms are not problematic since the functions are \( a \) and \( b \) are continuous and compactly supported.
c) Calderon–Zygmund commutator: Consider now \( a(z) = \phi(z/\alpha)/z^m \) for \( m \geq n \) which occurs when integrating by parts \( n \geq 1 \) times in (43). The dominating order in \( \alpha \) is indeed for \( m = 1 \). The Fourier integral of \( a \) is not absolutely convergent unless \( \phi(z) = 0 \), i.e., has only the meaning of a principal value. To compute its Fourier transform we use that \( \phi \in C^\infty \) and that \( \phi^{(m-1)}(0) = 0 \) for \( m > 1 \), due to the symmetry of \( \phi \) around zero. We get (see here also [32, pp.324]) with \( \psi(z) := e^{-i2\pi \alpha \xi z} \phi(z) \):

\[
\hat{a}(\xi) = \int e^{-i2\pi \xi z} \frac{\phi(z/\alpha)}{z^m} dz = \alpha^{1-m} \int \frac{\psi(z)}{z^m} dz (m>1) = \frac{\alpha^{1-m}}{(m-1)!} \int \frac{\psi^{(m-1)}(z)}{z} dz
\]

(58)

that the leading term is given by the Hilbert transform of \( \psi^{(m-1)} \).

In the exposition above we already stucked to \( n = 1 \), i.e. \( k(x, y) \) for \( m = 1 \) is the (smoothly truncated) kernel of the so called Calderon–Zygmund commutator (commutator of a multiplier \( b \) with the Hilbert transform having the kernel \( a \)). The Schatten class properties of such type of operators have been investigated by [33] and [34] and are related to the oscillatory characterization of the multiplier \( b \). Here we follow the lines of [30].

F. Schatten–Properties of Paracommutators and Besov Spaces

Recall that we need to evaluate the scaling of trace norms in (47). The corresponding operators \( T \) and \( T' \) can be decomposed into integrals (45) once the symbol \( \sigma(x, \omega) \) has sufficient decay in \( \omega \) (wideband case are therefore excluded). Each contribution to the integrals is a trace norm of restrictions of the operators \( T_\omega \) and \( T'_\omega \). As explained above, this can related to trace norms of para–commutators, i.e. Fourier integral operators of the form (48) with symbols \( b \) as defined (54). The most well–known example here is the Calderon–Zygmund commutator which can be written in terms of Hankel operator. Peller was the first who observed that such operators are nuclear (trace–class) if the symbol is in a particular Besov–Space [35].

a) Homogeneous Besov Spaces: Let \( \{\phi_k\}_{k=\infty}^\infty \) be the Paley–Littlewood decomposition, i.e., \( \phi = \phi_0 \in C_c^\infty \) with \( \text{supp}(\hat{\phi}) \subset \{ \frac{1}{2} \leq |\omega| \leq 2 \} \) and \( \hat{\phi}_k := \hat{\phi}(2^{-k} \cdot) \) with:

\[
\sum_k \hat{\phi}_k(\omega) = 1 \quad \text{for all} \quad 0 \neq \omega \in \mathbb{R}
\]

(59)

For \(-\infty < s < \infty \) and \( 0 < p, q \leq \infty \), the homogeneous Besov spaces of distributions (modulo polynomials) are defined by the quasi–norms:

\[
\|f\|_{\dot{B}^s_{p,q}} = \left( \sum_{k \in \mathbb{Z}} (2^{ks} \|\phi_k * f\|_p)^q \right)^{1/q} = \|\{2^{ks} \|\phi_k * f\|_p\} \}_{k \in \mathbb{Z}} \|_{\ell^q}
\]

(60)

The spaces are called homogeneous since for the whole \( (p, q, s) \)–range, given above, it follows that for all \( \alpha > 0 \) it holds (see for example [37, Proposition 3.8]):

\[
c_1 \|f\|_{\dot{B}^s_{p,q}} \leq \alpha^{-(1/p-s)} \|f(\cdot/\alpha)\|_{\dot{B}^0_{p,q}} \leq c_2 \|f\|_{\dot{B}^s_{p,q}}
\]

(61)

In [36] the spaces are denoted by \( \Lambda^s_{p,q} \).
with equality for $\alpha = 2^{-k}$ for some $k$ (see [38, Remark 4 on p.239], [39, Remark 2 on p.94] and also [36, Lemma 1.2 on p.288]). The Besov spaces $\dot{B}^s_{pq}$ for $s < 1/p$ or $s = 1/p$ with $q = 1$ can be regarded as subspaces of tempered distributions $S'$. Obviously, for $q' \leq q$ there holds the inclusion $\dot{B}^s_{pq} \subseteq \dot{B}^s_{p'q'}$ since $\ell_{q'} \leq \ell_q$ (see (60)). Furthermore, for $1 \leq p \leq p' \leq \infty$ there holds $\dot{B}^s_{pq} \subseteq \dot{B}^s_{p'q'}$ when $s - s' = 1/p - 1/p'$ (recall that we consider only dimension one).

We will further abbreviate some a fixed $q$ (here for $q = 1$) $\dot{B}^s_p := \dot{B}^s_{p1}$ and the scale–invariant Besov spaces (for $q = 1$) with $\dot{B}^s_p := \dot{B}^s_{p1/p}$.

In particular, $\dot{B}_p \cap L_\infty$ are (Quasi--) Banach algebras (see here [40, Remark 2 on p.148]) and we have for $s > 0$ [36, Lemma 1.5 on p.293] and also [37, Theorem 3.26]:

$$\|fg\|_{\dot{B}^s_{pq}} \leq c \left( \|f\|_{\dot{B}^s_{pq}} \|g\|_{L_\infty} + \|g\|_{\dot{B}^s_{pq}} \|f\|_{L_\infty} \right) \quad (62)$$

b) Trace–Class Results: We have to consider paracommutators $\hat{b}(\xi - \eta)A(\xi, \eta)$ with $A(\xi, \eta) = \hat{a}(\eta) - \hat{a}(\xi)$ where $b$ is given in (54) and $a(z) = \phi(z/(2\alpha))/z$. We will follow the notation in [30]. The function $A(\xi, \eta)$ is a uniformly bounded Schur multiplier and vanishes on the diagonal, i.e., $A(\xi, \xi) = 0$ (conditions A1 and A3(\infty) in [30]). The following theorem holds:

**Theorem 4** (Thm. 8.1 in [30]). Let $T_b$ be a paracommutator in the form (48) with $A(\xi, \eta) = \hat{a}(\eta) - \hat{a}(\xi)$ and $a$ as defined above. For $1 \leq p \leq \infty$ it holds:

$$\|T_b\|_{\dot{B}_p} \leq C\|b\|_{\dot{B}_p} \quad (63)$$

We have to investigate paracommutators $T_b$ with symbols $b$ given by equation (54), i.e., which depend on the symbol $\sigma(\cdot, \omega)$ for a particular $\omega$ and on the interval length $\alpha$. Thus, which conditions on the $x$–dependency of the symbols $\sigma(x, \omega)$ of $L_\sigma$ ensure a sublinear scaling in $\alpha$ of $\|b\|_{\dot{B}_p}$? According to (54) this involves the behavior of Besov norms with respect to pointwise multiplication with the smooth cutoff functions $\phi_\alpha$ and compositions of the form $\exp(i2\pi \nu \sigma)$. We will discuss both topics separately, yielding Lemma 5 and Lemma 6, and combine the results at the end as Theorem 7.

**G. Pointwise Multiplications on Besov Spaces**

In this part we will explicitly investigate the behavior of the homogeneous Besov norms with respect to pointwise multiplications. We need conditions on a distribution $g$ which allow for the following limit:

$$\frac{1}{\alpha} \|\phi(\cdot/\alpha) \cdot g\|_{\dot{B}_1} \rightarrow 0 \quad \text{for} \quad \alpha \rightarrow \infty \quad (64)$$

where $\phi$ is a smooth cutoff function and, in general, $g$ is not vanishing at infinity. The reason for this is that for $g$ we shall later use for example $\exp(i2\pi \sigma(\cdot, \omega))$. Obviously, to obtain non–zero transmission capacity the symbol

The meaning of (62) has to be taken with care since homogeneous Besov spaces are equivalent classes of distributions modulo polynomials. The precise statement can be found in [37, Theorem 3.26].
\( \sigma(x, \omega) \), somehow representing the “channel power” over time \( x \), can not vanish for \( x \to \infty \). Unfortunately, then - the algebra property (62) is not helpful in this context.

Instead a Hölder-type inequality for Besov spaces is required here which is known for inhomogeneous Besov spaces (see for example [39]). Using similar results for the homogeneous spaces in [41] we get the following theorem:

**Lemma 5.** Let \( \phi \in C^\infty_c \) and \( g \in \dot{B}^1_{\infty, \infty} \). Then

\[
\lim_{\alpha \to \infty} \frac{1}{\alpha} \| \phi(\cdot/\alpha) \cdot g \|_{\dot{B}^1_1} = 0
\]

We use Proposition and Remark 5 on p.11 in [41] for the case \( p = q = s = 1 \). Let \( 1 \leq r \leq \infty \) with \( \sigma > 0 \), \( \theta \geq 0 \), \( 0 < \delta \leq 1 \) and \( 0 \neq N \in \mathbb{N} \). It holds:

\[
\|fg\|_{\dot{B}^1_r} \leq (N^2 + 1)\|fg\|_{\dot{B}^1_{r+1, \delta}} + 2^{-N\delta}(N + 1)\left(\|fg\|_{\dot{B}^1_{r+1+\delta, \delta}} + \|fg\|_{\dot{B}^1_{r+1-\delta, \delta}}\right)
\]

We will apply this now on (64) where \( f = \phi(\cdot/\alpha) \). From the scaling property (61) we have that:

\[
\frac{1}{\alpha} \| \phi \|_{\dot{B}^1_r} \leq c \alpha^{1/r} \| \phi \|_{\dot{B}^1_r} = c \| \phi \|_{\dot{B}^1_r} / \alpha^{1/r}
\]

Intuitively, we would like to take \((r, \bar{r}) = (1, \infty)\) to support that \( g \) is non–vanishing at infinity and (68) implies then strictly–positive smoothness \( s > 0 \) such that (64) is possible. But, from the theorem above we have to take into account all \( s \in \{-\theta, -\sigma, \theta + 1, \sigma + 1 + \delta\} \) where \( \sigma > 0 \), \( \theta \geq 0 \) and \( 0 < \delta \leq 1 \). Thus, the critical – negative – exponents for \( 1/\alpha \) in (68) are \( s \in \{-\theta, -\sigma\} \) and therefore \((r, \bar{r}) = (0, \infty)\) is not directly possible. More precisely, the condition:

\[
\bar{r} < 1/ \max(\theta, \sigma)
\]

is necessary such (68) vanishes for \( \alpha \to \infty \).

Nevertheless, it possible to pose a dependency on \( \alpha \) such that the limit can approached and, above a certain \( \alpha_0 \), all requirements of the theorem are fulfilled for each finite \( \alpha \geq \alpha_0 \). To this end we set \( L = \log(\alpha) \) and choose \( r = (L + 1)/L > 1 \) and \( \bar{r} = L + 1 \) such that \((r, \bar{r}) \to (1, \infty)\) for \( \alpha \to \infty \). Next, we parametrize the smoothness parameters \( \theta, \sigma \) and \( \delta \). Fix some \( \epsilon > 1 \). Then there exists \( \alpha_0 \) such that:

\[
1 \geq \delta = \theta := 1/(L + 1) - 1/L^\epsilon > 0 \quad \text{for all} \quad \alpha \geq \alpha_0
\]

Summarizing:

\[
(r, \bar{r}, \theta, \sigma, \delta) \to (1, \infty, 0, 0, 0) \quad \text{for} \quad \alpha \to \infty
\]
H. Compositions on Besov Spaces $B^s_\infty$ for $s \geq 1$

Here we discuss now how to handle the composition problem for $\exp(i2\pi \nu \cdot) \circ \sigma(\cdot, \omega)$. We will use the property that for each $\omega$ the function $f : x \to \sigma(x, \omega)$ is bounded (since we already posed the assumptions $\sigma \in C^3$ and $\sigma \in L_\infty$). This means that $f \in B^s_\infty$ is in the inhomogeneous Besov space if $f \in \dot{B}^s_\infty$. Recall that in our case $s = 1 + \delta$ whereby $\delta > 0$ and the case $\delta = 0$ is still open.

**Lemma 6.** Let $f : \mathbb{R} \to \mathbb{R}$ be real–valued and $f \in B^s_\infty$ for $s \geq 1$. Then there exists a constant $c > 0$ depending on $f$ such that it holds:

$$
\|\exp(i2\pi \nu f)\|_{\dot{B}^s_\infty} \leq c(2\pi \nu)^s
$$

(72)

For $s = 1$ the constant $c$ is of the form $c = c'(1 + \|f\|_{B^1_\infty})$ for some other constant $c'$ (independent of $f$).

**Proof**: For $s > 1$ and $1 \leq q \leq \infty$ we have from Theorem 4 in [42] that composition operator $T_G : f \to G \circ f$ fulfills $T_G(B^s_{\infty,q}) \subseteq B^{s+1}_{\infty,q}$ if and only if $G \in B^{s+1}_{\infty,q}$. More precisely, there exists a continuous increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that it holds:

$$
\|G \circ f\|_{B^{s+1}_{\infty,1}} \leq \|G\|_{B^{s+1}_{\infty,1}} \psi(\|f\|_{B^s_{\infty,1}})
$$

(73)

for all $f, G \in B^s_{\infty,1}$. We have:

$$
\|\exp(i2\pi \nu f)\|_{\dot{B}^s_\infty,1} \leq \|\cos(2\pi \nu f)\|_{\dot{B}^s_\infty,1} + \|\sin(2\pi \nu f)\|_{\dot{B}^s_\infty,1}
$$

(61)

$$
\leq (2\pi \nu)^s (\|\cos(f)\|_{B^s_\infty,1} + \|\sin(f)\|_{B^s_\infty,1})
$$

(74)

where we switched to the inhomogeneous Besov spaces since $\cos$ and $\sin$ are bounded. From Theorem 5 in [42] it follows also that $\psi(x) = c'(1 + x)$ for $s = 1$.

I. Combining the Results

Here we will now combine the results so far. The following theorem is not the most general combination of the previous results. Instead we have preferred for the moment a straightforward enumeration of the statements.

**Theorem 7.** Let $\sigma$ be a real–valued symbol with (i) $\sigma(\cdot, \omega) \in B^s_\infty$ for $s > 1$ uniformly in $\omega$ and (ii) $\sigma(x, \cdot) \in L_1$ uniformly in $x$. Then it holds:

$$
\lim_{\alpha \to \infty} Q_\alpha(\nu)/\alpha = 0
$$

(75)

Again, at this point it is open whether the same can be obtained for $s = 1$.

**Proof**: Recall that from (47) we have:

$$
Q_\alpha(\nu) \leq c_1 \|PTP\|_{X_1} + \|PT'P\|_{X_1} + c_2 \sqrt{\alpha(1 + \log(\alpha))}
$$

(76)
and from (45)
\[ \|PT\|_{I_1} \leq \int \|PT_\omega P\|_{I_1} d\omega \quad \text{(same for } T') \] (77)

whereby the convergence of the integrals is ensured by sufficient decay of \( \sigma(x, \omega) \) in \( \omega \). For each \( \omega \) the operators \( T_\omega \) and \( T'_\omega \) can be replaced by smoothed para–commutators \( T_b \) of the form (48) with \( A(\xi, \eta) = \hat{a}(\eta) - \hat{a}(\xi) \) and \( a(z) = \phi(z/(2\alpha))/z \) and symbol \( b = b(\alpha, \sigma) \) depending on \( \alpha \) and \( \sigma \) according to formula (54). We apply now Lemma 5 and Lemma 6 to \( T_b \) for each \( \omega \).

IV. Conclusion

A new approach to the capacity of time–continuous doubly–dispersive Gaussian channels with periodic symbol has been established by proving a Szegö asymptotic for certain pseudo–differential operators. The result holds once the symbol \( \sigma(x, \omega) \) of the channel correlation operator \( L_\sigma \) has sufficient decay in frequency \( \omega \) and the time–oscillation have finite \( B_s^\infty \) Besov norm for \( s > 1 \), both is meant in a uniform sense.

ACKNOWLEDGMENT

This work is supported by Deutsche Forschungsgemeinschaft (DFG) grant JU 2795/1-1. The author would like to thank Holger Boche, Igor Bjelakovic, Winfried Sickel and Jan Vybiral.

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