THE WEIGHTED AM-GM INEQUALITY IS EQUIVALENT TO THE HÖLDER INEQUALITY

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Abstract. In this note, we investigate mathematical relations among the weighted AM-GM inequality, the Hölder inequality and the weighted power-mean inequality. Meanwhile, the detailed proofs of mathematical equivalence among weighted AM-GM inequality, weighted power-mean inequality and Hölder inequality is archived.

1. Introduction

In field of classical analysis, the weighted AM-GM inequality (see e.g. [2, pp. 74-75, Theorem 7.6]) is often inferred from the Jensen inequality, which is a more generalized inequality than AM-GM inequality, refer to [1–3] and references therein. In addition, the well-known Hölder inequality [3], found by Rogers (1888) and discovered independently by Otto Hölder (1889), is a basic inequality between integrals and an indispensable tool for the study of $L^p$ space, and is a extension form of Cauchy-Bunyakovsky-Schwarz inequality [4]. Hölder inequality is used to prove the Minkowski inequality, which is the triangle inequality (refer to [5–8]).

Weighted power means (also known as generalized means) $M_m^r(a)$ for a sequence $a = (a_1, a_2, \ldots, a_n)$ is defined as $M_m^r(a) = \left( m_1 a_1^r + m_2 a_2^r + \cdots + m_n a_n^r \right)^{\frac{1}{r}}$, which are a family of functions for aggregating sets of number, and plays a vital role in mathematical inequalities (see [3, 9, 10] for instance).

On the other hand, many researchers are interested in investigating the mathematical equivalence among some famous analytical inequalities, such as Cauchy-Schwarz inequality, Bernoulli inequality, Wielandt inequality, and Minkowski inequality; see [11–17] for the discussion of these issues. Motivated by these earlier mentioned pioneer work, in the present paper, the mathematical equivalence among these three well-known inequalities (i.e., Weighted AM-GM inequality, Hölder inequality, and Weighted power-mean inequality) is proved in detail, a generalization of a result in [18] is drawn.

The reminder of the present note is organized as follows. In the next section, we will give the detailed proofs of mathematical equivalence among three well-known mathematical inequalities. Finally, we end the paper with a few concluding remarks in Section 3.
2. MAIN RESULTS

Firstly, we will briefly review definitions of Weighted AM-GM inequality, Hölder inequality, and Weighted power-mean inequality. Then the main results of mathematical equivalence among these three well-known inequalities will be shown.

**Weighted AM-GM Inequality.** If \(c_1, c_2, \ldots, c_n\) are nonnegative real numbers, \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are nonnegative real numbers such that \(\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\), then
\[
\prod_{k=1}^{n} c_k^{\lambda_k} \leq \sum_{k=1}^{n} \lambda_k c_k.
\]

**Hölder Inequality.** If \(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\) are nonnegative real numbers, \(p\) and \(q\) are positive real numbers such that \(p^{-1} + q^{-1} = 1\), \(p > 1\), then
\[
\sum_{k=1}^{n} a_kb_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}.
\]

**Weighted Power-Mean Inequality.**[2 pp. 111-112, Theorem 10.5] If \(c_1, c_2, \ldots, c_n, \lambda_1, \lambda_2, \ldots, \lambda_n\) are nonnegative real numbers such that \(\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\), \(r\) and \(s\) are positive real numbers such that \(r \leq s\), then
\[
\left(\sum_{k=1}^{n} \lambda_k c_k^r\right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^{n} \lambda_k c_k^s\right)^{\frac{1}{s}}.
\]

The word “equivalence” between two statements \(A\) and \(B\), by convention, is understood as \(A\) implies \(B\) and \(B\) implies \(A\). Two true statements are equivalent. Thus the note just reveals a connection (in the sense of art) between these two well known facts.

**Theorem 1.** The Hölder inequality is equivalent to the Weighted AM-GM inequality.

**Proof.** To show \(2\) implies \(1\), we let \(a_k = (\lambda_k c_k)^{\frac{1}{p}}, b_k = (\lambda_k)^{\frac{1}{q}}\) in \(2\) for all \(k\), then
\[
\sum_{k=1}^{n} \left[(\lambda_k c_k)^{\frac{1}{p}}(\lambda_k)^{\frac{1}{q}}\right] \leq \left(\sum_{k=1}^{n} \lambda_k c_k^\frac{1}{p}\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \lambda_k^\frac{1}{q}\right)^{\frac{1}{q}}.
\]

Since \(p^{-1} + q^{-1} = 1\) and \(\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\), we obtain
\[
\sum_{k=1}^{n} \lambda_k c_k \geq \left(\sum_{k=1}^{n} \lambda_k c_k^\frac{1}{p}\right)^{p}.
\]

Now using inequality \(4\) successively, we obtain
\[
\sum_{k=1}^{n} \lambda_k c_k \geq \left(\sum_{k=1}^{n} \lambda_k c_k^\frac{1}{p}\right)^{p} \geq \left(\sum_{k=1}^{n} \lambda_k c_k^\frac{1}{p^2}\right)^{p^2} \geq \cdots \geq \left(\sum_{k=1}^{n} \lambda_k c_k^\frac{1}{p^m}\right)^{p^m} \geq \cdots.
\]
By L’Hospital’s rule, it is easy to see that
\[
\lim_{x \to 0^+} \ln \left( \sum_{k=1}^{n} \lambda_k c_k^x \right) / x = \sum_{k=1}^{n} \lambda_k \ln c_k.
\]
So
\[
\lim_{x \to 0^+} \left( \sum_{k=1}^{n} \lambda_k c_k^x \right)^{\frac{1}{x}} = \prod_{k=1}^{n} c_k^{\lambda_k}.
\]
Thus in (5), we can pass to the limit \( m \to +\infty \), giving
\[
\sum_{k=1}^{n} \lambda_k c_k \geq \prod_{k=1}^{n} c_k^{\lambda_k};
\]
hence (2) implies (1).

To show the converse, we only need a special case of (1),
\[
\lambda_1 c_1 + \lambda_2 c_2 \geq c_1^{\lambda_1} c_2^{\lambda_2}.
\]
Since \( p^{-1} + q^{-1} = 1 \), and by (6), we have
\[
\frac{1}{p} a_k^p \sum_{k=1}^{n} b_k^q + \frac{1}{q} b_k^q \sum_{k=1}^{n} a_k^p \geq a_k b_k \left( \sum_{k=1}^{n} b_k^q \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{q}}.
\]
For \( k = 1, 2, \ldots, n \). Summing over \( k \), we obtain
\[
\sum_{k=1}^{n} a_k^p \sum_{k=1}^{n} b_k^q \geq \sum_{k=1}^{n} a_k b_k \left( \sum_{k=1}^{n} b_k^q \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{q}}.
\]
Thus (2) follows. \( \square \)

**Theorem 2.** The Hölder inequality is equivalent to the Weighted power-mean inequality.

**Proof.** From Theorem 1 and according to inequality (2), we can obtain
\[
\sum_{k=1}^{n} \lambda_k x_k \geq \left( \sum_{k=1}^{n} \lambda_k x_k^p \right)^{\frac{1}{p}}.
\]
For \( r < s \), we can let \( p = \frac{s}{r} \) and \( x_k = c_k^r \) in (7) for all \( k \), so
\[
\sum_{k=1}^{n} \lambda_k c_k^r \geq \left( \sum_{k=1}^{n} \lambda_k c_k^s \right)^{\frac{1}{s}}.
\]
Thus, rewriting the above inequality (8) as the following.
\[
\left( \sum_{k=1}^{n} \lambda_k c_k^r \right)^{\frac{1}{r}} \leq \left( \sum_{k=1}^{n} \lambda_k c_k^s \right)^{\frac{1}{s}}.
\]
Now our task is to prove the converse, by inequality (3), we have
\[
\sum_{k=1}^{n} \lambda_k c_k \geq \left( \sum_{k=1}^{n} \lambda_k c_k^p \right)^{\frac{1}{p}} \geq \left( \sum_{k=1}^{n} \lambda_k c_k^q \right)^{\frac{1}{q}} \geq \cdots \geq \left( \sum_{k=1}^{n} \lambda_k c_k^r \right)^{\frac{1}{r}} \geq \cdots.
\]
By L’Hospital’s rule, it is easy to see that
\[ \sum_{k=1}^{n} \lambda_k c_k \geq \prod_{k=1}^{n} c_k^{\lambda_k}. \]

By using analogous methods from Theorem 1, we can prove
\[ \sum_{k=1}^{n} a_k^n \sum_{k=1}^{n} b_k^n \geq \sum_{k=1}^{n} a_k b_k \left( \sum_{k=1}^{n} b_k^n \right)^{\frac{1}{n}} \left( \sum_{k=1}^{n} a_k^n \right)^{\frac{1}{n}}. \]

**Theorem 3.** The Weighted power-mean inequality is equivalent to the Weighted AM-GM inequality.

**Proof.** To show (11) implies (3), we merely use a special case of (11),
\[ a_1^{\lambda_1} a_2^{\lambda_2} \leq \lambda_1 a_1 + \lambda_2 a_2. \]

Firstly, we denote \(U_n(a) = \lambda_1 a_1^n + \lambda_2 a_2^n + \cdots + \lambda_n a_n^n\), letting \(a_1 = \lambda_k a_k^n(U_n(a))^{-1}\), \(a_2 = \lambda_k\) and \(\lambda_1 = \frac{r}{s}, \lambda_2 = 1 - \frac{r}{s}\) in (11), then
\[ \lambda_k a_k^n(U_n(a))^{-\frac{1}{n}} \leq \frac{r}{s} \cdot \lambda_k a_k^n(U_n(a))^{-1} + \left(1 - \frac{r}{s}\right) \cdot \lambda_k. \]

For \(k = 1, 2, \ldots, n\). Summing over \(k\), we obtain
\[ \sum_{k=1}^{n} \lambda_k a_k^n(U_n(a))^{-\frac{1}{n}} \leq \sum_{k=1}^{n} \left[ \frac{r}{s} \cdot \lambda_k a_k^n(U_n(a))^{-1} + \left(1 - \frac{r}{s}\right) \cdot \lambda_k \right] = 1. \]

So
\[ \left( \sum_{k=1}^{n} \lambda_k c_k^r \right)^{\frac{1}{r}} \leq \left( \sum_{k=1}^{n} \lambda_k c_k^s \right)^{\frac{1}{s}}. \]

The converse is trivial from Theorem 2.

### 3. Concluding remarks

In this paper, the mathematical equivalence among Weighted AM-GM inequality, Hölder inequality, and Weighted power-mean inequality is investigated in detail. Meanwhile, we also generalize the interesting conclusion of Li’s paper [18]. At the end of the present study, it is remarked that the results on the equivalence of some well-known analytical inequalities can be summarized as follows,
- Equivalence of Hölder’s inequality and Minkowski inequality, see [13].
- Equivalence of Cauchy-Schwarz inequality and Hölder’s inequality, see [12].
- Equivalence of Cauchy-Schwarz inequality and Covariance-Variance inequality, see [11].
- Equivalence of Kantorovich inequality and Wielandt inequality, see e.g., [15].
- Equivalence of AM-GM inequality and Bernoulli inequality, see e.g., [14].
- Equivalence of Hölder inequality and Artin’s Theorem, refer to e.g., [16] pp. 657-663] for details.
- Equivalence of Hölder inequality and Weighted AM-GM inequality, refer to Theorem 1.
- Equivalence of Hölder inequality and Weighted power-mean inequality, refer to Theorem 2.
• Equivalence of Weighted power-mean inequality and Weighted AM-GM inequality, refer to Theorem 3.

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