1-convex extensions of partially defined cooperative games and the average value

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Abstract Partially defined cooperative games are a generalisation of classical cooperative games in which the worth of some of the coalitions is not known. Therefore, they are one of the possible approaches to uncertainty in cooperative game theory. The main focus of this paper is the class of 1-convex cooperative games under this framework.

For incomplete cooperative games with minimal information, we present a compact description of the set of 1-convex extensions employing its extreme points and its extreme rays. Then we investigate generalisations of three solution concepts for complete games, namely the $\tau$-value, the Shapley value and the nucleolus. We consider two variants where we compute the centre of gravity of either extreme games or of a combination of extreme games and extreme rays. We show that all of the generalised values coincide for games with minimal information and we call this solution concept the average value. Further, we provide three different axiomatisations of the average value and outline a method to generalise several axiomatisations of the $\tau$-value and the Shapley value into an axiomatisation of the average value.

We also briefly mention a similar derivation for incomplete games with defined upper vector and indicate several open questions.

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1 Introduction

In the theory of (cooperative) games, uncertainty is a long studied and very important problem. The reasons are both practical and theoretical. Regarding applications, inaccuracy in data is relatively common in real-world situations. The sources of such inaccuracies can be for example the lack of knowledge on the behaviour of others, corrupted data, signal noise or events such as voting or auctions for which we do not know full information.

Regarding the theoretical aspects, there is clearly no single ideal way how to tackle such problems under every possible setting. This stemmed various approaches to uncertainty, resulting in models differing in complexity, applicability, and other qualitative criteria. To name a few such models, let us highlight fuzzy cooperative games \[6,19,20\], multi-choice games \[6\], cooperative interval games \[10,24\], fuzzy interval games \[18\], games under bubbly uncertainty \[25\], ellipsoidal games \[26\], and games based on grey numbers \[24\].

In the classical cooperative game theory, groups of players, called coalitions, know the precise reward for the cooperation of its members. In partially defined cooperative games, this is no longer true, since only some of the coalitions know their values, while the others do not. This model was first introduced by Willson \[37\] in 1993. He gave the basic notion of incomplete games and a generalised definition of the Shapley value for such games. After more than two decades, Inuiguchi and Masuya continued in this line of research \[23\]. Their main focus was on superadditivity of possible extensions of the underlying incomplete game. Their research focused on a class of incomplete games with minimal information. Subsequently, Masuya further extended the results in \[21,22\], where he discussed more general classes of incomplete games and concentrated his efforts on generalisations of the Shapley value. Apart from that, Yu \[38\] introduced a generalisation of incomplete games to games with coalition structures and studied the proportional Owen value (which is in some sense a generalisation of the Shapley value for these games). Unfortunately for the general public, the paper of Yu was published only in Chinese. Very recently, Bok, Černý, Hartman, and Hladík \[5\] initiated study of convexity and positivity of extensions of incomplete games.

2 Preliminaries

We shall now introduce fundamentals of the theory of convex sets, classical cooperative games, and partially defined cooperative games. We present only the necessary background needed for our study of 1-convexity in the framework of partially defined cooperative games. We invite the interested reader to consult the following comprehensive sources on the theory of cooperative games: \[6,11,14,26\]. For more on applications of cooperative games, see e.g. \[6,10,17\].
We denote a real closed interval from $a$ to $b$, $a \leq b$, by $[a, b]$. For an inequality $L(x) \leq R(x)$, where $L(x)$ is the left-hand side in variable $x \in \mathbb{R}^n$ and $R(x)$ is the right-hand side in variable $x \in \mathbb{R}^n$, we distinguish two cases. For $x^* \in \mathbb{R}^n$, the inequality is strict (at $x^*$) if $L(x^*) < R(x^*)$ and it is tight or binding at $x^*$ if $L(x^*) = R(x^*)$. For the sake of brevity, we write $\pm$ (or $\mp$) in one inequality instead of two inequalities with + and −, e.g. $x \pm y \leq \mp z$ instead of $x + y \leq -z$ and $x - y \leq z$. Notice the difference between $\pm$ and $\mp$.

2.1 Convex sets

In partially defined cooperative games, we study subsets of complete games, so called C-extensions. All of the classes of C-extensions studied to date form convex sets. In this section, we revise the theory of convex sets and introduce tools for elegant and compact descriptions of C-extensions. We state all the results as facts and refer the reader to the book by Soltan [31] with exhaustive analysis of convex sets.

**Definition 1** A set $K \subseteq \mathbb{R}^n$ is called convex provided $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $\lambda \in [0, 1]$.

The convex sets we study are of a special form as they are intersections of closed halfspaces. A closed halfspace is the set $H := \{x \in \mathbb{R}^n \mid ax \leq b\}$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane $S$ is the set $S := \{x \in \mathbb{R}^n \mid ax = b\}$.

**Definition 2** A set $P \subseteq \mathbb{R}^n$ is called polyhedron if it is an intersection of finitely many closed halfspaces, say $H_1, \ldots, H_r$: $P = H_1 \cap \cdots \cap H_r$. The sets $\emptyset$ and $\mathbb{R}^n$ are polyhedrons. A bounded convex polyhedron is called polytope.

We say that a hyperplane $S_i$ corresponding to $H_i$ is supporting the set $P$. An important example of polyhedrons are sets $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Faces are the intersections of the convex set and its supporting hyperplanes. Extreme faces $F$ satisfy that whenever $\lambda x + (1 - \lambda)y \in F$ for $\lambda \in [0, 1]$, then $x, y \in F$. Extreme points and extreme rays are extreme faces of a special importance as they fully characterise the polyhedrons.

**Definition 3** Let $K$ be a convex set. A point $x \in K$ is an extreme point (or vertex) of $K$ if there is no way to express $x$ as a convex combination $\lambda y + (1 - \lambda)z$ such that $y, z \in K$ and $0 \leq \lambda \leq 1$, except for taking $y = z = x$.

**Theorem 1** Let $P \subseteq \mathbb{R}^n$ be a convex polyhedron. A point $e \in P$ is an extreme point (or vertex) if and only if for every $x \in \mathbb{R}^n$:

$$(e + x) \in P \land (e - x) \in P \implies x = 0.$$

To define extreme rays, we use halflines. A closed halfline (from $x$ to $y$) is the set $\{(1 - \lambda)x + \lambda y \mid \lambda \geq 0\}$ and an open halfline (from $x$ to $y$) is the set $\{(1 - \lambda)x + \lambda y \mid \lambda > 0\}$. Closed and open halflines form together halflines. We call $x$ to be the endpoint of the halfline. Notice that halflines form a special case of halfspaces.
Definition 4 An **extreme ray** of a convex set $K$ is a halfline $e \subseteq K$ which is an extreme face of $K$.

We shall make use of an alternative definition of extreme rays arising from the connection of extreme rays of a polyhedron and extreme rays of a specific **polyhedral cone** (defined further in the text as *recession cone*).

Definition 5 A nonempty set $C \subseteq \mathbb{R}^n$ is a **cone** with apex $s \in \mathbb{R}^n$ provided $s + \lambda(x - s) \in C$ whenever $x \in C$ and $\lambda \geq 0$. A **convex cone** is a cone which is a convex set. Further, a **polyhedral cone** $C$ is a convex cone which can be expressed as $C = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$ for some $A \in \mathbb{R}^{n \times n}$.

Setting $\lambda = 0$ in the definition of the cone, we can observe that the apex $s$ is always a part of the cone. One can reformulate this definition, stating that a nonempty set $C \subseteq \mathbb{R}^n$ is a cone with apex $s$ if and only if every halfline from $s$ to $x$ lies in $C$ whenever $x \in C \setminus \{s\}$. Hence a cone with apex $s$ is either the singleton $\{s\}$ or a union of closed halflines with the common endpoint $s$.

We will now introduce a connection between polyhedral cone and its extreme rays.

Definition 6 Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $y \in P$. The **recession cone** of $P$ (at $y$) is the set $R := \{d \in \mathbb{R}^n \mid y + \lambda d \in P \text{ for all } \lambda \geq 0\}$.

From its definition, the recession cone consists of all directions along which we can move indefinitely from $y$ and still remain in $P$. Notice that $y + \lambda d \in P$ for all $\lambda \geq 0$ if and only if $A(y + \lambda d) \leq b$ for all $\lambda \geq 0$ and this holds if and only if $Ad \leq 0$. Thus $R$ does not depend on a specific vector $y$.

We call the extreme rays of $R$ associated with $P$ the **extreme rays** of $P$. Note that the recession cone allows us to associate extreme rays of polyhedrons with extreme rays of convex cones. Any characterisation of extreme rays of polyhedral cones (including the following one) can be therefore also applied to extreme rays of polyhedrons.

Theorem 2 A nonzero element $x$ of a polyhedral cone $C \subseteq \mathbb{R}^n$ is an extreme ray if and only if there are $n - 1$ linearly independent constraints binding at $x$.

The following theorem gives a full description of pointed polyhedron (i.e. unbounded convex set with at least one extreme point) based only on extreme points and extreme rays.

Theorem 3 Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a nonempty polyhedron with at least one extreme point. Let $x_1, \ldots, x_r$ be its extreme points and $y_1, \ldots, y_\ell$ be its extreme rays. Then

$$P = \left\{ \sum_{i=1}^{r} \alpha_i x_i + \sum_{j=1}^{\ell} \beta_j y_i \mid \forall i, j : \alpha_i \geq 0, \beta_j \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\}.$$
2.2 Classical cooperative games

This subsection introduces the definition of cooperative games and the definition of the class of 1-convex games. We proceed with an introduction of solution concepts, namely the $\tau$-value, the nucleolus and the Shapley value and we review their properties for 1-convex games.

2.2.1 Main definitions and notation

**Definition 7** A cooperative game is an ordered pair $(N, v)$ where $N$ is a finite set of players (in this text $\{1, 2, \ldots, n\}$) and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function of the cooperative game. We further assume that $v(\emptyset) = 0$.

We denote the set of $n$-person cooperative games by $\Gamma_n$. Subsets of $N$ are called coalitions and $N$ itself is called the grand coalition. We often write $v$ instead of $(N, v)$ whenever there is no confusion over what the player set is. We shall often associate the characteristic functions $v: 2^N \rightarrow \mathbb{R}$ with vectors $v \in \mathbb{R}^{2^n}$. This will be more convenient for viewing sets of cooperative games as (possibly convex) sets of points.

We note that the presented definition assumes transferable utility (shortly TU). Therefore, by a cooperative game or a game we mean in fact a cooperative TU game.

To avoid cumbersome notation, we use the following abbreviations. We often replace singleton set $\{i\}$ with just $i$. Analogously, we use $S \cup \{i\}$ instead of $S - i$ and $S \setminus i$ instead of $S \setminus \{i\}$. We use $\subseteq$ for the relation of “being a subset of” and $\subsetneq$ for the relation “being a proper subset of”. To denote the sizes of coalitions e.g. $N, S, T$, we often use $n, s, t$, respectively.

We note that incomplete games can also be viewed as partial functions defined on a power set and, indeed, such structures were also extensively studied [29,34,35] (although in these cases without highlighting the connection to the theory of partially defined cooperative games). We refer to the excellent book of Grabisch [15] which discusses in a great detail connections of various types of set functions to entirely different parts of mathematics, with cooperative game theory being one of them.

The definition of 1-convex games relies on the notion of utopia (or upper) vector $b^i \in \mathbb{R}^n$. It captures each player’s marginal contribution to the grand coalition, i.e. $b^i = v(N) - v(N \setminus i)$. When there is no ambiguity, we use $b$ instead of $b^i$. The value $b^i$ is considered to be the maximal rightful value that player $i$ can claim when $v(N)$ is distributed among players. If he claims more, it is more advantageous for the rest of the players to form a coalition without player $i$.

**Definition 8** A cooperative game $(N, v)$ is called 1-convex game, if for all coalitions $\emptyset \neq S \subseteq N$, the inequality

$$v(S) \leq v(N) - b(N \setminus S)$$

(1)
holds and also
\[ b(N) \geq v(N). \] (2)

The set of 1-convex \( n \)-person games is denoted by \( C^n_1 \).

From (1), \((N, v)\) is 1-convex if even after every player outside the coalition \( S \) gets paid his utopia value, there is still more left of the value of the grand coalition for players from \( S \) than if they decided to cooperate on their own. This condition challenges the players to remain in the grand coalition and try to find a compromise in the payoff distribution. Also, in (2), the utopia vector sums to a value at least as large as the value of the grand coalition \( N \). This was motivated by the idea that the study of possible distributions is not interesting if every player can obtain his maximal rightful (utopia) value.

An equivalent formulation of 1-convexity is in terms of the *gap function*, defined as \( g^v(S) := b(S) - v(S) \). It captures the gap between the utopia distribution for coalition \( S \) and a possible distribution of the profit of \( S \).

**Theorem 4** \( \Box \) A game \((N, v)\) is 1-convex if and only if \( 0 \leq g^v(S) \leq g^v(S) \) for all coalitions \( S \subseteq N \).

Intuitively, the grand coalition is closest to the utopia distribution among possible coalitions. We can also rewrite conditions (1) and (2) in terms of the characteristic function as follows. For \( \emptyset \neq S \subseteq N \),
\[ v(S) + (n - s - 1)v(N) \leq \sum_{i \in N \setminus S} v(N \setminus i), \] (3)

and
\[ (n - 1)v(N) \geq \sum_{i \in N} v(N \setminus i). \] (4)

### 2.2.2 Solution concepts

The main task of cooperative game theory is to distribute the payoff of the grand coalition \( v(N) \) between the players. To be able to work with individual payoffs more easily, *payoff vectors* are introduced. Those are vectors \( x \in \mathbb{R}^n \) where \( x_i \) represents the individual payoff of player \( i \).

The definition of payoff vector is quite general, therefore, a suitable subset of payoff vectors, so called *imputations* are defined. Those are payoff vectors \( x \in \mathbb{R}^n \) such that \( x \) is *efficient*, i.e. \( \sum_{i \in N} x_i = v(N) \) and *individually rational*, i.e. \( x_i \geq v(i) \) for all \( i \in N \). This means an imputation distributes the worth of the grand coalition \( N \) between the players and only those payoffs where each player is at least as better off as he would be on his own are considered.

To choose between payoff vectors, different *solution concepts* are defined.

**Definition 9** Let \( \mathcal{C} \subseteq \Gamma^n \) be a class of \( n \)-person cooperative games. Then a function \( f : \mathcal{C} \to 2^{\mathbb{R}^n} \) is a *solution concept* (on class \( \mathcal{C} \)).
If the image $f(v)$ of every cooperative game $v \in C$ is exactly one vector, we write $f : C \to \mathbb{R}^n$ and we say $f$ is a one-point solution concept. Otherwise, we say $f$ is a multi-point solution concept.

We shall consider a generalization of two (actually three) solution concepts: the $\tau$-value, the nucleolus, and the Shapley value. We now introduce these solution concepts, stating their properties and different characterisations, which will be used for our generalisations to incomplete games.

The $\tau$-value is a known solution concept for 1-convex games (actually defined for a more general class of quasi-balanced games) defined originally by Tijs in 1981 [32]. We will follow his definition where he defines the $\tau$-value as a compromise between the utopia vector $b^v$ and the minimal right vector $a^v$ that is defined as $a^v := b^v - \lambda^v$ where $\lambda^v := \min_{S \subseteq N, i \in S} g^v(S)$ is the so-called concession vector.

The class of quasi-balanced games $Q^n$ is defined as

$$Q^n := \{(N, v) \mid \forall i \in N : a_i^v \leq b_i^v \text{ and } a^v(N) \leq v(N) \leq b^v(N)\}.$$ 

It holds that $C^n_1 \subseteq Q^n$.

**Definition 10** Let $(N, v) \in Q^n$. Then the $\tau$-value $\tau(v)$ of game $(N, v)$ is defined as the unique convex combination of $a^v$ and $b^v$ such that $\sum_{i \in N} \tau(v)_i = v(N)$.

For class $C^n_1$, the $\tau$-value can be expressed by a simple formula depending on the utopia vector and the gap function. The formula can be interpreted as follows. Every player receives his utopia value minus an equal share of the loss represented by the gap $g(N)$.

**Theorem 5** [12] Let $(N, v)$ be a 1-convex game. Then the $\tau$-value can be expressed as

$$\tau_i(v) = b_i^v - \frac{g^v(N)}{n}.$$ 

There are two known axiomatic characterisations of the $\tau$-value.

**Theorem 6** [33] The $\tau$-value is a unique function $f : Q^n \to \mathbb{R}^n$ satisfying

1. (efficiency) $\sum_{i \in N} f_i(v) = v(N)$,
2. (minimal right property) $f(v) = a^v + f(v - a^v)$, where $v - a^v(S) = v(S) - \sum_{i \in S} a_i^v$,
3. (restricted proportionality property) $f(v_\alpha) = \alpha b^{v_\alpha}$ where $\alpha \in \mathbb{R}$ and $(N, v_\alpha)$ is the zero-normalisation of $(N, v)$.

The second, axiomatic characterisation can be found in paper of Tijs [31]. It consists of five axioms, namely efficiency, translation equivalence, bounded aspirations, convexity, and restricted linearity.

On top of that, there are further results concerning axioms of the $\tau$-value, thus providing an even better comparison with other solution concepts. In the next theorem, we state several of them.
Theorem 7 For a 1-convex game \((N, v)\), the \(\tau\)-value \(\tau(v)\) satisfy

1. (individual rationality) \(\forall i \in N : \tau_i(v) \geq v(i)\),
2. (efficiency) \(\sum_{i \in N} \tau_i(v) = v(N)\),
3. (symmetry) For each permutation \(\pi : N \to N\) we have \(\tau(\pi \ast v) = \pi \ast (\tau(v))\),
4. (dummy player) \(i \in N, \forall S \subseteq N : v(S \cup i) = v(S) \implies \tau_i(v) = 0\),
5. (S-equivalence property) \(k \in [0, \infty], c \in \mathbb{R} : \tau(k \cdot v + c) = k \cdot \tau(v) + c\).

We note the \(\tau\)-value does not satisfy additivity which is crucial in our generalisation of this concept. Surprisingly, we show that our generalisation of the \(\tau\)-value satisfy a certain form of additivity on the class of incomplete games with minimal information.

The second solution concept is the nucleolus. Essential component of its definition is the excess \(e(S, x)\) which is a function dependent on a coalition \(S\) and an imputation \(x\) (a payoff vector which is both individually rational and efficient). It computes the remaining potential of \(S\) when the payoff is distributed according to \(x\), i.e. \(e(S, x) := v(S) - x(S)\). Further, \(\theta(x) \in \mathbb{R}^{|N|}\) is a vector of excesses with respect to \(x\) which is arranged in non-increasing order.

Definition 11 The nucleolus, \(\eta : \Gamma^n \to \mathbb{R}^n\), is the solution concept which assigns to a given game the minimal imputation \(x\) with respect to the lexicographical ordering \(\theta(x)\), defined as:

\[
\theta(x) < \theta(y) \text{ if } \exists k : \forall i < k : \theta_i(x) = \theta_i(y) \text{ and } \theta_k(x) < \theta_k(y).
\]

It is a basic result in cooperative game theory that the nucleolus is a one-point solution concept [26]. In general, the nucleolus can be computed by means of linear programming [16]. For 1-convex games, however, the notion of the nucleolus and the \(\tau\)-value coincide.

Theorem 8 Let \((N, v)\) be 1-convex game. Then \(\eta(v) = \tau(v)\).

In this text, we shall consider a generalisation of the \(\tau\)-value for \(C^n\)-extendable incomplete games, however, thanks to the theorem it can be also considered as a generalisation of the nucleolus.

Finally, we define the Shapley value.

Definition 12 The Shapley value is the unique function \(\phi : \Gamma^n \to \mathbb{R}^n\) such that

\[
\phi_i(v) := \sum_{S \subseteq N \setminus i \mid S} \frac{(|S| - 1)!(|n - |S|)!}{n!} (v(S) - v(S \setminus i)), \forall i \in N.
\]

There are alternative formulas for the Shapley value, including the one from the next theorem.

Theorem 9 The Shapley value for \((N, v)\) can be expressed as

\[
\phi_i(v) = \frac{1}{n} \sum_{S \subseteq N \setminus i} \binom{n - 1}{s}^{-1} (v(S \cup i) - v(S)).
\]
The Shapley value can be also characterised by means of axioms. The following is the characterisation proposed and proved by Shapley.

**Theorem 10** [30] The Shapley value is a unique function \( f : \Gamma^n \rightarrow \mathbb{R}^n \) with

1. (efficiency) \( \sum_{i \in N} f_i(v) = v(N) \),
2. (symmetry) \( i, j \in N, \forall S \subseteq N \setminus \{i, j\} : v(S + i) = v(S + j) \implies f_i(v) = f_j(v) \),
3. (null player) \( i \in N, \forall S \subseteq N : v(S) = v(S + i) \implies f_i(v) = 0 \),
4. (additivity) \( v, w \in \Gamma^n, v + w \in \Gamma^n : f(v + w) = f(v) + f(w) \).

Since the original introduction of the Shapley value, many alternative axiomatic characterisations of the Shapley value were given. Let us pin point the following few: [8,7,28,39]. As it would be an exhaustive task to investigate all of them at once, we considered only several of them (namely the second and the fourth mentioned). The Shapley value also satisfies all of the axioms from Theorem [7] except for individual rationality.

### 2.3 Partially defined cooperative games

**Definition 13** (*Incomplete game*) An incomplete game is a tuple \( (N, K, v) \) where \( N \) is a finite set of players (in this text \( \{1, \ldots, n\} \)), \( K \subseteq 2^N \) is the set of coalitions with known values and \( v : K \rightarrow \mathbb{R} \) is the characteristic function of the incomplete game. We further assume that \( \emptyset \in K \) and \( v(\emptyset) = 0 \).

We denote the set of \( n \)-person incomplete games with \( K \) by \( \Gamma^n(K) \). An incomplete game can be viewed from several perspectives. In one of the views, there is an underlying complete game \( (N, v) \) from a class of \( n \)-person games \( C \subseteq \Gamma^N \). The presence of \( (N, v) \) in \( C \) implies further properties of the characteristic function, e.g. superadditivity. Unfortunately, only partial information (captured by \( (N, K, v) \)) is known and there is no way to acquire more knowledge. The goal is then to reconstruct \( (N, v) \) as accurately as possible. This leads to the definition of \( C \)-extensions.

**Definition 14** Let \( C \) be a class of \( n \)-person games. A cooperative game \( (N, w) \in C \) is a \( C \)-extension of an incomplete game \( (N, K, v) \) if \( w(S) = v(S) \) for every \( S \in K \).

The set of all \( C \)-extensions of an incomplete game \( (N, K, v) \) is denoted by \( C(v) \). We write \( C(v) \)-extension whenever we want to emphasize the game \( (N, K, v) \). Also, if there is a \( C(v) \)-extension, we say \( (N, K, v) \) is \( C \)-extendable. Finally, the set of all \( C \)-extendable incomplete games with fixed \( K \) is denoted by \( C(K) \). In this text, we are mainly interested in \( C^n \)-extensions.

The sets of \( C \)-extensions studied in this text are always convex. One of the main goals of the model of partially defined cooperative games is to describe these sets using their extreme points and extreme rays whenever the description is possible. We refer to the extreme points as to **extreme games**.

If the structure of \( C(v) \) is too difficult to describe and it is bounded from either above or from below, we introduce the lower and the upper game.
Definition 15 (The lower game and the upper game of a set of \( \mathcal{C} \)-extensions)

Let \((N, \mathcal{K}, v)\) be a \( \mathcal{C} \)-extendable incomplete game. Then the lower game \((N, \underline{v})\) and the upper game \((N, \overline{v})\) are complete games such that for every \((N, w) \in \mathcal{C}(v)\) and every \(S \subseteq N\), it holds

\[
\underline{v}(S) \leq w(S) \leq \overline{v}(S),
\]

and for each \(S \subseteq N\), there are \((N, w_1), (N, w_2) \in \mathcal{C}(v)\) such that

\[
\underline{v}(S) = w_1(S) \text{ and } \overline{v}(S) = w_2(S).
\]

These games delimit the area of \(\mathbb{R}^{2|N|}\) that contains the set of \(\mathcal{C}\)-extensions. Even if we know the description of \(\mathcal{C}(v)\), the lower and the upper game are still useful as they encapsulate a range of possible profits \([\underline{v}(S), \overline{v}(S)]\) of coalition \(S\) across all possible \(\mathcal{C}\)-extensions.

In many situations in the cooperative game theory, full information on a cooperative game is not necessary for a satisfiable answer. For example, the \(\tau\)-value of a 1-convex cooperative game \((N, v)\) depends only on values \(v(N)\) and \(v(N \setminus i)\) for \(i \in N\). What if there are other satisfiable ways to distribute the payoff between players that can be computed only from partial information encoded by an incomplete game? Based on this question, we can generalise solution concepts to incomplete games.

Definition 16 Let \(\mathcal{C}(\mathcal{K})\) be a class of \(\mathcal{C}\)-extendable \(n\)-person incomplete games. Then function \(f: \mathcal{C}(\mathcal{K}) \rightarrow 2^{\mathbb{R}^n}\) is a solution concept (on class \(\mathcal{C}(\mathcal{K})\)).

If the image \(f(v)\) of every cooperative game \(v \in \mathcal{C}(\mathcal{K})\) is exactly one vector, we write \(f: \mathcal{C}(\mathcal{K}) \rightarrow \mathbb{R}^n\) and we say \(f\) is a one-point solution concept. Otherwise, we say \(f\) is a multi-point solution concept.

The model of partially defined cooperative games is still in its beginnings. One of the most significant downsides of classical cooperative games is the complexity of information required. For an \(n\)-person game, we have to consider \(2^n\) different coalitions with corresponding values of the characteristic function and to be able to apply the model, we often need all this information (the \(\tau\)-value of 1-convex games is rather an exception).

3 Incomplete games with minimal information

In this section, we focus on the subclass of incomplete games with minimal information and their \(\mathcal{C}\)-extensions. An incomplete game \((N, \mathcal{K}, v)\) is a game with minimal information if it contains (apart from \(\emptyset\)) only the grand coalition and singletons, i.e. \(\mathcal{K} = \{\{i\} \mid i \in N\} \cup \{\emptyset, N\}\). We also define the total excess as \(\Delta := v(N) - \sum_{i \in N} v(i)\) which will be widely used in this text.

In the first subsection, we derive a description of the set of \(\mathcal{C}\)-extensions. In the second subsection, we define different solution concepts and show they coincide on incomplete games with minimal information. Introduced as the average value, we investigate different axiomatisations of this solution concept in the third subsection.
3.1 Description of the set of $C^m_i$-extensions

The first step towards understanding the set of $C^m_i(v)$-extensions is to characterize when it is empty.

**Theorem 11** An incomplete game with minimal information $(N, K, v)$ is $C^m_i$-extendable if and only if $\Delta \geq 0$.

**Proof** Let $(N, w) \in C^m_i(v)$. Since it is 1-convex, it must hold for each $i \in N$, 

$$w(i) \leq w(N) - b(N \setminus i).$$

We sum the inequalities over all $n$ players to get

$$\sum_{i \in N} w(i) \leq n w(N) - \sum_{i \in N} b^w(N \setminus i).$$

We now expand expressions $b^w(N \setminus i)$ and rearrange the inequality into

$$\sum_{i \in N} w(i) + (n(n - 2)) w(N) \leq (n - 1) \sum_{i \in N} w(N \setminus j). \quad (5)$$

Since $b^w(N) \geq w(N)$ is equivalent to $\sum_{i \in N} w(N \setminus i) \leq (n - 1) w(N)$, we bound the right-hand side of (5) by $(n - 1)^2 w(N)$ and by rearranging, we conclude that $\Delta \geq 0$.

For the opposite direction, let us consider $C^m_i$-extensions $(N, v^i)$ for $i \in N$ defined as

$$v^i(S) := \begin{cases} v(S), & \text{if } S \in K, \\ v(N) - \sum_{j \not\in S} v(j), & \text{if } S \notin K \land i \in S, \\ v(N) - \sum_{j \not\in S} v(j) - \Delta, & \text{if } S \notin K \land i \notin S. \end{cases} \quad (6)$$

Notice that such games coincide on values of $S \in K$. We claim that for any $i \in N$, the game $v^i \in C^m_i(v)$. The condition $b^{v^i}(N) \geq v^i(N)$ holds since

$$b^{v^i}(N) = n v^i(N) - \sum_{j \notin N} v^i(N \setminus j) = n v(N) - nv(N) + \sum_{j \in N} v(j) + \Delta = \sum_{j \in N} v(j) + \Delta.$$ 

Furthermore, $v^i(N) = v(N) = \sum_{j \in N} v(j) + \Delta$ and hence the condition is clearly satisfied.

Now to verify the condition $v^i(S) \leq v^i(N) - b^{v^i}(N \setminus S)$ for each $S \subseteq N$, we distinguish two cases based on presence of $i$ in $S$.

For $i \in S$, $v^i(S) = v(N) - \sum v(j)$, which is equal to $v^i(N) - b^{v^i}(N \setminus S)$.

Therefore, the condition is satisfied and in fact, its upper bound is attained.

For $i \notin S$, $v^i(S) = v(N) = \sum v(j) - \Delta$ and $v^i(N) - b^{v^i}(N \setminus S) = v(N) - \sum v(j) - \Delta$. Again, the condition holds and the upper bound is attained. □
We note that if $\Delta = 0$, the set of $C^m_1$-extensions is rather simple and consists only of $(N, \pi)$ (the upper game defined in Theorem 12). Therefore, we are naturally more interested in situations when $\Delta > 0$.

The set of $C^m_1$-extensions is not bounded from below. For a $C^m_1$-extendable incomplete game $(N, K, v)$ and its $C^1_1(v)$-extension $(N, w)$, we can construct yet another $C^m_1(v)$-extension $(N, w_S)$ dependent on a coalition $S \subseteq N$ such that $1 < |S| < n - 1$. We set the characteristic functions of the two games to differ only in values of $S$, so that $w_S(S) < w(S)$. The 1-convexity of $(N, w_S)$ is easy to check from 1-convexity of $(N, w)$ and it can be immediately seen that any arbitrarily small number $\varepsilon$ satisfying $\varepsilon < v(S)$ could be chosen for the worth of coalition $S$ in $(N, w_S)$. While not bounded from below, the set of $C^m_1$-extensions is bounded from above.

**Theorem 12** Let $(N, K, v)$ be a $C^m_1$-extendable game with minimal information. Then the upper game $(N, \pi)$ has the following form:

$$\pi(S) := \begin{cases} v(S), & \text{if } S \in K, \\ v(N) - \sum_{i \notin S} v(i), & \text{if } S \notin K. \end{cases}$$

**Proof** To show that this is an upper bound for the value of each coalition $T \subseteq N$, we formulate the following optimization problem:

$$\begin{align*}
\max_{(N, w) \in C^1_1(v)} & \quad w(T) \\
\text{s.t.} & \quad w(N) \leq b^w(N), \\
& \quad w(S) \leq w(N) - b^w(N \setminus S) \quad \text{for } S \subseteq N, S \neq \emptyset.
\end{align*}$$

Clearly, the optimal value of the optimization problem (if it exists) is the value $\pi(T)$. Also notice that from the condition for $T$, i.e. $w(T) \leq w(N) - b^w(N \setminus T)$, that the upper bound of $w(T)$ is dependent only on value $w(N)$ (which is a constant since $N \in K$) and $n$ values $w(N \setminus i)$ for $i \in N$ (which are variables). The sum of these variables is bounded from above by $(n - 1)v(N)$ (since $w(N) \leq b^w(N) \iff \sum_{i \in N} v(N \setminus i) \leq (n - 1)v(N)$). From below, we have to consider only conditions $w(i) \leq w(N) - b^w(N \setminus i)$, because for $S \notin K$, we can always choose a $C^m_1$-extension such that the value $w(S)$ is small enough to satisfy $w(S) \leq w(N) - b^w(N \setminus S)$.

Therefore, we can simplify the optimization problem by:

1. removing conditions for $S \notin K$,
2. removing variables $w(S)$ for $S \notin K$, and
3. substituting objective function $w(T)$ for $w(N) - b^w(N \setminus T)$.

By these simplifications, we get the following optimization problem:

$$\begin{align*}
\max & \quad w(N) - b^w(N \setminus T) \\
\text{s.t.} & \quad w(i) \leq w(N) - b^w(N \setminus i) \\
& \quad i = 1, \ldots, n.
\end{align*}$$

(8)
The set of feasible solutions is now \( w \in \mathbb{R}^n \) where \( w_i = w(N \setminus i) \) and \( w(N) \) together with \( w(i) \) (for \( i \in N \)) are constants. A feasible solution \( w \in \mathbb{R}^n \) of problem (8) is equivalent to a feasible solution of problem (7) by setting \( w(S) := -(n - s - 1)w(N) + \sum_{k \in N \setminus S} w_k = w(N) - b^w(N \setminus S) \). Notice that the optimal values for both problems with corresponding feasible solutions equal.

We restate the problem in terms of the characteristic function \( w \) and we substitute \( w(N \setminus i) \) for \( w_i \), arriving at

\[
\max_{w \in \mathbb{R}^n} \sum_{i \in N \setminus S} w_i - (n - s - 1)w(N)
\]

s.t.

\[
\sum_{j \in N} w_j \leq (n - 1)w(N)
\]

\[
w(k) \leq \sum_{j \neq i} w_j - (n - 2)w(N)
\]

\[k = 1, \ldots, n.\]

Problem (9) is an instance of linear programming. Therefore, we can construct its dual program:

\[
\min_{y \in \mathbb{R}^{n+1}} \sum_{i \in N} \left[ (-(n - 2)w(N) - w(i))y_i \right] + (n - 1)w(N)y_{n+1} - (n - s - 1)w(N)
\]

s.t.

\[
- \sum_{j \neq i} y_j + y_{n+1} = 1 \text{ for } i \notin T
\]

\[
- \sum_{j \neq i} y_j + y_{n+1} = 0 \text{ for } i \in T
\]

\[y_i \geq 0 \text{ for all } i = 1, \ldots, n + 1.\]

Let us define the vector \( y^* \in \mathbb{R}^{n+1} \) as

\[
y^*_j = \begin{cases} 
0, & \text{if } j \in T, \\
1, & \text{if } j \notin T, \\
n - t, & \text{if } j = n + 1.
\end{cases}
\]

We deduce that

\[- y^*_j \geq 0 \text{ for all } j \in N,\]

\[- \sum_{j \neq i} y^*_j + y^*_{n+1} = -(n - t - 1)1 + (n - t) = 1 \text{ for } i \notin T,\]

\[- \sum_{j \neq i} y^*_j + y^*_{n+1} = -(n - t)1 + (n - t) = 0 \text{ for } i \in T.\]

Hence \( y^* \) is a feasible solution of (10). Furthermore, the value of the objective function for \( y^* \) equals \( w(N) - \sum_{i \notin T} w(i) + v(N) - \sum_{i \notin T} v(i) \). This means (from the duality of linear programming) that the primal program is feasible and the value of its objective function is bounded from above by this value.

To see that this upper bound is attained, take the game \((N, v^i)\) (from the proof of Theorem 11) such that \( i \notin T \). \(\square\)
It is important (and by our opinion interesting) that the upper game of the set of superadditive extensions of non-negative incomplete games with minimal information coincides with the upper game of $C_n^1$-extensions from Theorem 12.

The upper game $\pi$ is not 1-convex in general. For example, a 3-person incomplete game $(N, K, v)$ with minimal information with $v(N) = 1$ and $v(i) = 0$ for all $i \in N$ is $C_n^1$-extendable since $\Delta = 1$. However, $1 = \pi(N) \nleq b^\pi(N) = 0$. From the condition $v(N) \leq b^\pi(N)$, we can derive that $v(N) \leq \sum_{i \in N} v(i)$, and hence $\Delta = 0$. For $\emptyset \neq S \subset N$ and conditions $\pi(S) \leq \pi(N) - b^\pi(N \setminus S)$, we can easily derive from the definition of the upper game $(N, \pi)$ that

$$v(N) \leq \min_{\emptyset \neq S \subseteq N} \left\{ \frac{2}{n - s} \sum_{i \in N \setminus S} v(i) \right\}.$$

**Theorem 13** Let $(N, K, v)$ be an incomplete game with minimal information. Then it holds that the upper game $(N, \pi) \in C_n^1(v)$ if and only if $\Delta = 0$ and $v(N) \leq \min_{\emptyset \neq S \subseteq N} \left\{ \frac{2}{n - s} \sum_{i \in N \setminus S} v(i) \right\}$.

So far, we showed that the set $C_n^1(v)$ is a convex polyhedron since it can be described by a set of inequalities. It is bounded from above by $(N, \pi)$ and unbounded from below. Such polyhedrons (provided that they have at least one vertex) can be characterised by a set of extreme points and a cone of extreme rays (see Theorem 3).

We begin the derivation of the full description of the set of $C_n^1$-extensions by proving that games $(N, v^i)$ (defined as $(6)$) are actually extreme points of the set. To prove this, we use a characterisation of extreme points from Theorem 1.

**Theorem 14** For $C_n^1$-extendable game $(N, K, v)$ with minimal information, the games $(N, v^i)$ are extreme games of $C_n^1(v)$.

**Proof** Let $x \in \mathbb{R}^{2^{|N|}}$ be a vector such that both $(N, v^i \pm x) \in C_n^1(v)$. We will show that in such case, inevitably $x(S) = 0$ for all $S \subseteq N$, thus by Theorem 1 $(N, v^i)$ is an extreme game.

Define $f^+ := v^i + x$ and $f^- := v^i - x$. For $S \in K$, clearly $x(S) = 0$. It remains to show that for $S \notin K$, $x(S) = 0$.

For $S \notin K$ and $i \notin S$, for the sake of contradiction, suppose w.l.o.g. that $x(S) > 0$. Then $f^+(S) = v^i(S) + x(S) = \pi(S) + x(S) > \pi(S)$, therefore $(N, f^+) \notin C_n^1(v)$, a contradiction.

For $S \notin K$ and $i \in S$, again, suppose $x(S) = \delta > 0$. Because $(N, f^+)$ and $(N, f^-)$ are both 1-convex, conditions

$$f^+(S) + (n - s - 1)f^+(N) \leq \sum_{j \notin S} f^+(N \setminus j)$$
and 

$$f^-(S) + (n - s - 1)f^-(N) \leq \sum_{j \in S} f^-(N \setminus j)$$

must hold. We can rewrite both of the inequalities (and aggregate them by \( \pm \)) as

$$v'(S) + (n - s - 1)v(N) \pm x(S) \pm (n - s - 1)x(N) \leq \sum_{j \in S} v'(N \setminus j) \pm \sum_{j \in S} x(N \setminus j),$$

which is equivalent to

$$v(N) - \sum_{j \in S} v(j) - c + (n - s - 1)v(N) \pm x(S) \leq (n - s)v(N) - \sum_{j \in S} v(j) - c \pm \sum_{j \in S} x(N \setminus j)$$

or

$$\pm x(S) \leq \pm \sum_{j \in S} x(N \setminus j).$$

Since both inequalities hold, we conclude \( x(S) = \sum_{j \in S} x(N \setminus j) \). We already derived that \( x(N \setminus j) = 0 \) if \( i \in N \setminus j \) if and only if \( j \neq i \). But since \( i \notin S \) we conclude \( 0 < \delta = x(S) = \sum_{j \in S} x(N \setminus j) = 0 \), which is a contradiction.

We proved that \( x \) is necessarily a vector of zeroes and thus we conclude the proof by taking Theorem [I] into account. \( \square \)

Not only are games \((N, v')\) for \( i \in N \) the extreme games of \( C^\delta_1(v) \), they are also the only extreme games.

**Theorem 15** For a \( C^\delta_1 \)-extendable game \((N, K, v)\) with minimal information, the games \((N, v')\) are the only extreme games of \( C^\delta_1(v) \).

**Proof** We will prove this theorem by showing that any extreme game \((N, e)\) has the form of one of the \((N, v')\) games. Since there are \( n \) different games, we have to enforce that the game coincides with \((N, v')\) for a specific \( i \).

To do so, realise there is \( i \) such that \( e(N \setminus i) < \pi(N \setminus i) \). If there was no such \( i \), then \( \forall k : c(N \setminus k) \geq \pi(N \setminus k) \). The sum of these conditions leads to

$$\sum_{k \in N} e(N \setminus k) > \sum_{k \in N} \pi(N \setminus k) = n\pi(N) - \sum_{k \in N} v(k) = (n - 1)\pi(N) + \Delta \geq (n - 1)\pi(N).$$

But this is a contradiction, because the opposite inequality holds. Now we proceed to prove that \( e = v' \).

First, we show \( i \) is the unique coalition of size \( n - 1 \) with its coalition value \( e(N \setminus i) \) different from \( \pi(N \setminus i) \), i.e. there is no \( j \neq i \) such that \( e(N \setminus j) < \pi(N \setminus j) \).

For a contradiction, if there is such \( j \), denote \( \varepsilon_i = \pi(N \setminus i) - e(N \setminus i), \varepsilon_j = \pi(N \setminus j) - e(N \setminus j) \) and \( \varepsilon = \min\{\varepsilon_i, \varepsilon_j\} \). We define a non-trivial game \((N, x)\) such that both \((N, e + x) \in C^\delta_1(v)\) and \((N, e - x) \in C^\delta_1(v)\), contradicting (by Theorem [I]) that \((N, e)\) is an extreme game. The game \((N, x)\) can be described as

$$x(S) = \begin{cases} 
\varepsilon, & \text{if } S = N \setminus i \text{ or } S \notin K \land i \notin S \land j \in S, \\
-\varepsilon, & \text{if } S = N \setminus j \text{ or } S \notin K \land i \in S \land j \notin S, \\
0, & \text{otherwise}. 
\end{cases}$$
The condition (1) from Definition 8 for both \((N, e + x)\) and \((N, e - x)\) now reads as

\[
\sum_{k \in N} e(N \setminus k) \pm x(N \setminus i) \pm x(N \setminus i) \leq (n - 1)e(N)
\]

or equivalently

\[
\sum_{k \in N} e(N \setminus k) \pm \varepsilon \mp \varepsilon \leq (n - 1)e(N)
\]

is equivalent to the respective condition of \((N, e)\). Furthermore, for any nonempty coalition \(S\) such that \(i \notin S\) and \(j \in S\), the condition (2) from Definition 8 for both games is

\[
e(S) \pm x(S) - (n - s - 1)e(N) \leq \sum_{k \in N \setminus S} v(N \setminus k) \pm x(N \setminus i).
\]

Since \(x(S) = x(N \setminus i)\), it is equivalent to the respective condition of \((N, e)\).

The rest of the cases for non-empty coalition \(S\) can be dealt with in a similar manner, therefore both games \((N, e \pm x) \in C^0_t\). But this leads to a contradiction, because \((N, e)\) is an extreme game.

Second, we show that \(e(N \setminus i) = v(N) - v(i) - \Delta\). Clearly, for \(\alpha > 0\),

\[
(n - 1)e(N) = \sum_{k \in N} e(N \setminus k) = \alpha v(N) - \sum_{k \in N} v(k) - \alpha.
\]

We note the first equality holds, otherwise there is a game \((N, x)\) such that \(x(N \setminus j) \coloneqq \beta\) and for \(S, i \notin S: x(S) \coloneqq -\beta\) that leads to a contradiction with \((N, e)\) being an extreme game. Therefore \(\alpha = v(N) - \sum_{k \in N} v(k) = \Delta\).

Finally, it is elementary to prove that \(e(S) = \nu(N) - b(N \setminus S) = v'(S)\). If this was not true, yet another game \((N, x)\) with \(x(S) = e(N) - b(N \setminus S)\) and \(x(T) = 0\) would lead to a contradiction with extremality of \((N, e)\). \(\Box\)

Now let us proceed with the investigation of the extreme rays. For the game \((N, v + \lambda e)\) to be 1-convex (thus being in the recession cone of \(C^0_t(v)\)), the following conditions must hold for every nonempty \(S \subseteq N\):

\[
- b^v(N) + b^\lambda e(N) \geq v'(N) + \lambda e(N),
- v'(S) + \lambda e(S) \leq v'(N) + \lambda e(N) - b^v(N \setminus S) - b^\lambda e(N \setminus S).
\]

By Theorem 111 we have that

\[
- b^v(N) = v'(N),
- v'(S) = v'(N) - b^v(N \setminus S).
\]

We can therefore simplify the conditions, arriving at

\[
- b^\lambda e(N) \geq \lambda e(N),
- \lambda e(S) \leq \lambda e(N) - b^\lambda e(N \setminus S).
\]

Furthermore, we can factor out \(\lambda\) since it is non-negative. Notice that for each \(j \in N, e(j) = e(N) = 0\), otherwise \((N, v^i + \lambda e) \not\in C^0_t(v)\). Taking all this into consideration, we obtain the following conditions for \((N, e)\), representing an unbounded direction in \(C^0_t\):

\[
- b^\lambda e(N) \geq \lambda e(N),
- \lambda e(S) \leq \lambda e(N) - b^\lambda e(N \setminus S).
\]
1. \( b'(N) \geq e(N) \iff \sum_{j \in N} e(N \setminus j) \leq 0 \),
2. \( \forall S \subseteq N, S \neq 0 : e(S) \leq e(N) - b'(N \setminus S) \iff e(S) \leq \sum_{j \in N \setminus S} e(N \setminus j) \),
3. \( \forall k \in N : 0 \leq \sum_{j \in N \setminus k} e(N \setminus k) \),
4. \( \forall j \in N : e(j) = 0 \),
5. \( e(N) = 0 \).

Conditions 1 and 2 show that \( (N, e) \) itself has to be a 1-convex game. Moreover, if it is 1-convex, for any \( \lambda \geq 0 \), the game \( (N, \lambda e) \) is also 1-convex. Therefore, the game \( (N, e) \) is (not necessarily an extreme) ray of the recession cone of the set of \( C_1^n \)-extensions. It is a zero-normalised game with \( e(N) = 0 \) (thanks to conditions 4 and 5). Observe that condition 3 is a special case of condition 2 (take \( S = \{k\} \) for \( k \in N \)). We state it separately, since it will come in handy to refer just to this special case in further text. Notice an interesting fact: values of the game \( (N, e) \) do not depend on the value of \( (N, K, v) \). Therefore, the recession cone is the same for every incomplete game with minimal information.

Further, to simplify conditions 1 to 5, suppose that there is a \( C_1^n \)-extension \( (N, v^i + e) \) such that \( \sum_{j \in N} e(N \setminus j) < 0 \). Then there is \( k \in N \) such that \( \sum_{j \in N \setminus k} e(N \setminus j) < 0 \). But this is a contradiction with \( 0 \leq \sum_{j \in N \setminus k} e(N \setminus j) \).

Further, suppose that \( \sum_{j \in N} e(N \setminus j) = 0 \) and there is \( k \) such that \( e(N \setminus k) \neq 0 \). If \( e(N \setminus k) > 0 \), then \( e(N \setminus k) = -\sum_{j \in N \setminus k} e(N \setminus j) > 0 \), which is a contradiction because both \( 0 \leq \sum_{j \in N \setminus k} e(N \setminus j) \) and \( \sum_{j \in N \setminus k} e(N \setminus j) < 0 \). If \( e(N \setminus k) < 0 \), there is \( \ell \in N \) such that \( e(N \setminus \ell) > 0 \) and we arrive into a similar contradiction. Hence, it must hold for every \( i \in N \), that \( e(N \setminus i) = 0 \). We can now rewrite the conditions as

1. \( \forall S \subseteq N, S \neq 0 : e(S) \leq 0 \),
2. \( \forall i \in N : e(i) = e(N \setminus i) = e(N) = 0 \).

Let us now select the extreme rays. From Definition 4 all but one of conditions 1 or 2 have to be satisfied with equality for the game \( (N, v^i + e) \) to be an extreme ray. We see that the extreme rays are given by 1-convex games \( (N, e_T) \) for coalitions \( T \in E = 2^N \setminus \{0, N\} \cup \{N \setminus i \mid i \in N\} \cup \{\{i\} \mid i \in N\} \), where

\[
e_T(S) := \begin{cases} -1, & \text{if } S = T, \\ 0, & \text{if } S \neq T. \end{cases}
\] (11)

With such knowledge, we are ready to fully describe the set of \( C_1^n \)-extensions of games with minimal information.

**Theorem 16** For a \( C_1^n \)-extendable game \( (N, K, v) \) with minimal information, the set of \( C_1^n \)-extensions can be described as

\[
C_1^n(v) = \left\{ \sum_{i \in N} \alpha_i v^i + \sum_{T \in E} \beta_T e_T \mid \sum_{i \in N} \alpha_i v^i = 1, \alpha_i, \beta_T \geq 0 \right\}.
\]

**Proof** We have already proved that games \( (N, v^i) \) for \( i \in N \) from \( 9 \) are the extreme games of \( C_1^n(v) \) and games \( (N, e_T) \) for \( T \in E \) from \( 11 \) are the extreme rays of \( C_1^n(v) \). The rest of the proof follows from Theorem 8. \( \Box \)
3.2 Solution concepts

In this subsection, we present generalisations of the $\tau$-value and the Shapley value based on two ideas. The first idea is to consider solely the vertices of the set of $C^n_1$-extensions, compute their centre of gravity and for the resulting game, compute its $\tau$-value (see Definition 17) or its Shapley value (Definition 20). The second idea considers also the recession cone, which is completely neglected in the first approach (Definitions 19, 21). We show that from the symmetry of recession cone, both approaches for both generalisations of the $\tau$-value and the Shapley value lead to the same solution concept for games with minimal information. We call it the average value.

3.2.1 The average $\tau$-value

The first solution concept considers the centre of gravity of the extreme games, that is

$$\tilde{v} = \sum_{i \in N} \frac{v^i}{N}.$$

Note that $(N, \tilde{v})$ is 1-convex if $(N, v^i)$ is 1-convex for every $i \in N$. Since additivity does not hold for the $\tau$-value in general, $\tau(\tilde{v}) \neq \sum_{i \in N} \frac{\tau(v^i)}{N}$ in general. We consider both variants in the next definition.

**Definition 17** Let the games $(N, \tilde{v})$ and $(N, v^i)$ for $i \in N$ be the centre of gravity and the extreme games of $C^n_1(v)$, respectively.

1. The average $\tau$-value $\tilde{\tau}: C^n_1(K_{\min}) \to \mathbb{R}^n$ is defined as
   $$\tilde{\tau}(v) := \tau(\tilde{v}).$$

2. The solidarity $\tau$-value $\tau^s: C^n_1(K_{\min}) \to \mathbb{R}^n$ is defined as
   $$\tau^s(v) := \sum_{i \in N} \frac{\tau(v^i)}{n}.$$

The justification for the name of the solidarity $\tau$-value is given in the following theorem.

**Theorem 17** The average $\tau$-value $\tilde{\tau}: C^n_1(K_{\min}) \to \mathbb{R}^n$ and the solidarity $\tau$-value $\tau^s: C^n_1(K_{\min}) \to \mathbb{R}^n$ can be expressed as follows:

1. $\forall i \in N : \tilde{\tau}_i(v) = v(j) + \frac{\Delta}{n}$,
2. $\forall i \in N : \tau^s_i(v) = \frac{v(N)}{n}.$

**Proof** Both expressions can be easily derived from the definition of $\tilde{\tau}$, $\tau^s$ and extreme games $(N, v^i)$. First of all, the game $(N, \tilde{v})$ can be expressed as

$$\tilde{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N) - \left( \sum_{j \in N \setminus S} v(j) \right) - \frac{n-S}{n} \Delta, & \text{if } S \notin \mathcal{K}. \end{cases}$$
The values of its utopia vector are \( \tilde{v}_j = v(j) + \frac{n-1}{n} \Delta \), and by summing them together over all players in \( N \), we arrive at \( \tilde{v}(N) = \sum_{j \in N} v(j) + (n-1) \Delta = v(N) + (n-2) \Delta \). The gap function for \( N \) is \( g(\tilde{v}(N)) = n \Delta \), and finally (by Theorem 5), from the definition of \( \tau(\tilde{v}) \), using \( \tau(\tilde{v}) = \tau(\tilde{v}) \), we get

\[
\tau_j(\tilde{v}) = b_j^\tilde{v} - g(\tilde{v}(N)) = v(j) + \frac{n-1}{n} \Delta - \frac{n-2}{n} \Delta = v(j) + \frac{\Delta}{n}.
\]

The main reason behind the formula for the solidarity \( \tau \)-value is the fact that \( \tau(v^i) = b^{v^i} \), which immediately follows from \( g^{v^i}(N) = 0 \), and from the form of \( b^{v^i} \), which is

\[
b^{v^i}_j = \begin{cases} v(j), & \text{if } j = i, \\ v(j) + \Delta, & \text{if } j \neq i. \end{cases}
\]

By summing the values of vector \( b^{v^i} \), we get \( b^{v^i}(N) = \sum_{j \in N} v(j) + \Delta = v(N) \). Therefore, \( g^{v^i}(N) = v(N) - b^{v^i}(N) = 0 \). This implies the following equation:

\[
\tau_j^s = \frac{1}{n} \sum_{i \in N} \tau_j(v^i) = \frac{1}{n} \sum_{i \in N} b^{v^i}_j = \frac{1}{n} \sum_{i \in N} b^{v^i}_j = \frac{1}{n} \sum_{i \in N} \left( v(i) + \Delta \right) = v(N).
\]

Finally, we can rewrite \( \sum_{i \in N} b^{v^i}_j \):

\[
\sum_{i \in N} b^{v^i}_j = \sum_{i \in N} v(i) + \Delta = v(N).
\]

Using these equations, we arrive at the formula stated above. \( \Box \)

We immediately see that the solidarity \( \tau \)-value is not a very reasonable solution concept if we consider that under such value, every player should get an equal share of \( \frac{v(N)}{n} \) no matter his contribution.

### 3.2.2 The conic \( \tau \)-value

It might seem that the main downside of the previous two solution concepts is that we do not consider the recession cone of the set of \( C^n_1 \)-extensions. Here we provide an argument showing that with no further assumptions, this is not the case for games with minimal information. We define a solution concept dependent on the recession cone, which will serve as a foundation for the study of incomplete games with more general sets \( K \).

Let \( (N, v^i) \) for \( i \in N \) be the extreme games of \( C^n_1(v) \) and \( (N, e_T) \) for \( T \in E \) be the extreme rays of \( C^n_1(v) \). \( (N, \tilde{e}) \) denotes again the centre of gravity of extreme games and \( (N, \tilde{e}) \) the centre of gravity of extreme rays, i.e.

\[
\tilde{e} = \sum_{T \in E} \frac{e_T}{|E|} = \sum_{T \in E} \frac{e_T}{2n^2 - 2n - 2}.
\]

The conic \( \tau \)-value \( \tau^{\Delta} \), introduced in the following definition, is computed on the sum of these games. It considers the extreme games as well as extreme rays, thus the information from the shape of the conic cone is also included.
Definition 18 The conic $\tau$-value $\tau^{<}: C^p_1(K_{\min}) \rightarrow \mathbb{R}^n$ is defined as
\[
\tau^{<}(v) := \tau(\tilde{v} + \tilde{e}),
\]
where the games $(N, \tilde{v})$ and $(N, \tilde{e})$ are the centres of gravity of extreme points and of extreme rays of $C^p_1(v)$.

Surprisingly, the average $\tau$-value and conic $\tau$-value are the same function for incomplete games with minimal information. The reason is hidden in the symmetry of $(N, \tilde{e})$.

Theorem 18 The conic $\tau$-value $\tau^{<}: C^p_1(K_{\min}) \rightarrow \mathbb{R}^n$ can be expressed as follows:
\[
\forall i \in N : \tau^{<}_i(v) = v(i) + \frac{\Delta}{n},
\]
Proof The proof is a straightforward derivation from the definitions. First, we already know the description of $(N, \tilde{v})$ from the proof of Theorem 17. The description of $(N, \tilde{e})$ is
\[
\tilde{e}(S) = \begin{cases} 
\frac{1}{\varepsilon}, & \text{if } S \in \mathcal{K} \lor S = N \setminus j \text{ for } j \in N, \\
0, & \text{otherwise},
\end{cases}
\]
where $\varepsilon = 2^n - 2n - 2$. From this description we derive that $b^{\tilde{v} + \tilde{e}}_i = v(i) + \frac{n-1}{n} \Delta - \frac{1}{\varepsilon}$ as $b^{\tilde{v} + \tilde{e}}_i$ is equal to
\[
(\tilde{v} + \tilde{e})(N) - (\tilde{v} + \tilde{e})(N \setminus j) = v(N) - \tilde{v}(N \setminus i) - \tilde{e}(N \setminus i)
\]
and the right-hand side can be rewritten as $v(N) - (v(N) - v(i) - \frac{n-1}{n} \Delta) + \frac{1}{\varepsilon}$. Further, $b^{(\tilde{v} + \tilde{e})}(N) = \sum_{i \in N} v(i) + (n-1) \Delta + \frac{1}{\varepsilon} = v(N) + (n-2) \Delta + \frac{1}{\varepsilon}$. The last equality follows by using the fact that $\sum_{i \in N} v(i) + \Delta = v(N)$. The gap function
\[
g^{(\tilde{v} + \tilde{e})}(N) = b^{(\tilde{v} + \tilde{e})}(N) - v(N) = (n-2) \Delta + \frac{1}{\varepsilon}.
\]
Finally, by Theorem 5 $\tau^{<}_i(v) = b^{(\tilde{v} + \tilde{e})}_i - \frac{g^{(\tilde{v} + \tilde{e})}}{n} = v(i) + \frac{n-1}{n} \Delta + \frac{1}{\varepsilon} - \frac{(n-2) \Delta + \frac{1}{\varepsilon}}{n} = \frac{n-2}{n} \Delta + \frac{1}{\varepsilon}$, we have
\[
\tau^{<}_i(v) = v(i) + \frac{n-1}{n} \Delta + \frac{1}{\varepsilon} - \frac{n-2}{n} \Delta - \frac{1}{\varepsilon} = v(i) + \frac{\Delta}{n}.
\]
This completes the proof. \hfill \Box

As we already said, the reason for this (one might say surprising) result is the symmetry of $(N, \tilde{e})$. Actually, consider a more general setting in which take the expression
\[
\frac{1}{\gamma} \left( \beta \sum_{i \in N} v^i + \alpha \sum_{T \in E} v_T \right).
\]
(12)
This is a generalization of $\tilde{v} + \tilde{e}$ (as for $\beta = \frac{1}{\gamma}, \alpha = \frac{1}{\gamma},$ and $\gamma = 1$, we get $\tilde{v} + \tilde{e}$). Also, if $\beta \neq \gamma$, it can be shown that the game from (12) does not lie in $C^p_1(v)$. Fixing $\beta = \gamma$, the $\tau$-value of this expression is equal to $\tilde{v}$ for any $\alpha \in \mathbb{R}$.
On the other hand, if \((N, \tilde{e})\) would not be symmetric or would depend on values of \(v \in C_1^n(K_{\text{min}})\), the information about the cone might matter and it might be that \(\tau^<(v) \neq \tilde{\tau}(v)\). This motivates the following definition for games \((N, K, v)\) with a more general structure of \(K\).

**Definition 19** Let \(K \subseteq 2^N\) and suppose that \(\forall v \in C_1^n(K)\), the set \(C_1^n(v)\) is a polyhedron described by its extreme points and extreme rays. Then the \(\alpha\)-conic \(\tau\)-value \(\tau^\alpha: C_1^n(K) \to \mathbb{R}^n\) is defined as

\[
\tau^\alpha(v) := \tau(\tilde{v} + \alpha \tilde{e}),
\]

where \((N, \tilde{v}), (N, \tilde{e})\) are the centres of gravity of extreme points and of extreme rays of \(C_1^n(K)\), respectively.

As mentioned before, for games with minimal information, \(\tilde{\tau}(v) = \tau^\alpha(v)\). However, for more general sets \(K\), this definition might yield a different solution. This is supported by the investigation of incomplete games with defined upper vector in Section 4.2. In there, we show that a similar solution concept, the \(\alpha\)-conic Shapley value, does not coincide with the average Shapley value. The idea behind these solution concepts is the same as behind the average and the conic \(\tau\)-value.

Once more, the fact that additivity does not hold for the \(\tau\)-value in general leads to a question whether \(\tau(\tilde{v} + \tilde{e})\) and \(\tau(\tilde{v}) + \tau(\tilde{e})\) yield different functions. For games with minimal information, this is not the case since \(\tau(\tilde{e}) = 0\). Therefore, \(\tau(\tilde{v} + \tilde{e}) = \tau(\tilde{v}) = \tilde{\tau}(v)\).

### 3.2.3 The average Shapley value

The average Shapley value \(\tilde{\phi}\) was already studied by Masuya and Inuiguchi in [23] for the set of superadditive extensions of non-negative incomplete games with minimal information. We show that in the context of 1-convexity, the average Shapley value coincides with their definition, which is also equal to the average \(\tau\)-value. Yet again, the consideration of the recession cone (thanks to its symmetry) does not come to fruition.

**Definition 20** The average Shapley value \(\tilde{\phi}: C_1^n(K_{\text{min}}) \to \mathbb{R}^n\), is defined as

\[
\tilde{\phi}(v) := \phi(\tilde{v}),
\]

where \((N, \tilde{v})\) is the centre of gravity of extreme games of \(C_1^n(v)\).

**Theorem 19** The average Shapley value \(\tilde{\phi}: C_1^n(K_{\text{min}}) \to \mathbb{R}^n\) can be expressed as follows:

\[
\forall i \in N : \tilde{\phi}_i(v) = v(i) + \frac{\Delta}{n},
\]
Proof The proof is based on the characterisation of the Shapley value from Theorem 9 and the fact that for every \( S \subseteq N \setminus i \), \( \tilde{v}(S \cup i) - \tilde{v}(S) = v(i) + \frac{\Delta}{n} \). This holds as \( \tilde{v}(S \cup i) - \tilde{v}(S) \) is from the definition of \((N, \tilde{v})\) equal to

\[
v(N) - \sum_{j \in N \setminus (S \cup i)} v(j) - \frac{(n - (s + 1))}{n} \Delta - \left( v(N) - \sum_{j \in N \setminus S} v(j) - \frac{(n - s)}{n} \Delta \right),
\]

which can be rewritten to \( v(i) + \frac{\Delta}{n} \). Observe that \( v(i) + \frac{\Delta}{n} \) is independent of coalition \( S \). We know that \( \phi_i(\tilde{v}) = \phi_i(\tilde{v}) \) and substituting into the expression from Theorem 9, we get

\[
\phi_i(\tilde{v}) = \frac{1}{n} \sum_{S \subseteq N \setminus i} \left( \frac{n - 1}{s} \right)^{-1} (v(i) + \frac{\Delta}{n}) = \left( v(i) + \frac{\Delta}{n} \right) \frac{1}{n} \sum_{S \subseteq N \setminus i} \left( \frac{n - 1}{s} \right)^{-1}.
\]

Modifying the sum is an easy exercise using the following identity:

\[
\sum_{S \subseteq N \setminus i} \left( \frac{n - 1}{s} \right)^{-1} = \sum_{j=0}^{n-1} \left( \frac{n - 1}{j} \right) \left( \frac{n - 1}{j} \right)^{-1} = n.
\]

Combining together, we arrive at the desired formula. \( \square \)

Similarly to the investigation of the conic \( \tau \)-value, we arrive to a conclusion that any sensible integration of the recession cone in the definition of the generalised Shapley value does not yield a different result. This is since \( \phi(\tilde{v} + \alpha \hat{e}) = \phi(\tilde{v}) + \phi(\alpha \hat{e}) = \phi(\tilde{v}) \) as for symmetric game \((N, \alpha \hat{e})\), the Shapley value for any player \( i \) equals \( \phi_i(\alpha \hat{e}) = 0 \). Nonetheless, similar argument for the definition of \( \phi^n \) for games with general \( K \) holds. For the conic Shapley value of incomplete games with defined upper vector, we show in Subsection 4.2 that the two concepts do not coincide in general.

**Definition 21** Let \( K \subseteq 2^N \) and suppose that \( \forall v \in C^n_\pi(K) \), the set \( C^n_\pi(v) \) is a polyhedron described by its extreme points and extreme rays. Then the \( \alpha \)-conic Shapley-value \( \phi^n : C^n_\pi(K) \to \mathbb{R}^n \) is defined as

\[
\phi^n(v) := \phi(\tilde{v} + \alpha \hat{e}),
\]

where \( (N, \tilde{v}), (N, \hat{e}) \) are the centres of gravity of extreme points and of extreme rays, respectively.

To summarise, we considered generalisation of three values \( \tau, n, \phi \) of complete cooperative games in two variants (including/excluding the information from the recession cone of \( C^n_\pi \)-extensions) and showed that actually all of them coincide thanks to 1-convexity and symmetry of the recession cone of \( C^n_\pi(v) \). From now on, we will refer to this solution concept of incomplete games with minimal information as the *average value \( \tilde{\zeta} \).*
3.3 Axiomatization of the average value $\tilde{\zeta}$

In this subsection, we focus on axiomatization of the average value. We first consider known characterizations of the $\tau$-value and the Shapley value of complete games. We show how to generalize these characterization for the average value. This is done with taking into account the fact that the average value is both the $\tau$-value and Shapley value of a specific complete game $(N, \tilde{v})$. In the second part, we offer three axiomatisations where the axioms are defined in the context of values of $v \in C^n_1(K_{\min})$.

3.3.1 Generalisations of known axiomatisations

The idea behind generalisations of known axiomatisations is based on the fact that the average value is defined as either the $\tau$-value or the Shapley value of the centre of gravity $(N, \tilde{v})$. Since these solution concepts satisfy certain axioms, also the average value satisfies these axioms when restricted to $\tilde{v}$. The uniqueness of the average value is then given by the uniqueness of $\tilde{\zeta}(v)$ for each $v \in C^n_1(K_{\min})$. If we had a function $f$ satisfying the restricted axioms different from $\tilde{\zeta}$, we would have a game $v \in C^n_1(K_{\min})$ such that $\tilde{\zeta}(v) = \tau(\tilde{v}) \neq f(v)$. But this means that for $(N, \tilde{v})$, we have two solution concepts for complete games satisfying the axioms of the $\tau$-value (or the Shapley value) that differ in the imputation assigned to $(N, \tilde{v})$. This is a contradiction with the uniqueness of these values.

We demonstrate this proof method on two examples, generalising both an axiomatisation of the $\tau$-value and the Shapley value.

**Theorem 20** The average value $\tilde{\zeta}$ is the only function $f: C^n_1(K_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v \in C^n_1(K_{\min})$:

1. (efficiency) $\sum_{i \in N} f_i(v) = v(N),$
2. (restricted proportionality property of $\tilde{v}$) $f(v_0) = \alpha \tilde{v}_0,$
3. (minimal right property of $\tilde{v}$) $f(v) = a\tilde{v} + f(v - a\tilde{v}),$

where $\alpha \in \mathbb{R}$ and $(v - a\tilde{v})(S) := v(S) - \sum_{i \in S} a_i\tilde{v}$ for every $S \subseteq N$.

**Proof** To prove that the average value satisfies the mentioned properties, recall the definition $\tilde{\zeta}(v) = \tau(\tilde{v})$ and Theorem 6. Since $\tilde{\zeta}(S) = v(S)$ for $S \in K$, all three properties hold.

Regarding uniqueness, suppose there is a function $g: C^n_1(K_{\min}) \to \mathbb{R}^n$ such that the properties hold and there is a game $v \in C^n_1(K_{\min}), \tilde{\zeta}(v) \neq g(v)$. We can construct a function $\gamma: C^n_1 \to \mathbb{R}^n$ such that $\gamma(w) = \tau(w)$ for every $w \in C^n_1, w \neq \tilde{v}$ and $\gamma(\tilde{v}) := g(v)$. Clearly, $\gamma$ satisfies all axioms from Theorem 6 which leads (together with $\gamma(\tilde{v}) = g(v) \neq \tau(\tilde{v})$) to a contradiction with the uniqueness of the $\tau$-value. $\Box$

It can be showed that in the context of incomplete games the second axiom is equivalent to restricted proportionality property of $\tilde{v}$ where $\alpha = 1$. 
The alternative characterisation of the $\tau$-value was proposed in [9] and it can be generalised in a similar manner. Let us proceed with yet another example, generalising axiomatisation of the Shapley value.

**Theorem 21** The average value $\zeta$ is the only function $f: C^n_r(K_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v, w \in C^n_r(K_{\min})$:

1. (efficiency) $\sum_{i \in N} f_i(v) = v(N)$,
2. (symmetry of $\bar{v}$) $\forall i, j \in N$ and $\forall S \subseteq N \setminus \{i, j\}: v(S \cup i) = v(S \cup j) \implies f_i(v) = f_j(v)$,
3. (null player of $\bar{v}$) $\forall i \in N$ and $\forall S \subseteq N \setminus i: \bar{v}(S) = \bar{v}(S \cup i) \implies f_i(v) = 0$,
4. (additivity of $\bar{v}$) if $v + w \in C^n_r(K_{\min})$ then $f(v + w) = f(v) + f(w)$.

**Proof** Since the average value of $v \in C^n_r(K_{\min})$ acts as the Shapley value of $\bar{v} \in C^n_1$, the axioms are satisfied. We note that considering efficiency, $\sum_{i \in N} f_i(v) = v(N) = \bar{v}(N)$, therefore it is equivalent with $\phi(\bar{v}) = \bar{v}(N)$ and for additivity, $(\bar{v} + \bar{w}) = \bar{v} + \bar{w}$, where $(\bar{v} + \bar{w})$ is the centre of gravity of vertices of $v + w$ and $\bar{v} + \bar{w}$ is the sum of centres of gravity of $v$ and $w$. The uniqueness of $\zeta$ is given by the uniqueness of the Shapley value. \(\square\)

From the alternative characterisations of the Shapley value, we generalised those in [7, 39]. To do the same for the one by Roth [28] seems to be more challenging.

### 3.3.2 Axiomatisations in the context of values of $(N, K, v)$

The previously mentioned characterisations do not tell us anything new about the average value that we do not already know from its definition $\zeta(v) = \tau(\bar{v}) = \phi(\bar{v})$. In this subsection, we derive three axiomatisations in the context of values of $(N, K, v)$.

**Theorem 22** The average value $\zeta$ is the only function $f: C^n_r(K_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v \in C^n_r(K_{\min})$:

1. (efficiency) $\sum_{i \in N} f_i(v) = v(N)$,
2. (elementary symmetry) $\forall i, j \in N : i \neq j \land v(i) = v(j) \implies f_i(v) = f_j(v)$,
3. (zero-normalisation invariance) $\forall i \in N : f_i(v) = v(i) + f_i(v_0)$.

**Proof** Let us prove that $\zeta$ satisfies all three properties. First,

$$\sum_{i \in N} \zeta_i(v) = \sum_{i \in N} v(i) + v(N) = v(N).$$

Furthermore, for $v(i) = v(j)$, it holds $\zeta_i(v) = v(i) + \frac{\Delta}{n} = v(j) + \frac{\Delta}{n} = \zeta_j(v)$. For the third property, as $v(i) + \frac{\Delta}{n} = \zeta_i(v) = v(i) + \zeta_i(v_0)$ holds for every player $i$, it suffices to show that $\zeta_i(v_0) = \frac{\Delta}{n}$. For any $v \in C^n_r(K_{\min})$, the zero-normalisation $v_0$ can be described as

$$v_0(S) = \begin{cases} v(N) - \sum_{i \in N} v(i), & \text{if } S = N, \\ 0, & \text{if } S \neq N. \end{cases}$$
The total excess $\Delta_0$ of $(N, K, v_0)$ is equal to $v(N) - \sum_{i \in N} v(i) = \Delta$. Therefore $\tilde{\zeta}_i(v_0) = v_0(i) + \frac{\Delta_0}{n} = \frac{\Delta}{n}$, thus the third property also holds.

Now, let us prove that $f : C_1^0(K_{\min}) \rightarrow \mathbb{R}^n$ satisfying the three properties must be $\tilde{\zeta}$. First, from zero-normalisation invariance, it holds $\forall i : f_i(v) = v(i)$ for all $i \in N$, it suffices to prove that $f_i(v_0) = \frac{\Delta}{n}$ for any zero-normalisation $v_0$ of $v \in C_1^0(K_{\min})$. From the first property, we have $\sum_{i \in N} f_i(v_0) = \Delta = v_0(N)$. Also, $v_0(i) = v_0(j)$ for all pairs of players $i, j$ implies $f_i(v_0) = f_j(v_0)$. Combining both together, we get $f_i(v_0) = \sum_{\tau \subseteq N} f(\tau) = \frac{\Delta}{n}$. □

Another characterisation can be obtained by substituting the axiom of zero-normalisation invariance for additivity axiom. Such replacement has to be compensated by adding yet another axiom, because without it, they do not characterise the function uniquely (for example, the solidarity $\tau$-value $\tau^s$ also satisfies these three axioms). We deal with this by providing two different axioms: zero-excess axiom (employing the total excess $\Delta$) and a more familiar axiom of individual rationality.

**Theorem 23** The average value $\tilde{\zeta}$ is the only function $f : C_1^0(K_{\min}) \rightarrow \mathbb{R}^n$ such that the following properties hold for every $v, w \in C_1^0(K_{\min})$:

1. (efficiency) $\sum_{i \in N} f_i(v) = v(N)$,
2. (elementary symmetry) $\forall i, j \in N : i \neq j \land v(i) = v(j) \implies f_i(v) = f_j(v)$,
3. (elementary additivity) if $v + w \in C_1^0(K_{\min}) : f(v + w) = f(v) + f(w)$,
4. (zero-excess axiom) if $\Delta_v = 0 \implies \forall i : f_i(v) = v(i)$,
5. (individual rationality) $\forall i \in N : f_i(v) \geq v(i)$.

**Proof** We have already proved in Theorem 22 that the first two axioms are satisfied by $\tilde{\zeta}$. To prove additivity, we have $\tilde{\zeta}_i(v + w) = v(i) + w(i) + \frac{\Delta_{v+w}}{n}$ and

$$\tilde{\zeta}_i(v) + \tilde{\zeta}_i(w) = v(i) + \frac{\Delta_v}{n} + w(i) + \frac{\Delta_w}{n} = v(i) + w(i) + \frac{\Delta_v}{n} + \frac{\Delta_w}{n},$$

for any player $i$. Clearly, if $\Delta_{v+w} = \Delta_v + \Delta_w$, elementary additivity is satisfied. However,

$$\Delta_{v+w} = v(N) + w(N) - \sum_{i \in N} (v(i) + w(i)) = v(N) - \sum_{i \in N} v(i) + w(N) - \sum_{i \in N} w(i)$$

and

$$\Delta_v + \Delta_w = v(N) - \sum_{i \in N} v(i) + w(N) - \sum_{i \in N} w(i).$$

Zero-excess axiom is satisfied because for $\Delta_v = 0$ and any player $i$, $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta_v}{n} = v(i)$. Individual rationality is also satisfied as for any player $i$: $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta_v}{n} \geq v(i)$.

To substitute elementary additivity for zero-normalisation in our proof of uniqueness, we define a game with minimal information $(N, K, v_0)$ such that $v = v_0 + \Sigma$. We do so by setting $\Sigma(i) := v(i)$ and $\Sigma(N) := \sum_{i \in N} v(i)$. Notice
that $\Delta \Sigma = 0$ and thus, $\Sigma \in C^n_1(K_{\text{min}})$. Now, from elementary additivity $f_i(v) = f_i(v_0) + f_i(\Sigma)$. We already proved, that $f_i(v_0) = \frac{\Delta}{n}$, thus all that remains is to prove that for every player $i$, $f_i(\Sigma) = v(i)$.

From zero-excess axiom, this already holds as $\Delta \Sigma = 0$. Without zero-excess axiom, by efficiency $\sum_{i \in N} f_i(\Sigma) = \Sigma(N) = 0$ and individual rationality, each summand $f_i(\Sigma) \geq 0$, which leads together to the desired $f_i(\Sigma) = v(i)$.

We conclude this section with further axioms, which are connected with different definitions of the Shapley value [7,39].

Both of the following properties can be easily derived from the definition of the average value $\bar{\zeta}$.

**Theorem 24** For the average value $\bar{\zeta}: C^n_1(K_{\text{min}}) \rightarrow \mathbb{R}^n$, the following properties hold for every $v, w \in C^n_1(K_{\text{min}})$:

- **(elementary triviality)** $\bar{\zeta}(v_\emptyset) = 0$, where $v_\emptyset(S) := 0$ for $S \in K$,
- **(elementary fairness)** $\bar{\zeta}_i(v + w) - \bar{\zeta}_i(v) = \bar{\zeta}_j(v + w) - \bar{\zeta}_j(v)$ if $w(i) = w(j)$.

Notice that a property similar to null player cannot hold when $\Delta > 0$. That is because if $v(i) = 0$ for a player $i$, then $\bar{\zeta}_i = \frac{\Delta}{N} > 0$. This might seem surprising since in the characterisation of the Shapley value, the axiom of null player is satisfied. This corresponds with the idea that even though it might seem from the known information given by $K$ that the player does not have any worth in the game, since we cannot be sure, we act as if he has some.

4 Incomplete games with defined upper vector

By an incomplete game with defined upper vector, we mean a game $(N, K, v)$ such that $\{N\} \cup \{N \setminus i \mid i \in N\} \subseteq K$. Similarly to the previous section, we first derive a description of the set of $C^n_1$-extensions and in the following subsection, we focus on solution concepts. The results therein show that the average Shapley value and the conic Shapley value does not coincide in general for incomplete games with defined upper vector with at least four players.

4.1 Description of the set of $C^n_1$-extensions

Similarly to the previous section, we initiate the derivation of a description of $C^n_1(v)$ by a characterisation of $C^n_1$-extendability.

**Theorem 25** Let $(N, K, v)$ be an incomplete game with defined upper vector. It is $C^n_1$-extendable if and only if

$$\forall S \in K: v(S) \leq v(N) - b(N \setminus S)$$

(13)

and

$$b(N) \geq v(N).$$

(14)
Proof If the conditions hold, we can define a complete game \((N, \varpi)\) such that
\[
\varpi(S) = \begin{cases} 
v(S), & \text{if } S \in \mathcal{K}, \\
v(N) - b(N \setminus S), & \text{if } S \not\in \mathcal{K}.
\end{cases}
\]
The game is 1-convex, because \(v(S) \leq v(N) - b(N \setminus S)\) for \(S \not\in \mathcal{K}\) holds since the left-hand side is actually equal to the right-hand side. For \(S \in \mathcal{K}\) the conditions hold from the assumption as well as the condition \(b(N) \geq v(N)\). Therefore, it is a \(C_1^n\)-extension of \((N, \mathcal{K}, v)\).

If one of the conditions (13) or (14) fails, the condition does not hold for any extension, therefore the extension is not 1-convex.

We further define games \((N, e_T)\) for \(T \not\in \mathcal{K}\) as
\[
e_T(S) := \begin{cases} -1, & \text{if } S = T, \\0, & \text{otherwise}.
\end{cases}
\]
It is not hard to see that games \((N, \varpi + \alpha e_T)\) are \(C_1^n(v)\)-extensions for any \(\alpha \geq 0\), therefore \((N, e_T)\) are rays of \(C_1^n(v)\). Moreover, all but one conditions are satisfied for \((N, \varpi + e_T)\) with equality, therefore they are even the extreme rays. The following theorem shows they are the only extreme rays.

**Theorem 27** Let \((N, v)\) be a \(C_1^n\)-extendable incomplete game with defined upper vector. Then the set of \(C_1^n(v)\)-extensions can be described as
\[
C_1^n(v) = \left\{ \varpi + \sum_{T \not\in \mathcal{K}} \alpha_T e_T \mid \alpha_T \geq 0 \right\}.
\]
For a $C_1^n$-extension $(N, w)$, we show it can be expressed as a combination of the upper game and games $(N, e_T)$ for $T \not\in K$. Since $(N, w) \in C_1^n(v)$, it holds for every $T \not\in K$ that $w(T) \leq \overline{v}(T)$. Therefore, we define $\alpha_T := w(T) - \overline{v}(T)$. Immediately, it follows that

$$(w + \alpha_T e_T)(T) = w(T) - w(T) + \overline{v}(T) = \overline{v}(T).$$

Setting $\alpha_T$ for every $T \not\in K$ in this manner concludes the proof. \hfill \Box

4.2 Solution concepts

From the point of view of the $\tau$-value, a game with defined upper vector is equivalent with any of its $C_1^n$-extensions. This is because, for a complete game $v$, $\tau(v) = b^v - g^v(N)$ and both $b^v$ and $g^v(N)$ depend only on values $v(N)$ and $v(N \setminus i)$ for $i \in N$.

Incomplete games with defined upper vector are a good example for showing that in general, $\tilde{\phi}(v) \neq \phi^\alpha(v)$. From the definition of the conic Shapley value, additivity and $S$-equivalence axiom, we have

$$\phi^\alpha(v) = \phi(\tilde{v} + \alpha \tilde{e}) = \phi(\tilde{v}) + \alpha \phi(\tilde{e}).$$

Therefore, $\phi^\alpha(v) = \tilde{\phi}(v)$ for $\alpha > 0$ if and only if $\alpha \phi(\tilde{e}) = 0$, which is equivalent to $\phi(\tilde{e}) = 0$. In order to compute the Shapley value, we need to obtain the marginal contributions of player $i$ to all coalitions $S$, i.e. $\tilde{e}(S \cup i) - \tilde{e}(S)$ for $i \in N$ and $S \subseteq N \setminus i$.

**Lemma 1** Let $(N, K, v)$ be a $C_1^n$-extendable incomplete game with defined upper vector and for $C_1^n(v)$, let the game $(N, \tilde{e})$ be the centre of gravity of its extreme rays. Then we have

$$\tilde{e}(S \cup i) - \tilde{e}(S) = \begin{cases} 0, & \text{if } S \in K \text{ and } S \cup i \in K, \\ 0, & \text{if } S \not\in K \text{ and } S \cup i \not\in K, \\ \frac{1}{|E|}, & \text{if } S \not\in K \text{ and } S \cup i \in K, \\ -\frac{1}{|E|}, & \text{if } S \in K \text{ and } S \cup i \not\in K, \end{cases}$$

where $E = \{T \subseteq N \mid T \not\in 2^N \setminus K\}$ and thus $|E| = 2^{|N|} - |K|$.

**Proof** We denote by $\overline{K}$ the coalitions with unknown values, that is $\overline{K} := 2^N \setminus K$. From the definition of $(N, \tilde{e})$, we have

$$\tilde{e}(S \cup i) - \tilde{e}(S) = \frac{1}{|E|} \left( \sum_{T \in \overline{K}} e_T(S \cup i) - \sum_{T \in \overline{K}} v(S) \right).$$

Remember, that $e_T(S) = -1$ if and only if $T = S$, otherwise $e_T(S) = 0$. It means that if $S \in \overline{K}$, the sum $\sum_{T \in \overline{K}} e_T(S)$ is equal to zero and similarly for $S \cup i$. Let us now distinguish the following cases.
- If $S \in \mathcal{K}$ and $S \cup i \in \mathcal{K}$, we have
  \[
  \left( \sum_{T \in \mathcal{K}} e_T(S \cup i) - \sum_{T \in \mathcal{K}} v(S) \right) = 0 - 0 = 0.
  \]
- If $S \in \overline{\mathcal{K}}$ and $S \cup i \in \overline{\mathcal{K}}$, we have
  \[
  \left( \sum_{T \in \mathcal{K}} e_T(S \cup i) - \sum_{T \in \mathcal{K}} v(S) \right) = -1 - (-1) = 0.
  \]
- If $S \in \mathcal{K}$ and $S \cup i \in \overline{\mathcal{K}}$, we have
  \[
  \left( \sum_{T \in \mathcal{K}} e_T(S \cup i) - \sum_{T \in \mathcal{K}} v(S) \right) = 0 - (-1) = 1.
  \]
- If $S \in \overline{\mathcal{K}}$ and $S \cup i \in \overline{\mathcal{K}}$, then
  \[
  \left( \sum_{T \in \mathcal{K}} e_T(S \cup i) - \sum_{T \in \mathcal{K}} v(S) \right) = -1 - 0 = -1.
  \]

This case analysis concludes the proof. \(\square\)

**Lemma 2** Let $(N, \mathcal{K}, v)$ be $C_1^n$-extendable incomplete game with defined upper vector and for $C_1^n(v)$, game $(N, \tilde{\mathcal{E}})$ the centre of gravity of its extreme rays. Then it holds

\[
\phi_i(\tilde{\mathcal{E}}) = \frac{1}{|E|n!}\left( \sum_{\substack{S \subseteq N \setminus i \setminus S \subseteq \mathcal{K} \setminus S_{i} \cup j \in \mathcal{K} \setminus S_{i}}} s!(n-s-1)! \right) = \frac{1}{|E|n!}\left( \sum_{\substack{S \subseteq N \setminus i \setminus S \subseteq \mathcal{K} \setminus S_{i} \cup j \in \mathcal{K} \setminus S_{i}}} s!(n-s-1)! \right).
\]

**Proof** The result immediately follows from the definition of the Shapley value (Definition [12]) and Lemma [1], since $\phi_i(\tilde{\mathcal{E}}) = \frac{1}{|E|n!}\left( \sum_{\substack{S \subseteq N \setminus i \setminus S \subseteq \mathcal{K} \setminus S_{i} \cup j \in \mathcal{K} \setminus S_{i}}} s!(n-s-1)! \right)$ and substituting corresponding marginal contributions and dividing the sum into 4 sums according to presence of $S$ and $S \cup i$ in $\mathcal{K}$ yields the formula above. \(\square\)

From Lemma [2], we can conclude that for cooperative games with at most 3 players, $\phi$ and $\phi^\alpha$ always coincide. However, if we consider games with more players, the two solution concepts differ.

**Theorem 28** Let $(N, \mathcal{K}, v)$ be a $C_1^n$-extendable incomplete game with defined upper vector.

1. If $|N| \leq 3$, then $\tilde{\phi}(v) = \phi^\alpha(v)$.
2. If $|N| \geq 4$ and $\mathcal{K} = 2^N \setminus \{\{i\} \mid i \in N\}$, then $\tilde{\phi}(v) \neq \phi^\alpha(v)$.
Proof If $|N| \leq 3$, then if there is $S \notin K$, it is a singleton coalition $S = \{j\}$. This means, that in the case $S \notin K$ and $S \cup i \in K$, the element of the sum is $s!(n - s - 1)! = 0$. Also, if we consider the other sum where $T \in K$ and $T \cup i \notin K$, the only possibility is $T = \emptyset$, therefore, again $t!(n - t - 1)! = 0$. Thus $\phi_i(\vec{e}) = 0$ for any such game and any player $i$, leading to coincidence of $\tilde{\phi}$ and $\phi^\alpha$.

For $|N| \geq 4$ and $K = 2^N \setminus \{i \mid i \in N\}$, the coalition $S$ satisfying $S \in K$ and $S \cup i \notin K$ is only $S = \emptyset$, for $j \neq k$. Since there are $\binom{N}{2}$ coalitions $\{j, k\}$ for every $j$ and there are $n$ players, we get that the second sum is equal to $n(n-1)!(n-2)! = n!$. Further, the coalitions $S$ satisfying $S \notin K$ and $S \cup i \in K$ are only those satisfying $|S| = n - 2$. For every such coalition $s!(n - s - 1)! = (n - 2)!(n - (n - 2) - 1)! = (n - 2)!$ and there are $2\binom{n}{2} = 2\frac{n!}{(n-2)!2!} = n!$. Therefore, $\phi_i(v) \neq \phi^\alpha(v)$. \qed

5 Conclusion

This work is the first treatment of 1-convexity and $C^1_n$-extensions in the framework of partially defined games. We focused on two prominent classes of games with incomplete information: games with minimal information and games with defined upper vector. For these classes, we obtained the following.

- A characterisation of $C^1_n$-extendability, a description of the upper game, extreme games and of the set of $C^1_n$-extensions (Theorem 11, 12, 15 and 16 for minimal information, Theorem 25, 26, and 27 for games with defined upper vector).
- Regarding solution concepts of incomplete games, generalisations of the $\tau$-value, the Shapley value and the nucleolus were considered. We compared different variants and found that these values actually coincide on the set of $C^1_n$-extensions for games with minimal information (Theorems 17, 18, 19). This gave rise to the notion of the average value for which we have obtained several characterisations and axiomatisations involving some natural properties (Theorems 20, 21, 22, 23). Let us highlight that our work signalises that usefulness of such generalisations vary significantly depending on structures of $K$. This is exemplified in our work in Section 4 on games with defined upper vector.

A natural direction for future research is to go beyond the case of minimal information and upper defined vector. We have already seen that describing the set of $C^1_n$-extensions can be nontrivial even for these classes of incomplete games. Also, there is clearly no single right way to deal with one-point solution concepts and their analysis, namely different axiomatisations. We believe that the presented work can be seen as the first larger step towards understanding much richer structures of $K$ and related questions for 1-convexity. Indeed, some of the mentioned problems are now work in progress.
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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Alparslan Gök, S.Z.: Cooperative interval games. Ph.D. thesis, Middle East Technical University (2009)
2. Alparslan Gök, S.Z., Branzei, O., Branzei, R., Tijs, S.: Set-valued solution concepts using interval-type payoffs for interval games. Journal of Mathematical Economics 47(4), 621–626 (2011)
3. Bilbao, J.M.: Cooperative games on combinatorial structures, vol. 26. Springer Science & Business Media (2012)
4. Bok, J., Hladík, M.: Selection-based approach to cooperative interval games. In: Communications in Computer and Information Science, ICORES 2015 - International Conference on Operations Research and Enterprise Systems, Lisbon, Portugal, 10-12 January, 2015, vol. 577, pp. 40–53 (2015)
5. Bok, J., Černý, M., Hartman, D., Hladík, M.: Convexity and positivity in partially defined cooperative games. arXiv preprint arXiv:1812.05821 (2020)
6. Branzei, R., Dimitrov, D., Tijs, S.: Models in Cooperative Game Theory. Springer (2008)
7. van den Brink, J.: An axiomatization of the Shapley value using a fairness property. International Journal of Game Theory 30(3), 309–319 (2002)
8. van der Brink, J.R.: An axiomatization of the Shapley value using component efficiency and fairness. Tinbergen Discussion Paper (1995)
9. Calso, E., Tijs, S.H., Valenciano, F., Zarzuelo, J.M.: On axiomatization of the τ-value. TOP 3, 35–46 (1995)
10. Curiel, I.: Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems. Springer, Dordrecht (2013)
11. Driessen, T.: Cooperative Games, Solutions and Applications, Theory and Decision Library C, vol. 3. Kluwer, Dordrecht (1988)
12. Driessen, T.S.H.: Contributions to the theory of cooperative games: the τ-value and k-convex games. Ph.D. thesis, Radboud University, Nijmegen (1985)
13. Driessen, T.S.H.: Properties of 1-convex n-person games. OR Spektrum 7, 19–26 (1985)
14. Gilles, R.P.: The Cooperative Game Theory of Networks and Hierarchies, Theory and Decision Library C, vol. 44. Springer (2010)
15. Grabisch, M.: Set functions, games and capacities in decision making. Springer (2016)
16. Kopelowitz, A.: Computation of the kernels of simple games and the nucleolus of cooperative games as locuses in the strong $\epsilon$-core. Research Memo 31 (1967)
17. Lemaire, J.: Cooperative Game Theory and its Insurance Applications. Center for Research on Risk and Insurance, Wharton School of the University of Pennsylvania (1991)
18. Mallozzi, L., Scalzo, V., Tijs, S.: Fuzzy interval cooperative games. Fuzzy Sets and Systems 165(1), 98–105 (2011)
19. Mareš, M.: Fuzzy cooperative games: cooperation with vague expectations, vol. 72. Physica (2013)
20. Mareš, M., Vlach, M.: Fuzzy classes of cooperative games with transferable utility. Scientiae Mathematicae Japonica 2, 269–278 (2004)
21. Masuya, S.: The Shapley value on a class of cooperative games under incomplete information. In: International Conference on Intelligent Decision Technologies, pp. 129–139. Springer (2016)
22. Masuya, S.: The Shapley value and consistency axioms of cooperative games under incomplete information. In: I. Czarnowski, R.J. Howlett, L.C. Jain (eds.) Intelligent Decision Technologies 2017, pp. 3–12. Springer (2018)

23. Masuya, S., Inuiguchi, M.: A fundamental study for partially defined cooperative games. Fuzzy Optimization Decision Making 15(1), 281–306 (2016)

24. Palancı, O., Alparslan Gök, S.Z., Ergün, S., Weber, G.W.: Cooperative grey games and the grey Shapley value. Optimization 64(8), 1657–1668 (2015)

25. Palancı, O., Alparslan Gök, S.Z., Weber, G.W.: Cooperative games under bubbly uncertainty. Mathematical Methods of Operations Research 80(2), 129–137 (2014)

26. Peleg, B., Sudhölter, P.: Introduction to the Theory of Cooperative Games, vol. 34. Springer Science & Business Media (2007)

27. Peters, H.: The Shapley Value, vol. 28. Springer Berlin Heidelberg (2008)

28. Roth, A.G.: The Shapley value as a von Neumann-Morgenstern utility. Econometrica 45, 657–664 (1977)

29. Seshadhri, C., Vondrák, J.: Is submodularity testable? Algorithmica 69(1), 1–25 (2014)

30. Shapley, L.S.: A value for n-person game. Annals of Mathematical Studies 28, 307–317 (1953)

31. Soltan, V.: Lectures on Convex Sets. World Scientific (2015)

32. Tijs, S.H.: Bounds for the core of a game and the τ-value. Game Theory and Mathematical Economics pp. 123–132 (1981)

33. Tijs, S.H.: An axiomatization of the τ-value. Mathematical Social Sciences 13, 177–181 (1987)

34. Umang, B., Kumar, G.: Partial function extension with applications to learning and property testing. arXiv preprint arXiv:1812.05821 (2018)

35. Umang, B., Kumar, G.: The complexity of partial function extension for coverage functions. arXiv preprint arXiv:1907.07230 (2019)

36. Weber, G.W., Branzei, R., Alparslan Gök, S.Z.: On cooperative ellipsoidal games. In: 24th Mini EURO Conference-On Continuous Optimization and Information-Based Technologies in the Financial Sector, MEC EurOPT, pp. 369–372 (2010)

37. Willson, S.J.: A value for partially defined cooperative games. International Journal of Game Theory 21(4), 371–384 (1993)

38. Xiaohui, Y., Zhiping, D., Qiang, Z., Zhengxing, Z.: Proportional Owen value for the coalition structure cooperative game under the incomplete information. Systems Engineering - Theory & Practice 39(8), 2105 (2019). DOI 10.12011/1000-6788-2019-0113-11

39. Young, C.: A new axiomatization of the Shapley value. Games and Economic Behaviour 1, 119–130 (1989)