A Tiling Proof of Binomial Identities related to the Lucas cube

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Abstract

Using a cube tiling of $\mathbb{R}^n$ constructed by Lagarias and Shor a tiling proof of three well-known binomial identities related to the Lucas cube is given.

Key words: cube tiling, binomial identity, Lucas cube.

1 Introduction

A cube tiling of $\mathbb{R}^n$ is a family of cubes $[0,2)^n + T = \{[0,2)^n + t : t \in T\}$, where $T \subset \mathbb{R}^n$, which fill in the whole space without gaps and overlaps. In [4] Lagarias and Shor constructed a cube tiling code of $\mathbb{R}^n$. In this note we show how it can be used to prove the following well-known identities:

$$
\sum_{k \geq 0} \binom{n-k}{k} \frac{n}{n-k} 2^k = 2^n + (-1)^n, \quad (1.1)
$$

$$
\sum_{k \geq 0} \binom{n-k}{k} 2^k = \frac{1}{3} (2^{n+1} + (-1)^n) \quad (1.2)
$$

and

$$
\sum_{k \geq 0} \binom{n-k}{k} \frac{k}{n-k} 2^k = \frac{1}{3} (2^n + (-1)^n 2). \quad (1.3)
$$

The code of Lagarias and Shor is constructed as follows. Let $n \geq 3$ be an odd positive integer, and let $A$ be an $n \times n$ circulant matrix of the form
A = A(n) = \text{circ}(1, 2, 0, \ldots, 0). Let A^T be the transpose of A. By \( V(A) \) and 
\( V(A^T) \) we denote the sets of all the sums of the different rows in \( A \) and \( A^T \), respectively. Moreover, we add to these sets the vector \((0, \ldots, 0)\). Let 
\[ V = V_o(A) \cup (V_o(A^T) + (2, \ldots, 2)) \mod 4, \]
where \( V_o(A) \) denotes the set of all vectors in \( V(A) \) with an even number of 3s, and \( V_o(A^T) \) is the set of all vectors in \( V(A^T) \) with an odd number of 0s. We will refer to the code \( V \) as the Lagarias-Shor cube tiling code. This code has very interesting applications. Originally in \[4\] it was used to design a certain cube tiling of \( \mathbb{R}^n \) that was the basis for estimating distances between cubes in cube tilings of \( \mathbb{R}^n \). Recently in \[3\] the Lagarias-Shor cube tiling code was used to construct interesting partitions and matchings of an \( n \)-dimensional cube.

To obtain a cube tiling of \( \mathbb{R}^n \) from the code \( V \), let \( T = V - 1 + 4\mathbb{Z}^n \), where \( 1 = (1, \ldots, 1) \). It follows from \[4, \text{Proposition 3.1 and Theorem 4.1}\] that 
\[ [0, 2]^n + T \] is a cube tiling of \( \mathbb{R}^n \). (To be precisely, a tiling considered in \[4\] is of the form \([0, 1]^n + T', \) where \( T' = \frac{1}{2}V + 2\mathbb{Z}^n \), but \([0, 2]^n + T = [0, 2]^n + 2T' - 1). \)

All three identities (1.1)–(1.3) are related to the Lucas cube \( \Lambda_n \). This is a graph whose vertices are all elements of \( \{0, 1\}^n \) which do not contain two consecutive 1s as well as vertices having 1 at the first and last position simultaneously. It is known that \((\begin{pmatrix} n-k \\ k \end{pmatrix})\frac{n}{n-k}\) is the number of all vertices \( v \) in the Lucas cube \( \Lambda_n \) of weight \( k \), i.e., containing \( k \) 1s. This is also the number of all \( k \)-element subsets of the set \([n] = \{1, \ldots, n\}\) without two consecutive integers and which do not contain the pair 1, \( n \) (\[5\]). The number of all vertices in \( \Lambda_n \) of weight \( k \) which have 1 at the \( i \)th position, \( i \in [n] \), is equal to \((\begin{pmatrix} n-k \\ k \end{pmatrix})\frac{k}{n-k}\), while \((\begin{pmatrix} n-k \\ k \end{pmatrix})\) is the number of all vertices in \( \Lambda_n \) of weight \( k \) which have 0 at the \( i \)th position.

2 Tiling proofs

Since the Lagarias-Shor cube tiling code is defined for odd numbers, we prove identities (1.1)–(1.3) for odd and even positive integers separately.

Proof of (1.1) for \( n \geq 3 \) odd. We intersect the cube \([0, 2]^n\) with the cubes from the tiling \([0, 2]^n + T\). Let \( \mathcal{F}(n) = \mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + t) : t \in T\} \). Since \([0, 2]^n + T\) is a tiling, \( \mathcal{F} \) is a partition of the cube \([0, 2]^n\) into boxes. Let \( m(K) \) denote the volume of the box \( K \in \mathcal{F} \), and let \( m(\mathcal{F}) = \sum_{K \in \mathcal{F}} m(K) \). For every \( K \in \mathcal{F} \) we have \( m(K) = 2^k \), where \( k \) is the number of 1s in the vector \( v \in V \) such that \( K = [0, 2]^n \cap ([0, 2]^n + v - 1) \). Let \( M_k = \{|K \in \mathcal{F} : m(K) = 2^k\}| \). The family \( \mathcal{F} \) is a partition of \([0, 2]^n\) and therefore \( m(\mathcal{F}) = 2^n\). 

Proof of (1.2) for \( n \geq 3 \) odd. We intersect the cube \([0, 2]^n\) with the cubes from the tiling \([0, 2]^n + T\). Let \( \mathcal{F}(n) = \mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + t) : t \in T\} \). Since \([0, 2]^n + T\) is a tiling, \( \mathcal{F} \) is a partition of the cube \([0, 2]^n\) into boxes. Let \( m(K) \) denote the volume of the box \( K \in \mathcal{F} \), and let \( m(\mathcal{F}) = \sum_{K \in \mathcal{F}} m(K) \). For every \( K \in \mathcal{F} \) we have \( m(K) = 2^k \), where \( k \) is the number of 1s in the vector \( v \in V \) such that \( K = [0, 2]^n \cap ([0, 2]^n + v - 1) \). Let \( M_k = \{|K \in \mathcal{F} : m(K) = 2^k\}| \). The family \( \mathcal{F} \) is a partition of \([0, 2]^n\) and therefore \( m(\mathcal{F}) = 2^n\). 

Proof of (1.3) for \( n \geq 3 \) odd. We intersect the cube \([0, 2]^n\) with the cubes from the tiling \([0, 2]^n + T\). Let \( \mathcal{F}(n) = \mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + t) : t \in T\} \). Since \([0, 2]^n + T\) is a tiling, \( \mathcal{F} \) is a partition of the cube \([0, 2]^n\) into boxes. Let \( m(K) \) denote the volume of the box \( K \in \mathcal{F} \), and let \( m(\mathcal{F}) = \sum_{K \in \mathcal{F}} m(K) \). For every \( K \in \mathcal{F} \) we have \( m(K) = 2^k \), where \( k \) is the number of 1s in the vector \( v \in V \) such that \( K = [0, 2]^n \cap ([0, 2]^n + v - 1) \). Let \( M_k = \{|K \in \mathcal{F} : m(K) = 2^k\}| \). The family \( \mathcal{F} \) is a partition of \([0, 2]^n\) and therefore \( m(\mathcal{F}) = 2^n\). 

Proof of (1.4) for \( n \geq 3 \) odd. We intersect the cube \([0, 2]^n\) with the cubes from the tiling \([0, 2]^n + T\). Let \( \mathcal{F}(n) = \mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + t) : t \in T\} \). Since \([0, 2]^n + T\) is a tiling, \( \mathcal{F} \) is a partition of the cube \([0, 2]^n\) into boxes. Let \( m(K) \) denote the volume of the box \( K \in \mathcal{F} \), and let \( m(\mathcal{F}) = \sum_{K \in \mathcal{F}} m(K) \). For every \( K \in \mathcal{F} \) we have \( m(K) = 2^k \), where \( k \) is the number of 1s in the vector \( v \in V \) such that \( K = [0, 2]^n \cap ([0, 2]^n + v - 1) \). Let \( M_k = \{|K \in \mathcal{F} : m(K) = 2^k\}| \). The family \( \mathcal{F} \) is a partition of \([0, 2]^n\) and therefore \( m(\mathcal{F}) = 2^n\).
and
\[ m(\mathcal{F}) - 2 = \sum_{k \geq 1} M_k 2^k. \]

Note now that if \( v \in V \) contains 3 at some position \( i \in [n] \), then the cubes \([0, 2]^n + v - 1\) and \([0, 2]^n\) are disjoint. This means that these two cubes intersect if and only if \( v \in U \cup \{(0, \ldots, 0), (2, \ldots, 2)\} \), where \( U \) consists of all sums of non-adjacent rows of the matrix \( A \) where the first and last rows are treated as adjacent. Thus, \( M_k \) is the number of all \( k \)-element subsets of the set \([1, \ldots, n]\) without two consecutive integers and which do not contain the pair 1, \( n \). Hence, \( M_k = \binom{n-k}{k} \frac{n}{n-k} \). This completes the proof of (1.1) for \( n \geq 3 \) odd.

This proofs needs only the portion \( U = U(n) \) of the Lagarias-Shor cube tiling code, where the code \( U \) is as in the previous proof: it consists of all sums of non-adjacent rows of the matrix \( A(n) \), where the first and last rows are treated as adjacent. For \( n = 5 \) we have

\[
A(5) = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 2 \\
2 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad U(5) = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 2 \\
2 & 0 & 0 & 0 & 1 \\
1 & 2 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 1
\end{bmatrix}.
\]

This is easily seen that making in every vector \( v \in U \cup \{(0, \ldots, 0)\} \) the substitution \( 2 \rightarrow 0 \) we obtain the set of all vertices in the Lucas cube \( \Lambda_n \).

To prove (1.2) and (1.3) for \( n \geq 3 \) odd let \( \mathcal{F}_0^i, \mathcal{F}_2^i \) and \( \mathcal{F}_1^i, i \in [n] \) denote the sets of all boxes in \( \mathcal{F} \) which have the factors \([0, 1],[1, 2] \) and \([0, 2] \) at the \( i \)th position, respectively. Since \( \mathcal{F} = \{[0, 2]^n \cap ([0, 2]^n + v - 1) : v \in U \cup \{(0, \ldots, 0), (2, \ldots, 2)\}\} \), for every \( k \in \{0, 1, 2\} \) the set \( \mathcal{F}_k^i \) consists of all boxes in \( \mathcal{F} \) which are determined by the vectors \( v \in U \cup \{(0, \ldots, 0), (2, \ldots, 2)\} \) such that \( v_i = k \). Let

\[ m(\mathcal{F}_{02}^i) = \sum_{K \in \mathcal{F}_{02}^i} m(K) \text{ and } m(\mathcal{F}_1^i) = \sum_{K \in \mathcal{F}_1^i} m(K), \]

where \( \mathcal{F}_{02}^i = \mathcal{F}_0^i \cup \mathcal{F}_2^i \).
The partition $\mathcal{F}$ (Figure 1) has the structure which will be utilized below. Note that for every $i \in [n]$ the set $\bigcup F_i^0$, in an $i$-cylinder, i.e., for every line segment $l_i \subset [0, 2]^n$ of length 2 which is parallel to the $i$th edge of the cube $[0, 2)^n$ we have

$$l_i \subset \bigcup F_i^0 \text{ or } l_i \cap \bigcup F_i^0 = \emptyset.$$  

(2.1)

Obviously, the set $\bigcup F_i^1$ is an $i$-cylinder too.

![Fig. 1. The boxes in $\mathcal{F}$ are determined by the vectors $U = \{(1, 2, 0), (0, 1, 2), (2, 0, 1)\}$ (the three "long" boxes) and $\{(0, 0, 0), (2, 2, 2)\}$ (the two unit cubes).](image)

**Proof of (1.2) and (1.3) for $n \geq 3$ odd.** We will calculate $m(F_{02}^i)$ and $m(F_{12}^i)$. For every $v \in U$ we have $v_i = 1$ if and only if $v_{i+1} = 2$. Thus, $F_{2+1} = F_1^i \cup \{[1, 2]^n\}$ and then $m(F_{2+1}^i) = m(F_1^i) + 1$ (clearly, $n + 1$ is taken modulo $n$). It follows from (2.1) that $m(F_0^i) = m(F_2^i)$ (see Figure 1). Since $A$ is a circulant matrix, we have $m(F_i^j) = m(F_j^i)$ for $i, j \in [n]$, and $m(F) = m(F_0^i) + m(F_2^i) + m(F_1^i)$ because $F$ is a partition. Thus, $2^n = 3m(F_1^i) + 2$ and consequently

$$m(F_{02}^i) = 2/3(2^n + 1) \quad \text{and} \quad m(F_{12}^i) = 1/3(2^n - 2).$$  

(2.2)

As it was noted before the proof, the code $U \cup \{(0, \ldots, 0)\}$ can be identify with the set of all vertices in the Lucas cube $\Lambda_n$.

Since $\binom{n-k}{k} \frac{k}{n-k}$ is the number of all vertices of weight $k$ in $\Lambda_n$ with 1 at the first position, we have $|F_1^1| = \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k}$, and consequently

$$m(F_1^i) = \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k} 2^k,$$

which, by (2.2), gives (1.3) for $n \geq 3$ odd. Since

$$\sum_{k \geq 0} \binom{n-k}{k} \frac{n-k}{n-k} 2^k = \sum_{k \geq 0} \binom{n-k}{k} 2^k + \sum_{k \geq 0} \binom{n-k}{k} \frac{k}{n-k} 2^k,$$
the proof of the identity (1.2) for \( n \geq 3 \) odd is also completed. \qed

For \( n \geq 3 \) odd all three identities are strongly related to the partition \( \mathcal{F} \). The sums \( \sum_{k \geq 1} \binom{n-k}{k} 2^k \) and \( \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-k} 2^k \) are the total volumes of the boxes from the partition \( \mathcal{F} \) which belong to the sets \( \mathcal{F}_2 \) and \( \mathcal{F}_1 \), respectively (the number 2 in the first sum is the sum of the volumes of the boxes \([0, 1]^n\) and \([1, 2]^n\)). The summands \( \binom{n-k}{k} 2^k \) and \( \binom{n-k}{k} \frac{k}{n-k} 2^k \) for \( k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) are the total volumes of the boxes in \( \mathcal{F}_2 \) and \( \mathcal{F}_1 \), respectively which have exactly \( k \) factors \([0, 2]\).

The identities (1.1)–(1.3) for \( n - 1 \geq 2 \) even can be derived from the partitions \( \mathcal{F}(n) \) and \( \mathcal{F}(n - 2) \), where \( \mathcal{F}(1) = \{[0, 1], [1, 2]\} \).

Proofs of (1.1)–(1.3) for \( n - 1 \geq 2 \) even. Denote by \( r_1, \ldots, r_n \) the rows of the matrix \( A(n) \), and let \( \mathcal{G} = \mathcal{G}(n-1) \subset \mathcal{F}(n) \) be the set of all boxes which are determined by the vectors \( v \in U(n) \) which are sums of non-adjacent rows from the set \( \{r_1, \ldots, r_{n-1}\} \), where \( r_1 \) and \( r_{n-1} \) are treated as adjacent. Thus, the number \( |\mathcal{G}| \) is the same as the number of all vertices in the Lucas cube \( \Lambda_{n-1} \). Consequently

\[
m(\mathcal{G}) = \sum_{k \geq 1} \binom{n-1-k}{k} \frac{2^k}{n-1-k}.
\]

Since \( \mathcal{G}_0^n = \mathcal{F}_0^n \setminus \{[0, 1]_n\} \), it follows that \( m(\mathcal{G}_0^n) = m(\mathcal{F}_0^n) - 1 \), and by (2.2) and the fact that \( m(\mathcal{F}_0^n) = m(\mathcal{F}_2^n) \) we get

\[
m(\mathcal{G}_0^n) = 2/3(2^{n-1} - 1).
\]

We now calculate \( m(\mathcal{G}_2^n) \). Every box in \( \mathcal{G}_2^n \) is generated by a vector \( v \in U(n) \) which has 2 at the \( n \)th position. Therefore, \( v = r_{n-1} + \sum_{i \in I} r_i \) for some \( I \subset \{2, \ldots, n-3\} \). Let \( R \) be the set of all such sums \( \sum_{i \in I} r_i \). Every word in \( R \) is a sum of non-adjacent rows from the set \( \{r_2, \ldots, r_{n-3}\} \), where \( r_2 \) and \( r_{n-3} \) are not treated as adjacent. Let \( U_0^{n-2}(n-2) \) be the set of all vectors in \( U(n-2) \) having 0 at the last position. Observe now that the function \( b : R \to U_0^{n-2}(n-2) \) defined by the formula \( b(u) = \sum_{i \in I} h_i \), where \( h_1, \ldots, h_{n-2} \) are rows of the matrix \( A(n-2) \) and \( I - 1 = \{i-1 : i \in I\} \), is a bijection. Therefore, \( m(\mathcal{G}_2^n) = 2m(\mathcal{F}_0^{n-2}(n-2)) \) (recall that we add \( r_{n-1} \) to \( \sum_{i \in I} r_i \)). By (2.1), \( m(\mathcal{F}_0^{n-2}(n-2)) = m(\mathcal{F}_2^{n-2}(n-2)) \), and by (2.2),

\[
m(\mathcal{G}_2^n) = 1/3(2^{n-1} + 2).
\]

Thus, \( m(\mathcal{G}) = m(\mathcal{G}_0^n) + m(\mathcal{G}_2^n) = 2^{n-1} \) because \( \mathcal{G}_1^n = \emptyset \). This completes the proof of (1.1) for \( n - 1 \geq 2 \) even.
Since \( m(G_i^{i+1}) = m(G_i^1) \), \( m(G_i) = m(G_i^1) \) and \( m(G) = m(G_i^1) + m(G_{02}) \) for \( i, j \in [n-1] \), it follows that

\[
m(G_i^1) = 1/3(2^{n-1} + 2) \quad \text{and} \quad m(G_{02}) = 2/3(2^{n-1} - 1)
\]

for \( i \in [n-1] \).

By the definition of the set \( G \), we have \( |G_1| = \sum_{k \geq 1} \binom{n-k}{k} \frac{k}{n-1-k} \), and thus

\[
m(G_1) = \sum_{k \geq 1} \binom{n-1-k}{k} \frac{k}{n-1-k} 2^k
\]

which proves (1.3) for \( n - 1 \geq 2 \) even. Having this, in the same manner as for \( n \geq 3 \) odd we prove (1.2) for \( n - 1 \geq 2 \) even. \( \Box \)

**Remark 2.1** There are many tiling proofs that rely on counting the number of 1-dimensional tilings of a \( 1 \times n \) board by polyominoes (squares, dominoes etc.) (see [1, 2]). In our case we examine just one \( n \)-dimensional tiling of \( \mathbb{R}^n \) by translates of the cube \( [0, 2)^n \), and especially it is exploited the structure of that tiling.

At the end we show that for \( n \geq 3 \) odd the set of all vertices of the Lucas cube \( \Lambda_n \) is a selector of a discrete analogue of the partition \( \mathcal{P} \).

### 3 Vertices of the Lucas cube as a selector

Let \( L \cup \{(0, \ldots, 0), (1, \ldots, 1)\} \) be the code that arises from \( U \) by making in every vector \( v \in U \cup \{(0, \ldots, 0), (2, \ldots, 2)\} \) the following substitutions: \( 0 \to 0, 2 \to 1 \) and \( 1 \to * \) For example, for \( n = 5 \) we have

\[
L = \begin{bmatrix}
* & 1 & 0 & 0 & 0 \\
0 & * & 1 & 0 & 0 \\
0 & 0 & * & 1 & 0 \\
0 & 0 & 0 & * & 1 \\
1 & 0 & 0 & 0 & * \\
* & 1 & * & 1 & 0 \\
* & 1 & 0 & * & 1 \\
0 & * & 1 & * & 1 \\
1 & * & 1 & 0 & * \\
1 & 0 & * & 1 & *
\end{bmatrix}
\]
The set $L$ consists of all sums of non-adjacent rows of the matrix $\text{circ}(\ast, 1, 0, \ldots, 0)$, where the first and last rows are treated as adjacent. Therefore, making the substitution $\ast \to 0$ in every vector of $L \cup \{(0, \ldots, 0)\}$ we obtain the set $V(\Lambda_n)$ of all vertices in the Lucas cube $\Lambda_n$.

The code $L \cup \{(0, \ldots, 0), (1, \ldots, 1)\}$ induces a partition $\mathcal{L}$ of discrete box $\{0, 1\}^n$ into boxes which is a discrete analogue of the partition $\mathcal{F}$ from the previous section. The boxes $K(l) = K_1(l) \times \cdots \times K_n(l) \in \mathcal{L}$, where $l \in L \cup \{(0, \ldots, 0), (1, \ldots, 1)\}$, are of the form:

$$K_i(l) = \begin{cases} 
\{0\} & \text{if } l_i = 0, \\
\{1\} & \text{if } l_i = 1, \\
\{0, 1\} & \text{if } l_i = \ast 
\end{cases}$$

for $i \in [n]$.

Observe now that for every $n \geq 3\text{ odd}$ the set $V(\Lambda_n)$ of the vertices of the Lucas cube is a selector of the family of boxes $\mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$: for every $v \in V(\Lambda_n)$ there is exactly one $K(l) \in \mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$ such that

$$v \in K(l).$$

Indeed, let $K(l) \in \mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$ and pick a point $v = v_1 \cdot \cdots \cdot v_d \in K(l)$ in the following way:

$$v_i = \begin{cases} 
0 & \text{if } K_i(l) = \{0\}, \\
1 & \text{if } K_i(l) = \{1\}, \\
0 & \text{if } K_i(l) = \{0, 1\}.
\end{cases}$$

Since $l$ does not contain two consecutive 1s modulo $n$ and if $K_i = \{0, 1\}$, then $K_{i+1} = \{1\}$, it follows that $v \in V(\Lambda_n)$ and for every $w \in K(l)$, $w \neq v$, there is $i \in [n]$ such that $w_i = 1$ while $v_i = 0$. Thus, $K(l) \cap V(\Lambda_n) = \{v\}$.

References

[1] A.T. Benjamin and J.J. Quinn, Proofs That Really Count: The art of Combinatorial Proof, The Dolciani Mathematical Expositions, Vol. 27, Mathematica Association of America, Washington DC, 2003.

[2] K.S. Briggs, D.P Little and J.A. Sellers, Combinatorial proofs of various q-Pell identities via tilings, Ann. Combin. 14, (2010), 407–418.
[3] A. P. Kisielewicz, Partitions and balanced matchings of an $n$-dimensional cube, to appear in European J. Combin.

[4] J. C. Lagarias and P. W. Shor, Cube tilings and nonlinear codes, Discrete Comput. Geom. 11 (1994), 359–391.

[5] E. Munarini, C. Perelli Cippo, N. Zagaglia Salvi, On the Lucas cubes, Fibonacci Quart. 39 (2001) 12–21.