The Glueball content of $\eta_c$

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We carry out the first lattice QCD derivation of the mixing energy and the mixing angle of the pseudoscalar charmonium and glueball on gauge ensembles with $N_f = 2$ degenerate dynamical charm quarks. The mixing energy is determined to be 47(7) MeV, which seems insensitive to charm quark masses. By the assumption that $X(2370)$ is predominantly a pseudoscalar glueball, the mixing angle is approximately 4.3(5)$^\circ$, which results in a +3.9(5) MeV mass shift of the ground state pseudoscalar charmonium. In the mean time, the mixing can raise the total width of the pseudoscalar charmonium by approximately 7 MeV, which explains to some extent the relative large total width of the $\eta_c$ meson. Resultantly, the branching fraction of $\eta_c \rightarrow \gamma \gamma$ can be understood in this $c \bar{c}$-glueball framework. On the other hand, the seemingly discrepancy of the theoretical predictions and the experimental results of the partial width of $J/\psi \rightarrow \gamma \eta_c$ cannot be alleviated by the $c \bar{c}$-glueball mixing picture yet, which demands future precise experimental measurements of this partial width.

I. INTRODUCTION

The $\eta_c$ meson is usually assigned to be the $1^3S_0$ state of charmonium in the quark model. The latest review of the Particle Data Group (PDG) [1] gives its mass and the total width to be $m_{\eta_c} = 2983.9 \pm 0.4$ MeV and $\Gamma_{\eta_c} = 32.0 \pm 0.7$ MeV, respectively. Its width seems quite large among the charmonium states below the $D \bar{D}$ threshold, since its strong decays take place only through the Okubo-Zweig-Iizuka rule (OZI rule) [2–4] suppressed processes. This large width motivates the scenario that $\eta_c$ may have a sizable glueball component. Among the established flavor singlet pseudoscalar mesons, $\eta(1405)$ is usually taken as a candidate for the pseudoscalar glueball [5–7]. However, the quenched lattice QCD studies [8–10] predict that the mass of the pseudoscalar glueball is around 2.4-2.6 GeV, which is confirmed by lattice simulations with dynamical quarks [11–14]. This raised a question on $\eta(1405)$ as a glueball candidate because of its much lighter mass. On the other hand, there is also a theoretical analysis claiming that $\eta(1405)$ and $\eta(1475)$ can be the same state belong to the $q \bar{q}$-nonet in the 1.3-1.5 GeV mass region [15], such that there is no need of a pseudoscalar glueball state in this region. Given the mass of the pseudoscalar glueball predicted by lattice QCD, it is intriguing to study the possible mixing between the pseudoscalar charmonium and the glueball. Apart from the total width of $\eta_c$, this mixing scenario is also physically relevant to the understanding of $\eta_c$ properties in the $\eta_c \rightarrow \gamma \gamma$ [16–19] and $J/\psi \rightarrow \gamma \eta_c$ processes [20–25], where there exist more or less tensions between the experimental observations and the theoretical expectations. The phenomenological studies on this topic can be found in Refs. [26, 27].

In this work, we investigate the charmonium-glueball mixing relevant to $\eta_c$ in the lattice QCD formalism. There have been pioneering lattice studies of the mixing of the scalar glueball and $q \bar{q}$ mesons [28, 29]. This kind of study must be carried out by the lattice calculation with dynamical quarks, where the generic quark-gluon transition permits the mixing of glueballs and $q \bar{q}$ states. Therefore, in order to explore the possible mixing of $c \bar{c}$ state and the glueball with the same quantum number, the most crucial task is to generate the gauge configurations with charm sea quarks. If there are no charm sea quarks, the transition between the $c \bar{c}$ pair and gluons cannot take place dynamically to switch on the mixing between $c \bar{c}$ states and glueballs. Therefore, as an exploratory study and for the simplicity, we generated two ensembles of gauge configurations with degenerate $N_f = 2$ charm sea quarks. Large statistics are mandatory for glueballs to have good signal-to-noise ratios. Another crucial task is the calculation of the annihilation diagrams of charm quarks, which is highly computational demanding. For this we adopt the distillation method [30] which facilitates us to realize the gauge covariant smearing of quark fields and the all-to-all quark propagators simultaneously.

This paper is organized as follows: In Section II we describe the lattice setup, operator construction and formulation of correlation functions. Section III gives the theoretical formalism of the meson-glueball mixing, where the data analysis and the results can be found. The phenomenological implications of the mixing are discussed in Section IV. Section V summarizes the main results of this work.

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TABLE I. Parameters of two $N_f=2$ gauge ensembles with degenerate charm sea quarks.

| Ensemble | $L^3 \times T$ | $\beta$ | $a_s$(fm) | $\xi$ | $N_{cig}$ | $m_{J/\psi}$(MeV) |
|----------|----------------|--------|------------|------|---------|-------------------|
| I        | $16^3 \times 128$ | 2.8    | 0.1026     | 5    | $\sim 7000$ | 2743             |
| II       | $16^3 \times 128$ | 2.8    | 0.1026     | 5    | $\sim 6000$ | 3068             |

II. NUMERICAL DETAILS

A. Lattice Setup

As an exploratory study, we ignore the effect of light quarks and generate gauge configurations with $N_f=2$ flavors degenerate charm sea quarks on an $L^3 \times T = 16^3 \times 128$ anisotropic lattice with the aspect ratio being set to $\xi = a_s/a_t = 5$, where $a_t$ and $a_s$ are the temporal and spatial lattice spacing, respectively. The lattice spacing $a_s$ is determined to be $a_s = 0.1026$ fm through the static potential and $r_0 = 0.491$ fm. We use the tadpole improved anisotropic clover fermion action and tadpole improved gauge action, the details of lattice action can be found in [14]. To investigate the mass dependence of the mixing, we generate two gauge ensembles (denoted by Ensemble I and Ensemble II) with different bare charm quark masses. The parameters of the gauge ensembles are listed in Table I, where $m_{J/\psi}$ is the corresponding mass of the vector charmonium on these two ensembles. The charm quark mass on Ensemble II is close to the physical one with $m_{J/\psi} = 3068$ MeV, which is not far from the experimental $J/\psi$ mass 3097 MeV. The quark mass on Ensemble I is a little lighter that the physical charm quark mass. We would like to use these two ensembles to check quark mass dependence of our results. In order to get good signals of glueballs, we generate the gauge ensembles with high statistics in this study. In practice, the number of the configurations on Ensemble I and Ensemble II is approximately 7000 and 6000, respectively.

B. Operator Construction and Distillation Method

The principal goal of this work is to investigate the possible mixing of the pseudoscalar glueball and the pseudoscalar $c\bar{c}$ meson, therefore the annihilation diagrams of charm quark and antiquark should be taken care of. For this to be done, we adopt the distillation method [30], whose logic is briefed as follows. First, for each configuration and on each time slice, we calculate $N$ eigenvectors $v_n(t)$ of the gauge covariant three-dimensional lattice Laplacian operator

$$-\nabla^2_{xy}(t) = 6\delta_{xy} - \sum_{j=1}^{3} (U_j(x,t)\delta_{x+j,y} + U_j^\dagger(x-j,t)\delta_{x-j,y})$$

where $i,j$ are spatial indexes and $U_i(x)$ is the gauge link oriented in the $i$-th spatial direction starting from the lattice site $x$. After that, for a given source time slice $t'$ and the spinor index $\alpha = 1,2,3$ or $4$, we solve the linear equation array

$$M_{\beta\alpha}(z,t',x,t)S_{\delta\alpha}(x,t;y,t')v^{(n)}_y(t') = v^{(n')}_z(t')\delta_{\beta\alpha},$$

where $M$ is the Dirac matrix of the fermion action using in this work and $S = M^{-1}$. The solution of the above equation array gives the perambulator

$$z^{nn'}_{\delta\alpha}(t,t') = v^{(n)}_x(t)S_{\delta\alpha}(x,t;y,t')v^{(n')}_y(t').$$

The above procedure runs over all the time slices and all the spinor indexes. Let $V(t)$ denote the $3L^3 \times N$ matrix with each of its columns being one of the eigenvector $v^{(n)}(t)$ and the row number $3L^3$ referring to all the spatial points and color indexes, then theoretically the matrix $V(t)V^\dagger(t)$ defines a hermitian and gauge covariant smearing function $\Phi(x,y;t)$ for the charm quark field $c(x)$.

$$c^{(s)}(x,t) = [V(t)V^\dagger(t)]_{xy}c(y,t) \equiv \Phi(x,y;t)c(y,t) \quad (4)$$

where $c^{(s)}(x,t)$ is the smeared charm quark field, whose all-to-all propagator $S^{(c)}(x,y)$ can be expressed in terms of the perambulator and $v^{(n)}(t)$

$$S^{(c)}_{\alpha\beta}(x,t;y,t') = \Phi(x,w;t)S_{\alpha\beta}(w,t;z,t')\Phi(z,y;t') = v^{(n)}_w(t)\tau_{\alpha\beta}^{nn'}(t,t')v^{(n')}_{y}(t'). \quad (5)$$

Thus the interpolation field operator for the pseudoscalar charmonium can be defined through the smeared charm quark field $c^{(s)}(x)$.

Physically, there is only one flavor of the charm quark, while we have two degenerate flavors of charm quarks denoted by $c_1(x)$ and $c_2(x)$, which compose an ‘isospin’ doublet. Since glueballs are independent of quark flavors and can mix only with flavor singlet mesons, the pseudoscalar charmonium of interest in this work is only the flavor singlet state, whose interpolation field can be defined as

$$\mathcal{O}_T = \frac{1}{\sqrt{2}}(\gamma_1^c c_1^{(s)} \Gamma_1^{c_1} + \gamma_2^c c_2^{(s)} \Gamma_2^{c_2}), \quad (6)$$

where $\Gamma$ refers to $\gamma_5$ or $\gamma_5\gamma_4$. Based on the degeneracy of the two flavors of charm quarks, the correlation function of $\mathcal{O}_T$ can be expressed as

$$C_{CC}(t) = \frac{1}{T} \sum_{t_1=1}^{T} \sum_{xy} \langle \mathcal{O}_T(x,t+t_s)\mathcal{O}_T^\dagger(y,t_s) \rangle$$

$$= \frac{1}{T} \sum_{t_1=1}^{T} \sum_{xy} \langle C(x,t+t_s;y,t_s) + 2D(x,t+t_s;y,t_s) \rangle$$

$$\quad (7)$$
are the eigenstates of the vacuum state. For large operator $O_t$, namely
\begin{equation}
\langle x, t; t' \rangle = \text{Tr}[\Gamma^t S^{(c)}(x, t; t') \Gamma^t S^{(c)}(y, t'; x, t)]
\end{equation}

\end{quote}

In practice, we use $N = 50$ which turn out to be enough for good signals and the ground state enhancement in $C_{GC}(t)$. For the pseudoscalar glueball operator, we adopt the treatment in Ref. [9, 10] to get the optimized hermitian operator $O_G(t) = O_G^+(t)$ coupling mainly to the ground state glueball based on different prototypes of Wilson loops and gauge link smearing schemes. Thus we have the following correlation functions

\begin{equation}
C_{Gc}(t) = \langle O_G(t + t_s) O_G(t_s) \rangle
\end{equation}

\begin{equation}
C_{GC}(t) = \frac{1}{T} \sum_{t_s=1}^{T} \sum_{x} \langle O_T(x, t + t_s) O_T^\dagger(x, t_s) \rangle
\end{equation}

where the $\pm$ sign comes from the hermiticity of $O_T$ and takes the minus sign for $\Gamma = \gamma_5$ (anti-hermitian) and positive sign for $\gamma_5 \gamma_4$ (hermitian).

III. MIXING ANGLES

Strictly speaking, the hadronic states in lattice QCD are the eigenstates $|n\rangle$ of the lattice Hamiltonian $\hat{H}$, which are defined as $\hat{H}|n\rangle = E_n |n\rangle$. For a given quantum number, $|n\rangle$’s span an orthogonal and complete set, namely $\sum_n |n\rangle \langle n| = 1$ with the normalization condition $\langle m|n\rangle = \delta_{mn}$. Therefore, the correlation function $C_{XY}(t)$ of operator $O_X$ and $O_Y$ can be parameterized as

\begin{equation}
C_{XY}(t) = \frac{1}{Z_T} \text{Tr} \left[ e^{-\hat{H}(t-t_s)} O_X e^{-\hat{H}t} O_Y^\dagger \right]
= \frac{1}{Z_T} \sum_{m,n} e^{-E_m(t-t_s)} \langle m|O_X |n\rangle \langle n|O_Y^\dagger |m\rangle e^{-E_n t}
\end{equation}

For large $T$ and assuming the energy $E_0 = 0$ for the vacuum state $|0\rangle$, one has $Z_T \approx 1$ and

\begin{equation}
C_{XY} \approx \sum_{n \neq 0} \left[ \langle 0|O_X |n\rangle \langle n|O_Y^\dagger |0\rangle \left( e^{-E_n t} \pm e^{-E_n(T-t)} \right) \right]
\end{equation}

where the $\pm$ sign is for the same and opposite hermiticities of $O_X$ and $O_Y$, respectively.

On the other hand, for a unitary theory, one can choose another complete state set $\{|\alpha_i\rangle, i = 1, 2, \ldots \}$ as the state basis, such that an eigenstate $|n\rangle$ of $\hat{H}$ can be expressed in terms of $|\alpha_i\rangle$ as

\begin{equation}
|n\rangle = \sum_i C_{ni} |\alpha_i\rangle
\end{equation}

with $\sum_i |C_{ni}|^2 = 1$. In this sense, one can say that $|n\rangle$ is an admixture of states $|\alpha_i\rangle$ whose fraction are $|C_{ni}|^2$.

For the case of this work, our theoretical framework is unitary for charm quarks. As the primary assumption, we choose the state set $\{|\alpha_i\rangle, i = 1, 2, \ldots \}$ of flavor singlet pseudoscalars to be

\begin{equation}
|\alpha_i\rangle = |G_1\rangle, |(c\bar{c})_1\rangle, |G_2\rangle, |(c\bar{c})_2\rangle, \ldots ,
\end{equation}

where $|G_i\rangle$ and $|(c\bar{c})_i\rangle$ are the $i$-th pure gauge glueball state and the pure $c\bar{c}$ state, respectively. This might be physically meaningful since glueball states are well defined and turn out to exist in the quenched approximation, as well as that charmonium states are usually considered as $c\bar{c}$ bound states in the phenomenological studies. It should be emphasized that this assumption is the prerequisite of the following discussion, and is the common ansatz in the phenomenological mixing models.

Obviously, the mixing takes place only between glueball states and $c\bar{c}$ states. If the dynamics of the mixing can be treated as perturbations, then to the lowest order of the perturbation theory, it may be assumed that the mixing is dominated by that between the nearest glueball state and $c\bar{c}$ state. Thus the Hamiltonian $\hat{H}$ can be expressed as

\begin{equation}
H = \left( \begin{array}{ccc} m_{G_1} & x_1 & m_{(c\bar{c})_1} \\ x_1 & m_{G_2} & x_2 \\ m_{(c\bar{c})_1} & x_2 & m_{(c\bar{c})_2} \end{array} \right) \oplus \cdots
\end{equation}

where $m_{G_i}$ and $m_{(c\bar{c})_i}$ are the masses of the state $|G_i\rangle$ and $|(c\bar{c})_i\rangle$, respectively, and $x_i$ are their mixing energies. Thus the eigenstate $|n\rangle = |G_1\rangle, |\eta_c\rangle, |g_2\rangle, |\eta_2\rangle, \ldots$ can be expressed by

\begin{equation}
\begin{pmatrix} |g_1\rangle \\ |\eta_c\rangle \\ |g_2\rangle \\ |\eta_2\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \begin{pmatrix} |G_1\rangle \\ |(c\bar{c})_1\rangle \end{pmatrix}
\end{equation}

with the mixing angle $\theta_i$ being defined by

\begin{equation}
\sin \theta_i = \pm \sqrt{\frac{\delta_i - 1}{2 \delta_i}} \approx \pm \frac{x_i}{\Delta_i}
\end{equation}

where $\Delta_i = |m_{(c\bar{c})_i} - m_{G_i}|$, $\delta_i = \sqrt{1 + 4x_i^2/\Delta^2}$ and the $\pm$ sign comes from the sign of $m_{(c\bar{c})_i} - m_{G_i}$. Therefore, the key task is to extract $\theta_i$ that is the mixing angle of the ground state of the pseudoscalar glueball and the ground state of the pseudoscalar $c\bar{c}$ state.
A. The $\Gamma = \gamma_5$ case

Actually the mixing angle $\theta_i$ can be derived from the correlation function $C_{GG}(t)$ or $C_{CC}(t)$ for the $\Gamma = \gamma_5$ case if we assume boldly that $O_G$ couples almost exclusively with $|G_i\rangle$ and $O_T$ couples exclusively with $|(\overline{c}\overline{c})_i\rangle$, namely,

$$O_G^\dagger |0\rangle = \sum_{n\neq 0} \sqrt{Z_{G_i}} |G_i\rangle$$

$$O_{\gamma_5}^\dagger |0\rangle = \sum_{n\neq 0} \sqrt{Z_{(\gamma_5)_i}} |(\overline{c}\overline{c})_i\rangle \quad (17)$$

With this assumption and by utilizing Eq. (15) and Eq. (17), one has

$$C_{GC}(t) = -\sum_i \sqrt{Z_{G_i}Z_{(\gamma_5)_i}} \cos \theta_i \sin \theta_i \left( e^{-m_{g_i}t} - e^{-m_{g_i}(T-t)} - (e^{-m_{\eta_i}t} - e^{-m_{\eta_i}(T-t)}) \right). \quad (18)$$

Note that in the above equation, we also use that fact that $O_{\gamma_5}$ is anti-hermitian. When $T$ is large, the above parameterization of $C_{GC}(t)$ requires $C_{GC}(t = 0) \approx 0$, which is a direct consequence of the assumption of Eq. (17). The measured $C_{GC}(t)$’s from the ensemble I and II are shown in the right most column of Fig. 1, where one can see that this $C_{GC}(t = 0) \approx 0$ is meet. This manifests that the assumptions in Eq. (18) are reasonable.

In order for $\theta_i$’s to be extracted using Eq. (18), one has to know the parameters $m_{g_i}$, $m_{\eta_i}$, $Z_{G_i}$ and $Z_{(\gamma_5)_i}$, which, based on the assumptions of Eq. (17), are encoded in the correlation functions $C_{CC}(t)$ and $C_{GG}(t)$ as

$$C_{GG}(t) = \sum_i Z_{G_i} \left( e^{-m_{g_i}t} + e^{-m_{g_i}(T-t)} \right)$$

$$C_{CC}(t) = \sum_i Z_{(\gamma_5)_i} \left( e^{-m_{\eta_i}t} + e^{-m_{\eta_i}(T-t)} \right). \quad (19)$$

Therefore, we carry out a simultaneous fit to $C_{GG}(t)$, $C_{CC}(t)$ and $C_{GC}(t)$ through the correlated minimal-$\chi^2$ fitting procedure using the function forms in Eq. (18) and (19). Since we focus on $\theta_1$, in practice we only consider...
the contribution from the lowest two glueball states and two \(c\bar{c}\) states, namely, we use \(i = 1, 2\) in above functions to model the data and treat the second states to be the effective states that take account of the contribution of all the higher states. The calculated results and fit results are shown in Fig. 1. The data points in left most column show the effective masses \(m_{CC}^{\text{eff}}(t)\) of the correlation function \(C_{CC}(t)\) on the two ensembles (the upper panel is for ensemble I and the lower one is for ensemble II), which are defined by

\[
m_{CC}^{\text{eff}}(t) = \ln \frac{C_{CC}(t)}{C_{CC}(t+1)}. \tag{20}
\]

The effective mass \(m_{GG}^{\text{eff}}(t)\) on the two ensembles are shown as data points in the middle column. The right most two panels of Fig. 1 show the correlation functions \(C_{CC}(t)\) obtained on the two ensembles. The curves with error bands are plotted using the best fit parameters obtained through the fitting procedure mentioned above, where the colored bands illustrate the fitting time range. The fitted masses and the mixing angle \(\theta_1\) are collected in Table II with rows started with \(\gamma_5\) on two ensembles, where the masses are converted into the values in physical units through the lattice spacings in Table I along with the information of the fit time window and the related \(\chi^2\) per degree of freedom (\(\chi^2/\text{dof}\)). It is seen that the function forms of Eq. (18) and Eq. (19) describes the data very well with a reasonable \(\chi^2/\text{dof}\). On the ensemble I, the fitted \(\eta_8\) mass is around \(m_{\eta_8} \approx 2.7\) GeV, corresponding to the unphysical charm sea quark mass, while the result on ensemble II is \(m_{\eta_8} \approx 3.0\) GeV and close to the experimental value. On the two ensembles, the fitted pseudoscalar glueball mass is around 2.3 GeV and shows little dependence of the charm quark masses. From the fit, we get the mixing angle \(\theta_1 \approx 7.7(1.1)^\circ\) and \(\theta_1 \approx 4.9(6)^\circ\) on ensemble I and ensemble II respectively. According to Eq. (16) and using the mass differences \(\Delta_1 = m_{c\bar{c}} - m_{c\bar{c}}\) listed in Table. II, the mixing energy \(x_1\) can be derived to be 50(10) MeV and 58(9) MeV on these two ensembles.

### B. The \(\Gamma = \gamma_5\gamma_4\) case

As a cross check, we also carried out the similar calculation by using the \(\Gamma = \gamma_5\gamma_4\) for the interpolation field operator of the pseudoscalar \(c\bar{c}\) states. The corresponding correlation functions \(C_{CC}(t)\) and \(C_{GC}(t)\) are calculated using Eq. (9). The effective masses \(m_{CC}^{\text{eff}}(t)\), \(m_{GG}^{\text{eff}}(t)\) and \(C_{GC}(t)\) on the two ensembles are shown in Fig. 2. It is interesting to see that, in contrast to the case of \(\Gamma = \gamma_5\), the correlation function \(C_{GC}(t)\) does not go to zero when \(t \to 0\) now (see the right most column of Fig. 2). This implies that the assumptions in Eq. (17) may not apply here. If we insist the relation \(O_{\gamma\gamma}\) = \(\sum_{n \neq 0} \sqrt{Z_{G_i}}(G_i)\) still holds, then the second assumption in Eq. (17) should be modified.

Actually, the operator \(O_{\gamma\gamma}\) is the temporal component of the isoscalar axial vector current \(J^\mu_5 = \bar{c}\gamma_5\gamma_\mu c\) with \(c = (c_1, c_2)^T\) here (up to a normalization factor since the charm quark fields in \(O_{\gamma\gamma}\) are spatially smeared). According to the \(U_A(1)\) anomaly of QCD, \(J^\mu_5\) satisfies the following anomalous axial vector relation

\[
\partial_\mu J^\mu_5(x) = 2m_c\bar{c}(x)\gamma_5c(x) + q(x), \tag{21}
\]

where \(q(x) = \frac{g^2}{16\pi^2} \epsilon_{\alpha\beta\rho\sigma} G^\alpha_{\omega\beta} G^\rho_{\mu\sigma}\) is the anomalous term from the \(U_A(1)\) anomaly with \(q\) being the strong coupling constant and \(G^\alpha_{\omega\beta}\) being the strength of color fields. The first term on the right hand side of Eq. (21) is proportional to our operator \(O_{\gamma\gamma}\), thus based on the assumption in Eq. (17) we have

\[
\langle 0 | \partial_\mu J^\mu_5 | G_i \rangle \approx \langle 0 | q(x) | G_i \rangle. \tag{22}
\]

On the other hand, if we introduce the decay constant of the glueball state \(|G_i\rangle\) through the definition

\[
\langle 0 | J^\mu_5(x) | G_i, p = 0 \rangle = i f_{G_i} J^\mu e^{-ipx}, \tag{23}
\]

then we have

\[
\langle 0 | \partial_\mu J^\mu_5(0) | G_i, p = 0 \rangle = m_{G_i}^2 f_{G_i}, \tag{24}
\]

and therefore \(f_{G_i} = \frac{1}{m_{G_i}} \langle 0 | q(0) | G_i \rangle\). Thus we can estimate that

\[
\langle 0 | O_{\gamma\gamma} | G_i, p = 0 \rangle \propto \frac{1}{m_{G_i}} \langle 0 | q(0) | G_i \rangle. \tag{25}
\]

Previous lattice studies show that pseudoscalar states can be accessed by the operator \(q(x)\) \cite{10, 11}, thus the nozero matrix element \(\langle 0 | q(0) | G_i \rangle\) implies the coupling \(\langle 0 | O_{\gamma\gamma} | G_i \rangle \neq 0\). Consequently we have the following matrix elements

\[
\langle 0 | O_{\gamma\gamma} | G_i \rangle = \cos \theta_1 \langle 0 | O_{\gamma\gamma} | G_i \rangle - \sin \theta_1 \langle 0 | O_{\gamma\gamma} | (c\bar{c})_i \rangle
\]

\[
\langle 0 | O_{\gamma\gamma} | n_i \rangle = \sin \theta_1 \langle 0 | O_{\gamma\gamma} | G_i \rangle + \cos \theta_1 \langle 0 | O_{\gamma\gamma} | (c\bar{c})_i \rangle. \tag{26}
\]

As we have discussed above, the optimized glueball operator \(O_G\) couples dominantly to the ground state, which means \(\sqrt{Z_{G_i}} \sim 1 \gg \sqrt{Z_{G_i}}\) for \(i > 1\). Thus the correlation function \(C_{GC}(t)\) can be parameterized as
TABLE II. Ground state mass and mixing angle fitted from operators with $\Gamma = \gamma_5$ and $\Gamma = \gamma_4\gamma_5$ on ensemble I and ensemble II; rows started with avg. are the final weighted average results.

| ensemble | $\Gamma$ | $[t_i, t_h]_{\gamma_5}$ | $[t_i, t_h]_{\gamma_4\gamma_5}$ | $\chi^2$/dof | $m_{\eta_1}$ (MeV) | $m_{\eta_1}$ (MeV) | $\theta_1$ | $x_1$ (MeV) |
|----------|---------|----------------------|-------------------|-----------|-----------------|-----------------|---------|----------|
| I        | $\gamma_5$ | [10, 25]             | [2, 18]            | 10        | 2691(2)         | 2317(51)       | 7.7(1.1)$^\circ$ | 50(10)  |
|          | $\gamma_4\gamma_5$ | [10, 25]               | [2, 18]            | 0.98      | 2685(1)         | 2317(43)       | 6.8(8)$^\circ$ | 44(7)   |
|          | avg.     |                     |                   |           | 2686(1)         | 2317(46)       | 7.1(9)$^\circ$ | 46(8)   |
| II       | $\gamma_5$ | [13, 30]             | [3, 15]            | 1.0       | 2987(9)         | 2308(63)       | 4.9(6)$^\circ$ | 58(9)   |
|          | $\gamma_4\gamma_5$ | [11, 30]               | [2, 15]            | 1.1       | 3013(3)         | 2385(40)       | 4.2(3)$^\circ$ | 46(4)   |
|          | avg.     |                     |                   |           | 3010(1)         | 2363(47)       | 4.3(4)$^\circ$ | 49(6)   |

FIG. 2. Effective mass from two point functions $C_{CC}(t)$, $C_{GC}(t)$ and correlation function $C_{CC}(t)$ for operator with $\Gamma = \gamma_4\gamma_5$ using best fit parameters from $C_{GC}(t)$ of Eq. (18) and Eq. (29) and (27) on ensemble I (top) and ensemble II (bottom), where points with error bar are from simulation data with jackknife estimated error, the light gray band shows the fitted results with best fit parameters in Table II, and the color band indicates the fitting range.

$$C_{GC}(t) = \sqrt{Z_{G_i}(0|O_{\gamma_5\gamma_4}(G_1)} \cos^2 \theta_1 e^{-m_{\eta_1}} - \sum_{i=1}^{2} \sqrt{Z_{G_i} Z_{(\gamma_5\gamma_4),i}} \cos \theta_1 \sin \theta_1 \left( e^{-m_{\eta_1} t} + e^{-m_{\eta_1} (T-t)} - e^{-m_{\eta_1} t} + e^{-m_{\eta_1} (T-t)} \right),$$

(27)

where $\sqrt{Z_{(\gamma_5\gamma_4),i}} = \langle 0|O_{\gamma_5\gamma_4} |(\bar{c}c)_i \rangle$ has been defined.

It is easy to see that

$$C_{GC}(0) = \sqrt{Z_{G_i}(0|O_{\gamma_5\gamma_4}(G_1)} \cos^2 \theta_1.$$ 

(28)

We have obtained $\sin \theta_1 \approx 0.12$, on ensemble I, such that $\cos^2 \theta_1 \approx 1$. With $\sqrt{Z_{G_i}} \sim 1$, one has $\langle 0|O_{\gamma_5\gamma_4}(G_1) \sim C_{GC}(0) \approx 0.075$ (see Fig. 2), which is actually about 2-3 orders smaller than the matrix element $\sqrt{Z_{(\gamma_5\gamma_4),i}} \approx 50$ from $C_{CC}(t)$. Therefore, similar to the $\gamma_5$ case, we parameterize $C_{CC}(t)$ as

$$C_{CC}(t) = \sum_{i} Z_{(\gamma_5\gamma_4),i} \left( e^{-m_{\eta_1} t} + e^{-m_{\eta_1} (T-t)} \right).$$

(29)

Using the first equation of Eq. (19), Eq. (27) and Eq. (29), we carried out a simultaneous fit to $C_{GC}(t)$, $C_{GC}(t)$ and
prediction of the branching ratio of $J/\psi$ radiatively decaying into a pseudoscalar glueball, namely $\text{Br}(J/\psi \to \gamma \eta(2S) = 2.31(80) \times 10^{-4}$ [25]. Therefore, if $X(2370)$ can be taken to be dominated by the pseudoscalar glueball, then the mixing angle and the mass shift of the pseudoscalar charmonium due to the mixing with the pseudoscalar glueball can be estimated to the lowest order of the perturbation theory as

$$\sin \theta \approx \frac{x_1}{m_{\eta_c} - m_{X(2370)}} \approx 0.080(10)$$

$$\Delta m_{\eta_c} \approx \frac{x_1^2}{m_{\eta_c} - m_{X(2370)}} \approx 3.9(9) \text{ MeV.}$$ (34)

These results are relevant to the charmonium hyperfine splitting $\Delta_{\text{HFS}} = m_{J/\psi} - m_{\eta_c}$, which is usually used as a good quantity to calibrate the systematic uncertainties of lattice QCD calculations in charm physics. The PDG2020 result [1] gives $\Delta_{\text{HFS}} = 113.0(4)$ MeV. The latest lattice calculation carried out by the HPQCD collaboration finds $\Delta_{\text{HFS}} = 120.3(1.1)$ MeV at the physical point after considering the quenched QED effects [33]. Obviously, this result, with a much smaller error, still deviates the experimental value by $+7.3(1.2)$ MeV. The uncontrolled systematic uncertainties of this calculation are the charm quark annihilation effects and the possible mixing of pseudoscalar glueball and the pseudoscalar charmonium. As far as the charm annihilation effects are concerned, previous lattice studies show that they contribute little to the $J/\psi$ mass while move the $\eta_c$ mass upward by roughly 2 MeV [34]. We have also investigated this effects using the same ensembles in this work and obtained the mass shift of $\eta_c$ due to the charm annihilation effects is $+3.7(5)$ MeV. We are not sure whether this mass shift is theoretically equivalent to $\Delta m_{\eta_c}$ in Eq. (34) or they can be combined together to give the total mass shift of $\eta_c$. Anyway, these corrections to the $\eta_c$ mass are in the right direction. On the other hand, the effect of light sea quarks, which are not considered in this work, may push the $\eta_c$ mass upward further.

Secondly, we will give a tentative discussion on effects of the glueball-$\bar{c}c$ mixing on the total width of $\eta_c$. With the assumption that $X(2370)$ is predominantly a pseudoscalar glueball, we have approximately

$$\frac{|\mathcal{M}(X \to \text{LH})|}{|\mathcal{M}(\bar{c}c \to \text{LH})|} \approx \left( \frac{\Gamma_{\eta_c} \Gamma_{X}}{\Gamma_{\eta_c} \Gamma_{\bar{c}c}} \right)^{1/2}$$ (35)

where $\mathcal{M}$ refers to the transition amplitude of the pseudoscalar glueball or the pseudoscalar $\bar{c}c$ charmonium decaying into light hadrons. Thus we have

$$\frac{\Gamma_{\eta_c}}{\Gamma_{\bar{c}c}} \approx \left| \cos \theta + \sin \theta \frac{|\mathcal{M}(X \to \text{LH})|^2}{|\mathcal{M}(\bar{c}c \to \text{LH})|} \right|^2$$

$$\approx 1 + 2 \sin \theta \left( \frac{\Gamma_{\eta_c} \Gamma_{X}}{\Gamma_{\eta_c} \Gamma_{\bar{c}c}} \right)^{1/2} \left( \frac{\Gamma_{\eta_c}}{\Gamma_{\bar{c}c}} \right)^{1/2}. \quad (36)$$

If we take $\Gamma_{X} \approx 100$ MeV and use the PDG value $\Gamma_{\eta_c} \approx 32.0$ MeV, then the relative correction of the mixing to
\( \Gamma_{cc} \) is approximately \( \frac{\Gamma_{cc}}{\Gamma_{cc}} - 1 \approx 0.29(4) \), which implies \( \Gamma_{cc} \approx 24.9(8) \) MeV.

The decays of \( cc \) pseudoscalar meson into hadrons can be viewed as that the \( cc \) first decays into two gluons and then the two gluons are hadronized into light hadrons. In this sense, one can take the approximation \( \Gamma_{cc} \approx \Gamma(\bar{c}c \to gg) \). On the other hand, the radiative decay \( \eta_c \to \gamma \gamma \) is dominated by \( cc \to \gamma \gamma \). According to the running of the strong coupling constant \( \alpha_s(\mu) \), at \( \mu \approx m_c \approx 1.5 \) GeV, \( \alpha_s \) takes the value in the range \( 0.3 < \alpha_s < 0.35 \). If one takes \( \alpha = 1/134 \) at the charm quark mass scale, to the leading order QCD correction \[26, 35\] one has

\[
\frac{\Gamma(\bar{c}c \to \gamma \gamma)}{\Gamma(\bar{c}c \to gg)} \approx \frac{8 \alpha^2}{\alpha_s^2} \frac{1 - 3.4 \alpha_s}{\pi} = (1.6 \sim 2.5) \times 10^{-4}
\]

Experimentally, the PDG result of \( \text{Br}(\eta_c \to \gamma \gamma) = (1.61 \pm 0.12 \pm 1.0) \times 10^{-4} \) \[1\]. Considering the ratio \( \Gamma_{cc}/\Gamma_{cc} = 1.29(4) \), the experimental value implies \( \Gamma(\bar{c}c \to \gamma \gamma)/\Gamma(\bar{c}c \to gg) \approx (2.07 \pm 0.17) \times 10^{-4} \), which falls into the range of Eq. (37). Note that the above discussions are just tentative because of the range of the total width of the pseudoscalar glueball, which is predominantly a pseudoscalar glueball.

As for the decay width of \( J/\psi \to \gamma \eta_c \), however, the tension between the experiments and the theoretical predictions cannot be alleviated by the \( cc \)-glueball mixing. PDG gives the world average value \( \text{Br}(J/\psi \to \gamma \eta_c) = (1.7 \pm 0.4) \times 10^{-2} \) \[1\], which corresponds to the partial decay width \( \Gamma(J/\psi \to \gamma \eta_c) = 1.6 \pm 0.4 \) keV. The non-relativistic potential models predict the partial width to be 2.4-2.9 keV \[36, 37\]. The predictions of most of lattice QCD calculations, both quenched and full-QCD ones \[20–25\], are around 2.4-2.9 keV, which are in agreement with those of potential models. As addressed before, since the radiative production rate of the pseudoscalar glueball in the \( J/\psi \) decays is two orders of magnitude smaller than that of the pseudoscalar charmonium \[25\], and \( \eta_c \) has a very small fraction of the pseudoscalar glueball, the mixing cannot change the partial width of \( J/\psi \to \gamma \eta_c \). Hopefully, this discrepancy can be resolved by the future study of the BESIII collaboration using its large \( J/\psi \) event sample.

V. SUMMARY

We generate large gauge ensembles with \( N_f = 2 \) degenerate charm quarks on anisotropic lattices, such that the theoretical framework is unitary for charm quarks. The annihilation diagrams of charm quark are tackled through the distillation method. By calculating the correlation functions of the pseudoscalar quark bilinear operators and the pseudoscalar glueball operator, the mixing energy \( x \approx 49(6) \) MeV and the mixing angle \( \theta \sim 4.3(4)^{\circ} \) have been obtained for the first time through lattice QCD calculations.

The non-zero mixing energy and the mixing angle help to understand the properties of the \( \eta_c \) meson. If \( X(2370) \) observed by BESIII can be taken as predominantly a pseudoscalar glueball, then the \( cc \)-glueball mixing can result in a positive mass shift approximately 3.9(9) MeV of the ground state pseudoscalar charmonium, which serves to understand the discrepancy of lattice and the experimental results of the 1S hyperfine splitting of charmonia. In the mean time, the mixing implies that the total width of the pseudoscalar charmonium can be increased by approximately 7 MeV, which can explain to some extent the relatively large width of \( \eta_c \) in comparison to the theoretical expectations for a pure \( cc \) state. Resultantly, the branching fraction of \( \eta_c \to \gamma \gamma \) can be understood in this \( cc \)-glueball framework. It should be notified that, even though the assumption that \( X(2370) \) is predominantly a pseudoscalar glueball seems compatible with the discussion in this work, its justification should be clarified by future experimental and theoretical investigations. At last, the seemingly discrepancy of the theoretical predictions and the experimental results of the partial width of \( J/\psi \to \gamma \eta_c \) cannot be alleviated by the \( cc \)-glueball mixing picture, which demands future sophisticated experimental studies. The BESIII collaboration may take this mission by the help of its largest \( J/\psi \) event ensemble in the world.

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