ON THE $A_2$ INEQUALITY FOR CALDERÓN-ZYGmund OPERATORS

MICHAEL T. LACEY

Abstract. We prove that for an $L^2(\mathbb{R}^d)$-bounded Calderón-Zygmund operator and weight $w \in A_2$, that we have the inequality below due to Hytönen,

$$\|T\|_{L^2(w) \to L^2(w)} \leq C_T[w]_{A_2}.$$ 

Our proof will appeal to a distributional inequality used by several authors, adapted Haar functions, and standard stopping times.

1. Introduction: Main Theorem

We are interested in estimates for the norms of Calderón-Zygmund operators on weighted $L^p$-spaces, a question that has attracted significant interest recently; definitive estimates of this type were first obtained in [Hyt10], with a range of prior and subsequent contributions. In this paper, we will concentrate on $p = 2$, and give a new proof, more elementary than some of the preceding proofs.

Let $w$ be a weight on $\mathbb{R}^d$ with density also written as $w$. Assume $w > 0$ a.e. We define $\sigma = w^{-1}$, which is defined a.e., and set

$$[w]_{A_2} := \sup_Q \frac{w(Q) \sigma(Q)}{|Q|}.$$ 

We give a new proof of $1.1.$ Theorem. [Hyt10] Let $T$ be an $L^2$ bounded Calderón-Zygmund operator, and $w \in A_2$. It then holds that

$$\|Tf\|_{L^2(w)} \leq C_T[w]_{A_2} \|f\|_{L^2(w)}.$$ 

All proofs in this level of generality have used Hytönen’s random Haar shift representation from [Hyt10]. So does this proof. After this point, two strategies of prior proofs are (a) fundamental appeal to two-weight inequalities, an approach initiated in [LPR10] and further refined in [Hyt10, HLM+11, HP11, HPTV10], or (b) constructions of appropriate Bellman functions [Tre11], extending the works of [Pet07, Bez08]. In our approach, we borrow the distributional inequalities central to the two-weight approach, but then combine them with adapted Haar functions from the Bellman approach. Then, the familiar stopping time considerations of §4 are sufficient to conclude the proof. More detailed histories of this question can be found in the introductions to [HPTV10, Tre11, HP11].

Research supported in part by grant NSF-DMS 0968499.
As we will concentrate on the case of $L^2$ estimates, we will frequently use the notation $\|f\|_w := \left( \int f^2 w(dx) \right)^{1/2}$. At one or two points, an $L^1$ norm is needed, and this will be clearly indicated.

## 2. Haar Shift Operators

In this section, we introduce fundamental dyadic approximations of Calderón–Zygmund operators, the Haar shifts, and state reduction of the Main Theorem 1.1 to a similar statement, Theorem 2.3, in this dyadic model. In so doing, we are following the lead of [Hyt10].

### 2.1. Definition

A dyadic grid is a collection $\mathcal{D}$ of cubes so that for each $Q$ we have that

1. The set of cubes $\{Q' \in \mathcal{D} : |Q'| = |Q|\}$ partition $\mathbb{R}^d$, ignoring overlapping boundaries of cubes.
2. $Q$ is a union of cubes in a collection $\text{Child}(Q) \subset \mathcal{D}$, called the children of $Q$. There are $2^d$ children of $Q$, each of volume $|Q'| = 2^{-d}|Q|$.

We refer to any subset of a dyadic grid as simply a grid.

The standard choice for $\mathcal{D}$ consists of the cubes $2^k \prod_{s=1}^d [n_s, n_s + 1)$ for $k, n_1, \ldots, n_d \in \mathbb{Z}$.

But, the reduction we are stating here depends upon a random family of dyadic grids. This next definition is at slight variance with that of [HLM+11, HPTV10, Hyt10].

### 2.2. Definition

For integers $(m, n) \in \mathbb{Z}^2$, we say that a linear operator $S$ is a (generalized) Haar shift operator of complexity type $(m, n)$ if

$$Sf(x) = \sum_{Q \in \mathcal{D}} S_Q f(x) = \sum_{Q \in \mathcal{D}} \int_Q s_Q(x, y) f(y) \, dy$$

where here and throughout $\ell(Q) = |Q|^{1/d}$, and these properties hold.

1. $s_Q$, the kernel of the component $S_Q$, is supported on $Q \times Q$ and $\|s_Q\|_\infty \leq \frac{1}{|Q|}$. It is easy to check that
   $$\left| \sum_{Q \in \mathcal{D}} \int_Q s_Q(x, y) \right| \leq \frac{1}{|x - y|^d}.$$

2. The kernel $s_Q$ is constant on dyadic rectangles $R \times S \subset Q \times Q$ with $\ell(R) \leq 2^{-m}\ell(Q)$ and $\ell(S) \leq 2^{-n}\ell(Q)$.

3. For any subset $\mathcal{D}' \subset \mathcal{D}$, it holds that we have
   $$\left\| \sum_{Q \in \mathcal{D}'} S_Q f \right\|_2 \leq \|f\|_2.$$

We say that the complexity of $S$ is $\kappa := 1 + \max(m, n)$.

Note that the last property above is an statement about unconditionality of the sum in the operator norm. This is in fact a standard part of Calderón-Zygmund theory—and one that is automatic, depending upon exactly how the definition is formulated. This property is fundamental
to the proofs of this paper, and other results that we merely cite, justifying our inclusion of this property into the definition.

The main results of [HPTV10] (see [HPTV10, Theorem 4.1]; also [Hyt10, Theorem 4.2]) allows us to reduce the proof of the Main Theorem 1.1 to the verification of the following dyadic variant.

2.3. Theorem. Let $S$ be a Haar shift operator with complexity $\kappa$. For $w \in \mathcal{A}_2$, we then have the estimates

\begin{equation}
\|Sf\|_w \lesssim \kappa[w]_{\mathcal{A}_2} \|f\|_w
\end{equation}

Indeed, any polynomial dependence on the complexity parameter $\kappa$ would suffice for Theorem 1.1. (The linear bound in $\kappa$ was shown in [HLM+11, Theorem 2.10] in even greater generality in $L^p$ and maximal truncations. Later, and by different methods, it was shown by [Tre11] as stated above.)

In the remainder of this paper, $S$ will denote a Haar shift operator of complexity $\kappa$, with scales separated by $\kappa$. Namely, we have for a subset $D_\kappa \subset D$,

\begin{equation}
Sf(x) = \sum_{Q \in D_\kappa} \int_Q s_Q(x,y) f(y) \, dy,
\end{equation}

and $D_\kappa$ consists of all dyadic intervals with $\log_2 \ell(Q) = \ell \mod \kappa$, for some fixed integer $0 \leq \ell < \kappa$. In particular, if $Q', Q \in D_\kappa$ and $Q' \subsetneq Q$, then $\int_Q s_Q(x,y)f(y)$ is constant on $Q'$. The dual statement is also true.

3. The Basic Inequalities

We make a remark here about the formulation of the inequalities that we will consider below. Recalling the dual weight $\sigma = w^{-1}$ to an $\mathcal{A}_2$ weight, we will show that

\[ \|S(f\sigma)\|_w \lesssim \kappa[w]_{\mathcal{A}_2} \|f\|_\sigma \]

This is formally equivalent to the statement we are proving, namely (2.4), moreover the inequality above is the natural way to phrase the inequality as it dualizes in the natural way: Interchange the roles of $w$ and $\sigma$. Accordingly, we will especially in the next section, use the notation $\langle f, g \rangle_w$ for the natural inner-product on $L^2(w)$.

The arguments initiated in [LPR10], further refined in [Hyt10, HPTV10, HLM+11], yield the following estimates for Haar shifts on intervals.

3.1. Lemma. Let $w \in \mathcal{A}_2$, $S$ a Haar shift operator of complexity $\kappa$ as in (2.5). For a cube $Q$, and let $Q \subset D_\kappa$ be a collection of cubes contained in $Q$. We have

\begin{equation}
\int_Q |s_Q(1_Q)| w(dx) \lesssim [w]_{\mathcal{A}_2} |Q|,
\end{equation}

\begin{equation}
\int_Q s_Q(1_Q)^2 w(dx) \lesssim [w]_{\mathcal{A}_2}^2 \sigma(Q).
\end{equation}
There are two estimates, one of an $L^1(w)$ norm, with the right hand side being Lebesgue measure. The second is an $L^2(w)$ norm, with the right hand side being in terms of $\sigma$. Indeed, the papers \cite{LPR10, Hyt10, HLM+11, HPTV10} are argued such that the second estimate \eqref{3.3}, combined with a general two-weight theorem, imply the linear bound in $A_2$. Thus, the point of this paper is that the general two-weight theorems are not needed.

We suppress the proof of the $L^2$ estimate, which is based upon a corona decomposition, and distributional estimate. The $L^1$ estimate follows from the same line of attack. The reader can consult for instance \cite[Lemma 5.7]{HPTV10} or \cite[Section 11]{HLM+11}. Of course the $L^2$ estimate implies an estimate for the $L^1$ norm, and it is interesting to note that it is worse than what one gets by using the proof of the $L^2$ estimate.

4. Proof of the Weighted Estimate for the Haar Shift Operators

We will need the martingale difference operators associated with $D_\kappa$, and weight $\sigma$. For cube $Q \in D_\kappa$ consider the martingale difference operator

$$D_\sigma^Q f := \sum_{Q' \subset Q, \ell(Q')=2^{-\kappa} \ell(Q)} \mathbb{E}_Q^\sigma f_{Q'} - \mathbb{E}_Q^\sigma f.$$ 

Here, $\mathbb{E}_Q^\sigma f = \sigma(Q)^{-1} \int_Q f \sigma dx$. The operators $D_\sigma^Q f$ are self-adjoint contractions on $L^2(\sigma)$, and satisfy the fundamental orthogonality relationship is

$$\sum_{Q \in D_\kappa} \| D_\sigma^Q f \|_{\sigma}^2 \leq \| f \|_{\sigma}^2,$$

which holds under minimal assumptions on $\sigma$, satisfied for a weight with density non-negative almost everywhere.

Now, complexity shows that for any fixed $Q$, the components of the Haar shift operator are

$$\int_Q s_Q(x,y) f(y) \sigma(dy) = \mathbb{E}_Q^\sigma f \int_Q s_Q(x,y) \sigma(dy) + \int_Q s_Q(x,y) D_\sigma^Q f(y) \sigma(dy)$$

Note that the bilinear form $\langle S_\sigma f, g \rangle_w$ is the linear combination of the three terms below, and their duals.

$$\langle U_\sigma f, g \rangle_w := \sum_{Q \in D} \mathbb{E}_Q^\sigma f \int_Q \int_Q s_Q(x,y) g(x) \sigma(dy) w(dy)$$

$$\langle V_\sigma f, g \rangle_w := \sum_{Q \in D} \mathbb{E}_Q^\sigma f \int_Q \int_Q s_Q(x,y) D_\sigma^w f(y) \sigma(dy) w(dy) \quad \text{(4.2)}$$

$$\langle W(f, g) \rangle_w := \int_Q s_Q(x,y) D_\sigma^w f(y) D_\sigma^w g(x) \sigma(dy) w(dx)$$

By dual, we mean that the roles of $w$ and $\sigma$ are interchanged, which is relevant to $U_\sigma$ and $V_\sigma$ above. We will show that each of these three bilinear forms is bounded by $[\| w \|_{A_2}, \| f \|_{\sigma}, \| g \|_{\sigma}]$, which estimate also applies to their duals. Recall that we have assumed the separation of scales.
condition (2.5), hence under this condition we have proved Theorem 2.3 with absolute constant. This proves the Theorem as stated, with \( \kappa \) bound in terms of complexity.

We prove the difficult estimate first, the estimate for \( \mathcal{U}_\sigma \).

**The bound for \( \mathcal{U}_\sigma \).** The essential tool is this corona decomposition.

4.3. **Definition.** We say that \( F \subset D_\kappa \) is a set of \( f \)-stopping cubes if these conditions are met.

1. If \( F, F' \in \mathcal{F}, F' \subset F \) then \( \rho(F) := \mathbb{E}_F^f |f| > 4 \mathbb{E}_{Q_0}^f |f| \).
2. Every cube \( Q \in D_\kappa \) is contained in some \( F \in \mathcal{F} \).
3. Let \( D_F \) be those cubes for which \( F \) is the minimal element of \( \mathcal{F} \) containing \( Q \). For every \( Q \in D_F \), we have \( \mathbb{E}_Q^f |f| \leq 4 \mathbb{E}_F^f |f| \).

It is easy to recursively construct such a collection \( \mathcal{F} \), for \( \sigma \in A_2 \), which is the case we are considering.

A basic fact, a consequence of the maximal function estimate for general weights, that we have

\[
\left\| \sum_{F \in \mathcal{F}} \rho(F) 1_F \right\|_2^2 \leq \sum_{F \in \mathcal{F}} \rho(F)^2 \sigma(F) \leq \|f\|_\sigma^2, \quad \rho(F) = \mathbb{E}_F^f |f|^2.
\]

The collections \( D_F \) give a decomposition of \( \mathcal{U}_\sigma \) via

\[
\mathcal{U}_{\sigma,f} := \sum_{Q \in D_F} \mathbb{E}_Q^f \int_Q s_Q(x,y) \sigma(dy)
\]

\[
= \rho(F) \sum_{Q \in D_F} [\mathbb{E}_Q^f \cdot \rho(F)^{-1}] \int_Q s_Q(x,y) \sigma(dy).
\]

Note that the products \( |\mathbb{E}_Q^f \cdot \rho(F)^{-1}| \) are never more than 4, so by unconditionality of Haar shifts, the integral estimates of the previous section apply to the expressions above.

We abandon duality, expanding

\[
\| \mathcal{U}_{\sigma} f \|^2_w \leq \sum_{F \in \mathcal{F}} \| \mathcal{U}_{\sigma,f} f \|^2_w \leq I + 2 \mathbb{I}
\]

\[
I := \sum_{F \in \mathcal{F}} \| \mathcal{U}_{\sigma,f} f \|^2_w,
\]

\[
\mathbb{I} := \sum_{F \in \mathcal{F}} \sum_{F' \in \mathcal{F}'} \int_{F'} \left| \mathcal{U}_{\sigma,f} f \mathcal{U}_{\sigma,f'} f \right| w(dx).
\]

These are the diagonal and off-diagonal terms. The diagonal is immediate from (3.3) and (4.4):

\[
I \leq [w]_{A_2}^2 \sum_{F \in \mathcal{F}} \rho(F)^2 \sigma(F) \leq [w]_{A_2}^2 \|f\|_w^2.
\]
The off-diagonal is as follows. By the separation of scales hypothesis, note that in the definition of $II$, that $\mathbb{U}_{\sigma,f}$ is constant on $F'$ in the display below. Hence, by (3.2), we have

$$II \leq [w]_{A_2} \sum_{F \in \mathcal{F}} \sum_{F' \in \mathcal{F}} \mathbb{E}^\sigma_f |\mathbb{U}_{\sigma,f}f| \cdot \rho(F')|F'|$$

$$\leq [w]_{A_2} \int \sum_{F \in \mathcal{F}} |\mathbb{U}_{\sigma,f}f| \cdot \phi \, dx$$

where $\phi := \sum_{F \in \mathcal{F}} \rho(F)1_F$, and using the identity $w \cdot \sigma \equiv 1$,

$$= [w]_{A_2} \int \sum_{F \in \mathcal{F}} |\mathbb{U}_{\sigma,f}f| \cdot \phi \sqrt{w(x)\sigma(x)} \, dx \leq \|\sum_{F \in \mathcal{F}} \mathbb{U}_{\sigma,f}f \|_w \|\phi\|_\sigma.$$ 

We have however $\|\phi\|_\sigma \leq M \sigma f \|\sigma \leq \|f\|_\sigma$. Combining estimates, we see that we have proved

$$\|\sum_{F \in \mathcal{F}} \mathbb{U}_{\sigma,f}f \|_w^2 \leq [w]_{A_2}^2 \|f\|_\sigma^2 + [w]_{A_2} \|\sum_{F \in \mathcal{F}} \mathbb{U}_{\sigma,f}f \|_w \|f\|_\sigma$$

which immediately implies our linear bound in $A_2$ for the term $\mathbb{U}_{\sigma}$.

**The Remaining Estimates.**

*The bound for $\mathcal{V}_{\sigma,k}(f, g)$.* We consider $\mathcal{V}_{\sigma,k}(f, g)$, defined in (4.2). Using the orthogonality property of martingale differences (4.1), we see that

$$|\mathcal{V}_{\sigma,k}(f, g)| \leq \sum_{Q \in \mathcal{D}} |\mathbb{E}_{Q}^\sigma f| \cdot \left| \int_{Q} s_Q(x, y) \sigma(dy), D_{Q}^w g \right|_w$$

$$\leq \|g\|_w \left[ \sum_{Q \in \mathcal{D}} \left| \mathbb{E}_{Q}^\sigma f \cdot \int_{Q} s_Q(x, y) \sigma(dy) \right|_w^2 \right]^{1/2} \leq [w]_{A_2} \|f\|_\sigma \|g\|_w.$$ 

The last line follows from the bound already proved for the operator $\mathbb{U}_{\sigma}$ and the unconditionality. By a standard averaging over random choices of signs, we can deduce the linear in $A_2$ bound for the square function above. \(\square\)

*The bound for $\mathcal{W}$.* We insert $\sqrt{w\sigma}$ into the integrals below, and use the bound $|s_Q(x, y)| \leq |Q|^{-1}$, to see that

$$\left| \int_{Q} \int_{Q} s_Q(x, y)D_{Q}^w f(y)D_{Q}^w g(x) \, dx \, dy \right| \leq \frac{1}{|Q|} \int_{Q} |D_{Q}^w f(y)| \, dy \int_{Q} |D_{Q}^w g(x)| \, dx$$

$$\leq \|D_{Q}^w f\|_\sigma \|D_{Q}^w g\|_w \left[ \frac{\sigma(Q)w(Q)}{|Q|} \right]^{1/2}$$

$$\leq [w]_{A_2} \|D_{Q}^\sigma f\|_\sigma \|D_{Q}^w g\|_w,$$
since we always have \([w]_{A_2} \geq 1\). The martingale differences are pairwise orthogonal in \(L^2(\sigma)\), and \(L^2(w)\), so that a second application, in the variable \(Q\), of the Cauchy-Schwartz inequality finishes this case. □

4.5. Remark. Rather than consider the operators \(\mathbb{V}_\sigma\) in \((4.2)\), and the dual expression, we could have considered

\[
\tilde{\mathbb{V}}(f, g) := \sum_{Q \in \mathcal{D}} E_Q^2 f \cdot \int_Q \int_Q s_Q(x, y) \sigma(dy)w(dx) \cdot E_Q^w g.
\]

It follows from unconditionality of Haar shift operators, and the estimate \((3.2)\), that we have the uniform estimate

\[
\sum_{Q \in \mathcal{D}} \left| \int_Q \int_Q s_Q(x, y) \sigma(dy)w(dx) \right| \leq [w]_{A_2} |Q_0|, \quad Q_0 \in \mathcal{D}.
\]

From this, it is easy to see that

\[
|\tilde{\mathbb{V}}(f, g)| \leq [w]_{A_2} \int M^\sigma f \cdot M^w g \, dx = [w]_{A_2} \int M^\sigma f \cdot M^w g (w(x)\sigma(s))^{1/2} \, dx \leq [w]_{A_2} \|M^\sigma f\|_{\sigma} \|M^w g\|_w \leq [w]_{A_2} \|f\|_{\sigma} \|g\|_w.
\]

Compare to Section 4 of [RTV11]. But, we do not prefer this proof as it obscures the central role of the operator \(\mathbb{U}_\sigma\).

References

[Bez08] Oleksandra V. Beznosova, Linear bound for the dyadic paraproduct on weighted Lebesgue space \(L^2(w)\), J. Funct. Anal. 255 (2008), no. 4, 994–1007. MR2433959

[CUMP10a] David Cruz-Uribe, José María Martell, and Carlos Pérez, Sharp weighted estimates for approximating dyadic operators, Electron. Res. Announc. Math. Sci. 17 (2010), 12–19.

[CUMP10b] ______, Sharp weighted estimates for classical operators (2010), available at http://arxiv.org/abs/1001.4254.

[HMW73] Richard Hunt, Benjamin Muckenhoupt, and Richard Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176 (1973), 227–251.

[Hyt10] Tuomas Hytönen, The sharp weighted bound for general Calderon-Zygmund operators (2010), available at http://arxiv.org/abs/1007.4330.

[HLM+11] Tuomas Hytönen, Michael T. Lacey, Henri Martikainen, Tuomas Orponen, Maria Carmen Reguera, Eric T. Sawyer, and Ignacio Uriarte-Tuero, Weak And Strong Type Estimates for Maximal Truncations of Calderón-Zygmund Operators on \(A_p\) Weighted Spaces (2011), available at http://www.arxiv.org/abs/1103.5229.

[HP11] T. Hytönen and C. Pérez, Sharp weighted bounds involving \(A_\infty\) (2011), available at http://arxiv.org/abs/1103.5562.

[HPTV10] T. Hytönen, Carlos Pérez, S. Treil, and A. Volberg, Sharp weighted estimates of the dyadic shifts and \(A_2\) conjecture, ArXiv e-prints (2010), available at http://arxiv.org/abs/1010.0755.
[Lac11] Michael T. Lacey, *An $A_p - A_{\infty}$ inequality for the Hilbert Transform*, ArXiv e-prints (2011), available at http://arxiv.org/abs/1104.2199.

[LPR10] Michael T. Lacey, Stefanie Petermichl, and Maria Carmen Reguera, *Sharp $A_2$ inequality for Haar shift operators*, Math. Ann. 348 (2010), no. 1, 127–141.

[Ler10] Andrei K. Lerner, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. Lond. Math. Soc. 42 (2010), no. 5, 843–856. MR2721744

[Pet07] Stefanie Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic*, Amer. J. Math. 129 (2007), no. 5, 1355–1375.

[RTV11] Alexander Reznikov, Sergei Treil, and Alexander Volberg, *A sharp weighted estimate of dyadic shifts of complexity 0 and 1*, ArXiv e-prints (2011), available at http://arxiv.org/abs/1103.5347.

[Tre11] S. Treil, *Sharp $A_2$ estimates of Haar shifts via Bellman function* (2011), available at http://arxiv.org/abs/1105.2252.

School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332, USA

E-mail address: lacey@math.gatech.edu