A novel critical field theory for ferromagnetic quantum criticality in the strong coupling regime of the Hertz-Moriya-Millis theory

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We derive a critical field theory for ferromagnetic quantum criticality. An idea is to take the “spherical” coordinate instead of the “cartesian” coordinate in the order parameter space, where ferromagnetic spin fluctuations are decomposed into longitudinal (amplitude) and transverse (directional) spin fluctuations. Reformulating the Hertz-Moriya-Millis theory in terms of these canonical conjugate variables based on a strong coupling approach which diagonalizes the spin-fermion coupling term, we construct an effective field theory in terms of renormalized electrons, longitudinal and transverse spin fluctuations, gapless spin singlet excitations (gauge fluctuations), and their interactions, referred to as U(1) slave spin-rotor theory, which generalizes the U(1) slave rotor theory of charge dynamics to spin dynamics. Introducing quantum corrections into this effective field theory based on the Eliashberg approximation, we perform the scaling analysis to find a critical field theory. The resulting critical field theory turns out to differ from the Hertz-Moriya-Millis theory at least in this level of approximation, where more interaction vertices are marginal, involved with transverse spin fluctuations, while only the spin-fermion interaction vertex is marginal in the Hertz-Moriya-Millis theory. This implies the existence of a novel fixed point in the strong coupling regime of the Hertz-Moriya-Millis theory. This novel fixed point is characterized by localization of transverse spin fluctuations, originating from renormalization by the \( z = 3 \) critical dynamics of longitudinal spin fluctuations, where \( z \) is dynamical critical exponent. We show that the critical field theory differs from the Hertz-Moriya-Millis theory even in three dimensions, where locally critical transverse spin fluctuations are responsible for modifying the Hertz-Moriya-Millis theory.

I. INTRODUCTION

Hertz-Moriya-Millis theory is our standard critical field theory for quantum criticality from Fermi liquids, describing dynamics of local order parameter fluctuations coupled with renormalized electrons [1]. Dynamics of Fermi surface fluctuations gives rise to Landau damping for order parameter fluctuations, resulting in the dynamical critical exponent \( z \) larger than 1, for example, \( z = 2 \) for antiferromagnetic quantum criticality and \( z = 3 \) for ferromagnetic quantum criticality [2, 3]. As a result, the critical field theory turns out to live above the upper critical dimension, identifying a quantum critical point with a Gaussian fixed point. However, there exist dangerously irrelevant interaction vertices, responsible for breaking the hyperscaling relation, which does not allow \( \omega/T \) scaling [2, 3], where \( \omega \) is frequency and \( T \) is temperature.

This conclusion does not seem to be modified, even taking into account dynamics of both order parameter fluctuations and renormalized electrons on equal footing, where a critical field theory is given by

\[
Z = \int Dc_{\sigma\sigma} D\phi^k \exp \left[ -\int d\tau \int dx \int dy \left\{ \sum_{s=\pm} \sum_{\sigma=\uparrow,\downarrow} N_\sigma c_{\sigma\sigma}^\dagger \left( -i \frac{c}{N_\sigma} (-\partial_\tau^2) + iv_F (-\partial_x - \frac{v_F}{2\gamma} \partial_y^2) \right) c_{\sigma\sigma} \right. \right.
\]

\[
- \sum_{k=1}^3 \sum_{s=\pm} \sum_{\alpha,\beta=1}^{N_\sigma} \frac{g_{\phi}}{\sqrt{N_\sigma}} \phi^k c_{\sigma\alpha}^{\dagger} \sigma_{\alpha\beta} c_{\sigma\beta} + \phi^k \left( \gamma \sqrt{-\partial_\tau^2} \right) \left( -\phi^2 \right) \phi^k \left] \right. \right.
\]

(1)

for ferromagnetic quantum criticality in two dimensions, for example. \( c_{\sigma\sigma} \) is a low-energy electron field on a Fermi surface with a patch \( s = \pm \) in the \( \text{Sp}(\frac{1}{2}) \) representation [4], where the spin degeneracy is extended from \( \sigma = \uparrow, \downarrow \) to \( \sigma = 1, \ldots, N_\sigma \). \( x \) is a Fourier-transformed coordinate for a longitudinal momentum orthogonal to a Fermi surface, and \( y \) is that for a transverse momentum along the Fermi surface. \( sv_F \) with \( s = \pm \) is a Fermi velocity on the patch of \( s \) and \( \gamma \) is a Landau damping coefficient, identifying the curvature of a Fermi surface [5] along the direction of the Fermi surface. See Fig. 1. \(-i\frac{c}{N_\sigma} (-\partial_\tau^2)^{\dagger}\) appears from the self-energy of renormalized electrons with a numerical constant \( c \), which results from scattering with \( z = 3 \) critical spin fluctuations [5]. \( \phi^k \) with \( k = 1, 2, 3 \) is an order parameter field for ferromagnetism, where \( \gamma \sqrt{-\partial_\tau^2} \) comes from the polarization bubble of renormalized electrons, responsible for the...
FIG. 1: A schematic diagram of a Fermi surface in the double patch construction. Red-curved lines denote a pair of Fermi surfaces, connected by $2k_F$, inside which electrons are filled with. A coordinate system is defined as the figure for each patch of $s = \pm$.

$z = 3$ critical dynamics, referred to as Landau damping \[2, 3\]. Notice that only the $y$-coordinate dependence appears in dynamics of ferromagnetic order parameter fluctuations. $g_\phi$ is a spin-fermion coupling constant, normalized in the $\text{Sp}(\frac{N_\sigma}{2})$ representation, where $\sigma_{\alpha\beta}$ is a generator for the $\text{Sp}(\frac{N_\sigma}{2})$ rotation, which corresponds to a Pauli spin matrix in the case of $N_\sigma = 2$.

It is straightforward to see that Eq. (1) is a critical field theory, where the spin-fermion coupling constant $g_\phi$ is marginal under the scale transformation \[5\]

$$\tau = b^1\tau', \quad x = b^\frac{2}{3}x', \quad y = b^\frac{1}{3}y',$$ \tag{2}

where $b$ is a scale parameter. This marginality leads one to introduce the spin degeneracy $N_\sigma$, making the fixed point lie in the weak coupling region, which allows him/her to perform a controlled expansion, referred to as $1/N_\sigma$. Actually, it has been argued that vertex corrections give rise to higher order quantum corrections in $1/N_\sigma$, justifying the Hertz-Moriya-Millis theory in the $N_\sigma \to \infty$ limit, where the $z = 3$ critical spin dynamics is governed by the Gaussian fixed point \[2, 3\].

Recently, this problem has been revisited \[6–10\]. First of all, vertex corrections turn out to be not subleading in the $N_\sigma \to \infty$ limit \[9\], sometimes more singular in the $1/N_\sigma$ expansion \[10\]. This means that the critical field theory is strongly coupled even in the $N_\sigma \to \infty$ limit. Quite recently, two ways have been proposed for the controlled expansion: One is to introduce a parameter $x$ into the kinetic energy of order parameter fluctuations in addition to $N_\sigma$, where $x$ gives rise to a nonanalytic interaction potential between renormalized electrons which deviates from a standard Coulomb one \[11\], and the other is to consider a dimensional regularization in the scheme of the fermion renormalization group analysis \[12\], both keeping the fixed point within the weak coupling regime. Although the existence of a perturbative fixed point has been demonstrated quite nicely, it is still not clear how these perturbative fixed points reflect nonperturbative physics originating from self-consistent vertex corrections, which may allow the $\omega/T$ scaling physics for the susceptibility of order parameter fluctuations.

In this study we revisit the Hertz-Moriya-Millis theory, based on a strong coupling approach which diagonalizes the spin-fermion coupling term first \[13, 14\]. An idea is to take the “spherical” coordinate instead of the “cartesian” coordinate in the order parameter space, where ferromagnetic spin fluctuations are decomposed into longitudinal (amplitude) and transverse (directional) spin fluctuations. Actually, an effective theory based on the spherical coordinate of the order parameter space has been proposed for metal-insulator transitions, referred to as $U(1)$ slave rotor theory \[15\], describing charge dynamics in terms of the density-phase representation. Here, we apply the scheme of the $U(1)$ slave-rotor theory to spin dynamics, referred to as $U(1)$ slave spin-rotor theory, where the Hertz-Moriya-Millis theory for critical ferromagnetic spin fluctuations is reformulated in terms of critical longitudinal and transverse spin excitations \[14\]. Taking the strong coupling approach, we reformulate the effective field theory of Eq. (1) in terms of renormalized electrons, longitudinal and transverse spin fluctuations, gapless spin singlet excitations (gauge fluctuations), and their interactions. Introducing quantum corrections into this effective field theory based on the Eliashberg approximation, we find a renormalized field theory, where both longitudinal spin fluctuations and gauge fluctuations are described by $z = 3$ critical dynamics due to Landau damping as expected but dynamics of transverse spin fluctuations is characterized by nonlocal interactions in time, which originate from renormalization by the $z = 3$ critical spin dynamics. Performing the scaling analysis for the resulting renormalized field theory, we find a critical field theory in the strong coupling regime of the Hertz-Moriya-Millis theory, which looks different from the
FIG. 2: A schematic phase diagram for ferromagnetic quantum phase transitions. The $x$-axis is the inverse of a spin-fermion coupling constant, and the $y$-axis is temperature, constituting a conventional phase diagram. Here, we suggest to add an additional axis into this phase diagram, expected to control quantum fluctuations in dynamics of transverse spin excitations. Frankly speaking, it is not clear at all what kind of microscopic physics governs such quantum fluctuations while the spin degeneracy $N_{\sigma}$ is one possibility. “FM” and “FL” denote ferromagnetic ordering and Fermi liquid, respectively. “U1SSR FP” represents U(1) slave spin-rotor fixed point, and “HMM FP” expresses Hertz-Moriya-Millis fixed point (quantum criticality).

It is a conventional view that the fixed point of a ferromagnetic quantum critical point is described by the Hertz-Moriya-Millis theory. Applying the U(1) slave spin-rotor formulation to this ferromagnetic quantum phase transition, we reformulate the Hertz-Moriya-Millis theory in terms of renormalized electrons (holons), longitudinal and transverse spin fluctuations (spinons), gapless spin singlet excitations (gauge fields), and their interactions. This U(1) slave spin-rotor effective theory turns out to reproduce the HMM FP when quantum fluctuations in transverse spin dynamics become weak, giving rise to condensation of spinons. On the other hand, we find that there exists another fixed point, referred to as U1SSR FP, where the Hertz-Moriya-Millis theory becomes modified to contain other marginal interactions, involved with transverse spin fluctuations and spin-singlet excitations. A characteristic feature of this novel fixed point is that nonlocal interactions in dynamics of transverse spin fluctuations, which result from critical dynamics of longitudinal spin fluctuations, give rise to localization for the dynamics of transverse spin fluctuations, responsible for the marginality of interaction vertices involved with transverse spin fluctuations, while only the spin-fermion interaction vertex is marginal in the Hertz-Moriya-Millis theory.

Hertz-Moriya-Millis theory at least in this level of approximation. This implies the existence of a novel fixed point in the strong coupling regime of the Hertz-Moriya-Millis theory. An essential feature of this fixed point is that nonlocal interactions in dynamics of transverse spin fluctuations give rise to localization for the dynamics of transverse spin fluctuations, which turns out to be responsible for the fact that more interaction vertices are marginal, involved with transverse spin fluctuations, while only the spin-fermion interaction vertex is marginal in the Hertz-Moriya-Millis theory. It is quite appealing that the critical field theory differs from the Hertz-Moriya-Millis theory even in three dimensions for the case of the strong coupling regime, where locally critical transverse spin fluctuations modify the Hertz-Moriya-Millis theory. We summarize our physical picture in Fig. 2.
II. REVIEW ON U(1) SLAVE SPIN-ROTOR THEORY

In this section we introduce U(1) slave spin-rotor theory for a ferromagnetic quantum phase transition. Actually, this reformulation has been performed in our recent study \[14\]. Here, we rederive the U(1) slave spin-rotor theory within the double patch construction, which shows a smooth connection to an effective field theory at low energies. However, the procedure is essentially the same as before. One may skip this section if the reader is familiar with the U(1) slave spin-rotor theory.

A. A minimal model: Two patch construction

We start from the Hubbard model

\[
H = -t \sum_{ij} (c_{i\sigma}^\dagger c_{j\sigma} + H.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow},
\]

where \(c_{i\sigma}\) is an electron field with spin \(\sigma\) at site \(i\), and \(t\) and \(U\) are hopping and interaction parameters, respectively.

Focusing on ferromagnetic instability, we perform the Hubbard-Stratonovich transformation for the spin triplet channel and obtain an effective theory

\[
Z = \int Dc_{i\sigma} D\Phi_i \exp\left[-\int_0^{\beta} d\tau \left\{ \sum_i c_{i\sigma}^\dagger (\partial_{\tau} - \mu) c_{i\sigma} - t \sum_{ij} (c_{i\sigma}^\dagger c_{j\sigma} + H.c.) - \sum_i c_{i\sigma}^\dagger \Phi_i \cdot \sigma_{\alpha\beta} c_{i\beta} + \frac{1}{2g} \sum_i \Phi_i^2 \right\}\right],
\]

where \(\Phi_i\) is an order parameter of magnetization at site \(i\). \(g\) is an interaction parameter for the triplet channel, proportional to \(U\), and \(\mu\) is a chemical potential.

Representing the order parameter field in the spherical coordinate as follows

\[
\Phi_i \cdot \sigma_{\alpha\beta} = (m + \delta m_i) n_i \cdot \sigma_{\alpha\beta},
\]

where \(m\) is a uniform magnetization order parameter, \(\delta m_i\) is a longitudinal-fluctuation field, and \(n_i\) is a transverse-fluctuation field. Inserting this expression of the order parameter field into Eq. (4) and performing the Fourier transformation into momentum space, we obtain

\[
Z = \int Dc_{k\sigma} D\delta m_{k'} Dn_q \delta^2(\sum_q n_q n_{-q} - 1) \exp\left[-\int_0^{\beta} d\tau \left\{ \sum_k c_{k\sigma}^\dagger (\partial_{\tau} - \mu + \epsilon_k) c_{k\sigma} - \sum_k \frac{1}{L^d} \sum_q m_{k\sigma}^\dagger n_q \cdot \sigma_{\alpha\beta} c_{k+q\beta}^{\dagger} \Phi_i \cdot \sigma_{\alpha\beta} c_{k\alpha} \right\} + \frac{1}{2g} \sum_q \left( \delta m_q \delta m_{-q} + 2m\delta(q)\delta m_q \right) + L^d \frac{1}{2g} m^2 \right]\}
\]

Following the previous studies \[9,10\], we consider the double patch construction for a low-energy effective field theory, where low energy electrons near \(+k_F\) and \(-k_F\) of a Fermi surface are taken into account. Here, \(k_F\) is a Fermi momentum. Introducing the spinor of \(c_{k\sigma} = \left( \begin{array}{c} c_{k_F + k\sigma} \\ c_{k_F - k\sigma} \end{array} \right)\) for low energy electrons near each patch of the Fermi surface and performing the Fourier transformation, we obtain \(c_{i\sigma} = \left( \begin{array}{c} c_{i+\sigma} \\ c_{i-\sigma} \end{array} \right)\). We also introduce \(\delta m_{k'} = \left( \begin{array}{c} \delta m_{k'} \\ \delta m_{k'} + 2k_F \end{array} \right)\) and obtain \(\delta M_i = \left( \begin{array}{c} \delta m_{i1} \\ \delta m_{i2} \end{array} \right)\) after the Fourier transformation, where \(\delta m_{i1}\) and \(\delta m_{i2}\) represent “ferromagnetic” and “antiferromagnetic” amplitude fluctuations, respectively.

We keep only low energy degrees of freedom and construct a low-energy effective field theory in the double patch representation

\[
Z = \int Dc_{i\sigma} D\delta m_{i\sigma} Dn_{i\sigma} \delta(|n_{i\sigma}|^2 - 1) \exp\left[-\int_0^{\beta} d\tau \left\{ \sum_i c_{i\sigma}^\dagger (\partial_{\tau} - \mu) c_{i\sigma} - st \sum_{ij} (c_{i\sigma}^\dagger c_{j\sigma} + H.c.) - m \sum_i c_{i\sigma}^\dagger n_{i\sigma} \cdot \sigma_{\alpha\beta} c_{i\beta} - \sum_i \delta m_{i1} c_{i\sigma}^\dagger n_{i\sigma} \cdot \sigma_{\alpha\beta} c_{i\beta} - \sum_i \delta m_{i2} c_{i\sigma}^\dagger n_{i\sigma} \cdot \sigma_{\alpha\beta} c_{i-\beta} + \frac{1}{2g} \sum_i \left( \delta m_{i\sigma}^2 + 2m\delta m_{i\sigma} \right) \right\} + L^d \frac{1}{2g} m^2 \right]\]
\]
We emphasize that $c_{i\sigma}$ should be regarded as a low-energy electron field near the Fermi surface, defined on the patch of $s = \pm$. We note that ferromagnetic amplitude fluctuations are involved with low-energy electrons in the same patch while antiferromagnetic ones are associated with those in the opposite patch. Although we use the symbol of $\sum_i$ with a lattice index $i$, this should be regarded to be just formal. A continuum field theory will be constructed in the next section. We point out that the Fermi velocity has an opposite sign in each patch, denoted by $s \cdot t$ in the hopping term.

A problem is how to separate dynamics of transverse spin fluctuations from that of longitudinal ones. An idea is to take a strong coupling approach, which diagonalizes the spin-fermion coupling term first \[13^,14\]. We introduce the CP$^1$ representation \[16\]

$$n_i \cdot \sigma_{\alpha\beta} = U_{i\alpha\gamma}\sigma_{\gamma\delta}^3 U_{i\delta\beta}^\dagger,$$

where

$$U_i = \begin{pmatrix} z_{i\uparrow} & z_{i\downarrow}^* \\ z_{i\downarrow} & -z_{i\uparrow}^* \end{pmatrix}$$

is an SU(2) matrix field. $z_{i\sigma}$ is referred to as spinon. Then, the spin-fermion coupling term becomes diagonalized, introducing a fermion field $f_{i\sigma\beta}$ referred to as holon, where the holon field is related with the electron field through the unitary transformation

$$c_{i\sigma} = U_{i\alpha\beta} f_{i\alpha\beta}.$$

As a result, we obtain an effective theory in the strong coupling limit, reformulated in terms of spinon and holon fields

$$Z = \int Df_{i\sigma} D\delta m_{i\sigma} DU_{i\alpha\beta} \delta(\det U_i - 1) \exp \left[ -\int_0^\beta d\tau \left\{ \sum_i f^\dagger_{i\sigma\alpha} \left( \partial_\tau - \mu \right) \sigma_{\alpha\beta} U_{i\alpha\gamma} \sigma_{\gamma\delta}^3 U_{i\delta\beta}^\dagger f_{i\beta\delta} + m \sum_i f^\dagger_{i\sigma\alpha} \sigma_{\alpha\beta}^3 f_{i\beta\delta} - \sum_i \delta m_{i\alpha} f^\dagger_{i\alpha\beta} \sigma_{\gamma\delta} f_{i\beta\gamma} - \sum_i \delta m_{i\alpha} f^\dagger_{i\alpha\beta} \sigma_{\gamma\delta} f_{i\alpha\beta} - \frac{1}{2g} \sum_i \left( \delta m_{i\sigma}^2 + 2m\delta m_{i\sigma} \right) \right\} \right] + L \frac{1}{2g} m^2 \right].$$

The diagonalization for the spin-fermion coupling gives rise to mutual correlations in the hopping term. A conventional ansatz is to decompose the complex correlated hopping \[13^,14\], resorting to the Hubbard-Stratonovich transformation as follows. The time sector is given by

$$-f^\dagger_{i\sigma\alpha} U_{i\alpha\beta} \sigma_{\alpha\beta} U_{i\alpha\gamma} f_{i\gamma\beta} \rightarrow -x_i \sigma f^\dagger_{i\sigma\alpha} f_{i\alpha\sigma} + y_i z_{i\uparrow}^\dagger \partial_\tau z_{i\sigma} - x_i y_i$$

with two auxiliary field variables of $x_i$ and $y_i$, and the space part is given by

$$-st f^\dagger_{i\sigma\alpha} U_{i\alpha\gamma} U_{i\gamma\beta} f_{i\beta\sigma} = -st \left\{ f^\dagger_{i\sigma\alpha} \left( z_{i\uparrow}^* z_{i\downarrow} + f^\dagger_{i\sigma\alpha} \left( \epsilon_{\sigma\alpha} z_{i\sigma}^* f_{i\sigma\beta} + f_{i\sigma\beta}^\dagger \epsilon_{\sigma\alpha}^* f_{i\sigma\beta}^\dagger \right) \right) f_{i\beta\sigma} + f^\dagger_{i\sigma\alpha} \left( z_{i\downarrow}^* z_{i\sigma} + f^\dagger_{i\sigma\alpha} \left( \epsilon_{\sigma\alpha} z_{i\sigma}^* f_{i\sigma\beta} + f_{i\sigma\beta}^\dagger \epsilon_{\sigma\alpha}^* f_{i\sigma\beta}^\dagger \right) \right) f_{i\beta\sigma} \right\}$$

$$\rightarrow -t \left\{ f^\dagger_{i\sigma\alpha} \chi_{i\downarrow}^\dagger f_{i\sigma\beta} + f^\dagger_{i\sigma\alpha} \chi_{i\downarrow}^\dagger f_{i\sigma\beta}^\dagger \delta m_{i\sigma} - \chi_{i\downarrow}^\dagger \chi_{i\downarrow}^* f_{i\sigma\beta}^\dagger \delta m_{i\sigma} - \chi_{i\downarrow}^\dagger \chi_{i\downarrow}^* f_{i\sigma\beta} \right\}$$

with hopping parameters of $\chi_{i\downarrow}$ and $\chi_{i\uparrow}^{\pm}$, which renormalizes the band width of renormalized electrons (holons) and generates dynamics of spinons, respectively. Here, spin-singlet pairing fluctuations are assumed to be not relevant and neglected. Then, we follow the reading expression

$$Z = \int Df_{i\sigma\alpha} D\delta m_{i\sigma} Dx_i Dy_i D\chi_{i\downarrow}^\dagger D\chi_{i\uparrow}^\dagger D\lambda_i \exp \left[ -\int_0^\beta d\tau \left\{ \sum_i \left( f^\dagger_{i\sigma\alpha} \partial_\tau - \mu - \sigma m \right) f_{i\beta\sigma} + \sigma \left( \delta m_{i\alpha} + x_i f^\dagger_{i\sigma\alpha} f_{i\alpha\sigma} - \sigma \delta m_{i\alpha} f^\dagger_{i\alpha\sigma} f_{i\alpha\sigma} \right) \right\} \right] + \frac{1}{2g} \sum_i \left( \delta m_{i\sigma}^2 + 2m\delta m_{i\sigma} \right) \right] + L \frac{1}{2g} m^2 \right].$$
where \( \lambda_i \) incorporates the unimolecular constraint for the spinor field. Although spin-singlet pairing fluctuations are neglected, relevant approximations have not been made. In other words, integration for \( x_i, y_i, \chi^f_{ij}, \) and \( \chi^{\pm}_{ij} \) recovers Eq. (11) essentially.

Next, we perform integrals for \( x_i \) and \( y_i \). On the other hand, we determine hopping parameters of \( \chi^f_{ij} \) and \( \chi^{\pm}_{ij} \) in the saddle-point approximation, resulting in band renormalization for holons and creating dynamics of spinons. The saddle-point value of \( i\lambda_i \rightarrow \lambda \) plays the role of a mass in spinon excitations. Then, we obtain

\[
Z = \int D_{f\alpha\sigma} Dz_{\alpha\sigma} \exp \left[ -\int_0^\beta d\tau \left\{ \sum_i \left( \sum_{\alpha \sigma} \partial_{\tau} f_{i\alpha\sigma} - \sigma \delta m \sigma f_{i\alpha\sigma} - \sigma \sum_{i'} \chi^f_{ij} f_{i'\alpha\sigma} f_{i\alpha\sigma} \right) 
- st \sum_i \left( \sum_{\alpha \sigma} \partial_{\tau} z_{i\alpha\sigma} - \sigma \delta m \sigma z_{i\alpha\sigma} \right) \right\}, \right.
\]

where the saddle-point analysis for hopping parameters of spinons gives

\[
\chi^f_{ij} = \chi^{\pm}_{ij} = \chi^f_{ij}.
\]

We call this formulation U(1) slave-spin-rotor theory, the name of which is to benchmark U(1) slave-rotor theory for charge fluctuations \cite{13}. This U(1) slave-spin-rotor theory may be regarded as a reformulation of the spin-fermion model in the strong coupling regime. Unfortunately, the U(1) slave-spin-rotor theory turns out to be not stable in contrast with U(1) slave-charge-rotor theory. The positive sign in \( \frac{1}{2g} \sum_i (z^f_{i\alpha\sigma} \partial_{\tau} z_{i\alpha\sigma} - \sigma \delta m \sigma)^2 \) favors stronger directional fluctuations while it is negative in the U(1) slave-rotor theory, serving a parabolic potential for charge fluctuations. This difference originates from the opposite sign when the Hubbard-\( U \) term is decomposed into charge and spin channels.

### B. Integration of \( \delta m_{in} \)

An idea overcoming the inconsistency in dynamics of transverse spin fluctuations is to introduce quantum corrections from longitudinal spin fluctuations, renormalizing their dynamics. Taking the Luttinger-Ward functional approach \cite{17} within the Eliashberg approximation \cite{18}, we construct the partition function as follows

\[
Z = \int D_{f\alpha\sigma} Dz_{\alpha\sigma} \exp \left[ -\int_0^\beta d\tau \left\{ \sum_i \left( \sum_{\alpha \sigma} \partial_{\tau} f_{i\alpha\sigma} - \sigma \delta m \sigma f_{i\alpha\sigma} - st \sum_{i'} \chi^f_{ij} f_{i'\alpha\sigma} f_{i\alpha\sigma} + H.c \right) 
- \frac{1}{2} \int d\tau' \sum_i \sum_{\alpha \sigma} \left( \sigma f_{i\alpha\sigma}^\dagger \left( \sigma f_{i\alpha\sigma} + \frac{1}{g} [z^f_{i\alpha\sigma} \partial_{\tau} z_{i\alpha\sigma} - m] \right) D^{(1)}_{\alpha\sigma} (\tau - \tau'; m) \left( \sigma f_{i'\alpha'\sigma'}^\dagger f_{i'\alpha'\sigma'} + \frac{1}{g} [z^f_{i'\alpha'\sigma'} \partial_{\tau} z_{i'\alpha'\sigma'} - m] \right) \right) \right\}, \right.
\]

\[
- \frac{1}{2} \sum_{i\alpha\sigma} \left\{ \ln \left( \frac{1}{4g} - \Pi^{(1)} (q, i\Omega; m) \right) + \Pi^{(1)} (q, i\Omega; m) D^{(1)} (q, i\Omega; m) \right\}, \right.
\]

\[
- \frac{1}{2} \sum_{i\alpha\sigma} \left\{ \ln \left( \frac{1}{4g} - \Pi^{(2)} (q, i\Omega; m) \right) + \Pi^{(2)} (q, i\Omega; m) D^{(2)} (q, i\Omega; m) \right\}, \right.
\]

\[
- \beta \left\{ t \sum_{ij} \left( \chi^f_{ij} \chi^f_{ij} + H.c. \right) \right\} \right], \right.
\]

where \( D^{(n)} (q, i\Omega; m) = \frac{1}{4g} - \Pi^{(n)} (q, i\Omega; m) \) with \( n = 1, 2 \) is the propagator of ferromagnetic and antiferromagnetic amplitude fluctuations, respectively, and \( \Pi^{(n)} (q, i\Omega; m) \) is the self-energy of the amplitude-fluctuation propagator, given by the fermion polarization bubble.
Performing the Fourier transformation, we obtain

\[
Z = \int Df_{k\sigma}D\zeta_{k\sigma} \exp \left[ -\left( \sum_{i\omega_k} f_{k\sigma}^\dagger \left( -i\omega - \mu - zst\chi^f_\gamma k + \sigma m \frac{\Pi^{(1)}(m)}{4g} \right) \right) \right] 
\]

\[
-\frac{1}{2} \sum_{\Delta i} \beta \sum_{i\omega_k} \sum_{q} \sum_{k-k'} \sum_{\sigma} \sigma f^{\dagger}_{k\sigma}(i\omega_k + i\Omega)D^{(1)}(q, i\Omega; m) \left( \sigma' f^{\dagger}_{k'\sigma'}(i\omega_k)D^{(1)}(q, i\Omega; m) \right) 
\]

\[
-\frac{1}{2} \sum_{\Delta i} \beta \sum_{i\omega_k} \sum_{q} \sum_{k-k'} \sum_{\sigma} \sigma f^{\dagger}_{k\sigma}(i\omega_k + i\Omega)D^{(2)}(q, i\Omega; m) \left( \sigma' f^{\dagger}_{k'\sigma'}(i\omega_k)D^{(2)}(q, i\Omega; m) \right) 
\]

\[
+ \frac{1}{g} \sum_{\Delta i} \beta \sum_{i\omega_k} \sum_{q} \sum_{k-k'} \sum_{\sigma} \sigma f^{\dagger}_{k\sigma}(i\omega_k + i\Omega)D^{(1)}(q, i\Omega; m) \left( i\omega_k - \frac{1}{2}i\Omega \right) \left[ \zeta_{k'\sigma'}^\dagger(i\omega_k)\zeta_{k'\sigma'}(i\omega_k + i\Omega) \right] 
\]

\[
-\frac{1}{2} \sum_{\Delta i} \sum_{q} \left\{ \ln \left( \frac{1}{4g} - \Pi^{(1)}(q, i\Omega; m) \right) + \Pi^{(1)}(q, i\Omega; m)D^{(1)}(q, i\Omega; m) \right\} 
\]

\[
-\frac{1}{2} \sum_{\Delta i} \sum_{q} \left\{ \ln \left( \frac{1}{4g} - \Pi^{(2)}(q, i\Omega; m) \right) + \Pi^{(2)}(q, i\Omega; m)D^{(2)}(q, i\Omega; m) \right\} 
\]

\[
-\frac{1}{2g \beta} \sum_{i\omega_k} \sum_{q} \left\{ \frac{m}{2g} \frac{\Pi^{(1)}(m)}{4g} \left( \zeta_{\sigma\sigma'}^{\dagger} \partial_{\sigma\sigma'} \right) \right\} - \frac{1}{4g} \left( \zeta_{\sigma\sigma'}^{\dagger} \partial_{\sigma\sigma'} \right) \frac{\Pi^{(1)}(q, i\Omega; m)}{4g} \left( \zeta_{\sigma\sigma'}^\dagger \partial_{\sigma\sigma'} \right) - \frac{\Pi^{(1)}(q, i\Omega; m)}{4g} \left( \zeta_{\sigma\sigma'}^\dagger \partial_{\sigma\sigma'} \right) 
\]

\[
+ \left( \lambda - zt\chi^z_{\gamma k} \right) \left( \zeta_{\sigma\sigma'}^\dagger \right) \frac{\Pi^{(1)}(m)}{4g} \left( \zeta_{\sigma\sigma'} \right) - \frac{m^2}{8g} \frac{\Pi^{(1)}(m)}{4g} - \frac{\Pi^{(1)}(m)}{4g} \right] 
\]

where \( zt\chi^f_{\gamma k} \) is a Fourier-transformed expression of \( \chi^f_{ij} \) with a coordination number \( z \), taking \( \chi^f_{ij} = \chi^z \) in the saddle-point approximation. First of all, the inconsistency for dynamics of transverse spin fluctuations is resolved by the renormalization of ferromagnetic amplitude fluctuations. In the temporal part of transverse spin fluctuations, \(-\frac{1}{4g} \frac{\Pi^{(1)}(q, i\Omega; m)}{4g} < 0 \) to\(-\frac{1}{4g} \frac{\Pi^{(1)}(q, i\Omega; m)}{4g} \), we observe the sign change from \(-\frac{1}{4g} \frac{\Pi^{(1)}(q, i\Omega; m)}{4g} \), taking \( \Pi^{(1)}(q, i\Omega; m) > 0 \). As a result, dynamics of transverse spin fluctuations is well defined as it must be. It is also noticeable that the magnetization order parameter is renormalized to \( m \frac{\Pi^{(1)}(m)}{4g} \).

It is straightforward to obtain the Luttinger-Ward free-energy functional in the Eliashberg approximation [18].

\[
F(m, \lambda; \mu, g, T) = -N_\beta \sum_{i\omega_k} \sum_{s} \sum_{k} \left\{ \ln \left( \frac{1}{4g} - \Pi^{(1)}(q, i\Omega; m) \right) \right\} + \frac{1}{2} \sum_{i\omega_k} \ln \left( \frac{1}{4g} - \Pi^{(2)}(q, i\Omega; m) \right) + \frac{1}{2g} \beta \sum_{i\omega_k} \sum_{q} \ln \left( \frac{1}{4g} - \Pi^{(1)}(q, i\Omega; m) \right) 
\]

\[
+ \frac{1}{2g} \beta \sum_{i\omega_k} \sum_{q} \ln \left( \frac{1}{4g} - \Pi^{(2)}(q, i\Omega; m) \right) \right\} + L^d \left( \frac{2zt\chi^z}{\lambda} \right) - \frac{m^2}{8g} \frac{\Pi^{(1)}(m)}{4g} - \frac{\Pi^{(1)}(m)}{4g} \right] 
\]

where

\[
\Pi^{(1)}(q, i\Omega; m) = \frac{N_\beta}{\beta} \sum_{i\omega_k} \sum_{s} \sum_{k} G^{\dagger}_f(k + q, i\omega + i\Omega; m)G^z_f(k, i\omega; m),
\]

\[
\Pi^{(2)}(q, i\Omega; m) = \frac{N_\beta}{\beta} \sum_{i\omega_k} \sum_{s} \sum_{k} G^{\dagger}_f(k + q, i\omega + i\Omega; m)G^{z^*}_f(k, i\omega; m),
\]

\[
\Sigma^f_f(k, i\omega; m) = -\frac{1}{\beta} \sum_{q} \sum_{i\omega_k} D^{(1)}(q, i\Omega; m) - \frac{1}{\beta} \sum_{i\omega_k} \sum_{q} D^{(2)}(q, i\Omega; m),
\]

\[
\Sigma^z_f(k, i\omega; m) = -\frac{1}{\beta} \sum_{q} \sum_{i\omega_k} D^{(1)}(q, i\Omega; m) \left( i\omega + i\Omega/2 \right) \frac{\Pi^{(1)}(q, i\Omega; m)}{1 - 4g\Pi^{(1)}(q, i\Omega; m)}
\]

are self-energies of ferromagnetic, antiferromagnetic amplitude fluctuations, renormalized electrons, and transverse
FIG. 3: Feynmann diagrams in the Eliashberg approximation. Renormalized electrons $f_{\sigma}$ and longitudinal ferromagnetic fluctuations $\phi_1$ consist of the Hertz-Moriya-Millis theory while transverse spin fluctuations $z_{\sigma}$ and $U(1)$ gauge fields $a$ appear in the $U(1)$ slave spin-rotor formulation. Critical dynamics of longitudinal ferromagnetic fluctuations gives rise to not only the consistency (stability) for dynamics of transverse spin fluctuations but also nonlocal correlations in both space and time while longitudinal antiferromagnetic fluctuations $\phi_2$ are gapped, allowed to neglect them at low energies. All interaction vertices are given by diagrams with three lines, which characterize the $U(1)$ slave spin-rotor theory. Self-energy corrections of three bosonic fields, $\phi_1$, $\phi_2$, and $a$ are given by polarization bubbles of renormalized electrons and transverse spin fluctuations. Self-energy corrections of renormalized electrons and transverse spin fluctuations are given by typical one-loop diagrams with a bosonic line. Since longitudinal antiferromagnetic fluctuations are gapped, one may neglect all diagrams involved with $\phi_2$. In the lattice-model construction, dynamics of gauge fluctuations are neglected, not incorporated into the free energy.

spin fluctuations, respectively, and

\[
G_f^+(k, i\omega; m) = \frac{1}{-i\omega - \mu - zst\chi_f^f\gamma_k + \sigma m \frac{\Pi^{(+)}(m)}{\Pi^{(-)}(m)} + \Sigma_f^s(k, i\omega; m)},
\]

\[
D^{(n)}(q, i\Omega; m) = \frac{1}{\frac{1}{4g} - \Pi^{(n)}(q, i\Omega; m)}, \quad G_z(k, i\omega; m) = \frac{1}{\frac{m}{\frac{1}{4g} - \Pi^{(+)}(m)} i\omega - (\lambda - zt\chi^f\gamma_k) - \Sigma_z(k, i\omega; m)}
\]

are Green’s functions of holons, ferromagnetic ($n = 1$), antiferromagnetic ($n = 2$) amplitude fluctuations, and spinons, respectively. The self-energy of ferromagnetic (antiferromagnetic) amplitude fluctuations is given by the polarization bubble of renormalized electrons within the same (opposite) patch. The holon self-energy results from scattering with both amplitude fluctuations, and the spinon self-energy originates from the renormalization by ferromagnetic amplitude fluctuations. We summarize all quantum corrections up to the one-loop level in Fig. 3.

Performing the mean-field analysis for the magnetization order parameter $m$ and the mass parameter of spinons $\lambda$, one can find a phase diagram for a ferromagnetic quantum phase transition, where the condensation transition of spinons is expected to allow novel physics. Generally speaking, one may speculate that the ferromagnetic transition given by the formation of the ferromagnetic order parameter $m$ will not coincide with the condensation transition of spinons given by $\lambda = 0$. However, the existence of the frequency-linear term $\frac{m}{2g} \frac{\Pi^{(+)}(m)}{\Pi^{(-)}(m)} (-i\omega)$ in the ferromagnetic state does not allow this possibility at least within the level of the mean-field analysis since the boson condensation
should always occur at zero temperature in the case of a nonrelativistic dispersion. The only exotic possibility is that the spinon condensation may survive beyond the ferromagnetic quantum critical point given by $m = 0$, where the condensation transition occurs in the Fermi-liquid state slightly above the quantum critical point. However, our renormalization group analysis for an effective field theory shows that this is not possible, where the mass parameter of spinons also vanishes at the quantum critical point.

III. RENORMALIZATION GROUP ANALYSIS

A. A field theory for ferromagnetic quantum criticality approaching from the Fermi-liquid state

Following the patch construction of Ref. [9, 10], we write down an effective field theory in the U(1) slave spin-rotor representation

$$Z = \int Df_{\sigma}Dz_{\sigma}D\phi_{\sigma}Da \exp \left[ - \int_0^3 dx \int_{-\infty}^\infty dy \left\{ f_{\sigma}^* \left( \partial_x - isv_F \partial_x - \frac{v_F^2}{2\gamma} \partial_y^2 \right) f_{\sigma} 
+ \phi_1 \left( \partial_x - v_1^2 \partial_x^2 - v_1^2 \partial_y^2 + m_1^2 \right) \phi_1 + \frac{u_1}{2} \phi_1^2 + \phi_2 \left( -\partial_x^2 - v_2^2 \partial_x^2 - v_2^2 \partial_y^2 + m_2^2 \right) \phi_2 + \frac{u_2}{2} \phi_2^2 + a \left( -\partial_x^2 - v_3^2 \partial_x^2 - v_3^2 \partial_y^2 \right) a 
+ z_1 \left( -v_1^2 \partial_x^2 - v_2^2 \partial_y^2 + m_z^2 \right) z_1 + \frac{u_z}{2} |z_1|^2 - g_1 \phi_1 \phi_1 f_{\sigma}^* f_{\sigma} - g_2 \phi_2 \phi_2 f_{\sigma}^* f_{\sigma} - e_f s\sigma f_{\sigma}^* f_{\sigma} - g_z \phi_1 z_1 \partial_y z_1 
- ie_z a \left( z_1 \partial_x z_1 - \partial_z z_1 \right) \right\} \right]\right],$$

regarded to be a continuum version of Eq. (15). $f_{\sigma}$ is a low-energy renormalized electron field (holon) with spin $\sigma$ on the Fermi surface of a $s = \pm$ patch. Its dispersion relation is given by $\epsilon(k_{||}, k_\perp) = sv_F k_{||} + \frac{1}{2} \gamma k_\perp^2$, where $k_{||}$ is the longitudinal momentum out of the Fermi surface and $k_\perp$ is the transverse momentum along the Fermi surface. See Fig. 1. $v_F$ is a Fermi velocity and $\gamma$ is a Landau-damping coefficient [5]. $\phi_n$ with $n = 1, 2$ represent ferromagnetic and antiferromagnetic amplitude fluctuations, where their dispersion relations are given by non-relativistic $E_1(k_{||}, k_\perp) = v_1^2(k_{||}^2 + k_\perp^2) + m_1^2$ and relativistic $E_2(k_{||}, k_\perp) = \pm \sqrt{v_2^2(k_{||}^2 + k_\perp^2) + m_2^2}$, respectively, although these bare dispersions are not much relevant for their renormalized dynamics. Since only ferromagnetic amplitude fluctuations are allowed to be critical, we assume that antiferromagnetic amplitude fluctuations are gapped ($m_2 \neq 0$) at the ferromagnetic quantum critical point. $u_{n}$ with $n = 1, 2$ are their self-interaction constants. $a$ is a transverse gauge field with the relativistic dispersion $E_a(k_{||}, k_\perp) = u_a \sqrt{k_{||}^2 + k_\perp^2}$, describing phase (transverse) fluctuations ($a_{ij}$) of the hopping parameter given by $\chi^f_{ij} = \chi^f e^{ia_{ij}}$ and $\chi^z_{ij} = \chi^z e^{ia_{ij}}$, where amplitude (longitudinal) fluctuations are assumed to be gapped, not relevant. $z_{\sigma}$ is a transverse spin-fluctuation field (spinon), where the temporal part is given by the one-loop correction from critical ferromagnetic amplitude fluctuations. $v_1$ and $m_z$ are the velocity and mass of spinons, respectively. We show that both the velocity and mass of spinons become renormalized to vanish at the ferromagnetic quantum critical point of $m_1 = 0$, which may be identified with local quantum criticality. $u_z$ is the self-interaction parameter of spinons. Critical ferromagnetic amplitude fluctuations couple to both holons and spinons with coupling constants of $g_1$ and $g_2$, respectively. Gapped antiferromagnetic amplitude fluctuations couple to only holons with $g_2$ while gapless gauge fluctuations do to both holons and spinons with $e_f$ and $e_z$, respectively.

Next, we introduce quantum corrections into this field theory within the Eliashberg approximation as discussed in
the last section. Then, we obtain

$$Z = \int Df_{\sigma z} D\phi_{\sigma} D\phi_{\sigma} \exp \left[ -\frac{\int_{-\infty}^{\infty} d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{\parallel}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{\perp}}{2\pi} \left\{ f_{\sigma z}^{\dagger} \left( -i\omega - sv_F k_{\parallel} \right) \frac{v_F k_{\parallel}^2}{2\gamma_{\omega}^2 N_{\sigma}} - i\frac{c_{\text{sgn}}(\omega)}{N_{\sigma}} \right| \omega \right|^2 \right]$$

$$+ \left[ e_{\sigma z} \left( \Omega, q_{\parallel}, q_{\perp} \right) \sigma f_{\sigma z}^{\dagger} \left( \omega + \Omega, k_{\parallel} + q_{\parallel}, k_{\perp} + q_{\perp} \right) f_{\sigma z} \left( \omega, k_{\parallel}, k_{\perp} \right) \right] \bigg] ,$$

where $\sigma = \uparrow, \downarrow$, is generalized to $\sigma = 1, 2, \ldots, N_{\sigma}$. Both critical ferromagnetic amplitude fluctuations and gapless gauge fluctuations give rise to the $|\omega|^2$ self-energy correction with numerical constants $c_1$ and $c_6$ in holon dynamics, respectively, originating from the $z = 3$ critical dynamics, where $z$ is the dynamical critical exponent. On the other hand, the self-energy correction from antiferromagnetic amplitude fluctuations is marked with $\Sigma_2(\omega, k_{\parallel}, k_{\perp})$, the reason of which will be discussed below. The polarization bubble of $\Pi^{(1)}(q, \Omega)$ within the same patch gives rise to Landau damping for both ferromagnetic amplitude and gauge fluctuations, where $\gamma$ is a damping coefficient. On the other hand, the polarization bubble of $\Pi^{(2)}(q, \Omega)$ given by the opposite patch is expected to give the self-energy correction of $|\omega|^2$ to antiferromagnetic amplitude fluctuations, where the transverse momentum in the denominator of the Landau damping term is cut by $2k_F$. One may concern that the polarization bubble at $q = 2k_F$ will result in a more singular dependence when there exist critical fluctuations with $z = 3$, intensively discussed in Ref. 3. This issue will be more addressed below. Critical ferromagnetic amplitude fluctuations give rise to not only the consistency for the dynamics of transverse spin fluctuations as discussed in the last section but also nonlocal correlations in their temporal dynamics responsible for anomalous scaling in various interaction vertices involved with transverse spin fluctuations. They also generate the holon-spinon coupling term, which turns out to play an important role in our ferromagnetic quantum phase transition, modifying the Hertz-Moriya-Millis theory.

One may criticize that the self-energy correction $\Pi_2(\omega, k_{\parallel}, k_{\perp})$ for longitudinal spin (antiferromagnetic amplitude) fluctuations contains more singular dependence in frequency than the form of Landau damping since there exist $z = 3$ transverse gauge fluctuations. An antiferromagnetic ordering transition with an ordering wave vector $2k_F$ has been investigated in the U(1) spin-liquid state, described by fermionic spinons strongly coupled to U(1) gauge fluctuations of $z = 3$. This study addressed that the $2k_F$ susceptibility is possible to show a divergent behavior at low energies, expected to modify the dynamics of spin fluctuations in the Hertz-Moriya-Millis theory. First of all, we point out that a ferromagnetic transition from a Fermi-liquid state is being considered instead of from the U(1) spin-liquid phase. Of course, the $2k_F$ susceptibility does not show such a singular behavior in the Fermi-liquid state. Even if so, one may suspect that the existence of U(1) gauge fluctuations, which arises from the strong-coupling approach of the U(1) slave spin-rotor theory, can cause the similar singular behavior for the $2k_F$ susceptibility. However, we would like to argue that the correct way of renormalization for the dynamics of longitudinal spin fluctuations may differ from that in the U(1) spin-liquid state because there exist additional critical fluctuations with $z = 3$, where both U(1) gauge fluctuations and critical ferromagnetic amplitude fluctuations should be taken into account on equal
footing. We suspect the possibility that such a singular behavior in the $2k_F$ susceptibility may be canceled by the interplay between both gapless fluctuations, where the asymmetry in the sign difference between the holon-gauge and holon-\phi_1 interaction vertices would play an essential role. In this paper we assume that the divergence in the $2k_F$ susceptibility does not exist and thus, antiferromagnetic amplitude fluctuations are gapped. Such gapped excitations do not generate any singular self-energy corrections to the dynamics of holons, allowing us to neglect $\Sigma_2(\omega, k\parallel, k\perp)$.

Performing the Fourier transformation toward the real space, we find a consistent U(1) slave-spin-rotor effective field theory in terms of renormalized electrons, critical longitudinal spin (ferromagnetic amplitude) fluctuations, transverse spin fluctuations, U(1) gauge fluctuations, and their interactions, where quantum corrections are taken into account in the Eliashberg approximation near ferromagnetic quantum criticality.

$$Z = \int Df_{\sigma\sigma}D\phi Da \exp \left[ -\int_0^\beta d\tau \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \left\{ f_{\sigma\sigma}^\dagger \left( -i e N_\sigma \partial_x - \frac{v_F}{2}\partial_y \right) f_{\sigma\sigma} + \phi \left( \gamma \sqrt{-\partial_x^2 - \partial_y^2} \right) \phi + \frac{u_\phi}{2} \phi^4 + a \left( \gamma \sqrt{-\partial_x^2 - \partial_y^2} \right) a - \frac{g_\phi}{\sqrt{N_\sigma}} \phi f_{\sigma\sigma} f_{\sigma\sigma} - \frac{e_f}{\sqrt{N_\sigma}} s\sigma f_{\sigma\sigma} f_{\sigma\sigma} \right] \right].$$

(24)

Here, $\phi_1 \to \phi$, $c_1 + c_0 \to c$, $g_1 \to g_\phi$, $g_1g_2 \to g_c$, and $g^2/2 \to g_0$ have been performed for notational simplicity. Resorting to robustness of the Fermi surface, we keep dynamics along the transverse momentum for boson excitations $q$. In other words, boson dynamics along $-\partial_x^2$ are not relevant.

### B. Considering amplitude fluctuations only

We perform the scaling analysis for our renormalized field theory Eq. (24). Before we take into account all terms of this effective field theory, we focus on longitudinal spin fluctuations first in order to set our reference which corresponds to the Hertz-Moriya-Millis theory. Consider the field theory given by

$$Z = \int Df_{\sigma\sigma}D\phi Da \exp \left[ -\int_0^\beta d\tau \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \left\{ f_{\sigma\sigma}^\dagger \left( -i e N_\sigma \partial_x - \frac{v_F}{2}\partial_y \right) f_{\sigma\sigma} + \phi \left( \gamma \sqrt{-\partial_x^2 - \partial_y^2} \right) \phi + \frac{u_\phi}{2} \phi^4 + a \left( \gamma \sqrt{-\partial_x^2 - \partial_y^2} \right) a - \frac{g_\phi}{\sqrt{N_\sigma}} \phi f_{\sigma\sigma} f_{\sigma\sigma} - \frac{e_f}{\sqrt{N_\sigma}} s\sigma f_{\sigma\sigma} f_{\sigma\sigma} \right] \right].$$

(25)

Performing the Fourier transformation, we obtain

$$Z = \int Df_{\sigma\sigma}D\phi Da \exp \left[ -\int_0^\infty \frac{d\omega}{2\pi} \int_{-\infty}^\infty \frac{dk\parallel}{2\pi} \int_{-\infty}^\infty \frac{dk\perp}{2\pi} \left\{ f_{\sigma\sigma}^\dagger (\omega, k\parallel, k\perp) \left( -i e N_\sigma |\omega| \frac{\partial_t}{\partial_t} - \frac{v_F}{2}\partial_y \right) f_{\sigma\sigma} + \phi (\omega, k\parallel, k\perp) \left( \gamma \frac{|\omega|}{k\perp} + v_\phi^2 k_\perp^2 \right) \phi (\omega, -k\parallel, -k\perp) + a (\omega, k\parallel, k\perp) \left( \gamma \frac{|\omega|}{k\perp} + v_\phi^2 k_\perp^2 \right) a (-\omega, -k\parallel, -k\perp) - \frac{g_\phi}{\sqrt{N_\sigma}} \phi f_{\sigma\sigma} f_{\sigma\sigma} - \frac{e_f}{\sqrt{N_\sigma}} s\sigma f_{\sigma\sigma} f_{\sigma\sigma} \right] \right].$$

(26)

Assuming the robustness of fermion dynamics, we introduce the scale transformation of

$$\omega = b^{-1}\omega', \quad k\parallel = b^{-\frac{2}{3}}k\parallel', \quad k\perp = b^{-\frac{2}{3}}k\perp',$$

(27)

which leads all renormalized kinetic energies of holons, longitudinal spin fluctuations, and U(1) gauge fluctuations to be invariant under the transformation of

$$f_{\sigma\sigma}(\omega, k\parallel, k\perp) = b^{\frac{2}{3}}f_{\sigma\sigma}'(\omega', k\parallel', k\perp'), \quad \phi(\omega, k\parallel, k\perp) = b^{\frac{2}{3}}\phi'(\omega', k\parallel', k\perp'), \quad a(\omega, k\parallel, k\perp) = b^{\frac{2}{3}}a'(\omega', k\parallel', k\perp').$$

(28)
Then, both the spin-fermion coupling and holon-gauge interaction are marginal in two dimensions, shown from
\[ g_\phi = b \frac{-\pi^2}{4} q \phi, \quad e_f = b \frac{-\pi^2}{4} e_f \]
in \(d\)-dimensions. Eq. (25) is a critical field theory, which contains essentially the same structure as Eq. (1) in the introduction.

C. Considering the bosonic sector only

It is interesting to focus on the bosonic sector, given by
\[
Z = \int Dz_{\sigma} D\phi D\alpha \exp \left[ - \int_0^\beta d\tau \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \right. \\
\left. \phi \left( \gamma \sqrt{\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}} - v_a^2 \frac{\partial^2}{\partial x^2} - v_i^2 \frac{\partial^2}{\partial y^2} \right) \phi + \frac{u}{2} \phi^4 \right] \\
+ a \left( \gamma \sqrt{\frac{\partial^2}{\partial x^2}} - v_a^2 \frac{\partial^2}{\partial x^2} - v_i^2 \frac{\partial^2}{\partial y^2} \right) a - \frac{g_d}{N_\sigma} \left( z_\sigma^\dagger \partial_x z_\sigma \right) \frac{1}{\gamma \sqrt{\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}}} \left( z_\sigma^\dagger \partial_x z_\sigma \right) \\
+ z_\sigma^\dagger \left( -v_a^2 \frac{\partial^2}{\partial x^2} - v_i^2 \frac{\partial^2}{\partial y^2} + m^2 \right) z_\sigma + \frac{u}{2} |z_\sigma|^4 - \frac{g_z}{\sqrt{N_\sigma}} \phi z_\sigma^\dagger \partial_x z_\sigma - i \frac{e_z}{\sqrt{N_\sigma}} a \left[ \phi (\partial_x z_\sigma) - (\partial_x z_\sigma) \phi \right],
\]
where \(-\partial^2_z\) has been introduced since there does not exist a Fermi surface in this consideration. Performing the Fourier transformation, we obtain
\[
Z = \int Dz_{\sigma} D\phi D\alpha \exp \left[ - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{\|}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{\perp}}{2\pi} \right. \\
\left. \phi \left( \frac{\omega}{\sqrt{k_{\|}^2 + k_{\perp}^2}} + v_a^2 k_{\|}^2 + v_i^2 k_{\perp}^2 \right) \phi (-\omega, -k_{\|}, -k_{\perp}) \right] \\
+ a \left( \frac{|\omega|}{\sqrt{k_{\|}^2 + k_{\perp}^2}} + v_a^2 k_{\|}^2 + v_i^2 k_{\perp}^2 \right) a (-\omega, -k_{\|}, -k_{\perp}) - \frac{g_d}{N_\sigma} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{dk'_{\|}}{2\pi} \int_{-\infty}^{\infty} \frac{dk'_{\perp}}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{dq_{\|}}{2\pi} \\
\int_{-\infty}^{\infty} \frac{dq_{\perp}}{2\pi} \left( \omega + \Omega, k_{\|} + q_{\|}, k_{\perp} + q_{\perp} \right) z_\sigma (-\omega', k_{\|}, k_{\perp}) \left( \frac{i\omega + i\Omega/2}{\sqrt{q_{\|}^2 + q_{\perp}^2}} + v_a^2 q_{\|}^2 + v_i^2 q_{\perp}^2 \right) z_\sigma^\dagger (-\omega', k_{\|}, k_{\perp}) \\
\left. + z_\sigma^\dagger \left( v_a^2 k_{\|}^2 + v_i^2 k_{\perp}^2 + m^2 \right) z_\sigma (-\omega, k_{\|}, k_{\perp}) \right\}
\]
\[
- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{dq_{\|}}{2\pi} \int_{-\infty}^{\infty} \frac{dq_{\perp}}{2\pi} \left( \frac{g_z}{\sqrt{N_\sigma}} \left( i\omega + i\Omega/2 \right) \phi (\Omega, q_{\|}, q_{\perp}) z_\sigma (-\omega, k_{\|}, k_{\perp}) \right) \\
+ \frac{e_z}{\sqrt{N_\sigma}} a (\Omega, q_{\|}, q_{\perp}) \left( k_{\perp} + q_{\perp} \right) z_\sigma (-\omega, k_{\|}, k_{\perp}) \right\}
\]
(31)

Hinted from the \(z = 3\) critical dynamics, it is natural to take the scale transformation of
\[
\omega = b^{-1} \omega', \quad k_{\|} = b^{-\frac{3}{2}} k'_{\parallel}, \quad k_{\perp} = b^{-\frac{3}{2}} k'_{\perp}.
\]
(32)

Notice that the scale transformation for both momenta is isotropic. It is straightforward to check out that
\[
\phi (\omega, k_{\|}, k_{\perp}) = b^{\frac{3}{2}} \phi' (\omega', k'_{\parallel}, k'_{\perp}), \quad a (\omega, k_{\|}, k_{\perp}) = b^{\frac{3}{2}} a' (\omega', k'_{\parallel}, k'_{\perp})
\]
(33)
guarantee the scale invariance for their renormalized kinetic energies.

On the other hand, there appears uncertainty for the scale transformation in dynamics of spinons. First, we consider
\[
z_\sigma (\omega, k_{\|}, k_{\perp}) = b^{\frac{3}{2}} z_{\sigma'} (\omega', k'_{\parallel}, k'_{\perp}),
\]
which guarantees the scale invariance for their kinetic energy. However, it turns out that this transformation makes all interaction vertices involved with transverse spin fluctuations relevant, shown from
\[
g_d = b^{\frac{3}{2}} g_d', \quad g_z = b^{\frac{3}{2}} g_z', \quad e_z = b^{\frac{3}{2}} e_z'.
\]
(35)

As a result, this transformation rule does not give a fixed-point theory. We interpret this situation as follows. If we assume the scale invariance of the momentum sector in dynamics of transverse spin fluctuations, the frequency
term turns out to be relevant. Then, the spinon dynamics is expected to be static at low energies since only the zero-frequency sector is allowed. As a result, spinons are forced to condense at zero temperature. The condensation of spinons leads us to return back to the Hertz-Moriya-Millis description, where gauge fluctuations become gapped due to Anderson-Higgs mechanism. In this case the U(1) slave spin-rotor theory recovers the Hertz-Moriya-Millis fixed point.

Let us consider the second scale transformation for the spinon field, given by

\[ z_\sigma(\omega, k_\parallel, k_\perp) = b^{z_\sigma} z'_\sigma(\omega', k'_\parallel, k'_\perp), \]

which leads the nonlocal temporal-correlation term invariant. This scale transformation makes both the velocity and mass of spinons irrelevant, shown by

\[ v_z^2 = b^{-\frac{2}{3}} v_z'^2, \quad m_z^2 = b^{-\frac{2}{3}} m_z'^2. \]

As a result, dynamics of transverse spin fluctuations becomes locally critical at this ferromagnetic quantum critical point. Since the longitudinal momentum scales differently from the transverse momentum. It is straightforward to see that the interaction vertex between spinons and U(1) gauge fields is irrelevant, given by \( e_z = b^{-\frac{4}{3}} e'_z \), while the spin-boson coupling is marginal, i.e., \( g_z = g'_z \). As a result, we find a critical field theory

\[ Z = \int Dz_\sigma D\phi Da \exp\left[ -\int_0^\beta d\tau \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \left\{ \phi \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - v_z^2 \partial_x^2 - v_z^2 \partial_y^2 \right\} \phi \right] \]

where longitudinal spin fluctuations remain coupled with transverse spin excitations at the ferromagnetic quantum critical point. Since the \( \phi \quad \sigma \) sector may be identified with the Hertz-Moriya-Millis theory, this critical field theory implies a novel fixed point in the strong coupling regime of the Hertz-Moriya-Millis theory, where locally critical transverse spin fluctuations are expected to modify the \( z = 3 \) critical physics of the Hertz-Moriya-Millis theory.

### D. A novel critical field theory for ferromagnetic quantum criticality

Now, we consider Eq. (24). We take the scale transformation given by

\[ \omega = b^{-1} \omega', \quad k_\parallel = b^{-\frac{2}{3}} k_\parallel', \quad k_\perp = b^{-\frac{2}{3}} k_\perp'. \]

Notice that the longitudinal momentum scales differently from the transverse momentum. It is straightforward to see that

\[ f_{\sigma}(\omega, k_\parallel, k_\perp) = b^{\frac{2}{3}} f'_{\sigma}(\omega', k'_\parallel, k'_\perp), \quad \phi(\omega, k_\parallel, k_\perp) = b^{\frac{2}{3}} \phi'(\omega', k'_\parallel, k'_\perp), \quad a(\omega, k_\parallel, k_\perp) = b^{\frac{2}{3}} a'(\omega', k'_\parallel, k'_\perp) \]

lead all kinetic energies of holons, longitudinal spin fluctuations, and U(1) gauge fields invariant under the scale transformation.

Following the previous discussion, we consider

\[ z_\sigma(\omega, k_\parallel, k_\perp) = b^{z_\sigma} z'_\sigma(\omega', k'_\parallel, k'_\perp), \]

which makes the nonlocal temporal-correlation term of spinons invariant. This transformation rule causes both the velocity and mass of spinons irrelevant, given by

\[ v_z^2 = b^{-1} v_z'^2, \quad m_z^2 = b^{-\frac{4}{3}} m_z'^2. \]

It is quite interesting that all interaction vertices turn out to be marginal at this fixed point except for the spinon-gauge coupling, given by \( e_z = b^{-\frac{4}{3}} e'_z \).
Finally, we find a critical field theory, identifying a novel fixed point in the strong coupling regime of the Hertz-Moriya-Millis theory,

\[
Z = \int Df_{\sigma} Dz_{\sigma} D\phi Da \exp \left[ -\int_0^\beta d\tau \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \left\{ f_{\sigma}^\dagger \left( -i \frac{e}{N_\sigma} (-\partial^2)^{1/2} - \frac{v_F}{2\gamma} \partial_y^2 \right) f_{\sigma} + \phi \left( \gamma \frac{v_\phi}{\sqrt{v_\phi^2 - v_y^2 \partial^2}} - v_y^2 \partial^2 \right) \phi - \frac{g_{\phi}}{\sqrt{v_\phi}} \phi f_{\sigma}^\dagger f_{\sigma} + a \left( \gamma \frac{v_\phi}{\sqrt{v_\phi^2 - v_y^2 \partial^2}} - v_y^2 \partial^2 \right) a - \frac{e_f}{\sqrt{v_\phi}} s_{\pm} a f_{\sigma}^\dagger f_{\sigma} \\
- \frac{g_d}{N_\sigma} (z_{\sigma}^\dagger \partial_{\tau} z_{\sigma}) \frac{1}{\gamma \frac{v_\phi}{\sqrt{v_\phi^2 - v_y^2 \partial^2}} - v_y^2 \partial^2} (z_{\sigma}^\dagger \partial_{\tau} z_{\sigma}) - \frac{g_{\phi}}{N_\sigma} \sigma f_{\sigma}^\dagger f_{\sigma} + \frac{1}{\gamma \frac{v_\phi}{\sqrt{v_\phi^2 - v_y^2 \partial^2}} - v_y^2 \partial^2} (z_{\sigma}^\dagger \partial_{\tau} z_{\sigma}) - \frac{g_d}{\sqrt{v_\phi}} \phi z_{\sigma}^\dagger \partial_{\tau} z_{\sigma} \right] \right].
\]

It is straightforward to extend the present analysis into the three dimensional case, where the fermion self-energy is proportional to $|\omega|$ linearly. If the $z = 3$ quantum criticality of longitudinal spin fluctuations is forced to be protected, the $-\partial_x^2$ term in the dispersion of holons turns out to be irrelevant, which should be backup with $| - \partial_x^2 |^{3/2}$ as the next leading order for the curvature term. Then, the $z = 3$ quantum criticality leads all interactions marginal except for the spin-gauge coupling vertex denoted by $e_z$, irrelevant. As a result, we reach essentially the same expression as Eq. (43). The physical picture of our renormalization group analysis is presented in Fig. 4.

IV. SUMMARY AND DISCUSSION

Emergence of localized magnetic moments and their role in metal-insulator transitions have been central issues for strongly correlated electrons. In the present study we demonstrated that such localized magnetic moments may appear in magnetic quantum phase transitions of itinerant electrons. Of course, the interpretation for the emergence of localized magnetic moments at ferromagnetic quantum criticality should be checked out more carefully, where only transverse spin fluctuations are locally critical but the correlation length in longitudinal spin fluctuations is still diverging. However, it looks plausible that the strong coupling regime may not be described by the Hertz-Moriya-Millis theory. Instead, dynamics of spin fluctuations can be modified, here localized for transverse spin fluctuations due to strong correlations with both longitudinal spin fluctuations and itinerant electrons, while the dynamics of longitudinal spin fluctuations is still of Hertz-Moriya-Millis at least in the Eliashberg approximation. We believe that the emergence of localized magnetic moments at quantum criticality is not limited in ferromagnetism. Recently, we investigated an antiferromagnetic quantum phase transition with an ordering wave vector $2k_F$ based on the U(1) slave spin-rotor representation of the Hertz-Moriya-Millis theory, regarded to be essentially the same strong coupling approach as the present study [21]. There, we found that only transverse spin fluctuations but also longitudinal
spin fluctuations are locally critical, implying that this novel fixed point is described by a critical field theory in terms of emergent locally critical magnetic moments and renormalized electrons. Since dynamics of transverse spin excitations is locally critical, i.e., impurity-like, we expect the $\omega/T$ scaling physics beyond the weak coupling regime of the Hertz-Moriya-Millis theory. This $\omega/T$ scaling physics should be investigated more sincerely near future.

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