Efficient Minimax Optimal Estimators For Multivariate Convex Regression

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Abstract

We study the computational aspects of the task of multivariate convex regression in dimension \( d \geq 5 \). We present the first computationally efficient minimax optimal (up to logarithmic factors) estimators for the tasks of (i) \( L \)-Lipschitz convex regression (ii) \( \Gamma \)-bounded convex regression under polytopal support. The proof of the correctness of these estimators uses a variety of tools from different disciplines, among them empirical process theory, stochastic geometry, and potential theory. This work is the first to show the existence of efficient minimax optimal estimators for non-Donsker classes that their corresponding Least Squares Estimators are provably minimax sub-optimal; a result of independent interest.

1 Introduction and Main Results

In this paper, we consider the following well-specified regression model in the random design setting:

\[
Y = f^*(X) + \xi
\]

where \( f^* : \Omega \rightarrow \mathbb{R} \) lies in a known function class \( \mathcal{F} \), \( X \) is drawn from a known distribution \( \mathbb{P} \) on \( \Omega \), and \( \xi \) is a zero-mean noise with a finite variance \( \sigma^2 \).

In this task, given \((\mathcal{F}, \mathbb{P})\) and \( n \) i.i.d. observations \( D := \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \), we aim to estimate the underlying function \( f^* \) as well as possible with respect to the classical minimax risk (Tsybakov, 2003a). More precisely, define an estimator as a computable function that for any realization of the input \( D \), outputs some measurable function on \( \Omega \), denoted by \( \mathcal{M}(\Omega) \). The minimax risk of such an estimator \( \bar{f} : D \rightarrow \mathcal{M}(\Omega) \) is defined as

\[
\mathcal{R}_n(\bar{f}, \mathbb{P}, \mathcal{F}) := \sup_{f^* \in \mathcal{F}} \mathbb{E}_D \int (f^* - \bar{f})^2 d\mathbb{P},
\]

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and the minimax rate of $\mathcal{F}$ is defined by $\mathcal{M}_n(\mathbb{P}, \mathcal{F}) := \inf_{\hat{f}} \mathcal{R}_n(\hat{f}, \mathbb{P}, \mathcal{F})$. Our goal is to find an estimator that is minimax optimal up to logarithmic factors in $n$, i.e. its risk satisfies

$$\mathcal{R}_n(\hat{f}, \mathbb{P}, \mathcal{F}) = \mathcal{O}_{P, \mathcal{F}}(\mathcal{M}_n(\mathbb{P}, \mathcal{F})) := \mathcal{O}_{P, \mathcal{F}}(\mathcal{M}_n(\mathbb{P}, \mathcal{F}) \cdot \log(n)^{O_{P, \mathcal{F}}(1)}),$$

where $\mathcal{O}_{P, \mathcal{F}}$ denotes equality up to a multiplicative constant that depends only on $\mathbb{P}, \mathcal{F}$. We would also like our estimator to be efficiently computable, which for our purposes means that its runtime $R(f)(n)$ is polynomial in the number of samples: $R(f)(n) = \mathcal{O}_{P, \mathcal{F}}(n^{O_{P, \mathcal{F}}(1)})$.

In this paper, we take $\mathcal{F}$ to be one of the following two function classes, which are subsets of the class of convex functions on a domain:

1. $\mathcal{F}_L(\Omega)$, the class of convex $L$-Lipschitz functions supported on a convex domain $\Omega \subseteq B_d$, where $B_d$ denotes the unit (Euclidean) ball in dimension $d$.
2. $\mathcal{F}^\Gamma(P)$, the class of convex functions on a convex polytope $P \subseteq B_d$ with range contained in $[-\Gamma, \Gamma]$.

These tasks are known as $L$-Lipschitz convex regression (Seijo and Sen, 2011) and $\Gamma$-bounded convex regression (Han and Wellner, 2016), respectively. For reasons which will become apparent later, we always take $d \geq 5$. In our work, we take $\mathbb{P}$ to be the uniform distribution over a convex domain $\Omega$.

**Assumption 1.** $\mathbb{P}$ is uniformly bounded on its support by some positive constants $c(d), C(d)$ that only depend on $d$, i.e. $c(d) \leq \frac{d}{\mathcal{O}(d)}(x) \leq C(d)$, for all $x \in \Omega \subset B_d$.

Convex regression tasks have been a central concern in the “shape-constrained” statistics literature (Devroye and Lugosi, 2012), and have innumerable applications in a variety of disciplines, from economic theory (Varian, 1982) to operations research (Powell and Topaloglu, 2003) and more (Balázs, 2016). In general, convexity is extensively studied in pure mathematics (Artstein-Avidan et al., 2015), computer science (Lovász and Vempala, 2007), and optimization (Boyd et al., 2004). We remark that there is a density-estimation counterpart of the convex regression problem, known as log-concave density estimation (Samworth, 2018; Cule et al., 2010), and these two tasks are closely related (Kur et al., 2019; Kim and Samworth, 2016).

Due to the appearance of convex regression in various fields, it has been studied from many perspectives and by many different communities. For example, in the mathematical statistics literature the minimax rates of convex regression tasks and the risk of the maximum likelihood estimator (MLE) are the main areas of interest; an incomplete sample of works treating this problem is (Guntuboyina, 2012; Guntuboyina and Sen, 2013; Gardner, 1995; Gao and Wellner, 2017; Kur, Rakhlin, and Guntuboyina, Kur et al.; Han, 2019; Brunel, 2013; Diakonikolas et al., 2018, 2016; Carpenter et al., 2018; Kur et al., 2019; Balázs et al., 2015). In operations research, work has focused on the algorithmic aspects of convex regression, i.e., finding scalable and efficient algorithms; see, e.g., (Ghosh et al., 2021; Brunel, 2016; O’Reilly and Chandrasekaran, 2021; Soh and Chandrasekaran, 2021; Balázs, 2016; Mazumder et al., 2019; Chen and Mazumder, 2020; Bertsimas and Mandru, 2021; Siahkamari et al., 2021; Simchowitz et al., 2018; Hannah and Dunson, 2012; Blanchet et al., 2019; Lin et al., 2020; Chen et al., 2021). Initially, convex regression was mostly studied in the univariate case, which is now considered to be well-understood. Multivariate convex regression has only begun to be explored in recent years, and is still an area of active research.

The naïve algorithm for any variant of the convex regression task is the least squares estimator (LSE),
which is also the MLE under Gaussian noise, defined by
\[
\hat{f}_n := \arg\min_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^{n} (Y_i - f(X_i))^2,
\]
(1)
where \( \mathcal{F} = \mathcal{F}_L(\Omega) \) or \( \mathcal{F} = \mathcal{F}^\Gamma(P) \) in our convex regression tasks. From a computational point of view, the LSE can be formulated as a quadratic programming problem with \( O(n^2) \) constraints, and is thus efficiently computable in our terms (Seijo and Sen, 2011; Han and Wellner, 2016); however, the LSE has been seen empirically not to be scalable for large number of samples (Chen and Mazumder, 2020).

From a statistical point of view, the minimax rates of both of our convex regression tasks are \( \Theta_d(C(L, \sigma)n^{-\frac{d}{d+2}}) \) and \( \Theta_d(C(\Gamma, \sigma, P)n^{-\frac{\Gamma}{\Gamma+2}}) \) for all \( d \geq 1 \) (Gao and Wellner, 2017; Bronshtein, 1976; Yang and Barron, 1999). The LSE is minimax optimal only in low dimension, when \( d \leq 4 \) (Birgé and Massart, 1993), while for \( d \geq 5 \) it attains a sub-optimal risk of \( \Theta_d(n^{-\frac{d}{d+2}}) \) (Kur, Rakhlin, and Guntuboyina, Kur et al.; Kur et al., 2020). The poor statistical performance of the LSE for \( d \geq 5 \) has also been verified empirically (Gardner et al., 2006; Ghosh et al., 2021). There are known minimax optimal estimators when \( d \geq 5 \), yet all of them are computationally inefficient. Moreover, all of them are based on some sort of discretization of these function classes, i.e., they consider some \( \epsilon \)-nets (see Definition 29 below). In our tasks, these algorithms require examining nets of cardinality of at least \( O_d(\exp(\Theta_d(n^{\Theta(1)}))) \), and are thus perforce inefficient (Rakhlin et al., 2017; Guntuboyina, 2012).

The empirically-observed poor performance of the LSE and the computational intractability of known minimax optimal estimators have motivated the study of efficient algorithms for convex regression with better statistical properties than the LSE; an incomplete list of relevant works appears above. However, previously studied algorithms are either provably minimax sub-optimal or do not provide any statistical guarantees at all with respect to the minimax risk. We would however like to mention the “adaptive partitioning” estimator constructed in Hannah and Dunson (2013), which is the first provable computationally efficient estimator for convex regression which has been shown to be consistent in the \( L_\infty \) norm. The authors’ approach is somewhat related to our proposed algorithm, but it is unknown whether their algorithm is minimax optimal.

Our main results are the existence of computationally efficient minimax optimal estimators for the task of multivariate Lipschitz convex regression with arbitrary convex support and the (bounded) convex regression with a polytopal support. Specifically, we prove the following results:

**Theorem 2.** Let \( d \geq 5 \) and \( n \geq d + 1 \). Then, under Assumption 1, for the task of \( L \)-Lipschitz convex regression on the convex domain \( \Omega \subset B_d \), there exists an efficient estimator, \( \hat{f}_L \), with runtime of at most \( O_d(n^{O(d)}) \) such that
\[
\mathcal{R}_n(\hat{f}_L, \mathcal{F}_L(\Omega), \mathbb{P}) \leq O_d((\sigma + L)^2 n^{-\frac{d}{d+2}} \log(n)^{h(d)}),
\]
where \( h(d) \leq 3d \).

**Corollary 3.** Let \( d \geq 5 \) and \( n \geq d + 1 \). Then, under Assumption 1, for the task of \( \Gamma \)-bounded convex regression on the polytope \( P \subset B_d \), there exists an efficient estimator, \( \hat{f}^\Gamma \), with runtime of at most \( O_d(n^{O(d)}) \) such that
\[
\mathcal{R}_n(\hat{f}^\Gamma, \mathcal{F}^\Gamma(P), \mathbb{P}) \leq O_d(C(P)(\sigma + \Gamma)^2 n^{-\frac{\Gamma}{\Gamma+2}} \log(n)^{h(d)}),
\]
where \( h(d) \leq 3d \) and \( C(P) \) is a constant that only depends on \( P \).
As we mentioned earlier, for both of these two tasks the minimax rate is of order $n^{-\frac{d}{4}}$, and therefore up to logarithmic factors in $n$ the above estimators are minimax optimal. We note that in Corollary 3, the dependence of the constants on the polytope $P$ is unavoidable. This follows from the results of (Gao and Wellner, 2017; Han and Wellner, 2016), in which the authors showed the geometry of the support of the measure $P$ affects the minimax rate. For example, in the extreme case of $\Omega = B_d$ the minimax rate is of order $n^{-\frac{d}{2+\gamma}}$, which is asymptotically larger than the error rate for polytopes; thus, if we take a sequence of polytopes which approaches $B_d$, the sequence of constants $C(P_n)$ will necessarily blow up.

We consider our results as mainly a proof-of-concept for the existence of efficient estimators for the task of convex regression when $d \geq 5$. Due to their high polynomial runtime, in practice our estimators would probably not work well. However, as we mentioned above, the other minimax optimal estimators in the literature are computationally inefficient, and they all require consideration of some net of exponential size in $n$; our estimator is conceptually quite different. We hope that our insights from our algorithm can be used to construct a practical estimator with the same desirable statistical properties. From a purely theoretical point of view, our estimators are the first known minimax optimal efficient estimators for non-Donsker classes for which their LSE are provably minimax sub-optimal in $L_2$ (see Definition 31 and Remark 32 below). Prior to this work, there were efficient optimal estimators for non-Donsker classes such that their corresponding LSE (or MLE) is provably efficient and optimal (such as log-concave density estimation and Isotonic regression, cf. Kur et al. (2019); Han et al. (2019); Han (2019); Pananjady and Samworth (2022)). Our work should be contrasted with these earlier works. We show that it is possible to overcome the sub-optimality of the LSE with an efficient optimal algorithm in the non-Donsker regime - a result that was unknown before this work.

We prove Theorem 2 in Section 2. The proof of Corollary 3 uses the same method as that of Theorem 2, along with the main result of (Gao and Wellner, 2017, Thm 1.1). We sketch the requisite modifications to Section 4. We conclude this section with the following remarks:

**Remark 4.**

1. We conjecture that our estimators of Theorem 2 and Corollary 3 are minimax-optimal up to constants that only depend on $d, \sigma$, i.e. the $\log(n)$ factors are unnecessary. Our estimators runtime are of order $O_d(n^{O(d)})$, comparing to the sub-optimal convex LSE that has a better runtime of $O_d(n^{O(1)})$.

2. When $d \geq 5$, one can show that when $f^*$ is a max $k$-affine function (restricted to $P \subset B_d$), i.e. $f^* 1_P(x) = \max_{1 \leq i \leq k} a_i^T x + b_i$, our estimator attains a parametric rate, i.e.

$$
\mathbb{E} \int (\hat{f}^P - f^*)^2 dP \leq \tilde{O}_d \left( \frac{C(P, k)}{n} \right).
$$

When $d \leq 4$, Han and Wellner (2016) showed that $\Gamma$-bounded convex LSE, that is defined in Eq. (1) with $F = F^\Gamma(P)$, attains a parametric rate as well. However, when $d \geq 5$, the LSE attains a non-parametric error of $\tilde{O}_d(C(P, k)n^{-4/d})$ Kur et al. (2020) (or a more general result Kur and Rakhlin (2021)). Therefore, our algorithm has the proper adaptive rates when $d \geq 5$; see Ghosh et al. (2021) for more details.

3. An interesting property of our estimator is that the random design setting, i.e. the fact that data points $X_1, \ldots, X_n$ are drawn from $P$ rather than fixed, is essential to its success, a phenomenon not often observed when studying shape-constrained estimators. Usually these estimators also perform well on a "nice enough" fixed design set, for example when $\Omega = [-1/2, 1/2]^d$ and $X_1, \ldots, X_n$ are the regular grid points.
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2 The Proposed Estimator of Theorem 2

In this version of this manuscript (its conference version), we only prove Theorem 2 in the case that $\Omega$ is a polytope with at most $C(d)$ vertices or facets (where $C(d)$ is a constant that only depends on $d$). This covers many classical cases, such as the unit cube, the simplex, and the $\ell_1$ ball. The proof for an arbitrary convex body $\Omega$ involves no fundamentally new insights, but requires a more involved and technical detour through stochastic geometry, and is therefore omitted in this version. First, we state some notations, definitions and preliminary results:

2.1 Notations and preliminaries

Throughout this text, $C, C_1, C_2 \in (1, \infty)$ and $c, c_1, c_2, \ldots \in (0, 1)$ are positive absolute constants that may change from line to line. Similarly, $C(d), C_1(d), C_2(d), \ldots \in (1, \infty)$ and $c(d), c_1(d), c_2(d), \ldots \in (0, 1)$ are positive constants that only depend on $d$ that may change from line to line. We also often use expressions such as $g(n) \leq O_d(f(n))$ and the like to denote that there exists $C_d$ such that $g(n) \leq C_d f(n)$ for all $n$.

For any probability measure $Q$ and $m \geq 0$, we introduce the notation $Q_m$ for the random empirical measure of $Z_1, \ldots, Z_m \sim Q$, i.e. $Q_m = m^{-1} \sum_{i=1}^{m} \delta_{Z_i}$. Also, given a subset $A \subset \Omega$ of positive measure, we let $P_A$ denote the conditional probability measure on $A$. For a positive integer $k$, $[k]$ denotes $\{1, \ldots, k\}$.

Definition 5. A simplex in $\mathbb{R}^d$ is the convex hull of $d + 1$ points $v_1, \ldots, v_{d+1} \in \mathbb{R}^d$ which do not all lie in any hyperplane.

Definition 6. A convex function $f : \Omega \to \mathbb{R}$ is defined to be $k$-simplicial if there exists $\triangle_1, \ldots, \triangle_k \subset \mathbb{R}^{\dim(\Omega)}$ simplices such that $\Omega = \bigcup_{i=1}^{k} \triangle_i$ and for each $1 \leq i \leq k$, we have that $f : \triangle_i \to \mathbb{R}$ is affine.

Note that the definition is more restrictive than the usual definition of a $k$-max affine function (see Remark 2), since the affine pieces of a $k$-max affine function are not constrained to be simplices.

The following result from empirical process theory is a corollary of the peeling device (van de Geer, 2000, Ch. 5), (Bousquet, 2002) and Bronshtein’s entropy bound (Bronshtein, 1976).

Lemma 7. Let $d \geq 5$, $m \geq C^d$ and $Q$ be a probability measure on $\Omega' \subset B_d$. Suppose $Z_1, \ldots, Z_m$ are drawn independently from $Q$, then with probability at least $1 - C_1(d) \exp(-c_1(d)\sqrt{m})$, the following holds
uniformly for all \( f, g \in \mathcal{F}_L(\Omega') \):

\[
2^{-1} \int_{\Omega'} (f - g)^2 d\mathbb{Q} - CL^2 m^{-\frac{d}{2}} \leq \int_{\Omega'} (f - g)^2 d\mathbb{Q}_m \leq 2 \int_{\Omega'} (f - g)^2 d\mathbb{Q} + CL^2 m^{-\frac{d}{2}}.
\]

Next, we introduce a statistical estimator with the high-probability guarantees we shall need, based on a recent result that is presented in (Mourtada et al., 2021, Prop. 1). Its statistical aspects are proven in the seminal works (Tsybakov, 2003b; Lugosi and Mendelson, 2019), and guarantees on its runtime are given in (Hopkins, 2018; Depersin and Lecué, 2019; Hopkins et al., 2020).

**Lemma 8.** Let \( m \geq d + 1, \, d \geq 1, \, \delta \in (0,1) \) and \( \mathbb{Q} \) be a probability measure that is supported on \( \Omega' \subset B_d \). With a known covariance matrix \( \Sigma \). Consider the regression model: \( W = f^*(Z) + \xi \), where \( \| f^* \|_\infty \leq L \), and let \( Z_1, \ldots, Z_m \sim \mathbb{Q} \). Then, the exists an estimator \( \hat{f}_{R,\delta} \) that has an input of \((\Sigma, \{(Z_i, W_i)\}_{i=1}^m)\) and runtime of \( \tilde{O}_d(m) \) and outputs an \( L \)-Lipschitz affine function that satisfies with probability of at least \( 1 - \delta \)

\[
\int (\hat{f}_{R,\delta}(x) - w^*(x))^2 d\mathbb{Q}(x) \leq \frac{C(\sigma + L)^2(d + \log(1/\delta))}{m}
\]

where \( w^* = \argmin_w \text{affine } \int (w - f^*)^2 d\mathbb{Q} \).

### 2.2 Proof of Theorem 2

The first ingredient in our estimator is our new approximation theorem for convex functions:

**Theorem 9.** Let \( \Omega \subset B_d \) be a convex polytope, \( f \in \mathcal{F}_L(\Omega) \), and some integer \( k \geq (Cd)^{d/2} \), for some large enough \( C \geq 0 \). Then, there exists a convex set \( \Omega_k \subset \Omega \) and a \( k \)-simplicial convex function \( f_k : \Omega_k \to \mathbb{R} \) such that

\[
\mathbb{P}(\Omega \setminus \Omega_k) \leq C(\Omega) k^{-\frac{d+2}{d}} \log(k)^{d-1}.
\]

and

\[
\int_{\Omega} (f_k - f)^2 d\mathbb{P} \leq L^2 \cdot O_d(k^{-\frac{d}{2}} + C(\Omega)k^{-1} \log(k)^{d-1}).
\]

Note that the approximation error of the above theorem does not depend on \( \Omega \). Also note that both \( \Omega_k \), as well as \( f_k \), depend on \( f^* \). The bound of Eq. (4) is in fact tight, up to a a constant that only depends on \( d \), cf. (Ludwig et al., 2006).

Note that \( f_k \) is not necessarily an \( L \)-Lipschitz function, i.e., it may be an “improper” approximation to \( f \). We remark that the constant \( C(\Omega) \) depends on the flag number of the polytope \( \Omega \); for more details see (Reitzner et al., 2019). As we mentioned earlier, we assume that the number of vertices or facets of \( \Omega \) is bounded by \( C_d \), the definition of the flag number and the upper bound theorem of McMullen (McMullen, 1970) implies that we can assume \( C(\Omega) \leq C_1(d) \).

**Remark 10.** In the extended version of this manuscript, we prove the following approximation result: Let \( \Omega \subset B_d \) be a convex set, \( f \in \mathcal{F}_L(\Omega) \), and some integer \( k \geq (Cd)^{d/2} \), for some large enough \( C \geq 0 \). Then, there exists a convex set \( \Omega_k \subset \Omega \) and a \( k \)-simplicial convex function \( f_k : \Omega_k \to \mathbb{R} \) such that

\[
\mathbb{P}(\Omega \setminus \Omega_k) \leq Cdk^{-\frac{2(d+2)}{d+4}}.
\]
and
\[
\int_{\Omega} (f_k - f)^2 dP \leq L^2 \cdot O_d(k^{-\frac{d}{2}} \log(k)^{\frac{d+1}{2}}).
\]

We believe that \(\log(k)^{\frac{d+1}{2}}\) factor in the last equation is redundant, and can be removed with a more careful analysis. There are other results regarding that are not sufficient for our results. For example, in \(L_{\infty}\) squared error is significantly higher (Balázs, 2016, Lemma 5.2) or (Dümbgen and Walther, 1996) and its of order \(O_{d,L}(k^{-2/d})\). In \(L_1\) squared one can use the result of (Bárány and Larman, 1988), to show that the rate \(O_{d,L}(k^{-4/d})\). Our contribution is to show that up to logarithmic factors the same rate holds for \(L_2\) squared.

For simplicity, we shall assume that \(L = \sigma = 1\). Also, since we assume that function is 1-Lipschitz, we may assume that that \(f(0) = 0\), and since \(\Omega \subset B_d\), we may also assume that \(\|f^*\|_{\infty} \leq 1\). Finally, we can assume that \(P = U(\Omega)\), since we can always simulate \(\Theta_d(n)\) uniform samples using the method of rejection sampling from a \(P\) that satisfies Assumption 1 (cf. (Devroye, 1986)).

Fix \(n \geq (Cd)^{d/2}\), \(f^* \in F_1(\Omega)\), and set \(k(n) := n^{\frac{d}{d+2}}\). Let \(f_{k(n)} : \Omega_{k(n)} \rightarrow \mathbb{R}\) be the convex function whose existence is guaranteed by Theorem 9 for \(f = f^*\). We have
\[
\int (f^* - f_{k(n)})^2 dP \leq O_d(n^{-\frac{d+1}{d}}),
\]
and there exist \(\triangle_1, \ldots, \triangle_{k(n)} \subset \Omega\) simplices such that \(f_{k(n)}|_{\triangle_i}\) is affine on each \(i\).

If we were given the decomposition of \(\Omega\) into pieces on which \(f^*\) is near-affine, it would be relatively simple to estimate \(f^*\). In Appendix A, we give an algorithm for estimating \(f^*\) under these conditions; the estimator we construct there has better statistical and computational performance than the estimator we construct below, requires weaker assumptions, and is also quite intuitive. We recommend reading Appendix A, to get some intuition for our approach, before attempting the description and correctness proof for our “full” estimator below.

To overcome the fact that we do not know the simplices \(\triangle_i\) on which \(f_k\) is affine, we need another lemma, which says that if we randomly sample a set of points \(X_1, \ldots, X_n\) from \(\Omega\), there exists a triangulation of “most” of \(\Omega_k\) using the points of \(S\), with respect to which \(f_k\) is piecewise affine:

**Lemma 11.** Let \(n \geq d + 1\), and \(\triangle_1, \ldots, \triangle_{k(n)}\) that are defined above, and let \(X_1, \ldots, X_n \sim P\). Then, with probability at least \(1 - n^{-1}\), there exist \(k(n)\) disjoint sets, \(S_1^{(k)}, \ldots, S_n^{(k)}\) of simplices with disjoint interiors such that

1. The vertices of each simplex in \(\bigcup_{i=1}^{k(n)} S_i^{(k)}\) lie in \(\{X_1, \ldots, X_n\}\). Moreover, for each \(1 \leq i \leq k(n)\), we have that \(|S_i^{(k)}| \leq O_d(n^{d/(d+1)})\).

2. For each \(1 \leq i \leq k(n)\), we have that \(\mathbb{P}(\bigcup S_i^{(k)} \subset \triangle_i)\) and
\[
\mathbb{P}(\bigcup S_i^{(k)}) \geq \mathbb{P}(\triangle_i) - \min \left\{ O_d \left( \frac{\log(n) \log(n \mathbb{P}(\triangle_i))^{d-1}}{n} \right), \mathbb{P}(\triangle_i) \right\}.
\]

Essentially, this lemma states that we can triangulate “most” of each simplex \(\triangle_i\) with “few” simplices whose vertices lie among the data points \(X_1, \ldots, X_n\) which fall in \(\triangle_i\), so long as \(\triangle_i\) is large enough that enough points among the \(X_i\) fall inside it. The proof of this lemma appears in sub-Section 3.2.
We now sketch our algorithm. First, we condition on the high-probability event of Lemma 11. We are not given the simplices in \( S_X := \bigcup_{k=1}^{L(n)} S_{X_k} \), but we do know that they belong to the collection \( S \) of all simplices whose vertices lie in the set \( V := \{X_{n+1}, \ldots, X_{2n}\} \). For each simplex \( \Delta \in S \), we construct an affine approximation \( \hat{w}_\Delta \) to \( f^* \mid _\Delta \) by applying Lemma 8, with the data points of \( D^1 = \{(X_i, Y_i)\}_{i=1}^{n/2} \) that lie in \( \Delta \) as input.

For the next step of the algorithm we need to estimate (with high probability) the squared error of linear regression applied to each simplex in \( S \), i.e. \( \ell^2_\Delta = \|f^* - \hat{w}_\Delta\|_{L^2(\Delta)}^2 \). By Lemma 8, we know that with probability of at least \( 1 - n^{-2d} \) we should expect it to be at least \( Cd \log(n) / (\mathbb{P}(\Delta) n) \), a quantity that follows from the squared estimation error (i.e. \( \|w_* - \tilde{w}_\Delta\|^2_{L^2(\Delta)} \)), that is of order \( \mathcal{O}(d \log(n)) \). Unfortunately, \( f^* \) may not be affine on \( \Delta \), and the squared approximation error (i.e. \( \omega^2_\Delta := \|f^* - w_*\|^2_{L^2(\Delta)} \)) can possibly be significantly larger than the squared estimation error. When the later occurs, the estimation of \( \ell^2_\Delta \) by noisy samples is challenging. To see this, even under a sub-Gaussian noise assumption with a variance of \( \sigma \), one may try to use the natural estimator for \( \ell^2_\Delta \), that only depends on \( \sigma \), but we know that they belong to the collection \( S \).

\[
\frac{1}{\mathbb{P}(\Delta)n} \sum_{(X, Y) \in D^2, X \in \Delta} (Y - \tilde{w}_\Delta(X))^2 - \sigma^2,
\]

where \( D^2 = \{(X, Y)\}_{i=1}^{n/2+1} \). The problem is that the additive deviations are of order \( \Omega_d(1 / \sqrt{\mathbb{P}(\Delta)n}) \), and not of order \( \tilde{O}_d(1 / (\mathbb{P}(\Delta)n)) \). Therefore, in the range of \( [O_d(1/\mathbb{P}(\Delta)n), \Omega_d(1/\sqrt{\mathbb{P}(\Delta)n})] \) we cannot estimate \( \ell^2_\Delta \), and our algorithm cannot afford that. To this end, we first develop a new estimator for the \( L_1 \) norm of any convex function \( g \), that crucially uses its convexity.

**Lemma 12.** Let \( \delta \in (0, 1) \) and \( g : \Delta \to \mathbb{R} \) be a convex function, and \( m \) i.i.d. samples from the regression model \( Y = g(Z) + \xi \), where \( Z \sim U(\Delta) \). Then, there exists an estimator \( \hat{f}_m \), such that with probability of at least \( 1 - 3 \max\{\delta, e^{-c m}\} \) satisfies the following:

\[
c_1(\delta) \|g\|_1 - C(\delta) \|g\|_2 + \sigma \sqrt{\log(2/\delta) / m} \leq \hat{f}_m \leq C_1(\delta) \|g\|_1 + C(\delta) \|g\|_2 + \sigma \sqrt{\log(2/\delta) / m}.
\]

Note that the estimator of the last lemma gives an optimal estimation for the \( L_1(\mathbb{P}_\Delta) \) norm of any convex function \( g \) (with no restriction on its uniform bound or Lipschitz constant). Next, using the above estimator we also developed for the \( L_2 \) norm of a convex function. The formulation of the next lemma is added to the proof of Theorem 2.

**Lemma 13.** Let \( \delta \in (0, 1) \), and \( \Delta \subset \Omega \) be a simplex. Consider the following regression model \( Y = g(Z) + \xi \), where \( Z \sim \mathbb{P}_\Delta \) and \( \|g\|_\infty \leq L \). Furthermore, assume that \( \int_{\Delta} g^2 d\mathbb{P} \geq C d \log(n)^2 / (n L^2) \). Then, there exists an estimator \( \hat{f}_{E, \delta} \) that runs in time \( O_d(n^{O(1)}) \) and with probability of at least \( 1 - \log(n) \max\{\delta, n^{-4d}\} \) satisfies

\[
\|g\|_{L^2(\Delta)}^2 \leq \hat{f}_{E, \delta} \leq C(\delta) \log(n)^{2d-1} \left( \|g\|_{L^2(\Delta)}^2 + (L + \sigma)^2 \frac{\log(2/\delta)}{\mathbb{P}(\Delta)n} \right),
\]

where \( C(d) \) is a constant that only depends on \( d \).

Unfortunately, our estimators are far from being simple and it is based on both ideas from potential theory and the behaviour of the floating body of a simplex.
Note the our estimator outputs $\|g\|_{L_2(\triangle)}$ up to a multiplicative constants of $\tilde{\Theta}_d(1)$. We apply Lemma 13 with $g = f^*|_{\triangle} - \hat{w}_{\triangle}$, and $\delta = n^{-2d}$, and the data points of $D_2$ that fall in $\triangle$, and we denote this estimate by $\hat{\ell}_2^\triangle$. Note the definition of $\hat{\ell}_2^\triangle$ implies we must know some upper bound on $L$ and $\sigma$ (up to multiplicative constants that only depends on $d$). Both can be found using standard methods.

Given our regressors $\hat{w}_{\triangle}$ and squared error estimates $\hat{\ell}_2^\triangle$, we proceed to solve the quadratic program which encodes the conditions $\|\hat{f} - \hat{w}_{\triangle}\|_{L_2(\triangle)}^2 \leq \hat{\ell}_2^\triangle$ for all simplices with large enough volume. (We rely on the fact that the $L^2$-norm on each simplex can be approximated by the empirical $L^2$-norm, again using Lemma 7.) This program is feasible, since $f^*$ itself is a solution. $\hat{f}$ is close to $f^*$ on every simplex in our collection and in particular on the simplices restricted to which $f^*$ is near-affine (which we don’t know how to identify), which allows us to conclude that $\int_{\hat{\Omega}_{k(n)}} (\hat{f} - f^*)^2 d\mathbb{P} \leq O_d(n^{-d/4})$ with high probability, where $\hat{\Omega}_{k(n)}$ is the union of the simplices in Lemma 11.

So we have constructed a function $\hat{f}$ which closely approximates $f^*$ on $\hat{\Omega}_{k(n)}$, $\hat{\Omega}_{k(n)}$ is not known to us, but as we shall see $\Omega \backslash \hat{\Omega}_{k(n)}$ has asymptotically negligible volume, so the function $\min \{ \hat{f}, 1 \}$ turns out to be a minmax optimal improper estimator (up to logarithmic factors) of $f^*$ on all of $\Omega$. In order to transform this improper it to a proper estimator, i.e., one whose output is a convex function, we use a standard procedure (denoted by $MP$), as described in Appendix C. This concludes the sketch of our algorithm.

Pseudo-code for the algorithm is given in Algorithm 1. Note that the procedure $\hat{f}_{R,\delta(n)}$ is described in Lemma 8, $\hat{f}_{E,\delta(n)}$ is described in Lemma 13, and $MP$ is described in Appendix C. We define the convex (Lipschitz) LSE by as the estimator of Eq. (1) with $F = \mathcal{F}_L$.

**Algorithm 1** A Minimax Optimal For $L$-Lipschitz Multivariate Convex Regression

**Require**: $D = D^1 \cup D^2$

**Ensure**: A random $\hat{f}_L \in \mathcal{F}_L(\Omega)$ s.t. w.h.p. $\|\hat{f}_L - f^*\|_2 \leq \tilde{O}_d((L + \sigma)^2 n^{-d/4})$.

**Part I**: Draw $X_{n+1}, \ldots, X_{2n} \sim \mathbb{P}$,

$S \leftarrow \{ \text{conv}\{X_{n+i} : i \in S\} : S \subset [n], |S| = d + 1 \}$

**for** $S_1, \ldots, S_i, \ldots \in S$ **do**

Obtain a regressor $\hat{w}_i$ using the procedure $\hat{f}_{R,\delta(n)}$ with the data points $\{(X, Y) \in D^1 : X \in S_i\}$.

Apply $\hat{\ell}_i^2 := \min \{ \hat{f}_{E,\delta(n)}, 4 \}$ with the input $\{(X, Y - \hat{w}_i(X)) : (X, Y) \in D^2, X \in S_i\}$.

**end for**

**Part II**: For $i \in 1, \ldots, |S|$ **do**

Draw $Z_{i,1}, \ldots, Z_{i,n} \sim \mathbb{P}_{S_i}$

Define an inequality constraint $I_i := \frac{1}{n} \sum_{j=1}^n (f(Z_{i,j}) - \hat{w}_i(Z_{i,j}, 1))^2 \leq \hat{\ell}_i^2 + CL^2 \sqrt{\frac{\log(n)}{n}}$.

**end for**

Construct $\hat{f} \in \mathcal{F}_L(\Omega)$ satisfying the constraints $I_1, I_2, \ldots, I_{|S|}$ (cf. Eqs. (10)-(12))

Under our assumptions on $\Omega$ we just Return $MP(\min \{ \hat{f}, L \})$.

Define $\Omega_X := \text{conv}\{X_{n+1}^{4-\varepsilon k(n)} \frac{d+2}{d} \}$

Apply the Convex LSE $f_c$ on the samples $\{(X, Y) \in D : X \notin \Omega \backslash \Omega_X\}$.

**return** $MP(\hat{f}1_{\Omega_X} + f_c1_{\Omega \backslash \Omega_X})$. 

9
We now turn to the proof that Algorithm 1 succeeds with high probability. In the analysis, we assume for simplicity that \( L = \sigma = 1 \). Let \( S \) be as defined in Algorithm 1, and let \( S^T := \{ \triangle : \triangle \in S, \int_\triangle g^2 d\mathbb{P} \geq C d \log(n)^2 / n \} \), for some sufficiently large \( C \). Note that in particular we have that \( \mathbb{P}(S) \geq C_1(C) d \log(n)/n \) for all \( S \in S^T \). We first note that our samples may be assumed to be close to uniformly distributed on the simplices in \( S^T \). Indeed, by standard concentration bounds,

\[
\forall \triangle \in S^T, j \in \{1, 2, 3\} : \quad \frac{1}{2} \leq \frac{P^{(j)}(\triangle)}{P(\triangle)} \leq 2, \tag{6}
\]

where \( P^{(1)} = \frac{2}{n} \sum_{i=1}^{n/2} \delta_{X_i}, P^{(2)} = \frac{2}{n} \sum_{i=n/2+1}^{n} \delta_{X_i}, \) and \( P^{(3)} = \frac{1}{n} \sum_{i=n+1}^{2n} \delta_{X_i}, \) with probability \( 1 - 3n^{-3d} \) (see Lemma 16 in sub-Section 3.2). From now on, we condition on the intersection of the events of (6) and Lemma 11.

The first step in the algorithm is to apply the estimator of Lemma 8 for each \( S_i \in S^T \) with \( \mathbb{Q} := \mathbb{P}_{S_i} \), and \( \delta = n^{-(d+2)} \), using those points among of \( D^1 \) that fall in \( S_i \). (By the preceding paragraph, under our conditioning, we may assume \( \mathbb{P}(S_i) \) \( n \) of the points in \( X_1, \ldots, X_{2n} \) fall in each \( S_i \), up to absolute constants. We will silently use the same argument several more times below.) By the lemma and a union bound, we know that the following event has probability of at least \( 1 - n^{-1} \):

\[
\forall 1 \leq i \leq |S^T|, \quad \int_{S_i} (\hat{w}_i(x) - f^*(x))^2 d\mathbb{P}_{S_i} \leq \frac{2Cd \log(n)}{\mathbb{P}(S_i)n} + \int_{S_i} (\hat{w}_i(x) - f^*(x))^2 d\mathbb{P}_{S_i}, \tag{7}
\]

where \( \hat{w}_i = \arg\min_w \int_{S_i} (w(x) - f^*)^2 d\mathbb{P}_{S_i} \). We condition also on the event of (7). Next, we apply Lemma 13 (with \( \delta = n^{-(d+2)} \)) on each \( S_i \), with \( g = f^* - \hat{w}_i \), and using those points among of \( D^2 \) that fall in \( S_i \), and obtain that

\[
\forall 1 \leq i \leq |S| : \quad \int_{S_i} (\hat{w}_i - f^*)^2 d\mathbb{P}_{S_i} \leq \hat{\ell}_i^2, \tag{8}
\]

with \( \hat{\ell}_i^2 \) as defined in Algorithm 1. Note that \( S \in S \setminus S^T \), taking \( \hat{w}_i = 0 \) suffices, since \( f^* \) is bounded by 1, the loss is bounded by 4. Finally, we further condition on the event of the last equation.

We proceed to explain and analyze Part II of Algorithm 1. We first claim that conditioned on (8), the function \( f^* \) satisfies the constraints \( I_1, I_2, \ldots \) defined in the algorithm with probability at least \( 1 - n^{-1} \). Indeed, for each \( 1 \leq i \leq |S| \). Since \( \|f^* - \hat{w}_i\|_{L^\infty(S_i)} \leq 4 \), by Hoeffding’s inequality and (8) we know that with probability of at least \( 1 - n^{-(d+2)} \), we have that

\[
\frac{1}{n} \sum_{j=1}^{n} (f^*(Z_{i,j}) - \hat{w}_i(Z_{i,j}))^2 \leq \int_{S_i} (f^* - \hat{w}_i)^2 d\mathbb{P}_{S_i} + \sqrt{\frac{Cd \log(n)}{n}} \leq \hat{\ell}_i^2 + \sqrt{\frac{Cd \log(n)}{n}}. \tag{9}
\]

Taking a union over \( i \), we know that (9) holds for all \( i \) with probability at least \( 1 - n^{-1} \).

We also note (for later use) that applying Lemma 7 to the measures \( \mathbb{P}_{S_i} \) and using a union bound, it holds with probability at least \( 1 - Cn^d e^{-\sqrt{n}} \) that for all \( i \), the empirical measure \( \mathbb{P}_{S_i,n} = \frac{1}{n} \sum_{j=1}^{n} \delta_{Z_{i,j}} \) on \( S_i \) approximates \( \mathbb{P}_{S_i} \) in the sense of (2). We condition on the intersection of these two events as well.
We now explain how to construct \( \hat{f} \in \mathcal{F}_I(\Omega) \) satisfying all the constraints \( I_i \). The idea is to mimic the computation of the convex LSE (Seijo and Sen, 2011), by considering the values of the unknown function \( y_{i,j} = \hat{f}(Z_{i,j}) \) and the subgradients \( \xi_{i,j} \in \partial f(Z_{i,j}) \) at each \( Z_{i,j} \) as variables. More precisely, we search for \( y_{i,j} \in \mathbb{R} \) and \( \xi_{i,j} \in \mathbb{R}^d \) satisfying the following set of constraints (here \( L = 1 \))

\[
\forall i \leq |S| : \quad \frac{1}{n} \sum_{j=1}^{n} (y_{i,j} - \hat{w}_i(Z_{i,j}))^2 \leq \hat{\ell}_i^2 + \sqrt{\frac{Cd \log(n)}{n}} \quad (10)
\]

\[
\forall (i, j) \in [|S|] \times |n| : \quad \|\xi_{i,j}\|^2 \leq L^2 \quad (11)
\]

\[
\forall (i_1, j_1), (i_2, j_2) \in [|S|] \times |n| : \quad y_{i_2,j_2} \geq \langle \xi_{i_1,j_1}, Z_{i_2,j_2} - Z_{i_1,j_1} \rangle. \quad (12)
\]

For any feasible solution \( (y_{i,j}, \xi_{i,j})_{i,j} \) of (10)-(12), define the affine functions \( w_{i,j}(x) = y_{i,j} + \langle \xi_{i,j}, x - Z_{i,j} \rangle \). We claim that the function \( \hat{f} = \max_{i,j} w_{i,j} \) is a 1-Lipschitz convex function which satisfies the constraints \( I_i \).

Indeed, (12) guarantees that \( \hat{f}(Z_{i,j}) = y_{i,j} \) for each \( i \), so the \( I_i \) are satisfied due to (10); moreover, the \( w_{i,j} \) are convex and 1-Lipschitz (the latter because of (11)), so \( \hat{f} \) is convex as a maximum of convex functions and 1-Lipschitz as a maximum of 1-Lipschitz functions. (To prove the latter assertion, for any two points \( x, y \in \Omega \) take the line segment between \( x \) and \( y \) and break it up into segments \( L_k = (x_k, x_{k+1}) \) such that \( \hat{f}(x) \) coincides with one of the \( w_{i,j} \) on all of \( L_k \); then \( |f(x_{k+1}) - f(x_k)| \leq |x_{k+1} - x_k| \). Summing over \( k \) gives the desired inequality \( |f(y) - f(x)| \leq |y - x| \).

Conditioned on (9) there exists a feasible solution to the problem (10)-(12), namely that obtained by taking \( y_{i,j} = f^*(Z_{i,j}), \xi_{i,j} \in \partial f^*(Z_{i,j}) \) (where \( \partial f^*(x) \) denotes the subgradient set of a convex function \( f^* \) at the point \( x \)). Moreover, the constraints in (10)-(12) are either linear or convex and quadratic in \( f(Z_{i,j}), u_{i,j}, \) and hence the problem can be solved efficiently. For instance, it can be expressed as a second-order cone program (SOCP) with \( O_d(n^{2d+2}) \) variables and constraints, which can be solved in time \( O_d(n^{O(d)}) \) (see, e.g., Ben-Tal and Nemirovski (2001)).

Next, recall that under our conditions on the \( Z_{i,j} \), we have for each \( i \) that

\[
\int_{S_i} (\hat{f}(x) - f^*(x))^2 dP_{S_i} \leq \frac{1}{n} \sum_{j=1}^{n} (\hat{f}(Z_{i,j}) - f^*(Z_{i,j}))^2 + Cn^{-\frac{4}{d}}, \quad (13)
\]

since both \( f^* \) and \( \hat{f} \) lie in \( \mathcal{F}_1(\Omega) \). Recall also that under our conditioning, for each \( i \), the constraint

\[
\frac{1}{n} \sum_{j=1}^{n} (y_{i,j} - \hat{w}_i(Z_{i,j}))^2 \leq \hat{\ell}_i^2 + \sqrt{\frac{Cd \log(n)}{n}}
\]

holds whether we take \( y_{i,j} = \hat{f}(Z_{i,j}) \) or \( y_{i,j} = f^*(Z_{i,j}) \). Using this bound along with the inequality (\( f^* - \hat{f} \))^2 \leq 2(\hat{f} - \hat{w}_i)^2 + 2(\hat{w}_i - f^*)^2 \) in Eq. (13), we obtain

\[
\int_{S_i} (\hat{f}(x) - f^*(x))^2 dP_{S_i} \leq 2\hat{\ell}_i^2 + \sqrt{\frac{Cd \log(n)}{n}} + Cn^{-\frac{4}{d}} \leq 2\hat{\ell}_i^2 + C'n^{-\frac{4}{d}}, \quad (14)
\]

where we used our assumption of \( d \geq 5 \). Now, note that

\[
\hat{\ell}_i^2 \leq C_d \log(n)^{3d} \ell_i^2 \leq C_d \log(n)^{2d-1} \left( \|f^* - w_i^*\|_{L^2(S_i)}^2 + \frac{C(d) \log(n)}{P(S_i)n} \right).
\]
Thus, we obtain that
\[ S \]
where we used the fact that the cardinality of \( S \) of the
\[ \widetilde{\Omega} \]

error of doing so will be asymptotically negligible (\( n \)), we obtain
\[ f \]

Finally, since \( \mathbb{P}(\Omega \setminus \Omega_{k(n)}) \leq C(d)n^{-\frac{4d}{4d+1}} \log(n)^d \), we can estimate \( f^* \) simply by 1 on \( \Omega \setminus \Omega_{k(n)} \) and the error of doing so will be asymptotically negligible (\( n^{-\frac{4d}{4d+1}} \ll n^{-\frac{4d}{4d+1}} \)), so \( \min \{ \hat{f}, 1 \} \) is a minimax optimal estimator on all of \( \Omega \). (Recall that this is an improper estimator, and we can apply the procedure \( MP \) described in Appendix C to obtain a proper estimator.)

It is not hard to see that the runtime of the above algorithm is \( O_d(n^{O(d)}) \). The proof of Theorem 2 is complete.
3 Proofs of Missing Parts

3.1 Proof of Theorem 9

Since the squared $L^2$-error scales quadratically with the function to be estimated, it suffices to prove the theorem for the class of 1-Lipschitz functions. Since the range of a 1-Lipschitz function on a domain of diameter at most 1 is contained in an interval of length 1, it is no loss to assume that the range of $f^*$ is contained in $[0, 1]$.

The construction of a $k$-affine approximation to any convex 1-Lipschitz function $f^* : \Omega \to [0, 1]$, uses a combination of two tools: the theory of random polytopes in convex sets, and empirical processes. We use the notation $U(\cdot)$ to denote the uniform distribution.

Fix a convex body $K \subset \mathbb{R}^d$ and $n \geq d + 1$. The random polytope $K_n$ is defined to be the convex hull of $n$ random points $X_1, \ldots, X_n \sim U(K)$. It is well-known and easy to justify that $K_n$ is a simplicial polytope with probability 1: Indeed, if $X_1, \ldots, X_n$ form a facet of $K_n$, then in particular they lie in the same affine hyperplane, and if $k \geq d + 1$, the probability that $X_k$ lies in the affine hull $H$ of $X_1, \ldots, X_n$ is 0, since $K \cap H$ has volume 0. For future use we note that with probability 1, the projection of every facet of $K_n$ on the first $d - 1$ coordinates is a $(d - 1)$-dimensional simplex, by similar reasoning.

For $s \in \{0, 1, \ldots, d - 1\}$ and $P$ a polytope, we let $f_s(P)$ denote the number of $s$-dimensional faces of $P$. The first result regarding random polytopes that we need appears in (Bárány, 1988, Corollary 3):

**Theorem 14.** Let $d \geq 1$, $1 \leq s \leq d - 1$ and a convex body $K \subset \mathbb{R}^d$. Then, there exists $C(d, s) \leq C_1(d)$ such that

$$
\mathbb{E}[f_s(K_n)] \leq C(d, s)n^{\frac{d-1}{d+1}}.
$$

We will also use the following result that was derived in Dwyer (1988):

**Theorem 15.** Let $P \subset B_d$ be a polytope, and let $Y_1, \ldots, Y_m \sim \mathbb{P}_p$. Then, $P_m = \text{conv}(Y_1, \ldots, Y_m)$ is a simplicial polytope with probability 1, and the following holds:

$$
\mathbb{E}_P(P \setminus P_m) = O_d(C(P)m^{-1}\log(m)^{d-1}),
$$

The other result that we need from empirical processes appears as Lemma 7 in the main text.

We now describe our construction. Given a 1-Lipschitz function $f^* : \Omega \to [0, 1]$, define the convex body

$$
K = \{(x, y) : x \in \Omega, y \in [0, 2] \mid f^*(x) \leq y\}.
$$

In other words, $K$ is the epigraph of the function $f^*$, intersected with the slab $\mathbb{R}^d \times [0, 2]$. Note that $\text{vol}_{d-1}(K) \leq 2\text{vol}_d(K) \leq 2\text{vol}_{d-1}(\Omega)$, since $\text{Im} f^* \subset [0, 1]$.

Let $n = \lfloor kd + 1 \rfloor$, and consider the random polytope $K_n \subset K$. Let $\Omega_k$ be the projection of $K_n$ to $\mathbb{R}^d$, and define the function $f_k : \Omega_k \to [0, 2]$ by

$$
f_k(x) = \min\{y \in \mathbb{R} : (x, y) \in K_n\},
$$
i.e., $f_k$ is the lower envelope of $K_n$. In particular, since $K_n \subset K$, $f_k$ lies above the graph of $f^*$. We would like to show that with positive probability, $f_k$ satisfies the properties in the statement of the theorem. We treat each property in turn.

$f_k$ is $k$-simplicial with probability of at least $9/10$: Using Theorem 14, and Markov’s inequality $K_n$ has at most $10C(d)n^{\frac{d}{2}} = C'(d)k$ facets (recall that all facets of $K_n$ are simplices with probability 1). Letting $\triangle_1, \ldots, \triangle_F$ be the bottom facets of $K_n$, and letting $\pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be the projection onto the first factor, $\pi(\triangle_1), \ldots, \pi(\triangle_F)$ is a triangulation of $\Omega_k$ and for each $i = 1, \ldots, F$, $f_k|_{\triangle_i}$ is affine, as its graph is simply $\triangle_i$.

Bounding $\mathbb{P}(\Omega \setminus \Omega_k)$ with probability of at least $9/10$: Since $\Omega_k$ is the projection of $K_n$ to $\mathbb{R}^d$, it is equivalently defined as $\text{conv}(\pi(X_1), \ldots, \pi(X_n))$ where $X_1, \ldots, X_n$ are independently chosen from the uniform distribution on $K$, and $\pi$ is the projection onto the first $d$ coordinates as above. $\pi(X_i)$ is not uniformly distributed on $\Omega$, so we cannot apply Theorem 15 (and Markov’s inequality directly). Instead, we re-express $\pi(X_i)$ as a mixture of a uniform distribution and another distribution, and apply Theorem 15 to the points which come from the uniform distribution.

In more detail, note that we may write $K = K_1 \cup K_2$ where $K_1 = (\Omega \times [0, 1]) \cap \text{epi } f$ and $K_2 = \Omega \times [1, 2]$, since $f \leq 1$. Let $p = \frac{\text{vol}(K_2)}{\text{vol}(K)} \geq \frac{1}{2}$. The uniform distribution from $K$ can be sampled from as follows: with probability $p$, sample uniformly from $K_2$, and with probability $1 - p$ sample uniformly from $K_1$. Clearly, if $X$ is uniformly distributed from $K_2$ then $\pi(X)$ is uniformly distributed on $\Omega$. Hence, $\Omega_k$ can be constructed as follows: draw $M$ from the binomial distribution $B(n, p)$ with $n$ trials and success probability $p$, then sample $M$ points $X_1, \ldots, X_M$ uniformly from $\Omega$ and sample $k - M$ points $X_1', \ldots, X_M'$ from some other distribution on $\Omega$, which doesn’t interest us; then set $\Omega_k = \text{conv}(X_1, \ldots, X_M, X_1', \ldots, X_M')$. In particular, $\mathbb{P}(\Omega \setminus \Omega_k) \geq \mathbb{P}(\Omega \setminus \Omega_M)$, so it is sufficient to bound the RHS with high probability.

By the usual tail bounds on the binomial distribution, $M \geq \frac{np}{2} \geq \frac{n}{4}$ with probability $1 - e^{-\Omega(n)}$. Hence, by Theorem 15 we obtain

\[
\mathbb{E}\mathbb{P}(\Omega \setminus \Omega_M) \leq \mathbb{E}\mathbb{P}(\Omega \setminus \text{conv}(X_1, \ldots, X_n/4)) + C(d)e^{-c(d)n} \leq O_d(C(\Omega)n^{-1}\log(n)^{d-1})
\]

\[
\leq O_d(C(\Omega)k^{-\frac{d+2}{2}}\log(k)^{d-1}),
\]

and we obtain $\mathbb{P}(\Omega \setminus \Omega_M) \leq 10C(\Omega)k^{-\frac{d+2}{2}}\log(k)^{d-1}$ with probability at least $\frac{9}{10}$ by Markov’s inequality.

Bounding $\int (f - f_k)^2 d\mathbb{P}$ with probability of at least $9/10$: Finally, we wish to bound the $L^2(\mathbb{P})$-norm of $f^* - f_k$. To do this, we use the same strategy, that on average, $k$ of the points of $K_n$ can be thought of as drawn from the uniform distribution on a thin shell of width $k^{-\frac{2}{d}}$ lying above the graph of $f^*$, which automatically bounds the empirical $L^2$-norm $\int (f^* - f_k)^2 d\mathbb{P}_n$ and hence the $L^2$-norm by Lemma 7.

Now for the details. Set $\epsilon = k^{-\frac{2}{d}}$, and define

$K_\epsilon = \{(x, y) : x \in \mathbb{R}^d, y \in [0, 2] \mid f^*(x) \leq y \leq f^*(x) + \epsilon\},$

i.e., $K_\epsilon \subset K$ is just the strip of width $\epsilon$ lying above the graph of $f^*$. By Fubini, $K_\epsilon$ has volume $\epsilon \text{vol}(\Omega) \geq \frac{k}{2\text{vol}(K)}$, and if $X$ is uniformly distributed on $K_\epsilon$, $\pi(X)$ is uniformly distributed on $\Omega$. Hence, we can argue
precisely as in the preceding: with probability $1 - e^{-\Omega(n)}$,

$$L := |\{X_i : X_i \in K_r\}| \geq \frac{en}{4} = \frac{k}{4}.$$ 

Conditioning on $L$ for some $L \geq \frac{k}{4}$ and letting $X_1, \ldots, X_L$ be the points drawn from $K$ which lie in $K_r$, we have that $\pi(X_1), \ldots, \pi(X_L)$ are uniformly distributed on $\Omega$. Moreover, for any $i \in \{1, \ldots, n\}, X_i \in K_n$ and so it lies above the graph of $f_k$, but also $X_i \in K_r$ and so it lies below the graph of $f^* + \epsilon$. Combining these two facts yields

$$\forall 1 \leq i \leq L :\quad f_k(\pi(X_i)) \leq (X_i)_{d+1} \leq f^*(\pi(X_i)) + \epsilon,$$

where $(\cdot)_{d+1}$ denotes the $d + 1$ coordinate. Hence,

$$\forall 1 \leq i \leq L :\quad f^*(\pi(X_i)) \leq f_k(\pi(X_i)) \leq f^*(\pi(X_i)) + Ck^{-2/d}. \quad (18)$$

Thus, letting $\mathbb{P}_L = \frac{1}{L} \sum_{i=1}^L \delta_{\pi(X_i)}$ denote the empirical measure on $\pi(X_1), \ldots, \pi(X_L)$, we obtain

$$\int_{\Omega} (f^* - f_k)^2 \, d\mathbb{P}_L \leq \frac{1}{L} \sum_{i=1}^L \epsilon^2 = \epsilon^2 = k^{-\frac{d}{4}}.$$ 

Since the $\pi(X_i)$ are drawn uniformly from $\Omega$, if we knew that $f_k$ were $1$-Lipschitz it would follow from Lemma 7 that

$$\int_{\Omega} (f^* - f_k)^2 \, d\mathbb{P} \leq k^{-\frac{d}{4}} + CL^{-\frac{d}{2}} = C'k^{-\frac{d}{4}},$$

with high probability.

We do not know, however, that $f_k$ is $1$-Lipschitz. To get around this, define the function $\hat{f}_k$ as the function on $\Omega_k$ whose graph is $\text{conv}\{\{(\Pi(X_i), f^*(\Pi(X_i)))\}_{i=1}^L\}$. Unlike $f_k$, $\hat{f}_k$ is necessarily $1$-Lipschitz since $f^*$ is (see, e.g., the argument in the paragraph below equations (10)-(12)), so by Lemma 7, it follows that

$$\int_{\Omega} (f^* - \hat{f}_k)^2 \, d\mathbb{P} \leq C_1k^{-\frac{d}{4}}$$

with probability of at least $1 - C(d) \exp(-c(d)k)$. Also, by (18),

$$\forall 1 \leq i \leq L :\quad \hat{f}_k(\pi(X_i)) \leq f_k(\pi(X_i)) \leq \hat{f}_k(\pi(X_i)) + C_1k^{-2/d}.$$ 

It easily follows by the definitions of $f_k$ and $\hat{f}_k$ as convex hulls that on the domain $\Omega_{\Pi(X)} := \text{conv}\{\{(\Pi(X_i))\}_{i=1}^L\}$, we have

$$f^* \leq \hat{f}_k \leq f_k \leq \hat{f}_k + Ck^{-2/d}.$$ 

Hence, we conclude that

$$\int_{\Omega_{\Pi(X)}} (f_k - f^*)^2 \, d\mathbb{P} \leq 2\int_{\Omega_{\Pi(X)}} (\hat{f}_k - f^*)^2 \, d\mathbb{P} + 2\int_{\Omega_{\Pi(X)}} (f_k - \hat{f}_k)^2 \, d\mathbb{P} \leq 2\int_{\Omega_{\Pi(X)}} (\hat{f}_k - f^*)^2 \, d\mathbb{P} + 2\|f_k - \hat{f}_k\|_L^2 \leq C_2k^{-4/d}$$

15
with high probability.

Now, using Theorem 15 and Markov’s inequality, we also know that
\[ \mathbb{P}(\Omega \setminus \Omega_{\Pi(X)}) \leq 20 C(\Omega) k^{-1} \log(k)^{d-1}. \]
with probability of at least \( \frac{19}{20} \). Conditioned on this event, and using the fact that \( f_k \) is uniformly bounded by 1, we obtain
\[ \int_{\Omega \setminus \Omega_{\Pi(X)}} (f_k - f^*)^2 \leq C(\Omega) k^{-1} \log(k)^{d-1}. \]
On the intersection of the two events defined above, which has probability at least \( \frac{9}{10} \), we have
\[ \int_{\Omega \setminus \Omega_{\Pi(X)}} (f_k - f^*)^2 \leq C^3 k^{-\frac{d}{2}} + C(\Omega) k^{-1} \log(k)^{d-1}. \]

**Deriving the theorem** Since we have three events each of which hold with probability of at least \( \frac{9}{10} \), then the intersection of these events is not empty. Therefore, an \( f_k \) satisfying all the desired properties exists, and the theorem follows.

### 3.2 Proof of Lemma 11

We start with the following easy lemma:

**Lemma 16.** The following event holds with probability of at least \( 1 - n^{-3d} \)
\[ \forall 1 \leq i \leq k(n) \text{ s.t. } \mathbb{P}(\Delta_i) \geq C_3 d \log(n)/n : \quad 2^{-1} \mathbb{P}(\Delta_i) \leq \mathbb{P}_n(\Delta_i) \leq 2 \mathbb{P}(\Delta_i) \tag{19} \]

**Proof.** The lemma follows for the fact that \( n \cdot \mathbb{P}_n(S) \sim \text{Bin}(n, \mathbb{P}(S)) \), along with the concentration inequality (cf. (Boucheron et al., 2013)) for binomial random variables: for all \( \epsilon \in (0, 1) \)
\[ \Pr \left( \left| \frac{\mathbb{P}_n(S)}{\mathbb{P}(S)} - 1 \right| \leq \epsilon \right) \leq 2 \exp(-c \min\{\mathbb{P}(S), 1 - \mathbb{P}(S)\} n \epsilon^2). \]
By taking \( \epsilon = 1/2 \), and choosing \( C \) to be large enough, we conclude that for any particular \( \Delta_i \),
\[ \mathbb{P}(\Delta_i) \geq C_3 d \log(n)/n : \quad 2^{-1} \mathbb{P}(\Delta_i) \leq \mathbb{P}_n(\Delta_i) \leq 2 \mathbb{P}(\Delta_i) \]
with probability at least \( 1 - n^{-(3d+1)} \). Taking the union bound over all \( k(n) \) triangles, the claim follows. \( \square \)

The main step is the following lemma, which shows that for any given simplex \( \Delta_i \), if we draw \( C d \log n \) points from the uniform distribution on \( \Delta_i \) for sufficiently large \( C \), then there exists some subset \( S \) of these points whose convex hull \( P \) covers almost all of the simplex and can also be triangulated by a polylogarithmic number of simplices whose vertices lie in \( S \).

**Lemma 17.** Let \( S \subset \mathbb{R}^d \) be a simplex, and \( m \geq C_3 d \log(n) \), for some large enough \( C_3 \geq 0 \). Let \( Y_1, \ldots, Y_m \sim \mathbb{P}_S \). Then, with probability of at least \( 1 - n^{-3d} \) there exists a set \( A \) of simplices contained in \( S \) with disjoint interiors of cardinality \( |A| \leq C_d \log(m)^{d-1} \) such that
\[ \mathbb{P}_S(S \setminus \bigcup A) = O_d(m^{-1} \log(n) \log(m)^{d-1}). \]
Proof. For each \( s \in \{0, 1, \ldots, d-1\} \) and \( P \) a polytope, we let \( f_s(P) \) denote the number of \( s \)-dimensional faces of \( P \). We need the following result, which was first proven in (Dyer, 1988); for more details see the recent paper Reitzner et al. (2019).

**Theorem 18.** Let \( S \subseteq \mathbb{R}^d \) be a simplex, and let \( Y_1, \ldots, Y_m \sim \mathbb{P}_S \). Then, \( S_m = \text{conv}(Y_1, \ldots, Y_m) \) is a simplicial polytope with probability 1, and the following holds:

\[
\mathbb{E} \mathbb{P}_S(S \setminus S_m) = O_d(m^{-1} \log(m)^{d-1}),
\]

and

\[
\mathbb{E} f_{d-1}(S_m) = O_d(\log(m)^{d-1}).
\]

This theorem does not give us what we need directly, since it treats only expectation while we require high-probability bounds. (To the best of our knowledge, sub-Gaussian concentration bounds are not known for the random variables \( f_{d-1}(P_m), \mathbb{P}_S(S \setminus S_m) \) when \( S \) is a simplex, cf. (Vu, 2005).) This necessitates using a partitioning strategy. We divide our \( Y_1, \ldots, Y_m \) into \( C_1 d \log(n) \) blocks, for \( C_1 \) to be chosen later, each with \( m(n) := \ell \cdot \frac{m}{d \log(n)} \) samples drawn uniformly from \( \triangle \). Let \( P_1, \ldots, P_B \) be the convex hulls of the points in each block, each of which are independent realizations of the random polytope \( S_m(n) \). For each \( P_i \), Markov’s inequality and a union bound yield that with probability at least \( \frac{1}{4} \),

\[
\mathbb{P}_S(S \setminus P_i) \leq 3 \cdot \mathbb{E} \mathbb{P}_S(S \setminus P_i) \leq C_1(d) m(n)^{-1} \log(m(n))^{d-1} = \frac{C_2(d) \log(m)^{d-1} \log(n)}{n},
\]

and

\[
f_{d-1}(P_i) \leq 3 \mathbb{E} f_{d-1}(S_m(n)) \leq O_d(\log(m(n))^{d-1}) \leq O_d(\log(m)^{d-1}).
\]

Since there are \( C_1 d \log(n) \) independent \( P_i \), at least one of them will satisfy these conditions with probability \( 1 - \left(\frac{3}{4}\right)^C_1 d \log n \), and choose \( C_1 \) so that this is at least \( 1 - n^{-3d} \).

Conditioned on the existence of \( P_i \) satisfying (20) and (21), we take one such \( P_i \) and triangulate it by picking any point among the original \( Y_1, \ldots, Y_m \) lying in the interior of \( P_i \) and connecting it to each of the \( (d-1) \)-simplices making up the boundary of \( P_i \). The set \( A \) is simply the set of \( d \)-simplices in this triangulation. \( \square \)

Now, to obtain Lemma 11, we condition on the event of Lemma 16 and apply Lemma 17 to each \( \triangle_i \) such that \( \mathbb{P}(\triangle_i) \geq C d \log(n)/n \), with the \( Y_1, \ldots, Y_m \) taken to be the points of \( X_{n+1}, \ldots, X_{2n} \) drawn from \( \mathbb{P} \) which fall inside of \( \triangle_i \). Using the fact that \( \mathbb{P}_n(\triangle_i) \geq 0.5 \mathbb{P}(\triangle_i) \), we see that \( m \geq C d \log n \) for each \( \triangle_i \), so Lemma 17 is in fact applicable. In addition, the bounds on the cardinality of \( S_{\triangle_i} \) and on the volume of \( \triangle_i \) left uncovered by the simplices in \( S_{\triangle_i} \) follow immediately by substituting \( C \mathbb{P}(\triangle_i) \) for \( m \) in the conclusions of Lemma 17. For \( i \) such that \( \mathbb{P}(\triangle_i) \leq C d \log(n)/n \), we take \( S_{\triangle_i} \) to be the empty set.

### 3.3 Proof of Lemma 13

We will use the following auxiliary estimator for the mean of a random variable that is presented in (Devroye et al., 2016):
Lemma 19. Let $\delta \in (0, 1)$ and let $Z_1, \ldots, Z_k$ be i.i.d. samples from a distribution on $\mathbb{R}$ with finite variance $\sigma_Z^2$. There exists an estimator $\hat{f}_\delta : \mathbb{R}^k \to \mathbb{R}$ with a runtime of $O(k)$, such that with probability of at least $1 - \delta$,

$$
(\hat{f}_\delta(Z_1, \ldots, Z_k) - \mathbb{E}Z)^2 \leq \frac{8\sigma_Z^2 \cdot \log(2/\delta)}{k}.
$$

We will use the following notations: $\|g\|_p$ denotes the $L_p(U(\triangle))$ norm of some simplex $\triangle \subset B_d$. Our first step is to estimate the $L_1$ norm of any convex function $g$. In the following sub-sub-section, we will prove Lemma 12 that appears above.

The final step will be to estimate the $L_2$ norm of $g$ under the assumptions of Lemma 13, by using the Lemma 12 and the claim of Lemma 13 will follow.

3.3.1 Proof of Lemma 12

We will use the following facts, which can be extracted from the statements and proofs of (Gao and Wellner, 2017, Lemmas 2.6-2.7) that regards the regular simplex with radius one $S \subset B_d$, also for each $t \in [0, 1]$, denote by $S^t := (1 - t)S$.

Lemma 20. Let $g : S \to \mathbb{R}$ be a convex function. Then, the following holds:

- $g \geq -C_d \int_S |g|dU(x)$.
- For each $\delta \in (0, 1)$, we have that $g$ restricted to $S^\delta$ is $C_d \delta^{-(d+1)} \int_S |g|dU(x)$ Lipschitz, and that $g|_{S^\delta} \leq C_d \delta^{-(d+1)} \int_S |g|dU(x)$.

Using the lemma above, the almost immediate corollary follows:

Corollary 21. For any $l \in (0, 1)$, there exists a constant $\delta = \max\{c(l, d), c/d\}$ such that $g$ restricted to $S^\delta$ is uniformly bounded by $C_d \delta^{-(d+1)} \|g\|_1$, and moreover

$$
\int_{S \setminus S^\delta} |g^-|dU(x) \leq l\|g\|_1.
$$

Now, we will present our following result that may have independent of interest: Define the measure

$$
p_S(x) := \frac{1_{B_\mu(x)}(y)}{U(B_\mu(y))}dU(y),
$$

where we define $B_\mu$ to be the largest ball centered on $y$ which is contained in $S$. Next, for any simplex $\triangle$, we can define a measure $p_\triangle(x) := p_S(T^{-1}x)$, where $T$ is the affine transformation such that $\triangle = TS$.

Lemma 22. Let $g : \triangle \to [-M'\|g\|_{L_1(U(\triangle))}, M'\|g\|_{L_1(U(\triangle))}]$ be a convex $M\|g\|_{L_1(U(\triangle))}$ Lipschitz such that $\arg\min_w_{\text{affine}} \|g - w\|_{L_2(U(\triangle))} = 0$. Then, the following holds:

$$
c_1(M, M', d)\|g\|_{L_1(U(\triangle))} \leq \int g(x) p_\triangle(x) dx \leq C_1(M, M', d)\|g\|_{L_1(U(\triangle))},
$$

(22)
In addition, for every affine function \( w \),
\[
\int w p_\triangle \, dx = \int_\triangle w dU(x).
\]
Moreover,
\[
\max_{x \in \triangle} p_\triangle(x) \leq \alpha_d := \frac{2d^d + 1}{d-1} v_d^{-1}
\]
where \( v_d \) is the volume of the unit ball in dimension \( d \), and there exists an efficient algorithm to compute \( p_\triangle(x) \) for any \( x \in \triangle \).

We will prove this lemma below, note that by changing of variables and the fact that \( L_1 \) scales with the determinant of \( T \) such that \( S = T \triangle \), it is enough to prove this result for the regular simplex \( S \). The observation that leads to this measure is based on the following idea: If \( g \) is a convex function with zero mean and barycenter, and intuitively, such functions must be negative near the barycenter of \( \triangle \) and positive near the boundary. One can therefore hope that the integral of \( g \) in a neighborhood of the boundary gives a lower bound on the \( L^1 \)-norm of \( g \).

Using the above results we can estimate the \( L_1 \) norm \( g : \triangle \rightarrow [-1, 1] \). First, we define the unique affine transformation such that \( T \triangle = S \). Also, we define the shrunken simplex \( \triangle_\delta \), where \( \triangle_\delta := T^{-1}(T \triangle)^{d(l, d)} \), where \( l = 1/10 \) and \( \delta(l, d) \) is defined in Corollary 21. There are few cases that we are going to consider in this proof.

The first case is the following: \( \|g \mathbb{1}_{\triangle \setminus \triangle_\delta}\|_1 \geq 4^{-1}\|g\|_1 \), i.e. the \( L_1 \) norm of \( g \) is attained on the shell outside of \( \triangle_\delta \). Using Corollary 21, we know that
\[
\|g\|_1 \geq \int_{\triangle \setminus \triangle_\delta} g dU(x) = \int_{\triangle \setminus \triangle_\delta} g^+ dU(x) + \int_{\triangle \setminus \triangle_\delta} g^- dU(x) \geq (3/20) \cdot \|g\|_1.
\]
Therefore it is enough to estimate the mean (scaled by \( U(\triangle \setminus \triangle_\delta) \)) of the r.v. \( g(X) \) where \( X \sim U(\triangle \setminus \triangle_\delta) \). It can be done using the samples that lie in this set. Using Lemma 19 above, we conclude that using these samples we have an estimate \( \hat{f}(1) \) with probability of at least \( 1 - \delta \) such that
\[
|\hat{f}(1) - \int_{\triangle \setminus \triangle_\delta} g dU(x)|^2 \leq U(\triangle \setminus \triangle_\delta)^2 \frac{C_d(\sigma^2 + \|g\mathbb{1}_{\triangle \setminus \triangle_\delta}\|_2^2) \log(2/\delta)}{U(\triangle \setminus \triangle_\delta)m} \leq C_d \cdot \frac{(\sigma^2 + \|g\|_2^2) \log(2/\delta)}{m},
\]
where we used that fact that \( U(\triangle \setminus \triangle_\delta) \geq c_d \).

Next, consider the case that \( \|g \mathbb{1}_{\triangle_\delta}\|_1 \geq 4^{-1}\|g\|_1 \). Now, decompose \( g = w_g + (g - w_g) \), where \( w_g = \arg\min_w \mathbb{1}_{\text{affine}} \|g - w\|_{L_2(U(\triangle_\delta))} \). Assume that \( \|w_g \mathbb{1}_{\triangle_\delta}\|_1 \geq 4^{-1}\|g\|_1 \).

Using half of the samples that fall into \( \triangle_\delta \), we may apply Lemma 8, that gives us \( \hat{w}_g \) such that with probability \( 1 - \delta \)
\[
\|\hat{w}_g - w_g\|_2^2 \leq C_d(\sigma^2 + \|g\|_2^2) \frac{\log(2/\delta)}{m}
\]
Since, we can easily estimate \( \hat{f}(2) := \|\hat{w}_g\|_1 \), by the last inequality we conclude that
\[
4^{-1}\|g\|_1 - C_d\|g\|_1 \sqrt{\frac{\log(2/\delta)}{m}} \leq \hat{f}(2) \leq \|g\|_1 + C_d\|g\|_2 \sqrt{\frac{\log(2/\delta)}{m}}.
\]
Finally, we handle the last case, i.e. that \( \|g - w_g\|_1 \geq 4^{-1} \|g\|_1 \). For this purpose, we will use our Lemma 22. Note that by the definition of \( \Delta_\delta \), we may assume by Lemma 21 that \( \max\{M', M\} \leq C(d) \) (in Lemma 22). Therefore, we conclude that

\[
c_1(d) \|g - w_g\|_1 \leq \int (g - w_g) dp_{\Delta_\delta} \leq C_1(d) \|g - w_g\|_1.
\]

Now, using the \( \hat{w}_g \) and the last equation, it is easy to see that

\[
c_3(d) \|g\|_1 - C_d |\sigma + \|g\|_2| \sqrt{\frac{\log(2/\delta)}{m}} \leq \int (g - \hat{w}_g) dp_{\Delta_\delta}
\]

Therefore, it is enough to estimate the mean of \( \int (g - \hat{w}_g) dp_{\Delta_\delta} \), using the the second half of the samples that fall into \( \triangle_\delta \).

Next, we explain how to simulate sampling from \( p_{\Delta_\delta} \) given samples from \( U(\Delta_\delta) \) and their corresponding noisy samples of \( g - \hat{w}_g \). The idea is simply to use rejection sampling (Devroye, 1986): given a single sample \( X \sim U(\Delta_\delta) \), we keep it with probability \( p_{\Delta_\delta}(x) \cdot \frac{1}{c_d} \). Conditioned on keeping the sample, \( X \) is distributed according to \( p_{\Delta_\delta} \). If we are given \( m/2 \) i.i.d. samples from \( U(\Delta_\delta) \), then with probability \( 1 - e^{-c_d m} \) the random number of samples \( N \) we obtain from \( p_{\Delta_\delta} \) by this method is at least \( c(\alpha_d) \cdot m \geq c_1(d) m \), and conditioned on \( N \), these samples are i.i.d. \( p_{\Delta_\delta} \). We condition on this event going forward. Now, using these \( N \) samples, we conclude by Lemma 19, we conclude that there exists an estimator \( \bar{f}(3) \) such that

\[
|\bar{f}(3) - \int (g - \hat{w}_g) dp_{\Delta_\delta}| \leq C_d |\sigma + \|g\|_2| \sqrt{\frac{\log(2/\delta)}{m}}.
\]

with probability of at least \( 1 - \max\{\delta, e^{-cm}\} \). Now, note that both \( \bar{f}(1), \bar{f}(2), \bar{f}(3) \), are always bounded from above by \( C_d \|g\|_1 \) independently to their corresponding case that we applied them on. To see this, \( \bar{f}(1) \) is bounded by \( C_d \|g\|_1 \), since \( U(\Delta \setminus \Delta_\delta) \geq c_d \). Next, for \( \bar{f}(2) \) it follows from fact that

\[
\|w_g\|_{L_1(p_{\Delta_\delta})} \leq \|w_g\|_{L_2(p_{\Delta_\delta})} \leq \|g\|_{L_2(p_{\Delta_\delta})} \leq C(d) \|g\|_2 \leq C_1(d) \|g\|_1
\]

where we used the fact that \( \|p_{\Delta_\delta}\|_\infty \leq C_d \) and that \( \|g\|_{L_1(p_{\Delta_\delta})} \leq C_2(d) \|g\|_1 \). Next, for \( \bar{f}(3) \) the fact that \( p_{\Delta_\delta} \leq c_d \cdot \mathbb{P} \) gives the claim. Finally, by using \( \bar{f}(1), \bar{f}(2), \bar{f}(3) \), we conclude that with probability \( 1 - \max\{\delta, e^{-cm}\} \) of at least

\[
c_1(d) \|g\|_1 - C_d |\sigma + \|g\|_2| \sqrt{\frac{\log(2/\delta)}{m}} \leq \sum_{i=1}^{3} \bar{f}(i) \leq C_1(d) \|g\|_1 + C_d |\sigma + \|g\|_2| \sqrt{\frac{\log(2/\delta)}{m}},
\]

and the claim follows.

**Proof of Lemma 22.** Recall that it suffices to show that

\[
c_1(M, M') \leq \int_S g(x)p_S(x)dx \leq C_1(M, M').
\]

Note that the function \( g \) is convex and in particular subharmonic, i.e., for any ball \( B_x \) with center \( x \) contained in \( S \) we have

\[
\frac{1}{U(B_x)} \int_{B_x} gdx \geq g(x),
\]

20
where $U$ denotes the uniform measure on the regular simplex $S$. $g$ is non-affine and hence strictly subharmonic (as convex harmonic functions are affine), so there exists some $x$ such that for any ball $B_x \subset S$ centered on $x$, the above inequality is strict, since subharmonicity is a local property. As $g$ is convex and in particular continuous, the inequality is strict on some open set of positive measure. We obtain that for a non-affine convex function that
\[
\int_S g(x)p_S(x)\,dU(x) = \int_S \left( \int_{S(x)} g(x)\,dU(y) \right)\,dx > \int_S g(y)dU(y) = 0,
\]
i.e. we showed that for a non-harmonic $g$ that $\int_S g(x)p_S(x)\,dx > 0$.

Now, we show why Eq. (25) actually implies the lower bound of Eq. (23), which is certainly not obvious a priori. However, it follows from a standard compactness argument. The subset $C$ consisting of convex $M$-bounded that are also $M'$-Lipschitz functions of $L_2(S)$ with norm one that is orthogonal to the affine functions, is a closed set in the $L^\infty(S)$ and by the Lipschitz condition also equi-continuous. Hence, by the Arzela-Ascoli theorem it is also compact, we thus conclude that
\[
S = \left\{ \int_S g(x)p_S(x)dx : g \in C \right\}
\]
is compact; but (25) implies that $S \subset (0, \infty)$, which finally implies the existence of $c(M, M', d)$ such that $S \subset [c(M, M', d), \infty)$. As for the upper bound in (23), it follows immediately from the boundedness of $p_S$, which we prove below.

We claim that in this case (22) can be evaluated analytically as a function of $x$, though the formulas are sufficiently complicated that this is best left to a computer algebra system. Indeed, we note that $y \in S$ contributes to the integral at $x$ if and only if $x$ is closer to $y$ than $y$ is to the boundary of $S$. The regular simplex can be divided into $d+1$ congruent cells $C_1, \ldots, C_{d+1}$ such that the points in $C_i$ are closer to the $i$-th facet of the simplex than to any other facet (in fact, $C_i$ is simply the convex hull of the barycenter of $S$ and the $i$th facet); for any $y \in C_i$, $x \in B_y$ if and only if $x$ is closer to $y$ than $y$ is to the hyperplane $H_i$ containing $C_i$. But the locus of points equidistant from a fixed point $x$ and a hyperplane is the higher-dimensional analog of an elliptic paraboloid, for which it’s easy to write down an explicit equation. Letting $P_{i,x}$ be the set of points on $x$’s side of the paraboloid (namely, those closer to $x$ than to $H_i$), we obtain
\[
p_S(x) = \frac{1}{\Omega_d} \sum_{i=1}^{d+1} \int_{C_i \cap P_{i,x}} \frac{dy}{d(y, H_i)^d}.
\]
Each region of integration $C_i \cap P_{i,x}$ is defined by several linear inequalities and a single quadratic inequality, and the integrand can be written simply as $\frac{1}{\Omega_d}$ in an appropriate coordinate system. It is thus clear that the integral can be evaluated analytically, as claimed.

Finally, we need to show that $p_S(x)$ is bounded above by $\alpha_d$. By symmetry,
\[
p_S(x) \leq \frac{1}{\Omega_d} \sum_{i=1}^{d+1} \int_{C_i \cap P_{i,x}} dy = \frac{1}{\Omega_d} \sup_{x \in \mathbb{R}^n} \int_{P_{i,x}} dy = \alpha_d,
\]

(26)
Fix $x$, and choose coordinates such that $H_1 = \{x_1 = 0\}$ and $x = (x_0, 0, \ldots, 0)$ with $x_0 > 0$. Then for any $y = (t, z)$ with $t \in \mathbb{R}$, $z \in \mathbb{R}^{d-1}$, $y$ lies in $P_{1,x}$ if $t^2 \geq (x_0 - w)^2 + |z|^2$, or $2tx_0 - x_0^2 \geq |z|^2$. Hence,

$$\int_{P_{1,x}} \frac{dy}{d(y, H_1)^d} = \int_{\mathbb{R}^{d-1}} \frac{dt}{t^d} \int_{\mathbb{R}} \frac{1}{2} |z|^2 \leq 2tx_0 - x_0^2 \frac{2 + 1}{2} \leq \Omega_d \int_{\mathbb{R}} \frac{dt}{t^d} (2tx_0)^{2 + 1}$$

$$= \Omega_d \int_{\mathbb{R}} \frac{dt}{t^d} x_0^{d-1} \cdot \left( \frac{d-1}{2} \right) - 1 \left( \frac{x_0}{2} \right)^{2 + 1} = \Omega_d - 1 \cdot \frac{2d}{d-1},$$

and substituting in (26) gives the desired bound.

3.3.2 Estimating $\|g\|_2$

Recall that we may assume that $g : \triangle \to [-L, L]$ is a convex $L$-Lipschitz satisfies $\|g\|_\infty \leq L$ and $L = 1$. Our goal in this subsection is to estimate $\|g\|_2$ up to logarithmic factors given an estimate of $\|g\|_1$, where $g : \triangle \to [-1, 1]$. This part is only valid for functions that $\|g\|_2 \geq \frac{Cd^{1/2} \log n}{n^{1/2}}$.

For this section, we will need the following classical result about the floating body of a simplex (Bárány and Larman, 1988; Schütt and Werner, 1990).

**Lemma 23.** For a simplex $S$ and $\epsilon \in (0, 1)$, let $S\epsilon$ be its $\epsilon$-convex floating body, defined as

$$S\epsilon := \bigcap \{K : K \subset S \text{ convex}, \ vol(S \setminus K) \leq \epsilon \ vol(S)\},$$

and let $S(\epsilon) = S \setminus S\epsilon$ be the so-called wet part of $S$. Then $\vol(S(\epsilon)) \leq C_d \epsilon \log(\epsilon^{-1})^{d-1} \vol(S)$.

We also note that for any particular $\epsilon$ and $x \in S$ one can check in polynomial time whether $x \in S(\epsilon)$: indeed, letting

$$H^+_{x,u} = \{y \in \mathbb{R}^n : \langle y, u \rangle \geq \langle x, u \rangle\}$$

$$H^-_{x,u} = \partial H^+_{x,u} = \{y \in \mathbb{R}^n : \langle y, u \rangle = \langle x, u \rangle\},$$

the function $u \mapsto U(S \cap H^+_{x,u})$ is smooth on $S^d \setminus C_i$ outside of the closed, lower-dimensional subset $A$ where $H_{x,u}$ is not in general position with respect to some face of $u$, and, moreover, is given by an analytic expression in each of the connected components $C_i$ of $S^d \setminus A$. It can thus be determined algorithmically whether $\min_i \inf_{x \in C_i} U(S \cap H^+_{x,u}) \leq \epsilon$, i.e., whether $x \in S(\epsilon)$.

Let $v = \mathbb{P}(S)$, and let $i_{\min} = \min(\log_2(C_d \log(n)^{d+1}v^{-1}), 0)$; note that since $\|g\|_1 \geq v \geq \frac{\log(n)^d}{n}$, $|i_{\min}| \leq C \log n$. Set $V = g^{-1}((\infty, 2^{i_{\min}}])$, and for $i = i_{\min}, i_{\min} + 1, \ldots, 0$, set $U_i = g^{-1}((2^i, 1])$. Note that $V$ is convex, while each $U_i$, $i \geq 0$, is the complement of a convex subset of $S$.

We will use the following lemma:
**Lemma 24.** For \( g \) and \( V, U_i \) as defined above, at least one of the following alternatives holds:

1. \( c_d \log(n)^{d+1/2} \|g\|_2 \leq v^{-\frac{1}{3}} \|g\|_1 \leq \|g\|_2 \).
2. There exists \( i_0 \in [i_{\text{min}}, 0] \) such that \( 2^{-i_0} \geq C_d \log(n)^{d-1} g_{\|P(S)\|} \) and
   \[
   c \log(n)^{-1/2} \|g\|_2 \leq P(U_{i_0})^{-1/2} \int_{U_{i_0}} g \, dP. \tag{27}
   \]

The proof of this lemma appears at the end of this subsection.

If alternative (1) of the lemma holds, the \( L_1 \)-norm of \( g \) is only a polylogarithmic factor away from the \( L_2 \)-norm (up to normalizing by the measure of \( S \), which is known to us). Therefore, we may use the \( L_1 \)-estimator of the previous subsection and estimate the \( L_2 \) norm of \( g \), up to a larger polylogarithmic factor, as we will see below.

We must therefore consider what happens when alternative (2) of Lemma 24 holds. If we could estimate the integral of \( g \) over \( U_{i_0} \), we’d be done, but neither the index \( i_0 \) nor the set \( U_{i_0} \) are given to us. So we make use of the fact that each such \( U_{i_0} \), being the complement of a convex subset of \( \Delta \), is contained in the wet part \( S(P(U_i)) \), which has volume at most \( c_d \log(n)^{d-1} P(U_i) \) by Lemma 23. We will show in the next lemma that this replacement costs us a \( c_d \log(n)^{d/2} \) factor in the worst case.

More precisely, let \( \epsilon_j = 2^{-2j} \), and let \( S(\epsilon_j) \) be the corresponding wet part of \( S \), as defined in Lemma 23. Then we have the following:

**Lemma 25.** With \( g \) as above, we have

\[
\max_{j \in [i_{\text{min}}, 0]} P(S(\epsilon_j))^{-1/2} \int_{S(\epsilon_j)} g \, dP \leq \|g\|_2,
\]

and moreover, if alternative (2) of Lemma 23 holds, then there exists \( j \) such that \( P(S(\epsilon_j)) \geq \frac{C_d \log(n)}{n} \) and

\[
c_d \log(n)^{-d/2} \|g\|_2 \leq P(S(\epsilon_j))^{-1/2} \int_{S(\epsilon_j)} g \, dP.
\]

Using the last lemma, we can construct an estimator for \( \|g\|_2 \) that is at most a polylogarithmic factor away from the true value, whether we are in case (1) or case (2) of Lemma 24.

Indeed, note that we if alternative (2) holds, we have \( P(S_{\epsilon_j}) \geq \frac{C_d \log(n)}{n} \) and hence, as in Section 2, we can assume by a union bound that the number of sample points falling in \( S(\epsilon_j) \) is proportional to \( P(S(\epsilon_j)) \). Hence, for each \( j \) the estimation of \( \int_{S(\epsilon_j)} g \, dP \) can be done in a similar fashion as in §§3.3.1, with an additive deviation that is proportional to \( \sqrt{\frac{\log(2/\delta)}{nP(S(\epsilon_j))}} \). However, since we need to estimate \( P(S(\epsilon_j))^{-1/2} \int_{S(\epsilon_j)} g \, dP \), we can multiply it by \( \sqrt{P(S(\epsilon_j))} \), and obtain the correct deviation of \( O(\sqrt{\log(2/\delta)/P(S(\epsilon_j))}) \) (we also used the fact that for \( 0.5P(S(\epsilon_j)) \leq P_n(S(\epsilon_j)) \), with probability of at least \( 1 - n^{-2d_j} \).

We conclude that the max over \( j \in [-c_d \log(n) \leq i_{\text{min}}, 0] \), estimators of the means of the random variables \( P(S(\epsilon_j))^{-1/2} \int_{S(\epsilon_j)} g \, dP \) and the \( L_1 \) estimator of the above sub-sub section give the claim.
**Proof of Lemma 24.** Let \( g_{-} = \min(g, 0), g_{+} = \max(g, 0) \), so that \( \|g\|_{2}^{2} = \|g_{-}\|_{2}^{2} + \|g_{+}\|_{2}^{2} \).

We claim that alternative (1) holds if \( \|g_{-}\|_{2}^{2} \geq \frac{1}{4} \|g\|_{2}^{2} \). Indeed, by Lemma 20, \( g_{-} \geq -C_{d}v^{-1}\|g\|_{1} \), which immediately yields

\[
\int_{S} g_{-}^{2} d\mathbb{P} \leq -C_{d}v^{-1}\|g\|_{1} : \int_{S} g_{-} d\mathbb{P} \leq -C_{d}v^{-1}\|g\|_{1}^{2}
\]

i.e.,

\[
v^{-1/2}\|g\|_{1} \geq c_{d}\|g_{-}\|_{2} \geq c_{d}'\|g\|_{2}.
\]

Note that by Jensen’s inequality we have that \( v^{-1/2}\|g\|_{1} \leq \|g\|_{2} \). Otherwise, we have \( \|g_{+}\|_{2}^{2} \geq \frac{1}{4} \|g\|_{2}^{2} \). Let \( T_{i} = U_{i} \setminus U_{i+1} = g^{-1}\left( [2^{i}, 2^{i+1}] \right) \). We have

\[
\frac{1}{2} \|g\|_{2}^{2} \leq \|g_{+}\|_{2}^{2} \leq \sum_{i = 0}^{\varphi - \infty} 2^{2(i+2)}\mathbb{P}(T_{i}).
\]

By our assumption \( \|g\|_{2}^{2} \geq C\log \frac{n}{n} \) and the fact that \( v \leq 1 \), the terms in the sum with \( i \leq i_{\min} = \log \left( C\log \frac{n}{n} \right) + 2 \) cannot contribute more than half of the sum, so we have

\[
\frac{1}{4} \|g\|_{2}^{2} \leq \|g_{+}\|_{2}^{2} \leq \sum_{i = i_{\min} + 1}^{0} 2^{2(i+2)}\mathbb{P}(T_{i}).
\]

Hence there exists \( i_{0} \in [i_{\min}, 0] \) such that

\[
\frac{\|g\|_{2}^{2}}{4 \log n} \leq 2^{2(i_{0}+2)}\mathbb{P}(T_{i_{0}}) \leq 4 \min_{x \in U_{i}} g(x)^{2} \cdot \mathbb{P}(U_{i}) \leq 4\mathbb{P}(U_{i_{0}})^{-1} \left( \int_{U_{i}} g d\mathbb{P} \right)^{2},
\]

or

\[
c \log(n)^{-\frac{1}{2}}\|g\|_{2} \leq \mathbb{P}(U_{i_{0}})^{-\frac{1}{2}} \int_{U_{i_{0}}} g d\mathbb{P}.
\]

We consider two cases: either \( 2^{-i} \geq C(\log n)^{d-1}v^{-1}\|g\|_{1} \), or \( 2^{-i} \leq C(\log n)^{d-1}v^{-1}\|g\|_{1} \). The first case leads immediately to alternative (2), while in the second case we have

\[
c \log(n)^{-\frac{1}{2}}\|g\|_{2} \leq \mathbb{P}(U_{i_{0}})^{-\frac{1}{2}} \int_{U_{i_{0}}} g d\mathbb{P} \leq 4C(\log n)^{d-1}v^{-1}\|g\|_{1} \cdot \mathbb{P}(U_{i_{0}})^{\frac{1}{2}} \leq C(\log n)^{d-1}\|g\|_{1}v^{-\frac{1}{2}},
\]

which is another instance of alternative (1).

As for the right-hand inequality in alternative (1), this is simply Cauchy-Schwarz: \( \left( \int g d\mathbb{P} \right)^{2} \leq \int g^{2} d\mathbb{P} \cdot v. \)

**Proof of Lemma 25.** The first inequality is again Cauchy-Schwarz: for any subset \( A \) of \( S \), we have

\[
\int_{A} g d\mathbb{P} \leq \left( \int_{A} g^{2} d\mathbb{P} \right)^{\frac{1}{2}} \left( \int_{A} 1 d\mathbb{P} \right)^{\frac{1}{2}} \leq \|g\|_{2} \cdot \mathbb{P}(A)^{\frac{1}{2}}.
\]
As for the second statement, first note that since \( |g|_\infty \leq 1 \) and we have
\[
    c \log(n)^{-1/2} |g|_2 \leq \mathbb{P}(U_{i_0})^{-1/2} \int_{U_{i_0}} g \, d\mathbb{P} \leq \mathbb{P}(U_{i_0})^{1/2}.
\]
Let \( j = \lceil \log \mathbb{P}(U_{i_0}) \rceil \geq i_{\min}, \epsilon_j = 2^j, \) so that \( U_{i_0} \subset S(\epsilon_j) \) and
\[
    \mathbb{P}(S_{\epsilon_j}) \leq C \mathbb{P}(U_{i_0}) \log(\mathbb{P}(U_{i_0})^{-1})^{d-1} \leq C \mathbb{P}(U_{i_0})^d.\]
Recalling again that by (20), \( g \geq -C|g|_1 \mathbb{P}(S)^{-1} \), we have
\[
    \int_{S(\epsilon_j)} g \, d\mathbb{P} - 2^{-1} \int_{U_{i_0}} g \, d\mathbb{P} \geq 2^{-1} \int_{S(\epsilon_j)} g \, d\mathbb{P} + \int_{S(\epsilon_j) \setminus U_{i_0}} g \, d\mathbb{P} \\
    \geq \mathbb{P}(U_{i_0}) 2^i_0 - \mathbb{P}(S(\epsilon_j)) \cdot C |g|_1 \mathbb{P}(S)^{-1} \\
    \geq \mathbb{P}(U_{i_0}) \left( 2^i_0 - C_d (\log n)^{d-1} |g|_1 \mathbb{P}(S)^{-1} \right) \\
    \geq c \cdot \mathbb{P}(U_{i_0}) \cdot 2^i > 0,
\]
where we used our assumption on \( i_0 \) in the last line. Therefore, by the last two inequalities
\[
    \mathbb{P}(S(\epsilon_j))^{-\frac{1}{2}} \int_{S(\epsilon_j)} g \, d\mathbb{P} \geq c (\log n)^{-d_1} \mathbb{P}(U_{i_0})^{-\frac{1}{2}} \int_{U_{i_0}} g \, d\mathbb{P} \geq c (\log n)^{-\frac{d_1}{2}} |g|_2,
\]
as claimed. Finally, note that by the assumptions of \( |g|_2^2 \geq \frac{C_d (\log n)^2}{n} \) and \( |g|_\infty \leq 1 \), we obtain that
\[
    \mathbb{P}(S(\epsilon_j)) \geq \mathbb{P}(U_{i_0}) \geq (c \log(n)^{-\frac{1}{2}} |g|_2/\sqrt{|g|_\infty})^2 \geq C_1 \frac{d \log n}{n}.
\]

\[\Box\]

### 4 Sketch of the Proof of Corollary 3

The modifications of Algorithm 1 to work in this setting are minimal: we simply need to replace \( L \) by \( \Gamma \), and replace (11) with
\[
    \forall (i,j) \in |S| \times [n]: \quad |y_{i,j}| \leq \Gamma \quad (28)
\]
\[
    \forall 1 \leq i \leq |S| \quad \frac{1}{n} \sum_{j=1}^n (f(Z_{i,j}) - \hat{w}_i^\top (Z_{i,j}, 1))^2 \leq \bar{l}_i^2 + \Gamma \sqrt{\frac{C_d \log(n)}{n}}
\]
\[
    \forall (i,j) \in |S| \times [n]: \quad |f(Z_{i,j})| \leq \Gamma
\]
\[
    \forall (i_1, j_1), (i_2, j_2) \in |S| \times [n]: \quad f(Z_{i_2,j_2}) \geq \nabla f(Z_{i_1,j_1})^\top (Z_{i_2,j_2} - Z_{i_1,j_1}).
\]

For the correctness proof, we need some additional modifications. First, we replace Lemma 7 with a similar bound in the \( \Gamma \)-bounded setting. The following lemma is based on the \( L_4 \) entropy bound of (Gao and Wellner, 2017, Thm 1.1) and the peeling device (van de Geer, 2000, Ch. 5); it appears explicitly in (Han and Wellner, 2016):
Lemma 26. Let \( d \geq 5 \), \( m \geq C d \) and \( \mathbb{Q} \) be a uniform measure on a convex polytope \( P' \subset B_d \) and \( Z_1, \ldots, Z_m \sim \mathbb{Q} \). Then, the following holds uniformly for all \( f, g \in \mathcal{F}_\Gamma(P') \)

\[
2^{-1} \int_{P'} (f - g)^2 \, d\mathbb{Q} - C(P') \Gamma^2 m^{-\frac{4}{d}} \leq \int_{P'} (f - g)^2 \, d\mathbb{Q} + C(P') \Gamma^2 m^{-\frac{4}{d}},
\]

with probability of at least \( 1 - C_1(P') \exp(-c_1(P') \sqrt{m}) \).

Note that differently from Lemma 7, the constant before \( m^{-4/d} \) depends on the domain \( P' \), and this dependence is essential.

Since \( \mathcal{F}_\Gamma(P') \) has finite \( L^2 \)-entropy for every \( \epsilon \), it is in particular compact in \( L^2(P') \), which means that the proof of Lemma 13 in sub-Section 3.3 works for this class of functions as well.

The proof of Theorem 9 also holds for this case, by replacing the Lemma 7 by Lemma 26. But it be in the following version:

Theorem 27. Let \( P \subset B_d \) be a convex polytope, \( f \in \mathcal{F}_\Gamma(P) \), and some integer \( k \geq (Cd)^{d/2} \), for some large enough \( C \geq 0 \), there exists a convex set \( P_k \subset P \) and a \( k \)-simplicial convex function \( f_k : P_k \to \mathbb{R} \) such that

\[
\mathbb{P}(P \setminus P_k) \leq C(P) k^{-\frac{4d}{d-2}} \log(k)^{d-1}.
\]

and

\[
\int_{P_k} (f_k - f)^2 \, d\mathbb{P} \leq \Gamma^2 \cdot C(P) k^{-\frac{4}{d}}
\]

The remaining lemmas and arguments in the proof of Theorem 2, can easily be seen to apply in the setting of \( \Gamma \)-bounded regression under polytopal support \( P \).

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A Simplified version of our estimator

Like the estimator for our original problem, the simplified version of our estimator is based on the existence of a simplicial approximation \( f_k(n) : \Omega_k(n) \rightarrow [0, 1] \) to the unknown convex function \( f^* \) (Theorem 2). Here we demonstrate how to recover \( f^* \) to within the desired accuracy if we are given the simplicial structure of \( f_k(n) \), i.e., the set \( \Omega_k(n) \) and the decomposition \( \bigcup_{i=1}^{k(n)} S_i \) of \( \Omega_k(n) \) into simplices such that \( f_k(n)|_{S_i} \) is affine for each \( i \). In this case the performance of our algorithm is rather better: it runs in time \( O_d(n^{O(d)}) \) rather than \( O_d(n^{O(d)}) \), and minimax optimal up to a constant that depends on \( d \). We can also weaken the assumption: the distribution \( P \) just needs to be uniformly bounded above and below on its support, which is required for Theorem 2 to hold. Also, the assumption of that the variance \( \sigma^2 \) of the noise is not required.

We will use the following classical estimator (Györfi et al., 2002, Thm 11.3); it is quoted here with an improved bound which is proven in (Mourtada et al., 2021, Theorem A):

**Lemma 28.** Let \( m \geq d + 1, d \geq 1 \) and \( \mathcal{Q} \) be a probability measure that is supported on some \( \Omega' \subset \mathbb{R}^d \). Consider the regression model \( W = f^*(Z) + \xi \), where \( f^* \) is \( L \)-Lipschitz and \( \|f^*\|_\infty \leq L \), and \( Z_1, \ldots, Z_m \sim_{i.i.d.} \mathcal{Q} \). Then, the exists an estimator \( \hat{f}_R \) that has an input of \( \{(Z_i, W_i)\}_{i=1}^m \) and runtime of \( O_d(n) \) and outputs a function such that

\[
\mathbb{E} \int (\hat{f}_R(x) - f^*(x))^2 d\mathcal{Q}(x) \leq \frac{Cd(\sigma + L)^2}{m} + \inf_{w \in \mathbb{R}^d} \int (w^\top (x, 1) - f^*(x))^2 d\mathcal{Q}(x).
\]

Note that this estimator is distribution-free: it works irrespective of the structure of \( \mathcal{Q} \), nor does it require that \( \mathcal{Q} \) be known.

The first step of the simplified algorithm is estimating \( f^*|_{\triangle_i} \) on each \( \triangle_i \subset \Omega_k(n) \) (1 ≤ \( i \) ≤ \( k(n) \)) with the estimator \( \hat{f}_R \) defined in Lemma 28 (with respect to the probability measure \( P(\cdot | \triangle_i) \)) with the input of the data points in \( D \) that lie in \( \triangle_i \). We obtain independent regressors \( \hat{f}_1, \ldots, \hat{f}_k(n) \) such that

\[
\mathbb{E} \int_{\triangle_i} (\hat{f}_i(x) - f^*(x))^2 \frac{dP}{P(\triangle_i)} \leq \inf_{w \in \mathbb{R}^d} \int_{\triangle_i} (w^\top (x, 1) - f^*(x))^2 \frac{dP}{P(\triangle_i)} + \mathbb{E} \min\{\frac{Cd \cdot P(\triangle_i)}{P_n(\triangle_i)n}, 1\}, \tag{29}
\]

where the \( \min\{\cdot, 1\} \) part follows from the fact that when we have less than \( Cd \) points, we can always set \( \hat{f}_i \) to be the zero function.

Now, we define the function \( f'(x) := \sum_{i=1}^{k(n)} \hat{f}_i(x) 1_{x \in \triangle_i} \), and by multiplying the last equation by \( P(\triangle_i) \) for each 1 ≤ \( i \) ≤ \( k(n) \) and taking a sum over \( i \), we obtain that

\[
\mathbb{E} \int_{\Omega_k(n)} (f' - f^*)^2 dP \leq \sum_{i=1}^{k(n)} \inf_{w_i \in \mathbb{R}^d} \int_{\triangle_i} (w_i^\top (x, 1) - f^*)^2 dP + \sum_{i=1}^{k(n)} \min\{\frac{Cd \cdot P(\triangle_i)}{n \cdot P_n(\triangle_i)}, 1\}
\]

\[
\leq \int_{\Omega_k(n)} (f_k(n) - f^*)^2 dP + C_{1} dk(n) \cdot n^{-1} = O_d(n^{-\frac{d+2}{d+4}}),
\tag{30}
\]

where in the first equation, we used the the fact that \( n \cdot P_n(\triangle_i) \sim Bin(n, P(\triangle_i)) \) (for completeness, see Lemma 16), and in the last inequality we used Eq. (5). Next, recall that Theorem 9 implies that

\[
\mathbb{P}(\Omega \setminus \Omega_k(n)) \leq C(d)k(n)^{-\frac{d+2}{d+4}} \leq O_d(n^{-\frac{d+2}{d+4}}).
\]
Therefore, if we consider the (not necessarily convex) function \( \tilde{f} = f' \mathbb{1}_{\Omega_{k(n)}} + \mathbb{1}_{\Omega \setminus \Omega_{k(n)}} \), we obtain that

\[
E \int_{\Omega} (f' - f^*)^2 dP = E \int_{\Omega \setminus \Omega_{k(n)}} (f' - f^*)^2 dP + E \int_{\Omega_{k(n)}} (f' - f^*)^2 dP \leq O_d(n^{-\frac{d+4}{d+1}} + n^{-\frac{d}{d+1}})
\]

\[
\leq O_d(n^{-\frac{d}{d+1}}).
\]

Thus, \( \tilde{f} \) is a minimax optimal improper estimator. To obtain a proper estimator, we simply need to replace \( \tilde{f} \) by \( MP(\hat{f}) \), where \( MP \) is the procedure defined in Appendix C.

Finally, we remark that the runtime of this estimator is of order \( O_d(n^{O(1)}) \). Indeed, the procedure \( MP \) is essentially a convex LSE on \( n \) points, which can be formulated as a quadratic programming problem with \( O(n^2) \) constraints, and hence can be computed in \( O_d(n^{O(1)}) \) time (Seijo and Sen, 2011). In addition, the runtime of the other estimator we use, namely the estimator of Lemma 28 is linear in the number of inputs.

### B Definitions and Preliminaries

Here we collect definitions needed for the appendices below.

**Definition 29.** For a fixed \( \epsilon \in (0, 1) \), and a function class \( \mathcal{F} \) equipped with a probability measure \( Q \), an \( \epsilon \)-net is a set that has the following property: For each \( f \in \mathcal{F} \) there exists an element in this set, denoted by \( \Pi(f) \), such that \( \|f - \Pi(f)\|_Q \leq \epsilon \).

**Definition 30.** We denote by \( \mathcal{N}(\epsilon, \mathcal{F}, Q) \) the cardinality of the minimal \( \epsilon \)-net of \( \mathcal{F} \) (w.r.t to \( L_2(Q) \)).

Also denote by \( \mathcal{N}_\Gamma(\epsilon, \mathcal{F}, Q) \) the cardinality of the minimal \( \epsilon \)-net with bracketing, which is defined as a set that has the following property: For each \( f \in \mathcal{F} \) there exists two elements \( f_- \leq f \leq f_+ \) such that \( \|f_+ - f_-\|_Q \leq \epsilon \).

Next, we recall the definition of a \( \mathbb{P} \)-Donsker and non \( \mathbb{P} \)-Donsker classes for uniformly bounded \( \mathcal{F} \).

**Definition 31.** \((\mathcal{F}, \mathbb{P})\) is said to be \( \mathbb{P} \)-Donsker if for all \( \epsilon \in (0, 1) \), we have that \( \log \mathcal{N}(\epsilon, \mathcal{F}, \mathbb{P}) = \Theta_{\mathbb{P}, \mathcal{F}}(\epsilon^{-\alpha}) \) for \( \alpha \in (0, 2) \) and non \( \mathbb{P} \)-Donsker is \( \alpha \in (2, \infty) \).

**Remark 32.** It is shown in (Bronshtein, 1976; Gao and Wellner, 2017) that

\[
\log \mathcal{N}(\epsilon, \mathcal{F}_L(\Omega), \mathbb{P}) = \Theta_{\mathbb{P}, \mathcal{F}_L}(L/\epsilon^{d/2})
\]

and

\[
\log \mathcal{N}(\epsilon, \mathcal{F}^\Gamma(P), \mathbb{P}) = \Theta_{\mathbb{P}, \mathcal{F}^\Gamma}(C(P)/\epsilon)^{d/2}).
\]

Therefore, when the dimension \( d \geq 5 \), both \( \mathcal{F}_L(\Omega) \) and \( \mathcal{F}^\Gamma(P) \) are non-Donsker classes.

**Basic notions regarding polytopes** A quick but thorough treatment of the basic theory is given, e.g. (Schneider, 2014, §2.4). A set \( P \subset \mathbb{R}^d \) is called a polyhedral set if it is the intersection of a finite set of half-spaces, i.e., sets of the form \( \{x \in \mathbb{R}^d : x \cdot a \leq c\} \) for some \( a \in \mathbb{R}^d, c \in \mathbb{R} \). A polyhedral set \( P \) is called
a polytope if it is bounded and has nonempty interior; equivalently, a set \( P \) is a polytope if it is the convex hull of a finite set of points and has nonempty interior.

The affine hull of a set \( S \subset \mathbb{R}^d \) is defined as
\[
\text{aff } S = \bigcup_{k=1}^{\infty} \{ \sum_{i=1}^{k} a_i x_i : x_i \in K, a_i \in \mathbb{R} \mid \sum_{i=1}^{k} a_i = 1 \},
\]
which is the minimal affine subspace of \( \mathbb{R}^d \) containing \( S \). For a convex set \( K \), we define its dimension to be the linear dimension of its affine hull.

For any \( u \) that lie in the unit sphere, denoted by \( S^{d-1} \), and any convex set \( K \), the support set \( F(K, u) \) is defined as
\[
F(K, u) = \{ x \in K : x \cdot u = \max_{y \in K} y \cdot u \},
\]
where \( \max_{y \in K} y \cdot u = \infty \), then \( F(K, u) \) is defined to be the empty set.

Suppose \( P \) is a polyhedral set. For any \( u \in S^{m-1} \), \( F(P, u) \) is a polyhedral set of smaller dimension than \( K \). Any such \( F(P, u) \) is called a face of \( P \), and if \( F(P, u) \) has dimension \( m - 1 \), it is called a facet of \( P \). A polyhedral set \( P \) which is neither empty nor the whole space \( \mathbb{R}^d \) has a finite and nonempty set of facets, and every face of \( P \) is the intersection of some subset of the set of facets of \( P \). If \( P \) is a polytope, all of its faces, and in particular all of its facets, are bounded. A polytope is called simplicial if all of its facets are \((m - 1)\)-dimensional simplices, which is to say, each facet \( F \) of \( P \) is the convex hull of precisely \( m \) points in \( \text{aff } F \).

## C From an Improper to a Proper Estimator

The following procedure, which we named \( MP \), is classical and we give its description and prove its correctness here for completeness. However, note that we only give a proof for optimality in expectation; high probability bounds can be obtained using standard concentration inequalities.

The procedure \( MP \) is defined as follows: given an improper estimator \( \hat{f} \), draw \( X'_1, \ldots, X'_{k(n)} \sim_{i.i.d.} P \), and apply the convex LSE with the input \( \{(X'_i, \hat{f}(X'_i))\}_{i=1}^{k(n)} \), yielding a function \( \hat{f}_1 \). We remark that the convex LSE is only unique on the convex hull of the data-points \( X'_1, \ldots, X'_{k(n)} \), and not on the entire domain \( \Omega \) (Seijo and Sen, 2011), so we will show that any solution \( \hat{f}_1 \) of the convex LSE is optimal.

First off, we have
\[
\mathbb{E} \int_{\Omega} (\hat{f} - f^*)^2 d\mathbb{P}_{k(n)} = \mathbb{E} \int_{\Omega} (\hat{f} - f^*)^2 d\mathbb{P}
\]
(31)

Also recall the classical observation that for \( \hat{f}_1 \) that is defined above, we know that \( (\hat{f}_1(X'_1), \ldots, \hat{f}_1(X'_{k(n)})) \) is precisely the projection of \( (\hat{f}(X'_1), \ldots, \hat{f}(X'_{k(n)})) \) on the convex set
\[
\mathcal{F}_{k(n)} := \{(f(X'_1), \ldots, f(X'_{k(n)})) : f \in \mathcal{F}_1(\Omega) \} \subset \mathbb{R}^{k(n)},
\]
Now, the function $\Pi_{\mathcal{F}_{k(n)}}$ sending a point to its projection onto $\mathcal{F}_{k(n)}$, like any projection to a convex set, is a 1-Lipschitz function, i.e.,

$$\|\Pi_{\mathcal{F}_{k(n)}}(x) - \Pi_{\mathcal{F}_{k(n)}}(y)\| \leq \|x - y\| \quad \forall x, y \in \mathbb{R}^k(n).$$

We also know that $(\hat{f}_1(X'_i))_{i=1}^{k(n)} = \Pi_{\mathcal{F}_{k(n)}}(\hat{f})$ and $\Pi_{\mathcal{F}_{k(n)}}((f^*(X'_i))_{i=1}^{k(n)}) = (f^*(X'_i))_{i=1}^{k(n)}$, substituting in the preceding equation, we therefore obtain

$$E \int_{\Omega} (\hat{f}_1 - f^*)^2 d\mathbb{P}_k(n) \leq E \int_{\Omega} (\hat{f} - f^*)^2 d\mathbb{P}_k(n) = E \int_{\Omega} (\hat{f} - f^*)^2 d\mathbb{P};$$

since $\int (\cdot)^2 d\mathbb{P}_k(n)$ is just $\|\cdot\|^2 / k(n)$. In order to conclude the minimax optimality of $\hat{f}_1$, we know by Lemma 7 that for any function in

$$\mathcal{O} := \left\{ f \in \mathcal{F}_1 : \int_{\Omega} (f - \hat{f}_1)^2 d\mathbb{P}_{k(n)} = 0 \right\},$$

it holds that

$$E \int_{\Omega} (f - f^*)^2 d\mathbb{P} \leq 2E \int (\hat{f} - f^*)^2 d\mathbb{P}_{k(n)} + Ck(n)^{-\frac{1}{2}} \leq 2E \int (\hat{f} - f^*)^2 d\mathbb{P} + C_1 n^{-\frac{d+4}{d+2}},$$

where we used Eq. (31) and the fact that $k(n) = n^{\frac{d+4}{d+2}}$. Since we showed that $\hat{f}_1$ must lie in $\mathcal{O}$, the minimax optimality of this proper estimator follows.