DIRAC MONOPOLES AND HAWKING RADIATION
IN KOTTLER SPACETIME

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The natural extension of Schwarzschild metric to the case of nonzero cosmological constant \( \Lambda \) known as the Kottler metric is considered and it is discussed under what circumstances the given metric could describe the Schwarzschild black hole immersed in a medium with nonzero energy density. Under the latter situation such an object might carry topologically inequivalent configurations of various fields. The given possibility is analysed for complex scalar field and it is shown that the mentioned configurations might be tied with natural presence of Dirac monopoles on black hole under consideration. In turn, this could markedly modify the Hawking radiation process.

Keywords: Dirac monopoles; Hawking radiation; Kottler spacetime

1. Introductory remarks

At present time there are certain indications that formation and evolution of a significant part of black holes occur in active galactic nuclei (see, e.g. Ref.\(^1\) and references therein). Under the circumstances the notion of black hole as an isolated object being described, for example, by the Schwarzschild (SW) metric evidently becomes inadequate to the situation. In active galactic nuclei the regions filled with a nonzero energy density or strong magnetic fields are the rule rather than exception. As a result, the properties of black holes may be changed under these conditions and to take into account this change one should modify the description of black holes. One can note that solutions of the Einstein equations which can model the mentioned conditions are known for a long time. In particular, the Kottler metric which can be written down in the form

\[
ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv a dt^2 - a^{-1} dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2)
\]

with \( a = 1 - 2M/r + \Lambda r^2/3 \) is the solution of the Einstein equations with cosmological

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constant $\Lambda$ ($M$ is the black hole mass) and is known since 1918 (see, e. g., Refs.\(^2\)). Inasmuch as the cosmological constant $\Lambda$ can be associated with the nonzero energy density (in usual units) $\rho = \Lambda c^4/(8\pi G)$ (see e. g. Ref. 3) then the metric (1) may be interpreted as the one describing the SW black hole immersed in a medium with the nonzero energy density. It should be noted, however, that metric (1) is not asymptotically flat one and the given circumstance can lead to difficulties of formulating the scattering problem necessary for the Hawking radiation analysis (see below).

On the other hand, the nontrivial topological properties of spacetimes may play essential role while studying quantum geometry of fields on them (see, e. g., our reviews\(^4\)). The black hole spacetimes are not exceptions. Really, as was shown in a recent series of the papers\(^5\),\(^6\),\(^7\),\(^8\), the black holes can carry the whole spectrum of topologically inequivalent configurations (TICs) for miscellaneous fields, in the first turn, complex scalar and spinor ones. The mentioned TICs can essentially modify the Hawking radiation from black holes and also can give some possible statistical explanation of the black hole entropy\(^9\). Physically, the existence of TICs should be obliged to the natural presence of magnetic U(N)-monopoles (with $N \geq 1$) on black holes though the total (internal) magnetic charge (abelian or nonabelian) of black hole remains equal to zero.

In present paper we would like to clear up to what extent the mentioned features can be inherent to the black holes described by metric (1). For to simplify calculations our considerations will restrict themselves to the TICs of complex scalar field and, accordingly, to the Dirac U(1)-monopoles. Sec. 2 serves for obtaining the auxiliary relations necessary for the rest of paper. Sec. 3 is devoted to description of Dirac monopoles on the black holes under exploration and also to separation of variables for the corresponding wave equations while Sec. 4 contains the correct statement of the scattering problem for scalar particles interacting with the mentioned Dirac monopoles and description of the conforming $S$-matrix necessary to analyse the Hawking radiation process. The latter is discussed in Sec. 5 and Sec. 6 gives concluding remarks.

Throughout the paper we employ the system of units with $\hbar = c = G = 1$, unless explicitly stated. Finally, we shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold $F$ furnished with an integration measure.

2. Preliminary considerations

First of all it should be noted that the standard spacetime topology on which the metric (1) with arbitrary $a = a(r)$ can be realized in a natural way is of $\mathbb{R}^2 \times S^2$-form. In the given paper we shall, however, deal only with the Kottler metric where the form of $a$ was fixed above in Sec. 1. Let us introduce a number of quantities that for metric (1) with $\Lambda \neq 0$ will generalize the corresponding ones when $\Lambda = 0$, i. e., in the SW black hole case. Besides we shall consider $\Lambda > 0$ within the paper.

Let us consider the equation $a(r) = 1 - 2M/r + \Lambda r^2/3 = 0$ that can be rewritten as $r^3 + 3r/\Lambda - 6M/\Lambda = 0$. Discriminant of the latter equation is equal to $D =$
(1/Λ)^3 + (3M/Λ)^2 > 0 which means the mentioned equation to have one real and two complex conjugated roots. The only real root is

\[ r_+ = \frac{1}{\sqrt{\Lambda}} (\sqrt{\alpha} + \sqrt{\beta}) \]  

(2)

with \( \alpha = 3M\sqrt{\Lambda} - \sqrt{1 + 9M^2\Lambda} \) and \( \beta = 3M\sqrt{\Lambda} + \sqrt{1 + 9M^2\Lambda} \) and we have the expansion

\[ a(r) = \frac{\Lambda}{3r} (r - r_+) \left( r^2 + r_+ r + \frac{6M}{\Lambda r_+} \right) . \]  

(3)

It is clear that \( r_+ \to 2M \) when \( \Lambda \to 0 \) and we can see that horizon of the SW black hole is shifted to \( r_+ \) at \( \Lambda > 0 \). Under this situation when defining the black hole entropy as \( S = \pi r_+^2 \) we shall find the black hole temperature \( T \) from the relation \( 1/T = \partial S/\partial M \) which entails

\[ T = \frac{\Lambda\sqrt{1 + 9M^2\Lambda} (\alpha\beta)^{2/3}}{2\pi(\alpha^{1/3} + \beta^{1/3})[3\Lambda^2\alpha^{2/3} - \beta^{2/3}] + \sqrt{\Lambda}(1 + 9\Lambda^2(\alpha^{2/3} + \beta^{2/3}))} , \]  

(4)

so when \( \Lambda \to 0 \), \( T \to 1/8\pi M \), the SW black hole temperature.

Further, we define the variable \( r_* \) by the relation

\[ r_* = \int\frac{dr}{a} = \frac{r_+^2}{2(3M - r_+)} \ln \left( \frac{r}{r_+} - 1 \right) - \frac{r_+^2}{4(3M - r_+)} \ln \left[ 1 + \frac{\Lambda}{6M} (r^2 + rr_+) \right] \]

\[ + \left[ \frac{6M}{\Lambda(3M - r_+)} + \frac{r_+^2}{2(3M - r_+)} \right] \frac{1}{\sqrt{24M^2 - r_+^2}} \arctan \left( \frac{2r + r_+}{\sqrt{24M^2 - r_+^2}} - \frac{r_+}{2} \right) \]  

(5)

where we choose the integration constant for the expression (5) to pass on to the standard Regge-Wheeler variable \( r_* = r + 2M \ln |r/(2M) - 1| \) at \( \Lambda \to 0 \). At last, let us introduce the dimensionless quantities \( x = r_*/M \), \( y = r/M \), \( y_+ = r_+/M \) so that for them we shall have

\[ x = \frac{y_+^2}{2(3 - y_+)} \ln \left( \frac{y}{y_+} - 1 \right) - \frac{y_+^2}{4(3 - y_+)} \ln \left[ 1 + \frac{\Lambda y_+^2}{6} (y^2 + yy_+) \right] \]

\[ + \left[ \frac{6}{\Lambda M^2(3 - y_+)} + \frac{y_+^3}{2(3 - y_+)} \right] \frac{1}{\sqrt{24M^2 - y_+^2}} \arctan \left( \frac{2y + y_+}{\sqrt{24M^2 - y_+^2}} - \frac{y_+}{2} \right) \]  

(6)

i.e., \( y = y(x) \) is a reverse function to (6) and again at \( \Lambda \to 0 \) the relation is simplified to \( x = y + 2 \ln(0.5y - 1) \). At this point we face the difficulty that at \( y \to y_+ \), \( x \to -\infty \) while when \( y \to +\infty \), \( x \) some finite value which is different from the SW black hole case, where \( y_+ = 2 \leq y \leq +\infty \), whereas \( -\infty < x < +\infty \). Under this situation the correct statement of the scattering problem necessary for the Hawking radiation analysis is impossible so long as the latter problem should be considered on the whole \( x \)-axis.

To overcome this obstacle it should be noted that the region in universe with the nonzero energy density (and where the SW black hole can be located) has, generally
speaking, some finite sizes. Under the circumstances, if \( r_0 \) is such a characteristic size then the metric (1) should describe the black hole at \( r_+ \leq r \leq r_0 \), while at \( r_0 \leq r \leq +\infty \) the black hole should be described by metric (1) with \( \Lambda = 0 \), i. e., by the standard SW metric. As a result, the relation (6) should be applied when \( -\infty < x \leq x_0 \), otherwise, at \( x_0 \leq x < +\infty \) the corresponding \( y(x) \) should be determined from the equation \( x = y + 2 \ln(0.5y - 1) \) and one should put \( y_0 = r_0/M \) while \( x_0 = y_0 + 2 \ln(0.5y_0 - 1) \).

Now we shall have a function \( y(x) \) defined on the whole \( x \)-axis and the conforming scattering problem can be posed correctly (see Sec. 4). It is clear the size \( r_0 \) should be specified under the concrete conditions so in what follows we shall use it as a parameter in further considerations.

3. Dirac monopoles and separation of variables

As was discussed in Refs.5, 6, 7, the black holes can carry TICs of a complex scalar field that are conditioned by the availability of a countable number of complex line bundles over the \( \mathbb{R}^2 \times S^2 \)-topology underlying the 4D black hole physics. Each TIC corresponds to sections of a complex line bundle \( E \) while the latter can be characterized by its Chern number \( n \in \mathbb{Z} \) (the set of integers). TIC with \( n = 0 \) can be called untwisted, while the rest of the TICs with \( n \neq 0 \) should be referred to as twisted. Using the fact that all the mentioned line bundles can be trivialized over the chart of local coordinates \((t, r, \vartheta, \phi)\) covering almost the whole manifold \( \mathbb{R}^2 \times S^2 \) one can obtain a suitable wave equation on the given chart for TIC \( \phi \) with mass \( \mu_0 \) and Chern number \( n \in \mathbb{Z} \) in the form

\[
|g|^{-1/2}(\partial_{\mu} - ieA_\mu)[g^{\mu\nu}|g|^{1/2}(\partial_{\nu} - ieA_\nu)\phi] = -\mu_0^2\phi, \tag{7}
\]

where the specification depends on the gauge choice for connection \( A_\mu \) (vector-potential for the corresponding Dirac monopole) in the line bundle \( E \) with the Chern number \( n \). We shall here consider the gauge where one can put the above connection 1-form equal to \( A = A_\mu dx^\mu = -\frac{n}{r} \cos \vartheta d\varphi \). Then the curvature of the bundle \( E \) is \( F = dA = \frac{n}{r} \sin \vartheta d\vartheta \wedge d\varphi \). We can further introduce the Hodge star operator on 2-forms \( F \) of any \( k \)-dimensional (pseudo)riemannian manifold \( B \) provided with a (pseudo)riemannian metric \( g_{\nu\tau} \) by the relation (see, e. g., Ref.12)

\[
F \wedge *F = (g^{\rho\alpha}g^{\tau\beta} - g^{\rho\beta}g^{\tau\alpha})F_{\nu\tau}F_{\alpha\beta} \sqrt{|g|} dx^1 \wedge dx^2 \cdots \wedge dx^k \tag{8}
\]

in local coordinates \( x^\nu \). In the case of the metric (1) this yields

\[
*(dt \wedge dr) = \sqrt{|g|} g^{tt} g^{rr} d\vartheta \wedge d\varphi = -r^2 \sin \vartheta d\vartheta \wedge d\varphi, \nonumber
\]

\[
*(d\vartheta \wedge d\varphi) = \sqrt{|g|} g^{\vartheta\vartheta} g^{\varphi\varphi} dt \wedge dr = \frac{1}{r^2 \sin \vartheta} dt \wedge dr, \tag{9}
\]

so that \(*^2 = ** = -1\), as should be for the manifolds with lorentzian signature. After this we find \( *F = \frac{n}{er^2} dt \wedge dr \), and integrating over the surface \( t = \text{const} \),
\[ r = \text{const} \] with topology \( S^2 \) gives rise to the Gauss theorem

\[ \int_{S^2} *F = 0 \]  

and the Dirac charge quantization condition

\[ \int_{S^2} F = 4\pi \frac{n}{e} = 4\pi q \]  

with magnetic charge \( q \). Besides, the Maxwell equations \( dF = 0, d*F = 0 \) are clearly fulfilled with the exterior differential \( d = \partial_t dt + \partial_r dr + \partial_\theta d\theta + \partial_\varphi d\varphi \) in coordinates \( t, r, \theta, \varphi \). Also it should be emphasized that the total (internal) magnetic charge \( Q_m \) of black hole which should be considered as the one summed up over all the monopoles remains equal to zero because

\[ Q_m = \frac{1}{e} \sum_{n \in \mathbb{Z}} n = 0, \]  

so the external observer does not see any magnetic charge of the black hole though the monopoles are present on black hole in the sense described above.

### 3.1. Monopole masses

We can estimate the corresponding monopole masses so long as we are effectively in the asymptotically flat spacetime due to the remarks made late in Sec. 2 and one can use the \( T_{00} \)-component of the energy-momentum tensor for electromagnetic field

\[ T_{\mu\nu} = \frac{1}{4\pi} (-F_{\mu \alpha} F_{\nu \beta} g^{\alpha \beta} + \frac{1}{4} F_{\beta \gamma} F_{\alpha \delta} g^{\alpha \beta} g^{\gamma \delta} g_{\mu \nu}) \]  

(13)

to find the monopole masses according to

\[ m_{\text{mon}}(n) = \int_{\mathbb{R} \times S^2} T_{00} \sqrt{\gamma} d^3x = \int_{\mathbb{R} \times S^2} T_{00} r^2 \sin \vartheta a^{-1/2} d^3x \]  

(14)

with \( a(r) \) of (1) and \( \sqrt{\gamma} d^3x = r^2 \sin \vartheta a^{-1/2} d^3x \) is the volume element for the spatial part (with topology \( \mathbb{R} \times S^2 \)) of the metric (1) while \( T_{00} \)-component is proved to be equal to \( \frac{1}{16\pi} aF_{\vartheta \varphi}^2 g^{\vartheta \vartheta} g^{\varphi \varphi} \), with \( F_{\vartheta \varphi} = n \sin \vartheta / e \). This entails

\[ m_{\text{mon}}(n) = \frac{n^2}{4e^2} \left[ \int_{r_+}^{r_0} \sqrt{1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} r} + \int_{r_0}^{\infty} \sqrt{1 - \frac{2M}{r} r^{3/2}} \right] \]

\[ = \frac{n^2}{4e^2} \left[ \int_{r_+}^{r_0} \sqrt{1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} r} + \frac{1}{3M} \left( 1 - \frac{2M}{r_0} \right)^{3/2} \right]. \]  

(15)
It is clear that at $r_0 \to +\infty$, $m_{\text{mon}}(n) \to \infty$ and the conforming Compton wavelength $\lambda_{\text{mon}} = 1/m_{\text{mon}} \to 0$, i.e., $\lambda_{\text{mon}} \ll r_+ \text{ and the given monopoles might reside in black hole under discussion as quantum objects.}

3.2. Separation of variables

If returning to the Eq. (7), then, in the way analogous to that of Refs. [5,6,7], one can show the equation (7) to have in $L^2(\mathbb{R}^2 \times S^2)$ a complete set of solutions of the form

$$f_{\omega nml} = \frac{(2\pi\omega)^{-1/2}}{r} e^{i\omega t} Y_{nlm}(\vartheta, \varphi) R_{\omega nml}(r), \quad l = |n|, |n| + 1, ..., \quad |m| \leq l, \quad (16)$$

where the explicit form of the monopole (spherical) harmonics $Y_{nlm}(\vartheta, \varphi)$ can be found in Refs. [5,6,7]. As to the function $R_{\omega nml}(r)$, then it obeys the equation:

$$a\partial_r ar^2 \partial R_{\omega nml} \frac{r}{r} + \omega^2 r R_{\omega nml} = a \left[ \frac{\mu_0^2}{r^2} + \frac{l(l+1) - n^2}{r^2} \right] R_{\omega nml}, \quad (17)$$

which can be rewritten as

$$\frac{d^2 R_{\omega nml}}{dr_*^2} + (\omega^2 - \mu_0^2)R_{\omega nml} = \left[ \mu_0^2(a-1) + \frac{a da}{dr} + a \frac{l(l+1) - n^2}{r^2} \right] R_{\omega nml} \quad (18)$$

with the help of (5). At last, when denoting $k = M\omega$ and introducing the functions $\psi(x,k,n,l) = R_{\omega nml}[M y(x)]$, where $y(x)$ is a function reverse to (6), we shall obtain that $\psi(x,k,n,l)$ obeys the Schrödinger-like equation

$$\left[ \frac{d^2}{dx^2} + (k^2 - \mu_0^2 M^2) \right] \psi(x,k,n,l) = V(x,M,\Lambda,n,l) \psi(x,k,n,l), \quad (19)$$

where

$$V(x,M,\Lambda,n,l) = \left[ 1 - \frac{2}{y(x)} + \frac{\Lambda M^2}{3} y^2(x) \right] \left[ \frac{2}{y^3(x)} + \frac{2\Lambda M^2}{3} + \frac{l(l+1)-n^2}{y^2(x)} \right]$$

$$+ (\mu_0 M)^2 \left[ \Lambda M^2 \frac{3}{3} y^2(x) - \frac{2}{y^3(x)} \right], \quad -\infty \leq x \leq x_0, \quad (20)$$

and

$$V(x,M,\Lambda,n,l) = \left[ 1 - \frac{2}{y(x)} \right] \left[ \frac{2}{y^3(x)} + \frac{l(l+1)-n^2}{y^2(x)} \right] - (\mu_0 M)^2 \frac{2}{y(x)}, \quad x_0 \leq x \leq +\infty. \quad (21)$$

Now we can discuss how to pose the correct scattering problem for Eq. (19) because the corresponding solutions of this problem will be just necessary to analyse the Hawking radiation from black hole.

4. The scattering problem
One can notice that potential $V(x, M, \Lambda, n, l)$ of (20)–(21) satisfies the conditions explored in Refs.\ref{11} and referring for more details to those works we may here use their results to correctly formulate the scattering problem for Eq. (19). Namely, the correct statement of the scattering problem will consist in searching for two solutions $\psi^+(x, k^+, n, l)$, $\psi^-(x, k, n, l)$ of the equation (19) obeying the following conditions

\[
\psi^+(x, k^+, n, l) = \begin{cases} 
 e^{ikx} + s_{12}(k, l, n)e^{-ikx} + o(1), & x \to -\infty, \\
 s_{11}(k, l, n)w_{-i\mu, \frac{1}{2}}(-2ik^+ x) + o(1), & x \to +\infty,
\end{cases} \quad x \to -\infty,
\]

\[
\psi^-(x, k, n, l) = \begin{cases} 
 s_{22}(k, l, n)e^{-ikx} + o(1), & x \to -\infty, \\
 w_{i\mu, \frac{1}{2}}(2ik^+ x) + s_{21}(k, l, n)w_{-i\mu, \frac{1}{2}}(-2ik^+ x) + o(1), & x \to +\infty,
\end{cases} \quad x \to +\infty,
\]

(22)

where $k^+(k) = \sqrt{k^2 - (\mu_0 M)^2}$, $\mu = \frac{(\mu_0 M)^2}{(\mu_0 M)^2 + k^2}$ and the functions $w_{\pm i\mu, \frac{1}{2}}(\pm z)$ are related to the Whittaker functions $W_{\pm i\mu, \frac{1}{2}}(\pm z)$ (concerning the latter ones see e. g. Ref.\ref{13}) by the relation

\[
w_{\pm i\mu, \frac{1}{2}}(\pm z) = W_{\pm i\mu, \frac{1}{2}}(\pm z)e^{-\pi \mu/2},
\]

so that one can easily gain asymptotics (using the corresponding ones for Whittaker functions\ref{13})

\[
w_{i\mu, \frac{1}{2}}(-2ik^+ x) = e^{ik^+ x}e^{i\mu \ln|2k^+ x|}[1 + O(|k^+ x|^{-1})], \quad x \to +\infty,
\]

\[
w_{-i\mu, \frac{1}{2}}(2ik^+ x) = e^{-ik^+ x}e^{-i\mu \ln|2k^+ x|}[1 + O(|k^+ x|^{-1})], \quad x \to +\infty.
\]

(23)

We can see that there arises some $S$-matrix with elements $s_{ij}, i, j = 1, 2$ and it satisfies certain unitarity relations (for more details see Refs.\ref{11}). As will be seen below, for calculating the Hawking radiation we need the coefficient $s_{11}$, consequently, we need to have some algorithm for numerical computation of it insasmuch as the latter cannot be evaluated in exact form. The given algorithm can be extracted from the results of Refs.\ref{11}. To be more precise

\[
s_{11}(k, l, n) = 2ik/[f^-(x, k, l, n), f^+(x, k^+, l, n)],
\]

(24)

where $[,]$ signifies the Wronskian of functions $f^-, f^+$, the so-called Jost type solutions of Eq. (19). In their turn, these functions and their derivatives obey the certain integral equations. Since the Wronskian does not depend on $x$ one can take the following form of the mentioned integral equations

\[
f^-(x_0, k, l, n) = e^{-ikx_0} + \frac{1}{k} \int_{-\infty}^{x_0} \sin[k(x_0 - t)]V^-(t, l, n)f^-(t, k, l, n)dt,
\]

(25)

\[
(f^-)'_x(x_0, k, l, n) = -ike^{-ikx_0} + \int_{-\infty}^{x_0} \cos[k(x_0 - t)]V^-(t, l, n)f^-(t, k, l, n)dt.
\]

(26)
\[ f^+(x_0, k^+, l, n) = w_{i\mu, \frac{1}{2}}(-2ik^+x_0) + \]
\[ \frac{1}{k^+} \int_{x_0}^{+\infty} \text{Im}[w_{i\mu, \frac{1}{2}}(-2ik^+x_0)w_{-\mu, \frac{1}{2}}(2ik^+t)]V^+(t, l, n)f^+(t, k^+, l, n)dt, \quad (27) \]
\[ (f^+)_x'(x_0, k^+, l, n) = \frac{d}{dx}w_{i\mu, \frac{1}{2}}(-2ik^+x_0) + \]
\[ \frac{1}{k^+} \int_{x_0}^{+\infty} \text{Im}[\frac{d}{dx}w_{i\mu, \frac{1}{2}}(-2ik^+x_0)w_{-\mu, \frac{1}{2}}(2ik^+t)]V^+(t, l, n)f^+(t, k^+, l, n)dt, \quad (28) \]
where the potentials
\[ V^-(x, l, n) = \left[ 1 - \frac{2}{y(x)} + \frac{2\Lambda M^2}{3}y^2(x) \right] \left[ (\mu_0M)^2 + \frac{2}{y^2(x)} + \frac{2\Lambda M^2}{3} + \frac{l(l+1) - n^2}{y^2(x)} \right], \quad (29) \]
\[ V^+(x, l, n) = 2(\mu_0M)^2 \left[ \frac{1}{x} - \frac{1}{y(x)} \right] + \left[ 1 - \frac{2}{y(x)} \right] \left[ \frac{l(l+1) - n^2}{y^2(x)} + \frac{2}{y^3(x)} \right] \quad (30) \]
tend to 0 when, respectively, \( x \to -\infty \) or \( x \to +\infty \). The relations (24)–(30) can be employed for numerical calculation of \( s_{11} \) (see Refs. for the case \( \Lambda = 0 \)).

To summarize, we can now obtain the general solution of (19) in the class of functions restricted on the whole \( x \)-axis in the linear combination form
\[ \psi(x, k, n, l) = C^1_{n\ell}(k)\psi^+(x, k^+, n, l) + C^2_{n\ell}(k)\psi^-(x, k, n, l), \quad (31) \]
since a couple of the functions \( \psi^+(x, k^+, n, l) \), \( \psi^-(x, k, n, l) \) forms a fundamental system of solutions for the equation (19).

Having obtained all the above, we can discuss the Hawking radiation process for any TIC of complex scalar field.

5. Modification of Hawking radiation

As can be seen, the equation (7) corresponds to the lagrangian
\[ \mathcal{L} = (-g)^{1/2}(g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi - \mu_0^2\overline{\phi}\phi), \quad (32) \]
where the overbar signifies complex conjugation, \( D_\mu = \partial_\mu - ieA_\mu \) and, as a result, we have the conforming energy-momentum tensor for TIC with the Chern number \( n \)
\[ T_{\mu\nu} = \text{Re}[(D_\mu\phi)(D_\nu\phi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(D_\alpha\phi)(D_\beta\phi) - \mu_0^2g_{\mu\nu}\overline{\phi}\phi]. \quad (33) \]

When quantizing twisted TIC with the Chern number \( n \) we shall take the set of functions \( f_{\omega\ell m l} \) of (16) as a basis in \( L_2(\mathbb{R}^2 \times \mathbb{S}^2) \) so that \( |m| \leq l, l = |n|, |n| + 1,... \)
and we shall normalize the monopole spherical harmonics $Y_{nlm}(\vartheta, \varphi)$ to 1, i.e., by the condition
\[
\int_0^{2\pi} \int_0^{\pi} Y_{nlm}(\vartheta, \varphi) Y_{n'l'm'}(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi = \delta_{ll'} \delta_{mm'} .
\] (34)

After this we can evidently realize the procedure of quantizing TIC with the Chern number $n$, as usual, by expanding this TIC in the modes $f_{\omega nlm}$
\[
\phi = \sum_{l=|n|}^{\infty} \sum_{|m| \leq l} \int d\omega (a_{\omega nlm} f_{\omega nlm} + b_{\omega nlm}^* f_{\omega nlm}) ,
\]
\[
\phi^+ = \sum_{l=|n|}^{\infty} \sum_{|m| \leq l} \int d\omega (b_{\omega nlm}^* f_{\omega nlm} + a_{\omega nlm} f_{\omega nlm}) ,
\] (35)

so that $a_{\omega nlm}^+, b_{\omega nlm}^+$ should be interpreted as the corresponding creation and annihilation operators for charged scalar particle in both the gravitational field of black hole and the field of the conforming monopole with the Chern number $n$ and we have the standard commutation relations
\[
[a^-_i, a^+_j] = \delta_{ij}, \quad [b^-_i, b^+_j] = \delta_{ij}
\] (36)

and zero for all other commutators, where $i = \{\omega nlm\}$ is a generalized index. Under the circumstances one can speak about the Hawking radiation process for any TIC of complex scalar field. Actually, one should use the energy-momentum tensor of (33) for TIC with the Chern number $n$ in quantum form
\[
T_{\mu\nu} = \text{Re}[(\overline{\partial_\mu} \phi^*)(\partial_\nu \phi) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\overline{\partial_\alpha} \phi^*)(\partial_\beta \phi)].
\] (37)

To get the luminosity $L(n)$ with respect to the Hawking radiation for TIC with the Chern number $n$, we define a vacuum state $|0>$ by the conditions
\[
a^-_i |0> = 0, \quad b^-_i |0> = 0 ,
\] (38)

and then
\[
L(n) = \lim_{r \to \infty} \int_{S^2} <0|T_{tr}|0> \ d\sigma
\] (39)

with the vacuum expectation value $<0|T_{tr}|0>$ and the surface element $d\sigma = r^2 \sin \vartheta d\vartheta d\varphi$. Now with the help of (34), (35), (37), (38) and passing on to the quantities $x, k$ introduced in Sec. 3, we shall find
\[
L(n) = M^{-2} \lim_{x \to \infty} \text{Re} \left[ \frac{i}{2\pi} \sum_{l=|n|}^{\infty} (2l + 1) \int_{\mu_0 M}^{\infty} (\partial_x \psi(x, k, n, l) \psi(x, k, n, l)) dk \right] ,
\] (40)
where the relations
\[ \frac{d}{dr} = \frac{1}{a} \frac{d}{d r^*} = \frac{1}{aM} \frac{d}{dx} \] (41)
were used.

As a consequence, the choice of vacuum state can be defined by the choice of
the suitable linear combination of (31) for \( \psi(x,k,n,l) \). Under this situation the
Hawking radiation corresponds to the mentioned choice in the form
\[ C_{nl}^1(k) = \frac{1}{\sqrt{\exp(k/M) - 1}}, \quad C_{nl}^2(k) = 0, \] (42)
which at last with using asymptotics (22)–(23) entails (in usual units)
\[ L(n) = A \sum_{l=|n|}^{\infty} (2l + 1) \int_{\tilde{\mu}}^{\infty} \left| s_{11}(k,n,l) \right|^2 k^+ \frac{dk}{\exp(k/M) - 1}, \] (43)
where the black hole temperature \( T \) is given by (4), \( A = \frac{1}{2\pi\hbar} \left( \frac{h^3}{cM} \right)^2 \approx 0.273673 \cdot 10^{50} \text{ erg} \cdot \text{s}^{-1}. \ M^{-2}, \ \tilde{\mu} = (\mu_0 M)/\mu^2_{pl}, \ \mu_0 \) and \( M \) in g while \( \mu^2_{pl} = \hbar c/G \) is the Planck mass square in g².

We can interpret \( L(n) \) as an additional contribution to the Hawking radiation
due to the additional charged scalar particles leaving the black hole because of the
interaction with monopoles and the conforming radiation can be called the monopole
Hawking radiation. Under this situation, for the all configurations luminosity \( L \)
of black hole with respect to the Hawking radiation concerning the complex scalar
field to be obtained, one should sum up over all \( n \), i. e.
\[ L = \sum_{n \in \mathbb{Z}} L(n) = L(0) + 2 \sum_{n=1}^{\infty} L(n) \] (44)
since \( L(-n) = L(n) \).

As a result, we can expect a marked increase of Hawking radiation from black
holes under consideration. But for to get an exact value of this increase one should
apply numerical methods. In the case of the pure SW black hole (\( \Lambda = 0 \)), for
example, it was found that the contribution due to monopoles can be of order
11% of the total pion-kaon luminosity. So that it would be interesting enough to
evaluate similar increase in the case \( \Lambda \neq 0 \).

6. Concluding remarks

The results of the present paper show that nontrivial topological properties of
black holes may have essential influence on their physical properties even when
the black holes cannot be considered as the isolated objects. Though we dwelt only
upon the scalar particle case it is beyond doubt that similar considerations will hold
true for spinor particles too if employing the results of Refs. The correct statement
of the corresponding scattering problem for spinor particles on the SW black hole has been only recently explored and one can extend the results obtained here to the spinor case. Also the important case is that of black hole immersed in strong magnetic field. The conforming metric is the Schwarzschild-Ernst one which is defined on the topology $\mathbb{R}^2 \times S^2$ as well, consequently, one should investigate how the (internal) Dirac monopoles will behave under these conditions. We hope to continue the work along this line in future explorations.

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