One-dimensional heat equation with discontinuous conductance

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Abstract

We study a second-order parabolic equation with divergence form elliptic operator, having piecewise constant diffusion coefficients with two points of discontinuity. Such partial differential equations appear in the modelization of diffusion phenomena in medium consisting of three kind of materials. Using probabilistic methods, we present an explicit expression of the fundamental solution under certain conditions. We also derive small-time asymptotic expansion of the PDE’s solutions in the general case. The obtained results are directly usable in applications.

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1 Introduction

In many practical applications, one is encountered with heat propagations in heterogeneous media consisting of different kind of materials. Mathematically, such kind of heat propagation is modeled by heat equations with divergence form elliptic operators having discontinuous diffusion coefficients. In this paper we exam one-dimensional model where the medium consisting of three different kind of materials.

Let a, ai, ρi, i = 1, 2, 3 be positive constants and define

\[ A(x) = a_1 1_{\{x \leq 0\}} + a_2 1_{\{0 < x \leq a\}} + a_3 1_{\{a < x\}} \]

and

\[ \rho(x) = \rho_1 1_{\{x \leq 0\}} + \rho_2 1_{\{0 < x \leq a\}} + \rho_3 1_{\{a < x\}}. \]

Set

\[ \mathcal{L} = \frac{1}{2\rho(x)} \frac{d}{dx} \left( \rho(x) A(x) \frac{d}{dx} \right), \tag{1.1} \]

which is a self-adjoint operator in \( L^2(\mathbb{R}; \rho(x)dx) \). The parabolic equation

\[ \frac{\partial u(t, x)}{\partial t} = \mathcal{L} u(t, x) \tag{1.2} \]

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describes the heat propagation in the line that consists of three different kind of material. It is well-known that there is an $m$-symmetric diffusion process $X$ associated with $\mathcal{L}$, where $m(dx) := \rho(x)dx$. Since both $A(x)$ and $\rho(x)$ are bounded between two positive constants, it is known (see [7]) that $X$ possesses a jointly continuous transition density function $q(t,x,y)$ with respect to the measure $m(dx)$ on $\mathbb{R}$ that is symmetric in $x$ and $y$ and enjoys the following Aronson type Gaussian estimates: there exists positive constants $C_1, C_2 \geq 1$ so that for every $t > 0$ and $x, y \in \mathbb{R}$,

$$C_1^{-1}t^{-1/2} \exp(-C_2|x-y|^2/t) \leq q(t,x,y) \leq C_1 t^{-1/2} \exp(-|x-y|^2/(C_2 t)).$$ \hfill (1.3)

Using Dirichlet form theory, the following semimartingale representation of $X$ is derived in [10], Proposition 5:

$$dX_t = \sqrt{A(X_t)} dB_t + \frac{\rho_2 a_2 - \rho_1 a_1}{\rho_2 a_2 + \rho_1 a_1} d\hat{L}^0_t + \frac{\rho_3 a_3 - \rho_2 a_2}{\rho_3 a_3 + \rho_2 a_2} d\hat{L}^a_t,$$ \hfill (1.4)

where $B$ is one-dimensional Brownian motion, $\hat{L}^0$ and $\hat{L}^a$ are symmetric semimartingale local times of $X$ at 0 and $a$, respectively. Here for a semimartingale $Y$, its symmetric semimartingale local time $\hat{L}^a_w(Y)$ at $w \in \mathbb{R}$ is defined to be

$$\hat{L}^a_w(Y) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[w-\varepsilon,w+\varepsilon]}(Y_s) \, d\langle Y \rangle_s,$$ \hfill (1.5)

where $\langle Y \rangle$ is the predictable quadratic variation process of $Y$. We define the semimartingale local time for $Y$ at level $w \in \mathbb{R}$ to be

$$L^a_w(Y) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[w,w+\varepsilon]}(Y_s) \, d\langle Y \rangle_s.$$ \hfill (1.6)

It is known (see [2]) that equation (1.4) admits a unique strong solution. Moreover, $X$ is a strong solution to (1.4) if and only if it is a strong solution to

$$dX_t = \sqrt{A(X_t)} dB_t + \frac{1}{2} \left(1 - \frac{\rho_1 a_1}{\rho_2 a_2}\right) \, dL^0_t(X) + \frac{1}{2} \left(1 - \frac{\rho_2 a_2}{\rho_3 a_3}\right) \, dL^a_t(X).$$ \hfill (1.7)

The purpose of this paper is to study the parabolic equation (1.2) with initial value $u(0,x) = h(x)$ and solve it as explicitly as possible, where $h$ is a piecewise $C^{N+1}$ function on $\mathbb{R}$ of the form

$$h(x) = h_1(x) \mathbf{1}_{\{x \leq 0\}} + h_2(x) \mathbf{1}_{\{0 < x \leq a\}} + h_3(x) \mathbf{1}_{\{x > a\}}.$$ \hfill (1.8)

with $h_i$ being bounded with continuously differentiable up to order $N+1$, $i = 1, 2, 3$, for some integer $N \geq 1$. This kind of PDEs appears in the modelization of diffusion phenomena in many areas, for example, in geophysics [9], ecology [4], biology [11] and so on. The non-smoothness of the coefficients reflects the multilayered medium. The solution $u(t,x)$ of (1.2) with initial value $h(x)$ admits a probabilistic representation:

$$u(t,x) = \mathbb{E}_x[h(X_t)], \quad t \geq 0.$$ \hfill (1.9)

Here the subscript $x$ in $\mathbb{E}_x$ means the expectation is taken with respect to the law of the diffusion process $X$ that starts from $x$ at time $t = 0$. In the first part of this paper, we show that, when either $\rho_1 \sqrt{a_1} = \rho_2 \sqrt{a_2}$ or $\rho_2 \sqrt{a_2} = \rho_3 \sqrt{a_3}$, we can reduce the diffusion process of (1.3) to skew Brownian motion through transformations by scale functions. Thus in this case, we are able to derive an explicit formula for the transition density function $q(t,x,y)$ of the diffusion process $X$ with respect to the measure $m(dx)$. The kernel $q(t,x,y)$ is the fundamental solution (also called
heat kernel) $q(t, x, y)$ of $\mathcal{L}$. Skew Brownian motions are a subclass of diffusion processes $X$ of (1.3) with $a_1 = a_2 = a_3 = 1$ and $p_2 = p_3$. It is first introduced by Ito and McKean in 1963, and has since been studied extensively by many authors; see, for example, [1, 2, 13, 15] and the references therein. However we do not know the explicit formula for $q(t, x, y)$ in the general case. In [6], formulas for the fundamental solutions are derived in terms of inverse Fourier transform and Green functions, but they are not explicit.

Though we do not have explicit formula for its heat kernel, in the second part of this paper, we are able to derive asymptotic expansion in small time of solutions of (1.2) in the general case with piecewise $C^{N+1}$ initial data, by making use of the explicit heat kernel obtained under the special case $(a_2, p_2) = (a_3, p_3)$. The small-time asymptotic expansion we get in this paper is explicit and can be directly used in applications. This extends the work of [17] and of [18], where only one discontinuity is studied. In [5] and [10], numerical methods are proposed to study (1.4) with more than two discontinuities are mentioned at the end of the paper.

The rest of the paper is organized as follows. In the section 2, we derive explicit formula for the fundamental solution of $\mathcal{L}$ in the case when either $p_1\sqrt{a_1} = p_2\sqrt{a_2}$ or $p_2\sqrt{a_2} = p_3\sqrt{a_3}$.

We then study the small-time asymptotic expansion of the solution of (1.2) in the general case for any positive constants $a$, $a_1$ and $p_1$. Extensions to the more general piecewise constant coefficient case with more than two discontinuities are mentioned at the end of the paper.

## 2 Fundamental solution

In view of the probabilistic representation (1.9), properties of solution to the heat equation (1.9) can be deduced from the properties of the diffusion process $X$ of (1.7). For notational simplicity, let’s rewrite SDE (1.7) as

$$
\begin{aligned}
\begin{cases}
dY_t^x = \left( p1_{\{Y_t^x \leq 0\}} + q1_{\{0 < Y_t^x \leq a\}} + r1_{\{a < Y_t^x\}} \right) dB_t + \frac{a}{2} dL_0^x(Y_t^x) + \frac{a}{2} dL_i^x(Y_t^x), \\
Y_0^x = x \in \mathbb{R},
\end{cases}
\end{aligned}
$$

(2.1)

where $p, q, r$ are positive constants, $\alpha, \beta \in (-\infty, 1)$ and $B$ a one-dimensional Brownian motion. The superscript $x$ indicates the process $Y_t^x$ starts from $x$ at $t = 0$. In the particular case when $p = q = r = 1$ and $\beta = 0$, $Y_t^x$ is just a Skew Brownian motion of parameter $(2 - a)^{-1}$ (cf. [15]), and in the case when $p = q = r = 1$, $Y_t^x$ is the double-skewed Brownian motion (cf. [12]).

The next two theorems are known; see [2] Theorems 2.1 and 2.2 or [8]. We give a proof here not only for reader’s convenience but also some formulas in the proof will be used later to derive the transition density function of $Y$.

**Theorem 2.1** For every $\alpha < 1$ and $\beta < 1$, SDE (2.1) has a unique strong solution for every $x \in \mathbb{R}$.

**Proof.** Define

$$
s(x) = \begin{cases}
x/p & \text{for } x < 0, \\
x/q & \text{for } x \in [0, a], \\
(x - a)/r + a/q & \text{for } x > a.
\end{cases}
$$

(2.2)

Clearly, $s$ is strictly increasing and one-to-one. Let $\sigma$ denote the inverse of $s$:

$$
\sigma(x) = \begin{cases}
px & \text{for } x < 0, \\
qx & \text{for } x \in [0, s(a)], \\
r(x - s(a)) + s(a)q & \text{for } x > s(a).
\end{cases}
$$

(2.3)
Let $s'_t$ and $\sigma'_t$ denote the left hand derivative of $s$ and $\sigma$, respectively; that is,

$$
s'_t(x) = \begin{cases} 
1/p & \text{for } x \leq 0, \\
1/q & \text{for } x \in (0,a], \\
1/r & \text{for } x > a,
\end{cases}
\quad\text{and}\quad
\sigma'_t(x) = \begin{cases} 
p & \text{for } x \leq 0, \\
q & \text{for } x \in (0,s(a)], \\
r & \text{for } x > s(a),
\end{cases}
$$

The second derivative $\sigma''(dx) = (q-p)\delta_{(0)}(dx) + (r-q)\delta_{(s(a))}(dx)$. Here $\delta_{(z)}$ denotes the Dirac measure concentrated at point $z$.

Since both $\frac{q(\alpha-1)}{p} + 1$ and $\frac{r(\beta-1)}{q} + 1$ are strictly less than 1, by Theorem 2.1 of [2], the following SDE has a unique strong solution $Z = \{Z_t, t \geq 0\}$ with $Z_0 = s(x)$:

$$
dZ_t = dB_t + \frac{1}{2} \left( \frac{q(\alpha-1)}{p} + 1 \right) dL^0_t(Z) + \frac{1}{2} \left( \frac{r(\beta-1)}{q} + 1 \right) dL^{(a)}_t(Z). \tag{2.4}
$$

Let $Y = \sigma(Z)$. We will show that $Y$ is a solution to (2.1). Clearly $Y_0 = x$. Function $\sigma$ is a piecewise linear function and can be expressed as a difference of two convex functions. By Ito-Tanaka’s formula (see [13, Theorem VI.1.5]), $Y$ is a semimartingale and in fact, for $t \geq 0$,

$$
Y_t = \sigma(Z_0) + \int_0^t \sigma'_t(Z_s) dZ_s + \frac{1}{2} \int_0^t L^w_t(Z) \sigma''(dw) = x + \int_0^t \left( p1_{\{Y_t\leq 0\}} + q1_{\{0<Y_t\leq a\}} + r1_{\{Y_t> a\}} \right) dB_t + \frac{\alpha q}{2} L^0_t(Z) + \frac{\beta r}{2} L^{(a)}_t(Z). \tag{2.5}
$$

By Corollary VI.1.9 of Revuz and Yor [13], for $t \geq 0$,

$$
L^{(a)}_t(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[s(a),s(a)+\varepsilon]}(Z_s) d(Z_s) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[a,a+r\varepsilon]}(Y_s) d(Y_s) = L^a_t(Y)/r. \tag{2.6}
$$

A similar calculation shows that $L^0_t(Z) = L^0_t(Y)/q$. Thus we have from (2.5) that $Y = \sigma(Z)$ is a strong solution for (2.1) with $Y_0 = x$.

We now examine pathwise uniqueness. Suppose that $Y^x$ is a solution of (2.1). Since $s$ is the difference of two convex functions, we can apply Ito-Tanaka’s formula to get

$$
s(Y^x_t) = s(x) + \int_0^t s'_t(Y^x_s) dY^x_s + \frac{1}{2} \int L^w_t(Y^x) s''(dw). \tag{2.7}
$$

Let $Z = s(Y^x)$. A similar calculation as above shows that $Z$ satisfies SDE (2.4) with initial value $Z_0 = s(x)$. Since by Theorem 2.1 of [2] solutions to (2.4) is unique, hence so is solution $Y^x$ to (2.1).

\[ \Box \]

**Theorem 2.2** The process $Y^x$ is a strong solution of the SDE (2.1), if and only if $Y^x$ is a strong solution to

$$
dY^x_t = \left( p1_{\{Y^x_t \leq 0\}} + q1_{\{0<Y^x_t \leq a\}} + r1_{\{a<Y^x_t\}} \right) dB_t + \frac{\alpha}{2-\alpha} d\hat{L}^0_t(Y^x) + \frac{\beta}{2-\beta} d\hat{L}^{(a)}_t(Y^x) \tag{2.8}
$$

with $Y^x_0 = x$. \[ \Box \]
Proof. This is due to the fact that, for solution $Y^x$ of (2.11) (see the proof of Theorem 2.2 in [2]),

$$L_t^0(Y^x) = \frac{2}{2 - \alpha} \hat{L}_t^0(Y^x), \quad L_t^0(Y^x) = \frac{2}{2 - \beta} \hat{L}_t^0(Y^x),$$

(2.9)

and $L_t^w(Y^x) = \hat{L}_t^w(Y^x)$ for $w \notin \{0, a\}$. \hfill $\square$

Corollary 2.3 For every $p, q > 0, \alpha < 1$ and $x \in \mathbb{R}$, the following SDE has a unique strong solution for every $x \in \mathbb{R}$:

$$dX_t^x = \left(p1_{\{X_t^x \leq 0\}} + q1_{\{0 < X_t^x\}}\right) dB_t + \frac{\alpha}{2} dL_t^0(X^x),$$

$$X_0^x = x \in \mathbb{R}.$$  (2.10)

Moreover, the transition probability density of the diffusion $X^x$ is given by

$$p^x(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y \leq 0\}}}{p} + \frac{1_{\{y > 0\}}}{q}\right) \times \left\{ \exp\left(-\frac{(f(x) - f(y))^2}{2t}\right) + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \exp\left(-\frac{(|f(x)| + |f(y)|)^2}{2t}\right) \right\}$$

(2.11)

where $f(y) = \frac{y}{p}1_{\{y \leq 0\}} + \frac{y}{q}1_{\{y > 0\}}$.

Proof. Taking $q = r$ and $\beta = 0$, we have by Theorem 2.1 that for every $p, q > 0, \alpha < 1$ and $x \in \mathbb{R}$,

$$dX_t^x = \left(p1_{\{X_t^x \leq 0\}} + q1_{\{0 < X_t^x\}}\right) dB_t + \frac{\alpha}{2} dL_t^0(X^x),$$

(2.12)

has a unique strong solution with $X_t^x = x$. Let $s$ be the function defined by (2.2). We see from (2.3) that $Z := s(X^x)$ satisfies SDE (2.3) with $q = r$ and $\beta = 0$; that is,

$$dZ_t = dB_t + \frac{1}{2} \left(\frac{q(\alpha - 1)}{p} + 1\right) dL_t^0(Z).$$

(2.13)

Thus $Z$ is a skew Brownian, whose transition density function is explicitly known. Using Theorem 2.2 we can rewrite $Z$ in terms of its symmetric semimartingale local time $\hat{L}_t^0(Z)$:

$$dZ_t = dB_t + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} d\hat{L}_t^0(Z).$$

(2.14)

Thus by [13] (see also Exercise III.1.16 on page 82 of [13]), $Z$ has a transition density function $p^Z(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}$ given by

$$p^Z(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(x - y)^2}{2t}\right) + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \exp\left(-\frac{(|x| + |y|)^2}{2t}\right)\right).$$

The desired formula (2.11) now follows from the fact that

$$p^X(t, x, y) = \frac{1}{\sigma_t(s(y))} p^Z(t, s(x), s(y))$$

\hfill $\square$
Remark 2.4  (i) When $\alpha = 1 - (p^2/q^2)$, Corollary [2.3] in particular recovers the formula given in [1, Theorem 5.1] (see the proof of Proposition 5.1 of [1] for a more explicit one) for the transition density function of $X^x$ in (2.10). Note that in this case, the more compact formula (2.11) in this paper is equivalent to the form given in [1]. Using the connection to skew Brownian motion established in the proof of Corollary [2.3], one can derive the joint distribution of $X^x$ given by (2.8), its semimartingale (respectively, symmetric semimartingale) local time $L^0(X^x)$ (respectively, $L^0(X^x)$) at level 0 and its occupation time on $[0, \infty)$ from that of the corresponding skew Brownian motion. The latter can be found in [1, Theorem 1.2]. Same remark applies to the diffusion process $Y^x$ satisfying SDE (2.16) below.

(ii) The diffusion process $X$ of (2.10) can be called oscillating skew Brownian motion, or skewed oscillating Brownian motion as suggested by Wellner [16]. When $\alpha = 1 - (p^2/q^2)$, the marginal distribution of $X^x$ of (2.10) (characterized by its transition density function $p^X(t, x, y)$) is the Fechner distribution in statistics; see [16]. This corresponds exactly to the case when the density function $p^X(t, 0, y)$ is continuous in $y$. For generator $L$ in (1.4) or its associated SDE (1.7), the condition $\alpha = 1 - (p^2/q^2)$ corresponds precisely to the case of $\rho_2 = \rho_3$. □

The next theorem allows us to obtain the transition density function for solution $Y$ of (2.1) when $\beta = 1 - \frac{q}{r}$.

Theorem 2.5 Let $p, q, r, a > 0$ and $\alpha < 1$. For $x \in \mathbb{R}$, let $X^x$ denote the unique strong solution of (2.10) with initial value $X_0^x = x$. The process $Y$ defined by:

$$Y_t = \begin{cases} a + \frac{r}{q} (X^x_t - a)^+ - (X^x_t - a)^- & \text{if } Y_0 = x \leq a \\ a + \frac{r}{q} (X^x_t - a)^+ - (X^x_t - a)^- & \text{if } Y_0 = x > a. \end{cases}$$

(2.15)

is the unique strong solution of the stochastic differential equation:

$$dY_t^x = \left(p1_{Y_t^x \leq 0} + q1_{0 < Y_t^x \leq a} + r1_{a < Y_t^x}\right)dB_t + \frac{\alpha}{2}dL_t^0(Y^x) + \frac{1}{2} \left(1 - \frac{q}{r}\right)dL_t^a(Y^x)$$

(2.16)

with initial value $Y_0^x = x$.

Proof. We will present the proof just in the case where $x \leq a$. The other case is similar. Consider the bijective function defined by:

$$\varphi(x) = a + \frac{r}{q}(x-a)^+ - (x-a)^- \quad \text{for } x \in \mathbb{R}.$$ 

Then $Y = \varphi(X^x)$. Note that $\varphi(x) = x$ for $x \leq a$, and $\varphi(x) > a$ if and only if $x > a$. Since $\varphi$ is the difference of two convex functions, applying the Ito-Tanaka formula to $Y = \varphi(X^x)$, we obtain:

$$Y_t = Y_0 + \int_0^t \varphi''(X^x_s) dX^x_s + \frac{1}{2} \int_{\mathbb{R}} L^w_t(X^x) \varphi''(dx)$$

$$= x + \int_0^t \left(1_{X^x_s \leq a} + \frac{r}{q} 1_{X^x_s > a}\right) dX^x_s + \frac{1}{2} \left(\frac{r}{q} - 1\right) L^a_t(X^x)$$

$$= x + \int_0^t \left(p1_{X^x_s \leq 0} + q1_{0 < X^x_s \leq a} + r1_{X^x_s > a}\right) dB_t$$

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\[ + \frac{\alpha}{2} dL_t^0(X^x) + \frac{1}{2} \left( \frac{r}{q} - 1 \right) L_t^0(X^x) \]

\[ = x + \int_0^t \left( p1_{\{Y^x_\tau \leq 0\}} + q1_{\{0 < Y^x_\tau \leq a\}} + r1_{\{Y^x_\tau > a\}} \right) dB_t \]

\[ + \frac{\alpha}{2} dL_t^0(X^x) + \frac{1}{2} \left( \frac{r}{q} - 1 \right) L_t^0(X^x). \]  \hspace{1cm} (2.17)

By the same reasoning as that for (2.6), we have

\[ L_t^0(X^x) = L_t^0(Y) \quad \text{and} \quad L_t^0(X^x) = \frac{q}{r} L_t^0(Y). \]  \hspace{1cm} (2.18)

This together with (2.17) shows that \( Y \) is a strong solution to SDE (2.16) with \( Y_0 = x \). The uniqueness follows from Theorem 2.1.

**Corollary 2.6** When \( \beta = 1 - \frac{q}{r} \), the solution \( Y \) to SDE (2.1) has a transition probability density \( p_Y(t, x, y) \) with respect to the Lebesgue measure on \( \mathbb{R} \) given by

\[
p_Y^Y(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left( \frac{1}{p} 1_{\{y \leq 0\}} + \frac{1}{q} 1_{\{0 < y \leq a\}} + \frac{1}{r} 1_{\{y > a\}} \right) \times \left\{ \exp \left( - \frac{(s(x) - s(y))^2}{2t} \right) \right\}
\]

\[
+ \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \exp \left( - \frac{|s(x) - s(y)|^2}{2t} \right) \}
\]  \hspace{1cm} (2.19)

**Proof.** This follows immediately from Theorem 2.5 and Corollary 2.3.

**Remark 2.7** SDE (2.1) with \( \beta = 1 - \frac{q}{r} \) corresponds exactly to SDE (1.7) with \( \rho_2 \sqrt{a_2} = \rho_3 \sqrt{a_3} \).

If we use \( q(t, x, y) \) to denote the transition density function of \( Y \) with respect to its symmetrizing measure \( m(dx) = \rho(x)dx \), then clearly we have

\[ q(t, x, y) = p_Y(t, x, y) / \rho(y). \]

3 Small-time asymptotic expansion

In this section, we will give an explicit asymptotic expansion, in small time, of the function \( u(x, t) = \mathbb{E}(h(Y^x_t)) \), in the general case where \( p, q, r > 0 \) and \( \alpha, \beta \in (-\infty, 1) \).

**Definition 3.1** For every \( k \in \mathbb{N} \), we denote by \( \text{erfc}_k \) the function defined on \( \mathbb{R} \) by:

\[ \text{erfc}_k(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} u^k e^{-u^2} du, \quad z \in \mathbb{R}. \]

Observe that \( \text{erfc}_k(0) = \frac{1}{\sqrt{\pi}} \Gamma\left( \frac{k+1}{2} \right) \), where \( \Gamma \) is the Euler Gamma function. The following elementary properties are straightforward from its definition.

**Lemma 3.2** For every given integer \( n \geq 1 \),
(i) As \( z \rightarrow +\infty \), \( \text{erfc}_k(z) = o(z^{-n}) \).

(ii) As \( z \rightarrow -\infty \): \( \text{erfc}_k(z) = \frac{1}{\sqrt{\pi}} \left( 1 + (-1)^k \right) \Gamma \left( \frac{k+1}{2} \right) + o(|z|^{-n}) \).

Here the notation \( o(|z|^{-n}) \) as \( z \rightarrow +\infty \) (respectively, \( z \rightarrow -\infty \)) means that \( \lim_{z \rightarrow +\infty} \frac{o(|z|^{-n})}{|z|^{-n}} = 0 \) (respectively, \( \lim_{z \rightarrow -\infty} \frac{o(|z|^{-n})}{|z|^{-n}} = 0 \)). Similar notation will also be used for \( o(t^n) \) as \( t \rightarrow 0 \).

Note that by Corollary 2.3, SDE

\[
\begin{align*}
\frac{dS_t^x}{S_0^x} &= \left( q \mathbb{1}_{\{S_t^x \leq a\}} + r \mathbb{1}_{\{a < S_t^x\}} \right) dB_t + \frac{\beta}{2} dL_t^x(S^x) \\
S_0^x &= x \in \mathbb{R}
\end{align*}
\]

(3.1)

has a unique strong solution for every \( x \in \mathbb{R} \) and it has a transition probability density with respect to the Lebesgue measure on \( \mathbb{R} \) given by

\[
p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left( \frac{1_{\{y \leq a\}}}{q} + \frac{1_{\{y > a\}}}{r} \right) \times \left\{ \exp \left( -\frac{(g(x) - g(y))^2}{2t} \right) \right. \\
&+ \frac{q + r(\beta - 1)}{q - r(\beta - 1)} \text{sign}(y - a) \exp \left( -\frac{(|g(x)| + |g(y)|)^2}{2t} \right) \right\},
\]

(3.2)

where \( g(x) = \frac{x}{q} \mathbb{1}_{\{x \leq a\}} + \frac{x}{r} \mathbb{1}_{\{x > a\}} \).

The following estimation is a key in small-time asymptotic expansion for the solution of heat equation (1.2). It reduces the case to diffusions of the form (2.10) for which we have explicit knowledge about their heat kernels.

**Lemma 3.3** Define

\[
\Xi^x = \begin{cases} 
X^x, \text{ solution of sde (2.10)} & \text{if } x \leq \frac{a}{2} \\
S^x, \text{ solution of sde (3.1)} & \text{if } x > \frac{a}{2}
\end{cases}
\]

(3.3)

Then there exist positive constants \( c_1 \) and \( c_2 \) such that for every \( t > 0 \),

\[
\mathbb{P} \left( \Xi_s \neq Y^x_s \text{ for some } s \in [0, t] \right) \leq c_1 \exp \left( -c_2 a^2 / t \right),
\]

where \( Y^x \) is the diffusion process of (2.1).

**Proof.** We will prove the lemma in the case where \( x \leq \frac{a}{2} \). The other case is similar. For \( t > 0 \), let us denote by

\[
\tau_t = \inf\{t \geq 0; \Xi_s = a\} = \inf\{t \geq 0; X^x_s = a\},
\]

and

\[
A_t = \left\{ \omega \in \Omega : \Xi_s(\omega) \neq Y^x_s(\omega) \text{ for some } s \in [0, t] \right\} = \left\{ \omega \in \Omega : X^x_s(\omega) \neq Y^x_s(\omega) \text{ for some } s \in [0, t] \right\}.
\]

(3.4)
It is clear that, $A_t \subset \{ \tau t \leq t \}$ and consequently,

$$
\mathbb{P}(A_t) \leq \mathbb{P}(\tau t \leq t) \leq \mathbb{P}_x \left( \sup_{0 \leq s \leq t} |X^x_s - x| \geq |a - x| \right) \leq \mathbb{P}_x \left( \sup_{0 \leq s \leq t} |X^x_s - x| \geq a/2 \right).
$$

Here $\mathbb{P}_x$ is the probability law of $X^x$. Since $X$ is a diffusion whose transition density function $q(t, x, y)$ with respect to the measure $m(dx) = \rho(x)dx$ enjoys the Aronson type Gaussian estimate (1.3), we have by a similar argument as that for [14, Lemma II.1.2] that

$$
\mathbb{P}_x \left( \sup_{0 \leq s \leq t} |X^x_s - x| \geq a/2 \right) \leq c_1 e^{-c_2 a^2 / t}.
$$

This proves the lemma. \( \Box \)

**Corollary 3.4** Let $\Xi^x$ be defined by (3.3). Then for every integer $n \geq 1$ and time $t > 0$ small,

$$
\mathbb{E} \left[ h(Y^x_t) \right] = \mathbb{E} \left[ h(\Xi^x_t) \right] + o(t^n).
$$

**Proof.** Let us consider the case where $x \leq a/2$; the other case is similar. For all $t \geq 0$,

$$
\mathbb{E} \left[ h(Y^x_t) \right] = \mathbb{E} \left[ h(Y^x_t); A_t \right] + \mathbb{E} \left[ h(Y^x_t); A^c_t \right],
$$

where $A_t$ is the set defined in (3.4), and $A^c$ is the complementary of $A$. Since $X^x_s(w) = Y^x_s(w)$ for all $s \in [0, t]$ on $A^c_t$, we have

$$
\mathbb{E} \left[ h(Y^x_t) \right] = \mathbb{E} \left[ h(X^x_t); A^c_t \right] + \mathbb{E} \left[ h(Y^x_t); A_t \right] = \mathbb{E} \left[ h(X^x_t) \right] + \mathbb{E} \left[ h(Y^x_t) - h(X^x_t); A_t \right].
$$

The desired result now follows from Cauchy-Schwarz inequality, the fact that $h$ is bounded and Lemma 3.3. \( \Box \)

Let $h$ be a piecewise $C^{N+1}$ function of the form (1.8). By Taylor expansion, for every $x \in \mathbb{R} \setminus \{0, a\}$,

$$
h(y) = \sum_{j=1}^{N} \frac{h^{(j)}(x)}{j!} (y - x)^j + O(|y - x|^{N+1}),
$$

and for $x \in \{0, a\}$ and $i = 1, 2, 3$,

$$
h_i(y) = \sum_{j=1}^{N} \frac{h^{(j)}_i(x)}{j!} (y - x)^j + O(|y - x|^{N+1}).
$$

Here the notation $O(|y - x|^{N+1})$ means that there are constants $C, \delta > 0$ so that the term $O(|y - x|^{N+1})$ is no larger than $C|y - x|^{N+1}$ for any $y \in \mathbb{R}$ with $|y - x| < \delta$. Thus we have the following.
Proposition 3.5 If \( x \in \mathbb{R} \setminus \{0, a\} \),
\[
E \left[ h(\Xi_t^x) \right] = \sum_{j=0}^{N} \frac{1}{j!} h^{(j)}(x) E \left[ (\Xi_t^x - x)^j \right] + O \left( |\Xi_t^x - x|^{N+1} \right), \tag{3.5}
\]
and if \( x \in \{0, a\} \),
\[
E[h(\Xi_t^x)] = \sum_{j=0}^{N} \frac{1}{j!} h^{(j)}(x) E \left[ (\Xi_t^x - x)^j 1_{\{\Xi_t^x \leq 0\}} \right] + \sum_{j=0}^{N} \frac{1}{j!} h^{(j)}(x) E \left[ (\Xi_t^x - x)^j 1_{\{0 < \Xi_t^x \leq a\}} \right]
+ \sum_{j=0}^{N} \frac{1}{j!} h^{(j)}(x) E \left[ (\Xi_t^x - x)^j 1_{\{\Xi_t^x > a\}} \right] + O \left( |\Xi_t^x - x|^{N+1} \right). \tag{3.6}
\]

Using the transition probability densities of the diffusions \( X^x \) and \( S^x \) given in corollary 2.3 and in \( (3.2) \), we can compute the expectations in \( (3.5)-(3.6) \).

Proposition 3.6 Let \( k \geq 1 \) be an integer.

(i) For \( x < 0 \),
\[
E \left[ (\Xi_t^x - x)^k \right] = (-1)^k p^{k/2} \left(\frac{2q}{p} - 1\right)^{k/2} \text{erf}_k \left(\frac{x}{p \sqrt{2t}}\right) + \sum_{j=0}^{k} \binom{k}{j} 2^{j/2-1} t^{j/2} A_j(x) \text{erfc} \left(\frac{x}{p \sqrt{2t}}\right),
\]
where
\[
A_j(x) = \frac{x^{k-j}}{p - q(\alpha - 1)} \left( (p + q(\alpha - 1)) p^j (1 + 2q(1 - q)^{k-j} + 2pq^{k-j}) \right) \quad \text{and} \quad \binom{k}{j} = \frac{k!}{j!(k-j)!}.
\]

(ii) For \( 0 < x < a \),
\[
E \left[ (\Xi_t^x - x)^k \right] = 2^{k/2-1} q^k t^{k/2} \text{erf}_k \left(\frac{-x}{q \sqrt{2t}}\right) + \sum_{j=0}^{k} \binom{k}{j} 2^{j/2-1} t^{j/2} B_j(x) \text{erfc} \left(\frac{x}{q \sqrt{2t}}\right),
\]
where
\[
B_j(x) = \frac{x^{k-j}}{p - q(\alpha - 1)} \left( -2q(\alpha - 1) (-p)^j \left(\frac{p}{q} - 1\right)^k + (p + q(\alpha - 1)) q^j (-2)^{k-j} \right).
\]

(iii) For \( x > a \),
\[
E \left[ (\Xi_t^x - x)^k \right] = 2^{k/2-1} r^k t^{k/2} \text{erf}_k \left(\frac{a - x}{r \sqrt{2t}}\right) + \sum_{j=0}^{k} \binom{k}{j} 2^{j/2-1} t^{j/2} C_j(x) \text{erfc} \left(\frac{x-a}{r \sqrt{2t}}\right),
\]
where
\[
C_j(x) = \frac{x - a}{q - r(\beta - 1)} \left( -2r(\beta - 1)(-1)^j q^j \left(\frac{q}{r} - 1\right)^k + (q + r(\beta - 1)) r^j 2^{k-j} (-1)^{k-j} \right).
\]
As a consequence of the above Proposition, and by Lemma 3.2, we obtain the following expansion of \( E \left[ (\Xi_t^x - x)^k \right] \) for \( x \in \mathbb{R} \setminus \{0, a\} \) and small time \( t > 0 \).
Corollary 3.7 For any positive integers \( n > k \geq 1 \), \( x \in \mathbb{R} \setminus \{0, a\} \), as \( t \to 0^+ \),
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k \right] = 2^{k/2-1} D_k(x) t^{k/2} + o(t^n),
\]
where
\[
D_k(x) = \frac{1}{\sqrt{\pi}} (1 + (-1)^k) \Gamma \left( \frac{k+1}{2} \right) \left[ p^k 1_{\{x \leq 0\}} + q^k 1_{\{0 < x < a\}} + r^k 1_{\{x > a\}} \right].
\] (3.7)

Using again the transition probability densities of the diffusions \( X^x \) and \( S^x \), we get

Proposition 3.8 For \( x \in \{0, a\} \) and integer \( k \geq 1 \),
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{\Xi_t^x \leq 0\}} \right] = \begin{cases} 
\frac{q(1-\alpha)}{p-q(\alpha-1)} (1)^k p^k (2t)^{k/2} \sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) & \text{when } x = 0 \\
\frac{r(1-\beta)}{q-r(\beta-1)} (1)^k q^k (2t)^{k/2} \text{erfc} \left( \frac{a}{q\sqrt{2t}} \right) & \text{when } x = a,
\end{cases}
\]
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{0 < \Xi_t^x \leq a\}} \right] = A_k(x) q^k (2t)^{k/2} \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{k+1}{2} \right) - \text{erfc} \left( \frac{a}{q\sqrt{2t}} \right) \right),
\]
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{\Xi_t^x > a\}} \right] = \begin{cases} 
\frac{p}{p-q(\alpha-1)} q^k 2^{k/2} t^{k/2} \text{erfc} \left( \frac{a}{q\sqrt{2t}} \right) & \text{when } x = 0 \\
\frac{q}{q-r(\beta-1)} (1)^k r^k (2t)^{k/2} \sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) & \text{when } x = a.
\end{cases}
\]

Here
\[
A_k(x) = \begin{cases} 
p (p-q(\alpha-1)) & \text{when } x = 0 \\
r(1-\beta) (1)^k & \text{when } x = a.
\end{cases}
\] (3.8)

From Proposition 3.8 and Lemma 3.2 we deduce:

Corollary 3.9 For \( x \in \{0, a\} \) and integers \( n > k \geq 1 \),
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{\Xi_t^x \leq 0\}} \right] = \begin{cases} 
\frac{q(1-\alpha)}{p-q(\alpha-1)} (1)^k p^k 2^{k/2} t^{k/2} \sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) & \text{when } x = 0 \\
o(t^n) & \text{when } x = a;
\end{cases}
\]
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{0 < \Xi_t^x \leq a\}} \right] = A_k(x) q^k (2t)^{k/2} \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{k+1}{2} \right) + o(t^n),
\]
\[
\mathbb{E} \left[ (\Xi_t^x - x)^k 1_{\{\Xi_t^x > a\}} \right] = \begin{cases} 
o(t^n) & \text{when } x = 0 \\
\frac{q}{q-r(\beta-1)} (1)^k r^k 2^{k/2} t^{k/2} \sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) & \text{when } x = a,
\end{cases}
\]

where \( A_k(x) \) is defined in (3.8).

Combining Proposition 3.5 with Corollaries 3.7 and 3.9, we have

Theorem 3.10 For small time and \( x \in \mathbb{R} \),
\[
\mathbb{E} \left[ h(\Xi_t^x) \right] = \sum_{k=0}^{N} b_k(x) t^{k/2} + O(t^{(N+1)/2}),
\]
where

\[
b_k(x) = \begin{cases} 
2^{k/2-1} D_k(x) \frac{1}{k!} \frac{\partial^k h(x)}{\partial x^k} & \text{when } x \in \mathbb{R} \setminus \{0,a\} \\
\frac{\Gamma(k+1)}{k!(p-q(\alpha-1))} \sqrt{\pi} \frac{\partial^k h_1(0)}{\partial x^k} q(1-\alpha)(-1)^k p^k + \frac{\partial^k h_2(0)}{\partial x^k} p q^k & \text{when } x = 0 \\
\frac{\Gamma(k+1)}{k!(q-r(\beta-1))} \sqrt{\pi} \frac{\partial^k h_2(a)}{\partial x^k} r(1-\beta)(-1)^k q^k + \frac{\partial^k h_3(a)}{\partial x^k} p q r^k & \text{when } x = a
\end{cases}
\]

with the function \(D_k(x)\) given by (3.7).

By Theorem 3.10 and Corollary 3.4 we get:

**Corollary 3.11** For every \(x \in \mathbb{R}\), when \(t > 0\) is small,

\[
\mathbb{E}[h(Y_t^x)] = \sum_{k=0}^{N} b_k(x) t^{k/2} + O(t^{(N+1)/2}),
\]

where the function \(b_k(x)\) is defined by (3.9).

**Remark 3.12** (i) When \(a = 0\) and \(r = q\), we recover the expansion obtained in [17].

(ii) Employing the same approach used in this section, one can obtain without any difficulty a similar small-time expansion for the solution of the heat equation (1.2) where \(A(x)\) and \(\rho(x)\) are piecewise constant with \(n \geq 3\) points of discontinuity; that is, in case of the following SDE:

\[
\begin{align*}
\frac{dY_t^x}{dY_t^x} &= \sqrt{A(Y_t^x)} dB_t + \sum_{i=1}^{n} \frac{\alpha_i}{2} dL_t^{a_i}(Y_t^x), \\
Y_0 &= x \in \mathbb{R},
\end{align*}
\]

where \(a_1 = 0 < a_2 < ... < a_n\), \(p_i \in (0, +\infty)\), \(\alpha_i \in (-\infty, 1)\) for \(i = 1, 2, ..., n\) and

\[
A(Y_t^x) = p_0 1\{Y_t^x \leq a_1\} + \sum_{i=1}^{n-1} p_i 1\{a_i < Y_t^x \leq a_{i+1}\} + p_n 1\{a_n < Y_t^x\}.
\]

See [2] for the existence and uniqueness of strong solution of this SDE.

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