Insights into the superdiffusive dynamics through collision statistics in periodic Lorentz gas and Sinai billiard

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Abstract

We report on the stationary dynamics in classical Sinai billiard (SB) corresponding to the unit cell of the periodic Lorentz gas (LG) formed by square lattice of length \( L \) and dispersing circles of radius \( R \) placed in the center of unit cell. Dynamic correlation effects for classical particles, initially distributed by random way, are considered within the scope of deterministic and stochastic descriptions. A temporal analysis of elastic reflections from the SB square walls and circle obstacles is given for distinct geometries in terms of the wall-collision and the circle-collision distributions. Late-time steady dynamic regimes are explicit in the diffusion exponent \( z(R) \), which plays a role of the order-disorder crossover dynamical parameter. The ballistic \( (z_0 = 1) \) ordered motion in the square lattice \( (R = 0) \) switches to the superdiffusion regime with \( z_1 = 1.5 \), which is geometry-independent when \( R < L \sqrt{2}/4 \). This observed universal dynamics is shown to arise from long-distance particle jumps along the diagonal and nondiagonal Bleher cor-
ridors in the LG with the infinite horizon geometry. In the corresponding SB, this universal regime is caused by the long-time wall-collision memory effects attributed to the bouncing-ball orbits. The crossover nonuniversal behavior with $1.5 < z < 2$ is due to geometry with $L\sqrt{2}/4 \leq R < L/2$, when only the nondiagonal corridors remain open. All the free-motion corridors are closed in LG with finite horizon $(R \geq L/2)$ and the interplay between square and circle geometries results in the chaotic dynamics ensured by the normal Brownian diffusion $(z_2 = 2)$ and by the normal Gaussian distribution of collisions. PACs: 05.45.Gg, 05.40.Fb, 45.50.Tn. Key words: Sinai billiard, Lorentz gas, collision statistics, anomalous diffusion, order-disorder crossover, chaos. Corresponding author Valery B. Kokshenev, valery@fisica.ufmg.br
A mathematical exploration of two-dimensional Lorentz gas (LG) with periodic configuration of scatterers, related to dispersed billiards, was initiated [1] by Bunimovich and Sinai. The process of establishing equilibrium dynamics for randomly distributed, noninteracting classical particles driven by elastic collisions with the billiard walls and the scatterer of fixed geometry are commonly discussed in terms of the particle velocity-velocity and/or displacement-displacement correlation functions, which are related to a diffusion coefficient through the Einstein-Green-Kubo formula (see e.g. Ref. [2]). Besides the ergodicity, the entropy, the Lyapunov exponent, the mixing property, and among other interesting physical observables in LG with finite and infinite horizons, the enhanced diffusion studied numerically through the collision distribution function, has been expressed as of great importance in Ref. [3]. This stimulated our subsequent investigations of the collision statistics for nonescape particles observed through their survival probability in weakly open chaotic (Sinai billiard [4], (SB)) [5] and non-chaotic (circle and square billiards) [6] classical systems. Remarkably, collision statistics provided new insight into the delicate mathematical problem of the interplay between regular and irregular segments of the billiard boundary, as has been demonstrated for the case of the so-called almost-integrable systems, presented by open [7] and closed [8] rational polygons. In the current study we focus on the stationary superdiffusion dynamics in the closed SB. Analysis is given through the billiard-wall and the scatterer-disk random collision statistics and is based on the dynamical correspondence between SB and LG. Some of the findings of this study were preliminary communicated in Ref. [8].

The dynamics of particles (of unit mass and unit velocity) moving in two-dimensional closed region (billiard table) and dispersed by obstacles is governed by billiard-boundary geometry. The SB with the square-wall table of length \( L \) and the disk with radius \( R \) can be formally treated as the unit cell of a periodic LG, which is the two-dimensional periodic crystal formed by a regular set of circular scatterers (of radius \( R \)) centered at distances \( L \). This implies that SB and LG are dynamically equivalent classical systems. This statement
is justified by a step-to-step correspondence that can be established between a certain orbit in the closed SB and the corresponding trajectory in LG. For the case of the low-density LG, given by geometry with $R < L/2$, this correspondence is exemplified in inset A in Fig. 1. As seen for this LG configuration, there exist trajectories in which particles never collide with scatterers. Those particles, which move through such unbounded trajectories, have therefore an infinite horizon \[2,3\]. In turn, the free-motion trajectories belong to infinite corridors \[2\]. As shown by Bleher \[2\], the principal corridors are open for small disk radii limited by $0 < R < L\sqrt{2}/4$. This is illustrated by the diagonal and nondiagonal corridors in inset B in Fig.1. When the scatterer radius achieves the magnitude of $L\sqrt{2}/4$, the last nondiagonal corridor closes. With further increasing the radius, the principal corridors disappear at $L/2$ and scatterers start to overlap lattice cells. The particle trajectories become bounded thus having a finite horizon (see inset C in Fig.1).

In the infinite-horizon LG, the enhanced-diffusion motion regime was theoretically predicted in Ref. \[2\], through the asymptotic statistical behavior of the late-time particle displacements. But no description for the evolution of motion regimes with geometry was given (see also Ref. \[9\]). We therefore reformulate the problem of random particle displacements into that of random collisions. This yields a description of the late-time stationary dynamics in LG, as well as in the corresponding SB, through the diffusion exponent $z(R)$, which is continuous with $R$. The paper is organized as follows. In Sec. II we develop the billiard collision statistics based on the alternative deterministic and stochastic approaches. Also, we introduce the collision distribution function given in terms of the dynamic observables, which are available in simulation of SB with finite and infinite horizon geometries in the corresponding LG. Conclusion is drawn in Sec. III.
II. COLLISION STATISTICS

A. Distribution Function and Dynamic Characteristics

The trajectories of classical particles of unit mass moving with unit velocities in SB (or in LG) are preserved by the Liouville measure [10], namely

\[ d\mu(x) = \frac{1}{2\pi A} dxdy d\theta. \]  

This is introduced in the phase space through the billiard table area \( A \), given by the variable scatterer radius \( R \) and the fixed boundary side \( L \). The coordinate set \( x = (x, y, \theta) \) includes the particle position and the velocity launching angle \( \theta = [0, 2\pi] \), which is counted of the \( x \)-axis of the square billiard table. The mean collision time \( \tau_c(R) \), which is due to the two consequent elastic random collisions with the boundary wall or with the scatterer, are respectively defined [10] by

\[ \tau_c^{(w)}(R) = \frac{\pi A}{P_w} \text{ or } \tau_c^{(s)}(R) = \frac{\pi A}{P_s}. \]  

Here \( P_w = 4L \) and \( P_s = 2\pi R \) are corresponding collision perimeters of the table with accessible area \( A = L^2 - \pi R^2 \) (for \( R < L/2 \)).

The collision distribution function \( D(n,t) \) is a probability of a particle to collide \( n \) times with the fixed billiard boundary within a time \( t \) (for rigorous definition see e.g. Ref. [3]). Also, it can be introduced through the billiard mean collision-number equations, namely

\[ n_c(t) = \langle n(x,t) \rangle_c = \int n(x,t) d\mu(x) = \int_0^\infty n D(n,t)dn = \frac{t}{\tau_c}, \]  

where \( x \) stands for the boundary (wall and/or scatterer) position set and \( \tau_c(R) \) is given in Eq.(2). This results in

\[ D(n,t) = \left| \frac{d\mu[x(n,t)]}{dn} \right|, \]  

where the Liouville measure, defined in Eq. (1), is given by the inverse function to \( n(x,t) \).
The distribution (4) provides a rich information on the boundary-memory effects in chaotic SB with a fixed geometry. In other words, dynamic correlation effects can be characterized by the nonzero central moments of order \( m = 2, 4, \ldots \) defined for the random numbers \( n \) as

\[
\Delta^m n_c(t) = \int_0^\infty [n - n_c(t)]^m D(n, t)dn. \tag{5}
\]

\( \Delta^m n_c(t) \) describes \( m \)-order deviation from the mean collision number \( n_c(t) \) given in Eq.(3). Basically, we focus on low-order dynamic correlation effects presented by the variance of the random collision numbers 

\[
\Delta^2 n_c(t) = \frac{2}{\tau_c^2} \langle \Delta^2 r \rangle_c, \tag{6}
\]

expressed through the variance for particle displacements \( \langle \Delta^2 r \rangle_c \). This fundamental equation will be deduced below using the orbit-trajectory correspondence visualized in Fig. 1. Also, Eq.(6) permits one to employ the well known temporal variance-displacement equation (see e.g. Ref. [11])

\[
\langle \Delta^2 r \rangle_c \sim \ell_c^2 \left( \frac{t}{\tau_c} \right)^{2/z}, \text{ with } t \gg \tau_c. \tag{7}
\]

In this way, the diffusion exponent \( z(R) \) introduces a description for distinct stationary regimes in billiards with different \( R \). For particles of unit velocity, the mean free path \( \ell_c = \tau_c \) can be specified for the wall and the scatterer collisions with the help of Eq.(2).

The wall-collision and the scatterer-collision statistics is given by the corresponding distribution functions \( D^{(w)}(n, t) \) and \( D^{(s)}(n, t) \). These are defined by the equations \( n_c^{(w)}(t) = t/\tau_c^{(w)} \) and \( n_c^{(s)}(t) = t/\tau_c^{(s)} \), which extend Eq.(3), as well as Eqs. (4) and (5). Furthermore, from the same equations one has \( \tau_c^{(w)} n_c^{(w)} = \tau_c^{(s)} n_c^{(s)} = t \) that, with accounting for the total number of collisions \( n_c(t) = n_c^{(w)} + n_c^{(s)} \) with \( \tau_c n_c = t \), results in the overall mean collision frequency, namely

\[
\frac{1}{\tau_c} = \frac{1}{\tau_c^{(w)}} + \frac{1}{\tau_c^{(s)}} = \frac{P_w + P_s}{\pi A}. \tag{8}
\]

Here the mean wall \( (\tau_c^{(w)}) \) and scatterer \( (\tau_c^{(s)}) \) collision times are defined in Eq.(2).
B. Wall Collisions in Square Billiard

If one ignores splitting effects caused by $\pi/2$-angle vertices [7], particle in the closed square billiard are subjected to an ordered orbit motion driven by billiard walls. This implies that the velocity launching angle $\theta$ is the integral of motion for a given orbit, which can be therefore presented by a straight-line trajectory in the corresponding LG (see inset D in Fig. 1). A whole number of intersections of this line with the unit-cell boundaries, encountered in $x$ and $y$ directions in a time $t$, corresponds to the following square-billiard, wall-collision number, namely [12]

$$n_0^{(w)}(\theta, t) = \frac{t}{L} (\cos \theta + \sin \theta), \quad \text{(9)}$$

established for a given orbit by the velocity angle $\theta$. In view of the point symmetry of square lattice, the angle domain is reduced to $0 \leq \theta \leq \pi/4$. Extending Eq.(3) for the wall-ordered motion, the corresponding characteristic time $t_c^{(w)}(\theta)$ follows from equation $n_0^{(w)}(\theta, t) = t/t_c^{(w)}(\theta)$. This gives

$$\frac{1}{t_c^{(w)}(\theta)} = \frac{\cos \theta + \sin \theta}{L}, \quad \text{(10)}$$

obtained with the help of Eq.(9). Consequently, the mean collision number is

$$n_{c0}(t) \equiv \left\langle n_0^{(w)}(\theta, t) \right\rangle_c = \frac{4t}{\pi A} \int_0^{\pi/4} \frac{\cos \theta + \sin \theta}{L} d\theta \int_0^A dxdy = \frac{4t}{\pi L} = \frac{t}{\tau_{c0}^{(w)}}, \quad \text{(11)}$$

Here the averaging procedure is elaborated over all equivalent angles $\theta$. In turn, Eq.(11) defines the mean collision time $\tau_{c0}$ that agrees with Eq.(2), where $A = L^2$ and $P_w = 4L$. The variance for the random wall-collision number is

$$\Delta^2 n_{c0}(t) = \left\langle [n_0^{(w)}(\theta, t) - n_{c0}(t)]^2 \right\rangle_{c0} = \left( \frac{\pi^2}{16} + \frac{\pi}{8} - 1 \right) n_{c0}^2, \quad \text{(12)}$$

with $n_{c0}$ is obtained in Eq.(11). The function $\theta(n, t) = 0.5 \arcsin \left[(4\tau_{c0} n/\pi t)^2 - 1\right]$, inverse to the function $n(\theta, t)$ given in Eq.(9), results in the collision distribution function $D_0(n, t) = 4\pi^{-1} |\partial \theta(n, t)/\partial n|$ defined in Eq. (4). A straightforward estimation for the wall-collision distribution yields
$$D_0^{(w)}(n, t) = \frac{16}{\pi^2 n_c \sqrt{2}} \sin^{-1}\left(\frac{\pi}{4} - \frac{1}{2} \arcsin\left[\left(\frac{4 n}{\pi n_c}\right)^2 - 1\right]\right),$$

for $\pi/4 < n/n_c < \pi \sqrt{2}/4$, otherwise $D_0^{(w)}(n, t) = 0$. \hspace{1cm} (13)

One can verify that Eqs. (11), (12) and (13) are selfconsistent hence obey Eqs.(3) and (5).

As seen from Eqs. (7) and (12), the dynamical diffusion exponent $z_0 = 1$, i.e., the dynamic regime in the square billiard is ballistic. Meantime, ballistic trajectories were employed to deduce the square-billiard distribution (13).

In Fig. 2 the wall-collision distribution, which is predicted by Eq.(13), is compared with that simulated in square billiard at observation time $t_{obs} = 100 \tau_{c0}$ (for experimental details, see Ref. [6]). Analysis for whole-scale temporal evolution was also performed [13]. We therefore infer that the simulated root-mean-square deviation for collision numbers $\sqrt{\Delta^2 n_{c0}(t)}$ agrees with the stationary prediction given in Eq.(12), starting with times $t_{obs} \gg 30 \tau_{c0}$ (see the inset in Fig.2).

C. Random Walks in Lorentz Gas Lattice

1. Infinite Horizon Geometry

In the case of the scatterer radii $0 < R < L/2$, besides a free motion along the open corridors in the phase space of the chaotic SB, particles are dispersed by disks and in this way are involved in diffusive motion. Let us describe an evolution of a given trajectory, moving in the equivalent square-lattice LG, by random walking (see inset A in Fig. 1).

In a time $t$, a walker scattered by disks indicates random steps: $n(t) = s_x^+ + s_x^- + s_y^+ + s_y^-$. These steps are given by interceptions of a trajectory with the LG unit cells, that corresponds to reflections from the SB walls. More precisely, the walker under consideration does $s_x^+$ steps to the right, $s_x^-$ steps to the left in the $x$-direction as well as $s_y^+$ steps to the down, and $s_y^+$ steps to the up in the $y$-direction. The resultant particle displacement is therefore given by

$$\Delta r(t) = (s_x^+ - s_x^-)\ell_x e_x + (s_y^+ - s_y^-)\ell_y e_y.$$ \hspace{1cm} (14)
Here $\ell_x$ and $\ell_y$ are projections of the one-step, free-motion displacement that occurs between the two consequent intersections. For random walking in the $x$-direction, $< s_x^+ - s_x^- >_c = 0$ within the same approximation, and thus $< \Delta r >_c = 0$. For the mean squared displacement $< \Delta r \Delta r >_c$, one has

$$< \Delta^2 r >_c = 2 \ell_c^2 \left[ < s_i^2 >_c - < s_i s_k >_c (1 - \delta_{ik}) \right],$$

(15)

with the help of Eq.(14), where $\delta_{ik}$ is the Kronecker symbol. Here the mean $< \ell_x^2 + \ell_y^2 >_c = 2 < \ell_x^2 >_c = \ell_c^2$ is estimated in the isotropic approximation. The introduced indices $i, k = 1, 2, 3, 4$ count distinct random steps: $s_1 = s_x^+, s_2 = s_x^-, s_3 = s_y^+, s_4 = s_y^-$, which are dynamically equivalent: $< s_i >_c = < s_k >_c$, for any $i \neq k$. This observation permits one to describe the random-wall collisions through the random-walk steps, on the basis of the relation $n = \sum_{i=1}^{4} s_i$. This results in the mean $n_c$ and the variance $\Delta^2 n_c$, namely

$$n_c = \sum_{i=1}^{4} < s_i >_c = 4 < s_i >_c, \quad \Delta^2 n_c = \sum_{i=1}^{4} < \Delta^2 s_i >_c + 2 \sum_{i>k} < \Delta s_i \Delta s_k >_c,$$

(16)

introduced by the random step fluctuations $\Delta s_i = s_i - < s_i >$, discussed above in Eqs. (3) and (6).

In the stochastic approximation, the step fluctuations are independent and the last term in Eq.(16) is null. Moreover, one can see that the term $< \Delta^2 s_i >_c = (< s_i^2 >_c - < s_i >_c^2)$ is the same as in the square brackets in Eq.(15). This leads to Eq.(6), that in combination with Eq.(7) provides the desired relation for the variance for collision deviations, namely

$$\Delta^2 n_c(t) \sim \left( \frac{t}{\tau_c(R)} \right)^{2/z},$$

(17)

where the characteristic billiard collision time $\tau_c(R)$ is given in Eq.(2).

On the basis of Eq.(17), we have elaborated a numerical statistical analysis for the wall collisions in SB with $0 < R < L/2$. As seen from the upper inset in Fig. 3, the standard deviation indicates two superdiffusion motion regimes, which are well distinguished through the dynamic exponent $z(R) < 2$. For small scatterers, the $R$-independent and, therefore, universal regime is manifested by $z_1 = 1.50 \pm 0.05$ (shown by closed squares in Fig.3). Above
a crossover radius $\approx 0.35L$, the diffusion exponent continuously increase with $R$ from 1.5 to 2 (shown by open squares in Fig.3). A crossover from the universal to a transient dynamics starts at $R_1 = \sqrt{2}L/4 \approx 0.35L$, the point where a rearrangement of the LG lattice occurs with closing of the diagonal Bleher corridors (shown in inset B in Fig. 1).

We focus on the late-time asymptotic collision behavior. The observed distributions have been therefore reestimated in the reduced coordinates proposed in Ref. [3]. These coordinates can be formally introduced here by the relations

$$\tilde{D}(\tilde{n}) = \sqrt{\Delta^2 n_c(t)}D[n(\tilde{n},t),t] \text{ and } \tilde{n}(n,t) = \frac{n-n_c(t)}{\sqrt{\Delta^2 n_c(t)}},$$

where $n(\tilde{n},t)$ stands for the inverse function to $\tilde{n}(n,t)$. The distribution $D(n,t)$, the mean $n_c(t)$, and the variance $\Delta^2 n_c(t)$ are given in Eqs.(4), (3) and (17), respectively. Physically, the reduced collision number $\tilde{n}(n,t)$, as well as its distribution $\tilde{D}(\tilde{n})$, is expected to expose a stationary behavior when the late-time observation condition $t_{obs} \gg \tau_c(R)$ is satisfied (see also Eq.(7)). As follows from our temporal analysis shown in the insets in Fig. 3, the superdiffusion dynamic regime becomes steady starting from the observation times $t_{obs}^{(exp)} \gtrsim 50\tau_c$. The same can be referred to the steady collision distributions exemplified in Fig. 4.

As seen from Fig. 4, both the kinds of collision distributions are similar. With growth of the scatterer radius, they change their form from characteristic for the ordered motion (illustrated in Fig.2) to that which tends to the disordered Gaussian motion, shown by solid lines in Fig. 4. Furthermore, according to simulation studies carried out within the domain $0.05 < R/L < 0.35$, the late-time distribution functions fall down into the coinciding curves $\tilde{D}_1(\tilde{n})$, attributed to the universal superdiffusion regime introduced above by the exponent $z_1 = 1.50 \pm 0.05$. This regime is ensured by the open diagonal and nondiagonal corridors, that makes plausible to adopt that the mean collision time is caused mostly by the wall reflections. This implies that $\tau_{cl}^w \approx \tau_{cl}^{(w)} < \tau_{cl}^{(s)}$ and thus $n_{cl1}^{(w)} > n_{cl1}^{(s)}$ estimated for $R < R_1$ (see Eq.(8)). Consequently, the mean collision number $n_{cl1} = < n_1(x,t) >_c \approx t/\tau_{cl1}^{(w)}$, defined in Eq.(3), can be specified through random walks, namely
\[ n_1(x,t) = \sum_{i}^{n_{c1}} \frac{r_i}{t^{(w)}(\theta_i)} \approx \frac{r}{\tau^{(w)}}, \]  \hspace{1cm} (19)

with \( r = t = \sqrt{x^2 + y^2} \) defined by the position set \( x = (x, y, \theta) \). Eq.(19) describes the scatterer-scatterer collision, which occurs in the periodic LG during time \( t \) between two scatterers connected by the vector \( x \), with \( \sum_i x_i = x \); \( t^{(w)}(\theta_i) \) is defined by Eq.(10) for a given linear trajectory \( i \), which corresponds to the two consequent wall-to-wall collisions.

On the other hand, the superdiffusion regime under discussion can be treated through the two-dimensional Lévy jumps, which occur between two scatterers of distance \( r \). Therefore, the universal regime can be additionally characterized by the waiting-time probability distribution function

\[ \Psi(r,t) = \Lambda_s(r) \delta(r-t) \text{ with } \Lambda_s(r) \propto r^{\frac{4}{z}-5}, \]  \hspace{1cm} (20)

with \( 1 < z < 2 \) and the jump-length distribution \( \Lambda_s(r) \) function presented in the long-tail-distance asymptotic form. Eq.(20) represent Eq.(39) in Ref. [14], deduced with the help of Eq.(39) in Ref. [14] juxtaposed with Eq.(7). In turns, the asymptotic scatterer-collision distribution function

\[ D_{1}^{(s)}(n) = \Lambda_s[r(n)] \frac{dr}{dn} \propto \tau_{c1}^{(w)} \left( n\tau_{c1}^{(w)} \right)^{\frac{2}{z}-5}, \text{ with } r(n) = n\tau_{c1}^{(w)} \gg 1, \]  \hspace{1cm} (21)

follows from Eq.(20) and the last relation in Eq.(19). Numerical analysis of Eq.(21) is given in the reduced semi-log coordinates in the left inset in Fig. 4. The best fitting for the proposed \( D_{1}^{(s)}(n) \) with simulation data results in the derived dynamical exponent \( z_1 = 1.5 \), which is the same obtained in Fig. 3 by the variance-number analysis. This justifies the mechanism of long-range Lévy jumps in the Bleher principal corridors. The universal superdiffusion motion is also exposed by numerical analysis for the wall-collision distribution \( \tilde{D}_{1}^{(w)}(\tilde{n}) \), given in the right plot in Fig. 4.

The velocity dispersive parameter \( \sigma(R) \) was introduced in Ref. [5] to describe a disorder-to-order crossover in the weakly open SB in terms of the normal-to-wall-velocity \( (v_{\perp}) \) pseudo-Gaussian distribution function \( g_{\sigma}(v) \), with \( v = v_{\perp}/v \). The wall-collision statistics could
be also introduced [5] through the velocity-dependent random number given by \( n(v, t) = 2vt/\tau_c^{(w)} \), with the mean

\[
n_{c\sigma}(t) = \int_0^1 n(v, t)g_\sigma(v)dv = \frac{t}{\tau_c^{(w)}}, \quad \text{and} \quad g_\sigma(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(v - \frac{1}{2})^2/2\sigma^2}{\text{erf}(1/2\sqrt{2\sigma})}\right).
\] (22)

Here \( \text{erf}(x) \) is the standard error function. Two theoretical predictions were proposed for the dispersive parameter, namely

\[
\sigma(R) = \frac{\sqrt{1 - 2R/L}}{12\sqrt{5}(R_2/L)^2}, \quad \text{for} \quad R < R_2 = \frac{L}{2},
\] (23)

obtained in the simple "quasi-chaotic" (\( \sigma \ll 1 \)) approximation [5] and [13]

\[
\sigma(R) = \sigma(0) \sqrt{\frac{4\arcsin\left(\frac{1-2R/L}{\sqrt{1+(1-2R/L)^2}}\right)}{\pi}} \approx \sigma(0) \sqrt{\frac{1 - 2R/L}{1 - \pi(R/L)^2}},
\] (24)

which takes into consideration the long-living bouncing ball orbits and the square-billiard data [13] \( \sigma(0) = 0.29 \). In the right inset in Fig. 4, the simulation data for the weakly open SB is compared with the theoretical predictions made for \( \sigma(R) \). Remarkably, that similarly to \( z_1(R) \), the late-time dispersion parameter is \( R \)-independent, within the experimental error, and can be therefore characterized by \( \sigma_1(R) = 0.14 \pm 0.02 \) for the universal diffusion regime.

2. Finite Horizon Geometry

In the case of LG with \( R > L/2 \), when all the Bleher corridors are closed (see inset B in Fig.1), an application of the central limit theorem for random-walk displacements \( r(t) \) proves [1] its Gaussian distribution. By taking into account the established in Eq.(6) relation between the random displacements and collision numbers, one may expect the normal collision distribution, namely

\[
D_2^{(w)}(n, t) = \frac{1}{\sqrt{2\pi\Delta^2n_c(t)}} \exp\left[-\frac{(n - n_c(t))^2}{2\Delta^2n_c(t)}\right], \quad \text{for} \quad R \geq L/2
\] (25)

where \( n_c \) and \( \Delta^2n_c \) are the standard mean and deviation of the wall-collision number. Experimental justification of the asymptotic Gaussian distributions \( \tilde{D}_2^{(w)}(\tilde{n}) = \tilde{D}_2^{(s)}(\tilde{n}) = \)
\( (1/2\pi) \exp(-\tilde{n}^2/2) \), for both the wall and the scatterer collisions are given in Fig. 5. In general, our data on \( \widetilde{D}_2^{(s)}(\tilde{n}) \) are consistent with the first observation of the normal scatterer-collision distribution reported in Ref. [3]. Furthermore, our short-time analysis provides evidence that the Gaussian distribution, associated with chaotic dynamics, becomes steady at times \( t_{\text{obs}} \gtrsim 50\tau_c \).

It is noteworthy that no true Gaussian distribution was achieved for the late-time diffusion coefficient [2], when \( R \) tends to \( L/2 \) from below. With the aim to clarify a decay process of spatial correlations within this regime, we have analyzed the reduced fourth-order moments \( \zeta_4 \) defined as

\[
\zeta_4(R) = \frac{\Delta^4 n_c(t)}{\Delta^2 n_c(t)}, \quad \text{with} \quad \frac{t}{\tau_c} \gg 1,
\]

and obtained with the help of Eq.(5). The true Gaussian distribution prescripts \( \zeta_4 = 3 \). Our temporal analysis given for the case of \( R/L = 0.6 \), results in \( \zeta_4(t) = 3 \pm 0.01 \) for the observation times \( t \gtrsim 70\tau_c \). The distinct cases of the wall and scatter collisions are described by the characteristic times \( \tau_c \) given in Eq.(2). An approximation towards the Gaussian distributions, with the increasing of the disorder with \( R/L \), is exposed in the insets in Fig. 5. One can see that a chaotic motion in the SB with \( R > L/2 \) is established by both the normal collisions and by the normal diffusion. These universal regimes are observed with a good precision, which is guaranteed by, respectively, the higher-order central moment \( \zeta_4 = 3.00 \pm 0.04 \) and the diffusion exponent \( z_2 = 2.0 \pm 0.1 \).

III. CONCLUSION

We have discussed the superdiffusive behavior of SB, as well as of the dynamically equivalent periodic LG, in view of the interplay between the linear and the circular boundary geometries. The late-time correlations, driven by elastic reflections from the square walls and the dispersed disk, are shown to be steady for a given geometry at observation times 30-50\( \tau_c(R) \). This justifies a consideration of the statistical distributions for the random wall
and scatterer collisions. Besides the higher-order correlation effects, these collisions are characterized by their mean collision numbers $n_c^{(w)}$ and $n_c^{(s)}$, respectively. The relative collision numbers $n_c^{(w)}/n_c^{(s)} = \tau_c^{(s)}/\tau_c^{(w)} = 2L/\pi R$ follow from Eqs. (2) and (8).

When the scatterer is absent, the deterministic description results in the ballistic motion given by the diffusion exponent $z_0 = 1$. The wall-collision distribution is flat and asymmetric due to a sharp contribution from the bouncing-ball orbits [8] (see Fig. 2). At very small scatterer radius, when the wall collisions predominate ($n_c^{(w)} \gg n_c^{(s)}$), the late-time distribution tends to a smooth pseudo-Gaussian form, asymmetrically shifted from the center by the same long-living bouncing-ball orbits (see Fig. 4). When the radius of scatterers is relatively small, $R < R_1 = L\sqrt{2}/4$, all the principal free-motion corridors remain open in the corresponding LG, and the bouncing-ball trajectories evolve freely within the diagonal and nondiagonal corridors. This evolution, described by random walks driven by rare collisions with the scatterers, is revealed through the long-distance Lévy jumps along the diagonal corridors. This stochastic motion is shown to be radius-independent and characterized by the universal diffusion exponent $z_1 = 1.50 \pm 0.05$, as well as by the constant pseudo-Gaussian disorder-order parameter $\sigma_1 = 0.14 \pm 0.02$. It seems plausible to associate the underlying mechanism of the universal superdiffusion dynamics with the critical behavior of spatial correlations [15], with $z_{cr} = 3/2$, described by the unbounded orbits moving in the diagonal Bleher corridors. When the diagonal corridors become closed, one observes a transient motion regime for $R_1 \leq R < R_2$ and characterized by the nonuniversal diffusion with $z_1 < z(R) < 2$. For the geometry with $R \geq R_2 = L/2$, the scatterer overlaps the boundary walls and all the free-motion corridors close. In this case, the disordered motion is due to the bounded-orbit normal diffusion with $z_2 = 2$ and to the normal collisions with $n_c^{(w)} \approx n_c^{(s)}$. The bouncing-ball motion disappear and the central symmetry of the collision distribution is recovered. Despite of the fact that the normal correlations were earlier expected for this billiard geometry from rigorous theory [1] and observed in numerical experiments [3,8], we have proven that both the observed wall and scatterer collisions are true Gaussian.

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**FIGURES**

Fig. 1. Sinai billiard and the corresponding periodic Lorentz Gas model for different boundary geometries. *Case A*: reduction of a trajectory in the LG lattice to the corresponding orbit in SB is shown through the billiard wall-to-wall and wall-to-disk collision points 1, 2, 3, 5 and 4, 6, respectively. *Case B*: infinite horizon geometry for $0 < R < L\sqrt{2}/4$. Examples of the principal free-motion corridors by Bleher [2]: the nondiagonal (a) and diagonal (b) corridors. *Case C*: finite horizon geometry, $R \geq L/2$. *Case D*: reduction of the unbounded trajectory of launching angle $\theta$ in LG without scatterers to the corresponding orbit in the square billiard ($R = 0$).

Fig. 2. Analyses of the wall-collision dynamics in square billiard. Points correspond to simulation data on the collision function observed at $t_{obs} = 100\tau_{c0}$ and on the temporal evolution for the root-mean-square collision-number deviation, with $\tau_{c0} = \pi/4$. Solid lines are the corresponding theoretical predictions given by Eq.(13) and Eq.(12).

Fig. 3. Diffusion dynamic exponent against the reduced radius of the scatterer disk in Sinai billiard. Points are simulation data derived from the observed billiard-wall collisions through the standard deviation (17) and the estimated characteristic time $\tau_{c}^{(w)}(R)$ given in Eq.(2). Inserts: temporal evolution of the standard wall-collision-number deviation for different infinite ($R < L/2$, upper inset) and finite ($R > L/2$, lower inset) horizon geometries, in the log-log coordinates. The solid lines are the best linear fitting of the simulation data.

Fig. 4. Collision distributions against collision numbers in Sinai Billiard with the infinite
horizon geometry \( (R < L/2) \). Points are simulation data for the scatterer (left plot) and the wall (right plot) collisions represented in the reduced coordinates defined in Eq.(18). The solid lines are the Gaussian distributions \( \tilde{D}_2(\tilde{n}) = (1/2\pi) \exp(-\tilde{n}^2/2) \). Left inset: analysis of the late-time scatter-collision function for long-jump-scatterer collisions. Points are simulation data for the case \( R = 0.25L \) and \( t_{obs} = 200\tau_c(w) \). Solid line is the best fitting with the prediction given in Eq.(21), with \( \tilde{D}_1^{(s)}(\tilde{n}) = 0.5|\tilde{n}|^{-11/3} \). Right inset: velocity-dispersive parameter against reduced scatterer radius. Points are simulation asymptotic data [13] for the weakly open SB; dashed and solid lines are theoretical predictions given in Eqs. (23) and (24), respectively.

Fig. 5. Observation of the Gaussian collision statistics in Sinai Billiard with \( R > L/2 \). Points are simulation data for the scatterer (left plot) and the wall (right plot) collisions observed at \( t_{obs} = 200\tau_c \) for distinct geometries indicated in the legend. The lines are the same as in Fig. 4. In the left and right inserts, the analyses of the fourth moment (defined in Eq.(26)) for the scatterer and wall collisions are given at \( t_{obs} = 200\tau_c \), respectively. Points are simulation data and the dashed line corresponds to \( \zeta_4 = 3 \).