Primordial gravitational wave enhancement

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Abstract
We reconsider the enhancement of primordial gravitational waves that arises from a quantum gravitational model of inflation. A distinctive feature of this model is that the end of inflation witnesses a brief phase during which the Hubble parameter oscillates in sign, changing the usual Hubble friction to anti-friction. An earlier analysis of this model was based on numerically evolving the graviton mode functions after guessing their initial conditions near the end of inflation. The current study is based on an equation which directly evolves the normalized square of the magnitude. We are also able to make a very reliable estimate for the initial condition using a rapidly converging expansion for the sub-horizon regime. Results are obtained for the energy density per logarithmic wavenumber as a fraction of the critical density. These results exhibit how the enhanced signal depends upon the number of oscillatory periods; they also show the resonant effects associated with particular wavenumbers.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The case for a phase of accelerated expansion (inflation) during the very early universe is strong. One reason is that we can observe widely separated parts of the early universe which seem to be in thermal equilibrium with one another [1]. If one assumes the universe never underwent a period of inflation, there would not have been time for this thermal equilibrium to be established by causal processes. Without primordial inflation, the number of causally distinct regions in our past light cone at the time of recombination is over 10^3, and it would be 10^9 at the time of nucleosynthesis.

There is no strong indication for what caused primordial inflation. A natural mechanism for inflation can be found within gravitation—which, after all, plays the dominant role in shaping cosmological evolution—by supposing that the bare cosmological constant \( \Lambda \) is not...
unnaturally small but rather large and positive.\textsuperscript{4} Because $\Lambda$ is constant in *space*, no special initial condition is needed to start inflation. We also dispense with the need to employ a new, otherwise undetected scalar field. However, $\Lambda$ is constant in *time* as well, and classical physics can offer no natural mechanism for stopping inflation once it has begun \textsuperscript{[2]}. However, quantum physics can offer a natural mechanism: accelerated expansion continually rips virtual infrared gravitons out of the vacuum \textsuperscript{[3]} and these gravitons attract one another, thereby slowing inflation \textsuperscript{[4]}. This is a very weak effect for $G \Lambda \ll 1$, but a cumulative one, so inflation lasts a long time for no other reason than that gravity is a weak interaction \textsuperscript{[4]}.

This screening mechanism may be clear enough on the perturbative level but it has two frustrating features. The first is that, because inflationary particle production is a one-loop effect, the gravitational response to it is delayed until the two-loop order. The second frustration is that the two-loop effect becomes unreliable just when it starts to get interesting. The effective coupling constant is $GAHt$ and higher loops are insignificant as long as it is small. But all loops become comparable when $GAHt$ becomes of order 1, and the correct conclusion then is that the perturbation theory breaks down. The breakdown occurs not because any single graviton–graviton interaction gets strong but rather because there are so many of them.

We believe that it may be possible to derive a non-perturbative resummation technique by extending the stochastic method which Starobinsky devised for the same purpose in scalar potential models \textsuperscript{[5–7]}. However, generalizing this technique to gravity is a difficult problem \textsuperscript{[8]}. This paper is part of an effort which is based on the idea of guessing the most cosmologically significant part of the effective field equations of quantum gravity. While there is no chance of guessing the full effective field equations, it might be possible to guess just enough to correctly describe the evolution of the scale factor $a(t)$ for a homogeneous and isotropic geometry, using what we know from the perturbation theory about how the back-reaction effect scales. Such simple cosmological models were recently constructed \textsuperscript{[9, 10]} and are reviewed in section 2.

2. The cosmological model

In a previous paper \textsuperscript{[9]}, we proposed a phenomenological model which can provide evolution beyond the perturbation theory. In one sentence, we constructed an *effective* conserved stress–energy tensor $T_{\mu\nu}[g]$ which modifies the gravitational equations of motion:\textsuperscript{5}

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}[g],
\]

and which, we hope, contains the most cosmologically significant part of the full effective quantum gravitational equations.

What form to guess for $T_{\mu\nu}[g]$ was motivated by what we seek to do, and by what we know from the perturbation theory. We seek to describe cosmology, which implies homogeneous

\textsuperscript{4} Here ‘large’ means that a $\Lambda$ induced by a matter scale which can be as high as $10^{18}$ GeV. Then, the value of the dimensionless coupling constant can be as high as $G\Lambda \sim 10^{-4}$ rather than the putative value of $10^{-122}$.

\textsuperscript{5} Hellenic indices take on spacetime values, while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has a spacelike signature and our curvature tensor equals $R_{\mu\nu} = \Gamma_\eta^{\sigma}_{\mu\nu} + \Gamma_\mu^{\sigma}_{\nu} \Gamma_\nu^{\eta}_{\sigma} - (\mu \leftrightarrow \nu)$. The initial Hubble parameter is $3H_0^2 = \Lambda$. 

and isotropic geometries. When specialized to such a geometry, the full effective stress tensor must take the perfect fluid form and we lose nothing by assuming that generally

$$T_{\mu\nu}[g] = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \quad (2)$$

The relation between $p[g]$, $\rho[g]$ and $u_\mu[g]$ is heavily constrained by the stress–energy conservation, but it is possible to specify one function for free. It turns out that to be computationally simplest to take this free function to be the pressure $[9]$. We further require the pressure to be an ordinary function of some non-local scalar which grows like the number of e-foldings when specialized to de Sitter. If the pressure is to grow the way we know it does from the perturbation theory $[7]$, and to eventually end inflation, then a simple choice has the form $[9]$

$$p[g](x) = \Lambda^2 f[-G\Lambda X](x), \quad X \equiv \frac{1}{\Box} R, \quad (3)$$

where the function $f$ grows without bound and satisfies

$$f[-G\Lambda X] = -G\Lambda X + O((G\Lambda)^2), \quad (4)$$

and where the scalar d’Alember
tian

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu ) \quad (5)$$

is defined with retarded boundary conditions. The induced energy density $\rho[g]$ and the 4-velocity $u_\mu[g]$ are determined, up to their initial value data, from the stress–energy conservation:

$$D^\mu T_{\mu\nu} = 0. \quad (6)$$

The 4-velocity was chosen to be timelike and normalized:

$$g^{\mu\nu} u_\mu u_\nu = -1 \quad \implies \quad u^\mu u_{\mu;\nu} = 0. \quad (7)$$

The homogeneous and isotropic evolution$^6$ of this model—using a combination of numerical and analytical methods—revealed the following basic features:$^7$

- After the onset and during the era of inflation, the source $X(t)$ grows, while the curvature scalar $R(t)$ and the Hubble parameter $H(t)$ decrease.
- Inflationary evolution dominates roughly until we reach a critical point $X_{cr}$ defined by

$$1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] = 0. \quad (8)$$

- The epoch of inflation ends close to but before the universe evolves to the critical time. This is most directly seen from the deceleration parameter since initially $q(t = 0) = -1$, while at criticality $q(t = t_{cr}) = \frac{1}{2}$.
- Oscillations in $R(t)$ become significant as we approach the end of inflation; they are centred around $R = 0$, and their frequency equals

$$\omega = G\Lambda H_0 \sqrt{772\pi} \frac{f_{cr}}{f_{cr}^2}, \quad (9)$$

where $H_0$ is the constant inflationary Hubble parameter, and their envelope is linearly falling with time.

- During the oscillation era, although there is net expansion, the oscillations of $H(t)$ take it to small negative values for short time intervals—a feature conducive to rapid reheating; those of $H(t)$ take it to positive values for about half the time; and those of $a(t)$ are centred around a linear increase with time.

$^6$ The line element in co-moving coordinates is $ds^2 = -dt^2 + a^2(t) d\vec{x}^\cdot d\vec{x}$. In terms of the scale factor $a$, the Hubble parameter equals $H(t) = \dot{a}/a$ and the deceleration parameter equals $q(t) = -\ddot{a}/a - 2 = -1 - H \dot{H} / H^2 = -1 + \epsilon(t)$.

$^7$ In $[9]$, our analytical results were obtained for any function $f$ satisfying (4) and growing without bound, our numerical results for the choice $f(x) = \exp(x) - 1$. 

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Class. Quantum Grav. 30 (2013) 025004

M G Romania et al
A novel feature of this class of models is the existence of an oscillatory regime of short duration which commences towards the very end of the inflationary era. During this period, $H(t)$ is positive about half the time, which represents a violation of the weak energy condition. Such a violation cannot occur in classically stable theories [12] but it can be driven by quantum effects of the type we seek to model without endangering stability [13].

3. Linearized gravitons

In terms of the full metric field $g_{ij}(x)$, the fluctuating graviton field $h^{TT}_{ij}(x)$ is defined as

$$g_{ij}(t, x) = a^2(t)\left[\delta_{ij} + \sqrt{2\pi G} h^{TT}_{ij}(t, x)\right].$$

The free-field expansion of the graviton field is

$$h^{TT}_{ij}(t, x) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \{u(t, k) e^{ikx} \epsilon_{ij}(k, \lambda) \alpha(k, \lambda) + (c.c.)\},$$

where $(c.c.)$ denotes complex conjugation, the polarizations $\epsilon_{ij}(k, \lambda)$ and operators $\alpha(k, \lambda)$ obey

$$\epsilon_{ij}(k, \lambda) \epsilon^{ij}_{ij}(k, \lambda') = \delta_{\lambda\lambda'}, \quad \epsilon_{ij}(k, \lambda) = k_i \epsilon_{ij}(k, \lambda) = 0,$$

and the mode functions $u(t, k)$ satisfy

$$\ddot{u}(t, k) + 3H(t) \dot{u}(t, k) + \frac{k^2}{a^2(t)} u(t, k) = 0,$$

with the Wronskian associated with the two solutions of (14) equaling

$$\dot{u} u^* - \dot{u} u^* = ia^{-3}.$$ (15)

We shall be interested in the energy $E(t, k)$ at time $t$ of a mode with wavenumber $k$. The simplest way to derive this is to exploit the fact that the physical degrees of freedom of linearized gravitons have the same dynamics with those of a massless, minimally coupled scalar field $\phi(x)$. The scalar field Lagrangian density is

$$\mathcal{L}(x) = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{1}{2} a^3(t) \dot{\phi}^2 - \frac{1}{2} \nabla \phi \cdot \nabla \phi.$$ (16)

The Lagrangian diagonalizes in momentum space:

$$L(t) = \int d^3x \mathcal{L}(x) = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} a^3(t) |\dot{\phi}(t, k)|^2 - \frac{1}{2} a(t) k^2 |\tilde{\phi}(t, k)|^2 \right],$$ (17)

so that any mode with wavenumber $k$ evolves independently as a harmonic oscillator $q(t)$ with time-dependent mass $m(t) \equiv a^3(t)$ and angular frequency $\omega(t) \equiv ka^{-1}(t)$:

$$q(t) = u(t,k) A + u^*(t,k) A^*, \quad [A, A^*] = 1,$$ (18)

$$E_{\text{SHO}}(t) = \frac{1}{2} a^3(t) \dot{q}^2(t) + \frac{1}{2} a(t) k^2 q^2(t).$$ (19)

At any instant $t$, the minimum energy is $E_{\text{min}}(t, k) = \frac{1}{2} k a^{-1}(t)$. However since both the mass and angular frequency are time dependent, the state with minimum energy at one time instant is not the state with minimum energy at another time instant; there is particle production as time evolves. The Bunch–Davies vacuum $|\Omega\rangle$ is the minimum energy state in the distant past and the expectation value of the energy operator (19) in its presence equals

$$\langle \Omega | E(t, k) | \Omega \rangle = \frac{1}{2} a^3(t) |\dot{u}(t,k)|^2 + \frac{1}{2} k^2 a(t) |u(t,k)|^2.$$ (20)

8 The analogous computation within the linearized graviton theory should only make an $O(1)$ change to the result.
A fair measure of the excess energy $\Delta E(t, k)$ acquired during time evolution in any one wavenumber is obtained by subtracting the instantaneous minimum energy from (20):

$$\Delta E(t, k) \equiv \langle \Omega | E(t, k) | \Omega \rangle - E_{\text{min}}(t, k)$$

$$= \frac{1}{2} a^3(t) |\dot{u}(t, k)|^2 + \frac{1}{2} k^2 a(t) |u(t, k)|^2 - \frac{k}{2a}.$$  \hspace{1cm} (21)

4. The enhancement mechanism

The oscillatory phase is a very distinctive feature of these models and in [11], we investigated the possibility of gravitational wave enhancement due to its presence. There are two very plausible physical arguments that convinced us this is a worthwhile inquiry:

- During the oscillation era, the Hubble parameter $H(t)$ changes sign and this, in turn, changes the sign of the ‘friction’ term $3Hu$ in the evolution equation (14) obeyed by the mode functions $u(t, k)$. For $H(t) > 0$, this term tends to reduce $|\dot{u}(t, k)|$, whereas it tends to increase $|\dot{u}(t, k)|$ when $H(t) < 0$. What happens to the magnitude $|u(t, k)|$ depends upon where $u(t, k)$ is in its own oscillations when $H(t)$ changes sign but the change from ‘friction’ to ‘anti-friction’ can clearly strengthen the amplitude in some cases.

- The oscillation era is characterized by the frequency $\omega$ given by (9). Gravitational waves of frequency close to $\omega$ can resonate and their amplitude can increase.

The first effort to evolve (14) through the oscillatory phase was done in [11]. As expected, it is the near-horizon modes that experience enhancement: the natural timescale of their $u(t, k)$ is close to the inverse of the oscillatory frequency $\omega$ and we obtain a significant resonance response.9 When converted to current frequencies, the main conclusion of [11] is the enhancement of gravitational waves with frequencies somewhat less than $10^{10}$ Hz. For obtaining these results, however, certain assumptions were necessary since we do not possess exact forms for the two linearly independent solutions of (14) during the oscillatory regime, nor do we know which linear combination of these two solutions is the actual mode function as we do not know the linear combination coefficients.10 The latter are determined by the knowledge of the initial conditions at criticality. Because the post-inflationary scale factor effectively describes an overall linear expansion on which the oscillations are superimposed [9], in [11] we solved (14) for a linearly expanding $a(t)$—which can be done exactly—and then numerically superimposed the effect of the oscillations. We also had to make an ‘educated guess’ regarding the initial conditions at criticality.

In re-visiting the subject, we have developed a method—to be described in the next section—which is considerably more accurate and, therefore, leads to robust conclusions.

5. The evolution strategy

- **The variable $M(t, k)$**

We wish to derive an equation for the quantity $M(t, k)$:

$$M(t, k) \equiv u(t, k) u^*(t, k) = |u(t, k)|^2.$$  \hspace{1cm} (22)

9 Here and throughout, super-, sub- and near-horizons are with respect to the modes $k_{cr}$ whose first horizon crossing occurred at criticality, when the transition from the inflationary to the oscillating era occurred: $k_{cr} = H(t_{cr}) a(t_{cr})$.

10 The actual mode function is the coefficient of the annihilation operator in the free-field expansion of the graviton.
because it is directly related to the tensor power spectrum $\Delta_h^2(t, k)$:

$$
\Delta_h^2(t, k) \equiv \frac{32\pi G}{2\pi^2} \int d^3x \ e^{-ik\cdot x} \langle \Omega | h_{ij}^T(t, x) h_{ij}^{TT}(t, 0) | \Omega \rangle
$$

$$
= \frac{16\pi G}{\pi} k^3 M(t, k). \quad (23)
$$

From the definition of $M$, it follows that

$$
\dot{M} = \ddot{u} u^* + u \dot{u^*}, \quad (24)
$$

$$
\ddot{M} = \ddot{u} u^* + 2\dot{u} \dot{u^*} + u \dddot{u^*}. \quad (25)
$$

By using the fact that $\dddot{u}$ satisfies (14), we conclude

$$
\ddot{M} + 3\dot{M} + \frac{2k^2}{a^2} M = 2\dot{u} \dot{u^*} \quad (26)
$$

By subtracting the square of (15) from that of (24), we can express the right-hand side of (26) in terms of $M$ and $\dot{M}$:

$$
\dot{u} \dot{u^*} = \frac{1}{4M} \left[ M^2 + \frac{1}{a^6} \right], \quad (27)
$$

and obtain the desired equation

$$
\ddot{M} + 3\dot{M} + \frac{2k^2}{a^2} M = \frac{1}{2M} \left[ M^2 + \frac{1}{a^6} \right]. \quad (28)
$$

The goal is to find $M(t, k)$ such that (28) is obeyed. An exact solution is beyond our abilities but we can divide the full time evolution range into separate intervals and obtain reliable approximate expressions for $M(t, k)$ within each of these.

**The evolution of $M(t, k)$: inflation**

During the inflationary era, it makes sense to adopt a scheme that works accurately for any kind of mode and, at the same time, avoids numerical evolution for as long as possible. A method that seems optimal is the development of an asymptotic series expansion for $M(t, k)$ in powers of $H^2 a^2 / k^2$:

$$
M(t, k) = \frac{1}{2k a^2} \left[ 1 + \alpha(t) \left( \frac{Ha}{k} \right)^2 + \beta(t) \left( \frac{Ha}{k} \right)^4 + \ldots \right]. \quad (29)
$$

Substituting the above in (28) allows us to determine the leading coefficients $\alpha(t)$, $\beta(t)$ of the series. The final form for the asymptotic expansion of $M(t, k)$ becomes

$$
M(t, k) = \frac{1}{2k a^2} \left[ 1 + \left( 1 - \frac{\epsilon}{2} \right) \left( \frac{Ha}{k} \right)^2 + \left[ \frac{9}{4} \epsilon - \frac{21}{8} \epsilon^2 + \frac{3}{4} \epsilon^3 \right. \right.

$$

$$
\left. + \left( \frac{7}{4} - \frac{3\epsilon}{4} \right) \dot{\epsilon} \frac{Ha}{k}^2 + \left. \left( \frac{\epsilon}{8H^2} \right) \right] \left( \frac{Ha}{k} \right)^4 + \ldots \right]. \quad (30)
$$

As long as $\epsilon$ does not get large, the series (30) converges rapidly and we can use it to evolve all the way to within, say, two e-foldings before first horizon crossing.\textsuperscript{11} We shall, therefore, adopt this method and evolve very accurately: (i) any sub-horizon mode all the way to criticality, (ii) any near-horizon mode until, say, two e-foldings before criticality.

\textsuperscript{11} The error is in ignoring terms proportional to $(\frac{Ha}{k})^6$ and higher. Even when we reach two e-foldings before first horizon crossing that is very small: $(\frac{Ha}{k} = \epsilon^{-2})^6 = \epsilon^{-12}$. 

and (iii) any super-horizon mode until, say, two e-foldings before first horizon crossing. Afterwards, in all cases, equation (28) is evolved numerically. The important cosmological parameters in the inflationary era are [9, 11]

\[
a(t) = a_{cr} e^{-N},
\]

\[
H(t) = \frac{\dot{a}(t)}{a(t)} \simeq \frac{1}{3} \omega \sqrt{4N + \frac{4}{3}},
\]

\[
\dot{H}(t) \simeq -\frac{2H^2}{4N + \frac{4}{3}},
\]

\[
\epsilon(t) = \frac{\dot{H}(t)}{H^2(t)} \simeq \frac{2}{4N + \frac{4}{3}},
\]

where \( N \) is the number of e-foldings before criticality. The initial conditions used in the numerical analysis are those inherited from (30) at the appropriate time.

- The evolution of \( M(t, k) \): oscillations

During the oscillatory era, the important cosmological parameters are [9, 11]

\[
a(t) = a_{cr} C_2 [C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)],
\]

\[
H(t) = \frac{\omega [1 - \sqrt{2} \sin(\omega \Delta t + \phi)]}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)},
\]

\[
\dot{H}(t) = -\frac{\omega^2 \sqrt{2} \cos(\omega \Delta t + \phi)}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)},
\]

\[
\epsilon(t) = 1 + \frac{\sqrt{2} \cos(\omega \Delta t + \phi)}{[1 - \sqrt{2} \sin(\omega \Delta t + \phi)]^2} \frac{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)},
\]

where \( \Delta t \equiv t - t_{cr} \) measures time with respect to criticality. In this regime, we analyse equation (28) numerically. The parameters \( (\phi, C_1, C_2) \) in (35)–(38) are chosen to match the outcome from the inflationary epoch (31)–(34) at criticality, where \( N = 0 \) and \( \Delta t = 0 \):

\[
\phi = \arcsin \left( \frac{\sqrt{2} - \sqrt{2970}}{56} \right) \approx -\frac{\pi}{2},
\]

\[
C_1 = \frac{\sqrt{27}}{2} - \sqrt{\frac{27}{2}} \sin \phi - \sqrt{2} \cos \phi \approx \frac{11}{2},
\]

\[
C_2 = \frac{1}{C_1 + \sqrt{2} \cos \phi} \approx \frac{1}{6}.
\]

- An observable

To connect with physical measurements, consider the excess energy \( \Delta E(t, k) \) at time \( t \) of a mode with wavenumber \( k \). It is given by equation (21) or, equivalently, by

\[
\Delta E(t, k) = \frac{a^2 M^2}{8M} + \frac{1}{8a^2 M}[2k a^2 M - 1]^2,
\]

12 In the absence of time translation invariance, there is no unique definition of the energy. We have chosen to base our estimate on the energy density from the stress–energy tensor; other plausible definitions exist—see, for instance, [14]. While energy is not conserved in an expanding universe, the stress–energy is and this allows for fluctuations between the various components of \( T_{\mu \nu} \) which can lead to the oscillations described here.
where we have used (27), (22). We shall be interested in any total excess energy density \( \Delta \rho(t) \) acquired during time evolution through the oscillating regime:

\[
\Delta \rho(t) = \int \frac{d^3 k}{[2\pi a(t)]^3} \Delta E(t, k)
\]

(43)

\[
= \frac{1}{2\pi^2 a^3(t)} \int dk k^2 \Delta E(t, k).
\]

(44)

Perhaps of more relevance for gravity wave detectors is the amount of gravitational wave energy density \( \Delta \rho(t, k) \) per wavenumber \( k \), and divided by the critical density \( \rho_{\text{cr}} \):

\[
\frac{d}{d\ln k} \Omega_{gw}(t, k) \equiv \frac{1}{\rho_{\text{cr}}} \frac{d}{d\ln k} \Delta \rho(t, k)
\]

(45)

\[
= \frac{4Gk^3}{3\pi H^2_{\text{now}} a^3(t)} \Delta E(t, k).
\]

(46)

6. The results

- **Dimensionless variables**

It is convenient to define dimensionless variables:

\[
\tau \equiv \frac{\omega}{\Delta t}, \quad \kappa \equiv \frac{k}{\omega a_{\text{cr}}}, \quad \alpha \equiv \frac{a}{a_{\text{cr}}}, \quad \mathcal{H} \equiv \frac{H}{\omega}, \quad \mathcal{M} \equiv 2k a^2 M
\]

(47)

and re-express in terms of them the evolution equation (28):

\[
\frac{d^2 \mathcal{M}}{d\tau^2} + \mathcal{H} \frac{d\mathcal{M}}{d\tau} - \mathcal{H}^2 (4 - 2\epsilon) \mathcal{M} - \frac{2k^2}{\alpha^2} \left[ \frac{\mathcal{M}}{\mathcal{M}} - \mathcal{M} \right] = \frac{1}{2\mathcal{M}} \left( \frac{d\mathcal{M}}{d\tau} \right)^2
\]

(48)

the asymptotic series expansion (29):

\[
\mathcal{M}(t, k) = 1 + \left( 1 - \frac{\epsilon}{2} \right) \left( \frac{\mathcal{H}a}{k} \right)^2 + \left[ \frac{9}{4} - \frac{21}{8} \epsilon^2 + \frac{3}{4} \epsilon^3 + \left( \frac{7}{4} - \frac{3}{4} \epsilon^2 \right) \frac{\dot{\epsilon}}{H} + \frac{\ddot{\epsilon}}{8H^2} \right] \left( \frac{\mathcal{H}a}{k} \right)^4 + \cdots
\]

(49)

as well as the excess energy (42):

\[
\Delta E = \omega \left\{ \frac{\alpha \mathcal{H}^2}{4k \mathcal{M}} \left( \mathcal{M} \frac{1}{2\mathcal{H}} \frac{\partial \mathcal{M}}{\partial \tau} \right)^2 + \frac{\kappa}{4\alpha \mathcal{M}} [\mathcal{M} - 1]^2 \right\}. \quad \text{(50)}
\]

and the observable (46):

\[
\frac{d}{d\ln k} \Omega_{gw} = G \omega^2 \times \frac{4}{3\pi} \frac{1}{\mathcal{H}^2} \left( \frac{k}{\alpha} \right)^4 \\
\times \frac{1}{4\mathcal{M}} \left( \frac{\mathcal{H}a}{k} \right)^2 \left[ \mathcal{M} \frac{1}{2\mathcal{H}} \frac{\partial \mathcal{M}}{\partial \tau} \right]^2 + [\mathcal{M} - 1]^2 \right\}. \quad \text{(51)}
\]

- **Renormalization**

When \( \kappa \gg 1 \), the variable \( \mathcal{M} \) becomes essentially unity—as (49) shows—and the observable (51) simplifies considerably:

\[
\left( \frac{\mathcal{H}a}{k} \right)^2 \ll 1 \implies \mathcal{M} \approx 1 \implies \frac{d}{d\ln k} \Omega_{gw} \approx \# \times \frac{k^2}{\alpha^2}.
\]

(52)
At any fixed $\kappa$, the observable is proportional to $\alpha^{-2}$ and its value decreases accordingly as $\tau$ increases. Moreover, at any fixed time $\tau$, the value of the observable—being proportional to $\kappa^2$—will follow a parabola as $\kappa$ increases. This high-frequency ‘tail’ will inevitably lead to ultraviolet divergences. Ultimately, it is the correct ultraviolet theory of quantum gravity that will have to address the issue. Nonetheless, it is within the framework of ordinary perturbative quantum gravity that we must absorb the ultraviolet divergences of our observable.

The quantum field theory tells us how to renormalize the stress–energy tensor of the theory; for instance, its energy density component

$$T_{00}(t) = \Delta \rho(t) = \int \frac{dk}{k} \frac{d}{d\ln k} \Delta \rho(t, k).$$

There are two counterterms available to absorb the divergences:

$$\Delta L_1 = g_1 R \sqrt{-g}, \quad \Delta L_2 = g_2 R^2 \sqrt{-g}. \quad \text{(54)}$$

These counterterms induce the following stress–energy tensor contributions, respectively:

$$\Delta T^1_{\mu\nu} = 2g_1 \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right], \quad \text{(55)}$$

$$\Delta T^2_{\mu\nu} = 2g_2 \left[ 2R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2(g_{\mu\nu} \nabla^2 - D_\mu D_\nu) R \right]. \quad \text{(56)}$$

We are interested in the energy density component of the stress–energy tensor for cosmologically relevant (FRW) spacetimes; in that case,

$$\text{FRW} \implies \Delta T^1_{00} = 6 g_1 \omega^2 \mathcal{H}^2, \quad \text{(57)}$$

$$\text{FRW} \implies \Delta T^2_{00} = -144 g_2 \omega^4 \left[ \frac{3}{2} \epsilon - \frac{3}{4} \epsilon^2 + \frac{\dot{\epsilon}}{2\mathcal{H}} \right] \mathcal{H}^4. \quad \text{(58)}$$

The constants $g_1$ and $g_2$ are determined so that the divergences in (53) are absorbed by (57) and (58).

The quantum field theory does not tell us how to renormalize the observable (45) or the excess energy density $\Delta \rho(t, k)$ since we must integrate either of them over $k$ to obtain $T_{00}(t)$. There is no unique prescription which fixes how the constants $g_1$ and $g_2$ receive contributions from the various infinitesimal elements $dk$. We shall adopt a simple prescription for defining the renormalized observable:

$$\frac{d}{d\ln k} \Omega_{\text{gw}}(t, k) \bigg|_{\text{ren}} \equiv \frac{d}{d\ln k} \Omega_{\text{gw}}(t, k) \bigg|_{\text{subtr}} - \frac{d}{d\ln k} \Omega_{\text{gw}}(t, k) \bigg|_{\text{subtr}}, \quad \text{(59)}$$

where the subtracted observable is obtained by using (49) to expand (51) in powers of $(\frac{\mathcal{H}}{\alpha})^2$ until we reach ultraviolet convergence:

$$\frac{d}{d\ln k} \Omega_{\text{gw}}(t, k) \bigg|_{\text{subtr}} \equiv \frac{1}{\rho_{\text{cr}}} \times \frac{d}{d\ln k} \Delta \rho(t, k) \bigg|_{\text{subtr}} = \frac{1}{\rho_{\text{cr}}} \times \omega^4 \left\{ \mathcal{H}^2 \left( \frac{\kappa}{\alpha} \right)^2 + \left[ \frac{3}{2} \epsilon - \frac{3}{4} \epsilon^2 + \frac{\dot{\epsilon}}{2\mathcal{H}} \right] \mathcal{H}^4 \right\}. \quad \text{(60)}$$

To make the observable ultraviolet finite, two subtractions are needed: one to renormalize the $\kappa^2$ quadratically diverging term—associated with $\Delta T^1_{00}$ given by (57)—and one to
renormalize the constant in \( \kappa \) logarithmically diverging term—associated with \( \Delta T_{00}^2 \) given by (58). Subtracting (60) from (51), we arrive at the renormalized observable (59):

\[
\frac{d}{d \ln k} \Omega_{gw}(t, k) \bigg|_{\text{ren}} = G \omega^2 \times \frac{4}{3 \pi} \frac{1}{H^2} \left( \frac{\kappa}{\alpha} \right)^4 \left( \frac{H \alpha}{\kappa} \right)^2 \left( M - \frac{1}{2 H} \frac{\partial M}{\partial \tau} \right)^2 + [M - 1]^2 - \left( \frac{H \alpha}{\kappa} \right)^2 M \\
- \left[ \frac{3}{2} \epsilon - \frac{3}{4} \epsilon^2 + \frac{\dot{\epsilon}}{2H} \right] \left( \frac{H \alpha}{\kappa} \right)^4 M.
\]

(61)

**Numerical results**

We numerically integrated the evolution equation (48) using the program NDSolve of Mathematica and the results are presented in figures 1–9. It is important to note the following:

- The conditions used for initializing the numerical integration at two e-foldings before first horizon crossing are provided by evaluating (49) at this point.
- The conditions used for initializing the numerical integration at criticality are provided by matching the inflationary solution (31)–(34) with the oscillatory solution (35)–(38) at this point, so that the three parameters \((\phi, C_1, C_2)\) take on the values (39)–(41).
- The dimensionless wavenumber \( \kappa_{cr} \) which underwent first horizon crossing at \( t = t_{cr} \) and is the wavenumber differentiating super-horizon from sub-horizon modes equals

\[
\kappa_{cr} = \mathcal{H}_{cr} \alpha_{cr} = \frac{2}{\sqrt{27}} \approx 0.38.
\]

(62)

For creating figures 1 and 2, we have chosen 136 values of wavenumbers \( \kappa \) ranging from 0.05 (super-horizon) to 10 (sub-horizon).

- Inspection of figure 5 makes evident the time dependence of the observable for any fixed value of \( \kappa \). Different values of \( k \) show a similar structure, but with the curve
Figure 2. Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, from six (leftmost) to ten (rightmost) periods of oscillations. The vertical axis is expressed in units of $G \omega^2 \sim 4 \times 10^{-12}$, so that all results must be multiplied by this factor.

Figure 3. Time evolution of parameter $\epsilon = -\dot{H} H^{-2}$ during the oscillatory regime.

shifted so that what is a peak time for one wavenumber can be a low point for another. In general, some modes within the narrowband $0.05 < \kappa < 10$ are enhanced, but precisely which ones of these are depends upon the time. This raises the important question of what time to choose for the transition from the epoch of oscillations to the epoch of radiation domination. Because this is not fixed in our model, we have chosen instead to assume that the transition occurs when $\mathcal{H} > 0$ and $\epsilon = 2$, which determines the corresponding time $\tau$ to equal

\[
radiation \implies \mathcal{H} > 0 \text{ and } \epsilon = 2 \implies \tau \approx 2\pi N_{\text{osc}}. \tag{63}\n\]
Figure 4. Time evolution of the Hubble parameter $H$ during the oscillatory regime.

Figure 5. Time evolution of $H^2 \frac{d\Omega_{\text{osc}}}{d\ln k}$ for $\kappa = 2$ during the oscillatory regime. The vertical axis is expressed in units of $G\omega^2 \sim 4 \times 10^{-12}$, so that all results must be multiplied by this factor.

– To study the effect of the duration of the oscillatory regime on the enhancement, we have displayed the results for $N_{\text{osc}}$ number of oscillation periods within the regime. Figures 1 and 2 present the results—for $H > 0$ and $\epsilon = 2$—in ascending order of $N_{\text{osc}}$ values.

– The small negative values that the observable achieves are due to the subtractions and go away at late times.

- **Semi-analytical results for radiation domination**
  We shall now demonstrate that the effect built-up during the phase of oscillations persists during the subsequent phase of radiation domination. Figures 6–8 show that the amplitude $M(\tau, \kappa)$ oscillates between fixed positive values—with steadily increasing period—during
the epoch of radiation domination. We can understand this behaviour analytically by re-expressing equation (28) in terms of the variable $F$: 

$$M \equiv F^2 \implies \ddot{F} = -3H\dot{F} - \frac{k^2}{a^2}F + \frac{1}{4a^6F^3}.$$  

(64) 

The first term on the right-hand side is the ever-present Hubble friction, the second a linear restoring force and the third a nonlinear repulsive force. The nonlinear force starts playing a role when the two forces become equal:

$$-\frac{k^2}{a^2}F + \frac{1}{4a^6F^3} = 0 \implies F = \frac{1}{\sqrt{2k}\,a}.$$  

(65) 

It is reasonable to assume that in the radiation regime, $M$ starts with a value higher than its vacuum value, in which case we can ignore the nonlinear force:

$$M > \frac{1}{2ka^2} \implies \ddot{F} \simeq -3H\dot{F} - \frac{k^2}{a^2}F,$$  

(66) 

and arrive at an oscillatory solution:

$$F(t, k) \simeq \frac{F_0}{a(t)} \cos \left[k \int_0^t \frac{dt'}{a^{-1}(t')} \right].$$  

(67) 

showing that $F$ is falling like $a^{-1}$. As $F$ decreases in value with time, at some point the nonlinear term dominates over the linear term:

$$\ddot{F} \simeq -3H\dot{F} + \frac{1}{4a^6F^3} \implies \frac{1}{2}(a^3\dot{F})^2 \simeq -\frac{1}{8F^2} + C.$$  

(68) 

The integration constant $C$ is determined from the turnaround position $\dot{F}(t_1, k) = 0$. After the turnaround position and as $F(t, k)$ now increases, at some point the linear term dominates again and we go to the oscillating solution (67) again. The interplay between the two regimes goes on and confirms the numerical results displayed in figures 7 and 8.

13 During radiation domination $a(t) \sim t^{1/2}$ and $H(t) \sim t^{-1}$ so that $H < \frac{1}{2}$ and, therefore, $\ddot{F} \simeq \frac{k}{a^2}F$. 

Figure 6. Time evolution of the variable $M$ for $\kappa = 2$ from the oscillatory to the radiation regime.
Moreover, the same figures show that the time average of the dimensionless variable $\mathcal{M}$ behaves like a constant which is consistent with our semi-analytical results: $F$ falls like $a^{-1}$, so that $M = F^2$ falls like $a^{-2}$, so that the time average of $\mathcal{M} = 2ka^2M$ evolves as a constant.

Finally, the same figures show that the oscillation period $\Delta t$ increases with time. We shall confirm this behaviour semi-analytically by noting that the period is set by

$$k \int_t^{t+\Delta t} \frac{dr'}{a(r')} = 2\pi.$$  \hspace{1cm} (69)
Figure 9. Time evolution of $(d\omega_{gw}/d\ln k)$ for $\kappa = 2$ during the radiation regime. The vertical axis is expressed in units of $G\omega^2 \sim 4 \times 10^{-12}$, so that all results must be multiplied by this factor.

For radiation domination, (69) equals

$$k \int_{t-\Delta t}^{t+\Delta t} \frac{dt'}{a(t')} = k \left[ \frac{1}{H(t+\Delta t) a(t+\Delta t)} - \frac{1}{H(t) a(t)} \right] = 2\pi,$$

(70)

which implies that

$$\frac{2\pi}{k} H(t) a(t) = \sqrt{1 + \frac{\Delta t}{t}} - 1.$$

(71)

Since the left-hand side of (71) falls like $t^{1/2}$, we must have $\Delta t \approx \sqrt{t}$.

7. Physical consequences

The results of the previous section allow us to make the following remarks:

- The existence of the enhancement effect is confirmed by our analysis. In section 4, we argued, on physical grounds, that the effect is associated with sign changes of the Hubble parameter. This is explicitly seen in figures 4 and 5 where there is a synchronization among the strongest enhancement peaks of the observable (figure 5) and sign changes of the Hubble parameter (figure 4). We have also seen—both numerically and semi-analytically—that the time average of the variable $M$ behaves like a constant under time evolution (figures 7 and 8). Thus, the time average of the observable is essentially constant during radiation (figure 9) and this ensures that the enhancement effect survives.

- The far super-horizon modes left the horizon many e-foldings before criticality, their mode functions are ‘frozen’ thereafter and these modes are not affected much from the presence of the oscillating regime.

- The near-horizon modes show the enhancement due to resonances close to the oscillatory era frequency $\omega$. Note that as $N_{osc}$ increases, the peak enhancement magnitude increases and shifts towards higher values of $\kappa$. Thus, it is the near-sub-horizon modes that feel the biggest enhancement.
The small negative energy densities that appear in figures 1 and 2 are quite perplexing. They appear only after the two subtractions (57)–(58) have been performed to obtain the renormalized observable (61). We think they reflect the fact that—even in vacuum—there is some 'jiggling' of a gravity wave detector from quantum fluctuations. While this is an undetectable effect, if the measurement could be made with infinite accuracy, the signal must be present. The small negative energy density indicates that the amount of 'jiggling' an infinitely accurate gravity wave detector measures is a little lower than one expects from gravity waves which have existed forever in flat space.

After the transition to matter domination, the time-averaged amplitude $M$ remains constant. Therefore, during matter domination, the overall time dependence of the time-averaged observable arises from the prefactor $H^{-2} \alpha^{-4}$ appearing in (61) which is constant during radiation domination but falls like $a^{-1}$ during matter domination. As a result, the present value of the enhancement can be straightforwardly computed:

$$
\left( \frac{d}{d \ln k} \Omega_{gw} \right)_{\text{now}} \approx \left( \frac{d}{d \ln k} \Omega_{gw} \right)_{\text{matter}} \approx \left( \frac{a_{\text{matter}}}{a_{\text{now}}} \right) \left( \frac{d}{d \ln k} \Omega_{gw} \right)_{\text{osc}} G \omega^2 \approx 0.3 \times 10^{-3} \left( \frac{d}{d \ln k} \Omega_{gw} \right)_{\text{osc}} 4 \times 10^{-12}.
$$

The value of the observable in the oscillatory regime, for given $N_{\text{osc}}$ and $\kappa$, can be found in figures 1 and 2. The numerical value of the factor $G \omega^2$ follows from the measured value of the scalar power spectrum and the current limit of the tensor-to-scalar ratio [11]. Clearly, the remaining factor of about $10^{-15}$ in (74) makes the enhancement effect very small and presently unobservable.

The present frequency of the enhanced waves is given by

$$
f_{\text{now}} = \frac{k}{2 \pi a_{\text{now}}} = \frac{\omega \kappa}{2 \pi} \left( \frac{a_{\text{cr}}}{a_{\text{now}}} \right) \lesssim 10^9 \text{ Hz} \times e^{-\frac{1}{2} \Delta N} \lesssim 10^9 \text{ Hz}.
$$

For the estimate (75), we used the near-horizon value $\kappa \approx 1$ as well as [11]

$$
\left( \frac{a_{\text{cr}}}{a_{\text{now}}} \right) \lesssim e^{-63-\kappa/2} \Delta N \approx 10^{-28} \times e^{-\frac{1}{2} \Delta N},
$$

where $\Delta N$ is the number of oscillatory e-foldings which we expect to be small.

8. Epilogue

From figure 5, it is evident that the signal is peaked at a narrowband of very high frequencies and is negligible at significantly different frequencies. It would be challenging to detect gravitational radiation at such high frequencies but detectors in that range have been proposed [15]. As noted in the text, the phase of oscillations does not affect modes which experienced first horizon crossing more than a few e-foldings before the end of inflation. The wavelength of our effect is $\lambda = f^{-1} \gtrsim 0.3 \text{ m}$, whereas the smallest scale feature which is currently observed in the cosmic microwave radiation is about $10^{22} \text{ m}$ [15]! Our model does not change either how matter couples to gravity or the propagation of linearized gravitons, so it has no effect on the spin-down rate of the binary pulsars. The gravity waves we predict will certainly distort
how pulsar light propagates, but the short wavelength again seems to preclude a detectable
effect. LIGO is not sensitive above frequencies of 7000 Hz, which is far too low. The situation
is even worse with LISA’s high-frequency cutoff of 0.1 Hz [15].

Finally, it is worth mentioning that besides the graviton, the oscillations of the Hubble
constant should affect other fields as well and should be studied.

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