ON THE RIGIDITY OF GEOMETRIC AND SPECTRAL PROPERTIES OF GRASSMANNIAN FRAMES

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Abstract. We study the rigidity properties of Grassmannian frames: basis-like sets of unit vectors that correspond to optimal Grassmannian line packings. It is known that Grassmannian frames characterized by the Welch bound must satisfy the restrictive geometric and spectral conditions of being both equiangular and tight; however, less is known about the necessary properties of other types of Grassmannian frames. We examine explicit low-dimensional examples of orthoplectic Grassmannian frames and conclude that, in general, the necessary conditions for the existence of Grassmannian frames can be much less restrictive. In particular, we exhibit a pair of 5-element Grassmannian frames in \( \mathbb{C}^2 \) manifesting with differently sized angle sets and different reconstructive properties (i.e., only one of them is a tight frame). This illustrates the complexity of the line packing problem, as there are cases where a solution may coexist with another solution of a different geometric and spectral character. Nevertheless, we find that these "twin" instances still respect a certain rigidity, as there is a necessary trade-off between their tightness properties and the cardinalities of their angle sets. The proof of this depends on the observation that the traceless embedding of Conway, Hardin and Sloane sends the vectors of a unit-norm, tight frame to a zero-summing set on a higher dimensional sphere. In addition, we review some of the known bounds for characterizing optimal line packings in \( \mathbb{C}^2 \) and discuss several examples of Grassmannian frames achieving them.

1. Introduction

Due to their usefulness in numerous areas of science [1, 2, 3], engineering [4, 5, 6, 7] and mathematics [8, 9, 10], equiangular tight frames (ETFs) are a class of frames that have received much attention in recent years [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]; however, ETFs rarely exist [21]. Because a compactness argument shows that Grassmannian frames always exist [11], they often serve as ideal generalizations for ETFs in applications where low coherence is desired [22, 11, 23, 7, 24].

A complex Grassmannian frame is a set of \( N \) unit vectors that spans \( \mathbb{C}^M \) with the property that the maximal element of its angle set, or set of pairwise absolute inner products, is minimal. A Grassmannian frame is \( K \)-angular if the cardinality of its angle set is \( K \) and it is tight if it satisfies a scaled version of Parseval’s identity.
Equiangular tight frames are 1-angular Grassmannian frames characterized by achievement of the optimal lower bound of Welch \[24, 25\], but this is only possible if \(N \leq M^2\) \[10, 7\]. When \(N > M^2\), the orthoplex bound provides an alternative means for characterizing known examples of Grassmannian frames \[26, 27, 23\], but this bound can only be achieved if \(N \leq 2(M^2 - 1)\) \[26, 27, 23\].

For the special case of unit-norm frames in \(\mathbb{C}^2\), the isometric spherical embedding technique of Conway, Hardin, and Sloane \[26\] sends frame vectors to the unit sphere in \(\mathbb{R}^3\). As has been previously shown in \[28\], leveraging Tóth’s spherical cap packing bound for the unit sphere in \(\mathbb{R}^3\) \[29\] yields a lower bound for optimal coherence that is stronger than the orthoplex bound whenever \(N > 6\) (Theorem 4.4), and this saturates for the case \(N = 12\), as exemplified by a tight Grassmannian frame consisting of 12 vectors in \(\mathbb{C}^2\) (Example 4.6). As with the cases of Grassmannian frames consisting of \(N = 4\) and 6 vectors for \(\mathbb{C}^2\), it is striking that this example also embeds perfectly into the vertices of a Platonic solid, an icosahedron in this case. Furthermore, we observe that this example generates a complex projective 5-design and is thus relevant to combinatorial and quantum information literature.

Because Grassmannian frames that achieve the Welch bound are tight and have angle sets of minimal cardinality, it is natural to ask the following questions:

1.1. **Question.** Is every Grassmannian frame tight?

1.2. **Question.** If \(\Phi\) is a Grassmannian frame not characterized by the Welch bound, can we infer anything about the cardinality of its angle set or its spectral properties?

In \[30\], the authors answered Question 1.1 in the negative for the real case by showing that Grassmannian frames consisting of 5 vectors in \(\mathbb{R}^3\) are always equiangular but never tight; furthermore, their result suggests that a plausible answer to Question 1.2 is that the cardinality of the angle set of a real Grassmannian frame should satisfy a minimality condition.

By considering two distinct examples of Grassmannian frames consisting of 5 vectors in \(\mathbb{C}^2\), we find the answers to these questions for the complex case to be more complicated than we anticipated. Strictly speaking, the answer to Question 1.1 for the complex case is also in the negative, because there exists a non-tight, 2-angular Grassmannian frame with 5 vectors for \(\mathbb{C}^2\) (Example 4.3); however, this question may be the wrong one to ask, because a tight, 3-angular Grassmannian frame consisting of 5 vectors over \(\mathbb{C}^2\) also exists (Example 4.2).

Nevertheless, we find that one still encounters a certain amount of rigidity if one stipulates tightness in the frame’s design; in particular, we prove that every tight Grassmannian frame consisting of 5 vectors in \(\mathbb{C}^2\) must have an angle set with cardinality greater than 2 (Theorem 5.4). The proof
of this depends on basic properties about 2-angular, tight frames (Proposition 5.1 and Theorem 5.3) and the observation that whenever the spherical embedding technique of [26] is applied to a tight, unit-norm frame, then the embedded vectors in the higher-dimensional Euclidean sphere must sum to zero (Theorem 3.2).

In light of the coexistence of tight, 3-angular and non-tight, 2-angular Grassmannian frames in this scenario, we arrive at the following partial answer to the questions above. There exist cases where, for a given fixed number of vectors and fixed dimension, a Grassmannian frame may be constructed in multiple ways, where there is some trade-off between its spectral properties (tightness) and geometric properties (angle set).

The remainder of this paper is outlined as follows. In Section 2, we establish notation and terminology and collect a few basics facts about frame theory. In Section 3, we recall the spherical embedding technique of [26] and show that it embeds tight frames into zero-summing vectors. In Section 4, we recall the Welch bound and orthoplex bound, and we use the spherical embedding technique along with a result of Tóth to improve upon these bounds for the case of \( N > 6 \) vectors in \( \mathbb{C}^2 \). In this section, we also discuss several examples of Grassmannian frames that achieve these bounds. Finally, in Section 5, we prove a few basic facts about 2-angular tight frames and use these facts to prove that a tight Grassmannian frame consisting of 5 vectors in \( \mathbb{C}^2 \) can never be 2-angular.

2. Preliminaries

Let \( \{e_j\}_{j=1}^M \) denote the canonical orthonormal basis for \( \mathbb{F}^M \), where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \) and let \( I_M \) denote the \( M \times M \) identity matrix. A set of vectors \( \Phi = \{\phi_j\}_{j=1}^N \subset \mathbb{F}^M \) is a \textit{(finite) frame} if \( \text{span}\{\phi_j\}_{j=1}^N = \mathbb{F}^M \). It is often convenient to identify a frame \( \Phi = \{\phi_j\}_{j=1}^N \) in terms of its \textit{synthesis matrix}

\[
\Phi = [\phi_1 \phi_2 \ldots \phi_N],
\]

the \( M \times N \) matrix with columns given by the \textit{frame vectors}. Just as we have written \( \Phi = \{\phi_j\}_{j=1}^N \) and \( \Phi = [\phi_1 \phi_2 \ldots \phi_N] \) in the last sentence, we proceed with the tacit understanding that \( \Phi \) is both a matrix and a set of vectors. Furthermore, we reserve the symbols \( M \) and \( N \) to refer to the dimension of the span of a frame and the cardinality of a frame, respectively.

A frame \( \Phi = \{\phi_j\}_{j=1}^N \) is \textit{A-tight} if \( \Phi\Phi^* = \sum_{j=1}^N \phi_j\phi_j^* = AI_M \) for some \( A > 0 \) and it is \textit{unit-norm} if each frame vector has norm \( \|\phi_j\| = 1 \). If \( \Phi \) is unit-norm and \( A \)-tight, then \( A = \frac{N}{M} \) because

\[
N = \sum_{l=1}^M \sum_{j=1}^N |\langle e_l, \phi_j \rangle|^2 = \sum_{l=1}^M \sum_{j=1}^N \text{tr}(\phi_j\phi_j^*e_le_l^*) = A \sum_{l=1}^M \|e_l\|^2 = AM,
\]
in which case also we have the identity

$$\sum_{l=1}^{N} |\langle \phi_j, \phi_l \rangle|^2 = \frac{N}{M},$$

for every $j \in \{1, 2, ..., N\}$.

Given a unit-norm frame $\Phi = \{\phi_j\}_{j=1}^{N}$, its frame angles are the elements of the set

$$A_\Phi := \{|\langle \phi_j, \phi_l \rangle| : j \neq l\},$$

i.e. the angle set of $\Phi$, and we say that $\Phi$ is $K$-angular if $|A_\Phi| = K$ for some $K \in \mathbb{N}$. In the special cases that $K = 1, 2$ or $3$, we say that $\Phi$ is equiangular, biangular or triangular, respectively.

If $\Phi = \{\phi_j\}_{j=1}^{N}$ is $K$-angular with frame angles $c_1, c_2, ..., c_K$, then we say that $\Phi$ is equidistributed if there exist $m_1, m_2, ..., m_K \in \mathbb{N}$ such that

$$\left| \left\{ j' \in \{1, ..., N\} : j' \neq j, |\langle \phi_j, \phi_{j'} \rangle| = c_k \right\} \right| = m_k$$

for every $j \in \{1, 2, ..., N\}$ and every $k \in \{1, 2, ..., K\}$. In this case, we call the positive integers $m_1, m_2, ..., m_K$ the multiplicities of $\Phi$ and remark that $\sum_{j=1}^{K} m_j = N - 1$.

We let $\Omega_{N,M}(\mathbb{F})$ denote the space of unit norm frames consisting of $N$ vectors in $\mathbb{F}^M$. Given $\Phi = \{\phi_j\}_{j=1}^{N} \in \Omega_{N,M}(\mathbb{F})$, its coherence is defined by

$$\mu(\Phi) = \max_{j \neq l} |\langle \phi_j, \phi_l \rangle|$$

and we define the Grassmannian constant for the pair $(N, M)$ by

$$\mu_{N,M}(\mathbb{F}) = \inf_{\Phi \in \Omega_{N,M}(\mathbb{F})} \max_{j \neq l} |\langle \phi_j, \phi_l \rangle|.$$ 

We say that $\Phi \in \Omega_{N,M}(\mathbb{F})$ is a Grassmannian frame if

$$\mu(\Phi) = \mu_{N,M}(\mathbb{F}).$$

3. Spherical Embedding

In [26], the authors observed that a unit-norm frame can be isometrically embedded into a sphere in some high dimensional real Hilbert space.

3.1. Theorem. [Conway et al., [26]] Let $D = M^2 - 1$ if $\mathbb{F} = \mathbb{C}$ or let $D = \frac{(M+2)(M-1)}{2}$ if $\mathbb{F} = \mathbb{R}$. If $\Phi = \{\phi_j\}_{j=1}^{N} \in \Omega_{N,M}(\mathbb{F})$, then the frame vectors can be isometrically embedded into the unit sphere in $\mathbb{R}^D$ via

$$\phi_j \mapsto y_j \in \mathbb{R}^D$$

such that, for all $j, l \in \{1, ..., N\}$, we have

$$|\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{M} + \frac{M-1}{M} \langle y_j, y_l \rangle.$$
Proof. Let $\Phi = \{\phi_j\}_{j=1}^{N} \in \Omega_{N,M}(F)$. The frame vectors of $\Phi$ can be embedded into the "traceless" subspace of the $M \times M$ self-adjoint (or symmetric) matrices via the mapping

$$\phi_j \mapsto \phi_j \phi_j^* - \frac{1}{M} I_M,$$

which is isomorphic to $\mathbb{R}^D$ by a dimension counting argument. In particular, these embedded vectors all lie on a sphere of radius $\sqrt{\frac{M-1}{M}}$, because the Hilbert Schmidt norm gives $\|\phi_j \phi_j^* - \frac{1}{M} I_M\|_{H.S.}^2 = \frac{M-1}{M}$ for every $j \in \{1, \ldots, N\}$, and this embedding is distance preserving because for $j \neq l$

$$\|\phi_j \phi_j^* - \phi_l \phi_l^*\|_{H.S.}^2 = 2 \left(1 - tr((\phi_j \phi_j^* \phi_l \phi_l^*))\right)$$

$$= 2 \left(1 - |\langle \phi_j, \phi_l \rangle|^2\right).$$

By identifying $\phi_j \phi_j^* - \frac{1}{M} I_M$ and $\phi_l \phi_l^* - \frac{1}{M} I_M$ with vectors $\tilde{y}_j, \tilde{y}_l \in \mathbb{R}^D$ on a sphere of radius $\sqrt{\frac{M-1}{M}}$ and using that

$$\|y_j - y_l\|^2 = 2 \frac{M-1}{M} (1 - \langle \tilde{y}_j, \tilde{y}_l \rangle),$$

we can rewrite this equation as

$$|\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{M} + \frac{M-1}{M} \langle y_j, y_l \rangle,$$

where $y_j$ and $y_l$ are the unit vectors in the direction of $\tilde{y}_j$ and $\tilde{y}_l$, respectively. $\square$

We observe that whenever a unit-norm, tight frame is embedded into a higher dimensional sphere as above, then the embedded vectors are also zero-summing.

3.2. Corollary. Let $D = M^2 - 1$ if $F = \mathbb{C}$ or let $D = \frac{(M+2)(M-1)}{2}$ if $F = \mathbb{R}$. If $\Phi = \{\phi_j\}_{j=1}^{N} \in \Omega_{N,M}(F)$ is a tight frame, then the frame vectors can be isometrically embedded into the unit sphere in $\mathbb{R}^D$ via

$$\phi_j \mapsto y_j \in \mathbb{R}^D$$

such that, for all $j, l \in \{1, \ldots, N\}$, we have

$$(2)$$

$$|\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{M} + \frac{M-1}{M} \langle y_j, y_l \rangle$$

and

$$(3)$$

$$\sum_{j=1}^{N} y_j = 0.$$
Proof. Let \( \Phi = \{ \phi_j \}_{j=1}^N \in \Omega_{N,M}(\mathbb{F}) \). As in the proof of Theorem 3.1, we embed the frame vectors via
\[
\phi_j \mapsto \phi_j \phi_j^* - \frac{1}{M} I_M.
\]
Summing over these matrices and using the tightness property, we have
\[
\sum_{j=1}^N \left( \phi_j \phi_j^* - \frac{1}{M} I_M \right) = \frac{N}{M} I_M - \frac{N}{M} I_M = 0_M,
\]
where \( 0_M \) denotes the \( M \times M \) zero matrix. The claim then follows by mimicking the proof of Theorem 3.1. □

4. Lower Bounds for the Grassmannian Constant

The optimal lower bound for the Grassmannian constant is the Welch bound [31],
\[
\mu_{N,M}(\mathbb{F}) \geq \sqrt{\frac{N - M}{M(N - 1)}}.
\]
A Grassmannian frame achieves this lower bound if and only if it is an equiangular, tight frame [10, 25], but it is well-known that this cannot occur when \( N > M^2 \) if \( \mathbb{F} = \mathbb{C} \) or \( N > \frac{M(M+1)}{2} \) if \( \mathbb{F} = \mathbb{R} \) [26, 7].

By applying Theorem 3.1 to a sphere-packing result of Rankin [27], the authors of [10] (see also [32, 23]) extrapolated a lower bound for \( \mu_{N,M}(\mathbb{F}) \) that is stronger than the Welch bound whenever \( N > M^2 \) or \( N > \frac{M(M+1)}{2} \) for \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{R} \), respectively.

4.1. Theorem. [Orthoplex bound, [27, 10, 32, 23]] Let \( D = M^2 - 1 \) if \( \mathbb{F} = \mathbb{C} \) or let \( D = \frac{(M+2)(M-1)}{2} \) if \( \mathbb{F} = \mathbb{R} \). If \( N > D + 1 \), then
\[
\mu_{N,M}(\mathbb{F}) \geq \frac{1}{\sqrt{M}}.
\]

The following example comes from [23], where the authors constructed infinite families of tight, complex Grassmannian frames with coherence equal to the orthoplex bound, which they termed orthoplectic Grassmannian frames. Of particular interest to us, it is a tight, triangular Grassmannian frame in \( \Omega_{5,2}(\mathbb{C}) \).

4.2. Example. Let \( \omega = e^{2\pi i/3} \) and let
\[
\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2} & 0 & 1 & 1 & 1 \\
0 & \sqrt{2} & \omega & \omega^2
\end{bmatrix}.
\]
It is straightforward to check that \( \Phi \) is tight and has frame angles \( c_1 = \frac{1}{\sqrt{2}}, c_2 = \frac{1}{\sqrt{2}}, c_3 = 0 \), so it is triangular and it is a Grassmannian frame by Theorem 3.1.
The next example also achieves the orthoplex bound, so it is also a Grassmannian frame in $\Omega_{5,2}(\mathbb{C})$; however, unlike the preceding example, this one is biangular but not tight.

4.3. Example. Let

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & i & -i \end{bmatrix}. $$

It is straightforward to check that $\Phi$ has frame angles $c_1 = \frac{1}{\sqrt{2}}$ and $c_2 = 0$, so it is biangular and it is a Grassmannian frame by Theorem 4.1. However, $\Phi \Phi^* \neq \frac{5}{2}I_5$, so it is not tight.

Just as the Welch bound cannot saturate when a frame’s cardinality is too large, it is also known that a Grassmannian frame’s coherence must be greater than the orthoplex bound when $N > 2(M^2 - 1)$ in the complex case or $N > (M+2)(M-1)$ in the real case [27, 10, 32, 23]. Thus, the orthoplex bound can also be improved when there are too many vectors. In the special case that $F = \mathbb{C}$ and $M = 2$, the embedding from Theorem 3.1 sends points from the sphere in $\mathbb{C}^2$ to points on the sphere in $\mathbb{R}^3$, so the classical spherical cap packing result of Tóth [29] leads to an improved lower bound for $\mu_{N,2}(\mathbb{C})$ when $N \geq 7$, as has been previously noted in [28].

4.4. Theorem. [Tóth’s Bound, 29] Let $N \geq 3$. If $\{x_j\}_{j=1}^N$ is a set of unit vectors in $\mathbb{R}^3$, then

$$\max_{j \neq l} |\langle x_j, x_l \rangle| \geq \frac{1}{2} \csc^2 \left( \frac{N\pi}{6(N-2)} \right) - 1.$$

4.5. Theorem. [28] If $N \geq 3$, then

$$\mu_{N,2}(\mathbb{C}) \geq \frac{N}{2} \csc \left( \frac{N\pi}{6(N-2)} \right).$$

Proof. Let $\Phi = \{\phi_j\}_{j=1}^N \in \Omega_{N,2}(\mathbb{C})$. By Theorem 3.1 there exist points $\{y_j\}_{j=1}^N$ on the unit sphere in $\mathbb{R}^3$ such that

$$|\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{2} + \frac{1}{2} |\langle y_j, y_l \rangle|.$$

The claim then follows from Theorem 4.4 because

$$\max_{j \neq l} |\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{2} + \frac{1}{2} \max_{j \neq l} |\langle y_j, y_l \rangle|$$

$$\geq \frac{1}{2} + \frac{1}{2} \left[ \frac{1}{2} \csc^2 \left( \frac{N\pi}{6(N-2)} \right) - 1 \right].$$

□

Tóth’s bound from Theorem 4.4 saturates when $N = 3, 4, 6, 12$ [29]. When $N = 3$, the bound is obtained by the three vertices of an equilateral
triangle centered at the origin in $\mathbb{R}^3$. For $N = 4$, the bound is obtained by the vertices of a regular 3-simplex (i.e. a tetrahedron) centered at the origin. For $N = 6$, the bound is obtained by the vertices of an orthoplex (i.e. an $\ell^1$-ball or octahedron) centered at the origin and the case $N = 12$ corresponds to the twelve vertices of an icosahedron centered at the origin. Furthermore, for the cases $N = 3, 4, 6$, it is known that there exist tight Grassmannian frames in $\Omega_{N,2}(\mathbb{C})$ that not only achieve the lower bound in Theorem 4.5 but embed perfectly into the vertices of an equilateral triangle \cite{26}, regular tetrahedron \cite{2}, and regular octahedron \cite{33}, respectively. Next, we exhibit an example of a tight Grassmannian frame in $\Omega_{12,2}(\mathbb{C})$ that embeds perfectly into the vertices of a regular icosahedron.

\textbf{4.6. Example.} [Icosaplectic Grassmannian Frame] Let $a = \sqrt{\frac{5+\sqrt{5}}{10}}$ and $b = \sqrt{1-a^2}$ and let $\eta = e^{2\pi i/5}$ and $\omega = e^{2\pi i/10}$ be the primitive 5th and 10th roots of unity, respectively. A straightforward computation shows that

$$
\Phi = \begin{bmatrix}
1 & 0 & -b & -b & -b & -b & -b & a\omega & a\omega\eta & a\omega\eta^2 & a\omega\eta^3 & a\omega\eta^4 \\
0 & 1 & -a\omega & -a\omega\eta & -a\omega\eta^2 & -a\omega\eta^3 & -a\omega\eta^4 & b & b & b & b & b
\end{bmatrix}
$$

is an equidistributed, triangular, tight frame in $\Omega_{12,2}(\mathbb{C})$ with frame angles $c_1 = a > c_2 = b > c_3 = 0$ and corresponding multiplicities $m_1 = 5, m_2 = 5$ and $m_3 = 1$. It follows from elementary trigonometry that the lower bound in Theorem 4.5 equals $c_1$, showing that $\Phi$ is a Grassmannian frame in $\Omega_{12,2}(\mathbb{C})$. If $Y = \{y_j\}_{j=1}^{12}$ denotes the unit vectors in $\mathbb{R}^3$ obtained via the embedding from Theorem 3.1, then another computation using the identity

$$
|\langle \phi_j, \phi_l \rangle|^2 = \frac{1}{2} + \frac{1}{2} \langle y_j, y_l \rangle
$$

shows that the vectors of $Y$ must correspond to the vertices of a regular icosahedron.

Finally, we remark that the frame from Example 4.6 has some relevance for the combinatorial and quantum information literature. It is simple to check that

$$
\frac{1}{(12)^2} \sum_{j,l=1}^{N} |\langle \phi_j, \phi_l \rangle|^{10} = \frac{1}{6},
$$

so $\Phi$ generates an (equally) \textit{weighted complex projective 5-design} by Theorem 2.3 of \cite{33}. Such objects are known to be optimal for linear quantum state determination with respect to a fixed number of measurements \cite{34,33}.

\section{Grassmannian Frames for $\mathbb{C}^2$ Consisting of 5 vectors}

In this section, we show that although $\mu_{5,2}(\mathbb{C})$ can be achieved by both biangular and triangular frames, as in Examples 4.3 and 4.2 there is a necessary trade-off between the cardinality of the angle set and tightness. In order to do this, we collect a few basic facts about \textit{biangular, tight frames} (BTFs).
First, we show that every BTF is equidistributed. A specialized version of this result was shown for the case of 2-distance tight frames in \cite{4a}.

5.1. Proposition. If $\Phi = \{\phi_j\}_{j=1}^N$ is a biangular, tight frame for $\mathbb{F}^M$, then $\Phi$ is equidistributed.

Proof. If $c_1, c_2$ are the frame angles of $\Phi$, then Equation \ref{1} implies that for each $j \in \{1, 2, ..., N\}$, there exists a pair of positive integers $m_{1,j}$ and $m_{2,j}$ such that $m_{1,j} + m_{2,j} = N - 1$ and

\[
m_{1,j}c_1^2 + m_{2,j}(c_2^2 - c_1^2) = \sum_{l=1, l \neq j}^N |\langle \phi_j, \phi_l \rangle|^2 = \|\phi_j\|^2 \left( \frac{N}{M} - 1 \right) = \frac{N - M}{M},
\]

where the last equality follows from the unit-norm property. Next, for $j, l \in \{1, 2, ..., N\}$, we compute:

\[
(N - 1)c_1^2 + m_{2,j}(c_2^2 - c_1^2) = (m_{1,j} + m_{2,j})c_1^2 + m_{2,j}(c_2^2 - c_1^2)
\]

\[
= m_{1,j}c_1^2 + m_{2,j}c_2^2
\]

\[
= m_{1,j}c_1^2 + m_{2,j}c_2^2
\]

\[
= (m_{1,j} + m_{2,j})c_1^2 + m_{2,j}(c_2^2 - c_1^2)
\]

\[
= (N - 1)c_1^2 + m_{2,j}(c_2^2 - c_1^2).
\]

Since $c_1 \neq c_2$, it follows that $m_{2,j} = m_{2,l}$, which implies that $m_{1,j} = m_{1,l}$.

It is worth noting that, in general, K-angular tight frames are not equidistributed, so BTFs and ETFs are quite special in this regard. For instance, the frame from Example \ref{ex4.2} is tight and 3-angular, but it is not equidistributed. The second observation we need about BTFs concerns the multiplicities of their frame angles when $N$ is odd.

5.2. Lemma. Let $N \in \mathbb{N}$ be odd. If $\Phi = \{\phi_j\}_{j=1}^N$ is a biangular, equidistributed frame for $\mathbb{F}^M$ with frame angles $c_1, c_2$ and corresponding multiplicities $m_1, m_2$, then $m_1$ and $m_2$ are both even.

Proof. Let $M = (|\langle \phi_j, \phi_l \rangle|)_{j,l=1}^N$, the matrix obtained by taking the absolute values of the entries of the Gram matrix of $\Phi$. If $m_1$ is odd, then $Nm_1$ is odd, so $c_1$ occurs an odd number of times among the off-diagonal entries of $M$. However, $M$ is symmetric, so the number of occurrences of $c_1$ above the diagonal of $M$ equals the number of occurrences of $c_1$ below the diagonal, which implies that $Nm_1$ is even, a contradiction. Therefore, $m_1$ is even, so the fact that $N - 1 = m_1 + m_2$ is even implies that $m_2$ is also even.

5.3. Theorem. If $N$ is odd and $\Phi = \{\phi_j\}_{j=1}^N$ is a biangular, tight frame for $\mathbb{F}^M$, then $\Phi$ is equidistributed with even multiplicities.

Proof. This follows directly from Proposition \ref{prop5.1} and Lemma \ref{lem5.2}.

Finally, we show that a tight, biangular Grassmannian frame can never exist in $\Omega_{5,2}(\mathbb{C})$.
5.4. **Theorem.** If $\Phi$ is a tight Grassmannian frame in $\Omega_{5,2}(\mathbb{C})$, then $|A_\Phi| \geq 3$.

**Proof.** First, note that Examples 4.2 and 4.3 show that the lower bound in Theorem 4.1 is saturated in this setting. In particular, we know that $\mu_{5,2}(\mathbb{C}) = \frac{1}{\sqrt{2}}$.

By way of contradiction, suppose that $|A_\Phi| < 3$. If $|A_\Phi| = 1$, then $\Phi$ is an ETF, so it achieves the Welch bound, which means $\mu(\Phi) = \sqrt{\frac{3}{8}} < \frac{1}{\sqrt{2}}$, a contradiction. Therefore, it must be that $|A_\Phi| = 2$, meaning $\Phi$ is a BTF.

Let $c_1, c_2$ denote the frame angles of $\Phi$. Because $\Phi$ is a Grassmannian frame, we may assume with no loss of generality that $c_1 = \frac{1}{\sqrt{2}}$.

By Corollary 5.3, $\Phi$ is equidistributed with multiplicities $m_1 = m_2 = 2$. Because $\Phi$ is a $5/2$-tight, unit-norm frame, Equation 1 becomes

$$\frac{5}{2} = 1 + 2c_1^2 + 2c_2^2 = 1 + 2 \left( \frac{1}{\sqrt{2}} \right)^2 + 2c_2^2,$$

which implies that $c_2 = \frac{1}{2}$.

Next, let $Y = \{y_j\}_{j=1}^5$ be the zero-summing unit vectors obtained by embedding the vectors of $\Phi$ into $\mathbb{R}^3$, as in Corollary 3.2. Because $\Phi$ is equidistributed with multiplicities $m_1 = m_2 = 2$, it follows from Equation 2 that the statement

$$\mathcal{P}_j : |\{\langle y_j, y_l \rangle = 0 : l \neq j \}| = |\{\langle y_j, y_l \rangle = -1/2 = 0 : l \neq j \}| = 2$$

must be true for all $j \in \{1, 2, 3, 4, 5\}$. We will show that this contradicts the zero-summing property of $Y$.

After an appropriate choice of unitary rotation, there is no loss of generality in assuming that $y_1 = e_1$. The statement $\mathcal{P}_1$ implies that $y_1$ is orthogonal to two of the vectors of $Y$ and it has inner product $-\frac{1}{2}$ with the other two; therefore, we may further assume without loss of generality that, after an appropriate rotation, $y_2 = e_2$ and that $\langle y_1, y_3 \rangle = 0$ and $\langle y_1, y_4 \rangle = \langle y_1, y_5 \rangle = -\frac{1}{2}$. Viewing $Y$ as a $3 \times 5$ matrix, these assumptions mean that its first row and its first two columns are completely determined.

$$Y = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}.$$ 

We cannot have $\langle y_2, y_3 \rangle = 0$, because then the statements $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ force

$$\langle y_1, y_4 \rangle = \langle y_2, y_4 \rangle = \langle y_3, y_4 \rangle,$$

which in turn contradicts $\mathcal{P}_4$. Therefore, $\langle y_2, y_3 \rangle = -1/2$.

$$Y = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & * & * \\ 0 & 0 & * & * & * \end{bmatrix}.$$
ON THE RIGIDITY OF GEOMETRIC AND SPECTRAL PROPERTIES OF GRASSMANNIAN FRAMES

The statement $P_2$ then implies that either (i) $\langle y_2, y_4 \rangle = 0$ and $\langle y_2, y_5 \rangle = -1/2$ or (ii) $\langle y_2, y_4 \rangle = -1/2$ and $\langle y_2, y_5 \rangle = 0$. Since it is clear that these two cases are symmetric, we assume with no loss in generality that $\langle y_2, y_4 \rangle = 0$ and $\langle y_2, y_5 \rangle = -1/2$.

$Y = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & * & * & * \end{bmatrix}$.

Finally, the unit-norm condition means that the remaining entries of $Y$ must satisfy

$Y = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \pm\frac{\sqrt{3}}{2} & \pm\frac{\sqrt{3}}{2} & \pm\frac{\sqrt{2}}{2} \end{bmatrix}$,

but this contradicts the zero-summing condition, because there is no choice of signs for which

$\pm\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = 0$.

Therefore, $\Phi$ cannot exist, which completes the proof.

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