OPTIMAL SUP NORM BOUNDS FOR NEWFORMS ON GL₂ WITH MAXIMALLY RAMIFIED CENTRAL CHARACTER

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Abstract. Recently, the problem of bounding the sup norms of \( L^2 \)-normalized cuspidal automorphic newforms \( \phi \) on \( GL₂ \) in the level aspect has received much attention. However at the moment strong upper bounds are only available if the central character \( \chi \) of \( \phi \) is not too highly ramified. In this paper, we establish a uniform upper bound in the level aspect for general \( \chi \). If the level \( N \) is a square, our result reduces to \( \| \phi \|_\infty \ll N^{1/4+\epsilon} \), at least under the Ramanujan Conjecture. In particular, when \( \chi \) has conductor \( N \), this improves upon the previous best known bound \( \| \phi \|_\infty \ll N^{1/2+\epsilon} \) in this setup (due to Saha [14]) and matches a lower bound due to Templier [17], thus our result is essentially optimal in this case.

1. Introduction

Let \( \phi \) be a cuspidal automorphic form on \( GL₂(\mathbb{A}_Q) \) with conductor \( N = \prod_p p^{n_p} \) and central character \( \chi \). Assume in addition \( \phi \) is a newform, in the sense that there exists either a Maaß or holomorphic cuspidal newform \( f \) of weight \( k \) for \( \Gamma₁(N) \) such that for all \( g \in SL₂(\mathbb{R}) \) we have \( \phi(g) = j(g,i)^{-k}f(g·i) \), where as usual \( j(g,z) = cz + d \) for \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL₂(\mathbb{R}) \) and \( z \in \mathbb{H} \). In particular, \( \phi \) is bounded and has a finite \( L^2 \) norm, hence one may be interested in asking how its \( L^\infty \) and its \( L^2 \) norm relate. In the level aspect, one traditionally asks for bounds for \( \| \phi \|_\infty = \sup_g |\phi(g)| = \sup_z |y^{k/2}f(z)| \) depending on \( N \) as \( \| \phi \|_2 \) is fixed. Subsequent investigations have shown that it is relevant for this problem to also take into account the conductor \( C = \prod_p p^{c_p} \) of \( \chi \). Assuming that \( \phi \) is \( L^2 \)-normalized, the “trivial bound” is

\[
1 \ll \| \phi \|_\infty \ll N^{1/2+\epsilon}
\]

for any \( \epsilon > 0 \). Here and below, the implied constant may depend on \( \epsilon \) and on the archimedean parameters of \( \phi \). The upper bound in (1.1) does not appear to have been written down previously for general \( N \) and \( C \), but it can be deduced from the main result of [14] for instance.

For squarefree \( N \), the first non-trivial upper bound is due to Blomer and Holowinsky [4], and has been subject to several improvements by Harcos and Templier (and some unpublished work of Helfgott and Ricotta) culminating with the result of [6] which achieve the upper bound \( N^{1/4+\epsilon} \). For non-squarefree \( N \), the best result to date is due to Saha [14], but it significantly improves on the trivial bound only when \( \chi \) is not highly ramified (here and elsewhere we say \( \chi \) is highly ramified if \( c_p > \lceil \frac{n_p}{2} \rceil \) for some prime \( p \)). Indeed, if \( \chi \) is not highly ramified and \( N \) is a perfect square, then Saha’s result [14] gives an upper bound of

\[
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\]
Recent work of Hu and Saha (see [9], especially the last paragraph of their introduction) suggests that this bound may be further improved in the compact case. On the other hand, if $N = C$ and if $N$ is a perfect square, then Saha’s result [14] reduces to the trivial bound (1.1).

Templier was the first to provide evidence that the actual size of $\|\phi\|_\infty$ may depend on how ramified $\chi$ is. Namely, he proved in [17] that whenever $N = C$ we have

$$\|\phi\|_\infty \gg N^{-\epsilon} \prod_{p \mid N} p^{\frac{1}{8} \left\lceil \frac{N^2}{p} \right\rceil}.$$  

In particular, if $N$ is a square, then

$$\|\phi\|_\infty \gg N^{\frac{1}{4} - \epsilon}.$$  

We shall prove the following comparable upper bound, which improves on [14] when $\chi$ is highly ramified.

**Theorem 1.** Let $\pi$ be an unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\pi$. Let $N = \prod_p p^{n_p}$ be the conductor of $\pi$. Let $\phi \in \pi$ be an $L^2$-normalized newform. Then

$$\|\phi\|_\infty \ll_{\epsilon, \pi, \infty} N^{\delta + \epsilon} \prod_{p \mid N} p^{\frac{1}{8} \left\lceil \frac{N^2}{p} \right\rceil},$$

where $\delta$ is any bound towards the Ramanujan Conjecture for $\pi$.

Theorem 1 provides for the first time non-trivial upper bounds for general $N$ that do not get worse when the conductor $C$ varies. As a point of comparison, the main result of [14] had an additional factor of $\prod_p p^{\max\{0, n_p - \left\lceil \frac{N^2}{p} \right\rceil\}}$, which is larger than one precisely when $\chi$ is highly ramified. Furthermore, for $C = N$, in view of the lower bound (1.2) and assuming the Ramanujan Conjecture, our result is essentially optimal when $N$ is a square. Note that the Ramanujan Conjecture is known by work of Deligne and Serre for $\phi$ arising from a holomorphic cusp form, and otherwise $\delta = \frac{7}{64}$ is admissible [10].

**Remark 1.** In [14], the appeal to a bound towards the Ramanujan Conjecture is avoided by using Hölder inequality to estimate separately $L^2$ averages of the Whittaker newforms at primes at which the central character is ramified and moments of the coefficients $\lambda_{\pi}$ of the $L$-function attached to $\pi$. However, in our situation, we want to exploit the fact that the Whittaker coefficients are supported on arithmetic progressions of modulus $L$, say, as explained later. A similar technique as in [14] would thus lead us to estimate moments of $\lambda_{\pi}$ on these arithmetic progressions. One might expect that these moments are approximately $L$ times smaller than the full moments, but such a result does not seem to be available.

Hence, if we were to bound them by positivity by the full moments, we would expect an overestimate of same order as $L$. Since estimates are known by Rankin-Selberg theory up to the eighth moments, and, as we shall see, $L \leq \prod_{c_p > \frac{N}{2}} p^{\left\lceil \frac{N^2}{p} \right\rceil}$, one should be able to replace $N^\delta$ in Theorem 1 with $\prod_{c_p > \frac{N}{2}} p^{\frac{1}{8} \left\lceil \frac{N^2}{p} \right\rceil}$, similarly as in Theorem 1.1 of [8]. However, for the sake of brevity, we do not carry out this argument.

The lower bound (1.2) has been generalized by Saha in [13] and subsequently by Assing in [3]. When $\chi$ is not maximally ramified, there is still a gap between the best known lower bound and the upper bound from Theorem 1. Finally, let us mention that the hybrid bounds over $\mathbb{Q}$ in [14], which combines the Whittaker expansion with some amplification, still beats
our result when $\chi$ is not highly ramified. For hybrid bounds over general number fields, we refer to the work of Assing [1, 2].

The proof proceeds by using Whittaker expansion to reduce the problem of bounding $\phi$ to that of understanding the local newforms attached to $\phi$. By making use of the invariances of $\phi$, we can restrict ourselves to evaluate these local newforms in the Whittaker model on some convenient cosets. The values of these local newforms have been computed [2, 3] by using a “basic identity” derived from the Jacquet-Langlands local functional equations which was first expressed in this form in [3]. In the non maximally ramified case, local bounds are slightly weaker than needed to obtain our result, and we take advantage of strong $L^2$-bounds due to Saha [4] instead.

Actually, we are using the Whittaker expansion of a certain translate of $\phi$, the “balanced newform”. The main feature is that it is supported on arithmetic progressions, which enables us to get some savings. Though we are working adelically, this fact can also be seen classically by computing the Fourier expansion of the corresponding cusp form at cusps of large width. The situation is somewhat analogous to [5], where the authors also get Whittaker expansions supported on arithmetic progressions.

Let us explain this analogy in the maximally ramified case – in which we get optimal upper bounds. As we shall see, in this case each local representation with ramified central character is of the form $\chi_1 \boxtimes \chi_2$, where $\chi_1$ has exponent of conductor $n_p$ and $\chi_2$ is unramified. Then the local balanced newform for $\pi$ is a twist of the local balanced newform for $\chi_1^{-1} \boxtimes 1$. For representations of this type, the local balanced newform coincides with the $p$-adic microlocal lift as defined in [12]. Now as explained in [8], the microlocal lift is the split analogue of the minimal vectors used there. Therefore the fact that we get optimal sup norm bounds in this case is the direct analogue of Theorem 1.1 of [8] which gives an optimal sup norm bound for automorphic forms of minimal type.

It is worth noticing that [8], [15] as well as the present work provide instances of the seemingly general principle according to which when considering very localized vectors, one is able to establish very good and sometimes optimal upper bounds. This is even the case when a Whittaker expansion is not available, as in [15].

The analysis of local newforms is given in Section 2. The proof of Theorem 1 is given in Section 3.

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2. Local bounds

In this section, $F$ will denote a non-archimedean local field of characteristic zero with residue field $\mathbb{F}_q$. Let $\mathfrak{o}$ denote the ring of integers of $F$ and $\mathfrak{p}$ its maximal ideal with uniformizer $t_p$. The discrete valuation associated to $F$ will be denoted by $v_p$. We define $U(0) = \mathfrak{o}^\times$, and for $k \geq 1$, $U(k) = 1 + \mathfrak{p}^k$. We fix an additive unitary character $\psi$ of $F$ with conductor $\mathfrak{o}$. In the sequel, the Whittaker models given will be those with respect to $\psi$.

2.1. Generalities.
2.1.1. \textit{Double coset decomposition.} Let $G = \text{GL}_2(F)$, $K = \text{GL}_2(\mathfrak{o})$. For $x \in F$ and $y \in F^\times$, consider the following elements

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, a(y) = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}, n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, z(y) = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}.$$  

Then define the following subgroups

$$N = n(F), A = a(F^\times), Z = z(F^\times),$$

and, for $a$ an ideal of $\mathfrak{o}$,

$$K^{(1)}(a) = K \cap \left[ 1 + \frac{a}{\mathfrak{o}} \right], K^{(2)}(a) = K \cap \left[ \frac{\mathfrak{o}}{a} \frac{\mathfrak{o}}{a} \right].$$

Note that for $a = p^n$, with $n$ a non-negative integer, we have

$$K^{(2)}(p^n) = [p^n \ 1] K^{(1)}(p^n) [p^n \ 1]^{-1}.$$  

From [13, Lemma 2.13], for any integer $n \geq 0$ we have the following double coset decomposition

$$G = \bigsqcup_{m \in \mathbb{Z}} \bigsqcup_{\ell = 0}^{n} \bigsqcup_{\nu \in \mathfrak{o}^\times/(1 + p\ell n)} Z N g_{m,\ell,\nu} K^{(1)}(p^n),$$

where $\ell_n = \min \{ \ell, n - \ell \}$, and

$$g_{m,\ell,\nu} = a(t_p^m) u_n(t_p^{-\ell} \nu) = \begin{bmatrix} 0 & t_p^m \\ -1 & -t_p^{-\ell} \nu \end{bmatrix}. $$

\textbf{Definition 1.} Assume $n \geq 0$ is a fixed integer. Then for any $g \in G$ we define

$$(m(g), \ell(g), \nu(g)) \in \mathbb{Z} \times \{ 0, \ldots, n \} \times \mathfrak{o}^\times/(1 + p^{\ell(g)n})$$

as the unique triple such that

$$g \in Z N g_{m(g),\ell(g),\nu(g)} K^{(1)}(p^n).$$

\textbf{Remark 2.} Any $g \in \text{GL}_2(F)$ belongs to some $Z N a(y) \kappa$ where $\kappa = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathfrak{o})$. Then by Remark 2.1 of [14], we have $\ell(g) = \min \{ v_p(c), n \}$ and $m(g) = v_p(y) - 2\ell(g)$. In particular, if $g$ is already an element of $\text{GL}_2(\mathfrak{o})$, then $g$ is in a coset of the form $g_{-2j,j,*}$. 

Now we determine the double cosets corresponding to certain elements of interest for the global application.

\textbf{Lemma 1.} Consider two integers $0 \leq e \leq n$. Let $g \in \text{GL}_2(\mathfrak{o}) a(t_p^e)$. Then there exist a non-negative integer $\ell \leq n$ and $\nu \in \mathfrak{o}^\times$ such that one of the following holds

1. either $\ell \leq e$ and $g \in Z N g_{-e,\ell,\nu} K^{(1)}(p^n)$,
2. or $e < \ell \leq n$ and $g \in Z N g_{-2\ell + e,\ell,\nu} K^{(1)}(p^n)$,

where the subgroup $K^{(1)}(p^n)$ is defined in (2.1).
Proof. We know by (2.3) that \( g \in ZN g_{m, e, \nu} k_1 \) for some \( k_1 \in K^{(1)}(p^n) \) hence

\[
gk_1^{-1}a(t_p^{-e}) \in ZN g_{m, e, \nu} a(t_p^{-e}).
\]

Since \( g \in Ka(t_p^e) \), it follows that \( gk_1^{-1}a(t_p^{-e}) \in K \). By Remark 2, it is then in the coset of some \( g_{-2j, j, \ast} \) with \( 0 \leq j \leq n \). On the other hand,

\[
g_{m, e, \nu} a(t_p^{-e}) = a(t_p^m) w_n(t_p^{-e}) a(t_p^{-e})
\]

\[
= a(t_p^m) w a(t_p^{-e}) n(t_p^{-e} \nu)
\]

\[
= t_p^{-e} a(t_p^{m+e}) w_n(t_p^{e-\ell} \nu).
\]

If \( \ell \leq e \) then

\[
w_n(t_p^{e-\ell} \nu) = \begin{pmatrix} -1 & \frac{1}{-t_p^{e-\ell} \nu} \\ -1 & -t_p^{e-\ell} \nu \end{pmatrix} \in \text{GL}_2(\mathfrak{o})
\]

so by Remark 2, \( a(t_p^{m+e}) w_n(t_p^{e-\ell} \nu) \) is in the coset of \( g_{m+e, 0, \ast} \). So in this case, \( g_{-2j, j, \ast} = g_{m+e, 0, \ast} \) thus \( m = -e \) and we find that

\[
g \in ZN g_{-e, e, \nu} K^{(1)}(p^n).
\]

Otherwise \( a(t_p^{m+e}) w_n(t_p^{e-\ell} \nu) = g_{m+e, \ell, e, \nu} \), therefore \( g_{-2j, j, \ast} = g_{m+e, \ell, e, \nu} \) and we get \( m + e = -2(\ell - e) \), so

\[
g \in ZN g_{-2e+e, e, \nu} K^{(1)}(p^n).
\]

\( \square \)

2.1.2. Characters and representations. For \( \chi \) a character of \( F^\times \), we denote by \( a(\chi) \) the exponent of the conductor of \( \chi \), that is the least non-negative integer \( n \) such that \( \chi \) is trivial on \( U(n) \). For \( \pi \) an irreducible admissible representation of \( G \), we also denote by \( a(\pi) \) the exponent of the conductor of \( \pi \), that is the least non-negative integer \( n \) such that \( \pi \) has a \( K^{(1)}(p^n) \)-fixed vector. The central character of \( \pi \) will be denoted by \( \omega_\pi \).

2.1.3. The local Whittaker newform. Fix \( \pi \) a generic irreducible admissible unitarizable representation of \( G \). From now on, we fix \( n = a(\pi) \), and we shall assume that \( \pi \) is realized on its Whittaker model.

**Definition 2.** The normalized newform \( W_\pi \) attached to \( \pi \) is the unique \( K^{(1)}(p^n) \)-fixed vector such that \( W_\pi(1) = 1 \).

The normalized conjugate-newform \( W_\pi^* \) attached to \( \pi \) is the unique \( K^{(2)}(p^n) \)-fixed vector such that \( W_\pi^*(1) = 1 \).

**Remark 3.** By (2.2), the function

\[
g \mapsto W_\pi \left( g \begin{bmatrix} t_p^n & 0 \\ 0 & 1 \end{bmatrix} \right)
\]

is \( K^{(2)}(p^n) \)-invariant. Thus there exists a complex number \( \alpha_\pi \) such that

\[
W_\pi \left( \cdot \begin{bmatrix} t_p^n & 0 \\ 0 & 1 \end{bmatrix} \right) = \alpha_\pi W_\pi^*.
\]

In addition, we have \( W_\pi^*(g) = \omega_\pi(\det(g)) W_{\tilde\pi}(g) \), where \( \tilde\pi \) is the contragradient representation to \( \pi \). Altogether, we get that

\[
W_\pi \left( \cdot \begin{bmatrix} t_p^n & 0 \\ 0 & 1 \end{bmatrix} \right) = \alpha_\pi \omega_\pi(\det(g)) W_{\tilde\pi}(g).
\]
One can even show that $|\alpha_\pi| = 1$ (see [13, Lemma 2.17], or [13, Proposition 2.28] for an exact formula in terms of $\varepsilon$-factors). Also note the following identity

\begin{equation}
 n((t_p^\ell + m)^{-1})z((t_p^\ell - n)^{-1})g_{m,\ell,\nu} \left[ \begin{array}{c} t_p^\ell \\ t_p^n \end{array} \right] = g_{m+2\ell-n,n-\ell,\nu} \left[ \begin{array}{c} 1 \\ -\nu^{-2} \end{array} \right],
\end{equation}

which, combined with the above, enables one to restrict attention to those cosets satisfying $\ell \leq \frac{n}{2}$, at the price of changing $\pi$ to $\tilde{\pi}$.

Assing has computed the local Whittaker newforms in great generality, and estimated them using the $p$-adic stationary phase method [2, 3]. Let us briefly explain the basic ideas of his method. For any fixed $m \in \mathbb{Z}$ and $0 \leq \ell \leq n$ the function on $\mathfrak{o}^\times$ given by $\nu \mapsto W_{\tilde{\pi}}(g_{m,\ell,\nu})$ only depends on $\nu \mod (1 + p^\ell)$. Thus, by Fourier inversion, there exist complex numbers $c_{m,\ell}(\mu)$ such that

$$W_{\tilde{\pi}}(g_{m,\ell,\nu}) = \sum_{\mu \in \bar{X}(\ell)} c_{m,\ell}(\mu) \mu(\nu),$$

where $\bar{X}(\ell)$ is the set of characters $\mu$ satisfying $\mu(t_p) = 1$ and $a(\mu) \leq \ell$.

Then, one may reformulate the Jacquet-Langlands local functional equation as an equality of power series in the variable $q^s$ whose coefficients involve on one side the Fourier coefficients $c_{m,\ell}(\mu)$ one is interested in, and on the other side Gauss sums and values of the local newform at some diagonal matrices, both of which are known [16]. This is the content of [13, Proposition 2.23]. By identifying the coefficients of the power series appearing in both side, one is then able to compute inductively the coefficients $c_{m,\ell}(\mu)$, and, from there, the values of the local newform on each double coset.

This can be done for each local representation $\pi$, however Lemma 2 below (same as [13, Lemma 2.36]) will enable us to restrict ourselves to principal series representations. By Remark 3 we can further restrict ourselves to the situation $\ell \leq \frac{n}{2}$. Finally, as we mentioned earlier, in our global application we shall use Saha’s strong $L^2$-bound [14], so what we are really interested in this section is only the support of the local newforms.

**Lemma 2.** Assume $a(\omega_{\pi}) > \frac{a(\pi)}{2}$. Then $\pi = \chi_1 \boxplus \chi_2$, where $\chi_1$ and $\chi_2$ are unitary characters with respective exponents of conductors $a_1 = a(\omega_{\pi})$ and $a_2 = n - a(\omega_{\pi})$.

In the rest of this section, we shall only consider the case $a(\omega_{\pi}) > \frac{a(\pi)}{2}$, as the main point of our global application is to take advantage of primes at which the central character is highly ramified. Thus for our purpose, we only have to consider $\pi = \chi_1 \boxplus \chi_2$ with $a_2 < \frac{n}{2} < a_1$, where from now on we denote $a_1 = a(\chi_1)$ and $a_2 = a(\chi_2)$. We first state the case of maximally ramified principal series.

**Lemma 3.** Let $\pi$ be a generic irreducible admissible unitarizable representation of $G$ with exponent of conductor $a(\pi) = n > 1$. Assume $a(\omega_{\pi}) = a(\pi)$. Then there exists $\nu_1 \in \mathfrak{o}^\times$ such that for all $m \in \mathbb{Z}$ and for $0 \leq \ell \leq \frac{n}{2}$, we have

$$|W_{\tilde{\pi}}(g_{m,0,\nu})| = \mathbb{1}_{m \geq -n} q^{\frac{a_1}{2}},$$

$$|W_{\tilde{\pi}}(g_{-n-\ell,\nu})| = \begin{cases} q^\frac{a_1}{2} & \text{if } \nu \in \nu_1 + p^\ell, \\ 0 & \text{if } \nu \notin \nu_1 + p^\ell, \end{cases}$$

and if $0 < \ell < m$ and $m + \ell \neq -n$ then $W_{\tilde{\pi}}(g_{m,\ell,\nu}) = 0.$
Proof. This follows from Lemma 3.4 and proof of Lemma 5.8 in [3].

In particular, one sees that in this case the local Whittaker newform is essentially supported on arithmetic progressions. The case $1 \leq a_2 < \frac{n}{2} < a_1$ is a bit more complicated, but one may obtain a result similar in flavour. Work of Assing [2] gives precise bounds for the local newform, however these local bounds are slightly weaker than what we need for our global application. Consequently, we only give here statements regarding the support of the local newform, and we shall rely on strong bounds for the $L^2$ mass [11].

**Lemma 4.** Let $\pi$ be a generic irreducible admissible unitarizable representation of $G$ with exponent of conductor $n > 1$. Assume $\frac{n}{2} < a(\omega_\pi) < n$. Set $a_1 = a(\omega_\pi)$ and $a_2 = n - a_1$. Assume moreover $F = \mathbb{Q}_p$. There exists $\nu_1 \in \mathfrak{o}^\times$ such that if $m \in \mathbb{Z}$ and $0 \leq \ell \leq \frac{n}{2}$, then have $W_\pi(g_{m,\ell,\nu}) = 0$ unless one of the following holds:

1. $\ell < a_2$ and $m = -n$,
2. $\ell = a_2$ and $m \geq -n$,
3. $\ell > a_2$, $m = -a_1 - \ell$ and $\nu \in \nu_1^{-1} + t_p^{\ell-a_2} \mathfrak{o}^\times$

Proof. This follows almost directly from inspection of the cases in Lemma 3.4.12 in [2]. Since we are taking $F = \mathbb{Q}_p$, the quantity $\kappa_F$ defined in [2] equals one, so the only bothersome case is $a_2 < \ell \leq \frac{a_1+a_2}{2}$ when $a_2 = 1$. By [2 Lemma 3.3.9], for $a_2 < \ell < a_1$ we must have $m = -a_1 - \ell$, so it only remains to see that the congruence condition also holds. If $\ell \leq \frac{a_1}{2}$, this follows from Case I of the proof of Lemma [2 Lemma 3.4.12]. The only remaining case is thus $\ell = \frac{1+a_1}{2}$, which only occurs for $a_1$ odd, hence $a_1 \geq 3$, so $a_1 - a_2 \geq 2\kappa_F$. As seen from Case VI.2 of the proof, this last condition is enough to get the congruence condition. \[\Box\]

2.2. **Archimedean case.** The local representation at the infinite place is a generic irreducible admissible unitary representation $\pi$ of $\text{GL}_2(\mathbb{R})$. Let $\psi$ be the additive character of $\mathbb{R}$ given by $\psi(x) = e^{2i\pi x}$. The lowest weight vector in the Whittaker model with respect to $\psi$ is given by

$$W_\pi(n(x)a(y)) = e^{2i\pi x}\kappa(y),$$

where $\kappa$ is determined by the form of the representation $\pi$. We shall use that for $y \in \mathbb{R}^\times$

$$\kappa(y) \ll |y|^{-c}e^{-2\pi + \epsilon]|y|, \quad (2.6)$$

uniformly in $y$. To see this, let us examine the possibilities for $\pi$.

2.2.1. **Principal series representations.** If $\pi = \chi_1 \boxplus \chi_2$, where $\chi_i = \text{sgn}^{m_i} \cdot |s_i|$ with $0 \leq m_2 \leq m_1 \leq 1$ integers and $s_1 + s_2 \in i\mathbb{R}$ and $s_1 - s_2 \in i\mathbb{R} \cup (-1, 1)$ then the lowest weight vector is given by

$$\kappa(y) = \begin{cases} 
\text{sgn}(y)^{m_1}|y|^\frac{s_1+s_2}{2}K_{\frac{s_1-s_2}{2}}(2\pi|y|) & \text{if } m_1 = m_2 \\
|y|^\frac{s_1+s_2}{2}|y| \left(K_{\frac{s_1-s_2-1}{2}}(2\pi|y|) + \text{sgn}(y)K_{\frac{s_1-s_2+1}{2}}(2\pi|y|) \right) & \text{if } m_1 \neq m_2,
\end{cases}$$

where $K_\nu$ is the $K$-Bessel function of index $\nu$. By [5 Proposition 7.2], we have the following estimate.

**Lemma 5.** Let $\sigma > 0$. For $\Re(\nu) \in (-\sigma, \sigma)$ we have

$$K_\nu(u) \ll_{\nu} \begin{cases} 
u^{-\sigma+\epsilon} & \text{if } 0 < u \leq 1 + \frac{\pi}{2} \Im(\nu), \\
\nu^{-\frac{\sigma}{2}}e^{-u} & \text{if } u > 1 + \frac{\pi}{2} \Im(\nu).
\end{cases}$$

In particular, taking $\sigma = \frac{1}{2}$ if $m_1 = m_2$ and $\sigma = 1$ otherwise, (2.6) follows in this case.
2.2.2. **Discrete series representations.** If $\pi$ is the unique irreducible subrepresentation of $\chi_1 \boxplus \chi_2$, where $\chi_i = \text{sgn}^{m_i}|^\chi_i$ with $0 \leq m_2 \leq m_1 \leq 1$ integers and $s_1 + s_2 \in i\mathbb{R}$ and $s_1 - s_2 \in \mathbb{Z}_{>0}$, $s_1 - s_2 \equiv m_1 - m_2 + 1 \mod 2$, then the lowest weight vector is given by

$$\kappa(y) = |y|^{s_1+s_2} y^{\frac{s_1-s_2+1}{2}} (1 + \text{sgn}(y)) e^{-2\pi y},$$

and we see that it satisfies again the estimate (2.6).

3. **Global computations**

3.1. **Notations.** Let $A_{\mathbb{Q}}$ denote the ring of ad` eles of $\mathbb{Q}$ and let $\psi$ be the unique additive character of $A_{\mathbb{Q}}$ that is unramified at each finite place and equals $x \mapsto e^{2\pi i x}$ at $R$. For any local object defined in Section 2, we use the subscript $_p$ to denote this object defined over $\mathbb{Q}_p$. We also fix in all the sequel

$$(3.1) \quad \Gamma_\infty = \text{SO}_2(\mathbb{R}).$$

Let $\pi = \otimes_{p \leq \infty} \pi_p$ be a unitary cuspidal automorphic representation of $\text{GL}_2(A_{\mathbb{Q}})$ with central character $\omega_\pi$. Let $N = \prod_p p^{n_p}$ be the conductor of $\pi$ and let $C = \prod_p p^{c_p}$ be the conductor of $\omega_\pi$. In particular $C | N$. Let us introduce some notation to denote respectively the set of primes for which Lemma 2 do or do not apply, namely

$$(3.2) \quad \mathcal{H} = \{ p \mid N : c_p > \frac{n_p}{2} \} \quad \text{and} \quad \mathcal{L} = \{ p \mid N : c_p \leq \frac{n_p}{2} \}.$$ 

We also denote by $S_N$ the set of prime numbers dividing $N$, so that $S_N = \mathcal{H} \cup \mathcal{L}$.

Then according to Lemma 2, $\pi_p$ is an irreducible principal series representation for each prime $p \in \mathcal{H}$, and we have corresponding local exponents of conductors $a_1(p) = c_p$ and $a_2(p) = n_p - c_p$. Finally, for any set of primes $\mathcal{P}$, define $\Psi(\mathcal{P})$ to be the set of positive integers having all their prime divisors among $\mathcal{P}$. We shall use the following obvious result.

**Lemma 6.** Let $\mathcal{P}$ be a finite set of primes. Then for all $0 < \alpha \leq \frac{1}{\log(2)}$ we have

$$\sum_{s \in \Psi(\mathcal{P})} s^{-\alpha} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-\alpha}} \leq \left( \frac{2}{\alpha \log 2} \right)^{\#\mathcal{P}}.$$ 

3.2. **The Whittaker expansion.** Let $\phi \in \pi$ be an $L^2$-normalized newform. Define the global Whittaker newform on $\text{GL}_2(A_{\mathbb{Q}})$ by

$$W_\phi(g) = \int_{Q \backslash \text{Ad}} \phi(n(x)g) \psi(-x) dx.$$ 

It factors as

$$W_\phi(g) = c_\phi \prod_{p \leq \infty} W_p(g_p),$$

where $W_p$ are as defined in the first two sections, and $c_\phi$ is a constant that satisfies

$$2\zeta(2)c_\phi^2 \prod_{p \leq \infty} W_p^2 \text{reg} = 1,$$

with

$$\prod_{p \leq \infty} W_p^2 \text{reg} = L(\pi, \text{Ad}, 1) \prod_{p \leq \infty} \frac{\zeta_p(2)\|W_p\|_2}{\zeta_p(1)L_p(\pi, \text{Ad}, 1)},$$
see [11, Lemma 2.2.3]. In turn, we have the Whittaker expansion
\begin{equation}
\phi(g) = \sum_{q \in \mathbb{Q}^\times} W_\phi(a(q)g) = c_\phi \sum_{q \in \mathbb{Q}^\times} \prod_{p \leq \infty} W_p(a(q)g_p)
\end{equation}
for any $g \in \text{GL}_2(\mathbb{A}_\mathbb{Q})$. Our strategy to bound $\|\phi\|_\infty$ will be to bound for all $g$
\begin{equation}
|c_\phi| \sum_{q \in \mathbb{Q}^\times} \prod_{p \leq \infty} |W_p(a(q)g_p)| \geq |\phi(g)|,
\end{equation}
that is, we do not take advantage of the potential oscillations in the Whittaker expansion. First, we give a bound for the constant $c_\phi$ appearing here. By [7] we have
\begin{equation}
L(\pi, \text{Ad}, 1) \gg N^{-\epsilon}.
\end{equation}
For $p$ unramified,
\begin{equation}
\frac{\zeta_p(2\|W_p\|_2)}{\zeta_p(1)L_p(\pi, \text{Ad}, 1)} = 1.
\end{equation}
For $p$ ramified, we have
\begin{equation}
L_p(\pi, \text{Ad}, 1) \asymp 1 \text{ and } 1 \leq \|W_p\|_2 \leq 2
\end{equation}
(see [13, Lemma 2.16]). Consequently, $|c_\phi| \ll N^\epsilon$. We shall also use that for any integer $n$ coprime to $N$, we have
\begin{equation}
\prod_{p|N} W_p(a(n)) = n^{-\frac{1}{2}}\lambda_\pi(n),
\end{equation}
where $\lambda_\pi(n)$ is the $n$-th coefficient of the finite part of the $L$-function attached to $\pi$.

3.3. Generating domains. Using invariances of automorphic forms, we can restrict their argument to lie in some convenient set of representatives. We first describe such generating domains.

\textbf{Definition 3.} We denote by $\mathcal{D}_N$ be the set of $g \in \text{GL}_2(\mathbb{A}_\mathbb{Q})$ such that
- $g_\infty = n(x)a(y)$ for some $x \in \mathbb{R}$ and $y \geq \frac{\sqrt{3}}{2}$,
- $g_p = 1$ for all $p \nmid N$,
- $g_p \in \text{GL}_2(\mathbb{Z}_p)$ for all $p$.

\textbf{Lemma 7.} Let $\Gamma = \prod_{p \leq \infty} \Gamma_p$ be a subgroup of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $\Gamma_\infty = \text{SO}_2(\mathbb{R})$, for all finite $p$ the group $\Gamma_p$ is an open subgroup of $\text{GL}_2(\mathbb{Z}_p)$ whose image by the determinant map is $\mathbb{Z}_p^\times$, and $\Gamma_p = \text{GL}_2(\mathbb{Z}_p)$ for $p \nmid N$. Then the subset $\mathcal{D}_N$ of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ given by Definition 3 contains representatives of each double coset of $\mathbb{Z}(\mathbb{A}_\mathbb{Q}) \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_\mathbb{Q}) / \Gamma$.

\textbf{Proof.} By the strong approximation theorem, any $g \in \text{GL}_2(\mathbb{A}_\mathbb{Q})$ can be written as $g_\infty \gamma k$ with $g_\infty \in \text{GL}_2^+(\mathbb{R})$, $\gamma \in \text{GL}_2(\mathbb{Q})$, and $k \in \Gamma$. Multiplying on the left by $\gamma^{-1}$ and on the right by $k^{-1}$, we can first assume that $g_\infty = 1$ for all finite $p$. Next, let $z = g_\infty \cdot i$. Then there is $\sigma \in \text{SL}_2(\mathbb{Z})$ such that $\Im(\sigma \cdot z) \geq \frac{\sqrt{3}}{2}$. After multiplying on the left by $\sigma$ and on the right by $\prod_{p|N} \sigma^{-1}$, we can instead assume that $g_p = 1$ for all $p \nmid N$, $g_p \in \text{GL}_2(\mathbb{Z}_p)$ for $p|N$, and $\Im(g_\infty \cdot z) \geq \frac{\sqrt{3}}{2}$. Finally, multiplying by an element of $\text{SO}_2(\mathbb{R})$, we can assume that $g_\infty$ is of the form $n(x)a(y)$ with $y \geq \frac{\sqrt{3}}{2}$.

Instead of evaluating our newform $\phi$ on elements of our generating domain $\mathcal{D}_N$, we shall rather use it with a certain translate of $\phi$, the “balanced newform”.
Lemma 8. Consider the subgroup $K^{(1)}$ of $GL_2(\mathbb{A}_Q)$ defined by $K^{(1)} = \Gamma_\infty \prod_{p < \infty} K^{(1)}(p^{n_p} \mathbb{Z}_p)$, where the local subgroups $\Gamma_\infty$ and $K^{(1)}(p^{n_p} \mathbb{Z}_p)$ are defined in (3.1) and (2.1) respectively. For each prime $p$ dividing $N$, let $e_p$ be an integer with $0 \leq e_p \leq n_p$. Let $D_N$ be the subset of $GL_2(\mathbb{A}_Q)$ given by Definition 8. Then the set $D_N \prod_{p \mid N} a(p^{e_p}) \subset GL_2(\mathbb{A}_Q)$ contains representatives of each double coset of $Z(\mathbb{A}_Q) GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_Q) / K^{(1)}$.

Proof. Let

$$
\Gamma_p = GL_2(\mathbb{Z}_p) \cap \left[ 1 + \frac{p^{n_p} \mathbb{Z}_p}{p^{n_p - e_p} \mathbb{Z}_{p^e}} \right],
$$

and $\Gamma = \prod_{p \leq \infty} \Gamma_p$. Let $g \in GL_2(\mathbb{A}_Q)$. By Lemma 7 there exists $g_d \in D_N$ such that we have the following equality of double cosets

$$
Z(\mathbb{A}) GL_2(\mathbb{Q}) g \prod_p a(p^{-e_p}) \Gamma = Z(\mathbb{A}) GL_2(\mathbb{Q}) g_d \Gamma.
$$

In particular, for each $p \mid N$ there exists $k_p \in \Gamma_p$ such that

$$
Z(\mathbb{A}) GL_2(\mathbb{Q}) g \prod_p a(p^{-e_p}) = Z(\mathbb{A}) GL_2(\mathbb{Q}) g_d k_p.
$$

Now if

$$
k_p = \left[ \begin{array}{cc} 1 + ap^{n_p} & b p^{e_p} \\ cp^{n_p - e_p} & d \end{array} \right],
$$

then

$$
a(p^{-e_p}) k_p a(p^{e_p}) = \left[ \begin{array}{cc} 1 + ap^{n_p} & b \\ cp^{n_p} & d \end{array} \right] \in K^{(1)}(p^{n_p} \mathbb{Z}_p).
$$

Hence writing

$$
Z(\mathbb{A}) GL_2(\mathbb{Q}) g = Z(\mathbb{A}) GL_2(\mathbb{Q}) g_d \prod_{p \mid N} k_p a(p^{e_p})
$$

$$
= Z(\mathbb{A}) GL_2(\mathbb{Q}) (g_d \prod_{p \mid N} a(p^{e_p})) \prod_{p \mid N} a(p^{-e_p}) k_p a(p^{e_p}),
$$

we find that the double coset $Z(\mathbb{A}) GL_2(\mathbb{Q}) g K^{(1)}$ contains the element $g_d \prod_{p \mid N} a(p^{e_p}) \in D_N \prod_{p \mid N} a(p^{e_p})$. □

By Lemma 8 we can restrict ourselves to evaluate $|\phi|$ on $D_N \prod_{p} a(p^{e_p})$, where the exponents $e_p$ may be conveniently chosen. Of course, this is equivalent to evaluate its right translate by $\prod_{p} a(p^{e_p})$ on $D_N$. Now, by Lemma 1 of Section 2 we can describe this generating domain in terms of the explicit representatives corresponding to each local double coset decomposition.

Lemma 9. Let $D_N$ be the subset of $GL_2(\mathbb{A}_Q)$ given by Definition 8. Let $g \in D_N \prod_{p \mid N} a(p^{e_p}) \subset GL_2(\mathbb{A}_Q)$. Then $g$ satisfies the following.

- $g_\infty = n(x) a(y)$ for some $x \in \mathbb{R}$ and $y \geq \frac{\sqrt{3}}{2}$,
- $g_p = 1$ for all $p \nmid N$,
- Let $p \mid N$. If $\ell(g_p) \leq e_p$ then $m(g_p) = -e_p$, and if $\ell(g_p) > e_p$ then $m(g_p) = -2\ell(g_p) + e_p$, where we have used notations of Definition 7.

Proof. This follows immediately from Definition 8 and Lemma 7. □
Proof. For each $\| (3.5) $ where we have

Definition 4. Let $\mathcal{I}_N$ be the set of $g \in \text{GL}_2(\mathbb{A}_F)$ such that

- $g_x = n(x)a(y)$ for some $x \in \mathbb{R}$ and $y \geq \sqrt{2}$,
- $g_y = 1$ for all $p \nmid N$,
- for all $p \mid N$ we have $\ell(g_p) \leq \frac{p}{2}$ and $m(g_p) \in \{ -\frac{p}{2}, -\frac{p}{2} \}$.

Remark 4. Note that for $p \mid N$ we do not require $g_p \in \text{GL}_2(\mathbb{Z}_p)a(p^\nu)$, but only the stated conditions about $\ell(g_p)$ and $m(g_p)$.

Finally, let us state the quantity we shall actually bound.

Lemma 10. Recall notations from §[3.7]. For each $S \subset S_N$, define

$$ \phi^S(g) = \phi \left( g \prod_{p \in S} \left[ p^{n_p} 1 \right] \right). $$

Then

$$ \| \phi \|_{\infty} = \max_{S \subset S_N} \sup_{g \in \mathcal{I}_N} |\phi^S(g)|. $$

Moreover, for each subset $S \subset S_N$ and for every $g \in \mathcal{I}_N$ we have

$$ |\phi^S(g)| \leq |c_\phi| \sum_{q \in \mathbb{Q}^\times} \left| \prod_{p \mid N} W^S_p(a(q)g_m(g_p),\ell(g_p),\nu(g_p)) \prod_{p \nmid N} W_p(a(q))W_\infty(a(q)n(x)a(y)) \right|, $$

where $W^S_p = W_p$ if $p \notin S$, and $W^S_p$ is the normalized local newform attached to the counter-gradient $\pi_p$ if $p \in S$.

Proof. For each $p \mid N$, set $e_p = \lfloor \frac{p}{2} \rfloor$. For convenience, also set $e'_p = \lceil \frac{p}{2} \rceil$. Then by Lemma 8 we have

$$ \| \phi \|_{\infty} = \sup_{g \in \mathcal{D}_N \prod_{p \mid N} a(p^\nu)} |\phi(g)| $$

We now prove that

$$ \sup_{g \in \mathcal{D}_N \prod_{p \mid N} a(p^\nu)} |\phi(g)| \leq \max_{S} \sup_{g \in \mathcal{I}_N} |\phi^S(g)|. $$

By Lemma 6 we have

$$ \sup_{g \in \mathcal{D}_N \prod_{p \mid N} a(p^\nu)} |\phi(g)| \leq \sup_{x \in \mathbb{R}, y \geq \frac{\sqrt{2}}{2}} \left| \phi \left( n(x)a(y)\prod_{p \mid N} g_p \right) \right|. $$

By $2.4$ we have

$$ g^{-2\ell_p+e_p,\ell_p,\nu_p} \left[ 1 - p^{-2} \right] = n(-p^{e_p-\ell_p}p^{\nu_p}) n(-p^{e_p-\ell_p}p^{\nu_p}) g^{-e_p,\nu_p} \left[ p^{n_p} 1 \right]. $$
Using this identity at each prime belonging to the set $S$ of primes $p$ satisfying $\ell(g_p) > \frac{n_p}{2}$ in the right hand side of (3.8) we obtain by right-$K^{(1)}$ invariance of $\phi$ (3.9)

$$\sup_{x \in \mathbb{R}, y \geq \frac{x}{2}} \left| \phi \left( n(x)a(y) \prod_{p|N} g_p \right) \right| \leq \max_{S \subseteq S_N} \sup_{x \in \mathbb{R}, y \geq \frac{x}{2}} \left| \phi^S \left( n(x)a(y) \prod_{p|N} g_p \right) \right|.$$ 

Combining (3.8), (3.9) and the definition of $I_N$, we obtain the bound (3.7). From definition, it is clear that

$$\|\phi\|_{\infty} \geq \max_{S \subseteq S_N} \sup_{g \in I_N} |\phi^S(g)|,$$

so (3.6) follows.

The second claim follows from the Whittaker expansion (3.3). Observe that by Remark 3

$$\left| W_p \left( g_p \begin{pmatrix} p^{n_p} & 1 \end{pmatrix} \right) \right| = |\tilde{W}_p(g_p)|,$$

where $\tilde{W}_p$ is the normalized local newform attached to the contragradient $\tilde{\pi}_p$. The identity

$$a(q)n(x) = n(qx)a(q)$$

and the left invariance of the modulus of the local Whittaker newforms by $NZ$ give (3.6). □

As we shall be interested in the support of the Whittaker expansion, we make now the following definition.

**Definition 5.** Keep notations as in Lemma 10. For every $S \subseteq S_N$ and $g \in I_N$ we define

$$\text{Supp}(g; S) = \left\{ q \in \mathbb{Q}^\times : \prod_{p|N} W_p^S(a(q)g_m(g_p), \ell(g_p), \nu(g_p)) \prod_{p|N} W_p(a(q))W_\infty(a(q)n(x)a(y)) \neq 0 \right\}.$$

**Notation 1.** From now on we fix $g \in I_N$ and $S \subseteq S_N$ (in the notations of § 3.1 and Definition 4), and we define for each $p|N$, $\ell_p = \ell(g_p)$, $\epsilon_p = -m(g_p)$, $\epsilon'_p = n_p - \epsilon_p$, and $\nu_p = \nu(g_p)$. We then define the following integers

$$L = \prod_{p|N} p^{f_p}, \quad N_1 = \prod_p p^{f_p}, \quad N_2 = \prod_p p^{f'_p},$$

as well as the set of primes

(3.10) \hspace{1cm} \mathcal{H}_- = \left\{ p \in \mathcal{H} : \ell_p < a_2(p) \right\}, \quad \mathcal{H}_+ = \left\{ p \in \mathcal{H} : \ell_p = a_2(p) \right\}, \quad \mathcal{H}_+ = \left\{ p \in \mathcal{H} : \ell_p > a_2(p) \right\},$$

where $a_2(p) = n_p - e_p$ is the exponent of the conductor of the local character $\chi_2$ (note that in the case where $N = C$, we have $\mathcal{H}_- = \emptyset$ and $\mathcal{H}_+$ coincides with the set of primes dividing $N$ and not dividing $L$). If $M = \prod_p p^{m_p}$ is any integer, we may use the notation

$$M^* = \prod_{p \in \mathcal{H}_+} p^{m_p}$$

for $* \in \{+, -, =\}$. 

3.4. **Sup norms: maximally ramified case.** In this subsection, we are assuming $N = C$ and we prove Theorem 1 in this special case, as the proof becomes simpler. We first determine the support of the “Whittaker expansion” (3.6).

**Lemma 11.** Recall Notation 7. There is a map 
$$\Psi(\mathcal{H}_\sim) \to \{1, \cdots, L\}$$

such that
$$\text{Supp}(g; S) \subseteq \left\{ \frac{s}{N^2 L} (s + jL) : s \in \Psi(\mathcal{H}_\sim), j \in \mathbb{Z} \text{ with } t_s + jL \text{ coprime to } N \right\}.$$ 

**Proof.** Let $q = \prod q_p p^{q_p} \in \mathbb{Q}_\times$. Assume $q \in \text{Supp}(g; S)$ First, if $p \nmid N$ then we must have $q_p \geq 0$. So $\text{sgn}(q) \prod_{p|N} p^{q_p}$ is an integer. We shall see that it satisfies a certain congruence condition. Consider now a prime $p \mid N$, if $q = p^{q_p} u \in \mathbb{Q}_\times$ with $u \in \mathbb{Z}_{p}^\times$, we have

\begin{equation}
(3.11) \quad a(q) g_{-\epsilon_p, \ell_p, \nu_p} = g_{q_p-\epsilon_p, \ell_p, \nu_p} u^{-1} \left[ \frac{1}{u} \right] = g_{q_p-\epsilon_p, \ell_p, \nu_p} u^{-1} \left[ \frac{1}{q_p} \right].
\end{equation}

By Lemma 3 (applied either to $\pi_p$ if $p \notin S$ or to $\pi_p$ if $p \in S$), if $\ell_p = 0$ then $q_p - \epsilon_p \geq -n_p$, so $q_p \geq -\epsilon_p$. It follows that

$$s \doteq \prod_{p|N, p \nmid L} \prod_{p|N} p^{q_p+\epsilon_p} \in \Psi(\mathcal{H}_\sim).$$

On the other hand, if $\ell_p > 0$ then $q_p - \epsilon_p = -n_p - \ell_p$, so $q_p = -\epsilon_p - \ell_p$. Now fix a prime $p_0 \mid L$ (so $\ell_{p_0} > 0$), and write

$$\text{sgn}(q) \prod_{p|N} p^{q_p} = \text{sgn}(q) \prod_{p|N} p^{q_p} \prod_{p|L} p^{q_p+\epsilon_p}$$

$$= \text{sgn}(q) \prod_{p|N} p^{q_p} \prod_{p|N} p^{q_p+\epsilon_p} \prod_{p|L} p^{q_p+\epsilon_p+\ell_p}$$

$$= \text{sgn}(q) \prod_{p \neq p_0} p^{q_p} \prod_{p|N} p^{q_p} \prod_{p|L} p^\epsilon \prod_{p \neq p_0} p^{\epsilon_p+\ell_p}$$

$$= \left( p_0^{-q_{p_0}} \right) \left( \prod_{p|N} p^{q_p} \prod_{p|L} p^{q_p+\epsilon_p} \prod_{p \neq p_0} p^{\epsilon_p+\ell_p} \right).$$

By Lemma 3 and equality (3.11), $p_0^{-q_{p_0}} q$ satisfies a certain congruence condition modulo $p_0^{q_{p_0}} \mathbb{Z}_{p_0}$. In addition, $\prod_{p|N} p^{\epsilon_p} \prod_{p|L} p^{\epsilon_p+\ell_p}$ is clearly in $\mathbb{Z}_{p_0}^\times$. So we just showed that the integer $\text{sgn}(q) \prod_{p|N} p^{q_p}$ satisfies a certain congruence condition modulo $p_0^{q_{p_0}}$. Applying the same reasoning with each prime dividing $L$, we obtain by the Chinese remainder theorem a condition of the type

$$\text{sgn}(q) \prod_{p|N} p^{q_p} \equiv r_0 \mod L.$$
Since in addition $L$ and $s$ are coprime, we can write

\[(3.12) \quad \text{sgn}(q) \prod_{p \mid N} p^{q_p} = t_s + jL \]

for some integer $t_s \equiv r_0 s^{-1} \mod L$, and $j$ ranging over $\mathbb{Z}$. Finally,

\[ q = \text{sgn}(q) \prod_{p \mid L} p^{-\ell_p} \prod_{p \mid N} p^{q_p} \prod_{p \mid N} p^{q_p} = \frac{s}{N_2L}(t_s + jL). \]

We now compute the size of each term in "the Whittaker expansion" (3.6).

**Lemma 12.** Keep notations from Notation 7 and Lemma 10. Let $q = \frac{s}{N_2L}(t_s + jL)$ as in Lemma 11. Then we have

\[ \left| \prod_{p \mid N} W_p^S(a(q)g_{-\epsilon_p,\ell_p,\nu_p}) \prod_{p \mid N} W_p(a(q)) \right| = L^\frac{1}{2} s^{-\frac{1}{2}} |t_s + jL|^{-\frac{1}{2}} |\lambda_\pi(|t_s + jL|)|. \]

**Proof.** For $q$ of this form, using (3.12) and (3.4), we have

\[ \prod_{p \mid N} W_p(a(q)) = \prod_{p \mid N} W_p(t_s + jL) \]

\[ = (|t_s + jL|)^{\frac{1}{2}} \lambda_\pi(|t_s + jL|), \]

and Lemma 3 (observe that the contragradient representation $\bar{\pi}_p$ satisfies the same hypothesis as $\pi_p$) together with equality (3.11) give

\[ \left| \prod_{p \mid N} W_p^S(a(q)g_{-\epsilon_p,\ell_p,\nu_p}) \right| = \left| \prod_{p \mid N} W_p^S(g_{q_p-\epsilon_p,\ell_p,\nu_p}p^{\nu_p}q^{-1}) \right| \]

\[ = L^\frac{1}{2} \prod_{\ell_p=0} p^{-\epsilon_p} = L^\frac{1}{2} s^{-\frac{1}{2}}. \]

By Combining Lemmas 11 and 12 the “Whittaker expansion” (3.6) is thus bounded above by

\[ c_\phi L^\frac{1}{2} \sum_{s \in \Psi(H_m)} s^{-\frac{1}{2}} \sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2} + \delta} \kappa \left( \frac{t_s + jL}{N_2L} s \right). \]
Using estimate (2.6), we first evaluate the $j$-sum as follows:

$$\sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2} + \delta + \epsilon} \left( \frac{t_s + jL}{N_2L} sy \right)$$

$$\ll \left( \frac{sy}{N_2L} \right)^{-\epsilon} \sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2} + \delta + \epsilon} \exp \left( -2\pi + \epsilon \frac{|t_s + jL|}{N_2L} sy \right)$$

$$\ll \left( \frac{sy}{N_2L} \right)^{-\epsilon} \left( 1 + \int_{\mathbb{R}} |tL|^{-\frac{1}{2} + \delta + \epsilon} \exp \left( -2\pi + \epsilon \frac{|t|}{N_2L} sy \right) dt \right)$$

$$\ll \left( \frac{N_2L}{sy} \right)^{\epsilon} \left( 1 + \left( \frac{N_2}{Lsy} \right)^{\frac{1}{2}} \left( \frac{N_2L}{sy} \delta \right) \right).$$

Altogether, using Lemma 6 we get

$$|\phi(g)| \ll c_\phi \left( \frac{N_2L}{y} \right)^{\epsilon} \left( L^{\frac{1}{2}} + \frac{N_2^{\frac{1}{2} + \delta} L^\delta}{y} \right) \ll N^\epsilon \left( L^{\frac{1}{2}} + N_2^{\frac{1}{2}} N^\delta \right)$$

since $c_\phi \ll N^\epsilon$, $y \geq \frac{\sqrt{2}}{2}$ and $N_2L \leq N$. This establishes Theorem 1 when $N = C$ because we have $L \leq N^{\frac{1}{2}}$ and $N_2 \leq \prod_{p|N} p^{\left\lceil \frac{n_p}{2} \right\rceil}$.

### 3.5. Sup norms: general ramification

Finally, let us address the necessary modifications when we do not make any assumption about the conductor of $\chi$. The analysis of the local Whittaker newform $W_p$ is similar, but with more cases to take into account, depending on which of the sets (3.2) the prime $p$ belongs. In particular, it still holds that for all $p \in \mathcal{H}$ we have $\pi_p = \chi_1 \boxplus \chi_2$, but the exponents $a_2(p) = n_p - c_p$ of the conductor of the local characters $\chi_2$ may not all equal zero. We thus also get a Whittaker expansion supported on arithmetic progressions dictated by the primes at which the central character is highly ramified. The rest of our argument differs from the maximally ramified case, as we rather use strong $L^2$-averages of the local newforms, in the spirit of [14], instead of the local bounds. Of course, in the maximally ramified case, these $L^2$-averages follow immediately from the computation of the support of the local newform $W_p$ and the local bound, so the difference on the argument is mainly expository.

We first determine the support of the “Whittaker expansion” (3.6) in this more general case.

**Lemma 13.** Recall Notation [7]. There is a map

$$\Psi(\mathcal{H}_+) \times \Psi(\mathcal{L}) \to \left\{ 1, \cdots, \frac{L^+C^+}{N^+} \right\}$$

$$(s, u) \mapsto t_{su}$$

such that

$$\text{Supp}(g; S) \subseteq \left\{ su \frac{N^+}{N_2L^+C^+} \left( t_{su} + j \frac{L^+C^+}{N^+} \right), s \in \Psi(\mathcal{H}_+), u \in \Psi(\mathcal{L}), j \in \mathbb{Z} \right\}$$

$$\text{with } t_{su} + j \frac{L^+C^+}{N^+} \text{ coprime to } N.$$

**Remark 5.** It is immediate by unravelling the definitions that $\frac{L^+C^+}{N^+}$ is an integer.
Proof. The reasoning is quite similar to the proof of Lemma 11 but we use Proposition 2.10 for the primes in $L$ and Lemma 4 instead of Lemma 3 for those primes in $H$. Fix $q = \prod_p p^{\ell_p} \in \text{Supp}(g; S)$. As before, $\text{sgn}(q) \prod_{p \mid N} p^{\ell_p}$ is an integer and we shall see it satisfies some congruence condition. If $p \in H_-$ or $p \in L$ then examination of either Lemma 4 or Proposition 2.10 gives $q_p \geq -\epsilon_p$. So

$$su = \prod_{p \in L \cup H_-} p^{\ell_p^{p_p} + \epsilon_p} \in \Psi(L \cup H_-).$$

In addition Lemma 4 gives that for $p \in H_-$ we have $q_p = -\epsilon_p$, and for $p \in H_+$ we have $q_p = \epsilon_p - \ell_p - a_1(p)$. Fix $p_0 \in H_+$ and write

$$\text{sgn}(q) \prod_{p \mid N} p^{\ell_p} = \text{sgn}(q) \prod_{p \mid N} p^{\ell_p} \prod_{p \in L \cup H_-} p^{\ell_p^{p_p} + \epsilon_p} \prod_{p \in H_+} p^{\ell_p^{p_p} - (\epsilon_p - \ell_p - a_1(p))} = \text{sgn}(q) \prod_{p \neq p_0} p^{\ell_p} \prod_{p \in L \cup H_+ \cup H_-} p^{\epsilon_p} \prod_{p \in H_+} p^{\ell_p + a_1(p) - \epsilon_p}.$$

By Lemma 4 $\text{sgn}(q) \prod_{p \neq p_0} p^{\ell_p}$ satisfies a congruence condition modulo $p^{\ell_{p_0} - a_2(p_0)} \mathbb{Z}_{p_0}$. Then using the Chinese remainder theorem we see that $\text{sgn}(q) \prod_{p \mid N} p^{\ell_p}$ is an integer satisfying a congruence condition modulo $\frac{L^+ C^+}{N^+}$. It follows that we can write

$$\text{sgn}(q) \prod_{p \mid N} p^{\ell_p} = t_{su} + \frac{L^+ C^+}{N^+}.$$

Finally,

$$q = \text{sgn}(q) \prod_{p \mid N} p^{\ell_p} \prod_{p \in L \cup H_-} p^{\ell_p} \prod_{p \in H_+} p^{\ell_p - \ell_p - a_1(p)} = \left(t_{su} + j \frac{L^+ C^+}{N^+} \right) \frac{su}{\prod_{p \in L} p^{\ell_p^{p_p}} N_2 \frac{N^+}{N_2^+}} \frac{1}{L + C^+}.$$

If we were now to proceed following the exact same strategy as in the maximally ramified case, then we would get a worse estimate because of weaker local bounds for the local newform in the case $\ell_p = a_2(p)$ (see Lemma 5.10]). Instead, we rely on $L^2$-averages of the local new vectors established by Saha [14]. To this end, we make first the following trivial lemma.

Lemma 14. Suppose $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}$ are two families of positive real numbers such that $\sum_{n \in \mathbb{Z}} a_n b_n$ converges absolutely, and $a_n$ is periodic with period $T$. Let $M$ be such that

$$\sum_{n=0}^{T-1} a_n^2 \leq M.$$ 

1meaning the partial sums $\sum_{n \in J} |a_n b_n|$ indexed by finite sets $J \subset \mathbb{Z}$ are uniformly bounded.
Then we have

\[ \sum_{n \in \mathbb{Z}} a_n b_n \leq M^{\frac{1}{2}} \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^{T-1} b_{Tk+j}^2 \right)^{\frac{1}{2}}. \]

Next, we express the “Whittaker expansion” (3.6) so as to be tackled by previous lemma.

**Lemma 15.** Recall Notation 1. Then

\[ |\phi_S(g)| \leq |c_\phi| \sum_{s \in \Psi(H_+)} \sum_{u \in \Psi(L)} \sum_{n \in \mathbb{Z}} a_n b_n, \]

where \( a_n \) is periodic with period \( L \) and satisfies

\[ \sum_{n=0}^{L-1} a_n^2 \ll N^s L(su)^{-\frac{1}{2}} \]

and

\[ b_n = |n|^{-\frac{1}{2}} \lambda_\pi(n) \kappa \left( \frac{N^s \text{sun} y}{N_2 L + C^+} \right) \mathbf{1}_{n \equiv \ell_{su} \mod \frac{L + C^+}{\pi}}. \]

**Proof.** The claim will follow from the “Whittaker expansion” (3.6)

\[ |\phi_S(g)| \leq |c_\phi| \sum_{q \in \mathbb{Q}^\times} \prod_{p \mid N} W^S_p (g_{q^p - \epsilon_p, \ell_p, \nu_p}^{q^p - 1}) \prod_{p \mid N} W_p(a(q)) W_\infty(a(q)n(x) a(y))), \]

together with (2.5), (3.4) and Lemma 13 once we have shown that the sequence defined by

\[ a_n = \prod_{p \mid N} W^S_p \left( a \left( \frac{N^s \text{sun} y}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right) \]

satisfies the desired properties. For each \( p \mid N \), let us distinguish cases depending on which of the sets defined in (3.2) and (3.10) contains \( p \). For all \( v \in \mathbb{Z}_p^\times \) we have

\[ W^S_p \left( a \left( \frac{N^s \text{sun} y}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right) = \begin{cases} W^S_p \left( a(v) g_{u_p - \epsilon_p, \ell_p, \nu_p} \right) & \text{if } p \in \mathcal{L} \\ W^S_p \left( a(v) g_{-\epsilon_p, \ell_p, \nu_p} \right) & \text{if } p \in \mathcal{H}_- \\ W^S_p \left( a(v) g_{s_p - \epsilon_p, \ell_p, \nu_p} \right) & \text{if } p \in \mathcal{H}_+ \\ W^S_p \left( a(v) g_{-\epsilon_p - a_1(p), \ell_p, \nu_p} \right) & \text{if } p \in \mathcal{H}_+. \end{cases} \]

where each * is independent of \( v \). By [14] Proposition 2.10, we then get

\[ \int_{v \in \mathbb{Z}_p^\times} \left| W^S_p \left( a \left( \frac{N^s \text{sun} y}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right) \right|^2 d^\times v \ll \begin{cases} p^{\frac{T}{2}} & \text{if } p \in \mathcal{L} \\ 1 & \text{if } p \in \mathcal{H}_- \\ p^{\frac{T}{2}} & \text{if } p \in \mathcal{H}_+ \end{cases} \]

Now by [14] Remark 2.12, for each \( p \mid N \) and each fixed \( s \in \Psi(\mathcal{H}_- \cup \mathcal{L}) \), the map on \( \mathbb{Z}_p^\times \) given by

\[ v \mapsto \left| W^S_p \left( a \left( \frac{N^s \text{sun} y}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right) \right| \]

...
is $U_p(\ell_p)$-invariant. Hence by the Chinese remainder theorem, these give rise to a map on $(\mathbb{Z}/L\mathbb{Z})^\times$ given by

$$(r \mod L) \mapsto \prod_{p|N} W_p^S \left( a \left( \frac{N^+ suy}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right),$$

and by Lemma 13 if $a_n \neq 0$ then $n$ is coprime to $N$, thus the sum (3.13) is just

$$\sum_{r \in (\mathbb{Z}/L\mathbb{Z})^\times} \prod_{p|N} W_p^S \left( a \left( \frac{N^+ suy}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right)^2 = \phi(L) \prod_{p|N} \int_{v \in \mathbb{Z}^\times} W_p^S \left( a \left( \frac{N^+ suy}{N_2 L + C^+} \right) g_{-\epsilon_p, \ell_p, \nu_p} \right)^2 d^x v,$$

where $\phi$ is Euler’s totient.

By combining Lemmas 14 and 15 it follows

$$(3.14) \quad |\phi(g)| \ll N^* L^\frac{1}{2} \sum_{s \in \Psi(H_\cdot)} s^{-\frac{1}{4}} \sum_{u \in \Psi(L)} u^{-\frac{1}{4}} \sum_{k \in \mathbb{Z}} S_k^2,$$

where

$$(3.15) \quad S_k = \sum_{j=0}^{L-1} b_{Lk+j}^2,$$

and $b_n$ is defined in Lemma 15.

**Lemma 16.** For all $k \geq 1$ the sum (3.13) satisfies

$$S_k \ll \frac{N^+}{L^+ C^+} L^{2\delta} \left( \frac{N}{suy} \right)^{\epsilon} k^{-1+2\delta+\epsilon} \exp \left( -\pi \frac{N^+ suy L}{N_2 L + C^+} k \right),$$

and the same estimate holds for $k \leq -2$ upon replacing $k$ with $-k - 1$ in the right hand side. Finally,

$$S_0, S_{-1} \ll \left( \frac{N_2}{suy} \right)^{\epsilon} \left( 1 + \frac{N^+}{L^+ C^+} \left( \frac{N_2 L + C^+}{N^+ suy} \right)^{2\delta+\epsilon} \right).$$

**Proof.** For those intervals $[kL, (k+1)L]$ not containing zero we use estimate (2.6) then we bound $S_k$ by the number of terms multiplied by the largest term. Since $\frac{L^+ C^+}{N^+}$ divides $L$, the congruence condition on $n = Lk + j$ modulo $\frac{L^+ C^+}{N^+}$ is equivalent to the same congruence condition on $j$. We thus get, for $k \geq 1$

$$S_k \ll \frac{N^+}{L^+ C^+} L^{2\delta} \left( \frac{N}{suy} \right)^{\epsilon} k^{-1+2\delta+\epsilon} \exp \left( -\pi \frac{N^+ suy L}{N_2 L + C^+} k \right).$$

For $k = 0$ we have

$$S_0 \ll \left( \frac{N}{suy} \right)^{\epsilon} \left( 1 + \int_0^\infty \left( t_{su} L^+ C^+ N^+ \right)^{-1+2\delta+\epsilon} \exp \left( -\pi \frac{N^+ suy}{N_2 L + C^+} \left( t_{su} + t \right) \frac{L^+ C^+}{N^+} \right) dt \right)$$

$$\ll \left( \frac{N}{suy} \right)^{\epsilon} \left( 1 + \frac{N^+}{L^+ C^+} \left( \frac{N_2 L + C^+}{N^+ suy} \right)^{2\delta+\epsilon} \right).$$

The analogous results for $k < 0$ follow by changing $k$ to $-k - 1$ and $t_{su}$ to $\frac{L^+ C^+}{N^+} - t_{su}$. \qed
By a similar argument as in §3.4 Lemma 16 implies

$$\sum_{k \in \mathbb{Z}} S_k^+ \ll \left( \frac{N}{\text{supy}} \right)^\epsilon \left( 1 + \left( \frac{N^+}{L+C^+} \right)^{\frac{1}{2}} \left( \frac{N_2 L^+}{N^+ \text{supy}} \right)^{\delta + \epsilon} + \left( \frac{N_2}{L \text{supy}} \right)^{\frac{1}{2}} \left( \frac{N_2 L^+}{N^+ \text{supy}} \right)^{\delta + \epsilon} \right)$$

Substituting this into (3.14) and using Lemma 6 we obtain

$$|\phi(g)| \ll N^{\delta + \epsilon} \left( L^{\frac{1}{2}} + N_2^{\frac{1}{2}} \right).$$

Lemma 10 together with the results from Section 3.4 and 3.5 finishes the proof of Theorem 1.

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