Einstein equation and Hawking radiation govern Universe evolution

She-Sheng Xue

ICRANet, Piazzale della Repubblica, 10-65122, Pescara,
Physics Department, Sapienza University of Rome, P.le A. Moro 5, 00185, Rome, Italy

We present a possible understanding to the issues of cosmological constant, inflation, matter and coincidence problems based on the Einstein equation and Hawking pair production of particles and antiparticles. In this scenario, the cosmological constant is attributed to the spacetime horizon that generates matter via pair productions, in turn the matter contributes to the RHS of the Einstein equation, conversely governing the spacetime horizon. In such a way, the cosmological and matter terms are interacting via the spacetime horizon in their evolutions. As a result, the inflation naturally appears and results agree to observations. The CMB large-scale anomaly can be explained and the dark-matter acoustic wave is speculated. The cosmological term $\Omega_\Lambda$ tracks down the matter term $\Omega_M$ from the reheating to the radiation-matter equilibrium, then it varies very slowly, $\Omega_\Lambda \propto$ constant. Thus the cosmic coincidence problem can be possibly avoided. The relation between $\Omega_\Lambda$ and $\Omega_M$ is obtained and can be examined at large redshifts.

PACS numbers:

Keywords: Gravitational field theory; Cosmological evolution.
Contents

I. Introduction 4

II. Einstein equation and Bianchi identity 6
   A. The role of cosmological Λ-term 6
   B. Generalized equations in Friedmann Universe 7

III. Pair production from spacetime 9
   A. Matter from spacetime 9
   B. Back-reaction and energy-momentum tensor 9

IV. Cosmic inflation 10
   A. Numerical approach to pre-inflation epoch 11
   B. Analytical approach to inflation epoch 12
      1. Analytical expressions for pair productions 12
      2. Inflation epoch and its end 13
   C. Comparison with observations 14
   D. Area law and large-scale anomaly 16

V. Pairs production and annihilation 17
   A. Rate equation and equilibrium 18
   B. A preliminary discussion of the reheating 19

VI. Cosmic coincidence 20
   A. Two kinds of matter contributions 20
   B. Coupled equations for Ω_M and Ω_Λ evolutions 21
      1. Indirect interaction between matter and “dark energy” via horizon 22
      2. Massive pair decay impacts on “dark energy” 23
   C. Ω_Λ − Ω_M relation and coincidence 24
      1. Radiation dominant “dark” epoch 24
      2. Matter dominant “light” epoch 25
   D. Possible observations 27

VII. Summary and remarks 27
VIII. Acknowledgment

References
I. INTRODUCTION

In the standard model of modern cosmology (ΛCDM), the cosmological constant, inflation, normal/dark matter and coincidence problem have been long standing issues since decades. There are various models and many efforts, have been made to approach these issues, and readers are referred to review articles and professional books, for example, see Refs. [1–5]. We present here a possible understanding to these issues, on the basis of the Einstein equation and the Hawking pair-production of particles and antiparticles. In this theoretical framework: the cosmological Λ-term is attributed to the spacetime horizon, which generates matter term via the Hawking pair production of particles and antiparticles; in turn the matter term contributes to the right-handed side of the Einstein equation, conversely governing the spacetime horizon. In such a way, the cosmological and matter terms are interacting via the spacetime horizon in their evolutions.

As one of fundamental theories for interactions in Nature, the classical Einstein theory of gravity, which plays an essential role in the standard model of modern cosmology (ΛCDM), should be realized in the scaling-invariant domain of a fixed point of its quantum field theory, analogously to other renormalizable gauge field theories in the standard model of particle physics. It was suggested [6] that the quantum field theory of gravity regularized at an ultraviolet (UV) cutoff might have a non-trivial UV-stable fixed point and asymptotic safety, namely the renormalization group (RG) flows are attracted into the UV-stable fixed point with a finite number of physically renormalizable operators for the gravitational field. The evidence of such UV-stable fixed point has been found in the different short-distance regularization frameworks [7–14], as well as our approach to the quantum gravity [15, 16] that will be discussed below.

The Einstein gravity should be considered as an effective field theory of two physically relevant area operators of the Ricci scalar $R$ and cosmological Λ-term,

$$A_{EC}^{\text{eff}} = \int \frac{d^4 x}{16\pi G} \text{det}(-g)(R - 2\Lambda) + \cdots,$$

in the scaling invariant domain of fixed points, where irrelevant high-dimensional operators ($\cdots$) are suppressed. The gravitation constant $G \sim \ell_{\text{pl}}^2 = M_{\text{pl}}^{-2}$ is the smallest area at the Planck cutoff, while the cosmological constant $\Lambda$ should represent an intrinsic scale of the area operator $R$ for the spacetime. It is a difficult problem to find a viable field theory of quantum gravity regularized at the Planck scale, and its fixed points where the scaling-invariant effective action (1) is achieved, and quantum fluctuations and their operators at short distances $\ell_{\text{pl}}$ are irrelevant and can be averaged with respect to the correlation length $\xi \gg \ell_{\text{pl}}$ at large distance. There are many approaches to
this problem and we briefly mention one of them. As the Wilson loop $\sim \exp(ie \oint_C A_\mu dx^\mu)$ in gauge theories, the regularized action of quantum gravity can be given by the diffeomorphism and local Lorentz gauge-invariant holonomy field \[15\],

$$X_C(e, \omega) = P_C \text{tr} \exp \left\{ i \tilde{g} \oint_C v_{\mu\nu}(x)\omega^\mu(x)dx^\nu \right\}. \quad (2)$$

in terms of tetrad and spin-connection fields $(e, \omega)$, where the dimensionless gravitational coupling $\tilde{g} \equiv \tilde{G}/G$ for varying $\tilde{G}$. There are two fixed points: (i) the UV unstable fixed point $g_{ir} \approx 0$ could relate to the inflation epoch; (ii) the UV-stable fixed point $g_{uv} \approx (4/3)$ relates to the current epoch. In their scaling invariant domains, the cosmological $\Lambda$-term represents the correlation length square $\xi^2$ is the largest area at the Universe horizon $H$, namely $\Lambda \propto \xi^{-2} \sim H^2$ \[16, 17\]. This is a brief recall for one of possible explanations that the cosmological $\Lambda$-term represents the pure and intrinsic nature of the spacetime horizon, rather than the nature of exotic matter fields and their energies. If matter fields coupled to the spacetime are present in the regularized action $(2)$, we expect another fixed point, where the effective theory $(3)$ is achieved with the scaling law $h^2 = (H/H_0)^2 = h^2(\Omega_{\Lambda}, \Omega_{\Lambda})$ and the characteristic length scale $\xi \approx H_0^{-1}$ for the horizon size. However, the quantum aspect and origin of the cosmological $\Lambda$-term are not issues of this article.

In the present article, we show that via the Hawking process of pair production, the matter is produced from the spacetime horizon $H$, which is attributed to the cosmological $\Lambda$-term. In turn, the time evolution of the horizon $H$ is governed by the cosmological $\Lambda$-term and matter so produced, through the Einstein equation from the effective theory $(1)$ in the presence of the matter-field effective action $A_{EC}^{\text{matter}}$:

$$A_{EC}^{\text{total}} = \int \frac{d^4x}{16\pi G} \det(-g)(R - 2\Lambda) + A_{EC}^{\text{matter}}. \quad (3)$$

From this point of view, the evolutions of the Universe horizon $H$, cosmological $\Lambda$-term and matter term are closely related and coupled each others. It is the goal of this article to find the time-evolution of the Universe horizon $H$, cosmological $\Lambda$-term and matter term, as well as their relationships in various epochs of the Universe evolution, on the basis of the effective Einstein theory $(3)$ for the spacetime and semi-classical Hawking pair-production of particles and antiparticles for the matter content. Moreover if the so produced pair density is large enough, the matter content can be approximately described by a perfect fluid of the energy-momentum tensor,

$$T_{ab}^M = p_M g^{ab} + (\rho_M + p_M)U^aU^b, \quad (4)$$

where the energy density $\rho_M$ and pressure $p_M$ are in the comoving frame of the observer with the four velocity $U^a$ of the fluid.
We organize this article as follow. In Sec. II, we revisit the Einstein equation and Bianchi identity in the view of time-varying gravitational coupling $\tilde{g}$ and cosmological $\Lambda$-term interacting with the matter. In Sec. III, we present the discussions and calculations of the matter produced from the spacetime through the Hawking pair-production of particles and antiparticles. Based on these results and equations, assuming the absence of matter at the beginning of the Universe, we adopt numerical and analytical approaches to study the inflation epoch in connection with observations in Sec. IV. In this theoretical framework, we obtain how the horizon $H$, the cosmological $\Lambda$-term and matter term vary in time, and their relations in the Universe evolution, particularly focusing on the problem of cosmic coincidence in Sec. VI. A summary and remarks are given in the concluding section.

II. EINSTEIN EQUATION AND BIANCHI IDENTITY

A. The role of cosmological $\Lambda$-term

The effective action (3) yields the Einstein equation for the spacetime of Einstein tensor $G^{ab}$ coupling to the matter of energy-momentum tensor $T_{M}^{ab}$,

$$G^{ab} = -8\pi G T_{M}^{ab}; \quad (\Lambda) = G^{ab} - (1/2)g^{ab}R - \Lambda g^{ab}. \quad (5)$$

Its covariant differentiation and the Bianchi identity are

$$G_{;b}^{ab} = -8\pi [G T_{M}^{ab}]_{;b}, \quad [R^{ab} - (1/2)\delta^{ab}R]_{;b} \equiv 0, \quad (6)$$

which lead to the conservation law,

$$(\Lambda)_{;b} g^{ab} = 8\pi (G);b T_{M}^{ab} + 8\pi G (T_{M}^{ab})_{;b}, \quad (7)$$

with varying cosmological term $(\Lambda)_{;b} = (\Lambda)_{;b}$ and coupling $(G);b = (G)_{;b}$. Equation (7) clearly shows that the cosmological $\Lambda$-term explicitly interacts with the matter $T_{M}^{ab}$ via a varying gravitational coupling $G$. Moreover, as will be shown in Sec. III, the cosmological $\Lambda$-term implicitly couples with the matter $T_{M}^{ab}$ through the production and annihilation of particle-antiparticle pairs via the horizon $H$ of the spacetime, which is in turn governed by the Einstein equation (5).

Despite its essence of spacetime origin, rather than matter origin, the cosmological $\Lambda$-term in the Einstein spacetime tensor $G_{ab}$ can be moved to the RHS of Einstein equation (5), and formally expressed by using a symbol of energy-momentum tensor $T_{\Lambda}^{ab}$ analogously to the matter $T_{M}^{ab}$ of
Eq. (4),
\[ T^{ab}_{\Lambda} = p_{\Lambda} g^{ab} + (p_{\Lambda} + \rho_{\Lambda}) U^a U^b \equiv -\rho_{\Lambda} g^{ab}, \]
and implementing a negative mass density \( \rho_{\Lambda} = \Lambda / (8\pi G) \equiv -p_{\Lambda} \). This practical analogy between \( T^{ab}_{\Lambda} \) \(^{(8)}\) and \( T^{ab}_{\Lambda} \) \(^{(4)}\) is purely \textit{technical} in the sense of the convenience for calculations and expressions below. In so doing, we do not make any model to change the physical and geometrical nature of the cosmological \( \Lambda \)-term that are previously discussed in Sec. I. The interested readers are referred to the Ref. \(^{[18]}\) for the more detailed discussions on the cosmological \( \Lambda \)-term with respect to the vacuum energy of local field theories. In these notations, Equation (7) is equivalent to the total conservation law expressed by
\[ T^{ab}_{\Lambda} ; b \equiv (T^{ab}_M + T^{ab}_{\Lambda}) ; b = 0, \]
which is one of fundamental equations studied in the present article.

**B. Generalized equations in Friedmann Universe**

In the Robertson-Walker spacetime \( ds^2 = dt^2 - a^2(t)dx^2 \) of zero spatial curvature, defining variables of gravitational coupling \( g = G/M_{\text{pl}}^{-2} \) and Hubble rate \( H = \dot{a}/a \), Equations (5) and (7) become \(^{[17]}\),
\[ h^2 = g(\Omega_M + \Omega_{\Lambda}), \ h \equiv H/H_0, \ \Omega_{M,\Lambda} \equiv \rho_{M,\Lambda} / \rho_c, \]
and
\[ \frac{d}{dx} \left[ g(\Omega_M + \Omega_{\Lambda}) \right] = -3g(1 + \omega_M)\Omega_M. \]
where \( \omega_M = p_M / \rho_M, \ \ x = \ln(a/a_0), \ d(\cdot) / dt = H d(\cdot) / dx \). We define the \( \epsilon \)-rate of \( H \)-variation:
\[ \epsilon \equiv -\frac{H}{H^2} = -\frac{1}{H} \frac{dH}{dx}, \]
to characterize different epochs of Universe evolution. Moreover, we need the physical knowledge of two additional equations or formula to describe:

(i) how matter \( \Omega_M \) is produced and evolves, as well as its initial value \( \Omega^0_M \). This will be discussed in Sec. III.
(ii) how gravitational coupling $g$ varies and its initial value $g_0$. This can be obtained from the renormalization group equation at an adequate fixed point, see for example Refs. [17] and [19].

The initial conditions of these differential equations are given by the initial scales or values of Scaling factor $a_0$,

\[ H_0, \quad \text{Critical density} \quad \rho_c^0 = 3H_0^2/(8\pi M_{\text{pl}}^{-2}); \]  
\[ \text{Cosmological } \Omega^0, \quad \text{Matter content } \Omega^0_M; \]

and Gravitational coupling $g_0$, to describe the beginning of Universe evolution epochs under consideration. These initial scales (14) and (15) depend on not only a specific epoch of Universe evolution, but also the transitions from one epoch to another, and should eventually be determined by observations. On the other hand, these initial scales should be chosen in the range where the description of effective Einstein theory [4] for the Universe evolution is valid, namely the initial scales should be rather smaller than the Planck scale. This is important for the beginning of early Universe, inflation and reheating epochs that will be duly discussed.

In this article, for simplicity we only consider the constant gravitational coupling $g = 1$, namely, $G = M_{\text{pl}}^{-2}$ [20]. Equations (10,11) and (12) are recasted into two independent equations,

\[ h^2 = (\Omega_M + \Omega_\Lambda), \]  
\[ \frac{d}{dx} (\Omega_\Lambda + \Omega_M) = -3(1 + \omega_M)\Omega_M, \]

where, the first equation (16) is the Friedmann equation, i.e., 0−0 component of Einstein equation, and the second equation (17) is the energy conservation law of spacetime and matter, generalized from usual one $d\Omega_M/dx = -3(1 + \omega_M)\Omega_M$ for constant $\Omega_\Lambda$. Starting from the initial values $\Omega^0_\Lambda$ and $\Omega^0_M$ (15) at an adequate scale $H_0$ (15), $\Lambda(h)$ and $\rho_M(h)$ govern the varying spacetime horizon $h$. The variation $\Omega_M(h)$ dynamically leads to the variation of $h^2$ and $\Omega_\Lambda$ via Eq. (17), in turn $\Omega_M(h)$ changes via Eq. (16). This completely determines $h^2(x)$ and $\Omega_\Lambda(h)$ scaling in the Universe evolution, provided that $\Omega_M(h)$ can be calculated as a function of $h$.

In this article, as a convenient unit for calculations and expressions, we adopt the reduced Planck scale $m_{\text{pl}} \equiv (8\pi G)^{-1/2} = 1$, unless otherwise stated. Note that the reduced Planck scale $m_{\text{pl}} = (8\pi)^{-1/2}M_{\text{pl}} = 2.43 \times 10^{18}\text{GeV}$. 


III. PAIR PRODUCTION FROM SPACETIME

A. Matter from spacetime

In this article, the matter is produced from the spacetime by the Hawking pair production of particles $F$ and antiparticles $\bar{F}$:

$$S \Rightarrow F + \bar{F}. \quad (18)$$

Such pair production is a semi-classical process of producing particles and antiparticles in an external field $H$ of the spacetime, which obeys classical equation (16) and (17). We a priori assume that the $H$-field slowly varies, compared with the rates of pair-production and/or other microscopic processes, namely the $\epsilon$-rate (13) is very small ($\epsilon \ll 1$). Therefore, to calculate the matter content $\Omega_M(h)$, we consider the spontaneous pair production of massive spin-1/2 particles $F$ and antiparticles $\bar{F}$ from the exact De Sitter spacetime $S$ of the constant $H$ and scaling factor $a(t) = e^{Ht}$. The averaged number density of pairs produced from the initial time $t_o = 0$ to the final time $t \approx 2\pi H^{-1}$ is given by Refs. [21, 22]

$$n_M = \frac{H^3}{2\pi^2} \int_0^\infty dz z^2 \left| \beta_k^{(n)}(t) \right|^2$$

$$= \frac{H^3 e^{\pi \mu}}{16\pi} \int_0^\infty dz \frac{z^3}{\sqrt{z^2 + \mu^2}} F^{(n)}_{\nu}(z, \mu), \quad (19)$$

where $z \equiv kH^{-1}e^{-Ht}$, the particle mass $\mu = m/H$ and momentum $k$, the Bogolubov coefficient up to the $n$-th adiabatic order $|\beta_k^{(n)}(t)|_{k \rightarrow \infty} \sim O(1/k^{n+2})$ in the ultraviolet (UV) limit. Due to the exact De Sitter symmetry ($H = \text{const}$), the energy-momentum tensor of produced $F\bar{F}$ pairs is $T^{\mu\nu}_M \propto g^{\mu\nu}$ [21, 22]. On the other hand, the inverse process

$$F + \bar{F} \Rightarrow S \quad (20)$$

of particle $F$ and antiparticle $\bar{F}$ annihilation to the spacetime can take place as well.

B. Back-reaction and energy-momentum tensor

Because of pairs $F\bar{F}$ production (18) and annihilation (20), as well as their back reaction on the spacetime through the Eqs. (16) and (17), the $H$-field can not keep its constancy. Such back reaction processes lead to a possibly slowly decreasing $H$, as a result the exact De Sitter symmetry of ($H = \text{const}$) is broken. Therefore, we assume the energy-momentum tensor $T^{ab}_M$ of $F\bar{F}$ pairs to
be spatially homogenous and in the form (8) of the energy density

\[ \rho_M = \frac{2H^3}{2\pi^2} \int_0^\infty dz z^2 |\beta_k^{(n)}(t)|^2 \]

\[ = \frac{2H^4 e^{\pi \mu}}{16\pi} \int_0^\infty dz z^3 \mathcal{F}_\nu^{(n)}(z, \mu), \]  

(21)

and the pressure

\[ p_M = \frac{2H^3}{2\pi^2} \int_0^\infty dz z^2 \frac{(k/a)^2}{3\epsilon_k} |\beta_k^{(n)}(t)|^2 \]

\[ = \frac{\rho_M}{3} - 2 \frac{\mu^2 H^4 e^{\pi \mu}}{3 \times 16\pi} \int_0^\infty dz \frac{z^3}{z^2 + \mu^2} \mathcal{F}_\nu^{(n)}(z, \mu), \]  

(22)

where the spectrum of created particles \( \epsilon_k = a^{-1} [(k/a)^2 + m^2]^{1/2} \), and the equation of state is

\[ \omega_M = \frac{p_M}{\rho_M}. \]  

(23)

To ensure the UV finiteness of Eqs. (19), (21) and (22), the appropriate adiabatic order \( n \) is considered,

\[ \mathcal{F}_\nu^{(n)}(z, \mu) = \left| f_1^{(n)} \sigma_+ H_{\nu-1}^{(1)}(z) - i f_2^{(n)} \sigma_- H_{\nu}^{(1)}(z) \right|^2, \]  

(24)

where \( \sigma_\pm \equiv [(z^2 + \mu^2)^{1/2} \pm \mu]^{1/2} \), \( \nu = 1/2 - i \mu \), \( H_{\nu}^{(1)}(z) \) is the Hankel function of the first kind, and \( f_{1,2}^{(n)} = 1 + \sum_{i=1}^n f_{1,2}^{(i)} \) [21].

It is conceivable that the spacetime of the horizon \( H^{-1} \) could produce many particles and antiparticles (dark matter and normal matter) of different masses \( m > H \) and degeneracies \( g_d \), and their energy densities and pressures contribute to total energy density \( \rho_M \) and pressure \( p_M \). We simply introduce the unique mass scale parameter \( m \) to effectively characterize and describe the total contribution from all kinds of particle-antiparticle pairs to Eqs. (19), (21) and (22), and its value is determined by observations.

**IV. COSMIC INFLATION**

In this section, we study the inflation epoch on the basis of (i) the evolution equations (16) and (17) of the Universe in Sec. II (ii) the pair production from the spacetime described by densities \( n_M, \rho_M \) and pressure \( p_M \) of Eqs. (19), (21) and (22) in the previous Sec. III. We adopt that the initial conditions (14) and (15) at the beginning Universe is of pure spacetime nature without any matter content at the horizon \( H_0 \):

\[ h_0^2 = \Omega_\lambda^0 = 1, \quad \Lambda_0 = 3H_0^2, \quad \Omega_M^0 = 0, \]

(25)
and the critical density $\rho^c = 3H^2/\left(8\pi M_{\text{pl}}^{-2}\right)$. This means that the cosmological term $\Omega_\Lambda$ is dominant over the matter $\Omega_M$, that is completely negligible in the beginning of the Universe. Then, we calculate the matter content $\Omega_M(h)$ as a function of $h$ by the spontaneous pair production of particles and antiparticles. The nontrivial $\Omega_\Lambda(h) \neq 0$ governs the variation of the spacetime horizon $h$, dynamically leading to $h^2$ and $\Omega_\Lambda$ decrease via Eq. (17), in turn $\Omega_M(h)$ changes via Eq. (16). This completely determines the horizon $h^2(x)$, cosmological term $\Omega_\Lambda(h)$ and matter content $\Omega_M(h)$ scaling in the Universe evolution, provided that the initial horizon $H_o$ is given.

Needless to say, the initial value $H_o$ is bound to be much smaller than the Planck scale or the reduced Planck scale, where the effects and details of quantum gravity and/or Planck transition cannot be ignored, the semi-classical frameworks and equations from an effective Einstein theory in Secs. II and the pair-production in an external field in Sec. III are not valid. However, we do not have any alternative approach to this quantum regime of $H_o \lesssim M_{\text{pl}}$ being close to the Planck scale. Therefore, the cosmic inflation is divided into the pre-inflation epoch and inflation epoch. We present a numerical study of the pre-inflation epoch for the initial horizon $H_o \lesssim m_{\text{pl}}$ at the reduced Planck scale, in order to gain an insight into this epoch and its qualitative features. Instead, we present an analytical and quantitative study of the inflation epoch for the initial horizon $H_o = H_\star \ll m_{\text{pl}}$ being much smaller than the reduced Planck scale, in the connection with observations.

**A. Numerical approach to pre-inflation epoch**

For the pre-inflation epoch, selecting the initial scales at the horizon $H_o$,

$$H_o \lesssim m_{\text{pl}}, \quad h^2_0 = 1, \quad h^2_0 \gtrsim \Omega^c \gg \Omega^c_M,$$

and the critical density $\rho^c = 3H^2/\left(8\pi M_{\text{pl}}^{-2}\right)$ as the initial conditions (25), we numerically integrate Eqs. (16,17) and (21,22). As a result, we find that the cosmic inflation of very slowly decreasing $h^2$ and $\Omega_\Lambda(h)$ is indeed a solution, as illustrated in Fig. 1. The reason is that the pair production (19) is not so rapid that $\Omega_M$ are very small and slowly increases, thus $h^2$ and $\Omega_\Lambda$ decrease slowly, see Eq. (17), as a function of $e$-folding numbers $\ln(a/a_o)$. Consequently, we obtain the solution to the cosmological “constant”, slowly varying as an “area” law:

$$\Lambda = 3H^2_0 \Omega_\Lambda(h) \approx 3H^2 \quad \text{or} \quad \Omega_\Lambda(h) \approx h^2.$$

This result is consistent with Eq. (16) and negligible $\Omega_M(h)$ in this pre-inflation epoch. As shown in our numerical calculations, one of them plotted in Fig. 1, the pre-inflation epoch lasts longer than
FIG. 1: The inflation appears, as $h$ and $\Omega_{\Lambda}(h)$, $\omega_M = p_M/\rho_M$ slowly decrease in the $e$-folding number. In this illustration, we adopt $m = H_0 = 1$.

$\ln(a/a_0) > 10^{10}$, the horizon $h$ and $\Omega_{\Lambda}(h)$ monotonically decreases. This implies that quantum fluctuation modes in this regime are able to exit the horizon and reenter the horizon later on, imprinting their traces on the CMB power spectrum at larger scale, and nonlinear regimes of forming large scale structure and galaxies, even today. This is just a speculation, since we do not have a valid quantitative approach to the pre-inflation epoch. Nevertheless, we would like to point out the sign of the equation of state $\omega_M$ (23) monotonically decreasing in Fig. 1, which would relate to observational effects. We will come back to this point at the end of next section.

In terminology, we call this epoch as a pre-inflation epoch ($H_0 > H > H_\ast$) to distinguish from the inflation epoch ($H_\ast > H > H_{\text{end}}$), where the scales $H_\ast$ and $H_{\text{end}}$ are a priori supposed to be much smaller than the reduced Planck scale $m_{\text{pl}}$, namely $H_{\text{end}} < H_\ast \ll m_{\text{pl}}$ that will be duly discussed and become clear in the next section.

B. Analytical approach to inflation epoch

1. Analytical expressions for pair productions

Due to the continuous pair productions, $\Omega_M$ increasing, $H$ and $\Omega_{\Lambda}$ decreasing, Equations (16) and (17) for the Universe evolution run into the regime of smallness $H/m \ll 1$, where it is difficult to perform numerical calculations of Hankel functions [24] in Eqs. (19), (21) and (22) for pair productions as functions of $\mu = H/m$. Apart from these numerical difficulties, it is also important to note that the condition $H/m \ll 1$ is physical, in the scenes that the wavelengths $\lambda^{-1}$ of particles produced are smaller than the radius $H^{-1}$ of the Universe horizon, i.e., $\lambda = m^{-1} \ll H^{-1}$, therefore there particles are well inside the Universe horizon, and their energy-mass content contributes to
the Universe evolution. We seek for an analytical approach to these formulae for pair productions in this regime \((H/m \ll 1)\) and find asymptotic expressions:

\[
n_M \approx \chi m H^2, \tag{28}
\]

\[
\rho_M \approx 2\chi m^2 H^2(1 + s), \tag{29}
\]

\[
p_M \approx \left(\frac{s}{3}\right)\rho_M, \tag{30}
\]

where \(\chi \approx 1.85 \times 10^{-3}\), \(\omega_M = p_M/\rho_M \approx s/3\) and \(s \approx 1/2(H/m)^2 \ll 1\). Reference [21] shows that the number density (28) asymptotically approaches \(n_M \sim mH^2\), and we numerically determine \(\chi\) value in Eq. (28). In addition, we insert the damping factor \(e^{-\sigma(z^2+\mu^2)}\) into \(n_M \rightarrow n_M(\sigma)\) to estimate the \(s\)-term in Eqs. (29) and (30) by the saddle-point approximation. In the limit of \(H/m \ll 1\) and \(s \rightarrow 0\), \(\rho_M \approx mn_M\) and \(p_M \approx 0\), in the analogy with massive “non-relativistic” case.

The leading order of both \(n_M\) (28) and \(\rho_M\) (29) follows the area law \(\propto H^2\), rather than the volume law (19-22). The physical picture is the large number (or degeneracies \(g_d\)) \(N \sim H^{-1}/m^{-1} \gg 1\) of pairs produced mainly in the thin layer of the width \(1/m\) on the horizon surface area \(H^{-2}\). It will become clear that the regime \(H/m \ll 1\) is the physical regimes and the area law \(\propto H^2\) of pair productions has important physical consequences in the Universe evolution.

Another important quantity describing the pair-production process is the averaged pair-production rate

\[
\Gamma_M \approx dN/(2\pi dt) \approx (H/2\pi)dN/dx, \tag{31}
\]

where \(N = n_M H^{-3}/2\) is the number of particles. Using Eq. (28), we obtain

\[
\Gamma_M \approx -(\chi m/4\pi)(H^{-1}dH/dx) = (\chi m/4\pi)\epsilon \tag{32}
\]

where the \(\epsilon\)-rate for the Universe evolution is defined in Eq. (13). We note in advance that these analytical expressions for the regime \((H/m \ll 1)\), which approximately describe the Hawking process of pair-production of particles and antiparticles, are essential for our further analyzing each epoch of the Universe evolution.

2. Inflation epoch and its end

For the inflation epoch, we select the initial scales at the scaling factor \(a_*\), the horizon \(H_*\) and the critical density \(\rho_c^* = 3H_*^2/(8\pi M_{\text{pl}}^2) = 3H_*^2m_{\text{pl}}^2\)

\[
H_0 = H_* \ll m_{\text{pl}}, \quad \hbar_*^2 = 1,
\]
\[ \rho^*_{M} = 2\chi m^2 H_*^2, \quad \Omega^*_M \equiv \rho^*_M/\rho^*_c = (2/3)\chi (m/m_{pl})^2 \]  \hspace{1cm} (33)

\[ \Omega^*_\Lambda = h_*^2 - \Omega^*_M \gg \Omega^*_M, \quad \rho^*_\Lambda = \Lambda_s/(8\pi M_{pl}^{-2}), \quad \Omega^*_\Lambda \equiv \rho^*_\Lambda/\rho^*_c \]

as the initial conditions (14) (15) for evolution equations (16) and (17). The selected initial scale \( H_* \ll m_{pl} \) so that the details of quantum gravity and/or Planck transition could possibly be ignored in the inflation epoch and Eqs. (16) and (17) could be approximately hold, and we will verify the condition \( H_* \ll m_{pl} \) afterward.

The energy content of pairs produced in this epoch is

\[ \Omega_{M} \equiv \rho_{M}/\rho^*_c \approx (2/3)\chi (m/m_{pl})^2 (H/H_*)^2 (1 + s). \]  \hspace{1cm} (34)

Consequently, Eq. (17) becomes

\[ dH^2/dx \approx -2\chi m^2 H^2 (1 + \omega_M)(1 + s), \]  \hspace{1cm} (35)

yielding the solution

\[ H \approx H_* \exp -\chi m^2 x = H_* (a/a_*)^{-\chi m^2}, \]  \hspace{1cm} (36)

slowly decreasing for the dimensionless parameter

\[ \chi m^2 \equiv \chi (m/m_{pl})^2 \ll 1, \]  \hspace{1cm} (37)

that we define here and will be used henceforth.

Because of the continuous pair productions, \( \Omega_{M} \) increasing, \( H \) and \( \Omega_{\Lambda} \) decreasing, the inflation ends at \( a = a_{end} \) and \( H = H_{end} \). This can be estimated by the rate \( H_{end} \) being smaller than the averaged pair-production rate \( \Gamma_M \) (32), namely

\[ H_{end} < \Gamma_M. \]  \hspace{1cm} (38)

This inequality provides the upper bound \( H_{end} \) of the horizon \( H \) at the end of inflation. The value \( H_{end} \) should be determined more precisely from the epoch transition from the inflation to the reheating.

C. Comparison with observations

Let the initial scale \( H_* \) of the inflation correspond to the interested mode of the pivot scale \( k_\star \) crossed the horizon \( (c_s k_\star = H_* a_\star) \) for CMB observations, one calculates the scalar, tensor power
spectra and their ratio
\[ \Delta_{R}^2 = \frac{1}{8\pi^2 m_{pl}^2 c_s}, \quad \Delta_{h}^2 = \frac{2}{\pi^2 m_{pl}^2}; \]
\[ r \equiv \frac{\Delta_{h}^2}{\Delta_{R}^2} = 16\epsilon c_s, \tag{39} \]
where \( c_s < 1 \) due to the Lorentz symmetry broken by the time dependence of the background \cite{2}. Their deviations from the scale invariance is described by
\[ \Delta_{h,R}^{(n)} \equiv \frac{d^n \ln \Delta_{R,h}(k)}{d(k^n)} \bigg|_{k_*} \approx \frac{d^n \ln \Delta_{R,h}(k_*)}{dx^n}, \]
to the leading orders,
\[ n_s - 1 \equiv \Delta_{R}^{(1)} \approx -2\epsilon - \eta - \kappa, \quad \alpha_s \equiv \Delta_{R}^{(2)} \approx n'_s \tag{40} \]
\[ n_t \equiv \Delta_{h}^{(1)} \approx -2\epsilon, \quad \tilde{n}_t \equiv \Delta_{h}^{(2)} \approx n'_{t}, \tag{41} \]
and \( \tilde{\alpha}_s \equiv \Delta_{h}^{(3)} \approx \alpha'_s \), where \((\cdot\cdot\cdot)' \equiv d(\cdot\cdot\cdot)/dx\). We calculate the \( \epsilon \)-rate \cite{13} and its derivative,
\[ \epsilon \equiv -H'/H|_{k_*} \approx \chi m^2(1 + s), \tag{42} \]
\[ \eta \equiv \epsilon'/\epsilon|_{k_*} \approx -3\chi m^2 s \approx -3s\epsilon, \]
\[ \kappa = c's/c_s \] and their derivatives:
\[ \eta' = d\eta/dx \approx -3\eta\epsilon^2, \quad \epsilon'' \approx \eta^2\epsilon - 3\eta^3, \quad \eta'' \approx 9\eta^4 - 6\eta^2\epsilon^2, \]
which are evaluated at the pivot scale \( k_* \).

Based on two CMB observational values at the pivot scale \( k_* = 0.05\,\text{Mpc}^{-1} \) \cite{25}:

(i) \( n_s \approx 0.965 \), we use Eq. \cite{42} to estimate
\[ m = m_* \lesssim 3.08 m_{pl}, \tag{43} \]
by \( 2\epsilon \approx 2\chi m_*^2 \lesssim 1 - n_s \approx 0.035 \) for \( \epsilon \gg \eta \) and assuming \( 2\epsilon < \kappa \), where the dimensionless parameter \( \chi m_*^2 = \chi(m_*/m_{pl})^2 \) \cite{37};

(ii) \( \Delta_{R}^2 \approx 2.1 \times 10^{-9} \), we use Eq. \cite{39} to give the inflation scale
\[ H_* = 3.15 \times 10^{-5} (r/0.1)^{1/2} m_{pl}, \tag{44} \]
and Eq. \cite{32} to give the pair-production rate
\[ \Gamma_{M}^* = (\chi m_* / 4\pi)\epsilon = 7.9 \times 10^{-6} m_{pl}, \tag{45} \]
at the pivot scale \( k_* \) for the horizon crossing.
Note that we adopt the CMB observations to fix the value $m_\ast$ of the unique mass parameter $m$ introduced to represent the effective mass scale of pair productions and contributions.

The inflation ends at the necessary condition $\Gamma_M > H_{\text{end}}$ of Eq. (38), that might not be a sufficient condition. Nevertheless, we use this condition to give bounds on the tensor-to-scalar ratio $r$ and the $e$-folding numbers $N_{\text{end}}$ from the pivot scale $k_\ast$ to the inflation ending scale,

$$H_{\text{end}} = H_\ast \exp - (\epsilon N_{\text{end}}) .$$  \hspace{1cm} (46)

From Eqs. (45) and (46), we have for $\Gamma_M > H_{\text{end}},$

$$\left(\chi m_\ast / 4\pi\right) \epsilon > H_\ast \exp - (\epsilon N_{\text{end}}) ,$$  \hspace{1cm} (47)

yielding the number $N_{\text{end}}$ of $e$-folding before the inflation end

$$N_{\text{end}} = \ln \left(\frac{a_{\text{end}}}{a_\ast}\right) > \frac{2}{1 - n_s} \ln \left[ \frac{7.91 \times 10^{-4} (r/0.1)^{1/2}}{(1 - n_s) \chi (m_\ast / m_{\text{pl}})} \right] .$$  \hspace{1cm} (48)

This gives the results $r < 0.037, 0.052$ for $N_{\text{end}} = 50, 60$ in agreement with observations \[25\]. In addition, we calculate

$$n'_s < \epsilon^2 \approx (1 - n_s)^2/4, \quad n''_s < \epsilon^3 \approx (1 - n_s)^3/8 ,$$

and we need to know the parameter $\kappa = c'_s / c_\ast$ for further parameter constrains.

As a result, the scales $H_{\text{end}} < H_\ast \lesssim 3.15 \times 10^{-5} m_{\text{pl}}$ \[44\] and $m_\ast$ \[43\] confirm us a posteriori a consistent check of our assumption $H_{\text{end}} < H_\ast \ll m_{\text{pl}}$ and $H_\ast / m_\ast \lesssim 1.02 \times 10^{-5} \ll 1$. We have more confidence to adopt the semi-classical frameworks and equations presented in Secs. II and III. We expect to theoretically determine the values $N_{\text{end}} = \ln(a_{\text{end}}/a_\ast)$ and $H_{\text{end}}$ from theoretical framework and observations, which are however postponed to future studies.

D. Area law and large-scale anomaly

In this inflation epoch $H_\ast > H > H_{\text{end}}$, analogously to Eq. (27), the solution to the cosmological “constant” is given by the “area” law:

$$\Lambda = 3H_\ast^2 \Omega_\Lambda \propto H^2 ,$$  \hspace{1cm} (49)

obtained from the fact that $\Omega_\Lambda = (H/H_\ast)^2 - \Omega_M$ dominates over $\Omega_M \approx (\chi m_\ast^2 / 3)(H/H_\ast)^2$, i.e., the matter contribution is completely negligible compared with cosmological “constant” contributions to the inflation of Universe.
Using Eqs. (16) and (17), we recast Eqs. (39) and (13) to

\[ \Delta^2_R(k) = \frac{1}{12\pi^2} \frac{H^2 R^{-1}_M}{m^2_{pl}(1 + \omega_M)c_s}, \]

\[ \epsilon = \frac{3}{2}(1 + \omega_M)R_M, \]

where \( R_M = \Omega_M/(\Omega_\Lambda + \Omega_M) \). In the “pre-inflation” epoch \( H_0 > H > H_\ast \), the equation of state \( \omega_M \) varies from \( \sim 1/3 \) \((H/m \sim 1)\) to 0 \((H/m \ll 1)\), while \( H \) and \( \Omega_\Lambda, M \) slowly vary a few percent only, implying that \( \Delta^2_R(k) \) decreases 3/4 at most. This probably explains the large-scale anomaly of the low amplitude of the CMB power spectrum at low-\( \ell \) multipole, and implies that \( n_s \) decreases, \( \epsilon \) and \( r \) increase as \( k_\ast \) goes to large scales. Moreover, there could be the acoustic wave of dark-matter density perturbation \( \delta \rho_M/\rho_M \) in the “pre-inflation” epoch, described by the sound velocity \( c^M_s = \omega^{1/2}_M \neq 0 \). Analogously to baryon acoustic oscillations, these dark-matter sound waves should probably have imprinted in the both CMB and matter power spectra at large scales of \( k_\ast \sim 10^{-3}\text{Mpc}^{-1} \).

At the end of this section, we have to mention that \textit{a posteriori} the results of slowly varying \( H \) in the pre-inflation epoch (see Fig. 1) and the inflation epoch (36) in turn justify our approximate calculations (21) and (22) by using formulas for a constancy \( H \), i.e., the adiabatic approximation for the pair-production rate being much larger than the rate of the horizon variation. Before ending, we would like to emphasize that these results of pre-inflation and inflation are obtained without any extra field and/or exotic modeling beyond the effective Einstein equation and semi-classical Hawking pair production.

V. PAIRS PRODUCTION AND ANNIHILATION

The particle-antiparticle pair production (18) from the spacetime horizon \( H \) should be an entropically favorable process. On the other hand, particle-antiparticle pairs can in turn annihilate back to the spacetime (20). Such back and forth processes

\[ \mathcal{S} \leftrightarrow \bar{F}F, \]

can be regarded as particle emissions and absorptions of the spacetime. These processes can possibly provides a physically sensible concept of the spacetime entropy in terms of the entropy of particles, necessarily for understanding the thermal equilibrium at a temperature \( T_H \) or energy/states equipartition between pairs and the spacetime horizon \( H \). Postponing the detailed
studies of the entropy issue for future works, we present a brief discussion on the rate equation of pair productions and annihilations in this section.

A. Rate equation and equilibrium

In the inflation epoch $H > \Gamma_M$ and $\Omega_\Lambda \gg \Omega_M \neq 0$, $H$ and $\Omega_\Lambda$ slowly decrease, due to pair production. However the rate $\Gamma^\text{Anni}_M$ of pairs annihilating back to the spacetime is much smaller than the inflation rate $H$, i.e.,

$$\Gamma^\text{Anni}_M = \Gamma^\text{Prod}_M = \Gamma_M \ll H,$$  \hspace{1cm} (53)

where the pair-annihilation rate is the same as the pair-production rate \cite{53}, become the CPT symmetry of local field theories is held for local microscopic processes \cite{52}. Therefore the pairs are not only far from reaching an equilibrium or equipartition with the inflating spacetime, implying that the spacetime horizon temperature is much larger than the pair temperature $T_H \gg T_M$.

After the inflation epoch, $\Gamma_M > H$ implies that pairs have large density and rate to annihilate back to the spacetime. The back and forth processes \cite{52} undergo, and an equipartition or a thermal equilibrium between pairs $\bar{F}F$ and spacetime $S$ can be established. This epoch should be studied by integrating Eqs. (16) and (17) with the rate equation \cite{1} \cite{28}

$$\frac{dn_M}{dt} + 3Hn_M = \Gamma_M \left( n^{T_H}_M - n_M \right),$$  \hspace{1cm} (54)

where $n^{T_H}_M$ is the thermal density of pairs in a thermal equilibrium with the spacetime horizon at the temperature $T_H$ \cite{27}. If pairs reach a thermal equilibrium with the spacetime horizon $H$, namely $T_M = T_H$ and $n^{T_H}_M = n_M$ in Eq. (54),

$$dn_M/dx + 3n_M = 0 \text{ and } n_M \propto a^{-3}. \hspace{1cm} (55)$$

This means that in the Universe evolution, the spacetime horizon and pairs are in thermal equilibrium via the process \cite{52}.

Actually, at a certain point the pairs decay to light particles rather than annihilate to the spacetime, thus are out of the thermal equilibrium between pairs and spacetime, the thermal density $n^{T_H}_M$ exponentially decreases and $T_M > T_H$. 
B. A preliminary discussion of the reheating

The enormous matter entropy (temperature $\tilde{T}$) is generated (increased) by the decay of massive pairs of particles and antiparticles to light particles, when the decay rate

$$\Gamma_{\text{decay}} \propto g^2 Y \tilde{m} > \Gamma_M > H,$$

(56)

where $g_Y \sim \mathcal{O}(1)$ is the Yukawa coupling between the massive pairs and light particles, and $\Gamma_{\text{decay}} M$ also depends on the phase space of final particles, to which massive pairs decay. The term $\Gamma_{\text{decay}} M$ should be added to the RHS of the rate equation [54],

$$\frac{dn_M}{dt} + 3Hn_M = \Gamma_M \left(n_{\text{T}}^\Lambda - n_M\right) - \Gamma_{\text{decay}} M n_M,$$

(57)

and the particle number conservation law $(n_M U^a)_b = 0$ changes to

$$(n_M U^a)_b = -\Gamma_{\text{decay}} M n_M.$$

(58)

The detailed calculations of the horizon $H$, cosmological term $\Omega_\Lambda$ and matter term $\Omega_M$ variations in the reheating epoch are rather complex. The numerical approach is necessarily required and it is not scope of the present article.

In this article, in order to calculate in the next section the variations of the horizon $H$, cosmological term $\Omega_\Lambda$ and matter term $\Omega_M$ in the epoch of Standard Cosmology, we just assume that the reheating epoch is characterized by the scales

$$\tilde{a}, \quad \tilde{t}, \quad \tilde{H} \quad \rho_c = 3\tilde{H}^2 m_{\text{pl}}^2, \quad \tilde{T},$$

(59)

when an enormous amount of light particles (entropy) is produced and the temperature is increased to $\tilde{T}$. The matter term $\tilde{\Omega}_M$ dominates over the cosmological term $\tilde{\Omega}_\Lambda$, namely,

$$\tilde{h}^2 \gtrsim \tilde{\Omega}_M(\tilde{H}) \gg \tilde{\Omega}_\Lambda(\tilde{H}).$$

(60)

These are the initial conditions of the beginning of the Standard Cosmology epoch.

To end this section we would like to mention that in general the cosmological “constant” $\Omega_\Lambda$ decreases as the matter content $\Omega_M$ increases, the Universe stops acceleration $\ddot{a} > 0$ and starts deceleration $\ddot{a} < 0$ at $\ddot{a} = 0$ yielding

$$2\Omega_\Lambda = (1 + 3\omega_M) \Omega_M,$$

(61)

which is obtained from Eq. (11), i.e.,

$$\frac{dh^2}{dx} + 2h^2 = \frac{2\ddot{a}}{H_0^2 \dot{a}} = g \left[2\Omega_\Lambda - (1 + 3\omega_M) \Omega_M\right].$$

(62)

This tells us the balance point of the competition between the cosmological term $\Omega_\Lambda$ and matter term $\Omega_M$ in the Universe evolution.
VI. COSMIC COINCIDENCE

Analogously to studies presented in previous section, our goal is to find the variations of the horizon $H$, cosmological term $\Omega_\Lambda$ and matter term $\Omega_M$, as well as their relationships in the epoch of Standard Cosmology, by integrating the evolution equations (16) and (17) with the initial conditions (59) and (60). In order to do this, we first need to distinguish two different kinds of matter contributions (terms) to the evolution equations (16) and (17), because they follow different evolution laws and start from the different initial conditions.

A. Two kinds of matter contributions

In this theoretical framework, there are two kinds of matter contributions the evolution equations (16) and (17) in the Standard Cosmology epoch:

(i) The first kind is called the “Λ-coupled” matter and denoted by $\Omega^\Lambda_M(h)$ and $\omega^\Lambda_M$, indicating its origin from the spacetime horizon $H$ depending also on the cosmological Λ-term. Analogously to the matter of massive pairs produced in the inflation, This “Λ-coupled” matter is attributed to the Hawking process of particle and antiparticle ($F\bar{F}$) pair productions after the reheating. Their densities, pressure and equation of state are computed by Eqs. (19) and (21)-23) from the initial time $\tilde{t} = 0$ to the final time $t > \tilde{t}$, at another mass parameter $m = \tilde{m} \neq m_*$ [43]. This mass parameter $\tilde{m}$ is unique and introduced to represent the effective mass scale and degeneracy of pair productions in this epoch, and its value should be determined by observations.

In the case $H/\tilde{m} \ll 1$, where $H$ denote the horizon scale in the epoch under consideration, the “coupled” matter is approximately represented by the densities and equation of state,

\begin{equation}
\begin{aligned}
\tilde{n}^\Lambda_M &\approx 2\chi \tilde{m} H^2, & \tilde{\rho}^\Lambda_M &\approx 2\chi \tilde{m}^2 H^2, & \text{and} & \tilde{\omega}^\Lambda_M &\approx 0,
\end{aligned}
\end{equation}

analogously to Eqs. (28,29) and (30). We will check the validity and consistency of this approximation $H/\tilde{m} \ll 1$ in due course. Their initial values are

\begin{equation}
\begin{aligned}
\tilde{n}^\Lambda_M &\approx 2\chi \tilde{m} \tilde{H}^2, & \tilde{\rho}^\Lambda_M &\approx 2\chi \tilde{m}^2 \tilde{H}^2, & \text{and} & \tilde{\omega}^\Lambda_M &\approx 0,
\end{aligned}
\end{equation}

corresponding to the initial conditions [59] and [60] at the end of the reheating. In the unit of the critical density $\tilde{\rho}_c$ [59], we have

\begin{equation}
\begin{aligned}
\Omega^\Lambda_M(h^2) = \frac{\tilde{\rho}^\Lambda_M}{\tilde{\rho}_c} &\approx (2/3)\chi \tilde{m}^2 h^2, & \text{and} & \tilde{\Omega}^\Lambda_M &\approx (2/3)\chi \tilde{m}^2,
\end{aligned}
\end{equation}
where \( h^2 = (H/\dot{H})^2 \) and the dimensionless parameter \( \chi \dot{m}^2 = \chi (\dot{m}/m_{pl})^2 \ll 1 \), similar to Eq. (37).

(ii) The second kind is called the “usual” matter of all particles that have been produced by the end of the reheating, referring to the matter content \( \tilde{\Omega}_M(\dot{H}) \) in Eq. (60). This is exactly the same as the usual matter content well-studied in the Standard Cosmology. In order not to use too many notations, we henceforth use “usual” notations \( \Omega_M \) and \( \omega_M = 1/3, 0 \) to represent the “usual” matter of relativistic or non-relativistic particles, unless otherwise stated. These notations are the same as those used for the pairs of massive particles and antiparticles in the inflation, and readers should avoid confusions.

As will be immediately explained below, the “usual” matter \( \Omega_M \) approximately follows its own conservation law \( (x = \ln a/\tilde{a}) \),

\[
\frac{d\Omega_M}{dx} \approx -3(1 + \omega_M)\Omega_M, \quad \Omega_M(\tilde{a}) = \tilde{\Omega}_M \gg \tilde{\Omega}_\Lambda.
\]

(66)

and its evolution is then represented by

\[
\Omega_M \approx \tilde{\Omega}_M \exp -3(1 + \omega_M) x = \tilde{\Omega}_M \left( \frac{\tilde{a}}{a} \right)^{3(1 + \omega_M)}.
\]

(67)

This is the same as that in the Standard Cosmology.

**B. Coupled equations for \( \Omega_M \) and \( \Omega_\Lambda \) evolutions**

The total matter content should contain these two kinds of matter contributions,

\[
\Omega_{\text{tot}}^M = \Omega_M^\Lambda(h) + \Omega_M.
\]

(68)

The evolution equations (16) and (17) become

\[
h^2 = (\Omega_M + \Omega_M^\Lambda + \Omega_\Lambda),
\]

(69)

\[
\frac{d}{dx} \left( \Omega_M + \Omega_M^\Lambda + \Omega_\Lambda \right) = -3(1 + \omega_M)\Omega_M
\]

\[ -3(1 + \omega_M^\Lambda + \omega_{\text{decay}}^M)\Omega_M^\Lambda.
\]

(70)

To make notations short, we disregard for the moment the term \( \omega_{\text{decay}}^M \), that is due to the decay (58) of massive particle-antiparticle pairs \( \Omega_M^\Lambda \) into relativistic or nonrelativistic particles of the matter \( \Omega_M \). Equation (69) shows that the cosmological term \( \Omega_\Lambda \), matter contents \( \Omega_M^\Lambda(h) \) and \( \Omega_M \) are directly coupled together via the horizon \( h^2 \). Their variations depend on each other via Eq. (70).
1. Indirect interaction between matter and "dark energy" via horizon

We consider the epoch of the "usual" matter domination:

\[ \Omega_M \gg \Omega_\Lambda \gg 0, \] \hspace{1cm} \text{(60)}

and \( \Omega_M \gg \Omega^A_M \gg 0, \) \hspace{1cm} \text{(65)}.

At the leading order for \( (\Omega_\Lambda + \Omega^A_M)/\Omega_M \rightarrow 0 \), Equations (69) and (70) becomes \( h^2 \approx \Omega_M \) and Eq. (66). We then place the leading-order result \( \Omega_M \) back to Eqs. (69) and (70) to obtain the corrections from \( (\Omega_\Lambda + \Omega^A_M)/\Omega_M \ll 1 \) for the next leading order,

\[ h^2 \approx (\Omega_M + \Omega^A_M + \Omega_\Lambda), \] \hspace{1cm} \text{(72)}

\[ \frac{d}{dx} (\Omega^A_M + \Omega_\Lambda) \approx -3(1 + \omega^A_M)\Omega^A_M, \] \hspace{1cm} \text{(73)}

where \( \Omega^A_M = \Omega^A_M(h^2) \) is calculated by Eq. (65).

In fact, that Eq. (70) is split into Eqs. (66) and (73) implies the approximate conservation of the total matter produced up to the reheating end for the reasons:

(i) the Hawking pair production \( n^A_M \) of particles and antiparticles is negligible after the reheating,

(ii) the annihilation rate \( \Gamma_M \) of massive particles and antiparticles in the matter to the spacetime \( \bar{F}F \rightarrow S \) is much smaller than the rate \( \Gamma^\text{decay} \propto g^2 \bar{m} \) of massive particles and antiparticles decay to relativistic particles in this epoch.

Thus these effects have negligible impacts on the total matter and it evolution (66) or (67).

However, the total matter \( \Omega_M \) couples to the "dark energy" \( (\Omega^A_M + \Omega_\Lambda) \) through the horizon \( h^2 \) of Eq. (72), thus it has impacts on the evolution of the "dark energy" \( (\Omega^A_M + \Omega_\Lambda) \) by the \( \Omega^A_M(h^2) \) in the RHS of Eq. (73). As a result, in such a approximation, we obtain the coupled evolution equations (66) or (67) and

\[ h^2 \approx \Omega_M + \Omega_\Lambda, \] \hspace{1cm} \text{(74)}

\[ d\Omega_\Lambda/dx \approx -3(1 + \omega^A_M)\Omega^A_M(h^2), \] \hspace{1cm} \text{(75)}

where we rewrite \( (\Omega^A_M + \Omega_\Lambda) \) as a new notation \( \Omega_\Lambda \), since it overall represents "dark energy" in observations. Equations (66) and (75) show an indirect interaction of matter and "dark energy" through \( \Omega^A_M(h^2) \) and the varying horizon scale \( h^2 \). In other words, the "dark energy" \( \Omega_\Lambda \) evolution is governed by the \( \Omega_M \) evolution (67) and its initial value.
2. Massive pair decay impacts on “dark energy”

We now duly incorporate Eq. (58) for the decay of massive particle-antiparticle pairs $F\bar{F}$, namely $\Omega^\Lambda_M$ (65), to the matter of relativistic or non-relativistic particles in the radiation or matter dominate epoch. Straightforwardly following derivations in the previous section, Equation (75) is modified as below

$$d\Omega_\Lambda/dx \approx -3(1 + \omega_M^\Lambda + \omega_M^{\text{decay}})\Omega_M^\Lambda,$$

where we introduce the ratio

$$\omega_M^{\text{decay}} \equiv \Gamma_M^{\text{decay}}/H,$$

(77)

to effectively describe the massive particle-antiparticle pairs $F\bar{F}$ decay to relativistic or non-relativistic particles in the radiation or matter dominate epoch. As a result, Equations (74) and (75) yield

$$h^2 = (\Omega_M + \Omega_\Lambda),$$

(78)

and

$$d\Omega_\Lambda/dx \approx -3 (1 + \omega_M^\Lambda + \omega_M^{\text{decay}})\Omega_M^\Lambda(h).$$

(79)

The rate $\Gamma_M^{\text{decay}} \propto g_Y \tilde{m}$ (56) of massive pairs decay to particles depends not only on $g_Y$ and $\tilde{m}$, but also on the final states and phase space of particles that they subsequently decay. Therefore the rate $\Gamma_M^{\text{decay}}$ varies from the radiation dominate epoch to the matter dominate epoch. Beside the horizon size $H^{-1}$ increases in time. Thus, the ratio $\omega_M^{\text{decay}}(h)$ is a function of the Universe evolution $h$ in time, and we are not able to quantitatively calculate it.

To have an insight into the variation of the ratio $\omega_M^{\text{decay}}(h)$ (77) in the transition from the radiation dominate epoch to the radiation dominate epoch, we introduce its effective values in the transition respectively,

$$\omega_M^{\text{decay}} \approx \omega_M^{\text{decay},R}, \quad \text{the side of radiation dominate epoch}$$

(80)

where $\omega_M^{\text{decay},R}$ indicates the value $\omega_M^{\text{decay}}$ (77) for final decay products of massive pairs being relativistic particles, and

$$\omega_M^{\text{decay}} \approx \omega_M^{\text{decay},M}, \quad \text{the side of matter dominate epoch},$$

(81)

where $\omega_M^{\text{decay},M}$ indicates the value $\omega_M^{\text{decay}}$ (77) for final decay products of massive pairs being non-relativistic particles. The values of $\omega_M^{\text{decay},R}$ and $\omega_M^{\text{decay},M}$ are expected to be of the order of unity.
and vary smoothly, for the reasons that if the expansion rate is much larger than the decay rate
$H > \Gamma_M^{\text{decay}}$, spacetime generated particle-antiparticle pairs (65) have no enough time to undergo
the decay process, like a “decoupled” phenomenon, whereas the “coupled” phenomenon $\Gamma_M^{\text{decay}} \gtrsim H$
[1] so that the decay process can relevantly couples to the Universe evolution through Eqs. (16),
(17) and (58).

C. $\Omega_{\Lambda} - \Omega_M$ relation and coincidence

We are in the position to find the solution to the coupled equations (79), starting from the initial
conditions (59) and (60) at the end of the reheating. Inserting $\Omega_M^{\Lambda}(h^2)$ of Eq. (65) into Eq. (79),
we obtain

$$\frac{d\Omega_{\Lambda}}{dx} + \tau \Omega_{\Lambda} = -\tau \Omega_M, \quad \tau \equiv 2 \chi \tilde{m}^2 (1 + \omega_M^{\text{decay}}).$$

(82)

From the RHS of this equation, we notice that the coupling between the cosmological term and
matter term is not zero, but very small. In addition, the term $\tau \Omega_{\Lambda}$ shows that he initial conditions
for this differential equation crucially depend on the transitions from one epoch to another, as will
be shown below.

1. Radiation dominant “dark” epoch

The Universe starts the radiation dominate epoch starting after the reheating (66). As discussed,
$\omega_{M,R}^{\text{decay}}$ (80) smoothly varying in time, we have the approximate solution to Eqs. (82) and (67)
($x = \ln a/\tilde{a}$ and $\omega_M = 1/3$) is

$$\Omega_{\Lambda} = \frac{\tau_R \tilde{\Omega}_M}{4 - \tau_R} e^{-4x} + e^{-\tau_R x} \tilde{C} \equiv \frac{\tau_R}{4 - \tau_R} \Omega_M \ll \Omega_M, \quad (83)$$

$$\tau_R \approx 2 \chi \tilde{m}^2 [1 + \omega_{M,R}^{\text{decay}}] \ll 1; \quad \tilde{C} = 0.$$ 

(84)

In agreement with the conditions (59) and (60), here we choose the initial condition at $a = \tilde{a}$:

$$\tilde{C} = 0, \quad \tilde{\Omega}_{\Lambda} = \frac{\tau_R \tilde{\Omega}_M}{4 - \tau_R} / (4 - \tau_R) \ll \Omega_M, \quad (85)$$

for the reason that the transitions from the reheating to the beginning of standard cosmology are
radiation dominate and “continuous”, they should have the same values of $\omega_M = 1/3$ and $\omega_{M,R}^{\text{decay}}$.
However, this is just an argumentation, it needs a detailed study of the reheating epoch and its
transition to the radiation dominant epoch of the Standard Cosmology, since the $C^{\text{eq}} = 0$ (90) is
the integration over “continuous” transitions from the reheating epoch to the radiation dominant one.

Solution (83) shows that in a long dark epoch, $\Omega_\Lambda \ll \Omega_M$ and $\Omega_\Lambda$ tracks \[30\] down $\Omega_M$ (67) until the Universe reaches the radiation-matter equilibrium $a_{eq}$ and $H_{eq}$,

$$\Omega_\Lambda^{eq} = \frac{\tau_R}{4 - \tau_R} \Omega_M^{eq} \ll 1, \quad \Omega_M^{eq} = \Omega_M(a_{eq}) \lesssim 1, \quad (86)$$

in unit of the density $\rho_c^{eq} = 3H_{eq}^2$. The ratio $(a_{eq}/\tilde{a}) = (T/T_{eq})$ is estimated to be $\sim 10^{15}\text{GeV}/10\text{eV} \sim 10^{23}$. Because of $\Omega_M \gg \Omega_\Lambda$, Eq. (78) tells us $\Omega_M \lesssim h^2$, and Eq. (83) gives the cosmological constant being small and varying as an area law at the leading order $O(\tau_R)$,

$$\Omega_\Lambda \approx (\tau_R/4) h^2, \quad (87)$$

in this long and dark epoch. Here we see that in this long dark age of Universe, the $\Omega_\Lambda$ is much smaller than the total matter components $\Omega_M$, and it is negligible for $\tau_R \ll 1$. However, it is due to a such small horizon-coupling $\tau_R$ that the cosmological term $\Omega_\Lambda$ follows the matter dominant $\Omega_M$-evolution from the value $\tilde{\Omega}_\Lambda$ (85) to the value $\Omega_\Lambda^{eq}$ (86) for a long time period.

2. Matter dominant “light” epoch

We turn to the matter dominate epoch starting from the radiation-matter equilibrium point $a_{eq}$ (86) to the present time $a \simeq a_0$ and $(a/a_{eq}) \simeq (1 + z) \sim 10^4$. As $\omega_{decay}^{M,M}$ (81) smoothly varying in time, we have the approximate solution to Eqs. (82) and (67) ($x = \ln a/a_{eq}$ and $\omega_M = 0$)

$$\Omega_\Lambda = \frac{\tau_M}{3 - \tau_M} \Omega_M + e^{-\tau_M x} C^{eq} \quad (88)$$

$$\tau_M \approx 2 \chi \tilde{m}^2 [1 + \omega_{decay}^{M,M}] \quad (89)$$

The coefficient $C^{eq}$ has to be fixed by matching with the condition (86) at the radiation-matter equilibrium point

$$C^{eq} = 2 \chi \tilde{m}^2 \Delta \omega_{decay}^{M} \Omega_M^{eq}, \quad (90)$$

$$\Delta \omega_{decay}^{M} = \omega_{decay}^{M,R}/4 - \omega_{decay}^{M,M}/3 - 1/12. \quad (91)$$

The $\Delta \omega_{decay}^{M}$ represents the effective variation from $\omega_{decay}^{M,R}$ to $\omega_{decay}^{M,M}$, and the factor 1/12 is the variation from relativistic particles $\omega_M = 1/3$ to non-relativistic particles $\omega_M = 0$. The property $\Delta \omega_{decay}^{M} > 0$ is due to a larger and recursively generated phase space of final states of particles and their subsequent decays [29].
Actually, the $C_{\text{eq}} \neq 0$ (90) is the integration over “discontinuous” transitions from the radiation dominate epoch to the matter dominate one. The values $\tilde{C} = 0$ (85) and $C_{\text{eq}} \neq 0$ (90) however necessarily requires a detailed analysis that could be performed by numerical simulations. Nevertheless, we can be sure that the “continuous” transition between the reheating to the radiation epochs much be different the “discontinuous” transition between the radiation and matter epochs.

This essential difference could be the reason why the cosmological “constant” $\Omega_{\Lambda}$ evolves (88) in the light epoch very differently from its behavior (83) of tracking down $\Omega_M$ in the dark epoch. Otherwise, $\Delta \omega_{\text{decay}}^M = 0$ in Eq. (91) and $C_{\text{eq}} = \tilde{C} = 0$, the cosmological term $\Omega_{\Lambda}$ would have been following $\Omega_M$-evolution up to nowadays, $\Omega_{\Lambda} \ll \Omega_M \propto h^2$, which is inconsistent with the observations of current Universe acceleration.

In this light epoch, the solution (88) shows that the first term decreases as $\Omega_M \approx \Omega_M^{\text{eq}} (a/a_{\text{eq}})^{-3} = \Omega_M^{\text{eq}} (1 + z)^{-3}$, and $\Omega_{\Lambda}$ fails to track down $\Omega_M$, approaching to the second term of a slowly varying “constant” $e^{-\tau_M \pi C_{\text{eq}}}$. This means that up to the current epoch the cosmological term $\Omega_{\Lambda}$ has been almost “frozen” to its value (86) at the radiation-matter equilibrium. As a result, from Eqs. (88) and (90) we obtain the ratio

$$\frac{\Omega_{\Lambda}}{\Omega_M} \approx \frac{\tau_M}{3} + 2 \chi \tilde{m}^2 \Delta \omega_{\text{decay}}^M (1 + z)^3. \quad (92)$$

Using current observations $\Omega_{\Lambda}^0 \approx 0.7$ and $\Omega_M^0 \approx 0.3$, correspondingly the redshift $z + 1 = (a_0/a_{\text{eq}}) \sim 10^4$ at the radiation-matter equilibrium, we obtain

$$2 \chi \tilde{m}^2 \Delta \omega_{\text{decay}}^M \approx (1 + z)^{-3} \Omega_{\Lambda}/\Omega_M$$

$$\approx (1 + 10^4)^{-3} \Omega_{\Lambda}^0/\Omega_M^0 \approx 2.3 \times 10^{-12}. \quad (93)$$

As discussed at the end of Sec. VI B 2, $\Delta \omega_{\text{decay}}^{M,R}$, $\Delta \omega_{\text{decay}}^{M,M}$ and $\Delta \omega_{\text{decay}}^M$ are of the order of unity $\sim \mathcal{O}(1)$, we estimate the horizon $h$-coupling $\tau_{R,M} \sim \mathcal{O}(10^{-12})$ and the mass scale parameter $\tilde{m} \sim 10^{14}$ GeV coincides with the characteristic scale (temperature) $\tilde{T}$ of the reheating. This value is also consistent with our assumption $\tilde{m} \gg H$ for using the asymptotic solutions (28-30) for these epochs after the reheating.

This result seems natural without any fine tuning, since the light epoch of $z \sim 10^3$ is much shorter than the dark epoch of $(a^{\text{eq}}/a) \sim 10^{23}$, when the $\Omega_{\Lambda}$ tracks down $\Omega_M$. This gives us an insight into the cosmic coincidence. Otherwise we would have the cosmic coincidence problem of an incredibly fine tuning $\sim (10^{23})^4 \times (10^4)^3$ of the values $\tilde{\Omega}_{\Lambda}$ and $\tilde{\Omega}_M$ at the reheating end so as to reach their present observational values of the same order of magnitude.

To close this section, we have to mention that this is a preliminary scenario for understanding why the cosmological term is “constant” in current observations, and how the cosmic coincidence
problem can be possibly avoided, provided not only $\Omega_M - \Omega_M$ interactions, but also different transitions in various evolution epochs. However, more detailed studies are necessary to reach the conclusion.

D. Possible observations

The $\Omega_\Lambda - \Omega_M$ relation can be rewritten in terms of current observational values $\Omega_\Lambda^0$ and $\Omega_M^0$ in unit of the critical density $\rho_c^0 = 3H_0^2$ today,

$$\Omega_\Lambda \approx (\tau_M/3) \Omega_M^0 (1 + z)^3 + \Omega_\Lambda^0 (1 + z)^{3\tau_M}, \quad (94)$$

which can possibly be examined with observations. In particular, how to examine the $\Omega_\Lambda$-transition from the present “constant” $\sim (1 + z)^{3\tau_M}$, tracing back to the track-down evolution $\sim (1 + z)^3$ at the large redshift $z \sim 10^{3-4}$. We speculate that such $\Omega_\Lambda$-transition should induce the peculiar fluctuations of gravitational field that imprint on the CMB spectrum, analogously to the integrated Sachs-Wolfe effect. On the other hand, from the result that the current value of the cosmological term is the almost same as its value at the radiation-matter equilibrium, and $\rho_\Lambda^0 + \rho_M^0 = \rho_c^0$, the current value of “dark” energy density is

$$\rho_\Lambda^0 \approx \rho_{\Lambda_{\text{eq}}} = \frac{\tau_R}{4} \rho_{M_{\text{eq}}} = \frac{\tau_R}{4} \left( \frac{a_0}{a_{\text{eq}}} \right)^3 \rho_{M_{\text{eq}}} \approx \rho_c^0 = \frac{3}{8\pi G H_0^2}, \quad (95)$$

namely, the current value of the cosmological constant $\Lambda_0 \propto H_0^2$. In addition, Equations 61 and 62 give the turning point from deceleration $\ddot{a} < 0$ to acceleration $\ddot{a} > 0$, yielding

$$ (1 + z)_{\text{turning}} \approx \left( \frac{2\Omega_\Lambda^0}{\Omega_M^0} \right)^{1/(3+\tau_M)} \approx 1.67, \quad (96)$$

and $z_{\text{turning}} \approx 0.67$.

VII. SUMMARY AND REMARKS

In this article, we emphasize the cosmological $\Lambda$-term in the Einstein equation is attributed to the nature of the spacetime rather than the matter. The matter is produced, via the process of particle and antiparticle pair production, from the spacetime horizon that is mainly due to the cosmological $\Lambda$-term in early Universe evolution, and determined by both cosmological and matter terms in latter Universe evolution. Obeying the Einstein equation and conservation law, the cosmological $\Lambda$-term varies in time and couples the matter via the horizon of the spacetime.
In this theoretical framework, assuming zero matter content and nonzero cosmological \( \Lambda \)-term as initial conditions of Universe evolution, we derive the time evolution of Universe horizon \( H \), the cosmological term \( \Omega_\Lambda(H) \) and matter content \( \Omega_M(H) \). In the inflation epoch, we calculate the matter content \( \Omega_M(H) \) that is much smaller than \( \Omega_\Lambda(H) \propto H^2 \). The solution naturally leads to the inflation and results are consistent with observations, possibly shows the large scale anomaly of low amplitude of the CMB power spectrum, and the dark-matter acoustic wave. The studies of the baryogenesis and magnetogenesis in the reheating will be presented in the coming article [33]. The entropy issue in the Universe evolution deserves a very detailed study in future.

We further apply this theoretical framework to study the Universe evolution after the reheating, to reveal the indirect interaction between the cosmological \( \Lambda \)-term and the matter term through the pair production on the space time horizon \( H \). Such indirect interaction plays the role for the cosmological term \( \Omega_\Lambda \) evolution tracking down the matter \( \Omega_M \) evolution up to the point of the radiation-matter equilibrium. Afterward, the cosmological \( \Lambda \)-term, \( \Omega_\Lambda \approx \text{constant} \), varying very slowly up to the present time. Its current value is slightly smaller than the value at the transition from the radiation dominant epoch to the matter dominant epoch. This gives a possible explanation how the problem of cosmic coincidence can be avoided.

In conclusion, we provide a possible scenario to understand the issues of the cosmological constant, cosmic inflation, matter origin, and the cosmic coincidence problem in the framework that the cosmological term is an attribute of the spacetime horizon, which spontaneously undergoes the Hawking pair productions to generate the matter term, and both cosmological and matter terms couple each other via the horizon described by the Einstein equation. There are problems to solve in this theoretical framework, in particular, the problem of the reheating and entropy generation. Further studies are necessarily required and full numerical approach to this problem is also inviting.

To end this article, we mark that oppositely to the positive mass and negative gravitational potential of the matter, the \( \Omega_\Lambda \) physically represents a negative mass \( \Omega_\Lambda \), whose positive potential leads to the horizon and pair productions, drives the Universe acceleration as an entropy force. In turn, these pairs “screen” the positive potential, increase \( \Omega_M \) and deepen the negative potential. The positivity of total mass-energy \( M \) of the Universe should be expected.
VIII. ACKNOWLEDGMENT

Author thanks Dr. Yu Wang for an indispensable numerical assistance of using Python.

[1] E. W. Kolb and M. S. Turner, “The Early Universe”, Published by Westview press, 1994.
[2] For the review of inflation in effective theory, “Inflation and String theory”, D. Baumann and L. Meal-
lister, Cambridge University Press (2015), ISBN 978-1-107-08969-3, and references there in.
[3] Shuang Wang, Yi Wang, Miao Li, Physics Reports 696 (2017) 1-57, [arXiv:1612.00345]; Miao Li, Xiao-
Dong Li, Shuang Wang, Yi Wang, Commun. Theor. Phys. 56:525-604,2011 [arXiv:1103.5870];
J. Martin, C. Ringeval, V. Vennin, Phys.Dark Univ. 5-6 (2014) 75-235, [arXiv:1303.3787];
D. H. Lyth and A. Riotto, Phys.Rept.314:1-146,1999 [arXiv:hep-ph/9807278v4]
[4] A. A. Coley and G. F. R. Ellis, To appear as Topical Review in Class. Quant. Grav. [arXiv:1909.05346]
[5] M. Biagetti, Galaxies 2019, 7, 71 [arXiv:1906.12244]
[6] S. Weinberg, in Understanding the Fundamental Constituents of Matter, edited by A. Zichichi (Plenum
Press, New York, 1977);
H. Kleinert, “Multivalued Fields in Condensed Matter, Electromagnetism, and Gravita-
tion”, World Scientific Publishing (2008), ISBN-13 978-981-279-170-2, and http://users.physik.fu-
berlin.de/~kleinert/b6/psfiles/Chapter-28-csg.pdf.
[7] S. Weinberg, in General Relativity, edited by S.W. Hawking and W. Israel (Cambridge University Press,
Cambridge, England, 1979), p. 790; H. Kawai, Y. Kitazawa, and M. Ninomiya, Nucl. Phys. B404, 684
(1993); B467, 313 (1996); T. Aida and Y. Kitazawa, Nucl. Phys. B491, 427 (1997); M. Niedermaier,
Nucl. Phys. B673, 131 (2003).
There is also a nontrivial UV-stable fixed point in the quantum field theory of four-fermion operators,
for example the Einstein-Cartan gravity, She-Sheng Xue Phys. Lett. B737 (2014) 172177 and 744 (2015)
8894.
[8] L. Smolin, Nucl. Phys. B208, 439 (1982); R. Percacci, Phys. Rev. D 73, 041501 (2006).
[9] J. Ambjrn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. 93, 131301 (2004); 95, 171301 (2005); Phys.
Rev. D 72, 064014 (2005); 78, 063544 (2008); in Approaches to Quantum Gravity, edited by D. Oriti
(Cambridge University Press, Cambridge, England, 2009).
[10] M. Reuter Phys. Rev. D 57, 971 (1998); M. Reuter and F. Saueressig, Phys. Rev. D 65 065016 (2002),
Phys. Rev. D66, 125001 (2002), and New. J. Phys. 14 055022 (2012).
[11] D. Dou and R. Percacci, Classical Quantum Gravity 15, 3449 (1998); W. Souma, Prog. Theor. Phys.
102, 181 (1999); O. Lauscher and M. Reuter, Phys. Rev. D 65, 025013 (2001), D 66, 025026 (2002); R.
Percacci and D. Perini, Phys. Rev. D 67, 081503 (2003); 68, 044018 (2003); D. F. Litim, Phys. Rev.
Lett. 92, 201301 (2004); A. Codello and R. Percacci, Phys. Rev. Lett. 97, 221301 (2006), Phys. Lett. B
672, 280 (2009); P. F. Machado and F. Saueressig, Phys. Rev. D 77, 124045 (2008);

[12] M. R. Niedermaier, Phys. Rev. Lett. 103 (2009) 101303; Nucl. Phys. B 833 (2010) 226. M. R. Niedermaier and M. Reuter, Living Rev. Relativity 9, 5 (2006).

[13] A. Codello, R. Percacci, and C. Rahmede, Int. J. Mod. Phys. A 23, 143 (2008); Ann. Phys. (N.Y.) 324, 414 (2009); D. Benedetti, P. F. Machado, and F. Saueressig, Mod. Phys. Lett. A 24, 2233 (2009); Nucl. Phys. B824, 168 (2010); A. Bonanno, Phys. Rev. D85 (2012) 081503.

[14] M. Reuter, F. Saueressig, “Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety”, Cambridge Univ. Press (2019), DOI: 10.1017/9781316227596

[15] S.-S. Xue, “Detailed discussions and calculations of quantum Regge calculus of Einstein-Cartan theory”, Phys. Rev. D 82 (2010) 064039 [arXiv:0912.2435]; “Quantum Regge calculus of Einstein-Cartan theory”, Phys. Lett. B 682 (2009) 300 [arXiv:0902.3407].

[16] S.-S. Xue, “The phase and critical point of quantum Einstein-Cartan gravity”, Phys. Lett. B711 (2012) 404 [arXiv:1112.1328]. It is shown that the UV fixed point is the critical point of the second-order transition of Kosterlitz and Thouless type.

[17] S.-S. Xue, Nuclear Physics B897 (2015) 326; Int. J. Mod. Phys. 30 (2015) 1545003, https://arxiv.org/abs/1410.6152v3

[18] S.-S. Xue, IJMPA Vol. 24 (2009) 3865-3891, arXiv:hep-th/0608220

[19] Joan Solá, Int. J. Mod. Phys. A31 (2016) 1630035, arXiv:1612.02449

Javier Grande, Joan Sola, Julio C. Fabris, Ilya L. Shapiro, Classical and Quantum Gravity Classical and Quantum Gravity, Volume 27, Number 10 arXiv:1001.0259

[20] For the case of $G$ slowly varying, see for example, Refs. [17] and [19], it is straightforward to generalize the results of this article.

[21] A. Landete, J. Navarro-Salas, F. Torrenti, Phys. Rev. D89 (2014) 044030, https://arxiv.org/abs/1311.4958

[22] C. Stahl, E. Strobel, and S.-S. Xue, Phys. Rev. D 93, 025004 (2016), arXiv:1507.01686

[23] E. Mottola, Phys. Rev. D 31, 754 (1985).

[24] S. Murray, F. J. Poulin, S. Miller, & D. Zaslavsky. (2018, August 5). steven-murray/hankel: v0.3.6 Zenodo version (Version v0.3.6). Zenodo. http://doi.org/10.5281/zenodo.1336792

[25] Planck Collaboration, “Planck 2018 results. VI. Cosmological parameters”, https://arxiv.org/abs/1807.06209

[26] R. Ruffini, J. D. Salmonson, J. R. Wilson and S.-S. Xue, A&A 350, 334(1999)343, 359(2000) 855.

[27] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15 (1977) 27382751.

[28] see for example, B. W. Lee and S. Weinberg Phys. Rev. Lett. 39, (1977); Refs. [1, 26]

[29] PDG, Chap. 39, “Kinematics”, Revised January 2000 by J.D. Jackson (LBNL) and June 2008 by D.R. Tovey (Sheffield). http://pdg.lbl.gov/2011/reviews/rpp2011-rev-kinematics.pdf

[30] I. Zlatev, L. Wang, P. J. Steinhardt, Phys. Rev. Lett. 82, 896-899,1999;
P.J.E. Peebles and B. Ratra, Ap. J. Lett. 325, L17(1988), Phys. Rev. D37, 3406 (1988).
[31] For example, D. Bégué, C. Stahl and S.-S. Xue, Nuclear Physics B 940, 2019, 312-320.

[32] This is reminiscent of early works on the cosmological constant attributed to the vacuum-energy density of local field theories. Among them we mention “vacuum-energy” density $\rho^\text{vac}_\Lambda \approx \pi/(2\ell^2\text{pl}H_0^2)$, rather than $\rho^\text{vac}_\Lambda \propto 1/(\ell^4\text{pl})$, V. G. Gurzadyan and S.-S. Xue, IJMPA 18 (2003) 561-568, astro-ph/0105245.

[33] S.-S. Xue, “Baryogenesis, Magnetogenesis and cosmic horizon” in preparation.