Iterative Learning Control for High-Order Systems With Arbitrary Initial Shifts

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ABSTRACT

In this paper, two iterative learning control methods are proposed for the different high-order systems with arbitrary initial shifts. The tracking errors caused by nonzero initial shifts are easily detected when applying conventional learning algorithms. But this defect is overcome through applying a step-by-step rectifying controller with initial rectifying action introduced in a small interval. It demonstrates the improvement of tracking performance and shows the robustness with respect to the stochastic initial shifts. Finally, simulation results are presented to illustrate the effectiveness of the stated algorithms.

INDEX TERMS

Convergence, high-order systems, iterative learning control, step-by-step rectifying.

I. INTRODUCTION

Iterative learning control (ILC) is suitable for the repeated operation in a limited interval. It can improve the task execution in the next iteration by the observation of the past attempts and achieve error-free tracking [1], [2]. Because of its simple design and small online computation, ILC is applied in industrial robot control [3], [4], chemical process control [5] and other occasions [6]–[8].

Iterative learning control has a strict condition toward the robustness analysis, which requires that the actual initial state must be same to the desired initial state at each iteration. In fact, this condition is not available. This strict condition limits the application of iterative learning control. In order to solve this problem, scholars have done a lot of beneficial researches [9]–[25]. [11]–[13] are early research results. They present that systems only achieve asymptotic tracking. Up to now, the strict requirement has been relaxed because of introducing the initial rectifying action. The algorithm presented in [14] ensures that the system with relative degree can achieve tracking completely from a pre-specified moment. [15] presents a new controller that can make the high-order multi-agent systems achieve the uniform tracking in a given interval through initial states rectifying actions. But [14], [15] are only for the fixed initial shifts. A new method proposed in [16] is different from others. It is a useful attempt for the nonlinear continuous system with arbitrary initial shifts. But the approach can’t make the system achieve tracking completely through initial rectifying action. There is a steady error after finishing the initial rectifying action during the tracking process. So it only applies to the first-order systems. Hitherto, it is an open and important issue that the iterative learning control is applied to dynamical systems with non-fixed initial shifts by means of contraction mapping method.

Lyapunov-like method has brought great convenience in the process of convergence analysis. It has been used to focus on the ILC problem of dynamical systems with arbitrary initial shifts recently, and some results have been achieved [17]–[25]. The time-varying boundary layer method is presented in [20]. Because of the boundary layer monotonously decreasing with time and converging to zero, the tracking error decided by boundary layer converges to zero through learning. References [21]–[23] introduce initial-rectified-attractor, and achieve complete tracking over a specified interval. Reference [24] summarizes the convergence results about five types of initial conditions. Reference [25] presents a discrete adaptive ILC method aiming at a class of
discrete time-varying uncertain systems with arbitrary initial shifts and non-identical trajectories.

Initial rectifying actions introduced in the algorithms are combined into differential-differential (D-D) type feedback ILC technique for two classes of linear systems with non-fixed initial shifts. They can also guarantee the converged system output to achieve a desired trajectory with a smooth transient. Moreover, fast convergence of system errors is due to the application of compression mapping technology. At last, two numerical examples are given to illustrate the effectiveness of the proposed method.

II. PROBLEM FORMULATION

Consider the following class of high-order linear systems performing identical tasks repeatedly over a finite time interval, i.e., \( t \in [0, T] \).

\[
\begin{align*}
\dot{x}_{i,k}(t) &= x_{i+1,k}(t), \ i = 1, 2, \cdots, n-1 \\
\dot{x}_{n,k} &= Ax_{n,k} + Bu_k(t) \\
y_k(t) &= Cx_{1,k}(t)
\end{align*}
\]

where \( k \) denotes the \( k \)th repetitive operation; \( x_{i,k}(t) \in R^m (i = 1, 2, \cdots, n) \) are the states, and \( u_k \in R^r (r \geq m) \), \( y_k(t) \in R^n \) are control input and output vectors, respectively. \( A, B, C \) are matrices with appropriate dimensions.

Assumption 1: The initial states \( x_{i,k}(0)(i = 1, \cdots, n) \) are arbitrary and bounded in every iteration. The states \( x_{i,k}(t)(i = 1, \cdots, n) \) are measurable in a specified interval. But \( \dot{x}_{n,k}(t) \) must be measurable when \( t \in [0, T] \).

A. THE MATRIX B IS INVERTIBLE

The given desired states are \( x_{i,d}(t) \in R^m, i = 1, 2, \cdots, n \). From these, the tracking state errors are denoted as

\[
\begin{align*}
e_{x,k}(t) &= x_{1,d}(t) - x_{1,k}(t) \\
\dot{e}_{x,k}(t) &= x_{2,d}(t) - x_{2,k}(t) \\
& \vdots \\
e_{x,k}^{(n-1)}(t) &= x_{n,d}(t) - x_{n,k}(t) \\
e_{x,k}^{(n)}(t) &= \dot{x}_{n,d}(t) - \dot{x}_{n,k}(t)
\end{align*}
\]

In order to achieve complete tracking over the given interval, we present the following control law:

\[
\begin{align*}
u_{k+1}(t) &= u_k(t) \\
&= \begin{cases} \\
\Gamma_0 e_{x,k}^{(n)}(t) + \sum_{i=1}^{n-1} \Theta_i(t) \bar{E}_{i,k}(h_{i,1}) + \Theta_n(t) \bar{E}_{n,k}(0) \\
+ \Gamma_1 e_{x,k}^{(n)}(0) - B^{-1} A[x_{n,k+1}(t) - x_{n,k}(t)] \quad & t \in [0, 2^{n-1} T_p] \\
\Gamma_0 e_{x,k}^{(n)}(t) + \Gamma_1 e_{x,k}^{(n)}(t) & t \in [2^{n-1} T_p, T]
\end{cases}
\end{align*}
\]

among them, \( \Gamma_0 \) and \( \Gamma_1 \) are controller gains.

\[
\begin{align*}
\Theta_n(t) &= \begin{cases} 6 T_p(t_p - t) & t \in [0, t_p) \\
0 & t \in [t_p, T]
\end{cases} \\
\Theta_i(t) &= \begin{cases} 0 & t \in [0, h_{i,1}) \\
h_i t_i & t \in [h_{i,1}, h_{i,2}) \\
0 & t \in [h_{i,2}, T]
\end{cases} \\
h_{i,1} &= 2^{n-i-1} T_p \\
\bar{E}_{n,k}(0) &= \Gamma_0 e_{x,k}^{(n)}(0) + \Gamma_1 e_{x,k}^{(n)}(0) \\
+ B^{-1}(x_{n,k}(0) - x_{n,k+1}(0)) \\
\bar{E}_{i,k}(h_{i,1}) &= \Gamma_0 e_{x,k}^{(i-1)}(h_{i,1}) + \Gamma_1 e_{x,k}^{(i-1)}(h_{i,1}) \\
+ B^{-1}(x_{i,k}(h_{i,1}) - x_{i,k+1}(h_{i,1}))
\end{align*}
\]

where

\[
\begin{align*}
h_{i,2} &= 2 h_{i,1} \\
\bar{h}_i &= \frac{2 [N - (i + 1)]}{(N - i)!} \left( \frac{1}{N - i} \right)^{N-i} \\
\theta_i &= \binom{N-i}{i} \left( t - h_{i,1} \right)^{N-i} \left[ h_{i,2} - t \right]^{N-i} \\
N &> n \quad \text{is a positive integer and } t_p \text{ is given a positive real number. Obviously, } \theta_i \text{ is } (n-i)\text{th derivative of } (t^{N-i} - t) \text{ with respect to } t.
\end{align*}
\]

Remark 1: Obviously, this is a new D-D type feedback ILC rule with initial rectifying actions for high-order systems. Compared with the traditional ILC rule, the control law designed in this paper not only combines a feedback item, but also contains initial shifts rectifying actions. The purposes of the design are not only to accelerate convergence, but also to achieve tracking completely. In practice, it is difficult to obtain the high-order derivative of the current output, so the high-order derivative of the output at the previous moment can be used to replace the high-order derivative of the current moment output. Of course, the current derivative term can be discarded, that is, \( \Gamma_1 = 0 \).

There is a following theorem with respect to the function \( \Theta_i(t) \).

Theorem 1: For any function \( \Theta_i(t) \), when \( t_0 \in [h_{i,2}, T] \), the following formula is true.

\[
I_q = \int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t) dt_q \cdots dt_1
= \int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t_1) dt_q \cdots dt_1
\]

\[
\begin{align*}
&= \begin{cases} \\
1 & q = n - i + 1 \\
0 & 0 < q < n - i + 1
\end{cases}
\end{align*}
\]

Proof: 1) \( q = n - i + 1 \)

\[
I_q = \int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t_q) dt_q \cdots dt_1
= \int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t_q) dt_q \cdots dt_1
\]

\[
\int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t_q) dt_q \cdots dt_1
= \int_{h_i}^{t_{q-1}} \cdots \int_{h_i}^{t_{q-1}} \Theta_i(t_q) dt_q \cdots dt_1
\]
Let $\mu = t_1 - h_{i,1}$, then

$$I_q = \int_0^{h_{i,1}} \tilde{h}_i(\mu + h_{i,1})^{N-i} \mu^2(2N-i+1)[h_{i,1} - \mu]^{N-i} d\mu$$

$$= \frac{\tilde{h}_i}{\mu^2} \int_0^{h_{i,1}} (h_{i,1}^2 - \mu^2)^{N-i} \mu^2(2N-i+1) d\mu$$

$$= \frac{\tilde{h}_i}{\mu^2} \int_0^{h_{i,1}} (H - v)^{N-i} d\alpha$$

$$= \frac{\mu^2}{\tilde{h}_i} \int_0^{h_{i,1}} (H - v)^{2N-i} d\alpha$$

Because

$$\int_0^{\frac{\pi}{2}} (\cos\alpha)^{2N-i+1} d\alpha = \frac{2(N-i)}{2(N-i)+1} \cdot \frac{2(N-i-1)}{2(N-i)+1} \cdots \frac{2}{3}$$

then we can obtain that $I_q = 1$.

2) $0 < q < n - i + 1$

In this case, it is easy to obtain the following result.

$$I_q = \tilde{h}_i(t^{N-i}[h_{i,2} - t]^{N-i})^{N-i} d\alpha$$

$$= 0$$

So the conclusion is true and the proof is completed.

**Remark 2:** Theorem 1 indicates that each rectifying function $\Theta_i(t)$ has an effect similar to a pulse function after $(n-i+1)th$ integration. In fact, the pulse function don’t exist, so we can only find a similar function instead of the pulse function.

We call this controller as step-by-step rectifying controller, because the controller can achieve a reference trajectory $x_{n,d}(t)$ at the time $t_p$, $x_{i,d}(t)$ at the time $2^{i-1}t_p$, and so on $x_{1,d}$ at the time $2^{(n-1)}t_p$ that is to say, the system completes the tracking perfectly at time $2^{(n-1)}t_p$.

**B. THE PRODUCT OF MATRICES C AND B IS INVERTIBLE**

The given desired trajectory is $y_d(t) \in R^n, t \in [0, T]$. From this, we can define the tracking errors in the $kth$ iterative as

$$\begin{align*}
\tilde{y}_k(t) &= y_d(t) - y_k(t) \\
\tilde{y}_k(t) &= y_d(t) - y_{2,k}(t) \\
&\vdots \\
\tilde{y}_{(n-1),k}(t) &= y_d^{(n-1)}(t) - y_{n,k}(t) \\
\tilde{y}_{(n),k}(t) &= y_d^{(n)}(t) - \tilde{y}_{n,k}(t)
\end{align*}$$

In order to achieve complete tracking over the given interval, we present the following control law.

$$u_{k+1}(t) = u_k(t)$$

$$+ \begin{cases} 
\begin{align*}
\Theta_0(t) &\leq \sum_{i=1}^{n-1} \Theta_i(t) \mathcal{Z}_{i,k}(h_{i,1}) \\
+ \Theta_i(t) \mathcal{Z}_{n,k}(0) + \Gamma_1 \tilde{e}_{k+1}(t) \\
- (CB)^{-1} \mathcal{Z}_{n,k}(0) - \tilde{e}_{k+1}(t)
\end{align*}
\end{cases} \quad \t \in [0, 2^{(n-1)}t_p]
$$

among them, $\Gamma_0$ and $\Gamma_1$ are controller gains.

$$\mathcal{Z}_{n,k}(0) = \Gamma_0 \tilde{e}_{(n-1)}(0) + \Gamma_1 \tilde{e}_{k+1}(0)$$

$$+ (CB)^{-1} \mathcal{Z}_{n,k}(0) - \tilde{e}_{k+1}(0)$$

$$\mathcal{Z}_{n,k}(h_{i,1}) = \Gamma_0 \tilde{e}_{(n-1)}(h_{i,1}) + \Gamma_1 \tilde{e}_{k+1}(h_{i,1})$$

$$+ (CB)^{-1} \mathcal{Z}_{n,k}(h_{i,1}) - \tilde{e}_{k+1}(h_{i,1})$$

where the other variables are defined the same as before.

In order to facilitate the convergence analysis, we cite some definitions and lemmas in the reference [17] at first.

$$\|x(t)\|_{\lambda(t)} = \sup_{t \in [0,T]} e^{-\lambda t} \|x(t)\|, \lambda > 0$$

where $\|\cdot\|$ is some norm and its definition is following. If $Z(t)$ is an $n$-dimensional vector and $Z(t) = (z_1(t), \ldots, z_n(t))^T$, then $\|Z(t)\| = \max_{1 \leq i \leq n} |z_i(t)|$; if $Z(t)$ is a matrix and $Z(t) = [z_{ij}] \in R^{m \times n}$, then $\|Z(t)\| = \max_{1 \leq i \leq m, 1 \leq j \leq n} (\sum_{i=1}^{n} |z_{ij}(t)|)$. With respect to the $\lambda$-norm, there is a following lemma.

**Lemma 1:**

$$\sup_{t \in [0,T]} (e^{-\lambda t} \int_0^{h_{i,p}} \cdots \int_0^{h_{i,1}} \|Z(t_p)\| dt_p \cdots dt_1) \leq \frac{1}{\lambda^p} \|Z(t_p)\|_{\lambda}$$

**Lemma 2:** Supposing the series $b_k$ satisfy the following condition,

$$|b_k| \leq \rho \cdot |b_{k-1}| + q$$

if $0 \leq \rho < 1$, then

$$\lim_{k \to \infty} |b_k| \leq |q| \cdot \frac{1 - \rho}{1 - \rho}$$

**III. CONVERGENCE ANALYSIS**

In this section, the convergence of the system (1) after using the control law (3) and (8) will be considered separately in what follows.

**A. THE MATRIX B IS INVERTIBLE**

Regarding to the system (1), the convergence results of control law (3) are provided in Theorem 2.

**Theorem 2:** For the linear continuous system (1), if the initial states are arbitrary and bounded, and there exist matrices $B(B$ is invertible), $\Gamma_0$, $\Gamma_1$ such that

$$\| (I + B\Gamma_1)^{-1}(I - B\Gamma_0)\| < 1$$

References.
then the step-by-step rectifying control law (3) can make the system (1) track the desired trajectory. Especially the system can track perfectly when \( t \in [2^{n-1}p, T] \).

**Proof:** For the convenience of description, let’s define
\[
\Delta u_k(t) = u_{k+1}(t) - u_k(t)
\]
\[
\Delta x_i, k(t) = x_{i, k+1}(t) - x_{i, k}(t)
\]

The remaining proof consists of four steps. In step 1, we will obtain a conclusion that \( \lim_{k \to +\infty} x^{(n-1)}_{x, k+1}(t) |_{t=t_p} = 0 \).

In step 2, the following conclusions that \( \lim_{k \to +\infty} \|e^{(i-1)}_{x, k}(t)\| = 0, i = h_{j, 2}, j = i, i-1, \ldots, 1, i = n-1, \ldots, 1 \) will be obtained in turn due to the previous results.

On the basis of step 2, we can draw the conclusions that \( \lim_{k \to +\infty} \|e^{(i-1)}_{k}(h_{1, 2})\| = 0, i = n, n-1, \ldots, 1 \) in step 3, and the exact convergence of uniformly tracking error is analyzed when \( t \in [h_{j, 2}, T) \) in step 4.

**Step 1:** The proof of \( \lim_{k \to +\infty} e^{(n-1)}_{x, k+1}(t) |_{t=t_p} = 0 \)

When \( t \in [0, t_p) \), because of \( \Theta_i(t) = 0, i = 1, \ldots, n-1 \), using (1) and (3), we can obtain
\[
x_{n, k+1}(t) - x_n(t) = \int_0^t \[\dot{x}_{n, k+1}(t) - \dot{x}_n(t)\] dt
\]

Simplify further,
\[
(I + B\Gamma_i)[e^{(n-1)}_{x, k+1}(t)] - (1 - \int_0^t \Theta_i(t) dt) e^{(n-1)}_{x, k+1}(0)
\]
\[
= (I - B\Gamma_0)[e^{(n-1)}_{x, k}(t)] - (1 - \int_0^t \Theta_i(t) dt) e^{(n-1)}_{x, k}(0)
\]

Let’s denote the virtual state error as
\[
e^{(n-1), *}_{x, k}(t) = e^{(n-1)}_{x, k}(t) - (1 - \int_0^t \Theta_i(t) dt) e^{(n-1)}_{x, k}(0)
\]

According to the definition of \( e^{(n-1), *}_{x, k}(t) \), it is easy to obtain
\[
e^{(n-1), *}_{x, k+1}(t) = (I + B\Gamma_i)^{-1}(I - B\Gamma_0) e^{(n-1), *}_{x, k}(t)
\]

If \( \| (I + B\Gamma_i)^{-1}(I - B\Gamma_0) \| < 1 \), applying lemma (2), it is easy to produce
\[
\lim_{k \to +\infty} e^{(n-1), *}_{x, k}(t) = 0
\]

Accordingly, when \( t = t_p \), using (18) and Theorem 1, yield
\[
\lim_{k \to +\infty} e^{(n-1)}_{x, k+1}(t) = \lim_{k \to +\infty} e^{(n-1), *}_{x, k}(t) = 0
\]

**Remark 3:** We can know that \( \lim_{k \to +\infty} e^{(n-1), *}_{x, k}(t) = 0 \) when \( t \in [0, t_p] \) from the previous discussion. But it does not mean that \( \lim_{k \to +\infty} e^{(n-1)}_{x, k+1}(t) = 0 \). Actually, \( \lim_{k \to +\infty} e^{(n-1)}_{x, k+1}(t) = 0 \) only when \( t = t_p \) due to \( f_{p_{i}}^{t_p} \Theta_i(t) dt_1 = 1 \).

**Step 2:** The proofs of \( \lim_{k \to +\infty} \|e^{(i-1)}_{x, k}(t)\| = 0, i = h_{j, 2}, j = i, i-1, \ldots, 1, i = n-1, \ldots, 1 \)

Subsequently, we can obtain \( x_{i, k+1}(t) - x_i(t) \) when \( t \in [2^{n-i}t_p, 2^{n-i}t_p) \).
\[
x_{i, k+1}(t) - x_i(t) = \int_0^t \[\dot{x}_{i, k+1}(t) - \dot{x}_i(t)\] dt
\]

among them,
\[
\Omega_{i, k} = \int_{h_{i, 1}}^{h_{i, 0}} \int_{h_{i, 1}}^{h_{0, 1}} \Delta x_n(t) dt_1 \ldots dt_i
\]

(22)
Inserting (21) with (3),
\[ x_{i,k+1}(t) - x_{i,k}(t) = \Delta x_{i,k}(h_{i,1}) + \Omega_{i,k} \]
\[ + \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} B[\Gamma_{0} e_{x,k}^{(n)}(t_{n}) + \Theta(t_{n}) \Xi_{i,k}(h_{i,1})] \]
\[ + \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} \Gamma_{1} e_{x,k+1}(h_{i,1}) dt_{n} \cdots dt_{i} \]  
(23)
Using the formula of integration, we can obtain
\[ \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} B[\Gamma_{0} e_{x,k}^{(n)}(t_{n}) + \Theta(t_{n}) \Xi_{i,k}(h_{i,1})] \]
\[ = B\Gamma_{0} e_{x,k}^{(i-1)}(t) - B\Gamma_{0} e_{x,k}^{(i-1)}(h_{i,1}) - B\Gamma_{0} \omega_{i,k}^{1} \]  
(24)
and
\[ \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} \Gamma_{1} e_{x,k+1}(h_{i,1}) dt_{n} \cdots dt_{i} \]
\[ = B\Gamma_{1} e_{x,k+1}(h_{i,1}) - B\Gamma_{1} e_{x,k+1}(h_{i,1}) - B\Gamma_{1} \omega_{i,k+1}^{1} \]  
(25)
where
\[ \omega_{i,k}^{1} = \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} e_{x,k}^{(n-1)}(h_{i,1}) dt_{n-1} \cdots dt_{i} \]
\[ + \cdots + \int_{h_{i,1}}^{t_{i-1}} \int_{h_{i,1}}^{t_{k-1}} e_{x,k}^{(1)}(h_{i,1}) dt_{i} \]
\[ + \int_{h_{i,1}}^{t_{i-1}} e_{x,k}^{(i)}(h_{i,1}) dt_{i} \]  
(26)
Insert (23) with (24) and (25), then
\[ x_{i,k+1}(t) - x_{i,k}(t) \]
\[ = (I - B\Gamma_{0}) e_{x,k}^{(i-1)}(h_{i,1}) + B\Gamma_{0} e_{x,k}^{(i-1)}(t) \]
\[ + \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} B\Theta(t_{n}) \Xi_{i,k}(h_{i,1}) dt_{n} \cdots dt_{i} \]
\[ + B\Gamma_{1} e_{x,k+1}(h_{i,1}) - (I + B\Gamma_{1}) e_{x,k+1}(h_{i,1}) \]
\[ + \Omega_{i,k} - B\Gamma_{0} \omega_{i,k}^{1} - B\Gamma_{1} \omega_{i,k+1}^{1} \]  
(27)
Substitute \( \Xi_{i,k}(0) \) and insert (26) with (22), (28)
\[ (I + B\Gamma_{1}) e_{x,k}^{(i-1)}(t) - \omega_{i,k}^{1} \]
\[ - (1 - \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} \Theta(t_{n}) dt_{n} \cdots dt_{i}) e_{x,k}^{(i-1)}(h_{i,1}) \]
\[ = (I - B\Gamma_{0}) e_{x,k}^{(i-1)}(t) - \omega_{i,k}^{1} \]
\[ - (1 - \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} \Theta(t_{n}) dt_{n} \cdots dt_{i}) e_{x,k}^{(i-1)}(h_{i,1}) \]  
(28)
Similarly, let’s denote the virtual error as
\[ e_{x,k}^{(i-1),*}(t) = e_{x,k}^{(i-1)}(t) - \omega_{i,k}^{1} \]
\[ - (1 - \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{k-1}} \Theta(t_{n}) dt_{n} \cdots dt_{i}) e_{x,k}^{(i-1)}(h_{i,1}) \]  
(29)
According to the definition of \( e_{x,k}^{(i-1),*}(t) \), it is easy to obtain
\[ e_{x,k+1}^{(i-1),*}(t) = (I + B\Gamma_{1})^{-1}(I - B\Gamma_{0}) e_{x,k}^{(i-1),*}(t) \]  
(30)
If \( \| (I + B\Gamma_{1})^{-1}(I - B\Gamma_{0}) \| < 1 \), applying lemma (2), it is easy to produce
\[ \lim_{k \to \infty} \| e_{x,k}^{(i-2),*}(t) \| = 0 \]  
(31)
Accordingly, when \( i = n - 1, t = 2t_{p} \), yield
\[ \lim_{k \to \infty} \| e_{x,k}^{(i-2),*}(t) \| = 2t_{p} \]
\[ = \lim_{k \to \infty} \| e_{x,k}^{(i-2),*}(t) + \int_{h_{i,1}}^{t_{j-1}} e_{x,k}^{(i-1)}(t_{p}) dt_{p} + 1 \]
\[ - \int_{t_{p}}^{t} \Theta(t_{n}) dt_{n} + 1 \| e_{x,k}^{(i-2),*}(t) \| = 2t_{p} \]  
(32)
Using (20), (29), (31) and Theorem 1, it produces
\[ \lim_{k \to \infty} \| e_{x,k}^{(i-2),*}(t) \| = 2t_{p} = 0 \]  
(33)
And so on, we can draw the following conclusion when
\( i = n - 2, n - 3, \ldots, 1 \).
\[ \lim_{k \to \infty} \| e_{x,k}^{(i-1)}(t) \| = 0 \quad t = h_{j,2}, j = i, i - 1, \ldots, 0 \]  
(34)
Step 3: The transition from \( \lim_{k \to \infty} \| e_{x,k}^{(i-1)}(h_{j,2}) \| = 0, j = i, i - 1, \ldots, 1 \) to \( \lim_{k \to \infty} \| e_{x,k}^{(i-1)}(h_{j,2}) \| = 0 \) when \( i = n, n - 1, \ldots, 1 \).

From the above, we know that our control law can achieve tracking perfectly when \( t = h_{j,2} \) or if \( t = 2^{n-1}t_{p} \), \( k \to \infty \) and \( i = n, n - 1, \ldots, 1 \), i.e.,
\[ \lim_{k \to \infty} \| e_{x,k}^{(i-1)}(h_{j,2}) \| = 0, \quad i = n, n - 1, \ldots, 1 \]
\[ \lim_{k \to \infty} x_{i,k}(t) = x_{d}(t) \quad i = n, n - 1, \ldots, 1 \]  
(35)
Step 4: The exact convergence of uniformly tracking error
When \( t \in [h_{1,2}, T] \), using control law (3), it is easy to produce
\[
x_{i,k+1}(t) - x_{i,k}(t)
= (I - B\Gamma_1)e_{x,k}^{(i-1)}(h_{1,2}) + B\Gamma_1 e_{x,k}^{(i-1)}(t)
+ B\Gamma_1 e_{x,k+1}^{(i-1)}(t) - (I + B\Gamma_1) e_{x,k+1}^{(i-1)}(h_{1,2})
+ \Omega_{i,k}^* - B\Gamma_0 \omega_{i,k}^{1,*} - B\Gamma_1 \omega_{i,k+1}^{1,*}
+ \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} A\Delta x_{n,k}(t_n) dt_n \cdots dt_i
\]
where
\[
\Omega_{i,k}^* = \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} \Delta x_{n,k}(h_{1,2}) dt_n \cdots dt_i
\]
\[
\omega_{i,k}^{1,*} = \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} \omega_{i,k}(h_{1,2}) dt_n \cdots dt_i
\]
When \( k \rightarrow \infty \), using (35), we can get \( \Omega_{i,k}^* = 0 \) and \( \omega_{i,k}^{1,*} = 0 \). Then (36) can be further simplified.
\[
e_{x,k+1}^{(i-1)}(t)
= (I + B\Gamma_1)^{-1}(I - B\Gamma_0) e_{x,k}^{(i-1)}(t)
- (I + B\Gamma_1)^{-1} \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} A\Delta x_{n,k}(t_n) dt_n \cdots dt_i
\]
If \( \|(I + B\Gamma_1)^{-1}(I - B\Gamma_0)\| < 1 \), taking the \( \lambda \)-norm for the both sides of (37), and using Lemma 1 and 2, then the following result can be obtained.
\[
\lim_{k \rightarrow \infty} \|e_{x,k}^{(i-1)}(t)\|_{\lambda} \leq 0, \quad i = 1, 2, \ldots, n
\]
That is to say, uniform convergence of the system output to the desired trajectory is ensured on \([h_{1,2}, T]\) when \( k \rightarrow \infty \).

B. THE PRODUCT OF MATRICES C AND B IS INVERTIBLE
In this case, there is a following theorem about system (1) and control law (8).

Theorem 3: For a given desired trajectory \( y_d(t) \), if control law (8) is applied to system (1) under the Assumption 1, and there exist matrices \( C, B \) (The product of matrices C and B is invertible), \( \Gamma_0 \) and \( \Gamma_1 \) such that
\[
\|(I + CB\Gamma_1)^{-1}(I - CB\Gamma_0)\| < 1
\]
then the step-by-step rectifying control law (8) can guarantee that the tracking error \( e_k(t) \) is bounded. Especially, the tracking error \( e_k(t) \) can converges monotonically to zero when \( t \in [2^{n-1} T_p, T] \) and \( k \rightarrow \infty \).

Proof: Since the structure and proofs are similar to the previous one, this subsection contains only three steps, and step 3 in the last subsection is omitted.

Step 1: The proof of \( \lim_{k \rightarrow \infty} \|y_{d,k}^{(i-1)}(t)\|_{L_1} = 0 \)
When \( t \in [0, 2^{n-1} T_p] \), we suppose that the following assumption is realizable, where \( x_{n,k+1}(0) \) is arbitrary.
\[
y_{d,k}^{(i-1)}(t) = y_{d,k}^{(i-1)}(t) - (1 - \int_{0}^{t} \Theta_{n,k}(t_1) dt_1) e_{k}^{(n-1)}(0)
\]
Let’s denote the virtual error as
\[
ev_{k}^{(n-1)}(t) = y_{d,k}^{(i-1)}(t) - y_{k}^{(i-1)}(t)
\]
According to the definitions of \( \Xi_{i,k}(h_{1,2})(i = 1, 2, \ldots, n - 1) \), and using (1) and (8), we can obtain
\[
x_{n,k+1}(t) - x_{n,k}(t)
= \left[ x_{n,k+1}(0) - x_{n,k}(0) \right] + \int_{0}^{t} \left[ \dot{x}_{n,k+1}(t) - \dot{x}_{n,k}(t) \right] dt_1
= \Delta x_{n,k}(0) + \int_{0}^{t} B \Delta u_k(t_1) dt_1 + \int_{0}^{t} A \Delta x_{n,k}(t_1) dt_1
= \Delta x_{n,k}(0) + \int_{0}^{t} \Theta_{n,k}(t_1) B \Xi_{n,k}(0) dt_1
+ \int_{0}^{t} B \Gamma_0 \omega_{n,k}^{(n)}(t_1) dt_1 + \int_{0}^{t} B \Gamma_1 \omega_{n,k+1}^{(n)}(t_1) dt_1
- \int_{0}^{t} B \Gamma_1 e_{k+1}^{(n-1)}(t_1) dt_1
+ \int_{0}^{t} A \Delta x_{n,k}(t_1) dt_1
\]
According to the formula of integration, we have
\[
\int_{0}^{t} B \Gamma_0 e_{k}^{(n)}(t_1) dt_1 = B \Gamma_0 e_{k}^{(n-1)}(t) - B \Gamma_0 e_{k}^{(n-1)}(0)
\]
\[
\int_{0}^{t} B \Gamma_1 e_{k+1}^{(n)}(t_1) dt_1 = B \Gamma_1 e_{k+1}^{(n-1)}(t) - B \Gamma_1 e_{k+1}^{(n-1)}(0)
\]
Insert (41) with (42), and multiply with the matrix C at the both sides of (41).
\[
Cx_{n,k+1}(t) - Cx_{n,k}(t)
= C \Delta x_{n,k}(0) + \int_{0}^{t} \Theta_{n,k}(t_1) C B \Xi_{n,k}(0) dt_1
+ C B \Gamma_0 e_{k}^{(n-1)}(t) - C B \Gamma_0 e_{k}^{(n-1)}(0)
+ C B \Gamma_1 e_{k+1}^{(n-1)}(t) - C B \Gamma_1 e_{k+1}^{(n-1)}(0)
\]
Substituting $\Xi_{n,k}(0)$,
\[
C_{x_n,k+1}(t) - C_{x_n,k}(t) = e_k^{(n-1)}(t) - e_k^{(n-1)}(0)
\]
\[
= e_k^{(n-1)}(t) - e_k^{(n-1)}(0)
\]
\[
+ \int_0^t \Theta_n(t_1)CB\omega_k e_k^{(n-1)}(0)dt_1
- \int_0^t \Theta_n(t_1)\left[y_d^{(n-1)}(0) - y_k^{(n-1)}(0)\right]dt_1
+ CBT e_k^{(n-1)}(t) - CBT e_k^{(n-1)}(0)
+ \int_0^t \Theta_n(t_1)CB\omega_k e_k^{(n-1)}(0)dt_1
+ \int_0^t \Theta_n(t_1)y_d^{(n-1)}(t) - y_k^{(n-1)}(0)dt_1
\]
\[
= e_k^{(n-1)}(t) - e_k^{(n-1)}(0)
+ \int_0^t \Theta_n(t_1)CB\omega_k e_k^{(n-1)}(0)dt_1
- \int_0^t \Theta_n(t_1)y_d^{(n-1)}(0) - y_k^{(n-1)}(0)dt_1
\]
\[
\text{Simplifying (44), we can obtain}
\]
\[
(I - CBT)\omega_k e_k^{(n-1)}(t) = (I - CBT)\omega_k e_k^{(n-1)}(0)
\]
\[
= (I + CBT)\omega_k e_k^{(n-1)}(t) - (1 - \int_0^t \Theta_n(t_1)dt_1)e_k^{(n-1)}(0)
\]
(45)
\[
\text{According to the definition of } e_k^{(n-1),*}(t), \text{ it is easy to obtain}
\]
\[
e_k^{(n-1),*}(t) = y_d^{(n-1),*}(t) - y_k^{(n-1)}(t)
\]
\[
= y_d^{(n-1),*}(t) - y_d^{(n-1)}(t) + e_k^{(n-1)}(t)
\]
\[
= e_k^{(n-1)}(t) - (1 - \int_0^t \Theta_n(t_1)dt_1)e_k^{(n-1)}(0)
\]
(46)
\[
\text{Insert (45) with (46)}
\]
\[
e_k^{(n-1),*}(t) = (I + CBT)^{-1}(I - CBT)\omega_k e_k^{(n-1),*}(t)
\]
\[
\text{If } \| (I + CBT)^{-1}(I - CBT)\omega_k e_k^{(n-1),*}(t) \| < 1, \text{ applying lemma (2), it is easy to produce}
\]
\[
\lim_{k \to \infty} e_k^{(n-1),*}(t) = 0
\]
(47)
\[
\text{Accordingly, when } t = t_p, \text{ using Theorem 1, yield}
\]
\[
\lim_{k \to \infty} e_k^{(n-1),*}(t) = 0
\]
(48)
\[
\text{Step 2: The proofs of } \lim_{n \to \infty} \| e_k^{(i-1)}(t) \| = 0, t = h_{i,j}, j = i, i-1, \ldots, 1, i = n-1, \ldots, 1
\]
\[
\text{Subsequently, we can obtain}
\]
\[
x_{i,k+1}(t) - x_{i,k}(t)
= [x_{i,k+1}(h_{i,1}) - x_{i,k}(h_{i,1})] + \int_{h_{i,1}}^{t_{i-1}} \Delta x_{i,k}(t)dt_1
= \Delta x_{i,k}(h_{i,1}) + \Omega_{i,k}
+ \int_{h_{i,1}}^{t_{i-1}} \int_{h_{i,1}}^{t_{i-1}} \Delta x_{n,k}(t_2)dt_2 \cdots dt_1
\]
(49)
\[
\text{among them, the definition of } \Omega_{i,k} \text{ is the same as before.}
\]
\[
\text{Inserting (48) with (8),}
\]
\[
x_{i,k+1}(t) - x_{i,k}(t)
= \Delta x_{i,k}(h_{i,1}) + \Omega_{i,k}
+ \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{i-1}} B\omega_k e_k^{(n)}(t_1)
\]
\[
= \Theta(t_1)\Xi_{i,k}(h_{i,1}) + \gamma_{i+1}^{(n)}(h_{i,1}) + A\Delta x_{n,k}(h_{i,1})
+ (CB)^{-1}CA\Delta x_{n,k}(t_1)dt_1 \cdots dt_1
\]
(50)
\[
\text{Using the formula of integration, we can obtain}
\]
\[
\int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{i-1}} B\omega_k e_k^{(n)}(t_1)dt_1 \cdots dt_1
= \Theta(t_1)\Xi_{i,k}(h_{i,1}) + \gamma_{i+1}^{(n)}(h_{i,1}) + (CB)^{-1}CA\Delta x_{n,k}(t_1)dt_1 \cdots dt_1
\]
(51)
\[
\text{where}
\]
\[
\omega_{i,k}^2 = \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{i-2}} e_k^{(n-1)}(h_{i,1})dt_{n-1} \cdots dt_1
\]
\[
+ \cdots + \int_{h_{i,1}}^{t_{i-1}} \int_{h_{i,1}}^{t_{i-2}} e_k^{(n-1)}(h_{i,1})dt_{n-2} \int_{h_{i,1}}^{t_{i-1}} dt_1
\]
(52)
\[
\text{Insert (49) with (50) and (51), and multiply with the matrix } C \text{ at the both sides of (49)}
\]
\[
\Xi_{i,k} = C\Theta(t_1)\Xi_{i,k}(h_{i,1}) + C\gamma_{i+1}^{(n)}(h_{i,1})
\]
\[
+ C\Delta x_{n,k}(t_1)dt_1 \cdots dt_1
\]
\[
+ C\Omega_{i,k} - CBT\omega_{i,k}^2 - CB\gamma_{i+1}^{(n)}(h_{i,1})
\]
(53)
\[
\text{In fact, we can obtain the following equation.}
\]
\[
C\Omega_{i,k} - CBT\omega_{i,k}^2 - CB\gamma_{i+1}^{(n)}(h_{i,1})
= \int_{h_{i,1}}^{t_{i-1}} \cdots \int_{h_{i,1}}^{t_{i-2}} [(y_{n,d}(h_{i,1}) - y_{n,k}(h_{i,1}))
\]
\[
- (y_{n,d+k+1}(h_{i,1}))dt_{n-1} \cdots dt_1
\]
\[
+ \cdots + \int_{h_{i,1}}^{t_{i-1}} \int_{h_{i,1}}^{t_{i-2}} [(y_{i+2,d}(h_{i,1}) - y_{i+2,k}(h_{i,1}))
\]
(54)
Let’s denote the virtual error as:

\[ i \rightarrow \infty = (I - CB\Gamma_0)\omega_{k+1}^2 - (I + CB\Gamma_1)\omega_{k+1}^2 \]  

(53)

Substitute \( \Xi_{i,k}(0) \) and insert (52) with (53):

\[ (I + CB\Gamma_1)e_{k+1}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = (I - CB\Gamma_0)e_{k+1}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = (I - CB\Gamma_0)e_{k+1}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = (I - CB\Gamma_0)e_{k+1}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = (I - CB\Gamma_0)e_{k+1}^{(i-1)}(t) - \omega_{k+1}^2 \]

(54)

As before, we suppose that the following trajectory is realizable in kth iteration when \( t \in [2^{n-i-1}t_p, 2^{n-i}t_p) \), where \( e_{k+1}^{(i-1)}(h_{i,1}) \) is arbitrary due to the random \( e_{k+1}^{(0)}(0) \).

\[ y_{d}^{(i-1),*}(t) = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

\[ = y_{d}^{(i-1)}(t) - \omega_{k+1}^2 \]

(55)

Let’s denote the virtual error as:

\[ e_{k+1}^{(i-1),*}(t) = y_{d}^{(i-1)}(t) - y_i(t) \]

(56)

According to the definition of \( e_{k+1}^{(i-1),*}(t) \), it is easy to obtain:

\[ e_{k+1}^{(i-1),*}(t) = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

\[ = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

\[ = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

\[ = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

\[ = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

\[ = (I + CB\Gamma_1)e_{k+1}^{(i-1),*}(t) \]

(57)

If \( \| (I + CB\Gamma_1)^{-1} (I - CB\Gamma_0) \| < 1 \), applying lemma (2), it is easy to produce:

\[ \lim_{k \to \infty} \| e_{k+1}^{(i-1),*}(t) \| = 0 \]

(58)

Accordingly, when \( i = n - 1, t = 2t_p \), yield:

\[ \lim_{k \to \infty} \| e_{k}^{(n-2)}(t) \| = 2t_p \]

\[ = \lim_{k \to \infty} \| e_{k}^{(n-2),*}(t) + \int_{h_{i,1}}^{t} e_{x,k}^{(n-1)}(t_p)dt_1 + (1 - \int_{t_p}^{t} \int_{t_p}^{t_p} \Theta_{n-1}(t_n)dt_n \int_{t_p}^{t_p} e_{x,k}^{(n-2)}(t_p) \| = 2t_p \]

(59)

Using (58) and theorem (1), it produces:

\[ \lim_{k \to \infty} \| e_{k}^{(n-2)}(t) \| = 2t_p = 0 \]

(60)

Since it is similar to the last subsection, the following conclusions can be obtained:

\[ \lim_{k \to \infty} \| e_{k}^{(i-1)}(h_{i,2}) \| = 0, \quad i = n, n - 1, \cdots, 1 \]

\[ \lim_{k \to \infty} x_{i,k}(t) = x_{i,d}(t), \quad i = n, n - 1, \cdots, 1 \]

(61)

Step 3: The exact convergence of uniformly tracking error

When \( t \in [h_{1,2}, T] \), using control law (8), it is easy to produce:

\[ Cx_{i,k+1}(t) - Cx_{i,k}(t) \]

\[ = (I - CB\Gamma_0)e_{k+1}^{(i-1)}(h_{1,2}) + CB\Gamma_0 e_{k+1}^{(i-1)}(t) \]

\[ + CB\Gamma_1 e_{k+1}^{(i-1)}(t) - (I + CB\Gamma_1) e_{k+1}^{(i-1)}(h_{1,2}) \]

\[ + \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} A\Delta x_{n,k}(t_n)dt_n \cdots dt_1 \]

\[ + C\Omega_{i,k}^* - CB\Gamma_0 \omega_{k+1}^{2,*} - CB\Gamma_1 \omega_{k+1}^{2,*} \]

(62)

where:

\[ \omega_{k+1}^{2,*} = \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} e_{k}^{(n-1)}(h_{1,2})dt_{n-1} \cdots dt_1 \]

\[ + \cdots + \int_{h_{1,2}}^{t} \int_{h_{1,2}}^{t} e_{k}^{(i+1)}(h_{1,2})dt_1 \]

\[ + \int_{h_{1,2}}^{t} e_{k}^{(i)}(h_{1,2})dt_1 \]

When \( k \to \infty \), using (60), we can get \( \Omega_{i,k}^* = 0 \) and \( \omega_{k+1}^{2,*} = 0 \). Then (62) can be further simplified.

\[ e_{k+1}^{(i-1)}(t) \]

\[ = (I + CB\Gamma_1)^{-1}(I - CB\Gamma_0)e_{k+1}^{(i-1)}(t) - (I + CB\Gamma_1)^{-1} \int_{h_{1,2}}^{t} \cdots \int_{h_{1,2}}^{t} A\Delta x_{n,k}(t_n)dt_n \cdots dt_1 \]

(63)

If \( \| (I + CB\Gamma_1)^{-1}(I - CB\Gamma_0) \| < 1 \), taking the \( \lambda \)-norm for the both sides of (63), and using Lemma 1 and 2, then the result can be obtained:

\[ \lim_{k \to \infty} \| e_{k}^{(i-1)}(t) \|_\lambda \mid t \in [h_{1,2}, T] = 0, j = 1, 2, \cdots, n \]

That is to say, uniform convergence of the system output to the desired trajectory is ensured on \([h_{1,2}, T]\) when \( k \to \infty \).

Remark 4: Control law (3) and control law (8) have a sequential order in rectifying state errors. \( x_{n,k} \) has the highest priority and \( x_{1,k} \) has the lowest priority. In other words, both of them rectify the state error of \( x_{n,k} \) at first, then the errors of \( x_{n-1,k}, x_{n-2,k}, \cdots, x_{2,k}, x_{1,k} \) respectively, and the last rectifying actions are finished when time \( t = 2^{n-1}t_p \). From the definition of the functions \( \Theta_1 \), we can know that the actions of rectifying the errors of \( x_{i,k}(j = i + 1, i + 2, \cdots, n) \) have completed when rectifying the state \( x_{i,k} \), and the rectifying action of \( x_{i,k} \) will be complete at \( t = h_{i,2} \). Even though the rectifying action of \( x_{i,k} \) has been finished, it doesn’t mean that \( x_{i,k} \) has tracked the desired state \( x_{i,d} \) when \( t \in (h_{j,1}, h_{j,2}) \).
j = i - 1, i - 2, \ldots, 1 respectively. Because the equation (1) is true only when \( t = h_{j,j}, j = i - 1, i - 2, \ldots, 1 \) or \( t \in [2^{n-j}t_i, T] \) for any \( \Theta_j \). Once rectifying action of \( x_{i,k} \) is over, the effects upon \( x_{i,k} \) \( (j = i + 1, i + 2, \ldots, n) \) are also over and \( x_{i,k} \) \( (j = i + 1, i + 2, \ldots, n) \) return to the desired states \( x_{i,d} \) \( (j = i + 1, i + 2, \ldots, n) \) respectively. In one word, the essence of these algorithms is to add a pulse function for each \( x_{i,k} \) and there is an order when adding pulse functions. The first is for \( x_{n,k} \), then for \( x_{n-1,k}, \ldots, x_{2,k} \), and the last is for \( x_{1,k} \).

**Remark 5:** When rectifying the error of \( x_{i,k} \), the effects upon \( x_{j,k} \) \( (j = i, i + 1, \ldots, n - 1) \) are inevitable. Considering the limitations of control input, system rectifies the state errors in the following way. The smaller \( i \) is, the longer it takes when rectifying because the smaller \( i \) is, the more it effects. For example, rectifying action must have influenced on \( x_{2,k}, x_{3,k}, \ldots, x_{n,k} \) when rectifying \( x_{1,k} \).

**Remark 6:** The rectifying functions are not unique as long as they satisfied Theorem 1. For example, we can select the rectifying action must have influenced on \( x_{2,k}, x_{3,k}, \ldots, x_{n,k} \) when rectifying \( x_{1,k} \).

\[
\Theta_i(t) = \begin{cases} 
0 & t \in [0, h_{i,1}) \\
\frac{1}{\int_{h_{i,1}}^{h_{i,1}} \theta_i(\tau)d\tau} \theta_i(t)^{n-i} & t \in [h_{i,1}, h_{i,2}) \\
0 & t \in [h_{i,2}, T])
\end{cases}
\]

where \( \theta_i = t^{N-i}(t - h_{i,1})^{N-i}(h_{i,2} - t)^{N-i} \).

### IV. SIMULATION RESULTS

In this section, we will present two examples to demonstrate the effectiveness of the proposed methods. There is an invertible system gain matrix \( B \) in the first example, and the other is that the product of matrices \( C \) and \( B \) is nonsingular.

#### A. THE MATRIX \( B \) IS INVERTIBLE

We first consider the system (1) with matrices given by

\[
\begin{align*}
\dot{x}_{1,k}(t) &= x_{2,k}(t), \\
\dot{x}_{2,k}(t) &= \begin{pmatrix} -0.5 & 0.5 \\ 0 & -0.2 \end{pmatrix} x_{2,k}(t) + \begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.5 \end{pmatrix} u_k(t), \\
y_k(t) &= (0.5 \ 0.5)x_{1,k}(t)
\end{align*}
\]

The task interval of the system is \([0, 4]\) and the corresponding initial shifts rectifying interval is \([0, 0.8]\). The desired trajectories are such as

\[
\begin{align*}
x_{1,d}(t) &= \begin{pmatrix} \cos(\pi t/2) \\ \sin(\pi t/2) \end{pmatrix}, \\
x_{2,d}(t) &= \dot{x}_{1,d}(t) = \begin{pmatrix} \pi/2 \sin(\pi t/2) \\ \pi/2 \cos(\pi t/2) \end{pmatrix}
\end{align*}
\]

In the control law (3), the control gains and the candidate functions \( \Theta_i(t), i = 1, 2 \) are given as

\[
\Gamma_0 = \begin{pmatrix} 9.0000 & -9.0000 \\ -1.8000 & 3.6000 \end{pmatrix}; \quad \Gamma_1 = 0.5;
\]

In order to verify our ILC algorithm for a class of linear systems with stochastic initial shifts, we randomise the initial states of system (64) that \( x_{1,d}(0) = (1 - 0.2 \cdot \text{rand}; -0.1 \cdot \text{rand})^T \) and \( x_{2,d}(0) = (0.5 \cdot \text{rand}; \pi/2 + 0.5 \cdot \text{rand}) \), which rand function vary between 0 and 1 at different iterations of ILC process randomly. 10 iterations are performed during the simulation.

Via simulation, we obtain figures 1–5, which show that \( y_{1,k} \) and \( y_{2,k} \) completely track \( y_{1,d} \) and \( y_{2,d} \) in finite time which is specified in advance. The dashed, dashed—dotted and solid lines represent the system outputs as the control law (3) is iteratively executed 8, 9 and 10 times, respectively, and the dotted line represents the desired output in figures 1 and 3, and there are similar meanings in other figures.

Figures 1 and 2 demonstrate that the control law (3) does’t rectify the errors of the trajectory \( x_{1,k} \) in the interval \([0, 0.4]\), so the actual trajectories \( y_{1,k} \) doesn’t trend to the desired trajectory \( y_{1,d} \) in this process. the errors of \( x_{1,k} \) will be rectified in the second stage.

![Figure 1](image1.png)

**Figure 1.** Convergence performance of \( y_{1,k} \) in the presence of non-fixed initial shifts.

![Figure 2](image2.png)

**Figure 2.** Actual errors \( e_{1,k}(t) \) versus the non-fixed initial shifts.
From figures 3 and 4, we can know that $y_{2,k}$ has completely track $y_{2,d}$ when $t = 0.4$, but $y_{2,k}$ don’t follow up $y_{2,d}$ when $t \in (0.4, 0.8)$. The control law has completed rectifying the error of $x_{1,k}$ when $t = 0.8s$, and $y_{2,k}$ completely track $y_{2,d}$ when $t \in [0.8, 4]$

From figure 5, we can observe that two control inputs are continuous, because the functions $\Theta_i(t), i = 1, 2$ are continuous.

**B. THE PRODUCT OF MATRICES C AND B IS INVERTIBLE**

The second example is considered with the following form which the product of matrices $C$ and $B$ is nonsingular.

$$\begin{align*}
\dot{x}_{1,k}(t) &= x_{2,k}(t), \\
\dot{x}_{2,k}(t) &= x_{3,k}(t), \\
\dot{x}_{3,k}(t) &= \begin{pmatrix} -0.5 & 0.6 \\ 0.4 & -0.2 \end{pmatrix} x_{3,k}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\
y_k(t) &= (0.8 - 0.5)x_{1,k}(t)
\end{align*}$$

Transfer function of this system is $G(s) = (2 * (20 * s - 1))/(s ^ 2 * (50 * s ^ 2 + 35 * s - 7))$. Obviously, this system has a pole and a zero on the right half open plane of $s$. So it is a non-minimum phase system. One type of non-minimum phase system is a system containing non-minimum phase components or some local small loops are unstable systems; the other is a time-delay system. In the example, one of the eigenvalues of the system parameter matrix $A$ is greater than zero. So the system is unstable.

In the simulation, the task interval of the system is $[0, 8]$, and the corresponding initial shifts rectifying time interval is $[0, 3.2]$. The desired trajectories are given as

$$\begin{align*}
y_{1,d}(t) &= (0.8 - 0.5) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \\
y_{2,d}(t) &= \dot{y}_{1,d}(t) = (0.8 - 0.5) \begin{pmatrix} -\pi/2 \sin(t) \\ \pi/2 \cos(t) \end{pmatrix} \\
y_{3,d}(t) &= \dot{y}_{2,d}(t) = (0.8 - 0.5) \begin{pmatrix} \pi/4 - \pi/2 \cos(t) \\ \pi/4 - \pi/2 \sin(t) \end{pmatrix}
\end{align*}$$

In the control law (8), the control gains and the candidate functions $\Theta_i(t), i = 1, 2, 3$ are given as

$$\begin{align*}
\Gamma_0 &= 0.8; \quad \Gamma_1 = 0; \\
\Theta_1(t) &= \begin{cases}
0 & t \in [0, 1.6) \\
280 * 2^{14}/3.14^* t^2 *(t-1.6)^6 *(3.2 - t)^2 & t \in [1.6, 3.2) \\
*(41.6 - 26t) + (41.6t - 13t^2 - 3/2)* 3.2^2 & t  \\
*(t - 1.6)^5 *(3.2 - t) * (32t - 10t^2 - 3.2^2) & t \in [3.2, 8]
\end{cases}
\end{align*}$$

$$\begin{align*}
\Theta_2(t) &= \begin{cases}
0 & t \in [0, 0.8) \\
1024 * 60/(1.6^{10}) * t * (t-0.8)^4 & t \in [0.8, 1.6) \\
*(1.6-t) * (14.4 * t - 9 * t^2 - 2.56) & t \in [1.6, 8) \\
0 & t \in [8, 8]
\end{cases}
\end{align*}$$

$$\begin{align*}
\Theta_3(t) &= \begin{cases}
6/(0.8^3) * t * (0.8 - t) & t \in [0, 0.8) \\
0 & t \in [0.8, 8]
\end{cases}
\end{align*}$$

Similarly, for the sake of examining the robustness of our ILC algorithm for this kind of linear systems with variable initial shifts, we let the initial conditions be reset to $x_{1,k}(0) = (1 - 0.2* \text{rand}; 0.1* \text{rand})^T$, $x_{2,k}(0) = (0.5* \text{rand}; \pi/4 - 0.5* \text{rand})^T$ and $x_{3,k}(0) = (-3 * \text{rand}; 2 * \text{rand})$. 20 iterations are performed during the simulation.
We can conduct the simulation according to the above mentioned conditions and obtain figures 6 – 12, which show that $y_{1,k}$, $y_{2,k}$ and $y_{3,k}$ completely track $y_{1,d}$, $y_{2,d}$ and $y_{3,d}$ when $t \in [3.2, 8]$. Similarly, the dashed, dashed–dotted and solid lines represent the system outputs as the control law (8) is iteratively executed 18, 19 and 20 times, respectively, and the dotted line represents the desired output in figures 6, 8 and 10, and there are similar meanings in other figures.

Figures 6 and 7 demonstrate that the control law (8) doesn’t rectify the error of $y_{1,k}$ until $t \in [1.6, 3.2)$, so the actual trajectory $y_{1,k}$ doesn’t trend to the desired trajectory $y_{1,d}$ when $t \in [0, 1.6)$. The error of $y_{1,k}$ will be rectified in the third stage.

From figures 8 and 9, we can know that the control law has completed rectifying the error of $y_{2,k}$ when $t = 1.6s$, but $y_{2,k}$ don’t track $y_{2,d}$ perfectly when $t \in (1.6, 3.2)$. From figures 10 and 11, we can know that the control law has completed rectifying the error of $y_{2,k}$ and $y_{3,k}$ when $t = 1.6s$, but $y_{2,k}(t)$ and $y_{3,k}(t)$ don’t track $y_{2,d}(t)$ and $y_{3,d}(t)$ perfectly when $t \in (1.6, 3.2)$. Because it must effect tracking $y_{2,d}(t)$ and $y_{3,d}(t)$ when rectifying the state error $e_{1,k}(t)$. In fact, $y_{2,k}(t)$ and $y_{3,k}(t)$ change not only very large but also very fast in these process. It’s easy to understand, for instance, $y_{2,k}(t)$ and $y_{3,k}(t)$ actually represent the speed and acceleration, rectifying the errors of $y_{1,k}(t)$, i.e., changing displacement, it must need very large acceleration when spending little time. When $t = 3.2s$, rectifying the error of $y_{1,k}(t)$ is completed certainly, and the system has complete tracking $y_{1,d}(t)$, $y_{2,d}(t)$ and $y_{3,d}(t)$.

Figure 12 demonstrates that the control input are very large when $t \in (1.6, 3.2)$. Considering the limitations of system input in practice, it demands that the rectifying time can’t be too short.

There are two examples presented in this paper to demonstrate that the proposed step-by-step ILC methods for two classes of linear systems with stochastic initial shifts are...
very effective. They can overcome random initial shifts of ILC systems and can completely track the desired output trajectory over a small initial interval which is specified in advance.

V. CONCLUSION

The convergence and robustness properties of the step-by-step D-D type feedback iterative learning controllers have been presented in this paper. These controllers are suitable for the high-order systems with arbitrary initial shifts, and the different controllers are designed for the different types of systems. Applying one of these controllers, there is an order when they rectify the states shifts. After finishing rectifying the each state shift, the systems can achieve tracking completely. Compared with the conventional ILC methods, the robustness performances of the proposed algorithms versus arbitrary initial shifts are greatly improved by the initial rectifying actions. However, the control schemes only solve any initial value problem in iterative learning control theoretically, and there is still a gap from the actual application.

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