Counting Supershort Supermultiplets

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Abstract

We consider multiplet shortening for BPS solitons in $\mathcal{N}=1$ two-dimensional models. Examples of the single-state multiplets were established previously in $\mathcal{N}=1$ Landau-Ginzburg models. The shortening comes at a price of losing the fermion parity $(-1)^F$ due to boundary effects. This implies the disappearance of the boson-fermion classification resulting in abnormal statistics. To count such short multiplets we introduce a new index. We consider the phenomenon of shortening in a broad class of hybrid models which extend the Landau-Ginzburg models to include a nonflat metric on the target space. Our index turns out to be related to the index of the Dirac operator on the soliton moduli space. The latter vanishes in most cases implying the absence of shortening. We also generalize the anomaly in the central charge to take into account the target space metric.

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1 Introduction

Supersymmetry (SUSY) unites bosons and fermions into multiplets of degenerate states. It seems to imply that the minimal SUSY multiplet contains at least two states: one bosonic and one fermionic. Nevertheless, an example of the supermultiplet consisting of only one state is provided by a BPS soliton in $\mathcal{N}=1$ two-dimensional theories (see [1]). It is clear that this soliton is neither boson nor fermion — the fermion parity $(-1)^F$ becomes ill-defined in the soliton sector. The phenomenon is similar in many respects to the appearance of fractional charges for the 2D solitons [2].

The supersymmetry algebra in the $\mathcal{N}=1$ two-dimensional theories (often denoted as $\mathcal{N}=\{1,1\}$) is formed by two real supercharges $Q_\alpha$, ($\alpha = 1, 2$), the energy-momentum vector $P_\mu$, ($\mu = 1, 2$), and the central charge $Z$:

$$\{Q_\alpha, Q_\beta\} = 2 \left( \gamma^\mu P_\mu + \gamma^5 Z \right)_{\alpha\beta}, \quad [Q_\alpha, P_\mu] = [Q_\alpha, Z] = 0.$$

(1)

Here $\bar{Q}_\beta = Q_\alpha (\gamma^0)_{\alpha\beta}$ and $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, $\gamma^5 = i\gamma^0\gamma^1 = -i\sigma_1$ are purely imaginary two-by-two matrices. The central charge is nonvanishing in the soliton sector, and we define $Z > 0$ for the soliton ($Z < 0$ for the antisoliton).

In the rest frame of the soliton, $P_\mu = (M, 0)$, the algebra (1) takes the form

$$Q_1^2 = M + Z, \quad Q_2^2 = M - Z, \quad \{Q_1, Q_2\} = 0.$$

(2)

For the BPS soliton $Q_2 |\text{sol}\rangle = 0$ implying that its mass $M$ is equal to the central charge $Z$. Then it is clear that $Q_1$ equal to $\sqrt{2Z}$ (or $-\sqrt{2Z}$) leads to the irreducible representation of the superalgebra which is one-dimensional.

As was mentioned above in the one-dimensional representation the fermion parity $(-1)^F$ is not defined. Usually, it is implied, however, that $(-1)^F$ does exist. Indeed, microscopically we start with the local field theories where classification of the fields as bosonic and fermionic is explicit. For supercharges the fermion parity is $-1$ which makes the representation reducible — consisting of two one-dimensional representations [3].

How come that in the soliton sector $(-1)^F$ becomes ill-defined? This happens due to boundary effects. A technical signal is the existence of only one normalizable fermion zero mode on the BPS soliton. Another fermion zero mode (concentrated at the boundary) would appear if a finite box with proper boundary conditions were introduced. For physical measurements made far away from the boundary the fermion parity $(-1)^F$ is lost, and the one-dimensional multiplet becomes a physical reality. It is similar to the effect of the fractional charge: the total charge which includes boundaries stays integral but it is irrelevant for local experiments.

\[\text{Note, that in the same way the one-dimensional representation appears for the massless particles in 2+1 dimensions — the superalgebra there has the same form (1) with identification of } Z \text{ and } \gamma^5 \text{ as an extra component of } P_\mu \text{ and } \gamma^\mu. \text{ Maintaining } (-1)^F \text{ makes the representation two-dimensional and reducible, see e.g. [3].}\]
Once the boson-fermion classification for solitons is lost, it clearly implies certain abnormalities in the soliton statistics. In particular, the statistical counting of the multiplicity of states \( K \) (the multiplicity of states \( K \) is defined through the entropy per soliton of a gas of noninteracting solitons, \( S = \ln K \)) yields \( K = \sqrt{2} \). This unusual result was recently confirmed by Fendley and Saleur. In their stimulating work the Gross-Neveu model with \( N \) fermions was treated within the thermodynamical Bethe ansatz. For \( N = 3 \) the model coincides with the SUSY sine-Gordon model earlier considered by Tsvelik.

In this paper we address a few issues related to BPS solitons in \( \mathcal{N}=1 \) theories in 1+1 dimensions. We consider a class of \( \mathcal{N}=1 \) hybrid models where along with the superpotential \( \mathcal{W}(\phi) \) there is a nonflat metric \( g_{ab}(\phi) \) of the target space for the fields \( \phi^a \) (hybrids between the sigma models and Landau-Ginzburg models). Within these models we identify those in which the multiplet shortening does not take place. In fact, the existence of the BPS solitons belonging to one-dimensional supermultiplets turns out to be quite a rare occasion.

We introduce a new index which counts such short multiplets. Let us remind that the first SUSY index, \( \text{Tr} \ (-1)^F \), was introduced by Witten twenty years ago to count the number of supersymmetric vacua. About ten years ago, Cecotti, Fendley, Intriligator and Vafa introduced another index, \( \text{Tr}[F (-1)^F] \), counting the number of short multiplets in \( \mathcal{N}=2 \) theories in two dimensions. No index counting single-state multiplets in \( \mathcal{N}=1 \) theories in two dimensions (two supercharges) was known. This is probably not surprising, since it was always assumed that \( (-1)^F \) does exist.

Our task is to find such index for \( \mathcal{N}=1 \). We will show that the index is \( \{\text{Tr} \ Q_1\}^2/2Z \) — it vanishes for long multiplets and is equal to 1 for one-dimensional multiplets. If the value of this index does not vanish in the given \( \mathcal{N}=1 \) theory, short multiplets do exist with necessity.

Another issue we address is the generalization of the anomaly in the central charge found previously in the Landau-Ginzburg models, where the target space metric is flat, to the hybrid models with a nonflat metric. Our result for the anomaly in this case is a straightforward extension of and can be formulated as a substitution

\[
\mathcal{W}(\phi) \rightarrow \tilde{\mathcal{W}}(\phi) = \mathcal{W}(\phi) + \frac{1}{4\pi} \nabla^a \nabla_a \mathcal{W}(\phi)
\]

for the superpotential. The quantum anomaly is represented by term with the covariant Laplacian on the target space, \( \nabla^a \nabla_a \equiv g^{ab} \nabla_a \nabla_b \). The anomaly corrected superpotential enters into the energy-momentum tensor, the supercharges and the central charge. In particular, the operator of the central charge becomes

\[
Z = \tilde{\mathcal{W}}(\phi(z \to \infty)) - \tilde{\mathcal{W}}(\phi(z \to -\infty)).
\]

A more detailed discussion of the above issues and our results will be given elsewhere.

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3 The abnormal statistics were extensively discussed in the literature, see e.g. [7, 8].
2 The model

A generic hybrid model in two dimensions, with two supercharges contains \( k \) real bosonic fields \( \phi^a \ (a = 1, \ldots, k) \) and the same number of the real two-component fermionic fields \( \psi^a_\alpha \ (\alpha = 1, 2) \). The model is characterized by a metric \( g_{ab}(\phi) \) and a superpotential \( \mathcal{W}(\phi) \) defined on the target space for which \( \phi^a \) serve as coordinates.

We assume the target space to be an arbitrary Riemann manifold \( T \) and \( \mathcal{W}(\phi) \) an arbitrary function which has more than one critical point, i.e. points where \( \partial_a \mathcal{W}(\phi) = 0 \), on \( T \).

The general form of the Lagrangian is (for a review see Ref. [14])

\[
\mathcal{L} = \frac{1}{2} g_{ab} \left[ \partial_\mu \phi^a \partial^\mu \phi^b + \bar{\psi}^a i \gamma_\mu D_\mu \psi^b + F^a F^b \right] + \frac{1}{12} R_{abcd}(\bar{\psi}^a \psi^c)(\bar{\psi}^b \psi^d) + F^a \partial_a \mathcal{W} - \frac{1}{2} (\nabla_a \partial_b \mathcal{W}) \bar{\psi}^a \psi^b ,
\]

(5)

where \( F^a \) is the auxiliary field, \( F^a = -g^{ab} \partial_b \mathcal{W} \), and \( \partial_a \) and \( \nabla_a \) denote the conventional and covariant derivatives on the target space, e.g. for the vector, \( \nabla_a V_b = \partial_a V_b - \Gamma^c_{ab} V_c \). The covariantized space-time derivative \( D_\mu \) is

\[
D_\mu \psi^b = \partial_{x^\mu} \psi^b + \Gamma^b_{cd} \frac{\partial \phi^c}{\partial x^\mu} \psi^d .
\]

(6)

Furthermore, \( \Gamma^b_{cd}(\phi) \) and \( R_{abcd}(\phi) \) stand for the Christoffel symbols and the Riemann tensor on \( T \).

The \( \mathcal{N}=1 \) supersymmetry of the model is expressed by two supercharges,

\[
Q_\alpha = \int dz J^0_\alpha , \quad J^\mu = g_{ab} (\bar{\phi} \phi^a - i F^a) \gamma^\mu \psi^b ,
\]

(7)

where \( J^\mu \) is the conserved supercurrent. These supercharges form a centrally extended \( \mathcal{N} = 1 \) superalgebra (1) with the central charge

\[
Z_0 = \int dz \partial_z \phi^a \partial_a \mathcal{W} .
\]

(8)

The central charge does not vanish for classical solutions interpolating between different vacua of the theory, \( \phi = A \) at \( z = -\infty \) and \( \phi = B \) at \( z = \infty \). These vacua correspond to two distinct critical points of the superpotential \( \mathcal{W}(\phi) \). In the classical approximation for the soliton interpolating between the critical points \( A \) and \( B \) the central charge is equal to

\[
Z_0 = \mathcal{W}(B) - \mathcal{W}(A) .
\]

(9)

The BPS soliton satisfies the following equation:

\[
\frac{d\phi^a_{\text{sol}}}{dz} = g^{ab} \partial_b \mathcal{W}(\phi_{\text{sol}}) ,
\]

(10)

where \( z \) is the spatial coordinate.
3 Anomaly

Above, the subscript 0 marks a bare central charge $Z_0$, which, unlike $\mathcal{N}=2$ models, gets renormalized by quantum corrections. The model (3) is renormalizable provided the manifold $\mathcal{T}$ is symmetric, it is super-renormalizable when the metric is flat, e.g., $g_{ab} = \delta_{ab}$. In Ref. [12] it was shown that the super-renormalizable Landau-Ginzburg models with flat metric possess an anomaly in the central charge, a superpartner of the trace anomaly in the energy-momentum tensor. Parallelizing the derivations presented in Sec. III of [12] it is straightforward to include the target space metric. The result reduces to the substitution (3) for the superpotential. The anomalous part given by action of the covariant Laplacian can be written as

$$\frac{1}{4\pi} \nabla^a \nabla_a \mathcal{W} = \frac{1}{4\pi} g^{ab} \nabla_a \partial_b \mathcal{W}. \quad (11)$$

The expression for the central charge becomes

$$Z = \tilde{\mathcal{W}}(B) - \tilde{\mathcal{W}}(A), \quad (12)$$

where the anomaly corrected superpotential $\tilde{\mathcal{W}}$ defined by (3) takes place of $\mathcal{W}$. It is this expression which gives the mass of the BPS saturated soliton. Note that while the classical result (9) for the central charge $Z_0$ is metric independent a dependence on the target space metric emerges through the anomaly.

Omitting details of the derivation let us only note that the form of the anomaly in the hybrid models is constrained by the following considerations: (i) dimension and locality; (ii) general covariance in the target space; (iii) in the limit of flat metric it must coincide with the anomaly established in the Landau-Ginzburg models [12].

4 Tr $Q_1$ as an index

The construction of the BPS representations was discussed in the Introduction. In the soliton rest frame the centrally extended $\mathcal{N} = 1$ superalgebra takes the form (2). For the BPS soliton $M = Z$, and

$$Q_2|\text{sol}\rangle = 0, \quad (13)$$

while

$$Q_1|\text{sol}\rangle = \pm \sqrt{2Z} |\text{sol}\rangle. \quad (14)$$

Equation (14) implies that the fermion parity $(-1)^F$ is not defined for the irreducible one-dimensional supermultiplets.

Is there an index in the $\mathcal{N} = 1$ soliton problem which would count the BPS multiplets, i.e., states annihilated by the supercharge $Q_2$?

We assert that $\{ \text{Tr} Q_1 \}^2$ does the job. More exactly, the definition of the index is as follows

$$\text{Ind}_Z (Q_2/Q_1) = \frac{1}{2Z} \left\{ \lim_{\beta \to \infty} \text{Tr} \left[ Q_1 \exp(-\beta (Q_2)^2) \right] \right\}^2. \quad (15)$$
The exponential factor in Eq. (15) is introduced for the UV regularization. The necessity of taking the \( \beta \to \infty \) limit is due to continuous spectrum as is explained in [11].

This index vanishes for non-BPS multiplets. Indeed, as will be explained shortly, for non-BPS multiplets the fermion parity \((-1)^F\) can be consistently defined and \(\text{Tr} \; Q_1\) vanishes along with \(\text{Tr} \; (-1)^F\). For each irreducible BPS representation the index is unity,

\[
\text{Ind}_Z (Q_2/Q_1) [\text{irreducible BPS}] = 1.
\]

If a reducible representation contains a few irreducible BPS multiplets the index may or may not vanish. For the vanishing index one can introduce \((-1)^F\) and small deformations can destroy the BPS saturation. Note, that our index is not additive: it is not equal to the sum of the indices of the irreducible representations. An interesting example of a \(\mathcal{N}=1\) reducible BPS representation is provided by solitons in \(\mathcal{N}=2\) models. The \(\mathcal{N}=2\) BPS multiplet consists of two \(\mathcal{N}=1\) multiplets and the index (15) vanishes.

Thus, \(\{ \text{Tr} Q_1 \}^2/2Z\) counts the number of the BPS solitons annihilated by \(Q_2\), in the sector characterized by the value of the central charge \(Z\).

The definition (15) has a technical drawback — it refers to the soliton rest frame. It is simple to make it Lorentz invariant,

\[
\text{Ind}_Z (Q_2/Q_1) = \frac{1}{2Z^2} \left( \text{Tr} \; \bar{Q} \right) P \left( \text{Tr} \; Q \right),
\]

where the trace refers to the Hilbert space but not to the Lorentz indices of the supercharges \(Q_\alpha\) and \(\bar{Q}_\alpha = Q_\beta (\gamma^0)_{\alpha\beta}\). Here we have omitted the regularizing exponent.

Let us show now that for non-BPS representations the fermion-boson classification, based on \((-1)^F\), is well defined. For non-BPS solitons, with \(M > Z\), the irreducible representation of the algebra (2) is two-dimensional. For instance, one can choose

\[
Q_1 = \sigma_1 \sqrt{M + Z}, \quad Q_2 = \sigma_2 \sqrt{M - Z},
\]

where \(\sigma_1,2\) are the Pauli matrices. The boson-fermion classification in this two-dimensional representation is well known, the operator \((-1)^F\) (anticommuting with \(Q_\alpha\)) is represented by \(\sigma_3\).

Generically, for non-BPS supermultiplets one can define \((-1)^F\) in terms of supercharges as

\[
(-1)^F = \frac{Q \bar{Q}}{2 \sqrt{M^2 - Z^2}}.
\]

It is obvious that once \((-1)^F\) is defined the trace of the operators \(Q_\alpha\) connecting bosonic and fermionic states vanishes. The trace of \((-1)^F\) also vanishes since \(\text{Tr} \; \bar{Q} \; Q = -i \text{Tr} \; [Q_1, Q_2] = 0\).
Note, that for non-BPS multiplets not only the fermion parity but also $F$ as a generator of a U(1) symmetry can be defined, namely

$$F = \frac{1}{2} \left[ 1 - (-1)^F \right].$$

(19)

Let us emphasize, however, that unlike $\mathcal{N} = 2$ models, no local current associated with $F$ exists, and the fermion charge \([13]\) has no local representation.

5. **Tr $Q_1$ as an index of the Dirac operator**

Here is the central point: the index defined in Eq. (13) coincides with the square of the index of a Dirac operator on the (reduced) moduli space of solitons, which was studied by mathematicians. Thus, it is possible to determine in which $\mathcal{N} = 1$ models $\text{Ind}_z (Q_2/Q_1) = 0$, i.e. the multiplet shortening does not take place (in the general situation).

For every critical point $A$ the Morse index of this point $\nu(A)$ is defined as the number of the negative eigenvalues in the matrix of the second derivatives

$$H_{ab}(\phi) = \nabla_a \partial_b W(\phi)$$

(20)

at $\phi = A$. At the critical points the covariant derivative $\nabla_a$ coincides with the regular $\partial_a$. For solitons interpolating between two critical points, $\phi = A$ at $z \to -\infty$ and $\phi = B$ at $z \to \infty$ one can determine the relative Morse index $\nu_{BA}$,

$$\nu_{BA} = \nu(B) - \nu(A).$$

(21)

This relative Morse index counts the difference between the numbers of the zero modes of the operators $P$ and $P^\dagger$,

$$\nu_{BA} = \ker \{P\} - \ker \{P^\dagger\},$$

(22)

where $P$ and $P^\dagger$ are

$$P_{ab} = g_{ab}D_z - H_{ab}, \quad P^\dagger_{ab} = -g_{ab}D_z - H_{ab}.$$  

(23)

Here $D_z$ is defined in Eq. (6), and the field $\phi$ is taken to be $\phi_{\text{sol}}(z)$.

For the BPS soliton, satisfying Eq. (11), one zero mode certainly present in $P$ is the translational mode. It corresponds to the soliton center $z_0$, one of the coordinates in the soliton moduli space. The same zero mode of $P$ is the fermion zero mode — the corresponding modulus $\eta$ is the superpartner of $z_0$.

We will limit ourselves to the case when $\ker P^\dagger = 0$. (Note that even if that is not the case, one can get rid of the zero modes in $P^\dagger$ by small deformations of the superpotential). Then, the Morse index

$$\nu_{BA} \equiv n + 1 \geq 1$$

(24)
counts the dimension of the soliton moduli space $M^{n+1}$. Thus, we arrive at quantum mechanics of $n + 1$ bosonic and $n + 1$ fermionic moduli on $M^{n+1}$.

The simplest case $n = 0$ was analyzed in [1]. In this case the quantum moduli dynamics is trivial, and the single state BPS multiplet does exist, $\text{Ind}_Z (Q_2/Q_1) = 1$. We will see shortly that for odd $n$ the index $\text{Ind}_Z (Q_2/Q_1) = 0$, and the soliton supermultiplets are long (or reducible). We will start from a less trivial case of even $n$, only in this category can one expect to find $\text{Tr} Q_1 \neq 0$.

As was mentioned above one of $n + 1$ bosonic moduli is $z_0$, the coordinate of the soliton center. This is a cyclic coordinate conjugated to the generator $P_z$ of the spatial translations, $z_0 \in \mathbb{R}$. Note an ambiguity in $z_0$—one can add to $z_0$ an arbitrary function of other moduli. This ambiguity is fixed by the definition given below, see Eq. (29). Thus, the moduli space $M^{n+1}$ is a direct product

$$M^{n+1} = \mathcal{R} \otimes \mathcal{M}^n$$

of $\mathcal{R}$ and the manifold $\mathcal{M}^n$ with coordinates $m^1, \ldots, m^n$ describing internal degrees of freedom of the soliton. This manifold $\mathcal{M}^n$ can be called the reduced moduli space.

It is instructive to elucidate the factorization (25) in more detail. We must show that the moduli space metric $h_{ij}$,

$$h_{ij}(m) = \int dz g_{ab}(\phi_{sol}) \frac{\partial \phi^a_{sol}}{\partial m^i} \frac{\partial \phi^b_{sol}}{\partial m^j}, \quad i, j = 0, 1, 2, \ldots, n, \quad (26)$$

where $m^0 \equiv z_0$, has a block form, i.e. $h_{0j} = 0$ for $j = 1, 2, \ldots, n$. Indeed,

$$h_{0j}(m) = -\int dz \partial_\theta \mathcal{W}(\phi_{sol}) \frac{\partial \phi^\theta_{sol}}{\partial m^j} = -\frac{\partial}{\partial m^j} \int dz \left[ \mathcal{W}(\phi_{sol}) - \mathcal{W}(\phi_{sol})_{m=m_*} \right], \quad (27)$$

where we use the fact that the soliton solution depends on the spatial coordinate only through the combination $z - z_0$, to replace $\partial \phi^\theta_{sol}/\partial m^0$ by $\partial \phi^\theta_{sol}/\partial z$, which, in turn, can be replaced by $g^{ac} \partial_c \mathcal{W}(\phi_{sol})$ by virtue of Eq. (10). We also regularized the integral on the right-hand side of Eq. (27) by subtracting from the integrand the superpotential at some fixed values of the moduli $m = m_*$. Considering Eq. (27) for $h_{00}$ we get

$$h_{00} = -\frac{\partial}{\partial m^0} \int dz \left[ \mathcal{W}(\phi_{sol}) - \mathcal{W}(\phi_{sol})_{m=m_*} \right]. \quad (28)$$

Having in mind $h_{00} = Z$ we define the modulus $m^0$ as

$$m^0 = -\frac{1}{Z} \int dz \left[ \mathcal{W}(\phi_{sol}) - \mathcal{W}(\phi_{sol})_{m=m_*} \right]. \quad (29)$$

With this definition it is clear that

$$h_{0j} = Z \frac{\partial m^0}{\partial m^j} = 0, \quad (j = 1, \ldots, n). \quad (30)$$
Thus, the Lagrangian describing the moduli dynamics has the form

\[ L(M^{n+1}) = -\mathcal{Z} + \frac{\mathcal{Z}}{2} \left[ (\dot{z}_0)^2 + i \eta \dot{\eta} \right] + L(M^n), \]  

(31)

where \( L(M^n) \) is the Lagrangian of the internal moduli, both bosonic and fermionic, a sigma-model quantum mechanics on \( M^n \). We see, that the motion of the center of mass (together with its fermionic partner) is factored out, and we only need to consider the dynamics on \( M^n \).

Quantization of \( L(M^n) \) is standard. All operators act in the Hilbert space of the spinor wave functions \( \Psi_\alpha(m) \), where \( \alpha = 1, \ldots, 2^{n/2} \). The operators \( m^i \) act as multiplication, while \( \dot{m}_i \) become matrix-differential operators. The fermion moduli (their anticommutators form a Clifford algebra) become \( \gamma \) matrices of dimension \( 2^{n/2} \times 2^{n/2} \). Remember, \( n \) is even, so, there is \( \gamma^{n+1} = \prod_{i=1}^{n+1} \gamma^i \), an analog of \( \gamma^5 \) in four dimensions. On the moduli space \( M^n \) the supercharges (7) take the form

\[ Q_1 = \sqrt{2Z} \gamma^{n+1}, \quad Q_2 = -\frac{i}{\sqrt{2}} \gamma^j \nabla_j, \]  

(32)

where the covariant derivative \( \nabla_j \) includes spin connection (for more details see [13]). The expression for \( Q_2 \) is in fact the Dirac operator \( i \not\nabla \) on \( M^n \). Moreover, the Hamiltonian takes the form,

\[ H - \mathcal{Z} = Q_2^2 = \frac{1}{2} (i \not\nabla)^2 = -\frac{1}{2} \nabla^j \nabla_j + \frac{1}{8} \tilde{R}, \]  

(33)

where we used the famous Lichnerowicz formula, and \( \tilde{R} \) is the curvature in the soliton moduli space.

From Eqs. (32), (33) it is clear that the BPS soliton states are in correspondence with the zero modes of the Dirac operator \( i \not\nabla \) on \( M^n \). The index \( \text{Ind}_{\mathcal{Z}} (Q_2/Q_1) \) we defined in Eq. (15) becomes the square of the index of the Dirac operator

\[ \text{Ind}_{\mathcal{Z}} (Q_2/Q_1) = \left\{ \text{Ind} (i \not\nabla)_{M^n} \right\}^2, \]

\[ \text{Ind} (i \not\nabla)_{M^n} = \text{Tr} \left[ \gamma^{n+1} \exp \left( \beta \not\nabla \not\nabla \right) \right]_{M^n}. \]  

(34)

Equation (33) shows that if the curvature \( \tilde{R} \) is positive everywhere on the soliton moduli space the Dirac operator has no zero modes, its index vanishes, and so does the index \( \text{Ind}_{\mathcal{Z}} (Q_2/Q_1) \). Thus, there is no BPS solitons in this case. An explicit example [13] is provided by a sigma model on \( S^3 \). For a certain choice of the superpotential the soliton moduli space is the sphere \( S^2 \). Moreover, there exists a general mathematical assertion [15]: for any compact \( M^n \) with \( n \geq 2 \) the index of the Dirac operator vanishes. The proof due to P. Pushkar’ is outlined in Appendix of [13].

Thus, for \( n \geq 2 \) all soliton multiplets are long. If, for accidental or other reasons, they are still BPS saturated, they form a reducible representation. For example, in
$\mathcal{N}=2$ models the index $\text{Ind}_Z (Q_2/Q_1)$ vanishes while the BPS states do exist. From the standpoint of $\mathcal{N}=1$ they form a reducible representation for which $(-1)^F$ is well defined.

Summarizing the case of even $n$, we conclude that only for $n = 0$, when $\mathcal{M}^n$ reduces to a point, the index $\text{Ind}_Z (Q_2/Q_1) = 1$, and a single-state multiplet exists.

There is no general statement for noncompact $\mathcal{M}^n$. Noncompact geometry of $\mathcal{M}^n$ may emerge if there is an extra critical point $C$ such that $Z_{AB} = Z_{AC} + Z_{CB}$. Physically it means that the soliton $AB$ is a threshold bound state of lighter solitons $AC$ and $CB$.

Return now to the case of odd $n$. Now, there is no matrix $\gamma^{n+1}$ on $\mathcal{M}^n$; the number of spinorial components of the wave functions jumps by a factor of two compared to the previous even value of $n$, becoming $2 \times 2^{(n-1)/2}$. In this case the realization of the fermion moduli can be chosen as follows:

$$
\eta = \frac{1}{\sqrt{2}} I \times \sigma_3, \quad \eta^i = \frac{1}{\sqrt{2}} \gamma^i \times \sigma_1, \quad (i = 1, \ldots, n), \quad (35)
$$

where the matrices $\gamma^i$ are are of the same dimension as in the previous even $n$. Since $Q_1 \propto \eta$, the representation (35) clearly demonstrates that $\text{Tr} Q_1 = 0$.

### 6 Outlook

The first example of the single-state supersymmetric multiplet was suggested by Witten \[16\] in the context of 2+1 supergravity with the conic geometry. This example was thoroughly studied in Refs. \[17, 18\] where the BPS solitons were explicitly constructed. Out of four supercharges of the model two supercharges annihilate the BPS solitons. The other two supercharges produce the fermion zero modes. Without gravity these modes are normalizable which leads to two-state supermultiplet. With gravity switched on the fermion modes become non-normalizable, implying the single-state supermultiplet. This means that in the physical sector of the localized states all supercharges act on the soliton trivially.

In our $\mathcal{N}=1$ examples of the single-state supermultiplet one of two supercharges is realized nontrivially, $Q_1 = \pm \sqrt{2Z}$. In terms of modes there is one normalizable fermion mode. To compare with Witten’s example it is convenient to have in mind an infrared regularization — placing the system in a finite spatial box with supersymmetric boundary conditions. Then the normalizability is not a criterion, and the number of the fermion modes is always even. One can trace, however, the localization of the modes. In Witten’s case both zero modes are localized at the boundary. In our case one mode is localized on the soliton, while the other at the boundary, see Ritz \textit{et al.} in \[1\]. If one considers the entire system, including the boundary, the fermion parity $(-1)^F$ is defined. It is not defined, however, for localized states far away from the boundary. Similar run-away behavior of the modes occurs in fractional charge and other phenomena known in solid state physics.
A different way to maintain the fermion parity \((-1)^F\) in the soliton sector was suggested by Zamolodchikov \cite{5}. He studied a certain massive perturbation of the tricritical Ising model (in the field theory limit it leads to supersymmetric theory with the cubic superpotential). Rewriting the model in new variables Zamolodchikov arrives at three vacua instead of two. As a result, he doubles the multiplicity of the solitons, thus maintaining \((-1)^F\). Algebraically, it is a reducible representation of supersymmetry. From our point of view based on the quasiclassical quantization this doubling is unphysical (for localized particles) and should be eliminated. One of the possible ways to do it is through “orbifolding” (see Ref. \cite{19} for the relevant discussion in terms of lattice variables). Zamolodchikov’s idea was applied recently \cite{9} to the Gross-Neveu model (which for \(N = 3\) is equivalent to the supersymmetric sine-Gordon model).

Here we would like to stress a general nature of the above delocalization phenomenon which refers equally to minimal and extended supersymmetries. In any theory which is completely regularized in the infrared, the BPS shortening, strictly speaking, does not take place. Extra states living on the boundary make multiplets long, the BPS shortening is only recovering in the field-theoretic limit of the infinite volume.

In conclusion let us summarize our main points. We considered the previously established phenomenon of the multiplet shortening in a more general class of \(\mathcal{N}=1\) models in two dimensions. The generalization consists of introducing a non-flat metric on the target space. In most cases the shortening does not take place. In those rare cases when it does, it comes at a price of loosing the fermion parity \((-1)^F\), i.e. the disappearance of the boson-fermion classification.

To count such short multiplets we introduce a new index (15) (see also (16)). This index turns out to be related to the index of the Dirac operator on the reduced soliton moduli space. The latter vanishes for all compact moduli manifolds implying the absence of shortening. Finally, we generalize the anomaly in the central charge to take into account the target space metric.

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