Combining frequency-difference and ultrasound modulated electrical impedance tomography

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Abstract
Electrical impedance tomography (EIT) is highly affected by modeling errors regarding electrode positions and the shape of the imaging domain. In this work, we propose a new inclusion detection technique that is completely independent of such errors. Our new approach is based on a combination of frequency-difference and ultrasound modulated EIT measurements.

Keywords: electrical impedance tomography, frequency-difference EIT, ultrasound modulated EIT, anomaly detection, monotonicity, localized potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

The goal of electrical impedance tomography (EIT) is to image the conductivity inside a subject. To that end, electrodes are attached to the subject’s boundary, and one measures the voltages that are required to drive a specified static or time-harmonic current through different combinations of the attached electrodes. The potential advantages of EIT compared to other imaging technique are that conductivity values are typically of a high specificity, and that EIT devices are comparatively cheap and easily portable.

The inverse problem of reconstructing the conductivity from boundary voltage and current measurements is known to be highly non-linear and ill-posed. The measurements are
very insensitive to changes in the conductivity values away from the electrodes. They do, however, strongly depend on the measurement geometry, i.e., the electrode position and the shape of the imaging domain. In most applications, it is not feasible to precisely measure the geometry, and electrodes are frequently placed by hand. Hence, such modeling or geometry errors present a major challenge for practical EIT applications.

The main focus of EIT is often on the detection and localization of conductivity inclusions or anomalies (e.g., material faults or pathological regions) inside an otherwise more or less homogeneous medium.

In this work, we propose a new measurement setup for anomaly detection and describe a reconstruction method that is completely unaffected by geometrical modelling errors, as it does not require knowledge of the electrode position or the shape of the imaging domain.

The main idea of our new technique is to combine ultrasound-modulated EIT (UMEIT) measurements with frequency-difference EIT measurements. We focus an ultrasound wave on a small region inside the imaging domain to alter the conductivity in the focusing region. The resulting effect on the EIT measurements is then compared to the effect of a change in the electric current frequency. This comparison shows whether the focusing region lies inside a conductivity anomaly or not.

To decide whether the focusing region lies inside an anomaly, our method utilizes only the two sets of EIT measurements (with ultrasound-modulation and after the frequency change) and the ratio of the background conductivity before and after the frequency change. The latter can be estimated from comparing EIT measurements before and after the frequency change, as it is done in weighted frequency-difference EIT (see the references below). The method can be implemented using simple monotonicity tests, i.e., the taken voltage measurements are arranged in the form of a matrix and then compared in the sense of matrix definiteness (resp., in the idealized case of continuous boundary measurements, the measurements are interpreted as Neumann-to-Dirichlet operators and compared in the sense of definiteness of self-adjoint compact operators).

Our new method does not use any forward simulations, or explicitly known special solutions, that would depend on the geometry of the setup. It does not require any knowledge of the electrode position or the shape of the imaging domain, and is hence completely unaffected by modeling errors.

We give a complete proof for our method for the case of continuous boundary data, when the measurements are given by the Neumann-to-Dirichlet-operator. For the case of measurements on a finite number of electrodes, we prove that the method correctly identifies the case where the focusing region lies inside the anomaly. We also give a physical justification (in the spirit of [36]), that regions outside the anomaly will correctly be identified if enough electrodes are used for the measurements, see remark 3.3.

Let us now comment on related works and the origins of our approach. For a broad overview on EIT see [1, 6, 8, 10–13, 16, 39, 40, 60, 61, 64, 75]. For the task of anomaly detection in EIT, let us refer to Friedmann and Isakov [20, 21] for early works, Potthast [65] for an overview on non-iterative methods, and [35] for the recent result that shape information is invariant under linearization. Iterative anomaly detection methods are commonly based on level-set approaches, see, e.g. [18, 19, 52, 66, 72, 77]. Prominent non-iterative anomaly detection methods are the Factorization Method (see [7, 15, 17, 22, 24, 26–28, 31, 34, 36, 43, 54, 59, 63, 67, 68] and the recent overviews [30, 33, 55]), the enclosure method (see [14, 44, 45, 47–51, 76]), and the recently emerging monotonicity method (see the references below).

Our new method is based on a monotonicity-based comparison of weighted frequency-difference EIT (fdEIT) and UMEIT measurements. Monotonicity-based comparisons were
first considered as heuristic inclusion-detection methods and numerically tested by Tam- 
burrino and Rubinacci [73, 74]. Recently, the monotonicity method was rigorously justified 
[38] using the concept of localized potentials [23]. Weighted fDEIT has been introduced in 
order to improve the reconstruction stability with respect to modeling errors in settings where 
no reference (anomaly free) data is available, see [34, 36, 69]. The hybrid tomography 
technique UMEIT was introduced in [2, 79], see also [3–5, 9, 56–58, 62, 78] for more works 
on this subject. When the measurement geometry is known, UMEIT allows to measure the 
interior electrical energy of the subject by altering the conductivity with a focused ultrasound 
waves (see the related idea of impedance-acoustic tomography [25], where interior energy 
data is obtained from measuring expansion effects caused by electrical heating). Knowledge 
of this additional interior energy information eliminates the major cause of ill-posedness in 
the reconstruction process, which could greatly increase image resolution. Moreover, let us 
mention that combinations of EIT and ultrasound have been studied that rely on data-fusion 
rather than on coupled physics, e.g., by using ultrasound images as prior information for EIT 
reconstructions, see, e.g., [70, 71].

At this point, it has to be noted, that (up to the knowledge of the authors) the idea of 
using focused waves in UMEIT yet has to be experimentally validated. The results in this 
work are derived under the idealistic assumption of a perfectly focused ultrasound waves that 
changes the conductivity in a well-defined circular region. Of course, in reality, such a perfect 
focus cannot be realized, and the ultrasound wave will also affect the conductivity outside the 
focusing region. Moreover, the location of the focusing region will not be known exactly but 
depend on the measurement geometry. It is, however, widely accepted that in typical EIT 
applications, conductivity contrast is much higher than ultrasound contrast, while ultrasound 
resolution is much higher than EIT resolution. Therefore we believe that techniques relying 
on UMEIT are worth investigating despite the current lack of practical validation.

The paper is organized as follows. In section 2, we start with describing the general 
setting of complex conductivity EIT and ultrasound modulated EIT for continuous boundary 
data. Then we derive a monotonicity relation for complex conductivity EIT, and use this 
relation to develop an anomaly detection algorithm that is based on comparing EIT mea-
surements at a non-zero frequency with ultrasound-modulated dc measurements. Section 3 
contains the corresponding results for a setting with finitely many electrodes using the shunt 
electrode model. In section 4, we illustrate our new method with two- and three-dimensional 
numerical results. Section 5 concludes the paper with a discussion of our results.

2. Continuous boundary data

2.1. The setting

We start by describing the general setting of complex conductivity EIT and ultrasound 
modulated EIT with continuous boundary data. We consider a bounded imaging domain 
$\Omega \subset \mathbb{R}^n, n \geq 2$ with piecewise smooth boundary. For $x \in \Omega$, let 
$$
\gamma_\omega(x) = \sigma_\omega(x) + i\omega\epsilon_\omega(x)
$$
denote the body’s complex admittivity at frequency $\omega \geq 0$. We assume that 
$$
\Re(\gamma_\omega) = \sigma_\omega \in L^\infty(\Omega; \mathbb{R}), \quad \text{and} \quad \Im(\gamma_\omega) = \omega\epsilon_\omega \in L^\infty(\Omega; \mathbb{R}),
$$
where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary part, the subscript ‘+’ indicates 
functions with positive (essential) infima, and throughout this work all function spaces consist 
of complex valued functions if not stated otherwise.
Complex EIT measurements consist of applying time-harmonic currents to the surface of the imaging domain and measuring the resulting electric surface potential. In the so-called continuum model (see, e.g., [16]), these measurements are described by the Neumann-to-Dirichlet-Operator

\[ A(\gamma_w) : L^2_0(\partial\Omega) \rightarrow L^2_0(\partial\Omega), \quad g \mapsto u^{(e)}_{\gamma_w} |_{\partial\Omega}, \]

where \( u^{(e)}_{\gamma_w} \in H^1_0(\Omega) \) solves

\[ V \cdot \left( \gamma_w \nabla u^{(e)}_{\gamma_w} \right) = 0 \text{ in } \Omega \quad \text{and} \quad \gamma_w \partial_{\gamma_w} u^{(e)}_{\gamma_w} |_{\partial\Omega} = g. \]  

(1)

Here, the subspace of \( L^2(\partial\Omega) \) and \( H^1(\Omega) \)-functions with vanishing integral mean on \( \partial\Omega \) is denoted by \( L^2_0(\partial\Omega) \) and \( H^1_0(\Omega) \), respectively. \( \nu \) is the outer normal on \( \partial\Omega \). It is well known that \( A(\gamma_w) \) is a well-defined, linear and compact operator.

The idea of UMEIT is to focus an ultrasound wave on a small part \( B \subseteq \Omega \) in order to change the density of the material and thus its conductivity in \( B \), see [2]. A simple, very idealistic model is that the focused ultrasound wave changes the conductivity from \( \gamma_w \) to \( \gamma_w(1 + \beta \chi_B) \), where \( \beta > 0 \) depends on the strength of the ultrasound wave and \( \chi_B \) is the characteristic function of \( B \). Hence, UMEIT measurements can be modeled as

\[ A\left( \gamma_w(1 + \beta \chi_B) \right). \]

In this work, we will compare measurements at a non-zero frequency \( A(\gamma_w), \omega > 0 \), with ultrasound-modulated dc measurements \( A(\gamma_w(1 + \beta \chi_B)) \) in order to detect whether the ultrasound modulated part \( B \) lies inside a conductivity anomaly or not.

2.2. Monotonicity results for the continuous case

We will compare measurements in the sense of operator definiteness. Given a bounded linear operator \( A: L^2_0(\partial\Omega) \rightarrow L^2_0(\partial\Omega), \) we define its self-adjoint part by setting

\[ \Re(A) = \frac{1}{2}(A + A^*), \]

where \( A^*: L^2_0(\partial\Omega) \rightarrow L^2_0(\partial\Omega) \) is the adjoint of \( A \) with respect to the inner product of \( L^2_0(\partial\Omega) \), i.e.,

\[ \int_{\partial\Omega} g(Ah) ds = \int_{\partial\Omega} \left( \overline{Ah} \right) \overline{h} ds \quad \text{for all } g, h \in L^2_0(\partial\Omega). \]

Obviously, \( \Re(A) \) is a self-adjoint bounded linear operator.

For two self-adjoint bounded linear operators \( A, B: L^2_0(\partial\Omega) \rightarrow L^2_0(\partial\Omega), \) we write \( A \leq B \) if \( B - A \) is positive semidefinite, i.e.

\[ \int_{\partial\Omega} \overline{g}Ag ds \leq \int_{\partial\Omega} \overline{g}Bg ds \quad \forall \ g \in L^2_0(\partial\Omega). \]

For compact operators, this is equivalent to the fact that all eigenvalues of \( B - A \) are non-negative.

Note that, for all \( g, h \in L^2_0(\partial\Omega), \) the Neumann-to-Dirichlet-operator \( A(\gamma_w) \) satisfies

\[ \int_{\partial\Omega} \overline{g}A(\gamma_w)h ds = \int_{\partial\Omega} \overline{g}u^{(b)}_{\gamma_w} |_{\partial\Omega} ds = \int_{\Omega} \gamma_w \nabla u^{(e)}_{\gamma_w} \cdot \nabla u^{(b)}_{\gamma_w}, \]

\[ \int_{\partial\Omega} gA(\gamma_w)h ds = \int_{\Omega} \gamma_w \nabla u^{(e)}_{\gamma_w} \cdot \nabla u^{(b)}_{\gamma_w} = \int_{\partial\Omega} hA(\gamma_w)g ds. \]

In that sense, \( A(\gamma_w) \) is symmetric but generally (for complex \( \gamma_w \)) not self-adjoint.
In simple two-point conductivity measurement setups, there exists an obvious monotonicity relation. Given a larger conductivity we will require less voltage to drive the same current. Remarkably, this monotonicity relation extends to the case of continuous boundary measurements. For real-valued conductivity functions $\sigma_1, \sigma_2 \in L^\infty_\omega(\Omega, \mathbb{R})$ we have that, for all $g \in L^2_\omega(\partial \Omega)$,

$$\int_{\Omega} \frac{\sigma_2}{\sigma_1} (\sigma_1 - \sigma_2) \left|Vu_1^{(e)}(x)\right|^2 \, dx \leq \int_{\partial \Omega} g \left(\Lambda(\sigma_2) - \Lambda(\sigma_1)\right) g \, ds \leq \int_{\Omega} (\sigma_1 - \sigma_2) \left|Vu_2^{(e)}(x)\right|^2 \, dx,$$

(2)

where $u_e^{(e)}$ solves the EIT equation (1) with conductivity $\sigma_2$ and boundary currents $g$. Hence $\sigma_1 \leq \sigma_2$ implies that $\Lambda(\sigma_1) \geq \Lambda(\sigma_2)$, so that an imaging domain with larger conductivity yields to smaller measurements in the sense of operator definiteness. The monotonicity relation (2) goes back to Ikehata, Kang, Seo, and Sheen [46, 53]. It is the basis of many results on inclusion detection in EIT, see [33–36, 38, 45, 54].

The following lemma extends the relation (2) to complex-valued conductivities (see also [34, 36, 54] for similar results).

**Lemma 2.1.** Let $\gamma_1, \gamma_2 \in L^\infty(\Omega; \mathbb{R}) + iL^\infty(\Omega; \mathbb{R})$, $g \in L^2_\omega(\partial \Omega)$, and $u^{(g)}_1, u^{(e)}_2 \in H^1_\omega(\Omega)$ be the corresponding solutions of (1). Then

$$\int_{\Omega} \left|\frac{\Re(\gamma_2)}{\Re(\gamma_1)} \Re(\gamma_1 - \gamma_2) - \frac{\Im(\gamma_2)^2}{\Re(\gamma_1)}\right| \left|Vu_2^{(e)}(x)\right|^2 \, dx \leq \int_{\partial \Omega} g \left|\Lambda(\gamma_2) - \Lambda(\gamma_1)\right| g \, ds \leq \int_{\Omega} \left|\Re(\gamma_1 - \gamma_2) + \frac{\Im(\gamma_2)^2}{\Re(\gamma_1)}\right| \left|Vu_1^{(g)}(x)\right|^2 \, dx.$$

The proof of lemma 2.1 is postponed to the end of this section.

### 2.3. Detecting inclusions in the continuous case

We assume that the imaging domain $\Omega$ consists of a homogeneous background medium with one or several conductivity anomalies (inclusions) $D$. For simplicity, we will present our result for the case that the anomalies possess a constant admittance and that the conductivity $\sigma_0$ and the permittivity $\epsilon_0$ do not change with frequency. More precisely, we assume that $D \subset \Omega$ is a closed set with connected complement and that $\gamma_0$ and $\gamma_\infty$ are given by

$$\gamma_0(x) = \begin{cases} \gamma_0^{(g)} = \sigma_0 & \text{for } x \in \Omega \setminus D, \\ \gamma_0^{(e)} = \sigma_D & \text{for } x \in D \end{cases},$$

(3)

$$\gamma_\infty(x) = \begin{cases} \gamma_\infty^{(g)} = \sigma_0 + i\omega\epsilon_0 & \text{for } x \in \Omega \setminus D, \\ \gamma_\infty^{(e)} = \sigma_D + i\omega\epsilon_D & \text{for } x \in D, \end{cases}$$

(4)
with real-valued constants $\sigma_D$, $\sigma_0$, $\epsilon_D$, $\epsilon_D > 0$. We also assume that the anomaly fulfills
\[ \epsilon_D \sigma_D - \epsilon_0 \sigma_D \neq 0, \] (5)
which is the contrast condition required to detect inclusion in weighted fD-EIT, see [34, remark 2.3]. Our results can easily be extended to inclusions of spatially varying and frequency-dependent admittivities as long as the background conductivities are constant.

The ratio of the background conductivities is denoted by
\[ \alpha := \frac{\gamma'(\Omega)}{\gamma(\Omega)} = 1 + i \omega \frac{\epsilon_0}{\sigma_0}. \] (6)

Obviously, $\alpha \gamma(\gamma_0) = \alpha \gamma_0$.

We show that the anomaly $D$ can be detected from comparing (ratio-weighted) EIT measurements at a non-zero frequency $\omega > 0$ with ultrasound-modulated dc measurements, i.e. that we can detect $D$ from knowledge of $\alpha \gamma_0(1 + \beta \gamma_B)$ and the background ratio $\alpha$. (Note that, the background ratio $\alpha$ could also be estimated by additionally taking unmodulated dc measurements $\alpha \gamma_0$ and comparing them with $\alpha \gamma_0$ in the same way as in weighted fD-EIT, see [34, 36, 69].)

**Theorem 2.2.** Let $c := \epsilon_D \sigma_D - \epsilon_0 \sigma_D \neq 0$.

(a) If $c > 0$, then for sufficiently small $\beta > 0$ and every open set $B \subseteq \Omega$,
\[ B \subseteq D \text{ if and only if } \Re(\alpha \Lambda(\gamma_0)) \leq \Lambda(1 + \beta \gamma_B) \gamma_0. \] (7)

(b) If $c < 0$, then for sufficiently small $\beta > 0$ and every open set $B \subseteq \Omega$,
\[ B \subseteq D \text{ if and only if } \Re(\alpha \Lambda(\gamma_0)) \geq \Lambda(1 - \beta \gamma_B) \gamma_0. \] (8)

The modulation strength $\beta > 0$ is sufficiently small if
\[ \beta \leq \left\{ \begin{array}{ll}
\frac{\omega^2 |c|}{\sigma_D(\sigma_0^2 + \omega^2 \epsilon_0^2)} & \text{in case (a),} \\
\frac{\omega^2 |c|}{\sigma_D(\sigma_0 \sigma_D + \omega^2 \epsilon_D \sigma_0)} & \text{in case (b).}
\end{array} \right. \]

Theorem 2.2 shows that, for sufficiently small modulation strengths, the ultrasound-modulated dc measurements are larger ($c > 0$), resp., smaller ($c < 0$) than (the self-adjoint part of ratio-weighted) measurements taken at a non-zero frequency if and only if the focusing region lies inside the unknown inclusion $D$. The terms larger and smaller are to be understood in the sense of operator definiteness.

**Remark 2.3.** The monotonicity tests in 2.2 are stable in the following sense (see [38, remark 3.5]). Let $A^\delta$ be a (w.l.o.g. self-adjoint) approximation to the compact and self-adjoint operator $A$,
\[ \| A^\delta - A \|_{L^2(\Omega^2)} < \delta, \]
where $A := \Lambda(1 + \beta \gamma_B) \gamma_0 - \Re(\alpha \Lambda(\gamma_0))$ in case (a) of theorem 2.2 and $A := \Re(\alpha \Lambda(\gamma_0)) - \Lambda(1 - \beta \gamma_B) \gamma_0$ in case (b).
We consider the regularized definiteness test
\[ A^\delta \geq -\delta I. \quad (9) \]
If \( A \geq 0 \), then \( A^\delta \geq -\delta I \) will be fulfilled. On the other hand, if \( A \not\geq 0 \), then \( A \) must possess a negative eigenvalue \( \lambda < 0 \), so that \( A^\delta \not
\geq -\delta I \) for all \( \delta < -\frac{1}{\lambda} \).

Hence, in order to determine whether a given focusing region lies inside the unknown inclusion, it suffices to know the measurements up to a certain precision level \( \delta > 0 \). In that sense, also our arguably idealistic modeling of a perfectly focused ultrasound beam only has to be approximately valid.

2.4. Proof of lemma 2.1 and theorem 2.2

Our proof of theorem 2.2 relies on the monotonicity relation for complex conductivity EIT in lemma 2.1 and the concept of localized potentials developed by one of the authors in [23]. To prove lemma 2.1, we will first show the following auxiliary result that will also be useful for the case of electrode measurements.

**Lemma 2.4.** Let \( \gamma_1, \gamma_2 \in L^\infty(\Omega; \mathbb{R}) + iL^\infty(\Omega; \mathbb{R}), \ g \in L^2(\partial \Omega), \) and \( u_1, u_2 \in H^1(\Omega) \) fulfill
\[
\int_\Omega \gamma_1 |V_{u_1}|^2 dx = \int_\Omega \gamma_2 V_{u_2} \cdot \nabla u_1 dx,
\]
\[
\int_\Omega \gamma_2 |V_{u_2}|^2 dx = \int_\Omega \gamma_1 V_{u_1} \cdot \nabla u_2 dx.
\]

Then
\[
\int_\Omega \left( \frac{\Re(\gamma_2)}{\Re(\gamma_1)} \Re(\gamma_1 - \gamma_2) - \frac{\Im(\gamma_2)^2}{\Re(\gamma_1)} \right) |V_{u_2}|^2 dx
\]
\[
\leq \int_\Omega \Re(\gamma_2) |V_{u_2}|^2 dx - \int_\Omega \Re(\gamma_1) |V_{u_1}|^2 dx
\]
\[
\leq \int_\Omega \left( \Re(\gamma_1 - \gamma_2) + \frac{\Im(\gamma_1)^2}{\Re(\gamma_1)} \right) |V_{u_2}|^2 dx.
\]

**Proof.** Since
\[
0 \leq \int_\Omega \Re(\gamma_1) \left| V_{u_1} - \frac{\gamma_2}{\Re(\gamma_1)} V_{u_2} \right|^2
\]
\[
= \Re \left( \int_\Omega \gamma_1 |V_{u_1}|^2 dx - 2 \int_\Omega \gamma_2 V_{u_2} \cdot \nabla u_1 dx \right) + \int_\Omega \frac{|\gamma_2|^2}{\Re(\gamma_1)} |V_{u_2}|^2 dx
\]
\[
= \int_\Omega \Re(\gamma_2) |V_{u_2}|^2 dx - \int_\Omega \Re(\gamma_1) |V_{u_1}|^2 dx + \int_\Omega \left( \frac{|\gamma_2|^2}{\Re(\gamma_1)} - \Re(\gamma_2) \right) |V_{u_2}|^2 dx.
\]
the first inequality follows from
\[
\frac{|\gamma|}{\Re(\gamma)} - \Re(\gamma) = \frac{\Re(\gamma)^2 + \Im(\gamma)^2}{\Re(\gamma)} - \Re(\gamma) = \frac{\Re(\gamma)}{\Re(\gamma)}(\Re(\gamma) - \Im(\gamma)) + \frac{\Im(\gamma)^2}{\Re(\gamma)}.
\]
Likewise we obtain
\[
0 \leq \int_{\Omega} \Re(\gamma) \left| \nabla u_1 - \frac{\gamma}{\Re(\gamma)} \nabla u_2 \right|^2 \, dx
= \int_{\Omega} \Re(\gamma) \left| \nabla u_1 \right|^2 \, dx - 2\Re \left( \int_{\Omega} \frac{\gamma}{\Re(\gamma)} \nabla u_2 \cdot \nabla u_1 \, dx \right) + \int_{\Omega} \frac{|\gamma|^2}{\Re(\gamma)} \left| \nabla u_2 \right|^2 \, dx
= \int_{\Omega} \Re(\gamma) \left| \nabla u_1 \right|^2 \, dx - \int_{\Omega} \Re(\gamma) \left| \nabla u_2 \right|^2 \, dx + \int_{\Omega} \left| \frac{\gamma}{\Re(\gamma)} \left| \nabla u_2 \right|^2 \, dx - \Re(\gamma) \right| \left| \nabla u_2 \right|^2 \, dx,
\]
so that the second inequality follows from
\[
\frac{|\gamma|}{\Re(\gamma)} - \Re(\gamma) = \frac{\Re(\gamma)^2 + \Im(\gamma)^2}{\Re(\gamma)} - \Re(\gamma) = \Re(\gamma - \gamma) + \frac{\Im(\gamma)^2}{\Re(\gamma)}.
\]
We also require the following elementary computation:

**Lemma 2.5.** Let \(\gamma_0, \gamma_\omega: \Omega \rightarrow \mathbb{C} \), and \(\alpha \in \mathbb{C} \) be given by (3), (4), and (6). Then, for all \(\hat{\beta} \in \mathbb{R} \),
\[
\frac{\Re(\gamma_0)}{\Re(\gamma_\omega/\alpha - \gamma_0)} \Re(\gamma_\omega/\alpha - \gamma_0) = \begin{cases} 0 & \text{in } \Omega \setminus D, \\ \frac{\epsilon_\omega \sigma_D}{\sigma_D} C & \text{in } D, \end{cases}
\]
\[
\frac{\Re(\gamma_\omega/\alpha - \gamma_0)^2}{\Re(\gamma_\omega/\alpha)} = \begin{cases} 0 & \text{in } \Omega \setminus D, \\ \epsilon_\omega C & \text{in } D, \end{cases}
\]
\[
\Re \left( \frac{\gamma_\omega/\alpha - (1 + \hat{\beta}/\sigma_\omega)\gamma_0}{\gamma_\omega/\alpha} \right) = \begin{cases} -\hat{\beta}/\sigma_\omega x_\omega & \text{in } \Omega \setminus D, \\ \sigma_D \left( \frac{\epsilon_\omega}{\sigma_D} C - \hat{\beta}/\sigma_D x_\omega \right) & \text{in } D, \end{cases}
\]
\[
\Re \left( \frac{\gamma_\omega/\alpha - (1 + \hat{\beta}/\sigma_\omega)\gamma_0}{\gamma_\omega/\alpha} \right) = \begin{cases} -\hat{\beta}/\sigma_\omega x_\omega & \text{in } \Omega \setminus D, \\ \epsilon_\omega C - \hat{\beta}/\sigma_D x_\omega & \text{in } D, \end{cases}
\]
where
\[
C := \frac{\epsilon_\omega^2 \sigma_D \sigma_{\omega}}{\sigma_D \sigma_\omega + \omega^2 \epsilon_\omega \epsilon_\omega} \quad \text{and} \quad C' := \frac{\epsilon_\omega \sigma_D}{\sigma_D} \frac{\epsilon_\omega \sigma_\omega - \epsilon_\omega \sigma_\omega}{\sigma_D + \omega^2 \epsilon_\omega \epsilon_\omega}.
\]

**Proof.** Let
\[
\gamma_0 = \begin{cases} \gamma_0^{(\omega)} = \sigma_\omega & \text{in } \Omega \setminus D, \\ \gamma_0^{(\omega)} = \sigma_D & \text{in } D, \end{cases} \quad \gamma_\omega = \begin{cases} \gamma_\omega^{(\omega)} = \sigma_\omega + i \omega \epsilon_\omega & \text{in } \Omega \setminus D, \\ \gamma_\omega^{(\omega)} = \sigma_D + i \omega \epsilon_\omega & \text{in } D, \end{cases}
\]
with real-valued constants \(\sigma_\omega, \sigma_D, \epsilon_\omega, \epsilon_D > 0\), and let \(\alpha := \gamma_\omega^{(\omega)} / \gamma_0^{(\omega)} = 1 + i \omega \sigma_\omega^{-1} \epsilon_\omega \in \mathbb{C} \).
Then, by definition of $\alpha$,

$$\gamma_\omega/\alpha - \gamma_0 = 0 \quad \text{in} \ \Omega \setminus D, \quad \text{and} \quad \Im(\gamma_\omega/\alpha) = 0 \quad \text{in} \ \Omega \setminus D,$$

so that

$$\frac{\Re(\gamma_0)}{\Re(\gamma_\omega/\alpha)} \Re(\gamma_\omega/\alpha - \gamma_0) = 0 \quad \text{in} \ \Omega \setminus D,$$

$$\Re(\gamma_\omega/\alpha - \gamma_0) + \frac{\Im(\gamma_\omega/\alpha)^2}{\Re(\gamma_\omega/\alpha)} = 0 \quad \text{in} \ \Omega \setminus D,$$

$$\Re\left(\gamma_\omega/\alpha - (1 + \tilde{\beta} \chi_\Omega)\gamma_0\right) = -\tilde{\beta} \sigma_\Omega \chi_\Omega \quad \text{in} \ \Omega \setminus D,$$

$$\Re\left(\gamma_\omega/\alpha - (1 + \tilde{\beta} \chi_\Omega)\gamma_0\right) + \frac{\Im(\gamma_\omega/\alpha)^2}{\Re(\gamma_\omega/\alpha)} = -\tilde{\beta} \sigma_\Omega \chi_\Omega \quad \text{in} \ \Omega \setminus D.$$

In $D$, we have that

$$\Re(\gamma_0) = \sigma_D,$$

$$\Re(\gamma_\omega/\alpha) = \Re\left(\gamma_\omega(\gamma_\omega/\alpha)\right) = \sigma_D \Re\left(\frac{\sigma_D + i\omega \epsilon D}{\sigma_\Omega + i\omega \epsilon_\Omega}\right) = \sigma_D \frac{\sigma_D \sigma_\Omega + \omega^2 \epsilon_D \epsilon_\Omega}{\sigma_\Omega^2 + \omega^2 \epsilon_\Omega^2},$$

$$\Im(\gamma_\omega/\alpha) = \Im\left(\gamma_\omega(\gamma_\omega/\alpha)\right) = \omega \sigma_D \Im\left(\frac{\sigma_D + i\omega \epsilon D}{\sigma_\Omega + i\omega \epsilon_\Omega}\right) = \omega \sigma_D \frac{\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D}{\sigma_\Omega^2 + \omega^2 \epsilon_\Omega^2}.$$ 

Hence, in $D$,

$$\Re(\gamma_\omega/\alpha - \gamma_0) = \sigma_D \frac{\sigma_D \sigma_\Omega + \omega^2 \epsilon_D \epsilon_\Omega}{\sigma_\Omega^2 + \omega^2 \epsilon_\Omega^2} - \sigma_D = \omega \sigma_D \frac{\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D}{\sigma_\Omega^2 + \omega^2 \epsilon_\Omega^2} = \frac{\epsilon_D \sigma_\Omega}{\sigma_\Omega},$$

which shows that

$$\frac{\Re(\gamma_0)}{\Re(\gamma_\omega/\alpha)} \Re(\gamma_\omega/\alpha - \gamma_0) = \omega \sigma_D \frac{\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D}{\sigma_\Omega \left(\sigma_D \sigma_\Omega + \omega^2 \epsilon_D \epsilon_\Omega\right)} = \frac{\epsilon_D \sigma_\Omega}{\sigma_\Omega} C, \quad \frac{\epsilon_D \sigma_\Omega}{\sigma_\Omega} C,$$

and

$$\Re\left(\gamma_\omega/\alpha - (1 + \tilde{\beta} \chi_\Omega)\gamma_0\right) = \sigma_D \left(\frac{\epsilon_\Omega C}{\sigma_\Omega} - \tilde{\beta} \chi_\Omega\right).$$
The remaining two equalities follow from
\[
\Re\left(\frac{\gamma_0}{\alpha} - \gamma_0\right) + \frac{3}{\Re\left(\frac{\gamma_0}{\alpha}\right)}
\]
\[
= \omega^2 \epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2 \left(\frac{\epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2}{\sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2}\right)
\]
\[
= \omega^2 \epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2 \left(\frac{\epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2}{\sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2}\right)
\]
\[
= \omega^2 \epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2 \left(\frac{\epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2}{\sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2 + \omega^2 \epsilon_0^2 \sigma_0^2}\right)
\]
which also yields
\[
\Re\left(\frac{\gamma_0}{\alpha} - \left(1 + \beta \chi_B\right)\gamma_0\right) + \frac{3}{\Re\left(\frac{\gamma_0}{\alpha}\right)} = \epsilon_0^2 C - \beta \sigma_0 \chi_B.
\]

Now we are ready to prove lemma 2.1 and theorem 2.2.

**Proof of lemma 2.1.** For all \( g \in L^2_\omega(\partial\Omega) \) we have that
\[
\int_{\partial\Omega} \mathcal{A}(\gamma_1) g \, ds = \int_{\partial\Omega} \mathcal{A}(\gamma_2) g \, ds = \int_{\partial\Omega} \left| \nabla u_1^{(g)} \right|^2 \, dx = \int_{\partial\Omega} \left| \nabla u_2^{(g)} \right|^2 \, dx,
\]
so that the assertion of lemma 2.1 immediately follows from lemma 2.4.

**Proof of lemma 2.2.**

(a)

(i) Let \( c := \epsilon_0^2 \sigma_0^2 - \epsilon_0^2 \sigma_0^2 > 0 \), and \( B \subseteq D \).

We use the first inequality in lemma 2.1 with \( \gamma_2 := (1 + \beta \chi_B)\gamma_0 \) and \( \gamma_1 := \gamma_0/\alpha \) together with the third equality in lemma 2.5 with \( \tilde{\beta} := \beta \) to obtain that, for all \( g \in L^2_\omega(\partial\Omega) \),
\[
\int_{\partial\Omega} \mathcal{A}(\gamma_1) g \, ds \geq \int_{\partial\Omega} \left| \nabla u_1^{(g)} \right|^2 \, dx
\]
\[
= \int_{\partial\Omega} \left| \nabla u_2^{(g)} \right|^2 \, dx,
\]
where \( C' \) is defined by (10) in lemma 2.5. The right-hand side is non-negative if
so that, for sufficiently small \( \beta > 0 \),

\[
B \subseteq D \quad \text{implies} \quad \Re\left(\alpha A\left(\gamma_w\right)\right) \leq A\left(1 + \beta \chi_\Omega \right) \gamma_0.
\]

(ii) Now let \( c = c_2 \sigma_2 - c_2 \sigma_\Omega > 0 \), and \( B \not\subseteq D \).

We use the second inequality in lemma 2.1 with \( \gamma_2 := \gamma_0 \), and \( \gamma_1 := \gamma_w / \alpha \) together with the second equality in lemma 2.5 to obtain that, for all \( g \in L^2_\omega(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[ A\left(\gamma_0\right) - \Re\left(\alpha A\left(\gamma_w / \alpha\right)\right)\right] g \, ds
\]

\[
\leq \int_{\Omega} \left( \Re\left(\gamma_w / \alpha - \gamma_0\right) + \frac{3(\gamma_w / \alpha)^2}{\Re(\gamma_w / \alpha)} \right) \left| \nabla u_\gamma^{(v)} \right|^2 dx
\]

\[
= c_2 C \int_D \left| \nabla u_\gamma^{(v)} \right|^2 dx,
\]

where \( C \) is defined by (10) in lemma 2.5. The first inequality in lemma 2.1 with \( \gamma_2 := \gamma_0 \) and \( \gamma_1 := (1 + \beta \chi_\Omega) \gamma_0 \) yields that, for all \( g \in L^2_\omega(\partial \Omega) \),

\[
\int_{B} \frac{\beta}{1 + \beta} \gamma_0 \left| \nabla u_\gamma^{(v)} \right|^2 dx \leq \int_{\partial \Omega} \Re\left[ A\left(\gamma_0\right) - A\left(1 + \beta \chi_\Omega \right) \gamma_0\right] g \, ds.
\]

Combining both inequalities, we obtain that, for all \( g \in L^2_\omega(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[ A\left(1 + \beta \chi_\Omega \right) \gamma_0\right] - \Re\left(\alpha A\left(\gamma_w / \alpha\right)\right] g \, ds
\]

\[
\leq c_2 C \int_D \left| \nabla u_\gamma^{(v)} \right|^2 dx - \int_{B} \frac{\beta}{1 + \beta} \gamma_0 \left| \nabla u_\gamma^{(v)} \right|^2 dx.
\]

Now we apply the technique of localized potentials [23, 38] to show that the right-hand side of this inequality attains negative values. Since \( B \not\subseteq D \) we can choose a smaller open subset \( B' \subseteq B \) with \( B' \cap D = \emptyset \). Since \( D \subseteq \Omega \) and \( \Omega \setminus D \) is connected, we obtain from \([38 \text{ theorem 3.6}]\) a sequence of currents \( (g_k)_{k \in \mathbb{N}} \subseteq L^2_\omega(\partial \Omega) \), so that the solutions \( u^{(g_k)} \) fulfill

\[
\Delta u^{(g_k)} = 0, \quad \partial_n u^{(g_k)} \big|_{\partial \Omega} = g_k
\]

\[
\lim_{k \to \infty} \int_{B} \left| \nabla u^{(g_k)} \right|^2 dx = \infty \quad \text{and} \quad \lim_{k \to \infty} \int_{D} \left| \nabla u^{(g_k)} \right|^2 dx = 0.
\]

Since \( \gamma_0 \) is constant on \( \Omega \setminus D \), \([38 \text{ lemma 3.7}]\) yields that also the corresponding solutions \( (u^{(g_k)})_{k \in \mathbb{N}} \subseteq H^1_0(\Omega) \) of (1) fulfill

\[
\lim_{k \to \infty} \int_{B} \left| u^{(g_k)} \right|^2 dx = \infty \quad \text{and} \quad \lim_{k \to \infty} \int_{D} \left| u^{(g_k)} \right|^2 dx = 0.
\]

Hence, with this sequence of currents

\[
\int_{\partial \Omega} \Re\left[ A\left(1 + \beta \chi_\Omega \right) \gamma_0\right] g_k dx \to -\infty,
\]

which shows that, for all \( \beta > 0 \),
\[ B \nsubseteq D \] implies \[ \Re(\alpha A(\gamma_0)) \not\subseteq A\left((1 + \beta \chi_B)\gamma_0\right). \]

(b) \(\) Let \( c := \epsilon_D \sigma_2 - \epsilon_D \sigma_2 < 0 \), and \( B \nsubseteq D \).

We use the second inequality in lemma 2.1 with \( \gamma_2 : = (1 - \beta \chi_B)\gamma_0 \), and \( \gamma_1 : = \gamma_0/\alpha \) together with the fourth equality in lemma 2.5 with \( \hat{\beta} := -\beta \) to obtain that, for all \( g \in L^2_\partial(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[A\left((1 - \beta \chi_B)\gamma_0\right) - \Re(\alpha A(\gamma_0))\right]g \, ds \\
\leq \int_{\partial \Omega} \left|\frac{\Re(\gamma_0/\alpha)}{\Re(\gamma_0/\alpha)}\right| \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx. \\
= \int_{D} \left(\epsilon_D C + \beta \sigma \chi_B\right)\left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx.
\]

The right-hand side is non-positive if \( \beta \leq -\frac{\epsilon_D C}{\sigma_D} = \frac{\sigma_D}{\sigma_D(\sigma_D \sigma_2 + \omega^2 \epsilon_D \sigma_D)}, \)

so that, for sufficiently small \( \beta > 0 \),

\[ B \nsubseteq D \] implies \[ \Re(\alpha A(\gamma_0)) \not\subseteq A\left((1 - \beta \chi_B)\gamma_0\right). \]

(ii) Now let \( c : = \epsilon_D \sigma_2 - \epsilon_D \sigma_2 < 0 \), and \( B \nsubseteq D \).

We use the first inequality in lemma 2.1 with \( \gamma_2 : = \gamma_0 \), and \( \gamma_1 : = \gamma_0/\alpha \) together with the first equality in lemma 2.5 to obtain that, for all \( g \in L^2_\partial(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[A(\gamma_0) - \Re(\alpha A(\gamma_0))\right]g \, ds \\
\geq \int_{\Omega} \frac{\Re(\gamma_0/\alpha)}{\Re(\gamma_0/\alpha)} \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx. \\
= \frac{\epsilon_D \sigma_2}{\sigma_2} \int_{D} \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx.
\]

The second inequality in lemma 2.1 with \( \gamma_2 : = \gamma_0 \) and \( \gamma_1 : = (1 - \beta \chi_B)\gamma_0 \) yields that, for all \( g \in L^2_\partial(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[A(\gamma_0) - \Re(\alpha A(\gamma_0))\right]g \, ds \leq - \int_{B} \beta \gamma_0 \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx. \\
\]

Combining both inequalities, we obtain that, for all \( g \in L^2_\partial(\partial \Omega) \),

\[
\int_{\partial \Omega} \Re\left[A\left((1 - \beta \chi_B)\gamma_0\right) - \Re(\alpha A(\gamma_0))\right]g \, ds \\
\geq \frac{\epsilon_D \sigma_2}{\sigma_2} \int_{D} \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx + \int_{B} \beta \gamma_0 \left|V_{\gamma_0}^{(\epsilon)}\right|^2 \, dx. \\
\]

The same localized potentials argument as in part (a)(ii) shows that there exists a sequence of currents such that

\[
\int_{\partial \Omega} \Re\left[A\left((1 - \beta \chi_B)\gamma_0\right) - \Re(\alpha A(\gamma_0))\right]g \, ds \to \infty. \\
\]

Hence, for all \( \beta > 0 \),
3. Electrode measurements

3.1. The setting

In a realistic setting, the currents will be applied using a finite number of electrodes $E_l \subset \partial \Omega$, $l = 1, \ldots, m$, that are attached to the imaging domain’s surface. We assume that the electrodes are perfectly conducting and that contact impedances are negligible (the so-called shunt model, see, e.g., [16]). Driving a current $I_l \in \mathbb{C}$ through the $l$th electrode, with $\sum_{l=1}^{m} I_l = 0$, the electric potential is given by the solution $u_{\tau_{l}} \in H^1(\Omega)$ of

$$V \cdot \left( \tau_{l} V u_{\tau_{l}} \right) = 0 \quad \text{in} \quad \Omega,$$

$$\int_{E_l} \gamma_{\tau} \partial_{\nu} u_{\tau_{l}} \, ds = I_l \quad \text{for} \quad l = 1, \ldots, m, \quad (12)$$

$$\gamma_{\tau} \partial_{\nu} u_{\tau_{l}} = 0 \quad \text{on} \quad \partial \Omega \setminus \bigcup_{l=1}^{m} E_l, \quad (13)$$

$$u_{\tau_{l}} \big|_{E_l} = \text{const.} \quad \forall \quad j = 1, \ldots, m, \quad (14)$$

where $H^1(\Omega)$ is the subspace of $H^1$-functions that are locally constant on each $E_l$, $l = 1, \ldots, m$ and these constants sum up to zero.

We assume that the voltage-current-measurements are carried out in the following complete dipole–dipole configuration. Let $(j_r, k_r), \ r = 1, \ldots, N$ be a set of electrode pairs with $j_r \neq k_r$. For each of these pairs, $r = 1, \ldots, N$, a current of $I_{j_r} = 1$ and $I_{k_r} = -1$ is driven through the $j_r$th and the $k_r$th electrode, respectively. The other electrodes are kept insulated. The resulting electric potential inside the imaging domain is given by the solution $u_{l_{j_r, k_r}} \in H^1(\Omega)$ of

$(11)$–$(14)$ with $I_l = \delta_{j_r, k_r} - \delta_{j_r, k_r}, \ l = 1, \ldots, N$.

While the current is driven through the $r$th pair of electrodes, we measure the required voltage difference on all pairs of electrodes, i.e., between the $j_s$ and the $k_s$ electrode for all $s = 1, \ldots, N$. We collect these measurements in the matrix

$$R(\gamma_{\tau}) = \begin{pmatrix} u_{l_{j_s, k_s}} \big|_{E_s} - u_{l_{k_s, j_s}} \big|_{E_s} & \end{pmatrix}_{r,s=1,\ldots,N} \in \mathbb{C}^{N \times N}.$$ 

Let us comment on our use of the shunt electrode model. It seems to be widely accepted that the most accurate electrode model in EIT is the complete electrode model, see, e.g., [16], where not only the shunting effects but also contact impedances between the electrodes and the imaging domain are taken into account. The effect of contact impedances is often neglected in the case that voltages are not measured on current driven electrodes, but our method requires such measurements, see below. Contact impedances can also be neglected in the case of dc difference measurements on point electrodes, see [29]. Since both, the effect of an ultrasound modulation and the effect of a (weighted) frequency change on the measurements are widely analogous to using dc difference measurements, we believe that our use of the shunt model is justified for sufficiently small electrodes, though this has yet to be justified rigorously.

We also stress that our method relies on the matrix structure of the measurements $R$, which means that the same electrode pairs have to be used for measuring voltages and applying currents. In particular, we require voltage measurements on current driven electrodes (for the three main diagonals in $R$). The simultaneous measurement of voltage and current is
usually considered problematic and these measurements are avoided in traditional EIT approaches. Nevertheless, successful reconstructions have already been obtained in practical phantom experiments with methods requiring the full matrix such as the factorization method and monotonicity-based methods, see [36, 80]. Also, the recent preprint [32] studies the possibility of interpolating the voltages on current-driven electrodes from the measurements on current-free electrodes.

3.2. Monotonicity results for the shunt model

As in the continuous case, we will compare measurements in the sense of matrix definiteness. We define the self-adjoint part of a matrix \( A \in \mathbb{C}^{N \times N} \) by setting

\[
\mathcal{R}(A) = \frac{1}{2}(A + A^*),
\]

where \( A^* \in \mathbb{C}^{N \times N} \) is the adjoint (conjugate transpose) of \( A \), i.e.,

\[
g^*(Ah) = (Ag)^*h \quad \text{for all } g, h \in \mathbb{C}^N, \quad \text{and} \quad g^* = g^T.
\]

Obviously, \( \mathcal{R}(A) \) is self-adjoint.

For two self-adjoint matrices \( A, B \in \mathbb{C}^{N \times N} \), we write \( A \preceq B \) if \( BA - I \) is positive semi-definite, i.e.

\[
g^*Ag \preceq g^*Bg \quad \forall \ g \in \mathbb{C}^N.
\]

This is equivalent to the fact that all eigenvalues of \( B - A \) are non-negative.

Note that the entries of the measurement matrix \( R(\gamma_\omega) \) satisfy

\[
u^{(r)}_\omega \big|_{E_h} - u^{(r)}_\omega \big|_{E_h} = \int_{E_h} \gamma_\omega \partial_k u^{(r)}_\omega \, ds + \int_{E_h} \gamma_\omega \partial_k u^{(r)}_\omega \, ds = \int_{\Omega} \gamma_\omega \partial_k u^{(r)}_\omega \cdot V u^{(r)}_\omega = u^{(r)}_\omega \big|_{E_h} - u^{(r)}_\omega \big|_{E_h}.
\]

Hence \( R(\gamma_\omega) \) is a symmetric, but generally (for complex \( \gamma_\omega \)) not self-adjoint matrix. This also shows that the self-adjoint part of the measurement matrix \( \mathcal{R}(R(\gamma_\omega)) \) is identical to the matrix containing the real part of each voltage measurement

\[
\mathcal{R}(R(\gamma_\omega)) = \left\{ \mathcal{R}(\nu^{(r)}_\omega) \big|_{E_h} - \mathcal{R}(\nu^{(r)}_\omega) \big|_{E_h} \right\}_{r,s=1,\ldots,N} \in \mathbb{R}^{N \times N}.
\]

The monotonicity estimate from the continuous case can be extended to the case of electrode measurements.

**Lemma 3.1.** Let \( \gamma_1, \gamma_2 \in L^\infty(\Omega; \mathbb{R}) \) and \( g_1, g_2 \in L^\infty(\Omega; \mathbb{R}) \), \( g = (g_1)^N \) and \( u^{[\tau]}_l \in H^1_0(\Omega) \) \((\tau = 1, 2)\) denote the solution of

\[
\nabla \cdot \left( \gamma_\tau \nabla u^{[\tau]}_l \right) = 0 \quad \text{in } \Omega,
\]

\[
\int_{E_i} \gamma_\tau \partial_k u^{[\tau]}_l \, ds = \sum_{r: j, m = 1} g_j - \sum_{r: k, m = 1} g_k \quad \text{for all } l = 1, \ldots, m,
\]

\[
\gamma_\tau \partial_k u^{[\tau]}_l = 0 \quad \text{on } \partial \Omega \setminus \bigcup_{j=1}^{w} E_i,
\]

\[
u^{[\tau]}_l \big|_{E_i} = \text{const.} \quad \forall \ l = 1, \ldots, m.
\]
Then
\[
\int_{\Omega} \left( \frac{\Re(y_2)}{\Re(y_1)} \Re(y_1 - y_2) - \frac{2(y_2)\Re(y_1)}{\Re(y_1)} \right) |V_{\mu}|^2 \, dx \\
\leq g^* \Re \left[ R(y_2) - R(y_1) \right] g \leq \int_{\Omega} \left( \Re(y_1 - y_2) + \frac{2(y_2)^2}{\Re(y_1)} \right) |V_{\mu}|^2 \, dx.
\]

The proof of lemma 3.1 is postponed to the end of this section.

### 3.3. Detecting inclusions from electrode measurements

We make the same assumptions as for the continuous case in section 2.3. The inclusion (or anomaly) \(D \subset \Omega\) is assumed to be a closed set with connected complement. \(y_0\) and \(y_0\) are assumed to be given by

\[
\gamma_0(x) = \begin{cases} \gamma_0^{(0)} = \sigma_\Omega & \text{for } x \in \Omega, \\
\gamma_0^{(D)} = \sigma_D & \text{for } x \in D,
\end{cases}
\]

\[
\gamma_\omega(x) = \begin{cases} \gamma_\omega^{(0)} = \sigma_\Omega + i\omega\sigma_\Omega & \text{for } x \in \Omega, \\
\gamma_\omega^{(D)} = \sigma_D + i\omega\sigma_D & \text{for } x \in D,
\end{cases}
\]

with real-valued constants \(\sigma_\Omega, \sigma_D, \epsilon_\Omega, \epsilon_D > 0\). The anomaly is assumed to fulfill the contrast condition (5), i.e., \(\epsilon_\Omega\sigma_\Omega - \epsilon_\Omega\sigma_D \neq 0\), and

\[
\alpha \equiv \frac{\gamma_\omega^{(0)}}{\gamma_0^{(0)}} = 1 + i\omega\frac{\epsilon_D}{\epsilon_\Omega}.
\]

Denotes the ratio of the background conductivities. Obviously, \(aR(y_\omega) = R(y_\omega)/\alpha\).

As in section 2, the results in this section can easily be extended to inclusions of spatially varying and frequency-dependent admittivities as long as the background conductivities are constant.

Our results for continuous boundary data suggest to compare, for sufficiently small modulation strengths \(\beta > 0\), the matrix of ultrasound-modulated dc measurements \(R((1 + \beta y_\omega)\gamma_\omega)\) with the (self-adjoint part of the ratio-weighted) matrix of measurements taken at a non-zero frequency \(R(y_\omega)\). This comparison (in the sense of matrix definiteness) should yield information about whether the focusing region \(B\) lies inside the unknown inclusion \(D\). Indeed, we can prove the following theorem.

**Theorem 3.2.** Let \(c := \epsilon_\Omega\sigma_\Omega - \epsilon_\Omega\sigma_D \neq 0\).

(a) If \(c > 0\), then for sufficiently small \(\beta > 0\) and every open set \(B \subseteq \Omega\),

\[
B \subseteq D \quad \text{implies that} \quad \Re \left( aR(y_\omega) \right) \leq R \left( (1 + \beta y_\omega)\gamma_\omega \right).
\]

(b) If \(c < 0\), then for sufficiently small \(\beta > 0\) and every open set \(B \subseteq \Omega\),

\[
B \subseteq D \quad \text{implies that} \quad \Re \left( aR(y_\omega) \right) \geq R \left( (1 - \beta y_\omega)\gamma_\omega \right).
\]

The modulation strength \(\beta > 0\) is sufficiently small if
The converses of the implications (15) and (16) will generally not be true in the case of measurements with a finite number of electrodes. However, when we increase the number of electrodes used for the measurements, then we can expect that the measurement matrices $R(γ_0)$ and $R((1 + β_0)γ_0)$ more and more resemble their continuous counterparts, the Neumann-to-Dirichlet operators, see the works of Hakula, Hyvönen and Lechleiter [41, 42, 59]. In fact, we can give the following intuitive justification of the converses of the implications in theorem 3.2 for sufficiently many electrodes in the spirit of [36].

Remark 3.3. Let $B ⊈ D$ and $β > 0$. If there exists a current pattern $g = (g_r)_{r=1}^N ∈ C^N$ such that the resulting dc potential

$$u |_{L_1} = \sum_{r=1}^N g_r u \left( \frac{1}{N} \right)$$

possesses a very large energy in $B \setminus D$ and a very small energy in $D$, then

$$Re(aR(γ_0)) \not\subseteq R((1 + β_0)γ_0) \quad \text{if } c > 0$$

or

$$Re(aR(γ_0)) \not\supseteq R((1 - β_0)γ_0) \quad \text{if } c < 0.$$

3.4. Proof of lemma 3.1, theorem 3.2 and justification of remark 3.3

Proof of lemma 3.1. Let $g = (g_r)_{r=1}^N ∈ C^N$. First note that for $τ = 1, 2$, by linearity

$$u \left( \frac{1}{N} \right) = \sum_{r=1}^N g_r u \left( \frac{1}{N} \right)$$

and likewise

$$g^*R(γ_1)g = \sum_{s=1}^N \gamma_1 \frac{∂_1^s}{γ_1} | u |_{L_1} - u |_{L_1} = \sum_{s=1}^N \gamma_1 \frac{∂_1^s}{γ_1} | u |_{L_1} - \sum_{s=1}^N \gamma_1 \frac{∂_1^s}{γ_1} | u |_{L_1}$$

and

$$g^*R(γ_2)g = \int_Ω | u |_{L_1} = \int_Ω g^*V | u |_{L_1} \cdot | u |_{L_1} \, dx.$$
Proof of theorem 3.2. The proof is identical to that of theorem 2.2(a)(i) and (b)(i) with lemma 3.1 replacing lemma 2.1.

Justification of remark 3.3. As in theorem 2.2(a)(ii) and (b)(ii) (with lemma 3.1 replacing lemma 2.1), we obtain that, for all $g \in \mathbb{C}^N$,

$$g^* \left[ R \left( (1 + \beta b_{R}) \gamma_0 \right) - R(\alpha R(\gamma_w)) \right] g \leq \epsilon_D C \int_D \left| V_{u_{e}}^{[\epsilon]} \right|^2 dx - \int_B \frac{\beta}{1 + \beta^2} \left| V_{u_{e}}^{[\epsilon]} \right|^2 dx,$$

and

$$g^* \left[ R \left( (1 - \beta b_{R}) \gamma_0 \right) - R(\alpha R(\gamma_w)) \right] g \geq \frac{\epsilon_D \sigma_0}{\sigma_B} C \int_D \left| V_{u_{e}}^{[\epsilon]} \right|^2 dx + \int_B \beta \gamma_0 \left| V_{u_{e}}^{[\epsilon]} \right|^2 dx,$$

where $C$ is defined by (10) in lemma 2.5.

Hence, if there exists a current pattern $g = (g_r)_r = 1 \in \mathbb{C}^N$ such that the resulting dc potential $u_{e}^{[\epsilon]}$ possesses a very large energy in $B \setminus D$ and a very small energy in $D$, then for this $g$, we can expect that

$$g^* \left[ R \left( (1 + \beta b_{R}) \gamma_0 \right) - R(\alpha R(\gamma_w)) \right] g < 0,$$

resp.,

$$g^* \left[ R \left( (1 - \beta b_{R}) \gamma_0 \right) - R(\alpha R(\gamma_w)) \right] g > 0,$$

so that

$$R(\alpha R(\gamma_w)) \not\geq R \left( (1 + \beta b_{R}) \gamma_0 \right), \quad \text{resp.,} \quad R(\alpha R(\gamma_w)) \not\leq R \left( (1 - \beta b_{R}) \gamma_0 \right).$$

4. Numerical results

In this section, we numerically demonstrate our new method for the practically relevant electrode setting of section 3.1. In all of the following settings, $m$ electrodes $E_1, E_2, \ldots, E_m$ are numbered as shown in the corresponding figures, and adjacent-adjacent dipole driving patterns are used according to this numbering, i.e., in the notation of section 3.1

$$(j_r, k_r) := (r, r + 1) \quad \text{for } r = 1, \ldots, m - 1, \quad \text{and} \quad (j_m, k_m) := (m, 1).$$

The EIT measurements at zero and non-zero frequency, and with and without ultrasound-modulation, are simulated by solving the equations (11)–(14) using MATLAB® and the commercial FEM-software COMSOL®.

At this point, let us stress again, that in a practical application of our new method, all required quantities are measured and no numerical simulations have to be carried out.

Example 4.1. Consider the setting illustrated in figure 1. The imaging domain $\Omega$ is a two-dimensional circle with radius 10 centered at $(0, 0)$ and a circular anomaly $D$ (sketched in red in figure 1) with radius 1.5 is located at $(5, 0)$. On the boundary $\partial \Omega$, there are 16 electrodes $E_1, E_2, \ldots, E_{16}$ attached.
The dc and ac admittivities $\gamma_0$ and $\gamma_\omega$ are chosen as

$$\gamma_0 := 1, \quad \text{and} \quad \gamma_\omega := \begin{cases} 1 + i \omega & \text{in } \Omega \setminus D, \\ 1 + 2i \omega & \text{in } D, \end{cases}$$

with $\omega = 200\pi$, i.e. $\sigma_D = \sigma_0 = 1$, $\epsilon_D = 1$, and $\epsilon_0 = 2$. Hence, the ratio of the background conductivities is $\alpha = 1 + i\omega$, and the contrast assumption in theorem 3.2 is fulfilled with $c = \epsilon_0 \sigma_0 - \epsilon_D \sigma_D = 1$.

Theorem 3.2 guarantees that

$$B \subseteq D \quad \text{implies that} \quad \Re\{aR(\gamma_\omega)\} \leq R\left((1 + \beta\chi)\gamma_0\right),$$

i.e., that the ultrasound modulated dc measurements $R((1 + \beta\chi)\gamma_0)$ are larger (in the sense of matrix definiteness) than (the real part of ratio-weighted) ac measurements $\Re\{aR(\gamma_\omega)\}$ if the ultrasound-modulated focusing region $B$ lies inside the inclusion $D$, and the modulation strength $\beta > 0$ is small enough. Remark 3.3 suggests that the converse of (17) is true if enough electrodes are used. To test this numerically, we choose five circular focusing regions $B_1, \ldots, B_5$ (sketched in blue in figure 1) with radius 1.25. The modulation strength is chosen to be (see theorem 3.2)

$$\beta = \omega^2 |c| \frac{\epsilon_D}{\sigma_D (\sigma_D^2 + \omega^2 \epsilon_D^2)} \approx 0.9999.$$

Table 1 shows the eigenvalues of $R((1 + \beta\chi)\gamma_0) - \Re\{aR(\gamma_\omega)\}$ for $j \in \{1, \ldots, 5\}$. The numerical error ($\delta \approx 0.0005$) in table 1 was estimated by repeating the calculations on a finer FEM grid. Taking into account this estimated numerical error, the monotonicity test $R((1 + \beta\chi)\gamma_0) \geq \Re\{aR(\gamma_\omega)\}$ is only fulfilled for the focusing region $B_2$, which lies inside the inclusion.

**Example 4.2.** Now we consider the three-dimensional setting illustrated in figure 2. The imaging domain $\Omega$ is a cylindrical domain with

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \| (x_1, x_2, 0) \| < 10, \ 0 < x_3 < 5 \}.$$

And a ball-shaped anomaly $D$ with radius 1.5 is located at $(5, 0, 2.5)$. On the boundary $\partial \Omega$, there are 16 electrodes $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{16}$ attached.
The dc and ac admittivities $\gamma_0$ and $\gamma_\omega$ are chosen as

$$\gamma_0 := \begin{cases} 1, & \text{in } \Omega \setminus D, \\ 2, & \text{in } D, \end{cases}$$

and

$$\gamma_\omega := \begin{cases} 1 + i\omega, & \text{in } \Omega \setminus D, \\ 2 + i\omega, & \text{in } D, \end{cases}$$

with $\omega = 200\pi$, so that $\alpha = 1 + i\omega$ and $c = -1$. As in example 4.1 we check the monotonicity relation for five focusing regions $B_1, B_2, B_3, B_4$ and $B_5$. The regions are ball-shaped with radius 1.25 and centered at $(0, 0, 2.5)$, $(5, 0, 2.5)$, $(0, 5, 2.5)$, $(-5, 0, 2.5)$ and $(0, -5, 2.5)$, respectively. We choose $\beta$ according to theorem 3.2 as

$$\beta = \frac{\varepsilon D}{\sigma D (\varepsilon_0 \sigma_\omega + \omega^2 \varepsilon_1 \eta_2)} \approx 0.4999.$$  

Table 2 shows the eigenvalues of $R((1 - \beta x_0) y_0) - \Re(a R(y_\omega))$ for $j = 1, \ldots, 5$. The numerical error ($\delta \approx 0.14$) in table 2 was estimated by repeating the calculations on a finer grid.
FEM grid. Taking into account this estimated numerical error, the monotonicity test
\( R((1 - \beta \mathcal{E}_n)\gamma_0) - \Re(aR(\gamma_0)) \leq 0 \) is only fulfilled for the second focussing region, which lies inside the inclusion.

**Example 4.3.** In our last example we test a large number or small balls in order to demonstrate up to which extend the method is capable of determining the shape of an inclusion. We consider the two- and three-dimensional example shown in figures 3 and 4, respectively. In both settings
\[
\gamma_0 := 1, \quad \text{and} \quad \gamma_x := \begin{cases} 1 + 2i\omega & \text{in } \Omega \setminus D, \\ 1 + i\omega & \text{in } D, \end{cases}
\]
with \( \omega = 200\pi \), so that \( \alpha = 1 + 2i\omega \) and \( c = -1 \). In accordance with theorem 3.2, we choose
\[
\beta = \omega^2 |c| \frac{\varepsilon_D}{\sigma_D(\sigma_D\sigma_Q + \omega^2\varepsilon_D\varepsilon_Q)} \approx 0.4999.
\]

We now consider a large number of test balls \( B_j, j \in \{1, 2, \ldots, N\} \), and mark all balls for which
\[
R\left( (1 - \beta \mathcal{E}_n)\gamma_0 \right) - \Re(aR(\gamma_0)) \leq \delta I,
\]
where \( I \) is the identity matrix and \( \delta > 0 \) is a regularization parameter. In both examples, we used the heuristically chosen value \( \delta = 0.5 \cdot 10^{-7} \). Figures 5 and 6 show the test balls (in blue), the true inclusion (in red) and the balls for which (18) is fulfilled (in grey).

**5. Conclusion and discussion**

We have developed a new method to detect and localize conductivity anomalies by combining frequency-difference EIT with UMEIT. Our method is based on comparing (in terms of matrix definiteness) UMEIT measurements with (the real part of ratio-weighted) EIT
measurements at a non-zero frequency. We showed that this comparison determines whether the focusing region of the ultrasound wave lies inside a conductivity anomaly or not.

Remarkably, our new method merely utilizes the two sets of EIT measurements, and the background conductivity ratio which in turn can be estimated from EIT measurements. The method does not require any numerical simulations, forward calculations or geometry-dependent special solutions. It can be implemented without knowing the imaging domain shape or the electrode position, and is thus completely unaffected by modeling errors.

We gave a rigorous mathematical proof for our new method for the case of continuous boundary data, and we justified why the method can be expected to work also for realistic electrode measurements, provided that the number of electrodes is large enough.

The method is based on the assumption that the background conductivity is spatially constant, and that the anomalies fulfill the contrast condition that is required in frequency-difference EIT. Also, our method relies on the idealistic assumption of ultrasound modulated EIT, that it is possible to perfectly focus an ultrasound wave so that the conductivity changes only in a small test region. In real applications, background conductivities can be expected to be at least slightly inhomogeneous, and the ultrasound wave will also have some effect on the

Figure 3. Two-dimensional measurement setting of example 4.3.

Figure 4. Three-dimensional measurement setting of example 4.3.
conducivity outside the focusing region. The performance of our new method in such a setting has yet to be evaluated. Let us however note that the matrix definiteness comparisons, that are used by our method, are principally stable (see remark 2.3) so that our arguably idealistic modeling assumptions only have to be approximately valid. Moreover, monotonicity arguments also allow for worst-case testing and resolution guarantees (see [37]) which might be helpful in relaxing the idealistic assumptions in future studies.

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References

[1] Adler A, Gaburro R and Lionheart W 2011 Electrical impedance tomography Handbook of Mathematical Methods in Imaging (Berlin: Springer) pp 599–654
[2] Ammari H et al 2008 Electrical impedance tomography by elastic deformation SIAM J. Appl. Math. 68 1557–73
[3] Ammari H, Garnier J and Jing W 2012 Resolution and stability analysis in acousto-electric imaging Inverse Problems 28 084005
[4] Bal G 2013 Cauchy problem for ultrasound-modulated eit Anal. PDE 6 751–75
[5] Bal G, Naeter W, Scherzer O and Schotland J 2013 The levenberg-marquardt iteration for numerical inversion of the power density operator J. Inverse Ill-Posed Probl. 21 265–80
[6] Barber D and Brown B 1984 Applied potential tomography J. Phys. E. Sci. Instrum. 17 723–33
[7] Barth A, Harrach B, Hyvönen N and Mustonen L Detecting stochastic inclusions in electrical impedance tomography (Preprint, available online at http.mathematik.uni-stuttgart.de/oip)
[8] Bayford R 2006 Bioimpedance tomography (electrical impedance tomography) Annu. Rev. Biomed. Eng. 8 63–91
[9] Bonnetier E and Triki F 2014 A note on reconstructing the conductivity in impedance tomography by elastic perturbation The Impact of Applications on Mathematics (Berlin: Springer) pp 275–82
[10] Borcea L 2002 Electrical impedance tomography Inverse Problems 18 99–136
[11] Borcea L 2003 Addendum to electrical impedance tomography Inverse Problems 19 997–8
[12] Brown B 2003 Electrical impedance tomography (EIT): a review J. Med. Eng. Technol. 27 97–108
[13] Brown B and Seagar A 1987 The Sheffield data collection system Clin. Phys. Physiol. Meas. 8 91
[14] Brühl M and Hanke M 2000 Numerical implementation of two noniterative methods for locating inclusions by impedance tomography Inverse Problems 16 1029–42
[15] Chaulet N, Arridge S, Betcke T and Holder D 2014 The factorization method for three dimensional electrical impedance tomography Inverse Problems 30 045005
[16] Cheney M, Isaacson D and Newell J C 1999 Electrical impedance tomography SIAM Rev. 41 85–101
[17] Choi M K, Harrach B and Seo J K 2014 Regularizing a linearized eit reconstruction method using a sensitivity-based factorization method Inverse Problems Sci. Eng. 22 1029–44
[18] Chung E T, Chan T F and Tai X-C 2005 Electrical impedance tomography using level set representation and total variational regularization J. Comput. Phys. 205 357–72
[19] Dorn O and Lesselier D 2006 Level set methods for inverse scattering Inverse Problems 22 R67
[20] Friedman A 1987 Detection of mines by electric measurements SIAM J. Appl. Math. 47 201–12
[21] Friedman A and Isakov V 1989 On the uniqueness in the inverse conductivity problem with one measurement Indiana Univ. Math. J. 38 563–79
[22] Gebauer B 2006 The factorization method for real elliptic problems Z. Anal. Anwendungen 25 81–102
[23] Gebauer B 2008 Localized potentials in electrical impedance tomography Inverse Probl. Imaging 2 251–69
[24] Gebauer B and Hyvönen N 2007 Factorization method and irregular inclusions in electrical impedance tomography Inverse Problems 23 2159–70
[25] Gebauer B and Scherzer O 2008 Impedance-acoustic tomography SIAM J. Appl. Math. 69 565–76
[26] Haddar H and Migliorati G 2013 Numerical analysis of the factorization method for eit with a piecewise constant uncertain background Inverse Problems 29 065003
[27] Hakula H and Hyvönen N 2009 On computation of test dipoles for factorization method Inverse Problems 25 045009
[28] Hanke M and Brühl M 2003 Recent progress in electrical impedance tomography Inverse Problems 19 565–90
[29] Hanke M, Harrach B and Hyvönen N 2011 Justification of point electrode models in electrical impedance tomography Math. Models Methods Appl. Sci. 21 1395–413
[30] Hanke M and Kirsch A 2011 Sampling methods Handbook of Mathematical Models in Imaging ed O Scherzer (Berlin: Springer) pp 501–50
[31] Hanke M and Schappel B 2008 The factorization method for electrical impedance tomography in the half-space SIAM J. Appl. Math. 68 907–24
[32] Harrach B Interpolation of missing electrode data in electrical impedance tomography (Preprint, available online at www.mathematik.uni-stuttgart.de/oip)
[33] Harrach B 2013 Recent progress on the factorization method for electrical impedance tomography
Comput. Math. Methods Med. 2013 425184
[34] Harrach B and Seo J K 2009 Detecting inclusions in electrical impedance tomography without
reference measurements SIAM J. Appl. Math. 69 1662–81
[35] Harrach B and Seo J K 2010 Exact shape-reconstruction by one-step linearization in electrical
impedance tomography SIAM J. Math. Anal. 42 1505–18
[36] Harrach B, Seo J K and Woo E J 2010 Factorization method and its physical justification
in frequency-difference electrical impedance tomography IEEE Trans. Med. Imaging 29
1918–26
[37] Harrach B and Ullrich M Resolution guarantees in electrical impedance tomography IEEE Trans.
Med. Imaging 34 1513–21
[38] Harrach B and Ullrich M 2013 Monotonicity-based shape reconstruction in electrical impedance
tomography SIAM J. Math. Anal. 45 3382–403
[39] Henderson R and Webster J 1978 An impedance camera for spatially specific measurements of the
thorax IEEE Trans. Biomed. Eng. BME-25 250–4
[40] Holder D 2005 Electrical Impedance Tomography: Methods, History and Applications (Bristol:
IOP Publishing)
[41] Hyvönen N 2004 Complete electrode model of electrical impedance tomography: approximation
properties and characterization of inclusions SIAM J. Appl. Math. 64 902–31
[42] Hyvönen N 2009 Approximating idealized boundary data of electric impedance tomography by
electrode measurements Math. Models Methods Appl. Sci. 19 1185–202
[43] Hyvönen N, Hakula H and Pursiainen S 2007 Numerical implementation of the factorization
method within the complete electrode model of electrical impedance tomography Inverse Probl.
Imaging 1 299–317
[44] Ide T, Isozaki H, Nakata S and Siltanen S 2010 Local detection of three-dimensional inclusions in
electrical impedance tomography Inverse problems 26 035001
[45] Ide T, Isozaki H, Nakata S, Siltanen S and Uhlmann G 2007 Probing for electrical inclusions with
complex spherical waves Commun. Pure Appl. Math. 60 1415–42
[46] Ikehata M 1998 Size estimation of inclusion J. Inverse Ill-Posed Probl. 6 127–40
[47] Ikehata M 1999 How to draw a picture of an unknown inclusion from boundary measurements
Two mathematical inversion algorithms J. Inverse Ill-Posed Probl. 7 255–71
[48] Ikehata M 2000 Reconstruction of the support function for inclusion from boundary measurements
J. Inverse Ill-Posed Probl. 8 367–78
[49] Ikehata M 2002 A regularized extraction formula in the enclosure method Inverse Problems
18 435
[50] Ikehata M and Siltanen S 2000 Numerical method for finding the convex hull of an inclusion in
conductivity from boundary measurements Inverse Problems 16 1043–52
[51] Ikehata M and Siltanen S 2004 Electrical impedance tomography and Mittag–Leffler’s function
Inverse Problems 20 1325
[52] Ito K, Kunisch K and Li Z 2001 Level-set function approach to an inverse interface problem
Inverse problems 17 1225
[53] Kang H, Seo J K and Sheen D 1997 The inverse conductivity problem with one measurement:
stability and estimation of size SIAM J. Math. Anal. 28 1389–405
[54] Kirsch A 2005 The factorization method for a class of inverse elliptic problems Math. Nachr. 278
258–77
[55] Kirsch A and Grinberg N 2007 The Factorization Method for Inverse Problems (Oxford Lecture
Series in Mathematics and Its Applications vol 36) (Oxford: Oxford University Press)
[56] Kocyigit I 2012 Acousto-electric tomography and CGO solutions with internal data Inverse
Problems 28 125004
[57] Kuchment P 2012 Mathematics of hybrid imaging: a brief review The Mathematical Legacy of
Leon Ehrenpreis (Berlin: Springer) pp 183–208
[58] Kuchment P and Kunyansky L 2011 2D and 3D reconstructions in acousto-electric tomography
Inverse Problems 27 055013
[59] Lechleiter A, Hyvönen N and Hakula H 2008 The factorization method applied to the complete
electrode model of impedance tomography SIAM J. Appl. Math. 68 1097–121
[60] Lionheart W R B 2004 EIT reconstruction algorithms: pitfalls, challenges and recent developments
Physiol. Meas. 25 125–42
[61] Metherall P, Barber D C, Smallwood R H and Brown B H 1996 Three-dimensional electrical impedance tomography Nature 380 509–12
[62] Monard F and Bal G 2013 Inverse anisotropic conductivity from power densities in dimension $n \geq 3$ Commun. PDE 38 1183–207
[63] Nachman A I, Päivärinta L and Teirilä A 2007 On imaging obstacles inside inhomogeneous media J. Funct. Anal. 252 490–516
[64] Nguyen D, Jin C, Thiagalingam A and McEwan A 2012 A review on electrical impedance tomography for pulmonary perfusion imaging Physiol. Meas. 33 695–706
[65] Potthast R 2006 A survey on sampling and probe methods for inverse problems Inverse Problems 22 R1–47
[66] Rahmati P, Soleimani M, Pulletz S, Frerichs I and Adler A 2012 Level-set-based reconstruction algorithm for EIT lung images: first clinical results Physiol. Meas. 33 739
[67] Schmitt S 2009 The factorization method for EIT in the case of mixed inclusions Inverse Problems 25 065012
[68] Schmitt S and Kirsch A 2011 A factorization scheme for determining conductivity contrasts in impedance tomography Inverse Problems 27 095005
[69] Seo J K, Lee J, Zribi H, Kim S W and Woo E J 2008 Frequency-difference electrical impedance tomography (fdEIT): algorithm development and feasibility study Physiol. Meas. 29 929–44
[70] Soleimani M 2006 Electrical impedance tomography imaging using a priori ultrasound data Biomed. Eng. Online 5 8
[71] Steiner G, Soleimani M and Watzenig D 2008 A bio-electromechanical imaging technique with combined electrical impedance and ultrasound tomography Physiol. Meas. 29 S63
[72] Tai X-C and Chan T F 2004 A survey on multiple level set methods with applications for identifying piecewise constant functions Int. J. Numer. Anal. Model 1 25–47
[73] Tamburrino A 2006 Monotonicity based imaging methods for elliptic and parabolic inverse problems J. Inverse Ill-Posed Probl. 14 633–42
[74] Tamburrino A and Rubinacci G 2002 A new non-iterative inversion method for electrical resistance tomography Inverse Problems 18 1809–29
[75] Uhlmann G 2009 Electrical impedance tomography and calderón’s problem Inverse Problems 25 123011
[76] Uhlmann G and Wang J 2008 Reconstructing discontinuities using complex geometrical optics solutions SIAM J. Appl. Math. 68 1026
[77] van den Doel K and Ascher U M 2006 On level set regularization for highly ill-posed distributed parameter estimation problems J. Comput. Phys. 216 707–23
[78] Widlak T and Scherzer O 2012 Hybrid tomography for conductivity imaging Inverse Problems 28 084008
[79] Zhang H and Wang L V 2004 Acousto-electric tomography Proc. SPIE 5320 145–9
[80] Zhou L, Harrach B and Seo J K Monotonicity-Based electrical impedance tomography lung imaging (Preprint, available online at http://mathematik.uni-stuttgart.de/oip)