Representations of Two-parameter Quantum Orthogonal and Symplectic Groups

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Abstract. We investigate the finite-dimensional representation theory of two-parameter quantum orthogonal and symplectic groups that we found in [BGH] under the assumption that \( rs^{-1} \) is not a root of unity and extend some results [BW1, BW2] obtained for type A to types B, C and D. We construct the corresponding R-matrices and the quantum Casimir operators, by which we prove that the complete reducibility Theorem also holds for the categories of finite-dimensional weight modules for types B, C, D.

1. Preliminaries: Two-parameter Quantum Groups for Classical Types

Let \( \mathbb{K} \supset \mathbb{Q}(r, s) \) denote an algebraically closed field, where the two-parameters \( r, s \) are nonzero complex numbers satisfying \( r^2 \neq s^2 \).

In this section, we recall the definitions of the two-parameter quantum groups \( U_{r,s}(g) \) for \( g = \mathfrak{sl}_{n+1} \) from [BW1], and for \( g = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n} \) and \( \mathfrak{so}_{2n} \) from [BGH].

Let \( \Psi \) be a finite root system of a simple Lie algebra \( g \) of rank \( n \) with \( \Pi \) a base of simple roots. Regard \( \Psi \) as a subset of a Euclidean space \( E = \mathbb{R}^n \) with an inner product \( (\cdot, \cdot) \). Let \( \epsilon_1, \cdots, \epsilon_n \) denote an orthonormal basis of \( E \). We need the following data on (prime) root systems.

Type A:

\[
\Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n \}, \\
\Psi = \{ \pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n+1 \}.
\]

Type B:

\[
\Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n \} \cup \{ \alpha_n = \epsilon_n \}, \\
\Psi = \{ \pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n \} \cup \{ \pm\epsilon_i \mid 1 \leq i \leq n \}.
\]
Type C:  
\[ \Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n \} \cup \{ \alpha_n = 2\epsilon_n \}, \]
\[ \Psi = \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n \} \cup \{ 2\epsilon_i \mid 1 \leq i \leq n \}. \]

Type D:  
\[ \Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n \} \cup \{ \alpha_n = \epsilon_{n-1} + \epsilon_n \}, \]
\[ \Psi = \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n \}. \]

In the cases of type A, C and D, we set \( r_i = r^{(\alpha_i, \alpha_i)}, s_i = s^{(\alpha_i, \alpha_i)} \); while for type B, we set \( r_i = r^{(\alpha_i, \alpha_i)}, s_i = s^{(\alpha_i, \alpha_i)} \).

Assigned to \( \Pi \), there are two sets of mutually-commutative symbols \( W = \{ \omega_i^{\pm 1} \mid 1 \leq i \leq n \} \) and \( W' = \{ \omega_i^{r,\pm 1} \mid 1 \leq i \leq n \} \). Define a pairing \( \langle \cdot, \cdot \rangle : W' \times W \to K \) as follows:

\[
\begin{align*}
(1A) \quad & \langle \omega'_i, \omega_j \rangle = r^{(\epsilon_j, \alpha_i)} s^{(\epsilon_{j+1}, \alpha_i)}, & i \leq n+1, j \leq n, & \text{for } sl_{n+1}, \\
(1B) \quad & \langle \omega'_i, \omega_j \rangle = \begin{cases} r^{2(\epsilon, \alpha_i)} s^{2(\epsilon_{j+1}, \alpha_i)}, & i \leq n, j < n, \\ r^{2(\epsilon, \alpha_i)} s^{(\epsilon, \alpha_{n})}, & i < n, j = n, \\ r^{(\epsilon, \alpha_{n})} s^{-(\epsilon, \alpha_{n})}, & i = j = n. \\ \end{cases} & \text{for } so_{2n+1}, \\
(1C) \quad & \langle \omega'_i, \omega_j \rangle = \begin{cases} r^{(\epsilon, \alpha_{n})} s^{(\epsilon_{j+1}, \alpha_{n})}, & i \leq n, j < n, \\ r^{2(\epsilon, \alpha_{n})}, & i < n, j = n, \\ r^{(\epsilon, \alpha_{n})} s^{-(\epsilon, \alpha_{n})}, & i = j = n. \\ \end{cases} & \text{for } sp_{2n}, \\
(1D) \quad & \langle \omega'_i, \omega_j \rangle = \begin{cases} r^{(\epsilon, \alpha_{n})} s^{(\epsilon_{j+1}, \alpha_{n})}, & i \leq n, j < n, \\ r^{(\epsilon, \alpha_{n})} s^{-(\epsilon, \alpha_{n})}, & i \neq n-1, j = n, \\ r^{(\epsilon, \alpha_{n-1})} s^{-(\epsilon, \alpha_{n-1})}, & i = n-1, j = n. \\ \end{cases} & \text{for } so_{2n}, \\
(2) \quad & (\omega_i^{r,\pm 1}, \omega_j^{-1}) = (\omega_i^{r,\pm 1}, \omega_j^{-1}) = (\omega_i^{r,\pm 1}, \omega_j)^{\mp 1}, & \text{for any } g. 
\end{align*}
\]

**Lemma 1.1.** For the prime root systems of the Lie algebras \( g = sl_{n}, so_{2n+1}, so_{2n}, \) and \( sp_{2n}, \) there hold the identities:

\[
\begin{align*}
(\epsilon_{j+1}, \alpha_i) = -(\epsilon_i, \alpha_j), & \quad (i, j < n), & \text{for any } g. \\
(\epsilon_{j+1}, \alpha_n) = \begin{cases} -(\epsilon_n, \alpha_j), & (j < n), \\ -2(\epsilon_n, \alpha_j), & (j \leq n) \\ \end{cases} & \text{for } g = so_{2n+1}, \\
(\epsilon_j, \alpha_n) = \begin{cases} -(\epsilon_n, \alpha_{j-1}), & (j \leq n, j \neq n-1), \\ (\epsilon_{n-1}, \alpha_{n-1}), & (j = n-1) \\ \end{cases} & \text{for } g = sp_{2n}. 
\end{align*}
\]

Observe that Lemma 1.1 ensures the compatibility of the defining relations of the two-parameter quantum groups defined below.

Let \( U_{r,s}(g) \) be the unital associative algebra over \( K \) generated by symbols \( \epsilon_i, f_i, \omega^{r,\pm 1}_i, \omega^{s,\pm 1}_i \) (1 \( \leq i \leq n \)), subject to the following relations (X1) – (X4):

\[
\begin{align*}
(X1) \quad & \omega_i^{r,\pm 1} \omega_j^{r,\pm 1} = \omega_j^{r,\pm 1} \omega_i^{r,\pm 1}, & \omega_i^{s,\pm 1} \omega_i^{s,\mp 1} = 1 = \omega_i^{s,\mp 1} \omega_i^{s,\pm 1}. 
\end{align*}
\]
(X2) For \(1 \leq i, j \leq n\), we have
\[
\omega_i e_i \omega_j^{-1} = \langle \omega_i, \omega_j \rangle e_i, \quad \omega_j f_i \omega_j^{-1} = \langle \omega_i, \omega_j \rangle^{-1} f_i,
\]
\[
\omega'_i e_i \omega'_j^{-1} = \langle \omega'_i, \omega'_j \rangle^{-1} e_i, \quad \omega'_j f_i \omega'_j^{-1} = \langle \omega'_i, \omega'_j \rangle f_i.
\]
(X3) For \(1 \leq i, j \leq n\), we have
\[
[e_i, f_j] = \delta_{ij} \omega_i - \omega_j^\prime \frac{r_i}{r_i - s_i}.
\]
(X4) For any \(i \neq j\), we have the \((r, s)\)-Serre relations:
\[
(ad_l e_i)^{1-a_{ij}} (e_j) = 0,
\]
\[
(ad_r f_i)^{1-a_{ij}} (f_j) = 0,
\]
where the definitions of the left-adjoint action \(ad_l e_i\) and the right-adjoint action \(ad_r f_i\) are given in the following sense:
\[
ad_l a(b) = \sum_{(a)} a_{(1)} b S(a_{(2)}), \quad ad_r a(b) = \sum_{(a)} S(a_{(1)}) b a_{(2)}, \quad \forall a, b \in U_{r,s}(g),
\]
where \(\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}\) is given by Proposition 1.2 below.

The following fact is straightforward.

**Proposition 1.2.** The algebra \(U_{r,s}(g) (g = sl_{n+1}, so_{2n+1}, sp_{2n}, or so_{2n})\) is a Hopf algebra under the comultiplication, the counit and the antipode defined below:

\[
\Delta(\omega^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \quad \Delta(\omega'_i^{\pm 1}) = \omega'_i^{\pm 1} \otimes \omega'_i^{\pm 1},
\]
\[
\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i,
\]
\[
\varepsilon(\omega^{\pm 1}) = \varepsilon(\omega'_i^{\pm 1}) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,
\]
\[
S(\omega^{\pm 1}) = \omega_i^{\mp 1}, \quad S(\omega'_i^{\pm 1}) = \omega'_i^{\mp 1},
\]
\[
S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i \omega_i^{-1}.
\]

**Remark 1.3.** When \(r = s^{-1} = q\), Hopf algebra \(U_{r,s}(g)\) modulo the Hopf ideal generated by the elements \(\omega'_i - \omega_i^{-1} (1 \leq i \leq n)\), is just the quantum groups \(U_q(g)\) of Drinfel’d-Jimbo type.

**Definition 1.4.** A skew-dual pairing of two Hopf algebras \(A\) and \(U\) is a bilinear form \(\langle , \rangle : U \times A \rightarrow K\) such that
\[
\langle f, 1_A \rangle = \varepsilon_U(f), \quad \langle 1_U, a \rangle = \varepsilon_A(a),
\]
\[
\langle f, a_1 a_2 \rangle = \langle \Delta^{op}_U(f), a_1 \otimes a_2 \rangle, \quad \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta_A(a) \rangle,
\]
for all \(f, f_1, f_2 \in U\), and \(a, a_1, a_2 \in A\), where \(\varepsilon_U\) and \(\varepsilon_A\) denote the counits of \(U\) and \(A\), respectively, and \(\Delta_U\) and \(\Delta_A\) are their respective comultiplications.

Let \(B = B(g)\) (resp. \(B' = B'(g)\)) denote the Hopf subalgebra of \(U = U_{r,s}(g)\) generated by \(e_j, \omega^{\pm 1}_j\) (resp. \(f_j, \omega'_i^{\pm 1}\)) with \(1 \leq j \leq n\) for \(g = sl_{n+1}\), and with \(1 \leq j \leq n\) for \(g = so_{2n+1}, so_{2n}, and sp_{2n}\), respectively. The following result was obtained for the type \(A\) case by [BW1], and for the types \(B, C\) and \(D\) cases by [BGH].
Proposition 1.5. There exists a unique skew-dual pairing $\langle \cdot, \cdot \rangle : B' \times B \rightarrow \mathbb{K}$ of the Hopf subalgebras $B$ and $B'$ in $U_{r,s}(g)$, for $g = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$, such that $\langle f, e_j \rangle = \delta_{ij}$, and the conditions (1X) (where $X = A$, $B$, $C$, or $D$) and (2) are satisfied, and all other pairs of generators are 0. Moreover, we have $\langle S(a), S(b) \rangle = \langle a, b \rangle$ for $a \in B'$, $b \in B$. \hfill $\square$

Definition 1.6. For any two skew-paired Hopf algebras $A$ and $U$ by a skew-dual pairing $\langle \cdot, \cdot \rangle$, one may form the Drinfel'd double $D(A, U)$ as in [KS, 8.2], which is a Hopf algebra whose underlying coalgebra is $A \otimes U$ with the tensor product coalgebra structure, and whose algebra structure is defined by

$$\langle S(1) \otimes f \rangle(a' \otimes f') = \sum \langle S_U((1) \otimes f), (a')_1 \rangle \langle (3), a_2 \rangle a_2 (2) \otimes f(2) f', \tag{3}$$

for $a, a' \in A$ and $f, f' \in U$. The antipode $S$ is given by

$$S(a \otimes f) = (1 \otimes S_U(f)) (S_A(a) \otimes 1).$$

Clearly, both mappings $\mathcal{D} \ni a \mapsto a \otimes 1 \in \mathcal{D}(A, U)$ and $U \ni f \mapsto 1 \otimes f \in \mathcal{D}(A, U)$ are injective Hopf algebra homomorphisms. Let us denote the image $a \otimes 1$ (resp. $1 \otimes f$) of $a$ (resp. $f$) in $\mathcal{D}(A, U)$ by $\hat{a}$ (resp. $\hat{f}$). By (3), we have the following cross commutation relations between elements $\hat{a}$ (for $a \in A$) and $\hat{f}$ (for $f \in U$) in the algebra $\mathcal{D}(A, U)$:

$$\hat{f} \hat{a} = \sum \langle S_U((1) \otimes f), (a(1)) \rangle \langle (3), a(2) \rangle \hat{a}(2) \hat{f}(2), \tag{4}$$

$$\sum \langle (1) \otimes f, (a(1)) \rangle \hat{f}(2) \hat{a}(2) = \sum \hat{a}(1) \hat{f}(1) \langle f(2), a(2) \rangle. \tag{5}$$

In fact, as an algebra the double $D(A, U)$ is the universal algebra generated by the algebras $A$ and $U$ with cross relations (4) or, equivalently, (5).

Theorem 1.7 ([BW1, BGH]). The two-parameter quantum group $U = U_{r,s}(g)$ is isomorphic to the Drinfel’d quantum double $D(B, B')$, for $g = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$. \hfill $\square$

Let us denote $U_{r,s}(n)$ (resp. $U_{r,s}(n^-)$) the subalgebra of $B$ (resp. $B'$) generated by $e_i$ (resp. $f_i$) for all $i \leq n$. Let

$$U^0 = \mathbb{K}[\omega_1, \omega_2, \omega_n], \quad U_0 = \mathbb{K}[\omega_1 \pm 1, \omega_2 \pm 1, \ldots, \omega_n \pm 1],$$

$$U' = \mathbb{K}[\omega_1 ^{\pm 1}, \omega_2 ^{\pm 1}, \ldots, \omega_n ^{\pm 1}].$$

denote the respective Laurent polynomial subalgebras of $U_{r,s}(g)$, $B$, and $B'$. Clearly, $U^0 = U_0 U_0 = U_0 U_0$. Thus, by definition, we have $B = U_{r,s}(n) \ltimes U_0$, and $B' = U_0 \ltimes U_{r,s}(n^-)$, such that the double $D(B, B') \cong U_{r,s}(n) \otimes U^0 \otimes U_{r,s}(n^-)$, as vector spaces.

Let $\langle \cdot, \cdot \rangle : B \times B' \rightarrow \mathbb{K}$ denote the skew-dual pairing given by $\langle b, b' \rangle_0 = \langle S(b'), b \rangle$. Then, via a variation of its Drinfel’d double structure, we obtain the standard triangular decomposition of $U_{r,s}(g)$ in [BGH, Corollary 2.6] as follows.

Corollary 1.8. $U_{r,s}(g) \cong U_{r,s}(n^-) \otimes U^0 \otimes U_{r,s}(n)$, as vector spaces. In particular, it induces $U_q(g) \cong U_q(n^-) \otimes U^0 \otimes U_q(n)$, as vector spaces. \hfill $\square$

Let $Q = \mathbb{Z}Q$ denote the root lattice and set $Q^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$. Then for any

$$\zeta = \sum_{i=1}^n \zeta_i \alpha_i \in Q,$$ we denote

$$\omega_{\zeta} = \omega_1^{\zeta_1} \cdots \omega_n^{\zeta_n}, \quad \omega_{\zeta} = (\omega_1^{\zeta_1}) \cdots (\omega_n^{\zeta_n}).$$


The following $Q$-graded structure on $U_{r,s}(g)$ is necessary to develop to its weight representation theory discussed in the sequel.

**Corollary 1.9 ([BGH, Corollary 2.7]).** For any $\zeta = \sum_{i=1}^{n} \zeta_i \alpha_i \in Q$, the defining relations (X2) in $U_{r,s}(g)$ take the form below:

\[
\omega_i e_i \omega_i^{-1} = \langle \omega', \omega \rangle e_i, \quad \omega_i f_i \omega_i^{-1} = \langle \omega', \omega \rangle^{-1} f_i,
\]
\[
\omega' e_i \omega'^{-1} = \langle \omega'_r, \omega \rangle^{-1} e_i, \quad \omega' f_i \omega'^{-1} = \langle \omega'_r, \omega_i \rangle f_i.
\]

Then $U_{r,s}(n^\pm) = \bigoplus_{\eta \in Q^+} U_{r,s}^\eta(n^\pm)$ is $Q^\pm$-graded, where

\[
U_{r,s}^\eta(n^\pm) = \left\{ a \in U_{r,s}(n^\pm) \mid \omega_\eta a \omega_\eta^{-1} = \langle \omega'_\eta, \omega \rangle a, \ \omega'_\eta a \omega'^{-1} = \langle \omega'_{\eta}, \omega_i \rangle^{-1} a \right\},
\]

for $\eta \in Q^+ \cup Q^-$. Moreover, $U = \bigoplus_{\eta \in Q} U_{r,s}^\eta(g)$ is $Q$-graded such that

\[
U_{r,s}^\eta(g) = \left\{ \sum F_r \omega_r \omega_r E_r \in U \mid \omega_\eta (F_r \omega_r \omega_r E_r) \omega_\eta^{-1} = \langle \omega_r, \omega \rangle F_r \omega_r \omega_r E_r, \right. \]
\[
\omega'_{\eta} (F_r \omega_r \omega_r E_r) \omega'^{-1}_{\eta} = \langle \omega'_{\eta}, \omega_r \rangle^{-1} F_r \omega_r \omega_r E_r, \text{ with } \beta - \alpha = \eta \}
\]

where $F_r$ (resp. $E_r$) is a certain monomial $f_1 \cdots f_i$ (resp. $e_1 \cdots e_j$) such that $\alpha_i + \cdots + \alpha_{ij} = \beta$.

2. Finite-Dimensional Weight Representation Theory and Category $\mathcal{O}$

As we know, the standard triangular decomposition of $U_{r,s}(g)$ suggests that $U_{r,s}(g)$ possesses highest weight representation theory. Indeed, this has been developed by Benkart and Witherspoon in [BW2] for $g = gl_n$ or $sl_n$. In principle, one can expect the same theory to be valid as well for $g = so_{2n+1}$, $so_{2n}$, and $sp_{2n}$. To establish this, we will follow Benkart and Witherspoon's main ideas. However, to treat these cases in a unified fashion, we need to have better insights here and there in order to generalize the techniques used in the type $A$ case. Throughout the article, we assume that $K$ is an algebraically closed field containing $\mathbb{Q}(r,s)$ as a subfield and $rs^{-1}$ is not a root of unity.

Let $\Lambda$ be the weight lattice of $g$ for $g = so_{2n+1}$, $so_{2n}$, or $sp_{2n}$, respectively. We adopt similar notions and notations in [BW1]. Associated to any $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra $U^0$ over $K$ generated by the elements $\omega_i, \omega'_i (1 \leq i \leq n)$ to $K$ given by

\[
\hat{\lambda}(\omega_i) = \langle \omega_i', \omega \rangle, \quad \hat{\lambda}(\omega'_i) = \langle \omega_i, \omega \rangle^{-1},
\]

here we extend the definition of $\langle \cdot, \cdot \rangle$ from $\lambda \in Q$ to $\lambda \in \Lambda$ via taking appropriate half-integer powers when necessary, observing that $\Lambda \subseteq \bigoplus_{i=1}^{n} \mathbb{Z} \alpha_i \subseteq \bigoplus_{i=1}^{n} \frac{1}{2} \mathbb{Z} \epsilon_i$.

Let $M$ be a $U$-module of dimension $d < \infty$ where $U = U_{r,s}(g)$. As $K$ is algebraically closed, by linear algebra, we have

\[
M = \bigoplus_{\chi} M_\chi,
\]

where each $\chi : U^0 \rightarrow K$ is an algebra homomorphism, and $M_\chi$ is the generalized eigenspace given by

\[
M_\chi = \left\{ m \in M \mid (\omega_i - \chi(\omega_i))1^d m = 0 = (\omega_i' - \chi(\omega_i))1^d m, \ \forall i \right\}.
\]
When $M_\chi \neq 0$ we say that $\chi$ is a weight and $M_\chi$ is the corresponding weight space. In the case when $M$ decomposes into genuine eigenspaces relative to $U^0$, we say that $U^0$ acts semisimply on $M$.

Relations in (X2) imply

$$e_j M_\chi \subseteq M_{\chi - \alpha_j}, \quad f_j M_\chi \subseteq M_{\chi + \alpha_j},$$

where $\alpha_j$ as in (1), and $\chi \cdot \psi$ is the homomorphism with values $(\chi \cdot \psi)(\omega_i) = \chi(\omega_i)\psi(\omega_i)$ and $(\chi \cdot \psi)(\omega'_i) = \chi(\omega'_i)\psi(\omega'_i)$. In fact, if $(\omega_i - \chi(\omega_i)\psi(\omega_i))^k m = 0$, then $(\omega_i - \chi(\omega_i))\psi(\omega'_i) 1^k e_j m = 0$, and similarly for $\omega'_i$ and for $f_j$. On the one hand, (3) means that the sum of the eigenspaces is a submodule of $M$, and so if $M$ is simple, the sum must be $M$ itself, meanwhile we may replace the power $d$ in (2) by 1, that is, $U^0$ acts semisimply on each simple $M$. On the other hand, a direct consequence of (3) is that for each simple $M$ there is a homomorphism $\chi$ so that all the weights of $M$ are of the form $\chi \cdot \zeta$, where $\zeta \in Q$.

When all the weights of a module $M$ are of the form $\lambda$, where $\lambda \in \Lambda$, we say that $M$ has weights in $\Lambda$. Any simple $U$-module having one weight in $\Lambda$ has all its weights in $\Lambda$.

The observation below, which arises from Benkart and Witherspoon [BW2, Proposition 3.5] in the case when $g = gl_n$, or $sl_n$, also holds in our cases when $g = so_{2n+1}$, $so_{2n}$ and $sp_{2n}$.

**Lemma 2.1.** For $g = sl_n$, $so_{2n+1}$, $so_{2n}$ and $sp_{2n}$, suppose that $\zeta = \eta$, where $\zeta, \eta \in \Lambda$. Assume that $rs^{-1}$ is not a root of unity, then $\zeta = \eta$.

**Proof.** The proof for $g = sl_n$ was given in [BW1, Proposition 3.5]. We now give the proof case by case for $g = so_{2n+1}$, $sp_{2n}$ and $so_{2n}$, respectively.

For $\zeta = \sum_{i=1}^n \zeta_i \alpha_i \in \Lambda$, by definition, we have

(B) $\hat{\zeta}(\omega_i) = \langle \omega'_i, \omega_i \rangle = \begin{cases} r^{2(\epsilon_i,\zeta)}s^{2(\epsilon_i+1,\zeta)}, & i < n, \\ r^{2(\epsilon_i,\zeta)}(rs)^{-\zeta_n}, & i = n, \end{cases}$

(C) $\hat{\zeta}(\omega'_i) = \langle \omega'_i, \omega'_i \rangle^{-1} = \begin{cases} r^{2(\epsilon_i+1,\zeta)}s^{2(\epsilon_i,\zeta)}, & i < n, \\ s^{2(\epsilon_i,\zeta)}(rs)^{-\zeta_n}, & i = n. \end{cases}$

(D) $\hat{\zeta}(\omega_i) = \langle \omega'_i, \omega_i \rangle = \begin{cases} r^{(\epsilon_i,\zeta)}s^{(\epsilon_i+1,\zeta)}, & i < n, \\ r^{(\epsilon_i,\zeta)}(rs)^{-\zeta_n}, & i = n; \end{cases}$

Denote $\mu = \zeta - \eta$, from $\hat{\zeta}(\omega_n) = \hat{\eta}(\omega_n)$ and $\hat{\zeta}(\omega'_n) = \hat{\eta}(\omega'_n)$, in the type B or C case, we get $r^{2(\epsilon_n,\mu)}(rs)^{-\mu_n} = 1$, $s^{2(\epsilon_n,\mu)}(rs)^{-\mu_n} = 1$; or $r^{2(\epsilon_n,\mu)}(rs)^{-2\mu_n} = 1$, $s^{2(\epsilon_n,\mu)}(rs)^{-2\mu_n} = 1$. So $(rs^{-1})^{2(\epsilon_n,\mu)} = 1$ which, together with the assumption, means the integer $2(\epsilon_n,\mu) = 0$, that is,

$$\mu_{n-1} = \mu_n, \quad \text{(for type B)}, \quad \text{or} \quad \mu_{n-1} = 2\mu_n, \quad \text{(for type C)}.$$
Again from \( \hat{\zeta}(\omega_{n-1}) = \hat{\eta}(\omega_{n-1}) \) and \( \hat{\zeta}(\omega'_{n-1}) = \hat{\eta}(\omega'_{n-1}) \), in the type B or C case, we get \( (\alpha_{n-1}, \mu) = 0 \), that is,

\[
\mu_{n-2} = \mu_n, \quad (\text{for type B}), \quad \text{or} \quad \mu_{n-2} = 2\mu_n, \quad (\text{for type C}).
\]

But similar to the deduction in the case of type A (see [BW1]), noting \( \mu_0 = 0 \), we have

\[
\mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1} = 0, \quad (i = 1, 2, \cdots, n - 2),
\]

\[
(7) \quad \mu_{2k} = k\mu_2, \quad \mu_{2k+1} = k\mu_2 + \mu_1.
\]

Thus, by (4), (5) & (6), we get \( \mu_n = \mu_{n-1} = \cdots = \mu_1 = \mu_0 = 0 \) in the type B case. For the type C case, if \( n = 2m \), by (5) & (7), we get \( \mu_{n-2} = (m-1)\mu_2 = 2\mu_n = 2m\mu_2 \), i.e., \( \mu_2 = 0 \), so \( \mu_n = 0 \); if \( n - 1 = 2m \), then by (4), (5), & (7), we get \( m\mu_2 = \mu_{n-2} = (m-1)\mu_2 + \mu_1 \), i.e., \( \mu_2 = \mu_1 \), again by (4) & (7), we get \( \mu_2 = 0 \), so \( \mu_n = 0 \), which is reduced to the precondition of the proof in the type A case. Hence, using the same argument as in the case of type A ([BW1]), we have \( \mu = 0 \). Therefore, \( \zeta = \eta \) in both cases B and C.

For the type D case, from \( \hat{\zeta}(\omega_i) = \hat{\eta}(\omega_i) \) and \( \hat{\zeta}(\omega'_i) = \hat{\eta}(\omega'_i) \) for \( i = n - 1, n \), we have \( (rs^{-1})^{(\alpha_{n-1}, \mu)} = 1 \) and \( (rs^{-1})^{(\alpha_n, \mu)} = 1 \), that means, together with the assumption, the integers \( (\alpha_{n-1}, \mu) = 0 \) and \( (\alpha_n, \mu) = 0 \). So we get \( \mu_{n-2} = 2\mu_{n-1} = 2\mu_n \). If \( n = 2m \), then \( (m-1)\mu_2 = \mu_{n-2} = 2m\mu_2 \), i.e., \( \mu_2 = 0 \). If \( n - 1 = 2m \), applying (7) to \( \mu_{n-1} = \mu_n \), we get \( \mu_1 = 0 \); applying (7) to \( \mu_{n-2} = 2\mu_{n-1} \), we get \( \mu_2 = 0 \). So we have \( \mu_n = 0 \) for any \( n \). Using the same proof as in the case of type A, we obtain \( \mu = 0 \), i.e., \( \zeta = \eta \).

\[\square\]

**Remark 2.2.** Lemma 2.1 indicates that under the assumption that \( rs^{-1} \) is not a root of unity, we may simplify the notation by writing \( M_\lambda \) for the weight space rather than writing \( M_\lambda \) for \( \lambda \in \Lambda \). So it makes sense to let (3) take the classical form: \( e_i M_\lambda \subseteq M_{\lambda + \alpha_i} \), and \( f_j M_\lambda \subseteq M_{\lambda - \alpha_j} \).

Similar to the proof of [BW2, Corollary 3.14], we have

**Corollary 2.3.** Let \( M \) be a finite-dimensional \( U_{r,s}(\mathfrak{g}) \)-module for \( \mathfrak{g} = \mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \) or \( \mathfrak{osp}_{2n} \). Assume that \( rs^{-1} \) is not a root of unity, then the elements \( e_i, f_i \) (\( 1 \leq i \leq n \)) act nilpotently on \( M \).

\[\square\]

Obviously, when \( rs^{-1} \) is not a root of unity, a finite-dimensional simple \( U \)-module is a highest weight module by Corollary 2.3 and (3).

We state the definition of the category \( \mathcal{O} \) of weight \( U \)-modules as in [BW1, Section 4].

**Definition 2.4.** Let \( \mathcal{O} \) denote the category of modules \( M \) for \( U_{r,s}(\mathfrak{g}) \) (where \( \mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \) or \( \mathfrak{osp}_{2n} \)) which satisfy the following conditions:

\((O1)\) \( U^0 \) acts semisimply on \( M \), and the set \( \text{wt}(M) \) of weights of \( M \) belongs to \( \Lambda : \ M = \bigoplus_{\lambda \in \text{wt}(M)} M_\lambda \), where \( M_\lambda = \{ m \in M \mid \omega_i m = \langle \omega'_i, \omega_i \rangle m, \omega'_i m = \langle \omega'_i, \omega_i \rangle^{-1} m \} \);

\((O2)\) \( \dim_k M_\lambda < \infty \) for all \( \lambda \in \text{wt}(M) \);

\((O3)\) \( \text{wt}(M) \subseteq \cup_{\mu \in F} (\mu - Q^+) \) for some finite set \( F \subseteq \Lambda \).

The morphisms in \( \mathcal{O} \) are \( U \)-module homomorphisms.

Actually, the category \( \mathcal{O} \) just focuses on the class of the so-called type 1 \( U \)-modules like in the case of Drinfel’d-Jimbo quantum groups (see [J], [Jo], [KS]),
which is closed under taking sub-object or sub-quotient object, making finite direct sum and taking tensor product.

Let $V^\psi$ be the one-dimensional $B$-module on which $e_i$ acts as multiplication by 0 ($1 \leq i \leq n$), and $U^0$ acts via $\psi$, an algebra homomorphism from $U^0$ to $K$. As usual, we can define the Verma module $M(\psi)$ with highest weight $\psi$ to be the $U$-module induced from $V^\psi$, that is,

$$M(\psi) = U \otimes_B V^\psi.$$  

Set $v_\psi = 1 \otimes v \in M(\psi)$, where $v(\neq 0) \in V^\psi$. Then $e_i v_\psi = 0$ ($1 \leq i \leq n$) and $a v_\psi = \psi(a) v_\psi$ for any $a \in U^0$ by construction. By Corollary 1.8, $M(\psi) \cong U_{r,s}(n^-) \otimes v_\psi$. Corollary 1.9 indicates that each Verma module $M(\psi) \in \text{Ob}(O)$ if and only if $\psi \in \Lambda$.

Let $N'$ be a proper submodule of $M(\psi)$, then (3) implies that

$$N' \subset \sum_{\mu \in \mathcal{Q}^+ - \{0\}} M(\psi)(\zeta_{\mu}),$$

as $M(\psi)_\psi = K v_\psi$ generates $M(\psi)$. Hence, $M(\psi)$ has a unique maximal submodule $N$, namely the sum of all proper submodules, and a unique simple quotient, $L(\psi)$. Actually, all finite-dimensional simple $U$-modules are of this form, as the Theorem below indicates (which was proved by Benkart and Witherspoon [BW2, Theorem 2.1] in the case when $g = \mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{sp}_n$, but still holds with the same proof for our cases of $g$).

**Theorem 2.5.** For $g = \mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, let $M$ be a $U_{r,s}(g)$-module, on which $U^0$ acts semisimply and which contains an element $m \in M_\psi$ ($\psi \in \text{Hom}_B(U^0, K)$) such that $e_i m = 0$ for all $i$. Then there is a unique homomorphism of $U_{r,s}(g)$-modules $F : M(\psi) \rightarrow M$ with $F(v_\psi) = m$. In particular, if $rs^{-1}$ is not a root of unity and $M$ is a finite-dimensional simple $U_{r,s}(g)$-module, then $M \cong L(\psi)$ for some weight $\psi$. \hfill $\square$

As in [BW2, Lemma 2.3], it is easy to verify the commutation relations below.

**Lemma 2.6.** For $m \geq 1$, set $[m]_i = \frac{r_i^m - s_i^m}{r_i - s_i}$. Then for $1 \leq i \leq n$, we have

$$e_i f_i^m = f_i^m e_i + [m]_i f_i^{m-1} \frac{r_i^{1-m} \omega_i - s_i^{1-m} \omega_i'}{r_i - s_i},$$

$$e_i^m f_i = f_i e_i^m + [m]_i e_i^{m-1} \frac{s_i^{1-m} \omega_i - r_i^{1-m} \omega_i'}{r_i - s_i}. \hfill \square$$

Set $\alpha^\vee = \frac{2 \alpha}{(\alpha, \alpha)}$, for any simple root $\alpha \in \Pi$, then for any $\lambda \in \Lambda$, $(\lambda, \alpha^\vee) \in \mathbb{Z}$ by definition. Let $\Lambda^+ \subset \Lambda$ be the subset of dominant weights, that is, $\Lambda^+ = \{ \lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0, \text{ for } 1 \leq i \leq n \}$. Similar to [BW2, Lemma 2.4] in the type A case, we have

**Lemma 2.7.** For $g = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, assume that $rs^{-1}$ is not a root of unity. Let $M$ be a nonzero finite-dimensional $U_{r,s}(g)$-module on which $U^0$ acts semisimply. Suppose there is some nonzero vector $v \in M_\lambda$ with $\lambda \in \Lambda$ such that $e_i v = 0$ for all $i$ ($1 \leq i \leq n$). Then $\lambda \in \Lambda^+$.  

PROOF. It suffices to prove that \( (\lambda, \alpha'_j) \geq 0 \), as the proof of \( (\lambda, \alpha'_i) \geq 0 \) (1 \( \leq i < n \)) is the same as that of [BW2, Lemma 2.4].

Since \( f_n \) acts nilpotently on \( M \) by Corollary 2.3, there is some integer \( m \geq 0 \) such that \( f_n^{m+1}v = 0 \) but \( f_n^m v \neq 0 \). Applying \( e_n \) to \( f_n^{m+1}v = 0 \), using Lemma 2.6 and the fact that \( e_nv = 0 \), we get \( r^{-m}_n\lambda(w_n) = s^{-m}_n\lambda(w'_n) \). Equivalently,

\[
\begin{align*}
\left(r^{-m}_n,\lambda\right)_{(rs)}^{-\lambda_n} &= s^{-m}_n\left(2e,\lambda\right)_{(rs)}^{-\lambda_n}, \quad \text{(for type B)} \\
r^{-m}_n\left(2e,\lambda\right)_{(rs)}^{-2\lambda_n} &= s^{-m}_n\left(2e,\lambda\right)_{(rs)}^{-2\lambda_n}, \quad \text{(for type C)} \\
r^{-m}_n\left(e_{n-1},\lambda\right)s^{-\left(e,\lambda\right)_{(rs)}^{-2\lambda_n-1}} &= s^{-m}_n\left(-e,\lambda\right)s^{-\left(e_{n-1},\lambda\right)_{(rs)}^{-2\lambda_n-1}}, \quad \text{(for type D)}
\end{align*}
\]

or equivalently,

\[
(r_n s_{n-1}^{-1})^{-m+\left(\lambda,\alpha'_n\right)} = 1, \quad \text{(for types B, C, D)}.
\]

The assumption of \( rs^{-1} \) forces \( (\lambda, \alpha'_n) = m \geq 0 \). Therefore, \( \lambda \in \Lambda^+ \).

**Theorem 2.8.** For \( g = so_{2n+1}, so_{2n} \) and \( sp_{2n} \), assume that \( rs^{-1} \) is not a root of unity, then any finite-dimensional simple \( U_{r,s}(g) \)-module with weights in \( \Lambda \) is isomorphic to \( L(\lambda) \) for some \( \lambda \in \Lambda^+ \).

The representation theory of \( U_{r,s}(sl_2) \), developed by Benkart and Witherspoon in [BW2], plays a crucial role in the classification of finite-dimensional simple modules for \( U_{r,s}(sl_n) \) (see [BW2, Section 2]) like in the classical case of the simple Lie algebras or in the quantized case of the Drinfel’d-Jimbo quantum groups. Note the observation arising from the structure constants of \( U \) for types \( B, C, D \), respectively, \( \langle \omega'_1, \omega'_i \rangle = r_is_{i-1}^{-1} \) always holds. This fact guarantees that even in the two-parameter quantum orthogonal or symplectic groups \( U_{r,s}(g) \), there exist isomorphic copies of \( U_{r,s}(sl_2) \) as well. This suggests that these quantum groups possess a familiar finite-dimensional (weight) representation theory provided that \( rs^{-1} \) is not a root of unity.

Now let us recall the representation theory for \( U_{r,s}(sl_2) \). The first two assertions of the following Proposition comes from [BW2, Proposition 2.8 (i)], the last one may be regarded as an intrinsic generalization of [BW2, Proposition 2.8 (ii)] with a deep insight.

**Proposition 2.9.** Assume that \( rs^{-1} \) is not a root of unity. For \( U = U_{r,s}(sl_2) \) generated by \( e, f, \omega \) and \( \omega' \), for a given \( \phi \in \text{Hom}_{Alg}(U^0, \mathbb{K}) \), set \( \phi = \phi(\omega) \), \( \phi' = \phi(\omega') \), and in the Verma module \( M(\phi) \), put \( v_j = f^j/[j]! \otimes v_\phi \) for \( j \geq 0 \). Then

(i) \( M(\phi) \) is a simple \( U \)-module if and only if \( \phi \cdot r^{-j} - \phi' \cdot s^{-j} \neq 0 \) for any \( j \geq 0 \).

(ii) If \( \phi(\omega') = \phi(\omega)(rs^{-1})^{-m} \) for some integer \( m \geq 0 \), then \( \text{Span}_K \{ v_j \mid j \geq m+1 \} \cong M(\phi - (m+1)\alpha) \) is the unique maximal submodule of \( M(\phi) \). The quotient is the \((m+1)\)-dimensional simple module \( L(\phi) \) spanned by vectors \( v_0, v_1, \cdots, v_m \) and having \( U \)-action given by

\[
\begin{align*}
\omega.v_j &= \phi \cdot (rs^{-1})^{-j}v_j, & \omega'.v_j &= \phi \cdot (rs^{-1})^{-(m-j)}v_j, \\
e.v_j &= \phi \cdot r^{-m}[m+1-j]v_{j-1}, & (v_{-1} = 0) \\
f.v_j &= [j+1]v_{j+1}, & (v_{m+1} = 0)
\end{align*}
\]

Any \((m+1)\)-dimensional simple \( U \)-module is isomorphic to \( L(\phi) \) for some such \( \phi \).
(iii) If $\nu = \nu_1 \lambda_1 + \cdots + \nu_n \lambda_n \in \Lambda^+$, where $\lambda_i$ is the $i$-th fundamental weight for $\mathfrak{g}$, then $\hat{\nu}(\omega'_i) = \hat{\nu}(\omega_i)(r_i s_i^{-1})^{-\nu_i}$, and the $U_i$-module $L(\nu \lambda_i)$ is $(\nu_i + 1)$-dimensional and has $U_i$-action given by (8) with $\hat{\phi}_i = \hat{\nu}(\omega_i)$, where $U_i$ is the copy of $U_{r,s}(\mathfrak{sl}_2)$ in $U_{r,s}(\mathfrak{g})$ corresponding to the $i$-th vertex of the Dynkin diagram.

**Proof.** For the proof of the last assertion, it suffices to show that there hold

$$\hat{\nu}(\omega'_i) = (r_i s_i^{-1})^{-\alpha_i^\vee} = \frac{\hat{\nu}_i \lambda_i(\omega'_i)}{\nu_i \lambda_i(\omega_i)}, \quad \text{(for any } i)$$

for $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$.

In the type $A$ case, we have $\lambda_i = \epsilon_1 + \cdots + \epsilon_i$ for $1 \leq i \leq n$ and $\lambda_n = 0$. By definition,

$$\hat{\nu}(\omega'_i) = \frac{s^{(\epsilon_i,\nu)} r^{(\epsilon_i+1,\nu)}}{r^{(\epsilon_i,\nu)} s^{(\epsilon_i+1,\nu)}} = (rs^{-1})^{-(\alpha_i^\vee)} = (rs^{-1})^{-\nu_i}$$

$$= \frac{s^{(\epsilon_i,\nu)} r^{(\epsilon_i+1,\nu)}}{r^{(\epsilon_i,\nu)} s^{(\epsilon_i+1,\nu)}} = \frac{\hat{\nu}_i \lambda_i(\omega'_i)}{\nu_i \lambda_i(\omega_i)}$$

For types $B$, $C$ and $D$, it suffices to consider types $B_2$, $C_2$ and $D_4$, respectively.

In the type $B_2$ case, we have $\lambda_1 = \epsilon_1$, $\lambda_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)$. By the defining formula (B) in Lemma 2.1, for $i = 1$, it follows directly from the argument in the type $A$ case; while for $i = 2$, we get

$$\hat{\nu}(\omega'_2) = \frac{s^{(\epsilon_2,\nu)} r^{(\epsilon_2+1,\nu)}}{r^{(\epsilon_2,\nu)} s^{(\epsilon_2+1,\nu)}} = (rs^{-1})^{-(\alpha_2^\vee)} = \frac{\hat{\nu}_2 \lambda_2(\omega'_2)}{\nu_2 \lambda_2(\omega_2)}$$

In the type $C_2$ case, we have $\lambda_1 = \epsilon_1$, $\lambda_2 = \epsilon_1 + \epsilon_2$. It suffices to consider the case $i = 2$. Similarly, we have

$$\hat{\nu}(\omega'_2) = \frac{s^{(\epsilon_2,\nu)} r^{(\epsilon_2+1,\nu)}}{r^{(\epsilon_2,\nu)} s^{(\epsilon_2+1,\nu)}} = (rs^{-1})^{-(\alpha_2^\vee)} = \frac{s^{(\epsilon_2,\epsilon_2+\lambda_2)}}{r^{(\epsilon_2,\epsilon_2+\lambda_2)}} = \frac{\hat{\nu}_2 \lambda_2(\omega'_2)}{\nu_2 \lambda_2(\omega_2)}$$

In the type $D_4$ case, we have $\lambda_1 = \epsilon_1$, $\lambda_2 = \epsilon_1 + \epsilon_2$, $\lambda_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$, $\lambda_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$. It suffices to consider the cases $i = 3, 4$. By the formula (D) in Lemma 2.1, we have

$$\hat{\nu}(\omega'_i) = (rs^{-1})^{-(\alpha_i^\vee)} = \frac{\hat{\nu}_i \lambda_i(\omega'_i)}{\nu_i \lambda_i(\omega_i)},$$

for $i = 3, 4$.

The proof is completed. \(\square\)

**Corollary 2.10.** Assume that $rs^{-1}$ is not a root of unity and $\lambda \in \Lambda^+$, set $\nu_i = (\lambda, \alpha_i^\vee)$, then each vector $f_i^{\nu_i+1}.\nu_\lambda$ in the Verma $U$-module $M(\lambda)$ generates the Verma submodule $M(\lambda - (\nu_i + 1)\alpha_i)$ for all $i$, where $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$.

**Proof.** It follows from a direct calculation of $e_i f_i^{\nu_i+1}.\nu_\lambda = 0$ by Lemma 2.6 and (9). \(\square\)

More generally, we have
Proposition 2.11. Let $M(\lambda)$ be a Verma module with $\lambda \in \Lambda^+$. Then for every element $\omega$ of the Weyl group $W$ of $\mathfrak{g}$, there exists a Verma submodule in $M(\lambda)$ with highest weight
\begin{equation}
\lambda_\omega = \omega(\lambda + \rho) - \rho,
\end{equation}
where $\rho$ is the half-sum of all positive roots of $\mathfrak{g}$. Every simple $U$-module as a composition factor of $M(\lambda)$ determines a highest weight module in $\mathcal{O}$. These highest weights are of the form (10).

Proof. The proof of this proposition is analogous to that of the corresponding assertion in the classical theory (see Dixmier [D]). \hfill \Box

Lemma 2.12. For any simple $U$-module $L(\lambda)$ with $\lambda \in \Lambda^+$, take any $\beta = \sum_{i=1}^n n_i \alpha_i \in \mathbb{Q}^+$ such that $n_i \leq (\lambda, \alpha_i^\vee)$, $\forall i$, then the linear mapping $U_{r,s}^{-\beta}(n^-) \ni x \mapsto x.v_\lambda$ is injective.

Proof. By the definition of the Verma module, it is enough to show that $\lambda - \beta$ is not a weight of the maximal $U$-submodule $N$. This follows from Proposition 2.11, because no set of weights $\{ \lambda - \sum_{i=1}^n n_i \alpha_i | n_i \in \mathbb{Z}^+ \}$, $\omega \in \mathcal{W} - \{1\}$ contains $\lambda - \beta$. \hfill \Box

Lemma 2.13. If an element $a \in U_{r,s}^{-\beta}(n^-)$ satisfies the relations $e_i a = a e_i$ for $i = 1, 2, \ldots, n$, then we have $a = 0$. If $f_i b = b f_i$, $i = 1, 2, \ldots, n$, for some $b \in U_{r,s}^{-\beta}(n)$, then $b = 0$.

Proof. Write $\beta = \sum_{i=1}^n n_i \alpha_i \in \mathbb{Q}^+$, and take a dominant weight $\lambda \in \Lambda^+$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq m_i$ for all $i$. Consider the simple $U$-module $L(\lambda)$ with highest weight vector $v_\lambda$. Since $(e_i a).v_\lambda = (a e_i).v_\lambda = 0$ for all $i$, the vector $a.v_\lambda$ generates a proper submodule of $L(\lambda)$. Thus $a.v_\lambda = 0$, as $L(\lambda)$ is simple. Hence $a = 0$ by Lemma 2.12.

In order to prove the second assertion, we introduce a $\mathbb{Q}$-algebra isomorphism $\theta : U_{r,s}(\mathfrak{g}) \rightarrow U_{r,s}(\mathfrak{g})$ defined by
\begin{equation}
\begin{align*}
\theta(r) &= s^{-1}, & \theta(s) &= r^{-1}, \\
\theta(\omega_i) &= \omega_i', & \theta(\omega_i') &= \omega_i, \\
\theta(e_i) &= f_i, & \theta(f_i) &= (r_i s_i) e_i.
\end{align*}
\end{equation}
In fact, we can find that the image of $\theta$ is $\mathbb{Q}$-algebraically isomorphic to the associated quantum group $U_{s^{-1},r^{-1}}(\mathfrak{g})$, i.e., $\text{Im}(\theta) \cong (U_{s^{-1},r^{-1}}(\mathfrak{g}), \langle \rVert \rangle)$, where the pairing $\langle \omega'_i \rVert \omega_j \rangle$ is defined via substituting $(r, s)$ by $(s^{-1}, r^{-1})$ in the defining formula for $\langle \omega'_i \rVert \omega_j \rangle$ (see formulae (1.1) and (2) in Section 1).

Now applying the $\mathbb{Q}$-algebra isomorphism $\theta$ to the equation $f_i b = b f_i$, we get $\theta(b) = 0$, by the first assertion. Hence, $b = 0$. \hfill \Box

Returning to the pairing $\langle \rVert \rangle : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{K}$ in Proposition 1.5, and combining with the $\mathbb{Q}$-gradation on $U$ (see Corollary 1.9), we have

Proposition 2.14. For any $\beta \in \mathbb{Q}^+$, the restriction of the pairing $\langle \rVert \rangle$ in Proposition 1.5 to $\mathcal{B}_{s^{-\beta}} \times \mathcal{B}^\beta$ is nondegenerate.

Proof. We have to show that for any $a \in \mathcal{B}_{s^{-\beta}}$ such that $\langle a, b \rangle = 0$ for some $b \in \mathcal{B}^\beta$, implies that $b = 0$. This will be proved by induction with respect to the usual ordering of $\mathbb{Q}_+$. If $\beta$ is a simple root, then it is true by formula (2) in Section
1. Let $\beta > 0$ with $\text{ht}(\beta) > 1$ and suppose that it holds for all $\gamma \in Q^+$ such that $\beta - \gamma \in Q^+$.

Note that using the defining properties of skew-dual pairing and the comultiplication in $U$ (see Proposition 1.2), we may check by induction:

\begin{align}
\langle c, \omega'_{\alpha}, \omega_{\beta} \rangle d &= \langle \omega'_{\alpha}, \omega_{\beta} \rangle \langle c, d \rangle, \quad \forall c \in U_{r,s}(n^-), \ d \in U_{r,s}(n), \\
\langle c, d \rangle &= 0, \quad c \in U_{r,s}(n^-), \ d \in U_{r,s}(n), \ \sigma, \delta \in Q^+, \ \sigma \neq \delta.
\end{align}

It suffices to assume that $b \in U_{r,s}^\beta(n)$. By Proposition 1.2, we can write

\begin{align}
\Delta(b) &= \sum_{0 \leq \gamma \leq \beta} (\omega_{\gamma} \otimes 1) b_{\gamma}, \quad b_{\gamma} \in U_{r,s}^\gamma(n) \otimes U_{r,s}^{\beta-\gamma}(n),
\end{align}

where $b_0 = b \otimes 1$ and $b_{\beta} = 1 \otimes b$. Let $\gamma \in Q^+, \ 0 < \gamma < \beta, \ x \in B^{r,-\gamma}$ and $y \in B^{s,-(\beta-\gamma)}$.

By (2), (12) & (13), we have

\begin{align}
0 &= \langle xy, b \rangle = \langle x \otimes y, \Delta(b) \rangle = \langle x \otimes y, (\omega_{\gamma} \otimes 1) b_{\gamma} \rangle = \langle x \otimes y, b_{\gamma} \rangle.
\end{align}

By assumption, for any $\gamma' < \beta$ the restriction of $(,)$ to $B^{r,-\gamma'} \times B^{s,-\gamma'}$ is nondegenerate, so its extension to a bilinear form on $[B^{r,-\gamma} \otimes B^{s,-(\beta-\gamma)}] \times [B^{s} \otimes B^{s,-(\beta-\gamma)}]$. Hence it follows from (15) that $b_{\gamma} = 0$. Because of (14) this means that $\Delta(b) = b \otimes 1 + \omega_{\beta} \otimes b$.

By (13), together with $\Delta(f_i) = f_i \otimes 1 + f_i \otimes \omega'_i$, we get $f_i b = b f_i$, and then $f_i b = b f_i$ for any $i$, after using $\varphi$ (see the proof of [BGH, Theorem 2.5]). Thus, by Lemma 2.12, $b = 0$.

Similar reasoning indicates that for any $b \in B^{\beta}$ such that $\langle a, b \rangle = 0$ for some $a \in B^{r,-\beta}$ implies that $a = 0$. \(\square\)

In what follows, we consider the finite-dimensionality question of the simple $U_{r,s}(g)$-modules $L(\lambda)$ with $\lambda \in \Lambda^+$. This problem has been solved by Benkart and Witherspoon in [BW2, Section 2] in the case when $g = \mathfrak{gl}_n$, or $\mathfrak{sl}_n$. The same idea can be used to prove that $M(\lambda)$ has a $U_{r,s}(g)$-submodule $M'(\lambda)$ of finite codimension, as $L(\lambda)$ is the quotient of $M(\lambda)$ by its unique maximal submodule, where $M'(\lambda)$ is defined by

\begin{align}
M'(\lambda) = \sum_{i=1}^n U_{r,s}(g) f_i^{k_i+1} v_{\lambda} \cong \sum_{i=1}^n M(\lambda - (k_i+1) \alpha_i),
\end{align}

where $k_i = (\lambda, \alpha_i^\vee)$ for all $i$. That is, to prove the module $L'(\lambda) = M(\lambda)/M'(\lambda)$ is nonzero and finite-dimensional, $L'(\lambda) \neq 0$ is clear, since any weight in $M'(\lambda)$ is less than or equal to $\lambda - (k_i+1) \alpha_i$ for some $i$, $v_\lambda \notin M'(\lambda)$.

**Lemma 2.15.** (i) ([BW2, Lemma 2.10]) The elements $e_j, f_j$ $(1 \leq j \leq n)$ act locally nilpotently on $U_{r,s}(g)$-module $L'(\lambda)$.

(ii) ([BW2, Lemma 2.11]) Assume that $rs^{-1}$ is not a root of unity, $V = \bigoplus_{j \in \mathbb{Z}_+} \mathcal{V}_{\lambda-j\alpha} \in \text{Ob}(\mathcal{O})$ is a $U_{r,s}(\mathfrak{sl}_2)$-module for some weight $\lambda \in \Lambda$. If $e, f$ act locally nilpotently on $V$, then $\dim_k V < \infty$, and the weights of $V$ are preserved under the simple reflection taking $\alpha$ to $-\alpha$.

**Proof.** The proof of (i) is parallel to the type $A$ case; the second part assertion is direct from [BW2]. \(\square\)

**Proposition 2.16.** Assume that $rs^{-1}$ is not a root of unity. Then for the $U_{r,s}(g)$-module $L'(\lambda) \in \text{Ob}(\mathcal{O})$ with $\lambda \in \Lambda^+$, we have $\dim_k L'(\lambda) < \infty$, so $\dim_k L(\lambda) < \infty$. 

PROOF. Consider $L'(\lambda)$ as a $U_i$-module, where $U_i$ is the copy generated by $e_i, f_i, \omega_i, \omega'_i$. For $\mu$ a weight of $L'(\lambda)$, applying Lemma 2.15 to the $U_i$-module

$$L'_i(\mu) = U_i L'(\lambda)_\mu = \bigoplus_{j \in \mathbb{Z}^+} L'_i(\mu)_{\lambda' - j\alpha_i}$$

for some weight $\lambda' \leq \lambda$, we get that the simple reflection $w_i$ preserves the weights of $L'_i(\mu)$, so $w_i(\mu)$ is a weight of $L'(\lambda)$. That is, the Weyl group $W$ of $\mathfrak{g}$ preserves the set of weights of $L'(\lambda)$. From Lie theory, we know that each $W$-orbit only contains one dominant weight. But there are only finitely many dominant weights $\leq \lambda$, and as each weight space of $L'(\lambda)$ is of finite-dimension, we have $\dim_{\mathbb{K}} L'(\lambda) < \infty$. Thereby, $\dim_{\mathbb{K}} L(\lambda) < \infty$. □

For $\mathfrak{g} = \mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, Corollary 2.8 and Proposition 2.16 imply the following

COROLLARY 2.17. A finite-dimensional simple object in the category $\mathcal{O}$ is precisely a $U_{r,s}(\mathfrak{g})$-module $L(\lambda)$ for some $\lambda \in \Lambda^+$, and $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$. □

Finite-dimensional simple (weight) modules of generic type. As noted in [BW2, Section 2], for $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{sl}_n$, Benkart and Witherspoon gave a description of a classification of finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules. We find that a similar structural feature for finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules also holds when $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$, after modifying some of the treatments.

Given a one-dimensional $U_{r,s}(\mathfrak{g})$-module $L$, Theorem 2.5 indicates that $L = L(\chi)$ for some $\chi \in \text{Hom}_{\mathbb{K}}(U^0, \mathbb{K})$ with the elements $e_i, f_i$ ($1 \leq i \leq n$) trivially acting on $L(\chi)$. Relation (X3) ($X = B, C, D$) gives

$$\chi(\omega_i) = \chi(\omega'_i), \quad (1 \leq i \leq n).$$

Conversely, if $\chi \in \text{Hom}_{\mathbb{K}}(U^0, \mathbb{K})$ satisfies the equation (17), then Proposition 2.9 (ii) guarantees $\dim_{\mathbb{K}} L(\chi) = 1$. We denote by $L_\chi$ the one-dimensional $U_{r,s}(\mathfrak{g})$-module $L(\chi)$.

The following Lemma was proved by Benkart and Witherspoon in the case of type $A$. We will give a unified proof for the classical types of $\mathfrak{g}$ based on an intrinsic observation in Proposition 2.9 (ii) & (iii).

LEMMA 2.18. Assume $rs^{-1}$ is not a root of unity. Given a finite-dimensional simple $U_{r,s}(\mathfrak{g})$-module $L(\psi)$ with highest weight $\psi$, there exists a pair $(\chi, \lambda)$, where $\chi \in \text{Hom}_{\mathbb{K}}(U^0, \mathbb{K})$ such that (17) holds, and $\lambda \in \Lambda^+$, so that $\psi = \chi \cdot \lambda$, and $\text{wt}(L(\psi)) \leq \chi \cdot \Lambda$.

PROOF. As $L(\psi)$ is finite-dimensional and simple, for each pair of eigenvalues $(\psi(\omega_i), \psi(\omega'_i))$ when considering $L(\psi)$ as a $U_i$-module (where $U_i$ is a $U_{r,s} (\mathfrak{sl}_2)$-copy of $U_{r,s}(\mathfrak{g})$), Proposition 2.9 (ii) tells us that there exists a nonnegative integer $\nu_i$ for each index $i$ such that $\psi(\omega'_i) = \psi(\omega_i)(rs^{-1})^{-\nu_i}$. Set $\lambda = \sum_{i=1}^n \nu_i \lambda_i$ where $\lambda_i$ is the $i$th fundamental weight of $\mathfrak{g}$, then $\lambda \in \Lambda^+$. 

Now we take \( \chi(\omega_i) = \psi(\omega_i)\hat{\lambda}_i(\omega_i)^{-1} \) and \( \chi(\omega_i') = \psi(\omega_i')\hat{\lambda}_i(\omega_i')^{-1} \), that is, \( \chi = \psi \cdot \hat{\lambda}_i^{-1} \in \text{Hom}_{\text{Alg}}(U^0, \mathbb{K}) \) and satisfies
\[
\begin{align*}
\chi(\omega_i') &= \psi(\omega_i')(\hat{\lambda}_i^{-1}(\omega_i')) = \psi(\omega_i)(r_is_i^{-1})^{-v_i}\hat{\lambda}_i^{-1}(\omega_i') \\
&= \psi(\omega_i)\hat{\lambda}_i^{-1}(\omega_i) \quad (\text{by (9)}) \\
&= \chi(\omega_i),
\end{align*}
\]
as required. The last assertion that \( \text{wt}(L(\psi)) \subseteq \chi \cdot \hat{\Lambda} \) is quite clear. \( \square \)

Similar to [BW2, Theorem 2.19], for \( g = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n} \), or \( \mathfrak{sp}_{2n} \), we have the classification Theorem for finite-dimensional simple \( U \)-module isomorphisms as follows.

**Theorem 2.19.** Let \( rs^{-1} \) be a non-root of unity. Each finite-dimensional simple \( U \)-module \( L(\psi) \) with \( \psi \in \text{Hom}_{\text{Alg}}(U^0, \mathbb{K}) \) is isomorphic to \( L_\chi \otimes L(\lambda) \), where \( \chi \in \text{Hom}_{\text{Alg}}(U^0, \mathbb{K}) \) with \( \chi(\omega_i) = \chi(\omega_i') \) \( (1 \leq i \leq n) \) and \( \lambda \in \Lambda^+ \). \( \square \)

### 3. R-matrices, Quantum Casimir Operators, Complete Reducibility

For any two objects \( M, M' \in \text{Ob}(\mathcal{O}) \), Benkart and Witherspoon in [BW1, Section 4] constructed a \( U_{r,s}(\mathfrak{g}) \)-module isomorphism
\[
R_{M', M} : M' \otimes M \rightarrow M \otimes M'
\]
by a remarkable method due to Jantzen [J, Chap. 7] for the quantum groups \( U_q(\mathfrak{g}) \) of Drinfel’d-Jimbo type.

The aim of this section is to generalize this result to the setting of \( g = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n} \).

Noting that the weight lattice
\[
\Lambda \subseteq \bigoplus_{i=1}^n \frac{1}{2}Z\alpha_i \subseteq \bigoplus_{i=1}^n \frac{1}{2}Z\epsilon_i,
\]
as it was done in formula (1) of Section 2, for \( \lambda \in \Lambda \), we have an algebra homomorphism \( \hat{\lambda} \in \text{Hom}_{\text{Alg}}(U^0, \mathbb{K}) \). Furthermore, we extend the pairing \( \langle , \rangle \) to \( \Lambda \times \Lambda \), such that for any \( \lambda = \sum_{i=1}^n p_i\alpha_i, \mu = \sum_{i=1}^n q_i\alpha_i \in \Lambda \) with \( p_i, q_i \in \frac{1}{2}Z \), we define
\[
\langle \omega_\lambda', \omega_\mu \rangle = \Pi_{i=1}^n (\hat{\lambda}(\omega_i))^{q_i}, \tag{1}
\]
which is well-defined in the algebraically closed field \( \mathbb{K} \).

Now we define the map \( f : \Lambda \times \Lambda \rightarrow \mathbb{K}^* \) by
\[
\langle \lambda, \mu \rangle = f(\lambda, \mu) = \langle \omega_\lambda', \omega_\mu \rangle^{-1}, \tag{2}
\]
which satisfies
\[
\begin{align*}
f(\lambda + \mu, \nu) &= f(\lambda, \nu) f(\mu, \nu), \\
f(\lambda, \mu + \nu) &= f(\lambda, \mu) f(\lambda, \nu), \\
f(\alpha_i, \mu) &= \langle \omega_\mu', \omega_\alpha_i \rangle^{-1}, \\
f(\lambda, \alpha_i) &= \langle \omega_\lambda', \omega_\alpha_i \rangle^{-1}.
\end{align*}
\]
And we define the linear transformation \( \hat{f} = \hat{f}_{M, M'} : M \otimes M' \rightarrow M \otimes M' \) by \( \hat{f}(m \otimes m') = f(\lambda, \mu)(m \otimes m') \) for \( m \in M_\lambda \) and \( m' \in M'_\mu \).
Owing to $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$, we have $\Delta(x) = \sum_{0 \leq \nu \leq \zeta} U_{r,s}^{\zeta - \nu}(n) \omega_\nu \otimes U_{r,s}^{\nu}(n)$, for all $x \in U_{r,s}^{\zeta}(n)$, by induction. For each $i$, the expression of $\Delta(x)$ defines two skew-derivations $\partial_i, i \partial : U_{r,s}^{\zeta}(n) \rightarrow U_{r,s}^{\zeta - \alpha_i}(n)$ such that

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^n \partial_i(x) \omega_i \otimes e_i + \text{the rest},$$

$$\Delta(x) = \omega_\zeta \otimes x + \sum_{i=1}^n e_i \omega_{\zeta - \alpha_i} \otimes i \partial(x) + \text{the rest},$$

where in each case “the rest” refers to terms involving products of more than one $e_j$ in the second (resp. first) factor. More precisely, parallel to [BW1, Lemma 4.6] or comparing with [KS, Lemmas 6.14, 6.17], we have

**Lemma 3.1.** For all $x \in U_{r,s}^{\zeta}(n)$, $x' \in U_{r,s}^{\zeta}(n)$, and $y \in U_{r,s}(n^*)$, the following hold:

(i) $\partial_i(x x') = \langle \omega'_i, \omega_i \rangle \partial_i(x) x' + x \partial_i(x')$.

(ii) $i \partial(x x') = i \partial(x) x' + \langle \omega'_i, \omega_i \rangle i \partial(x')$.

(iii) $\langle y, x_i \rangle = \langle f_i, e_i \rangle \langle \partial_i(y), x \rangle = (s_i - r_i)^{-1} \langle y, i \partial(x) \rangle$.

(iv) $\langle y f_i, x \rangle = \langle f_i, e_i \rangle \langle y, \partial_i(x) \rangle = (s_i - r_i)^{-1} \langle y, \partial_i(x) \rangle$.

(v) $f_i x - x f_i = (s_i - r_i)^{-1} (\partial_i(x) \omega_i - \omega'_i i \partial(x))$. \hfill $\square$

Also, for each $i$, the expression of $\Delta(y)$ for $y \in U_{r,s}^{-\zeta}(n^-)$ defines two skew-derivations $\partial_i, i \partial : U_{r,s}^{-\zeta}(n^-) \rightarrow U_{r,s}^{-\zeta + \alpha_i}(n^-)$ such that

$$\Delta(y) = y \otimes \omega'_\zeta + \sum_{i=1}^n \partial_i(y) \otimes f_i \omega_{\zeta - \alpha_i} + \text{the rest},$$

$$\Delta(y) = 1 \otimes y + \sum_{i=1}^n f_i \otimes i \partial(y) \omega'_i + \text{the rest}.$$

Parallel to [BW1, Lemma 4.8], we have

**Lemma 3.2.** For all $y \in U_{r,s}^{-\zeta}(n^-)$, $y' \in U_{r,s}^{-\zeta}(n^-)$, and $x \in U_{r,s}(n)$, the following hold:

(i) $\partial_i(y y') = \partial_i(y) y' + \langle \omega'_i, \omega_i \rangle y \partial_i(y')$.

(ii) $i \partial(y y') = i \partial(y) y' + \langle \omega'_i, \omega_i \rangle i \partial(y')$.

(iii) $\langle y, x_i \rangle = \langle f_i, e_i \rangle \langle \partial_i(y), x \rangle = (s_i - r_i)^{-1} \partial_i(y, x)$.

(iv) $\langle y, x e_i \rangle = \langle f_i, e_i \rangle \langle i \partial(y), x \rangle = (s_i - r_i)^{-1} \langle i \partial(y), x \rangle$.

(v) $e_i y - y e_i = (s_i - r_i)^{-1} (\partial_i(y) \omega_i - \omega'_i i \partial(y))$. \hfill $\square$

By Proposition 2.14, the spaces $U_{r,s}^{\zeta}(n)$ and $U_{r,s}^{-\zeta}(n^-)$ are non-degenerately paired. We may select a basis $\{u_k^\zeta\}_{k=1}^{d_\zeta}$, $(d_\zeta = \dim U_{r,s}^{\zeta}(n))$, for $U_{r,s}^{\zeta}(n)$ and a dual basis $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ for $U_{r,s}^{-\zeta}(n^-)$. Then for each $x \in U_{r,s}^{\zeta}(n)$ and $y \in U_{r,s}^{-\zeta}(n^-)$, we have

$$x = \sum_{k=1}^{d_\zeta} \langle v_k^\zeta, x \rangle u_k^\zeta, \quad y = \sum_{k=1}^{d_\zeta} \langle y, u_k^\zeta \rangle v_k^\zeta.$$

For $\zeta \in Q^+ = \bigoplus_{i=1}^n \mathbb{Z}^+ \alpha_i$, we define

$$\Theta_\zeta = \sum_{k=1}^{d_\zeta} v_k^\zeta \otimes u_k^\zeta.$$
Set $\Theta_\zeta = 0$ if $\zeta \notin Q^+$. Similar to [BW1, Lemma 4.10], for the cases when $g = so_{2n+1}$, so$_{2n}$ and sp$_{2n}$, we also have

**Lemma 3.3.** For $1 \leq i \leq n$, the following relations hold

(i) $(\omega_i \otimes \omega_i) \Theta_\zeta = \Theta_\zeta (\omega_i \otimes \omega_i)$, $(\omega'_i \otimes \omega'_i) \Theta_\zeta = \Theta_\zeta (\omega'_i \otimes \omega'_i)$;

(ii) $(e_i \otimes 1) \Theta_\zeta + (\omega_i \otimes e_i) \Theta_{\zeta - \omega_i} = \Theta_\zeta (e_i \otimes 1) + \Theta_{\zeta - \omega_i} (\omega'_i \otimes e_i)$;

(iii) $(1 \otimes f_i) \Theta_\zeta + (f_i \otimes \omega'_i) \Theta_{\zeta - \omega_i} = \Theta_\zeta (1 \otimes f_i) + \Theta_{\zeta - \omega_i} (f_i \otimes \omega_i)$.

Now we define

(8) \[ \Theta = \sum_{\zeta \in Q^+} \Theta_\zeta. \]

Given $U_{r,s}(g)$-module $M$ and $M'$ in $O$, we apply $\Theta$ to their tensor product:

$\Theta = \Theta_{M,M'} : M \otimes M' \to M \otimes M'$.

Note that $\Theta_\zeta : M_\lambda \otimes M'_\mu \to M_\lambda - \zeta \otimes M'_{\mu + \zeta}$ for all $\lambda, \mu \in \Lambda$, and there are only finitely many $\zeta \in Q^+$ such that $M'_{\mu + \zeta} \neq 0$, thanks to condition (O3). So $\Theta$ is a well-defined linear transformation on $M \otimes M'$. After appropriately ordering the chosen countable bases of weight vectors for both $M$ and $M'$, we see that each $\Theta_\zeta$ with $\zeta > 0$ has a strictly triangular matrix, while $\Theta_0 = 1 \otimes 1$ acts as the identity transformation on $M \otimes M'$, hence $\Theta_{M,M'}$ is an invertible transformation.

**Theorem 3.4.** Let $M$ and $M'$ be $U_{r,s}(g)$-modules in $O$ where $g = so_{2n+1}$, so$_{2n}$ or sp$_{2n}$. Then the map

$R_{M',M} = \Theta \circ \tilde{f} \circ P : M' \otimes M \to M \otimes M'$

is an isomorphism of $U_{r,s}(g)$-modules, where $P : M' \otimes M \to M \otimes M'$ is the flip map such that $P(m' \otimes m) = m \otimes m'$ for any $m \in M$, $m' \in M'$.

**Proof.** Obviously, $R_{M',M}$ is invertible. It remains to show that $R_{M',M}$ is a $U_{r,s}(g)$-module homomorphism, that is, to check that

(9) \[ \Delta(a) R_{M',M} (m' \otimes m) = R_{M',M} \Delta(a) (m' \otimes m) \]

holds for all $a \in U_{r,s}(g)$, $m \in M_\lambda$ and $m' \in M'_{\mu}$. It suffices to verify (9) for the generators $e_n, f_n, \omega_n, \omega'_n$ because the subalgebra generated by the first $4(n-1)$ generators $e_i, f_i, \omega_i, \omega'_i \ (1 \leq i < n)$ is isomorphic to $U_{r,s}(sl_n)$, and this can be reduced to the proof of the type $A$ case (see [BW1, Theorem 4.11]). We will present the computation just for $a = f_n$. Using Lemma 3.3 (iii), we get

LHS of (9) \[ = f(\lambda, \mu) \Delta(f_n) \Theta (m \otimes m') \]
\[ = f(\lambda, \mu) (1 \otimes f_n) (\sum \Theta_\zeta) (m \otimes m') \]
\[ + f(\lambda, \mu) (f_n \otimes \omega'_n) (\sum \Theta_{\zeta - \alpha_n}) (m \otimes m') \]
\[ = f(\lambda, \mu) (\sum \Theta_\zeta) (1 \otimes f_n) (m \otimes m') \]
\[ + f(\lambda, \mu) (\sum \Theta_{\zeta - \alpha_n}) (f_n \otimes \omega_n) (m \otimes m') \]
\[ = f(\lambda, \mu) (\omega'_n, \omega_n) (\sum \Theta_\zeta) (\omega'_n m \otimes f_n m') \]
\[ + f(\lambda, \mu) (\omega'_n, \omega_n) (f_n m \otimes \omega'_n m'). \]
On the other hand, we have

\[
\text{RHS of (9)} = R_{M',M}(m' \otimes f_nm + f_n m' \otimes \omega'_n m)
\]
\[
= (\Theta \circ f')(f_n m \otimes m' + \omega'_n m \otimes f_n m')
\]
\[
= f(\lambda - \alpha_n, \mu)\Theta(f_n m \otimes m') + f(\lambda, \mu - \alpha_n)\Theta(\omega'_n m \otimes f_n m')
\]
\[
= f(\lambda - \alpha_n, \mu)(\sum \Theta(\xi))(f_n \otimes 1)(m \otimes m')
\]
\[
+ f(\lambda, \mu - \alpha_n)(\sum \Theta(\xi))(\omega'_n \otimes f_n)(m \otimes m').
\]

Thus (3) indicates that (9) holds. \hfill \Box

**Remark 3.5.** Similar to the treatment in [BW1, Section 5] for the type A case, we can prove the maps \( R_{M',M} \) satisfy the quantum Yang-Baxter equation for our cases. That is, given three \( U_{r,s}(g) \)-modules \( M, M', M'' \) in \( \mathcal{O} \), we have \( R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23} \) as maps from \( M \otimes M' \otimes M'' \) to \( M'' \otimes M' \otimes M \) (see [BW1, Theorem 5.4]). On the other hand, we also can prove the hexagon identities (see [BW1, Theorem 5.7]) for the maps \( R_{M',M} \) by the same approach. Consequently, \( \mathcal{O} \) is a braided monoidal category with braiding \( R = R_{M',M} \) for each pair of modules \( M', M \) in \( \mathcal{O} \).

**Quantum Casimir operators and complete reducibility.** The \( U_{r,s}(g) \)-module isomorphisms \( R_{M',M} \) constructed in Theorem 3.4, which are called the \( R \)-matrices, are mainly determined by \( \Theta \). For the expression (7) of \( \Theta(\xi) \), we set

\[
\Omega = \sum_{\zeta \in \mathbb{Q}^+} \Omega_{\xi}, \quad \Omega' = \sum_{\zeta \in \mathbb{Q}^+} \Omega'_{\xi},
\]

where \( \theta \) is the \( \mathbb{Q} \)-algebra isomorphism of \( U_{r,s}(g) \) into its associated quantum group \( U_{r^{-1},s^{-1}}(g) \) (for definition, see [BGG]) introduced in the formula (11) in Section 2. Obviously, \( \Theta(\xi), \Omega, \Omega' \) are independent of the choice of bases \( \{ u_k \} \) and \( \{ v_k \} \). \( \Omega \) preserves the weight spaces of any \( M \in \mathcal{O} \).

**Definition 3.6.** The element \( \Omega \) is called a quantum Casimir element for the two-parameter quantum group \( U_{r,s}(g) \).

**Proposition 3.7.** Let \( \psi \) and \( \varphi \) be the algebra automorphisms of \( U_{r,s}(g) \) such that \( \psi(\omega_i) = \omega_i, \psi'(\omega_i) = \omega'_i, \psi(e_i) = \omega'_i \omega^{-1}_i e_i, \psi(f_i) = f_i \omega^{-1}_i \omega_i \) and \( \varphi(\omega_i) = \omega_i = \omega'_i, \varphi(e_i) = e_i \omega^{-1}_i \omega'_i, \varphi(f_i) = \omega_i \omega'^{-1}_i f_i \). Then

\[
\psi(a) \Omega = \Omega a, \quad \varphi(a) \Omega' = \Omega' a, \quad \text{for } a \in U_{r,s}(g).
\]

**Proof.** Since \( \psi \) is an algebra automorphism, it is enough to prove the first assertion for the generators \( a = \omega_i, \omega'_i, e_i, f_i \). For \( a = \omega_i \) or \( \omega'_i \), it is obviously true. Applying the mapping \( m \circ (S \otimes 1) \) to both sides of Lemma 3.3 (ii) & (iii) (where \( m \) is the product of \( U_{r,s}(g) \) and \( S \) is its antipode) and summing over \( \zeta \in \mathbb{Q}^+ \) we obtain \( \Omega e_i = \omega'_i \omega^{-1}_i e_i \Omega \) and \( \Omega f_i = f_i \omega'^{-1}_i \omega_i \Omega \). This means that \( \Omega e_i = \psi(e_i) \Omega \) and \( \Omega f_i = \psi(f_i) \Omega \). Applying the automorphism \( \theta \) we get the assertion for \( \Omega' \). \hfill \Box

**Corollary 3.8.** For \( M \in \text{Ob}(\mathcal{O}) \), assume that \( m \in M_\lambda \). Then

(i) \( \Omega e_i m = (r_i s_i^{-1})(\lambda + \alpha_i, \alpha'_i) e_i \Omega m \),

(ii) \( \Omega f_i m = (r_i s_i^{-1})(\lambda, \alpha'_i) f_i \Omega m \).
Using formulas (B), (C), & (D) in Lemma 2.1, we can conclude the required result.

Remark 3.9. According to Section 1, we have made a convention: we have $r_i = r^{(\alpha_i,\alpha_i)}$, $s_i = s^{(\alpha_i,\alpha_i)}$ only in the type $B$ case, so $(r_i s_i^{-1})^{(\lambda,\alpha_i)} = (r s_i^{-1})^{(2\lambda,\alpha_i)}$ for any $i$. However, for any other case, we always have $(r_i s_i^{-1})^{(\lambda,\alpha_i)} = (r s_i^{-1})^{(\lambda,\alpha_i)}$ for any $i$ since $r_i = r^{(\alpha_i,\alpha_i)}$, $s_i = s^{(\alpha_i,\alpha_i)}$. Based on this observation, we make the following definition.

Definition 3.10. For $M \in \text{Ob}(\mathcal{O})$, define a linear operator $\omega : M \rightarrow M$ by setting

$$(\omega.v)_\mu = (rs_i^{-1})^\Delta X,B (\mu + \rho, \lambda + \rho) v_\mu, \quad \text{for } v_\mu \in M_\mu,$$

where $\rho$ is the half-sum of all positive roots of $\mathfrak{g}$, and $\Delta X,B = 2$ if $X = B$, otherwise, $\Delta X,B$ will take value 1.

Proposition 3.11. Assume that the Verma module $M(\lambda) \in \text{Ob}(\mathcal{O})$, then the operator $\Omega \omega$ is a multiple of the identity operator, that is,

$$\Omega \omega = (rs_i^{-1})^\Delta X,B (\lambda + \rho, \lambda + \rho) I.$$

Proof. Let $v_\lambda$ be a highest weight vector of the Verma module $M(\lambda)$. Then $M(\lambda) = U_{rs_i}(n^-)v_\lambda = \sum_{\beta \in Q^+} U_{rs_i}^{-\beta}(n^-)v_\lambda$. For $f_\beta \in U_{rs_i}^{-\beta}(n^-)$, denote $v_{\lambda-\beta} := f_\beta v_\lambda$, which is a weight vector of weight $\lambda - \beta$. We claim that

$$\Omega \omega f_i.v_{\lambda-\beta} = f_i \Omega \omega.v_{\lambda-\beta},$$

for any $\beta \in Q^+$ and any $i$. Indeed, noting that

$$\frac{1}{2} \left[ (\sigma - \alpha_i + \rho, \sigma - \alpha_i + \rho) - (\sigma + \rho, \sigma + \rho) \right] + (\sigma, \alpha_i) = \frac{1}{2} \left[ (\alpha_i, \alpha_i) - 2(\alpha_i, \rho) \right] = 0,$$

and setting $\lambda - \beta = \sigma$, we have

$$\Omega \omega f_i.v_{\lambda-\beta} = (\Omega f_i)(rs_i^{-1})^\Delta X,B \omega.v_{\lambda-\beta}$$

$$= (r^{-1} f_i \omega_i^{-1} \Omega)(rs_i^{-1})^\Delta X,B \omega.v_{\lambda-\beta}$$

$$= f_i (rs_i^{-1})^\Delta X,B (\lambda - \beta, \alpha_i) (rs_i^{-1})^\Delta X,B \Omega \omega.v_{\lambda-\beta}$$

$$= f_i \Omega \omega.v_{\lambda-\beta},$$

where $c = \frac{1}{2} \left[ (\lambda - \beta - \alpha_i + \rho, \lambda - \beta - \alpha_i + \rho) - (\lambda - \beta + \rho, \lambda - \beta + \rho) \right]$. (15) yields

$$\Omega \omega f_\beta.v_\lambda = f_\beta \Omega \omega.v_\lambda$$

$$= (rs_i^{-1})^\Delta X,B (\lambda + \rho, \lambda + \rho) f_\beta \Omega v_\lambda$$

$$= (rs_i^{-1})^\Delta X,B (\lambda + \rho, \lambda + \rho) f_\beta \Omega_0 v_\lambda$$

$$= (rs_i^{-1})^\Delta X,B (\lambda + \rho, \lambda + \rho) f_\beta v_\lambda, \quad (\Omega_0 = 1).$$

So the relation (14) follows. □
COROLLARY 3.12. (i) For the simple $U_{r,s}(\mathfrak{g})$-module $L(\lambda) \in \text{Ob}(\mathcal{O})$, there holds
\[ \Omega \omega = (rs^{-1})^{\frac{\Delta X, B}{2}}(\lambda + \rho, \lambda + \rho)I. \]
(ii) For each finite-dimensional $M \in \text{Ob}(\mathcal{O})$, the eigenvalues of the operator $(\Omega \omega)|_M$ are integral powers of $(rs^{-1})^{\frac{1}{2}}$.

PROOF. (i) is evident. For (ii), as $M \in \text{Ob}(\mathcal{O})$ is finite-dimensional, it has a composition series whose factors are finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules in $\mathcal{O}$, on which $\Omega \omega$ acts as multiplication by $(rs^{-1})^{\frac{\Delta X, B}{2}}(\mu + \rho, \mu + \rho)$ for some $\mu \in \Lambda^+$, as indicated by (i) and Corollary 2.8. After taking an appropriate basis of $M$ compatible with a chosen composition series, the acting matrix of $(\Omega \omega)|_M$ has the required property.

From Corollary 3.8 and Definition 3.10, we have a further result as follows.

THEOREM 3.13. The operator $\Omega \omega : M \rightarrow M$ commutes with the action of $U_{r,s}(\mathfrak{g})$ on any module $M \in \text{Ob}(\mathcal{O})$, where $\mathfrak{g} = \mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$.

PROOF. At first, it needs to show that $\Omega \omega$ commutes with $e_i, f_i$ $(1 \leq i \leq n)$. For $m \in M_\mu$, by Corollary 3.8 and Definition 3.10, we get
\[
\Omega \omega. (e_i.m) = (rs^{-1})^{\frac{\Delta X, B}{2}}(\mu + \alpha_i + \rho, \mu + \alpha_i + \rho) \Omega e_i.m
= (rs^{-1})^{\frac{\Delta X, B}{2}}(\mu + \rho, \mu + \rho) e_i \Omega.m
= e_i. (\Omega \omega.m).
\]
\[
\Omega \omega. (f_i.m) = (rs^{-1})^{\frac{\Delta X, B}{2}}(\mu - \alpha_i + \rho, \mu - \alpha_i + \rho) \Omega f_i.m
= (rs^{-1})^{\frac{\Delta X, B}{2}}(\mu + \rho, \mu + \rho) f_i \Omega.m
= f_i. (\Omega \omega.m).
\]
Obviously, $\Omega \omega$ commutes with the action of $\omega_i, \omega'_i$ $(1 \leq i \leq n)$, for it preserves the weight spaces of $M$. \qed

The following Lemma is due to [BW2, Lemma 3.7] for the case of $\mathfrak{g} = \mathfrak{sl}_{n+1}$, or $\mathfrak{so}_{n+1}$, which is still valid in our cases.

LEMMA 3.14. Assume that $rs^{-1}$ is not a root of unity. Let $M$ be a nonzero finite-dimensional quotient of the Verma $U_{r,s}(\mathfrak{g})$-module $M(\lambda) \in \text{Ob}(\mathcal{O})$. Then $M$ is simple. In particular, $L'(\lambda) = L(\lambda)$ for $\lambda \in \Lambda^+$.

PROOF. Lemma 2.6 means $\lambda \in \Lambda^+$. The proof is based on the counter-evidence method and Proposition 3.11, which is the same as that of [BW2, Lemma 3.7], with slight differences: for the function $g(\lambda)$ used in the proof there we use $(rs^{-1})^{\frac{\Delta X, B}{2}}(\lambda + \rho, \lambda + \rho)$ instead, noting the fact from Lie algebra theory (see [D], or [K]) that for any weight $\mu \leq \lambda$ where $\lambda \in \Lambda^+$, $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$ if and only if $\mu = \lambda$. \qed
Based on the above results, using a similar argument due to Kac [K] in the proof of complete reducibility of category $\mathcal{O}$ for affine Kac-Moody Lie algebras (or comparing with the proof of [BW2, Theorem 3.8] in the spirit of Lusztig [L1]), we have

**Theorem 3.15.** Assume that $rs^{-1}$ is a non-root of unity. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, let $M$ be a nonzero finite-dimensional $U_{r,s}(\mathfrak{g})$-module on which $U^0$ acts semisimply. Then $M$ is completely reducible. $\square$

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