RIGIDITY OF THE FIRST BETTI NUMBER VIA RICCI FLOW SMOOTHING

SHAOSAI HUANG AND BING WANG

ABSTRACT. The Colding-Gromov gap theorem asserts that an almost non-negatively Ricci curved manifold with unit diameter and maximal first Betti number is homeomorphic to the flat torus. In this paper, we prove a parametrized version of this theorem, in the context of collapsing Riemannian manifolds with Ricci curvature bounded below: if a closed manifold with Ricci curvature uniformly bounded below is Gromov-Hausdorff close to a (lower dimensional) manifold with bounded geometry, and has the difference of their first Betti numbers equal to the dimensional difference, then it is diffeomorphic to a torus bundle over the one with bounded geometry. We rely on two novel technical tools: the first is an effective control of the spreading of minimal geodesics with initial data parallel transported along a short geodesic segment, and the second is a Ricci flow smoothing result for certain collapsing initial data with Ricci curvature bounded below.

CONTENTS

1. Introduction 1
2. Background and preliminary discussion 6
3. First homology classes generated by short loops 9
4. Effective distance control of initially nearby geodesics 15
5. The first Betti numbers and dimensional difference 29
6. Ricci flow smoothing and the rigidity case 31
7. Further discussions 36
References 37

1. Introduction

The classical Bochner technique (see \cite{2, 3}) implies that a closed Riemannian manifold with non-negative Ricci curvature has its first Betti number bounded above by its dimension, with the equality case only achieved by the flat torus. This is a rarely found topological rigidity theorem for Riemannian manifolds with Ricci curvature bounded below.

After more than three decades since the birth of Bochner’s technique, Gromov \cite[Page 75]{28} conjectured a quantitative gap phenomenon, expecting the existence of a small dimensional constant $\delta_G > 0$, such that if the Ricci curvature of a closed Riemannian manifold (with unit diameter)
has its lowest eigenvalue bounded below by $-\delta_G$, then the first Betti number does not exceed the dimension, while equality warrants the toral structure of the manifold. This conjecture was later proven by Colding [21] based on his renowned volume continuity theorem; see also [12].

The Colding-Gromov gap theorem is akin to Gromov’s almost flat manifold theorem (see [27, 51]), the latter asserting the existence of a dimensional gap $\delta_{AF} > 0$ such that if a closed Riemannian manifold (with unit diameter) has sectional curvature bounded by $\delta_{AF}$ in absolute value, then it is diffeomorphic to an infranil manifold.

While the almost flat manifold theorem is beautiful it only detects a special class of manifolds; it is the study of the collapsing geometry with bounded curvature by Cheeger, Fukaya and Gromov (see [14, 15, 24, 25, 16]) that fits this theorem in a much broader context — as suggested by [24, Main Theorem], if a Riemannian manifold with uniformly bounded sectional curvature is Gromov-Hausdorff close (collapsing) to a lower dimensional one with bounded geometry, then it is a fiber bundle over the lower dimensional manifold, with fibers being infranil manifolds. Putting it another way, one could think of the collapsing manifold as a collection of infranil manifolds smoothly parametrized by the collapsing limit space (assumed to be a lower dimensional manifold).

The purpose of the current paper is then to report a parametrized version of the Colding-Gromov gap theorem, describing the collapsing behavior of certain Riemannian manifolds with Ricci curvature bounded below. Before stating our theorem, let us fix some notations. We let $\mathcal{M}_{Rc}(m)$ denote the collection of $m$-dimensional complete Riemannian manifolds $(M, g)$ with $Rc_g \geq -(m-1)g$. We also let $\mathcal{M}_{Rm}(k, D, v)$ denote the collection of $k$-dimensional closed Riemannian manifolds with sectional curvature at any point not exceeding 1 in absolute value, diameter bounded above by $D \geq 1$ and volume bounded below by $v > 0$.

With these notations, our main result states as

**Theorem 1.1** (Rigidity of the first Betti number). *Given the data $m \in \mathbb{N}$, $D \geq 1$ and $v > 0$, there is a constant $\delta_B(m, D, v) \in (0, 1)$ such that if for some $(M, g) \in \mathcal{M}_{Rc}(m)$ and some $(N, h) \in \mathcal{M}_{Rm}(k, D, v)$ (with $k \leq m$) it holds $d_{GH}(M, N) < \delta_B$, then

(1) $b_1(M) - b_1(N) \leq m - k$; and

(2) if the equality holds, then $M$ is diffeomorphic to an $(m - k)$-torus bundle over $N$.*

**Remark 1.** Of course, $(N, h) \in \mathcal{M}_{Rm}(k, D, v)$ is just one way to describe that $(N, h)$ has “bounded geometry”. Alternative descriptions include assuming that $\text{diam}(N, h) \leq D$, $\text{Rc}_h \geq -(k-1)h$ and the $C^{1, \frac{1}{2}}$ harmonic radii at all points of $N$ are bounded below by $\iota \in (0, 1)$ — in fact, $\delta_B(m, D, v)$ directly depends on the $C^{1, \frac{1}{2}}$ harmonic radii lower bound, obtained in [41] for manifolds in $\mathcal{M}_{Rm}(k, D, v)$.

While the collapsing phenomena of sequences of Riemannian manifolds with bounded sectional curvature is well-understood thanks to the works of Cheeger, Fukaya, Gromov and Rong [14, 15, 24, 25, 16, 49, 17, 18, 29, 32, 44], the behavior of metrics when collapsing with only Ricci curvature lower bound is much more complicated and much less understood. For instance, even when a sequence of Ricci flat manifolds collapse to very regular limit spaces, there may be no uniform curvature bound for the collapsing sequence, as shown by examples in [29, 32, 44]. Beyond these recently discovered examples, Theorem [1.1] provides a definite result that helps us better understand the collapsing geometry with only Ricci curvature bounded below. The strength of Theorem [1.1] lies in the fact that while the assumption on the first Betti numbers is only numerical, the outcome provides a much more detailed structural description.
The torus fiber bundle structure predicted by Theorem 1.1 is even simpler than the infranil fibration structure expected from the general theory of collapsing geometry with bounded sectional curvature (see [24][25][16]) — it is the assumption on the first Betti numbers that drastically reduces the topological complexity. We believe that the methods in proving Theorem 1.1, when further localized, should shed some light on our understanding of the collapsing geometry of Ricci flat Kähler manifolds, especially the SYZ conjecture [55].

We notice that the equality case in Claim (2) of Theorem 1.1 does not apply to Berger’s sphere, as \( b_1(\mathbb{S}^2) = b_1(\mathbb{S}^3) = 0 \) — in fact, when \( M \) is almost non-negatively Ricci curved, we expect that \( M \simeq N \times \mathbb{T}^{m-k} \) in the equality case of Theorem 1.1; compare Theorem 2.1. On the other hand, examples of manifolds satisfying Claim (2) of Theorem 1.1 include, but are not limited to, non-positively curved compact manifolds that collapse to lower dimensional manifolds, as discussed by Cao, Cheeger and Rong in [9].

When \( b_1(M) - b_1(N) = \dim M - \dim N \) in Theorem 1.1, since \( N \) is a smooth manifold, \( M \) admits a pure \( Cr \)-structure of rank \( m - k \), which is a special type of \( F \)-structure \textit{à la} Cheeger and Gromov [14][15]; see [4][5][6] and [9, Section 4] for the definition of \( Cr \)-structure. It is easily seen that we could construct an invariant metric with respect to such structure; see also [46]:

**Corollary 1.2.** In the equality case of Theorem 1.1 on \( M \) there is a Riemannian metric \( g' \) which defines a distance function close to the original one induced by \( g \), and a regular Riemannian foliation on \( (M, g') \) with leaves generated by \( m - k \) commuting Killing vector fields. Moreover, by shrinking \( g' \) on the leaf directions, there admits a family of Riemannian metrics on \( M \) that (volume) collapse with uniformly bounded diameter and sectional curvature.

A basic concept in studying the local geometry of Riemannian manifolds in \( \mathcal{M}_{Rc}(m) \), as discussed in [26][42][47], is the fibered fundamental group, which takes into consideration those very short loops based at a given point, and allowed to be deformed in a definite geodesic ball centered at that point. More precisely, given \( (M, g) \in \mathcal{M}_{Rc}(m) \), for any \( \delta \in (0, 1) \) and any \( p \in M \), the fibered fundamental group at \( p \) is defined as

\[
\Gamma_\delta(p) := \text{Image}[\pi_1(B_g(p, \delta), p) \to \pi_1(B_g(p, 2), p)].
\]

For suitably small \( \delta \), it is known, through the work of Kapovitch and Wilking ([42 Theorem 1]), that \( \Gamma_\delta(p) \) is an almost nilpotent group with nilpotency rank bounded above by \( m = \dim M \). In the setting of Theorem 1.1, \( M \) is \( \delta \)-Gromov-Hausdorff close to some \( (N, h) \in \mathcal{M}_{Rm}(k, D, v) \), and the work of Naber and Zhang [47] provides more information: by [47] Theorem 2.27 we know that \( \text{rank} \ \Gamma_\delta(p) \leq \dim M - \dim N \) when \( \delta > 0 \) is sufficiently small, and [47] Proposition 5.9] tells that when the equality holds, the universal covering space of \( B_g(p, 2) \) is uniformly non-collapsing.

From this point of view, Theorem 1.1 could also be seen as a global version of the above mentioned results on the fibered fundamental groups, in a more natural situation — notice that the conditions on the fibered fundamental groups are purely local, and could hardly be checked at each and every single point, whereas our considerations on the first Betti numbers in Theorem 1.1 are global and topological. In fact, much of our effort is devoted to “localizing” the information encoded in the first Betti numbers to control the nilpotency rank of the fibered fundamental groups.

This “localization” is carried out by first locating those very short loops in \( M \). We collect all the first homology classes that could be generated by loops of lengths not exceeding 10\( \delta \) in \( H_1(M; \mathbb{Z}) \), which clearly is a subgroup of the abelian group \( H_1(M; \mathbb{Z}) \), and we will show that \( b_1(M) - b_1(N) = \text{rank} \ H_1^*(M; \mathbb{Z}) \) in Proposition 3.7. Notice that if \( \gamma' \) is a geodesic loop based at
some \( p_0 \in M \), representing a torsion-free generator of \( H^1(M; \mathbb{Z}) \) with \( |\gamma'| \leq 10\delta \), we could perturb it in its free homotopy class to find a shortest representative \( \gamma : [0, 1] \to M \) — this does not alter the homology class of \( \gamma' \) although in general \( \gamma \) may no longer be a loop passing through \( p_0 \in M \).

The advantage of \( \gamma \) is that it is a closed geodesic, rather than just being a geodesic loop. In the second step, we will basically show that for \( \delta \) sufficiently small, if we slide \( \gamma \) along a minimal geodesic initially perpendicular to \( \gamma \), it will then end up with being a geodesic loop of length comparable to \( \delta \). In this way, if \( \gamma \) generates a torsion-free class in \( H^1(M; \mathbb{Z}) \), then sliding it to another point \( p \in M \backslash \gamma([0, 1]) \) will produce a loop contained in \( B_{\delta}(p, \mathcal{C}(m, D)\delta) \), with \( D \geq \text{diam} M \) and \( m = \text{dim} M \). Making \( \delta \) sufficiently small, we could make sure that any torsion-free class in \( H^1(M; \mathbb{Z}) \) defines a torsion-free homotopy class in \( \hat{\Gamma}_{C_{\delta}}(p) \), therefore bounding \( \text{rank} \hat{\Gamma}_{C_{\delta}}(p) \) from below by \( \text{rank} H^1(M; \mathbb{Z}) \), which is shown to be equal to \( b_1(M) - b_1(N) \) — here \( \hat{\Gamma}_{\delta}(p) \) denotes the pseudo-local fundamental group, which is defined for any \( \delta \in (0, 1) \) and \( p \in M \) as

\[
\hat{\Gamma}_{\delta}(p) := \text{Image}(\pi_1(B_{\delta}(p, \delta)) \to \pi_1(M, p)).
\]

Roughly speaking, this group considers those very short loops based at the given point, but are allowed to deform within the entire manifold. The pseudo-local fundamental group is an intermediate concept that interpolates between the \( \delta \)-small first homology classes \( H^1_\delta(M; \mathbb{Z}) \), which is entirely global, and the purely local concept \( \Gamma_{\delta}(p) \). In Lemma 2.2 we will check that under the assumption of Theorem 1.1, each \( \hat{\Gamma}_{\delta}(p) \) is almost nilpotent with \( \text{rank} \hat{\Gamma}_{\delta}(p) \leq \text{dim} M - \text{dim} N \), as long as \( \delta \) is sufficiently small. This will lead to the first claim in Theorem 1.1.

We now present our first major technical input, which is an effective control of the geodesic spreading. To set up the context, for \((M, g) \in M_{Ke}(m) \) and \( \Sigma \subset M \) a closed embedded submanifold, we let \( r_{\Sigma} : M \to \mathbb{R} \) denote the distance function to \( \Sigma \). This is a Lipschitz function and is almost everywhere smooth (see §4.1). It defines a smooth vector field \( \nabla r_{\Sigma} \) almost everywhere on \( M \). Notice that any minimal geodesic realizing the distance between a point and \( \Sigma \) is an integral curve of \( \nabla r_{\Sigma} \) with initial value in \( T\Sigma \), the normal bundle of \( \Sigma \subset M \). We now state

**Theorem 1.3.** For any positive numbers \( D \geq 1, \beta < 10^{-2} \) and \( m \in \mathbb{N} \), there are constants \( \bar{\gamma} \in (0, 1) \) and \( \bar{C} > 1 \) solely determined by \( m, D, \) and \( \beta \), to the following effect: let \((M, g) \in M_{Ke}(m) \) and let \( \Sigma \subset M \) be a closed embedded submanifold, and let \( \sigma_0, \sigma_1 : [0, 1] \to M (\frac{1}{4} \leq l \leq D) \) be two minimal geodesics of unit speed that are also integral curves of \( \nabla r_{\Sigma} \) with \( \sigma_0(0), \sigma_1(0) \in \Sigma \), if \( d_{g}(\sigma_0(\beta l), \sigma_1(\beta l)) \leq \bar{\gamma}, \) then we have

\[
\forall t \in [\beta l, (1 - \beta)l], \quad d_{g}(\sigma_0(t), \sigma_1(t)) \leq \bar{C}d_{g}(\sigma_0(\beta l), \sigma_1(\beta l)).
\]

In the application, if \( \gamma : [0, 1] \to M \) is a closed geodesic generating a torsion-free class in \( H^1(M; \mathbb{Z}) \) with \( |\gamma| \leq 10\delta \), then it lifts to the universal covering \( \tilde{M} \) (equipped with the covering metric) and becomes a complete geodesic \( \tilde{\gamma} : \mathbb{R} \to \tilde{M} \). Regarding \( \gamma \) as an isometric action on \( \tilde{M} \), we understand that bounding the size of \( \gamma \) slid along a minimal geodesic \( \sigma \) realizing \( d_{g}(p, \gamma([0, 1])) \) amounts to estimating the distance between the two lifted minimal geodesics \( \sigma_0 = \bar{\sigma} \) and \( \sigma_1 = \gamma \bar{\sigma} \) in \( \tilde{M} \) — here we notice that \( \tilde{\gamma}(\mathbb{R}) \subset \tilde{M} \) is a closed embedded smooth submanifold and that both \( \sigma_0 \) and \( \sigma_1 \) are integral curves of \( \nabla r_{\tilde{\gamma}(\mathbb{R})} \) — Theorem 1.3 applies to the pair \((\tilde{M}, \tilde{\gamma}(\mathbb{R}))\); see Figure 1.

We remark that the proof of Theorem 1.3 is inspired by Colding and Naber’s original work [22], where the Hölder continuity (in the Gromov-Hausdorff sense) of geodesic balls centered along the middle of a minimal geodesic is proven. Colding and Naber [22] developed ingenious and powerful arguments that enable us to pass the metric properties along the middle of a minimal
geodesic beyond the local scale, and we expect applications in many other settings. For instance, in [39] their arguments are adapted to show that any pointed Gromov-Hausdorff limit of a sequence of Ricci shrinkers with a uniform $\mu$-entropy lower bound is a conifold Ricci shrinker; see also [45].

The rigidity case in Theorem 1.1, i.e. when $b_1(M) - b_1(N) = \dim M - \dim N$, is then a relatively straightforward consequence of our second major technical tool, a Ricci flow smoothing result:

**Theorem 1.4.** Given positive constants $D \geq 1$, $m \in \mathbb{N}$, $\alpha < 10^{-2}m^{-1}$ and $\iota < \min\{1, 10^{-2}D\}$, there are positive constants $\delta_{RF}(m, D, \alpha, \iota) < 1$ and $\varepsilon_{RF}(m, D, \alpha, \iota) < 1$ to the following effect: if $(M, g) \in \mathcal{M}_{Rc}(m)$ and $(N, h) \in \mathcal{M}_{Rm}(k, D, v)$ (with $k \leq m$) satisfy

1. $d_{GH}(M, N) < \delta$ for some $\delta \leq \delta_{RF}$, and
2. $b_1(M) - b_1(N) = \dim M - \dim N$,

then there is a Ricci flow solution $g(t)$ defined on $M$ with $g(0) = g$, existing for a period no shorter than $\varepsilon_{RF}^2$, such that

$$\forall \ t \in (0, \varepsilon_{RF}^2), \ \sup_M |Rm|_{g(t)} \leq \alpha t^{-1} + \varepsilon_{RF}^{-2}. \quad (1.2)$$

This theorem grows out of a program initiated by the first named author in [38] to investigate the behavior of Ricci flows with possibly collapsing initial data. While in the setting of Theorem 1.4 one could always start a Ricci flow as $M$ is a closed manifold (see [30]), the emphasis here is the uniform lower bound on the existence time, a crucial aspect when applying Ricci flows as means of smoothing. In dimensions at least three, all known results on the short-time existence of Ricci flows with initial Ricci curvature lower bound (see [53, 33, 31, 54, 37]) rely on the initial uniform non-collapsing assumption to bound from below the existence time. In contrast, Theorem 1.4 (when $k < m$) enables one to start the Ricci flow from collapsing initial data for a definite period of time. A localized version of Theorem 1.4 with singular collapsing limit could be found in [40].

In fact, by (1.2) it is not hard to check that the smoothing metric $g(\varepsilon_{RF}^2)$, obtained from Theorem 1.4 defines a distance function that is equivalent to the original distance specified by $g = g(0)$,
and thus \((M, g(e_{RF}^2))\) is also sufficiently Gromov-Hausdorff close to the (lower dimensional) manifold \((N, h)\) at a fixed scale. By the fibration theorems in \([16, 36]\), we know that \(M\) is an infranil fiber bundle over the smooth manifold \(N\). Relatively simple arguments involving the Hurewicz theorem and the first Betti number then show that the fibers must be tori.

After discussing the background and pointing out the technical difficulties in proving Theorem \([1.1]\) in §2, we will quantify the first Betti number difference by short loop homology (cf. Proposition \([8.7]\)) in §3. We will then prove Theorem \([1.3]\) in §4, and consequently Claim (1) of Theorem \([1.1]\) in §5. The proof of Claim (2) in Theorem \([1.1]\) will follow once Theorem \([1.4]\) is established in §6, and some further remarks will be left in the final section.

2. Background and preliminary discussion

In this section we explain the rationale and the technical difficulties in the proof of Theorem \([1.1]\). We begin our discussion with a much simpler case.

2.1. A precursor for non-negative Ricci curvature. For closed manifolds with non-negative Ricci curvature, the Bochner technique tells that \(b_1(M) \leq \dim M\), and the following theorem, which is due to Yau \([57\text{ Theorem 3}]\), reveals the structural information encoded in \(b_1(M)\):

**Theorem 2.1.** (Yau, 1972) If \((M, g)\) is a closed oriented Riemannian manifold with non-negative Ricci curvature and dimension at least three, then there is a closed oriented Riemannian manifold \((F, h)\) with non-negative Ricci curvature, such that \(M\) is isometric to an \(F\)-bundle over \(\mathbb{R}^{b_1(M)}\).

Just as the approach of Bochner’s original theorem (see \([2, 3]\)), Yau’s proof relied on the existence of harmonic vector fields. The assumed non-negative Ricci curvature plays a key role in the proof: firstly, it implies that the harmonic vector fields are parallel, therefore integrating to **lines** in the universal covering space; and secondly, it allows the application of the de Rham (or the Cheeger-Gromoll) splitting theorem, to isometrically produce an \(\mathbb{R}^{b_1(M)}\)-factor of the universal covering space. Moving from the case of non-negative Ricci curvature to the more general setting of Ricci curvature bounded (negatively) from below usually involves non-trivial localization and quantification, as exemplified by the Colding-Gromov gap theorem \([21]\) and the Cheeger-Colding almost splitting theorem \([11]\). In order to facilitate such localization and quantification, we will realize the torsion-free first homology classes by closed geodesics.

By the Hurewicz theorem, for each torsion-free generator in \(H_1(M; \mathbb{Z})\), we could find a continuous loop \(\gamma' : [0, 1] \to M\) such that the homotopy class \([\gamma'] \in \pi_1(M, \gamma'(0))\) is also torsion-free. Minimizing the length functional within the free homotopy class of \(\gamma'\), we could find a loop \(\gamma : [0, 1] \to M\) of minimal length in the class. Clearly \([\gamma'] = \|\gamma'\| \in H_1(M; \mathbb{Z})\), and \(\gamma\) is in fact a closed geodesic (\([23\text{ §12.2}]\)), i.e. \(\gamma\) is a smooth geodesic and \((\gamma(0), \dot{\gamma}(0)) = (\gamma(1), \dot{\gamma}(1)) \in TM\).

Let \(\pi : \tilde{M} \to M\) denote the universal covering map, and equip \(\tilde{M}\) with the pull-back metric \(\pi^* g\). Now lifting \(\gamma\) to \(\tilde{M}\), since \(\gamma\) is a closed geodesic, we know that the lifted curve extends over both ends as a smooth geodesic. Moreover, since \([\gamma]\) is torsion-free, it is of infinite order — we could therefore extend the lifted curve infinitely towards both directions and obtain a smooth geodesic \(\tilde{\gamma} : \mathbb{R} \to \tilde{M}\). In particular, \(\tilde{\gamma}(\mathbb{R}) \subset \tilde{M}\) is a closed embedded submanifold.

In general, \(\tilde{\gamma}\) is not a line; even if it were a line its existence in a complete manifold with a negative Ricci curvature lower bound does not guarantee the desired isometric splitting. We
therefore needs to rely on Theorem 1.3 to quantitatively and uniformly control the size of the action induced by a small isometric action along the curve \( \tilde{y} \). On the other hand, the existence of such small isometric actions is a consequence of the “collapsing” assumption in Theorem 1.1—the Gromov-Hausdorff closeness of \( M \) to (the lower dimensional) \( N \) enables us to find the short closed geodesics that generate small isometric actions along their lifts in the universal covering space \( \tilde{M} \).

2.2. Outlining the proof of Theorem 1.1. As previously mentioned, we will need to “localize” and “quantify” the information encoded in the first Betti number.

Given \((M, g) \in \mathcal{M}_{\text{g}}(m)\), we recall that the pseudo-local fundamental group \( \tilde{\Gamma}_\delta(p) \) is defined for any \( \delta \in (0, 1) \) and \( p \in M \) as

\[
\tilde{\Gamma}_\delta(p) = \text{Image}[\pi_1(B_g(p, \delta), p) \to \pi_1(M, p)].
\]

Notice that \( \tilde{\Gamma}_\delta(p) \) could be generated by geodesic loops \( \gamma : (S^1, 1) \to (B_g(p, \delta), p) \) with length not exceeding \( 2\delta \). On the other hand, considering the induced action of \( \gamma \) on \((M, \pi^*g)\) — here \( \pi : \tilde{M} \to M \) is the universal covering and \( \pi^*g \) is the covering metric — we clearly see that whenever \( \delta > 0 \) is sufficiently small, \( \tilde{\Gamma}_\delta(p) \) is almost nilpotent with nilpotency rank not exceeding \( m - k \), for any \( p \in M \).

Lemma 2.2. In the setting above, there is a constant \( \delta_{\text{Nil}} > 0 \) determined by \( \iota \) and \( m \), such that if \( d_{GH}(M, N) < \delta \) for some \( \delta \leq 10^{-1}\delta_{\text{Nil}} \), then \( \text{rank} \tilde{\Gamma}_{\delta_{\text{Nil}}}(p) \leq \dim M - \dim N \) for any \( p \in M \).

Proof. For any \((N, h) \in \mathcal{M}_{\text{g}}(k, D, v)\), by [41] we understand that there are uniform constants \( C_{hr}(k, D, v) > 0 \) and \( \iota_{hr}(k, D, v) \in (0, 1) \) such that the \( C^{1, \frac{1}{2}} \) harmonic radii at all points in \( N \) is bounded below by \( \iota_{hr} \). We let \( \iota_{hr}(m, D, v) := \min_{0 \leq k \leq m} \iota_{hr}(k, D, v) \). On the other hand, there is a constant \( \epsilon_{NZ}(m) := \min_{0 \leq k \leq m} \epsilon_0(m, B^k(1)) \), with each \( \epsilon_0(m, B^k(1)) \in (0, 1) \) denoting the uniform constant obtained in [47, Theorem 4.25]. Now by the \( C^{1, \frac{1}{2}} \) harmonic radius lower bound, we have a uniform radius \( t_0 \) (max over \( k \)) such that

\[
(2.1) \quad \forall \tilde{p} \in N, \quad d_{GH}(B_{\tilde{h}}(\tilde{p}, t_0), B^k(t_0)) < 10^{-1}\epsilon_{NZ}(m)t_0.
\]

We now set \( \delta_{\text{Nil}}(m, D, v) := 2^{-1}\epsilon_{NZ}(m)t_0 \) and assume that \( \delta \leq 10^{-1}\delta_{\text{Nil}} \).

If \( d_{GH}(M, N) < \delta \), let \( \Phi : M \to N \) denote a \( \delta \)-Gromov-Hausdorff approximation and we have

\[
d_{GH}(B_g(p, t_0), B^k(t_0)) \leq d_{GH}(M, N) + d_{GH}(B_h(\Phi(p), t_0), B^k(t_0)) \leq 5^{-1}\epsilon_{NZ}t_0
\]

for every \( p \in M \).

Now performing the rescaling \( g \mapsto 4t_0^{-2}g =: \tilde{g} \) and \( h \mapsto 4t_0^{-2}h =: \tilde{h} \), the above estimate becomes

\[
(2.2) \quad d_{GH}(B_{\tilde{g}}(p, 2), \tilde{B}^k(2)) < \epsilon_{NZ}.
\]
On the other hand, we notice that the universal covering \( \pi : \tilde{M} \to M \) is a normal covering with deck transformation group \( \pi_1(M, p) \), and the same conditions hold for its restriction to the local covering \( \pi_p : \pi^{-1}(B_{\gamma}(p, 2)) \to B_{\tilde{g}}(p, 2) \) — the rescaled metric \( \tilde{g} \) is pulled back to the universal covering space \( \tilde{M} \). We then see that

\[
\tilde{G}_{\delta_{Nil}}(p) = \{ \gamma \in \pi_1(M, p) : d_{\pi^g}(\gamma.p, \tilde{p}) \leq 2\delta_{Nil} \} = \{ \gamma \in \pi_1(M, p) : d_{\pi^\tilde{g}}(\gamma.p, \tilde{p}) \leq 2\varepsilon_{NZ} \}.
\]

Consequently, we appeal to \([47, \text{Theorem 4.25}]\) to see that \( \tilde{G}_{\delta_{Nil}}(p) \) is almost nilpotent with nilpotency rank bounded above by \( m - k \). But as we have already seen that \( \tilde{\Gamma}_{\delta_{Nil}}(p) \equiv \tilde{G}_{\delta_{Nil}}(p) \) for any \( p \in M \), we know \( \tilde{\Gamma}_{\delta_{Nil}}(p) \) is almost nilpotent, with rank \( \tilde{\Gamma}_{\delta_{Nil}}(p) \leq \dim M - \dim N \). \( \square \)

For any \( \delta < \delta_{Nil} \), once we have bounded rank \( \tilde{\Gamma}_{\delta_{Nil}}(p) \) by the dimensional difference, our goal would be to show that \( b_1(M) - b_1(N) \leq \text{rank} \ \tilde{\Gamma}_{\delta_{Nil}}(p) \) for any \( p \in M \).

To extract the homological information and get the desired control on the pseudo-local fundamental group, we collect the first homology classes in \( M \) generated by short loops in the group

\[
H^1_\delta(M; \mathbb{Z}) := \langle \|\gamma\| : |\gamma| \leq 10\delta \rangle.
\]

Clearly, \( H^1_\delta(M; \mathbb{Z}) \) is an abelian subgroup of \( H_1(M; \mathbb{Z}) \), and in Proposition 5.2 we will show, under the assumption of Theorem 4.1 that

\begin{equation}
\text{rank} \ H^1_\delta(M; \mathbb{Z}) = b_1(M) - b_1(N).
\end{equation}

So our discussion will now be to compare rank \( H^1_\delta(M; \mathbb{Z}) \) and rank \( \tilde{\Gamma}_{\delta_{Nil}}(p) \) for any \( p \in M \). While the Hurewicz theorem tells that

\[
\forall p \in M, \quad H_1(M; \mathbb{Z}) \cong \pi_1(M, p)/[\pi_1(M, p), \pi_1(M, p)],
\]

the same reasoning may not directly lead to the realization of \( H^1_\delta(M; \mathbb{Z}) \) as (a sub-group of)

\[
\tilde{\Gamma}_{\delta_{Nil}}(p)/\left( [\pi_1(M, p), \pi_1(M, p)] \cap \tilde{\Gamma}_{\delta_{Nil}}(p) \right)
\]

for every \( p \in M \). This is because the definition of \( \tilde{\Gamma}_{\delta_{Nil}}(p) \) not just requires the generating loops in consideration to be very short, but also to be based at the given point \( p \in M \). A \( \delta \)-small generator in \( H^1_\delta(M; \mathbb{Z}) \) may, however, be located anywhere in \( M \), not necessarily passing through the given point \( p \in M \). In contrast, the Hurewicz theorem holds because in \( H_1(M; \mathbb{Z}) \) the size of the generators are allowed to be arbitrarily large — though not exceeding \( 2\text{diam}(M, g) \).

To remedy the situation, we would start from the \( \delta \)-small generators of \( H^1_\delta(M; \mathbb{Z}) \), and estimate its size when slide to other points. More specifically, denoting rank \( H^1_\delta(M; \mathbb{Z}) =: l_M \), we could find geodesic loops \( \gamma_1, \ldots, \gamma_{l_M} \) of length not exceeding \( 10\delta \), such that \( \|\gamma'_1\|, \ldots, \|\gamma'_{l_M}\| \) generate the torsion-free part of \( H^1_\delta(M; \mathbb{Z}) \), which is a rank \( l_M \) free \( \mathbb{Z} \)-module. For each \( i = 1, \ldots, l_M \), we may then perturb \( \gamma'_i \) within its free homotopy class to some \( \gamma_i \), achieving the minimal possible length. Then each \( \gamma_i \) becomes a closed geodesic with \( \|\gamma_i\| = \|\gamma'_i\| \) and \( |\gamma_i| \leq 10\delta \). Notice that each \( \gamma_i \in \tilde{\Gamma}_{5\delta}(\gamma_i(0)) \equiv \tilde{G}_{5\delta}(\gamma_i(0)) \), and we will examine the effect of the action \( \gamma_i \in \text{Isom}(\tilde{M}, \pi^\tilde{g}) \) on \( \pi^{-1}(p) \), for any \( p \notin \gamma_i([0, 1]) \).

Fix some \( \gamma_i \) \((i = 1, \ldots, l_M)\), by straightforward volume comparison we get an estimate of the form \( d_{\pi^g}(\gamma_i.p, \tilde{p}) \leq Cd_i(p, \gamma_i)^m\delta^{-m} \), for any \( \tilde{p} \in \pi^{-1}(p) \) with \( p \notin \gamma_i([0, 1]) \); compare the constants in \([47, \text{Lemma 5.2}]\). While this estimate may be useful when \( p \) and \( \gamma_i([0, 1]) \) are within a distance
also, for any curve $\Phi$ to assume that

Moreover, for any curve $c = [\delta, \sigma]$ among homology classes $\{\delta, \sigma\}$ we say that two curves $c, d$ any minimal geodesic realizing $\sigma$ on $Colding and Naber’s original arguments and results in [22 Theorem 1.1], where, say, for a minimal geodesic $\gamma : [0,1] \rightarrow M$ such that $\gamma(t_0) = \sigma(\epsilon)$ and $|\sigma|_{[\epsilon,1]} = d_g(p, \gamma([0,1]))$, we could lift it to a minimal geodesic $\tilde{\gamma}$ in $M$ with $\tilde{\gamma}(\epsilon) = \gamma(t_0)$ and see

$$d_{GH}(\tilde{\gamma}(\epsilon), r) \leq C(m, D)\epsilon^{-1}r,$$

for $\epsilon, r > 0$ sufficiently small, with $\tilde{\gamma} = \sigma(1) \in \pi^1(p)$. Let $\tilde{\Phi} : B_{\pi^1}(\tilde{\sigma}(\epsilon), r) \rightarrow B_{\pi^1}(\tilde{\rho}, r)$ denote the Gromov-Hausdorff approximation obtained from the proof of Colding and Naber’s theorem ([22 Theorem 1.1]). While (2.4) provides certain control on $d_{\pi^1}(\tilde{\Phi}(\gamma_i, \tilde{\sigma}(\epsilon)), \tilde{\Phi}(\tilde{\sigma}(\epsilon)))$ in terms of $d_{\pi^1}(\gamma_i, \tilde{\sigma}(\epsilon), \tilde{\sigma}(\epsilon))$, the problem is that $\tilde{\Phi}$ is not almost equivariant with respect to the action of $\gamma_i$ — in general we have no comparison between $d_{\pi^1}(\tilde{\Phi}(\gamma_i, \tilde{\sigma}(\epsilon)), \tilde{\Phi}(\tilde{\sigma}(\epsilon)))$ and $d_{\pi^1}(\gamma_i, \tilde{\sigma}(1), \tilde{\sigma}(1))$. See Figure 2 for an illustration.

This explains the necessity of developing Theorem 1.3 whose proof in §5 essentially relies on Colding and Naber’s original arguments and results in [22]. With this theorem at hand, we could slide $\gamma_i$ to any $p \in M$ and obtain a geodesic loop of length smaller than $\delta_{Nil}$, producing an element of $\tilde{\Gamma}_{\delta_{Nil}}(p)$ — in fact, the “slided loop” at $p$ is defined as the projection under $\pi$ of any minimal geodesic realizing $d_{\pi^1}(\gamma_i, \tilde{\sigma}(1), \tilde{\sigma}(1))$ in the setting above. The $\mathbb{Z}$-independence of the homology classes $[\gamma_1], \ldots, [\gamma_{l_M}]$ then guarantees the new loops at $p$ to define independent torsion-free elements of $\tilde{\Gamma}_{\delta_{Nil}}(p)$, proving $\text{rank } \tilde{\Gamma}_{\delta_{Nil}}(p) \geq l_M = \text{rank } H^1_{\mathbb{Z}}(M; \mathbb{Z})$ — here we obviously need to assume that $\delta << \delta_{Nil}(m)$ is sufficiently small.

3. First homology classes generated by short loops

The goal of this section is to prove the equality $\text{rank } H^1_{\mathbb{Z}}(M; \mathbb{Z}) = b_1(M) - b_1(N)$ for manifolds $(M, g)$ and $(N, h)$ satisfying the assumption of Theorem 1.1 and $\delta \leq 10^{-3}\delta_{h}$. In this section, we let $M \in \mathcal{M}_0(m)$ and $N \in \mathcal{M}_0(k, D, \nu)$, and assume that there is a $10^{-1}\delta$-Gromov-Hausdorff approximation $\Phi : M \rightarrow N$ with $\delta \in (0, 10^{-3}\delta_{h})$. We put the following notations for any $\epsilon > 0$: we say that two curves $c_0, c_1 : [0,1] \rightarrow M$ are $\epsilon$-close to each other if $\sup_{t \in [0,1]} d(c_0(t), c_1(t)) < 2\epsilon$; also, for any curve $c : [0,1] \rightarrow M$ we let $c^{-1}$ denote inverse curve $c^{-1}(t) := c(1-t) : [0,1] \rightarrow M$. Moreover, for any curve $c : [0,1] \rightarrow M$, we say that a curve $\tilde{c} : [0,1] \rightarrow N$ is $\delta$-approximating...
if \( \sup_{\epsilon \in [0,1]} d_\epsilon ( \tilde{c}(t), \Phi(c(t))) < \delta \) — notice that \( \Phi \) is not necessarily continuous and so we cannot directly take \( \Phi(c) \) as a \( \delta \)-approximating loop, but for any curve in \( M \), it is easy to see that a \( \delta \)-approximating curve in \( N \) always exists.

To see this, we just let \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) be a fine enough partition of \([0,1]\) so that \( |c_{[t_{i-1}, t_i]}| \leq 10^{-1} \delta \) for each \( i = 1, \ldots, n \), pick \( y_i \in B_\delta(\Phi(c(t_i))), 10^{-1} \delta \) and let \( \bar{\mu}_i \) be a minimal geodesic connecting \( y_{i-1} = \bar{\mu}_i(0) \) to \( y_i = \bar{\mu}_i(1) \) (we could choose \( \bar{\mu}_i(0) = \bar{\mu}_n(1) \) when \( c \) is a loop); it is easy to see that

\[
(3.1) \quad |\bar{\mu}_i| \leq d_\delta (y_{i-1}, \Phi(c(t_{i-1}))) + d_\delta (y_i, \Phi(c(t_i))) + d_\delta (\Phi(c(t_{i-1})), \Phi(c(t_i))) \\
\leq \frac{3}{10} \delta + d_\delta (c(t_{i-1}), c(t_i)) \leq \frac{3}{10} \delta + |c_{[t_{i-1}, t_i]}| \leq \frac{2}{5} \delta 
\]

forming the loop \( \bar{c} := \bar{\mu}_1 * \bar{\mu}_2 * \cdots * \bar{\mu}_n \), it is easily seen that for any \( t \in [0,1] \), say, \( t \in [t_{i-1}, t_i] \),

\[
(3.2) \quad d_\delta (\Phi(c(t)), \bar{c}(t)) = d_\delta (\Phi(c(t)), \bar{\mu}_i(t)) \leq d_\delta (\Phi(c(t)), y_{i-1}) + |\bar{\mu}_i| \\
\leq |c_{[t_{i-1}, t_i]}| + \frac{3}{5} \delta \leq \frac{7}{10} \delta.
\]

Therefore, \( \bar{c} \) is the desired \( \delta \)-approximating curve of \( c \). Since the harmonic radii at all points of \( N \) are bounded below by \( \overline{\rho}_h \geq 10^3 \delta \), we also notice that two \( \delta \)-approximating loops for a given approximating loop determines the same homology class in \( H_1(N; \mathbb{Z}) \).

We now discuss the finitely generated abelian group \( H_1^\phi(M; \mathbb{Z}) \), consisting of homology classes generated by geodesic loops of length not exceeding \( 10^6 \delta \). As a basic property, we notice that loops that are \( \delta \)-close to each other in \( M \) define the same first homology class modulo \( H_1^\phi(M; \mathbb{Z}) \):

**Lemma 3.1.** Let \( \gamma_0 : [0,1] \to M \) be a loop formed by connecting geodesic segments of lengths not exceeding \( \delta \), and let \( \gamma_1 : [0,1] \to M \) be another loop which is \( 2\delta \)-close to \( \gamma_0 \), then

\[
\|\gamma_0\| \equiv \|\gamma_1\| \mod H_1^\phi(M; \mathbb{Z}).
\]

**Proof.** Suppose \( \gamma_0 = \mu_1 * \mu_2 * \cdots * \mu_l \) with \( \mu_j \) being minimal geodesics in \( M \), connecting \( \gamma_0(s_{j-1}) \) to \( \gamma_0(s_j) \) for \( j = 1, \ldots, l \), and \( |\mu_j| = |\gamma_0|_{[s_{j-1}, s_j]} \leq \delta \); clearly \( \gamma_0(s_0) = \gamma_0(s_l) \). We also subdivide each \([s_{j-1}, s_j]\) sufficiently fine by inserting \( t_i \) so that \( |\gamma_1|_{[t_{i-1}, t_i]} \leq \delta \). We denote \( t_i = s_j \), and set

\[
I_j := \{ 0 \leq i \leq n : t_i \in [s_{j-1}, s_j) \} \quad \text{for} \quad j = 1, \ldots, l.
\]

So our notation becomes

\[
0 = t_0 = s_0 < t_1 < \cdots < t_{i_j-1} = s_{j-1} < t_{i_j} < \cdots < t_{i_j-1} < t_{i_j-i} < t_{i_j} = s_j < \cdots < t_l = t_n = s_l = 1,
\]

showing \( I_j = \{ i_{j-1}, i_{j-1} + 1, \ldots, i_j - 1 \} \) in the middle.

For each \( j = 1, \ldots, l \) and \( i \in I_j \cup \{ i_{j-1} - 1 \} \) (with \( i_0 - 1 = i_l - 1 \)), connect \( \gamma_0(s_{j-1}) =: \mu_{j-1,i}(0) \) to \( \gamma_1(t_i) =: \mu_{j-1,i}(1) \) by a minimal geodesic \( \mu_{j-1,i} \), and we have

\[
|\mu_{j-1,i}| \leq d_\delta (\gamma_0(s_{j-1}), \gamma_0(t_i)) + d_\delta (\gamma_0(t_i), \gamma_1(t_i)) \leq \max \{ |\mu_{j-1,i}|, |\mu_j| \} + 2 \delta \leq 3 \delta.
\]

Notice that for \( j = 1, \ldots, l \), each \( i_{j-1} - 1 \) is “double-booked” — \( \mu_{j-1,i-1}(1) = \gamma_1(t_{i-1}) = \mu_{j,i-1}(1) \).

With the convention \( i_{-1} = 0 \), we then define a family of singular 1-cycles as

\[
\sigma_j := \mu_j * \mu_{j,j-1} * \mu_{j-1,i-1} \quad \text{for each} \quad j = 1, \ldots, l;
\]

\[
\sigma_{i_j} := \mu_{j-i, j-1} * \mu_{j-1,i-1} \quad \text{for each} \quad i \in I_j.
\]
Clearly we have $|\sigma_j| \leq 7\delta$ and $|\sigma_{i,j}| \leq 7\delta$ for all possible $i$ and $j$. Moreover, from the construction it is clear that

$$\gamma_0 - \gamma_1 = \sum_{j=1}^{l} \left( \sigma_j + \sum_{i\in I_j} \sigma_{i,j} \right),$$

implying the claim of the lemma, as the right-hand side defines an element in $H^1_\delta(M; \mathbb{Z})$. \qed

This lemma enables us to replace any loop in $M$ with one that we could construct bare handedly with an error in $H^1_\delta(M; \mathbb{Z})$. Actually, our basic principle predicts that for any loop $\gamma$ in $M$, if its $\delta$-approximating loop is trivial in $H_1(N; \mathbb{Z})$, then we must have $\|\gamma\| \in H^1_\delta(M; \mathbb{Z})$. Intuitively speaking, we expect deformations of $\delta$-approximating loops in $N$ to produce corresponding deformations of the original loops in $M$ modulo loops of lengths comparable to the Gromov-Hausdorff distance between $M$ and $N$. We now explain a relatively simple case:

**Lemma 3.2.** If $\gamma_0$ and $\gamma_1$ are two geodesic loops in $M$, such that the $\delta$-approximating loops $\bar{\gamma}_0$ and $\bar{\gamma}_1$ are homotopic to each other, then $\|\gamma_0\| \equiv \|\gamma_1\|$ mod $H^1_\delta(M; \mathbb{Z})$.

**Proof.** Let $\bar{\gamma}_0$ and $\bar{\gamma}_1$ be $\delta$-approximating loops of $\gamma_0$ and $\gamma_1$, respectively. Let $H : [0, 1]^2 \to N$ be a homotopy between $\bar{\gamma}_0$ and $\bar{\gamma}_1$, with $H(0, -) = \bar{\gamma}_0$, $H(1, -) = \bar{\gamma}_1$ and $H(-, 0) = H(-, 1)$. By the compactness of $[0, 1]^2$, we may let $n$ be so large that $\text{diam}_H([\frac{j-1}{n}, \frac{j}{n}] \times [\frac{i-1}{n}, \frac{i}{n}]) < 10^{-1}\delta$. Let us define the paths in $N$ by $\bar{\mu}_{i,j}(t) := H(\frac{j-1}{n}, \frac{i}{n} + \frac{t}{n})$ and $\bar{\nu}_{i,j}(t) := H(\frac{j-1}{n}, \frac{i}{n})$ for $i, j = 0, 1, \ldots, n - 1$. Now we could find $p_{i,j} \in M$ such that $\Phi(p_{i,j}) \in B_\delta(H(i, j), 10^{-1}\delta)$; clearly, we have

$$d_k \left( p_{i,j}, p_{k,l} \right) \leq d_h \left( \Phi(p_{i,j}), \Phi(p_{k,l}) \right) + 10^{-1}\delta$$

$$\leq d_h \left( H(i, j), H(k, l) \right) + d_h \left( H(i, j), \Phi(p_{i,j}) \right) + d_h \left( H(k, l), \Phi(p_{k,l}) \right) + 10^{-1}\delta$$

$$< \frac{2}{5}\delta$$

as long as $\max(|i - k|, |j - l|) \leq 1$. Since $\gamma_0$ and $\gamma_1$ already exist in $M$, we could assume that $\{p_{0,j}\} \subset \gamma_0([0, 1])$ and $\{p_{n,j}\} \subset \gamma_1([0, 1])$.

We could moreover find minimal geodesics $\mu_{i,j}$ connecting $\mu_{i,j}(0) = p_{i,j}$ to $\mu_{i,j}(1) = p_{i,j+1}$, as well as $\nu_{i,j}$ with $\nu_{i,j}(0) = p_{i,j}$ and $\nu_{i,j}(1) = p_{i+1,j}$. We now form the loops $\sigma_{i,j} := \mu_{i,j} * \nu_{i,j+1} * \gamma_{i,j}^{-1}$ for all $i, j = 0, 1, \ldots, n - 1$. By (3.3) it is clear that

$$|\sigma_{i,j}| \leq |\mu_{i,j}| + |\nu_{i,j+1}| + |\mu_{i+1,j}| + |\nu_{i,j}| \leq \frac{8}{5}\delta.$$

Moreover, it is easily seen that the loop $\sigma_{i,j}$ have its image contained within $B_\delta(p_{i,j}, \frac{4}\delta)$, for all $i, j = 0, 1, \ldots, n - 1$.

We also notice that the boundary loop $\gamma'_0 := \mu_{0,0} * \mu_{0,1} * \cdots * \mu_{0,n-1}$ is $2\delta$-close to the original loop $\gamma_0$, as we now check: For any $t \in [0, 1]$, say $t \in [\frac{j-1}{n}, \frac{j}{n}]$, then by (3.3) and the choice that
If $N$ is simply connected, then $H_0(N)$ is contractible, as the harmonic radii of all points are at least $\bar{r}$ by Corollary 3.4. We could also see that the first homology of $N$ is determined by $N$ up to $H_1^s(M; \mathbb{Z})$:

**Corollary 3.3.** If $N$ is simply connected, then $H_1(M; \mathbb{Z}) = H_1^s(M; \mathbb{Z})$.

We could also see that the first homology of $M$ at scale $\bar{r}$ is determined by $N$ up to $H_1^s(M; \mathbb{Z})$:

**Corollary 3.4.** If $\gamma_0, \gamma_1 : [0, 1] \to M$ are two $\frac{1}{4}\bar{r}$-close loops, then $\|\gamma_0\| = \|\gamma_1\|$ mod $H_1^s(M; \mathbb{Z})$.

**Proof.** This is because $d_h(\Phi(\gamma_0(t)), \Phi(\gamma_1(t))) < \delta + \frac{\bar{r}}{\delta}$, the assumption $\delta \leq 10^{-3}\bar{r}$, and that the $\bar{r}$-balls in $N$ are contractible, as the harmonic radii of all points are at least $\bar{r}$ on $N$. □

For a loop $c$ defined on $[0, 1]$, we let $c^k$ denote the $k$-fold concatenation of itself, as a loop defined on $[0, k]$. Clearly, for any loop $c$, the homology class $k[c]$ can be represented by the loop $c^k$. We now upgrade Lemma 3.2 by showing the same results for homologous loops in $N$:

**Lemma 3.5.** If $\gamma_1, \ldots, \gamma_l$ are geodesic loops in $M$ with $\delta$-approximating loops $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$ in $N$, such that there is a vanishing $\mathbb{Z}$-linear combination $k_1\|\tilde{\gamma}_1\| + \cdots + k_l\|\tilde{\gamma}_l\| = 0 \in H_1(N; \mathbb{Z})$, then we have

$$k_1\|\gamma_1\| + \cdots + k_l\|\gamma_l\| \in H_1^s(M; \mathbb{Z}).$$
Proof. Clearly we only need to consider the case $k_1^2 + \cdots + k_l^2 \neq 0$. By the assumption we could find singular 2-simplicies $\bar{\omega}_{abc} : \Delta^2 \to N$ such that as singular 1-cycles,

\[ k_1 \gamma_1 + \cdots + k_l \gamma_l = \sum_{a,b,c} \partial \bar{\omega}_{abc}. \]

By covering $\bar{\omega}_{abc}(\Delta^2)$ by $20^{-1}\delta$-balls and the compactness of $\Delta^2$, we may perform barycentric subdivision and guarantee that each diam$_b \bar{\omega}_{abc}(\Delta^2) < 10^{-1}\delta$. Each $\partial \bar{\omega}_{abc}$ is a 1-boundary in $N$, and we denote $\partial \bar{\omega}_{abc} = \bar{\mu}_{ab} + \bar{\mu}_{bc} - \bar{\mu}_{ac}$, with the singular 1-simplicies $\bar{\mu}_{ab}, \bar{\mu}_{bc}, \bar{\mu}_{ac} : [0, 1] \to N$ connecting the corresponding vertices $\bar{p}_a, \bar{p}_b, \bar{p}_c \in N$ oriented in accordance with the subscripts. Clearly, $d_b(\bar{p}_a, \bar{p}_b) < 10^{-1}\delta$, and notice that the above equation becomes

\[ (3.7) \quad k_1 \gamma_1 + \cdots + k_l \gamma_l = \sum_{a,b,c} \bar{\mu}_{ab} + \bar{\mu}_{bc} - \bar{\mu}_{ac}, \]

and each $\|\bar{\mu}_{ab} + \bar{\mu}_{bc} - \bar{\mu}_{ac}\| = 0$ in $H_1(N; \mathbb{Z})$ thanks to the singular 2-simplex $\bar{\omega}_{abc}$. Moreover, for each loop $\gamma_i^{k_i} : [0, k_i] \to N$, we may assume it is a concatenation of some $\bar{\mu}_{ab}$, i.e. there is a sub-collection $\{\bar{\mu}_{a_1b_1}, \ldots, \bar{\mu}_{a_n b_n}\}$ of the singular 1-simplicies appeared on the right-hand side of $\bar{\omega}_{abc}$, such that $\gamma_i^{k_i} = \bar{\mu}_{a_1b_1} \ast \cdots \ast \bar{\mu}_{a_n b_n}$ with $\bar{p}_{\bar{d}_0} = \bar{p}_{\bar{d}_n}$.

We could then work as before to find $\{p_a\} \subset M$ such that $d_a(\Phi(p_a), \bar{p}_a) < 10^{-1}\delta$, and minimal geodesics $\mu_{ab}$ with $\mu_{ab}(0) = p_a$ and $\mu_{ab}(1) = p_b$. By the same argument leading to $(3.3)$, we know that $|\mu_{ab}| \leq \frac{1}{2}\delta$ for all indices $a, b$. Moreover, let $\gamma_i'$ be the loop in $M$ formed by concatenating those $\mu_{ab}$’s such that $\bar{p}_a, \bar{p}_b$ are in $\gamma_i^{k_i}$, then by the same way leading to the estimate $(3.5)$, we know that each loop $\gamma_i'$ is $2\delta$-close to the loop $\gamma_i^{k_i}$. Consequently, we have $\|\gamma_i'\| \equiv k_i \|\gamma_i\|$ mod $H_1^0(M; \mathbb{Z})$ for each $i = 1, \ldots, l$, thanks to Lemma 3.1.

We now consider the geodesic triangles $\sigma_{abc} := \mu_{ab} \ast \mu_{bc} \ast \mu_{ac}^{-1}$ as singular 1-cycles in $M$. Clearly each $|\sigma_{abc}| \leq 2\delta$, and has its image contained in in $B_\delta(p_a, \delta)$. Moreover, the above combinatorial relation $(3.7)$ implies that

\[ \|\gamma_1\| + \cdots + \|\gamma_l\| = \sum_{a,b,c} \|\sigma_{abc}\| \in H_1^0(M; \mathbb{Z}), \]

whence the claim of the lemma, as $k_i \|\gamma_i\| - \|\gamma_i'\| \in H_1^0(M; \mathbb{Z})$ for each $i = 1, \ldots, l$. \qed

We also have a certain inverse to this lemma:

**Lemma 3.6.** If $\gamma_1, \ldots, \gamma_l : [0, 1] \to M$ are geodesic loops, such that there is a $\mathbb{Z}$-linear relation

\[ k_1 \|\gamma_1\| + \cdots + k_l \|\gamma_l\| \in H_1^0(M; \mathbb{Z}), \]

then their $\delta$-approximating loops $\bar{\gamma}_1, \ldots, \bar{\gamma}_l$ in $N$, as constructed at the very beginning of the subsection, satisfy

\[ k_1 \|\bar{\gamma}_1\| + \cdots + k_l \|\bar{\gamma}_l\| = 0 \in H_1(N; \mathbb{Z}). \]

**Proof.** We could find a sufficiently large $n \in \mathbb{N}$ such that $\|\gamma_i|_{\frac{1}{k_i n}}\| < 10^{-1}\delta$ for each $i = 1, \ldots, l$ and $j = 1, \ldots, n$. For each $i = 1, \ldots, l$, we also let $p_{i,j} := \gamma_i^{k_i}(\frac{j}{k_i n})$, for $j = 0, 1, \ldots, k_i n$. Notice that
By the Hurewicz theorem, we could find \( \bar{\text{lengths}} \) either
\[ H \]
The shortest representatives of the torsion-free generators of \( H \) of length \( \geq d_{\delta} \). Obviously, \( \mu_{d_{j-1}a_{j}} \) connects from \( p_{d_{j-1}} = p_{i,j-1} \) to \( p_{d_{j}} = p_{i,j} \) and \( \gamma_{i} = \mu_{d_{j}a_{j}} \cdots \mu_{d_{n-1}a_{n}} \).

We now pick \( \bar{a} \subset N \) with \( d_{\delta}(\Phi(p_{\bar{a}}), \bar{a}) < 10^{-1}\delta \), and let \( \bar{a} \) be a minimal geodesic connecting \( \bar{a} = \bar{a}(0) \) to \( \bar{b} = \bar{a}(1) \). Here we insist that if \( p_{d_{1}} = p_{d_{2}} \in M \), then we pick \( \bar{a} = \bar{a}(1) \). By the assumption that \( |\mu_{ab}| \leq 10^{-1}\delta \), we could argue as in (3.1) to see that \( |\bar{a}| \leq \frac{\delta}{2} \). For each \( i = 1, \ldots, l \) we define \( \bar{a}_{i} := \mu_{d_{j_{i}}a_{j_{i}}} \cdots \mu_{d_{j_{i}n-1}a_{j_{i}}} \) with \( a_{i} = a_{0} \), and (3.2) shows that \( \gamma_{i} \) is also \( \delta \) approximating loop of \( \gamma_{i} \). Notice that \( \gamma_{i} := \bar{a}_{i} \cdots \bar{a}_{i} \) is also \( \delta \) approximating loop of \( \gamma_{i} \).

Now for any triple \( (a, b, c) \), if \( \sigma_{abc} \) appear on the right hand side of (3.8), we form the geodesic triangles \( \bar{a} := \bar{a}_{ab} \bar{b} \bar{b} \bar{a}_{ab} \) in \( N \), and regard each such \( \bar{a} \) as a singular 1-cycle in \( N \). According to (3.3), we then write the following equation of singular 1-cycles in \( N \):
\[
\gamma_{1} + \cdots + \gamma_{l} = \sum_{a,b,c} \sigma_{abc}.
\]
Moreover, it is clear that \(|\bar{a}_{abc}| \leq 2\delta|, ensuring each \( \bar{a}_{abc} \subset B_{\delta}(\bar{a}_{abc}) \) as \( \delta \leq 10^{-1}\delta \). But since the harmonic radius at \( \bar{a} \) is less than \( \delta \), \( B_{\delta}(\bar{a}, \delta) \) is homeomorphic to an Euclidean ball, which is contractible. By the Poincaré lemma, \( \bar{a}_{abc} \) is a 1-boundary, i.e. \( \bar{a}_{abc} = \delta \bar{a}_{abc} \) for some singular 2-simplex \( \bar{a}_{abc} : \Delta \rightarrow B_{\delta}(\bar{a}, \delta) \). Therefore, the right-hand side of (3.9) vanishes in \( H_{1}(N; \mathbb{Z}) \).

With the above understanding, we could now compare rank \( H_{1}(M; \mathbb{Z}) \) and \( b_{1}(M) - b_{1}(N) \).

**Proposition 3.7.** The shortest representatives of the torsion-free generators of \( H_{1}(M; \mathbb{Z}) \) have lengths either \( \leq 10\delta \) or \( \geq 10^{-1}\delta h \). Among these loops, we have a total number of \( b_{1}(N) \) loops of length \( \geq 10^{-1}\delta h \), representing distinct torsion-free homology classes in \( H_{1}(M; \mathbb{Z}) \), and making rank \( H_{1}(M; \mathbb{Z})/H_{1}^{s}(M; \mathbb{Z}) = b_{1}(N) \). Consequently, we have
\[
\text{rank } H_{1}^{s}(M; \mathbb{Z}) = b_{1}(M) - b_{1}(N).
\]

**Proof.** If \( \gamma \) represents a generator of \( H_{1}(M; \mathbb{Z}) \) and \( |\gamma| < 10^{-1}\delta h \), then it has a \( \delta \) approximating loop \( \tilde{\gamma} \) in \( B_{\delta}(\Phi(\gamma(0), \delta)h) \), since \( \delta + 10^{-1}\delta h < \frac{1}{2}h \). Since the harmonic radius of \( \Phi(\gamma(0)) \) is at least \( \delta h \), it means that \( B_{\delta}(\Phi(\gamma(0), \delta)h) \) is contractible, and thus \( \gamma = \gamma_{\delta} \equiv_{N} \Phi(\gamma(0)) \), the constant loop based at \( \Phi(\gamma(0)) \). By Lemma 3.2 we must have \( \|\gamma\| \in H_{1}^{s}(M; \mathbb{Z}) \).

Since \( H_{1}(M; \mathbb{Z}) \) is a finitely generated abelian group, so are \( H_{1}^{s}(M; \mathbb{Z}) \) and their quotients. To compute the rank of the quotient group \( H_{1}(M; \mathbb{Z})/H_{1}^{s}(M; \mathbb{Z}) \), we notice that a coset \( \|\gamma\| + H_{1}^{s}(M; \mathbb{Z}) \) defines a torsion element in the quotient group \( if and only if \ k\|\gamma\| \in H_{1}^{s}(M; \mathbb{Z}) \) for some \( k \in \mathbb{Z} \). By the Hurewicz theorem, we could find \( \tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{b_{1}(N)} : [0, 1] \rightarrow N \), representing the distinct generators of \( H_{1}(N; \mathbb{Z})/\text{Torsion} \), which is a free \( \mathbb{Z} \)-module of rank \( b_{1}(N) \). Subdividing \([0, 1]\) by \( 0 = t_{0} < t_{1} < \cdots < t_{n} = 1 \) sufficiently fine, we could ensure \( |\tilde{\gamma}_{i}(t_{j})| \leq 10^{-1}\delta \) for each \( j = 1, \ldots, n \).

We then find \( p_{i,j} \subset M \) such that \( d_{\delta}(\Phi(p_{i,j}), \tilde{\gamma}_{i}(t_{j})) < 10^{-1}\delta \) for each \( i \) and \( j \), let \( \mu_{i,j} \) be a minimal
geodesic connecting \( p_{i,j-1} = \mu_{i,j}(0) \) to \( p_{i,j} = \mu_{i,j}(1) \), with \( \mu_{i,0}(1) = \mu_{i,1}(0) \), and form the loops 
\( \gamma_i := \mu_{i,1} * \mu_{i,2} * \cdots * \mu_{i,n} \) for each \( i = 1, \ldots, b_1(N) \). Just done in (3.3) we see that \( |\mu_{i,j}| \leq \frac{1}{5} \delta_0 \), and for 
\( t \in [t_{i,j-1}, t_j] \) we have 
\[
d_b(\Phi(\gamma_i(t)), \tilde{\gamma}_i(t)) \leq |\mu_{i,j}| + |\tilde{\gamma}_i| + \frac{1}{5} \delta \leq \frac{7}{10} \delta.
\]
This implies that each \( \tilde{\gamma}_i \) is \( \delta \)-approximating \( \gamma_i \). We notice that \( |\gamma_i| \geq 10^{-1} t_{hr} \), because otherwise we must have \( \|\gamma_i\| \in \mathcal{H}_1^0(M; \mathbb{Z}) \), making \( \|\tilde{\gamma}_i\| = 0 \in H_1(N; \mathbb{Z}) \) by Lemma 3.6, contradicting the choice of \( \|\gamma_i\| \) as a generator.

In the same vein, we could show that the homology classes \( \|\gamma_1\|, \ldots, \|\gamma_{b_1(N)}\| \) are indeed \( \mathbb{Z} \)-independent modulo \( \mathcal{H}_1^0(M; \mathbb{Z}) \): if there is some \( \mathbb{Z} \)-linear relation such that 
\[
k_1 \|\gamma_1\| + \cdots + k_{b_1(N)} \|\gamma_{b_1(N)}\| \in \mathcal{H}_1^0(M; \mathbb{Z}),
\]
then by Lemma 3.6 we have \( k_1 \|\tilde{\gamma}_1\| + \cdots + k_{b_1(N)} \|\tilde{\gamma}_{b_1(N)}\| = 0 \in H_1(N; \mathbb{Z}) \), and the \( \mathbb{Z} \)-independence of the chosen classes in \( H_1(N; \mathbb{Z}) \) forces \( k_1 = \cdots = k_{b_1(N)} = 0 \). Moreover, each \( \gamma_i \) \( (i = 1, \ldots, b_1(N)) \) is torsion free again by Lemma 3.6: if \( \|\gamma_i\| \) defines a torsion element in \( H_1(M; \mathbb{Z})/\mathcal{H}_1^0(M; \mathbb{Z}) \), then \( n \|\gamma_i\| \in \mathcal{H}_1^0(M; \mathbb{Z}) \) for some \( n \in \mathbb{Z} \), implying that \( n \|\tilde{\gamma}_i\| = 0 \in H_1(N; \mathbb{Z}) \), contradicting the choice of \( \|\gamma_i\| \) as a generator of \( H_1(N; \mathbb{Z})/\text{Torsion} \). This proves \( \text{rank } H_1(M; \mathbb{Z})/\mathcal{H}_1^0(M; \mathbb{Z}) \geq b_1(N) \).

Conversely, by the Hurewicz theorem, \( H_1(M; \mathbb{Z})/\text{Torsion} \) also has a collection of generators represented by geodesic loops. If \( \gamma : [0, 1] \to M \) is such a representing loop with \( |\gamma| \geq 10^{-1} t_{hr} \), we consider a \( \delta \)-approximating loop \( \tilde{\gamma} : [0, 1] \to N \), and we have 
\[
\|\tilde{\gamma}\| = \sum_{i=1}^{b_1(N)} k_i \|\gamma_i\| + \sum_j \|\tilde{\gamma}_j^{tor}\| \in H_1(N; \mathbb{Z}),
\]
where \( \|\tilde{\gamma}_j^{tor}\| \in H_1(N; \mathbb{Z}) \) are torsion elements, represented by geodesic loops (by the Hurewicz theorem). Since we could argue as before to obtain loops \( \gamma_j^{tor} : [0, 1] \to M \) so that each \( \tilde{\gamma}_j^{tor} \) is \( \delta \)-approximating to \( \gamma_j^{tor} \), by Lemma 3.5 we know that 
\[
\|\gamma\| = \sum_{i=1}^{b_1(N)} k_i \|\gamma_i\| + \sum_j \|\gamma_j^{tor}\| \mod \mathcal{H}_1^0(M; \mathbb{Z}).
\]
But since certain finite multiple of \( \|\tilde{\gamma}_j^{tor}\| \) vanishes in \( H_1(N; \mathbb{Z}) \), by Lemma 3.5 we know that the coset \( \sum_j \|\gamma_j^{tor}\| + \mathcal{H}_1^0(M; \mathbb{Z}) \) defines a torsion element in \( H_1(M; \mathbb{Z})/\mathcal{H}_1^0(M; \mathbb{Z}) \). Consequently, the coset \( \|\gamma\| + \mathcal{H}_1^0(M; \mathbb{Z}) \) is generated by those of \( \|\gamma_1\|, \ldots, \|\gamma_{b_1(N)}\| \).

The above discussion implies that \( \text{rank } H_1(M; \mathbb{Z})/\mathcal{H}_1^0(M; \mathbb{Z}) \leq b_1(N) \). Moreover, since \( H_1(M; \mathbb{Z}) \) and \( \mathcal{H}_1^0(M; \mathbb{Z}) \) are finitely generated \( \mathbb{Z} \)-modules, we have 
\[
\text{rank } \mathcal{H}_1^0(M; \mathbb{Z}) = \text{rank } H_1(M; \mathbb{Z}) - \text{rank } H_1(M; \mathbb{Z})/\mathcal{H}_1^0(M; \mathbb{Z}) = b_1(M) - b_1(N),
\]
which is the desired equality for this section. \( \square \)

4. Effective distance control of initially nearby geodesics

In this section we prove Theorem 1.3 which is the major technical ingredient in proving the first claim Theorem 1.1. Our proof is inspired by the work of Colding and Naber 22 — in fact, once we have set up the most basic estimates, i.e. the Laplacian comparison for the distance function
to a closed embedded submanifold, and the local control of the spreading of minimal geodesics, the rest of Colding and Naber’s original argument works directly. While this may seem to be obvious to experts, we will fill in the necessary details that bridge our considerations to Colding and Naber’s original results in [22 §2 and §3].

Let \( \Sigma \) be a smoothly embedded submanifold of a complete Riemannian manifold \((M^m, g)\) such that \( \Sigma = \overline{\Sigma} \), i.e. \( \Sigma \) is closed but not necessarily bounded. Let \( r_\Sigma : M \to \mathbb{R} \) denote the distance to \( \Sigma \), i.e. \( r_\Sigma(q) := \inf_{q \in \Sigma} d_g(q, y) \). By the completeness of \((M, g)\) and the closedness of \( \Sigma \), we know that for any \( q \in M \setminus \Sigma \), \( r_\Sigma(q) > 0 \) is always realized by some unit speed smooth geodesic \( \sigma \) with \( \sigma(0) = p \in \Sigma \), \( \sigma(r_\Sigma(q)) = q \) and \( |\sigma| = r_\Sigma(q) \). This tells, by the triangle inequality, that \( r_\Sigma \) is a 1-Lipschitz function on \( M \). We will check that \( r_\Sigma \) is in fact smooth almost everywhere on \( M \) in Lemma 4.1. Consequently, the gradient vector field \( \nabla r_\Sigma \) is smoothly defined almost everywhere on \( M \), and so is its gradient flow \( \psi^s_\Sigma \) for each \( s \geq 0 \).

Assuming \( \text{Rc}_g \geq -(m - 1)g \) and\( \text{diam}(M, g) \leq D \), we will check that \( \Delta r_\Sigma \leq C(m, D)r^{-1}_\Sigma \) in the distributional sense in the first subsection, and then locally control the spreading of the flow lines of \( \nabla r_\Sigma \) in the second subsection. Once these are done, we could directly appeal to the estimates in [22 §2] to obtain uniform \( C^1_{\text{loc}}, \text{H}^2_{\text{loc}} \) control of the parabolic approximation of \( r_\Sigma \) in the third subsection, and finally, we follow the argument in [22 §3] to effectively control the spreading of the flow lines of \( \nabla r_\Sigma \).

4.1. Laplacian comparison for distance to submanifolds. In this subsection, we will obtain an upper bound of \( \Delta r_\Sigma \) in (4.1), in the barrier sense a la Calabi [7]. Though being a simple estimate, we surprisingly notice its absence in the literature, and the purpose of this subsection is to fill this gap; also compare [20 Lemma 2.1 and Lemma 2.2] for the case of non-negative Ricci curvature. The first order of business is to understand the regularity of \( r_\Sigma \).

Lemma 4.1. The function \( r_\Sigma \) is almost everywhere smooth on \( M \).

Proof. We beginning with considering a point \( q \in M \setminus \Sigma \) which is not a focal point of \( \Sigma \) (see [23 §10.4]), and which is connected to \( \Sigma \) by a unique minimal geodesic \( \sigma \) of unit speed, such that \( |\sigma| = r_\Sigma(q) =: l \), \( \sigma(0) = p \in \Sigma \) and \( \sigma(l) = q \). We will show that \( r_\Sigma \) is smooth in a neighborhood around \( q \).

Since \( q \in M \) is not a focal point of \( \Sigma \), the initial data \((\sigma(0), \dot{\sigma}(0))\) is a regular point of the normal exponential map \( \exp^\perp : T^\perp \Sigma \to M \), where \( T^\perp \Sigma \) is the normal bundle of \( \Sigma \) within \( TM \), and \( \exp^\perp \) is nothing but the restriction of the usual exponential map restricted to \( T^\perp \Sigma \). Since \( \dim T^\perp \Sigma = \dim M \) and \( q \) is not a singular point of \( \exp^\perp \), there is an open neighborhood \( U_0 \subset T^\perp \Sigma \) where \( \exp^\perp \) restricts to be a diffeomorphism onto its image \( W_0 := \exp^\perp(U_0) \). We now make the following

Claim: There is a smaller neighborhood \( W \subset W_0 \) of \( q \) such that for any \( q' \in W \), \( r_\Sigma(q') \) is uniquely realized by the geodesic \( t \mapsto \exp_{q'} t \vec{v}, \) for some \((p', \vec{v}) \in U = (\exp^\perp)^{-1}(W) \subset U_0 \).

Proof of the Claim. Suppose otherwise, that there is a sequence \( q_i \to q \in W_0 \) with distinct initial data \((\tau_i(0), \dot{\tau}_i(0)) \in T^\perp \Sigma \) and \((\sigma_i(0), \dot{\sigma}_i(0)) \in U_0 \), such that \( |\tau_i| = d_g(q_i, \Sigma) \leq |\sigma_i| \), \( \tau_i(|\tau_i|) = q_i \), and \( \exp_{\sigma_i(0)} t_i \dot{\sigma}_i(0) = q_i \) with \( t_i \geq d_g(q_i, \Sigma) \). (Here we only work with unit tangent vectors.) Since \( \exp^\perp \big|_{U_0} \) is bijective, we have for all \( i \) large enough \((\tau_i(0), \tau_i(0)) \notin U_0 \). On the other hand, for all \( i \) sufficiently large, since

\[
d_g(q, \tau_i(0)) \leq d_g(q, q_i) + d_g(q_i, \tau_i(0)) < 3d_g(q, \Sigma),
\]
by taking subsequence if necessary, we know that \((\tau_i(0), \tau_i(0)) \to (p', \bar{v}) \in T^\perp \Sigma \setminus U_0\) for some \(p' \in \Sigma\) and \(\bar{v} \in T_{p'} M\) with unit length. However, denoting the limit geodesic \(t \mapsto \exp_{p'} t\bar{v}\) by \(\tau\), we notice that it has length
\[
|\tau| = \lim_{i \to \infty} |\tau_i| = \lim_{i \to \infty} d_g(q_i, \Sigma) = r_\Sigma(q),
\]
and this contradicts our uniqueness assumption on \(\sigma\), which has length \(|\sigma| = r_\Sigma(q)| and initial data \((\sigma(0), \dot{\sigma}(0)) \in U_0\).

Now let \(\tau(0) > 0\) and \(\bar{v} \in U\) be a minimal geodesic realized by a minimal geodesic \(\sigma\). Therefore, we understand that a possibly non-smooth point \(q \in M\) of \(r_\Sigma\) must fall into one of the following two categories:

1. \(q\) is a focal point of \(\Sigma\); or
2. there are multiple minimal geodesics that realizes \(r_\Sigma(q)\).

Notice that the focal points of \(\Sigma\) are characterized by the critical values of the normal exponential map (see [23] §10.4, Proposition 4.4]), which, by Sard’s theorem, is a null set in \(M\). On the other hand, if (2) is the case, say, there are geodesics \(\sigma_1\) and \(\sigma_2\) realizing \(r_\Sigma(q)\), with \(\sigma_1(0), \sigma_2(0) \in \Sigma\) and \(\sigma_1(1) = \sigma_2(1) = q\), but \((\sigma_1(0), \dot{\sigma}_1(0)) \neq (\sigma_2(0), \dot{\sigma}_2(0))\), then we must have \(\dot{\sigma}_1(1) \neq \dot{\sigma}_2(1)\), and \(r_\Sigma\) fails to be differentiable at \(q\). However, since \(r_\Sigma\) is a Lipschitz function, its non-differentiable points form a measure 0 subset of \(M\). Therefore, all non-smooth points of \(r_\Sigma\) fall into the union of two null subsets of \(M\), which has to be of measure 0.

Before checking the desired Laplacian bound for \(r_\Sigma\) at its smooth points, we define the notation \(F_m(r)\) as a function for \(r > 0\) with
\[
F_m(r) := (m - 1)r \coth r = (m - 1)\frac{\sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!}}{\sum_{n=0}^{\infty} \frac{r^{2n}}{(2n+1)!}}.
\]
Clearly \(F_m(r) > m - 1\) and \(\lim_{r \to 0} F_m(r) = m - 1\). We also put \(C_{F_m}(l) := \max_{r \in [0, l]} F_m(r)\).

**Lemma 4.2** (Laplacian comparison at smooth points). Let \(\Sigma \subset M\) be a closed and smoothly embedded submanifold. Suppose that the Riemannian manifold \((M, g)\) satisfies \(R_{\Sigma} \geq -(m - 1)g\), and let \(r_\Sigma\) denote the distance function to \(\Sigma\). Then at the smooth points of \(r_\Sigma\), we have
\[
\Delta r_\Sigma \leq \frac{F_m(r_\Sigma)}{r_\Sigma}.
\]

The basic observation here is that the level set of \(r_\Sigma\) is always less convex than the geodesic sphere touching it, as illustrated in Figure 3.

**Proof.** Suppose \(r_\Sigma(q) > 0\) and \(r_\Sigma\) is a smooth function around \(q \in M\). We also assume that \(r_\Sigma(q)\) is realized by a minimal geodesic \(\sigma : [0, r_\Sigma(q)] \to M\) of unit speed, with \(\sigma(0) = p \in \Sigma\) and \(\sigma(1) = q\). Now let \(d_p\) denote the distance function to \(p\), i.e. \(d_p(x) := d_g(x, p)\). Then \(d_p\) is smooth around \(q \in M\) with \(d_p(q) = r_\Sigma(q)\) and \(d_p(q)\) is also realized by the minimal geodesic \(\sigma\).
Now pick an orthonormal basis $E_1, \ldots, E_m$ of $T_qM$ such that $E_1 = \dot{\sigma}(1)$. Let $\gamma_i : (-\varepsilon, \varepsilon) \to M$ be minimal geodesics with $\gamma_i(0) = q$ and $\dot{\gamma}_i(0) = E_i$ ($i = 2, \ldots, m$). Here we choose $\varepsilon > 0$ sufficiently small (smallness depending on $q \in M$) so that both functions $r$ and $d_p$ are smooth when restricted to each $\gamma_i$. For each $i = 2, \ldots, m$, let $f_i, g_i : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be defined as $f_i(t) := r(\gamma_i(t))$ and $g_i(t) := d_p(\gamma_i(t))$, then we see that
\[
f_i(0) = g_i(0) = r(\Sigma(q)), \quad \text{and} \quad f'_i(0) = g'_i(0) = \langle E_i, \dot{\sigma}(1) \rangle = 0.
\]
Moreover, since for each $i = 2, \ldots, m$ we have
\[
\forall t \in (-\varepsilon, \varepsilon), \quad f_i(t) = \inf_{y \in \Sigma} d_y(\gamma_i(t), y) \leq d_y(\gamma_i(t), p) = g_i(t),
\]
comparing the Taylor polynomials of $f_i$ and $g_i$ expanded around $t = 0$, we get the estimates
\[
f''_i(0) \leq g''_i(0) \quad \text{for $i = 2, \ldots, m$.}
\]
On the other hand, for each $i = 2, \ldots, m$ simple computation gives
\[
f''_i(0) = Hess_r(\Sigma(q), E_i, E_i) \quad \text{and} \quad g''_i(0) = Hess_d_p(E_i, E_i),
\]
and thus (4.2) leads to
\[
\Delta r(\Sigma(q)) \leq \Delta d_p(q).
\]
Since the usual Laplace comparison for the distance function to a point gives
\[
\Delta d_p \leq (m - 1) \coth d_p
\]
whenever $d_p$ is smooth, and $d_p(q) = r(\Sigma(q))$, by (4.3) we especially have at $q \in M$ that
\[
\Delta r(\Sigma(q)) \leq \frac{F_m(r(\Sigma(q)))}{r(\Sigma(q))}.
\]
Since $q \in M$ is an arbitrary smooth point the function $r(\Sigma)$, we have derived (4.1) wherever $r(\Sigma)$ is smooth.\qed
In a similar spirit, we could in fact show that (4.1) holds everywhere on \( M \) in the barrier sense:

**Proposition 4.3** (Global Laplacian comparison). As assumed in Lemma 4.2, we have (4.1) holding everywhere on \( M \) in the barrier sense, i.e. for any \( q \in M \) and every \( \varepsilon > 0 \) small enough, there is an open neighborhood \( U \) of \( q \) and a function \( h_{q, \varepsilon} \in C^2(U) \), such that

1. \( r_\Sigma(q) = h_{q, \varepsilon}(q) \),
2. \( h_{q, \varepsilon} \geq r_\Sigma \) in \( U \), and
3. \( \Delta h_{q, \varepsilon} \leq \frac{F_m(r_\Sigma(q))}{r_\Sigma(q)} + \varepsilon \).

Consequently, (4.1) holds everywhere on \( M \) in the distributional sense.

**Proof.** As in the proof of Lemma 4.2, we let \( \sigma \) denote a unit speed minimal geodesic such that \( \sigma(0) = p \in \Sigma, \sigma(r_\Sigma(q)) = q \) and \( |\sigma| = r_\Sigma(q) \), without assuming \( q \in M \) being a smooth point of \( r_\Sigma \). Now for any positive \( \varepsilon \)

\[
\varepsilon \leq \min \left\{ 10^{-1} \sqrt{r_\Sigma(q)}, (m - 1)^{-1} \left( 2(m - 1)^{-1} C_{F_m}(r_\Sigma(q))^2 r_\Sigma(q)^{-2} - 1 \right)^{-1} \right\},
\]

we consider the function

\[
h_{q, \varepsilon}(x) := d_g(x, \sigma(\varepsilon^2)) + \varepsilon^2.
\]

Then clearly \( r_\Sigma(q) = h_{q, \varepsilon}(q) \). On the other hand, by the triangle inequality we have

\[
\forall x \in M, \quad h_{q, \varepsilon}(x) \geq d_g(x, p) \geq r_\Sigma(x).
\]

Moreover, \( h_{q, \varepsilon} \) is smooth in some open neighborhood around \( q \), as \( q \) is not in the cut locus of \( p \in M \), and thus for some \( s \in [r_\Sigma(q) - \varepsilon^2, r_\Sigma(q)] \) we have

\[
\Delta h_{q, \varepsilon}(q) \leq (m - 1) \coth d_g(q, \sigma(\varepsilon^2))
\]

\[
= (m - 1) \coth r_\Sigma(q) + (m - 1) \varepsilon^2 \left( (m - 1)^{-1} s^{-2} F_m(s)^2 - 1 \right)
\]

\[
\leq \frac{F_m(r_\Sigma(q))}{r_\Sigma(q)} + \varepsilon,
\]

by the constraint imposed on \( \varepsilon > 0 \). Therefore, we have constructed an upper barrier function \( h_{q, \varepsilon} \) for \( r_\Sigma \) satisfying all requirements in Calabi’s Laplacian comparison in the barrier sense (see [7]). It is well-known that this guarantees (4.1) to hold in the distributional sense.

Slightly modifying the proof of [22, Lemma 3.2], which is originated from [8], we obtain the following uniform Hessian \( L^2 \) estimate along the interior of a minimal geodesic, as a straightforward consequence of the above Laplacian comparison for \( r_\Sigma \):

**Lemma 4.4.** Suppose \( \Sigma \) is a closed embedded submanifold of a Riemannian manifold \( (M, g) \) with \( \text{Rc} \geq -(m - 1) g \), and let \( r_\Sigma \) denote the distance function to \( \Sigma \), as discussed above. If \( q \in M \setminus \Sigma \) and \( \sigma \) is a minimal geodesic of unit speed realizing the value \( l := r_\Sigma(q) > 0 \), then

\[
\int_{\beta l}^{(1-\beta)l} \left| \text{Hess}_{r_\Sigma} \right|^2(\sigma(t)) \, dt \leq \frac{C_H^2(m, l)}{\beta l},
\]

for any \( \beta \in (0, 10^{-2}) \), with \( C_H^2(m, l) := (m - 1)l^2 + C_{F_m}(l) \).
Proof. We put \( d_q(x) := d_q(q, x) = d_q(\sigma(l), x) \), then \( d_q \) is smooth away from \( q = \sigma(l) \), and around \( p = \sigma(0) \in \Sigma \). Now the usual Laplace comparison for the distance function to a point tells that

\[
\Delta d_q \leq \frac{F_m(d_q)}{d_q}
\]

in a neighborhood around \( \sigma((0, (1 - \beta)l)) \), where \( d_q \geq \beta l \). Consequently, we have

\[
\sup_{0 < t \leq (1 - \beta)l} \Delta d_q(\sigma(t)) \leq \frac{C_{F_m}(l)}{\beta l}.
\]

By the previous Laplace comparison (4.1) for \( r_\Sigma \), we also have

\[
\sup_{\beta l \leq t \leq l} \Delta r_\Sigma(\sigma(t)) \leq \frac{C_{F_m}(l)}{\beta l},
\]
as \( r_\Sigma \geq \beta l \) when restricted on \( \sigma([\beta l, l]) \).

On the other hand, since on \( \sigma([\beta l, (1 - \beta)l]) \) both functions \( d_q \) and \( r_\Sigma \) are smooth, and their sum \( d_q + r_\Sigma \) achieves the minimum value \( l \) by the triangle inequality, we must have \( \Delta(d_q + r_\Sigma) \geq 0 \) when restricted to \( \sigma([\beta l, (1 - \beta)l]) \). Therefore, we have for any \( t \in [\beta l, (1 - \beta)l] \),

\[
-\frac{C_{F_m}(l)}{\beta l} \leq -\Delta d_q(\sigma(t)) \leq \Delta r_\Sigma(\sigma(t)) \leq \frac{C_{F_m}(l)}{\beta l}.
\]

We could now apply the Weitzenböck formula to \( r_\Sigma \) to see

\[
\partial_t r_\Sigma(\sigma(t)) + |Hess r_\Sigma|^2(\sigma(t)) \leq m - 1,
\]
and integrating along \( \sigma \) from \( \beta l \) to \( (1 - \beta)l \) we have

\[
\int_{\beta l}^{1 - \beta l} \left| Hess r_\Sigma \right|^2(\sigma(t)) \, dt \leq (m - 1)(1 - 2\beta l) + \Delta r_\Sigma(\sigma(\beta l)) - \Delta r_\Sigma(\sigma((1 - \beta)l)),
\]
which leads to the desired estimate (4.4) if we put \( C_2^l(m, l) = (m - 1)l^2 + C_{F_m}(l) \).

4.2. Local control of the geodesic spreading. Recall that we would like to control the spreading of the flow lines of \( \nabla r_\Sigma \). In this subsection, we do this locally around a smooth flow line of \( \nabla r_\Sigma \) that connects a smooth point back to \( \Sigma \).

Now let \( q \in M \setminus \Sigma \) with the minimal geodesic \( \sigma \) realizing \( r_\Sigma(q) =: l \) (with \( \sigma(0) = p \in \Sigma \)), then for any \( t \in (0, l) \), \( \sigma(t) \) is a smooth point of \( r_\Sigma \). We fix some \( \beta \in (0, 10^{-2}) \), and cover \( [(\frac{\beta}{2} l, (1 - \frac{\beta}{2})l)] \) by finitely many open sets \( W_i \subset M \) as obtained by the Claim in the proof of (4.1) and let \( U_i \subset T^* \Sigma \) be the corresponding open subsets of initial values. By the compactness of \( [(\frac{\beta}{2} l, (1 - \frac{\beta}{2})l)] \), \( \{W_i\} \) can be reduced to a finite covering and we could let \( U := \cup U_i \) which is an open neighborhood of \( \{(p, t\sigma(0)) : t \in [0, (1 - \frac{\beta}{2})l]\} \) in \( T^* \Sigma \). We also let \( W := \exp^l U \) which is an open subset of \( M \). Notice that the geodesics \( \sigma_{(p', \vec{v})} : t \mapsto \exp_{p'} t\vec{v} \) uniquely realizes \( r_\Sigma(\sigma_{(p', \vec{v})}(t)) \) for any \( t \in [0, 1] \), whence being an integral curve of \( \nabla r_\Sigma \) with initial value \( (p', \vec{v}) \in U \).

With the previous Hessian \( L^2 \) estimate along a minimal geodesic in Lemma (4.4) we now aim to control the spreading of the integral curves of \( \nabla r_\Sigma \) more effectively in a small tubular neighborhood.
around $\sigma$. To be precise, for any $t \in [\beta l, l - \beta l]$ fixed, and for each $r \in [0, \beta/10]$, we consider the following core neighborhood of $\sigma(t)$:

$$H^r(\sigma) := \left\{ y \in B_\rho(\sigma(t), r) : \forall s \in [0, (1 - \beta)l - t], \frac{d_\rho(\psi^\sigma_s(y), \sigma(t + s))}{d_\rho(y, \sigma(t))} \leq \exp \left( \frac{2C_H(m, l)}{\sqrt{\beta l}} \sqrt{s} \right) \right\}. $$

Intuitively speaking, such a neighborhood of $\sigma(t)$ consists of points in $B_\rho(\sigma(t), r)$ that are carried by the gradient flow $\psi^\sigma_s$ up to a controllable distance for all $s \leq (1 - \beta)l - t$. When the ambient manifold $M$ has a uniform Ricci curvature lower bound, we could in fact conclude that almost every point of $B_\rho(\sigma(t), r)$ are in $H^r(\sigma)$, provided that $r > 0$ is sufficiently small:

**Lemma 4.5.** With the same assumptions as in Lemma 4.4 we fix $\beta \in (0, 10^{-2})$ and $t \in [\beta l, l - \beta l]$. For some $r > 0$ sufficiently small, we have $H^r(\sigma) = B_\rho(\sigma(t), r)$.

**Proof.** Let $W \subset M$ denote the open neighborhood of $\sigma([\beta l, (1 - \beta)l])$ where $r_\Sigma$ is smooth and let $r$ satisfy

$$r \leq \frac{1}{10} \min \left\{ \beta l, \min_{s \in [\beta l, (1 - \beta)l]} d_\rho(\sigma(s), M \setminus W), \inf_j d_\rho(\sigma(t)) \right\},$$

where $\inf_j d_\rho(\sigma(t))$ denotes the injectivity radius at $\sigma(t)$. Further shrinking $r$ if necessary, we also required that $\inf_{B_\rho(\sigma(0), 2r)} r_\Sigma \geq \frac{\beta l}{2}$. By the compactness of $\sigma([\beta l, (1 - \beta)l])$ and the Lipschitz continuity of $r_\Sigma$, it is clear that $r > 0$.

For any smooth point $y \in B_\rho(\sigma(t), r)$, there is a unique $\bar{v} \in T_{\sigma(t)}M$ such that $\exp^\sigma_{\sigma(t)} \bar{v}_0 = y$. We let $\tau(s, u) := \psi^\sigma_s(\exp^\sigma_{\sigma(t)}uv) : [0, (1 - \beta)l - t] \times [-2, 2] \to M$ be a parametrized 2-dimensional submanifold in $M$. Notice that for any $u \in [-2, 2]$, $s \mapsto \tau(s, u)$ is an integral curve of $\nabla r_\Sigma$ and thus a smooth geodesic. This implies that the variation $\tau$ is by geodesics and thus $J(s, u) := \frac{\partial}{\partial u} \tau(s, u)$ is a Jacobi field along the geodesic $s \mapsto \tau(s, u)$. Since for each $s$ fixed, $u \mapsto \tau(s, u)$ furnishes a curve connecting $\sigma(t + s) = \tau(s, 0)$ and $\psi^\sigma_s(y) = \tau(s, 1)$, we have

$$\forall s \in [0, (1 - \beta)l - t], \quad d_\rho(\psi^\sigma_s(y), \sigma(t + s)) \leq |\tau(s, -)_{[0,1]}|.$$ 

Since $|\tau(s, -)_{[0,1]}| = \int_0^1 |J(s, u)| du$, we would like to compare $|J(0, u)|$ and $|J(s, u)|$. As $\mathcal{L}_{\tau_{\Sigma}} J = 0$, we have

$$\forall s \in [0, (1 - \beta)l - t], \forall u \in [-2, 2] \quad \partial_u |J(s, u)|^2 = 2 Hess_{r_\Sigma}(J(s, u), J(s, u)), $$

and thus

$$|\partial_u \log |J(s, u)|^2| \leq 2 |Hess_{r_\Sigma}|(\tau(s, u)).$$

Integrating with respect to $s$ we see for any $s_1 \in [0, (1 - \beta)l - t]$ that

$$\log \frac{|J(t + s_1, u)|^2}{|J(t, u)|^2} \leq 2 \int_0^{s_1} |Hess_{r_\Sigma}|(\tau(t + s, u)) ds \leq 2 \left( \int_{\frac{s_1}{\beta l}}^{(1 - \beta)l} |Hess_{r_\Sigma}|(\tau) \right)^\frac{1}{2} \sqrt{s_1},$$

since the geodesic $u \mapsto \tau(0, u) = \exp^\sigma_{\sigma(t)}(uv)$ is at least $\frac{\beta l}{4}$ away from $\Sigma$. Consequently, we have

$$\exp \left( -2 \left( \int_{\frac{s_1}{\beta l}}^{(1 - \beta)l} |Hess_{r_\Sigma}|(\tau) \right)^\frac{1}{2} \sqrt{s_1} \right) \leq \frac{|J(t + s_1, u)|^2}{|J(t, u)|^2} \leq \exp \left( 2 \left( \int_{\frac{s_1}{\beta l}}^{(1 - \beta)l} |Hess_{r_\Sigma}|(\tau) \right)^\frac{1}{2} \sqrt{s_1} \right).$$
and by Lemma 4.4 we have
\[ \forall s_1 \in [0, (1 - \beta)l - t], \forall u \in [0, 1], \quad |J(t + s_1, u)| \leq e^{-\frac{2G(\rho, t)}{\sqrt{s}} \sqrt{t}} |J(t, u)|. \]
Integrating \( u \) from 0 to 1 we get
\[ |\tau(s_1, -)|_{[0, 1]} \leq e^{-\frac{2G(\rho, t)}{\sqrt{s}} \sqrt{t}} |v|, \]
and as \( s_1 \) is arbitrary in \([0, (1 - \beta)l - t]\), we have
\[ \forall s \in [0, (1 - \beta)l - t], \quad d_g\left(\psi^\Sigma(t, \tau(s)), \tau(t + s)\right) \leq e^{-\frac{2G(\rho, t)}{\sqrt{s}} \sqrt{t}} d_g(y, \tau(t)). \]
By the definition of \( H^l(\tau) \) and the choice of \( r \), we see that \( H^l(\sigma) = B_g(\sigma(t), r) \).

From the proof of this lemma, we could clearly see that \( H^l(\sigma) \) depends on the specific manifold \( M \) and geodesic \( \sigma \), rather than being a uniform neighborhood that we wish to find. In fact, it is impossible to get such a neighborhood in a uniform way; however, we notice that once the good neighborhood \( W \) is specified, the actual distance estimate (4.8) only depends on the \( L^2 \) Hessian control of \( r_\Sigma \) along the interior of \( \sigma \). Based on this observation, we will see in the sequel that there is a subset \( T^l_{\eta}(\sigma) \subset B_g(\sigma(t), r) \) of sufficiently large measure, that resembles the key property of \( H^l(\sigma) \): the gradient flow lines of \( \nabla r_\Sigma \) with initial data in \( T^l_{\eta}(\sigma) \) does not spread too far away from \( \sigma \). Moreover, \( T^l_{\eta, \sigma}(\sigma) \) is defined analytically and its properties depend on the estimates uniformly.

4.3. Parabolic approximation and effective estimates. In order to define the desired subset that stays close to a given flow line of \( \nabla r_\Sigma \), we need to uniformly estimate the behavior of \( r_\Sigma \), especially bounding \( \text{Hess}_r \). While impossible to control \( \text{Hess}_r \) in the \( C^0 \) sense, Colding and Naber observed in [22] that by parabolically smoothing \( r_\Sigma \) (with \( \Sigma \) being a single point in their setting), an \( L^2_{\text{loc}} \) estimate around a given flow line of \( \nabla r_\Sigma \) is indeed possible, and is sufficient for the purpose. For a general closed embedded submanifold \( \Sigma \), we notice that once the Laplacian comparison (4.1) for \( r_\Sigma \) is ready at hand, then all the estimates in [22] §2 go through without any change, for the parabolic approximation of \( r_\Sigma \). In this subsection we summarize the relevant estimates and refer directly to the corresponding ones in [22] §2.

Fix a minimal geodesic \( \sigma : [0, l] \to M \) such that \( d_g(\sigma(t), \Sigma) = t \) for all \( t \in [0, l] \), we let \( p := \sigma(0) \in \Sigma, q := \sigma(l) \) and denote \( d^+(x) := t - d_g(q, x) \) for all \( x \in M \). We also put the notation
\[ M_{r,s} := \{ x \in M : r < l - l^{-1}r_\Sigma(x) < s \text{ and } r < l^{-1}d_g(q, x) < s \}. \]
Now we consider the excess function \( e^\Sigma : M \to \mathbb{R} \) defined as \( e^\Sigma := r_\Sigma - d^+ \). Since \( r_\Sigma(x) \leq d_g(x, p) \) for any \( x \in M \), we always have
\[ e^\Sigma(x) \leq e_{p,q}(x), \]
where \( e_{p,q}(x) = d_g(p, x) - d^+(x) \) is the original excess function defined for a minimal geodesic connecting the two end points. By the excess function estimate due to Abresch and Gromoll [1], we have for any \( t \in (0, l - r) \) (with \( r > 0 \) sufficiently small),
\[ \sup_{B_g(\sigma(t), r)} e^\Sigma \leq \sup_{B_g(\sigma(t), r)} e_{p,q} \leq C_{\text{AG}}(m) r^{1 + \alpha_{\text{AG}}(m)}, \]
where \( C_{\text{AG}}(m) > 1 \) and \( \alpha_{\text{AG}}(m) \in (0, 1) \) are dimensional constants. By Proposition 4.3, we see that
\[ \Delta e^\Sigma \leq \frac{2C_{F_u}(l)}{r_\Sigma}. \]
Consequently, we could invoke [22, Corollary 2.4] to obtain the following estimate, which is a version of [22, Theorem 2.8]:

**Lemma 4.6** (Average excess estimate). For any \( \beta \in (0, 10^{-2}) \) and \( D \geq 1 \), there are constants \( C_{E_\delta}(m, D, \beta) > 1 \) and \( e_{E_\delta}(m, D, \beta) \in (0, 1) \) such that if \( x \in M_{\beta,2} \) satisfies \( e(x) \leq e^2 l \leq e^{2}_{E_\delta} l \), then

\[
\int_{B_{g}(x,\varepsilon)} e^x \leq C_{E_\delta} e^2 l.
\]

Now we let \( \psi^\pm : M \rightarrow \mathbb{R} \) be the cut-off function given by [22, Lemma 2.6] such that for some \( \beta \in (0, 10^{-2}) \) we have

\[
\psi^-(x) = \begin{cases} 1 & \text{if } \frac{\beta l}{4} < r_\Sigma(x) < 8l, \\ 0 & \text{if } r_\Sigma(x) \leq \frac{\beta l}{16} \text{ or } r_\Sigma(x) > 16l; \\ \end{cases} \quad \psi^+(x) = \begin{cases} 1 & \text{if } \frac{\beta l}{4} < d_g(q, x) < 8l, \\ 0 & \text{if } d_g(q, x) \leq \frac{\beta l}{16} \text{ or } d_g(q, x) > 16l. \\ \end{cases}
\]

We now put \( \psi := \psi^+ \psi^- \) and evolve \( \psi r_\Sigma, \psi d^+, \text{ and } \psi e^\Sigma \) by the heat equation to obtain smooth functions \( h_t, d^+_t \) and \( e^\Sigma_t \) on \( M \), i.e. we have

\( (\partial_t - \Delta) h_t = 0 \) with \( h_0 = \psi r_\Sigma, \) \( (\partial_t - \Delta) d^+_t = 0 \) with \( d^+_0 = \psi d^+ \), and \( (\partial_t - \Delta) e^\Sigma_t = 0 \) with \( e^\Sigma_0 = \psi e^\Sigma \).

By uniqueness we clearly have \( e^\Sigma_t = h_t - d^+_t \).

Now by (4.1) and [22, Lemma 2.6], we could estimate

\[
\Delta h_0 = r_\Sigma \Delta \psi + 2(\nabla \psi, \nabla r_\Sigma) + \psi \Delta r_\Sigma \leq C(m, D, \beta) l^{-1},
\]

where \( C(m, D, \beta) \) depends on \( C_{F_{\text{loc}}}(D) \). Similarly, \( \Delta d^+_0 \geq -C(m, D, \beta) l^{-1} \) and \( \Delta e^\Sigma_0 \leq C(m, D, \beta) l^{-1} \). Moreover, \( \Delta h_0, \Delta d^+_0 \) and \( \Delta e^\Sigma_0 \) are supported in \( M_{\beta,16} \). Consequently, by the proof of [22, Lemma 2.10] we see that for some positive constant \( C(m, D, \beta) > 0 \),

\[
\max \{ \Delta h_t, -\Delta d^+_t, \Delta e^\Sigma_t \} \leq C(m, D, \beta) l^{-1}.
\]

We could then plug this estimate into [22, Lemma 2.11 and Lemma 2.13] to obtain some new constants \( C_{C_\delta}(m, D, \beta) > 0 \) and \( \bar{e}_{C_\delta}(m, D, \beta) > 0 \) such that for any \( \varepsilon \in (0, \bar{e}_{C_\delta}) \),

\[
\sup_{M_{\beta,4}} |h_{2e^\Sigma} - r_\Sigma| \leq C_{C_\delta} \left( e^2 l + e^\Sigma \right).
\]

Moreover, since for any \( \varepsilon \)-geodesic \( \sigma' \) connecting \( p \) and \( q \), it holds \( e(\sigma(t)) < e^2 l \), and so does \( e^\Sigma_{\beta,4}(\sigma(t)) \) by (4.9), we have, by [22, Corollary 2.16] that

\[
\sup_{\sigma \in M_{\beta,4}} |h_{2e^\Sigma} - r_\Sigma| \leq C_{C_\delta} e^2 l, \quad \text{and} \quad \sup_{t \in (\frac{\beta l}{4}, (1 - \frac{\beta}{4})l)} |h_{e^\Sigma(\sigma(t))} - l| \leq C_{C_\delta} e^2 l.
\]

The gradient upper bound of \( h_t \) could be obtained by the Bochner formula and Li-Yau heat kernel upper bound (see [43]), as done in [22, Lemma 2.17]:

\[
\sup_{M_{\beta,4}} |\nabla h_{e^\Sigma}| \leq 1 + C_{C_\delta}(m, D, \beta) e^2 l^2.
\]

This estimate, together with (4.14) and [22, Lemma 2.1], then implies an \( H^1_{\text{loc}} \) estimate of \( h_{e^\Sigma} \) around the interior of the geodesic curve \( \sigma \) as in [22, Theorem 2.18]. Integration by parts along \( \sigma \) and in time, we could then obtain an \( H^1_{\text{loc}} \) estimate of \( h_{e^\Sigma} \), as done in [22, Theorem 2.19 and Lemma 2.20] — the proofs are identical since \( h_t \) satisfies exactly the same estimate as \( h^r_t \) in [22, §2], and we only record the needed estimates:
Proposition 4.7. For each \( \beta \in (0, 10^{-2}) \) and \( D \geq l \), there are positive constants \( C_{\beta,p}(m, D, \beta) > 1 \) and \( r_{\beta,p}(m, D, \beta) \leq \epsilon_{E_x} \) to the following effects: \( \forall \epsilon \in (0, r_{\beta,p}), \exists c \in [1, 2], \) such that

\[
(4.16) \quad \int_{\beta l}^{(1-\beta)l} \left( \int_{B_\epsilon(x,|d|)} \left| \text{Hess} h_{2,\epsilon} \right|^2 \right) \, dV_g \, ds \leq C_{\beta,p} \epsilon^{-2};
\]

moreover, for any smooth point \( x \in M_{2,2} \) with \( \epsilon^2(x) \leq \epsilon^2 l \), let \( \sigma_x \) denote the integral curve of \( \nabla r_{\Sigma} \) passing through \( x \), then

\[
(4.17) \quad \forall \beta r_{\Sigma}(x) \leq s < t \leq r_{\Sigma}(x), \quad \int_s^t \left| \nabla h_{\epsilon^2} - \nabla r_{\Sigma}(\sigma_x(u)) \right| \, du \leq C_{\beta,p} \epsilon \sqrt{t-s}. \]

4.4. Effective control of the geodesic spreading. In this subsection, we prove the desired estimate of the distance between two integral curves of \( \nabla r_{\Sigma} \) in Theorem 1.3. This relies on the existence of some subset that remains (up to certain time) close to a given flow line of \( \nabla r_{\Sigma} \). The definition of such a set is due to Colding and Naber [22]. While our argument mimics the original one in [22, §3], it is much simplified thanks to [22] Proposition 3.6 and Corollary 3.7. In fact, [22 Proposition 3.6] is the major technical input in Colding and Naber’s work, utilizing all estimates obtained from the parabolic approximation in [22, §2] — the proof of Theorem 1.3 not just borrows from Colding and Naber’s arguments, but also relies on their results.

As in the last subsection, we consider a closed embedded submanifold \( \Sigma \subset M \). Fixing any \( q \in M \setminus \Sigma \), we let \( \sigma \) be a unit-speed minimal geodesic realizing \( r_{\Sigma}(q) = l \), and let \( q := \sigma(l) \) and \( p := \sigma(0) \in \Sigma \). Since \( \sigma \) is a minimal geodesic connecting its two end points \( p \) and \( q \), we could apply [22 Proposition 3.6 and Corollary 3.7] to \( \sigma \) and see

Lemma 4.8 (Interior volume comparison). Suppose \( \frac{1}{2} \leq l \leq D \), then there exist positive constants \( \epsilon_{CN}(m, D, \beta) < 1 \) and \( r_{CN}(m, D, \beta) < 1 \) such that if \( s, t \in [\beta l, (1-\beta)l] \) satisfy \( |s-t| < \epsilon_{CN} l \), then for any \( r \in (0, r_{CN}] \),

\[
(4.18) \quad \frac{1}{2} \leq \frac{B_\epsilon(\sigma(s), r)}{B_\epsilon(\sigma(t), r)} \leq 2.
\]

Remark 2. In [22, §3], this result is proven for \( l = 1 \), and the constants there only depend on the dimension. When \( l \approx D > 1 \), the constants \( r_{CN} \) and \( \epsilon_{CN} \) are affected by \( C_{F_n}(D) \) in the Laplacian comparison, while the lower bound \( l \geq \frac{1}{2} \) is required essentially due to (4.16).

We now make the notation for the scale \( \tilde{r}_0(m, D, \beta) := \frac{1}{4} \min \left\{ 10^{-2} \beta, r_{\beta,p}, r_{CN} \right\} \), and define for any \( r \in (0, \tilde{r}_0] \) the subset

\[
(4.19) \quad \mathcal{A}_{\epsilon}(\sigma, r) := \left\{ z \in B_\epsilon(\sigma(t), r) : \forall u \in [0, sl], \psi_{\epsilon u}(z) \in B_\epsilon(\sigma(t+u), 2r) \right\}.
\]

Clearly, \( \mathcal{A}_{\epsilon}(\sigma, r) = B_\epsilon(\sigma(t), r) \), since \( \psi_0 \) is the identity map; also notice that when \( r, s > 0 \) are very small, \( H_{\epsilon u}(\sigma) \in \mathcal{A}_{\epsilon}(\sigma, r) \) by (4.8). We also let \( \chi_{\sigma(t)}^{sl} \) be the characteristic function of \( \mathcal{A}_{\epsilon}(\sigma, r) \times \mathcal{A}_{\epsilon}(\sigma, r) \) in \( B_\epsilon(\sigma(t), r) \times B_\epsilon(\sigma(t), r) \), then for any \( s \in [0, l - \beta l - t] \) and \( \eta \in (0, 10^{-2}) \), we define quantities

\[
(4.20) \quad F_{\sigma}(x, y; s) := \int_0^s \chi_{\sigma(t)}^{ul}(x, y) \left( \int_{[\phi_{\epsilon u}(x), \phi_{\epsilon u}(y)]} \left| \text{Hess} h_{2,\epsilon} \right| \right) \, du,
\]

and

\[
(4.21) \quad I_{\sigma}(x, y) := \int_{B_\epsilon(\sigma(t), r) 	imes B_\epsilon(\sigma(t), r)} F_{\sigma}(x, y; s) \, dV_\epsilon(x) \, dV_\epsilon(y),
\]
where the constant $c^2 \in [\frac{1}{4}, 2]$ depends on $r \leq r_{Ap,l}$ and is guaranteed to exist by Proposition 4.7.

We also define the subsets (notice that we omit writing the dependence on $t \in [\beta l, (1 - \beta)l]$)

$$(4.22) \quad T_{\eta,s}(\sigma) := \left\{ x \in B_g(\sigma(t), r) : e^x(x) \leq \frac{C_E r^2}{\eta l}, \int_{B_g(\sigma(t), r)} F'_{\sigma}(x, y; s) dV_g(y) \leq \frac{I'_l(\sigma, r)}{\eta} \right\},$$

and for each $x \in T_{\eta,s}(\sigma)$ we define

$$(4.23) \quad T_{\eta,s}(\sigma, x) := \left\{ y \in B_g(\sigma(t), r) : e^y(y) \leq \frac{C_E r^2}{\eta l}, \int_{B_g(\sigma(t), r)} F'_{\sigma}(x, y; s) \leq \frac{I'_l(\sigma, r)}{\eta^2} \right\}. $$

By the average excess function estimate in Lemma 4.6 applied to $B_g(\sigma(t), r)$ and Chebyshev’s inequality, we have

$$(4.24) \quad \frac{|T_{\eta,s}(\sigma)|}{|B_g(\sigma(t), r)|} \geq 1 - 2\eta, \quad \forall x \in T_{\eta,s}(\sigma), \quad \frac{|T_{\eta,s}(\sigma, x)|}{|B_g(\sigma(t), r)|} \geq 1 - 2\eta.$$

Notice that these estimates are uniform, and we would like to first understand how the analytic conditions defining $T_{\eta,s}(\sigma)$ could affect the spreading of the flow lines of $\nabla r_g$:

**Lemma 4.9** (Effective distance estimate). Fix $\eta \in (0, 10^{-2})$ and $D \geq 1$ such that $\frac{1}{4} \leq l \leq D$, then there are constants $C_0(m, D, \beta) > 0$ and $\epsilon_0(m, D, \beta, \eta) \in (0, 1)$ such that for any $s \in [0, \epsilon_0]$, and any $r \in (0, \bar{r}_0]$, every pair of smooth points $x_1 \in T_{\eta,s}(\sigma)$ and $x_2 \in T_{\eta,s}(\sigma, x) \cap \mathcal{A}_s(\sigma, \xi_r)$ (where we set the notation $\xi := \frac{l}{2}$),

$$(4.25) \quad \left| d_g\left(\psi^\Sigma_{\epsilon(1)}(x_1), \psi^\Sigma_{\epsilon}(x_2)\right) - d_g(x_1, x_2) \right| \leq C_0 \eta^{-2} r \sqrt{s/l}.$$

Especially, $|T_{\eta,s}(\sigma) \setminus \mathcal{A}_s(\sigma, r)| = 0$ for all $s \in [0, \epsilon_0]$.

**Proof.** Recalling that $\bar{r}_0(m, D, \beta) = \frac{1}{4} \min \left\{ 10^{-2} \beta, r_{Ap,l}, r_{CN} \right\}$, we fix for any $r \in (0, \bar{r}_0]$ a smooth point $x_1 \in T_{\eta,s}(\sigma)$ and denote

$$\epsilon(1) := \sup \left\{ s \leq l - \beta l - t : \forall u \in [0, s], \psi^\Sigma_{\epsilon}(x_1) \in B_g(\sigma(t + u), 2r) \right\}.$$

Without loss of generality, we may assume that $\epsilon(1) \leq \epsilon_{CN}(m, D, \beta)$. Clearly, when $s \leq \epsilon(1)$, $x_1 \in \mathcal{A}_s(\sigma, r)$; moreover, $\mathcal{A}_s(\sigma, \xi_r) \subset \mathcal{A}_s(\sigma, r)$. We want to understand how $d_g(\psi^\Sigma_{\epsilon}(x_1), \sigma(t + s))$ is controlled by the properties of $T_{\eta,s}(\sigma)$. By the continuity of the mapping $u \mapsto \psi^\Sigma_{\epsilon}(x_1)$ and the maximality of $\epsilon(1)$, we see that

$$(4.26) \quad \psi^\Sigma_{\epsilon(1)}(x_1) \notin B_g(\sigma(t + \epsilon(1)), \frac{3}{2} r).$$

In fact, we will show that $\epsilon(1) \geq \epsilon_0$ for suitably chosen $\epsilon_0$. Fix any $x_2 \in T_{\eta,s}(\sigma, x_1) \cap \mathcal{A}_s(\sigma, r)$ which is also a smooth point of $\nabla r_g$ clearly $\chi^\Sigma_{\epsilon}(x_1, x_2) = 1$ for $s \leq \epsilon(1)$. We let $\sigma_1$ and $\sigma_2$ denote the integral curves of $\nabla r_g$ starting from $x_1$ and $x_2$, respectively. These are smooth geodesics. Since $r \leq \frac{1}{4} r_{Ap,l} \leq r_{Ap,l}$, there is some $c^2 \in [\frac{1}{4}, 2]$ so that (4.16) holds. Now integrating (3.6) in [22, Lemma 3.4] for $s \leq \epsilon(1)$, we have

$$(4.27) \quad \left| d_g(\psi^\Sigma_{\epsilon}(x_1), \psi^\Sigma_{\epsilon}(x_2)) - d_g(x_1, x_2) \right| \leq \int_0^s \left| \nabla h_{\epsilon,\Sigma} - \nabla r_g \right| (\sigma_2(u)) \, du$$

$$+ \int_0^s \left| \nabla h_{\epsilon,\Sigma} - \nabla r_g \right| (\sigma_2(u)) \, du + F'_{\sigma}(x_1, x_2; s).$$
We now estimate each term in the right-hand side of (4.27). By the bound on \( r \), the estimate (4.17) in Proposition 4.7 and the choice of \( x_1 \) and \( x_2 \), we see for \( i = 1, 2 \),

\[
\forall s \in [0, \varepsilon(x_1)], \quad \int_0^s |\nabla h_{z,2,2} - \nabla r_2| (\sigma(u)) \, du \leq \sqrt{2}C_{Ap}rl^{-1} \sqrt{s/l}.
\]

The last term on the right-hand side of (4.27) is by definition bounded by \( \eta^{-2}I'_s(\sigma, r) \). By the segment inequality in [22, Lemma 3.5] and the definition of \( \mathcal{J}_i'(\sigma, r) \), for any \( s \in [0, \varepsilon(x_1)] \) we could estimate \( I'_s(\sigma, r) \) as:

\[
I'_s(\sigma, r) \leq \int_0^s \left( \frac{1}{|B_g(\sigma(t), r)|} \right)^2 \left( \int_{\mathcal{J}_{z,x}'} (Hess_{h_{z,2,2}}) \right) dV_g \, du \\
\leq \int_0^s \left( 10r C_{Seg}(m) \right) \left( \int_{B_g(\sigma(t), r)} \left( Hess_{h_{z,2,2}} \right) dV_g \right) \, du \\
\leq \int_0^s \left( 10r C_{Seg}(m) \right) \left( \int_{B_g(\sigma(t) + u, 2r)} \left( Hess_{h_{z,2,2}} \right) dV_g \right) \, du.
\]

We rely on the Bishop-Gromov volume comparison and Lemma 4.8 to compare \( |B_g(\sigma(t), r)| \) and \( |B_g(\sigma(t) + u, 2r)| \) for \( u \leq \varepsilon(x_1) \); by (4.18) we have

\[
I'_s(\sigma, r) \leq \int_0^s \left( 10r C(m, \tilde{r}_0) \left( \frac{B_g(\sigma(t) + u, r)}{|B_g(\sigma(t), r)|} \right)^2 \right) \left( \int_{B_g(\sigma(t) + u, 5r)} Hess_{h_{z,2,2}} \right) dV_g \, du \\
\leq 40r C(m, \tilde{r}_0) \left( \int_{\beta} Hess_{h_{z,2,2}} \right) \, du \\
\leq 40C(m, \tilde{r}_0) \sqrt{C_{Ap}rl^{-1}} \sqrt{s},
\]

where \( C(m, \tilde{r}_0) \) is the multiple of \( C_{Seg}(m) \) by the doubling constant on the space form of sectional curvature \(-1\), up to scale \( \tilde{r}_0 \), so \( C(m, \tilde{r}_0) \) is ultimately determined by \( m, D \) and \( \beta \).

Now (4.27), (4.28) and (4.29) together imply that for every pair of smooth points \( x_1 \in T_{\eta,1}^r(\sigma) \) and \( x_2 \in T_{\eta,1}^r(\sigma, x_1) \cap \mathcal{J}_i'(\sigma, r) \),

\[
\forall s \in [0, \varepsilon(x_1)], \quad \left| d_g(\psi^x_2(x_1), \psi^x_2(x_2)) - d_g(x_1, x_2) \right| \leq C_0 \eta^{-2} r \sqrt{s/l},
\]

where \( C_0 := 8 \sqrt{2}C_{Ap} + 80C(m, \tilde{r}_0) \sqrt{C_{Ap}} \) only depends on \( m, D \) and \( \beta \); compare Remark 2. In proving this estimate we only needed \( x_1 \in T_{\eta,1}^r(\sigma) \cap \mathcal{J}_i'(\sigma, r) \) and \( x_2 \in T_{\eta,1}^r(\sigma, x_1) \cap \mathcal{J}_i'(\sigma, r) \), and we emphasize that the stronger assumption \( x_2 \in \mathcal{J}_i'(\sigma, \xi r) \) is only used later to bound \( \varepsilon_0 \).

Now we put \( \varepsilon_0 := \min \left\{ \varepsilon_{CN}, \eta^3/(16C_0^2) \right\} \) — notice that \( \varepsilon_0 \) only depends on \( m, D, \beta \) and \( \eta \). Suppose, for the purpose of a contradiction argument, that the inequalities \( \varepsilon(x_1) < \varepsilon_0 \) hold, then since actually \( x_2 \in \mathcal{J}_i'(\sigma, \xi r) \), we have

\[
d_g(\psi^x_2(x_1), \sigma(t + s)) \leq 2 \xi r \leq \frac{r}{10},
\]

whenever \( s \in [0, \varepsilon(x_1)] \), and the triangle inequality implies that

\[
\forall s \in [0, \varepsilon(x_1)], \quad d_g(\psi^x_2(x_1), \sigma(t + s)) \leq \frac{7}{5} r,
\]
contradicting \((4.26)\) at \(s = \varepsilon(x_i)\). Therefore it must hold that \(\varepsilon(x_i) \geq \varepsilon_0 l\), and \((4.28)\) is valid for all \(s \in [0, \varepsilon_0 l]\). Moreover, \((4.31)\) tells that \(x_1 \in \mathcal{A}_l(\sigma, r)\) whenever \(s \leq \varepsilon_0 l\). \(\Box\)

We are now ready to effectively control the spreading, under the diffeomorphisms \(\psi_s^\Sigma\), of the set \(T_{\eta,\delta_0}^r(\sigma)\), for any integral curve \(\sigma\) of \(\nabla r_\Sigma\), and for uniformly controlled \(\varepsilon > 0\) and \(r > 0\).

**Lemma 4.10** (Controlling \(T_{\eta,\delta_0}^r(\sigma)\) under \(\psi_s^\Sigma\)). For the closed embedded submanifold \(\Sigma \subset M\) and for any \(l\) with \(\frac{1}{2} \leq l \leq D\), there is a constant \(\tilde{\varepsilon}_0(m, D, \beta) \in (0, 1)\) such that \(|T_{\eta,\delta_0}^r(\sigma) \setminus \mathcal{A}_l^r(\sigma, r)| = 0\) for any \(r \in [0, \tilde{r}_0]\).

**Proof.** We begin with recalling that by Lemma 4.5, there is a small \(r' = r'(M, \sigma) > 0\) and a core neighborhood specified by \(H_{r'}^*(\sigma)\), a full measure subset of \(B_\delta(\sigma(t), r')\), such that flow lines of \(\nabla r_\Sigma\) initiating from it stay uniformly close to \(\sigma\). Let us now fix this neighborhood of \(\sigma(t)\), which depends on the specific \(M\) and \(\sigma\). Notice that if we set \(\varepsilon_1 := (\ln 2)^2 \beta/(4C_{H}(m, D))^2\), then by the definition of \(H_{r'}^*(\sigma)\) and the proof of Lemma 4.5, we have

\[
\forall s \in [0, \varepsilon_1 l], \forall x \in H_{r'}^*(\sigma),\quad d_g(\psi_s^\Sigma(x), \sigma(t + s)) \leq 2d_g(x, \sigma(t)).
\]

(4.32)

For any \(r \in (0, \tilde{r}_0]\), we set \(r_i := \xi^i r\) for \(i = 0, 1, 2, \ldots, I\), where \(I := \left\lfloor \log_{\xi^I} \frac{\varepsilon_1}{2} \right\rfloor\) is defined to be the first natural number such that \(r_I \leq r'/2\).

We then put \(\tilde{\varepsilon}_0 = \min \{\varepsilon_0, \varepsilon_1\}\) and pick a smooth point \(x_0 \in T_{\eta,\delta_0}^r(\sigma)\) with \(r \leq \tilde{r}_0\). We “connect” it to \(H_{r'}^*(\sigma)\) by selecting \(\{x_i\}_{i=0}^I\) inductively: suppose \(x_i\) is chosen, then pick any smooth point \(x_{i+1} \in T_{\eta,\delta_0}^{r_i}(\sigma, x_i) \cap T_{\eta,\delta_0}^{r_{i+1}}(\sigma)\). This is doable because \((4.24)\) is independent of \(r\) — as long as we choose \(\eta := \min \left\{10^{-2}, \frac{C(m, D, \beta)}{\Lambda_1(m, D, \beta)}\right\}\), where

\[
C(m, D, \beta) := \min \left\{\sup_{r \in [0, \bar{r}]} \frac{\Lambda_1^m(\xi r)}{\Lambda_1^m(r)}, 1\right\}
\]

is determined by \(m, D\) and \(\beta\) — whence the sole dependence of \(\eta\) on \(m, D\) and \(\beta\). We now have

\[
\left|T_{\eta,\delta_0}^{r_i}(\sigma, x_i)\right| + \left|T_{\eta,\delta_0}^{r_{i+1}}(\sigma)\right| \geq (1 - 2\eta) \left(\left|B_\delta(\sigma(t), r_i)\right| + \left|B_\delta(\sigma(t), r_{i+1})\right|\right)
\]

\[
\geq (1 - 2\eta)(1 + C(m, D, \beta)) \left|B_\delta(\sigma(t), r_i)\right|
\]

\[
> \left|B_\delta(\sigma(t), r_i)\right|
\]

i.e. \(T_{\eta,\delta_0}^{r_i}(\sigma, x_i) \cap T_{\eta,\delta_0}^{r_{i+1}}(\sigma)\) has positive measure, and especially there are smooth points of \(r_\Sigma\) in the intersection. We denote by \(\sigma_i\) the integral curve of \(\nabla r_\Sigma\) with initial value \(x_i\). Clearly we could select \(x_i\) so that each \(\sigma_i\) is a smooth geodesics on \([0, \tilde{\varepsilon}_0 l]\).

According to \((4.32)\), \(x_i \in \mathcal{A}_l^r(\sigma, r_i) = \mathcal{A}_l^r(\sigma, \xi r_{i-1})\) whenever \(s \in [0, \varepsilon_1 l]\). Therefore, applying Lemma 4.9 to \(x_{i-1} \in T_{\eta,\delta_0}^{r_{i-1}}(\sigma)\) and \(x_i \in T_{\eta,\delta_0}^{r_{i+1}}(\sigma, x_{i-1}) \cap \mathcal{A}_l^r(\sigma, r_i)\), we could obtain

\[
\forall s \in [0, \tilde{\varepsilon}_0 l],\quad d_g(\psi_s^\Sigma(x_i), \psi_s^\Sigma(x_{i-1})) \leq \left\lfloor \frac{\xi}{4} + \xi \right\rfloor r_{i-1}.
\]
This further implies that for any \( s \leq \bar{\varepsilon}_0l \),
\[
d_g\left(\psi_{x}^\Sigma(x_{l-1}), \sigma(t+s)\right) \leq d_g\left(\psi_{x}^\Sigma(x_i), \sigma(t+s)\right) + \left(\frac{5}{4} + \xi\right) r_{l-1} \\
\leq \left(\frac{5}{4} + 3\xi\right) r_{l-1}.
\]
(4.33)

Especially, \( x_{l-1} \in \mathcal{A}_{\bar{r}_0}(\sigma, r_{l-1}) \).

We could then apply Lemma \[4.9\] to the pair of smooth points \( x_{l-2} \) and \( x_{l-1} \), and conclude that \( x_{l-2} \in \mathcal{A}_{\bar{r}_0}(\sigma, r_{l-2}) \). Repeating the same argument another \( I - 2 \) steps and by the choice of \( \xi = 10^{-2} \), we get for any \( s \leq \bar{\varepsilon}_0l \),
\[
d_g\left(\psi_{x}^\Sigma(x_0), \sigma(t+s)\right) \leq d_g\left(\psi_{x}^\Sigma(x_i), \sigma(t+s)\right) + \left(\frac{5}{4} + \xi\right) r \sum_{i=0}^{l-1} \xi^i \\
< 2r.
\]

Especially, this implies that \( x_0 \in \mathcal{A}_{\bar{r}_0}(\sigma, r) \). But since the collection of smooth points of \( r_\Sigma \) is a full measure subset of \( M \), we have \(|T^r_{\eta,\bar{r}_0}(\sigma) \setminus \mathcal{A}_{\bar{r}_0}(\sigma, r)| = 0 \) for any \( r \in (0, \bar{r}_0] \).

We could now control the distance of two minimal geodesics emanating from closely parallel initial data along \( \Sigma \), as promised in Theorem \[1.3\].

**Proof of Theorem \[1.3\]** Let fix some \( \theta' \in (0, 1) \) as the largest number such that
\[
\forall r \in [0, \bar{r}_0], \quad \frac{\Lambda_{-1}^m((1 - \theta')r)}{\Lambda_{-1}^m((1 + \theta')r)} \geq \frac{3}{4}.
\]
Notice that by the continuity of \( \Lambda_{-1}^m(s) \) in \( s \), such \( \theta' \) exists, and is determined by \( m, D \) and \( \beta \), due to the dependence of \( \bar{r}_0 \) on these parameters.

Now for any pair of flow lines \( \sigma_0 \) and \( \sigma_1 \) of \( \nabla r_\Sigma \), with parallel initial data along \( \Sigma \) such that \( d_g^\Sigma(\sigma_0(t), \sigma_1(t)) \leq 2\theta' \bar{r}_0 \), let us put \( r := (2\theta')^{-1} d_g^\Sigma(\sigma_0(t), \sigma_1(t)) \), and let \( A := B_g(\sigma_0(t), r) \cap B_g(\sigma_1(t), r) \) denote the intersection. For \( i = 0, 1 \), the volume comparison tells that
\[
|A| \geq \frac{3}{4} |B_g(\sigma_i(t), r)|.
\]
On the other hand, by (4.24) we have for each \( i = 0, 1 \),
\[
|T^r_{\eta,\bar{r}_0}(\sigma_i) \cap A| \geq \left(\frac{3}{4} - 2\eta\right) |B_g(\sigma_i(t), r)|.
\]
Consequently, by the assumption \( \eta \leq 10^{-2} \) we have
\[
|T^r_{\eta,\bar{r}_0}(\sigma_0) \cap T^r_{\eta,\bar{r}_0}(\sigma_1)| \geq \frac{1}{4} |A| > 0.
\]
(4.34)

By Lemma \[4.10\] we know that for \( i = 0, 1 \), \( \left|T^r_{\eta,\bar{r}_0}(\sigma_i) \setminus \mathcal{A}_{\bar{r}_0}(\sigma_i, r)\right| = 0 \), and thus by the definition (4.19) of \( \mathcal{A}_{\bar{r}_0}(\sigma_i, r) \), we have
\[
\forall s \in [0, \bar{\varepsilon}_0l], \quad \left|\psi_{x}^\Sigma(T^r_{\eta,\bar{r}_0}(\sigma_i)) \setminus B_g(\sigma_i(t + s), 2r)\right| = 0 \quad \text{for} \quad i = 0, 1.
\]
(4.35)
Since both $\sigma_0$ and $\sigma_1$ are integral curves of $\nabla_{\xi}$, which is smoothly defined almost everywhere on $B_g(\sigma_0(t), r) \cup B_g(\sigma_1(t), r)$ — the very reason that we reworked Colding and Naber’s original proof to suit $\nabla_{\xi}$ — we have
\[
\forall s \in [0, \bar{\varepsilon}_0], \quad \psi_s^\xi(T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_0) \cap T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_1)) \subseteq \psi_s^\xi(T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_0) \cap T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_1)).
\]
Especially, by (4.34) and (4.35) we clearly see that
\[
\forall s \in [0, \bar{\varepsilon}_0], \quad \left| B_g(\sigma_0(t+s), 2r) \cap B_g(\sigma_1(t+s), 2r) \right| \geq \left| \psi_s^\xi(T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_0) \cap T_{\eta, \bar{\varepsilon}_0}^\rho(\sigma_1)) \right| > 0.
\]
Consequently, we could give the distance bound for the geodesics
\[
(4.36) \quad \forall s \in [0, \bar{\varepsilon}_0], \quad d_g(\sigma_0(t+s), \sigma_1(t+s)) \leq 4(2\theta)^{-1} d_g(\sigma_0(t), \sigma_1(t)).
\]
Therefore, we could start from $t = \beta l$ and iterate the above estimate along $\sigma_0$ and $\sigma_1$, to see that as long as $d_g(\sigma_0(\beta l), \sigma_1(\beta l)) \leq (2^{-1}\theta)^{1+2\bar{\varepsilon}_0^{-1}} \bar{r}_0 =: \bar{r}$, then
\[
(4.37) \quad \forall t \in [\beta l, (1-\beta)l], \quad d_g(\sigma_0(t), \sigma_1(t)) \leq \bar{C} d_g(\sigma_0(\beta l), \sigma_1(\beta l)),
\]
where $\bar{C} := (2^{-1}\theta)^{1+2\bar{\varepsilon}_0^{-1}}$ and $\bar{r} \in (0, 1)$ clearly only depend on $m, D$ and $\beta$.

**Remark 3.** With suitable controls on the sectional curvature and the second fundamental form of $\Sigma$ around $\sigma_0(0)$ and $\sigma_1(0)$, we may find some uniform $\bar{C} > 0$ depending on these data, such that
\[
d_g(\sigma_0(t), \sigma_1(t)) \leq \bar{C} d_g(\sigma_0(0), \sigma_1(0)).
\]

5. The first betti numbers and dimensional difference

With the heuristic discussion in §2 and technical preparation in §3 and §4, we are now ready to fill in the details in proving the first claim of Theorem 1.1. We assume that $(M, g) \in M_{Rm}(m)$ and $(N, h) \in M_{Rm}(k, D, v)$ (with $k \leq m$) satisfy $d_{GH}(M, N) \leq 10^{-1}\delta$ for some $\delta \in (0, 1)$, and our task would be to determine the range of $\delta$ uniformly according to $m, D$ and $v$, so that the first claim of Theorem 1.1 holds. We also let $\Phi : M \to N$ denote a $10^{-1}\delta$-Gromov-Hausdorff approximation.

Since we have known that $b_1(M) - b_1(N) = \text{rank } H_1^\rho(M; \mathbb{Z}) =: l_M$ whenever $\delta < 10^{-2}\bar{\varepsilon}_0(k, D, v)$, as mentioned before, we would like to “localize” the torsion-free generators of $H_1^\rho(M; \mathbb{Z})$ to each point $p \in M$, as torsion-free generators of $\tilde{\Gamma}_\rho^\eta(p)$. By the Hurewicz theorem, we could find a total number of $b_1(M)$ geodesic loops whose homology classes generate $H_1(M; \mathbb{Z})/\text{Torsion}$. By Proposition 3.7, we know that exactly $b_1(N)$ of these loops are of lengths at least $10^{-1}\bar{\varepsilon}_0 r$, and the rest of these loops have lengths not exceeding $10\delta$. Since $H_1^\rho(M; \mathbb{Z})$ is a subgroup of $H_1(M; \mathbb{Z})$ and $\text{rank } H_1^\rho(M; \mathbb{Z}) = b_1(M) - b_1(N)$, we know that there are geodesic loops $\gamma'_1, \ldots, \gamma'_{l_M}$ in $M$, with lengths not exceeding $10\delta$, and that $[\gamma'_1], \ldots, [\gamma'_{l_M}]$ generate $H_1^\rho(M; \mathbb{Z})/\text{Torsion}$.

For each $i = 1, \ldots, l_M$, we now let $\gamma_i$ be a length minimizer in the free homotopy class of $\gamma'_i$. Clearly, $[\gamma_i] = [\gamma'_i] \in H_1(M; \mathbb{Z})$, and $|\gamma_i| \leq 10\delta$. Moreover, each $\gamma_i : [0, 1] \to M$ is a closed geodesic. Letting $\pi : \widetilde{M} \to M$ denote the universal covering and equipping $\widetilde{M}$ with the covering metric $\pi^*g$, we see that each $\gamma_i$ lifts to a complete geodesic $\tilde{\gamma}_i$ in $\widetilde{M}$. We denote $\Sigma_i := \tilde{\gamma}_i(\mathbb{R})$, which is clearly a closed embedded smooth submanifold of $\widetilde{M}$. Also each $\gamma_i$ acts on $(\widetilde{M}, \pi^*g)$ isometrically, while restricting to a translation along $\Sigma_i$ by distance $|\gamma_i| \leq 10\delta$.

We now fix an arbitrary $p \in M$, and for each $i = 1, \ldots, l_M$, let $\sigma_i : [0, d_i] \to M$ be a unit speed minimal geodesic that realizes $d_g(p, \gamma_i([0, 1])) =: d_i$, with $\sigma_i(0) = \gamma_i(t_i)$ for some $t_i \in [0, 1]$ and
\[ \sigma_i(d_i) = p. \] Clearly \( \sigma_i(0) \perp \gamma_i(t_i) \). Fixing some \( \tilde{p} \in \pi^{-1}(p) \) in the universal covering space \( \tilde{M} \), we could uniquely lift each \( \sigma_i \) \((i = 1, \ldots, l_M)\) to a minimal geodesic \( \tilde{\sigma}_i : [0, d_i] \to \tilde{M} \) of unit speed with \( \tilde{\sigma}(d_i) = \tilde{p} \). Clearly \( \tilde{\sigma}_i(0) \in \Sigma_i \) and we could parametrize \( \tilde{\gamma}_i : \mathbb{R} \to \tilde{M} \) so that \( \tilde{\gamma}_i(t_i) = \tilde{\sigma}_i(0) =: \tilde{q}_i \), and that \( \tilde{\sigma}_i(0) \perp \Sigma_i \).

Notice that for each \( i = 1, \ldots, l_M \), the isometric action \( \gamma_i \) sends \( \tilde{\sigma}_i \) to another minimal geodesic \( \gamma_i \tilde{\sigma}_i \) in \( \tilde{M} \), which realizes the distance \( d_{\pi^g}(\gamma_i \tilde{p}, \Sigma_i) = d_{\pi^g}(\gamma_i \tilde{p}, \gamma_i \tilde{q}_i) = d_i \). Moreover, we have \( (\gamma_i' \tilde{\sigma}_i)'(0) \perp \tilde{\gamma}_i(t_i + n|\gamma_i|) \in T_{\tilde{\gamma}_i(0)} \Sigma_i \) for any \( n \in \mathbb{Z} \). Now we would like to estimate \( d_{\pi^g}(\gamma_i' \tilde{p}, \tilde{p}) \) for a suitable positive power \( n \).

We assume \( \delta < \delta_1 \) with \( \delta_1 := \min \{10^{-1} \Psi_{NZ}(\delta_{Nil}, 1, m), \tilde{r} \}, 1, m \}, 10^{-2} \bar{t}_{hu}(k, D, v) \}, \) where the uniform constants \( \tilde{C}(m, 2D, \beta) > 1 \) and \( \tilde{r}(m, 2D, \beta) \in (0, 1) \) are obtained from Theorem 1.3 by setting \( \beta := \min \{10^{-3}, (4D)^{-1} \} \), the uniform constant \( \Psi_{NZ}(\varepsilon, 1, m) \in (0, \varepsilon) \) is obtained from [47] Lemma 5.2 for any \( \varepsilon \in (0, 1) \) given, and the uniform constants \( \bar{t}_{hu}(m, D, v) \) and \( \delta_{Nil}(m, D, v) \) in (0, 1) are determined in Lemma 2.2.

If \( d_i < 1 \), since \( |\gamma_i| \leq 10\delta < \Psi_{NZ}(\min \{\tilde{C}(\tilde{C}^{-1} \Psi_{NZ}(\delta_{Nil}, 1, m), \tilde{r} \}, 1, m \}, 10^{-2} \bar{t}_{hu}(k, D, v) \} \), by [47] Lemma 5.2 we know that there is some uniform \( N_{NZ}(m, D, \varepsilon) \in \mathbb{N} \) such that for some \( k_i \leq N_{NZ} \),

\[
(5.1) \quad d_{\pi^g}(\gamma_i^k \tilde{\sigma}_i(d_i), \tilde{\sigma}_i(d_i)) \leq \tilde{C}^{-1} \Psi_{NZ}(\delta_{Nil}, 1, m) < \delta_{Nil}.
\]

Then obviously \( \gamma_i \in \tilde{G}_{\delta_{Nil}}(p) \) by definition.

If \( d_i \geq 1 \) instead, we will apply Theorem 1.3 to \( (\tilde{M}, \pi^g) \), with \( \Sigma_i := \tilde{\gamma}_i \). Since \( \beta d_i \leq \beta D < \frac{1}{2} \), from the previous case we have some \( k_i \leq N_{NZ} \) so that

\[
(5.2) \quad d_{\pi^g}(\gamma_i^k \tilde{\sigma}_i((1 - \beta)d_i), \tilde{\sigma}_i((1 - \beta)d_i)) \leq \Psi_{NZ}(\delta_{Nil}, 1, m).
\]

Now since \( d_{\pi^g}(\tilde{p}, \tilde{\sigma}_i((1 - \beta)d_i)) = |\tilde{\sigma}_i|_{d((1 - \beta)d_i)} = \beta d_i < 1 \), we could apply [47] Lemma 5.2 again to see that for some \( k_i' \leq N_{NZ} \),

\[
(5.3) \quad d_{\pi^g}(\gamma_i^{k_i'} \tilde{p}, \tilde{p}) = d_{\pi^g}(\gamma_i^{k_i'} \tilde{\sigma}_i(d_i), \tilde{\sigma}_i(d_i)) \leq \delta_{Nil}.
\]

This shows that \( \gamma_i^{k_i'} \in \tilde{G}_{\delta_{Nil}}(p) \) for each \( i = 1, \ldots, l_M \).

We now connect \( \tilde{p} \) to each \( \gamma_i \) by a minimal geodesic \( \gamma_{i,p} \). Since the curve

\[
\tilde{\sigma}_i^{-1} \ast \tilde{\gamma}_i |_{[t_0, t_0 + k_i']} \ast \left( \gamma_i^{k_i'} \tilde{\sigma}_i \right) \ast \left( \gamma_{i,p} \right)^{-1}
\]

is actually a loop based at \( \tilde{p} \) in \( \tilde{M} \), it is null homotopic in \( \tilde{M} \) as \( \pi_1(\tilde{M}, \tilde{p}) = 0 \). Consequently, the loop \( \gamma_{i,p} := \pi \circ \tilde{\gamma}_{i,p} \) in \( M \) is based at \( p \in M \) and is free homotopic to the loop \( \gamma_i^{k_i'} := \pi \circ \tilde{\gamma}_i |_{[t_0, t_0 + k_i'] |_{\gamma_i}} \), along the curve \( \sigma_i \). Especially, \( \| \gamma_{i,p} \| = k_i' \| \gamma_i \| \in H_1(M; \mathbb{Z}) \). On the other hand, by the estimate \( |\tilde{\gamma}_{i,p}| = d_{\pi^g}(\gamma_i^{k_i'} \tilde{p}, \tilde{p}) \leq \delta_{Nil} \), we know that \( [\gamma_{i,p}] \in \tilde{G}_{\delta_{Nil}}(p) \leq \delta_{Nil} \). The loop \( \gamma_{i,p} \) is the “slided” loop of \( \gamma_i^{k_i'} \) to \( p \in M \), as mentioned in the introduction.
We could therefore regard evolved metric is solvable up to some Shi’s estimate blows out of control when time depending on specific $(M, g)$, for $j = 1, \ldots, l$; but for $i = 1, \ldots, l_M$, letting $k_i \in \mathbb{Z}$ denote the (oriented) number of copies $\gamma_{i,p}$ appeared in the above vanishing homotopic equation, we have the corresponding homological relation
\[
\begin{align*}
\delta d_1[\gamma_{1,p}] + \cdots + \delta d_l[\gamma_{l_M,p}] &= \tilde{k}_1[k_1'][\gamma_{1,p}] + \cdots + \tilde{k}_l[k_l'][\gamma_{l_M,p}] \in \text{Tor}(H_1(M; \mathbb{Z})),
\end{align*}
\]
and this contradicts our choice of the homology classes $[\gamma_{1,p}], \ldots, [\gamma_{l_M,p}]$ as a minimal set of generators of $H_1^0(M; \mathbb{Z})/\text{Torsion}$, unless $\tilde{k}_1 = \cdots = \tilde{k}_l = 0$. Therefore, when $\delta < \delta_1$ we have shown that $\text{rank} \ G_{\delta_n}(p) \geq l_M = \text{rank} H_1^0(M; \mathbb{Z})$, and consequently, by Lemma 2.2 and Proposition 3.7 we have shown the first claim of Theorem 1.1 $m - n \geq b_1(M) - b_1(N)$.

Remark 4. Since $G_{\delta_n}(p) \leq \pi_1(M, \mathbb{Z})$, we could also abelianize $G_{\delta_n}(p)$ to obtain a subgroup of $H_1(M; \mathbb{Z})$, according to the Hurewicz theorem. In fact, the above argument defines an injective group homomorphism

$\varphi_p : \bigoplus_{i=1}^{l_M} \mathbb{Z}[\gamma_i] \to G_{\delta_n}(p)/([\pi_1(M, p), \pi_1(M, p)] \cap G_{\delta_n}(p)),$

sending $[\gamma_i]$ to $[\gamma_i]^{k_i'} \cdot ([\pi_1(M, p), \pi_1(M, p)] \cap G_{\delta_n}(p))$. Here $\bigoplus_{i=1}^{l_M} \mathbb{Z}[\gamma_i] \leq H_1^0(M; \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $l_M$. Consequently, we see that the abelianization of $G_{\delta_n}(p)$, as a finitely generated abelian group, has rank at least $l_M$.

6. Ricci flow smoothing and the rigidity case

In this section we prove Theorem 1.4 and consequently the equality case of Theorem 1.1. Throughout this section we assume that $(M, g) \in M\mu_k(m)$ and $(N, h) \in M\mu_{k_1}(k, D, \nu)$ with $k \leq m$, and that $d_{GH}(M, N) \leq \delta$ for some $\delta \in (0, 10^{-1}\delta_1)$. We have shown that $b_1(M) - b_1(N) \leq m - k$, and in this section we further assume that $b_1(M) - b_1(N) = m - k$. We would like to find some positive $\delta_R \leq 10^{-1}\delta_1$ so that the conclusion of Theorem 1.4 holds, and another positive $\delta_B \leq \delta_R$ so that when $\delta \leq \delta_B$, we could see $M$ as a $T^{m-k}_\mu$-bundle over $N$.

6.1. Starting Ricci flows with collapsing initial data. Since $(M, g)$ is a complete manifold and $\text{diam}(M, g) \leq D + 2\delta < \infty$, we know that $M$ has to be a closed manifold. Therefore, we could appeal to Hamilton’s short-time existence result in [30] to see that the initial value problem to the Ricci flow equation

$$
\begin{cases}
\frac{\partial t}{\partial g(t)} = -2\text{Rc}_{g(t)} & \text{when } t \geq 0, \\
g(0) = g,
\end{cases}
$$

is solvable up to some positive time depending on specific $(M, g)$. By Shi’s estimate [52], the evolved metric $g(t)$ has much improved regularity for any $t > 0$ fixed:

$$
\forall t \in \mathbb{N}, \quad \sup_M |\nabla' Rm_{g(t)}|_{g(t)} \leq C_S(t)^{-1}. \quad \text{(5.1)}
$$

We could therefore regard $g(t)$ as a “smoothing metric” to the original metric $g = g(0)$. However, Shi’s estimate blows out of control when $t \searrow 0$, and if one would like to find a smoothing metric
with uniform regularity control, a uniform lower bound on the existence time of the Ricci flow is desired. Here the uniformity refers to the dependence on the data \( m, D \) and \( v \).

A typical approach in obtaining a uniform lower bound on the maximal existence time of a Ricci flow solution is to rely on Perelman’s pseudo-locality theorem (see [48, Theorem 10.1]), whose proof in the complete non-compact setting could be found in [10, §8]. Since the initial data we consider have a uniform Ricci curvature lower bound, we will invoke the version of the pseudo-locality theorem due to Tian and the second named author ([56, Proposition 3.3]):

**Proposition 6.1** (Pseudo-locality for Ricci flows). For any \( \alpha \in (0, 10^{-2}m^{-1}) \), there are positive constants \( \delta_p = \delta_p(m, \alpha) < 1 \) and \( \varepsilon_p = \varepsilon_p(m, \alpha) < 1 \), such that for any \( m \)-dimensional complete non-compact Ricci flow solution \((M, g(t))\) defined on \( t \in [0, T) \), if each time slice has bounded sectional curvature, then for any \( x \in M \) satisfying

\[
\begin{align*}
\text{Rc}_{g(0)} & \geq -\delta_p^2(m-1)g(0) \text{ on } B_{g(0)}(x, \delta_p^{-1}) \quad \text{and} \quad \delta_p^m B_{g(0)}(x, \delta_p^{-1})_{g(0)} \geq (1 - \delta_p)\omega_m,
\end{align*}
\]

we have the following curvature bound for any \( t \in [0, T) \cap (0, \varepsilon_p^{-2}) \):

\[
\left| Rm_{g(t)} \right|_{g(t)}(x) \leq \alpha t^{-1} + \varepsilon_p^{-2}.
\]

**Proof.** Notice that the assumed initial Ricci curvature lower bound in (6.1) implies an initial scalar curvature lower bound. Therefore, if the theorem were to fail, then the proof of [10, Theorem 8.1] provides a contradicting sequence which validates [56, (15) and (16)]. Notice that up to this stage, the assumption on the isoperimetric constant in [10, Theorem 8.1] has never been used. Starting from [56, (15) and (16)], the rest of the proof of [56, Proposition 3.1] goes through verbatim, producing a contradiction and concluding the proof. \( \Box \)

For those initial data satisfying the assumptions of Proposition 6.1 at every point, we could obtain the uniform existence time lower bound by a contradiction argument: were the existence time \( T \) of the Ricci flow shorter than \( \varepsilon_p^{-2} \), then for some sequence \( t_i \not\to T \) we could observe points \( x_i \in M \) such that \( \lim_{t_i \to T} \left| Rm_{g(t_i)} \right|_{g(t_i)}(x_i) \to \infty \); especially, we will get \( \left| Rm_{g(t_i)} \right|_{g(t_i)}(x_i) > 2\alpha T^{-1} + \varepsilon_p^{-2} \) for all \( i \) large enough, contradicting the conclusion (6.2) since \( T > 0 \) is fixed.

In the setting of Theorem 1.4, however, we could not directly apply the pseudo-locality theorem to the Ricci flow obtained from Hamilton’s short-time existence result [30], as the almost Euclidean volume ratio assumption in (6.1) may fail drastically for the initial data \((M, g)\) in our consideration. In order to overcome this difficulty, we will pull the initial metric back to the universal covering space, which could be shown to be non-collapsing under the assumption \( b_1(M) - b_1(N) = \dim M - \dim N \). By Colding’s volume continuity theorem [21], we then expect to improve the lower bound for the volume ratio of the universal covering space:

**Lemma 6.2** (Almost Euclidean condition for the universal covering space). For any \( \varepsilon \in (0, 1) \) fixed, there are \( \delta_{AE} \in (0, 1) \) and \( r_{AE} \in (0, 1) \), solely determined by \( \varepsilon, m, D \) and \( v \), to the following effect: if \((M, g) \in \mathcal{M}_{R_{AE}}(m)\) and \((N, h) \in \mathcal{M}_{R_{AE}}(k, D, v)\) with \( k \leq m \) satisfy

1. \( d_{GH}(M, N) < \delta \) for some \( \delta \leq \delta_{AE} \), and
2. \( b_1(M) - b_1(N) = m - k \),

then for any \( r \in (0, r_{AE}) \) and \( \tilde{p} \in \widetilde{M} \) we have

\[
\left| B_{\sigma_r \tilde{g}}(\tilde{p}, r) \right|_{\sigma_r \tilde{g}} \geq (1 - \varepsilon)\omega_m r^m,
\]
where $\pi : \widetilde{M} \to M$ is the universal covering and we equip $\widetilde{M}$ with the covering metric $\pi^*g$.

Proof. Fixing $\varepsilon \in (0, 1)$, we let $r_1 = r_1(\varepsilon) \in (0, 1)$ be the constant such that

$$\forall r \in (0, r_1), \quad (1 - 10^{-3} \varepsilon)\omega_mr^m \leq V^m_{\varepsilon^{-1}}(r) \leq (1 + 10^{-3} \varepsilon)\omega_dr^d,$$

where $V^m_{\varepsilon^{-1}}(r)$ is the volume of geodesic $r$-ball in the space form of sectional curvature equal to $-1$.

By Colding's volume continuity theorem, [21 Main Lemma 2.1], we obtain the corresponding positive constants $\delta_C = \delta_C(10^{-3} \varepsilon) < 1$, $\Lambda_C = \Lambda_C(10^{-3} \varepsilon) < 1$ and $R_C = R_C(10^{-3} \varepsilon) > 1$ for $10^{-3} \varepsilon$. We then put $\varepsilon' := r_1 \Lambda_C \delta_C R_C^{-1}$ in [47, Proposition 5.4] to obtain some uniform positive constant $\delta_{NZ}(\varepsilon') < 1$ and $r' := r_{NZ}(\varepsilon') \in (\delta_{NZ}(\varepsilon'), 1)$. On the other hand, by the uniform $C^1, \frac{1}{2}$ harmonic radius lower bound for manifold $(N, h) \in \mathcal{M}_{Nm}(k, D, \nu)$, there is some uniform constant $\bar{\delta}_1(m, \varepsilon', \max_{0 \leq k \leq m} C_{hr}(k, D, \nu)) \in (0, \bar{i}_0)$ such that

$$\forall \tilde{p} \in N, \quad d_{GH}(B_{\bar{\delta}_1}(\tilde{p}, \bar{i}_1), \mathbb{E}^k(\bar{i}_1)) < 10^{-1} \lambda_1 \bar{i}_1,$$

where $\lambda_1 := 2^{-1} \min \{\delta_{NZ}(\varepsilon'), \epsilon_{NZ}(m)\}$ — notice that $\lambda_1 \bar{i}_1 \leq \delta_{Nil}$. Following the proof of Claim (1) of Theorem [1.1] in the last section, we define (compare the definition of $\delta_1$ there)

$$\delta_{AE} := 10^{-1} \min \{10^{-1} \Psi_{NZ} \left( \min \{\bar{C}^{-1} \Psi_{NZ}(2 \lambda_1 \bar{i}_1, 1, m), \bar{r'} \}, 1, m \}, \lambda_1 \bar{i}_1, 10^{-3} \bar{i}_{hr} \} \in (0, 1)$$

where $\bar{C}(m, D, \beta) > 1$ and $\bar{r}(m, D, \beta) \in (0, 1)$ are the uniform constants obtained from Theorem [1.5] by setting $\beta = \min \{10^{-3}, (4D)^{-1} \}$, and as before, $\Psi_{NZ}$ is obtained from [47, Lemma 5.2].

If $(M, g)$ and $(N, h)$ satisfy (1) and (2) in the assumption with the $\delta_{AE}$ just defined, we first understand the implication of the (2) on the nilpotency rank of the pseudo-local fundamental group at each point of $M$. It is easily seen that the estimates [5.1], [5.2] and [5.3] hold with $2\lambda_1 \bar{i}_1$ in place of $2\delta_{Nil}$ — for any $p \in M$ and any $\tilde{p} \in \pi^{-1}(p)$, we have

$$d_{\pi^*g}(\gamma_i^{k''}, \tilde{p}, \bar{i}_1) \leq 2 \lambda_1 \bar{i}_1 \leq 2 \delta_{Nil}$$

for each $\gamma_i$ ($i = 1, \ldots, b_1(M) - b_1(N)$) obtained there as a torsion-free short generator of $H^1(M; \mathbb{Z})$, with some $k'' \leq 2N_{NZ}(m, D, t)$. In particular, $\gamma_i^{k''} \in \widetilde{G}_{\lambda_1 \bar{i}_1}(p)$ for each $i = 1, \ldots, b_1(M) - b_1(N)$. Lemma [2.2], Proposition [3.7] and the proof of Claim (1) in Theorem [1.1] then lead to

$$b_1(M) - b_1(N) = H^1(M; \mathbb{Z}) \leq \text{rank} \ G_{\lambda_1 \bar{i}_1}(p) \leq \text{rank} \ G_{\delta_{Nil}}(p) \leq m - k,$$

while assumption (2) forces

$$\text{rank} \ G_{\lambda_1 \bar{i}_1}(p) = m - k.$$

We now examine the effect of further assuming (1). By the choice of $\delta_{AE}$ and $\bar{i}_1$, we have

$$\forall p \in M, \quad d_{GH}(B_{\bar{\delta}_1}(p, \bar{i}_1), \mathbb{E}^k(\bar{i}_1)) \leq d_{GH}(M, N) + d_{GH}(B_{\bar{\delta}_1}(\Phi(p), \bar{i}_1), \mathbb{E}^k(\bar{i}_1)) \leq \frac{2}{5} \lambda_1 \bar{i}_1.$$

Moreover, performing the rescaling $g \mapsto 4\bar{i}_1^{-2}g =: \bar{g}_1$, we see that

$$\forall p \in M, \quad d_{GH}(B_{\bar{\delta}_1}(p, 2), \mathbb{E}^k(2)) < 2 \lambda_1.$$

On the other hand, fixing any lift $\tilde{p} \in \pi^{-1}(p)$, we could see as in the proof of Lemma [2.2] that

$$\widetilde{G}_{\lambda_1 \bar{i}_1}(p) = \left\{ \gamma \in \pi_1(M, p) : d_{\pi^*g}(\gamma, \tilde{p}, \bar{i}_1) \leq 2 \lambda_1 \bar{i}_1 \right\}$$

(6.6)\footnotesize

$$\gamma \in \pi_1(M, p) : d_{\pi^*g}(\gamma, \tilde{p}, \bar{i}_1) \leq 4 \lambda_1.$$
Since \( \pi : (\tilde{M}, \tilde{\rho}) \to (M, \rho) \) is a normal covering with deck transformation group being \( \pi_1(M, p) \), the same holds for the restriction \( \pi_p : \pi^{-1}(B_{\tilde{g}_p}(\rho, 2)) \to B_{\tilde{g}_p}(\rho, 2) \).

Applying \([47, \text{Proposition 5.4}]\) to the normal covering \( \pi_p : \pi^{-1}(B_{\tilde{g}_p}(\rho, 2)) \to B_{\tilde{g}_p}(\rho, 2) \) and the subgroup \( \tilde{G}_{i_1} \) of the deck transformation group \( \pi_1(M, p) \), we conclude, thanks to \((6.5), (6.6), (6.7)\) and the choice of \( \lambda < \delta_{NZ}(\varepsilon') \), that
\[
d_{GH}(B_{\tilde{g}_p}(\tilde{\rho}, r'), \mathbb{B}^m(r')) \leq \varepsilon r'.
\]

We now further rescale the metric \( \tilde{g}_2 := \lambda_2^{-2} \pi^* \tilde{g}_1 \) with
\[
(6.8) \quad \lambda_2(\varepsilon) := \min \{ r_1, \Lambda_C, r'^{-1} \}.
\]
then for any \( p \in M \) and any \( \tilde{\rho} \in \pi^{-1}(p) \), we have
\[
d_{GH}(B_{\tilde{g}_2}(\tilde{\rho}, R_C), \mathbb{B}^m(R_C)) < \delta_C,
\]
and we have the Ricci curvature lower bound
\[
Rc_{\tilde{g}_2} \geq -(m - 1)\Lambda_C^2 \tilde{g}_2.
\]
Consequently, applying \([21, \text{Main Lemma 2.1}]\) we have
\[
\tilde{\rho} \in \tilde{M}, \quad \left| B_{\tilde{g}_2}(\tilde{\rho}, 1) \right|_{\tilde{g}_2} \geq (1 - 10^{-2}\varepsilon)\omega_m.
\]
By the volume ratio comparison \((6.4)\) we have
\[
(6.9) \quad \forall \tilde{\rho} \in \tilde{M}, \forall r \in (0, 1], \quad \left| B_{\tilde{g}_2}(\tilde{\rho}, r) \right|_{\tilde{g}_2} \geq (1 - \varepsilon)\omega_m r^m.
\]
Notice the scaling invariance of the estimate.

Now we scale back to the original metric \( \pi^* g \) and the estimate \((6.9)\) remain valid for geodesic balls centered anywhere in \( \tilde{M} \), with radii not exceeding \( \frac{1}{2}i_1\Lambda_2 \). By \((6.8)\) and the bound of \( r' \geq \delta_{NZ} \) in \([47, \text{Proposition 5.8}]\), we have \( \frac{1}{2}i_1\Lambda_2 \) always bounded below by
\[
(6.10) \quad r_{AE} := \frac{1}{2}i_1(\varepsilon') \min \{ r_1, \Lambda_C, \delta_{NZ}(\varepsilon')R_C^{-1} \},
\]
with \( \Lambda_C, R_C \) and \( \varepsilon' \) determined by \( 10^{-2}\varepsilon \) via Colding’s volume continuity theorem, and \( \varepsilon_{NZ}(m) \) described in the proof of Lemma \([2, 2]\). Clearly, \( \delta_{AE} \) and \( r_{AE} \) are determined by \( \varepsilon, m, D \) and \( v \). \( \square \)

With the help of this lemma, we could then apply Proposition \([6.1]\) to the rescaled covering flow \((\tilde{M}, \pi^* g(t)) \) to bound the existence time uniformly from below.

Proof of Theorem \([4, 4]\) Given \( \alpha \in (0, 10^{-2}m^{-1}) \), let \( \delta_{AE}(\alpha) \in (0, 1) \) be the almost Euclidean threshold required in \((6.1)\). Given \( (M, g) \) and \( (N, h) \) as in the assumption, we know that \( (M, g) \) is a closed Riemannian manifold as it is complete with finite diameter. Therefore, Hamilton’s short-time existence result applies and there is a Ricci flow solution \( (M, g(t)) \) for \( t \in [0, T) \) with \( g(0) = g \). For \( \delta < \delta_{RF} := \delta_{AE}(\delta_p) \) (omitting the dependence on \( m, D \) and \( t \)), we consider the covering Ricci flow \((\tilde{M}, \tilde{g}(t)) \) with initial data \((\tilde{M}, \pi^* g) \). Notice that the time slices of the covering flow are complete, and satisfy \( \|Rm_{\tilde{g}(t)}\|_{L^\infty(\tilde{M}, \tilde{g}(t))} = \|Rm_{g(t)}\|_{L^\infty(M, g(t))} < \infty \) for any \( t < T \), and by Lemma \([6, 2]\) we have
\[
\forall \tilde{\rho} \in \tilde{M}, \quad \left| B_{\pi^* \tilde{g}(t)}(\tilde{\rho}, r_{AE}) \right|_{\pi^* \tilde{g}} \geq (1 - \delta_p)\omega_m r^m_{\pi^* \tilde{g}}.
\]
Rescaling \( g \mapsto r_{AE}^2 \delta_p^{-2} g =: \tilde{g} \) and \( t \mapsto r_{AE}^{-2} \delta_p^{-2} t =: \tilde{t} \), we could apply Proposition \([6.1]\) to the Ricci flow \((\tilde{M}, \tilde{g}(t)) \) and conclude that the flow exists at least up to \( \tilde{t} = \varepsilon_p^2(\alpha) \). Now scaling back, we see
that the original Ricci flow exists up to \( T > \varepsilon^2_{RF} := \varepsilon^2_{RF} \delta^2_{\mathcal{P}} \), and \((1.2)\) follows directly from \((6.2)\).

We notice that both \( \delta_{RF} \) and \( \varepsilon_{RF} \) are solely determined by \( m, D \) and \( v \), besides \( \alpha \).

\[ \square \]

In order to apply Theorem \((1.4)\) as a smoothing tool, we need to keep track of the distance change by running the Ricci flow. We have the following distance distortion estimate, which is a rewording of \([36, \text{Lemma 1.11}]\):

**Lemma 6.3** (Distance distortion). For any \( \alpha \in (0, 10^{-2} m^{-1}) \), under the assumptions of Theorem \((1.4)\) there is some \( \Psi_D(\alpha|m) \in (0, 1) \) with \( \lim_{\alpha \to 0} \Psi_D(\alpha|m) = 0 \), such that for any \( t \in (0, \varepsilon^2_{RF}) \), and for any \( x, y \in M \) with \( d_g(x, y) \leq \sqrt{t} \), we have

\[
(6.11) \quad |d_{g(t)}(x, y) - d_g(x, y)| \leq \Psi_D(\alpha|m) \sqrt{t}.
\]

**Proof.** As in the proof of Theorem \((1.4)\) we consider the universal covering \( \pi : \tilde{M} \to M \) and we have a Ricci flow solution \( \pi^*g(t) \) on \( \tilde{M} \). Notice that for each \( t \in [0, \varepsilon^2_{RF}] \), the fundamental group \( \pi_1(M) \) acts on \( (\tilde{M}, \pi^*g(t)) \) by free and totally discontinuous isometries and the Ricci flow \( g(t) \) on \( M \) is the quotient flow \( (\tilde{M}, \pi^*g(t))/\pi_1(M) \). Recall that \( (\tilde{M}, \pi^*g(t))/\pi_1(M) \) is the quotient space in every point, after suitable rescaling (making \( r_{AE} \to 1 \)). By the scaling invariance, the original estimate in \([36, \text{Lemma 1.11}]\) descends to the flow \((M, g(t))\) and proves \((6.11)\).

\[ \square \]

6.2. **Rigidity of the first Betti number.** With the help of Theorem \((1.4)\) and Lemma \(6.3\) we now prove Claim \((2)\), the equality case, of Theorem \(1.1\). Recalling that by \([16, \text{Theorem 2.6}]\) and \([36, \text{Theorem 2.2}]\), we have some dimensional constants \( \varepsilon_F(m) \in (0, 1) \) and \( C_F(m) \geq 1 \), such that if \( m \)-dimensional manifold \( X \) with sectional curvature bounded by 1 in absolute value is \((1, \varepsilon_F)-\text{Gromov-Hausdorff}\) close to a \( k \)-dimensional \((k \leq m)\) manifold \( Y \) with the same sectional curvature bound and unit injectivity radius lower bound, then there is a fibration \( F : X \to Y \), which is also a \((2^{-1}, C_F \varepsilon_F)\)-\text{Gromov-Hausdorff}\) approximation. Here an \((r, \delta)\)-\text{Gromov-Hausdorff}\) approximation is a \( \delta \)-dense map whose restriction to each geodesic \( r \)-ball is a \( \delta \)-\text{Gromov-Hausdorff}\) approximation. We now pick the largest \( \alpha_B(m, D, v) \in (0, 10^{-2} m^{-1}) \) so that \( \Psi_D(\alpha_B|m) \leq 4^{-1} \min[\varepsilon_F, C_F^{-1} \delta_{Nil}] \) — here the dependence of \( \alpha_B \) on \( D \) and \( v \) is due to \( \delta_{Nil}(m, D, v) \), obtained in Lemma \(2.2\).

If \( d_{GH}(M, N) < \delta_{RF}(\alpha_B) \) and \( b_1(M) - b_1(N) = m - k \), then we could run the Ricci flow with initial data \((M, g)\) by Theorem \((1.4)\) to obtain a smoothing metric \( g(T_B) \) with \( T_B := \min[\varepsilon_{RF}(\alpha_B)^2, \varepsilon^2_{RF}] \), satisfying \( \|Rm_{g(T_B)}\|_{L^\infty(M, g(T_B))} \leq 2T_B^{-1} \). On the other hand, by Lemma \(6.3\) we know that \((M, 2T_B^{-1} g(T_B))\) and \((M, 2T_B^{-1} g)\) are \((1, \Psi_D(\alpha_B))\)-\text{Gromov-Hausdorff}\) close, meaning that the identity map restricts to a \( \Psi_D(\alpha_B)\)-\text{Gromov-Hausdorff}\) approximation on any geodesic unit ball in \((M, 2T_B^{-1} g(T_B))\) or \((M, 2T_B^{-1} g)\). Therefore, setting

\[
\delta_B(m, D, v) := \frac{1}{10} \min \left\{ \delta_{RF}(\alpha_B), \varepsilon_F T_B^{\frac{1}{2}}, C_F^{-1} \delta_{Nil} \right\},
\]

we know that \((M, 2T_B^{-1} g(T_B))\) and \((N, 2T_B^{-1})\) are \((1, \frac{1}{2} \min[\varepsilon_F, 2C_F^{-1} \delta_{Nil}])\)-\text{Gromov-Hausdorff}\) close to each other, whenever \( d_{GH}(M, N) = \delta < \delta_B \). Moreover, both \((M, 2T_B^{-1} g(T_B))\) and \((N, 2T_B^{-1})\) have sectional curvature uniformly bounded by 1 in absolute value, and \((N, 2T_B^{-1})\) has injectivity radius everywhere bounded below by 1. Now applying \([36, \text{Theorem 2.2}]\), we obtain an infranil fibration \( F : M \to N \), which is also a \((2^{-1}, 2^{-1} \min[C_F \varepsilon_F, \delta_{Nil}])\)-\text{Gromov-Hausdorff}\) approximation. Especially, for any \( p \in M \), the fiber \( F_p \) has extrinsic diameter \( \text{diam}_{2T_B^{-1} g(T_B)} F_p \leq 2^{-1} \delta_{Nil} \). Consequently, \( \text{diam}_{2T_B^{-1} g} F_p \leq \delta_{Nil} \), and as \( T_B < 1 \), we have \( \text{diam}_g F_p \leq \delta_{Nil} \) for any \( p \in M \).
Notice that each $F$-fiber is diffeomorphic to an infranil manifold of dimension $m - k$, and we are yet to check that the fibers are actually toral. We now fix an arbitrary $p \in M$. By the fiber bundle structure $F : M \to N$ and the assumption $\delta < \delta_B$, we know that $\tilde{G}_{\delta_{Nil}}(p) \cong \tilde{\Gamma}_{\delta_{Nil}}(p) \cong \pi_1(F_p, p)$, since $\text{diam}_q F_p \leq \delta_{Nil}$ and the base $N$ is homotopically trivial at the scale $\delta_{Nil}$. On the other hand, by Remark 4 we know that the abelianization of $\tilde{G}_{\delta_{Nil}}(p)$ has rank at least $b_1(M) - b_1(N)$. We have $\text{rank} \tilde{G}_{\delta_{Nil}}(p) \leq m - k$ by Lemma 2.2 and by the structure of the finitely generated almost nilpotent groups we have

$$b_1(M) - b_1(N) \leq \frac{\text{rank} \tilde{G}_{\delta_{Nil}}(p)}{([\pi_1(M, p), \pi_1(M, p)] \cap \tilde{G}_{\delta_{Nil}}(p))} \leq \frac{\text{rank} \tilde{G}_{\delta_{Nil}}(p)}{\pi_1(M, p)} \leq m - k.$$ 

Now the assumption $b_1(M) - b_1(N) = m - k$ forces the almost nilpotent group $\tilde{G}_{\delta_{Nil}}(p)$ to have the same rank as its abelianization, which is the case only when $\tilde{G}_{\delta_{Nil}}(p)$ is a finitely generated abelian group. Therefore, we have shown that each fiber $F_p$ has abelian fundamental group. As it is an infranil manifold, we thus know that $F_p$ is diffeomorphic to an $(m - k)$-torus $\mathbb{T}^{m-k}$. Therefore, the proof of Claim (2) of Theorem 1.1 is complete.

**Remark 6.4.** After finishing this work, a fiber bundle theorem for collapsing manifolds with the so-called “local bounded Ricci covering geometry” appears in [35]. While the collapsing manifolds in Theorem 1.1 with maximal first Betti number differences are shown through Sections 3 and 4 to satisfy the assumptions in [35] Theorem 0.3, this theorem fails to provide the structure of the fibers. The proof of [35] Theorem 0.3] could neither be applied to the case when the collapsing limit is a singular orbifold, as done in [40]. We recently learn that an upcoming work of Rong [50] will provide an alternative proof of Theorem 1.1 purely relying on techniques from metric Riemannian geometry and independent of the Ricci flow smoothing technique.

7. **Further discussions**

The torus fibration structure in Theorem 1.1 is not the end of the journey. Under what extra conditions can we simplify the topological structure of the collapsing manifolds? In fact, if $\pi_1(N) = 0$ and $b_1(M) = \dim M - \dim N$, it is easy to see that $M \cong N \times \mathbb{T}^{m-k}$ as smooth manifolds. This can be done purely by a topological argument. The general discussion of product structure under the assumptions in Theorem 1.1 will appear elsewhere.

On the other hand, it is natural to extend Theorem 1.1 for generic collapsing limit spaces — we notice that if $X$ is a compact Ricci limit space, the generalized first Betti number $b_1(X)$ is well-defined; see [34] Remark 7.22]. In this direction, the study of local Ricci bounded covering geometry pioneered by Rong [36] should provide useful tools, and the localization of the short first homology group, as well as Theorem 1.4 will be inevitable; see also [40] for a local Ricci flow smoothing result for collapsing manifolds near lower-dimensional orbifold limits.

**Acknowledgements.** We would like to thank Xiaochun Rong for enlightening discussions and his warm encouragement. The first named author thanks Song Sun for several valuable comments. The second named author is grateful to Xin Peng for reading the early version of the manuscript and pointing out a mistake in “further discussions” and many typos. The second named author is partially supported by YSBR-001, the General Program of the National Natural Science Foundation of China (Grant No. 11971452) and a research fund of USTC.
References

[1] Uwe Abresch and Detlef Gromoll, On complete manifolds with nonnegative Ricci curvature. *J. Amer. Math. Soc.* 3 (1990), no. 2, 355-374.

[2] Salomon Bochner, Vector fields and Ricci curvature. *Bull. Amer. Math. Soc.* 52 (1946), 776-797.

[3] Salomon Bochner and Kentaro Yano, Curvature and Betti numbers. Annals of Mathematics Studies, No. 32. *Princeton University Press, Princeton, N.J.*, 1953.

[4] Sergei Buyalo, Volume and the fundamental group of a manifold of non-positive curvature. *Math. USSR Sbornik* 50 (1985), 137-150.

[5] Sergei Buyalo, Collapsing manifolds of non-positive curvature I. *Leningrad Math. J.* 1 (1990), no. 5, 1135-1155.

[6] Sergei Buyalo, Collapsing manifolds of non-positive curvature II. *Leningrad Math. J.* 1 (1990), no. 6, 1371-1399.

[7] Eugenio Calabi, An extension of E. Hopf’s maximum principle with an application to Riemannian geometry. *Duke Math. J.* 25 (1958), 45-56.

[8] Eugenio Calabi, On Ricci curvature and geodesics. *Bull. Amer. Math. Soc.* 52 (1946), 776-797.

[9] Jianguo Cao, Jeff Cheeger and Xiaochun Rong, Splittings and C^r-structures for manifolds with nonpositive sectional curvature. *Invent. Math.* 144 (2001), 139-167.

[10] Albert Chau, Luen-Fai Tam and Chengjie Yu, Pseudolocality for the Ricci flow and applications. *Canad. J. Math.* 63 (2011), no. 1, 55-85.

[11] Jeff Cheeger and Tobias Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math.* 144 (1996), no. 1, 189-237.

[12] Jeff Cheeger and Tobias Colding, On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.* 46 (1997), no. 3, 406-480.

[13] Jeff Cheeger and Detlef Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Differential Geom.* 6 (1971), no. 1, 119-128.

[14] Jeff Cheeger and Mikhail Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded. I. *J. Differential Geom.* 23 (1986), no. 3, 309-346.

[15] Jeff Cheeger and Mikhail Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded. II. *J. Differential Geom.* 32 (1990), no. 1, 269-298.

[16] Jeff Cheeger, Kenji Fukaya and Mikhail Gromov, Nilpotent structures and invariant metrics on collapsed manifolds. *J. Amer. Math. Soc.* 5 (1992), no. 2, 327-372.

[17] Jeff Cheeger and Xiaochun Rong, Collapsed Riemannian manifolds with bounded diameter and bounded covering geometry. *Geom. Funct. Anal.* 5 (1995), no. 2, 141-163.

[18] Jeff Cheeger and Xiaochun Rong, Existence of polarized F-structures on collapsed manifolds with bounded curvature and diameter. *Geom. Funct. Anal.* 6 (1996), no. 3, 411-429.

[19] Shiu Yuen Cheng and Shing-Tung Yau, Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.* 28 (1975), no. 3, 333-354.

[20] Jaigyoung Choe and Ailana Fraser, Mean curvature in manifolds with Ricci curvature bounded from below. *Comment. Math. Helv.* 93 (2018), no. 1, 55-69.

[21] Tobias H. Colding, Ricci curvature and volume convergence. *Ann. of Math. (2)* 145 (1997), no. 3, 477-501.

[22] Tobias H. Colding and Aaron C. Naber, Sharp H"older continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. *Ann. of Math. (2)* 176 (2012), no. 2, 1173-1229.

[23] Manfredo Perdigão do Carmo, Riemannian geometry. Translated from the second Portuguese edition by Francis Flaherty. *Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA*, 1992. xiv+300 pp. ISBN: 0-8176-3490-8

[24] Kenji Fukaya, Collapsing Riemannian manifolds to ones of lower dimensions. *J. Differential Geom.* 25 (1987), no. 1, 139-156.

[25] Kenji Fukaya, A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. *J. Differential Geom.* 28 (1988), no. 1, 1-21.

[26] Kenji Fukaya and Takao Yamaguchi, The fundamental groups of almost non-negatively curved manifolds. *Ann. of Math. (2)* 136 (1992), no. 2, 253-333.

[27] Mikhail Gromov, Almost flat manifolds. *J. Differential Geom.* 13 (1978), no. 2, 231-241.

[28] Mikhail Gromov, Structures métriques pour les variétés Riemanniennes, Editions Cedic, Paris (1981).
[29] Mark Gross and Perelom M. H. Wilson, Large complex structure limits of $K^3$ surfaces. *J. Differential Geom.* 55 (2000), no. 3, 475-546.
[30] Richard Hamilton, Three-manifolds with positive Ricci curvature. *J. Differential Geom.* 17 (1982), no. 2, 255-306.
[31] Fei He, Existence and applications of Ricci flows via pseudolocality. *Preprint*, arXiv: 1610.01735.
[32] Hans-Joachim Hein, Song Sun, Jeff Viaclovsky and Ruobing Zhang, Nilpotent structures and collapsing Ricci-flat metrics on $K^3$ surfaces. *Preprint*, arXiv: 1807.09367.
[33] Raphael Hochard, Short-time existence of the Ricci flow on complete, non-collapsed 3-manifolds with Ricci curvature bounded from below. *Preprint*, arXiv: 1603.08726.
[34] Shouhei Honda, Existence and applications of Ricci flows via pseudolocality. *Preprint*, arXiv: 1610.01735.
[35] Shaochuang Huang, A note on existence of exhaustion functions and its applications. *J. Geom. Anal.* 29 (2019), no. 2, 1649-1659.
[36] Shaochun Rong, The existence of polarized $F$-structures on volume collapsed 4-manifolds. *Geom. Funct. Anal.* 3 (1993), no. 5, 474-501.
[37] Xiaochun Rong, Collapsed manifolds with local Ricci bounded covering geometry. *Front. Math. China* 15 (2020), 69-89.
[38] Xiangang Tan, Almost flat manifolds. *J. Differential Geom.* 17 (1982), no. 1, 1-14.
[39] Xiannan Wang, Deforming the metric on complete Riemannian manifolds. *J. Differential Geom.* 30 (1989), no. 1, 223-301.
[40] Miles Simon, Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below. *J. Reine Angew. Math.* 662 (2012), 59-94.
[41] Miles Simon and Peter Topping, Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces. *Preprint*, arXiv: 1706.09490.
[42] Andrew Strominger, Shing-Tung Yau and Eric Zaslow, Mirror symmetry is T-duality. *Nuclear Phys. B* 479 (1996), no. 1-2, 243-259. MR1429831
[43] Gang Tian and Bing Wang, On the structure of almost Einstein manifolds. *J. Amer. Math. Soc.* 28 (2015), no. 4, 1169-1209.
[44] Shing-Tung Yau, Compact flat manifolds. *J. Differential Geom.* 6 (1972), 395-402.
Shaosai Huang, Two Morneau Shepell Centre Suite 915, 895 Don Mills Road, Toronto, ON, M3C 1W3, Canada

Email address: arthur.ut.ca@gmail.com

Bing Wang, Institute of Geometry and Physics, and School of Mathematical Sciences, University of Science and Technology of China, 96 Jizhai Road, Hefei, Anhui Province, 230026, China

Email address: toppspin@ustc.edu.cn