Laughlin-Type Topological Order on a Fractal Lattice with a Local Hamiltonian

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Anyons are mainly studied and considered in two spatial dimensions. For fractals, the scaling dimension that characterizes the system can be non integer and can take values between that of a standard one-dimensional or two-dimensional system. Generating Hamiltonians that meet locality conditions and support anyons is not a simple task. Here, we construct a local Hamiltonian on a fractal lattice which realizes physics similar to the fractional quantum Hall effect. The fractal lattice is obtained from a second generation Sierpinski carpet, which has 64 sites, and is characterized by a Hausdorff dimension of 1.89. We demonstrate that the proposed local Hamiltonian acting on the fractal geometry has Laughlin-type topological order by creating anyons and then studying their charge and braiding statistics. We also find that the energy gap between the ground state and the first excited state is approximately three times larger for the fractal lattice than for a standard square lattice with 64 sites, and the model on the fractal lattice is significantly more robust against disorder. We propose a scheme to implement fractal lattices and our proposed local Hamiltonian for ultracold atoms in optical lattices. The discussed scheme could also be utilized to study integer quantum Hall phases and the physics of other quantum systems on fractal lattices.

Topologically ordered quantum systems harbor fractionalized excitations that are neither fermions nor bosons, but anyons. Phases hosting anyons have been realized experimentally in solid state systems in strong magnetic fields displaying the fractional quantum Hall effect. Fractional quantum Hall phases also exist in systems defined on two-dimensional lattices, where the physical magnetic field is replaced by an artificial magnetic field. In this way, the effective magnetic field strength can be increased significantly. Due to their unique degree of tunability, realizing fractional quantum Hall physics with ultracold atoms in optical lattices would give unique possibilities for investigating the effect in great detail, and there are currently several efforts towards achieving this for systems with few atoms. The key components of artificial magnetic fields and topological band structures have already been prepared in several experiments.

In the past few years, interest has grown in studying topological models on fractal lattices. Most of these studies consider non-interacting systems and those have revealed new and interesting properties, including modifications of the Hofstadter butterfly and the presence of inner edge states. Much less is currently known about how fractal lattices affect the properties of topologically ordered phases of interacting systems. Initial steps have been taken by constructing Laughlin and Moore-Read trial states on fractal lattices, but the derived parent Hamiltonians of these states are nonlocal and involve many different types of interactions making them difficult to realize. The study of models on fractal lattices is also motivated by experimental developments, such as the preparation of fractal models in molecules on surfaces. It is desirable to realize the fractal models with ultracold atoms due to their ability to reach the regime of strongly interacting quantum systems and achieve single-site resolution.

Here, we show that a model with only nearest-neighbor complex hopping and hardcore interactions on a finite generation fractal lattice can give rise to a bosonic Laughlin phase hosting anyons, and we propose a scheme to implement the Hamiltonian experimentally with ultracold atoms in optical lattices. Specifically, we construct a fractal lattice with 64 sites from the second generation of a Sierpinski carpet. The phases of the hopping terms are chosen such that they mimic a magnetic flux penetrating each lattice site. We demonstrate the topological nature of this model by creating anyons and studying their charge and braiding properties under adiabatic time evolution.

Comparing the model on the fractal lattice to a similar topological model on a square lattice with the same number of sites, we observe that the gap between the ground state and the first excited state is approximately three times larger for the model on the fractal lattice. We also observe that the model on the fractal lattice is more robust with respect to disorder. This robustness could be advantageous in future applications that utilize anyons separate from the experimental proposal we consider.

The proposed protocol to implement the Hamiltonian involves single-site addressing and laser-assisted hopping tuned to achieve the desired phase factors. We expect that the setup could also be used to generate integer quantum Hall phases on fractal lattices. The main challenge in realizing fractional quantum Hall phases in ultracold atoms in optical lattices is to reach the ground state. The larger gap for the fractal lattice could mean that less cooling is required. Generating optical, fractal lattices as described below also opens the door for studying various phenomena of quantum systems on fractal...
lattices more generally.

Model—The typical ingredients required to obtain fractional quantum Hall physics include interactions and a magnetic flux perpendicular to the plane that breaks time reversal symmetry. In lattice systems, the magnetic flux is often translated into corresponding complex hopping terms through the Peierls substitution [30]. Here, we start from a second generation Sierpinski carpet [Fig. 1(a)] and obtain the considered fractal lattice by putting a lattice site in the center of each of the small squares. We denote the positions of the lattice site in the center of each of the small squares with \( z_j \) with \( j \in \{1, \ldots, N\} \) and consider \( M = 4 \) bosons on the lattice. The Hamiltonian

\[
H = -J \sum_{\langle jk \rangle} c_j^\dagger c_k e^{i \phi_{jk}} + U \sum_{l} n_l(n_l - 1), \quad U \gg J, \quad (1)
\]

consists of complex hopping terms of strength \( J \) and an interaction term of strength \( U \). The bracket \( \langle \ldots \rangle \) refers to nearest-neighbor sites, \( c_k \) annihilates a boson on the \( k \)th lattice site, \( n_k = c_k^\dagger c_k \), and \( \phi_{jk} \) is the phase the wavefunction acquires when a particle hops from \( z_k \) to \( z_j \). In the computations below, we assume that \( U/J \) is so large that one can neglect the possibility to have more than one boson on a site, i.e. we work with hardcore bosons.

The particular form of \( \phi_{jk} \) is determined from the magnetic flux configuration in the system and can give rise to different models. In two-dimensional fractional quantum Hall models, the magnetic field is often either chosen to be uniform or to go through the lattice sites only. For a fractal lattice, it is similarly natural to let the magnetic field go through the lattice sites only, so that the pattern of magnetic flux also forms a fractal. One can think of this as each site being equipped with an infinitely long and thin solenoid perpendicular to the plane. We measure the flux \( \alpha \) going through the cross section of one solenoid in terms of the magnetic flux unit. The flux in one solenoid gives rise to the vector potential \( \vec{A}(r) = \alpha \vec{r}/r \) at a distance \( r \) from the solenoid, where \( \vec{r} \) is the azimuthal unit vector with respect to a coordinate system, where the solenoid is at the origin. When a particle hops from \( z_k \) to \( z_j \), the acquired phase due to the magnetic flux through \( z_i \) is

\[
\int_{\vec{z}_k}^{\vec{z}_j} \vec{A}(r) \cdot d\vec{l} = \alpha \beta_i, \quad (2)
\]

where \( d\vec{l} \) is an infinitesimal vector along the hopping direction and \( \beta_i \) is the angle between the vectors \( z_k - z_i \) and \( z_j - z_i \).

The total phase \( \phi_{jk} \), which the many-body state acquires while a particle hops from \( z_k \) to \( z_j \), is obtained by summing over the contributions from all the fluxes, i.e.

\[
\phi_{jk} = \alpha \sum_{i(\neq j \neq k)} \beta_i, \quad (3)
\]

where the sum \( i(\neq k \neq j) \) is over \( i \). If a particle encircles a single lattice site the phase acquired is \( 2\pi \alpha \). We consider the case where \( M/(\alpha N) = 1/2 \). Hence, if the system is topological, we expect it to be in a bosonic Laughlin phase with quasiholes of charge \( 1/2 \).

The model described above can also be defined on a square lattice, which we shall use for comparison. In particular, we consider the 64 site quadratic lattice to allow for a proper comparison to the fractal lattice.

Energy gap—The energy gap, \( \delta E \), between the ground state and the first excited state is an important property of the model. This is due to the gap being related to the state’s stability, both in terms of robustness to disorder and feasibility of experimental implementations. We find that the energy gap is substantially larger on the fractal lattice \((\delta E = 0.313J)\) than that on the square lattice \((\delta E = 0.105J)\).

Creation of anyons—Laughlin-type quasiholes give rise to local reductions of the particle density, and therefore local potentials tend to trap anyons [31]. To obtain anyons in our model, we hence add local trapping potentials of the form

\[
H_V = V n_l + V n_m, \quad l \neq m, \quad V \gg J, \quad (4)
\]

to the Hamiltonian \( H \). If the model is in a topological phase, we expect the potentials to trap one anyon at site \( l \) and one anyon at site \( m \). The anyons have a finite width and hence also modify the densities on nearby sites. To show that anyons are indeed formed, we compute the charge and statistics of the anyons.

The anyon density profile

\[
\rho(z_i) = \langle n_i \rangle_{H+H_V} - \langle n_i \rangle_H \quad (5)
\]

is the difference between the particle densities \( \langle n_i \rangle_{H+H_V} \) and \( \langle n_i \rangle_H \) at \( z_i \) in the presence or absence of anyons in the system. We sum this quantity over a local region \( \sigma_k \) around the \( k \)th anyon position to obtain the change in the number of particles within \( \sigma_k \). The region should be large enough to enclose the complete anyon, and here we take \( \sigma_k \) to be the sites inside the dashed lines in Figs. 1(b) and 1(c). Taking the charge of a particle to be \(-1\), we define the anyon charge as

\[
Q_k = - \sum_{z_i \in \sigma_k} \rho(z_i), \quad k \in \{1, 2\}. \quad (6)
\]

We find that the anyon charges are close to 0.5 (see \( h = 0 \) in Fig. 2) for both the fractal and square lattices.

Fractional statistics—We calculate the anyon exchange statistics to further characterize the type of anyons present in the system. We do this by adiabatically exchanging two anyons. This exchange results in the ground state \( |\Psi\rangle \) of the Hamiltonian \( H + H_V \) acquiring a Berry phase \( \exp(i\theta) \), defined by

\[
\theta = i \oint_{\gamma} (\Psi|\nabla_w|\Psi) dw + \text{c.c.}, \quad (7)
\]
These values are close to the expected value of \( \theta \) and the statistical phase of the anyons are found to be \( \theta \) itselfs. Therefore we have \( \theta = \theta_{\text{AB}} + \theta_s \). The value of \( \theta_{\text{AB}} \) is obtained by circulating one anyon and keeping the other anyon fixed at a position sufficiently outside the moving anyon’s path. The particular value of \( \theta_s \) (\( \neq [0,1] \)) characterizes the type of anyons present in a given topological order.

To adiabatically move a trapping potential from the site \( l \) to the nearby site \( l' \), we follow the procedure in Ref. [31] and consider the Hamiltonian

\[
H + (1 - \gamma)V_{n_l} + \gamma V_{n_{l'}} + V_{n_m},
\]

We exchange the anyons in the anti-clockwise direction by adiabatically reducing the strength of the potential on one lattice site while increasing it on its neighboring lattice site. We move one anyon at a time while keeping the other anyon fixed at its position to minimize overlap during the driving. We vary \( \gamma \) by taking the ramp

\[
\gamma = \frac{\delta r}{r} - \frac{1}{2\pi} \sin \left( \frac{2\pi \delta r}{r} \right),
\]

with \( r \) a number of steps sufficiently large to maintain adiabaticity and \( \delta r \in [0, 1, \ldots, r] \) as the individual step. We choose the exchange path for the square and fractal lattices shown in Fig. 1. The statistical phase of the anyons are found to be \( \theta_s = 0.4589 \) on the square lattice and \( \theta_s = 0.5089 \) on the Sierpinski carpet fractal lattice. These values are close to the expected value of \( \theta_s = 1/2 \) and the differences are likely due to finite size effects. Therefore, we conclude that the local Hamiltonian proposed here generates the correct phase for topological order. We also note that if we instead choose a uniform magnetic field on the fractal lattice, we do not obtain a statistical phase close to 1/2.

Effect of disorder—We next study the robustness of the topological properties of the models with respect to weak disorder. We add a disordered potential at each lattice site and write the Hamiltonian as

\[
H' = H + \sum_i h_i n_i,
\]

where \( h_i \in [-h, h] \) is drawn from a uniform random distribution with \( h \) the disorder strength. We again calculate the topological properties through the charge and statistics. We plot in Fig. 2 the energy gap \( \delta E/J \), averaged over random disorder realizations, between the ground state and the first excited state as a function of \( h/J \) for both the models on the square lattice and on the fractal lattice. The model acquires a larger gap on the fractal lattice than that on the square lattice, which could be significant for its realization. This gap reduces with the introduction of disorder but is still substantially larger than that of the square lattice over large ranges of disorder.

We calculate anyon density profiles \( \rho(z_i) \) via Eq. (5), but using \( H' \) instead of \( H \). We compute anyon charges \( Q_1, Q_2 \) using Eq. (6), averaged over random disorder realizations. We plot the respective anyon charges as a function of the disorder strength in Fig. 2. We find that the values of \( Q_1, Q_2 \) are close to the expected 1/2 even in the presence of disorder. Importantly, we observe that the charges observed for the fractal geometry are not only closer to the expected value but more stable against disorder.

We also compute braiding statistics of the anyons in the presence of disorder in the system. This is a com-

FIG. 1. (a) The model is defined on a second generation Sierpinski carpet (red squares). The lattice sites are marked by circles, and the bonds connecting the sites illustrate the hopping terms. A magnetic flux goes through each lattice site in the direction perpendicular to the plane. (b-c) Anyons trapped on the square and fractal lattices. The colors of the sites show the anyon charges and the bonds connecting the sites illustrate the hopping terms. A magnetic flux goes through each lattice site in the direction perpendicular to the plane. (b) Anyons trapped on the square and fractal lattices. The colors of the sites show the anyon charges and the bonds connecting the sites illustrate the hopping terms. A magnetic flux goes through each lattice site in the direction perpendicular to the plane. (c) Anyons trapped on the square and fractal lattices. The colors of the sites show the anyon charges and the bonds connecting the sites illustrate the hopping terms. A magnetic flux goes through each lattice site in the direction perpendicular to the plane.
FIG. 2. The anyon charges $Q_1$ and $Q_2$ and the energy gap $\delta E/J$ between the ground state and the first excited state as a function of the disorder strength $h/J$ for the models on the square lattice and the Sierpinski carpet fractal lattice. For the charges (energy gap), we average over 2000 (400) statistically independent disorder realizations for each $h$ to ensure convergence (the error bar is of order $10^{-3}$ for all data points). The energy gap is seen to be significantly larger for the model on the fractal lattice. The anyon charges are in the vicinity of the ideal value $1/2$, which is marked by the horizontal line.

| Disorder $h/J$ | $\theta_s$ on square lattice | $\theta_s$ on fractal lattice |
|---------------|-----------------------------|-----------------------------|
| $h/J = 0$     | 0.4589                      | 0.5089                      |
| $h/J = 0.25$  | 0.0570                      | 0.4803                      |
| $h/J = 0.5$   | 0.2615                      | 0.5034                      |
| $h/J = 1$     | 0.0977; 0.0994              | 0.5171; 0.5479              |

TABLE I. Exchange statistics for a single disorder realization on the square and fractal lattices (the ideal value of $\theta_s$ is $1/2$). The two numbers for the disorder strength $h/J = 1$ are for two different disorder realizations.

... computational expensive task. Therefore, we consider the case of $h = 0$ and three $h \neq 0$ values where we take a single or two disorder realizations. We find that the statistics of the anyons are approximately of the ideal value $\theta_s = 1/2$ on the fractal lattice for the considered disorder strengths, see Table I. However, for the square lattice we find that the anyon statistics is destroyed already for $h/J = 0.25$. This does not necessarily mean that the model is not topological. It could also be that we do not get the expected statistics because the anyons overlap at least for some parts of the trajectory. The fractal lattice helps to trap the anyons due to the smaller scaling dimension allowing for there to be less chance of overlap. Therefore, it allows for the robust realization of topological order in a smaller system.

Proposal for implementing the Hamiltonian—An experimental demonstration of the Sierpinski carpet fractal lattice in a cold-atom system requires two components: efficient preparation of the lattice system with the desired filling factor and generation of the required site-to-site hopping phases. For the former, we assume that we start by loading a single plane of a 3D cubic lattice in a conventional quantum-gas microscopy system capable of imaging atoms with single-site resolution using a high-numerical aperture (NA) microscope objective. Using spin-addressing techniques, a set number of atoms can be loaded into the lattice.

We now discuss the problem of generating the desired site-to-site hopping magnitudes and phases via light-assisted tunneling. In general, tunneling is inhibited in the system if there exists an energy gradient along $x$ and $y$ giving rise to a bias $\Delta$ between each adjacent site. Light-assisted tunneling between adjacent sites can be restored if a pair of running-wave beams is added to the system. Given that the frequency difference between the running waves satisfies the relation $\omega = \omega_1 - \omega_2 = \pm \Delta/h$, atoms are again allowed to tunnel between adjacent sites. The two light fields need only be present where the Wannier functions overlap significantly (that is, between adjacent lattice sites). Therefore, we can control the magnitude of the effective tunneling parameter through control of the amplitudes of the two running waves we project onto the system. In Refs. [29, 32], the hopping phases were controlled via the relative directions of the two running waves. Here, however, we propose to control the amplitude and phase of the tunneling parameter by locally shaping one of these two running waves using a spatial light modulator (SLM).

The required tunneling phases are controlled via projection of two counterpropagating light potentials from the top and bottom of the lattice, respectively, with both beams running orthogonal to the lattice axes. The first laser acts as a light sheet onto the atoms from one direction and does not require high-resolution capabilities. Then, through the high-resolution objective, one can project a second light-based potential with a phase and amplitude pattern mapped onto it via a SLM. In this way, one can engineer the local tunneling properties by carefully configuring the system so that light is present only between the adjacent lattice sites where tunneling is desired. If the resolution of the objective is high enough such that the point-spread function of the projection system is comparable to (or smaller than) the distance between lattice sites (see, e.g., the system in Ref. [36]), one can project these light potentials onto the lattice with minimal crosstalk between sites. Even in the presence of small amounts of crosstalk, the SLM-generated light field can readily be modified such that the desired field amplitudes and phases are generated at each lattice site. Finally, given that crosstalk is small, SLMs can also be used to project (again through the high-resolution objective) the local trapping potentials required for anyon
technologies hub for quantum computing and simulation through a semper ardens grant. work at the university by the independent research fund denmark under grant fractal lattices that could support topological order. For the fractal lattice, we have found that the topological properties remain in the presence of disorder. In fact, the fractal lattice enhances the robustness of the topological state against the effects of disorder. We hypothesise that this enhancement is due to the different non-integer scaling dimension that characterises the fractal lattice, and future research in this direction is warranted. We have proposed the experimental implementation of the local model introduced here with ultracold atoms in optical lattices. The experimental realization of the local model studied could allow for the consideration of different geometries of the lattice, including potentially different fractal lattices that could support topological order.

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