THE LINEARISED EINSTEIN EQUATIONS AS A GAUGE THEORY

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Abstract. We linearise the Einstein vacuum equations with a cosmological constant via the Calabi operator from projective differential geometry.

1. Introduction

This article concerns the vacuum Einstein equations with a cosmological constant

\[ R_{ab} = \lambda g_{ab}, \]

where \( g_{ab} \) is a (semi-)Riemannian metric and \( R_{ab} \) its Ricci tensor. By the contracted Bianchi identity, the smooth function \( \lambda \) is obliged to be constant. To investigate these equations, one often linearises around the flat metric, i.e. considers a metric of the form

\[ \eta_{ab} + \epsilon h_{ab}, \]

where \( \eta_{ab} \) is the Euclidean or Minkowski metric and \( h_{ab} \) is symmetric tensor. For \( h_{ab} \) fixed, this is non-singular for \( \epsilon \) sufficiently small and one can compute its various curvatures, including the Ricci tensor. The linearised Einstein equations arise by keeping only the linear terms in \( \epsilon \) (e.g. [6, §5.7] or [7, §7.5]).

The aim of this article is to linearise around a general vacuum solution, i.e. to consider a metric of the form

\[ \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}, \]

where the ‘background metric’ \( g_{ab} \) satisfies (1). Writing \( \bigodot^2 \Lambda^1 \) for the bundle of symmetric covariant 2-tensors, we shall show that there is a second order linear differential operator

\[ \mathcal{P} : \bigodot^2 \Lambda^1 \to \bigodot^2 \Lambda^1, \]

whose kernel consists of \( h_{ab} \) whose corresponding perturbed metric (2) also satisfies (1) to first order in \( \epsilon \). Furthermore, if \( \mathcal{K} : \Lambda^1 \to \bigodot^2 \Lambda^1 \) is the Killing operator \( \nabla_a X_b \mapsto \nabla_{(a} X_{b)}, \) then we shall find that

\[ \Lambda^1 \xrightarrow{\mathcal{K}} \bigodot^2 \Lambda^1 \xrightarrow{\mathcal{P}} \bigodot^2 \Lambda^1 \]

is a complex, i.e. the composition \( \mathcal{P} \circ \mathcal{K} \) vanishes. The range of \( \mathcal{K} \) can be interpreted as the infinitesimal changes in metric of the form \( \mathcal{L}_X g_{ab} \), where \( \mathcal{L}_X \) denotes the Lie derivative along a vector field \( X^a \), i.e. the infinitesimal changes in metric merely due to infinitesimal coordinate changes (e.g. [6 (5.7.11)] or [7 (C.2.17)]). Thus, the complex (3) gives a local ‘potential/gauge’ description of the linearised Einstein equations.

It it not at all obvious that a complex of the form (3) should exist. Indeed, for an arbitrary background metric one naïvely might expect a complex of the form

\[ \Lambda^1 \xrightarrow{\mathcal{K}} \bigodot^2 \Lambda^1 \xrightarrow{\mathcal{L}} \bigodot^2 \Lambda^1 \]

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where \( \mathbb{H} \) denotes the bundle of tensors \( X_{abcd} \) with Riemann symmetries \( X_{abcd} = X_{[ab][cd]} \) and \( X_{[abcd]} = 0 \) but, as we shall see, such a complex exists only on a background of constant sectional curvature.

The notation and conventions in this article follow [6]. In particular, indices are always abstract and do not entail any choice of local coordinates. Round brackets on indices denote symmetrisation whilst square brackets denote skew-symmetrisation (as in the upcoming formula (6)). We shall denote the cotangent bundle by \( \wedge^1 \) and our convention for the curvature \( R_{abcd} \) of a torsion-free affine connection \( \nabla_a \) is so that
\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)X^c = R_{ab}{}^e_d X^d,
\]
for all vector fields \( X^c \).

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2. The Calabi operator

There is a natural candidate for the operator \( C \) in (4). In [4] it is shown that, suitably interpreted, the Killing operator \( X_b \mapsto \nabla(a X_b) \) is projectively invariant and that there is a projectively invariant operator \( C : \bigodot^2 \wedge^1 \to \mathbb{H} \) given by
\[
h_{ab} \mapsto \nabla(a \nabla_c) h_{bd} - \nabla(b \nabla_c) h_{ad} - \nabla(a \nabla_d) h_{bc} + \nabla(b \nabla_d) h_{ac} - R_{ab}{}^e_d h_{ce} - R_{cd}{}^e_a h_{be},
\]
which we shall refer to as the Calabi operator. As observed in [3], the composition \( C \circ K \) is given by
\[
X_a \mapsto 2R_{ab}{}^e_a \mu_{de} + 2R_{cd}{}^e_a \mu_{be} - (\nabla^e R_{abcd}) X_e, \quad \text{where} \ \mu_{ab} \equiv \nabla(a X_b).
\]
In particular, this composition vanishes if and only if \( g_{ab} \) has constant sectional curvature. Indeed, Calabi showed [1] that the complex (4) is locally exact in this case (cf. [3]). The Calabi operator therefore has the optimal symbol and it follows that there are no better curvature terms.

3. The deformation operator

Even so, the Calabi operator is not exactly what we obtain by deforming the metric. The following proposition is taken from [3].

**Proposition 1.** If \( \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab} \), then the corresponding Riemann curvature tensor is
\[
\tilde{R}_{ab}{}^c_d = R_{ab}{}^c_d - \frac{\epsilon}{2} \left[ \nabla(a \nabla_c) h_{bd} - \nabla(b \nabla_c) h_{ad} - \nabla(a \nabla_d) h_{bc} + \nabla(b \nabla_d) h_{ac} \right] + O(\epsilon^2).
\]
**Proof.** A straightforward computation. \( \square \)

Despite the unqualified assertion in [4], notice the opposite sign in front of the curvature terms in comparison to (6). For the Ricci tensor, we obtain:

**Proposition 2.** If \( \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab} \), then the corresponding Ricci curvature tensor is
\[
\tilde{R}_{bd} = R_{bd} + \epsilon \left[ R_{(b}{}^{c}h_{d)e} - \frac{1}{2} g^{ac} (Ch)_{abcd} \right] + O(\epsilon^2).
\]
**Proof.** The inverse of the perturbed metric is
\[
\tilde{g}^{ab} = g^{ab} - \epsilon h^{ab} + O(\epsilon^2),
\]
where \( h^{ab} \equiv g^{ac}g^{bd}h_{cd} \), in line with the usual conventions for ‘raising and lowering indices,’ always with respect to the background metric \( g_{ab} \). We must compute \( \tilde{R}_{bd} \equiv \tilde{g}^{ac}\tilde{R}_{abcd} \) and for this let us write (8) as

\[
\tilde{R}_{abcd} = R_{abcd} - \epsilon \left[ \frac{1}{2} (Ch)_{abcd} + R_{abc}^\epsilon h_{d\epsilon} + R_{cde}^\epsilon [a h_{b\epsilon}] \right] + O(\epsilon^2).
\]

Then

\[
\tilde{R}_{bd} = R_{bd} - \epsilon \left[ h^{ac}R_{abcd} + \frac{1}{2} g^{ac}(Ch)_{abcd} + g^{ac}(R_{abc}^\epsilon h_{d\epsilon} + R_{cde}^\epsilon [a h_{b\epsilon}]) \right] + O(\epsilon^2).
\]

However,

\[
g^{ac}(R_{abc}^\epsilon h_{d\epsilon} + R_{cde}^\epsilon [a h_{b\epsilon}]) = -R_{b}^{\epsilon d} h_{d\epsilon} - h^{ac}R_{abcd}
\]

and there is some cancellation to yield (9). \( \square \)

4. IMPOSING THE EINSTEIN EQUATIONS

**Proposition 3.** If \( g_{ab} \) satisfies (1) and \( h_{ab} = \nabla(a X_b) \) for some 1-form \( X_a \), then

\[
g^{ac}(Ch)_{abcd} = 0.
\]

**Proof.** If we write \( \mu_{ab} \equiv \nabla(a X_b) \), then it follows from (7) that

\[
g^{ac}(Ch)_{abcd} = g^{ac}[2R_{abc}^\epsilon [c \mu_{d\epsilon}] + 2R_{cde}^\epsilon [a \mu_{b\epsilon}] - (\nabla^e R_{abcd})X_e] = -2R_{b}^{\epsilon d} \mu_{d\epsilon} - (\nabla^e R_{bd})X_e.
\]

Now, from (1), both of these terms vanish. \( \square \)

**Proposition 4.** If \( g_{ab} \) satisfies (1) and \( \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab} \), then

\[
\tilde{R}_{ab} = \lambda \tilde{g}_{ab} - \frac{1}{2} \epsilon g^{ac}(Ch)_{abcd} + O(\epsilon^2).
\]

In particular, the perturbed metric \( \tilde{g}_{ab} \) satisfies the vacuum Einstein equations \( \tilde{R}_{ab} = \lambda \tilde{g}_{ab} \) to first order in \( \epsilon \) if and only \( g^{ac}(Ch)_{abcd} = 0 \).

**Proof.** Immediate from (9). \( \square \)

5. THE EINSTEIN DEFORMATION COMPLEX

Let us define \( \mathcal{P} : \bigodot^2 \Lambda^1 \to \bigodot^2 \Lambda^1 \) by

\[
h_{ab} \mapsto g^{ac}(Ch)_{abcd},
\]
equivalently, using (5), by

\[
h_{ab} \mapsto \Delta h_{bd} - 2\nabla^e \nabla(b h_{d\epsilon}) + \nabla_b \nabla_d h + 2R_{b}^{\epsilon d} h_{d\epsilon}, \quad \text{where} \quad \Delta \equiv \nabla^a \nabla_a \text{ and } h \equiv h^a_a.
\]

**Theorem 1.** If \( g_{ab} \) satisfies the vacuum Einstein equations (1), then (3) is a complex. The kernel of \( \mathcal{P} \) consists of those symmetric tensors \( h_{ab} \) such that \( \tilde{g}_{ab} \equiv g_{ab} + \epsilon h_{ab} \) satisfies the vacuum Einstein equations \( \tilde{R}_{ab} = \lambda \tilde{g}_{ab} \) to first order in \( \epsilon \). The range of \( \mathcal{K} \) may be regarded as those \( h_{ab} \) arising from infinitesimal changes of coordinate.

**Proof.** The statements regarding \( \mathcal{P} \) follow from Propositions 3 and 4. Regarding \( \mathcal{K} \), it remains to observe that for any 1-form \( X_a \), the Lie derivative \( \mathcal{L}_X \) along the corresponding vector field \( X^a \), when applied to the metric \( g_{ab} \), gives

\[
\mathcal{L}_X g_{ab} = X^c \nabla_c g_{ab} = \nabla^c (\nabla_c X^b) g_{ab} + (\nabla_a X^c) g_{cb} + (\nabla_b X^c) g_{ac} = \nabla_a X_b + \nabla_b X_a = 2\nabla(a X_b),
\]

and we are done. \( \square \)
In the Ricci-flat case $R_{ab} = 0$, the kernel of (10) appears as [2 (2.6)] and is formulated there in terms of spinors when the dimension is four. The Ricci-flat equation

$$\Delta h_{bd} - 2\nabla^c \nabla_e (b h_{de}) + \nabla_b \nabla_d h = 0$$

also appears as [6 (5.7.14)] and [7 (7.5.15)] and, with non-zero Ricci tensor, is equivalent to [5 (2.5)].

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