Symmetry Analysis of Holes Localized on a Skyrmion in a Doped Antiferromagnet

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Abstract

We use the low-energy effective field theory for holes coupled to the staggered magnetization in order to investigate the localization of holes on a Skyrmion in a square lattice antiferromagnet. When two holes get localized on the same Skyrmion, they form a bound state. The quantum numbers of the bound state are determined by the quantization of the collective modes of the Skyrmion. Remarkably, for p-wave states the quantum numbers are the same as those of a hole-pair bound by one-magnon exchange. Two holes localized on a Skyrmion with winding number $n = 1$ or 2 may have s- or d-wave symmetry as well. Possible relations with pre-formed Cooper pairs of high-temperature superconductors are discussed.
1 Introduction

In the cuprates, high-temperature superconductivity is separated from antiferromagnetism by a pseudo-gap regime. It has been conjectured that the relevant low-energy degrees of freedom in the pseudo-gap regime are responsible for superconductivity as well. Reliably identifying those degrees of freedom by theoretical investigations is a highly non-trivial task, because unbiased first principles analytic or numerical calculations in microscopic systems such as the Hubbard or t-J model are presently out of reach. In lightly doped antiferromagnets, on the other hand, the situation is more favorable. First of all, precise numerical simulations of undoped antiferromagnets [1-4], such as the Heisenberg model, are possible with the loop-cluster algorithm [5], and individual doped holes can also be simulated reliably [6, 7]. Second, the low-energy dynamics of lightly doped antiferromagnets can be described with a systematic effective field theory for magnons and holes. The pure magnon effective field theory has been developed in [8-11] and is completely analogous to chiral perturbation theory for the Goldstone pions in QCD [12]. In the past few years, the systematic effective field theory for magnons and doped holes has been constructed [13, 14] in complete analogy to baryon chiral perturbation theory — the effective theory for pions and nucleons [15-18]. In contrast to previous attempts to construct effective theories for magnons and holes in a square lattice antiferromagnet [19-23], the construction of [14] is based on a systematic symmetry analysis and provides a complete set of all terms contributing to the effective action at leading and sub-leading order. As a result, the predictions of the effective theory are exact, order by order in a systematic derivative expansion. In particular, the low-energy physics of any lightly doped antiferromagnet is described quantitatively once some low-energy parameters (such as the spin stiffness or the spinwave velocity of the underlying microscopic system) have been fixed either by experiment or by numerical simulations. The effective theory has been used in systematic studies of magnon-mediated two-hole bound states [24] and of spiral phases [25]. Earlier (but somewhat less systematic) studies had been presented in [23, 26, 27]. Systematic effective field theories have also been constructed for antiferromagnets on a honeycomb lattice [28, 29], as well as for lightly electron-doped antiferromagnets [30].

Unfortunately, before one enters the high-temperature superconductor or even just the pseudo-gap regime, both antiferromagnetism and the systematic effective theory that describes it break down. While one might expect that one can hence not learn anything about high-temperature superconductivity or the pseudo-gap regime from the effective theory, the situation may not be entirely hopeless. In particular, the effective theory still contains information about what objects may form when the theory is about to break down. In this way, we can identify new candidate low-energy degrees of freedom for which another effective theory with an extended validity range can be constructed. In this paper, we do not yet attempt to construct an effective field theory for the pseudo-gap regime. Instead, we concentrate on the identification of new low-energy objects that may form when antiferromagnetism is about to break
down.

When antiferromagnetism is weakened, the spin stiffness $\rho_s$ is reduced. In particular, if antiferromagnetism is ultimately destroyed in a second order phase transition, $\rho_s$ vanishes at the transition. A small value of $\rho_s$ favors topological excitations in the staggered magnetization — the order parameter for antiferromagnetism. In $(2 + 1)$ dimensions, the topological excitations of the staggered magnetization vector are Skyrmions which carry a topologically conserved winding number $n \in \Pi_2[S^2] = \mathbb{Z}$ in the second homotopy group of the order parameter manifold $S^2$. The coset space $S^2 = SU(2)_s/U(1)_s$ arises because in an antiferromagnet the $SU(2)_s$ spin symmetry is spontaneously broken down to the subgroup $U(1)_s$. The possible role of Skyrmions as relevant excitations in quantum antiferromagnets has been discussed in several publications [31–45]. Haldane was first to realize that Skyrmions in an antiferromagnet are associated with a geometric phase [31]. When Skyrmions proliferate, antiferromagnetic order is destroyed. Read and Sachdev showed that on a square lattice the Skyrmion’s geometric phase then implies a competing valence bond solid order with 4-fold degeneracy [32]. The interplay of geometric phases and competing orders has been discussed in detail in [43]. The suppression of Skyrmions has been related to unconventional deconfined quantum critical points [38, 39]. It has also been argued that a hole localized near a dopant stabilizes a Skyrmion texture in the staggered magnetization [33–37]. The analogies between pions in QCD and magnons in ferro- and antiferromagnets have been investigated in detail in [40, 41]. In particular, it was argued that Skyrmions endowed with fermion number 2 may act as preformed Cooper pairs of high-temperature superconductivity. Experimental evidence for Skyrmions in the lightly doped insulating antiferromagnet La$_2$Cu$_{1-x}$Li$_x$O$_4$ in an external magnetic field has been reported in [44]. Furthermore, the possible role of Skyrmions for the superconductivity of Fe based pnictides and chalogenides has been discussed in [45]. In this paper, for the first time, we investigate the localization of holes on a Skyrmion using the low-energy effective theory for lightly hole-doped antiferromagnets on a square lattice. In particular, we carefully quantize the Skyrmion’s collective modes, which allows us to unambiguously determine the quantum numbers of single holes as well as hole pairs localized on a Skyrmion.

At the classical level, the mass of a Skyrmion is given by $4\pi\rho_s$. When $\rho_s$ becomes small, these excitations hence become energetically favorable. Skyrmions are beyond reach of the systematic derivative expansion of the low-energy effective theory for magnons and holes. Indeed, when Skyrmions become relevant low-energy degrees of freedom, antiferromagnetism as well as the effective theory that describes it are about to break down. Still, the effective theory correctly describes the way in which holes couple to a Skyrmion excitation in the staggered magnetization order parameter. In particular, holes may get localized on a Skyrmion. When two holes get localized on the same Skyrmion, they form a bound state which may represent a relevant low-energy degree of freedom even when antiferromagnetism gives way to the pseudo-gap phase. In particular, such bound states are a potential candidate for preformed pairs whose
condensation may ultimately lead to high-temperature superconductivity. In order to
decide whether this is a viable scenario, in this paper we investigate the symmetry
properties of Skyrmion-hole bound states in great detail. We find that the p-wave
states of two holes localized on a Skyrmion with winding number \( n = 1 \) transform
exactly like the two-hole states weakly bound by one-magnon exchange. Two holes
localized on a Skyrmion may also have s- or d-wave symmetry. Which of these states
is energetically most favorable depends on the details of the dynamics, and will remain
a subject for future investigations.

The rest of the paper is organized as follows. In Section 2 the effective theory
for the staggered magnetization order parameter is introduced and Skyrmions are
discussed as classical solutions. The Hopf term is introduced and the collective modes
of a rotating Skyrmion are then quantized. In Section 3 doped holes are added to
the effective theory. In Section 4 states of single holes as well as a pair of holes
(residing in two different hole pockets) localized on a static or rotating Skyrmion
are constructed and their symmetry properties are investigated. Possible relations to
the mechanism responsible for high-temperature superconductivity are also discussed.
Section 5 contains our conclusions. Finally, the case of two holes residing in the same
hole pocket is investigated in Appendix A.

2 Skyrmions in the Effective Theory for the Staggered Magnetization

In this section we discuss the collective mode quantization of Skyrmions in the low-
energy effective theory for antiferromagnetic magnons.

2.1 Effective Action and its Symmetries

Magnons are the Goldstone bosons of a spontaneously broken spin symmetry \( SU(2)_s \)
with an unbroken subgroup \( U(1)_s \). Consequently, magnons are described by a 3-
component unit-vector field \( \vec{e}(x) \in S^2 \) in the coset space \( S^2 = SU(2)_s/U(1)_s \). Here
\( x = (x_1, x_2, t) \) is a point in \( (2 + 1) \)-d Euclidean space-time and \( \vec{e}(x) \) represents the
direction of the local staggered magnetization vector — the order parameter for the
spontaneously broken spin symmetry. To leading order in a systematic derivative
expansion, the Euclidean low-energy effective action for the magnons is given by

\[
S[\vec{e}] = \int d^2x \, dt \, \frac{\rho_s}{2} \left( \partial_t \vec{e} \cdot \partial_t \vec{e} + \frac{1}{c^2} \partial_\nu \vec{e} \cdot \partial_\nu \vec{e} \right),
\]

Here \( \rho_s \) is the spin stiffness and \( c \) is the spinwave velocity. The vacuum configuration of
the effective theory is described by a constant staggered magnetization vector which
can be chosen to point in the 3-direction, i.e. \( \vec{e}(x) = (0, 0, 1) \). Magnons are small fluctuations around the vacuum configuration. It should be noted that, in contrast to a ferromagnet, antiferromagnetic magnons have a “relativistic” dispersion relation.

The most important symmetry of the action is the spontaneously broken spin symmetry \( SU(2)_s \). In the following, global transformations in the unbroken subgroup \( U(1)_s \) will play an important role. Introducing

\[
\vec{e}(x) = (\sin \theta(x) \cos \varphi(x), \sin \theta(x) \sin \varphi(x), \cos \theta(x)),
\]

these transformations take the form

\[
I(\gamma)\vec{e}(x) = (\sin \theta(x) \cos(\varphi(x) + \gamma), \sin \theta(x) \sin(\varphi(x) + \gamma), \cos \theta(x)).
\]

It should be pointed out that the \( SU(2)_s \) spin symmetry plays the role of an internal symmetry (analogous to chiral symmetry in particle physics). Consequently, its unbroken \( U(1)_s \) subgroup (which is analogous to isospin in particle physics) should also be viewed as an internal symmetry. Because of the analogy with isospin, we denote transformations in the unbroken subgroup \( U(1)_s \) by \( I(\gamma) \).

In addition to the \( SU(2)_s \) spin symmetry, the effective action has other symmetries as well. First of all, due to the relativistic dispersion relation of antiferromagnetic magnons, the leading terms in the effective action have an emergent accidental Poincaré symmetry which is not present in the underlying Hubbard or \( t-J \) model, and which will thus be explicitly broken by higher-order terms in the effective action containing a larger number of derivatives. The remaining symmetries are the discrete translations and rotations of the underlying quadratic lattice. Similar to the spin symmetry, the displacements \( D_i \) by one lattice spacing in the \( i \)-direction are also spontaneously broken in an antiferromagnet. They act on the staggered magnetization field as

\[
D_i \vec{e}(x) = -\vec{e}(x).
\]

Since the shift symmetries \( D_i \) are spontaneously broken in an antiferromagnet, it is convenient to also introduce modified shift symmetries \( D'_i \) which combine \( D_i \) with an \( SU(2)_s \) spin rotation \( g = i\sigma_2 \) such that

\[
D'_i \vec{e}(x) = (e_1(x), -e_2(x), e_3(x)).
\]

Spatial translations by an even number of lattice spacings, on the other hand, remain unbroken. Such translations \( D(x_0) \) by a distance vector \( x_0 = (x_{01}, x_{02}, 0) \) act as

\[
D(x_0) \vec{e}(x) = \vec{e}(x - x_0).
\]

Similarly, parametrizing \( x = (r \cos \chi, r \sin \chi, t) \), spatial rotations by an angle \( \beta \) act as

\[
O(\beta) \vec{e}(x) = \vec{e}(O(\beta)x), \ O(\beta)x = (r \cos(\chi + \beta), r \sin(\chi + \beta), t),
\]
and a spatial reflection at the $x_1$-axis is represented by

$$ R \vec{e}(x) = \vec{e}(Rx), \quad Rx = (x_1, -x_2, t) = (r \cos \chi, -r \sin \chi, t). \quad (2.8) $$

Finally, time reversal, which changes the direction of a spin, acts as

$$ T \vec{e}(x) = -\vec{e}(Tx), \quad Tx = (x_1, x_2, -t) = (r \cos \chi, r \sin \chi, -t). \quad (2.9) $$

The effective action of eq. (2.1) is invariant under all these symmetries.

### 2.2 Classical Skyrmion Solutions

In particle physics Skyrmions arise as topological excitations in the pion effective field theory for the strong interactions \[46\], which takes the form of a $(3 + 1)$-d $SU(2)_L \times SU(2)_R = O(4)$ model. In order to distinguish them from their particle physics analogs, the topological excitations in the $(2+1)$-d $O(3)$ model are sometimes denoted as baby-Skyrmions. For simplicity, here we also refer to them just as Skyrmions. Skyrmions are topologically non-trivial classical solutions of the magnon effective theory with integer winding number

$$ n[\vec{e}] = \frac{1}{8\pi} \int d^2 x \varepsilon_{ij} \vec{e} \cdot [\partial_i \vec{e} \times \partial_j \vec{e}] \in \Pi_2[S^2] = \mathbb{Z}, \quad (2.10) $$

in the second homotopy group of the sphere $S^2$. Correspondingly, there is a topological current

$$ j_\mu(x) = \frac{1}{8\pi} \varepsilon_{\mu
u\rho} \vec{e}(x) \cdot [\partial_\nu \vec{e}(x) \times \partial_\rho \vec{e}(x)], \quad (2.11) $$

which is conserved, i.e. $\partial_\mu j_\mu(x) = 0$, irrespective of the classical equations of motion. The winding number $n[\vec{e}] = \int d^2 x \ j_t(x)$ is just the integrated topological charge density. Under the various symmetries the topological charge density transforms as

\begin{align*}
U(1)_s : & \quad I^{(\gamma)} j_t(x) = j_t(x), \\
D_i : & \quad D^i j_t(x) = -j_t(x), \\
D'_i : & \quad D'^i j_t(x) = -j_t(x), \\
O(\beta) : & \quad O(\beta) j_t(x) = j_t(O(\beta)x), \\
R : & \quad R j_t(x) = -j_t(Rx), \\
T : & \quad T j_t(x) = -j_t(Tx). \quad (2.12)
\end{align*}

In particular, the winding number changes sign under the displacements $D_i$ and $D'_i$ as well as under the reflection $R$ and under the time reversal $T$.

Let us consider static classical solutions for which the energy

$$ E[\vec{e}] = \int d^2 x \frac{\rho}{2} \partial_i \vec{e} \cdot \partial_i \vec{e} \quad (2.13) $$
is minimized. We can write
\[0 \leq \int d^2x \, (\partial_i \vec{e} \pm \varepsilon_{ij} \partial_j \vec{e} \times \vec{e})^2\]
\[= \int d^2x \, (2\partial_i \vec{e} \cdot \partial_i \vec{e} \pm 2\varepsilon_{ij} \partial_j \vec{e} \cdot (\partial_i \vec{e} \times \partial_j \vec{e})) = \frac{4}{\rho_s} E[\vec{e}] \pm 16\pi n[\vec{e}], \tag{2.14}\]
which implies the Schwarz inequality
\[E[\vec{e}] \geq 4\pi \rho_s |n[\vec{e}]|. \tag{2.15}\]

Skyrmions are minima of the energy in the topological sector with \(n[\vec{e}] = 1\), while anti-Skyrmions have \(n[\vec{e}] = -1\). At the classical level both have a rest energy of \(Mc^2 = 4\pi \rho_s\). (Anti-)Skyrmions satisfy the previous inequality as an equality which is possible only if they satisfy the (anti-)self-duality equation
\[\partial_i \vec{e} + \sigma \varepsilon_{ij} \partial_j \vec{e} \times \vec{e} = 0. \tag{2.16}\]
Here \(\sigma = \pm 1\) distinguishes between Skyrmions and anti-Skyrmions. It is worth mentioning that static (anti-)Skyrmions are mathematically equivalent to (anti-)instantons of the 2-d \(O(3)\) model \([47]\). Using polar coordinates \((x_1, x_2) = r(\cos \chi, \sin \chi)\), a particular (anti-)Skyrmion configuration is given by
\[\vec{e}_{\sigma, n, \rho}(r, \chi) = \left(\frac{2r^n \rho^n}{r^{2n} + \rho^{2n}} \cos(n\chi), \frac{2r^n \rho^n \sigma}{r^{2n} + \rho^{2n}} \sin(n\chi), \frac{r^{2n} - \rho^{2n}}{r^{2n} + \rho^{2n}}\right). \tag{2.17}\]

Depending on the sign of \(\sigma\), this configuration describes a Skyrmion or anti-Skyrmion of winding number \(n[\vec{e}] = \sigma n\) (with \(n \in \mathbb{N}_0\)) and size \(\rho\) centered at the origin. It should be noted that there are many other multi-Skyrmion configurations with different Skyrmions located in different positions. Such configurations would be important in investigations of a Skyrmion gas or liquid. Here we concentrate on a Skyrmion centered at a single point, possibly with a larger winding number than just \(n = 1\). The winding is chosen to arise from the angular \(\chi\)-dependence which influences the rotational symmetry of the Skyrmion and not from the radial \(r\)-dependence which only influences the finer details of the dynamics.

The Skyrmion configurations of eq. (2.17) have a number of zero-modes. In particular, their energy remains unchanged when they are shifted to an arbitrary position \(x\), when they are spatially rotated by an arbitrary angle \(\beta\), or when they are \(U(1)_s\) spin-rotated by an arbitrary angle \(\gamma\). Interestingly, spatial rotations and \(U(1)_s\) spin rotations act on a Skyrmion in a similar manner, i.e.
\[O(\beta) \vec{e}_{\sigma, n, \rho}(r, \chi) = \left(\frac{2r^n \rho^n}{r^{2n} + \rho^{2n}} \cos(n(\chi + \beta)), \frac{2r^n \rho^n \sigma}{r^{2n} + \rho^{2n}} \sin(n(\chi + \beta)), \frac{r^{2n} - \rho^{2n}}{r^{2n} + \rho^{2n}}\right),\]
\[I(\sigma\gamma) \vec{e}_{\sigma, n, \rho}(r, \chi) = \left(\frac{2r^n \rho^n}{r^{2n} + \rho^{2n}} \cos(n\chi + \gamma), \frac{2r^n \rho^n \sigma}{r^{2n} + \rho^{2n}} \sin(n\chi + \gamma), \frac{r^{2n} - \rho^{2n}}{r^{2n} + \rho^{2n}}\right), \tag{2.18}\]
such that

\[ I(\sigma) \vec{e}_{\sigma,n,\rho}(r,\chi) = O(\gamma/n) \vec{e}_{\sigma,n,\rho}(r,\chi). \]  

(2.19)

Another zero-mode is related to dilations. Indeed, the energy of a Skyrmion also remains invariant under changes of the scale parameter \( \rho \). A family of Skyrmion configurations is obtained by spin-rotating the original Skyrmion of eq.(2.17) by an angle \( \sigma \gamma \) and then shifting it by a distance-vector \( x \) such that

\[ \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = D(x) \left[ I(\sigma) \vec{e}_{\sigma,n,\rho}(r,\chi) \right]. \]  

(2.20)

Under the various unbroken symmetry transformations the configuration of eq.(2.20) transforms as

\[ U(1)_s : \quad I(\sigma_{\gamma_{\rho}}) \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi), \]
\[ D'_i : \quad D'_i \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = \vec{e}_{-\sigma,n,\rho,x,\gamma}(r,\chi), \]
\[ D : \quad D(x_0) \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = \vec{e}_{\sigma,n,\rho,x+x_0,\gamma}(r,\chi), \]
\[ O(\beta) : \quad O(\beta) \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = \vec{e}_{\sigma,n,\rho,O(\beta)x,\gamma+n(\beta,\rho)}(r,\chi), \]
\[ R : \quad R \vec{e}_{\sigma,n,\rho,x,\gamma}(r,\chi) = \vec{e}_{-\sigma,n,\rho,Rx,\gamma}(r,\chi). \]  

(2.21)

In particle physics Skyrmions play an interesting role in the effective theory for the strong interactions. In particular, Skyrmions arise as topological excitations in the pion field \[46\]. While Skyrmions are outside the validity range of the systematic low-energy expansion of chiral perturbation theory, they have been used to model baryons phenomenologically \[48\]. Remarkably, the \( \Pi_3[S^3] \) topological winding number of the Skyrmions of the strong interactions has the same symmetry properties as the baryon number, and is indeed identified with it. The identification of Skyrmions as baryons can even be established within the framework of chiral perturbation theory, by investigating the electromagnetic interactions of pions which are affected by a Goldstone-Wilczek current \[40, 41, 49, 50\]. Since the underlying QCD theory has a conserved baryon number current, the conservation of the topological Skyrme current is guaranteed beyond the semi-classical regime.

It is natural to ask whether the winding number \( n[\vec{e}] \in \Pi_2[S^2] \) of the Skyrmions in an antiferromagnet can also be identified with a conserved quantity of an underlying microscopic system, such as the Hubbard model. In particular, in analogy to particle physics, one might suspect that the winding number can be identified with the fermion number of doped holes. However, this is not the case because the winding number and the fermion number have different symmetry properties. In particular, the winding number changes sign under a shift \( D_i \) by one lattice spacing, while the fermion number does not. Hence, unlike in particle physics, in an antiferromagnet the conservation of the topological current is not protected by the underlying microscopic dynamics and may thus be limited to the semi-classical regime. Interestingly, when holes get localized on a Skyrmion, they endow the Skyrmion with their conserved fermion number, which may stabilize the Skyrmion beyond the semi-classical regime.
The conservation of the topological current also plays a central role in the scenario of deconfined quantum criticality [39] in which dynamically generated gauge fields and deconfined spinons are conjectured to appear at a new type of quantum phase transition outside the realm of the standard Ginsburg-Landau-Wilson paradigm. In fact, the suppression of Skyrmion number violating (so-called monopole) events has been argued to change the universality class of the phase transition in the \((2 + 1)\)-d \(O(3)\) model [38]. A better understanding of the role of Skyrmions would thus also be useful for addressing the issue of deconfined quantum criticality.

2.3 The Hopf Term

The integer winding number \(n[\vec{e}]\) is defined at any instant of time and is conserved for topological reasons. Interestingly, there is another topological invariant — the Hopf number \(H[\vec{e}]\) — which characterizes the topology of the order parameter field \(\vec{e}(x)\) as a function of both space and time. The integer-valued Hopf number \(H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}\) is an element of the third homotopy group of the sphere \(S^2\). In order to construct the Hopf term, it is most convenient to introduce the \(\mathbb{C}P(1)\) representation

\[
P(x) = \frac{1}{2} (1 + \vec{e}(x) \cdot \vec{\sigma})
\]

of the staggered magnetization field. Here \(\vec{\sigma}\) are the Pauli matrices and, as a result, \(P(x)\) is a Hermitean \(2 \times 2\) projector matrix that obeys

\[
P(x)^\dagger = P(x), \quad P(x)^2 = P(x), \quad \text{Tr}P(x) = 1.
\]

Under a spin rotation \(g \in SU(2)_s\) the matrix \(P(x)\) transforms as

\[
P(x)' = g P(x) g^\dagger.
\]

The matrix \(P(x)\) can be diagonalized by a unitary transformation \(u(x) \in SU(2)_s\), i.e.

\[
u(x) P(x) u(x)^\dagger = \frac{1}{2}(1 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{11}(x) \geq 0.
\]

We demand that \(u_{11}(x)\) is real and positive, which fixes a \(U(1)_s\) gauge ambiguity and uniquely determines \(u(x)\) as

\[
u(x) = \frac{1}{\sqrt{2(1 + e_3(x))}} \begin{pmatrix} 1 + e_3(x) & e_1(x) - ie_2(x) \\ -e_1(x) - ie_2(x) & 1 + e_3(x) \end{pmatrix} = \begin{pmatrix} \cos \left( \frac{1}{2} \theta(x) \right) & \sin \left( \frac{1}{2} \theta(x) \right) \exp(-i\varphi(x)) \\ -\sin \left( \frac{1}{2} \theta(x) \right) \exp(i\varphi(x)) & \cos \left( \frac{1}{2} \theta(x) \right) \end{pmatrix} = \cos \left( \frac{1}{2} \theta(x) \right) + i \sin \left( \frac{1}{2} \theta(x) \right) \vec{e}_\varphi(x) \cdot \vec{\sigma}.
\]

\(\square\)
where the unit-vector $\vec{e}_x(x)$ is given by
\[ \vec{e}_x(x) = (-\sin \varphi(x), \cos \varphi(x), 0). \] (2.27)
Under a global $SU(2)_s$ transformation $g$, the diagonalizing field $u(x)$ transforms as
\[ u(x)' = h(x)u(x)g^\dagger, \quad u_{11}' \geq 0, \] (2.28)
which implicitly defines the nonlinear symmetry transformation
\[ h(x) = \exp(i\alpha(x)\sigma_3) = \begin{pmatrix} \exp(i\alpha(x)) & 0 \\ 0 & \exp(-i\alpha(x)) \end{pmatrix} \in U(1)_s. \] (2.29)
In this way, the global transformations $g \in SU(2)_s$ of the spontaneously broken non-Abelian spin symmetry “disguise” themselves as local transformations $h(x) \in U(1)_s$ of the unbroken subgroup. The global subgroup transformations $I(\gamma)$ introduced in eq.(2.3) simply lead to $\alpha(x) = -\gamma/2$.

The diagonalizing matrix $u(x)$ maps space-time onto the group manifold $S^3$ of $SU(2)_s$. When the $(2 + 1)$-d space-time is also compactified to $S^3$, one can relate the Hopf number $H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}$ to the topological winding number $W[u] \in \Pi_3[SU(2)_s] = \Pi_3[S^3] = \mathbb{Z}$, i.e.
\[ H[\vec{e}] = W[u] = \frac{1}{2\pi^2} \int dt \, d^2x \, \varepsilon_{\mu\rho\sigma} \text{Tr} \left[ \left( u^\dagger \partial_\mu u \right) \left( u^\dagger \partial_\nu u \right) \left( u^\dagger \partial_\rho u \right) \right]. \] (2.30)
It should be noted that the evaluation of eq.(2.30) requires some care. In particular, due to the $U(1)_s$ gauge fixing $u_{11}(x) \geq 0$, $u(x)$ covers only an $S^2$ subspace of the $SU(2)_s$ group manifold $S^3$. This may seem to imply that the winding number $W[u]$, which counts the number of times the map $u(x)$ covers $S^3$, should vanish. However, this is not the case because $u(x)$ in eq.(2.20) is singular at the Skyrminon center where $e_3(x) = -1$ (i.e. $\theta(x) = \pi$). The singularities which lie on a vortex line encircled by $\vec{e}_x(x)$ contribute non-trivially to eq.(2.30). Alternatively, one may remove the singularities in $u(x)$ by undoing the $U(1)_s$ gauge fixing $u_{11}(x) \geq 0$, which implies that $u(x)$ extends to all of $S^3$. Then eq.(2.30) can be evaluated in a straightforward manner. The Hopf term is $SU(2)_s$-invariant because
\[ W[u] = W[hu g^\dagger] = W[h] + W[u] - W[g] = W[u]. \] (2.31)
Here we have used $W[g] = 0$, which follows because $g$ is constant, and $W[h] = 0$, which follows because the Abelian gauge transformations $h(x) \in U(1)_s$ are topologically trivial in three dimensions, i.e. $\Pi_3[U(1)_s] = \Pi_3[S^1] = \{0\}$.

Under the various relevant symmetries the Hopf number transforms as
\[
\begin{align*}
U(1)_s : & \quad H[^{(\gamma)}] \vec{e} = H[\vec{e}], \\
D_4 : & \quad H[^{D_4}] \vec{e} = H[\vec{e}], \\
D_4' : & \quad H[^{D_4'}] \vec{e} = H[\vec{e}], \\
O(\beta) : & \quad H[^{O(\beta)}] \vec{e} = H[\vec{e}], \\
R : & \quad H[^{R}] \vec{e} = -H[\vec{e}], \\
T : & \quad H[^{T}] \vec{e} = -H[\vec{e}].
\end{align*}
\] (2.32)
The Hopf term gives rise to an additional factor $\exp(i\Theta H[\vec{e}])$ in the Euclidean path integral with $\Theta$ being the anyon statistics angle. In systems with reflection or time-reversal symmetry, the value of $\Theta$ is hence limited to 0 or $\pi$. As we will see, in these cases Skyrmions are quantized as bosons or fermions, respectively. In systems without reflection and time-reversal symmetry, arbitrary values of $\Theta$ are allowed, and then the Skyrmions may have any (neither integer nor half-integer) spin. By investigating field configurations in which two Skyrmions interchange their positions, one can also show that Skyrmions pick up a phase $\exp(i\Theta)$ and thus obey anyon statistics [51]. It should be noted that the Hopf term is expected to be absent in doped cuprates [31, 32, 52–54], while it is known to be present, for example, in quantum Hall ferromagnets [40, 57–59]. In order to keep the discussion as general as possible, we will include the Hopf term, although in the cuprates one expects $\Theta = 0$.

2.4 Collective Mode Quantization of the Skyrmion

Let us now consider the collective mode quantization of the Skyrmion. The main goal is to understand the quantum numbers of the quantized Skyrmion, first of all in an undoped system. It should be pointed out that Skyrmions in an undoped antiferromagnet are heavy objects whose pair-creation is suppressed at low temperatures. When antiferromagnetism is weakened by hole doping, the Skyrmion mass is reduced and, in addition, the holes may lower their mass by getting localized on a Skyrmion. This favors Skyrmion formation in doped antiferromagnets. A central goal of this paper is to understand the quantum numbers of the Skyrmion-hole bound states. In this subsection, we consider the collective mode quantization of a Skyrmion in the undoped system.

In order to perform the collective mode quantization, we consider the zero-mode parameters $\rho(t)$, $x(t)$, and $\gamma(t)$ as functions of time. We now evaluate the Euclidean action (including the Hopf term) for a time-dependent Skyrmion and (after a somewhat lengthy but straightforward calculation) we obtain

$$S[\vec{e}_{\sigma,n,\rho,x,\gamma}] + i\Theta H[\vec{e}_{\sigma,n,\rho,x,\gamma}] = \int dt \left( \mathcal{M} c^2 + \frac{M}{2} x^2 + \frac{D(\rho)}{2} \dot{\rho}^2 + \frac{I(\rho)}{2} \dot{\gamma}^2 + i n \frac{\Theta}{2\pi} \dot{\gamma} \right).$$

Here the Skyrmion’s rest energy is given by

$$\mathcal{M} c^2 = \rho_s \int d^2 x \frac{4\pi^2 \rho^{2n} r^{2n-2}}{(r^{2n} + \rho^{2n})^2} = 4\pi \rho_s n,$$

which confirms that for self-dual solutions the Schwarz inequality of eq. (2.15) is obeyed as an equality. For $n > 1$ the Skyrmion’s inertia against dilations takes the form

$$D(\rho) = \frac{\rho_s}{c^2} \int d^2 x \frac{4\pi^2 \rho^{2n} r^{2n-2}}{(r^{2n} + \rho^{2n})^2} = \frac{\pi \mathcal{M}}{n \sin(\pi/n)}.$$
For $n = 1$ the integral is logarithmically infrared divergent. In a finite volume or in a system with a finite density of Skyrmions, the infrared divergence may be regularized because the volume available to each Skyrmion becomes effectively finite. Indeed such effects are known to arise in the instanton gas of the 2-d $O(3)$ model [55, 56]. In this paper, we do not attempt to decide whether the same happens in the $(2 + 1)$-d $O(3)$ model that is relevant here. We just regularize $\mathcal{D}(\rho)$ by an infra-red cut-off $R$ which may or may not be infinite such that for $n = 1$

$$\mathcal{D}(\rho) = \frac{4\pi\rho_s}{c^2} \int_0^R dr \frac{2r^3}{(r^2 + \rho^2)^2} = \mathcal{M} \left( \log \frac{R^2 + \rho^2 - R^2}{R^2 + \rho^2} \right). \quad (2.36)$$

Finally, the moment of inertia of the Skyrmion is given by

$$I(\rho) = \frac{\rho_s c^2}{c^2} \int d^2 x = \frac{4n^2 \rho_{2n}^2}{(\rho_{2n}^2 + \rho^2)^2} = \frac{\mathcal{D}(\rho)\rho^2}{n^2}, \quad (2.37)$$

which is affected by the same infrared divergence as $\mathcal{D}(\rho)$. Hence, although the Skyrmion has a finite mass (and can thus undergo translational motion), in the limit of an infinite infra-red cut-off $R$ it has an infinite moment of inertia $I(\rho)$ and can thus not rotate.

From the Euclidean action of eq.(2.33) we read off the real-time Lagrange function as

$$L = \frac{\mathcal{M}}{2} \dot{x}^2 + \frac{\mathcal{D}(\rho)}{2} \left( \dot{\rho}^2 + \frac{\rho^2}{n^2} \dot{\gamma}^2 \right) - n \frac{\Theta}{2\pi} \dot{\gamma} - \mathcal{M}c^2. \quad (2.38)$$

In the next step, we consider the canonically conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \mathcal{M} \dot{x}_i, \quad p_{\rho} = \frac{\partial L}{\partial \dot{\rho}} = \mathcal{D}(\rho) \dot{\rho}, \quad p_{\gamma} = \frac{\partial L}{\partial \dot{\gamma}} = \frac{\mathcal{D}(\rho)\rho^2\dot{\gamma}}{n^2} - n \frac{\Theta}{2\pi}. \quad (2.39)$$

It should be noted that, at the classical level, the $\Theta$-term is suppressed because relative to $p_{\gamma}$ it is of order $\hbar$ (which we have put to 1). The canonically conjugate momenta lead to the classical Hamilton function

$$\mathcal{H} = p_{i} \dot{x}_{i} + p_{\rho} \dot{\rho} + p_{\gamma} \dot{\gamma} - L = \mathcal{M}c^2 + \frac{p_{i}^2}{2\mathcal{M}} + \frac{1}{2\mathcal{D}(\rho)} \left[ p_{\rho}^2 + \frac{n^2}{\rho^2} \left( p_{\gamma} + n \frac{\Theta}{2\pi} \right)^2 \right]. \quad (2.40)$$

The momentum $p_{i}$, the spin $p_{\gamma}$, and the energy

$$E = \frac{1}{2\mathcal{D}(\rho)} \left[ p_{\rho}^2 + \frac{n^2}{\rho^2} \left( p_{\gamma} + n \frac{\Theta}{2\pi} \right)^2 \right] = \frac{\mathcal{D}(\rho)}{2} \dot{\rho}^2 + \frac{n^2}{2\mathcal{D}(\rho)\rho^2} \left( p_{\gamma} + n \frac{\Theta}{2\pi} \right)^2 \quad (2.41)$$

of the coupled rotational and dilational motion are conserved quantities. The last equality determines the size $\rho(t)$ of the Skyrmion as a function of time

$$t = \int_{\rho(0)}^{\rho(t)} dp \left[ \frac{2E}{\mathcal{D}(\rho)} - \frac{n^2}{\mathcal{D}(\rho)^2 \rho^2} \left( p_{\gamma} + n \frac{\Theta}{2\pi} \right)^2 \right]^{-1/2}. \quad (2.42)$$
When the Skyrmion is rotating (i.e. when \( p_{\gamma} + n\frac{\Theta}{2\pi} \neq 0 \)), centrifugal forces lead to an unlimited increase of \( \rho(t) \).

Upon canonical quantization the momentum \( p_i \) and the spin \( p_{\gamma} \) turn into the operators

\[
p_i = -i\partial_{x_i}, \quad p_{\gamma} = -i\partial_{\gamma},
\]

while the classical Hamilton function \( \mathcal{H} \) turns into the quantum mechanical Hamiltonian

\[
\mathcal{H} = \mathcal{M}c^2 - \frac{1}{2\mathcal{M}}\partial_{x_i}^2 - \frac{1}{\sqrt{2D(\rho)}}\left( \frac{\partial^2}{\rho} + \frac{1}{\rho} \partial_{\rho} \right) \frac{1}{\sqrt{2D(\rho)}} - \frac{n^2}{2D(\rho)\rho^2} \left( \partial_{\gamma} + in\frac{\Theta}{2\pi} \right)^2.
\]

The collective mode wave function of a Skyrmion or anti-Skyrmion with winding number \( \sigma n \), momentum \( p_i \), and spin \( p_{\gamma} = \sigma m \in \mathbb{Z} \) takes the form

\[
\Psi_{p,\sigma,n,m}(x, \rho, \gamma) = \exp(ip_i x_i) \exp(i\sigma m \gamma) \psi(\rho).
\]

The dilational part of the wave function solves the Schrödinger equation

\[
\left[ -\frac{1}{\sqrt{2D(\rho)}}\left( \frac{\partial^2}{\rho} + \frac{1}{\rho} \partial_{\rho} \right) \frac{1}{\sqrt{2D(\rho)}} + \frac{n^2}{2D(\rho)\rho^2} \left( m + n\frac{\Theta}{2\pi} \right)^2 \right] \psi(\rho) = E\psi(\rho),
\]

which may again lead to an instability of a rotating Skyrmion against unlimited increase of its size \( \rho \). As we will see later, localized holes prevent the increase of \( \rho \) and thus stabilize the Skyrmion.

In the presence of the Hopf term the spin operator of the Skyrmion (which is analogous to isospin in particle physics) is given by

\[
I = \sigma \left( p_{\gamma} + n\frac{\Theta}{2\pi} \right) = \sigma \left( -i\partial_{\gamma} + n\frac{\Theta}{2\pi} \right).
\]

The state \( \Psi_{p,\sigma,n,m}(x, \rho, \gamma) \) hence has the “isospin”

\[
I\Psi_{p,\sigma,n,m}(x, \rho, \gamma) = \left( m + \sigma n\frac{\Theta}{2\pi} \right) \Psi_{p,\sigma,n,m}(x, \rho, \gamma).
\]

In particular, for \( \Theta = 0 \) the “isospin” is an integer, while for \( \Theta = \pi \) it is a half-integer for odd \( n \).

Let us also investigate the quantum numbers of the Skyrmion with respect to spatial rotations. As a consequence of eq. (2.19), the angular momentum \( J \) is given by

\[
J = \sigma n I = n \left( p_{\gamma} + n\frac{\Theta}{2\pi} \right) = n \left( -i\partial_{\gamma} + n\frac{\Theta}{2\pi} \right),
\]

such that

\[
J\Psi_{p,\sigma,n,m}(x, \rho, \gamma) = n \left( \sigma m + n\frac{\Theta}{2\pi} \right) \Psi_{p,\sigma,n,m}(x, \rho, \gamma).
\]
Hence, for $\Theta = 0$ the Skyrmion has integer angular momentum and thus is a boson, while for $\Theta = \pi$ the angular momentum is a half-integer and the Skyrmion is a fermion. Interestingly, in $(2 + 1)$ dimensions it is possible to have particles of any (neither integer nor half-integer) angular momentum — the anyons which arise for $\Theta \neq 0$ or $\pi$.

By construction, the Skyrmion state is also an eigenstate of the momentum operator with eigenvalue $p_i$. Under the modified shift symmetries $D_i'$ and under the reflection $R$ the Skyrmion state transforms as

$$U_{D_i'}\Psi_{p,\sigma,n,m}(x,\rho,\gamma) = \Psi_{p,-\sigma,n,m}(x,\rho,\gamma),$$

$$U_R\Psi_{p,\sigma,n,m}(x,\rho,\gamma) = \Psi_{Rp,-\sigma,n,m}(x,\rho,\gamma),$$

(2.51)

where $Rp = (p_1,-p_2)$ is the spatially reflected momentum. Here $U_{D_i'}$ and $U_R$ are unitary transformations representing the corresponding discrete symmetries in the Hilbert space of the collective modes of the Skyrmion. It should be noted that shifted or reflected Skyrmions (which have $\sigma = 1$) are actually anti-Skyrmions (with $\sigma = -1$).

3 Effective Action for Doped Holes

In order to make the paper self-contained, in this section we review the main features of the effective field theory constructed in [14] which couples doped holes to the staggered magnetization order parameter.

3.1 Nonlinear Realization of the $SU(2)_s$ Symmetry

In order to couple holes to the staggered magnetization order parameter, a nonlinear realization of the spontaneously broken $SU(2)_s$ symmetry has been constructed in [13]. The global $SU(2)_s$ symmetry then manifests itself as a local $U(1)_s$ symmetry in the unbroken subgroup. This is analogous to baryon chiral perturbation theory in which the spontaneously broken $SU(2)_L \otimes SU(2)_R$ chiral symmetry of QCD is implemented on the nucleon fields as a local $SU(2)_{L=R}$ transformation in the unbroken isospin subgroup.

The definition of the nonlinear realization of the $SU(2)_s$ symmetry is based on the diagonalizing matrix $u(x)$ defined in eq.(2.26), which transforms as

$$D_i' u(x) = u(x)^*, \quad (3.1)$$

under the modified displacement symmetry $D_i'$. Introducing the traceless anti-Hermitian field

$$v_\mu(x) = u(x)\partial_\mu u(x)^\dagger, \quad (3.2)$$

14
one obtains the following transformation rules

\[ SU(2)_s : \quad v_\mu(x)' = h(x)[v_\mu(x) + \partial_\mu h(x)], \]
\[ D'_i : \quad D'_i v_\mu(x) = v_\mu(x)^*, \]
\[ O : \quad O v_1(x) = \epsilon_{ij} v_j(Ox), \quad O v_2(x) = v_1(Ox), \]
\[ R : \quad R v_1(x) = v_1(Rx), \quad R v_2(x) = -v_2(Rx), \quad R v_1(x) = v_1(Rx). \]

Writing

\[ v_\mu(x) = i v^a_\mu(x) \sigma_a, \quad v_\pm^\mu(x) = v^1_\mu(x) \mp i v^2_\mu(x), \]

the field \( v_\mu(x) \) decomposes into an Abelian “gauge” field \( v^3_\mu(x) \) and two “charged” vector fields \( v^\pm_\mu(x) \).

Using eq. (2.20), for a Skyrmion \( \vec{e}_{\sigma,n,\rho,0,\gamma}(r,\chi) \) centered at \( x = 0 \) one obtains

\[
\begin{align*}
v^3_1(r, \chi) &= -\frac{\sigma n \rho^{2n}}{r^{2n} + \rho^{2n}} \sin \chi, \\
v^3_2(r, \chi) &= \frac{\sigma n \rho^{2n}}{r^{2n} + \rho^{2n}} \cos \chi, \\
v^3_t(r, \chi) &= \frac{\sigma \rho^{2n}}{r^{2n} + \rho^{2n}} \dot{\gamma}, \\
v^+_1(r, \chi) &= \mp i \frac{nr^{n-1} \rho^n}{r^{2n} + \rho^{2n}} \exp(\mp i \sigma [(n + 1) \chi + \gamma]), \\
v^+_2(r, \chi) &= \frac{\sigma n r^{n-1} \rho^n}{r^{2n} + \rho^{2n}} \exp(\mp i \sigma [(n + 1) \chi + \gamma]), \\
v^+_t(r, \chi) &= \frac{\sigma r^n \rho^n}{r^{2n} + \rho^{2n}} \exp(\mp i \sigma (n \chi + \gamma)) \dot{\gamma}.
\end{align*}
\]

In principle, when holes get localized on a Skyrmion, they affect the radial profile of the Skyrmion. Here we neglect this effect and concentrate on symmetry considerations which are independent of such details of the dynamics.

### 3.2 Hole Fields and their Transformation Properties

As discussed in detail in [14] the holes are described by Grassman-valued fields \( \psi_f^\pm(x) \). Here \( f \in \{\alpha, \beta\} \) is a flavor index which specifies the momentum space pocket in which the hole resides, and the subscript \( \pm \) denotes the spin of the hole relative to the direction of the local staggered magnetization. Under the various relevant symmetries
Figure 1: Elliptically shaped hole pockets centered at $(\pm \frac{\pi}{2a}, \pm \frac{\pi}{2a})$. Two half-pockets combine to form the pockets for the flavors $f = \alpha, \beta$.

of the underlying antiferromagnet on a square lattice, the hole fields transform as

\[
\begin{align*}
SU(2)_s & : \quad \psi_{\pm}^f(x)' = \exp(\pm i\alpha(x))\psi_{\pm}^f(x), \\
U(1)_{Q} & : \quad Q\psi_{\pm}^f(x) = \exp(i\omega)\psi_{\pm}^f(x), \\
D'_{i} & : \quad D'_{i}\psi_{\pm}^f(x) = \pm \exp(i k_{i} a)\psi_{\pm}^f(x), \\
O & : \quad O_{\psi_{\pm}^f(x)} = \mp \psi_{\pm}^\beta(Ox), \quad O_{\psi_{\pm}^\alpha(x)} = \psi_{\pm}^\alpha(Ox), \\
R & : \quad R\psi_{\pm}^\alpha(x) = \psi_{\pm}^\beta(Rx), \quad R\psi_{\pm}^\beta(x) = \psi_{\pm}^\alpha(Rx). 
\end{align*}
\] (3.6)

The $U(1)_Q$ symmetry is just fermion number, while $k^\alpha = (\frac{\pi}{2a}, \frac{\pi}{2a})$ and $k^\beta = (\frac{\pi}{2a}, -\frac{\pi}{2a})$ (with $a$ being the lattice spacing) point to the centers of the two hole pockets illustrated in figure 1. It is interesting that in the effective theory momentum indices of the underlying microscopic dynamics turn into internal flavor quantum numbers.

3.3 Effective Action for Holes coupled to the Staggered Magnetization

Based on the above symmetry properties, the leading and sub-leading terms of the effective action for an antiferromagnet on a square lattice have been constructed systematically in [14]. Here we restrict ourselves to the leading terms. We also make the simplifying (but somewhat unrealistic) assumption that the momentum-space hole pockets have a circular shape, which enables us to perform large parts of the following calculations analytically. It would be straightforward to take into account the more realistic elliptic shape of the hole pockets, but this would require some numerical
work. Here we concentrate foremost on the symmetry properties of holes localized on a Skyrmion on which the simplifying assumption of spherical hole pockets has no effect. The total action of the coupled system including doped holes then takes the form

\[
S[\psi_+^f, \psi_-^f, \mathbf{e}] = \int d^2x \, dt \left\{ \frac{\rho_s}{2} \left( \partial_t \mathbf{e} \cdot \partial_t \mathbf{e} + \frac{1}{c^2} \partial_i \mathbf{e} \cdot \partial_i \mathbf{e} \right) \right. \\
+ \sum_{f=\alpha,\beta} \left. \left[ M \psi_s^{f\dagger} \psi_s^f + \psi_s^{f\dagger} D_t \psi_s^f + \frac{1}{2M'} D_t \psi_s^{f\dagger} D_t \psi_s^f + \Lambda \left( \psi_s^{f\dagger} v_1^s \psi_s^f + \sigma_f \psi_s^{f\dagger} v_2^s \psi_s^f \right) \right] \right\}.
\]

(3.7)

Here \( M \) and \( M' \) are the rest energy and the kinetic mass of a hole, and \( \Lambda \) is the hole-one-magnon coupling constant. The sign \( \sigma_f \) is + for \( f = \alpha \) and – for \( f = \beta \). The covariant derivatives are given by

\[
D_\mu \psi_{\pm}^f(x) = \left[ \partial_\mu \pm iv_3^\mu(x) \right] \psi_{\pm}^f(x).
\]

(3.8)

Remarkably, the Shraiman-Siggia term in the action, which is proportional to \( \Lambda \), contains just a single (uncontracted) spatial derivative. Due to the nontrivial rotation properties of flavor, this term is still 90 degrees rotation invariant. Due to the small number of derivatives it contains, this term dominates the low-energy dynamics. In particular, it alone is responsible for one-magnon exchange between hole pairs \([14, 24]\) as well as for potential spiral phases in the staggered magnetization order parameter \([25]\). It is interesting to note that a similar term is absent in lightly electron-doped antiferromagnets \([30]\), such that spiral phases do not arise in these systems.

4 Hole Localization on a Skyrmion

In this section we apply the effective theory of the previous section to the localization of holes on a Skyrmion. First, we consider the localization of a single hole first on a static and then on a rotating Skyrmion. Then the localization of two holes on the same Skyrmion is considered, and the symmetry properties of the resulting two-hole bound states are analyzed.

4.1 Single Hole Localized on a Static Skyrmion

As we have seen, the moment of inertia \( \mathcal{I}(\rho) \) of a Skyrmion with \( n = 1 \) is logarithmically divergent in the infra-red. Unless the divergence is regularized due to a finite spatial volume or the presence of other Skyrmions, the Skyrmion then cannot rotate. In the interest of analytic solubility, and because we want to focus on symmetry aspects, we will no longer consider the translational and dilatational motion of
the Skyrmion. Instead, we fix the Skyrmion center at the origin $x = 0$ and we fix
the Skyrmion size to a constant $\rho$. As we will see later, in the presence of holes, the
energy of the Skyrmion-hole bound states is minimized for a particular value of $\rho$.

The wave function of a single hole localized on a Skyrmion takes the form

$$
\Psi^f_{\sigma,n}(r, \chi) = \begin{pmatrix}
\Psi^f_{\sigma,n,+}(r, \chi) \\
\Psi^f_{\sigma,n,-}(r, \chi)
\end{pmatrix},
$$

(4.1)

Omitting the constant rest energy $M$ of the holes, which just amounts to a constant
energy shift, the corresponding Hamiltonian resulting from the action of eq.(3.7) is given by

$$
H^f = \begin{pmatrix}
H^f_{++} & H^f_{+-} \\
H^f_{-+} & H^f_{--}
\end{pmatrix},
$$

$$
H^f_{++} = -\frac{1}{2M'} \left[ \partial_t + iv^3_i(x) \right]^2 = -\frac{1}{2M'} \left[ \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \left( \partial_\chi + i \frac{\sigma n \rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right],
$$

$$
H^f_{+-} = \Lambda (v^3_i(x) + \sigma f_v^+(x)) = \sqrt{2} \Lambda \sigma f \frac{nr^2 - 1}{r^{2n} + \rho^{2n}} \exp \left( -i \sigma \left[ (n+1) \chi + \gamma + \sigma f \frac{\pi}{4} \right] \right),
$$

$$
H^f_{-+} = \Lambda (v^3_i(x) + \sigma f_v^-(x)) = \sqrt{2} \Lambda \sigma f \frac{nr^2 - 1}{r^{2n} + \rho^{2n}} \exp \left( i \sigma \left[ (n+1) \chi + \gamma + \sigma f \frac{\pi}{4} \right] \right),
$$

$$
H^f_{--} = -\frac{1}{2M'} \left[ \partial_t - iv^3_i(x) \right]^2 = -\frac{1}{2M'} \left[ \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \left( \partial_\chi - i \frac{\sigma n \rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right].
$$

(4.2)

Using the explicit form of $v^3_i(x)$ and $v^\pm_i(x)$ for the Skyrmion of eq.(3.5) and making
the ansatz

$$
\Psi^f_{\sigma,m_+,m_-}(r, \chi) = \begin{pmatrix}
\psi_{m_+,m_-,+}(r) \exp \left( i \sigma [m_+ \chi - \frac{\gamma}{2} - \sigma f \frac{\pi}{8}] \right) \\
\sigma f \psi_{m_+,m_-,+}(r) \exp \left( i \sigma [m_- \chi + \frac{\gamma}{2} + \sigma f \frac{\pi}{8}] \right)
\end{pmatrix},
$$

(4.3)

with $m_- - m_+ = n + 1$, after some algebra one obtains the radial Schrödinger equation

$$
H_r \psi_{m_+,m_-}(r) = \begin{pmatrix}
H_{r++} & H_{r+-} \\
H_{r-+} & H_{r--}
\end{pmatrix} \begin{pmatrix}
\psi_{m_+,m_+,+}(r) \\
\psi_{m_+,m_-,+}(r)
\end{pmatrix} = E_{m_+,m_-} \psi_{m_+,m_-}(r),
$$

(4.4)

with

$$
H_{r++} = -\frac{1}{2M'} \left[ \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \left( m_+ + \frac{n \rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right],
$$

$$
H_{r+-} = H_{r-+} = \sqrt{2} \Lambda \sigma f \frac{nr^2 - 1}{r^{2n} + \rho^{2n}},
$$

$$
H_{r--} = -\frac{1}{2M'} \left[ \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \left( m_- - \frac{n \rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right].
$$

(4.5)
It should be noted that the resulting radial Schrödinger equation is the same for Skyrmions and anti-Skyrmions as well as for both flavors \( f = \alpha, \beta \). Interestingly, for odd \( n \) and \( m_- = -m_+ = (n + 1)/2 \), the two equations decouple. The equation that leads to a localized hole takes the form

\[
\left[-\frac{1}{2M'} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( \frac{n+1}{2} - \frac{n\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right) - \frac{\sqrt{2\Lambda}nr^{n-1}\rho^n}{r^{2n} + \rho^{2n}} \right] \psi(r) = E\psi(r),
\]

where \( \psi(r) \) is the linear combination

\[
\psi(r) = \frac{1}{\sqrt{2}} \left( \psi_{m+,m-,+}(r) - \psi_{m+,m-,+}(r) \right).
\]

For even winding number \( n \), on the other hand, the two equations do not decouple. In the following we will be most interested in Skyrmions (or anti-Skyrmions) with winding number \( n = 1 \).

In this paper, we concentrate on the symmetry properties of holes localized on a Skyrmion, not paying much attention to finer details of the dynamics. Hence, here we do not solve the radial equation, which would be straightforward using numerical methods. Still, we want to obtain at least a rough estimate for the ground state energy of a hole localized on a Skyrmion. For \( n = 1 \) the radial Schrödinger equation takes the form

\[
\left[-\frac{1}{2M'} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + V(r) \right) \right] \psi(r) = E\psi(r),
\]

with the potential given by

\[
V(r) = \frac{1}{2M'} \frac{r^2}{(r^2 + \rho^2)^2} - \frac{\sqrt{2\Lambda}\rho}{r^2 + \rho^2}.
\]

At short distances, the potential can be approximated by a harmonic oscillator

\[
V_{\text{approx}}(r) = -\frac{\sqrt{2\Lambda}}{\rho} + \frac{M'}{2} \left( \frac{1}{M'^2\rho^4} + \frac{2\sqrt{2\Lambda}}{M'\rho^3} \right) r^2 + O(r^4),
\]

and hence, in a rather crude harmonic approximation, the ground state energy takes the form

\[
E_0 = -\frac{\sqrt{2\Lambda}}{\rho} + \frac{1}{M'^2\rho^4} + \frac{2\sqrt{2\Lambda}}{M'\rho^3} = M'\Lambda^2 \left( \sqrt{x^2 + 2\sqrt{2}x - 2} \right), \quad x = \frac{1}{M'\Lambda\rho}.
\]

Minimizing the energy as a function of \( x \) yields \( x^3 + 3\sqrt{2}x^2 + 4x = \sqrt{2} \), which is solved by

\[
x = \sqrt{2} \left[ \left( \frac{3\sqrt{3}}{4} + \frac{\sqrt{11}}{4} \right)^{1/3} + \left( \frac{3\sqrt{3}}{4} + \frac{\sqrt{11}}{4} \right)^{-1/3} \right] - \sqrt{2} \approx 0.271 \Rightarrow
\]

\[
\rho \approx \frac{1}{0.271M'\Lambda}.
\]
This shows that the presence of the hole explicitly breaks the scale invariance that led to the dilational instability of the pure Skyrmion. The resulting bound state with the strongest binding energy has

$$E_0 = M' \Lambda^2 x \left( \sqrt{x^2 + 2\sqrt{2}x} - \sqrt{2} \right) \approx -0.135 M' \Lambda^2.$$  \hspace{1cm} (4.13)

The potential $V(r)$ is shown in figure 2 together with its harmonic approximation and the corresponding ground state energy $E_0$. The figure implies that the true ground state energy is smaller than the harmonic approximation suggests.

### 4.2 Single Hole Localized on a Rotating Skyrmion

In this subsection we consider a single hole localized on a rotating Skyrmion. When the moment of inertia $\mathcal{I}(\rho)$ diverges (as it is the case for $n = 1$ and $R = \infty$) the fixed orientation $\gamma$ of the Skyrmion explicitly breaks the $U(1)_s$ symmetry and the analysis of Section 4.1 applies. Here we assume that $\mathcal{I}(\rho) = \mathcal{D}(\rho) \rho^2 / n^2$ is finite. This is actually the case when the feedback of the localized hole on the radial structure of the Skyrmion is taken into account. When $\mathcal{I}(\rho)$ is finite, the Skyrmion can rotate and thus $\gamma$ becomes a dynamical variable. The $\gamma$-dependent terms in the Lagrange
function for the rotational motion are given by

\[ L = \frac{\mathcal{D}(\rho)^2}{2n^2} \dot{\gamma}^2 - n \frac{\Theta}{2\pi} \dot{\gamma} + \int d^2x \sum_{f=\alpha,\beta, s} \psi_s^f \nabla^2 \psi_s^f. \]  

(4.14)

Using eq. (3.5), the momentum canonically conjugate to \( \gamma \) thus takes the form

\[ p_\gamma = \frac{\mathcal{D}(\rho)^2 \dot{\gamma}}{n^2} - n \frac{\Theta}{2\pi} + \int d^2x \sigma \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \sum_{f=\alpha,\beta, s} \psi_s^f \nabla^2 \psi_s^f, \]  

(4.15)

which leads to the corresponding Hamiltonian

\[ H^\gamma = \frac{1}{2I(\rho)} (-i \partial_\gamma - A_\gamma)^2, \]  

(4.16)

with the Berry gauge field

\[ A_\gamma = \int d^2x \sum_{f=\alpha,\beta, s} \Psi_s^f \sigma \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \Psi_s^f - n \frac{\Theta}{2\pi}. \]  

(4.17)

Combining the results, one sees that while the off-diagonal elements of the Hamiltonian (4.12) remain the same, the diagonal elements receive additional contributions such that now

\[
\begin{align*}
H_{++} & = -\frac{1}{2M} \left[ \partial_t + iv_1^3(x) \right]^2 - \frac{n^2}{2\mathcal{D}(\rho)\rho^2} \left( \partial_\gamma + in\frac{\Theta}{2\pi} - i\sigma \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2, \\
H_{+-} & = \Lambda(v_1^3(x) + \sigma f v_2^3(x)) \\
H_{-+} & = \sqrt{2} \Lambda \sigma f \frac{n^{n-1} \rho^n}{r^{2n} + \rho^{2n}} \exp \left( -i\sigma \left[ (n+1)\chi + \gamma + \sigma f \frac{\pi}{4} \right] \right), \\
H_{-} & = \Lambda(v_1^3(x) + \sigma f v_2^3(x)) \\
H_{-} & = \sqrt{2} \Lambda \sigma f \frac{n^{n-1} \rho^n}{r^{2n} + \rho^{2n}} \exp \left( i\sigma \left[ (n+1)\chi + \gamma + \sigma f \frac{\pi}{4} \right] \right), \\
H_{-} & = -\frac{1}{2M} \left[ \partial_t - iv_1^3(x) \right]^2 - \frac{n^2}{2\mathcal{D}(\rho)\rho^2} \left( \partial_\gamma + in\frac{\Theta}{2\pi} + i\sigma \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2, \\
H_{-} & = -\frac{1}{2M} \left[ \partial_t - iv_1^3(x) \right]^2 - \frac{n^2}{2\mathcal{D}(\rho)\rho^2} \left( \partial_\gamma - in\frac{\Theta}{2\pi} - i\sigma \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2.
\end{align*}
\]  

(4.18)
We now make the ansatz
\[
\Psi^f_{\sigma,m_-,m_+}(r,\chi,\gamma) = \left( \begin{array}{c} \psi_{\sigma,m_+,m_-,m_+}(r) \exp\left(i\sigma\left[m_+\chi - \sigma f \frac{\Theta}{2\pi}\right]\right) \\ \sigma \sigma f \psi_{\sigma,m_+,m_-,m_-}(r) \exp\left(i\sigma\left[m_-\chi + \sigma f \frac{\Theta}{2\pi}\right]\right) \end{array} \right) \exp(i\sigma(m - \frac{1}{2})\gamma)
\]
with \(m_- - m_+ = n + 1\). In order to ensure \(2\pi\)-periodicity of the wave function in the variable \(\gamma\), \(m\) must now be one half of some odd integer. This is in contrast to the rotating Skyrmion without a hole that was discussed in Subsection 2.4, for which \(m\) was an integer. The radial Schrödinger equation is then given by
\[
H_r \psi_{\sigma,m_+,m_-,m_-}(r) = \left( \begin{array}{cc} H_{r++} & H_{r+-} \\ H_{r-+} & H_{r--} \end{array} \right) \left( \begin{array}{c} \psi_{\sigma,m_+,m_-,m_+}(r) \\ \psi_{\sigma,m_+,m_-,m_-}(r) \end{array} \right) = E_{\sigma,m_+,m_-,m_-} \psi_{\sigma,m_+,m_-,m_-}(r).
\]
(4.20)

In this case, the four matrix elements of the radial Hamiltonian \(H_r\) take the form
\[
H_{r++} = -\frac{1}{2M} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( m_+ + \frac{n\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right] + \frac{n^2}{2D(\rho)\rho^2} \left( m_+ + \frac{n\Theta}{2\pi} - \frac{1}{2} \right) - \frac{n^2}{2D(\rho)\rho^2} \left( m_- - \frac{n\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2,
\]
\[
H_{r+-} = H_{r-+} = \sqrt{2\Lambda} \frac{nr^{n-1}\rho}{r^{2n} + \rho^{2n}}.
\]
\[
H_{r--} = -\frac{1}{2M} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( m_- - \frac{n\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2 \right] + \frac{n^2}{2D(\rho)\rho^2} \left( m_- + \frac{n\Theta}{2\pi} + \frac{1}{2} + \frac{\rho^{2n}}{r^{2n} + \rho^{2n}} \right)^2.
\]
(4.21)

### 4.3 Symmetry Properties of a Single Hole Localized on a Skyrmion

Let us again consider the spin operator (which generates an internal symmetry and is thus analogous to isospin in particle physics)
\[
I = \left( \begin{array}{cc} -i\sigma \partial_\gamma + \frac{n\Theta}{2\pi} + \frac{1}{2} & 0 \\ 0 & -i\sigma \partial_\gamma + \frac{n\Theta}{2\pi} - \frac{1}{2} \end{array} \right),
\]
(4.22)
which commutes with the Hamiltonian, i.e. \([H^f, I] = 0\). The wave function \(\Psi^f_{\sigma,m_+,m_-,m_-}\) is indeed an eigenstate of \(I\), i.e.
\[
I \Psi^f_{\sigma,m_+,m_-,m_-}(r,\chi,\gamma) = \left( m + \frac{n\Theta}{2\pi} \right) \Psi^f_{\sigma,m_+,m_-,m_-}(r,\chi,\gamma).
\]
(4.23)
Since \( m \) is half of an odd integer, the rotating Skyrmion with one hole localized on it has half-integer spin (or “isospin”), at least for vanishing anyon statistics parameter \( \Theta = 0 \).

The various symmetries such as the displacements \( D'_1 \) and \( D'_2 \), the 90 degrees rotation \( O \), as well as the reflection \( R \), act on the wave function

\[
\Psi^f_{\sigma,n}(r, \chi, \gamma) = \begin{pmatrix}
\Psi^f_{\sigma,n,+}(r, \chi, \gamma) \\
\Psi^f_{\sigma,n,-}(r, \chi, \gamma)
\end{pmatrix},
\]

of a single hole localized on a rotating (anti-)Skyrmion with winding number \( \sigma n \) as follows

\[
D'_i \Psi^f_{\sigma,n}(r, \chi, \gamma) = \exp(ik_i a) \begin{pmatrix}
\Psi^f_{\sigma,n,-}(r, \chi, \gamma) \\
-\Psi^f_{\sigma,n,+}(r, \chi, \gamma)
\end{pmatrix},
\]

\[
O \Psi^f_{\sigma,n}(r, \chi, \gamma) = \begin{pmatrix}
\sigma f \Psi^f_{\sigma,n,+}(r, \chi + \frac{\pi}{2}, \gamma - n \frac{\pi}{2}) \\
\Psi^f_{\sigma,n,-}(r, \chi + \frac{\pi}{2}, \gamma - n \frac{\pi}{2})
\end{pmatrix},
\]

\[
R \Psi^f_{\sigma,n}(r, \chi, \gamma) = \begin{pmatrix}
\Psi^f_{\sigma,n,+}(r, -\chi, -\gamma) \\
\Psi^f_{\sigma,n,-}(r, -\chi, -\gamma)
\end{pmatrix}.
\]

For energy eigenstates this then implies

\[
D'_i \Psi^f_{\sigma,m_+,m_-,m}(r, \chi, \gamma) = \sigma \sigma_f \exp(ik_i a) \Psi^f_{-\sigma,-m_-,m_+,m}(r, \chi, \gamma),
\]

\[
O \Psi^\alpha_{\sigma,m_+,m_-,m}(r, \chi, \gamma) = \exp(i[\sigma m_+ + m_- - 2 - 2nm] \frac{\pi}{4}) \Psi^\beta_{\sigma,m_+,m_-,m}(r, \chi, \gamma),
\]

\[
O \Psi^\beta_{\sigma,m_+,m_-,m}(r, \chi, \gamma) = -\exp(i[\sigma m_+ + m_- - 2nm] \frac{\pi}{4}) \Psi^\alpha_{\sigma,m_+,m_-,m}(r, \chi, \gamma),
\]

\[
R \Psi^\alpha_{\sigma,m_+,m_-,m}(r, \chi, \gamma) = \Psi^\beta_{-\sigma,m_+,m_-,m}(r, \chi, \gamma),
\]

\[
R \Psi^\beta_{\sigma,m_+,m_-,m}(r, \chi, \gamma) = \Psi^\alpha_{-\sigma,m_+,m_-,m}(r, \chi, \gamma).
\]

It should be noted that for \( \Theta \neq 0 \) or \( \pi \), the reflection symmetry \( R \) is explicitly broken by the Hopf term. Assuming appropriate phase conventions for the radial wave functions, in the considerations of the shift symmetries \( D'_i \), we have used

\[
\psi_{-m_-,m_-,m_+}(r) = \psi_{m_+,m_-,m_-}(r), \quad \psi_{-m_-,m_+,m_-}(r) = \psi_{m_+,m_-,m_+}(r),
\]

which follows from the behavior of eq. \((1.21)\) under the replacement of \( m_+ \rightarrow m'_+ = -m_- \), \( m_- \rightarrow m'_- = -m_+ \), and \( m \rightarrow m' = -m \). It is worth noting that after this replacement the constraint

\[
m'_- - m'_+ = -m_+ + m_- = n + 1
\]

remains satisfied.
4.4 Schrödinger Equation for a Pair of Holes of Different Flavor Localized on a Rotating Skyrmion

Let us now consider bound states of two holes localized on the same Skyrmion. Both a hole of flavor $\alpha$ and another hole of flavor $\beta$ can occupy the same single-particle ground state in a Skyrmion. For holes of the same flavor this would be forbidden by the Pauli principle. Since we are most interested in the lowest energy states, we consider two holes of different flavor. The case of two holes with the same flavor is discussed in Appendix A. The Hamiltonian for two holes of different flavor $\alpha$ and $\beta$ is given by

$$H = H^\alpha + H^\beta + H^\gamma,$$

where $H^\alpha$ and $H^\beta$ are the Hamiltonians for a hole of flavor $\alpha$ and $\beta$, respectively. Explicitly one has

$$H^\alpha = \begin{pmatrix} H^\alpha_{++} & 0 & H^\alpha_{+-} \\ 0 & H^\alpha_{++} & 0 \\ H^\alpha_{-+} & 0 & H^\alpha_{--} \end{pmatrix}, \quad H^\beta = \begin{pmatrix} H^\beta_{++} & 0 & 0 \\ H^\beta_{+-} & H^\beta_{--} & 0 \\ 0 & 0 & H^\beta_{++} \end{pmatrix},$$

$$H^\gamma = \begin{pmatrix} H^\gamma_{++++} & 0 & 0 \\ 0 & H^\gamma_{++--} & 0 \\ 0 & 0 & H^\gamma_{--++} \end{pmatrix},$$

with

$$H^\alpha_{++} = -\frac{1}{2M}(\partial_t + iv_1^\alpha(x))^2, \quad H^\alpha_{+-} = \Lambda(v_1^+(x) + v_2^+(x)),$$

$$H^\alpha_{-+} = -\frac{1}{2M}(\partial_t - iv_1^\alpha(x))^2, \quad H^\alpha_{++} = \Lambda(v_1^-(x) + v_2^-(x)),$$

$$H^\beta_{++} = -\frac{1}{2M}(\partial_t + iv_1^\beta(x))^2, \quad H^\beta_{+-} = \Lambda(v_1^+(x) - v_2^+(x)),$$

$$H^\beta_{-+} = -\frac{1}{2M}(\partial_t - iv_1^\beta(x))^2, \quad H^\beta_{++} = \Lambda(v_1^-(x) - v_2^-(x)),$$

$$H^\gamma_{++++} = -\frac{n^2}{2D(\rho)^2} \left( \partial_t + i\frac{\Theta}{2\pi} - i\frac{\rho_2^2}{r_\alpha^2 + \rho^2} - i\frac{\rho_2^2}{r_\beta^2 + \rho^2} \right)^2,$$

$$H^\gamma_{++--} = -\frac{n^2}{2D(\rho)^2} \left( \partial_t + i\frac{\Theta}{2\pi} - i\frac{\rho_2^2}{r_\alpha^2 + \rho^2} + i\frac{\rho_2^2}{r_\beta^2 + \rho^2} \right)^2,$$

$$H^\gamma_{--++} = -\frac{n^2}{2D(\rho)^2} \left( \partial_t + i\frac{\Theta}{2\pi} + i\frac{\rho_2^2}{r_\alpha^2 + \rho^2} - i\frac{\rho_2^2}{r_\beta^2 + \rho^2} \right)^2,$$

$$H^\gamma_{----} = -\frac{n^2}{2D(\rho)^2} \left( \partial_t + i\frac{\Theta}{2\pi} + i\frac{\rho_2^2}{r_\alpha^2 + \rho^2} + i\frac{\rho_2^2}{r_\beta^2 + \rho^2} \right)^2.$$
We now make the following ansatz for a two-hole energy eigenstate without holes, in this case,

\[ \Psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m} (r_\alpha, r_\beta, \chi_\alpha, r_\gamma, \chi_\beta) = \]

\[ \begin{align*}
\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,++}(r_\alpha, r_\beta) \exp \left( i \sigma \left[ m^\alpha_+ \chi_\alpha + m^\beta_+ \chi_\beta \right] \right) \\
-\sigma \psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,+}(r_\alpha, r_\beta) \exp \left( i \sigma \left[ m^\alpha_+ \chi_\alpha + m^\beta_+ \chi_\beta - \frac{\pi}{4} \right] \right) \\
\sigma \psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,-}(r_\alpha, r_\beta) \exp \left( i \sigma \left[ m^\alpha_+ \chi_\alpha + m^\beta_+ \chi_\beta + \frac{\pi}{4} \right] \right) \\
-\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,-}(r_\alpha, r_\beta) \exp \left( i \sigma \left[ m^\alpha_+ \chi_\alpha + m^\beta_+ \chi_\beta \right] \right) 
\end{align*} \]

(4.32)

Again, this solves the Schrödinger equation only if \( m^-_\alpha - m^+_\beta = n + 1 \). As for the Skyrmion without holes, in this case, \( m \) is again an integer. The resulting radial Schrödinger equation then takes the form

\[ H_r \psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m}(r_\alpha, r_\beta) = E_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m} \psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m}(r_\alpha, r_\beta), \]

(4.33)

with

\[ \psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m}(r_\alpha, r_\beta) = \begin{pmatrix}
\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,++}(r_\alpha, r_\beta) \\
\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,+}(r_\alpha, r_\beta) \\
\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,-}(r_\alpha, r_\beta) \\
\psi_{\sigma,m^+_\alpha,m^-_\alpha,m^+_\beta,m^-_\beta,m,-}(r_\alpha, r_\beta)
\end{pmatrix}. \]

(4.34)

The radial Hamiltonian is given by

\[ H_r = H_r^\alpha + H_r^\beta + H_r^\gamma, \]

(4.35)

with

\[ H_r^\alpha = \begin{pmatrix}
H_{r++}^\alpha & 0 & H_{r+-}^\alpha & 0 \\
0 & H_{r++}^\alpha & 0 & H_{r+-}^\alpha \\
H_{r+-}^\alpha & 0 & H_{r--}^\alpha & 0 \\
0 & H_{r+-}^\alpha & 0 & H_{r--}^\alpha
\end{pmatrix}, \]

\[ H_r^\beta = \begin{pmatrix}
H_{r++}^\beta & 0 & 0 & 0 \\
H_{r+-}^\beta & H_{r--}^\beta & 0 & 0 \\
0 & 0 & H_{r++}^\beta & H_{r+-}^\beta \\
0 & 0 & H_{r+-}^\beta & H_{r--}^\beta
\end{pmatrix}, \]

\[ H_r^\gamma = \begin{pmatrix}
H_{r++++}^\gamma & 0 & 0 & 0 \\
0 & H_{r+++-}^\gamma & 0 & 0 \\
0 & 0 & H_{r+-++}^\gamma & 0 \\
0 & 0 & 0 & H_{r----}^\gamma
\end{pmatrix}. \]

(4.36)
The matrix elements of the fermionic part of the radial Hamiltonian are

\[ H^f_{r++} = -\frac{1}{2M'} \left[ \partial^2_{r_f} + \frac{1}{r_f} \partial_{r_f} - \frac{1}{r^2_f} \left( m'_+ + \frac{n\rho^{2n}}{r^2_f + \rho^{2n}} \right)^2 \right], \]

\[ H^f_{r+-} = H^f_{r-+} = \sqrt{2}\Lambda \rho \frac{n^{n-1}}{r^2_f + \rho^{2n}}, \]

\[ H^f_{r--} = -\frac{1}{2M'} \left[ \partial^2_{r_f} + \frac{1}{r_f} \partial_{r_f} - \frac{1}{r^2_f} \left( m'_- - \frac{n\rho^{2n}}{r^2_f + \rho^{2n}} \right)^2 \right], \quad (4.37) \]

while the rotational Skyrmion contributions are given by

\[ H^\gamma_{r++++} = \frac{n^2}{2D(\rho)\rho^2} \left( m + \sigma n \frac{\Theta}{2\pi} - 1 - \frac{\rho^{2n}}{r^2_\alpha + \rho^{2n}} - \frac{\rho^{2n}}{r^2_\beta + \rho^{2n}} \right)^2, \]

\[ H^\gamma_{r+++-} = \frac{n^2}{2D(\rho)\rho^2} \left( m + \sigma n \frac{\Theta}{2\pi} - \frac{\rho^{2n}}{r^2_\alpha + \rho^{2n}} + \frac{\rho^{2n}}{r^2_\beta + \rho^{2n}} \right)^2, \]

\[ H^\gamma_{r---+} = \frac{n^2}{2D(\rho)\rho^2} \left( m + \sigma n \frac{\Theta}{2\pi} + \frac{\rho^{2n}}{r^2_\alpha + \rho^{2n}} - \frac{\rho^{2n}}{r^2_\beta + \rho^{2n}} \right)^2, \]

\[ H^\gamma_{r----} = \frac{n^2}{2D(\rho)\rho^2} \left( m + \sigma n \frac{\Theta}{2\pi} + 1 + \frac{\rho^{2n}}{r^2_\alpha + \rho^{2n}} + \frac{\rho^{2n}}{r^2_\beta + \rho^{2n}} \right)^2. \quad (4.38) \]

### 4.5 Symmetry Properties of a Pair of Holes with Different Flavors Localized on a Skyrmion

It is worth noticing that the spin operator \( I \), which commutes with the two-hole Hamiltonian \( H \), is given by

\[
I = \begin{pmatrix}
-\sigma \partial_\gamma + \sigma n \frac{\Theta}{2\pi} + 1 & 0 & 0 & 0 \\
0 & -\sigma \partial_\gamma + \sigma n \frac{\Theta}{2\pi} & 0 & 0 \\
0 & 0 & -\sigma \partial_\gamma + \sigma n \frac{\Theta}{2\pi} & 0 \\
0 & 0 & 0 & -\sigma \partial_\gamma + \sigma n \frac{\Theta}{2\pi} - 1
\end{pmatrix}, \quad (4.39)
\]

such that

\[
I \Psi_{\sigma,m_\alpha^\sigma,m_\beta^\sigma,m_\rho^\sigma,m_\rho^\rho,m}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \\
(m + \sigma n \frac{\Theta}{2\pi}) \Psi_{\sigma,m_\alpha^\sigma,m_\beta^\sigma,m_\rho^\sigma,m_\rho^\rho,m}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma). \quad (4.40)
\]

Since \( m \) is an integer, as expected, for \( \Theta = 0 \) the state with two holes localized on a Skyrmion has integer spin (which plays the role of “isospin”).
The symmetries \( D_i, O, \) and \( R \) act on a general two-hole wave function

\[
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \begin{pmatrix}
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma)
\end{pmatrix}
\]  

(4.41)

as follows

\[
\begin{align*}
D_i \Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \exp(i(k_1^\alpha + k_1^\beta) a)
\begin{pmatrix}
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
-\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
-\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma)
\end{pmatrix}, \\
O \Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \begin{pmatrix}
-\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, \chi_\beta + \frac{\pi}{2}, r_\alpha, \chi_\alpha + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
-\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, \chi_\beta + \frac{\pi}{2}, r_\alpha, \chi_\alpha + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, \chi_\beta + \frac{\pi}{2}, r_\alpha, \chi_\alpha + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, \chi_\beta + \frac{\pi}{2}, r_\alpha, \chi_\alpha + \frac{\pi}{2}, \gamma - n\frac{\pi}{2})
\end{pmatrix}, \\
R \Psi_{\sigma,n}^{\alpha\beta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \begin{pmatrix}
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, -\chi_\beta, r_\alpha, -\chi_\alpha, -\gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, -\chi_\beta, r_\alpha, -\chi_\alpha, -\gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, -\chi_\beta, r_\alpha, -\chi_\alpha, -\gamma) \\
\Psi_{\sigma,n}^{\alpha\beta}(r_\beta, -\chi_\beta, r_\alpha, -\chi_\alpha, -\gamma)
\end{pmatrix}. 
\end{align*}
\]

(4.42)

It is straightforward to show that for the two-hole energy eigenstates this implies

\[
\begin{align*}
D_i \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \\
D_2 \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= -\Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \\
O \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \exp\left(i(a|m_+^\alpha + m_-^\beta - mn + \frac{\pi}{2})\right) \\
&\times \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \\
R \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) &= \Psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma). 
\end{align*}
\]

(4.43)

Here we have assumed an appropriate phase convention for the radial wave function \( \psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta) \). In the context of the shift symmetries \( D_i \) we have used

\[
\begin{align*}
\psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta) &= \psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta), \\
\psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta) &= \psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta), \\
\psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta) &= \psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta), \\
\psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta) &= \psi_{\sigma,m_+^\alpha,m_+^\beta,m_+^\gamma,m_+^\delta,m_-^\alpha,m_-^\beta,m_-^\gamma,m_-^\delta}(r_\alpha, r_\beta). 
\end{align*}
\]

(4.44)
These relations follow from the symmetries of the radial Schrödinger equation \[4.33\]. Similarly, in the context of the rotation \( O \) we have used

\[
\begin{align*}
\psi_{\sigma,m^\alpha_+,m^\beta_+,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,m^\alpha_+,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_+,m^\beta_+,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_+,m^\beta_+,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_+,m^\beta_-,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha).
\end{align*}
\]

Finally, in the context of the reflection symmetry \( R \) we have used

\[
\begin{align*}
\psi_{\sigma,m^\alpha_+,m^\beta_+,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,-m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_+,m^\beta_-,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,-m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_-,m^\beta_+,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,-m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha), \\
\psi_{\sigma,m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_-}(\vec{r}_\beta, r_\alpha) &= \psi_{\sigma,-m^\alpha_-,m^\beta_-,m^\alpha_-,m^\beta_+}(\vec{r}_\beta, r_\alpha).
\end{align*}
\]

The relations in eq. \[4.46\] follow from the symmetries of the radial Schrödinger equation \[4.33\] for \( \Theta = 0 \). For \( \Theta \neq 0 \) or \( \pi \), the Hopf term explicitly breaks the reflection symmetry.

### 4.6 Comparison with Two-Hole States Bound by One-Magnon Exchange

In [14] states of two holes bound by one-magnon exchange in a square lattice antiferromagnet have been investigated in great detail. Here we summarize as well as extend some of the relevant results. In the rest frame, the Schrödinger equation for two holes of flavor \( \alpha \) and \( \beta \) takes the form

\[
( -\frac{1}{M^2} \Delta - V^{\alpha\beta}(\vec{r}) ) \begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix} = E \begin{pmatrix} \Psi_1(\vec{r}) \\ \Psi_2(\vec{r}) \end{pmatrix}. \tag{4.47}
\]

The components \( \Psi_1(\vec{r}) \) and \( \Psi_2(\vec{r}) \) are probability amplitudes for the spin-flavor combinations \( \alpha_+ \beta_- \) and \( \alpha_- \beta_+ \), respectively. The potential

\[
V^{\alpha\beta}(\vec{r}) = \frac{\Lambda^2}{2\pi \rho_s} \frac{\cos(2\varphi)}{r^2}
\]

couples the two channels because magnon exchange is accompanied by a spin-flip. Here \( \vec{r}_+ - \vec{r}_- \) is the distance vector between the two holes of spin \( + \) and \( - \) and \( \varphi \) is the angle between \( \vec{r} \) and the \( x \)-axis. Magnon exchange is attractive between holes of opposite spin, and hence magnon-mediated two-hole bound states are invariant under the unbroken subgroup \( U(1)_s \). We make the ansatz

\[
\Psi_1(\vec{r}) \pm \Psi_2(\vec{r}) = R(r) \chi_{\pm}(\varphi). \tag{4.49}
\]
For the angular part of the wave function this implies
\[ -\frac{d^2\chi_\pm(\varphi)}{d\varphi^2} \pm \frac{M'\Lambda^2}{2\pi\rho_s} \cos(2\varphi)\chi_\pm(\varphi) = -\lambda\chi_\pm(\varphi). \] (4.50)

This is a Mathieu equation whose solution with the lowest eigenvalue \(-\lambda_1\) is given by
\[ \chi^1_\pm(\varphi) = \frac{1}{\sqrt{\pi}} ce_0 \left( \varphi, \pm \frac{M'\Lambda^2}{4\pi\rho_s} \right), \quad \lambda_1 = \frac{1}{2} \left( \frac{M'\Lambda^2}{4\pi\rho_s} \right)^2 + \mathcal{O}(\Lambda^8). \] (4.51)

The first excited state and its eigenvalue \(-\lambda_2\) is given by
\[
\begin{align*}
\chi^2_+(\varphi) &= \frac{1}{\sqrt{\pi}} se_1 \left( \varphi, \frac{M'\Lambda^2}{4\pi\rho_s} \right), \\
\chi^2_-(\varphi) &= \frac{1}{\sqrt{\pi}} se_1 \left( \varphi - \frac{\pi}{2}, \frac{M'\Lambda^2}{4\pi\rho_s} \right) = -\frac{1}{\sqrt{\pi}} ce_1 \left( \varphi, -\frac{M'\Lambda^2}{4\pi\rho_s} \right), \\
\lambda_2 &= -1 + \frac{M'\Lambda^2}{4\pi\rho_s} + \frac{1}{8} \left( \frac{M'\Lambda^2}{4\pi\rho_s} \right)^2 - \frac{1}{64} \left( \frac{M'\Lambda^2}{4\pi\rho_s} \right)^3 + \mathcal{O}(\Lambda^8). \quad (4.52)
\end{align*}
\]

For small \(\Lambda\), \(\lambda_2 < 0\), which (as we will see) implies that the corresponding two-hole state is unbound. For \(M'\Lambda^2/4\pi\rho_s > 0.908046\), on the other hand, \(\lambda_1, \lambda_2 > 0\), such that then both states are bound. The periodic Mathieu functions \(ce_0(\varphi, M'\Lambda^2/4\pi\rho_s)\) and \(se_1(\varphi, M'\Lambda^2/4\pi\rho_s)\) [60] are shown in figure 3. The corresponding radial Schrödinger equation is given by
\[
-\left[ \frac{d^2 R_i(r)}{dr^2} + \frac{1}{r} \frac{dR_i(r)}{dr} \right] - \frac{\lambda_i}{r^2} R_i(r) = M' E_i R_i(r), \quad i \in \{1, 2\}. \quad (4.53)
\]
Figure 4: Probability distribution for two holes with flavors α and β. Left panel: the ground state with p-wave symmetry. Right panel: excited states with s- or d-wave symmetry, but with identical probability densities \( (M'\Lambda^2/4\pi \rho_s = 1.25, r_0 = a) \).

The short-distance repulsion between two holes can be incorporated by a hard core of radius \( r_0 \), i.e. we require \( R_i(r_0) = 0 \). The radial Schrödinger equation for the bound states is solved by a Bessel function

\[
R_i(r) = A_i K_\nu(\sqrt{M' |E_i|} r), \quad k = 1, 2, 3, \ldots, \quad \nu = i \sqrt{\lambda_i}.
\] (4.54)

The energy (determined from \( K_\nu(\sqrt{M' |E_i|} r_0) = 0 \)) is then given by

\[
E_{ik} \sim -(M' r_0^2)^{-1} \exp(-2\pi k/\sqrt{\lambda_i})
\] (4.55)

for large \( n \). Magnon exchange mediates weak attractive forces that lead to a small binding energy.

The two lowest energy states with angular part \( \chi_1^1(\varphi) \) and \( \chi_1^1(\varphi) \) are degenerate in energy. Linearly combining the two states to two eigenstates of the rotation \( O \), one obtains

\[
\Psi_{\pm}(\vec{r}) = R_1(r) \begin{pmatrix} \chi_1^1(\varphi) \mp i \chi_1^1(\varphi) \\ \chi_1^1(\varphi) \pm i \chi_1^1(\varphi) \end{pmatrix}.
\] (4.56)

The corresponding probability density is illustrated in figure 4 (left panel). While the probability density seems to resemble \( d_{x^2-y^2} \) symmetry, unlike for an actual d-wave, the wave function is suppressed, but not equal to zero, along the lattice diagonals. In fact, as one operates on the states \( \Psi_{\pm}(\vec{r}) \) with the 90 degrees rotation \( O \), one obtains the eigenvalues \( \pm i \), which shows that they actually have p-wave symmetry.
Under the discrete symmetries $D', O,$ and $R$, the ground states $\Psi^1_\pm(\vec{r})$, which are bound by magnon exchange, transform as

\[
D'_1 \Psi^1_\pm(\vec{r}) = R_1(r) \left( \frac{\chi^1_+(\varphi) \pm i \chi^1_-\varphi)}{\chi^1_+\varphi) \mp i \chi^1_-\varphi} \right) = \Psi^1_\pm(\vec{r}),
\]
\[
D'_2 \Psi^1_\pm(\vec{r}) = -R_1(r) \left( \frac{\chi^1_+(\varphi) \pm i \chi^1_-\varphi)}{\chi^1_+\varphi) \mp i \chi^1_-\varphi} \right) = -\Psi^1_\pm(\vec{r}),
\]
\[
O \Psi^1_\pm(\vec{r}) = R_1(r) \left( \frac{\chi^1_+(\varphi) \pm i \chi^1_-\varphi)}{\chi^1_+\varphi) \mp i \chi^1_-\varphi} \right) = \Psi^1_\pm(\vec{r}),
\]
\[
R \Psi^1_\pm(\vec{r}) = R_1(r) \left( \frac{\chi^1_+(\varphi) \pm i \chi^1_-\varphi)}{\chi^1_+\varphi) \mp i \chi^1_-\varphi} \right) = \Psi^1_\pm(\vec{r}). \quad (4.57)
\]

It should be noted that in [14] there are two typos in the last line of the previous equation for the reflection symmetry $R$ (eq.(6.20) in [14]).

Remarkably, the magnon-mediated two-hole ground states $\Psi^1_\pm(\vec{r})$ transform exactly as the two-hole states localized on a rotating Skyrmion with $n = 1$, provided that we associate $\Psi^1_\pm(\vec{r})$ with the corresponding two-hole-Skyrmion wave function $\Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma)$ with the quantum numbers $\sigma = \pm$, $m^\alpha_+ = m^\beta_+ = -1$, $m^\alpha_- = m^\beta_- = 1$, and $m = 0$. Indeed, according to eq.(4.43) one obtains

\[
D'_1 \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma),
\]
\[
D'_2 \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = -\Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma),
\]
\[
O \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \pm i \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma),
\]
\[
R \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \Psi^{\alpha\beta}_{\pm,-1,1,-1,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma). \quad (4.58)
\]

Just as the magnon-mediated bound states, these states are also invariant under $U(1)_s$ and they have fermion number 2. One may argue that the two-hole-Skyrmion states, in addition, have Skyrmion number as a conserved topological quantum number. However, as we discussed before, Skyrmion number has no analog in the underlying microscopic Hubbard or $t$-$J$ models and is just an accidental symmetry of the effective theory. We thus conclude that, in their ground state, two holes bound by magnon exchange indeed have exactly the same quantum numbers as two holes localized on a rotating Skyrmion with $n = 1$. This implies that these sets of states may evolve into each other upon doping. In this way, two holes weakly bound by magnon exchange at small doping may evolve into a strongly correlated preformed pair of holes localized on a Skyrmion. However, as we have just seen, these bound states actually have p-wave symmetry.
Let us also consider the excited states, bound by magnon exchange, with angular part $\chi^\pm_\alpha(\phi)$ and $\chi^\pm_\beta(\phi)$, which are again degenerate. Linearly combining these two states to two eigenstates of the rotation $O$, one obtains

$$\Psi^2_\pm(\vec{r}) = R_2(r) \left( \begin{array}{c} \chi^2_\pm(\phi) \mp \chi^2_\mp(\phi) \\ \chi^2_\mp(\phi) \mp \chi^2_\pm(\phi) \end{array} \right).$$

Operating on the states $\Psi^2_\pm(\vec{r})$ with the 90 degrees rotation $O$, one now obtains the eigenvalues $\pm 1$, which implies that $\Psi^2_\pm(\vec{r})$ represents an $s$-wave, while $\Psi^2_\pm(\vec{r})$ actually has $d$-wave symmetry. As a consequence of an interplay of the various symmetries, the two states are exactly degenerate. The corresponding probability density is illustrated in figure 4 (right panel). Interestingly, although the states have different symmetries, their probability densities are identical.

Under the discrete symmetries $D'_i$, $O$, and $R$, the excited states $\Psi^2_\pm(\vec{r})$, which are bound by magnon exchange, transform as

$$D'_i \Psi^2_\pm(\vec{r}) = -R_2(r) \left( \begin{array}{c} \chi^2_\pm(\phi) \mp \chi^2_\mp(\phi) \\ \chi^2_\mp(\phi) \mp \chi^2_\pm(\phi) \end{array} \right) = -\Psi^2_\pm(\vec{r}),$$

$$D'_i \Psi^2_\pm(\vec{r}) = R_2(r) \left( \begin{array}{c} \chi^2_\pm(\phi) \mp \chi^2_\mp(\phi) \\ \chi^2_\mp(\phi) \mp \chi^2_\pm(\phi) \end{array} \right) = \Psi^2_\pm(\vec{r}),$$

$$O \Psi^2_\pm(\vec{r}) = R_2(r) \left( \begin{array}{c} \chi^2_\pm(\phi + \frac{\pi}{2}) \pm \chi^2_\pm(\phi - \frac{\pi}{2}) \\ -\chi^2_\pm(\phi + \frac{\pi}{2}) \pm \chi^2_\pm(\phi - \frac{\pi}{2}) \end{array} \right)$$

$$= R_2(r) \left( \begin{array}{c} -\chi^2_\pm(\phi) \mp \chi^2_\pm(\phi) \\ \chi^2_\pm(\phi) \pm \chi^2_\pm(\phi) \end{array} \right) = \pm \Psi^2_\pm(\vec{r}),$$

$$R \Psi^2_\pm(\vec{r}) = R_2(r) \left( \begin{array}{c} \chi^2_\pm(-\phi) \pm \chi^2_\pm(-\phi) \\ \chi^2_\pm(-\phi) \pm \chi^2_\pm(-\phi) \end{array} \right)$$

$$= R_2(r) \left( \begin{array}{c} -\chi^2_\pm(\phi) \mp \chi^2_\pm(\phi) \\ -\chi^2_\pm(\phi) \mp \chi^2_\pm(\phi) \end{array} \right) = -\Psi^2_\pm(\vec{r}).$$

States with $d$-wave symmetry can also be constructed for two holes localized on a Skyrmion. For example, for $n = 1$, the states $\Psi^{\alpha\beta}_{\pm,1,1,0,2,0}(r_\alpha, r_\beta, \chi_\alpha, \chi_\beta, \gamma$) and $\Psi^{\alpha\beta}_{\pm,0,2,-1,1,0}(r_\alpha, r_\beta, \chi_\alpha, \chi_\beta, \gamma$) have $d$-wave symmetry. Under the symmetries $D'_i$ and $D'_2$ they transform into $\Psi^{\alpha\beta}_{\pm,-1,1,0,2,0}(r_\alpha, r_\beta, \chi_\alpha, \chi_\beta, \gamma$) and $\Psi^{\alpha\beta}_{\pm,-2,0,-1,1,0}(r_\alpha, r_\beta, \chi_\alpha, \chi_\beta, \gamma$), which have $s$-wave symmetry. According to eq. (4.43), under the various symmetries

32
the d-wave states transform as

\[
D_1' \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_2' \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
O \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
R \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_1' \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_2' \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
O \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
R \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma). \quad (4.61)
\]

Similarly, the s-wave states transform as follows

\[
D_1' \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_2' \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
O \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
R \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,2,0,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_1' \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_2' \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
O \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
R \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,0,2,1,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma). \quad (4.62)
\]

Hence, just as for the magnon-mediated excited states, as a consequence of the interplay of the various symmetries, s- and d-wave states are again degenerate.

Alternatively, d-wave states also arise for two holes localized on a Skyrmion with winding number \( n = 2 \). For example, the two states \( \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) \) have d-wave symmetry, and they transform into the states \( \Psi_{\pm,2,1,2,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) \), which again have s-wave symmetry, under \( D_1' \) and \( D_2' \). According to eq. \((1.43)\), the d-wave states transform as

\[
D_1' \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
D_2' \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
O \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = -\Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \\
R \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma) = \Psi_{\pm,1,2,1,0}^\alpha (r_\alpha, \chi_\alpha, r_\beta, \chi_\beta; \gamma), \quad (4.63)
\]
while the s-wave states transform as
\[ D^\prime_1 \Psi^{\alpha\beta}_{\pm,-2,1,-1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \Psi^{\alpha\beta}_{\pm,-1,2,-1,2,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \]
\[ D^\prime_2 \Psi^{\alpha\beta}_{\pm,-2,1,-2,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = -\Psi^{\alpha\beta}_{\pm,-1,2,-1,2,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \]
\[ O \Psi^{\alpha\beta}_{\pm,-2,1,-2,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \Psi^{\alpha\beta}_{\pm,-2,1,-2,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma), \]
\[ R \Psi^{\alpha\beta}_{\pm,-2,1,-2,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma) = \Psi^{\alpha\beta}_{\pm,-2,1,-2,1,0}(r_\alpha, \chi_\alpha, r_\beta, \chi_\beta, \gamma). \] (4.64)

Depending on the details of the dynamics, whose investigation goes beyond the scope of the present paper, it may be possible that the degenerate s- and d-wave states have a lower energy than the p-wave states discussed earlier.

### 4.7 Possible Implications for Cooper Pair Formation in High-Temperature Superconductors

As we have seen, two holes, one of flavor \( \alpha \) and one of flavor \( \beta \), can both get localized in the ground state of a single rotating Skyrmion with \( n = 1 \), which turns out to have p-wave symmetry. Alternatively, the holes may get localized on a rotating \( n = 1 \) or 2 Skyrmion with s-wave or d-wave symmetry. As discussed in Appendix A, two holes of the same flavor can also get localized on a Skyrmion. It will be the subject of a subsequent publication to decide which of the various states is energetically most favorable.

While in this paper we have concentrated on a detailed symmetry analysis, we also want to get at least a crude estimate of the binding energy of two-hole states localized on an \( n = 1 \) Skyrmion. Ignoring contact interactions between the two holes, the total energy of the bound state of two holes and a Skyrmion can then be estimated as
\[ E_{\text{tot}} = 2M + 4\pi \rho_s + 2E_0, \] (4.65)
while two free holes (not localized on a Skyrmion) just have their rest energy \( 2M \). Using the result of eq.(4.13), the perturbative vacuum thus becomes unstable against the formation of two-hole-Skyrmion bound states when
\[ 4\pi \rho_s + 2E_0 < 0 \Rightarrow 0.270M'\Lambda^2 > 4\pi \rho_s. \] (4.66)

Hence, for sufficiently small spin stiffness \( \rho_s \), the instability will indeed arise. Similar instabilities are related to the formation of spiral phases in the staggered magnetization order parameter. In particular, in [25] we have shown that the ground state with a spatially constant staggered magnetization becomes unstable against the formation of a 45 degrees spiral phase for \( M'\Lambda^2 > 4\pi \rho_s \). Since antiferromagnetism is weakened upon doping, \( \rho_s \) is expected to eventually go to zero. Before this happens, pairs of holes will get localized on a Skyrmion.
In order to get at least a rough idea of the involved energy scales, let us estimate the values of the relevant low-energy parameters for realistic lightly doped quantum antiferromagnets. By comparison with \[23, 26, 27\], where a generalized \(t\)-\(J\) model on a square lattice with spacing \(a\) was considered at \(J/t \approx 0.3\), one obtains the rough estimate
\[
M' \approx \frac{1}{ta^2} \approx 0.3 J a^2, \quad \Lambda \approx 2.5 J a.
\] (4.67)

It would be interesting and definitely feasible to extract these parameters with high precision from numerical simulations. In this way, in the Heisenberg model (i.e. the undoped \(t\)-\(J\) model) very accurate numerical results have been obtained for the spin stiffness, the spinwave velocity, and the staggered magnetization per lattice site \[1, 3, 4\]
\[
\rho_s = 0.18081(11) J, \quad c = 1.6586(3) J a, \quad M_s = 0.30743(1)/a^2.
\] (4.68)

Hence, one obtains \(0.270 M' \Lambda^2 \approx 0.5 J\) compared to \(4\pi \rho_s = 2.2721(1) J\), which implies that two-hole-Skyrmion bound states are still far from being energetically favorable at zero doping. The exchange coupling of undoped La\(_2\)CuO\(_4\) is \(J = 1540(60)\) K \[1\]. A high transition temperature of \(T_c \approx 50\) K, and hence \(T_c \approx 0.03 J\), would thus require a two-hole-Skyrmion bound state energy of about
\[
4\pi \rho_s + 2E_0 = 4\pi \rho_s - 0.270 M' \Lambda^2 \approx -0.03 J \Rightarrow \rho_s \approx 0.04 J.
\] (4.69)

If doping reduces \(\rho_s\) by a factor of about 4 or 5 (and assuming for simplicity that the other parameters remain unchanged), the estimated energy scales should indeed be of the right magnitude in order to make two-holes localized on a rotating Skyrmion a viable candidate for a preformed Cooper pair of a high-temperature superconductor. Using eq. (4.12), one can estimate the radius of the Skyrmion, which sets the scale for the size of the candidate Cooper pair, as \(\rho \approx 1/(0.271 M' \Lambda) \approx 5a\), which again seems reasonable.

It may involve some wishful thinking to assume that the d-wave state of two holes localized on an \(n = 1\) or \(2\) Skyrmion will not only turn out to be energetically favorable, but also ready to condense at sufficiently large doping. However, we think that it is worthwhile to take this possibility seriously. Deciding whether the radial dynamics favors these states as promising candidates for a preformed Cooper pair in the pseudo-gap phase is the natural next step. The question of condensation is another important issue.

5 Conclusions

We have performed a detailed study of the localization of holes on a Skyrmion in an square lattice antiferromagnet. When two holes get localized on the same Skyrmion, they form a bound state. Interestingly, in some cases, the quantum numbers of these topologically non-trivial bound states are the same as those of the topologically trivial
bound states resulting from one-magnon exchange between two holes. The ground state of two holes weakly bound by one-magnon exchange has p-wave symmetry and may evolve into a strongly bound state of two holes localized on an $n = 1$ Skyrmion at strong coupling.

Magnon-mediated two-hole bound states which are excited in the angular motion have s- or d-wave symmetry. Remarkably, s- and d-wave states are degenerate due to an interplay of the various symmetries. Similarly, there are strongly bound states of two holes localized on an $n = 1$ or 2 Skyrmion which also have s- or d-wave symmetry, and are again degenerate. Which of these states is energetically most favorable will be an interesting subject for future studies. If a d-wave state turns out to be the ground state at sufficiently strong doping, two holes localized on a Skyrmion are a promising candidate for a preformed Cooper pair in the pseudo-gap regime. Interestingly, the effective theory provides detailed predictions for the anatomy of these objects. In particular, their angular structure follows unambiguously from our symmetry analysis, and is insensitive to the details of the radial dynamics.

Understanding the dynamical mechanism responsible for high-temperature superconductivity has proved to be one of the most challenging problems in theoretical physics. While hole pair localization on a rotating Skyrmion may ultimately turn out not to be the relevant mechanism, it seems rather promising. Beyond the symmetry analysis presented here, studying its dynamics in more detail is certainly worthwhile.

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A Hole Pairs of the Same Flavor

In this appendix, we consider a pair of holes in a square lattice antiferromagnet residing in the same hole pocket. First, we investigate two holes localized on a rotating Skyrmion, and then we compare the results with the corresponding two-hole magnon-mediated bound states.

A.1 Schrödinger Equation for a Pair of Holes of the Same Flavor Localized on a Rotating Skyrmion

Let us consider bound states of two holes of the same flavor $f$ localized on a rotating Skyrmion. In this case, as a consequence of the Pauli principle, the holes cannot occupy the same quantum state. We distinguish the holes by an unphysical label 1 or 2. In order to satisfy the Pauli principle, the wave function must be anti-symmetric under the exchange of the two labels.

The Hamiltonian for two holes of the same flavor $f$ is then given by

$$H = H^1 + H^2 + H^\gamma,$$

(A.1)

where

$$H^1 = \begin{pmatrix}
H^f_{++} & 0 & H^f_{+-} & 0 \\
0 & H^f_{++} & 0 & H^f_{+-} \\
H^f_{-+} & 0 & H^f_{--} & 0 \\
0 & H^f_{-+} & 0 & H^f_{--}
\end{pmatrix},$$

$$H^2 = \begin{pmatrix}
H^f_{++} & H^f_{+-} & 0 & 0 \\
H^f_{+-} & H^f_{--} & 0 & 0 \\
0 & 0 & H^f_{++} & H^f_{+-} \\
0 & 0 & H^f_{+-} & H^f_{--}
\end{pmatrix},$$

(A.2)

with $H^f_{++}$ and $H^f_{++}$ given in eq.(4.31).

Before anti-symmetrizing the wave function in the artificial labels 1 and 2, we ignore the Pauli principle, and make the following ansatz for an energy eigenstate of two holes (distinguished by the labels 1 and 2)

$$\psi_{\sigma, m_1^+, m_2^+, m_1^-, m_2^-} (r_1, r_2, \chi_1, \chi_2; \gamma) =$$

$$\begin{pmatrix}
\psi_{\sigma, m_1^+, m_1^+, m_2^-, m_2^-} (r_1, r_2) \exp\left(i\sigma \left[m_1^+ \chi_1 + m_2^- \chi_2 - \sigma f_{\frac{\pi}{4}}\right]\right) \exp(i\sigma(m - 1)\gamma) \\
\sigma \sigma f \psi_{\sigma, m_1^+, m_1^-, m_2^+, m_2^-} (r_1, r_2) \exp\left(i\sigma \left[m_1^+ \chi_1 + m_2^- \chi_2\right]\right) \exp(i\sigma m\gamma) \\
\sigma \sigma f \psi_{\sigma, m_1^+, m_1^-, m_2^+, m_2^-} (r_1, r_2) \exp\left(i\sigma \left[m_1^+ \chi_1 + m_2^- \chi_2\right]\right) \exp(i\sigma m\gamma) \\
\psi_{\sigma, m_1^+, m_1^-, m_2^-, m_2^-} (r_1, r_2) \exp\left(i\sigma \left[m_1^- \chi_1 + m_2^+ \chi_2 + \sigma f_{\frac{\pi}{4}}\right]\right) \exp(i\sigma(m + 1)\gamma)
\end{pmatrix}$$

(A.3)
As before, this solves the Schrödinger equation only if $m_i^i - m_i^+ = n + 1$, $i = 1, 2$. In this case, $m$ is again an integer. The resulting radial Schrödinger equation now takes the form

\[ H_r \psi_{\sigma,m^+,m^-,m^2,m}(r_1, r_2) = E_{\sigma,m^+,m^-,m^2,m} \psi_{\sigma,m^+,m^-,m^2,m}(r_1, r_2), \quad (A.4) \]

with

\[ \psi_{\sigma,m^+,m^-,m^2,m}(r_1, r_2) = \begin{pmatrix} \psi_{\sigma,m^+,m^2,m^2,m,m,+}(r_1, r_2) \\ \psi_{\sigma,m^+,m^2,m^2,m,m,-}(r_1, r_2) \\ \psi_{\sigma,m^+,m^2,m^2,m,-}(r_1, r_2) \\ \psi_{\sigma,m^+,m^2,m^2,m,-}(r_1, r_2) \end{pmatrix}. \quad (A.5) \]

The radial Hamiltonian is given by

\[ H_r = H^1_r + H^2_r + H^\gamma_r, \quad (A.6) \]

with

\[ H^1_r = \begin{pmatrix} H^{1+}_{r++} & 0 & H^{1-}_{r+-} & 0 \\ 0 & H^{1+}_{r+-} & 0 & H^{1-}_{r-} \\ H^{1+}_{r+} & 0 & H^{1-}_{r-} & 0 \\ 0 & H^{1+}_{r-} & 0 & H^{1-}_{r-} \end{pmatrix}, \]

\[ H^2_r = \begin{pmatrix} H^{2+}_{r++} & H^{2+}_{r+-} & 0 & 0 \\ H^{2+}_{r+} & H^{2-}_{r-} & 0 & 0 \\ 0 & 0 & H^{2+}_{r+} & H^{2-}_{r-} \\ 0 & 0 & H^{2+}_{r-} & H^{2-}_{r-} \end{pmatrix}, \]

\[ H^\gamma_r = \begin{pmatrix} H^{\gamma}_{r++} & 0 & 0 & 0 \\ 0 & H^{\gamma}_{r+-} & 0 & 0 \\ 0 & 0 & H^{\gamma}_{r+} & 0 \\ 0 & 0 & 0 & H^{\gamma}_{r-} \end{pmatrix}. \quad (A.7) \]

The matrix elements of the fermionic part of the radial Hamiltonian are given by

\[ H^i_{r++} = -\frac{1}{2M'} \left[ \partial^2_{r_i} + \frac{1}{r_i} \partial_{r_i} - \frac{1}{r_i^2} \left( m^i + \frac{n\rho^{2n}}{r_i^{2n} + \rho^{2n}} \right)^2 \right], \]

\[ H^i_{r+-} = H^i_{r-+} = \sqrt{2} \Lambda \frac{nn_i^{n-1} \rho^n}{r_i^{2n} + \rho^{2n}}, \]

\[ H^i_{r-} = -\frac{1}{2M'} \left[ \partial^2_{r_i} + \frac{1}{r_i} \partial_{r_i} - \frac{1}{r_i^2} \left( m_i - \frac{n\rho^{2n}}{r_i^{2n} + \rho^{2n}} \right)^2 \right], \quad (A.8) \]
while the rotational Skyrmion contributions are given by

\[
H_{\gamma}^\gamma = \frac{n^2}{2D(p)\rho^2} \left( m + \frac{\Theta}{2\pi} n \right) - \frac{\rho^{2n}}{r_1^{2n} + \rho^{2n}} - \frac{\rho^{2n}}{r_2^{2n} + \rho^{2n}} \right)^2, \\
H_{\gamma}^\gamma = \frac{n^2}{2D(p)\rho^2} \left( m + \frac{\Theta}{2\pi} n \right) - \frac{\rho^{2n}}{r_1^{2n} + \rho^{2n}} + \frac{\rho^{2n}}{r_2^{2n} + \rho^{2n}} \right)^2, \\
H_{\gamma}^\gamma = \frac{n^2}{2D(p)\rho^2} \left( m + \frac{\Theta}{2\pi} n \right) + \frac{\rho^{2n}}{r_1^{2n} + \rho^{2n}} - \frac{\rho^{2n}}{r_2^{2n} + \rho^{2n}} \right)^2, \\
H_{\gamma}^\gamma = \frac{n^2}{2D(p)\rho^2} \left( m + \frac{\Theta}{2\pi} n \right) + 1 + \frac{\rho^{2n}}{r_1^{2n} + \rho^{2n}} + \frac{\rho^{2n}}{r_2^{2n} + \rho^{2n}} \right)^2. \tag{A.9}
\]

A.2 Symmetry Properties of a Pair of Holes with the Same Flavor Localized on a Skyrmion

The spin operator \( I \) is again given by eq. (4.39), such that

\[
I\Psi^{\text{ff}}_{\sigma,m_1^-,m_2^-,m}(r_1, r_2, \chi, \gamma) = \left( m + \frac{\Theta}{2\pi} n \right) \Psi^{\text{ff}}_{\sigma,m_1^-,m_2^-,m}(r_1, r_2, \chi, \gamma).
\tag{A.10}
\]

Since \( m \) is an integer, at least for \( \Theta = 0 \), the state with two holes of the same flavor localized on a Skyrmion again has integer spin.

The symmetries \( D' \), \( O \), and \( R \) act on the two-hole wave function

\[
\Psi^{\text{ff}}_{\sigma,n}(r_1, r_2, \chi, \gamma) = \begin{pmatrix}
\Psi^{\text{ff}}_{\sigma,n,++}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,+--}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,----}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,-----}(r_1, r_2, \chi, \gamma)
\end{pmatrix}
\tag{A.11}
\]

as follows

\[
D'_{\sigma} \Psi^{\text{ff}}_{\sigma,n}(r_1, r_2, \chi, \gamma) = \exp(2i\frac{D'}{\hbar} a) \begin{pmatrix}
\Psi^{\text{ff}}_{\sigma,n,++}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,+--}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,----}(r_1, r_2, \chi, \gamma) \\
\Psi^{\text{ff}}_{\sigma,n,-----}(r_1, r_2, \chi, \gamma)
\end{pmatrix},
\]

\[
O \Psi^{\text{ff}}_{\sigma,n}(r_1, r_2, \chi, \gamma) = \begin{pmatrix}
\Psi^{\text{ff}}_{\sigma,n,++}(r_1, \chi_1 + \frac{\pi}{2}, r_2, \chi_2 + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
\sigma_f \Psi^{\text{ff}}_{\sigma,n,++}(r_1, \chi_1 + \frac{\pi}{2}, r_2, \chi_2 + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
\sigma_f \Psi^{\text{ff}}_{\sigma,n,--}(r_1, \chi_1 + \frac{\pi}{2}, r_2, \chi_2 + \frac{\pi}{2}, \gamma - n\frac{\pi}{2}) \\
\Psi^{\text{ff}}_{\sigma,n,----}(r_1, \chi_1 + \frac{\pi}{2}, r_2, \chi_2 + \frac{\pi}{2}, \gamma - n\frac{\pi}{2})
\end{pmatrix},
\]

\[
R \Psi^{\text{ff}}_{\sigma,n}(r_1, r_2, \chi, \gamma) = \begin{pmatrix}
\Psi^{\text{ff}}_{\sigma,n,++}(r_1, -\chi_1, r_2, -\chi_2, -\gamma) \\
\Psi^{\text{ff}}_{\sigma,n,++}(r_1, -\chi_1, r_2, -\chi_2, -\gamma) \\
\Psi^{\text{ff}}_{\sigma,n,--}(r_1, -\chi_1, r_2, -\chi_2, -\gamma) \\
\Psi^{\text{ff}}_{\sigma,n,----}(r_1, -\chi_1, r_2, -\chi_2, -\gamma)
\end{pmatrix}. \tag{A.12}
\]
For the two-hole energy eigenstates this implies

\[
D'_i \Psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma) = -\Psi_{-\sigma,-m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
O \Psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma) = -\exp \left( i\sigma [m_1^+ + m_2^- - mn] \right)^{\frac{\pi}{2}} \Psi_{\sigma,m_1^-,m_2^-,m}^{\beta\beta}(r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
R \Psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma) = \Psi_{-\sigma,m_1^-,m_2^-,m}^{\beta\beta}(r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
R \Psi_{\sigma,m_1^-,m_2^-,m}^{\beta\beta}(r_1, \chi_1, r_2, \chi_2, \gamma) = \Psi_{-\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma). \quad (A.13)
\]

Here we have again assumed an appropriate phase convention for the radial wave function \(\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2)\). In the context of the shift symmetries \(D'_i\) we have used

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,-m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,-m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,-m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,-m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2). \quad (A.14)
\]

These relations follow from the symmetries of the radial Schrödinger equation \([A.4]\). In the context of the reflection symmetry \(R\) we have used

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2),
\]

\[
\psi_{\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2) = \psi_{-\sigma,m_1^-,m_2^-,m}^{\alpha\alpha}(r_1, r_2). \quad (A.15)
\]

The relations in eq. \([A.15]\) follow from the symmetries of the radial Schrödinger equation \([A.4]\) for \(\Theta = 0\). As before, for \(\Theta \neq 0\) or \(\pi\), the Hopf term explicitly breaks the reflection symmetry.

Let us now impose the Pauli principle by explicitly anti-symmetrizing the wave function in the artificial indices 1 and 2. For this purpose we act with the pair permutation \(P\), i.e.

\[
P \Psi_{\sigma,n}^{\alpha\alpha}(r_1, \chi_1, r_2, \chi_2, \gamma) = \begin{pmatrix}
\psi_{\sigma,n,++}^{\alpha\alpha}(r_2, \chi_2, r_1, \chi_1, \gamma) \\
\psi_{\sigma,n,-+}^{\alpha\alpha}(r_2, \chi_2, r_1, \chi_1, \gamma) \\
\psi_{\sigma,n,-+}^{\alpha\alpha}(r_2, \chi_2, r_1, \chi_1, \gamma) \\
\psi_{\sigma,n,--}^{\alpha\alpha}(r_2, \chi_2, r_1, \chi_1, \gamma)
\end{pmatrix} \quad (A.16).
\]
For an energy eigenstate this implies
\[ P_{\sigma,m_1^-,m_2^-,m,m}(r_1,\chi_1,\chi_2,\gamma) = \Psi_{\sigma,m_1^+,m_2^+,m_1^-,m_2^-,m}(r_1,\chi_1,\chi_2,\gamma). \] (A.17)

Here we have assumed a symmetric radial wave function, i.e.
\[ \psi_{\sigma,m_1^-,m_2^-,m,m,+}(r_1, r_1) = \psi_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, r_2), \]
\[ \psi_{\sigma,m_1^-,m_2^-,m,m,-}(r_1, r_1) = \psi_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, r_2), \]
\[ \psi_{\sigma,m_1^-,m_2^-,m,m,+}(r_2, r_1) = \psi_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, r_2), \]
\[ \psi_{\sigma,m_1^-,m_2^-,m,m,-}(r_2, r_1) = \psi_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, r_2). \] (A.18)

The properly anti-symmetrized wave function now takes the form
\[ \tilde{\Psi}_{\sigma,n}(r_1, \chi_1, r_2, \chi_2, \gamma) = \frac{1}{\sqrt{2}} [\Psi_{\sigma,n}(r_1, \chi_1, r_2, \chi_2, \gamma) - P_{\sigma,n}(r_1, \chi_1, r_2, \chi_2, \gamma)]. \] (A.19)

For an energy eigenstate this implies
\[ \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = \frac{1}{\sqrt{2}} \left[ \Psi_{\sigma,m_1^+,m_2^+,m_1^-,m_2^-,m}(r_1, \chi_1, r_2, \chi_2, \gamma) - \Psi_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, \chi_1, r_2, \chi_2, \gamma) \right]. \] (A.20)

As expected, in order to obtain a non-vanishing wave function, the two sets of quantum numbers \( m_1^+, m_2^+ \) and \( m_1^-, m_2^- \) must be different, because otherwise two identical fermions would occupy the same single particle state. If one would consider an anti-symmetric radial wave function, one could allow \( m_1^+ = m_2^- \) and \( m_1^- = m_2^+ \).

Based on eq. (A.13), the properly anti-symmetrized two-hole energy eigenstates transform as follows
\[ D_{\sigma} \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = -\tilde{\Psi}_{\sigma,-m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma), \]
\[ O_{\sigma} \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = -\exp \left( i\sigma [m_1^- + m_2^- - mn] \frac{\pi}{2} \right) \times \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma), \]
\[ R_{\sigma} \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = \tilde{\Psi}_{\sigma,-m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma), \]
\[ R_{\sigma} \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = \tilde{\Psi}_{\sigma,-m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma). \] (A.21)

In order to show this for the rotation \( O \), we have used \( m_1^+ + m_2^- = m_2^+ + m_1^- \).

Finally, let us combine states with flavors \( \alpha \alpha \) and \( \beta \beta \) to eigenstates of \( O \),
\[ \tilde{\Psi}_{\sigma,m_1^-,m_2^-,m,m}(r_1, \chi_1, r_2, \chi_2, \gamma) = \frac{1}{\sqrt{2}} \left[ \Psi_{\sigma,m_1^+,m_2^+,m_1^-,m_2^-,m}(r_1, \chi_1, r_2, \chi_2, \gamma) \pm i\tilde{\Psi}_{\sigma,m_1^-,m_2^-,m_1^+,m_2^-,m}(r_1, \chi_1, r_2, \chi_2, \gamma) \right]. \] (A.22)
which transform as

\[
D'_\sigma \tilde{\Psi}^\pm_{\sigma,m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma) = -\tilde{\Psi}^-_{-\sigma,-m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
O \tilde{\Psi}^+_{\sigma,m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma) = \pm i \exp \left( i \sigma [m_1^+ + m_2^- - m_1^-] \frac{\pi}{2} \right) \times \tilde{\Psi}^+_{-\sigma,m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
R \tilde{\Psi}^+_{\sigma,m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma) = \pm i \tilde{\Psi}^+_{-\sigma,m_1^-,m_2^-,m} (r_1, \chi_1, r_2, \chi_2, \gamma).
\] (A.23)

The lowest energy states in the same flavor channel are expected to correspond to \( m_1^+ = -1, m_1^- = 1, m_2^+ = -2, m_2^- = 0, m = 0 \) or \( m_1^+ = -1, m_1^- = 1, m_2^+ = 0, m_2^- = 2, m = 0 \). These states transform as

\[
D'_\sigma \tilde{\Psi}^\pm_{-\sigma,-1,-2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) = -\tilde{\Psi}^-_{\sigma,-1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
O \tilde{\Psi}^\pm_{-\sigma,-1,-2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) = \pm \sigma \tilde{\Psi}^\pm_{-\sigma,-1,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
R \tilde{\Psi}^\pm_{-\sigma,-1,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) = -\tilde{\Psi}^\pm_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
D'_\sigma \tilde{\Psi}^\pm_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) = \mp \sigma \tilde{\Psi}^\pm_{\sigma,-1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma),
\]

\[
R \tilde{\Psi}^\pm_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) = \pm i \tilde{\Psi}^\pm_{\sigma,-1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma).
\] (A.24)

This implies that the states \( \tilde{\Psi}^\pm_{-\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^-_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^+_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^-_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) \) are s-waves, while the states \( \tilde{\Psi}^\pm_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^-_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^+_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma), \tilde{\Psi}^-_{\sigma,-1,1,0,2,0} (r_1, \chi_1, r_2, \chi_2, \gamma) \) are d-waves.

### A.3 Comparison with Magnon-Mediated Two-Hole Bound States of the Same Flavor

In [14] states of two holes of the same flavor bound by one-magnon exchange have also been investigated. Here we summarize as well as extend some of the relevant results. We consider two holes of the same flavor \( f \) with opposite spins + and −. In the rest frame the wave function depends on the distance vector \( \vec{r} \) which points from the spin + hole to the spin − hole. Since magnon exchange is accompanied by a spin-flip, the vector \( \vec{r} \) changes its direction in the magnon exchange process. The Schrödinger equation thus takes the form

\[
-\frac{1}{M} \Delta \Psi (\vec{r}) + V_{ff}(\vec{r}) \Psi (-\vec{r}) = E \Psi (\vec{r}).
\] (A.25)

The one-magnon exchange potential for two holes of the same flavor is given by

\[
V^{aa}(\vec{r}) = \frac{\Lambda^2}{2\pi \rho_s} \frac{\sin(2\varphi)}{r^2}, \quad V^{\beta \beta}(\vec{r}) = -\frac{\Lambda^2}{2\pi \rho_s} \frac{\sin(2\varphi)}{r^2}.
\] (A.26)
Figure 5: Angular wave functions $c_{0}^{(\phi - \pi/4, \frac{1}{2}M'\gamma)}$ (solid curve) and $s_{1}^{(\phi - \pi/4, \frac{1}{2}M'\gamma)}$ (dashed curve) as well as angle-dependence $\sin(2\phi)$ of the potential (dotted curve) for two holes of flavor $\alpha$ residing in a circular hole pocket ($M'\Lambda^2/4\pi\rho_s = 1.25$).

We make a separation ansatz

$$\Psi(\vec{r}) = R'(r)\chi'(|\phi|).$$  \hspace{1cm} (A.27)

The ground state is even with respect to the reflection of $\vec{r}$ to $-\vec{r}$, i.e.

$$\chi'^1(\phi + \pi) = \chi'^1(\phi).$$  \hspace{1cm} (A.28)

The angular part of the Schrödinger equation then takes the form

$$-\frac{d^2\chi'^1_{\pm}(\phi)}{d\phi^2} \pm \frac{M'\Lambda^2}{2\pi\rho_s} \sin(2\phi)\chi'^1_{\pm}(\phi) = -\lambda_1\chi'^1_{\pm}(\phi).$$  \hspace{1cm} (A.29)

Here $+$ and $-$ are associated with an $\alpha\alpha$ and a $\beta\beta$ pair, respectively. Again, eq.\((A.29)\) is a Mathieu equation. The ground state with eigenvalue $-\lambda_1$ takes the form

$$\chi'^1_{\pm}(\phi) = \chi_{\pm}^{\pm}(\phi - \frac{\pi}{4}) = \frac{1}{\sqrt{\pi}} c_{0}^{(\phi - \frac{\pi}{4}, \frac{1}{2}M'\gamma)} \pm \frac{M'\Lambda^2}{4\pi\rho_s},$$

$$\lambda_1 = \frac{1}{2} \left( \frac{M'\Lambda^2}{4\pi\rho_s} \right)^2 + O(\Lambda^8).$$  \hspace{1cm} (A.30)

The first excited states are odd with respect to the reflection of $\vec{r}$ to $-\vec{r}$, i.e.

$$\chi'^2_{\pm}(\phi + \pi) = -\chi'^2_{\pm}(\phi).$$  \hspace{1cm} (A.31)
and the angular part of the Schrödinger equation now reads

\[
-d^2 \lambda^2(\varphi) + \frac{M' \Lambda^2}{2\pi \rho_s} \sin(2\varphi) \lambda^2(\varphi) = -\lambda_2 \lambda^2(\varphi).
\]  

(A.32)

Now — and + are associated with an \(\alpha\alpha\) and a \(\beta\beta\) pair, respectively. The excited states with eigenvalue \(-\lambda_2\) are given by

\[
\lambda^2(\varphi) = \lambda^2(\varphi - \frac{\pi}{4}) = \frac{1}{\sqrt{\pi}} \text{se}_1(\varphi - \frac{\pi}{4}, M' \Lambda^2),
\]

\[
\lambda^2(\varphi) = \lambda^2(\varphi - \frac{\pi}{4}) = -\frac{1}{\sqrt{\pi}} \text{ce}_1(\varphi - \frac{\pi}{4}, -M' \Lambda^2),
\]

\[
\lambda_2 = -1 + \frac{M' \Lambda^2}{4\pi \rho_s} + \frac{1}{8} \left( \frac{M' \Lambda^2}{4\pi \rho_s} \right)^2 - \frac{1}{64} \left( \frac{M' \Lambda^2}{4\pi \rho_s} \right)^3 + O(\Lambda^8).
\]  

(A.33)

The angular wave functions for the ground state and for the first excited state together with the angular dependence of the one-magnon exchange potential are shown in figure 5.

As before, the radial Schrödinger equation takes the form of eq.(4.53). Again, the short-distance repulsion between two holes is modeled by a hard core of radius \(r_0'\), i.e. \(R'(r_0') = 0\). The value of \(r_0'\) may, however, differ from \(r_0\) in the \(\alpha\beta\) case. The radial wave functions are thus given by

\[
R'_k(r) = A'_k K_\nu \left( \sqrt{M' E'_{ik}^2 r} \right), \quad k = 1, 2, 3, \ldots, \nu = i \sqrt{\lambda},
\]  

(A.34)

and the energy is determined from \(K_\nu \left( \sqrt{M' E'_{ik}^2 r_0'} \right) = 0\).

There are two degenerate states — one for an \(\alpha\alpha\) and one for a \(\beta\beta\) pair, which are eigenstates of flavor related to each other by a 90 degrees rotation. The two degenerate states can be combined to eigenstates of the rotation symmetry \(O\). For this purpose, we construct the 2-component wave functions

\[
\Psi^\alpha_\pm(r) = R'_1(r) \begin{pmatrix} \lambda^\alpha_\pm(\varphi) \\ \pm i \lambda^\alpha_\pm(\varphi) \end{pmatrix}, \quad \Psi^\beta_\pm(r) = R'_2(r) \begin{pmatrix} \lambda^\beta_\pm(\varphi) \\ \pm i \lambda^\beta_\pm(\varphi) \end{pmatrix},
\]

(A.35)

whose first component represents the \(\alpha\alpha\) and whose second component represents the \(\beta\beta\) pair. Under the various symmetries, the two degenerate ground states transform as

\[
D' \Psi^\alpha_\pm(r) = R'_1(r) \begin{pmatrix} \lambda^\alpha_\pm(\varphi + \pi) \\ \pm i \lambda^\alpha_\pm(\varphi + \pi) \end{pmatrix} = R'_1(r) \begin{pmatrix} \lambda^\alpha_\pm(\varphi) \\ \pm i \lambda^\alpha_\pm(\varphi) \end{pmatrix} = \Psi^\alpha_\pm(r),
\]

\[
O \Psi^\alpha_\pm(r) = R'_1(r) \begin{pmatrix} \pm i \lambda^\alpha_\pm(\varphi + \frac{\pi}{2}) \\ - \lambda^\alpha_\pm(\varphi + \frac{\pi}{2}) \end{pmatrix} = R'_1(r) \begin{pmatrix} \pm i \lambda^\alpha_\pm(\varphi) \\ - \lambda^\alpha_\pm(\varphi) \end{pmatrix} = \pm i \Psi^\alpha_\pm(r),
\]

\[
R \Psi^\alpha_\pm(r) = R'_1(r) \begin{pmatrix} \pm i \lambda^\alpha_\mp(-\varphi) \\ \lambda^\alpha_\mp(-\varphi) \end{pmatrix} = R'_1(r) \begin{pmatrix} \pm i \lambda^\alpha_\pm(\varphi) \\ \lambda^\alpha_\pm(\varphi) \end{pmatrix} = \pm i \Psi^\alpha_\pm(r).
\]  

(A.36)
Figure 6: Probability distribution for bound states of two holes with flavors $\alpha \alpha$ or $\beta \beta$, combined to an eigenstate of the 90 degrees rotation symmetry $O$. Left panel: the ground state with p-wave symmetry. Right panel: excited states with s- or d-wave symmetry, but with identical probability densities ($M'\Lambda^2/4\pi r_s = 1.25, r'_0 = a$).

Again, the corresponding eigenvalues of the 90 degrees rotation $O$ are $o = \pm i$, and hence, as for $\alpha \beta$ pairs, the symmetry is actually p-wave. Similarly, the two degenerate first excited states transform as

\[
D'_i \Psi^2_{\pm}(\vec{r}) = -R'_2(r) \begin{pmatrix} \chi^2_+(\varphi) \\ \pm \chi^2_-(\varphi) \end{pmatrix} = -\Psi^2_{\pm}(\vec{r}),
\]

\[
O \Psi^2_{\pm}(\vec{r}) = R'_2(r) \begin{pmatrix} \pm \chi^2_+(\varphi + \frac{\pi}{2}) \\ -\chi^2_-(\varphi + \frac{\pi}{2}) \end{pmatrix} = R'_2(r) \begin{pmatrix} \mp \chi^2_-(\varphi) \\ \chi^2_+(\varphi) \end{pmatrix} = \mp \Psi^2_{\pm}(\vec{r}),
\]

\[
R \Psi^2_{\pm}(\vec{r}) = R'_2(r) \begin{pmatrix} \pm \chi^2_+(-\varphi) \\ \chi^2_-(\varphi) \end{pmatrix} = R'_2(r) \begin{pmatrix} \pm \chi^2_-(\varphi) \\ \chi^2_+(\varphi) \end{pmatrix} = \pm \Psi^2_{\pm}(\vec{r}).
\]  

(A.37)

Again, the first excited states transform as s- or d-waves. The resulting probability distributions, which resemble $d_{xy}$ symmetry, are illustrated in figure 6 for the ground state (left panel) and the first excited state (right panel). Unlike for an $\alpha \beta$ pair, in the same flavor case the lowest energy bound states localized on a Skyrmion have a different transformation behavior than the magnon-mediated two-hole bound states.
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