KOSZUL HOMOLOGY OF QUOTIENTS BY EDGE IDEALS

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Abstract

We show that the Koszul homology algebra of a quotient by the edge ideal of a forest is generated by the lowest linear strand. This provides a large class of Koszul algebras whose Koszul homology algebras satisfy this property. We obtain this result by constructing the minimal graded free resolution of a quotient by such an edge ideal via the so called iterated mapping cone construction and using the explicit bases of Koszul homology given by Herzog and Maleki. Using these methods we also recover a result of Roth and Van Tuyl on the graded Betti numbers of quotients of edge ideals of trees.

1 Introduction

Let $k$ be a field and let $R = \bigoplus_{i \geq 0} R_i$ be a standard graded $k$-algebra. Let $K(R)$ be the Koszul complex on a minimal set of generators of $R_1$. It is well-known that the differential graded algebra structure on $K(R)$ induces a $k$-algebra structure on its homology, $H(R)$, see for example [1, 1.3]. This algebra structure on Koszul homology holds important information about the ring $R$. For example, $R$ is a complete intersection if and only if $H(R)$ is generated by $H_1(R)$ as a $k$-algebra [6, Thm 2.3.11], $R$ is Gorenstein if and only if $H(R)$ satisfies Poincare duality [6, Thm 3.4.5], and $R$ is Golod if and only if $K(R)$ admits a trivial Massey operation [1, Thm 5.2.2].

Another property of $R$ which has strong connections to the structure of $H(R)$ when $R$ is Koszul, is the Koszul property. $R$ is said to be Koszul if it has a linear resolution over $R$. To discuss the connections between $R$ and $H(R)$ when $R$ is Koszul, one views the Koszul homology algebra $H(R) = \bigoplus_{i,j} H_i(R)_j$ as a bigraded algebra, where $i$ is the homological degree and $j$ is the internal degree given by the grading on $R$. If $R$ is Koszul, then it is known that $H_i(R)_j = 0$ for all $j > 2i$ [2, Thm 3.1], that $H_i(R)_{2i} = (H_1(R)_{2i})^i$ [3, Thm 5.1], and that $H_i(R)_{2i-1} = (H_1(R)_{2i})^{i-1}H_2(R)_{3i} + H_2(R)_{2i}$ [4, Thm 3.1]. These results show that certain parts of $H(R)$ are generated by the lowest linear strand when $R$ is Koszul.

Avramov asked the following question regarding this behavior.

Question 1.1. If $R$ is Koszul, is the Koszul homology algebra generated as a $k$-algebra by the lowest linear strand? That is, is $H(R) = \bigoplus_i H_i(R)_{i+1}$?

The answer to this question is negative in general. The authors of [4] show that the Koszul homology algebra of the quotient by the edge ideal of an $n$-cycle where $n \equiv 1(\text{mod} 3)$ is not generated by the lowest linear strand. However, interest lies in determining for which Koszul algebras, this question has a positive answer. The answer is positive for the Koszul homology algebra of the quotient by the edge ideal of an $n$-path [4, Thm 3.15] and for the Koszul homology algebra of the second Veronese algebra [8, Cor 2.4]. Still the question remains open for many classes of algebras known to be Koszul.

In this paper, we give a positive answer to this question for a large class of edge ideals. Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field $k$ and let $I$ be an edge ideal associated to a tree. We show that the Koszul homology algebra of the quotient $R = Q/I$ is generated by the lowest linear strand. This result extends easily to edge ideals of forests and our result recovers [4, Thm 3.15]. To obtain this result, we utilize the so called iterated mapping cone construction and the explicit $k$-bases of each $H_i(R)$ given by Herzog and Maleki in [12, Thm 1.3].
We now outline the contents of this paper. In Section 2, we recall some important terminology which we use throughout the paper and we discuss the main tools we use in our results, including the iterated mapping cone construction, multiplicative structures on resolutions, and explicit bases for the Koszul homology modules from [12]. In Section 3, we construct the minimal graded free resolution of \( Q/I \) over \( Q \) which we use in the proof of our main result. We also recover a result of Roth and Van Tuyl on the Betti numbers of such quotients \( Q/I \). In Section 4, we state and prove the main result.

2 Preliminaries

In this section, we set up the basic terminology which we will use throughout the paper and discuss the main tools we will use to obtain our results. Let \( Q = k[x_1, ..., x_n] \) be a standard graded polynomial ring over a field \( k \).

We begin by recalling the notion of graded Betti numbers. We consider the minimal graded free resolution \( F \):

\[
\ldots \rightarrow \bigoplus_j Q(-j)^{\beta_{i,j}} \rightarrow \bigoplus_j Q(-j)^{\beta_{i-1,j}} \rightarrow \ldots
\]

of a \( Q \)-module \( M \). The \( i \)-th graded Betti number of internal degree \( j \) is \( \beta_{i,j} \). The Betti table of \( M \) is given by

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & \ldots \\
0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \ldots \\
1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} & \ldots \\
2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} & \ldots \\
3 & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \beta_{3,6} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Now we recall the following basic isomorphism, which we will use throughout this paper. Let \( I \) be any ideal in \( Q \) and let \( R = Q/I \). Throughout this paper, we denote the homology of the Koszul complex \( K(x_1, ..., x_n; R) \) by \( H(R) \). If \( F \) is the minimal graded free resolution of \( R \) over \( Q \), then there is an isomorphism of \( k \)-algebras

\[
\Phi : F \otimes k \rightarrow H(R).
\]  

(1)

Thus, given a basis \( e_1, ..., e_b \) of \( F \), we have that the elements \( \Phi(e_j \otimes 1) \) for \( j = 1, ..., b \), form a basis for \( H_i(R) \). Furthermore, if \( \deg e_j = k \), then \( \Phi(e_j \otimes 1) \in H_i(R)_j \). Given this isomorphism we can represent the Koszul homology algebra of \( R \) as a table

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & \ldots \\
0 & H_{0,0} & H_{1,1} & H_{2,2} & H_{3,3} & \ldots \\
1 & H_{0,1} & H_{1,2} & H_{2,3} & H_{3,4} & \ldots \\
2 & H_{0,2} & H_{1,3} & H_{2,4} & H_{3,5} & \ldots \\
3 & H_{0,3} & H_{1,4} & H_{2,5} & H_{3,6} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

where \( H_{i,j} = H_i(R)_j \). In this paper we often discuss the lowest linear strand of \( H(R) \), which is the second row (i.e. row 1) in the above table.
2.1 Edge Ideals

Let \( Q = k[x_1, \ldots, x_n] \) be a standard graded polynomial ring over a field \( k \). We begin this subsection by recalling the notion of an edge ideal.

**Definition 2.1.** Let \( G \) be a simple graph (that is, with no loops nor multiple edges) on vertices \( x_1, \ldots, x_n \). The \textit{edge ideal} associated to \( G \) is the ideal

\[
I_G = (x_i x_j \mid x_i x_j \text{ is an edge in } G).
\]

The class of edge ideals we focus on in this paper is that of trees.

**Definition 2.2.** Let \( G \) be a simple graph. \( G \) is a \textit{tree} if \( G \) is connected and contains no cycle. Equivalently, \( G \) is a \textit{tree} if every vertex in \( G \) is connected by exactly one path. A \textit{leaf} is a vertex in \( G \) of degree 1. A \textit{forest} is a disjoint union of trees and a \textit{subforest} of \( G \) is a subgraph of \( G \) which is a forest.

We illustrate the above definitions with the following example.

**Example 2.3.** Let \( Q = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7] \) be a polynomial ring. The edge ideal associated to the tree \( G \) shown in Figure 1 is \( I_G = (x_1 x_2, x_2 x_3, x_2 x_4, x_2 x_5, x_3 x_6, x_4 x_7) \).

![Figure 1: The graph G is a tree.](image)

We make the following easy remarks about trees that will be useful throughout this paper.

**Remark 2.4.**

(i) By definition, a tree \( G \) must have a leaf, otherwise \( G \) would contain a cycle.

(ii) It is easy to see that the subgraph of a tree \( G \) obtained by deleting any vertex and all adjacent edges is a subforest.

In the following subsection, we discuss a way to obtain the minimal graded free resolution of a quotient by the edge ideal of a tree.

2.2 Iterated Mapping Cones

In this subsection, we discuss one of the main tools we use to obtain the results in this paper, namely the iterated mapping cone construction. We begin by recalling the notion of a mapping cone.

**Definition 2.5.** Let \( (F, \partial^F) \) and \( (G, \partial^G) \) be two complexes of finitely generated \( Q \)-modules and let \( \phi : F \to G \) be a map of complexes. The \textit{mapping cone} of \( \phi \), denoted cone \( (\phi) \), is the complex \( (\text{cone} \ (\phi), \partial) \) with

\[
\partial_i = \begin{bmatrix}
\partial^G_i & \phi_{i-1} \\
0 & -\partial^F_{i-1}
\end{bmatrix}.
\]

\[\text{cone} \ (\phi)_i = G_i \oplus F_{i-1}\]
By the definition of the differential on cone(φ), it is easy to see the following fact.

**Remark 2.6.** If φ : F → G, then G is a subcomplex of cone(φ).

Mapping cones can be used to build free resolutions of quotients by monomial ideals in the following way. See for example [14, Constr 27.3] for more details.

**Construction 2.7.** Let Q be a graded polynomial ring and let I be the ideal minimally generated by monomials m_1, ..., m_r. Denote by d_i the degree of the monomial m_i and by I_i the ideal generated by m_1, ..., m_i. For each i ≥ 1, we have the following graded short exact sequence

\[ 0 \rightarrow Q/I : m_{i+1})(-d_{i+1}) \rightarrow Q/I_i \rightarrow Q/I_{i+1} \rightarrow 0. \]

Note that we have shifted the first term by the degree of the monomial m_{i+1} to make the multiplication by m_{i+1} a degree zero map. Thus, given graded free resolutions G_i of Q/I_i and F_i of Q/(I_i : m_{i+1}), there is a map of complexes φ_i : F_i → G_i induced by multiplication by m_{i+1}, which we will call the comparison map. The mapping cone of the comparison map is a graded free resolution of Q/I = cone(φ_i) of Q/I_{i+1}. Applying this construction for each i = 1, ..., r - 1 to obtain a graded free resolution of Q/I = Q/I_r, is called the iterated mapping cone construction.

We make the following important remarks about the iterated mapping cone construction.

**Remark 2.8.** Using the notation from Construction 2.7, we note the following.

(i) The resolution of I = (m_1, ..., m_r) produced by the mapping cone construction depends on the given order of the monomials. We illustrate this remark in Example 2.12 below.

(ii) For any i ≥ 1, cone(φ_i) need not be minimal, even if the given free resolutions F_i and G_i are minimal. Thus, the resolution of I produced by the iterated mapping cone construction need not be minimal. We illustrate this remark in Example 2.10 below.

We now recall the following theorem that follows from results of Hà and Van Tuyl in [10] and was proved independently by Bouchat in [5, Thm 3.0.16]. It will be useful in the proofs of our results.

**Theorem 2.9.** Let Q = k[x_1, ..., x_n] and let G be a simple graph on vertices x_1, ..., x_n such that x_n is a vertex of degree 1 and is connected by an edge to the vertex x_{n-1}. Then the mapping cone construction applied to the map

\[ Q/(I_G \setminus x_n : x_{n-1}x_n)(-2) \xrightarrow{x_{n-1}x_n} Q/I_G \setminus x_n \]

is a minimal graded free resolution of Q/I_G.

We sketch another proof of this result because we wish to clearly demonstrate the need for the degree requirement on x_n for later constructions in this paper.

**Proof.** Let (F^c, ∂^c) and (F, ∂^F) be the minimal free resolutions of Q/(I_G \setminus x_n : x_{n-1}x_n) and Q/I_G \setminus x_n respectively and let φ be the comparison map induced by multiplication by x_{n-1}x_n. We will show that Im φ_i ⊆ (x_n)F_i by induction on i. This is clear for i = 0, since φ_0 is given by multiplication by x_{n-1}x_n.

Now suppose that Im φ_{i-1} ⊆ (x_n)F_{i-1}. Since φ is a map of complexes, we have that

\[ \text{Im} (\partial_i^F φ_i) = \text{Im} (φ_{i-1} \partial_i^c) \subseteq \text{Im} φ_{i-1} \subseteq (x_n)F_{i-1}. \]

Thus, viewing the maps ∂_i^F and φ_i as matrices, we see that we can choose φ_i such that for every j, k, and ℓ we have (φ_i)_j,ℓ(k) ∈ (x_n) and thus for each ℓ either (∂_i^F)_j,ℓ ∈ (x_n) or (φ_i)_j,ℓ ∈ (x_n). However, since the degree of the vertex x_n is 1, I_G \setminus x_n is generated by monomials in x_1, ..., x_{n-1}. Thus, x_n cannot possibly divide any entry of ∂_i^F, see for example [14, Cor 26.9]. Therefore, Im φ_i ⊆ (x_n)F_{i-1}, as desired. □
The following example shows that Theorem 2.9 need not be true if the graph $G$ has no vertex of degree 1.

Example 2.10. Let $G$ be the 5-cycle shown in Figure 2 and consider its associated edge ideal $I_G = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Applying the iterated mapping cone construction to resolve $Q/I_G$, we get the following comparison map in the last iteration.

We see that the cone of this comparison map will produce a non-minimal resolution.

By Remark 2.4, Theorem 2.9 provides an inductive method for finding the minimal graded free resolution of $Q/I_G$, where $G$ is any tree. We state this as a corollary.

Corollary 2.11. If $G$ is a tree, then, in some order, the iterated mapping cone construction gives the minimal graded free resolution of $Q/I_G$ over $Q$.

The following example illustrates the importance of the order in which the iterated mapping cone construction is applied.

Example 2.12. Let $G$ be the tree shown in Figure 3 and consider its associated edge ideal $I_G = (x_1x_3, x_2x_3, x_3x_4, x_4x_5)$. Applying the iterated mapping cone construction to resolve $Q/I_G$, we get the following comparison map in the last iteration.

Figure 2: The graph $G$ is a 5-cycle.
However, if we instead order the minimal generators of the edge ideal as $I_G = (x_1x_3, x_2x_3, x_4x_5, x_3x_4)$ and apply the iterated mapping cone construction, we get the following comparison map in the last iteration.

\[
\begin{array}{c}
0 \\ x_1 \\ -x_2 \\ x_5
\end{array} \to
\begin{array}{c}
Q(-5) \\
Q(-4)^3 \\
Q(-3)^3 \\
Q(-2)
\end{array}
\]

\[
\begin{pmatrix}
-x_4 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

\[
\begin{array}{c}
0 \\ x_4 \\ 0 \\
0 \\ 0 \\
0 \\ x_3 \\
x_3x_4
\end{array}
\]

It is clear that applying the mapping cone construction in these two orders produce different resolutions. We note that the second resolution is not minimal. If it was, it would have to be isomorphic to the first one.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{The graph $G$ is a tree}
\end{figure}

We use the iterated mapping cone construction in Section 3 to explicitly build the minimal graded free resolution of $Q/I_G$, where $G$ is a tree. This resolution is an important ingredient in our proof of the main result in Section 4.

2.3 Multiplicative Structures on Resolutions

Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field $k$ and let $I$ be a monomial ideal of $Q$. Let $F$ be the minimal graded free resolution of $Q/I$ over $Q$. In this section we recall the notion of a multiplicative structure on $F$; see for example [13].

**Definition 2.13.** A $Q$-linear map $F \otimes Q F \to F$ sending $a \otimes b$ to $a \cdot b$ is a *multiplication* if it satisfies the following conditions for $a, b \in F$:

(i) it extends the usual multiplication on $F_0 = Q$

(ii) it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$

(iii) it is homogeneous with respect to the homological grading: $|a \cdot b| = |a| + |b|

(iv) it is graded commutative: $a \cdot b = (-1)^{|a||b|}b \cdot a$

Notice we do not require that a multiplication is associative. The following fact is due to Buchsbaum and Eisenbud.

**Proposition 2.14.** [7, Prop 1.1] The resolution $F$ admits a multiplication.

This fact will be useful in our results.
2.4 Explicit Bases for Koszul Homology

Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field $k$, let $I$ be a homogeneous ideal of $Q$, and let $R = Q/I$. In this section, we discuss explicit bases of the Koszul homology modules $H_i(R)$ given by Herzog and Maleki in [12]. In order to describe these bases explicitly, we first set up some notation.

Herzog and Maleki define operators on $Q$ as follows. For $f \in (x_1, \ldots, x_n)$ and for $r = 1, \ldots, n$, let

$$d^r(f) = \frac{f(0, \ldots, 0, x_r, \ldots, x_n) - f(0, \ldots, 0, x_{r+1}, \ldots, x_n)}{x_r}.$$

It is clear that the operators $d^r : Q \to Q$ are $k$-linear maps and that they depend on the order of the variables. In this paper, we apply these operators to monomials. The following basic lemma describes how $d^r$ behaves in this context.

**Lemma 2.15.** Let $f$ be the monomial $x_{k_1} \cdots x_{k_r}$ with $k_1 \leq \ldots \leq k_r$. Then

$$d^r(f) = \begin{cases} x_{k_1} \cdots x_{k_r} & r = k_1 \\ 0 & \text{otherwise} \end{cases}.$$

**Proof.** If $r = k_1$, then by definition $d^r(f) = \frac{x_{k_1} \cdots x_{k_r} - 0}{x_{k_1}} = x_{k_1} \cdots x_{k_r}$. If $r < k_1$, then $d^r(f) = \frac{x_{k_1} \cdots x_{k_r}}{x_r} = 0$. If $r > k_1$, then $d^r(f) = \sum_{i=1}^{k_1-1} x_i = 0$. \hfill \Box

This simple fact will be useful in the proof of our main result. The following theorem due to Herzog and Maleki describes explicit bases for the Koszul homology modules. Setting notation for the theorem, let $F$ be a minimal graded free resolution of $Q/I$ over $Q$ and let $b_i$ be the rank of $F_i$ for each $i$. For each $i$, fix a basis $e_1^i, \ldots, e_{b_i}^i$ of $F_i$ and let $\partial(e_1^i) = \sum_{k=1}^{b_{i-1}} f_{k,j}^i e_k^{i-1}$.

**Theorem 2.16.** [12, Thm 1.3] For each $i = 1, \ldots, n$, a $k$-basis of $H_i(R)$ is given by $[\bar{z}_{i,j}]$ for $j = 1, \ldots, b_i$, where

$$z_{i,j} = \sum_{1 \leq k_1 < \ldots < k_i \leq n} \sum_{j_{i-1} = 1}^{b_{i-1}} \cdots \sum_{j_1 = 1}^{b_1} d^{k_i}(f_{j_{i-1},j_i}^i) \cdots d^{k_2}(f_{j_2,j_3}^i) d^{k_1}(f_{j_1,j_2}^i) dx_{k_i} \cdots dx_{k_1}.$$ 

In the proof of Theorem 2.16 Herzog and Maleki show that the isomorphism (1) is given explicitly by

$$\Phi : F \otimes k \to H(R)$$

$$\Phi(e_1^i \otimes \bar{1}) = [\bar{z}_{i,j}].$$

We conclude this section with the following remark.

**Remark 2.17.** We note that in [11], Herzog gives a different description of bases for $H_i(R)$ under the assumption that $k$ is a field of characteristic zero. In [9], the author gives explicit bases in a more general case, namely the case of the Koszul complex on any full regular sequence, which recover Herzog’s bases in characteristic zero. In this paper, we use the description of the bases given by Herzog and Maleki as they hold in any characteristic.

3 Construction of the Resolution

Let $G$ be a tree and let $Q$ be a standard graded polynomial ring over any field $k$ with variables given by the vertices in $G$. In this section we construct the minimal graded free resolution of $Q/I_G$.
over $Q$. We begin by setting the notation to be used throughout the section.

We name the vertices in $G$ as follows. Since $G$ is a tree, it has at least one vertex of degree 1; call it $x_1$ and call the vertex it is connected to by an edge $x_2$. Call the other vertices to which $x_2$ is connected to by an edge, $x_{2,1}, \ldots, x_{2,r}$. For each $\ell = 1, \ldots, r$, call the other vertices to which $x_{2,\ell}$ is connected to by an edge, $x_{2,\ell,1}, \ldots, x_{2,\ell,m_{\ell}}$. Note that $G \setminus \{x_1, x_{2,1}, \ldots, x_{2,r}\}$ is a subforest. In particular, it is the disjoint union of $r \cdot m_{\ell}$ trees, call them $T_1, \ldots, T_{m_{\ell}}$.

We aim to resolve $Q/I_Q$ minimally. By Theorem 2.9 this can be done by applying the iterated mapping cone construction as long as at each iteration, we add a vertex of degree one. We choose the following order to apply the iterated mapping cone construction:

$$I_G = (x_2x_{2,1}, \ldots, x_2x_{2,r}, \{x_{2,\ell}x_2,_{\ell,p}\}_{\ell=1, r, p=1, \ldots, m_{\ell}}) \equiv (T_1) \equiv (T_{m_{\ell}}), x_1x_2)$$

where, abusing notation, we write $e(T_i)$ to mean the set of relations coming from the edges of the tree $T_i$, taking the edge connecting $T_i$ to the corresponding $x_{2,\ell,p}$ to be the first one. We do not impose any further ordering on the relations coming from each of these trees, but we know that there is an ordering which will preserve minimality in the iterated mapping cone construction by Corollary 2.11. In this way, we obtain the minimal graded free resolution of $Q/I_G$. Throughout the remainder of this section, we will write this resolution more explicitly in order to obtain our main result in the next section.

Denote by $G_1$ the graph $G \setminus x_1$ and by $C$ the colon ideal $(I_G : x_1x_2)$. It is easy to see that

$$C = (x_2, \ldots, x_{2,r}, e(T_1), \ldots, e(T_{m_{\ell}})) = (x_2, \ldots, x_{2,r}) + \sum_{i=1}^{m_{\ell}} I_{T_i}.$$ 

The following fact is a key ingredient in our results.

**Lemma 3.1.** The minimal graded free resolution of $Q/C$ over $Q$ is

$$F_C = K(x_2, \ldots, x_{2,r}; Q) \otimes \bigoplus_{i} F_i \otimes \bigoplus_{i} Q,$$

where $F_i$ is the minimal graded free resolution of $Q/I_{T_i}$ for each $i$.

**Proof.** Let $A$ denote the subring, $k[x_2, \ldots, x_{2,r}]$ of $Q$ and let $B$ denote the polynomial subring on all other variables in $Q$ so that $Q = A \otimes_k B$. Let $\tilde{K}$ be the minimal graded free resolution of $A/J$ over $A$, where $J = (x_2, \ldots, x_{2,r})$, and let $\tilde{F}$ be the minimal graded free resolution of $B/L$ over $B$, where $L = \sum_{\ell=1}^{m_{\ell}} I_{T_i}$. Then we have that $K = \tilde{K} \otimes_k B$ and $F = A \otimes_k \tilde{F}$ are minimal graded free resolutions of $Q/J$ and $Q/L$, respectively, over $Q$. We note that $K$ is precisely the Koszul complex $K(x_2, \ldots, x_{2,r}; Q)$.

Now we have that

$$K \otimes F = (K \otimes B) \otimes (A \otimes \tilde{F}) = (K \otimes B) \otimes (A \otimes \tilde{F}) = (K \otimes A \otimes B) \otimes (A \otimes B \otimes \tilde{F})$$

$$= \tilde{K} \otimes (A \otimes B) \otimes \tilde{F} = \tilde{K} \otimes \tilde{F}.$$ 

Thus, taking homology, we see that

$$H_n(K \otimes F) = H_n(\tilde{K} \otimes \tilde{F}) = \bigoplus_{i+j=n} \left( H_i(\tilde{K}) \otimes H_j(\tilde{F}) \right),$$

where the last equality follows from the Künneth Formula over $k$; see for example [16, Cor 10.84]. Thus $K \otimes Q F$ is exact in all positive degrees and $H_0(K \otimes Q F) = A/J \otimes_k B/L = Q/(I + J) = Q/C$.

Minimality is clear, so we have that

$$K(x_2, \ldots, x_{2,r}; Q) \otimes F$$

8
is the minimal graded free resolution of $Q/C$. Noticing that $T_1, ..., T_{r_m}$ involve disjoint sets of variables, we can apply a similar argument repeatedly to conclude that $F \cong F_1 \otimes_Q \ldots \otimes_Q F_{r_m}$, thus giving the desired result.

Now we have that the minimal graded free resolution $F$ of $Q/I$ over $Q$ is the cone of $\phi$, where $\phi$ is a comparison map given by

$$
\begin{array}{c}
F_C(-2) \longrightarrow Q/C(-2) \\
\downarrow \phi \\
F_{G_1} \longrightarrow Q/I_{G_1}
\end{array}
$$

Thus, we have that $F$ has modules $F_i = F_i^{G_1} \oplus F_i^{C}(-2)$, and differentials,

$$
\partial_i = \begin{bmatrix} \partial_i^{G_1} & \phi_i-1 \\ 0 & -\partial_i^{C} \end{bmatrix}.
$$

We make the following remark about the resolutions $F_1, ..., F_{r_m}$ in Lemma 3.1.

**Remark 3.2.** The resolutions $F_1, ..., F_{r_m}$ are subcomplexes of $F_i^{G_1}$. Indeed, a minimal resolution of $Q/I_{G_1}$ can be obtained from each $F_q$ by the iterated mapping cone construction, thus by Remark 2.6 each $F_q$ is a subcomplex of $F_i^{G_1}$.

We now aim to give an explicit description of the map $\phi$. To obtain such a description, we first observe that it is enough to define $\phi$ on elements of the form $\alpha \otimes 1 \otimes \ldots \otimes 1$.

**Lemma 3.3.** If $\overline{\phi}$ is a comparison map

$$
\begin{array}{c}
K(x_{2,1}, ..., x_{2,r}; Q)(-2) \longrightarrow Q/(x_{2,1}, ..., x_{2,r})(-2) \\
\downarrow \overline{\phi} \\
F_{G_1} \longrightarrow Q/I_{G_1}
\end{array}
$$

then $\phi(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{r_m}) = \overline{\phi}(\alpha) \cdot \left( \beta_1 \cdot (\ldots \cdot (\beta_{r_m-1} \cdot \beta_{r_m}) \ldots) \right)$ defines a comparison map in (3).

Before giving a proof, we note that each $\beta_i$ is a basis element of $F_i^{G_1}$ by Remark 3.2. We also note that the multiplication appearing in the lemma is a multiplication on the resolution $F_i^{G_1}$; it has one by Proposition 2.14. Thus this definition of $\phi$ makes sense.

**Proof.** We must check that $\phi$ is a chain map. Thus we compute

$$
\begin{align*}
\phi(\partial_i^C(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{r_m})) &= \phi(\partial^K(\alpha) \otimes \beta_1 \otimes \ldots \otimes \beta_{r_m} + \sum_{i=1}^{r_m} (-1)^{|\alpha|+\ldots+|\beta|} \alpha \otimes \beta_1 \otimes \ldots \otimes \partial_i^{F_i}(\beta_i) \otimes \ldots \otimes \beta_{r_m}) \\
&= \overline{\phi}(\partial^K(\alpha)) \cdot \left( \beta_1 \cdot (\ldots \cdot (\beta_{r_m-1} \cdot \beta_{r_m}) \ldots) \right) \\
&+ \sum_{i=1}^{r_m} (-1)^{|\alpha|+\ldots+|\beta|} \overline{\phi}(\alpha) \cdot \left( \beta_1 \cdot (\ldots \cdot (\partial_i^{F_i}(\beta_i) \cdot (\ldots \cdot (\beta_{r_m-1} \cdot \beta_{r_m}) \ldots) \ldots) \ldots) \right).
\end{align*}
$$
On the other hand, we have that
\[
\partial^{G_1}(\phi(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_i})) = \partial^{G_1}\left(\tilde{\phi}(\alpha) \cdot \left(\beta_1 \cdot (\ldots (\beta_{rm_i-1} \cdot \beta_{rm_i})\ldots)\right)\right)
\]
\[
= \partial^{G_1}(\tilde{\phi}(\alpha)) \cdot \left(\beta_1 \cdot (\ldots (\beta_{rm_i-1} \cdot \beta_{rm_i})\ldots)\right) + (-1)^{\tilde{\phi}(\alpha)} \cdot \phi(\partial^{G_1}(\beta_1 \cdot (\ldots (\beta_{rm_i-1} \cdot \beta_{rm_i})\ldots))).
\]
Applying the Leibniz rule repeatedly, and by our assumption that \(\tilde{\phi}\) is a chain map, we see that \(\phi(\partial^{G_1}(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_i})) = \partial^{G_1}(\phi(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_i}))\), which completes the proof. \(\square\)

Now we work towards defining \(\tilde{\phi}(\alpha) = \phi(\alpha \otimes 1 \otimes \ldots \otimes 1)\). To accomplish this, we need to examine \(F^{G_1}\) more closely. We apply the iterated mapping cone construction in the order given in \([2]\). We observe that the resolution of \(Q/(x_{2x_2,1}, \ldots, x_{2x_2,r})\) produced by the iterated mapping cone procedure is precisely the Taylor resolution (see for example \([14, \text{Constr 26.5}]\)), which we write as follows. Let \(E\) be the exterior algebra over \(k\) on basis elements \(e_1, \ldots, e_r\). Then the minimal graded free resolution of \(Q/(x_{2x_2,1}, \ldots, x_{2x_2,r})\) over \(Q\) is \(F\), where \(F_i = Q \otimes E_i\) and the differentials are given by
\[
\partial^F(e_{j_1}, \ldots, e_{j_i}) = \sum_{\ell=1}^{i} (-1)^{\ell-1} \frac{\text{lcm}(x_{2x_2,j_1}, \ldots, x_{2x_2,j_{\ell}})}{\text{lcm}(x_{2x_2,j_1}, \ldots, x_{2x_2,j_{\ell}}, \ldots, x_{2x_2,j_i})} e_{j_1} \cdots \hat{e}_{j_\ell} \cdots e_{j_i},
\]
\[
= \begin{cases} \sum_{\ell=1}^{i} (-1)^{\ell-1} x_{2x_2,j_{\ell}} e_{j_1} \cdots \hat{e}_{j_\ell} \cdots e_{j_i}, & i \geq 2 \\ x_{2x_2,j_i} & i = 1 \end{cases}.
\]
Recall that by Remark \([2, 6]\) \(F\) is a subcomplex of \(F^{G_1}\). We are now ready to define \(\phi\). To set up notation, we write the Koszul complex \(K(x_{2,1}, \ldots, x_{2,r}; Q)\) as \(Q(a_1, \ldots, a_r|\partial^K(a_j) = x_{2,j})\).

**Proposition 3.4.** Define \(\phi : F^C \rightarrow F^{G_1}\) by

\[
\phi(a_{j_1}, \ldots, a_{j_i} \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_i}) = x_1 e_{j_1} \cdots e_{j_i} \cdot \left(\beta_1 \cdot (\ldots (\beta_{rm_i-1} \cdot \beta_{rm_i})\ldots)\right).
\]

Then \(\phi\) is a comparison map for

\[
\begin{array}{ccc}
F_C(-2) & \longrightarrow & Q/C(-2) \\
\downarrow \phi & & \downarrow x_1 x_2 \\
F_{G_1} & \longrightarrow & Q/I_{G_1}
\end{array}
\]

**Proof.** By Lemma \([3, 3]\) it suffices to check that
\[
\partial^{G_1}(\phi(a_{j_1}, \ldots, a_{j_i} \otimes 1 \otimes \ldots \otimes 1)) = \phi(\partial^{C}(a_{j_1}, \ldots, a_{j_i} \otimes 1 \otimes \ldots \otimes 1))
\]
for all \(i\). So, for \(i \geq 2\), we compute
\[
\phi(\partial^{C}(a_{j_1}, \ldots, a_{j_i} \otimes 1 \otimes \ldots \otimes 1)) = \phi\left(\sum_{\ell=1}^{i} (-1)^{\ell-1} x_{2x_2,j_{\ell}} a_{j_1} \cdots \hat{a}_{j_\ell} \cdots a_{j_i} \otimes 1 \otimes \ldots \otimes 1\right)
\]
\[
= \sum_{\ell=1}^{i} (-1)^{\ell-1} x_1 x_{2x_2,j_1} e_{j_1} \cdots \hat{e}_{j_\ell} \cdots e_{j_i}
\]
\[
= \partial^{G_1}(x_1 e_{j_1} \cdots e_{j_i})
\]
\[
= \partial^{G_1}(\phi(a_{j_1}, \ldots, a_{j_i} \otimes 1 \otimes \ldots \otimes 1)).
\]
And for any \( j \), we have
\[
\phi(\partial^G(a_j \otimes 1 \otimes ... \otimes 1)) = \phi(x_{2,j}) = x_1x_2x_{2,j} = \partial^{G_1}(\phi(a_j \otimes 1 \otimes ... \otimes 1))
\]
which completes the proof.

To summarize the discussions in this section, we can think of the resolution \( F \) as the cone of the diagram,

\[
\begin{align*}
\rightarrow Q(-(r+2)) \oplus \tilde{F}_r^C & \rightarrow Q(-4) \oplus \tilde{F}_2^C & \rightarrow Q(-3) \oplus \tilde{F}_1^C & \rightarrow Q(-2) \\
\begin{bmatrix} x_1 \ast \\ 0 \ast \end{bmatrix} & \begin{bmatrix} x_1 \ast \\ 0 \ast \end{bmatrix} & \begin{bmatrix} x_1 \ast \\ 0 \ast \end{bmatrix} & x_1,x_2 \\
\rightarrow Q(-(r+1)) \oplus \tilde{F}_r^{G_1} & \rightarrow Q(-3) \oplus \tilde{F}_2^{G_1} & \rightarrow Q(-2) \oplus \tilde{F}_1^{G_1} & \rightarrow Q
\end{align*}
\]

where each \( F_i^C(-2) = K_i(x_{2,1},...,x_{2,r}; Q)(-2) \oplus \tilde{F}_r^C \) and \( F_i^{G_1} = F_i \oplus \tilde{F}_i^{G_1} \). The following corollary is an immediate consequence of the construction and diagram above.

**Corollary 3.5.** The elements \( \Phi(\alpha \otimes 1 \otimes ... \otimes 1) \) are generators of \( H_\ell(R) \) which lie on the lowest linear strand, \( H_\ell(R)_{\ell+1} \), where \( |\alpha| = \ell - 1 \).

We conclude this section by noting that the above constructions are based on resolving the quotient of an edge ideal of a tree, adding a leaf, and then using the mapping cone construction to resolve the quotient of that new edge ideal. Thus the entire construction is inductive by nature. One could copy the above construction for each vertex, define a comparison map as we did in Proposition 3.4 and come up with a corresponding diagram for the resolution at each step. This provides a way of counting the Betti numbers on the linear strand of \( F \), and equivalently the generators on the lowest linear strand of \( H(R) \). With this procedure in mind, we recover the following result of Roth and Van Tuyl [15, Cor 2.6].

**Corollary 3.6.** Let \( G \) be a tree. Then \( \beta_{1,2}(Q/I_G) = |e(G)| \) and
\[
\beta_{i,i+1}(Q/I_G) = \sum_{v \in G} \binom{\deg(v)}{i}
\]
for all \( i \geq 2 \).

**Proof.** For each vertex \( v \in G \), we can repeat the above construction and draw a similar diagram for \( F \) (the one we drew above is for the vertex \( x_2 \)). We count the Betti numbers on the linear strand of each diagram as follows. From our diagram above, we see that the number of copies of \( Q(-(i+1)) \) in homological degree \( i \) is
\[
\binom{r}{i} + \binom{r}{i-1} = \binom{r+1}{i} = \binom{\deg x_2}{i},
\]
where the first equality is just the identity called Pascal’s Rule. Summing over each vertex of \( G \), we get the desired formula.
The Main Result

In this section, we show that Question 1.1 has a positive answer for edge ideals of trees, and thus also for forests. Throughout this section we let $G$ be a tree and let $Q$ be a standard graded polynomial ring over any field $k$ with variables given by the vertices in $G$. Denote by $N$ the set of vertices, so that $Q = k[\{x_n\}_{n \in N}]$. Let $\Phi_{G_1}$ be the isomorphism $F_{G_1} \otimes_Q k \to H(Q/I_{G_1})$ as in (1).

Lemma 4.1. The map of $k$-algebras

$$\theta : H(Q/I_{G_1}) \to H(R)$$

satisfies the equality $\theta(\Phi_{G_1}(e \otimes \bar{1})) = \Phi(e \otimes \bar{1})$ for any $e \in F_{G_1}$.

Proof. We note that the quotient map $Q/I_{G_1} \to Q/I = R$ induces the map of DG algebras

$$K(Q/I_{G_1}) \to K(R)$$

that sends $dx_i$ to $dx_i$ for all $i$. The induced map

$$\theta : H(Q/I_{G_1}) \to H(R)$$

on homology is a map of $k$-algebras. Since $F_{G_1}$ is a subcomplex of $\mathbb{F}$ (see Remark 2.6), it is clear that the equality $\theta(\Phi_{G_1}(e \otimes \bar{1})) = \Phi(e \otimes \bar{1})$ holds.

Now we show that the Koszul homology algebra of the quotient by $I_G$ is generated by the lowest linear strand.

Theorem 4.2. If $R = Q/I_G$, then $H(R) = \bigoplus_i H_i(R)_{i+1}$ as $k$-algebras.

Proof. We use induction on the number of edges in $G$. For the base case, we consider the tree with one edge. In this case, $I_G = (x_1 x_2)$ and the minimal graded free resolution $F$ of $R$ is

$$0 \to Q(-2) \xrightarrow{x_1 x_2} Q \to 0.$$ 

Thus applying the isomorphism $\Phi$ from (1) to $F \otimes k$, we see that the only basis element of $H_1(R)$ lies in $H_1(R)_2$. Hence $H(R)$ is trivially generated by the lowest linear strand.

Now take $G$ to be any tree and assume that the result is true for every tree with strictly fewer edges. Let $\mathbb{F}$ be the minimal graded resolution of $R$ over $Q$ constructed in Section 3 and fix the basis of each $H_i(R)$ given in Theorem 2.16. It is enough to show that each basis element of $H_i(R)$ is in $\bigoplus_j H_j(R)_{j+1}$. Thus, we take $h$ to be any basis element of $H_i(R)$. Then $h = \Phi(e \otimes \bar{1})$, for some basis element $e$ of $F_i$. We have that $\mathbb{F}$ is the cone of the map

$$F^C \xrightarrow{\phi} F^{G_1}.$$ 

Thus $F_i = F^{G_1}_i \oplus F^C_{i-1}$ and $e$ must either be a basis element of $F^{G_1}_i$ or of $F^C_{i-1}$.

We first consider the case where $e$ is a basis element of $F^{G_1}_i$. By Lemma 4.1, $h = \theta(\Phi_{G_1}(e \otimes \bar{1}))$, but by the induction hypothesis, $H(Q/I_{G_1})$ is generated by the lowest linear strand. So we have that

$$\Phi_{G_1}(e \otimes \bar{1}) = \sum_{L,M} c_{L,M} \prod_{\ell,m} \Phi_{G_1}(e^{m+1} \otimes \bar{1})$$ 

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where for \( \ell = 1, \ldots, b_{m,m+1} \) the elements \( e^{m,m+1}_\ell \) are basis elements of \( F^G_m \) of internal degree \( m+1 \), and where \( c_{L,M} \in k \). Now we have that

\[
h = \vartheta \left( \sum_{L,M} c_{L,M} \prod_{\ell,m} \Phi_G(e^{m,m+1}_\ell \otimes 1) \right) = \sum_{L,M} c_{L,M} \prod_{\ell,m} \vartheta(\Phi_G(e^{m,m+1}_\ell \otimes 1)) \\
= \sum_{L,M} c_{L,M} \prod_{\ell,m} \Phi(e^{m,m+1}_\ell \otimes 1)
\]

by Lemma 4.1. The elements \( \Phi(e^{m,m+1}_\ell \otimes 1) \) are basis elements of \( H_m(R) \) that are in \( H_m(R)_{m+1} \), thus \( h \) is generated in the subalgebra generated by the linear strand.

Now we assume that \( e \) is a basis element of \( F^C_{m+1} \). Then by Lemma 3.1, \( e = \alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_\ell} \), for some basis elements \( \alpha \) of \( K_\ell(x_{2,1}, \ldots, x_{2,r}) \) and \( \beta_p \) of \( (F_p)_p \), where \( \ell + r_1 + \ldots + i_{rm_\ell} = i - 1 \). By Theorem 2.16, \( h = [\bar{g}] \), where

\[
g = \sum_{(k_1 < \ldots < k_i) \subseteq \mathbb{N}} \sum_{j_1=1}^{b_1} \ldots \sum_{j_i=1}^{b_{i-1}} d^k_i(f_{j_{i-1},j_i}) \ldots d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) dx_{k_1} \ldots dx_{k_i}
\]

and each \( f_{i,j} \) is the \((i,j)\)-th entry in the \( k \)th differential of \( F \) when viewing the differentials as matrices.

Recall that, the operators \( d^k \) depend on the order of the variables, so we fix an ordering on the variables in \( Q \) as follows

\[
x_1 < x_2 < x_2, \ell < x_2,\ell, p < v(T_1) < \ldots < v(T_{rm_\ell})
\]

for all \( \ell \) and \( p \), where by \( v(T_q) \) we mean the variables given by the vertices in the tree \( T_q \).

Now we analyze the terms in (5) more carefully in order to remove the initial sum. We note that since the differentials of \( F \) are given by (4), we have that for each set \( \{j_1, \ldots, j_{i-1}\} \), there is some \( m \) such that \( f_{j_{i-1},j_i}^{j_{i-1},j_i}, \ldots, f_{j_{m},j_{m+1}}^{j_{m},j_{m+1}} \) are entries of \( \partial^C \), \( f_{j_{m-1},j_m}^{j_{m-1},j_m} \) is an entry of \( \phi_{m-1} \), and \( f_{j_{m},j_{m+1}}^{j_{m},j_{m+1}} \) are entries of \( \partial^C \). Then, in particular, we have that \( f_{j_{m-1},j_m}^{j_{m-1},j_m} \in (x_1) \) by Proposition 3.4. Thus by Lemma 2.15 we have that \( d^{k_m}(f_{j_{m-1},j_m}^{j_{m-1},j_m}) = 0 \) unless \( k_m = 1 \). So, every term in the sum with \( k_m \neq 1 \) vanishes, giving

\[
g = \sum_{(k_2 < \ldots < k_i) \subseteq \mathbb{N}} \sum_{j_1=1}^{b_1} \ldots \sum_{j_i=1}^{b_{i-1}} d^k_i(f_{j_{i-1},j_i}) \ldots d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) dx_{k_2} \ldots dx_{k_i}
\]

with \( f_{j_{i-1},j_i}^{j_{i-1},j_i}, \ldots, f_{j_{1},j_2}^{j_{1},j_2} \) entries of \( \partial^C \) and \( f_{j_{1},j_2}^{j_{1},j_2} \) is an entry of \( \phi_0 \). But \( \phi_0 \) is given by multiplication by \( x_1 x_2 \), thus \( f_{j_{1},j_2}^{j_{1},j_2} = x_1 x_2 \).

Similarly by Lemma 2.15, for each \( f_{j_{p-1},j_p}^{j_{p-1},j_p} \) with \( p > 1 \), there is only one value of \( k_p \) such that \( d^{k_p}(f_{j_{p-1},j_p}^{j_{p-1},j_p}) \) is nonzero. Thus, we see that for each set \( \{j_1, \ldots, j_{i-1}\} \), there exist unique \( k_p = k_q(j_1, \ldots, j_{i-1}) \), for \( p = 2, \ldots, i \), such that

\[
g = \sum_{j_1=1}^{b_1} \ldots \sum_{j_i=1}^{b_{i-1}} d^k_i(f_{j_{i-1},j_i}) \ldots d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) d^k_i(f_{j_{1},j_2}) dx_{k_1} \ldots dx_{k_i}
\]

with \( f_{j_{i-1},j_i}^{j_{i-1},j_i}, \ldots, f_{j_{1},j_2}^{j_{1},j_2} \) entries of \( \partial^C \).

Furthermore, since

\[
\partial^C(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_\ell}) = \partial^K(\alpha) \otimes \beta_1 \otimes \ldots \otimes \beta_{rm_\ell} + \sum_{i=1}^{rm_\ell} (-1)^{|\alpha|+\ldots+|\beta_i|} \alpha \otimes \beta_1 \otimes \ldots \otimes \partial^K(\beta_i) \otimes \ldots \otimes \beta_{rm_\ell},
\]
This induces an isomorphism on Koszul homology
\[ f_{j_1, j_1}^i, \ldots, f_{j_1, j_m+1}^{i-1} \] are entries of \( \partial^F \)
\[ f_{j_1, j_1}^{i+1}, \ldots, f_{j_1, j_2}^{i+2} \] are entries of \( \partial^K \).

We also notice that (7) can be written as
\[
g = \sum_{j_{\ell+1}=1}^{b_{\ell+1}} \sum_{j_{\ell-1}=1}^{b_{\ell-1}} d^{k_{\ell+1}}(f_{j_{\ell+1}, j_{\ell}}^{i+1}) \cdots d^{k_{\ell+2}}(f_{j_{\ell+2}, j_{\ell+2}}^{i+2}) \left( \sum_{j_{\ell+1}=1}^{b_{\ell+1}} \sum_{j_{\ell-1}=1}^{b_{\ell-1}} d^{k_{\ell+1}}(f_{j_{\ell+1}, j_{\ell}}^{i+1}) \cdots d^{k_{\ell+2}}(f_{j_{\ell+2}, j_{\ell+2}}^{i+2}) \right) dx_{k_{\ell+2}} \cdots dx_{k_{\ell+1}}.
\]

We note the sum over \( j_{\ell+1} \) can be removed. Indeed \( f_{j_{\ell+1}, j_{\ell+2}}^{i+2} \) is an entry of \( \partial^F \) and thus there is only one index \( j_{\ell+1} \) producing nonzero terms. Thus (4) can be written as
\[
h = \Phi(\alpha \otimes \bar{1}) \cdot \Phi(\beta_1 \otimes \bar{1}) \cdot \ldots \Phi(\beta_{r_{m+1}} \otimes \bar{1})
\]
by Remark 3.2. Since \( \beta_1, \ldots, \beta_{r_{m+1}} \) are basis elements of \( F_{G_1} \), a similar inductive argument to the one given in the first case implies that
\[
h = \Phi(\alpha \otimes \bar{1}) \cdot \left( \sum_{L, M} c_{L, M} \prod_{\ell, m} \Phi((\beta_1^m)_{\ell}^{m+1} \otimes \bar{1}) \right) \cdot \ldots \cdot \left( \sum_{L, M} c_{L, M} \prod_{\ell, m} \Phi((\beta_{r_{m+1}}^m)_{\ell}^{m+1} \otimes \bar{1}) \right)
\]
where each \((\beta_q^m)_{\ell}^{m+1}\) is a basis element of \( F_m \) of internal degree \( m+1 \). We note that by Corollary 3.5 \( \Phi(\alpha \otimes \bar{1}) \) is a generator of \( H_{\ell+1}(R) \) that is in \( H_{\ell+1}(R)_{\ell+2} \), thus \( h \) is in the subalgebra generated by the lowest linear strand.

Since paths are trees, this recovers [4, Thm 3.15]. Now consider a forest \( G \). By definition, \( G \) is a disjoint union of trees, \( T_1, \ldots, T_m \). Thus, the quotient of the edge ideal of \( G \) is of the form
\[
Q/I_G = Q_1/I_{T_1} \otimes_k \ldots \otimes_k Q_m/I_{T_m}
\]
where \( Q_1, \ldots, Q_m \) are polynomial rings on disjoint sets of variables such that \( Q = Q_1 \otimes_k \ldots \otimes_k Q_m \). This induces an isomorphism on Koszul homology
\[
H(Q/I_G) \cong H(Q_1/I_{T_1}) \otimes_k \ldots \otimes_k H(Q_m/I_{T_m}),
\]
thus yielding the following corollary as a direct consequence of Theorem 4.2.

**Corollary 4.3.** If \( R = Q/I_G \) and \( G \) is a forest, then \( H(R) = \bigoplus_i H_i(R)_{i+1} \) as \( k \)-algebras. \( \square \)
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