Constraining torsion in maximally symmetric (sub)spaces

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Abstract
We look into the general aspects of space-time symmetries in presence of torsion, and how the latter is affected by such symmetries. Focusing in particular to space-times which either exhibit maximal symmetry on their own, or could be decomposed to maximally symmetric subspaces, we work out the constraints on torsion in two different theoretical schemes. We show that at least for a completely antisymmetric torsion tensor (for example the one motivated from string theory), an equivalence is set between these two schemes, as the non-vanishing independent torsion tensor components turn out to be the same.

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1. Introduction
The geometry of Riemann–Cartan (RC) space-time has been of some importance both in the context of local field theories and in the effective scenarios originating from string theory. Such a space-time is characterized by an asymmetric (but metric-compatible) affine connection, the antisymmetrization of which in at least two of its indices gives rise to a third rank tensor field known as torsion [1–5]. The incorporation of torsion is a natural modification of General Relativity (GR), especially from the point of view that at the simplest level a classical background can be provided for quantized matter fields (with definite spin), typically below the Planck scale [4–6].

There are several predictions of the observable effects of torsion, whose origin could be traced from various sources. For instance, the massless antisymmetric Kalb–Ramond (KR) field in closed string theory has been argued to be the source of a completely antisymmetric torsion in the background [7, 8]. Such a torsion has its significance in the effects produced on a number of astrophysical phenomena, which have been explored in detail both in the
usual four dimensional framework and in the context of compact extra dimensional theories [9–14]. Investigations have also been carried out for the observable effects of gravitational parity violation, which is shown to be plausible in presence of torsion [15]. Another subject of some relevance has been the study of torsion in local conformal frames, particularly in the context of non-minimal metric-scalar theories of gravitation [5, 16]. Certain non-minimal scalar–torsion couplings have been proposed [5, 16], which may assign scalar field sources to some of the irreducible torsion modes and thereby produce interesting physical effects [17, 18]. Non-minimal couplings of torsion with spinor fields are also of some importance, as specific bounds on certain torsion components for such couplings have been extracted from modern experimental searches for Lorentz violation [19]. In recent years, with major advances in the formal aspects of the Poincaré gauge theory of gravity [20], the effects of the associated torsion modes have been explored in the context of inflationary cosmology [21] as well as for the problem of dark energy in the universe [22, 23]. Moreover, certain modified versions of the teleparallel (torsion without curvature) theories, known as $f(T)$ theories, are found to have interesting implications in cosmology and astrophysics [24–28].

Now, it is crucial to emphasize here that the estimation of any of the observable effects of torsion is subject to a clear understanding of how torsion is affected by, and in turn does affect, the symmetries of space-time. In other words, given a space-time metric structure, the foremost requirement for any observable prediction of torsion is the complete determination of the admissible torsion degrees of freedom, depending on the symmetries that are exhibited. Let us limit our attention to the scenarios of natural interest in any gravitational theory, viz. to space-times which either exhibit maximal symmetry on their own or could be decomposed to maximally symmetric subspaces. If the maximal symmetry is to be preserved in presence of torsion, then one has to ascertain the existent torsion tensor components, taking note of the fact that these components would back-react on the metric and hence affect the geometric structure of space-time.

One may recall that in GR, a tensor (of specific rank and symmetry properties) can in principle be constrained if it is invariant in form under the infinitesimal isometries of the metric of a given space. On the other hand, an $n$-dimensional space is said to be maximally symmetric if its metric admits the maximum number $(n(n+1)/2)$ of independent Killing vectors. Form-invariance of a tensor under maximal symmetry therefore implies the vanishing Lie derivative of the tensor with respect to each of these $n(n+1)/2$ Killing vectors. In presence of torsion however, one needs to be careful in dealing with the concepts of space-time symmetries and in particular the maximal symmetry. In fact, the geometric nature of torsion may reveal in the form-invariance of the torsion tensor with respect to all isometries if there is a mathematical principle which entails the space-time to have a definite structure, viz. either maximally symmetric in entirety or could be decomposed into maximally symmetric subspaces. Refer, for example, to the (large scale) homogeneity and isotropy of the universe—the so-called cosmological principle. If such a principle is to be obeyed in presence of torsion then the entire four dimensional (RC) manifold should consist of space-like three dimensional maximally symmetric subspaces identified as hypersurfaces of constant cosmic time, and like any other cosmic tensor field torsion may be form-invariant under the isometries of the metric of such subspaces [29]. This is logical at least when the torsion modes are specified in terms of some other degrees of freedom in the theory, for e.g. a scalar potential [25, 30, 31] or a second rank tensor potential [6, 7, 32, 33]. However, for the theories of rather conventional type, in which torsion is an independent variable that contributes either algebraically to the action (as in the original Einstein–Cartan formulation [1, 2, 4]) or is propagating [8, 34, 35], there is no concrete reason in support of the form-invariance of torsion. In such cases, the fundamental question:
What is the precise meaning of maximal symmetry in presence of torsion?

has its relevance to the more appropriate question in all possible circumstances in which
definite space-time structures involving torsion have to be maintained:

How is the torsion tensor constrained either by virtue of its form-invariance under maximal
symmetry, or in the course of defining maximal symmetry in its presence?

From the technical point of view, the (manifestly covariant) Killing equation is given
by the vanishing anticommutator of the covariant derivatives of the Killing vectors. In GR,
this equation directly follows from the isometry condition, viz. vanishing Lie derivative of
the metric tensor with respect to the Killing vectors [36]. In the RC space-time however, this
equivalence no longer exists in general, as the covariant derivatives now involve torsion.
Moreover, the equations relevant for the integrability of the Killing equation1 are also
manifestly covariant and hence get altered when expressed in terms of the covariant derivatives
involving torsion in the RC space-time. So, one needs to sort out whether the preservation
of the Killing equation or(and) the Killing integrability criterion is(are) absolutely necessary
in space-times with torsion. If so, then what are the constraints on the torsion tensor? What
are the constraints otherwise, under the demand that torsion is form-invariant in maximally
symmetric spaces?

A study of literature reveals that one can in principle resort to two different schemes (from
contrasting viewpoints) in order to access the underlying aspects of space-time symmetries
(and in particular, of maximal symmetry) in presence of torsion [29, 38–42]:

Scheme I: from a somewhat weaker viewpoint, maximal symmetry is to be understood
solely from the metric properties of space-time. Therefore, in presence of torsion a
maximally symmetric $n$-dimensional space is still the one which admits the maximum
number $n(n+1)/2$ of Killing vectors, the latter satisfying the usual (general relativistic)
Killing equation and the equations relevant for its integrability. However, as torsion is a
characteristic of space-time, maximal symmetry has its significance only when it leaves
torsion form-invariant—a condition that imposes constraints on the torsion tensor, just as
it would on any other third rank tensor (with the specific antisymmetry property similar
to torsion) [29, 38, 39].

Scheme II: from a stronger viewpoint, maximal symmetry has to be understood in presence
of torsion by explicitly taking into account torsion’s effect on the Killing and other
relevant equations, and demanding that the form of these equations should remain intact.
This would however constrain the torsion tensor itself so that a maximally symmetric
$n$-dimensional space (exhibiting torsion) would not only be the space which admits the
maximum number of Killing vectors, but also that this maximum number would precisely
be $n(n+1)/2$ (as in GR) [40–42].

Either of these schemes may be useful for a self-consistent implementation of the concept
of maximal symmetry in presence of torsion. However, the scheme I, which actually suppresses
the influence of torsion on such symmetry, is primarily applicable for torsion modes that are
derived from some other degrees of freedom in the theory (as is common, for e.g., in many
effective scenarios originating from string theory). As such, from the phenomenological point
of view the scheme I is generally favored in spite of the fact that one has to comply with
the lack of appropriate covariant generalization of the equations of relevance (the Killing

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1 In GR, the successive application of two equations satisfied by the curvature tensor (viz. its definition in terms of the
anticommutator of two covariant derivatives of a vector, and its cyclicity property) actually leads to the integrability
criterion, from which it is inferred that an $n$-dimensional maximally symmetric space has the maximum number
$n(n+1)/2$ of Killing vectors.
equation inclusive) in presence of torsion [17, 18, 29, 38, 39]. Scheme II, on the other hand, gives a clearer geometric picture and wider applicability, but at the same time seems a bit too idealistic as it requires stringent conditions on the torsion tensor for apparently no reason other than to define maximal symmetry in the exact analogy of that in GR$^2$ [40–42]. Of course, the ambiguity in choosing which of the two schemes to follow could be resolved if it turns out that the outcome (in the form of the set of constraints on torsion) is the same.

The objective of this paper is to make a systematic study of the independent, non-vanishing torsion tensor components in a manifold of given dimensionality, say $d$, which is either entirely maximally symmetric or consist of maximally symmetric subspaces of dimensionality $n (< d)$. A major portion of the paper deals with the scheme I mentioned above. The analysis is carried out along the lines of Tsimparlis [29], in which the relevant components of the torsion tensor have been found under the demand of its form-invariance in a maximally symmetric subspace of dimensionality $n = 3$ or more, with emphasis on the homogeneous and isotropic cosmologies. We extend Tsimparlis’ work to include all possible scenarios ($n \geq 2$) and also find the existent irreducible modes of torsion, viz. the trace, the completely antisymmetric (pseudo-trace) part, and the (pseudo-)trace-free part, in a $d$-dimensional space-time. The remainder of our paper determines the constraints on the torsion tensor under the scheme II. Although our initial approach is similar to that in [41], while clarifying the meaning of the maximal symmetry we concentrate only on the essential restrictions on torsion. As such, the constraints we find for the scheme II are in general different from those in [41]. We also make a comparison of the outcome of the schemes I and II, in order to establish a correlation, and possibly an equivalence between them. We actually observe that such an equivalence exists at least in the case of a completely antisymmetric torsion tensor.

The organization of the paper is as follows: in section 2, we discuss the basic concepts related to space-time symmetries (viz. isometries, Killing vectors, etc) and how these concepts could possibly be understood in space-times admitting torsion. In section 3, taking the approach based on the scheme I we determine the existent components of the torsion tensor and its irreducible modes in a $d$-dimensional space-time manifold $\mathcal{M}$ with a maximally symmetric sub-manifold $\mathcal{M}'$ of dimensionality $n$ ($< d$). In section 4, we look for the constraints imposed on torsion in the formal development of theories based on the scheme II. A comparative study of the results obtained in the two schemes is done in section 5, by resorting to certain scenarios of physical relevance. We conclude with a summary and some open questions in section 6. The general aspects of a $d$-dimensional RC space-time, viz. the definition of torsion, its irreducible modes, the covariant derivatives, geodesics and auto-parallels, etc are reviewed in the appendix.

2. Symmetric (sub)spaces and torsion

Let us look into the basic concepts related to symmetric (sub)spaces and the role of torsion in influencing or preserving the symmetries. The starting point is the condition

$$g_{AB}(x) = \frac{\partial x^M}{\partial x'^I} \frac{\partial x^N}{\partial x'^J} g_{MN}(x'),$$

(1)

Moreover, a strict enforcement of both the minimal coupling prescription and the principle of general covariance is implied in the scheme II. Minimal coupling ensures that in presence of torsion the Riemannian covariant derivatives get replaced by the RC ones, so that all equations that involve covariant derivatives of tensors (of rank $\geq 1$) are in general altered. General covariance, on the other hand, suggests that in order to understand maximal symmetry one needs to take into account only the (altered) Killing and other relevant equations, as they are manifestly covariant in the RC space-time.
from the requirement that certain coordinate transformations $x \rightarrow x'$, known as \textit{isometries}, would leave the metric tensor form-invariant, $g_{AB}(x') = g_{AB}(x')$ [36]. For infinitesimal isometries, viz. $x^M = x^M + \xi^M(x)$, the above condition (1) reads (to the first order in $\xi^M$ and its derivatives):

$$\Omega_\xi g_{AB}(x) = 0 = \xi^M \partial_M g_{AB} + g_{MB} \partial_A \xi^M + g_{AN} \partial_B \xi^N,$$

(2)

where $\Omega_\xi$ denotes the Lie derivative with respect to the vector $\xi^M$. In a similar manner, form-invariance of any tensor, for e.g. torsion $T_{ABC}$, under the infinitesimal isometries of the metric would imply the vanishing Lie derivative of the tensor with respect to $\xi^M$, i.e. $\Omega_\xi T_{ABC} = 0$. Equation (2) is of crucial importance in GR for understanding the aspects of symmetries, and in particular the maximal symmetry of a given space or subspaces.

\subsection*{2.1. Maximal symmetry in absence of torsion}

In the Riemannian space-time (without torsion), equation (2) can be expressed in a covariant form—the so-called \textit{Killing equation} in GR:

$$\nabla_A \xi_B + \nabla_B \xi_A = 0,$$

(3)

and any vector field which satisfies this equation is said to form a \textit{Killing vector} of the metric $g_{AB}(x)$. Hence the infinitesimal isometries of the metric are essentially determined by the space of vector fields spanned by the Killing vectors [36]. Moreover, when a metric space admits the maximum possible number of linearly independent Killing vectors, the space is said to be \textit{maximally symmetric}. Now, what is this maximum possible number? To determine this, one uses the following two relations in GR:

(i) the commutator of two covariant derivatives of a vector in terms of the product of the vector and the Riemann curvature tensor:

$$(\nabla_A \nabla_B - \nabla_B \nabla_A) \xi_C = -R^M_{\ ABC} \xi_M,$$

(4)

(ii) the cyclic sum rule for the Riemann curvature tensor:

$$R^M_{\ ABC} + R^M_{\ BCA} + R^M_{\ CAB} = 0.$$

(5)

Adding with equation (4) its two cyclic permutations in the indices $A, B, C$, and using equations (5) and (3), we get

$$\nabla_A \nabla_B \xi_C - \nabla_B \nabla_A \xi_C - \nabla_C \nabla_B \xi_A = 0,$$

(6)

whence the equation (4) becomes

$$\nabla_C \nabla_B \xi_A = -R^M_{\ CAB} \xi_M.$$

(7)

This is the \textit{integrability criterion} for the Killing vectors, which implies that any particular Killing vector $\xi_M(x)$ of the metric $g_{AB}(x)$ is specified uniquely by the values of the Killing vector and its covariant derivative at any particular point $X$, i.e. by $\xi_M(X)$ and $\nabla_N \xi_M|_{x=X}$. As a result, in an $n$-dimensional metric space there can be at most $n(n+1)/2$ Killing vectors $\xi_M(x)$ which do not satisfy any linear relation of the form $\sum_q c_q \xi_M^{(q)}(x) = 0$, with constant coefficients $c_q$ [36].

For spaces with maximally symmetric subspaces, the analysis is similar to the above. However, the maximum number of independent and non-vanishing Killing vectors that are admitted for a family of such subspaces, say of dimensionality $n$, are only $n(n+1)/2$, and not $d(d+1)$, if the entire space-time is of dimensionality $d$. However, these Killing vectors are of the metric of the subspaces only. Hence the constraints on a tensor due to its form-invariance under the infinitesimal isometries of the metric, in the case of spaces with maximally symmetric
subspaces, would in general be different from those in the scenario where the entire space-time is maximally symmetric. We shall look into this rather explicitly in what follows. However, for the time being, let us first concentrate on how to go about understanding maximal symmetry when the space-time admits torsion, in the next subsection.

2.2. Maximal symmetry in presence of torsion

Torsion being a geometric entity, it is fair to argue that the form-invariance of the torsion tensor (in addition to that of the metric) is a requirement for the preservation of maximal symmetry. But how do we conceive maximal symmetry after all, in presence of torsion? The question amounts to justify which should be taken to be fundamental—the equation (2) giving the condition of form-invariance of the metric tensor under infinitesimal isometries, or the explicitly covariant Killing equation (3) which is in general modified in presence of torsion (and so are the other relevant equations (4) and (5)). As mentioned earlier, there are two different schemes for the implementation of maximal symmetry in space-times with torsion:

- The one which sets aside the general covariance (and also minimal coupling), and considers equation (2) to be the most fundamental, as this follows straightaway from the first principles. Maximal symmetry is then to be realized precisely in the same way as in GR, i.e. the analysis in the previous subsection would go through even in presence of torsion. The only objective that remains is to determine the constraints on the torsion tensor on account of its form-invariance under the maximal symmetry.

- The other which takes the general covariance (and the minimal coupling) in a serious note, and hence considers the modified version of equation (3), and of the follow-up equations (4) and (5) in torsioned spaces, to be fundamental. In fact, the demand is to be that all these equations should retain their forms, even when the covariant derivatives involve torsion (i.e. $\nabla_A$ is replaced by $\tilde{\nabla}_A$, see the appendix for notations and definitions). The analysis in the previous subsection would then again go through, however at the expense of constraining the torsion tensor severely for such restoration of forms of the above equations.

In the next two sections we work out the independent non-vanishing components of the torsion tensor (and also of its irreducible modes), by taking into account one-by-one the constraining equations on torsion in the above two schemes. Thereafter, resorting to some specific scenarios of physical importance, we make a comparison of the allowed torsion degrees of freedom in these two schemes. In particular, we look for the cases in which the independent torsion components allowed by the two schemes turn out to be the same. That would at least partially resolve the somewhat conflicting issue of maximal symmetry in a given theory involving torsion.

3. Scheme I: constraints on a maximally form-invariant torsion

In this section we consider the usual (general relativistic) definition of maximal symmetry in presence of torsion, and work out the constraints on the torsion tensor due to its form-invariance under such symmetry. This is the scheme I mentioned above, in which the fundamental aspects of symmetries of metric spaces are supposedly governed by equation (2) that follows from the first principles. We proceed along the lines of Tsimparlis [29] to determine the non-vanishing independent components of the torsion tensor in maximally symmetric (sub)spaces. Let us first consider the following picture:

- $\mathcal{M}$ is a $d$-dimensional (‘bulk’) manifold, with a non-degenerate symmetric metric $g_{AB}(x)$, where $x := \{x^a\}$ is the set of bulk coordinate labels and $A, B, \ldots$ are the bulk indices (each of which runs over all the $d$ values $0, 1, 2, \ldots, d - 1$).
• $\mathcal{M}$ is a maximally symmetric $n$-dimensional sub-manifold of $\mathcal{M}$, with metric $\bar{g}_{ab}(\vec{v})$, where $\vec{v} := \{\vec{v}^\alpha\}$ is the set of coordinate labels in $\mathcal{M}$, and each of the corresponding indices $\vec{a}, \vec{b}, \ldots$ runs over $n$ of the $d$ values.

• The quotient $\mathcal{M}/\mathcal{M}$ is in general not maximally symmetric, and is just a topological space, with topology induced from $\mathcal{M}$. Although not necessarily a manifold, we assume $\mathcal{M}/\mathcal{M}$ to be such (but in general not a sub-manifold of $\mathcal{M}$), with metric $g_{ab}(v)$, where $v := \{v^\nu\}$ is the corresponding set of coordinate labels, and each of the indices $a, b, \ldots$ runs over the remaining $(d-n)$ of the $d$ values.

In fact, the assumption that $\mathcal{M}/\mathcal{M}$ is a manifold amounts to mentioning that we are restricting our analysis to an open set $\mathcal{U} \subset \mathcal{M}$, which is diffeomorphic to a direct product

$$\mathcal{U} = \bar{\mathcal{U}} \times \hat{\mathcal{U}}, \quad \bar{\mathcal{U}} \subset \mathcal{M}, \quad \hat{\mathcal{U}} \subset \mathcal{M}/\mathcal{M},$$

(8)

where $\bar{\mathcal{U}}$ and $\hat{\mathcal{U}}$ are also open sets. Choosing then local coordinates adapted to this diffeomorphism, one may split the coordinates on $\mathcal{M}$ as $x^A := (\vec{v}^\alpha, v^\nu)$. This is reasonable as the gravitational field equations are of local nature. One solves them in open sets and then, through a mechanism of gluing (if necessary), they can be extended to other, intersecting open sets to construct a global manifold.

We shall look into all possible scenarios $2 \leq n \leq d$, the special case $n = d$ of course deals with a maximally symmetric bulk manifold $\mathcal{M}$. For $n < d$ however, we shall resort to a rather complicated picture, viz. that of the $d$-dimensional manifold $\mathcal{M}$ being decomposed in a family of maximally symmetric $n$-dimensional sub-manifolds $\mathcal{M}$. Then the above splitting $x^A := (\vec{v}^\alpha, v^\nu)$ of the bulk coordinates has a succinct implication, and there are stringent conditions on the bulk metric $g_{ab}(x)$ [36]:

• The sub-manifolds $\mathcal{M}$, the coordinate functions on which are $\vec{v}^\alpha$ with $\vec{a}$ taking $n$ of the $d$ values $0, 1, 2, \ldots, d-1$, are distinguished from one another by the coordinate labels $v^\nu$, where the index $a$ can take the remaining $(d-n)$ of the $d$ values.

• The subspaces with constant $v^\nu$ are maximally symmetric subject to the condition that the bulk metric $g_{ab}(x)$ is invariant under the infinitesimal transformations:

$$\vec{v}^\alpha \longrightarrow \vec{v}^\alpha + \xi^\alpha(\vec{v}), \quad v^\nu \longrightarrow v^\nu \equiv v^\nu,$$

(9)

where $\xi^\alpha$ are the Killing vectors in $\mathcal{M}$. There are $n(n+1)/2$ such Killing vectors $\xi^\alpha$ which are linearly independent. As the transformations (9) leave the coordinates $v^\nu$ invariant, the Killing vectors $\xi^\alpha(\vec{v}, v)$ are all zero.

• Finally, there is a powerful theorem which states that: it is always possible to choose the coordinates $\vec{v}^\alpha$ such that the bulk metric $g_{ab}$ is decomposed as

$$g_{ab} dx^a dx^b = g_{ab}(v) dv^a dv^b + f(v) g_{ab}(\vec{v}) dv^a dv^b,$$

(10)

where $f(v)$ is some specific function of the $v$-coordinates only. Equation (10) implies that there are no mixed elements of the form $g_{va}$, and the Killing vectors do not depend on the $v$-coordinates, i.e. $\xi^\alpha = \xi^\alpha(\vec{v})$.

Now, the form-invariance of torsion under the isometries of the metric in $\mathcal{M}$ is given by

$$\hat{\Sigma}_\alpha T_{ABC} = 0 = \xi^\alpha \partial_\alpha T_{ABC} + T_{ABC} \partial_\alpha \xi^\alpha + T_{ABC} \partial_\alpha \xi^\beta + T_{ABC} \partial_\alpha \xi^\gamma,$$

(11)

On the other hand, the maximal symmetry in $\mathcal{M}$ implies that a Killing vector $\xi^\alpha$ be so chosen that

(i) it vanishes at any given point $\bar{U}$ in $\mathcal{M}$, i.e. $\xi(\bar{U}) = 0$, whereas
(ii) its covariant derivative (defined in terms of the Christoffel connection) at that point, i.e. $\nabla_\alpha \xi^\alpha |_{x_\alpha = \vec{v}^\alpha}$, forms an arbitrary matrix, which is of course antisymmetric because of the Killing equation $\nabla_\alpha \xi^\alpha + \nabla_\beta \xi^\beta = 0$. 


Therefore at \( u = U \), equation (11) gives
\[
(\delta^p_A T^q_{BC} + \delta^p_B T^q_{AC} + \delta^p_C T^q_{AB}) \nabla^p \xi^q = 0,
\]
and \( \nabla^p \xi^q \) being arbitrary and antisymmetric, its coefficient is symmetric in \( p \) and \( q \):
\[
\delta^p_A T^q_{BC} + \delta^p_B T^q_{AC} + \delta^p_C T^q_{AB} = \delta^q_A T^p_{BC} + \delta^q_B T^p_{AC} + \delta^q_C T^p_{AB}.
\]
This equation should hold everywhere, since the point \( U \) is arbitrary as well.

### 3.1. Allowed independent components of the torsion tensor

In view of the antisymmetry property of torsion, viz. \( T^A_{BC} = -T^A_{CB} \), we have the following six conditions [29]:

(i) \( \mathcal{L}\xi T_{abc} = 0 \),
(ii) \( \mathcal{L}\xi T_{aba} = 0 \),
(iii) \( \mathcal{L}\xi T_{aab} = 0 \),
(iv) \( \mathcal{L}\xi T_{a}^{ab} = 0 \),
(v) \( \mathcal{L}\xi T_{a}^{ab} = 0 \),
(vi) \( \mathcal{L}\xi T_{abc} = 0 \).

Let us now examine the outcome of these conditions in detail.

- **For the condition (i)**, equation (13) gives
  \[
  \delta^p_a T^q_{ba} + \delta^p_b T^q_{a} + \delta^p_c T^q_{ab} = \delta^q_a T^p_{ba} + \delta^q_b T^p_{a} + \delta^q_c T^p_{ab}.
  \]
  Contracting \( p \) with \( a \) gives
  \[
  (n-1)T^q_{ba} + T^q_{ba} + T^q_{ba} = \delta^q_a T^p_{ba} + \delta^q_b T^p_{a} + \delta^q_c T^p_{ab}.
  \]
  Contracting further \( q \) with \( b \) results in \( T^q_{ba} = 0 \), which when substituted back in equation (16) yields finally
  \[
  T^q_{ba} = \begin{cases} 
  T^q_{ba}(v), & n = 3 < d \\
  0, & n \neq 3.
  \end{cases}
  \]

- **For the condition (ii)**, equation (13) gives
  \[
  \delta^p_a T^q_{ba} + \delta^p_b T^q_{b} + \delta^p_c T^q_{ab} = \delta^q_a T^p_{ba} + \delta^q_b T^p_{a} + \delta^q_c T^p_{ab}.
  \]
  Contracting \( p \) with \( a \) we get
  \[
  (n-1)T^q_{ba} + T^q_{ba} + T^q_{ba} = \delta^q_a T^p_{ba} + \delta^q_b T^p_{a} + \delta^q_c T^p_{ab}.
  \]
  Now interchanging \( \bar{q} \) and \( \bar{b} \), then multiplying both sides by \((n-1)\), and subtracting the resulting equation from equation (19), one concludes that
  \[
  T^q_{ba} = -T^q_{ba} = \begin{cases} 
  T^q_{ba}(v) - \frac{1}{2} \delta^q_a T^p_{ba}(v), & n = 2 \\
  -\frac{1}{n} \delta^q_a T^p_{ba}(v), & 3 \leq n < d.
  \end{cases}
  \]

- **For the conditions (iii) and (iv)**, one gets in a similar manner the following
  \[
  T_{abc} = 0, \quad T_{aba} = 0, \quad (\forall n \geq 2),
  \]
  whereas the condition (v) leads to
  \[
  T_{a}^{ab} = \begin{cases} 
  T_{a}^{ab}(v), & n = 2 < d \\
  0, & n \neq 2.
  \end{cases}
  \]
3.2. Constraints on the completely antisymmetric part of torsion.

The independent components of the tensor $A_{ABC}$ are: $A_{[abc]}$, $A_{[ab]a}$, $A_{[ab]b}$, and $A_{[abc]}$. An analysis similar to the above reveals that the existent ones are

The condition (vi) is actually redundant, i.e. it does not provide any additional information about the component $T_{abc}$. However, since the indices $a$, $b$, $c$ can only run over $(d - n)$ values, and $T_{abc}$ is antisymmetric in $b$ and $c$, we have

$$T_{abc} = T_{a[bc]}(v), \quad (\forall n \leq d - 2).$$

(23)

One may note that in the case $n = 2$, equation (20) implies that $T_{[abc]}$ is antisymmetric in $a$ and $b$ when $a \neq b$. But $T_{[abc]}$ is antisymmetric in the last two indices as well. Therefore, $T_{[abc]}$ must be completely antisymmetric, i.e. $T_{[abc]} = T_{[bac]}$ for $n = 2$ when it is pre-assigned that $a \neq b$. Moreover, a completely antisymmetric $T_{[abc]}$ is also equal to $T_{abc}$. Thus in a $d$-dimensional manifold $\mathcal{M}$, the non-vanishing independent torsion tensor components that preserve the maximal symmetry of the sub-manifolds $\overline{\mathcal{M}}$ of dimensionality $n$, are:

$$T_{[abc]} = \epsilon_{abc} \beta(v); \quad (\text{only for } n = 3 \leq d),$$

(24)

$$T_{a[bc]} = \begin{cases} T_{[abc]}(v) - \frac{1}{2} \delta_{bc} \alpha_a(v) & (\text{for } n = 2 < d) \\ \frac{1}{2} \delta_{bc} \alpha_a(v) & (\text{for } 3 \leq n < d), \end{cases}$$

(25)

$$T_{abc} = T_{d[bc]}(v); \quad (\text{for } 2 \leq n \leq d - 2).$$

(26)

In the above, $\beta(v)$ is a pseudo-scalar and $\alpha_a(v)$ is $T_{[abc]}(v)$ is a $(d - n)$-space vector.

The particular case $n = d$, which refers to a maximally symmetric bulk manifold $\mathcal{M}$ finds the torsion tensor having only its completely antisymmetric part $T_{[ABC]} = T_{[abc]}$ non-vanishing (and constant). Moreover, the dimensionality of the bulk is $d = 3$.

3.2. Allowed independent components of the irreducible torsion modes

Let us refer to equation (A.7) in the appendix, for the irreducible decomposition of the torsion tensor $T_{ABC}$ in a $d$-dimensional space-time. The irreducible modes are the trace of torsion $T_{(a} := T_{ab}^a$, the totally antisymmetric part (or the pseudo-trace) $A_{ABC} := T_{[ABC]}$, and the (pseudo-)trace-free part $Q_{ABC}$ that satisfies the conditions (A.10).

3.2.1. Constraints on the trace of torsion. The components of the torsion trace vector $T_a$ are given by

$$T_a := T_{[abc]} = T_{[abc]}^c + T_{a[bc]}^c, \quad \text{and} \quad T_a := T_{(a}^b = T_{(a}^b + T_{b(a}^b).$$

(27)

Now, $T_{[abc]}$ is either completely antisymmetric (for $n = 3$) or zero (for $n \neq 3$). Therefore, its trace $T_a^c = 0(\forall n)$. Also, since $T_{abc} = 0(\forall n)$, we have $T_a^c = 0(\forall n)$. Hence,

$$T_a = 0, \quad (\forall n \geq 2).$$

(28)

Similarly, it can be shown that

$$T_a = \begin{cases} \alpha_a(v), & n = d - 1 \\ \alpha_a(v) + \gamma_a(v), & n < d - 1, \end{cases}$$

(29)

where $\alpha_a = T_{[abc]}^b$ (as before), and we have defined another vector $\gamma_a := T_{ab}^b$ in $\mathcal{M}/\overline{\mathcal{M}}$.  

3.2.2. Constraints on the completely antisymmetric part of torsion. The independent components of the tensor $A_{ABC}$ are: $A_{[abc]}$, $A_{[ab]a}$, $A_{[ab]b}$ and $A_{[abc]}$. An analysis similar to the above reveals that the existent ones are
\[ A_{\sigma \beta \gamma} = \epsilon_{\sigma \beta \gamma} \beta(v); \quad (\text{only for } n = 3 \leq d), \]  
(30)

\[ A_{\sigma \nu} = T_{[\sigma \nu]}(v); \quad (\text{only for } n = 2 < d), \]  
(31)

\[ A_{abc} = T_{[abc]}(v); \quad (\text{for } 2 \leq n \leq d - 3), \]  
(32)

where once again \( \beta \) is a pseudo-scalar that depends only on the coordinates \( \nu^{\alpha} \) of \( \mathcal{M}/\overline{\mathcal{M}} \).

### 3.2.3. Constraints on the (pseudo-)trace-free part of torsion

The tensor \( Q_{ABC} \) given by (see the appendix)

\[ Q_{ABC} = T_{ABC} - \frac{1}{d-1} (g_{AC} T_B - g_{AB} T_C) - A_{ABC}, \]  
(33)

satisfies the conditions (A.10). In general, the independent components are: \( Q_{\sigma \nu} \), \( Q_{\sigma ab} \), \( Q_{ab} \), \( Q_{abc} \) and \( Q_{a b c} \). However, only the following can exist:

\[ Q_{\sigma ab} = \delta_{\sigma ab}[\left(\frac{1}{d-1} - \frac{1}{n}\right) \alpha_{ab}(v) + \left(\frac{1}{d-1}\right) \gamma_{ab}(v)]; \quad (n < d - 1), \]  
(34)

\[ Q_{abc} = W_{\sigma abc}(v) \quad (\forall n), \]  
(35)

\[ W_{\sigma abc} = \frac{4}{3} T_{\sigma abc} + \frac{2}{d-1} g_{ab} T_c, \]  
(36)

\[ = \begin{cases} 
2 \frac{d-1}{d-1} g_{ab} \alpha_c, & (n = d - 1) \\
\frac{4}{3} T_{\sigma abc} + \frac{2}{d-1} g_{ab} [\alpha_c + \gamma_c], & (n < d - 1). 
\end{cases} \]

### 4. Scheme II: torsion in a generally covariant maximal symmetric set-up

Let us now look into the concept of maximal symmetry in presence of torsion when the principle of general covariance is strictly obeyed. Under the minimal coupling prescription, the covariant derivatives of a \( d \)-dimensional Riemannian space \( (R^d) \) are generalized to those of a space admitting torsion \((U_d)\), i.e. \( \nabla_A \rightarrow \overline{\nabla}_A \). We demand that the Killing equation (3) should be preserved in form, when expressed in terms of these new covariant derivatives \( \overline{\nabla}_A \), i.e.

\[ \overline{\nabla}_A \xi_B + \overline{\nabla}_B \xi_A = 0. \]  
(37)

Such a demand is actually based on the argument that the Killing vectors determine the constants associated with the motion along the affine curves with properties defined by the metric (or) and the connection. In the space-times admitting torsion such curves are the auto-parallel curves (sometimes called the affine geodesics) which transport their tangent vectors parallelly to themselves. These curves are in general different from the usual (metric) geodesics which extremize the separation between events and depend only on the metric properties of space-time (see the appendix for details). If a vector \( \nu^{A} = dx^{A}/d\sigma \) is tangent to an auto-parallel curve affinely parameterized by \( \sigma \), then the constants of motion are determined from the relation [43]

\[ \frac{d}{d\sigma} (\nu^{A} \xi_A) = 0. \]  
(38)

However, in the \( U_d \) space-time one has

\[ \frac{d}{d\sigma} (\nu^{A} \xi_A) = \nu^{B} \overline{\nabla}_B (\nu^{A} \xi_A) = \xi_A \nu^{B} \overline{\nabla}_B \nu^{A} + \nu^{A} \nu^{B} \overline{\nabla}_B \xi_A. \]  
(39)
The first term on the right hand side of course vanishes by virtue of the auto-parallel equation \( u_\lambda \tilde{\nabla}_\lambda v^\alpha = 0 \) (cf equation (A.14) in the appendix), but the second term would vanish only when we assert that the Killing vectors satisfy the above equation (37). Moreover, since the Killing vectors also satisfy the relation \( \nabla_A \xi_B + \nabla_B \xi_A = 0 \) (cf equation (3)), they determine the constants of motion along the metric geodesics as well. That is to say, if \( u^\lambda = d\xi^\lambda/d\lambda \) is tangent to a metric geodesic parameterized by \( \lambda \), then we have

\[
\frac{d}{d\lambda} (u^\lambda \xi_\lambda) = u^\beta \nabla_B (u^\lambda \xi_\lambda) = \xi_\lambda u^\beta \nabla_B u^\lambda + u^\lambda u^\beta \nabla_B \xi_\lambda = 0,
\]

(40)
as in Riemannian space-time. However, the parameter \( \lambda \) may not be an affine parameter in space-times involving torsion (except in the case of a completely antisymmetric torsion tensor for which the metric geodesics are identical with the auto-parallels, and one may verify that equations (3) and (37) are also the same).

Now, for the above Killing equation (37) to hold along with the equation (3), the torsion tensor should satisfy

\[
(T_{ABC} + T_{BAC}) \xi^C = 0,
\]

(41)
and if we proceed exactly as in GR (see section 2.1), we first encounter the equation

\[
(\tilde{\nabla}_A \tilde{\nabla}_B - \tilde{\nabla}_B \tilde{\nabla}_A) \xi_C = -\tilde{R}^M_{\phantom{M}CA} \xi^M + T^M_{\phantom{M}AB} \tilde{\nabla}^M \xi_C,
\]

(42)
which is of course the generalization of equation (4). Here, \( \tilde{R}^M_{\phantom{M}CA} \) is the \( U_d \) analogue of the Riemannian curvature tensor \( R^M_{\phantom{M}CA} \):

\[
\tilde{R}^M_{\phantom{M}CA} = R^M_{\phantom{M}CA} + T^M_{\phantom{M}CB}.
\]

(43)

Now, adding with equation (42), its two cyclic permutations in the indices \( A, B \) and \( C \), then using the Killing equation (37) and the cyclicity condition (5) for \( R^M_{\phantom{M}BCD} \), we get an equation similar to equation (6):

\[
\tilde{\nabla}_A \tilde{\nabla}_B \xi_C - \tilde{\nabla}_B \tilde{\nabla}_A \xi_C - \tilde{\nabla}_C \tilde{\nabla}_B \xi_A = 0,
\]

(45)
under the condition

\[
(\tilde{R}^M_{\phantom{M}ABC} + \tilde{R}^M_{\phantom{M}BCA} + \tilde{R}^M_{\phantom{M}CAB}) \xi_M = -(T^M_{\phantom{M}AB} \tilde{\nabla}_C \xi_M + T^M_{\phantom{M}BC} \tilde{\nabla}_A \xi_M + T^M_{\phantom{M}CA} \tilde{\nabla}_B \xi_M).
\]

(46)
Substituting equation (45) back in equation (42), and using the Killing equation (37) once more, we obtain

\[
\tilde{\nabla}_C \tilde{\nabla}_B \xi_A = -\tilde{R}^M_{\phantom{M}CAB} \xi^M - T^M_{\phantom{M}AB} \tilde{\nabla}_C \xi_M.
\]

(47)
This equation is not entirely similar to equation (7) above, because of the second term on the right hand side. However, one may still use this as the integrability criterion for the Killing vectors in a space-time with torsion. That is to say, all the arguments that follow in GR, after getting equation (7), would be the same here as well, once the equation (47) is set. Given the values of \( \xi_N \) and \( \tilde{\nabla}_M \xi_N \) at some point \( X \), equation (47) gives the second derivative, and successive differentiations of equation (47) yield the corresponding higher derivatives of \( \xi_N \) at \( X \). Consequently, a particular Killing vector \( \xi_N^{(q)}(x) \), is only linearly dependent on the initial values \( \xi_N^{(q)}(X) \) and \( \tilde{\nabla}_M \xi_N^{(q)}(x) \mid_{x=X} \):

\[
\xi_N^{(q)}(x) = A^M_{\phantom{M}N}(x, X) \xi_N^{(q)}(X) + B^M_{\phantom{M}LM}(x, X) \tilde{\nabla}_L \xi_N^{(q)}(x) \mid_{x=X}.
\]

(48)

It is to be noted that for the scheme I, the Killing vectors can determine the constants of motion along the (metric) geodesics but not in general along the auto-parallels. So this scheme is primarily applicable when the torsion modes could be traded away with some other fields in the theory (via constraint equations, as torsion is auxiliary). In such cases the RC action effectively reduces to the Riemannian one coupled with other fields.
where the coefficients $A_{MN}^M$ and $B_{LM}^M$ depend on the metric and the torsion, and are the same for all Killing vectors. Now, in an $n$-dimensional space, for every $q$, there can be at most $n$ independent quantities $\xi_{N}^{(q)}(X)$ and $n(n-1)/2$ independent quantities $\nabla_{M}\xi_{N}^{(q)}|_{X=X}$ (by virtue of the Killing equation (37)). So, any linearly independent set of Killing vectors, in $n$ dimensions, can consist of a maximum number of $n + n(n-1)/2 = n(n+1)/2$ of such vectors. Accordingly, one may say that an $n$-dimensional space which admits all of the $n(n+1)/2$ independent Killing vectors is maximally symmetric in presence of torsion, as long as the torsion tensor satisfies the above two conditions (41) and (46).

As to the maximal symmetry of subspaces of a bulk space-time involving torsion, the arguments are be similar to the above. However, one requires the prior assumption that the bulk metric is decomposed exactly in the same way as in GR, viz. the equation (10) holds. We of course make this assumption here, without attempting the rigorous proof of equation (10) in presence of torsion\(^4\).

Let us now turn our attention to the conditions (41) and (46), and see to what extent they can constrain the torsion tensor components. In the following two subsections, we shall treat separately the cases of (i) the entire (bulk) space-time being maximally symmetric, and of (ii) the maximally symmetric subspaces of the bulk.

4.1. Constraints on torsion due to the maximal symmetry of the bulk

When the $d$-dimensional bulk space-time admits the maximum number of $d(d + 1)/2$ independent Killing vectors $\xi^A$, which are of course arbitrary, the condition (41) implies that the torsion tensor should be antisymmetric in the first two indices, i.e. $T_{ABC} = -T_{BAC}$. But the torsion tensor is antisymmetric in its last two indices as well. So one infers that it should be completely antisymmetric: $T_{ABC} = T_{[ABC]}$. Moreover, recalling that maximal symmetry implies the Killing vectors $\xi_A$ be chosen such that at a given point $X$, they vanish and their covariant derivatives $\tilde{\nabla}_B \xi_A$ are arbitrary (and of course antisymmetric because of the Killing equation (37)). So, at $X$, the left hand side of the condition (46) could be made to vanish, which means that on the right hand side the coefficient of the antisymmetric tensor $\tilde{\nabla}_N \xi_M$ is symmetric in $N$ and $M$:

$$\delta^N_C T_M^{AB} + \delta^N_A T_B^{MC} + \delta^N_B T_M^{CA} = \delta^M_C T^N_{AB} + \delta^M_A T^N_{BC} + \delta^M_B T^N_{CA}. \quad (49)$$

This holds everywhere as the point $X$ is also arbitrary. Contraction of $M$ with $C$ yields

$$(d - 3)T^N_{AB} = \delta^N_A T_B - \delta^N_B T_A. \quad (50)$$

As $T_{ABC}$ is completely antisymmetric, its trace $T_A = T^B_{AB}$ is zero, i.e. the right hand side of equation (50) vanishes. Therefore, $T_{ABC}$ could be non-vanishing only when the bulk has dimensionality $d = 3$. Moreover, $T_{ABC}$ is a constant since it cannot depend on any of the maximally symmetric bulk coordinates.

This is of course a known result, which has been demonstrated in different contexts previously [40–42]. We, in this section, have taken the route of [41] in which the authors have made a comprehensive study of maximal symmetry in presence of a completely antisymmetric torsion, i.e. the one for which equation (41) is satisfied automatically. However, the condition (46) which we find here is not the same as the conditions imposed in [41] on the completely antisymmetric torsion due to the maximal symmetry of the entire space-time. In fact, the authors in [41] have demanded that the second term on the right hand side of equation (42) should vanish altogether, and also the part $R^M_{\ CAB}$ of $\tilde{\nabla}^M_{CAB}$ should have the same cyclicity property as exhibited by the Riemann curvature tensor $R^M_{CAB}$ (viz. Equation (5)). But these

\(^4\) For the proof of equation (10) in a torsionless scenario, see [36].
restrictions on torsion are not essential for obtaining equation (45) and carrying out the analysis thereafter in a similar manner as in GR. What is sufficient is the condition (46) that we have here.

4.2. Constraints on torsion in maximally symmetric subspaces of the bulk

When the \(d\)-dimensional bulk space-time is not maximally symmetric on the whole, but can be decomposed into subspaces of dimensionality say \(n\) \((<d)\) which are maximally symmetric, the torsion tensor can be constrained in the following way.

Adopting the same notations and conventions as in section 3, we have the only surviving Killing vectors to be \(\xi_\mathfrak{p}\), with \(\mathfrak{p}\) taking \(n\) values corresponding to the coordinate labels of an \(n\)-dimensional maximally symmetric sub-manifold \(\mathcal{M}\). The total number of linearly independent such Killing vectors \(\xi_\mathfrak{p}\) is \(n(n+1)/2\). The Killing vectors \(\xi_\mathfrak{q}\) are all zero. Also, as mentioned above, we assume that the metric \(g_{\mathfrak{A}\mathfrak{B}}\) of the \(d\)-dimensional bulk manifold \(\mathcal{M}\) is decomposed as in equation (10), so that the elements \(g_{\mathfrak{p}\mathfrak{q}}\) do not exist, and the Killing vectors \(\xi_\mathfrak{p}\) are functions only of the coordinates \(\vec{\mathfrak{p}}\) of \(\mathcal{M}\), i.e. \(\xi_\mathfrak{p} = \xi_\mathfrak{p}(\vec{\mathfrak{p}})\). The Killing equation is now required to be given by

\[
\tilde{\nabla}_\mathfrak{q}\xi_\mathfrak{p} + \tilde{\nabla}_\mathfrak{p}\xi_\mathfrak{q} = 0, \tag{51}
\]

as a generalization of \(\nabla_\mathfrak{q}\xi_\mathfrak{p} + \nabla_\mathfrak{p}\xi_\mathfrak{q} = 0\) when there was no torsion. Thus, instead of equation (41) we have the condition

\[
(T_{\mathfrak{p}\mathfrak{q}} + T_{\mathfrak{q}\mathfrak{p}}) \xi_\mathfrak{r} = 0. \tag{52}
\]

Moreover, among the quantities \(\nabla_\mathfrak{A}\xi_\mathfrak{P}\), the existent ones are \(\tilde{\nabla}_\mathfrak{q}\xi_\mathfrak{p}\). So equation (46) becomes

\[
(\mathcal{R}^{ABC}_{\mathfrak{A}} + \mathcal{R}^{BCA}_{\mathfrak{A}} + \mathcal{R}^{CAB}_{\mathfrak{A}})\xi_\mathfrak{p} = - (\delta_\mathfrak{c}^\mathfrak{a} T_\mathfrak{a}^{\mathfrak{p}\mathfrak{q}} \mathfrak{B} + \delta_\mathfrak{a}^\mathfrak{q} T_\mathfrak{q}^{\mathfrak{p}\mathfrak{c}} \mathfrak{B} + \delta_\mathfrak{b}^\mathfrak{p} T_\mathfrak{b}^{\mathfrak{c}\mathfrak{a}} \mathfrak{C}) \tilde{\nabla}_\mathfrak{p}\xi_\mathfrak{q}. \tag{53}
\]

Once again, we can make the choice that at a given point \(\vec{n} = \overline{U}\), \(\xi_\mathfrak{p}\) vanishes and \(\tilde{\nabla}_\mathfrak{p}\xi_\mathfrak{q}\) is an arbitrary antisymmetric tensor. Therefore, the coefficient of \(\tilde{\nabla}_\mathfrak{p}\xi_\mathfrak{q}\) is symmetric at \(\overline{U}\) (and of course, everywhere, since the point \(\overline{U}\) is also arbitrary):

\[
\delta_\mathfrak{c}^\mathfrak{a} T_\mathfrak{a}^{\mathfrak{p}\mathfrak{q}} \mathfrak{B} + \delta_\mathfrak{a}^\mathfrak{q} T_\mathfrak{q}^{\mathfrak{p}\mathfrak{c}} \mathfrak{B} + \delta_\mathfrak{b}^\mathfrak{p} T_\mathfrak{b}^{\mathfrak{c}\mathfrak{a}} \mathfrak{C} \equiv \delta_\mathfrak{c}^\mathfrak{a} T_\mathfrak{a}^{\mathfrak{q}\mathfrak{p}} \mathfrak{C} + \delta_\mathfrak{a}^\mathfrak{q} T_\mathfrak{q}^{\mathfrak{c}\mathfrak{p}} \mathfrak{C}. \tag{54}
\]

This is analogous, but not identical, to equation (13) for the form-invariance of torsion under maximal symmetry defined conventionally in scheme I (see section 3). Accordingly, the constraints on torsion here may differ in general from those obtained in section 3. Let us work out these constraints by applying the equations (52) and (54) on each of the six independent torsion tensor components \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{r}}\), \(T_{\mathfrak{p}\mathfrak{r}\mathfrak{q}}\), \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{a}}\), \(T_{\mathfrak{a}\mathfrak{b}\mathfrak{p}}\), \(T_{\mathfrak{a}\mathfrak{p}\mathfrak{b}}\) and \(T_{\mathfrak{b}\mathfrak{a}\mathfrak{p}}\).

- For \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{r}}\): the condition (52) implies antisymmetry in the first two indices (as \(\xi_\mathfrak{r}\) is arbitrary).
- But \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{r}}\) is antisymmetric in the last two indices as well. So, we infer that it should be completely antisymmetric, i.e. \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{r}} = T_{\mathfrak{q}\mathfrak{p}\mathfrak{r}}\). Moreover, the condition (54) suggests the dimensionality of the maximally symmetric sub-manifold to be \(n = 3\), in the same way as in the previous subsection. Hence, one can express

\[
T_{\mathfrak{p}\mathfrak{q}\mathfrak{r}} = \epsilon_{\mathfrak{p}\mathfrak{q}\mathfrak{r}} \beta(v), \quad (n = 3 \text{ only}), \tag{55}
\]

where \(\beta(v)\) is a pseudo-scalar.

- For \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{a}}\): the condition (52) is not applicable, whereas the condition (54) gives

\[
\delta_\mathfrak{c}^\mathfrak{a} T_\mathfrak{a}^{\mathfrak{p}\mathfrak{q}} \mathfrak{C} = \delta_\mathfrak{p}^\mathfrak{c} T_\mathfrak{c}^{\mathfrak{q}\mathfrak{a}} + \delta_\mathfrak{q}^\mathfrak{c} T_\mathfrak{c}^{\mathfrak{a}\mathfrak{p}}. \tag{56}
\]

Contracting \(\mathfrak{p}\) with \(\mathfrak{n}\), and using the fact that \(T_{\mathfrak{p}\mathfrak{q}\mathfrak{a}} = -T_{\mathfrak{q}\mathfrak{p}\mathfrak{a}}\), we get

\[
T_{\mathfrak{p}\mathfrak{q}\mathfrak{a}} = -T_{\mathfrak{q}\mathfrak{p}\mathfrak{a}} = \frac{1}{n-2} \delta_{\mathfrak{p}\mathfrak{q}} \delta_{\mathfrak{a}\mathfrak{u}} \beta(v); \quad (\forall n \neq 2), \tag{57}
\]
where $\alpha_a = T^a_{\sigma\tau}$. For $n = 2$ not much can be said about $T^a_{\sigma\tau}$ except that it is trace-free ($\alpha_a = 0$), i.e. at best we can express

$$T^a_{\sigma\tau} = T^a_{\sigma\tau}(v) + Q^a_{\sigma\tau}(v); \quad (n = 2),$$

(58)

where $Q^a_{\sigma\tau}$ is the (pseudo-)trace-free irreducible mode of torsion (see the appendix for general definition) which satisfies the condition $Q^a_{\sigma\tau} + Q^a_{\tau\sigma} + Q^a_{\sigma\tau} = 0$.

- For $T^a_{\sigma\tau}$: the condition (52) is again not applicable, whereas (54) gives

$$T^a_{\sigma\tau} = 0; \quad (\forall n \geq 2).$$

(59)

- For the rest ($T_{ab\sigma}$, $T_{c\sigma\tau}$ and $T_{abc}$): the above conditions (52) and (54) yield nothing, however, once again the component $T_{abc}$ can be expressed as in equation (23), because of the antisymmetry in its last two indices, and of course due to fact that the indices $a$, $b$, $c$ can only take $(d - n)$ values.

We thus see that not all types of components of torsion could be restricted in this scheme.

In the next section, we shall compare these components with those allowed by the scheme I, resorting to some particular cases.

5. A comparison between torsion components allowed in the two schemes

From the analysis for the schemes I and II in the previous two sections, we observe:

- If the entire bulk manifold $M$ is maximally symmetric, then both the schemes allow for a totally antisymmetric torsion $T_{ABC} = T_{[ABC]}$, provided the dimension of the bulk is $d = 3$, so that torsion is determined by only one constant parameter.

- If instead, the maximal symmetry is exhibited only by the sub-manifolds $\mathcal{M}$, then

  - For a totally antisymmetric torsion, the outcome of the two schemes are again the same. Torsion is non-vanishing if the sub-manifold $\mathcal{M}$ dimensionality is either $n = 2$ or $n = 3$. Whereas for $n = 2$ the only surviving component is $T^a_{\sigma\tau}$, for $n = 3$ the only torsion degree of freedom (DoF) is a pseudo-scalar $\beta$ which is a function of the coordinates $v^\sigma$.

  - For a generic torsion (antisymmetric only in a pair of indices) however, the results of the two schemes differ in general. Whereas scheme I constrains all types of independent torsion components except one ($T_{abc}$), scheme II can at most restrict three types $T^a_{\sigma\tau}$, $T^a_{\sigma\tau}$ and $T_{ab\sigma}$.

Let us now consider, as illustrations, some particular cases of physical importance.

5.1. Relevant scenarios in four dimensions

$M$ is the bulk manifold of dimension $d = 4$, with coordinates $x^A := x^0, x^1, x^2, x^3$ (i.e. the bulk indices $A, B, \ldots$ run over 0, 1, 2, 3). We have the following scenarios, for which the allowed torsion tensor components in the schemes I and II are shown in table 1:

5 It should also be noted that following an analysis similar to that in section 3, one can constrain some of the components of the torsion irreducible modes. However, for brevity, we are not showing them here.
Table 1. A comparison of the allowed torsion components in the two schemes, for a given manifold of dimension $d = 4$ with maximally symmetric sub-manifolds of dimension $n = 2, 3$. All the components are in general functions of the $(d - n)$ coordinates, i.e. of $x^0, x^1$ for $n = 2$, and of $x^0$ for $n = 3$.

| Sub-manifold dimensionality | Scheme I | Scheme II |
|-----------------------------|----------|-----------|
|                             | Allowed components | DoF | Allowed components | DoF |
| $n = 2$                     | Type $T_{abc}$: $T_{220} = T_{330} = -\frac{1}{2} \alpha_0$, $T_{221} = T_{331} = -\frac{1}{2} \alpha_1$, $T_{330} = T_{[230]}$, $T_{331} = T_{[231]}$. | 4 | Type $T_{abc}$: $T_{230}, T_{220}$, $T_{231}, T_{221}$, $T_{330} = T_{[230]}$, $T_{331} = T_{[231]}$. | 4 |
| $[a, b, \ldots = 0, 1]$    | Type $T_{abc}$: $T_{[001]}, T_{[110]}$. | 2 | Type $T_{abc}$: $T_{[001]}, T_{[110]}$. | 2 |
| $n = 3$                     | Type $T_{abc}$: $T_{121} = T_{[123]} = \beta$ | 1 | Type $T_{abc}$: $T_{123} = T_{[123]} = \beta$ | 1 |
| $[a, b, \ldots = 0]$       | Type $T_{abc}$: $T_{110} = T_{220} = T_{330} = -\frac{1}{2} \alpha_0$. | 1 | Type $T_{abc}$: $T_{110} = T_{220} = T_{330} = \alpha_0$. | 1 |
| $[a, b, \ldots = 1, 2, 3]$ | Type $T_{abc}$: $T_{[001]}, T_{[002]}, T_{[003]}$. | 3 | Type $T_{abc}$: $T_{[001]}, T_{[002]}, T_{[003]}$. | 3 |

(i) The maximally symmetric sub-manifolds $\mathcal{M}$ are of dimension $n = 2$, with coordinates $\vec{\pi}^0 := x^2, x^3$ say (i.e. the indices $\vec{\pi}, \vec{b}, \ldots = 2, 3$). Then $d - n = 2$, and the coordinates $\nu^a := x^0, x^1$ (i.e. the indices $a, b, \ldots = 0, 1$).

Example: spherically symmetric space-time ($x^0, x^1 = t, r$ and $x^2, x^3 = \vartheta, \varphi$).

(ii) The maximally symmetric sub-manifolds $\mathcal{M}$ are of dimension $n = 3$, with coordinates $\vec{\pi}^0 := x^1, x^2, x^3$ say (i.e. the indices $\vec{\pi}, \vec{b}, \ldots = 1, 2, 3$). Then $d - n = 1$, and the coordinate $\nu^a := x^0$ (indices $a, b, \ldots = 0$).

Example: homogeneous and isotropic space-time ($x^0 = t$ and $x^1, x^2, x^3 = r, \vartheta, \varphi$).

We observe that the torsion DoF are in general different in the two schemes. In fact, scheme II allows more torsion DoF than scheme I for both the cases $n = 2$ and $n = 3$. Even when the number of DoF for the components of a particular type are the same in the two schemes, the components themselves are different. For instance, there are four allowed components of the type $T_{abc}$ for $n = 2$ in either scheme, but these components are not the same. For $n = 3$ also, the allowed components of the type $T_{abc}$ are $T_{110} = T_{220} = T_{330}$ in both the schemes, but the values of these components are different. So the schemes I and II are not in general equivalent. However, there is an interesting point to note. If the torsion tensor is completely antisymmetric in its indices, then for $n = 2$ both the schemes allow two DoF and the components are also the same, viz. $T_{230}$ and $T_{231}$. Similarly, for $n = 3$ we have only one torsion DoF, viz. the pseudo-scalar $\beta$ (which is a function of $x^0$), in both the schemes. Thus, with the additional property of complete antisymmetry in the indices, the torsion tensor apparently does not distinguish between the schemes I and II. The reason for this could be traced to the fact that a completely antisymmetric torsion covariantly preserves the Killing equation (see equation (41)), although it alters the equations relevant for the integrability of the latter.
5.2. A higher dimensional example

For simplicity, let us consider the following:

- \( M \): bulk manifold of dimension \( d = 5 \), coordinates \( x^A := x^0, x^1, x^2, x^3, y \) (indices \( A, B, \ldots = 0, 1, 2, 3, y \)).
- \( \mathcal{M} \): maximally symmetric sub-manifold of dimension \( n = 4 \), coordinates \( \bar{u}^a := x^0, x^1, x^2, x^3 \) (indices \( a, b, \ldots = 0, 1, 2, 3 \)).
- \( M/\mathcal{M} \): quotient manifold of dimension \( d - n = 1 \), coordinate \( v^a := y \) (indices \( a, b, \ldots = y \)).

The ‘extra’ (fifth) coordinate \( y \) is presumably compact, and the chosen scheme of compactification may be, for example, the Randall–Sundrum (RS) \( S^1/\mathbb{Z}_2 \) orbifolding [44].

The minimal version of the two-brane RS model assumes the bulk geometry to be anti-de Sitter, with the hidden and the visible branes located at two orbifold fixed points \( y = 0 \) and \( y = r_c \pi \) respectively, \( r_c \) being the brane separation. We can consider this to be true here as well, alongwith the supposition that torsion co-exists with gravity in the bulk. The RS five dimensional line element

\[
d s^2 = e^{-2\sigma(y)} \eta_{\bar{u}u}(\bar{u}) \, d\bar{u}^ad\bar{u}^b + dy^2, \tag{60}
\]
describes a non-factorizable geometry with an exponential warping, given by the warp factor \( \sigma(y) \), over a four dimensional flat (Minkowski) metric \( \eta_{\bar{u}u}(\bar{u}) \). One can see that equation (60) shows a structural breakup similar to that in equation (10) for the metric of any given manifold (say of dimension \( d = 5 \)) with a maximally symmetric sub-manifold (say of dimension \( n = 4 \)). The objective of the RS model is to provide a resolution to the well known fine-tuning problem of the Higgs mass against radiative corrections due to the gauge hierarchy. In the usual (torsionless) picture, the solution for the warp factor \( \sigma(y) \) turns out to be linear in \( |y| \) [44]. Therefore, applying the boundary conditions one finds that the four dimensional Planck mass \( M_p \) is related to the five dimensional Planck mass \( M \) as

\[
M_p^2 = \frac{M^3}{k} (1 - e^{-2k r_c \pi}), \quad [k \sim M]. \tag{61}
\]

That is, the Planck-electroweak hierarchy could effectively be made to subside on the visible brane (our observable four dimensional world) by appropriate adjustment of the parameters in the exponential factor \( e^{-2k r_c \pi} \). In fact, setting \( k r_c \simeq 12 \) achieves the desired stabilization of the Higgs mass. In presence of the bulk torsion however, the solution for the warp factor would be altered. That is torsion would back-react on the RS warping. Such a backreaction would have its immediate effect on the stability of the RS model, in the sense that the existent torsion DoF would describe the dynamics of the radion, i.e. the field which governs the fluctuations in the brane separation \( r_c \) [45]. Now, the torsion tensor components that can take part in the backreaction, and as such in the radion stabilization, are shown in table 2 for the schemes I and II.

We clearly see that neither of the schemes allow for a completely antisymmetric torsion. However, for a generic torsion (antisymmetric in the last two indices), one DoF is allowed in scheme I allows whereas scheme II allows as many as \( 1 + 4 + 6 = 11 \) DoF. So, the warping (and hence the overall aspect of, for e.g., the radion stabilization) is expected to be affected in different ways for the two schemes\(^6\).

\(^6\) It is worth mentioning here that one should not get confused with the result in [13] that the massless mode arising from a bulk torsion field is heavily suppressed by the exponential RS warping in the visible brane. The authors in [13] assumed torsion to be completely antisymmetric (being induced by the KR field in a string-inspired picture) and did not consider restricting it on account of maximal symmetry exhibited by the four dimensional flat (Minkowski) sub-manifold.
Table 2. A comparison of the allowed torsion components in the two schemes, for a given manifold of dimension $d = 5$ with a maximally symmetric sub-manifold of dimension $n = 4$. All the components are in general functions of the coordinate $y$.

| Sub-manifold dimensionality | Scheme I                          | Scheme II                         |
|-----------------------------|-----------------------------------|-----------------------------------|
|                             | Allowed components DoF            | Allowed components DoF            |
| $n = 4$                     | $T_{00} = T_{11} = T_{22}$, $T_{33} = -\frac{1}{4} \alpha_i(y)$ | $T_{00} = T_{11} = T_{22}$                   |
| $[a, b, \ldots = y]$        | 1                                 | 1                                 |
| $[\bar{a}, \bar{b}, \ldots = 0, 1, 2, 3]$ | $T_{01}, T_{02}, T_{03}, T_{12}, T_{23}, T_{31}$ | $T_{01}, T_{02}, T_{03}$ |

6. Conclusions

We have thus addressed certain conceptual issues related to the basic understanding of symmetries of space-times admitting torsion. In particular, we have concentrated on determining the independent torsion degrees of freedom that are allowed for the preservation of maximal symmetry of either the entire bulk manifold or of its subspaces. This is of importance in implicating torsion’s role in a variety of physical scenarios and in a number of observable phenomena. In fact, one may realize that the whole concept of maximal symmetry deserves a proper clarification in presence of torsion. This has been addressed in a few earlier works [29, 38–42] either in a direct way or in some specific contexts. The ideas put forward may be summed up into two different schemes of implementing the symmetry concepts in space-times with torsion. In the first of these (scheme I), the maximal symmetry of (sub)spaces is supposedly being sensed in the usual way (as in GR), the only demand is that torsion should be form-invariant under the infinitesimal isometries of the metric of such (sub)spaces. We have made a careful examination of the torsion components, thus constrained, in all possible scenarios. Scheme II is more robust as this requires a complete covariant generalization of the GR conception of maximal symmetry in presence of torsion. Under strict enforcement of the minimal coupling prescription ($\nabla \rightarrow \tilde{\nabla}$), such a generalization amounts to the preservation of the Killing equation (which is in general modified by torsion), and the essential outcome of the integrability of the same. Unlike in [41], we have looked for the conditions of absolute necessity to be thus imposed on torsion. Such conditions enabled us to identify the allowed independent torsion components (for the scheme II), which we have compared with those that are allowed in scheme I. Although these components are in general different, we find that in the special cases of a maximally symmetric bulk manifold or (and) a completely antisymmetric torsion, they are identical in the two schemes. In fact, if the entire bulk is maximally symmetric only a completely antisymmetric torsion is allowed, and that also when the bulk dimensionality is only $d = 3$. Thus, at least for the completely antisymmetric torsion, we can uniquely identify its components that can preserve the maximal symmetry of given (sub)spaces.

We have made illustrations of particular cases of physical interest in the context of both four and higher dimensional theories. For the four dimensional bulk space-time, we have explored the relevant cases, viz. maximally symmetric sub-manifolds of dimensionality $n = 2$ and 3. These cases correspond respectively to, for e.g., a spherically symmetric space-time and a homogeneous and isotropic (cosmological) space-time. So our analysis may be useful in examining (say) the viability of the spherically symmetric black-hole solutions in presence
of torsion, or the role of torsion in the context of cosmological inflation or the problem of dark energy. As to the higher dimensional example, we have considered a five dimensional bulk manifold (which admits torsion) with a maximally symmetric four dimensional sub-manifold. The general structure of the bulk metric has resemblance with that of the RS (two-brane model [44], which aims to resolve the fine-tuning of the Higgs mass due to the Planck-electroweak hierarchy. However, the bulk torsion is expected to back-react on the RS warp factor, and we have actually been able to figure out which of the torsion components would be responsible for that. Such a backreaction can have its significance in, for e.g., the radion stabilization [45]. So our analysis of determining the allowed torsion components would enable one to examine the role of torsion (if at all conceivable) in the stability of the RS brane-world.

Some issues remain open in the context of this paper. Firstly, one has to prove rigorously whether the general structural breakup of the metric, viz. equation (10), is indeed valid if one adopts the line of approach of scheme II. That should be a consistency check for the scheme II. Secondly, it has to be verified whether the scheme II at all provides a unique way of integrating the Killing equation in space-times with torsion. That is to say, whether it is absolutely necessary in a covariant generalization in presence of torsion that one should maintain the exact GR analogy at every crucial step. Thirdly, what would happen with relevance to say the non-minimal coupling of the torsion modes to scalar or tensor fields? What would be the consequential effects in cosmology, astrophysics, brane-world scenarios, string-motivated phenomenology? Works are under way to explore some of these issues [17, 18] which we hope to report soon.

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Appendix. General Characteristics of Riemann–Cartan space-time

In a $d$-dimensional Riemannian space-time $(\mathbb{R}_d)$, the formulation of General Relativity (GR) is based on two essential requirements:

(i) symmetry of the affine connection $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$, and

(ii) metricity of the covariant derivative $\nabla_M g_{\alpha\beta} = 0$, where $g_{\alpha\beta}$ is the metric tensor.

By virtue of these, the standard expression of the covariant derivative of any arbitrary tensor $V^{\alpha_1...\alpha_n}_{\beta_1...\beta_m}$, viz.

$$\nabla_M V^{\alpha_1...\alpha_n}_{\beta_1...\beta_m} := \partial_M V^{\alpha_1...\alpha_n}_{\beta_1...\beta_m} + \Gamma^{\alpha_1}_{\rho\sigma} V^{\rho\sigma}_{\beta_1...\beta_m} + \cdots - \Gamma^{\beta_1}_{\rho\sigma} V^{\alpha_1}_{\rho\sigma\beta_2...\beta_m} - \cdots$$  \hspace{1cm} (A.1)

leads to a unique solution for $\Gamma^{\alpha}_{\beta\gamma}$ in the form of the Christoffel symbol

$$\Gamma^{\alpha}_{\beta\gamma} := \frac{1}{2} g^{\alpha\lambda} (\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\beta\lambda} - \partial_\lambda g_{\beta\gamma}).$$  \hspace{1cm} (A.2)

However, referring back to equation (A.1) we see that its left hand side transforms as a tensor even when one adds an arbitrary tensor $K^{\alpha}_{\beta\gamma}$ to any given connection $\Gamma^{\alpha}_{\beta\gamma}$ (which itself is of course not a tensor). That is, there is an ambiguity in the definition of the affine connection right from the beginning, and only the above requirements make the connection uniquely determined in GR. Relaxation of even the first requirement (i.e. symmetry property of the
connection) leads to the formulation of one of the most simplest and natural modifications of GR, in the $d$-dimensional Riemann–Cartan ($U_d$) space-time [1, 2, 4, 5]. Such a space-time is characterized by an asymmetric affine connection $\tilde{\Gamma}^A_{BC}$ which is related to the Christoffel symbol as

$$\tilde{\Gamma}^A_{BC} = \Gamma^A_{BC} + K^A_{BC}. \quad \text{(A.3)}$$

Although this new connection is still non-tensorial, the antisymmetrization of its last two indices gives rise to a tensor, referred to as torsion:

$$T^A_{BC} := 2\tilde{\Gamma}^A_{[BC]} - \tilde{\Gamma}^A_{[CB]} = 2\tilde{\Gamma}^A_{BC} - \tilde{\Gamma}^A_{CB}. \quad \text{(A.4)}$$

Moreover, assuming that the metricity condition would still hold for the new covariant derivatives $\tilde{\nabla}$ in terms of $\tilde{\Gamma}^A_{BC}$, i.e.

$$\tilde{\nabla}^M g_{AB} = 0,$$

one can express the tensor $K_{ABC}$ (known as contorsion) in the form:

$$K^A_{ABC} = \frac{1}{2}(T^A_{BC} - T^A_{CB} - T^A_{BC}). \quad \text{(A.5)}$$

Torsion being antisymmetric in the last two indices, one may verify that the contorsion is antisymmetric in the first two indices

$$T_{ABC} = T_{A[BC]}, \quad K_{ABC} = K_{[ABC]}.$$

The torsion tensor can further be decomposed into three irreducible components as [5, 16, 46]:

$$T_{ABC} = \frac{1}{d-1}(g_{MC}T_B - g_{MB}T_C) + A_{ABC} + Q_{ABC}, \quad \text{(A.7)}$$

where

- $T_C$ is the torsion trace vector, given by
  $$T_C := T^A_{CA} = -T^A_{AC}. \quad \text{(A.8)}$$

- $A_{ABC}$ is the completely antisymmetric part of torsion, given by
  $$A_{ABC} := T_{[ABC]} = \frac{1}{2}(T_{ABC} + T_{BCA} + T_{CAB}). \quad \text{(A.9)}$$

  which in four dimensions ($d = 4$) is expressed as $A_{\alpha\beta\gamma} = \frac{1}{6} \varepsilon_{\alpha\beta\gamma\delta} A^\delta$ ($A^\delta$ being the torsion pseudo-trace).

- $Q_{ABC}$ is the (pseudo-)traceless part of torsion, which satisfies the conditions
  $$Q_{ABC} = Q_{A[BC]}, \quad Q^A_{CA} = 0, \quad Q_{ABC} + Q_{BCA} + Q_{CAB} = 0. \quad \text{(A.10)}$$

Covariant derivatives in the Riemann–Cartan ($U_d$) space-time are analogous to those in the Riemannian ($R_d$) space-time. For any tensor field $V^A..._B...$ the $U_d$ covariant derivative is given by

$$\tilde{\nabla}_M V^A..._B... := \partial_M V^A..._B... + \tilde{\Gamma}^A_{NM} V^N..._B... + \cdots = \tilde{\nabla}_M V^A..._B... + K^A_{NM} V^N..._B... + \cdots - \tilde{\Gamma}^A_{BM} V^{A...}_L... - \cdots,$$

where $\nabla_M$ denotes the Riemannian covariant derivative (in terms of the Christoffel symbols). Such a definition is of course based on the notion of parallel transport along the distinguished curves whose properties are defined solely by the metric or (and) the connection. However, in the Riemann–Cartan space-time one has to make the distinction between two types of such curves, viz. the geodesics and the auto-parallel. The geodesics, or more appropriately the metric geodesics, are the curves which extremize the infinitesimal separation between events, i.e. $ds^2 = g_{AB} dx^A dx^B$, and depend only on the metric properties of the space-time. The equation of a geodesic is the same as that in the Riemannian geometry:

$$u^B \nabla_B u^A = \frac{du^A}{d\lambda} + \Gamma^A_{BC} du^B du^C = 0. \quad \text{(A.12)}$$
where \( \Gamma^A_{BC} \) is the usual Christoffel connection, and \( \lambda \) parameterizes the geodesic, the tangent vectors to which are denoted by \( u^A = dx^A/d\lambda \). The parameter \( \lambda \) could be an affine parameter when it is fixed up to an affine transformation \( \lambda \rightarrow \lambda' = a\lambda + b \) where \( a \) and \( b \) are constants and one may choose \( \lambda \) as \( s \) (or \( \tau \)), the proper distance (or time) for space-like (or time-like) geodesics. However, the geodesics in the Riemann–Cartan (\( U_d \)) space-time are not affine (except in some special cases). For e.g. the functional
torsion (see [31] for further clarification). The auto-parallels, on the other hand, are the curves which transport their tangent vectors parallelly along themselves. In the \( U_d \) space-time, the equation of an auto-parallel curve, parameterized by \( \sigma \), is given by [4, 31]

\[
\frac{Du^A}{d\sigma} = v^B \nabla_B u^A = \frac{dv^A}{d\sigma} + \tilde{\Gamma}^A_{BC} d\sigma^B d\sigma^C = 0,
\]

where \( v^A = dx^A/d\sigma \) denotes the tangent vector to the auto-parallel. The parameter \( \sigma \) is truly affine, since up to an affine transformation it can be identified with the space-time interval \( s \), given by

\[
\left( \frac{d\sigma}{ds} \right)^2 = g_{AB} \frac{dx^A}{ds} \frac{dx^B}{ds}
\]

which remains constant under parallel transport along the auto-parallel curve in the Riemann–Cartan space-time [31]. Hence the auto-parallels are often referred to as the affine geodesics and these are the curves which are of importance in the context of defining the Riemann–Cartan covariant derivatives. It is to be noted that the (metric) geodesics coincide with the affine geodesics. However, the geodesics in the Riemann–Cartan (\( U_d \)) space-time are not affine (except in some special cases). For e.g. the functional
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One may also verify that for any scalar field \( \phi \), we have the following expression

\[
(\nabla_M \nabla_N - \nabla_N \nabla_M) \phi = T^L_{MN} \partial_L \phi,
\]

which implies that torsion could be sensed even by a scalar field. For a vector field \( V^A \), one has

\[
(\nabla_M \nabla_N - \nabla_N \nabla_M)V^A = T^L_{MN} \nabla_L V^A + \tilde{R}^A_{LMN} V^L
\]

where \( \tilde{R}^A_{BCD} \) is the curvature tensor defined in the Riemann–Cartan (\( U_d \)) space-time, in analogy with the Riemannian curvature tensor \( R^A_{BCD} \):

\[
\tilde{R}^A_{BCD} = \partial_C \tilde{\Gamma}^A_{BD} - \partial_D \tilde{\Gamma}^A_{BC} + \tilde{\Gamma}^A_{LC} \tilde{\Gamma}^L_{BD} - \tilde{\Gamma}^A_{LD} \tilde{\Gamma}^L_{BC} - \tilde{\Gamma}^A_{BCD} = R^A_{BCD} + \nabla_C K^A_{BD} - \nabla_D K^A_{BC} + K^A_{LC} K^L_{BD} - K^A_{LD} K^L_{BC}.
\]

The \( U_d \) analogues of the Ricci tensor \( R_{AB} \) and the Ricci scalar curvature \( R \) of the Riemannian geometry, are given respectively by

\[
\tilde{R}_{AB} = \tilde{R}^M_{AMB} = R_{AB} + \nabla_M K^M_{AB} - \nabla_B K_A + K_L K^L_{AB} - K^M_{LB} K^L_{AM},
\]

where \( K_A = R^B_{AB} = T_A \) is the trace of the contorsion tensor.

\[
\tilde{R} = g^{AB} \tilde{R}_{AB} = R - 2\nabla_A K^A - K_A K^A + K_{ABC} K^{ABC},
\]
One may note that unlike $R_{AB}$, the tensor $\tilde{R}_{AB}$ is not symmetric in $A$ and $B$. Moreover, the cyclicity property of the Riemann curvature tensor $R_{ABCD}$ is not preserved for $\tilde{R}_{ABCD}$ in the $U_d$ space-time

$$\tilde{R}_{BCD} + \tilde{R}_{CDB} + \tilde{R}_{DBC} \neq 0.$$ (A.21)

In terms of the irreducible torsions components given above, $\tilde{R}$ is expressed as [16]

$$\tilde{R} = R - 2\nabla_A T^A - \frac{d}{d-1} T_A T^A - \frac{1}{4} A_{ABC} A^{ABC} + Q_{ABC} Q^{ABC},$$ (A.22)

and this is generally taken as the Lagrangian density for gravity (plus torsion) in the Riemann–Cartan space-time.

References

[1] Hehl F W, von der Heyde P, Kerlick G and Nester J 1976 Rev. Mod. Phys. 48 393
Hehl F W, McCrea J D, Mielke E W and Ne’eman Y 1995 Phys. Rep. 258 1
Hehl F W and Obukhov Y N 2001 Lect. Notes Phys. 562 479

[2] Raychaudhuri A K 1979 Theoretical Cosmology (Oxford: Clarendon Press)

[3] Trautman A 1973 Nature Phys. Sci. 242 7

[4] de Sabbata V and Gasperini M 1985 Introduction to Gravitation (Singapore: World Scientific)
de Sabbata V and Sivaram C 1994 Spin Torsion and Gravitation (Singapore: World Scientific)

[5] Shapiro I L 2001 Phys. Rep. 357 113 and references therein

[6] Hammond R T 1991 Gen. Rel. Grav. 23 1195
Hammond R T 2000 Gen. Rel. Grav. 32 2007
Hammond R T 2002 Rep. Prog. Phys. 65 599

[7] Majumdar P and SenGupta S 1999 Class. Quantum Grav. 16 L89

[8] Saa A 1996 Gen. Rel. Grav. 29 205

[9] Kar S, SenGupta S and Sur S 2003 Phys. Rev. D 67 044005
SenGupta S and Sur S 2001 Phys. Lett. B 521 350
SenGupta S and Sur S 2003 J. Cosmol. Astropart. Phys. 12(2003)001
SenGupta S and Sur S 2004 Europhys. Lett. 65 601
SenGupta S and Sinha A 2001 Phys. Lett. B 514 109
Maity D, SenGupta S and Sur S 2005 Eur. Phys. J. C 42 453

[10] Kar S, Majumdar P, SenGupta S and Sinha A 2002 Eur. Phys. J. C 23 357
Kar S, Majumdar P, SenGupta S and Sur S 2002 Class. Quantum Grav. 19 677
Das P, Jain P and Mukherji S 2001 Int. J. Mod. Phys. A 16 4011
Maity D and SenGupta S 2004 Class. Quantum Grav. 21 3379
Rubilar G F, Obukhov Y N and Hehl F W 2003 Class. Quantum Grav. 20 L185–92
Maity D, SenGupta S and Sur S 2005 Phys. Rev. D 72 066012
Alexandre J, Mavromatos N E and Tanner D 2008 Phys. Rev. D 78 066001

[11] Maity D, Majumdar P and SenGupta S 2004 J. Cosmol. Astropart. Phys. JCAP06(2004)005

[12] Chatterjee A and Majumdar P 2005 Phys. Rev. D 72 066013
Bhattacharjee S and Chatterjee A 2011 Phys. Rev. D 83 106007

[13] Mukhopadhyaya B, Sen S and SenGupta S 2002 Phys. Rev. Lett. 89 121101
Mukhopadhyaya B, Sen S and SenGupta S 2002 Phys. Rev. Lett. 89 259902 (erratum)

[14] Sur S, Das S and SenGupta S 2005 J. High Energy Phys. 10(2005)064
Deshpande A and SenGupta S 2005 Phys. Rev. D 71 104005
Ghosh T and SenGupta S 2004 Phys. Rev. D 70 024045
Ghosh T and SenGupta S 2008 Phys. Rev. D 78 124005
Ghosh T and SenGupta S 2009 Phys. Lett. B 678 112
Balakin A B and Ni W-T 2010 Class. Quantum Grav. 27 055003

[15] Mukhopadhyaya B and SenGupta S 1999 Phys. Lett. B 458 8
Mukhopadhyaya B, SenGupta S and Sur S 2002 Mod. Phys. Lett. A 17 43
Mukhopadhyaya B, Sen S, SenGupta S and Sur S 2004 Eur. Phys. J. C 35 129

[16] Helayel-Neto J A, Penna-Firme A and Shapiro I L 2000 Phys. Lett. B 479 411

[17] Sur S Fluctuations over A CDM in metric-scalar-torsion cosmology (in preparation)

[18] Sur S and Bhatia A S Phase plane analysis of dark energy models in metric-scalar-torsion theories (in preparation)
[19] Kostelecky V A, Russell N and Tasson J D 2008 Phys. Rev. Lett. 100 111102
Heckel B R et al 2008 Phys. Rev. D 78 092006

[20] Blagojevic M 2002 Gravitation and Gauge Symmetries (Bristol: Institute of Physics) See for example

[21] Yo H-J and Nester J M 2007 Mod. Phys. Lett. A 22 2057
Nester J M, So L L and Vargas T 2008 Phys. Rev. D 78 044035
Baekler P, Hehl F W and Nester J M 2011 Phys. Rev. D 83 024001

[22] Minkevich A V 2009 Phys. Lett. B 678 423
Minkevich A V, Garkun A S and Kudin V I 2007 Class. Quantum Grav. 24 5835
Minkevich A V 1980 Phys. Lett. A 80 232

[23] Shie K-F, Nester J M and Yo H-J 2008 Phys. Rev. D 78 023522
Li X-Z, Sun C-B and Xi P 2009 Phys. Rev. D 79 027301

[24] Capozziello S, Gonzalez P A, Saridakis E N and Vasquez Y 2013 J. High Energy Phys. JHEP02(2013)039
Cardone V F, Radicella N and Camera S 2012 Phys. Rev. D 85 124007
Capozziello S, Cianci R, Stornaiolo C and Vignolo S 2007 Class. Quantum Grav. 24 6417

[25] Flanagan É E and Rosenthal E 2007 Phys. Rev. D 75 124016

[26] Iorio L and Saridakis E N 2012 Mon. Not. R. Astron. Soc. 427 1555

[27] B¨ohmer C G, Mussa A and Tamanini N 2011 Class. Quantum Grav. 28 215011
Hojman S, Rosenbaum M and Ryan M P 2011 Class. Quantum Grav. 28 215011

[28] Li B, Sotiriou T P and Barrow J D 2011 Phys. Rev. D 83 104017

[29] Minkevich A V, Garkun A S and Kudin V I 2007 Class. Quantum Grav. 24 5835

[30] Sezgin E 1981 Phys. Rev. D 24 1677

[31] Carroll S M and Field G B 1994 Phys. Rev. D 50 3867

[32] Sezgin E 1980 Adv. Stud. Theor. Phys. 2 741
Zecca A 2008 Adv. Stud. Theor. Phys. 2 751

[33] Gangopadhyay D and SenGupta S 2009 Class. Quantum Grav. 26 075005

[34] Kachru S, Schulz M B and Silverstein E 2000 Phys. Rev. D 62 045021
Maity D, SenGupta S and Sur S 2006 Phys. Rev. D 75 107505
Das S, Dey A and SenGupta S 2008 Europhys. Lett. 83 51002
Dey A, Maity D and SenGupta S 2007 Phys. Rev. D 75 107505
Koley R, Mitra J and SenGupta S 2009 Europhys. Lett. 85 41001 and references therein

[35] Maity D, SenGupta S and Sur S 2009 Class. Quantum Grav. 26 055003

[36] Cardone V F, Radicella N and Camera S 2012 Phys. Rev. D 85 124007

[37] Minkevich A V, Garkun A S and Kudin V I 2007 Class. Quantum Grav. 24 5835

[38] Poisson E 2004 A Relativist's Toolkit (Cambridge: Cambridge University Press)

[39] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690

[40] Bloomer I 1978 Gen. Rel. Grav. 9 765

[41] Goldberger W D and Wise M B 1999 Phys. Rev. D 60 107505

[42] Multam¨aki T, Vainio J and Vilja I 2009 Class. Quantum Grav. 26 075005

[43] Poisson E 2004 A Relativist’s Toolkit (Cambridge: Cambridge University Press)

[44] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690

[45] Dey A, Maity D and SenGupta S 2007 Phys. Rev. D 75 107901
Koley R, Mitra J and SenGupta S 2009 Europhys. Lett. 85 41001 and references therein

[46] Capozziello S, Lambiase G and Stornaiolo C 2001 Ann. Phys. 30 713