On the Fractal Dimension of the Visible Universe

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Estimates of the fractal dimension $D$ of the set of galaxies in the universe, based on ever improving data sets, tend to settle on $D \approx 2$. This result raised a raging debate due to its glaring contradiction with astrophysical models that expect a homogeneous universe. A recent mathematical result indicates that there is no contradiction, since measurements of the dimension of the visible subset of galaxies is bounded from above by $D = 2$ even if the true dimension is anything between $D = 2$ and $D = 3$. We demonstrate this result in the context of a simple fractal model, and explain how to proceed in order to find a better estimate of the true dimension of the set of galaxies.

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The value of the (fractal) dimension $D$ of the galaxy distribution in the universe is an important open question in cosmology. Steadily improving observations are available, giving scientists hope that enough data will allow finally to decide the highly debated issue of whether $D$ is 3 or substantially lower (usually stated to be about 2). For example, in the recent book [1] on the “Discovery of Cosmic Fractals” it is emphatically declared that “The megarecords— the cosmic continents, archipelagos and islands— were the news brought home by the modern explorers of the cosmos, exotic, but truths nevertheless about the worlds overseas. Even if the fractal dimension and the maximum scale are still debated, megarecords cry for explanation. Their origin is the number one challenge for cosmological physics.” What these authors refer to are mainly results of fractal analysis of the data sets of galaxies which indicate the fractal dimension $D$ of the set of galaxies is about 2 [2, 3, 4]. In this Letter we build on a recent theorem of fractal mathematics [5] which indicates that there is no contradiction, since measurements of the dimension of the visible subset of galaxies is bounded from above by $D = 2$ even if the true dimension is anything between $D = 2$ and $D = 3$. We demonstrate this result in the context of a simple fractal model, and explain how to proceed in order to find a better estimate of the true dimension of the set of galaxies.

The question is then whether this is really an indication that the set of all galaxies is of dimension $D \approx 2$. We argue first that this may not be the case. In a recent paper [6] the following theorem was established: let $F$ be a fractal set in $\mathbb{R}^3$ with dimension $D > 2$. The visible part of the set $F$ from a point $P$ is the subset $F_V$ of those points lit by a spotlight at $P$. Then the part $F_V$ that is visible to an observer can in general not have a dimension more than 2 [6].

We stress that this result is not about the projection of the set onto the celestial sphere, but about those observations in which the distance of each point (galaxy) is given along with its celestial coordinates (such data sets are called 3-dimensional catalogs). The meaning of the theorem is that it is in fact impossible to determine the dimension of the set of galaxies from measurements of the visible subset if the dimension of the full set is larger than 2. The basic reason for this impossibility is that galaxies “hide” behind each other when the dimension is above 2. This issue will not go away with improving the catalogs. Rather, it will become more and more important as better and better catalogs become available.
We now illustrate some aspects of this problem, and in particular show that there might be some lower bound on the true dimension when taking into account finite size effects (which are absent in mathematical treatments of fractals, but are an evident necessity of any real-life experiment).

We first note that the catalogs provide measurement of the positions of galaxies away from us. In other words, we should consider a relatively small sphere around \( P \) and look with radial rays issuing from the sphere. In [2] it is shown that looking from a plane defines an equivalent problem, and we prefer that formulation. To further simplify the discussion in our examples we will consider a fractal embedded in 2 rather than 3 dimensions, illuminated by rays perpendicular to a randomly given baseline. In Fig. 1 we present a simple model of a fractal universe which is constructed hierarchically. At the \( n \)th level of the construction we see \( 4^n \) balls of size \( \lambda^n \) which are supposed to contain galaxies. At the \((n+1)\)th level each ball is further subdivided to 4 balls of size \( \lambda^{n+1} \). To avoid non-generic effects we rotate the new group of balls with a random angle at each step of construction. Fig. 1 shows the set of balls at the 4th level each ball is further subdivided to 4 balls of size \( \lambda \). The fractal dimension of this example is \( D = -\log 4/\log \lambda \approx 1.51 \).

The last term is smooth and its contribution to the dimension is negligible. Covering the graph with balls leads to the well known result \( D = 2 - |\log a|/\log b \). But, in our analogous case, \( a = \lambda \), \( b = 1/\lambda \), leading to a 1-dimensional graph. Loosely speaking the stretching is not very strong in the \( y \)-direction, and the dimension of the graph (and hence of the visible set) remains 1. In fact, the same argument explains why in dimension \( D < 1 \) the visible part of a fractal has the same dimension as the fractal itself.

To illustrate these issues we consider first a fractal of dimension smaller than 2 (cf. Fig. 2). Here the visible and full sets will have the same dimension, as is demonstrated in Fig. 3, where we plot \( \log C(r) \) vs. \( \log r \) for both \( F \) and 

\[
g(x) = a^{-1}g(bx) - a^{-1}\sin x . \tag{2}
\]
Fig. 3: Upper panel: The graphs of $C(r)$ for the visible part and the full fractal of dimension $D = 0.7$, at level 6. The top curve is the binned number of pairs of points whose pairwise distance falls in the bin (in equal bins on the logarithmic scale) for the full set (multiplied by 2 to shift the curve up). The lower curve is the same for the visible part. The least square fits for the measured dimensions are $D = 0.7095 \pm 0.0036$ and $D = 0.71023 \pm 0.0038$. Lower panel: $C(r)$ for the full fractal versus $C(r)$ for the visible part at level 6. A least square fit gives a slope of 1.0002. Note that this does not at all mean that all disks are visible!

$F_V$ (upper panel). Evidently the slope is the same for these sets. To demonstrate this fact further we present in the lower panel of Fig. 3 a plot of log $C(r)$ for $F$ vs. log $C(r)$ for $F_V$. The slope of this line is unity, stressing the fact that the bulk of $F$ is revealed in the visible subset $F_V$.

The results change qualitatively when the dimension of the set is higher than 2. In the upper panel of Fig. 4 we present the double logarithmic plot of the correlation integral vs distance for $F$ and $F_V$ of the set of dimension 1.5 of Fig. 1. Clearly they do not scale in the same way, with the visible set settling on dimension 1 when larger and larger $r$ are taken into account. This is underlined again by the results shown in the lower panel where the pointwise slopes of the curves in the upper panel are shown. Obviously the correlation integral for $F$ settles nicely on dimension $D \approx 1.5$, whereas the local slope of the correlation integral for $F_V$ tends to $D \approx 1$ as $r$ increases. We stress that subdividing the set further in the hierarchic construction will not cure the problem. Quite on the contrary, it will make the visible set $F_V$ a rela-
tively smaller subset of the full set $F$. Unfortunately going deeper in the hierarchic construction is analogous to studying larger and deeper catalogs, so we cannot expect that newer and better data on the galaxy distributions may automatically cure the problem. We thus conclude that the results of the fractal analysis presented so far do not exclude a homogeneous universe with the fractal dimension of the full set of galaxies being as high as 3.

Lastly, we should investigate whether all is lost, or whether there is a way to probe the true dimension of the full set $F$ from the knowledge of the visible set $F_V$. A modest way out is offered by the observation that the slopes of the curves in the upper panel of Fig. 4 are very close at small distances. This observation is underlined by the pointwise slopes of the curve in Fig. 4 at small distances. This is clearly a finite size effect which can be understood by looking again at Fig. 1. Due to the finite size of the smallest balls at this level of construction, many of the visible balls appear in groups of 4. This is due to the balls that were visible for the previous level of construction (4th in this case), mainly near the edge, but not only, which remain visible also after one step of refinement. With less degree of conviction one can also observe groups of 16 balls, or almost 16 balls, that are visible mainly near the edge. This finite size phenomenon will go away at the present small length scales when we subdivide many steps further, but will remain observable at the smallest available scales forever. This observation rationalizes why we get the “correct” dimension of the full set $F$ from the smaller scales of the correlation integral.

These observations indicate that despite the mathematical impossibility of measuring the true dimension, its value may be gleaned from the behavior of the correlation integral at small scales. We stress that this possibility is not only due to close points (or galaxies) near the visible edge—also points that are far away from the observation line (or point $P$) contribute. Balls that are lighted at level $n$ have high probability to give rise to a full set of lighted balls also in the next level, but not so for many subdivisions. Thus lighted balls will count the “right” dimension only with regard to small pair-wise distances close to where they are. Once we try to count larger pairwise distances we unavoidable run to the problems explained above. Indeed, interestingly enough, it appears that the data analysis presented in [2,3,4] indicates a slight increase in the apparent dimension for smaller scales. We suggest that this increase may very well point to the true answer, namely, that the dimension of the set of galaxies is considerably larger than 2, and maybe even 3-dimensional in agreement with the expectations expressed in [5].

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[6] In fact, there are really two possibilities of which the second is even more dramatic for any measurement than the first: Either the part $F_V$ that is visible to an observer at $P$ can in general not have a dimension greater than 2 or it may happen that the dimension of $F_V$ depends in an essential way on the observation point $P$. All the known examples belong to the first class.
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