Ordered abelian groups over a CW complex

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Abstract
If $X$ is a CW complex, one can assign to each point of $X$ an ordered abelian group of finite rank whose subset of positive elements depends continuously on the points of $X$. A locally trivial bundle which arises in this way we denote by $E_X$. In the present work we establish a topological classification of such bundles in terms of the first cohomology group of $X$ with coefficients in the ring $\mathbb{Z}_2$.

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Introduction
In this note we prove the following theorem.

**Theorem** Let $X$ be a finite CW complex. If $E_X$ is a bundle of the totally ordered abelian groups of rank $k$ over $X$, then $E_X$ is classified by elements of the first cohomology group $H^1(X; \mathbb{Z}_2)$. (See Section 2 for an exact formulation of this theorem.)

An interest to $E_X$ arises from the study of characteristic classes of regular foliations on the hyperbolic manifolds, see Section 4 of this note. Let us briefly remind this link.

To a noncommutative ring $R$ one can relate a semi-group of equivalence classes of the finitely generated projective modules over $R$. The completion of this semi-group to an abelian group is known as the Grothendieck, or $K_0$-group, of $R$. 

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To ‘restore’ the ring $R$ from a $K_0$-group one is obliged to introduce an extra structure on this group, called the order. Roughly speaking, the order on $K_0$ takes care of the initial semi-group of $K_0$. This order can be total or partial.

Let $X$ be a finite CW complex endowed with a foliation, $\mathcal{F}$. The $C^*$-algebra of A. Connes of $\mathcal{F}$ has natural structure of a noncommutative ring, $R$. The $K_0$-group of $R$ is isomorphic to the first Betti group of $X$. This fact was established in [5], see also Ch. 10 of [6]. The order on $K_0$ comes from the asymptotic behaviour of $\mathcal{F}$ at the universal cover of $X$.

The homotopy class of $\mathcal{F}$ consists of foliations obtained by continuous rotation of the tangent planes that preserves the Frobenius (integrability) condition. There exist a parametrization of the family of such foliations by the points of $X$ (Section 4). Thus, we have a bundle, $E_X$, of ordered abelian groups whose ‘positive cone’ depends continuously on the points $x \in X$. The problem of classification of such bundles has an independent interest and intrinsic beauty.

Theorem 1 of Section 2 establishes a classification of $E_X$ in terms of the cohomology invariants of $X$. Surprisingly, these invariants coincide with the elements of first cohomology group of $X$ with coefficients in the ring $\mathbb{Z}_2$. One of applications of this theorem is calculation of the characteristic classes of regular foliations (Corollary 2). The characteristic classes have been extensively studied in the past by R. Bott, A. Haefliger, J. Mather, W. Thurston and others, see a remarkable survey of Lawson [3]. This problem is known to be unsolved in the majority of cases, cf Haefliger p. 193 of [2].

The paper is organized as follows. In Section 1 we fix notation and terminology to be used throughout the paper. We formulate and prove the main result in Sections 2 and 3, respectively. An application of Theorem 1 to the characteristic classes of regular foliations is discussed in Section 4. An Appendix on the Stiefel-Whitney classes of vector bundles is attached (Section 5). Finally, in Section 6 we suggest a generalization of Theorem 1 and Corollary 2.

1 Notation

The partially ordered abelian groups make a category which is in a sense dual to the category of noncommutative rings. This duality appears if instead of usual isomorphisms between the rings one considers rings which are Morita equivalent. The left (right) projective $R$-modules generate an abelian (Grothendieck) group with an order structure. This order structure defines a ring $R$ up to the Morita equivalence. An excellent introduction to the area is the monograph of Goodearl [1].

1.1 Ordered abelian groups with interpolation

Let $G$ be an additive abelian group. A partial order on $G$ is any reflexive, antisymmetric, transitive relation $\leq$ on $G$. If any pair $x, y \in G$ is comparable by this relation, the order is called total. $G$ is a partially ordered abelian group.
if for any $x, y, z \in G$ such that $x \leq y$ it follows that $x + z \leq y + z$. The set of all positive elements $G^+ \subset G$ is a cone, i.e. a subset of $G$ containing 0 and closed under the addition. If $G$ is a totally ordered abelian group, then

$$G = G^+ \cup (-G^+).$$

An order unit in a partially ordered abelian group $G$ is a positive element $u \in G^+$ such that for any $x \in G$ there exists a positive integer $n$ such that $x \leq nu$. Usually there are more than one choices of the order unit in a fixed group $G$. The partially ordered abelian group $G$ with an order unit $u$ is denoted by $(G, u)$.

Ideals of ordered groups. Let $H \subseteq G$ be a subgroup of a partially ordered abelian group $G$. $H$ is an order ideal in $G$ if for every $x, y \in H$ and $z \in G$ such that $x \leq z \leq y$, it follows that $z \in H$. The family of order ideals in a partially ordered abelian group is a complete lattice under the relation of inclusion (Corollary 1.10 of [1]). If $G$ has no order ideals except $\{0\}$ and $G$, then $G$ is called simple.

**Lemma 1** For any order ideal $H \subseteq G$ the factor-group $G/H$ is a partially ordered abelian group with the positive cone

$$(G/H)^+ \defeq (G^+ + H)/H.$$ (2)

**Proof.** See Goodearl [1]. □

Interpolation and Elliott groups. The interpolation and Elliott groups represent two valuable classes of the partially ordered abelian groups. They were introduced in connection with the study of linear operators in the Hilbert space by F. Riesz and classification of the AF $C^*$-algebras by G. A. Elliott, respectively.

The partially ordered abelian group $G$ is called an interpolation group if $G$ satisfies the Riesz interpolation property: for any $x_1, x_2, y_1, y_2 \in G$ such that $x_1 \leq y_1, x_2 \leq y_1, x_1 \leq y_2, x_2 \leq y_2$, there exists $z \in G$ such that $x_1 \leq z \leq y_1, x_2 \leq z \leq y_1$ and $x_2 \leq z \leq y_2$. In other words, there exist infinitely many elements of the order structure lying ‘between’ any two elements. The interpolation property is hereditary with respect to the order ideals $H$ and their quotients $G/H$ (Proposition 2.3 of [1]).

$G$ is called unperforated if for any positive integer $n$ and the element $nx \geq 0$ it holds $x \geq 0$. By the dimension (Elliott) group one understands an unperforated partially ordered abelian group $G$ satisfying the Riesz interpolation property. The order ideals and quotients of the Elliott groups are also the Elliott groups (Proposition 3.1 of [1]).

Simplicial groups. The simplicial groups are opposite to the simple partially ordered abelian groups. Such groups have an abundance of the order ideals. On the other hand every Elliott group can be represented as the direct limit of a sequence of the simplicial groups.
A simplicial group is a partially ordered abelian group $\mathbb{Z}^n$ whose positive cone consists of the vectors with nonnegative components:

$$G^+ = \sum_{i=1}^{n} \mathbb{Z}^+ x_i,$$

where $X = \{x_1, \ldots, x_n\}$ is a basis in $\mathbb{Z}^n$. Every subset of $X$ generates an order ideal in the simplicial group (Proposition 3.8 of [1]). On the other hand, there exist infinitely many possibilities to take a basis in $\mathbb{Z}^n$ so is the number of the order ideals in $G$.

**Lemma 2** Any countable Elliott group is order isomorphic to a direct limit of a countable sequence of simplicial groups.

*Proof*. See Goodearl [1]. □

### 1.2 $S(G, u)$

Let $(G, u)$ be a partially ordered abelian group with the order unit. By a state on $(G, u)$ one understands a normalized positive homomorphism from $(G, u)$ to the reals:

$$s : G \rightarrow \mathbb{R}, \quad s(G^+) \subseteq \mathbb{R}^+, \quad s(u) = 1.$$

The set of all states on $(G, u)$ is called a state space and is denoted by $S(G, u)$. This is known to be a compact convex set in the space $\mathbb{R}^G$ of all functions from $G$ to $\mathbb{R}$. As a convex set $S(G, u)$ is determined by the set of extreme points and faces that correspond to certain states on $G$. Much of the theory of partially ordered abelian groups can be formulated in terms of the compact convex sets, cf Goodearl [1].

**Discrete states.** The homomorphism $s$ defines an additive subgroup in the group of real numbers. This subgroup can be either a cyclic group or a dense subset of $\mathbb{R}$ (Lemma 4.21 of [1]). By a discrete state one understands a state whose $s(G)$ is a cyclic subgroup of $\mathbb{R}$. A link between the discrete states and simplicial groups is given by the following lemma.

**Lemma 3** Let $s$ be a state on the interpolation group $(G, u)$. The set $H = \{x - y \mid x, y \in (\text{Ker } s)^+\}$ is an order ideal of $G$. If $s$ is a discrete state, then $G/H$ is a simplicial group.

*Proof*. See Goodearl [1]. □

The extreme points of the space $S(G, u)$ can be the discrete states. By a rational convex combination of the extreme points $x_1, \ldots, x_n$ one understands a linear combination $\alpha_1 x_1 + \ldots + \alpha_n x_n$ such that $\alpha_i$ are non-negative rationals and $\sum_{i=1}^{n} \alpha_i = 1$. Every discrete state on $(G, u)$ can be presented as a rational convex combination of the discrete extreme states (Proposition 6.22 of [1]).
The space \( \text{Aff} \ S(G, u) \). The state space \( S(G, u) \) has been derived from a partially ordered set. There exists a dual functor which ‘returns’ \( S(G, u) \) to a partially ordered space, \( \text{Aff} S(G, u) \), consisting of affine continuous real-valued functions on \( S(G, u) \). This functor gives a useful ‘representation’ of the group \( (G, u) \).

The mapping \( f : K_1 \rightarrow K_2 \) between two compact convex sets \( K_1 \) and \( K_2 \) is said to be affine if it preserves the convex combinations of points of \( K_1 \). Let \( K_1 = S(G, u) \) and \( K_2 = \mathbb{R} \). A space \( \text{Aff} S(G, u) \) is defined as the space of all affine continuous real-valued functions on \( S(G, u) \).

Let us define a map \( \hat{x} : S(G, u) \rightarrow \mathbb{R} \) so that \( \hat{x}(s) = s(x) \) for all states \( s \in S(G, u) \). There is no difficulty to check that \( \hat{x} \) is affine and continuous mapping on \( S(G, u) \). The function

\[
\phi : G \rightarrow \text{Aff} S(G, u),
\]

acting by the rule \( \phi(x) = \hat{x} \), is known as a natural map from \( G \) to \( \text{Aff} S(G, u) \). (Note that \( \phi \) is not necessarily a surjective map.) The following lemma gives a partial answer to a ‘representation problem’ for the partially ordered abelian groups.

**Lemma 4** Let \( (G, u) \) be an interpolation group satisfying a general comparability condition. Let \( \phi \) be the natural map of form (5). Let us set

\[
A = \{ p \in \text{Aff} S(G, u) \mid p(s) \in s(G) \text{ for all discrete } s \in \partial_e S(G, u) \},
\]

where \( \partial_e \) is the boundary of \( S(G, u) \) consisting of the extreme states. Then \( \phi(G) \) is a dense subgroup of \( A \).

**Proof.** See Goodearl [1]. \( \Box \)

### 1.3 Ordered abelian groups of finite rank

The results on ordered groups can be precised if one restricts to the groups whose state space is a Choquet simplex. The representation theory for such groups (Lemma [1]) is relatively full developed area, see Goodearl [1] for the details. In the case \( G \) is a simple Elliott group of finite rank, a complete classification is possible.

**Choquet simplex.** A *Choquet simplex* is a compact simplex whose inductive dimension is allowed to be infinity. For example, if \( (G, u) \) is an interpolation group, then \( S(G, u) \) is a Choquet simplex (Theorem 10.17 of [1]). The fundamental advantage is that any face of such a simplex is a simplex (of lower dimension). In particular, every Choquet simplex of finite dimension coincides with the usual simplex.

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1This condition means that the state space of \( (G, u) \) is ‘rich enough’, see [1] for an exact definition. For example, every totally ordered abelian groups and simplicial groups satisfy this condition.
Until the end of this paragraph we assume that \((G, u)\) is simple (i.e. has only trivial order ideals). In this case any nonzero element of \(G^+\) can be chosen as the order unit \(u\) (Lemma 14.1 of [1]). The ‘majority’ of simple groups cannot afford any discrete states because of the following lemma.

**Lemma 5** Let \((G, u)\) be a simple interpolation group. Then the following three conditions are equivalent: (i) there exists a discrete state on \((G, u)\); (ii) \(G\) is a cyclic abelian group; (iii) \((G, u)\) is order isomorphic to \((\mathbb{Z}, m)\) for some \(m \in \mathbb{N}\).

**Proof.** See Goodearl [1]. □

As a corollary of Lemma 5 one obtains the representation theory of simple interpolation groups without perforation (Elliott groups). Namely, the image \(\phi(G)\) of the natural map \((\mathbb{R}, u)\) is a dense subgroup of \(\text{Aff}(G, u)\) and \(\phi(G^+)\) is dense in \(\text{Aff}(G, u)^+\) (Corollary 13.7 and Theorem 14.14 of [1]).

To obtain a consistent classification one imposes restrictions either on the state space \(S(G, u)\) or on the group \(G\) itself. One reasonable assumption is that \(S(G, u)\) is finite-dimensional. In this case the space \(\text{Aff}(G, u)\) is isomorphic to \(\mathbb{R}^k\) for some \(k\) and the classification of \((G, u)\) is reduced to classification of all dense subgroups of \(\mathbb{R}^n\), cf Goodearl [1].

We take the second option making restrictions on \(G\). For the topological applications \(G\) is the Betti group of a CW complex (Section 4). Such groups are known to be free abelian of a finite rank \(k \geq 2\). Surprisingly, this assumption implies that the state space \(S(G, u)\) is finite-dimensional. (Thus we come back to the first case.)

**Lemma 6 (D. Handelman, unpublished)** Let \((G, u)\) be a simple interpolation group where \(G\) is a free abelian group of finite rank \(k \geq 2\). Then the dimension of \(S(G, u)\) is at most \(k - 2\).

**Proof.** Denote by \(\{x_1, \ldots, x_k\}\) a basis for \(G\) as a free abelian group. Let \(\{p_1, \ldots, p_k\}\) be the set of projections on the corresponding coordinates (i.e. a dual basis). This set is a basis for the real vector space \(\text{Hom}_\mathbb{Z}(G, \mathbb{R})\).

Let to the contrary, the dimension of \(S(G, u)\) is greater or equal to \(k - 1\). Then \(S(G, u)\) contains \(n\) affine independent states which we denote by \(s_1, \ldots, s_k\). It is not hard to see that these states are also linearly independent. Hence the vectors \(\{s_1, \ldots, s_k\}\) span \(\text{Hom}_\mathbb{Z}(G, \mathbb{R})\). Therefore, there exists a nondegenerate \(k \times k\) matrix \((\alpha_{ij})\) with real entries such that \(p_i = \sum_{j=1}^{k} \alpha_{ij} s_j\).

Fix a number \(m > k \max |\alpha_{ij}|\). Since \(G\) is a free abelian group of rank \(k > 1\), \(G\) cannot be cyclic. Then there exists an element \(y \in G\) such that \(y > 0\) and \(m y < u\) (\(u\) is the order unit). Therefore \(0 \leq s_j(y) \leq 1/m\) for any \(j\). Thus for all \(i\) the following inequality is true:

\[
|p_i(y)| \leq \sum_{j=1}^{k} |\alpha_{ij}| |s_j(y)| \leq \sum_{j=1}^{k} \frac{|\alpha_{ij}|}{m} < \sum_{j=1}^{k} \frac{1}{k} = 1.
\]

Since \(p_i(G) = \mathbb{Z}\) for every \(i\) we conclude that \(p_i(y) = 0\) and therefore \(y = 0\). This is a contradiction which proves Lemma 6. □
Corollary 1 Let \((G, u)\) be as in Lemma 4. Then \((G, u)\) is totally ordered if and only if \(S(G, u)\) consists of a unique state. In this case the positive cone \(G^+\) is defined by a hyperplane passing through the origin of the space \(\mathbb{R}^k\).

Proof. If \((G, u)\) is totally ordered then \(S(G, u)\) consists of a point (Corollary 4.17 of [1]). If there exists a unique state \(s : (G, u) \to (\mathbb{R}, 1)\) then \(s\) is given by a linear function \(f : \mathbb{R}^k \to \mathbb{R}\). The set \(f^{-1}(0)\) is a hyperplane in \(\mathbb{R}^k\) which splits \(\mathbb{R}^k\) into two parts. Placing \(G \cong \mathbb{Z}^k\) in \(\mathbb{R}^k\) one gets a total order by the formula (1). □

2 Main result

The space of totally ordered abelian groups of rank \(k\) can be made to a bundle space, \(E_X\), over a CW complex \(X\). Surprisingly enough, the classification of \(E_X\) reduces to classification of certain subbundles of a vector bundle \(\xi\) of rank \(k\) over \(X\). An invariant responsible for such a classification turns out to be the Stiefel-Whitney class \(w_1(\xi)\). (In [7] bundles \(E_X\) were called the characteristic fibrations.)

Let \((G, u)\) be a totally ordered simple abelian group of rank \(k\). By Corollary 1 \((G, u)\) is defined by a hyperplane in the Euclidean space \(\mathbb{R}^k\). Let us denote by \(\mathcal{S}(G, u)\) the set of all such groups of a fixed rank \(k\). If \(V_m(\mathbb{R}^k)\) and \(G_m(\mathbb{R}^k)\) are the Stiefel and Grassmann varieties, respectively, then by \(SG_m(\mathbb{R}^k)\) we understand a subvariety of \(G_m(\mathbb{R}^k)\) consisting of \(m\)-dimensional oriented planes in \(\mathbb{R}\). The manifold \(SG_m(\mathbb{R}^k)\) is known to be a double cover of the Grassmann manifold \(G_m(\mathbb{R}^k)\). There exists a natural function

\[
f : V_m(\mathbb{R}^k) \longrightarrow SG_m(\mathbb{R}^k),
\]

which identifies the \(m\)-frame \((v_1, \ldots, v_m)\) with the linear space \(\langle v_1, \ldots, v_m \rangle\) that it spans and which discerns the ‘left’ and ‘right’ orientation of \(m\)-frames. Mapping \(f\) is surjective since every \(m\)-plane has a frame. The topology of \(SG_m(\mathbb{R}^k)\) coincides with the topology induced by the mapping \(f\).

Let \(m = k - 1\) be a hyperplane (with orientation) in \(\mathbb{R}^k\). By Corollary 1 and discussion of Section 1, to an everywhere dense \(\Omega\) subset, \(\Omega\), of such hyperplanes there correspond a set of totally ordered simple abelian groups of rank \(k\). Taking a closure of \(\Omega\) in \(SG_{k-1}(\mathbb{R}^k)\), one gets an induced topology on the set \(\mathcal{S}(G, u)\). The closure of \(\mathcal{S}(G, u)\) is denoted by \(\bar{\mathcal{S}}(G, u)\). If \(X\) is a CW complex of dimension \(n\), then to every point of \(X\) one can associate a dimension group \((G, u) \in \bar{\mathcal{S}}(G, u)\). In other words, we have a continuous map

\[
X \longrightarrow \bar{\mathcal{S}}(G, u).
\]

\(^2\)In effect, this set is of ‘full measure’ in the space of ordered abelian groups of rank \(k\) (not necessarily simple). The residue set consists of groups with the order ideals. Roughly speaking, such groups correspond to ‘rational directions’ of the hyperplane. Even if the direction is ‘irrational’ it may happen that the corresponding group is not totally ordered. However, such a possibility is also exceptional (i.e. belongs to a residue set). This was shown in our work Geometry of the Bratteli diagrams, Preprint FI-NP2000-002, Fields Inst., 2000.
Map (7) is cross-section of a locally trivial bundle, which we denote by \( E_X \). Globally there exists a variety of topologically distinct bundles \( E_X \), since the pieces \( U \times \bar{S}_k(G, u) \) can be glued together in a variety of ways. The following theorem establishes classification of such bundles.

**Theorem 1** Let \( X \) be a finite CW complex. Then the topological class of bundle \( E_X \) is determined by the elements of group \( H^1(X; \mathbb{Z}_2) \).

**3 Proof of Theorem 1**

The idea of the proof is a stepwise reduction of \( E_X \) to a subbundle of the vector bundle over \( X \) whose topological class is determined by the Stiefel-Whitney invariants. Of course, there is no hope whatsoever that the obtained invariants are sufficient, i.e. that they completely discern the topological classes of \( E_X \). But if \( E_X \) and \( E'_X \) have different set of invariants, then they are topologically distinct.

Let \( X, V_m(\mathbb{R}^k) \) and \( SG_m(\mathbb{R}^k) \) be the manifolds introduced in Section 2. Consider the following commutative diagram:

\[
\begin{array}{ccc}
V_m(\mathbb{R}^k) & \xrightarrow{f} & SG_m(\mathbb{R}^k) \\
\downarrow{p} & & \downarrow{\sigma} \\
X & \xrightarrow{\sigma} & X
\end{array}
\]

where \( p \) is surjective and \( \sigma \) is a cross-section of the bundle \( E_X \), see (7). The map \( p \) is defined so that for any point \( x \in SG_m(\mathbb{R}^k) \), \( p(f^{-1}(x)) = y \in X \) such that \( \sigma(y) = x \). In other words, \( p \) sends the \( m \)-frames which define an oriented \( m \)-plane to the point of \( X \) to which this plane is attached. Of course, map \( p \) is not injective since to each \( m \)-plane in \( SG_m(\mathbb{R}^k) \) one can relate a bunch of \( m \)-frames that span this plane. To make \( p \) injective, one must choose one \( m \)-frame in every \( m \)-plane. The following lemma will be helpful.

**Lemma 7** To every \( m \)-plane \( \alpha \in SG_m(\mathbb{R}^k) \) one can relate a (left) \( m \)-frame of the space \( V_m(\mathbb{R}^k) \) defined uniquely by \( \alpha \).

**Proof.** Denote by \( O_n \) the orthogonal group of the real Euclidean space \( \mathbb{R}^n \). The Stiefel manifold \( V_m(\mathbb{R}^k) \) can be identified with the factor space \( O_k/O_{k-m} \), see e.g. N. Steenrod, *The topology of fibre bundles*, Princeton, N.J., 1951, §7.5-7.10. In this setting, the Grassmann manifold \( G_m(\mathbb{R}^k) \cong O_k/(O_m \times O_{k-m}) \) what means that the Stiefel manifold is a bundle over \( G_m(\mathbb{R}^k) \) with the fibre \( O_m \).
Let us construct a standard $m$-frame in the fibre $O_m$. For that let us decompose $O_m$ into a sequence of inclusions:

$$O_1 \subset O_2 \subset \ldots \subset O_{m-1} \subset O_m.$$  \hfill (8)

Fix a unit vector $v_1 \in \mathbb{R}^1$. The orthogonal group $O_1$ is cyclic of order 2, so it flips $v_1$ relatively the origin. We choose $v_2 \in \mathbb{R}^2$ so that $O_2 \supset O_1$ keeps invariant $v_1$ and complements $v_1$ to a left orthonormal 2-frame. The procedure can be prolonged by induction so that it stops with the unit vector $v_m$. Thus, we have constructed a left $m$-frame which is invariant under the action of group $O_{m+1}$. This frame we call standard and it was obtained from sequence (8) by a Gram-Schmidt process. Lemma is proved. \hfill $\square$

Put $m = k - 1$ in Lemma 7 and apply the commutative diagram that we discussed earlier. Equation (7) in this case takes the form:

$$X \to V_{k-1}(\mathbb{R}^k),$$  \hfill (9)

where $V_{k-1}(\mathbb{R}^k)$ is a Stiefel variety of $(k - 1)$-frames in $\mathbb{R}^k$. Map (8) is the cross-section to a rank $k$ vector bundle, $\mathcal{E}_X$, which admits in total $k - 1$ linearly independent cross-sections over $X$. Two (topologically) distinct bundles $\mathcal{E}_X \neq \mathcal{E}'_X$ will result in two distinct bundles $E_X \neq E'_X$. The following lemma is basic for the classification of bundles $\mathcal{E}_X$.

**Lemma 8** Let $\mathcal{E}_X$ and $\mathcal{E}'_X$ be two rank $k$ vector bundles over a CW complex $X$ which admit $k - 1$ linearly independent cross-sections. If the first Stiefel-Whitney classes $w_1(\mathcal{E}_X)$ and $w_1(\mathcal{E}'_X)$ are distinct, then the bundles $\mathcal{E}_X$ and $\mathcal{E}'_X$ are topologically distinct.

**Proof.** Since the vector bundle $\mathcal{E}_X$ has $k - 1$ linearly independent cross-section, the higher Stiefel-Whitney classes of $\mathcal{E}_X$ vanish (Lemma 12 of Appendix):

$$w_2(\mathcal{E}_X) = w_3(\mathcal{E}_X) = \ldots = w_k(\mathcal{E}_X) = 0.$$  \hfill (10)

Let us show that the remaining class $w_1(\mathcal{E}_X)$ is non-trivial. Indeed, denote by $\eta$ a rank 1 vector bundle over $X$ such that $\mathcal{E}_X \oplus \eta = \varepsilon$, where $\varepsilon$ is a trivial vector bundle of rank $k + 1$. By equations (11) the total Stiefel-Whitney classes of $\mathcal{E}_X$ and $\eta$ are as follows:

$$w(\mathcal{E}_X) = 1 + w_1(\mathcal{E}_X), \quad w(\eta) = 1 + w_1(\eta).$$  \hfill (11)

Since the Whitney sum $\mathcal{E}_X \oplus \eta$ is trivial, one can apply formula (17) (cf. Appendix) so that $w(\eta) = \bar{w}(\mathcal{E}_X)$, where $\bar{w}(\mathcal{E}_X) = 1 + w_1(\mathcal{E}_X) + \ldots$, see Milnor-Stasheff. Thus we have

$$w_1(\mathcal{E}_X) = w_1(\eta) \neq 0,$$  \hfill (12)

since there exists a plenty of non-trivial line bundles over $X$. 

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(Note that the same result can be proved by the methods of obstruction theory, see §5.2. Namely, the Stiefel lemma (Lemma 13) says that for $m = k - 1$ the map $f : X \to V_{k-1}(\mathbb{R})$ can always be continuously extended to a 1-dimensional skeleton, $K^1$, of $X$. This is exactly what equation (12) suggests.)

Lemma is proved. □

To finish the proof of Theorem 1 it remains to apply Lemma 8 and axiom (ii) of the Stiefel-Whitney classes, see §5.2.

Theorem follows. □

4 Characteristic classes of regular foliations

The bundles of ordered abelian groups were of modest interest had they no application outside noncommutative algebra. Fortunately, they arise in the study of characteristic classes of regular foliations. In particular, Theorem 1 defines the characteristic classes of such foliations. This section is reserved for the discussion of some geometric consequences of Theorem 1. For a full treatment, see [7].

4.1 Regular foliations

Let us fix the following notation:

- $X$ $n$-dimensional hyperbolic manifold;
- $k$ first Betti number of $X$;
- $\mathbb{H}^n$ hyperbolic $n$-space $\{x \in \mathbb{R}^n | x_n > 0\}$ endowed with metric $ds = |dx|/x_n$;
- $\partial \mathbb{H}^n$ absolute $\{x \in \mathbb{H}^n | x_n = 0\}$;
- $G$ Möbius group, i.e a discrete subgroup of $SL(2,\mathbb{C})/\pm I$;
- $X = \mathbb{H}^n/G$ representation of $X$ by actions of $G$ at the universal cover $\mathbb{H}^n$;
- $\tau$ tessellation of $\mathbb{H}^n$ by the fundamental domains of group $G$;
- $\mathbb{Z}^k \subseteq \tau$ abelianized fundamental group of $X$, see details in [8].

Let $\mathcal{F}$ be a foliation on $X$ of the codimension 1. Such foliations always exist on 3-dimensional manifolds and in the higher dimensions good examples are known (Lawson [3]). Foliation $\mathcal{F}$ on $X$ will be called regular if it has no Reeb components and either of the following conditions is satisfied:

(i) all leaves of $\mathcal{F}$ are compact, i.e. the limit set of every leaf is the leaf itself;
(ii) all leaves of $\mathcal{F}$ are dense in $X$, i.e. the limit set of every leaf is the whole $X$, or a Cantor set in $X$.

(One can usefully think of regular foliations as a higher-dimensional analog of the Kronecker foliations on the two-dimensional torus. These can either
be “rational”, i.e. with all leaves compact, or “irrational”, so that each leaf is everywhere dense on the torus. An exceptional “Denjoy” case adds to the picture when the leaves tend to a Cantor set.)

Asymptotic of regular foliations. Let \( \mathcal{F} \) be a regular foliation. Denote by \( \tilde{\mathcal{F}} \) the preimage of \( \mathcal{F} \) on the universal cover \( \mathbb{H}^n \). The asymptotic properties of leaves of \( \mathcal{F} \) were studied by S. Fenley, D. Gabai and W. Thurston. They proved that any leaf \( \tilde{l} \) of such foliation is a quasi-isometric immersion into \( \mathbb{H}^n \) and the limit set of \( \tilde{l} \) is either a ‘circle’ \( S^{n-2} \) of the absolute or the whole absolute. (See S. Fenley, *Foliations with good geometry*, J. Amer. Math. Soc. 12, 1999, 619-676.)

4.2 \((\mathbb{Z}^k)^+\)

Every regular foliation generates a total order on the abelianized fundamental group \( \mathbb{Z}^k \) of \( X \). For that take a leaf \( \tilde{l} \) which has the limit ‘circle’ \( S^{n-2} \) at the absolute. This circle defines a geodesic hemisphere \( S^{n-1} \) in the hyperbolic \( n \)-space which is either rational or irrational depending on property (i) or (ii) of \( \mathcal{F} \). A positive cone of \( \mathbb{Z}^k \subseteq \tau \) is defined as

\[
(\mathbb{Z}^k)^+ = \{ z \in \mathbb{Z}_k \mid z \in \text{Int } S^{n-1} \},
\]

where \( \text{Int } S^{n-1} \) denotes the interior points of \( \mathbb{H}^n \) separated by the hemi-sphere \( S^{n-1} \). The group \( (\mathbb{Z}^k, (\mathbb{Z}^k)^+) \) is called an ordered abelian group associated to \( \mathcal{F} \). This group is simple if \( \mathcal{F} \) has property (ii) and has non-trivial order ideals otherwise. A link between \( (\mathbb{Z}^k, (\mathbb{Z}^k)^+) \) and the ring structure of a foliation C*-algebra introduced by A. Connes is given by the following lemma.

**Lemma 9** *The \( K_0 \)-group of the C*-algebra associated to a regular foliation \( \mathcal{F} \) is order-isomorphic to the totally ordered abelian group \( (\mathbb{Z}^k, (\mathbb{Z}^k)^+) \).*

*Proof. See [5]-[7]. □*

4.3 \(E_X\)

Given a regular foliation \( \mathcal{F} \) on the manifold \( X \) one can look at the integrable distribution which is a set of planes tangent to \( \mathcal{F} \) at every point of \( X \). We allow to rotate every plane of this distribution with a single restriction that the newly obtained distributions are integrable and correspond to regular foliations. The set of regular foliations obtained by a continuous rotation from \( \mathcal{F} \) we call a homotopy class of \( \mathcal{F} \). The elements of this homotopy class can be put in bijection with the points of manifold \( X \) as follows.

Consider a restriction of the regular foliation \( \mathcal{F} \) to the unit cylinder \( \text{Cyl}_h = D^{n-1} \times [0, h], ||D|| = 1, h > 0 \) of height \( h \). This restriction looks like a level set of the function \( h = \text{Const} \). Let us rotate the distribution in \( \text{Cyl}_h \) so that all the tangent planes remain parallel. In this way, the rotation is parametrized by a vector \( v \in \mathbb{R}^n \) normal to all planes. We set \( ||v|| = h \). In this way local deformations are parametrized by vectors \( v \) whose endpoints \( x \in \mathbb{R}^n \) we identify
with the points of manifold $X$ covered by a local chart in $0$. Gluing together the local homotopy deformations one gets a global one by identifying deformations on the overlapping domains.

(The reader should firmly understand that even local homotopy deformations change dramatically the structure of foliations. For example, an irrational foliation on the torus can be made rational, and vice versa, by a local modification near any point of torus; see also examples of Ch. Pugh and others.)

**Definition 1** By a bundle $E_X$ over manifold $X$ one understands a continuous field of totally ordered abelian groups over $X$. Given $x \in X$ we relate a group $(\mathbb{Z}^k, (\mathbb{Z}^k)^+)$ if and only this group corresponds to a regular foliation parametrized by $x$. In this way, the homotopy (characteristic) classes of regular foliations correspond to topologically distinct types of $E_X$.

**Corollary 2** (Characteristic classes of regular foliations) Let $X$ be a hyperbolic manifold whose first Betti number does not vanish. Then the characteristic classes of regular foliations on $X$ coincide with the elements of the first cohomology group $H^1(X; \mathbb{Z}_2)$ of $X$.

**Proof.** The manifold $X$ admits a regular triangulation to a CW complex for which we keep the same notation. Since $k \neq 0$, one applies Theorem Corollary follows. □

### 5 Appendix: Stiefel-Whitney invariants

The characteristic classes were created by H. Hopf, E. Stiefel and H. Whitney to extend the Euler characteristic of a manifold. Namely, the number and type of singularities of a vector field is an invariant of the manifold (Euler number). Similarly, the singularities of $n$-tuples of linearly independent vector fields is an invariant of the vector bundle over the manifold (characteristic class). For an introduction in the area we recommend the monograph of Milnor and Stasheff and the original papers of Stiefel and Whitney.

#### 5.1 Stiefel-Whitney class of a vector bundle

We fix the following notation:

- $X$ finite-dimensional CW complex;
- $\xi$ vector bundle of rank $k$ over $X$ with total space $E$;
- $\xi \oplus \eta$ Whitney sum of the vector bundles $\xi$ and $\eta$;
- $V_m(\mathbb{R}^k)$ Stiefel manifold of the orthogonal $m$-frames in the Euclidean space $\mathbb{R}^k$;
- $G_m(\mathbb{R}^k)$ Grassmann manifold of the $m$-dimensional linear subspaces in $\mathbb{R}^k$;
- $H^m(X; G)$ singular cohomology group of $X$ of order $m$ with coefficients in the ring $G$. 

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Let $G \cong \mathbb{Z}_2$ be the cyclic group of order 2. By a Stiefel-Whitney class of the vector bundle $\xi$ one understands a sequence of the singular cohomology classes $\{w_i \in H^i(X; \mathbb{Z}_2) \mid i = 0, 1, 2, \ldots\}$ such that the following axioms are satisfied:

(i) $w_0(\xi)$ is a unit element of the group $H^0(X; \mathbb{Z}_2)$ and $w_i(\xi) = 0$ for $i > k$ where $k$ is the rank of vector bundle $\xi$;

(ii) if a bundle map $\xi \to \eta$ covers the map $X(\xi) \to X(\eta)$ then $w_i(\xi) = f^*w_i(\eta)$;

(iii) cohomology class of the Whitney sum is calculated by the formula

$$w_i(\xi \oplus \eta) = \sum_{j=1}^{i} w_j(\xi) \cup w_{i-j}(\eta); \quad (14)$$

(iv) Stiefel-Whitney class $w_1$ of the line bundle over projective line is non-trivial.

The characteristic classes satisfying axioms (i)-(iv) do exist. A proof of this fact can be found in Milnor-Stasheff [4], §8. The rest of this subsection is devoted to some implications of axioms (i)-(iv).

**Lemma 10** Let $\varepsilon$ be a trivial vector bundle. Then $w_i(\varepsilon) = 0$ whenever $i > 0$.

**Proof.** There exists a bundle map from $\varepsilon$ to a bundle $\eta$ whose base $X$ is a singleton. Lemma follows from axiom (ii). □

**Lemma 11** Let $\varepsilon$ be a trivial vector bundle. Then $w_i(\varepsilon \oplus \eta) = w_i(\eta)$.

**Proof.** This follows from Lemma 10 and formula (14). □

**Lemma 12** Let $\xi$ be vector bundle of rank $k$. If $\xi$ admits $m$ cross-sections $s : X \to E$ which define an $m$-frame at every point $p \in X$, then

$$w_{k-m+1}(\xi) = w_{k-m+2}(\xi) = \ldots = w_k(\xi) = 0. \quad (15)$$

In particular, if $\xi$ has a cross-section, then $w_k(\xi) = 0$.

**Proof.** Let $\xi$ have $m$ linearly independent cross-sections. Then one can decompose $\xi = \varepsilon \oplus \varepsilon'$, where $\varepsilon$ is a trivial bundle of rank $m$ and $\varepsilon'$ is an orthogonal vector bundle of rank $k - m$. By Lemma 11 $w_i(\varepsilon \oplus \varepsilon') = w_i(\varepsilon')$. On the other hand, rank $\varepsilon' = k$. By axiom (i) lemma follows. □

**Ring of invariants.** Formula (14) can be simplified by considering a ring $H^*(X; \mathbb{Z}_2)$

$$H^0(X; \mathbb{Z}_2) \times H^1(X; \mathbb{Z}_2) \times \ldots$$ of formal infinite series $w_0 + w_1 + \ldots$ endowed with the usual (polynomial) multiplication. The sum of two classes of order $m$ denotes the union of two $m$-cocycles and the product of order $m$ and $m'$ classes means an $mm'$-cocycle which is the topological product of $m$- and $m'$-cocycle.

$^3$In particular, if $\xi$ and $\eta$ are topologically equivalent then $w_i(\xi) = w_i(\eta)$. 

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By a total Stiefel-Whitney class of a vector bundle $\xi$ one understands an element $w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \ldots$ of the ring $H^*(X; \mathbb{Z}_2)$. In this notation formula (14) reduces to:

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$  \hfill (16)

Equation (16) can be resolved uniquely relatively $w(\eta)$. For this one introduces an 'inverse' element $\bar{w}(\xi) = 1 + \bar{w}_1 + \bar{w}_2 + \ldots$, where $\bar{w}_1 = w_1$, $\bar{w}_2 = w_2^2 + w_2$, etc, see [4]. In particular, if the bundle $\xi \oplus \eta$ is trivial, then

$$w(\eta) = \bar{w}(\xi).$$  \hfill (17)

### 5.2 Obstruction theory

The Stiefel-Whitney classes appear as a topological ‘obstruction’ to continuous extension of a field of $m$-frames from the skeleton $K^{\nu-1}$ to the skeleton $K^\nu$ of a CW complex $X$. Such an extension is possible if and only if the cohomology classes of certain cocycles in $X$ vanish. The group of coefficients in this cohomology is either $\mathbb{Z}$ or $\mathbb{Z}_2$. Milnor and Stasheff showed that these cohomology classes define (and are completely defined by) the Stiefel-Whitney classes of vector bundles over $X$ (Milnor-Stasheff [4], pp. 139-143).

**Stiefel manifolds.** Let us denote by $V_m(\mathbb{R}^k)$ a set of the orthonormal $m$-frames with centre at the origin of the Euclidean space $\mathbb{R}^k$. The space $V_m(\mathbb{R}^k)$ is endowed with a topology in which two $m$-frames are ‘close’ if and only if the endpoints of the respective basis vectors on the unit sphere $S^{k-1}$ are close. This topology turns $V_m(\mathbb{R}^k)$ to a compact manifold of dimension $m(2k - m - 1)/2$ ([8], p.311) which is called a *Stiefel manifold*. The homotopy and homology groups of $V_m(\mathbb{R}^k)$ are as follows (Stiefel [8], §1.4):

$$\pi_{k-m-1}(V_m(\mathbb{R}^k)) = 0,$$

$$H_{k-m}(V_m(\mathbb{R}^k)) = \begin{cases} 
\mathbb{Z}, & \text{if} \ k - m \text{ is even} \\
\mathbb{Z}_2, & \text{if} \ k - m \text{ is odd.}
\end{cases}$$  \hfill (19)

Let $X$ be a CW complex of dimension $n$. Consider the representation of $X = \cup_{\nu=0}^n K^\nu$ as a sum of $\nu$-dimensional skeletons. The extension of an $m$-frame field $f : K^{\nu-1} \to V_m(\mathbb{R}^k)$ to the skeleton $K^\nu$ depends essentially on extension of $f$ from the boundary $\partial E^\nu$ of elementary cell $E^\nu$ to the whole $E^\nu$. Since $\partial E^\nu$ is homeomorphic to the unit $(\nu-1)$-sphere, the problem is to extend the map

$$f : S^{\nu-1} \to V_m(\mathbb{R}^k),$$  \hfill (20)

to the interior of the ball bounded by $S^{\nu-1}$. The answer to this question depends drastically on $\nu$.

**Lemma 13 (Stiefel)** Let $f$ be a continuous map (20). Let $\alpha$ be a characteristic number defined from the cohomology equation $f(S^{\nu-1}) = \alpha Z^{\nu-1}_\nu$, where $Z^{\nu-1}_\nu$ is a $(\nu-1)$-dimensional cycle in $V_m(\mathbb{R}^k)$. If $0 \leq \nu \leq k - m$ then $f$ can always be continuously extended to the interior of $S^{\nu-1}$. If $k - m + 1 \leq \nu \leq n$ then $f$ can
be continuously extended to the interior of $S^\nu-1$ if and only $\alpha \in G$ is zero. In the later case

(i) $G \cong \mathbb{Z}$ if and only if $\nu$ is odd;
(ii) $G \cong \mathbb{Z}_2$ if and only if $\nu$ is even.

**Proof.** The first part of lemma follows from the equation (18) since every $(k - m - 1)$-dimensional sphere in $V_m(\mathbb{R}^k)$ is simply connected.

To prove the second part of lemma it is enough to notice that the image $f(S^\nu-1)$ of the sphere $S^\nu-1$ is a cycle in $V_m(\mathbb{R}^k)$ which is proportional to a ‘basic’ cycle $Z_{\nu-1}$. By formula (19) the coefficient, $\alpha$, in this proportion belongs to $\mathbb{Z}$ if $\nu - 1$ is even, or to $\mathbb{Z}_2$ if $\nu - 1$ is odd. □

**Corollary 3** For every value of $\nu$ such that $k - m + 1 \leq \nu \leq n$ there exists an obstruction class

$$O_\nu \in H^\nu(X; G),$$

(21)

where $G$ is as indicated in Lemma 13. The field of $m$-frames on the skeleton $K_{\nu-1}$ can be extended to $K^\nu$ if and only if $O_\nu$ vanishes. Moreover, the obstruction classes $O_\nu$ are equal to the Stiefel-Whitney classes $w_\nu(\xi)$ of a vector bundle of rank $k$ over $X$.

**Proof.** The proof of this statement coincides with those of Theorem 12.1 and remarks on p. 143 of Milnor-Stasheff [4]. □

### 6 Conclusions

The assumption on regular foliation to be of codimension 1 can be eventually dropped. There is no difficulty to present any foliation of codimension $p$ as a transversal ‘product’

$$\mathcal{F}_1 \cap \ldots \cap \mathcal{F}_p,$$

(22)

where each of $\mathcal{F}_i$, $1 \leq i \leq p$ is a regular codimension 1 foliation. The theory of characteristic classes of such foliations can be constructed by similar methods. The following conjecture is probably true.

**Conjecture 1** Let $X$ be a manifold specified in Corollary 3. Then the characteristic classes of codimension $p$ foliations on $X$ coincide with the elements of the product cohomology group

$$H^1(X; \mathbb{Z}_2) \times \ldots \times H^p(X; \mathbb{Z}_2).$$

(23)

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