Research Article

Topologically Transitive and Mixing Properties of Set-Valued Dynamical Systems

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We introduce and study two properties of dynamical systems: topologically transitive and topologically mixing under the set-valued setting. We prove some implications of these two properties for set-valued functions and generalize some results from a single-valued case to a set-valued case. We also show that both properties of set-valued dynamical systems are equivalence for any compact intervals.

1. Introduction

In dynamical systems, one of the most important research topics is to determine the chaotic behaviour of the system. Various definitions of chaos have been introduced by mathematicians in the past (see [1–5]) as there is no universally accepted definition of chaos. Definitions of chaos are constructed based on some topological properties. One of the commonly used properties is topologically transitive. The concept of topologically transitive was introduced by Birkhoff [6] in 1920. Dynamical systems with a topologically transitive property contain at least one point which moves under iteration from one arbitrary neighborhood to any other neighborhood. This property has been studied intensively by mathematicians since it is a global characteristic in the dynamical system. Some prefer to study topologically mixing of dynamical systems as it is a notion stronger than topologically transitive.

Numerous studies related to the transitivity and mixing properties of the dynamical systems especially in a one-dimensional system have been done; see [7–12]. As we all know, in general, dynamical systems are studied in the view of a single point. However, knowing how the points of the system move is not sufficient as there are problems and applications that require one to know how the subsets of the system move. In recent years, several works and research on the topological dynamics of set-valued dynamical systems can be found (see [13–20]). However, there are many properties for the dynamics of set-valued dynamical systems yet to be discovered. Loranty and Pawlak [21] studied the connection between transitivity and a dense orbit for multifunction in generalized topological spaces. Information about topologically transitive for the single-valued dynamical systems can be found in [22–25].

In this paper, we will introduce and study the notion of topologically transitive and topologically mixing for set-valued functions. We prove some elementary results of these two properties. Some of the results are generalization from the single-valued case (e.g., [11, 26–28]). We also prove that the definitions of these two properties for a set-valued function on compact intervals are equivalence. This paper is organized as follows. In Section 2, we give some background settings and define topologically transitive and topologically mixing for set-valued functions. In Section 3, we present some elementary implication results of topologically transitive and topologically mixing. In Section 4, we prove the
equivalence of topologically transitive and topologically mixing of set-valued function on an arbitrary compact interval. In Section 5, present some conclusions.

2. Preliminaries

Let $X$ be a compact metric space. We denote $2^{X}$ as the collection of all nonempty closed subsets of $X$. We call a function $F: X \to 2^{X}$ as set-valued function. If $A \subseteq X$, then $F(A) = \{ y \in X : \text{there is a point } x \in A \text{ such that } y \in F(x) \}$. $F$ is said to be upper semicontinuous at $x \in X$ if for any open subset $V$ of $X$ containing $F(x)$ there is an open subset $U$ of $X$ containing $x$ such that for every $t \in U$, $F(t) \subseteq V$. $F$ is upper semicontinuous if it is upper semicontinuous at every point of $X$. Throughout the paper, we assume the set-valued function $F$ is upper semicontinuous unless explicitly stated.

Since $X$ is compact, by [29] the hyperspace $2^{X}$ is compact. Therefore, every element of $2^{X}$ is a nonempty closed subset of $X$. The pair $(X, F)$ is called as set-valued dynamical system. $2^{X}$ is denoted as the identity on $X$ and $F^{n} = F \circ F^{n-1}$ for all integers $n > 0$. The inverse set-valued function is defined as below.

Definition 1 [30]. Let $F: X \to 2^{X}$ be a set-valued function; then, the inverse set-valued function $F^{-1}: X \to 2^{X}$ is defined by $F^{-1}(y) = \{ x \in X : y \in F(x) \}$ for all $y \in X$.

Recall that in the single-valued dynamical system $(X, f)$ where $f: X \to X$ represent a continuous function, for any point $x \in X$, we define the orbit of $x$ under $f$ as $(x_{i})_{i=0}^{\infty}$ where $x_{0} = x$ and $x_{i+1} = f(x_{i})$ for all integers $i \geq 0$. The point $x \in X$ is said to be a periodic point of $f$ with period $n$ provided $f^{n}(x) = x$ and $f^{j}(x) \neq x$ for all integers $0 < j < n$. If the point $x$ has period $n = 1$, then, it is called as a fixed point. We extend these definitions to a set-valued case.

Definition 2 [31]. Let $(X, F)$ be a set-valued dynamical system. For any point $x \in X$, an orbit of $x$ is a sequence $(x_{i})_{i=0}^{\infty}$ such that $x_{0} = x$ and $x_{i+1} \in F(x_{i})$ for all integers $i \geq 0$. The collection of all orbits of $x$ is called as the complete orbit of $x$, denoted by $CO(x)$.

Definition 3 [30, 31]. For a set-valued dynamical system $(X, F)$, let $x \in X$ and let $(x_{i})_{i=0}^{\infty}$ be an orbit of $x$. The orbit is said to be a periodic orbit if there exists $m \in \mathbb{N}$ such that $x_{i} = x_{i+m}$ for all integers $i \geq 0$. The point $x$ is a periodic point if it has at least one periodic orbit. The period of $x$ is the smallest number $m \in \mathbb{N}$ satisfying $x_{j} = x_{j+m}$ for all integers $i \geq 0$. If $m = 1$, then, $x$ is said to be a fixed point.

In Definitions 2 and 3, we can see that in set-valued dynamical systems, the orbits of $x$ under $F$ is no longer uniquely determined. The following example shows that the orbit $(x_{i})_{i=0}^{\infty}$ is not necessarily periodic even if there exists $j \in \mathbb{N}$ such that $x = x_{0} = x_{j}$.

Example 4. Let $X = \{0, 1\}$ and let the set-valued function $F: X \to 2^{X}$ defined by $F(1) = \{0, 1\}$ and $F(0) = \{1\}$. Let $x = 0$; then, one of the orbit of $x$ under $F$ is $(0, 1, 1, 0, 1, 0, 1, \cdots)$. We can see that $x = x_{3}$ but $x_{j} \neq x_{j+3}$ for some $j > 0$. Therefore, the orbit $(0, 1, 1, 0, 1, \cdots)$ is not a periodic orbit of $x$.

Next, we define topologically transitive of set-valued functions. For topologically mixing of set-valued functions, we adopt the definition which has been defined by [31].

Note that the product set-valued function $F \times F: X \times X \to 2^{X \times X}$ is defined by $(F \times F)(x, x') = \{ (y, y') \in X \times X : y \in F(x) \text{ and } y' \in F(x') \}$ for all $x, x' \in X$.

Definition 5. Let $(X, F)$ be a set-valued dynamical system. The set-valued function $F$ is topologically transitive if for any nonempty open subsets $U$ and $V$ of $X$, there exists $m \in \mathbb{N}$ and $x \in U$ with an orbit $(x_{i})_{i=0}^{\infty}$ such that $x_{m} \in V$.

Definition 6 [31]. A set-valued function $F$ is topologically mixing if for any nonempty open sets $U$ and $V$ in $X$, there is an $M \in \mathbb{N}$ such that for any $m > M$, there is an $x_{0} \in U$ with an orbit $(x_{i})_{i=0}^{\infty}$ such that $x_{m} \in V$.

Definition 7. Let $F$ be a set-valued function of a compact metric space $X$. Then $F$ is said to be

(1) topologically bitransitive if $F^{2}$ is topologically transitive

(2) totally transitive if $F^{n}$ is topologically transitive for all $n \in \mathbb{N}$

(3) topologically weakly mixing if the product set-valued function $F \times F$ is topologically transitive

We end this section by recalling some basic concepts from topology. The interior of $U$ denoted by $\text{int}(U)$ is the union of all open subsets of $U$. The closure of $U$ denoted by $\text{cl}(U)$ is the intersection of all closed subsets of $X$ containing $U$. A set $U$ is said to be dense in $X$ if $\text{cl}(U) = X$. In other words, we can say that $U$ is dense in $X$ if every open subset of $X$ contains at least a point of $U$ (see [32, 33]).

3. Topologically Transitive and Mixing of Set-Valued Functions

In a single-valued case, there are two commonly used definitions for topologically transitive: one is defined by using open sets and another one is defined by using points with a dense orbit (see [25]). Block [26] showed that both definitions coincide when the space is compact. But in general, both characterizations of transitivity are not equivalent as shown in [34, 35]. On a compact metric space with a set-valued function, we show that if there is a point with a dense orbit, then it will imply the transitivity of the set-valued function.

Proposition 8. Let $F: X \to 2^{X}$ be a set-valued function. If there exists at least a point $x \in X$ with an orbit $(x_{i})_{i=0}^{\infty} \in CO(x)$ such that the orbit is dense in $X$, then, $F$ is topologically transitive.
If the set-valued function $F$ we will have at least a point $y_k \in U$. Now, we try to show that there is a positive integer $m$ such that $y_{k+m} \in V$. Then, the proof is done by letting $x = y_k$, so we have $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ where $x_m = y_{k+m} \in V$.

Since the orbit $(y_i)_{i=0}^{\infty}$ of $y$ is dense, $V$ contains at least one iterate of $y$. Suppose that there are only finitely many iterates of $y$ in $V$. Let $z$ be any element of $V$ such that $z$ is not an iterate of $y$ and let $\epsilon = \min \{d(z, y_i) : j = 0, 1, 2, 3, \ldots\}$ where $d$ is the metric on $X$. We have $\epsilon > 0$ and the neighborhood $N_{\epsilon/2}(z)$ does not contain any iterates of $y$. This implies that the orbit $(y_i)_{i=0}^{\infty}$ of $y$ is not dense in $X$, a contradiction. Therefore, $V$ must contain infinitely many iterates of $y$. Since there are only finitely many positive integers less than $k$, there exists an integer $n > k$ such that $y_n \in V$. We let $m = n - k$ and the proof is complete.

With Proposition 8, we obtain the following theorem.

**Theorem 9.** A set-valued function $F : X \to 2^X$ is topologically transitive if and only if there exists at least a point $x \in X$ with an orbit $(x_i)_{i=0}^{\infty} \in CO(x)$ such that the orbit is dense in $X$.

**Proof.** Clearly by the definition of topologically transitive, we will have at least a point $x \in X$ with an orbit $(x_i)_{i=0}^{\infty} \in CO(x)$ such that the orbit is dense in $X$. For the converse part, we have proved in Proposition 8.

Similar to the single-valued case (see [36]), it is easy to see that by the definition, if the set-valued function $F$ is topologically mixing, then it implies that $F$ is topologically transitive. We show that the converse is not true in the following example.

**Example 11.** Let us consider the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $R_z : S^1 \to S^1$ be an irrational rotation of $S^1$. We define a set-valued function $R$ of $S^1$ by $R(z) = [z, R_z(z)]$. Since all the points of $S^1$ have dense orbits, for an open subset $U$ of $S^1$, it will intersect with other open subset $V$ of $S^1$ for some iterates under $R$. Therefore, $R$ is topologically transitive. But $R_z$ is an irrational rotation, so there exists at least one further iterate of $U$ under $R$ that did not intersect with $V$. Hence, $R$ is not topologically mixing.

Next, we discuss some connections between Definitions 5, 6, and 7 in Section 2.

**Proposition 12.** Let $(X, F)$ be a set-valued dynamical system. If the set-valued function $F$ is topologically transitive, then $F$ is topologically weakly mixing.

**Proof.** Assume that $F$ is topologically mixing. Let $W_1, W_2$ be any two nonempty open sets in $X \times X$. There exist nonempty open sets $U_1, U_2, V_1,$ and $V_2$ in $X$ such that $U_1 \times U_2 \subset W_1$ and $V_1 \times V_2 \subset W_2$. Since $F$ is topologically mixing, there exist $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, there is a $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_n \in V_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, there is a $y \in U_2$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $y_n \in V_2$. Let $M = \max \{N_1, N_2\}$. Then, for all $n \geq M$, there exists $(x, y) \in W_1$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_M, y_n) \in W_2$. Hence, we conclude that $F$ is topologically weakly mixing.

**Proposition 14.** Let $(X, F)$ be a set-valued dynamical system. If the set-valued function $F$ is topologically mixing, then $F$ is topologically bitransitive.

**Proof.** Assume that $F$ is topologically mixing. Then, for any two nonempty open sets $U$ and $V$ of $X$, there is a $M \in \mathbb{N}$ such that for any positive integer $m > M$, there is an $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_n \in V$. If we take $m$ to be an even positive integer greater than $M$, i.e., $m = 2n > M$ where $n$ is a positive integer, then, there exists an $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_{2n} \in V$. This implies that $F^2$ is topologically transitive and hence $F$ is topologically bitransitive.

**Lemma 16.** Let $(X, F)$ be a set-valued dynamical system, and the iterations of any open set are an open set of $X$. If the set-valued function $F$ is topologically weakly mixing, then the product set-valued dynamical system $(X^n, F \times \cdots \times F)$ is topologically transitive for all integers $n \geq 1$.

**Proof.** For all open sets $U$, $V$ in $X$, we define $N(U, V) = \{n \geq 1 : \text{there exists } x \in U \text{ with an orbit } (x_i)_{i=0}^{\infty} \text{ such that } x_n \in V\}$.

Let $U_1, U_2, V_1,$ and $V_2$ be nonempty open sets in $X$. Since $F$ is topologically weakly mixing, there exists a natural number $m$ such that there is a point $(x, y) \in U_1 \times U_2$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_m, y_n) \in V_1 \times V_2$. This means that there is a point $x \in U_1$ with orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in V_1$ and there is a point $y \in U_2$ with orbit $(y_i)_{i=0}^{\infty}$ such that $y_n \in V_2$. We can say that for any $G, H$ nonempty open sets in $X, N(G, H) \neq \emptyset$.

Now, we are going to show that there exist nonempty open sets $U, V$ in $X$ such that $N(U, V) \subseteq N(U_1, V_1) \cap N(U_2, V_2)$. Let us define the open sets $U = U_1 \cap F^n(U_2)$ and $V = V_1 \cap F^n(V_2)$ as follow:

$$U = \{x : x \in U_1 \text{ with orbit } (x_i)_{i=0}^{\infty} \text{ such that } x_0 = x \text{ and } x_n \in U_2\},$$

$$V = \{x : x \in V_1 \text{ with orbit } (x_i)_{i=0}^{\infty} \text{ such that } x_0 = x \text{ and } x_n \in V_2\}.$$  \hfill (1)

We have already shown that these sets are not empty. Let $k \in N(U, V)$. This integer exists and satisfies the condition of there exists an $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_k = x$ and $x_k \in V$. This means that there is a point $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_0 = x \in U_1, x_k \subset U_2, x_k \subset V_1$ and $x_{k+n} \subset V_2$. Then, we can deduce that there is a $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_k \subset V_1$ and there is a $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_k \subset V_2$. Therefore, $x$ belongs to $N(U_1, V_1) \cap N(U_2, V_2)$ and hence $N(U, V) = N(U_1, V_1) \cap N(U_2, V_2)$.
that $x_k \in V_2$. Therefore, we obtain $k \in N(U_1, V_1)$ and $k \in N(U_2, V_2)$ and this implies that $N(U, V) \subseteq N(U_1, V_1) \cap N(U_2, V_2)$. By using the principle of mathematical induction, we were able to see that for all nonempty open sets $U_1, \cdots, U_n, V_1, \cdots, V_n$ in $X$, there exist nonempty open sets $U, V$ in $X$ such that

$$N(U, V) \subseteq N(U_1, V_1) \cap N(U_2, V_2) \cap \cdots \cap N(U_n, V_n).$$ \hspace{1cm} (2)

Hence, we conclude that $(X^n, F \times \cdots \times F)$ is topologically transitive.

**Theorem 18.** Let $(X, F)$ be a set-valued dynamical system and the iterations of any open set is an open set of $X$. If the set-valued function $F$ is topologically weakly mixing, then, $F$ is totally transitive.

**Proof 19.** Assume that $F$ is topologically weakly mixing. Let $n \geq 1$ be an arbitrary fixed positive integer and $U', V', V''$ be nonempty open sets in $X$. We define two open sets $W, W'$ in $X^{2n}$ as follows:

$$W = \left\{ (x_0, x_1, \cdots, x_{n-1}, y_0, y_1, \cdots, y_{n-1}) : x_i \in F'(U), \right.$$ \hspace{1cm} (3)

$$y_j \in F'(V) \text{ for all } i = 0, 1, \cdots, n - 1 \} ,$$

$$W' = \left\{ (x_0', x_1', \cdots, x_{n-1}', y_0', y_1', \cdots, y_{n-1}') : x_i' \in U', \right.$$ \hspace{1cm} (3)

$$y_j' \in V' \text{ for all } i = 0, 1, \cdots, n - 1 \} .$$

By Lemma 16, $(X^{2n}, F \times F \times \cdots \times F)$ is topologically transitive. So, there exists a positive integer $m$ such that there is a $(x_0, x_1, \cdots, x_{n-1}, y_0, y_1, \cdots, y_{n-1}) \in W$ with an orbit $(x_0^{(j)}, x_1^{(j)}, \cdots, x_{n-1}^{(j)}, y_0^{(j)}, y_1^{(j)}, \cdots, y_{n-1}^{(j)} ; j = 0, \infty)$ such that

$$(x_0, x_1, \cdots, x_{n-1}, y_0, y_1, \cdots, y_{n-1}) = (x_0^{(0)}, x_1^{(0)}, \cdots, x_{n-1}^{(0)}, y_0^{(0)}, y_1^{(0)}, \cdots, y_{n-1}^{(0)}) \in W,$$ \hspace{1cm} (4)

$$(x_0^{(m)}, x_1^{(m)}, \cdots, x_{n-1}^{(m)}, y_0^{(m)}, y_1^{(m)}, \cdots, y_{n-1}^{(m)}) \in W'.$$

This implies that for all $i = 0, 1, \cdots, n - 1$, there is a $x_i \in F'(U)$ with an orbit $(x_i^{(j)} ; j = 0, \infty)$ such that $x_i^{(0)} \in F'(U)$ and $x_i^{(n)} = U'$ and there is a $y_i \in F'(V)$ with an orbit $(y_i^{(j)} ; j = 0, \infty)$ such that $y_i^{(n)} \in V'$ and $y_i^{(m)} = V''$. Since $x_i \in F'(U)$ and $y_i \in F'(V)$ for all $i = 0, 1, \cdots, n - 1$, there exists $u \in U$ such that $x_i \in F'(u)$ and $v \in V$ such that $y_i \in F'(v)$ for all $i = 0, 1, \cdots, n - 1$. So, we can write that for all $i = 0, 1, \cdots, n - 1$, there is a $u \in U$ with an orbit $(u_i^{(j)} ; j = 0, \infty)$ such that $u = u_0 \in U$ and $u_{i+m} = u' \in U'$ and there is a $v \in V$ with an orbit $(v_i^{(j)} ; j = 0, \infty)$ such that $v_0 \in V$ and $v_{i+m} = v'$. Now, we choose an $i \in \{0, 1, \cdots, n - 1\}$ such that $m + i = np$ where $p$ is a positive integer. We deduce that there is a point $(u, v) \in U \times V$ with an orbit $(u_i, v_i^{(j)} ; j = 0, \infty) \subset F^n \times V'$ that $(u_m, v_{np}) \in U' \times V'$, this shows that $F^n$ is topologically weakly mixing which implies that $F^n$ is topologically transitive. Since $n$ is chosen arbitrary, $F^n$ is topologically transitive for all $n \geq 1$, and hence, $F$ is totally transitive.

**Proposition 17.** Let $(X, F)$ be a set-valued dynamical system. If the set-valued function $F$ is totally transitive, then, $F$ is topologically bitransitive.

**Proof 18.** Let $U, V$ be any two nonempty open sets of $X$. Since $F$ is totally transitive, $F^n$ is topologically transitive for all positive integers $n \geq 1$. For each $n \geq 1$, there exists $m \in N$ such that there is an $x \in U$ with an orbit $(x_i^{(j)} ; j = 0, \infty)$ such that $x_m = V$. When we take $n = 2$, there exists $m \in N$ such that there is an $x \in U$ with an orbit $(x_i^{(j)} ; j = 0, \infty)$ such that $(x_2)_m \in V$. This implies that $F^2$ is topologically transitive, hence $F$ is topologically bitransitive.

The implications between the various conditions of topologically transitive and mixing in this section are summarized as follows:

$$\text{mixing} \Rightarrow \text{weakly mixing} \Rightarrow \text{totally transitive} \Rightarrow \text{bitransitive} \Rightarrow \text{transitive}. \hspace{1cm} (5)$$

### 4. Transitivity and Mixing Properties of Set-Valued Functions on Compact Intervals

In this section, we investigate the properties of topologically transitive and topologically mixing of set-valued functions on arbitrary compact interval $I = [a, b]$, where $a, b \in \mathbb{R}$ such that $a < b$. Note that for a single-valued case, the set of periodic points is dense in $I$ if the function is topologically transitive [26]. First, we present a lemma that will be used in the following proposition.

**Lemma 19.** Let $(I, F)$ be a set-valued dynamical system and $J$ be a subinterval of $I$ which contains no periodic point of $F$. Suppose that, $x \in J$ has an orbit $(x_i^{(j)} ; j = 0, \infty)$ such that $x_m = \infty$ for some integers $m > 0$ and $y \in J$ has an orbit $(y_i^{(j)} ; j = 0, \infty)$ such that $y_n = \infty$ for some integers $n > 0$. If $x < x_m$, then $y < y_n$ and if $x > x_m$, then $y > y_n$.

**Proof 20.** Without loss of generality, suppose that $x < x_m$. Let $G = F^n$; then, the subinterval $[x, x_m]$ of $I$ contains no periodic point of $F$. If $x < x_m$ for some $k \geq 1$, then, $x(k+1)m > x_m$ as there is no fixed point of $G$ inside the interval $[x, x_m]$. Clearly by mathematical induction, $x_{km} > x$ for all $k \geq 1$. So, in particular, we have $x_{km} > x$.

Assume that $y > y_n$. With similar argument as the above (the order of inequality is reversed), we yield $y > y_m$. Consequently, there exists a point $z \in J$ lying in between $x$ and $y$ such that $x_m = z$. This leads to a contradiction as $J$ contains no periodic point of $F$. Therefore, we conclude that $y < y_n$.

**Proposition 21.** If the set-valued function $F : I \rightarrow 2^I$ is topologically transitive, then, the set of periodic points of $F$ is dense in $I$.
there exists a positive integer nondegenerate subintervals concludes that we refer the notation from [28] to denote mixing for set-valued function on compact interval. Recall J for set of points with an orbit \( v \) such that \( v_{q-p} = u_q \). Then, we obtain the following inequality:

\[
x < v_{q-p} < v < u_m < y.
\]

But this is impossible as Lemma 19 is applied to the open interval \((x, y)\). Thus, we conclude that the set of periodic points of \( F \) is dense in \( I \).

Next, we prove an elementary property of topologically mixing for set-valued function on compact interval. Recall that an interval is degenerate [28] if it is either empty or reduced to a single point and it is nondegenerate otherwise. We refer the notation from [28] to denote \((a, b)\) as the smallest interval containing \([a, b]\), that is, \(\langle a, b \rangle = [a, b] \) if \(a \leq b\) and \(\langle a, b \rangle = (a, b) \) if \(a > b\).

Proposition 23. Let \( I = [a, b] \) and \( F : I \to \mathbb{I}^2 \) be a set-valued function. Then, \( F \) is topologically mixing if and only if for all nondegenerate subintervals \( J \subseteq I \) and any pair \( c, d \in \operatorname{int}(I) \), there exists a positive integer \( M \) such that \( \langle c, d \rangle \in F^n(J) \) for all \( n \geq M \).

Proof 24. Suppose that \( F \) is topologically mixing. Let \( U = (a, c) \) and \( V = (d, b) \). If \( J \) is a nonempty open subinterval of \( I \), then, there exists \( N_1 \in \mathbb{N} \) such that there is an \( x \in J \) with an orbit \( \langle x_i \rangle_{i=0}^\infty \) such that \( x_n \in U \) for all \( n \geq N_1 \). Similarly, there exists \( N_2 \in \mathbb{N} \) such that there is a \( y \in J \) with an orbit \( \langle y_i \rangle_{i=0}^\infty \) such that \( y_n \in V \) for all \( n \geq N_2 \). Let \( M = \max\{N_1, N_2\} \). Then, we have an \( x \in J \) with an orbit \( \langle x_i \rangle_{i=0}^\infty \) such that \( x_n \in U \) and we have a \( y \in J \) with an orbit \( \langle y_i \rangle_{i=0}^\infty \) such that \( y_n \in V \) for all \( n \geq M \). This implies that \( \langle c, d \rangle \in F^n(J) \) by connectedness of \( \langle c, d \rangle \). If \( J \) is a nondegenerate subinterval, the same result holds by considering the nonempty open interval \( J \).

Conversely, suppose that for any pair \( c, d \in \operatorname{int}(I) \) and a nondegenerate subinterval \( J \subseteq I \), there is a positive integer \( M \) such that \( \langle c, d \rangle \in F^n(J) \) for all \( n \geq M \). Let \( U, V \) be two nonempty open subintervals of \( I \). Choose two nonempty open subintervals \( K, L \) such that \( K \subseteq U, L \subseteq V \) and neither \( a \) nor \( b \) is an endpoint of \( L \). There exists a pair \( c, d \in \operatorname{int}(I) \) such that \( L \subseteq \langle c, d \rangle \). By assumption, there exists a positive integer \( M \) such that \( \langle c, d \rangle \in F^n(K) \) for all \( n \geq M \). Therefore, we have \( L \subseteq \langle c, d \rangle \subseteq F^n(K) \) for all \( n \geq M \) and this implies that there is an \( x \in U \) with an orbit \( \langle x_i \rangle_{i=0}^\infty \) such that \( x_n \in V \) for all \( n \geq M \). We conclude that \( F \) is topologically mixing.

When topologically mixing is replaced with topologically bi-transitive for set-valued functions, a weaker version of Proposition 23 can be obtained with the help of the following lemma.

Lemma 25. Let \( F : I \to \mathbb{I}^2 \) be a set-valued function. Let \( J \) be a subinterval of \( I \) which contains a fixed point \( p \) of \( F \) and a periodic point \( q \) of \( F \) with least period \( m \geq 3 \). Let \( q \) represent the iterates of \( q \) in the periodic orbit where \( i \in \{0, 1, \ldots, m - 1\} \). Then, one of the following holds:

1. \( F^n(J) \supseteq \{q, q_1, \ldots, q_{m-1}\} \) for all \( n \geq 2m - 2 \)
2. \( m \) is even, set \( \{q, q_2, \ldots, q_m\} \) and \( \{q_1, q_3, \ldots, q_{m-1}\} \) lie on the opposite sides of \( p \), and \( F^{2m}(J) \supseteq \{q, q_2, \ldots, q_{m-1}\} \) for all \( n \geq (m/2) - 1 \)

Proof 26. Let \( u \) and \( v \) be two points in the periodic orbits of \( q \) such that \( v' \leq u < v \leq v' \) where \( v' = F(v) \) and both \( u', v' \) are contained in the same periodic orbit of \( q \). Let \( i, j \in \{0, 1, \ldots, m - 1\} \) such that \( u = q_i \) and \( v = q_j \). If both point \( q_k \) and \( q_{k+1} \) lie on the same side of the fixed point \( p \) for some \( k \in \{0, 1, \ldots, m - 1\} \), then, we have four possible cases:

\[
\begin{align*}
q_i &= u < v < p, \\
q_k &< u < p < v, \\
u < p < v < q_k, \\
p < u < v = q_j.
\end{align*}
\]

For case 1, we have \( F(J) \supseteq [u, v] \). For case 4, we have \( F(J) \supseteq [u, v] \). For cases 2 and 3, the set \( F(J) \) contains the compact interval \([p, q_k] \). Thus, we have \( F^{k+1}(J) = F(F(J)) \supseteq [u, v] \).

In both cases, since \( F([u, v]) \supseteq [u, v] \), we obtain \( F^{m-2}[u, v] \supseteq \{q, q_1, \ldots, q_{m-1}\} \), and, since \( F^{m-2}[u, v] \supseteq \{q, q_1, \ldots, q_{m-1}\} \), we have \( F^{m-2}(J) \supseteq \{q, q_2, \ldots, q_{m-1}\} \) for all \( n \geq (m/2) - 2 \).

Otherwise, if \( q_i \) and \( q_{i+1} \) lie on the opposite sides of \( p \) for all \( i \in \{0, 1, \ldots, m - 1\} \), clearly \( m \) is an even integer. Let \( r \) be the even integer from the set \([0, 1, \ldots, m - 1]\) such that \( q_r \) is the point which stays on the same side of \( p \) as \( q \) and lies most far from \( p \). Then, \( F^{r+1}(J) \) contains the set \( \{q_m, q_{m-1}, q_1, \cdots, q_{m-2}\} \) for all \( n \geq 0 \). Therefore, we conclude that \( F^{2r}(J) \supseteq \{q, q_2, \ldots, q_{m-1}\} \) for all \( n \geq (m/2) - 1 \).

Theorem 27. Let \( I = [a, b] \) and \( F : I \to \mathbb{I}^2 \) be a set-valued function. If \( F \) is topologically bitransitive, then, for any nondegenerate subinterval \( J \subseteq I \) and any pair \( c, d \in \operatorname{int}(I) \) such that \( a < c < d \), there exists a positive integer \( M \) such that \( \langle c, d \rangle \subseteq F^n(J) \) for all \( n \geq M \).

Proof 28. Let \( z \) be a fixed point of \( F \) and \( J \) be any nondegenerate subinterval of \( I \). Without the loss of generality, we may assume that \( z < x \) for all \( x \in J \). Since \( F \) is topologically bitransitive, it means that \( F^2 \) is topologically transitive and by Theorem 9, there exists a point \( u \in J \) with an orbit \( \langle u_i \rangle_{i=0}^\infty \) with respect to \( F^2 \) which is dense in \( I \).

Let \( L = \langle c, d \rangle \) be a compact subinterval in \( I \) where \( c, d \in \operatorname{int}(I) \) and \( c < d \). Since the orbit \( \langle u_i \rangle_{i=0}^\infty \) with respect to \( F^2 \) is
dense in \( I \), for some positive integers \( n_1 < n_2 \), we obtain

\[
u_{n_1} < \min \{ z, \min L \} < \max \{ z, \max L \} < u_{n_2}.
\]

By Proposition 21, we know that the set of periodic points is dense in \( I \). Since \( u \in \text{int} (J) \), there exists a periodic point \( p \in J \) with period \( m \) which is close to point \( u \) and the set contains all points of the periodic orbit of \( p \) which is contained in \( \text{int} (I) \). Then, we have

\[
u_{n_1} \approx p_{n_1} < \min \{ z, \min L \} < \max \{ z, \max L \} < p_{n_2} \approx u_{n_2}.
\]

Consequently, there is a positive integer \( s \) where the interval \( F^s (J) \) contains the fixed point \( z \) and the periodic point \( p^s \) with period \( m \). For the orbit \( \{ p_{j}^{\circ} \}_{j=0}^{s} \), we can see that the even iterates are distributed on both sides of \( z \). Hence, by Lemma 25, we have \( F^s (F^s (J)) > \{ p, p_1, \ldots, p_{m-1} \} \) for all \( j \geq 2m - 2 \). Therefore, \( F^n (J) > \{ \min \{ \{ p, p_1, \ldots, p_{m-1} \}, \max \{ p, p_1, \ldots, p_{m-1} \} \} \} \) \( \ni L \) for all \( n \geq s + 2m - 2 \) and the proof is complete.

The following theorem gives an overview on the relation between topologically transitive and topologically mixing for set-valued functions of compact intervals. In fact, for compact intervals, both definitions are equivalent, which is similar to the results in the single-valued case (see [26]).

**Theorem 29.** Let \( I = [a, b] \) and \( F : I \rightarrow 2^I \) be a set-valued function. If \( F \) is topologically transitive, then the following statements are equivalent:

1. \( F \) is topologically bitransitive
2. \( F \) is totally transitive
3. \( F \) is topologically weakly mixing
4. \( F \) is topologically mixing
5. For any nondegenerate subinterval \( J \subset I \) and any pair \( c, d \in \text{int} (J) \) such that \( a < c < d < b \), there exists a positive integer \( M \) such that \( [c, d] \subset F^n (J) \) for all \( n \geq M \)

**Proof.** It follows from Theorem 27 that (1) \( \Rightarrow \) (5). By Proposition 23, we have (5) \( \Rightarrow \) (4) for the case \( c < d \). Finally, by the implication diagram at the end of Section 3, we have (4) \( \Rightarrow \) (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1).

**5. Conclusion**

In this paper, we studied the two properties of dynamical systems which are topologically transitive and topologically mixing under the setting of a set-valued case. An implication diagram to show the connection between various conditions of transitivity and mixing is provided in Section 3. We also investigated transitivity and mixing properties of set-valued functions for compact intervals and showed that both definitions are equivalent. Some results are similar to a single-valued case, but it will serve as a stepping stone for future research in a set-valued case.

**Data Availability**

All data required for this research is included within the paper.

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this article.

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