Stability of differential equations associated with a class of one dimensional maps

M. C. Valsakumar¹,² A. Rajan Nambiar² and P Rameshan³

¹Materials Science Division, IGCAR, Kalpakkam, India, Pin 603102
²Dept. of Physics, Govt. Arts & Science College, Kozhikode, India, Pin 673018
³Department of Physics, University of Calicut, Calicut University PO, India, Pin 673635

Abstract. Discrete time evolution of one-dimensional maps is embedded in continuous time by truncating the Taylor series expansion of the time evolution operator to a finite order N. Truncations with N > 4 leads to unconditional instability. Generalization of the truncated models with N = 3 and 4 shows dynamical behaviour characteristic of systems with a riddled parameter space

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§ To whom correspondence should be addressed (valsa@igcar.ernet.in)
1. Introduction

Time development of systems evolving continuously in time is modelled with differential equations, whereas difference equations (maps) are used to describe systems that evolve in discrete time. Here we try to study connection, if any, between solutions of discrete maps and a set of ordinary differential equations that are derived from the map using an ansatz described below. Basically, we truncate the Taylor series expansion of the time evolution operator corresponding to the discrete map at a finite order $N$ to get an ordinary differential equation (ODE) of order $N$. Contrary to the expectation that the solution of such an ODE should converge to that of the discrete map (when $t$ = an integer) in the limit $N \to \infty$ we find surprising results. In particular, all these ODEs with $N \geq 5$ are seen to be unconditionally unstable for almost all initial conditions regardless of the details of the underlying map. The nature of the solutions for truncation at $N = 3$ and 4 do depend on the details of the map. The 3rd and 4th order ODEs obtained for the logistic map show period doubling bifurcations leading to chaos, reverse bifurcations, instability in some regions etc. Unlike the logistic map, these equations show sudden switching from regular to chaos or to unstable behaviour by a slight variation in the parameter, which is an indication of structural instability that was discussed in the context of systems with a riddled parameter space (Cazelles 2001, Kapitaniak et. al. 2000, Kim and Lim 2001, Krawiecki and Matyjasek 2001, Lai and Winslow 1994, Lai 2000, Lai and Andrade 2001, Madvinsky et. al. 1999, Woltering and Markus 2000, Yang 2000, Yang 2001).

2. Ordinary differential equations corresponding to discrete maps

Consider a general 1-D map

$$x_{n+1} = f(x_n).$$

(1)

This discrete time evolution can be embedded in continuous time $t$ by considering the equation

$$\hat{T}x(t) = f(x(t))$$

(2)

where $\hat{T}$ is the unit time evolution operator

$$\hat{T} = \exp \left( \frac{d}{dt} \right)$$

(3)

We consider the stability of the sequence of equations

$$\sum_{j=0}^{N} \left( \frac{1}{j!} \right) \left( \frac{d^j}{dt^j} \right) x(t) = f(x(t))$$

(4)

obtained by truncating the formal power series expansion of the time evolution operator $\hat{T}$ to order $N$. 
3. Stability analysis

As mentioned in the introduction, one of the objectives of this paper is to examine the nature of solutions of the ordinary differential equations (4) for various values of $N$. For any map $f(x)$ that is continuous and differentiable, we prove that the solutions of (4) are linearly unstable for almost all initial conditions if $N \geq 5$.

Consider the stability of solutions of Eq. (4) about an arbitrary reference point (not necessarily a fixed point) $x_*$. Writing $x(t) = x_* + \delta x(t)$, one gets

$$\sum_{j=0}^{N} \left( \frac{1}{j!} \right) \left( \frac{d^j}{dt^j} \right) \delta x(t) + \alpha(x_*) \delta x(t) = \beta(x_*)$$

(5)

to first order in $\delta x(t)$. In Eq. (5), $\alpha(x_*) = 1 - \frac{d}{dx}f(x)|_{x=x_*}$ and $\beta(x_*) = f(x_*) - x_*$. Clearly, $\beta(x_*) = 0$ if $x_*$ is a fixed point of the map. It is easy to show (see appendix A for details) that the stability property of the linear inhomogeneous equation (5) is the same as that of the homogeneous part

$$\sum_{j=0}^{N} \left( \frac{1}{j!} \right) \left( \frac{d^j}{dt^j} \right) \delta x(t) + \alpha(x_*) \delta x(t) = 0.$$ 

(6)

Since the above equation is linear we look for a solution of the form

$$\delta x(t) = c \exp(\mu t)$$

(7)

to get

$$\sum_{j=0}^{N} \left( \frac{1}{j!} \right) \mu^j + \alpha(x_*) = 0.$$ 

(8)

We now show that there will exist at least one $\mu$ with positive real part irrespective of the value of $\alpha(x_*)$, if $N \geq 5$. This would imply that the solution of the equation (4) is unstable for any finite $N \geq 5$ for almost any initial condition, regardless of the specific functional form of $f(x)$.

Now, Eq. (8) is of the form

$$a_0 \mu^N + a_1 \mu^{N-1} + \ldots + a_{N-1} \mu + a_N = 0,$$

(9)

with $a_j = \frac{1}{(N-j)!}$ for $0 \leq j \leq (N-1)$ and $a_N = \alpha(x_*)$.

Nature of zeros of Eq. (9) can be examined by using the well known Routh-Hurwitz theorem (Korn and Korn 1961). Define

$$U_0 = a_0, \quad U_1 = a_1, \quad U_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad U_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \quad U_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}, \quad \text{etc.}$$ 

(10)
Then Routh-Hurwitz theorem states that the number of roots of Eq. (9) with positive real parts is equal to the number of sign changes in the sequence \( \{U_j\} \). If at least one sign change occurs in the sequence \( \{U_j\} \), then Eq. (9) has at least one root (\( \mu \)) with positive real part and hence the solution of Eq. (4) is unstable in the neighbourhood of the reference point \( x^\star \). If this happen for almost any \( x^\star \), then the solution is unstable for any initial condition, globally.

We now calculate the sequence \( \{U_j\} \) when \( N > 5 \). Substituting for \( a_j \) in Eq. (10), and on simplifying, we get

\[
U_0 = \frac{1}{N!}, \quad U_1 = \frac{1}{(N-1)!}, \quad U_2 = \frac{2}{N!(N-2)!}, \quad U_3 = -\frac{2(N-5)}{N!(N-1)!(N-3)!},
\]

which are positive for all positive \( N \) and

\[
U_3 = -\frac{2(N-5)}{N!(N-1)!(N-3)!},
\]

which is negative for all \( N > 5 \). Since there is at least one sign change in the sequence \( \{U_j\} \), it follows that the solutions of Eq. (10) are linearly unstable for \( N > 5 \) irrespective of the details of the mapping function \( f(x) \), so long as it is continuous and differentiable. Since the reference point \( x^\star \) does not explicitly appear in arriving at this conclusion, it is evident that the solutions of Eq. (4) are unstable for almost any initial condition for \( N > 5 \).

We now turn to the case \( N = 5 \). Explicit calculation shows that

\[
U_0 = \frac{1}{5!}, \quad U_1 = \frac{1}{4!}, \quad U_2 = \frac{2}{5!3!}, \quad U_3 = \frac{\alpha(x^\star) - 1}{5!4!} \quad \text{and} \quad U_4 = -\frac{(\alpha(x^\star) - 5/3)^2 + 20/9}{5!^2} \tag{13}
\]

From Eq. (13), it is clear that there is a sign change at \( U_3 \) and no further sign change at \( U_4 \), if \( \alpha(x^\star) < 1 \). On the other hand, if \( \alpha(x^\star) > 1 \), \( U_3 > 0 \) and \( U_4 < 0 \). Thus there is at least one sign change in the sequence \( \{U_j\} \), regardless of the details of the mapping function \( f(x) \) for \( N = 4 \), as well. No such general conclusions can be drawn for the cases \( N = 1, 2, 3 \) and 4.

We now turn to the specific case of truncation of chaotic maps. Since a minimum of three dimensions is required for occurrence of chaos in ordinary differential equations, truncations to order \( N = 1 \) and 2 are uninteresting from this perspective. We also now know that the truncations to order \( N \geq 5 \) lead to unconditional instability. Therefore, in what follows, we concentrate on truncations with \( N = 3 \) and 4. It is found (Rajan Nambiar 2003) that the solutions of these equations show many features which are not shared by the solutions of the original discrete map. For concreteness, we present the results for the logistic map \( f(x) = px(1-x), \; x \in [0,1] \).

4. Truncation at N=3 and 4

For this case, Eq. (10) becomes

\[
\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 6(x - f(x)) = 0 \tag{14}
\]
Linear stability analysis of the above equation, with \( f(x) = px(1-x) \) shows that both fixed points (0 and 1 − 1/p) are stable for \( p < 4 \). Thus the truncation at \( N = 3 \) leads to regular behaviour for all \( p \in [0,4] \), the range of \( p \) for which the logistic map is map of the unit interval to itself. This has to be contrasted with the occurrence of chaos in the logistic map when \( p \in [3.66..., 4] \). However, the system of equations \((14)\) do show chaotic behaviour when \( p > 4 \).

Using the scaled variables \( X = (2p/9)x \) and \( \tau = t/3 \), Eq. \((14)\) can be rewritten as

\[
\frac{d^3X}{d\tau^3} + \frac{d^2X}{d\tau^2} + \nu \frac{dX}{d\tau} - \lambda X + X^2 = 0
\]

where \( \nu = 2/3 \) and \( \lambda = 2(p - 1)/9 \). In fact, the generalization of Eq. \((15)\) obtained by allowing \( \nu \) to take arbitrary values is equivalent to the equations studied by Coulett et. al. (Coulett et. al. 1979) and Arneodo et. al. (Arneodo et. al. 1985) which exhibited striking features in its solution. The model shows regular, chaotic and or unstable behaviour for certain choices of the parameters \( \nu \) and \( \lambda \). For example, for a fixed \( \nu \), as \( \lambda \) is increased, we observe sequences of either finite or infinite number of period doubling bifurcations and reverse bifurcations with unstable regions in between. The bifurcation diagram is substantially different for a neighbouring \( \nu \). Thus the significant aspect of this equation is that the nature of solution changes drastically for very small changes in the values of the parameters. It is as if the parameter space is riddled. The solution of ODE obtained by truncating at \( N = 4 \) also show similar features.

5. Conclusions

The present work shows that the nature of solutions of the ordinary differential equations, obtained by truncating the power series of the time evolution operator corresponding to discrete maps, are very different from those of the discrete maps. In particular, the solutions with \( N \geq 5 \) are unconditionally unstable regardless of the details of the map. For the specific case of the logistic map, truncations at \( N = 3 \) and 4 lead to solutions characteristic of systems with a riddled parameter space.

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References

Arneodo A, Coulett P H, Spiegel E A and Tresser C 1985 *Physica D* 14 327.
Cazelles B 2001 *Phys. Rev. E* 64 032901.
Coulett P, Tresser C and Arneodo A 1979 *Phys. Lett. A* 72 268.
Kapitaniak T, Maistrenko Y and Popovych S 2000 *Phys. Rev. E* 62 1972.
Kim S T and Lim W 2001 *Phys. Rev. E* 63 026217.
Appendix A. Solution of the inhomogeneous equation (5)

Let us define an $N$-dimensional column vector $\xi$

$$\xi = (\xi_1, \ldots, \xi_N)^T,$$

with $\xi_1 = \delta x(t)$, $\xi_{j+1} = \frac{d}{dt}\xi_j$, $j = 1, \ldots, (N - 1)$ (A.1)

Equation (5) can then be rewritten as

$$\frac{d}{dt}\xi(t) = M\xi(t) + v$$

where

$$M = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_N & -a_{N-1} & -a_{N-2} & -a_{N-3} & \cdots & -a_1
\end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix}
0 \\
0 \\
\vdots \\
\beta(x_*)
\end{pmatrix}$$

The formal solution of Eq. (A.2) is given by

$$\xi(t) = \exp [Mt]\xi_0 + \int_0^tdt'\exp [M(t-t')v(t')]$$

(A.4)

Using the fact that $v$ is independent of time and redefining the variable of integration, we get

$$\xi(t) = \exp [Mt]\xi_0 + \left(\int_0^tdt'\exp [Mt']\right)v$$

(A.5)

Let $S$ be the similarity transformation matrix that diagonalises $M$ so that

$$S^{-1}MS = D = \text{Diag}(\mu_1, \ldots, \mu_N)$$

(A.6)

It then follows that

$$\xi(t) = S\text{Diag}(e^{\mu_1t}, \ldots, e^{\mu_Nt})\xi_0 + S\text{Diag}\left(\frac{e^{\mu_1t} - 1}{\mu_1}, \ldots, \frac{e^{\mu_Nt} - 1}{\mu_N}\right)S^{-1}v$$

(A.7)

The first term on the right hand side of Eq. (A.7) corresponds to the solution of the homogeneous part (see Eq. (6)). It is clear that if at least one of the $\{\mu_j\}$ has a positive real part so that the solution of the homogeneous equation is unstable, so is the solution of the inhomogeneous equation.