Precise analytic treatment of Kerr and Kerr-(anti) de Sitter black holes as gravitational lenses

G V Kraniotis

Department of Physics, Section of Theoretical Physics, The University of Ioannina, GR-451 10 Ioannina, Greece

E-mail: gkraniot@cc.uoi.gr

Received 15 October 2010, in final form 3 February 2011
Published 4 April 2011
Online at stacks.iop.org/CQG/28/085021

Abstract
The null geodesic equations that describe the motion of photons in Kerr spacetime are solved exactly in the presence of the cosmological constant $\Lambda$. The exact solution for the deflection angle for generic light orbits (i.e. non-polar, non-equatorial) is calculated in terms of the generalized hypergeometric functions of Appell and Lauricella. We then consider the more involved issue in which the black hole acts as a ‘gravitational lens’. The constructed Kerr black hole gravitational lens geometry consists of an observer and a source located far away and placed at arbitrary inclination with respect to the black hole’s equatorial plane. The resulting lens equations are solved elegantly in terms of Appell–Lauricella hypergeometric functions and the Weierstraß elliptic function. We then, systematically, apply our closed form solutions for calculating the image and source positions of generic photon orbits that solve the lens equations and reach an observer located at various values of the polar angle for various values of the Kerr parameter and the first integrals of motion. In this framework, the magnification factors for generic orbits are calculated in closed analytic form for the first time. The exercise is repeated with the appropriate modifications for the case of a non-zero cosmological constant.

PACS numbers: 04.20.Jb, 04.70.—s, 04.70.Bw, 98.62.Sb, 95.30.Sf, 98.80.Jk

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The issue of the bending of light (and the associated phenomenon of gravitational lensing) from the gravitational field of a celestial body (planet, star, black hole, galaxy) has been a very active and fruitful area of research for fundamental physics [1, 2].

Despite the importance of the gravitational bending of light, in unravelling the nature of the gravitational field and its cosmological implications not many exact analytic results for the
deflection angle of light orbits from the gravitational field of important astrophysical objects are known in the literature.

Recently, progress has been achieved [5] in obtaining the closed form (strong-field) solution for the deflection angle of an equatorial light ray in the Kerr gravitational field (spinning black hole, rotating mass), and thus, going beyond the corresponding calculation for the static gravitational field of a Schwarzschild black hole [3, 4]. More specifically, the closed form solution for the gravitational bending of light for an equatorial photon orbit in Kerr spacetime was derived and expressed elegantly in terms of Lauricella’s hypergeometric function $F_D$ [6]. It was then applied to calculate the deflection angle for various values of the impact parameter and the spin of the galactic centre black hole Sgr A*. The results clearly exhibited the strong dependence of the gravitational bending of light on the spin of the black hole for small values of the impact parameter (frame dragging effects) [5]. In addition, in [5], the exact solution for (unstable) spherical bound polar and non-polar photonic orbits was derived. However, the closed form analytic solution for the important class of generic (i.e. non-polar and non-equatorial) unbound light orbits was left out of the discussion in [5].

One of the unsolved related important problems so far was the full analytic treatment of the Kerr and Kerr-de Sitter black holes as gravitational lenses. The closed form solution of this problem is imperative since the Kerr black hole acts as a very strong gravitational lens and we may probe general relativity, through the phenomenon of the bending of light induced by the spacetime curvature of a spinning black hole, at the strong gravitational field regime.

It is therefore the purpose of this paper to calculate the exact solution for the deflection angle for a generic photon orbit in the asymptotically flat Kerr spacetime thereby generalizing the results in [5] and solve in closed analytic form the more involved problem of treating the rotating black hole as a gravitational lens. The constructed Kerr black hole gravitational lens geometry consists of an observer and a source located far away and placed at arbitrary inclination with respect to the black hole’s equatorial plane.

More specifically, we solve for the first time in closed analytic form, the resulting lens equations in the Kerr geometry, in terms of the Weierstraß elliptic function $\wp(z)$, equation (90), and in terms of the generalized hypergeometric functions of Appell–Lauricella equations (94), (87), (63), (104).

In addition, we calculate for the first time exactly the resulting magnification factors for generic light orbits in terms of the hypergeometric functions of Appell and Lauricella [6]. Our closed form solutions for the source and image positions of the lens equations and the corresponding magnification factors represent an important progress step in the extraction of the phenomenological and astrophysical implications of spinning black holes.

The resulting theory should be of interest for the galactic centre studies given the strong experimental evidence we have from the observation of stellar orbits (in particular from the orbits of the S-stars in the central arcsecond of the Milky Way) and flares, that the Sagittarius A* region, at the galactic centre of the Milky Way, harbours a supermassive rotating black hole with mass of 4 million solar masses [7, 8].

Previous efforts on the issue of gravitational lensing from a Kerr black hole were concentrated on various approximations as well as on numerical techniques using formal integrals [10, 11].

This paper is organized as follows: in section 2, we present the null geodesics in Kerr spacetime with a cosmological constant. In section 3, we describe the Kerr lens geometry and relate the first integrals of motion to the observer’s image plane coordinates. In section 4, we derive a formal expression for the magnification using the Jacobian that relates the observer’s image plane coordinates to the source position. This expression involves derivatives of the lens equations in the Kerr geometry and in our contribution we shall calculate in closed form.
these derivatives in terms of the generalized hypergeometric functions of Appell–Lauricella. In section 5, we derive constraints from the condition that a photon escapes to infinity and it is not caught in an (unstable) spherical orbit. These constraints on Carter’s constant and impact factor define a region usually called the shadow of the rotating black hole. For values of the initial conditions inside the region enclosed by the boundary of the shadow and the line with null value for Carter’s constant, there is no lensing effect since the photons cannot escape and reach an observer. We also discuss constraints arising from the polar motion. In section 7, we derive for the first time the closed form solution for the angular integrals involved in the gravitational Kerr lens, in terms of the generalized hypergeometric functions of Appell–Lauricella. The full exact solution for a light ray which originates from the source’s polar position and involves $m$-polar inversions before reaching the polar coordinate of the observer is derived, equations (67), (63). In section 7, we perform the analytic computation of the radial integrals involved in the Kerr lens in terms of the hypergeometric functions of Appell and Lauricella. In the same section, we derive the closed form solution for the source polar position in terms of the Weierstraß elliptic function $\wp(z, g_2, g_3)$ that implements the constraint that arises from the first lens equation (4). In section 8 (subsection 8.3), we apply our exact solutions for the calculation of the source and image positions for various values of the spin of the black hole and the first integrals of motion, for an equatorial observer and an observer located at a polar angle of $\pi/3$, respectively. We exhibit the image positions on the observer’s image plane. In appendix A, we collect the definition and the integral representation of Lauricella’s multivariable hypergeometric function $\,_{\!3}F_2$. In addition, in appendix A, we prove in the form of propositions, some mathematical results concerning the transformation properties of the function $\,_{\!3}F_2$ which are used in the main text. Finally, in appendix B we integrate exactly, the null geodesic equations for the time coordinate for the Kerr black hole in terms of the hypergeometric functions of Appell–Lauricella.

2. Null geodesics in a Kerr-(anti) de Sitter black hole

Taking into account the contribution from the cosmological constant $\Lambda$, the generalization of the Kerr solution [12] is described by the Kerr-de Sitter metric element which in Boyer–Lindquist (BL) coordinates is given by [13, 14]

$$ds^2 = \frac{\Delta_r}{\Xi^2 \rho^2} (c \, dt - a \sin^2 \theta \, d\phi)^2 - \frac{\rho^2}{\Delta_r} \, dr^2 - \frac{\rho^2}{\Delta_\theta} \, d\theta^2$$

$$- \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} \left( ac \, dt - (r^2 + a^2) \, d\phi\right)^2$$

$$\Delta_r := 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta, \quad \Xi := 1 + \frac{a^2 \Lambda}{3}, \quad \rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta_\theta := \left(1 - \frac{\Lambda}{3} r^2\right) (r^2 + a^2) - 2 \frac{GM}{c^2 r}$$

We denote by $a$ the rotation (Kerr) parameter and by $M$ the mass of the spinning black hole.

The relevant null geodesic differential equations for the calculation of the gravitational lensing effects (lens equation) and for the calculation of the deflection angle are [5]

$$\int^{\prime} \frac{dr}{\pm \sqrt{R}} = \int^{\prime} \frac{d\theta}{\pm \sqrt{\Theta}}$$

$$\int^{\prime} \frac{dr}{\pm \sqrt{R}} = \int^{\prime} \frac{d\theta}{\pm \sqrt{\Theta}}$$

$$= \int^{\prime} \frac{dr}{\pm \sqrt{R}} = \int^{\prime} \frac{d\theta}{\pm \sqrt{\Theta}}$$

3
\[ \Delta \phi = \int_{\Delta \phi} \frac{\Xi^2}{\pm \Delta_\theta \sin^2 \theta} \frac{(a \sin^2 \theta - \Phi)}{\sqrt{\Theta}} + \int^r_{\Delta r} \frac{a \Xi^2}{\pm \Delta_r \left[ (r^2 + a^2) - a \Phi \right]} \frac{dr}{\sqrt{R}}, \]

where

\[ R := \left\{ \Xi^2 \left[ (r^2 + a^2) - a \Phi \right]^2 - \Delta_r \left[ \Xi^2 \left( \Phi - a \right)^2 + \Omega \right] \right\} \]

and

\[ \Theta := \left\{ \left\{ \Omega + \left( \Phi - a \right)^2 \Xi^2 \right\} \Delta_\theta - \frac{\Xi^2 \left( a \sin^2 \theta - \Phi \right)^2}{\sin^2 \theta} \right\}. \]

We also derive the equation related to time-delay:

\[ ct = \int^r_{\Delta r} \left\{ \Xi^2 (r^2 + a^2) \left[ (r^2 + a^2) - a \Phi a \right] \frac{dr}{\pm \Delta_r \sqrt{R}} - \int^\theta_0 \frac{a \Xi^2 (a \sin^2 \theta - \Phi)}{\pm \Delta_\theta \sqrt{Theta}} \right\} d\theta \]

The parameters \( \Phi, \Omega \) are associated with the first integrals of motion [5]. The former is the impact parameter and the latter is related to the hidden first integral (due to the separation of variables in the corresponding Hamilton–Jacobi partial differential equation (PDE)).

3. The Kerr black hole as a gravitational lens

3.1. Observer’s image plane

Assume without the loss of generality that the observer’s position is at \((r_0, \theta_0, 0)\). Likewise, for the source we have \((r_s, \theta_s, \phi_s)\). We also assume in this section that \( \Delta = 0 \). In the observer’s reference frame, an incoming light ray is described by a parametric curve \( x(r), y(r), z(r) \), where \( r^2 = x^2 + y^2 + z^2 \). For large \( r \), this is the usual radial BL coordinate. At the location of the observer, the tangent vector to the parametric curve is given by \((dx/dr)|_{r_0}, \hat{\mathbf{x}} + (dy/dr)|_{r_0}, \hat{\mathbf{y}} + (dz/dr)|_{r_0}, \hat{\mathbf{z}} \). This vector describes a straight line which intersects the \((\alpha, \beta)\) plane or the observer’s image plane as it is usually called [9–11] at \((\alpha_i, \beta_i)\) see figure 1.

The point \((\alpha_i, \beta_i)\) is the point \((-\beta_i \cos \theta_0, \alpha_i, \beta_i \cos \theta_0)\) in the \((x, y, z)\) system. Our purpose now is to relate the \(\alpha_i, \beta_i\) variables to the first integrals of motion \(\Phi, \Omega\). For this we need to use the equation of straight line in space. A straight line can be defined from a point \(P_1(x_1, y_1, z_1)\) on it and a vector \(\mathbf{e}(e_1, e_2, e_3)\) parallel to it. The analytic equations of the straight line are then

\[ \frac{x - x_1}{e_1} = \frac{y - y_1}{e_2} = \frac{z - z_1}{e_3}. \]  

Applying (9) we derive the equations

\[ \frac{-\beta_i \cos \theta_0 - r_0 \sin \theta_0}{r_0 \cos \theta_0 \frac{d \theta_0}{dr}|_{r=r_0} + \sin \theta_0} = \frac{\alpha_i}{r_0 \sin \theta_0 \frac{d \theta_0}{dr}|_{r=r_0}} = \frac{-\beta_i \cos \theta_0 - r_0 \cos \theta_0}{\cos \theta_0 - r_0 \sin \theta_0 \frac{d \theta_0}{dr}|_{r=r_0}}. \]

Solving for \(\alpha_i, \beta_i\) we obtain the equations

\[ \alpha_i = -r_0^2 \sin \theta_0 \frac{d \phi}{dr}|_{r=r_0} \]  
\[ \beta_i = r_0^2 \frac{d \theta}{dr}|_{r=r_0} \]  

Now we have from the null geodesics that

\[ \frac{d \theta}{dr}|_{r=r_0} = \frac{\Theta(\theta_0)^{1/2}}{R(r_0)^{1/2}} \]

\[ \frac{d \phi}{dr}|_{r=r_0} = \frac{\Theta(\theta_0)^{1/2}}{R(r_0)^{1/2}} \]
Figure 1. The Kerr black hole gravitational lens geometry. The reference frame is chosen so that, as seen from infinity, the black hole is rotating around the $z$-axis.

and

$$\frac{d\phi}{dr}\bigg|_{r=r_O} = \Phi \frac{1}{\sqrt{R(r_O)}} + \frac{2aGM}{r_O^2} - \frac{a^2}{r_O^2} \frac{1}{\sqrt{R(r_O)}}. \quad (14)$$

Using equations (13), (14) and assuming the large observer’s distance $r_O$ (i.e. $r_O \rightarrow \infty$) we derive simplified expressions relating the coordinates $(\alpha_i, \beta_i)$ on the observer’s image plane to the integrals of motion:

$$\Phi \simeq -\alpha_i \sin \theta_O \quad (15)$$

$$Q \simeq \beta_i^2 + (\alpha_i^2 - a^2) \cos^2(\theta_O). \quad (16)$$

We can also express the position of the source on the observer’s sky in terms of its coordinates $(r_S, \theta_S, \phi_S)$ and the observer coordinates. Indeed, the equation for a straight line can be determined by two points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \quad (17)$$

Thus applying the above formula for the straight line connecting the observer and the source yields the equations

$$\alpha_S = \frac{r_or_S \sin \theta_S \sin \phi_S}{r_O - r_S (\cos \theta_S \cos \theta_O + \sin \theta_O \sin \theta_S \cos \phi_S)}$$

$$\beta_S = \frac{-r_or_S (\sin \theta_S \cos \theta_S - \sin \theta_O \sin \theta_S \cos \phi_S \cos \theta_O)}{r_O - r_S (\cos \theta_S \cos \theta_O + \sin \theta_O \sin \theta_S \cos \phi_S)}. \quad (18)$$
4. Magnification factors and the positions of images

In the following sections, we perform a detailed novel calculation of the lens effect for the deflection of light produced by the gravitational field of a rotating (Kerr) black hole and a cosmological Kerr black hole (i.e. for the non-zero cosmological constant $\Lambda$).

The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta \omega$ it subtends on the sky. Since the former quantity is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in the absence of the lens is given by

$$\mu = \frac{\Delta \omega}{(\Delta \omega)_0} = \frac{1}{|J|},$$  \hspace{1cm} (19)

where 0-subscripts denote undeflected quantities \[2\] and $J$ is the Jacobian of the transformation $(x_S, y_S) \rightarrow (x_i, y_i)$. Writing $x_S = x_S(x_i, y_i), \; y_S = y_S(x_i, y_i)$ we can find expressions for the partial derivatives appearing in the Jacobian by differentiating equations (4) and (5). Indeed, the Jacobian is given by the expression

$$J = xw - Zy, \hspace{1cm} (20)$$

where we defined $x := \frac{\partial x_S}{\partial x_i}, \; y := \frac{\partial x_S}{\partial y_i}, \; z := \frac{\partial y_S}{\partial x_i}, \; w := \frac{\partial y_S}{\partial y_i}$. Writing equations (4) and (5) as follows:

$$R_1(x_i, y_i) - A_1(x_i, y_i, x_S, y_S, m) = 0$$

$$\Delta \phi(x_S, y_S, n) - R_2(x_i, y_i) - A_2(x_i, y_i, x_S, y_S, m) = 0,$$

we set up the following system of equations:

$$\beta_1 = -\alpha_1 x - \alpha_2 z, \hspace{1cm} (22)$$

$$\beta_2 = -\alpha_1 y - \alpha_2 w$$

$$\beta_3 = \alpha_3 x + \alpha_4 z$$

$$\beta_4 = \alpha_3 y + \alpha_4 w, \hspace{1cm} (25)$$

where \[
\begin{align*}
\alpha_1 &= \frac{\partial A_1}{\partial x_i}, \; \alpha_2 = \frac{\partial A_2}{\partial y_i}, \; \alpha_3 = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_2}{\partial x_i}, \; \alpha_4 = -\frac{\partial \phi}{\partial y_i} - \frac{\partial A_2}{\partial y_i}, \\
\beta_1 &= \frac{\partial R_1}{\partial x_i} - \frac{\partial A_1}{\partial x_i}, \; \beta_2 = \frac{\partial R_1}{\partial y_i} - \frac{\partial A_1}{\partial y_i}, \; \beta_3 = \frac{\partial R_2}{\partial x_i} + \frac{\partial A_2}{\partial x_i}, \; \beta_4 = \frac{\partial R_2}{\partial y_i} + \frac{\partial A_2}{\partial y_i}.
\end{align*}
\]

Solving for $x, y, z, w$ we obtain

$$\mu = \frac{1}{|J|} = \left| \frac{\alpha_1 \alpha_4 - \alpha_2 \alpha_3}{\beta_1 \beta_4 - \beta_2 \beta_3} \right|. \hspace{1cm} (26)$$

The parameters $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ are the number of windings around the $z$-axis and the number of turning points in the polar coordinate $\theta$, respectively. We shall discuss the latter in detail in the section that follows.

\[1\] Recall in the small angle approximation $\rho_i \approx r_0 \rho_i, \; \beta_i \approx r_0 y_i$. Also we define $x_S := \frac{x}{r_0}, \; y_S := \frac{y}{r_0}.$
5. The boundary of the shadow of the rotating black hole and constraints on the parameter space

The condition for a photon to escape to infinity, which is also the condition for the spherical photon orbits in Kerr spacetime [5], is given by the vanishing of the quartic polynomial \( R(r) \) and its first derivative (also in this case \( \frac{dR}{dr} |_{r=r_j} > 0 \)). Implementing these two conditions, the expressions for the parameter \( \Phi \) and Carter’s constant \( Q \) are obtained [5, 15]:

\[
\Phi = \frac{a^2 \frac{GM}{c^2} + a^2 r - 3 \frac{GM}{c^2} r^2 + r^3}{a \left( \frac{GM}{c^2} - r \right)}, \quad Q = -r^3 \left( -4a^2 \frac{GM}{c^2} + r \left( \frac{3GM}{c^2} + r \right)^2 \right) \left( \frac{GM}{c^2} - r \right)^2
\] (27)

The perturbed, from the radius \( r = r_{\text{min}} \) of unstable spherical null orbits in Kerr spacetime, and thus escaped photon, will be detected on the observer’s image plane, at the coordinates \( y_i = \frac{\pm \sqrt{-r^3 \left[ r \left( r - \frac{3GM}{c^2} \right)^2 - 4a^2 \frac{GM}{c^2} \right] - 2a^2 r \left( 2a^2 \frac{GM}{c^2} + r^3 - 3r \frac{GM^2}{c^4} \right) z_O - a^4 \left( r - \frac{GM}{c^2} \right)^2 \right] z_O}{r_o \sin \theta_o a \left( r - \frac{GM}{c^2} \right)} \). (28)

Equations (28) were derived by plugging into equations (15), (16) the values of the parameters \( Q, \Phi \) that correspond to the conditions for the photon to escape to infinity, equations (27). A photon will be detected when the argument of the square root in equation (28) is positive. In equation (28), \( z_O := \cos^2 \theta_o \).

With the help of equations (15) and (16) we derive

\[
a^2 + \beta^2 = \Phi^2 + Q + a^2 z_o.
\] (29)

Apart from the constraints expressed by equations (27) we also derive constraints for the motion of light from the allowed polar region; \( \theta_{\text{min}} \leq \theta_j, \theta_o \leq \theta_{\text{max}} \). Indeed using the variable \( z_j := \cos^3 \theta_j \), we have \( z_m \geq z_o \) where \( z_m \) is the positive root of

\[-a^2 z_m^2 + (a^2 - Q - \Phi^2)z_m + Q = 0.
\]

Let us see how this can be understood. Defining \( z_m := z_o - x \) we derive the quadratic equation for \( x \):

\[-a^2 x^2 - x(a^2 - Q - \Phi^2 - 2a^2 z_o) - a^2 z_o (a^2 - Q - \Phi^2) + a^2 z_o = 0,
\] (30)

with roots

\[
x_{1,2} = \frac{-a^2 + Q + 2a^2 z_o + \Phi^2 \mp \sqrt{4a^2 Q + (-a^2 + Q + \Phi^2)^2}}{2a^2} = \frac{(a^2 + \beta^2) - \sqrt{((a^2 + \beta^2) - a^2 w_o)^2 + 4a^2 \beta^2 w_o}}{2a^2}.
\] (31)

where \( w_o := \sin^2 \theta_o \). The ‘radius’ \( \Phi^2 + Q \) must be greater than or equal to the boundary of the photon region defined by equations (27) and the line \( Q = 0 \). The minimum of this value is reached when \( Q = 0 \) and \( a \rightarrow 1 \). The actual minimum value is \( (\Phi^2(r) + Q(r))_{\text{min}} = 4 \). Thus, by equation (29) we have that \( a^2 + \beta^2 \geq 4 \), and since \( 0 \leq a^2 w_o \leq 1 \), it follows the inequality \( a^2 w_o - (a^2 + \beta^2) < 0 \) and consequently, \( x \leq 0 \). Thus we conclude that \( z_m \geq z_o \). Similar arguments ensure that when \( z_s > z_o \) it follows \( z_m \geq z_s \) [11].
6. Closed form solution for the angular integrals

Let us now perform the exact computation of the angular integrals which occur in the generic photon orbits in Kerr spacetime thereby generalizing the results of [5]. In the case under investigation, we have to take into account the turning points in the polar coordinate. A generic angular polar integral can be written as

\[ \pm \int_{\theta_1}^{\theta_2} \int_{\min(z_1,z_2)}^{\max(z_1,z_2)} + [1 - \text{sign}(\theta_1 \circ \theta_2)] \int_0^{\min(z_1,z_2)} , \] (32)

where

\[ \theta_1 \circ \theta_2 := \cos \theta_1 \cos \theta_2. \] (33)

Indeed, using the variable \( z = \cos^2 \theta \) we derive

\[ -\frac{1}{2} \frac{dz}{\sqrt{z}} \frac{1}{\sqrt{1-z}} = \text{sign} \left( \frac{\pi}{2} - \theta \right) d\theta. \] (34)

This is the result of the fact that in the interval \( 0 \leq \theta \leq \frac{\pi}{2} \), \( \cos \theta \geq 0 \) and \( \sin \theta \geq 0 \), while in the interval \( \frac{\pi}{2} \leq \theta \leq \pi \), \( \sin \theta \geq 0, \cos \theta \leq 0 \). The angular integration in the polar variable includes the terms

\[
\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{z_{max} - z}} + \cdots + \frac{d\theta}{\sqrt{z_{min} - z}}. \]

The roots \( z_m, z_3 \) (of \( \Theta(\theta) = 0 \)) are expressed in terms of the integrals of motion and the cosmological constant by the expressions

\[ z_m, z_3 = \frac{Q + \Phi^2 \Xi^2 - H^2 \pm \sqrt{(Q + \Phi^2 \Xi^2 - H^2)^2 + 4H^2Q}}{-2H^2} \] (36)

and

\[ H^2 := \frac{a^2 \Lambda}{3} [Q + (\Phi - a)^2 \Xi^2] + a^2 \Xi^2. \] (37)

For \( \Lambda = 0 \), the turning points take the form

\[ z_m = \frac{a^2 - Q - \Phi^2 + \sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2}}{2a^2}. \] (38)

where the subscript ‘m’ stands for ‘min/max’. The corresponding angles are

\[ \theta_{\min/\max} = \arccos(\pm \sqrt{z_m}). \] (39)

Now for \( \theta_j \) and \( \theta_{\min/\max} \) in the same hemisphere

\[ \int_{\theta_j}^{\theta_{\min/\max}} d\theta = \frac{1}{2|a|} \int_{z_j}^{z_m} \frac{dz}{\sqrt{z(z_m - z)(z - z_j)}} \equiv I_3. \] (40)

Let us now calculate the elliptic integral in equation (40) in closed analytic form. Applying the transformation

\[ z = z_m + \xi^2(z_j - z_m), \] (41)

our integral is calculated in closed form in terms of Appell’s generalized hypergeometric function \( F_1 \) of two variables:

\[ I_3 = \frac{1}{2|a| \sqrt{z_m(z_m - z_j)}} F_1 \left( \begin{array}{c} 1 \ 1 \ 1 \ 3 \ z_m - z_j \ z_m - z_j \\ 2 \ 2 \ 2 \ 2 \ \frac{z_m - z_j}{z_m - z_j} \ z_m - z_j \ z_m - z_j \ z_m - z_j \end{array} \right) \frac{\Gamma \left( \frac{1}{2} \right) \Gamma(1)}{\Gamma(3/2)}. \] (42)
On the other hand using the transformation
\[ \zeta \equiv \frac{u z_j z_m - z_j z_m}{u z_j - z_m}, \] (43)
we calculate in closed form
\[ \frac{1}{2|a|} \int_{0}^{z_j} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}} = \frac{1}{|a|} \sqrt{\frac{z_j - z_m}{z_m z_j (z - z_j)}} F_1 \left( 1, 1 \right) \frac{1}{3} \frac{z_j - z_m}{z_m (z_m - z_3)} F_1 \left( 1, 2, \frac{3}{2}, \frac{2}{2} \right) \frac{z_j (z_m - z_3)}{z_j (z_m - z_3)} \] (44)
In going from the second line to the third of (44) we made use of the following identity of Appell’s first generalised hypergeometric function of two variables:
\[ F_1(\alpha, \beta, \gamma, x, y) = (1 - x)^{-\alpha}(1 - y)^{-\beta}(1 - x)^{-\gamma}(1 - z)^{-\alpha} \]
Likewise we derive the closed form solution for the following integral:
\[ \frac{1}{2|a|} \int_{0}^{z_j} \frac{dz}{(1 - z)\sqrt{z(z_m - z)(z - z_3)}} = \frac{z_j}{z_m |a|} \frac{1}{1 - z_j} \sqrt{\frac{z_j - z_m}{z_j (z - z_j)}} \]
\[ \times F_D \left( 1, 1 - \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \frac{z_j (z_m - z_3)}{z_j (z_m - z_3)} \]
\[ = \frac{1}{|a|} \sqrt{\frac{z_j - z_m}{z_j z_m}} F_D \left( 1, 1 - \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \] (46)
Producing the last line of equation (46) we used the following formula for the Lauricella function \( F_D \):
**Proposition 1.**
\[ F_D(\alpha, \beta, \gamma, x, y, z) = (1 - y)^{-\alpha}(1 - x)^{-\beta}(1 - z)^{-\gamma} \]
\[ \times F_D \left( \gamma - \alpha, \beta, \gamma - \beta - \alpha, \gamma, x - 1, y, z - 1 \right). \]
**Proof.** Applying the transformation
\[ u = \frac{1 - y}{1 - vy} \] (47)
on the integral
\[ IR_{F_D} = \int_{0}^{1} u^{y-a-1} (1 - u)^{-\alpha}(1 - u\alpha)^{-\beta}(1 - u\alpha)^{-\gamma}(1 - u\alpha)^{-\alpha} \] (48)
we derive
\[ (1 - u)^{-\alpha-1} = \left( \frac{v(1 - y)}{1 - vy} \right)^{-\alpha}, \quad (1 - u\alpha)^{-\beta} = \left( \frac{1 - v(1 - y)}{1 - vy} \right)^{-\beta} \] (49)
\[ (1 - u\alpha)^{-\gamma} = \left( \frac{1 - y}{1 - vy} \right)^{-\gamma}, \quad (1 - u\alpha)^{-\alpha} = \left( \frac{1 - z}{1 - vy} \right)^{-\alpha} \]
and thus we obtain the result
\[
IR_{F_D} = (1 - y)^{\gamma - \alpha} (1 - x)^{-\beta} (1 - y)^{-\beta} (1 - z)^{-\beta'} \int_0^1 dv v^{\gamma - \alpha - 1} \\
\times (1 - v)^{\alpha'} (1 - vy)^{-(\gamma - \beta - \beta' - \beta'') (1 - v \frac{x - y}{x - 1})^{-\beta}} (1 - v \frac{z - y}{z - 1})^{-\beta''} \tag{50}
\]
or
\[
F_D(\alpha, \beta, \beta', \beta'', \gamma, x, y, z) = (1 - y)^{\gamma - \alpha} (1 - x)^{-\beta} (1 - z)^{-\beta'} \\
\times F_D \left( \gamma - \alpha, \beta, \gamma - \beta - \beta', \beta'', \gamma, \frac{x - y}{x - 1}, \frac{z - y}{z - 1} \right) \tag{51}\]

Likewise applying (41),
\[
I_4 := \frac{\Phi}{2|a|} \int_{\mathcal{S}_0} \frac{dz}{(1 - z) \sqrt{z(z_m - z)(z - z_3)}} \\
= \frac{\Phi}{2|a|} \int_{\mathcal{S}_0} \frac{1}{z_m \sqrt{(z_m - z)(z - z_3)}} \frac{1}{(1 - z_m)} \\
\times F_D \left( \frac{1}{2}, 1; 1, 1, 3, \frac{z_m - z_j}{1 - z_m}, \frac{z_m - z_j}{z_m - z_3}, \frac{z_m - z_j}{z_m - z_3} \right) \tag{52}\]

At this stage it is of convenience that we start making use of the compact notation of the multivariable Lauricella’s fourth hypergeometric function $F_D$ (see also appendix A), namely the notation $F_D(\alpha, \beta, \gamma, z)$ in which we use bold type to denote the $m$-tuples of beta parameters and variables of the function, i.e. $\beta = (\beta_1, \ldots, \beta_m), z = (z_1, \ldots, z_m)$.

For this purpose, we define the following tuples of numbers for the beta parameters and the variables of the function $F_D$ that will occur in our closed form solutions in the rest of the main body of the paper:

\[
z_j^1 = \left( \frac{z_j - z_m}{1 - z_m}, \frac{z_m - z_j}{z_m - z_3} \right), \quad j = 1, 2, \quad z_1^1 \equiv z_2^1, \quad z_3^1 \equiv z_5^1;
\]

\[
z_j^2 = \left( \frac{z_S - z_m}{1 - z_m}, \frac{z_m - z_S}{z_m - z_3}, \frac{z_m - z_S}{z_m - z_3} \right), \quad j = 2, 3, \quad z_1^2 \equiv z_2^2, \quad z_3^2 \equiv z_5^2;
\]

\[
z_0^2 = \left( \frac{z_S - z_m}{1 - z_m}, \frac{z_m - z_0}{z_m - z_3}, \frac{z_m - z_0}{z_m - z_3} \right), \quad z_0^2 \equiv z_3^2;\]

\[
z_0^3 = \left( \frac{z_S(1 - z_3)}{z_S - z_3}, \frac{z_S(z_m - z_3)}{z_m(z_m - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_m - z_3)} \right);
\]

\[
\beta_j^1 = \left( 2, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_j^2 = \left( 1, \frac{3}{2}, \frac{1}{2} \right), \quad \beta_j^3 = \left( 1, \frac{1}{2}, \frac{3}{2} \right);\]

\[
\beta_1^4 = \left( 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_3^4 = \left( 2, \frac{1}{2}, \frac{1}{2} \right);
\]

\[
\beta_1^5 = \left( 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_2^5 = \left( 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_3^5 = \left( 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_3^5 = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).\]
\[ \beta_4^{10} = \left( -2, 2, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_4^{11} = \left( -1, 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_4^{A1} = \left( 1, 1, \frac{1}{2}, \frac{1}{2} \right). \]

\[ \beta_4^{A2} = \left( 1, 1, -\frac{3}{2}, \frac{1}{2} \right), \quad \beta_4^{A3} = \left( -1, \frac{3}{2}, \frac{1}{2}, 1 \right). \]

and the corresponding 2-tuples for the two-variable Appell’s first hypergeometric function \( F_1 \):

\[ z_{A1} = \left( \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m} \right), \quad z_{A0} = \left( \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m} \right), \]

\[ z_{AaS} = \left( \frac{z_O - z_m}{z_m}, \frac{z_O - z_m}{z_m} \right), \quad z_{AaO} = \left( \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m} \right). \]

\[ z_{AO}^{1/4} = \left( \frac{z_m}{z_m}, \frac{z_m}{z_m} \right), \quad \beta_a^{1/4} = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_A^{a1} = \left( \frac{3}{2}, \frac{1}{2} \right). \]

Thus the term \( I_4 \), in equation (51), in the compact form just introduced, is given by the expression

\[ I_4 = \frac{\Phi}{2[a|} \int_0^{z_m} \frac{1}{\sqrt{(z_m - z)}} \frac{1}{2} F_D \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - z_m, 1, \frac{z_m}{z_m - z_3} \right) \tag{54} \]

Let us now compute exactly the term \( \pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} \) in (35):

\[ \pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2 \int_0^{z_m} \tag{55} \]

since \( \cos^2 \theta_{\min} = z_m \) and \( \theta_{\min} \circ \theta_{\max} = -z_m \).

Equation (51) for \( z_j = 0 \) becomes

\[ \Phi \frac{1}{2[a|} \int_0^{z_m} \frac{1}{\sqrt{(z_m - z)}} \frac{2}{(1 - z_m)} F_D \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1 - z_m, 1, \frac{z_m}{z_m - z_3} \right) \]

\[ = \Phi \frac{1}{2[a|} \int_0^{z_m} \frac{1}{\sqrt{(z_m - z)}} \frac{1}{(1 - z_m)} \frac{\pi}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - z_m, \frac{z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \]

\[ = \Phi \frac{1}{2[a|} \int_0^{z_m} \frac{1}{\sqrt{(z_m - z)}} \frac{1}{(1 - z_m)} \frac{\pi}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - z_m, \frac{z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \]

\[ = \Phi \frac{1}{2[a|} \int_0^{z_m} \frac{1}{\sqrt{(z_m - z)}} \frac{1}{(1 - z_m)} \frac{\pi}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - z_m, \frac{z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \]

\[ \times \left( F \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - z_m, \frac{z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \right) \tag{56} \]
On the other hand the angular integrals of the form \( \pm \int_{\theta_S}^{\theta_m} \) in equation (5) are solved in closed analytic form as follows:

\[
\pm \int_{\theta_S}^{\theta_m} = \frac{\Phi}{2 |a|} \sqrt{z_m - z_S} \frac{1}{z_m - z_S} \frac{2}{(1 - z_m)} F_D \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, z_S \right) \\
+ \left[ 1 - \text{sign}(\theta_S \circ \theta_m) \right] \frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{1}{z_S - z_3} \sqrt{z_S - z_m} (z_3 - z_S) \\
\times F_D \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, z_S \right). \tag{57}
\]

An equivalent expression for the above integral is

\[
\pm \int_{\theta_S}^{\theta_m} = \frac{\Phi}{2 |a|} \sqrt{z_m - z_S} \frac{1}{z_m - z_S} \frac{2}{(1 - z_m)} F_D \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, z_S \right) \\
+ \left[ 1 - \text{sign}(\theta_S \circ \theta_m) \right] \frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{1}{z_S - z_3} \sqrt{z_S - z_m} (z_3 - z_S) \\
\times \left[ -\frac{z_3}{1 - z_3} F_D \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, z_S \right) + \frac{1}{1 - z_3} F_1 \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, z_S \right) \right]. \tag{58}
\]

In going from equation (57) to equation (58) we made use of the functional equation of Lauricella’s hypergeometric function \( F_D \), proposition 2 (A.8), and proposition 3 which are proved in appendix A.

Now, for a light trajectory that encounters \( m \) turning points (\( m \geq 1 \)) in the polar motion we have

\[
\pm \int_{\theta_S}^{\theta_m} = \pm \int_{\theta_S}^{\theta_m} \pm \int_{\theta_S}^{\theta_m} \pm \cdots \pm \int_{\theta_S}^{\theta_m}, \tag{59}
\]

\[
= \int_{z_S}^{z_m} + \int_{z_S}^{z_m} + \int_{z_S}^{z_m} + \int_{z_S}^{z_m} + \cdots + \int_{z_S}^{z_m} \\
+ \int_{z_o}^{z_m} + \int_{z_o}^{z_m} + \int_{z_o}^{z_m} + \int_{z_0}^{z_m} \\
+ 2(m - 1) \int_{z_0}^{z_m}, \tag{60}
\]

where

\[
\theta_m O := \arccos(\text{sign}(y_i) \sqrt{z_m}) = \arccos(\text{sign}(\beta_i) \sqrt{z_m}); \tag{61}
\]

here \( y_i \) is the possible position of the image and

\[
\theta_m S := \left\{ \begin{array}{ll}
\theta_m O, & \text{m odd} \\
\pi - \theta_m O, & \text{m even}.
\end{array} \right. \tag{62}
\]

2 Recall the constraints in section 5.
Thus we have that

\[ A_2(x_i, y_i, x_s, y_s, m) = 2(m - 1) \times \left[ \Phi \frac{1}{[a] \sqrt{(z_m - z_s)} (1 - z_m)} \right] \]

\[ \times F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, -z_m, \frac{z_m}{1 - z_m}, \frac{z_m}{z_m - z_s} \right) \]

\[ + \Phi \frac{1}{2[a]} \sqrt{\frac{z_m - z_s}{z_m}} \frac{2}{z_m} (1 - z_m) \times F_D \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_m}{2}, z_s \right) \]

\[ + [1 - \text{sign}(\theta_3 \circ \theta_{m3})] \frac{\Phi}{[a]} \frac{z_m - z_s}{z_m} \frac{1}{z_m} \times F_D \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_m}{2}, z_0 \right) \]

\[ F_D \left( 1, \beta_3^A, 3, \frac{z_m}{2}, z_s \right) + \Phi \frac{1}{2[a]} \sqrt{\frac{z_m - z_s}{z_m}} \frac{2}{z_m} (1 - z_m) \times F_D \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_0}{2}, z_s \right) \]

\[ + [1 - \text{sign}(\theta_0 \circ \theta_{m0})] \frac{\Phi}{[a]} \frac{z_0 - z_m}{z_m} \frac{1}{z_m} \times F_D \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_m}{2}, z_0 \right) \]

\[ \times F_D \left( 1, \beta_3^A, 3, \frac{z_0}{2}, z_0 \right). \]  

(63)

We now calculate in closed form the angular term \( A_1(x_i, y_i, x_s, y_s, m) \) which appears in equations (4), (21).

Indeed, the angular integrals of the form \( \pm \int_{\theta_s}^{\theta_{smax}} \frac{d\theta}{\sqrt{\Theta}} \), in equation (4), are computed in closed analytic form in terms of Appell’s generalized hypergeometric function of two variables as follows:

\[ \pm \int_{\theta_s}^{\theta_{smax}} \frac{d\theta}{\sqrt{\Theta}} = \frac{1}{2[a]} \sqrt{\frac{z_m - z_s}{z_m}} F_1 \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_s}{2}, z_0^A \right) \frac{\Gamma \left( \frac{1}{2} \right) \Gamma(1)}{\Gamma(3/2)} \]

\[ + [1 - \text{sign}(\theta_3 \circ \theta_{m3})] \frac{1}{[a]} \sqrt{\frac{z_m - z_s}{z_m}} \frac{\Gamma(1/2) \Gamma(1)}{\Gamma(3/2)} \]

\[ \times F_1 \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_s}{2}, z_0^A \right) \]

(64)

Also integral (40) is calculated for \( z_j = 0 \) in terms of the ordinary Gauß’s hypergeometric function

\[ 2(m - 1) \int_0^{\frac{z_m}{\sqrt{z_m - z_0} (z_m - z_3)}} \frac{dz}{\sqrt{z_m - z_3}} \]

\[ = \frac{2(m - 1)}{2[a]} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} \pi F \left( \frac{1}{2}, 1, \frac{1}{2}, -\frac{z_m}{z_m - z_3} \right). \]

(65)

(66)

Thus we obtain

\[ A_1(x_i, y_i, x_s, y_s, m) = 2(m - 1) \frac{1}{2[a]} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} \pi F \left( \frac{1}{2}, 1, -\frac{z_m}{z_m - z_3} \right) \]

\[ + \frac{1}{2[a]} \sqrt{\frac{z_m - z_s}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_s}{2}, z_0^A \right) \frac{\Gamma(1/2) \Gamma(1)}{\Gamma(3/2)} \]

\[ + [1 - \text{sign}(\theta_3 \circ \theta_{m3})] \frac{1}{[a]} \sqrt{\frac{z_m - z_s}{z_m(z_m - z_3)}} \pi F \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_s}{2}, z_0^A \right) \]

\[ \times F_1 \left( \frac{1}{2}, \beta_3^A, 3, \frac{z_s}{2}, z_0^A \right). \]
\[ + \frac{1}{2|a|} \sqrt{\frac{z_m - z_o}{z_m - z_3}} F_1 \left( \frac{1}{2}, \beta_A^a, \frac{3}{2}, z_{A1}^{10} \right) \Gamma \left( \frac{1}{2} \right) \Gamma(1) \frac{\sqrt{\frac{z_m(z_m - z_1)}{z_m(z_m - z_3)}}}{\Gamma(3/2)} + [1 - \text{sign}(\theta_O \circ \theta_m O)] \frac{1}{|a|} \sqrt{\frac{z_m - z_3}{z_m - z_3}} \times F_1 \left( \frac{1}{2}, \beta_A^a, \frac{3}{2}, z_{A1}^{10} \right). \]  

For \( m = 0 \), i.e. for no turning points in the polar coordinate, the exact solutions for the angular integrals in equations (4), (5) become

\[ A_1(x_i, y_i, x_s, y_s) = \pm \int_{\theta_o}^{\theta_o} \int_{\theta_o}^{\theta_o} (1 - \text{sign}(\theta_S \circ \theta_O)) \int_{0}^{z_1} \int_{z_1}^{z_1} - \int_{z_2}^{z_2} (1 - \text{sign}(\theta_S \circ \theta_O)) \int_{0}^{z_1} \int_{z_3}^{z_3} \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_3)}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2} \beta_A^a \frac{3}{2} z_{A1} \right) 2 \]

and

\[ A_2(x_i, y_i, x_s, y_s) = \Phi \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_3)}{z_m}} \frac{1}{2} \frac{z_m}{z_m - z_3} \frac{1}{z_m - z_m} F_D \left( \frac{1}{2}, \beta_3^a, \frac{3}{2}, z_{LD1} \right) \]

\[ - \Phi \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_3)}{z_m}} \frac{1}{2} \frac{z_m}{z_m - z_3} \frac{1}{z_m - z_m} F_D \left( \frac{1}{2}, \beta_3^a, \frac{3}{2}, z_{LD2} \right) \]

\[ \times F_D \left( \frac{1}{2}, \beta_3^a, \frac{3}{2}, z_{LD1} \right) \]

\[ + [1 - \text{sign}(\theta_S \circ \theta_O)] \Phi \frac{1}{|a|} \sqrt{\frac{z_1}{z_m}} \frac{z_m - z_3}{z_1 - z_3} \frac{1}{z_m - z_3} \frac{1}{z_m - z_3} F_D \left( \frac{1}{2}, \beta_3^a, \frac{3}{2}, z_{LD3} \right) \]

where

\[ z_{A1} = \left( \frac{z_m - z_j}{z_m - z_3}, \frac{z_m - z_j}{z_m - z_3}, \right), \quad j = 1, 2, \]

\[ z_{A3} = \left( \frac{z_m(z_m - z_3)}{z_m - z_3}, \frac{z_m(z_m - z_3)}{z_m - z_3}, \right), \]

\[ z_{LDj} = \left( \frac{z_j - z_m}{1 - z_m}, \frac{z_j - z_m}{1 - z_m}, \frac{z_m - z_1}{z_m - z_1}, \frac{z_j - z_m}{z_m - z_1}, \right), \quad j = 1, 2, \]

\[ z_{LD3} = \left( \frac{z_j - z_m}{z_j - z_3}, \frac{z_j - z_m}{z_j - z_3}, \frac{z_j - z_m}{z_j - z_3}, \frac{z_j - z_m}{z_j - z_3}, \right) \]

and \( z_1 := \min(z_S, z_O), z_2 := \max(z_S, z_O). \)

Equations (67), (63) for \( m \geq 1 \) turning points and (68), (69) for \( m = 0 \) turning points constitute our exact results for the angular integrals which appear in (21) for the case of the vanishing cosmological constant \( \Lambda \). It is time to turn our attention to the exact computation of the radial integrals which appear in the lens equations of the Kerr black hole.
7. Closed form solution for the radial integrals

We now perform the radial integration assuming \( \Lambda = 0 \).

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form

\[
\int_0^r \to \int_\alpha^r + \int_{a_0}^r \simeq 2 \int_\alpha^\infty.
\]  

(71)

For instance, in the calculation of the azimuthal coordinate (5) the following radial integral is involved:

\[
\int_\alpha^\infty \frac{a}{\Delta} \left[(r^2 + a^2) - a \Phi\right] \frac{dr}{\sqrt{R}}.
\]  

(72)

where \( \Delta := r^2 + a^2 - 2GMc^2 \). In order to calculate the contribution to the deflection angle from the radial term we need to integrate the above equation from the distance of the closest approach (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic polynomial \( R \) (equation (6) for \( \Lambda = 0 \)) by \( \alpha, \beta, \gamma, \delta : \alpha > \beta > \gamma > \delta \). We manipulate first the terms

\[
\int_\alpha^\infty \frac{a}{\Delta} \left(\frac{r^2 + a^2}{\sqrt{R}} - \frac{a^2}{\Phi}\right) \frac{dr}{\Delta}. 
\]  

(73)

It is enough to proceed with the term

\[
\int_\alpha^\infty \frac{a}{\Delta} \frac{2GMr}{c^2} - \frac{a^2}{\Phi} \frac{dr}{\Delta}. 
\]  

(74)

Expressing the roots of \( \Delta \) as \( r_+, r_- \), which are the radii of the event horizon and the inner or Cauchy horizon, respectively, and using partial fractions we derive the expression

\[
\int_\alpha^\infty \frac{a}{\Delta} \left(\frac{2GMr}{c^2} - \frac{a^2}{\Phi}\right) \frac{dr}{\Delta} = \int_\alpha^\infty \frac{A^{\alpha\alpha}_+}{(r-r_+)\sqrt{R}} \frac{dr}{\Delta} + \int_\alpha^\infty \frac{A^{\alpha\alpha}_-}{(r-r_-)\sqrt{R}} \frac{dr}{\Delta}
\]  

\[
= \int_\alpha^\infty \frac{A^{\alpha\alpha}_+}{(r-r_+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} dr + \int_\alpha^\infty \frac{A^{\alpha\alpha}_-}{(r-r_-)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} dr
\]  

(75)

where \( A^{\alpha\alpha}_{\pm} \) are given by the equation

\[
A^{\alpha\alpha}_{\pm} = \pm \frac{r_{\pm}a2GM}{c^2 - \Delta} (r_{\pm} - r_-).
\]  

(76)

For polar orbits, \( \Phi = 0 \) and the coefficients in (76) reduce to those calculated in [5].

We organize all roots in ascending order of magnitude as follows:

\[
\alpha_{\mu} > \alpha_{\nu} > \alpha_i > \alpha_{\rho}.
\]  

(77)

3 The radial term \( \int_\alpha^\infty \frac{a}{\Delta} \frac{2GMr}{c^2} \) is cancelled from the angular term \( \int_\alpha^\infty \frac{a}{\Delta} \frac{dr}{\sqrt{\Theta}} \).

4 We have the correspondence \( \alpha_{\mu+1} = \alpha, \alpha_{\mu+2} = \beta, \alpha_{\mu-1} = r_1 = \alpha_{\mu-2}, \alpha_{\mu-3} = \gamma, \alpha_{\mu} = \delta \).
where \( \alpha_\mu = \alpha_{\mu+1}, \alpha_v = \alpha_{\mu+2}, \alpha_\rho = \alpha_\mu \) and \( \alpha_i = \alpha_{\mu-i}, i = 1, 2, 3 \), and we have that \( \alpha_{\mu-1} \geq \alpha_{\mu-2} > \alpha_{\mu-3} \). By applying the transformation

\[
 r = \frac{\omega z \alpha_{\mu+2} - \alpha_{\mu+1}}{\omega z - 1}
\]

(78)

or equivalently

\[
 z = \left( \frac{\alpha_\mu - \alpha_{\mu+2}}{\alpha_\mu - \alpha_{\mu+1}} \right) \left( \frac{r - \alpha_{\mu+1}}{r - \alpha_{\mu+2}} \right),
\]

(79)

where

\[
 \omega := \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_\mu - \alpha_{\mu+2}}.
\]

(80)

we can bring our radial integrals into the familiar integral representation of Lauricella’s \( F_D \) and Appell’s hypergeometric function \( F_1 \) of three and two variables, respectively. Indeed, we derive

\[
 \Delta \phi_\epsilon^o = 2 \left[ \int_0^{1/\omega} - A_+^o \omega (\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{dz}{H^+ \sqrt{z(1-z)(1-\kappa_+^2 z)}} \right. \\
 + \int_0^{1/\omega} A_+^o \omega^2 (\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{dz}{H^+ \sqrt{z(1-z)(1-\kappa_+^2 z)}} \\
 + \int_0^{1/\omega} - A_-^o \omega (\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{dz}{H^- \sqrt{z(1-z)(1-\kappa_-^2 z)}} \\
 \left. + \int_0^{1/\omega} - A_-^o \omega^2 (\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{dz}{H^- \sqrt{z(1-z)(1-\kappa_-^2 z)}} \right],
\]

(81)

where the moduli \( \kappa_+^2, \mu^2 \) are

\[
 \kappa_+^2 = \left( \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_\mu - \alpha_{\mu+2}} \right) \left( \frac{\alpha_{\mu+2} - \alpha_{\mu-1}}{\alpha_{\mu+1} - \alpha_{\mu-1}} \right), \quad \mu^2 = \left( \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_\mu - \alpha_{\mu+2}} \right) \left( \frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} \right).
\]

(82)

Also

\[
 H^\pm = \sqrt{\omega} (\alpha_{\mu+1} - \alpha_{\mu+2})(\alpha_{\mu+1} - \alpha_{\mu-1})^{1/2} \sqrt{\omega} \alpha_{\mu+1} - \alpha_{\mu+2} - \alpha_{\mu-3}
\]

(83)

and \( \alpha_{\mu-1} = r^\pm \). By defining a new variable \( \zeta := \omega z \) we can express the contribution \( \Delta \phi_\epsilon^o \), to the deflection angle, from the above radial terms in terms of Lauricella’s hypergeometric function \( F_D \):

\[
 \Delta \phi_\epsilon^o = 2 \left[ \frac{-2 A_+^o \sqrt{\omega} (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} F_D \left( \frac{1}{2}, \beta_+^o, \frac{3}{2}, z_+^r \right) \right. \\
 + \frac{A_+^o \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} F_D \left( \frac{3}{2}, \beta_+^o, \frac{5}{2}, z_+^r \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \\
 + \frac{-2 A_-^o \sqrt{\omega} (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} F_D \left( \frac{1}{2}, \beta_+^o, \frac{3}{2}, z_-^r \right) \\
 + \frac{A_-^o \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} F_D \left( \frac{3}{2}, \beta_+^o, \frac{5}{2}, z_-^r \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \left] \right.
\]

(84)

where we defined

\[
 z_\pm = \left( \frac{1}{\omega}, \kappa_\pm^2, \mu^2 \right).
\]

(85)
and the variables of the function $F_D$ are given in terms of the roots of the quartic and the radii of the event and Cauchy horizons by the expressions

$$\frac{1}{\omega} = \frac{\alpha_{\mu} - \alpha_{\mu+2}}{\alpha_{\mu} - \alpha_{\mu+1}} = \delta - \beta,$$
$$\kappa^2 = \frac{\alpha_{\mu+2} - \alpha_{\mu+1}}{\alpha_{\mu+1} - \alpha_{\mu+1}} = \beta - r_\pm,$$
$$\mu^2 = \frac{\alpha_{\mu+2} - \alpha_{\mu+3}}{\alpha_{\mu+1} - \alpha_{\mu+1}} = \frac{\beta - \gamma}{\alpha - \gamma}.$$  \hfill (86)

An equivalent expression is as follows:

$$\Delta \phi_{10} = 2 \left[ -2 \alpha^o \sqrt{\omega} (\alpha_{\mu+1} - \alpha_{\mu+2}) \right] + \frac{A^o \sqrt{\omega} (\alpha_{\mu+1} - \mu_{\mu+2})}{H^+} \left( -\frac{1}{\kappa^2} \right) F_1 \left( \frac{1}{2}, \beta \rho, \frac{3}{2}, \frac{R}{2} \right) \right]$$
$$\equiv R(\xi, y),$$  \hfill (87)

where $R(\xi, y) = \left( \frac{1}{\omega}, \mu^2 \right)$. In going from (84) to (87) we used the identity proven in [5] (equation (52)).

Finally, the term $\int_0^\infty \frac{dr}{\sqrt{R}}$ is calculated in closed form in terms of Appell’s first hypergeometric function of two variables:

$$\int_0^\infty \frac{dr}{\sqrt{R}} = \frac{1}{(\alpha - \gamma)(\alpha - \delta)} \frac{\Gamma(1/2) \Gamma(3/2)}{\Gamma(3/2)} F_1 \left( \frac{1}{2}, \beta \rho, \frac{3}{2}, \frac{R}{2} \right).$$  \hfill (88)

We exploit further the lens equations (21). Indeed

$$R_1(x_i, y_i) = 2(m - 1) \left( \frac{1}{2} \right) \frac{\sqrt{z_m(z_m - z_3)}}{z_m(z_m - z_3)} \pi F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m(z_m - z_3)}{3} \right)$$
$$+ \cdots = \int_{\xi_S}^{\xi_0} \frac{d\xi}{\sqrt{4 \xi^3 - g_2 \xi - g_3}}.$$  \hfill (89)

Inverting

$$\xi_S = \frac{2(88)}{1} \left( \frac{1}{2} \right) \frac{\sqrt{z_m(z_m - z_3)}}{z_m(z_m - z_3)} \pi F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m(z_m - z_3)}{3} \right) + \cdots + \epsilon),$$  \hfill (90)

while

$$- \phi_S = R_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m).$$  \hfill (91)
where \( \wp(z) \) denotes the Weierstrass elliptic function (which is also a meromorphic Jacobi modular form of weight 2) and the Weierstrass invariants are given in terms of the initial conditions by

\[
\begin{align*}
g_2 &= \frac{1}{12} (\alpha + \beta)^2 - \frac{Q\alpha}{4}, \\
g_3 &= \frac{1}{216} (\alpha + \beta)^3 - \frac{Q\alpha^2}{48} - \frac{Q\alpha\beta}{48}.
\end{align*}
\]

Also \( \alpha := -a^2, \beta := Q + \frac{\varepsilon}{a^2} \) and \( \epsilon \) is a constant of integration.

To recapitulate our exact solutions of the lens equations (21) are given by

\[
2 \int_{a}^{\infty} \frac{1}{\sqrt{R}} \, dr = A_1(x_i, y_i, x_S, y_S, m) \Leftrightarrow \frac{2}{\sqrt{(\alpha - y)(\alpha - z)}} F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right)
\]

\[
\Gamma(1/2) \Gamma(3/2) \Gamma(1/2) \Gamma(3/2) \Gamma(1/2) \Gamma(3/2)
\]

\[
\begin{align*}
&+ [1 - \text{sign} (\theta_S \circ \theta_m S)] \frac{1}{|a|} \sqrt{\frac{z_m(z_m - z_3)}{z_m - z_3}} F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) \\
&+ [1 - \text{sign} (\theta_O \circ \theta_m O)] \frac{1}{|a|} \sqrt{\frac{z_m(z_m - z_3)}{z_m - z_3}} F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right)
\end{align*}
\]

and equation (90).

In the subsequent sections we shall apply our exact solutions for the lens equations in the Kerr geometry expressed by (94), (90), (95) to particular cases which include (a) an equatorial observer: \( \theta_O = \pi/2 \Rightarrow z_O = 0 \) and (b) a generic observer located at \( \theta_O = \pi/3 \Rightarrow z_O = \frac{1}{4} \).

### 8. Positions of images, source and resulting magnifications for an equatorial observer in a Kerr black hole

In this case (\( \theta_O = \pi/2 \)), equations (15), (16) become

\[
\Phi \simeq -\alpha_i \sin \theta_O = -\alpha_i
\]

\[
Q \simeq \beta_i^2 + (a_i^2 - a^2) \cos^2 \theta_O = \beta_i^2.
\]

Thus the length of the vector on the observer’s image plane equals

\[
\sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\Phi^2 + Q}.
\]

Furthermore, we derive the equations

\[
\frac{x_S}{r_O} = \frac{r_S \sin \theta_S \sin \phi_S}{r_O - r_S \sin \theta_S \cos \phi_S}
\]
\[
\frac{y_S}{r_0} = -r_S \cos \theta_S
\]

or equivalently
\[
\frac{\alpha_S}{\beta_S} = -\tan \theta_S \sin \phi_S.
\]

8.1. Solution of the lens equation and the computation of \( \theta_S, \phi_S, \alpha, \beta \).

We now describe how we solve the lens equations (21) using the properties of the Weierstraß Jacobi modular form \( \wp(z) \) (equation (90)) and the computation of the radial and angular integrals in terms of the Appell–Lauricella hypergeometric functions (equations (88), (87), (67), (63)), respectively.

For a choice of initial conditions \( a, \Phi, \mathcal{Q} \) we determine values for the observer image plane coordinates \( \alpha, \beta \), see equation (96). Subsequently we determine the value of \( z_S \) and therefore of \( \theta_S \) that satisfies the equation\(^5\)

\[
2 \int_a^\infty \frac{1}{\sqrt{R}} \, dr = A_1(x_1, y_1, x_S, y_S, m) \Leftrightarrow \frac{2}{\sqrt{(\alpha - \gamma)(\alpha - \delta)}} \Gamma(1/2) \Gamma(3/2) F_1 \left( \frac{1}{2}, \beta^o \cdot \frac{3}{2}, z^2_A \right)
\]

\[
= 2(m - 1) \frac{1}{2|\alpha|} \sqrt{z_m(z_m - z_3)} \pi F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right)
\]

\[
+ \frac{1}{2|\alpha|} \sqrt{z_m(z_m - z_3)} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z^2_A \right)
\]

\[
+ [1 - \text{sign}(\theta_S \circ \theta_m)] \frac{1}{|\alpha|} \sqrt{z_m(z_m - z_3)} \pi F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right)
\]

\[
\text{using our exact solution for } z_S \text{ in terms of the Weierstraß elliptic function equation (90). For this we need to know at which regions of the fundamental period parallelogram the Weierstraß function takes real and negative values. Indeed, the function of Weierstraß takes the required values at the points } x = \frac{\pi}{4} + \omega', \ l \in \mathbb{R}, \text{ of the fundamental region } \wp\left( \frac{\pi}{4} + \omega'; g_2, g_3 \right) \in \mathbb{R}^+). \text{ Thus as the parameter } l \text{ varies we determine the value of } z_S \text{ that satisfies equations (90), (101).}\n\]

The quantities \( \omega, \omega' \) denote the Weierstraß half-periods. In the case under investigation \( \omega \) is a real half-period while \( \omega' \) is pure imaginary. For positive discriminant \( \Delta_c = g_2^2 - 27g_3^2 \), all three roots \( e_1, e_2, e_3 \) of \( 4z^3 - g_2z - g_3 \) are real and if the \( e_i \) are ordered so that \( e_1 > e_2 > e_3 \) we can choose the half-periods as

\[
\omega = \int_{e_1}^\infty \frac{dr}{\sqrt{4t^3 - g_2t - g_3}}, \ \omega' = i \int_{e_1}^{e_2} \frac{dr}{\sqrt{-4t^3 + g_2t + g_3}}. \]

The period ratio \( \tau \) is defined by \( \tau = \omega'/\omega \). An alternative expression for the real half-period \( \omega \) of the Weierstraß elliptic function is given by the hypergeometric function of Gauss\(^6\):

\[
\omega = \frac{1}{\sqrt{e_1 - e_3}} \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{e_2 - e_1}{e_1 - e_3} \right)
\]

\(^5\) Which is the closed form solution of the radial and angular integrals of the first of lens equations in equation (21).

\(^6\) The three roots are given in terms of the first integrals of motion by the expressions \( e_1 = \frac{1}{2}(-a^2 + Q + \Phi^2 + 3\sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2}), e_2 = \frac{1}{2}(a^2 - Q - \Phi^2), e_3 = \frac{1}{2}(-a^2 + Q + \Phi^2 - 3\sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2}) \).
Table 1. Solution of the lens equations in the Kerr geometry and the predictions for the source and image positions for an observer at $\theta_0 = \pi/2$, $\phi_0 = 0$. The number of turning points in the polar variable is 3. The values for the Kerr parameter and the impact factor $\Phi$ are in units of $GM/c^2$ while those of Carter’s constant $\mathcal{Q}$ are in units of $(GM/c)^2$.

| $a=0.6$, $\mathcal{Q}=24.64563$, $\Phi=-2.719110$ | $a=0.6$, $\mathcal{Q}=0.128$, $\Phi=3.839$ |
|---|---|
| $\alpha_i \left( \frac{GM}{c^2} \right)$ | 2.719 110 | $-3.839$ |
| $\beta_i \left( \frac{GM}{c^2} \right)$ | $-4.9644365239$ | 0.357 770 876 399 |
| $x_i \left( \frac{GM}{c^2} \right)$ | 1.359 555 | $-1.9195$ |
| $y_i \left( \frac{GM}{c^2} \right)$ | $-2.48221826$ | 0.178 885 |
| $m$ | 3 | 3 |
| $z_S$ | 0.316 100 791 499 2452 | 0.002 614 581 864 |
| $\theta_S$ | 55.79° | 87.069° |
| $\Delta \phi$ (rad) | $-11.086$ | 7.094 41 |
| $\phi_S$ | 95.1794° | 133.52° |
| $\omega$ | 0.554 534 199 020 150 3500 | 0.824 718 843 878 947 |
| $\omega'$ | 1.327 866 936 603 256 7973i | 2.940 082 845 914 9726i |

Table 2. Solution of the lens equations in the Kerr geometry and the predictions for the source and image positions for an observer at $\theta_0 = \pi/2$, $\phi_0 = 0$ for a high value of the spin of the black hole. The number of turning points in the polar variable is 3. The values for the Kerr parameter and the impact factor $\Phi$ are in units of $GM/c^2$ while those of Carter’s constant $\mathcal{Q}$ are in units of $(GM/c)^2$.

| $a=0.9939$, $\mathcal{Q}=27.0220588123$, $\Phi=-2.29885534$ |
|---|
| $\alpha_i \left( \frac{GM}{c^2} \right)$ | 2.298 855 34 |
| $\beta_i \left( \frac{GM}{c^2} \right)$ | 5.198 274 599 547 431 |
| $x_i \left( \frac{GM}{c^2} \right)$ | 1.149 427 67 |
| $y_i \left( \frac{GM}{c^2} \right)$ | 2.599 137 299 773 7154 |
| $m$ | 3 |
| $z_S$ | 0.013 784 351 851 09 |
| $\theta_S$ | 83.2575° |
| $\Delta \phi$ (rad) | $-11.243$ |
| $\phi_S$ | 104.177° |
| $\omega$ | 0.550 543 397 095 0226 |
| $\omega'$ | 1.128 870 829 886 0726i |

Having determined $\theta_S$ by the procedure we just described we determine the azimuthal position of the source $\phi_S$ by the second equation of (21):

$$-\phi_S = R_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m).$$  \hspace{1cm} (104)

Let us give an example. For the choice $\mathcal{Q} = 24.64563 \frac{GM^2}{c^2}$, $\Phi = -2.719110 \frac{GM}{c^2}$, $a = 0.6 \frac{GM}{c}$ we determine $z_S = 0.316 100 791 499 2452$, $m = 3$ and $\Delta \phi = -11.086$, $\phi_S = 95.1794^\circ$. Keeping fixed the value of the Kerr parameter we solved the lens equations for different values of Carter’s constant $\mathcal{Q}$ and impact factor $\Phi$. We exhibit our results in table 1.\(^7\)

Let us at this point, present a solution with a higher value for the Kerr parameter in table 2.\(^7\)

\(^7\) Assuming that the galactic centre region, SgrA*, is a Kerr black hole with mass: $M_{BH} = 4.06 \times 10^6 M_\odot$ and a distance from the observer to the Galactic centre: $r_0 = 8 \text{ Kpc}$, the second solution in table 1 will require an angular resolution of 19.3102$\mu$arcsec. This is in the range of experimental accuracy for both the TMT and GRAVITY experiments.
The two images of table 1 on the observer’s image plane. The value of the Kerr parameter is $a = 0.6 \frac{GM}{c^2}$, while the observer is located at $\theta_O = \pi/2$. In red is the image solution, first column of table 1, and in green is the image solution, second column of table 1.

Figure 2. The two images of table 1 on the observer’s image plane. The value of the Kerr parameter is $a = 0.6 \frac{GM}{c^2}$, while the observer is located at $\theta_O = \pi/2$. In red is the image solution, first column of table 1, and in green is the image solution, second column of table 1.

The positions of the images of tables 1 and 2 on the observer’s image plane are displayed in figures 2 and 3, respectively. In the same figures the boundary of the shadow of the spinning black hole is also displayed.

8.2. Closed form calculation for the magnifications

We outline in this subsection the closed form calculation of the resulting magnification factors. It turns out that the derivatives involved in the expression for the magnification are elegantly
computed using the beautiful property of the hypergeometric functions, namely, that the
derivatives of the hypergeometric functions of Appell–Lauricella are again hypergeometric
functions of the same type with a different set of parameters. It is a powerful property of our
formalism which we exploit to the full in what follows:
\[
\frac{\partial (58)}{\partial x_S} = \frac{\partial (58)}{\partial z_S} \frac{\partial z_S}{\partial x_S}
\]
\[
\frac{\partial (58)}{\partial z_S} = \frac{\Phi}{2 |a|} \frac{1}{z_m} \frac{1}{(1 - z_m)} \frac{1}{\sqrt{z_m - z_3}} \left( \frac{z_m - z_S}{z_m} \right)^{-1/2}
\times F_D \left( \frac{1}{2}, \beta^4, \frac{3}{2}, z_S \right)
\]
\[
+ \left( -\frac{\Phi}{2 |a|} \frac{\sqrt{(z_m - z_S)}}{z_m} \frac{1}{\sqrt{z_m - z_3}} \frac{2}{(1 - z_m)} \right) \times \left\{ F_D \left( \frac{3}{2}, \beta^4, \frac{5}{2}, z_S \right) \frac{1}{1 - z_m} \right\}
\]
\[
+ F_D \left( \frac{1}{2}, \beta^4, \frac{3}{2}, z_S \right) \frac{1}{z_m} + F_D \left( \frac{3}{2}, \beta^4, \frac{5}{2}, z_S \right) \frac{-1}{z_m - z_3}
\]
\[
\times \frac{z_S}{z_m} \frac{z_S - z_m}{z_m (1 - z_S)} \frac{1}{\sqrt{z_S(z_m - z_3)(z_3 - z_S)}} \times F_D \left( 2, \beta^4, \frac{5}{2}, z_S \right) \frac{1 - z_m}{z_m (1 - z_S)^2}
\]
\[
+ \frac{1}{2 |a|} \frac{\Gamma(1) \Gamma(1/2)}{\Gamma(3/2)} \left( \frac{z_m - z_S}{z_m (z_m - z_3)} \frac{1}{\sqrt{z_m - z_3}} \right) F_1 \left( \frac{1}{2}, \beta^4 a, \frac{3}{2}, z_A \right)
\]
\[
+ \frac{1}{2 |a|} \frac{\Gamma(1) \Gamma(1/2)}{\Gamma(3/2)} \left( \frac{z_m - z_S}{z_m (z_m - z_3)} \frac{1}{\sqrt{z_m - z_3}} \right) \times F_1 \left( \frac{3}{2}, \beta^4 a, \frac{5}{2}, z_A \right) \frac{1}{z_m}
\]
\[
+ F_1 \left( \frac{1}{2}, \beta^4 a, \frac{3}{2}, z_A \right) \frac{1}{z_m - z_3} \right\}
\]
Thus,
\[
\frac{\partial (58)}{\partial x_S} = (105) \times \left( -2 \cos \theta_S \sin \theta_S \frac{r_1^2 \sin \theta_1 \cos \theta_1 \sin \phi_1}{r_2 \cos \phi_2 \sin \phi_1} \right) J_1.
\]

Now we calculate the term \( \frac{\partial (64)}{\partial z_S} \). Indeed, calculating the derivatives w.r.t. \( z_S \) we derive the expression
\[
\frac{\partial (64)}{\partial z_S} = \frac{1}{2 |a|} \frac{\Gamma(1) \Gamma(1/2)}{\Gamma(3/2)} \left( \frac{1}{2 \sqrt{z_m (z_m - z_3) \sqrt{z_m - z_3}}} \right) F_1 \left( \frac{1}{2}, \beta^4 a, \frac{3}{2}, z_A \right)
\]
\[
+ \frac{1}{2 |a|} \frac{\Gamma(1) \Gamma(1/2)}{\Gamma(3/2)} \left( \frac{z_m - z_S}{z_m (z_m - z_3)} \frac{1}{\sqrt{z_m - z_3}} \right) \times F_1 \left( \frac{3}{2}, \beta^4 a, \frac{5}{2}, z_A \right) \frac{1}{z_m}
\]
\[
+ F_1 \left( \frac{1}{2}, \beta^4 a, \frac{3}{2}, z_A \right) \frac{1}{z_m - z_3} \right\}
\]
\[
+ \left[ 1 - \text{sign}(\theta_S \circ \theta_{ms}) \right] \left\{ \frac{1}{2 |a|} \frac{\Gamma(1) \Gamma(1/2)}{\Gamma(3/2)} \left( \frac{z_S(z_m - z_S)}{z_m (z_m - z_3)} \right)^{-1/2} \left( \frac{(-z_3) \sqrt{z_m - z_3}}{z_m (z_m - z_3)^2} \right) \right\}
\]
\[ F_{1} \left( \frac{1}{2}, \beta, \beta, \frac{3}{2}, z_{A} \right) + \frac{1}{|\alpha|} \sqrt{z_{m} - z_{3}} \]
\[ \times \left[ F_{1} \left( \frac{3}{2}, \beta, \frac{5}{2}, z_{A} \right) \left( \frac{z_{3}}{(z_{m} - z_{3})^{2}} \right) \right. \]
\[ + F_{1} \left( \frac{3}{2}, \beta, \frac{5}{2}, z_{A} \right) \left( \frac{z_{m} - z_{3}}{(z_{m} - z_{3})^{2}} \right) \right]. \tag{107} \]

Now
\[ \alpha_{1} = \frac{\partial A_{1}}{\partial x_{s}} = \left( 107 \right) \times \frac{\partial z_{s}}{\partial x_{s}} = \left( 107 \right) \times \left( -2 \cos \theta_{s} \sin \theta_{s} \times \frac{r_{s}^{2} \sin \theta_{s} \cos \theta_{s} \sin \phi_{s}}{J_{1}} \right) \tag{108} \]

and
\[ \alpha_{2} = \frac{\partial A_{2}}{\partial y_{s}} = \left( 107 \right) \times \frac{\partial z_{s}}{\partial y_{s}} = \left( 107 \right) \times \left( -2 \cos \theta_{s} \sin \theta_{s} \times \frac{-\left[ r_{s} \sin \theta_{s} \cos \phi_{s} - r_{s}^{3} \sin^{2} \theta_{s} \right]}{J_{1}} \right). \tag{109} \]

while for the \( \alpha_{3}, \alpha_{4} \) terms which contribute to the expression for the magnification, equation (26), we derive the expressions
\[ \alpha_{3} = -\frac{\partial \theta_{s}}{\partial x_{s}} \tag{110} \]
\[ = -\frac{\partial A_{2}}{\partial x_{s}} = \left( \frac{r_{s} \sin \theta_{s} \cos \phi_{s}}{J_{1}} \right) \]
\[ \alpha_{4} = \frac{\partial \phi_{s}}{\partial y_{s}} \tag{111} \]
\[ = -\frac{\partial A_{2}}{\partial y_{s}} = \left( \frac{-r_{s} \sin \theta_{s} \cos \phi_{s}}{J_{1}} \right) \]

where \( J_{1} \) denotes the Jacobian
\[ J_{1} = \frac{\partial(x_{s}, y_{s})}{\partial(\theta_{s}, \phi_{s})} \tag{112} \]

and
\[ \frac{\partial \theta_{s}}{\partial x_{s}} = \frac{\left( r_{s}^{2} \sin \theta_{s} \cos \theta_{s} \sin \phi_{s} \right) / \left( (r_{s} \sin \theta_{s} \cos \phi_{s})^{2} \right)}{J_{1}} \]
\[ \frac{\partial \theta_{s}}{\partial y_{s}} = \frac{-\left[ r_{s} \sin \theta_{s} \cos \phi_{s} - r_{s}^{3} \sin^{2} \theta_{s} \right] / \left( (r_{s} \sin \theta_{s} \cos \phi_{s})^{2} \right)}{J_{1}} \]
\[ \frac{\partial \phi_{s}}{\partial x_{s}} = \frac{-\left( r_{s} \sin \theta_{s} \cos \phi_{s} \right) / \left( (r_{s} \sin \theta_{s} \cos \phi_{s})^{2} \right)}{J_{1}} \]
\[ \frac{\partial \phi_{s}}{\partial y_{s}} = \frac{r_{s} \sin \theta_{s} \cos \phi_{s} / \left( (r_{s} \sin \theta_{s} \cos \phi_{s})^{2} \right)}{J_{1}} \tag{113} \]

In producing the results exhibited in equations (105), (107) in our calculations for the magnification factors we made use of the important identity of Appell’s hypergeometric function \( F_{1} \):
\[ \frac{\partial^{m+n}}{\partial x^{m} \partial x^{n}} F_{1}(a, \beta, \gamma, x, y) = \frac{(a, m + n)(b, m)(\beta', n)}{J_{1}} \times F_{1}(a + m + n, b + m, \beta' + n, y + m + n, x, y) \tag{114} \]
and its corresponding generalization for the fourth hypergeometric function of Lauricella. Similar calculations that we do not exhibit in this paper lead to the derivation of the coefficients $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$ in terms of the generalized hypergeometric functions of Appel–Lauricella.

A phenomenological analysis of our exact solutions for the magnifications in Kerr spacetime will be a subject of a separate publication [16].

8.3. Source and image positions for an observer located at $\theta_O = \frac{\pi}{3}$

In this case, the coordinates on the observers image plane are related to the first integrals of motions as follows:

$$\Phi = -a_i \frac{\sqrt{3}}{2}, \quad Q = \beta|^2 + \left(\frac{4\Phi^2}{3} - a^2\right) \frac{1}{4}; \tag{115}$$

Furthermore, our solution for the first lens equation (94) takes the form

$$2 \int_a^\infty \frac{1}{\sqrt{R}} \, dr = A_1(x_1, y_1, x_S, y_S, m) \Leftrightarrow \frac{2}{(\alpha - \gamma)(\alpha - \delta)} \Gamma(1) \left[ \frac{1}{2}, \frac{\beta^{\alpha}{\gamma}{\delta}}{2}, \frac{3}{2}, z_A \right] \Gamma(3/2) \left[ \frac{1}{2}, \frac{1}{2}, 2, z_m - z_3 \right]$$

$$+ \frac{1}{2} \sqrt{(z_m - z_3)^2 - \frac{2}{a}} F_1 \left[ \frac{1}{2}, \frac{\beta^{\alpha}{\gamma}{\delta}}{2}, \frac{3}{2}, z_A \right] \Gamma \left( \frac{1}{2} \right) \Gamma(1) \Gamma(3/2)$$

$$+ \left[ 1 - \text{sign}(\theta_m \circ \theta_O) \right] \frac{1}{2} \sqrt{(z_m - z_3)^2 - \frac{2}{a}} F_1 \left[ \frac{1}{2}, \frac{\beta^{\alpha}{\gamma}{\delta}}{2}, \frac{3}{2}, z_A \right] \Gamma \left( \frac{1}{2} \right) \Gamma(1) \Gamma(3/2)$$

$$+ \left[ 1 - \text{sign}(\theta_m \circ \theta_O) \right] \frac{1}{2} \sqrt{(z_m - z_3)^2 - \frac{2}{a}} F_1 \left[ \frac{1}{2}, \frac{\beta^{\alpha}{\gamma}{\delta}}{2}, \frac{3}{2}, z_A \right] \Gamma \left( \frac{1}{2} \right) \Gamma(1) \Gamma(3/2) \tag{116}$$

Let us now give examples of solutions for the images and source positions of equations (116), (90).

For the initial conditions $Q = 25.64563$, $a = 0.9939$, $\Phi = -3.11$ we calculate the parameter $z_S$ of the source latitude position to be $z_S = 0.090978 208 48 (\theta_O = 72.4447^\circ)$ for $m = 3$. The position at the fundamental period parallelogram that provides the above value of $z_S$ as a solution of (116), (90) for three turning points in the polar variable is located at

$$\omega = 0.527 923 388 586 882 28, \quad \omega' = 1.119 903 617 249 492 i \tag{117}$$

Also the azimuthal position of the source was calculated to be, using (95) and the calculated value of $z_S$, $\phi_S = 2.672 315 89 \text{ rad} = 153.112^\circ = 551.205^\circ$.

The first solution is shown on the image plane of the observer, figure 4. We observe that the solution lies close to the boundary of the shadow of the black hole.

---

[8] $\Delta \phi = R_2(x_1, y_1) + R_3(x_1, y_1, x_3, y_3, m)$ was calculated to be $\Delta \phi = -12.0971 \text{ rad}$ so that the photons perform more than one loop and a half around the black hole.
Figure 4. The solution, first column of table 3, as it will be detected on the observer image plane by an observer at $\theta_O = \pi/3, \phi_O = 0$. The boundary of the shadow of the black hole is also exhibited.

Table 3. Solution of the lens equations in the Kerr geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/3, \phi_O = 0$. The number of turning points in the polar variable is 3. The values for the Kerr parameter and the impact factor are in units of $\frac{GM}{c^2}$.

|       |       |       |
|-------|-------|-------|
|       | $a = 0.9939, Q = 25.64563, \alpha_i(\frac{GM}{c^2})$ | $a = 0.52, Q = 23.64563, \Phi = -2.85$ |
| $\Phi$ | $3.591 118 674$ | $3.290 89$ |
| $\beta_i(\frac{GM}{c^2})$ | $-4.761 150 6980$ | $-4.583 200 84657$ |
| $x_i(\frac{1}{r_0} \frac{GM}{c^2})$ | $1.795 56$ | $1.645 45$ |
| $y_i(\frac{1}{r_0} \frac{GM}{c^2})$ | $-2.380 58$ | $-2.291 6004$ |
| $m$ | $3$ | $3$ |
| $z_S$ | $0.090 978 208 48$ | $0.598 017 107 2414$ |
| $\theta_S$ | $72.4447^\circ$ | $39.3474^\circ$ |
| $\Delta \phi$ (rad) | $-12.0971$ | $-11.8577$ |
| $\phi_S$ | $153.112^\circ$ | $139.395^\circ$ |
| $\omega$ | $0.527 923 388 586 882 28$ | $0.557 102 642 750 1503$ |
| $\omega'$ | $1.119 903 617 249 492i$ | $1.389 041 935 594 241i$ |

9. Exact solution of the angular integrals in the presence of the cosmological constant $\Lambda$.

There has been a discussion in the literature as to whether or not the cosmological constant contributes to the gravitational lensing. However, the debate has been restricted to the Schwarzschild-de Sitter spacetime [17–19]. Let us now discuss the more general case of gravitational lensing in the Kerr-de Sitter spacetime.
The generalized solution for the angular integral (57) in the presence of $\Lambda$ is given by
\[
\pm \int_{\phi_s}^{\phi_{\text{max}}/\Lambda} \frac{Z^2}{2|H|} \frac{z_m - z_S}{\lambda_1} \left( 1 - \eta z_m \right) \sqrt{z_m (z_m - z_3)(z_m - z_1)}
\]
\[
\times \left\{ + \frac{\Phi}{(1 - z_m)} F_D \left( \frac{1}{2}, \beta_4^1 A 3 \right) \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} - \alpha F_D \left( \frac{1}{2}, \beta_3^2 A 3 \right) \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \right\}
\]
\[
+ (1 - \text{sign}(\theta_S \circ \theta_{\text{max}})) \left[ \frac{Z^2}{2|H|} \frac{z_m - z_S}{\lambda_1} \left( 1 - \eta z_S \sqrt{z_S (z_S - z_m)(z_S - z_3)} \right)
\]
\[
\times \left\{ - \alpha F_D \left( 1, \beta_3^2 A 3 \right) + \frac{\Phi}{1 - z_S} F_D \left( 1, \beta_4^1 A 3 \right) \right\} \right],
\]
(118)
where
\[
\eta := - \frac{a^2 \Lambda}{3}, \quad \mu = \frac{z_S - z_m}{z_m - z_S}, \quad \lambda = \frac{z_S - z_m}{1 - \eta z_m}, \quad \nu = \frac{z_S - z_m}{1 - z_S}
\]
(119)
and
\[
z^{a1}_A := \left( \eta z_S - z_m \right) \frac{z_S - z_m}{1 - \eta z_m}, \quad z^{a1}_A := \frac{z_m - z_S}{z_m - z_3}, \quad z^{a2}_A := \frac{z_m - z_S}{z_m - z_3}, \quad z^{a3}_A := \frac{z_S - z_m}{z_m - z_3},
\]
\[
z^{a4}_A := \left( \lambda, \nu, \frac{z_S - z_m}{z_m} \right), \quad z^{a4}_A := \left( \lambda, \nu, \frac{z_S - z_m}{z_m} \right).
\]
(120)

Also the integrals $\pm \int_{\phi_{\text{max}}/\Lambda}^{\phi_{\text{max}}/\Lambda} = 2 \int_0^{z_m} \frac{Z^2}{2|H|} \left( 1 - \eta z_m \right) \sqrt{z_m (z_m - z_3)(z_m - z_1)}
\]
\[
\times \left\{ + \frac{\Phi}{(1 - z_m)} F_D \left( \frac{1}{2}, \beta_4^1 A 3 \right) \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} - \alpha F_D \left( \frac{1}{2}, \beta_3^2 A 3 \right) \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \right\}
\]
\[
+ (1 - \text{sign}(\theta_S \circ \theta_{\text{max}})) \left[ \frac{Z^2}{2|H|} \frac{z_m - z_S}{\lambda_1} \left( 1 - \eta z_S \sqrt{z_S (z_S - z_m)(z_S - z_3)} \right)
\]
\[
\times \left\{ - \alpha F_D \left( 1, \beta_3^2 A 3 \right) + \frac{\Phi}{1 - z_S} F_D \left( 1, \beta_4^1 A 3 \right) \right\} \right] \right],
\]
(121)
where the tuples of numbers $z^{a1}_A z^{a2}_A$ appearing in (121) are defined by setting $z_S = 0$ in the tuples of numbers $z^{a1}_A, z^{a2}_A$, respectively. Note that for $\Lambda = 0$ this reduces to equation (56).

9.1. Closed form solution for the radial integrals in the presence of the cosmological constant $\Lambda$.

Assume that $\Lambda > 0$. We need to calculate the radial integrals of the form
\[
\int \frac{a^2 \Xi^2 (r^2 + a^2 - a \Phi)}{\Delta r} dr.
\]
(122)
We use the technique of partial fractions from integral calculus
\[
\frac{a^2 \Xi^2 (r^2 + a^2 - a \Phi)}{\Delta r} = \frac{A^1}{r - r^*_A} + \frac{A^2}{r - r^*_A} + \frac{A^3}{r - r^*_A} + \frac{A^4}{r - r^*_A}
\]
(123)
where \( r_1^\pm, r_2^\pm, r_3, r_4 \) are the four real roots of \( \Delta_r \).

For instance, for \( r_0, r_\omega < r_\Lambda^+ \) one of the integrals we need to calculate is

\[
\frac{1}{\sqrt{\frac{1}{2}(Q \Lambda + 3 \Xi^2(1 + \frac{1}{a}(\alpha - \Phi)^2))}} \int_{r_o}^{r_{1/2}} \frac{A^1 dr}{(r - r_1^\pm)(r - r_2^\pm)(r - \gamma)(r - \delta)}.
\]

(124)

Indeed, we compute in closed form

\[
\int_{r_o}^{r_{1/2}} \frac{A^1 dr}{(r - r_1^\pm)(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)} = \frac{\rho_1}{\sqrt{\rho_1}} H^+_{\Lambda^*}
\]

\times F_D \left( \frac{1}{2}, \frac{\beta_1}{\Lambda^*}, \frac{3}{2}, \frac{z_{\Lambda^*}}{r} \right) \Gamma(1/2) \Gamma(3/2),
\]

(125)

where

\[
\rho_1 := \frac{r_1^+ - r_1^- - 2\alpha}{r_1^- - r_1^+ - 2\beta},
\]

\[
z_{\Lambda^*} := \left( r_1^+ - 2\alpha \beta - \gamma - r_1^- - 2\alpha \beta - \delta - r_1^+ - 2\alpha \beta - r_1^- - 2\beta \right) \left( r_1^- - 2\alpha \beta - \gamma - r_1^+ - 2\alpha \beta - \delta - r_1^+ - 2\alpha \beta - r_1^- - 2\beta \right)
\]

\[
H^+_{\Lambda^*} := \frac{\alpha - \beta}{\beta - \alpha} \frac{1}{r_{\Lambda^*} - \beta \sqrt{\omega(\gamma - \alpha)(\delta - \alpha)}}.
\]

Also the radial integral involved in the lhs in the ‘balance’ lens equation (4) is computed exactly in terms of the hypergeometric function of Appell \( F_1 \):

\[
\frac{1}{\sqrt{\frac{1}{2}(Q \Lambda + 3 \Xi^2(1 + \frac{1}{a}(\alpha - \Phi)^2))}} \int_{r_o}^{r_{1/2}} \frac{dr}{\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}}
\]

\[
= \frac{\rho_1}{\sqrt{\mathcal{E}}} \frac{1}{\sqrt{\omega(\gamma - \alpha)(\delta - \alpha)}} \Gamma(1/2) \Gamma(3/2) F_1 \left( \frac{1}{2}, \frac{\beta_1}{\Lambda^*}, \frac{3}{2}, \frac{z_{\Lambda^*}}{r} \right),
\]

(127)

where \( \mathcal{E} := \frac{1}{2}(Q \Lambda + 3 \Xi^2(1 + \frac{1}{a}(\alpha - \Phi)^2)) \), \( \omega := \frac{z_{\Lambda^*}}{r}, z_{\Lambda^*} := \left( \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\beta - \delta}{\alpha - \delta}, \frac{\beta - r_1^+}{\alpha - r_1^+}, \frac{\beta - r_1^-}{\alpha - r_1^-} \right) \) and \( \alpha, \beta, \gamma, \delta \) denote the roots of the quartic polynomial \( R \) in the presence of \( \Lambda \) equation (6).

Likewise, the generalization of equation (90) is given by

\[
\xi_s = \varphi(2 \times (127) + \cdots + \epsilon),
\]

(128)

where the Weierstrass invariants take the form

\[
g_2 = \frac{1}{12} (\alpha_\Lambda + \beta_\Lambda)^2 - Q \frac{\alpha_\Lambda}{4},
\]

\[
g_3 = \frac{1}{216} (\alpha_\Lambda + \beta_\Lambda)^3 - Q \frac{\alpha_\Lambda}{48} - \frac{Q \alpha_\Lambda \beta_\Lambda}{48}
\]

(129)

and

\[
\alpha_\Lambda := -H^2, \quad \beta_\Lambda := Q + \Phi^2 \Xi^2.
\]

(130)

A complete phenomenological analysis of our exact solutions in the presence of the cosmological constant \( \Lambda \) will be a subject of a separate publication [16]. Nevertheless, it is evident from the closed form solutions we derived in this work that the cosmological constant does contribute to the gravitational bending of light.
10. Conclusions

In this work the precise analytic treatment of Kerr and Kerr-de Sitter black holes as gravitational lenses has been achieved. A full analytic strong-field calculation of the source, image positions and the resulting magnification factors has been performed. A full blend of important functions from mathematical analysis such as the Weierstraß elliptic function \( \wp \) and the generalized multivariable hypergeometric functions of Appell–Lauricella \( F_D \) were deployed in deriving the closed form solution of the gravitational lens equations. From the exact solution of the radial and angular Abelian integrals which are involved in the lens equations we concluded that \( \Lambda \) does contribute to the gravitational bending of light. A full quantitative phenomenological analysis of gravitational lensing by a Kerr deflector in the presence of \( \Lambda \) is beyond the scope of this work and will appear elsewhere [16]. We provided examples of image–source configurations that solve the gravitational Kerr lens equations and exhibited their appearance on the observer’s image plane as they will be detected by an equatorial observer \((\theta_O = \pi/2, \phi_O = 0)\) and an observer located at \((\theta_O = \pi/3, \phi_O = 0)\), for various values of the Kerr parameter \( a \), and the first integrals of motion \( \Phi, Q \). The resulting solutions lie close to the boundary of the shadow of the black hole.

The theory produced in this work based on the exact solution of the null geodesic equations of motion in Kerr spacetime will have an important application to the Sgr A∗ galactic centre supermassive black hole [16]. It may serve the important goal of probing general relativity at the strong-field regime through the phenomenon of the gravitational bending of light induced by the spacetime curvature. It is complementary to other investigations which have the ambition to probe gravitation at the strong-field regime through the relativistic effects of periastron precession and frame dragging [20].

If, the now under development, near-infrared interferometric GRAVITY experiment [8] and the proposed thirty metre telescope (TMT) [7] reach the aimed accurcy of 10 \( \mu \)arcs and in combination with very long baseline interferometry (VLBI) observations, then it may be possible for these experiments to detect the effects of the strong light bending (the shadow) by the galactic centre black hole [8]. The observation of this shadow would be direct evidence of an event horizon.

Another interesting application of our work would be to investigate the polarization vector and the rotation of the plane of polarization of light rays passing near the spinning black hole. In order to study the propagation of light polarization, the solution of parallel transport problem for the null geodesics is required [21]. Marck [23] has constructed a quasi-orthonormal tetrad that is propagated parallel along the null geodesics in Kerr spacetime. His approach makes use of the Killing–Yano tensor discovered by Penrose and Floyd [22]. The authors in [24] have investigated the rotation of the polarization plane for specific light rays in the weak-field approximation (slow rotation) in the Kerr metric by applying the null tetrad formalism of Newman–Penrose [25]. Also the authors of [26] performed a perturbative calculation of the rotation of the polarization plane in Kerr spacetime utilizing the Penrose–Walker constant [21]. However, an investigation of the problem of light polarization for generic null orbits beyond the weak-field limit is still lacking and it will require the exact (non-perturbative) solutions we derived in our paper. The resolution of this interesting problem is beyond the scope of the current work and will be the theme of a separate publication.

There is a fruitful synergy of various fields of science: general relativity, astronomy, cosmology and pure mathematics.
Acknowledgments

The author is obliged to C E Vayonakis for discussions and comments on the manuscript. He would like to thank L Perivolaropoulos and P Kanti for their interest in his work. He warmly thanks his colleagues and his undergraduate students at the Physics department, University of Ioannina, for a stimulating academic environment. He is grateful to the referees for their constructive comments which improved the presentation of this work. In addition, he thanks G Kakarantzas for discussions and his friendship. Last but not the least, he thanks his family for moral support during the early stages of this work.

Appendix A. Transformation properties of Lauricella’s hypergeometric function \( F_D \)

In this appendix we prove the useful transformation properties of Lauricella’s hypergeometric function \( F_D \). We first introduce the function and its integral representation

Lauricella’s 4th hypergeometric function of \( m \)-variables.

\[
F_D(\alpha, \beta, \gamma, z) = \sum_{n_1, n_2, \ldots, n_m=0}^{\infty} \frac{(\alpha)^{n_1+\cdots+n_m}}{(\gamma)^{n_1+\cdots+n_m}} \frac{(\beta_1)_{n_1}}{(1)^{n_1}} \cdots \frac{(\beta_m)_{n_m}}{(1)^{n_m}} z_1^{n_1} \cdots z_m^{n_m},
\]
(A.1)

where

\[ z = (z_1, \ldots, z_m), \]
\[ \beta = (\beta_1, \ldots, \beta_m). \]
(A.2)

The Pochhammer symbol \((\alpha)_m = (\alpha, m)\) is defined by

\[
(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } m = 0 \frac{\Gamma(\alpha + 1) \cdots (\alpha + m - 1)}{\Gamma(\alpha)} & \text{if } m = 1, 2, 3. \end{cases}
\]
(A.3)

With the notations \( z^n := (z_1^n, \ldots, z_m^n), \) \( (\beta)_n := (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}, \) \( n! = n_1! \cdots n_m!, \) \( |n| := n_1 + \cdots + n_m \) for \( m \)-tuples of numbers in (A.2) and of non-negative integers \( n = (n_1, \ldots, n_m) \) the Lauricella series \( F_D \) in compact form is

\[
F_D(\alpha, \beta, \gamma, z) := \sum_{n} \frac{(\alpha)_m(\beta)_n}{(\gamma)_m n!} z^n.
\]
(A.4)

The series admits the following integral representation:

\[
F_D(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 u^{\alpha-1} (1 - z t)^{-\beta_1} \cdots (1 - z_m t)^{-\beta_m} \frac{u^{\gamma-a-1}}{1 - u} du,
\]
(A.5)

which is valid for \( \text{Re}(\alpha) > 0, \text{Re}(\gamma - \alpha) > 0 \). It converges absolutely inside the \( m \)-dimensional cuboid:

\[
|z_j| < 1, \quad j = 1, \ldots, m.
\]
(A.6)

For \( m = 2, F_D \) in the notation of Appell becomes the two variable hypergeometric function \( F_1(\alpha, \beta, \beta', \gamma, x, y) \) with integral representation

\[
\int_0^1 u^{\alpha-1} (1 - u)^{\gamma-a-1} (1 - ux)^{-\beta} (1 - uy)^{-\beta'} du = \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y).
\]
(A.7)
Proposition 2. The following holds:

\[ F_D(\alpha, \beta, \beta', \beta'', \gamma, x, y, z) = (1 - z)^{-\alpha} \]

\[ \times F_D\left(\alpha, \beta, \beta', \gamma - \beta - \beta' - \beta'', \gamma, \frac{z - x}{z - 1}, \frac{z - y}{z - 1}, \frac{z}{z - 1}\right) \]  

(A.8)

Proof. Applying the transformation in equation (48)

\[ u = \frac{v}{1 - z + vz} = \frac{v}{(1 - z)\left[1 - \frac{vz}{z - 1}\right]} \]  

(A.9)

we obtain

\[ 1 - u = \frac{1 - v}{1 - \frac{vz}{z - 1}}, \quad (1 - ux)^{-\beta} = \left(\frac{1 - \frac{v(z - 1)}{z - 1}}{1 - \frac{vz}{z - 1}}\right)^{-\beta} \]

\[ (1 - uy)^{-\beta'} = \left(\frac{1 - \frac{v(z - 1)}{z - 1}}{1 - \frac{vz}{z - 1}}\right)^{-\beta'}, \quad (1 - uz)^{-\beta''} = \left(\frac{1}{1 - \frac{vz}{z - 1}}\right)^{-\beta''} \]  

(A.10)

Thus

\[ IRF_D = (1 - z)^{-\alpha} \times \]

\[ \int_0^1 dv^v u^{-1}(1 - v)^{\gamma - a - 1}\left(1 - \frac{v - x}{z - 1}\right)^{-\beta}\left(1 - \frac{v - y}{z - 1}\right)^{-\beta'}\left(1 - \frac{v - z}{z - 1}\right)^{-\beta''}(\gamma - \beta - \beta' - \beta'') \]  

(A.11)

and proposition follows. □

Proposition 3. The following identity holds:

\[ \frac{1}{|d|} \left\{ \frac{z_m - z_3}{z_m} \frac{1}{z_j - z_3} \frac{1}{\sqrt{z_m - z_3}} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, \frac{z_j (1 - z_3)}{z_j - z_3}, \frac{z_j (z_m - z_3)}{z_j - z_3}, \frac{z_j}{z_j - z_3}\right) \right\} \]

\[ = \frac{1}{|d|} \left\{ -\frac{z_3}{1 - z_3} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, \frac{z_j (1 - z_3)}{z_j - z_3}, \frac{z_j (z_m - z_3)}{z_j - z_3}, \frac{z_j}{z_j - z_3}\right) \right\} \]

\[ + \frac{1}{1 - z_3} F_1\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, \frac{z_j (z_m - z_3)}{z_j - z_3}, \frac{z_j}{z_j - z_3}\right) \]  

(A.12)

Proof. We start with the integral representation of Lauricella’s hypergeometric function

\[ F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2, x_1, x_2, x_3\right) = \int_0^1 du u^{1/2}(1 - ux_1)^{-1/2}(1 - ux_2)^{-1/2}(1 - ux_3)^{1/2} = \frac{\Gamma(3/2)}{\Gamma(1/2)} \]  

\[ = \frac{\Gamma(3/2)}{\Gamma(1/2)} \int_0^1 du \left[ \frac{1}{\sqrt{u}}\frac{1}{1 - ux_1}\frac{1}{\sqrt{1 - ux_2}}\frac{1}{\sqrt{1 - ux_3}} \right] \]

\[ = \frac{\Gamma(3/2)}{\Gamma(1/2)} \left[ 1 - \int_0^1 du \left(\frac{1}{\sqrt{u}(1 - ux_1)}\frac{1}{\sqrt{1 - ux_2}}\frac{1}{\sqrt{1 - ux_3}} \right) \right] \]  

(A.13)

with \( x_1 := \frac{z_j (1 - z_3)}{z_j - z_3}, x_2 := \frac{z_j (z_m - z_3)}{z_j - z_3}, x_3 := \frac{z_j}{z_j - z_3}. \) □
Appendix B. Time-delay assuming vanishing $\Lambda$

For the time-delay, in the case of the vanishing cosmological constant, we derive the equation

$$\tau = \int \frac{r^2 (a^2 + a'^2)}{\pm \Delta \sqrt{R}} \, dr + \int \frac{2GMr}{\pm c^2 \Delta \sqrt{R}} (a^2 - \Phi) \, dr + \int \frac{a^2 \cos^2 \theta \, d\phi}{\pm \sqrt{\Theta}}. \quad (B.1)$$

It is convenient to define

$$z_{zm}^t = \left( \frac{\beta - r_s r_s - \alpha}{\alpha - r_s r_s - \beta} \right), 
\quad z_{zm}^r = \left( \frac{\beta - r_s r_s - \alpha}{\alpha - r_s r_s - \beta} \right), 
\quad z_{zm}^\delta = \left( \frac{\beta - r_s r_s - \alpha}{\alpha - r_s r_s - \beta} \right), 
\quad z_{zm}^\gamma = \left( \frac{\beta - r_s r_s - \alpha}{\alpha - r_s r_s - \beta} \right). \quad (B.2)$$

In calculating the last angular term in (B.1) and using the variable $z = \cos^2 \theta$, one of the integrals we need to calculate is

$$\frac{1}{2} \int_0^1 \frac{a^2}{|a|} \int_0^{z_j} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}}. \quad (B.3)$$

Indeed, its calculation in closed analytic form gave us the result

$$\frac{1}{2} \frac{a^2}{|a|} \int_0^{z_j} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}} = \frac{1}{2} \frac{a^2}{|a|} \frac{z_j^2 (z_m - z_j)}{z_m \sqrt{(z_m - z)(z_3 - z_j)}} \times F_1 \left( 1, \frac{3}{2}, \frac{5}{2}, \frac{1}{2}; \frac{3}{2}, \frac{5}{2}; \frac{z_j}{z_m}, \frac{z_j}{z_3} \right) \frac{\Gamma(1) \Gamma(3/2)}{\Gamma(5/2)}. \quad (B.4)$$

In total we derive for the angular integrals in (B.1)

$$\int_0^\theta \frac{a^2 \cos^2 \theta \, d\theta}{\pm \sqrt{\Theta}} = A_{\text{time-delay}} = \frac{1}{2} \frac{a^2}{|a|} \frac{z_m(z_m - z_3)}{z_m \sqrt{(z_m - z)(z_3 - z_j)}} \times F_1 \left( 1, \frac{3}{2}, \frac{5}{2}, \frac{1}{2}; \frac{3}{2}, \frac{5}{2}; \frac{z_m(z_m - z_3)}{z_m \sqrt{(z_m - z_3)(z_m - z_j)}} \frac{\Gamma(1) \Gamma(3/2)}{\Gamma(5/2)} \right), \quad (B.5)$$

We now turn our attention to the calculation of the radial contribution to time-delay in equation (B.1). Assume $r_0 = r_s$. The first term can be written as
\[ \int_a^{r_s} \frac{r^2 (r^2 + a^2)}{\Delta \sqrt{R}} \, dr = \int_a^{r_S} \frac{r^2 \, dr}{\sqrt{R}} + \int_a^{r_S} \frac{2GMr}{c^2 \sqrt{R}} \, dr - \int_a^{r_S} \frac{2a^2 GMr \, dr}{c^2 \Delta \sqrt{R}} \]
\[ + \frac{4G^2 M^2}{c^4} \int_a^{r_S} \left( \frac{1 - \frac{a^2}{\Delta \sqrt{R}}}{\sqrt{R}} \right) \, dr. \] (B.6)

In total for this radial term the exact integration yields the result
\[ \int_a^{r_S} \frac{r^2 (r^2 + a^2)}{\Delta \sqrt{R}} \, dr = \frac{\alpha^2 \Omega'' z_S}{r_{S-a}} \times F_D \left( \frac{1}{2}, \beta_4^o \cdot \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \]
\[ + \frac{\alpha \Omega'' z_S \cdot 2GM}{c^2} \times F_D \left( \frac{1}{2}, \beta_4^o \cdot \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \frac{\Gamma(3/2)}{\Gamma(3/2)} \]
\[ - \frac{z_S \Omega''}{(r_s - a) \sqrt{r_{S-a}}} \left( A_{s=0} - \frac{4G^2 M^2}{c^4} A_{s=1} \right) \]
\[ \times \left\{ F_D \left( \frac{1}{2}, \beta_4^o \cdot \frac{3}{2}, \frac{3}{2} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} - \frac{r_s - a}{r_S - \beta} F_D \left( \frac{3}{2}, \beta_4^o \cdot \frac{5}{2}, \frac{5}{2} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\} \]
\[ - \frac{z_S \Omega''}{(r_s - a) \sqrt{r_{S-a}}} \left( A_{s=0} - \frac{4G^2 M^2}{c^4} A_{s=1} \right) \]
\[ \times \left\{ F_D \left( \frac{1}{2}, \beta_4^o \cdot \frac{3}{2}, \frac{3}{2} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} - \frac{r_s - a}{r_S - \beta} F_D \left( \frac{3}{2}, \beta_4^o \cdot \frac{5}{2}, \frac{5}{2} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\} \]
\[ + \frac{4G^2 M^2 \Omega'' \cdot z_S}{c^4} \times \left( \frac{1}{2}, \beta_4^o \cdot \frac{3}{2}, \frac{3}{2} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \] (B.7)

In equation (B.7) \( \Omega'' \) is defined by
\[ \Omega'' := \frac{\omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{\sqrt{\omega (\alpha_{\mu+2} - \alpha_{\mu+1}) (\alpha_{\mu+2} - \alpha_{\mu+1}) (\alpha_{\mu-3} - \alpha_{\mu+1}) (\alpha_{\mu} - \alpha_{\mu+1})}} \]
\[ = \frac{\sqrt{(\beta - \alpha)(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha)}}{\omega (\alpha - \beta)(\alpha - \beta)(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha)} \]
\[ = \frac{\delta - \alpha}{\beta - \alpha} \frac{\alpha - \beta}{\beta - \alpha} \sqrt{\beta - \alpha} \frac{\gamma - \alpha}{\delta - \alpha}. \] (B.8)

The quantity \( z_S \) is defined in terms of the source’s position and the roots of the quartic polynomial as follows
\[ z_S := \frac{r_S - \alpha}{r_S - \beta} \equiv \frac{r_S - \alpha}{\delta - \beta} \] (B.9)

Also we defined the coefficients
\[ A_{s=0}^{ld} = \pm \frac{2a^2 GM}{c^2} \frac{r_{S-a}}{r_s - r_a} \] (B.10)

and
\[ A_{s=1}^{ld} = \pm \frac{\sqrt{2GM} \sqrt{r_{S-a} - a^2}}{r_s - r_a}. \] (B.11)
The exact computation of the integral \( \int_a^r \frac{r dr}{\Delta \sqrt{R}} \) (which stems from the middle radial term in (B.1)) yields the result

\[
E_2 \int_a^r \frac{r dr}{\Delta \sqrt{R}} = \frac{A_{2h}^d}{r_s - \alpha} \sqrt{\frac{r_s - \alpha}{r_s - \beta}} \times
\]

\[
\frac{\Gamma(1/2)}{\Gamma(3/2)} - \left( \frac{r_s - \alpha}{r_s - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)}
\]

\[
\times \left( \frac{F_D \left( 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \beta \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} - \left( \frac{r_s - \alpha}{r_s - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right)
\]

\[
+ E_2 \frac{-A_{2h}^d}{(r_+ - \alpha)} \sqrt{\frac{r_+ - \alpha}{r_+ - \beta}} \times
\]

\[
\frac{\Gamma(1/2)}{\Gamma(3/2)} - \left( \frac{r_+ - \alpha}{r_+ - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)}
\]

\[
\times \left( \frac{F_D \left( 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \beta \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} - \left( \frac{r_+ - \alpha}{r_+ - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right).
\]

(B.12)

where \( E_2 := \frac{2GM}{c^2} (\alpha^2 - \Phi) \) and

\[ A_{2h}^d = \frac{\pm r_h}{r_+ - r_-}. \]

(B.13)

References

[1] Einstein A 1915 Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie, Sitzungsberichte der Preussischen Akademie der Wissenschaften p 831

[2] Schneider P, Ehlers J and Falco E E 1992 Gravitational Lenses A&A Library (Berlin: Springer)

[3] Einstein A 1936 Science 84 306

[4] Olhainan H C 1987 Am. J. Phys. 55 428–32

[5] Kranoti G V 2005 Frame dragging and bending of light in Kerr and Kerr-(anti) de Sitter spacetimes Class. Quantum Grav. 22 4391–424

[6] Lauricella G 1893 Sulle funzioni ipergeometriche a più variabili Rend. Circ. Mat. Palermo 7 111–58

[7] Ghez A M et al 2008 Measuring distance and properties of the Milky Way’s central supermassive black hole with stellar orbits Astrophys. J. 689 1044 (arXiv:0808.2870)

Ghez A M et al 2003 Astrophys. J. 586 L127–31

Ghez A M et al 2009 The galactic center: a laboratory for fundamental astrophysics and galactic nuclei Astro2010 Science White Paper (arXiv:0903.0383v1 [astro-ph.GA])

Ghez A M et al 2010 Increasing the scientific return of stellar orbits at the galactic center arXiv:1002.1729 [astro-ph.GA]

Yelda S et al 2010 Improving galactic center astrometry by reducing the effects of geometric distortion Astrophys. J. 725 331–52 (arXiv:1010.0064)

[8] Eisenhauer F et al 2005 arXiv:astro-ph/0508607

Gillessen S et al 2010 arXiv:1007.1612

Genzel R et al 2010 Rev. Mod. Phys. 82 3121–95

[9] Cunningham C T and Bardeen J M 1973 Astrophys. J. 183 237

[10] Bray I 1986 Phys. Rev. D 34 367

Sereno M and Luca F De 2006 Phys. Rev. D 74 123009 (arXiv:astro-ph/0609435v2)

Sereno M and De Luca F 2008 Phys. Rev. D 78 023008

Bozza V, De Luca F and Scarpetta G 2006 Phys. Rev. D 74 063001

Bozza V 2008 Phys. Rev. D 78 063014

Bozza V 2009 arXiv:0911.2187

[11] Vázquez S E and Esteban E P 2004 Nuovo Cimento B 119 489

[12] Kerr R P 1963 Phys. Rev. Lett. 11 237

[13] Stuchlík Z and Calvani M 1991 Gen. Rel. Grav. 23 507–19

33
[14] Carter B 1968 Commun. Math. Phys. 10 280–310
Demianski M 1973 Acta Astron. 23 197–231
Hawking S W, Hunter C J and Taylor-Robinson M M 1999 Phys. Rev. D 59 064005
[15] Teo E 2003 Gen. Rel. Grav. 35 1909
[16] Kraniotis G V in preparation
[17] Lake K 2007 arXiv:0711.0673v2 [gr-qc]
[18] Sereno M 2008 Phys. Rev. D 77 043004
[19] Rindler W and Ishak M 2007 Phys. Rev. D 76 043006
[20] Kraniotis G V 2007 Class. Quantum Grav. 24 1775–808 (arXiv:gr-qc/0602056)
Will C M 2008 Astrophys. J. 674 L25
Merritt D, Alexander T, Mikkola S and Will C M 2010 Phys. Rev. D 81 062002
Iorio L 2011 Mon. Not. R. Astron. Soc. 411 453 (arXiv:1008.1720v4 [gr-qc])
Jaroszyński M 1998 Acta Astron. 48 653
Rubilar G F and Eckart A 2001 Astron. Astrophys. 374 95
Fragile P C and Mathews G J 2000 Astrophys. J. 542 328
Weinberg N N, Milosavljević M and Ghez A M 2005 Astrophys. J. 622 878
Preto M and Saha P 2009 Astrophys. J. 703 1743
[21] Walker M and Penrose R 1970 Commun. Math. Phys. 18 265
[22] Penrose R 1973 Ann. New York Acad. Sci. 224 125
[23] Marck J-A 1983 Phys. Lett. A 97 140
[24] Pineault S and Roeder R C 1977 Astrophys. J. 212 541
[25] Newman E and Penrose R 1962 J. Math. Phys. 3 566
[26] Connors P A, Piran T and Stark R F 1980 Astrophys. J. 235 224
Ishihara H, Takahashi M and Tomimatsu A 1988 Phys. Rev. D 38 472