ON THE EXTREMAL EXTENSIONS OF A NON-NEGATIVE JACOBI OPERATOR

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Dedicated with deep respect to Professor Ya.V. Mykytyuk on the occasion of his 60th birthday.

Abstract. We consider minimal non-negative Jacobi operator with \( p \times p \) matrix entries. Using the technique of boundary triplets and the corresponding Weyl functions, we describe the Friedrichs and Krein extensions of the minimal Jacobi operator. Moreover, we parameterize the set of all non-negative self-adjoint extensions in terms of boundary conditions.

1. Introduction

Let \( A \) be a densely defined non-negative symmetric operator in the Hilbert space \( \mathcal{H} \). Since \( A \) is non-negative, then by Friedrichs-Krein theorem, it admits non-negative self-adjoint extensions. Qualified description of all non-negative self-adjoint extensions of \( A \) and also criterion of uniqueness of non-negative self-adjoint extension of \( A \) were first given by Krein in [16]. His results were generalized in numerous papers (see [3, 9, 11] and references therein).

Among all non-negative self-adjoint extensions of \( A \), two (extremal) extensions are particularly interesting and important enough to have a name. The Friedrichs extension (so-called "hard" extension) \( A_F \) is the "greatest" one in the sense of quadratic forms. It is given by restriction of \( A^* \) to the domain

\[
\text{dom}(A_F) = \left\{ u \in \text{dom}(A^*): \exists u_k \in \text{dom}(A) \text{ such that } \|u - u_k\|_{\mathcal{H}} \to 0 \text{ as } k \to \infty \text{ and } (A(u_j - u_k), u_j - u_k)_{\mathcal{H}} \to 0 \text{ as } j, k \to \infty \right\}.
\]

In other words, \( A_F \) is the self-adjoint operator associated with the closure of the symmetric form

\[
t[u, v] = (Au, v)_{\mathcal{H}}, \quad u, v \in \text{dom}(A).
\]

The Krein extension ("soft" extension) \( A_K \) is defined to be restriction of \( A^* \) to the domain

\[
\text{dom}(A_K) = \left\{ u \in \text{dom}(A^*): \exists u_k \in \text{dom}(A) \text{ such that } \|A^*u - Au_k\|_{\mathcal{H}} \to 0 \text{ as } k \to \infty \text{ and } (u_j - u_k, A(u_j - u_k))_{\mathcal{H}} \to 0 \text{ as } j, k \to \infty \right\}.
\]

If \( A \) is positive definite, \( A \geq \varepsilon I > 0 \), then (1) takes the form

\[
\text{dom}(A_K) = \text{dom}(A) + \ker(A^*).
\]

Krein proved in [16] that all the non-negative self-adjoint extensions \( \tilde{A} \) of \( A \) lie between \( A_F \) and \( A_K \), i.e.,

\[
((A_F + aI)^{-1}u, u)_{\mathcal{H}} \leq ((\tilde{A} + aI)^{-1}u, u)_{\mathcal{H}} \leq ((A_K + aI)^{-1}u, u)_{\mathcal{H}}, \quad u \in \mathcal{H},
\]

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Moreover, we assume that matrices $A_j, B_j \in \mathbb{C}^{p \times p}$. Moreover, we assume that matrices $A_j$ are self-adjoint and matrices $B_j$ are invertible for each $j \geq 0$ (see [5] Chapter VII, §2). We consider semi-infinite Jacobi matrix with

$$
J = \begin{pmatrix}
A_0 & B_0 & O_p & \ldots \\
B_0^* & A_1 & B_1 & \ldots \\
O_p & B_1^* & A_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

where $O_p$ is zero $p \times p$ matrix. Given a sequence $u = (u_j), u_j \in \mathbb{C}^p$, $Ju$ is again a sequence of column vectors. If we set $B_{-1} = O_p$,

$$(Ju)_j = B_ju_{j+1} + A_ju_j + B_{j-1}^*u_{j-1}, \quad j \geq 0.$$ 

The maximal operator $T_{\max}$ is defined by

$$(T_{\max}u)_j = (Ju)_j, \quad j \geq 0$$

on the domain

$$\text{dom}(T_{\max}) = \{ u \in l_2^p : Ju \in l_2^p \}.$$ 

The minimal operator $T_{\min}$ is the closure in $l_2^p$ of the preminimal operator $T$ which is the restriction of $T_{\max}$ to the domain

$$\text{dom}(T) = \{ u \in l_2^p : u_j = 0 \text{ for all but a finite number of values of } j \}.$$ 

It is straightforward to see that $T_{\min}$ is a densely defined symmetric operator and

$$T_{\min}^* = T_{\max}, \quad T_{\max}^* = T = T_{\min}.$$ 

Deficiency indices $n_+(T_{\min}) = \dim(\ker(T_{\max} \pm zI)), z \in \mathbb{C}_+$, satisfy the inequality $0 \leq n_+(T_{\min}), n_-(T_{\min}) \leq p$ (see [5] Chapter VII, §2). In the following, we shall assume that $n_+(T_{\min}) = p$ (completely indefinite case takes place) and $T$ is non-negative. Note that non-negativity of $T$ implies non-negativity of $A_j, j \geq 0$.

Examples of symmetric block Jacobi matrices generating symmetric operators with arbitrary possible values of the deficiency numbers were constructed in [13].

The problem of description of the extremal extensions $T_F$ and $T_K$ in the scalar case ($p = 1$) was studied in the number of papers. Description of the Friedrichs domain in terms of a weighted Dirichlet sum was obtained in [4].

In [20], Simon showed that certain matrix operators that approximate the Friedrichs and Krein extensions converge in the strong resolvent norm. In [7] Brown and Christiansen (assuming that $T$ is positive definite) obtained the description of $T_F$ and $T_K$ using the concept of the so-called minimal solution (see also [18]).

The purpose of this work is to generalize at least partially the results obtained in [7] to the case of arbitrary $p \in \mathbb{N}$ and non-negative operator $T$. We use an abstract description of extremal non-negative extensions obtained by V. Derkach and M. Malamud in the framework of boundary triplets and the corresponding Weyl functions approach (see [9 14] and also Section 2 for the precise definitions). In particular, we show that mentioned results from [7] might be expressed in terms of boundary triplets theory.

**Notation.** In what follows $\mathbb{C}^{p \times p}$ denotes the set of $p \times p$ complex-valued matrices; $l_2^p$ denotes Hilbert space of infinite sequences $u = (u_j), u_j \in \mathbb{C}^p$ equipped with inner
product \( (u, v)_\mathcal{H}^2 = \sum_{j=0}^{\infty} v_j^* u_j \). The set of closed (bounded) operators in the Hilbert space \( \mathcal{H} \) is denoted by \( \mathcal{C}(\mathcal{H}) \) (respectively, \( \mathcal{B}(\mathcal{H}) \)).

2. Linear relations, boundary triplets and Weyl functions

Let \( A \) be a closed densely defined symmetric operator in the separable Hilbert space \( \mathcal{H} \) with equal deficiency indices \( n_\pm(A) = \dim \ker(A^* \pm iI) \leq \infty \).

**Definition 1.** [13] A triplet \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is called a boundary triplet for the adjoint operator \( A^* \) of \( A \) if \( \mathcal{H} \) is an auxiliary Hilbert space and \( \Gamma_0, \Gamma_1 : \text{dom}(A^*) \to \mathcal{H} \) are linear mappings such that

(i) the second Green identity,

\[
(A^* f, g)_{\mathcal{H}} - (f, A^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}},
\]

holds for all \( f, g \in \text{dom}(A^*) \), and

(ii) the mapping \( \Gamma := (\Gamma_0, \Gamma_1)^{\top} : \text{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H} \) is surjective.

With each boundary triplet \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) one associates two self-adjoint extensions \( A_j := A^* \upharpoonright \ker(\Gamma_j) \), \( j \in \{0, 1\} \).

**Definition 2.**

(i) A closed linear relation \( \Theta \) in \( \mathcal{H} \) is a closed subspace of \( \mathcal{H} \oplus \mathcal{H} \). The domain, the range, and the multivalued part of \( \Theta \) are defined as follows

\[
\text{dom}(\Theta) := \{ f : \{ f', f'' \} \in \Theta \}, \quad \text{ran}(\Theta) := \{ f' : \{ f, f' \} \in \Theta \}, \quad \text{mul}(\Theta) := \{ f' : \{ 0, f' \} \in \Theta \}.
\]

(ii) A linear relation \( \Theta \) is symmetric if

\[
(f', h)_{\mathcal{H}} - (f, h')_{\mathcal{H}} = 0 \quad \text{for all} \quad \{ f, f' \}, \{ h, h' \} \in \Theta.
\]

(iii) The adjoint relation \( \Theta^* \) is defined by

\[
\Theta^* = \{ \{ h, h' \} : (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \quad \text{for all} \quad \{ f, f' \} \in \Theta \}.
\]

(iv) A closed linear relation \( \Theta \) is called self-adjoint if both \( \Theta \) and \( \Theta^* \) are maximal symmetric, i.e., they do not admit symmetric extensions.

For the symmetric relation \( \Theta \subseteq \Theta^* \) in \( \mathcal{H} \) the multivalued part \( \text{mul}(\Theta) \) is orthogonal to \( \text{dom}(\Theta) \) in \( \mathcal{H} \). Setting \( \mathcal{H}_{\text{op}} := \text{dom}(\Theta) \) and \( \mathcal{H}_{\infty} := \text{mul}(\Theta) \), one verifies that \( \Theta \) can be rewritten as the direct orthogonal sum of a self-adjoint operator \( \Theta_{\text{op}} \) (operator part of \( \Theta \)) in the subspace \( \mathcal{H}_{\text{op}} \), and a “pure” relation \( \Theta_{\infty} = \{ \{ 0, f' \} : f' \in \text{mul}(\Theta) \} \) in the subspace \( \mathcal{H}_{\infty} \).

**Proposition 1.** [13] Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). Then the mapping

\[
(3) \quad \text{Ext}_A \ni \tilde{A} := A_\Theta \to \Theta := \Gamma(\text{dom}(\tilde{A})) = \{ \{ \Gamma_0 f, \Gamma_1 f \} : f \in \text{dom}(\tilde{A}) \}
\]

establishes a bijective correspondence between the set of all closed proper extensions \( \text{Ext}_A \) of \( A \) and the set of all closed linear relations \( \mathcal{C}(\mathcal{H}) \) in \( \mathcal{H} \). Furthermore, the following assertions hold.

(i) The equality \( (A_\Theta)^* = A_{\Theta^*} \) holds for any \( \Theta \in \mathcal{C}(\mathcal{H}) \).

(ii) The extension \( A_\Theta \) in (3) is symmetric (self-adjoint) if and only if \( \Theta \) is symmetric (self-adjoint).

(iii) If, in addition, extensions \( A_{\Theta} \) and \( A_0 \) are disjoint, i.e., \( \text{dom}(A_{\Theta}) \cap \text{dom}(A_0) = \text{dom}(A) \), then (3) takes the form

\[
A_\Theta = A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0), \quad B \in \mathcal{C}(\mathcal{H})
\]
Remark 1. In the case \( n_\pm(A) = n < \infty \), any proper extension \( A_\Theta \) of the operator \( A \) admits representation (see [12])
\[
A_\Theta := A_{C,D} = A^* \mid \ker(D\Gamma_1 - C\Gamma_0), \quad C, D \in \mathcal{B}(\mathcal{H}).
\]
Moreover, according to the Rofe-Beketov theorem [19] (see also [2, Theorem 125.4]), \( A_{C,D} \) is self-adjoint if and only if \( C, D \), satisfy the following conditions
\[
CD^* = DC^* \quad \text{and} \quad 0 \in \rho(CC^* + DD^*).
\]

Definition 3. [9] Let \( \Pi = CD \) (5)
\[
\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad \text{for all } f_z \in \ker(A^* - zI), \quad z \in \rho(A_0),
\]
is called the Weyl function, corresponding to the triplet \( \Pi \).

Proposition 2. [9] Let \( A \) be a densely defined nonnegative symmetric operator with finite deficiency indices in \( \mathfrak{S} \), and let \( \Pi = \{ H, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). Let also \( M(\cdot) \) be the corresponding Weyl function. Then the following assertions hold.

(i) There exist strong resolvent limits
\[
M(0) := s - R - \lim_{x \to 0^+} M(x), \quad M(-\infty) := s - R - \lim_{x \downarrow -\infty} M(x).
\]

(ii) \( M(0) \) and \( M(-\infty) \) are self-adjoint linear relations in \( \mathcal{H} \) associated with the semi-bounded below (above) quadratic forms
\[
t_0[f] = \lim_{x \to 0^+} (M(x)f, f) \geq \beta ||f||^2, \quad t_{-\infty}[f] = \lim_{x \downarrow -\infty} (M(x)f, f) \leq \alpha ||f||^2,
\]
and
\[
dom(t_0) = \{ f \in \mathcal{H} : \lim_{x \to 0^+} |(M(x)f, f)| < \infty \} = dom((M(0)_{\text{op}} - \beta)^{1/2}),
\]
\[
dom(t_{-\infty}) = \{ f \in \mathcal{H} : \lim_{x \downarrow -\infty} |(M(x)f, f)| < \infty \} = dom((\alpha - M(-\infty)_{\text{op}})^{1/2}).
\]
Moreover,
\[
dom(A_K) = \{ f \in \text{dom}(A^*) : (\Gamma_0 f, \Gamma_1 f) \in M(0) \},
\]
\[
dom(A_F) = \{ f \in \text{dom}(A^*) : (\Gamma_0 f, \Gamma_1 f) \in M(-\infty) \}.
\]

(iii) Extensions \( A_0 \) and \( A_K \) are disjoint (\( A_0 \) and \( A_F \) are disjoint) if and only if
\[
M(0) \in C(\mathcal{H}) \quad (M(-\infty) \in C(\mathcal{H}), \text{ respectively}).
\]
Moreover,
\[
dom(A_K) = \text{dom}(A^*) \mid \ker(\Gamma_1 - M(0)\Gamma_0)
\]
\[
dom(A_F) = \text{dom}(A^*) \mid \ker(\Gamma_1 - M(-\infty)\Gamma_0), \text{ respectively}.
\]

(iv) \( A_0 = A_K (A_0 = A_F) \) if and only if
\[
\lim_{x \to 0^+} (M(x)f, f) = +\infty, \quad f \in \mathcal{H} \setminus \{0\}
\]
\[
(\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty, \quad f \in \mathcal{H} \setminus \{0\}, \text{ respectively}).
\]

(v) If, in addition, \( A_0 = A_F (A_0 = A_K) \), then the set of all non-negative self-adjoint extensions of \( A \) admits parametrization [3], where \( \Theta \) satisfies
\[
\Theta - M(0) \geq 0 \quad (\Theta - M(-\infty) \leq 0, \text{ respectively}).
\]
Moreover, if [7] does not hold, the number of negative eigenvalues of arbitrary self-adjoint extension \( \kappa_- (A_\Theta) \) is given by
\[
\kappa_- (A_\Theta) = \kappa_- (\Theta - M(0)) \quad (\kappa_- (A_\Theta) = \kappa_- (M(-\infty) - \Theta), \text{ respectively}).
\]
Remark 2. We should mention that the existence of the limits in (9) follows from finiteness of deficiency indices of $A$.

Remark 3. Note that if the lower bound of $A$ is zero and the spectrum of $AF$ is purely discrete, then $AF$ and $AK$ are not disjoint. In this case $M(0)$ is a linear relation if $A_0 \geq 0$.

Corollary 1. Let assumptions of Proposition 2 hold and $A_0 = AK$. Let also $ACD$ be self-adjoint extension of $A$ defined by (4). Then $ACD$ is non-negative if and only if

$$CD^* - DM(-\infty)D^* \leq 0.$$  

Moreover, if $\mathbf{8}$ does not hold, the number of negative eigenvalues of $ACD$ (counting multiplicities) coincides with the number of positive eigenvalues of the linear relation $CD^* - DM(-\infty)D^*$, i.e.,

$$\kappa_-(ACD) = \kappa_+(CD^* - DM(-\infty)D^*).$$

Corollary 2. Suppose that $AF$ has purely discrete spectrum. Then the Krein extension $AK$ is given by

$$\operatorname{dom}(AK) = \operatorname{dom}(\overline{A}) + \ker(A^*).$$

Moreover, the spectrum of $AK \setminus \ker(A^*)$ is purely discrete.

3. EXTREMAL EXTENSIONS OF $T_{\min}$

As usual, we denote by $(P_j(z))$ the solution to the matrix equation

$$(JU)_j = zU_j, \quad j \geq 0$$

with the initial conditions $P_0(z) = I_p, P_1(z) = B_0^{-1}(zI_p - A_0)$. Here $I_p \in \mathbb{C}^{p \times p}$ is the identity matrix. Furthermore, we denote by $(Q_j(z))$ the solution to

$$(JU)_j = zU_j, \quad j \geq 1$$

with $Q_0(z) = O_p$ and $Q_1(z) = B_0^{-1}$. The matrix functions $P_j(z)$ and $Q_j(z)$ are polynomials in the complex variable $z$ of degree $j$ and $j - 1$, respectively, with matrix coefficients. We mention that $(P_j(z))$ and $(Q_j(z))$ are called matrix polynomials of the first and second kind, respectively.

Following [10], we define a boundary triplet $\Pi = \{H, \Gamma_0, \Gamma_1\}$ for $T_{\max}$ by setting

$$H = \mathbb{C}^p, \quad \Gamma_1u = (Q(0))^*T_{\max}u - P_0u, \quad \Gamma_0u = (P(0))^*T_{\max}u,$$

where $u \in \operatorname{dom}(T_{\max})$ and $P_0$ is orthoproyection in $\bigoplus_{j=0}^{\infty} H_j$ onto $H_0$, in which $H_j = \mathbb{C}^p$.

We should note that the mappings $(P(0))^*$ and $(Q(0))^*$ act as infinite $p \times \infty$ matrices, i.e., $(P(0))^*, (Q(0))^*: l_2^p \to \mathbb{C}^p$. Each their row is "constructed" by the corresponding rows of $P_j^*(0)$ and $Q_j^*(0)$, respectively.

It is easily seen that

$$\Gamma_1(P_j(z)) = z \sum_{j=1}^{\infty} Q_j^*(0)P_j(z) - I_p, \quad \Gamma_0(P_j(z)) = z \sum_{j=0}^{\infty} P_j^*(0)P_j(z),$$

and, by Definition 3 we get

$$M(z) = \Gamma_1(P_j(z))|\Gamma_0(P_j(z)))^{-1}$$

$$= \left(z \sum_{j=1}^{\infty} Q_j^*(0)P_j(z) - I_p \right) \cdots \left(z \sum_{j=0}^{\infty} P_j^*(0)P_j(z) \right)^{-1}.$$  

Applying Proposition 2 to the operator $T_{\min}$, we obtain the following result.
**Theorem 1.** Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be the boundary triplet for \( T_{\text{max}} \) given by (9) and let \( M(\cdot) \) be the corresponding Weyl function. Then the following assertions hold.

(i) The Krein extension \( T_K \) coincides with \( T_0 \), i.e.,

\[
\text{dom}(T_K) = \text{dom}(T_{\text{max}}) \upharpoonright \ker(\Gamma_0) = \{ u \in T_{\text{max}} : (P(0))^* T_{\text{max}} u = 0 \}.
\]

(ii) Self-adjoint extension \( T_\Theta \) is non-negative if and only if it admits representation

\[
T_\Theta = T_{C,D} = T_{\text{max}} \upharpoonright \ker(D\Gamma_1 - C\Gamma_0),
\]

where \( C, D \) satisfy (5) and (3).

(iii) The number of negative eigenvalues of \( T_{C,D} = T_{C,D}^* \) (defined by (41)) coincides with the number of positive eigenvalues of the linear relation \( CD^* - DM(-\infty)D^* \), i.e.,

\[
\kappa_-(T_{C,D}) = \kappa_+(CD^* - DM(-\infty)D^*).
\]

**Proof.** (ii) The statement might be proven at least in two different ways.

1. Since \( M(x) \) is holomorphic and increasing in \( (-\varepsilon, 0) \) (see [11]), the strong limit \( s - \lim_{x \uparrow 0}(M(x) + \gamma)^{-1} \) exists for any \( \gamma > 0 \). Namely,

\[
M_\gamma(0) := s - \lim_{x \uparrow 0}(M(x) + \gamma)^{-1} = s - \lim_{x \uparrow 0}\left( x \sum_{j=0}^{\infty} P_j^*(0) P_j(x) \right)^{-1} = O_p.
\]

Indeed, since \( n_\pm(T_{\text{min}}) = p \) (see [15]), the series \( \sum_{j=0}^{\infty} \|P_j(z)\|^2 \) and \( \sum_{j=0}^{\infty} \|Q_j(z)\|^2 \) converge uniformly on each bounded subset \( \mathbb{C} \) (see, for instance, [15 Theorem 1]). Therefore, we can pass to the limit in (10) under the sum sign as \( x \to 0 \). Hence \( M_\gamma(0) = \{0, \mathcal{H}\} = \{\{0, f\} : f \in \mathcal{H}\} \).

Now, \( \text{dom}(T_0) = \{ u \in \text{dom}(T_{\text{max}}) : \Gamma_0 u = 0 \} = \{ u \in \text{dom}(T_{\text{max}}) : \{\Gamma_0 u, \Gamma_1 u\} \in \{0, \mathcal{H}\} \} \), by Proposition 2 (ii), we arrive at \( T_K = T_0 \).

2. Suppose, in addition, that \( T \geq \varepsilon I > 0 \). Using the equality \( \text{dom}(T_0) = \text{dom}(T_{\text{max}}) \upharpoonright \ker(\Gamma_0) \), we easily get from (9) that \( \ker(T_{\text{max}}) \supset \text{dom}(T_0) \). Therefore, (2) implies the inclusion \( \text{dom}(T_0) \supset \text{dom}(T_K) \). Since \( T_0 \) and \( T_K \) are self-adjoint, we arrive at \( T_K = T_0 \).

(iii) and (iii) easily follow from Corollary 4 and assertion (i). \( \square \)

Assume now that \( p = 1 \) and \( A_j, B_j \) are positive real numbers. It is known that \( T_{\text{min}} \) is connected with some Stiltjes moment problem, see [1]. Briefly, a Stiltjes moment problem has a following description. Given a sequence \( \gamma_0, \gamma_1, \gamma_2, \ldots \) of reals. When is there a measure, \( d\mu \) on \( [0, \infty) \) so that

\[
\gamma_n = \int_0^\infty x^n d\mu(x)
\]

and if such a \( \mu \) exists, is it unique?

The operator \( T_{\text{min}} \) is self-adjoint if and only if associated Stiltjes moment problem is determinate, i.e., it has unique solution. Since \( n_\pm(T_{\text{min}}) = 1 \), the determinacy does not take place and, therefore, sequence \( \frac{Q_j(0)}{P_j(0)} \) converges (see [1] Theorem 0.4, p. 293 or [6] Section 3).
The existence of this limit is the key fact for the description of the Friedrichs extension which we are going to present. In particular, the limit

\[(11) \quad \alpha := \lim_{j \to \infty} \frac{Q_j(0)}{P_j(0)}\]

is negative. Indeed, since all the zeros of the polynomials \(P_j(x)\) and \(Q_j(x)\) lie in the interval \([0, \infty)\) (see, [8] Chapter I), \(P_j(\cdot)\) and \(Q_j(\cdot)\) do not change the sign in \((-\infty, 0)\). Noticing that \(P_j(x)/Q_j(x) < 0\) for \(x < 0\) large enough (the degrees of \(P_j(\cdot)\) and \(Q_j(\cdot)\) are \(j\) and \(j - 1\), respectively, and leading coefficients equal \(B_{j-1}^{-1} \cdots B_0^{-1} > 0\)), we get the negativity of \(\alpha\).

**Theorem 2.** Assume \(p = 1\). Let also \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\) be the boundary triplet for \(T_{\text{max}}\) defined by \((\mathcal{H})\) and \(M(\cdot)\) be the corresponding Weyl function. The domain of the Friedrichs extension is given by

\[(12) \quad \text{dom}(T_F) = \{u \in \text{dom}(T_{\text{max}}) : (\Gamma_1 - \alpha \Gamma_0)u = 0\},\]

where \(\alpha\) is defined by \((\mathcal{H})\).

**Proof.** To prove the statement we use Proposition\((\mathcal{H})\)(iii). Namely, it is sufficient to show that

\[(13) \quad M(-\infty) = \lim_{x \downarrow -\infty} M(x) = \lim_{x \downarrow -\infty} \frac{x \sum_{j=1}^{\infty} Q_j(0)P_j(x) - 1}{x \sum_{j=0}^{\infty} P_j(0)P_j(x)} = \alpha.\]

Since the orthogonal polynomials do not change the sign in \((-\infty, 0)\),

\[(14) \quad P_j(0)P_j(x) = |P_j(0)P_j(x)| \quad \text{and} \quad Q_j(0)P_j(x) = -|Q_j(0)P_j(x)|.\]

Thus, the sequence \(\sum_{j=0}^{n} P_j(0)P_j(x) = \sum_{j=0}^{n} |P_j(0)P_j(x)|\) increases as \(n \to \infty\) and, therefore,

\[
\lim_{x \downarrow -\infty} M(x) = \lim_{x \downarrow -\infty} \frac{x \sum_{j=1}^{\infty} Q_j(0)P_j(x) - 1}{x \sum_{j=0}^{\infty} P_j(0)P_j(x)} = -\lim_{x \downarrow -\infty} \frac{\sum_{j=1}^{\infty} |Q_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|}.
\]

It follows from \((\mathcal{H})\) that for any small \(\delta > 0\) there exists \(N = N(\delta)\) such that estimate

\[\alpha - \delta < \frac{Q_j(0)}{P_j(0)} < \alpha + \delta\]

holds for \(j > N(\delta)\). Combining this inequality with \((\mathcal{H})\), we get

\[\alpha - \delta < \frac{Q_j(0)P_j(x)}{|P_j(0)P_j(x)|} < \alpha + \delta, \quad x \in (-\infty, 0), \quad j \geq N(\delta).\]

The latter is equivalent to

\[(\alpha - \delta)|P_j(0)P_j(x)| < Q_j(0)P_j(x) < (\alpha + \delta)|P_j(0)P_j(x)|, \quad x \in (-\infty, 0), \quad j \geq N(\delta).\]
we get from (15)–(17) the following inequality
\[
(\alpha - \delta) \sum_{j=N(\delta)}^{\infty} |P_j(0)P_j(x)| < \sum_{j=1}^{\infty} Q_j(0)P_j(x) - \sum_{j=1}^{N(\delta)-1} Q_j(0)P_j(x)
\]
\[
< (\alpha + \delta) \sum_{j=0}^{\infty} |P_j(0)P_j(x)|, \quad x \in (-\infty, 0), \quad j \geq N(\delta).
\]
Since
\[
\sum_{j=0}^{\infty} P_j(x)P_j(0) = \sum_{j=0}^{\infty} |P_j(x)P_j(0)| > |P_{N(\delta)}(x)P_{N(\delta)}(0)|,
\]
and \( P_j \) is polynomial of degree \( j \), we get
\[
0 \leq \lim_{x \to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} |P_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} < \lim_{x \to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} |P_j(0)P_j(x)|}{|P_{N(\delta)}(x)P_{N(\delta)}(0)|} = 0.
\]
Similarly we obtain
\[
\lim_{x \to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} Q_j(0)P_j(x)}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} = 0.
\]
Taking into account that
\[
\sum_{j=0}^{\infty} |P_j(0)P_j(x)| = \sum_{j=N(\delta)}^{\infty} |P_j(0)P_j(x)| + \sum_{j=0}^{N(\delta)-1} |P_j(0)P_j(x)|,
\]
we get from (15)–(17) the following inequality
\[
\alpha - \delta \leq \lim_{x \to -\infty} \frac{\sum_{j=1}^{\infty} Q_j(0)P_j(x)}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} \leq \alpha + \delta
\]
for any arbitrary small \( \delta \). Thus, equality (13) takes place. \( \Box \)

**Remark 4.** We should mention that the description of the Krein and Friedrichs extensions given in Theorems 1 and 2 in the scalar case coincides with one obtained earlier by Brown and Christiansen in [7].

**Remark 5.**

(i) The condition \( B_j > 0 \) can be dropped. Indeed, it is easy to show that scalar Jacobi matrix with arbitrary real \( B_j \) is unitarily equivalent to the Jacobi matrix with positive \( B_j \). The unitary equivalence is established by the diagonal matrix with 1 and \(-1\) on the diagonal. Besides, if \( B_j < 0 \), then 1 and \(-1\) have to stand next to each other in the same rows as \( B_j \).

(ii) The fact that all zeroes of \( P_j(\cdot) \) belong to \([0, \infty)\) might be also derived from the holomorphicity of the Weyl function, corresponding to another boundary triplet (see [11] Proposition 10.1(2)), on \((-\infty, 0)\).
(iii) In [2], authors obtained the description [12] by a different method. Namely, they essentially used the fact that the minimal (or principal) solution $u = (u_j)$ of the equation $(Ju)_j = 0$, $j \geq 1$, has the form $u = (u_j) = (P_j(0) - \alpha Q_j(0))$ and belongs to the domain $\text{dom}(T_F)$.

(iv) In [17] Chapter 5, §3 (see also [7]), it was noted that all solutions $\tilde{\mu}$ of the Stieltjes moment problem associated with $T_{\text{min}}$ lie between the solutions $\mu_K$ and $\mu_F$ coming from the Friedrichs and Krein extensions in the following sense

$$\int_0^\infty \frac{d\mu_K(t)}{x-t} \leq \int_0^\infty \frac{d\tilde{\mu}(t)}{x-t} \leq \int_0^\infty \frac{d\mu_F(t)}{x-t}, \quad x < 0.$$ 

Proposition 2(v) leads to the description of all non-negative self-adjoint extensions of $T$ in the scalar case.

**Corollary 3.** Assume $p = 1$. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\text{max}}$ given by (9) and let $M(\cdot)$ be the corresponding Weyl function. The set of all non-negative self-adjoint extensions $T_h$ of the operator $T_{\text{min}}$ is parameterized as follows

$$\text{dom}(T_h) = \{u \in \text{dom}(T_{\text{max}}) : (\Gamma_1 - h\Gamma_0)u = 0\}, \quad h \in [-\infty; \alpha] \quad \text{where} \quad \alpha \text{ is defined by (11)}. \quad \text{In particular,}$$

$$\text{dom}(T_{-\infty}) = \{u \in \text{dom}(T_{\text{max}}) : \Gamma_0 u = 0\}.$$

**Proof.** First note that for $h = \alpha$ and $h = -\infty$ the statement was proved above. Indeed,

$$\text{dom}(T_\alpha) = \{u \in \text{dom}(T_{\text{max}}) : (\Gamma_1 - \alpha\Gamma_0)u = 0\} = \text{dom}(T_F)$$

and

$$\text{dom}(T_{-\infty}) = \text{dom}(T_{\text{max}}) \upharpoonright \text{ker}(\Gamma_0) = \text{dom}(T_K).$$

Thus, it remains to prove that for $h < \alpha$ formula (18) defines non-negative self-adjoint extension. The result is implied by combining Proposition 2(v) with Theorem 2.

Indeed, consider a new boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$

$$\tilde{\mathcal{H}} = \mathbb{C}, \quad \tilde{\Gamma}_0 f = \Gamma_1 f - \alpha \Gamma_0 f, \quad \tilde{\Gamma}_1 f = -\Gamma_0 f,$$

where $\Gamma_0, \Gamma_1$ are given by (3). One easily obtains that the corresponding Weyl function is $\tilde{M}(z) = (\alpha - M(z))^{-1}$. Taking into account the above information about ”limit values” of the Weyl function $M(\cdot)$, we get that $\tilde{M}(-\infty)$ is ”pure” linear relation, i.e., $\tilde{M}(-\infty) = \{0, \tilde{H}\}$, and $\tilde{M}(0) = 0$. Hence, by Proposition 2(ii), $T_0 := T_{\text{max}} \upharpoonright \text{ker}(\Gamma_0) = T_F$. Equation (18) in terms of the new boundary triplet takes the form

$$\text{dom}(T_h) = \{u \in \text{dom}(T_{\text{max}}) : (\tilde{\Gamma}_1 - \frac{1}{\alpha - h} \tilde{\Gamma}_0)u = 0\}.$$

Applying Proposition 2(v), we get that $T_h \geq 0$ if and only if $\frac{1}{\alpha - h} > \tilde{M}(0) = 0$ or $h < \alpha$. □

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