BASES OF THE CONTACT-ORDER FILTRATION OF DERIVATIONS OF COXETER ARRANGEMENTS

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Abstract. In a recent paper, we constructed a basis for the contact-order filtration of the module of derivations on the orbit space of a finite real reflection group acting on an ℓ-dimensional Euclidean space. Recently M. Yoshinaga constructed another basis for the contact-order filtration. In this note we give an explicit formula relating Yoshinaga’s basis to the basis we constructed earlier. The two bases turn out to be equal (up to a constant matrix).

1. The setup and the main result

Let V be an ℓ-dimensional Euclidean vector space with inner product I. Then its dual space V∗ is equipped with the inner product I∗, which is induced by I. Let S be the symmetric algebra of V∗ over R. Identify S with the algebra of polynomial functions on V. Let DerS be the S-module of R-linear derivations of S. When X1, · · · , Xℓ denote a basis for V∗, the partial derivations ∂i := ∂/∂X i with respect to X i (1 ≤ i ≤ ℓ) naturally form a basis for DerS over S. Let K be the field of quotients of S and DerK be the K-vector space of R-linear derivations of K. Then the partial derivations ∂i (1 ≤ i ≤ ℓ) naturally form a basis for DerK over K.

Let W be a finite irreducible orthogonal reflection group (a Coxeter group) acting on V. The Coxeter group W naturally acts on V∗, S and DerS. The W-invariant subring of S is denoted by R. Then it is classically known [1, V.5.3, Theorem 3] that there exist algebraically independent homogeneous polynomials P1, · · · , Pℓ ∈ R with deg P1 ≤ · · · ≤ deg Pℓ, which are called basic invariants, such that

\[ R = \mathbb{R}[P_1, \cdots, P_\ell]. \]

The primitive derivation D ∈ DerR is characterized by

\[ DP_i = \begin{cases} 1 & \text{for } i = \ell, \\ 0 & \text{otherwise.} \end{cases} \]
Write $\mathbf{P} = (P_1, \ldots, P_t)$. Let $J(\mathbf{P})$ denote the Jacobian matrix $[\partial P_i/\partial X_j]_{ij}$. Define $\Delta := \det J(\mathbf{P})$. Then $\Delta$ is an anti-invariant \cite{1} V.5.4 and $\Delta^2 \in R$. Let

$$I : \text{Der}_R \times \text{Der}_R \to \frac{1}{\Delta^2} R$$

be the symmetric $R$-bilinear form induced by $I$. Let

$$\nabla : \text{Der}_R \times \text{Der}_R \to \frac{1}{\Delta^2} \text{Der}_R,$$

$$(X,Y) \mapsto \nabla_X Y$$

be the Levi-Civita connection with respect to $I$. Since $\frac{\partial}{\partial P_i} \in \frac{1}{\Delta} \text{Der}_S$ by Cramer’s rule, one can embed $\text{Der}_R$ into $\text{Der}_K$. Extend the Levi-Civita connection naturally to

$$\nabla : \text{Der}_K \times \text{Der}_K \to \text{Der}_K.$$ 

Note that $\nabla_{\xi}(\eta) = \sum_i \xi(\eta(X_i))\partial_i$ because each $I(\partial_i, \partial_j)$ is constant.

In \cite{3}, we introduced derivations $\xi^{(m)}_1, \ldots, \xi^{(m)}_k \in \text{Der}_S$ for $m \geq 0$ by

$$\langle \xi^{(m)}_1, \ldots, \xi^{(m)}_k \rangle = \begin{cases} \left( \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_t} \right) J(\mathbf{P})^T A J(D^k[X])^{-1} & \text{if } m = 2k, \\ \left( \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_t} \right) J(\mathbf{P})^T A J(D^k[X])^{-1} J(\mathbf{P}) & \text{if } m = 2k + 1. \end{cases}$$

Here $A := [I^*(X_i, X_j)]_{ij}$ and $J(D^k[X]) := [\partial(D^k(X_j)/\partial X_i)]_{ij}$ ($D^k := D \circ D \circ \cdots \circ D$ ($k$ times)). Let $\mathcal{A}$ be the Coxeter arrangement determined by the Coxeter group $W$. $\mathcal{A}$ is the set of reflecting hyperplanes. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. The derivations $\xi^{(m)}_1, \ldots, \xi^{(m)}_k \in \text{Der}_S$ were constructed in \cite{3} so that they may form a basis for the $S$-module \cite{1}

$$\mathcal{D}^{(m)}(\mathcal{A}) := \{ \theta \in \text{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^m \text{ for any } H \in \mathcal{A} \}$$

for each nonnegative integer $m$. The filtration

$$\text{Der}_S = \mathcal{D}^{(0)}(\mathcal{A}) \supset \mathcal{D}^{(1)}(\mathcal{A}) \supset \mathcal{D}^{(2)}(\mathcal{A}) \supset \cdots$$

of $\text{Der}_S$ is called the contact-order filtration. Define

$$\mathcal{H}^{(k)} = \mathcal{D}^{(2k-1)}(\mathcal{A}) \cap \text{Der}_R$$

for each $k \geq 1$. Let $\mathcal{H}^{(0)} = \text{Der}_R$. Then we have a filtration

$$\text{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \mathcal{H}^{(2)} \supset \cdots$$

of $\text{Der}_R$. This filtration is known \cite{7} to be equal to the Hodge filtration introduced by K. Saito \cite{1}. The derivations $\xi^{(2k-1)}_1, \ldots, \xi^{(2k-1)}_k \in \text{Der}_R$ form an $R$-basis for $\mathcal{H}^{(k)}$ for $k \geq 1$. Therefore the contact-order filtration, restricted to $\text{Der}_R$, gives the Hodge filtration. Define

$$T := \{ f \in R \mid Df = 0 \} = \mathbb{R}[P_1, \ldots, P_t].$$

Then the covariant derivative $\nabla_D : \text{Der}_K \to \text{Der}_K$ is $T$-linear. The Hodge filtration was originally defined so that the $T$-linear map

$$\nabla^{(k)} : \nabla_D \circ \cdots \circ \nabla_D (k \text{ times}) : \mathcal{H}^{(k)} \to \mathcal{H}^{(0)} = \text{Der}_R$$

is bijective \cite{3}. Thus we may define $\nabla^{-k} \xi \in \mathcal{H}^{(k)}$ for any $\xi \in \text{Der}_R$ and $k \geq 0$. In \cite{8} M. Yoshinaga proved the following theorem.
Theorem 1.1. (\cite{6} Theorem 6) Let $k \geq 0$ and

$$E := \sum_i X_i \partial_i = \sum_i (\deg P_i)^{-1} P_i (\partial/\partial P_i)$$

be the Euler derivation. Suppose that $\xi_1, \ldots, \xi_\ell$ are a basis for $D^{(1)}(A)$. Then

1. the derivations $\nabla_{\xi_1} \nabla_D^{-k} E, \ldots, \nabla_{\xi_\ell} \nabla_D^{-k} E$ form a basis for $D^{(2k-1)}(A)$ over $S$, and

2. the derivations $\nabla_{\partial_1} \nabla_D^{-k} E, \ldots, \nabla_{\partial_\ell} \nabla_D^{-k} E$ form a basis for $D^{(2k)}(A)$ over $S$.

It thus seems natural to ask how the basis above constructed by Yoshinaga is related to the basis $\xi^{(m)}_1, \ldots, \xi^{(m)}_\ell$ given in \cite{6}. The following theorem answers this question:

Theorem 1.2. Let $k \geq 0$. Then

$$(\xi^{(2k+1)}_1, \ldots, \xi^{(2k+1)}_\ell) = (-1)^k (\nabla_{\xi^{(1)}_1} \nabla_D^{-k} E, \ldots, \nabla_{\xi^{(1)}_\ell} \nabla_D^{-k} E)$$

and

$$(\xi^{(2k)}_1, \ldots, \xi^{(2k)}_\ell) = (-1)^k (\nabla_{\partial_1} \nabla_D^{-k} E, \ldots, \nabla_{\partial_\ell} \nabla_D^{-k} E) A.$$

So the two bases turn out to be equal up to a constant matrix.

The significance of Theorem 1.2 is as follows: Recently M. Yoshinaga \cite{9} affirmatively settled the Edelman-Reiner conjecture, which asserts that the cones of the extended Shi/Catalan arrangements are free. In other words, there exist basic derivations for each extended Shi/Catalan arrangement. However, a formula for basic derivations is still unknown. Since the derivations $\xi^{(m)}_i (i = 1, \ldots, \ell)$ are the principal (= highest degree) parts of basic derivations, Theorem 1.2 can be interpreted as a differential-geometric formula for the “principal parts.” So it may suggest the existence of a differential-geometric formula for the whole basic derivations, including the “non-principal part.” One may also regard Theorem 1.2 as a very explicit algebraic description of the derivations in the right-hand side. They are, as mentioned above, when $m$ is odd, bases for the Hodge filtration, which is the key to defining the flat structure on the orbit space $V/W$ in \cite{5}. The flat structure is called the Frobenius manifold structure from the viewpoint of topological field theory \cite{2}.

2. Proof

We will prove Theorem 1.2 in this section. First we show the following.

Lemma 2.1. For $k \geq 1$ and $\xi \in \text{Der}_K$, we have

$$\nabla_D^k \circ \nabla_\xi - \nabla_\xi \circ \nabla_D^k = k \nabla_D^{k-1} \circ \nabla_{[D,\xi]}.$$

Proof. We use an induction on $k$. When $k = 1$, the lemma asserts that

$$\nabla_D \circ \nabla_\xi - \nabla_\xi \circ \nabla_D = \nabla_{[D,\xi]},$$

which is the integrable property of the Levi-Civita connection $\nabla$. Let $k > 1$. We have

$$\nabla_D^k \circ \nabla_\xi = \nabla_D^{k-1} \circ (\nabla_\xi \circ \nabla_D + \nabla_{[D,\xi]})$$

$$= (\nabla_D^{k-1} \circ \nabla_\xi) \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D,\xi]}$$

$$= (\nabla_\xi \circ \nabla_D^{k-1} + (k-1)\nabla_D^{k-2} \circ \nabla_{[D,\xi]}) \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D,\xi]}$$

$$= \nabla_\xi \circ \nabla_D + (k-1)\nabla_D^{k-2} \circ \nabla_{[D,\xi]} \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D,\xi]}$$
by using the induction assumption. Since 
\[ [D, [D, \xi]] = 0, \]
we obtain
\[ \nabla_D \circ \nabla_{[D, \xi]} = \nabla_{[D, \xi]} \circ \nabla_D. \]
This implies that
\[ \nabla_D^k \circ \nabla_{\xi} = \nabla_{\xi} \circ \nabla_D^k + k \nabla_D^{k-1} \circ \nabla_{[D, \xi]} \]
□

Recall the \( W \)-invariant inner product \( I^* : V^* \times V^* \rightarrow \mathbb{R} \). Let \( \Omega^1_R \) denote the \( R \)-module of Kähler differentials. Let
\[ I^* : \Omega^1_R \times \Omega^1_R \rightarrow \mathbb{R} \]
be the symmetric \( R \)-bilinear form induced by \( I^* \). Let
\[ G := [I^*(dP_i, dP_j)]_{ij} = J(P)^T A J(P). \]
Define
\[ B^{(k)} := -J(P)^T A J(D^k[X]) J(D^{k-1}[X])^{-1} J(P) \]
for \( k \geq 1 \) as in \[6\]. Then we have

**Lemma 2.2.** (1) Every entry of \( B^{(k)} \) lies in \( T \); \( D[B^{(k)}] = 0 \),
(2) \( \det B^{(k)} \in \mathbb{R}^* \),
(3) \( D[G] = B^{(1)} + (B^{(1)})^T \),
(4) \( B^{(k+1)} = B^{(1)} + kD[G] \).

**Proof.** By \[6\] Lemmas 3.3–3.6, Remark 3.7. □

The following proposition is one of the main results in \[7\]:

**Proposition 2.3.** For \( k \geq 1 \),
\[ (\nabla_D^{(2k-1)} \xi_1, \ldots, \nabla_D^{(2k-1)} \xi_\ell) = (-1)^{k-1} (\frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_\ell}) B^{(k)}. \]

In the rest of this note, let \((\xi_1, \ldots, \xi_\ell)\) denote \((\xi^{(1)}_1, \ldots, \xi^{(1)}_\ell)\) for simplicity. Note that
\[ (\xi_1, \ldots, \xi_\ell) = (\frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_\ell}) G. \]

**Lemma 2.4.** \([D, \xi_1] \ldots [D, \xi_\ell] = (\frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_\ell}) D[G] = (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell) (B^{(1)})^{-1} D[G]. \)

**Proof.** The first equality easily follows from
\[ [D, \xi_j](P_i) = D \circ \xi_j(P_i) - \xi_j \circ D(P_i) = D \circ \xi_j(P_i) = D[I^*(dP_i, dP_j)]. \]
For the second equality, apply Proposition 2.3 when \( k = 1 \). □

**Lemma 2.5.** For any \( \eta \in \text{Der}_K \),
\[ (\nabla_{\xi_1} \eta, \ldots, \nabla_{\xi_\ell} \eta) = (\nabla_{\partial_1} \eta, \ldots, \nabla_{\partial_\ell} \eta) A J(P). \]
Proof. Since both sides are additive with respect to \( \eta \), we may assume \( \eta = f \partial_j \) for some \( j \) and \( f \in K \). Then we have

\[
(\nabla_{\xi_1} \eta, \ldots, \nabla_{\xi_\ell} \eta) = (\nabla_{\xi_1} (f \partial_j), \ldots, \nabla_{\xi_\ell} (f \partial_j))
= (\xi_1(f) \partial_j, \ldots, \xi_\ell(f) \partial_j)
= ((\partial_1 f) \partial_j, \ldots, (\partial_\ell f) \partial_j) AJ(P)
= (\nabla_{\partial_1} (f \partial_j), \ldots, \nabla_{\partial_\ell} (f \partial_j)) AJ(P)
= (\nabla_{\partial_1} \eta, \ldots, \nabla_{\partial_\ell} \eta) AJ(P).
\]

\( \square \)

Proof of Theorem 1.2. When \( k = 0 \), the formulas are obviously true. Let \( k \geq 1 \). Let \( \xi \in \{\xi_1, \ldots, \xi_\ell\} \). Apply the identity in Lemma 2.4 to \( \nabla_D^{-k} E \) to get

\[
(\nabla_D^k \circ \nabla_\xi)(\nabla_D^{-k} E) - \nabla_\xi E = k(\nabla_D^{k-1} \circ \nabla_{[D,\xi]})(\nabla_D^{-k} E) = k \nabla_D^{-1} (\nabla_{[D,\xi]} E).
\]

Since \( \nabla_\eta E = \eta \) for any \( \eta \in \text{Der}_K \), one obtains

\[
\nabla_D^k(\nabla_\xi \nabla_D^{-k} E) - \xi = k \nabla_D^{-1}[D,\xi]
\]

and thus

\[
\nabla_D^{k+1}(\nabla_\xi \nabla_D^{-k} E) = \nabla_D \xi + k [D,\xi].
\]

On the other hand, Lemma 2.4 asserts

\[
([D,\xi_1], \ldots, [D,\xi_\ell]) = (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell)(B(1))^{-1} D[G].
\]

So we have

\[
(\nabla_D^{k+1}(\nabla_\xi_1 \nabla_D^{-k} E), \ldots, \nabla_D^{k+1}(\nabla_\xi_\ell \nabla_D^{-k} E))
= (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell) + k (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell)(B(1))^{-1} D[G]
= (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell)(B(1))^{-1} B(1) + k D[G]
= (\nabla_D \xi_1, \ldots, \nabla_D \xi_\ell)(B(1))^{-1} B(k+1)
= (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) B(k+1)
= (-1)^k (\nabla_D^{k+1} \xi_1^{(2k+1)}, \ldots, \nabla_D^{k+1} \xi_\ell^{(2k+1)})
\]

by Lemma 2.2 and Proposition 2.3. This proves the first formula.

For the second formula, compute

\[
(\nabla_{\xi_1} \nabla_D^{-k} E, \ldots, \nabla_{\xi_\ell} \nabla_D^{-k} E) = (\nabla_{\partial_1} \nabla_D^{-k} E, \ldots, \nabla_{\partial_\ell} \nabla_D^{-k} E) AJ(P)
\]

by Lemma 2.6. Applying the first formula, we get

\[
(\xi_1^{(2k)}, \ldots, \xi_\ell^{(2k)}) J(P) = (\xi_1^{(2k+1)}, \ldots, \xi_\ell^{(2k+1)})
= (-1)^k (\nabla_{\xi_1} \nabla_D^{-k} E, \ldots, \nabla_{\xi_\ell} \nabla_D^{-k} E)
= (-1)^k (\nabla_{\partial_1} \nabla_D^{-k} E, \ldots, \nabla_{\partial_\ell} \nabla_D^{-k} E) AJ(P).
\]

This completes the proof of Theorem 1.2. \( \square \)
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