On Pole-Zero Assignment of Vibratory Systems by Multi-Input Feedback Control

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Abstract

In this paper, we consider the pole-zero assignment problem for vibratory systems via multi-input feedback control. We propose a multi-step two-stage approach for solving the multi-input pole-zero assignment problem. We first reformulate the assignment problem as a multi-step single-input pole-zero assignment problem. Then, in each step, we propose a two-stage approach for solving the single-input pole-zero assignment problem. In the first stage, based on the measured receptances, we replace the selected zeros of the prescribed open-loop point receptance to the desired locations, where we need to solve a small underdetermined linear equation for finding the corresponding columns of the feedback matrices. In the second stage, by using linear matrix inequalities, the complete closed-loop poles are assigned to the prescribed subregion of the complex left-hand plane. Finally, we give some numerical examples to demonstrate the effectiveness of our method.

Keywords. Vibrating system, pole and zero assignment, receptance method, linear matrix Inequalities

1 Introduction

The active vibration control is an important strategy for vibration attenuation of various vibratory structures such as flexible cantilever plate [31], space antenna reflector [36], beam [42], large flexible space structures [3, 6, 20], mechanical descriptor systems [25], damped-gyroscopic system control [18], earthquake engineering control [10], etc. In structural dynamics [11, 16], by using the finite element modeling, a vibratory structure is often discretized as the following system of second-order differential equations:

\[ M \ddot{x}(t) + C \dot{x}(t) + K x(t) = f(t), \]  

(1.1)

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where the $n$-by-$n$ real symmetric matrices $M, C, K$ mean, respectively, the mass, damping, and stiffness matrices with $M$ being positive definite and $K$ being positive semidefinite while the $n$-vectors $x(t)$ and $f(t)$ denotes the displacement state vector and the external force vector dependent on the time $t$, respectively.

It is well-known that the natural frequency of vibration and the mode shape of the system (1.1) is closely related to the eigenvalue and eigenvector of the following quadratic eigenvalue problem [40]

$$P(\lambda)x := (\lambda^2 M + \lambda C + K)x = 0,$$

where $P(\lambda) := \lambda^2 M + \lambda C + K$ is called the open-loop quadratic pencil.

To reduce the vibration, it is desired to assign the poles and zeros (natural frequencies and antiresonances) for various vibratory structures. An active vibration control aims to find a feedback control force $f(t)$ in the form of

$$f(t) = Bu(t) \quad \text{with} \quad u(t) = F^T \dot{x}(t) + G^T x(t),$$

such that the unwanted natural frequencies are replaced by desired ones and antiresonances are moved such that the vibration response vanishes with chosen coordinates and frequencies [26, 27]. Here, $B \in \mathbb{R}^{n \times m}$ is the control force distribution, $u(t) \in \mathbb{R}^m$ is the control force, and $F, G \in \mathbb{R}^{n \times m}$ are feedback matrices. In applications, a nonzero entry of $B$ means that an actuator is employed and a nonzero entry in $G$ or $F$ means that a sensor is used to collect the feedback information.

By substituting (1.3) into (1.1), we obtain the following closed-loop second-order control system

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = B(F^T \dot{x} + G^T x(t)).$$

There is a large literature on the pole assignment for the second-order control system (1.4). In particular, it was established in [43] that the poles of a first-order control system can be assigned via state feedback if and only if the system is controllable. In [21], some numerical algorithms were developed for the robust state-feedback pole assignment problem in the first-order control system. For other robust and minimum norm algorithms for the complete pole assignment in the first-order control system, one may refer to [22, 41]. While the partial quadratic eigenvalue assignment problem (PQEAP) for second-order control systems was originally considered in 1997 by Datta et al. [12] in quadratic setting, where the no spillover property was preserved. Since then, the PQEAP has attracted much attention of many researchers (see for instance [1, 8, 9, 11, 13, 32] and the references therein).

Recently, the measured receptances was employed for the pole and zero assignment in the second-order control system. In particular, in [33], the partial pole and zero assignment was solved for the single-input second-order control system based on the measured receptances without the knowledge of the system matrices $M, C, K$. In [28], the pole and zero assignment was solved for the multi-input second-order control system based on the measured receptances without the knowledge of the system matrices $M, C, K$, where the symmetry and definiteness properties of the closed-loop system matrices are preserved. For various numerical methods for solving the PQEAP based on the measured receptances, one may refer to [2, 29, 33, 35, 39] and the references therein. In these methods, the poles were assigned to point locations of the complex left-hand plane.
In this paper, we consider the pole-zero assignment problem for the multi-input second-order control system (1.4). The open-loop receptance matrix is defined by

\[ H(s) := (s^2M + sC + K)^{-1}, \quad \forall s \in \mathbb{C}, \]

(1.5)

which can be obtained from the measured \( \mathbb{H}(\Im(s)i) \) (\( i := \sqrt{-1} \)) by fitting rational fraction polynomials \([15, 28]\), where \( \Im(s) \) means the imaginary part of \( s \). Define the closed-loop receptance matrix \( H^c(s) \) by

\[ H^c(s) := (s^2M + s(C - BF^T) + (K - BG^T))^{-1}, \quad \forall s \in \mathbb{C}. \]

(1.6)

Then the zero assignment problem for the multi-input closed-loop system (1.4) aims to find the two feedback matrices \( F, G \in \mathbb{R}^{n \times m} \) such that the selected zeros of the prescribed closed-loop \( pq \)-th point receptance \( H^c_{pq}(s) \) are assigned to the desired ones, where \( H^c_{pq}(s) \) means the \((p,q)\) entry of \( H^c(s) \). While the pole assignment problem for the multi-input closed-loop system (1.4) aims to find the two feedback matrices \( F, G \in \mathbb{R}^{n \times m} \) such that the complete poles of the multi-input closed-loop system (1.4) are assigned to the prescribed subregion \( D \) of the complex left-half plane. Therefore, the multi-input pole-zero assignment problem (PZAP) for the system (1.4) can be stated as follows:

**PZAP:** Given \( M, C, K \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \), the open-loop receptance matrix \( H(s) \), and a self-conjugate set of \( r \) complex numbers \( \{\mu_1, \ldots, \mu_r\} \), find the two feedback matrices \( F, G \in \mathbb{R}^{n \times m} \) such that the prescribed closed-loop \( pq \)-th point receptance \( H^c_{pq}(s) \) has the desired zeros \( \{\mu_1, \ldots, \mu_r\} \) and the complete poles of the multi-input closed-loop system (1.4) lie in the prescribed subregion \( D \) of the complex left-half plane.

We shall introduce a multi-step two-stage approach for solving the PZAP in the multi-input second-order control system based on the measured receptances, the system matrices \( M, C, K \) and linear matrix inequalities (LMIs). This is motivated by the two papers due to Ram and Elhay [32] and Belotti et al. [4]. In [32], Ram and Elhay proposed a multi-step solution method for the multi-input PQEAP. In [4], based on the receptance method and LMIs, Belotti et al. presented a two-stage method for assigning the dominant poles to the desired locations and keep the remaining ones within a prescribed region in the complex plane for the single-input second-order control system. To our knowledge, there is no numerical method available for solving the PZAP for the multi-input second-order control system (1.4). By employing the measured receptances, the system matrices \( M, C, K \), and LMIs, we provide a multi-step two-stage approach for solving the PZAP. We reformulate the PZAP as a multi-step successive PZAP for a series of single-input second-order control systems (see the MS-PZAP below). Then, at each step, we propose a two-stage approach for solving the corresponding single-input PZAP. In the first stage, we assign the selected zeros of the prescribed open-loop point receptance to the desired point locations, where, based on the measured receptances, a small underdetermined linear system of equations is formulated for finding the corresponding columns of the feedback matrices. In the second stage, we replace the complete open-loop poles by the prescribed region of the complex left-hand plane, where a semidefinite programming (SDP) is formulated for find the corresponding columns of the feedback matrices based on the system matrices and LMIs. Finally, some numerical examples are reported to illustrate the effectiveness of the proposed method for solving the PZAP.

The main contribution of this paper are as follows:
a) The PZAP is reformulated as a multi-step single-input PZAP based on the measured receptances, the selected zeros of the open-loop point receptance and the prescribed desired closed-loop zeros;

b) A two-stage approach is proposed for solving the pole and zero assignment problem at each step of the multi-step single-input PZAP;

c) In the first stage, the general expressions of the feedback column vectors are provided for the zero assignment problem for the multi-step single-input PZAP at the $k$-th step using the measured receptances;

d) In the second stage, based on the system matrices $M, C, K$, a LMI-based SDP is proposed for solving the pole assignment problem for the multi-step single-input PZAP at the $k$-th step.

The remaining part of this paper is organized as follows. In Section 2 we give some necessary preliminaries. In Sections 3 we propose a multi-step two-stage approach for solving the PZAP by using the receptance method, the system matrices, and the LMI. In Section 4 we present some numerical examples to indicate the effectiveness of our algorithm. Finally, we give some concluding remarks in Section 5.

2 Preliminaries

Throughout this paper, we need the following notation. The symbols $\mathbb{C}^{m \times n}$ and $\mathbb{R}^{m \times n}$ denote the set of complex $m \times n$ matrices and the set of real $m \times n$ matrices, respectively, and $\mathbb{C}^n = \mathbb{C}^{n \times 1}$ and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$. Denote by $\Re(\cdot)$ and $\Im(\cdot)$ the real part and the imaginary part of a complex matrix, respectively. Let $I_n$ be the identity matrix of order $n$. Let $e_p$ be the $p$-th column of $I_n$. Let $|\cdot|$ denote the modulus of a complex number or the absolute value of a real number. The superscripts $\cdot^T$, $\cdot^{-1}$, and $\cdot^\dagger$ denote the transpose, the inverse, and the Moore-Penrose inverse of a matrix accordingly. The symbol $\otimes$ means the Kronecker product. Denote by $\|\cdot\|$ the Euclidean vector norm or its induced matrix norm. For two real square matrix $A_1, A_2$, $A_1 \succeq 0$ ($A_1 \succ 0$, respectively) denotes that $A_1$ is symmetric and positive semidefinite (symmetric and positive definite, respectively), $A_1 \succeq A_2$ ($A_1 \succ A_2$, respectively) means that $A_1 - A_2$ is symmetric and positive semidefinite (symmetric and positive definite, respectively), $A_1 \preceq 0$ ($A_1 \prec 0$, respectively) means that $-A_1 \succeq 0$ ($-A_1 \succ 0$, respectively), and $A_1 \preceq A_2$ ($A_1 \prec A_2$, respectively) means that $-A_1 \succeq -A_2$ ($-A_1 \succ -A_2$, respectively).

Let $\{\nu_j\}_{j=1}^{r}$ be the unwanted zeros of the $pq$-th point open-loop receptance $H_{pq}(s)$. We assume that the set $\{\nu_j\}_{j=1}^{r}$ is closed under complex conjugation, $\{\nu_j\}_{j=1}^{r} \cap \{\mu_j\}_{j=1}^{\tau} = \emptyset$, and the matrix $B$ is full column rank.

3 A multi-step two-stage approach

In this section, we first reformulate the PZAP as a multi-step single-input PZAP. Then we successively solve a series of single-input PZAPs. At each step, we provide a two-stage approach for the corresponding single-input PZAP, where the first stage aims to solve the zero assignment
problem for a closed-loop point receptance (see (3.4) below) while the second stage aims to solve
the pole assignment problem for the corresponding single-input closed-loop system (see (3.2)
below).

3.1 A multi-step reformulation

In this subsection, we shall reformulate the PZAP as a multi-step single-input PZAP. We first
observe from (1.5) that, for any \( s \in \mathbb{C} \),

\[
H(s) = \frac{1}{\det(s^2M + sC + K)} \text{adj}(s^2M + sC + K),
\]

where \( \text{adj}(\cdot) \) means the adjugate of a square matrix. The open-loop \( pq \)-th point receptance
\( H_{pq}(s) \) is the \((p,q)\) entry of \( H(s) \), whose zeros are determined by the following characteristic
equation:

\[
\det(s^2M_{qp} + sC_{qp} + K_{qp}) = 0,
\]

where \( M_{qp}, C_{qp}, K_{qp} \in \mathbb{R}^{(n-1) \times (n-1)} \) are obtained by deleting the \( q \)-th row and the \( p \)-column of \( M, C, K \).

In the following, we reformulate the PZAP as a multi-step single-input PZAP. To do so, let
\( C_1 := C, K_1 := K \), and define

\[
C_k := C - \sum_{i=1}^{k-1} b_i f_i^T \quad \text{and} \quad K_k := K - \sum_{i=1}^{k-1} b_i g_i^T,
\]

(3.1)

for \( k = 2, \ldots, m \), where \( b_j, f_j, \) and \( g_j \) are the \( j \)-th columns of \( B, F, \) and \( G \) accordingly. Then, based on the multi-input closed-loop control system (1.4), we can obtain the following \( m \)
successive single-input closed-loop feedback control systems:

\[
M\ddot{x}(t) + C_k \dot{x}(t) + K_k x(t) = b_k (f_k^T \dot{x}(t) + g_k^T x(t)),
\]

(3.2)

for \( k = 1, \ldots, m \). To define a PZAP for the \( k \)-th single-input closed-loop feedback control system
(3.2), as in [32], for \( 1 \leq j \leq r \), we consider the a segment line \( \Sigma_j \) in the complex plane including
both endpoints \( \nu_j \) and \( \mu_j \). Then we define the points, which divide \( \Sigma_j \) into \( m \) equal parts, by

\[
\eta_{jk} := \nu_j + \frac{k}{m} (\mu_j - \nu_j), \quad k = 1, \ldots, m.
\]

(3.3)

Then, for any \( 2 \leq k \leq m \), the closed-loop \( pq \)-th point receptance \( H_{pq}^{(k)}(s) \) is the \((p,q)\) entry of the closed-loop receptance matrix \( H^{(k)}(s) \), which is defined by

\[
H^{(k)}(s) := (s^2M + sC_k + K_k)^{-1}, \quad \forall s \in \mathbb{C}.
\]

(3.4)

Based on the above analysis, the PZAP can be reformulated as the following multi-step
single-input PZAP.

**MS-PZAP.** Given \( M, C, K \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), a self-conjugate set of \( \{\mu_j\}_{j=1}^r \), and the open-
loop receptance matrices \( \{H(\eta_{jk})\}_{j=1}^r \) with \( \{\eta_{jk}\} \) defined by (3.3), find the two feedback vectors
\(f_k\) and \(g_k\) successively such that the closed-loop \(pq\)-th point receptance \(H_{pq}^{(k)}(s)\) has the desired zeros \(\{\eta_{1k}, \ldots, \eta_{rk}\}\) and the poles of the single-input closed-loop system \([3.2]\) lie in the prescribed subregion \(\mathcal{D}\) of the complex left-half plane for \(k = 1, \ldots, m\).

In the following two subsections, we present a two-stage approach for solving the MS-PZAP at the \(k\)-th step.

### 3.2 Zero assignment problem for the MS-PZAP at the \(k\)-th step

In this subsection, we provide the general expressions of the two feedback vectors \(f_k\) and \(g_k\) such that the closed-loop \(pq\)-th point receptance \(H_{pq}^{(k)}(s)\) has the desired zeros \(\{\eta_{1k}, \ldots, \eta_{rk}\}\) at the \(k\)-th step.

Using the Sherman-Morrison formula \([47]\), we have

\[
H^{(k)}(s) = H^{(k-1)}(s) + \frac{H^{(k-1)}(s)b_k(g_k^T + sf_k^T)H^{(k-1)}(s)}{1 - (g_k^T + sf_k^T)H^{(k-1)}(s)b_k},
\]

where \(H^{(0)}(s) = H(s)\). It follows from \([3.5]\) that the closed-loop \(pq\)-th point receptance \(H_{pq}^{(k)}(s)\) is given by

\[
H_{pq}^{(k)}(s) = H_{pq}^{(k-1)}(s) + \frac{e_p^T(H^{(k-1)}(\eta_{jk})b_k(g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk}))e_q}{1 - (g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk})b_k}.
\]

The zero assignment of the MS-PZAP at the \(k\)-th step aims to find the gains \(f_k\) and \(g_k\) such that the \(pq\)-th point receptance \(H_{pq}^{(k)}(s)\) has the desired zeros \(\{\eta_{1k}, \ldots, \eta_{rk}\}\). Using \([3.6]\) we have

\[
0 = H_{pq}^{(k)}(\eta_{jk}) = H_{pq}^{(k-1)}(\eta_{jk}) + \frac{e_p^T(H^{(k-1)}(\eta_{jk})b_k(g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk}))e_q}{1 - (g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk})b_k},
\]

or

\[
H_{pq}^{(k-1)}(\eta_{jk}) = H_{pq}^{(k-1)}(\eta_{jk})b_k(g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk})b_k + e_p^T H^{(k-1)}(\eta_{jk})b_k(g_k^T + \eta_{jk}f_k^T)H^{(k-1)}(\eta_{jk})e_q,
\]

for \(j = 1, \ldots, r\).

Based on the above analysis, we have the following theorem on the zero assignment for the MS-PZAP at the \(k\)-th step.

**Theorem 3.1** Let

\[
G_k := \begin{bmatrix} \eta_{1k}T_{1k} & T_{1k} \\ \eta_{2k}T_{2k} & T_{2k} \\ \vdots & \vdots \\ \eta_{rk}T_{rk} & T_{rk} \end{bmatrix} \in \mathbb{C}^{r \times (2n)} \quad \text{and} \quad \varphi_k := \begin{bmatrix} H_{pq}^{(k-1)}(\eta_{1k}) \\ H_{pq}^{(k-1)}(\eta_{2k}) \\ \vdots \\ H_{pq}^{(k-1)}(\eta_{rk}) \end{bmatrix} \in \mathbb{C}^r,
\]

where

\[
t_{jk} = H_{pq}^{(k-1)}(\eta_{jk})H^{(k-1)}(\eta_{jk})b_k - (e_p^T H^{(k-1)}(\eta_{jk})b_k) H^{(k-1)}(\eta_{jk})e_q.
\]
If the following underdetermined linear equation
\[ G_k y_k \equiv G_k \begin{bmatrix} f_k \\ g_k \end{bmatrix} = \varphi_k \] (3.8)
admits a real solution \( y_k = [f_k^T, g_k^T]^T \), then the feedback vectors \( f_k \) and \( g_k \) are such that the closed-loop \( pq \)-th point receptance \( H_{pq}^{(k)}(s) \) has the desired zeros \( \{ \mu_1^k, \ldots, \mu_r^k \} \).

**Remark 3.2** We observe from Theorem 3.1 that, in the \( k \)th step, we need \( H^{(k-1)}(s) \) for solving (3.8). Since \( H^{(0)}(s) = H(s) \) is available from the physical tests, it follows from (3.5) that for any \( k \geq 2 \), \( H^{(k-1)}(s) \) can be computed recursively based on \( H(s) \), \( \{ f_i \}^k_{i=1} \), and \( \{ g_i \}^k_{i=1} \).

Next, we discuss the solvability of (3.8). We first recall the following lemmas.

**Lemma 3.3** [38, Lemma 1.3] Let \( A_1 \in \mathbb{C}^{l \times p_1} \), \( A_2 \in \mathbb{C}^{p_2 \times q_1} \), and \( A_3 \in \mathbb{C}^{l \times q_1} \). Define
\[ \mathcal{L} := \{ E \in \mathbb{C}^{p_1 \times p_2} : A_1 E A_2 = A_3 \} \]
Then \( \mathcal{L} \neq \emptyset \) if and only if \( A_1, A_2, A_3 \) satisfy
\[ A_1 A_1^\dagger A_3 A_2^\dagger = A_3, \]
and in case of \( \mathcal{L} \neq \emptyset \), any \( E \in \mathcal{L} \) can be expressed as:
\[ E = A_1^\dagger A_3 A_2^\dagger + W - A_1^\dagger A_1 W A_2 A_2^\dagger, \]
where \( W \in \mathbb{C}^{p_1 \times p_2} \) is arbitrary.

**Lemma 3.4** [19, Theorem 4.4] Let \( U \in \mathbb{C}^{p_1 \times p_2} \) and \( h \in \mathbb{C}^{p_1} \). Then \( z \in \mathbb{C}^{p_2} \) solve the following least squares problem
\[ \min_{z \in \mathbb{C}^{p_2}} \| U z - h \| \]
if and only if
\[ z = U^\dagger h + v \quad \text{for some } v \in \ker(U), \]
where \( \ker(\cdot) \) denotes the kernel of a matrix.

Based on Lemma 3.3, we obtain the following result on the solvability of (3.8).

**Proposition 3.5** The linear equation (3.8) is solvable if and only if
\[ G_k G_k^\dagger \varphi_k = \varphi_k. \]
In this case, the general solution of (3.8) is given by
\[ y_k = G_k^\dagger \varphi_k + w - G_k^\dagger G_k w, \]
where \( w \in \mathbb{C}^{2n} \) is arbitrary.
In practice, only real gains $f_k$ and $g_k$ are needed. By Proposition \ref{prop:real-gains}, we have the following result. The proof follows the similar arguments in \cite[Theorem 3]{33}. Hence, we omit it here.

**Theorem 3.6** Suppose the set $\{\eta_{jk}\}_{j=1}^r$ is closed under complex conjugation. If $G_k G_k^\dagger \varphi_k = \varphi_k$, then the gains $f_k$ and $g_k$ determined by (3.8) can be chosen to be real. In this case, we can obtain the gains $f_k$ and $g_k$ by solving the following linear equation

$$
\begin{bmatrix}
\Re(G_k) \\
\Im(G_k)
\end{bmatrix} y_k = \begin{bmatrix}
\Re(\varphi_k) \\
\Im(\varphi_k)
\end{bmatrix}
$$

(3.9)

for $y_k := [f_k^T, g_k^T]^T$ and the general solution of (3.9) is given by

$$
y_k = \begin{bmatrix}
\Re(G_k) \\
\Im(G_k)
\end{bmatrix}^\dagger \begin{bmatrix}
\Re(\varphi_k) \\
\Im(\varphi_k)
\end{bmatrix} + V_{kR} \psi_{kR} \equiv \tilde{y}_{kR} + V_{kR} \psi_{kR},
$$

(3.10)

where $\psi_{kR} \in \mathbb{R}^{2n-k}$ is arbitrary with $k = \text{rank}(G_k)$. Here, the column vectors of $V_{kR}$ form an orthonormal basis for the kernel of $[\Re(G_k)^T, \Im(G_k)^T]^T$. For simplicity, we let $V_{kR} := \text{ker}([\Re(G_k)^T, \Im(G_k)^T]^T)$.

**Remark 3.7** Suppose the set $\{\mu_j\}_{j=1}^r$ is closed under complex conjugation. We observe from Theorem 3.6 that, if the set $\{\mu_j\}_{j=1}^r$ is such that $G_k$ is full row rank, then (3.8) must admit real solutions. This is guaranteed in practice (see the later numerical tests) since for $k = 1, \ldots, m$, the receptance data $\{H(\eta_{jk})\}_{j=1}^r$ can be measured precisely \cite{17} and $\{H^{(k-1)}(\eta_{jk})\}_{j=1}^r$ can be computed exactly via (3.5). If (3.8) has no solution, then we see from Lemma 3.4 that the gains $f_k$ and $g_k$ defined by (3.10) is a least squares solution to the following least squares problem:

$$
\min_{y_k \in \mathbb{R}^{2n}} \|G_k y_k - \varphi_k\| \quad \text{or} \quad \min_{y_k \in \mathbb{R}^{2n}} \left\| \begin{bmatrix}
\Re(G_k) \\
\Im(G_k)
\end{bmatrix} y_k - \begin{bmatrix}
\Re(\varphi_k) \\
\Im(\varphi_k)
\end{bmatrix} \right\|.
$$

In this case, we must check whether the feedback vectors $f_k$ and $g_k$ are such that the closed-loop pq-th point receptance $H_{pq}^{(k)}(s)$ has the desired zeros $\{\mu_1, \ldots, \mu_r\}$ or their acceptable approximations.

### 3.3 Pole assignment problem for the MS-PZAP at the $k$-th step

In this subsection, we assign the complete poles of the $k$-th single-input closed-loop system (3.2) in the prescribed subregion $\mathcal{D}$ of the complex left-half plane. We first recall the stability region and its LMI formulation (see for instance \cite{7}). Then we propose the LMI method for assigning the closed-loop poles of the system (3.2) to the given subregion $\mathcal{D}$.

We recall the definition of the subregion stability for a first-order dynamical system \cite{7}.

**Definition 3.8** Given a subregion $\mathcal{D}$ of the complex left-half plane, the first-order dynamical system $\ddot{q}(t) = Aq(t)$ is called $\mathcal{D}$-stable if all its poles lie in $\mathcal{D}$. In this case, we call $A$ is $\mathcal{D}$-stable.

The following LMI region was also introduced in \cite{7}. 

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**Definition 3.9** Let $\mathcal{D}$ be a subregion of the complex plane. Then $\mathcal{D}$ is called an LMI region if there exist a real symmetric matrix $R$ and a real square matrix $Z$ such that

$$\mathcal{D} = \{ c \in \mathbb{C} \mid R + cZ + \bar{c}Z^T < 0 \}.$$  

**Remark 3.10** Let $\mathcal{D}_1(\alpha)$ be a vertical strip defined by

$$\mathcal{D}_1(\alpha) := \{ c \in \mathbb{C} \mid \Re(c) < -\alpha < 0 \}.$$  

Let $\mathcal{D}_2(\tau_{\text{min}}, \theta)$ be a conic sector defined by

$$\mathcal{D}_2(\tau_{\text{min}}, \theta) := \{ c \in \mathbb{C} \mid \tan(\theta)\Re(c) < -|\Im(c)| \}$$

with a minimum damping ratio $\tau_{\text{min}} = \cos(\theta)$. By Definition 3.9, we have $R = 2\alpha$ and $Z = 1$ for $\mathcal{D}_1(\alpha)$ and

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix}$$

for $\mathcal{D}_2(\tau_{\text{min}}, \theta)$. The vertical strip, conic sector, and their intersection, as shown in Figure 3.1, are often desired for the pole assignment problem of the closed-loop system (1.4).

![Image of vertical strip, conic sector, and their intersection](image)

**Figure 3.1:** Vertical strip (left), conic sector (middle), and their intersection (right).

---

On the $\mathcal{D}$-stability of a real square matrix $A$, we have the following lemma from [7, Theorem 2.2].

**Lemma 3.11** Let $A$ be a real square matrix. Then $A$ is $\mathcal{D}$-stable if and only if there exists a real symmetric matrix $X$ such that

$$\Phi_\mathcal{D}(A, X) < 0 \quad \text{and} \quad X > 0,$$

where

$$\Phi_\mathcal{D}(A, X) := R \otimes X + Z \otimes (AX) + Z^T \otimes (AX)^T.$$  

In the following, we develop a LMI-based SDP for solving the pole assignment problem for the MS-PZAP at the $k$-th step. We first give the first-order model of the $k$-th closed-loop feedback control system (3.2) as follows:

$$\dot{q}(t) = \tilde{A}_k q(t) + \tilde{b}_k u_k(t), \quad (3.11)$$
where
\[ A_k = \begin{bmatrix} -M^{-1}C_k & -M^{-1}K_k \\ I & 0 \end{bmatrix}, \quad b_k = \begin{bmatrix} M^{-1}b_k \\ 0 \end{bmatrix}, \quad u_k(t) = [f_k^T, g_k^T]q(t), \quad q(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}. \]

Using (3.10) and (3.11) we have the following first-order model of the \( k \)-th closed-loop control system (3.2):
\[ \dot{q}(t) = \left( A_k + b_k(\bar{y}_{kR} + V_{kR}\psi_{kR})^T \right)q(t), \]
where \( \psi_{kR} \in \mathbb{R}^{2n-\kappa} \) is arbitrary. Let
\[ A_k^{(0)} := A_k + b_k\psi_{kR}^TV_{kR}^T. \]
Then we have
\[ \dot{q}(t) = (A_k^{(0)} + b_k\psi_{kR}^TV_{kR}^T)q(t), \quad (3.12) \]
where \( \psi_{kR} \in \mathbb{R}^{2n-\kappa} \) is arbitrary.

By Definition 3.8 the first-order system (3.12) is called \( \mathcal{D} \)-stable if all the eigenvalues of \((A_k^{(0)} + b_k\psi_{kR}^TV_{kR}^T)\) lie in \( \mathcal{D} \). By Definition 3.9 an LMI region of the first-order system (3.12) is characterized by the characteristic matrix
\[ \Phi_{\mathcal{D}}(A_k^{(0)}, \psi_{kR}, X) = R \otimes X + Z \otimes \left( (A_k^{(0)} + b_k\psi_{kR}^TV_{kR}^T)X \right) + Z^T \otimes \left( (A_k^{(0)} + b_k\psi_{kR}^TV_{kR}^T)X \right)^T, \quad (3.13) \]
where the real symmetric matrix \( R \) and the real square matrix \( Z \) describe the LMI regions \( \mathcal{D} \) (see for instance Remark 3.10). Then, using Lemma 3.11 the first-order system (3.12) is \( \mathcal{D} \)-stable if and only if there exists a real symmetric matrix \( X \) such that
\[ \Phi_{\mathcal{D}}(A_k^{(0)}, \psi_{kR}, X) < 0 \quad \text{and} \quad X > 0. \quad (3.14) \]

We observe from (3.13) and (3.14) that \( \Phi_{\mathcal{D}}(A_k^{(0)}, \psi_{kR}, X) < 0 \) is a bilinear matrix inequality with respect to \( \psi_{kR} \) and \( X \). To employ LMI-based solvers, we let \( \zeta_{kR} := V_{kR}\psi_{kR} \) and \( \omega_{kR} := X\zeta_{kR} \). Then we have
\[ \Phi_{\mathcal{D}}(A_k^{(0)}, \omega_{kR}, X) = R \otimes X + Z \otimes \left( A_k^{(0)}X + b_k\omega_{kR}^T \right) + Z^T \otimes \left( A_k^{(0)}X + b_k\omega_{kR}^T \right)^T. \]
This shows that \( \Phi_{\mathcal{D}}(A_k^{(0)}, \omega_{kR}, X) < 0 \) is a LMI with respect to \( \omega_{kR} \) and \( X \). In this case, the \( \mathcal{D} \)-stability condition in (3.14) becomes
\[ \Phi_{\mathcal{D}}(A_k^{(0)}, \omega_{kR}, X) < 0, \quad \text{and} \quad X > 0. \quad (3.15) \]

To reduce the energy consumption and noise influence, it is desired to find the feedback matrices with their norms as small as possible. From (3.10) we see that the norms of \( f_k \) and \( g_k \) can be minimized if the norm of \( V_{kR}\psi_{kR} \) is minimized since \( ||y_k|| = ||\bar{y}_{kR}|| + ||V_{kR}\psi_{kR}|| \). Sparked by the LMIIs (3.15), we can minimize the feedback norm in the sense of minimizing the norm of \( \omega_{kR} \). We note that \( ||\omega_{kR}||_2 = ||XV_{kR}\psi_{kR}||_2 \equiv ||V_{kR}\psi_{kR}||_X, \) which is a weighted norm in terms of the real symmetric and positive definite matrix \( X \). Therefore, we can solve the pole
assignment problem for the MS-PZAP at the $k$-th step by finding an optimal solution $(\omega_{kR}, X)$ to the following SDP:

$$
\begin{align*}
\min & \quad \frac{1}{2} \|\omega_{kR}\|_2^2 \\
\text{subject to (s.t.)} & \quad \Phi_D(\tilde{A}_k^{(0)}, \omega_{kR}, X) < 0, \\
& \quad \quad \quad X > 0.
\end{align*}
$$

(3.16)

For numerical reliability, instead of problem (3.16), one may solve the following SDP:

$$
\begin{align*}
\min & \quad \frac{1}{2} \|\omega_{kR}\|_2^2 \\
\text{s.t.} & \quad \Phi_D(\tilde{A}_k^{(0)}, \omega_{kR}, X) \preceq -\delta_1 I, \\
& \quad \quad \quad X \succeq \delta_2 I,
\end{align*}
$$

(3.17)

where $\delta_1$ and $\delta_2$ are two positive scalars.

When we get the optimal solution $(\omega_{kR}, X)$ of the SDP (3.17), we can compute $\psi_{kR}$ in the following two ways. By hypothesis, $V_{kR}$ is full row rank. Then we can compute $\psi_{kR}$ via

$$
\psi_{kR} = V_{kR}^\dagger X^{-1} \omega_{kR}.
$$

(3.18)

If $V_{kR}$ is ill-conditioned, then the vector $\psi_{kR}$ determined by (3.18) may not guarantee the LMI constraint $\Phi_D(\tilde{A}_k^{(0)}, \psi_{kR}, X) < 0$ holds. In this case, one may find $\psi_{kR} \in \mathbb{R}^{2n-\kappa}$ such that the distance $\|V_{kR}\psi_{kR} - X^{-1} \omega_{kR}\|_2$ is minimized subject to $\Phi_D(\tilde{A}_k^{(0)}, \psi_{kR}, X) < 0$. This forms a SDP as follows:

$$
\begin{align*}
\min & \quad \frac{1}{2} \|V_{kR}\psi_{kR} - X^{-1} \omega_{kR}\|_2^2 \\
\text{s.t.} & \quad \tilde{\Phi}_D(\tilde{A}_k^{(0)}, \psi_{kR}) \equiv \Phi_D(\tilde{A}_k^{(0)}, \psi_{kR}, X) \preceq -\delta_3 I,
\end{align*}
$$

(3.19)

where $\delta_3$ is a positive scalar.

Finally, for demonstration purpose, the multi-step two-stage algorithm is described in Algorithm 3.1. The total computational cost of Algorithm 3.1 is $O(m(n^3 + n^2 r + nr^2 + \pi))$, where $\pi$ stands for the cost for solving the SDP (3.18) or (3.19) by using a LMI-based solver (see the later numerical tests).

4 Numerical examples

In this section, we give some numerical examples to show that Algorithm 3.1 is effective for solving the PZAP. The numerical experiments were implemented in MATLAB R2020a and run on a workstation with an Intel Xeon CPU E5-2687W of 3.10 GHz and 32 GB of RAM.

In our numerical tests, it is desired to assign the poles of the closed-loop system (3.2) to a vertical strip $D_1(\alpha)$, a conic sector $D_2(\tau_{\min}, \theta)$, or their intersection $D(\alpha, \tau_{\min}, \theta) := D_1(\alpha) \cap D_2(\tau_{\min}, \theta)$ as noted in Remark 3.10. By Lemma 3.11, a real square matrix $A$ is $D(\alpha, \tau_{\min}, \theta)$-stable if and only if there exists a real symmetric matrix $X$ such that

$$
\begin{align*}
\Phi_{D_1(\alpha)}(A, X) & := AX + XA^T + 2\alpha X < 0, \\
\Phi_{D_2(\tau_{\min}, \theta)}(A, X) & := \begin{bmatrix}
\sin \theta((AX + XA^T) & \cos \theta(A + XA^T) \\
\cos \theta(XA^T - AX) & \sin \theta(A + XA^T)
\end{bmatrix} < 0, \\
X & > 0.
\end{align*}
$$
Algorithm 3.1 A multi-step two-stage algorithm for the PZAP

**Input:**
1. The symmetric matrices $M, C, K \in \mathbb{R}^{n \times n}$ with $M \succ 0$;
2. The control matrix $B \in \mathbb{R}^{n \times m}$ ($m \leq n$);
3. The zeros $\{\nu_j\}_{j=1}^r$ of the open-loop $pq$-th point receptance $H_{pq}(s)$;
4. The prescribed self-conjugate set $\{\mu_j\}_{j=1}^r$;
5. The measured receptance data $H^{(0)}(\eta_{jk}) = H(\eta_{jk})$ with $\eta_{jk}$ defined by (3.3) for $j = 1, \ldots, r$ and $k = 1, \ldots, m$;
6. The prescribed subregion $D$ of the complex left-half plane;
7. The parameters $\delta_1, \delta_2, \delta_3 > 0$.

**Step 1** Compute $G_1$ and $\varphi_1$ by (3.7) with $k = 1$.

**Step 2** Compute $\hat{y}_{1R}$ by (3.10) with $k = 1$ and $V_{1R} = \ker([\Re(G_1)^T, \Im(G_1)^T]^T)$.

**Step 3** Solve the SDP (3.17) with $k = 1$ for $(\bar{\omega}_1, \bar{X})$. If $V_{1R}$ is well-conditioned, then compute $\psi_{1R}$ by (3.18) with $k = 1$; Otherwise, solve the SDP (3.19) with $k = 1$ for $\psi_{1R}$.

**Step 4** Form $f_1$ and $g_1$ by (3.10) with $k = 1$.

**Step 5** For $k = 2, \ldots, m$

1. **Step 5.1** Compute $H^{(k-1)}(\eta_{jk})$ recursively by (3.5) using $H^{(0)}(\eta_{jk}), \{f_i\}_{i=1}^{k-1}$, and $\{g_i\}_{i=1}^{k-1}$ for $j = 1, \ldots, r$.
2. **Step 5.2** Compute $G_k$ and $\varphi_k$ by (3.7).
3. **Step 5.3** Compute $\hat{y}_{kR}$ by (3.10) and $V_{kR} = \ker([\Re(G_k)^T, \Im(G_k)^T]^T)$.
4. **Step 5.4** Solve the SDP (3.17) for $(\bar{\omega}_k, \bar{X})$. If $V_{kR}$ is well-conditioned, then compute $\psi_{kR}$ by (3.18); Otherwise, solve the SDP (3.19) for $\psi_{kR}$.
5. **Step 5.5** Form $f_k$ and $g_k$ by (3.10).

**Output:** The real feedback matrices $F = [f_1, \ldots, f_m]$ and $G = [g_1, \ldots, g_m]$ such that the closed-loop $pq$-th point receptance $\hat{H}_{pq}(s)$ has the desired zeros $\{\mu_1, \ldots, \mu_r\}$ and the poles of the closed-loop system (1.4) lie in the prescribed subregion $D$. This is done as follows: For $k = 1, \ldots, m$, find the two feedback vectors $f_k$ and $g_k$ successively such that the closed-loop $pq$-th point receptance $H^{(k)}_{pq}(s)$ has the desired zeros $\{\eta_{1k}, \ldots, \eta_{rk}\}$ and the poles of the closed-loop system (3.2) lie in the prescribed subregion $D$. 

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In this case, the SDPs (3.17) and (3.19) can be written as

\[
\begin{align*}
\min & \quad \frac{1}{2}\|\omega_k R\|^2_2 \\
\text{s.t.} & \quad \Phi_{D_1(\alpha)}(\tilde{A}^{(0)}_k, \omega_k R, X) \preceq -\delta_1 I, \\
& \quad \Phi_{D_2(\tau_{\min}, \theta)}(\tilde{A}^{(0)}_k, \omega_k R, X) \preceq -\delta_1 I, \\
& \quad X \succeq \delta_2 I
\end{align*}
\]  \tag{4.1}

and

\[
\begin{align*}
\min & \quad \frac{1}{2}\|V_k R \psi_k R - X^{-1} \omega_k R\|^2_2 \\
\text{s.t.} & \quad \Phi_{D_1(\alpha)}(\tilde{A}^{(0)}_k, \psi_k R) = \Phi_{D_1(\alpha)}\left((\tilde{A}^{(0)}_k + \tilde{b}_k \psi_k R V_k R, X) \right) \preceq -\delta_3 I, \\
& \quad \Phi_{D_2(\tau_{\min}, \theta)}(\tilde{A}^{(0)}_k, \psi_k R) = \Phi_{D_2(\tau_{\min}, \theta)}\left((\tilde{A}^{(0)}_k + \tilde{b}_k \psi_k R V_k R, X) \right) \preceq -\delta_3 I,
\end{align*}
\]  \tag{4.2}

respectively, where

\[
\Phi_{D_1(\alpha)}(\tilde{A}^{(0)}_k, \omega_k R, X) = W(\tilde{A}^{(0)}_k, \omega_k R, X) + W^T(\tilde{A}^{(0)}_k, \omega_k R, X) + 2\alpha X,
\]

\[
\Phi_{D_2(\tau_{\min}, \theta)}(\tilde{A}^{(0)}_k, \omega_k R, X) = \\
\begin{bmatrix}
\sin \theta\left(W(\tilde{A}^{(0)}_k, \omega_k R, X) + W^T(\tilde{A}^{(0)}_k, \omega_k R, X)\right) & \cos \theta\left(W(\tilde{A}^{(0)}_k, \omega_k R, X) - W^T(\tilde{A}^{(0)}_k, \omega_k R, X)\right) \\
\cos \theta\left(W^T(\tilde{A}^{(0)}_k, \omega_k R, X) - W(\tilde{A}^{(0)}_k, \omega_k R, X)\right) & \sin \theta\left(W^T(\tilde{A}^{(0)}_k, \omega_k R, X) + W(\tilde{A}^{(0)}_k, \omega_k R, X)\right)
\end{bmatrix}
\]

with \(W(\tilde{A}^{(0)}_k, \omega_k R, X) := \tilde{A}^{(0)}_k X + \tilde{b}_k \omega_k R\).

To solve the SDPs (3.17), (3.19), (4.1), or (4.2), we use the MATLAB-based toolbox YALMIP \cite{yalmip} by interfacing the Mosek solver, where the LMI{s} can be carried out readily.

We consider the following numerical examples.

**Example 4.1** Consider the second-order control system (1.4) with \(n = 3\) and \(m = 2\), where

\[
M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = 0.01 \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 9 & -3 \\ 0 & -3 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

We wish replace a pair of complex conjugate zeros \(\nu_{1,2} = -0.0100 \pm 2.4495i\) of the 32-th open-loop point receptance \(H_{32}(s)\) by \(\mu_{1,2} = -0.0005 \pm 2.0000i\) and assign the 6 open-loop poles \(\lambda_{1,2} = -0.0212 \pm 3.5694i, \lambda_{3,4} = -0.0060 \pm 1.8958i, \text{and} \lambda_{5,6} = -0.0128 \pm 2.7685i\) to the region \(\mathcal{D}(0.01, 0.001, \theta)\).

By using Algorithm 3.1 to Example 4.1 with \(\delta_1 = \delta_2 = \delta_3 = 10^{-5}\), we obtain the following feedback matrices

\[
F = \begin{bmatrix} 0.0095 & -0.0101 \\ -0.0262 & -0.0244 \\ -0.0130 & -0.0316 \end{bmatrix}, \quad G = \begin{bmatrix} 1.0505 & -0.9499 \\ -0.0011 & 0.0004 \\ -0.0006 & -0.0005 \end{bmatrix}.
\]
with \( \|F\|_F = 0.0514 \) and \( \|G\|_F = 1.1463 \). Table 4.1 lists the open-loop zeros of \( H_{32}(s) \) and closed-loop zeros of \( H_{32}^{(m)}(s) \), the open-loop and closed-loop poles of the closed-loop system (1.4) for Example 4.1. The pole-zero map is depicted in Figure 4.2, where we observe from Table 4.1 and Figure 4.2 that, as expected, the desired closed-loop poles and zeros are assigned via Algorithm 3.1.

Table 4.1: Numerical results for Example 4.1

| open-loop zeros | desired closed-loop zeros | closed-loop zeros |
|-----------------|---------------------------|-------------------|
| \( \mu_{1,2} = -0.0005 \pm 2.0000i \) | \( \lambda_{1,2} = -0.0205 \pm 3.5777i \) | \( \mu_{1,2} = -0.0005 \pm 2.0000i \) |
| \( \lambda_{3,4} = -0.0123 \pm 1.7848i \) | \( \lambda_{5,6} = -0.0182 \pm 2.6387i \) | \( \lambda_{3,4} = -0.0123 \pm 1.7848i \) |
| \( \lambda_{5,6} = -0.0128 \pm 2.7685i \) | \( \lambda_{5,6} = -0.0182 \pm 2.6387i \) | \( \lambda_{3,4} = -0.0123 \pm 1.7848i \) |

Consider the second-order control system (1.4) with \( n = 3 \) and \( m = 2 \), where

\[
M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix}, \quad K = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

We report our numerical results for (a) replace the two pairs of complex conjugate zeros \( \nu_{1,2} = -0.0247 \pm 1.7549i \) and \( \nu_{3,4} = -0.0170 \pm 0.9590i \) of the 22-th open-loop point receptance \( H_{22}(s) \) by \( \mu_{1,2} = -0.0370 \pm 2.0000i \) and \( \mu_{3,4} = -0.0250 \pm 1.2000i \) and assign the 6 open-loop poles \( \lambda_{1,2} = -0.0509 \pm 2.2638i \), \( \lambda_{3,4} = -0.0374 \pm 1.6016i \), and \( \lambda_{5,6} = -0.0033 \pm 0.5513i \) to the region \( \mathcal{D}(0.01, 0.001, \theta) \); (b) replace a pair of complex conjugate zeros \( \nu_{1,2} = -0.0126 \pm 1.8696i \) of the
32-th open-loop point receptance $H_{32}(s)$ by $\mu_{1,2} = -0.0250 \pm 2.0000i$ and assign the complete open-loop poles to the region $D(0.01, 0.001, \theta)$.

We apply Algorithm 3.1 to Example 4.2(a) with $\delta_1 = \delta_2 = \delta_3 = 10^{-5}$ and Example 4.2(b) with $\delta_1 = \delta_2 = \delta_3 = 10^{-3}$. For Example 4.2(a), we obtain the following feedback matrices

$$F = \begin{bmatrix} -0.0407 & 0.0701 \\ 0.0000 & -0.0000 \\ -0.1591 & -0.0610 \end{bmatrix}, \quad G = \begin{bmatrix} -1.3833 & -2.6498 \\ 0.0000 & -0.0000 \\ -3.5097 & -2.2453 \end{bmatrix}$$

with $\|F\|_F = 0.1887$ and $\|G\|_F = 5.1278$. While for Example 4.2(b), the computed feedback matrices are given by

$$F = \begin{bmatrix} -0.0480 & 0.0621 \\ -0.0466 & 0.0000 \\ -0.1428 & 0.0000 \end{bmatrix}, \quad G = \begin{bmatrix} -0.5006 & 0.5033 \\ 0.0015 & 0.0000 \\ -0.0020 & 0.0000 \end{bmatrix}$$

with $\|F\|_F = 0.1695$ and $\|G\|_F = 0.7099$. Table 4.2 displays the numerical results for Example 4.2. The pole-zero map is depicted in Figure 4.3. The total computing time is about 0.388 seconds for (a) and 0.369 seconds for (b). We observe from Table 4.2 and Figure 4.3 that our method can effectively assign the selected closed-loop zeros to desired positions and assign the closed-loop poles to the prescribed region.

Table 4.2: Numerical results for Example 4.2

| open-loop zeros | desired closed-loop zeros | closed-loop zeros |
|-----------------|---------------------------|-------------------|
| $\mu_{1,2} = -0.0247 \pm 1.7549i$ | $\mu_{1,2} = -0.0370 \pm 2.0000i$ | $\mu_{1,2} = -0.0370 \pm 2.0000i$ |
| $\mu_{3,4} = -0.0170 \pm 0.9590i$ | $\mu_{3,4} = -0.0250 \pm 1.2000i$ | $\mu_{3,4} = -0.0250 \pm 1.2000i$ |

| open-loop poles | desired closed-loop poles | closed-loop poles |
|-----------------|---------------------------|-------------------|
| $\lambda_{1,2} = -0.0509 \pm 2.2638i$ | $\Re(\lambda_{1,2}) \leq -0.01$ | $\lambda_{1,2} = -0.0583 \pm 2.4014i$ |
| $\lambda_{3,4} = -0.0374 \pm 1.6016i$ | $\Re(\lambda_{3,4}) \leq -0.01$ | $\lambda_{3,4} = -0.0402 \pm 1.5572i$ |
| $\lambda_{5,6} = -0.0033 \pm 0.5513i$ | $\Re(\lambda_{5,6}) \leq -0.01$ | $\lambda_{5,6} = -0.0136 \pm 1.1151i$ |

| open-loop zeros | desired closed-loop zeros | closed-loop zeros |
|-----------------|---------------------------|-------------------|
| $\mu_{1,2} = -0.0126 \pm 1.8609i$ | $\mu_{1,2} = -0.0250 \pm 2.0000i$ | $\mu_{1,2} = -0.0250 \pm 2.0000i$ |

| open-loop poles | desired closed-loop poles | closed-loop poles |
|-----------------|---------------------------|-------------------|
| $\lambda_{1,2} = -0.0509 \pm 2.2638i$ | $\Re(\lambda_{1,2}) \leq -0.01$ | $\lambda_{1,2} = -0.0543 \pm 2.4014i$ |
| $\lambda_{3,4} = -0.0374 \pm 1.6016i$ | $\Re(\lambda_{3,4}) \leq -0.01$ | $\lambda_{3,4} = -0.0402 \pm 1.5572i$ |
| $\lambda_{5,6} = -0.0033 \pm 0.5513i$ | $\Re(\lambda_{5,6}) \leq -0.01$ | $\lambda_{5,6} = -0.0136 \pm 1.1151i$ |

Example 4.3 [14] [30] Consider the second-order control system (1.4) with $n = 3$ and $m = 2$, where

$$\begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$
We wish replace a pair of complex conjugate zeros \( \nu_{1,2} = \pm 2.0000i \) of the 32-th open-loop point receptance \( H_{32}(s) \) by \( \mu_{1,2} = -0.2500 \pm 1.6000i \) and assign the 6 open-loop poles \( \lambda_{1,2} = \pm 3.6039i \), \( \lambda_{3,4} = \pm 2.4940i \), and \( \lambda_{5,6} = \pm 0.8901i \) to the region \( D(0.01, 0.001, \theta) \).

We use Algorithm 3.1 to Example 4.3 with \( \delta_1 = \delta_2 = \delta_3 = 10^{-8} \) and the computed feedback matrices are given by

\[
F = \begin{bmatrix}
-2.5000 & 2.2940 \\
-0.0000 & 1.3364 \\
-0.0000 & -1.0031
\end{bmatrix}, \quad G = \begin{bmatrix}
7.4437 & -6.6699 \\
0.0000 & 0.7558 \\
-0.0000 & -0.5674
\end{bmatrix}
\]

with \( \|F\|_F = 3.7821 \) and \( \|G\|_F = 10.039 \). Table 4.3 lists the open-loop zeros of \( H_{32}(s) \) and closed-loop zeros of \( H_{32}^{(m)}(s) \), the open-loop and closed-loop poles of the closed-loop system (1.4) for Example 4.3. The pole-zero map is depicted in Figure 4.4. The total computing time is about 0.505 seconds. We observe from Table 4.3 and Figure 4.4 that our method works very effectively for solving the PZAP.

### Table 4.3: Numerical results for Example 4.3

| open-loop zeros | desired closed-loop zeros | closed-loop zeros |
|-----------------|--------------------------|-------------------|
| \( \nu_{1,2} = \pm 2.0000i \) | \( \mu_{1,2} = -0.2500 \pm 1.6000i \) | \( \mu_{1,2} = -0.2500 \pm 1.6000i \) |
| open-loop poles | desired closed-loop poles | closed-loop poles |
| \( \lambda_{1,2} = \pm 3.6039i \) | \( \Re(\lambda_{1,2}) \leq -0.01 \) \( \tau(\lambda_{1,2}) \geq 0.001 \) | \( \lambda_{1,2} = -0.0395 \pm 3.6177i \) |
| \( \lambda_{3,4} = \pm 2.4940i \) | \( \Re(\lambda_{3,4}) \leq -0.01 \) \( \tau(\lambda_{3,4}) \geq 0.001 \) | \( \lambda_{3,4} = -0.0946 \pm 2.3844i \) |
| \( \lambda_{5,6} = \pm 0.8901i \) | \( \Re(\lambda_{5,6}) \leq -0.01 \) \( \tau(\lambda_{5,6}) \geq 0.001 \) | \( \lambda_{5,6} = -0.0411 \pm 0.7178i \) |

### Example 4.4

Consider a five degrees-of-freedom (dof) lumped parameter system (1.4) with \( n = 5 \) and \( m = 2 \), which was shown in Figure 4.5. Here,
Figure 4.4: Pole-zero map for Example 4.3.

Figure 4.5: Five-dof lumped parameter system.

\[ M = \text{diag}(m_1, m_2, m_3, m_4, m_5), \quad C = 0, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}^T, \]

\[ K = 10^3 \begin{bmatrix} k_{g1} + k_{12} & -k_{12} & 0 & 0 & 0 \\ -k_{12} & k_{g2} + k_{12} + k_{23} & -k_{23} & 0 & 0 \\ 0 & -k_{23} & k_{g3} + k_{23} + k_{34} & -k_{34} & 0 \\ 0 & 0 & -k_{34} & k_{g4} + k_{34} + k_{45} & -k_{45} \\ 0 & 0 & 0 & -k_{45} & k_{g5} + k_{45} \end{bmatrix}, \]

where the five lumped masses \( m_1 = 1.727 \text{kg}, m_2 = 5.123 \text{kg}, m_3 = 8.214 \text{kg}, m_4 = 2.609 \text{kg}, \) and \( m_5 = 1.339 \text{kg}, \) which connect to the frame by linear springs of stiffness \( k_{g1} = k_{g2} = k_{g3} = k_{g4} = k_{g5} = k_g = 94.26 \text{kN/m}. \) Moreover, all two adjacent masses are connected by the springs \( k_{12} = 75.14 \text{kN/m}, k_{23} = 67.74 \text{kN/m}, k_{34} = 75.47 \text{kN/m}, k_{45} = 83.40 \text{kN/m} \) accordingly. We wish replace the two pairs of complex conjugate zeros \( \nu_{1,2} = \pm 155.1i \) and \( \nu_{3,4} = \pm 404.4i \) of the 22-th open-loop point receptance \( H_{22}(s) \) by \( \mu_{1,2} = \pm 100.0i \) and \( \mu_{3,4} = -5.000 \pm 405.0i \) and assign the 10 open-loop poles to the region \( \mathcal{D}_1(3). \)

By applying Algorithm 4.1 to Example 4.4 with \( \delta_1 = \delta_2 = \delta_3 = 10^{-8}, \) we obtain the following
feedback matrices

\[ F = \begin{bmatrix}
-427.6 & -12.594 \\
-100.5 & 0.0024 \\
68.888 & 427.1 \\
396.2 & -1289.5 \\
-158.0 & -665.0
\end{bmatrix}, \quad G = \begin{bmatrix}
1854 & 0.4804 \\
940.3 & -0.8418 \\
173681 & 5.3530 \\
-109944 & -12.212 \\
-260257 & -3.7252
\end{bmatrix} \]

\[ \|F\|_F = 1633 \text{ and } \|G\|_F = 332292. \]

Table 4.4 lists the open-loop zeros of \( H_{22}(s) \) and closed-loop zeros of \( H_{22}^{(m)}(s) \), the open-loop and closed-loop poles of the closed-loop system (1.4) for Example 4.4. The pole-zero map is depicted in Figure 4.6. The total computing time is about 0.563 seconds. We observe from Table 4.4 and Figure 4.6 that our method solves the PZAP very effectively.

Table 4.4: Numerical results for Example 4.4

| open-loop zeros | desired closed-loop zeros | closed-loop zeros |
|-----------------|---------------------------|------------------|
| \( \nu_{1,2} = \pm 155.1i \) | \( \mu_{1,2} = \pm 100.0i \) | \( \mu_{1,2} = \pm 100.0i \) |
| \( \nu_{3,4} = \pm 404.4i \) | \( \mu_{3,4} = \pm 5.000 \pm 405.0i \) | \( \mu_{3,4} = \pm 5.000 \pm 405.0i \) |

| open-loop poles | desired closed-loop poles | closed-loop poles |
|-----------------|---------------------------|------------------|
| \( \lambda_{1,2} = \pm 137.4i \) | \( \Re(\lambda_{1,2}) \leq -3 \) | \( \lambda_{1} = -931.1, \lambda_{2} = -104.9 \) |
| \( \lambda_{3,4} = \pm 201.9i \) | \( \Re(\lambda_{3,4}) \leq -3 \) | \( \lambda_{3,4} = -7.4995 \pm 62.852i \) |
| \( \lambda_{5,6} = \pm 266.9i \) | \( \Re(\lambda_{5,6}) \leq -3 \) | \( \lambda_{5,6} = -6.2700 \pm 206.0i \) |
| \( \lambda_{7,8} = \pm 329.5i \) | \( \Re(\lambda_{7,8}) \leq -3 \) | \( \lambda_{7,8} = -78.029 \pm 292.8i \) |
| \( \lambda_{9,10} = \pm 404.4i \) | \( \Re(\lambda_{9,10}) \leq -3 \) | \( \lambda_{9,10} = -5.2471 \pm 404.9i \) |

Figure 4.6: Pole-zero map for Example 4.4

5 Concluding remarks

We have proposed a multi-input two-stage approach for solving the pole-zero assignment problem for second-order vibratory systems with multi-input feedback control. The pole-zero assignment problem is transformed into a multi-step single-input pole-zero assignment problem. In each
step, a two-stage approach is presented: in the first stage, the zeros of the closed-loop point receptance are assigned to the desired positions via the solution of a small linear equation for the feedback vectors; in the second stage, the complete closed-loop poles are assigned to the prescribed region of the complex left-hand plane via solving a LMI-based SDP. Numerical results are reported to illustrate the effectiveness of the proposed approach. An interesting question is how to extend the proposed approach to the case of time delay. This needs further study.

References

[1] Z. J. Bai, B. N. Datta, and J. W. Wang, Robust and minimum norm partial quadratic eigenvalue assignment in vibrating systems: A new optimization approach, Mech. Syst. Signal Process., 24 (2010) 766–783.

[2] Z. J. Bai, M. Lu, and Q. Y. Wan, Minimum norm partial quadratic eigenvalue assignment for vibrating structures using receptances and system matrices, Mech. Syst. Signal Process., 112 (2018) 265–279.

[3] M. Balas, Trends in large space structure control theory: Fondest hopes, wildest dreams, IEEE Trans. Automat. Control, 27 (1982) 522–535.

[4] R. Belotti, D. Richiedei, I. Tamellin, and A. Trevisani, Pole assignment for active vibration control of linear vibrating systems through linear matrix inequalities, Appl. Sci., 10 (2020) 5494.

[5] R. Belotti, D. Richiedei, A. Trevisani, Optimal design of vibrating systems through partial eigenstructure assignment, J. Mech. Des., 138 (2016) 071402.

[6] A. Bhaya and C. Desoer, On the design of large flexible space structures, IEEE Trans. Automat. Control, 30 (1985) 1118–1120.

[7] M. Chilali and P. Gahinet, $H_\infty$ design with pole placement constraints: an LMI approach, IEEE Trans. Autom. Control, 41 (1996) 358–367.

[8] E. K. Chu, Pole assignment for second-order systems, Mech. Syst. Signal Process., 16 (2002) 39–59.

[9] E. K. Chu and B. N. Datta, Numerically robust pole assignment for second-order systems, Int. J. Control, 64 (1996) 1113–1127.

[10] R. W. Clough and S. Mojtahedi, Earthquake response analysis considering nonproportional damping, Earthquake Engrg. Structural Dynam., 4 (1976) 489–496.

[11] B. N. Datta, Numerical Methods for Linear Control Systems Design and Analysis, Elsevier Academic Press, New York, 2003.

[12] B. N. Datta, S. Elhay, and Y. M. Ram, Orthogonality and partial pole assignment for the symmetric definite quadratic pencil, Linear Algebra Appl., 257 (1997) 29–48.
[13] B. N. Datta, W. W. Lin, and J. N. Wang, Robust partial pole assignment for vibrating structures with aerodynamic effect, IEEE Trans. Automat. Control, 51 (2006) 1979–1984.

[14] B. N. Datta and F. Rincón, 1993, Feedback stabilisation of a second-order system: a non-modal approach, Linear Algebra Appl., 188-189 (1993) 135–161.

[15] D. J. Ewins, Modal Testing: Theory, Practice and Application, Research Studies Press, 1998.

[16] M. I. Friswell and J. E. Mottershead, Finite Element Model Updating in Structural Dynamics, Kluwer Academic Publishers, Dordrecht, 1995.

[17] N. J. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Philadelphia, PA, USA, 2nd edition, 2002.

[18] D. W. Ho and H. C. Chan, Feedback Stabilization of Damped-Gyroscopic Second-Order Systems, Research report, Report No. MA-93-14, Faculty of Science and Technology, City University, Hong Kong, 1993.

[19] X. Q. Jin and S. W. Vong, An Introduction to Applied Matrix Analysis, World Scientific and Higher Education Press, Beijing, 2016.

[20] S. Joshi, Control of Large Flexible Space Structures, Lecture Notes in Control and Inform. Sci. 131, Berlin and New York, Springer-Verlag, 1989.

[21] J. Kautsky, N. K. Nichols, and P. Van Dooren, Robust pole assignment in linear state feedback, International Journal of Control, 41 (1985) 1129–1155.

[22] L. H. Keel, J. A. Fleming, and S. P. Bhattacharya, Minimum norm pole assignment via Sylvester’s equation, Contemporary Mathematics, 47 (1985).

[23] Z. Liu, W. Li, H. Ouyang, and D. Wang, Eigenstructure assignment in vibrating systems based on receptances, Arch. Appl. Mech., 85 (2015) 713–724.

[24] J. Löfberg, YALMIP: a toolbox for modeling and optimization in MATLAB, In: 2004 IEEE International Symposium on Computer Aided Control Systems Design, pp. 284–289. Taipei, 2004.

[25] P. C. Mueller, Linear quadratic control of mechanical descriptor systems, in Systems and Networks: Mathematical Theory and Applications, Vol. II, U. Helmke, R. Mennicken, and J. Saurer, eds., Akademie-Verlag, Birkhaeuser, Berlin, 1994, pp. 361–366.

[26] J. E. Mottershead, A. Kyprianou, and H. Ouyang, Structural modification, part 1: rotational receptances, J. Sound Vibration, 284 (2005) 249-265.

[27] J. E. Mottershead, M. G. Tehrani, D. Stancioiu, S. James, and H. Shahverdi, Structural modification of a helicopter tailcone, J. Sound Vibration, 298 (2006) 366–384.

[28] J. E. Mottershead, M. G. Tehrani, S. James, and Y. M. Ram, Active vibration suppression by pole-zero placement using measured receptances, J. Sound Vibration, 311 (2008) 1391–1408.
[29] J. E. Mottershead, M. G. Tehrani, and Y. M. Ram, *Assignment of eigenvalue sensitivities from receptance measurements*, Mech. Syst. Signal Process., 23 (2009) 1931–1939.

[30] N. K. Nichols and J. Kautsky, Robust eigenstructure assignment in quadratic matrix polynomials: nonsingular case, SIAM J. Matrix Anal. Appl., 23 (2001) 77–102.

[31] Z. C. Qiu, X. M. Zhang, H. X. Wu, and H. H. Zhang, *Optimal placement and active vibration control for piezoelectric smart flexible cantilever plate*, J. Sound Vibration, 301 (2007) 521-543.

[32] Y. Ram and S. Elhay, *Pole assignment in vibratory systems by multi-input control*, J. Sound Vibration, 230 (2000) 309–321.

[33] Y. M. Ram and J. E. Mottershead, *Receptance method in active vibration control*, AIAA J., 45 (2007) 562–567.

[34] Y. M. Ram and J. E. Mottershead, *Multiple-input active vibration control by partial pole placement using the method of receptances*, Mech. Syst. Signal Process., 40 (2013) 727–735.

[35] Y. M. Ram, J. E. Mottershead, and M. G. Tehrani, *Partial pole placement with time delay in structures using the receptance and the system matrices*, Linear Algebra Appl., 434 (2011) 1689–1696.

[36] A. Sharma, R. Kumar, R. Vaish, V. S. Chauhan, *Active vibration control of space antenna reflector over wide temperature range*, Composite Structures, 128 (2015) 291-304.

[37] J. Sherman and W. J. Morrison, *Adjustment of an inverse matrix corresponding to a change in one element of a given matrix*, Ann. Math. Statist., 21 (1950) 124–127.

[38] J. G. Sun, *Backward perturbation analysis of certain characteristic subspaces*, Numer. Math., 65 (1993), 357–382.

[39] M. G. Tehrani, J. E. Mottershead, A. T. Shenton, and Y. M. Ram, *Robust pole placement in structures by the method of receptances*, Mech. Syst. Signal Process., 25 (2011) 112–122.

[40] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev., 43 (2001) 235–286.

[41] A. Varga, *A numerically reliable approach to robust pole assignment for descriptor systems*, Future Generation Computer Systems, 19 (2003) 1221–1230.

[42] C. M. A. Vasques and J. Dias Rodrigues, *Active vibration control of smart piezoelectric beams: Comparison of classical and optimal feedback control strategies*. Computers & Structures, 84 (2006) 1402–1414.

[43] W. M. Wonham, *On pole assignment in multi-input controllable linear systems*, IEEE Transactions on Automatic Control, AC-12 (1967) 660–665.