A non-incremental solution procedure for elastoplastic problems in structural mechanics

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In this contribution, we present a non-incremental solution procedure for the efficient treatment of elastoplastic problems. To this end, all time history data are decoupled into space and time, and the solution is obtained using the fixed-point algorithm.

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1 Static equilibrium in the incremental framework

In static equilibrium, the terms of inertia are neglected, and the external force \( F(t) \) on the system is equal to the resulting internal force \( F_{\text{int}}(u) \):

\[
F(t) - F_{\text{int}}(u) = 0.
\]

Here, \( u \) denotes the unknown structural displacement. The equation of equilibrium is rewritten as [1]:

\[
K(u)\Delta u = F - F_{\text{int}}(u) = R(u).
\]

In this equation, \( R(u) \) denotes the residual, \( K(u) \) is the tangential stiffness matrix dependent on displacement and \( \Delta u \) is the displacement increment. This formulation allows for an iterative solution algorithm, such as the Newton Raphson method [2], where a sequence of linear algebraic systems of equations is solved. Thus, one proceeds in a step-by-step manner forward in time to evaluate the unknown displacement time history.

2 Static equilibrium in the non-incremental framework

In order to overcome this incremental approach, we present a new version of equation (2) that fulfills static equilibrium during the whole time history at once:

\[
(K_x \otimes k_t) \left( \Delta u_x^{(m)} \otimes \Delta u_t^{(m)} \right) = \sum_{j=1}^{Tr} s^{(j)} R_x^{(j)} \otimes R_t^{(j)}.
\]

In this formulation, \( K_x \) denotes the spatial stiffness matrix and \( k_t \) the temporal stiffness vector. Both quantities are constant in this paper. Thus, the proposed algorithm constitutes the space-time version of the modified Newton-Raphson algorithm [2]. Using the singular value decomposition [3,4], the time history of the residual can be represented by the sum of a dyadic product of left singular vectors \( R_x^{(j)} \) and right singular vectors \( R_t^{(j)} \) multiplied by the corresponding singular value \( s^{(j)} \). The space time equivalent of the displacement increment \( (\Delta u_x^{(m)} \otimes \Delta u_t^{(m)}) \) is represented by the dyadic product of spatial and temporal enrichment modes \( \Delta u_x^{(m)} \) and \( \Delta u_t^{(m)} \), while the total displacement time history is evaluated as [5]:

\[
U = \sum_{i=1}^{m-1} \Delta u_x^{(i)} \otimes \Delta u_t^{(i)} + \Delta u_x^{(m)} \otimes \Delta u_t^{(m)}.
\]

The space-time equation of equilibrium (3) is solved using the fixed-point algorithm. To this end, Equation (3) is simplified and projected into the \( u^{(m)}_t \) -direction to obtain the time equation:

\[
\tilde{K}_t \Delta u_x^{(m)} = \tilde{R}_x, \quad \tilde{K}_t = \left( \Delta u_t^{(m)} \right)^T K_t \Delta u_t^{(m)} K_x, \quad \tilde{R}_x = \sum_{j=1}^{Tr} s^{(j)} \left( \Delta u_t^{(m)} \right)^T R_x^{(j)} R_t^{(j)} \left( R_t^{(j)} \right)^T.
\]
and projected into the $u^{(m)}_x$-direction to obtain the space equation:

$$\hat{K}_x \Delta u^{(m)}_x = \hat{R}_x, \quad \hat{K}_x = \left( \Delta u^{(m)}_y \right)^T K_x \Delta u^{(m)}_y K_t, \quad \hat{R}_x = \sum_{j=1}^{T_r} s^{(j)} \left( \Delta u^{(m)}_y \right)^T R^{(j)} x \left( R^{(j)}_x \right)^T .$$

(6)

This equation pair is solved simultaneously using the fixed-point algorithm. To this end, $u^{(m)}_t$ is fixed, while $u^{(m)}_y$ is evaluated using Equation (5). Subsequently, $u^{(m)}_t$ is fixed, while $u^{(m)}_x$ is obtained using Equation (6). This procedure is repeated until convergence is achieved.

### 3 Numerical example

We introduce the geometry of a frame structure with two stories and one bay (height and width are 8m), subjected to a force time history $F(t)$, as shown in Figures 1a and 1b. The parameters are chosen as: initial stiffness $E_1 = 240$ GPa, yield stress $\sigma_Y = 220$ MPa, post yielding stiffness $E_2 = 0.1E_1$, Poisson’s ratio 0.3. All cross-sections of the columns and beam are modeled by quadratic hollow rectangular shapes. The dimensions are 0.3 m × 0.1 m. The thickness is $t = 10$ mm.

The solution is obtained using the new non-incremental approach as well as the conventional Newton-Raphson method for comparison. The response time history and the hysteresis, i.e. displacement versus force plot, are depicted in Figures 1c and 1d. Every enrichment is presented in a different color, while the benchmark solution is represented by the black dashed lines. After the first enrichment, the linear solution is obtained. With an increasing number of enrichments, plasticity increases, and the non-incremental algorithm converges to the benchmark solution.

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