POLYMATROIDS SATISFYING THE ONE-SIDED STRONG SYMMETRIC EXCHANGE PROPERTY

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ABSTRACT. In this paper we introduce discrete polymatroids satisfying the one-sided strong exchange property and show that they are sortable (as a consequence their base rings are Koszul) and that they satisfy White’s conjecture. Since any pruned lattice path polymatroid satisfies the one-sided strong exchange property, this result provides an alternative proof for one of the main theorems of J. Schweig in [9], where it is shown that every pruned lattice path polymatroid satisfies White’s conjecture. In addition, for two classes of pruned lattice path polymatroidal ideals $I$ and their powers we determine their depth and their associated prime ideals, and furthermore determine the least power $k$ for which $\text{depth} S/I^k$ and $\text{Ass}(S/I^k)$ stabilize. It turns out that $\text{depth} S/I^k$ stabilizes if and only if $\text{Ass}(S/I^k)$ stabilizes.

INTRODUCTION

Let $K$ be a field and let $\mathcal{P}$ be a discrete polymatroid on the ground set $[n]$ with $B$ as set of bases. The base ring $K[B]$ is the subring of $K[t_1, \ldots, t_n]$ generated by monomials $t^u = t_1^{u_1} \cdots t_n^{u_n}$ with $u \in B$. Let $T$ be the polynomial ring $K[x_u : u \in B]$ and let $I_B$ be the kernel of the $K$-algebra homomorphism $\phi : T \to K[B]$ with $\phi(x_u) = t^u$. There are some obvious generators in $I_B$. Indeed, let $u, v \in B$ with $u(i) > v(i)$. Then there exists $i$ such that $u(j) < v(j)$ and such that $u - e_i + e_j$ and $v + e_i - e_j$ belong to $B$. It is clear that $x_u x_v - x_{u-e_i+e_j} x_{v+e_i-e_j} \in I_B$. Such relations are called symmetric exchange relations. White [12] conjectured that for a matroid the symmetric exchange relations generate $I_B$. It is natural to conjecture that this also holds for discrete polymatroids.

Conjecture (White). Let $\mathcal{P}$ be a discrete polymatroid on the ground set $[n]$ with $B$ as set of bases. Then $I_B$ is generated by symmetric exchange relations.

In [3] it was shown that if White’s conjecture holds for all matroids, then it holds for all discrete polymatroids as well, and that any discrete polymatroid satisfying the strong symmetric exchange property satisfies White’s conjecture.

In [9] J. Schweig introduced pruned lattice path polymatroids and proved that they satisfy White’s conjecture. He actually proved that the symmetric exchange relations form a Gröbner basis of $I_B$ for such discrete polymatroids.

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In Section 1 we introduce discrete polymatroids satisfying the one-sided strong symmetric exchange property (see Definition 1.1) and show that they are sortable and that they satisfy White’s conjecture. It is known by [3, Lemma 5.2] that the sorting relations form a Gröbner basis of the defining ideal $I_B$. As a consequence, the base ring of a discrete polymatroid satisfying the one-sided strong symmetric exchange property is Koszul.

A pruned lattice path polymatroid may be described in terms of inequalities of its set of the bases as follows:

**Definition.** Let $n, d$ be positive integers and given vectors $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{Z}_+^n$ such that $\alpha_1 \leq \cdots \leq \alpha_n = d$, $\beta_1 \leq \cdots \leq \beta_n = d$ and $\alpha_i \leq \beta_i$. A discrete polymatroid $P$ on the ground set $[n]$ is called a **pruned lattice path polymatroid** (of type $(a, b | \alpha, \beta)$), if the set $B$ of its bases consists of vectors $u \in \mathbb{Z}_+^n$ such that

$$a_i \leq u_i \leq b_i \quad \text{for } i = 1, \ldots, n,$$

and

$$\alpha_i \leq u_1 + u_2 + \cdots + u_i \leq \beta_i \quad \text{for } i = 1, \ldots, n.$$

Taking advantage of this definition we prove in Section 2 that every pruned lattice path polymatroid satisfies the two-sided strong symmetric exchange property. We find an example of discrete polymatroid satisfying the two-sided strong symmetric exchange property which is not a pruned lattice path polymatroid. However this example is isomorphic to a pruned lattice path polymatroid. On the other hand, we show for some special polymatroids that the one-sided strong exchange property implies the property of being a pruned lattice path polymatroid. The precise relationship between polymatroids satisfying the two-sided strong symmetric exchange property and pruned lattice path polymatroid remains to be revealed.

It was pointed out in [9] that a lattice path polymatroid is transversal. In Section 3, we explicitly express a lattice path polymatroidal ideal as a product of monomial prime ideals and also characterize those transversal polymatroidal ideals which are lattice path polymatroidal ideals.

In Section 4, we deduce a formula to compute the depth for a pruned lattice path polymatroidal ideal, see [3]. This formula plays a key role in the last two sections. As an application, we determine in Proposition 4.6 the associated prime ideals of a lattice path polymatroidal ideal. This result is repeatedly used in what follows.

In the remaining two sections we consider special classes of pruned lattice path polymatroidal ideals where the questions concerning depth and associated prime ideals of powers of ideals have complete answers.

The ideals considered in Section 5 are called left pruned lattice path polymatroidal ideals. We say that a pruned lattice path polymatroidal ideal of type $(a, b | \alpha, \beta)$ is a **left pruned lattice path polymatroidal ideal**, if there exists $k \in [1, n - 1]$ such that $a_i = 0, b_i \geq d$ for all $i \geq k + 1$ and $\alpha_i = 0, \beta_i = \beta_{k+1}$ for $i \leq k$. We first show
in Proposition 5.2 that \( \text{depth } S/I = |\{k + 1 \leq i \leq n - 1: \alpha_i = \beta_i\}| \), and thus the depth function \( \text{depth } S/I^k \) stabilizes from the very beginning.

The main tool to determine the associated prime ideals of our ideals is localization. By using this technique it suffices to characterize when a suitable localization of the ideal is of depth zero. This can be checked with formula (3). Given a subset \( A \) of \([1, n]\), let \( P_A \) denote the prime ideal \((x_i: i \in A)\). It turns out, see Corollary 5.5, that \( P_A \in \text{Ass}(S/I) \) only if \( A \) is an interval contained in \([k + 2, n]\) or \( A = B \cup [k + 1, n] \) for some subset \( B \) of \([1, k]\) with \( k \) as in the definition of left pruned lattice path polymatroidal ideals. In Theorem 5.8 the precise set of associated prime ideals is determined. As a consequence we obtain in Corollary 5.9 that all powers of a left pruned lattice path polymatroidal ideal have the same set of associated prime ideals.

In Section 6, we consider right pruned lattice path polymatroidal ideals. A pruned lattice path polymatroidal ideal of type \((a, b|\alpha, \beta)\) is called a right pruned lattice path polymatroidal ideal, if there exists \( k \in [1, n - 1] \) such that \( a_i = 0 \) and \( b_i \geq d \) for all \( i \leq k \), and \( \alpha_i = \alpha_k \) and \( \beta_i = d \) for all \( k + 1 \leq i \leq n - 1 \). To determine the depth and the associated prime ideals for this class of ideals is more complicated than in the case of left one. For example, if \( I \) is a left pruned lattice path polymatroidal ideal then \( \text{Ass}(S/I^k) \) is not stable from the very beginning. The precise power when this happens is given in Theorem 6.11. In Proposition 6.9 the set \( \text{Ass}^\infty(S/I) \), which describes the set of associated prime ideals of all large powers of \( I \), is determined. A formula for the depth of \( I \) is given in Theorem 6.4 and the powers for which the depth stabilizes is given in Corollary 6.5.

As observed in [8], every polymatroidal ideal is of strong intersection type. Thus we can use the results obtained in Section 5 and Section 6 to provide an irreducible primary decomposition for any left or right pruned lattice path polymatroidal ideal.

Since the left and right pruned lattice path polymatroidal ideal have shown such a different algebraic behaviour it is not expected that there is a nice and uniform description of the depth and the set of associated prime ideals for arbitrary pruned lattice path polymatroidal ideals.

1. The one-sided strong symmetric exchange property and the conjecture of White

In this section we introduce the concept of the one-sided strong symmetric exchange property for discrete polymatroids and show that such discrete polymatroids are sortable and satisfy White’s conjecture. Note that if \( u \) is a vector we use \( u(i) \) to denote the \( i \)th entry of \( u \) in this section.

**Definition 1.1.** Let \( \mathcal{P} \) be a discrete polymatroid on the ground set \([n]\) with \( B \) as set of bases. Then we say that \( B \) (or \( \mathcal{P} \)) satisfies the left-sided strong symmetric exchange property, if for any pair \( u, v \in B \) such that \( u(i) > v(i) \) and \( u(1) + \cdots + u(i - 1) < v(1) + \cdots + v(i - 1) \), there exists \( j \leq i - 1 \) such that \( u(j) < v(j) \) and both \( u - \varepsilon_i + \varepsilon_j \) and \( v + \varepsilon_i - \varepsilon_j \) belong to \( B \). Here \( \varepsilon_i \) denotes the \( i \)th canonical basis vector of \( \mathbb{R}^n \).
Similarly, the right-sided strong symmetric exchange property is defined. If $B$ satisfies both the left-sided and right-sided strong symmetric exchange property, we say that $B$ satisfies the two-sided strong symmetric exchange property. We also say that a monomial ideal $I$ satisfies the left-, right- or two-sided strong symmetric exchange property, if it is the polymatroidal ideal of a polymatroid which has this property.

In the following theorem we will show if $B$ satisfies one-sided (left-sided or right-sided) strong symmetric exchange property, then $B$ is sortable and $I_B$ is generated by symmetric exchange relations.

For the proof of this result we need some preparations. Let $u,v$ be elements in $B$ with $|u(i) - v(i)| \leq 1$ for $i = 1, \ldots, n$. As in [3, Page 253] one associates $(u,v)$ with a sequence $s(u,v)$ of signs $+$ and $-$ depending on $u - v$. For example, if $u - v = (0,0,-1,0,1,1,-1,0)$, then $s(u,v) = -,-,+,-$. Note that the sequence $s(u,v)$ always contains as many $+$ as $-$ signs, since $\sum_{i=1}^{n} u(i) = \sum_{i=1}^{n} v(i)$ and $|u(i) - v(i)| \leq 1$ for $i = 1, \ldots, n$.

We observe the following easy fact: if $u(i) - v(i) = 1$ and $u(j) - v(j) = -1$, then $s(u - \varepsilon_i + \varepsilon_j, v + \varepsilon_i - \varepsilon_j)$ is obtained from $s(u,v)$ by exchanging corresponding signs. For instance for $u$ and $v$ with $u - v$ as before, we get $s(u + \varepsilon_3 - \varepsilon_5, v - \varepsilon_3 + \varepsilon_5) = +,-,+,-$ which is obtained from $s(u,v)$ by exchanging the first and second sign in the sequence.

**Theorem 1.2.** Assume $B$ satisfies the one-sided (left-sided or right-sided) strong symmetric exchange property. Then

(a) $B$ is sortable and sorting relations form a Gröbner base of $I_B$;
(b) $I_B$ is generated by symmetric exchange relations.

**Proof.** (a) Without loss of generality we assume that $B$ satisfies right-sided strong symmetric exchange property. We denote by $E_B$ the ideal (contained in $I_B$) which is generated by the symmetric exchange relations.

Let $(u,v) \in B \times B$. By [3, Lemma 5.4] there exists $(u_1,v_1) \in B \times B$ such that $x_ux_v - x_{u_1}x_{v_1} \in E_B$ and $|u_1(i) - v_1(i)| \leq 1$ for $i = 1, \ldots, n$. We claim that there exists $(u_2,v_2) \in B \times B$ such that $x_{u_1}x_{v_1} - x_{u_2}x_{v_2} \in E_B$ and $(u_2,v_2)$ is sorted. Recall that $(u_2,v_2)$ is sorted if and only if $s(u_2,v_2)$ is a sequence of alternating signs by [3, Lemma 5.1].

The sign at position $i$ of the sequence $s(u,v)$ will be denoted by $s_i(u,v)$. Without loss of generality we may suppose that $s_1(u_1,v_1) = +$. Let $i \geq 2$ be the smallest integer with the property that $s_{i-1}(u_1,v_1) = s_i(u_1,v_1)$. For convenience, we denote this number $i$ by $c(u_1,v_1)$ and set $c(u_1,v_1) = \infty$ if no such $i$ exists. There are two cases to consider.

Let $i = c(u_1,v_1)$ and suppose first that $s_i(u_1,v_1) = +$. By definition, $s_i(u_1,v_1)$ corresponds to the sign of say the $i_1$-th entry of $u_1 - v_1$ for some $i_1 \geq i$. Note that our assumptions imply that $u_1(i_1) - v_1(i_1) = 1$ and $u_1(i_1 + 1) + \cdots + u(n) < v_1(i_1 + 1) + \cdots + v(n)$. Therefore, since $B$ satisfies the right-sided strong symmetric exchange property, there exist $j_1 > i_1$ such that $u_1(j_1) - v_1(j_1) = -1$ and that $u_1 - \varepsilon_{i_1} + \varepsilon_{j_1}$ and $v_1 + \varepsilon_{i_1} - \varepsilon_{j_1}$ belong to $B$. Since $s(u_1 - \varepsilon_{i_1} + \varepsilon_{j_1}, v_1 + \varepsilon_{i_1} - \varepsilon_{j_1})$ is
obtained from \( s(u, v) \) by exchanging \( s_i(u, v) \) and \( s_j(u, v) \) for suitable \( j > i \), it follows that 
\[
c(u_1 - \varepsilon_{i_1} + \varepsilon_{j_1}, v_1 + \varepsilon_{i_1} - \varepsilon_{j_1}) \geq i + 1.
\]

Next we assume that \( s_i(u_1 - v_1) = -1 \). Similarly as the case above, let \( i_1 \) be the entry of \( u_1 - v_1 \) corresponding to \( s_i(u_1, v_1) \). Then, since \( u_1(i_1) - v_1(i_1) = -1 \) and 
\[
u_1(i_1 + 1) + \cdots + u(n) > v_1(i_1 + 1) + \cdots + v(n),
\]
the right-sided symmetric exchange property implies that there exists \( j_1 > i_1 \) such that 
\[
u_1(j_1) - v_1(j_1) = 1\quad \text{and that}
\]
\[
u_1 + \varepsilon_{i_1} - \varepsilon_{j_1} \quad \text{and} \quad v_1 - \varepsilon_{i_1} + \varepsilon_{j_1} \quad \text{belong to} \quad B.
\]
Again, we have 
\[
c(u_1 + \varepsilon_{i_1} - \varepsilon_{j_1}, v_1 - \varepsilon_{i_1} + \varepsilon_{j_1}) \geq i + 1.
\]

Thus in both cases we obtain a pair \((u', v') \in B \times B\) such that \( x_{u_1}x_{v_1} - x_{u'}x_{v'} \in E_B\) and 
\[
c(u', v') > c(u_1, v_1).
\]
Thus the claim follows by induction.

Now let \((u_2, v_2)\) be as in the claim. Since \( x_{u_1}x_{v_1} - x_{u_2}x_{v_2} \in E_B\), we have \( u_1 + v_1 = u_2 + v_2\), and so 
\[
sort(u_1, v_1) = sort(u_2, v_2),
\]
But \((u_2, v_2)\) is sorted, hence 
\[
sort(u, v) = (u_2, v_2) \in B \times B.
\]
This implies \( B \) is sortable and so \( I_B \) has a Gröbner base consisting of sorting relations by \([3, \text{Lemma 5.4}])\.

(b) From the proofs of (a) and \([3, \text{Lemma 5.4}]\), we see that for any \((u, v) \in B \times B\), we have 
\[
x_{u}x_{v} - x_{u'}x_{v'} \in E_B,
\]
where \((u', v') = sort(u, v)\). This implies that all sorting relations belong to \( E_B \). By \([3, \text{Lemma 5.2}]\), we have \( I_B = E_B\), as required.

2. Pruned Lattice path Polymatroids

In this section we will give an alternative definition of a pruned lattice path polymatroid and show that this class of discrete polymatroids satisfies the two-sided strong symmetric exchange property. In addition, we present some properties of this class of discrete polymatroids. Note that if \( u \) is a vector we use \( u_i \) to denote the \( i \)-entry of \( u \) in the remaining part of this paper except in the proof of Proposition \([2.9])\.

Let \( B = \{m(\sigma) : \alpha \succeq \sigma \succeq \beta\} \) be the set of bases of a lattice path polymatroid as defined in \([10]\). Here \( \alpha, \sigma, \beta \) are paths from \((1, 1)\) to \((n, d)\). Given a path \( \sigma \) from \((1, 1)\) to \((n, d)\), we denote by \( \sigma_i \) the maximal integer \( j \) such that \((i, j) \in \sigma\). Identifying the elements of \( B \) with the corresponding monomials, then for 
\[
m(\sigma) = x_{u_1}^n \cdots x_{u_n}^n,
\]
the \( u_i, i = 1, \ldots, n \) satisfy \( u_1 + \cdots + u_i = \sigma_i \) for \( i = 1, \ldots, n \). Hence a monomial \( x^a \) belongs to \( B \) if and only if
\[
\alpha_i \leq u_1 + \cdots + u_i \leq \beta_i, i = 1, \ldots, n.
\]

**Definition 2.1.** Let \( n, d \) be positive integers and given vectors \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \) in \( \mathbb{Z}_+^n \) such that 
\[
\alpha_1 \leq \cdots \leq \alpha_n = d, \quad \beta_1 \leq \cdots \leq \beta_n = d \quad \text{and} \quad \alpha_i \leq \beta_i.
\]
A discrete polymatroid \( P \) on the ground set \([n]\) is called a pruned lattice path polymatroid (of type \((a, b|\alpha, \beta)\)), if the set \( B \) of its bases consists of vectors \( u \in \mathbb{Z}_+^n \) such that
\[
(1) \quad \alpha_i \leq u_i \leq b_i \quad \text{for} \; i = 1, \ldots, n,
\]
and
\[
(2) \quad \alpha_i \leq u_1 + u_2 + \cdots + u_i \leq \beta_i \quad \text{for} \; i = 1, \ldots, n.
\]

In this definition, if \( \alpha_i = 0 \) and \( \beta_i = d \) for all \( i \leq n - 1 \), then \( P \) is a discrete polymatroid of Veronese type. If all \( \alpha_i = 0 \) and all \( b_i \geq d \), then \( P \) is a lattice path
polymatroid. If this is the case, we say that $P$ is a lattice path polymatroid of type $(\alpha, \beta)$.

We also say that a monomial ideal $I$ is a pruned lattice path polymatroidal ideal (of type $(a, b|\alpha, \beta)$) if it is a polymatroidal ideal of a pruned lattice path polymatroid (of type $(a, b|\alpha, \beta)$) and that a monomial ideal $I$ is a lattice path polymatroidal ideal (of type $(\alpha, \beta)$) if it is a polymatroidal ideal of a lattice path polymatroid (of type $(\alpha, \beta)$).

As an example, consider the graphic matroid $M$ of the graph $G$ shown in Figure 1.

![Figure 1.](image)

We will show that $M$ is a pruned lattice path polymatroid. Let $B$ be the set of bases of $M$. Every element in $B$ is identified with a 0-1 vector of dimension 10. Since a set of 7 edges of $G$ forms a spanning tree of $G$ if and only if it does not contain a cycle, a 0-1 vector $u$ belongs to $B$ if and only if $u$ satisfies the following system of inequalities:

\[
\begin{align*}
4 \sum_{i=1}^{4} u_i &\leq 3, \\
7 \sum_{i=1}^{7} u_i &\leq 5, \\
7 \sum_{i=4}^{10} u_i &\leq 3, \\
10 \sum_{i=4}^{10} u_i &\leq 3, \\
10 \sum_{i=1}^{10} u_i &= 7.
\end{align*}
\]

One can check that this set of inequalities is equivalent to the following set of inequalities:

\[
\begin{align*}
2 \leq \sum_{i=1}^{3} u_i, \\
4 \sum_{i=1}^{4} u_i &\leq 3, \\
4 \leq \sum_{i=1}^{6} u_i, \\
7 \sum_{i=1}^{7} u_i &\leq 5, \\
10 \sum_{i=1}^{10} u_i &= 7.
\end{align*}
\]

This shows that $M$ is indeed a pruned lattice path polymatroid.

As another example, consider the transversal ideal $(x_1, x_2)(x_3, x_4)(x_1, x_3)$. It is a monomial ideal generated by $x^u \in k[x_1, \ldots, x_4]$ with $u$ satisfying: $0 \leq u_i \leq 2, i = 1, 3, 0 \leq u_i \leq 1, i = 2, 4, 1 \leq u_1 + u_2 \leq 2$ and $u_1 + u_2 + u_3 + u_4 = 3$. Hence it is a pruned lattice path polymatroidal ideal.

Pruned lattice path polymatroids can be characterized by its ground set rank function. Let $\rho$ be the ground set rank function of the discrete polymatroid $P$ (more accurately of the integer polymatroid conv($P$) ). If $A = \{i_1, \ldots, i_k\}$, we denote $\rho(A)$ by $\rho(i_1, \ldots, i_k)$. Following [3], a subset $\emptyset \neq A \subseteq \{n\}$ is called $\rho$-closed if $\rho(A) < \rho(B)$ for any subset $B \subseteq A$ which contains $A$ properly and a subset $\emptyset \neq A \subseteq \{n\}$ is called $\rho$-separable if there exist nonempty subsets $A_1, A_2$ with $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$ such that $\rho(A) = \rho(A_1) + \rho(A_2)$. 

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Proposition 2.2. Let $\mathcal{P}$ be a discrete polymatroid on the ground set $[n]$. Then $\mathcal{P}$ is a pruned lattice path polymatroids if and only if it follows that any nonempty subset $C$ of $[n]$, different from $[k, n]$ or $[1, k]$ with $|C| \neq 1, n-1$ is either not $\rho$-closed or is $\rho$-separated.

Proof. The assertion follows from [3, Proposition 7.2] or from [1]. □

Proposition 2.3. Let $\mathcal{P}$ be a pruned lattice path polymatroid with $B$ as set of bases. Then $\mathcal{P}$ satisfies the two-sided strong symmetric exchange property.

Proof. We only need to prove that $B$ satisfies the left-sided strong symmetric exchange property, since the other case is treated similarly. Let $u, v$ be two generators of $I$ such that $u_i < v_i$, where $1 < i \leq n$ and $u_1 + \cdots + u_{i-1} > v_1 + \cdots + v_{i-1}$. Then there is $1 \leq k \leq i-1$ such that $u_k > v_k$. Let $j$ be the largest number $k$ with this property. We will show that both $u - \epsilon_j + \epsilon_i$ and $v + \epsilon_j - \epsilon_i$ belong to $B$. Indeed, write $u - \epsilon_j + \epsilon_i = (t_1, t_2, \ldots, t_n)$, that is, $t_k = u_k$ if $k \neq j$, $i$ and $t_j = u_j - 1$, $t_i = u_i + 1$. Since $\mathcal{P}$ is a pruned lattice path polymatroid, we may assume that the elements of $B$ satisfy the inequalities of Definition 2.1.

It is clear $a_k \leq t_k \leq b_k$ for all $k = 1, \ldots, n$. If $k \leq j - 1$ or $k \geq i$, then $t_1 + \cdots + t_k = u_1 + \cdots + u_k$, and in particular, $\alpha_k \leq t_1 + \cdots + t_k \leq \beta_k$. Fix $k \in [j, i-1]$. Then $t_1 + \cdots + t_k = u_1 + \cdots + u_k - 1 \leq \beta_k$. Note that $u_{k+1} \leq v_{k+1}, \ldots, u_{i-1} \leq v_{i-1}$ by the choice of $j$, it follows that $u_1 + \cdots + u_k > v_1 + \cdots + v_k$ and so $t_1 + \cdots + t_k = u_1 + \cdots + u_k - 1 \geq v_1 + \cdots + v_k \geq \alpha_k$. Hence $u - \epsilon_j + \epsilon_i \in B$. Similarly $v + \epsilon_j - \epsilon_i \in B$, as required. □

Example 2.4. Let $I = (x_1, x_3)(x_2, x_4)$. Then $I$ is a polymatroidal ideal satisfying the two-sided strong symmetric exchange property. If $I$ is a pruned lattice path polymatroidal ideal, then there exist $a_i, b_i, i = 1, \ldots, 4$ and $\alpha_2, \beta_2$ such that $I$ is generated by monomials $x^u$ with $u$ satisfying

$$a_i \leq u_i \leq b_i, i = 1, \ldots, 4, \quad \alpha_2 \leq u_1 + u_2 \leq \beta_2, \quad u_1 + \cdots + u_4 = 2.$$ 

Note that the inequality $\alpha_3 \leq u_1 + u_2 + u_3 \leq \beta_3$ does not appear in the conditions since it is equivalent to the inequality $d - \beta_3 \leq u_4 \leq d - \alpha_3$. Since $u_1$ can be 1, it follows that $b_1 \geq 1$. Proceeding in this way, we have $a_i = 0, b_i \geq 1$ for $i = 1, \ldots, 4$ and $\alpha_2 = 0, \beta_2 \geq 2$. Therefore we have $x_1x_3 \in I$, a contradiction. Hence $I$ is not a pruned lattice path polymatroidal ideal. However, $I$ is isomorphic to the lattice path polymatroidal ideal $(x_1, x_3)(x_3, x_4)$. Till now, we cannot find a discrete polymatroid satisfying the one-sided symmetric exchange property which is not isomorphic to a pruned lattice path polymatroid.

For two classes of discrete polymatroids we will show that the one-sided strong symmetric exchange property implies the property of being a pruned lattice path polymatroid.

Let $P_{[a,b]}$ denote the prime ideal generated by variables $x_i$ with $i \in [c, d]$. We first show

Lemma 2.5. Let $B$ be the set of bases of a discrete polymatroid $\mathcal{P}_1$ and $C$ be the set of bases of the discrete polymatroid $\mathcal{P}_2$ corresponding to the transversal polymatroidal
ideal \( P_{[a,b]}P_{[c,d]} \) with \( c < a < b < d \). Then
\[
B + C = \{ u + v : u \in B, v \in C \}
\]
satisfies neither the right- nor the left-sided symmetric exchange property.

**Proof.** Let \( \rho_1, \rho_2, \rho \) be the ground set rank functions of \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P} \) respectively, where \( \mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 \) is the discrete polymatroid whose set of bases is \( B + C \). By [3, Lemma 3.2], there exists \( w \in B \) such that \( w_c + w_d = \rho_1(c,d) \). Let \( u = \varepsilon_c + \varepsilon_b \) and \( v = \varepsilon_a + \varepsilon_d \). Then \( u, v \in C \). Assume \( B + C \) satisfies the right-sided symmetric exchange property. Since \( (w + u)_b > (w + v)_b \) and \( \sum_{i>b}(w + u)_i < \sum_{i>b}(w + v)_i \), the right-sided symmetric exchange property implies that \( w + u - \varepsilon_b + \varepsilon_d = w + \varepsilon_c + \varepsilon_d \in B + C \). Hence \( \rho(c, d) \geq \rho_1(c, d) + 2 \), which is impossible, since \( \rho(c, d) \leq \rho_1(c, d) + \rho_2(c, d) \) and \( \rho_2(c, d) = 1 \). Similarly, \( B + C \) does not satisfy the left-sided symmetric exchange property. \( \square \)

**Proposition 2.6.** Let \( \mathcal{P} \) be the discrete polymatroid corresponding to the transversal ideal \( I = P_{[s_1,t_1]} \cdots P_{[s_k,t_k]} \). Then the following conditions are equivalent:

(a) \( \mathcal{P} \) satisfies the one-sided symmetric exchange property;
(b) after a suitable rearrangement of the factors of \( I \), we have
\[
s_1 \leq s_2 \leq \cdots \leq s_k \quad \text{and} \quad t_1 \leq t_2 \leq \cdots \leq t_k;
\]
(c) \( \mathcal{P} \) is a lattice path polymatroid.

**Proof.** (a) \( \Rightarrow \) (b): We can rearrange the order of prime ideals in the expression of \( I = P_{[s_1,t_1]} \cdots P_{[s_k,t_k]} \) in this way: if \( s_i < s_j \), then \( P_{[s_i,t_i]} \) is placed before \( P_{[s_j,t_j]} \); if \( s_i = s_j \) and \( t_i \leq t_j \), then \( P_{[s_i,t_i]} \) is placed before \( P_{[s_j,t_j]} \). After this rearrangement, we have \( s_1 \leq s_2 \leq \cdots \leq s_k \) and \( t_1 \leq t_j \) for any \( i < j \) with \( s_i = s_j \). It suffices to prove that \( t_i \leq t_j \) if \( s_i < s_j \). But this follows from Lemma 2.5.

(b) \( \Rightarrow \) (c) follows from Proposition 3.1 and (c) \( \Rightarrow \) (a) follows from Proposition 2.3. \( \square \)

**Remark 2.7.** It was proved in [3] that if \( \rho \) is the ground rank function of a discrete polymatroid \( \mathcal{P} \), then \( u \in \mathbb{Z}_+^n \) is a base of \( \mathcal{P} \) if and only if \( \sum_{i \in A} u_i \leq \rho(A) \) for any subset \( A \) of [n] and \( \sum_{i \in [n]} u_i = \rho([n]) \).

**Proposition 2.8.** Let \( \mathcal{P} \) be a discrete polymatroid on the ground set \([4]\) and assume \( \rho(i) < \min\{\rho(i, 3), \rho(i, 4)\} \) for \( i = 1, 2 \). Then \( \mathcal{P} \) satisfies the left-sided symmetric exchange property if and only if it is a pruned lattice path polymatroid.

**Proof.** If \( \rho(i) = 0 \) for some \( i \), \( \mathcal{P} \) is isomorphic to a discrete polymatroid on the ground set \([n]\) with \( n \leq 3 \). Therefore \( \mathcal{P} \) is of Veronese type (see [3, Example 2.6]) and there is nothing to prove. Hence we may assume that \( \rho(i) > 0 \) for \( i = 1, \ldots, 4 \).

The proof of the “if” part follows from Proposition 2.3.

Proof of the “only if” part: by the same argument as given in the proof of [3, Theorem 1.1] we may assume that \( \rho([4] \setminus i) = d \) for all \( i \), where \( d = \rho([4]) \).

Assume that \( \mathcal{P} \) is not a pruned lattice path polymatroid. Then, by Proposition 2.2 there is one pair \( \{i, j\} \in \{\{2,3\}, \{1,4\}, \{1,3\}, \{2,4\}\} \) such that \( \rho(i, j) < \rho(i) + \rho(j) \) and \( \rho(i, j) < d \).
If \( \{i, j\} = \{1, 4\} \), the vector \( u = (\rho(1, 4) - \rho(4), d - \rho(1, 4), 0, \rho(4)) \) and the vector \( v = (\rho(1, 4) - \rho(4) + 1, d - \rho(1, 4) - 1, 1, \rho(4) - 1) \) belong to \( B \), see Remark 2.7. Here we used that \( \rho(1) < \rho(1, 3) \), by assumption. Now by the left-sided strong symmetric exchange property of \( B \), it follows that the vector \( u' = (\rho(1, 4) - \rho(4) + 1, d - \rho(1, 4) - 1, 0, \rho(4)) \) belongs to \( B \), a contradiction since \( u'_1 + u'_4 = \rho(1, 4) + 1 \).

If \( \{i, j\} = \{1, 3\} \), consider vectors \( u = (\rho(1, 3) - \rho(3), d - \rho(1, 3), \rho(3), 0) \) and \( v = (\rho(1, 3) - \rho(3) + 1, d - \rho(1, 3) - 1, \rho(3) - 1, 1) \).

If \( \{i, j\} = \{2, 3\} \), consider vectors \( u = (d - \rho(2, 3), \rho(2, 3) - \rho(3), \rho(3), 0) \) and \( v = (d - \rho(2, 3) - 1, \rho(2, 3) - \rho(3) + 1, \rho(3) - 1, 0) \).

Finally, if \( \{i, j\} = \{2, 4\} \), consider vectors \( u = (d - \rho(2, 4), \rho(2, 4) - \rho(4), 0, \rho(4)) \) and \( v = (d - \rho(2, 4) - 1, \rho(2, 4) - \rho(4) + 1, 1, \rho(4) - 1) \).

In each of these cases, we obtain a contradiction as in the case when \( \{i, j\} = \{1, 4\} \). This completes the proof.

In view of Example 2.4, the condition that \( \rho(i) < \min\{\rho(i, 3), \rho(i, 4)\} \) for \( i = 1, 2 \) in Proposition 2.8 cannot be skipped in general.

In the following proposition we will show that any power of a pruned lattice path polymatroidal ideal is again a pruned lattice path polymatroidal ideal.

**Proposition 2.9.** Let \( I \) be a pruned lattice path polymatroidal ideal of type \( (a, b|\alpha, \beta) \). Then \( I^k \) is a pruned lattice path polymatroidal ideal of type \( (ka, kb|k\alpha, k\beta) \) for any \( k > 1 \).

**Proof.** Let \( J \) be the pruned lattice path polymatroidal ideal of type \( (ka, kb|k\alpha, k\beta) \).

It is clear that \( I^k \subseteq J \). Let \( x^u \) be a minimal generator in \( J \). We will show there are minimal generators \( x^{v_1}, \ldots, x^{v_k} \) of \( I \) such that \( x^u = x^{v_1} \cdots x^{v_k} \), that is, \( u = v_1 + \cdots + v_k \). The \( i \)-th entry of \( v_j \) is denoted by \( v_j(i) \). Note that \( i \)-th entry of \( u \) is still denoted by \( u_i \).

For each \( i \), there exist \( s_i, t_i \) such that \( u_1 + \cdots + u_i = ks_i + t_i \) and \( 0 \leq t_i < k \). For \( j = 1, \ldots, k \) we define \( v_j \) by

\[
v_j(1) + \cdots + v_j(i) = s_i + 1, \quad \text{for} \quad 1 \leq j \leq t_i, \quad 1 \leq i \leq n,
\]

and

\[
v_j(1) + \cdots + v_j(i) = s_i, \quad \text{for} \quad t_i + 1 \leq j \leq k, \quad 1 \leq i \leq n.
\]

Since \( ks_i + t_i = ks_i + t_i \), it follows that \( v_j(1) + \cdots + v_j(i + 1) \geq v_j(1) + \cdots + v_j(i) \) for all \( i, j \), and so \( v_j \) belong to \( \mathbb{Z}_+ \). Since

\[
\sum_{1 \leq j \leq k} v_j(1) + \cdots + \sum_{1 \leq j \leq k} v_j(i) = ks_i + t_i, \quad i = 1, \ldots, n,
\]

we have \( u = v_1 + \cdots + v_k \).

We have \( \alpha_i \leq v_j(1) + \cdots + v_j(i) \leq \beta_i \) for \( 1 \leq j \leq k \) and \( 1 \leq i \leq n \), since \( k\alpha_i \leq ks_i + t_i \leq k\beta_i \).

It remains to be shown that \( a_i \leq v_j(i) \leq b_i \) for \( 1 \leq j \leq k \) and \( 1 \leq i \leq n \). If \( t_i > t_{i-1} \), then \( v_j(i) \) is either \( s_i + 1 - s_{i-1} \) or \( s_i - s_{i-1} \). Note that \( u_i = k(s_i - s_{i-1}) + t_i - t_{i-1} \). Therefore we have \( a_i \leq s_i - s_{i-1} + \frac{k}{k}(t_i - t_{i-1}) \leq b_i \), and hence \( a_i \leq v_j(i) \leq b_i \) for \( 1 \leq j \leq k \).
Similarly one shows that \(a_i \leq v_j(i) \leq b_i\) for \(1 \leq j \leq k\) when \(t_i = t_{i-1}\) or \(t_i < t_{i-1}\). This completes the proof of the proposition. \(\square\)

We do not know if the product of pruned lattice polymatroidal ideals is always a pruned lattice polymatroidal ideal. However there is an example to show the product of lattice path polymatroidals may be not a lattice path polymatroidal.

**Example 2.10.** Set \(I_1 = (x_1 x_2^2 x_3, x_1 x_3 x_2^3, x_2^3 x_3^2, x_1^4 x_3)\) and set \(I_2 = (x_1, x_2, x_3)\). Then \(I_1\) and \(I_2\) are lattice path polymatroidal ideals of type \(((0, 3, 5), (1, 4, 5))\) and \(((0, 0, 1), (1, 1, 1))\) respectively. We claim that \(I_1I_2\) is not a lattice path polymatroidal ideal. If it is, there are \(\alpha, \beta, i = 1, 2\) such that \(I_1I_2\) is generated by \(x^u\) with \(u\) satisfying \(\alpha_1 \leq u_1 \leq \beta_1, \alpha_2 \leq u_2 \leq \beta_2, u_1 + u_2 + u_3 = 6\). One have \(\alpha_1 = 0, b_n \geq 2\) and \(a_2 \leq 3, b_2 \geq 5\). This implies \(x_1^2 x_2 x_3^2 \in I_1I_2\), a contradiction.

3. The Transversality of Lattice Path Polymatroids

It was pointed out in [9] that a lattice path polymatroid is transversal. In this section, we explicitly write a lattice path polymatroidal ideal as a product of monomial prime ideals and also characterize those transversal polymatroidal ideals which are lattice path polymatroidal ideals.

**Proposition 3.1.** Let \(I\) be the transversal polymatroidal ideal \(P_{[s_1, t_1]} \cdots P_{[s_d, t_d]}\) with \(1 \leq s_1 \leq \cdots \leq s_d \leq n, 1 \leq t_1 \leq \cdots \leq t_d \leq n\) and \(s_i \leq t_i\) for \(i = 1, \ldots, d\). Then \(I\) is a lattice path polymatroidal ideal of type \((\alpha, \beta)\), where \(\alpha_i = \max\{i : t_i \leq k\}\) and \(\beta_i = \max\{i : s_i \leq k\}\) for all \(i = 1, \ldots, n\).

**Proof.** Let \(J\) be the lattice path polymatroidal ideal of type \((\alpha, \beta)\). Fix \(w = x_1^{u_1} \cdots x_d^{u_d} \in G(J)\) and write \(w = x_{j_1} x_{j_2} \cdots x_{j_d}\) with \(j_1 \leq j_2 \leq \cdots \leq j_d\). Fix \(1 \leq k \leq d\) and set \(i = j_k\). Then \(u_1 + \cdots + u_{i-1} + 1 \leq k \leq u_1 + \cdots + u_i\). Since \(k \leq \max\{j : s_j \leq i\}\) and since \(s_i\) is non-decreasing, it follows that \(s_k \leq i\). On the other hand, since \(k \leq \max\{j : t_j \leq i - 1\}\) and \(t_j\) is non-decreasing, it follows that \(t_k \geq i\). In conclusion, \(s_k \leq j_k \leq t_k\), and so \(x_{j_k}\) is in \(P_{[s_k, t_k]}\) for each \(k\). Hence \(w \in I\).

Conversely, let \(w\) be a minimal generator in \(I\). Then \(w = x_{j_1} \cdots x_{j_d}\) with \(x_{j_k} \in P_{[s_k, t_k]}\) for \(k = 1, \ldots, d\). Let \(i_1 \leq \cdots \leq i_d\) be such that \(\{i_1, \ldots, i_d\} = \{j_1, \ldots, j_d\}\).

We claim \(x_{i_k} \in P_{[s_k, t_k]}\) for \(k = 1, \ldots, d\). Indeed, assume \(i_1 = j_a\) for some \(1 \leq a \leq d\). Then \(s_1 \leq i_1 \leq t_1\) and \(s_a \leq j_1 \leq t_a\). Hence \(x_{i_1} \in P_{[s_1, t_1]}\) and \(x_{j_2} \cdots x_{j_{a-1}} x_{j_1} x_{j_{a+1}} \cdots x_{j_d} \in P_{[s_2, t_2]} \cdots P_{[s_d, t_d]}\). It follows that \(x_{i_2} \in P_{[s_2, t_2]}\) since \(i_2\) is the smallest number in the set \(\{j_2, j_{a+1}, j_{a+1}, \ldots, j_d\}\). Proceeding in this way we show by induction on \(k\) that \(x_{i_k} \in P_{[s_k, t_k]}\) for \(k = 1, \ldots, d\), as claimed.

It follows that \(w = x_{i_1} \cdots x_{i_d}\) with \(x_{i_k} \in P_{[s_k, t_k]}\) for \(k = 1, \ldots, n\) and \(i_1 \leq \cdots \leq i_d\). We now write \(w\) in the form \(w = x_1^{u_1} \cdots x_n^{u_n}\). Fix \(1 \leq i \leq n\) and put \(u_1 + \cdots + u_i = k\). Then \(i_k = i\) and \(i_{k+1} = i + 1\), and it follows that \(s_k \leq i\) and \(t_{k+1} > i\). Hence \(\max\{j : t_k \leq i\} \leq k \leq \max\{j : s_j \leq i\}\), as required. \(\square\)

Given positive integers \(n\) and \(d\), we set
\[
S_{n,d} = \{(s_1, \ldots, s_d) : 1 \leq s_1 \leq s_2 \leq \cdots \leq s_d \leq n\}
\]
and
\[ T_{n,d} = \{(\alpha_1, \ldots, \alpha_n): \ 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = d\}. \]
Then \(|S_{n,d}| = |T_{n,d}| = \binom{d+n-1}{d}. \)

We define a map \( f : S_{n,d} \to T_{n,d} \) by the rule that if \( f(s_1, \ldots, s_d) = (\alpha_1, \ldots, \alpha_n) \), then \( \alpha_i = \max\{k : s_k \leq i\}, i = 1, \ldots, n \) and \( g : T_{n,d} \to S_{n,d} \) by the rule that if \( g(\alpha_1, \ldots, \alpha_n) = (s_1, \ldots, s_d) \), then \( s_i = \min\{k: \alpha_k \geq i\}, i = 1, \ldots, d. \) Then \( f \circ g = id_{S_{n,d}} \) and \( g \circ f = id_{T_{n,d}}. \)

**Corollary 3.2.** Let \( I \) be a lattice path polymatroidal ideal of type \((\alpha, \beta)\). Then \( I = P_{[s_1, t_1]} \cdots P_{[s_d, t_d]} \), where \((t_1, \ldots, t_d) = g(\alpha_1, \ldots, \alpha_n)\) and \((s_1, \ldots, s_d) = g(\beta_1, \ldots, \beta_n)\).

### 4. Linear quotients for pruned lattice path polymatroidal ideals

From the proofs of [4, Theorem 12.6.2 and Theorem 12.7.2], we see that a polymatroidal ideal \( I \) has linear quotients if its minimal generators are arranged in either the lexicographical order or the reverse lexicographical order. In this section we will show if \( I \) is a pruned lattice path polymatroidal ideal, then its linear quotients are more easily trackable. We will use this result in the following sections.

**Remark 4.1.** Let \( I \) be a pruned lattice path polymatroidal ideal of type \((a, b|\alpha, \beta)\). We may always assume \( a_1 = \cdots = a_n = 0. \) In fact, let \( J \) be the ideal generated by monomials \( x_1^{a_1} \cdots x_n^{a_n} \) satisfying
\[ 0 \leq u_i \leq b_i - a_i, \forall i = 1, \ldots, n \]
and
\[ \alpha_i - \sum_{j=1}^{i} a_j \leq u_1 + \cdots + u_i \leq \beta_i - \sum_{j=1}^{i} a_j, \forall i = 1, \ldots, n. \]

Then, since \( I = x_1^{a_1} \cdots x_n^{a_n} J, \) the ideals \( I \) and \( J \) are isomorphic as modules and hence \( I \) and \( J \) have the same projective dimension and the same depth.

**Proposition 4.2.** Let \( I \) be a pruned lattice path polymatroidal ideal of type \((a, b|\alpha, \beta)\) with \( a_1 = \cdots = a_n = 0, \) and let \( G(I) = \{m_1, \ldots, m_r\} \), where \( m_1 > m_2 > \cdots > m_r \), with respect to the lexicographical order. Fix \( 1 \leq q \leq r \) and let \( J = (m_1, \ldots, m_{q-1}). \) Write \( m_q = x_1^{t_1} \cdots x_n^{t_n}. \) Then

(a) \( x_n \) is not in \( J : m_q \) for any \( 1 \leq q \leq r; \)

(b) for any \( 1 \leq i \leq n-1, \) \( x_i \in J : m_q \) if and only if \( t_i < b_i \) and \( t_1 + \cdots + t_i < \beta_i. \)

**Proof.** If \( x_i \in J : m_q, \) then there is \( l < q \) with \( x_i = m_l/[m_l : m_q]. \) (Here \([u, v]\) denotes the greatest common divisor of the monomials \( u \) and \( v. \)) Since \( m_l > m_q, \) we have \( m_l = x_1^{t_1} \cdots x_i^{t_i+1} \cdots x_j^{t_j-1} \cdots x_n^{t_n} \) for some \( j > i. \) Hence \( x_n \notin J : m_q. \) This proves (a).

Proof of (b): Let \( i < n \) and \( x_i \in J : m_q. \) Then as in part (a) of the proof there exists \( m_l \in G(I) \) with \( m_l = x_1^{t_1} \cdots x_i^{t_i+1} \cdots x_j^{t_j-1} \cdots x_n^{t_n} \) for some \( j > i. \) It follows that \( t_i < t_i + 1 \leq b_i \) and \( t_1 + \cdots + t_i < t_1 + \cdots + t_i + 1 \leq \beta_i. \)

Conversely let \( i < n \) and assume \( t_i < b_i \) and \( t_1 + \cdots + t_i < \beta_i. \) Let \( j \) be the smallest integer \( k \) with the property that \( k > i \) and \( t_1 + \cdots + t_k = \beta_k. \) Note that such a \( k \) exists since \( t_1 + \cdots + t_n = \beta_n. \) By the choice of \( j, \) we have \( t_j > \beta_j - \beta_{j-1} \geq 0. \)
Set \( m = x_1^{t_1} \cdots x_i^{t_i+1} \cdots x_j^{t_j-1} \cdots x_n^{t_n} \). Then \( m \) belongs to \( G(I) \) with \( m > m_q \) and \( m/[m, m_q] = x_i \). Hence \( x_i \in J : m_q \), and this completes the proof of (b). \( \square \)

For any \( x^u = x_1^{u_1} \cdots x_n^{u_n} \in G(I) \), let
\[
N(u) = \{1 \leq i \leq n-1 : u_i < b_i, u_1 + \cdots + u_i < \beta_i\}.
\]

Then \( \operatorname{depth} S/I = n - 1 - \max\{N(u) : x^u \in G(I)\} \) by [4, Corollary 8.2.2]. In general it is not easy to find \( u \) for which \( N(u) \) is maximal.

In the remaining part of this section we will determine the associated prime ideals of lattice path polymatroidal ideals. First, as an immediate consequence of Proposition 4.5 we obtain

**Proposition 4.3.** Let \( I \) be a lattice path polymatroidal ideal of type \((\alpha, \beta)\). Then the following statements are equivalent:

(a) \( \operatorname{depth} S/I = 0 \);
(b) \( m \in \operatorname{Ass}(S/I) \);
(c) \( \alpha_i < \beta_i \) for \( i = 1, \ldots, n - 1 \).

**Proof.** The equivalence between (a) and (b) is well-known and it holds for all monomial ideals.

(a) \( \iff \) (c): Note that in this special case, \( N(u) = \{|i|u_1 + \cdots + u_i < \beta_i\} \) for any \( u \) with \( x^u \in G(I) \). If \( \alpha_i < \beta_i \) for \( i = 1, \ldots, n - 1 \), then \( v = (\alpha_1, \alpha_2 - \alpha_1, \ldots, \alpha_n - \alpha_{n-1}) \) belongs to \( \{u : x^u \in G(I)\} \) and \( N(v) = n - 1 \). Hence \( \operatorname{depth} S/I = 0 \). For the converse part, we only need to note that \( N(v) \leq \#\{1 \leq i \leq n - 1 : \alpha_i < \beta_i\} \) for any \( v \) with \( x^v \in G(I) \).

\( \square \)

**Lemma 4.4.** Let \( I \) be a lattice path polymatroidal ideal. Then every associated prime ideal in \( I \) is of the form \( P_{[s,t]} \) for some \( s \leq t \).

**Proof.** The assertion follows from Corollary 3.2 together with [3, Theorem 3.7]. \( \square \)

Following [4], we denote by \( I(P) \) the monomial localization of \( I \) at \( P \), and \( S(P) \) the polynomial ring in the variables in \( P \). Recall \( I(P) \subseteq S(P) \) is the monomial ideal obtained from \( I \) as the image of the map \( \phi : S \to S(P) \) defined by \( \phi(x_i) = x_i \) if \( x_i \in P \) and \( \phi(x_i) = 1 \), otherwise. Let \( A \) be a subset of \( [n] \) and \( P_A = (x_i : i \in A) \). Then we denote \( I(P_A) \) by \( I(A) \) and \( S(P_A) \) by \( S(A) \) for short. One can check that if \( I, J \) are monomial ideals then \( (IJ)(A) = I(A)J(A) \), and if \( A \subseteq B \) then \( I(A) = I(B)(A) \).

In the later proofs we need the following facts.

**Proposition 4.5.** (a) Let \( I \) be a monomial ideal. Then \( P \in \operatorname{Ass}(S/I) \) if and only if \( \operatorname{depth} S(P)/I(P) = 0 \).

(b) Let \( I \) be a polymatroidal ideal and let \( a_i = \max\{u_i : x^u \in G(I)\} \). Then \( I([n]\setminus i) \) is again a polymatroidal ideal generated by monomials \( x^u/x^{a_i} \) with \( x^u \in G(I) \) and \( u_i = a_i \).

**Proof.** (a) has been observed in [2, Lemma 2.11].

(b) see [4, Proposition 2.1] and its proof. \( \square \)
Let $I$ be a lattice path polymatroidal ideal of type $(\alpha, \beta)$. By Proposition 4.5(b) we have $I([s, n])$ is generated by $x^u$ with $u$ satisfying
\[
\max\{\alpha_i - \beta_{s-1}, 0\} \leq u_s + \cdots + u_i \leq \beta_i - \beta_{s-1} \quad \text{for } i = s, \ldots, n,
\]
and $I([1, t])$ is generated by $x^u$ with $u$ satisfying
\[
\alpha_i \leq u_1 + \cdots + u_i \leq \min\{\alpha_i, \beta_i\} \quad \text{for } i = 1, \ldots, t.
\]
Hence $I([s, t])$ is generated by $x^u$ with $u$ satisfying
\[
\max\{\alpha_i - \beta_{s-1}, 0\} \leq u_s + \cdots + u_i \leq \min\{\max\{\alpha_i - \beta_{s-1}, 0\}, \beta_i - \beta_{s-1}\}
\]
for $i = s, \ldots, t$.

**Proposition 4.6.** Let $I$ be a lattice path polymatroidal ideal of type $(\alpha, \beta)$. Then $P_{[s, t]}$ belongs to $\text{Ass}(S/I)$ if and only if $\alpha_i < \beta_i$ for $i = s, \ldots, t - 1$ and $\beta_{s-1} < \beta_s$, $\beta_{t-1} < \alpha_t$, $\alpha_{t-1} < \alpha_t$. In particular, $(x_i) \in \text{Ass}(S/I)$ if and only if $\alpha_t > \beta_{t-1}$. Here we use the convention that $\beta_0 = 0$.

**Proof.** In view of Lemma 4.5(a) and Proposition 4.3, we see that $P_{[s, t]}$ belongs to $\text{Ass}(S/I)$ if and only if $\max\{\alpha_i - \beta_{s-1}, 0\} < \max\{\alpha_i - \beta_{s-1}, 0\}$ and $\max\{\alpha_i - \beta_{s-1}, 0\} < \beta_i - \beta_{s-1}$ for $i = s, \ldots, t - 1$, which is equivalent to the conditions given in Proposition 4.6.

5. The depth and the associated prime ideals of left pruned lattice path polymatroidal ideals

A pruned lattice path polymatroidal ideal $I$ is called a left pruned lattice path polymatroidal ideal, if $I$ is generated by monomials $x^u$ such that
\[
0 \leq u_i \leq b_i \quad \text{for } 1 \leq i \leq k,
\]
and
\[
\alpha_{k+i} \leq u_1 + \cdots + u_{k+i} \leq \beta_{k+i} \quad \text{for } i = 1, \ldots, n - k,
\]
where $1 \leq k \leq n - 1$ and $b_i > 0$, $\alpha_{k+1} \leq \cdots \leq \alpha_n = d$, $\beta_{k+1} \leq \cdots \leq \beta_n = d$. Hence a pruned lattice path polymatroidal ideal of type $(a, b, \alpha, \beta)$ is a left pruned lattice path polymatroidal ideal, if there exists $k \in [2, n - 1]$ such that $a_i = 0$, $b_i \geq d$ for all $i \geq k + 1$ and $\alpha_i = 0$, $\beta_i = \beta_{k+1}$ for all $i \leq k$.

In this section we will investigate the depth and the associated prime ideals of this class of ideals.

Let $x^u = x_1^{u_1} \cdots x_n^{u_n}$. The support of $u$, denoted by $\text{supp}(u)$, is the set $\{i \mid u_i \neq 0\}$. If $I$ is a monomial ideal, then the support of $I$, denoted by $\text{supp}(I)$, is the set $\bigcup_{u \in \text{supp}(I)} \text{supp}(u)$.

**Lemma 5.1.** Let $I_1, I_2, \ldots, I_t \subset S$ be monomial ideals whose supports are pairwise disjoint. For $j = 1, \ldots, t$, let $S_j$ be a polynomial ring whose set of variables $V_j$ contains the set $\{x_i \mid i \in \text{supp}(I_j)\}$. We further assume that $V_1, \ldots, V_t$ are pairwise disjoint and that $V = V_1 \cup V_2 \cup \ldots \cup V_t$ is the set of variables of $S$. For $j = 1, \ldots, t$, let $J_j = S_j \cap I_j$.

(a) depth $S/I_1 \cdots I_t = \sum_{j=1}^t \text{depth } S_j/J_j + t - 1$. 

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(b) $\mathrm{Ass}(S/I_1 \cdots I_t) = \bigcup_{1 \leq i \leq t} \mathrm{Ass}(S/I_i)$.

Proof. (a) Let $J, K$ be nonzero monomial ideals with disjoint supports. Then

$$\mathrm{Tor}_1(S/J, S/K) = \frac{J \cap K}{JK} = 0$$

and so $\mathrm{Tor}_i(S/J, S/K) = 0$ for all $i > 0$ by the rigidity of the functor $\mathrm{Tor}$. This implies that the tensor product of the minimal free resolutions of $J$ and $K$ is the minimal free resolution of $JK$. In particular, $\mathrm{proj dim} \ JK = \mathrm{proj dim} J + \mathrm{proj dim} K$.

Now (a) follows by induction on $t$ by using the Auslander-Buchsbaum formula.

(b) Let $J, K$ be as above. From the short exact sequence

$$0 \to S/JK \to S/J \oplus S/K \to S/(J + K) \to 0,$$

we obtain

$$\mathrm{Ass}(S/JK) \subseteq \mathrm{Ass}(S/J) \cup \mathrm{Ass}(S/K) \subseteq \mathrm{Ass}(S/JK) \cup \mathrm{Ass}(S/(J + K)).$$

Since $\mathrm{Ass}(S/(J + K)) = \{P_1 + P_2 : P_1 \in \mathrm{Ass}(S/J), P_2 \in \mathrm{Ass}(S/K)\}$, it follows that $(\mathrm{Ass}(S/J) \cup \mathrm{Ass}(S/K)) \cap \mathrm{Ass}(S/(J + K)) = \emptyset$. This implies that $\mathrm{Ass}(S/J) \cup \mathrm{Ass}(S/K) = \mathrm{Ass}(S/JK)$. Now (b) follows by the induction on $t$. \hfill $\square$

In the following I denotes the left pruned lattice path polymatroidal ideal as given in the begin of this section.

**Proposition 5.2.** $\mathrm{depth} \ S/I = |\{i : \alpha_i = \beta_i, k + 1 \leq i \leq n - 1\}|$.

Proof. Set $t = |\{i : \alpha_i = \beta_i, k + 1 \leq i \leq n - 1\}|$. Define a vector $u$ by $u_i = 0$ for $i = 1, \ldots, k, \ u_{k+1} = \alpha_{k+1}$, and $u_i = \alpha_i - \alpha_{i-1}$ for $i = k + 2, \ldots, n$. Then $x^n \in G(I)$ and $N(u) = n - 1 - t$ (see (3) for the definition of $N(u)$). Since $N(v) \leq n - 1 - t$ for any $v$ with $x^v \in G(I)$, we have depth $S/I = t$. \hfill $\square$

By definition, dstab($I$) (resp. astab($I$)) is the smallest integer $k$ with the property that

$$\mathrm{depth} \ S/I^k = \mathrm{depth} \ S/I^\ell \quad \text{(resp.} \ \mathrm{Ass}(S/I^k) = \mathrm{Ass}(S/I^\ell))$$

for all $\ell \geq k$.

**Corollary 5.3.** dstab($I$) = 1.

Proof. The assertion follows from Proposition 5.2 and Proposition 2.9. \hfill $\square$

**Lemma 5.4.** Let $P_A \in \mathrm{Ass}(S/I)$ and suppose $i \notin A$ for some $i \geq k + 1$. Then either $[1, i-1] \cap A = \emptyset$ or $[i+1, n] \cap A = \emptyset$.

Proof. Assume on the contrary that there exist $1 \leq \ell < i < j \leq n$ such that $\ell \in A$ and $j \in A$. Let $B = [1, n] \setminus \{i\}$. Note that $A \subseteq B$. Let $a_i = \max\{u_i : x^u \in G(I)\}$. There are two cases to consider.

The case when $i = k + 1$: Since $a_{k+1} = \beta_{k+1}$ and since $u_1 = \cdots = u_k = 0$ for any $x^u \in G(I)$ with $u_{k+1} = \beta_{k+1}$, we have $\mathrm{supp}(I(B)) \subseteq [k + 1, n]$ by Lemma 4.5(b), and in particular, $\mathrm{supp}(I(A)) \subseteq [k + 1, n]$. It follows that $\ell \notin \mathrm{supp}(I(A))$, and so depth $S(A)/I(A) \neq 0$. Hence $P_A \notin \mathrm{Ass}(S/I)$, a contradiction.

The case when $i > k + 1$: Note that $a_i = \beta_i - \alpha_{i-1}$, and for any $x^u \in G(I)$ with $u_i = a_i$, we have $u_1 + \cdots + u_{i-1} = \alpha_{i-1}$. This implies $I(B) = JK$ by Lemma 4.5(b).
Here $J$ is generated by the monomials $x_1^{u_1} \cdots x_{i-1}^{u_{i-1}}$ such that the $u_1, \ldots, u_{i-1}$ satisfy $0 \leq u_1 \leq b_1, \ldots, 0 \leq u_k \leq b_k, \alpha_{k+1} \leq u_1 + \cdots + u_{k+1} \leq \beta_{k+1}, \ldots, u_1 + \cdots + u_{i-1} = \alpha_{i-1}$, and $K$ is generated by the monomials $x_i^{u_i} \cdots x_n^{u_n}$ such that the $u_i, \ldots, u_n$ satisfy $\text{max}\{\alpha_{i+1} - \beta_i, 0\} \leq u_{i+1} + \cdots + u_i \leq d - \beta_i$. It follows that $I(A) = J(A)K(A)$. Since $\ell \in A$ and $j \in A$, $A$ contains $\text{supp}(K(A))$ and $\text{supp}(J(A))$ properly. By this fact and since $J(A)$ and $K(A)$ have disjoint supports, we have $\text{depth}(S(A)/I(A)) \neq 0$ by Lemma 5.1 and so $P_A \notin \text{Ass}(S/I)$, a contradiction again.

Corollary 5.5. If $P_A \in \text{Ass}(S/I)$, then either $A$ is a subinterval of $[k+2, n]$, or $A = B \cup [k+1, t]$ for some $B \subseteq [1, k]$ and some $t \geq k+1$.

Proof. Let $P_A \in \text{Ass}(S/I)$. In view of Lemma 5.4, we have $A \cap [k+1, n]$ is either empty or an interval. We claim $A \cap [k+1, n] \neq \emptyset$. In fact, as seen in the proof of Lemma 5.4, $\text{supp}(I([1, n] \setminus \{k+1\})) \subseteq [k+2, n]$. Thus, for any $B \subseteq [1, k]$, we have $\text{supp}(I(B)) \subseteq [k+2, n]$ and, in particular, $P_B \notin \text{Ass}(S/I)$. This proves the claim.

Let $j$ be the smallest integer in $A \cap [k+1, n]$. If $j \geq k+2$, then $k+1 \notin A$ and it follows that $A \cap [1, k] = \emptyset$ by Lemma 5.4 and so $A$ is an interval of $[k+2, n]$. If $j = k+1$, then $A \cap [k+1, n] = [k+1, t]$ for some $t \geq k+1$ and so $A = B \cup [k+1, t]$ and some $B \subseteq [1, k]$.

Proposition 5.6. Let $B \subseteq [1, k]$ and $t \in [k+1, n]$. Then $P_{B \cup [k+1, t]} \in \text{Ass}(S/I)$ if and only if

$$\sum_{j \in \overline{B}} b_j < \min\{\alpha_t, \beta_{k+1}\}, \quad \alpha_{t-1} < \alpha_t$$

and

$$\alpha_i < \beta_i, \quad \text{for all } i = k+1, \ldots, t-1.$$  

Here $\overline{B} = [1, k] \setminus B$ and $\alpha_k = 0$.

Proof. For any $t \geq k+1$, Lemma 4.5(b) implies that $I([1, t])$ is generated by $x^u \in S([1, t])$ with $u$ satisfying $0 \leq u_i \leq b_i$ for $i = 1, \ldots, k$ and $\alpha_i \leq u_1 + \cdots + u_i \leq \min\{\alpha_t, \beta_i\}$ for $i = k+1, \ldots, t$. Let $B \subseteq [1, k]$. We claim that if $\sum_{j \in \overline{B}} b_j \geq \min\{\alpha_t, \beta_{k+1}\}$, then $P_{B \cup [k+1, t]} \notin \text{Ass}(S/I)$, equivalently, $\text{depth}(S([1, t] \setminus \overline{B})/I([1, t] \setminus \overline{B})) \neq 0$. For this, we use induction on $|\overline{B}|$.

If $|\overline{B}| = 1$, say $\overline{B} = \{i_1\}$, then, since $\text{max}\{u_{i_1} : x^u \in I([1, t])\} = \min\{\alpha_t, \beta_{k+1}\}$, we have $k+1 \notin \text{supp}(I(B \cup [k+1, t]))$ by Lemma 4.5(b). In particular, $P_{B \cup [k+1, t]} \notin \text{Ass}(S/I)$.

Assume $|\overline{B}| > 1$. We assume there exists $i_1 \in \overline{B}$ such that $b_{i_1} < \min\{\alpha_t, \beta_{k+1}\}$, because we can argue as in the paragraph before. Then $I([1, t] \setminus \{i_1\})$ is generated by $x^u \in S([1, t] \setminus \{i_1\})$ with $u$ satisfying $0 \leq u_i \leq b_{i_1}$ for $i \neq i_1$ and $1 \leq i \leq k$ and $\text{max}\{u_{i_1} - b_{i_1}, 0\} \leq u_1 + \cdots + u_i \leq \min\{\alpha_{i_1}, \beta_{i_1}\} - b_{i_1}$ for $i = k+1, \ldots, t$. Let $T = [1, t] \setminus \{i_1\}$ and $C = B \cup \{i_1\}$. Note that $|C| < |\overline{B}|$. By induction hypothesis,

$$\text{depth}(S([1, t] \setminus \overline{B})/I([1, t] \setminus \overline{B})) = \text{depth}(S(T \setminus \overline{C})/I(T \setminus \overline{C})) \neq 0,$$

that is, $P_{B \cup [k+1, t]} \notin \text{Ass}(S/I)$. This completes the proof of our claim.
Now assume $\sum_{j \in B} b_j < \min\{\alpha_t, \beta_{k+1}\}$. Then $I(B \cup [k+1, t])$ is generated by $x^u \in S(B \cup [k+1, t])$ with $u$ satisfying $0 \leq u_i \leq b_i$ for each $i \in B$ and

$$\max\{\alpha_i - \sum_{j \in B} b_j, 0\} \leq \sum_{j \in B} u_j + u_{k+1} + \cdots + u_i \leq \min\{\alpha_t, \beta_i\} - \sum_{j \in B} b_j$$

for $i = k+1, \ldots, t$.

In view of Proposition 5.2, we see that depth $S(B \cup [k+1, t])/I(B \cup [k+1, t]) = 0$ if only if $\alpha_i < \min\{\alpha_t, \beta_i\}$ for $i = k+1, \ldots, t-1$. This is the case if and only if $\alpha_i < \beta_i$ for $i = k+1, \ldots, t-1$ and $\alpha_t-1 < \alpha_t$, as required.

**Proposition 5.7.** For any interval $[s, t] \subseteq [k+2, n]$, we have $P_{[s,t]} \in \Ass(S/I)$ if and only if $\alpha_i < \beta_i$ for $i = s, \ldots, t-1$, $\beta_{s-1} < \beta_s$, $\alpha_{t-1} < \alpha_t$ and $\beta_{s-1} < \alpha_t$.

**Proof.** Note that $I([1, n] \setminus \{k+1\})$ is generated by $x^u \in S([1, n] \setminus \{k+1\})$ with $u$ satisfying $u_i = 0$ for $i = 1, \ldots, k$, and

$$\max\{\alpha_{k+2} - \beta_{k+1}, 0\} \leq u_{k+2} \leq \beta_{k+2} - \beta_{k+1}, \ldots, \sum_{k+2}^n u_i = d - \beta_{k+1}.$$ 

Hence $I([k+2, n])$ is generated by $x^u \in S([k+2, n])$ with $u$ satisfying the inequalities in (4). Applying Proposition 4.6 to this case, the result follows.

Combining last three results, the associated prime ideals of $I$ are determined.

**Theorem 5.8.** $P_A \in \Ass(S/I)$ if and only if either

(a) $A = B \cup [k+1, t]$, where $B \subseteq [1, k], t \geq k+1$ and

$$\sum_{j \in B} b_j < \min\{\alpha_t, \beta_{k+1}\}, \quad \alpha_{t-1} < \alpha_t, \quad \alpha_i < \beta_i \text{ for all } i = k+1, \ldots, t-1,$$

or

(b) $A = [s, t]$, where $k+2 \leq s \leq t \leq n$ and

$$\beta_{s-1} < \beta_s, \quad \alpha_{t-1} < \alpha_t, \quad \beta_{s-1} < \alpha_t, \quad \alpha_i < \beta_i \text{ for all } i = s, \ldots, t-1.$$

As an application we obtain

**Corollary 5.9.** $\aststab(I) = 1$.

**Proof.** The assertion follows from Theorem 5.8 together with Proposition 2.9.

It was proved in [8] that every polymatroidal ideal is of *strong intersection type*, that is, if $I$ is a polymatroidal ideal, then $\bigcap_{P \in \Ass(S/I)} P^{d_P}$ is an irreducible primary decomposition of $I$, where $d_P$ is the degree of generators in $I(P)$. From the proofs of Propositions 5.6 and 5.7, we see that if $P_{B \cup [k+1, t]} \in \Ass(S/I)$ then $d_{P_{B \cup [k+1, t]}} = \alpha_t - \sum_{j \in B} b_j$ and if $P_{[s,t]} \in \Ass(S/I)$ then $d_{P_{[s,t]}} = \alpha_t - \beta_{s-1}$.

**Example 5.10.** Let $I$ be an ideal generated by $x^u$ such that $u$ satisfies $0 \leq u_i \leq 2$ for $i = 1, 2$, $0 \leq u_3 \leq 3$, $2 \leq \sum_{i=1}^4 u_i \leq 4$ and $\sum_{i=1}^5 u_i = 5$. Then $I = P_{[1,5]}^5 \cap P_{[1,4]}^2 \cap P_{[1,5] \setminus \{3\}}^3 \cap P_{[2,5]}^3 \cap P_{[1,5] \setminus \{2\}}^3 \cap P_5$. Here $P_5 = \langle x_5 \rangle$. 

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6. The depth and the associated prime ideals of right pruned lattice path polymatroidal ideals

In this section we will investigate another class of pruned lattice path polymatroidal ideals, which we call right pruned lattice path polymatroidal ideals. A pruned lattice path polymatroidal ideal of type \((a,b|\alpha,\beta)\) is called a right pruned lattice path polymatroidal ideal, if there exists \(k \in [1,n-1]\) such that \(a_i = 0\) and \(b_i \geq d\) for all \(i \leq k\), and \(\alpha_i = \alpha_k\) and \(\beta_i = d\) for all \(k + 1 \leq i \leq n - 1\). Hence a right pruned lattice path polymatroidal ideal \(I\) is generated by the monomials \(x^u\) with \(u\) satisfying

\[
\alpha_i \leq u_1 + \cdots + u_i \leq \beta_i, \text{ for } i = 1, \ldots, k, \\
0 \leq u_i \leq b_i, \text{ for } i = k + 1, \ldots, n,
\]

and

\[
u_1 + \cdots + u_n = d.
\]

Here \(\alpha_1 \leq \cdots \leq \alpha_k \leq \beta_1 \leq \cdots \leq \beta_k \leq d, \alpha_i \leq \beta_i\) for \(i = 1, \ldots, k\) and \(b_i > 0\) for \(i = k + 1, \ldots, n\). To ensure that \(I \neq 0\) we always assume that \(d \leq \beta_k + b_{k+1} + \cdots + b_n\).

In this section, the ideal \(I\) always denotes the right pruned lattice path polymatroidal ideal as given above. Note that if \(k = 1\) then \(I\) is of Veronese type. First we characterize the case when \(\text{depth } S/I = 0\). For convenience we set \(b(B) = \sum_{i \in B} b_i\) for \(B \subseteq [k+1,n]\)

**Lemma 6.1.** Suppose \(\text{depth } S/I = 0\) if and only if \(d \leq \beta_k + b([k+1,n]) - n + k\) and \(\alpha_i < \beta_i\) for \(i = 1, \ldots, k\).

**Proof.** Suppose \(\text{depth } S/I = 0\). Then there is \(x^u \in G(I)\) such that \(N(u) = n - 1\) (see (3) for the definition of \(N(u)\)). This implies that \(u_1 + \cdots + u_k \leq \beta_k - 1\) and \(u_i \leq b_i - 1\) for \(i = k + 1, \ldots, n - 1\). Therefore we have \(d = u_1 + \cdots + u_n \leq \beta_k - 1 + b_{k+1} + \cdots + b_n\). Since \(u_1 + \cdots + u_i \leq \beta_i\), we have \(\alpha_i < \beta_i\) for \(i = 1, \ldots, k\).

Conversely, since \(0 < d - \beta_k + 1 \leq (b_{k+1} - 1) + \cdots + (b_n - 1) + b_n\), there exist integers \(c_{k+1}, \ldots, c_n\) such that \(c_{k+1} + \cdots + c_n = d - \beta_k + 1, 0 \leq c_i \leq b_i - 1\) for \(i = k + 1, \ldots, n - 1\) and \(c_n \leq b_n\). Let \(u = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_k - \beta_k - 1, c_{k+1}, \ldots, c_n)\). Then \(x^u \in G(I)\) and \(N(u) = n - 1\), which implies \(\text{depth } S/I = 0\). \(\square\)

**Lemma 6.2.** Suppose that \(\alpha_i < \beta_i\) for \(i = 1, \ldots, k\). Then

\[
\text{depth } S/I = \max\{0, d - \beta_k - b([k+1,n]) + n - k\}.
\]

**Proof.** Let \(t = d - \beta_k - b([k+1,n]) + n - k\). The case when \(t \leq 0\) follows from Lemma 6.1. Assume \(t > 0\). Set \(j = n - 1 - t\). Note that \(j \geq k - 1\), since \(d \leq \beta_k + b([k+1,n])\). There are several cases to consider.

If \(j = k - 1\), we let \(u = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_k - \beta_k - 1, b_{k+1}, \ldots, b_n)\). Then \(x^u \in G(I)\) and \(N(u) = k - 1\). It follows that \(\text{depth } S/I \leq n - k = t\).

If \(j = k\), we let \(u = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_k - b_{k+1}, \ldots, b_n)\). Then \(x^u \in G(I)\) and \(N(u) = k\). It follows that \(\text{depth } S/I \leq n - k - 1 = t\).

If \(j \geq k + 1\), we let \(u = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_k - b_{k+1}, 1, \ldots, b_j - 1, b_{j+1}, \ldots, b_n)\). Then \(u \in G(I)\) and \(N(u) = j\). Hence \(\text{depth } S/I \leq n - 1 - j = t\).
It remains to be shown that \( N(u) \leq j \) if \( x^u \in G(I) \). Fix \( u \) with \( x^u \in G(I) \). We set
\[
j_1 = |\{i: \ u_1 + \cdots + u_i < \beta_i, 1 \leq i \leq k\}| \quad \text{and} \quad j_2 = |\{i: \ u_i < b_i, k + 1 \leq i \leq n - 1\}|.
\]
Then \( N(u) = j_1 + j_2 \). If \( j_1 = k \), then \( u_1 + \cdots + u_k \leq \beta_k - 1 \) and so \( u_{k+1} + \cdots + u_n \geq d - \beta_k + 1 \). On the other hand, \( u_{k+1} + \cdots + u_n \leq b([k + 1, n]) - j_2 \). This implies that \( d - \beta_k + 1 \leq b([k + 1, n]) - j_2 \), and so \( N(u) = k + j_2 \leq n - 1 - t = j \); If \( j_1 < k \), then \( d - \beta_k \leq b([k + 1, n]) - j_2 \), and so \( N(u) \leq j_2 + k - 1 \leq n - 1 - t = j \). This completes the proof. \( \square \)

**Remark 6.3.** We now consider the general case in which we do not require the strict inequalities \( \alpha_i < \beta_i \) for \( i = 1, \ldots, k \). For this, let \( i_1 < \cdots < i_s \) be such that \( \{i: \ \alpha_i = \beta_i, 1 \leq i \leq k\} = \{i_1, \ldots, i_s\} \). We then observe that \( I \) can be written as the product of ideals \( I_1, \ldots, I_s \) and \( J \) which are given as follows. The ideal \( I_1 \) is generated by \( x_1^{u_1} \cdots x_i^{u_i} \), and \( \alpha_i \leq u_1 + \cdots + u_i \leq \beta_i \) for \( i = 1, \ldots, i_1 \).

If \( i_s = k \), that is, if \( \alpha_k = \beta_k \), then the ideal \( I \) is generated by \( x_1^{u_1} \cdots x_k^{u_k} \) with \( u_{k+1}, \ldots, u_n \) satisfying \( 0 \leq u_{k+1} \leq b_{k+1}, \ldots, 0 \leq u_n \leq b_n, u_{k+1} + \cdots + u_n = d - \beta_k \).

If \( i_s < k \), that is, if \( \alpha_k \neq \beta_k \), then the ideal \( J \) is generated by \( x_1^{u_1} \cdots x_n^{u_n} \) with
\[
u_{i_s+1} + \cdots + u_{k+1} + \cdots + u_n = d - \beta_{i_s}.
\]
Note that these ideals \( I_1, \ldots, I_s, J \) have pairwise disjoint supports.

For \( 1 \leq t \leq s \), let \( S_t \) be the polynomial ring in variables \( x_i \) with \( i \in \text{supp}(I_t) \), then \( \text{depth} S_t/(I_t \cap S_t) = 0 \) by Proposition 6.3.

Let \( a \in \mathbb{R} \). We denote by \( \lfloor a \rfloor \) the lower integer part and by \( \lceil a \rceil \) the upper integer part of \( a \).

**Theorem 6.4.** Let \( s = |\{i: \ \alpha_i = \beta_i, 1 \leq i \leq k\}|. \) Then
\[
\text{depth} S/I = s + \max\{0, d - \beta_k - b([k + 1, n]) + n - k - \lfloor \frac{\alpha_k}{\beta_k} \rfloor \}.
\]

**Proof.** Let \( R \) be the polynomial ring in variables \( x_{i_1+1}, \ldots, x_n \). By Remark 6.3 and Lemma 5.1 we have \( \text{depth} S/I = s + \text{depth} R/R \cap J \). If \( \alpha_k = \beta_k \), then \( \text{depth} R/R \cap J = \max\{0, d - \beta_k - b([k + 1, n]) + n - k - 1\} \) by [7, Corollary 4.7]. If \( \alpha_k < \beta_k \), then \( \text{depth} R/R \cap J = \max\{0, d - \beta_k - b([k + 1, n]) + n - k\} \) by Lemma 6.2. Hence our formula for the depth follows. \( \square \)

**Corollary 6.5.**
(a) If \( d = \beta_k + b([k + 1, n]) \), then \( \text{dystab}(I) = 1 \).
(b) Set \( \delta = \lfloor \frac{n-k-d}{\beta_k} \rfloor \). If \( d < \beta_k + b([k + 1, n]) \), then \( \text{dystab}(I) = \lceil \frac{n-k-d}{\beta_k + b([k + 1, n])} \rceil \).
Proof. (a) If \( d = \beta_k + b([k + 1, n]) \), then \( I = x_k^{b_k+1} \cdots x_n^{b_n} K \), where \( K \) is generated by \( x_1^{u_1} \cdot \cdots \cdot x_k^{u_k} \) such that \( u_1, \ldots, u_k \) satisfies \( \alpha \leq u_1 + \cdots + u_i \leq \beta_i \) for \( i = 1, \ldots, k-1 \) and \( u_1 + \cdots + u_k = \beta_k \). Let \( S_1 = k[x_1, \ldots, x_k] \) and \( S_2 = k[x_k+1, \ldots, x_n] \). By Lemma 5.1 and since \( \text{depth} S_2/(x_k^{b_k+1} \cdots x_n^{b_n}) = n - k - 1 \) for all \( m \), we have
\[
\text{depth} S/I^m = \text{depth} S_1/(S_1 \cap K)^m + n - k.
\]

Note that \( S_1 \cap K \) is a lattice path polymatroidal ideal. Therefore, by Proposition 4.6 we have \( \text{dstab}(S_1 \cap K) = 1 \), and so \( \text{dstab}(I) = 1 \).

(b) If \( d < \beta_k + b([k + 1, n]) \), then \( \text{depth} S/I^n = s \) for \( n \gg 0 \), by Theorem 6.4. It follows that \( \text{dstab}(I) \) is the smallest integer \( i \) such that \( i(d - \beta_k - b([k + 1, n])) + n - k - \delta \leq 0 \), which certainly is \( \lceil \frac{n-k-\delta}{\beta_k+b([k+1,n])-d} \rceil \).

We now want to determine \( \text{Ass}(S/I) \). By using a similar argument as in Lemma 5.4 one obtain

**Lemma 6.6.** Let \( P_A \in \text{Ass}(S/I) \). If there is \( 1 \leq i \leq k \) such that \( i \notin A \), then either \( A \cap [1, i-1] \cap A = \emptyset \) or \( A \cap [i+1, n] \cap A = \emptyset \).

As an immediate consequence we obtain

**Corollary 6.7.** Let \( P_A \in \text{Ass}(S/I) \). Then \( A \) is an interval contained in \([1, k]\), or \( A \) is a subset of \([k + 1, n]\), or \( A = [s, k] \cup B \) for some \( s \leq k \) and some subset \( B \) of \([k + 1, n]\).

**Proposition 6.8.** Set \( \beta_0 = 0 \). Suppose \( \alpha_i < \beta_i \) for \( i = 1, \ldots, k \). Given \( 1 \leq s \leq t \leq k \) and a subset \( B \) of \([k + 1, n]\), we have

(a) The following statements are equivalent:

1. \( P_{[s, t]} \in \text{Ass}(S/I) \)
2. \( \alpha_k \leq d - b([k + 1, n]) \leq \beta_k \), \( \beta_{s-1} < d - b([b + 1, n]) \), \( \beta_{s-1} < \beta_s, \beta_{s-1} < \alpha_t \) and \( \alpha_{t-1} < \alpha_t \).

(b) The following statements are equivalent:

1. \( P_B \in \text{Ass}(S/I) \),
2. \( b([k + 1, n] \setminus B) + \beta_k < d \leq \beta_k + b([k + 1, n]) - |B| \).

(c) The following statements are equivalent:

1. \( P_{[s, k] \cup B} \in \text{Ass}(S/I) \),
2. \( b([k + 1, n] \setminus B) + \beta_{s-1} + \max\{0, \alpha_k - \beta_{s-1}\} < d \leq \beta_k + b([k + 1, n]) - |B| \) and \( \beta_{s-1} < \beta_s \).

Moreover, \( \text{Ass}(S/I) \) consists exactly of these monomial prime ideals which satisfy the conditions described in (a), (b) or (c).

Proof. (a) Note that \( I([1, k]) \) is generated by monomials \( x^u \in S([1, k]) \) with \( u \) satisfying \( \alpha_1 \leq u_1 \leq \beta_1, \ldots, \alpha_k \leq u_1 + \cdots + u_k \leq \beta_k \), and \( u_1 + \cdots + u_k = d - b([k + 1, n]) \). Hence \( I([1, k]) \neq 0 \) if and only if \( \alpha_k \leq d - b \leq \beta_k \). If this is the case, then \( I([1, k]) \) is the lattice path polymatroidal ideal generated by monomials \( x^u \in S([1, k]) \) with \( u \) satisfying
\[
\alpha_1 \leq u_1 + \cdots + u_i \leq \min\{d - b([k + 1, n]), \beta_i\}
\]
for \(i = 1, \ldots, k - 1\), and

\[
u_1 + \cdots + u_k = d - b([k + 1, n]).
\]

Applying Proposition 4.6 to this case, the result follows.

(b) By using Proposition 4.5(b) repeatedly, we have \(I([k + 1, n])\) is generated by monomials \(x^u \in S([k + 1, n])\) with \(u\) satisfying \(0 \leq u_i \leq b_i\) for \(i = k + 1, \ldots, n\) and \(u_{k+1} + \cdots + u_n = d - \beta_k\). In a similar way as in Proposition 5.6, we see that if \(d - \beta_k - b([k + 1, n] \setminus B) \leq 0\) then \(P_B \notin \text{Ass}(S/I)\), moreover if \(d - \beta_k - b([k + 1, n] \setminus B) > 0\), then \(I(B)\) is generated by monomials \(x^u \in S(B)\) with \(u\) satisfying

\[
0 \leq u_i \leq b_i, \text{ for all } i \in B \quad \text{and} \quad \sum_{i \in B} u_i = d - \beta_k - b([k + 1, n] \setminus B).
\]

Applying Lemma 6.1 and Proposition 4.3(a), the result follows.

(c) Note that \(I([s, k] \cup B)\) is generated by monomials \(x^u \in S([s, k] \cup B)\) with \(u\) satisfying

\[
\max\{\alpha_i - \beta_{s-1}, 0\} \leq u_s + \cdots + u_i \leq \beta_i - \beta_{s-1} \quad \text{for } i = s, \ldots, k
\]

\[
0 \leq u_i \leq b_i \text{ for } i \in B \quad \text{and} \quad u_s + \cdots + u_k + \sum_{i \in B} u_i = d - \beta_{s-1} - b([k + 1, n] \setminus B).
\]

Hence \(I([s, k] \cup B) \neq 0\) if and only if \(d - \beta_{s-1} - b([k + 1, n] \setminus B) > \max\{0, \alpha_k - \beta_{s-1}\}\).

If this is the case, we let \(\hat{d} = d - \beta_{s-1} - b([k + 1, n] \setminus B)\). We describe \(I([s, k] \cup B)\) in the form as given in the beginning of this section: \(I([s, k] \cup B)\) is generated by monomials \(x^u \in S([s, k] \cup B)\) with \(u\) satisfying

\[
\max\{\alpha_i - \beta_{s-1}, 0\} \leq u_s + \cdots + u_i \leq \min\{\hat{d}, \beta_i - \beta_{s-1}\} \quad \text{for } i = s, \ldots, k
\]

\[
0 \leq u_i \leq b_i \text{ for } i \in B \quad \text{and} \quad u_s + \cdots + u_k + \sum_{i \in B} u_i \leq \hat{d}.
\]

Note that \(\max\{\alpha_i - \beta_{s-1}, 0\} < \min\{\hat{d}, \beta_i - \beta_{s-1}\}\) for \(i = s, \ldots, k\) if and only if \(\beta_{s-1} < \beta_s\). By Lemma 6.1 and Proposition 4.3(a), we have \(P_{[s, k] \cup B} \in \text{Ass}(S/I)\) if and only if \(\beta_{s-1} < \beta_s\) and \(\hat{d} \leq \min\{\hat{d}, \beta_k - \beta_{s-1}\} + b(B) - |B|\). One can check the last inequality is equivalent to \(d \leq \beta_k + b([k + 1, n]) - |B|\), as required.

Finally the last statement follows from Corollary 6.7.

By definition, \(\text{Ass}^\infty(S/I)\) consists of prime ideals \(P\) for which \(P \in \text{Ass}(S/I^m)\) for all \(m \gg 0\).

**Proposition 6.9.** Suppose \(\alpha_i < \beta_i\) for \(i = 1, \ldots, k\) and \(d < \beta_k + b([k + 1, n])\). Given \(1 \leq s \leq t \leq k\) and a subset \(B\) of \([k + 1, n]\), we have

\[
\text{(a) } P_{[s, t]} \in \text{Ass}^\infty(S/I) \iff P_{[s, t]} \in \text{Ass}(S/I).
\]

\[
\text{(b) } P_B \in \text{Ass}^\infty(S/I) \iff b([k + 1, n] \setminus B) + \beta_k < d.
\]

\[
\text{(c) } P_{[s, k] \cup B} \in \text{Ass}^\infty(S/I) \iff b([1, k] \setminus B) + \beta_{s-1} + \max\{0, \alpha_k - \beta_{s-1}\} < d, \quad \beta_{s-1} < \beta_s.
\]

Moreover, \(\text{Ass}^\infty(S/I)\) consists exactly of these monomial prime ideals which satisfy the conditions described in (a), (b) or (c).
Proof. By the assumption that \( d < \beta_k + b([k + 1, n]) \) we have \( md \leq m\beta_k + mb([k + 1, n]) - |B| \) for all \( m \gg 0 \). By this fact and in view of Proposition 6.8 the assertions (a), (b) and (c) follows. Finally the last statement follows from the last one of Proposition 6.8. \( \square \)

Corollary 6.10. Suppose \( \alpha_i < \beta_i, i = 1, \ldots, k \) and \( d < \beta_k + b([k + 1, n]) \). Then
\[
\astab(I) = \lfloor \frac{n - k}{\beta_k + b([k + 1, n]) - d} \rfloor.
\]

Proof. Set \( \Delta = \lfloor \frac{n - k}{\beta_k + b([k + 1, n]) - d} \rfloor \). In view of Proposition 6.9(c) and since \( \alpha_k < \beta_k \), we have \( P_{[s, n]} = P_{[s, k] \cup [k + 1, n]} \in \Ass^\infty(S/I) \). Here \( s \) is the smallest integer such that \( \beta_s \neq 0 \). Moreover, \( P_{[s, n]} \in \Ass(S/I^m) \) if and only if \( m \geq \Delta \) by Proposition 6.8(c). This implies \( \astab(I) = \Delta \).

It remains to be shown that \( P \in \Ass(S/I^m) \) for any \( m \geq \Delta \) and for any \( P \in \Ass^\infty(S/I) \). But this is clear by checking it for three classes of ideals in \( \Ass^\infty(S/I) \) given in Proposition 6.9 respectively. \( \square \)

We remark that if \( k = 1 \), then Corollary 6.10 is equivalent to [7, Corollary 4.6]. In the proof of our final result we will use the following fact: if \( I \) is a monomial ideal in \( S = k[x_1, \ldots, x_n] \), then \( \Ass(T/IT) = \{PT: P \in \Ass(S/I)\} \) for any polynomial ring \( T = k[x_1, \ldots, x_m] \) with \( m \geq n \). As before we set \( \delta = \lfloor \frac{a_k}{\beta_k} \rfloor \).

Theorem 6.11. (a) If \( d = \beta_k + b([k + 1, n]) \), then \( \astab(I) = 1 \).

(b) If \( d < \beta_k + b([k + 1, n]) \), then \( \astab(I) = \lfloor \frac{n - k}{\beta_k + b([k + 1, n]) - d} \rfloor \).

Proof. (a) As in the proof of Theorem 6.4 \( I = x_{k+1}^{b_{k+1}} \cdots x_{n}^{b_{n}} J \), where \( J \) is a lattice path polymatroidal ideal. By Proposition 4.6 we have \( \astab(J) = 1 \) and so \( \astab(I) = 1 \).

(b) Let \( I_1, \ldots, I_s, J \) and \( S_1, \ldots, S_s, R \) be as in Remark 6.3. Since \( S_i \cap I_i \) is a lattice path polymatroidal ideal, we have \( \astab(S_i \cap I_i) = 1 \) for \( i = 1, \ldots, s \). It follows that \( \astab(I) = \astab(J \cap R) \) by Lemma 5.1(b). There are two cases to consider.

In the case when \( \alpha_k = \beta_k \) we have \( \astab(R \cap J) = \lfloor \frac{n - k}{\beta_k + b([k + 1, n]) - d} \rfloor \) by [7, Corollary 4.6], and in the case when \( \alpha_k < \beta_k \) we have \( \astab(R \cap J) = \lfloor \frac{n - k}{\beta_k + b([k + 1, n]) - d} \rfloor \) by Corollary 6.10. This yields the desired formula. \( \square \)

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