The spectrum and the phase transition of models solvable through the full interval method

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Abstract. The most general exclusion single species reaction–diffusion models with nearest-neighbor interactions on a one-dimensional lattice are investigated, for which the evolution of full intervals are closed. Using a generating function method, the probability that \( n \) consecutive sites be full is investigated. The stationary values of these probabilities, as well as the spectrum of the time translation generator (Hamiltonian) of these are obtained. It is shown that, depending on the reaction rates, the model could exhibit a dynamical phase transition.

Keywords: classical phase transitions (theory), exact results
1. Introduction

Different methods have been used to study non-equilibrium statistical models, including analytical and asymptotic methods, mean field methods, and large-scale numerical methods. Although mean field solutions may be suitable for higher dimensions, in low-dimensional cases fluctuation effects may cause mean field results to differ from the real ones. One-dimensional models, which are usually easier to investigate, help us gain more knowledge on systems far from equilibrium [1]–[13]. Many techniques are used to obtain exact results in one-dimensional models. Among them is the empty interval method (EIM), or its equivalent the full interval method (FIM). Using the EIM, the aim is to calculate the probability that $n$ consecutive sites be empty, $E_n$, (or alternatively the probability that $n$ consecutive sites be full, $F_n$). This technique has been used to analyze the one-dimensional dynamics of diffusion-limited coalescence [14]–[17]. There some of the reaction rates have been taken infinite and the EIM has been used to analyze the models. These models have been worked out on a continuum and some approximate solutions have been obtained. This method has been used to study a reaction–diffusion process with three-site interactions [18]. The EIM has been also generalized to study the kinetics of the $q$-state one-dimensional Potts model in the zero-temperature limit [19]. In [20], all the one-dimensional reaction–diffusion models with nearest-neighbor interactions that can be exactly solved by the EIM were found and studied. Conditions were obtained for systems with finite reaction rates to be solvable via the EIM, and then the equations of the EIM were solved. There, solvability means that the evolution equation for $E_n$ be closed, and it turned out that certain relations between the reaction rates are needed, so that the system is solvable via the EIM. When these conditions between reaction rates are met, the time derivative of $E_n$s would be linear combinations of $E_n$s. It was shown that if certain reactions are absent, namely reactions that produce particles in two adjacent empty sites, the coefficients of the empty intervals in the evolution equation of the empty intervals are $n$-independent, so that the evolution equation can be easily solved. The criteria for solvability and the solution of the empty interval equation were generalized to cases of multi-species systems and multi-site interactions in [21]–[23]. In [24], models were studied which were solvable through the EIM, but did include interaction which produce particles.
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in two adjacent empty sites. There, these models were investigated in continuum, although
some terms in the evolution equation were missed, as will be discussed in the present paper.
In [25], the conventional EIM has been extended to a more generalized form. Using this
extended version, a model has been studied which cannot be solved by the conventional
EIM.

Recently, the coagulation–diffusion process on a one-dimensional chain has been
studied using the empty interval method [26]. There, the behavior of the time-dependent
double-empty-interval probability has been studied. In [27], the exact two-time correlation
and response functions for a one-dimensional coagulation–diffusion process have been
studied using the EIM.

In a recent article, the most general exclusion single species one-dimensional
reaction–diffusion models with nearest-neighbor interactions were studied, which are both
autonomous and can be solved exactly through the full interval method [28]. There, using a
generating function method, the general solution for $F_n$, the probability that $n$ consecutive
sites be full, and some other correlation functions of number operators were explicitly
obtained.

In this paper we relax the condition of autonomy and study the most general exclusion
single species reaction–diffusion models with nearest-neighbor interactions on a one-
dimensional lattice, which can be solved exactly through the full interval method. The
change of empty interval to full interval is, of course, not important, as a simple interchange
of particles and holes would do that. The scheme of the paper is as follows. In section 2,
the most general exclusion single species one-dimensional reaction–diffusion models with
nearest-neighbor interactions are introduced, which can be solved exactly through the
FIM. In section 3, a generating function method is used to obtain the time-independent
solutions to the evolution of full intervals. In section 4, a similar generating function
method is used to study the spectrum of the time translation generator (Hamiltonian)
of the full intervals and it is shown that the model may exhibit a dynamical phase
transition.

2. Full interval equation

Consider a one-dimensional lattice, any site of which is either occupied by a single particle
or empty. Defining the full interval $F_n$ as the probability that $n$ consecutive sites be full

$$F_n := P(\circ \cdots \circ)$$

where an empty (occupied) site is denoted by $\circ$ ($\bullet$), it is found (for example, similar to [24]),
that the most general nearest-neighbor interactions for which the evolution equations
governing $F_n$'s are closed are

\[
\begin{align*}
\circ \bullet & \rightarrow \begin{cases} & \circ \circ, \quad q_1, \\
& \bullet \circ, \quad r_1, \end{cases} \\
\bullet \circ & \rightarrow \begin{cases} & \circ \circ, \quad q_2, \\
& \circ \bullet, \quad r_2, \end{cases}
\end{align*}
\]
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\[
\begin{align*}
\circ \to & \begin{cases} 
\circ, & r_1 \\
\circ, & r_2,
\end{cases} \\
\cd \to & \begin{cases} 
\circ, & w_1 \\
\circ, & w_2 \\
\circ \to & w.
\end{cases}
\end{align*}
\]

The equality of some rates means that the rates of the reactions \((\circ \to \cd)\) and \((\cd \to \circ)\) are the same and the rates of the reactions \((\circ \to \circ)\) and \((\cd \to \cd)\) are the same as well.

As in, for example, [24] it can be seen that the time evolution equation for \(F_n\) is

\[
\frac{dF_n}{dt} = (r_1 + r_2) (F_{n-1} + F_{n+1} - 2 F_n) - (q_1 + q_2) (F_n - F_{n+1}) \\
- (n - 1) (w_1 + w_2 + w) F_n - (w_1 + w_2 + 2 w) F_{n+1}, \quad n \geq 2
\]

(2)

It is seen that defining

\[F_0 := 1,\]

the evolution equation for \(F_1\) takes a form similar to that of other \(F_n\)s:

\[
\frac{dF_1}{dt} = (r_1 + r_2) (1 + F_2 - 2 F_1) - (q_1 + q_2) (F_1 - F_2) \\
- (w_1 + w_2 + 2 w) F_2.
\]

(3)

The difference of this from what was obtained in [24], apart from the obvious interchange of particles and vacancies, is that the last term was missing in [24]. In [28], it is assumed the models are autonomous, which leads to removing the term \(F_{n+1}\) from (5).

Rescaling the time by \((w_1 + w_2 + w)\):

\[\hat{t} := (w_1 + w_2 + w) t,\]

the evolution equation becomes

\[
\frac{dF_n}{dt} = b F_{n-1} - (a + n - 1) F_n + c F_{n+1},
\]

(4)

where

\[
\begin{align*}
a & := 2 (r_1 + r_2) + q_1 + q_2, \\
b & := \frac{w + w_1 + w_2}{r_1 + r_2}, \\
c & := \frac{r_1 + r_2 + q_1 + q_2 - 2 w - w_1 - w_2}{w + w_1 + w_2}.
\end{align*}
\]

(5)

The case \((w_1 + w_2 + w) = 0\) will be dealt with separately. From now on, the rescaled time \(\hat{t}\) is denoted by \(t\), so that the evolution equation is written as

\[
\frac{dF_n}{dt} = b F_{n-1} - (a + n - 1) F_n + c F_{n+1}.
\]

(6)

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A generating function $F$ is also defined, which will be used later:

$$F(x) := \sum_{n=0}^{\infty} \frac{F_n x^n}{n!}. \quad (10)$$

It is seen that

$$F_n = F^{(n)}(0), \quad (11)$$

where $F^{(n)}$ is the $n$th derivative of $F$.

### 3. The time-independent equation

The time-independent solution $F^\text{st}$ satisfies

$$b F^\text{st}_{n-1} - (a + n - 1) F^\text{st}_n + c F^\text{st}_{n+1} = 0, \quad (12)$$

which yields

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} [c F^\text{st}_{n+1} + b F^\text{st}_{n-1} - (a + n - 1) F^\text{st}_n] = 0, \quad (13)$$

so that

$$(c - x) \frac{d^2 F^\text{st}}{dx^2} - a \frac{d F^\text{st}}{dx} + b F^\text{st} = 0. \quad (14)$$

Defining

$$F^\text{st}(x) := z^{(1-a)} G^\text{st}(z), \quad z := 2 \sqrt{b(c-x)}, \quad (15)$$

one arrives at

$$z^2 \frac{d^2 G^\text{st}}{dz^2} + z \frac{d G^\text{st}}{dz} + [z^2 - (1-a)^2] G^\text{st} = 0. \quad (16)$$

So,

$$G^\text{st}(z) = \alpha J_{a-1}(z) + \beta J_{1-a}(z), \quad (17)$$

where $J_{\nu}$ is the Bessel function of order $\nu$, and $\alpha$ and $\beta$ are constants. So,

$$F^\text{st}(x) = \alpha \left[ 2 \sqrt{b(c-x)} \right]^{(1-a)} J_{a-1} \left[ 2 \sqrt{b(c-x)} \right]$$

$$+ \beta \left[ 2 \sqrt{b(c-x)} \right]^{(1-a)} J_{1-a} \left[ 2 \sqrt{b(c-x)} \right]. \quad (18)$$

As $F^\text{st}_n$ is in $[0,1]$, the convergence radius of the series defining $F^\text{st}$ is infinity. So the generating function $F^\text{st}$ as a function of $x$ is analytic on the entire complex plane. Using

$$J_{\nu}(z) = \frac{1}{\Gamma(\nu+1)} \left( \frac{z}{2} \right)^\nu + \cdots, \quad |z| \ll 1, \quad (19)$$

the analyticity of $F^\text{st}$ at $(x = c)$ demands that

$$\beta = 0. \quad (20)$$

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As
\[ F_{st}^0 = 1, \]  
(21)

one has
\[ F_{st}^0(0) = 1, \]  
(22)

which can be exploited to obtain \( \alpha \). So,
\[ F_{st}(x) = \left[ 2 \sqrt{b(c-x)} \right]^{(1-a)} J_{a-1} \left( 2 \sqrt{b(c-x)} \right) \]  
(23)

Using (11), one can find \( F_n \). One has
\[ \left( \frac{d}{dx} \right)^n F_{st}(x) = \frac{1}{(2 \sqrt{bc})^{1-a} J_{a-1}(2 \sqrt{bc})} \left( \frac{2b}{z} \frac{d}{dz} \right)^n [z^{1-a} J_{a-1}(z)], \]  
(24)

which results in
\[ F_{st}^n = \left( \frac{b}{c} \right)^{n/2} \frac{J_{a-1+n}(2 \sqrt{bc})}{J_{a-1}(2 \sqrt{bc})}, \]  
(25)

There are two limiting cases to be studied separately:

(i) \( b = 0 \).

This can be studied as the limit \( b \to 0 \) of the general case. Using (25) and the limiting behavior (19), one arrives at
\[ F_{st}^n = \frac{\Gamma(a)}{\Gamma(a+n)} b^n, \quad b \ll 1, \]  
(26)

which leads to
\[ \lim_{b \to 0} F_{st}^n = \delta_n^0. \]  
(27)

This is the case if \( a \) is nonzero. If \( a \) and \( b \) both vanish, the result could be obtained directly from (12) to be
\[ F_{st}^n = \begin{cases} \rho, & n = 1 \\ 0, & n > 1 \end{cases}, \]  
(28)

where
\[ 0 \leq \rho \leq \frac{1}{2}, \]  
(29)

and it has been assumed that at least one of the rates is nonvanishing. One notes that \( \rho \) is in fact the density of the particles, and the restriction on its value results from the fact that in the stationary configuration no two adjacent sites are full.

These results are expected. If \( b \) vanishes but \( a \) does not, there are no reactions which produce particles but there are reactions which annihilate particles, whether the particles are adjacent to holes or other particles. So at large time the lattice becomes empty. If both \( a \) and \( b \) vanish, there are no reactions which produce particles, but there are reactions
which annihilate particles, only if there are two neighboring particles. So particles will be annihilated, but when there are particles with empty neighboring sites, they will survive.

In fact, regarding the stationary solution as the large-time solution, one can obtain an expression for $\rho$ in terms of the initial conditions. As both $a$ and $b$ vanish, one has

$$\frac{dF_n}{dt} = -(n - 1) F_n + c F_{n+1}. \quad (30)$$

Defining

$$\mathcal{F}_n(t) := \exp[(n - 1) t] F_n(t), \quad (31)$$

equation (30) is recast to

$$\frac{d\mathcal{F}_n}{dt} = c \exp(-t) \mathcal{F}_{n+1}, \quad (32)$$

which results in

$$\mathcal{F}_n(t) = \sum_{k=0}^{\infty} \frac{[c - c \exp(-t)]^k}{k!} \mathcal{F}_{n+k}(0), \quad (33)$$

so that

$$F_n(t) = \exp[-(n - 1) t] \sum_{k=0}^{\infty} \frac{[c - c \exp(-t)]^k}{k!} F_{n+k}(0), \quad (34)$$

from which,

$$F_1(\infty) = \sum_{k=0}^{\infty} \frac{c^k}{k!} F_{1+k}(0), \quad (35)$$

or

$$\rho = \sum_{k=0}^{\infty} \frac{c^k}{k!} F_{1+k}(0). \quad (36)$$

One notes that when $a$ and $b$ both vanish, $c$ is nonpositive.

(ii): $w_1 + w_2 + w = 0$.

In this case all of the rates $w_1$, $w_2$, and $w$ should vanish. One arrives at

$$(r_1 + r_2) F_{n-1}^{\text{st}} - (2 r_1 + 2 r_2 + q_1 + q_2) F_n^{\text{st}} + (r_1 + r_2 + q_1 + q_2) F_{n+1}^{\text{st}} = 0, \quad (37)$$

$$F_0^{\text{st}} = 1.$$ 

The solution to (37) is

$$F_n^{\text{st}} = \zeta + (1 - \zeta) \left( \frac{r_1 + r_2}{r_1 + r_2 + q_1 + q_2} \right)^n, \quad (38)$$

where

$$0 \leq \zeta \leq 1, \quad (39)$$

provided that of $q_1$ and $q_2$, at least one is nonvanishing. Otherwise

$$F_n^{\text{st}} = 1, \quad (40)$$

which corresponds to a full lattice.
4. Relaxation towards the time-independent equation

To study the spectrum of the time translation generator (Hamiltonian) of the full intervals, again the generating function is used. Defining

\[ F^\text{dy}_n := F_n - F^\text{st}_n, \]  

(41)

one arrives at

\[ \frac{dF^\text{dy}_n}{dt} = b F^\text{dy}_{n-1} - (a + n - 1) F^\text{dy}_n + c F^\text{dy}_{n+1}, \]  

(42)

with the boundary condition

\[ F^\text{dy}_0 = 0. \]  

(43)

Equation (42) is of the form

\[ \frac{dF^\text{dy}_n}{dt} = (h F^\text{dy})_n, \]  

(44)

where \( h \) is a linear operator. To find this relaxation time, one should obtain the eigenvalues of \( h \). The eigenvalue with the largest real part determines the relaxation time. Denoting the eigenvector of \( h \) corresponding to the eigenvalue \( E \) by \( \psi_E \), one has

\[ E \psi_E(n) = b \psi_E(n-1) - (a + n - 1) \psi_E(n) + c \psi_E(n+1), \]  

(45)

where \( E \) is the corresponding eigenvalue. This is similar to (12), with \( a \) replaced by \((a + E)\).

So repeating similar arguments, one arrives at

\[ \psi_E(x) = \alpha (c - x)^{(1-a-E)/2} J_{a+E-1} \left[ 2\sqrt{b(c - x)} \right]. \]  

(46)

But now the boundary condition is

\[ \psi_E(0) = 0, \]  

(47)

which results in

\[ J_{a+E-1}(2\sqrt{bc}) = 0. \]  

(48)

This gives the spectrum of \( h \).

In the case \( b = 0 \), one can find more explicit forms for the eigenvalues and eigenvectors. Starting from (45), one arrives at

\[ \psi_{E_{n+1}} = \frac{E + a + n - 1}{c} \psi_{E_n}. \]  

(49)

This recursive relations shows that \( \psi_{E_n} \) tends to infinity as \( n \) tends to infinity, unless there is a \( k \) so that \( \psi_{E_n}s \) vanish for \( n > k \). This happens if \( E \) is equal to one of the \( E_k \)s, where

\[ E_k = 1 - a - k, \]  

(50)

and \( k \) is a positive integer. Denoting the corresponding eigenvector by \( \psi_k \) instead of \( \psi_E \), one would arrive at

\[ \psi_{kn} = \frac{(-c)^{k-n}}{\Gamma(k + 1 - n)} \psi_{kk}. \]  

(51)

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The relaxation time of the system is obtained from the largest real part of the eigenvalues, which in this case is \((-a\)). So

\[ \tau = \frac{1}{a}, \quad (bc) = 0. \]  \(52\)

Defining

\[ \varepsilon := E + a - 1, \]  \(53\)

it is seen from \((48)\) that \(\varepsilon\) depends on only the product \((bc)\). So the expression \((50)\) holds for the case \(c = 0\) as well. The case \(c = 0\) has already been discussed in greater detail in \([28]\).

From \((8)\) is it seen that \(b\) is nonnegative, while \(c\) can change sign. These expressions also show that

\[ -1 \leq (bc), \]  \(54\)

but there is no upper limit for \((bc)\).

For \((bc)\) near zero, one can find the leading correction to \((51)\) as follows. Defining

\[ \delta_k := \varepsilon_k + k, \]  \(55\)

equation \((48)\) becomes

\[ J_{-k+\delta_k} (2 \sqrt{bc}) = 0, \]  \(56\)

which is, up to the leading order, equivalent to

\[ 1 - \frac{(bc)^k}{k! (k-1)!} \delta_k = 0, \]  \(57\)

showing that

\[ \delta_k = \frac{(bc)^k}{k! (k-1)!} + \cdots, \quad |bc| \ll k^2. \]  \(58\)

So one has,

\[ \tau = \frac{1}{a - bc} + \cdots, \quad |bc| \ll 1. \]  \(59\)

If \((bc)\) is positive, there is an inner product with respect to which \(h\) is Hermitian. So if \((bc)\) is positive, the spectrum of \(h\) is real. Increasing \((bc)\) from zero, the values of the \(\varepsilon_k\)s are also increased. For large values of \((bc)\) one can find the values of the \(\varepsilon_k\)s using various asymptotic expressions of the Bessel functions. One arrives at

\[ \varepsilon_k = \begin{cases} 
2 \sqrt{bc} - a_k (bc)^{1/6} + \cdots, & k \ll (bc)^{1/3} \\
\frac{4 \sqrt{bc}}{\pi} + \frac{3}{2} - 2k + \cdots, & |\varepsilon_k| \ll (bc)^{1/2} \\
-k + \frac{(bc)^k}{k! (k-1)!} + \cdots, & k \gg (bc)^{1/2}, 
\end{cases} \]  \(60\)
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Figure 1. The plot of $(b c)$ versus $\varepsilon$, corresponding to the four largest eigenvalues of $h$.

where $(-a_k)$ is the $k$th zero of the Airy function. The largest of the $\varepsilon_k$s determines the relaxation time. One then has

$$
\tau = \frac{1}{a - 1 + 2 \sqrt{bc}} + \cdots, \quad (b c) \gg 1.
$$

(61)

From (8) is it seen that, for nonnegative $c$,

$$(a - 1) \geq 2 \sqrt{bc}.
$$

(62)

For a negative $(b c)$, however, there are cases where the spectrum of $h$ is not real. A plot of $(b c)$ in terms of (real) $\varepsilon$ for

$$J_c(2 \sqrt{bc}) = 0,$
$$

(63)

shows that there is one minimum for $(b c)$ for each interval $\varepsilon \in (-2n, -2n+1)$, where $n$ is a positive integer, figure 1. However, only the minimum corresponding to $\varepsilon \in (-2, -1)$ is larger than $(-1)$. So there is a critical value for $(b c)$, at which two of the eigenvalues of $h$ (the largest and the next largest) become equal, and if $(b c)$ is less than that critical value, two of the eigenvalues of $h$ become nonreal (complex conjugates of each other). Denoting that critical value of $(b c)$ by $\gamma$, and the corresponding value of $\varepsilon$ by $\varepsilon_{\text{tr}}$, one arrives at the following approximate expression for $\varepsilon$ and $(b c)$ near their critical value.

$$
(b c) = \gamma + \frac{(\varepsilon - \varepsilon_c)^2}{\nu},
$$

(64)

where the numerical values of the constants in the above equation are

$$
\gamma = -0.401 873, \quad \nu = 0.754 464, \quad \varepsilon_c = -1.697 524.
$$

(65)
One then arrives at the following expression for $\varepsilon_1$, for $(b\,c)$ near the critical value $\gamma$:

$$\varepsilon_1 = \varepsilon_c + \sqrt{\nu (b\,c - \gamma)},$$

resulting in

$$\tau = \begin{cases} 
\frac{1}{a - 1 - \varepsilon_c - \sqrt{\nu (b\,c - \gamma)}}, & (b\,c) \gtrsim \gamma \\
\frac{1}{a - 1 - \varepsilon_c}, & (b\,c) \lesssim \gamma.
\end{cases}$$

(67)

So the derivative of the relaxation time with respect to $(b\,c)$ is infinite for $(b\,c) \to \gamma^+$ and zero for $(b\,c) \to \gamma^-$. This model shows a dynamical phase transition.

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References

[1] Schütz G M, Exactly solvable models for many-body systems far from equilibrium, 2000 Phase Transitions and Critical Phenomena vol 19, ed C Domb and J Lebowitz (London: Academic)
[2] Henkel M, Hinrichsen H and Lübeck S, 2008 Non-equilibrium Phase Transitions (Absorbing Phase Transitions vol 1) (Berlin: Springer)
[3] Henkel M and Pleimling M, 2010 Non-equilibrium Phase Transitions (Ageing and Dynamical Scaling Far from Equilibrium vol 2) (Berlin: Springer)
[4] Alcaraz F C, Droz M, Henkel M and Rittenberg V, 1994 Ann. Phys., NY 230 250
[5] Krebs K, Pfannmuller M P, Wehefritz B and Hinrichsen H, 1995 J. Stat. Phys. 78 1429
[6] Simon H, 1995 J. Phys. A: Math. Gen. 28 6585
[7] Privman V, Cadilhe A M R and Glasser M L, 1995 J. Stat. Phys. 81 881
[8] Henkel M, Orlandini E and Schütz G M, 1995 J. Phys. A: Math. Gen. 28 6335
[9] Henkel M, Orlandini E and Santos J, 1997 Ann. Phys. 259 163
[10] Lushnikov A A, 1986 Zh. Eksp. Teor. Fiz. 91 1376

Lushnikov A A, 1986 Sov. Phys.—JETP 64 811 (Engl. transl.)
[11] Alimohammadi M, Karimipour V and Khorrami M, 1998 Phys. Rev. E 57 6370
[12] Alimohammadi M, Karimipour V and Khorrami M, 1999 J. Stat. Phys. 97 373
[13] Aghamohammadi A and Khorrami M, 2000 J. Phys. A: Math. Gen. 33 7843
[14] Burschka M A, Doering C R and Ben-Avraham D, 1989 Phys. Rev. Lett. 63 700
[15] Ben-Avraham D, 1995 Mod. Phys. Lett. B 9 895
[16] Ben-Avraham D, 1997 Nonequilibrium Statistical Mechanics in One Dimension (Cambridge: Cambridge University press) pp 29–50
[17] Ben-Avraham D, 1998 Phys. Rev. Lett. 81 4756
[18] Henkel M and Hinrichsen H, 2001 J. Phys. A: Math. Gen. 34 1561–8
[19] Masser T and Ben-Avraham D, 2000 Phys. Lett. A 275 382
[20] Alimohammadi M, Khorrami M and Aghamohammadi A, 2001 Phys. Rev. E 64 056116
[21] Khorrami M, Aghamohammadi A and Alimohammadi M, 2003 J. Phys. A: Math. Gen. 36 345
[22] Aghamohammadi A, Alimohammadi M and Khorrami M, 2003 Eur. Phys. J. B 31 371
[23] Aghamohammadi A and Khorrami M, 2004 Int. J. Mod. Phys. B 18 2047
[24] Aghamohammadi A and Khorrami M, 2005 Eur. Phys. J. B 47 583
[25] Mobilia M and Bares P A, 2001 Phys. Rev. E 64 066123
[26] Durang X, Fortin J-Y, Del Biondo D, Henkel M and Richert J, 2010 J. Stat. Mech. P04002
[27] Durang X, Fortin J-Y and Henkel M, 2011 J. Stat. Mech. P02030
[28] Khorrami M and Aghamohammadi A, 2012 Eur. Phys. J. B 85 134

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