Heuristic Rating Estimation Method for the incomplete pairwise comparisons matrices

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Abstract

The Heuristic Rating Estimation Method enables decision-makers to decide based on existing ranking data and expert comparisons. In this approach, the ranking values of selected alternatives are known in advance, while these values have to be calculated for the remaining ones. Their calculation can be performed using either an additive or a multiplicative method. Both methods assumed that the pairwise comparison sets involved in the computation were complete.

In this paper, we show how these algorithms can be extended so that the experts do not need to compare all alternatives pairwise. Thanks to the shortening of the work of experts, the presented, improved methods will reduce the costs of the decision-making procedure and facilitate and shorten the stage of collecting decision-making data.

Keywords: decision-making methods, MCDM, pairwise comparisons, HRE, AHP, Heuristic rating estimation

1. Introduction

Decision making based on pairwise comparisons of alternatives has a long history. Although the first well-documented use of the pairwise comparison method is attributed to Llull \cite{12}, we may suspect that the idea itself is much older. The ancient Egyptian Osiris judgment ceremony \cite{10}, whose essence was to compare the deeds of the deceased (symbolized by the heart) against the appropriate values (symbolized by an ostrich feather - the sign of the goddess Maat) on a scale, may be a clue. Llull and his successors such as Cusanus, Condorcet and Copeland \cite{13, 2, 51} used qualitative pairwise comparisons. This means that they did not specify the strength of preferences, but only their nature:

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who is the winner and who is the loser. Quantitative use of pairwise comparisons can be found in twentieth-century researcher Thurstone who dealt with the quantitative description of the social values [37]. Perhaps the best-known example of the use of quantitative pairwise comparison for decision making is the Analytic Hierarchy Process (AHP) method proposed by Saaty [36] in the second half of the twenty century. In addition to AHP, the idea of comparing alternatives in pairs can be found in many other decision-making methods including BWM [33], ELECTRE, PROMETHEE [16], MACBETH [3] and techniques such as multiobjective optimization [17] or multiple criteria sorting [19]. Despite its long history the pairwise-comparisons method is still the subject of research and development. Starting from considering the original AHP method [14, 4, 20] through examining the inconsistency of a set of comparisons [8, 2, 30, 25], uncertainty handling [32, 5, 33] incomplete PC matrix processing [26, 4, 11, 12] and others. A broader overview of the topics related to the pairwise comparison method and AHP can be found in the books [27, 34].

The Heuristic Rating Estimation (HRE) method is one of the decision making techniques based on comparing alternatives in pairs [21]. At the core of HRE there is an assumption that the priorities of some (referential) alternatives are initially known. Hence, priorities need to be calculated only for non-referential alternatives with previously unknown ranking values. Initially, we defined HRE assuming that the priority of a single alternative should correspond to the weighted arithmetic mean of the other alternatives [21, 23, 24]. This assumption makes the HRE similar to the Eigenvalue method (EVM) of calculating the priority in AHP [29]. Later we defined the geometric HRE [28]. Both arithmetic and geometric HRE, however, assume the existence of all pairwise comparisons, i.e. between any two of the alternatives considered. As with AHP, this can make it difficult to collect the data needed to calculate the ranking [18]. Therefore, in this paper, we propose procedures to calculate the ranking for an incomplete set of comparisons. Thanks to this solution, for a certain number of pairs of alternatives, we do not have to compare to get a ranking. This solution reduces the working time of experts, speeds up the decision-making process and lowers its cost.

2. Preliminaries

2.1. Pairwise comparisons

The pairwise comparisons (PC) method is the process designed to transform the set of comparisons into a ranking of alternatives. Let \( A = \{a_1, \ldots, a_n\} \) denote a set of alternatives while \( C = [c_{ij}] \) means set of comparisons in the form of \( n \times n \) matrix, where \( c_{ij} \in \mathbb{R}_+ \) for \( i, j = 1, \ldots, n \). Each \( c_{ij} \) means the result of direct comparison between \( a_i \) and \( a_j \). When the result of the given comparison \( c_{ij} \) is unknown we write that \( c_{ij} = ? \).

**Definition 1.** A PC matrix \( C \) is said to be reciprocal if \( c_{ij} = 1/c_{ji} \) for all \( i, j = 1, \ldots, n \) except when \( a_{ij} = ? \).
The purpose of PC method is to calculate the ranking.

**Definition 2.** Let the weight function for $A$ be $w : A \rightarrow \mathbb{R}_+$, such that if $w(a_i) > w(a_j)$ for some $0 < i, j \leq n$ then $a_i$ is more preferred than $a_j$.

The above function prioritizes the various alternatives. The more preferred alternatives have higher weights. The function $w$ is usually represented as a priority vector in the form:

$$ w = (w(a_1), w(a_2), \ldots, w(a_n))^T. $$

The fact that $c_{ij}$ corresponds to the ratio of the preferential strength of alternatives $a_i$ and $a_j$ implies that one may expect that $c_{ij} = w(a_i)/w(a_j)$. This in turn results in a postulate of transitivity $\text{i.e.} \ c_{ij} = c_{ik}c_{kj}$ for every triad $i, j, k = 1, \ldots, n$. Unfortunately, because the vector $w$ is computed based on all pairwise comparisons, in practice we may expect only that $c_{ij} \approx w(a_i)/w(a_j)$, thus $c_{ij} \approx c_{ik}c_{kj}$.

**Definition 3.** A $n \times n$ PC matrix $C$ is said to be inconsistent if there exist $i, j, k = 1, \ldots, n$ such that $c_{ij} \neq c_{ik}c_{kj}$.

It is easy to prove that if $c_{ij} = c_{ik}c_{kj}$ then $c_{ij} = w(a_i)/w(a_j)$ [27, p. 96] regardless of the prioritization method providing of course that $c_{ij}$ means the ratio between preferential strength of $a_i$ and $a_j$.

There are at least a dozen methods for computing a vector $w$ [11, 27]. The two most popular are the Eigenvalue Method (EVM) and Geometric Mean Method (GMM). The first one was originally proposed by Saaty in his seminal paper [36] is based on the concept of a vector and the eigenvalue of the matrix $C$. So, let $C = [c_{ij}]$ be a PC matrix containing expert judgments for $n$ alternatives, and $w_{\text{max}}$ be a principal eigenvector of $C$ i.e.

$$ Cw_{\text{max}} = \lambda_{\text{max}}w_{\text{max}}, $$

where $\lambda_{\text{max}}$ is a principal eigenvector (spectral radius) of $C$. Thus, the priority vector $w_{\text{ev}}$ [11] is a rescaled version of $w_{\text{max}}$ i.e.

$$ w_{\text{ev}} = \frac{1}{\sum_{i=1}^{n} w_{\text{max}}(a_i)}w_{\text{max}}. $$

The second method, although it is based on similar premises [29], is easier to calculate. In GMM the priority of the $i$-th alternative is the appropriately rescaled geometric mean of all its direct comparisons. I.e.

$$ w_{\text{gm}}(a_i) = \alpha \left( \prod_{j=1}^{n} c_{ij} \right)^{1/n}, $$

\footnote{As $c_{ik} = w(a_i)/w(a_k)$ and $c_{kj} = w(a_k)/w(a_j)$.}
Both methods EVM and GMM have their incomplete counterparts. Extending the EVM to incomplete matrices was proposed by Harker [18]. Similar extension of GMM for incomplete PC matrices can be found in [26] or equivalently for Logarithmic Least-Squares method in [4, 38].

2.2. Heuristic Rating Estimation Method

The main assumption of the HRE method is the existence of two subsets of alternatives. The first \( A_U \), consisting of alternatives with unknown preferential values and the second \( A_K \) consisting of reference alternatives for which the preferences are known. We assume, without loss of generality, that \( A_U = \{ a_1, \ldots, a_k \} \) and \( A_K = \{ a_{k+1}, \ldots, a_n \} \). Thus, the set of alternatives is given as \( A = A_U \cup A_K \) where \( A_U \cap A_K = \emptyset \). For \( a_\tau \in A_K \) the ranking value \( w(a_\tau) \in \mathbb{R}_+ \) is known and fixed at the beginning of the decision process. It follows that also comparisons \( c_{ij} \) where \( i, j = k+1, \ldots, n \) are known and fixed. Thus, the aim of the priority deriving procedure in the HRE method is to calculate \( w(a_1), \ldots, w(a_k) \) based on the pairwise comparisons matrix and the already known priorities of the reference alternatives.

As for AHP, two priority deriving methods were defined: arithmetic [22, 23] and the geometric one [28]. To calculate the ranking in the arithmetic HRE we need to solve the equation:

\[
Aw = b,
\]

where

\[
A = \begin{bmatrix}
1 & -\frac{1}{n-1}c_{21} & \cdots & -\frac{1}{n-1}c_{1k} \\
-\frac{1}{n-1}c_{21} & 1 & \cdots & -\frac{1}{n-1}c_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n-1}c_{k1} & \cdots & -\frac{1}{n-1}c_{k,k-1} & 1
\end{bmatrix}
\]

is \( k \times k \) auxiliary matrix and

\[
b = \begin{bmatrix}
\frac{1}{n-1}\sum_{j=k+1}^{n} c_{1j}w(a_j) \\
\frac{1}{n-1}\sum_{j=k+1}^{n} c_{2j}w(a_j) \\
\frac{1}{n-1}\sum_{j=k+1}^{n} c_{kj}w(a_j)
\end{bmatrix}
\]

is a constant term vector. In the case of a geometric approach the equation

\[
\hat{A}\hat{w} = \hat{b}
\]

is solved, where
\[
\hat{A} = \begin{bmatrix}
(n-1) & -1 & \cdots & -1 \\
-1 & (n-1) & \cdots & -1 \\
-1 & -1 & \ddots & -1 \\
-1 & -1 & \cdots & (n-1)
\end{bmatrix}
\]

and
\[
\hat{b} = \begin{bmatrix}
(n-1)\hat{w}(a_1) - \sum_{i=2}^{k} \hat{w}(a_i) \\
(n-1)\hat{w}(a_2) - \sum_{i=1, i \neq 2}^{k} \hat{w}(a_i) \\
\vdots \\
(n-1)\hat{w}(a_k) - \sum_{i=1}^{k-1} \hat{w}(a_i)
\end{bmatrix}
\]

Since \(\hat{w}(a_i)\) is the logarithmized value of \(w(a_i)\) i.e. \(\hat{w}(a_i) = \log_e w(a_i)\) then the final ranking vector is obtained from \(\hat{w}\) by the exponential transformation \(w(a_i) = e^{\hat{w}(a_i)}\) for \(i = 1, \ldots, k\). The solution to a geometric HRE always exists and is optimal [28]. In the case of the arithmetic approach, a sufficiently small local inconsistency is a sufficient condition [24].

3. Arithmetic Incomplete Heuristic Rating Estimation

3.1. Method definition

As in the original method, we assume that the set of alternatives \(A = \{a_1, \ldots, a_n\}\) consists of two subsets \(A_U = \{a_1, \ldots, a_k\}\) and \(A_K = \{a_{k+1}, \ldots, a_n\}\). \(A_K\) contains references (i.e. alternatives with the known priority values) and \(A_U\) is composed of alternatives whose ranking has yet to be calculated.

Since it is natural to expect that \(w(a_i)\) is similar\(^2\) to \(c_{ij}w(a_j)\), i.e.

\[w(a_i) \approx c_{ij} w(a_j)\]  \(2\)

then, we may approximate \(w(a_i)\) by the weighted average of all other priority values:

\[w(a_i) = \frac{1}{n-1} \sum_{\substack{j=1 \atop j \neq i}}^{n} c_{ij} w(a_j), \text{ for } i = 1, \ldots, n.\]  \(3\)

However, when \(c_{ij}\) is undefined i.e. \(c_{ij} = \_\) we assume, following [2], that its value is

\[c_{ij} = \frac{w(a_i)}{w(a_j)}.\]

Thus, we assume that the missing values are completely consistent with the final, yet unknown, ranking result. As a result [3] takes the form:

\(^2\)Ideally \(w(a_i) = c_{ij} w(a_j)\). This is when the PC matrix \(C\) is consistent.
3.1 Method definition

\[
w(a_i) = \frac{1}{n-1} \left( \sum_{j=1, i \neq j}^{c_{ij}} c_{ij} w(a_j) + \sum_{j=1, i \neq j}^{c_{ij}} w(a_i) \right), \quad \text{for } i = 1, \ldots, n
\]

Thus, let \( s_i \) be the number of undefined comparisons in the \( i \)-th row, hence the above takes the form

\[
w(a_i) = \frac{1}{n-1} \left( \sum_{j=1, i \neq j}^{c_{ij}} c_{ij} w(a_j) + s_i w(a_i) \right), \quad \text{for } i = 1, \ldots, n
\]

By multiplying both sides by \( n-1 \) we get

\[
(n-1)w(a_i) = \sum_{j=1, i \neq j}^{c_{ij}} c_{ij} w(a_j) + s_i w(a_i), \quad \text{for } i = 1, \ldots, n
\]

i.e.

\[
(n-s_i-1)w(a_i) = \sum_{j=1, i \neq j}^{c_{ij}} c_{ij} w(a_j), \quad \text{for } i = 1, \ldots, n
\]

finally

\[
w(a_i) = \frac{1}{n-s_i-1} \sum_{j=1, i \neq j}^{c_{ij}} c_{ij} w(a_j), \quad \text{for } i = 1, \ldots, n
\]

The above can be written in the form of an equation system:

\[
\begin{align*}
w(a_1) &= \frac{1}{n-s_1-1} (d_{1,1} w(a_1) + \ldots + d_{1,n} w(a_n)) \\
w(a_2) &= \frac{1}{n-s_2-1} (d_{2,1} w(a_1) + d_{2,2} w(a_2) + \ldots + d_{2,n} w(a_n)) \\
&\phantom{=} \vdots \\
w(a_k) &= \frac{1}{n-s_k-1} (d_{k,1} w(a_1) + \ldots + d_{k,k-1} w(a_{k-1}) + d_{k,k+1} w(a_{k+1}) + d_{k,n} w(a_n))
\end{align*}
\]

where

\[
d_{ij} = \begin{cases} c_{ij} & \text{if } c_{ij} \neq 0 \\ 0 & \text{if } c_{ij} = 0 \end{cases}
\]

The values \( w(a_{k+1}), \ldots, w(a_n) \) are known and constant (\( a_{k+1}, \ldots, a_n \) belongs to \( A_K \) known alternatives). For the same reason, all the comparisons \( c_{ij} \) where \( i, j = k+1, \ldots, n \) do not need to be determined by experts. Their values are:

\[
c_{k+1,k+2} = \frac{w(a_{k+1})}{w(a_{k+2})}, \ldots, c_{n,n-1} = \frac{w(a_n)}{w(a_{n-1})}.
\]
Therefore, all the components in the row composed of known elements can be summed up together. Let us denote:

\[ b_j = \frac{1}{n-s_j-1} c_{j,k+1} w(a_{k+1}) + \ldots + \frac{1}{n-s_j-1} c_{j,n} w(a_n). \]

Thus, the linear equations system can be rewritten as:

\[
\begin{align*}
    w(a_1) &= \frac{1}{n-s_1-1} d_{1,2} w(a_2) + \ldots + \frac{1}{n-s_1-1} d_{1,k} w(a_k) + b_1 \\
    w(a_2) &= \frac{1}{n-s_2-1} d_{2,1} w(a_1) + \frac{1}{n-s_2-1} d_{2,3} w(a_3) + \ldots + \frac{1}{n-s_2-1} d_{2,k} w(a_k) + b_2 \\
    \vdots & \vdots \\
    w(a_k) &= \frac{1}{n-s_k-1} d_{k,1} w(a_1) + \frac{1}{n-s_k-1} d_{k,k-1} w(a_{k-1}) + b_k
\end{align*}
\]

The matrix form of the above equation system is given as

\[ \overline{C} w = b, \quad (4) \]

where

\[
\overline{C} = \begin{pmatrix}
    1 & -\frac{1}{n-s_1-1} d_{1,2} & \ldots & -\frac{1}{n-s_1-1} d_{1,k} \\
    -\frac{1}{n-s_2-1} d_{2,1} & 1 & \ldots & -\frac{1}{n-s_2-1} d_{2,k} \\
    \vdots & \vdots & \ddots & \vdots \\
    -\frac{1}{n-s_k-1} d_{k,1} & \ldots & -\frac{1}{n-s_k-1} d_{k,k-1} & 1
\end{pmatrix},
\]

the vector of constant terms is

\[
b = \begin{pmatrix}
    \frac{1}{n-s_1-1} c_{1,k+1} w(a_{k+1}) + \ldots + \frac{1}{n-s_1-1} c_{1,n} w(a_n) \\
    \frac{1}{n-s_2-1} c_{2,k+1} w(a_{k+1}) + \ldots + \frac{1}{n-s_2-1} c_{2,n} w(a_n) \\
    \vdots \\
    \frac{1}{n-s_k-1} c_{k,k+1} w(a_{k+1}) + \ldots + \frac{1}{n-s_k-1} c_{k,n} w(a_n)
\end{pmatrix}, \quad (6)
\]

and values that need to be determined are denoted as:

\[ w = \begin{pmatrix}
    w(a_1) \\
    \vdots \\
    \vdots \\
    w(a_k)
\end{pmatrix}. \quad (7) \]

4. Multiplicative Incomplete Heuristic Rating Estimation

With the same assumptions as to the sets \( A, A_U \) and \( A_K \) we may want to request that \( w(a_i) \) should be a geometric mean of components in the form of \( c_{ij} w(a_j) \) for \( j = 1, \ldots, n \). This leads to a nonlinear system of equations:
\[
\forall i = 1, \ldots, n
\]
By logarithmic transformation of (11) we get

\[
\begin{align*}
(n - s_1 - 1)\hat{w}(a_1) &= \hat{d}_{1,2} + \ldots + \hat{d}_{1,k} + \hat{g}_1 \\
(n - s_2 - 1)\hat{w}(a_2) &= \hat{d}_{2,1} + \ldots + \hat{d}_{2,k} + \hat{g}_2 \\
&\quad\vdots \\
(n - s_k - 1)\hat{w}(a_k) &= \hat{d}_{k,1} + \ldots + \hat{d}_{k,k-1} + \hat{g}_k
\end{align*}
\]

where \(\log_\xi w(a_i) = df\hat{w}(a_i)\), \(\log_\xi c_{ij} = df\hat{c}_{ij}\), \(\log_\xi d_{ij} = df\hat{d}_{ij}\) and \(\log_\xi g_j = df\hat{g}_j\) for some real constant \(\xi \in \mathbb{R}_+\). Note that if \(c_{ij} = \)? then corresponding \(\hat{d}_{ij} = 0\). When \(c_{ij} = \)?, then \(\hat{d}_{ij} = \hat{w}(a_j) + \hat{c}_{ij}\).

Let us group the constant values in each row. As a result, we obtain:

\[
\begin{align*}
(n - s_1 - 1)\hat{w}(a_1) - \sum_{j=2, c_{1,j} = \)?\}^k \hat{w}(a_j) &= b_1 \\
(n - s_2 - 1)\hat{w}(a_2) - \sum_{j=1, j \neq c_{2,j} = \)?\}^k \hat{w}(a_j) &= b_2 \\
&\quad\vdots \\
(n - s_k - 1)\hat{w}(a_k) - \sum_{j=1, c_{k,j} = \)?\}^{k-1} \hat{w}(a_j) &= b_k
\end{align*}
\]

where \(b_i = df\sum_{j=1, i \neq j, c_{ij} = \)?\}^k \hat{c}_{ij} + \hat{g}_i\) for \(i = 1, \ldots, k\).

The above linear equation system can be written down in the form of the following matrix equation:

\[
\hat{C}\hat{w} = b,
\]

where

\[
\hat{C} = \begin{bmatrix}
(n - s_1 - 1) & q_{1,2} & \cdots & q_{1,k} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
q_{k,1} & q_{k,2} & \cdots & (n - s_k - 1)
\end{bmatrix},
\]

with

\[
q_{ij} = \begin{cases} 
-1 & \text{if } c_{ij} \neq \)\
0 & \text{if } c_{ij} = \)
\end{cases}
\]

and

\[
\hat{w} = \begin{bmatrix}
\hat{w}(a_1) \\
\hat{w}(a_2) \\
\vdots \\
\hat{w}(a_k)
\end{bmatrix}, \quad \text{and } b = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_k
\end{bmatrix}.
\]

By solving (12) we get the solution of the original nonlinear equation (9). Indeed, when computed, the original priority vector is given as:

\[
w = \begin{bmatrix}
\xi\hat{w}(a_1) \\
\vdots \\
\xi\hat{w}(a_k)
\end{bmatrix}.
\]
5. Summary

In the presented work, we defined two extensions of the Heuristic Rating Estimation (HRE) method. Both the arithmetic and geometric incomplete significantly reduce the number of pairwise comparisons needed to calculate a ranking. Thus, they reduce the costs of the decision-making process and shorten the working time of experts. They also make it possible to calculate the ranking when the set of pairwise comparisons is incomplete for reasons beyond the control of the body conducting the decision-making process. Therefore, we hope that the HRE method, extended with the ability to process incomplete PC matrices, may interest a broad group of practitioners and researchers.

6. References

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