AUTOMORPHISMS AND DERIVATIONS OF ALGEBRAS OF INFINITE MATRICES

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Abstract. We describe automorphisms and derivations of several important associative and Lie algebras of infinite matrices over a field.

1. Introduction

Let \( F \) be a ground field and let \( A \) be an associative \( F \)-algebra. Recall that an associative algebra \( A \) gives rise to the Lie algebra \( A^{(-)} = (A, [a, b] = ab - ba) \). For an arbitrary element \( a \in A \) the operator \( \text{ad}(a) : x \rightarrow [a, x] \) is an (inner) derivation of the associative algebra \( A \) and of the Lie algebra \( A^{(-)} \). If \( J \) is an ideal of the algebra \( A \) then \( \text{ad}_J(a) \) denotes the restriction of the derivation \( \text{ad}(a) \) to \( J \).

An associative algebra \( A \) with 1 gives rise to the group \( G(A) \) of invertible elements of \( A \). For an arbitrary invertible element \( a \in G(A) \) the conjugation \( x \rightarrow a^{-1}xa, x \in A \), is an (inner) automorphism of the algebra \( A \) and, hence, of the algebra \( A^{(-)} \).

Let \( I \) be an infinite set and let \( \mathbb{N}, \mathbb{Z} \) denote the set of positive integers and the set of integers, respectively.

Example 1. Let \( M_\infty(I, F) \) be the associative algebra of all \((I \times I)\)-matrices over \( F \) having finitely many nonzero entries (finitary matrices). The Lie algebras

\[
\mathfrak{gl}_\infty(I, F) = M_\infty(I, F)^{(-)}, \quad \mathfrak{sl}_\infty(I, F) = [\mathfrak{gl}_\infty(I, F), \mathfrak{gl}_\infty(I, F)]
\]

and their representation attracted considerable attention in the literature (see, [12, 17]).

A linear transformation \( * : A \rightarrow A \) of an associative algebra \( A \) is called an involution if \( (a^*)_* = a, (ab)^* = b^*a^* \) for arbitrary elements \( a, b \in A \). The subspace of skew-symmetric elements \( K(A,^*) = \{a \in A \mid a^* = -a\} \) is a subalgebra of the Lie algebra \( A^{(-)} \). The algebra

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$M_\infty(I, F)$ is equipped with the transpose and the symplectic involutions that give rise to Lie algebras $\mathfrak{o}_\infty(I, F)$ and $\mathfrak{sp}_\infty(I, F)$ of skew-symmetric elements, respectively.

**Example 2.** Let $M_{rcf}(I, F)$ be the associative algebra of all $(I \times I)$-matrices over $F$ having finitely many nonzero entries in each row and in each column. It is easy to see that $M_\infty(I, F)$ is an ideal of the algebra $M_{rcf}(I, F)$. The algebra $M_{rcf}(I, F)$ gives rise to the Lie algebra $\mathfrak{gl}_{rcf}(I, F) = M_{rcf}(I, F)(-)$ and to the group $GL_{rcf}(I, F) = G(M_{rcf}(I, F))$.

The transpose and the symplectic involutions on $M_\infty(I, F)$ naturally and uniquely extend to involutions of $M_{rcf}(I, F)$.

**Example 3.** Let $M(I, F)$ denote the associative algebra of $(I \times I)$-matrices over the field $F$ having finitely many nonzero entries in each column. If $V$ is a vector space over $F$ of dimension $|I|$ then the algebra $\text{End}_F(V)$ of all linear transformations of $V$ is isomorphic to $M(I, F)$.

Let $\mathfrak{gl}(I, F) = M(I, F)(-)$, The group $G(M(I, F))$ is isomorphic to the group $GL(V)$ of all invertible linear transformations of $V$.

**Example 4.** Let $MJ(F)$ be the algebra of Jacobi matrices, that is, $(\mathbb{Z} \times \mathbb{Z})$-matrices having finitely many nonzero diagonals. In other worlds, a matrix $a = (a_{ij})_{i,j \in \mathbb{Z}}$ lies in $MJ(F)$ if there exists $n \in \mathbb{N}$ such that $a_{ij} = 0$ whenever $|i - j| > n$.

The Lie algebras $\mathfrak{gl}_J(F) = MJ(F)(-)$ found applications in the theory of solitones [9, 11].

**Example 5.** Let $\text{End}_{fin}(V)$ be the subalgebra of $\text{End}_F(V)$ that consists of all linear transformations of finite range.

N. Jacobson [14], Chap. 9, Sec. 11, Th. 7) showed that for an arbitrary associative algebra $A$, $M_\infty(I, F) \subseteq A \subseteq M(I, F)$, every automorphism $\varphi$ of $A$ is a conjugation by an invertible element from $M(I, F)$, $\varphi(a) = x^{-1}ax$, $a \in A$.

K.-H. Neeb [16] and N. Stumme [19] described derivations and automorphisms of Lie algebras $\mathfrak{sl}_\infty(I, F)$, $\mathfrak{o}_\infty(I, F)$, $\mathfrak{sp}_\infty(I, F)$ over a field of zero characteristic.

2. **Main results**

Theorems [2] and [11] below describe all derivations of the associative $F$-algebras $M_\infty(I, F)$, $\text{End}_{fin}(V)$, $M_{rcf}(I, F)$, $MJ(F)$, $M(I, F)$ (Theorem [2]) and all automorphisms of the associative $F$-algebras $M_\infty(I, F)$, $\text{End}_{fin}(V)$, $M_{rcf}(I, F)$, $M(I, F)$ (Theorem [4]). Theorem [5] describes
anti-automorphisms of the algebras $M_{\infty}(I, F)$, $M_{rcf}(I, F)$, $MJ(F)$ and shows that the algebras $\text{End}_{\text{fin}}(V)$, $M(I, F)$ do not have any anti-automorphisms.

Finally, using the proof of Herstein's conjectures [13] by K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale (see, [2, 3, 4]) we describe $M$-anti-automorphisms of the algebras $M$.

Theorem 1. For an arbitrary infinite set $I$ we have:

(a) $M(I, F) = [M(I, F), M(I, F)]$,
(b) $M_{rcf}(I, F) = [M_{rcf}(I, F), M_{rcf}(I, F)]$,
(c) $MJ(F) = [MJ(F), MJ(F)]$,
(d) $\mathfrak{o}_{\infty}(I, F) = [\mathfrak{o}_{\infty}(I, F), \mathfrak{o}_{\infty}(I, F)]$,
$\mathfrak{sp}_{\infty}(I, F) = [\mathfrak{sp}_{\infty}(I, F), \mathfrak{sp}_{\infty}(I, F)]$.

In particular, the algebras $\mathfrak{gl}(I, F)$, $\mathfrak{gl}_{rcf}(I, F)$ and $\mathfrak{gl}_{f}(F)$ are perfect.

Theorem 2. (a) An arbitrary derivation of the algebra $M_{\infty}(I, F)$ is of the type
$\text{ad}_{M_{\infty}(I, F)}(a)$, where $a \in M_{rcf}(I, F)$;
(b) an arbitrary derivation of the algebra $\text{End}_{\text{fin}}(V)$ is of the type
$\text{ad}_{\text{End}_{\text{fin}}(V)}(a)$, where $a \in \text{End}_{F}(V)$;
(c) all derivations of the algebras $M_{rcf}(I, F)$, $MJ(F)$, $M(I, F)$ are inner.

Theorem 3. Let $F$ be a field of the characteristic not equal to 2.
(a) An arbitrary derivation of the Lie algebra $\mathfrak{sl}_{\infty}(I, F)$ is of the type
$\text{ad}_{\mathfrak{sl}_{\infty}(I, F)}(a)$, where $a \in \mathfrak{gl}_{rcf}(I, F)$;
(b) an arbitrary derivation of the Lie algebra $\mathfrak{o}_{\infty}(I, F)$ (resp. $\mathfrak{sp}_{\infty}(I, F)$) is of the type
$\text{ad}_{\mathfrak{o}_{\infty}(I, F)}(a)$ (resp. $\text{ad}_{\mathfrak{sp}_{\infty}(I, F)}(a)$),
where $a \in K(M_{rcf}(I, F), t)$ (resp. $K(M_{rcf}(I, F), s)$;
(c) all derivations of Lie algebras $\mathfrak{gl}(I, F)$, $\mathfrak{gl}_{rcf}(I, F)$, $\mathfrak{gl}_{f}(F)$ are inner.

Theorem 4. (a) An arbitrary automorphism $\varphi$ of the algebra $M_{\infty}(I, F)$ is a conjugation by an element from $GL_{rcf}(I, F)$. In
other words, there exists an element \( x \in GL_{ref}(I, \mathbb{F}) \) such that
\( \varphi(a) = x^{-1}ax \) for all \( a \in M_{\infty}(I, \mathbb{F}) \);
(b) an arbitrary automorphism of the algebra \( \text{End}_{fin}(V) \) is a conjugation by an element from \( GL(V) \);
(c) all automorphisms of the algebras \( M_{ref}(I, \mathbb{F}), M(I, \mathbb{F}) \) are inner.

Given an algebra \( A \) an invertible linear transformation \( \varphi : A \to A \) is called an anti-automorphism if \( \varphi(ab) = \varphi(b)\varphi(a) \) for arbitrary elements \( a, b \in A \). It is easy to see that an anti-automorphism is an isomorphism \( A \to A^{op} \) with the opposite algebra \( A^{op} = (A, a*b = ba) \). The transpose transformations \( (a_{ij})^t = (a_{ji}), a_{ij} \in \mathbb{F} \), are anti-isomorphisms of the algebras \( M_{\infty}(I, \mathbb{F}), M_{ref}(I, \mathbb{F}), MJ(\mathbb{F}) \).

**Theorem 5.**
(a) An arbitrary anti-automorphism of one of the algebras \( M_{\infty}(I, \mathbb{F}), M_{ref}(I, \mathbb{F}), MJ(\mathbb{F}) \) is a composition of the transpose and an automorphism (see, Theorem 4);
(b) algebras \( \text{End}_{fin}(V) \) and \( M(I, \mathbb{F}) \) do not have anti-isomorphisms. In other words, these algebras are not isomorphic to their opposite algebras.

**Theorem 6.** Let \( \mathbb{F} \) be a field of the characteristic not equal to 2.
(a) An arbitrary automorphism \( \varphi \) of the Lie algebra \( L = \mathfrak{sl}_{\infty}(I, \mathbb{F}) \) is of the type \( \varphi(a) = x^{-1}ax, a \in L \), or of the type \( \varphi(a) = -x^{-1}a^t x \), \( a \in L \), where \( x \in GL_{ref}(I, \mathbb{F}) \);
(b) an arbitrary automorphism \( \varphi \) of the Lie algebra \( L = \mathfrak{o}_{\infty}(I, \mathbb{F}) \) (resp. \( \mathfrak{sp}_{\infty}(I, \mathbb{F}) \)) is of the type \( \varphi(a) = x^{-1}ax, a \in L \), where \( x \in GL_{ref}(I, \mathbb{F}) \) and \( xx^t \in \mathbb{F} \) (resp. \( xx^s \in \mathbb{F} \));
(c) an arbitrary automorphism \( \varphi \) of the Lie algebra \( \mathfrak{gl}_{ref}(I, \mathbb{F}) \) is of the type \( \varphi(a) = x^{-1}ax, a \in \mathfrak{gl}_{ref}(I, \mathbb{F}) \), or of the type \( \varphi(a) = -x^{-1}a^t x, a \in \mathfrak{gl}_{ref}(I, \mathbb{F}) \), where \( x \in G(M_{ref}(I, \mathbb{F})) \);
(d) an arbitrary automorphism \( \varphi \) of the Lie algebra \( \mathfrak{gl}(I, \mathbb{F}) \) is of the type \( \varphi(a) = x^{-1}ax, a \in \mathfrak{gl}(I, \mathbb{F}) \), where \( x \in G(M(I, \mathbb{F})) \);
(e) an arbitrary automorphism \( \varphi \) of the Lie algebra \( \text{End}_{fin}(V) \) is of the type \( \varphi(a) = x^{-1}ax, a \in \text{End}_{fin}(V) \), where \( x \in G(\text{End}_{fin}(V)) \).

**Remark.** We don’t have a description of automorphisms of the algebra \( \mathfrak{gl}_f(\mathbb{F}) \).

3. DERIVATIONS OF ASSOCIATIVE ALGEBRAS OF INFINITE MATRICES

Recall the definition of the Tychonoff topology. Let \( X, Y \) be arbitrary sets. Let \( \text{Map}(X, Y) \) be the set of mappings \( X \to Y \). For distinct
elements $a_1, \ldots, a_n \in X$ and arbitrary elements $b_1, \ldots, b_n \in Y$, $n \geq 1$, consider the subset
\[ M(a_1, \ldots, a_n; b_1, \ldots, b_n) = \{ f : X \to Y \mid f(a_i) = b_i, \ 1 \leq i \leq n \} \]
of \( \text{Map}(X, Y) \). The Tykhonoff topology on \( \text{Map}(X, Y) \) is generated by all open sets of this type. In other words, \( \text{Map}(X, Y) \) as a topological space is homeomorphic to the space \( Y^{|X|} \), the Tykhonoff product of \( |X| \) copies of \( Y \), where \( Y \) is equipped with the discrete topology.

An associative \( \mathbb{F} \)-algebra \( A \) is called a \textit{locally matrix algebra} if for each finite subset of \( A \) there exists a subalgebra \( B \subset A \) containing this subset and isomorphic to the algebra \( M_n(\mathbb{F}) \) for some positive integer \( n \). For properties and theory of locally matrix algebras, see [1] [5] [6] [7] [8] [15].

If \( A \) is an associative \( \mathbb{F} \)-algebra and \( M \) is an \( A \)-bimodule then the space \( C^1(A, M) \) of all \textit{bimodule derivations} \( A \to M \) lies in \( \text{Map}(X, Y) \) and, therefore, is equipped with the Tykhonoff topology. Let \( B^1(A, M) \) be the space of all \textit{inner bimodule derivations} \( A \to M \). The factor-space \( H^1(A, M) = C^1(A, M)/B^1(A, M) \) is called the first cohomology space (for details see, [10] [18]).

We will need the following generalization of Theorem 1(1) from [5].

**Lemma 1.** If \( A \) is a locally matrix algebra then for an arbitrary \( A \)-bimodule \( M \) the subspace \( B^1(A, M) \) is dense in the space \( C^1(A, M) \) in the Tykhonoff topology.

**Proof.** We need to show that for arbitrary elements \( a_1, \ldots, a_n \in A \) and an arbitrary derivation \( d : A \to M \) there exists an element \( x \in M \) such that \( [x, a_i] = d(a_i), \ 1 \leq i \leq n \).

There exists a subalgebra \( A_1 \subset A \) such that \( a_1, \ldots, a_n \in A_1 \) and \( A_1 \cong M_k(\mathbb{F}) \). Then
\[ M_1 = d(A_1) + A_1d(A_1) + d(A_1)A_1 + A_1d(A_1)A_1 \]
is a finite-dimensional \( A_1 \)-submodule of \( M \) and the restriction of the derivation \( d \) to \( A_1 \) is a derivation \( A_1 \to M_1 \). Since every bimodule derivation of a matrix algebra over a field is inner it follows that there exists an element \( x \in M_1 \) such that \( d(a) = [x, a] \) for all elements \( a \in A_1 \).

This completes the proof of the lemma. \( \Box \)

Let \( (I \times I)(\mathbb{F}) \) denote the vector space of all \( (I \times I) \)-matrices over \( \mathbb{F} \). Since for arbitrary matrices \( a \in (I \times I)(\mathbb{F}), b \in M_\infty(\mathbb{F}) \) the products \( ab, ba \) are well defined it follows that the vector space \( (I \times I)(\mathbb{F}) \) is a bimodule over \( M_\infty(\mathbb{F}) \). We will start with the following proposition.

**Proposition 1.** \( H^1(M_\infty(I, \mathbb{F}), (I \times I)(\mathbb{F})) = (0) \).
Proof. Let \( d : M_\infty(I, \mathbb{F}) \rightarrow (I \times I)(\mathbb{F}) \) be a bimodule derivation. By Lemma 1 for an arbitrary nonempty finite subset \( J \subseteq I \) there exists an element \( y_J \in (I \times I)(\mathbb{F}) \) such that \( d(a) = [y_J, a] \) for all elements \( a \in M_{|J|}(\mathbb{F}) \).

Divide the matrix \( y_J \) into blocks

\[
y_J = \begin{pmatrix} y_J(11) & y_J(12) \\ y_J(21) & y_J(22) \end{pmatrix},
\]

where \( y_J(11) \in M_{|J|}(\mathbb{F}) ; y_J(12) \) is a \((J \times (I \setminus J))\)-matrix over \( \mathbb{F} \); \( y_J(21) \) is a \(((I \setminus J) \times J)\)-matrix over \( \mathbb{F} \); and \( y_J(22) \) is a \(((I \setminus J) \times (I \setminus J))\)-matrix over \( \mathbb{F} \). For an arbitrary element \( a \in M_{|J|}(\mathbb{F}) \) we have

\[
[y_J, a] = \begin{pmatrix} [y_J(11), a] & -a y_J(12) \\ y_J(21)a & 0 \end{pmatrix}.
\]

Let \( J_1 \) be another finite subset of \( I \), \( J \subseteq J_1 \). Then \([y_J, a] = [y_{J_1}, a]\) for all elements \( a \in M_{|J_1|}(\mathbb{F}) \). Hence the \((J \times J)\)-minor of \( y_{J_1} \) differs from \( y_J(11) \) by a scalar matrix.

We define a \((I \times I)\)-matrix \( y \) as follows. Fix \( i_0 \in I \). Let \( y_{i_0,i_0} = 0 \). All matrices \( y_J \), where \( J \) runs over finite subsets of \( I \), \( i_0 \in J \), can be selected so that \((y_J)_{i_0,i_0} = 0\).

For \( i, j \in I \) choose a finite subset \( J \subseteq I \) such that \( i, j, i_0 \in J \). Define \( y_{ij} = (y_J)_{ij} \). Let \( J_1 \) be a finite subset of \( I \), \( J \subseteq J_1 \). Since \( y_J(11) \) and the \((J \times J)\)-minor of \( y_{J_1} \) differ by a scalar and \((y_J)_{i_0,i_0} = (y_{J_1})_{i_0,i_0} = 0\) it follows that \( y_{ij} = (y_{J_1})_{ij} \). Hence \( y_{ij} \) does not depend on a choice of the subset \( J \).

For an arbitrary element \( a \in M_\infty(I, \mathbb{F}) \) the \((J \times J)\)-minors of \((I \times I)\)-matrices \( d(a) = [y_J, a] \) and \([y, a]\) coincide as long as \( J \subseteq I \) is a finite subset, \( i_0 \in J \), and \( a \in M_{|J|}(\mathbb{F}) \). This implies that \( d(a) = [y, a] \) and completes the proof of the proposition. \( \square \)

**Lemma 2.** Let \( y \in (I \times I)(\mathbb{F}) \).

(a) The inclusion \([y, M_\infty(I, \mathbb{F})] \subseteq M(I, \mathbb{F})\) implies \( y \in M(I, \mathbb{F}) \).

(b) The inclusion \([y, M_\infty(I, \mathbb{F})] \subseteq M_{rcf}(I, \mathbb{F})\) implies \( y \in M_{rcf}(\mathbb{F}) \).

**Proof.** Let \( e_{pq}, p, q \in I \), denote the matrix unit having 1 at the position \((p, q)\) and zeros elsewhere. We have

\[
[y, e_{jj}] = ye_{jj} - e_{jj} y, \quad [y, e_{jj}]_{ij} = \begin{cases} y_{ij}, & i \neq j, \\ 0, & i = j. \end{cases}
\]
If \( y, e_{ij} \in M(I, F) \) then \( y_{ij} \neq 0 \) for only finitely many \( i \in I \). Hence \( y \in M(I, F) \). Similarly,

\[
[y, e_{ii}]_{ij} = \begin{cases} 
-y_{ij}, & i \neq j, \\
0, & i = j.
\end{cases}
\]

If \( y, e_{ii} \in M_{rcf}(I, F) \) then \( y_{ij} \neq 0 \) for only finitely many \( j \in I \). This completes the proof of the lemma. \( \square \)

**Proof of Theorem**

(a) Let \( d : M_{\infty}(I, F) \to M(I, F) \) be a bimodule derivation. By Proposition there exists an \((I \times I)\)-matrix \( y = (y_{ij})_{i,j \in I} \) such that \( d(a) = [y, a] \) for an arbitrary element \( a \in M_{\infty}(I, F) \). By Lemma (a), the matrix \( y \) lies in \( M(I, F) \). Hence

\[
H^1(M_{\infty}(I, F), M(I, F)) = (0).
\]

Let \( d : M_{\infty}(I, F) \to M_{\infty}(I, F) \) be a derivation. By the above, there exists an \((I \times I)\)-matrix \( y \) such that \( d(a) = [y, a] \) for an arbitrary element \( a \in M_{\infty}(I, F) \). By Lemma (b), we have \( y \in M_{rcf}(I, F) \).

(b) Choose a basis \( v_i, i \in I \), in the vector space \( V \). Linear transformations from \( \text{End}_{fin}(V) \) have matrices lying in the subalgebra \( M_{r-fin}(I, F) \) of \( M(I, F) \) that consists of \((I \times I)\)-matrices that are column-finite and have finite range. These two conditions are equivalent to matrices from \( M_{r-fin}(I, F) \) having finitely many nonzero rows.

We have

\[
M_{\infty}(I, F) \subset M_{r-fin}(I, F) \subset M(I, F).
\]

Let \( d : M_{r-fin}(I, F) \to M_{r-fin}(I, F) \) be a derivation. The restriction of \( d \) to \( M_{\infty}(I, F) \) is a bimodule derivation from \( M_{\infty}(I, F) \) to \( M(I, F) \). By (1), there exists a matrix \( y \in M(I, F) \) such that \( d(a) = [y, a] \) for an arbitrary element \( a \in M_{\infty}(I, F) \).

Consider the derivation

\[
d' = d - \text{ad}_{M_{r-fin}(I, F)}(y)
\]

of the algebra \( M_{r-fin}(I, F) \). We have \( d'(M_{\infty}(I, F)) = (0) \). For an arbitrary element \( a \in M_{r-fin}(I, F) \) and arbitrary indices \( i, j \in I \) we have

\[
d'(e_{ii} a e_{jj}) = e_{ii} d'(a) e_{jj} = 0.
\]

Hence, \( d'(a)_{ij} = 0 \). We showed that \( d' = 0 \), which completes the proof of the part (b).

(c) Let \( d : M_{rcf}(I, F) \to M_{rcf}(I, F) \) be a derivation. There exists a matrix \( y \in M(I, F) \) such that \( d(a) = [y, a] \) for an arbitrary element \( a \in M_{\infty}(I, F) \). By Lemma (2) the inclusion

\[
[y, M_{\infty}(I, F)] \subseteq M_{rcf}(I, F)
\]
implies \( y \in M_{\text{ref}}(I, \mathbb{F}) \). Consider the derivation
\[
d' = d - \text{ad}(y) \quad \text{of} \quad M_{\text{ref}}(I, \mathbb{F}), \quad d'(M_\infty(I, \mathbb{F})) = (0).
\]
As in the proof of Theorem 2(b), for arbitrary indices \( i, j \in I \) we have
\[
d'(e_{ii} M_{\text{ref}}(I, \mathbb{F}) e_{jj}) = e_{ii} d'(M_{\text{ref}}(I, \mathbb{F})) e_{jj} = 0
\]
which implies \( d' = 0, \ d = \text{ad}(y) \).

Let \( d : MJ(\mathbb{F}) \to MJ(\mathbb{F}) \) be a derivation. Since
\[
M_\infty(\mathbb{Z}, \mathbb{F}) \subset MJ(\mathbb{F}) \subset M_{\text{ref}}(\mathbb{Z}, \mathbb{F}),
\]
by Lemma 2 there exists a matrix \( y \in M_{\text{ref}}(\mathbb{Z}, \mathbb{F}) \) such that \( d(a) = [y, a] \) for an arbitrary element \( a \in M_\infty(\mathbb{Z}, \mathbb{F}) \). Consider the bimodule derivation
\[
d' = d - \text{ad}(y), \quad d' : MJ(\mathbb{F}) \to M_{\text{ref}}(\mathbb{Z}, \mathbb{F}), \quad d'(M_\infty(\mathbb{Z}, \mathbb{F})) = (0).
\]
Let us show that \( d'(MJ(\mathbb{F})) = (0) \). Let \( a \in MJ(\mathbb{F}), a = (a_{ij})_{i,j \in \mathbb{Z}}, a_{ij} \in \mathbb{F} \). There exists \( k \geq 1 \) such that \( a_{ij} = 0 \) whenever \( |i - j| > k \).

Let \( n \geq 1 \). Consider matrices \( a'(n), a''(n) : \)
\[
a'(n)_{ij} = \begin{cases} a_{ij} & \text{if } |i|, |j| \leq n \\ 0 & \text{otherwise}, \end{cases} \quad a''(n)_{ij} = \begin{cases} a_{ij} & \text{if } |i| > n \text{ or } |j| > n \\ 0 & \text{otherwise}. \end{cases}
\]
Clearly, \( a = a'(n) + a''(n) \). If \( a''(n+k)_{ij} \neq 0 \) then \( |i| > n \text{ and } |j| > n \), that is,
\[
a''(n + k) \in \left( \begin{array}{c|c} 0_{n \times n} & 0 \\ \hline 0 & 0_{n \times n} \end{array} \right),
\]
where \( 0_{n \times n} \) denote zero \((n \times n)\)-matrix. Let us show that
\[
d'(a''(n + k)) \in \left( \begin{array}{c|c} 0_{n \times n} & 0 \\ \hline 0 & 0_{n \times n} \end{array} \right)
\]
as well. Indeed, if \( i \leq n \) or \( j \leq n \) then
\[
d'(a''(n + k))_{ij} = d'(e_{ii} a''(n + k) e_{jj}) = 0
\]
since \( e_{ii} a''(n + k) e_{jj} = 0 \). We have \( d'(a'(n + k)) \) = 0. Hence
\[
d'(a) \in \left( \begin{array}{c|c} 0_{n \times n} & 0 \\ \hline 0 & 0_{n \times n} \end{array} \right)
\]
for any \( n \geq 1 \). Hence \( d'(a) = 0, \ d(a) = [y, a] \) for an arbitrary \( a \in MJ(\mathbb{F}) \).

Consider the matrix
\[
E_1 = \sum_{i \in \mathbb{Z}} e_{i,i+1} \in MJ(\mathbb{F}).
\]
It is easy to see that if \( y \not\in MJ(\mathbb{F}) \) then \([y, E_1] \not\in MJ(\mathbb{F})\) as well. This implies that \( y \in MJ(\mathbb{F}) \).

Finally, let \( d : M(I, \mathbb{F}) \to M(I, \mathbb{F}) \) be a derivation. The ideal \( M_{r-fin}(I, \mathbb{F}) \) is invariant with respect to all derivations. By Theorem \( 2(b) \), there exists an element \( y \in M(I, \mathbb{F}) \) such that

\[
d \mid M_{r-fin}(I, \mathbb{F}) = \text{ad}_{M_{r-fin}(I, \mathbb{F})} (y).
\]

Let \( d' = d - \text{ad}(y) \). Then \( d'(M_{r-fin}(I, \mathbb{F})) = 0 \). Arguing as above, we get \( d' = 0 \). This completes the proof of Theorem \( 2 \). \( \square \)

4. Lie algebras of infinite matrices

We will start with the proof of Theorem \( 1 \).

Let \( I \) be an arbitrary infinite set. It follows that \( |I| = |\mathbb{N} \times I| \). That is why without loss of generality, we will assume that \( I \) is a direct product of \( \mathbb{N} \) with another set \( J \), \( |J| = |I| \), \( I = \mathbb{N} \times J \).

Therefore, we can view an arbitrary \((I \times I)\)-matrix as an \((\mathbb{N} \times \mathbb{N})\)-matrix over \((J \times J)(\mathbb{F})\). For two positive integers \( i, j \in \mathbb{N} \) and a \((J \times J)\)-matrix \( x \) let \( e_{ij}(x) \) denote an \((I \times I)\)-matrix having the block \( x \) at the position \((i, j)\) and zeros elsewhere.

Consider the matrix

\[
E = \sum_{i=1}^{\infty} e_{i,i+1}(\text{Id}) \in (I \times I)(\mathbb{F}),
\]

where \( \text{Id} \) is the identity \((J \times J)\)-matrix. The \((I \times I)\)-matrix \( E \) contains \( \leq 1 \) nonzero elements in each row and in each column. Therefore, for an arbitrary \((I \times I)\)-matrix \( x \) the products \( Ex, xE \) make sense.

Let \( a = (a_{ij})_{i,j \in \mathbb{N}}, a_{ij} \in (J \times J)(\mathbb{F}) \). Consider the \((I \times I)\)-matrix

\[
\tilde{a} = (\tilde{a}_{ij})_{i,j \in \mathbb{N}}, \quad \tilde{a}_{ij} = \sum_{k=0}^{\infty} a_{i-1-k,j-k},
\]

where we let \( a_{ij} = 0 \) for \( i \leq 0 \) or \( j \leq 0 \). It is easy to see that for \( a \in M(I, \mathbb{F}) \) the matrix \( \tilde{a} \) also lies in \( M(I, \mathbb{F}) \).

Lemma 3. \( [E, \tilde{a}] = a \).

Proof. We have

\[
E \tilde{a} = \left( \sum_{i=1}^{\infty} e_{i,i+1}(1) \right) \left( \sum_{j,k=1}^{\infty} e_{jk}(\tilde{a}_{jk}) \right) = \sum_{i,k=1}^{\infty} e_{ik}(\tilde{a}_{i+1,k});
\]

1V.V. Sergeichuk (2020), personal communication.
\[ \tilde{a} \ E = \left( \sum_{j,k=1}^{\infty} e_{jk}(\tilde{a}_{jk}) \right) \left( \sum_{i=1}^{\infty} e_{i,i+1}(1) \right) = \sum_{j,i=1}^{\infty} e_{j,i+1}(\tilde{a}_{ji}) = \sum_{i \geq 1, k \geq 2} e_{ik}(\tilde{a}_{i,k-1}). \]

Finally,
\[ [E, \tilde{a}] = \sum_{i,k=1}^{\infty} e_{ik}(\tilde{a}_{i+1,k} - \tilde{a}_{i,k-1}), \]
where \( \tilde{a}_{i,0} = 0 \). It remains to verify that
\[ (3) \quad \tilde{a}_{i+1,j} - \tilde{a}_{i,j-1} = a_{ij} \]
for all \( i, j \in \mathbb{N} \).
Define \( a_{ij} = 0 \) whenever \( i \leq 0 \) or \( j \leq 0 \). Then
\[ \tilde{a}_{ij} = \sum_{k=0}^{\infty} a_{i-1-k,j-k}. \]
Now,
\[ \tilde{a}_{i+1,j} - \tilde{a}_{i,j-1} = \sum_{k=0}^{\infty} a_{i-k,j-k} - \sum_{k=0}^{\infty} a_{i-1-k,j-1-k} = a_{ij}, \]
which completes the proof of the lemma. \( \square \)

Proof of Theorem [1]. (a) An arbitrary \((I \times I)\)-matrix \( a \) over the field \( \mathbb{F} \) can be divided into blocks \( a = (a_{ij})_{\mathbb{N} \times \mathbb{N}} \), where each block \( a_{ij} \) is a \((J \times J)\)-matrix. If \( a \) lies in \( \mathfrak{gl}(I, \mathbb{F}) \) then each block \( a_{ij} \) lies in \( \mathfrak{gl}(J, \mathbb{F}). \) The reverse statement is not true.

Suppose that \( a \in \mathfrak{gl}(I, \mathbb{F}) \). We will show that the matrix \( \tilde{a} \) (see above) also lies in \( \mathfrak{gl}(I, \mathbb{F}) \). Choose a column indexed by \((n, \alpha) \in \mathbb{N} \times J, n \in \mathbb{N}, \alpha \in J \). We need to verify that the \( \alpha \)-th column of the \((I \times J)\)-matrix
\[ \begin{pmatrix} \tilde{a}_{1n} \\ \tilde{a}_{2n} \\ \vdots \end{pmatrix} \]
contains finitely many nonzero entries. Since the \((I \times I)\)-matrix \( a \) has finitely many nonzero entries in each column it follows that for an arbitrary \( l \in \mathbb{N} \) there exists a positive integer \( N(l, \alpha) \) such that the \((J \times J)\)-matrix \( a_{kl} \) has zero \( \alpha \)-th column for \( k > N(l, \alpha) \).

Let \( i > n + \max(N(1, \alpha) + \cdots + N(n, \alpha)) = s \). Then \( i - 2 \geq n - 1 \). By the equality (2), we have
\[ \tilde{a}_{in} = a_{i-1,n} + \cdots + a_{i-n,1}. \]
All summands on the right hand side have zero $\alpha$-th column. Hence the matrix $\tilde{a}_{in}$ has zero $\alpha$-th column. The matrix
\[
\begin{pmatrix}
\tilde{a}_{1n} \\
\vdots \\
\tilde{a}_{sn}
\end{pmatrix}
\]
has finitely many nonzero entries in the $\alpha$-th column since the $(J \times J)$-matrices $\tilde{a}_{1n}, \ldots, \tilde{a}_{sn}$ have finitely many nonzero entries in each column.

We showed that the matrix $\tilde{a}$ lies in $\mathfrak{gl}(I, \mathbb{F})$. As above, let
\[
E = \sum_{i=1}^{\infty} e_{i,i+1}(\text{Id}_J),
\]
where $\text{Id}_J$ is the identity $(J \times J)$-matrix. By Lemma 3, we have
\[
a = [E, \tilde{a}] \in [\mathfrak{gl}(I, \mathbb{F}), \mathfrak{gl}(I, \mathbb{F})].
\]
Hence, we proved Theorem (a).

(b) We need to show that for an arbitrary matrix $a \in M_{rcf}(I, \mathbb{F})$ the matrix $\tilde{a}$ also lies in $M_{rcf}(I, \mathbb{F})$. Consider indices $(k, j), (k', j') \in \mathbb{N} \times J$. The entry $\tilde{a}_{(k,j),(k',j')}$ of the matrix $\tilde{a}$ is the $(j,j')$-th entry of the block $\tilde{a}_{k,k'}$. We have
\[
\tilde{a}_{k,k'} = \sum_{t=0}^{k-2} a_{k-1-t,k'-t}.
\]
Hence,
\[
\tilde{a}_{(k,j),(k',j')} = (\tilde{a}_{k,k'})_{j,j'} = \sum_{t=0}^{k-2} (a_{k-1-t,k'-t})_{j,j'} = \sum_{t=0}^{k-2} a_{(k-1-t,j),(k'-t,j')}.
\]
We showed that the $(k, j)$-th row of the matrix $\tilde{a}$ is a sum of permuted $(1, j)$-th, $\ldots$, $(k-1, j)$-th rows of the matrix $a$. Hence, every row of the matrix $\tilde{a}$ contains finitely many nonzero elements, $\tilde{a} \in M_{rcf}(I, \mathbb{F})$. This proves the part (b) of the theorem.

(c) Consider a $(\mathbb{Z} \times \mathbb{Z})$-matrix
\[
E_J = \sum_{i \in \mathbb{Z}} e_{i,i+1}(1).
\]
For a $(\mathbb{Z} \times \mathbb{Z})$-matrix $a = (a_{ij})_{i,j \in \mathbb{Z}}$ consider the matrix $\tilde{a} = (\tilde{a}_{ij})_{i,j \in \mathbb{Z}}$, where
\[
\tilde{a}_{ij} = \begin{cases} 
a_{i-1,j} + \cdots + a_{1,2-i+j} & \text{for } i \geq 2 \\
0 & \text{for } i = 1 \\
-a_{i,j+1} - \cdots - a_{0,j-i+1} & \text{for } i \leq 0.
\end{cases}
\]
We claim that \([E_J, \tilde{a}] = a\). Indeed, a straightforward computation shows that

1. \([E_J, \tilde{a}]_{ij} = \tilde{a}_{i+1,j} - \tilde{a}_{i,j+1},\)
2. \(\tilde{a}_{i+1,j} - \tilde{a}_{i,j+1} = a_{ij}\)

for all \(i, j \in \mathbb{Z}\).

It remains to check that for an arbitrary matrix \(a \in MJ(\mathbb{F})\) the matrix \(\tilde{a}\) also lies in \(MJ(\mathbb{F})\). Suppose that \(a_{ij} = 0\) whenever \(|i - j| > k\). The expression for \(\tilde{a}_{ij}\) implies that \(\tilde{a}_{ij} = 0\) whenever \(|i - j| > k + 1\). Hence \(\tilde{a} \in MJ(\mathbb{F})\). This completes the proof of the part (c).

(d) The Lie algebra \(\mathfrak{o}_\infty(I, \mathbb{F})\) (resp., \(\mathfrak{sp}_\infty(I, \mathbb{F})\)) is simple since it has a local system of simple finite-dimensional subalgebras \(\mathfrak{o}_\infty(I_0, \mathbb{F})\) (resp., \(\mathfrak{sp}_\infty(J_0, \mathbb{F})\)), where \(I_0, J_0\) run over all nonempty finite subsets of \(I\), the order of \(J_0\) is even. This implies \(\mathfrak{o}_\infty(I, \mathbb{F}) = [\mathfrak{o}_\infty(I, \mathbb{F}), \mathfrak{o}_\infty(I, \mathbb{F})]\) and \(\mathfrak{sp}_\infty(I, \mathbb{F}) = [\mathfrak{sp}_\infty(I, \mathbb{F}), \mathfrak{sp}_\infty(I, \mathbb{F})]\). \(\square\)

In \([13]\), I.N. Herstein formulated a series of conjectures about links between derivations and automorphisms of an associative algebra \(A\) and derivations and automorphisms of Lie algebras \([A, A], K(A, *)\). These conjectures were proved by K. Beidar, M. Brešar, M. Chebotar and W. Martindale in \([2, 3, 4]\).

We will formulate here and in Sec. 5 only particular cases of their results that are directly related to this work:

**Theorem I (K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale; see, Cor. 1.4(b), [4])** Let \(A\) be a simple associative algebra with the center \(Z\). The algebra \(A\) is not finite-dimensional over \(Z\). Let \(d\) be a derivation of the Lie algebra \([A, A]/[A, A] \cap Z\). Then there exists a derivation \(\tilde{d} : A \to A\) of the associative algebra \(A\) such that \(d(a) - \tilde{d}(a) \in Z\) for an arbitrary element \(a \in [A, A]\).

**Theorem II (K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale; see, Cor. 1.9(b), [4])** Let \(A\) be a simple associative algebra with an involution \(* : A \to A\). Let \(Z\) be the center of the algebra \(A\). The algebra \(A\) is not finite-dimensional over \(Z\). Let \(d\) be a derivation of the Lie algebra \([K, K]/[K, K] \cap Z\), where \(K = K(A, *) = \{a \in A | a^* = -a\}\). Then there exists a derivation \(\tilde{d} : A \to A\) of the associative algebra \(A\) such that \(d(a) - \tilde{d}(a) \in Z\) for an arbitrary element \(a \in [K, K]\).

**Proof of Theorem 3** (a) The algebra \(A = M_\infty(I, \mathbb{F})\) is simple and has zero center. Let \(d\) be a derivation of the Lie algebra \(\mathfrak{sl}_\infty(I, \mathbb{F}) = [A, A]\). By Theorem I above, there exists a derivation \(\tilde{d}\) of the algebra \(A\) that
coincides with \( \tilde{d} \) on \([A,A]\). By Theorem 2(a), the derivation \( \tilde{d} \) looks as \( \text{ad}_{\mathfrak{sl}_\infty(I,F)}(a) \), where \( a \in M_{rcf}(I,F) \).

(b) Now, let \( \ast \) be the transpose involution or the symplectic involution of the algebra \( A = M_\infty(I,F) \). Then \( \varnothing_\infty(I,F) \) (resp., \( \mathfrak{sp}_\infty(I,F) \)) is the Lie algebra of skew-symmetric elements \( K = K(A,\ast) \). By Theorem II above, there exists a derivation \( \tilde{d} \) of the algebra \( A \) that coincides with \( d \) on \([K,K]\). By Theorem 1(d), \([K,K] = K \). Theorem 2(a) implies that the derivation \( \tilde{d} \) is the restriction of a derivation \( \text{ad}_{M_\infty(I,F)}(a) \), where \( a \in M_{rcf}(I,F) \).

Both the transpose and the symplectic involution on the algebra \( M_\infty(I,F) \) extend to \( M_{rcf}(I,F) \). Let

\[
\begin{align*}
a &= a_h + a_k, \quad \text{where} \quad a_h^* = a_h, \quad a_k^* = -a_k.
\end{align*}
\]

For an arbitrary element \( b \in K \) we have \([a,b] = [a_h,b] + [a_k,b] \in K\). However,

\[
\begin{align*}
[a_h,b]^* &= [a_h,b], \quad [a_k,b]^* = -[a_k,b].
\end{align*}
\]

Hence \([a_h,K] = (0), \quad [a,b] = [a_k,b], \quad \text{that is,} \quad \text{ad}_K(a) = \text{ad}_K(a_k), \quad \text{where} \quad a_k \in K(M_{rcf}(I,F),\ast)\).

(c) Let \( d \) be a derivation of the Lie algebra \( \mathfrak{gl}(I,F) = M(I,F)^{-} \). By Theorem I of Beidar-Brešar-Chebotar-Martindale [4], there exists a derivation \( \tilde{d} \) of the associative algebra \( M(I,F) \) such that \( d(a) - \tilde{d}(a) \) lies in the center \( Z \) of \( M(I,F) \) for all elements \( a \in [M(I,F),M(I,F)] \). By Theorem 1(a), \([M(I,F),M(I,F)] = M(I,F) \), hence \( d(a) - \tilde{d}(a) \in Z \) for all elements \( a \in M(I,F) \).

Now, for arbitrary elements \( a, b \in M(I,F) \) we have

\[
\begin{align*}
d([a,b]) &= [d(a),b] + [a,d(b)] = [\tilde{d}(a),b] + [a,\tilde{d}(b)] = \tilde{d}([a,b]).
\end{align*}
\]

Hence, \( d = \tilde{d} \) on \([M(I,F),M(I,F)] \). Again by Theorem 1(a), we get \( d = d \). It remains to refer to Theorem 2(c). This proves the assertion about derivations of \( \mathfrak{gl}(I,F) \).

Let \( d \) be a derivation of the Lie algebra \( \mathfrak{gl}_{rcf}(I,F) \). The subalgebra \( \mathfrak{sl}_\infty(I,F) \) is an ideal of \( \mathfrak{gl}_{rcf}(I,F) \). For arbitrary elements \( a, b \in \mathfrak{sl}_\infty(I,F) \) we have

\[
\begin{align*}
d([a,b]) &= [d(a),b] + [a,d(b)] \in [\mathfrak{gl}_{rcf}(I,F),\mathfrak{sl}_\infty(I,F)] \subseteq \mathfrak{sl}_\infty(I,F).
\end{align*}
\]

Hence the ideal \( \mathfrak{sl}_\infty(I,F) \) is differentially invariant. By Theorem 3(a), there exists an element \( a \in \mathfrak{gl}_{rcf}(I,F) \) such that \( d(x) = [a,x] \) for an arbitrary element \( x \in \mathfrak{sl}_\infty(I,F) \). Consider the derivation \( d' = d - \text{ad}(a) \) of the algebra \( \mathfrak{gl}_{rcf}(I,F) \). We have

\[
d'(\mathfrak{sl}_\infty(I,F)) = (0).
\]
For arbitrary elements $x \in \mathfrak{sl}_\infty(I, \mathbb{F})$, $y \in \mathfrak{gl}_{rcf}(I, \mathbb{F})$ we have
\[
0 = d'(x, y) = [x, d'(y)].
\]
Hence $d'(\mathfrak{gl}_{rcf}(I, \mathbb{F}))$ lies in the centralizer of $\mathfrak{sl}_\infty(I, \mathbb{F})$. It is easy to see that the centralizer of $\mathfrak{sl}_\infty(I, \mathbb{F})$ in the algebra $\mathfrak{gl}_{rcf}(I, \mathbb{F})$ is the space of scalar matrices $\alpha \cdot \text{Id}$, where $\alpha \in \mathbb{F}$ and Id is the identity $(I \times I)$-matrix. Hence
\[
d'(\mathfrak{gl}_{rcf}(I, \mathbb{F})) \subseteq \mathbb{F} \cdot \text{Id}.
\]
For arbitrary elements $a, b \in \mathfrak{gl}_{rcf}(I, \mathbb{F})$ we have $d'([a, b]) = [d'(a), b] + [a, d'(b)] = 0$. Hence,
\[
d'([\mathfrak{gl}_{rcf}(I, \mathbb{F}), \mathfrak{gl}_{rcf}(I, \mathbb{F})]) = (0).
\]
By Theorem \ref{thm:ad}(a),
\[
[\mathfrak{gl}_{rcf}(I, \mathbb{F}), \mathfrak{gl}_{rcf}(I, \mathbb{F})] = \mathfrak{gl}_{rcf}(I, \mathbb{F}),
\]
which implies $d' = 0$, $d = \text{ad}(a)$. This completes the proof of the part concerning $\mathfrak{gl}_{rcf}(I, \mathbb{F})$.

Let $d : \mathfrak{gl}_f(\mathbb{F}) \to \mathfrak{gl}_f(\mathbb{F})$ be a derivation. As above, we conclude that the ideal $\mathfrak{sl}_\infty(\mathbb{Z}, \mathbb{F})$ of the algebra $\mathfrak{gl}_f(\mathbb{F})$ is differentially invariant. By Theorem \ref{thm:main}(a), there exists a matrix $a \in M_{rcf}(\mathbb{Z}, \mathbb{F})$ such that $d(x) = [a, x]$ for an arbitrary element $x \in \mathfrak{sl}_\infty(\mathbb{Z}, \mathbb{F})$. Consider the bimodule derivation
\[
d' : \mathfrak{gl}_f(\mathbb{F}) \to \mathfrak{gl}_{rcf}(\mathbb{Z}, \mathbb{F}), \quad d'(x) = d(x) - [a, x], \quad x \in \mathfrak{gl}_f(\mathbb{F}).
\]
We have $d'([\mathfrak{sl}_\infty(\mathbb{Z}, \mathbb{F})]) = (0)$. As above, for arbitrary elements $x \in \mathfrak{sl}_\infty(\mathbb{Z}, \mathbb{F})$, $y \in \mathfrak{gl}_f(\mathbb{F})$ we have
\[
0 = d'([x, y]) = [x, d'(y)].
\]
Hence $d'([\mathfrak{gl}_f(\mathbb{F})])$ lies in the centralizer of $\mathfrak{sl}_\infty(\mathbb{Z}, \mathbb{F})$ in the algebra $\mathfrak{gl}_{rcf}(\mathbb{Z}, \mathbb{F})$. In other words,
\[
d'(\mathfrak{gl}_f(\mathbb{F})) \subseteq \mathbb{F} \cdot \text{Id},
\]
where Id is the identity $(\mathbb{Z} \times \mathbb{Z})$-matrix. By Theorem \ref{thm:ad}(a), $\mathfrak{gl}_f(\mathbb{F}) = [\mathfrak{gl}_f(\mathbb{F}), \mathfrak{gl}_f(\mathbb{F})]$. So, by Proposition \ref{prop:ideal}(2), we conclude that $d' = 0$. Hence $d(x) = [a, x]$ for an arbitrary element $x \in \mathfrak{gl}_f(\mathbb{F})$.

It was already mentioned in the proof of Theorem \ref{thm:main}(c) that if a $(\mathbb{Z} \times \mathbb{Z})$-matrix $a$ does not lie in $\mathfrak{gl}_f(\mathbb{F})$ then the matrix $[a, E_1]$, $E_1 = \sum_{i \in \mathbb{Z}} e_{i,i+1}(1)$, also does not lie in $\mathfrak{gl}_f(\mathbb{F})$. Hence $a \in \mathfrak{gl}_f(\mathbb{F})$, which completes the proof of Theorem \ref{thm:main}. \hfill \Box
5. Automorphisms of algebras of infinite matrices

Let $V$ be an infinite-dimensional vector space and let $V^*$ be the dual space of $V$. A subspace $W \subseteq V^*$ is called total if

$$ \bigcap_{w \in W} \ker w = (0). $$

Recall the action of the algebra $\text{End}_F(V)$ on the dual space $V^*$: for a linear transformation $a \in \text{End}_F(V)$ and a linear functional $f \in V^*$ we define $(fa)(v) = f(a(v))$, $v \in V$.

For a total subspace $W \subseteq V^*$ consider the subalgebra $\text{End}_F(V|W) = \{ a \in \text{End}_F(V) \mid Wa \subseteq W \}$. Then the subalgebra $\text{End}_{fin}(V|W) = \text{End}_F(V|W) \cap \text{End}_{fin}(V)$ is a locally matrix algebra that can be identified with the tensor product $V \otimes_F W$. Moreover, $\text{End}_{fin}(V|W)$ is dense in $\text{End}_F(V)$ in the Tykhonoff topology (see, [14]).

Let $E = \{ e_i, i \in I \}$ be a basis of the vector space $V$ and let $E^* = \{ e^*_i, i \in I \} \subset V^*$ be the dual basis: $e^*_i(e_j) = \delta_{ij}$. It is easy to see that the subspace $\text{Span}_F(E^*)$ is total. In the basis $E$ the algebra $\text{End}_F(V|\text{Span}_F(E^*))$ corresponds to the matrix algebra $M_{rcf}(I, F)$. The subalgebra $\text{End}_{fin}(V|\text{Span}_F(E^*))$ corresponds to $M_{\infty}(I, F)$.

If $W = V^*$ then $\text{End}_F(V|W) = \text{End}_F(V)$, $\text{End}_{fin}(V|W) = \text{End}_{fin}(V)$.

N. Jacobson ([14], Chap. 9, Sec. 11, Th. 7) proved that for an arbitrary algebra $A$,

$$ \text{End}_{fin}(V|W) \subseteq A \subseteq \text{End}_F(V|W) $$

and an arbitrary automorphism $\varphi$ of $A$ there exists an element $x \in G(\text{End}_F(V|W))$ such that $\varphi(a) = x^{-1}ax$, $a \in A$.

Substituting $W = \text{Span}_F(E^*)$ or $W = V^*$ we obtain Theorem 4.

In the proof of Theorem 6 on automorphisms of Lie algebras, we will again rely on the work [14] of K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale on Herstein’s Conjectures.

The results in [14] are quite general. As in Sec. 4, we will use only a small part of them that is directly related to our proof.

Theorem III (K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale; see, Cor. 1.2 (c), [14]) Let $\text{char} F \neq 2$ and let $A$ be a simple associative $F$-algebra with the center $Z$ and $A$ is not finite-dimensional over $Z$. If $\alpha$ is an automorphism of the Lie algebra $[A, A]$ then there exist mappings $\varphi, \psi : A \to A$ such that

(i) $\varphi$ is either an automorphism or the negative of an anti-automorphism of $A$,
(ii) \( \psi(A) \subseteq Z, \psi([A, A]) = (0) \),
and \( \alpha(a) = \varphi(a) + \psi(a) \) for all \( a \in [A, A] \).

Theorem IV (K.I. Beidar, M. Brešar, M. Chebotar and W.S. Martindale; see, Cor. 1.7, [4]) Let \( \text{char} \mathbb{F} \neq 2 \) and Let \( A \) be a simple associative \( \mathbb{F} \)-algebra with an involution and let \( K \) be the Lie algebra of skew-symmetric elements. Suppose that the algebra \( A \) is not finite-dimensional over \( Z \). Then for an arbitrary automorphism \( \alpha \) of the Lie algebra \([K, K]\) there exists an automorphism \( \varphi \) of the algebra \( A \) such that \( \alpha(a) = \varphi(a) \) for all \( a \in [K, K] \).

Theorem III above implies that in order to describe Lie automorphisms of a simple associative algebra we need to describe its anti-automorphisms first.

Proof of Theorem 5. (a) The algebras \( M_\infty(I, \mathbb{F}), M_{\text{ref}}(I, \mathbb{F}), MJ(\mathbb{F}) \) are closed with respect to the transpose \( t \). If \( * \) is an anti-automorphism then the composition \( \varphi \) of \( * \) and \( t \) is an automorphism, \( (a^t)^* = \varphi(a) \) for an arbitrary element \( a \). Hence, \( a^* = \varphi(a^t) \).

(b) N. Jacobson ([14], Chap. IX.12, Th. 8) showed that if an algebra \( A, \text{End}_{fin}(V|W) \subseteq A \subseteq \text{End}(V|W) \), has an anti-isomorphism then \( \dim \mathbb{F} V = \dim \mathbb{F} W \). P. Erdos and J. Kaplansky (see, [14], Chap. IX.5, Th. 2) showed that \( \dim \mathbb{F} V^* = |\mathbb{F}|^{\dim \mathbb{F} V} > \dim \mathbb{F} V \).

Hence, the algebras \( \text{End}_{fin}(V) = \text{End}_{fin}(V|V^*) \) and \( M(I, \mathbb{F}) \cong \text{End}_{\mathbb{F}}(V) = \text{End}(V|V^*) \) do not have anti-isomorphisms. \( \square \)

Proof of Theorem 6. The part (a) immediately follows from Theorem III above applied to the algebra \( A = M_\infty(I, \mathbb{F}) \) and Theorem II(a).

(b) Let \( * \) be the transpose or the symplectic involution on the algebra \( A = M_\infty(I, \mathbb{F}) \). Then \( K = K(A, *) = o_\infty(I, \mathbb{F}) \) or \( sp_\infty(I, \mathbb{F}) \). By Theorem II(d), we have \( K = [K, K] \). By Theorem IV above, an arbitrary automorphism \( \varphi \) of the Lie algebra \( K \) lifts to an automorphism of the algebra \( A \). By Theorem II(a), there exists an element \( x \in G(M_{\text{ref}}(I, \mathbb{F})) \) such that \( \varphi(a) = x^{-1}ax \) for all elements \( a \in K \). Hence
\[
(x^{-1}ax)^* = -x^*a(x^*)^{-1} = -x^{-1}ax,
\]
which implies
\[
(xx^*)a(xx^*)^{-1} = a.
\]
We proved that the element $xx^*$ commutes with all elements from $K$. Since the algebra $A$ is generated by the subspace $K$ it follows that the element $xx^*$ commutes with all elements from $M_\infty(I,F)$ and, therefore, is a scalar matrix $xx^* = \alpha \cdot \text{Id}$, $0 \neq \alpha \in F$. This completes the proof of the part (b).

The part (c) immediately follows from Theorem III above, Theorem 1, Theorem 4(c), and parts (d),(e) follow from Theorem IV above, Theorem 5(b) and Theorem 4(b),(c). This completes the proof of Theorem 6. □

**REFERENCES**

[1] Baranov A.A., Classification of the direct limits of involution simple associative algebras and the corresponding dimension groups, *Journal of Algebra*, 381 (2013), P.73-95.

[2] Beidar K.I., Brešar M., Chebotar M.A., Martindale 3rd W.S., On Herstein’s Lie map conjectures I, *Trans. Amer. Math. Soc.*, 353 (2001), P.4235-4260.

[3] Beidar K.I., Brešar M., Chebotar M.A., Martindale 3rd W.S., On Herstein’s Lie map conjectures II, *J. Algebra*, 238 (2001), P.239-264.

[4] Beidar K.I., Brešar M., Chebotar M.A., Martindale 3rd W.S., On Herstein’s Lie map conjectures III, *J. Algebra*, 249 (2002), P.59-94.

[5] Bezushchak O., Derivations and automorphisms of locally matrix algebras, *J. Algebra*, 576 (2021), P.1-26.

[6] Bezushchak O., On the Lie structure of locally matrix algebras, *Carpathian Math. Publ.*, 12 (2020), no.2, P.311–316.

[7] Bezushchak O. and Oliynyk B., Unital locally matrix algebras and Steinitz numbers, *J. Algebra Appl.*, 19 (2020), no.9., Doi:10.1142/S0219498820501807.

[8] Bezushchak O. and Oliynyk B., Primary decompositions of unital locally matrix algebras, *Bull. Math. Sci.*, 10 (2020), no.1, Doi:10.1142/S166436072050006X.

[9] Date E., Jimbo M., Kashiwara M., Miwa N., Transformation groups for soliton equations. III. Operator approach to the Kadomtsev-Petrvashvili equation, *J. Phys. Soc. Japan*, 50 (1981), no.11, P.3806-3812.

[10] Drozd Yu.A., Kirichenko V.V., *Finite Dimensional Algebras*, Springer-Verlag, Berlin-Heidelberg-New York, (1994).

[11] Feigin B.L., Tsygan B.L., Cohomologies of Lie algebras of generalized Jacobi matrices, *Funktional. Anal. i Prilozhen.*, 17:2 (1983), P.86-87; *Funct. Anal. Appl.*, 17:2 (1983), P.153-155.

[12] Frenkel I., Penkov I., Serganova V., A categorification of the boson-fermion correspondence via representation theory of $\mathfrak{sl}(\infty)$, *Comm. Math. Phys.*, 3 (2016), P.911-931.

[13] Herstein I.N., Lie and Jordan structures in simple associative rings, *Bull. Amer. Math. Soc.*, 67 (1961), P.517-531.

[14] Jacobson N., *Lectures in abstract algebra*. Graduate Texts in Mathematics, Vol.2, Linear algebra, Springer-Verlag, Berlin-Heidelberg-New York, (1975).

[15] Kurosh A., Direct decompositions of simple rings, *Rec. Math. [Mat. Sbornik] N.S.*, 11(53) (1942), no.3, P.245-264.
[16] Neeb K.-H., Derivations of locally simple Lie algebras, *J. Lie Theory*, 15 (2005), P.589-594.
[17] Penkov I., Serganova V., Categories of integrable $\mathfrak{sl}(\infty)$-, $\mathfrak{o}(\infty)$-, $\mathfrak{sp}(\infty)$-modules, in Representation Theory and Mathematical Physics, *Contemporary Math.*, *Amer. Math. Soc.*, 557 (2011), P.335-357.
[18] Pierce R.S., *Associative Algebras*, Graduate Texts in Mathematics, 88, Springer-Verlag, Berlin-Heidelberg-New York, (1982).
[19] Stumme N., Automorphisms and conjugacy of compact real forms of the classical infinite dimensional matrix Lie algebras, *Forum. Math.* 13, (2001), P.817-851.

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