Modelling the motion of a non-free system of rigid bodies using the Lagrange equations of the first kind

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Abstract. A technique for modelling the motion of a non-free system of rigid bodies is proposed (the mutual position of bodies is limited by superimposed holonomic constraints) using Lagrange equations of the first kind. In the paper, the proposed technique was used to simulate the compression of the amortization strut of the aircraft landing gear.

1. Introduction

Traditionally, Lagrange equations of the second kind in generalized coordinates were used to solve the motion of a non-free system of rigid bodies, which made it possible to reduce the dimensionality of the system of equations being solved. The main drawback of this approach was the fact that, depending on the choice of generalized coordinates, the type of the system being solved changed, that is, the equations were determined by an individual approach to the problem being solved. In the case of a change (complication) in the system of rigid bodies, it was necessary to obtain again resolving equations. Modeling with the proposed approach allows representing the model of the system as a set of objects: solids, power factors and mechanical connections, which ensure the modularity and extensibility of the model. This in turn makes it possible to automate the process of constructing the model [1–5]. Calculations were carried out on the work capacity of the amortization struts during the drop testing of the landing gear of specific aircraft in Siberian Aeronautical Research Institute named after S. A. Chaplygin (Tu-204SM, MS-21, Ka-62). When compared with the results of measurements of the main parameters carried out during the tests, the high accuracy of the analytical calculations obtained using the model was confirmed.

2. Mathematical foundations

The motion of an arbitrary system of rigid bodies with holonomic constraints relative to the fixed coordinate system \(Oxyz\) is considered. For each body, a mobile coordinate system \(C_\xi\eta\zeta\) is defined, rigidly connected with the body, starting at the center of mass of the body and with axes coinciding with the principal axes of inertia of the body.

The inertial characteristics of the body are given by the mass \(m_i\) and the inertia tensor:

\[
I_{ij} = \begin{pmatrix}
I^{(\xi)}_{ij} & 0 & 0 \\
0 & I^{(\eta)}_{ij} & 0 \\
0 & 0 & I^{(\zeta)}_{ij}
\end{pmatrix}
\]
The center of mass of the body is determined by its position vector:
\[ \mathbf{r}_i = (x_i, y_i, z_i)^T \]
and the rotation quaternion:
\[ q_i = (\cos(\alpha_i/2), u_i, \sin(\alpha_i/2)) \]
where \( \alpha_i \) is the angle of rotation of the body, \( u_i \) is the axis of rotation (vector of unit length).

The angular position of the body can also be represented as a rotation matrix:
\[
\mathbf{R}_i = \begin{bmatrix}
1 - 2b_i^2 - 2c_i^2 & 2a_i b_i - 2w_i c_i & 2a_i c_i + 2w_i b_i \\
2a_i b_i + 2w_i c_i & 1 - 2a_i^2 - 2c_i^2 & 2b_i c_i - 2w_i a_i \\
2a_i c_i - 2w_i b_i & 2b_i c_i + 2w_i a_i & 1 - 2a_i^2 - 2b_i^2
\end{bmatrix},
\]
where \( w_i \) is the scalar (real) part of the rotation quaternion; \( a_i, b_i \) and \( c_i \) are components of the vector (imaginary) part of the rotation quaternion.

The position of the body can be represented as a transition matrix in homogeneous coordinates:
\[
\mathbf{T}_i = \begin{bmatrix}
\mathbf{R}_i & \mathbf{r}_i \\
0 & 1
\end{bmatrix}.
\]

Equations of motion of the center of mass of the body:
\[
d\mathbf{v}_i/dt = m_i^{-1} (\mathbf{f}_i + \mathbf{f}_r),
\]
where \( \mathbf{v}_i \) is the velocity of the center of mass; \( \mathbf{f}_i \) is the total vector of active forces acting on the body; \( \mathbf{f}_r \) is the total vector of the constraint reactions acting on the body.

Equations of rotational motion (dynamic Euler equations) in a fixed coordinate system:
\[
d\mathbf{\omega}_i/dt = \mathbf{I}^{-1}_i (\mathbf{\tau}_i - \mathbf{\omega}_i \times (\mathbf{I}_i \mathbf{\omega}_i) + \mathbf{\tau}_r),
\]
\[
\mathbf{I}_i = \mathbf{R}_i \mathbf{I}_0 \mathbf{R}_i^T,
\]
where \( \mathbf{\omega}_i \) is the angular velocity of the body; \( \mathbf{\tau}_i \) is the total vector of the moments of the active forces acting on the body; \( \mathbf{\tau}_r \) is the total vector of constraint reaction moments acting on the body.

Equations of motion of the system of rigid bodies can be written in the following form:
\[
d\mathbf{V}/dt = \mathbf{M}^{-1} (\mathbf{F} + \mathbf{F}_r),
\]
\[
\mathbf{V} = \begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{\omega}_1 \\
\vdots \\
\mathbf{v}_n \\
\mathbf{\omega}_n
\end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix}
m_1 \mathbf{E}_3 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots \\
0 & 0 & \cdots & m_n \mathbf{E}_3 & 0 \\
0 & 0 & \cdots & 0 & 1_n
\end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix}
\mathbf{f}_1 \\
\mathbf{\tau}_1 - \mathbf{\omega}_1 \times (\mathbf{I}_1 \mathbf{\omega}_1) \\
\vdots \\
\mathbf{f}_n \\
\mathbf{\tau}_n - \mathbf{\omega}_n \times (\mathbf{I}_n \mathbf{\omega}_n)
\end{bmatrix}, \quad \mathbf{F}_r = \begin{bmatrix}
\mathbf{f}_r \\
\mathbf{\tau}_r
\end{bmatrix},
\]
where \( n \) is the number of rigid bodies in the system; \( \mathbf{E}_3 \) is the unit matrix of dimensions 3x3.

Ideal holonomic constraints are superposed on the positions of the bodies. One such constraint between \( i \)th and \( j \)th bodies can be described by the scalar function \( C_k(\mathbf{r}_i, q_i, \mathbf{r}_j, q_j) \). Constraint functions for the system of rigid bodies are combined into a column vector \( \mathbf{C} \) with a dimension \( d \) equal to the number of constraints in the system.

Unilateral constraint, restricting movement in only one direction, is given as [5]:
\[
C_k(\mathbf{r}_i, q_i, \mathbf{r}_j, q_j) \geq 0.
\]

Bilateral constraint is given as:
\[
C_k(\mathbf{r}_i, q_i, \mathbf{r}_j, q_j) = 0.
\]

Consider the \( k \)th constraint function that restricts the displacement of \( i \)th and \( j \)th bodies by a given distance \( L \) between two points of these bodies:
\[ \mathbf{C}_c = \left( (\mathbf{p}_j - \mathbf{p}_i)^2 - L^2 \right) / 2, \]

where \( \mathbf{p}_i \) and \( \mathbf{p}_j \) are position vectors of the points in the fixed coordinate system.

Time derivative of the constraint function:

\[
d\mathbf{C}_c / dt = \left( \left( \mathbf{p}_j - \mathbf{p}_i \right) \left( \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) \right) - \mathbf{v}_i - \omega_i \times \left( \mathbf{p}_i - \mathbf{r}_i \right) \right) \frac{d\mathbf{r}_j}{dt}. \]

or

\[
d\mathbf{C}_c / dt = -\mathbf{d}^T \left( \left( \mathbf{p}_j - \mathbf{r}_j \right) \times \mathbf{d} \right)^T \left( \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) \right) \mathbf{d}. \]

Denoting \( \mathbf{J}_{ij} = \left( \left( \mathbf{p}_j - \mathbf{r}_j \right) \times \mathbf{d} \right)^T \). \( \mathbf{J}_{ij} = \mathbf{d}^T \left( \left( \mathbf{p}_j - \mathbf{r}_j \right) \times \mathbf{d} \right) \), we obtain:

\[
d\mathbf{C}_c / dt = \mathbf{J}_{ij} \mathbf{v}_j \mathbf{v}^T. \]

Generalizing for any constraint function:

\[
d\mathbf{C} / dt = \mathbf{J} \mathbf{v}, \quad \mathbf{J} = \left[ \begin{array}{ccc} \mathbf{J}_{11} & \cdots & \mathbf{J}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{J}_{d1} & \cdots & \mathbf{J}_{dn} \end{array} \right], \]

where \( \mathbf{J} \) is the Jacobi matrix for the vector \( \mathbf{C} \) of constraint functions.

The vector of constraint reactions:

\[
\mathbf{F}_c = \mathbf{J} \lambda. \]

where \( \lambda = \left( \lambda_1, \ldots, \lambda_d \right)^T \) is the vector of indeterminate Lagrange multipliers.

Consider the \( k \)th constraint function that bounds the position of the point \( \mathbf{p}_i \) of the \( i \)th body by the plane of the \( j \)th body passing through the point \( \mathbf{p}_j \):

\[ \mathbf{C}_i = \mathbf{n}_j \left( \mathbf{p}_j - \mathbf{p}_i \right), \]

where \( \mathbf{n}_j \) is the normal to the given plane of the \( j \)th body.

Time derivative of the constraint function:

\[
d\mathbf{C}_i / dt = \mathbf{n}_j \left( \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) \right) - \mathbf{v}_i - \omega_i \times \left( \mathbf{p}_i - \mathbf{r}_i \right) + \left( \omega_j \times \mathbf{n}_j \right) \left( \mathbf{p}_j - \mathbf{p}_i \right), \]

\[
d\mathbf{C}_i / dt = \mathbf{d}^T \left( \left( \mathbf{p}_j - \mathbf{r}_j \right) \times \mathbf{d} \right) \left( \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) \right) \frac{d\mathbf{r}_j}{dt}. \]

\[
\mathbf{J}_{ij} = \left( \mathbf{n}_j - \left( \mathbf{p}_j - \mathbf{r}_j \right) \times \mathbf{d} \right)^T \frac{d\mathbf{r}_j}{dt}. \]

Consider the \( k \)th vector constraint function that limits the position of the point of the \( i \)th body to the point of the \( j \)th body (ball joint):

\[ \mathbf{C}_k = \mathbf{p}_j - \mathbf{p}_i, \]

where \( \mathbf{p}_i \) and \( \mathbf{p}_j \) are position vectors of the points in the fixed coordinate system.

Time derivative of the constraint function:

\[
d\mathbf{C}_k / dt = \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) - \mathbf{v}_i - \omega_i \times \left( \mathbf{p}_i - \mathbf{r}_i \right), \]

\[
d\mathbf{C}_k / dt = \left[ -\mathbf{E}_j - \left[ \mathbf{p}_j - \mathbf{r}_j \right] \right] \left[ \mathbf{p}_j - \mathbf{r}_j \right] \frac{d\mathbf{r}_j}{dt} \left( \mathbf{v}_j + \omega_j \times \left( \mathbf{p}_j - \mathbf{r}_j \right) \right) \frac{d\mathbf{r}_j}{dt}. \]

\[
\mathbf{J}_{ki} = \left[ -\mathbf{E}_j - \left[ \mathbf{p}_j - \mathbf{r}_j \right] \right], \quad \mathbf{J}_{ji} = \left[ \mathbf{E}_j - \left[ \mathbf{p}_j - \mathbf{r}_j \right] \right], \]
The numerical integration of the equations of motion of a system of solids will be considered using the fourth-order Runge-Kutta method.

Equations of motion:
\[
dV/dr = M^{-1}F + M^{-1}J^T \lambda, \\
dr_i/dr = v_i, \\
dq_i/dr = \omega_i q_i/2, \quad \forall i \in [1, n].
\]

The velocity vector at the next step of integration is calculated as:
\[
V(t_e + \Delta t) = V(t_e) + \Delta t (k_1 + 2k_2 + 2k_3 + k_4)/6,
\]
where \(k_1, k_2, k_3\) and \(k_4\) are the right-hand sides of the differential equation calculated in four steps.

Introduce notations:
\[
r = (r_1, \ldots, r_n)^T, \quad r = (q_1, \ldots, q_n)^T, \quad r^{(i)} = r(t_e), \quad q^{(i)} = q(t_e), \quad V^{(i)} = V(t_e),
\]
\[
F^{(i)} = F(t_e, r^{(i)}, q^{(i)}, V^{(i)}), \quad J^{(i)} = J(r^{(i)}, q^{(i)}),
\]
then
\[
k_i = M^{-1}F^{(i)} + M^{-1} (J^{(i)})^T \lambda^{(i)}.
\]

To determine \(k_i\), it is necessary to calculate the values of the indeterminate Lagrange multipliers \(\lambda^{(i)}\). Let's write the velocity expression for the second stage:
\[
v^{(2)} = V^{(i)} + \Delta t (k_1)/2.
\]

Using the expression for the derivative of constraint functions vector, we get:
\[
J^{(0)}M^{-1} (J^{(0)})^T \lambda^{(0)} = dC/dt - J^{(0)} (2V^{(0)}/\Delta t + M^{-1}F^{(0)}).
\]

This system of algebraic equations can be solved by the Gauss-Seidel iterative method, using additional conditions:
\[
dC_i/dt = 0 \quad \text{for bilateral constraint},
\]
\[
dC_i/dt \geq 0 \quad \text{for unilateral constraint}.
\]

Then \(k_1, v^{(2)}, q^{(2)}, F^{(2)}, J^{(2)}\) is calculated:
\[
r^{(2)} = r^{(1)} + \Delta t v^{(1)}/2, \quad q^{(2)} = q^{(1)} + \Delta t \omega^{(1)} q^{(1)}/4,
\]
\[
F^{(2)} = F(t_e + \Delta t/2, r^{(2)}, q^{(2)}, V^{(2)}), \quad J^{(2)} = J(r^{(2)}, q^{(2)}).
\]

The following steps are calculated in a similar way:
\[
k_2 = M^{-1}F^{(2)} + M^{-1} (J^{(2)})^T \lambda^{(2)},
\]
\[
r^{(3)} = r^{(2)} + \Delta t v^{(2)}/2, \quad q^{(3)} = q^{(2)} + \Delta t \omega^{(2)} q^{(2)}/4, \quad V^{(3)} = V^{(2)} + \Delta t (k_2)/2.
\]
\[
F^{(3)} = F(t_e + \Delta t/2, r^{(3)}, q^{(3)}, V^{(3)}), \quad J^{(3)} = J(r^{(3)}, q^{(3)}).
\]
\[
k_3 = M^{-1}F^{(3)} + M^{-1} (J^{(3)})^T \lambda^{(3)},
\]
\[
r^{(4)} = r^{(3)} + \Delta t v^{(3)}/2, \quad q^{(4)} = q^{(3)} + \Delta t \omega^{(3)} q^{(3)}/2, \quad V^{(4)} = V^{(3)} + \Delta t (k_3),
\]
\[
F^{(4)} = F(t_e + \Delta t, r^{(4)}, q^{(4)}, V^{(4)}), \quad J^{(4)} = J(r^{(4)}, q^{(4)}).
\]
\[
k_4 = M^{-1}F^{(4)} + M^{-1} (J^{(4)})^T \lambda^{(4)},
\]
\[
r(t_e + \Delta t) = r^{(4)} + \Delta t V^{(4)} + 2V^{(2)} + 2V^{(3)} + V^{(4)})/6,
\]
\[
q(t_e + \Delta t) = q^{(4)} + \Delta t \omega^{(4)} q^{(4)} + 2\omega^{(2)} q^{(2)} + 2\omega^{(3)} q^{(3)} + \omega^{(4)} q^{(4)})/12,
\]
\[
V(t_e + \Delta t) = V^{(4)} + \Delta t (k_1 + 2k_2 + 2k_3 + k_4)/6.
\]
3. Model of the main landing gear leg of the Tu-204SM aircraft

Figure 1 shows the model of the main landing gear leg of the Tu-204SM aircraft [6]. The model consists of six rigid bodies: two pairs of wheels 1 and 2, the bogie 3, the shock absorber rod 4, the piston of the second shock absorber chamber 6 and the shock absorber cylinder together with the weight of the airframe per one leg 5.

![Diagram of the main landing gear leg](image)

**Figure 1.** Model of the main landing gear leg.

Joints “a” and “b”, connecting pairs of wheels with the bogie, as well as the joint “c” connecting the bogie with the shock absorber rod, are hinges. Each hinge connection restricts the two degrees of freedom of the system and is determined by two bilateral constraints. Sliding joint “d” connect the shock absorber rod and joint “e” connect the piston of the second chamber with the cylinder. A shock absorber cylinder is fixed in the sliding fit “f”. The sliding joint also restricts the two degrees of freedom of the system and is determined by two bilateral constraints. Stops “g” and “h” restrict one degree of freedom each and are modeled by the unilateral constraints. Thus, the model includes 14 constraints.

The following six active forces are introduced in the model: two forces of compression $P_w$, force $P_{st}$ by a stabilizing damper and three forces in the shock absorber – $P_1$, $P_2$ and $P_3$. The force $P_1$ includes the frictional force in the axle-box and the force of the hydrodynamic resistance when the liquid flows from chamber 3 to chamber 1 (Figure 2). The force $P_2$ is the force of compression of the gas spring of the chamber 1. The force $P_3$ includes the force of compression of the gas spring of the chamber 2 and the frictional force of the piston.
Figure 2. Scheme of the shock absorber.

Figure 3 shows theoretical and experimental load curves for damping systems of the strut. The parameters of the landing impact of the aircraft landing gear calculated by the proposed method agree with the test data within the experimental error, which confirms the predictive capabilities of the mathematical model.

Figure 3. Vertical load diagram.
4. Conclusion

The method proposed in this paper differs from the existing methods in that it is, first of all, universal. If another system of rigid bodies is considered, one does not need to rewrite the equations of motion in the generalized coordinates and determine these coordinates by excluding the “redundant” coordinates using the constraint equations. The dimensionality of the system changes, but the form of equations remains the same. This universal approach is more algorithmic and simple to implement numerically. As an advantage of the method, the method can be used to solve a wide range of problems of rigid body system dynamics.

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