ON DUAL TIMELIKE MANNHEIM PARTNER CURVES
IN $D^3_1$

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Abstract

The first aim of this paper is to define the dual timelike Mannheim partner curves in Dual Lorentzian Space $D^3_1$, the second aim of this paper is to obtain the relationships between the curvatures and the torsions of the dual timelike Mannheim partner curves with respect to each other and the final aim of this paper is to get the necessary and sufficient conditions for the dual timelike Mannheim partner curves in $D^3_1$.

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1 INTRODUCTION

In the differential geometry, special curves have an important role. Especially, the partner curves, i.e., the curves which are related each other at the corresponding points, have drawn attention of many mathematicians so far. The well-known of the partner curves is Bertrand curves which are defined by the property that at the corresponding points of two space curves principal normal vectors are common. Bertrand partner curves have been studied in ref. [2,3,5,7,17,22] Ravani and Ku have transported the notion of bertrand curves to the ruled surfaces and called Bertrand offsets [16]. Recently, Liu and Wang have defined a new curve pair for space curves. They called these new curves as Mannheim partner curves: Let $\alpha$ and $\beta$ be two curves in th three dimensional Euclidean space. If there exists a correspondence between the space curves $\alpha$ and $\beta$ such that, at the corresponding points of the curves, the principal normal lines of $\alpha$ coincides with the binormal lines of $\beta$, then $\alpha$ is called a Mannheim curve, and $\beta$ is called a Mannheim partner curve of $\alpha$. The pair $\{\alpha, \beta\}$ is said to be a Mannheim pair. They showed that the curve $\alpha(s)$ is the Mannheim...
partner curve of \( \beta(s^*) \) if and only if the curvature \( k_1 \) and the torsion \( k_2 \) of \( \beta(s^*) \) satisfy the following equation

\[
k_2' = \frac{dk_2}{ds^*} = \frac{k_1}{\lambda} (1 + \lambda^2 k_2^*)
\]

for some non-zero constants \( \lambda \). They also studied the Mannheim partner curves in the Minkowski 3-space and obtained the necessary and sufficient conditions for the Mannheim partner curves in \( E^3_1 \) [See 8 and 22 for details]. Moreover, Oztekin and Ergut [15] studied the null Mannheim curves in the same space. Orbay and Kasap gave [13] new characterizations of Mannheim partner curves in Euclidean 3-space. They also studied [12] the Mannheim offsets of ruled surfaces in Euclidean 3-space. The corresponding characterizations of Mannheim offsets of timelike and spacelike ruled surfaces have been given by Onder and et al [9,10]. New characterizations of Mannheim partner curves are given in Minkowski 3-space by Kahraman and et al [6].

In this paper, we study the dual timelike Mannheim partner curves in dual Lorentzian space \( D^3_1 \). Furthermore, we show that the Mannheim theorem is not valid for Mannheim partner curves in \( D^3_1 \). Moreover, we give some new characterizations of the Mannheim partner curves by considering the spherical indicatrix of some Frenet vectors of the curves.

2 PRELIMINARY

By a dual number \( A \), we mean an ordered pair of the form \((a, a^*)\) for all \( a, a^* \in \mathbb{R} \). Let the set \( \mathbb{R} \times \mathbb{R} \) be denoted as \( D \). Two inner operations and an equality on \( ID = \{(a, a^*)|a, a^* \in \mathbb{R}\} \) are defined as follows:

(i) \( \oplus : D \times D \rightarrow D, A \oplus B = (a, a^*) \oplus (b, b^*) = (a + b, a^* + b^*) \) is called the addition in \( D \),

(ii) \( \odot : D \times D \rightarrow D, A \odot B = (a, a^*) \odot (b, b^*) = (ab, ab^* + a^*b) \) is called the multiplication in \( D \),

(iii) \( A = B \) iff \( a = b \), \( a^* = b^* \).

If the operations of addition, multiplication and equality on \( D = \mathbb{R} \times \mathbb{R} \) with set of real numbers \( \mathbb{R} \) are defined as above, the set \( D \) is called the dual numbers system and the element \((a, a^*)\) of \( D \) is called a dual number. In a dual number \( A = (a, a^*) \in D \), the real number \( a \) is called the real part of \( A \) and the real number \( a^* \) is called the dual part of \( A \). The dual number \( 1 = (1, 0) \) is called the unit element of multiplication operation \( D \) with respect to multiplication and denoted by \( \varepsilon \). In accordance with the definition of the operation of multiplication, it can be easily seen that \( \varepsilon^2 = 0 \). Also, the dual number \( A = (a, a^*) \in D \) can be written as \( A = a + \varepsilon a^* \).

The set \( D = \{A = a + \varepsilon a|a, a^* \in \mathbb{R}\} \) of dual numbers is a commutative ring according to the
operations,

i) \((a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)\)

ii)\((a + \varepsilon a^*)(b + \varepsilon b^*) = ab + \varepsilon(ab^* + ba^*)\).

The dual number \(A = a + \varepsilon a^*\) divided by the dual number \(B = b + \varepsilon b^*\) provided \(b \neq 0\) can be defined as

\[
\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^* - ab^*}{b^2}.
\]

Now let us consider the differentiable dual function. If the dual function \(f\) expansions the Taylor series then we have

\[f(a + \varepsilon a^*) = f(a) + \varepsilon a^*f'(a)\]

where \(f'(a)\) is the derivation of \(f\). Thus we can obtain

\[
sin(a + \varepsilon a^*) = \sin a + \varepsilon a^*\cos a
\]

\[
cos(a + \varepsilon a^*) = \cos a - \varepsilon a^*\sin a
\]

The set of \(D^3 = \{\overrightarrow{A} | \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}^3\}\) is a module on the ring \(D\). For any \(\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*, \overrightarrow{B} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^* \in D^3\), the scalar or inner product and the vector product of \(\overrightarrow{A}\) and \(\overrightarrow{B}\) are defined by, respectively,

\[
\langle \overrightarrow{A}, \overrightarrow{B} \rangle = \langle \overrightarrow{a}, \overrightarrow{b} \rangle + \varepsilon (\langle \overrightarrow{a}, \overrightarrow{b}^* \rangle + \langle \overrightarrow{a}^*, \overrightarrow{b} \rangle),
\]

\[
\overrightarrow{A} \wedge \overrightarrow{B} = \varepsilon (\overrightarrow{a} \wedge \overrightarrow{b} + \overrightarrow{a}^* \wedge \overrightarrow{b}^*).
\]

If \(\overrightarrow{a} \neq 0\), the norm \(\|\overrightarrow{A}\|\) of \(\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*\) is defined by

\[
\|\overrightarrow{A}\| = \sqrt{\langle \overrightarrow{A}, \overrightarrow{A} \rangle} = \|\overrightarrow{a}\| + \varepsilon \|\overrightarrow{a}^*\|, \quad \|\overrightarrow{a}\| \neq 0.
\]

A dual vector \(\overrightarrow{A}\) with norm 1 is called a dual unit vector. The set \(S^2 = \{\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in D^3|\|\overrightarrow{A}\| = (1, 0), \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}^3\}\) is called the dual unit sphere with the center \(\overrightarrow{O}\) in \(D^3\).

Let \(\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))\) and \(\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))\) be real valued curves in \(E^3\). Then \(\overrightarrow{a}(t) = \alpha(t) + \varepsilon \alpha^*(t)\) is a curve in \(D^3\) and it is called dual space curve. If the real valued functions \(\alpha_i(t)\) and \(\alpha_i^*(t)\) are differentiable then the dual space curve \(\overrightarrow{a}(t)\) is differentiable in \(D^3\). The real part \(\alpha(t)\) of the dual space curve \(\overrightarrow{a}(t)\) is called indicatrix. The dual arc-length of real dual space curve \(\overrightarrow{a}(t)\) from \(t_1\) to \(t\) is defined by

\[
\overrightarrow{s} = \int_{t_1}^{t} \|\alpha'(t)\|dt = \int_{t_1}^{t} \|\overrightarrow{a}'(t)\|dt + \varepsilon = \int_{t_1}^{t} (\overrightarrow{a}'(t), (\overrightarrow{a}'(t))^{'*})dt = s + \varepsilon s^{*}
\]

\(\overrightarrow{s}\) is unit tangent vector of the indicatrix \(\alpha(t)\) which is a real space curve in \(IE^3\). From now on we will take the arc length \(s\) of \(\alpha(t)\) as the parameter instead of \(t\).

The Lorentzian inner product of dual vectors \(\overrightarrow{A}, \overrightarrow{B} \in D^3\) is defined by

\[
\langle \overrightarrow{A}, \overrightarrow{B} \rangle = \langle \overrightarrow{a}, \overrightarrow{b} \rangle + \varepsilon (\langle \overrightarrow{a}, \overrightarrow{b}^* \rangle + \langle \overrightarrow{a}^*, \overrightarrow{b} \rangle)
\]

with the Lorentzian inner product \(\overrightarrow{a} = (a_1, a_2, a_3)\) and \(\overrightarrow{b} = (b_1, b_2, b_3) \in \mathbb{R}^3\)

\[
\langle \overrightarrow{a}, \overrightarrow{b} \rangle = -a_1b_1 + a_2b_1 + a_3b_3.
\]

Thus, \(D^3, \langle \cdot \rangle\) is called the dual Lorentzian space and denoted by \(D^3\). We call the elements of \(D^3\) as the dual vectors. For \(\overrightarrow{A} \neq 0\). the norm \(\|\overrightarrow{A}\|\) of \(\overrightarrow{A}\) is defined by

\[
\|\overrightarrow{A}\| = \sqrt{\langle \overrightarrow{A}, \overrightarrow{A} \rangle}.
\]
vector $\tilde{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*}$ is called dual spacelike vector if $\langle \tilde{A}, \tilde{A} \rangle > 0$ or $\tilde{A} = 0$, dual timelike vector if $\langle \tilde{A}, \tilde{A} \rangle < 0$, dual lightlike vector if $\langle \tilde{A}, \tilde{A} \rangle = 0$ for $\tilde{A} \neq 0$. The dual Lorentzian cross-product of $\tilde{A}, \tilde{B} \in D^3$ is defined by

$$\tilde{A} \wedge \tilde{B} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon \left( \overrightarrow{a} \wedge \overrightarrow{b^*} + \overrightarrow{a^*} \wedge \overrightarrow{b} \right)$$

where $\overrightarrow{a} \wedge \overrightarrow{b} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_3b_1 - a_1b_2 - a_2b_1) \overrightarrow{a}, \overrightarrow{b} \in \mathbb{R}^3$ is the Lorentzian cross product.

Dual number $\Phi = \theta + \varepsilon \theta^*$ is called dual angle between $\overrightarrow{A}$ ve $\overrightarrow{B}$ unit dual vectors. Then we was

$$\sinh (\theta + \varepsilon \theta^*) = \sinh \theta + \varepsilon \theta^* \cosh \theta$$

$$\cosh (\theta + \varepsilon \theta^*) = \cosh \varphi + \varepsilon \theta^* \sinh \theta.$$  

Let $\{T(s), N(s), B(s)\}$ be the moving Frenet frame along the curve $\tilde{a}(s)$. Then $T(s), N(s)$ and $B(s)$ are dual tangent, the dual principal normal and the dual binormal vector of the curve $\tilde{a}(s)$, respectively. Depending on the casual character of the curve $\tilde{a}$, we have the following dual Frenet formulas:

If $\tilde{a}$ is a dual timelike curve:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad (2.1)$$

where $\langle T, T \rangle = -1, \langle N, N \rangle = \langle B, B \rangle = 1, \langle T, N \rangle = \langle N, B \rangle = \langle T, B \rangle = 0$.

We denote by $\{V_1(s), V_2(s), V_3(s)\}$ the moving Frenet frame along the curve $\tilde{\beta}(s)$. Then $V_1(s), V_2(s)$ and $V_3(s)$ are dual tangent, the dual principal normal and the dual binormal vector of the curve $\tilde{\beta}(s)$, respectively. Depending on the casual character of the curve $\tilde{\beta}$, we have the following dual Frenet – Serret formulas:

If $\tilde{\beta}$ is a dual timelike curve:

$$\begin{pmatrix} V'_1 \\ V'_2 \\ V'_3 \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad (2.2)$$

where $\langle V_1, V_1 \rangle = -1, \langle V_2, V_2 \rangle = 1, \langle V_3, V_3 \rangle = 1, \langle V_1, V_2 \rangle = \langle V_2, V_3 \rangle = \langle V_1, V_3 \rangle = 0$.

If the curves are unit speed curve, then curvature and torsion calculated by,
\[ \begin{align*}
\kappa &= \|T'\|, \\
\tau &= \langle N', B \rangle, \\
P &= \|V_1\|, \\
Q &= \langle V_2, V_3 \rangle.
\end{align*} \]

(2.3)

If the curves are not unit speed curve, then curvature and torsion calculated by,

\[ \begin{align*}
\kappa &= \frac{\|\tilde{\alpha}' \wedge \tilde{\alpha}''\|}{\|\tilde{\alpha}'\|^3}, \\
\tau &= \frac{\det(\tilde{\alpha}', \tilde{\alpha}'', \tilde{\alpha}''')}{\|\tilde{\alpha}' \wedge \tilde{\alpha}''\|^2}, \\
P &= \frac{\|\tilde{\beta}' \wedge \tilde{\beta}''\|}{\|\tilde{\beta}\|^3}, \\
Q &= \frac{\det(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')}{\|\tilde{\beta} \wedge \tilde{\beta}''\|^2}.
\end{align*} \]

(2.4)

**Definition 2.1. a) Dual Hyperbolic angle:** Let \( \tilde{A} \) and \( \tilde{B} \) be dual timelike vectors in \( D^3_1 \). Then the dual angle between \( \tilde{A} \) and \( \tilde{B} \) is defined by \( \langle \tilde{A}, \tilde{B} \rangle = -\frac{\|\tilde{A}\| \|\tilde{B}\| \cosh \Phi}{\|\tilde{A}\|} \) \( \cos \Phi \). The dual number \( \Phi = \theta + \varepsilon \theta^* \) is called the dual hyperbolic angle.

**b) Dual Central angle:** Let \( \tilde{A} \) and \( \tilde{B} \) be spacelike vectors in \( D^3_1 \) that span a dual timelike vector subspace. Then the dual angle between \( \tilde{A} \) and \( \tilde{B} \) is defined by \( \langle \tilde{A}, \tilde{B} \rangle = -\frac{\|\tilde{A}\| \|\tilde{B}\| \cosh \Phi}{\|\tilde{A}\|} \) \( \cosh \Phi \). The dual number \( \Phi = \theta + \varepsilon \theta^* \) is called the dual central angle.

**c) Dual Spacelike angle:** Let \( \tilde{A} \) and \( \tilde{B} \) be dual spacelike vectors in \( D^3_1 \) that span a dual spacelike vector subspace. Then the dual angle between \( \tilde{A} \) and \( \tilde{B} \) is defined by \( \langle \tilde{A}, \tilde{B} \rangle = -\frac{\|\tilde{A}\| \|\tilde{B}\| \cosh \Phi}{\|\tilde{A}\|} \) \( \cosh \Phi \). The dual number \( \Phi = \theta + \varepsilon \theta^* \) is called the dual spacelike angle.

**d) Dual Lorentzian timelike angle:** Let \( \tilde{A} \) be a dual spacelike vector and \( \tilde{B} \) be a dual timelike vector in \( ID^3_1 \). Then the dual angle between \( \tilde{A} \) and \( \tilde{B} \) is defined by \( \langle \tilde{A}, \tilde{B} \rangle = -\frac{\|\tilde{A}\| \|\tilde{B}\| \sinh \Phi}{\|\tilde{A}\|} \) \( \sinh \Phi \). The dual number \( \Phi = \theta + \varepsilon \theta^* \) is called the dual Lorentzian timelike angle [18, 19, 20].

### 3 DUAL TIMELIKE MANNHEIM PARTNER CURVE IN \( D^3_1 \)

In this section, we define dual timelike Mannheim partner curves in \( D^3_1 \) and we give some characterization for dual timelike Mannheim partner curves in the same space. Using these relationships, we will comment again Shell’s and Mannheim’s theorems.

**Definition 3.1.** Let \( \tilde{\alpha} : I \to ID^3_1, \tilde{\alpha}(s) = \alpha(s) + \varepsilon \alpha^*(s) \) and \( \tilde{\beta} : I \to ID^3_1, \tilde{\beta}(s) = \beta(s) + \varepsilon \beta^*(s) \) be dual timelike curves. If there exists a corresponding
relationship between the dual timelike curve \( \tilde{\alpha} \) and the dual timelike curve \( \tilde{\beta} \) such that, at the corresponding points of the curves, the dual binormal lines of \( \tilde{\alpha} \) coincides with the dual principal normal lines of \( \tilde{\beta} \), then \( \tilde{\alpha} \) is called a dual timelike Mannheim curve, and \( \tilde{\beta} \) is called a dual Mannheim partner curve of \( \tilde{\alpha} \). The pair \( \{ \tilde{\alpha}, \tilde{\beta} \} \) is said to be dual timelike Mannheim pair. Let \( \{ T, N, B \} \) be the dual Frenet frame field along \( \tilde{\alpha} = \tilde{\alpha}(s) \) and let \( \{ V_1, V_2, V_3 \} \) be the Frenet frame field along \( \tilde{\beta} = \tilde{\beta}(s) \). On the way \( \Phi = \theta + \varepsilon \theta^* \) is dual angle between \( T \) and \( V_1 \), there is an following equations between the Frenet vectors and their derivative;

\[
\begin{pmatrix}
V'_1 \\
V'_2 \\
V'_3
\end{pmatrix} =
\begin{pmatrix}
\cosh \Phi & \sinh \Phi & 0 \\
0 & 0 & 1 \\
-\sinh \Phi & \cosh \Phi & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B
\end{pmatrix},
\]

(3.1)

**Theorem 3.1.** The distance between corresponding dual points of the dual timelike Mannheim partner curves in \( D^3_1 \) is constant.

**Proof:** From the definition of dual spacelike Mannheim curve, we can write

\[
\tilde{\beta}(s^*) = \tilde{\alpha}(s) + \lambda(s)B(s)
\]

(3.2)

By taking the derive of this equation with respect to \( s \) and applying the Frenet formulas, we get

\[
V_1 \frac{ds^*}{ds} = T - \lambda\tau N + \lambda' B
\]

(3.3)

where the superscript (’) denotes the derivative with respect to the arc length parameter \( s \) of the dual curve \( \tilde{\alpha}(s) \). Since the dual vectors \( B \) and \( V_2 \) are linearly dependent, the equation (3.2) can be written as

\[
\left< V_1 \frac{ds^*}{ds}, B \right> = \left< T, B \right> - \lambda\tau \left< N, B \right> + \lambda' \left< B, B \right>
\]

and \( \lambda' = 0 \)

If we take \( \lambda = \lambda_1 + \varepsilon \lambda_1^* \), we get \( \lambda_1' = 0 \) ve \( \lambda_1^* = 0 \). From here, we can write \( \lambda_1 = c_1 \) and \( \lambda_1^* = c_2 \), \( c_1, c_2 = \text{cons.} \)

Then we get \( \lambda = c_1 + \varepsilon c_2 \). On the other hand, from the definition of distance function between \( \tilde{\alpha}(s) \) and \( \tilde{\beta}(s) \) we can write

\[
d\left( \tilde{\alpha}(s), \tilde{\beta}(s) \right) = \left\| \tilde{\beta}(s) - \tilde{\alpha}(s) \right\| = |\lambda_1| + \varepsilon |\lambda_1^*| = |c_1| + \varepsilon c_2
\]

This is completed the proof.

**Theorem 3.2.** For a dual timelike curve \( \tilde{\alpha} \) in \( D^3_1 \), there is a dual timelike curve \( \tilde{\beta} \) so that \( \{ \tilde{\alpha}, \tilde{\beta} \} \) is a dual timelike Mannheim pair.

**Proof:** Since the dual vectors \( V_2 \) and \( B \) are linearly dependent, the equation (3.2) can be written as
\[ \tilde{\alpha} = \tilde{\beta} - \lambda V_2 \]  
\[ (3.4) \]

Since \( \lambda \) is a nonzero constant, there is a dual timelike curve \( \tilde{\beta} \) for all values of \( \lambda \).

Now, we can give the following theorem related to curvature and torsion of the dual timelike Mannheim partner curves.

**Theorem 3.3.** Let \( \{\tilde{\alpha}, \tilde{\beta}\} \) be a dual timelike Mannheim pair in \( D^3_1 \). If \( \tau \) is dual torsion of \( \tilde{\alpha} \) and \( P \) is dual curvature and \( Q \) is dual torsion of \( \tilde{\beta} \), then

\[ \tau = \frac{P}{\lambda Q} \]  
\[ (3.5) \]

**Proof:** By taking the derivate of equation (3.3) with respect to \( s \) and applying the Frenet formulas, we obtain

\[ V_1 \frac{ds^*}{ds} = T - \lambda \tau N \]  
\[ (3.6) \]

Let \( \Phi = \theta + \varepsilon \theta^* \) be dual angle between the dual tangent vectors \( T \) and \( V_1 \), we can write

\[ \left\{ \begin{array}{l} V_1 = \cosh \Phi T + \sinh \Phi N \\ V_3 = \sinh \Phi T + \cosh \Phi N \end{array} \right. \]  
\[ (3.7) \]

From (3.6) and (3.7), we get

\[ \frac{ds^*}{ds} = \frac{1}{\cosh \Phi} , \quad -\lambda \tau = \sinh \Phi \frac{ds^*}{ds} \]  
\[ (3.8) \]

By taking the derivate of equation (3.4) with respect to \( s \) and applying the Frenet formulas, we obtain

\[ T = (1 - \lambda P) V_1 \frac{ds^*}{ds} - \lambda Q V_3 \frac{ds^*}{ds} \]  
\[ (3.9) \]

From equation (3.7) we can write

\[ \left\{ \begin{array}{l} T = \cosh V_1 - \sinh \Phi V_3 \\ N = -\sinh \Phi V_1 + \cosh \Phi V_3 \end{array} \right. \]  
\[ (3.10) \]

where \( \Phi \) is the dual angle between \( T \) and \( V_1 \) at the corresponding points of the dual curves of \( \tilde{\alpha} \) and \( \tilde{\beta} \). By taking into consideration equations (3.9) and (3.10), we get

\[ \cosh \Phi = (1 - \lambda P) \frac{ds^*}{ds} , \quad \sinh \Phi = \lambda Q \frac{ds^*}{ds} \]  
\[ (3.11) \]
Substituting $ds^* / ds$ into (3.11), we get

$$\cosh^2 \Phi = -(1 + \lambda P), \quad \sinh^2 \Phi = \lambda^2 \tau Q$$

(3.12)

From the last equation, we can write

$$\tau = \frac{P}{\lambda Q}$$

If the last equation is separated into the dual and real parts, we can obtain

$$\left\{
\begin{array}{l}
k_2 = \frac{P}{cq} \\
k_2^* = \frac{P^* pq^*}{cq^*}
\end{array}
\right.$$  

(3.13)

**Corollary 3.1.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim pair in $D_1^3$. Then, the dual product of torsions $\tau$ and $Q$ at the corresponding points of the dual timelike Mannheim partner curves is not constant.

Namely, Schell’s theorem is invalid for the dual timelike Mannheim curves. By considering Theorem 3.3 we can give the following results.

**Corollary 3.2.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim pair in $D_1^3$. Then, torsions $\tau$ and $Q$ has a negative sign.

**Theorem 3.4.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim pair in $D_1^3$. Between the curvature and the torsion of the dual timelike curve $\tilde{\beta}$, there is the relationship

$$\mu Q + \lambda P = 1$$

(3.14)

where $\mu$ and $\lambda$ are nonzero dual numbers.

**Proof:** From equation (3.11), we obtain

$$\frac{\cosh \Phi}{1 - \lambda P} = \frac{\sinh \Phi}{\lambda Q}$$

arranging this equation, we get

$$\tanh \Phi = \frac{1 - \lambda P}{\lambda Q}$$

and if we choose $\mu = \lambda \tanh \Phi$ for brevity, we see that

$$\mu Q + \lambda P = 1.$$  

**Theorem 3.5.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim pair in $D_1^3$. There are the following equations for the curvatures and the torsions of the curves $\tilde{\alpha}$ and $\tilde{\beta}$

$$\begin{align*}
i \kappa &= -\frac{d\Phi}{ds^*} \\
ii \tau &= -P \sinh \Phi \frac{d\alpha^*}{ds} - Q \cosh \Phi \frac{d\alpha^*}{ds} \\
iii P &= \tau \sinh \Phi \frac{d\alpha^*}{ds^*} \\
iv Q &= -\tau \cosh \Phi \frac{d\alpha^*}{ds^*}
\end{align*}$$
Proof: i) By considering equation (3.7), we can easily that \( \langle T, V_1 \rangle = - \cosh \Phi \). Differentiating this equality with respect to \( s \) by considering equation (2.1), we have
\[
\langle T', V_1 \rangle + \langle T, V_1' \rangle = - \sinh \Phi \frac{ds}{ds},
\]
from equations (2.1) and (2.2), we can write
\[
\langle \kappa N, V_1 \rangle + \langle T, P V_2 \rangle = - \sinh \Phi \frac{ds}{\Phi ds},
\]
from equations (3.10), we get
\[
\kappa = - \frac{ds}{ds}.
\]
If the last equation is separated into the dual and real part, we can obtain

ii) By considering equation (3.7), we can easily that \( \langle N, V_2 \rangle = 0 \). Differentiating of this equality with respect to \( s \) and by considering equation (2.1), we have
\[
\langle N', V_2 \rangle + \langle N, V_2' \rangle = 0,
\]
From equations (2.1) and (2.2), we can write
\[
\langle \kappa T + \tau B, V_2 \rangle + \left( - \sinh \Phi V_1 + \cosh \Phi V_3, (P V_1 + Q V_3) \frac{ds}{ds} \right) = 0,
\]
From equations (3.10), we get
\[
\tau = - P \sinh \Phi \frac{ds}{ds} - Q \cosh \Phi \frac{ds}{ds},
\]

iii) By considering equation (3.7), we can easily that \( \langle B, V_1 \rangle = 0 \). Differentiating of this equality with respect to \( s \) and by considering equation (2.1), we have
\[
\langle B', V_1 \rangle + \langle B, V_1' \rangle = 0,
\]
From equations (2.1), (2.2) and (3.10) we can write
\[
\langle \tau, (\kappa V_2 + \tau B) \rangle + \left( - \tau \sinh \Phi V_1 + \cosh \Phi V_3, \left( B, P V_2 \right) \frac{ds}{ds} \right) = 0,
\]
\[
\tau = \tau \sinh \Phi \frac{ds}{ds},
\]

iv) By considering equation (3.7), we can easily that \( \langle B, V_3 \rangle = 0 \). Differentiating of this equality with respect to \( s \) by considering equation (2.1), we have
\[
\langle B', V_3 \rangle + \left( B, V_3' \right) = 0,
\]
From equations (2.1), (2.2) and (3.10) we can write
\[
\langle \tau, (\kappa V_3 - \tau B) \rangle + \left( - \tau \sinh \Phi V_1 + \cosh \Phi V_3, \left( B, - Q V_2 \right) \frac{ds}{ds} \right) = 0,
\]
\[
Q = - \tau \cosh \Phi \frac{ds}{ds}.
\]

Corollary 3.3. Let \( \{ \tilde{a}, \tilde{\beta} \} \) be a dual timelike - spacelike Mannheim pair in \( D^3_1 \). If the statements of Theorem 3.5 is separated into the dual and real part, we can obtain

\[
i \left\{ \begin{array}{ll}
k_2 = - p \sinh \theta \frac{ds}{ds} - q \cosh \theta \frac{ds}{ds} \\
k_2^* = - (p^* \sinh \theta + p \theta^* \cosh \theta) \frac{ds}{ds} - (q^* \cosh \theta + q \theta^* \sinh \theta) \frac{ds}{ds}
\end{array} \right.
\]

\[
ii \left\{ \begin{array}{ll}
p = k_2 \sinh \theta \frac{ds}{ds} \\
p^* = (k_2^* \sinh \theta + k_2 \theta^* \cosh \theta) \frac{ds}{ds},
\end{array} \right.
\]

\[
iii \left\{ \begin{array}{ll}
q = - k_2 \cosh \theta \frac{ds}{ds} \\
q^* = - (k_2^* \cosh \theta + k_2 \theta^* \cosh \theta) \frac{ds}{ds}.
\end{array} \right.
\]

By considering the statements iii and iv) of Theorem 2.5 we can give the following results.
**Corollary 3.4.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim pair in $D^3_{1}$. Then there exist the following relation between curvature and torsion of $\tilde{\beta}$ and torsion of $\tilde{\alpha}$:

$$Q^2 - P^2 = \tau^2 \left( \frac{ds}{ds^*} \right)^2$$  \hspace{1cm} (3.15)

**Theorem 3.6.** A dual timelike space curve in $ID^3_{1}$ is a dual timelike Mannheim curve if and only if its curvature $P$ and torsion $Q$ satisfy the formula

$$\lambda \left( P^2 - Q^2 \right) = P$$  \hspace{1cm} (3.16)

where $\lambda$ is never pure dual constant.

**Proof:** By taking the derivative of the statement $\tilde{\alpha} = \tilde{\beta} - \lambda V_2$ with respect to $s$ and applying the Frenet formulas we obtain

$$T \frac{d\alpha}{ds} = V_1 - \lambda (PV_1 + QV_3),$$
$$\kappa N \left( \frac{d\alpha}{ds} \right) + T \frac{d^2\alpha}{ds^2} = PV_2 - \lambda \left( P'V_1 + Q'V_3 + (P^2 - Q^2) V_2 \right)$$

Taking the inner product the last equation with $B$, we get

$$\lambda \left( P^2 - Q^2 \right) = P.$$

If the last equation is separated into the dual and real part, we can obtain

$$\begin{cases}
    p = \lambda \left( p^2 - q^2 \right) \\
    p^* = 2\lambda \left( pp^* - qq^* \right)
\end{cases}$$  \hspace{1cm} (3.17)

where $\lambda = c_1 + \varepsilon c_2$.

**Theorem 3.7.** Let $\{\tilde{\alpha}, \tilde{\beta}\}$ be a dual timelike Mannheim partner curves in $D^3_{1}$. Moreover, the dual points $\tilde{\alpha}(s)$, $\tilde{\beta}(s)$ be two corresponding dual points of $\{\tilde{\alpha}, \tilde{\beta}\}$ and $M$ ve $M^*$ be the curvature centers at these points, respectively. Then, the ratio

$$\frac{\left\|\tilde{\beta}(s) M\right\|}{\left\|\tilde{\alpha}(s) M\right\|}, \frac{\left\|\tilde{\beta}(s) M^*\right\|}{\left\|\tilde{\alpha}(s) M^*\right\|} = (1 + \kappa P)(1 + \lambda P) \neq constant.$$  \hspace{1cm} (3.18)

**Proof:** A circle that lies in the dual osculating plane of the point $\tilde{\alpha}(s)$ on the dual timelike curve $\tilde{\alpha}$ and that has the centre $M = \tilde{\alpha}(s) + \frac{1}{\kappa} N$ lying on the dual principal normal $N$ of the point $\tilde{\alpha}(s)$ and the radius $\frac{1}{\kappa}$ far from $\tilde{\alpha}(s)$, is called dual osculating circle of the dual curve $\tilde{\alpha}$ in the point $\tilde{\alpha}(s)$. Similar definition can be given for the dual curve $\tilde{\beta}$ too.

Then, we can write

$$\|\tilde{\alpha}(s) M\| = \left\| \frac{1}{\kappa} N \right\| = \frac{1}{\kappa},$$

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\[ \|\tilde{\alpha}(s)M^*\|=\|\lambda B+\frac{1}{P}V_2\|=\frac{1}{P}+\lambda, \]
\[ \|\tilde{\beta}(s)M^*\|=\|\frac{1}{P}V_2\|=\frac{1}{P}, \]
\[ \|\tilde{\beta}(s)M\|=\|\lambda V_3+\frac{1}{P}N\|=\frac{1}{P}+\lambda \]

Therefore, we obtain
\[ \frac{\|\tilde{\beta}(s)M\|}{\|\tilde{\alpha}(s)M\|}:\frac{\|\tilde{\beta}(s)M^*\|}{\|\tilde{\alpha}(s)M^*\|}=(1+\lambda P)\sqrt{1-\lambda^2}\kappa^2 \neq \text{cons.} \]

Thus, we can give the following

**Corollary 3.5.** Mannheim’s Theorem is invalid for the dual timelike Mannheim partner curve \( \{\tilde{\alpha},\tilde{\beta}\} \) in \( D_3^1 \).

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