FURSTENBERG ENTROPY VALUES FOR NONSINGULAR ACTIONS OF GROUPS WITHOUT PROPERTY (T)

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ABSTRACT. Let $G$ be a discrete countable infinite group that does not have Kazhdan’s property (T) and let $\kappa$ be a generating probability measure on $G$. Then for each $t > 0$, there is a type $\text{III}_1$ ergodic free nonsingular $G$-action whose $\kappa$-entropy (or the Furstenberg entropy) is $t$.

0. Introduction

Let $G$ be a discrete countable infinite group. A probability measure $\kappa$ on $G$ is called generating if the support of $\kappa$ generates $G$ as a semigroup. Let $T = (T_g)_{g \in G}$ be a nonsingular action of $G$ on a standard probability space $(X, \mathcal{B}, \mu)$. The Furstenberg entropy (or $\kappa$-entropy) of $T$ is defined by

$$h_{\kappa}(T, \mu) := -\sum_{g \in G} \kappa(g) \int_X \log d\mu \circ T_g(x) d\mu(x)$$

(see [Fu]). Jensen’s inequality implies that $h_{\kappa}(T, \mu) \geq 0$ and that (for generating measures) equality holds if and only if $\mu$ is invariant under $T$. Of course, the $\kappa$-entropy is invariant under conjugacy. If $\sum_{g \in G} \kappa(g) \frac{d\mu T_g}{d\mu}(x) = 1$ for a.e. $x \in X$ then $T$ is called $\kappa$-stationary. Furstenberg entropy realization problem is to describe all values that $\kappa$-entropy takes on the set of $\kappa$-stationary actions. The problem appears quite difficult. Some progress was achieved in recent papers [NeZi], [Ne], [Bo], [HaTa]. To state one of the results on the entropy realization problem we first recall that $G$ has Kazhdan’s property (T) if every unitary representation of $G$ which has almost invariant vectors admits a nonzero invariant vector (see [Be–Va]). It was shown in [Ne] that if $G$ has property (T) then for every generating measure $\kappa$, the pair $(G, \kappa)$ has an entropy gap, i.e. there exists some constant $\epsilon = \epsilon(G, \kappa) > 0$ such that $h_{\kappa}(T) > \epsilon$ for each purely infinite ergodic stationary $G$-action $T$. We recall that an ergodic action is called purely infinite if it does not admit an equivalent invariant probability measure. In [Bo–Ta] the converse statement was proved: if $T$ does not have property (T) then for each generating measure $\kappa$,

$$\inf \{h_{\kappa}(T, \mu) \mid T \text{ is purely infinite, ergodic, } \mu\text{-nonsingular action of } G \} = 0.$$  

We note that the authors of [Bo–Ta] consider $\kappa$-entropy values on arbitrary (not only stationary as in the other aforementioned papers) purely infinite nonsingular actions. They also show that the entropy gap for $(G, \kappa)$ established in [Ne] for the stationary actions holds also for all (purely infinite) ergodic nonsingular actions.
In this connection we note that if a purely infinite ergodic $G$-action is stationary then the space of the action is non-atomic. However, in the general (non-stationary) case considered in [Bo–Ta], there exist purely infinite transitive $G$-actions on purely atomic measure spaces. In particular, the action of $G$ on itself via rotations is free, nonconservative, ergodic and purely infinite. We consider such actions as pathological. Unfortunately, the proof the main result from [Bo–Ta] does not exclude appearance of pathological actions in (0-1).

Our purpose in the present paper is to refine the main result from [Bo–Ta] in two aspects: to examine all possible values for the $\kappa$-entropy and “get rid” of possible pathological actions on which such values are attained. In fact, we show more.

**Main Theorem.** Let $G$ do not have property (T). Let $\kappa$ be a generating measure on $G$. Then the following are satisfied.

1. For each real $t \in (0, +\infty)$, there is a type $III_1$ ergodic free nonsingular action $T = (T_g)_{g \in G}$ on a standard probability space $(X, \mu)$ such that $h_\kappa(T, \mu) = t$.
2. For each real $t \in (0, +\infty)$, there is $\lambda \in (0, 1)$ and a type $III_\lambda$ ergodic free nonsingular action $T = (T_g)_{g \in G}$ on a standard probability space $(X, \mu)$ such that $h_\kappa(T, \mu) = t$.

The proof is based on the measurable orbit theory (see [FeMo], [Sc1] and a survey [DaSi]) and cohomology properties of non-strongly ergodic actions [Sc2], [JoSc].

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1. Some background on orbit theory

Let $T$ be an ergodic free nonsingular action of $G$ on a standard nonatomic probability space $(X, \mathcal{B}, \mu)$. Denote by $\mathcal{R}$ the $T$-orbit equivalence relation on $X$. We recall that the full group $[\mathcal{R}]$ of $\mathcal{R}$ consists of all one-to-one nonsingular transformations $r$ of $(X, \mu)$ such that the graph of $r$ is a subset of $\mathcal{R}$. Given a locally compact second countable group $H$, denote by $\lambda_H$ a left Haar measure on $H$. A Borel map $\alpha : \mathcal{R} \to H$ is called a cocycle of $\mathcal{R}$ if $\alpha(x, y) = \alpha(x, z)\alpha(z, y)$ for all points $x, y, z$ from a $\mu$-conull subset of $X$ such that $x \sim \mathcal{R} y \sim \mathcal{R} z$. By $T(\alpha) = (T(\alpha)_g)_{g \in G}$ we denote the $\alpha$-skew product extension of $T$, i.e. a $G$-action on the product space $(X \times H, \mu \times \lambda_H)$:

$$T(\alpha)_g(x, h) = (T_g x, \alpha(T_g x, x) h).$$

It is obvious that $T(\alpha)$ is $(\mu \times \lambda_H)$-nonsingular. We say that $\alpha$ is ergodic if $T(\alpha)$ is ergodic. Consider the $H$-action on $(X \times H, \mu \times \lambda_H)$ by rotations (from the right) along the second coordinate. It commutes with $T(\alpha)$. The restriction of this action to the sub-$\sigma$-algebra of $T(\alpha)$-invariant Borel subsets is called the action of $H$ associated with $\alpha$. It is ergodic. It is trivial if and only if $\alpha$ is ergodic. If $H = \mathbb{R}_+^*$ and $\alpha(T_g x, x) := \log \frac{d\mu_{T_g x}}{d\mu}(x)$ at a.e. $x$ for each $g \in G$ then $\alpha$ is called the Radon-Nikodym cocycle of $\mathcal{R}$. It does not depend on the choice of nonsingular group action generating $\mathcal{R}$. The corresponding associated action of $\mathbb{R}_+^*$ is called the associated flow of $T$. If two group actions are orbit equivalent then their associated flows are isomorphic. The associated flow is transitive and free if and only if $T$ admits an $\sigma$-finite invariant $\mu$-equivalent measure. In this case $T$ is said to be of type $II_1$. If the invariant measure
is infinite, $T$ is said to be of type $II_\infty$. If $T$ does not admit an invariant equivalent measure then $T$ is said to be of type $III$. Type $III$ admits further classification into subtypes $III_\lambda$, $0 \leq \lambda \leq 1$. If the associated flow of $T$ is periodic with period $-\log \lambda$ for some $\lambda \in (0, 1)$ then $T$ is said to be of type $III_\lambda$. If the associated flow is trivial (on a singletone) then $T$ is said to be of type $III_1$. Equivalently, $T$ is of type $III_1$ if and only if the Radon-Nikodym cocycle of $T$ is ergodic. If $T$ is of type $III$ but not of type $III_\lambda$ for any $\lambda \in (0, 1]$ then $T$ is said to be of type $III_0$.

**Lemma 1.1.** Let $\mu$ be invariant under $T$. Let $H$ be discrete and countable. Let $\alpha : \mathcal{R} \to H$ be an ergodic cocycle. Then the following holds.

(i) The subrelation $\mathcal{R}_0 := \{(x, y) \in \mathcal{R} | \alpha(x, y) = 1\}$ of $\mathcal{R}$ is ergodic.

(ii) For each $h \in H$, there is an element $r_h \in [\mathcal{R}]$ such that $\alpha(r_h x, x) = h$ for $\mu$-a.e. $x \in X$.

**Idea of the proof.** Pass to the ergodic skew product extension $T(\alpha)$ and use the following Hopf lemma: if $D = (D_h)_{h \in H}$ is an ergodic $H$-action of type $II$ and $\lambda$ is a $D$-invariant equivalent measure then for all subsets $A$ and $B$ with $\lambda(A) = \lambda(B)$, there is a Borel bijection $\tau : A \to B$ such that the graph of $\tau$ is a subset of the $D$-orbit equivalence relation.

Let $S$ be a nonsingular $H$-action on a standard probability space $(Y, \mathcal{F}, \nu)$. Given a cocycle $\alpha : \mathcal{R} \to H$, we can form a skew product action $T(\alpha, S) = (T(\alpha, S)_g)_{g \in G}$ of $G$ on the product space $(X \times Y, \mu \times \nu)$ by setting

$$T(\alpha, S)_g(x, y) := (T_g x, S_{\alpha(T_g x, x)} y).$$

Then $T(\alpha, S)$ is $(\mu \times \nu)$-nonsingular.

**Lemma 1.2.** Let $T, \mu, H$ be as in Lemma 1.1. If $\alpha$ is ergodic and $S$ is ergodic then $T(\alpha, S)$ is also ergodic. The associated flow of $T(\alpha, S)$ is isomorphic to the associated flow of $S$. In particular, the type of $T(\alpha, S)$ equals the type of $S$.

**Proof.** Let $F : X \times Y \to \mathbb{R}$ be a Borel function. If $F$ is $T(\alpha, S)$-invariant then $F(x, y) = F(x', y)$ if $(x', x) \in \mathcal{R}_0$ for a.e. $y$. By Lemma 1.1(i), $F(x, y) = f(y)$ for some Borel function $f : Y \to \mathbb{R}$. Lemma 1.1(ii) now yields that $f$ is invariant under $S$. Since $S$ is ergodic, $f$ is constant mod $\nu$. Thus $F$ is constant mod $\mu \times \nu$. Hence $T(\alpha, S)$ is ergodic.

The second claim of the lemma follows from Lemma 1.1, the fact that $\mathcal{R}$ is generated by $\mathcal{R}_0$ and the family of transformations $(r_h)_{h \in H}$ and that $r_h[\mathcal{R}_0]r_h^{-1} = [\mathcal{R}_0]$ for each $h \in H$.

We now recall the definition of strongly ergodic actions (see [CoWe], [JoSc] and references therein). Let $T$ be an ergodic nonsingular $G$-action on non-atomic probability space $(X, \mathcal{B}, \mu)$. A sequence $(B_n)_{n \in \mathbb{N}}$ in $\mathcal{B}$ is called asymptotically invariant if $\lim_{n \to \infty} \mu(B_n \triangle T_y B_n) = 0$ for every $g \in G$. If every asymptotically invariant sequence $(B_n)_{n \in \mathbb{N}}$ is trivial, i.e. $\lim_{n \to \infty} \mu(B_n)(1 - \mu(B_n)) = 0$ then $T$ is called strongly ergodic. We note the the strong ergodicity is invariant under the orbit equivalence. We will need the following lemma.

**Lemma 1.3.**

(i) If $G$ does not have property $(T)$ then there is an ergodic probability preserving free action $T$ of $G$ which is not strongly ergodic [CoWe].
(ii) If $T$ is an ergodic nonsingular free action of $G$ which is not strongly ergodic
then for each countable discrete Abelian group $A$, there is an ergodic cocycle
of the $T$-orbit equivalence relation with values in $A$ (see [Sc, Corollary 1.5]
and Theorem A2 below$^1$).

2. Proof of the main result

The following lemma is almost a literal repetition of [Bo-Ta, Lemma 4.1], where
it was proved under an additional assumption that $T$ is measure preserving.

Lemma 2.1 (Entropy addition formula). Let $\kappa$ be a probability on $G$ and let
$T$ be a nonsingular action of $G$ on a standard probability space $(X, \mathcal{B}, \mu)$. Given
a discrete countable group $H$ and a nonsingular action $S = (S_h)_{h \in H}$ of $H$ on a
standard probability space $(Y, \mathcal{F}, \nu)$, let $\kappa_x$ denote the pushforward of $\kappa$ under the
map $G \ni g \mapsto \alpha(T_g x, x) \in H$ for each $x \in X$. Then

$$h_\kappa(T(\alpha), \mu \times \nu) = h_\kappa(T, \mu) + \int_X h_{\kappa_x}(S, \nu) d\mu(x).$$

Proof.

$$h_\kappa(T(\alpha), \mu \times \nu) = -\sum_{g \in G} \kappa(g) \int_{X \times Y} \log \left( \frac{d(\mu \times \nu) \circ T_g(\alpha)}{d(\mu \times \nu)}(x, y) \right) d\mu(x)d\nu(y)$$

$$= h_\kappa(T, \mu) - \int_X \sum_{g \in G} \kappa(g) \int_Y \log \left( \frac{d\nu \circ S_{\alpha(T_g x, x)}(y)}{d\nu}(y) \right) d\nu(y)d\mu(x)$$

$$= h_\kappa(T, \mu) - \int_X \sum_{h \in H} \kappa_x(h) \int_Y \log \left( \frac{d\nu \circ S_h(y)}{d\nu}(y) \right) d\nu(y)d\mu(x)$$

$$= h_\kappa(T, \mu) + \int_X h_{\kappa_x}(S, \nu) d\mu(x). \quad \square$$

Proof of Main Theorem. We will proceed in two steps. On the first step, for each
$\epsilon > 0$, we construct an ergodic nonsingular $G$-action of type $III_1$ (or of type $III_\lambda$
for some $\lambda \in (0, 1)$) whose $\kappa$-entropy is less then $\epsilon$. On the second step we show
how to change the quasiinvariant measure for the action constructed on the first
step with appropriate equivalent measures to forse the $\kappa$-entropy to attain all the
values from the interval $(\epsilon, +\infty)$.

Step 1. Fix $\epsilon > 0$ and $\lambda \in (0, 1)$. Fix an enumeration $G = \{g_n \mid n \in \mathbb{N}\}$ and a
sequence of integers $1 = l_1 \leq l_2 \leq \cdots$ such that $l_{n+1} - l_n \leq 1$ for all $n \in \mathbb{N}$, $l_n \to \infty$
and

$$\sum_{n=1}^{\infty} \kappa(g_n)(l_n + 1) < 2. \quad (2-1)$$

By Lemma 1.3(i), there is a measure preserving free action $T$ of $G$ on a standard
probability space $(X, \mathcal{B}, \mu)$ which is not strongly ergodic. Denote by $\mathcal{R}$ the $T$-orbit

$^1$Since the proof of [Sc, Corollary 1.5] was not completed there we provide a complete proof of
it in Appendix A.
equivalence relation. Let \( F := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} \). We consider the elements of \( F \) as \( \mathbb{Z}/2\mathbb{Z} \)-valued functions on \( \mathbb{N} \) with finite support. Given \( f \in F \), we let

\[
\|f\| := \max\{j \in \mathbb{N} \mid f(j) \neq 0\}.
\]

By Lemma 1.3(ii), there exists an ergodic cocycle \( \alpha : \mathcal{R} \to F \). For each \( n \in \mathbb{N} \), we can choose \( M_n \in \mathbb{N} \) such that

\[
\mu\left( \left\{ x \in X \mid \max_{1 \leq t \leq n} \|\alpha(T_g^n x, x)\| < M_n \right\} \right) > 1 - \frac{1}{n2^n}.
\]

Without loss of generality we may assume that \( M_n = M_{n+1} \) if and only if \( l_n = l_{n+1} \) for each \( n \in \mathbb{N} \). Let \( N := \{ f \in F \mid f(M_n) = 0 \text{ for each } n \in \mathbb{N} \} \). Then \( N \) is a subgroup of \( F \). The quotient group \( F/N \) is identified naturally with the “complimentary to \( F \)” subgroup \( \{ f \in F \mid f(n) = 0 \text{ for each } n \neq M_1, M_2, \ldots \} \) which is, in turn, isomorphic to \( F \) in a natural way. Hence passing from \( \alpha \) to the quotient cocycle

\[
\alpha + N : \mathcal{R} \ni (x, y) \mapsto \alpha(x, y) + N \in F/N
\]

means that we may assume without loss of generality that \( M_n = l_n \) for each \( n \in \mathbb{N} \) in (2-2). (We use here a simple fact that \( \alpha + N \) is ergodic whenever \( \alpha \) is.) Therefore applying (2-2) we obtain that

\[
\int_X \|\alpha(T_g^n x, x)\| d\mu(x) = \sum_{s=1}^{\infty} s \mu(\{ x \in X \mid \|\alpha(T_g^n x, x)\| = s \})
\]

\[
\leq l_n + \sum_{s > l_n} s \mu(\{ x \in X \mid \|\alpha(T_g^n x, x)\| = s \})
\]

\[
\leq l_n + \sum_{s > l_n} \frac{1}{2^s}
\]

\[
\leq l_n + 1.
\]

The second inequality here follows from the fact that for each \( s > l_n \), we have \( s = l_m \) for some \( m \geq n \) and hence

\[
\mu(\{ x \in X \mid \|\alpha(T_g^n x, x)\| = s \}) \leq \frac{1}{m2^m} \leq \frac{1}{s2^s}
\]

because \( m \geq l_m = s \).

Now we consider \( F \) as a (dense) subgroup of the compact Abelian group \( K := (\mathbb{Z}/2\mathbb{Z})^\mathbb{N} \) of all \( \mathbb{Z}/2\mathbb{Z} \)-valued functions on \( \mathbb{N} \). Denote by \( S \) the action of \( H \) on \( K \) by translations. Let \( \nu_n \) denote the distribution on \( \mathbb{Z}/2\mathbb{Z} \) such that

\[
\nu_n(0) = \frac{1}{1 + e^{\epsilon_n}}, \quad \nu_n(1) = \frac{e^{\epsilon_n}}{1 + e^{\epsilon_n}}
\]

for some sequence \( (\epsilon_n)_{n \in \mathbb{N}} \) of reals such that \( \lim_{n \to \infty} \epsilon_n = 0 \), \( \sum_{n \in \mathbb{N}} \epsilon_n^2 = \infty \) and \( \max\{|\epsilon_n| \mid n \in \mathbb{N} \} < \epsilon \). Let \( \nu = \bigotimes_{n \in \mathbb{N}} \nu_n \). Then \( S \) is \( \nu \)-nonsingular, ergodic and of type \( III_1 \) [ArWo]. Therefore by Lemma 1.2, the skew product \( G \)-action \( T(\alpha, S) \) on
$(X \times K, \mu \times \nu)$ is ergodic and of type $\text{III}_1$. Hence $h_\kappa(T(\alpha, S)) > 0$. To estimate $h_\kappa(T(\alpha, S))$ from above, we first let $N_f := \{ n \in \mathbb{N} \mid f(n) \neq 0 \}$ for $f \in F$. It is obvious that $\#N_f \leq \|f\|$. Since

$$-\int_Y \log \left( \frac{d\nu \circ S_f}{d\nu} \right)(y) d\nu(y) = -\sum_{n \in N_f} \int_{\mathbb{Z}/2\mathbb{Z}} \log \left( \frac{\nu_n(y_n + 1)}{\nu_n(y_n)} \right) d\nu_n(y_n)$$

$$= \sum_{n \in N_f} (\nu_n(1) - \nu_n(0)) \log \frac{\nu_n(1)}{\nu_n(0)},$$

we obtain that for each probability $\xi$ on $F$,

$$h_\xi(S, \nu) = \sum_{f \in F} \xi(f) \sum_{n \in N_f} (\nu_n(1) - \nu_n(0)) \log \frac{\nu_n(1)}{\nu_n(0)}$$

(2-4)

$$\leq \sum_{f \in F} \xi(f) \sum_{n \in N_f} |\epsilon_n|$$

$$\leq \epsilon \|f\| \sum_{f \in F} \xi(f).$$

Since $T$ preserves $\kappa$, it follows that $h_\kappa(T, \mu) = 0$. Then Lemma 2.1, (2-4) and (2-3) yield that

$$h_\kappa(T(\alpha, S), \mu \times \nu) \leq \epsilon \int_X \sum_{f \in F} \kappa(x)(f) \|f\| d\mu(x)$$

$$= \epsilon \sum_{g \in G} \kappa(g) \int_X \|\alpha(T_g x, x)\| d\mu(x)$$

$$\leq \epsilon \sum_{n=1}^\infty \kappa(g_n)(l_n + 1).$$

It now follows from (2-1) that $h_\kappa(T(\alpha, S), \mu \times \nu) \leq 2\epsilon$. Hence

$$\inf\{h_\kappa(A) \mid A \text{ is a type } \text{III}_1 \text{ ergodic free action of } G\} = 0.$$

In a similar way we may show that

(2-5) $$\inf\{h_\kappa(A) \mid A \text{ is a type } \text{III}_\lambda \text{ ergodic free action of } G, \lambda \in (0, 1)\} = 0.$$  

For that we argue as above but with a different measure $\nu$. Indeed, let $\nu_n$ denote the distribution on $\mathbb{Z}/2\mathbb{Z}$ such that

$$\nu_n(0) = \frac{1}{1 + e^\epsilon}, \quad \nu_n(1) = \frac{e^\epsilon}{1 + e^\epsilon}.$$

Let $\nu = \bigotimes_{n \in \mathbb{N}} \nu_n$. Then $S$ is $\nu$-nonsingular, ergodic and of type $\text{III}_{e^{-\epsilon}}$ [ArWo]. Therefore by Lemma 1.2, the skew product $G$-action $T(\alpha, S)$ on $(X \times K, \mu \times \nu)$ is ergodic and of type $\text{III}_{e^{-\epsilon}}$. Hence $h_\kappa(T(\alpha, S)) > 0$. As in the $\text{III}_1$-case considered above, we obtain that $h_\kappa(T(\alpha, S), \mu \times \nu) < 2\epsilon$ and hence (2-5) follows.
Step 2. Given $\epsilon > 0$, let $\nu$ be a measure on $K$ such that
\[
(2-5) \quad h_{\kappa}(T(\alpha, S), \mu \times \nu) < \epsilon.
\]
We choose $n_0 > 0$ such that
\[
(2-6) \quad \int_X \kappa_x(\{f \in F \mid f(n_0) \neq 0\}) \, d\mu(x) > 0.
\]
It exists because otherwise we would have that $\kappa_x$ is supported at 0 for a.e. $x \in X$. The latter yields that $\alpha(T_g x, x) = 0$ at a.e. $x$ for all $g$ from the support of $\kappa$. Since $\kappa$ is generating, it follows that $\alpha$ is trivial, a contradiction.

Let $\omega$ be a probability on $\mathbb{Z}/2\mathbb{Z}$ supported at $\frac{1}{2}$. For each $\theta \in (0, 1]$, we let
\[
\nu^\theta_n = \begin{cases} 
\nu_n, & \text{if } n \neq n_0 \\
\theta \nu_n + (1 - \theta) \omega, & \text{if } n = n_0
\end{cases}
\]
and $\nu^\theta := \bigotimes_{n \in \mathbb{N}} \nu^\theta_n$. Then $\nu^\theta$ is equivalent to $\nu$ and hence $\mu \times \nu^\theta$ is equivalent to $\mu \times \nu$. Therefore the dynamical systems $(T(\alpha, S), \mu \times \nu^\theta)$ and $(T(\alpha, S), \mu \times \nu)$ are of the same Krieger’s type. It follows from the equality in (2-4) that
\[
h_{\kappa_x}(S, \nu) - h_{\kappa_x}(S, \nu^\theta) = \kappa_x(\{f \in F \mid f(n_0) \neq 0\})(\Phi(\nu_n(0)) - \Phi(\nu^\theta_n(0))),
\]
where $\Phi(t) := (1 - 2t) \log \frac{1-t}{t}$, if $t \in (0, 1)$. Therefore the map
\[
(0, 1] \ni \theta \mapsto h_{\kappa}(T(\alpha, S), \mu \times \nu^\theta) = \int_X h_{\kappa_x}(S, \nu^\theta) \, d\mu(x) \in \mathbb{R}
\]
is continuous. In view of (2-6), this map goes to infinity as $\theta \to 0$. Since $\nu^1 = \nu$ and (2-5) holds, it follows that
\[
\{h_{\kappa}(T(\alpha, S), \mu \times \nu^\theta) \mid \theta \in (0, 1]\} \supset (\epsilon, \infty),
\]
as desired. \hfill \Box

Appendix A

Let $\mathcal{R}$ be an ergodic measure preserving countable equivalence relation on a nonatomic standard probability space $(X, \mathcal{B}, \mu)$ and let $G$ be a locally compact second countable group. A cocycle $\rho : \mathcal{R} \to G$ is called regular if the action of $G$ associated with $\rho$ is transitive. For instance, an ergodic cocycle is regular. A coboundary is also regular.

Proposition A1. Let $A$ be an amenable discrete countable group and let $H$ be a locally compact second countable amenable group. Let $\alpha : \mathcal{R} \to A$ be a cocycle. If $\alpha$ is not regular then there is an ergodic cocycle of $\mathcal{R}$ with values in $H$.

Proof. Let $\mathcal{A}$ stand for the “transitive” equivalence relation on $A$, i.e. $\mathcal{A} = A \times A$. Let $\lambda$ be a probability measure on $A$ which is equivalent to Haar measure. Then the equivalence relation $\mathcal{R} \times \mathcal{A}$ is an ergodic equivalence relation on the probability $\{0, 1\}$ with addition mod 2.\footnote{We consider the group $\mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$ with addition mod 2.}
space \((X \times A, \mu \times \lambda)\). Let \(V = (V_a)_{a \in A}\) denote the nonsingular action of \(A\) on \((X \times A, \mu \times \lambda)\) by right rotations along the second coordinate. Then \(R \times A\) is generated by a subrelation \(R(\alpha)\) and \(V\), i.e., two points \(z_1, z_2 \in X \times A\) are \((R \times A)\)-equivalent if and only if the points \(V_a \cdot z_1\) and \(V_a \cdot z_2\) are \(R(\alpha)\)-equivalent for some \(a_1, a_2 \in A\). Let \(W = (W_a)_{a \in A}\) stand for the action of \(A\) associated with \(\alpha\). Denote by \((\Omega, \nu)\) the space of this action. Then we can assume that there is a Borel map \(\pi : X \times A \to \Omega\) such that \(\nu = (\mu \times \lambda) \circ \pi^{-1}\) and \(\pi \circ V_a = W_a \circ \pi\) for each \(a \in A\). We observe that \(\pi\) is the \(R(\alpha)\)-ergodic decomposition of \(X \times A\). Since \(A\) is amenable, the \(W\)-orbit equivalence relation \(\mathcal{I}\) on \((\Omega, \nu)\) is hyperfinite [Co–We]. It is ergodic. Since \(\alpha\) is not regular, \(\mathcal{I}\) is non-transitive. Hence there is an ergodic cocycle \(\beta : \mathcal{I} \to H\) (see [He] and [GoSi]). We now define a cocycle \(\beta^* : R \times A \to H\) by setting

\[
\beta^*(z_1, z_2) := \beta(\pi(z_1), \pi(z_2)).
\]

Then \(\beta^*\) is well defined. Since \(\pi\) is the \(R(\alpha)\)-ergodic decomposition, it follows that \(\beta^*\) is ergodic. Restricting \(R \times A\) and \(\beta^*\) to the subset \(X \times \{1_A\}\) we obtain \(R\) and a cocycle of \(R\) with values in \(H\) respectively. Of course, this cocycle is also ergodic. \(\square\)

Let \(A\) be an Abelian locally compact noncompact second countable group. For a cocycle \(\alpha : R \to A\), we denote by \(E(\alpha) \subset A \sqcup \{\infty\}\) the essential range of \(\alpha\) (see [Sc1, Definition 3.1]). The following theorem provides a complete proof of [Sc2, Corollary 1.5] (it was assumed additionally in [Sc2] that \(A\) is Abelian).

**Theorem A2.** Let \(T = (T_g)_{g \in G}\) be an ergodic measure preserving action of \(G\) on a standard nonatomic probability space \((X, \mathcal{B}, \mu)\). Let \(A\) be a countable amenable group. If \(T\) is not strongly ergodic then there is an ergodic cocycle of the \(T\)-orbit equivalence relation \(R\) with values in \(A\).

**Proof.** Let \(R\) denote the \(T\)-orbit equivalence relation. Let \(K, F, S, N_f\) and \(\|\cdot\|\) denote the same objects as in the proof of Main Theorem. Let \(\lambda_K\) stand for the Haar measure on \(K\). For each \(n \in \mathbb{N}\), denote by \(f_n\) the element of \(F\) such that \(N_{f_n} = \{n\}\). By [JoSc, Lemma 2.4], there are a Borel map \(\pi : X \to K\) and a sequence \((V_n)_{n \in \mathbb{N}}\) of transformations in \([R]\) such that \(\mu \circ \pi^{-1} = \lambda_K\),

\[
\{\pi(T_gx) \mid g \in G\} = \{S_f \pi(x) \mid f \in F\}
\]

and \(\pi(V_nx) = S_{f_n} \pi(x)\) for a.a. \(x \in X\). Denote by \(S\) the \(S\)-orbit equivalence relation on \(K\). We define a cocycle \(\beta : S \to F\) by setting \(\beta(S_{f,y}) := f, y \in Y, f \in F\). By [Sc1, Proposition 3.15], \(E(\beta) = \{0, +\infty\}\). We now define a cocycle \(\beta^* : R \to F\) by setting

\[
\beta^*(T_gx, x) := \beta(\pi(T_gx), \pi(x)),
\]

\(x \in X, g \in G\). Since \(E(\beta^*) \subset E(\beta)\), we obtain that either \(E(\beta^*) = \{0, \infty\}\) or \(E(\beta^*) = \{0\}\). In the latter case, \(\beta^*\) is a coboundary [Sc1]. Hence there is a Borel map \(\xi : X \to K\) such that \(\beta^*(V_nx, x) = \xi(V_nx) - \xi(x)\) for each \(n\) at a.e. \(x \in X\). There is a constant \(C > 0\) and a subset \(X_C \subset X\) such that \(\mu(X_C) > 3/4\) and \(\|\xi(x)\| < C\) for all \(x \in X_C\). It follows that for each \(n\),

\[
\sup_{x \in X_C \cap V_n^{-1}X_C} \|\beta^*(V_nx, x)\| < C.
\]

This contradicts to the fact that \(\|\beta^*(V_nx, x)\| = n\) for a.a. \(x \in X\) and \(n \in \mathbb{N}\). Hence \(E(\beta^*) = \{0, \infty\}\). This yields that \(\beta^*\) is not regular. It remains to apply Proposition A1. \(\square\)
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