SZEGŐ KERNEL EXPANSION AND EQUIVARIANT EMBEDDING OF CR MANIFOLDS WITH CIRCLE ACTION

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ABSTRACT. Let $X$ be a compact strongly pseudoconvex CR manifold with a transversal CR $S^1$-action. In this paper, we establish the asymptotic expansion of Szegő kernels of positive Fourier components and by using the asymptotics, we show that $X$ can be equivariant CR embedded into some $\mathbb{C}^N$ equipped with a simple $S^1$-action. An equivariant embedding of quasi-regular Sasakian manifold is also derived.

1. INTRODUCTION

Let $X$ be a compact strongly pseudoconvex CR manifold. The question of whether or not $X$ admits a CR embedding into a complex Euclidean space has attracted a lot of attention. By a theorem of Boutet de Monvel [6], any compact strongly pseudoconvex CR manifold of dimension greater than or equal to five can be CR embedded into $\mathbb{C}^N$ for some $N$. The classical example of non-embeddable three dimensional strongly pseudoconvex CR manifold given by Rossi [33] (see also [2]) showed that an arbitrary real analytic deformation of the standard CR structure on the three-sphere may fail to be embeddable. There exists an extensive literature on the embeddability of deformed CR structures. For this subject, we refer the reader to [3, 28, 14, 10, 11] and the references therein. Given a compact strongly pseudoconvex CR manifold equipped with a locally free transversal CR $S^1$-action, it was shown in [28, 29] and also [27] that $X$ can always be CR embedded into some complex space.

In this work, we attack the embedding problem for compact strongly pseudoconvex CR manifolds equipped with a transversal CR circle action from a pure analytic point of view. More precisely, we develop an asymptotic expansion for the Szegő kernels $(S_m)$ concerning CR functions which lie in the space of positive Fourier components $(H^0_{b,m}(X))$, see Definition [18]. Our main results involving that features are Theorem [2,6] and Theorem [2,7] which we deduce from a general result about Szegő kernel expansion (see Theorem [2,4]) using partially the machinery developed by the second and third named authors in [22, 23]. Theorem [2,6] describes the expansion on the regular part, i.e. the part of the manifold where $S^1$ acts globally free. Here, the expansion works out well. Difficulties occur on the complement of the regular part, but we can still prove some expansion as

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stated in Theorem 2.7. Roughly speaking, the problem is that the leading term of the expansion does not change smoothly (even not continuously) from the regular part to its complement.

Inspired by a work of the second named author [19], our second main result in this paper is using Theorem 2.6 and Theorem 2.7 to construct a CR embedding which is equivariant with respect to a simple $S^1$-action on $\mathbb{C}^N$ (see Theorem 1.2). Recently, Hsiao-Li-Marinescu [24] established the Kodaira embedding theorem for CR manifolds with circle action which complements the results of this paper with the study of the embedding in the presence of a positive line bundle but without the hypothesis of strict pseudoconvexity.

Before we state the precise embedding result, let us see some examples of compact strongly pseudoconvex CR manifolds in $\mathbb{C}^N$ to get an idea how such simple $S^1$-actions could look like.

**Example I:** Let $X = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots + |z_n|^2 = 1\}$ which is a CR manifold with a transversal CR $S^1$-action (see Definition 1.5):
$$e^{i\theta} \circ (z_1, z_2, \ldots, z_n) = (e^{im_1 \theta}z_1, e^{im_2 \theta}z_2, \ldots, e^{im_n \theta}z_n),$$
where $(m_1, \ldots, m_n) \in \mathbb{N}^n$.

**Example II:** $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_2^2 + z_2 |^2 + |z_3^3 + z_3 |^6 = 1\}$. Then $X$ admits a transversal CR locally free $S^1$-action:
$$e^{i\theta} \circ (z_1, z_2, z_3) = (e^{i\theta}z_1, e^{i2\theta}z_2, e^{i6\theta}z_3).$$

**Definition 1.1.** We say that an $S^1$-action $e^{i\theta}$ on $\mathbb{C}^N$ is simple if
$$e^{i\theta} \circ (z_1, \ldots, z_N) = (e^{im_1 \theta}z_1, \ldots, e^{im_N \theta}z_N), \quad \forall (z_1, \ldots, z_N) \in \mathbb{C}^N, \quad \forall \theta \in [0, 2\pi),$$
where $(m_1, \ldots, m_N) \in \mathbb{N}^N$. The minimal weight of a simple $S^1$-action on $\mathbb{C}^N$ is given by $\min_{1 \leq j \leq N} m_j$.

Our goal is to prove the following equivariant embedding theorem for compact strongly pseudoconvex CR manifolds which admit a transversal CR $S^1$-action.

**Theorem 1.2.** Let $(X, T^{1,0}X)$ be a compact strongly pseudoconvex CR manifold with a transversal CR locally free $S^1$-action $e^{i\theta}$. Then, for any $m_0 \in \mathbb{N}$ we can find $N \in \mathbb{N}$, a simple $S^1$-action on $\mathbb{C}^N$ with minimal weight bigger than $m_0$ and an equivariant CR embedding
$$\Phi : X \to \mathbb{C}^N, \quad x \mapsto (\Phi_1(x), \ldots, \Phi_N(x)).$$

More precisely, $\Phi$ is an embedding, $\Phi_1, \ldots, \Phi_N$ are CR functions and there exist $m_1, \ldots, m_N \in \mathbb{N}$, $m_j > m_0$ for $j = 1, \ldots, N$, such that
$$\Phi(e^{i\theta} \circ x) = (e^{im_1 \theta} \Phi_1(x), \ldots, e^{im_N \theta} \Phi_N(x)) = e^{i\theta} \circ \Phi(x), \quad \forall x \in X, \quad \forall \theta \in [0, 2\pi),$$
where the simple $S^1$-action on $\mathbb{C}^N$ is given by $e^{i\theta} \circ (z_1, \ldots, z_N) = (e^{im_1 \theta}z_1, \ldots, e^{im_N \theta}z_N)$.

The equivariance, where a lower bound for the minimal weight of the simple $S^1$-action on the embedding space can be chosen arbitrarily large, distinguishes Theorem 1.2 from the embedding results mentioned at the beginning of this section.

An application of Theorem 1.2 is an equivariant embedding result for quasi-regular Sasakian manifolds. Sasakian manifolds have gained prominence in physics
and algebraic geometry, especially in String theory (see [7, 8]). Let $X$ be a compact smooth manifold of dimension $2n - 1, n \geq 2$. The triple $(X, g, \alpha)$ where $g$ is a Riemannian metric and $\alpha$ a real 1-form is called a Sasakian manifold if the cone $C(X) = \{(x, t) \in X \times \mathbb{R}_{>0}\}$ is a Kähler manifold with complex structure $J$ and Kähler form $t^2 \alpha + 2t dt \wedge \alpha$ which are compatible with the metric $t^2 g + dt \otimes dt$ (see [4], [8], [16], [31], [32]). As a consequence, $X$ is a compact strongly pseudoconvex CR manifold and the Reeb vector field $\xi$, defined by $\alpha(\cdot) = g(\xi, \cdot)$, induces a transversal CR action on $X$. If the orbits of $\xi$ are closed, the Sasakian structure is called quasi-regular. In this case, the Reeb vector field generates a locally free transversal $\mathbb{S}^1$-action on $X$. We thus can identify a compact quasi-regular Sasakian manifold with a compact strongly pseudoconvex CR manifold $(X, T^{1,0}X)$ with a transversal CR locally free $S^1$-action where the induced vector field of the $S^1$-action coincides with the Reeb vector field on $X$. From Theorem 1.2, we get

**Theorem 1.3.** Let $X$ be a quasi-regular Sasakian manifold which admits a transversal CR $S^1$-action induced by the Reeb vector field. For any $m_0 \in \mathbb{N}$, there is an equivariant CR embedding of $X$ into some $\mathbb{C}^N$ equipped with a simple $S^1$-action with minimal weight bigger than $m_0$.

The paper is organized as follows. In the rest of this section we will give an outline of the idea of the proof of Theorem 1.2 and introduce some terminology. Section 2 contains the proofs of the results on Szegő kernel expansion for positive Fourier components. The embedding result (Theorem 1.2) is proven in Section 3.

### 1.1. The idea of the proof of Theorem 1.2.

We refer the reader to Section 1.2, Section 1.3, and Section 1.4 for some notations and terminology used here. Assume that $(X, T^{1,0}X)$ is a compact connected strongly pseudoconvex CR manifold of dimension $2n - 1, n \geq 2$, with a transversal CR locally free $S^1$-action $e^{i\theta}$. Let $T$ denote the vector field induced by the $S^1$-action and let $\overline{\partial}_b$ be the tangential Cauchy-Riemann operator on $X$. For every $m \in \mathbb{N}$, let $H^0_{b,m}(X) := \{u \in C^\infty(X) : \overline{\partial}_b u = 0, Tu = imu\}$ be the $m$-th positive Fourier component of the space of global smooth CR functions. The main inspiration of this paper is the following: In [23] the second and third-named author have shown that $\dim H^0_{b,m}(X) \approx m^{n-1}$ as $m \to \infty$. Hence, the space of CR functions which lie in the positive Fourier components is very large and we therefore ask whether $X$ can be CR embedded into some $\mathbb{C}^N$ by CR functions which lie in the positive Fourier components. In this work we give an affirmative answer to this question and as a corollary, we deduce Theorem 1.2. More precisely, we will prove

**Theorem 1.4.** Let $X$ be a compact connected strongly pseudoconvex CR manifold with a locally free transversal $S^1$-action. Then $X$ can be CR embedded into some complex space by the CR functions which lie in the positive Fourier components.

Motivated by the second-named author’s work on the Kodaira embedding theorem ([19], [20], [21]), we will use the asymptotic expansion of the Szegő kernel with respect to $H^0_{b,m}(X)$ to prove Theorem 1.2. For every $k \in \mathbb{N}$, let $X_k$ and $X_{reg}$ be defined as in (1.3). Let $\{f_j\}_{j=1}^{dm} \subset H^0_{b,m}(X)$ be an orthonormal basis. The $m$-th
Szegő kernel $S_m(x, y)$ is given by $S_m(x, y) := \sum_{j=1}^{d_m} f_j(x) \overline{f_j(y)}$. Let us first consider

$$\Psi_m : X \to \mathbb{C}^{d_m}, x \mapsto (f_1(x), \ldots, f_{d_m}(x)).$$

We notice that $S_m(x, y) = 0$ on $X_k$ if $k \nmid m$. From this observation, we see that $X \setminus X_{\text{reg}} \neq \emptyset$ implies that $\Psi_m$ can not be an embedding even if $m$ is large. Suppose $X = X_1 \cup X_2 \cup \cdots \cup X_l$. For $1 \leq k \leq l$, let $\{f_j^{(k)}\}_{j=1}^{d_{km}}$ be an orthonormal basis of $H^0_{b,km}(X)$. We next consider the map

$$\Psi_m : X \to \mathbb{C}^{N_m},$$

$$x \mapsto (f_1(x), \ldots, f_{d_{km}}^{(k)}(x), f_1^2(x), \ldots, f_{d_{km}}^2(x), \ldots, f_1^{d_m}(x), \ldots, f_{d_{km}}^{d_m}(x)),$$

where $N_m = d_m + d_{2m} + \cdots + d_{km}$. In Section 2.3, we will show that on a canonical coordinate patch $D \subset X_{\text{reg}}$ with canonical coordinates $x = (z, \theta)$, we have

$$S_m(x, y) \equiv \frac{1}{2\pi} e^{im(x_{2m-1} - y_{2m-1} + \Phi(z, \theta))} \hat{b}(z, w, m) \mod O(m^{-\infty}),$$

$$\hat{b}(z, w, m) \sim \sum_{j=0}^{\infty} m^{n-1-j} \hat{b}_j(z, w), \hat{b}_j(z, w) \in C^\infty(D \times D), \quad j = 0, 1, 2, \cdots,$$

(see Theorem 2.6). Moreover, for fixed $x_0 \in X_k$, $k > 1$, one finds that $k \nmid m$ implies $S_m(x, x_0) = 0$ and that $k \mid m$ leads to

$$S_m(x, x_0) \equiv \frac{k}{2\pi} e^{im(x_{2m-1} + \Phi(z, 0))} \hat{b}(z, 0, m) \mod O(m^{-\infty})$$

for some canonical coordinate patch $D$ with canonical coordinates $x = (z, \theta)$, $x_0 \in D$, $(z(x_0), \theta(x_0)) = (0, 0)$ (see Theorem 2.7). It should be mentioned that (1.1) and (1.2) are based on Boutet de Monvel-Sjöstrand’s classical result on Szegő kernels [9] (after the seminal work [15] of Fefferman) and the complex stationary phase formula of Melin-Sjöstrand [30]. From (1.1) and (1.2), we can check that $\Psi_m$ is an immersion if $m$ is large. But $\Psi_m$ is not globally injective: in general, assume that $m$ is even, then we can not separate the points $p \in X_k$ and $e^{i\theta} \circ p$, where $k > 1$. To overcome this difficulty, let $\{g_j^{(k)}\}_{j=1}^{d_{km+1}}$ be an orthonormal basis of $H^0_{b,k(m+1)}(X)$ for $1 \leq k \leq l$. For any $k$, $1 \leq k \leq l$, we define a CR map from $X$ to Euclidean space as follows

$$\Phi_m^k : X \to \mathbb{C}^{d_{km} + d_{k(m+1)}}, x \mapsto (f_1^k(x), \ldots, f_{d_{km}}^k(x), g_1^k(x), \ldots, g_{d_{k(m+1)}}^k(x)),$$

and we set

$$\Phi_m : X \to \mathbb{C}^{N_m}, x \mapsto (\Phi_m^1(x), \ldots, \Phi_m^l(x)),$$

where $N_m = \sum_{k=1}^{l} (d_{km} + d_{k(m+1)})$. We thus try to prove that $\Phi_m$ is globally injective.

It is not difficult to see that $\Phi_m$ can separate the points $p \in X_k$ and $e^{i\theta} \circ p$, where $p \neq e^{i\theta} \circ p$, if $m$ is large enough. But another difficulty comes from the fact that the expansion (1.1) converges only locally uniformly on $X_{\text{reg}}$ and on $X \setminus X_{\text{reg}}$, we can only get an expansion of $S_m(x, x_0)$ for fixed $x_0 \in X \setminus X_{\text{reg}}$ and this causes that $\Phi_m$ could not be globally injective. To overcome this difficulty, we analyze carefully the behavior of the Szegő kernel $S_m(x, y)$ near the complement of $X_{\text{reg}}$ and in
Section 3.2 we can construct many CR functions $h_1, \ldots, h_K$ with large potentials near the complement of $X_{\text{reg}}$ which lie in the positive Fourier components such that the map
\[ x \in X \to (\Phi_m(x), h_1(x), \ldots, h_K(x)) \in \mathbb{C}^{N_m+K} \]
is an embedding if $m$ is large (see Theorem 3.3). This finishes the proof of Theorem 1.4.

1.2. Set up and terminology. Let $(X, T^{1,0}X)$ be a compact connected orientable CR manifold of dimension $2n - 1$, $n \geq 2$, where $T^{1,0}X$ is the CR structure of $X$. That is $T^{1,0}X$ is a subbundle of the complexified tangent bundle $\mathbb{C}TX$ of rank $n - 1$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$ and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$.

We assume that $X$ admits an $S^1$-action: $S^1 \times X \to X$, $(e^{i\theta}, x) \to e^{i\theta} \circ x$. Here we use $e^{i\theta}$ to denote the $S^1$-action. Set $X_{\text{reg}} = \{x \in X : \forall e^{i\theta} \in S^1, \text{ if } e^{i\theta} \circ x = x, \text{ then } e^{i\theta} = \text{id}\}$. For every $k \in \mathbb{N}$, put
\[ (1.3) \quad X_k := \left\{ x \in X : e^{i\theta} \circ x \neq x, \forall \theta \in (0, \frac{2\pi}{k}), e^{i\frac{2\pi}{k}} \circ x = x \right\}. \]
Thus, $X_{\text{reg}} = X_1$. In this paper, for simplicity we always assume that $X_1 \neq \emptyset$.

Actually, one can re-normalize the $S^1$-action by lifting such that the new $S^1$-action satisfies $X_1 \neq \emptyset$, (see [13, Remark 1.14]).

Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the $S^1$-action given as follows
\[ (1.4) \quad (Tu)(x) = \frac{\partial}{\partial \theta} (u(e^{i\theta} \circ x)) \bigg|_{\theta = 0}, u \in C^\infty(X). \]

**Definition 1.5.** We say that the $S^1$-action $e^{i\theta}$ ($0 \leq \theta < 2\pi$) is CR if
\[ [T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X), \]
where $[,]$ is the Lie bracket between the smooth vector fields on $X$. Furthermore, we say that the $S^1$-action is transversal if for each $x \in X$ one has
\[ CT(x) \oplus T^{1,0}_xX \oplus T^{0,1}_xX = CT_xX. \]

We throughout assume that $(X, T^{1,0}X)$ is a compact connected CR manifold with a transversal CR locally free $S^1$-action and we denote by $T$ the global vector field induced by the $S^1$-action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form uniquely determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

We recall

**Definition 1.6.** For $x \in X$, the Levi-form $\mathcal{L}_x$ associated with the CR structure is the Hermitian quadratic form on $T^{1,0}_xX$ defined as follows. For any $U, V \in T^{1,0}_xX$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(x) = U, \mathcal{V}(x) = V$. Set
\[ \mathcal{L}_x(U, V) = \frac{1}{2i} \langle [\mathcal{U}, \mathcal{V}](x), \omega_0(x) \rangle \]
where $[,]$ denotes the Lie bracket between smooth vector fields. Note that $\mathcal{L}_x(U, V)$ does not depend on the choice of $\mathcal{U}$ and $\mathcal{V}$.
Definition 1.7. The CR structure on $X$ is called pseudoconvex at $x \in X$ if $L_x$ is semi-positive definite. It is called strongly pseudoconvex at $x$ if $L_x$ is positive definite. If the CR structure is (strongly) pseudoconvex at every point of $X$, then $X$ is called a (strongly) pseudoconvex CR manifold.

Denote by $T^{1,0}X$ and $T^{0,1}X$ the dual bundles of $T^1X$ and $T^0X$, respectively. Define the vector bundle of $(0, q)$-forms by $\Lambda^q T^{0,1}X$. Let $D \subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $\Lambda^q T^{0,1}X$ over $D$.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$d e^{i \theta_0} : \mathbb{C}T_xX \to \mathbb{C}T_{e^{i \theta_0}x}X$$

denote the differential map of $e^{i \theta_0} : X \to X$. By the properties of transversal CR $S^1$-actions, we can check that

$$de^{i \theta_0} : T^{1,0}e^{i \theta_0}x \to T^{1,0}x,$$

$$d e^{i \theta_0} : T^{0,1}e^{i \theta_0}x \to T^{0,1}x,$$

$$d e^{i \theta_0}(T(x)) = T(e^{i \theta_0}x).$$

Let $(e^{i \theta_0})^* : \Lambda^q(\mathbb{C}T^*X) \to \Lambda^q(\mathbb{C}T^*X)$ be the pull back of $e^{i \theta_0}$, $q = 0, 1, \ldots, n - 1$. From (1.5), we can check that for every $q = 0, 1, \ldots, n - 1$

$$d e^{i \theta_0} = \Lambda^q e^{i \theta_0}T^{0,1}X \to \Lambda^q T^{0,1}X.$$

Let $u \in \Omega^{0,q}(X)$ be a section. The Lie derivative of $u$ along the direction $T$ is denoted by $T u$. From (1.5) we have $T u \in \Omega^{0,q}(X)$ for all $u \in \Omega^{0,q}(X)$.

Let $\bar{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. From (1.5), it is straightforward to deduce

$$\bar{T} \bar{\partial}_b = \bar{\partial}_b T \text{ on } \Omega^{0,q}(X).$$

For every $m \in \mathbb{Z}$, put $\Omega^{0,q}_m(X) := \{u \in \Omega^{0,q}(X) : T u = i m u\}$. We denote by $\bar{\partial}_{b,m}$ the restriction of $\bar{\partial}_b$ to $\Omega^{0,q}_m(X)$. From (1.7) we have the $\bar{\partial}_{b,m}$-complex for every $m \in \mathbb{Z}$:

$$\bar{\partial}_{b,m} : \cdots \to \Omega^{0,q-1}_m(X) \to \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X) \to \cdots.$$  

For $m \in \mathbb{Z}$, the $q$-th $\bar{\partial}_{b,m}$-cohomology is given by

$$H^q_{b,m}(X) := \frac{\text{Ker } \bar{\partial}_b : \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X)}{\text{Im } \bar{\partial}_b : \Omega^{0,q-1}_m(X) \to \Omega^{0,q}_m(X)}.$$  

Moreover, we have (see Theorem 1.13 in [23])

$$\dim H^q_{b,m}(X) < \infty, \text{ for all } q = 0, \ldots, n - 1.$$  

Definition 1.8. A function $u \in C^\infty(X)$ is a Cauchy-Riemann function (CR function for short) if $\bar{\partial}_b u = 0$, that is $\bar{Z} u = 0$ for all $Z \in C^\infty(X, T^1X)$. For $m \in \mathbb{N}$, $H^0_{b,m}(X)$ is called the $m$-th positive Fourier component of the space of CR functions.
1.3. Hermitian CR geometry.

**Definition 1.9.** Let $D$ be an open set and let $V \in C^\infty(D, \mathbb{C}T^*X)$ be a vector field over $D$. We say that $V$ is rigid if
\[
\text{de}^i\theta_0(V(x)) = V(e^{i\theta_0}x)
\]
holds for any $x \in D$ and $\theta_0 \in [0, 2\pi)$ satisfying $e^{i\theta_0} \circ x \in D$.

**Definition 1.10.** Let $\langle \cdot | \cdot \rangle$ be a Hermitian metric on $\mathbb{C}T^*X$. We say that $\langle \cdot | \cdot \rangle$ is rigid if for all rigid vector fields $V, W$ on $D$, where $D$ is any open set, we have
\[
\langle V(x) | W(x) \rangle = \langle (\text{de}^i\theta_0(V)) (e^{i\theta_0} \circ x) | (\text{de}^i\theta_0(W)) (e^{i\theta_0} \circ x) \rangle, \forall x \in D, \theta_0 \in [0, 2\pi).
\]

**Lemma 1.11** (Theorem 9.2 in [21]). Let $X$ be a compact CR manifold with a transversal CR $S^1$-action. There always exists a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$ and $\langle u | v \rangle$ is real if $u, v$ are real tangent vectors.

From now on, we fix a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$ satisfying all the properties in Lemma 1.11. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$ induces by duality a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$-forms $\Lambda^q T^{1,0}X$, $q = 0, 1, \ldots, n - 1$. We will denote all these induced metrics by $\langle \cdot | \cdot \rangle$. For every $v \in \Lambda^q T^{1,0}X$, we write $|v|^2 := \langle v | v \rangle$. We have the pointwise orthogonal decompositions:
\[
\mathbb{C}T^*X = T^{*1,0}X \oplus T^{0,1}X \oplus \{\lambda \omega_0 : \lambda \in \mathbb{C}\}, \quad \mathbb{C}T^*X = T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T : \lambda \in \mathbb{C}\}.
\]

For $p \in X$, locally there is an orthonormal frame $\{U_1, \ldots, U_{n-1}\}$ of $T^{1,0}X$ such that the Levi-form $\mathcal{L}_p$ is diagonal with respect to this frame. That is, $\mathcal{L}_p(U_i, U_j) = \lambda_j \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. The entries $\{\lambda_1, \ldots, \lambda_{n-1}\}$ are called the eigenvalues of the Levi-form at $p$ with respect to the rigid Hermitian metric $\langle \cdot | \cdot \rangle$. Moreover, the determinant of $\mathcal{L}_p$ is defined by $\det \mathcal{L}_p = \lambda_1(p) \cdots \lambda_{n-1}(p)$.

1.4. Canonical local coordinates. In this work, we need the following result due to Baouendi-Rothschild-Treves, (see [11]).

**Theorem 1.12.** Let $X$ be a compact CR manifold of $\dim X = 2n - 1$, $n \geq 2$ with a transversal CR $S^1$-action. Let $\langle \cdot | \cdot \rangle$ be a rigid Hermitian metric on $X$ as in Lemma 1.11. For $x_0 \in X$, there exists a local patch $D$ and local coordinates $(x_1, \ldots, x_{2n-1}) = (z, \theta) = (z_1, \ldots, z_{n-1}, \theta)$, $z_j = x_{2j-1} + i x_{2j}, 1 \leq j \leq n - 1, x_{2n-1} = \theta$, centered at $x_0$. In terms of these coordinates, $D = \{(z, \theta) \in \mathbb{C}^{n-1} \times \mathbb{R} : |z| < \varepsilon, |	heta| < \delta\}$ and on $D$

\[
T = \frac{\partial}{\partial \theta}, \quad Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial \varphi(z)}{\partial z_j} \frac{\partial}{\partial \theta}, j = 1, \ldots, n - 1,
\]

where $\{Z_j(x)\}_{j=1}^{n-1}$ form a basis of $T_x^{1,0}X$, for each $x \in D$ and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of $\theta$. Moreover, on $D$, $\varphi(z) = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z|^3)$, $\forall (z, \theta) \in D$, where $\{\lambda_j\}_{j=1}^{n-1}$ are the eigenvalues of the Levi-form of $X$ at $x_0$ with respect to the given rigid
Hermitian metric on $X$. We call $D$ a canonical local patch and $(z, \theta, \varphi)$ canonical coordinates centered at $x_0$.

In order to make use of the $S^1$-action locally, we need coordinates which contain nearly whole orbits. On the regular part, this is possible without any restrictions while on the irregular part we just find such coordinates approximately. This issue is stated in the following two lemmas.

**Lemma 1.13** ([23], Lemma 1.17). Fix $x_0 \in X_{\text{reg}}$. Then we can find canonical coordinates $(z, \theta, \varphi)$ centered at $x_0$ and defined on a canonical local patch $D = \{(z, \theta) : |z| < \varepsilon_0, |\theta| < \pi\}$.

**Lemma 1.14** ([23], Lemma 1.18). Let $x_0 \in X_k$ be a point, where $k \in \mathbb{N}, k > 1$. For every $\varepsilon > 0$, $\varepsilon$ small, we can find canonical coordinates $(z, \theta, \varphi)$ centered at $x_0$ and defined on a canonical local patch $D_\varepsilon = \{(z, \theta) : |z| < \varepsilon_0, |\theta| < \pi/\varepsilon\}$.

**Lemma 1.15** ([23], Lemma 1.19). Fix $x_0 \in X$. Let $(z, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical chart $D = \bar{D} \times (-\delta, \delta)$. We denote by $dv_X$ the volume form associated with the rigid Hermitian metric. Then on $D$ one has $dv_X = \lambda(z)dv(z)d\theta$ with $\lambda(z) \in C^\infty(\bar{D}, \mathbb{R})$ which does not depend on $\theta$ and $dv(z) = 2^{n-1}dx_1 \cdots dx_{2n-2}$.

2. Szegő Kernel Expansion

2.1. Some standard notations. First, we introduce some standard notations and definitions. We shall use the following notations: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An element $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ will be called a multiindex and the size of $\alpha$ is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $x = (x_1, \cdots, x_n)$, $\partial^\alpha_x = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\partial_x = \partial_x^{\alpha}$. Let $z = (z_1, \cdots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n$ be the coordinates of $\mathbb{C}^n$. We write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\partial^\alpha_z = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$, $\partial_z = \partial_z^{\alpha}$. In this section, we will study the semi-classical asymptotic expansion of the Szegő kernel of positive Fourier components. We recall some notations in semi-classical analysis.

**Definition 2.1.** Let $W$ be an open subset of $\mathbb{R}^N$. Let $S(1; W) = S(1)$ be the set of $a \in C^\infty(W)$ such that for every $\alpha \in \mathbb{N}_0^N$, there exists constant $C_\alpha$ such that $|\partial^\alpha_x a(x)| \leq C_\alpha$ on $W$. If $a = a(x, k)$ depends on $k \in (1, \infty)$, we say that $a(x, k)$ is uniformly bounded in $S(1)$ when $k$ varies in $(1, \infty)$ for every $\chi(x) \in C_0^\infty(W)$. For $m \in \mathbb{R}$, we put $S^m_{\text{loc}}(1; W) = S^m_{\text{loc}}(1) = k^m S_{\text{loc}}(1)$. If $a \in S^0_{\text{loc}}(1)$, $a_j \in S^m_{\text{loc}}(1)$, $m_j < -\infty$, we say that $a \sim \sum_{j=0}^{\infty} a_j$ in $S^m_{\text{loc}}(1)$ if $a - \sum_{j=0}^{N_0} a_j \in S^{m-N_0+1}_{\text{loc}}(1)$ for every $N_0$.

Let $W_1, W_2$ be two open subsets of $\mathbb{R}^N$. If $A : C^\infty_0(W_1) \to \mathcal{D}'(W_2)$ is continuous, by the Schwartz kernel theorem (Theorem 5.2.1 in [17]) we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of $A$. The following two statements are equivalent

(a) $A$ can be extended to an continuous operator $: \mathcal{E}'(W_1) \to C^\infty(W_2)$,
(b) $A(x, y) \in C^\infty(W_1 \times W_2)$.

If $A$ satisfies (a) or (b), we say that $A$ is smoothing.
A $k$-dependent continuous operator $A_k : C^{0}_0(W_1) \to \mathcal{D}'(W_2)$ is called $k$-negligible if $A_k$ is smoothing and the kernel $A_k(x,y)$ of $A_k$ satisfies $|\partial^\alpha_x \partial^\beta_y A_k(x,y)| = O(k^{-m})$ locally uniformly on every compact set in $W_1 \times W_2$, for all multi-indices $\alpha, \beta \in \mathbb{N}_0^N$ and all $m \in \mathbb{N}_0$. Let $C_k : C^{0}_0(W_1) \to \mathcal{D}'(W_2)$ be another $k$-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ if $A_k - C_k$ is $k$-negligible. We write $A_k = C_k + O(k^{-\infty})$ if $A_k \equiv C_k \mod O(k^{-\infty})$.

Similarly, we write $B_k(x) \equiv 0 \mod O(k^{-\infty})$ for any $k$-dependent smooth function $B_k(x) \in \mathcal{C}^\infty(W)$ if $|\partial^\alpha_x B_k(x)| = O(k^{-m})$ locally uniformly on every compact subset of $W$ for all $\alpha \in \mathbb{N}_0^N$ and all $m \in \mathbb{N}_0$.

2.2. Asymptotic Szegő kernel expansion. Let $(\cdot, \cdot)$ be the inner product on $\Omega^{0,0}(X)$ induced by $dv_x$. Let $L^2(X)$ (resp. $L^2_m(X)$) be the completions of $\Omega^{0,0}(X)$ (resp. $\Omega^{0,0}_m(X)$) with respect to $(\cdot, \cdot)$. By elementary Fourier analysis one has $L^2_m(X) \perp L^2_{m'}(X)$ for $m \neq m', m, m' \in \mathbb{Z}$. For $m \in \mathbb{Z}$, let $Q_m : L^2(X) \to L^2_m(X)$ be the orthogonal projection with respect to $(\cdot, \cdot)$.

From now on we assume $m \in \mathbb{N}$. Let $S_m : L^2(X) \to H^0_{b,m}(X)$ be the orthogonal projection with respect to $(\cdot, \cdot)$. We call $S_m$ the $m$-th Szegő projection.

From (1.10), one finds $\dim H^0_{b,m}(X) < \infty$. Let $\{f_j\}_{j=1}^m$ be an orthonormal basis of $H^0_{b,m}(X)$. Then the $m$-th Szegő kernel function is given by $S_m(x) = \sum_{j=1}^m |f_j(x)|^2$. Let $S_m(x,y)$ be the distribution kernel with respect to the operator $S_m$ which is given by $S_m(x,y) = \sum_{j=1}^m f_j(x)f_j(y)$. The goal of this section is to study the semiclassical asymptotic expansion of $S_m(x,y)$.

We extend $\overline{\partial}_b$ to $L^2(X)$ in the sense of distribution and denote its kernel by $\text{Ker}(\overline{\partial}_b) = \{u \in L^2(X) : \overline{\partial}_b u = 0\}$ which is a closed subspace of $L^2(X)$. Let $S : L^2(X) \to \text{Ker}(\overline{\partial}_b)$ be the usual Szegő projection. We denote by $S(x,y)$ the distribution kernel of the Szegő projection.

**Lemma 2.2.** With the notations above, we have

\[(2.1)\] $H^0_{b,m}(X) = \text{Ker}(\overline{\partial}_b) \cap L^2_m(X)$

and

\[(2.2)\] $S_mu = SQ_mu = Q_m S u, \forall u \in C^\infty(X)$.

**Proof.** It is obvious that $H^0_{b,m}(X) \subset \text{Ker}(\overline{\partial}_b) \cap L^2_m(X)$. The converse is a direct corollary from following subelliptic estimate (see theorem 1.12 in [22])

\[(2.3)\] $\|u\|_s \leq C_{s,m}(\|\overline{\partial}_b u\|_{s-1} + \|u\|), \forall u \in H^s(X) \cap L^2_m(X), s \geq 1,$

where $H^s(X)$ is the usual Sobolev space on $X$, $\|u\|_s$ is the usual Sobolev norm of order $s$ and $C_{s,m}$ is a constant.

For any $u \in C^\infty(X)$, write $u = u_1 + u_2$, $u_1 \in H^0_{b,m}(X), u_2 \in H^0_{b,m}(X)^\perp$. For any $v \in H^0_{b,m}(X)$, we have

\[ (S_m u|v) = (u_1|v) = (u|v) = (Q_mu|v) = (SQ_mu|v). \]

For any $v \in L^2(X) \cap H^0_{b,m}(X)^\perp$, we have

\[ (S_m u|v) = 0 = (SQ_mu|v) \]

since $S_m u, SQ_mu \in H^0_{b,m}(X)$. This implies $S_m u = SQ_mu$ for all $u \in C^\infty(X)$. Similarly, we have $S_m u = Q_m S u$ for all $u \in C^\infty(X)$.  \qed
Fix $x_0 \in X$. Let $(z, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical local patch $D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \delta_1\}$. Choose $D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta \} \subset D_1$.

Choose two cut-off functions $\chi, \chi_1 \in \mathcal{C}_0^\infty(D_1)$ such that $\chi = 1$ in some small neighborhood of $\overline{D}$ and $\chi_1 = 1$ in some small neighborhood of $\text{supp}\chi$. By Lemma 2.2, we have $S_m = S_Q m$ and hence

$$\chi S_m = \chi S_Q m = \chi S \chi_1 Q_m + \chi S(1 - \chi)Q_m.$$ 

We write $F = \chi S(1 - \chi_1)$ and $F_m = \chi S(1 - \chi_1)Q_m$ and denote by $F(x, y)$ and $F_m(x, y)$ the distribution kernels of $F$ and $F_m$, respectively. We will show

**Lemma 2.3.** $F_m : \mathcal{C}_0^\infty(D) \to \mathcal{E}'(D_1)$ is $m$-negligible.

**Proof.** Since $\text{supp}\chi \cap \text{supp}(1 - \chi_1) = \emptyset$, by a result of Boutet de Monvel-Sjöstrand [9] (see also [18] and [25]) we know that $F$ is smoothing. Let $\cup_{j=1}^{n_0} U_j$ be a finite covering of $X$. Assume that all the $U_j$s, $1 \leq j \leq n_0$ are canonical local patches. Choose a partition of unity $\{\rho_j\}_{j=1}^{n_0}$ with $\text{supp}\rho_j \subset U_j$, $1 \leq j \leq n_0$, and $\sum_{j=1}^{n_0} \rho_j = 1$ on $X$. Then for all $u \in \mathcal{C}_0^\infty(D)$ we have

$$F_m u = F Q_m u = F \left( \sum_{j=1}^{n_0} \rho_j Q_m u \right) = \sum_{j=1}^{n_0} F(\rho_j Q_m u). \tag{2.4}$$

For $1 \leq j \leq n_0$, let $y = (w, y_{2n-1})$ be canonical coordinates in $U_j$. Then on $U_j$ one finds

$$\rho_j Q_m u = \rho_j(y) (Q_m u)(y) = \rho_j(y) \hat{u}_m(w) e^{imy_{2n-1}}.$$ 

Set $F_j(x, y) = F(x, y) \rho_j(y)$ for $x \in D, y \in U_j$. Then on $D$, by a direct calculation we have

$$F(\rho_j Q_m u)(x) = -\frac{1}{2\pi mi} \int_{U_j} \left( \int_0^{2\pi} \frac{\partial F_j}{\partial y_{2n-1}} (x, e^{i\theta} \circ y) e^{im\theta} d\theta \right) u(y) \lambda(w) dv(w) dy_{2n-1}. \tag{2.5}$$

By (2.4), (2.5) and the induction method, we have $F_m(x, y) = O(m^{-N})$ locally uniformly for all $N \in \mathbb{N}$ and similarly for the derivatives. Thus, the lemma follows. \hfill \Box

Set $G = \chi S \chi_1$ and $G_m = \chi S \chi_1 Q_m$. Write $D_1 = \tilde{D}_1 \times (-\delta_1, \delta_1)$ and $D = \tilde{D} \times (-\delta, \delta)$ with $\tilde{D}_1 = \{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_1\}$ and $\tilde{D} = \{z \in \mathbb{C}^{n-1} : |z| < \varepsilon\}$. Assume that on $D_1$, $\chi_1(y) = \tilde{\chi}_1(w) \tilde{\chi}_2(y_{2n-1})$ holds with $\tilde{\chi}_1(w) \in C_0^\infty(\tilde{D}_1), \tilde{\chi}_2(y_{2n-1}) \in C_0^\infty(-\delta_1, \delta_1)$ and $\tilde{\chi}_1(w) = 1$ in some small neighborhood of $\overline{\tilde{D}}$ and $\tilde{\chi}_2 = 1$ in some small neighborhood of $[-\delta, \delta]$. Let $u \in \mathcal{C}_0^\infty(D)$ be a smooth function. On $D_1$, we write $(Q_m u)(y) = \hat{u}_m(w) e^{imy_{2n-1}}, \hat{\tilde{u}}_m(w) \in C^\infty(\tilde{D}_1)$. Then on $D$ we have

$$G_m u(x) = \int_{\tilde{D}_1} \tilde{\chi}_1(w) \hat{\tilde{u}}_m(w) \chi(x) S(x, w, y_{2n-1}) \tilde{\chi}_2(y_{2n-1}) e^{imy_{2n-1}} dy_{2n-1} dw. \tag{2.6}$$

In order to calculate the integral with respect to $dy_{2n-1}$ in (2.6), we need the following result due to Boutet de Monvel and Sjöstrand [9], [18] and Hsiao-Marinescu [26].
Theorem 2.4. Let $X$ be a compact strongly pseudoconvex CR manifold with a transversal CR $S^1$-action. For any $x_0 \in X$, let $D_1$ be the canonical local patch defined as in Theorem 1.12 with canonical coordinates $(z, \theta, \varphi)$ centered at $x_0$. Then on $D_1 \times D_1$ the distribution kernel $S(x, y)$ of the Szegő projection $S : L^2(X) \to \text{Ker}(\partial \bar{\partial})$ satisfies

$$(2.7) \quad S(x, y) = \int_0^\infty e^{i\Psi(x,y)t}b(x, y, t)dt$$

in the sense of oscillatory integrals, where

$$(2.8) \quad \Psi(x, y) \in C^\infty(D_1 \times D_1), \Psi(x, y) = x_{2n-1} - y_{2n-1} + \Phi(z, w),$$

$$\Phi(z) = -\overline{\Phi}(w, z), \exists \ c > 0 : \text{Im} \Phi \geq c|z - w|^2, \Phi(z, w) = 0 \Leftrightarrow z = w,$$

$$\Phi(z) = i(\varphi(z) + \varphi(w)) - 2i \sum_{|\alpha|+|\beta| \leq N} \frac{\partial^{|\alpha|+|\beta|} \varphi}{\partial z^\alpha \partial \overline{w}^\beta}(0) \frac{\alpha! \beta!}{\alpha! \beta!} + O(|(z, w)|^{N+1}), \forall N \in \mathbb{N},$$

$$b(x, y, t) \sim \sum_{k=0}^\infty b_k(x, y)t^{n-1-k} \text{in } S^{-n-1}_{\text{loc}}(1; D_1 \times D_1),$$

$$b_j(x, y) \in C^\infty(D_1 \times D_1), j = 0, 1, \cdots ,$$

$$b_0(x, y) = \frac{1}{2\pi^n} |\det L_x|, \forall x \in D_1.$$

By Theorem 2.4, the integral with respect to $dy_{2n-1}$ in (2.6) can be computed by making use of the stationary phase formula due to Melin-Sjöstrand [30]. Substituting (2.7) and (2.8) to (2.6) and changing coordinates $t = m\sigma$, where $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}_+$, we have

$$(2.9) \quad \int_{-\delta_1}^{\delta_1} \chi(x)S(x, w, y_{2n-1})\tilde{\chi}_2(y_{2n-1})e^{imy_{2n-1}}dy_{2n-1}$$

$$= m \int_{-\delta_1}^{\delta_1} \int_0^\infty e^{im[(x_{2n-1} - y_{2n-1})\sigma + \Phi(z, w)\sigma + y_{2n-1}]\chi(x)b(x, y, m\sigma)\tilde{\chi}_2(y_{2n-1})d\sigma dy_{2n-1}.$$
one can apply the stationary phase formula of Melin and Sjöstrand [30] to carry out the $d\sigma dy_{2n-1}$ integration in (2.9):

$$
m\int_{-\delta_1}^{\delta_1} \int_0^{\infty} e^{im\tilde{\Psi}(x,w,y_{2n-1},\sigma)} \chi(x) b(x,y,m\sigma) \tilde{\chi}_2(y_{2n-1}) d\sigma dy_{2n-1}
$$

(2.10) 

$$= m\int_{-\delta_1}^{\delta_1} \int e^{im\tilde{\Psi} \tau(\sigma)} \chi(x) b(x,y,m\sigma) \tilde{\chi}_2(y_{2n-1}) d\sigma dy_{2n-1} + m\int_{-\delta_1}^{\delta_1} \int e^{im\tilde{\Psi} (1 - \tau(\sigma))} \chi(x) b(x,y,m\sigma) \tilde{\chi}_2(y_{2n-1}) d\sigma dy_{2n-1},$$

where $\tau(\sigma) \in C^\infty(\mathbb{R})$ with $\text{supp} \tau \subset (\frac{1}{2}, \frac{3}{2})$ and $\tau = 1$ near $\sigma = 1$.

First we show that on $D_1 \times \hat{D}_1$, the second term on the right-hand side of (2.10) satisfies the following identity

$$m\int_{-\delta_1}^{\delta_1} \int e^{im\tilde{\Psi}(x,w,y_{2n-1},\sigma)(1-\tau(\sigma))} \chi(x) b(x,y,m\sigma) \tilde{\chi}_2(y_{2n-1}) d\sigma dy_{2n-1} \equiv 0 \mod O(m^{-\infty}).$$

This is a direct corollary of the following formula

$$e^{im\tilde{\Psi}} = \frac{1}{im(1-\sigma)} \frac{\partial}{\partial y_{2n-1}} e^{im\tilde{\Psi}}$$

and the integration by parts with respect to the variable $y_{2n-1}$. For convenience we denote by $H_m(x,w)$ the left-hand side of (2.11).

Making use of Melin-Sjöstrand’s stationary phase formula [30], the first term on the right-hand side of (2.10) becomes

$$m\int_{-\delta_1}^{\delta_1} \int e^{im\tilde{\Psi} \tau(\sigma)} \chi(x) b(x,y,m\sigma) \tilde{\chi}_2(y_{2n-1}) d\sigma dy_{2n-1} \equiv e^{im(x_{2n-1}+\Phi(z,w))} \chi(x) \hat{b}(x,w,m) \mod O(m^{-\infty}),$$

(2.12)

where

$$\hat{b}(x,w,m) \sim \sum_{j=0}^{\infty} \hat{b}_j(x,w) m^{n-1-j} \in \mathcal{E}^{n-1}_{\text{loc}}(1; D_1 \times \hat{D}_1),$$

(2.13)

$$\hat{b}_j(x,w) \in C^\infty(D_1 \times \hat{D}_1), j = 0, 1, 2, \cdots.$$ In particular, one has

$$\hat{b}_0(x,w) = (2\pi) \tilde{b}_0(x,w,x_{2n-1} + \Phi(z,w)), \quad \hat{b}_0(x,z) = \pi^{1-n} |\text{det} \mathcal{L}_x|,$$

(2.14)

where $\hat{b}_0$ denotes an almost analytic extension of $b_0$, that is $\tilde{b}_0(\bar{x}, \bar{y}) \in C^\infty(U_1 \times U_1)$ with $\tilde{b}_0|_{D_1 \times D_1} = b_0$ and $|\partial_x \tilde{b}_0(\bar{x}, \bar{y})| + |\partial_y \tilde{b}_0(\bar{x}, \bar{y})| \leq C_N (|\text{Im } \bar{x}|^N + |\text{Im } \bar{y}|^N)$, for every $N > 0$ where $C_N > 0$ is a constant. Here $U_1$ is an open set in $\mathbb{C}^{2n-1}$ with $U_1 \cap \mathbb{R}^{2n-1} = D_1$ (we identify $D_1$ with an open set in $\mathbb{R}^{2n-1}$) and $\bar{x}, \bar{y}$ are complex coordinates of $\mathbb{C}^{2n-1}$. Substituting (2.11) and (2.12) to (2.6) one has

$$G_m u = \int_{D_1} \tilde{\chi}_1(w) \hat{u}_m(w) e^{im(x_{2n-1}+\Phi(z,w))} \chi(x) \hat{b}(x,w,m) \lambda(w) dv(w)
$$

(2.15)

$$+ \int_{D_1} \tilde{\chi}_1(w) \hat{u}_m(w) H_m(x,w) \lambda(w) dv(w)$$
with $H_m(x, w) \equiv 0 \mod O(m^{-\infty})$ on $D_1 \times \tilde{D}_1$.

Choose $\eta(y_{2n-1}) \in C^\infty_0(-\delta_1, \delta_1)$ such that $\int_{-\delta_1}^{\delta_1} \eta(y_{2n-1})dy_{2n-1} = 1$. Then the first term on the right-hand side of (2.15) is equal to

$$(2.16) \int_{D_1} (Q_m u)(y) \hat{\chi}_1(w) \eta(y_{2n-1}) e^{im(x_{2n-1} - y_{2n-1}) + \Phi(z, w)} \chi(x) \hat{b}(x, w, m) \lambda(w)dw dy_{2n-1} = \chi(x) \int_{D_1} (Q_m B_m)(x, y) u(y) \lambda(w)dy.$$ 

Here, we have set $L(2.18)$

From (2.17), (2.19), (2.20) and (2.21) we have

$$(2.17) B_m(x, y) = e^{im(x_{2n-1} - y_{2n-1}) + \Phi(z, w)} \hat{b}(x, w, m) \hat{\chi}_1(w) \eta(y_{2n-1})$$

and $(Q_m B_m)(x, y)$ denotes that $Q_m$ acts on $B_m(x, y)$ seen as a function in the variable $y$. Combining (2.15), (2.16), (2.17) and Lemma 2.3, we have

$$S_m(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_m(x, e^{i\theta} \circ y) e^{im\theta} d\theta + A_m(x, y), \forall x, y \in D \times D,$$

where $A_m(x, y) \equiv 0 \mod O(m^{-\infty})$. On the other hand, we have $S_m(x, y) = \sum_{j=1}^{d_m} f_j(x) f_j(y)$. On $D$, we can write $f_j(x) = f_j(z) e^{imx_{2n-1}}$ which leads to $S_m(x, y) = \sum_{j=1}^{d_m} f_j(z) f_j(w) e^{im(x_{2n-1} - y_{2n-1})}$. Thus,

$$(2.18) e^{-imx_{2n-1}} S_m(x, y) = \sum_{j=1}^{d_m} f_j(z) f_j(w) e^{im(-y_{2n-1})}$$

does not depend on $x_{2n-1}$. We get

$$(2.19) e^{-imx_{2n-1}} S_m(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n-1}} B_m(x, e^{i\theta} \circ y) e^{im\theta} d\theta + e^{-imx_{2n-1}} A_m(x, y).$$

Choose $\chi_0(x_{2n-1}) \in C^\infty_0(-\delta, \delta)$ such that $\int_{-\delta}^{\delta} \chi_0(x_{2n-1})dx_{2n-1} = 1$. From (2.18) and (2.19) we have

$$(2.20) e^{-imx_{2n-1}} S_m(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \chi_0(x_{2n-1}) e^{-imx_{2n-1}} B_m(x, e^{i\theta} \circ y) e^{im\theta} dx_{2n-1} d\theta + C_m(z, y),$$

where $C_m(z, y) = \int_{-\delta}^{\delta} A_m(x, y) e^{-imx_{2n-1}} \chi_0(x_{2n-1})dx_{2n-1}, C_m(z, y) \equiv 0 \mod O(m^{-\infty})$. Set

$$(2.21) \hat{S}_m(x, y) = e^{imx_{2n-1}} \int_{-\delta}^{\delta} \chi_0(x_{2n-1}) e^{-imx_{2n-1}} B_m(x, y) dx_{2n-1}.$$ 

From (2.17), (2.19), (2.20) and (2.21) we have

**Theorem 2.5.** Let $X$ be as in Theorem 2.4 Consider the orthogonal projection $S_m : L^2(X) \rightarrow H^0_m(X)$. We denote by $S_m(x, y)$ the distribution kernel of $S_m$. For $x_0 \in X$, let $(\zeta, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical local
patch $D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \delta_1\}$. For any $D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\} \Subset D_1$, we have

$$S_m(x, y) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{S}_m(x, e^{i\theta} \circ y) e^{im\theta} d\theta \mod O(m^{-\infty})$$
onumber

on $D \times D$, where

(2.22) \hspace{1cm} \hat{S}_m(x, y) = e^{im(x_{2n-1} - y_{2n-1} + \Phi(z, w))} b(z, w, m) \tilde{\chi}_1(w) \eta(y_{2n-1}), \Phi(z, w) = i(\varphi(z) + \varphi(w)) - 2i \sum_{|\alpha| \neq |\beta| \leq N} \frac{\partial^{i|\alpha| + |\beta|}}{\partial z^\alpha \overline{w}^\beta} \varphi(0) \frac{z^n}{\alpha! \beta!} + O(|(z, w)|^N + 1), \nonumber

\hat{b}(z, w, m) \sim \sum_{k=0}^{\infty} m^n \hat{b}_k(z, w) \text{ in } S^{n-1}_{\text{loc}}(1; \tilde{D} \times \tilde{D}), \tilde{D} = \{z \in \mathbb{C}^n : |z| < \varepsilon\}, \hat{b}_0(z, w) = (2\pi) \int_{-\delta}^{\delta} \hat{b}_0(z, x_{2n-1}, w, x_{2n-1} + \Phi(z, w)) \chi_0(x_{2n-1}) dx_{2n-1}, \hat{b}_0(z, z) = \pi^{-(n-1)} |\det L_x|, x = (z, 0), \forall z \in \tilde{D}, \nonumber

and

\hat{b}_j(z, w) \in C^\infty(\tilde{D} \times \tilde{D}), \forall j; \chi_0(x_{2n-1}) \in C^\infty_0(\delta, \delta), \int_{-\delta}^{\delta} \chi_0(x_{2n-1}) dx_{2n-1} = 1; \chi_1(w) \in C^\infty_0(\tilde{D}_1), \chi_1 = 1 \text{ in a neighborhood of } \tilde{D}; \nonumber

\eta(y_{2n-1}) \in C^\infty_0(\delta_1, \delta_1), \int_{-\delta_1}^{\delta_1} \eta(y_{2n-1}) dy_{2n-1} = 1. \nonumber

Here, $\hat{b}_0$ is as in (2.14).

2.3. Asymptotic Szeg\'o kernel expansion on $X_{\text{reg}}$. Recall that $X_{\text{reg}}$ is the regular part of $X$, that is $X_{\text{reg}} = \{x \in X : \forall e^{i\theta} \in S^1, \text{ if } e^{i\theta} \circ x = x, \text{ then } e^{i\theta} = \text{id}\}$, and that we assume $X_{\text{reg}} \neq \emptyset$. Fix $x_0 \in X_{\text{reg}}$. Let $(z, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical patch $D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \pi/2\}$. Set $D = \{(z, \theta) \in \mathbb{C}^n \times \mathbb{R} : |z| < \varepsilon, |\theta| < \pi/2\}$ with $\varepsilon < \varepsilon_1$. From Theorem 2.5 it follows that

(2.23) \hspace{1cm} S_m(x, y) \equiv e^{-imy_{2n-1}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{S}_m(x, (w, \theta)) e^{im\theta} d\theta \mod O(m^{-\infty}) \nonumber

holds on $D \times D$. Substituting (2.22) to (2.23), we have

(2.24) \hspace{1cm} S_m(x, y) \equiv \frac{1}{2\pi} e^{im(x_{2n-1} - y_{2n-1} + \Phi(z, w))} b(z, w, m) \mod O(m^{-\infty}), \nonumber

(2.24) \hspace{1cm} S_m(x, x) \equiv \frac{1}{2\pi} \hat{b}(z, z, m) \mod O(m^{-\infty}). \nonumber

Thus, from (2.24) we have

Theorem 2.6. Let $X$ be as in Theorem 2.4. For $x_0 \in X_{\text{reg}}$, let $(z, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical patch $D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \pi/2\}$.
\[ \varepsilon_1, |\theta| < \pi \}. \text{ Set } D = \{(z, \theta) \in \mathbb{C}^{n-1} \times \mathbb{R} : |z| < \varepsilon, |\theta| < \frac{\pi}{k} \} \subset D_1. \text{ Then on } D \times D, \text{ we have} \]

\[
S_m(x, y) = \frac{1}{2\pi} e^{im(x_{2n-1} - y_{2n-1} + \Phi(z, w))} \hat{b}(z, w, m) \mod O(m^{-\infty}),
\]

where

\[
\hat{b}(z, w, m) \sim \sum_{j=0}^{\infty} m^{-n-1-j} \hat{b}_j(z, w) \text{ in } S^{-1}_{\text{loc}}(1, \tilde{D} \times \tilde{D}),
\]

(2.25)

\[
\hat{b}_j(z, w) \in C^\infty(\tilde{D} \times \tilde{D}), \quad j = 0, 1, 2, \ldots,
\]

(2.27)

\[
\hat{b}_0(z, z) = \pi^{-(n-1)} |\det L_x|, \quad x = (z, 0), \text{ all } z \in \tilde{D}.
\]

Here, we set \( \tilde{D} = \{ z \in \mathbb{C}^{n-1} : |z| < \varepsilon \}. \) In particular, we have

\[
S_m(x, x) = \frac{1}{2\pi} \hat{b}(z, z, m) \mod O(m^{-\infty}).
\]

2.4. Asymptotic Szegő kernel expansion on the complement of \( X_{\text{reg}}. \) In this section, we try to get the asymptotic expansion of the Szegő kernel on the complement of \( X_{\text{reg}}. \) Fix \( x_0 \in X_k \) for some \( k > 1, \) where \( X_k \) is defined in (1.3). Let \((z, \theta, \varphi)\) be canonical coordinates centered at \( x_0 \) and defined on a canonical chart \( D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \frac{\pi}{k} - \epsilon\}. \) It is straightforward to see that there is a small neighborhood \( D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\} \subset D_1 \) of \( x_0 \) such that

\[
e^{i\theta} \circ (0, 0) \neq (z, \tilde{\theta}), \quad \forall \theta \in [0, 2\pi), \quad (z, \tilde{\theta}) \in D, \quad z \neq 0.
\]

From Theorem 2.5, we have

\[
S_m(x, x_0) = \frac{1}{2\pi} \sum_{s=1}^{k} \int_{\frac{2\pi}{k}(s-1)}^{\frac{2\pi}{k}s} \hat{S}_m(x, e^{i\theta} \circ (0, 0))e^{im\theta} d\theta \mod O(m^{-\infty})
\]

(2.28)

\[
= \frac{1}{2\pi} \sum_{s=1}^{k} e^{i\frac{2\pi}{k}(s-1)m} \int_{0}^{\frac{2\pi}{k}} \hat{S}_m(x, e^{i\theta} \circ (0, 0))e^{im\theta} d\theta \mod O(m^{-\infty})
\]

for any \( x \in D. \) By a direct calculation, we find

\[
\sum_{s=1}^{k} e^{i\frac{2\pi}{k}(s-1)m} = \begin{cases} k, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m. \end{cases}
\]

From (2.27), we can check that

\[
\frac{k}{2\pi} \int_{-\frac{2\pi}{k}}^{\frac{2\pi}{k}} \hat{S}_m(x, e^{i\theta} \circ (0, 0))e^{im\theta} d\theta = \frac{k}{2\pi} \int_{-\frac{2\pi}{k}}^{\frac{2\pi}{k}} \hat{S}_m(x, (0, \theta))e^{im\theta} d\theta
\]

holds. Substituting (2.29) to (2.28) for \( k \mid m \) and using (2.30), we have

\[
S_m(x, x_0) = \frac{k}{2\pi} \int_{-\frac{2\pi}{k}}^{\frac{2\pi}{k}} \hat{S}_m(x, (0, \theta))e^{im\theta} d\theta \mod O(m^{-\infty}).
\]

(2.31)

Substituting (2.22) to (2.31) yields

\[
S_m(x, x_0) = \frac{k}{2\pi} e^{im(x_{2n-1} + \Phi(z, 0))} \hat{b}(z, 0, m) \mod O(m^{-\infty}).
\]

Summing up, we obtain
Theorem 2.7. Let $X$ be as in Theorem 2.4. Assume $x_0 \in X_k, k > 1$. Let $D_{1, D}$ and $(z, \theta, \varphi)$ be as above with $\frac{\pi}{k} - \epsilon$ replaced by $\delta_1$. For $k \nmid m$ we have $S_m(x, x_0) = 0$ for all $x \in D$. For $k \mid m$ we have

$$S_m(x, x_0) = \frac{k}{2\pi} e^{im(x_{2n-1} + \Phi(z,0))} \tilde{b}(z, 0, m) \mod O(m^{-\infty})$$

on $D$. In particular, given $k \mid m$ and $x = x_0$, we have

$$S_m(x_0, x_0) = \frac{k}{2\pi} \tilde{b}(0, 0, m) + O(m^{-\infty})$$

and

$$\tilde{b}(0, 0, m) \sim \tilde{b}_0(0, 0)m^{n-1} + \tilde{b}_1(0, 0)m^{n-2} + \cdots$$

in the sense that for any $N \in \mathbb{N}_0$ there exists $C_N > 0$ independent of $m$ such that

$$\left| \tilde{b}(0, 0, m) - \sum_{j=0}^{N} \tilde{b}_j(0, 0)m^{n-1-j} \right| \leq C_N m^{n-2-N}$$

holds for all $m \in \mathbb{N}$.

3. EQUIVARIANT EMBEDDING OF CR MANIFOLDS

Let $X$ be a compact strongly pseudoconvex CR manifolds with a locally free transversal CR $S^1$-action. Now we use the Szegő kernel expansion we have established in Section 2 to get the equivariant embedding of $X$.

3.1. Immersion of CR manifold. We assume $X = X_1 \cup X_2 \cup \cdots \cup X_l$, $X_1 \neq \emptyset$, where $X_k$, $1 \leq k \leq l$, is defined in (1.3). For $1 \leq k \leq l$ let $\{f_j^k\}_{j=1}^{d_{km}}$ and $\{g_j^k\}_{j=1}^{d_{km+1}}$ be orthonormal bases of $H^0_{b, km}(X)$ and $H^0_{b, km+1}(X)$, respectively. Now for $1 \leq k \leq l$ we can define a CR map from $X$ to Euclidean space as follows

$$\Phi_m^k : X \to \mathbb{C}^{d_{km} + d_{km+1}}, x \mapsto (f_1^k(x), \cdots, f_{d_{km}}^k(x), g_1^k(x), \cdots, g_{d_{km+1}}^k(x)).$$

Combining the $\Phi_m^k$’s, $1 \leq k \leq l$, we define a CR map

$$\Phi_m : X \to \mathbb{C}^{N_m}, x \mapsto (\Phi_m^1(x), \cdots, \Phi_m^l(x)),$$

where $N_m = \sum_{k=1}^{l} (d_{km} + d_{km+1})$. If the transversal CR $S^1$-action on $X$ is globally free, then $X = X_1 = X_{reg}$ and Epstein [14] showed that $\Phi_m^1$ is an CR embedding when $m$ is large. However, if the transversal CR $S^1$-action on $X$ is just locally free the CR functions in $H^0_{b, cm}(X) \bigoplus H^0_{b, cm+1}(X)$ are not enough for the embedding. The reason is that the space $H^0_{b, cm}(X) \bigoplus H^0_{b, cm+1}(X)$ will be not enough to separate the points in $X \setminus X_{reg}$.

Now we use the asymptotic Szegő kernel expansion in Section 2 to establish

Lemma 3.1. The map $\Phi_m : X \to \mathbb{C}^{N_m}$ is an immersion when $m$ is large.

Proof. For $x_0 \in X_k$, let $(z, \theta, \varphi)$ be canonical coordinates centered at $x_0$ and defined on a canonical local patch $D = \{(z, \theta) : |z| < \epsilon, |\theta| < \delta \} = D \times (-\delta, \delta)$. Assume that $k \mid m$. Let $\{f_j\}_{j=1}^{d_m} \subset H^0_{b, cm}(X)$ be an orthonormal basis. Since $S_m(x, y) = $
Choose cut-off functions \( \chi \in C_0^\infty(\mathbb{C}^{n-1}) \), \( \chi_2 \in C_0^\infty(-\delta, \delta) \) such that \( \text{supp} \chi \subseteq \{ w \in \mathbb{C}^{n-1} : |w| < 1 \} \) and \( \int_{-\delta}^\delta \chi_2(y_{n-1})dy_{2n-1} = 1 \). For \( j = 1, \ldots, n-1 \), set

\[
(3.2) \quad u_j(y) = w_j \chi \left( \frac{\sqrt{mw}}{\log m} \right) \chi_2(y_{2n-1})e^{imy_{2n-1}}e^{imRe\Phi(w,0)},
\]

where \( \Phi \) is as in Theorem 2.5. Then \( u_j \in C_0^\infty(X) \) has its support in \( D \) if \( m \) is sufficiently large. Define \( v_j = S_m u_j, j = 1, \ldots, n-1 \). Then from Theorem 2.5 and (3.1) we have

\[
(3.3) \quad \frac{\partial \hat{S}_m}{\partial \bar{z}_j}(y_{n+1}) = \frac{\partial \hat{S}_m}{\partial \bar{z}_j}(y_{n+1})|_{x=x_0}.
\]

Then by (3.3) and a direct calculation we have

\[
(3.4) \quad \frac{\partial S_m u_j}{\partial \bar{z}_j}(x_0) = \frac{k}{2\pi} \int_D \int_{-\pi}^\pi \frac{\partial \hat{S}_m}{\partial \bar{z}_j}(y,(\theta,0))e^{im\theta}d\theta u_j(y)dv_X + O(m^{-\infty})
\]

which leads to

\[
(3.5) \quad \frac{\partial \hat{S}_m}{\partial \bar{z}_j}(y,(\theta,0)) = e^{im(y_{2n-1} - \theta + \Phi(w,0))}\eta(\theta) \times \left[ \frac{im}{\partial \bar{z}_j} \hat{b}(w,0,m,\chi_1(0) + \frac{\partial \hat{b}(w,0,m)}{\partial \bar{z}_j} \chi_1(0) + \hat{b}(w,0,m) \frac{\partial \tilde{\chi}_1}{\partial \bar{z}_j}(0) \right] \]

Substituting (3.5) to (3.4), we have

\[
(3.6) \quad \frac{\partial S_m u_j}{\partial \bar{z}_j}(x_0) = \frac{k}{2\pi} \int_D e^{im(y_{2n-1} + \Phi(w,0))} \left[ 2m(\lambda_j w_j + O(|w|^2))\hat{b}(w,0,m) + \frac{\partial \hat{b}(w,0,m)}{\partial \bar{z}_j} \right] \times \frac{\partial S_m u_j}{\partial \bar{z}_j}(x_0)dv_X + O(m^{-\infty}).
\]
Substituting (3.2) to (3.6) and taking the coordinate transformation \( w \to \frac{w}{\sqrt{m}} \), we have

\[
\frac{\partial S_m u_j}{\partial z_j}(x_0) = \frac{k}{2\pi} \int_{|w| \leq \log m} e^{-\lambda |w|^2} 2\lambda |w_j|^2 \tilde{b}_0(0, 0) dw(w) = c_j \neq 0,
\]

and hence

\[
\lim_{m \to \infty} \frac{\partial S_m u_j}{\partial z_j}(x_0) = \frac{k}{2\pi} \int_{\mathbb{C}^{n-1}} e^{-\lambda |w|^2} 2\lambda |w_j|^2 \tilde{b}_0(0, 0) dw(w) = c_j \neq 0,
\]

where \( \lambda |w|^2 = \sum_{j=1}^{n-1} \lambda_j |w_j|^2 \) and \( c_j \) is a non-zero real number.

For \( j \neq k \), we can repeat the procedure above and get

\[
\frac{\partial S_m u_j}{\partial z_k}(x_0) = \frac{k}{2\pi} \int_{|w| \leq \log m} e^{-\lambda |w|^2} 2\lambda k |w_k|^2 \tilde{b}_0(0, 0) dw(w) = 0.
\]

Letting \( m \to \infty \), we get

\[
\lim_{m \to \infty} \frac{\partial S_m u_j}{\partial z_k}(x_0) = \frac{k}{2\pi} \int_{\mathbb{C}^{n-1}} e^{-\lambda |w|^2} 2\lambda k |w_k|^2 \tilde{b}_0(0, 0) dw(w) = 0.
\]

Similarly, one computes

\[
\frac{\partial S_m u_j}{\partial z_k}(x_0) = \frac{k}{2\pi} \int_{|w| \leq \log m} e^{-\lambda |w|^2} 2\lambda k |w_k|^2 \tilde{b}_0(0, 0) dw(w) = 0.
\]

Letting \( m \to \infty \), we have

\[
\lim_{m \to \infty} \frac{\partial S_m u_j}{\partial z_k}(x_0) = \frac{k}{2\pi} \int_{\mathbb{C}^{n-1}} e^{-\lambda |w|^2} 2\lambda k |w_k|^2 \tilde{b}_0(0, 0) dw(w) = 0.
\]

Given \( j = n \), choose \( \chi_3(y_{2n-1}) \in C^\infty_0(-\delta_1, \delta_1) \) satisfying \( \int_{-\delta_1}^{\delta_1} y_{2n-1} \chi_3(y_{2n-1}) = 1 \). Set

\[
u_n = m y_{2n-1} \chi_3(m y_{2n-1}) e^{im y_{2n-1} \chi_3 \left( \frac{\sqrt{m \log m}}{\log m} \right) e^{i m \text{Re}(w, 0)}}.
\]
Then by (3.3) and the same argument as in (2.28) we have

\[
\frac{\partial S_m u_n}{\partial x_{2n-1}}(x_0) = \frac{k}{2\pi} \int_D \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \frac{\partial S_m}{\partial x_{2n-1}}(y, e^{i\theta} \circ x_0) e^{i\theta y} \, d\theta \, u_n(y) \, dv_X + O(m^{-\infty}).
\]

By a direct calculation, we have

\[
\frac{\partial S_m}{\partial x_{2n-1}}(y, 0, \theta) = e^{i\text{Im}(y_{2n-1} - \theta + \Phi(w,0))} \hat{b}(w, 0, m) \left[ -i\text{Im}(\theta + \frac{\partial \eta(\theta)}{\partial \theta} \right].
\]

Substituting (3.14) to (3.13) and using the fact that \( \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \frac{\partial \eta(\theta)}{\partial \theta} d\theta = 0 \), we find

\[
\frac{k}{2\pi} \int_D \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \frac{\partial S_m}{\partial x_{2n-1}}(y, e^{i\theta} \circ x_0) e^{i\theta y} \, d\theta \, u_n(y) \, dv_X = -i k \int_{|w| \leq \log m} m^{-(n-1)} \hat{b}(\frac{w}{\sqrt{m}}, 0, m) e^{-m \text{Im}(\frac{w}{\sqrt{m}})} \lambda(w) \, dv(w).
\]

Substituting (3.15) to (3.13) and letting \( m \to \infty \), we have

\[
\lim_{m \to \infty} \frac{\partial S_m u_n}{\partial x_{2n-1}}(x_0) = -i k \int_{C_n} e^{-\lambda |w|^2} \, dv(w) = ic_n \neq 0,
\]

where \( c_n \) is a nonzero real number.

On the other hand, for \( j = 1, \cdots, n-1 \) by a similar calculation we have

\[
\frac{\partial S_m u_n}{\partial x_j}(x_0) = \frac{k}{2\pi} \int_{|w| \leq \log m} e^{-m \text{Im}(\frac{w}{\sqrt{m}})} [2(\lambda \frac{w_j}{\sqrt{m}} + \frac{1}{m} O(|w|^2)) \hat{b}(\frac{w}{\sqrt{m}}, 0, m) \]
\[
+ \frac{1}{m} \frac{\partial \hat{b}}{\partial x_j}(\frac{w}{\sqrt{m}}, 0, m)] \chi(\frac{w}{\log m}) \lambda(\frac{w}{\sqrt{m}}) m^{-(n-1)} \, dv(w).
\]

By (3.17) we get

\[
\left| \frac{\partial S_m u_n}{\partial x_j}(x_0) \right| \leq C \frac{1}{\sqrt{m}},
\]

where \( C \) is a constant which does not depend on \( x_0 \) and \( m \). Similarly, we find

\[
\left| \frac{\partial S_m u_n}{\partial x_j}(x_0) \right| \leq C \frac{1}{\sqrt{m}}.
\]

Set \( v_j = \alpha_{2j-1} + i \alpha_{2j}, j = 1, \cdots, n \). Then combining the above arguments there are positive constants \( c, C \) independent of \( x_0 \) and \( m \) and a sequence \( \{\varepsilon_m\} \) which does not depend on \( x_0 \in X \) with \( \varepsilon_m \to 0 \) as \( m \to \infty \) such that the following estimates hold

\[
\left| \frac{\partial \alpha_j}{\partial x_j}(x_0) \right| \geq c, \ \left| \frac{\partial \alpha_{2n}}{\partial x_{2n}}(x_0) \right| \geq c, j = 1, \cdots, 2n - 2,
\]

\[
\left| \frac{\partial \alpha_j}{\partial x_j}(x_0) \right| \leq \varepsilon_m, j \neq k, j, k = 1, \cdots, 2n - 2,
\]

\[
\left| \frac{\partial \alpha_{2n}}{\partial x_j}(x_0) \right| \leq C \frac{1}{\sqrt{m}}, j = 1, \cdots, 2n - 2.
\]
From (3.20) the real Jacobian matrix of \( \Phi_m \) is non-degenerate at any \( x_0 \in X \) when \( m \) is large enough which implies that \( \Phi_m \) is an immersion. Thus, we get the conclusion of the lemma. \( \square \)

3.2. Analysis near the complement of \( X_{\text{reg}} \). In order to get the global embedding of CR manifolds by CR functions which lie in the positive Fourier components we need the following

**Proposition 3.2.** Fix \( x_0 \in X \setminus X_{\text{reg}} \). Without loss of generality, we assume \( x_0 \in X_{k_0} \) for some \( k_0 > 1 \). We have

1. There exist a positive integer \( m_0 \) and a neighborhood \( U(x_0) \) of \( x_0 \) such that \( \Phi_{m_0} : U(x_0) \to \mathbb{C}^{d_{k_0} + d_{k_0} + 1} \) is an embedding and \( S_{k_0m_0}(x, x_0) \neq 0 \), \( S_{k_0m_0}(x, x_0) \neq 0 \), for all \( x \in U(x_0) \).

2. There exist positive constants \( \varepsilon_0, \delta_0 \) and a neighborhood \( V(x_0) \) of \( x_0 \) with \( V(x_0) \subseteq U(x_0) \) such that

\[
ed^\theta \circ V(x_0) \subset U(x_0), \forall \theta \in I(x_0, \varepsilon_0),
\]

and

\[
-1 \leq \cos k_0 \theta \leq 1 - \delta_0, \forall \theta \notin I(x_0, \varepsilon_0), 0 \leq \theta < 2\pi
\]

holds, where \( I(x_0, \varepsilon_0) \) is given by

\[
I(x_0, \varepsilon_0) = \{ \theta : 0 \leq \theta < \varepsilon_0 \} \cup \{ \theta : |\theta - \frac{2\pi}{k_0}| < \varepsilon_0 \} \cup \{ \theta : |\theta - \frac{4\pi}{k_0}| < \varepsilon_0 \} \cup \cdots
\]

\[
\cup \{ \theta : |\theta - \frac{2(k_0 - 1)\pi}{k_0}| < \varepsilon_0 \} \cup \{ \theta : 2\pi - \varepsilon_0 < \theta < 2\pi \}.
\]

3. Fix \( 0 < \sigma < \frac{\delta_0}{100} \), where \( \delta_0 > 0 \) is as in (2). There exist a positive integer \( m_1 \) and a neighborhood \( W(x_0) \) of \( x_0 \) with \( W(x_0) \subseteq V(x_0) \) such that \( S_{k_0m_1}(x, x_0) \neq 0 \) for all \( x \in W(x_0) \) and the real part of \( \frac{S_{k_0m_1}(x, x_0)}{S_{k_0m_1}(x, x_0)} \) denoted by \( \mathcal{R}_{k_0m_1}(x) \) satisfies

\[
|1 - \mathcal{R}_{k_0m_1}(x)| < \sigma, \forall x \in W(x_0).
\]

The imaginary part of \( \frac{S_{k_0m_1}(x, x_0)}{S_{k_0m_1}(x, x_0)} \) denoted by \( \mathcal{I}_{k_0m_1}(x) \) satisfies the following inequality

\[
|\mathcal{I}_{k_0m_1}(x)| < \frac{\sigma}{8}, \forall x \in W(x_0).
\]

4. For any positive constant \( c > 0 \), there exist a positive integer \( m_2 \) and a neighborhood \( W(x_0) \subseteq W(x_0) \) of \( x_0 \) such that

\[
|S_{k_0m_2}(x, x_0)| > \frac{c}{2}, \forall x \in W(x_0)
\]

and

\[
|S_{k_0m_2}(y, x_0)| < \frac{c}{8}, \forall y \notin \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W(x_0)
\]

hold.
Proof. Fix \( x_0 \in X_{k_0} \) and let \( D \) be the canonical local patch given in Theorem 1.12. From (2.32), we have for any \( D' \in D \) and \( N \in \mathbb{N} \), there exists a constant \( C_{D', N} \) such that

\[
|S_{k_0 m}(x, x_0)| \geq \frac{k_0}{2\pi} |\hat{b}(z, 0, k_0 m)| e^{-k_0 \text{Im}(\Phi(z, 0)) - C_{D', N} m^{-N}} \quad m \gg 1
\]

holds. Given \( x = (z, \theta) \) with \( |z| \leq \frac{1}{m}, |\theta| \leq \frac{1}{m} \), one has \( |S_{k_0 m}(x, x_0)| > 0 \) when \( m \gg 1 \). Thus, there is a \( \lambda_0 > 0 \) such that for all \( m \geq \lambda_0 \) we have \( |S_{k_0 m}(x, x_0)| > 0 \) for all \( x \in U_m(x_0) \), where \( U_m(x_0) = \{(z, \theta) : |z| < \frac{1}{m}, |\theta| < \frac{1}{m}\} \). Moreover, from the proof of Lemma 3.1, we see that there is a \( \lambda_1 > 0 \) such that for all \( m \geq \lambda_1 \), there is a small neighborhood \( U_m(x_0) \) of \( x_0 \) such that \( \Phi_m : \tilde{U}_m(x_0) \to C^{d_{k_0 m} + \delta_{k_0 m} + 1} \) is an embedding. Taking \( m_0 \geq \lambda_0 + \lambda_1 \) and setting \( U(x_0) = U_{m_0}(x_0) \cap U_{m_0 + 1}(x_0) \cap \tilde{U}_{m_0}(x_0) \), we get (1).

Since \( x_0 \in X_{k_0} \), we have \( e^{i \frac{\pi}{m_0}} \circ x_0 = x_0 \) for \( 0 \leq j \leq k_0, j \in \mathbb{Z} \). Then for any \( \varepsilon_0 \) we define \( I(x_0, \varepsilon_0) \) as in (3.21). When \( \varepsilon_0 \) is sufficiently small there exists a small neighborhood of \( x_0 \) denoted by \( V(x_0) \subseteq U(x_0) \) such that \( e^{i \theta} \circ V(x_0) \subseteq U(x_0) \) for \( \theta \in I(x_0, \varepsilon_0) \). For \( \theta \notin I(x_0, \varepsilon_0) \), \( 0 \leq \theta < 2\pi \), we have \( |k_0 \theta - 2 \pi j| \geq \varepsilon_0 k_0 \) for every \( j = 0, 1, \ldots, k_0 \) which implies that there exists a constant \( \delta_0 \) depending on \( \varepsilon_0 \) such that \( -1 \leq \cos k_0 \theta \leq 1 - \delta_0 \) for \( \theta \notin I(x_0, \varepsilon_0) \). Thus we get the conclusion of (2) in this proposition.

From the proof of (1), there is an \( \tilde{m}_1 > 0 \) such that for every \( m \geq \tilde{m}_1 \), there is a neighborhood \( W_m(x_0) \) of \( x_0 \) such that \( S_{k_0 m}(x, x_0) \neq 0 \) and \( S_{k_0 (m+1)}(x, x_0) \neq 0 \). We assume that \( m \geq \tilde{m}_1 \) and \( x \in W_m(x_0) \). By (2.32), we have

\[
S_{k_0 m}(x, x_0) \equiv \frac{k_0}{2\pi} e^{i k_0 m (x_{2n-1} + \Phi(z, 0))} \hat{b}(z, 0, m) \quad \text{mod} \ O(m^{-\infty}),
\]

(3.23)

\[
S_{k_0 (m+1)}(x, x_0) \equiv \frac{k_0}{2\pi} e^{i k_0 (m+1) (x_{2n-1} + \Phi(z, 0))} \hat{b}(z, 0, m+1) \quad \text{mod} \ O(m^{-\infty}),
\]

\[
\hat{b}(z, 0, m) \sim \sum_{j=0}^{\infty} \hat{b}_j(z, 0) m^{n-1-j} \quad \text{in} \ S_{\text{hc}}^{n-1}(1; D).
\]

Write

\[
\frac{S_{k_0 (m+1)}(x, x_0)}{S_{k_0 m}(x, x_0)} = R_{k_0 m}(x) + i I_{k_0 m}(x).
\]

Since \( \hat{b}(0, 0, 0) \neq 0 \) (see Theorem 2.6), we have \( \hat{b}(0, 0, m) \neq 0 \) for \( m \geq 0 \) and this implies that \( \hat{b}(z, 0, m) \neq 0 \) when \( |z| \) is sufficiently small. We assume that \( \hat{b}(z, 0, m) \neq 0 \) for every \( m \geq \tilde{m}_1 \) and every \( (z, 0) \in W_m(x_0) \). Set

\[
a_m(x) = \frac{k_0}{2\pi} e^{i k_0 m (x_{2n-1} + \Phi(z, 0))} \hat{b}(z, 0, m), b_m(x) = S_{k_0 m}(x, x_0) - a_m(x).
\]

From (3.23), for any \( D' \subseteq V(x_0) \subseteq D \) and any \( N \in \mathbb{N} \) there exists a positive constant \( C_{D', N} \) such that

\[
\sup_{x \in D'} |S_{k_0 m}(x) - a_m(x)| \leq C_{D', N} m^{-N}, \quad m \gg 1,
\]

holds. For any \( m \geq \tilde{m}_1 \), define \( V_m(x_0) = \{(z, \theta) \in D : |z| < \frac{1}{m}, |\theta| < \frac{1}{m}\} \cap W_m(x_0) \), then \( V_m(x_0) \subseteq D' \) when \( m \) is sufficiently large. Then on \( V_m(x_0) \),
we have
\begin{equation}
|b_{m+1}(x)| \leq C_{D,N} \frac{1}{(m+1)^{N}}, \quad |b_{m}(x)| \leq C_{D,N} \frac{1}{m^{N}}.
\end{equation}

On the other hand, we have \(|a_{m}(x)| = \frac{k_0}{2\pi} e^{-k_0 m \Im \Phi(z,0)} \hat{b}(z,0,m)\). From \ref{2.3}, by a direct calculation we get \(\Im \Phi(z,0) = \lambda |z|^2 + O(|z|^3)\). So we assume \(D'\) to be sufficient small such that on \(D'\) we have
\[c_1 |z|^2 \leq \Im \Phi(z,0) \leq c_2 |z|^2\]
for some constants \(c_1, c_2\). Then
\begin{equation}
|a_{m}(x)| \geq \hat{c} m^{n-1}, \forall x \in V_m(x_0), \quad \frac{a_{m+1}(x)}{a_{m}(x)} \approx 1, \forall x \in V_m(x_0),
\end{equation}
holds for some positive constant \(\hat{c}\) when \(m\) is sufficiently large. Since
\[
\frac{S_{k_0(m+1)}(x, x_0)}{S_{k_0 m}(x, x_0)} = \frac{b_{m+1} + a_{m+1}}{b_m + a_m} = \frac{\frac{b_{m+1}}{a_{m+1}} + \frac{a_{m+1}}{a_m}}{\frac{b_m}{a_m} + 1},
\]
\ref{3.24} and \ref{3.25} imply
\[
\frac{S_{k_0(m+1)}(x, x_0)}{S_{k_0 m}(x, x_0)} \approx 1, \forall x \in V_m(x_0)
\]
for \(m >> 1\). Then for any fixed \(0 < \sigma < \frac{k_0}{100}\), we can choose \(m_1\) sufficiently large such that \(W(x_0) = \{(z, \theta) : |z| < \frac{1}{m_1}, |\theta| < \frac{1}{m_1}\}\) satisfies \(W(x_0) \subset V(x_0)\) and on \(W(x_0)\) we have
\begin{equation}
|1 - R_{k_0 m_1}(x)| < \sigma, |I_{k_0 m_1}(x)| < \frac{\sigma}{8}.
\end{equation}
Thus, we get the conclusion of (3) in the proposition.

Choose a neighborhood \(W_1(x_0)\) of \(x_0\) such that \(W_1(x_0) \subset W(x_0)\) holds. Following the same arguments as in the proof of Lemma \ref{2.3}, we have
\begin{equation}
S_{k_0 m}(x_0, y) \equiv 0 \mod O(m^{-\infty}), \forall y \notin \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W_1(x_0).
\end{equation}
Since \(X \setminus \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W(x_0) \subset X \setminus \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W_1(x_0)\), \ref{3.27} implies that for any \(N > 0\) there exists a constant \(C_N\) satisfying
\[
|S_{k_0 m}(x_0, y)| \leq C_N m^{-N} \forall m >> 1, \forall y \in X \setminus \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W(x_0).
\]
Thus, for any \(c > 0\), there exists \(n_0\) such that for any \(m > n_0\) we have \(|S_{k_0 m}(x_0, y)| < \frac{c}{8}\) for all \(y \notin \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W(x_0)\). Then following the same arguments as in the proof of (1) in the proposition, there exists a positive integer \(m_2\) and a neighborhood \(W(x_0) \subset W_1(x_0) \subset W(x_0)\) such that \(|S_{k_0 m_2}(x, x_0)| > \frac{c}{2}\) holds for all \(x \in W(x_0)\) and moreover \(|S_{k_0 m_2}(x_0, y)| < \frac{c}{8}\) holds for all \(y \notin X \setminus \bigcup_{0 \leq \theta < 2\pi} e^{i\theta} \circ W(x_0)\). Thus, we get the conclusion of (4) in this proposition. \qed
3.3. **Embedding of CR manifold by positive Fourier components.** Now, we are going to establish the global embedding of the CR manifolds, which have a locally free transversal CR $S^1$-action, by positive Fourier components.

Since $X \setminus X_{\text{reg}} \subset X$, there exist finite $W(x_i) \in W(x_i) \subset V(x_i) \subset U(x_i)$ and positive constants $m_0(x_i), m_1(x_i), m_2(x_i)$ with respect to the points $x_i, 0 \leq i \leq n_0$ satisfying the properties in Proposition 3.2 and moreover $X \setminus X_{\text{reg}} = \bigcup_{i=1}^{n_0} W(x_i)$. Without loss of generality, we assume that $x_i \in X_{\text{reg}}, 0 \leq i \leq n_0$. For every $i = 0, 1, \ldots, n_0$, set

$$H_{x_i} = \bigoplus_{j=0}^{2} \left( H_{b,k,m_j(x_i)}^0(X) \bigoplus H_{b,k,(m_j(x_i)+1)}^0(X) \right),$$

$$H_m = \bigoplus_{k=1}^{i} \left( H_{b,k,m}^0(X) \bigoplus H_{b,k,(m+1)}^0(X) \right) \bigoplus_{i=0}^{n_0} H_{x_i}.$$

Recall that $X = X_1 \cup X_2 \cup \cdots \cup X_l, X_{\text{reg}} = X_1 \neq \emptyset$. Put $N_m = \dim H_m$ and let $\{f_j\}_{j=1}^{N_m}$ be an orthonormal basis of $H_m$ with respect to its decomposition. Define a map

$$\Phi_m : X \to \mathbb{C}^{N_m}, x \mapsto (f_1(x), \ldots, f_{N_m}(x)).$$

We will prove the following

**Theorem 3.3.** Let $X$ be a compact connected strongly pseudoconvex CR manifold with a locally free transversal CR $S^1$-action. Then $\Phi_m$ is an embedding when $m$ is large.

**Proof.** By Lemma 3.1, we know that $\Phi_m$ is an immersion when $m$ is large. Now we show that $\Phi_m$ is injective when $m$ is large by seeking a contradiction. We assume that there exist two sequences $\{\hat{y}_m\}, \{\hat{z}_m\} \subset X, \hat{y}_m \neq \hat{z}_m$ such that $\Phi_m(\hat{y}_m) = \Phi_m(\hat{z}_m)$. Since $X$ is compact, there exist subsequences of $\{\hat{y}_m\}, \{\hat{z}_m\}$ which are also denoted by $\{\hat{y}_m\}, \{\hat{z}_m\}$ such that $\hat{y}_m \to \hat{y}, \hat{z}_m \to \hat{z}$ for $m \to \infty$.

First we assume that $\hat{y}, \hat{z} \in X \setminus X_{\text{reg}}$.

Case I: \(\hat{y} = e^{i\theta_0} \circ \hat{z}, \hat{z} \in X_k\) for some $k > 1$ and $\hat{z} \in U(x_i)$ for some $i$. By assumption of $\hat{y}_m, \hat{z}_m$ we have that

$$S_{k,m_0(x_i)}(\hat{y}, x_i) = S_{k,m_0(x_i)}(\hat{z}, x_i),$$

$$S_{k,(m_0(x_i)+1)}(\hat{y}, x_i) = S_{k,(m_0(x_i)+1)}(\hat{z}, x_i).$$

In the following context, we will omit $x_i$ in $m_j(x_i), j = 0, 1, 2$ for brevity if it makes no confusion. Then (3.28) implies

$$e^{ik_0m_0\theta_0} S_{k,m_0}(\hat{z}, x_i) = S_{k,m_0}(\hat{z}, x_i),$$

$$e^{ik_0(m_0+1)\theta_0} S_{k,(m_0+1)}(\hat{z}, x_i) = S_{k,(m_0+1)}(\hat{z}, x_i).$$

By (1) in Proposition 3.2 we have $e^{ik_0\theta_0} = 1$. Then $\theta_0 = \frac{2\pi}{k_0}m$ holds for some $m \in \mathbb{Z}$. The rigid Hermitian metric on $X$ implies that $e^{i\theta} : X \to X$ is an isometric map for each $\theta$. Thus, we have

$$\text{dist}(\hat{y}, x_i) = \text{dist}(e^{i\frac{2\pi}{k_0}m} \circ \hat{z}, x_i) = \text{dist}(e^{i\frac{2\pi}{k_0}m} \circ \hat{z}, e^{i\frac{2\pi}{k_0}m} \circ x_i) = \text{dist}(\hat{z}, x_i).$$

This implies $\hat{y} \in U(x_i)$ if the $U(x_i)$ we chose is a geodesic ball centered at $x_i$. This is a contradiction since $\Phi_m$ is an embedding on $U(x_i)$. 

\[\]
From (3.31) we have

\[ S_{k,m_1}(\hat{y}, x_i) = S_{k,m_1}(\hat{\theta}, x_i), \quad S_{k,(m_1+1)}(\hat{\theta}, x_i) = S_{k,(m_1+1)}(\hat{\theta}, x_i), \]

we find

\[ \frac{S_{k,(m_1+1)}(\hat{\theta}, x_i)}{S_{k,m_1}(\hat{\theta}, x_i)} = e^{i\delta s} \frac{S_{k,(m_1+1)}(\hat{\theta}, x_i)}{S_{k,m_1}(\hat{\theta}, x_i)}. \]

From (3.30) we have

\[ \mathcal{R}_{k,m_1}(\hat{\theta}) + i \mathcal{I}_{k,m_1}(\hat{\theta}) = (\cos k_i \hat{\theta} + i \sin k_i \hat{\theta})(\mathcal{R}_{k,m_1}(\hat{\theta}) + i \mathcal{I}_{k,m_1}(\hat{\theta})) \]

which leads to

\[ \mathcal{R}_{k,m_1}(\hat{\theta}) = \mathcal{R}_{k,m_1}(\hat{\theta}) \cos k_i \hat{\theta} - \mathcal{I}_{k,m_1}(\hat{\theta}) \sin k_i \hat{\theta}. \]

Then we get

\[ 1 - \mathcal{R}_{k,m_1}(\hat{\theta}) = 1 + (1 - \mathcal{R}_{k,m_1}(\hat{\theta})) \cos k_i \hat{\theta} - \mathcal{I}_{k,m_1}(\hat{\theta}) \sin k_i \hat{\theta}. \]

From (3.31) we have

\[ |1 - \mathcal{R}_{k,m_1}(\hat{\theta})| \geq 1 - \cos k_i \hat{\theta} - |1 - \mathcal{R}_{k,m_1}(\hat{\theta})| - |\mathcal{I}_{k,m_1}(\hat{\theta})|. \]

By (3) in Proposition 3.2 we have

\[ \sigma \geq |1 - \mathcal{R}_{k,m_1}(\hat{\theta})| \geq 1 - (1 - \delta_0) - \sigma - \frac{\sigma}{8}, \]

that is

\[ (2 + \frac{1}{8})\sigma \geq \delta_0. \]

This is a contradiction with \( 0 < \sigma < \frac{\delta_0}{100} \). Thus we get the conclusion of the claim.

From the above claim and by (4) in Proposition 3.2, we have

\[ |S_{k,m_2}(\hat{\theta}, x_i)| > \frac{c}{2} |S_{k,m_2}(\hat{\theta}, x_i)| < \frac{c}{8}. \]

This is a contradiction with

\[ S_{k,m_2}(\hat{\theta}, x_i) = S_{k,m_2}(\hat{\theta}, x_i). \]

Next, we assume \( \hat{\theta}, \hat{\theta} \in X_{\text{reg}} \).

Case III: \( \hat{\theta}, \hat{\theta} \in X_{\text{reg}} \) and \( \hat{\theta} = e^{i\theta} \circ \hat{\theta} \) for some \( \hat{\theta} \in [0, 2\pi) \). Choose canonical coordinates \((z, \theta, \varphi)\) centered at some \( \hat{\theta}_0 \in X \) and defined on a canonical local patch

\[ D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \pi\} \]

such that in terms of the canonical coordinates,

\[ \hat{z} = (0, \theta_1), \quad \hat{y} = (0, \theta_2). \]

Then we have \( \theta_2 - \theta_1 = \hat{\theta}. \) Let \( \{f_j\}_{j=1}^{d_n} \) and \( \{g_j\}_{j=1}^{d_m+1} \) be orthonormal bases of \( H_{b,m}^0(X) \) and \( H_{b,m+1}^0(X) \), respectively. Then by the assumptions on \( \hat{y}_m \) and \( \hat{z}_m \) we have

\[ S_m(\hat{z}_m, \hat{y}_m) = S_m(\hat{z}_m, \hat{y}_m), \quad S_{m+1}(\hat{z}_m, \hat{y}_m) = S_{m+1}(\hat{z}_m, \hat{z}_m). \]
Without loss of generality, we assume $\hat{z}_m, \hat{y}_m \in D$ for each $m$. Then in terms of canonical local coordinates, we write $\hat{z}_m = (z_m, \theta_m)$ and $\hat{y}_m = (w_m, \eta_m)$. By Theorem 2.6 we have

$$S_m(z_m, y_m) = \frac{1}{2\pi} e^{i\theta_m - \eta_m + \Phi(z_m, w_m)} \hat{b}(z_m, w_m, m) + O(m^{-\infty}),$$

$$S_{m+1}(\hat{z}_m, \hat{y}_m) = \frac{1}{2\pi} e^{i(m+1)\theta_m - \eta_m + \Phi(z_m, w_m)} \hat{b}(z_m, w_m, m + 1) + O(m^{-\infty}),$$

$$S_m(\hat{z}_m, \hat{z}_m) = \frac{1}{2\pi} \hat{b}(z_m, z_m, m) + O(m^{-\infty}),$$

$$S_{m+1}(\hat{z}_m, \hat{z}_m) = \frac{1}{2\pi} \hat{b}(z_m, z_m, m + 1) + O((m + 1)^{-\infty}).$$

(3.32)

We assume \( \lim_{m \to \infty} m \Im \Phi(z_m, w_m) = M \) (\( M \) can be infinite).

(a): Assume \( \lim_{m \to \infty} m \Im \Phi(z_m, w_m) = M \in (0, \infty) \).

From \( S_m(\hat{z}_m, \hat{y}_m) = S_m(\hat{z}_m, \hat{z}_m) \) and (3.32) we have

$$e^{i\theta_m - \eta_m + \Phi(z_m, w_m)} \hat{b}(z_m, w_m, m) = \hat{b}(z_m, z_m, m) + O(m^{-\infty}).$$

Then we get

$$m^{-(n-1)} |\hat{b}(z_m, w_m, m)| e^{-m \Im \Phi(z_m, w_m)} = m^{-(n-1)} |\hat{b}(z_m, z_m, m) + O(m^{-\infty})|.$$ 

Letting \( m \to \infty \), we have

$$\hat{b}(0, 0) = e^{-M} \hat{b}(0, 0),$$

that is \( \hat{b}(0, 0) = 0 \). Thus, we get a contradiction.

(b): Assume

$$\lim_{m \to \infty} m \Im \Phi(z_m, w_m) = 0.$$ 

From \( S_{m+1}(\hat{z}_m, \hat{y}_m) - S_m(\hat{z}_m, y_m) = S_{m+1}(\hat{z}_m, \hat{z}_m) - S_m(\hat{z}_m, \hat{z}_m) \) combined with (3.32) we have

$$m^{-(n-1)} \left| e^{i\theta_m - \eta_m + \Phi(z_m, w_m)} \left[ e^{i\theta_m - \eta_m + \Phi(z_m, w_m)} \hat{b}(z_m, w_m, m + 1) - \hat{b}(z_m, w_m, m) \right] \right| = m^{-(n-1)} \left| \hat{b}(z_m, z_m, m + 1) - \hat{b}(z_m, z_m, m) \right| + O(m^{-\infty}).$$

Letting \( m \to \infty \) and using (3.33), we have

$$|e^{i\theta} \hat{b}(0, 0) - \hat{b}(0, 0)| = 0.$$ 

Hence, \( \theta = 0 \) and \( \hat{z} = \hat{y} \). Put

$$f_m(t) = \frac{|S_m(t \hat{z}_m + (1-t) \hat{y}_m, \hat{y}_m)|^2}{S_m(t \hat{z}_m + (1-t) \hat{y}_m, t \hat{z}_m + (1-t) \hat{y}_m) S_m(\hat{y}_m, \hat{y}_m)}.$$ 

We have

$$f_m(0) = \frac{S_m(\hat{y}_m, \hat{y}_m)^2}{S_m(\hat{y}_m, \hat{y}_m)^2} = 1,$$

(3.34)

$$f_m(1) = \frac{|S_m(\hat{z}_m, \hat{y}_m)|^2}{S_m(\hat{z}_m, \hat{z}_m) S_m(\hat{y}_m, \hat{y}_m)} = \frac{S_m(\hat{y}_m, \hat{y}_m)^2}{S_m(\hat{y}_m, \hat{y}_m) S_m(\hat{y}_m, \hat{y}_m)} = 1.$$
By the Schwartz inequality, we get $0 \leq f_m(t) \leq 1$. Then (3.34) implies that there is a $t_m \in (0, 1)$ such that $f'_m(t_m) = 0$ and $f''_m(t_m) \geq 0$ holds. Hence, we get

$$\lim_{m \to \infty} \inf_{t \in [0, 1]} \frac{f'_m(t_m)}{|z_m - w_m|^2} \geq 0.$$  

Then, making use of the same arguments as in [19]((4.22) in Theorem 4.7), we have that (3.35) is impossible under the assumption (3.33).

Case IV: $\hat{z}, \hat{y} \in X_{\text{reg}}$, $\hat{y} \neq e^{i \theta} \hat{z}$ for any $\theta \in [0, 2\pi)$. Choose a canonical local patch $D(\hat{z})$ with canonical coordinates $(z, \theta, \varphi)$ centered at $\hat{z}$. Since $\hat{z} \in X_{\text{reg}}$, we can apply Lemma [1.13] and have that $D(\hat{z})$ can be chosen such that $D(\hat{z}) = \{(z, \theta) : |z| < \varepsilon, |\theta| < \pi\}$ holds in terms of canonical coordinates. Let $\varepsilon$ be sufficiently small such that $D(\hat{z})$ is $S^1$-invariant. Since $\hat{y} \neq e^{i \theta} \hat{z}$ for all $\theta \in [0, 2\pi)$, for $\varepsilon$ small enough we can choose a canonical local patch $D(\hat{y})$ such that $D(\hat{y}) \cap D(\hat{z}) = \emptyset$ holds.

Choose two functions $\chi, \chi_1 \in C_0^\infty(X)$ satisfying $\chi = 1$ in a small neighborhood of $D(\hat{z})$ and $\chi_1 = 1$ in a small neighborhood of $\text{supp} \chi$ and $\text{supp} \chi_1 \cap D(\hat{y}) = \emptyset$. Choose $\chi_0(w) \in C_0^\infty(\mathbb{C}^{n-1})$ such that $\text{supp} \chi_0(w) \subset \{w : |w| < 1\}$ and $\int_{\mathbb{C}^{n-1}} \chi_0(w) dv(w) = 1$ hold. Furthermore, choose $\eta_0(y_{2n-1}) \in C_0^\infty(-\pi, \pi)$ with $\int_{-\pi}^{\pi} \eta_0(y_{2n-1}) dy_{2n-1} = 1$. For any $m \in \mathbb{N}$, set

$$u_m(y) = m^{n-1} e^{im(y_{2n-1} - \theta_m - \text{Re}(\hat{z}, w))} \eta_0(y_{2n-1}) \chi_0(m(w - z_m)) \in C_0^\infty(D(\hat{z})).$$

Then we have

$$S_m u_m(\hat{y}_m) = \chi S_m u_m(\hat{y}_m) + (1 - \chi) S_m u_m(\hat{y}_m) = (1 - \chi) S_m u_m(\hat{y}_m)$$

and

$$S_m u_m(\hat{y}_m) = (1 - \chi) S_m u_m(\hat{y}_m) + (1 - \chi) S_1 Q_m u_m(\hat{y}_m) + (1 - \chi) S(1 - \chi_1) Q_m u_m(\hat{y}_m).$$

Since $D(\hat{z})$ is an $S^1$-invariant subset and $\text{supp} u_m \subset D(\hat{z})$, we have $\text{supp} Q_m u_m \subset D(\hat{z})$. This implies

$$S_m u_m(\hat{y}_m) = 0.$$

Then by the same arguments as in the proof of Lemma [2.3], we get

$$S_m u_m(\hat{y}_m) = O(m^{-\infty}).$$

Combining (3.37), (3.38), (3.39) and (3.40), we conclude

$$S_m u_m(\hat{y}_m) = O(m^{-\infty}).$$

On the other hand, we have

$$S_m u_m(\hat{z}_m) = \frac{m^{n-1}}{2\pi} \int_X e^{-m\text{Im}\Phi(z_m, w)} \hat{b}(z_m, w, m) \chi_0(m(w - z_m)) \lambda(w) dv(w) + O(m^{-\infty})$$

$$= \frac{1}{2\pi} \int_{\{w \in \mathbb{C}^{n-1} : |w| < 1\}} e^{-m\text{Im}\Phi(z_m, \frac{w}{m} + z_m)} \hat{b}(z_m, \frac{w}{m} + z_m, m) \chi_0(w) \lambda \left( \frac{w}{m} + z_m \right) m^{-(n-1)} dv(w) + O(m^{-\infty}).$$

Since $\text{Im}\Phi(z_m, \frac{w}{m} + z_m) \geq c_0 |\frac{w}{m}|^2$ for some constant $c_0$, we have $-m\text{Im}\Phi(z_m, \frac{w}{m} + z_m) \to 0$ uniformly on $\{w \in \mathbb{C}^{n-1} : |w| < 1\}$ as $m \to \infty$. Letting $m \to \infty$ we have

$$\lim_{m \to \infty} S_m u_m(\hat{z}_m) = \frac{1}{2\pi} \hat{b}(0, 0) \neq 0.$$
This is a contradiction with the assumption $S_m u_m (\hat{z}_m) = S_m u_m (\hat{y}_m)$.

Case V: $\hat{z} \in X_{reg}, \hat{y} \not\in X_{reg}$. We have that $\hat{y} \neq e^{i\theta} \circ \hat{z}$ for all $\theta \in [0, 2\pi)$. Following the same arguments as in Case IV, we find that this is impossible.

Thus, we get the conclusion of Theorem 3.3. □

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