UNIQUENESS OF THE FOURIER TRANSFORM ON CERTAIN LIE GROUPS

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ABSTRACT. In this article, we prove that if the group Fourier transform of certain integrable functions on the Heisenberg motion group (or step two nilpotent Lie groups) is of finite rank, then the function is identically zero. These results can be thought as an analogue to the Benedicks theorem that dealt with the uniqueness of the Fourier transform of integrable functions on the Euclidean spaces.

1. Introduction

In an interesting article, M. Benedicks [3] had extended the classical Paley-Wiener theorem for compactly supported function to the class of integrable functions. In other words, support of an integrable function and its Fourier transform both cannot be of finite measure simultaneously. Thereafter, a series of analogous results to the Benedicks theorem has been explored in various contexts, including the Heisenberg group and the Euclidean motion groups (see [12, 14, 18]). In article [12], an analogous result on the Heisenberg group has worked out for the partial compactly supported functions in terms of finite rank of Fourier transform of the function. Further, Vemuri [22] has relaxed the compact support condition on the functions by finite Lebesgue measure.

In this article, we explore analogous results to the Amrein-Berthier and Benedicks theorem on the Heisenberg motion group and step two nilpotent Lie groups. We prove that if the group Fourier transform of finitely supported certain integrable functions on the Heisenberg motion group (or step two nilpotent Lie groups) is of finite rank, then the function has to vanish identically. However, it would be a reasonable to consider the case when the spectrum of the Fourier transform of an integrable function will be supported on a thin uncountable set.

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2. Preliminaries on the Heisenberg motion group

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that equipped with the group law

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w}) \right).$$

By Stone-von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. That is, each of $\lambda \in \mathbb{R}^*$ defines a Schrödinger representation $\pi_\lambda$ of $\mathbb{H}^n$ by

$$\pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t} e^{i \lambda (x \xi + \frac{1}{2} x y)} \phi(\xi + y),$$

where $z = x + iy$ and $\phi \in L^2(\mathbb{R}^n)$. Let

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}.$$ 

Then $\{T, X_j, Y_j : j = 1, \ldots, n\}$ forms a basis for the Lie algebra $\mathfrak{h}^n$ consists of all left-invariant vector fields on $\mathbb{H}^n$ and the representation $\pi_\lambda$ induces a representation $\pi_\lambda^*$ of $\mathfrak{h}^n$ on the space of $C^\infty$ vectors in $L^2(\mathbb{R}^n)$ via

$$\pi_\lambda^*(X) f = \frac{d}{dt} \big|_{t=0} \pi_\lambda(\exp tX) f.$$ 

It is easy to see that $\pi_\lambda^*(X_j) = i \lambda x_j$ and $\pi_\lambda^*(Y_j) = \frac{\partial}{\partial x_j}$. Hence for the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$, it follows that $\pi_\lambda^*(\mathcal{L}) = -\Delta_x + \lambda^2 |x|^2 =: H_\lambda$, the scaled Hermite operator. Let $\phi_\alpha^\lambda(x) = |\lambda|^\frac{n}{2} \phi_\alpha(\sqrt{|\lambda|} x), \quad \alpha \in \mathbb{Z}^n_+$, where $\phi_\alpha$ are the Hermite functions on $\mathbb{R}^n$. Then $\phi_\alpha^\lambda$’s are the eigenfunctions of $H_\lambda$ with eigenvalue $(2|\alpha| + n)|\lambda|$. Hence the entry functions $E_{\alpha \beta}^\lambda$’s of the representation $\pi_\lambda$ are eigenfunctions of the sub-Laplacian $\mathcal{L}$ satisfying

$$\mathcal{L} E_{\alpha \beta}^\lambda = (2|\alpha| + n)|\lambda| E_{\alpha \beta}^\lambda,$$

where $E_{\alpha \beta}^\lambda(z, t) = \langle \pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$. Since $E_{\alpha \beta}^\lambda(z, t) = e^{i \lambda t} \langle \pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$, the eigenfunctions $E_{\alpha \beta}^\lambda$’s are not in $L^2(\mathbb{H}^n)$. However, for a fix $t$, they are in $L^2(\mathbb{C}^n)$. Now, define an operator $L_\lambda$ by $\mathcal{L} (e^{i \lambda t} f(z)) = e^{i \lambda t} L_\lambda f(z)$. Then the special Hermite functions

$$\phi_{\alpha \beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}} \langle \pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$$

are eigenfunctions of $L_\lambda$ with eigenvalue $2|\alpha| + n$. We summarize by noting that the special Hermite functions $\phi_{\alpha \beta}^\lambda$’s forms an orthonormal basis for $L^2(\mathbb{C}^n)$ (see [21], Theorem 2.3.1).

Heisenberg motion group $G$ is the group of isometries of $\mathbb{H}^n$ that leaves invariant the sub-Laplacian $\mathcal{L}$. Since the action of the unitary group $K = U(n)$ defines a group of automorphism on $\mathbb{H}^n$ via $k \cdot (z, t) = (kz, t)$, where $k \in K$, we summarize by noting that the special Hermite functions $\phi_{\alpha \beta}^\lambda$’s forms an orthonormal basis for $L^2(\mathbb{C}^n)$ (see [21], Theorem 2.3.1).
the group law on $G$ can be expressed as the semidirect product of $\mathbb{H}^n$ and $K$. Hence the group law on $G$ can be understood by
\[
(k_1, z, t) \cdot (k_2, w, s) = \left( k_1 k_2, z + k_1 w, t + s - \frac{1}{2} \text{Im}(k_1 w \cdot \bar{z}) \right).
\]
Since a right $K$-invariant function on $G$ can be thought as a function on $\mathbb{H}^n$, we infer that the Haar measure on $G$ can be written as $dg = dk dz dt$, where $dk$ and $dz dt$ are the normalized Haar measure on $K$ and $\mathbb{H}^n$ respectively.

For $k \in K$, define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda,k}(z, t) = \pi_{\lambda}(k z, t)$. Since $\pi_{\lambda,k}$ agrees with $\pi_{\lambda}$ on the center of $\mathbb{H}^n$, it follows by the Stone-Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda,k}$ is equivalent to $\pi_{\lambda}$. Hence there exists an intertwining operator $\mu_{\lambda}(k)$ satisfying
\[
(2.1) \quad \pi_{\lambda}(k z, t) = \mu_{\lambda}(k) \pi_{\lambda}(z, t) \mu_{\lambda}(k)^*.
\]
The operator-valued function $\mu_{\lambda}$ can be thought as a unitary representation of the group $K$ on $L^2(\mathbb{R}^n)$ and it is known as metaplectic representation. Since for $\lambda \in \mathbb{R}^*$, the set $\{\phi_{\alpha}^{\lambda} : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$, let $P_m = \{\phi_{\alpha}^\lambda : |\alpha| = m \}$.

Then $\mu_{\lambda} |_{P_m}$ is an irreducible representation of $K$ and the action of $\mu_{\lambda}$ on $L^2(\mathbb{R}^n)$ can be realized by
\[
(2.2) \quad \mu_{\lambda}(k) \phi_{\alpha}^{\lambda} = \sum_{|\gamma| = |\alpha|} \eta_{\alpha \gamma}^{\lambda}(k) \phi_{\gamma}^{\lambda}.
\]

For more details about the metaplectic representations and the spherical functions on $\mathbb{H}^n$, we refer the article by Benson et al. [4]. Let $(\sigma, H_\sigma)$ be an irreducible unitary representation of $K$ and $H_\sigma = \text{span}\{e_{ij}^\sigma : 1 \leq i, j \leq d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \hat{K}$ are defined by
\[
\varphi_{\sigma ij}^\sigma(k) = \langle \sigma(k) e_{ij}^\sigma, e_{ij}^\sigma \rangle.
\]

Define a bilinear form $\phi_{\alpha}^\lambda \otimes e_i^\sigma$ on $L^2(\mathbb{R}^n) \times H_\sigma$ by $\phi_{\alpha}^\lambda \otimes e_i^\sigma = \phi_{\alpha}^\lambda e_i^\sigma$. Then the set $\{\phi_{\alpha}^\lambda \otimes e_i^\sigma : 1 \leq i, \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes H_\sigma$.

Denote $H_\sigma^2 = L^2(\mathbb{R}^n) \otimes H_\sigma$.

For $\lambda \neq 0$, we define a representation $\rho_{\lambda}^\lambda$ of $G$ on the space $H_\sigma^2$ by
\[
\rho_{\lambda}^\lambda(z, t, k) = \pi_{\lambda}(z, t) \mu_{\lambda}(k) \otimes \sigma(k).
\]
In the article [17], it has been shown that $\rho_{\lambda}^\lambda$ are the only irreducible unitary representations of $G$ which appears in the Plancherel formula. Thus, in view of the above argument, we denote the partial dual of the group $G$ by $G' \cong \mathbb{R}^* \times \hat{K}$.

Now, we define the Fourier transform of the function $f \in L^1(G)$ by
\[
\hat{f}(\lambda, \sigma) = \int_K \int_{\mathbb{R}^-} \int_{\mathbb{C}^*} f(z, t, k) \rho_{\lambda}^\lambda(z, t, k) dz dt dk.
\]
Let \( f^\lambda \) be the inverse Fourier transform of the function \( f \) in \( t \) variable. Then

\[
f^\lambda(z, k) = \int_{\mathbb{R}} f(z, t, k) e^{i\lambda t} dt.
\]

Thus,

\[
\hat{f}(\lambda, \sigma) = \int_{K} \int_{\mathbb{C}^n} f^\lambda(z, k) \rho^\lambda_\sigma(z, k) dz dk,
\]

where \( \rho^\lambda_\sigma(z, k) = \rho^\lambda_\sigma(z, 0, k) \). For \( f \in L^1 \cap L^2(G) \), the following Plancherel formula derived in \cite{17}.

\[
\int_{K} \int_{\mathbb{H}^n} f(z, t, k) dz dt dk = (2\pi)^{-n} \sum_{\sigma \in \hat{K}} \int_{\mathbb{R}\setminus\{0\}} |\hat{f}(\lambda, \sigma)|^2 \| \lambda \|^2 d\lambda.
\]

Further, the set \( \{ \phi^\lambda_\alpha \otimes e^\sigma_i : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma \} \) forms an orthonormal basis for \( \mathcal{H}_\sigma^2 \), we can write

\[
\hat{f}(\lambda, \sigma)(\phi^\lambda_\alpha \otimes e^\sigma_i) = \sum_{|\alpha|=|\gamma|} \int_{K} \eta^\lambda_\alpha_\gamma(k) \int_{\mathbb{C}^n} f^\lambda(z, k) (\pi_\lambda(z) \phi^\lambda_\alpha \otimes \sigma(k) e^\sigma_i) dz dk.
\]

### 3. Uniqueness results on the Heisenberg motion group

In this section, we work out some of the results pertaining to the uniqueness of the Fourier transform on the Heisenberg motion group \( G = \mathbb{H}^n \ltimes K \). Those results can be thought as an analogue to the Benedicks theorem.

**Weyl transform.** For proving the main result of this section, we need to derive some of the properties of the Weyl type transform on \( G^\times = \mathbb{C}^n \times K \). For more details on the Wely transform on the Heisenberg group, see \cite{20}.

For \( (\lambda, \sigma) \in G^\times \), we define the Weyl transform \( W^\lambda_\sigma \) on \( L^1(G^\times) \) by

\[
W^\lambda_\sigma(F) = \int_{K} \int_{\mathbb{C}^n} F(z, k) \rho^\lambda_\sigma(z, k) dz dk.
\]

Now, we define the \( \lambda \)-twisted convolutions of \( F, H \in L^1 \cap L^2(G^\times) \) by

\[
F \times_\lambda H(g) = \int_{G^\times} F(g g'^{-1}) H(g') e^{-\frac{i}{2} \text{Im}(kw \cdot z)} dg',
\]

where \( g = (z, k) \) and \( g' = (w, s) \). For \( \lambda = 1 \), we simply call the \( \lambda \)-twisted convolutions as twisted convolutions and denote it by \( F \times H \). We derive the following properties of the Weyl transform \( W^\lambda_\sigma \).

**Proposition 3.1.** If \( F, H \in L^1 \cap L^2(G^\times) \), then

(i) \( W^\lambda_\sigma(F^*) = W^\lambda_\sigma(F)^* \), where \( F^*(z, k) = \overline{F((z, k)^{-1})} \),

(ii) \( W^\lambda_\sigma(F \times_\lambda H) = W^\lambda_\sigma(F)W^\lambda_\sigma(H) \).
Proof. By the scaling argument, it is enough to prove these results for the case \( \lambda = 1 \).

(i) If \( \phi, \psi \in \mathcal{H}_2^\lambda \), then we have

\[
\langle W_\sigma(F^*)\phi, \psi \rangle = \int_K \int_{\mathbb{C}^n} F^*(z, k) \langle \rho_\sigma(z, k)\phi, \psi \rangle d z d k
\]

\[
= \int_K \int_{\mathbb{C}^n} \langle \phi, F((z, k)^{-1})\rho_\sigma((z, k)^{-1})\psi \rangle d z d k
\]

\[
= \langle \phi, W_\sigma(F)\psi \rangle = (W_\sigma(F)^*\phi, \psi) .
\]

(ii) Let \( dg = dzdk \), then

\[
\langle W_\sigma(F)W_\sigma(H)\phi, \psi \rangle = \int_{G^x} F(z, k) \langle \rho_\sigma(z, k)W_\sigma(H)\phi, \psi \rangle dg
\]

\[
= \int_{G^x} \int_{G^x} F(g)H(g')e^{-\frac{i}{2}\text{Im}(kw, z)} \langle \rho_\sigma(z + kw, ks)\phi, \psi \rangle dg' dg
\]

\[
= \int_{G^x} \int_{G^x} F(gg'^{-1})H(g'e^{-\frac{i}{2}\text{Im}(kw, z)}) \langle \rho_\sigma(g)\phi, \psi \rangle dg' dg
\]

\[
= \int_{G^x} (F \times H)(z, k) \langle \rho_\sigma(z, k)\phi, \psi \rangle dg
\]

\[
= \langle W_\sigma(F \times H)\phi, \psi \rangle .
\]

Next, we derive the Plancherel formula for the Weyl transform \( W_\sigma^\lambda \) on \( L^2(G^x) \) corresponding to \( \lambda = 1 \).

Proposition 3.2. If \( F \in L^2(G^x) \), then the following holds.

\[
\sum_{\sigma \in K} d_\sigma \| W_\sigma(F) \|_{HS}^2 = (2\pi)^n \int_K \int_{\mathbb{C}^n} |F(z, k)|^2 d z d k.
\]

Proof. Since \( L^1 \cap L^2(G^x) \) is dense in \( L^2(G^x) \), it is enough to prove the result for \( L^1 \cap L^2(G^x) \). For the sake of convenience, let \( \phi_{\alpha, i}^\sigma = \phi_\alpha^\sigma \otimes e_i^\sigma \) and \( \phi_{\alpha\beta} = (2\pi)^\frac{n}{2} \phi_{\alpha\beta}^\lambda \) when \( \lambda = 1 \). Then the set \( \{ \phi_{\gamma, i}^\sigma : \gamma \in \mathbb{N}^n, 1 \leq i \leq d_\sigma \} \) forms an orthonormal basis for \( \mathcal{H}_2^\sigma \). By the Parseval identity, we have

\[
\| W_\sigma(F)\phi_{\gamma, i}^\sigma \|_{\mathcal{H}_2^\sigma}^2 = \sum_{\beta \in \mathbb{N}^n} \sum_{j=1}^{d_\sigma} |\langle W_\sigma(F)\phi_{\gamma, i}^\sigma, \phi_{\beta, j}^\sigma \rangle|^2 =
\]

\[
(2\pi)^n \sum_{\beta \in \mathbb{N}^n} \sum_{j=1}^{d_\sigma} \left| \sum_{|\alpha| = |\gamma|} \eta_{\alpha\gamma}(k) \int_K F(z, k) \phi_{\alpha\beta}(z)\varphi_{\beta, j}^\sigma(k) dzdk \right|^2.
\]
It is easy to see that the matrix coefficients \( \eta_{\alpha \gamma} \) of the representation \( \mu_\lambda \) satisfy the identity

\[
(3.1) \quad \sum_{|\alpha|=m} \left| \sum_{|\gamma|=m} c_\alpha \eta_{\alpha \gamma}(k) \right|^2 = \sum_{|\alpha|=m} |c_\alpha \eta_{\alpha \gamma}(k)|^2,
\]

where \( k \in K \) and \( c_\alpha \in \mathbb{C} \). Now, by Plancherel theorem for the compact group \( K \) and the identity (3.1), we infer that

\[
\sum_{\sigma \in \hat{K}} d_\sigma \|W_\sigma(F)\|_{HS}^2 = (2\pi)^n \sum_{\beta, \gamma \in \mathbb{N}^n} \int_K \left| \sum_{|\alpha|=|\gamma|} \eta_{\alpha \gamma}(k) \int_{\mathbb{C}^n} F(z, k) \phi_{\alpha \beta}(z) dz \right|^2 dk
\]

\[
= (2\pi)^n \sum_{\alpha, \beta \in \mathbb{N}^n} \int_K \left| \int_{\mathbb{C}^n} F(z, k) \phi_{\alpha \beta}(z) dz \right|^2 dk
\]

\[
= (2\pi)^n \int_K \int_{\mathbb{C}^n} |F(z, k)|^2 dz dk.
\]

For \( \sigma \in \hat{K} \), we defining a Fourier-Wigner type transform \( V_{\gamma}^\sigma \) of functions \( f, g \in \mathcal{H}_\sigma^2 \) on \( G^\times \) by

\[
V_{\gamma}^\sigma(z, k) = \langle \rho_\sigma(z, k) f, g \rangle.
\]

**Lemma 3.3.** For \( f_l, g_l \in \mathcal{H}_\sigma^2 \), \( l = 1, 2 \), the following identity holds.

\[
\int_K \int_{\mathbb{C}^n} V_{\gamma, f_1}^\sigma(z, k) \overline{V_{\gamma, f_2}^\sigma(z, k)} dz dk = (2\pi)^n \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle.
\]

**Proof.** Since the set \( \{ \phi_\alpha \otimes e_\sigma^i : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma \} \) form an orthonormal basis for \( \mathcal{H}_\sigma^2 \) and \( f_l, g_l \in \mathcal{H}_\sigma^2 \), we can write

\[
f_l = \sum_{\gamma \in \mathbb{N}^n} \sum_{1 \leq i \leq d_\sigma} f_{l,i}^1 \phi_\gamma \otimes e_\sigma^i, \quad g_l = \sum_{\beta \in \mathbb{N}^n} \sum_{1 \leq j \leq d_\sigma} g_{l,j}^1 \phi_\beta \otimes e_\sigma^j, \quad l = 1, 2,
\]

where \( f_{l,i}^1 \) and \( g_{l,j}^1 \) are constants. Thus,

\[
V_{\gamma, f_l}^\sigma(z, k) = (2\pi)^{\frac{n}{2}} \sum_{\alpha, \beta \in \mathbb{N}^n} \sum_{1 \leq i,j \leq d_\sigma} \sum_{|\gamma|=|\alpha|} f_{l,i}^1 \overline{g_{l,j}^1} \eta_{\alpha \gamma}(k) \phi_{\alpha \beta}(z) \varphi_{ji}^\sigma(k),
\]

By the orthogonality of the special Hermite functions \( \phi_{\alpha \beta} \) together with the identity (3.1), it follows that

\[
\int_{\mathbb{C}^n} V_{\gamma, f_1}^\sigma(z, k) \overline{V_{\gamma, f_2}^\sigma(z, k)} dz =
\]

\[
(2\pi)^n \sum_{\gamma, \beta \in \mathbb{N}^n} \left[ \sum_{i,j=1}^{d_\sigma} \left( f_{l,i}^1 \overline{g_{l,j}^1} \right) \phi_{ji}^\sigma(k) \sum_{i,j=1}^{d_\sigma} \left( f_{l,j}^1 \overline{g_{l,i}^1} \right) \overline{\phi_{ji}^\sigma(k)} \right].
\]

Finally, by integrating both the sides with respect to \( k \), we get
\[
\int_K \int_{\mathbb{C}^n} V_{f_1}^g(z,k) V_{f_2}^g(z,k) \, dz \, dk = (2\pi)^n \left( \sum_{\gamma} \sum_{1 \leq i \leq d_\sigma} f_{\gamma,i} f_{\gamma,i}^* \left( \sum_{\beta} \sum_{1 \leq j \leq d_\sigma} g_{\beta,j}^2 g_{\beta,j}^* \right) \right) = (2\pi)^n \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle.
\]

□

Notice that, if \( f, g \in \mathcal{H}_\sigma^2 \), then as particular case of Lemma 3.3, it follows that \( V_g^f \in L^2(G^\times) \). Let \( V_\sigma = \text{span} \{ V_f^g : f, g \in \mathcal{H}_\sigma^2 \} \). Since the set \( B_\sigma = \{ \psi_{\sigma,i} : \alpha, e_i : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma \} \) form an orthonormal basis for \( \mathcal{H}_\sigma^2 \), by Lemma 3.3 we infer that the set

\[
V_{B_\sigma} = \left\{ V_{\psi_{\alpha,i}^\sigma} : \psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma \in B_\sigma \right\}
\]

is an orthonormal basis for \( V_\sigma \). Next, we recall the Peter-Weyl theorem which is crucial for the proof of Proposition 3.5. For more details, see [19].

**Theorem 3.4.** (Peter-Weyl). Let \( \hat{K} \) be the unitary dual of the compact Lie group \( K \). Then the set \( \{ \sqrt{d_\sigma} \phi_{ij}^\sigma : 1 \leq i, j \leq d_\sigma, \sigma \in \hat{K} \} \) is an orthonormal basis for the space \( L^2(K) \).

**Proposition 3.5.** The set \( \{ V_{B_\sigma} : \sigma \in \hat{K} \} \) is an orthonormal basis for \( L^2(G^\times) \).

**Proof.** By Theorem 3.4 it follows that \( \{ V_{B_\sigma} : \sigma \in \hat{K} \} \) is an orthonormal set. It only remains to prove the completeness. For this, suppose \( F \in V_{B_\sigma} \), then

\[
\langle W_\sigma(F) \psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma \rangle = \int_K \int_{\mathbb{C}^n} F(z,k) V_{\psi_{\alpha,i}^\sigma}^\sigma(z,k) \, dz \, dk = \langle F, V_{\psi_{\beta,j}^\sigma} \psi_{\alpha,i}^\sigma \rangle = 0,
\]

whenever \( \psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma \in B_\sigma \). Hence, it follows that \( W_\sigma(F) = 0 \) for all \( \sigma \in \hat{K} \). Thus, by Proposition 3.2 we conclude that \( F = 0 \). □

Moreover, by using the fact that \( V_{B_\sigma} \) is an orthonormal basis for \( V_\sigma \), as a corollary to Proposition 3.5 we infer that \( L^2(G^\times) = \bigoplus_{\sigma \in \hat{K}} V_\sigma \).

Now, we state our main result of this section. Let A and B are Lebesgue measurable subsets of \( \mathbb{R}^n \) such that \( 0 < m(A), m(B) < \infty \), where \( m \) denotes the Lebesgue measure on \( \mathbb{R}^n \).

**Theorem 3.6.** Let \( F \in L^1 \cap L^2(G) \) be supported on \( (\Sigma \times \mathbb{R}) \times K \).

(i) If \( \Sigma \) has finite Lebesgue measure and \( \hat{F}(\lambda, \sigma) \) is a rank one operator for all \( (\lambda, \sigma) \in \mathbb{R}^* \times \hat{K} \), then \( F = 0 \).
Theorem 3.7. For $\phi, \psi \in L^2(\mathbb{R}^n)$, write $X = T(\phi, \psi)$. If $\{z \in \mathbb{C}^n : X(z) \neq 0\}$ has finite Lebesgue measure, then $X = 0$.

In view of Theorem 3.7, we prove the following analogous result for the Fourier-Wigner transform. In fact, it says that the Fourier-Wigner transform of a pair of non-zero functions cannot be finitely supported.

Proposition 3.8. For $f_j \in \mathcal{H}_\sigma^2; j = 1, 2$, denote $F = V f_j^1$. If $\{z \in \mathbb{C}^n : F(z,k) \neq 0\}$ has finite Lebesgue measure for all $k \in K$, then $F = 0$.

Proof. Since $f_j \in \mathcal{H}_\sigma^2$, we can express $f_j = \phi_j \otimes h_j$, where $\phi_j \in L^2(\mathbb{R}^n)$ and $h_j \in \mathcal{H}_\sigma$. Then

$$F(z,k) = \langle \rho_\sigma(z,k) f_1, f_2 \rangle = \langle \pi(z) \mu(k) \otimes \sigma(k) (\phi_1 \otimes h_1), \phi_2 \otimes h_2 \rangle = \langle \pi(z) \mu(k) \phi_1, \phi_2 \rangle \langle \sigma(k) h_1, h_2 \rangle = \langle \pi(z) \psi_1, \phi_2 \rangle \langle \sigma(k) h_1, h_2 \rangle = X(z) \langle \sigma(k) h_1, h_2 \rangle,$$

where $\psi_1 = \mu(k) \phi_1$ and $X = T(\psi_1, \phi_2)$. If $\langle \sigma(k) h_1, h_2 \rangle = 0$ for some $k \in K$, then $F(.,k) = 0$. On the other hand, if $\langle \sigma(k) h_1, h_2 \rangle \neq 0$, then $X$ is a non-zero function that supported on a set of finite Lebesgue measure. Thus, in view of Theorem 3.7, we conclude that $F = 0$. \[]

Next, we prove that if for $F \in L^1 \cap L^2(G^\times)$, the operator $W_\sigma(F)$ is of finite rank for each $\sigma \in K$, then $F = 0$. For proving this, we require the following crucial results.

For $k \in K$, define $b_j(k) = \langle \sigma(k) \psi_1, \phi_j \rangle$, where $\psi_j \in H_\sigma$. Then $b_j(e) = ||\psi_j||^2$. Set $||\psi_j|| = \alpha_j$.

Proposition 3.9. For $\phi_j \in L^2(\mathbb{R}^n); j \in \{1, \ldots, N\}$, define the function $\psi$ on $\mathbb{C}^n$ by $\psi(z,k) = \sum_{j=1}^N b_j(k) \langle \pi(z) \mu(k) \phi_j, \phi_j \rangle$, where $k \in K$. If $\psi$ is supported on a subset $\mathcal{E} \times \mathcal{F}$ of $\mathbb{C}^n$ such that $0 < m(\mathcal{E}), m(\mathcal{F}) < \infty$, then $\psi \equiv 0$.

Proof. Let $e$ be the identity element of the group $K$. Then $\mu(e) = I$ is the identity operator on $L^2(\mathbb{R}^n)$. For $z = x + iy \in \mathbb{C}^n$, we write $\psi_e(x) = \psi(z,e)$. Since $\phi_j \in L^2(\mathbb{R}^n)$, there exists a set $A$ of measure zero such that $|\phi_j|$ is finite.
on $\mathbb{R}^n \setminus A$. Denote $K_y(\xi) = \sum_{j=1}^{N} \alpha_j^2 \phi_j(\xi + y) \overline{\phi_j(\xi)}$ for almost all $\xi \in \mathbb{R}^n$. Then by the hypothesis, $\psi$ can be expressed as

$$\psi_y(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \frac{1}{2}x \cdot y)} K_y(\xi) \, d\xi. \quad (3.2)$$

Since $\psi$ is supported on $\mathcal{E} \times \mathcal{F}$ of finite Lebesgue measure, it follows that $\psi_y = 0$ for all $y \in \mathbb{R}^n \setminus \mathcal{F}$. Hence we infer that $K_y = 0$, whenever $y \in \mathbb{R}^n \setminus \mathcal{F}$.

Define the function $\chi$ on $\mathbb{R}^n \setminus A$ by $\chi = (\alpha_1 \phi_1, \ldots, \alpha_N \phi_N)$. If $\chi = 0$ on $\mathbb{R}^n \setminus A$, then result will follow. Suppose $\chi \neq 0$, then there exists $\xi_1 \in \mathbb{R}^n \setminus A$ such that $\chi(\xi_1) \neq 0$.

Now, if it happens that $\chi = 0$ on $\mathbb{R}^n \setminus (B(\xi_1) \cup A)$, where $B(\xi_1)$ is the set $\xi_1 + (\mathcal{F} \cup \{0\})$, then $\phi_j$’s are finitely supported.

Otherwise, we can choose $\xi_l \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l-1} B(\xi_i) \cup A$, where $B(\xi_i) = \xi_i + (\mathcal{F} \cup \{0\})$ such that $\chi(\xi_l) \neq 0$, whenever $l \leq N$. For $l \neq m$, we have $\xi_l - \xi_m \notin \mathcal{F}$.

By the hypothesis, $K_{\xi_l - \xi_m}(\xi) = \sum_{j=1}^{N} \alpha_j^2 \phi_j(\xi + \xi_l - \xi_m) \overline{\phi_j(\xi)} = 0$, whenever $\xi \in \mathbb{R}^n \setminus A$. Hence it follows that $\chi(\xi_l)$ and $\chi(\xi_m)$ are orthogonal. Thus, the set $S = \{\chi(\xi_1), \ldots, \chi(\xi_N)\}$ is an orthogonal set in $\mathbb{C}^N$.

Therefore, if $\xi \in \mathbb{R}^n \setminus \bigcup_{l=1}^{N} (B(\xi_l) \cup A)$, then $\chi(\xi) \perp S$, and hence $\chi(\xi) = 0$. Thus, each of $\phi_j$ is supported on a set of finite Lebesgue measure.

Now, for $k \in K$, $\psi$ can be expressed as

$$\psi_y(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \frac{1}{2}x \cdot y)} \left( \sum_{j=1}^{N} b_j(k) \chi_j(\xi + y) \overline{\phi_j(\xi)} \right) \, d\xi,$$

where $\chi_j = \mu(k) \phi_j \in L^2(\mathbb{R}^n)$. Let $H_y(\xi) = \sum_{j=1}^{N} b_j(k) \chi_j(\xi + y) \overline{\phi_j(\xi)}$. Then $H_y$ is finitely supported for all $y \in \mathbb{R}^n$. By the Benedicks theorem, $H_y$ and its Fourier transform both cannot be finitely supported simultaneously. Hence we conclude that $\psi_y \equiv 0$ for all $y \in \mathbb{R}^n$. \qed

**Remark 3.10.** Instead of the rectangle $\mathcal{E} \times \mathcal{F}$ in $\mathbb{R}^{2n}$ if we consider a set $E$ of finite Lebesgue measure in $\mathbb{C}^n$, then the projection of $E$ on $\mathbb{R}^n$ need not be a set of finite measure. Hence the above proof of Proposition 3.9 will not work.

Let $\mathcal{E}$ and $\mathcal{F}$ are Lebesgue measurable subsets of $\mathbb{R}^n$ such that $0 < m(\mathcal{E}), m(\mathcal{F}) < \infty$ and $\Sigma = \mathcal{E} \times \mathcal{F}$.

**Theorem 3.11.** Let $F \in L^1 \cap L^2(G^\Sigma)$ be supported on $\Sigma \times K$. If for each $\sigma \in K$, the operator $W_{\sigma}(F)$ has finite rank, then $F = 0$. 


Proof. Let $\bar{\tau} = F^* \times F$, where $F^*(v) = \overline{F(v^{-1})}$. Then $W_\sigma(\bar{\tau}) = W_\sigma(F)^* W_\sigma(F)$ is a positive, finite rank operator on $H_\sigma^2$. By the spectral theorem, it follows that

$$W_\sigma(\bar{\tau})f = \sum_{j=1}^N a_j \langle f, f_j \rangle f_j,$$

where $\{f_1, \ldots, f_N\}$ is an orthonormal basis for the range of $W_\sigma(\bar{\tau})$ which satisfies $W_\sigma(\bar{\tau})f_j = a_j f_j$ with $a_j \geq 0$. Now, for $f, g \in H_\sigma^2$, we have

$$\langle W_\sigma(\bar{\tau})f, g \rangle = \sum_{j=1}^N a_j \langle f, f_j \rangle \langle f_j, g \rangle = (2\pi)^{-n} \sum_{j=1}^N a_j \int_K \int_{\mathbb{C}^n} V^g_f(z, k) \overline{V^f_{f_j}(z, k)} \, dz \, dk.$$

Since $\tau \in L^2(G^\times)$, by Proposition 3.5, we can write $\tau = \bigoplus_{\sigma \in \hat{K}} \tau_\sigma$. In view of the above decomposition and by the definition of $W_\sigma(\bar{\tau})$, we can write

$$\langle W_\sigma(\bar{\tau})f, g \rangle = \int_K \int_{\mathbb{C}^n} \bar{\tau}(z, k) \langle \rho_\sigma^1(z, k), f \rangle \, dz \, dk.$$

Hence, by comparing (3.4) with (3.5) in view of the orthogonality relation for the Fourier-Wigner transform as in Lemma 3.3, it follows that

$$\tau_\sigma = \sum_{j=1}^N V^h_{f_j},$$

where $h_j = (2\pi)^{-n/2} \sqrt{a_j} f_j \in H_\sigma^2$. Now, let $h_j = \phi_j \otimes \psi_j$ for some $\phi_j \in L^2(\mathbb{R}^n)$ and $\psi_j \in H_\sigma$. Then from (3.6) we have

$$\tau_\sigma(z, k) = \sum_{j=1}^N \langle \rho_\sigma(z, k) h_j, h_j \rangle = \sum_{j=1}^N \langle \pi(z) \mu(k) \phi_j, \phi_j \rangle \langle \sigma(k) \psi_j, \psi_j \rangle$$

$$= \sum_{j=1}^N b_j(k) \langle \pi(z) \chi_j, \phi_j \rangle,$$

where $\chi_j = \mu(k) \phi_j \in L^2(\mathbb{R}^n)$ and $b_j(k) = \langle \sigma(k) \psi_j, \psi_j \rangle$. Since $\bar{\tau}$ is finitely supported in $\mathbb{C}^n$ variable, by Proposition 3.9 it follows that $\tau_\sigma = 0$, whenever $\sigma \in \hat{K}$. In view of Plancherel formula for the Weyl transform as mentioned in Proposition 3.2, we conclude that $F = 0$. \hfill \Box

Next, we prove Theorem 3.6 in the following two cases.
Proof of Theorem 3.6. (i). Since $F \in L^1 \cap L^2(G)$, we can write

\begin{equation}
\hat{F}(\lambda, \sigma) = \int_K \int_{C^n} \int_{\mathbb{R}} F(z, t, k) \rho_\sigma(z, t, k) dt dz dk
\end{equation}

\begin{equation}
= \int_K \int_{C^n} F^\lambda(z, k) \rho_\sigma(z, k) dz dk
\end{equation}

\begin{equation}
= W_\sigma(F^\lambda).
\end{equation}

Suppose the operator $W_\sigma(F^\lambda)$ has rank one. Then it is enough to show that $F^\lambda = 0$. Consider the case when $\lambda = 1$. Since by hypothesis, $W_\sigma(F^1)$ has rank one, there exist $f_j \in \mathcal{H}_2^\sigma; j = 1, 2$ such that $W_\sigma(\bar{\tau}) f = \langle f, f_1 \rangle f_2$ for all $f \in \mathcal{H}_2^\sigma$, where $\bar{\tau} = F^1$. Hence for $f, g \in \mathcal{H}_2^\sigma$, Lemma 3.3 yields

\begin{equation}
\langle W_\sigma(\bar{\tau}) f, g \rangle = \langle f, f_1 \rangle \langle f_2, g \rangle
\end{equation}

\begin{equation}
= (2\pi)^{-n} \int_K \int_{C^n} V^g(z, k) V^{f_2}_f(z, k) dz dk.
\end{equation}

Let $\tau = \bigoplus_{\sigma \in \hat{K}} \tau_\sigma$, where $\tau_\sigma \in V_{B_\sigma}$. Then by definition of $W_\sigma(\bar{\tau})$, it follows that

\begin{equation}
\langle W_\sigma(\bar{\tau}) f, g \rangle = \int_K \int_{C^n} \tau_\sigma(z, k) V^g_f(z, k) dz dk.
\end{equation}

Now, by comparing (3.8) with (3.9) in view of Proposition 3.5 we infer that $\tau_\sigma = (2\pi)^{-n} V^{f_2}_{f_1}$. Finally, by Proposition 3.5 it follows that $\tau_\sigma = 0$ for all $\sigma \in \hat{K}$. That is, $\tau = 0$ and hence we conclude that $F = 0$.

(ii). Suppose the operator $W_\sigma(F^\lambda)$ has finite rank. We prove the result for $\lambda = 1$ and the general case will be followed by the scaling argument. Since $\hat{F}(1, \sigma) = W_\sigma(F^1)$, by Theorem 3.11 it follows that $F^1 = 0$. Similarly, it can be shown that $F^\lambda = 0$ for all $\lambda \in \mathbb{R}^*$. Thus, we conclude that $F = 0$.

4. Preliminaries on step two nilpotent group

In this section, we prove an analogous result of the Benedick’s theorem for the Euclidean Fourier transform on the step two nilpotent Lie groups. However, for the sake simplicity, we derive the result for the class of groups introduced by G. Métivier (see [10]). These groups are step two nilpotent Lie groups when quotiented with the hyperplane in the center becomes the Heisenberg group. The Heisenberg-type groups introduced by A. Kaplan (see [8]) are examples of Métivier group. However, there are Métivier groups which are distinct from the Heisenberg-type groups. For more details, see [11].

Let $G$ be a connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the orthogonal decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective. Thus, $G$ can be parameterized by $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, endowed with the exponential coordinates.
Let \{V_i : i = 1, \ldots, m\} and \{Z_j : j = 1, \ldots, k\} be orthonormal bases of \(b\) and \(\mathfrak{z}\) respectively. Then for \(V + Z \in b \oplus \mathfrak{z}\), we can identify \(g \in G\) with the point \((V, Z) \in \mathbb{R}^m \times \mathbb{R}^k\) such that \(g = \exp(V + Z)\). Since \([b, b] \subset \mathfrak{z}\) and \([\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}\), by the Baker-Campbell-Hausdorff formula, the group law on \(G\) can be expressed as

\[
(V, Z) (V', Z') = \left( V + V', Z + Z' + \frac{1}{2} [V, V'] \right).
\]

Let \(dV\) and \(dZ\) be the Lebesgue measures on \(b\) and \(\mathfrak{z}\) respectively. Then the left-invariant Haar measure on \(G\) can be expressed as \(dg = dVdZ\).

Now, for \(\omega \in \mathfrak{z}^*\), consider the skew-symmetric bilinear form \(B_\omega\) on \(b\) by

\[
B_\omega(X, Y) = \omega ([X, Y]).
\]

Let \(m_\omega\) be the orthogonal complement of \(r_\omega = \{ X \in b : B_\omega(X, Y) = 0, \forall Y \in b \}\) in \(b\). Then \(B_\omega\) is called a non-degenerate bilinear form when \(r_\omega\) is trivial. If \(B_\omega\) is non-degenerate for all \(\omega \neq 0\), then \(G\) is called Métivier group.

Since \(m_\omega\) is invariant under the skew-symmetric bilinear form \(B_\omega\), it follows that the dimension of \(m_\omega\) is even. Let \(\Lambda = \{ \omega \in \mathfrak{z}^* : \dim m_\omega\text{ is maximum} \}\). Then \(\Lambda\) is a Zariski open subset of \(\mathfrak{z}^*\) and for \(\omega \in \Lambda\), there exists an orthonormal almost symplectic basis \(\{ X_i(\omega), Y_j(\omega) : i = 1, \ldots, n \}\) of \(b\) and \(d_i(\omega) > 0\) such that

\[
\omega[X_i(\omega), Y_j(\omega)] = \left\{ \begin{array}{ll} 
\delta_{ij}d_i(\omega), & \text{when } X \neq Y; \\
0, & \text{otherwise.}
\end{array} \right.
\]

Let \(\zeta_\omega = \text{span}\{ X_i(\omega) : i = 1, \ldots, n \}\) and \(\eta_\omega = \text{span}\{ Y_j(\omega) : j = 1, \ldots, n \}\). Then we can write \(b = \zeta_\omega \oplus \eta_\omega\) and each \((X, Y, Z) \in G\) can be represented by

\[
(X, Y, Z) = \sum_{i=1}^{n} x_i(\omega)X_i(\omega) + \sum_{i=1}^{n} y_i(\omega)Y_i(\omega) + \sum_{i=1}^{k} t_i(\omega)Z_i(\omega).
\]

Hence a typical element of \(G\) can be written as \((x, y, t)\), where \(x, y \in \mathbb{R}^n\) and \(t \in \mathbb{R}^k\). For more details, we refer to [5, 9, 10].

Next, we briefly describe the irreducible representation of the Métivier group \(G\) which can be parameterized by \(\Lambda\). That is, each \(\omega \in \Lambda\) induces an irreducible unitary representation \(\pi_\omega\) of \(G\) by

\[
(\pi_\omega(x, y, t)\phi)(\xi) = e^{i \sum_{j=1}^{k} \omega_j t_j+i \sum_{j=1}^{n} d_j(\omega)(x_j \xi_j + \frac{1}{2} x_j y_j)} \phi(\xi + y),
\]

whenever \(\phi \in L^2(\eta_\omega)\). For the sake of simplicity, we write \(v = (x, y)\). Then the group Fourier transform of \(f \in L^1(G)\) can be defined by

\[
\hat{f}(\omega) = \int_{\mathfrak{g}} \int_{b} f(v, t) \pi_\omega(v, t)dvdt,
\]
where $\omega \in \Lambda$. Now, we define the Fourier inversion of $f$ in the $t$ variable by

$$f^\omega(v) = e^{i \sum_{j=1}^k \omega_j t_j} f(v, t) dt.$$ 

Then for the suitable functions $f$ and $g$ on $b$, we can define the $\omega$-twisted convolution of $f$ and $g$ by

$$f * \omega g(v) = \int_b f(v - v') g(v') e^{\frac{i}{2} \omega(v, v')} dv'.$$

Here it is immediate that $(f * g)^\omega = f^\omega * \omega g^\omega$. Let $p(\omega) = \prod_{i=1}^n d_i(\omega)$ be the symmetric function of degree $n$ corresponding to $B_\omega$. For $f \in L^1 \cap L^2(G)$, the operator $\hat{f}(\omega)$ is a Hilbert-Schmidt operator that satisfies

$$p(\omega) \|\hat{f}(\omega)\|_{HS}^2 = (2\pi)^n \int_b |f^\omega(v)|^2 dv.$$

Denote $\pi_\omega(v) = \pi_\omega(v, o)$. Then the Fourier inversion $f^\omega$ can be determined by the formula

$$f^\omega(v) = (2\pi)^{-n} p(\omega) tr(\pi_\omega(v)^* \hat{f}(\omega)).$$

5. Uniqueness results on step two nilpotent group

For $\omega \in \Lambda$ and $h \in L^1 \cap L^2(b)$, the Weyl transform $W_\omega(h)$ is defined by

$$(5.1) \quad W_\omega(h) = \int_b h(v) \pi_\omega(v) dv.$$ 

The Weyl transform $W_\omega(h)$ is a Hilbert-Schmidt operator on $L^2(\eta_\omega)$ that satisfies the following Plancherel formula, (see [15]).

**Theorem 5.1.** For $h \in L^2(b)$, the following equality holds:

$$p(\omega) \|W_\omega(h)\|_{HS}^2 = (2\pi)^n \int_b |h(v)|^2 dv.$$ 

**Proposition 5.2.** For $h \in L^1 \cap L^2(b)$, we have the identities:

(i) $W_\omega(h^*) = W_\omega(h)^*$, where $h^*(v) = \overline{h(v^{-1})}$,

(ii) $W_\omega(h^* \omega h) = W_\omega(h)^* W_\omega(h)$.

**Proof.** (i) For $\phi, \psi \in L^2(\eta_\omega)$, we can write

$$\langle W_\omega(h^*)\phi, \psi \rangle = \int_b h^*(v) \langle \pi_\omega(v)\phi, \psi \rangle dv = \int_b \langle \phi, h(v^{-1}) \pi_\omega(v^{-1}) g \rangle dv = \langle \phi, W_\omega(h)\psi \rangle = \langle W_\omega(h)^* \phi, \psi \rangle.$$
(ii) Further, we have
\[
\langle W_\omega(h)^*W_\omega(h)\phi, \psi \rangle = \int_b \int_b h^*(v)h(v') \langle \pi_\omega(v)\pi_\omega(v')\phi, \psi \rangle dv dv'
= \int_b \int_b h^*(v - v')h(v')e^{\frac{i}{2}\omega([v,v'])} \langle \pi_\omega(v)\phi, \psi \rangle dv dv'
= \int_b (h^**_\omega h)(v) \langle \pi_\omega(v)\phi, \psi \rangle dv
= \langle W_\omega(h^**_\omega h)\phi, \psi \rangle.
\]
\[
\square
\]

Now, we state our main result of this section. Let A and B are Lebesgue measurable subsets of \(\mathcal{Z}\) and \(\eta_\omega\) respectively such that \(0 < m(A), m(B) < \infty\), where \(m\) denotes the Lebesgue measure.

**Theorem 5.3.** Suppose \(f \in L^1(G)\) is supported on the set \(\Sigma \oplus 3\), where \(\Sigma\) is a subset of \(b\).

(i) If \(\Sigma\) has finite Lebesgue measure and \(\hat{f}(\omega)\) is a rank one operator for all \(\omega \in \Lambda\), then \(f = 0\).

(ii) If \(\Sigma = A \times B\) and \(\hat{f}(\omega)\) has finite rank for all \(\omega \in \Lambda\), then \(f = 0\).

In order to prove Theorem 5.3 we need the following crucial results. Let \(\phi, \psi \in L^2(\eta_\omega)\). Then the Fourier-Wigner transform of \(\phi\) and \(\psi\) is a function on \(b\) defined by
\[
T(\phi, \psi)(v) = \langle \pi_\omega(v)\phi, \psi \rangle.
\]
As a consequence of the Schur’s orthogonality relation, these functions \(T(\phi, \psi)\)’s are orthogonal among themselves. For more details, we refer to Wolf [23].

**Lemma 5.4.** [23] Let \(\phi_j, \psi_j \in L^2(\eta_\omega); \ j = 1, 2\). Then
\[
\int_b T(\phi_1, \psi_1)(v)T(\phi_2, \psi_2)(v)dv = c(\omega) \langle \phi_1, \phi_2 \rangle \langle \psi_1, \psi_2 \rangle,
\]
where \(c(\omega) = (2\pi)^n p(\omega)^{-1}\).

We observe that these functions \(T(\phi, \psi)\)’s generate an orthonormal basis for \(L^2(b)\). Let \(\{\varphi_j : j \in \mathbb{N}\}\) be an orthonormal basis for \(L^2(\eta_\omega)\).

**Proposition 5.5.** The set \(\{T(\varphi_i, \varphi_j) : i, j \in \mathbb{N}\}\) is an orthonormal basis for \(L^2(b)\).

**Proof.** In view of Lemma 5.4, it is clear that \(\{T(\varphi_i, \varphi_j) : i, j \in \mathbb{N}\}\) is an orthonormal set. Now, it only remains to verify the completeness. For this, let \(f \in L^2(b)\) be such that \(\langle f, T(\varphi_i, \varphi_j) \rangle = 0\), whenever \(i, j \in \mathbb{N}\). Then
\[
(5.2) \quad \langle W_\omega(f)\phi_i, \phi_j \rangle = \int_b f(v) \langle \pi_\omega(v)\phi_i, \phi_j \rangle dv
= \langle f, T(\phi_i, \phi_j) \rangle = 0.
\]
Hence, we infer that \( W_\omega(\tilde{f}) = 0 \). Thus, by the Plancherel Theorem 5.1 we conclude that \( f = 0 \). \( \square \)

**Proposition 5.6.** Let \( F = T(\phi, \psi) \), where \( \phi, \psi \in L^2(\eta_\omega) \). If the set \( \{ v \in b : F(v) \neq 0 \} \) has a finite Lebesgue measure, then \( F \) has to vanish identically.

**Proof.** We would like to mention that the proof of Proposition 5.6 is almost similar to Theorem 3.7 and hence we omit it here. \( \square \)

Let \( E \) and \( F \) be finite measure subset of \( \zeta_\omega \) and \( \eta_\omega \) respectively such that \( 0 < m(E), m(F) < \infty \) and \( \Sigma = E \times F \).

**Lemma 5.7.** For \( h_j \in L^2(\eta_\omega) \), write \( K_y(\xi) = \sum_{j=1}^N h_j(\xi + y)h_j(\xi) \), where \( y \in \eta_\omega \).

If \( K_y(\xi) = 0 \) for all \( y \in \eta_\omega \setminus F \) and for almost all \( \xi \in \eta_\omega \), then each of \( h_j \) is finitely supported.

**Proof.** Since \( h_j \in L^2(\eta_\omega) \), there exists a set \( A \) of Lebesgue measure zero such that \( |h_j| \) is finite on \( \eta_\omega \setminus A \). Define a function \( \chi \) on \( \eta_\omega \setminus A \) by

\[
\chi = (h_1, \ldots, h_N).
\]

If \( h_j \) is non-vanishing on \( \eta_\omega \setminus A \) for some \( j \), then we can choose \( \xi_1 \in \eta_\omega \setminus A \) such that \( \chi(\xi_1) \neq 0 \). Let \( B(\xi_1) \) be the set \( \xi_1 + (F \cup \{0\}) \). If \( \chi \) vanishes on \( \eta_\omega \setminus B(\xi_1) \cup A \), then the result follows. Otherwise, by induction, we can choose \( \xi_j \in \eta_\omega \setminus \bigcup_{i=1}^{j-1} (B(\xi_i) \cup A) \) such that \( \chi(\xi_j) \neq 0 \), whenever \( j \leq N \), where \( B(\xi_i) = \xi_i + (F \cup \{0\}) \). Thus by the hypothesis, the set \( S = \{ \chi(\xi_j) : j = 1, 2, \ldots, N \} \) is an orthogonal set in \( \mathbb{C}^N \). Now, if \( \xi \in \eta_\omega \setminus \bigcup_{j=1}^N (B(\xi_j) \cup A) \), then \( \chi(\xi) \in S^\perp \), and hence \( \chi(\xi) = 0 \). \( \square \)

**Proposition 5.8.** Let \( h \in L^1 \cap L^2(b) \) be supported on \( \Sigma \) in \( b \). If \( W_\omega(h) \) is a finite rank operator, then \( h = 0 \).

**Proof.** Let \( \tilde{h} = h^* \ast_\omega h \), where \( h^*(v) = \overline{h(v^{-1})} \). Then \( W_\omega(\tilde{h}) = W_\omega(h)^\ast W_\omega(h) \) is a positive and finite rank operator on \( L^2(\eta_\omega) \). By the spectral theorem, there exist an orthonormal set \( \{ \phi_j \in L^2(\eta_\omega) : j = 1, \ldots, N \} \) and scalars \( a_j \geq 0 \) such that

\[
W_\omega(\tilde{h})\phi = \sum_{j=1}^N a_j \langle \phi, \phi_j \rangle \phi_j,
\]

whenever \( \phi \in L^2(\eta_\omega) \). Now, for \( \psi \in L^2(\eta_\omega) \), we have

\[
\langle W_\omega(\tilde{h})\phi, \psi \rangle = \sum_{j=1}^N a_j \langle \phi, \phi_j \rangle \langle \phi_j, \psi \rangle
\]

\[
(5.3) = c(\omega)^{-1} \sum_{j=1}^N a_j \int_b T(\phi, \psi)(v)\overline{T(\phi_j, \phi_j)(v)}dv.
\]
Further, by definition of $W_\omega(\bar{\tau})$, we have
\begin{align}
\langle W_\omega(\bar{\tau})\phi, \psi \rangle &= \int_\mathfrak{b} \bar{\tau}(v)T(\phi, \psi)(v)dv.
\end{align}
Hence, by comparing (5.3) with (5.4) in view of Proposition 5.5, it follows that
\begin{align}
\tau &= \sum_{j=1}^{N} T(h_j, h_j),
\end{align}
where $h_j = c(\omega)^{-\frac{1}{2}}\sqrt{a_j} \phi_j \in L^2(\eta_\omega)$. Now, for $v = (x, y)$, write $\tau_y(x) = \tau(x, y)$. Then Equation (5.5) becomes
\begin{align}
\tau_y(x) &= \int_{\eta_\omega} e^{i \sum_{j=1}^{n} d_j(\omega)(x_j \xi_j + \frac{1}{2}x_j y_j)} K_y(\xi)d\xi.
\end{align}
Since $\bar{\tau}$ is supported on $\mathcal{E} \times \mathcal{F}$, it follows that $K_y(\xi) = 0$ for almost every $\xi$ and for all $y \in \eta_\omega \smallsetminus \mathcal{F}$. Then in view of Lemma 5.7, it follows that each of $h_j$ is finitely supported and hence each of $K_y$ is finitely supported. Since $\tau_y$ is is supported on $\mathcal{E}$, whenever $y \in \eta_\omega$, we infer that $\tau_y$ is zero for all $y \in \eta_\omega$. Now, by Plancherel Theorem 5.1, we conclude that $h = 0$. □

Proof of Theorem 5.3 (i). By a simple calculation, we get
\begin{align}
\hat{f}(\omega) &= \int_0^b f^\omega(v)\pi_\omega(v)dv = W_\omega(f^\omega).
\end{align}
Since $f^\omega$ is finitely supported and the operator $W_\omega(f^\omega)$ has finite rank, by Proposition 5.8, it follows that $f^\omega = 0$, whenever $\omega \in \Lambda$. Hence we infer $f = 0$.

(ii). It is enough to prove that if $W_\omega(f^\omega)$ has rank one, then $f^\omega = 0$. Let $W_\omega(f^\omega)$ be a rank one operator. Then there exist $\phi_j \in L^2(\eta_\omega)$; $j = 1, 2$ such that
\begin{align}
W_\omega(\bar{\tau})\phi &= \langle \phi, \phi_1 \rangle \phi_2
\end{align}
for all $\phi \in L^2(\eta_\omega)$, where $\bar{\tau} = f^\omega$. Thus, for $\psi \in L^2(\eta_\omega)$, it follows that
\begin{align}
\langle W_\omega(\bar{\tau})\phi, \psi \rangle &= \langle \phi, \phi_1 \rangle \langle \phi_2, \psi \rangle
\end{align}
\begin{align}
&= c(\omega)^{-1} \int_{\eta_\omega} \int_{\zeta_\omega} T(\phi, \psi)(v)\overline{T(\phi_1, \phi_2)(v)}dv.
\end{align}
Further, by definition, we get
\begin{align}
\langle W_\omega(\bar{\tau})\phi, \psi \rangle &= \int_{\zeta_\omega} \int_{\eta_\omega} \bar{\tau}(v)T(\phi, \psi)(v)dv.
\end{align}
Hence by comparing (5.7) with (5.8) in view of Lemma 5.4, we infer that $\tau(v) = c(\omega)^{-1}T(\phi_1, \phi_2)(v)$. Thus, from Proposition 5.6, it follows that $\tau \equiv 0$. □
Concluding remarks:

If the Fourier transform of a compactly supported function $f$ on $\mathbb{H}^n \ltimes U(n)$ (or step two nilpotent Lie groups) lands into the space of compact operators, then $f$ might be zero. However, it would be a good question to consider the case when the spectrum of the Fourier transform of a compactly supported function is supported on a thin uncountable set.

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