Klein-Gordon equation for a particle in brane model

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Brane model of universe is considered for a free particle. Conservation laws on the brane are obtained using the symmetry properties of the brane. Equation of motion is derived for a particle using variation principle from these conservation laws. This equation has a form of Klein-Gordon equation. Comparison of squared Dirac-Fok-Ivanenko equation for a spin particle with Klein-Gordon equation in curved space has given an expression for chiral spin current variation through the derivative of spin connectivity. This chiral spin current is anomalous spin current corresponding to spontaneous chiral symmetry breaking of massive particle in the space of KG equation solutions.

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I. INTRODUCTION

There is Dirac-Fok-Ivanenko (DFI) equation [1] describing fermions behavior being valid when the supersymmetry is broken. Klein-Gordon (KG) equation for a particle describes behavior of both bosons and fermions and is valid for the case of global symmetry where currents analogous to Dirac chiral currents are conserved [2]. Squaring of DFI equation yields KG equation in curved space [3, 4, 6]. KG equation is also obtained from variation principle in works [5, 7, 8].

In present paper, we will derive starting from the symmetry properties of the brane [9–11] the equation of motion for a particle in the framework of brane model. This equation has a form of Klein-Gordon equation. We perform comparison of squared DFI equation with KG equation containing free term in concrete form corresponding to brane symmetry approach. We derive by that an additional relation - the expression for gamma five matrix through the derivative of spin connection. Then we use it for determination of chiral spin current manifesting chiral anomaly at spontaneous symmetry breaking.

II. ENERGY CONSERVATION LAW

Let’s consider our space as four dimensional hyper-surface that is embedded in the space of higher dimension. Then interval for a moving particle in normal Gauss coordinates can be written as

\[ ds = \sqrt{g_{ij}dx^i dx^j - c^2 dt^2}, \]  

where \( g_{ij} \) is metric tensor, \( dx^i, dx^j \) are differentials of coordinates \( i,j = 0, 1, 2, 3 \) on brane, \( dt \) is differential of universal time that is proportional to extra dimensional coordinate \( dx^4 = c dt \) and is the same for all points on brane. Greek symbols will denote all indexes \( \alpha = 0, 1, 2, 3, 4 \) including extra dimension.

We take the convention that the metric signature is \((+−−−)\). Then the action can be written as

\[ S = mc \int_0^T ds = \int_0^T L dt \]  

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where \( m \) is mass of particle, \( c \) is speed of light,

\[
L = \sqrt{g_{ij}(\dot{mx}^i)(\dot{mx}^j) - m^2c^2},
\]

(3)
is Lagrangian, \( T \) is current value of universal time in multidimensional space (proportional to the brane radius).

Let’s introduce the symmetry of configuration space as single parametric transformation group \( f(q, \varepsilon) \):

\[
t \rightarrow t + \varepsilon, x_i \rightarrow x_i(t + \varepsilon),
\]

\[
f(q, \varepsilon) = x_i(t + \varepsilon), f(q, 0) = x_i(t)
\]

(4a)

conserving Lagrangian \([3]\). According to Noether’s theorem, first integral

\[
I = \frac{\partial L}{\partial \dot{x}_i} h^i,
\]

(5)

where

\[
h^i = \frac{\partial f^i}{\partial \varepsilon},
\]

(6)
can be put in correspondence to each symmetry. Then

\[
I = \frac{1}{L} g_{ij} m^2 \dot{x}^i \left( \frac{\partial x_i(t + \varepsilon)}{\partial (t + \varepsilon)} \frac{\partial (t + \varepsilon)}{\partial \varepsilon} \right) \bigg|_{\varepsilon=0},
\]

(7)
or

\[
g_{ij} m \dot{x}^i m \dot{x}^j = \text{const},
\]

(8)

If the particle moves uniformly and rectilinearly on the background of Lorenz’s metrics than we can choose reference system where \( \dot{x}^{(1)} = \dot{x}^{(2)} = \dot{x}^{(3)} = 0 \) and \( \dot{x}^{(0)} = 0 \) when brane is expanding with velocity \( c \). Then \([3]\) yields

\[
g_{ij} m \dot{x}^i m \dot{x}^j = m^2c^2,
\]

(9)

The same equation can be derived in the framework of quantum mechanical treatment. The wave function of particle in quasi-classical approximation is

\[
\psi = ae^{\frac{iS}{\hbar}},
\]

(10)

where \( a \) is slowly varying amplitude, \( S \) is action expressed by formulas \([2, 3]\). Let’s differentiate both sides of expression \([10] \) by \( T \) neglecting the dependence of amplitude on universal time

\[
\frac{\partial \psi}{\partial T} = a \frac{i}{\hbar} e^{\frac{iS}{\hbar}} \frac{\partial S}{\partial T} = i \frac{c}{\hbar} \sqrt{g_{ij} p^i p^j - m^2 c^2} \psi
\]

(11)

where \( p^i = mx^i \).

If evolution of particle in brane does not depend on brane radius then \( \frac{\partial \psi}{\partial T} = \frac{\partial S}{\partial T} = 0 \) and

\[
g_{ij} p^i p^j = m^2 c^2.
\]

(12)

III. KLEIN-GORDON EQUATION

Expression \([12]\) can be rewritten in the following form for wave function in Hilbert space:

\[
p_i p^i \psi = m^2 c^2 \psi.
\]

(13)
Let's consider functional variation of relation (13) in the vicinity of $x$. Complete variation of momentum vector can be written as the sum of functional variation $\delta p$ of vector $p$ at the comparison of $p(x')$ with $p(x'')$ in the vicinity of $p(x)$ at the parallel transfer of momentum vector in universal space and ordinary variation $dp$. Then, it can be written that

$$\Delta p = \frac{1}{2} (p(x') - p(x'')) = \frac{1}{2} (p(x') - \tilde{p}(x') + \tilde{p}(x') - p(x'')) = \frac{1}{2} \delta p + \frac{1}{2} dp,$$

where

$$\delta p = p'(x') - \tilde{p}(x')$$

and

$$dp = \tilde{p}(x') - p(x''),$$

$\tilde{p}(x')$ is momentum vector at its parallel transfer in the universal space from point $x'' = x - \delta x$ to point $x' = x + \delta x$. If trajectory of particle is geodetic one then

$$dp_i = \frac{\partial p_i}{\partial x^k} dx^k = 0,$$

$$\delta p_i = \tilde{p}_k \Gamma^k_{i\alpha} \delta x^\alpha,$$

where $\delta x^\alpha = \frac{1}{2} (x'^\alpha - x''^\alpha)$. Rome indexes numerate here coordinates of usual four-coordinate space and Greek indexes enumerate coordinates of universal five-coordinate space. It was assumed at formulation of (18) that $\tilde{p}_4 = 0$.

Then, it can be written, omitting stroked index of momentum vector,

$$p(x') = p(x) + \frac{1}{2} \delta p.$$  \hspace{1cm} (19)

At the transform $x \to x'$, relation (13) is transforming accounting (19) to the following form:

$$\left\{ p_i p^i + \frac{1}{2} (p_i \delta p^i + \delta p_i p^i) + \frac{1}{4} \delta p_i \delta p^i \right\} \psi = m^2 c^2 \psi.$$  \hspace{1cm} (20)

Let's pass in relation (20) to operators acting in Hilbert space of wave functions $\psi(x)$. We represent for this sake the components of vector $p$ as

$$p_i = -i\hbar \frac{\partial}{\partial x^i},$$  \hspace{1cm} (21)

and rewrite relation (15) as

$$\delta p_i = -i\hbar \left\{ \Gamma^k_{i\alpha} \delta x^\alpha \right\}_{;j} \psi,$$

assuming that $\tilde{p}_k$ is a covariant derivative.

Let's consider the first term in the left side of equation (20). For this purpose, we represent it in the form

$$p_i p^i \psi = p_i g^{ij} p_j \psi.$$  \hspace{1cm} (23)

Using expression (21), we get

$$p_i p^i \psi = -\hbar^2 \left( \frac{\partial g^{ij}}{\partial x^i} \frac{\partial}{\partial x^j} + g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right) \psi.$$  \hspace{1cm} (24)
Let’s use well known relation

\[
\frac{\partial g^{ij}}{\partial x^k} = -\Gamma^i_{mk}g^{mj} - \Gamma^j_{mk}g^{im}.
\] (25)

Then

\[
p_i p^i \psi = -\hbar^2 \left( g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - g^{mj} \Gamma^i_{mi} \frac{\partial}{\partial x^j} - g^{im} \Gamma^j_{mi} \frac{\partial}{\partial x^i} \right) \psi.
\] (26)

Changing indexes of summation, we get

\[
p_i p^i \psi = -\hbar^2 g^{ij} \left( \frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \psi.
\] (27)

Let’s consider second term in the left side of equation (14), rewriting it in the form

\[
p_i \delta p^i \psi = p_i g^{ij} \delta p^j \psi.
\] (28)

Using formula (25), we get

\[
p_i \delta p^i \psi = \left\{ g^{ij} (p_i \delta p_j) + i\hbar \left( g^{ij} \Gamma^m_{im} + g^{im} \Gamma^j_{im} \right) \delta p_j + g^{ij} \delta p^j p_i \right\} \psi.
\] (29)

Let’s write in its direct form the covariant derivative in the expression (22):

\[
\delta p_j = -i\hbar \left( \frac{1}{2} \Gamma^k_{jk} + \frac{\partial \Gamma^k_{jk}}{\partial x^\alpha} \delta x^\alpha - \Gamma^k_{jk} \Gamma^l_{j\alpha} \delta x^\alpha + \Gamma^k_{jk} \Gamma^l_{j\alpha} \delta x^\alpha \right).
\] (30)

We get from formula (30)

\[
\delta p_j = -i\hbar \left( \frac{1}{2} \Gamma^k_{jk} + R_{j\alpha} \delta x^\alpha + \frac{\partial \Gamma^k_{jk}}{\partial x^\alpha} \delta x^\alpha \right).
\] (31)

If \( \frac{\partial \Gamma^k_{jk}}{\partial x^\alpha} = 0 \) for \( \delta x^\alpha = \delta x^4 \), \( \Gamma^k_{jk} = 0 \) and \( \delta x^\alpha = 0 \) after taking derivatives then

\[
p_i \delta p^i \psi = \frac{1}{2} \hbar^2 g^{ij} R_{ij} \psi.
\] (32)

and \( p_i \delta p^i \psi = 0, \delta p_i \delta p^i \psi = 0 \). Using equations (20, 27, 32), we get

\[
g^{ij} \nabla_i \nabla_j \psi = \left\{ \frac{1}{4} R - \left( \frac{mc}{\hbar} \right)^2 \right\} \psi.
\] (33)

Thus, we have obtained Klein-Gordon equation in Schrodinger form.

**IV. APPROXIMATE SOLUTIONS.**

Assuming that the metrics of space-time is almost Galileo’s one, we can rewrite equation (33) in single dimensional approximation for brane as

\[
\left\{ (g^{11}_0 + \hbar^{11}) \frac{\partial^2}{\partial x^2} + 2h^{10} \frac{\partial^2}{\partial x \partial t} + a \right\} \psi = -\left( g^{00}_0 + \hbar^{00} \right) \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}
\] (34)
where
\[
a = \left(\frac{mc}{\hbar}\right)^2 - \frac{1}{4}(g_{01}^1 + h^{11})R_{11},
\]
(35)

Taking \(g_{00}^0 = 1, g_{01}^1 = -1, h^{11} = h, h^{10} = h^{00} = 0\), we get
\[
(1 - h) \frac{\partial^2 \psi}{\partial x^2} - a \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}.
\]
(36)

Looking for the solution in the form of plane wave
\[
\psi = A e^{i(kx + \omega t)},
\]
(37)
we obtain the following dispersion equation:
\[
(1 - h)k^2 + a - \frac{\omega^2}{c^2} = 0.
\]
(38)

It has a solution
\[
k = \sqrt{\frac{\omega - a}{1 - h}}.
\]
(39)

Assuming that \(h\) and \(a\) are small, we approximately get
\[
k = \frac{\omega}{c} + \frac{1}{2c} \left( \frac{\omega h - \frac{ac^2}{2\omega}}{1 - h} \right).
\]
(40)

So, we have for the frequency shift
\[
\Delta \omega = \omega - \omega_0 = \frac{1}{2} \omega h - \frac{1}{4} \frac{ac^2}{\omega}.
\]
(41)

Using expressions (35, 41), we get
\[
\Delta \omega = \frac{1}{2} \omega h + \frac{1}{4} \frac{c^2}{\omega} \left( \frac{1}{4} R_{11} - \left(\frac{mc}{\hbar}\right)^2 \right).
\]
(42)

For a photon at \(m = R_{11} = 0\) we have
\[
\Delta \omega = \frac{1}{2} \omega h.
\]
(43)
that is usual gravitational shift.

Also, we get from the formula (41) the expression for the group speed of a particle
\[
\frac{\partial \omega}{\partial k} = \frac{c}{1 - \frac{c^2}{4\omega} \left( \frac{1}{4} R_{11} - \left(\frac{mc}{\hbar}\right)^2 \right)}
\]
(44)

For a photon in flat space, we have \(\frac{\partial \omega}{\partial k} = c\) or ordinary group speed of light.

\[\text{V. SQUARED DIRAC-FOK-IVANENKO EQUATION.}\]

We have derived Klein-Gordon equation for a brane. Solutions of Klein-Gordon equation have global gauge symmetry that is connected with conservation of chiral current [2]. Thus, brane had chiral symmetry after Big Bang. But then, spontaneous symmetry breaking had occurred and we must use Dirac-Fok-Ivanenko equation [1] after it that do not provide conservation of chirality for massive particles. We shall compare Klein-Gordon equation and squared Dirac-Fok-Ivanenko equation. Let’s consider Dirac-Fok-Ivanenko equation
\[
i \gamma^i (\nabla_i + \Gamma_i) \psi = m \psi.
\]
(45)
with Dirac gamma matrices $\gamma^i$ and spin connection $\Gamma_j$. Squaring of this equation yields

$$\left(\gamma^i \gamma^j \nabla_i \nabla_j + \left\{ \gamma^i (\nabla_i \gamma^j) + \gamma^i (\Gamma_j \gamma^j) \right\} \nabla_j + \gamma^i \nabla_i (\gamma^j \Gamma_j) + (\gamma^i \Gamma_i)(\gamma^j \Gamma_j) \right) \psi = -m^2 c. \tag{46}$$

If

$$\frac{1}{2} \{ \gamma^i, \gamma^j \} = g^{ij}, \tag{47}$$

$$\gamma^i(\partial_i \gamma^j) + \{ (\gamma^i \Gamma_i), \gamma^j \} = 0, \tag{48}$$

and

$$\gamma^i \nabla_i (\gamma^j \Gamma_j) + (\gamma^i \Gamma_i)(\gamma^j \Gamma_j) = -\frac{1}{4} R, \tag{49}$$

we come Klein-Gordon equation (34).

Relations (47,48) coincide with that of [14]. Taking into account expression from [6]

$$\partial_i \gamma^j + [\Gamma_i, \gamma^j] + \Gamma^k_{ij} \gamma^k = 0, \tag{50}$$

we come as in [14] from (47) and (48) to the following relation between spin connection and Christoffel symbol

$$\Gamma^j = \frac{1}{2} g^{ik} \Gamma^j_{ik}, \tag{51}$$

Let’s consider relation (49). Using (50) again we get

$$\gamma^i \gamma^j \nabla_i \Gamma_j + \gamma^i \gamma^j \Gamma_j \Gamma_j = -\frac{1}{4} R, \tag{52}$$

where $\nabla_i \Gamma_j = \partial_i \Gamma_j - \Gamma^k_{ij} \Gamma_k$ is covariant derivative. Therefore we obtain that

$$\gamma^i \gamma^j D_i \Gamma_j = -\frac{1}{4} R, \tag{53}$$

where $D_i = \nabla_i + \Gamma_i$ is generalized covariant derivative.

VI. CHIRAL CURRENT.

Let’s consider the first term in (53). It can be decomposed into commutator and anticommutator parts

$$\gamma^i \gamma^j D_i \Gamma_j = \frac{1}{2} \{ \gamma^i, \gamma^j \} D_i \Gamma_j + \frac{1}{2} [\gamma^i, \gamma^j] \left( \frac{1}{2} (D_i \Gamma_j + D_j \Gamma_i) + \frac{1}{2} (D_i \Gamma_j - D_j \Gamma_i) \right). \tag{54}$$

It can be rewritten as

$$\gamma^i \gamma^j D_i \Gamma_j = g^{ij} D_i \Gamma_j + \frac{1}{4} [\gamma^i, \gamma^j] \left( \nabla_i \Gamma_j - \nabla_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i \right). \tag{55}$$

Introducing spin curvature

$$\Phi_{ij} = \nabla_i \Gamma_j - \nabla_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = D_i \Gamma_j - D_j \Gamma_i \tag{56}$$

we get from (53) the expression

$$[\gamma^i, \gamma^j] \Phi_{ij} = -4 g^{ij} D_i \Gamma_j - R. \tag{57}$$

that can be used for determination of $[\gamma^i, \gamma^j]$.

Another variant of this formula is derived as following. Spin curvature can be expressed as [6]
Substitution of (58) into (57) gives
\[ \Phi_{ij} = -\frac{1}{8}[\gamma^i, \gamma^j]R_{ijkl}. \] (58)

Using symmetry properties of metric tensor and Bianchi identity we get
\[ \gamma^i \gamma^j \gamma^k \gamma^l R_{tdjk} = 4g^{ij}D_i \Gamma_j, \] (60)
The left hand side of (60) can be rewritten as
\[ \gamma^i \gamma^j \gamma^k \gamma^l R_{tdjk} = \delta_{klmn}^t \gamma^i \gamma^m \gamma^n R_{isjp}, \] (61)
where tensor \( \delta_{klmn}^t \) is the generalized Kronecker symbol. Further, we use the identity
\[ \delta_{klmn}^t = \frac{1}{4!} \varepsilon_{klmn} \gamma^k \gamma^l \gamma^m \gamma^n. \] (62)

Eventually, we get from (60) and (63) the following expression for:
\[ \gamma^5 = 4i \tilde{R}^{-1} D_i \Gamma_j, \] (66)
or
\[ \gamma^5 = 4i \tilde{R}^{-1} D_i \Gamma_j. \] (67)
\( \gamma^5 \) can be used for the determination of chiral spin current according to the formula
\[ \gamma^k 5 = \overline{\psi} \gamma^k \gamma^5 \psi, \] (68)
where \( \psi \) is spinor wave function. Substitution of (66) into (68) yields
\[ j^k 5 = 4i \overline{\psi} \gamma^k \tilde{R}^{-1} D_i \Gamma^i \psi. \] (69)
Compare it with the expression for topologically nontrivial vacuum current of abnormal parity
\[ j^k = \frac{1}{8\pi} \epsilon^k ij EF_{ij}. \] (70)
where \( F_{ij} \) is electromagnetic field strength, in the theory of fractional Hall effect [14]. Both formulas are relativistic, and in both cases current emerge due spin-orbit interaction in nontrivial field topology. The difference is that whereas, in the second case, we deal with a magnetic field, in the first case, we have gravitation affine gauge field
\[ G_{ij} = \nabla_i G_j + \frac{1}{2} g_{\beta \rho} \Gamma^\beta_{ij} g^{\beta \iota} \Gamma_{\iota j}^\rho. \] (71)
corresponding by its structure to that of works [15, 16] and yielding the chiral current
\[ j^k 5 = i \overline{\psi} \gamma^5 \tilde{R}^{-1} g^{ij} (G_{ij} - G_{ji}) \psi. \] (72)
in its symmetric form.
VII. CONCLUSION

Thus, we have derived Klein-Gordon equation for a particle on brane using variation principle. Solutions of Klein-Gordon equation have chiral symmetry and thus brane after its emergence due Big Bang supported chiral symmetry. But then this symmetry was spontaneously broken and we must deal with Dirac equation. It can be verified that the Dirac decomposition of obtained Klein-Gordon equation yields Dirac-Fock-Ivanenko equation that can be solved with Hilbert-Einstein equation. Squaring Dirac-Fock-Ivanenko equation gives wave equation [2], [3], [17] coinciding with that obtained in the present paper.

Solution of modified Dirac-Fock-Ivanenko equation for a photon on the background of almost Galileo’s metrics yields the changes in frequency and speed of particle’s wave packet that can be verified experimentally.

We have derived the expression for gamma five matrix through the derivative of spin connection by comparison of squared DFI equation with KG equation and used it for determination of chiral spin current. This chiral spin current is anomalous spin current corresponding to spontaneous chiral symmetry breaking of mass particle in the space of KG equation solutions. Derived formula for chiral spin current in affine gravitational gauge field has the structure analogous to that of anomalous Hall current [14] and topological axial current in dense matter induced by magnetic flux [17].

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