GEOMETRIC DEFORMATIONS OF CURVES IN THE MINKOWSKI PLANE

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Abstract

In this paper, we extend the method developed in [17, 18] to curves in the Minkowski plane. The method proposes a way to study deformations of plane curves taking into consideration their geometry as well as their singularities. We deal in detail with all local phenomena that occur generically in 2-parameters families of curves. In each case, we obtain the geometry of the deformed curve, that is, information about inflections, vertices and lightlike points. We also obtain the behavior of the evolute/caustic of a curve at especial points and the bifurcations that can occur when the curve is deformed.

1. Introduction

Consider the orthogonal projections of a curve $C$ in $\mathbb{R}^3$ to planes. The $A$-classification of the singularities of such projections was obtained by David in [8]. The equivalence relation by the group $A$ is very important when we study singularities of projections of a space curve. However, diffeomorphisms destroy the geometry of the projected curve. In [9, 15, 22] is studied the geometry of a germ or multi-germ of a curve together with its contact with lines.

However it is also of interest to study the geometry of curves that are deformed. This lead to the following question [18]: is there a deformations theory which takes into account the deformations of singularity of a plane curve as well as its geometry? This question is still an open problem. In [17, 18], Salarinoghabi and Tari proposed a method to study such deformations of curves in the Euclidian plane. They named such method Flat and Round Singularity theory for plane curves (FRS). See also [22] for an alternative approach using invariants.

In this paper, we extend the method introduced in [17, 18] to deal with curves in the Minkowski plane. We study all the phenomena of codimension less or equal to 2. In section 2 we present some preliminary concepts about the Minkowski plane; in section 3 we study the stable behavior of the evolute/caustic; in section 4 we define the concepts of $FRLS$-equivalence and $FRLS$-genericity (F for flat, R for round, L for lightlike and S for singular). In the rest of this paper we study the geometric deformations of plane curve at a vertex of order 2, an inflection of order 2 and 3 (section 5), a lightlike ordinary inflection and a lightlike inflection of order 2 (section 6), a non-lightlike ordinary cusp

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(section 7), a lightlike ordinary cusp (section 8) and a non-lightlike ramphoid cusp (section 9).

2. Curves in the Minkowski plane

The Minkowski plane, denoted by \( \mathbb{R}^2 \), is the plane \( \mathbb{R}^2 \) endowed with the metric induced by the pseudo scalar product \( \langle u, v \rangle = -u_1v_1 + u_2v_2 \), where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). We say that a non-zero vector \( u \in \mathbb{R}^2_1 \) is spacelike if \( \langle u, u \rangle > 0 \), lightlike if \( \langle u, u \rangle = 0 \) and timelike if \( \langle u, u \rangle < 0 \). Given a vector \( u = (u_1, u_2) \in \mathbb{R}^2_1 \), we denote by \( u^\perp \) the vector given by \( u^\perp = (u_2, u_1) \) which is orthogonal to \( u \). If \( u \) is lightlike, then \( u^\perp = \pm u \) and if \( u \) is spacelike (resp. timelike), then \( u^\perp \) is timelike (resp. spacelike).

Let \( \gamma : I \to \mathbb{R}^2_1 \) be a smooth and regular curve. We say that \( \gamma \) is spacelike (resp. timelike) if \( \gamma'(t) \) is a spacelike (resp. timelike) vector for all \( t \in I \). A point \( \gamma(t) \) is called a lightlike point if \( \gamma'(t) \) is a lightlike vector.

We denote by \( T(t) \) the unit tangent vector of \( \gamma \) at a non-lightlike point \( t \in I \) and consider \( N(t) \) the unit normal vector of \( \gamma \) at \( t \in I \) such that \( \{ T(t), N(t) \} \) is a positively oriented basis of \( \mathbb{R}^2_1 \), that is,

\[
T(t) = \frac{\gamma'(t)}{\| \gamma'(t) \|} \quad \text{and} \quad N(t) = \frac{\pm \gamma'(t)^\perp}{\| \gamma'(t)^\perp \|},
\]

where we use + if \( \gamma(t) \) is timelike and - if \( \gamma(t) \) is spacelike. Note that if \( \gamma(t) \) is spacelike (resp. timelike), then \( N(t) \) is timelike (resp. spacelike).

The curvature function \( \kappa : I \to \mathbb{R}^2_1 \) of \( \gamma \) at \( t \) is given by

\[
\kappa(t) = \frac{\langle \gamma''(t), \gamma'(t)^\perp \rangle}{\| \langle \gamma'(t), \gamma'(t) \rangle \|^2}.
\]

We say that a point \( \gamma(t_0) \) is a vertex of \( \gamma \) if \( \kappa'(t_0) = 0 \). Moreover, a vertex is called an ordinary vertex if \( \kappa'(t_0) = 0 \) and \( \kappa''(t_0) \neq 0 \) and a vertex of order \( k \) (with \( k \geq 2 \)) if \( \kappa'(t_0) = \ldots = \kappa^{(k)}(t_0) = 0 \) and \( \kappa^{(k+1)}(t_0) \neq 0 \). A point \( \gamma(t_0) \) is called an inflection of \( \gamma \) if \( \kappa(t_0) = 0 \). Moreover, an inflection is called an ordinary inflection if \( \kappa(t_0) = 0 \) and \( \kappa'(t_0) \neq 0 \) and an inflection of order \( k \) (with \( k \geq 2 \)) if \( \kappa(t_0) = \ldots = \kappa^{(k-1)}(t_0) = 0 \) and \( \kappa^{(k)}(t_0) \neq 0 \).

The contact of \( \gamma \) with lines orthogonal to a non-zero vector \( v \in \mathbb{R}^2_1 \) is captured by the singularities of the height function \( h_v : I \to \mathbb{R} \) given by \( h_v(t) = \langle \gamma(t), v \rangle \). We say that a curve \( \gamma \) has an \( A_k \)-contact (resp. \( A_{\geq k} \)-contact) with the line \( l_v(t_0) = \{ p \in \mathbb{R}^2_1 ; \langle p, v \rangle = \langle \gamma(t_0), v \rangle \} \) at \( t_0 \) if \( h_v \) has an \( A_k \)-singularity (resp. \( A_{\geq k} \)-singularity) at \( t_0 \).

Using the contact of \( \gamma \) with its tangent line, we can define an inflection in a general way (including at lightlike points) as follows. A point \( t_0 \) is called an inflection if \( \gamma \) has an \( A_{\geq 2} \)-contact with its tangent line at \( t_0 \). Moreover, an inflection is called an ordinary inflection (resp. lightlike ordinary inflection) if the contact is of order \( 2 \) and \( t_0 \) is not (resp. is) a lightlike point and an inflection of order \( k \) (resp. lightlike inflection of order \( k \)), with \( k \geq 2 \), if the contact is of order \( k + 1 \) and \( t_0 \) is not (resp. is) a lightlike point. The two definitions are equivalents at non-lightlike points.
The evolute of $\gamma$ at a non-lightlike point and away from inflections is the curve in $\mathbb{R}^2_1$ parametrized by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)} N(t).$$

**Proposition 1.** Let $\gamma : I \to \mathbb{R}^2_1$ be a smooth and regular curve and let $t_0 \in I$ be a point which is neither a lightlike point nor an inflection point. Then, the evolute of $\gamma$ lies locally in the half-plane determined by the tangent line of $\gamma$ at $t_0$ which does not contain the curve $\gamma$.

**Proof.** It follows similarly to that of curves in the Euclidean case ([21], p. 47). $\square$

The family of distance squared function on a smooth and regular curve $\gamma : I \to \mathbb{R}^2_1$ is the family of maps $d : I \times \mathbb{R}^2_1 \to \mathbb{R}$ given by $d(t, u) = \langle \gamma(t) - u, \gamma(t) - u \rangle$. The caustic of $\gamma$ is the bifurcation set of the family $d$, that is

$$Bif(d) = \{ u \in \mathbb{R}^2_1; \exists t \in I \text{ such that } d'_u(t) = d''_u(t) = 0 \}$$

and is a locus of centers of pseudo circles which have an $A_{>2}$-contact with $\gamma$. For more details about caustics and its singularities see, for example, [1, 2, 19, 20, 23].

We have

$$Bif(d) = \{ \gamma(t) - \lambda \gamma'(t)^\perp; \text{ such that } \lambda \langle \gamma'(t)^\perp, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle = 0, t \in I, \lambda \in \mathbb{R} \}.$$

The caustic of $\gamma$ is well defined at lightlike and singular points and coincide with the evolute away from such points ([12, 20]).

### 3. Stable configurations of the caustic

The stable phenomena on curves in the Minkowski plane are lightlike points, ordinary vertices and inflections. In this section we describe the configurations of the caustic at such points. All figures are draw in blue and red, blue for the curve and red for its caustic. Moreover, we represent lightlike points, vertices and inflections, respectively, by stars, discs and squares.

We start with lightlike points. For this we consider $\Omega$ the set of smooth and regular curves $\gamma : S^1 \to \mathbb{R}^2_1$ which satisfies $\langle \gamma''(t), \gamma'(t) \rangle \neq 0$ when $\langle \gamma'(t), \gamma'(t) \rangle = 0$, that is, their lightlike points are not lightlike inflections.

**Proposition 2.** ([20]) Let $\gamma \in \Omega$. Then the caustic of $\gamma$ is a regular curve at the lightlike point of $\gamma$ and it has an ordinary tangency with $\gamma$ at such points. Moreover, $\gamma$ and its caustic lie locally at opposite sides of the common tangent line at the lightlike point.

The following result gives the configuration of the evolute at an ordinary vertex.

**Proposition 3.** Let $\gamma : I \to \mathbb{R}^2_1$ be a smooth and regular curve. Suppose $t_0 \in I$ is an ordinary vertex of $\gamma$ which is not an inflection. Then, locally at $t_0$, the configuration of $\gamma$ and its evolute is as in Figure 1 left if $\kappa(t_0)\kappa''(t_0) < 0$ and right if $\kappa(t_0)\kappa''(t_0) > 0$. 
Proof. We take $\gamma$ parametrized by arc-length. Differentiating the parametrization $e(t) = \gamma(t) - \frac{1}{\kappa(t)}N(t)$ of the evolute three times and evaluating at $t_0$ we get 

\[ e''(t_0) = \kappa''(t_0) N(t_0) \quad \text{and} \quad e'''(t_0) = 2\kappa''(t_0) T(t_0) + \kappa'''(t_0) \frac{N(t_0)}{2}. \]

Hence, in the coordinate system with basis $\{T(t_0), N(t_0)\}$, we can write the 3-jet of $e(t) - e(t_0)$ at $t_0$ in the form

\[ j^3(e(t) - e(t_0)) = (\kappa''(t_0)(t - t_0)^3, \kappa''(t_0)2(t - t_0)^2 + \kappa'''(t_0)6(t - t_0)^3). \]

We have two possibilities at the cusp of the evolute: turning towards the curve or away from it (see Figure 1). The cusp of the evolute turns towards (resp. away from) the curve when $\kappa(t_0)\kappa''(t_0) > 0$ (resp. $\kappa(t_0)\kappa''(t_0) < 0$).

\[ \square \]

Definition 1. An ordinary vertex $t_0 \in I$ is called an inward vertex if $\kappa(t_0)\kappa''(t_0) > 0$ and an outward vertex if $\kappa(t_0)\kappa''(t_0) < 0$.

The next result gives the configuration of the evolute at an ordinary inflection.

Proposition 4. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a smooth and regular curve. Suppose that $\gamma$ has an ordinary inflection at $t_0$. Then the evolute of $\gamma$ tends to infinity asymptotically along the normal to $\gamma$ at $t_0$ and can be modeled by $xy = 1$ in some coordinate system. The two components of the evolute lie in the two quadrants determined by the tangent and normal lines of $\gamma$ at $t_0$ which does not contain the curve $\gamma$, see Figure 2.

Proof. We consider the case when $\gamma$ is a timelike curve, the spacelike case is analogous and is omitted. We can suppose that $t_0 = 0$ and $\gamma(t) = (t, f(t)) = (t, c_2t^2 + c_3t^3 + O(t^4))$. As $t_0 = 0$ is an ordinary inflection of $\gamma$, we have $\kappa(0) = 0$ and $\kappa'(0) \neq 0$, that is, $c_2 = 0$ and $c_3 \neq 0$. Thus, the expression of the evolute is

\[ e(t) = (\frac{1}{2}t + O(t^2), \frac{1}{6c_3t}(-1 + O(t))), \]

and the results follows. \[ \square \]
We present a method for studying the geometry of deformations of plane curves in the Minkowski plane, which we denote by FRLS-deformations (F for Flat, R for Round, L for Lightlike and S for Singular). The method is an extension of that in [18]; here we need to consider, additionally, the lightlike points.

**Definition 2.** (Compare with Definition 1.1 in [18]) Consider two germs of \( m \)-parameter deformations \( \gamma_s \) and \( \alpha_u \) of a curve in \( \mathbb{R}^2 \) and consider the parameter space of \( \gamma_s \) endowed with a stratification \( S_1 \) such that if \( s_1 \) and \( s_2 \) belong to the same stratum, then \( \gamma_{s_1} \) and \( \gamma_{s_2} \)

(i) are diffeomorphics;
(ii) have the same number of inflections, vertices and lightlike points;
(iii) have the same relative position of their singularities, self-intersection points, inflections, vertices and lightlike points.

Consider also the parameter space of \( \alpha_u \) endowed with another stratification \( S_2 \) satisfying properties (i)-(iii). We say that \( \gamma_s \) and \( \alpha_u \) are FRLS-equivalents if there is a germ of homeomorphism \( k : \mathbb{R}^m, (S_1, 0) \rightarrow \mathbb{R}^m, (S_2, 0) \) such that \( \gamma_{s(k)} \) satisfies (i)-(iii).

We can define the notion of FRLS-equivalence of deformations of a plane curve with different number of parameters. We say that an \( m \)-parameter deformation \( \gamma_s \) and an \( n \)-parameter deformation \( \alpha_u \) (say \( n \geq m \)) of a plane curve are FRLS-equivalent if \( \tilde{\gamma}_s \) and \( \alpha_{u} \) are FRLS-equivalents in the above sense, where \( \tilde{\gamma}_s \) is an \( n \)-parameter deformation given by \( \tilde{\gamma}_s(t, s_1, ..., s_m, \bar{s}_{m+1}, ..., \bar{s}_n) = \gamma_s(t, s_1, ..., s_m) \).

In each case that we deal with, the stratification of the parameter space is obtained from a stratification in the jet or multi-jet space. The stratification of the jet or multi-jet space is given by equations which determine each phenomenon. The notation that we use for local strata are \( I(k), V(k), LI(k) \) for inflections, vertices and lightlike inflections of order \( k \), respectively, (or \( I, V, LI \) for the ordinary case), \( L \) for lightlike points, \( C \) for cusps, \( LC \) for lightlike cusps and \( RC \) for ramphoid cusps. The multi-local strata
are denoted by $IT, VT, LT$ for points of transverse intersection between two branches where one of them is an inflection, a vertex, a lightlike point, respectively, and $Tc$ (Tacnode) for points of tangential intersection between two branches. To obtain such stratification, we use the Monge-Taylor map, which we define as follows.

Let $\gamma(t) = (\alpha(t), \beta(t))$ be a plane curve. At each point $t_0$ in the source of $\gamma$ we write

$$j^k_{t_0} \gamma(t) = (\alpha(t_0) + \alpha'(t_0)(t - t_0) + \frac{1}{2!}\alpha''(t_0)(t - t_0)^2 + \frac{1}{k!}\alpha^{(k)}(t_0)(t - t_0)^k;$$

$$\beta(t_0) + \beta'(t_0)(t - t_0) + \frac{1}{2!}\beta''(t_0)(t - t_0)^2 + \frac{1}{k!}\beta^{(k)}(t_0)(t - t_0)^k).$$

We take $\gamma(t_0) = (\alpha(t_0), \beta(t_0)) = (0, 0)$. We define the Monge-Taylor map as the map $j^k_\gamma : \mathbb{R}, 0 \rightarrow J^k(1, 2)$ given by $j^k_\gamma(t) = (j^k_\alpha, j^k_\beta).$ Identifying $J^k(1, 2)$ with $\mathbb{R}^k \times \mathbb{R}^k$, we write

$$(1) \quad j^k_\gamma(t) = (\alpha'(t), \frac{1}{2!}\alpha''(t), \ldots, \frac{1}{k!}\alpha^{(k)}(t); \beta'(t), \frac{1}{2!}\beta''(t), \ldots, \frac{1}{k!}\beta^{(k)}(t)).$$

We can simplify the map $j^k_\gamma$ depending on the case in consideration:

1. If $t = 0$ is a regular but not a lightlike point of $\gamma$, then we take $\gamma(t) = (t, \beta(t))$, with $\beta'(0) = 0$, for the timelike case and $\gamma(t) = (\alpha(t), t)$, with $\alpha'(0) = 0$, for the spacelike case. The Monge-Taylor is taken as the map $j^k_\gamma : \mathbb{R}, 0 \rightarrow J^k(1, 1)$ given by

   $$(2) \quad j^k_\gamma(t) = \frac{1}{2!}\beta''(t), \ldots, \frac{1}{k!}\beta^{(k)}(t), \quad \text{if } \gamma \text{ is timelike}$$

   $$(3) \quad j^k_\gamma(t) = \frac{1}{2!}\alpha''(t), \ldots, \frac{1}{k!}\alpha^{(k)}(t), \quad \text{if } \gamma \text{ is spacelike}$$

2. If $t = 0$ is a lightlike regular point of $\gamma$ we can take $\gamma(t) = (t, \beta(t))$, with $\beta'(0) = \pm 1$. In this case, the Monge-Taylor map is taken as the map $j^k_\gamma : \mathbb{R}, 0 \rightarrow J^k(1, 1)$ given by

   $$(3) \quad j^k_\gamma(t) = (\beta'(t), \frac{1}{2!}\beta''(t), \ldots, \frac{1}{k!}\beta^{(k)}(t)).$$

3. If $\gamma$ is singular at $t = 0$, then the Monge-Taylor map is taken as in (1).

The stratification in the space $2J^k(1, 2) \subset J^k(1, 2) \times J^k(1, 2)$ of bi-jets (see [13], p. 47) is used when $\gamma_s$ has self-intersection points. We define the Monge-Taylor bi-jet map as the map $j^k_\gamma : \mathbb{R} \times \mathbb{R}, (0, 0) \rightarrow 2J^k(1, 2)$ given by

$$(4) \quad j^k_\gamma(t_1, t_2) = ((\alpha(t_1), \alpha'(t_1), \ldots, \frac{1}{k!}\alpha^{(k)}(t_1); \beta(t_1), \beta'(t_1), \ldots, \frac{1}{k!}\beta^{(k)}(t_1),$$

We can define in the analogous way the Monge-Taylor n-jet map, for $n \geq 3$.

**Definition 3.** We say that a germ of an $m$-parameter deformation $\gamma_s$ of a plane curve $\gamma = \gamma_0$ is an FRLS-generic deformation if the family of Monge-Taylor maps induced by $\gamma_s$ is transverse to the stratification of the jet space determined by the $\mathcal{A}$-invariant strata and the geometric strata $I(k), V(k), LI(k)$, and their intersections.
Theorem 5. In $J^n(1,2)$, the stratum of lightlike inflections of order $k$ is contained in the stratum of vertices of order $2k$, that is, $LI(k) \subset V(2k)$, for any integer $k \geq 1$. Moreover, $2k$ is the largest integer with such property, that is, $LI(k) \subset V(2k) \setminus V(2k+1)$.

Proof. Let $\gamma(t) = (t, f(t))$ with a lightlike inflection of order $k$ at $t_0 = 0$. Consider $\gamma_s$ an $m$-parameter deformation of $\gamma = \gamma_0$ which we can write in the form $\gamma_s(t) = (t, f_s(t))$, and let $g_s(t) = f_s'''(t)(1 - f_s'(t)^2) + 3f_s''(t)f_s''(t)^2$ be the numerator of $\kappa_s$. Then,

$$LI(k) : \quad a_1 = 1, a_2 = \cdots = a_{k+1} = 0$$

$$V(2k) : \quad g_s = g'_s = \cdots = g_{s(2k-1)} = 0$$

with $a_i = f_s^{(i)}(0)/(i!)$.

The proof follows by induction on $k \geq 1$. For $k = 1$, we have $LI \subset V(2)$.

Suppose $LI(k) \subset V(2k)$. Considering $a_1 = 1$ and $a_2 = \cdots = a_{k+2} = 0$, then clearly we have $g_s = \cdots = g_{s(2k-1)} = 0$. To show that $g_{s(2k)} = g_{s(2k+1)} = 0$ we use the following formula obtained differentiating $g_s$ successively

$$g_s^{(n)} = (n + 3)!a_{n+3}$$

$$- \sum_{i=0}^{n} \binom{n}{i} (i + 3)!(a_i) \sum_{l=0}^{n-i} \binom{n-i}{l} (n-i-l+1)!(l+1)!a_{n-i-l+1}a_{l+1}$$

$$+ 3 \sum_{j=0}^{n} \binom{n-j}{j} (n-j+1)!a_{n-j+1} \sum_{l=0}^{j} \binom{j}{l} (j-l+2)!(l+2)!a_{j-l+2}a_{l+2}.$$  \hspace{1cm} (5)

For $n = 2k$ in (5) we get $g_{s(2k)} = (2k + 3)!a_{2k+3} - (2k + 3)!a_{2k+3} = 0$. For $n = 2k+1$ we get $g_{s(2k+1)} = (2k + 4)!a_{2k+4} - (2k + 4)!a_{2k+4} = 0$. Therefore, $LI(k+1) \subset V(2(k+1))$.

To prove that $LI(k) \subset V(2k) \setminus V(2k+1)$, as $LI(k) \subset V(2k)$, it is enough to show that $g_{s(2k)} \neq 0$ when $a_1 = 1, a_2 = \cdots = a_{k+1} = 0$ and $a_{k+2} \neq 0$. Setting $n = 2k$ in (5) we get $g_{s(2k)} = (2k)!((k+2)^2(k+1)(k+3)a_{k+2}^2$. As $a_{k+2} \neq 0$, we have $g_{s(2k)} \neq 0$.  \hspace{1cm} □

5. Geometric deformations of regular curves at non-lightlike points

Here, the Monge-Taylor map does not intersect the strata involving singularities and lightlike points, so an FRLS-generic family (see Definition 3) will be called FR-generic (we drop the letters S and L). We study FR-generic deformations of a plane curve at an inflection of finite order and the bifurcations in the curve and in its evolute at an inflection of order 2 and 3.

Away from inflections, the deformations of a regular curve at a vertex of finite order can be studied using the family of distance squared functions. If the family is an $\mathcal{R}$-versal deformation, we get the deformations of the evolute whose singularities gives the vertices of the curve (see [2, 5]).

The deformations of a regular curve at an inflection of finite order can be studied using the family of height functions. If the family is an $\mathcal{R}$-versal deformation, then we have a model of its discriminant and consequently a model for the dual curve whose singularities give the inflections (see [5]). However, if the inflection is of order $k \geq 2$, then $\kappa = \kappa' = 0$ at that point, so there is a vertex concentrated at the inflection. In
this way, when we deform an inflection of order \( \geq 2 \) we also need to consider how the vertices appear on the deformed curve. Before that, we need the following result.

**Proposition 6.** Let \( \gamma \) be a smooth and regular curve. Suppose that \( t_0 \) is a vertex of order 2 which is not an inflection. Then the evolute of \( \gamma \) has an \( E_6 \) singularity at \( t_0 \) (see Figure 3 center). Moreover, if \( \gamma_s \) is a generic 1-parameter family with \( \gamma_0 = \gamma \), then the evolute of \( \gamma_s \) undergoes the swallowtail transitions as shown in Figure 3. We have an inward and an outward vertices on one side of the transition and none on the other.

![Figure 3. Bifurcations of the evolute at a vertex of order 2.](image)

**Proof.** It is shown in [16] that the evolute of \( \gamma \) has an \( E_6 \) singularity at \( t_0 \), that is, \( \gamma \) is \( A \)-equivalent to \((t^3, t^4)\). As \( \gamma_s \) is a generic family, it follows that the family of distance squared function \( d \) of \( \gamma_s \) is an 3-parameter \( R^+ \)-versal deformation of the \( A_1 \) singularity of \( d_0 \). Hence the bifurcation set of the family is a swallowtail and the sections are generic. Consequently, the evolute of \( \gamma_s \) corresponds to the generic sections of the swallowtail (see [4]).

The next result gives conditions for a deformation to be FR-generic at an inflection of order \( k \).

**Theorem 7.** Let \( \gamma_s \) be an \( m \)-parameter deformation of a regular curve \( \gamma_0 = \gamma \) which has an inflection of order \( n \geq 2 \) at \( t_0 \). Suppose that \( t_0 \) is not a lightlike point. Then, the family of height functions on \( \gamma_s \) is an \( R^+ \)-versal deformation of the height function on \( \gamma_0 \) at \( t_0 \) if, and only if, the family of Monge-Taylor maps \( j^k \Phi \), \( k \geq n+1 \), is transverse to the strata \( I, I(2), \ldots, I(2n+1), V, V(2), \ldots, V(n) \) at \( j^k \phi_{\gamma_0}(0) \).

**Proof.** We suppose \( \gamma \) timelike at \( t_0 = 0 \) (the spacelike case is analogous) and take \( \gamma = \gamma_0 \) and \( \gamma_s \), respectively, in the form \( \gamma(t) = (t, f(t)) \) and \( \gamma_s(t) = (t, f_s(t)) \), with \( f_s(0) = f'_s(0) = 0, f''_s(0) = \cdots = f^{(n+1)}_s(0) = 0, f^{(n+2)}_s(0) = (n+2)!c_{n+2} \neq 0 \). The stratum of inflections of order \( n \) is given by \( I(n) : a_2 = \cdots = a_{n+1} = 0 \) and the stratum of vertices of order \( n \) is given by \( V(n) : g_s = \cdots = g_s^{(n-1)} = 0 \), where \( g_s(t) = f''_s(t)(1 - f'_s(t)^2) + 3f'_s(t)f''_s(t)^2 \).

It is clear that \( I(n) \subset V(n-1) \). Hence, its enough to verify the transversality of the family of Monge-Taylor maps to the stratum \( I(n) \). Note that the tangent space of the stratum \( I(n) \) in \( J^k(1,1) \) is \( \mathbb{R} \cdot \{ e_1, e_{n+2}, \ldots, e_k \} \), where \( e_i \) is the \( i \)-th vector of the canonical basis of \( \mathbb{R}^k \). Therefore, \( j^k \Phi \) is transversal to the stratum \( I(n) \) if, and only if, \( e_2, \ldots, e_{n+1} \in \text{Im}(dj^k \Phi(0,0)) + \mathbb{R} \cdot \{ e_1, e_{n+2}, \ldots, e_k \} \).
Note that \( j^k \Phi(t, s) = (f'_j(t), \frac{1}{j!} f''_j(t), \ldots, \frac{1}{k!} f^{(k)}_j(t)) \) and consequently the jacobian matrix of \( j^k \Phi \) at \((0, 0)\) is

\[
\begin{pmatrix}
0 & \frac{1}{j!} \frac{\partial^{j+1} f}{\partial t^{j+1}}(0, 0) \\
(l + 1)c_{l+1} & \frac{1}{n!} \frac{\partial^{j+1} f}{\partial s^{j+1}}(0, 0)
\end{pmatrix}_{k \times (m+1)}
\]

with \( j = 1, \ldots, n, l = n + 1, \ldots, k \) and \( i = 1, \ldots, m \). Taking \( u = (\frac{1}{e^{n+1}}, 0, \ldots, 0) \in \mathbb{R}^{m+1} \) we have \( d_j^k \Phi(0, 0)(u) = e_{n+1} + v \), where \( v \in \mathbb{R} \cdot \{e_{n+2}, \ldots, e_k\} \). Thus, \( e_2, \ldots, e_n \in \text{Im}(d_j^k \Phi(0, 0)) + \mathbb{R} \cdot \{e_1, e_{n+2}, \ldots, e_k\} \) if, and only if, the matrix \( (\frac{1}{j!} \frac{\partial^{j+1} f}{\partial t^{j+1}}(0, 0))_{(n-1) \times m} \) has rank \( n - 1 \), which is exactly the condition for \( \mathcal{R}^+ \)-versality of the family of height function on \( \gamma_s \).

**Corollary 8.** A deformation \( \gamma_s \) of a regular curve \( \gamma \) at an inflection of finite order is \( \mathcal{R} \)-generic if, and only if, the family of height functions on \( \gamma_s \) is an \( \mathcal{R}^+ \)-versal deformation of the singularity of the height function on \( \gamma \) along the normal direction.

**Remark 1.** Denote by \( H_{(u, s)} \), \( \kappa_s \) and \( \kappa'_s \), the family of height functions on \( g_s \), the curvature function and the derivatives of the curvature function of \( \gamma_s \), respectively. By direct computations we can show, in the hypothesis of the Theorem 7, that

(i) \( H_{(u, s)} \) is an \( \mathcal{R}^+ \)-versal deformation of \( H_{(u_0, 0)} \) if, and only if, \( \kappa_s \) is an \( \mathcal{R} \)-versal deformation of \( \kappa_0 \).

(ii) If \( \kappa_s \) is an \( \mathcal{R} \)-versal deformation of \( \kappa_0 \), then \( \kappa'_s \) is an \( \mathcal{R} \)-versal deformation of \( \kappa'_0 \).

In the rest of this section we study the \( \mathcal{S} \)-bifurcations of the curve and its evolute at an inflection of order 2 and 3.

**Theorem 9.** Let \( \gamma \) be a regular curve with an inflection of order 2 at \( t_0 \), with \( t_0 \) a non-lightlike point.

(i) The evolute goes to infinity asymptotically along the normal line of \( \gamma \) at \( t_0 \) and can be modeled by \( x^2y = 1 \) in some coordinate system. The two components of the evolute lie in the two quadrants determined by the tangent and normal lines of \( \gamma \) at \( t_0 \) which do not contain the curve \( \gamma \) (see Figure 4 center).

(ii) Let \( \gamma_s \) be an \( \mathcal{S} \)-generic 1-parameter family with \( \gamma_0 = \gamma \). Then the bifurcations in the evolute are as shown in Figure 4. We have two ordinary inflections and an outward vertex on one side of the transition and an inward vertex and no inflections on the other side of the transition.

(iii) Any \( \mathcal{S} \)-generic 1-parameter family of \( \gamma \) is \( \mathcal{S} \)-equivalent to the model family \( \alpha_u(t) = (t, t^4 + ut^2) \).

**Proof.** We deal with the case \( \gamma \) a timelike curve, the spacelike case follows analogously.

(i) As \( t_0 \) is an inflection of order 2 of \( \gamma \), we take \( t_0 = 0 \) and \( \gamma(t) = (t, f(t)) = (t, c_4 t^4 + O(t^5)) \), with \( c_4 \neq 0 \). We have \( e(t) = (\frac{3}{2} t + O(t^2), \frac{1}{12c_4^2}(1 + O(t))) \), which can be modeled by \( x^2y = 1 \) in some coordinate system.
(ii) As $\gamma_s$ is FR-generic, we can take $\gamma_s(t) = (t, f_s(t)) = (t, st^2 + c_3(s)t^3 + (c_4 + c_5(s))t^4 + O_s(t^5))$, where $c_3(0) = 0$, with $i = 3, 4$, (see Corollary 8).

The family $\kappa_s$ is an $R$-versal deformation of the $A_1$ singularity of $\kappa_0$ (Remark 1). Therefore, $\kappa_s$ is equivalent to $t^2 + u$. Then on one side of the transition $\kappa_s$ has a critical point and no zeros, while on the other side it has one critical point between two zeros.

At critical points of $\kappa_s$ on both sides of the transition the sign of $\frac{\partial^2 \kappa}{\partial t^2}(t, s)$ is the same, but the sign of $\kappa_s$ changes. Therefore, on the side of the transition which does not have inflections, the vertex of $\gamma_s$ is inward and on the other side it is outward.

The bifurcations of $\gamma_s$ and of its evolute are as shown in Figure 4 (where we also used Proposition 3 and 4).

(iii) The calculations in (ii) depend only on the fact that the curve has an inflection of order 2 and on the family being an FR-generic deformation. The family $\alpha_u$ satisfies these conditions and can be taken as an FR-model.

\begin{proof}
We deal with the case $\gamma$ a timelike curve, the spacelike case follows analogously.
\end{proof}

**Theorem 10.** Let $\gamma$ be a regular curve with an inflection of order 3 at $t_0$.

(i) The evolute goes to infinity asymptotically along the normal line of $\gamma$ at $t_0$ and can be modeled by $x^3y = 1$ in some coordinate system. The two components of the evolute lie in the two quadrants determined by the tangent and normal lines of $\gamma$ at $t_0$ which do not contain the curve $\gamma$ (see Figure 6).

(ii) Let $\gamma_s$ be an FR-generic 2-parameter family with $\gamma_0 = \gamma$. Then the bifurcations in the evolute are as shown in Figure 6.

(iii) Any FR-generic 2-parameter family of $\gamma$ is FR-equivalent to the model family $\alpha_u(t) = (t, t^5 + u_1t^2 + u_2t^3)$.

\begin{proof}
We deal with the case $\gamma$ a timelike curve, the spacelike case follows analogously.
\end{proof}
(i) As $t_0$ is an inflection of order 3 of $\gamma$, we can take $t_0 = 0$ and $\gamma(t) = (t, f(t)) = (t, c_5 t^5 + O(t^6))$, with $c_5 \neq 0$. We have $e(t) = \left(\frac{3}{4} t + O(t^2), -\frac{1}{20c_5 t^3}(1 + O(t))\right)$, which can be modeled by $x^3 y = 1$ in some coordinate system, see Figure 6.

(ii) As $\gamma_s$ is FR-generic, we can take $\gamma_s(t) = (t, f_s(t)) = (t, s_1 t^2 + s_2 t^3 + c_4(s) t^4 + (c_5 + c_4(s)) t^5 + O_s(t^6))$, where $c_4(0) = 0$, for $i = 4, 5$ and $s = (s_1, s_2)$, (see Corollary 8).

The strata $I(2)$ and $V(2)$ are, respectively, the discriminant and the bifurcation sets of $\kappa_s$. We find that $I(2)$ consists of the germ of the curve $\left(\frac{70}{3} c_5 t^3 + O(t^4), -10 c_5 t^2 + O(t^3)\right)$ with $t \in \mathbb{R}, 0$ and $V(2)$ is given by the germ of the curve $(s_1, \frac{2c_4^2 s_2}{5c_5} + O(s_1^3))$ with $s_1 \in \mathbb{R}, 0$. Thus, the stratification of the parameter space $(s_1, s_2)$ of $\gamma_s$ is as shown in Figure 6 center.

As $\gamma_s$ is FR-generic, by Remark 1, the families $\kappa_s$ and $\kappa'_s$ are $\mathcal{R}$-versal deformations of an $A_2$ and an $A_1$ singularity, respectively. The bifurcations in the graph of the curvature function $\kappa_s$ are as shown in Figure 5, where we indicate the inflections ($\kappa_s = 0$) and vertices ($\kappa'_s = 0$) of $\gamma_s$.

![Figure 5. Bifurcations in the graph of $\kappa_s$. The horizontal line is the zero level.](image)

Using the behavior of the evolute at an inflection of order 3, the number and relative position of vertices and inflections of $\gamma_s$ obtained above and the previous results we conclude that the bifurcations in $\gamma_s$ and its evolute are as shown in Figure 6.

(iii) The computations in (ii) depend only on the fact that the curve has an inflection of order 3 and on the family being an FR-generic deformation. The family $\alpha_u$ satisfies these conditions and can be taken as an FR-model.

6. **Geometric deformations of a regular curve at a lightlike point**

In this section, we deal with lightlike inflections of finite order of regular curves, so an FRLS-generic family (see Definition 3) will be called FRL-generic. We suppose that the
Figure 6. FR-generic deformations of a curve and its evolute at an inflection of order 3. The central figure is the stratification of the parameter space.

Tangent direction is along (1, 1) at the origin and take \( \gamma_s \) in the form \( \gamma_s(t) = (t, f_s(t)) \), with \( \gamma_0 = \gamma \), \( f_0 = f \), \( f(0) = 0 \) and \( f'(0) = 1 \).

When deforming the curve, the lightlike points on \( \gamma_s \) are points of tangency with a line parallel to \( y = x \). To capture this contact, we use the subgroup \( K^* \) of the contact group \( K \) formed by changes of coordinates in the source that preserve parallel lines. In order to simplify the calculations we apply the transformation \( T(x, y) = (x, x - y) \) and consider \( K^* \) preserving horizontal lines. (For more details about the group \( K^* \) and the classification of function germs \( \mathbb{R}^2, 0 \to \mathbb{R}, 0 \) under its action, see [7, 14].)

Applying \( T \) on \( \gamma_s \) gives \( \gamma_s(t) = (t, t - f_s(t)). \) We can consider, locally, \( \gamma_s = G_s^{-1}(0) \), where \( G : \mathbb{R}_1^2 \times \mathbb{R}, (0, 0) \to \mathbb{R} \) is given by \( G(x, y, s) = y - x + f_s(x) \).

The extended tangent space \( L K^*_e \cdot g \), with \( g \in \mathcal{M}_2 \), is defined as \( L K^*_e \cdot g = \mathcal{E}_2 \cdot \{ \frac{\partial y}{\partial x} \} + \mathcal{E}_y \cdot \{ \frac{\partial y}{\partial y} \} + g^*(\mathcal{M}_1) \cdot \mathcal{E}_2 \), where \( \mathcal{E}_y \) denotes the set of germs in \( \mathcal{E}_2 \) which depend only on \( y \). As \( K^* \) is a geometric subgroup of \( K \) ([7]), we have the following result.

**Theorem 11.** ([7]) An \( m \)-parameter deformation \( G \) of a function germ \( g \in \mathcal{M}_2 \) is a \( K^*_e \)-versal deformation if, and only if, \( L K^*_e \cdot g + \mathbb{R} \cdot \{ G_1, \ldots, G_m \} = \mathcal{E}_2 \), where \( G_i \) is the derivative of \( G \) in relation to the \( i \)-th parameter.

Consider \( \gamma \) a regular curve with a lightlike inflection of order \( k - 1 \) at \( t_0 = 0 \), with \( k \geq 2 \), and \( \gamma_s \) an \( m \)-parameter family of regular curves, with \( \gamma_0 = \gamma \).
Theorem 12. Let $\gamma_0$ be a regular curve with a lightlike inflection of order $k - 1$ at $t_0$, with $k \geq 2$, and $\gamma_s$ be an $m$-parameter family of regular curves containing $\gamma_0$, with $m \geq k - 1$. Then, the family $G$ associated to $\gamma_s$ is a $\mathcal{K}_e^*$-versal deformation of $G_0$ at $t_0$ if, and only if, $j^n\Phi$ is transverse to the stratum of lightlike inflections of order $k - 1$ in $J^n(1, 1)$, with $n \geq k$.

Proof. We start by proving the theorem for $\gamma_s$ an $(k - 1)$-parameter family.

The stratum $LI(k - 1)$ is given by $a_1 = 1, a_2 = a_3 = \cdots = a_k = 0$ and its tangent space by $\mathbb{R}\cdot\{e_{k+1}, \ldots, e_n\}$. We have $j^n\Phi(0, 0) = (1, 1, \ldots, 1, c_{k+1}, c_{k+2}, \ldots, c_n)$, with $c_{k+1} \neq 0$, where $\gamma_0(t) = (t, t + c_{k+1}t^{k+1} + c_{k+2}t^{k+2} + O(t^{k+3}))$. Hence, $j^n\Phi$ is transverse to $LI(k - 1)$ if, and only if, $Im(d_{(0, 0)}j^n\Phi) + \mathbb{R}\cdot\{e_{k+1}, \ldots, e_n\} = \mathbb{R}^n$.

Note that the Jacobian matrix of $j^n\Phi$ at $(0, 0)$ is given by

$$Jac(j^n\Phi)(0, 0) = \begin{pmatrix}
0 & \frac{1}{n} \frac{\partial^{j+1}f}{\partial s^{j+1}t^l}(0, 0) \\
(j + 1)c_{j+1} & \frac{1}{j!} \frac{\partial^{j+1}f}{\partial s^{j+1}t^l}(0, 0)
\end{pmatrix}_{n \times k},$$

with $l = 1, \ldots, k - 1$ and $j = k, \ldots, n$. We can show that $e_1, \ldots, e_k \in Im(d_{(0, 0)}j^n\Phi) + \mathbb{R}\cdot\{e_{k+1}, \ldots, e_n\}$ if, and only if, the matrix $\left(\frac{1}{j!} \frac{\partial^{j+1}f}{\partial s^{j+1}t^l}(0, 0)\right)$ has rank $k - 1$, which is equivalent to $\gamma_s$ be a $\mathcal{K}_e^*$-versal deformation of $\gamma_0$.

The proof when $\gamma_s$ is an $m$-parameter family, with $m \geq k - 1$, follows in a similar way. \[\square\]

Note that the stratum $LI(k - 1)$ is contained in the strata $I, I(2), \ldots, I(k - 1), L, LI, LI(2), \ldots, LI(k - 2), V, V(2), \ldots, V(2(k - 1))$. Therefore, if $j^n\Phi$ is transverse to the stratum $IL(k - 1)$ it will also be transverse to these strata.

Corollary 13. A deformation $\gamma_s$ of a regular curve $\gamma$ at a lightlike inflection of finite order is FRL-generic if, and only if, the family $G$ associated to $\gamma_s$ is a $\mathcal{K}_e^*$-versal deformation of $G_0$ at $t_0$.

Theorem 14. Let $\gamma$ be a regular curve with a lightlike ordinary inflection at $t_0$.

(i) The caustic of $\gamma$ is the union of the tangent line of $\gamma$ at $t_0$ and a smooth curve with a lightlike ordinary inflection at $\gamma(t_0)$ which lies in the two quadrants determined by the lightlike lines which do not contain the curve $\gamma$, as shown in Figure 7 center.

(ii) For an FRL-generic 1-parameter family $\gamma_s$ with $\gamma_0 = \gamma$, the bifurcations in the caustic are as shown in Figure 7. We have two lightlike points and an ordinary inflection on one side of the transition and two outward vertices and an ordinary inflection on the other side of the transition.

(iii) Any FRL-generic 1-parameter family of $\gamma$ is FRL-equivalent to the model family $\alpha_u(t) = (t, (1 + u)t + t^3)$.

Proof. (i) This is a consequence of Theorem 3.6 in [16].
(ii) As $\gamma_s$ is an FRL-generic deformation of $\gamma$, we write it in the form $\gamma_s(t) = (t, f_s(t))$ with $f_s(t) = (1 + s)t + c_2(s)t^2 + (c_3 + c_3) t^3 + O_s(t^4)$, where $c_i(0) = 0$ for $i = 2, 3$ and $c_3 \neq 0$.

The family of distance squared function $d$ is an $R^+\text{-versal deformation}$ of an $A_3$ singularity. That implies that the bifurcation set $Bif(d)$ of $d$ is locally diffeomorphic to a cuspidal edge. Hence the individual caustics of $\gamma_s$ are obtained by taking sections of the cuspidal edge (see [2]). One can show that the plane $s = 0$ has a Morse contact with the singular set of $Bif(d)$ and intersects $Bif(d)$ along two tangential curves. Thus the caustics undergoes the beaks transitions. It follows that $\gamma_s$ has two vertices on one side of the transition and none on the other.

To obtain the inflections and lightlike points on $\gamma_s$, we consider a 2-dimensional section of the jet space given by $(a_3, \ldots, a_k) = (c_3, \ldots, c_k)$. In such section, $V$ is a regular curve given by $(1 + \frac{2}{c_3} a_2^2 + O(a_3^3), a_2)$. Moreover, for $s = 0$, we can write the image of the restriction of the Monge-Taylor map to this section in the form $(1 + \frac{1}{3c_3} t^2 + O(t^3), t)$. Comparing such curves we conclude that this section is as described in Figure 8 center.

We conclude that on one side of the transition we have an inflection between two lightlike points and on the other side we have an inflection between two vertices (see Figure 8 center).
Figure 8). Moreover, when \( \gamma_s \) has vertices, the curvature function has a maximum and a minimum between a zero. Consequently, \( \kappa_s \kappa''_s < 0 \) at both critical points of \( \kappa_s \), that is, both vertices of \( \gamma_s \) are outward.

(iii) The computations in (ii) depend only on the fact that the curve has a lightlike ordinary inflection and on the family being an FRL-generic deformation. The family \( \alpha_u \) satisfies these conditions and can be taken as an FRL-model. \( \square \)

**Theorem 15.** Let \( \gamma \) be a regular curve with a lightlike inflection of order 2 at \( t_0 \).

(i) The caustic of \( \gamma \) is the union of the tangent line of \( \gamma \) at \( t_0 \) and a smooth curve with a lightlike inflection of order 2 which lies in the semi-plane determined by the lightlike line that contains the curve \( \gamma \), see Figure 9(1).

(ii) For an FRL-generic 2-parameter family \( \gamma_s \) with \( \gamma_0 = \gamma \), the bifurcations in the caustic are as shown in Figure 9.

(iii) Any FRL-generic 2-parameter family of \( \gamma \) is FRL-equivalent to the model family \( \alpha_u(t) = (t, (1 + u_1)t + u_2t^2 + t^4) \).

**Proof.** As \( t_0 \) is a lightlike inflection of order 2, we take \( \gamma(t) = (t, t + c_4t^4 + c_5t^5 + c_6t^6 + ct^7 + O(t^8)) \), with \( c_4 \neq 0 \), at \( t_0 = 0 \).

(i) The result is a consequence of the Theorem 3.6 in [16].

(ii) First, we analyze the deformations with respect to the lightlike points and inflections. As \( \gamma_s \) is an FRL-generic family, we can write \( \gamma_s(t) = (t, f_s(t)) \), with \( f_s(t) = (1 + s_1)t + s_2t^2 + \overline{c}_3(s)t^3 + (c_4 + c_4(s))t^4 + O(s(t^5)) \) and \( \overline{c}_3(0) = 0 \) for \( i = 3, 4 \). The zeros of the function \( g_s(t) = 1 - f'_s(t) \) give the lightlike points of \( \gamma_s \) and the singular points of \( g_s \) give the inflections of \( \gamma_s \). We show that \( g_s \) is an \( \mathcal{R}^+ \) and \( \mathcal{R} \)-versal deformation of an \( A_2 \) singularity. Thus, we have models for the zero, discriminant, singular and bifurcation sets of \( g_s \). We have

\[
\mathcal{D}_{g_s} : \quad \{(8c_4t^3 + O(t^4), -6c_4t^2 + O(t^3)) \in \mathbb{R}^2; \ t \in \mathbb{R}, 0\}
\]

\[
\text{Bif}(g_s) : \quad \{(s_1, \frac{3c_4^{30}c_7}{4c_4^3} + O(s_1^3)) \in \mathbb{R}^2; \ s_1 \in \mathbb{R}, 0\}.
\]

where \( \mathcal{D}_{g_s} \) denotes the discriminant set of \( g_s \). Note that \( \mathcal{D}_{g_s} \) and \( \text{Bif}(g_s) \) correspond, respectively, to the strata \( LI \) and \( I(2) \) of \( \gamma_s \).

We obtain deformations of the lightlike points and inflections using the deformations in the graph of \( g_s \) (Figure 5).

For the vertices we use the family \( G(t, s_1, s_2) = f'''(t, s_1, s_2)(1 - (f'(t, s_1, s_2))^2) + 3f''(t, s_1, s_2)(f''(t, s_1, s_2))^2 \), whose discriminant set is the stratum \( V(2) \). The function \( G_0 \) has an \( A_3 \) singularity at \( t_0 = 0 \). However, \( G \) is not an \( \mathcal{R} \)-versal deformation of \( G_0 \) at \( t_0 \), it is induced by the \( \mathcal{R} \)-versal deformation \( F(x, u, v, w) = x^4 + ux^2 + vx + w \), that is, there are germs of maps \( a : \mathbb{R} \times \mathbb{R}^2, (0, 0) \to \mathbb{R} \) and \( b : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0 \) such that \( a(t, 0) = \varphi(t) \) is a diffeomorphism and

\[
G(t, s) = F(a(t, s), b(s)) = (a(t, s))^4 + b_1(s)(a(t, s))^2 + b_2(s) + b_3(s),
\]

where \( b = (b_1, b_2, b_3) \). Note that if \( s \) belongs to the discriminant set of \( G \), then \( b(s) \) belongs to the discriminant set of \( F \) which is a swallowtail.
We can set, \( b_1(s_1, s_2) = s_2 \), \( b_2(s_1, s_2) = s_1 \) and we get
\[
b_3(s_1, s_2) = \beta_1 s_1^2 + \beta_2 s_1 s_2 s + \frac{5}{4} s_2^2 + \beta_3 s_2^3 + \beta_4 s_1^2 s_2 + \beta_5 s_1 s_2^2 - \left(\frac{88c_4c_6 - 149c_5^2}{16c_4^2}\right)s_2^3 + O(s_1, s_2).
\]

Therefore, the image of \( b \) is the graph of the function \( b_3(s_2, s_1) \), and is also the zero set of the function \( h(x, y, z) = z - b_3(x, y) \). We can use the classification of submersions from \( \mathbb{R}^3 \) to \( \mathbb{R} \) up to diffeomorphisms in the source which preserve the swallowtail \( X = D_F \) (the \( R(X) \)-classification) obtained in [2, 6, 10] and show that \( h \) is \( R(X) \)-equivalent to
\[
\overline{h}(x, y, z) = z - \frac{5}{4} x^2 + \left(\frac{88c_4c_6 - 149c_5^2}{16c_4^2}\right)x^3.
\]
Thus the intersection of the swallowtail and the fiber $h^{-1}(0)$ is diffeomorphic to
\{(2t^2, -8t^3, 5t^4); t \in \mathbb{R}\} \cup \{(-\frac{2}{3}t^2, -\frac{8}{3}t^3, \frac{2}{3}t^4); t \in \mathbb{R}\}
which consists of two tangential curves at the origin.

From Theorem 5, the parametrization of $LI$ is part of the parametrization of $V(2)$.

Looking the intersection of the fiber of $\overline{h}$ and the swallowtail we obtain the number
of zeros of $G_{(s_1, s_2)}$, that is, the number of vertices in each stratum. We conclude that
the FRL-deformations of $\gamma$ are as shown in Figure 9.

(iii) The statement follows from the fact that $\alpha_u$ is an FRL-generic deformation at a
lightlike inflection of order 2 and the calculations in (ii) depend only on the fact that the
curve has a lightlike inflection of order 2 and on the deformation being FRL-generic. \hfill \Box

7. Geometric deformations of curves with a non-lightlike ordinary cusp

Suppose that $\gamma(t) = (\alpha(t), \beta(t))$ has a cusp singularity at the origin and that the
limiting tangent direction to $g$ at that point is timelike (the spacelike case follows
analogously). We call this singularity a non-lightlike cusp. Here, the Monge-Taylor map
does not intersect the stratum involving lightlike point, so an FRS-generic family will
be called FRS-generic. Denote by $(a_1, \ldots, a_k; b_1, \ldots, b_k)$ the coordinates in $J^k(1, 2)$.
The strata of interest are:

- **Cusp** $(C)$: $a_1 = 0$ and $b_1 = 0$
- **Lightlike** $(L)$: $a_1 \pm b_1 = 0$
- **Inflections** $(I)$: $a_1 b_2 - a_2 b_1 = 0$
- **Vertices** $(V)$: $(a_1^2 - b_1^2)(a_1 b_3 - a_3 b_1) + 2(b_1 b_2 - a_1 a_2)(a_1 b_2 - a_2 b_1) = 0$

Suppose $\gamma$ has an ordinary cusp singularity at $t_0 = 0$, that is, $\gamma$ is $A$-equivalent to
$(t^2, t^3)$. We can take, without loss of generality, $\gamma''(0)$ parallel to $(1, 0)$ and write
$\gamma(t) = (t^2, \beta(t)) = (t^2, c_3 t^3 + O(t^4))$, with $c_3 \neq 0$. The strata $I, V$ and $L$ are submanifolds
of $J^k(1, 2)$ of codimension 1 which contain the submanifold $C$ of codimension 2. The
stratum $I$ is smooth along $C$, the stratum $L$ is the union of two transverse components
$L_1$ and $L_2$, represented, respectively, by the equations $a_1 + b_1 = 0$ and $a_1 - b_1 = 0$, and
the stratum $V$ is the union of two smooth submanifolds $V_1$ and $V_2$ which intersects
transversally along $C$.

We can write any 1-parameter family of curves $\gamma_s$ with $\gamma = \gamma_0$ in the form

$$\gamma_s(t) = (t^2, \beta(t, s)) = (t^2, \overline{c}_1(s)t + \overline{c}_2(s)t^2 + \overline{c}_3(s)t^3 + O_s(t^4)),$$

with $\overline{c}_1(0) = \overline{c}_2(0) = 0$, $\overline{c}_3(0) = c_3 \neq 0$, $\overline{c}_k(0) = c_k$ for $k \geq 4$ and $s \in \mathbb{R}, 0$.

Our goal is to find conditions to the FRS-genericity of $\gamma_s$. Note that if the image of
d$(j^k \Phi)(0, 0)$ is transverse to the stratum $C$, then it will be transverse to $I, L$ and $V$.
The family of Monge-Taylor maps $j^k \Phi$ is transverse to the stratum $C$ if, and only if,
$\frac{\partial^2 \beta}{\partial s \partial t}(0, 0) \neq 0$, which is exactly the condition for $\gamma_s$ to be an $A_e$-versal deformation of
the ordinary cusp singularity of $\gamma$.

**Corollary 16.** Let $\gamma$ be a germ of a non-lightlike ordinary cusp and $\gamma_s$ be an 1-
parameter family of curves with $\gamma_0 = \gamma$. Then $\gamma_s$ is an FRS-generic deformation
of $\gamma$ if, and only if, $\gamma_s$ is an $A_e$-versal deformation of $\gamma$. 

Theorem 17. Let $\gamma$ be a germ of a non-lightlike ordinary cusp at $t_0$.

(i) The caustic of $\gamma$ is the union of the limiting normal line of $\gamma$ at $t_0$ and a smooth curve which has ordinary tangency at $\gamma(t_0)$ with the limiting normal line of $\gamma$ at $t_0$. Moreover, that curve lies in the semi-plane determined by the normal line of $\gamma$ at $t_0$ which does not contain the curve $\gamma$, see Figure 10 center.

(ii) For an FRS-generic 1-parameter family $\gamma_s$ with $\gamma_0 = \gamma$, the bifurcations in the curve and its caustic are as shown in Figure 10.

(iii) Any FRS-generic 1-parameter family of $\gamma$ is FRS-equivalent to the model family $\alpha_u(t) = (t^2, ut + t^3)$.

Proof. (i) We have

$$d_u(t) = -(t^2 - u_1)^2 + (\beta(t) - u_2)^2 = -t^4 + 2u_1 t^2 - u_1^2 + \beta^2(t) - 2u_2 \beta(t) + u_2^2.$$  

Isolating $u_1$ in $d'_u = 0$ and substituting in $d'_u = 0$ we get

$$t^2(4t + u_2 \frac{-\beta'(t) + \beta''(t)t}{t^2} + \frac{\beta(t)(\beta'(t) - \beta''(t)t) - \beta^2(t)t}{t^2}) = 0.$$  

As $t = 0$ is a root of the above equation, the line $u_1 = 0$ is part of the caustic of $\gamma$.

For $t \neq 0$, we have $u_2 = \frac{4t^3 + (-\beta(t)\beta''(t) - \beta'^2(t)t + \beta(t)\beta''(t))}{\beta'(t) - \beta''(t)t}.$

Therefore, the caustic of $\gamma$ is the union of the line $u_1 = 0$ and the curve parametrized by $t \mapsto (-t^2 + O(t^3), \frac{-4}{3c_3} t + O(t^2)).$

(ii) We take $\gamma_s(t) = (t^2, \beta(t, s)) = (t^2, st + c_2)(s^2 + c_3(s))^3 + O_s(t^4)).$

We first take the 2-dimensional section of the stratification of $J^k(1, 2)$ by $C, L, I, V$ obtained by fixing $(a_2, a_3, \ldots, a_k; b_2, b_3, \ldots, b_k) = (1, 0, \ldots, 0; 0, c_3, \ldots, c_k)$, with $c_3 \neq 0$. In this section the stratum $I$ is given by $b_1 = 0$ and $V$ by $a_1 = 0$ union $\lambda(a_1^2 - b_1^2) + 2b_1 = 0$. The second component of $V$ is a regular curve with tangent direction at the origin is parallel to the vector $(1, 0)$. Therefore, the stratification of the 2-dimensional section is as shown in Figure 11 center. Moreover, $(f^k \phi_{t_0})'(0) = (2, 0, \ldots, 0; 0, 3c_3, \ldots, (k + 1)c_{k+1})$, which is tangent to the strata $I$ and $V_1$ and it is transversal to $V_2$ and $L$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10}
\caption{FRS-generic deformations of a curve and its caustic at a non-lightlike ordinary cusp.}
\end{figure}
Therefore, the relative position of \( j^k \phi_{\gamma_s} \) with respect to the strata \( C, L, I, V \) is given in Figure 11.

\[ L_2 \quad I \quad V_1 \quad L_1 \quad V_2 \]

Figure 11. Transversal 2-dimensional section of the stratification of \( J^k(1,2) \) by \( C, L, I, V \) and the relative position of the image of the Monge-Taylor map of \( \gamma_s \) for \( s < 0 \) in left, \( s = 0 \) in center and \( s > 0 \) in right (case \( c_3 > 0 \)).

From Figure 11, we get the number and the relative position of lightlike points, inflections and vertices, but we do not obtain their relative position with respect to the self-intersection points. The self-intersection points of \( \gamma_s \) are given by \( t_1^2 = t_2^2 \) and \( \beta_s(t_1) = \beta_s(t_2) \) with \( t_1 \neq t_2 \), that is, \( t_1 = -t_2 = t \) and \( \beta_s(t) = \beta_s(-t) \). Consider
\[
h(t, s) = \frac{1}{2}(\beta_s(t) - \beta_s(-t)) = st + c_3(s)t^3 + O_s(t^5) = t(s + c_3(s)t^2 + O_s(t^4)) = \bar{h}(t, s).
\]

Hence, the self-intersection points of \( \gamma_s \) are determined by the zeros of \( \bar{h} \). As \( c_3 \neq 0 \), \( t^2 = -\frac{1}{c_3}s + O(s^2) \) and the two solutions of this equation are symmetric and give the values of \( t_1 \) and \( t_2 \) such that \( \gamma_s(t_1) = \gamma_s(t_2) \). Therefore, \( \gamma_s \) has a self-intersection point in the side of the transition which contains one vertex and two lightlike points.

The lightlike points are given by \( \beta'_s(t) = \pm 2t \). Considering the families \( h^+(t, s) = \beta'_s(t) + 2t = s + 2(c_2(s) + 1)t + 3c_3(s)t^2 + O_s(t^3) \), we obtain that the zeros of \( h^+_s \) and, consequently, the lightlike points of \( \gamma_s \), are given by \( t = \pm \frac{1}{2}s + O(s^2) \).

Comparing the defining equations of the lightlike and self-intersection points, we conclude that the lightlike points are between the self-intersection points.

Note that for \( s > 0 \), the curve \( \gamma_s \) has 2 inflections (\( \kappa_s = 0 \)), 3 vertices (\( \kappa'_s = 0 \)) and 2 lightlike points (where \( \kappa_s \) goes to infinity). Moreover, the signal of \( \kappa_s \) is the same of \( \langle \gamma'_s, \gamma''_s \rangle \). As \( t = 0 \) is between the lightlike points and the signal of \( \langle \gamma''_s, \gamma''_s \rangle \) does not change when we cross a lightlike point, then \( \kappa_s > 0 \) between the inflections and \( \kappa_s < 0 \) outside. Hence, the graph of \( k_s \) is as in Figure 12 right. Therefore, the central vertex is inward and the others are outward.

For \( s = 0 \), we have \( \kappa_0(t) = \frac{6c_3t^2 + O(t^3)}{|4t^2 - 9c_3^2t^4 + O(t^5)|^2} = \frac{3b_3}{4} \frac{1}{|t|}(1 + O(t)) \). Thus, the graph of \( \kappa_0 \) is as in Figure 12 center.

For \( s < 0 \), the curve \( \gamma_s \) has 1 vertex, 2 lightlike points and no inflections. As \( k_s(0) < 0 \), then \( k_s < 0 \). Hence, the graph of \( \kappa_s \) is as in Figure 12 left. We have \( \kappa_s \kappa''_s > 0 \) so the vertex is inward.
From the above analysis and the results in the previous sections we conclude that the bifurcations in the caustic of $\gamma_s$ are as in Figure 10.

(iii) The results in (ii) depend only on the fact that the curve has a non-lightlike ordinary cusp and on the family being FRS-generic. The family $\alpha_a$ satisfies such conditions, consequently, it is a model for FRS-generic deformations.

\[ \text{Figure 12. Transitions in the graph of the curvature function of } \gamma_s. \]

8. Geometric deformations of curves with a lightlike ordinary cusp

Consider $\gamma$ a plane curve with a lightlike ordinary cusp singularity at $t_0 = 0$ (i.e., its limiting tangent direction at $t_0$ is lightlike). Here we need to consider multilocal strata and stratify the multi-jet space $2J^k(1,2) \subset J^k(1,2) \times J^k(1,2)$. We take the Monge-Taylor bi-jet map as in (4). Denote by $((a_0, a_1, \ldots, a_k; b_0, b_1, \ldots, b_k); (a'_0, a'_1, \ldots, a'_k; b'_0, b'_1, \ldots, b'_k))$ the coordinates in $J^k(1,2) \times J^k(1,2)$.

The local strata in $J^k(1,2)$ are viewed product strata in $2J^k(1,2)$, consequently, it is enough to compute them in $J^k(1,2)$. We have:

- $LC$: $a_1 = 0$, $b_1 = 0$, $a_2 - b_2 = 0$
- $C$: $a_1 = 0$, $b_1 = 0$
- $I(2)$: $a_1b_2 - a_2b_1 = 0$, $a_1b_3 - a_3b_1 = 0$
- $LI$: $a_1b_2 - a_2b_1 = 0$, $a_1 \pm b_1 = 0$
- $V(2)$: $(a_1^2 - b_1^2)(a_1b_3 - a_3b_1) + 2(b_1b_2 - a_1a_2)(a_1b_2 - a_2b_1) = 0$, $-a_1^2 + b_1^2)(2(a_1b_4 - a_4b_1) + a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1)(b_1b_2 - a_1a_2) - (a_1b_2 - a_2b_1)(3(b_1b_3 - a_1a_3) + 2(b_2 - a_2^2)) = 0$
- $IT$: $a_0 - a'_0 = 0$, $b_0 - b'_0 = 0$, $a_1b_2 - a_2b_1 = 0$
- $LT$: $a_0 - a'_0 = 0$, $b_0 - b'_0 = 0$, $a_1 \pm b_1 = 0$
- $Tc$: $a_0 - a'_0 = 0$, $b_0 - b'_0 = 0$, $a_1b'_1 - a'_1b_1 = 0$
- $VT$: $a_0 - a'_0 = 0$, $b_0 - b'_0 = 0$, $(a_1^2 - b_1^2)(a_1b_3 - a_3b_1) + 2(b_1b_2 - a_1a_2)(a_1b_2 - a_2b_1) = 0$

We take $\gamma(t) = (t^2, t^2 + c_3t^3 + c_4t^4 + O(t^5))$, with $c_3 \neq 0$.

Let $\gamma_s$ be a 2-parameter deformation of $\gamma$ which we can take in the form

\[ \gamma_s(t) = (t^2, c_1(s)t + (c_2(s) + 1)t^2 + c_3(s)t^3 + c_4(s)t^4 + O_s(t^5)), \]

with $c_1(0) = c_2(0) = 0$, $c_3(0) = c_4 \neq 0$ and $c_k(0) = c_k$, for $k \geq 4$.

Theorem 18. Consider $\gamma_s$ a 2-parameter deformation of $\gamma$ as in (9). Then the family of Monge-Taylor bi-jet maps $2J^k\Psi$ is transverse to the strata $LC$, $C$, $I(2)$, $LI$, $V(2)$, $IT$. 

\( \text{LT, VT and Tc if, and only if, } \frac{\partial c}{\partial \gamma_1}(0,0) \neq 0 \text{ and } \frac{\partial c}{\partial \gamma_1}(0,0) \frac{\partial c}{\partial \gamma_2}(0,0) - \frac{\partial c}{\partial \gamma_2}(0,0) \frac{\partial c}{\partial \gamma_1}(0,0) \neq 0. \)

**Proof.** The proof follows by standard lengthy calculations and is omitted (see [11] for details).

The condition \( \frac{\partial c}{\partial \gamma_1}(0,0) \neq 0 \) is exactly the condition to \( \gamma_s \) be an \( \mathcal{A}_c \)-versal deformation of \( \gamma. \)

**Corollary 19.** Let \( \gamma \) be a lightlike ordinary cusp and \( \gamma_s \) be a 2-parameter deformation of \( \gamma_0 = \gamma \) as in (9). Then \( \gamma_s \) is an FRLS-generic deformation if, and only if, \( \gamma_s \) is an \( \mathcal{A}_c \)-versal deformation of \( \gamma_0 \) and \( \frac{\partial c}{\partial \gamma_1}(0,0) \frac{\partial c}{\partial \gamma_2}(0,0) - \frac{\partial c}{\partial \gamma_2}(0,0) \frac{\partial c}{\partial \gamma_1}(0,0) \neq 0. \)

By Corollary 19, if \( \gamma_s \) is a 2-parameter deformation as in (9) and is FRLS-generic, we can take \( \gamma_s(t) = (t^2, \beta_s(t)), \) with

\[
\beta_s(t) = s_1 t + (s_2 + 1)t^2 + \overline{c_3}(s)t^3 + \overline{c_4}(s)t^4 + O_s(t^5).
\]

**Theorem 20.** Let \( \gamma_s \) be an FRLS-generic 2-parameter deformation of a lightlike ordinary cusp \( \gamma \) as in (10). Then, the stratification of the parameter space \( s = (s_1, s_2) \in \mathbb{R}^2, 0 \) of \( \gamma_s \) (see Figure 13, center) consists of the origin \( (LC) \) and the following curves:

- **C:** \( s_1 = 0 \)

- **LI:** \( (s_1, s_2) = (3c_3 t^2 + O(t^3), -3c_3 t + O(t^2)) \)

- **V(2):** \( (s_1, s_2) = \left( \frac{-2\sqrt{5} - 5}{25c_3} t^2 + O(t^3), t + O(t^2) \right) \)

- **LT:** \( (s_1, s_2) = (-3c_3 t^2 + c_3 c_301 t^3 + O(t^4), -c_3 t + (c_3 c_301 - 2c_4) t^2 + O(t^3)) \)

- **VT:** \( (s_1, s_2) = (-3c_3 t^2 + c_3 c_301 t^3 + O(t^4), -c_3 t + (c_3 c_301 + 2c_3 - 2c_4) t^2 + O(t^3)) \)

where \( c_301 \) is the coefficient of \( s_2 \) in \( \overline{c_3}(s). \) The strata \( I(2), \text{LT and Tc are empty.} \)

**Proof.** The stratification of the parameter space of \( \gamma_s \) is the projection to that space of the pre-image by the Monge-Taylor map of the stratification of the jet space by phenomena of codimension \( \leq 2. \) We present here the calculations for some stratum, the others follows in a similar way.

Consider the stratum \( LI \) given by \( a_1 \pm b_1 = 0 \) and \( a_1 b_2 - a_2 b_1 = 0, \) that is, \( b_1 = \mp a_1 \) and \( a_1 (b_2 \pm a_2) = 0. \) If \( a_1 = 0 \) then \( b_1 = 0 \) and, consequently, we are on the cusp stratum. Hence, \( b_2 \pm a_2 = 0 \) so \( LI = \{ a_1 + b_1 = 0, a_2 + b_2 = 0 \} \cup \{ a_1 - b_1 = 0, a_2 - b_2 = 0 \}. \) Note that \( 2j^k \Phi \) does not intersect the first component of \( LI. \) Taking the pre-image of the second component by \( 2j^k \Phi \) we get \( \{ 2t - \beta'_s(t) = 0, 1 - \frac{1}{2} \beta'_s(t) = 0 \} \) and this gives \( LI \) parametrized by \( (s_1, s_2) = (3c_3 t^2 + O(t^3), -3c_3 t + O(t^2)). \)

For the stratum \( LT \) we take its pre-image by \( 2j^k \Phi \) and we get \( t_1^2 - t_2^2 = 0, \beta_s(t_1) - \beta_s(t_2) = 0 \) and \( 2t_1 \pm \beta'_s(t_1) = 0, \) that is, \( t_1 = -t_2, \beta_s(t) - \beta_s(-t) = 0 \) and \( 2t \pm \beta'_s(t) = 0. \) A calculation shows now that \( s_1 = -c_3 t^2 + c_3 c_301 t^3 + O(t^4) \) and \( s_2 = -c_3 t + (c_3 c_301 - 2c_4) t^2 + O(t^3). \)

The stratum \( V(2) \) is given by the zeros of multiplicity 2 of the family \( G(t, s) \) obtained by taking the numerator of \( \kappa'_s(t). \) As \( G \) is a deformation of an \( A_3 \) singularity, \( G \) is \( R \)-induced by \( F(t, u_0, u_1, u_2) = u_0 + u_1 t + u_2 t^2 + t^4, \) that is, we can write \( G(t, s_1, s_2) = \)
Let $\gamma$ be a germ of a lightlike ordinary cusp at $t_0$.

(i) The caustic of $\gamma$ is the union of the tangent line of $\gamma$ at $t_0$ and a curve with a lightlike ordinary cusp singularity at $\gamma(t_0)$ as shown in Figure 14. 

(ii) The bifurcations of an FRLS-generic 2-parameter family $\gamma_s$ with $\gamma_0 = \gamma$ are as shown in Figure 13.

(iii) For an FRLS-generic 2-parameter family $\gamma_s$ with $\gamma_0 = \gamma$, the bifurcations in the caustic are as shown in Figure 14.

(iv) Any FRLS-generic 2-parameter family of $\gamma$ is FRLS-equivalent to the model family $\alpha_u(t) = (t^2, u_1t + (1 + u_2)t^2 + t^3)$.

Proof. (i) We take $\gamma(t) = (t^2, \beta(t)) = (t^2, t^2 + c_3t^3 + c_4t^4 + O(t^5))$ at $t=0$. Thus, $d_u(t) = -(t^2 - u_1)^2 + (\beta(t) - u_2)^2$. Equations $d_u'' = 0$ and $d_u' = 0$ give

$$t^2 (4t + \frac{(\beta''(t)t - \beta'(t))^2}{t^2})u_2 - \frac{(\beta''(t)t - \beta'(t))^2}{t^2} \beta(t) - \frac{\beta'(t)^2t}{t^2} = 0.$$ 

As $t = 0$ is a root of the above equation, the line $u_1 = u_2$ is part of the caustic of $\gamma$.

For $t \neq 0$, we get $u_2 = \frac{-4t^3 + (\beta''(t)t - \beta'(t))\beta(t) + \beta'(t)^2t}{\beta''(t)t - \beta'(t)}$. Therefore, the caustic of $\gamma$ is the union of the lightlike line $u_1 = u_2$ and the curve parametrized by $(5t^2 + (9c_3 - \frac{16c_4}{3c_3})t^3 + O(t^4), 5t^2 + (4c_3 - \frac{16c_4}{3c_3})t^3 + O(t^4))$. Comparing this lightlike ordinary cusp with $\gamma$ we conclude that the relative position of the two curves is as shown in Figure 14.

(ii) The lightlike points are given by $h^\pm(t, s) = \beta''_s(t) \pm 2t = 0$. The family $h^+$ is regular with $h^+(0, 0, 0) = 0$, then $h^+$ has one zero for any $s$. The family $h^-$ is an $R$-versal deformation of an $A_1$ singularity and, consequently, it has at most 2 zeros. Therefore, $\gamma_s$ has at most 3 lightlike points.

For inflections, we use the numerator of $k_s$ given by $G(t, s) = -2s_1 + 6s_3(t^2 + O_s(t^3))$ which is a deformation of an $A_1$ singularity. Thus $\gamma_s$ has at most 2 inflections.
For vertices, we note that the numerator of $\kappa'_s$ is given by $H(t, s) = -8t^3 \beta'''_s(t) + 2t\beta'_s(t)^2 \beta''_s(t) - 24t\beta'_s(t) + 24t^2 \beta''_s(t)^2 \beta''_s(t) - 6t\beta'_s(t) \beta''_s(t)^2$ which is a deformation of an $A_3$ singularity. Then \( \gamma_s \) has at most 4 vertices.

To obtain the configuration of the lightlike points, inflections and vertices in \( \gamma_s \) we use the results obtained in the previous sections and the fact that changes occur only when we cross a curve in the stratification of the parameter space given in Theorem 20.
Firstly, we find the configuration on the cusp stratum, that is, when $s_1 = 0$. In this case, $h^+(t, 0, s_2) = t(4 + 2s_2 + 3\sqrt[3]{s_2}t + O(s_2(t^2)))$ and $h^-(t, 0, s_2) = t(2s_2 + 3\sqrt[3]{s_2}t + O(s_2(t^2)))$. 

FIGURE 14. FRLS-generic deformations of the caustic of a curve with a light-like ordinary cusp.
of \( O(s_s(t^2)) \). Therefore, using \( h^- \), we get \( t = -\frac{2}{\delta s^3} s_2 + O(s^2) \) as the unique lightlike point of \( \gamma_s \) on the cusp stratum.

For inflections, we have \( G(t, 0, s_2) = t^2 (6c_3 s_2 + O(s_2(t))) \) which implies that the two inflections of \( \gamma_s \) are concentrated at the cusp point.

For vertices, we have \( H(t, 0, s_2) = t^3 (-48c_3 s_2 - 180 c_3^2 t + O_2(t, s_2)) \), that is, \( \gamma_s \) has 3 vertices concentrated at the cusp point and another given by \( t = -\frac{4}{15 c_3} s_2 + O(s_2^2) \).

The points are positioned in the following order \( L - V - C \) when \( s_2 > 0 \) and \( C - V - L \) when \( s_2 < 0 \). Moreover, the vertex is inward, since \( \text{sgn}(\kappa_{s_2}(t)) = \text{sgn}(G(t, s_2)) = \text{sgn}(c_3) \), for all \( t \) sufficiently close to 0.

Going around the origin in the \((s_1, s_2)\)-space and using the results of the sections 5, 6 and 7 we obtain the configuration of \( \gamma_s \) in each stratum as shown in Figure 13.

(iii) The statement follows from (ii) and the results in sections 5, 6 and 7.

(iv) From Theorem 20, the stratifications of the parameter spaces of \( \gamma_s \) and \( \alpha_u \) are homeomorphic. Moreover, the calculations in (ii) and (iii) depend only on the fact that the curve has a lightlike ordinary cusp and the family is FRLS-generic, conditions that \( \alpha_u \) satisfies.

9. Geometric deformations of curves with a ramphoid cusp singularity

Consider \( \gamma \) a curve with a non-lightlike ramphoid cusp singularity \( A_h \)-equivalent to \((t^2, t^4 + t^5)\) (see [9] for the \( A_h \)-equivalence). The strata of interest are:

- **RC**: \( a_1 = 0 \), \( b_1 = 0 \), \( a_2 b_3 - a_3 b_2 = 0 \)
- **C**: \( a_1 = 0 \), \( b_1 = 0 \)
- **I(2)**: \( a_1 b_2 - a_2 b_1 = 0 \), \( a_1 b_3 - a_3 b_1 = 0 \)
- **V(2)**: \( (a_1^2 - b_1^2)(a_1 b_3 - a_3 b_1) + 2(b_1 b_2 - a_1 a_2)(a_1 b_2 - a_2 b_1) = 0 \), \( 2(a_1 b_2 - a_2 b_1)[-(-a_1^2 + b_1^2)(-a_1^2 + b_2^2) + 5(b_1 b_2 - a_1 a_2)]^2 + (-a_2^2 + b_1^2)(4a_2 b_3 + 4a_3 b_2) a_1^2 - 9(a_2 a_3 + b_2 b_3) a_1 b_1 + (4a_2 b_3 + 5 a_3 b_2) b_1^2] + 2(-a_1^2 + b_1^2)^2 (a_1 b_4 - a_1 b_1) = 0 \)
- **L(1)**: \( a_1 \pm b_1 = 0 \), \( a_1 b_2 - a_2 b_1 = 0 \)
- **I(1)**: \( a_0 - a_0 = 0 \), \( b_0 - b_0 = 0 \), \( a_1 b_2 - a_2 b_1 = 0 \)
- **V(1)**: \( a_0 - a_0 = 0 \), \( b_0 - b_0 = 0 \), \( (a_1^2 - b_1^2)(a_1 b_3 - a_3 b_1) + 2(b_1 b_2 - a_1 a_2)(a_1 b_2 - a_2 b_1) = 0 \)
- **L(0)**: \( a_0 - a_0 = 0 \), \( b_0 - b_0 = 0 \), \( a_1 \pm b_1 = 0 \)
- **T(1)**: \( a_0 - a_0 = 0 \), \( b_0 - b_0 = 0 \), \( a_1 b_1 - a_1 b_1 = 0 \).

Suppose the limiting tangent direction of \( \gamma \) at the ramphoid cusp is timelike, the spacelike case is similar. We take \( \gamma(t) = (t^2, \beta(t)) = (t^2, c_4 t^4 + c_5 t^5 + O(t^6)) \), with \( c_4, c_5 \neq 0 \). Let \( \gamma_s \) be a 2-parameter deformation of \( \gamma \) which can be taken in the form \( \gamma_s(t) = (t^2, \beta_s(t)) \), where

\[
\beta_s(t) = \tau_1(s) t + \tau_2(s) t^2 + \tau_3(s) t^3 + \tau_4(s) t^4 + \tau_5(s) t^5 + O_s(t^6)
\]

with \( \tau_1(0) = \tau_2(0) = \tau_3(0) = 0 \) and \( \tau_4(0) = c_k \), with \( k \geq 4 \).

The strata **RC, C** and **Tc** are all the strata in \( J^k(1, 2) \) of codimension \( \leq 3 \) that come from the \( A \)-equivalence. Hence, from the Mather’s Theorem, to ensure the transversality of \( J^k_0(1, 2) \) with respect to such strata we need the \( A_\epsilon \)-versality of \( \gamma_s \).

We can show that \( \gamma_s \) as in (11) is an \( A_\epsilon \)-versal deformation of \( \gamma \) if, and only if, \( \frac{\partial \tau_1}{\partial s_1} (0) \frac{\partial \tau_2}{\partial s_2}(0) - \frac{\partial \tau_3}{\partial s_2}(0) \frac{\partial \tau_1}{\partial s_1}(0) \). Therefore, supposing \( \gamma_s \) an \( A_\epsilon \)-versal deformation of \( \gamma \), we can take \( \beta_s(t) = s_2 t + \tau_2(s) t^2 + s_1 t^3 + \tau_3(s) t^4 + \tau_5(s) t^5 + O_s(t^6) \).
Theorem 22. Let $\gamma_s$ be an $A_c$-versal 2-parameter deformation of a ramphoid cusp $\gamma A_k$-equivalent to $(t^2, t^4 + t^5)$. Then, the family of Monge-Taylor bi-jet maps $\phi_j$ is transverse to the above strata and the stratification of the parameters space of $\gamma_s$ (given in Figure 15 center) consists of the origin (RC) and the following curves:

$$C: \quad s_2 = 0$$

$$I(2): \quad (s_1, s_2) = (-4c_4t + O(t^2), -4c_4t^3 + O(t^4))$$

$$V(2): \quad (s_1, s_2) = (10c_5t^2 + 30c_0t^3 + O(t^4), 5c_5t^4 + 24c_0t^5 + O(t^6))$$

$$IT: \quad (s_1, s_2) = (-2c_4t + O(t^2), 2c_4t^3 + O(t^4))$$

$$VT: \quad (s_1, s_2) = (2c_5t^2 + 8c_0t^3 + O(t^4), -3c_5t^4 - 8c_0t^5 + O(t^6))$$

$$Tc: \quad (s_1, s_2) = (-2c_5t^2 + O(t^3), c_5t^4 + O(t^5)).$$

The strata LI and LT are empty.

Proof. It follows similarly to the proof of Theorem 18 and 20 (see [11] for details). □

From Theorem 22, the strata $V(2)$ and $VT$ have a ramphoid cusp singularity if $c_6 \neq 0$. This is a new condition that will be required for the concept of FRS-genericity.

Corollary 23. Consider the curve $\gamma(t) = (t^2, \beta(t)) = (t^2, c_4t^4 + c_5t^5 + c_6t^6 + O(t^7))$ with $c_4 \neq 0, c_5 \neq 0$ and $c_6 \neq 0$. Then a 2-parameter deformation $\gamma_s$ of $\gamma$ is FRS-generic if, and only if, $\gamma_s$ is an $A_c$-versal deformation of the ramphoid cusp singularity of $\gamma$ at $t = 0$.

Theorem 24. Let $\gamma$ be a germ of a non-lightlike ramphoid cusp $A_k$-equivalent to $(t^2, t^4 + t^5)$ at $t_0$.

(i) The caustic of $\gamma$ is the union of the tangent line of $\gamma$ at $t_0$ and a smooth curve with an ordinary inflection as shown in Figure 16 (1).

(ii) The bifurcations of an FRS-generic 2-parameter family $\gamma_s$ with $\gamma_0 = \gamma$ are as shown in Figure 15.

(iii) For an FRS-generic 2-parameter family $\gamma_s$ with $\gamma_0 = \gamma$, the bifurcations in the caustic are as shown in Figure 16.

(iv) Any FRS-generic 2-parameter family of $\gamma$ is FRS-equivalent to the model family $\alpha_u(t) = (t^2, u_2t + u_1t^3 + t^4 + t^5 + t^6)$.

Proof. (i) The proof is similar to that of item (i) of the Theorem 21. We obtain that the caustic of $\gamma$ is the union of the line $u_1 = 0$ and a curve given by

$$\left(\frac{5c_5}{8c_4}t^3 + O(t^4), -\frac{1}{2c_4} + \frac{15c_5}{16c_4}t + O(t^2)\right)$$

as shown in Figure 16 (1).

(ii) The lightlike points of $\gamma_s$ are given by $h^\pm(t, s) = \beta'_l(t) \pm 2t = 0$, which are regular. Note that $h^+$ and $h^-$ have only one zero each for each fixed $s \in \mathbb{R}^2, 0$, given by $t_1 = -\frac{1}{2}s_2 + O_2(s_1, s_2)$ and $t_2 = \frac{1}{2}s_2 + O_2(s_1, s_2)$, respectively. Therefore, $\gamma_s$ has two lightlike points for all $(s_1, s_2) \in \mathbb{R}^2, 0$, with $s_2 \neq 0$.

For inflections we consider the numerator of $\kappa_s$ given by $G(t, s) = -2s_2 + 6s_1t^2 + 8c_0(s)t^3 + 15c_5(s)t^4 + 24c_0(s)t^5 + O_s(t^6)$ which is a deformation of an $A_2$ singularity. Thus $\gamma_s$ can have at most 3 inflections.
For vertices we consider the numerator of $\kappa'_s$ given by $H(t, s) = 2t\beta''''_s(t)(-4t^2 + \beta'_s(t)^2) - 3(-2\beta'_s(t) + 2t\beta''_s(t))(-4t + \beta'_s(t)\beta''_s(t))$ which is a deformation of an $A_4$ singularity. Then $\gamma_s$ can have at most 5 vertices.

To obtain the relative position of lightlike points, inflections and vertices in $\gamma_s$ we use the results in sections 5 and 7 and the fact that changes only occur when we cross a stratum in the parameters space in Theorem 22. We start by considering the configuration of $\gamma_s$ on the cusp stratum, that is, when $s_2 = 0$. 
In this case \( h^+(t,s_1,0) = (2 + 2c^2(s_1) + 3s_1 + 4c^4(s_1)t^2 + O(s_1^3)) \) and \( h^-(t,s_1,0) = (2 - 2c^2(s_1) + 3s_1 + 4c^4(s_1)t^2 + O(s_1^3)) \). Thus, \( \gamma_s \) does not have lightlike points outside the cusp point on the cusp stratum. We have \( G(t,s_1,0) = t^2(6s_1 + 8c^2(s_1)t^2 + 15c^4(s_1)t^4 + O(s_1^3)) \), that is, \( \gamma_s \) has two inflections concentrated at the cusp point and another one given by \( t_n = -\frac{3c^4(s_1)}{8s_1 + O(s_1)} \).
For vertices, we have \( H(t, s_1, 0) = 12t^3(2s_1 + O(s_1)s_1 - 10c_5t^2 + O(s_1)t^2 + O(s_1)(t^3)) \) which implies that the vertices of \( \gamma_s \) are given by \( s_1 = 5c_5t^2 + O(t^3) \). Hence, we have 2 vertices for \( s_1 > 0 \) and none for \( s_1 < 0 \). Moreover, as \( c_4 \) and \( c_5 \) are fixed (and suppose, without loss of generality, to be positives), then \( \frac{3s_1}{8c_4} < \sqrt{\frac{s_1}{5c_5}} \), which implies that the relative position of the relevant points is V-I-C-V for \( s_1 > 0 \) and C-I for \( s_1 < 0 \). The first vertex is outward and the second is inward. In fact, the curvature function is given by \( \kappa_s(t) = \frac{6s_1t^2 + O(s_1)(t^3)}{4t^2(-1 + \frac{1}{c_5}s_1)) + O(s_1)(t^3)|t|} = \frac{3s_1}{4} \frac{1}{|t|}(1 + O(s_1)(t)) \) and as \( s_1 > 0 \), the curvature function goes to infinity when \( t \) tends to 0. As the inflections of \( \gamma_s \) are zeros of \( \kappa_s \) and vertices are maximum or minimum points of \( \kappa_s \), then we can conclude that the vertices are minimum points of \( \kappa_s \) (that is, \( \kappa''_s > 0 \)) and, consequently, we get \( \kappa_s\kappa''_s < 0 \) at the first vertex and \( \kappa_s\kappa''_s > 0 \) at the second.

Going around the origin in the \((s_1, s_2)\)-space and using the configuration on the cusp stratum and the results in sections 5 and 7 we get the configuration of \( \gamma_s \) in each stratum as shown in Figure 15.

(iii) It follows from (ii) and the results in sections 5 and 7.

(iv) From Theorem 22, the stratifications in the parameters spaces of \( \gamma_s \) and \( \alpha_u \) are homeomorphic, see Figure 15 center. Moreover, the calculations in (ii) and (iii) depend only on the fact that the curve has a ramphoid cusp \( A_h \)-equivalent to \((t^2, t^4 + t^5)\) and the family is FRS-generic, conditions that \( \alpha_u \) satisfies.  

\( \square \)

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