Sharp decay estimates and asymptotic behaviour for 3D magneto-micropolar fluids

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Abstract. We characterize the $L^2$ decay rate of solutions to the 3D magneto-micropolar system in terms of the decay character of the initial datum. Due to a linear damping term, the microrotational field has a faster decay rate. We also address the asymptotic behaviour of solutions by comparing them to solutions to the linear part. As a result of the linear damping, the difference between the microrotational field and its linear part also decays faster. As part of the proofs of these results, we prove estimates for the derivatives of solutions which might be of independent interest.

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1. Introduction

The Navier–Stokes equations are one of the main tools for the mathematical study of the evolution of incompressible, homogeneous fluids. When the fluid has more properties or structure arising from the physical model studied, it is necessary to couple these equations to others describing the new features. Recently, there has been a surge of activity on the study of the magneto-micropolar system

\begin{align}
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla p = (\mu + \chi) \Delta u + \chi \nabla \times w + (b \cdot \nabla) b, \\
\partial_t w + (u \cdot \nabla) w = \gamma \Delta w + \nabla (\nabla \cdot w) + \chi \nabla \times u - 2\chi w, \\
\partial_t b + (u \cdot \nabla) b = \nu \Delta b + (b \cdot \nabla) u, \\
\nabla \cdot u(\cdot, t) = \nabla \cdot b(\cdot, t) = 0,
\end{cases}
\end{align}

(1.1)

with initial data $z_0 = (u_0, w_0, b_0) \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2_\sigma(\mathbb{R}^3)$. From now on, we denote $z = (u, w, b)$. This system, introduced by Ahmadi and Shahinpoor [1] to study the stability of solutions to (1.1) in bounded domains (see also Galdi and Rionero [15]), models the evolution in time of a 3D homogeneous, conducting, incompressible fluid with velocity $u$, pressure $p$ and magnetic field $b$, which possesses some “microstructure” described by a microrotational velocity $w$. This microstructure may correspond to rigid microparticles suspended or diluted in the fluid, as may be the case for liquid crystals or polymer solutions. The positive constants $\mu, \gamma$ in (1.1) correspond to the kinematic and angular viscosity, respectively; $\nu$ is the inverse of the magnetic Reynolds number and $\chi$ is the microrotational viscosity. Note that (1.1) reduces to the Navier–Stokes equations, when $b \equiv w \equiv 0$; to the MHD system, when $w \equiv 0$; and to the micropolar system, when $b \equiv 0$.

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Equations (1.1) are introduced by Ahmadi and Shahinpoor [1], who based their model on the theory of micropolar fluids developed by Eringen [10] and studied the stability of solutions in bounded domains (see also Galdi and Rionero [15]). In this context of bounded domains, many results have been obtained concerning different aspects of the study of (1.1), as existence of weak and strong solutions [3, 30, 33, 34], stability or blowup of solutions [6, 25, 26], asymptotic behaviour [21, 28, 35, 48], numerical methods for (1.1) [29, 32] and properties of stochastic versions of (1.1) [50–53].

Plenty of results have been obtained for (1.1) in $\mathbb{R}^3$, as the problems studied and the techniques used to solve them are inspired on those for the Navier–Stokes equations. Amongst the many articles on the magneto-micropolar system recently published, we should mention those on the existence of weak and strong solutions in different function spaces [24, 43, 56], Beale-Kato-Majda criteria and blowup results [7, 14, 41, 44, 45, 57], regularity criteria [13, 16, 40, 42, 47, 54, 55, 58], and properties of stochastic versions of (1.1) [49].

In this article, we are mainly concerned with the $L^2$ norm decay and asymptotic behaviour of solutions to (1.1). Guterres et al. [18] proved that the norm of Leray solutions tends to zero, i.e., for $z_0 \in L^2$
\[
\lim_{t \to \infty} \|z(t)\|_{L^2} = 0. \tag{1.2}
\]
Moreover, when $\chi > 0$, they obtained a sharper result for the microrotational field $w$, namely
\[
\lim_{t \to \infty} t^{\frac{1}{\chi}} \|w(t)\|_{L^2} = 0. \tag{1.3}
\]
For initial data $z_0 \in (L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^3$, Li and Shang [23] used the classical Fourier splitting method to prove that the decay has algebraic rate, i.e.,
\[
\|z(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}. \tag{1.4}
\]
With the same initial data, Cruz and Novais [9] proved that the decay rate for the microrotational field can be improved to
\[
\|w(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}. \tag{1.5}
\]
Using a method based on estimates for decay of equations on Sobolev spaces with negative indices, Tan et al. [39] proved that for initial data $z_0$ which is small in $H^N(\mathbb{R}^3)$, for $N \in \mathbb{Z}, N \geq 3$ and which also belongs to either $\dot{H}^{-s}(\mathbb{R}^3)$ or $\dot{B}^{-s}_{2,\infty}(\mathbb{R}^3)$, for $0 \leq s < \frac{3}{2}$, then
\[
\|z(t)\|_{L^2} \leq C(1+t)^{-s}. \tag{1.6}
\]
As a Corollary of this result, they proved that if $z_0 \in H^N(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, with $1 \leq p \leq 2$, $N \geq 2$ and small $H^N$ norm, then
\[
\|z(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}(\frac{3}{p}-1)}. \tag{1.7}
\]
Note that when $p = 1$ this recovers the result in (1.4), albeit the conditions imposed on the initial data to obtain (1.7) are stronger.

**Remark 1.1.** For results concerning decay of other norms, or of norms of derivatives of solutions, see Guterres et al. [18], Perusato et al. [31] and Tan et al. [39].

Our main goal in this article is to improve estimates (1.2)–(1.7) by either proving sharper results or by disposing of unnecessary hypotheses, using an unified approach. The main tools we use are the Fourier splitting method and the decay character of initial data. The Fourier splitting method was developed by Schonbek [36–38] to prove that the $L^2$ norm of solutions to viscous conservation laws and to Navier–Stokes equations decay with algebraic rate when initial data have the form of that that leads to bounds as in (1.4) and (1.7). The decay character was introduced by Bjorland and Schonbek [2] and refined by Niche and Schonbek [27] and Brandolese [4] and associates to initial data $z_0$ in $L^2$ a number $r^* = r^*(z_0)$ which characterizes the decay of solutions to a large family of linear systems which such initial data.
This, in turn, allows to prove decay estimates for nonlinear equations. For details concerning the decay character and decay of linear systems, see Sect. 2.

Our first result concerning the decay of (1.1) is the following.

**Theorem 1.2.** Let \( z \) be a weak solution to (1.1), with \( 32 \chi (\mu + \chi + \gamma) > 1, \nu > 0 \). Let \( r^*(z_0) = r^* \) be the decay character of \( z_0 \), with \(-\frac{3}{2} < r^* < \infty\). Then, for all \( t > 0 \)

\[
\|z(t)\|_{L^2}^2 \leq C(1 + t)^{-\min\left\{ \frac{7}{2} + r^*, \frac{5}{2} \right\}}.
\]

For \(-\frac{3}{2} < r^* \leq 1\), we have that

\[
\|z(t)\|_{L^2}^2 \geq C(1 + t)^{-\left( \frac{3}{2} + r^* \right)}.
\]

As computed in Example 2.6 in Ferreira et al. [12], for \( z_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \) for \( 1 \leq p < 2 \), we have that \( r^*(z_0) = -3 \left( 1 - \frac{1}{p} \right) \). Then, through Theorem 1.2 we recover estimates (1.4) and (1.7). Note, however, that Tan et al. [39] need \( r^*(z_0) \) to be small in some Sobolev space for (1.7) to hold, a hypothesis we do not need in our result. Guterres et al. [18] proved that the norm of Leray solutions goes to zero, see (1.2). In Theorem 1.2, we are able to provide a rate for this decay, as long as the initial datum obeys \(-\frac{3}{2} < r^* < \infty\). As a consequence of the results by Brandolese [4], our result also extends and improves estimate (1.6) by disposing of the small norm in \( H^N \) hypothesis and also by extending the range for which it is valid to \( 0 \leq s \leq \frac{3}{2} \). We discuss this fact in Sect. 2.3.

The equation for \( w \) in (1.1) has a feature that distinguishes it from those for \( \bar{u} \) and \( \bar{b} \) in that contains a linear damping term \( 2\chi w \). Linear equations or systems of that form use to have exponential decay, so we expect this to improve the decay of \( w \) with respect to that in Theorem 1.2.

**Theorem 1.3.** Consider the same hypothesis as in Theorem 1.2. Then, we have the improved decay estimate

\[
\|w(t)\|_{L^2}^2 \leq C(1 + t)^{-\min\left\{ \frac{7}{2} + r^*, \frac{5}{2} \right\}},
\]

for all \( t > 0 \).

Thus, the result in this theorem improves and extends the decays in (1.3) by Guterres et al. [18] and in (1.5) by Cruz and Novais [9]. Note that for any algebraic decay rate, by Theorem 1.2 we can always find initial data with appropriate \( r^* \) that leads to a solution \( z \) with decay slower than this given one. However, from Theorem 1.3 the decay of \( \|w(t)\|_{L^2} \) will be at least of order \( (1 + t)^{-\frac{3}{2}} \) for any initial datum \( w_0 \). This is a consequence of the exponential decay of the linear part of the equation for \( w \) caused by the linear damping.

We now address first-order asymptotics, by studying the decay of the difference between the full solution \( z(t) \) and \( \bar{z}(t) \), the solution of its linear part

\[
\begin{align*}
\ddot{u}_t &= (\mu + \chi) \Delta \dot{u} + \chi \nabla \times \dot{w}, \\
\ddot{w}_t &= \gamma \Delta \dot{w} + \nabla (\nabla \cdot \dot{w}) + \chi \nabla \times \dot{u} - 2\chi \dot{w}, \\
\ddot{b}_t &= \nu \Delta \dot{b}
\end{align*}
\]

with the same initial data.

**Theorem 1.4.** Let \( z \) be a weak solution to (1.1), with \( 32 \chi (\mu + \chi + \gamma) > 1, \nu > 0 \). Let \( r^*(z_0) = r^* \) be the decay character of \( z_0 \), with \(-\frac{3}{2} < r^* < \infty\). Then,

\[
\|z(t) - \bar{z}(t)\|_{L^2}^2 \leq C(1 + t)^{-\min\left\{ \frac{7}{2} + 2r^*, \frac{5}{2} \right\}}, \quad \forall t > 0
\]

and

\[
\|w(t) - \dot{w}(t)\|_{L^2}^2 \leq C(1 + t)^{-\min\left\{ \frac{7}{2} + 2r^*, \frac{5}{2} \right\}}, \quad \forall t > 0.
\]

As in the previous results, the exponential decay of the linear part of the equation for \( w \) leads to a faster decay in the corresponding asymptotic behaviour.
This work is organized as follows: In Sect. 2, we gather all definition and results concerning the decay character and its use for establishing decay for linear systems. More precisely, in Sect. 2.1 we define the decay character and state Theorem 2.3 (from Niche and Schonbek [27]) in which sharp upper and lower bounds are proved for “diagonalizable” systems. In Sect. 2.2, we specifically apply the results from the previous section to the linear part of (1.1). To wit, we first establish a relation between the decay character of \( x_0 \) and those of \( u_0, w_0 \) and \( b_0 \). Then, we prove a Lemma that allows us to effectively use Theorem 2.3. Finally, in Sect. 2.3 we carefully discuss the work of Brandolese [4], which we use to show that Theorem 1.2 extends some previously known estimates. In Sect. 3, we prove our results. We point out that some gradient estimates proved in this section, more specifically Lemmas 3.1, 3.2 and 3.3, may be of independent interest.

2. Decay character and decay of linear part

2.1. Decay character and linear operators

In order to establish sharp decay rates for the linear part in (1.1), we recall the idea of decay character, as defined and developed by Bjorland and Schonbek [2], Niche and Schonbek [27] and Brandolese [4].

As the long-time evolution of the norm of solutions is determined by its low frequencies, it is expected that the small frequencies of the initial datum provide insight into the decay of the \( L^2 \) or Sobolev norms of linear systems. Roughly speaking, the decay character compares \( |\hat{w}_0(\xi)|^2 \) to \( f(\xi) = |\xi|^{2r} \) near \( \xi = 0 \).

**Definition 2.1.** Let \( v_0 \in L^2(\mathbb{R}^n) \). For \( r < \left( -\frac{n}{2}, \infty \right) \), we define the decay indicator \( P_r(v_0) \) corresponding to \( v_0 \) as

\[
P_r(v_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\hat{v}_0(\xi)|^2 \, d\xi,
\]

provided this limit exists. In the expression above, \( B(\rho) \) denotes the ball at the origin with radius \( \rho \).

**Definition 2.2.** The decay character of \( v_0 \), denoted by \( r^* = r^*(v_0) \), is the unique \( r \in \left( -\frac{n}{2}, \infty \right) \) such that \( 0 < P_r(v_0) < \infty \), provided that this number exists. If such \( P_r(v_0) \) does not exist, we set \( r^* = -\frac{n}{2} \), when \( P_r(v_0) = \infty \) for all \( r \in \left( -\frac{n}{2}, \infty \right) \) or \( r^* = \infty \), if \( P_r(v_0) = 0 \) for all \( r \in \left( -\frac{n}{2}, \infty \right) \).

The decay character can be explicitly computed in many cases. For example as pointed out in the Introduction, when \( v_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) for \( 1 \leq p < 2 \), we have that \( r^*(v_0) = -n \left( 1 - \frac{1}{p} \right) \), see Example 2.6 in Ferreira et al. [12]. For more details, see Example 2.5 in Niche and Schonbek [27].

We now use the decay character for establishing upper and lower bounds for decay rates of energy for solutions to a large family of dissipative linear operators. For a Hilbert space \( X \) on \( \mathbb{R}^n \), we consider a pseudodifferential operator \( \mathcal{L} : X^n \to (L^2(\mathbb{R}^n))^n \), with symbol \( \mathcal{M}(\xi) \) such that

\[
\mathcal{M}(\xi) = P^{-1}(\xi) D(\xi) P(\xi), \quad \xi \text{ - a.e.} \quad (2.1)
\]

where \( P(\xi) \in O(n) \) and \( D(\xi) = -c_1|\xi|^{2\alpha} \delta_{ij} \), for \( c_1 > c > 0 \) and \( 0 < \alpha \leq 1 \). Taking the Fourier transform of the linear equation

\[
v_t = \mathcal{L}v, \quad (2.2)
\]

multiplying by \( \hat{v} \), integrating in space and then using (2.1), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\hat{v}(t)\|_{L^2}^2 \leq -C \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{v}|^2 \, d\xi, \quad (2.3)
\]
which is the key inequality for using the Fourier splitting method. The vectorial fractional Laplacian and the operator
\[ \mathcal{L}u = \Delta u + \nabla \text{div} u \] (2.4)
obey (2.1), so they are amenable to our analysis, see Examples 2.8 and 2.9 in Niche and Schonbek [27].

We now state the theorem that describes decay in terms of the decay character for linear operators as in (2.1).

**Theorem 2.3.** (Theorem 2.10, Niche and Schonbek [27]) Let \( v_0 \in L^2(\mathbb{R}^n) \) have decay character \( r^*(v_0) = r^* \). Let \( v(t) \) be a solution to (2.2) with data \( v_0 \), where the operator \( \mathcal{L} \) is such that (2.1) holds. Then, if \(-\frac{n}{2} < r^* < \infty\), there exist constants \( C_1, C_2 > 0 \) such that
\[ C_1(1 + t)^{-\frac{1}{2}(\frac{n}{2} + r^*)} \leq \|v(t)\|_{L^2} \leq C_2(1 + t)^{-\frac{1}{2}(\frac{n}{2} + r^*)}. \]

**2.2. Decay characterization for the linear part of (1.1)**

We now study the linear system associated with (1.1), namely
\[
\begin{align*}
\partial_t \tilde{u} &= (\mu + \chi)\Delta \tilde{u} + \chi \nabla \times \tilde{w}, \\
\partial_t \tilde{w} &= \chi \nabla \cdot \tilde{w} \times \nabla \times \tilde{u} - 2\chi \tilde{w}, \\
\partial_t \tilde{b} &= \nu \Delta \tilde{b}
\end{align*}
\] (2.5)

with initial data \( \tilde{z}_0 = z_0 = (u_0, w_0, b_0) \) \( \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \), where we set \( \tilde{z} = (\tilde{u}, \tilde{w}, \tilde{b}) \subset L^2(\mathbb{R}^3) \), for simplicity.

We first address the relation between \( r^*(z_0), r^*(u_0), r^*(w_0) \) and \( r^*(b_0) \).

**Lemma 2.4.** Let \( r^*(u_0), r^*(w_0), r^*(b_0) \in (-\frac{3}{2}, \infty) \). Then,
\[ r^*(z_0) = \min\{r^*(u_0), r^*(w_0), r^*(b_0)\}. \]

**Proof.** Let \( \lambda = \min\{r^*(u_0), r^*(w_0), r^*(b_0)\} \). In order to fix ideas, suppose \( \lambda = r^*(u_0) \) and \( \lambda < r^*(w_0), r^*(b_0) \). Note that
\[ P_\lambda(z_0) = P_\lambda(u_0) + P_\lambda(w_0) + P_\lambda(b_0) \]
and that \( P_\lambda(u_0) > 0 \). Now
\[ P_\lambda(w_0) = \lim_{\rho \to 0} \rho^{-(2\lambda + 3)} \int_{B(\rho)} |\tilde{w}_0(\xi)|^2 \, d\xi \]
\[ = \lim_{\rho \to 0} \rho^{-2(\lambda + (r^*(w_0) - \lambda))} \rho^{2(r^*(w_0) - \lambda)} \rho^{-3} \int_{B(\rho)} |\tilde{w}_0(\xi)|^2 \, d\xi \]
\[ = \lim_{\rho \to 0} \rho^{2(r^*(w_0) - \lambda)} \rho^{-(2r^*(w_0) + 3)} \int_{B(\rho)} |\tilde{w}_0(\xi)|^2 \, d\xi \]
\[ = \lim_{\rho \to 0} \rho^{2(r^*(w_0) - \lambda)} r^*(w_0) = 0 \]
because \( r^*(w_0) > \lambda \). The same argument proves that \( P_\lambda(b_0) = 0 \), and hence, \( P_\lambda(z_0) = P_\lambda(u_0) \), which leads to the result. \( \square \)
In order to use Theorem 2.3, we pass to frequency space, where after taking the Fourier transform of (2.5) we obtain
\[ \partial_t \hat{x} = M(\xi) \hat{x}, \]
where \( M = M(\xi) \) is the matrix of symbols given by
\[
M = \begin{pmatrix}
-(\mu + \chi)|\xi|^2 I_{3 \times 3} & i\chi R_3(\xi) & 0_{3 \times 3} \\
i\chi R_3(\xi) & -(\gamma|\xi|^2 + 2\chi)I_{3 \times 3} - \xi_1 \xi_j & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & -\nu|\xi|^2 I_{3 \times 3}
\end{pmatrix}. \tag{2.6}
\]
Here \( I_{3 \times 3} \) and \( 0_{3 \times 3} \) are the \( 3 \times 3 \) identity and zero matrices, respectively, and \( iR_3(\xi) \) is the rotation matrix
\[
iR_3(\xi) = \begin{pmatrix}
0 & \xi_3 & -\xi_2 \\
-\xi_3 & 0 & \xi_1 \\
\xi_2 & -\xi_1 & 0
\end{pmatrix}.
\]
As \( M \) is self-adjoint, it is diagonalizable and \( M(\xi) = P^{-1}(\xi)D(\xi)P(\xi), \) where \( P \in U(n) \) and \( D(\xi) \) is a diagonal matrix. To use Theorem 2.3, we would need to compute the eigenvalues of \( M(\xi) \), which is a cumbersome task. Instead, we will prove the following lemma, which provides an estimate for the largest eigenvalue. This immediately leads to (2.3) and allows us to use Theorem 2.3.

**Lemma 2.5.** Let \( 32\chi(\mu + \chi + \gamma) > 1. \) Then, for \( M = M(\xi) \) we have that
\[
\lambda_{\max}(M) \leq -C|\xi|^2, \quad C = C(\mu, \chi, \gamma, \nu) > 0.
\]

**Proof.** We follow the ideas in Section 3.1 in Ferreira and Villamizar-Roa [11]. From the Rayleigh–Ritz theorem, we know that for any Hermitian matrix \( M \in M_n(\mathbb{C}) \), the inequality
\[
R_M(v) = v^* M v \leq \lambda_{\max}(M)
\]
holds, for all \( \|v\|_2 = 1. \)

A simple computation shows that the matrix \( iR_3(\xi) \) has spectrum \( \sigma(iR_3(\xi)) = \{ -|\xi|, 0, |\xi| \} \), with associated orthonormal eigenvectors \( v_1, v_2, v_3 \). With these, we construct the orthonormal basis \( B = (b_1, \cdots, b_n) \) for \( \mathbb{C}^9 \), where
\[
B = \left\{ \frac{1}{2}(v_1, v_1, 0), \frac{1}{2}(v_3, -v_3, 0), \frac{1}{2}(v_2, v_2, 0), \frac{1}{2}(v_2, -v_2, 0), \frac{1}{2}(v_3, v_3, 0), \frac{1}{2}(v_1, -v_1, 0), e_7, e_8, e_9 \right\}
\]
where \( e_7, e_8, e_9 \) are the last three vectors in the canonical base in \( \mathbb{C}^9 \).

We now write \( M(\xi) = M_1(\xi) + M_2(\xi) + M_3(\xi), \) where
\[
M_1(\xi) = \begin{pmatrix}
-(\mu + \chi)|\xi|^2 I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & -(\gamma|\xi|^2 + 2\chi)I_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & -\nu|\xi|^2 I_{3 \times 3}
\end{pmatrix}
\]
\[
M_2(\xi) = \begin{pmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & -\xi_1 \xi_j & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix}, \quad M_3(\xi) = \begin{pmatrix}
0_{3 \times 3} & i\chi R_3(\xi) & 0_{3 \times 3} \\
i\chi R_3(\xi) & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix}.
\]

As the eigenvalues of \( M_2 \) are 0 and \( -|\xi| \), we have that \( v^* M_2(\xi)v \leq 0. \) Now take \( v = \sum_{i=1}^9 c_i b_i \). This leads to
\[
v^* M_1 v = -\frac{1}{2} \left( c_1^2 + c_2^2 + c_5^2 + c_6^2 \right) ((\mu + \chi)|\xi|^2 + \gamma|\xi|^2 + 2\chi) - \frac{1}{2} c_4^2 (c_7^2 + c_8^2 + c_9^2)\nu|\xi|^2
\]

\[ - \frac{1}{2} c_4^2 (\mu + \chi)|\xi|^2 - \frac{1}{2} c_4^2 (\gamma|\xi|^2 + 2\chi) - (c_7^2 + c_8^2 + c_9^2)\nu|\xi|^2. \]
and to
\[ v^* M_3 v = - \frac{1}{4} (c_1^2 + c_6^2) |\xi| + \frac{1}{4} (c_2^2 + c_5^2) |\xi|. \]
Then,
\[ v^* M_1 v + v^* M_3 v = - (c_1^2 + c_6^2) \left( \frac{1}{2} (\mu + \chi + \gamma) |\xi|^2 + \frac{1}{4} |\xi| + \chi \right) \]
\[ \quad - (c_2^2 + c_5^2) \left( \frac{1}{2} (\mu + \chi + \gamma) |\xi|^2 - \frac{1}{4} |\xi| + \chi \right) \]
\[ \quad - \frac{1}{2} c_3^2 (\mu + \chi) |\xi|^2 - \frac{1}{2} c_4^2 (\gamma |\xi|^2 + 2\chi) - (c_7^2 + c_8^2 + c_9^2) \nu |\xi|^2 \]
\[ \leq - \|v\|_2^2 \min \{ (\mu + \chi + \gamma) |\xi|^2 - \frac{1}{2} |\xi| + 2\chi, (\mu + \chi) |\xi|^2, \gamma |\xi|^2 + 2\chi, 2\nu |\xi|^2 \}. \]
If \(32(\mu + \chi + \gamma) > 1\), then \((\mu + \chi + \gamma)|\xi|^2 - \frac{1}{2} |\xi| + 2\chi > 0\) for any \(|\xi|\) and the result follows. \(\square\)

The estimate obtained in Lemma 2.5 leads to (2.3). We can now use Theorem 2.3 to obtain

**Theorem 2.6.** Let \(\tilde{z}_0 \in L^2(\mathbb{R}^3)\) have decay character \(r^*(\tilde{z}_0) = r^*\). Then if \(-\frac{n}{2} < r^* < \infty\), there exist constants \(C_1, C_2 > 0\) such that
\[ C_1 (1 + t)^{-\left(\frac{n}{2} + r^*\right)} \leq \|\tilde{z}(t)\|_{L^2} \leq C_2 (1 + t)^{-\left(\frac{n}{2} + r^*\right)}. \]

As the symbol matrix (2.6) is diagonalizable, the linear system (2.5) decouples. The fact that the decay character of \(z_0\) is the minimum of the decay characters of \(u_0, w_0, b_0\) implies that decay of solutions \(\tilde{z}(t)\) to (2.5) is the slowest of the decays of \(\tilde{u}(t), \tilde{w}(t), \tilde{b}(t)\).

### 2.3. The work of Brandolese [4] and estimate (1.6)

The decay character of initial data \(v_0 \in L^2(\mathbb{R}^n)\) is used to prove sharp upper and lower bounds for decay of “diagonalizable” linear systems, see Theorem 2.3 and its application to (2.5) in Theorem 2.6. In Definitions 2.1 and 2.2, introduced by Bjorland and Schonbek [2] and extended by Niche and Schonbek [27], the existence of a limit and a positive \(P_\gamma(u_0)\) is assumed. However, this need not be the case for all of \(v_0 \in L^2(\mathbb{R}^n)\). Brandolese [4] constructed initial data in \(L^2\), highly oscillating near the origin, for which the limit in Definition 2.1 does not exist for some \(r\). As a result of this, the decay character does not exist. Then, Brandolese gave a slightly different definition of decay character, more general than that in Definitions 2.1 and 2.2, but which produces the same result when these hold.

In this same article, Brandolese proved that the decay character \(r^*\) (in his more general version) exists for \(v_0 \in L^2(\mathbb{R}^n)\) if and only if \(v_0\) belongs to a certain subset \(\tilde{A}_{2,\infty}^{-\left(\frac{n}{2} + r^*\right)} \subset \tilde{B}_{2,\infty}^{-\left(\frac{n}{2} + r^*\right)}\). Moreover, for diagonalizable linear operators \(\mathcal{L}\) as in (2.1), solutions to the linear system (2.2) with initial data \(v_0\) obey
\[ C_1 (1 + t)^{-\frac{1}{2}\left(\frac{n}{2} + r^*\right)} \leq \|v(t)\|_{L^2}^2 \leq C_2 (1 + t)^{-\frac{1}{2}\left(\frac{n}{2} + r^*\right)}, \]
if and only if the decay character \(r^* = r^*(v_0)\) exists. This provides a sharp characterization of algebraic decay rates for such systems and provides a key tool for studying decay for nonlinear systems.

Now, let us recall estimate (1.6), proved by Tan et al. [39]. By taking \(s = -\left(\frac{n}{2} + r^*\right)\), where \(-\frac{n}{2} \leq r^* < 0\), their estimate reads
\[ \|z(t)\|_{L^2}^2 \leq C(1 + t)^{-\left(\frac{n}{2} + r^*\right)}, \]
(2.7) for \(z_0 \in H^N(\mathbb{R}^3) \cap \tilde{B}_{2,\infty}^{-\left(\frac{n}{2} + r^*\right)}(\mathbb{R}^3)\) with small \(H^N\) norm, for some \(N \geq 3\). As a result of Brandolese’s results discussed above, existence of (our version of) the decay character implies that using Theorem 1.2
we obtain (2.7) without the necessity of assuming \( z_0 \in H^N(\mathbb{R}^3) \). Moreover, our result shows that (2.7) also holds for \( \frac{3}{2} \leq s \leq \frac{5}{2} \).

### 3. Proofs

#### 3.1. Proof of Theorem 1.2

**Proof.** As usual when using the Fourier splitting, we prove decay for regular enough solutions to an approximate nonlinear problem obtained through spectral cut-off, as in Li and Shang [23] for the magnetomicrofluidic equations or through retarded mollifiers (see Cafarelli, Kohn and Nirenberg [8]), as in the case of the micropolar fluid equations, see Braz e Silva et al. [5]. The decay for weak solutions is obtained through a standard limiting process; for full details, see Braz e Silva et al. [5] for the micropolar fluid equations, Lemarié-Rieusset [22] and Appendix in Wiegner [46] for the Navier–Stokes equations case.

We now proceed formally. As we have seen before

\[
\partial_t \|z(t)\|^2_{L^2} \leq -C \|
abla z(t)\|^2_{L^2}.
\]

Let \( B(t) = \{ \xi \in \mathbb{R}^3 : |\xi| \leq g(t) \} \), for a nonincreasing, continuous \( g \) with \( g(0) = 1 \). Then

\[
- C \| \nabla z(t)\|^2_{L^2} = -C \int_{B(t)} |\xi|^2 |\hat{z}(\xi, t)|^2 \, d\xi - C \int_{B(t)^c} |\xi|^2 |\hat{z}(\xi, t)|^2 \, d\xi
\]

\[
\leq -C \int_{B(t)} |\xi|^2 |\hat{z}(\xi, t)|^2 \, d\xi \leq -C g^2(t) \int_{B(t)^c} |\hat{z}(\xi, t)|^2 \, d\xi
\]

which leads to

\[
\frac{d}{dt} \left( \exp \left( \int_0^t Cg^2(s) \, ds \right) \|z(t)\|^2_{L^2} \right) \leq g^2(t) \left( \exp \left( \int_0^t Cg^2(s) \, ds \right) \right) \int_{B(t)} |\hat{z}(\xi, t)|^2 \, d\xi.
\]  
(3.1)

We now need a pointwise estimate for

\[
\hat{z}(\xi, t) = e^{tM(\xi)} \hat{z}_0(\xi) - \int_0^t e^{(t-s)M(\xi)} G(\xi, s) \, ds
\]

where \( M \) is as in (2.6) and

\[
G(\xi, s) = \mathcal{F} \left( NL(u, w, b) \right)(\xi, s)
\]

for

\[
NL(u, w, b) = ((b \cdot \nabla)b - (u \cdot \nabla)u - \nabla p, -(u \cdot \nabla)w, -(u \cdot \nabla)b + (b \cdot \nabla)u).
\]  
(3.2)

For \( F = u, b \) and \( G = u, w, b \), we have that

\[
\mathcal{F} ((F \cdot \nabla)G) = \mathcal{F} (\nabla (F \otimes G)) = i\xi \cdot \left( \mathcal{F} (\otimes G) \right),
\]

so

\[
|\mathcal{F} ((F \cdot \nabla)G)(\xi)| \leq |\xi| \|F\|_{L^2} \|G\|_{L^2}.
\]  
(3.3)

By taking divergence in the first equation in (1.1), we obtain

\[
\Delta p = \text{div} (b \cdot \nabla)b - \text{div} (u \cdot \nabla)u
\]
from which we get
\[- |\xi|^2 \tilde{p}(\xi) = i \sum_{j,k} \xi_j \xi_k \tilde{b}_j \tilde{b}_k + i \sum_{j,k} \xi_j \xi_k \tilde{u}_j \tilde{u}_k \leq |\xi|^2 \left( \|\tilde{b}(t)\|_{L^2}^2 + \|\tilde{u}(t)\|_{L^2}^2 \right). \tag{3.4} \]

Then, from (3.2), (3.3) and (3.4), we obtain
\[|G(\xi, t)| \leq C|\xi|\|\tilde{z}(t)\|_{L^2}^2. \tag{3.5} \]

Thus,
\[
\left| \int_0^t e^{(t-s)M(\xi)} G(\xi, s) \, ds \right| \leq C \int_0^t e^{-C(t-s)|\xi|^2} |\xi|\|\tilde{z}(s)\|_{L^2}^2 \, ds \\
\leq C|\xi| \left( \int_0^t \|\tilde{z}(s)\|_{L^2}^2 \, ds \right),
\]

where we used the estimate in Lemma 2.5.

Suppose now that \(\|\tilde{z}(t)\|_{L^2}^2 \leq C(1+t)^{-\alpha}\), for some \(0 \leq \alpha\). As a result of this
\[
\int_{B(t)} \left( \int_0^t e^{(t-s)M(\xi)} G(\xi, s) \, ds \right)^2 \, d\xi \leq C|\xi|^5(1+t)^{2(1-\alpha)},
\]

which leads, after choosing \(g^2(t) = A(1+t)^{-1}\) and for large enough \(A > 0\), to
\[
\int_{B(t)} |\tilde{z}(\xi, t)|^2 \, d\xi \leq C \int_{B(t)} |e^{tM(\xi)} \tilde{z}_0|^2 \, d\xi + C \int_{B(t)} \left( \int_0^t e^{(t-s)M(\xi)} G(\xi, s) \, ds \right)^2 \, d\xi \\
\leq C\|e^{tM(\xi)} \tilde{z}_0\|_{L^2}^2 + Cg^5(t)(1+t)^{2(1-\alpha)} \\
\leq C(t+1)^{-\left(\frac{3}{2}+r^*\right)} + C(1+t)^{-\left(\frac{1}{2}+2\alpha\right)} \\
\leq C(t+1)^{-\min\left\{\frac{1}{2}+2\alpha, \frac{3}{2}+r^*\right\}}, \tag{3.6}
\]

where \(r^* = r^*(\tilde{z}_0)\) and we used Theorem 2.6 for the decay of the linear part. From (3.1), (3.6) and our choice of \(g\), we obtain
\[
\frac{d}{dt} \left( (t+1)^A \|\tilde{z}(t)\|_{L^2}^2 \right) \leq C(t+1)^{A-1}(t+1)^{-\min\left\{\frac{1}{2}+2\alpha, \frac{3}{2}+r^*\right\}}. \tag{3.7}
\]

We start with \(\alpha = 0\); this is the apriori estimate \(\|\tilde{z}(t)\|_{L^2}^2 \leq C\). In (3.7), we consider the two cases \(\frac{3}{2} + r^* \leq \frac{1}{2}\) and \(\frac{1}{2} \leq \frac{3}{2} + r^*\). In the first case, i.e., when \(r^* \leq -1\), we have
\[
\|\tilde{z}(t)\|_{L^2}^2 \leq C(t+1)^{-\left(\frac{3}{2}+r^*\right)}. \tag{3.8}
\]

In the second case, we obtain
\[
\|\tilde{z}(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{1}{2}},
\]

which is the slower decay. Hence, we improved our rate to an exponent at least as large as \(\alpha = \frac{1}{2}\). We use this estimate to bootstrap in (3.7) and we have to separate again the study in two cases, namely \(\frac{3}{2} + r^* \leq \frac{3}{2}\) and \(\frac{3}{2} \leq \frac{3}{2} + r^*\). In the first case, which corresponds to \(r^* \leq 0\), we obtain (3.8) again. In the second situation, i.e., when \(r^* \geq 0\), we have improved to \(\alpha = \frac{3}{2}\). But then
\[
\int_0^t \|\tilde{z}(s)\|_{L^2}^2 \, ds \leq C,
\]
Lemma 3.1. Let
\[
\int_{B(t)} \left( \int_{0}^{t} e^{(t-s)M(\xi)} G(\xi, s) \, ds \right)^{2} \, d\xi \leq C g^{5}(t).
\]
Then, (3.6) becomes
\[
\int_{B(t)} |\tilde{z}(\xi, t)|^{2} \, d\xi \leq C \int_{B(t)} |e^{tM(\xi)} \tilde{z}_{0}|^{2} \, d\xi + C \int_{B(t)} \left( \int_{0}^{t} e^{(t-s)M(\xi)} G(\xi, s) \, ds \right)^{2} \, d\xi
\]
\[
\leq C \|e^{tM(\xi)} \tilde{z}_{0}\|_{L^{2}}^{2} + C g^{5}(t)
\]
\[
\leq C (t+1)^{-\left(\frac{3}{2} + r\right)} + C (1 + t)^{-\frac{\alpha}{2}} \leq C (t+1)^{-\min\left\{ \frac{3}{2} + r, \frac{\alpha}{2} \right\}}.
\]
Using this in (3.1) yields the upper bound for decay.

The reverse triangle inequality leads to
\[
\|z(t)\|_{L^{2}} \geq \|\tilde{z}(t)\|_{L^{2}} - \|z(t) - \tilde{z}(t)\|_{L^{2}},
\]
where \(\tilde{z}(t)\) is the solution to the linear part of (1.1), i.e., system (2.5). By Theorem 1.4, we only have upper bounds for the decay of \(\|z(t) - \tilde{z}(t)\|_{L^{2}}\), so the upper bound we have just proved and Theorem 1.4 lead to the lower bound only when the decay of linear part is slower than that of the difference. \(\square\)

3.2. Proof of Theorem 1.3

Our proof follows that of Theorem 3.2 in Braz et al. [5], where an analogous result is proved for the micropolar system for \(z_{0} \in L^{1}(\mathbb{R}^{3}) \cap L^{2}(\mathbb{R}^{3})\).

In the proof of the following Lemmas and Theorems, we will need the standard heat kernel estimate in \(\mathbb{R}^{3}\)
\[
\|\nabla e^{t\Delta} f\|_{L^{q}} \leq K \|f\|_{L^{r}} \, t^{-\frac{3}{2}} (\frac{1}{\frac{1}{r} - \frac{1}{2}}), \quad \forall t > 0,
\]
for \(1 \leq r \leq q \leq \infty\) (see Kato [19]). We will also need the following gradient estimate.

Lemma 3.1. Let \(z\) be a weak solution to (1.1), with \(32 \chi(\mu + \chi + \gamma) > 1, \nu > 0\). Let \(r^{*}(z_{0}) = r^{*}\) be the decay character of \(z_{0}\), with \(-\frac{3}{2} < r^{*} < \infty\). Then,
\[
\|\nabla z(t)\|_{L^{2}}^{2} \leq C (1 + t)^{-\min\left\{ \frac{3}{2} + r^{*}, \frac{\alpha}{2} \right\}}, \quad \forall t \geq t_{0},
\]
for an appropriate, large enough \(t_{0} = t_{0}(\|z_{0}\|_{L^{2}})\).

Proof. We follow the ideas in Braz et al. [5] and Gutierrez et al. [18]. Let \(\alpha(r^{*}) = \min\left\{ \frac{3}{2} + r^{*}, \frac{\alpha}{2} \right\}\). Taking \(\delta > 0\), multiplying (1.1) by \((1 + t)^{\alpha(r^{*}) + \delta} (u, w, b)\) and integrating on \(\mathbb{R}^{3} \times (t_{0}, t)\), we obtain
\[
(1 + t)^{\alpha(r^{*}) + \delta} \|z(t)\|_{L^{2}}^{2} + 2 \min\{\mu, \gamma, \nu\} \int_{t_{0}}^{t} (1 + s)^{\alpha(r^{*}) + \delta} \|\nabla z(s)\|_{L^{2}}^{2} \, ds
\]
\[
\leq C \int_{t_{0}}^{t} (1 + s)^{\alpha(r^{*}) + \delta - 1} \|z(s)\|_{L^{2}}^{2} \, ds
\]
\[
\leq C (1 + t)^{\delta},
\]
where we used Theorem 1.2 on the right-hand side.
We now use the notation \(D_k = \partial_{x_k}, D^2 = \sum_{i,j} D_i D_j\). Taking \(D_k\) in the first three equations in (1.1), multiplying by \((1 + t)^{\alpha(r^*)+\delta+1}(D_k u, D_k w, D_k b)\) and summing up, after integrating in \(\mathbb{R}^3 \times (t_0, t)\) we obtain

\[
(1 + t)^{\alpha(r^*)+\delta+1}\|\nabla z\|_{L^2}^2 + 2(\mu + \chi) \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\|D^2 u(s)\|_{L^2}^2 \, ds
\]

\[
+ 2\gamma \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\|D^2 w(s)\|_{L^2}^2 \, ds + 2\nu \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\|D^2 b(s)\|_{L^2}^2 \, ds
\]

\[
+ 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\|\nabla (\nabla \cdot w)(s)\|_{L^2}^2 \, ds + 4\chi \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\|\nabla w(s)\|_{L^2}^2 \, ds
\]

\[
= \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta}\|\nabla z(s)\|_{L^2}^2 \, ds + 4\chi \sum_{k=1}^3 \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\langle \nabla \times D_k u(s), D_k w(s) \rangle \, ds
\]

\[
+ 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\int_{\mathbb{R}^3} \sum_{i,j,k} D_j D_k u_i(x, s) \cdot D_k (u_j(x, s) u_i(x, s)) \, dx \, dx
\]

\[
- 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\int_{\mathbb{R}^3} \sum_{i,j,k} D_j D_k b_i(x, s) \cdot D_k (b_j(x, s) b_i(x, s)) \, dx \, dx
\]

\[
+ 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\int_{\mathbb{R}^3} \sum_{i,j,k} D_j D_k u_i(x, s) \cdot D_k (u_j(x, s) w_i(x, s)) \, dx \, dx
\]

\[
+ 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\int_{\mathbb{R}^3} \sum_{i,j,k} D_j D_k u_i(x, s) \cdot D_k (u_j(x, s) b_i(x, s)) \, dx \, dx
\]

\[
- 2\int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\int_{\mathbb{R}^3} \sum_{i,j,k} D_j D_k b_i(x, s) \cdot D_k (b_j(x, s) u_i(x, s)) \, dx \, dx.
\]

By Cauchy–Schwarz

\[
4\chi \sum_{k=1}^3 \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1}\langle \nabla \times D_k u(s), D_k w(s) \rangle \, ds
\]

\[
\leq 4\chi \int_{t_0}^t (1 + s)^{\alpha(r^*)+\delta+1} \left(\|\nabla w(s)\|_{L^2}^2 + \|D^2 u(s)\|_{L^2}^2\right) \, ds.
\]

For \(D_j D_k f_i(x, s) \cdot D_k (g_j(x, s) h_i(x, s))\), where \(f, g, h \in \{u, w, b\}\) we have that

\[
\int_{\mathbb{R}^3} D_j D_k f_i(x, s) \cdot D_k (g_j(x, s) h_i(x, s)) \, dx \leq C \|z(s)\|_{L^\infty} \|\nabla z(s)\|_{L^2} \|D^2 z(s)\|_{L^2}.
\]
We then obtain
\[
(1 + t)^{\alpha(r^*) + \delta + 1} \| \nabla z(t) \|^2_{L^2} + C \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta + 1} \| D^2 z(s) \|^2_{L^2} ds
\]
\[
\leq \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta} \| \nabla z(s) \|^2_{L^2} ds
\]
\[
+ C \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta + 1} \| z(s) \|_L \| \nabla z(s) \|_L \| D^2 z(s) \|_L ds
\]
\[
\leq \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta} \| \nabla z(s) \|^2_{L^2} ds
\]
\[
+ C \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta + 1} \| \nabla z(s) \|^2_{L^2} ds,
\]
where we used that \( \| z \|_L \| \nabla z \|_L \leq \| z \|_{L^2}^{1/3} \| \nabla z \|_{L^2}^{1/3} \| D^2 z \|_L \) (see Kreiss et al. [20]) and that \( \| z(t) \|_L \leq C \). For some large enough \( t_0 \), we have that if \( t > t_0 \), then \( \| \nabla z(t) \|_L \) is small enough for the last term on the right-hand side to absorb by the second term in the left-hand side. Hence,
\[
(1 + t)^{\alpha(r^*) + \delta + 1} \| \nabla z(t) \|^2_{L^2} + \min\{\mu, \gamma, \nu\} \int_{t_0}^t (1 + s)^{\alpha(r^*) + \delta + 1} \| \nabla^2 z(s) \|^2_{L^2} ds \leq C(1 + t)^\delta \tag{3.11}
\]
which is the result we wanted to prove.

\[\square\]

**Proof of Theorem 1.3.** As in the proof of Theorem 1.2, we assume solutions are regular enough. We note that the equation for \( w \) in (1.1) can be written as:
\[
\partial_t w = \mathbb{L} w + \chi \nabla \times u - (u \cdot \nabla) w,
\]
where \( \mathbb{L} = \gamma \Delta w + \nabla (\nabla \cdot w) - 2 \chi w \). Let \( t_0 > 0 \) be as in Lemma 3.1. Then, for \( \mathcal{L} = \gamma \Delta w + \nabla (\nabla \cdot w) \), we have that
\[
w(x, t) = e^{-2\chi(t-t_0)} e^{\mathcal{L}(t-t_0)} w(x, t_0)
\]
\[
- \int_{t_0}^t e^{-2\chi(t-s)} e^{\mathcal{L}(t-s)} \left( \nabla \times u - (u \cdot \nabla) w \right) (x, s) ds. \tag{3.12}
\]
First, as \( \| e^{\mathcal{L} t} \|_{L^2} \leq C \| v \|_{L^2} \), we have that
\[
\| e^{-2\chi(t-t_0)} e^{\mathcal{L}(t-t_0)} w(t_0) \|_{L^2} \leq C e^{-2\chi(t-t_0)} , \quad C = C \left( \| w_0 \|_{L^2} \right). \tag{3.13}
\]
As \( \| \nabla \times u \|_{L^2} \leq C \| \nabla u \|_{L^2} \), then
\[
\| e^{\mathcal{L}(t-s)} \nabla \times u(s) \|_{L^2} \leq C \| \nabla u(s) \|_{L^2} \leq C(1 + s)^{-\frac{1}{2} \min\{\frac{3}{2} + r^*, \frac{7}{4}\}}, \tag{3.14}
\]
because of Lemma 3.1. Also
\[
\| e^{\mathcal{L}(t-s)} (u \cdot \nabla) w(s) \|_{L^2} \leq \| e^{\Delta(t-s)} (u \cdot \nabla) w(s) \|_{L^2} \leq \| e^{\Delta(t-s)} \|_{L^2} \| (u \cdot \nabla) w(s) \|_{L^1} \leq \| e^{\Delta(t-s)} \|_{L^2} \| u(s) \|_{L^2} \| \nabla w(s) \|_{L^2} \leq C(t - s)^{-\frac{3}{4}} (1 + s)^{-\frac{1}{2} \min\{\frac{3}{2} + r^*, \frac{7}{4}\}}, \tag{3.15}
\]
where we used (3.9) and Lemma 3.1. Now, from (3.12)–(3.15), we obtain
\[ \| w(t) \|_{L^2} \leq C e^{-2\chi(t-t_0)} + \int_{t_0}^{t} e^{-2\chi(t-s)} (1 + s)^{-\min\{\frac{5}{4}, r^*, \frac{7}{4}\}} (1 + (t - s)^{-\frac{3}{2}}) \, ds \]
\[ \leq \tilde{C}(1 + t)^{-\frac{1}{2} \min\{\frac{5}{4}, r^*, \frac{7}{4}\}}, \quad \forall t > t_0. \]
We now conclude the proof, showing that the estimate also holds in \( 0 < t < t_0 \). Indeed, letting \( M = \max_{0 \leq \tau \leq t_0} \{(1 + \tau)^n \| w(\tau) \|_{L^2} \} \), where we set \( \kappa = \min\{\frac{5}{4}, r^*, \frac{7}{4}\} \), we clearly get \( \| w(t) \|_{L^2} \leq C(1 + t)^{-\kappa} \), for all \( t > 0 \), with \( C = \max\{M, \tilde{C}\} \).

### 3.3. Proof of Theorem 1.4

We will need a gradient decay estimate.

**Lemma 3.2.** Let \( z \) be a weak solution to (1.1), with \( 32 \chi(\mu + \chi + \gamma) > 1, \nu > 0 \). Let \( r^*(z_0) = r^* \) be the decay character of \( z_0 \), with \( -\frac{3}{2} < r^* < \infty \). Then,
\[ \| \nabla w(t) \|_{L^2} + \| D^2 z(t) \|_{L^2} + (1 + t) \| D^3 z(t) \|_{L^2} \leq C(1 + t)^{-\min\{\frac{7}{4}, r^*, \frac{9}{4}\}}, \quad \forall t > t_0, \]
for some \( t_0 = t_0(\| z_0 \|_{L^2}) \).

**Proof.** We follow the ideas in the proof of Lemma 3.1. Differentiating (1.1) with respect to \( x_{\ell_1} \) and \( x_{\ell_2} \), multiplying by \( (1 + t)^{\alpha(r^*)+\delta+2} D_{\ell_1} D_{\ell_2} (u, w, b) \), integrating the result on \( \mathbb{R}^3 \times [t_0, t] \) and summing up, we obtain
\[
(1 + t)^{\alpha(r^*)+\delta+2} \| D^2 z(t) \|_{L^2} + C \int_{t_0}^{t} (1 + s)^{\alpha(r^*)+\delta+2} \| D^3 z(s) \|_{L^2} \, ds \\
\leq C \int_{t_0}^{t} (1 + s)^{\alpha(r^*)+\delta+1} \| D^2 z(s) \|_{L^2} \, ds \\
+ C \int_{t_0}^{t} (s - t_0)^2 \| D^3 z(s) \|_{L^2} \| z(s) \|_{L^{\infty}} \| D^2 z(s) \|_{L^2} \, ds \\
+ C \int_{t_0}^{t} (s - t_0)^2 \| D^3 z(s) \|_{L^2} \| \nabla z(s) \|_{L^{\infty}(\mathbb{R}^3)} \| \nabla z(s) \|_{L^2} \, ds \\
\leq C \int_{t_0}^{t} (1 + s)^{\alpha(r^*)+\delta+1} \| D^2 z(s) \|_{L^2} \, ds \\
+ \int_{t_0}^{t} (1 + s)^{\alpha(r^*)+\delta+2} \| z(s) \|_{L^2}^{1/2} \| \nabla z(s) \|_{L^2}^{1/2} \| D^2 z(s) \|_{L^2} \, ds,
\]
where we used that \( \| z \|_{L^2} \| \nabla^2 z \|_{L^2} \leq C \| z \|_{L^2}^{1/2} \| \nabla^2 z \|_{L^2}^{1/2} \| \nabla^3 z \|_{L^2} \). By (3.11), we have
\[
(1 + t)^{\alpha(r^*)+\delta+2} \| D^2 z(t) \|_{L^2} + \min\{\mu, \gamma, \nu\} \int_{t_0}^{t} (1 + s)^{\alpha(r^*)+\delta+1} \| D^3 z(s) \|_{L^2}^{2} \, ds \leq C(1 + t)^{\delta}. \quad (3.16)
\]
Now, applying the same argument as in the previous Lemma (see also Guterres et al. [17], for more details), we obtain

\[(1 + t)^{\alpha(r^*) + \delta + 3} \| D^3 z(t) \|_{L^2}^2 + C \int_{t_0}^{t} (1 + s)^{\alpha(r^*) + \delta + 3} \| D^4 z(s) \|_{L^2}^2 ds \leq C(1 + t)^{\delta}\]

which yields

\[\| \nabla^2 z(t) \|_{L^2}^2 + (1 + t) \| \nabla^3 z(t) \|_{L^2}^2 \leq C(1 + t)^{-\min\{\frac{7}{2} + r^*, \frac{7}{2}\}}.\]

Furthermore, by standard Sobolev embeddings, we obtain

\[\| \nabla^j z(t) \|_{L^4}^2 \leq C(1 + t)^{-\alpha(r^*) - 3/4 - j}, \quad \text{(3.17)}\]

for all sufficiently large $t > 0$ and each $0 \leq j \leq 2$. This particular bound allows us to estimate the gradient of the microrotational field. Indeed, for $\mathcal{L} = \gamma \Delta + \nabla (\nabla \cdot)$ we have

\[
\| \nabla w(t) \|_{L^2} \leq e^{-2 \chi(t-t_0)} \| \nabla e^{\mathcal{L}(t-t_0)} z(t_0) \|_{L^2} \\
+ \chi \int_{t_0}^{t} e^{-2 \chi(t-s)} \| \nabla e^{\mathcal{L}(t-s)} (\nabla \times u(s)) \|_{L^2} ds \\
+ \int_{t_0}^{t} e^{-2 \chi(t-s)} \| e^{\mathcal{L}(t-s)} (\nabla (u \cdot \nabla w)(s)) \|_{L^2} ds \\
\leq C e^{-2 \chi(t-t_0)} + \chi \int_{t_0}^{t} e^{-2 \chi(t-s)} \| e^{\Delta(t-s)} \nabla^2 u(s) \|_{L^2} ds \\
+ C \int_{t_0}^{t} e^{-2 \chi(t-s)} \| e^{\Delta(t-s)} \{\nabla (u \cdot \nabla w)(s)\} \|_{L^2} ds \\
\leq C e^{-2 \chi(t-t_0)} + \chi \int_{t_0}^{t} e^{-2 \chi(t-s)} \| \nabla^2 u(s) \|_{L^2} ds \\
+ C \int_{t_0}^{t} e^{-2 \chi(t-s)} \| \{\nabla (u \cdot \nabla w)(s)\} \|_{L^2} ds.
\]

By (3.17) and Lemma 3.1, we have

\[
\| \nabla w(t) \|_{L^2} \leq C e^{-2 \chi(t-t_0)} + C \int_{t_0}^{t} e^{-2 \chi(t-s)} (1 + s)^{-\frac{\alpha(r^*)+1}{2}-1} ds \\
+ C \int_{t_0}^{t} e^{-2 \chi(t-s)} \left( \| z(s) \|_{L^4} \| \nabla^2 z(s) \|_{L^4} + \| \nabla z(s) \|_{L^4} \right) ds \\
\leq C e^{-2 \chi(t-t_0)} + C \int_{t_0}^{t} e^{-2 \chi(t-s)} (1 + s)^{-\frac{\alpha(r^*)+1}{2}-1} ds
\]
Identity, we obtain Proof of Theorem 1.4. which concludes the proof.

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Using the heat kernel estimate (3.9), we estimate the convolution term in two different ways, namely

\[ \begin{align*}
&\text{For} t \leq t_0, \\
&\text{and} (e^{A t})_{t \geq t_0} \text{ is the semigroup associated with the linear system (2.5). Using Lemma 2.5 and Plancherel identity, we obtain}
\end{align*} \]

\[ \int_{t_0}^{t} \| e^{A (t-\tau)} Q(\tau) \|_{L^2} d\tau \leq \int_{t_0}^{t} \| e^{C A (t-\tau)} Q(\tau) \|_{L^2} d\tau. \quad (3.19) \]

Using the heat kernel estimate (3.9), we estimate the convolution term in two different ways, namely

\[ \| e^{A (t-\tau)} Q(\tau) \|_{L^2} = \| \nabla e^{A (t-\tau)} \nabla \mathcal{L}(u, w, b)(\tau) \|_{L^2} \]

\[ \leq C \| e^{A (t-\tau)} \|_{L^2} \| \nabla \mathcal{L}(u, w, b)(\tau) \|_{L^1} \]

\[ \leq C (t - \tau)^{-\frac{5}{4}} \| z(\tau) \|_{L^2} \quad (3.20) \]

where \( Q(t) = \nabla \mathcal{L}(u, w, b) \), and

\[ \| e^{A (t-\tau)} Q(\tau) \|_{L^2} \leq C \| e^{A (t-\tau)} \|_{L^2} \| Q(\tau) \|_{L^1} \]

\[ C \leq (t - \tau)^{-3/4} \| z(\tau) \|_{L^2} \| \nabla z(\tau) \|_{L^2} \quad (3.21) \]

Then (3.18)–(3.21) imply

\[ \| z(t) - \bar{z}(t) \|_{L^2} \leq \int_{t_0}^{t_0 + t} (t - \tau)^{-\frac{5}{4}} \| z(\tau) \|_{L^2}^2 d\tau + \int_{t_0 + t}^{t} (t - \tau)^{-\frac{3}{4}} \| z(\tau) \|_{L^2} \| \nabla z(\tau) \|_{L^2} d\tau \]

\[ = \mathcal{I}_1(t) + \mathcal{I}_2(t). \]

Using Theorem 1.2, a straightforward calculation leads to

\[ \mathcal{I}_1(t) \leq C (1 + t)^{-\frac{5}{4} + r^*}, \text{ for } r^* < -\frac{1}{2}, \quad \mathcal{I}_1(t) \leq C (1 + t)^{-\frac{5}{4}}, \text{ for } r^* > -\frac{1}{2}. \]

For \( r^* = -\frac{1}{2} \), we have that

\[ \mathcal{I}_1(t) \leq C (1 + t)^{-\frac{5}{4} + \delta}, \quad \forall \delta > 0. \]
Analogously, from Theorem 1.2 and Lemma 3.1

\[ I_2(t) \leq C(1 + t)^{-\min\{\frac{15}{2} + 2r^*, \frac{3}{2}\}}, \quad -\frac{3}{2} < r^* < \infty. \]

As a result of this, the estimate is true for all \( t > t_0 \). An argument similar to that in page 239 in Kreiss et al. [20] allows to extend the bound to all \( t > 0 \).

In order to prove (1.9), we will need the following estimate.

**Lemma 3.3.** Let \( z \) be a weak solution to (1.1), with \( 32 \chi (\mu + \chi + \gamma) > 1, \nu > 0 \). Let \( r^*(z_0) = r^* \) be the decay character of \( z_0 \), with \(-\frac{3}{2} < r^* < \infty\). Then,

\[ \| \nabla z(t) - \nabla z(t) \|_{L^2} \leq C(1 + t)^{-\min\{\frac{3}{2} + 2r^*, \frac{3}{2}\}}, \quad \forall t > t_0, \]

for some \( t_0 = t_0(\|z_0\|_{L^2}) \).

**Proof of Lemma 3.3.** We have that

\[ \| \nabla z(t) - \nabla z(t) \|_{L^2} \leq \int_{t_0}^{t} \| \nabla e^{\Delta(t-\tau)} Q(\tau) \|_{L^2} d\tau. \]

As before

\[ \| \nabla e^{\Delta(t-\tau)} Q(\tau) \|_{L^2} = \| \nabla^2 e^{\Delta(t-\tau)} NL(u, w, b)(\tau) \|_{L^2} \]

\[ \leq C \| \nabla^2 e^{\Delta(t-\tau)} \|_{L^2} \| NL(u, w, b)(\tau) \|_{L^1} \]

\[ \leq C (t - \tau)^{-\frac{7}{2}} \| z(\tau) \|_{L^2}^2, \]

where \( Q(t) = \nabla NL(u, w, b) \), and

\[ \| \nabla e^{\Delta(t-\tau)} Q(\tau) \|_{L^2} \leq C \| \nabla e^{\Delta(t-\tau)} \|_{L^2}^4 \| Q(\tau) \|_{L^2}^4 \]

\[ \leq C (t - \tau)^{-\frac{7}{8}} \| z(\tau) \|_{L^4} \| \nabla z(\tau) \|_{L^2} \]

\[ \leq C \leq (t - \tau)^{-\frac{7}{8}} \| z(\tau) \|_{L^2}^\frac{1}{2} \| \nabla z(\tau) \|_{L^2}^\frac{7}{2}. \]

Then

\[ \| \nabla z(t) - \nabla z(t) \|_{L^2} \leq \int_{t_0}^{t} (t - \tau)^{-\frac{7}{8}} \| z(\tau) \|_{L^2}^\frac{1}{2} d\tau + \int_{t_0}^{t} (t - \tau)^{-\frac{7}{8}} \| z(\tau) \cdot \nabla z(\tau) \|_{L^2}^\frac{1}{2} d\tau \]

\[ \leq \int_{t_0}^{t} (t - \tau)^{-\frac{7}{8}} \| z(\tau) \|_{L^2}^\frac{1}{2} d\tau + \int_{t_0}^{t} (t - \tau)^{-\frac{7}{8}} \| z(\tau) \|_{L^2}^\frac{7}{2} \| \nabla z(\tau) \|_{L^2}^\frac{7}{2} d\tau \]

\[ = I_1(t) + I_2(t) \]

As in the previous Lemma, from Theorem 1.2 we obtain

\[ I_1(t) \leq C(1 + t)^{-\frac{7}{8}}, \quad \text{for } r^* < -\frac{1}{2}, \quad \text{for } r^* > -\frac{1}{2}. \]

For \( r^* = -\frac{1}{2} \), we have that

\[ I_1(t) \leq C(1 + t)^{-\frac{7}{8}+\delta}, \quad \forall \delta > 0. \]

From Theorem 1.2 and Lemma 3.1

\[ I_2(t) \leq C(1 + t)^{-\min\{\frac{13}{2} + r^*, \frac{17}{2}\}}, \quad -\frac{3}{2} < r^* < \infty. \]

Then, the decay is given by \( I_1(t) \forall t > t_0 \). \( \square \)
We return to the proof of (1.9). By using Duhamel’s principle, we take advantage of the damping term $-2\chi w$ to get, after a few computations

\[
\|w(t) - \bar{w}(t)\|_{L^2} \leq \chi \int_{t_0}^{t} e^{-2\chi (t-s)}|e^{C\mathcal{L}(t-s)}\{\nabla \times (u(s) - \bar{u}(s))\}|_{L^2} ds + \int_{t_0}^{t} e^{-2\chi (t-s)}|e^{C\Delta(t-s)}(u \cdot \nabla w)(s)|_{L^2} ds
\]

\[
\leq \chi \int_{t_0}^{t} e^{-2\chi (t-s)}|e^{C\Delta(t-s)}(\nabla \times (u(s) - \bar{u}(s))|_{L^2} ds + \int_{t_0}^{t} e^{-2\chi (t-s)}|e^{C\Delta(t-s)}(u \cdot \nabla w)(s)|_{L^2} ds
\]

\[
= J_1(t) + J_2(t).
\]

If $-\frac{3}{2} < r^* < -\frac{1}{2}$, by Lemma 3.3 we have

\[
J_1(t) \leq \int_{t_0}^{(t_0+t)/2} e^{-2\chi (t-s)}(1 + s)^{-\frac{3}{4} - r^*} ds + \int_{(t_0+t)/2}^{t} e^{-2\chi (t-s)}(1 + s)^{-\frac{5}{4} - r^*} ds = J_{11}(t) + J_{12}(t).
\]

We now observe that $J_{11}(t) \leq Ce^{-\chi t}$ and $J_{12}(t) \leq C(1 + t)^{-\frac{5}{4} - r^*}$. For the term $J_2(t)$, we proceed as follows. By using Lemma 3.2, we have

\[
J_2(t) \leq \int_{t_0}^{(t_0+t)/2} e^{-2\chi (t-s)}(t-s)^{-\frac{3}{4}}(1 + s)^{-\frac{5}{2} - r^*} ds + \int_{(t_0+t)/2}^{t} e^{-2\chi (t-s)}(t-s)^{-\frac{3}{2}}(1 + s)^{-\frac{5}{2} - r^*} ds = J_{21}(t) + J_{22}(t).
\]

But $J_{21}(t) \leq Ce^{-\chi t}t^{-3/4}$ and

\[
J_{22}(t) \leq C(1 + t)^{-\frac{5}{2} - r^*} \int_{(t_0+t)/2}^{t} e^{-2\chi (t-s)}(t-s)^{-3/4} ds \leq C(1 + t)^{-\frac{5}{2} - r^*} \Gamma(1/4),
\]

where $\Gamma$ is the Gamma function. Hence, $J_1(t) + J_2(t) \leq C(1 + t)^{-\frac{5}{2} - r^*}$. If $-\frac{1}{2} \leq r^* < 1$, we similarly obtain $J_1(t) \leq e^{-\chi t} + C(1 + t)^{-\frac{5}{2}}$ and $J_2(t) \leq Ce^{-\chi t}t^{-3/4} + C(1 + t)^{-\frac{5}{2} - r^*}$ which leads to $J_1(t) + J_2(t) \leq C(1 + t)^{-\frac{5}{2}}$, since $-\frac{1}{2} \leq r^* < 1$. Finally, when $r^* \geq 1$, we immediately get $J_1(t) + J_2(t) \leq C(1 + t)^{-\frac{5}{2}}$ which concludes the proof for $t > t_0$ and repeating the argument given in the proof of Theorem 1.3 or (1.8), the upper bound holds also for $0 < t < t_0$ concluding the demonstration.

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References

[1] Ahmadi, G., Shahinpoor, M.: Universal stability of magneto-micropolar fluid motions. Int. J. Eng. Sci. 12, 657–663 (1974)
[2] Bjorland, C., Schonbek, M.E.: Poincaré’s inequality and diffusive evolution equations. Adv. Differ. Equ. 14(3–4), 241–260 (2009)
[3] Boldrini, J.L., Durán, M., Rojas-Medar, M.A.: Existence and uniqueness of strong solution for the incompressible micropolar fluid equations in domains of $\mathbb{R}^3$. Ann. Univ. Ferrara Sez. VII Sci. Mat. 56(1), 37–51 (2010)
[4] Brandolese, L.: Characterization of solutions to dissipative systems with sharp algebraic decay. SIAM J. Math. Anal. 48(3), 1616–1633 (2016)
[5] Braze Silva, P., Cruz, F.W., Freitas, L.B.S., Zingano, P.R.: On the $L^2$ decay of weak solutions for the 3D asymmetric fluids equations. J. Differ. Equ. 267(6), 3578–3609 (2019)
[6] Braze Silva, P., Friz, L., Rojas-Medar, M.A.: Exponential stability for magneto-micropolar fluids. Nonlinear Anal. 143, 211–223 (2016)
[7] Braze Silva, P., Melo, W.G., Zingano, P.R.: Lower bounds on blow-up of solutions for magneto-micropolar fluid systems in homogeneous Sobolev spaces. Acta Appl. Math. 147, 1–17 (2017)
[8] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[9] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[10] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[11] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[12] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[13] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[14] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[15] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[16] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[17] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[18] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[19] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[20] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[21] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[22] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[23] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[24] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[25] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[26] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[27] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[28] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[29] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
[30] Ortega-Torres, E.E., Rojas-Medar, M.A.: Magneto-micropolar fluid motion: global existence of strong solutions. Abstr. Appl. Anal. 4(2), 109–125 (1999)

[31] Perusato, C.F., Melo, W.G., Gutes, R.H., Nunes, J.R.: Time asymptotic profiles to the magneto-micropolar system. Appl. Anal. 99, 2680–2693 (2020)

[32] Rojas-Medar, M.A.: Magneto-micropolar fluid motion: on the convergence rate of the spectral Galerkin approximations. Z. Angew. Math. Mech. 77(10), 723–732 (1997)

[33] Rojas-Medar, M.A.: Magneto-micropolar fluid motion: existence and uniqueness of strong solution. Math. Nachr. 188, 301–319 (1997)

[34] Rojas-Medar, M.A., Luiz Boldrini, J.: Magneto-micropolar fluid motion: existence of weak solutions. Rev. Mat. Complut. 11(2), 443–460 (1998)

[35] Sadowski, W.: Upper bound for the number of degrees of freedom for magneto-micropolar flows and turbulence. Int. J. Eng. Sci. 41(8), 789–800 (2003)

[36] Schonbek, M.E.: Decay of solutions to parabolic conservation laws. Commun. Partial Differ. Equ. 5(7), 449–473 (1980)

[37] Schonbek, M.E.: $L^2$ decay for weak solutions of the Navier–Stokes equations. Arch. Rational Mech. Anal. 88(3), 209–222 (1985)

[38] Schonbek, M.E.: Large time behaviour of solutions to the Navier–Stokes equations. Commun. Partial Differ. Equ. 11(7), 733–763 (1986)

[39] Tan, Z., Wenpei, W., Zhou, J.: Global existence and decay estimate of solutions to magneto-micropolar fluid equations. J. Differ. Equ. 266(7), 4137–4169 (2019)

[40] Wang, Y.: Regularity criterion for a weak solution to the three-dimensional magneto-micropolar fluid equations. Bound. Value Probl. 2013, 58 (2013)

[41] Wang, Y.: Blow-up criteria of smooth solutions to the three-dimensional magneto-micropolar fluid equations. Bound. Value Probl. 2015, 118 (2015)

[42] Wang, Y., Gu, L.: Global regularity of 3D magneto-micropolar fluid equations. Appl. Math. Lett. 99, 105980, 9 (2020)

[43] Wang, Y., Wang, K.: Global well-posedness of 3D magneto-micropolar fluid equations with mixed partial viscosity. Nonlinear Anal. Real World Appl. 33, 348–362 (2017)

[44] Wang, Y.-Z., Hu, L., Wang, Y.-X.: A Beale-Kato-Majda criterion for magneto-micropolar fluid equations with partial viscosity. Bound. Value Probl. 2011, 128614 (2011)

[45] Wang, Y.-Z., Li, Y., Wang, Y.-X.: Blow-up criterion of smooth solutions for magneto-micropolar fluid equations with partial viscosity. Bound. Value Probl. 2011, 1–11 (2011)

[46] Wiegerinck, M.: Decay results for weak solutions of the Navier–Stokes equations on $\mathbb{R}^n$. J. London Math. Soc. (2) 35(2), 303–313 (1987)

[47] Xiang, Z., Yang, H.: On the regularity criteria for the 3D magneto-micropolar fluids in terms of one directional derivative. Bound. Value Probl. 2012, 139 (2012)

[48] Yamaguchi, N.: Existence of global strong solution to the micropolar fluid system in a bounded domain. Math. Methods Appl. Sci. 28(13), 1507–1526 (2005)

[49] Yamazaki, K.: 3-D stochastic micropolar and magneto-micropolar fluid systems with non-Lipschitz multiplicative noise. Commun. Stoch. Anal. 8(3), 413–437 (2014)

[50] Yamazaki, K.: Exponential convergence of the stochastic micropolar and magneto-micropolar fluid systems. Commun. Stoch. Anal. 10(3), 271–295 (2016)

[51] Yamazaki, K.: Large deviation principle for the micropolar, magneto-micropolar fluid systems. Discrete Contin. Dyn. Syst. Ser. B 23(2), 913–938 (2018)

[52] Yamazaki, K.: Gibbsian dynamics and ergodicity of stochastic micropolar fluid system. Appl. Math. Optim. 79(1), 1–40 (2019)

[53] Yamazaki, K.: Irreducibility of the three, and two and a half dimensional Hall-magnetohydrodynamics system. Phys. D 401, 132199 (2020)

[54] Yuan, B.: Regularity of weak solutions to magneto-micropolar fluid equations. Acta Math. Sci. Ser. B (Engl. Ed.) 30(5), 1469–1480 (2010)

[55] Yuan, B., Li, X.: Regularity of weak solutions to the 3D magneto-micropolar equations in Besov spaces. Acta Appl. Math. 163, 207–223 (2019)

[56] Yuan, J.: Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations. Math. Methods Appl. Sci. 31(9), 1113–1130 (2008)

[57] Zhang, H., Zhao, Y.: Blow-up criterion for strong solutions to the 3D magneto-micropolar fluid equations in the multiplier space. Electron. J. Differ. Equ. 188, 1–7 (2012)

[58] Zhang, Z., Yao, Z., Wang, X.: A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces. Nonlinear Anal. 74(6), 2220–2225 (2011)
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