Quantum Mechanics and the Principle of Least Radix Economy

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A new variational method, the principle of least radix economy, is formulated. It generalizes the principle of least action yielding two classes of physical solutions: least action paths and quantum wavefunctions. The Schrödinger and Klein-Gordon equations and the breaking of the commutativity of spacetime geometry are then derived from this method, which is shown to include the Boltzmann’s principle of classical statistical thermodynamics as well.

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INTRODUCTION

It has been conjectured that all natural processes can be understood as the result of computation \textsuperscript{[1, 2, 5]}. This statement is contained in Wolfram’s principle of computational equivalence \textsuperscript{[1]} which is closely related to the Church-Turing thesis \textsuperscript{[4]}: A computable function (expressing e.g. a law of physics) is also effectively calculable (i.e. its values can be found by some purely mechanical process). This thesis is the base for digital physics, in which the universe is modeled as a giant computer \textsuperscript{[1, 2, 5]} processing the information contained in it. Ideas of digital physics had been independently advanced in the 60’s in the context of quantum mechanics by Bastin at al. \textsuperscript{[6, 7]}, and then by Noyes and Kauffman \textsuperscript{[8–10]}. In the context of nonlinear dynamics, McCauley and Palmore showed how converting real numbers into finite strings of digits from a finite alphabet can capture all dynamical features of chaotic deterministic systems. Very recently, these latter ideas have been systematically explored \textsuperscript{[13]} by means of B-calculus \textsuperscript{[14]}, which constitutes a mathematical formalism for rule-based dynamical systems (examples being cellular automata \textsuperscript{[14, 16]} and substitution systems \textsuperscript{[17]}).

If one accepts the view of the universe as a computer as described above, several questions can be raised. First of all, since computations are assumed to be made with symbols of a finite alphabet, one can ask what the size (cardinal) \( \eta \) of the alphabet should be and what physical meaning is to be attributed to the alphabet’s size (note that in this view computations are intended to directly map physical processes). In performing computations, the size of the alphabet coincides with the radix (base) in which a number is expressed. We shall henceforth use the latin word \textit{radix} as a synonym for “base” to avoid confusion with other uses of the latter word in physics. Because of its obvious, useful connection with boolean algebra, the binary radix has long been considered in the research of discrete physics \textsuperscript{[1, 3, 8]}

An important objection to the use of purely discrete physical models is how they can model the continuum

spacetime and the observable physics. For example, although discrete models employing simple cellular automata are able to describe fermionic fields \textsuperscript{[18]}, the fact that cellular automata evolve on a lattice make them unsuitable to describe the ubiquitously observed rotational symmetry of physical laws \textsuperscript{[18]}.

In this note we present a new approach to quantum mechanics inspired in digital physics giving answers to the above questions and objections. We follow, however, a \textit{semi-discrete} approach, a possibility that is opened by recent mathematical methods \textsuperscript{[13, 14]} that allow to handle real numbers as infinite strings of digits from a finite alphabet to arbitrary precision. The non-commutativity of the continuum spacetime at the quantum level is derived from our approach. We claim that \textit{nature makes dynamically the most effective choice for the radix in which its computations take place}. It is then shown that classical and quantum physics merge together from a single variational principle. A simple integer function of the dimensionless Lagrangian action \(|S/h|\) (here \(x\) denotes the floor function (lower closest integer) of \(x\), \(S\) is the Lagrangian action and \(h\) is the Planck’s constant) is interpreted as the radix in which the computations that implement the laws of nature take place. \textit{By simply demanding that this radix works most efficiently physical laws are derived}. The principle of least abbreviated action (leading to Euler-Lagrange equations), the Schrödinger and Klein-Gordon equations, and Boltzmann’s entropy are then obtained from the principle of least radix economy. The Lagrangian action can thus be understood as a key quantity for the effectiveness of mathematics in the natural sciences \textsuperscript{[19]}.

RADIX ECONOMY AND THE LAGRANGIAN ACTION

The laws of physics are not scale invariant and we usually distinguish the physical impact of numbers in terms of the orders of magnitude that they involve. A related (but so far unexplored) approach is to consider
the radix in which numbers are expressed. The decimal radix which we always adopt for giving numbers in physics is just a tacit convention which is not necessarily the most effective one. All numbers with physical meaning are radix dependent. For example, if we consider a length of 123 meter, it is understood that we mean \(1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0\) m, i.e. that we give the number in the decimal radix. Had we used another radix, 50 say, this number would be \(1 \cdot 50^2 + 50^1 + 3 \cdot 50^0\) m and with the same digits we would mean 2603 m instead. We all conventionally agree to mean all our numbers in the same radix so that we can compare them (we see that by changing the radix we change, e.g., a length). Let \(b\) (a natural number) denote an arbitrary radix. We can express any real number \(A\) in radix \(b\) as

\[
A = \sum_{m=-\infty}^{[1+\log_b A]} b^{m-1} d_b(m, A)
\]

where the upper bound in the sum \([\log_b A + 1]\) is the total number of integer digits of \(A\) in radix \(b\) \([13]\, and \(d_b(m, A)\) is an integer function that returns the digit \(A\) that accompanies the \((m-1)\)th-power of \(b\) when \(A\) is written in base \(b\). This latter function \([13]\) (see also \([20]\)), yields an integer between 0 and \(b-1\) and is defined as

\[
d_b(m, A) = \left[ \frac{A}{b^{m-1}} \right] - b \left[ \frac{A}{b^m} \right]
\]

An important means to quantify the effectiveness of radix \(b\) to express the number \(A\) is the radix economy \(C(b, A)\), also called digit capacity \([21]\).

\[
C(b, A) = b \left[ 1 + \log_b A \right]
\]

This quantity is related to the hardware complexity in circuits with multiple valued logic \([21]\). When this quantity is a minimum, we say that radix \(b\) expresses most efficiently \(A\) or that \(b\) has the least radix economy in expressing \(A\). For \(b = \lfloor A \rfloor > 1\) we have

\[
C(\lfloor A \rfloor, A) = \lfloor A \rfloor \left[ 1 + \log_{\lfloor A \rfloor} A \right]
\]

\[
= \lfloor A \rfloor \left[ 1 + \frac{\ln A}{\ln \lfloor A \rfloor} \right] = 2 \lfloor A \rfloor
\]

since we have \(\lfloor A \rfloor = |A| + \epsilon\) with \(0 \leq \epsilon < 1\) and thus

\[
1 \leq \frac{\ln A}{\ln \lfloor A \rfloor} = \frac{\ln(|A| + \epsilon)}{\ln |A|} \approx 1 + \frac{\epsilon}{|A| \ln |A|} < 2
\]

We have seen above that if a number \(A_1\) has the same digits in radix \(b_1\) as it has \(A_2\) in radix \(b_2\), then, if \(b_1 < b_2\) so is also \(A_1 < A_2\). We note also that a real number \(A\) in base \(\lfloor A \rfloor\) has always 10 as integer part since \(\lfloor A \rfloor < |A| + 1\) and, therefore, \(A = 1 \cdot \lfloor A \rfloor^1 + 0 \cdot \lfloor A \rfloor^0 + \{A\}\), where \(\{A\}\) denotes the fractional part of \(A\). Thus, crazy as it may seem, 10 m can denote any arbitrarily large length if we tune the radix in which 10 is expressed (note, however, that 1 m is always 1 m regardless of the radix used!).

We read in Dirac’s book \([22]\) (p. 3): \(\text{So long as big and small are relative concepts, it is no help explaining the big in terms of the small. It is therefore necessary to modify classical ideas in such a way so as to give an absolute meaning to size.}\)

We show now how the dimensionless Lagrangian action \(S/h\), where \(h\) is the Planck constant, can be used to that purpose. Along a path \(\gamma\) connecting two points ‘1’ and ‘2’, \(S\) is an scalar functional given by

\[
S(q(t)) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) \, dt
\]

where \(L\) is the Lagrangian, \(t\) denotes time, and \(q(t)\) and \(\dot{q}(t)\) are the generalized position and velocity vectors evaluated along the path. When \(S/h\) is large the physical trajectories are governed by the principle of least action

\[
D \varepsilon S(q(t)) = \frac{d}{d\varepsilon} S(q(t) + \varepsilon f) \bigg|_{\varepsilon=0} = 0
\]

where \(D\varepsilon\) denotes the first-variation operator, \(\varepsilon\) is a scalar and \(f\) is an arbitrary function. The extremization of the action leads to the Euler-Lagrange equations describing the physical trajectories

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad i \in [1, N]
\]

where \(N\) is the total number of degrees of freedom in the system and \(q_i\) and \(\dot{q}_i\) are the \(i\)-th component of the generalized position and velocity vectors, respectively.

An equivalent formulation is provided by the Hamilton-Jacobi equation, which implies the following relationships involving the Hamiltonian \(H\) and generalized momenta \(p\) as

\[
H = -\frac{\partial S}{\partial t} \quad p = \nabla S
\]

When \(S/h\) is small, there are no longer well defined unique trajectories along which the motion of the system takes place. Rather, infinitely many choices are possible and one is forced to speak about the probability of finding a certain physical state. This is a most striking fact in quantum mechanics: When the action \(S\) is of the order of \(h\) the physical laws seem very different to when \(S/h\) is large. Thus, the (generally real) number \(S/h\) suggests a way to give an absolute meaning to size, breaking the relativity of big and small.

**THE PRINCIPLE OF LEAST RADIX ECONOMY**

We then postulate that the physical radix (a natural dimensionless number) is given by

\[
\eta \equiv \lfloor S/h \rfloor
\]
(thus we have $S/h = \eta + (S/h)$). The physical paths are then all those for which this radix is most efficient, i.e. those paths for which the radix economy $C((S/h), S/h)$ is the least. Since from Eqs. (4) and (10) we have

$$C(\eta, S/h) = \eta \left\lfloor \log_{\eta} (S/h) + 1 \right\rfloor = 2\eta = 2 \lfloor S/h \rfloor$$

(11)

the above statement amounts to replace the principle of least action Eq. (7) by

$$D_z C(\eta, S/h) = 2D_z \lfloor S/h \rfloor = 0$$

(12)

This is the principle of least radix economy for the dimensionless Lagrangian action $S/h$. When the number to be expressed in base $\eta$ is precisely $S/h$ the physical paths are those that minimize the radix $\eta$, i.e. those paths for which the action is first minimized and then a value $nh$ is attributed to the least action trajectories. This guarantees that the classical limit governed by Eq. (7) is asymptotically approached from Eq. (14) for $S/h$ large through the semiclassical quantization rule, as was already observed in the matrix formulation of quantum mechanics [24]. In this limit $S/h \sim |S/h|$, Eq. (15) does not hold ($\lfloor S/h \rfloor$ can be neglected compared to $S/h$) and Eq. (14) reduces to the least action principle, Eq. (7).

From Eq. (13), the Dirichlet kernel $D_\infty(2\pi S/h)$ is always a valid physical solution of the least radix economy principle for any $S/h$ (if $S/h$ is an integer, the Dirichlet kernel is still a solution if the value of $S$ corresponds to the path with the least action as well). Note also that, as are Eqs. (13) and (14), the kernel is invariant under a change

$$S \rightarrow S + nh$$

(16)

with $n$ integer, i.e. $D_\infty(2\pi(S + nh)/h) = D_\infty(2\pi S/h)$. The state of the physical system can thus be specified by a function $\psi(S)$ whose values are related through the Dirichlet kernel by convolution as

$$\psi(S) = \frac{1}{h} \int_{-h/2}^{h/2} \psi(S') D_\infty(2\pi(S - S'/h)) \, dS'$$

(17)

This function satisfies $\psi(S) = \psi(S + nh)$, therefore having the required symmetry Eq. (16), and must then have the general form

$$\psi(S) = \sum_{k=-\infty}^{\infty} \tilde{\psi}(k) e^{ikS/h}$$

(18)

i.e. a linear superposition of plane waves, where $\tilde{\psi}(k)$ is a complex-valued function of $k$. It is well-known that the $L^1$ norm of the Dirichlet kernel $D_n$ diverges as $\|D_n\|_{L^1} \sim \log n$ when $n \rightarrow \infty$. Therefore, in order for the sum in Eq. (18) to converge pointwise (so that Eq. (17) makes also sense) this automatically demands that $\psi(S)$ belongs to the set of square-summable functions and, hence, that the sum converges in the Hilbert space sense (that is, in the $L^2$ mean) [25]. Hence, furthermore, the $\tilde{\psi}(k)$’s are terms of a sequence of real functions $\tilde{\psi}(k)$ such that the series $\sum_{k=-\infty}^{\infty} |\tilde{\psi}(k)|^2$ converges to a real, positive value $\mathcal{N}$. Since $S$ is a scalar functional and can be thought as dependent on the normalized position vector $q$ and time $t$, appropriate orthonormal functions other than the $e^{ikS/h}$ can be chosen and this norm will always be conserved from Parseval’s theorem. Furthermore, since the spectral power density obtained from
\( \psi(q,t) \psi^*(q,t) = |\psi(q,t)|^2 \) (with the asterisk denoting complex conjugation) will yield positive real values for any value of \( q \) and \( t \), and normalization is guaranteed, this shows indeed the full consistency with the Copenhagen interpretation of \( |\psi(q,t)|^2/N \) as a probability. In the case \( S/h \) integer, each term in the sum in Eq. 15 describes a de Broglie standing wave in the quantum wave function.

Note thus that type (1) paths above lead to trajectories defined by the Euler-Lagrange differential equations Eq. 8, the state of the system (a point particle) being specified by the vector \( (t, q) \). Paths of type (2) are described by the integral equation Eq. 17 in terms of the Dirichlet kernel, which acts as a propagator, and the state is described by the (scalar or vectorial) function \( \psi(S) \) with the discrete symmetry of Eq. 16. In the classical limit \( S/h \) large only paths (1) are relevant. In the quantum regime, the fact that the two different kinds of paths co-exist in phase space is consistent with the wave-particle duality.

**DERIVATION OF SCHRÖDINGER AND KLEIN-GORDON EQUATIONS**

From Eq. 15 it is now straightforward to derive the Schrödinger and Klein-Gordon equations. We can now define \( S_k = kS \), \( p_k = kp \) and \( E_k = kH \). Then Eq. 9 implies

\[
E_k = -\frac{\partial S_k}{\partial t} \quad p_k = \nabla S_k
\]

and, therefore, from Eq. 15

\[
\frac{\partial \psi}{\partial t} = \sum_{k=-\infty}^{\infty} \tilde{\psi}(k) \frac{\partial e^{iS_k/h}}{\partial t} = \frac{i}{\hbar} \sum_{k=-\infty}^{\infty} \tilde{\psi}(k)e^{iS_k/h} \frac{\partial S_k}{\partial t}
\]

\[
= \frac{i}{\hbar} \sum_{k=-\infty}^{\infty} E_k \tilde{\psi}(k)e^{iS_k/h} = -\frac{i}{\hbar} \hat{H} \psi
\]

\[
\nabla \psi = \sum_{k=-\infty}^{\infty} \tilde{\psi}(k) \nabla e^{iS_k/h} = \frac{i}{\hbar} \sum_{k=-\infty}^{\infty} \tilde{\psi}(k)e^{iS_k/h} \nabla S_k
\]

\[
= \frac{i}{\hbar} \sum_{k=-\infty}^{\infty} p_k \tilde{\psi}(k)e^{iS_k/h} = \frac{i}{\hbar} \hat{p} \psi
\]

\[
\nabla^2 \psi = \nabla \cdot \nabla \psi = \frac{1}{\hbar^2} \sum_{k=-\infty}^{\infty} \tilde{p}_k^2 \tilde{\psi}(k)e^{iS_k/h}
\]

\[
= -\frac{1}{\hbar^2} \hat{p} \cdot \hat{p} \psi
\]

In Eqs. (20) and (21) the Hamiltonian \( \hat{H} = \hbar \frac{\partial}{\partial t} \) and momentum operators \( \hat{p} = -i\hbar \nabla \) have been defined. From Eqs. (20) and (22) we now have

\[
\begin{align*}
\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi &= \sum_{k=-\infty}^{\infty} \left(E_k - \frac{\hbar^2}{2m} \right) \tilde{\psi}(k)e^{iS_k/h} \\
&= \sum_{k=-\infty}^{\infty} V(q) \tilde{\psi}(k)e^{iS_k/h} = V(q)\psi
\end{align*}
\]

(23)

where \( V(q) \) is the potential energy and it has been used that \( E_k = \frac{p_k^2}{2m} + V(q) \) (conservation of energy). We thus obtain the time-dependent Schrödinger equation

\[
\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar^2}{2m} \nabla^2 + V(q) \right) \psi = \hat{H} \psi
\]

(24)

If we consider a free particle \( (V(q) = 0) \) we obtain from Eqs. 15 and 24 the following solution for \( \psi(S) \)

\[
\psi(S) = \sum_{k=-\infty}^{\infty} \tilde{\psi}(k)e^{iS_k/h} = A e^{i(\mathbf{p} \cdot \mathbf{q} - Et)/\hbar}
\]

(25)

with \( A \) being a constant. This plane wave corresponding to the free particle can be interpreted from this latter expression as a mean-field (averaged) complex order parameter of an (infinite) collection of “oscillators” \( \tilde{\psi}(k)e^{iS_k/h} \). Each point in space-time can thus be assumed to contain such an infinite collection of oscillators which are not to be considered as hidden variables: Only their mean field is physically relevant and, furthermore, only the power spectral density of the order parameter is physically observable.

Had we considered the relativistic expression \( E_k^2 = p_k^2 c^2 + m^2 c^4 \), we would obtain, from Eqs. (20) to (24)

\[
-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} + \hbar^2 c^2 \nabla^2 \psi = \sum_{k=-\infty}^{\infty} \left(E_k^2 - p_k^2 c^2 \right) \tilde{\psi}(k)e^{iS_k/h}
\]

\[
= \sum_{k=-\infty}^{\infty} m^2 c^4 \tilde{\psi}(k)e^{iS_k/h} = m^2 c^4 \psi
\]

(26)

which is the Klein-Gordon equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = \frac{m^2 c^4}{\hbar^2} \psi
\]

(27)

**BREAKING OF SPACETIME COMMUTATIVITY**

If we define the position operator as \( \hat{q} \equiv q \), we have, for two components \( i \) and \( j \) \( [\hat{q}_i, \hat{p}_j] = (\hat{q}_i \hat{p}_j - \hat{p}_j \hat{q}_i) = -i\hbar \delta_{ij} \), we have

\[
\hbar \left( \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right) \psi = -i\hbar \delta_{ij} \psi
\]

(28)

\( \delta_{ij} \) being the Kronecker delta). In general, for two conjugate variables \( \alpha \) and \( \beta \) that satisfy \( \beta_\alpha = \partial S_k/\partial \alpha \), the operator \( \hat{\beta} = -i\hbar \partial/\partial \alpha \) can be defined through the straightforward generalization of Eq. 21. Then, if one considers the operator \( \hat{\alpha} \equiv \alpha \), the commutation relationship \([\hat{\alpha}, \hat{\beta}] = i\hbar\)

(29)
holds. This implies the Heisenberg uncertainty principle \(26,27\). The breaking of geometric commutativity can be also directly understood from the concepts introduced in this note. If we now use \(\eta\) given by Eq. \(10\) as radix in Eq. \(1\), we have, for \(A = S/h\)
\[
\frac{S}{h} = 1 \cdot \eta^1 + 0 \cdot \eta^0 + \sum_{m=-\infty}^{0} \eta^{m-1} d_\eta(m, S/h) \tag{28}
\]
For classical paths for which the action is large the sum in the last term can be neglected: Eq. \(28\) means the same as \(S/h = \eta + \{S/h\}\), and \(\{S/h\}\) can be neglected for \(\eta\) large. In this classical limit we are left with only two digits at integer positions: All digits after the decimal point are zero. From Eq. \(2\) we have, for each of these two digits, \(d_\eta(1, S/h) = 0 = \{S/h\} = \eta \left(\frac{S/h}{\eta}\right)\)
\[
d_\eta(2, S/h) = 1 = \left\lfloor\frac{S/h}{\eta}\right\rfloor - \eta \left\lfloor\frac{S/h}{\eta}\right\rfloor = \left\lfloor\frac{S/h}{\eta}\right\rfloor. \tag{29}
\]
The former of these equations means that the operations of dividing by \(\eta\) and taking the floor brackets \([\ldots]\) commute, and hence \(\{S/h/\eta\} = \{S/h\}/\eta\). The latter equation means that \(S/h\) is proportional to \(\eta\) with prefactor 1. Let us now assume, however, than we are in the quantum regime so that Eq. \(15\) is satisfied. This means that \(S/h \sim \{S/h\}\) and thus the fractional part cannot be neglected (the sum in Eq. \(28\) contains non-zero terms). Let us assume that in the fractional part of \(S/h\) the digit \(d_\eta(-|m'|, S/h)\) accompanying the power \(\eta^{-|m'|-1}\) with \(m' \leq 0\) in Eq. \(28\) is the first, most significant, nonzero digit. Then from Eq. \(2\) we have
\[
d_\eta(-|m'|, S/h) = \left\lfloor \eta^{1+|m'|} S/h \right\rfloor - \eta \left\lfloor \eta^{|m'|} S/h \right\rfloor \tag{29}
\]
This latter expression means that the operations of multiplying by \(\eta\) and taking the floor function of the quantity \(\eta^{|m'|} S/h\) do not commute. The classical limit (where these operations do commute) occurs in the limit \(|m'| \to \infty\) (i.e. when the fractional part \{S/h\} is negligible). In such limit, the principle of least radix economy reduces to the principle of least action and commutativity is regained. Note that this non-commutative relationship for the action is not a conventional Heisenberg-like as the ones discussed above since \(d_\eta(-|m'|, S/h)\) in Eq. \(29\) is an integer number in \([0, \eta - 1]\).

In atomic models, the physical radix \(\eta\) coincides with the so-called principal quantum number \(n\). Then it must also be remarked that \(d_\eta(m, S/h)\) for any \(m \leq 0\) is a non-negative integer in \([0, \eta - 1]\). This suggests that \(d_\eta(m', S/h)\) (i.e. the most significant digit of the fractional part of the action \{S/h\}) corresponds to the azimuthal quantum number \(\ell\) describing the orbitals (electronic subshells). The necessary existence of such quantum numbers comes directly from the variational principle presented in this note without solving any further equation. When the angular momentum is important as a further conservation law coming from the semiclassical problem, the splitting of the main quantum shells into orbital subshells predicted by the Schrödinger equation, Eq. \(24\), can also be understood from this new point of view as the increased significance that the digit \(d_\eta(-|m'|, S/h)\) acquires on the expansion of the lagrangian action \(S/h\) in the radix \(\eta\), Eq. \(28\). That this digit is responsible for the breaking of the commutativity of the spacetime, from Eq. \(29\) is also made spatially evident, since orbitals are countable discrete objects that arise out of a continuum and commutative spacetime.

**BOLTZMANN’S PRINCIPLE**

The Second Law of Thermodynamics becomes important when the total number of degrees of freedom \(N\) in Eq. \(8\) is huge. The Second Law finds an elegant interpretation in Statistical Thermodynamics through the Boltzmann’s principle, which establishes that the equilibrium thermodynamic entropy \(S_B\) for an isolated system given by
\[
S_B = k \ln \Omega \tag{30}
\]
(where \(k\) is the Boltzmann constant) attains its maximum at equilibrium. \(\Omega\) in Eq. \(30\) is interpreted as the space of configurations of a finite system. For a classical Hamiltonian system where energy is conserved one has \(28,29\)
\[
\Omega = \int \frac{d^3N p d^3N q}{h^{3N}N!} \delta (E - H(q, p)) \tag{31}
\]
This number corresponds to the total number of attainable microstates in the constant energy surface. Eq. \(30\) is connected to the Gibbs canonical ensemble through the Laplace transform of \(\Omega\). Why Eq. \(30\) or, equivalently, the Gibbs ensemble, describes indeed thermodynamic equilibrium is a mystery. We quote Ruelle \(31\):

*The problem of why the Gibbs ensemble describes the thermal equilibrium (at least for “large systems”) [...] is deep and incompletely clarified.*

From the principle of least radix economy we can now derive Boltzmann’s entropy. To see how, let us remark that \(\Omega\) is just a huge number corresponding to all possible configurations of the conservative Hamiltonian system: There is also a huge variety of finite paths with dimensionless action \(S/h\) (and radix \(\eta = \{S/h\}\)) that are contained in the constant energy surface and we can express the number \(\Omega\) in terms of the radix corresponding to any of these paths. At equilibrium, the concept of typicality was coined to describe the paths with the largest probability measure. The principle of least radix economy, as we show now, provides both the right expression for the equilibrium entropy and an estimate of the length of the typical paths. To show this, first observe that the
number to be expressed now in the radix \( \eta = |S/h| \) is \( \Omega \) which is fixed and given by Eq. \( (31) \). Thus, we have
\[
D_{\varepsilon}C (\eta, \Omega) = D_{\varepsilon} (\eta [1 + \log_{\eta} \Omega]) \approx D_{\varepsilon} (\eta \log_{\eta} \Omega)
\]
\[
= D_{\varepsilon} \left( \frac{\eta}{\ln \eta} \right) \ln \Omega = \frac{d}{d\eta} \left( \frac{\eta}{\ln \eta} \right) \ln \Omega \frac{D_{\varepsilon} \eta}{\eta} = 0
\]
(32)
where Eq. \( (3) \) has been used together with the fact that \( \Omega \) is very large and so is its logarithm compared to unity, i.e.
\[
C (\eta, \Omega) \approx \eta \log_{\eta} \Omega = \frac{\ln \Omega}{\ln \eta}
\]
(33)
We see from Eq. \( (32) \) that each least radix path obeying Eq. \( (7) \) (i.e. satisfying \( D_{\varepsilon} \eta = 0 \)) is a local minimum. There is, however, a global minimum as well, which occurs when
\[
\frac{d}{d\eta} \left( \frac{\eta}{\ln \eta} \right) \bigg|_{\eta_{\text{min}}} = 0
\]
(34)
The function \( x/\ln x \) has a minimum at \( x = e \). Therefore, since \( \eta = |S/h| \) is an integer the minimum is at \( \eta_{\text{min}} = 3 (\approx e) \) \[21\] which gives the typical paths. Such paths in the constant energy surface are characteristic of thermodynamic equilibrium and are tiny because of the effect of thermalization in phase space (i.e. the principle of the equipartition of energy) at equilibrium where the global minimum is attained. The constant energy surface is homogeneously filled by an ensemble of paths and those which are the typical ones have the most significant contribution to the average.

From Eqs. \( (30), (32) \) and \( (34) \) we observe that
\[
C (\eta_{\text{min}}, \Omega) = C (3, \Omega) = \frac{3}{k \ln 3} \ln \Omega = \frac{3}{k \ln 3} S_B
\]
(35)
which shows how Boltzmann entropy naturally arises from the principle of least radix economy. Furthermore, since from Eq. \( (34) \) we have
\[
\frac{kC (3, \Omega)}{S_B} = \frac{3}{\ln 3} \leq \frac{\eta}{\ln \eta}
\]
(36)
and then
\[
S_B \geq \frac{k \ln \eta}{\eta} C (3, \Omega) = \frac{3 \ln \eta}{\eta \ln 3} k \ln \Omega
\]
(37)
This suggests to define the nonequilibrium path-dependent entropy
\[
S(\eta) \equiv \frac{3 \ln \eta}{\eta \ln 3} k \ln \Omega
\]
(38)
and thus, we have, from Eq. \( (37) \)
\[
S_B \geq S(\eta)
\]
(39)
where the equality only holds for \( \eta = \eta_{\text{min}} \) (i.e. at equilibrium). In a nonequilibrium situation, a significant proportion of paths is different to the typical ones. In its evolution to equilibrium, the average characteristic lengths of the possible paths on the constant energy surface changes with time. Finally, a situation is reached when typicality is most relevant and this corresponds to a situation where equipartition of the energy has taken place. Such a situation is described by Boltzmann entropy \( S_B = S(\eta_{\text{min}}) \) and this entropy is a maximum compared to any other path-dependent nonequilibrium entropy. This is the second law of thermodynamics. Explicitly, the nonequilibrium entropy Eq. \( (38) \) is
\[
S(\eta) = \frac{3k}{\ln 3} \left[ \frac{1}{\hbar} \int_{t_1}^{t_2} L dt \right] \ln \left[ \int \frac{3N}{h} \prod_{k=1}^{3N} dq_k dq_k \frac{\partial L}{\partial q_k} \delta \left( E + \sum_i q_i \frac{\partial L}{\partial q_i} \right) \right]
\]
(40)
and can thus be calculated, given any Lagrangian, for any specific path. The paths which satisfy
\[
\left[ \frac{1}{\hbar} \int_{t_1}^{t_2} L dt \right] = 3
\]
(41)
are typical at equilibrium. In a nonequilibrium situation, there are flow structures in phase space with well defined characteristic lengths (if one thinks for example in the coexistence between KAM tori and a chaotic sea in the weakly chaotic regime of a nonlinear Hamiltonian system, the KAM tori define islands of regular motion with well defined characteristic dimensions). These characteristic lengths can be described by corresponding paths in phase space and the entropy associated to any of these paths will be lower than the entropy associated to a typical segment (which provides the characteristic length for a “thermalized” path within the chaotic sea).
CONCLUSIONS

In this note we have presented a new variational method: the principle of least radix economy, Eq. (12). The dimensionless integer quantity \( \eta = \lfloor S/h \rfloor \) has been interpreted as the most efficient radix in which physical laws are expressed. Minimizing the radix economy \( C(\eta, S/h) \) has been shown to yield two different classes of solutions: least action paths and quantum wavefunctions. The Schrödinger and Klein-Gordon equations have been derived from this approach, as well as Heisenberg uncertainty relationships. The breaking of the commutativity of spacetime geometry and the existence of quantum numbers has then also been elucidated. Finally, Boltzmann’s principle of statistical thermodynamics has also been derived from the principle of least radix economy.

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