Wolf space coset spectrum in the large $\mathcal{N} = 4$ holography

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Abstract
After reviewing the four eigenvalues (the conformal dimension, two $SU(2)$ quantum number, and $U(1)$ charge) in the minimal (and higher) representations in the Wolf space coset where the $\mathcal{N} = 4$ superconformal algebra is realized by 11 currents in nonlinear way, these four eigenvalues in the higher representations up to two boxes (of Young tableaux) are examined in detail. The eigenvalues associated with the higher spin-1, 2, 3 currents in the (minimal and) higher representations up to two boxes are studied. They are expressed in terms of the two finite parameters $(N, k)$ where the Wolf space coset contains the group $SU(N + 2)$ and the affine Kac–Moody spin 1 current has the level $k$. Under the large $(N, k)$ ’t Hooft-like limit, they are simply linear combinations of the eigenvalues in the minimal representations. As a by product, the three-point functions of the higher spin currents with two scalar operators can be determined at finite $(N, k)$.

Keywords: AdS/CFT, higher spin symmetry, conformal symmetry
1. Introduction

It is known in [1] that the conformal dimension of a coset $\frac{G}{H}$ primary can be described as

$$h(\Lambda_+; \Lambda_-) = \frac{C^{(N+2)}(\Lambda_+)}{(k+N+2)} - \frac{C^{(N)}(\Lambda_-)}{(k+N+2)} - \frac{u^2}{N(N+2)(k+N+2)} + n.$$  \hfill (1.1)

The quantity $C^{(N+2)}(\Lambda_+)$ is the quadratic Casimir of $SU(N+2)$ of $G$ on the representation $\Lambda_+$. Similarly, the quantity $C^{(N)}(\Lambda_-)$ is the quadratic Casimir of $SU(N)$ of $H$ on the representation $\Lambda_-$. These Casimirs depend on $N$ and have explicit forms for any representations. The $\hat{u}$ charge is related to the $U(1)$ (of $H$) current of large $\mathcal{N} = 4$ ‘linear’ superconformal algebra [2–6]. Finally, the last quantity $n$ is known as the excitation number which can be positive integer or half-integer. That is, $n = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. When the representation $\Lambda_-$ appears in the branching of $\Lambda_+$, then this excitation number vanishes. Otherwise (when the representation occurs in the product of $(\Lambda_+; 0)$ and $(0; \Lambda_-)$), in general, the excitation number is nonzero. Because the $\hat{g}$ charge behaves as $N$ under the large $(N, k)$ ‘t Hooft-like limit and due to the fact that the denominator behaves as cubic term, the third term in (1.1) behaves differently when one compares to the first two terms of (1.1). Note that the quadratic Casimir behaves as $N$ under the large $(N, k)$ ‘t Hooft-like limit. The quantity $k$ appearing in (1.1) is the level of spin 1 current having the adjoint index of $SU(N+2)$.

In this paper, we would like to examine the above conformal dimensions closely, obtain them at finite $(N, k)$, identify the BPS representations (as well as non BPS representations), and observe some relation between the above formula (1.1) and the BPS bound [7–9], up to two boxes of Young tableaux. The BPS bound is described in terms of two $SU(2)$ quantum numbers $I^\pm$ corresponding to six spin 1 currents of the large $\mathcal{N} = 4$ ‘nonlinear’ superconformal algebra [7, 10–12]. Therefore, the four quantum numbers, $h$ corresponding to the stress energy tensor of the large $\mathcal{N} = 4$ nonlinear superconformal algebra, $I^\pm$, and $\hat{u}$ will be presented explicitly. One of the reasons why one should reexamine these quantum numbers is that there is no known formula like as (1.1) for the higher spin currents. In particular, when the excitation number is nonzero (the representation appears in the product of $(\Lambda_+; 0)$ and $(0; \Lambda_-)$), then the conformal dimensions and other eigenvalues corresponding to the higher spin currents are not sum of the ones in $(\Lambda_+; 0)$ and the ones in $(0; \Lambda_-)$. Instead, there is a contribution, at finite $(N, k)$, from the commutator between the zeromode of spin 2 current (or the zeromode of higher spin current) and other mode from the multiple product of spin $\frac{1}{2}$ currents having the adjoint index of $SU(N+2)$. After one makes sure that the two approaches, from the conformal formula (1.1) directly and from the collection of three contributions above, give the same answer, one can go to the eigenvalues for the higher spin currents.

Because there is a higher spin symmetry in the Wolf space coset [13–16], it is natural to ask what are the eigenvalues for the higher spin currents on the (minimal and) higher representations. How one can obtain the eigenvalues of the higher spin zero modes acting on the higher representations? Recall that in the above eigenvalues, $h$, $I^\pm$ and $\hat{u}$, the first three quantum numbers for the any representation remains the same by going to its complex conjugated representation. However, the last one, $\hat{u}$ charge can have an extra minus sign under this process. In particular, when one considers the vanishing excitation number (when the representation $\Lambda_-$ appears in the branching of $\Lambda_+$), the $SU(N+2)$ generators play the crucial role in the calculations of these eigenvalues. One can associate the quadratic $SU(N+2)$ generators with the zeromodes of the three quantities (corresponding to $h$ and $I^\pm$) while linear $SU(N+2)$ generators with the zeromode of spin 1 current corresponding to $\hat{u}$. Then it is obvious to see that
the former does not change the sign and the latter does change the sign when one changes the sign of SU(N + 2) generators because the complex conjugated representation is the minus of the original representation. We will see that some states contain the same quantum number \( h \) but they will have different quantum numbers from the higher spin currents.

Recall that the lowest 16 currents of the \( \mathcal{N} = 4 \) multiplet have the spin contents, (1, \( (\frac{1}{2})^4 \), 2\( \delta \), \( (\frac{1}{2})^4 \), 3). For the higher spin 1 and 3 currents which transform as \( SO(4) \) singlet, one expects that there should be minus sign between the representation and its complex conjugated representation. On the other hand, there is no sign change for the higher spin 2 currents, which transform as two \( SU(2) \) adjoints, between these two representations. One can make two \( SU(2) \) singlets by squaring of each higher spin 2 current and summing over the adjoint indices, in analogy of two quantities corresponding to the quantum numbers \( I^\pm \). Of course, their conformal dimensions are 4. For the minimal representations, \((f;0)\) (or \((\bar{f};0))\) and \((0;f)\) (or \((0;\bar{f}))\), the corresponding eigenvalues for the higher spin currents were obtained in [17]. In this paper, the eigenvalues for the higher spin currents in the higher representations, which are obtained from the various products of the minimal representations (which will correspond to single or mult particle states in the AdS3 bulk theory in the large \((N,k)\) ’t Hooft like limit), are determined explicitly by considering the eigenvalues for each minimal representation (and some additional eigenvalue contributions)\(^3\).

One of the reasons why one should obtain the eigenvalues for the higher spin currents is that one can determine the three-point functions (in the higher representations) for the higher spin (which is fixed) current with two scalar operators at finite \((N,k)\). It will turn out that although the various three-point functions for the higher spin currents with two scalar operators depend on \((N,k)\) explicitly (so far, it is not known how to write them down in terms of the data of the representations, contrary to (1.1)), the large \((N,k)\) ’t Hooft like limit of them leads to very simple form. That is, the ’t Hooft coupling dependent piece of the three-point functions in the higher representations (the coefficient of the two point function of two scalar operators) is a multiple of the ones in the minimal representations. For example, for the higher spin 3 current, the fundamental quantities are given by the eigenvalues of the representations, \((0;f)\) (or \((0;\bar{f}))\), \((f;0)\) (or \((\bar{f};0)\)) and \((f;f)\) (or \((\bar{f};\bar{f}))\). The first two of these behave as the functions of ’t Hooft coupling while the last one behaves as \( \frac{1}{N} \). Although we consider the two boxes (of Young tableaux) in this paper, we expect that the above behavior holds for any boxes of the Young tableaux\(^4\).

In section 2, the work of Gaberdiel and Gopakumar [1] is reviewed. The four eigenvalues mentioned in the abstract are obtained.

In section 3, further eigenvalues are determined for the other higher representations\(^5\) up to two boxes of Young tableaux. In particular, we summarize the conformal dimensions we have found under the large \((N,k)\) ’t Hooft like limit with table.

In section 4, the eigenvalues for the higher spin currents in the minimal representations [17] are reviewed.

\(^3\)In the coset model, we can obtain expressions with finite \((N,k)\), where a large \((N,k)\) limit corresponds to a classical higher spin theory. This implies that we can learn something about quantum effects in the dual higher spin theory from the exact expressions in the coset. The operators with the minimal and higher representations are dual to bulk scalar fields and their bound states, respectively. The information obtained in this paper is expected to be useful to examine the quantum corrections for bound states in the dual higher spin theory.

\(^4\)Recently [18], by analyzing the BPS spectra of string theory and supergravity theory on \( AdS_3 \times S^1 \times S^3 \times S^1 \), it has been found that the BPS spectra of both descriptions agree (where the world sheet approach is used). See also [19–21] for further studies along this direction. It would be interesting to see how the large \( \mathcal{N} = 4 \) superconformal higher spin and CFT duality arises in the context of these world sheet approaches.

\(^5\)The higher representations can be obtained from the minimal representations and have more than two boxes of Young tableaux. The currents of the \( \mathcal{N} = 4 \) nonlinear superconformal algebra can act on the minimal representations as well as the higher representations.
In section 5, as in sections 2 and 3, the eigenvalues for the higher spin currents in the higher representations up to two boxes of Young tableaux are obtained. This section is one of the main results of this paper. Some tables under the large $(N,k)$ ’t Hooft like limit summarize this section.

In section 6, some open problems related to this paper are presented. Some expectations for the particular eigenvalues for the higher spin currents are given.

In appendix, some detailed explanations in section 5 are given.

The mathematica [22] package by Thielemans [23] is used all the time.

In the arXiv version, further detailed calculations can be found.

2. Review of the work of Gaberdiel and Gopakumar

The unitary Wolf space coset [13–16] is given by

$$ Wolf = \frac{G}{H} = \frac{SU(N + 2)}{SU(N) \times SU(2) \times U(1)}. $$

The adjoint group indices in the complex basis are divided into

$$ G \text{ indices} : a, b, c, \cdots = 1, 2, \cdots, \frac{1}{2}([N + 2]^2 - 1], 1^*, 2^*, \cdots, \frac{1}{2}([N + 2]^2 - 1]^*; $$

$$ \frac{G}{H} \text{ indices} : \bar{a}, \bar{b}, \bar{c}, \cdots = 1, 2, \cdots, 2N, 1^*, 2^*, \cdots, 2N^*. \quad (2.1) $$

For given $(N + 2) \times (N + 2)$ unitary matrix, one can associate the above $4N$ Wolf space coset indices as follows [1]:

$$ \begin{pmatrix}
\begin{array}{cccc}
* & * & \cdots & * \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * \\
\end{array}
\end{pmatrix}
\quad (N+2)\times(N+2) $$

Note that the adjoint subgroup $H$ indices run over $1, 2, \cdots, \frac{1}{2}[N^2 + 3], 1^*, 2^*, \cdots, \frac{1}{2}[N^2 + 3]^*$.

The operator product expansions (OPEs) between the spin-1 current $V^a(z)$ and the spin-$\frac{1}{2}$ current $Q^a(z)$ are described as [24]

$$ V^a(z) V^b(w) = \frac{1}{(z - w)^2} k g^{ab} - \frac{1}{(z - w)} f^{abc} V^c(w) + \cdots, $$

$$ Q^a(z) Q^b(w) = -\frac{1}{(z - w)} (k + N + 2) g^{ab} + \cdots, $$

$$ V^a(z) Q^b(w) = + \cdots. \quad (2.3) $$

The positive integer $k$ is the level of the spin 1 current. The metric $g_{ab}$ in (2.3) is given by $g_{ab} = \text{Tr}(T_a T_b)$ and the structure constant $f_{abc}$ is given by $f_{abc} = \text{Tr}(T_c [T_a, T_b])$ where $T_a$ is the $SU(N+2)$ generator. Note that the nonvanishing metric components are given by $g_{AA^*} = g_{A^*A} = 1$ where $A = 1, 2, \cdots, \frac{1}{2}([N + 2]^2 - 1]$. Then by raising the $SU(N + 2)$ adjoint
lower index $A$, one has the $SU(N + 2)$ adjoint upper index $A^*$ and vice versa. Note that the above adjoint indices $a, b, \cdots$ for the group $G$ are further divided into index $A$ and index $A^*$.

The explicit 11 currents of large $\mathcal{N} = 4$ nonlinear superconformal algebra with (2.1) are given by

$$G^0(z) = \frac{i}{(k + N + 2)} Q_a V^a(z), \quad G^i(z) = \frac{i}{(k + N + 2)} h^i_{ab} Q^a V^b(z),$$

$$A^{+i}(z) = -\frac{1}{4N} f^{ab} h^i_{ab} V^c(z), \quad A^{-i}(z) = \frac{1}{4(k + N + 2)} h^i_{ab} Q_a^b \bar{Q}^c(z),$$

$$T(z) = \frac{1}{2(k + N + 2)^2} \left[(k + N + 2) V_a V^a + k Q_a \partial Q^a + f_{abc} Q^a \bar{Q}^b \bar{Q}^c \right](z) - \frac{1}{(k + N + 2)} \sum_{i=1}^{3} (A^{+i} + A^{-i})^2(z),$$

(2.4)

where $i = 1, 2, 3$. The $G^{0\pm i}(z)$ with $\mu = (0, i)$ currents are four supersymmetry currents, $A^{\pm i}(z)$ are six spin-1 generators of $SU(2)_k \times SU(2)_N$ and $T(z)$ is the spin-2 stress energy tensor. The three almost complex structures $h^i_{ab}$ are given by $4N \times 4N$ matrices as in [17]. Note that the spin-1 current $A^{+i}(z)$ with level $k$ depends on the spin-1 current $V^a(z)$ only while the spin-1 current $A^{-i}(z)$ with level $N$ depends on the spin-$\frac{1}{2}$ current $Q^a(z)$ only. Then it is obvious that the OPEs between them are regular.

2.1. The minimal representations in the $\mathcal{N} = 4$ Wolf space coset

Let us describe the eigenvalues of

(1) the zero mode of the stress energy tensor spin-2 current: $T_0$,
(2) the zero mode of sum of the square of spin-1 current: $-\left[\sum_{i=1}^{3} (A^{+i})^2\right]_0$,
(3) the zero mode of sum of the square of spin-1 current: $-\left[\sum_{i=1}^{3} (A^{-i})^2\right]_0$,
(4) the zero mode of other spin-1 current: $\frac{1}{2} U_0$,

acting on the minimal representations [1]. The corresponding eigenvalues with the help of (2.4) are denoted by $h$, $I^+ (I^+ + 1)$, $I^- (I^- + 1)$ and $\tilde{u}$ respectively. Note that the $U(1)$ current $U(z)$ is equivalent to the one $U(z)$ of the large $\mathcal{N} = 4$ linear superconformal algebra. That is, $U(z) = 2\sqrt{N(N + 2)} U(z)$ [17].

2.1.1. The $(0; f)$ representation. One of the minimal representations is given by the fundamental representation $(f)$ of $SU(N)$ living in the denominator of the Wolf space coset. The corresponding state is given by [1]

$$|0; f\rangle = \frac{1}{\sqrt{k + N + 2}} Q^{A^*}_{\frac{1}{2}} |0\rangle, \quad A^* = 1^*, 2^*, \cdots, 2N^*.$$

(2.5)

They correspond to the rectangular $N \times 2$ unitary matrix inside of $(N + 2) \times (N + 2)$ unitary matrix in (2.2). Under the decomposition of $SU(N+2)$ into the $SU(N) \times SU(2)$, the adjoint representation of $SU(N + 2)$ contains $(\mathbf{N}, 2)$ which corresponds to this $(0; f)$ representation. Furthermore, the representation $(0; \tilde{f})$ corresponds to $(\mathbf{N}, 2)$ associated with $\frac{1}{\sqrt{k + N + 2}} Q^{A}_{\frac{1}{2}} |0\rangle$ with $A = 1, 2, \cdots, 2N$ in the other rectangular $2 \times N$ unitary matrix in (2.2).
The four eigenvalues are summarized by [1]
\[ h(0; \square) = \frac{(2k + 3)}{4(N + k + 2)}, \]
\[ l^+(l^+ + 1)(0; \square) = 0, \quad l^-(l^- + 1)(0; \square) = \frac{3}{4}, \]
\[ \tilde{u}(0; \square) = \frac{(N + 2)}{2}. \]  
(2.6)

The eigenvalue \( h \) can be obtained by using the relation (1.1). The excitation number \( n \) is given by \( \frac{1}{2} \). Note that the excitation number for the fermion is given by \( \frac{1}{2} \) and for the multiple fermions, one can have the excitation number which is given by the number of fermions divided by two. Or one can calculate the OPE between the ‘reduced’ \( T(z) \), where the spin-1 current \( V^0(z) \) dependence is ignored completely, and \( Q^{A+}(w) \) and read off the second-order pole.

For the \( l^- \) quantum number, the eigenvalue is obtained from the OPE between the corresponding operator \( \sum_{l=1}^{N} (\bar{A}^{-1})^2(z) \) and \( Q^{A+}(w) \) and read off the second-order pole. Therefore, one obtains \( l^- = \frac{1}{2} \) which is consistent with the fact that above state transforms as a doublet under the \( SU(2) \).

For the \( l^+ \) quantum number, the relevant spin-1 current is given by \( A^{\pm}(z) \) which contains only the spin-1 current \( V^0(z) \). Therefore, the corresponding eigenvalue vanishes and \( l^+ = 0 \). Similarly, the \( U(1) \) charge \( \tilde{u} \) can be determined by calculating the first order pole in the OPE between the \( U(1) \) current \( \frac{1}{2} U(z) \) and \( Q^{A+}(w) \).

It is known that the BPS bound for the conformal dimension [7–9] is
\[ \frac{1}{(N + k + 2)} \left( k + 1 \right) l^+ + (N + 1) l^+ + (l^+ - l^-)^2 \right]. \]  
(2.7)

One can easily check that by substituting the quantum numbers \( l^+ = 0 \) and \( l^- = \frac{1}{2} \), the above conformal dimension becomes the BPS bound. The large \( (N, k) \) ’t Hooft limit for the BPS bound can be obtained.

2.1.2. The \( (f; 0) \) representation. Other minimal representation is given by \( (f; 0) \) representation. In other words, the singlets with respect to the \( SU(N) \) can be obtained from the fundamental representation in the \( SU(N + 2) \). It is known that the branching rule for the fundamental representation in the \( SU(N + 2) \) with respect to the \( SU(N) \times SU(2) \) is characterized by
\[ \square \to (\square, 1) + (1, 2)_{-\frac{1}{2}} \]  
(2.8)

The indices \( 1 \) and \( -\frac{1}{2} \) are the \( U(1) \) charges \( \tilde{u} \) [25, 26]. Then the state \( \left| (f; 0) \right\rangle \) corresponds to \( (1, 2)_{-\frac{1}{2}} \).

The four eigenvalues are described as [1]
\[ h(\square; 0) = \frac{(2N + 3)}{4(N + k + 2)}, \]
\[ l^+(l^+ + 1)(\square; 0) = \frac{3}{4}, \quad l^-(l^- + 1)(\square; 0) = 0, \]
\[ \tilde{u}(\square; 0) = -\frac{N}{2}. \]  
(2.9)

One can obtain the conformal dimension using the formula (1.1). Or after substituting the \( SU(N + 2) \) generator \( T^\mu \) into the zero mode of spin-1 current \( V^0_\mu \) in the quadratic of the
reduced spin-2 current $T(z)$ where all the $Q^I$ dependent terms are ignored, one obtains $(N + 2) \times (N + 2)$ unitary matrix acting on the state $| f; 0 \rangle$. Then the conformal dimension for this state can be determined by the diagonal elements of the last $2 \times 2$ subdiagonal matrix.

For the $I^+$ quantum number, as described in the conformal dimension, one can find $(N + 2) \times (N + 2)$ unitary matrix for the zero mode of $-\sum_{i=1}^{3}(A^+)^2$ which contains only the spin-1 current $V^A(z)$. It turns out that the diagonal element of the last $2 \times 2$ subdiagonal matrix is given by $\frac{1}{2}$ which implies $I^+ = \frac{1}{2}$. This is consistent with the fact that this state is a doublet under the $SU(2)_h (1, 2)_{-\frac{1}{2}}$.

For the $I^-$ quantum number, because the spin-$\frac{1}{2}$ current $Q^I(z)$ does not contribute to the eigenvalue equation associated with this state, one has $I^- = 0$.

For the $U(1)$ charge, one can construct $(N + 2) \times (N + 2)$ unitary matrix for the zero mode of the reduced $\frac{1}{2} U(z)$ where the spin-$\frac{1}{2}$ current $Q^I(z)$ dependence is removed. Then the diagonal element of the last $2 \times 2$ subdiagonal matrix is given by $-\frac{N}{2}$.

One can check the above conformal dimension satisfies the BPS bound by substituting $I^+ = \frac{1}{2}$ and $I^- = 0$.

For the state $| f; 0 \rangle$, one can obtain similar eigenvalues where the only difference appears in the eigenvalue $\tilde{u}$. That is, $\tilde{u}(| f; 0 \rangle) = \frac{N}{2}$.

2.2. The higher representations

Let us describe the four eigenvalues for the higher representations which arise in the various products of the above minimal representations $(0; f), (f; 0), (0; \bar{f})$ and $| f; 0 \rangle$.

2.2.1. The $(f; f)$ representation. According to the previous branching rule in (2.8), this higher representation corresponds to $(\square, \square)_{1}$ transforming as a fundamental representation $f$ (or $\square$) under the $SU(N)$ and a singlet under the $SU(2)_h$. The nonzero $\tilde{u}$ charge is given by 1 from the subscript.

One can summarize the following eigenvalues [1]

\[
\begin{align*}
    h(\square; \square) &= \frac{1}{(N + k + 2)}, \\
    I^+(I^++1)(\square; \square) &= 0, \\
    I^-(I^-+1)(\square; \square) &= 0, \\
    \tilde{u}(\square; \square) &= 1.
\end{align*}
\]

In this case, the conformal dimension can be determined by reading off the diagonal elements in the $N \times N$ subdiagonal unitary matrix inside of $(N + 2) \times (N + 2)$ unitary matrix obtained in previous subsection. Or the formula in (1.1) can be used also. From the diagonal elements in the $N \times N$ subdiagonal unitary matrix inside of $(N + 2) \times (N + 2)$ unitary matrix for the zero mode of $-\sum_{i=1}^{3}(A^+)^2$, one can determine the $I^+$ quantum number which is equal to 0. This is consistent with the singlet under the $SU(2)_h$ in $(\square, \square)_{1}$. For the $I^-$ quantum number, it is the same as before and $I^- = 0$. From the diagonal elements in the $N \times N$ subdiagonal unitary matrix inside of $(N + 2) \times (N + 2)$ unitary matrix for the zero mode of the reduced $\frac{1}{2} U(z)$, the above $\tilde{u}$ charge can be obtained. It is easy to see that the above representation does not satisfy the BPS bound because the conformal dimension for the BPS bound is equal to 0 by substituting $I^+ = I^- = 0$. 

2.2.2. The \( (f; f) \) representation. This representation can be obtained from the product between the minimal representations \( (f; 0) \) and \((0; f)\). Then one can write down the state as \( \frac{1}{\sqrt{k+N+2}} Q^A_{\frac{1}{2}} (f; 0) \) where \( A = 1, 2, \cdots, 2N \). The four eigenvalues are given by [1]

\[
h(\square; \square) = \frac{1}{2},
\]
\[
l^+(l^+ + 1)(\square; \square) = \frac{3}{4}, \quad l^-(l^- + 1)(\square; \square) = \frac{3}{4},
\]
\[
\hat{u}(\square; \square) = -N - 1.
\]

For the conformal dimension, there exists other contribution in addition to the sum of \( h(\square; 0) \) and \( h(0; \square) \) (which is equal to \( h(0; \square) \)). The extra contribution coming from the lower order pole in the commutator \([T_0, Q^A_{\frac{1}{2}}]\) takes the form

\[
\begin{pmatrix}
\frac{1}{N(N+k+2)} I_{N \times N} \\
0 \\
-rac{1}{2(N+k+2)} I_{2 \times 2}
\end{pmatrix}.
\]

By realizing the diagonal elements in the lower \(2 \times 2\) subdiagonal matrix as the above extra contribution, one can write down the final conformal dimension from (2.9) and (2.6) as

\[
\frac{2N+3}{4(N+k+2)} + \frac{2k+3}{4(N+k+2)} - \frac{1}{2(N+k+2)} = \frac{1}{2}.
\]

Of course, this counting can be seen from the formula (1.1). For the \( l^\pm \) quantum numbers, one can add each contribution from (2.9) and (2.6). For the \( \hat{u} \) charge, one can add each contribution and it turns out that \( -\frac{N}{2} - \frac{N+k+2}{2} = -N - 1 \). By substituting \( l^\pm = \frac{1}{2} \), the above conformal dimension satisfies the BPS bound.

2.2.3. The \((\text{symm}; 0)\) representation. This representation can be obtained from the product of minimal representations \((f; 0)\) and \((f; 0)\). By taking the product between the two identical branching rules in (2.8), we obtain [1, 25, 26]

\[
\begin{aligned}
\square \otimes \square &= \square + \square \\
&= \begin{bmatrix} (\square, 1)_1 + (1, 2)_{-\frac{N}{2}} \end{bmatrix} \otimes \begin{bmatrix} (\square, 1)_1 + (1, 2)_{-\frac{N}{2}} \end{bmatrix} \\
&= \begin{bmatrix} (\square, 1)_2 + (\square, 2)_{1-\frac{N}{2}} + (1, 3)_{-N} \end{bmatrix} \\
&+ \begin{bmatrix} (\square, 1)_2 + (\square, 2)_{1-\frac{N}{2}} + (1, 1)_{-N} \end{bmatrix}.
\end{aligned}
\]

Then one can identify the following branching rules under the \( SU(N)_k \times SU(2)_k \times U(1) \)

\[
\begin{aligned}
\square &\rightarrow (\square, 1)_2 + (\square, 2)_{1-\frac{N}{2}} + (1, 3)_{-N}, \\
\square &\rightarrow (\square, 1)_2 + (\square, 2)_{1-\frac{N}{2}} + (1, 1)_{-N}.
\end{aligned}
\]

(2.10)

The \((\text{symm}; 0)\) representation, which is the singlet in \( SU(N)_k \), corresponds to \((1, 3)_{-N}\). Then one can describe the four eigenvalues as follows:
One can calculate the conformal dimension by using the formula or by substituting the $SU(N + 2)$ generator $T_{a'}$ into the zero mode of spin-1 current $V_{0}^{a}$ in the reduced spin-2 current $T(z)$, one obtains $\frac{1}{2}(N + 2)(N + 3)$ unitary matrix acting on the state $|\text{symm}; 0\rangle$. Then the conformal dimension for this state can be determined by the diagonal elements of the last $3 \times 3$ subdiagonal matrix. The role of remaining diagonal elements in the $\frac{1}{2}N(N + 1) \times \frac{1}{2}N(N + 1)$ unitary matrix or $2N \times 2N$ unitary matrix will be explained in next subsection.

From the whole unitary matrix for the zero mode of $-\sum_{i=1}^{3} (A_{i}^{+})^{2}$, the diagonal element of the last $3 \times 3$ subdiagonal matrix (triplet under $SU(2)_k$) implies $l^+ = 1$. This is consistent with the fact that this state is a triplet under the $SU(2)_k (1, 3)_{-N}$. For the $U(1)$ charge, one can construct the whole unitary matrix for the zero mode of the reduced $\frac{1}{2}U(z)$. Then the diagonal element of the last $3 \times 3$ subdiagonal matrix is given by $-N$ which is simply the sum of two $U(1)$ charge $-\frac{1}{2}$ for $\Box$. One can check the above conformal dimension satisfies the BPS bound by substituting $l^+ = 1$ and $l^- = 0$. For the state $|\text{antisymm}; 0\rangle$, one can obtain similar eigenvalues where the only difference appears in the eigenvalue $\hat{u}$. That is, $\hat{u}(\Box; 0) = N$.

### 2.2.4. The (antisymm; 0) representation

The other representation can arise from the product of minimal representations $(f; 0)$ and $(f; 0)$. The (antisymm; 0) representation, which is the singlet in $SU(N)_k$, corresponds to $(1, 1)_{-N}$ in the branching rule (2.10).

Then the four eigenvalues are given as follows:

$$
l^+(l^+ + 1)|\Box; 0\rangle = 0, \quad l^-(l^- + 1)|\Box; 0\rangle = 0
$$

The conformal dimension formula can be used here or by substituting the $SU(N + 2)$ generator $T_{a'}$ into the zero mode of spin-1 current $V_{0}^{a}$ in the reduced spin-2 current $T(z)$, one obtains $\frac{1}{2}(N + 2)(N + 3)$ unitary matrix acting on the state $|\text{antisymm}; 0\rangle$. The conformal dimension for this state can be obtained by the last diagonal element (singlet of $SU(2)_k$). The detailed descriptions of remaining diagonal elements in the $\frac{1}{2}N(N - 1) \times \frac{1}{2}N(N - 1)$ unitary matrix or $2N \times 2N$ unitary matrix will be given in next subsection.

By reading off the last diagonal element in the whole matrix for the zero mode of the operator corresponding to $l^+$ quantum number, one obtains $l^+ = 0$ which is a singlet under the $SU(2)_k (1, 1)_{-N}$. As before, one has a trivial $l^- = 0$ quantum number. The similar analysis for the $\hat{u}$ charge can be done. One can check the above conformal dimension does not satisfy the BPS bound by substituting $l^+ = 0$. For the state $|\text{antisym}; 0\rangle$, one has $\hat{u}(\Box; 0) = N$. 

$$
l^+(l^+ + 1)|\Box; 0\rangle = 2, \quad l^-(l^- + 1)|\Box; 0\rangle = 0,
$$

$$
\hat{u}(\Box; 0) = -N.
$$

(2.11)
2.2.5. The \((0; \text{symm})\) representation. One can also construct the product of two minimal representations \((0; f)\) and \((0; f)\). The \((0; \text{symm})\) representation can arise. Let us focus on \(N = 3\) case for simplicity. One can visualize the spin-\(\frac{1}{2}\) current \(Q^a(z)\) in the following \(5 \times 5\) unitary matrix

\[
\begin{pmatrix}
0 & 0 & 0 & Q^{13} & Q^{16} \\
0 & 0 & 0 & Q^{14} & Q^{17} \\
0 & 0 & 0 & Q^{15} & Q^{18} \\
Q^1 & Q^2 & Q^3 & 0 & 0 \\
Q^4 & Q^5 & Q^6 & 0 & 0
\end{pmatrix}.
\] (2.13)

The \(Q^{13}, \ldots, Q^{16}\) appear in (2.5).

Let us construct the symmetric combinations between the spin-\(\frac{1}{2}\) currents \(Q^I(z)\). There are six states and the corresponding operators are as follows:

\[
\begin{align*}
Q^{13}Q^{16}(z), & \quad (Q^{13}Q^{17} + Q^{14}Q^{16})(z), & \quad (Q^{13}Q^{18} + Q^{15}Q^{16})(z), \\
Q^{14}Q^{17}(z), & \quad (Q^{14}Q^{18} + Q^{15}Q^{17})(z), & \quad Q^{15}Q^{18}(z),
\end{align*}
\] (2.14)

up to some overall normalizations. First of all, the two \(SU(2)\) indices should be different from each other because of the fermionic property of \(Q^a(z)\). One should have one factor from the elements, \(Q^{13}(z), Q^{14}(z)\) or \(Q^{15}(z)\) and the other factor from the elements, \(Q^{16}(z), Q^{17}(z)\) or \(Q^{18}(z)\). When the \(SU(N = 3)\) indices are equal (i.e. if one takes two operators in the same row of the matrix (2.13)), the interchange of these indices gives the original term. Then one obtains the single terms in (2.14). When the \(SU(N = 3)\) indices are not equal to each other, then there are extra terms. Then we have the remaining terms in (2.14).

The four eigenvalues are given by

\[
\begin{align*}
h(0, \square) &= \frac{k}{(N + k + 2)}, \\
l^+ \{l^+ + 1\}(0, \square) &= 0, \quad l^- \{-l^- + 1\}(0, \square) = 0, \\
\hat{u}(0, \square) &= N + 2.
\end{align*}
\]

One can interpret the conformal dimension by calculating the following second order pole of the OPE

\[
T(z)Q^{13}Q^{16}(w) \bigg|_{\frac{1}{(z-w)^2}} = \frac{k}{(N + k + 2)} Q^{13}Q^{16}(w).
\]

Of course, the stress energy tensor spin-2 current does not contain the spin-1 current \(V^a(z)\). The \(N\)-dependence appearing in the denominator can be easily generalized from \(N = 3\) result. It is easy to check that the similar calculations for other symmetric combinations in (2.14) lead to the same results.

For the \(l^+\) quantum number, we have trivial \(l^+ = 0\). For the \(l^-\) quantum number, one can compute the following OPE and read off the coefficient of second order pole

\[
-\sum_{i=1}^{3} (A^{-i})^2(z) Q^{13}Q^{16}(w) \bigg|_{\frac{1}{(z-w)^2}} = 0.
\]

Similarly, the \(\hat{u}\) charge can be added and leads to
\[ i\sqrt{N(N+2)} U(z) Q^{13} Q^{16}(w) \bigg|_{z-w} = (N+2) Q^{13} Q^{16}(w). \]

One can check the above conformal dimension does not satisfy the BPS bound by substituting \( l^+ = 0 \).

### 2.2.6. The \((0; \text{antisymm})\) representation

Let us construct the antisymmetric combinations between the spin-\( \frac{1}{2} \) currents \( Q^i(z) \). There are six states and the corresponding operators are as follows:

\[
Q^{13} Q^{14}(z), \quad Q^{13} Q^{15}(z), \quad Q^{14} Q^{15}(z), \\
Q^{16} Q^{17}(z), \quad Q^{16} Q^{18}(z), \quad Q^{17} Q^{18}(z). \tag{2.15}
\]

The two \( SU(2) \) indices should be equal to each other because of the fermionic property of \( Q^{\bar{a}}(z) \). One should take the two operators in (2.13) from the same column.

The four eigenvalues are summarized by

\[
\begin{align*}
h(0; \underline{B}) &= \frac{(k + 2)}{(N + k + 2)}, \\
l^+(l^+ + 1)(0; \underline{B}) &= 0, \quad l^-(l^- + 1)(0; \underline{B}) = 2, \\
\hat{u}(0; \underline{B}) &= N + 2.
\end{align*}
\]

For the conformal dimension, one can calculate the following OPE

\[
T(z) Q^{13} Q^{14}(w) \bigg|_{z-w} = \frac{(k + 2)}{(N + k + 2)} Q^{13} Q^{14}(w).
\]

Similarly, one can check that the conformal dimension for other operators in (2.15) leads to the same quantum number. The denominator for general \( N \) can be expected.

For \( l^+ \) quantum number, we have trivial \( l^+ = 0 \). For the \( l^- \) quantum number, one can compute the following OPE and read off the second order pole

\[
- \sum_{i=1}^{3} (A^{-i})^2(z) Q^{13} Q^{14}(w) \bigg|_{z-w} = 2 Q^{13} Q^{14}(w).
\]

This implies that the \( l^- \) quantum number is \( l^- = 1 \). Similarly, the \( \hat{u} \) charge can be obtained from the following OPE

\[
i\sqrt{N(N+2)} U(z) Q^{13} Q^{14}(w) \bigg|_{z-w} = (N+2) Q^{13} Q^{14}(w).
\]

By substituting \( l^+ = 0 \) and \( l^- = 1 \) into the BPS bound for the conformal dimension, one sees that this state satisfies the BPS bound.
2.3. Summary of this section

Let us summarize what has been done in this section. The conformal dimension for any representation can be encoded in the formula (1.1). For the representation $(\Lambda^+_e; 0)$, the $\hat{u}$ charge is additive and is given by $-N_\Sigma$ (which is the $\hat{u}$ charge in $(\square; 0)$) times the number of boxes. Its $l^+$ quantum number is trivial $l^+ = 0$. For the $l^-$ quantum number, the maximum number, which is the $\frac{1}{2}$ times the number of boxes, can arise in the symmetric representation.

For the representation $(0; \Lambda_-)$, the $\hat{u}$ charge is additive and is given by $\frac{1}{2}(N + 2)$ (which is the $\hat{u}$ charge in $(0; \square)$) times the number of boxes. Its $l^+$ quantum number is trivial $l^+ = 0$. For the $l^-$ quantum number, the maximum number, which is the $\frac{1}{2}$ times the number of boxes, can arise in the symmetric representation.

For the representation $(\Lambda^+_e; \Lambda_-)$, there are two cases.

(1) The case where the representation $\Lambda_-$ appears in the branching of $\Lambda^+_e$ under the $SU(N)_k \times SU(2)_k \times U(1)$. There is a trivial $l^- = 0$ quantum number. For the $l^+$ quantum number and $\hat{u}$ charge can be read off from the multiple product of $(\square, 1)_1 + (1, 2)_\pm$ in (2.8).

(2) The case where the representation arises in the product of $(\Lambda^+_e; 0)$ and $(0; \Lambda_-)$. The $l^\pm$ and $\hat{u}$ quantum numbers are additive. In other words, the $l^+$ quantum number of $(\Lambda^+_e; \Lambda_-)$ comes from the one of $(\Lambda^+_e; 0)$ while the $l^-$ quantum number of $(\Lambda^+_e; \Lambda_-)$ comes from the one of $(0; \Lambda_-)$. The $\hat{u}$ charge of $(\Lambda^+_e; \Lambda_-)$ is the sum of the one in $(\Lambda^+_e; 0)$ and the one in $(0; \Lambda_-)$. Note that this is not true for the conformal dimension because there exists the extra contribution.

3. More eigenvalues for the higher representations in the \( SU(N+2) \times SU(N) \times SU(2) \times U(1) \) Wolf space coset

In this section, we would like to construct the four eigenvalues for the zero modes of previous stress energy tensor spin-2 current, the sum of square of spin-1 current, the sum of square of other spin-1 current and other spin-1 current, acting on other higher representations by considering the multiple products of $(0; f)$, $(f; 0)$, $(0; \bar{f})$ or $(\bar{f}; 0)$.

3.1. The symmetric representations $\Lambda_+$ with two boxes

One of the simplest higher representation is given by the symmetric representation symm $\square \square$. It is known that the product of the minimal representation $(f; 0)$ and itself $(f; 0)$ implies that there are symmetric and antisymmetric representations. From the branching rule for the $SU(N+2)$ under the $SU(N)_k \times SU(2)_k \times U(1)$, one can identify the following branching rules

\[
\square \otimes \square = \square + \Box \\
\rightarrow \left[ (\square, 1)_1 + (1, 2)_\pm \frac{\lambda}{2} \right] \otimes \left[ (\square, 1)_1 + (1, 2)_\pm \frac{\lambda}{2} \right] \\
= \left[ (\square \square, 1)_2 + (\Box \square, 1)_1 \frac{\lambda}{2} + (1, 3)_N \right] \\
+ \left[ (\square \Box, 1)_2 + (\Box \Box, 1)_1 \frac{\lambda}{2} + (1, 1)_N \right].
\]
The subscript stands for the $U(1)$ charge $\hat{u}$. The final $\hat{u}$ charge is obtained by adding each $\hat{u}$ charge. Then one obtains the following branching rules for the symmetric and antisymmetric representations under the $SU(N)_k \times SU(2)_k \times U(1)$ (again from (2.10))

\[
\begin{align*}
\square & \quad \rightarrow \quad (\square, 1)_2 + (\square, 2)_{1 - \frac{N}{2}} + (1, 3)_{-N}, \\
\square & \quad \rightarrow \quad (\square, 1)_2 + (\square, 2)_{1 - \frac{N}{2}} + (1, 1)_{-N}.
\end{align*}
\tag{3.1}
\]

Note that the last representations in (3.1) correspond to the symmetric and antisymmetric representations respectively.

The two-index symmetric parts of the $SU(N + 2)$ representation can be obtained from the generators of the fundamental representation of $SU(N + 2)$ by using the projection operator

\[
\frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \quad \text{where} \quad i \leq j \quad \text{and} \quad k \leq l \quad \text{and} \quad i, j, k, l = 1, 2, \cdots, (N + 2) [27, 28].
\]

Then by acting on the space $T_a \otimes \mathbf{1}_{(N+2)\times(N+2)} + \mathbf{1}_{(N+2)\times(N+2)} \otimes T_a$, one has the generators for the symmetric representation for the $SU(N + 2)$

\[
(T_a)_{ik} \delta_{jl} + (T_a)_{jk} \delta_{il} + \delta_{ik} (T_a)_{jl} + \delta_{jk} (T_a)_{il}.
\]

For $N = 3$, one has $\frac{1}{2}(N + 2)(N + 3) \times \frac{1}{2}(N + 2)(N + 3) = 15 \times 15$ unitary matrix, and the row and columns are characterized by the following double index notations

11, 12, 13, 22, 23, 33; 14, 15, 24, 25, 34, 35; 44, 45, 55.

The first six elements correspond to the symmetric representation for $SU(3)$. The next six elements correspond to the fundamental representation of $SU(3)$ with $SU(2)_k$ doublet. The last three elements correspond to the singlet of $SU(3)$ with $SU(2)_k$ triplet according to (3.1).

Let us calculate the zero mode for the reduced stress energy tensor spin-2 current acting on the state $\mathbf{1}_{15\times15}$. It turns out that the $15 \times 15$ matrix is given by

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 17 & 0 \\
0 & 0 & 5
\end{pmatrix}_{3\times3}.
\tag{3.2}
\]

There are three block diagonal elements. The last block diagonal elements correspond to the eigenvalue on the state $\langle \mathbf{1}_{15\times15} \rangle$. We will describe the detailed quantum numbers for the other eigenvalues soon.

One can also calculate the zero mode for the sum of the square for the spin-1 current with minus sign acting on the above state $\langle \mathbf{1}_{15\times15} \rangle$ and the explicit result is given by

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 21
\end{pmatrix}_{3\times3}.
\tag{3.3}
\]

The diagonal elements correspond to the eigenvalues and in particular, the last one is the eigenvalue for the state $\langle \mathbf{1}_{15\times15} \rangle$ which behaves as a singlet under the $SU(3)$.

Similarly, one can also compute the zero mode for the spin-1 current acting on the state $\langle \mathbf{1}_{15\times15} \rangle$ and one obtains

\[
\begin{pmatrix}
21 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 31
\end{pmatrix}_{3\times3}.
\tag{3.4}
\]

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In this case, also the last elements are the eigenvalues for the state $|\square; 0\rangle$.

### 3.1.1. The (symm; symm) representation

Let us consider the higher representation where the symmetric representation in $SU(N)$ $|\square; 1\rangle_2$ survives in the branching of (3.1). The four eigenvalues are given by

$$h(\square; \square) = \frac{2}{(N + k + 2)}$$

$$l^+(l^+ + 1)(\square; \square) = 0, \quad l^-(l^- + 1)(\square; \square) = 0,$$

$$\hat{u}(\square; \square) = 2.$$

One observes that the above conformal dimension does not satisfy the vanishing BPS bound with $l^\pm = 0$.

### 3.1.2. The (symm; f) representation

Let us consider the higher representation where the fundamental representation in $SU(N)$ $|\square, 2\rangle_{1-2}$ survives in the branching of (3.1). The four eigenvalues can be summarized by

$$h(\square; \square) = \frac{(2N + 11)}{4(N + k + 2)},$$

$$l^+(l^+ + 1)(\square; \square) = \frac{3}{4}, \quad l^-(l^- + 1)(\square; \square) = 0,$$

$$\hat{u}(\square; \square) = -\frac{N}{2} + 1.$$

One observes that the above conformal dimension does not satisfy the BPS bound.

### 3.1.3. The (symm; 0) representation

The eigenvalues in this higher representation are given in the section 2.2.3. One sees that there are eigenvalues in (2.11). From the three matrices in (3.2)–(3.4), the relevant eigenvalues are given by $(\frac{5}{k+1})$, 2 and $-3$ for the nontrivial quantum numbers. They are generalized to $(\frac{N+2}{k+N+2})$, 2 and $-N$ respectively. The conformal dimension can be checked from $\frac{(N+2)(k+N+2)}{2(N+2)(k+N+2)} - \frac{N^2}{N(N+2)(k+N+2)}$ also.

### 3.1.4. The (symm; 1) representation

Let us consider the higher representation which arises from the product of $|\square; 0\rangle$ and $|0; \square\rangle$. The former occurs in the section 2.2.3 and the latter occurs in the section 2.1.1 together with the complex conjugation.

In this case, the corresponding four eigenvalues are described by

$$h(\square; \square) = \frac{(4N + 2k + 7)}{4(N + k + 2)},$$

$$l^+(l^+ + 1)(\square; \square) = 2, \quad l^-(l^- + 1)(\square; \square) = \frac{3}{4},$$

$$\hat{u}(\square; \square) = -\frac{3N}{2} - 1. \quad (3.5)$$

First of all, one can obtain the following $15 \times 15$ matrix by calculating the commutator $[T_0, Q_{-1}^A]$ as in the section 2.2.2.
\[
\begin{pmatrix}
\frac{2}{3(N+k+2)}I_{6 \times 6} & 0 & 0 \\
0 & -\frac{1}{6(5+k)}I_{6 \times 6} & 0 \\
0 & 0 & -\frac{1}{(5+k)}I_{3 \times 3}
\end{pmatrix}.
\tag{3.6}
\]

The last three eigenvalues (the \(N\) generalization is straightforward to obtain) appearing in the last block diagonal matrix in (3.6) provide the extra contribution as well as the sum of conformal dimensions of \(\boxtimes; 0\) and \(0; \square\). They are given in (2.11) and (2.6) respectively. Then one obtains the final conformal dimension by adding the above contribution appearing in (3.6) as follows

\[
\frac{(N+2)}{(N+k+2)} + \frac{(2k+3)}{4(N+k+2)} - \frac{1}{4(N+k+2)} = \frac{(4N+2k+7)}{4(N+k+2)},
\]

as in (3.5). It is easy to see that the above conformal dimension satisfies the BPS bound by substituting \(l^+ = 1\) and \(l^- = \frac{1}{2}\).

3.1.5. The \((\text{symm}; \text{antisymm})\) representation. Let us consider the higher representation which arises from the product of \(\boxtimes; 0\) and \(0; \square\). The former occurs in the section 2.2.3 and the latter occurs in the section 2.2.6.

The four eigenvalues can be summarized by

\[
h(\boxtimes; \square) = \frac{(N+k+6)}{(N+k+2)},
\]

\[
l^+(l^+ + 1)(\boxtimes; \square) = 2, \quad l^-(l^- + 1)(\boxtimes; \square) = 2,
\]

\[
\bar{u}(\boxtimes; \square) = 2.
\]

One can easily see that the above conformal dimension does not lead to the BPS bound with \(l^\pm = 1\).

3.1.6. The \((\text{symm}; \text{symm})\) representation. Let us consider the higher representation which arises from the product of \(\boxtimes; 0\) and \(0; \square\). The former occurs in the section 2.2.3 while the latter occurs in the section 2.2.5 with complex conjugation.

The four eigenvalues are given by

\[
h(\boxtimes; \boxtimes) = \frac{(k+N)}{(k+N+2)},
\]

\[
l^+(l^+ + 1)(\boxtimes; \boxtimes) = 2, \quad l^-(l^- + 1)(\boxtimes; \boxtimes) = 0,
\]

\[
\bar{u}(\boxtimes; \boxtimes) = -2N - 2.
\]

One can easily see that the above conformal dimension does not lead to the BPS bound with \(l^+ = 1\) and \(l^- = 0\).

3.2. The antisymmetric representations \(\Lambda_+\) with two boxes

The two-index antisymmetric parts of the \(SU(N+2)\) representation can be obtained from the generators of the fundamental representation of \(SU(N+2)\) by using the projection operator \(\frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})\) where \(i < j\) and \(k < l\) and \(i, j, k, l = 1, 2, \ldots, (N+2)\) [27, 28]. Then by acting
on the space $T_a \otimes \mathbf{1}_{(N+2)\times(N+2)} + \mathbf{1}_{(N+2)\times(N+2)} \otimes T_a$, one has the generators for the antisymmetric representation for the $SU(N+2)$

\[(T_a)_{\bar{a}b} \delta_{\bar{a}c} - (T_a)_{\bar{b}c} \delta_{\bar{a}d} + \delta_{\bar{a}b} (T_a)_{\bar{c}d} - \delta_{\bar{a}d} (T_a)_{\bar{c}b}.
\]

For $N = 3$, one has $\frac{1}{2} (N+2)(N+1) \times \frac{1}{2} (N+2)(N+1) = 10 \times 10$ unitary matrix, and the row and columns are characterized by the following double index notations

\[12, 13, 23; 14, 15, 24, 25, 34, 35, 45.\]

The first three elements correspond to the antisymmetric representation for $SU(3)$. The next six elements correspond to the fundamental representation of $SU(3)$ with $SU(2)_k$ doublet. The last element corresponds to the singlet of $SU(3)$ with $SU(2)_k$ singlet.

Let us calculate the zero mode for the reduced stress energy tensor spin-2 current acting on the state $|\mathbb{\Box}; 0\rangle$. It turns out that the $10 \times 10$ matrix is given by

\[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 9 & 0 & 0 \\
0 & 0 & 3 & 0 \\
(3+\kappa)
\end{pmatrix}
\]

The last block diagonal element gives us the eigenvalue on the state $|\mathbb{\Box}; 0\rangle$ and its $N$ generalization is straightforward.

One can also calculate the zero mode for the spin-1 current acting on the state $|\mathbb{\Box}; 0\rangle$ and one obtains

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(3+\kappa)
\end{pmatrix}
\]

Again, the last block diagonal element gives us the eigenvalue associated with the above operator on the state $|\mathbb{\Box}; 0\rangle$ and its $N$ generalization is straightforward.

Similarly, one can also compute the zero mode for the spin-1 current acting on the state $|\mathbb{\Box}; 0\rangle$ and it turns out that

\[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 3 \\
(3+\kappa)
\end{pmatrix}
\]

The last block diagonal element gives us the eigenvalue associated with the above spin-1 operator on the state $|\mathbb{\Box}; 0\rangle$ and its $N$ generalization is straightforward.

3.2.1. The (antisym; antisym) representation. Let us consider the higher representation where the antisymmetric representation in $SU(N)$ $\mathbb{\Box}_2$ survives in the branching of (3.1). The four eigenvalues are given by

\[
\begin{align*}
\kappa(\mathbb{\Box}\mathbb{\Box}) &= \frac{2}{(N+k+2)}, \\
t^{+}(t^{+}+1)(\mathbb{\Box}\mathbb{\Box}) &= 0, \\
t^{-}(t^{-}+1)(\mathbb{\Box}\mathbb{\Box}) &= 0, \\
u(\mathbb{\Box}\mathbb{\Box}) &= 2.
\end{align*}
\]

The above conformal dimension does not satisfy the vanishing BPS bound with $t^k = 0$. 

3.2.2. The (antisymm; f) representation. Let us consider the higher representation where the fundamental representation in $SU(N)$ $(\square, 2)_{1 - \frac{N}{2}}$ survives in the branching of (3.1). The four eigenvalues can be summarized by

\[ h(\square) = \frac{(2N + 3)}{4(N + k + 2)} \]
\[ l^+(l^++1)(\square) = \frac{3}{4}, \quad l^-(l^-+1)(\square) = 0, \]
\[ \psi(\square) = -\frac{N}{2} + 1. \]

One observes that the above conformal dimension does satisfy the BPS bound.

3.2.3. The (antisymm; 0) representation. The eigenvalues in this higher representation are given in the section 2.2.4. One sees that there are eigenvalues in (2.12). From the three matrices in (3.7)–(3.9), the relevant eigenvalues are given by $\frac{3}{4(N + k + 2)}$, 0 and $-3$ for the nontrivial quantum numbers. They are generalized to $\frac{2N(N+2+1)}{2(N+2)(N+k+2)}$ and $-N$ respectively. The conformal dimension can be checked from $\frac{2N(N+2+1)}{2(N+2)(N+k+2)}$ also.

3.2.4. The (antisymm; f) representation. Let us consider the higher representation which arises from the product of $(\square, 0)$ and $(0; \square)$. The former occurs in the section 2.2.4 and the latter occurs in the section 2.1.1 together with the complex conjugation.

In this case, the corresponding four eigenvalues are described by

\[ h(\square) = \frac{(4N + 2k - 1)}{4(N + k + 2)} \]
\[ l^+(l^++1)(\square) = 0, \quad l^-(l^-+1)(\square) = \frac{3}{4}, \]
\[ \psi(\square) = -\frac{3N}{2} - 1. \]

One can obtain the following $10 \times 10$ matrix by calculating the commutator $[T_0, Q^A_{\pm \frac{1}{2}}]$ as in the section 2.2.2

\[ \begin{pmatrix}
\begin{array}{ccc}
\frac{2}{3(N+k+2)} & 0 & 0 \\
0 & \frac{1}{6(5+k)} & 0 \\
0 & 0 & -\frac{1}{(5+k)}
\end{array}
\end{pmatrix}. \tag{3.10}
\]

The last eigenvalue (the $N$ generalization is straightforward to obtain) appearing in the last diagonal element in (3.10) provides the extra contribution as well as the sum of conformal dimensions of $(\square, 0)$ and $(0; \square)$. They are given in (2.12) and (2.6) respectively. Then one obtains the final conformal dimension by adding the above contribution appearing in (3.10) as follows

\[ \frac{N}{(N + k + 2)} + \frac{(2N + 3)}{4(N + k + 2)} - \frac{1}{(N + k + 2)} = \frac{(4N + 2k - 1)}{4(N + k + 2)}. \]

as described in (3.5). It is easy to see that the above conformal dimension does not satisfy the BPS bound by substituting $l^+ = 0$ and $l^- = \frac{1}{2}$. 

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3.2.5. *The (antisymm; symm) representation.* Let us consider the higher representation which arises from the product of $\text{(antisymm; 0)}$ and $(0; \text{symm})$. The former occurs in the section 2.2.4 and the latter occurs in the section 2.2.5.

The four eigenvalues can be summarized by

\[
\begin{align*}
\lambda_0 &= 1, \\
\lambda_1 &= (l^+ + 1), \\
\lambda_2 &= 0, \\
\lambda_3 &= 2.
\end{align*}
\]

One can easily see that the above conformal dimension does not lead to the BPS bound with $l^+ = 0$.

3.2.6. *The (antisymm; antisymm) representation.* Let us consider the higher representation which arises from the product of $(\text{antisymm; 0)}$ and $(0; \text{antisymm})$. The former occurs in the section 2.2.4 while the latter occurs in the section 2.2.6 with complex conjugation.

The four eigenvalues are given by

\[
\begin{align*}
\lambda_0 &= \frac{k + N}{k + N + 2}, \\
\lambda_1 &= (l^+ + 1), \\
\lambda_2 &= 0, \\
\lambda_3 &= -2N - 2.
\end{align*}
\]

One can easily see that the above conformal dimension does not lead to the BPS bound with $l^+ = 0$ and $l^- = 1$.

3.3. **Summary of this section**

Let us describe the conformal dimension for the higher representation arising from the product of $(\text{symm; 0)}$ and $(0; \text{antisymm})$ where the number of box for the symm representation is given by $p$ and the number of box for the antisymm representation is given by $q$. Then by substituting the quadratic Casimirs into the formula [29] and the excitation number is given by $q$, one obtains the following expression

\[
\frac{(N + 1)p(N + p + 2)}{2(N + 2)(k + N + 2)} = \frac{(N + 1)q(N - q)}{2N(k + N + 2)} - \frac{(\frac{1}{2}(N + 2)(-q - \frac{Np}{2})^2 + q}{N(N + 2)(k + N + 2)} + \frac{q}{2} - \frac{2kp + 2pq + 2p + q^2 + 2q}{4(k + N + 2)}.
\]

According to the conditions $p = 2l^+$, $q = 2l^-$, the above expression (3.12) reduces to the BPS bound in (2.7).

Now one can classify the possible combinations as follows:
Furthermore, there are also complex conjugated representations for (3.13). The quadratic Casimirs do not change and the $\hat{u}^2$ does not change.

Let us describe the conformal dimension for the higher representation where the representation $\Lambda^-$ appears in the branching rule of $\Lambda^+$. The representation arises from $(\text{antisymm}; \text{antisymm})$ where the number of box for the first antisymm representation is given by $p$ and the number of box for the second antisymm representation is given by $q$. Then the formula implies

$$\frac{(N + 2 + 1)p(N - p + 2)}{2(N + 2)(k + N + 2)} = \frac{(N + 1)q(N - q)}{2N(k + N + 2)} - \frac{(q - \frac{N}{2})^2}{N(N + 2)(k + N + 2)}.$$

(3.14)

Under the further condition $q = p - 1$, the above result (3.14) reduces to $\frac{(2N + 1)}{4(k + N + 2)}$, which is equal to the BPS bound with $l^+ = \frac{1}{2}$ and $l^- = 0$:

$$p = 1, 2, 3; \quad (\boxtimes 0), \quad (\boxtimes), \quad (\boxtimes).$$

(3.13)

In this case also, the complex conjugated representations are possible. The quadratic Casimirs do not change and the $\hat{u}^2$ does not change.

In summary of this section, the conformal dimensions for the higher representations up to two boxes are described explicitly. In next table, its large $(N,k)$ ’t Hooft-like limit is written and the particular ones with ‘BPS’ notation are specified. The large $(N,k)$ ’t Hooft-like limit is defined by

$$N, k \to \infty, \quad \lambda \equiv \frac{(N + 1)}{(N + k + 2)} \text{ fixed},$$

(3.15)

which will be used in tables 1–4.

4. Review of eigenvalues in the minimal representations with the higher spin-1, 2, 3 currents in the $SU(N + 2)/SU(N) \times SU(2) \times U(1)$ Wolf space coset

Let us describe the eigenvalues of

(1) the zero mode of the higher spin-1 current: $(\Phi_0^{(1)})_0$,
(2) the zero mode of sum of the square of higher spin-2 current: $(V^+)_0$,
(3) the zero mode of sum of the square of other higher spin-2 current: $(V^-)_0$,
(4) the zero mode of the higher spin-3 current: $(\Phi_2^{(1)})_0$. 

where the higher spin-2 currents $V^{\pm i}(z)$ in the $SU(2)_k \times SU(2)_N$ basis are related to the ones $\Phi_k^{(1),\mu\nu}(z)$ in the $SO(4)$ basis
\[
V^{\pm 1}(z) \equiv i(\Phi_k^{(1),14} \mp \Phi_k^{(1),23})(z),
\]
\[
V^{\pm 2}(z) \equiv -i(\Phi_k^{(1),24} \pm \Phi_k^{(1),13})(z),
\]
\[
V^{\pm 3}(z) \equiv i(-\Phi_k^{(1),34} \pm \Phi_k^{(1),12})(z).
\]
Then one can construct the following two quantities by summing over each $SU(2)$ adjoint indices
\[
V^+(z) \equiv \sum_{i=1}^{3} (V^+)^2(z), \quad V^-(z) \equiv \sum_{i=1}^{3} (V^-)^2(z),
\]
which have the conformal dimension (or spin) of 4. The corresponding eigenvalues are described by $\phi_0^{(1)}, \phi^+, \phi^-$ and $\phi_2^{(1)}$ respectively. See also the relevant works in [30, 31].

The higher spin 1 current is described as [32] (see also [33–35] for fixed $N$)
\[
\Phi_0^{(1)}(z) = -\frac{1}{2(k+N+2)} d^{\mu}_{ab} f^{ab}_{\rho} V^{\rho}(z) + \frac{k}{2(k+N+2)^2} d^{\rho}_{ab} Q^a Q^b(z), \quad (4.1)
\]
where the antisymmetric $d$ tensor of rank 2 is given by $4N \times 4N$ matrix as follows:
\[
d^{\rho}_{ab} = \begin{pmatrix}
 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & -1 \\
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0
\end{pmatrix}.
\]
Each element is $N \times N$ matrix. The locations of the nonzero elements of this matrix are the same as the previous almost complex structure $h^{1,1}_{\alpha\beta}$ but numerical values are different from each other. Note that the summation over $c$ index in (4.1) runs over the whole range of $SU(N+2)$ adjoint indices.

This higher spin 1 current plays the role of the ‘generator’ of the next higher spin currents because one can construct them using the OPEs between the spin $\frac{3}{2}$ currents of the large $\mathcal{N} = 4$ nonlinear superconformal algebra and the higher spin 1 current. That is, from the first order pole of the OPE between $G^{\mu}(z)$ and the higher spin 1 current $\Phi_0^{(1)}(w)$, one obtains $\Phi_{\frac{3}{2}}^{(1),\mu}(w)$ with minus sign [36]. After that, one can calculate the OPE between $G^{\mu}(z)$ and the higher spin $\frac{5}{2}$ current $\Phi_{\frac{5}{2}}^{(1),\nu}(w)$. Then the first order pole will provide the next higher spin 2 current $\Phi_{\frac{5}{2}}^{(1),\mu\nu}(w)$. One can go further. The OPE between $G^{\mu}(z)$ and the higher spin 2 current $\Phi_{\frac{5}{2}}^{(1),\nu\rho}(w)$ contains the first order pole where the next higher spin $\frac{7}{2}$ current $\delta^{\nu\mu} \Phi_{\frac{7}{2}}^{(1),\rho}(w)$ occurs. Finally, one can calculate the OPE between $G^{\mu}(z)$ and the higher spin $\frac{5}{2}$ current $\Phi_{\frac{5}{2}}^{(1),\nu}(w)$ and then the first order pole gives us to the last higher spin 3 current term $\delta^{\nu\mu} \Phi_{\frac{7}{2}}^{(1)}(w)$. Once the normalization of the higher spin 1 current is fixed, then the normalization for the higher spin 3 current can be fixed in this way.

So far, although the complete closed form for the higher spin currents in terms of the adjoint spin 1 and $\frac{3}{2}$ currents is not known, but their expressions for several $N$ values are known explicitly. They (which are written explicitly for $N = 3, 5, 7, 9, 11$ and maybe for $N = 13$ for some other cases) are enough to obtain all the results of this paper.
4.1. The eigenvalues in the \((0; f)\) and \((0; \bar{f})\) representations

The relevant subsection is given by the section 2.1.1. The above four eigenvalues associated with one of the minimal representations can be summarized by

\[
\begin{align*}
\phi_0^{(1)}(0; \square) &= -\frac{k}{(N + k + 2)}, \\
\nu^+(0; \square) &= \frac{24k}{(k + N + 2)^2}, \\
\nu^-(0; \square) &= \frac{12k(5k + 4N + 2)}{(k + N + 2)^2}, \\
\phi_2^{(1)}(0; \square) &= \frac{4k(12 + 28N + 5N^2 + 14k + 39kN + 6kN^2 + 4k^2 + 12k^2N)}{3(2 + k + N)^2(4 + 5k + 5N + 6kN)}.
\end{align*}
\]

(4.2)

For the first eigenvalue, one should calculate the OPE between the ‘reduced’ \(\Phi_0^{(1)}(z)\) and \(Q^{13}(w)\) (in \(SU(5)\)) and read off the first order pole. See also (2.5) and (2.13). The coefficient of \(Q^{13}(w)\) in the right hand side of this OPE is the corresponding eigenvalue. In other words, the zero mode of \(d_{ab}^{(0)} Q^a Q^b\) in (4.1) acting on this state gives \(-2(N + k + 2)\). For the second and third eigenvalues, one calculates the OPEs between the ‘reduced’ \(V^{(\pm)}(z)\) and \(Q^{13}(w)\) (in \(SU(5)\)) and read off the fourth order pole respectively. The coefficients of \(Q^{13}(w)\) in the right hand side of these OPEs are the corresponding eigenvalues respectively. For the last eigenvalue, one computes the OPE between the ‘reduced’ \(\Phi_2^{(1)}(z)\) and \(Q^{13}(w)\) (in \(SU(5)\)) and read off the third order pole. Of course, all the higher spin currents do not contain the spin-1 currents \(V^a(z)\).

By counting the highest powers of \(k\) or \(N\) (the sum of powers in \(k\) and \(N\) for the expressions containing both dependences) in the numerators and the denominators appearing in the above eigenvalues, one can observe the behaviors under the large \((N,k)\) \(\neq \) \(\text{t Hooft like limit.} \) Except the \(\nu^+\) eigenvalue having \(\frac{1}{N}\) dependence, the remaining three eigenvalues approach to the finite \(\lambda\) dependent values\(^6\).

Similarly, the other four eigenvalues can be also obtained from

\[
\begin{align*}
\phi_0^{(1)}(0; \bar{\square}) &= \frac{k}{(N + k + 2)}, \\
\nu^+(0; \bar{\square}) &= \frac{24k}{(k + N + 2)^2}, \\
\nu^-(0; \bar{\square}) &= \frac{12k(5k + 4N + 2)}{(k + N + 2)^2}, \\
\phi_2^{(1)}(0; \bar{\square}) &= -\frac{4k(12 + 28N + 5N^2 + 14k + 39kN + 6kN^2 + 4k^2 + 12k^2N)}{3(2 + k + N)^2(4 + 5k + 5N + 6kN)}.
\end{align*}
\]

(4.3)

Because the generators for the complex conjugated (antifundamental) representation \(\bar{\square}\) have an extra minus sign compared to the fundamental representation \(\square\), the eigenvalue for the odd higher spin currents (corresponding to the first and the last ones) have an extra minus sign and the ones for the even higher spin currents (corresponding to the middle one) remain the same compared to the results of the previous section in (4.2).

More explicitly, one can obtain the OPEs between the ‘reduced’ higher spin currents and the \(\frac{1}{2}\) current \(Q^1(w)\). By reading off the corresponding coefficients in the appropriate poles, the

---

\(^6\)Note that we make some boldface notation for the highest power of \((N,k)\) in the numerator of the higher spin 3 current in (4.2). We will observe that they will play the role of the fundamental quantity in the sense that the eigenvalues of the higher spin 3 current for any representation \((0,\Lambda)\) will be a multiple of this quantity, under the large \((N,k)\) \(\neq \) \(\text{t Hooft like limit.} \)
above eigenvalues can be determined. One can also analyze the large \((N,k)\) ’t Hooft like limit for these eigenvalues.

4.2. The eigenvalues in the \((f;0)\) and \((\bar{f};0)\) representations

The relevant subsection is given by the section 2.1.2. The four eigenvalues can be described as

\[
\begin{align*}
\phi_0^{(1)}(\square;0) &= -\frac{N}{(N+k+2)}, \\
\nu^+(\square;0) &= \frac{12N(4k+5N+2)}{(k+N+2)^2}, \quad \nu^-(\square;0) = \frac{24N}{(k+N+2)^2}, \\
\phi_2^{(1)}(\square;0) &= -\frac{4N(12+28k+5k^2+14N+39kN+6k^2N+4N^2+12kN^2)}{3(2+k+N)^2(4+5k+5N+6kN)}.
\end{align*}
\]

One obtains these eigenvalues by substituting the \(SU(N+2)\) generators \(T_{ab}\) into the zero mode of the spin 1 current \(V_0^a\) in the corresponding ‘reduced’ higher spin currents where all the \(Q(z)\) dependent terms are ignored. Then one has the unitary matrix acting on the corresponding state and the diagonal elements of the last 2 \(\times 2\) subdiagonal matrix provide the above eigenvalues. From the explicit form for the higher spin 1 current in (4.1), the corresponding eigenvalue implies that the zero mode of \(d^{a_1a_2}_ab_{c_1c_2}V^c\) acting on this state is equal to \(2N\). The large \((N,k)\) ’t Hooft like limit can be analyzed similarly.

As observed in [17], under the symmetry \(N \leftrightarrow k\) and \(0 \leftrightarrow \square\), the eigenvalues become \(\phi_0^{(1)}(\square;0) \rightarrow \phi_0^{(1)}(0;\square), \quad \nu^+(\square;0) \rightarrow \nu^-(0;\square), \quad \nu^-(\square;0) \rightarrow \nu^+(0;\square)\) and \(\phi_2^{(1)}(\square;0) \rightarrow -\phi_2^{(1)}(0;\square)\).

When we consider the complex conjugated representation, the following results hold

\[
\begin{align*}
\phi_0^{(1)}(\square;0) &= \frac{N}{(N+k+2)}, \\
\nu^+(\square;0) &= \frac{12N(4k+5N+2)}{(k+N+2)^2}, \quad \nu^-(\square;0) = \frac{24N}{(k+N+2)^2}, \\
\phi_2^{(1)}(\square;0) &= \frac{4N(12+28k+5k^2+14N+39kN+6k^2N+4N^2+12kN^2)}{3(2+k+N)^2(4+5k+5N+6kN)}.
\end{align*}
\]

According to the previous analysis, the first eigenvalue (corresponding to the higher spin 1 current) and the last eigenvalue (corresponding to the higher spin 3 current) have the extra minus signs compared to the ones in (4.4).

5. Eigenvalues for the higher representations with the higher spin-1, 2, 3 currents in the \(SU(N+2)\) \(SU(N)\times SU(2)\times U(1)\) Wolf space coset

In this section, there are 7 subsections and the remaining 16 representations are presented in appendix.

5.1. The \((f;f)\) representation

The relevant subsection on this higher representation is given by section 2.2.1. In this case, when one takes the \(N \times N\) subdiagonal unitary matrix inside of \((N+2) \times (N+2)\) unitary
matrix, the corresponding diagonal elements for the higher spin currents provide the following four eigenvalues

\[ \phi_0^{(1)}(\square; \square) = \frac{2}{(k + N + 2)}, \quad \nu^+(\square; \square) = \frac{96k}{(k + N + 2)^2}, \quad \nu^-(\square; \square) = \frac{96N}{(k + N + 2)^2}. \]

\[ \phi_2^{(1)}(\square; \square) = \frac{8(k - N)(6kN + 5k + 5N + 16)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}. \]

(5.1)

Under the symmetry \( N \leftrightarrow k \) (with \( \square \leftrightarrow \Box \)), the first eigenvalue in (5.1) remains the same, the second eigenvalue becomes the third one, the third eigenvalue becomes the second one and the last eigenvalue remains the same with an extra sign change. By power counting of \( N \) and \( k \), one sees that the above eigenvalues behave as \( \frac{1}{N} \) dependence under the large \( (N, k) \)'t Hooft like limit. However, this will play the role of next leading order and moreover this term will be the fundamental quantity in the sense that the eigenvalues of the higher spin 3 current for any representation \( (\Lambda_+; \Lambda_+) \) will be a multiple of this quantity. Here \( \Lambda_+ \) is the symmetric or antisymmetric representation and the number of boxes is arbitrary. Note the presence of factor \( (k - N) \) in the above.

5.2. The \( (t; \overline{T}) \) representation

The relevant subsection on this higher representation is given by section 2.2.2. The four eigenvalues corresponding to the zero modes of the higher spin currents of spins 1, 4, 4 and 3 which act on the representation \( (\square; \square) \) can be summarized by

\[ \phi_0^{(1)}(\square; \square) = \phi_0^{(1)}(\square; 0) + \phi_0^{(1)}(0; \square) = \frac{(k - N)}{(k + N + 2)}, \]

\[ \nu^+(\square; \square) = \frac{12(4kN + 2k + 5N^2 - 16N + 5)}{(k + N + 2)^2}, \quad \nu^-(\square; \square) = \frac{12(5k^2 + 4kN - 28k + 2N + 5)}{(k + N + 2)^2}. \]

\[ \phi_2^{(1)}(\square; \square) = \frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \times (6kN + 2k^3 + 6k^2N^2 + k^2N - 11k^2 + 6kN^3 + 7kN^2 - 32kN - 42k + 2N^3 - 11k^2 - 30N - 24). \]

(5.2)

The previous relations in (4.3) and (4.4) are used. Note that the eigenvalue in (5.2) for the higher spin 1 current does not have any contribution from the commutator \([\Phi_0^{(1)}_0, \mathcal{Q}_4^{\frac{1}{2}}]\) because the OPE between the corresponding higher spin 1 current and the spin \( \frac{1}{2} \) current has only the first order pole. See also (2.5). This provides only the eigenvalue for the representation \( (0; \square) \). Also the term \( \mathcal{Q}_4^{\frac{1}{2}} \Phi_0^{(1)}_0 \) acting on the representation \( (\square; 0) \) gives the eigenvalue \( \phi_0^{(1)}(\square; 0) \) with \( \mathcal{Q}_4^{\frac{1}{2}} \) acting on the state \((\square; 0)\). By inserting the overall factor into this state, one has the final state associated with the representation \((\square; \square)\). Therefore, one arrives at the above eigenvalue for the higher spin 1 current.

For the eigenvalues corresponding to the remaining higher spin currents, there are the contributions from the lower order poles appearing in the commutators, \([V^+)_0, \mathcal{Q}_4^{\frac{1}{2}}\), \([V^-)_0, \mathcal{Q}_4^{\frac{1}{2}}\) and \([\Phi_0^{(1)}_i, \mathcal{Q}_4^{\frac{1}{2}}] \). They can be summarized by

\[ \text{They all contribute to the eigenvalue.} \]

\[ \text{In other words, for example, one has } [\Phi_0^{(1)}_i, \mathcal{Q}_4^{\frac{1}{2}}] = \left( f_i \frac{4(1 + i)}{\pi w^{i + (i - 1)} f_{i'} \frac{4(1 + (i - 1))}{\pi w^{(i - 1)} \Phi_0^{(1)}_i (z)} \mathcal{Q}_4^{i}(w) \right). \]

One sees that the OPE between \( \Phi_0^{(1)}_i (z) \) and \( \mathcal{Q}_4^{i}(w) \) contains the first and second order poles as well as the third order pole. They all contribute to the eigenvalue.
\[
\begin{align*}
\delta v^+ (\square; \square) &= -\frac{12(18N - 5)}{(k + N + 2)^2}, \\
\delta v^- (\square; \square) &= -\frac{60(6k - 1)}{(k + N + 2)^2}, \\
\delta \phi^{(1)}_2 (\square; \square) &= \frac{8(7k^2N + 6k^2 + 5kN^2 + 20kN + 16k + 6N^2 + 12N + 8)}{(k + N + 2)^2(kN + 5N + 4)}. 
\end{align*}
\]

Once again, the large \((N, k)\)'t Hooft like limits for these extra contributions lead to the \(\frac{1}{N}\) behavior. Then the above eigenvalues are obtained from the relations,

\[
\begin{align*}
v^\pm (\square; \square) &= v^\pm (\square; 0) + v^\pm (0; \square) + \delta v^\pm (\square; \square), \\
\phi^{(1)}_2 (\square; \square) &= \phi^{(1)}_2 (\square; 0) + \phi^{(1)}_2 (0; \square) + \delta \phi^{(1)}_2 (\square; \square)
\end{align*}
\]
respectively. The previous relations (4.4) and (4.3) can be used.

5.3. The \((f; \text{symm})\) representation

This higher representation can be obtained from the product of \((\square; 0)\) and \((0; \square)\). It turns out that the four eigenvalues are given by

\[
\begin{align*}
\phi^{(1)}_0 (\square \square) &= \phi^{(1)}_0 (\square 0) + \phi^{(1)}_0 (0; \square) = -\frac{(N + 2k)}{(N + k + 2)}, \\
v^+ (\square \square) &= \frac{12(4kN + 8k + 5N^2 + 38N + 20)}{(k + N + 2)^2}, \\
v^+ (\square \square) &= \frac{12(4kN + 8k + 5N^2 + 38N + 20)}{(k + N + 2)^2}, \\
\phi^{(1)}_2 (\square \square) &= \frac{4}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \\
&\times (24k^3N - 4k^3 + 6k^2N^2 + 73k^2N - 14k^2 - 12kN^2 - 17kN^2 \\
&- 56kN - 72k - 4N^3 - 32N^3 - 156N - 96). 
\end{align*}
\]

The relations in (4.4) and appendix (A.3) which will appear later are used. Note that the eigenvalue (5.3) for the higher spin 1 current does not have any contribution from the commutator \([\phi^{(1)}_0, \mathcal{Q}_{-\frac{1}{2}}^{13} \mathcal{Q}_{-\frac{1}{2}}^{16}]\) (see also the section 2.2.5) because the OPE between the corresponding higher spin 1 current and the product of spin 1/2 currents has only the first order pole. Therefore, one arrives at the above eigenvalue for the higher spin 1 current.

5.4. The \((f; \text{symm})\) representation

This higher representation can be obtained from the product of \((\square; 0)\) and \((0; \square)\). It turns out that the four eigenvalues are given by

\[
\begin{align*}
\phi^{(1)}_0 (\square \square) &= \phi^{(1)}_0 (\square 0) + \phi^{(1)}_0 (0; \square) = -\frac{(N - 2k)}{(N + k + 2)}, \\
v^+ (\square \square) &= \frac{12(4kN + 8k + 5N^2 - 34N - 20)}{(k + N + 2)^2}, \\
v^+ (\square \square) &= \frac{12(4kN + 8k + 5N^2 - 34N - 20)}{(k + N + 2)^2}, \\
\phi^{(1)}_2 (\square \square) &= -\frac{4}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \\
&\times (24k^3N - 4k^3 + 18k^2N^2 - 61k^2N - 50k^2 + 12kN^3 - 11kN^2 \\
&- 108kN - 36k + 4N^3 - 4N^3 + 48N + 48). 
\end{align*}
\]
The relations (4.4) can be used. Note that the eigenvalue for the higher spin 1 current in (5.4) does not have any contribution from the commutator $[[\Phi_1^{(1)},Q_1^1],Q_1^1]$ (see also the section 2.2.5) because the OPE between the corresponding higher spin 1 current and the spin $\frac{1}{2}$ current has only the first order pole. Therefore, one arrives at the above eigenvalue for the higher spin 1 current.

5.5. The (f; antisymm) representation

It turns out that the four eigenvalues are given by

$$\phi_0^{(1)}(\square) = \phi_0^{(1)}(\square,0) + \phi_0^{(1)}(0;\square) = -\frac{(N + 2k)}{(N + k + 2)},$$

$$\nu^+(\square) = 12\frac{(4kN + 8k + 5N^2 + 38N + 20)}{(k + N + 2)^2},$$

$$\nu^-(\square) = 8\frac{(16k^2 + 12kN + 108k + 3N + 20)}{(k + N + 2)^2},$$

$$\phi_2^{(1)}(\square) = 4\frac{3(k + N + 2)^2(6kN + 5k + 5N + 4)}{(24k^3N - 4k^3 + 6k^2N^2 + 145k^2N - 2k^2 - 12kN^3 - 17kN^2 + 196kN + 156k - 4N^3 + 40N^2 + 24N + 48).}$$

See also the section 2.2.6. The first and the second eigenvalues in (5.5) are the same as the ones in (5.3) respectively. This implies that there is no difference between $\square$ or $\square$ as long as these eigenvalues are concerned. Furthermore, the last eigenvalue has common behavior with the one in (5.3) because they contain $(24k^3N + 6k^2N^2 - 12kN^3)$ in the numerators. Note that the eigenvalue for the higher spin 1 current does not have any contribution from the commutator $[[\Phi_1^{(1)},Q_1^1],Q_1^1]$ because the OPE between the corresponding higher spin 1 current and the product of spin $\frac{1}{2}$ currents has only the first order pole. Therefore, one arrives at the above eigenvalue for the higher spin 1 current.

5.6. The (f; antisymm) representation

Similarly, the four eigenvalues are given by

$$\phi_0^{(1)}(\square) = \phi_0^{(1)}(\square,0) + \phi_0^{(1)}(0;\square) = -\frac{(N + 2k)}{(N + k + 2)},$$

$$\nu^+(\square) = 12\frac{(4kN + 8k + 5N^2 - 34N - 20)}{(k + N + 2)^2},$$

$$\nu^-(\square) = 8\frac{(16k^2 + 12kN - 100k + 3N + 20)}{(k + N + 2)^2},$$

$$\phi_2^{(1)}(\square) = 4\frac{3(k + N + 2)^2(6kN + 5k + 5N + 4)}{(24k^3N - 4k^3 + 18k^2N^2 + 11k^2N - 38k^2 + 12kN^3 - 11kN^2 - 144kN - 168k + 4N^3 - 76N^2 - 132N - 96).}$$

(5.6)
Table 1. The eigenvalue \( h \) under the large \((N,k)\) \('t\) Hooft-like limit (3.15). All the eigenvalues described in the section 3 are presented in this table. Some eigenvalues in the section 3 are denoted by the boldface notation. The subscript \('bps'\) stands for the conformal dimension satisfying the BPS bound (1.1) at finite \((N,k)\). According to the branching rule (and its complex conjugated one), there is no singlet under the \(SU(N)\). Therefore, there are no higher representations corresponding to the blanks that can be obtained from the product of zero and \((0; Λ−)\) in this Table. Of course, there are higher representations where the representation \(Λ+\) appears in the branching of \(Λ+ = \text{antisymm or antisym} \) with two boxes in this table.

| \((Λ+; Λ−)\) | 0 | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| 0 | 0 | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |
| | □ | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |
| | □ | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |
| | □ | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |
| | □ | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |
| | □ | \((\frac{1}{2})_{bps}\) | \((\frac{3}{2})_{bps}\) | \((1)_{bps}\) | \((1)_{bps}\) |

Table 2. The eigenvalue \(ε_{0}^{(1)}\) under the large \((N,k)\) \('t\) Hooft-like limit (3.15). The eigenvalue with \(Λ+ = Λ−\) can be written in terms of the multiple of the eigenvalue of \((f; f)\) or \((\overline{f}; \overline{f})\). The \(\frac{1}{2}\) behavior in this case is written explicitly in this table and next ones. The general structure for the eigenvalue with \(Λ+ \neq Λ−\) (in the product of \((Λ+; 0)\) and \((0; Λ−)\)) is given by the linear combinations of the one of \((0; f)\) (or \((0; \overline{f})\)) and the one of \((f; 0)\) (or \((\overline{f}; 0)\)). Then each coefficient depends on the number of boxes in \(Λ+\) and \(Λ−\). When the representation \(Λ−\) appears in the branching of \(Λ+\), the eigenvalue leads to the representation \(|Λ+| − |Λ−|; 0\) where \(|Λ±|\) denotes the number of boxes. We also present the eigenvalues in terms of the \('t\) Hooft coupling constant \(λ\).

| \((Λ+; Λ−)\) | 0 | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| 0 | 0 | −(1 − λ) | (1 − λ) | −2(1 − λ) | −2(1 − λ) | 2(1 − λ) |
| | □ | −λ | \((\frac{2}{N})\) | \((λ; 0)\) | \((λ; 0)\) | \((λ; 0)\) |
| | □ | −2λ | −λ | \((\frac{4}{N})\) | \((λ; 0)\) | \((λ; 0)\) |
| | □ | −2λ | −λ | \((\frac{4}{N})\) | \((λ; 0)\) | \((λ; 0)\) |
| | □ | 2λ | −(1 − 3λ) | λ | −2(1 − 2λ) | −2(1 − 2λ) |
| | □ | 2λ | −(1 − 3λ) | λ | −2(1 − 2λ) | −2(1 − 2λ) |

The relations in (4.4) are needed. The first and the second eigenvalues in (5.6) are the same as the ones in (5.4). Furthermore, the last eigenvalue has common behavior with the one in (5.4) because they contain \((24k^{3}N + 18k^{2}N^{2} + 12kN^{3})\) in the numerators. See also the section 2.2.6. Note that the eigenvalue for the higher spin 1 current does not have any contribution from the commutator \([Φ_{0}^{(1)}]_{0}, O_{−\frac{1}{2}} O_{−\frac{1}{2}}^{3}\) because the OPE between the corresponding higher
Table 3. The eigenvalue $\phi_2^{(1)}$ under the large $(N,k)$ ’t Hooft-like limit (3.15). The general structure for the eigenvalue with $\Lambda_+ \neq \Lambda_-$ in the product of $(\Lambda_+;0)$ and $(0;\Lambda_-)$ is given by the linear combinations of the one of $(0;f)$ (or $(0;\bar{f})$) and the one of $(f;0)$ (or $(\bar{f};0)$). Then each coefficient depends on the the number of boxes in $\Lambda_+$ and $\Lambda_-$. When the representation $\Lambda_-$ appears in the branching of $\Lambda_+$, the eigenvalue leads to the representation $((\Lambda_+;0) - |\Lambda_-;0|)$ where $|\Lambda_\pm|$ denotes the number of boxes. The eigenvalue with $\Lambda_+ = \Lambda_-$ can be written in terms of the multiple of the eigenvalue of $(f;f)$ or $(\bar{f};\bar{f})$. The behavior in this case is written explicitly in this table and next one. We also present the eigenvalues in terms of the ’t Hooft coupling constant $\lambda$.

| $(\Lambda_+;\Lambda_-)$ | 0 | □ | □ |
|------------------------|---|----|----|
| 0                      | $\phi_2^{(1)}(0;\square) = -\frac{4}{3}\lambda(\lambda + 1)$ | $\phi_2^{(1)}(\square;0) = -\frac{8}{3\lambda}(2\lambda - 1)$ | $-\phi_2^{(1)}(0;\square) = -\frac{4}{3}(1 - \lambda)(2 - \lambda)$ |
| □                      | $\phi_2^{(1)}(\square;0) = -\frac{4}{3}\lambda(\lambda + 1)$ | $\phi_2^{(1)}(\square;\square) = -\frac{8}{3\lambda}(2\lambda - 1)$ | $-\phi_2^{(1)}(\square;0) = -\frac{4}{3}(1 - \lambda)(2 - \lambda)$ |
| □                      | $-\phi_2^{(1)}(\square;0) = \frac{4}{3}\lambda(\lambda + 1)$ | $-\phi_2^{(1)}(\square;0) - (0;\square) = \frac{8}{3}(\lambda^2 - \lambda + 1)$ | $-\phi_2^{(1)}(\square;\square) = \frac{8}{3\lambda}(2\lambda - 1)$ |
| □                      | $2\phi_2^{(1)}(\square;0) = -\frac{4}{3}\lambda(\lambda + 1)$ | $\phi_2^{(1)}(\square;\square) = -\frac{4}{3}\lambda(\lambda + 1)$ | $2\phi_2^{(1)}(\square;0) - (0;\square) = -\frac{4}{3}(3\lambda^2 - \lambda + 2)$ |
| □                      | $2\phi_2^{(1)}(\square;0) = -\frac{4}{3}\lambda(\lambda + 1)$ | $\phi_2^{(1)}(\square;\square) = -\frac{4}{3}\lambda(\lambda + 1)$ | $2\phi_2^{(1)}(\square;0) - (0;\square) = -\frac{4}{3}(3\lambda^2 - \lambda + 2)$ |
| □                      | $-2\phi_2^{(1)}(\square;0) = \frac{4}{3}\lambda(\lambda + 1)$ | $-\phi_2^{(1)}(2\square;0) - (0;\square) = \frac{4}{3}(3\lambda^2 - \lambda + 2)$ | $-\phi_2^{(1)}(\square;0) = \frac{4}{3}\lambda(\lambda + 1)$ |
| □                      | $-2\phi_2^{(1)}(\square;0) = \frac{4}{3}\lambda(\lambda + 1)$ | $-\phi_2^{(1)}(2\square;0) - (0;\square) = \frac{4}{3}(3\lambda^2 - \lambda + 2)$ | $-\phi_2^{(1)}(\square;0) = \frac{4}{3}\lambda(\lambda + 1)$ |
Table 4. The (continued) eigenvalue $\phi_2^{(1)}$ under the large $(N,k)$ 't Hooft-like limit (3.15).

|               | 0    | □    | □    | □    |
|---------------|------|------|------|------|
| $2\phi_2^{(1)}(0;\square) = \frac{7}{12}(1 - \lambda)(2 - \lambda)$ | $2\phi_2^{(1)}(0;\square) = \frac{7}{12}(1 - \lambda)(2 - \lambda)$ | $-2\phi_2^{(1)}(0;\square) = -\frac{7}{12}(1 - \lambda)(2 - \lambda)$ | $-2\phi_2^{(1)}(0;\square) = -\frac{7}{12}(1 - \lambda)(2 - \lambda)$ |
| $\phi_2^{(1)}[(\Box;0) + 2(0;\Box)] = \frac{1}{2}(\lambda^2 - 7\lambda + 4)$ | $\phi_2^{(1)}[(\Box;0) + 2(0;\Box)] = \frac{1}{2}(\lambda^2 - 7\lambda + 4)$ | $\phi_2^{(1)}[(\Box;0) - 2(0;\Box)] = -\frac{1}{4}(3\lambda^2 - 5\lambda + 4)$ | $\phi_2^{(1)}[(\Box;0) - 2(0;\Box)] = -\frac{1}{4}(3\lambda^2 - 5\lambda + 4)$ |
| $-\phi_2^{(1)}[(\Box;0) - 2(0;\Box)] = \frac{1}{2}(3\lambda^2 - 5\lambda + 4)$ | $-\phi_2^{(1)}[(\Box;0) - 2(0;\Box)] = \frac{1}{2}(3\lambda^2 - 5\lambda + 4)$ | $-\phi_2^{(1)}[(\Box;0) + 2(0;\Box)] = -\frac{1}{4}(\lambda^2 - 7\lambda + 4)$ | $-\phi_2^{(1)}[(\Box;0) + 2(0;\Box)] = -\frac{1}{4}(\lambda^2 - 7\lambda + 4)$ |
| $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ |
| $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}(2\lambda - 1)$ | $2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}(2\lambda - 1)$ |
| $-2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = -\frac{16}{3}\lambda(2\lambda - 1)$ |
| $-2\phi_2^{(1)}[(\square;\square)] = \frac{16}{3}(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = \frac{16}{3}(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = \frac{16}{3}(2\lambda - 1)$ | $-2\phi_2^{(1)}[(\square;\square)] = \frac{16}{3}(2\lambda - 1)$ |
spin 1 current and the spin $\frac{1}{2}$ current has only the first order pole. Therefore, one arrives at the above eigenvalue for the higher spin 1 current.

5.7. Summary of this section

In summary of this section, we present the relevant tables where the large $(N, k)$ ’t Hooft limit for the eigenvalues is taken. We observe that for the linear case, the corresponding eigenvalues behave as exactly same as these tables under the large $(N, k)$ ’t Hooft like limit. At finite $(N, k)$, the eigenvalues, $\phi_0^{(1)}$ and $\nu^\pm$ in linear case [37, 38] coincide with the ones in the nonlinear case. Only the eigenvalues $\phi_2^{(1)}$ are different from each other at finite $(N, k)$.

For the eigenvalue for the higher spin 1 current on the representation $(\Lambda^+; \Lambda^-)$ which can be obtained the product of $(\Lambda^+; 0)$ and $(0; \Lambda^-)$, one obtains

$$
\phi_0^{(1)}(\Lambda^+; \Lambda^-) = \phi_0^{(1)}(\Lambda^+; 0) + \phi_0^{(1)}(0; \Lambda^-) = \left[ \mp \frac{|\Lambda^+| N \mp |\Lambda^-| k}{(N + k + 2)} \right] (\Lambda^+; \Lambda^-). \quad (5.7)
$$

There are four cases depending on whether the representation $\Lambda_\pm$ is given by the multiple product of $\square$ or $\overline{\square}$. We have minus (plus) sign for the former (latter) in (5.7). We also denote the number of boxes as $|\Lambda_\pm|$.

6. Conclusions and outlook

We present the first table where one can find the conformal dimensions for the spin 2 current acting on the various representations in the coset under the large $(N, k)$ ’t Hooft like limit (up to two boxes). The other quantum numbers $l^\pm$ and $\hat{u}$ can be made in table but we did not do it. We have found other quantum numbers associated with the higher spin currents on some part of representations in the above table. The explicit large $(N, k)$ behaviors of those eigenvalues for the higher spin 3 currents are summarized in the last two tables. One realizes that although the conformal dimensions are equal to each other for different two representations, the eigenvalues for the higher spin 3 current for those two representations are different from each other under the large $(N, k)$ ’t Hooft like limit. For example, for the representations $(\square; \square)$ and $(\overline{\square}; \square)$, the eigenvalues $\phi_2^{(1)}$ are different but their conformal dimensions $h$ are the same under the large $(N, k)$ ’t Hooft like limit. For the eigenvalue $h$, the corresponding conformal dimension for the spin 2 current is 2 while for the eigenvalue $\phi_2^{(1)}$, the corresponding conformal dimension for the higher spin 3 current is 3. Under the complex conjugation, the former remains the same (even spin) but the latter changes the sign (odd spin).

Let us present some related and open problems in the near future.

- The three-point functions

As mentioned in the abstract, one can determine the three-point functions [39] of the higher spin currents with two scalar operators at finite $(N, k)$. From appendix (A.1),

$$
\langle \overline{\square \square}(0) \square \square(0) \Phi^{(1)} \rangle = \frac{8 N (6 k^2 N + 5 k^2 + 12 k N^2 + 45 k N + 43 k - 2 N^2 - N + 12)}{3 (k + N + 2)^2 (6 k N + 5 k + 5 N + 4)} \langle \overline{\square \square}(0) \square \square(0) \rangle.
$$

$$
\rightarrow \frac{8}{3} \lambda (\lambda + 1) \langle \overline{\square \square}(0) \square \square(0) \rangle.
$$
The large \((N,k)\) ’t Hooft like limit (3.15) in the final expression is taken. It is straightforward to write down all the three-point functions we have found in this paper.

- The higher spin 3 current in terms of adjoint spin 1 and spin \(\frac{1}{2}\) currents

In this paper, we used the higher spin currents for several \(N\) values which are written in terms of adjoint spin 1 and spin \(\frac{1}{2}\) currents. In principle, one can find the explicit expression for the higher spin 3 current (by hand) as described in section 4. Although it will be rather complicated to obtain this form because that all the calculations on the OPEs should be checked step by step, it will be worthwhile to determine this full expression. Once this will be found, then it will be an open problem to obtain the corresponding eigenvalues associated with any representations. See also the relevant work in [40].

- The spectrum for the higher spin 4 current

What happens for the higher spin current with different spin? For example, the next 16 higher spin current contains the higher spin 4 current \(\Phi^{(s=2)}_2(z)\). One expects that the behavior of large \((N,k)\) ’t Hooft-like limit in the eigenvalues on this higher spin 4 current looks similar to the results of this paper. The general structure for the eigenvalue with \(\Lambda_+ \neq \Lambda_-\) in the product of \((\Lambda_+;0)\) and \((0;\Lambda_-)\) is given by the linear combinations of the one of \((0;\Box)\) (or \(0;\Box\)) and the one of \((\Box;0)\) (or \(\Box;0)\)). Then each coefficient depends on the the number of boxes in \(\Lambda_+\) and \(\Lambda_-\). Also one has plus sign for the fundamental representation while minus sign for the complex conjugated (anti fundamental) representation. The corresponding basic eigenvalues with an appropriate normalization are found in [36]

\[
\phi^{(2)}_2(\Box;0) = \frac{12}{5} \lambda(1 + \lambda)(2 + \lambda), \quad \phi^{(2)}_2(0;\Box) = -\frac{12}{5} (1 - \lambda)(2 - \lambda)(3 - \lambda).
\]

Their complex conjugated ones remain the same because this higher spin current has the conformal spin 4. When the representation \(\Lambda_-\) appears in the branching of \(\Lambda_+\), one expects that the eigenvalue leads to the representation \(\{|\Lambda_+| - |\Lambda_-|;0\}\) where \(|\Lambda_\pm|\) denotes the number of boxes. The eigenvalue with \(\Lambda_+ = \Lambda_-\) can be written in terms of the multiple of the eigenvalue of \((\Box;\Box)\) or \((\Box;\Box)\). It would be interesting to observe whether these behaviors occur. Similarly, it is an open problem to obtain the eigenvalues for the higher spin 2 current \(\Phi^{(s=2)}_0(z)\).

- The three boxes in \(\Lambda_+\)

In this paper, the boxes for \(\Lambda_+\) in the eigenvalues of the higher spin currents are limited to 2.

One considers the case where \(\Lambda_+\) contains the three boxes: \(\Box\Box\Box\), \(\Box\Box\), \(\Box\Box\), and \(\Box\Box\) (and its conjugated ones). At least, one needs to have the \(SU(N + 2)\) generators with \(N = 3, 5, 7, 9, 11\) in these higher representations in order to extract the eigenvalues. It is known that the dimensions for the above higher representations are given by \(\frac{1}{2}(N + 2)(N + 3)(N + 4), \frac{1}{2}(N + 2)(N + 3)(N + 5), \) and \(\frac{1}{2}(N + 2)(N + 3)(N + 5)\). In particular, for \(N = 11\), these become 455, 728, and 286. This implies that the 84 generators of \(SU(13)\) should be written in terms of \(455 \times 455\) matrices, \(728 \times 728\) matrices, and \(286 \times 286\) matrices respectively. It is rather difficult to obtain 84 generators in the mixed representation. It is an open problem to find out the systematic way to read off the complete 84 generators which are \(728 \times 728\) matrices by using the general formula.

- The three-point functions from the decomposition of the four-point functions of scalar operators with Virasoro conformal blocks

Recently [41], using the decomposition of the scalar four-point functions by Virasoro conformal blocks, the three-point functions including \(\frac{1}{2}\) corrections in the two dimensional (bosonic) \(W_N\) minimal model were obtained using the result of [42] (see also the
works of [43, 44]). The $\frac{1}{\sqrt{N}}$ corrections for the conformal dimension 6, 7, 8 were new. It would be interesting to obtain the eigenvalues for the higher spin currents in the higher representations of the $W_N$ minimal model. As observed in [41], it is an open problem to obtain the three-point functions from the decomposition of four-point functions in the large $\mathcal{N} = 4$ holography.

- The orthogonal Wolf space coset spectrum

One can ask what happens for the orthogonal Wolf space coset spectrum. The relevant previous works are given by [45, 46]. It is an open problem to obtain the eigenvalues for the higher spin currents in the higher representations. One should obtain the generators of $SO(N + 4)$ in various higher representations explicitly and obtain the higher spin currents (where the spins are 2, 3 or 4) for several $N$ values.

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**Appendix. Further higher representations**

The 16 remaining eigenvalues are obtained.

- The (symm; 0) representation

  The relevant subsection on this higher representation is given by section 2.2.3. This higher representation can be obtained from the product of the minimal representation $(f; 0)$ and itself. The four eigenvalues with this representation can be described as

  \[
  \phi^{(1)}_0 ([\square, 0]) = -\frac{2N}{(k + N + 2)}, \quad \nu^+([\square, 0]) = \frac{32N(3k + 4N + 1)}{(k + N + 2)^2}, \\
  \nu^-([\square, 0]) = \frac{96N}{(k + N + 2)^2}, \\
  \phi^{(1)}_2 ([\square, 0]) = -\frac{8N(6k^2N + 5k^2 + 12kN^2 + 45kN + 43k - 2N^2 - N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}.
  \] (A.1)

  Furthermore, if one sees the last eigenvalue closely, one observes that the highest power terms in the numerator are given by $-8N(6k^2N + 12kN^2)$ which is the twice of the ones in (4.4).

- The (antisymm; 0) representation

  The relevant subsection on this higher representation is given by section 2.2.4. The remaining higher representation obtained from the product of the minimal representation $(f; 0)$ and itself is given by this higher representation and the four eigenvalues can be summarized by

  \[
  \phi^{(1)}_0 ([\square, 0]) = -\frac{2N}{(k + N + 2)}, \quad \nu^+([\square, 0]) = \frac{96N(k + N - 1)}{(k + N + 2)^2}, \quad \nu^-([\square, 0]) = \frac{96N}{(k + N + 2)^2}, \\
  \phi^{(1)}_2 ([\square, 0]) = -\frac{8N(6k^2N + 5k^2 + 12kN^2 + 9kN - 11k - 2N^2 - 7N - 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}.
  \] (A.2)
Although the exact expressions in both cases are different from each other at finite $(N, k)$, the highest power terms in the numerator given by $-8N(6k^2N + 12kN^2)$ are the same as the ones in appendix (A.1). In other words, they are twice of the ones in (4.4).

- The $(0; \text{symm})$ representation

The relevant subsection on this higher representation is given by section 2.2.5. This higher representation can be obtained from the product of the minimal representation $(0; f)$ and itself. The four eigenvalues with this representation can be described as

\[
\phi_{\text{symm}}^{(1)}(0, \quad) = -\frac{2k}{(N + k + 2)^1}, \quad \nu^+(0, \quad) = \frac{96k}{(k + N + 2)^3}, \quad \nu^-(0, \quad) = \frac{96k(k + N - 1)}{(k + N + 2)^3}.
\]

Furthermore, one observes that the highest power terms in the numerator are given by $8k(12k^2N + 6kN^2)$ which is the twice of the ones in (4.2).

- The $(0; \text{antisymm})$ representation

The relevant subsection on this higher representation is given by section 2.2.6. The remaining higher representation obtained from the product of the minimal representation $(0; f)$ and itself is given by this higher representation and the four eigenvalues can be summarized by

\[
\phi_{\text{antisymm}}^{(1)}(0, \quad) = -\frac{2k}{(N + k + 2)^1}, \quad \nu^+(0, \quad) = \frac{96k}{(k + N + 2)^3}, \quad \nu^-(0, \quad) = \frac{32k(4k + 3N + 1)}{(k + N + 2)^3}.
\]

The last eigenvalue in appendix (A.4) shares the common behavior with the one in appendix (A.3). Although the exact expressions in both cases are different from each other at finite $(N, k)$, the highest power terms in the numerator given by $8k(12k^2N + 6kN^2)$ are the same as the ones in (A.3).

- The $(\text{symm}; \text{symm})$ representation

The relevant subsection on this higher representation is given by section 3.1.1. One can describe the following eigenvalues

\[
\phi_{\text{symm}, \text{symm}}^{(1)}(\quad, \quad) = \frac{4}{(N + k + 2)^1}, \quad \nu^+(\quad, \quad) = \frac{192(k - 1)}{(k + N + 2)^3}, \quad \nu^-(\quad, \quad) = \frac{192(N + 1)}{(k + N + 2)^3}.
\]

The last eigenvalue in appendix (A.5) remains the same with an extra sign change if we consider only the case where the total power of $N$ and $k$ is given by 3: $(6k^2N - 6kN^2) = 6(k - N)kN$. Then one can see that the eigenvalue of the higher spin 3 current is the twice of the one in (5.1) under the large $(N, k)$ ’t Hooft like limit.

- The $(\text{symm}; f)$ representation

The relevant subsection on this higher representation is given by section 3.1.2. The following eigenvalues in this higher representation can be determined by
One observes that the first, the second and the last eigenvalues in (A.6) coincide with the ones in (4.4) if one takes the higher order terms in the numerators respectively.

- The \((\text{symm}; f)\) representation

The relevant subsection on this higher representation is given by section 3.1.4. The four eigenvalues corresponding to the zero modes of the higher spin currents of spins 1, 4, 4 and 3 which act on the representation \((\Box; \Box)\) can be summarized by

\[
\phi_0^{(1)}(\Box; \Box) = \phi_0^{(1)}(\Box; 0) + \phi_0^{(1)}(0; \Box) = \frac{(k - 2N)}{(k + N + 2)},
\]

\[
v^+(\Box; \Box) = \frac{8(12kN + 3k + 16N^2 - 52N + 20)}{(k + N + 2)^2},
\]

\[
v^-(\Box; \Box) = \frac{12(5k^2 + 4kN - 58k + 8N + 20)}{(k + N + 2)^2},
\]

\[
\phi_2^{(1)}(\Box; \Box) = -\frac{4}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \times \left( 12k^3N + 4k^3 + 18k^2N^2 - 35k^2N - 64k^2 + 24kN^3 + 35kN^2 - 132kN - 180k - 4N^3 - 62N^2 - 120N - 96 \right). \tag{A.7}
\]

All the eigenvalues survive even under the large \((N, k)\)'t Hooft like limit. As expected, one can rewrite the highest power terms in terms of \((24kN^3 + 12k^3N^2)\) and \(k(12kN + 6kN^2)\).

- The \((\text{symm}; \text{antisymm})\) representation

The relevant subsection on this higher representation is given by section 3.1.5. The four eigenvalues are characterized by

\[
\phi_0^{(1)}(\Box; \Box) = \phi_0^{(1)}(\Box; 0) + \phi_0^{(1)}(0; \Box) = -\frac{2(k + N)}{(k + N + 2)},
\]

\[
v^+(\Box; \Box) = \frac{32(3kN + 3k + 4N^2 + 29N + 20)}{(k + N + 2)^2},
\]

\[
v^-(\Box; \Box) = \frac{32(4k^2 + 3kN + 53k + 3N + 20)}{(k + N + 2)^2},
\]

\[
\phi_2^{(1)}(\Box; \Box) = \frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \times \left( 12k^3N - 2k^3 + 100k^2N - 7k^2 - 12kN^3 - 28kN^2 + 132kN + 144k + 2N^3 + 67N^3 + 24N + 48 \right).
\]
For the highest power of \((N,k)\) in the last eigenvalue, the \(12k^3N\) is the same as the one in (5.5) while the \(-12k^3N\) is the twice of the one in (5.5).

* The \((\text{symm}; \text{antisymm})\) representation

The four eigenvalues are given by

\[
\begin{align*}
\phi_0^{(1)} (\boxed{\text{symm}}, \boxed{\text{antisymm}}) &= \phi_0^{(1)} (\boxed{\text{symm}}, 0) + \phi_0^{(1)} (0; \boxed{\text{antisymm}}) = -\frac{2(-k + N)}{(k + N + 2)}, \\
v^-(\boxed{\text{symm}}, \boxed{\text{antisymm}}) &= \frac{32(3kN + 3k + 4N^2 - 27N + 20)}{(k + N + 2)^2}, \\
v^- (\boxed{\text{symm}}, \boxed{\text{antisymm}}) &= \frac{32(4k^2 + 3kN - 51k + 3N + 20)}{(k + N + 2)^2}, \\
\phi_2^{(1)} (\boxed{\text{symm}}, \boxed{\text{antisymm}}) &= -\frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \\
&\times (12k^3N - 2k^3 + 12k^2N^2 - 4k^2 + 12N^2 - 10kN^3 - 18kN - 180k - 2N^3 - 79N^2 - 132N - 96). \quad (A.8)
\end{align*}
\]

The third eigenvalue of appendix \((A.8)\) in the highest power of \((N,k)\) is the same as the one in \((5.6)\). The highest power terms in the last eigenvalue of appendix \((A.8)\) are twice of the ones in \((5.2)\).

* The \((\text{symm}; \text{symm})\) representation

The relevant subsection on this higher representation is given by section 3.1.6. The four eigenvalues are

\[
\begin{align*}
\phi_0^{(1)} (\boxed{\text{symm}}, \boxed{\text{symm}}) &= \phi_0^{(1)} (\boxed{\text{symm}}, 0) + \phi_0^{(1)} (0; \boxed{\text{symm}}) = \frac{2(-N + k)}{(k + N + 2)}, \\
v^+(\boxed{\text{symm}}, \boxed{\text{symm}}) &= \frac{32(3kN + 3k + 4N^2 - 27N + 20)}{(k + N + 2)^2}, \\
v^- (\boxed{\text{symm}}, \boxed{\text{symm}}) &= \frac{96(k^2 + kN - 9k + N)}{(k + N + 2)^2}, \\
\phi_2^{(1)} (\boxed{\text{symm}}, \boxed{\text{symm}}) &= -\frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \\
&\times (12k^3N - 2k^3 + 12k^2N^2 - 4k^2 + 12N^2 - 10kN^3 - 10kN^2 - 79N^2 - 7N^2 + 48N + 48). \quad (A.9)
\end{align*}
\]

The last eigenvalue with boldface notation in appendix \((A.9)\) is the same as the one in appendix \((A.8)\).

* The \((\text{antisymmm}; \text{antisymmm})\) representation

The relevant subsection on this higher representation is given by section 3.2.1. The four eigenvalues are given by

\[
\begin{align*}
\phi_0^{(1)} (\boxed{\text{antisymmm}}, \boxed{\text{antisymmm}}) &= \frac{4}{(k + N + 2)}, \quad v^+(\boxed{\text{antisymmm}}, \boxed{\text{antisymmm}}) = \frac{192(k + 1)}{(k + N + 2)^2}, \quad v^- (\boxed{\text{antisymmm}}, \boxed{\text{antisymmm}}) = \frac{192(N - 1)}{(k + N + 2)^2}, \\
\phi_2^{(1)} (\boxed{\text{antisymmm}}, \boxed{\text{antisymmm}}) &= \frac{16(6k^2N + 5k^2 - 6kN^2 + 18kN + 43k - 5N^2 - 13N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}. \quad (A.10)
\end{align*}
\]
Under the symmetry $N \leftrightarrow k$ (with $\boxed{\text{symm}} \leftrightarrow \boxed{\text{antisymm}}$), the first eigenvalue in appendix (A.10) remains the same, and the second eigenvalue becomes the third one, and the third eigenvalue becomes the second one by ignoring the constant term in the numerator. The last eigenvalue in appendix (A.10) remains the same with an extra sign change if we consider only the case where the total power of $N$ and $k$ is given by 3: $$(6k^2N - 6kN^2).$$

- The (antisymm; $f$) representation
The relevant subsection on this higher representation is given by section 3.2.2. The four eigenvalues are

\[
\phi_0^{(1)}(\boxed{\text{ antisymm}; f}) = -\frac{(N - 2)}{(k + N + 2)}, \quad \nu^+(\boxed{\text{ antisymm}; f}) = \frac{12(4kN + 5N^2 - 10N + 12)}{(k + N + 2)^2},
\]

\[
\nu^-(\boxed{\text{ antisymm}; f}) = \frac{24(5N - 4)}{(k + N + 2)^2},
\]

\[
\phi_2^{(1)}(\boxed{\text{ antisymm}; f}) = -\frac{4(N - 2)(6k^2N + 5k^2 + 12kN^2 + 39kN + 28k + 4N^2 + 14N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}.
\]

The large $(N, k)$ behavior of the eigenvalue for the higher spin 3 current is the same as the one in appendix (A.6).

- The (antisymm; $f$) representation
The relevant subsection on this higher representation is given by section 3.2.4. The four eigenvalues are given by

\[
\phi_0^{(1)}(\boxed{\text{ antisymm}; f}) = \phi_0^{(1)}(\boxed{0}; f) + \phi_0^{(1)}(\boxed{0}; f) = \frac{(k - 2N)}{k + N + 2},
\]

\[
\nu^+(\boxed{\text{ antisymm}; f}) = \frac{24(4kN + k + 4N^2 - 20N)}{(k + N + 2)^3}, \quad \nu^-(\boxed{\text{ antisymm}; f}) = \frac{12(5k^2 + 4kN - 5k + 8N + 20)}{(k + N + 2)^2},
\]

\[
\phi_2^{(1)}(\boxed{\text{ antisymm}; f}) = -\frac{3(k + N + 2)^2(6kN + 5k + 5N + 4)}{4}
\times \frac{(12k^2N + 4k^3 + 18k^2N^2 - 35k^3N - 40k^3 + 24kN^3 - 37kN^2 - 192kN - 120k - 4N^3 - 74N^2 - 108N - 48)}{(A.11)}.
\]

The relations in appendix (A.2) and (A.3) are used. The large $(N, k)$ behavior of the eigenvalue in appendix (A.11) for the higher spin 3 current is the same as the one in appendix (A.7).

- The (antisymm; symm) representation
The relevant subsection on this higher representation is given by section 3.2.5. The four eigenvalues are summarized by

\[
\phi_0^{(1)}(\boxed{\text{ antisymm}; \text{ symm}}) = \phi_0^{(1)}(\boxed{0}; \text{ symm}) + \phi_0^{(1)}(\boxed{0}; \text{ symm}) = \frac{2(k + N)}{k + N + 2},
\]

\[
\nu^+(\boxed{\text{ antisymm}; \text{ symm}}) = \frac{96(kN + k + N^2 + 7N)}{(k + N + 2)^4}, \quad \nu^-(\boxed{\text{ antisymm}; \text{ symm}}) = \frac{96(k^2 + kN + 7k + N)}{(k + N + 2)^4},
\]

\[
\phi_2^{(1)}(\boxed{\text{ antisymm}; \text{ symm}}) = \frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}
\times \frac{(12k^2N - 2k^3 + 64k^2N + 11k^2 - 12kN^2 + 8kN^2 + 36kN + 2N^3 + 72N + 48)}{(A.12)}.
\]

Again the relations appendix (A.2) and (A.3) are used. The third eigenvalue in appendix (A.12) in the highest power of $(N, k)$ is the twice of the one in (5.3).

- The (antisymm; symm) representation
The four eigenvalues are
\[ \phi_0^{(1)}(\text{antisymm}) = \phi_0^{(1)}(0, \text{antisymm}) = \frac{2(-k + N)}{k + N + 2}, \]

\[ v^+ = \frac{96(kN + k + N^2 - 9N)}{(k + N + 2)^2}, \quad v^- = \frac{96(k^3 + kN - 9k + N)}{(k + N + 2)^2}, \]

\[ \phi_1^{(1)}(\text{antisymm}) = -\frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \times (12k^3N - 2k^3 + 12k^2N^2 - 70k^2N - 25k + 12kN - 64kN^2 - 94kN + 36k - 2N^3 - 13N^2 + 84N + 96). \] (A.13)

The last eigenvalue with boldface notation in appendix (A.13) is the same as the one in appendices (A.8) or (A.9).

• The (antisymm; antisymm) representation

The relevant subsection on this higher representation is given by section 3.2.6. The four eigenvalues are

\[ \phi_0^{(1)}(\text{antisymm}) = \phi_0^{(1)}(0; \text{antisymm}) = \frac{2(k - N)}{k + N + 2}, \]

\[ v^+ = \frac{96(kN + k + N^2 - 9N)}{(k + N + 2)^2}, \quad v^- = \frac{32(4k^3 + 3kN - 51k + 3N + 20)}{(k + N + 2)^2}, \]

\[ \phi_2^{(1)}(\text{antisymm}) = -\frac{8}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \times (12k^3N - 2k^3 + 12k^2N^2 - 34k^2N - 19k^2 + 12kN^3 - 46kN^2 - 184kN - 120k - 2N^3 - 85N^2 - 96N - 48). \] (A.14)

The last eigenvalue with boldface notation in appendix (A.14) is the same as the one in appendices (A.8), (A.9) or (A.13). See also (3.11).

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