Bridging over $p$-wave $\pi$-production and weak processes in few-nucleon systems with chiral perturbation theory

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Abstract

I study an aspect of chiral perturbation theory ($\chi$PT) which enables one to “bridge” different reactions. That is, an operator fixed in one of the reactions can then be used to predict the other. For this purpose, I calculate the partial wave amplitude for the $p$-wave pion production ($pp \rightarrow pn\pi^+$) using the pion production operator from the lowest and the next nonvanishing orders. The operator includes a contact operator whose coupling has been fixed using a matrix element of a low-energy weak process ($pp \rightarrow de^+\nu_e$). I find that this operator does not reproduce the partial wave amplitude extracted from experimental data, showing that the bridging over the reactions with significantly different kinematics is not necessarily successful. I study the dependence of the amplitude on the various inputs such as the $NN$ potential, the $\pi N\Delta$ coupling, and the cutoff. I argue the importance of a higher order calculation. In order to gain an insight into a higher order calculation, I add a higher order counter term to the operator used above, and fit the couplings to both the low-energy weak process and the pion production. The energy dependence of the partial wave amplitude for the pion production is described by the operator consistently with the data. However, I find a result which tells us to be careful about the convergence of the chiral expansion for the $pp \rightarrow pn\pi^+$ reaction.

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I. INTRODUCTION

Applications of the chiral perturbation theory (χPT) to few-nucleon systems have been of great interest in the last decade. For reviews, see, e.g., Refs. [1]. Particularly, a comprehensive review for the π production is given in Ref. [2], and for the electroweak processes in Ref. [3]. A powerful aspect of χPT is that it provides us with a bridge between different reactions in a model-independent way. This means that coupling constants (the so-called low-energy constants, LECs) fixed using experimental data for one of the reactions can then be used in the other. An interesting interaction in this context is

\[ \mathcal{L} = -2d_1 N^\dagger S \cdot uNN^\dagger N, \]  

with

\[ f_\pi u_\mu = -\tau_a \partial_\mu \pi_a - \epsilon_{3ab} V_\mu \pi_a \tau_b + f_\pi A_\mu + \cdots. \]  

The spin operator is \( S \), and the external vector (axial) current is \( V_\mu (A_\mu) \). The constant \( d_1 \) is a LEC. This contact interaction between the nucleons contributes to the three-nucleon force [4], the \( p \)-wave π-production (\( pp \rightarrow pn\pi^+ \)), the radiative pion capture on the deuteron (\( \pi^-d \rightarrow \gamma nn \)), and the weak processes in few-nucleon systems, such as tritium β-decay and the proton-proton fusion (\( pp \rightarrow d\pi^+ \)), the neutrino-induced disintegration of the deuteron (\( \nu_e d \rightarrow e^- pp, \nu d \rightarrow \nu pn \)), and the muon capture by the deuteron (\( \mu^-d \rightarrow \nu_\mu nn \)). The contact term in Eq. (1) makes the connection among these reactions, which may be referred to as the two-body analog of the Goldberger-Treiman relation, as stated in Ref. [6]. If all couplings except for \( d_1 \) have been fixed using experimental data for \( \pi N \) and \( NN \) elastic scattering, then one can fix \( d_1 \) using one of the above reactions, and predict the others.

There have been several such calculations which I will refer to as the “bridging program”. One of them was done by Park et al. [8], where they fixed \( d_1 \) using the experimental tritium β-decay rate, and calculated, with no free parameters, the weak proton capture by a proton (or \( ^3\text{He} \)). In this case, all the reactions are low-energy weak processes, and the kinematics are relatively similar. Therefore, the bridging program is expected to work well. In another work due to Hanhart et al. [5], the authors calculated the partial wave amplitude for the \( p \)-wave π-production (\( pp \rightarrow pn\pi^+ \)), and showed that the use of \( d_1 \) fixed by three-nucleon observables also consistently reproduces the partial wave amplitude extracted from experimental data [11]. Although this result seems to be satisfactory, the bridging program in this work was not done fully consistently. This is because they used a nuclear force which is different from the nuclear force used for fixing \( d_1 \) in Ref. [4]. Because of the short-range nature of the \( d_1 \) term, the \( d_1 \) value is largely dependent on the choice of the nuclear force. Hanhart et al. also showed that the \( p \)-wave π-production amplitude is rather largely
dependent on the contribution from the $d_1$ term. This indicates the importance of careful
treatment of the $d_1$ term in the calculation; the nuclear force and the $d_1$ value have to be
consistent. Finally, I mention the bridging program done by Gärdestig et al. [6], where they
fixed the $d_1$ value using the matrix element of $pp \rightarrow de^+ \nu_e$ and then used it in calculating
observables for $\pi^- d \rightarrow \gamma nn$. The two reactions are similar in kinematics of the $NN$ sector,
but are rather different in the momentum transfer, which might have to be taken care of.
They are interested in extracting the neutron-neutron scattering length from the $\pi^- d \rightarrow \gamma nn$
reaction and, for that purpose, they are interested in the shape of the spectrum rather than
the absolute value. They found that the use of the $d_1$ term fixed in the above manner signific-
antly reduces the dependence of the shape on $R$ which is the matching point between the
long range one-pion-exchange potential and the short range square well potential. Although
this result is a success of the bridging program, it is still interesting to study the absolute
value of the cross section in order to more rigorously test the power of $\chi$PT.

In this work, I would like to more seriously investigate how reliably the bridging program,
an important aspect of $\chi$PT, works. I believe that my investigation is important because
there has been sometimes an argument which supposes that the bridging program works,
even though it has not been quantitatively confirmed that a bridging program over reactions
with considerably different kinematics works. For this purpose, I calculate the partial wave
amplitude for the $p$-wave $\pi$-production in $NN$ collision ($pp \rightarrow pn\pi^+$), with $d_1$ fixed by
an observable of a low-energy weak process. This obviously provides a stringent test of
$\chi$PT, because the two reactions are strong and weak processes, and are quite different in
kinematics. My program is as follows. I extend the operator of Ref. [8] by including the $\Delta$,
which is known to play an important role in the $p$-wave $\pi$-production, and re-fit $d_1$ to an
observable of a weak process. Here, I fit $d_1$ to a matrix element of the axial vector current
used in Ref. [8]: the kinematics is that for the $pp \rightarrow de^+ \nu_e$ reaction. The $\pi$-production
operators I use are the same as those used in Ref. [5], except that the $\Delta$ is included in the nuclear wave
functions [12]. I use nuclear wave functions including only the nucleons as dynamical degrees
of freedom. I take account of the $\Delta$ by including it in the $\pi$-production operator. [1]
The $\pi$-production operator I use is based on a counting rule proposed in Ref. [5, 13], which is
different from Weinberg’s counting [14], and the large initial on-shell momentum [$\sim \sqrt{m_Nm_{\pi}}$,
$m_N$ ($m_{\pi}$): the nucleon (pion) mass] is considered as a characteristic scale of the system. I
consider the operators up to next-to-leading order [$NLO$, $O(m_{\pi}/m_N)$]. [2]

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1 Thus I treat the $\Delta$ perturbatively and do not fully take account of some non-perturbative effect of the
$\Delta$ in wave functions. A calculation with a fuller account of the $\Delta$ might be worthwhile doing to see a
difference.

2 The operator used in Ref. [8] is based on Weinberg’s counting while the operator used in Ref. [5] is based
on the counting proposed in Ref. [5, 13]. Therefore, one may wonder whether the two operators contain
It is very interesting to see whether the operator constructed using the low-energy weak process can reasonably describe the \( \pi \) production. I will use several combinations of the inputs (the \( \pi N \Delta \) coupling, the \( NN \) potential, cutoff) in my calculation. Even though the \( d_1 \) value is fixed for each combination of the inputs so that the low-energy weak process is reproduced, one may expect a dependence of the \( \pi \) production amplitude on the combination because of the rather different kinematics between the two reactions. I will study such a dependence. In fact, we will see that this bridging program is not successful when working with the NLO operators; the partial wave amplitude of the \( \pi \) production based on \( \chi PT \) is not consistent with the data. I will argue the importance of going to a higher order calculation.

In order to, even roughly, explore a result of a higher order calculation, I add a higher order counter term and study a consequence.

This paper is organized as follows. In Sec. II I present the chiral operator for the \( p \)-wave \( \pi \)-production up to the order I work with. I discuss the determination of the \( d_1 \) value using the low-energy weak process in Sec. III. In Sec. IV I perform a multipole expansion of the operator, and express the cross section in terms of the partial wave amplitudes. Then, in Sec. V I present my result for the \( p \)-wave \( \pi \) production amplitudes and compare them with the data. The result obtained with the higher order counter term is also presented. Finally I summarize this work in Sec. VI.

II. CHIRAL OPERATOR FOR \( p \)-WAVE \( \pi \)-PRODUCTION

I present expressions of chiral operators which contribute to the \( p \)-wave \( \pi \)-production. I basically follow Ref. [5], except for the treatment of the \( \Delta \) degree of freedom, and the way how the high momentum components are cut off. I start with the chiral interaction Lagrangian given in Ref. [15]. By keeping terms relevant to my calculation, I have

\[
\mathcal{L}^{(0)}_{\text{int}} = -\frac{1}{4f_{\pi}^2} N^\dagger \tau \cdot (\pi \times \bar{\pi}) N + \frac{g_A}{2f_{\pi}} N^\dagger (\tau \bar{\sigma} \cdot \vec{\nabla} \pi) N + \frac{h_A}{2f_{\pi}} N^\dagger (\vec{T} \cdot \vec{\nabla} \pi) \Delta + \text{h.c.} \, ,
\]

and

\[
\mathcal{L}^{(1)}_{\text{int}} = \left( \frac{i}{8m_N f_{\pi}^2} \right) ^N N^\dagger \tau \cdot (\pi \times \bar{\nabla} \bar{\pi}) \cdot \bar{\nabla} N - \frac{c_3}{f_{\pi}^2} N^\dagger (\bar{\nabla} \bar{\pi}) N + \left( \frac{h_A}{4m_N f_{\pi}} \right) N^\dagger (\bar{\nabla} \bar{\pi}) \Delta - \frac{d_1}{f_{\pi}} N^\dagger \tau \bar{\sigma} \cdot \bar{\nabla} \bar{\pi} N N^\dagger N
\]

different mechanisms and thus the \( d_1 \) value is different in each case. I will argue in Sec. III that the \( d_1 \) value should be the same in the both cases, up to the order I am working.
\[ -\frac{d_2}{2f_\pi} \vec{\nabla}_\pi \times N^\dagger \bar{\sigma} \tau N \cdot N^\dagger \bar{\sigma} \tau N + \text{h.c.}, \]

with \( c_4 = c_4 + \frac{1}{4m_N} \). The nucleon’s spin (isospin) operator is \( \bar{\sigma} (\tau) \), while the \( N\Delta \) transition spin (isospin) operator is \( \vec{S} (T) \). I use the pion decay constant \( f_\pi = 93 \) MeV and the nucleon mass, \( m_N = 939 \) MeV. Regarding the parameters \( (g_A, h_A, c_3, c_4) \), I follow Ref. [16], where Krebs et al. constructed a chiral nuclear force including the \( \Delta \) explicitly. The axial coupling constant is \( g_A (\approx 1.27) \). For the \( \pi N\Delta \) coupling, I use two choices: \( h_A = 2.10 \) from the \( \Delta \)-decay width [17], and \( h_A = 3g_A/\sqrt{2} = 2.68 \) from large \( N_C \). In Ref. [16], the authors calculated the \( s \)- and \( p \)-wave \( \pi N \) scattering threshold parameters at next-to-leading order with taking account of the \( \Delta \), and fit the couplings \( (c_3, c_4 \text{ and others}) \) to the analysis of Matsinos [18]. The result is \( c_3 = -1.87 \) GeV\(^{-1} \), \( c_4 = 1.87 \) GeV\(^{-1} \) for \( h_A = 2.10 \), and \( c_3 = -0.79 \) GeV\(^{-1} \), \( c_4 = 1.33 \) GeV\(^{-1} \) for \( h_A = 2.68 \). The remaining unknown LECs are \( d_1 \) and \( d_2 \) which will be determined in the next section.

I use the following \( \pi \)-production operator derived from the above interaction Lagrangian. The leading order [LO, \( \mathcal{O}(1) \)] operator is the one-body direct production of the pion with the isospin state \( a \) off the nucleon [Fig. (1a)],

\[ O_{1B} = i \frac{g_A}{2f_\pi} (2\pi)^3 \delta^{(3)} (p'_2 - p_2) \bar{\tau}_1^a \sigma_1 \cdot q_\pi + (1 \leftrightarrow 2), \]

where \( q_\pi \) is the momentum of the emitted pion. The quantity \( p_i \) (\( p'_i \)) is the momentum of the incoming (outgoing) \( i \)-th nucleon. Another LO mechanism I consider is the \( \Delta \)-excitation followed by the \( \pi \) emission. In Ref. [5], the authors used the wave function which explicitly includes the \( \Delta \) component, and considered the one-body operator which produces the pion with the \( \Delta \) deexcited to the nucleon. Because I use nuclear wave functions with only

\[^{3}\text{The definition of the } \pi N\Delta \text{ coupling constant here is different from that of Ref. [16] by a factor of 2: } h_A \text{ (this work)} = 2h_A \text{ (Ref. [16]).}\]
the nucleonic degrees of freedom, I alternatively use a two-body operator in which the $\Delta$ is excited either by the $\pi$-exchange [Fig.1(b)], or by a contact interaction between the nucleons [Fig.1(c)]. The former is given by

$$O_{\Delta \pi} = \frac{i}{36} \frac{g_A h_A^2}{f_\pi} \frac{\vec{\sigma}_2 \cdot \vec{k}}{m_\pi^2 + k^2} \frac{(4 \tau^2 \vec{k} - (\vec{\tau}_1 \times \vec{\tau}_2)^a \vec{\sigma}_1 \times \vec{k}) \cdot \vec{q}_\pi}{m_\Delta - m_N - p_o^2/m_N + (p' + \vec{q}_\pi/2)^2/2\mu} + (1 \leftrightarrow 2) \; ,$$

where $m_\Delta$ (= 1232 MeV) is the mass of the $\Delta$, and $p_o$ is the initial on-shell relative $NN$ momentum. The quantities, $\mu$, $k$, $p'$ are respectively defined by $\mu \equiv m_N m_\Delta/(m_N + m_\Delta)$, $k \equiv p_2 - p'_2$ and $p' \equiv (p'_1 - p'_2)/2$. I assume the equal energy sharing between the nucleons and $m_\pi^2 \equiv (3m_\pi^2 - q_\pi^2)/4$, $m_\pi = 138$ MeV. In Appendix, I explain how I treat the energy denominator in Eq. (6) in my calculation. An expression for the diagram in Fig.1(c) is

$$O_{\Delta CT} = \frac{i}{9} \frac{h_A C_2^{NN\Delta}}{f_\pi} \frac{(4 \tau^2 \vec{\sigma}_2 \times \vec{k}) \times \vec{k}_j - (\vec{\tau}_1 \times \vec{\tau}_2)^a \vec{\sigma}_1 \times [(\vec{\sigma}_2 \times \vec{k}) \times \vec{k}]}{m_\Delta - m_N - p_o^2/m_N + (p' + \vec{q}_\pi/2)^2/2\mu} \; ,$$

(7)

where I have used the contact $NN \rightarrow N\Delta$ interaction with two derivatives,

$$\mathcal{L}_{NN \rightarrow N\Delta} = -iC_2^{NN\Delta} \Delta^\dagger \mathbf{T} \left( \vec{S} \cdot (\vec{\nabla} + \vec{\nabla}) \vec{\sigma} - \vec{S} \vec{\sigma} \cdot \vec{\nabla} \right) N \cdot N^\dagger \left( \vec{\sigma} \times (\vec{\nabla} + \vec{\nabla}) \right) \tau N \; .$$

(8)

It is noted that a contact $NN \rightarrow N\Delta$ interaction without derivative does not contribute to the transition under consideration. I use the $C_2^{NN\Delta}$ value taken from the resonance saturation of the $\rho$-exchange $NN \rightarrow N\Delta$ potential used in several phenomenological models: 25

$$h_A C_2^{NN\Delta} = \frac{18}{25} \frac{g_{\rho NN}^2}{m_N^2} \frac{1 + \kappa_\rho}{m_\rho^2} \; ,$$

(9)

where the $\rho NN$ vector coupling is $g_{\rho NN}$ ($g_{\rho NN}^2/4\pi = 0.5$) and the tensor coupling is $\kappa_\rho$ (= 6.6). The factor at the end, $1/m_\rho^2$ ($m_\rho = 770$ MeV), is the static limit of the $\rho$-meson propagator. Although Eq. (3) is not a general contact $NN \rightarrow N\Delta$ interaction with two derivatives, one may take account of the most important part of the $NN \rightarrow N\Delta$ contact interaction by invoking a meson-exchange model, and the remaining part may be phenomenologically absorbed by the contact $d_i$ ($i = 1, 2$) terms. Obviously, my treatment of the $\Delta$ is rather phenomenological. However, a construction of a nuclear force from a chiral Lagrangian with the $\pi$, $N$ and $\Delta$, which is yet to be done, is necessary to determine the $C_2^{NN\Delta}$ value. I believe that my treatment is one of what one can do best for the moment, and expect a fully consistent calculation in future.

Next I discuss next-to-leading order [NLO, $\mathcal{O}(m_\pi/m_N)$] terms which consist of four types. One of them is the recoil correction to the LO terms. The recoil corrections to the one-body term [Eq. (5)], $\pi\Delta$ term [Eq. (5)], and contact-$\Delta$ term [Eq. (7)] are respectively given by

$$O_{1B,\text{recoil}} = -i \frac{g_A \omega_\pi}{4m_N f_\pi} (2\pi)^3 \delta^{(3)}(p'_2 - p_2) \tau^a_1 \vec{\sigma}_1 \cdot (p_1 + p'_1) + (1 \leftrightarrow 2) \; ,$$

(10)

Such a nuclear force for the peripheral wave has been constructed recently. 16
For a convenience, I decompose the last term as
\[ O_{\Delta \pi, \text{recoil}} = - \frac{i}{72} \frac{\omega_\pi}{m_N} \frac{g_A h_A^2}{f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} \times \frac{4 \tau_2^a \mathbf{k} \cdot (\mathbf{p}_1 + \mathbf{p}_1') - (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a [ (\mathbf{\sigma}_1 \times \mathbf{k}) \cdot (\mathbf{p}_1 + \mathbf{p}_1') + 2i k^2]}{m_\Delta - m_N - p_o^2/m_N + (p' - q_\pi)^2/2\mu} + (1 \leftrightarrow 2) , \]  
\[ O_{\Delta \text{CT}, \text{recoil}} = \frac{i}{18} \frac{\omega_\pi}{m_N} \frac{h_A C_{2N}^\Delta}{f_\pi} \frac{1}{m_\Delta - m_N - p_o^2/m_N + (p' - q_\pi)^2/2\mu} \times \left\{ \tau_2^a \left[ -2 k^2 \mathbf{k} \cdot (\mathbf{\sigma}_1 \times \mathbf{\sigma}_2) + 4((\mathbf{\sigma}_2 \times \mathbf{k}) \times (\mathbf{p}_1 + \mathbf{p}_1')) \right] - (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a (\mathbf{\sigma}_1 \times ((\mathbf{\sigma}_2 \times \mathbf{k}) \times (\mathbf{p}_1 + \mathbf{p}_1'))) \right\} + (1 \leftrightarrow 2) , \]  
where \( \omega_\pi \) is the energy of the emitted pion. The second type of the NLO operator is a pion rescattering through either the \( c_3 \) term, or the \( c_4 \) term or the Galilean correction to the Weinberg-Tomozawa term [the first term in Eq. (11)]. They are graphically represented by Fig. 1(d), and their expressions are given by
\[ O_{c_3} = -i \frac{c_3 g_A}{f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} \frac{\mathbf{q}_\pi \tau_2^a}{\mathbf{k} \cdot \mathbf{q}_\pi} + (1 \leftrightarrow 2) , \]  
\[ O_{c_4} = -i \frac{c_4 g_A}{2f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} (\mathbf{\sigma}_1 \times \mathbf{k}) \cdot (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a + (1 \leftrightarrow 2) , \]  
\[ O_{\text{WT, Galilean}} = \frac{g_A}{16m_N f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} (\mathbf{q}_\pi - \mathbf{k}) \cdot (\mathbf{p}_1 + \mathbf{p}_1') (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a + (1 \leftrightarrow 2) . \]  
For a convenience, I decompose the last term as \( O_{\text{WT, Galilean}} = O_{\text{WT1(G)}} + O_{\text{WT2(G)}} \) with
\[ O_{\text{WT1(G)}} = \frac{g_A}{16m_N f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} (2 \mathbf{p}_1' + \mathbf{k}) \cdot (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a + (1 \leftrightarrow 2) , \]  
\[ O_{\text{WT2(G)}} = \frac{g_A}{16m_N f_\pi^3} \frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2 + k^2} (-k^2 - 2 \mathbf{p}_1' \cdot \mathbf{k}) (\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a + (1 \leftrightarrow 2) . \]  
\[ ^5 \text{When convoluted with the wave functions, } O_{\text{WT, Galilean}} \text{ gives a non-vanishing amplitude in the soft pion limit. (The tree amplitude vanishes in the soft pion limit.) This is not consistent with chiral symmetry. This problem is similar to that found in Ref. [20] in the context of the } s\text{-wave pion production in the } NN \text{ collision. The solution to this problem was given in Ref. [21]. Probably, the problem here is also resolved by the same mechanism found in Ref. [21]; e.g., one loop diagram formed by } O_{\text{WT, Galilean}} \text{ and the one-pion-exchange potential is (partly) canceled out by other irreducible pion loop diagrams, leaving a contribution consistent with the chiral symmetry. One may take some prescription to maintain the chiral symmetry. However, I use Eq. (15) without modification because this term gives a small contribution (a few percents) to an amplitude for the } pp \rightarrow pn\pi^+ \text{ reaction; the modification will not significantly change results.} \]
The third type of the NLO terms is a pion rescattering via the Weinberg-Tomozawa term:

$$O_{\text{WT}} = -\frac{3g_A\omega_\pi}{16f_\pi^3}\frac{\mathbf{\sigma}_2 \cdot \mathbf{k}}{m_\pi^2+k^2}(\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a + (1 \leftrightarrow 2).$$  \hfill (18)

Finally, the fourth type is a pion emission from the contact terms [the $d_1$ and $d_2$ terms in Eq. (4), Fig. 1(e)]:

$$O_d = -i \left( \frac{d_1}{f_\pi}\sigma_1 \tau_1^a + \frac{d_2}{2f_\pi}(\mathbf{\sigma}_1 \times \mathbf{\sigma}_2)(\mathbf{\tau}_1 \times \mathbf{\tau}_2)^a \right) \cdot \mathbf{q}_\pi + (1 \leftrightarrow 2).$$  \hfill (19)

Starting with the operators presented above, I take the following procedure to obtain the momentum space ($p$-space) operator contributing to the $^1S_0 \rightarrow ^3S_1 - ^3D_1$ transition. At first, I perform Fourier transformation of the above operators to represent them in the coordinate space ($r$-space). In the Fourier transformation, I multiply a Gaussian cutoff function, $\exp \left( -k^2/\Lambda^2_G \right)$, to the operators other than the one-body operators; I use $\Lambda_G = 2$ GeV. In the $r$-space, I perform the multipole expansion of the operators, and evaluate the matrix elements for the spin, isospin and angular parts of the operators. The remaining radial part of the operators is converted to the $p$-space using Fourier transformation, and I obtain the radial part of the $p$-space operator, $O(p', p)$, where $p$ ($p'$) is off-shell relative momentum of the incoming (outgoing) two nucleons. I classify the operators into two groups:

Group I : $O_{\Delta \pi}, O_{\Delta \text{CT}}, O_{c_3}, O_{c_4}, O_d$

Group II : $O_{1\text{B}}, O_{1\text{B, recoil}}, O_{\Delta \pi, \text{recoil}}, O_{\Delta \text{CT, recoil}}, O_{\text{WT, Galilean}}, O_{\text{WT}}$

I introduce a sharp cutoff $\Lambda$ for the operators belonging to Group I such that

$$O_\Lambda (p', p) = O(p', p), \quad \text{for } p \leq \Lambda \text{ and } p' \leq \Lambda,$$

$$O_\Lambda (p', p) = 0, \quad \text{otherwise}.$$  \hfill (20)

I do not apply the sharp cutoff to the operators of Group II. As we will see in the next section where the LECs $d_i$ are determined using low-energy weak processes, the renormalization of $d_i$ only takes care of high momentum components (larger than $\Lambda$) of the operators in Group I. The LECs $d_i$ contain the same physics for both the reactions bridged. Therefore, I retain the high momentum components of the operators in Group II in my calculation. \footnote{One may also apply the sharp cutoff to the Group II operators, which amounts to cutting off a higher order effect.} As representatives, I choose $\Lambda = 500, 600$ and $800$ MeV. I use the $p$-space operator explained above because of its usefulness for the renormalization group (RG) analysis which will be done later.
For the purpose of the multipole expansion of the $\pi$ production operator, which will be done in Sec. IV, I express the above $\pi$ production operators as follows:

$$O_X = -\vec{O}_X \cdot \vec{Q}, \quad X = 1B, \Delta \pi, \Delta CT, c_3, c_4, WT1(G), d$$

$$O_Y = O_Y^0, \quad Y = \text{recoil terms}, \ WT2(G), \ WT$$

where $\vec{Q} = i\vec{q}_\pi/2f_\pi$ and $Q^0 = i\omega_\pi/2f_\pi$. I will refer to $\vec{O}_X$ as the spatial component of the $\pi$ production operator while $O_Y^0$ as the time component.

III. DETERMINATION OF $d_i$

In this section, I determine the LECs $d_i$ in Eq. (19) using an observable of a low-energy weak process. The $d_i$ terms contribute to the $^1S_0 \rightarrow ^3S_1$ transition and appear as a single linear combination, $\tilde{d} \equiv d_1 + 2d_2$. Because $d_1$ and $d_2$ are not separately determined by considering the $^1S_0 \rightarrow ^3S_1$ transition only, I determine $\tilde{d}$ rather than $d_1$ and $d_2$ individually.

In Ref. [8], the authors fixed $\tilde{d}$ using the experimental data of the tritium $\beta$-decay rate. They derived the axial vector current contributing the tritium $\beta$-decay from the chiral Lagrangian. They did not explicitly consider the $\Delta$ as a degree of freedom. The spatial component of the axial vector currents used in Ref. [8] are obtained from the one-body operator [Eq. (5)] and Group I operators without the $\Delta$ [Eqs. (13), (14) and (19)], with the factor of $(i\vec{q}_\pi/2f_\pi)$ being eliminated and $m_\pi'$ replaced by $m_\pi$. Accordingly, the parameters $(c_i)$ used in Ref. [8] are different from those presented in the previous section. Although they additionally included some other operators which give a negligible contribution, I do not consider them in the following. They multiply a Gaussian cutoff function, $\exp \left(-k^2/\Lambda^2\right)$, to the operators, with $\Lambda = 500, 600$ and $800$ MeV. For each choice of $\Lambda$, they fixed $\tilde{d}$ so that the tritium $\beta$-decay rate is reproduced. They used the AV18 NN potential [22] supplemented by the Urbana-IX three-nucleon potential [23] when calculating the matrix element for the tritium $\beta$-decay.

Here, I need to re-fit the value of $\tilde{d}$ in “my” axial current operator. My operator is the axial current operator used in Ref. [8] plus the $\Delta$-excitation current. The $\Delta$-excitation current is obtained from Eqs. (6) and (7) by eliminating the factor of $(i\vec{q}_\pi/2f_\pi)$ and $m_\pi' \rightarrow m_\pi$. The way of cutting off the high momentum states is also different from that used in Ref. [8], as has been discussed in the previous section. I do not directly use the tritium $\beta$-decay rate but take an indirect way to fix $\tilde{d}$, as discussed in the next paragraph.

I start with a benchmark calculation to which the $\tilde{d}$ value in my operator is fitted. For that purpose, I use the same spatial axial current operator used in Ref. [8] (the same couplings}

\footnote{The authors fixed dimensionless constant $\hat{d}_R$ rather than $\tilde{d}$. The two quantities are related through $\hat{d}_R = (m_N f^2_g/g_A)\tilde{d} + \tilde{c}_3/3 + 2\tilde{c}_4/3 + 1/6$, with $\tilde{c}_{3,4} = \tilde{c}_{3,4}/m_N$.}
and the same cutoff but without the negligible operators) to calculate a matrix element for the $^1S_0 \to ^3S_1 - ^3D_1$ transition in two-nucleon system. This matrix element is the benchmark to which $\tilde{d}$ is re-fitted so that “my” operator reproduces the same matrix element. In the benchmark calculation, I use the AV18 potential for consistency, and choose a kinematics with $T_{\pi NN} = 0.5$ MeV ($T_{\pi NN}^i$: the initial on-shell kinetic energy of the relative motion), the deuteron final state, and $q = 2.5$ MeV ($q$: the momentum transfer from the two-nucleon system to the external current). This kinematics is for the low-energy $pp \to d\bar{e} + \nu_e$ reaction.

I use the proton-neutron interaction to generate the initial wave functions, and therefore do not consider the Coulomb interaction and the isospin violation effect. I calculate the matrix element with different choices of the Gaussian cutoff, $\Lambda_\chi = 500, 600$ and $800$ MeV. I average the three matrix elements which have a slight cutoff dependence, and regard the average as the benchmark. When I calculate the matrix element of my operator, I use several combinations of the $NN$ potential, the $\pi N\Delta$ coupling ($h_A$) and the corresponding $c_i$ values, and the sharp cutoff value ($\Lambda$). I use the following $NN$ potentials: the CD-Bonn [24], the AV18 [22], the Reid93 [25], the Nijmegen I [25] and the chiral $N\!^3\!LO$ [26] potentials. For each combination of these inputs, the $\tilde{d}$ value is fitted to the benchmark, and the result is given in Table I. In the table, I show the dimensionless coupling $\hat{d}(\equiv m_Nf_\pi^2 \tilde{d})$. Although the $\tilde{d}$ value is adjusted so that the benchmark is reproduced irrespective of changing the inputs, it would be expected that the $\pi$-production amplitude, evaluated in a significantly different kinematics, have the dependence on them. I will study the dependence later.

Before closing this section, I discuss the issue mentioned in footnote 2. In order for the bridging program to be meaningful, the $\tilde{d}$ term has to (implicitly) include the same mechanisms (up to the external probe) for both of the reactions bridged. If the $\tilde{d}$ term for one of the reactions includes a mechanism which is explicitly considered in the other, the $\tilde{d}$ value

8 I take the hybrid approach where a matrix element of the chiral operator is sandwiched by wave functions obtained with a phenomenological nuclear force. This approach is also referred to as more effective effective field theory (MEEFT). An argument for employing MEEFT is given in Ref. [27].
should be different in each case. In the case under consideration, the axial current operator and the pion production operator are based on different counting schemes, and therefore there is a concern that the two operators explicitly contain different mechanisms. In the following, I will argue that the \( \tilde{d} \) term includes the same physics for both of the operators, up to the order I am working. As we have seen in the above, all of the mechanisms for the axial vector current are included in the pion production operator (up to the external probe). However, the pion production operator contains some mechanisms which are not included in the axial vector current; the recoil corrections, \( \text{WT} \) and \( \text{WT}(G) \). Among them, \( \text{WT} \) and \( \text{WT}(G) \) do not contribute to the weak process, and thus the \( \tilde{d} \) term fixed in the weak process does not contain these mechanisms. The recoil correction of the one-body axial current [the counterpart of Eq. (10)] has been considered in Ref. [8], which means that the \( \tilde{d} \) term fixed in Ref. [8] does not contain this mechanism. Although I did not consider this mechanism when fixing \( \tilde{d} \), I am safe because this mechanism gives only a negligible contribution to the matrix element considered. The recoil corrections of the \( \Delta \) axial current [the counterpart of Eqs. (11) and (12)] have not been considered in Ref. [8]. However, the \( \tilde{d} \) value does not depend on the inclusion of these mechanisms because they give a negligible contribution to the matrix element. (In fact, these recoil corrections should be captured by another higher order counter term.) Therefore, up to the order I am working, the \( \tilde{d} \) term contains the same physics for both of the reactions bridged, and I can use the \( \tilde{d} \) term fixed in this section for calculating the pion production amplitude.

IV. PARTIAL WAVE AMPLITUDE AND CROSS SECTION

In this section, I will express the cross section for the \( pp \rightarrow pn\pi^+ \) reaction in terms of partial wave amplitudes. \(^9\) For this purpose, I perform the standard multipole expansion of the \( \pi \) production operator\(^{28}\), and a partial wave expansion of the initial and final \( NN \) scattering wave functions. Then, I use the partial wave amplitudes to express the transition amplitude, and subsequently the cross section. I perform these expansions in the \( r \)-space, which will be followed by the conversion into the \( p \)-space.

A. Partial wave amplitude

The multipole operator for the time component of a \( \pi \) production operator is defined by

\[
T_{C}^{JM}(O) = \int d\mathbf{x} j_j(x) Y_{JM}(\hat{x}) O_0(x), \tag{23}
\]

\(^9\) I will work with the center-of-mass system of the initial \( pp \) system throughout this work.
where \( q \equiv -q_\pi (q = |q|) \), \( j_J(qx) \) is the spherical Bessel function of order \( J \), and \( \hat{x} \equiv x/|x| \). An \( r \)-space operator is \( \mathcal{O}^0(x) \), the dependence on the center-of-mass coordinate being eliminated; e.g., \( \mathcal{O}^0(x) = \mathcal{O}^0(x)|_{R=0} \). For the spatial component of the \( \pi \) production operator, I show only the longitudinal multipole operator because the electric and magnetic multipole operators give vanishing contribution for the case in question:

\[
T_{J M}^L (\mathcal{O}) = \frac{i}{q} \int \! d\mathbf{x} \nabla [j_J(qx) Y_{J M} (\hat{x})] \cdot \mathcal{O}(\mathbf{x}). \tag{24}
\]

The transition amplitude for the \( pp \rightarrow pn\pi^+ \) reaction is written by

\[
T_{fi} = \langle \psi_f \int \! d\mathbf{x} \, e^{i\mathbf{q} \cdot \mathbf{x}} \left[ \sum_Y \mathcal{O}^0_Y (x) Q^0 - \sum_X \mathcal{O}_X (x) \cdot \hat{Q} \right] | \psi_i \rangle , \tag{25}
\]

where \( \psi_i (\psi_f) \) is the nuclear wave function for the initial (final) state, whose \( r \)-space representation will be given later. In the summations in Eq. (23), \( X (Y) \) takes various components of the \( \pi \) production operator specified in Eq. (21) [Eq. (22)]. The four-vector \( Q \) has been defined in Eqs. (21) and (22). Using the multipole operators presented above, I can rewrite \( T_{fi} \) as

\[
T_{fi} = \sum_{J,M_a} \frac{4\pi i^{J_a} (-1)^{M_a} \langle \psi_f | \left[ T_{C, J = 0}^0 Q_C^{J = 0} + T_{L, J = 0}^L Q_L^{J = 0} \right] | \psi_i \rangle , \tag{26}
\]

with

\[
Q_C^{JM} = Y_{JM}(\hat{q}) \, Q^0, \tag{27}
\]

\[
Q_L^{JM} = \left( \sqrt{\frac{J}{2J+1}} Y_{J-1JM}(\hat{q}) - \sqrt{\frac{J+1}{2J+1}} Y_{J+1JM}(\hat{q}) \right) \cdot \mathbf{Q} , \tag{28}
\]

and \( Y_{JLM}(\hat{q}) \) are the vector spherical harmonics.

Now I proceed to the partial wave expansion of the \( NN \) wave function. An \( NN \) scattering wave function with the relative momentum \( p \), the third component of the spin (isospin) of \( i \)-th nucleon being \( s_i (\tau_i) \) is expanded as follows:

\[
\psi(\mathbf{r}) = \sum_{\alpha, \mu} 4\pi (1/2, s_1, 1/2, s_2 | S \mu)(1/2, \tau_1, 1/2, \tau_2 | T, T^3)(L m S \mu | J M) i^L Y_{L,m}^*(\hat{p}) \psi_\alpha(\mathbf{r}), \tag{29}
\]

where the index \( \alpha \) collectively denotes the quantum numbers of a partial wave; \( \alpha = \{ J, L, S, T \} \) where \( J, L, S \) are the total, orbital, total spin angular momenta of the \( NN \) system, respectively and \( T \) is the total isospin. The partial wave function \( (\psi_\alpha) \) is expressed as

\[
\psi_\alpha(\mathbf{r}) = \frac{1 - (-1)^{L+S+T}}{\sqrt{2}} \sum_{L'} \mathcal{Y}_{L' S J} (\hat{r}) \, R_{L', L; S}(r) \, \eta_{T, T^3} , \tag{30}
\]

\[
\mathcal{Y}_{L S J}(\hat{r}) = [Y_L(\hat{r}) \otimes \chi_{S}(J)] , \tag{31}
\]
where the two-nucleon spin (isospin) wave function is denoted by $\chi_S$ ($\eta_T$). The radial part of the above wave function is normalized, in the plane wave limit, to be

$$R^{f}_{L',L;S}(r) \rightarrow j_L(pr)\delta_{L,L'}.$$ (32)

With the multipole operators and the partial waves presented above, I express the transition amplitude in terms of the partial wave amplitudes:

$$T_{fi} = \sum_{\alpha_i,m_i;\alpha_f,m_f} \langle J_f, M_f | T^{J o,M o}_{a} | J_i, M_i \rangle = \sum_{\alpha_i,m_i;\alpha_f,m_f} (-1)^{M_o} \times (1/2, s_{1,i}, 1/2, s_{2,i}) (1/2, \tau_{1,i}, 1/2, \tau_{2,i} | T_i | T_i^3) (L_i m_i S_i | J_i M_i) \times (1/2, s_{1,f}, 1/2, s_{2,f} | S_f M_f) (1/2, \tau_{1,f}, 1/2, \tau_{2,f} | T_f, T_f^3) (L_f m_f S_f | J_f M_f) \times (J_i M_i J_o M_o | J_f M_f) \sum_{a=C,L} (T^J_\alpha) Q_a^{J_o-M_o} ,$$ (33)

with the suffix $i$ ($f$) indicates the initial (final) state. In the above equation, I used the abbreviation

$$\langle T^J_\alpha \rangle = \langle \psi_{\alpha_f} | T^J_\alpha | \psi_{\alpha_i} \rangle ,$$ (34)

for the reduced matrix element defined by

$$\langle J_f, M_f | T^{J o,M o}_{a} | J_i, M_i \rangle = \frac{1}{\sqrt{2J_f + 1}} (J_i M_i J_o M_o | J_f M_f) \langle J_f | T^{J o}_a | J_i \rangle .$$ (35)

**B. Cross section**

The unpolarized cross section for the $pp \rightarrow pn\pi^+$ reaction is

$$d\sigma = \sum_{i,f} \frac{1}{v_{rel}} \frac{1}{(2\pi)^5} \frac{1}{2\omega_\pi} \delta^{(4)}(P_i - q_\pi - P_f) \langle T_{fi} \rangle \frac{d^2q_\pi dp_{1,f} dp_{2,f}}{d^2q_\pi dp_{1,f} dp_{2,f}} ,$$ (36)

where $\sum_{i,f}$ indicates the average (summation) over the initial (final) spin and isospin states of the two nucleons. The quantities $P_i$ and $P_f$ are the initial and final four total momentum of the two-nucleon system; $P_\mu^2 = (P^0)^2 - \mathbf{P}^2$. The relative velocity between the initial two nucleons is denoted by $v_{rel}$.

I derive the pion angular distribution, retaining only partial wave transition amplitudes of interest. I am primarily interested in the $^1S_0 \rightarrow ^3S_1$ transition amplitude, $\langle \psi_{S_1} | T^{J o}_a | \psi_{S_0} \rangle$, where the $\bar{d}$ term plays an important role. [It is noted that $\psi_{S_1}$ contains the $^3D_1$ component, as seen in Eq. (30), and thus the $^1S_0 \rightarrow ^3D_1$ transition is also included in $\langle \psi_{S_1} | T^{J o}_a | \psi_{S_0} \rangle$.] However, I also retain the $^1D_2 \rightarrow ^3S_1$ transition amplitude for a later convenience. I retain
multipole operators with rank one \((J_o = 1)\) which dominantly induce the transition. I integrate over the final two nucleon momenta to obtain:

\[
\frac{d\sigma}{d\Omega_\pi} = \int_0^{q_{\pi}^{\text{max}}} dq_\pi \frac{E_p \sqrt{q_\pi^2 + P_j^2 \mu P_j^2 q_\pi^2}}{16\pi p \omega_\pi} |M|^2 ,
\]

with

\[
|M|^2 = \frac{1}{4} \sum_{\alpha_0=1}^{3} |\langle \psi_{3S_1} | T_{C}^{\dagger} - q_\pi T_{L}^{\dagger} | \psi_{\alpha_0} \rangle |^2 + P_2(\cos \theta_\pi) \left( \frac{1}{4} |\langle \psi_{3S_1} | T_{C}^{\dagger} - q_\pi T_{L}^{\dagger} | \psi_{D_2} \rangle |^2 
- \frac{1}{\sqrt{2}} \text{Re} \left[ \langle \psi_{3S_1} | T_{C}^{\dagger} - q_\pi T_{L}^{\dagger} | \psi_{D_2} \rangle \langle \psi_{3S_1} | T_{C}^{\dagger} - q_\pi T_{L}^{\dagger} | \psi_{S_0} \rangle^* \right] \right) .
\]

The scattering angle of the pion with respect to the direction of the initial proton is denoted by \(\theta_\pi\), and \(P_2(x)\) is the Legendre function of degree two. The initial one nucleon energy is \(E_p\) and the maximum magnitude of the pion momentum is given by

\[
q_{\pi}^{\text{max}} = \sqrt{(E_p + m_N + m_\pi/2)(E_p + m_N - m_\pi/2)(E_p - m_N + m_\pi/2)(E_p - m_N - m_\pi/2)} / E_p .
\]

I will compare my calculation with “partial wave amplitudes” extracted from experimental data in Ref. [11]. However, the “partial wave amplitudes” are actually different from the ordinary one, \(\langle T_{C}^{\dagger} - q_\pi T_{L}^{\dagger} \rangle\). Therefore, I establish the relation between them so that the comparison makes sense, which will be done in the next paragraph.

In Ref. [11], six lowest partial wave amplitudes are assumed to contribute to the \(pp \rightarrow pn\pi^+\) reaction in the energy region under investigation, and are extracted from the data. In extracting the amplitudes, they treated the final \(NN\) system as the “pseudo-deuteron” which is the \(NN\) scattering state with the relative motion integrated over; the pseudo-deuteron may have the angular momentum different from the deuteron. This means that they parameterized the data using a formula in which the \(pp \rightarrow pn\pi^+\) reaction was regarded as the \(pp \rightarrow "d"\pi^+\) reaction; “\(d\)” is the pseudo-deuteron. More specifically, they parameterized the pion angular distribution in the \(pp \rightarrow pn\pi^+\) reaction using the formula:

\[
\frac{d\sigma}{d\Omega_\pi} = C_0 + C_2 P_2(\cos \theta_\pi) ,
\]

with

\[
C_0 = \frac{|a_0|^2}{4} + \frac{|a_1|^2}{4} + \frac{|a_2|^2}{4} + \frac{|b_0|^2}{4} + \frac{|b_1|^2}{4} + \frac{|b_2|^2}{4} ,
\]

\[
C_2 = \frac{|a_2|^2}{4} - \frac{\text{Re}[a_0a_2^*]}{\sqrt{2}} ,
\]

where \(a_0\) (\(a_2\)) is the partial wave amplitude for \(1S_0 \rightarrow 3S_1\) (\(1D_2 \rightarrow 3S_1\)); see Table IV of Ref. [11] for the other partial wave amplitudes. By comparing Eq. (40) with Eq. (37), I can
find the relation between the two differential cross section formula. To see the relation more clearly, I denote the theoretical counterpart to \( a_0 \) (\( a_2 \)) by \( \tilde{a}_0 \) (\( \tilde{a}_2 \)) and rewrite Eq. (37) as

\[
\frac{d\sigma}{d\Omega} = \frac{|\tilde{a}_0|^2}{4} + \frac{|\tilde{a}_2|^2}{4} + \left( \frac{|\tilde{a}_2|^2}{4} - \frac{\text{Re} \{ \tilde{a}_0 \tilde{a}_2^* \}}{\sqrt{2}} \right) P_2(\cos \theta),
\]

with

\[
|\tilde{a}_0|^2 = \int_{q_0}^{q_{\text{max}}} \frac{E_p \sqrt{q_0^2 + P_f^2}}{16\pi p \omega_f f_\pi^2} |\langle \psi_{S_0} | T_{C}^1 - q_\pi T_{L}^1 | \psi_{S_0} \rangle|^2,
\]

\[
|\tilde{a}_2|^2 = \int_{q_0}^{q_{\text{max}}} \frac{E_p \sqrt{q_0^2 + P_f^2}}{16\pi p \omega_f f_\pi^2} |\langle \psi_{S_1} | T_{C}^1 - q_\pi T_{L}^1 | \psi_{D_2} \rangle|^2,
\]

\[
\text{Re} \{ \tilde{a}_0 \tilde{a}_2^* \} = \int_{q_0}^{q_{\text{max}}} \frac{E_p \sqrt{q_0^2 + P_f^2}}{16\pi p \omega_f f_\pi^2} \times \text{Re} \left[ \langle \psi_{S_1} | T_{C}^1 - q_\pi T_{L}^1 | \psi_{D_2} \rangle \langle \psi_{S_0} | T_{C}^1 - q_\pi T_{L}^1 | \psi_{S_0} \rangle^* \right].
\]

I distinctly used \( \tilde{a} \) and \( \tilde{a} \) because they are not necessarily the same.\(^{10}\) Which (\( \tilde{a} \) or \( \tilde{a} \)) should be compared with \( a_0 \) and \( a_2 \) from the experimental data? I take the following way to find a solution.

At first, I factorize out the phase coming from the initial state interaction as

\[ \tilde{a}_0 = e^{i\delta_0} \tilde{a}_0', \quad \tilde{a}_2 = e^{i\delta_2} \tilde{a}_2', \]

where \( \delta_0 \) (\( \delta_2 \)) is the phase shift of the \( ^1S_0 \) (\( ^1D_2 \)) partial wave scattering. I choose \( \tilde{a}_0' \) and \( \tilde{a}_2' \) to be real. The same factorization applies to \( \tilde{a}_0 \) and \( \tilde{a}_2 \). Then, as Step (i), I calculate \( |\tilde{a}_0'| \) and \( |\tilde{a}_2'| \) from Eq. (44) and Eq. (45), respectively. Next, as Step (ii), I solve a coupled equation consisting of Eq. (46) and \( |\tilde{a}_0|^2 + |\tilde{a}_2|^2 = |\tilde{a}_0'|^2 + |\tilde{a}_2'|^2 \) [the r.h.s. is from Step (i)], thereby finding a set of solutions, \( \tilde{a}_0' \) and \( \tilde{a}_2' \). Step (ii) is closer to the way how \( a_0 \) and \( a_2 \) are extracted from the data; each amplitude is extracted from the sum of the amplitudes (data). My numerical result shows that the two sets of the solutions obtained in Step (i) and (ii) are essentially the same in the absolute value. [Step (i) cannot fix the sign.]\(^{11}\) Therefore, I will employ the solution from Step (ii), in which the relative phase between \( \tilde{a}_0 \) and \( \tilde{a}_2 \) can also be fixed. (Hereafter, I do not distinguish between \( \tilde{a} \) and \( \tilde{a} \), and denote them by \( \tilde{a} \).) Still, the overall phase of \( \tilde{a} \) has not been fixed. In order to make the comparison with the data

\(^{10}\) I use the symbol \( a \) (\( \tilde{a} \)) in a generic sense, referring to both \( a_0 \) and \( a_2 \) (\( \tilde{a}_0 \) and \( \tilde{a}_2 \)).

\(^{11}\) Strictly speaking, \( \tilde{a} \) obtained in my analysis is not necessarily the same as \( a \) from Ref. [11]. In order to obtain \( a \) theoretically, one calculates the analyzing power and the pion angular distribution for the \( pp \to pn\pi^+ \) reaction, taking account of all non-negligible partial wave amplitudes. Then \( a \) is extracted from those observables in the same way as done in Ref. [11]. However, the fact that \( \tilde{a} \) obtained in Step (i) and (ii) are essentially the same would indicate that my result would not change drastically even if I took the “ideal” procedure to calculate \( a \).
FIG. 2: The transition amplitude $\tilde{a}_2$ for $pp \rightarrow pn\pi^+$. The chiral NLO $\pi$ production operator and the CD-Bonn $NN$ potential are used. The solid curve is obtained with $h_A = 2.10$ while the dashed one with $h_A = 2.68$. Experimental data are from Ref. [11].

meaningful, one needs to choose the phase convention for $\tilde{a}$ to be the same as that for $a$. I use $a_2$ and $\tilde{a}_2$ to match the phase conventions of the experiment and the theory. I calculate $\tilde{a}_2$ using the same operator used in calculating $\tilde{a}_0$; the $\tilde{d}$ term does not contribute here. I will fix the phase convention in this way in the next section, followed by the comparison between $a_0$ and $\tilde{a}_0$.

V. RESULT

A. $^1D_2 \rightarrow ^3S_1$ transition amplitude

I calculate the $^1D_2 \rightarrow ^3S_1$ transition amplitude, $\tilde{a}_2$, with the operators presented in Sec. II. Then, I compare $\tilde{a}_2$ with $a_2$ from the data [11]. In calculating $\tilde{a}_2$, I set the sharp cutoff $\Lambda = \infty$ because there is no counter term at this order which takes care of the high momentum components of the operators integrated out. In Fig. 2 I show $\tilde{a}_2$ obtained with the CD-Bonn $NN$ potential, as a function of $\eta \equiv q_{\pi}^{\text{max}}/m_\pi$. As stated in Sec. III, I consider neither the Coulomb interaction nor isospin violation effects. I use the CD-Bonn

12 Because of the finite cutoff, the $\tilde{d}$ term gives a very small contribution, which I safely neglect here.
potential for the proton-neutron channel. The solid curve is obtained with $h_A = 2.10$ (and the corresponding $c_i$'s) while the dashed curve with $h_A = 2.68$. I choose the overall sign for $\tilde{a}_2$ such that $a_2$ and $\tilde{a}_2$ have the same sign, thereby fixing the phase convention. The phase convention does not change if I use a finite cutoff such as $\Lambda = 800$ MeV in calculating $\tilde{a}_2$. As observed in Fig. 2, $\tilde{a}_2$ with $h_A = 2.10$ falls exactly on the experimental data. We see the dependence of $\tilde{a}_2$ on the choice of the $NN$ potential in Fig. 3. I used only $h_A = 2.10$ in Fig. 3. Although there is some dependence on the $NN$ potential, all $\tilde{a}_2$ are on the vicinity of the data. $\chi$PT gives a successful description for $a_2$ at this order. Although $a_2$ is significantly larger than $a_0$ in magnitude, I stop the discussion on $a_2$ here and will give a detailed discussion on $a_0$ in the following subsections. This is because I am primarily concerned with the validity of the bridging program in this work, and $a_0$ is the one to be examined for this purpose.

B. $^1S_0 \to ^3S_1$ transition amplitude

Now I move on to the $^1S_0 \to ^3S_1$ transition amplitude, $\tilde{a}_0$. Because the relative phase between $\tilde{a}_2$ and $\tilde{a}_0$ is fixed within the theory, and the overall phase has been fixed by comparing $a_2$ and $\tilde{a}_2$, I am now able to compare $a_0$ and $\tilde{a}_0$ with the same phase convention. It is noted that my phase convention for $\tilde{a}_0$ is the same as the convention used in Ref. [5].

At first, in order to see the importance of the $\Delta$, I mention a result obtained with the $\pi$-production operator without the $\Delta$. I use the operators given in Eqs. (5), (10), (13)-(19) with the parameters taken from Ref. [8]. The cutoff function is also the same as that used in Ref. [8] (the Gaussian cutoff). I found a very large cutoff dependence of $\tilde{a}_0$ calculated with the $\Delta$-less $\chi$PT. Depending on the cutoff ($\Lambda_\chi = 500, 600$ and $800$ MeV), the contributions from the two-body operators to $\tilde{a}_0$ are different by a factor of 4, which demonstrates the failure of the $\Delta$-less $\chi$PT in describing the $p$-wave $\pi$-production. I note that this result is quite different from the situation of the low-energy weak process where the $\Delta$-less $\chi$PT gives the cross sections with a small cutoff dependence.

Next I present results obtained with the operator including the $\Delta$. The operators have been presented in Sec. III and the LEC $\tilde{d}$ has been fixed using the low-energy weak process in Sec. IV. I am interested in how reasonably and reliably one can predict $\tilde{a}_0$ for the $pp \to pn\pi^+$ reaction using this $\chi$PT-based operator. For this purpose, I examine the dependence of $\tilde{a}_0$ on several inputs, $\Lambda$, $h_A$, $C_2^{N\Delta}$, and the $NN$ potential. At first, in Fig. 4 I present a result obtained with the CD-Bonn $NN$ potential and two choices of the $h_A$ value. The partial wave amplitude $\tilde{a}_0$ is rather different from the experimental counterpart, $a_0$. For comparison, I also show a result obtained without the $\tilde{d}$ term. This result with $\tilde{d} = 0$ is similar to the case with $\delta = 0$ in Fig. 3 of Ref. [5] where a negative $\tilde{d}$ value (e.g., $\delta = -0.2$) brings the theoretical amplitude into the agreement with the experimental one. In my calculation, however, the
FIG. 4: The transition amplitude $\tilde{a}_0$ for $pp \rightarrow pn\pi^+$. The chiral NLO $\pi$ production operator and the CD-Bonn $NN$ potential are used. The solid and dashed curves correspond to $h_A = 2.10$ and 2.68, respectively; $\Lambda = 800$ MeV. The upper (lower) curves are obtained with (without) the $\tilde{d}$ term. Experimental data are from Ref. [11].

FIG. 5: The transition amplitude $\tilde{a}_0$ for $pp \rightarrow pn\pi^+$. The CD-Bonn $NN$ potential is used. The solid, dashed and dotted curves correspond to $\Lambda = 800$, 600 and 500 MeV, respectively; $h_A = 2.10$. The upper (lower) three curves are obtained with (without) the $\tilde{d}$ term. Experimental data are from Ref. [11].

positive $\tilde{d}$ value (see the last row of the second and third columns in Table I) separates $\tilde{a}_0$ and $a_0$ furthermore; even the sign of $\tilde{d}$ fixed by the low-energy weak process is inconsistent with the experimental data of the $\pi$-production. I change the values of $\Lambda$ and $h_A$ and examine the dependence of $\tilde{a}_0$ on these inputs. The cutoff dependence is shown in Fig. 5. The situation of the disagreement does not change. 13 This result shows that the bridging program among reactions with quite different kinematics is not necessarily successful. This is understandable if we recall another case. For example, a chiral nuclear force [26, 29] describes the elastic $NN$ scattering over a fairly wide energy region, partly because the LECs in it have been fixed using data from the same energy region. Therefore, it is no wonder to find that the operator fixed in the low-energy process cannot describe well the intermediate-energy process. In order to accurately describe the two reactions in the different energy regions simultaneously, data from both of the energy regions would be necessary to fix the LECs. It is also expected that higher order terms are necessary to accurately reproduce the data from the wide energy region, as in the case of the nuclear force.

We find from Fig. 5 the cutoff dependence ($\sim$10% level for $\tilde{a}_0$; the $\tilde{d}$ term included) which is much reduced compared with the $\Delta$-less case. We also find a certain amount of dependence on the choice of $h_A$ (Fig. 4), even if the $\tilde{d}$ value has been adjusted to eliminate the dependence

13 For some combinations of $h_A$, $\Lambda$ and $NN$ potential, the sign of $\tilde{d}$ is consistent with the data. However, the strength is not enough to bring $\tilde{a}_0$ into agreement with $a_0$. 

18
TABLE II: Contribution from each component of the operator to the matrix element for $pp \rightarrow d\pi^+$ (second row) and $pp \rightarrow de^-\nu_e$ (third row). Each contribution is divided by the “sum”, and therefore, “1B” actually gives the matrix elements with the same sign for both cases. The CD-Bonn potential, $h_A = 2.10$, and $\Lambda = 800$ MeV are used. Some operators, not shown here, contribute to the $\pi$ production by a small amount; at most, “WT” contributes by $\sim 0.08$.

| kinematics  | 1B   | $\Delta \pi$ | $\Delta CT$ | $c_3$ | $c_4$ | $\tilde{d}$ | sum (arb.units) |
|-------------|------|--------------|-------------|-------|-------|------------|----------------|
| $pp \rightarrow d\pi^+$ | -0.60 | -0.54 | 0.97 | -1.11 | 0.71 | 1.57 | -0.0251 |
| $pp \rightarrow de^-\nu_e$ | 0.996 | 0.009 | -0.008 | 0.015 | 0.005 | -0.017 | 0.757 |

on it at the low-energy kinematics. \(^{14}\) This means that the $\Delta$ operator and the $\tilde{d}$ contact operator have quite different dependence on the kinematics, and it is important to take care of each component of the operator individually. The dependence on the kinematics is also found in Table III where I tabulated contributions from each component of the operator to $\tilde{a}_0$ for two kinematics; one for low-energy weak process ($pp \rightarrow de^-\nu_e$) where $\tilde{d}$ is fixed, and the other for the $\pi$-production ($pp \rightarrow d\pi^+$, $\eta = 0.5$). This situation is in contrast to the bridging program done in Ref. \(^8\). In Ref. \(^8\), the operator fixed by a low-energy weak process (\textit{i.e.}, the tritium $\beta$-decay) was used in another weak process which takes place in a relatively similar kinematics, and the result was given with a small cutoff dependence. I might say that it is not important to take care of each component of the operator individually in this case. Rather, one needs to take care of the sum of each component. This is also a reason why the pionless theory, whose two-body operator is just a contact operator, can reasonably describe several weak processes in a low-energy region\(^{30}\).

From Table III we also find that the contributions from the NLO operators is comparable to those from the LO operator. The LO contributions mostly cancel each other. For this reaction, there is no sign of the convergence in the chiral expansion up to this order. Furthermore, there is also a significant cancellation among the NLO contributions, which may make the amplitude sensitive to the higher order contributions. This situation reminds us of the result in Ref. \(^{31, 32}\) where the $pp \rightarrow pp\pi^0$ reaction near threshold was studied with $\chi$PT. They found that some higher order two-pion-exchange mechanism contributes more than lower order one-pion-exchange mechanism, leading to a poor convergence of the chiral expansion. It would be important to do a higher order calculation of the $p$-wave $\pi$ production to see the convergence of the chiral expansion.

We find in Table III that the contribution from the $\tilde{d}$ term is substantial. This result may

\(^{14}\) The variation of $h_A$ should be compensated for by the change of $c_i$ and other higher order one-pion rescattering diagrams; not by the change of $\tilde{d}$. I will come back to this point later.
FIG. 6: The transition amplitude $\tilde{a}_0$ for $pp \rightarrow pn\pi^+$. The solid, dashed and dotted curves are respectively obtained with the CD-Bonn, AV18 and N$^3$LO $NN$ potentials; $h_A = 2.10$ and $\Lambda = 800$ MeV. The upper (lower) three curves are obtained with (without) the $\tilde{d}$ term. Data are from Ref. [11].

FIG. 7: The transition amplitude $\tilde{a}_0$ for $pp \rightarrow pn\pi^+$. The solid, dashed, dotted and dash-dotted curves are respectively obtained with the CD-Bonn, AV18, Reid93 and Nijmegen I $NN$ potentials; $h_A = 2.10$ and $\Lambda = 500$ MeV. The N$^3$LO potential almost gives the solid curve. The $\tilde{d}$ term is included. Data are from Ref. [11].

also indicate the importance of a higher order calculation, which I will argue in the following. The large contribution from the $\tilde{d}$ term means that its dependence on the kinematics is rather influential on $\tilde{a}_0$. Because I am working at the NLO, two-pion-exchange (TPE) mechanisms are not explicitly considered but mimicked by the $\tilde{d}$ term. If the TPE and the $\tilde{d}$ term have rather different dependence on the kinematics, $\tilde{a}_0$ given by the NLO and NNLO calculations may be significantly different.

I show results obtained with various $NN$ potentials to see the dependence of $\tilde{a}_0$ on it. The result is shown in Fig. 6. We find a considerable dependence on the $NN$ potential, even though all $NN$ potentials give almost the same amplitude in the low-energy region where the $\tilde{d}$ value has been fixed. The $\tilde{d}$ term is quite sensitive to the short-distance behavior of the wave function because of its point-like nature. Therefore, when the cutoff is relatively large and thus the short-distance behavior of the wave function is very dependent on the $NN$ potential, the $\tilde{d}$ value is also largely dependent on the $NN$ potential. In case of $\Lambda = 800$ MeV, for example, a part of the dependence of $\tilde{a}_0$ on the $NN$ potential is ascribable to that the $\tilde{d}$ term alone is too simple to compensate for the difference in the short distance physics. However, I consider that there is a more influential source of the dependence on the $NN$ potential, because we still find the dependence in the case of $\Lambda = 500$ MeV (Fig. 7). Let me explain more on this in the following. In the rest of this paragraph, I discuss the case with $\Lambda = 500$ MeV. At this cutoff, the details of the short distance physics have been integrated out substantially, and therefore there is no significant difference among the wave functions.
for different $NN$ potentials any more. Recall that low-momentum $NN$ potentials obtained from various phenomenological $NN$ potentials are very similar at $\Lambda \sim 400$ MeV\cite{33, 34}. In this situation, the $\tilde{d}$ values for different $NN$ potentials should be almost the same to give almost the same $\tilde{a}_0$. Contrary to this expectation, I obtained quite different $\tilde{d}$ as seen in the second row of TABLE I leading to rather different $\tilde{a}_0$.\footnote{In the next subsection where I include one more contact term, we will see that the expectation is realized.} I suspect that the $\tilde{d}$ term alone is too simple to simulate the operator to be simulated,\footnote{Even though we do not know, there should exist an operator which the $\tilde{d}$ term tries to simulate. The operator should include all non-negligible chiral operators other than those explicitly considered already. I refer to the operator as the operator to be simulated.} and that my procedure of determining $\tilde{d}$, discussed in the previous section, gives the $\tilde{d}$ term which is far from being as approximate to the operator to be simulated as it can be. I suspect that this is the main source of the $NN$ dependence of $\tilde{a}_0$. The relatively large $h_A$-dependence of $\tilde{d}$ (and $\tilde{a}_0$) is also likely to have the same origin. In order to improve the situation, one would need to do a higher order calculation, and include a few more contact operators so that a better simulation can be done.\footnote{One might suspect that the dependence on the $NN$ potential is due to a difference in the phase shift of the $^1S_0$ partial wave in the energy region above the pion production threshold. Among the $NN$ potentials I use, the phase shifts from the Reid93 and the N$^3$LO potentials are noticeably different from the others at this energy. However, I do not consider the difference in the phase shift to be influential because the inclusion of one more contact term results in that all of the $NN$ potentials with $\Lambda = 500$ MeV give essentially the same $\tilde{a}_0$ over the entire energy region under consideration. Note that the contact term does not reflect the difference in the long range behavior (phase shift) of the wave function.}

Because I treated the contact $NN \rightarrow N\Delta$ interaction in the phenomenological manner, as discussed in Sec. III it is informative to study an impact of changing the coupling on the amplitude $\tilde{a}_0$. For this purpose, I change the $C_2^{N\Delta}$ value by $\pm 10\%$, re-fit the $\tilde{d}$ value in the way discussed in Sec. III and calculate $\tilde{a}_0$. I found that $\tilde{a}_0$ is less dependent on the variation of $C_2^{N\Delta}$ than that of $h_A$. This result may be understood as follows. The contact-induced $\Delta$-excitation mechanism [Fig. I(c)] is similar to the $\tilde{d}$ term [Fig. I(e)] in the sense that Fig. I(c) is reduced to Fig. I(e) in the $\Delta$-less theory. Therefore, one may expect that the variation of $C_2^{N\Delta}$ is fairly well compensated for by the change of $\tilde{d}$.

C. $^1S_0 \rightarrow ^3S_1$ transition amplitude with one more counter term

As seen in the previous section, the bridging program was not successful; the $\chi$PT-based operator with $\tilde{d}(\equiv m_N f_n^2 \tilde{d})$ fixed by the low-energy weak process does not reproduce the partial wave amplitude for the $\pi$-production, $a_0$, extracted from the data. Because I have pointed out several reasons for expecting a higher order calculation, here I try to see what happens there by doing a simple extension of my calculation. The extension is to add a
TABLE III: Dimensionless contact couplings, $\hat{d}$ and $\hat{e}$. The first column is the sharp cutoff value, and the others are values of the LECs. The 2–4th rows are for the $\hat{d}$ values while 5–7th rows for the $\hat{e}$ values. For each $NN$ potential, the left side is the LECs for $h_A = 2.10$ while the right side for $h_A = 2.68$.

| $\Lambda$ | CD-Bonn | AV18 | Reid93 | Nij I | $N^3$LO |
|-----------|---------|------|--------|-------|---------|
| 500       | 0.57    | 0.57 | 0.45   | 0.46  | 0.43    | 0.44    | 0.51    | 0.51    | $-0.22$ | $-0.21$ |
| 600       | 0.49    | 0.48 | 0.20   | 0.19  | 0.15    | 0.14    | 0.37    | 0.36    | $-4.12$ | $-4.08$ |
| 800       | $-1.15$ | $-1.34$ | 3.06   | 3.30  | 3.12    | 3.35    | $-5.41$ | $-6.14$ | 8.42    | 8.35    |
| 500       | 0.85    | 0.85 | 0.94   | 0.94  | 0.96    | 0.95    | 0.90    | 0.90    | 1.44    | 1.43    |
| 600       | 1.10    | 1.11 | 1.35   | 1.36  | 1.40    | 1.41    | 1.20    | 1.22    | 5.15    | 5.12    |
| 800       | 3.39    | 3.60 | $-1.55$ | $-1.84$ | $-1.62$ | $-1.89$ | 8.38    | 9.23    | $-7.83$ | $-7.75$ |

higher order counter term to the NLO $\chi$PT-based operator used in the previous section. I use the following counter term of $\mathcal{O}[(m_\pi/m_N)^2]$ \textsuperscript{18}:

$$
\mathcal{L}_{\text{CT}}^{(2)} = -\frac{\hat{e}}{m_N f^2_N \Lambda^2} N^\dagger \tau \cdot \bar{\sigma} \cdot \vec{\nabla} \pi N \left(N^\dagger \vec{\nabla}^2 N + \text{h.c.}\right),
$$

where the dimensionless LEC is denoted by $\hat{e}$. This is not a general form of the counter terms at this order, and there are other counter terms with different spin-isospin and derivative structures. However, it is sufficient to consider only this counter term for my purpose of gaining a rough insight into a higher order calculation.

Now I have the two independent LECs: $\hat{d}$ and $\hat{e}$. I fix these two LECs so that the following two conditions are satisfied. The first condition is the same as that used in Sec. III for fixing $\hat{d}$. This condition is from the low-energy weak process. The second condition is due to $a_0$ extracted from the experimental data for the $pp \rightarrow pn\pi^+$ reaction\textsuperscript{11}. I choose the LECs so that the central value of the empirical amplitude, $a_0(\eta = 0.5) = -0.418 \text{ }\mu\text{b}^{1/2}$, is reproduced. As seen in the previous section, one cannot reproduce the $\pi$-production partial wave amplitude ($a_0$) by using $\hat{d}$ fitted to the amplitude of the low-energy weak process alone. The disagreement between $a_0$ and $\tilde{a}_0$ is rather serious; even the sign of $\hat{d}$ is inconsistent with the data in some cases. Therefore, it is not obviously expected that the addition of the $\hat{e}$ term with a natural strength brings $\tilde{a}_0$ into agreement with $a_0$. However, such a set of $\hat{d}$ and $\hat{e}$ does exist, as presented in Table III.

\textsuperscript{18} There exist $\mathcal{O}[(m_\pi/m_N)^{3/2}]$ operators coming from pion-loop diagrams. For a simple analysis, I use the $\mathcal{O}[(m_\pi/m_N)^2]$ counter term.
FIG. 8: The transition amplitude \( \tilde{a}_0 \) for \( pp \rightarrow pn\pi^+ \). The chiral NLO \( \pi \) production operator plus the counter term [Eq. (48)] and the CD-Bonn \( NN \) potential are used. The solid, dashed and dotted curves correspond to \( \Lambda = 800, 600 \) and 500 MeV, respectively, and \( h_A = 2.10 \). Experimental data are from Ref. [11].

I calculate \( \tilde{a}_0 \) with the parameter sets in Table III. The results for the CD-Bonn and the AV18 \( NN \) potentials are shown in Figs. 8 and 9 respectively. I show results obtained with \( h_A = 2.10 \) only. The dependence of \( \tilde{a}_0 \) on \( h_A \) is small in this case. By construction, the calculated \( \tilde{a}_0 \) goes through the central value of the data at \( \eta = 0.5 \). The \( \eta \) dependence of \( \tilde{a}_0 \) is consistent with the data for all \( NN \) potentials and for all \( \Lambda \). Particularly, all \( NN \) potentials give essentially the same \( \tilde{a}_0 \) for \( \Lambda = 500 \) MeV. From Table III we also observe that the couplings \( \hat{d} \) and \( \hat{e} \) for \( \Lambda = 500 \) MeV are similar for all the \( NN \) potentials, except for the \( N^3\text{LO} \) potential. In fact, this result has been expected; see footnote 15. For \( \Lambda = 800 \) MeV, a relatively large dependence of \( \tilde{a}_0 \) (two types of the behavior) on the \( NN \) potential is observed, within the consistency with the data. Regarding the CD-Bonn and the Nijmegen I potentials, \( \tilde{a}_0 \) (\( \Lambda = 800 \) MeV) is very similar to \( \tilde{a}_0 \) (\( \Lambda = 500 \) or 600 MeV); see Fig. 8. On the other hand, the other \( NN \) potentials give \( \tilde{a}_0 \) (\( \Lambda = 800 \) MeV) whose \( \eta \)-dependence is quite different from \( \tilde{a}_0 \) (\( \Lambda = 500 \) or 600 MeV); see Fig. 9. From a viewpoint of the renormalization group, the operators with \( \Lambda = 500, 600 \) and 800 MeV should be related by integrating out the high momentum states whose effects are simulated by the renormalization of \( \hat{d} \) and \( \hat{e} \). If this relation is realized, these operators should give the same observables, to the extent that the contact operators simulate the high energy modes integrated out. From this viewpoint, the results for the CD-Bonn (Fig. 8) and the Nijmegen I are understandable, while the others (Fig. 9) are not. However, the situation may change in a correct higher order calculation where the TPE is explicitly considered. This is because the model space
employed here probably has a resolution at which the contact operators cannot accurately simulate the intermediate-range mechanism such as TPE. If the TPE is explicitly considered, then the contact operators do not have to mimic the intermediate-range mechanism, and more accurately describe the shorter-range mechanism.

I look into the contact operators for different $NN$ potentials, and understand the similar (different) $\eta$-dependence of $\tilde{a}_0$ for $\Lambda = 500$ (800) MeV among different $NN$ potentials. As representatives, I show the contact operators for the CD-Bonn and the AV18 potentials in Figs. 10 and 11, respectively. We can find that the contact operators, including both $\hat{d}$ and $\hat{e}$, for $\Lambda = 500$ (800) MeV are very similar (different), leading to the quite similar (different) $\eta$ dependence of $\tilde{a}_0$. Starting with the operators for $\Lambda = 800$ MeV, one can reduce $\Lambda$ to examine the running of the operators using the Wilsonian RG equation. I refer the readers to Refs. [35, 36] for a detailed explanation of how I reduce $\Lambda$, and just show the result here. In Fig. 12, the RG running of the contact operator for the CD-Bonn potential is given. For a comparison, the contact operator for $\Lambda = 500$ MeV, whose LECs are given in Table III, are also shown. We can see that the contact operators ($\Lambda = 500$ MeV) derived in the different two ways, one from directly fitting to the data and the other from the RG running, are fairly similar. Although the figure shows the result for the diagonal components of the operators, the same level of the agreement is confirmed for off-diagonal components. For the AV18 potential, however, the two contact operators ($\Lambda = 500$ MeV) with the different origins are quite different as shown in Fig. 13. This result has been expected by observing Fig. 9 where the contact operators with $\Lambda = 500$ and 800 MeV do not give the same result, indicating that the two operators are not equivalent.

I compare the contact operator including both $\hat{d}$ and $\hat{e}$ (Table III), and those with only $\hat{d}$ (Table I) in Figs. 10 and 11. A naive expectation is that a contact operator with $\hat{d}$ only (a dashed curve) is an approximation of the one with $\hat{d}$ and $\hat{e}$ (the solid curve). In the two figures, this expectation is not always the case, and therefore we have to be careful about the convergence of the chiral expansion of the short distance physics. In any case, the $\hat{d}$ term alone cannot be a good approximation of the $\hat{d}$ plus $\hat{e}$ terms and, for that matter, not a good approximation of the operator to be simulated. It is recalled that this observation has been used in the previous subsection to argue over a reason for the dependence of $\tilde{a}_0$ on $h_A$ and the $NN$ potential. By including the $\hat{e}$ term, the situation is much improved in this point. We found the very small dependence of $\tilde{a}_0$ on the choice of $h_A$ and on the $NN$ potential for $\Lambda = 500$ MeV. Regarding the $\hat{d}$ and $\hat{e}$ values, as seen in Table III, they hardly depend on $h_A$. This is quite consistent with the discussion given in footnote 14.
VI. SUMMARY

I determined the LEC $\hat{d}$ using the low-energy weak process and then used it to predict the partial wave amplitude, $a_0 \left( ^1S_0 \rightarrow ^3S_1 \right)$, for the $pp \rightarrow pn\pi^+$ reaction. Through this work, I tried to explore the power of $\chi$PT that enables one to bridge different reactions. My investigation is more stringent test of this aspect of $\chi$PT than similar analyses in the literature because the reactions connected through $\chi$PT here take place under significantly different kinematics.

I started with the chiral Lagrangian including the nucleon, pion and $\Delta$. It is mandatory to include the $\Delta$ explicitly for describing the $p$-wave $\pi$ production. With the Lagrangian, I constructed, up to NLO of the chiral expansion proposed in Ref. [5, 13], a set of operators which describes the Gamow-Teller transition in low-energy weak processes and the $p$-wave $\pi$-production. I fixed the remaining unknown LEC $\hat{d}$ (indirectly) using the experimental data of the low-energy weak process. Then I calculated the partial wave amplitudes ($\tilde{a}$) for the $pp \rightarrow pn\pi^+$ reaction. I chose the phase convention such that $\tilde{a}_2 \left( ^1D_2 \rightarrow ^3S_1 \right)$ has the same phase as $a_2$ extracted from the data in Ref. [11]. My prediction of $\tilde{a}_0$ using the NLO operator does not agree with the data. I used several different sets of the inputs such as the $NN$ potential, $h_A$ and $\Lambda$. Even though there is some dependence of $\tilde{a}_0$ on these inputs, all results differ from the data in the similar manner. Unfortunately, even the sign of $\hat{d}$, fixed by the weak process, is sometimes not consistent with the data. This result indicates that the
FIG. 12: RG running of contact operators for the CD-Bonn potential. The solid curves are the same as those in Fig. 10. (The curve for $\Lambda = 600$ MeV is not shown.) The dashed and dotted curves are derived from the solid curve for $\Lambda = 800$ MeV by solving the RG equation. The dashed (dotted) curve is obtained with the kinematics for the $pp \rightarrow d\pi^+$ ($pp \rightarrow d\pi^+\nu\bar{\nu}$) reaction.

bridging program between the two reactions with quite different kinematics is not always successful. In the literature, we have sometimes found an argument which supposes that the bridging program works. Given the result here, it is clear that we need to seriously study a feasibility of the bridging program, particularly for reactions with different kinematics. This conclusion may be disappointing, but still understandable if we recall the success of the chiral nuclear force. The chiral nuclear force accurately describes the $NN$ scattering over a wide energy region, partly because the LECs included have been fixed using data from the same energy region. In fact, there are several reasons to expect a higher order calculation to significantly improve the situation. First of all, one may naively think that the NLO, tree level, operator is too simple to describe the $\pi$ production. We know that two-pion-exchange mechanism and higher order contact terms are necessary for accurately describing the $NN$ elastic scattering near the $\pi$ production threshold. Second, the $\hat{d}$ term largely contributes to $\tilde{a}_0$, and therefore it would be important to extract the TPE mechanism from the $\hat{d}$ term, thereby describing the intermediate-range mechanism more elaborately and reducing the role of the $\hat{d}$ term. Third, $\tilde{a}_0$ is considerably dependent on the choice of the $NN$ potential, which means that the single contact term is too simple to cancel out different short distance physics for different $NN$ potentials. I also argued that a substantial part of the $NN$-dependence (and $h_A$-dependence) of $\tilde{a}_0$ is likely to be ascribable to the fact that

FIG. 13: RG running of contact operators for the AV18 potential. The solid curves are the same as those in Fig. 11. The other features are the same as Fig. 12.
the $\hat{d}$ term alone is too simple to simulate the operator to be simulated. Meanwhile, a higher order calculation is also desirable to see the convergence of the chiral expansion; regarding the $pp \to pn\pi^+$ reaction, we found no sign of the convergence up to this order.

In order to explore, even roughly, a result of a higher order calculation, I added a higher order counter term, with the LEC $\hat{e}$, to the NLO operator. I fitted the LECs $\hat{d}$ and $\hat{e}$ to both the Gamow-Teller amplitude for the low-energy weak process and $\tilde{a}_0$ for the $\pi$ production. I found a set of the LECs with the natural strength. The LECs are mostly independent of the choice of $h_A$ as they should; without the $\hat{e}$ term, however, $\hat{d}$ is rather dependent on $h_A$. With this parameter set, $\eta$-dependence of $\tilde{a}_0$ is described in a way consistent with experimental data, irrespective of the choices of the $NN$ potential, $h_A$ and $\Lambda$. This result would be an indication that a higher order calculation is promising. I found that the single $\hat{d}$ term is not always a good approximation of the $\hat{d}$ plus $\hat{e}$ terms. I also showed using the RG analysis that the contact operators with different cutoff are not always equivalent. These findings tell us to be careful about the convergence of the chiral expansion, and also the importance of going to a higher order calculation.

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APPENDIX A: MULTIPOLAR EXPANSION OF $O_{\Delta\pi}$ OPERATOR

I present our calculational procedure for the $O_{\Delta\pi}$ operator [Eq. (6)]. I start with the Fourier transform of $O_{\Delta\pi}$:

$$
\int \frac{dk^3}{(2\pi)^3} e^{-ik \cdot r} \frac{k \cdot \sigma_2 \cdot k}{m_\pi^2 + k^2} \frac{f(k)}{m_\Delta - m_N - p_0^2/m_N + (p' + q_{\pi}/2)^2/2\mu},
$$

(A1)

where I only consider one term in $O_{\Delta\pi}$; the other terms and a constant factor are omitted. The function $f(k)$ is a cutoff function: $f(k) = \exp(-k^2/\Lambda_G^2)$ with $\Lambda_G = 2$ GeV. In this equation, I expand the energy denominator as follows:

$$
D(p', q_{\pi}) = \frac{1}{m_\Delta - m_N - p_0^2/m_N + (p' + q_{\pi}/2)^2/2\mu} = 4\pi \sum_\ell (-1)^\ell \sqrt{2\ell + 1} [Y_\ell(p') \otimes Y_\ell(q_{\pi})]_{(0)}^0 D_\ell(p', q_{\pi}),
$$

(A2)
where $D_\ell(p', q_\pi)$ is defined by

\[
D_\ell(p', q_\pi) = \left( \frac{p' q_\pi}{2\mu} \right) Q_\ell(\beta) ,
\]

(A3)

with

\[
\beta \equiv \frac{m_\Delta - m_N - p^2_\pi/m_N + (p^2 + q^2_\pi/4)/2\mu}{p' q_\pi/2\mu} ,
\]

(A4)

and $z \equiv \hat{p}' \cdot \hat{q}_\pi$. The function $Q_\ell(\beta)$ is the Legendre function of the second kind, and is given by

\[
Q_\ell(\beta) = \frac{1}{2} \int_{-1}^{1} P_\ell(z) \frac{dz}{\beta - z} .
\]

(A5)

I write Eq. (A1) using the expanded form Eq. (A2). When I retain only the first term ($\ell = 0$) of the expansion, I obtain

\[
\int \frac{dk^3}{(2\pi)^3} e^{-i\bm{k} \cdot \bm{r}} \frac{\bm{k} \cdot \sigma_2 \cdot \bm{k}}{m^2_\pi + k^2} \frac{2\mu}{p' q_\pi} Q_0(\beta) f(k) = \left( \frac{1}{3} \sigma_2 F_0(r) + \frac{\sqrt{8\pi}}{3} [Y_2(\hat{r}) \otimes \sigma_2]_{(1)} F_2(r) \right) \frac{2\mu}{p' q_\pi} Q_0(\beta) ,
\]

(A6)

with

\[
F_0(r) = \frac{1}{2\pi^2} \int_0^\infty dk \frac{k^4}{k^2 + m^2_\pi} j_0(kr) f(k) ,
\]

\[
F_2(r) = \frac{1}{2\pi^2} \int_0^\infty dk \frac{k^4}{k^2 + m^2_\pi} j_2(kr) f(k) .
\]

(A7)

I take a matrix element of Eq. (A1) after setting $q_\pi = 0$, and compared it with the matrix element of Eq. (A6) in which $q_\pi (\neq 0)$ is fixed by the kinematics. In the kinematical region of interest, I found a small correction ($\sim 1.5\%$). The use of Eq. (A6) may be regarded as an inclusion of the lowest order correction [$O(q^2_\pi)$] from finite $q_\pi$, even though there is still another $O(q^2_\pi)$ correction. I do not consider the higher order ($\ell \geq 1$) terms in Eq. (A2) to be influential on our result because: the $O(q^2_\pi)$ correction from Eq. (A6) is small; the expansion in Eq. (A2) may be regarded as an expansion in terms of $z/\beta$, and $z/\beta \ll 1$ in most of the kinematical region of interest. I use Eq. (A6) in our calculation.

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