Macroscopic Einstein equations for a system of interacting particles and their cosmological applications.

A. V. Zakharov

Dept. Gen. Rel. & Grav.
Kremljevskaja Str. 18
Kazan Stata University
420008 Kazan, Russia
tel.: (+7)(8432)315586
e-mail: Alexei.Zakharov@ksu.ru

SUMMARY

One of the possible applications of macroscopic Einstein equations has been considered. So, the nonsingular isotropic and uniform cosmological model is built. The cosmological consequences of this model are agree with conclusions of standard hot model of the Universe.
Macroscopic Einstein equations for a system of interacting particles and their cosmological applications.

A. V. Zakharov

1. Macroscopic system of Einstein equations

As it is known the macroscopic Maxwell equation for continuous media can be obtained also from the microscopic Maxwell equations by ensemble averaging the latter (refer to.[1]-[3]).

The Einstein equations, which right-hand sides contain the energy-momentum tensor of matter, are phenomenological equations. It is natural to suppose that the Einstein equations (or their generalizations) for continuous media can be also obtained from the microscopic Einstein equations, i.e., Einstein equations which right-hand side contain the sum of the energy-momentum tensors of individual particles. However, due to the nonlinearity of the Einstein equations left-hand side, the averaging of the microscopic Einstein equations is more complicated than one of the microscopic Maxwell equations (refer to. [4] - [6]).

In Ref. [7] - [9] were obtained the macroscopic Einstein equations for a system of interaction particles up to second order in the interaction constant. They have the forms:

\[ G_{ij} + Z_{ij} = \chi T_{ij}, \]  

(1)

Here \( G_{ij} \) is the Einstein’s tensor of the Riemannian space with macroscopic metric \( g_{ij} \), \( T_{ij} \) is the macroscopic energy-momentum tensor, \( \chi = 8\pi k/c^4 \) is Einstein constant, \( k \) is the gravitational constant, \( c \) is the velocity of light.

The Einstein equations of the gravitational field for continuum media differ from the classical Einstein equations by the presence of additional tensor \( Z_{ij} \) in the left-hand side. It caused by particle interaction.

The form of this tensor is given in Ref.[8] in the case of gravitationally interaction of particles and in Ref.[9] in the case of electromagnetically interaction of particles.

The tensor \( Z_{ij} \) is the traceless tensors with zero divergence:

\[ g^{ij} Z_{ij} = 0, \quad Z_{ij}^{ij} = 0. \]  

(2)

where the semicolon denote a covariant derivative in a space with the macroscopic metric \( g_{ij} \).

In the case of gravitationally interacting particles the tensor \( Z_{ij} \) has the form (Ref. [8]):

\[ Z_{ij} = \phi_{ij;k}^k + \mu_{ij}, \]  

(3)
\[ \varphi_{ij} = -\frac{\chi^b \chi^c}{8(2\pi)^3} \int \frac{d^3p'}{p'^0 \sqrt{(-g)}} \int \frac{d^3p''}{p''^0 \sqrt{(-g)}} \left[ \frac{1}{2} g^f_{ij} u''_i u''_j + u'^k (u' u'') (\delta_i^f u''_i + \delta_j^f u''_j) \right] (u' u'')^2 - \frac{1}{2} \]

\[ K_{f\alpha}(u', u'') \left( F^\alpha_c \frac{\partial F^{\prime\prime}_{b}}{\partial p'_{\alpha}} - F^\alpha_b \frac{\partial F^{\prime\prime}_{c}}{\partial p'_{\alpha}} \right), \quad (4) \]

\[ \mu_{ij} = -\frac{\chi^b \chi^c}{8(2\pi)^3} \int \frac{d^3p'}{p'^0 \sqrt{(-g)}} \int \frac{d^3p''}{p''^0 \sqrt{(-g)}} \left\{ \left[ \left( z^2 + \frac{1}{2} \right) (u''_i u''_j + u'_i u'_j) + (z^2 - \frac{1}{2}) \delta_i^f \delta_j^f \right] g^{\prime\prime} - 2(z^2 - \frac{1}{2}) \delta_i^f \delta_j^f \right\} \]

\[ F''_c \frac{\partial}{\partial p''_{f}} \left\{ F'_b \left[ (z^2 - \frac{1}{2}) \delta_{j}^{m} f + (z^2 + \frac{1}{2}) u''_i u''_j - 2 z u''_i u''_j \right] \right\} J_{\lambda \mu \rho} (u', u''). \quad (5) \]

Here \( p'' \) is the momentum of particles of species \( b \), \( p'' \) is the momentum of particles of species \( c \) \( u''_i = p''_i / m_b c \), \( u''_i = p''_i / m_c c \), \( F'_b \) is the one - particle distribution function of particles of species \( b \), \( F''_c \) is the one - particle distribution function of species \( c \), \( z = u''_i u''_j \), \( m_b \) and \( m_c \) are the mass of particles of species \( b \) and \( c \) respectively.

\[ \frac{d^3p'}{p'^0 \sqrt{(-g)}} \quad \text{and} \quad \frac{d^3p''}{p''^0 \sqrt{(-g)}} \]

are the invariant volume elements in tree - dimensional momentum space of particles species "b" and "c" respectively.

The greek index \( \alpha \) in (4) takes the values 1,2 and 3 only (the spartial index). The derivative with respect to \( p'_{\alpha} \) in (5) should be calculated as all four components of momentum are independent. The dependence of \( p'_0 \) on \( p'_{\alpha} \) is taken into account after differentiation with respect \( p'_{\alpha} \) is completed.

The tensors \( K_{ij}(u', u'') \) and \( J_{ijk}(u', u'') \) have the form:

\[ K_{ij}(u', u'') = \frac{4 \pi^2}{k_{\min}^2 [(u' u'')]^2 - 1} \{ - [(u' u'')]^2 - 1 \} g_{ij} - u''_i u''_j + u''_i u''_j + (u' u'')(u'_i u'_j + u''_i u''_j) \}

\[ J_{ijk}(u', u'') = A \left[ (g_{ij} u''_k + g_{ik} u''_j + g_{jk} u''_i) - z (g_{ij} u''_k + g_{ik} u''_j + g_{jk} u''_i) - (u''_i u''_k + u''_i u''_k + u''_i u''_k) + 3 z u''_i u''_k u''_k \right] + C \left[ u''_i u''_ju''_k - z (u''_i u''_j u''_k + u''_i u''_j u''_k + u''_i u''_j u''_k) + z^2 (u''_i u''_j u''_k + u''_i u''_j u''_k + u''_i u''_j u''_k) - z u''_i u''_j u''_k \right] \]

\[ (6) \]

\[ (7) \]
where
\[ A = - \frac{2\pi \sqrt{2}}{k_{\text{min}}} \left[ \frac{(z-2)}{(z-1)^2(z+1)^{1/2}} + \frac{(2z-1)}{(z+1)(z-1)^{5/2}} \ln\left( z + \sqrt{z^2-1} \right) \right] \]  
(8)

\[ C = - \frac{2\pi \sqrt{2}}{k_{\text{min}}} \left[ \frac{(z-6)}{(z-1)^3(z+1)^{3/2}} + \frac{(6z-1)}{(z+1)^2(z-1)^{7/2}} \ln\left( z + \sqrt{z^2-1} \right) \right] \]  
(9)

In these expressions
\[ k_{\text{min}} = \frac{1}{r_{\text{max}}} \]
where \( r_{\text{max}} \) is the size of the correlation region in the case of gravitationally interacting particles. In Refs. [10], [11] there are estimates for \( r_{\text{max}} \) in the case where the average metric \( g_{ij} \) is the metric of isotropic cosmological model.

2. Applications of the theory. Nonsingular isotropic and uniform cosmological models in macroscopic theory of gravity.

Let’s consider received equations for the ambience, base at local thermodynamic balance. In this cause the distribution function of each sorts of particles has the type (refer to. [12]):

\[ F_a(p_\alpha) = A_a \exp\left[ -c(v_i p^i)/(k_B T) \right] \]  
(10)

Here \( v_i \) is the macroscopic four-velocities of ambience, \( k_B \) is the Boltzman’s constant, \( T \) is the temperature, \( A_a \) is the normalizing constant.

In this case tensor \( \varphi^k_{ij} \) becomes zero, but tensor \( \mu_{ij} \) has the form:

\[ \mu_{ij} = \chi \epsilon_1 \left( \frac{4}{3} v_i v_j - \frac{1}{3} g_{ij} \right) \]  
(11)

This is correct both for gravitational and electromagnetic interactions.

If move \( \mu_{ij} \) from left-hand side to the right-hand side, the macroscopic equation effectively change into usual Einstein equations with the additional tensor of energy-momentum in the right-hand side. Moreover the ”additional tensor of energy-momentum” has the form of energy-momentum tensor of ideal liquid with the state equation \( P_1 = \epsilon_1/3 \), but with negative ”density of energy” \( (-\epsilon_1) \) and negative ”pressure” \( (-P_1) \). Via \( \epsilon_1 \) and \( P_1 \) their absolute values are marked.

In the case of gravitational interactions in nonrelativistic limit when \( mc^2 >> k_B T \), the absolute value of this ”negative density of energy” is

\[ \epsilon_1 = \sum_{ab} \frac{4k^2 r_{\text{max}}}{k_B T} m_a^2 m_b^2 N_a N_b \]  
(12)

where \( N_a \) and \( N_b \) is the number of particles of species \( a \) and \( b \) repectively densities, \( m_a \) and \( m_b \) are their masses.
It is proportional to the square of gravitational constant and square of number of particles density. Consequently the additional terms can play significant role in ambiences of sufficiently high density. Such density is possible on the early stages of the Universe evolution. So, naturally, one can use the first exhibits of the received equations in the theories of the early stages of the Universe evolution.

Thereby there is a real possibility of manifestation an interaction on early stages of the Universe evolution.

Let’s turn to the building of uniform and isotropic cosmological models within the framework of macroscopic theory of gravity.

Let’s write the metricses of these models in a form:

\[
(ds)^2 = a^2(\eta) \left( (d\eta)^2 - (dr)^2 - \phi^2(r)((d\theta)^2 + \sin^2(\theta) (d\varphi)^2) \right).
\]

(13)

Here \( \phi(r) = r \), \( \phi(r) = \sin r \), \( \phi(r) = \sinh r \) for the flat, close and open models respectively.

The system of Einstein equations for these metricses is reduced to two equations (refer, for instance, [13]):

\[
a'^2 + \xi a^2 = \frac{8\pi k}{3c^4} a^4 \tilde{\epsilon},
\]

(14)

\[
\frac{d\tilde{\epsilon}}{\tilde{\epsilon} + \tilde{P}} = -3\frac{da}{a}.
\]

(15)

Here stroke under \( a \) marks derivative on time variable \( \eta \); \( \tilde{\epsilon} \) and \( \tilde{P} \) is a density of energy and pressure of matter, \( \xi = 0, +1, -1 \) for the flat, closed and open models respectively.

The macroscopic Einstein equations are distinguish from (14) - (13) by following. In the first, \( \tilde{\epsilon} \) in the right-hand side of (14) is replased by the difference of usual energy density \( \epsilon \) and absolute value "density of energy" \( \epsilon_1 \), stipulated by the particles interaction ofambience.

At the present moment of the Universe evolution the expression (12) for \( \epsilon_1 \) is correct.

In the second, the equation (13) is changed by two similar:

\[
\frac{d\epsilon}{\epsilon + P} = -3\frac{da}{a},
\]

(16)

\[
\frac{d\epsilon_1}{\epsilon_1 + P_1} = -3\frac{da}{a}.
\]

(17)

Here \( P \) is a usual pressure of matter, \( P_1 = \epsilon_1/3 \) is the "pressure", stipulated by the interaction of particles.

Equations (16) and (17) follows from the fact, that divergency of the usual energy - momentum tensor and divergency of the "additional energy - momentum tensor", stipulated by the interaction, equals to zero.
From (12) it follow that in present the value $\epsilon_1$ and the relict radiation energy density are of the same order of value (see to below). So, if we take into account a contribution $\epsilon_1$ to $\tilde{\epsilon}$, we must take into account a contribution of the relict radiation energy density to $\tilde{\epsilon}$ density.

So, let’s put in the right part of (14)

$$\tilde{\epsilon} = \epsilon_m - \tilde{\epsilon}.$$  \hspace{1cm} (18)

Here $\epsilon_m$ is the density of energy of material disregarding density of energy of relict radiation, but under $\tilde{\epsilon}$ we shall understand a difference between the absolute value ”density of energy”, stipulated by the interaction, and density of energy of relict radiation. This is suitable, since equations of condition for relict radiation and for ”energy - momentum tensor”, stipulated by the interaction alike: pressure is one third from density of energy. So that the equation (16) and (17) caused from the following $\tilde{\epsilon}$ dependency on the scale factor:

$$\tilde{\epsilon} = \frac{3c^4a_1^2}{8\pi ka^4}.$$ \hspace{1cm} (19)

Here $a_1 = const$.

For density of energy of rest material from (16) we have:

$$\epsilon_m = \frac{3c^4a_0}{4\pi ka^2},$$ \hspace{1cm} (20)

where $a_0 = const$.

We substitute (18), (19) and (20) into (14). As a result we obtain the equation:

$$a^2 + \xi a^2 = 2a_0a - a_1^2.$$ \hspace{1cm} (21)

The solution of this equation for the scale factor $a(\eta)$ one can write in the form:

$$a = a_m + \frac{1}{2}a_0\eta^2$$ \hspace{1cm} (22)

for flat ($\xi = 0$) models ( here $a_m = a_1^2/2a_0$),

$$a = a_m + (a_0 - a_m)(1 - \cos \eta)$$ \hspace{1cm} (23)

for closed ($\xi = +1$) models (here $a_m = a_0 - \sqrt{(a_0^2 - a_1^2)}$), and

$$a = a_m + (a_0 + a_m)(\cosh \eta - 1)$$ \hspace{1cm} (24)

for open ($\xi = -1$) models (here $a_m = \sqrt{(a_0^2 + a_1^2) - a_0}$).
As it is seen, all three models are nonsingular: under $\eta = 0$ the scale factor $a(\eta)$ does not apply to the zero as in standard cosmological models, but takes a minimum value $a_m$.

Let us calculate a value of density $\rho_m$ of nonrelativistic matter at the moment, when scale factor takes a minimum value $a_m$:

$$\rho_m = \rho_0 \frac{a^3(\eta_0)}{a_m^3}. \quad (25)$$

Here $\rho_0$ is the value of usual matter density at present moment.

This moment corresponds a time coordinate value $\eta_0$. Substituting (22) — (24) into (25) we obtain the expression for $\rho_m$ via $\rho_0$, $\eta_0$ and ratio $a_0/a_m$.

Let us write down the expressions for $\Omega = (\epsilon_m - \bar{\epsilon})_{\eta=\eta_0}/(c^2 \rho)$, and $\lambda = (\bar{\epsilon}/\epsilon_m)_{\eta=\eta_0}$ via $\eta_0$ and ratio $a_0/a_m$.

Here $\rho = 3H_0^2/8\pi k$ is the critical density at the present moment, $\Omega$ is the densities parameter, $\lambda$ is a ratio of ”energy density”, stipulated by the interaction absolute value, and density of relict radiation energy difference to the density of usual matter energy at the present moment.

From (14), (18) - (20) we obtain

$$\Omega = \left(1 + \xi a^2/a^2\right)_{\eta=\eta_0} \quad \lambda = \frac{a^2}{2a_0a(\eta_0)} \quad (26)$$

Substituting (22) — (24) into (26) we obtain the expression for $\Omega$ and $\lambda$ via, $\eta_0$ and ratio $a_0/a_m$.

Expressing from received correlations $\eta_0$ and $a_0/a_m$ via $\Omega$ and $\lambda$ and then substituting then received expressions for $\eta_0$ and $a_0/a_m$ in (25), we obtain the following result:

$$\rho_m = \frac{\rho_0}{\lambda^3} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda(1 - \lambda)\left(\frac{\Omega - 1}{\Omega}\right)}\right)^3. \quad (27)$$

Result (27) is correct for flat ($\Omega = 1$), open ($\Omega < 1$), and closed ($\Omega > 1$) models.

The received models can be realized, if absolute value of ”energy density” (12), stipulated by the interaction, exceeds at present moment the energy density of relict radiation.

The ”energy density” (12), stipulated by interaction, is created basically galaxies accumulations with the mass $m \sim 10^{15} - 10^{16} M_\odot$. This ”energy density” is estimated as

$$\frac{12k^2t_0\rho_0^2m}{<v>}. $$

When getting this correlation we put $3k_BT = m < v >^2$, where $<v>$ is an average chaotic velocity of galaxies accumulations, $N = \rho_0/m$, but parameter $r_{max}$ is estimated as $<v > t_0$, where $t_0$ is the modern age of the Universe (refer to. [10], [11]).
Substituting into the obtained ratio the numerical values of fundamental constants, cosmological time, average density of matter in the Universe at present moment (refer to, for instance, [14]), we obtain under $<v>/c = 10^{-6}$, $m \sim 10^{15}M_{\odot}$:

$$\epsilon_1 \sim 7 \cdot 10^{-13} \text{erg/cm}^3$$

It is approximately the value of relict radiation energy density we have at present moment (refer to [14]).

Consequently, realization of such situation is possible, when absolute value of "energy density", stipulated by interaction, exceeds the density of relict radiation energy. And it is the case considered in the paper when $\lambda > 0$.

On initial stage of the Universe evolution the density $\rho_m \sim \rho_0/\lambda^3$ is sufficiently great, so usual scenario of the Universe hot models preserves. For a moment when scale factor takes a minimum value and the density of matter is maximum let us introduce the following.

If at this moment a temperature does not exceed the temperatures of nucleon - antinucleons couple annihilation, the density of nonrelativistic matter can be evaluated as

$$\frac{m_p 10^{-8} \sigma T^4}{k_B T},$$

where $\sigma$ is the constant of Stefan - Boltzman, $m_p$ is the mass of proton.

Comparing this expression with $\rho_0/\lambda^3$ one can make a conclusion, that minimum value of scale factor will be reach on leptones stage of the Universe evolution when parameter $\lambda$ is within $10^{-10} - 10^{-12}$, and the temperature will be about $10^{10}K^o - 10^{12}K^o$.

3. Domain of the theory application.

When concluding the macroscopic Einstein equations we suggest the distant collision to be the prevalent. It is correct if

$$\gamma = \frac{E_p}{E_k} \ll 1,$$

where $E_p$ is the potential energy of two particles interaction, $E_k$ is the kinetic energy of particles.

For the system of electromagnetical interacting particles in a condition of local thermodynamic balance we can put $E_k \sim kT$, $E_p \sim e^2 n^{1/3}$.

So the condition of validity takes the form

$$\gamma = \frac{e^2 n^{1/3}}{kT} \ll 1. \quad (28)$$

Here $e, n, T, k$ is the charge, density, temperature and Boltzman constant correspondingly.
Let us verify (28) in constructing model under $t = t_m$, when the Universe reaches the maximum density. The majority of electromagnetical interacting particles on this stage compose the electron-positron pairs. It's density is compared on the order of value with the one of relict photons and can be estimated as $\sigma T^4/kT = \sigma T^3/k$, where $\sigma$ is the Stephan-Boltzman constant. Then for parameter $\gamma$ we have estimation:

$$\gamma = \frac{e^2 \sigma^{1/3}}{k^{4/3}} \sim 10^{-2} \ll 1.$$

Consequently the application condition of the macroscopic Einstein equations in this model is not violated.

References

[1] H. A. Lorentz. Versush einer Theorie der Elektrischen und Optischen Ercheinungen in Bewegten Korpern. Leiden, (1895)

[2] H. A. Lorentz. The Theory of Electrons. Leipzig: Teubner, (1916)

[3] Yu. L. Klimontovich, Kinetic Theory of Electromagnetic Processes [in Russian],Nauka, Moscow (1980)

[4] Shirokov M. F., I. Z. Fisher, Astron. Zh. 39, 899 (1962) [in Russian]

[5] G. F. R. Ellis. General Relativity and Gravitation, ed B. Bertotti, F. de Felici and a\' Pascolini. Dortrecht: Reidel. (1984)

[6] R. M. Zalaletdinov, Gen Relativ. Gravit. 25,673 (1993)

[7] A. V. Zakharov, Zh. Eksp. Teor. Fiz. 110, 3 - 10 (1996) [Sov. Phys. JETP 83, 1 - 8, (1996)].

[8] A. V. Zakharov, Zh. Eksp. Teor. Fiz. bf 112, 1153 - 1166 (1997) [Sov. Phys. JETP 85, 627 - 634 (1997)].

[9] A. V. Zakharov. Macroscopic Einstein equations taking into account the electromagnetic interaction of particles of medium. //In : Noveishie problemy teprii polya. Edit: prof. Aminova A.V., Kazan 1998 p.118-124 (in Russian)

[10] G. S. Bisnovatij - Kogan, I. G. Shukhman, Zh. Eksp. Teor. Fiz. 82, 3 (1982).

[11] A. V. Zakharov, Zh. Eksp. Teor. Fiz. 99, 769 - 782 (1989). [Sov. Phys. JETP, 69(3), 437 - 443, (1989)].
[12] N. A. Chernikov, Nauchn. docl. visch. shcol. Fiz. mat. N 1, 168 (1959).

[13] L. D. Landau, E. M. Lifshitz. Teorija polja. Izd-vo "Nauka" M., (1973). (in Russian)

[14] Ja. B. Zeldovich, I. D. Novikov. Stroenie i evoljutzija Vselelnoi.
    Izd-vo "Nauka" M., (1975).(in Russian)