Length of parallel curves

E. Macías-Virgós

Abstract

We prove that the length difference between a closed periodic curve and its parallel curve at a sufficiently small distance $\varepsilon$ equals $2\pi \varepsilon$ times the rotation index. As an application, the rotation index of a curve could be estimated by means of Cauchy-Crofton’s formula.

INTRODUCTION. The aim of this note is to prove the following result. Let $\alpha$ be a closed periodic regular curve, let $\beta$ be the parallel curve at distance $\varepsilon \geq 0$. Assume that $\varepsilon$ is small enough to not exceed the radius of curvature of $\alpha$ when $\kappa > 0$ ($\kappa$ is the signed curvature of $\alpha$). Let $\omega$ be the rotation index of $\alpha$.

Theorem 1 The length difference $L(\alpha) - L(\beta)$ equals $2\pi \varepsilon \omega$.

I think this result is new or at least is not well known in differential geometry. Although elementary, it seems interesting because it actually finds the exact difference and shows that a relatively sophisticated invariant like the rotation index can be determined by a much simpler invariant, namely, the length of a curve. Of course, the converse is also a useful observation: computing the length by the rotation index since this is a regular homotopy invariant [1, p.330]. As a corollary, the difference of the length of a curve and its $\varepsilon$-parallel curve is a regular homotopy invariant. Possibly this could also be directly proved by a variational argument.

BASIC DEFINITIONS AND NOTATIONS. Let $\alpha(t)$ be a differentiable plane curve, defined on the interval $[a, b]$. The length of the curve is given by

$$L(\alpha) = \int_a^b |\alpha'(t)| \, dt. \quad (1)$$

Suppose that the curve is regular, which means that the speed $|\alpha'|$ never vanishes. Then the arc-length parameter $s(t)$, defined by $ds = |\alpha'| dt$ and $s(a) = 0$, serves to reparametrize the curve with unit speed.

The (signed) curvature of $\alpha$ is the function

$$\kappa = \det(\alpha', \alpha'')/|\alpha'|^3. \quad (2)$$
If the parameter is arc-length, the absolute value of the curvature is $|\kappa| = |\ddot{\alpha}|$, the module of the second derivative.

When $\kappa \neq 0$, the unitary normal vector $\vec{n} = \ddot{\alpha}/|\ddot{\alpha}|$ is well defined. It is perpendicular to the tangent direction and it points inwards the curve. For an arbitrary parameter $t$, the vector $\alpha''$ is not collinear to $\ddot{\alpha}$, but both are on the same side of the tangent line.

**PARALLEL CURVES.** Let $\alpha(t)$ be an arbitrary regular parametrization. At each point $\alpha(t)$ we take a unitary vector $\vec{e}(t)$ orthogonal to $\alpha'(t)$ and such that $\det(\alpha', \vec{e}) > 0$. In other words, $\vec{e}$ is obtained by rotating in the counter-clockwise sense the unitary tangent vector $\vec{t} = \dot{\alpha} = \alpha'/|\alpha'|$ (in Alfred Gray’s book [3], $\vec{e}$ is denoted by $J\alpha'$).

Then $\vec{e} = +\vec{n}$ when $\kappa > 0$ (the curve turns left) and $\vec{e} = -\vec{n}$ when $\kappa < 0$ (the curve turns right).

**Definition 1** We define the (left) parallel curve to $\alpha$ at distance $\varepsilon \geq 0$ as the curve $\beta = \alpha + \varepsilon \vec{e}$.

**Remark:** It is unnecessary to consider the case $\varepsilon \leq 0$, as we can always reparametrize the curve $\alpha$ in the opposite direction.

We now discuss the regularity of $\beta$. For that we have to take into account the radius of curvature $1/|\kappa|$ and the evolute of $\alpha$, which is the geometric locus of the centers of curvature $\alpha + (1/|\kappa|) \vec{n} = \alpha + (1/\kappa) \vec{e}$.

By differentiating with respect to the arc-length parameter $s$ of $\alpha$, we obtain $\dot{\vec{e}} = -\kappa \vec{t}$, which is just a reformulation of the usual Frénet formula $\vec{n} = -|\kappa| \vec{t}$ [1] [2] [3]. Hence $d\beta/ds = (1 - \varepsilon \kappa) \dot{\alpha}$ and

$$|d\beta/ds| = |1 - \varepsilon \kappa|.$$  \hspace{1cm} (3)

It follows that the parallel $\beta$ has a singularity each time $\varepsilon$ equals $1/\kappa$. This can only occur (as we are taking $\varepsilon \geq 0$) when $\kappa > 0$ and $\varepsilon$ equals the radius of curvature, i.e. the parallel $\beta$ touches
the evolute of $\alpha$ at corresponding points (see figure 2).

Remark: The evolute itself has singularities at the places where the curvature attains a critical value; this is a consequence of the fact that the tangent vector to the evolute points in the normal direction to $\alpha$.

By applying definition (1) to formula (3) we obtain the length of $\beta$.

![Figure 2: The same curve, its evolute (dashed) and one parallel with two singularities](image)

**Theorem 2** The length of the left parallel curve $\beta$ at distance $\varepsilon \geq 0$ to $\alpha$ is given by

$$L(\beta) = \int_0^{L(\alpha)} |1 - \varepsilon \kappa(s)| \, ds.$$ 

In Corollary 3 we shall emphasize two particular cases of Theorem 2.

**Definition 2** The total curvature of the curve $\alpha$ is the number

$$K = \int_0^{L(\alpha)} \kappa(s) \, ds = \int_a^b \kappa(t)|\alpha'(t)| \, dt.$$ 

**Corollary 3**

1. If $\kappa \leq 1/\varepsilon$ then $L(\beta) = L(\alpha) - \varepsilon K$;

2. If $\kappa \geq 0$ and $\varepsilon \geq 1/\kappa$ then $L(\beta) = \varepsilon K - L(\alpha)$.

**Example:** Let $\alpha(t) = (R \cos t, R \sin t)$, $0 \leq t \leq \pi$, be a half-circle with a big radius $R > 0$. It has global curvature $K = \pi$. The parallel curve at distance $R + 1$ is a small half-circle of radius 1 which goes backwards. Its length is $(R + 1)\pi - \pi R = \pi$.

**Remark:** From (2) it follows that the curvature of the parallel curve $\beta = \alpha + \varepsilon \overline{e}$ is given by $\kappa_\beta = \kappa/|1 - \varepsilon \kappa|$, see [3, p. 117]. Then, when $\kappa < 1/\varepsilon$, $\beta$ has the same evolute that $\alpha$. 

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Let $\alpha(t)$ be a regular curve defined in $[a, b]$. From now on we shall assume that our curve is closed and periodic, i.e. it satisfies $\alpha(a) = \alpha(b)$ and $\alpha'(a) = \alpha'(b)$.

Let us recall the notion of rotation index (also called turning number). For simplicity, we parametrize $\alpha$ by the arc-length $s \in [0, L(\alpha)]$, so the tangent vector $\bar{\mathbf{t}} = \dot{\alpha}$ has module 1. Write $\dot{\alpha} = (\cos \theta, \sin \theta)$. Then

$$\kappa = \det(\dot{\alpha}, \ddot{\alpha}) = d\theta/ds,$$

which proves that it is always possible to choose the angle $\theta$ in a differentiable way (unique for any preassigned value of $\theta(0)$). Namely

$$\theta(s) = \theta(0) + \int_0^s \kappa. \quad (4)$$

Clearly $\theta$ does not depend on the parametrization of $\alpha$. Moreover, since the curve is periodic, the difference $\theta(b) - \theta(a)$ equals $2\pi \omega$, for some integer number $\omega$.

**Definition 3 ([3, p. 159])** The integer $\omega$ is called the rotation index of $\alpha$. It measures how many times the curve turns with respect to a fixed direction.

**Example:** The rotation index of the Pascal Snail in Figure 1 is $\omega = \pm 2$ depending on the sense of rotation.

The following result is immediate from (4).

**Proposition 4** The total curvature $K$ of a closed periodic curve with rotation index $\omega$ equals $2\pi \omega$.

Finally, if $\varepsilon$ is not too large, directly from Corollary [3] we obtain Theorem [1].

In addition, we have the following consequence of the Hopf theorem on turning tangents [1, p. 333].

**Corollary 5** For a simple closed curve in the plane (i.e., one without selfintersections), the length of the $\varepsilon$-parallel curve minus the length of the original curve is always $\pm 2\pi \varepsilon$, for $\varepsilon$ small enough.

**Remark:** The following is a very well-known fact, which seems quite striking to non-mathematicians. Imagine that we surround the earth by the equator with a cable at ground level. If we next wanted the cable to stand a metre above ground level, how much extra cable would we need? The answer is: a little more than 6 metres. The reason is: $2\pi(R + 1) - 2\pi R = 2\pi$. Of course this is a very particular case of our result.

**ESTIMATION OF THE ROTATION INDEX** An estimation of the rotation index of the closed curve $\alpha$ can be obtained by applying Cauchy-Crofton’s formula [4] in order to estimate the lengths of $\alpha$ and the offset curve $\beta$, then applying Theorem [1].
References

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Enrique Macías-Virgós
Institute of Mathematics
Department of Geometry and Topology
University of Santiago de Compostela
15782- SPAIN
xtquique@usc.es
http://www.usc.es/imat/quique