ANALYSIS OF THE OPTIMAL EXERCISE BOUNDARY OF
AMERICAN OPTIONS FOR JUMP DIFFUSIONS

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Abstract. In this paper we show that the optimal exercise boundary / free boundary of the American put option pricing problem for jump diffusions is continuously differentiable (except at the maturity). This differentiability result has been established by Yang et al. (European Journal of Applied Mathematics 17(1):95-127, 2006) in the case where the condition \( r \geq q + \lambda \int_{\mathbb{R}} (e^z - 1) \nu(dz) \) is satisfied. We extend the result to the case where the condition fails using a unified approach that treats both cases simultaneously. We also show that the boundary is infinitely differentiable under a regularity assumption on the jump distribution.

1. Introduction

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space hosting a Wiener process \( W = \{W_t; t \geq 0\} \) and a Poisson random measure \( N \) on \( \mathbb{R}_+ \times \mathbb{R} \) with the mean measure \( \lambda dt \nu(dz) \) (in which \( \nu \) is a probability measure on \( \mathbb{R} \)) independent of the Wiener process. Let \( \mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]} \) be the (augmented) natural filtration of \( W \) and \( N \). We will consider a Markov process \( S = \{S_t; t \geq 0\} \), which follows the dynamics
\[
dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t + S_t \int_{\mathbb{R}} (e^z - 1) N(dt, dz),
\]
as the stock price process. We will take \( \mu \triangleq r - q + \lambda - \lambda \xi \), in which
\[
\xi \triangleq \int_{\mathbb{R}} e^z \nu(dz) < \infty,
\]
as a standing assumption. We impose this condition on \( \xi \) so that the discounted stock prices are martingales. The constant \( r \geq 0 \) is the interest rate, \( q \geq 0 \) is the dividend. The volatility \( \sigma(S, t) \) is assumed to be continuously differentiable in both \( S \) and \( t \). Moreover, there are positive constants \( \delta \) and \( \Delta \) such that
\[
0 < \delta \leq \sigma(S, t) \leq \Delta, \quad \text{for all } S, t \geq 0.
\]
We should note that at the time of a jump the stock price moves from \( S_{t-} \) to \( S_{t-} e^Z \) in which \( Z \) is a random variable whose distribution is given by \( \nu \). When \( Z < 0 \) the stock price jumps down, when \( Z > 0 \) the stock price jumps up. In the classical Merton jump diffusion model, \( Z \) is a Gaussian random variable.

In this framework, we will study the American put option pricing problem. The value function of the American put option is defined by
\[
V(S, t) \triangleq \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}\{e^{-r\tau}(K - S_\tau)^+|S_0 = S\},
\]

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in which $S_{0,T-t}$ is the set of stopping times (with respect to the filtration $\mathbb{F}$) taking values in $[0,T-t]$. The value function $V$ is the classical solution of a free boundary problem (see Proposition 2.1). The main goal of this paper is to analyze the regularity of the free boundary. We will show that the free boundary is $C^1$ except at the maturity $T$, and $C^\infty$ with an appropriate regularity assumption on the jump distribution $\nu$. For notational simplicity we will first change variables and transform the value function $V$ into $u$ and its free boundary $s$ into $b$ (see (2.6)) and state our results in terms of $u$ and $b$.

While the continuity of the free boundary of the American put option in jump models has been studied extensively, for example, by Pham [1997], Yang et al. [2006] and Lamberton and Mikou [2008], the differentiability of the free boundary was left as an open problem. Even when the geometric Brownian motion is the underlying process the differentiability is difficult to establish (see the discussion on page 172 of Peskir [2005]) and has only recently been fully analyzed by Chen and Chadam [2006/07]. In the jump diffusion case, Yang et al. [2006] proved that the free boundary is continuously differentiable before the maturity when the parameters satisfy

\begin{equation}
(1.5) \quad r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1) \nu(dz).
\end{equation}

When the condition (1.5) is violated, the free boundary of the American option for jump diffusions exhibits a discontinuity at the maturity (see Theorem 5.3 in Yang et al. [2006] and equation (3.20) in this paper). This behavior of the free boundary was also observed by Levendorskiï [2004] and Lamberton and Mikou [2008] in the exponential Lévy models. The purpose of our paper is to extend the regularity results of the free boundary to the case where (1.5) is not satisfied. We will see that the boundary is differentiable even when (1.5) is violated.

There are two critical points in showing the differentiability properties without the condition (1.5): 1) to show the Hölder continuity of the free boundary, 2) to show that $\partial^2 V(S,t)$ is strictly larger than 0 when the point $(S,t)$ is close to the free boundary in the continuation region. We achieve these two results in Theorem 3.1 and Corollary 3.1 respectively. Combining these two properties and a generalization of the result in Cannon et al. [1974] (see Lemma 4.1), we upgrade the regularity of the free boundary from Hölder continuity to continuous differentiability in Theorem 4.1. Then we analyze the higher order regularity of the free boundary making use of a technique Schaeffer [1976] used for the free boundary of a one dimensional Stefan problem on a bounded domain.

In order to show that the free boundary is continuously differentiable, it is essential that the value function $V(S,t)$ is the unique classical solution of the free boundary problem and has a continuous second derivative (see (4.5)). In the jump diffusion models, this has been shown by Pham [1997] under condition (1.5). This condition was removed in Yang et al. [2006] and also in Bayraktar [2008]. Moreover, continuous differentiability of the free boundary requires the continuity of the cross derivatives of the value function. In the Lévy models with infinite activity jumps, the value function is not expected to be a classical solution in general. Yet in the literature different notions of generalized solutions were explored. For example, Pham [1998] showed that the value function is a viscosity solution, Achdou [2008] showed that the value function is the solution in the Sobolev sense and Lamberton and Mikou [2008] proved that the value function is the solution in the distribution sense. Moreover, the smooth-fit property (see (2.4)) is also necessary in our analysis (see Theorem 4.1 and equation (5.1)). While this property may not hold for general pay-off functions (see Peskir [2007]), it has been shown to hold for the put option pay-off in Zhang [1997], Pham [1997] and Bayraktar [2008] in the jump diffusion models. The analysis in this paper also applies to the pay-off functions which are continuously differentiable, bounded, convex on $[0, +\infty)$ and equal to zero in $[K, +\infty)$. In fact, the singularity at the strike of the put option pay-off is the source of the technical difficulties. Therefore, we will focus on the put option pay-off in this paper and leave the investigation of the boundary behavior for general pay-off functions to future work.
The rest of the paper is organized as follows: In Section 2, after changing variables we will collect several useful properties of the function $u$, which will be crucial in establishing our main results in the next three sections about the regularity of its free boundary. In Section 3, we will introduce an auxiliary function and use it to show that the free boundary is Hölder continuous. In Section 4, we will prove the continuous differentiability of the free boundary. In Section 5, we will upgrade the regularity of the boundary curve and show that it is infinitely differentiable under an appropriate regularity assumption on the jump distribution. Finally, in Section 6, we will show that the approximation free boundary, constructed in Bayraktar [2008], have the similar regular properties with the original free boundary. Proofs of some auxiliary results are presented in the Appendix.

Our main results are Theorems 3.1, 4.1 and 5.1. In Figure 1 we show the logical flow of the paper, i.e. we show how several results proved in the paper are related to each other.

2. Properties of the value function

The value function $V(S,t)$ of the American put option for jump diffusions solves a free boundary problem with the free boundary $s(t)$. In particular, Theorem 4.2 of Yang et al. [2006] and Theorem 3.1 of Bayraktar [2008] state the following:

**Proposition 2.1.** $V(S,t)$ is the unique classical solution of the following boundary value problem:

$$
\begin{align}
\frac{\partial V}{\partial t} + & \frac{\bar{\sigma}(S,t)^2 S^2 \frac{\partial^2 V}{\partial S^2}}{2} + \mu S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda \int_R V(Se^z,t)\nu(dz) = 0, \quad S > s(t), \\
V(s(t),t) &= K - s(t), \quad t \in [0,T), \\
V(S,T) &= (K - S)^+, \quad S \geq s(T).
\end{align}
$$

Moreover, the smooth fit property is satisfied, i.e.

$$
\frac{\partial}{\partial S} V(s(t),t) = -1, \quad t \in [0,T).
$$
In the region \( \{(S, t) : S < s(t), t \in [0, T]\} \), \( V(S, t) \) also satisfies the following inequality:
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}(S, t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda \int_{\mathbb{R}} V(Se^z, t)\nu(dz) \leq 0.
\]

In the following, let us first change the variables to state (2.1) - (2.5) in a more convenient form:
\[
x = \log(S), \quad u(x, t) = V(S, T - t), \quad b(t) = \log(s(T - t)) \quad \text{and} \quad \sigma(x, t) = \tilde{\sigma}(S, t).
\]
It is clear from the assumptions of \( \tilde{\sigma}(S, t) \) that
\[
\sigma \text{ is continuously differentiable in both variables and}
\]
there are positive constants \( \delta \) and \( \Delta \) such that \( 0 < \delta < \sigma(x, t) < \Delta \) for all \((x, t) \in \mathbb{R} \times [0, T]\).

While the first part of (2.7) will be used in (4.3) and Lemma 4.1, the second part, which makes sure that the differential operators involved are uniformly parabolic, will be necessary for Lemma 2.3, Corollary 5.1 and Theorem 5.1. For the simplicity of the notation, we will omit the variables of \( \sigma \) in the sequel. In terms of the new variables introduced in (2.6), (2.1) - (2.5) reduce to the uniformly parabolic boundary value problem
\[
\mathcal{L}u \triangleq \frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} - \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{\partial u}{\partial x} + (r + \lambda)u - \lambda \int_{\mathbb{R}} u(x + z, t)\nu(dz) = 0, \quad x > b(t),
\]
\[
u(b(t), t) = K - e^{b(t)}, \quad t \in (0, T],
\]
\[
u(x, 0) = (K - e^x)^+, \quad x \geq b(0),
\]
\[
\frac{\partial}{\partial x}u(b(t), t) = -e^{b(t)},
\]
\[
\mathcal{L}u(x, t) \geq 0, \quad x < b(t), t \in (0, T].
\]

Let us define the continuation region \( \mathcal{C} \) and the stopping region \( \mathcal{D} \) as follows
\[
\mathcal{C} \triangleq \{ (x, t) | b(t) < x < +\infty, 0 < t \leq T \}, \quad \mathcal{D} \triangleq \{ (x, t) | -\infty < x \leq b(t), 0 < t \leq T \}.
\]

From Proposition 2.1, it is clear that the boundary value problem (2.8) - (2.10) has a unique classical solution \( u(x, t) \) in \( \mathcal{C} \).

**Remark 2.1.** The integral term in (2.8) can also be considered as a driving term, then the integro-differential equation (2.8) can be viewed as the following parabolic differential equation with a driving term \( f(x, t) = \lambda \int_{\mathbb{R}} u(x + z, t)\nu(dz) \):
\[
\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} - \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{\partial u}{\partial x} + (r + \lambda)u = f(x, t).
\]

This point of view will be useful in the proof of some results in later sections.

In this section, we will study the properties of \( u \) in both the continuation and the stopping regions. Let us start from the following proposition from Yang et al. [2006]. It shows that the time derivative of \( u \) is continuously differentiable across the free boundary.

**Proposition 2.2.** \( \partial_t u(x, t) \) is a continuous function in \( \mathbb{R} \times (0, T] \). In particular, for any \( t \in (0, T] \),
\[
\lim_{x \downarrow b(t)} \frac{\partial}{\partial t} u(x, t) = 0.
\]

**Proof.** The proof is given in Theorem 5.1 in Yang et al. [2006], which summarized Lemmas 2.8 and 2.11 in the same paper and used a special case of Lemma 4.1 in page 239 of Friedman [1976].

Moreover, we will show in the following that \( t \to u(x, t) \) is strictly increasing function in the continuation region.
Proposition 2.3.

\begin{equation}
\frac{\partial u}{\partial t}(x, t) > 0, \quad (x, t) \in \mathcal{C}.
\end{equation}

Proof. The inequality (2.15) is proved in Proposition 4.1 in Yang et al. [2006] using the Maximum Principle for the integro-differential equations, which can be found in Theorem 2.7 in Chapter 2 of Garroni and Menaldi [1992]. However, it can be proved using the ordinary Maximum Principle for parabolic differential equations (see Corollary 7.4 in Lieberman [1996]). We know that \( u = \frac{\partial w}{\partial t} \) satisfies the following equation in \( \mathcal{C} \),

\begin{equation}
\mathcal{L}_D w = \lambda \int_{\mathbb{R}} w(x + z, t) \nu(dz),
\end{equation}

\begin{equation}
(2.17)
\end{equation}

Since \( w = \partial_t u \geq 0 \) in \( \mathbb{R} \times (0, T) \), (2.16) implies that \( \mathcal{L}_D w \geq 0 \). If there is a point \( (x_0, t_0) \in \mathcal{C} \) such that \( w(x_0, t_0) = 0 \) (i.e. \( w \) achieves its non-positive minimum at \( (x_0, t_0) \)), it follows from the strong Maximum Principle that \( w(x, t) = 0 \) in \( \mathcal{C} \cap \mathbb{R} \times \{ 0 < t \leq t_0 \} \). Together with the fact that \( w(x, t) = 0 \) in \( \mathcal{D} \), we have that \( w(x, t) = 0 \) in \( \mathbb{R} \times \{ 0 < t \leq t_0 \} \). As a result, from

\[
u(x_0, t_0) - u(x_0, 0) = \int_0^{t_0} w(x_0, s) ds = 0,
\]

we obtain \( u(x_0, t_0) = (K - e^{x_0})^+ \). This contradicts with the definition of the free boundary \( b(t) \), because \( b(t_0) = \max \{ x \in \mathbb{R} : u(x, t_0) = (K - e^x)^+ \} \) and \( x_0 > b(t_0) \).

\[ \square \]

Combining Propositions 2.2 and 2.3 with the Hopf’s Lemma for parabolic integro-differential equations (see Theorem 2.8 in page 78 of Garroni and Menaldi [1992]), we obtain that the free boundary is strictly decreasing.

Lemma 2.1. The function \( t \rightarrow b(t) \) is strictly decreasing for \( t \in (0, T] \).

Proof. The proof is given in Theorem 5.4 in Yang et al. [2006].

\[ \square \]

In order to investigate the regularity of the free boundary in the later sections, we need more properties of \( u \), which we will develop in the following three lemmas. Since the results of these lemmas are intuitive but proofs are technical, we will list the proofs of these lemmas in the Appendix A.1.

It is well known that \( S \rightarrow V(S, t) \) is uniformly Lipschitz in \( \mathbb{R}^+ \) and \( t \rightarrow V(S, t) \) is uniformly semi-Hölder continuous in \( [0, T] \) (see Pham [1997]). The following lemma shows the same properties also holds for \( u(x, t) \), the function that we obtained after the change of variables in (2.6). (The globally Lipschitz continuity with respect to \( x \) is not a priori clear and one needs to check whether \( \partial_x u(x, t) \) is bounded.)

Lemma 2.2. Let \( u(x, t) \) be the solution of equation (2.8) - (2.10), then we have

\begin{equation}
|u(x, t) - u(y, t)| \leq C|x - y|, \quad x, y \in \mathbb{R}, t \in [0, T],
\end{equation}

(2.18)

\begin{equation}
|u(x, t) - u(x, s)| \leq D|t - s|^\frac{1}{2}, \quad x \in \mathbb{R}, 0 \leq t, s \leq T,
\end{equation}

(2.19)

where \( C \) and \( D \) are positive constants independent of \( x \) and \( t \).

Proof. See Appendix A.1.

\[ \square \]

In the rest of this section, we will investigate the boundness of \( \partial_t u(x, t) \) and its behavior when \( x \rightarrow +\infty \). These two properties will be useful to show several results in Sections 4 and 5 (see e.g. (4.2), proof of Lemma 4.1 and Remark 5.1). Let us first recall the definition of the Hölder spaces on page 7 of Ladyženskaja et al. [1968].
Definition 2.1. Let $\Omega$ be a domain in $\mathbb{R}$, $Q_T = \Omega \times (0, T)$. We denote $\overline{Q_T}$ the closure of $Q_T$. For any positive nonintegral real number $\alpha$, $H^{\alpha, \alpha/2}(\overline{Q_T})$ is the Banach space of functions $v(x, t)$ that are continuous in $\overline{Q_T}$, together with continuous derivatives of the form $\partial_t^v \partial_x^s v$ for $2r + s < \alpha$, and have a finite norm

$$
||v||^{(\alpha)} = |v|_x^{(\alpha)} + |v|_t^{(\alpha/2)} + \sum_{2r+s \leq [\alpha]} ||\partial_t^r \partial_x^s v||^{(0)},
$$

in which

$$
|v|_x^{(\alpha)} = \sum_{2r+s=[\alpha]} < \partial_t^r \partial_x^s v > x^{(\alpha-[\alpha])}, \quad |v|_t^{(\alpha/2)} = \sum_{\alpha-2 < 2r+s < \alpha} < \partial_t^r \partial_x^s v > t^{(\alpha-2s-[\alpha])};
$$

$$
\begin{align*}
<v>^\beta_x &= \sup_{(x,t),(x',t) \in Q_T, |x-x'| \leq \rho_0} \frac{|v(x,t)-v(x',t)|}{|x-x'|^\beta}, \quad 0 < \beta < 1, \\
<v>^\beta_t &= \sup_{(x,t),(x',t) \in Q_T, |t-t'| \leq \rho_0} \frac{|v(x,t)-v(x,t')|}{|t-t'|^\beta}, \quad 0 < \beta < 1,
\end{align*}
$$

where $\rho_0$ is a positive constant.

On the other hand, $H^\alpha(\overline{\Omega})$ is the Banach space whose elements are continuous functions $f(y)$ on $\overline{\Omega}$ that have continuous derivatives up to order $[\alpha]$ and the following norm finite

$$
||f||^{(\alpha)} = \sum_{j \leq [\alpha]} ||d^j_y f||^{(0)} + ||d^{|\alpha|}_y f||^{(\alpha-[\alpha])},
$$

in which

$$
|f|^{(\beta)} = \sup_{y,y' \in \Gamma, |y-y'| \leq \rho_0} \frac{|f(y)-f(y')|}{|y-y'|^\beta}.
$$

Here $d^j_y f$ is the $j$th derivative of $f$. These Hölder norms depend on $\rho_0$, but for different $\rho_0 > 0$, the corresponding Hölder norms are equivalent hence their dependence on $\rho_0$ will not be noted in the sequel.

Using the Hölder spaces and regularity results for parabolic equations, we have the following result.

Lemma 2.3. For any $\epsilon > 0$, $\partial_t u(x, t)$ is uniformly bounded on $\mathbb{R} \times [\epsilon, T]$.

Proof. See Appendix A.1. □

Remark 2.2. (i) In the statement of Lemma 2.3, $t = 0$ cannot be included, i.e., $\lim_{t \to 0} \partial_t u(x, t)$ is not uniformly bounded in $x \in \mathbb{R}$, because $\partial_t u = \frac{1}{2}\sigma^2 \partial_x^2 u + (\mu - \frac{1}{2}\sigma^2) \partial_x u - (r + \lambda) u + \lambda \int_y u(x+z,t) \nu(dz)$ and $\lim_{t \to 0} \partial_x^2 u(x, t)$ is not bounded as a result of non-smoothness of the initial value at $x = \log K$.

In the following, we will use the previous lemma to analyze the behavior of $\partial_t u(x, t)$ as $x \to +\infty$.

Lemma 2.4.

$$
\lim_{x \to +\infty} \partial_t u(x, t) = 0, \quad t \in (0, T].
$$

Proof. See Appendix A.1. □
Remark 2.3. Given the result in Lemma 2.2, it is clear from the differential equation (2.13) that $\partial_z^2u$ is uniformly bounded in $\mathbb{R} \times [\epsilon, T]$, since $\partial_z u$ is uniformly bounded (see Lemma 2.2). Combining with semi-Hölder continuity of $u(x, \cdot)$ in Lemma 2.2, Lemma 3.1 in page 78 of Ladyženskaja et al. [1968] now tells us that $\partial_z u(x, \cdot) \in H^{1/2}([\epsilon, T])$. Therefore, combining with the smooth fit property and Proposition 2.2, we have

$$u \in C^1(\mathbb{R} \times (0, T)).$$

In the following three sections we will use the properties of the value function we have shown in this section to investigate the regularity of the free boundary $b(t)$.

3. The free boundary is Hölder continuous

3.1. An auxiliary function. Before we begin to analyze the regularity of the free boundary, let us introduce the following important auxiliary function, which was also used in Lamberton and Mikou [2008] to prove the continuity of the free boundary in an exponential Lévy model:

$$(3.1) \quad J(x, t) \triangleq qe^{x} - rK + \lambda \int_{\mathbb{R}} [u(x + z, t) + e^{x+z} - K] \nu(dz), \quad x \in \mathbb{R}, \ t \in [0, T].$$

As a result of the assumption (1.2), $J < \infty$. Moreover, $J$ is closely related to the behavior of the value function $u$ in the stopping region, since one can check that

$$(3.2) \quad \mathcal{L}u(x, t) = -J(x, t), \quad \text{for } x < b(t), \ t \in (0, T],$$

$$(3.3) \quad \mathcal{L}g(x) = \mathcal{L}u(x, 0) = -\left[qe^{x} - rK + \lambda \int_{\mathbb{R}} (e^{x+z} - K)^{+} \nu(dz)\right] = -J(x, 0), \quad \text{for } x < \log K,$n

in which $g(x) \triangleq (K - e^{x})^{+}$. As we shall see in the rest of this section, the function $J(x, 0)$ is of special importance. We rename it as $J_0(x)$, i.e.,

$$(3.4) \quad J_0(x) \triangleq qe^{x} - rK + \lambda \int_{\mathbb{R}} (e^{x+z} - K)^{+} \nu(dz).$$

Let us analyze the properties of $J$.

Lemma 3.1. (i) $J(x, t) \geq -rK$, $\lim_{x \to -\infty} J(x, t) = -rK$ and $\lim_{x \to +\infty} J(x, t) = +\infty$,

(ii) $J(x, t) \in C^1(\mathbb{R} \times (0, T]) \cap C(\mathbb{R} \times [0, T])$,

(iii) The functions $x \to J(x, t)$ and $t \to J(x, t)$ are non-decreasing. If either either $q > 0$ or

$$(3.5) \quad \nu((M, +\infty)) > 0, \quad \text{for any } \ M > 0;$$

then $x \to J(x, t)$ is a strictly increasing function. On the other hand, if

$$(3.6) \quad \nu((0, \infty)) > 0$$

$$(3.7) \quad \partial_t J(x, t) > 0, \quad x \geq b(t), \ t \in (0, T].$$

Proof. (i) The first statement follows from $u(x + z, t) \geq (K - e^{x+z})^{+} \geq K - e^{x+z}$. The two limit statements follow from the Bounded Convergence Theorem.

(ii) The continuity of $u(x, t)$ on $\mathbb{R} \times [0, T]$ implies that $J$ is continuous on the same region. For the differentiability,
since $\partial_x u$ and $\partial_t u$ are uniformly bounded in $\mathbb{R} \times [\epsilon, T]$ for any $\epsilon > 0$ (see Lemmas 2.2 and 2.3), the Bounded Convergence Theorem gives us

$$
\frac{\partial}{\partial x} J(x,t) = q e^x + \lambda \int_{\mathbb{R}} \left[ \frac{\partial}{\partial x} u(x+z,t) + e^{x+z} \right] \nu(dz) < +\infty,
$$

(3.8)

$$
\frac{\partial}{\partial t} J(x,t) = \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x+z,t) \nu(dz) < +\infty.
$$

These partial derivatives are also continuous in $\mathbb{R} \times [\epsilon, T]$ as a result of Remark 2.3. Then the statement in (ii) follows since the choice of $\epsilon$ is arbitrary.

(iii) It is clear that the functions $x \rightarrow J(x,t)$ and $t \rightarrow J(x,t)$ are nondecreasing functions since $x \rightarrow u(x,t) + e^x$ and $t \rightarrow u(x,t)$ are nondecreasing.

The condition (3.5) means that the support of the measure $\nu$ is not bounded from above. As a result we have that the set $A = \{ z : x + z \in C \}$ has positive measure, i.e., $\nu(A) > 0$ for any $x \in \mathbb{R}$. For any $z \in A$ we have that $\partial_z u(x + z,t) + e^{x+z} > 0$, which is equivalent to $\partial_z V(Se^x,t) + 1 > 0$. The latter follows from the convexity of the function $V$ and (2.4). If $z \notin A$, then clearly $\partial_z u(x + z,t) + e^{x+z} = 0$. Using these facts in the first equation in (3.8), we see that (3.5) yields $\partial_x J(x,t) > 0$ in $\mathbb{R} \times [0,T]$. On the other hand, when $q > 0$ the condition assumed on $\nu$ can be dropped.

Moreover, when $x \geq b(t)$ (3.6) ensures that $\nu(A) > 0$. Then (3.7) follows from Proposition 2.3.

In the rest of the paper, we will assume either (3.5) or $q > 0$ and (3.6) are satisfied. Indeed, in the two well-known examples of jump diffusions, Kou’s model and Merton’s model (see Cont and Tankov [2004] p.111), in which $\nu$ is the double exponential and normal distribution respectively, condition (3.5) is fulfilled.

As the consequence of Lemma 3.1, the level curve

$$
B(t) \triangleq \{ x : J(x,t) = 0, t \in [0,T] \}.
$$

(3.9)

is well defined. $B(0)$, which is the unique solution of the integral equation,

$$
J_0(x) = q e^x - r K + \lambda \int_{\mathbb{R}} (e^{x+z} - K)^+ \nu(dz) = 0.
$$

(3.10)

will be crucial in describing the behavior of $b(t)$ close to 0 (see Section 3.2).

Remark 3.1. When $r=0$, Lemma 3.1 (i) implies that $B(t) = -\infty$. On the other hand, the proof in the following lemma tell us that $B(t) \geq b(t)$. Therefore $b(t) = -\infty$ in this case. We will assume $r > 0$ in the rest of the paper to exclude this trivial case.

This level curve $B(t)$ will be crucial in analyzing the regularity properties of the free boundaries in the rest of this section. Let us analyze its properties first.

Lemma 3.2. (i) $B(t)$ is non-increasing,

(ii) $B(t) \in C^1(0,T) \cap C[0,T]$,

(iii) $B(t) > b(t)$ for $t \in (0,T)$. Here $b(t)$ is the free boundary in (2.8) - (2.10).

Proof. (i) The proof follows from Lemma 3.1 (iii).

(ii) We have the continuity of $B$ because $J(x,t)$ is continuous and strictly increasing in $x$ (see Lemma 3.1 (ii) and (iii)). Let us focus on the differentiability in the following. It follows from Lemma 3.1 (ii) that $J(x,t)$ is a $C^1$ function in $\mathbb{R} \times (0,T]$. Moreover, it follows from (3.7) and $B(t) \geq b(t)$ (which we will prove in the Step 1 in (iii)) that

$$
\partial_t J(x,t_0)|_{x=B(t_0)} > 0, \quad t_0 \in (0,T_0].
$$
Therefore, the Implicit Function Theorem implies that there exists an open set $U$ containing $t_0$ such that

$$B(t) \in C^1(U).$$

Then the statement in (ii) follows after pasting different neighborhoods for all points $t \in (0, T]$ together.

(iii) The proof consists of two steps:

**Step 1:** First we show that $B(t) \supseteq b(0)$, if $B(t_0) < b(t_0)$, from the definition of $B(t)$ and the fact that $x \rightarrow J(x, t)$ is strictly increasing, we obtain $J(x, t_0) > 0$ for all $x \in (B(t_0), b(t_0))$. Combining with (3.2), we have

$$\mathcal{L}u(x, t_0) < 0, \quad \text{for any } x \in (B(t_0), b(t_0)),$$

which contradicts with (2.12).

**Step 2:** Second, we show that $B(t) \neq b(t), t \in (0, T]$. Since $b(t) < \log K$ (thanks to Lemma 2.1) and $t \rightarrow B(t)$ is non-increasing, it is clear that $B(t) > b(t)$ for any $t \in (0, t^*)$ where $t^* = T \wedge \sup\{t \in \mathbb{R}^+ : B(t) = \log K\}$. Hence we only need to focus on the region where $B(t) < \log K$. If there is a $t_0 \in (0, T]$ such that $B(t_0) = b(t_0)$, we will derive a contradiction in the following.

First, let us define the region $\Omega \triangleq \{(x, t) \mid B(t) < x < \log K, t \in (0, T]\}$. Because of the result in Step 1, $\Lambda \subset C$. Hence $u(x, t)$ satisfies

$$\mathcal{L}_D u(x, t) = \lambda \int_{\mathbb{R}} u(x + z, t) \nu(dz), \quad (x, t) \in \Omega.$$

Let us define $\xi \triangleq x - B(t), \tilde{u}(\xi, t) \triangleq u(x, t)$ and $\tilde{g}(\xi, t) \triangleq (K - e^{\xi + B(t)})^+ = g(x)$. In the region $\tilde{\Omega} \triangleq \{(\xi, t) \mid 0 < \xi < \log K - B(t), t \in (0, T]\}$ we have

$$\tilde{\mathcal{L}}_{D} \tilde{u} \triangleq \frac{\partial \tilde{u}}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{u}}{\partial \xi^2} - \left(\mu + B'(t) - \frac{1}{2}\sigma^2\right) \frac{\partial \tilde{u}}{\partial \xi} + (r + \lambda)\tilde{u} = \lambda \int_{\mathbb{R}} \tilde{u}(\xi + z, t) \nu(dz),$$

since $B(t) \in C^1(0, T]$. On the other hand,

$$\tilde{\mathcal{L}}_{D} \tilde{g} = -e^{\xi + B(t)} B'(t) + \frac{1}{2}\sigma^2 e^{\xi + B(t)} + \left(\mu + B'(t) - \frac{1}{2}\sigma^2\right) e^{\xi + B(t)} + (r + \lambda) \left(K - e^{\xi + B(t)}\right)$$

$$= - \left[ q e^{\xi + B(t)} - r K + \lambda \int_{\mathbb{R}} \left( e^{\xi + B(t) + z} - K \right) \nu(dz) \right].$$

Therefore, we obtain from (3.11) and (3.12) that

$$\tilde{\mathcal{L}}_{D} (\tilde{u} - \tilde{g}) (\xi, t) = q e^{\xi + B(t)} - r K + \lambda \int_{\mathbb{R}} \left[ \tilde{u}(\xi + z, t) + e^{\xi + B(t) + z} - K \right] \nu(dz) = J(\xi + B(t), t),$$

for $(\xi, t) \in \tilde{\Omega}$. Note that $J(x, t) > 0$ when $x > B(t)$. Therefore (3.13) yields

$$\tilde{\mathcal{L}}_{D} (\tilde{u} - \tilde{g}) > 0, \quad (\xi, t) \in \tilde{\Omega}. \quad (3.14)$$

On the other hand, from our assumption $\xi_0 \triangleq b(t_0) - B(t_0) = 0$. Moreover, there clearly exists a ball $B \subset \tilde{\Omega}$ such that $1) \overline{B}(\xi = 0) = (\xi_0, 0) \cap \{\xi < 0\}$ and $2) (\tilde{u} - \tilde{g})(\xi, t) > (\tilde{u} - \tilde{g})(\xi_0, t_0) = 0$ for all $(\xi, t) \in B$, since $(\tilde{u} - \tilde{g})(\xi, t) = (u - g)(x, t) > 0$ when $x > B(t) \geq b(t)$. Now applying Hopf’s Lemma (see Theorem 17 in page 49 of Friedman [1964]) to $\tilde{u} - \tilde{g}$ in $B$, we obtain

$$\frac{\partial}{\partial \xi} (\tilde{u} - \tilde{g})(\xi_0, t_0) > 0, \quad (3.15)$$

which contradicts with the smooth fit property at $(\xi_0, t_0)$, i.e., $\partial_t (u - g)(\xi_0, t_0) = \partial_x (u - g)(b(t_0), t_0) = 0$. \qed
Remark 3.2. In the proof of Lemma 3.2 (iii), the reason we work with the domain $\tilde{\Omega}$ instead of the domain $\Omega$ is that $\Omega$ may not satisfy the interior ball condition (see Theorem 17 in page 49 of Friedman [1964]), which is a crucial assumption of the Hopf Lemma. If one can show $B(t) \in C^2$, the interior ball condition automatically holds for $\Omega$ (see Remark in page 330 of Evans [1998]). However, $B(t) \in C^2$ does not follow directly from the Implicit Function Theorem, because $J(x,t)$ is not expected to be a $C^2$ function in a neighborhood of the point $(b(t_0), t_0)$, for any $t_0$, as a result of the discontinuity of $\partial^2_x u(x,t)$ across the free boundary $b(t)$ (see the following corollary).

As a corollary of Lemma 3.2 (iii), $\partial^2_x u(x,t)$ does not cross the free boundary continuously.

Corollary 3.1.

\begin{equation}
\frac{\partial^2}{\partial x^2} u(b(t)+, t) \equiv \lim_{x \to b(t)} \frac{\partial^2}{\partial x^2} u(x,t) > -e^{b(t)}, \quad t \in (0,T].
\end{equation}

(This is equivalent to $\lim_{S \uparrow b(t)} \partial^2_x V(S,t) > 0, t \in [0,T]$.)

Proof. On the one hand, since $B(t) > b(t)$ and $x \to J(x,t)$ is strictly increasing, we have

\begin{equation}
J(b(t),t) < 0, \quad t \in (0,T],
\end{equation}

On the other hand, from the continuity of $u$, (2.11), (2.8) and Proposition 2.2, it follows that

\begin{equation}
0 = \lim_{x \to b(t)} \mathcal{L} u(x,t) = -\frac{1}{2} \sigma^2 \lim_{x \to b(t)} \frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{2} \sigma^2 e^{b(t)} - \left\{ q e^{b(t)} - r K + \lambda \int_{\mathbb{R}} \left[ u(b(t) + z, t) + e^{b(t)+z} - K \right] \nu(dz) \right\}
= -\frac{1}{2} \sigma^2 \lim_{x \to b(t)} \frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{2} \sigma^2 e^{b(t)} - J(b(t),t).
\end{equation}

The inequality (3.16) now follows from combining (3.17) and (3.18). \hfill \square

3.2. The behavior of the free boundary close to maturity. We are ready to analyze the regularity of the free boundaries. The continuity of the free boundaries for differential equations with or without integral terms have been studied intensively, see e.g. Friedman [1975], Pham [1997], Yang et al. [2006] and Lamberton and Mikou [2008]. For the American option in jump diffusions, Pham [1997] showed the continuity of the free boundary under the technical condition

\begin{equation}
r > q + \lambda \int_{\mathbb{R}^+} (e^{z} - 1) \nu(dz).
\end{equation}

In Yang et al. [2006], this condition was removed in the proof of the continuity. Moreover, in their Theorem 5.3, they showed that

\begin{equation}
b(0+) \equiv \lim_{t \to 0^+} b(t) = \min \{ \log K, B(0) \} = \begin{cases}
\log K, & r \geq q + \lambda \int_{\mathbb{R}^+} (e^{z} - 1) \nu(dz) \\
B(0), & r < q + \lambda \int_{\mathbb{R}^+} (e^{z} - 1) \nu(dz)
\end{cases}
\end{equation}

in which $B(0)$ is the unique solution of (3.10). The same result has been shown for the exponential Lévy models in Lamberton and Mikou [2008].

3.3. Hölder continuity of the free boundary. In the following, the function $J_0(x)$ in (3.4) and the Maximum Principle will play a crucial role in showing that $t \to b(t)$ is Hölder continuous.

Lemma 3.3. Let $b(t)$ be the free boundary in Lemma 2.1. For any $\epsilon > 0$, if there exists $\delta > 0$ such that for any $t_1$ and $t_2$ satisfying $\epsilon \leq t_1 < t_2 \leq T$ and $t_2 - t_1 \leq \delta$ one has

\begin{equation}
u(b(t_1), t) - u(b(t_1), t_1) \leq C_\epsilon (t_2 - t_1)\alpha, \quad t_1 \leq t \leq t_2,
\end{equation}
in which \(0 < \alpha \leq 1\) and \(C_\epsilon\) is a constant that does not depend on \(t_1\) and \(t_2\), then there exists \(\delta' \in (0, \delta]\) such that
\[
b(t_1) - b(t_2) \leq C'_\epsilon(t_2 - t_1)^{\frac{\alpha}{2}}, \quad 0 \leq t_2 - t_1 \leq \delta',
\]
in which \(C'_\epsilon\) is another positive constant that is independent of \(t_1\) and \(t_2\).

Proof. This proof is motivated by Lemma 5.1 in Friedman and Shen [2002]. For any \(t_1\) and \(t_2\) such that \(\epsilon \leq t_1 < t_2 \leq T\) and \(t_2 - t_1 \leq \delta\), let us consider the domain \(D \doteq \{(x, t) : b(t) < x < b(t_1), t_1 < t < t_2\}\). (In what follows, we will choose \(t_1\) and \(t_2\) close to each other, i.e. we will find an appropriate \(\delta'\) such that \(t_2 - t_1 \leq \delta'\).) Let \(\partial D\) be the closure of the domain \(D\).

In the following, we will show that the function
\[
\chi(x) = \left\{\left[\sqrt{C_\epsilon(t_2 - t_1)^{\frac{\alpha}{2}}} + \beta(x - b(t_1))\right]^2\right\}, \quad b(t_2) \leq x \leq b(t_1)
\]
satisfies \(\chi(x) \geq (u - g)(x, t)\) on the domain \(D\) for suitably chosen positive constant \(\beta\).

It is clear that \(\chi(x) = 0\), when \(x \leq b(t_1) - \frac{\alpha}{\beta} \epsilon (t_2 - t_1)^{\frac{\alpha}{2}} \equiv \xi\). We also have \(\chi(b(t_1)) = C_\epsilon (t_2 - t_1)^{\alpha} \geq u(b(t_1), t) - g(b(t_1))\) for \(t_1 \leq t \leq t_2\) because of the assumption (3.21). On the other hand, \(\chi(b(t)) \geq 0 = u(b(t), t) - g(b(t))\). Therefore on the parabolic boundary of the domain \(D\), we have that \(\chi \geq u - g\). We will show that this holds for all \((x, t) \in D\). To this end, we will compare \(L\chi\) with \(L(u - g)\) using the Maximum Principle. Note that \(\chi\) is carefully chosen so that it has a continuous first derivative and a bounded second derivative. These properties of \(\chi\) makes the application of the Maximum Principle for weak solutions (see e.g. Corollary 7.4 in Lieberman [1996]) possible.

First, for \((x, t) \in D\) let us estimate the integral term:
\[
\lambda \int_{\mathbb{R}} \chi(x + z) \nu(dz) = \lambda \int_{\mathbb{R}} \chi(x + z) \nu(dz) \\
\leq \lambda \int_{\mathbb{R}} \chi(x + z) \nu(dz) \\
\leq 2\lambda \int_{\mathbb{R}} \chi(x + z) \nu(dz)
\]
for a sufficiently large positive constant \(M\) independent of \(t_1\) and \(t_2\). To obtain the first inequality, we used \(x < b(t_1)\) for \((x, t) \in D\). The third inequality follows, because \(\int_{\mathbb{R}} e^z \nu(dz) = +\infty\) in (1.2) and \(z\) is bounded from below.

With the estimate (3.24), we can calculate \(L\chi\) inside the domain \(D\).
\[
L\chi(x) = \left[-\sigma^2 \beta^2 - \left(\mu - \frac{1}{2} \sigma^2\right) \beta \chi^{\frac{1}{2}} + (r + \lambda) \chi\right] 1_{\{x > \xi\}} - \lambda \int_{\mathbb{R}} \chi(x + z, t) \nu(dz)
\]
\[
\geq - \left[\frac{(\mu - \sigma^2/2)^2}{r + \lambda} + \sigma^2\right] \beta^2 1_{\{x > \xi\}} - 2\lambda \left[C_\epsilon (t_2 - t_1)^{\alpha} + \beta^2 M\right]
\]
in which \(E \doteq \frac{(\mu - \sigma^2/2)^2}{r + \lambda} + \sigma^2 + 2\lambda M\) and \(F \doteq 2\lambda C_\epsilon\) are positive constants.

Recall that for any \(\epsilon > 0\), \(b(\epsilon) < \min\{\log K, B(0)\}\) and that the strictly increasing function \(J_0\) defined in (3.4) satisfies \(J_0(x) < 0\) for \(x < B(0)\). Using these observations and (3.3) it can be seen that for any \(x \leq b(\epsilon)\) we have
\[
Lg(x) = -J_0(x) \geq -J_0(b(\epsilon)) > 0.
\]
Now choosing
\[
c = -J_0(b(\epsilon)) > 0
\]
and \( \delta' = \min\{(\frac{C}{\beta})^{1/\alpha}, \delta\} \) and \( \beta \leq \sqrt{\frac{C}{\beta}} \), we have that
\[
L \chi(x)(x) \geq -c \geq L(u - g)(x,t), \quad (x,t) \in D.
\]
Considering \( \Psi = \chi - u + g \), we have \( L \Psi \geq 0 \) in \( D \) and \( \Psi \geq 0 \) on the parabolic boundary of \( D \). It follows from the Maximum Principle for weak solutions that \( \Psi \geq 0 \) in \( D \), i.e.,
\[
(3.28) \quad \chi(x) \geq (u - g)(x,t), \quad (x,t) \in D.
\]
Observe that \( (u - g)(x,t) = 0 \) if \( x \leq \xi \). For any \((x,t) \in D \), since \( (u - g)(x,t) > 0 \), we can see that \( x > \xi \). This gives us
\[
(3.29) \quad \inf_{t_1 \leq t \leq t_2} b(t) \geq b(t_1) - \frac{\sqrt{C_L}}{\beta} (t_2 - t_1)^{\frac{1}{2}}, \quad 0 < t_2 - t_1 \leq \delta'.
\]
We have shown the free boundary \( b(t) \) is continuous and strictly decreasing in Lemma 2.1. Along with this fact, the inequality (3.29) gives us (3.22) with \( C'_e = \sqrt{C_e}/\beta \).

Now we are ready to state the main result of this section.

**Theorem 3.1.** Let \( b(t) \) be the free boundary in problem (2.8) - (2.10), then for any \( \epsilon > 0 \) if \( \epsilon \leq t_1 < t_2 \leq T \), and \( t_2 - t_1 \) is sufficiently small, then
\[
(3.30) \quad b(t_1) - b(t_2) \leq C_e (t_2 - t_1)^{\frac{3}{2}},
\]
in which \( C_e \) is a positive constant independent of \( t_1 \) and \( t_2 \).

**Proof.** The proof will follow by applying Lemma 3.3 twice. The first application will show that \( b(t) \) is Hölder continuous with exponent \( \frac{1}{2} \). Applying Lemma 3.3 for the second time we will upgrade the Hölder exponent to \( \frac{3}{2} \).

As a result of Propositions 2.2 and 2.3 for any \( \epsilon > 0 \), \( t_1 \) and \( t_2 \) satisfying \( \epsilon \leq t_1 < t_2 \leq T \) we have that
\[
(3.31) \quad u(b(t_1), t) - u(b(t_1), t_1) \leq \max_{t_1 \leq s \leq t} \frac{\partial u}{\partial t}(b(t_1), s)(t - t_1) \leq C_1(t_2 - t_1),
\]
where \( C_1 = \max_{t_1 \leq s \leq T} \partial_t u(b(t_1), s) \) is a positive constant. Now as a result of Lemma 3.3, we know that there exists a sufficiently small constant \( \delta_1 \in (0, T - \epsilon) \) such that
\[
(3.32) \quad b(t_1) - b(t_2) \leq C'_1(t_2 - t_1)^{\frac{3}{2}}, \quad 0 \leq t_2 - t_1 \leq \delta_1,
\]
in which \( C'_1 \) is a positive constant that does not depend on \( t_1 \), \( t_2 \) and \( \delta_1 \).

It follows from Lemmas 2.8 and 2.11 in Yang et al. [2006] and the Sobolev Embedding Theorem (see also (A-27) in Appendix A.3) that for any \( a < b < \log K \) and \( t \in [t_1, t_2] \),
\[
(3.33) \quad \left| \frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(\overline{x}, t) \right| \leq \tilde{C} |x - \overline{x}|^{\frac{1}{2}}, \quad x, \overline{x} \in (a, b),
\]
in which \( \tilde{C} \) is a positive constant that does not depend on \( t \). Taking \( x = b(t_1) \) and \( \overline{x} = b(t) \) in (3.33) and using Proposition 2.2, we obtain
\[
(3.34) \quad 0 \leq \frac{\partial u}{\partial t}(b(t_1), t) \leq \tilde{C} |b(t_1) - b(t)|^{\frac{1}{2}} \leq \tilde{C} |b(t_1) - b(t_2)|^{\frac{1}{2}}, \quad t_1 \leq t \leq t_2,
\]
where the third inequality follows from \( b(t) \) being strictly decreasing in Lemma 2.1. Combining (3.32) and (3.34), we get
\[
(3.35) \quad 0 \leq \frac{\partial u}{\partial t}(b(t_1), t) \leq C_2(t_2 - t_1)^{\frac{1}{2}}, \quad t_1 \leq t \leq t_2, \quad 0 \leq t_2 - t_1 \leq \delta_1.
\]
As a result
\[(3.36) \quad u(b(t_1), t) - u(b(t_1), t_1) \leq \max_{t_1 \leq s \leq t_2} \frac{\partial u}{\partial t}(b(t_1), s)(t_2 - t_1) \leq C_2(t_2 - t_1)^{\frac{2}{3}}.\]
Applying Lemma 3.3 for the second time, we know that there exists \(\delta_2 \in (0, \delta_1]\) such that
\[(3.37) \quad b(t_1) - b(t_2) \leq C_\epsilon(t_2 - t_1)^{\frac{2}{3}}, \quad 0 \leq t_2 - t_1 \leq \delta_2,
\]
where \(C_\epsilon\) is a positive constant that does not depend on \(t_1, t_2\) and \(\delta_2\).

\[\square\]

4. The free boundary is continuously differentiable

In this section, we will investigate the continuous differentiability of the free boundary. In Theorem 5.6 in Yang et al. [2006], the authors have shown that \(b(t) \in C^1(0, T]\), with the extra condition
\[(4.1) \quad r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1) \nu(dz).\]
Thanks to Corollary 3.1 and Theorem 3.1, we can show the continuous differentiability of the free boundary without imposing this extra condition.

Remark 4.1. If condition (4.1) is not satisfied, we can see from (3.20) that there is a gap between \(\lim_{t \to 0^+} b(t)\) and \(b(0) = \log K\). Therefore it is impossible to have \(b(t)\) to be even continuous at \(t = 0\). But we shall see that it is continuously differentiable for all \(t \in (0, T]\).

Let us consider the time derivative \(\partial_t u(x, t)\). Recall that \(u(x, t)\) is the solution of (2.8) - (2.10). Using the assumption (2.7), the time derivative \(w = \partial_t u(x, t)\) satisfies the following partial differential equation
\[(4.2) \quad L_D w = h(x, t), \quad x > b(t), \quad t \in (0, T],
\]
\[w(b(t), t) = 0, \quad \lim_{x \to +\infty} w(x, t) = 0, \quad t \in (0, T],
\]
\[w(x, 0) = \lim_{t \to 0} \partial_t u(x, t), \quad x \geq b(0),
\]
in which
\[(4.3) \quad h(x, t) \triangleq \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x + z, t) \nu(dz) + \sigma \cdot (\partial_x \sigma) \cdot \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right).\]
When \(x < b(t)\), we also have \(w(x, t) = 0\). Given \(u(x, t)\) and \(b(t)\), (4.2) is a parabolic differential equation for \(w(x, t)\). In this equation, the boundary conditions for \(w(x, t)\) along \(b(t)\) and at the infinity follow from Proposition 2.2 and Lemma 2.4.

In order to show the differentiability of the free boundary, we need to study the behavior of \(\frac{\partial^2}{\partial x \partial t} u\) at the free boundary (by first making sure that the cross derivatives exist in the classical sense), which is carried out in the following lemma.

Lemma 4.1. (i) As a function of \(t\), \(\frac{\partial^2}{\partial x \partial t} u(b(t)+, t) \triangleq \lim_{x \to b(t)} \frac{\partial^2}{\partial x \partial t} u(x, t)\) is continuous on \((0, T]\).

(ii) Moreover, the function \(\frac{\partial^2}{\partial x \partial t} u(x, t)\) is continuous for \(x > b(t), t \in (0, T]\).

This lemma is a slight generalization of the result in Cannon et al. [1974] to the parabolic integro-differential equation (4.2). Considering the integral term \(h\) in (4.2) as the driving term, this lemma follows from using the same technique presented in Section 1 of Chapter 8 in Friedman [1964]. We will postpone this proof to the Appendix A.2. We are now ready to state and prove the main theorem of this section.

Theorem 4.1. Let \(b(t)\) be the free boundary in the boundary value problem (2.8) - (2.10), then \(b(t) \in C^1(0, T]\).
Proof. First, we will show \( b(t) \) is differentiable at \( t_0 \in (0, T] \). Let us define \( \rho = \partial_x^2 u(b(t_0)+, t_0) + e^{b(t_0)} \). Corollary 3.1 implies that \( \rho > 0 \).

For sufficiently small \( \epsilon > 0 \), it follows from (2.11) that
\[
\frac{1}{\epsilon} \left[ \frac{\partial}{\partial x} u(b(t_0), t_0) - \frac{\partial}{\partial x} u(b(t_0 - \epsilon), t_0 - \epsilon) + e^{b(t_0)} - e^{b(t_0-\epsilon)} \right] = 0.
\]
Applying the Mean Value Theorem yields
\[
(4.4) \quad \left( \frac{\partial^2}{\partial x^2} u(b(t_0) + y, t_0) + e^{b(t_0)+y} \right) \frac{b(t_0) - b(t_0 - \epsilon)}{\epsilon} = - \frac{\partial^2}{\partial x \partial t} u(b(t_0 - \epsilon), t_0 - \tau),
\]
for some \( y \in (0, b(t_0 - \epsilon) - b(t_0)) \) and \( \tau \in (0, \epsilon) \). Letting \( \epsilon \to 0 \) in (4.4) and using Lemma 4.1 (ii), we obtain
\[
(4.5) \quad \lim_{\epsilon \to 0} \frac{b(t_0) - b(t_0 - \epsilon)}{\epsilon} = - \frac{\partial^2}{\partial x^2} u(b(t_0)+, t_0) + \frac{\partial^2}{\partial x \partial t} u(b(t_0)+, t_0) + e^{b(t_0)},
\]
which implies that \( b(t) \) is differentiable since \( \rho > 0 \). Moreover, from (2.13) and Proposition 2.2, we have
\[
\frac{\partial^2}{\partial x^2} u(b(t)+, t) = \frac{2(r + \lambda)}{\sigma(b(t), t)^2} K + \left( \frac{2(\mu - r - \lambda)}{\sigma(b(t), t)^2} - 1 \right) e^{b(t)} - \frac{2}{\sigma(b(t), t)^2} f(b(t), t),
\]
which is clearly a continuous function of \( t \) on \( t \in (0, T] \), since \( b(t) \) is a continuous function and \( \sigma(x, t) \) is continuous from our assumption (2.7). Along with Lemma 4.1 (i), we can see from (4.5) that \( b(t) \in C^1(0, T] \). \( \square \)

5. Higher order regularity of the free boundary

In the previous section, we have proved that the free boundary \( b(t) \) is continuously differentiable. In this section, we will upgrade their regularity. Throughout this section, for the simplicity of the notation, we will assume that \( \sigma \) is a positive constant. In this case, \( h(x, t) = \lambda \int_\mathbb{R} \frac{\partial}{\partial x} u(x + z, t) v(dz) \), which is bounded thanks to Lemma 2.3. More generally, if \( \sigma = \sigma(x, t), h(x, t) \) is given in (4.3). If we assume \( \sigma(x, t) \in C^\infty(\mathbb{R} \times [0, T]) \) with all its derivatives bounded and \( \delta \leq \sigma \leq \Delta \) for some positive constants \( \delta \) and \( \Delta \), the same arguments in this section can still be carried through. Because of Lemmas 2.2 and 2.3, we can see from the equation (2.8) that \( \partial_x^2 u(x, t) \) is also bounded in \( \mathbb{R} \times [\epsilon, T] \) for any \( \epsilon > 0 \). Hence, \( h(x, t) \) is also bounded in this general case.

First, let us derive an identity for \( b'(t) \). Since \( b(t) \) is differentiable, taking derivative with respect to \( t \) on both sides of (2.11), we have
\[
(5.1) \quad \frac{\partial^2}{\partial x^2} u(b(t)+, t)b'(t) + \frac{\partial^2}{\partial x \partial t} u(b(t)+, t) = -e^{b(t)}b'(t).
\]
The term \( \partial_x^2 u(b(t)+, t) \) can be represented as
\[
(5.2) \quad \frac{\partial^2}{\partial x^2} u(b(t)+, t) = \left( \frac{2(\mu - r - \lambda)}{\sigma^2} - 1 \right) e^{b(t)} + \frac{2(r + \lambda)}{\sigma^2} K - \frac{2}{\sigma^2} f(b(t), t).
\]
Plugging (5.2) back into (5.1) and recalling \( w = \partial_t u \), we obtain
\[
(5.3) \quad b'(t) = -\frac{\frac{\sigma^2}{2} \frac{\partial}{\partial x} w(b(t)+, t)}{(\mu - r - \lambda)e^{b(t)} + (r + \lambda)K - f(b(t), t)}, \quad t \in (0, T].
\]

We can see from equations (4.2) that \( w(x, t) \) is the solution of a formal Stefan problem in the unbounded continuation regions \( C. \text{ Schaeffer [1976]} \) gave a proof of the infinite differentiability of the free boundary of a one dimensional Stefan problem in a bounded domain. By introducing the new variable \( \xi = \frac{x}{b(t)} \), he reduced the problem into a fixed boundary problem on a bounded domain. However, if we apply the same change of variables
we will have unbounded coefficients in the corresponding fixed boundary problem. Instead, similar to the change of variables in the proof of Lemma 3.2 (iii), we will define
\[ \xi \triangleq x - b(t), \quad v(\xi, t) \triangleq w(x, t), \]
in which \( b(t) \) is the free boundary in (2.8) - (2.10). The function \( v(\xi, t) \) satisfies the following fixed boundary equation,
\begin{align*}
\frac{\partial v}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial \xi^2} + \left( \mu + b'(t) - \frac{1}{2} \sigma^2 \right) \frac{\partial v}{\partial \xi} + (r + \lambda)v &= h(\xi + b(t), t), \quad (\xi, t) \in (0, +\infty) \times (0, T], \\
v(0, t) &= 0, \quad t \in (0, T], \\
v(\xi, 0) &= w(\xi + b(0), 0), \quad \xi \geq 0.
\end{align*}
Moreover, we have the following identity
\begin{equation}
(5.7) \quad b'(t) = - \frac{\sigma^2 \frac{\partial v(0, t)}{\partial \xi}}{(\mu - r - \lambda)e^{b(t)} + (r + \lambda)K - f(b(t), t)}, \quad t \in (0, T].
\end{equation}

**Remark 5.1.** Since \( b(t) \in C^1(0, T] \), so for any \( \epsilon > 0 \), \( b'(t) \) is continuous and bounded in \([\epsilon, T]\). On the other hand, since \( \partial_t u \) is bounded by Lemma 2.3, so \( h(\xi + b(t), t) = \lambda \int_R \partial_t u(\xi + b(t) + z, t)\nu(dz) \) is also bounded when \( (\xi, t) \in [0, +\infty) \times [\epsilon, T] \). As a result, it follows from Theorem 2.6 in page 19 of Ladyženskaja et al. [1968] that the parabolic differential equation (5.4) with the initial condition \( v(\xi, \epsilon) = w(\xi + b(\epsilon), \epsilon) \) instead of (5.6) has at most one bounded classical solution. It follows from the proof of Lemma 4.1 (i) that \( \partial_t u(x, t) \) is a bounded classical solution, so it is the unique bounded solution of (5.4).

The following result for parabolic differential equations will be an essential tool in the proof of the main result in this section.

**Lemma 5.1.** Let us assume \( w(\xi, t) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T]) \) (for some \( \alpha \) and \( \delta > 0 \)) satisfies the following equation
\begin{align*}
(5.8) \quad &\frac{\partial w}{\partial t} - a \frac{\partial^2 w}{\partial \xi^2} + \ell \frac{\partial w}{\partial \xi} + cw = d \int_R \phi(\xi + z, t)\nu(dz), \quad (\xi, t) \in ((0, +\infty) \times (\delta, T)) \\
(5.9) \quad &w(0, t) = g(t), \quad t \in [\delta, T].
\end{align*}

We assume that \( d \int_R \phi(\xi + z, t)\nu(dz) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T]) \) and that coefficients \( a, \ell, c \) also belong to \( H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T]) \) with \( \delta \leq a \leq \Delta \) for some positive constants \( \delta \) and \( \Delta \), moreover \( g(t) \in H^{1+\alpha}([\delta, T]) \). Then \( w(\xi, t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta', T]), \) for any \( \delta' > \delta \).

**Proof.** Consider a cut-off function \( \eta(t) \in C_0^\infty((0, T]), \) such that \( \eta(t) = 0 \) when \( t \in (0, \delta) \) and \( \eta(t) = 1 \) for \( t \in [\delta', T] \). The function \( \bar{w}(\xi, t) = \eta(t)w(\xi, t) \) satisfies
\begin{align*}
&\frac{\partial \bar{w}}{\partial t} - a \frac{\partial^2 \bar{w}}{\partial \xi^2} + \ell \frac{\partial \bar{w}}{\partial \xi} + c\bar{w} = d \int_R \eta(t)\phi(\xi + z, t)\nu(dz) + \frac{\partial \eta}{\partial t} w(\xi, t), \quad (\xi, t) \in (0, +\infty) \times (\delta, T], \\
&\bar{w}(0, t) = \eta(t)g(t), \quad t \in [\delta, T], \\
&\bar{w}(\xi, \delta) = 0, \quad \xi \geq 0.
\end{align*}
From our assumptions we have that
\begin{align*}
d \int_R \eta(t)\phi(\xi + z, t)\nu(dz) + \frac{\partial \eta}{\partial t} w(\xi, t) &\in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T]), \\
\eta(t)g(t) &\in H^{1+\alpha}([\delta, T]).
\end{align*}
Moreover, the coefficients of the above differential equation are all inside space $H^{2α,α}([0, +∞) \times [δ, T])$. In addition, this equation is uniformly parabolic as the result of $0 < δ ≤ a ≤ Δ$. It follows from regularity estimation for parabolic differential equations (see Theorem 5.2 in page 320 of Ladyženskaja et al. [1968]) that $\tilde{w}(ξ, t) \in H^{2+2α,1+α}([0, +∞) \times [δ, T])$, which implies $w(ξ, t) \in H^{2+2α,1+α}([0, +∞) \times [δ', T])$ by the choice of $η(t)$.

**Remark 5.2.** We will apply the previous lemma to $w(x, t) = \partial_x u(x, t)$. Because the initial condition for $w(x, t)$, $\lim_{t→0} \partial_t u(x, t)$, is not smooth, we can not apply Theorem 5.2 in page 320 of Ladyženskaja et al. [1968] to upgrade the regularity of $w$ directly. This is the reason we work with $\tilde{w}$ in the proof of the previous lemma.

In order to apply Lemma 5.1 to (5.4) - (5.7), we need Hölder continuous coefficients and value functions. Let us first show that the coefficients in equation (5.4) are Hölder continuous.

**Lemma 5.2.** Let $b(t)$ be the free boundary in (2.8) - (2.10). Then $b(t) \in H^{1+α}([δ, T])$ with $0 < α < \frac{1}{2}$ for any $δ > 0$.

**Proof.** For any $δ > 0$, since $b(t) \in C^1([0, T]$ by Theorem 4.1, the coefficients in equation (5.4) are bounded and continuous in $[δ, T]$. On the other hand, because $\partial_x u(x, t)$ is bounded in $R \times [δ, T]$ by Lemma 2.3, the function $h(ξ + b(t), t) = \lambda \int_δ^∞ R u(ξ + b(t) + z, t)υ(dz)$ is bounded when $(ξ, t) \in [0, +∞) \times [δ, T]$. It follows from Theorem 9.1 in page 341 of Ladyženskaja et al. [1968] that equation (5.4) has a unique solution $v(ξ, t) \in W^{2,1}_q([0, M] \times [δ, T])$ for any $q > 1$ and $M > 0$.

By the Sobolev Embedding Theorem (see, for example, Theorem 2.1 in page 61 of Ladyženskaja et al. [1968]), for $q > 3$, we have $v(ξ, t) \in H^{2,β/2}([0, M] \times [δ, T])$ with $β = 2 - \frac{4}{q} (1 < β < 2)$. As a result, we have

$$
\frac{∂}{∂ξ} v(0, t) \in H^{β-1}([δ, T]), \quad \text{with } 0 < \frac{β-1}{2} < \frac{1}{2}.
$$

Let us analyze the terms in the denominator on the right hand side of (5.7). We have that $b(t) \in C^1([δ, T])$ and that

$$
f(b(t), t) = λ \int_δ^∞ R u(b(t) + z, t)υ(dz) \in C^1([δ, T]),
$$

since $u(x, t) \in C^1(R \times [δ, T])$ (see Remark 2.3). Moreover, this denominator is also bounded away from 0, because

$$(μ - r - λ)σ^2 \epsilon^{b(t)} + (r + λ)K - f(b(t), t) = \frac{σ^2}{2} \left( \frac{∂^2}{∂z^2} u(b(t), t) + e^{b(t)} \right) > 0, \quad t ∈ [δ, T],
$$

where the last inequality follows from Corollary 3.1. It is clear from (5.7) and (5.10) that,

$$b'(t) \in H^{2α-1}([δ, T]).$$

As a corollary of Lemmas 5.1 and 5.2, we can improve the regularity of the functions $u(x, t)$.

**Corollary 5.1.** Let $u(x, t)$ be the classical solution of the boundary value problem (2.8) - (2.10). Then $u(ξ + b(t), t) \in H^{2+2α,1+α}([0, +∞) \times [δ', T])$ for any $δ' > 0$, with $α \in (0, 1/2)$.

**Proof.** Let $ξ = x - b(t)$, $κ(ξ, t) = u(x, t)$ and $φ(ξ + z, t) = u(ξ + b(t) + z, t)$. Then $κ(ξ, t)$ satisfies a differential equation of the form (5.8) and (5.9) in Lemma 5.1 with $g(t) = K - e^{b(t)}$ (in fact $κ$ satisfies (5.4) when $h$ in the driving term is replaced by $f$). Moreover, by Lemma 5.2, the coefficients in this equation (5.8) are inside space $H^{α}([δ, T])$ for any $δ > 0$, and $g(t) \in H^{1+α}([δ, T])$. In addition, thanks to the assumption (2.7), the equation (5.8) is uniformly parabolic.
On the other hand, since $u(x, t)$ is uniformly Lipschitz in $x \in \mathbb{R}$ and uniformly semi-Hölder continuous in $t \in [0, T]$ (see Lemma 2.2), and $b(t)$ is continuously differentiable, it is not hard to see that $\int_{\mathbb{R}} u(\xi + b(t) + z, t) \nu(dz) \in H^{2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$. Moreover, $u(\xi + b(t), t) \in H^{2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$ again because of Lemma 2.2. Now, the statement follows directly from Lemma 5.1.

Armed with Lemmas 5.1, 5.2 and Corollary 5.1, we can state and prove the main theorem of this section.

**Theorem 5.1.** Let $b(t)$ be the free boundary in (2.8) - (2.10). Assume that $\nu$ has a density, i.e. $\nu(dz) = \rho(z)dz$. Let $\alpha \in (0, 1/2)$. If $\rho(z)$ satisfies $\int_{-\delta}^{\infty} \rho(z)dz \in H^{2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$, then $b(t) \in H^{1+2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$. On the other hand, if $\rho(z) \in H^{1+2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$ for $\ell \geq 1$, then $b(t) \in H^{1+\ell+2\alpha, \alpha}(\mathbb{R} \times [\delta, T])$, for any $\epsilon > 0$.

**Proof.** The proof consists of four steps.

**Step 1.** From Lemma 5.2 and Corollary 5.1, we have that $b(t) \in H^{1+\alpha}(\mathbb{R} \times [\delta, T])$ and that $u(\xi + b(t), t) \in H^{2+2\alpha, 1+\alpha}(\mathbb{R} \times [\delta', T])$ for any $\delta' > \delta > 0$ with $\alpha \in (0, 1/2)$, which implies that $\partial_t u(\xi + b(t), t) \in H^{2\alpha, \alpha}(\mathbb{R} \times [\delta', T])$ (see Definition 2.1).

**Step 2.** Assume that there is a positive noninteger real number $\beta$ with $2\beta \leq 2\alpha + 1$, such that

\begin{align*}
(5.11) & \quad b(t) \in H^{1+\beta}(\mathbb{R} \times [\delta, T]), \\
(5.12) & \quad \frac{\partial}{\partial t} u(\xi + b(t), t) \in H^{2^{\beta}, 2\beta}(\mathbb{R} \times [\delta', T]), \\
(5.13) & \quad u(\xi + b(t), t) \in H^{2+2\beta, 1+\beta}(\mathbb{R} \times [\delta', T]),
\end{align*}

for $\delta' > \delta > 0$. We will upgrade the regularity exponent from $\beta$ to $1/2 + \beta$, in steps 2 and 3.

Let us analyze $\partial_t u(\xi + b(t), t)$. For any integers $r, s \geq 0$, $2r + s < 2\beta$, since $\partial_t u(\xi + b(t) + z, t) = 0$ when $z \leq -\xi$, we have

\begin{align*}
(5.14) & \quad \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) = \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{-\xi}^{+\infty} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \rho(z)dz \\
& \quad = 1_{\{r \geq 1\}} \sum_{i=0}^{s-1} \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial \xi} u(\xi + b(t) + z, t) \bigg|_{z=-\xi} \frac{d^{s-1-i}}{dz^{s-1-i}} \rho(-\xi) \\
& \quad + \int_{-\xi}^{+\infty} \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} u(\xi + b(t) + z, t) \rho(z)dz,
\end{align*}

for any $\xi \geq 0$.

When $t$ is fixed, in the following, we will show

\begin{align*}
(5.15) & \quad \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) \in H^{2^{\beta} - [2\beta]}(\mathbb{R} \times [\delta, T]), \quad \text{for } 2r + s = [2\beta].
\end{align*}

For any $\xi_1 > \xi_2 \geq 0$ such that $\xi_1 - \xi_2 \leq \rho_0$, we have

\begin{align*}
(5.16) & \quad \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi_2 + b(t) + z, t) \nu(dz) - \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi_1 + b(t) + z, t) \nu(dz) \right| \\
& \leq 1_{\{s \geq 1\}} \sum_{i=0}^{s-1} \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t^r} \left| u(\xi_2 + b(t) + z, t) - u(\xi_1 + b(t) + z, t) \right| \rho(z)dz \\
& \quad + \int_{-\xi_2}^{+\infty} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial \xi} u(\xi_1 + b(t) + z, t) - \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial \xi} u(\xi_2 + b(t) + z, t) \right| \rho(z)dz \\
& \quad + \int_{-\xi_1}^{-\xi_2} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial \xi} u(\xi_1 + b(t) + z, t) - \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial \xi} u(\xi_2 + b(t) + z, t) \right| \rho(z)dz.
\end{align*}
Let us analyze the right hand side of \((5.16)\) term by term. When \(s > 1\), since \(s - 1 < 2\beta - 1 \leq 2\alpha + \ell - 1\), we have \(\rho(z) \in H^{2\beta-1}(\mathbb{R}_-)\), which implies

\[
\sum_{i=0}^{s-1} \left| \frac{\partial^i}{\partial \xi^i} \frac{\partial}{\partial t} u(\xi + b(t) + z,t) \right|_{\xi=\xi_1} \leq C \|\partial u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]},
\]

where \(\rho(z) \in H^{[2\beta]}(\mathbb{R}_-)\) for \(\ell \geq 1\) or \(\int_{-\infty}^{u} \rho(z)dz \in H^{2\alpha}(\mathbb{R}_-)\) for \(\ell \geq 0\). In particular, using \(2\beta < 2\alpha + \ell\), we can see \(\int_{-\infty}^{u} \rho(z)dz \in H^{2\beta - [2\beta]}(\mathbb{R}_-)\). As a result,

\[
\int_{-\xi_2}^{-\xi_1} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial}{\partial t} u(\xi + b(t) + z,t) \right| \rho(z)dz \leq \tilde{C} \|\partial u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]},
\]

where \(\tilde{C}\) is also a positive constant. Plugging the estimates \((5.17) - (5.19)\) into \((5.16)\), we observe that \((5.15)\) holds.

When \(\xi\) is fixed, using \((5.14)\), it directly follows from \((5.11)\) and \((5.12)\) that

\[
\frac{\partial^s}{\partial \xi^s} \frac{\partial}{\partial t} u(\xi + b(t) + z,t) \in H^{\beta - \frac{2s+\ell}{s}}([\beta', T]), \quad \text{for} \ 2\beta - 2 < 2r + s < 2\beta.
\]

Now, \((5.15)\) and \((5.20)\) imply that

\[
\int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z,t) \nu(z)dz \in H^{\beta - \frac{2s+\ell}{s}}([\beta', T]),
\]

Let \(v(\xi, t)\) be a bounded solution of the boundary value problem \((5.4)\) with the initial condition \(v(\xi, \delta') = \partial_t u(\xi + b(\delta'), t)\). The uniqueness in Remark \(5.1\) implies that

\[
v(\xi, t) = \frac{\partial}{\partial t} u(\xi + b(t), t), \quad (\xi, t) \in [0, +\infty) \times [\delta', T].\]

As a result, the assumption \((5.12)\) implies that

\[
v(\xi, t) \in H^{\beta - \frac{2s+\ell}{s}}([\beta', T]),
\]

We will apply Lemma \(5.1\) to \((5.4) - (5.6)\) with \(\phi(\xi + z,t) = \partial_t u(\xi + b(t) + z, t)\), \(a = \sigma^2/2\), \(\ell = -(\mu + b'(t) - \sigma^2/2)\), \(c = r + \lambda \) and \(d = \lambda\). Thanks to \((5.11)\), the coefficient \(l\) belongs to \(H^{\beta}([\delta, T])\). The other coefficients already happen to reside there since they are constants. Along with \((5.21)\) and \((5.23)\), Lemma \(5.1\) yields

\[
v(\xi, t) \in H^{\beta - \frac{2s+\ell}{s}+1}([\delta'', T]) \quad \text{for any} \ \delta'' > \delta' > \delta,
\]

which implies that

\[
\frac{\partial}{\partial \xi} v(0, t) \in H^{\frac{3}{2}+\beta}([\delta'', T])
\]
and
\begin{equation}
\frac{\partial}{\partial t} u(\xi + b(t), t) \in H^{2+2\beta, 1+\beta}([0, +\infty) \times [\delta'', T]),
\end{equation}
by (5.22).

Using (5.7) and (5.25), we will improve the regularity of \(b(t)\) in the following. From (A-1) we have
\begin{equation}
f(b(t), t) = \lambda \int_{\mathbb{R}} u(b(t) + z, t) \nu(dz)
= \lambda \int_{-\infty}^{+\infty} u(b(t) + z, t) \nu(dz) + \lambda \int_{-\infty}^{0} (K - e^{b(t)+z}) \nu(dz).
\end{equation}
Along with (5.11) and (5.13), we can see from (5.27) that
\begin{equation}
f(b(t), t) \in H^{1+\beta}([\delta'', T]).
\end{equation}
Together with (5.11), (5.25) and (5.28), we can see from the identity (5.7) that \(b'(t) \in H^{\frac{1}{2}+\beta}([\delta'', T])\) for any \(\delta'' > \delta'\). It in turn implies that
\begin{equation}
b(t) \in H^{\frac{1}{2}+\beta}([\delta', T]).
\end{equation}

**Step 3.** Let us investigate \(u(\xi + b(t), t)\). For any \(r, s \geq 0, 2r + s < 2 + 2\beta\), we have
\begin{equation}
\frac{\partial^r}{\partial \xi^s} \frac{\partial^r}{\partial t} \int_{\mathbb{R}} u(\xi + b(t) + z, t) \nu(dz)
= \frac{\partial^r}{\partial \xi^s} \frac{\partial^r}{\partial t} \int_{-\infty}^{+\infty} u(\xi + b(t) + z, t) \rho(z) dz + \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t} \int_{-\infty}^{-\xi} u(\xi + b(t) + z, t) \rho(z) dz
= 1_{\{s \geq 1\}} \sum_{i=0}^{s-1} \left[ \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t} u(\xi + b(t) + z, t) \right]_{z=\xi} - \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t} u(\xi + b(t) + z, t) \right]_{z=-\xi}
\end{equation}
for any \(\xi \geq 0\). It is worth noticing that \(\frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t} u(\xi + b(t) + z, t)\) is \(0\), for any \(i \leq r\).

Following the same arguments that lead up to (5.21), we can show
\begin{equation}
\int_{\mathbb{R}} u(\xi + b(t) + z, t) \nu(dz) \in H^{2+2\beta, 1+\beta}([0, +\infty) \times [\delta', T]),
\end{equation}
given \(1 + 2\beta \leq 2\alpha + \ell - 1\).

Now, we can apply Lemma 5.1 to the differential equation \(u(\xi + b(t), t)\) satisfies, taking (5.13) and (5.29) into account. This results in
\begin{equation}
u(\xi + b(t), t) \in H^{1+2\beta, \frac{1}{2}+\beta}([0, +\infty) \times [\delta'', T]),
\end{equation}
for any \(\delta'' > \delta''\). As a result, we have improved the regularities from (5.11), (5.12) and (5.13) to (5.29), (5.26) and (5.31), respectively.

**Step 4.** For any \(\epsilon > 0\), we apply Steps 2 and 3 inductively starting from \(\beta = \alpha\) in Step 1. Let \(n\) be the number of time we apply Steps 2 and 3. Let \(\delta'_n = \delta'\), in which \(\delta' > 0\) is as in Step 1. Running Step 2 and 3 once, we obtain two constants \(\delta''_n\) and \(\delta'''_n\) such that (5.29), (5.31) hold with \(\beta = \alpha\). In the \(n\)-th time, \(n \geq 2\), we choose \(\delta'_n = \delta''_{n-1}\) and \(\delta''_n = \delta'''_n\), such that \(\delta''_n < \epsilon\) for any \(n\) so that \([\epsilon, T] \subset [\delta'''_n, T]\).

The application of Step 2 for the \(n\)-th time will give us that \(b(t) \in H^{1+\alpha+\frac{\ell}{2}}([\epsilon, T])\). Applying Step 2 for \(\ell + 1\) and Step 3 for \(\ell \) times the result follows. □
Remark 5.3. (i) The previous proof has also shown the higher order regularity of $u(x,t)$, i.e. $u(\xi + b(t),t) \in H^{2+2\alpha+\ell,1+\alpha+\frac{\ell}{2}}([0,\infty) \times [\xi,T])$, for any $\epsilon > 0$, under the assumptions of Theorem 5.1.

(ii) Note that $b(t) \in C^1([0,T])$ without any assumption on the density $\rho(z)$. If $\rho(z) \in H^{2m-1+2\alpha}(\mathbb{R}_-)$ for some $m \geq 1$, then $b(t) \in H^{2+m+\alpha}([\xi,T])$. From Definition 2.1 and the arbitrary choice of $\epsilon$, we have that $b(t) \in C^{m+1}((0,T])$ under this assumption.

As a corollary of Theorem 5.1, we have the following sufficient condition for the infinitely differentiability of $b(t)$.

Corollary 5.2. Let $b(t)$ be the free boundary in (2.8) - (2.10). Assume that $\nu$ has a density, i.e. $\nu(dz) = \rho(z)dz$. If $\rho(z) \in C^\infty(\mathbb{R}_-)$ with $\frac{d\rho}{dz}(z)$ bounded for each $\ell \geq 1$, but not necessarily uniformly, then $b(t) \in C^\infty((0,T])$.

Proof. For any $m \geq 1$ with $\rho(z) \in C^{2m+1}(\mathbb{R}_-)$ and derivatives of $\rho(z)$ up to order $2m + 1$ are bounded, it follows from Definition 2.1 that $\rho(z) \in H^{2m-1+2\alpha}(\mathbb{R}_-)$. As a result of Remark 5.3 (ii), we have $b(t) \in C^{m+1}((0,T])$.

Remark 5.4. There are two well-known examples of jump diffusion models in the literature, Kou’s model and Merton’s model (see Cont and Tankov [2004], p.111), in which the density $\rho(z)$ is double exponential and normal, respectively. For both of these densities, it is easy to see that the conditions for Corollary 5.2 are satisfied. Therefore, the free boundaries in both models are infinitely differentiable.

6. The boundaries of the approximating free boundary problems introduced by Bayraktar [2008]

In this section, we want to show that the approximating free boundaries $b_n(t)$, constructed in Bayraktar [2008], have regularity properties similar to the free boundary $b(t)$.

Bayraktar [2008] constructed a monotone increasing sequence $\{u_n\}_{n \geq 0}$ that converges to the unique solution $u(x,t)$ of the parabolic integro-differential equation (2.8) - (2.10), uniformly. In this sequence, $u_0(x,t) = (K - e^x)^+$, and each $u_n(x,t)$ ($n \geq 1$) is the unique classical solution of the following parabolic differential equation:

\begin{align}
(6.1) & \quad \mathcal{L}_Du_n \triangleq \frac{\partial u_n}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u_n}{\partial x^2} - \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{\partial u_n}{\partial x} + (r + \lambda) u_n = f_n(x,t), \quad x > b_n(t), \\
(6.2) & \quad u_n(b_n(t),t) = K - e^{b_n(t)}, \quad t \in (0,T], \\
(6.3) & \quad u_n(x,0) = (K - e^x)^+, \quad x \geq b_n(0),
\end{align}

in which

\begin{align}
(6.4) & \quad f_n(x,t) \triangleq \lambda \int_{\mathbb{R}} u_{n-1}(x + z,t) \nu(dz),
\end{align}

and the free boundary $b_n(t) \triangleq \log(s_n(T-t))$ is defined in terms of $s_n(.)$, which is the approximating free boundary in Bayraktar [2008]. Moreover, the smooth fit property is also satisfied for each $u_n$, i.e.

\begin{align}
(6.5) & \quad \frac{\partial}{\partial x} u_n(b_n(t),t) = -e^{b_n(t)}, \quad t \in (0,T].
\end{align}

In the region $\{(x,t)| x < b_n(t), t \in (0,T]\}$, one also has that

\begin{align}
(6.6) & \quad \mathcal{L}_Du_n(x,t) - f_n(x,t) \geq 0.
\end{align}

We can define the approximating continuation regions $C_n$ and the stopping regions $D_n$ as follows

$C_n \triangleq \{(x,t) | b_n(t) < x < +\infty, 0 < t \leq T\}, \quad D_n \triangleq \{(x,t) | -\infty < x \leq b_n(t), 0 < t \leq T\}, \quad \text{for all } n \geq 1.$

Since $\{u_n\}_{n \geq 0}$ is a monotone increasing sequence, the approximating free boundary $\{b_n\}_{n \geq 1}$ is a monotone decreasing sequence. As a result, we have $\cup_{n \geq 1} C_n = \mathcal{C}$ and $\cap_{n \geq 1} D_n = \mathcal{D}.$
The approximating sequences \( \{u_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) have the similar properties with the value function \( u \) and its free boundary \( b \). Proposition 2.3, Lemmas 2.1, 2.2 and 2.4 have their analogous versions for \( u_n \) and \( b_n \) via the same proofs only replacing the integral term \( f \) by \( f_n \) in (6.4). Proposition 2.2 and Lemma 2.3, on the other hand, can be slightly modified as follows:

**Proposition 6.1.** For all \( n \geq 1 \),

(i) If \( \partial_t u_n(x,t) \) is bounded in \( \mathbb{R} \times [\epsilon, T] \) for any \( \epsilon > 0 \), then \( \partial_t u_n(x,t) \) is continuous in \( \mathbb{R} \times (0,T] \) and

\[
\lim_{x \downarrow b_n(t)} \frac{\partial}{\partial t} u_n(x,t) = 0. \tag{6.7}
\]

(ii) On the other hand, if \( \lim_{x \downarrow b_n(t)} \partial_t u_n(x,t) = 0 \) for \( t \in (0,T] \) and \( \partial_t u_n(x,t) \) is continuous in \( \mathbb{R} \times (0,T] \), then \( \partial_t u_n(x,t) \) is uniformly bounded in \( \mathbb{R} \times [\epsilon, T] \), for any \( \epsilon > 0 \).

**Proof.** See Appendix A.3 for the proof of (i). Under the assumption that \( \lim_{x \downarrow b_n(t)} \partial_t u_n(x,t) = 0 \) for \( t \in (0,T] \), we have \( \partial_t u(x,t) \) is bounded in the domain \( \{(x,t) \mid b_n(t) \leq x \leq x_0, \epsilon \leq t \leq T\} \) for any \( \epsilon \geq 0 \) and \( x_0 > \log K \). Then the rest of the proof of (ii) is similar to the proof of Lemma 2.3. \( \square \)

**Remark 6.1.** To show that assumptions in both (i) and (ii) are satisfied for all \( u_n \), \( n \geq 1 \), we need to walk through (i) and (ii) successively. Starting from \( \partial_t u_0(x,t) = 0 \) (since \( u_0(x,t) = (K - e^x)^+ \)), (i) tells us that \( \lim_{x \downarrow b_n(t)} \partial_t u_1(x,t) = 0 \) and \( \partial_t u_1(x,t) \) is continuous in \( \mathbb{R} \times (0,T] \). Then it follows from (ii) that \( \partial_t u_1(x,t) \) is bounded in \( \mathbb{R} \times [\epsilon, T] \) for any \( \epsilon > 0 \). This result feeds back to (i). Now, as a result of an induction argument it can be seen that assumptions in both (i) and (ii) are satisfied for all \( n \).

Results similar to Lemmas 3.1, 3.2 and Corollary 3.1 can also be shown to hold for each \( u_n \), \( n \geq 1 \). Defining

\[
J_n(x,t) \triangleq qe^x - rK + \lambda \int_{\mathbb{R}} [u_{n-1}(x + z,t) + e^{z+\nu} - K] \nu(dz), \quad x \in \mathbb{R}, t \in [0,T],
\]

\[
B_n(t) \triangleq \{ x : J_n(x,t) = 0, t \in [0,T]\}.
\]

we obtain the following:

\[
\mathcal{L}_D u_n(x,t) - \lambda \int_{\mathbb{R}} u_{n-1}(x + z,t) \nu(dz) = -J_n(x,t), \quad x < b_n(t), t \in [0,T], \tag{6.8}
\]

\[
 x \to J_n(x,t) \text{ is strictly increasing and } t \to J_n(x,t) \text{ is non-decreasing for } (x,t) \in \mathbb{R} \times [0,T], \tag{6.9}
\]

\[
B_n(t) > b_n(t), \quad t \in (0,T], \tag{6.10}
\]

\[
\lim_{x \downarrow b_n(t)} \frac{\partial^2}{\partial x^2} u_n(x,t) > -e^{b_n(t)}, \quad t \in (0,T]. \tag{6.11}
\]

Moreover, as we can see in the following Proposition, the approximating free boundaries \( b_n \) have the same critical value as \( b \) at 0.

**Proposition 6.2.** For the approximating sequence \( b_n(t) \), we have

\[
b_n(0+) \triangleq \lim_{t \to -0^+} b_n(t) = \min \{ \log K, B(0) \} = \begin{cases} \log K, & r \geq g + \lambda \int_{\mathbb{R}} (e^z - 1) \nu(dz) \\ B(0), & r < g + \lambda \int_{\mathbb{R}} (e^z - 1) \nu(dz) \end{cases}, \tag{6.12}
\]

in which \( B(0) \) the unique solution of (3.10).

**Proof.** When \( x < b_n(t)(t > 0) \), it follows from (6.6), (6.8) and (6.9) that

\[
0 \leq \mathcal{L}_D u_n(x,t) - \lambda \int_{\mathbb{R}} u_{n-1}(x + z,t) \nu(dz) = -J_n(x,t) \leq -J_n(x,0) = -J_0(x).
\]
The fact that $J_0(B(0)) = 0$ and $x \rightarrow J_0(x)$ is strictly increasing tells us that $x \leq B(0)$. Hence $b_n(t) \leq B(0)$ thanks to the choice of $x$. It is also clear that $b_n(t) \leq \log K$. Then we obtain

\begin{equation}
(6.13) \quad b_n(0+) \leq \min\{\log K, B(0)\}.
\end{equation}

Now, the corollary results from combining (3.20) and (6.13), since $\{b_n\}_{n \geq 1}$ is a decreasing sequence of functions. □

Furthermore, the Hölder continuity in Theorem 3.1 also holds for $b_n$, $n \geq 1$. In the proof of Lemma 3.3, we only need to replace $c$ in (3.27) by $\min \{-2/\sigma^2 J_n(x, t)\} b_n(t) < x < B_n(t), \epsilon \leq t \leq T\} > 0$. On the other hand, results in Lemma 4.1 also hold for $\partial_x u_n, n \geq 1$. Therefore, combining with (6.11), we have from (6.5) that

**Proposition 6.3.** $b_n(t) \in C^1(0, T), n \geq 1$.

Finally, using the following representation

\begin{equation}
(6.14) \quad b_n(t) = \frac{\sigma^2 \partial_x^2 u_n(b_n(t)+, t)}{(\mu - r - \lambda) e^{b_n(t)+} + (r + \lambda) K - f_n(b_n(t), t)}, \quad t \in (0, T],
\end{equation}

one can follow the proof of Lemma 5.2 to show that there is $\alpha \in (0, 1/2)$ such that

$$b_n(t) \in H^{1+\alpha}(\delta, T], \quad \text{for any } \delta > 0.$$
for some positive constant $c$, (see Theorem 16.3 in page 413 of Ladyženskaja et al. [1968]). Since $\int_{\mathbb{R}} dy \exp(-c(x-y)^2) \leq d (t-s)^{\frac{d}{2}}$ for some other positive constant $d$, we have that

$$\int_0^t ds \int_{\mathbb{R}} dy |\partial_x G(x, t; y, s)| \leq \int_0^t ds \partial_t \tilde{c}(t-s)^{-\frac{d}{2}} = 2\tilde{c}t^{\frac{d}{2}},$$

Using this estimate and the boundness of $f$ and $\tilde{f}$, the Dominated Convergence Theorem implies that

$$\partial_x v(x, t) = \int_0^t ds \int_{\mathbb{R}} dy \partial_x G(x, t; y, s)(f(y, s)\eta(y) + \tilde{f}(y, s)),$$

which is uniformly bounded. On the other hand, $\partial_x v = \eta' u + \eta \partial_x u$. By our choice of $\eta(x)$, we have that $\partial_x u(x, t)$ is uniformly bounded on $[X, +\infty) \times [0, T]$.

Moreover, in the stopping region $\mathcal{D}$, we have $\partial_x u(x, t) = -e^x$. This implies that $0 > \partial_x u(x, t) \geq -e^{b(t)} \geq -K$. On the other hand, since it is continuous $\partial_x u$ is also bounded in the compact closed domain $\{(x, t)|b(t) \leq x \leq X, 0 \leq t \leq T\}$. As a result we have that $\partial_x u(x, t)$ is uniformly bounded in $\mathbb{R} \times [0, T]$.

**Proof of Lemma 2.3.** Let us choose $X_0$ such that $X_0 > \log K$. We will first prove that $\partial_x u(x, t)$ is uniformly bounded in the domain $[X_0, +\infty) \times [0, T]$. Let $k(x, t) \in C^\infty_0(\mathbb{R} \times [0, T])$ be such that

$$\partial_x k(x, t)|_{x=X_0} = \partial_x u(x, t)|_{x=X_0}, \quad t \in [0, T],$$

and that $k(x, 0) = 0, x \in \mathbb{R}$. These two conditions on $k$ are consistent since $\partial_x u(x, 0)|_{x=X_0} = 0$. The function $v(x, t) \triangleq u(x, t) - k(x, t)$ satisfies

(A-3)

$$\partial_x v(x, t)|_{x=X_0} = 0,$$

and

(A-4)

$$\mathcal{L}_D v(x, t) = f(x, t) + g(x, t), \quad x > b(t), t \in (0, T],$$

in which $g(x, t) = -\mathcal{L}_D k(x, t)$ and $f$ is given by (A-1). Let us define the even extension of $v(x, t)$ with respect to the line $x = X_0$ as

(A-5)

$$\hat{v}(x, t) \triangleq \begin{cases} v(x, t) & x \geq X_0, \\ v(2X_0 - x, t) & x < X_0. \end{cases}$$

We similarly define $\hat{f}(x, t)$ and $\hat{g}(x)$. From (A-3) and (A-5), we have $\hat{v}(x, t) \in C^{2,1}(\mathbb{R} \times (0, T])$ and that it satisfies the equation

$$\mathcal{L}_D \hat{v} = \hat{f}(x, t) + \hat{g}(x, t), \quad (x, t) \in \mathbb{R} \times (0, T],$$

$$\hat{v}(x, 0) = 0, \quad x \in \mathbb{R}.$$

Here the initial condition follows from (2.10) and the choice of $X_0$ and $k(x, t)$.

It follows from (2.18) and (2.19) that $f(x, t)$ is uniformly Lipschitz in $x$ and semi-Hölder continuous in $t$. So for any $x_1 < x_2$, if we have either $x_2 \leq X_0$ or $X_0 \leq x_1$, then

$$|\hat{f}(x_1, t) - \hat{f}(x_2, t)| \leq \lambda C(x_2 - x_1),$$

for the same constant $C$ as in (2.18). On the other hand, if $x_1 < X_0 < x_2$, then

$$|\hat{f}(x_1, t) - \hat{f}(x_2, t)| \leq |\hat{f}(x_1, t) - \hat{f}(X_0, t)| + |\hat{f}(X_0, t) - \hat{f}(x_2, t)|$$

$$\leq \lambda C(X_0 - x_1) + \lambda C(x_2 - X_0) = \lambda C(x_2 - x_1).$$
As a result of the last two equations we observe that \( \hat{f}(x, t) \) is uniformly Lipschitz in its first variable. It is also clear that \( \hat{f}(x, t) \) is semi-Hölder continuous in its second variable. Thus, it follows from Definition 2.1 that

\[
\hat{f}(x, t) \in H^{\alpha, \frac{2}{\alpha}}(\mathbb{R} \times [0, T]), \quad \text{for some } 0 < \alpha < 1.
\]

On the other hand, \( \hat{g}(x, t) \in H^{\alpha, \alpha/2}(\mathbb{R} \times [0, T]) \), because \( k(x, t) \in C_0^\infty(\mathbb{R} \times [0, T]) \). Combining with the assumption (2.7) on \( \alpha \), the regularity property of parabolic differential equation (see Theorem 5.1 in page 320 of Ladyženskaja et al. [1968]) implies that

\[
\hat{v}(x, t) \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R} \times [0, T]).
\]

In particular, \( u(x, t) \in H^{2+\alpha, \alpha/2}([X_0, +\infty) \times [0, T]) \). As a result, in \([X_0, +\infty) \times [0, T], \partial_t u(x, t)\) is uniformly bounded by the Hölder norm of \( u(x, t) \). Now, the result follows from the continuity of \( \partial_t u(x, t) \) inside domain \([x, t] | b(t) \leq x \leq X_0, \epsilon \leq t \leq T \) for any \( \epsilon > 0 \) (see Proposition 2.2). \( \square \)

**Proof of Lemma 2.4.** Let \( X_0 > \log K \) be the same as in the proof of Lemma 2.3, again choose a cut-off function \( \eta(x) \in C^\infty(\mathbb{R}) \), such that \( \eta(x) = 1 \) when \( x \geq 2X_0 \) and \( \eta(x) = 0 \) when \( x \leq X_0 \). Then formally the function \( \eta(x)\partial_t u(x, t) \) satisfies the following Cauchy problem

\[
L_D w = \eta(x)h(x, t) + \bar{h}(x, t), \quad (x, t) \in \mathbb{R} \times [t_0, T],
\]

where

\[
h(x, t) = \lambda \int \partial_t u(x + z, t)\nu(dz), \quad \bar{h}(x, t) = -\frac{1}{2}\sigma^2 (2\eta'\partial_x \eta + \eta''\partial_t \eta) - \left( \mu - \frac{1}{2}\sigma^2 \right) \eta'\partial_t u,
\]

and we choose \( \eta(x)\partial_t u(x, t_0) \), for some \( t_0 \in [0, T) \), as the initial condition. It follows from Theorem 3.1 in page 346 of Garreni and Menaldi [1992] that this Cauchy problem has an unique classical solution, we call it \( w \). On the other hand, we have \( w(x, t) = \eta(x)\partial_t u(x, t) \). Indeed, it is easy to check that \( \int_{t_0}^t w(x, s)ds \) is the unique classical solution of the Cauchy problem

\[
L_D v = \int_{t_0}^t ds \left( \eta(x)h(x, s) + \bar{h}(x, s) \right) + \eta(x)\partial_t u(x, t_0), \quad v(x, t_0) = 0.
\]

Note that \( \eta(x) [u(x, t) - u(x, t_0)] \) is another classical solution. Therefore \( w(x, t) = \eta(x)\partial_t u(x, t) \) by the uniqueness.

Using the Green function \( G(x, t; y, s) \) corresponding to the differential operator \( L_D \), the solution \( w(x, t) \) can be represented as

\[
(A-6) \quad w(x, t) = \int_{\mathbb{R}} dy G(x, t; y, t_0)w(y, t_0) + \int_{t_0}^t ds \int_{\mathbb{R}} dy G(x, t; y, s)(\eta(y)h(y, s) + \bar{h}(y, s)),
\]

for all \((x, t) \in \mathbb{R} \times (t_0, T)\). Since the Green function satisfies

\[
|G(x, t; y, s)| \leq C(t - s)^{-\frac{\alpha}{2}} \exp \left( -\frac{c(x - y)^2}{t - s} \right), \quad (y, s) \in \mathbb{R} \times [0, t).
\]

The first term in (A-6) is bounded, as long as \( w(y, t_0) \) is uniformly bounded. The contribution of \( \eta'\partial_x \partial_t u \) (in the expression for \( \bar{h} \)) to \( w \) is given by

\[
- \int_{\mathbb{R}} dy \left( G(x, t; y, s)\eta'(y) \frac{\partial^2}{\partial y \partial s} u(y, s) \right) = \int_{\mathbb{R}} dy \frac{\partial}{\partial y} \left[ G(x, t; y, s)\eta'(y) \right] \frac{\partial}{\partial s} u(y, s).
\]

Now it follows from Lemma 2.3 that both \( w(x, t_0) \) and \( h(x, t) \) are uniformly bounded for \( x \in \mathbb{R}, t \in [t_0, T] \). We also have that \( \eta' \) and \( \eta'' \) vanish outside \([X_0, 2X_0]\). Since \( \lim_{x \to \pm \infty} G(x, t; y, s) = 0 \) and it can easily be shown that \( \lim_{x \to \pm \infty} \partial_y G(s, t; y, s) = 0 \), the Dominated Convergence Theorem implies that

\[
\lim_{x \to \pm \infty} w(x, t) = 0, \quad t \in (t_0, T].
\]

Then the statement follows from the choice of \( \eta \). \( \square \)
A.2. Proof of Lemma 4.1. We will first establish a one to one correspondence between solutions of (4.2) and solutions of an integral equation of Volterra type.

Lemma A-1. (i) Let $G(x, t; y, s)$ be the Green function associated to the differential operator $\mathcal{L}_D$ and let us consider the following nonlinear integral equation of Volterra type,

$$
(A-7) \quad \left( 1 + \frac{1}{4} \sigma^2(b(t), t) \right) v(t) = -\int_{t_0}^t ds v(s) \frac{1}{2} \sigma^2(b(s), s) \partial_x G(b(t), t; b(s), s) + \sum_{i=1}^{2} N_i(t), \quad t_0 \leq t \leq T,
$$

where $N_1(t) = \int_{b(t)}^{+\infty} dy \partial_x G(b(t), t; y, t_0) w(y, t_0)$ and $N_2(t) = \int_{t_0}^t ds \int_{b(s)}^{+\infty} dy \partial_x G(b(t), t; y, s) h(y, s)$. There exists a unique solution $v$ to (A-7). The function $v(t)$ is continuous.

(ii) Let $w(x, t)$ be a classical solution of (4.2) on $[t_0, T]$ with the initial condition $w(x, t_0) = \partial_t w(x, t_0)$, such that $t \to \partial_x w(b(t)+, t)$ is continuous. Then there is a one to one correspondence between $w(x, t)$ and $v(t)$. Moreover $\partial_x w(b(t)+, t) = v(t)$, $t_0 \leq t \leq T$.

The initial value of (4.2) may not be smooth. This is the reason we take $w(x, t_0) = \partial_t u(x, t_0)$, $0 < t_0 < T$, as the initial condition of (4.2) and consider the differential equation on $t \in [t_0, T]$.

Remark A-1. The correspondence in Lemma A-1 is well known for the Stefan problem on heat equation with Lipschitz continuous free boundary (see Section 1 Chapter 8 of Friedman [1964]). Along Friedman’s line of proof, we will extend the correspondence to our parabolic differential equation with Hölder continuous free boundary.

Proof of Lemma A-1. Proof of (i). First, because $G(b(t), t; b(s), s)$ and $\sigma(b(s), s)$ are continuous for $s \in (0, t)$ (see (2.7)), it follows from the classical result on Volterra equations (see Rust [1934]) that the integral equation (A-7) has a unique solution $v(t)$ and it is continuous with respect to $t \in [t_0, T]$, as long as $N_i(t)$, $i = 1, 2$, are continuous with respect to $t$. It is not hard to show these functions are indeed continuous, using the continuity of $b(t)$ and the following estimates on the Green function $G$ and its derivatives:

$$
|\partial^\ell_x G(x, t; y, s)| \leq C(t-s)^{-\frac{\ell+\mu}{2}} \exp \left( -c\frac{|x-y|^2}{t-s} \right),
$$

$$
|\partial_x G(x, t; y, s) - \partial_x G(x, t; \tilde{x}, s)| \leq C(t-\tilde{t})^\frac{\mu}{2} (t-s)^{-\frac{\ell+\mu}{2}} \exp \left( -c\frac{|x-y|^2}{t-s} \right),
$$

$$
|\partial_x G(x, t; y, s) - \partial_x G(x, t; \tilde{x}, s)| \leq C|x-\tilde{x}|^\alpha (t-s)^{-\frac{\ell+\mu}{2}} \exp \left( -c\frac{|x''-y|^2}{t-s} \right),
$$

where $\ell = 0, 1$, $s < \tilde{t} < t$, $|x''-y| = |x-y| \land |\tilde{x}-y|$, $0 < \alpha < 1$, $C$ and $c$ are positive constants. These estimates are from Theorem 16.3 in page 413 of Ladyženskaja et al. [1968].

Proof of (ii) Let us assume that $w(x, t)$ is a classical solution of (4.2). As a result, the following Green’s identity (see page 27 of Friedman [1964]) is satisfied

$$
(A-8) \quad \frac{\partial}{\partial y} \left( \frac{1}{2} \sigma^2(y, s) G(x, t; y, s) \right) w(y, s) - \frac{\partial}{\partial s} (G(x, t; y, s) w(y, s)) + \frac{\partial}{\partial y} \left( \left( \mu - \frac{1}{2} \sigma^2(y, s) \right) G(x, t; y, s) w(y, s) \right) = -G(x, t; y, s) h(y, s),
$$

where \( t_0 \leq s < t \leq T, x > b(t) \) and \( y > b(s) \). Integrating both hand side of (A-8) over the domain \( b(s) < y < +\infty, t_0 < s < t - \epsilon \), we obtain

\[
\int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} \frac{1}{2} \sigma^2(y, s) \partial_y w(y, s) G(x, t; y, s) - \int_{t_0}^{t-\epsilon} ds \frac{1}{2} \sigma^2(b(s), s) \partial_y w(b(s)+, s) G(x, t; b(s), s) \\
- \int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} \frac{1}{2} \sigma^2(y, s) w(y, s) \partial_y G(x, t; y, s) + \int_{t_0}^{t-\epsilon} ds \frac{1}{2} \sigma^2(b(s), s) \partial_y G(x, t; b(s), s) \\
- \int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} w(y, s) G(x, t; y, s) \sigma \partial_y G(x, t; b(s), s) + \int_{t_0}^{t-\epsilon} ds w(b(s), s) G(x, t; b(s), s) \sigma \partial_y G(x, t; b(s), s) \\
\int_{b(t-\epsilon)}^{b(t)} dy [G(x, t; y, t-\epsilon) - G(x, t; y, t_0)w(y, t_0)] \\
+ \int_{t_0}^{t-\epsilon} ds \left[ \lim_{y \to +\infty} \left( \mu - \frac{1}{2} \sigma^2(y, s) \right) w(y, s) G(x, t; y, s) - \left( \mu - \frac{1}{2} \sigma^2(b(s), s) \right) w(b(s), s) G(x, t; b(s), s) \right] \\
= - \int_{t_0}^{t-\epsilon} ds \int_{b(s)}^{+\infty} dy G(x, t; y, s) h(y, s).
\]

In the seventh term on the left of (A-9), we used \( w(x, t) = 0 \) when \( x < b(t) \). Using the boundary and initial conditions for \( w(x, t) \) and the facts that \( \lim_{y \to +\infty} G(x, t; y, s) = 0 \) and \( \lim_{y \to +\infty} \partial_y G(x, t; y, s) = 0 \), letting \( \epsilon \to 0 \), we can write

\[
w(x, t) = - \int_{t_0}^{t} ds \partial_x w(b(s)+, s) \frac{1}{2} \sigma^2(b(s), s) G(x, t; b(s), s) + \int_{b(t)}^{+\infty} dy G(x, t; y, t_0)w(y, t_0) \\
+ \int_{t_0}^{t} ds \int_{b(s)}^{+\infty} dy G(x, t; y, s) h(y, s) \\
\triangleq - M_0(x, t) + M_1(x, t) + M_2(x, t).
\]

Before differentiating both sides of (A-10) with respect to \( x \), let us recall the jump identity: if \( \rho(t), t_0 \leq t \leq T \), is a continuous function and \( b(t) \) is the Hölder continuous with Hölder exponent \( \alpha > \frac{1}{2} \), then for every \( t_0 \leq t \leq T \),

\[
\lim_{x \downarrow b(t)} \frac{\partial}{\partial x} \int_{t_0}^{t} ds \rho(s)G(x, t; b(s), s) = \frac{1}{2} \rho(t) + \int_{t_0}^{t} ds \rho(s) \partial_x G(x, t; b(s), s)|_{x=b(t)}.
\]

This identity can be proved in the similar way as in Lemma 1 in Chapter 8 of Friedman [1964]. As commented in the paragraph after Lemma 4.5 in Friedman [1975], the proof of Lemma 1 can go through when we replace Lipschitz free boundary with Hölder continuous free boundary with the Hölder exponent \( \alpha > \frac{1}{2} \).

Now we will take the derivative of (A-10) with respect to \( x \) to obtain

\[
\frac{\partial}{\partial x} w(x, t) = \sum_{i=0}^{2} \frac{\partial}{\partial x} M_i(x, t)
\]

and let \( x \downarrow b(t) \). Since \( \partial_x w(b(s)+, s) \) and \( \sigma(b(s), s) \), \( t_0 \leq s < t \), are continuous and \( b(t) \) is Hölder continuous with exponent \( \alpha > \frac{1}{2} \) (see Theorem 3.1), taking \( \rho(s) = \frac{1}{2} \sigma^2(b(s), s) \partial_x w(b(s)+, s) \) in (A-11), we obtain

\[
\lim_{x \downarrow b(t)} \frac{\partial}{\partial x} M_0(t) = \lim_{x \downarrow b(t)} \frac{\partial}{\partial x} \int_{t_0}^{t} ds \frac{1}{2} \sigma^2(b(s), s) \partial_x w(b(s)+, s) G(x, t; b(s), s) \\
= \frac{1}{4} \sigma^2(b(t), t) \partial_x w(b(t)+, t) + \int_{t_0}^{t} ds \frac{1}{2} \sigma^2(b(s), s) \partial_x w(b(s)+, s) \partial_x G(b(t), t; b(s), s).
\]
On the other hand, by Lemmas 2.2 and 2.3, \( w(y, t_0) \) and \( h(y, s) \) are bounded in \( \mathbb{R} \times [t_0, T] \). Using the Dominated Convergence Theorem we get

\[
\lim_{x \uparrow b(t)} \frac{\partial}{\partial x} M_1(x, t) = \int_{b(t)}^{+\infty} dy \partial_x G(b(t), t; y, t_0) w(y, t_0) \equiv N_1(t),
\]

(A-14)

\[
\lim_{x \uparrow b(t)} \frac{\partial}{\partial x} M_2(x, t) = \int_{t_0}^t ds \int_{b(s)}^{+\infty} dy \partial_x G(b(t), t; y, s) h(y, s) \equiv N_2(t),
\]

(A-15)

It follows from (A-12) - (A-15) that \( \partial_x w(b(t)+, t) \) satisfies (A-7).

Let us prove the converse. For any solution \( v(t) \) of the integral equation (A-7), we can define \( w(x, t) \) as follows

\[
w(x, t) := -\int_{t_0}^t ds \int_{b(t)}^{+\infty} dy \frac{1}{2} \sigma^2(b(s), s) G(x, t; b(s), s) + \int_{b(t)}^{+\infty} dy G(x, t; y, t_0) w(y, t_0) + \int_{t_0}^t ds \int_{b(s)}^{+\infty} dy G(x, t; y, s) h(y, s),
\]

(A-16)

and \( w(x, t_0) := \partial_t u(x, t_0) \). We will show in the following that \( w(x, t) \) is a classical solution of (4.2) and that \( t \to \partial_x w(b(t)+, t) \) is continuous.

Now we will show that \( w(x, t) \) defined in (A-16) is a classical solution of (4.2) on \([t_0, T] \) with initial condition \( \partial_t u(x, t_0) \). By definition \( w(x, t_0) = \partial_t u(x, t_0) \). On the other hand we have that \( \lim_{x \to +\infty} w(x, t) = 0 \), which follows from the facts that \( \lim_{x \to +\infty} G(x, t; y, t_0) = 0 \) and \( \sigma, v(s), w(y, t_0) \) and \( h(y, s) \) are all bounded. Furthermore, using the properties of the Green function and the definition of \( w \) (see A-16), we also have that \( L Dw(x, t) = h(x, t) \) for \( x > b(t), t \in [t_0, T] \). Observe that \( \partial_t w \), \( \partial_x w \) and \( \partial^2_x w \) all exist and are all continuous in this domain.

In the following we will show that \( \partial_x w(b(t)+, t) = v(t) \), which implies the continuity of \( \partial_x w(b(t)+, t) \). We differentiate \( w(x, t) \) with respect to \( x \) and let \( x \downarrow b(t) \). Since \( v(t) \) and \( \sigma \) are continuous and \( b(t) \) is Holder continuous with exponent \( \alpha > \frac{1}{2} \), we can apply the jump identity (A-11) with \( \rho(s) = \frac{1}{2} \sigma^2(b(s), s) v(s) \). Following the steps that lead to (A-7) in the first part of the proof, we obtain

\[
\partial_x w(b(t)+, t) = -\frac{1}{4} \sigma^2(b(t), t) v(t) - \int_{t_0}^t ds \int_{b(t)}^{+\infty} dy \frac{1}{2} \sigma^2(b(s), s) \partial_x G(b(t), t; b(s), s) + \sum_{i=1}^{2} N_i(t).
\]

(A-17)

Comparing (A-17) to (A-7), we see that \( \partial_x w(b(t)+, t) = v(t), t_0 \leq t \leq T \).

Then it remains to show that \( w(b(t), t) = 0, t_0 \leq t \leq T \). To this end, since we have already shown \( L Dw = h, w \) satisfies the Green’s identity given by (A-8). Integrating the identity (A-8) and using (A-16) and the fact that \( \lim_{x \to +\infty} w(x, t) = 0 \) we can write

\[
\int_{t_0}^t ds \int_{b(t)}^{+\infty} dy \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma \sigma_x (b(s), s) \right) \partial_y G(x, t; b(s), s) \partial_y G(x, t; b(s), s) - \left( \mu - \frac{1}{2} \sigma^2(b(s), s) \right) G(x, t; b(s), s) = 0,
\]

(A-18)

\[x > b(t), t_0 \leq t \leq T.\]

Let \( x > b(t) \). Integrating both sides of (A-18) on \([x, +\infty) \) and using the fact that \( \partial_x G = -\partial_y G \), we obtain

\[
0 = \int_{t_0}^t ds \int_{b(t)}^{+\infty} dy \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma \sigma_x \right) \partial_x G(u, t; b(s), s) - \left( \mu - \frac{1}{2} \sigma^2(b(s), s) \right) \int_{x}^{+\infty} du G(u, t; b(s), s)
\]

\[= \int_{t_0}^t ds \int_{b(t)}^{+\infty} dy \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma \sigma_x \right) G(x, t; b(s), s) - \left( \mu - \frac{1}{2} \sigma^2(b(s), s) \right) \int_{x}^{+\infty} du G(u, t; b(s), s).\]
Taking the derivative with respect to \( x \), letting \( x \downarrow b(t) \) and using the jump identity \((A-11)\) with 
\[
\rho(s) = \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma_x(b(s), s) \right) \ w(b(s), s),
\]
we arrive at
\[
(A-19) \\
\frac{1}{2} \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma_x(b(s), s) \right) w(b(t), t) \\
= \int_{t_0}^t ds \ w(b(s), s) \left[ \left( \frac{1}{2} \sigma^2(b(s), s) + \sigma_x(b(s), s) \right) \partial_y G(b(t), t; b(s), s) - \left( \mu - \frac{1}{2} \sigma^2(b(s), s) \right) G(b(t), t; b(s), s) \right].
\]
Since \( b(t) \) is Hölder continuous with exponent \( \alpha > 1/2 \), we have
\[
|\partial_y G(b(t), t; b(s), s)| \leq \frac{C}{(t-s)^{\frac{\alpha}{2}}},
\]
Therefore both \( \partial_y G(b(t), t; b(s), s) \) and \( G(b(t), t; b(s), s) \) are integrable. Consequently, it follows from \((A-18), (A-19)\) and the Dominated Convergence Theorem that \( w(b(t), t) = 0, \ t_0 \leq t \leq T. \)

**Proof of Lemma 4.1.** Proof of (i). Let \( v(t) \) be the unique continuous solution of the Volterra equation \((A-7)\). Define \( w(x, t) \) as in \((A-16)\). The Lemma A-1 shows that \( w(x, t) \) is a classical solution to equation \((4.2)\). Let us define
\[
\tilde{u}(x, t) = u(x, t_0) + \int_{t_0}^t w(x, s)ds, \quad x \geq b(t), t_0 \leq t \leq T.
\]
It is easy to check that \( \tilde{u}(x, t) \) is a classical solution of the equation \((2.8) - (2.10)\) with initial condition \( u(x, t_0) \). Since \((2.8) - (2.10)\) has a unique solution, we conclude that \( u(x, t) = \tilde{u}(x, t), x \geq b(t) \) and \( t_0 \leq t \leq T. \) Lemma A-1 also implies that
\[
\partial_x \partial_t \tilde{u}(b(t)+, t) = \partial_x w(b(t)+, t) = v(t), \quad t_0 \leq t \leq T,
\]
which implies that \( \partial_x \partial_t u(b(t)+, t), t_0 \leq t \leq T, \) is continuous. The statement follows since \( t_0 > 0 \) is arbitrary.

Proof of (ii). Let \((x, t)\) be such that \( x > b(t) \). Choosing \( t_0 < t \) such that \( b(t_0) < x \), we can see that \( \int_{t_0}^t ds \partial_x G(x, t; b(s), s) < +\infty \). As a result, we have
\[
\frac{\partial}{\partial x} M_0(x, t) = \int_{t_0}^t ds \left( \frac{1}{2} \sigma^2(b(s), s) \partial_x w(b(s)+, s) \partial_x G(x, t; b(s), s). \right.
\]
We have shown in part (i) that \( \partial_x w(b(s)+, s) \) is continuous with respect to \( s \). It is easy to show \( \partial_x M_0(x, t) \) is continuous around a sufficiently small neighborhood of \((x, t)\). One can also show that the functions \( \partial_x M_i(x, t), i \in \{1, 2\} \) are also continuous by similar means. Thus, it is clear from \((A-12)\) that \( \partial_x \partial_t u(x, t) \) is continuous in this small neighborhood around \((x, t)\). Therefore, the part (ii) of Lemma 4.1 follows, because of the arbitrary choice of \( x \) and \( t \).

**A.3. Proof of Proposition 6.1 (i).** We will use the following result in Lemma 4.1 in page 239 of Friedman [1976]:

**Lemma A-2.** For any \( a < b < \log K, 0 < t_1 < t_2 < T, \) if both \( u(x, t) \) and \( \partial_t u(x, t) \) belong to \( L^2((t_1, t_2); L^2(a, b)) \), then \( u(t) \) belongs to \( C((t_1, t_2); L^2(a, b)) \).

In this lemma, \( L^2((t_1, t_2); L^2(a, b)) \) is the class of \( L^2 \) maps which map \( t \in (t_1, t_2) \) to the Hilbert space \( L^2(a, b) \). On the other hand \( C((t_1, t_2); L^2(a, b)) \) is the class of continuous maps which map \( t \in (t_1, t_2) \) to \( L^2(a, b) \).

The proof of (6.7) is similar to that of (2.14): First, we will study the penalty problem associated to the free boundary problem (6.1) - (6.5). Then, we will list some key estimates for the solution of the penalty problem. And finally using Lemma A-2 we will conclude. We will give a sketch of this proof below.
Let us consider the following penalty problem

\[ \mathcal{L}_D u_n^\epsilon + \beta_\epsilon (u_n^\epsilon - g_\epsilon) = f_n^\epsilon (x, t), \quad x \in \mathbb{R}, \ 0 < t < T, \]
\[ u_n^\epsilon (x, 0) = g_\epsilon (x), \quad x \in \mathbb{R}, \]

in which \( 0 < \epsilon < 1 \), \( g_\epsilon (x) \in C^\infty (\mathbb{R}) \) such that \( g_\epsilon (x) = (K - \epsilon^2)^+ \) when \( x \) satisfies \( |K - \epsilon^2| \geq \epsilon \). We define \( f_n^\epsilon (x, t) = \zeta_\epsilon * f_n(x, t) \), where \( \zeta_\epsilon \) is the standard mollifier in \( x \) and \( t \) (see Evans [1998] Appendix C4 in page 629). As a result, we have \( f_n^\epsilon (x, t) \in C^\infty (\mathbb{R} \times (0, T)) \). Moreover, because \( f_n(x, t) \) is continuous, \( f_n^\epsilon (x, t) \) uniformly converge to \( f_n(x, t) \) on any compact domains as \( \epsilon \to 0 \). On the other hand, from our assumption that \( \partial_t u_{n-1}(x, t) \) is bounded for any \( \epsilon > 0 \) and \( \nu \) is a probability measure on \( R \), we obtain that

\[ \partial_t f_n(x, t) \text{ is bounded in } \mathbb{R} \times [\epsilon, T], \quad \text{for any } \epsilon > 0. \]

Thanks to (A-21), it is easy to see that \( \partial_t f_n^\epsilon (x, t) \) are uniformly bounded for any \( \epsilon > 0 \). The penalty functions \( \beta_\epsilon (x) \) is a sequence of infinitely differentiable, negative, increasing and concave functions such that \( \beta_\epsilon (0) = -C_\epsilon \leq -(r + \lambda)K - r_\epsilon \). The limit of the sequence is

\[ \lim_{\epsilon \to 0} \beta_\epsilon (x) = \begin{cases} 0, & x \geq 0, \\ -\infty, & x < 0. \end{cases} \]

It is well known that the penalty problem has a classical solution (see page 1009 of Friedman and Kinderlehrer [1974/75]). Moreover, a proof similar to that of the proof of Theorem 2.1 of Yang et al. [2006] shows that \( u_n^\epsilon (x, t) \in C^\infty (\mathbb{R} \times (0, T)) \) \( \cap L^\infty (\mathbb{R} \times (0, T)) \).

On the other hand, \( u_n^\epsilon (x, t) \) satisfy the following estimates for any \( a < b < \log K, \ 0 < t_1 < t_2 \leq T, \)

\[ \int_a^b \left( \frac{\partial u_n^\epsilon}{\partial t} \right)^2 (x, t)dx \leq C, \quad t \in [t_1, t_2], \]
\[ \int_{t_1}^{t_2} \int_a^b \left( \frac{\partial^2 u_n^\epsilon}{\partial x \partial t} \right)^2 dx dt \leq C, \]
\[ \int_{t_1}^{t_2} \int_a^b \left( \frac{\partial^2 u_n^\epsilon}{\partial t^2} \right)^2 dx dt + \int_a^b \left( \frac{\partial^2 u_n^\epsilon}{\partial x \partial t} \right)^2 (x, t)dx \leq C, \quad t \in [t_1, t_2], \]

in which \( C \) is a constant independent of \( \epsilon \). These estimates use similar techniques to the ones used in the proofs of Lemmas 2.8, 2.10 and 2.11 in Yang et al. [2006], since \( f_n(x, t) \) satisfies (A-21). (Similar estimates can also be found in Friedman and Kinderlehrer [1974/75]). We will give the proof for the inequality (A-24) below. The other inequalities can be similarly obtained.

**Proof of inequality (A-24).** Let us consider \( w_n(x, t) = \partial_t u_n^\epsilon (x, t) \). Since \( u_n^\epsilon (x, t) \in C^\infty (\mathbb{R} \times (0, T)) \), it follows from (A-20) that \( w_n(x, t) \) satisfies

\[ \mathcal{L}_D w_n + \beta_\epsilon'(u_n^\epsilon - g_\epsilon)w_n = \frac{\partial}{\partial t} f_n^\epsilon (x, t). \]

Let \( \eta(x, t) \in C_0^\infty (\mathbb{R} \times (0, T)) \), such that \( \eta(x, t) = 1 \) for \( (x, t) \in [a, b] \times [t_1, t_2] \), and \( \eta(x, t) = 0 \) outside a small neighborhood of \( [a, b] \times [t_1, t_2] \). Multiplying both sides of (A-25) by \( \eta^2 \partial_t w_n \) and integrating over the domain
\( \Omega_t = \mathbb{R} \times (0, t) \) in which \( t_1 \leq t \leq t_2 \), we obtain

\[
0 = \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds - \int \int_{\Omega_t} \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 w_n}{\partial x^2} \frac{\partial w_n}{\partial x} \partial_t dx \, ds - \int \int_{\Omega_t} \left( \mu - \frac{1}{2} \sigma^2 \right) \eta^2 \frac{\partial w_n}{\partial x} \frac{\partial w_n}{\partial t} \, dx \, ds \\
+ \big( r + \lambda \big) \int \int_{\Omega_t} \eta^2 \frac{\partial w_n}{\partial t} \, dx \, ds + \int \int_{\Omega_t} \eta^2 \beta'(u_n - g) w_n \frac{\partial w_n}{\partial t} \, dx \, ds - \int \int_{\Omega_t} \eta^2 \frac{\partial w_n}{\partial x} \frac{\partial f_n(x, s)}{\partial t} \, dx \, ds \\
\triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]

where \( I_j \) is the \( j \)-th term on the left and \( \sigma = \sigma(x, t) \) satisfying the assumption (2.7). In the following, we will estimate each \( I_j \) separately. In deriving these estimates we will make use of the inequality

(A-26) \[ \frac{1}{6} A^2 + AB + \frac{9}{6} B^2 \geq 0, \]

for any \( A, B \in \mathbb{R} \). In the following estimations, \( C \) will represent different constants independent of \( \epsilon \).

\[
I_2 = \frac{1}{2} \int \int_{\Omega_t} \sigma^2 \eta^2 \frac{\partial^2 w_n}{\partial x^2} \frac{\partial w_n}{\partial x} \, dx \, ds + \frac{1}{2} \int \int_{\Omega_t} \sigma^2 \eta^2 \frac{\partial^2 w_n}{\partial x^2} \frac{\partial w_n}{\partial x} \, dx \, ds + \int \int_{\Omega_t} \sigma \eta \frac{\partial w_n}{\partial x} \frac{\partial w_n}{\partial t} \, dx \, ds \\
= \frac{1}{4} \int \int_{\Omega_t} \sigma^2 \eta^2 \frac{\partial \frac{\partial w_n}{\partial x}}{\partial t} \, dx \, ds + \int \int_{\Omega_t} \sigma \eta \frac{\partial w_n}{\partial x} \frac{\partial w_n}{\partial t} \, dx \, ds \\
\geq \frac{\delta^2}{4} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial x} \right)^2 \, dx \, ds - \frac{9}{6} \int \int_{\Omega_t} \sigma \eta \frac{\partial w_n}{\partial x} \frac{\partial w_n}{\partial t} \, dx \, ds - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds
\]

The first four equalities follow from integration by part. The first inequality follows from the assumption (2.7) and the inequality (A-26) with \( A = \eta \frac{\partial w_n}{\partial t} \) and \( B = \sigma \frac{\partial w_n}{\partial x} \). The last inequality follows from estimation (A-23).

For \( I_i \) (\( i = 3, 4, 5 \)), a similar procedure yields

\[
I_3 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds, I_4 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds, I_5 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds.
\]

For \( I_6 \), we have

\[
I_6 = - \int \int_{\Omega_t} \eta^2 \frac{\partial w_n}{\partial t} \frac{\partial f_n}{\partial t} \, dx \, ds \geq - \frac{9}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial f_n}{\partial t} \right)^2 \, dx \, ds - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds
\]

The first inequality can be obtained using (A-26), whereas to obtain the last inequality, we use the fact that \( \partial_t f_n(x, t) \) is uniformly bounded. Combining all these estimates for \( I_j \), we obtain

\[
\frac{1}{6} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial t} \right)^2 \, dx \, ds + \frac{\delta^2}{4} \int \int_{\Omega_t} \eta^2 \left( \frac{\partial w_n}{\partial x} \right)^2 \, dx \, ds \leq C.
\]

This completes the proof of (A-24). \( \square \)

Using a similar proof to that of Lemma 2.2 of Yang et al. [2006], we can show that \( u_n^\epsilon(x, t) \) is uniformly bounded. Thus there is a subsequence that \( \{ u_n^\epsilon \} \) converges weakly to \( u_n \) in \( L^2((a, b); L^2(t_1, t_2)) \) for any \( a < b < \log K \).
0 < t_1 < t < t_2 < T (see Appendix D in Evans [1998] for an account of the concept of weak convergence). On the other hand, it follows from the estimates in (A-22) - (A-24) that \( \frac{\partial u_n}{\partial t} \) and \( \frac{\partial^2 u_n}{\partial x \partial t} \) are uniformly bounded in \( L^2(a, b) \), \( \frac{\partial^2 u_n}{\partial x^2} \) and \( \frac{\partial^2 u_n}{\partial t^2} \) are uniformly bounded in \( L^2((t_1, t_2); L^2(a, b)) \). Therefore there exists a further subsequence satisfying

\[
\frac{\partial u_n}{\partial t}, \frac{\partial^2 u_n}{\partial x \partial t}, \frac{\partial^2 u_n}{\partial x^2} \rightarrow \frac{\partial^2 u}{\partial t^2},
\]

where derivatives of \( u_n \) are defined in weak sense (see Appendix D in Evans [1998]). Here, the convergences are weak convergences. Since \( ||u|| \leq \lim \inf_j ||u^j|| \) (see Appendix D in Evans [1998] ) (A-22) - (A-24) imply that

\[
\frac{\partial u_n}{\partial t} \in L^\infty((t_1, t_2); L^2(a, b)), \quad \frac{\partial^2 u_n}{\partial x \partial t} \in L^2((t_1, t_2); L^2(a, b)).
\]

Then it follows from Lemma A-2 that the derivative \( \partial_t u_n \) exists and is inside the space \( C((t_1, t_2); L^2(a, b)) \). On the other hand, for fixed \( t \in [t_1, t_2] \), it also follows from (A-22) and (A-24) and the Sobolev Embedding Theorem (see, for example, Theorem 4 in page 266 of Evans [1998]) that

\[
(A-27) \quad \left| \frac{\partial u_n}{\partial t}(x, t) - \frac{\partial u_n}{\partial t}(\bar{x}, t) \right| \leq C|x - \bar{x}|^{1/2}, \quad x, \bar{x} \in (a, b),
\]

in which \( C \) is a positive constant that does not depend on \( t \). We already know that \( \partial_t u_n(\cdot, t) \) is a continuous map with respect to \( t \), therefore (A-27) implies that

\[
\frac{\partial u_n}{\partial t} \in C((a, b) \times (t_1, t_2)).
\]

Therefore \( \partial_t u_n \in C(\mathbb{R} \times (0, T]) \) because the choice of \( a, b, t_1 \) and \( t_2 \) are arbitrary and \( \partial_t u_n \in C([\log K, +\infty) \times (0, T]) \) since \( [\log K, +\infty) \times (0, T)] \in C_n \). Moreover, we have

\[
(A-28) \quad \lim_{x \to b(t_0)} \frac{\partial u_n}{\partial t}(x, t_0) = \lim_{t \to t_0^-} \frac{\partial u_n}{\partial t}(b(t_0), t) = 0,
\]

because \( (b_n(t_0), t) \) is inside the stopping region for \( t < t_0 \) as \( b_n(t) \) is decreasing. 

\[\square\]

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