ON THE REDUCED CANONICAL QUANTIZATION OF THE
INDUCED 2D-GRAVITY†

José Navarro-Salas\textsuperscript{1,2}, Miguel Navarro\textsuperscript{2,3,4},
C.F. Talavera\textsuperscript{1,2} and Víctor Aldaya\textsuperscript{2,4}

1.- Departamento de Física Teórica, Burjassot-46100, Valencia, Spain.
2.- IFIC, Centro Mixto Universidad de Valencia-CSIC, Burjassot-46100, Valencia, Spain.
3.- The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom.
4.- Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias,
Universidad de Granada, Campus de Fuentenueva, 18002, Granada, Spain.

Abstract

The quantization of the induced 2d-gravity on a compact spatial section is
carried out in three different ways. In the three approaches the supermomentum
constraint is solved at the classical level but they differ in the way the
hamiltonian
constraint is imposed. We compare these approaches establishing an isomorphism
between the resulting Hilbert spaces.

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1 Introduction

Generally covariant theories in a two-dimensional space-time collect the advantages of both being much simpler than the corresponding theories in 3+1 and 2+1 dimensions, and of having a sufficiently rich structure which can shed light on the issues that appear in quantizing higher dimensional theories.

Several years ago Jackiw and Teitelboim [1, 2] proposed the equation

\[ R + \frac{\Lambda}{2} = 0 \]  \hspace{1cm} (1)

as the natural analogue of the vacuum Einstein equations with a cosmological term. This equation can be obtained from a local variational principle if a scalar field, playing the role of a lagrangian multiplier, is incorporated in the theory. The above equation can also be derived from the induced 2d gravity [3]:

\[ S = \frac{c}{96\pi} \int \sqrt{-g} (R - \frac{1}{2} R + \Lambda) . \]  \hspace{1cm} (2)

This action is non-local but it is preferable to convert it into a local one by introducing an auxiliary scalar field \( \Phi \). The action can be written as

\[ S = \frac{1}{2} \int \sqrt{-g} (g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2R\Phi + \Lambda) . \]  \hspace{1cm} (3)

The aim of this paper is to carry out a canonical analysis of the induced 2d-gravity theory (2) in three different ways (we shall restrict ourselves to the case of a compact spatial section). The first one is presented in section 3 and it was spelled out in [4]. It is based on the covariant formulation of the canonical formalism [5]. The reduced phase space of the theory turns out to be a two-dimensional cotangent bundle, and the corresponding (geometric) quantization permits to determine the Hilbert space. In section 3 we introduce the ADM formulation of the theory. By gauge fixing and imposing the supermomentum constraint we can reduce the theory to a finite-dimensional system. At this point one can choose different ways to quantize the theory. One way is to look for the reduced hamiltonian (this requires a complete gauge fixing) and then to impose the corresponding Schödinger equation. This is our second approach and it is developed in section 4. The third approach (section 5) is based on the (reduced) Wheeler-DeWitt equation. Along the paper we set up the equivalence of these approaches establishing an isomorphism between the corresponding Hilbert spaces.

2 Covariant phase-space quantization

The covariant definition of the reduced phase-space [5] has been very useful to determine the phase space of a variety of field theories [6, 7]. In this approach,
the reduced phase space is defined as the set of all solutions of the classical theory, modulo gauge transformations (see below). The symplectic form is defined as follows.

Let us consider a field theory with fields $\Psi^\alpha$ and Lagrangian $L$. If we vary the fields in the Lagrangian we get

$$\delta L = \partial_\mu j^\mu + (E - L)_a \delta \Psi^a .$$ \hfill (4)

If we regard now $\delta$ as an exterior derivative operator in the space of classical solutions $\Psi^\alpha(x)$, we can, in a natural way, pullback (4) to the space of all solutions of the equations of motion, $(E-L)=0$, and consider $j^\mu$ as a vector-valued one-form on this space.

Since $\delta j^\mu$ is a conserved current, $\partial_\mu \delta j^\mu = 0$, it is natural to define the (pre-)symplectic form $\omega$ as the corresponding conserved charge,

$$\omega = - \int_\Sigma \delta j^\mu \mathrm{d} \sigma_\mu$$ \hfill (5)

($\Sigma$ is any spatial hypersurface of the space-time). With this definition, we prevent $\omega$ from depending on $\Sigma$ or on the time co-ordinate. Because $\delta j^\mu$ is exact, so is $\omega$ and thus closed.

The only property we can not assure for $\omega$ is nondegenerateness since $\omega$ as defined above can have a non-trivial kernel. We define now the gauge transformations as the ones generated by the kernel of $\omega$. If now, in the space of all solutions, we take modulus by the gauge transformations we get a symplectic space which is called the reduced (or physical) phase space.

Let us apply the program above to the induced 2d-gravity. The equations of motion obtained from (2) imply the vanishing of the stress-tensor $T_{\mu\nu}$ that is given by

$$T_{\mu\nu} = - \nabla_\mu \Phi \nabla_\nu \Phi + 2 \nabla_\mu \nabla_\nu \Phi + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Phi \nabla_\alpha \Phi$$

$$- g_{\mu\nu} (2R + \frac{1}{2} \Lambda) ,$$ \hfill (6)

and the relation of $\Phi$ with the curvature:

$$\Box \Phi = R .$$ \hfill (7)

If we use now the gauge invariance of the metric under diffeomorphisms to bring it to a conformally flat form,

$$\mathrm{d} s^2 = -2e^\rho \mathrm{d} x^+ \mathrm{d} x^- ,$$ \hfill (8)

($x^+ = t + x, \ x^- = t - x$ are the light-cone coordinates) the equations of motion split into the relation of $\Phi$ with the curvature,

$$\Box \Phi \equiv 2e^{-\rho} \partial_+ \partial_- \Phi = R \equiv -2e^{-\rho} \partial_+ \partial_- \rho ,$$ \hfill (9)
the Liouville equation,

\[ 0 = T_{++} = 2 \partial_+ \partial_- \rho - \frac{\Lambda}{2} e^\rho , \]  

(10)

and the constraints

\[ 0 = T_{+} = - (\partial_+ \Phi)^2 + 2 \partial_+^2 \Phi - 2 \partial_+ \rho \partial_+ \Phi = 0 , \]  

(11)

\[ 0 = T_{-} = - (\partial_- \Phi)^2 + 2 \partial_-^2 \Phi - 2 \partial_- \rho \partial_- \Phi = 0 . \]  

(12)

The general solution for the metric field \( g_{\mu \nu} \) and the dilaton field \( \Phi \) can be easily found [4], and can be written as follows:

\[ ds^2 = -2 \frac{\partial_+ A \partial_- B}{(1 - \frac{A}{B} AB)^2} d^+ dx^- , \]  

(13)

\[ \Phi = \ln \lambda \frac{(1 - \frac{A}{B} AB)^2}{((d - a) A + b) (\frac{A}{B})^2} , \]  

(14)

and

\[ \Phi = \ln \lambda \frac{(1 - \frac{A}{B} AB)^2}{(-b \frac{A}{B} + (d - a))^2} , \]  

(15)

where \( a, b, d \) are such that \( M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) belong to the affine subgroup of \( PSL(2, R) \), i.e., \( a = d^{-1} \), and \( A = A(x^+) , B = B(x^-) \) verify the monodromy transformations properties (we choose the length of the circle equal to unity)

\[ A(y + 1) = \frac{a A(y) + b}{d} \equiv M(A(y)) , \]  

(16)

\[ -\frac{\Lambda}{8} B(y - 1) = \frac{-d \frac{A}{B} B(y)}{b \frac{A}{B} B(y) + a} \equiv M^{-1T} \left( -\frac{\Lambda}{8} B(y) \right) . \]  

(17)

If we choose the spatial hypersurface as the one defined by \( t = t_0 \) the symplectic form can be written as:

\[ \omega = \int_x^{x^+} [ - \delta \Phi \delta (\partial_+ + \partial_-)(\Phi + \rho) + \delta (\partial_+ + \partial_-) \Phi \delta \rho ] . \]  

(18)

The projection onto the space of classical solutions takes an special form

\[ \omega = \frac{1}{2} \int_x^{x^+} (\partial_+ - \partial_-) W , \]  

(19)
where $W$ is given by

$$ W = \frac{1}{2} \left\{ \delta \ln \frac{\partial A}{\partial B} \left( \frac{\Lambda}{8} B \right)^2 \right\} \delta \ln \left( \frac{1 - \frac{\Lambda}{8} A B}{(d - a) A + b} \right)^2 \right\} \delta \ln \left( \frac{1 - \frac{\Lambda}{8} A B}{((d - a) A + b)^2 \left( \frac{\Lambda}{8} B \right)^2} \right) + \delta \ln \left( \frac{1 - \frac{\Lambda}{8} A B}{((d - a) A + b)^2 \left( \frac{\Lambda}{8} B \right)^2} \right) \right\} , $$

(20)

for the solution (14) for $\Phi$, and by

$$ W = \frac{1}{2} \left\{ \delta \ln \frac{\partial A}{\partial B} \left( \frac{\Lambda}{8} B - (d - a) \right)^2 \right\} \delta \ln \left( \frac{(1 - \frac{\Lambda}{8} A B)^2}{(\frac{\Lambda}{8} B - (d - a))^2} \right) \right\} \delta \ln \left( \frac{(1 - \frac{\Lambda}{8} A B)^2}{(\frac{\Lambda}{8} B - (d - a))^2} \right) \right\} , $$

(21)

for the solution (15) for $\Phi$.

In any case, since $\omega$ does not depend on the coordinate $x$ in (19), it cannot depend on either of the functions $A$ or $B$. So $\omega$ will depend only on the classes of monodromy transformations to which the functions $A$ and $B$ belong, and on the parameter $\lambda$ in (14,15). Moreover, if we transform the functions $A$ and $B$ as

$$ A \rightarrow h(A) , $$

$$ -\frac{\Lambda}{8} B \rightarrow h^{-1T}(-\frac{\Lambda}{8} B) , $$

(22)

(23)

where $h$ is a constant affine matrix acting as a M"{o}bius transformation, we get the same solution of the equations of motion. Under the transformation (22,23) the monodromy parameters transform as

$$ M \rightarrow h M h^{-1} . $$

(24)

Thus, two solutions which differs on a transformation of the type (24) are the same point of the reduced phase space. The only invariant quantity under the transformation (24), and hence the only allowed monodromy dependence in $\omega$, is the parameter $a$.

A direct computation from (20, 21) leads to:

$$ \omega = 2 \frac{\delta \lambda}{\lambda} \frac{\delta a}{a} , $$

(25)
for the solutions in (20), and

\[ \omega = -2 \frac{\delta \lambda}{\lambda} \frac{\delta a}{a}. \]  

(26)

for the solutions in (21).

From (25, 26) and previous considerations, we are tempted to assume that the reduced phase space is of the form

\[ T^* (G/\text{ad} G) \cup T^* (G/\text{ad} G), \]  

(27)

where \( G \) is the affine subgroup of \( PSL(2, R) \). This would lead us to a Hilbert space of the form

\[ \mathcal{H} = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}, \]  

(28)

where

\[ \mathcal{H}^{(+)} = \mathcal{H}^{(-)} = L^2(R^+) \oplus C^2. \]  

(29)

The result (28), in which the Hilbert space has a continuum and a discrete sector is in accordance with some results obtained by BRST methods [9]. However, there are also some evidence that the discrete sector cannot be endowed with a well defined inner product. This result is achieved here by showing that, in fact, the discrete sector actually does not appear. This is a consequence of the additional symmetry \( A \to -A, \ B \to -B, \ a \to a, \ b \to -b \) that identify the otherwise distinct parabolic \((a = 1)\) solutions.

Let us write the classical solutions (13, 14, 15) in a more explicit form. To this end we should completely fix the space-time coordinates by imposing an additional “gauge-type” condition. From (9) we observe that

\[ \partial_+ \partial_- (\Phi + \rho) = 0. \]  

(30)

Therefore we can (and we will) choose a spatially homogeneous conformal gauge by imposing

\[ \Phi + \rho = \epsilon + 2pt, \]  

(31)

where \( \epsilon \) and \( p \) are constant parameters.

If \( a \neq 1 \), i.e., if the monodromy class is hyperbolic, the solutions take the form:

\[ ds^2 = -2 \frac{8}{|\Lambda|} \frac{p^2 e^{2pt}}{(1 - \text{sgn} \Lambda e^{2pt})^2} dx^+ dx^-, \]  

(32)
\( \Phi = \ln \lambda \frac{(1 - sgn \Lambda e^{2pt})^2}{4(\sinh \frac{p}{2})^2 e^{4pt}} , \) \hfill (33) \\
\( \Phi = \ln \lambda \frac{(1 - sgn \Lambda e^{2pt})^2}{4(\sinh \frac{p}{2})^2} , \) \hfill (34)

where \( e^p = a = d^{-1} \). On the other hand if \( a = 1 = d \), the unique parabolic solution takes the form

\[
\begin{align*}
    ds^2 &= -\frac{2}{\Lambda t^2} dx^+ dx^- , \\
    \Phi &= \ln 4 \lambda t^2 .
\end{align*}
\] \hfill (35) \hfill (36)

We can easily see that (33) and (34) transform into each other when we make the replacement \( a \leftrightarrow a^{-1} = d \), under which also (24) and (25) transform into each other. Moreover, when \( \Lambda > 0 \), (35) has the right signature and can be obtained from (32-34) in the limit \( p \to 0 \) \( (a \to 1) \). So we can conclude that the phase space for \( \Lambda > 0 \) is just

\[
    T^* (R) \cup T^* (\mathbb{R}) ,
\] \hfill (37)

with the symplectic form

\[
\omega = 2 \delta (\ln \lambda) \delta p .
\] \hfill (38)

The two sectors in (37) correspond to whether the scalar field is expanding or contracting.

The cotangent bundle structure of the phase space makes it easy to determine the Hilbert space of the quantum theory: it will be given by the square integrable functions on the configuration space. Hence in this case we shall have

\[
\mathcal{H} = L^2 (R, dp) \oplus L^2 (R, dp). 
\] \hfill (39)

For \( \Lambda < 0 \) neither (35) is positive definite nor (34) can be obtained from (33,34) as a limiting case. So that the phase space is given by

\[
    T^* (R^+) \cup T^* (R^+) ,
\] \hfill (40)

with the symplectic form

\[
\omega = 2 \delta \ln \lambda \delta p .
\] \hfill (41)

The Hilbert space should be now of the form

\[
\mathcal{H} = L^2 \left( \mathbb{R}^+, \frac{dp}{p} \right) \oplus L^2 \left( \mathbb{R}^+, \frac{dp}{p} \right). 
\] \hfill (42)

Although it is difficult to figure out how the Hilbert spaces (42) can be actually realized, we shall see in the next sections that this prediction for the Hilbert space is consistent with other quantization approaches.
3 ADM formulation.

In section 2 we saw explicitly that the classical solutions of the theory are spatially homogeneous. As has been shown in [10] for a wide class of 2d dilaton gravity models, this is so because the theory (3) has a Killing vector the flow of which determines a natural coordinate system on the cylinder where the metric and the scalar fields take an homogeneous form. The existence of the Killing vector requires the metric equations of motion be satisfied. At this point it is important to remark that one can indeed reduce the theory to a finite number of degrees of freedom by imposing the supermomentum constraint only.

To this end let us now present the basic ingredients of the ADM formulation of the induced 2d-gravity (see also [11]). First, we introduce the standard parametrization of the two-dimensional metric

\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + N_1 N^1 & N_1 \\ N_1 & a^2 \end{pmatrix}, \]

where \( N \) and \( N^1 \) are the lapse and shift functions respectively. To derive the canonical form of the action we can use the two-dimensional identity

\[ \sqrt{-g} R = -2 \partial_t (aK) + 2 \partial_x (a(KN^1 - a^{-2}N^1)) , \]

where \( K \) is the extrinsic curvature scalar

\[ K = \frac{1}{a^2 N} (N|1_1 - a\dot{a}). \]

Removing total time derivatives we arrive at

\[ S = \int d^2 x (\pi_a \dot{a} + \pi_{\Phi} \dot{\phi} - NC - N^1 C_1) , \]

where the canonical momenta are

\[ \pi_a = \frac{4}{N} (\Phi'N^1 - \Phi), \]

\[ \pi_{\Phi} = \frac{2a}{N} (\Phi'N^1 - \Phi) + \frac{4}{N} \left( (aN^1)' - \dot{a} \right), \]

and the supermomentum and hamiltonian constraints are given by

\[ C_1 = \Phi' \pi_{\Phi} - \pi_{\phi} a, \]

\[ C = \frac{1}{16} a^2 \pi_a^2 - \frac{1}{4} \pi_a \pi_{\Phi} - a\Lambda - \frac{1}{a} \Phi'^2 + 4(a\Phi')'. \]

Making use of the spatial diffeomorphism invariance of the theory we can fix the space coordinate and assume that
In addition to this, and due to the time reparametrization invariance, we can also make a choice of time. All the above considerations suggest the following class of spatially homogeneous definitions of the internal time variable

$$ a = a(t) . \quad (51) $$

where \( \chi \) is a generic function. This implies that

$$ \Phi = \Phi(t) . \quad (53) $$

and also \( N = N(t) \).

The momentum \( \pi_\Phi \) is still a function of both \( t \) and \( x \). However we can integrate the action in (46) with respect to the compact coordinate and the resulting expression is

$$ S = \int dt \left( \pi_a \dot{a} + \dot{\Phi} \int dx \pi_\Phi - NC \right) , \quad (55) $$

where now

$$ C = \frac{1}{16} a \pi_a^2 - \frac{1}{4} \pi_a \int dx \pi_\Phi - a \Lambda . \quad (56) $$

From now on \( \pi_\Phi \) stands for the momentum conjugated to \( \Phi(t) \), i.e., \( \pi_\Phi(t) = \int dx \pi_\Phi(t, x) \). Although (55) corresponds to a minisuperspace approach to the theory, it must be regarded instead as a reduced form of the theory in an appropriate gauge choice and not as a mere approximation to the theory.

For the sake of completeness we write down the equations of motion and the symplectic form obtained from (55)

$$ \dot{\pi}_\Phi = 0 , \quad \dot{\pi}_a = -N \left[ -\frac{1}{16} \pi_a^2 + \frac{\Lambda}{2} \right] , \quad (57) $$

$$ \dot{\Phi} = N \pi_a , \quad \dot{a} = N \left[ \frac{1}{8} a \pi_a - \frac{1}{4} \pi_\Phi \right] , $$

$$ \omega = \delta \pi_\Phi \delta \Phi + \delta \pi_a \delta a . \quad (58) $$

\section{4 Reduced phase-space quantization in the conformal choice of time.}

In this section we shall develop a genuine hamiltonian quantization of the reduced theory (55). In this approach the choice of time is done before quantize and the
constraint \( C = 0 \) is solved classically (see for instance the review [12]). In this context, the choice of time is nothing other but a gauge fixing condition. This gauge fixing is required to be complete in the sense that no further gauge freedom must be left, but also we must not lose information, i.e., actual solutions to the equation of motion.

Let us choose the conformal gauge

\[ N = a , \tag{59} \]

that implies, according to the equations of motion (57) (see also (32)), the following implicit definition of the time variable:

\[ a^2 = 4 \frac{\pi^2 \Phi}{|\Lambda|} \frac{e^{\pi \phi t}}{(1 - \text{sgn} \Lambda e^{\pi \phi t})^2} . \tag{60} \]

Solving now the constraint \( C = 0 \) for \( a\pi_a \) we find the solutions

\[ a\pi_a = 4\pi \Phi \frac{1}{1 + \text{sgn} \Lambda e^{\pi \phi t}} , \tag{61} \]

and

\[ a\pi_a = 4\pi \Phi \frac{1}{1 + \text{sgn} \Lambda e^{-\pi \phi t}} , \tag{62} \]

which remind us the classical twofold solution for the field \( \Phi \).

Once the choice of time has been done, the effective Hamiltonian associated with it, i.e., the function that gives the proper classical time evolution for the remaining fields, is (minus) the conjugate momentum of time. Substituting (61,62) into (58) we find

\[ \omega = \delta \pi \Phi \delta \Phi - 2\pi \Phi \frac{1}{1 + \text{sgn} \Lambda e^{\pm \pi \phi t}} \delta \pi \Phi \delta t . \tag{63} \]

Since this two-form must project down to the (reduced) symplectic form of the model, the hamiltonian flow of the vector field in the kernel of (63) should provide the remaining trajectories of motion (see for instance [13]). Therefore, the effective Hamiltonian should fit the expression

\[ \omega = \delta \pi \Phi \delta \Phi - \delta H \delta t . \tag{64} \]

So, we obtain

\[ H = \int \pi \Phi \frac{1}{1 + \text{sgn} \Lambda e^{\pm \pi \phi t}} \]

\[ = \pi^2 - (\pm) \frac{2\pi \Phi}{t} \ln(1 + \text{sgn} \Lambda e^{\pm \pi \phi t}) - 2 \frac{1}{t^2} \text{Polylog}(2, -\text{sgn} \Lambda e^{\pm \pi \phi t}) . \tag{65} \]
The Hamiltonians in (65) can be converted into each other by means of the change \( \pi_\Phi \leftrightarrow -\pi_\Phi \) or \( t \leftrightarrow -t \). Thus, in this system, reversing the arrow of time is equivalent to changing the sign of the momentum \( \pi_\Phi \).

The quantum system will be described by the wave functions \( \Psi(\pi_\Phi, t) \) that obey a time-dependent Schrödinger equation:

\[
\frac{i\hbar}{\partial t} \Psi(\pi_\Phi, t) = H(\pi_\Phi, t)\Psi(\pi_\Phi, t) .
\] (66)

Since the Hamiltonian functions at different times commute, the Schrödinger equation (66) can be solved immediately to give

\[
\Psi(\pi_\Phi, t) = \Psi(\pi_\Phi) e^{-\frac{i}{\hbar} \int^t d\tau H(\pi_\Phi, \tau)} .
\] (67)

The scalar product of two wave function \( \Psi(\pi_\Phi, t) \) and \( \varphi(\pi_\Phi, t) \) will be taken as the natural one:

\[
< \Psi | \varphi > = \int d\pi_\Phi \Psi^*(\pi_\Phi, t)\varphi(\pi_\Phi, t) = \int d\pi_\Phi \Psi^*(\pi_\Phi) \varphi(\pi_\Phi) .
\] (68)

The Hilbert space for \( \Lambda > 0 \) is hence given by

\[
\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-
\]

\[
= L^2(\mathbb{R}, d\pi_\Phi) \oplus L^2(\mathbb{R}, d\pi_\Phi) .
\] (69)

The two sectors correspond to the double sign of the effective Hamiltonian (65) and represent whether the two-dimensional universe is expanding or contracting. We recover thus the result of section 2. Note that the monodromy parameter \( a = e^{2\pi} \) in section 2 must be identified with the constant of motion \( e^{\pi_\Phi} \).

However, for \( \Lambda < 0 \) we must prevent the wave functions to take any non null value in \( \pi_\Phi = 0 \) since at this point the gauge fixing condition (60) is not well defined. So, we must impose on the wave functions the restriction of vanishing at \( \pi_\Phi = 0 \),

\[
\Psi(\pi_\Phi = 0, t) = 0 ,
\] (70)

restriction that is preserved by the time evolution. Therefore, for \( \Lambda < 0 \), the Hilbert space will be given by

\[
\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-
\]

\[
= L^2(\mathbb{R}^+, d\pi_\Phi) \oplus L^2(\mathbb{R}^+, d\pi_\Phi) .
\] (71)

which can be identified with (12).
5 Quantization via the Wheeler-DeWitt equation.

In this section we shall quantize the reduced theory (55) without any identification of time prior to quantization. This essentially means to impose the operator version of the classical hamiltonian constraint, i.e., the Wheeler-DeWitt equation. To propose the Wheeler-DeWitt operator $\hat{C}$ for (56) we face at once the problem of the operator ordering ambiguities and the inequality $a > 0$ of the scale variable. The second difficulty can be solved by using the affine algebra $[\hat{a}, \hat{p}_a] = i\hbar \hat{a}$ ($\hat{p}_a = -i\hbar \frac{\partial}{\partial a}$), instead of the Heisenberg- Weyl algebra, as the basic one to define the quantization [14]. The reason is that the operator $\hat{\pi}_a = -i\hbar \frac{\partial}{\partial a}$ fails to be self-adjoint on $L^2(\mathbb{R}^+, da)$, whereas the affine operator $\hat{p}_a = -i\hbar a \frac{\partial}{\partial a}$ is self-adjoint in $L^2(\mathbb{R}^+, da)$.

Imposing that the Wheeler-DeWitt operator be self-adjoint with respect to the measure $\frac{da}{a} d\Phi$ we can write the following expression for $\hat{C}$:

$$\hat{C} = \left[ \frac{1}{16} \hat{a}^{\alpha+i\beta} \hat{p}_a \hat{a}^{-1-2\alpha} \hat{p}_a \hat{a}^{-i\beta} + \right.$$
$$- \left. \frac{1}{8} \left( \hat{a}^{\gamma+i\sigma} \hat{p}_a \hat{a}^{-\gamma-i\sigma} + \hat{a}^{\gamma+i\sigma-1} \hat{p}_a \hat{a}^{-\gamma-i\sigma} \right) \hat{\pi}_\Phi + \Lambda \hat{a} \right], \quad (72)$$

where $\alpha, \beta, \gamma$ and $\sigma$ are arbitrary factor-ordering parameters.

We can separate variables in the Wheeler-DeWitt equation by expanding the wave function $\Psi$ in $\hat{\pi}_\Phi$ eigenstates

$$\Psi = \int dq e^{i\Phi q} \Psi_q(a) \ . \quad (73)$$

Inserting (73) into the equation $\hat{C}\Psi = 0$, where $\hat{C}$ is given by (72), we obtain that the functions $\Psi_q(a)$ obey the equation

$$\left( \frac{d^2}{da^2} + \frac{1}{a} (1-2\zeta) \frac{d}{da} + \frac{1}{a^2} (\zeta^2 - \nu^2) + 4 \frac{\Lambda}{\hbar^2} \right) \Psi_q(a) = 0 \ , \quad (74)$$

where

$$\zeta = \frac{1}{2} \left( 1 + 4 \frac{iq}{\hbar} + 2i\gamma \right) \ , \quad (75)$$
$$\nu^2 = \frac{1}{4} \left( 1 - 16 \frac{q^2}{\hbar^2} - 8\gamma^2 + 4\alpha(\alpha + 1) + 16 \frac{q}{\hbar} (\sigma - \gamma) \right) \ . \quad (76)$$

The solutions of the above equation are

$$\Psi_q(a) = a^{\zeta} Z_{\nu} \left( \frac{2|\Lambda|^{\frac{1}{2}}}{\hbar} a \right) \ , \quad (77)$$
where \( Z_\nu \) are ordinary (modified) Bessel functions for \( \Lambda > 0 \) (\( \Lambda < 0 \)) with order \( \nu \).

In constructing the Wheeler-DeWitt operator we required hermiticity with respect to the standard inner product

\[
< \Psi_1 | \Psi_2 > = \int \frac{da}{a} d\Phi \Psi_1^* \Psi_2.
\] (78)

In canonical quantum gravity it is therefore natural to propose (78) as the scalar product for the solutions of the Wheeler-DeWitt equation. This proposal for the scalar product is problematic in the sense that we are integrating over one of the configuration variables that could have been defined as the “internal” time variable \([12]\). However we shall insist on using it but having in mind that (78) could be divergent and require, therefore, some sort of regularization.

Let us analyse now the situation for the case of negative cosmological constant. We can expand the general solution to the Wheeler-DeWitt equation in terms of the modified Bessel and Hankel functions \( I_\nu \) and \( K_\nu \). However, due to the exponential behaviour of the functions \( I \) for large \( x \) (\( x \equiv 2|\Lambda \|^\frac{1}{2}a \)), they do not lead to normalizable wavefunctions and, therefore, should then be excluded from the physical Hilbert space. The physical wave functions should be of the form

\[
\Psi = \int dq e^{iqx} a^\xi C(q)K_\nu(x),
\] (79)

To determine the Hilbert space we should find out the range of variation of the order \( \nu \). Due to the small \( x \) behaviour or the modified Hankel functions, the wave functions will be normalizable when

\[
\nu^2 < \frac{1}{4}.
\] (80)

To obtain the maximum range of variation for \( \nu^2 \) as \( q \) varies over the real line we should choose the factor ordering parameters in such a way that (76) turns out to be of the form

\[
\nu^2 = \frac{1}{4}(1 - 16\left(\frac{q - q_0}{\hbar}\right)^2).
\] (81)

The constant shift \( q_0 \) of \( q \) in (81) can be chosen according to the classical theory. On the covariant phase space the constant of motion \( \pi_\Phi \) is proportional to the monodromy parameter \( \ln a \). Owing to the absence of classical solutions for \( a = 1 \) the constant \( q_0 \) should vanish to exclude the quantum solution \( \pi_\Phi = 0 \). Therefore we are finally led to the expression

\[
\nu^2 = \frac{1}{4}(1 - 16\left(\frac{q^2}{\hbar^2}\right)),
\] (82)

which corresponds to \( \alpha = \beta = \gamma = \sigma = 0 \) in (76).
Now we want to determine the Hilbert space when the cosmological constant is positive. According to (77) the general solution to the Wheeler-DeWitt equation can be expanded as (\(\text{Re}\nu \geq 0, \text{Im}\nu \geq 0\))

\[
\Psi = \int dq a^{k+2i\frac{q}{\hbar}} e^{i\frac{qx}{\hbar}} (A(q)J_\nu(x) + B(q)N_\nu(x)),
\]

where \(A(q)\) and \(B(q)\) are arbitrary complex functions and \(\nu\) is given by (82). The norm of the wave function (83) with respect to (78) is given by (82).

\[
<k|\Psi|> = \frac{\hbar}{k} \int_{-\infty}^{+\infty} dq \int_{0}^{+\infty} dx \left[ |A(q)|^2 |J_\nu(x)|^2 + |B(q)|^2 |N_\nu(x)|^2 + A^*(q)B(q)J_\nu^*(x)N_\nu(x) + A(q)B^*(q)J_\nu(x)N_\nu^*(x) \right].
\]

Due to the asymptotic behaviour of the Bessel functions for large \(x\) the above integral are divergent. We can define a regularized scalar product by substituting the integration measure \(dx\) in (84) by \(dx/x^\epsilon\) (\(\epsilon > 0\)). In the limit \(\epsilon \to 0\) the new inner product turns out to be

\[
<k|\Psi> \sim \frac{\hbar}{\epsilon \pi^2} \frac{\Gamma(\epsilon)}{2^\epsilon} \int_{-\infty}^{+\infty} dq \left\{ [\cos(\pi\nu) |A|^2 + |B|^2] \right.
\]

\[
+ \sin(\pi\nu)(B^*A - A^*B)|\Theta(-\nu^2)
\]

\[
+ [A^2 + |B|^2] \Theta(\nu^2) \right\},
\]

where \(\Theta\) is the step function. We can eliminate the overall divergent factor \(\Gamma(\epsilon)/2^\epsilon\) to define the physical scalar product. The elementary normalizable solutions with respect to the regularized scalar product can be classified immediately. They are \(J_\nu\) for \(\nu \in [0, \frac{1}{2}] \) or \(\text{Re} \nu = 0\), and \(N_\nu\) for \(\nu \in [0, \frac{1}{2}] \) or \(\text{Re} \nu = 0\). Note that the unique normalizable solution for \(\nu = \frac{1}{2}\) is \(J_\nu\).

Next we would like to relate the quantization obtained via the Wheeler-DeWitt equation with the approach developed in previous sections. The main point is to see how the Hilbert space \(L^2(R^+) \oplus L^2(R^+)\) (or \(L^2(R) \oplus L^2(R)\), depending on the sign of \(\Lambda\)), obtained from the covariant and reduced phase-space quantizations, can be realized in terms of the normalizable solutions of the Wheeler-DeWitt equations. Let us first consider the case of negative cosmological constant. Any normalizable solutions \(\Psi\) of the form (73) can be decomposed as \(\Psi = \Psi^+ + \Psi^-,\) where (we have redefined the function \(C(q)\))

\[
\Psi^+ = \left( \frac{k}{\pi \hbar} \Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu) \right)^{\frac{1}{2}} \int_{0}^{+\infty} dq a^{k+2i\frac{q}{\hbar}} e^{i\frac{qx}{\hbar}} C^+(q)K_{|\nu|}(x),
\]

\[
\Psi^- = \left( \frac{k}{\pi \hbar} \Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu) \right)^{\frac{1}{2}} \int_{-\infty}^{0} dq a^{k+2i\frac{q}{\hbar}} e^{i\frac{qx}{\hbar}} C^-(q)K_{|\nu|}(x).
\]
The scalar product takes the form
\[
\langle \Psi | \Psi \rangle = \int_{\infty}^{0} dq |C^{(+)}(q)|^2 + \int_{-\infty}^{0} dq |C^{(-)}(q)|^2 ,
\]
and this shows the coincidence with the Hilbert space derived in sections 2 and 4.

When the cosmological constant is positive, the Hilbert space of normalizable solutions of the Wheeler-DeWitt equation can also be decomposed into two orthogonal subspaces. Any normalizable solution \( \Psi \) of the form \( (83) \) can be split as \( \Psi = \Psi^{(+)} + \Psi^{(-)} \), where

\[
\Psi^{(+)} = \left( \frac{k}{2\hbar} \right)^{1/2} \int_{-\infty}^{+\infty} dq a^{1/2 + 2i/\hbar A} \left[ \Theta(q) J_{\nu}(x) \right. \\
+ \left. \Theta(-q) \left( \cos(\pi Im(\nu)) \right)^{1/2} N_{\nu}(x) \right]
\]

\[
\Psi^{(-)} = \left( \frac{k}{2\hbar} \right)^{1/2} \int_{-\infty}^{+\infty} dq a^{1/2 + 2i/\hbar A} \left[ \Theta(-q) J_{\nu}(x) \right. \\
+ \left. \Theta(q) \left( \cos(\pi Im(\nu)) \right)^{1/2} N_{\nu}(x) \right]
\]

The resulting expression for the scalar product turns out to be
\[
\langle \Psi | \Psi \rangle = \int_{-\infty}^{+\infty} dq \left( |A^{(+)}(q)|^2 + |A^{(-)}(q)|^2 \right)
\]
and shows the equivalence between the Hilbert space derived from the Wheeler-DeWitt equation and the one obtained from the covariant and reduced phase-space approach.

6 Final comments

In this paper we have constructed the quantum theory of the induced 2d-gravity in three different ways: i) The covariant phase space quantization; ii) The reduced ADM phase-space quantization and iii) The reduced Wheeler-DeWitt equation. We have explicitly shown the coincidence of the Hilbert space of these approaches. The first approach is based on the space of classical solutions and it permits to determine the “size” of the Hilbert space. The other approaches lead to two different realizations of the Hilbert space. The comparison between the different approaches allows to understand the role played by the classical solutions in the quantum theory. The absence of normalizable wavefunctions for \( \nu^2 = \frac{1}{4} \).
when $\Lambda < 0$, can be understood as the absence of classical parabolic solutions. Furthermore, the existence of an unique wavefunction for $\nu^2 = \frac{1}{4}$, when $\Lambda > 0$, find its classical counterpart in the existence of an unique classical parabolic solution (up to and additive constant for the scalar field).

Finally we want to remark that one could obtain an inequivalent quantization if both the hamiltonian and the supermomentum constraints were imposed at the quantum level. Solving the supermomentum constraint classically prevent the emergence of the “Schwinger term” in the algebra of surface deformations generated by $C_1$ and $C$ (the central extension involves both hamiltonian and supermomentum constraints [2]).

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