1 Introduction - the origin of the problem.

We consider the heat equation in $[0, 1] \subset \mathbb{R}$:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \Delta u(t, x) \quad t \geq 0, \quad x \in [0, 1], \\
u(0, x) &= u_0(x) \quad x \in [0, 1], \\
u(t, 0) &= u(t, 1) = 0 \quad t \geq 0,
\end{align*}
\]

where the initial condition $u_0(x)$ belongs to $L^2(0, 1) = H$. Let \{\textbf{e}_n\} be a complete orthonormal basis of $L^2(0, 1)$. Then, for the initial condition $u_0(x) = \sum_{n=1}^{\infty} a_n \textbf{e}_n$, the solution $u(t, x)$ is given by the semigroup \{\textbf{T}_t\}:

\[
\textbf{T}_t u_0(x) = \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 t) \textbf{e}_n,
\]

which traces out the trajectory of the solution of (1) on $H$.

If we go along the trajectories of solutions to the direction negative in time, we remark that there are two kinds of situations: (i) the trajectory which continues until $t = -\infty$; and (ii) the trajectory which continues backward only for finite period of time. The initial condition $u_0(x)$ of the latter
trajectory, which stops at finite \( t \leq 0 \) can be written \( u_0(x) = T_t u_1(x) \), where \( u_1(x) \) is the initial condition from which the backward trajectory does not exist. Thus, the case of (ii) can be studied by the initial value \( u_1 \), at which the solution trajectory is irreversible to the past. For the case of (ii), we can characterize the initial function \( u_0(x) = \sum_{n=1}^{\infty} a_n e_n \) from which the solution trajectory goes back to \( t = -\infty \), by:

\[
D = \{ \sum_{n=1}^{\infty} a_n e_n | \sum_{n=1}^{\infty} a_n^2 \exp(2n^2 \pi^2 t) < \infty \text{ for } \forall t \geq 0 \}.
\]

On the other hand, the initial function \( u_0(x) = \sum_{n=1}^{\infty} a_n e_n \) from which the solution trajectory cannot be defined for any \( t < 0 \) satisfies:

\[
Z = \{ \sum_{n=1}^{\infty} a_n e_n | \sum_{n=1}^{\infty} a_n^2 < \infty, \sum_{n=1}^{\infty} a_n^2 \exp(2n^2 \pi^2 t) = \infty \text{ for } \forall t \geq 0 \}.
\]

If \( u_0 \in D \), then

\[
T_{-t} u_0(x) = \sum_{n=1}^{\infty} a_n \exp(n^2 \pi^2 t) e_n \in H = L^2(0,1).
\]

For \( u_0 \in Z \), though we can formally write

\[
T_{-t} u_0(x) = \sum_{n=1}^{\infty} a_n \exp(n^2 \pi^2 t) e_n,
\]

\( T_{-t} u_0 \) is not in \( H \), and should belong to a functional space larger than \( H = L^2(0,1) \). For example, for the following Hilbert space

\[
H_t = \{ \sum_{n=1}^{\infty} a_n e_n | \sum_{n=1}^{\infty} a_n^2 \exp(-2n^2 \pi^2 t) < \infty \},
\]

equipped with the norm

\[
\| \sum_{n=1}^{\infty} a_n \exp(n^2 \pi^2 t) e_n \|_t = \| \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 t) e_n \|,
\]

where the \( \| \cdot \| \) is the norm in \( H \). We have

\[
H \subset H_t, \quad T_{-t} u_0(x) \in H_t \text{ if } u_0(x) \in Z.
\]
Thus, the solution trajectory which stops at a point in the Hilbert space $H$ is extendable in the larger space $H_t$. Therefore, the heat semigroup $T_t$ defined by (1), is extendable backward in time, by considering the larger functional spaces equipped with the above kind of norms $\| \cdot \|_t$. This construction of the sequence of spaces $\{H_t\}$ comes from the explicit expression:

$$T_tu_0(x) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n t)e_n, \quad l_n < 0, \quad t < 0 \quad (u_0(x) = \sum_{n=1}^{\infty} a_ne_n),$$

where $\lambda_n = -(n\pi)^2$, eigenvalues of $-\Delta$:

$$-\Delta = \begin{pmatrix}
-\pi^2 & -(2\pi)^2 & \cdots \\
-(2\pi)^2 & \cdots \\
\cdots & \cdots & \cdots \\
-(n\pi)^2 & \cdots & \cdots 
\end{pmatrix}.$$

The above consideration leads to the following question. If an infinitesimal generator $A$ of a semigroup $T_t$ in a functional space $X$ does not have the spectrum decomposition as above, is there a larger functional space $X_t$ in which any solution trajectory which starts from any point in $X$ continues backward in time uniformly until $-t < 0$? Moreover, is there an extended space $E$ of $X$, in which any solution trajectory starting from any point in $E$, can go back to the past until $t = -\infty$? In other words, we are interested in the following question.

**Problem** Is there an extended functional space in which a semigroup becomes a group? Find the condition for such an extension.

In this paper, under the following two conditions (I) and (II), we show the existence of the functional space $E$ described in the above problem.

(I) The set

$$D = \{x \in X \mid \forall t > 0 \quad \exists y = y(t) \in X \quad \text{s.t.} \quad T_t y = x\}$$

is dense in $X$.

(II) The semigroup $T_t$ has the backward uniqueness property, i.e.

if $T_t x = T_t y$ for some $t > 0$, then $x = y$. 

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The following is the plan of this paper: in §2, a sufficient condition such that $D$ satisfies (1) is given, as well as some properties of $D$; in §3, we present the extended space $E$ explicitly; and in §4, the structure of $E$ and the relationship between $E$ and $D$ are investigated.

This paper was written in 1989 as my master’s thesis, by receiving valuable advises from Professor Yukio Komura.

2 The set of points from which trajectories continue negative in time until $t = -\infty$.

From this section, we denote

$$D = \{ x \in X | \forall t > 0 \exists y = y(t) \in X \text{ s.t. } T_t y = x \},$$

for the set of points from which trajectories continue negative in time until $t = -\infty$, and

$$D_t = \{ x \in X | \exists y \in X \text{ s.t. } T_t y = x \},$$

for the set of points from which trajectories continue negative in time at least until $-t < 0$.

The following is a sufficient condition such that $D$ is dense in $X$.

Lemma 1.

Let $T_t (t \geq 0)$ be a linear semigroup defined on $X$. Then, the necessary and sufficient condition such that $D$ is dense in $X$ is:

$$\forall t > 0 \quad D_t = T_t X \text{ is dense in } \ X.$$ 

Proof. The necessary condition is clear. We only show the part of the sufficient condition. It is enough to prove for any $x_0 \in X$ and for any $\varepsilon_0 > 0$,

$$U_{\varepsilon_0}(x_0) \cap D \neq \emptyset.$$ 

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Since $D_1$ is dense in $X$, there exists $x_1 \in X$ such that $T_1x_1 \in U_{\varepsilon_0}(x_0)$. By repeating this, for any $\varepsilon_k > 0$ there exists $x_{k+1} \in X$ such that $T_1x_{k+1} \in U_{\varepsilon_k}(x_k)$. In particular, we can take $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that the sequence $\{T_nx_n\}$ becomes a Cauchy sequence in $X$, since

$$||T_nx_n - T_{n-1}x_{n-1}|| \leq M_{n-1} ||T_1x_n - x_{n-1}|| \leq M_{n-1} \varepsilon_{n-1},$$

where $M_t > 0$ is a number such that $||T_t|| \leq M_t$, more precisely $M_t = Me^{wt}$ $(M > 0$ a constant$)$. For such a choice of $\{x_n\}$, there exists a limit

$$\lim_{n \to \infty} T_nx_n = \exists x_{\infty, 0}.$$

Moreover, for each $k$, since

$$||T_{n-k}x_n - T_{n-1-k}x_{n-1}|| \leq M_{n-1-k} ||T_1x_n - x_{n-1}|| \leq Me^{-kw}e^{(n-1)w} \varepsilon_n,$$

there exists a limit

$$\lim_{n \to \infty} T_{n-k}x_n = \exists x_{\infty,k}.$$

Thus,

$$T_kx_{\infty,k} = \lim_{n \to \infty} T_nx_n = x_{\infty,0},$$

and $x_{\infty,0} \in D$. Since

$$||x_{\infty,0} - x_0|| = \lim_{n \to \infty} ||T_nx_n - x_0||,$$

and

$$||T_nx_n - x_0|| \leq ||T_nx_n - T_{n-1}x_{n-1}|| + ||T_{n-1}x_{n-1} - T_{n-2}x_{n-2}|| + \cdots + ||T_1x_1 - x_0|| \leq \sum_n Me^{wt} \varepsilon_n,$$

by choosing $\varepsilon_n > 0$ appropriately, we have $||x_{\infty,0} - x_0|| < \varepsilon_0$, which proves the claim.

Lemma 1 holds also for the nonlinear semigroup $T_1$. Since, in general, the nonlinear semigroup is defined on $\overline{D(A)}$ ($A$ is the infinitesimal generator of $T_t$), not necessarily on whole $X$, we rewrite the lemma as follows.

**Lemma 2.**

Let $T_t$ be a semigroup (possibly nonlinear) defined on $\overline{D(A)}$, satisfying

$$||T_t x - T_t y|| \leq Me^{wt} ||x - y|| \quad \forall x, y \in \overline{D(A)},$$

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where $M, w$ are positive constants independent on $t > 0$. Then, the necessary and the sufficient condition that $D$ is dense in $\overline{D(A)}$ is that $D_t$ is dense in $\overline{D(A)}$ for any $t > 0$.

**Proof.** As before, it is enough to prove the part of the sufficient condition. As in the proof of Lemma 1, for any $x_0 \in \overline{D(A)}$ and for any $\varepsilon_0 > 0$, we can take inductively a sequence of numbers $\varepsilon_k > 0$ ($k \geq 1$), and a sequence of points $x_k \in \overline{D(A)}$ ($k \geq 1$) such that

$$T_1 x_{k+1} \in U_{\varepsilon_k}(x_k), \quad \lim_{n \to \infty} T_n x_n = \exists x_{\infty,0} \in D,$$

where $\|x_{\infty,0} - x_0\| \leq \varepsilon_0$. This proves the claim.

By using Lemmas 1 and 2, we can now give sufficient conditions such that $D$, the set of points in $\overline{D(A)}$ from which the trajectories continue until $t = -\infty$, is dense in $\overline{D(A)}$.

**Proposition 1.**

Let $T_t$ be a linear holomorphic semigroup on a Banach space $X$. That is, for any $t_0 > 0$, there exists a neighborhood $U$ of $t_0$, and the following Taylor development:

$$T_t = \sum_{k=0}^{\infty} (t - t_0)^k A_k \quad t \in U.$$

Then, the set

$$D = \{ x \in X \mid \forall t > 0 \exists y = y(t) \text{ s.t. } T_t y = x \}$$

is dense in $X$.

**Proof.** Put

$$D_{t_1} = \{ x \in X \mid \exists y \in X \quad T_{t_1} y = x \}.$$

We use the argument by contradiction. From Lemma 1, we assume that there exists $t_1 > 0$ such that $D_{t_1}$ is not dense in $X$, and we shall lead to a contradiction. If $\overline{D_{t_1}}$, closed and convex, is not $X$, by the Hahn-Banach theorem there exists $f \in X'$ (dual space of $X$), such that

$$f \neq 0, \quad < x, f > = 0 \quad \forall x \in \overline{D_{t_1}}.$$
Since $T_t$ is holomorphic, there exists a neighborhood $U$ of $t_1$ such that

$$< T_t x, f >= < \sum_{k=0}^{\infty} (t - t_1)^k A_k x, f > = \sum_{k=0}^{\infty} (t - t_1)^k < A_k x, f >.$$  \hspace{1cm} (2)

For $t > t_1$, $T_t x \in D_{t_1}$, and $< T_t x, f >= 0$. Since the right-hand side of (2) is a holomorphic function, from the unique continuation theorem,

$$< T_t x, f >= 0 \hspace{1cm} \forall t \in U.$$

By repeating this argument, we have

$$< T_t x, f >= 0 \hspace{1cm} \forall t > 0, \hspace{1cm} \forall x \in X.$$

Moreover, from

$$< x, f >= \lim_{t \downarrow 0} < T_t x, f > = 0 \hspace{1cm} \forall x \in X,$$

we get $f = 0$. This contradicts to our previous assumption, and thus we have proved the claim.

**Corollary 1.**

Let $A$ be an infinitesimal generator of a linear holomorphic semigroup. Consider the following evolution equation in $X$

$$\frac{dx}{dt} + Ax = f(t) \hspace{1cm} x(0) = x_0,$$

where $f$ is a locally H"older continuous function from $(0, \infty)$ to $X$, such that

$$\int_0^\rho \|f(t)\|dt < +\infty \hspace{1cm} \text{for some} \hspace{1cm} \rho > 0,$$

and assume that there exists a global solution $x(t)$:

$$x(t) = T_t x_0 + \int_0^t T_{t-s} f(s)ds.$$

Let $S_t$ be the semigroup

$$S_t x_0 = x(t).$$
Then, the set
\[ D = \{ x \in X \mid \forall t > 0 \ \exists y = y(t) \text{ s.t. } S_t y = x \} \]
is dense in \( X \).

Proof. From Lemma 2, it is enough to show that for any \( t > 0 \), the set
\[ D_t = \{ x \in X \mid \exists y \in X \text{ s.t. } T_t y + \int_0^t T_{t-s} f(s) ds = x \} \]
is dense in \( X \). However, it is clear, because
\[ D_t = T_t X + \int_0^t T_{t-s} f(s) ds, \]
where \( + \) is the parallel translation.

Remark. The example of the heat equation in §.1 is a special case of Corollary 1.

3 Other properties of the set D.

The next result will be used in §4, when we mention the relationship between the set \( D \) and the set \( E \) (the definition will be given in below).

Proposition 2.

Let \( T_t \) be a linear semigroup in a Banach space \( X \), and assume that it has the backward uniqueness property. Put
\[ D = \{ x \in X \mid \forall t > 0 \ \exists y = y(t) \text{ s.t. } T_t y = x \}. \]
Then, \( D \) is a Fréchet space, equipped with the following countable norms.
\[ \| x \|_0 = \| x \|, \cdots, \| x \|_n = \| T_{-n} x \|, \cdots \ n \in \mathbb{N} \cup \{0\}, \]
where \( \| \cdot \| \) denotes the norm in \( X \).
Proof. First, we see that \( \| \cdot \|_n \) \((n = 0, 1, \ldots)\) are seminorms. In fact,
\[
\| x \|_n \geq 0, \quad \| x + y \|_n = \| T_{-n}(x + y) \| = \| T_{-n}x + T_{-n}y \| \leq \| T_{-n}x \| + \| T_{-n}y \| = \| x \|_n + \| y \|_n.
\]
Next, we show that \( \| \cdot \|_n \) is separable, by a contradiction argument. Assume that there exists \( x \neq 0 \) such that \( \| x \|_n = \| T_{-n}x \| = 0 \). However, this reads \( T_{-n}x = 0 \), and \( x = T_n(T_{-n}x) = 0 \), which is a contradiction. Thus, \( \| x \|_n > 0 \) for any \( x \neq 0 \), and \( \| \cdot \|_n \) is separable.
Finally, we confirm that \( D \) is complete with this topology. Let \( \{ y_m \} \subset D \) be a Cauchy sequence with respect to the above locally compact topology. That is, for any \( n \),
\[
\| y_m - y_{m'} \|_n = \| T_{-n}y_m - T_{-n}y_{m'} \| \to 0 \quad \text{as} \quad m, m' \to \infty.
\]
Thus, there exists \( z_n \in D \) such that
\[
\lim_{m \to \infty} T_{-n}y_m = z_n \quad \text{in} \quad X.
\]
It is easy to see that
\[
z_{n+1} = T_{-n}z_n,
\]
for \( z_{n-1} = \lim_{m \to \infty} T_{-(n-1)}y_m = \lim_{m \to \infty} T_1T_{-n}y_m = T_1z_n \). This leads to
\[
T_{n+1}z_{n+1} = T_nT_1z_n = T_nz_n = z_0 \quad z_0 \in D.
\]
We have now \( \lim_{m \to \infty} T_{-n}y_m = T_{-n}z_0 \) in \( X \). Therefore, \( y_m \) converges to \( z_0 \in D \), and we have proved the claim.

If \( D \) is not dense in \( X \) (or \( D(A) \)), neither is \( D_t \) \((\forall t > 0)\) dense in \( X \) (or \( D(A) \)).

Proposition 3.

If there exists \( t > 0 \) such that
\[
D_t = \{ x \in X \mid \exists y \in X \quad \text{s.t.} \quad T_1y = x \}
\]
is not dense in \( D(A) \), then
\[
\inf \{ t \mid D_t \text{ is not dense in } D(A) \} = 0.
\]

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Proof. We assume that
\[ t_0 = \inf \{ t \mid D_t \text{ is not dense in } \overline{D(A)} \} \neq 0, \]
and we shall lead to a contradiction. For any \( x \in \overline{D(A)} \), and for any \( \varepsilon > 0 \), there exists \( y \in \overline{D(A)} \) such that \( T_{\frac{2t_0}{3}} y \in U_\varepsilon(x) \). For this \( y \), and for any \( \delta > 0 \), there exists \( z \in \overline{D(A)} \) such that \( T_{\frac{2t_0}{3}} z \in U_\delta(y) \). Since
\[
\| T_{\frac{t_0}{3}} z - x \| \leq \| T_{\frac{t_0}{3}} z - T_{\frac{2t_0}{3}} y \| + \| T_{\frac{2t_0}{3}} y - x \|
\leq \| T_{\frac{2t_0}{3}} \| \| T_{\frac{2t_0}{3}} z - y \| + \varepsilon \leq \| T_{\frac{2t_0}{3}} \| \delta + \varepsilon,
\]
by tending \( \delta \to 0 \), we get
\[
\| T_{\frac{t_0}{3}} z - x \| \leq \varepsilon.
\]
However, since \( x \) is an arbitrary point, this contradicts to the definition of \( t_0 \). And thus, \( t_0 = 0 \) must hold.

Proposition 4.

If there exists \( t > 0 \) such that
\[
D_t = \{ x \in \overline{D(A)} \mid \exists y \in \overline{D(A)} \text{ s.t. } T_t y = x \}
\]
is not dense in \( \overline{D(A)} \), then there exists an open set \( U \) in \( X \) such that
\[
U \cap \overline{D(A)} \neq \emptyset, \quad U \cap \overline{D(A)} \cap D_t = \emptyset,
\]
and
\[
\inf \{ t \mid U \cap D_t = \emptyset \} = \min \{ t \mid U \cap D_t = \emptyset \}. \tag{3}
\]

Proof. Assume that (3) does not hold, and put \( t_0 = \inf \{ t \mid U \cap D_t = \emptyset \} \). We have
\[
U \cap \overline{D(A)} \cap D_{t_0} \neq \emptyset,
\]
and for any \( \varepsilon > 0 \) \( U \cap \overline{D(A)} \cap D_{t_0+\varepsilon} = \emptyset \). \( T_{-t_0} U = \{ x \in \overline{D(A)} \mid T_{t_0} x \in U \} \neq \emptyset \) is an open set. Thus, there exists \( x \in T_{-t_0} U \) such that \( T_{t_0} x \in T_{-t_0} U \) for some \( \varepsilon > 0 \). However, this reads \( T_{t_0+\varepsilon} x \in U \), which is a contradiction to the definition of \( t_0 \). Therefore, (3) must be true.
4 The extended space $E$.

Assume that there exists a semigroup $T_t$ on a Banach space $X$, and that it
defines solution trajectories in $X$. Assume also that
\[ D = \{ x \in X \mid \forall t > 0 \ \exists y = y(t) \in X \ s.t. \ T_t y = x \} \]
is dense in $X$. Then, for an arbitrary point $z \in X$, we can take a sequence
of points $\{z_n\}$ in $D$, such that
\[ \lim_{n \to \infty} z_n = z \text{ in } X. \]
Remark that the solution trajectory which passes $z_n$ continues backward until
$t = -\infty$. In general, we cannot expect that the solution trajectory which
passes $z$ continues until $t = -\infty$. For any $t > 0$, we know that $T_{-t} z_n$ exists,
but we do not know if the sequence $\{T_{-t} z_n\}$ is a Cauchy sequence or not.
Here, our idea is that we regard the sequence $\{T_{-t} z_n\}$ as ”a point” in an
extended space, and we define
\[ T_{-t} z = \{ T_{-t} z_n \}. \]
This consideration leads us to define the following extended space $E$, on
which the semigroup is extendable to the group.
\[ E = \{(z_n) \mid z_n \in D, \ \exists t > 0 \ \lim_{n \to \infty} T_t z_n \in X/\sim, \quad (4) \]where $\sim$ represents the equivalence defined by :
\[ (z_n) \sim (z_n') \text{ if and only if } \lim_{n \to \infty} T_t z_n = \lim_{n \to \infty} T_t z_n' \text{ for some } t > 0. \]

Theorem 1.

Let $T_t$ be a semigroup defined in a Banach space $X$. Assume that the
following two conditions hold.

(I) The set
\[ D = \{ x \in X \mid \forall t > 0 \ \exists y = y(t) \in X \ s.t. \ T_t y = x \} \]
is dense in $X$.
(II) $T_t$ has the backward uniqueness property, i.e. if $T_t x = T_t y$ for some
$t > 0$, then $x = y$.

Let $E$ be the set defined in (4). Then, the following holds.
• $X \subset E$.

• There exists a group $\mathcal{T}_t$ ($t \in \mathbb{R}$) on $E$, such that

$$\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}, \quad \mathcal{T}_0 = I \quad (I \text{ identity map}).$$

•

$$\mathcal{T}_t x = T_t x \quad x \in X \quad (\text{whenever } T_t x \text{ exists}).$$

**Proof.** The relationship $X \subset E$ comes from the identification of $x \in X$ to $(x_n) \in E$, where the sequence $\{x_n\}$ in $X$ satisfies $\lim_{n \to \infty} x_n = x$ in $X$. We define the group $\mathcal{T}_t$ in the following way: for $(x_n) \in E$

$$\mathcal{T}_t(x_n) = (T_t x_n) \quad \forall t \in \mathbb{R},$$

where the right-hand side is well-defined from the boundedness of $T_t$. Then,

$$\mathcal{T}_{t+s}(x_n) = (T_{t+s} x_n) = (T_t T_s x_n) = \mathcal{T}_t \mathcal{T}_s(x_n);$$

$$\mathcal{T}_0(x_n) = (x_n), \quad \text{if} \quad \lim_{n \to \infty} x_n = x, \quad \text{then} \quad \mathcal{T}_t(x_n) = (T_t x).$$

The semigroup $T_t$ on $X$ is extended to the group $\mathcal{T}_t$ on $E$, and the solution trajectory defined by $T_t$ is extended to $\mathcal{T}_t$ on $E$, backward in time until $t = -\infty$.

**Remarks**

1. For the case that $T_t$ is a nonlinear semigroup, the above theorem holds by replacing $X$ to $D(A)$, too.

2. Even if $T_t$ does not have the backward uniqueness property, $\mathcal{T}_t$ constructed as above has the backward uniqueness property on $E$ defined by (i). This is the reason why we assumed the condition (ii).

3. If a semigroup $T_t$ corresponds to a group $\mathcal{T}_t$ by $\mathcal{T}_t(x_n) = (T_t x_n)$, the infinitesimal generator $A$ of $T_t$ corresponds to the infinitesimal generator $A$ defined by

$$A(x_n) = (Ax_n) \quad \text{whenever the right-hand side exists.}$$
5 The relationship between $E$ and $D$.

In this section, we study some structures of $E$ introduced in §3, and the relationship between $E$ and $D$, the set of initial points from which the solution trajectory continues until $t = -\infty$ in $X$.

The space $E$ is not in general a Banach space, but the intermediate spaces between $X$ and $E$ are Banach spaces.

**Proposition 5.**

Let $T_t$ ($t \geq 0$) be a semigroup on a Banach space. For $t > 0$, put

$$E_{-t} = \{(z_n) | \quad z_n \in D \lim_{n \to \infty} T_t z_n \in X\},$$

the set of points in $X$ from which the solution trajectory exists backward in time at least for $t$. Then, $E_{-t}$ is a Banach space equipped with the norm:

$$\| (z_n) \|_{-t} = \| \lim_{n \to \infty} T_t z_n \|.$$

**Proof.** It is clear that $E_{-t}$ is a linear space. To see that $E_{-t}$ is a normed space, we confirm the following.

$$\| (z_n) + (y_n) \|_{-t} = \| \lim_{n \to \infty} T_t (z_n + y_n) \| \leq \| \lim_{n \to \infty} T_t z_n \| + \| \lim_{n \to \infty} T_t y_n \| = \| (z_n) \|_{-t} + \| (y_n) \|_{-t}.$$

Also, it is obvious that $\| (z_n) \|_{-t} \geq 0$, and that if $\| (z_n) \|_{-t} = 0$, then $(z_n) = 0$, for $\| \lim_{n \to \infty} T_t z_n \| = 0$ implies $\lim_{n \to \infty} T_t z_n = 0$ and thus $(z_n) \sim (0)$.

Moreover, for any $\alpha \in \mathbb{R}$,

$$\| \alpha (z_n) \|_{-t} = \| (\alpha z_n) \|_{-t} = \| \lim_{n \to \infty} T_t (\alpha z_n) \| = |\alpha| \| (z_n) \|_{-t}.$$

Finally, to see that $E_{-t}$ is complete with respect to the norm $\| \cdot \|_{-t}$, let $\{(z_n^m)\}_{m}$ be a Cauchy sequence with respect to the norm $\| \cdot \|_{-t}$, i.e.

$$\| (z_n^m) - (z_n^{m'}) \|_{-t} \to 0 \quad \text{as} \quad m, m' \to +\infty.$$
If we put \( c_m = \lim_{n \to \infty} T_t z_n^m = c_m \), then
\[
\|c_m - c_m'\| \to 0 \quad \text{as} \quad m, m' \to +\infty.
\]
Since \( \{c_m\} \) is a Cauchy sequence with respect to the norm \( \| \cdot \| \), there exists \( c_0 \in X \) such that \( \lim_{m \to \infty} c_m = c_0 \) in \( X \). Since \( D \) is dense in \( X \), there exists \( \{y_n\} \subset D \) such that \( \lim_{n \to \infty} y_n = c_0 \) in \( X \). Therefore,
\[
\|(z_n^m) - (T_{-t}y_n)\|_{-t} = \|c_m - c_0\| \to 0 \quad m \to \infty.
\]
From the above argument, we have shown that \( E_{-t} \) is a Banach space.

**Remarks.**

1. Similarly, we can see that the intermediate spaces between \( D \) and \( E \):
\[
D_t = T_t X = \{ x \in X \mid \exists y \in X, \ T_t y = x \}
\]
is also a Banach space with respect to the norm \( \|x\|_t = \|T_{-t}x\| \).

2. We have the following relationship:
\[
D \subset D_t \subset X \subset E_{-t} \subset E.
\]

3. Let \( S_t \) be the semigroup defined in Corollary 1 (§2.1), i.e.
\[
S_t x_0 = x(t) = T_t x_0 + \int_0^t T_{t-s} f(s) ds,
\]
where \( T_t = e^{-tA} \). Define the norm of \( E_{-t} \) by
\[
\|x\|_{-t} = \|S_t x - \int_0^t T_{t-s} f(s) ds\|.
\]
Then, \( E_{-t} \) is a Banach space with respect to the norm \( \| \cdot \|_{-t} \).

4. Let \( T_t \) be a non-expansive nonlinear semigroup defined on \( \overline{D(A)} \), a closed convex subset in a Banach space \( X \). Assume that the set
\[
D = \{ x \in \overline{D(A)} \mid \forall t > 0 \ \exists y = y(t) \ \text{s.t.} \ T_t y = x \}
\]
is dense in \( \overline{D(A)} \). Put
\[
\overline{D(A)}_{-t} = \{ (z_n) \mid \lim_{n \to \infty} T_t z_n \in \overline{D(A)}, \ z_n \in D \}.
\]
Then, \( \overline{D(A)}_{-t} \) is a complete metric set with respect to the norm :
\[
\|(z_n)\|_{-t} = \|\lim_{n \to \infty} T_t z_n\|.
\]
However, in general, we do not know whether it is possible to linearize $D(A)_{-t}$, and extend it to a larger Banach space containing the original space $X$. It is because, differently from the linear case, there is no longer the structure of a Banach space.

Next, by assuming that $X = H$ a Hilbert space, we consider the relationship between $E$ and $D$.

**Definition 1.** We say that two linear vector spaces $V$ and $W$ are dual, with respect to $\langle \cdot , \cdot \rangle_{V \times W}$, when the following holds.

$$\langle x, y \rangle_{V \times W} = 0 \quad \forall y \in W \quad \text{implies} \quad x = 0.$$  

**Proposition 6.**

The linear vector spaces $V$ and $W$ are dual, with respect to the following bilinear functional: for any $x \in D$ and any $(z_n) \in E$

$$\langle x, (z_n) \rangle_{D \times E} = (T_{-t}x, \lim_{n \to \infty} T_t z_n),$$

where $(\cdot, \cdot)$ is the inner product of $H$, and

$$t = \inf \{ s \mid \lim_{n \to \infty} T_s z_n \in H \}. $$

**Proof.** For any $(z_n) \in E$, let $t = \inf \{ s \mid \lim_{n \to \infty} T_s z_n \in H \}$, and assume that

$$(T_{-t}x, \lim_{n \to \infty} T_t z_n) = 0.$$  

In particular, if for any $(z) \in E$ such that $z \in D$ the above holds, then from the density of $D$ in $H$, we get $x = 0$. Thus, we proved the claim.

**Remark.** In Proposition 2, we have seen that $D$ is a Fréchet space with respect to the countable seminorms $\| \cdot \|_n (\|x\|_n = \|T_{-n}x\|)$. Concerning with this topology, the linear functional:

$$\langle \cdot , (z_n) \rangle_{D \times E} = (T_{-t} \cdot, \lim_{n \to \infty} T_t z_n)$$

is an element in $D^*$. This shows the existence of a one to one map from $E$ to $D^*$. 

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Finally, let us introduce a topology by the countable number of norms : $\|x\|_n = \|T_n x\|_n$.

**Proposition 7.**

Let $H$ be a Hilbert space, and $\{e_n\}$ be a complete orthonormal bases of $H$. If a semigroup $T_t$ on $H$ has the expression

$$T_t u_0 = \sum_{n=1}^{\infty} \exp(\lambda_n t) a_n e_n \quad t \geq 0, \quad u_0 = \sum_{n=1}^{\infty} a_n e_n,$$

then $D^* = E$.

**Proof.** It is clear that $E \subset D^*$. For $f \in D^*$, put $b_k = \langle e_k, f \rangle_{D \times D^*}$, and see

$$\sum_{n=1}^{\infty} a_n b_n < \infty \quad \text{if} \quad \sum_{n=1}^{\infty} a_n e_n \in D.$$

Since the sequence $(a_n)$ satisfies $(a_n \exp \lambda t_e) \in (l^2)$ for any $t \in \mathbb{R}$, we can write

$$b_n = \beta_n \exp \lambda_n t \quad (\beta_n) \in (l^2),$$

where

$$t = \sup \{s \mid \exists (\beta_n) = (\beta_n(s)) \in (l^2) \quad \text{s.t.} \quad b_n = \beta_n \exp \lambda_n s\}.$$

Here, $(\sum_{k=1}^{n} \beta_k \exp \lambda_k t) \in E$. If $t \geq 0$,

$$\langle e_1, (\sum_{k=1}^{n} \beta_k \exp \lambda_k t e_k) \rangle_{D \times E} = \beta_1 \exp \lambda_1 t = b_1,$$

and if $t < 0$,

$$\langle e_1, (\sum_{k=1}^{n} \beta_k \exp \lambda_k t e_k) \rangle_{D \times E} = \langle \exp \lambda t e_1, \sum_{k=1}^{\infty} \beta_k e_k \rangle = \beta_1 \exp \lambda_1 t = b_1.$$

Thus,

$$f = (\sum_{k=1}^{n} \beta_k \exp \lambda_k t e_k) \in E,$$

and we proved the claim.
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