Abstract

The existence two S-dual descriptions of (N,1) string bound states suggests that the strong coupling behaviour of electric flux lines in large N 2D SYM theory has a dual description in terms of weakly coupled IIB string theory. In support of this identification, we propose a dual interpretation of the SYM loop equation as a perturbative string Ward identity, expressing the conformal invariance of the corresponding boundary interaction. This correspondence can be viewed as a weak coupling check of the matrix string conjecture.

Introduction

Consider the large N limit of 1+1-dimensional \( \mathcal{N} = 8 \) supersymmetric Yang-Mills theory with gauge group U(N), described by the Lagrangian (here and in most of this paper we omit all fermionic fields)

\[
S = \int d^2x \, \text{tr} \left[ \frac{1}{g_{YM}^2} F_{\alpha\beta}^2 + (D_{\alpha} \Phi^i)^2 - g_{YM}^2 \Phi^i \Phi^j \right].
\]

with \( i = 2, \ldots, 9 \). This model can be viewed as the dimensional reduction of ten dimensional \( \mathcal{N} = 1 \) SYM theory, where \( \Phi^i \) represent the 8 transverse components of the ten-dimensional gauge potential \( A_{\mu} \). We will be interested in giving a string interpretation of the loop equation of motion \([1, 2]\) of the Wilson loop average

\[
W[X] = \langle \text{tr} P \exp \oint_C dS \dot{X}^\nu A_{\mu}(X(S)) \rangle
\]

in this 2D SYM theory. Here \( S = (s, \theta) \) denotes the 1D super-space coordinate along the loop \( C \). The index \( \mu \) in \( (2) \) runs from 0 to 9, so that the path-ordered exponential is in fact defined along a path in ten dimensions, where the \( \mathcal{N} = 1 \ D = 10 \) gauge field \( A_{\mu} = (A_\alpha, \Phi^i) \) is restricted to depend on \( X_0 \) and \( X_1 \) only. We will assume that the \( X^1 \)-direction is compact, so that we can choose \( X_1 \) to be periodic modulo \( 2\pi R \).
The above SYM model arises in ten-dimensional IIB string theory as the zero slope $\alpha' \to 0$ description of a bound state consisting of N D-strings. As emphasized in [3], the invariance under the gauge transformations $B \to B + d\Lambda$ and $A \to A + \Lambda$ of the NS anti-symmetric $B$-field implies that in this context the Wilson loop (2) is necessarily accompanied by a IIB string world-sheet that fills up its interior. The path-ordered exponential in (2) indeed defines a reparametrization invariant functional of the trajectory $\{X(s)\}$, and provides a conformally invariant boundary interaction for the IIB string, provided the gauge field $A$ satisfies the appropriate equation of motion. To leading order in $\alpha'$ this is just the super Yang-Mills equation.

The location of the string boundary divides the D-string world sheet into a vacuum outside region and an inside region with non-zero electric flux. As shown in [3], the gauge sector carrying one fundamental unit of electric flux $\int \text{tr} E_1 = 1$ can indeed be thought of as a bound state where one single type IIB string has been embedded into the $N$ D-string world sheet. In the limit of large $N$ (and fixed string coupling), the energy per unit length of the electric flux is much smaller than the fundamental string tension. This reduction of tension was interpreted in [3] as a manifestation of the binding force between the two types of strings. Intuitively what happens is that, due to higher loop interactions, the IIB worldsheet inside the Wilson line develops holes, and starts to look like a Feynman graph of the SYM field theory. In fact, it is possible to take a combined large $N$ and zero slope limit, while keeping the effective string tension of the electric flux finite, and in this limit the identification of the multi-holed string worldsheet with a multi-loop Feynman graph becomes exact.

Given the recent progress in string duality, it seems natural to look for a dual formulation of the theory in which the electric flux string can be studied in a weak coupling language. There is by now a large body of evidence that IIB string theory possesses an exact duality symmetry that interchanges the roles of the D-strings and the fundamental strings. We can apply this symmetry to our context, and reinterpret the two-dimensional SYM model as a “matrix string” description of a collection of $N$ fundamental IIB strings stretched in the $X_1$-direction. In this new identification, the eigenvalues of the Higgs fields $\Phi^i$ parametrize the location of the fundamental IIB strings, while the SYM electric flux becomes identified with the D-string charge. Unless stated otherwise, we will from now on apply this switch in terminology!

The Yang-Mills coupling $g_{YM}$ is expressed in terms of the IIB string coupling and slope parameter as

$$\frac{1}{g_{YM}^2} = g_s^2 \alpha'$$

(3)

A direct way to obtain this relation is by equating the tension of a single unit of electric flux with the effective tension of a single D-string embedded inside $N$ IIB strings. The tension of the combined $(N,1)$ string bound state reads

$$T = \frac{1}{\alpha'} \sqrt{N^2 + \frac{1}{g_s^2}} \approx \frac{N}{\alpha'} + \frac{1}{2Ng_s^2 \alpha'}$$

(4)

The second term must be compared with the energy per unit length of one unit of electric flux, given by $E/R = g_{YM}^2 / 2N$. We note that, similarly as before, the embedded D-string has reduced

*For notational simplicity we will assume that the target space coordinate $X^1$ is identified with the worldsheet coordinate $X^1$. Hence the radius $R$ in fact denotes the compactification radius of the target space $S^1$.
its string tension to \( T = 1/\alpha'_{\text{eff}} \) with

\[
\alpha'_{\text{eff}} = 2 N g_s^2 \alpha'
\]  

which amounts to a renormalization by a factor of \( 2N g_s \) relative to the tension \( T = (g_s \alpha')^{-1} \) of a free D-string. In the following we will combine the large \( N \) limit with a rescaling of \( g_s \) and \( \alpha' \), such that the effective string scale (5) of the electric flux string remains finite.

We can compare this dual representation of the (N,1) string with the conventional description in terms perturbative IIB string theory [3]. From the latter viewpoint, the worldsheet dynamics of the D-string is induced by integrating out the open IIB strings attached to it. The general boundary interaction reads

\[
\oint d\sigma \left[ \tilde{A}_\alpha(X) \partial_\tau X^\alpha + Y_i(X) \partial_\sigma X^i \right]
\]

with \( \alpha = 0, 1 \). The fields \( Y_i(X) \) parametrize the transverse motion of the D-string, while \( \tilde{A}_1(X) \) specifies an abelian gauge field defined on its worldsheet. Standard perturbative methods [6] show that their effective dynamics is described by the BI lagrangian

\[
\mathcal{L}_{BI} = \frac{1}{g_s \alpha'} \sqrt{\det(G_{\alpha\beta} + \alpha' \tilde{F}_{\alpha\beta})}
\]  

where \( G_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha Y^i \partial_\beta Y_i \) denotes the induced metric on the D-string worldsheet.

Working in the \( A_0 = 0 \) gauge, we have \( \tilde{F}_{01} = \partial_0 \tilde{A}_1 \). Due to the presence of large gauge transformations, the potential \( \tilde{A}_1 \) defines a periodic variable. The abelian electric field \( \tilde{E}_1 \), defined as the canonical momentum conjugate to \( \tilde{A}_1 \)

\[
\tilde{E}_1 = \frac{\tilde{F}_{01}}{g_s \sqrt{\det(G + \alpha' \tilde{F})}},
\]

therefore has quantized total flux

\[
\frac{1}{R} \oint dx_1 \tilde{E}_1 = N
\]

which according to [3] needs to be interpreted as the number of IIB strings bound to the D-string. The tension (4) of the total bound state is equal to the Hamiltonian density \( \mathcal{H} = \tilde{E}_1 - \mathcal{L}_{BI} \) evaluated in the ground state of the flux sector (5), see e.g. [7].

It will be useful to visualize this dual representation of the large \( N \) limit. Taking \( N \) to infinity in this case corresponds to the limit in which the BI field strength \( \tilde{F}_{01} = \partial_0 \tilde{A}_1 \) approaches its critical value \( F_{01} = 1/\alpha' \). The physical meaning of this critical limit is that the end points of the open IIB strings, which are electrically charged, are pulled very far away from each other under influence of the critical field. Hence the IIB worldsheet is stretched to a very large size compared to the original string scale set by \( \alpha' \). Mathematically, the boundary interaction (3) in a constant electric field gives rise to a term \( F_{01} \oint x^0 \partial X^1 \), which equals \( F_{01} \) times the 2D area covered by the IIB string world sheet. In the critical limit, this contribution almost cancels the
standard Nambu-Goto area. As a result, the IIB string tension is effectively reduced by a factor $1 - \alpha' F_{01}$, leading to a new effective tension $^{\dagger}$

$$T_{\text{eff}} \simeq \frac{1}{N^2 g_s^2 \alpha'}$$

The corresponding new string scale is very large, even relative to the scale $^{5}$ of the SYM electric flux string.

We note that the renormalization of the string scale $^{5}$ are with the same factor $2 N g_s$. In fact, an identical factor also appears if one computes the disk vacuum amplitude in the near critical electric field. One finds

$$\langle 1 \rangle_\Sigma \simeq \frac{1}{2 N g_s^2 \alpha'} \int d^2 x.$$

This should be compared with the standard normalization of the open string vacuum amplitude without electric field, which is $1/g_s \int d^2 x$. In $^{11}$ we introduced the notation $\langle \ldots \rangle_\Sigma$ for the tree-level expectation value of the open IIB string moving in the electric field $E_1 = N$. The result $^{11}$ will be of relevance later on $^{7}$.

**Loop equation as conformal Ward identity**

A priori, the U(N) SYM representation of the (N,1) string can be motivated only as a dual, effective description at strong coupling. According to the matrix string conjecture $^{4, 5}$, however, this description is supposed to be valid all the way into the perturbative string regime (provided we send N to infinity). Though quite convincing, the quantitative evidence supporting the equivalence of these two dual descriptions is still rather limited, however. Quite generally, perturbative IIB string theory is expected to arise as an effective description of the infra-red SYM dynamics. As shown in $^{5}$, in the leading IR limit the SYM degrees of freedom indeed reduce to that of a second quantized gas of freely propagating strings.$^{\S}$ In addition, evidence was presented that the perturbative splitting and joining interactions between the strings is also reproduced. We would like to find a similar correspondence in the present context. In particular, we would like to see how the gauge theory equation of motion is reproduced in the dual language. These equations of motion are expressed in terms of the loop average $\langle \ldots \rangle$ via the well-known loop equation $^{1, 2}$. A short summary is given in Appendix A.

Since the SYM electric flux translates into D-string winding under the duality, the Wilson loop now marks the boundary of a piece of open D-string attached to the N fundamental strings. In the IR regime, the IIB strings formed by the Higgs eigenvalues should therefore contain sector of open strings, whose boundaries are confined to the interior of the Wilson loop. Although it

---

$^{1}$ I thank I. Klebanov and A. Polyakov for valuable discussions on this point.

$^{2}$ It is suggestive that the result $^{11}$ can be written as $g_s^2 T_{\text{eff}}$ with $g_s^2 = 1/N$ and $T_{\text{eff}}$ as in $^{11}$. Notice further that this single string effective tension $T_{\text{eff}}$ is smaller by a factor of (order) $N$ than that of the electric flux string, which suggests that the electric flux string is made up from (of order) $N$ dual open strings.

$^{\S}$ The discussion in $^{5}$ concerned the matrix representation $^{4, 5}$ of type IIA strings, which is related via T-duality along the $X^1$-direction to the present IIB set-up. We will briefly discuss this T-duality map in the concluding section.
should be relatively straightforward to exhibit this open string sector in the SYM language (along the lines of [5]), we will at this point simply assume that this sector exists and that its properties are exactly as described above in terms of strings in a near-critical U(1) electric field.

We will assume that the fundamental string scale (set by $\alpha'$) is much smaller than the size of the Wilson loop (set by $\alpha_{\text{eff}}$ in [5]). Nonetheless, the critical electric field stretches the perturbative open string worldsheet until it essentially fills the whole interior region of the loop $C$. It is therefore reasonable to expect that the interaction between the Wilson loop and the open IIB strings translates into an effective boundary condition, that in the IR limit will flow to a conformally invariant fixed point. In the following we will investigate the associated string equations of motion, and exhibit a detailed correspondence with the large N SYM loop equation.

Instead of the position representation (2) of the Wilson loop, it will turn out to be convenient to introduce a generalized fourier transform of the loop average, formally defined as (cf. [2])

$$W_n(\epsilon_i, k_i) = \int [dX] W[X] \prod_{i=1}^{n} V_{\epsilon_i}(k_i)$$  \hspace{1cm} (12)

Here

$$V_{\epsilon_i}(k_i) = \int_C dS_i V_{\epsilon_i}(k_i, S_i)$$  \hspace{1cm} (13)

with

$$V_{\epsilon_i}(k_i, S_i) = \epsilon_\mu \dot{X}^\mu(S_i) e^{ik_i X(S_i)}.$$  \hspace{1cm} (14)

The integrand on the right-hand side is reparametrization invariant, and the functional integration over the paths $X(S)$ needs to be gauge fixed accordingly. The SYM loop equation of the position loop $W[X]$ implies a recursion relation for the transformed amplitudes $W_n(\epsilon_i, k_i)$, which we have written in equation (30) in Appendix A. The form of this recursion relation (30) is very suggestive of a non-linear Ward identity expressing the cut-off independence of a collection of string amplitudes. In the following we will summarize the reasoning that supports this interpretation.

To start with, we imagine that the transformed Wilson averages (12) can be given a dual representation in the perturbative IIB string theory by means of an analogous expectation value

$$W_n(\epsilon_i, k_i) = \left\langle \prod_j V_{\epsilon_j}(k_j) \right\rangle_\Sigma$$  \hspace{1cm} (15)

of local photon vertex operators (14). Usually one only considers string amplitudes of vertex operators $V_{\epsilon}(k)$ that create on-shell asymptotic states. The transformed loop average $W_n(\epsilon_i, k_i)$ defined above, however, is clearly an off-shell quantity. It will therefore be crucial for our set-up that the vertex operators $V_{\epsilon}(k)$ are not exactly on-shell. Nonetheless, as we will argue, it is possible to impose the condition of conformal invariance on the amplitudes and find non-trivial solutions. Under an infinitesimal reparametrization $S \to S + \xi(S)$ we have

$$\delta_\xi \left\langle \prod_j V_{\epsilon_j}(k_j) \right\rangle_\Sigma = \sum_i \left\langle \prod_{j\neq i} V_{\epsilon_j}(k_j) \left[ L[\xi], V_{\epsilon_i}(k_i) \right] \right\rangle_\Sigma$$  \hspace{1cm} (16)
where

\[ L[\xi] = \int dS \xi^\Vert(s) T_\perp^\Vert(s) \]  

(17)

is a tangential component of the world-sheet stress-energy tensor. The conformal transformation law of \( V_\mu(k, S) \) reads

\[
\left[ L[\xi], V_\mu(k, S) \right] = \frac{1}{2} \left( k^2 \delta_{\mu\nu} - k_\mu k_\nu \right) \xi(s) V_\nu(k; S) \\
+ D \left( \xi(s) k_\mu V_T(k; S) + \xi(s) V_\mu(k, S) \right)
\]

with \( D = \partial_\theta + \theta \partial_s \) and

\[ V_T(k, S) = e^{ikX(S)}. \]  

(18)

The first two terms of the expression on the right-hand side of (18) can be re-written as

\[- \frac{1}{2} \oint dS' \xi(s') \mathcal{L}(s') V_\mu(k, S) \]

where \( \mathcal{L}(s) \) denotes the operator (25) that appears in the large N loop equation (see also eqn (29) in Appendix A).

\[ \mathcal{L}(s) = \int^{\epsilon}_{-\epsilon} ds' \frac{\delta^2}{\delta X_\mu(s + s') \delta X_\mu(s)}. \]  

(20)

This correspondence is a first indication that the loop equation can be interpreted as a condition of conformal invariance.

As seen in equations (18), the insertion of the stress-tensor gives rise to two types of contributions: besides the anomalous conformal dimension of the vertex operators, we distinguish a total derivative term. Very naively, one could drop this total derivative term and arrive at the usual linear on-shell condition for the vertex operators as the condition for conformal invariance. String theory, however, is a non-linear theory, and the linear on-shell condition can receive corrections from string interactions. These corrections arise from the total derivative term, which can give contributions from the boundary of the \( S_i \) integration domain (that is, when two or more vertex operators collide). These additional factorization terms will turn the Ward identity into an non-linear recursion relation.

The moduli space \( \mathcal{M}_{D,n} \) of the disk \( D \) with \( n \) boundary points is not simply equal to the product of \( n \) copies of the disk boundary \( \partial D \). In particular, it has a boundary \( \partial \mathcal{M}_{D,n} \), which is reached when points start to collide. Naively one would think this boundary parametrizes configurations with only two points colliding, since collisions of more than two points appear to be of higher co-dimension. However, a more natural compactification of \( \mathcal{M}_{D,n} \) is obtained by using the full conformal group of the complex plane to transform the collision between two or more boundary points on the disk into a conformally equivalent situation, in which the single disk \( D \) is about to split into two smaller disks \( D_1 \) and \( D_2 \). There is one such boundary component for each way of dividing the boundary points into two smaller groups. Thus, schematically

\[ \partial \mathcal{M}_{D,n} = \sum_{m=2}^{n-2} \mathcal{M}_{D,m+1} \times \mathcal{M}_{D,n-m+1} \]  

(21)
Due to this form of the moduli space, it not entirely correct to represent the integral over the positions $S_i$ of the photon vertex operators as contour integrations over the disk boundary. A more correct definition of the amplitude, that extends also to the boundary of $\mathcal{M}_{D,n}$, is given in Appendix B. There we also outline the BRST derivation of the identity that expresses the decoupling of the stress-energy tensor from physical correlators.

In this Ward identity, at each given component of $\partial \mathcal{M}_{D,n}$, one finds a factorization term whenever a physical state propagates through the small strip separating the two disks. Since the momenta $k_i$ that we consider are assumed small compared to the fundamental string scale set by $\alpha'$, only the massless photon states can become on-shell. (All excited string states still have masses of the order of $1/\sqrt{\alpha'}$.) Combining all possible terms, we indeed arrive at a non-linear Ward identity of the same form as the equation (30) derived from the loop equation.

$$\frac{1}{2} \sum_{j=1}^{n} \epsilon_j^\mu (k_j^2 \delta_{\mu\nu} - k_\mu k_\nu) \left\langle V_\nu(k_j, 0) \prod_{i \neq j} V_{\epsilon_i}(k_i) \right\rangle_\Sigma =$$

$$\sum_{I \cup J = \{1, \ldots, n\}} \left\langle \prod_{i \in I} V_{\epsilon_i}(k_i) V_{\nu}(q_I, 0) \right\rangle_\Sigma \left\langle V_{\nu}(-q_I) \prod_{j \in J} V_{\epsilon_j}(k_j) \right\rangle_\Sigma$$

with $q = - \sum_{i \in I} k_i$.

Another way of obtaining this result is to consider the behaviour of the amplitudes (15) under conformal transformations near the disk boundary. In general, the expectation value must be regulated and this introduces a dependence on a short-distance cut-off. If we define this cut-off in terms of some fixed small coordinate difference, this results in a dependence on the local coordinate $S$ that parametrizes the boundary $C$. One source of this coordinate dependence is the anomalous scale dimension of the vertex operators $V_{\epsilon_i}(k_i)$. Another cut-off dependence arises because, due to the singular OPE's between the boundary vertex operators, the integral over the $S_i$ must be regulated near the boundary $\partial \mathcal{M}$. We can again specify this cut-off in terms of a fixed small coordinate distance. As a result, the amplitude receives an additional non-linear coordinate dependence that should cancel the linear dependence due to the anomalous scale dimension of the vertex operators. The non-linear recursion equation that expresses this cancelation again takes the form as given in (22).

As a final important technical comment, we note that the result (11) for the normalization of the disk vacuum amplitude ensures that the relative normalization of the left and right-hand side of this non-linear Ward identity (22) is in accordance with that of the transformed loop-equation (30) given in Appendix A.

**Concluding remarks**

We have used the existence two S-dual descriptions of $(N,1)$ string bound states to relate the strong coupling dynamics of electric flux lines in large $N$ 2D SYM theory to that of an open piece of D-string in weakly coupled IIB string theory. In support of this identification, we have shown that the SYM loop equation can be written in the form of a string Ward identity expressing the cut-off (in)dependence of the corresponding boundary interaction. Such a relation between a string Ward identity and the YM loop equation has been suggested on many occasions by A. Polyakov. Our recursion relation is indeed very similar to the non-linear $\beta$-function equations discussed in [9]. Our results indicate that the particular example of 2D SYM theory may provide a concrete realization of these ideas. A crucial ingredient seems to be that the dual IIB string
propagates in a near critical electric field. This fact allows one to consider string worldsheets large relative to the fundamental string scale. Clearly, however, many aspects of this long distance string correspondence with large N SYM theory still need to be clarified.

Finally, let us comment on how this result may be viewed as a possible check of the matrix string formalism of IIA string theory \[1, 8, 5\]. To arrive in the IIA context, we just need to apply a T-duality transformation along the \(X_1\) direction. Instead of IIB strings in a critical electric field, the perturbative description then becomes that of open IIA strings attached to a boosted D-particle with light-cone momentum proportional to \(N\). Furthermore, the \(k_1\)-components of the momenta \(k_i\) in the transformed loop averages \(W_n(\epsilon_i, k_i)\) get reinterpreted as shifts of the \(X_1\)-position of this D-particle. In other words, the boundary of the open IIA string worldsheet now forms a closed contour in the \((X_1, k_0)\)-plane, making discrete jumps at the location of the vertex operators. The analysis of the non-linear conformal Ward-identity of these amplitudes should proceed analogously as described above, provided we again restrict to energies \(k_0\) small compared to the string scale \(\alpha'\) (so that only photon states can contribute in the factorization terms). The formal correspondence of this Ward identity with the SYM loop equation appears to provide new support for the matrix string conjecture.

**Acknowledgements**

It is a pleasure to thank D. Gross, A. Mikhailov, A. Migdal, R. Dijkgraaf, I. Klebanov, S. Ramgoolam, E. Verlinde and especially A. Polyakov for very useful discussions. This research is partly supported by a Pionier Fellowship of NWO, a Fellowship of the Royal Dutch Academy of Sciences (K.N.A.W.), the Packard Foundation and the A.P. Sloan Foundation.

**Appendix A: The loop equation in large N SYM theory**

Consider the path-ordered exponential

\[ W[x] = \text{tr} P \exp \oint_C dS \hat{X}^\mu A_\mu(X(S)) \]  

Here \(A_\mu = (A_\alpha, \Phi^i)\) denotes the dimensional reduction to two dimensions of the U(N) gauge potential of ten-dimensional \(N = 1\) SYM theory. It is well-known that \(W[x]\) satisfies the following identity

\[ \mathcal{L}(S)W[x] = \text{tr} \left( D^\mu F_{\mu\nu}(X(S)) \hat{X}^\nu(S) \right) \text{P exp} \oint_{C_S} ds' \hat{X}^\mu A_\mu[X(s')] \]  

where \(\mathcal{L}(S)\) denotes the functional differential operator

\[ \mathcal{L}(S) = \int_{-\epsilon}^{\epsilon} ds' \frac{\delta^2}{\delta X_\mu(S + s') \delta X_\mu(S)}. \]

So if the gauge potential \(A\) satisfies its Yang-Mills equation of motion, the classical Wilson loop satisfies \(\mathcal{L}(S)W[x] = 0\). If we replace the classical Wilson line \(\hat{2}\) by the corresponding quantum
and the above linear equation for $W[X]$ gets replaced by a non-linear identity for the Wilson loop average. This is the famous loop equation for the quantum mechanical Wilson loop [2, 1]. In leading order at large $N$ it factorizes into a closed non-linear recursion relation

$$
\mathcal{L}(0) W[X] = \frac{g^2_{YM}}{N} \int_0^1 dS \hat{X}_\mu(0) \hat{X}_\mu(s) \delta(X(s)-X(0)) W[X]^N S W[X]_S\delta
$$

where $W[X]$ now denotes the SYM expectation value of (2). A loop equation of this form can be derived for general YM theories. Although strictly speaking it is only defined in a regulated theory, the high degree of supersymmetry in our case may be enough to ensure that this unrenormalized form of the equation remains valid in the continuum theory. In particular, it is known that, by expanding $W[X]$ in powers of the gauge coupling, one can recover from (27) the standard large $N$ SYM perturbation series. In two dimensions one finds the leading order perturbative solution

$$
W[X] \simeq \exp\left[\frac{g^2_{YM}}{2N} \int dS \int dS' \hat{X}_\mu(s) \hat{X}_\mu(s') \log |X(s)-X(s')| \right],
$$

which shows the characteristic area law behaviour of a string with tension $T = \frac{g^2_{YM}}{2N}$.

It is straightforward to rewrite the loop equation (2) as a recursion relation for the transformed loop averages $W_n(k_i)$ defined in (12). For this we need the identity

$$
\mathcal{L}(0) V_\mu(k, s) = -\delta(s)(k^2\delta_{\mu\nu} - k_\mu k_\nu) V_\nu(k, 0) + 2 D \left[\delta(s)ik_\mu e^{ikX(s)}\right]
$$

with $D = \partial_\theta + \theta \partial_s$ and where the $V_\nu(k, s)$ the photon vertex operator defined in (14). The second, total derivative term in (29) will drop out of the loop equation after the integration over $S$. Using this result (and the fact that the delta-function on the right-hand side of (27) can be expanded in terms of the photon vertices) one finds that the loop equation can be rewritten in the form of a closed, non-linear recursion relation of the transformed loop averages

$$
\sum_{j=1}^n \epsilon_j^\nu (k^2\delta_{\mu\nu} - k_\mu k_\nu) \left\langle V_\nu(k_j, 0) \prod_{i \neq j} V_{\epsilon_i}(k_i) \right\rangle_W = \frac{g^2_{YM}}{2N} \sum_{I \cup J = \{1, \ldots, n\}} \left\langle \prod_{i \in I} V_{\epsilon_i}(k_i) V_{\nu}(q_I, 0) \right\rangle_W \left\langle V_\nu(-q_I) \prod_{j \in J} V_{\epsilon_j}(k_j) \right\rangle_W
$$

with

$$
q = - \sum_{i \in I} k_i.
$$
Here we used the notation
\[
\langle \ldots \rangle_W = \int [dX] W[X] \langle \ldots \rangle.
\] (31)

The above form (30) is quite suggestive of a string Ward identity. In the main text we try to make this correspondence more precise.

**Appendix B: BRST derivation of the Ward identity**

A covariant definition of the scattering amplitude is given by separating the anti-ghost fields, that provide the measure of integration on \( M_{D,n} \) from the vertex operators
\[
\langle \prod_j V_j(k_j) \rangle \Sigma = \int_{M_{D,n}} \prod_i dS_i \langle \prod_i V^{(-1)}_i(k_i, S_i) \prod_i b_i \delta(\beta_i) \rangle
\] (32)
with
\[
V^{(-1)}_c(k, S) = c(S) \delta(\gamma(S)) e^\mu \psi e^{ikX}
\] (33)
the photon emission vertex in the -1-picture. The condition of conformal invariance is that the worldsheet stress-tensor \( T(z) \) must decouple from physical amplitudes
\[
\langle T(z) \prod_j V_j(k_j) \rangle \Sigma = 0
\] (34)
The insertion of the stress-energy tensor can be replaced by
\[
T(z) = \{ Q_{\text{brst}}, b(z) \}.
\] (35)
In deriving the conformal Ward identity, we can make use of the BRST-invariance of the expectation value to perform a “partial integration” and transpose the action of \( Q_{\text{brst}} \) to the vertex operator insertions. Their transformation law reads
\[
\{ Q_{\text{brst}}, V^{(-1)}_\mu(k, S) \} = \frac{i}{2} k^\mu \delta_{\mu\nu} - k_k^\mu k_k^\nu \dot{c} V^{(-1)}_\nu(k, S) + i k_\mu \dot{\gamma}(S) V^{(-1)}_T(k, S)
\] (36)
with \( V^{(-1)}_T \) the tachyon vertex (18) in the -1-picture. After picture changing to make \( V \) appropriate for integration over the moduli \( S_i \), the right-hand side becomes
\[
\frac{i}{2} (k^\mu \delta_{\mu\nu} - k_k^\mu k_k^\nu) \dot{c} V^{(-1)}_\nu(k, S) + D(\dot{c} k_\mu V^{(-1)}_T(k))
\] (37)
The total derivative term will drop out of (GSO projected) physical amplitudes. In addition, however, there is a contribution that arises because the BRST-charge acts as the exterior derivative on the moduli space: the BRST-commutator with the anti-ghost insertions in (32) combines into a total derivative on \( M_{D,n} \). This total derivative will integrate to a non-zero result due to boundary contributions at \( \partial M_{D,n} \), which arise whenever an intermediate photon state gets on-shell. A rather standard analysis shows that the resulting non-linear recursion relation, that expresses the decoupling of the stress-energy tensor, takes the form as given in (22).
References

[1] A. Polyakov, *Gauge Fields and Strings*, Harwood Academic Press, 1987. Some recent progress is reported in: A. Polyakov, “Confining Strings”, Nucl.Phys. **B486** (1997) 23-33 *hep-th/9607049*

[2] A. Migdal, “Loop Equations and 1/N Expansion,” Phys. Rept. **102** (1983) 199-290; For a recent discussion of the (supersymmetric) momentum loop equation, see A. Migdal, “Second Quantization of the Wilson Loop,” *hep-th/9411100* and “Hidden Symmetries of Large N QCD”, *hep-th/9610126*

[3] E. Witten, “Bound States of Strings and p-Branes,” *Nucl. Phys. B460* (1996) 335–350, *hep-th/9510135*.

[4] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, “M Theory as a Matrix Model: A Conjecture,” *hep-th/9610043*.

[5] R. Dijkgraaf, E. Verlinde, and H. Verlinde, “Matrix String Theory,” *hep-th/9703030*.

[6] E. Fradkin and A. Tseytlin, Phys. Lett. **B 158** (1985) 316.

[7] C. Callan and I. Klebanov, “D-Brane Boundary State Dynamics,” Nucl.Phys. **B465** (1996) 473, *hep-th/9511173*.

[8] L. Motl, “Proposals on Nonperturbative Superstring Interactions,” *hep-th/9701025*; T. Banks and N. Seiberg, “Strings from Matrices,” *hep-th/9702187*.

[9] A. Polyakov, “A Few Projects in String Theory,” *hep-th/9304146*

[10] S. Gukov, I. R. Klebanov and A. M. Polyakov, “Dynamics of (n,1) strings,” Phys. Lett. **B423**, 64 (1998) [hep-th/9711112].