INTEGRABLE DISCRETE GEOMETRY: 
THE QUADRILATERAL LATTICE, 
ITS TRANSFORMATIONS AND REDUCTIONS

Adam Doliwa\(^1\), Paolo Maria Santini\(^2,3,5\)

\(^1\)Instytut Fizyki Teoretycznej, Uniwersytet Warszawski
ul. Hoża 69, 00-681 Warszawa, Poland

\(^2\)Istituto Nazionale di Fisica Nucleare, Sezione di Roma
P.le Aldo Moro 2, I–00185 Roma, Italy

\(^3\)Dipartimento di Fisica, Università di Roma ”La Sapienza”
P.le Aldo Moro 2, I–00185 Roma, Italy

\(^4\)e-mail: Adam.Doliwa@fuw.edu.pl
\(^5\)e-mail: Paolo.Santini@roma1.infn.it

Abstract

We review recent results on Integrable Discrete Geometry. It turns out that most of the known (continuous and/or discrete) integrable systems are particular symmetries of the quadrilateral lattice, a multidimensional lattice characterized by the planarity of its elementary quadrilaterals. Therefore the linear property of planarity seems to be a basic geometric property underlying integrability. We solve the initial value problem for the quadrilateral lattice and we present the geometric meaning of its \(\tau\)–function, as the potential connecting its forward and backward data. We present the theory of transformations of the quadrilateral lattice, which is based on the discrete analogue of the theory of rectilinear congruences. In particular, we discuss the discrete analogues of the Laplace, Combescure, Lévy, radial and fundamental transformations and their interrelations. We also show how the sequence of Laplace transformations of a quadrilateral surface is described by the discrete Toda system in three dimensions. We finally show that these classical transformations are strictly related to the basic operators associated with the quantum field theoretical formulation of the multicomponent Kadomtsev–Petviashvili hierarchy. We review the properties of quadrilateral hyperplane lattices, which play an interesting role in the reduction theory, when the introduction of additional geometric structures allows to establish a connection between point and hyperplane lattices. We present and fully characterize some geometrically distinguished reductions of the quadrilateral lattice, like the symmetric, circular and Egorov lattices; we review also basic geometric results of the theory of quadrilateral lattices in quadrics, and the corresponding analogue of the Ribaucour reduction of the fundamental transformation.

We finally remark that the equations characterizing the above lattices are relevant in physics, being integrable discretizations of equations arising in hydrodynamics and in quantum field theory.
1 Quadrilateral Lattices

1.1 The Quadrilateral Lattice

Our first goal is to build a lattice which is integrable by construction. For the sake of simplicity, we consider first a 2-dimensional lattice \( \mathbb{Z}^2 \rightarrow \mathbb{R}^M \). Given the three arbitrary points \( \mathbf{x}(n_1, n_2), \mathbf{x}(n_1 + 1, n_2), \mathbf{x}(n_1, n_2 + 1) \), the forth point \( \mathbf{x}(n_1 + 1, n_2 + 1) \) is fixed prescribing a rule (the one defining the lattice); since we want an integrable lattice, this rule must be linear. We shall impose the simplest linear rule: planarity; i.e., the forth point \( \mathbf{x}(n_1 + 1, n_2 + 1) \) will belong to the plane generated by the first three points \( \mathbf{x}(n_1, n_2), \mathbf{x}(n_1 + 1, n_2) \) and \( \mathbf{x}(n_1, n_2 + 1) \) or, in algebraic terms,

\[
\Delta_1 \Delta_2 \mathbf{x} = (T_1 A_{12}) \Delta_1 \mathbf{x} + (T_2 A_{21}) \Delta_2 \mathbf{x},
\]

where \( T_i \) is the translation operator in the \( i \) direction and \( \Delta_i = T_i - 1 \) is the corresponding difference operator. Given two discrete initial curves (two sequences of points) \( \{ \mathbf{x}_1^{(0)} \}, \{ \mathbf{x}_2^{(0)} \} \) and two arbitrary sets of functions \( A_{12}(n_1, n_2), A_{21}(n_1, n_2) : \mathbb{Z}^2 \rightarrow \mathbb{R} \), the planarity constraint allows one to construct uniquely a two-dimensional lattice, which we have called quadrilateral (or planar). The quadrilateral surface was first proposed in [27] as the proper discrete analogue of a conjugate net on a surface, without any connection with integrability.

Indeed, at the moment, the above construction does not seem to have anything to do with integrability; to show its profound connection with the familiar integrability schemes it is necessary to generalize the picture to higher dimensions, imposing the planarity constraint on each two-dimensional \((ij)\) surface of the lattice [12].

**Definition 1** An \( N \)-dimensional lattice \( \mathbf{x} : \mathbb{Z}^N \rightarrow \mathbb{R}^M \) is a quadrilateral lattice (QL) iff its elementary quadrilaterals \( \{ \mathbf{x}, T_i \mathbf{x}, T_j \mathbf{x}, T_i T_j \mathbf{x} \} \) are planar; i.e., iff

\[
\Delta_i \Delta_j \mathbf{x} = (T_j A_{ij}) \Delta_i \mathbf{x} + (T_i A_{ji}) \Delta_j \mathbf{x}, \quad i \neq j, \quad i, j = 1, \ldots, N.
\]

The planarity constraints (2) are compatible only for the special class of data \( A_{ij} \) satisfying the nonlinear system

\[
\Delta_k A_{ij} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik} - (T_k A_{ij}) A_{ik}, \quad i \neq j \neq k \neq i.
\]

These equations characterize the quadrilateral lattice and we refer to them as the QL equations. They were first derived in [3] as integrable discrete analogues of the Darboux equations for conjugate nets, but without any geometric characterization.

We remark that, if a quadrilateral lattice exists, it is integrable by construction, since it is built out of the linear constraints (3). Indeed, in the language of the theory of integrable systems, the planarity constraints correspond to the
set of linear spectral problems (2) and the resulting QL and its characterizing
equations (3) correspond to the integrable nonlinear equations arising as the
compatibility condition for such spectral problems.

Actually, in connection with integrability, certain constructions based on the
quadrilateral surfaces were already known: the integrable discrete analogue of
isothermic surfaces [3] and the Laplace sequence of quadrilateral surfaces [9],
which provides the geometric meaning for the Hirota equation (see also Sec-
tion 2.2 and [10] for more details).

It is often convenient to reformulate equations (2) as a first order system
[12]. To do so we introduce the suitably scaled tangent vectors $X_i, i = 1, \ldots, N$:

$$\Delta_i x = (T_i H_i) X_i,$$

in such a way that the $j$-th variation of $X_i$ is proportional to $X_j$ only:

$$\Delta_j X_i = (T_j Q_{ij}) X_j, \quad i \neq j.$$

The compatibility condition for the system (3) gives the following new form
of the QL equations

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i \neq j \neq k \neq i$$

and the scaling factors $H_i$, called the Lamé coefficients, solve the linear equations

$$\Delta_i H_j = (T_i H_i) Q_{ij}, \quad i \neq j,$$

whose compatibility gives equations (3) again; moreover $A_{ij} = \frac{\Delta_i H_i}{H_i}$, $i \neq j$.

The solution of the initial value problem for the QL is contained in the
following result [12].
Theorem 1 (The Initial Value Problem for the QL) Given \( N \) initial discrete curves \( \{x_i^{(0)}\}, i = 1, \ldots, N \), assigning the initial data \( A_{ij}^{(0)}(n_i, n_j), A_{ji}^{(0)}(n_i, n_j) \), \( i, j = 1, \ldots, N, i \neq j \) (or, equivalently, the initial data \( H_i^{(0)}(n_i), Q_{ij}^{(0)}, Q_{ji}^{(0)} \)), one constructs the initial \( N(N - 1)/2 \) quadrilateral surfaces of the lattice applying equations (2) (or, equivalently, equations (5)). Then the quadrilateral lattice \( \mathbf{x} \) follows uniquely from the planarity constraint.

Figure 2.

In the continuous limit:

\[
\Delta_i \sim \varepsilon \frac{\partial}{\partial u_i} = \varepsilon \partial_i, \quad Q_{ij} \sim \varepsilon \beta_{ij}, \quad 0 < \varepsilon \ll 1 \quad (8)
\]

the QL reduces to an \( N \) dimensional conjugate net in \( \mathbb{R}^M \), characterized by the famous Darboux equations (4)

\[
\partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k \neq i. \quad (9)
\]

1.2 The backward representation of the quadrilateral lattice and the geometric meaning of the \( \tau \) function

In this Section we define the backward data \( \tilde{X}_i, \tilde{H}_i, \tilde{Q}_{ij} \) of the quadrilateral lattice. It turns out that the relation between the forward data \( X_i, H_i, Q_{ij} \) and the backward data is given in terms of the \( \tau \)-function, which is one of central objects of the theory of integrable systems.

In the previous Section the quadrilateral lattice was built through a forward construction; it is of course possible to build the lattice also through a backward construction. The backward tangent vectors \( \tilde{X}_i \) and the backward Lamé
coefficients $\bar{H}_i, i = 1, \ldots, N$ are defined by the equations

$$
\bar{\Delta}_i \bar{x} = (T_i^{-1} \bar{H}_i) \bar{x}_i, \quad \text{or} \quad \Delta_i \bar{x} = \bar{H}_i (T_i \bar{x}_i); \quad (10)
$$
in terms of the backward difference operator $\bar{\Delta}_i := 1 - T_i^{-1}$. The backward Lamé coefficients are again chosen in such a way that the $\bar{\Delta}_i$ variation of $\bar{x}_j$ is proportional to $\bar{x}_i$ only. We define the backward rotation coefficients $\bar{Q}_{ij}$ as the corresponding proportionality factors

$$
\bar{\Delta}_i \tilde{X}_j = (T_i^{-1} \bar{Q}_{ij}) \tilde{X}_i, \quad \text{or} \quad \Delta_i \tilde{X}_j = (T_i \tilde{X}_i) \bar{Q}_{ij}, \quad i \neq j . \quad (11)
$$

Comparing equations (7) and (11) we see immediately that the new functions $\tilde{Q}_{ij}$ satisfy the MQL equations (6) as well. Moreover the new scaling factors $\bar{H}_i$ satisfy the following system of linear equations

$$
\Delta_j \bar{H}_i = (T_j \bar{Q}_{ij}) \bar{H}_j, \quad i \neq j , \quad (12)
$$
whose compatibility condition gives again the QL equations (6).

An easy consequence of equations (10), (11) and (12) is the following

**Proposition 1** The vector function $x : Z^N \rightarrow R^M$ representing a quadrilateral lattice satisfies the backward Laplace equation

$$
\bar{\Delta}_i \bar{\Delta}_j x = (T_i^{-1} \bar{A}_{ij}) \bar{\Delta}_i x + (T_j^{-1} \bar{A}_{ji}) \bar{\Delta}_j x, \quad i \neq j , \quad (13)
$$

where, in the notation of this Section, $\bar{A}_{ij} = \frac{\bar{\Delta}_i \bar{H}_j}{\bar{H}_i}$. 


The forward and backward rotation coefficients $Q_{ij}$ and $\tilde{Q}_{ij}$ describe the same lattice $\mathbf{x}$ from different points of view, therefore they are not independent. Indeed, defining the functions $\rho_i : \mathbb{Z}^N \to \mathbb{R}$ as the proportionality factors between $X_i$ and $T_i \tilde{X}_i$ (both vectors are proportional to $\Delta_i \mathbf{x}$):

$$X_i = -\rho_i (T_i \tilde{X}_i), \quad T_i H_i = -\frac{1}{\rho_i} \tilde{H}_i, \quad i = 1, \ldots, N,$$

we have the following

**Proposition 2** The forward and backward data of the lattice $\mathbf{x}$ are related through the following formulas

$$\rho_j T_j \tilde{Q}_{ij} = \rho_i T_i Q_{ji},$$

and the factors $\rho_i$ are first potentials satisfying equations

$$\frac{T_j \rho_1}{\rho_i} = 1 - (T_i Q_{ji})(T_j Q_{ij}), \quad i \neq j.$$  

**Remark 1** Since $Q_{ij}$ and $\tilde{Q}_{ij}$ are both solutions of the QL equations (13), then equations (14), (15) describe a special symmetry transformation of equations (13).

The RHS of equation (14) is symmetric with respect to the interchange of $i$ and $j$; this implies the existence of a potential $\tau : \mathbb{Z}^N \to \mathbb{R}$, such that

$$\rho_i = \frac{T_i \tau}{\tau} ;$$

therefore equation (15) defines the second potential $\tau$:

$$\frac{(T_i T_j \tau) \tau}{(T_i \tau)(T_j \tau)} = 1 - (T_i Q_{ji})(T_j Q_{ij}), \quad i \neq j.$$  

The potential $\tau$ connecting the forward and backward data:

$$T_j (\tau \tilde{Q}_{ij}) = T_i (\tau Q_{ji}),$$

$$T_i (\tau \tilde{X}_i) = \tau X_i ,$$

$$\tau \tilde{H}_i = T_i (\tau H_i),$$

is the famous $\tau$-function of the quadrilateral lattice. In terms of the $\tau$ function, the QL equations read as follows:

$$(T_i T_j \tau) \tau = (T_i \tau) T_j \tau - (T_i \tau_{ji}) T_j \tau_{ij},$$

$$(T_k \tau_{ij}) \tau = (T_k \tau) \tau_{ij} + (T_k \tau_{ik}) \tau_{kj},$$

where

$$\tau_{ij} = \tau Q_{ij}.$$
2 Transformations of quadrilateral lattices

We present the basic ideas and results of the theory of the Darboux type transformations of the multidimensional quadrilateral lattice. Our approach follows the spirit of the book [17], which summarizes the theory of transformations of (two dimensional) conjugate nets. We also name the transformations of the quadrilateral lattice according to the classical geometric terminology of the transformations of conjugate nets.

All the results of this section can be found in [14], to whom we also refer for proofs and more detailed explanation.

2.1 Transformations and congruences

To define the transformations we need another geometric object – the congruence – which serves as the link between two quadrilateral lattices.

Definition 2 An $N$-dimensional rectilinear congruence (or, simply, congruence) is a mapping $l: \mathbb{Z}^N \to \text{Gr}_A(2, M + 1)$ from the integer lattice to the space of lines in $\mathbb{R}^M$ such that every two neighboring lines $l$ and $T_i l$, $i = 1, ..., N$, are coplanar.

We remark that with any $N$ dimensional quadrilateral lattices we may naturally associate $N$ congruences as follows.

Definition 3 Given an $N$-dimensional quadrilateral lattice $x$, its $i$-th tangent congruence $t_i(x)$ consists of the lines passing through the points $x$ of the lattice and directed along the tangent vectors $\Delta_i x$.

Reversing that process, one can associate with any $N$-dimensional congruence $N$ lattices which are, in general, quadrilateral lattices.

Definition 4 The $i$-th focal lattice $y_i(l)$ of a congruence $l$ is the lattice constructed out of the intersection points of the lines $l$ with $T_i^{-1} l$.

There exists a simple (but basic for the theory of transformations) relation between quadrilateral lattices and congruences.

Definition 5 A quadrilateral lattice and a congruence are called conjugate if there exists a one-to-one correspondence between points of the lattice and lines of the congruence such that lines pass through the corresponding points.

Corollary 1 Focal lattices of congruences conjugate to quadrilateral lattices are quadrilateral lattices.

The above introduced notions allow to define all the transformations of quadrilateral lattices.
Definition 6 Two quadrilateral lattices are related by the fundamental transformation \( F \) of Jonas when they are conjugate to the same congruence, which is called the congruence of the transformation.

Usually one assumes that, in the fundamental transformation, the congruence is \textit{transversal} to the lattice, i.e., non-tangent to the lattice in corresponding points. Otherwise we get the following reductions of the fundamental transformation

- the congruence of the transformation is the \( i \)-th tangent congruence of the original lattice – the \textit{Lévy transformation} \( \mathcal{L}_i \)
- the congruence of the transformation is the \( i \)-th tangent congruence of the new lattice – the \textit{adjoint Lévy transformation} \( \mathcal{L}_i^* \)
- both lattices are focal lattices of the congruence of the transformation – the \textit{Laplace transformation}

In addition, there are two useful special transformations

- the \textit{projective} or \textit{radial transformation}, when all the lines of the congruence meet in a single point
- the \textit{Combescure transformation}, when the transformed lattice is parallel to the original one

Let us present some formulas for the transformations, the details can be found in [14].

Theorem 2 The fundamental transformation \( F(x) \) of the quadrilateral lattice \( x \) is given by

\[
F(x) = x - \frac{\phi}{\phi_C} x_C ,
\]

(25)
where

i) \( \phi : \mathbb{Z}^N \to \mathbb{R} \) is a new solution of the Laplace equation (2) of the lattice \( \mathbf{x} \)

ii) \( \mathbf{x}_C \) is the Combescure transformation vector, which is a solution of the equations

\[
\Delta_i \mathbf{x}_C = (T_i \sigma_i) \Delta_i \mathbf{x} ,
\]

where, due to the compatibility of the system (26), the functions \( \sigma_i \) satisfy

\[
\Delta_j \sigma_i = A_{ij} (T_j \sigma_j - T_j \sigma_i) , \quad i \neq j ;
\]

moreover

iii) \( \phi_C \) is a solution, corresponding to \( \phi \), of the Laplace equation of the lattice \( \mathbf{x}_C \), i.e.

\[
\Delta_i \phi_C = (T_i \sigma_i) \Delta_i \phi .
\]

The Combescure transformation vector \( \mathbf{x}_C \) is used to construct the congruence of the transformation, and the function \( \phi \) serves to place points of the new lattice on the lines of the congruence. The definitions (25) and (26) -(27) of the fundamental and Combescure transformations were independently presented in [20].

**Remark 2** Notice that, given \( \mathbf{x}_C \) and \( \phi \), then equation (28) determines \( \phi_C \) uniquely, up to a constant of integration.

**Corollary 2** One can stop the transformation of the lattice \( \mathbf{x} \) at the intermediate level of construction of the vector \( \mathbf{x}_C \) to obtain the Combescure transformation

\[
\mathcal{C}(\mathbf{x}) = \mathbf{x} + \mathbf{x}_C ,
\]

characterized by the property that the tangent lines to both lattices in corresponding points are parallel.

**Definition 7** The \( i \)-th Lévy transform \( \mathcal{L}_i(\mathbf{x}) \) of the quadrilateral lattice \( \mathbf{x} \) is a quadrilateral lattice conjugate to the \( i \)-th tangent congruence of \( \mathbf{x} \).

Since the congruence of the transformation is known, to construct the transformation we need only the solution \( \phi \) of the Laplace equation of the lattice \( \mathbf{x} \).

**Proposition 3** The Lévy transform \( \mathcal{L}_i(\mathbf{x}) \) of the quadrilateral lattice \( \mathbf{x} \) is given by

\[
\mathcal{L}_i(\mathbf{x}) = \mathbf{x} - \frac{\phi}{\Delta_i \phi} \Delta_i \mathbf{x} .
\]

Another way to reduce the fundamental transformation is to fix the way we place the new lattice on the congruence.
Definition 8 The \( i \)-th adjoint Lévy transform \( L_i^*(x) \) of the quadrilateral lattice \( x \) is the \( i \)-th focal lattice of a congruence conjugate to \( x \).

To find the adjoint Lévy transformation we need only the congruence of the transformation, which can be find via the Combescure transformation vector \( x_C \).

Proposition 4 The adjoint Lévy transform of the lattice \( x \) is given by

\[
L_i^*(x) = x - \frac{1}{\sigma_i} x_C,
\]

where the functions \( \sigma_i \) are defined in Theorem 4.

The deepest reduction of the fundamental transformation is the Laplace transformation where both the congruence of the transformation and the way to place the new lattice on the congruence are fixed.

Definition 9 The Laplace transform \( L_{ij}(x) \) of the quadrilateral lattice \( x \) is the \( j \)-th focal lattice of its \( i \)-th tangent congruence.

Proposition 5 The Laplace transformation of the quadrilateral lattice \( x \) is given by

\[
L_{ij}(x) = x - \frac{1}{A_{ji}} \Delta_i x.
\]

Corollary 3 The superpositions of Laplace transformations satisfy the following identities

\[
L_{ij} \circ L_{ji} = \text{id},
\]

\[
L_{jk} \circ L_{ij} = L_{ik},
\]

\[
L_{ki} \circ L_{ij} = L_{kj}.
\]
Finally, the notion of radial (or projective) transformation is related to the so-called radial congruence, whose focal lattices are degenerated to a single point, which can be taken as the origin.

**Definition 10** The radial (or projective) transform $P(x)$ of the quadrilateral lattice $x$ is a quadrilateral lattice conjugate to the radial congruence of $x$.

**Proposition 6** The radial transform $P(x)$ is given by
\[
P(x) = \frac{1}{\phi}x, \quad (36)
\]
where $\phi: \mathbb{Z}^N \rightarrow \mathbb{R}$ is a solution of the Laplace equation (3) of the lattice $x$.

The degeneration of the fundamental transformation to the Lévy, adjoint Lévy, the Combescure, radial, and the Laplace transformations can be shown also on the analytic level. From another point of view, the fundamental transformation can be decomposed into the superposition of the Lévy and adjoint Lévy transformations, or into the superposition of two Combescure and one radial transformation. Moreover the fundamental transformation can be put into the scheme of the Laplace transformation (preceeded by building a new level on the original lattice).

### 2.2 The Hirota equation and the Laplace transformations of quadrilateral surfaces

The Laplace transformations of two dimensional conjugate nets generates the two dimensional Toda system \[2, 24\]. Analogously, one can associate \[2\] (see also \[3\] for more details and new interpretations) the integrable discrete version of the Toda system (the Hirota equation \[18\]) with the Laplace transformations of a two dimensional quadrilateral lattice.

Given a quadrilateral surface $x: \mathbb{Z}^2 \rightarrow \mathbb{R}^M$, consider the sequence $x^{(k)}$ of quadrilateral surfaces obtained from $x$ by recursive application of the Laplace transformation $L_{12}$:
\[
x^{(k)} = (L_{12})^k(x), \quad k \in \mathbb{Z}. \quad (37)
\]

The above sequence, which is the proper analogue of the Laplace sequence of conjugate nets, is well defined due to formula (33). The coefficients $A_{12}^{(k)}$, $A_{21}^{(k)}$ of the Laplace equations of the sequence of lattices satisfy the following system
\[
\begin{align*}
\frac{A_{12}^{(k+1)} + 1}{T_1 A_{12}^{(k)} + 1} &= \frac{A_{21}^{(k)}}{T_2 A_{21}^{(k)}}, & (38) \\
\frac{A_{21}^{(k-1)} + 1}{T_2 A_{21}^{(k)} + 1} &= \frac{A_{12}^{(k)}}{T_1 A_{12}^{(k)}}. & (39)
\end{align*}
\]
Notice that one can consider the parameter $k \in \mathbb{Z}$ along the Laplace sequence as the third discrete variable. Actually, the Laplace sequence of quadrilateral surfaces may be viewed as a deep reduction of the three dimensional quadrilateral lattice in the same sense like the Laplace transformations are reductions of the fundamental transformation. An elementary quadrilateral of such reduced (or degenerated) lattice with $T_3 = L_{12}$ is shown below.

Figure 6.

Define the function $K : \mathbb{Z}^2 \to \mathbb{R}$ as the cross-ratio of four collinear points: $x, L_{12}(x), T_1 x$ and $T_j L_{12}(x)$. Simple derivation shows that

$$K = \frac{A_{21}(T_1 A_{12} + 1) - T_2 A_{21}}{(1 + T_1 A_{12})(1 + A_{21})},$$

and equations (38)-(39) imply that $K$ satisfies the gauge invariant form of the Hirota equation

$$T_2 \left( \frac{K^{(l+1)} + 1}{K^{(l)} + 1} \right) T_1 \left( \frac{K^{(l-1)} + 1}{K^{(l)} + 1} \right) = \frac{(T_1 T_2 K^{(l)}) K^{(l)}}{(T_1 K^{(l)})(T_2 K^{(l)})}.$$

### 2.3 Superposition of fundamental transformations

Consider $K \geq 1$ fundamental transformations $F_k(x)$, $k = 1, \ldots, K$, of the quadrilateral lattice $x$, which are built from

i) $K$ solutions $\phi^k$, $k = 1, \ldots, K$ of the Laplace equation of the lattice $x$;

ii) $K$ Combescure transformation vectors $x_{C,k}$, where

$$\Delta_i x_{C,k} = (T_i \sigma_{i,k}) \Delta_i x, \quad i = 1, \ldots, N, \quad k = 1, \ldots, K,$$

and $\sigma_{i,k}$ satisfy equations

$$\Delta_j \sigma_{i,k} = A_{ij}(T_j \sigma_{j,k} - T_j \sigma_{i,k}), \quad i \neq j;$$

12
iii) $K$ functions $\phi_{c,k}^k$, which satisfy

$$\Delta_i \phi_{c,k}^k = (T_i \sigma_{i,k}) \Delta_i \phi^k.$$  (44)

We arrange functions $\phi^k$ in the $K$ component vector $\Phi = (\phi^1, \ldots, \phi^K)^T$; similarly, we arrange the Combescure transformation vectors $x_{c,k}$ into the $M \times K$ matrix $X_C = (x_{c,1}, \ldots, x_{c,K})$; moreover we introduce

iv) the $K \times K$ matrix $\Phi_C = (\Phi_{C,1}, \ldots, \Phi_{C,K})$, whose columns are the $K$ component vectors $\Phi_{C,k} = (\phi_{c,k}^1, \ldots, \phi_{c,k}^K)^T$, which are the Combescure transforms of $\Phi$

$$\Delta_i \Phi_{c,k} = (T_i \sigma_{i,k}) \Delta_i \Phi.$$  (45)

**Remark 3** The diagonal part of $\Phi_C$ is fixed by the initial fundamental transformations. To find the off-diagonal part of $\Phi_C$ we integrate equations (45) introducing $K(K-1)$ arbitrary constants.

One can show that the vectorial fundamental transformation $F(x)$ of the quadrilateral lattice $x$, which is defined as

$$F(x) = x - X_C \Phi_C^{-1} \Phi,$$  (46)

is again a quadrilateral lattice. Moreover, the vectorial transformation is the superposition of the fundamental transformations

$$F(x) = (F_{k_1} \circ F_{k_2} \circ \ldots \circ F_{k_K})(x), \quad k_i \neq k_j \quad \text{for} \quad i \neq j$$  (47)

and does not depend on the order in which the transformations are taken. In applying the fundamental transformations at the intermediate stages, the transformation data should be suitably transformed as well.

To prove the superposition and permutability statements, it is important to notice the following basic fact.

**Theorem 3 (Permutability theorem)** Assume the following splitting of the data of the vectorial fundamental transformation

$$\Phi = \left( \begin{array}{c} \Phi^{(1)} \\ \Phi^{(2)} \end{array} \right), \quad X_C = (X_{C(1)}, X_{C(2)}) , \quad \Phi_C = \left( \begin{array}{cc} \Phi_{C(1)}^{(1)} & \Phi_{C(1)}^{(2)} \\ \Phi_{C(2)}^{(1)} & \Phi_{C(2)}^{(2)} \end{array} \right) .$$  (48)

associated with the partition $K = K_1 + K_2$. Then the vectorial fundamental transformation $F(x)$ is equivalent to the following superposition of vectorial fundamental transformations:

1. Transformation $F_{(1)}(x)$ with the data $\Phi^{(1)}$, $X_{C(1)}$, $\Phi_{C(1)}^{(1)}$:

$$F_{(1)}(x) = x - X_{C(1)} \left( \Phi_{C(1)}^{(1)} \right)^{-1} \Phi^{(1)}. \quad \text{(49)}$$
2. Application on the result obtained in point 1., transformation $F_2$ with the data transformed by the transformation $F_1$ as well

$$F_2(F_1(x)) = F_1(x) - F_1(X_{C(2)}) \left( F_1(\Phi^{(2)}_{C(2)}) \right)^{-1} F_1(\Phi^{(2)}) ,$$  \hspace{1cm} (50)

where

$$F_1(X_{C(2)}) = X_{C(2)} - X_{C(1)} \left( \Phi^{(1)}_{C(1)} \right)^{-1} \Phi^{(1)}_{C(2)} \hspace{1cm} (51)$$

$$F_1(\Phi^{(2)}) = \Phi^{(2)} - \Phi^{(2)}_{C(1)} \left( \Phi^{(1)}_{C(1)} \right)^{-1} \Phi^{(1)} \hspace{1cm} (52)$$

$$F_1(\Phi^{(2)}_{C(2)}) = \Phi^{(2)}_{C(2)} - \Phi^{(2)}_{C(1)} \left( \Phi^{(1)}_{C(1)} \right)^{-1} \Phi^{(1)}_{C(2)} \hspace{1cm} (53)$$

Another important feature of the fundamental transformation is that its recursive application generates new dimensions of the quadrilateral lattice. In particular, when applied to the continuous limit of the quadrilateral lattice – the conjugate net – this procedure explains, on the geometric level, the power of using the Darboux type transformations in discretizing the integrable equations.

![Figure 7.](image)

2.4 Vertex operators as transformations of conjugate nets

The continuous limit of the above theory reduces to the classical theory of finite transformations of $N$ dimensional conjugate nets. It was recently shown in [16] that these transformations of conjugate nets are strictly related to the quantum field theoretical formulation of the multicomponent Kadomtsev–Petviashvili hierarchy.
The $b$-$c$ system of quantum fields, which appears as the system of ghost fields in string theory, is constructed in terms of the anticommutation relations

$$\{b_i(z), c_j(z')\} = \delta_{ij} \delta(z - z'), \quad (54)$$

$$\{b_i(z), b_j(z')\} = \{c_i(z), c_j(z')\} = 0, \quad (55)$$

where $b_i(z)$ and $c_i(z)$, $i = 1, \ldots, N$, are free charged fermion fields defined on the unit circle $S^1$, and $\delta(z - z')$ is the Dirac distribution on $S^1$.

The Clifford algebra generated by the $b$-$c$ system admits a representation in terms of bosonic variables. In this representation the fields act on the Fock space $F$ of complex-valued functions

$$\tau = \sum_\ell \tau(\ell, t) \lambda^\ell,$$

with

$$\ell := (\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N,$$

$$t := (t_1, \ldots, t_N) \in \mathbb{C}^{N,\infty}, \quad t_i := (t_{i,1}, t_{i,2}, \ldots) \in \mathbb{C}^\infty,$$

$$\lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N, \quad \lambda^\ell := \lambda_1^{\ell_1} \ldots \lambda_N^{\ell_N}. \quad (56)$$

The representation of the $b$-$c$ generators takes the form:

$$b_i(z)\tau(\ell, t) = (-1)^{\sum_{j>i} \ell_j} z^{\ell_i - 1} \exp(\xi(z, t_i)) \tau(\ell - e_i, t - [1/z] e_i), \quad (57)$$

$$c_i(z)\tau(\ell, t) = (-1)^{\sum_{j>i} \ell_j} z^{-\ell_i} \exp(-\xi(z, t_i)) \tau(\ell + e_i, t + [1/z] e_i), \quad (58)$$

where

$$\xi(z, t_i) := \sum_{n=1}^\infty z^n t_{i,n}, \quad [1/z] := \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots\right),$$

and $\{e_i\}_{i=1}^N$ are the canonical generators of $\mathbb{C}^N$.

The Fock space decomposes into a direct sum of charge sectors

$$F = \bigoplus_{q \in \mathbb{Z}} F_q, \quad F_q = \{\tau \in F : Q\tau = q\tau\},$$

where the total charge operator $Q := \sum_{i=1}^N Q_i$ is the sum of $N$ commuting partial charges $Q_i = \lambda_i \partial/\partial \lambda_i$, $i = 1, \ldots, N$; they correspond to the $N$ different flavours of fermions of the model.

The $N$-component KP hierarchy can be formulated as the following bilinear identity

$$B(\tau \otimes \tau) = 0, \quad \tau \in F_0, \quad (59)$$

15
where
\[
\mathcal{B} := \int_{S^1} \frac{dz}{z} \sum_{i=1}^{N} b_i(z) \otimes c_i(z).
\]

In terms of the components \( \tau(\ell, t) \) we have
\[
\int_{S^1} \frac{dz}{z} \sum_{m=1}^{N} (-1)^{\sum_{j>m} \ell_j + \ell'_j} \exp(\xi(z, t_m - t'_m)) z^{\ell_m - \ell'_m - 2} \times \tau(\ell - e_m, t - [1/z] e_m) \tau(\ell' + e_m, t' + [1/z] e_m) = 0,
\]
for any \( t, t', \) and \( \ell, \ell' \) such that \( \ell_1 + \ldots + \ell_N - 1 = \ell'_1 + \ldots + \ell'_N + 1 = 0. \)

Denote by
\[
\tau_{ij}(\ell, t) = S_{ij} \tau(\ell, t) := \tau(\ell + e_i - e_j, t);
\]
the shift operators \( S_{ij} \) correspond to the so-called Schlesinger transformations in monodromy theory \([8, 19]\) and do not alter the neutral character of the assembly of fermions. Moreover, \( S_{ij} \) are obvious symmetries of (60).

The \( N \times N \) matrix Baker function \( \psi \) and its adjoint \( \psi^* \) can be defined in terms of the \( \tau \) function as
\[
\psi_{ij}(z, t) = \epsilon_{ij} z^{\delta_{ij} - 1} \frac{\tau_{ij}(t - [1/z] e_j)}{\tau(t)} \exp(\xi(z, t_j)), \quad (62)
\]
and
\[
\psi^*_{ij}(z, t) = \epsilon_{ji} z^{\delta_{ij} - 1} \exp(-\xi(z, t_i)) \frac{\tau_{ij}(t + [1/z] e_i)}{\tau(t)}, \quad (63)
\]
where \( \epsilon_{ij} := \text{sgn}(j - i), j \neq i \) (\( \epsilon_{ii} := 1 \)).

The connection of the \( N \) component hierarchy with the conjugate nets is given in the following result \([16]\).

**Proposition 7** The solutions of the \( N \)-component KP hierarchy describe \( N \)-dimensional conjugate nets with coordinates \( u_i = t_i, 1 \leq i \leq N \), while the remaining times \( t_{i,k}, k > 1 \), describe integrable iso-conjugate deformations of the nets.

In particular, the rotation coefficients are given by
\[
\beta_{ij}(t) = \epsilon_{ij} \frac{\tau_{ij}(t)}{\tau(t)}, \quad i \neq j, \quad i, j = 1, \ldots, N, \quad (64)
\]
the tangent vectors \( X_i \) are constructed from the rows of the matrix
\[
X(t) = \int_{S^1} dz \psi(z, t) f(z), \quad (65)
\]
for some distribution matrix \( f(z) \in M_{N \times M}(C) \); the Lamé coefficients are given by the entries of the row matrix
\[
H(t) = \int_{S^1} dz g(z) \psi^*(z, t), \quad i = 1, \ldots, N, \quad (66)
\]
for some distribution row matrix \( g(z) \in C^N \).
As one can expect, various elements of the KP theory have their counterparts on the level of the conjugate nets.

**Theorem 4**

i) The Schlesinger transformation $S_{ij}$ gives rise to the Laplace transformation $L_{ij}$ of the conjugate nets.

ii) The action of the (vertex) operators $b_i(p)$ and $c_i(q)$ gives rise to the Lévy and the adjoint Lévy transformations $L_i$ and $L_i^*$. Correspondingly

iii) the soliton vertex operator generated by the superposition $b_i(p)c_i(q)$ gives rise to the fundamental transformation.

## 3 Quadrilateral Hyperplane Lattices

In this Section we introduce and study the properties of quadrilateral lattices in the dual space (hyperplane lattices) [13]. These considerations turn out to be relevant in the reduction theory of the quadrilateral lattices, when the introduction of the additional geometric structure will allow to establish a direct connection between point lattices and hyperplane lattices.

Generic hyperplanes can be represented by the co-vectors $a^* \in (\mathbb{R}^M)^*$ and consist of points $x$ satisfying the equation

$$\langle a^* | x \rangle = a^*_1 x^1 + \ldots + a^*_M x^M = 1 ;$$

the equation of the hyperplane passing through $0$ and parallel to that represented by $a^*$ can be written as $\langle a^* | x \rangle = 0$ (the co-vector $0^*$ represents the hyperplane at infinity).

The linear mappings between points and hyperplanes are called *correlations*. Given the sphere of radius $R$ centered at the origin, it defines the corresponding correlation $P$ (called *polarity* with respect to that sphere). The image $P(v)$ of a point $v \in \mathbb{R}^M$ is called the *polar hyperplane* of $v$, and consists of points $x$ satisfying equation $v^T x = R^2$, i.e., $P(v) = \frac{v}{R}$. The notions dual to the parallelism of two lines and to the planarity of four points are contained in the following

**Definition 11**

i) Two subspaces of co-dimension 2 are called "co-parallel" if there exists a hyperplane passing through the origin and containing them.

ii) Four hyperplanes are "co-planar" if the fourth hyperplane contains the intersection of the first three.

A *hyperplane lattice* is simply a lattice $y^* : \mathbb{Z}^N \rightarrow (\mathbb{R}^M)^*$, $N \leq M$, in the space of hyperplanes. From Definition 11 we obtain the following geometric object dual to the quadrilateral lattice.

**Definition 12** *The hyperplane lattice $y^* : \mathbb{Z}^N \rightarrow (\mathbb{R}^M)^*$ is "quadrilateral" ("co-planar") if, for any $i, j = 1, \ldots, N, i \neq j$, the hyperplane $T_i T_j y^*$ contains the subspace $y^* \cap T_i y^* \cap T_j y^*$.*
This property reduces again to the Laplace equation:

$$\Delta_i \Delta_j y^* = (T_i A^*_{ij}) \Delta_i y^* + (T_j A^*_{ji}) \Delta_j y^*, \quad i \neq j.$$  

(68)

A convenient way to construct quadrilateral hyperplane lattices makes use of systems of parallel quadrilateral point lattices. We recall that two quadrilateral lattices $x, x'$ are parallel (Comberscure related) iff $\Delta_i x' \sim \Delta_i x$; in this case the scaled tangent vectors of the two lattices can be chosen to be equal, then the rotation coefficients $Q_{ij}, Q'_{ij}$ of both lattices coincide and the Lamé coefficients $H_i$ and $H'_i$ are solutions of the same equation.

**Definition 13** Consider a system of $M$ parallel and linearly independent point lattices $x(k), \ k = 1, \ldots, M$. Denote by $y^*_k$ the hyperplane lattice made out of the hyperplanes passing through the (corresponding) points of $x(k)$ and spanned by the vectors $x(l), l \neq k$, i.e.,

$$\langle y^*_k | x(l) \rangle = \delta_{kl}. \quad (69)$$

We call the system $\{y^*_k\}$ of hyperplane lattices the "dual system" to the system of parallel point lattices $\{x(k)\}$.

If we arrange the parallel system $x(k), \ k = 1, \ldots, M$ of point lattices as columns of the matrix $\Omega$ of the system:

$$\Omega = (x(1), \ldots, x(M)), \quad (70)$$

then we have the following results.

1. The dual system $\{y^*_k\}$ consists of co-parallel quadrilateral hyperplane lattices and is given by the rows of the matrix $\Omega^{-1}$.

2. The rows of the matrix $\Omega$ define a system $\{x^*_k\}$ of co-parallel quadrilateral hyperplane lattices, which we call the "adjoint system" to $x(k)$, characterized by the property that the forward rotation coefficients of the system $\{x^*_k\}$ are the backward rotation coefficients of the system $\{x^*_{(k)}\}$:

$$Q_{ij} = \tilde{Q}^*_{ij}. \quad (71)$$

**4 Reductions of the Quadrilateral Lattice**

We now study some geometrically distinguished reductions of the QL which possess additional geometric properties that, once imposed on the initial surfaces, "propagate" everywhere through the construction of the lattice. Since the quadrilateral lattice is integrable, these reductions will inherit its integrability properties.
4.1 The Symmetric Lattice

Forward and backward data allow one to introduce the first basic reduction of the quadrilateral lattice [13].

**Definition 14** A quadrilateral lattice $\mathbf{x}$ is symmetric iff its forward rotation coefficients are also its backward rotation coefficients:

$$\tilde{Q}_{ij} = Q_{ij}. \quad (71)$$

The symmetric lattice admits the following different characterizations [13].

1. A quadrilateral lattice is symmetric iff, for a given set of rotation coefficients $Q_{ij}$, there exists a $\tau$–function of the lattice such that

$$T_i(\tau Q_{ji}) = T_j(\tau Q_{ij}), \quad i \neq j, \quad (72)$$

or equivalently, in terms of the corresponding potentials $\rho_i$,

$$\rho_i T_i Q_{ji} = \rho_j T_j Q_{ij}. \quad (73)$$

2. A quadrilateral lattice is symmetric iff, for a given set of forward tangent vectors $X_i$ of the lattice, there exists a complementary set of the backward tangent vectors $\tilde{X}_i$ such that the parallelograms $P(T_i \tilde{X}_i, T_j \tilde{X}_j)$ and $P(\Delta_i X_j, \Delta_j X_i)$ are similar.

3. A quadrilateral lattice is symmetric iff, for different indices $i, j, k$, its rotation coefficients satisfy the following constraint

$$(T_i Q_{ji})(T_j Q_{kj})(T_k Q_{ik}) = (T_j Q_{ij})(T_i Q_{ki})(T_k Q_{jk}). \quad (74)$$

We remark that, unlike the others, the characterization (74) is local.

![Diagram](image-url)

Figure 8.
As we have anticipated, the constraints discussed in this paper allow one to establish a connection between quadrilateral point lattices and their duals, the quadrilateral hyperplane lattices. The following proposition describes this connection in the case of the symmetry constraint.

**Proposition 8**  Given a system of parallel quadrilateral lattices \( \{x_{(k)}\}_{k=1}^{M} \) and the associated matrix \( \Omega \), then the following properties are equivalent.

i) The matrix \( \Omega \) of the system is symmetric:

\[
\Omega = \Omega^T. \quad (75)
\]

ii) The polar hyperplane \( P(x_{(k)}) \) of the point lattice \( x_{(k)} \) coincides with the hyperplane lattice \( x^*_i(k) \):

\[
P(x_{(k)}) = x^*_i(k), \quad k = 1, \ldots, M. \quad (76)
\]

iii) The lattices \( x_{(k)}, k = 1, \ldots, M \) are symmetric. Furthermore the associated tangent vectors \( X_i \) and \( X^*_i \) are related in the following way

\[
X^T_i = \rho_i(T_iX^*_i), \quad i = 1, \ldots, N. \quad (77)
\]

**Corollary 4** A quadrilateral lattice \( x \) is symmetric iff it is adjoint to its own polar.

In the continuous limit (8) the symmetric quadrilateral lattice reduces to a symmetric conjugate net, for which the rotation coefficients \( \beta_{ij} \) satisfying the Darboux equations (9) are symmetric

\[
\beta_{ij} = \beta_{ji}. \quad (78)
\]

### 4.2 The Circular Lattice

The discrete analogue of an \( N \)-dimensional orthogonal system of coordinates is the circular lattice \( \mathbb{C} \).

**Definition 15** An \( N \)-dimensional lattice is circular if and only if its elementary quadrilaterals are inscribed in circles.

This notion was first proposed in \( [23, 25] \) for \( N = 2, M = 3 \), as a discrete analogue of surfaces parametrized by curvature lines (see also \( [6] \)); later in \( [2] \) for \( N = M = 3 \) and, finally, for arbitrary \( N \leq M \) in \( [5] \); subsequently a convenient set of equations characterizing the circular lattices in \( \mathbb{E}_3 \) was found in \( [20] \). A geometric proof of the integrability of the circular lattice was first given in \( [5] \) while the analytic proof of its integrability was given in \( [15] \) through the \( \partial \) method.

An elementary characterization of circular quadrilaterals states that, if a circular quadrilateral is convex, then the sum of its opposite angles is \( \pi \); when the quadrilateral is skew, then its opposite angles are equal. Other convenient characterizations are the following.
1. A quadrilateral lattice is circular if and only if:
\[
\cos \angle(X_i, T_i X_j) + \cos \angle(X_j, T_j X_i) = 0 ,
\]
(79)

2. A quadrilateral lattice is circular if and only if [17]:
\[
X_i \cdot T_i X_j + X_j \cdot T_j X_i = 0 , \quad i \neq j .
\]
(80)

3. A quadrilateral lattice \( x \) is circular iff the scalars
\[
v_i := (T_i x \cdot x_i) \cdot x_i , \quad i = 1, \ldots, N
\]
(81)
solve the linear system (5) or, equivalently, iff the function \(|x|^2\) (the square of the norm of \( x \)) satisfies the Laplace equation (8) of \( x \). This characterization was found in [20] and explained geometrically in [11].

In addition, the circularity constraint (80) implies the following formula [15]:
\[
\frac{T_i |X_j|^2}{|X_j|^2} = 1 - (T_i Q_{ji})(T_i Q_{ji}) ;
\]
(82)
which, compared with equations (16)-(18), allows to fix, without loss of generality, the backward formulation of the circular lattice in the following way:
\[
|X_j|^2 = \rho_i = \frac{T_i \tau}{\tau} \quad \Rightarrow \quad |T_i \ddot{X}_i|^2 = 1/\rho_i = \frac{\tau}{T_i \tau}.
\]
(83)
A distinguished sub-class of circular lattices corresponds to the particular case in which the lattice points \( \mathbf{x} \) belong to the sphere of radius \( R \): \( |\mathbf{x}| = R \). In this case there exists, like for the symmetric reduction, an elegant relation between point lattices and hyperplane lattices \[13\].

**Proposition 9** Given a system of parallel quadrilateral lattices \( \{\mathbf{x}(k)\}_{k=1}^M \) and the associated matrix \( \Omega \) of the system, then the following properties are equivalent.

i) The matrix \( \Omega/R \) is orthogonal:

\[
\Omega \Omega^T = \Omega^T \Omega = R^2 I, \quad \Omega^T = R^2 \Omega^{-1}.
\]

ii) The polar hyperplane \( P(\mathbf{x}(k)) \) coincides with the dual hyperplane \( R^2 y^*_k \):

\[
P(\mathbf{x}(k)) = R^2 y^*_k, \quad k = 1, \ldots, M.
\]

iii) The quadrilateral lattices \( \mathbf{x}(k)/R, k = 1, \ldots, M \) form an orthonormal basis:

\[
\mathbf{x}(i) \cdot \mathbf{x}(j) = R^2 \delta_{ij}, \quad i, j = 1, \ldots, M.
\]

In addition, the associated tangent vectors \( \mathbf{X}_i, \mathbf{X}^*_i, i = 1, \ldots, N \), are related by the following formulas

\[
\mathbf{X}_i = \frac{\rho_i}{2R^2} T_i (\Omega \mathbf{X}^*_i)^T = -\frac{\rho_i}{2R^2} \Omega (T_i \mathbf{X}^*_i)^T, \quad i = 1, \ldots, N,
\]

\[
T_i \mathbf{X}^*_i = -\frac{2}{\rho_i} \mathbf{X}^*_i \Omega, \quad i = 1, \ldots, N,
\]

with

\[
|\mathbf{X}_i|^2 = \rho_i, \quad T_i |\mathbf{X}^*_i|^2 = \frac{4R^2}{\rho_i}
\]

and satisfy the circularity constraint \[80\] and its adjoint

\[
C_{ij}^{os} := \mathbf{X}^*_i \cdot T_i^{-1} \mathbf{X}^*_j + \mathbf{X}^*_j \cdot T_j^{-1} \mathbf{X}^*_i = 0.
\]

In the continuous limit, equations \[81\] become the orthogonality conditions

\[
\mathbf{X}_i \cdot \mathbf{X}_j = 0, \quad i \neq j
\]

and the circular lattice reduces to an orthogonal conjugate net.

### 4.3 The Egorov Lattice

In the previous sections we introduced two integrable reductions of the quadrilateral lattice: the symmetric and the circular lattice. Imposing both properties on the initial surfaces of the lattice, they propagate through the lattice in the unique construction determined by the planarity condition and the corresponding lattice will be an integrable deeper reduction of the quadrilateral lattice. We call this lattice a *Egorov lattice* since, in the continuous limit, it reduces to a Egorov system of coordinates on submanifolds \[13\].
Definition 16 A Egorov lattice is a circular and symmetric lattice.

A Egorov lattice can be characterized as follows.

Proposition 10 A quadrilateral lattice is a Egorov lattice iff the internal angles corresponding to the vertices $T_i x$ and $T_j x$ are right angles; i.e.:
\[
X_i \cdot T_i X_j = 0, \quad i \neq j.
\] (92)

This property was first announced in [28] without details; it was later shown in [13] to be equivalent to Definition 16.

If $N = M$, then it turns out that the Egorov lattice is a circular lattice invariant under translations along the main diagonal [13], i.e., such that:
\[
TX_i = X_i \Rightarrow TQ_{ij} = Q_{ij},
\] (93)

where $T$ is the total shift: $T := \prod_{i=1}^{N} T_i$.

We remark that this invariance property implies that the lattice depends effectively on $N - 1$ parameters, since it depends on the differences of the variables $n_i$:
\[
X_i = X_i(n_1 - n_2, n_2 - n_3, \ldots, n_{N-1} - n_N).
\] (94)

We have been recently informed of a work on the finite gap formulation of the circular and Egorov lattices [21].

The continuous limits of the above equations:
\[
\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad \beta_{ij} = \beta_{ji}, \quad X_i \cdot X_j = 0, \quad i \neq j
\] (95)

characterize submanifolds parametrized by Egorov systems of conjugate coordinates (Egorov nets). At last, the invariance property (93) reduces to
\[
\sum_{\ell=1}^{N} \partial_{\ell} \beta_{ij} = 0, \quad \sum_{\ell=1}^{N} \partial_{\ell} X_i = 0,
\] (96)

implying that $\beta_{ij} = \beta_{ij}(u_1 - u_2, \ldots, u_{N-1} - u_N)$. For $N = 3$, we recover a classical characterization of the Egorov net [1, 6].

4.4 Quadratic reductions and the Ribaucour transformation

Quadrilateral lattices in quadrics generalize (in a sense which will be explained in Proposition 11) the circular lattices. Their continuous analogs – the so called quadratic conjugate nets (see [17] and references therein) were the subject of intensive study by Ribaucour, Darboux, Bianchi and Eisenhart. The particular type of fundamental transformation of conjugate nets which preserves the quadratic constraint is called the Ribaucour transformation. In this section we
present the basic geometric results concerning quadrilateral lattices in quadrics (see [11] for details); in particular, we give the analogue of the Ribaucour transformation.

Consider the quadrilateral lattice $\mathbf{x}$ in the quadric $Q$ defined by the equation

$$\mathbf{x}^T Q \mathbf{x} + \mathbf{a}^T \mathbf{x} + c = 0 ,$$

where $Q$ is a symmetric matrix, $\mathbf{a}$ is a constant vector, $c$ is a number.

As it was shown in [11], such a constraint is compatible with the geometric integrability scheme of the quadrilateral lattice. This integrability statement can be generalized to quadrilateral lattices in spaces obtained by intersection of many quadric hypersurfaces like, for example, lattices in the Grassmann manifolds and in Segré or Veronese varieties.

To show a simple example, consider a quadrilateral lattice in the standard $M$ dimensional sphere $S^M$. The intersection of the plane of any elementary quadrilateral of the lattice $\mathbf{x} : \mathbb{Z}^N \rightarrow S^M$ with the sphere $S^M$ is a circle. In the stereographic projection $S^M \rightarrow \mathbb{E}^M \cup \{\infty\}$ circles of the sphere $S^M$ are mapped into circles (or straight lines, i.e., circles passing through the infinity point) of $\mathbb{E}^M$, therefore we have:

**Proposition 11** Quadrilateral lattices in the sphere $S^M$ are mapped during the stereographic projection into multidimensional circular lattices in $\mathbb{E}^M$; conversely, any circular lattice in $\mathbb{E}^M$ can be obtained in this way.

Theorem 2 states that, in order to construct the fundamental transformation of the lattice $\mathbf{x}$, we need three new ingredients: $\phi$, $\mathbf{x}_C$ and $\phi_C$. In looking for the Ribaucour reduction of the fundamental transformation, we can use the following additional information:

i) the initial lattice $\mathbf{x}$ satisfies the quadratic constraint (97),

ii) the final lattice $R(\mathbf{x})$ should satisfy the same constraint as well.

This should allow to reduce the number of the necessary data and, indeed, to find the Ribaucour transformation, it is enough to know the Combescure transformation vector $\mathbf{x}_C$ only.

**Proposition 12** The Ribaucour reduction $R(\mathbf{x})$ of the fundamental transformation of the quadrilateral lattice $\mathbf{x}$ subjected to quadratic constraint (97) is determined by the Combescure transformation vector $\mathbf{x}_C$, provided that $\mathbf{x}_C$ is not annihilated by the bilinear form $Q$ of the constraint

$$\mathbf{x}_C^T Q \mathbf{x}_C \neq 0 .$$

The functions $\phi$ and $\phi_C$ entering in formula (25) are then given by

$$\phi = 2 \mathbf{x}^T Q \mathbf{x}_C + \mathbf{a}^T \mathbf{x}_C ,$$

$$\phi_C = \mathbf{x}_C^T Q \mathbf{x}_C .$$
As it was shown in [11], the Ribaucour transformation for circular lattices in $E^M$, defined algebraically in [20], can be obtained combining the stereographic projection with the Ribaucour transformation of the quadrilateral lattices in $S^M$.

To present the Ribaucour reduction of the vectorial fundamental transformation we use results and notation of Section 2, together with equations (99) and (100), to obtain

$$\phi^k = 2x^t_k Qx^C_{\ell,k} + a^t x^C_{\ell,k},$$

$$\phi^C_{\ell,k} = x^C_{\ell,k} Qx^C_{\ell,k}.$$  (101)

Equations (45) and (101) lead to

$$\Delta_i (\phi^C_{\ell,k} + \phi^C_{\ell,k}) = (T_i x^C_{\ell,k} + x^C_{\ell,k}) Q(\Delta_i x_k) + (T_i x^C_{\ell,k} + x^C_{\ell,k}) Q(\Delta_i x_k)$$  (103)

which implies that

$$\phi^C_{\ell,k} + \phi^C_{\ell,k} = 2x^C_{\ell,k} Qx^C_{\ell,k};$$  (104)

the constant of integration was found from the condition $(R_k \circ R_\ell)(x) \subset Q$.

**Proposition 13** The vectorial Ribaucour transformation $R$, i.e., the reduction of the vectorial fundamental transformation (46) compatible with the quadratic constraint (97), is given by the following constraints

$$\Phi^T = 2x^t Qx + a^t X_c,$$  (105)

$$\Phi_c + \Phi^T_c = 2X^t Qx.$$  (106)

One can also consider the superposition of several Ribaucour transformations and show the corresponding permutability theorem.

**Theorem 5** Assume the following splitting of the data of the vectorial Ribaucour transformation

$$\Phi = \left( \begin{array}{c} \Phi^{(1)} \\ \Phi^{(2)} \end{array} \right), \quad X_c = (X^{(1)}_c, X^{(2)}_c), \quad \Phi_c = \left( \begin{array}{c} \Phi^{(1)}_c^{(1)} \\ \Phi^{(1)}_c^{(2)} \\ \Phi^{(2)}_c^{(1)} \\ \Phi^{(2)}_c^{(2)} \end{array} \right),$$  (107)

associated with the partition $K = K_1 + K_2$. Then the vectorial Ribaucour transformation $R(x)$ is equivalent to the following superposition of vectorial Ribaucour transformations:

1. Transformation $R^{(1)}(x)$ with the data $\Phi^{(1)}$, $X^{(1)}_c$, $\Phi^{(1)}_c$.
2. Application on the result obtained in point 1. the transformation $R^{(2)}(x)$ with the data $R^{(1)}(X^{(2)}_c)$, $R^{(1)}(\Phi^{(2)}_c)$, $R^{(1)}(\Phi^{(2)}_c)$ given by $R$-analogous of formulas (51)–(53).

Combining the stereographic projection with the vectorial Ribaucour transformation of the quadrilateral lattices in $S^M$, one can obtain the vectorial Ribaucour transformation for circular lattices in $E^M$ and prove its permutability property. This result was also found independently in [22], directly on the level of circular lattices in $E^M$.  

25
References

[1] L. Bianchi, *Lezioni di Geometria Differenziale*, 3-a ed., Zanichelli, Bologna, 1924.

[2] A. Bobenko, *Discrete conformal maps and surfaces*, in: *Symmetries and Integrability of Difference Equations*, eds. P. Clarkson and F. Nijhoff, pp. 97–108, Cambridge University Press, 1999.

[3] A. Bobenko and U. Pinkall, *Discrete isothermic surfaces*, J. reine angew. Math. 475 (1996) 187–208.

[4] L. V. Bogdanov and B. G. Konopelchenko, *Lattice and q-difference Darboux–Zakharov–Manakov systems via ∂ method*, J. Phys. A: Math. Gen. 28 (1995) L173-L178.

[5] J. Cieśliński, A. Doliwa and P. M. Santini, *The Integrable Discrete Analogues of Orthogonal Coordinate Systems are Multidimensional Circular Lattices*, Phys. Lett. A 235 (1997) 480–488.

[6] G. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, 2-ème éd., complétée, Gauthier – Villars, Paris, 1910.

[7] G. Darboux, *Leçons sur la théorie générale des surfaces I–IV*, Gauthier – Villars, Paris, 1887–1896.

[8] E. Date and M. Kashiwara and M. Jimbo and T. Miwa, *Transformation groups for soliton equations*, [in:] *Nonlinear integrable systems — classical theory and quantum theory*, Proc. of RIMS Symposium, M. Jimbo and T. Miwa (Eds.), World Scientific, 1983, Singapore, pp. 39-119.

[9] A. Doliwa, *Geometric discretisation of the Toda system*, Phys. Lett. A 234 (1997) 187–192.

[10] A. Doliwa, *Lattice geometry of the Hirota equation*, solv-int/9907013.

[11] A. Doliwa, *Quadratic reductions of quadrilateral lattices*, J. Geom. Phys. 30 (1999) 169–186.

[12] A. Doliwa and P. M. Santini, *Multidimensional Quadrilateral Lattices are Integrable*, Phys. Lett. A 233 (1997) 365–372.

[13] A. Doliwa and P. M. Santini, *The symmetric and Egorov reductions of the quadrilateral lattice*, solv-int/9907012.

[14] A. Doliwa, P. M. Santini and M. Mañas, *Transformations of Quadrilateral Lattices*, solv-int/9712017.
[15] A. Doliwa, S. V. Manakov and P. M. Santini, $\bar{\partial}$-reductions of the Multi-
dimensional Quadrilateral Lattice: the Multidimensional Circular Lattice, Comm. Math. Phys. 196 (1998) 1-18.

[16] A. Doliwa, M. Mañas, L. Martínez Alonso, E. Medina and P. M. Santini, Charged Free Fermions, Vertex Operators and Transformation Theory of Conjugate Nets, J. Phys. A 32 (1999) 1197–1216.

[17] L. P. Eisenhart, Transforms of Surfaces, Princeton University Press, 1923.

[18] R. Hirota, Discrete Analogue of a Generalized Toda Equation, J. Phys. Soc. Jpn. 50 (1981), 3785–3791.

[19] J. van de Leur, Schlesinger–Bäcklund transformation for N-component KP, J. Math. Phys. 39 (1998), 2833–2847.

[20] B. G. Konopelchenko and W. K. Schief, Three-dimensional integrable lattices in Euclidean spaces: Conjugacy and orthogonality, Proc. Roy. Soc. London A 454 (1998) 3075–3104.

[21] I. M. Krichever, seminar given at the Dept. of Mathematics, University of Rome "La Sapienza", June 1998.

[22] Q. P. Liu and M. Mañas, Superposition Formulae for the Discrete Ribaucour Transformations of Circular Lattices, Phys. Lett. A 249 (1998) 424–430.

[23] R. R. Martin, J. de Pont and T. J. Sharrock, Cyclic surfaces in computer aided design, in: The Mathematics of Surfaces, ed. J. A. Gregory, pp.253–268, Oxford, Clarendon Press, 1986.

[24] A. V. Mikhailov, Integrability of a two-dimensional generalization of the Toda chain, JETP Lett. 30 (1979) 414–418.

[25] A. W. Nutbourne, The solution of a frame matching equation, in: The Mathematics of Surfaces, ed. J. A. Gregory, pp.233–252, Oxford, Clarendon Press, 1986.

[26] D. Pedoe, Geometry, a comprehensive course, Dover Publications, New York, 1988.

[27] R. Sauer, Differenzengeometrie, Springer, Berlin, 1970.

[28] W. K. Schief, talk given at the Workshop: Nonlinear systems, solitons and geometry, Oberwolfach, October 1997.