DIMENSION BOUND FOR DOUBLY BADLY APPROXIMABLE AFFINE FORM

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Abstract. We prove that for all $b$, the Hausdorff dimension of the set of $m \times n$ matrices $\epsilon$-badly approximable for the target $b$ is not full. The doubly metric case follows.

It was known that for almost every matrix $A$, the Hausdorff dimension of the set $\text{Bad}_A(\epsilon)$ of $\epsilon$-badly approximable target $b$ is not full, and that for real numbers $\alpha$, $\dim_H \text{Bad}_\alpha(\epsilon) = 1$ if and only if $\alpha$ is singular on average. We show that if $\dim_H \text{Bad}_A(\epsilon) = m$, then $A$ is singular on average.

1. Introduction

Diophantine approximation for irrational numbers has been generalized to studying vectors, linear forms, and more generally matrices, and are classical subjects in number theory. In this article, we mainly consider the inhomogenous Diophantine approximation: for rational numbers, it corresponds to approximating a real number $b$ by $aq + p$ for some integers $p$ and $q$.

Let $M_{m,n}(\mathbb{R})$ be the set of $m \times n$ real matrices, and let $\tilde{M}_{m,n}(\mathbb{R}) := M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$. Let $\langle v \rangle = \inf_{p \in \mathbb{Z}^k} ||v - p||$ be the distance from $v$ to the nearest integral vector.

We call $A$ $\epsilon$-bad for $b \in \mathbb{R}^m$ if

\begin{equation}
\liminf_{q \in \mathbb{Z}^n, ||q|| \to \infty} ||q||^{n/m} \langle Aq - b \rangle \geq \epsilon,
\end{equation}

Denote

\[
\text{Bad}(\epsilon) \overset{\text{def}}{=} \left\{ (A, b) \in \tilde{M}_{m,n}(\mathbb{R}) : A \text{ is } \epsilon \text{-bad for } b \right\},
\]

\[
\text{Bad}_A(\epsilon) \overset{\text{def}}{=} \{ b \in \mathbb{R}^m : A \text{ is } \epsilon \text{-bad for } b \}, \text{Bad}_A \overset{\text{def}}{=} \bigcup_{\epsilon > 0} \text{Bad}_A(\epsilon),
\]

\[
\text{Bad}^b(\epsilon) \overset{\text{def}}{=} \{ A \in M_{m,n}(\mathbb{R}) : A \text{ is } \epsilon \text{-bad for } b \}, \text{Bad}^b \overset{\text{def}}{=} \bigcup_{\epsilon > 0} \text{Bad}^b(\epsilon).
\]

The set $\text{Bad}^0$ can be considered as the set of badly approximable systems of $m$ linear forms in $n$ variables. This set is of Lebesgue measure zero [Gro38], but has full Hausdorff dimension $mn$ [Sch69].

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For general $b$, $\text{Bad}^b$ also has zero Lebesgue measure [Sch66] and full Hausdorff dimension for every $b$ [ET11]. Indeed, [ET11] showed that $\text{Bad}^b$ is a winning set and [HKS] further showed that it is a hyperplane winning set. On the other hand, the set $\text{Bad}_A$ also has full Hausdorff dimension for every $A$ [BHKV10], but can have positive Lebesgue measure.

The set $\text{Bad}^b$ and $\text{Bad}_A$ are unions of subsets $\text{Bad}^b(\epsilon)$ and $\text{Bad}_A(\epsilon)$ over $\epsilon > 0$, respectively, thus a more refined question is about the Hausdorff dimension of $\text{Bad}^b(\epsilon)$, $\text{Bad}_A(\epsilon)$. For the homogeneous case $b = 0$, [BK13] estimated the Hausdorff dimension of $\text{Bad}^0(\epsilon)$, and [Sim18] recently estimated the bound more precisely. These results directly imply that the Hausdorff dimension of $\text{Bad}^0(\epsilon)$ is strictly less than full dimension $mn$. Thus, a natural question is whether $\text{Bad}^b(\epsilon)$ can have full Hausdorff dimension for general $b$.

Our main result answers the above question.

**Theorem 1.1.** For any $\epsilon > 0$, there exists $\delta > 0$ such that for all $b \in \mathbb{R}^m$,

$$\dim_H \text{Bad}^b(\epsilon) < mn - \delta.$$  

This result directly implies a similar result for doubly metric case.

**Corollary 1.2.** For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\dim_H \text{Bad}(\epsilon) < mn + m - \delta.$$  

It was known that the set $\text{Bad}_A(\epsilon)$ is less than the full Hausdorff dimension for almost every $A$ [LSS]. The argument for Theorem 1.1 can be applied similarly to improve the result for $\text{Bad}_A(\epsilon)$ in [LSS] in terms of the exceptional set.

**Theorem 1.3.** For any $\epsilon > 0$, there exists $\delta > 0$ such that the Hausdorff dimension of the set of $A$ satisfying $\dim_H \text{Bad}_A(\epsilon) \geq m - \delta$ is strictly less than $mn$.

To state our last result, let us introduce more notations. For $d = m + n$, let $G(\mathbb{R}) = \text{ASL}_d(\mathbb{R})$ be the set of area-preserving affine transformations. Let $G(\mathbb{Z}) = \text{ASL}_d(\mathbb{Z}) = \text{SL}_d(\mathbb{Z}) \times \mathbb{Z}^d = \text{Stab}_{G(\mathbb{R})}(\mathbb{Z}^d)$ be the stabilizer of $\mathbb{Z}^d$. For the 1-parameter diagonal subgroup

$$\{a_t = \Delta(e^{nt}1_m, e^{-nt}1_n)\}_{t \in \mathbb{R}}$$

in $\text{SL}_d(\mathbb{R})$, we take a lift of this group to $G(\mathbb{R}) \subset \text{SL}_{d+1}(\mathbb{R})$ given by $a_t \rightarrow \begin{pmatrix} a_t & 0 \\ 0 & 1 \end{pmatrix}$ and we denote it again by $a_t$ by abuse of notation. Let $a = a_1$ be the time-one map of the diagonal flow $a_t$. We denote by
Let $Y = G(\mathbb{R})/G(\mathbb{Z})$ and let $\mathbf{d}(\cdot,\cdot)$ be a right invariant metric on $Y$. We may assume that the locally defined maps (on balls of radius, say $r_0$) log and exponential maps between $U$ and its Lie algebra $\mathfrak{u}$ are Lipschitz. For the norm of Lie algebra $\mathfrak{u}$, we take the Euclidean norm $|| \cdot ||_{\mathbb{R}^{m}}$. Similarly, we may assume that log and exponential maps between $W$ and its Lie algebra $\mathfrak{w}$ are bi-Lipschitz if we take the Euclidean norm $|| \cdot ||_{\mathbb{R}^{m}}$ for the norm of Lie algebra $\mathfrak{w}$. Observe that $Ad_a u = e^{(m+n)\mathbf{t}} u$ holds for any $u \in \mathfrak{u}$ and $Ad_a w = e^{nt} w$ for any $w \in \mathfrak{w}$.

Let $X = SL(d, \mathbb{R})/SL(d, \mathbb{Z})$ be the space of unimodular lattices. By a unimodular grid $y$ in $\mathbb{R}^d$, we mean a coset $x + v$ of a lattice $x \in X$ where $v \in \mathbb{R}^d$. One can view a unimodular grid $y$ as an element in $Y$. There is a natural projection $\pi: Y \to X$ defined by $\pi\left( \begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix} G(\mathbb{Z}) \right) = B \cdot SL(d, \mathbb{Z})$ for $B \in SL(d, \mathbb{R})$ and $v \in \mathbb{R}^d$.

For $A \in M_{m,n}$, we associate a point $x_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} SL(d, \mathbb{Z})$ in $X$, and for $(A,b) \in \tilde{M}_{m,n}(\mathbb{R})$, we associate a point $x_{A,b} = \begin{pmatrix} I_m & A & -b \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} x_0$ in $Y$, where $x_0$ is the identity coset $SL_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$.

We say that a point $x \in X$ has $\delta$-escape of mass on average (with respect to the diagonal flow $a_t$) if

$$\liminf_{N \to \infty} \frac{1}{N} \left| \{ l \in \{1, \ldots, N\} : a_l x \notin Q \} \right| \geq \delta$$

for any compact set $Q$ in $X$. In [LSS], it was shown that $\dim_H \text{Bad}_A(\epsilon) < m$ for all $\epsilon > 0$ if $x_A$ is heavy which is a condition similar to no escape of mass on average. Note that $x_A$ is heavy for almost every $A \in M_{m,n}(\mathbb{R})$. A $m \times n$ matrix $A$ is called singular on average if for any $\epsilon > 0$

$$\lim_{N \to \infty} \frac{1}{N} \left| \left\{ l \in \{1, \ldots, N\} : \exists q \in \mathbb{Z}^n \text{ s.t. } \langle Aq \rangle < \epsilon 2^{-\frac{l}{m}} \text{ and } 0 < ||q|| < 2^l \right\} \right| = 1.$$

This property is equivalent to the fact that the corresponding point $x_A$ has 1-escape of mass on average (with respect to the diagonal flow $a_t$) by Dani’s correspondence.

For $m = n = 1$, $A$ is singular on average if and only $\text{Bad}_A(\epsilon)$ has full Hausdorff dimension for some $\epsilon > 0$ [BKLR]. For $m, n \neq 1$, nothing was known about $\dim_H \text{Bad}_A(\epsilon)$ for $A$ which has $\eta$-escape of mass on average.
for some $0 < \eta < 1$. The next theorem generalizes the necessary part of the previous result.

**Theorem 1.4.** Let $A \in M_{m,n}(\mathbb{R})$. If $\dim_H \text{Bad}_A(\epsilon) = m$ for some $\epsilon > 0$, then $A$ is singular on average.

We remark that the set of matrices which are singular on average has Hausdorff dimension at most $mn - \frac{m+n}{mn}$ [KKLM].

2. Measures with large entropy

2.1. Correspondence with dynamics. For $y \in Y$, $\Gamma_y$ denotes the corresponding unimodular grid in $\mathbb{R}^d$. Let

$$L_\epsilon := \left\{ y : \Gamma_y \cap B_{\epsilon}^{d}(0) = \emptyset \right\},$$

which is a (non-compact) closed subset of $Y$.

**Proposition 2.1.** For any $(A, b) \in \text{Bad}(\epsilon)$, one of the following statements holds:

1. there exists some $q \in \mathbb{Z}^n$ such that $\langle Aq - b \rangle = 0$.

2. the point $x_{A,b}$ is eventually in $L_\epsilon$, i.e., there exists $T \geq 0$ such that $a_t x_{A,b} \in L_\epsilon$ for all $t \geq T$.

**Proof.** Assume that both of the statements do not hold. Then there exist arbitrarily large $t$’s satisfying $a_t x_{A,b} \notin L_\epsilon$. As

$$a_t x_{A,b} = \begin{pmatrix} e^{nt} I_m & 0 & 0 \\ 0 & e^{-mt} I_m & 0 \\ 0 & 0 & 1 \end{pmatrix} x_{A,b} = \begin{pmatrix} e^{nt} A & -e^{nt} b \\ 0 & e^{-mt} I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} x_0,$$

the vectors in the grid $\Gamma_{a_t x_{A,b}}$ can be represented as $\begin{pmatrix} e^{nt} (Aq + p - b) \\ e^{-mt} q \end{pmatrix}$ for integer vectors $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$. Therefore $a_t x_{A,b} \notin L_\epsilon$ implies that for some $q$, $\langle Aq - b \rangle e^{nt} < \epsilon^{m+n}$ and $e^{-mt} ||q|| < \epsilon^{m+n}$, thus $||q|| \frac{m}{n} \langle Aq - b \rangle < \epsilon$. Since $\langle Aq - b \rangle \neq 0, \forall q$, we use the condition $\langle Aq - b \rangle e^{nt} < \epsilon^{m+n}$ for arbitrarily large $t$ to conclude that $||q|| \frac{m}{n} \langle Aq - b \rangle < \epsilon$ holds for infinitely many $q$’s. This is a contradiction to the assumption that $A \in \text{Bad}^0(\epsilon)$. \qed

We claim that for a fixed $b \in \mathbb{R}^m$, the subset $\text{Bad}_{(1)}^b(\epsilon)$ of $\text{Bad}^0(\epsilon)$ satisfying (2.1) is a subset of $\text{Bad}^0(\epsilon)$. Indeed, if $A \in \text{Bad}^0(\epsilon)$ satisfies (2.1), $\langle Aq_0 - b \rangle = 0$ for some $q_0 \in \mathbb{Z}^m$ and $\liminf_{||q|| \to \infty} ||q||^{1/m} \langle Aq - b \rangle \geq \epsilon$, thus $\liminf_{||q|| \to \infty} ||q||^{1/m} \langle A(q - q_0) \rangle \geq \epsilon$. Therefore, for every $b \in \mathbb{R}^m$,

$$\dim_H \text{Bad}_{(1)}^b(\epsilon) \leq \dim_H \text{Bad}^0(\epsilon) = mn - c_{m,n} \epsilon^m + o(\epsilon^m) < mn.$$
for some constant \( c_{m,n} > 0 \) [Sim18].

For a fixed \( A \in M_{m,n}(\mathbb{R}) \), the subset of \( \text{Bad}_A(e) \) satisfying (2.1) is of the form \( Ag + p \) for some \( q, p \in \mathbb{Z}^m \) thus has Hausdorff dimension zero.

In the rest of the paper, we will focus on \( x_{A,b} \) that are eventually in \( L_c \).

2.2. Constructing measure with entropy lower bound. In this subsection, we construct an \( a_t \)-invariant measure on \( Y \) with a lower bound on the entropy. We will use entropy contribution for the subgroup \( U \), with respect to the Borel \( \sigma \)-algebra \( \mathcal{B}_Y^U \) generated by the collection of \( U \)-invariant Borel sets in \( Y \). For any countable partition \( \mathcal{P} \) of \( Y \), \( H_\mu(\mathcal{P}|\mathcal{B}_Y^U) \) will denote the relative entropy of \( \mathcal{P} \) with respect to the \( \sigma \)-algebra \( \mathcal{B}_Y^U \). Also denote by \( h_\mu(a|\mathcal{B}_Y^U) \) the relative entropy of the transformation \( a \) for \( \mu \). Before constructing the desired measure, we recall the following theorem about escape of mass.

**Theorem 2.2** (KKLM, Remark 2.1). For any \( x \in X \), the set
\[
\{ u \in U | \text{ux has } \delta \text{-escape of mass on average} \}
\]
has Hausdorff dimension at most \( mn - \frac{\delta(m+n)}{mn} \).

**Remark 2.3.** For the definition of escape of mass, we follow the definition of singularity on average in [DFSU18], Section 1.4. [KKLM] uses lim instead of lim inf for the definition of escape of mass.

For any compact set \( Q \subset X \) and positive integer \( k > 0 \), and any \( 0 < \eta < 1 \), let
\[
E_\eta \overset{\text{def}}{=} \{ A \in M_{m,n}(\mathbb{R}) : x_A \text{ has } \eta \text{-escape of mass on average} \}
\]
\[
F_{\eta,Q} \overset{\text{def}}{=} \left\{ A \in M_{m,n}(\mathbb{R}) : \frac{1}{k} \sum_{i=0}^{k-1} \delta_{a_i x_A}(X \setminus Q) < \eta \text{ for infinitely many } k \right\}
\]
\[
F_{\eta,Q}^k \overset{\text{def}}{=} \left\{ A \in M_{m,n}(\mathbb{R}) : \frac{1}{k} \sum_{i=0}^{k-1} \delta_{a_i x_A}(X \setminus Q) < \eta \right\}
\]
Take a sequence of increasing compact sets \( \{Q_j\} \) exhausting \( X \). Observe that \( M_{m,n}(\mathbb{R}) \setminus E_\eta = \bigcup_{j=1}^{\infty} F_{\eta,Q_j} \).

We denote by \( \bar{Y} \) the one-point compactification of \( Y \) with \( \sigma \)-algebra \( \mathcal{B}_Y \) generated by \( \mathcal{B}_Y \) and \( \{x\} \). The diagonal action \( a_t \) is extended to the action on \( \bar{Y} \) by \( a_t(x) = \infty \) for \( t \in \mathbb{R} \). For a finite partition \( \mathcal{P} = \{P_1, \cdots, P_N, P_\infty\} \) of \( Y \) which has only one non-compact element \( P_\infty \), denote by \( \mathcal{P}(q) \) the finite partition \( \{P_1, \cdots, P_N, \mathcal{P}_{\infty}^{\text{def}} = P_\infty \cup \{x\}\} \) of \( \bar{Y} \). Note that \( \mathcal{P}(q) = \mathcal{P}_{\infty}^{(q)} \) for \( q \in \mathbb{N} \) and \( H_\mu(\mathcal{P}(q)|\mathcal{B}_Y^U) = H_\mu(\mathcal{P}(q)|\mathcal{B}_Y^U) \) for \( \mu \in \mathcal{P}(Y) \). Here, \( \mathcal{B}_Y^U \) is the Borel \( \sigma \)-algebra generated by the collection of \( U \)-invariant Borel sets in \( \bar{Y} \). For the rest of the section, we construct the desired measure on \( \bar{Y} \) in Proposition 2.4. The construction will basically follow the construction in the [LSS],
Section 2. However, the additional step using Theorem 2.2 is necessary to control the escape of mass since we will allow a small amount of escape of mass.

**Proposition 2.4.** Assume that \( \dim_H \text{Bad}^b(\varepsilon) > \dim_H \text{Bad}^b(\varepsilon) \). For \( \gamma > 0 \), let \( \eta = \frac{mn}{m+n}(mn - \dim_H \text{Bad}^b(\varepsilon) + \gamma) > 0 \). Then there exist an \( a \)-invariant probability measure \( \mu^b \in \mathcal{P}(Y) \) satisfying:

1. \( \text{Supp} \mu^b \subseteq \mathcal{L}_\varepsilon \),
2. \( \mu^b(Y \setminus Y) \leq \eta \),
3. Let \( K < Y \) be a compact set. If \( \mathcal{P} \) is any finite partition of \( Y \) satisfying:
   - \( \mathcal{P} \) contains an atom \( P_\varepsilon \) where \( K \subseteq Y \setminus P_\varepsilon \),
   - \( \forall P \in \mathcal{P} \setminus \{ P_\varepsilon \}, \text{diam } P < r_0 \) for some \( 0 < r_0 < \frac{1}{2} \),
   - \( \forall P \in \mathcal{P}, \mu^b(\partial P) = 0 \),
then, for all \( q \geq 1 \),

\[
\frac{1}{q} H_{\mu^b}(\mathcal{P}(q)|B_q^q) \geq (1 - \mu^b(Y \setminus K)^\frac{1}{\gamma})(m+n)(\dim_H \text{Bad}^b(\varepsilon) - \gamma - mn \mu^b(Y \setminus K)^\frac{1}{\gamma}).
\]

**Proof.** Denote by \( R^b \) the set \( \text{Bad}^b(\varepsilon) \setminus \text{Bad}^b(1)(\varepsilon) \), and let \( R^{b,T} \) be the increasing set \( \{ A \in M_{m,n} | \forall t \geq T, a_t x_{A,b} \in \mathcal{L}_\varepsilon \} \). By Proposition 2.1, \( R^b = \bigcup_{T=1}^{\infty} R^{b,T} \).

Since \( \dim_H \text{Bad}^b(\varepsilon) \geq \dim_H \text{Bad}^b(1)(\varepsilon), \dim_H R^b = \dim_H \text{Bad}^b(\varepsilon) \). For any \( \gamma > 0 \), there exists \( T_\gamma > 0 \) satisfying \( \dim_H R^{b,T_\gamma} > \dim_H \text{Bad}^b(\varepsilon) - \gamma \). By Theorem 2.2, we obtain \( \dim_H E_\eta \leq mn - \frac{\eta(m+n)}{mn} = \dim_H \text{Bad}^b(\varepsilon) - \gamma \). From the fact that \( M_{m,n}(\mathbb{R}) \setminus E_\eta = \bigcup_{j=1}^{\infty} F_{\eta,Q_j} \), we obtain

\[
\dim_H \left( \bigcup_{j=1}^{\infty} (R^{b,T_\gamma} \cap F_{\eta,Q_j}) \right) = \dim_H (R^{b,T_\gamma} \cap E_\eta) > \dim_H \text{Bad}^b(\varepsilon) - \gamma,
\]

thus we can take a compact set \( Q^b \subset X \) from the above sequence of increasing compact sets \( \{Q_j \} \to X \) which satisfies

\[
\dim_H (R^{b,T_\gamma} \cap F_{\eta,Q^b}) > \dim_H \text{Bad}^b(\varepsilon) - \gamma.
\]

Since \( F_{\eta,Q^b} = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} F_{\eta,Q^k} = \limsup_{k \to \infty} F_{\eta,Q^k}, \)

\[
\dim_H (R^{b,T_\gamma} \cap F_{\eta,Q^k}) > \dim_H \text{Bad}^b(\varepsilon) - \gamma
\]

for an increasing sequence of positive integers \( \{k_i \} \) [Fal03].

For a bounded subset \( S \subseteq Y \), let \( N_\delta(S, \delta) \) be the maximal cardinality of a \( \delta \)-separated subset of \( S \) for the metric \( d \). Then

\[
\dim_H S \leq \liminf_{\delta \to 0} \frac{\log N_\delta(S, \delta)}{\log \frac{1}{\delta}}
\]
(See section 2.2 of [LSS]). Let \( \phi_b : M_{m,n} \to Y \) be the function defined by \( \phi_b(A) = x_{A,b} \). For each \( k_i \geq T_\gamma \), let \( S_i \) be a maximal \( e^{-k_i(m+n)} \)-separated subset of \( S_i^c \) with respect to the metric \( d \), where \( S_i^c = \phi_b(R^{b,T_\gamma} \cap F_{\eta,Q^b}) \subset Y \). Then

\[
\liminf_{i \to \infty} \frac{\log |S_i|}{k_i} \geq (m + n) \liminf_{\delta \to 0} \frac{\log N_d(S_i^c, \delta)}{\log \frac{1}{\delta}} \geq (m + n) \dim_H(K^{b,T_\gamma} \cap F_{\eta,Q^b}) \geq (m + n)(\dim_H \Bad^b(\epsilon) - \gamma)
\]  

(2.2)

holds since \( \phi_b \) is bi-Lipschitz from the bi-Lipschitz property between \( d \) and \( \| \cdot \|_{\mathbb{R}^m} \).

Let \( \nu_i \defeq \frac{1}{|S_i|} \sum_{y \in S_i} \delta_y \) be the normalized counting measure on \( S_i \) and let

\[
\mu_i \defeq \frac{1}{k_i} \sum_{k=0}^{k_i-1} a^k \nu_i \overset{w^*}{\to} \mu^b \in \mathcal{P}(Y)
\]

By extracting a subsequence if necessary there exists a probability measure \( \mu^b \) which is a weak*-accumulation point of \( \{\mu_i\} \). Now we prove that the measure \( \mu^b \) is the desired measure. The measure \( \mu^b \) is clearly an \( \alpha \)-invariant measure.

(1) For any \( y \in S_i \subseteq \phi_b(R^{b,T_\gamma}) \), \( a^T y \in \mathcal{L}_\epsilon \) holds for \( T > T_\gamma \). Thus

\[
\mu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{k_i-1} a^k \nu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{T_\gamma} a^k \nu_i(Y \setminus \mathcal{L}_\epsilon) \leq \frac{T_\gamma}{k_i}
\]

and we obtain item (1) by taking limit for \( k_i \to \infty \).

(2) Let \( K^b \subset Y \) be the compact set which is defined as \( K^b = \pi^{-1}(Q^b) \).

Since \( y \in S_i \)'s are contained in \( \phi_b(F^{k_i}_{\eta,Q^b}) \), \( \frac{1}{k_i} \sum_{k=0}^{k_i-1} \delta_{a^k y}(Y \setminus K^b) < \eta \) holds for all \( i \in \mathbb{N} \). Therefore for all \( i \in \mathbb{N} \),

\[
\mu_i(Y \setminus K^b) = \frac{1}{|S_i|} \sum_{y \in S_i} \frac{1}{k_i} \sum_{k=0}^{k_i-1} \delta_{a^k y}(Y \setminus K^b) < \eta,
\]

thus

\[
\mu^b(\bar{Y} \setminus Y) \leq \mu^b(\bar{Y} \setminus K^b) = \lim_{i \to \infty} \mu_i(Y \setminus K^b) \leq \eta.
\]

For the rest of the proof, we check the condition (3).

(3) Let \( \rho > 0 \) be a small positive real number, then

\[
\mu^b(Y \setminus K) + \rho > \mu_i(Y \setminus K) = \frac{1}{k_i |S_i|} \sum_{y \in S_i, 0 \leq k < k_i} \delta_{a^k y}(Y \setminus K)
\]

holds for large enough \( i \). For simplicity, let \( \beta \defeq \mu^b(Y \setminus K) + \rho \). The above inequality means that there exist at most \( \beta k_i |S_i| \) numbers of \( a^k y \)'s in \( Y \setminus K \). Let
$S_i' \subset S_i$ be the set of $y$’s satisfying $a^ky \in K$ for some $k \in [(1-\beta^\frac{1}{2})k_i, k_i)$. Then we have $|S_i'| \geq (1 - \beta^\frac{1}{2})|S_i|$ by Pigeonhole principle. Let $\nu'_i \overset{\text{def}}{=} \frac{1}{|S_i'|} \sum_{y \in S_i'} \delta_y$ be the normalized counting measure on $S_i'$, then $\nu_i(A) \geq \frac{|S_i'|}{|S_i|} \nu'_i(A)$ holds for all measurable set $A \subseteq Y$. There exist some constant $C_1$ such that for any arbitrary countable partition $\mathcal{P}$, 

\[ H_{\nu_i}(\mathcal{P}) \geq H_{|S_i'|/|S_i|\nu'_i}(\mathcal{P}) - C_1 = \frac{|S_i'|}{|S_i|} H_{\nu'_i}(\mathcal{P}) + \frac{|S_i'|}{|S_i|} \log \frac{|S_i'|}{|S_i|} - C_1 \]

\[ \geq (1 - \beta^\frac{1}{2})H_{\nu'_i}(\mathcal{P}) - (C_1 + \frac{1}{e}). \]

If $P$ is any non-empty atom of $\mathcal{P}^{(k_i)}$, fixing any $y_0 \in P$, any $y' \in S_i' \cap P = S_i' \cap [y_0]_{\mathcal{P}^{(k_i)}}$ satisfies

\[ r_0 > d(a^ky_0, a^ky') \geq C'e^{(m+n)k}d(y_0, y) \geq C'e^{(m+n)(1-\beta^\frac{1}{2})k_i}d(y_0, y) \]

for some constant $C' > 0$. Here, we used the right invariant property of $d$ and bi-Lipschitz property between $d$ and $|\cdot|$. Thus $S_i' \cap P$ can be covered by one ball of $C'^{-1}r_0e^{-(m+n)(1-\beta^\frac{1}{2})k_i}$-radius for metric $d$ as well as by $C_2e^{-(m+n)mn\beta^\frac{1}{2}k_i}$ many balls of $r_0e^{-(m+n)k_i}$-radius for the metric $d$ and some constant $C_2 > 0$. Since $S_i'$ is $e^{(m+n)k_i}$-separated with respect to $d$, we get

\[ \text{Card}(S_i' \cap [y_0]_{\mathcal{P}^{(k_i)}}) \leq C_2e^{(m+n)mn\beta^\frac{1}{2}k_i}, \]

and therefore

\[ H_{\nu'_i}(\mathcal{P}^{(k_i)}) \geq \log |S_i'| - (m + n)mn\beta^\frac{1}{2}k_i - \log C_2. \]

Now we can estimate the lower bound of the entropy. For $q \geq 1$, write the Euclidean division of large enough $k_i - 1$ by $q$ as

\[ k_i - 1 = qk' + s \quad \text{with} \quad s \in \{0, \ldots, q - 1\}. \]

By subadditivity of the entropy with respect to the partition, for each $p \in \{0, \ldots, q - 1\}$,

\[ H_{\nu_i}(\mathcal{P}^{(k_i)}|B_Y^{(p)}) \leq H_{\nu_i}(\mathcal{P}^{(q)}|B_Y^{(p)}) + \cdots + H_{\nu_i}(\mathcal{P}^{(q)}|B_Y^{(p)}) + 2q \log |\mathcal{P}|. \]

Summing those inequalities for $p = 0, \ldots, q - 1$, and using the concave property of entropy with respect to the measure, we obtain

\[ qH_{\nu_i}(\mathcal{P}^{(k_i)}|B_Y^{(p)}) \leq \sum_{k=0}^{k_i-1} H_{\nu_i}(\mathcal{P}^{(q)}|B_Y^{(p)}) + 2q^2 \log |\mathcal{P}| \]

\[ \leq k_i H_{\nu_i}(\mathcal{P}^{(0)}|B_Y^{(p)}) + 2q^2 \log |\mathcal{P}| \]
and therefore
\[
\frac{1}{q} H_{\mu_i}(\mathcal{P}(q)|\mathcal{B}_Y^U) \geq \frac{1}{k_i} H_{\nu_i}(\mathcal{P}(k_i)|\mathcal{B}_Y^U) - \frac{2q \log |P|}{k_i} \\
\geq \frac{1}{k_i} \left\{ (1 - \beta^\frac{4}{3}) H_{\nu_i}(\mathcal{P}(k_i)) - (C_1 + \frac{1}{e}) - 2q \log |P| \right\} \\
\geq \frac{1}{k_i} \left\{ (1 - \beta^\frac{4}{3})(\log |S'_1| - mn(m + n)\beta^\frac{4}{3}k_i - \log C_2) - (C_1 + \frac{1}{e}) - 2q \log |P| \right\}
\]

Here, for the second inequality, we used inequality (2.3) and \( H_{\nu_i}(\mathcal{P}(q)|\mathcal{B}_Y^U) = H_{\nu_i}(\mathcal{P}(q)) \) from the fact that \( \nu_i \) is supported on an atom of \( \mathcal{B}_Y^U \). Now we can take \( i \to \infty \) because the atoms \( P \) of \( \mathcal{P} \) and hence of \( \mathcal{P}(q) \), satisfy \( \mu^b(\partial P) = 0 \). Thus we obtain the inequality
\[
\frac{1}{q} H_{\mu^b}(\mathcal{P}(q)|\mathcal{B}_Y^U) \geq (1 - \beta^\frac{4}{3})(m + n)\{(\dim_H \text{Bad}^b(\cdot) - \gamma) - mn\beta^\frac{4}{3}\},
\]
from the inequality (2.2) and finally get the inequality
\[
\frac{1}{q} H_{\mu^b}(\mathcal{P}(q)|\mathcal{B}_Y^U) \geq (1 - \mu^b(Y \setminus K)^\frac{4}{3})(m + n)\{(\dim_H \text{Bad}^b(\cdot) - \gamma) - mn\mu^b(Y \setminus K)^\frac{4}{3}\}
\]
we desired by taking \( \rho \to 0 \). \( \square \)

3. Proof of main results

3.1. Maximal entropy implies invariance. To deduce the invariance property of the measure we constructed in Section 2, we need the following proposition about maximal entropy.

**Proposition 3.1** (Maximal entropy implies \( U \)-invariance). Let \( \mu \) be an \( a_t \)-invariant probability measure on \( Y \). Then
\[
h_\mu(a|\mathcal{B}_Y^U) \leq \log |\det(Ad_{a|u})|
\]
with equality if and only if \( \mu \) is \( U \)-invariant.

Note that for our situation \( a_t = \Delta(t^mt \cdot 1_m, e^{-mt}1_n) \) in \( SL_d(\mathbb{R}) \), the restriction of the adjoint map \( Ad_{a|u} \) can be considered as the map \( A \rightarrow e^{-(m + n)}A \) for \( A \in M_{m,n}(\mathbb{R}) \) by identifying \( u \simeq M_{m,n}(\mathbb{R}) \), so the maximal entropy is \( \log |\det(Ad_{a|u})| = (m + n)mn \).

**Definition 3.2** (7.25. of [EL10]). Let \( G^{-} \overset{\text{def}}{=} \{ g \in G|a_t g a_t^{-1} \rightarrow e \text{ as } t \to \infty \} \) be the stable horospherical subgroup associated to \( a \). Let \( \mu \) be an \( a \)-invariant measure on \( Y \) and \( U < G^{-} \) be a closed \( a \)-normalized subgroup.

1. We say that a countably generated \( \sigma \)-algebra \( \mathcal{A} \) is subordinate to \( U^+ \) (mod \( \mu \)) if for \( \mu \)-a.e. \( y \), there exists \( \delta > 0 \) such that
\[
B_{\delta}^{U^+} \cdot y \subset [y]_\mathcal{A} \subset B_{\delta^{-1}}^{U^+} \cdot y.
\]
2. We say that \( \mathcal{A} \) is \( a \)-descending if \( a^{-1}\mathcal{A} \subset \mathcal{A} \).
The proof of Proposition 3.1 is based on the following theorem applied to $a^{-1}$ so that $U < G^-$. To obtain following theorem, it is enough to consider the case that $\mu$ is ergodic by using ergodic decomposition. Combining Proposition 7.34 and Theorem 7.9 of [EL10], we obtain the following statement under the ergodicity assumption.

**Theorem 3.3** (Einsiedler-Lindenstrauss). Let $\mu$ be an $\alpha_t$-invariant ergodic probability measure on $Y$. If $A$ is a countably generated sub-$\sigma$-algebra of the Borel $\sigma$-algebra which is $a$-descending and $U$-subordinate, then

$$H_\mu(A|a^{-1}A) \leq -\log |\det(Ad_a|_U)|.$$

For detailed proof of Proposition 3.1 using Theorem 3.3, we refer the reader to [LSS], Chapter 3. The only difference here is that we calculate relative entropy with respect to the $\sigma$-algebra $B_Y^U$ instead of the $\sigma$-algebra $\pi^{-1}(B_X)$ in [LSS].

3.2. **Proof of Theorem 1.1.** In this subsection, to prove Theorem 1.1, we investigate the closed set $L_\epsilon$. Measures supported on the set $L_\epsilon$ only admit few invariance properties, as stated in the following proposition.

**Proposition 3.4.** Let $\mu \in \mathcal{P}(Y)$ be a measure which is $\alpha_t$-invariant and $U$ (or $W$)-invariant. Then $\mu$ cannot be supported on $L_\epsilon$ for any $\epsilon > 0$.

**Proof.** Assume that $\text{Supp} \mu \subseteq L_\epsilon$ for some $\epsilon > 0$, then $\mu(Y \setminus L_\epsilon) = 0$ holds. First we claim that for any $y \in Y$, there exist $t \in \mathbb{R}$ and $u \in U$ satisfying $a_tuy \in Y \setminus L_\epsilon$. Let $y = \begin{pmatrix} X_1 & X_2 & b_1 \\ X_3 & X_4 & b_2 \\ 0 & 0 & 1 \end{pmatrix} G(\mathbb{Z})$, then for $u = u(A)$ and $\alpha_t$, we have

$$a_tuy = \begin{pmatrix} e^{nt} & 0 & 0 \\ 0 & e^{-mt} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & A & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & b_1 \\ X_3 & X_4 & b_2 \\ 0 & 0 & 1 \end{pmatrix} G(\mathbb{Z}).$$

We consider two cases to prove the claim:

**Case 1** $b_2 \neq 0$.
In this case, we can find a matrix $A \in M_{m,n}(\mathbb{R})$ satisfying $b_1 + Ab_2 = 0$. Then the translation vector of the grid from the origin, which is the first $m + n$ entries of the last column, is $(0_m, e^{-mt}b_2)$. Thus the vector $(0_m, e^{-mt}b_2)$ is contained in the grid $\Gamma_{a_tuy}$. By taking $t \to \infty$, we can choose $t >> 0$ with $||e^{-mt}b_2|| < \epsilon m+n$, which gives $\Gamma_{a_tuy} \cap B_{\epsilon m+n}^d(0) \neq \phi$.

**Case 2** $b_2 = 0$.
In this case, the last column is $(e^{nt}b_1, 0_n)$. We can make this translation vector small enough by taking $t \to -\infty$. Thus we get $\Gamma_{a_tuy} \cap B_{\epsilon m+n}^d(0) \neq \phi$. 


as in the Case 1.

Combining these two cases, we arrived at the claim.

Similarly, for any \( y \in Y \), there exist \( t \in \mathbb{R} \) and \( w \in W \) satisfying \( a_tw + cy \in Y \setminus \mathcal{L}_\epsilon \).

Indeed, for \( w \in W \), let \( w = \begin{pmatrix} I_m & 0 & c \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \), then

\[
a_tw = \begin{pmatrix} e^{mt}X_1 & e^{mt}X_2 & e^{mt}(b_1 + c) \\ e^{-mt}X_4 & e^{-mt}b_2 \\ 0 & 0 & 1 \end{pmatrix} G(\mathbb{Z})
\]

holds and we can apply previous argument by taking \( c = -b_1 \) and \( t \to \infty \).

Since \( \mathcal{L}_\epsilon \) is closed, for every \( y \in \mathbb{R} \), there exist \( r_y > 0 \), \( t \in \mathbb{R} \), and \( u \in U \) such that the \( d \)-ball \( B_{r_y}(a_tw + cy) \subseteq Y \setminus \mathcal{L}_\epsilon \). Choose \( r_y' > 0 \) such that \( B_{r_y'}(y) \subseteq (a_tu)^{-1}B_{r_y}(a_tw + cy) \). Then for \( a_t \)-invariant and \( U \)-invariant measure \( \mu \),

\[
\mu(B_{r_y'}(y)) \leq \mu((a_tu)^{-1}B_{r_y}(a_tw + cy)) = \mu(B_{r_y}(a_tw + cy)) \leq \mu(Y \setminus \mathcal{L}_\epsilon) = 0.
\]

Covering \( Y \) by balls \( B_{r_y'}(y) \), we obtain

\[
\mu(Y) = \mu\left( \bigcup_{y \in Y} B_{r_y'}(y) \right) \leq 0,
\]

which gives a contradiction. We also prove the \( W \)-invariant case by replacing \( u \in U \) with \( w \in W \).

\[ \square \]

Proof of Theorem 1.1. Suppose that for some fixed \( \epsilon > 0 \), there exist a sequence \( \{b_j\} \) such that \( \dim_H \text{Bad}^{b_j}(\epsilon) > \dim_H \text{Bad}^{b}(\epsilon) \) and

\[
\dim_H \text{Bad}^{b_j}(\epsilon) \geq mn - \frac{1}{j}
\]

so that \( \lim_{j \to \infty} \dim_H \text{Bad}^{b_j}(\epsilon) = mn \). Take a sequence of real numbers \( \{\gamma_j\} \) which converge to zero. Now apply Proposition 2.4 for \( b_j \in \mathbb{R}^m \) and \( \gamma_j > 0 \). Note that the corresponding \( \eta_j = \frac{mn}{m+n}(mn - \dim_H \text{Bad}^{b_j}(\epsilon) + \gamma_j) \) goes to zero as \( j \to \infty \). Then we have a sequence of probability measures \( \{\mu^{b_j}\} \) in \( \mathcal{P}(\bar{Y}) \) satisfying the four conditions in Proposition 2.4. Now, there exists a probability measure \( \mu \in \mathcal{P}(\bar{Y}) \) which is a weak*-accumulation point of \( \{\mu^{b_j}\} \). Let

\[
\mu^{b_j} \overset{w^*}{\longrightarrow} \mu \in \mathcal{P}(\bar{Y})
\]

by extracting a subsequence if necessary. Clearly \( \mu \) is \( a \)-invariant and \( \text{Supp} \mu \subseteq \mathcal{L}_\epsilon \). Since \( \mu(Y \setminus \bar{Y}) = \lim_{j \to \infty} \mu^{b_j}(Y \setminus \bar{Y}) = \lim_{j \to \infty} \eta_j = 0 \), we may consider \( \mu \) as a probability measure on \( Y \).

Let us show that \( \mu \) has maximal entropy. For any compact set \( K \subset Y \), we can build a finite partition \( \mathcal{P} \) satisfying:

1. \( \mathcal{P} \) contains an atom \( P_{\epsilon} \) where \( K \subseteq Y \setminus P_{\epsilon} \),
2. \( \forall P \in \mathcal{P} \setminus \{P_{\epsilon}\}, \text{diam } P < r_0 \) for some \( 0 < r_0 < \frac{1}{7} \),
3) \( \forall P \in \mathcal{P} \) and \( j \in \mathbb{N} \), \( \mu^{b_j}(\partial P) = 0 \).

It is possible to build this partition to satisfy the part (3) because the collection of measure \( \{\mu^{b_j}\} \) is countable. Thus we have

\[
\frac{1}{q} H_{\mu^{b_j}}(\mathcal{P}^{(q)}|\mathcal{B}_Y^{(q)}) \geq (1-\mu^{b_j}(Y\setminus K)^{\frac{1}{2}})(m+n)(\dim_{\mathcal{H}} \text{Bad}_{b_j}(\epsilon) - \gamma_j - mn\mu^{b_j}(Y\setminus K)^{\frac{1}{2}})
\]

for all \( q, j \in \mathbb{N} \). Since \( \mathcal{P}^{(q)} \) is finite, we can take \( j \to \infty \) to obtain

\[
\frac{1}{q} H_{\mu}(\mathcal{P}^{(q)}|\mathcal{B}_Y^{(q)}) \geq (1-\mu(Y\setminus K)^{\frac{1}{2}})^2(m+n)mn
\]

for all \( q \in \mathbb{N} \). Thus \( h_{\mu}(a|\mathcal{B}_Y^{(q)}) \geq (1-\mu(Y\setminus K)^{\frac{1}{2}})^2(m+n)mn \) holds for any compact set \( K \subset Y \), and eventually we have \( h_{\mu}(a|\mathcal{B}_Y^{(q)}) = (m+n)mn \) which is the maximal entropy with respect to \( \mathcal{B}_Y^{(q)} \). It follows that \( \mu \) is \( U \)-invariant by Proposition 3.1. We arrived at the desired contradiction by using proposition 3.4 since we showed that \( \mu \) is a probability measure on \( Y \) which is \( a_t \)-invariant, \( U \)-invariant, and supported on \( \mathcal{L}_c \). \( \square \)

### 3.3. Proof of Theorem 1.3 and 1.4.

Before we start the proof of Theorem 1.3 and 1.4, we construct a measure on \( Y \) with large relative entropy as we did in Proposition 2.4. However, in this case, we will calculate entropy relative to the factor \( X \), instead of the \( \sigma \)-algebra \( \mathcal{B}_Y^{(q)} \) we used in previous chapters. For any countable partition \( \mathcal{P} \) of \( Y \), \( H_{\mu}(\mathcal{P}|X) \) will denote the relative entropy of \( \mathcal{P} \) with respect to the \( \sigma \)-algebra \( \pi^{-1}(\mathcal{B}_X) \) where \( \mathcal{B}_X \) is the Borel \( \sigma \)-algebra on \( X \). Similarly, for \( \mu \in \mathcal{P}(\hat{Y}) \), \( H_{\mu}(\mathcal{P}|\hat{X}) \) will denote the relative entropy with respect to the \( \pi^{-1}(\mathcal{B}_X) \) where \( \mathcal{B}_\hat{X} \) is the Borel \( \sigma \)-algebra on \( \hat{X} \). Note that \( H_{\mu}(\mathcal{P}^{(q)}|X) = H_{\mu}(\mathcal{P}^{(q)}|\hat{X}) \) for \( q \in \mathbb{N} \) and \( \mu \in \mathcal{P}(\hat{Y}) \). Also denote by \( h_{\mu}(a|X) \) and \( h_{\mu}(a|\hat{X}) \) the relative entropy of the transformation \( a \) with respect to \( X \) and \( \hat{X} \), respectively. The following proposition is analogous to Proposition 2.4 as well as its proof except a slight difference on the construction of measure.

**Proposition 3.5.** Let \( \eta_0 = \sup \{ \eta : x_A \text{ has } \eta \text{-escape of mass on average} \} \) for \( A \in M_{m,n}(\mathbb{R}) \). Then for any \( \epsilon > 0 \) and \( \gamma > 0 \), there exist an \( a_t \)-invariant probability measure \( \mu^\gamma \in \mathcal{P}(\hat{Y}) \) satisfying:

1. \( \text{Supp } \mu^\gamma \subseteq \mathcal{L}_c \),
2. \( \mu^\gamma(Y\setminus \hat{Y}) = \eta_0 \),
3. If \( \mathcal{P} \) is any finite partition of \( Y \) satisfying:
   - \( \mathcal{P} \) contains an atom \( P_\infty \) of the form \( \pi^{-1}(P_\infty^0) \), where \( X \setminus P_\infty^0 \) has compact closure,
   - \( \forall P \in \mathcal{P} \setminus \{P_\infty\} \), \( \text{diam } P < r_0 \) for some \( 0 < r_0 < \frac{1}{2} \),
   - \( \forall P \in \mathcal{P}, \mu^\gamma(\partial P) = 0 \),
then, for all \( q \geq 1 \),

\[
\frac{1}{q} H_{\mu^\gamma}(\mathcal{P}^{(q)}|\hat{X}) \geq n(\dim_{\mathcal{H}} \text{Bad}_A(\epsilon) - \gamma) - mn\mu^\gamma(P_\infty).
\]
Proof. Let $R^{A,T} := \{ b \in \mathbb{R}^m \mid \forall t \geq T, a_t x_{A,b} \in \mathcal{L}_e \} \cap \text{Bad}_A(\epsilon)$. By Proposition 2.1, \( \bigcup_{T=1}^{\infty} R^{A,T} \) has Hausdorff dimension equal to \( \dim_H \text{Bad}_A(\epsilon) \), thus there exists \( T_\gamma > 0 \) satisfying \( \dim_H R^{A,T_\gamma} \geq \dim_H \text{Bad}_A(\epsilon) - \gamma \). Since $A$ has $\eta$-escape of mass on average for $\eta < \eta_0$, we may fix an increasing sequence of integers \( \{k_i\} \) such that

\[
\frac{1}{k_i} \sum_{k=0}^{k_i-1} \delta_{a^k x_A} \overset{w^*}{\to} \mu_A \in \mathcal{P}(\tilde{X})
\]

with \( \mu_A(X) = 1 - \eta_0 \). Let \( \phi_A : \mathbb{R}^m \to Y \) be the function defined by \( \phi_A(b) = x_{A,b} \). For each \( k_i \geq T_\gamma \), let \( S_i \) be a maximal \( e^{-k_i n} \)-separated subset of \( \phi_A(R^{A,T_i}) \) with respect to the metric \( d \). Then

\[
\liminf_{i \to \infty} \frac{\log |S_i|}{k_i} \geq n \dim_H(R^{A,T_i}) \geq n(\dim_H \text{Bad}_A(\epsilon) - \gamma)
\]

holds from the bi-Lipschitz property between \( d \) and \( || \cdot ||_{\mathbb{R}^m} \).

Let \( \nu_i \overset{\text{def}}{=} \frac{1}{|S_i|} \sum_{y \in S_i} \delta_y \) be the normalized counting measure on \( S_i \) and let

\[
\mu_i \overset{\text{def}}{=} \frac{1}{k_i} \sum_{k=0}^{k_i-1} a^k \nu_i \underset{w^*}{\to} \mu^\gamma \in \mathcal{P}(\tilde{Y}).
\]

we prove that \( \mu^\gamma \) is the desired measure.

(1) For any \( y \in S_i \subseteq \phi_A(R^{A,T_i}) \), \( a^T y \in \mathcal{L}_e \) holds for \( T > T_\gamma \). Thus

\[
\mu_i(Y \setminus \mathcal{L}_e) = \frac{1}{k_i} \sum_{k=0}^{k_i-1} a^k \nu_i(Y \setminus \mathcal{L}_e) = \frac{1}{k_i} \sum_{k=0}^{T_\gamma} a^k \nu_i(Y \setminus \mathcal{L}_e) \leq \frac{T_\gamma}{k_i}
\]

and we obtain item (1) by taking limit for \( k_i \to \infty \).

(2) Since \( \pi_* \nu_i = \delta_{x_A} \) for all \( i \geq 1 \), \( \pi_* \mu^\gamma = \mu_A \) holds. Thus

\[
\mu^\gamma(\tilde{Y} \setminus Y) = \mu_A(\tilde{X} \setminus X) = \eta_0.
\]

(3) Let \( \rho > 0 \) be a small positive real number, then

\[
\mu^\gamma(P_\infty) + \rho > \mu_i(P_\infty) = \frac{1}{k_i |S_i|} \sum_{y \in S_i, 0 \leq k < k_i} \delta_{a^k y}(P_\infty) = \frac{1}{k_i} \sum_{0 \leq k < k_i} \delta_{a^k x_A}(P_\infty)
\]

holds for large enough \( i \). For simplicity, let \( \beta \overset{\text{def}}{=} \mu^\gamma(P_\infty) + \rho \). The above inequality means that there exist at most \( \beta k_i \) number of \( a^k x_A \)'s in \( P_\infty^0 \), thus there exists some \( k \in [(1 - \beta)k_i, k_i) \) such that \( a^k y \in Y \setminus P_\infty \) for all \( y \in S_i \). If
$P$ is any non-empty atom of $\mathcal{P}(k_i)$, fixing any $y_0 \in P$, any $y \in S_i \cap P = S_i \cap [y_0]_{\mathcal{P}(k_i)}$ satisfies

$$r_0 > d(a^ky_0, a^ky) \geq C''e^{nk}d(y_0, y) \geq C''e^{n(1-\beta)k}d(y_0, y)$$

for some constant $C'' > 0$. Here, we used the right invariant property of $d$ and bi-Lipschitz property between $d$ and $|| \cdot ||$. Thus $S_i \cap P$ can be covered by one ball of $C''e^{-n(1-\beta)k_i}$-radius for metric $d$ as well as by $C_2e^{mn\beta k_i}$ many balls of $r_0e^{-k_i}$-radius for the metric $d$ and some constant $C_2 > 0$. Since $S_i$ is $e^{-k_i}$-separated with respect to $d$, we get

$$\text{Card}(S_i \cap [y_0]_{\mathcal{P}(k_i)}) \leq C_2e^{mn\beta k_i},$$

and therefore

$$H_{\nu_i}(\mathcal{P}(k_i)) \geq \log |S_i| - mn\beta k_i - \log C_2.$$ 

Now we can estimate the lower bound of the entropy. For $q \geq 1$, write the Euclidean division of large enough $k_i - 1$ by $q$ as

$$k_i - 1 = qk' + s \text{ with } s \in \{0, \ldots, q - 1\}.$$ 

By subadditivity of the entropy with respect to the partition, for each $p \in \{0, \ldots, q - 1\}$,

$$H_{\nu_i}(\mathcal{P}(k_i)|X) \leq H_{a^p\nu_i}(\mathcal{P}(q)|X) + \cdots + H_{a^{p+qk'}\nu_i}(\mathcal{P}(q)|X) + 2q \log |\mathcal{P}|.$$ 

Summing those inequalities for $p = 0, \ldots, q - 1$, and using the concave property of entropy with respect to the measure, we obtain

$$qH_{\nu_i}(\mathcal{P}(k_i)|X) \leq \sum_{k=0}^{k_i-1} H_{a^k\nu_i}(\mathcal{P}(q)|X) + 2q^2 \log |\mathcal{P}|$$

and therefore

$$\frac{1}{q}H_{\mu_i}(\mathcal{P}(q)|X) \geq \frac{1}{k_i}H_{\nu_i}(\mathcal{P}(k_i)|X) - \frac{2q \log |\mathcal{P}|}{k_i}$$

$$\geq \frac{1}{k_i}\left\{(\log |S_i| - mn\beta k_i - \log C_2) - 2q \log |\mathcal{P}|\right\}.$$ 

Here, for the second inequality, we used $H_{\nu_i}(\mathcal{P}(q)|X) = H_{\nu_i}(\mathcal{P}(q))$ from the fact that $\nu_i$ is supported on an atom of $\pi^{-1}(\mathcal{B}_X)$. Now we can take $i \to \infty$ because the atoms $P$ of $\overline{P}$ and hence of $\overline{\mathcal{P}}(q)$, satisfy $\mu^\gamma(\partial P) = 0$. Thus we obtain the inequality

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{P}}(q)|X) \geq n(\dim_H \text{Bad}_A(\epsilon) - \gamma) - mn\beta,$$

and finally get the inequality

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{P}}(q)|X) \geq n(\dim_H \text{Bad}_A(\epsilon) - \gamma) - mn\mu^\gamma(\overline{P}_x)$$

we desired by taking $\rho \to 0$. $$\square$$
Now we prove Theorem 1.3 and Theorem 1.4.

**Proof of Theorem 1.3.** We use proof by contradiction again. Suppose that for any $\delta > 0$, the set of $A$ satisfying $\dim_H \text{Bad}_A(\epsilon) \geq m - \delta$ has full Hausdorff dimension. The set of $A$ satisfying $\dim_H \text{Bad}_A(\epsilon) \geq m - \delta$ has full Hausdorff dimension and for any $\eta > 0$, the Hausdorff dimension of the set of $A$ with $\eta$-escape of mass on average is strictly less than full dimension by Theorem 2.2. Thus there exists $A \in M_{m,n}(\mathbb{R})$ which does not have $\eta$-escape of mass on average and satisfies $\dim_H \text{Bad}_A(\epsilon) \geq m - \delta$ for any $\eta > 0$.

For $k \in \mathbb{N}$, fix $\delta = \eta = \frac{1}{k}$ and take a sequence of real numbers $\{\gamma_j\}$ which converges to zero. Then there exists a sequence of $a_t$-invariant probability measures $\{\mu_{\gamma_j} \in \mathcal{P}(Y)\}$ described in Proposition 3.5. By taking weak*-limit of this sequence, we can construct the measure $\mu_k \in \mathcal{P}(Y)$ such that $\text{Supp}\, \mu_k \subset \mathcal{L}_e$, $\mu_k(\bar{Y}\setminus Y) = \eta_0 \leq \eta$, $\frac{1}{q}H_{\mu_k}(\mathcal{P}(q)|\bar{X}) \geq mn - \frac{n}{k} - mn\mu^*(\mathcal{P}_x)$

for all $q \geq 1$, and $\mathcal{P}$ satisfying the condition (3) in Proposition 3.5. By taking weak*-limit of sequence $\{\mu_k\}$ again, eventually we obtain the measure $\mu \in \mathcal{P}(Y)$ supported on $\mathcal{L}_e$ such that $\frac{1}{q}H_{\mu}(\mathcal{P}(q)|X) \geq (1 - \mu(P_x))mn$. By taking an appropriate partition $\mathcal{P}$ to make $\mu(P_x)$ arbitrarily small, $h_{\mu}(a|X) = mn$, which is the maximal relative entropy, thus $\mu$ is a $W$-invariant measure by Proposition 3.1 of [LSS]. We obtain the desired contradiction by Proposition 3.4 since $\mu$ is a probability measure on $Y$ which is $a_t$-invariant, $W$-invariant, and supported on $\mathcal{L}_e$. This proves Theorem 1.3. \qed

**Proof of Theorem 1.4.** We use proof by contradiction. Suppose that $A \in M_{m,n}(\mathbb{R})$ is not singular on average and satisfies $\dim_H \text{Bad}_A(\epsilon) = m$. Let $\eta_0 = \sup \{\eta : x_A \text{ has } \eta \text{-escape of mass}\}$, then $\eta_0 < 1$ since $x_A$ is not singular on average. Take a sequence of real numbers $\{\gamma_j\}$ which converges to zero. Then there exists a sequence of $a_t$-invariant probability measures $\{\mu_{\gamma_j} \in \mathcal{P}(Y)\}$ described in Proposition 3.5. By taking weak*-limit of this sequence, we can construct the measure $\mu \in \mathcal{P}(Y)$ such that $\text{Supp}\, \mu \subset \mathcal{L}_e \cup (\bar{Y}\setminus Y)$, $\frac{1}{q}H_{\mu}(\mathcal{P}(q)|\bar{X}) \geq mn(1 - \mu(P_x))$

for all $q \geq 1$, and $\mathcal{P}$ satisfying the condition (3) in Proposition 3.5. By taking the compact set $X \subset P_X^0$ large enough, we obtain $h_{\mu}(a|X) \geq mn(1 - \eta_0)$. On the other hand, the measure $\mu \in \mathcal{P}(Y)$ can be represented by the linear combination $\mu = (1 - \eta_0)\mu_0 + \eta_0\delta_x$, where $\delta_x$ is the dirac delta measure on $\bar{Y}\setminus Y$ and $\mu_0 \in \mathcal{P}(Y)$. Since $s = \mu(\bar{Y}\setminus Y) \leq \eta_0$ and the entropy is a linear function with respect to the measure, $h_{\mu_0}(a|X) = \frac{1}{1 - \eta_0}h_{\mu}(a|X) = mn$, which is the maximal relative entropy, thus $\mu_0$ is a $W$-invariant measure by Proposition 3.1 of [LSS]. We obtain the desired contradiction by Proposition
3.4 since $\mu_0$ is a probability measure on $Y$ which is $a_r$-invariant, $W$-invariant, and supported on $L_r$. □

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