Deformed multi-variable Fokker-Planck equations

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In this paper new multi-variable deformed Fokker-Planck (FP) equations are presented. These deformed FP equations are associated with the Ruijsenaars-Schneider-van Diejen (RSvD) type systems in the same way that the usual one variable FP equation is associated with the one particle Schrödinger equation. As the RSvD systems are the “discrete” counterparts of the celebrated exactly solvable many-body Calogero-Sutherland-Moser systems, the deformed FP equations presented here can be considered as “discrete” deformations of the ordinary multi-variable FP equations.

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I. INTRODUCTION

The Fokker-Planck (FP) equation is one of the most important tools to deal with fluctuation phenomena in various kinds of systems [1]. Recent discovery of anomalous diffusion in fractal and disordered media have prompted new developments in the FP theory, the major one being fractional FP equations, where the ordinary spatial derivatives are replaced by fractional derivatives [2]. Nonlinear FP equations [3] and diffusion equation based on \(q\)-derivatives have also been considered [4].

In [5], based on the well-known relation between FP equations and Schrödinger equations of quantum mechanics, we have proposed new types of deformed FP equations which are associated with the Schrödinger equations of the “discrete” quantum mechanics considered in [6, 7, 8]. The latter is a natural discretization of quantum mechanics and its Schrödinger equations are difference instead of differential equations. The eigenfunctions of some exactly solvable “discrete” quantum mechanics include various deformations of the classical orthogonal polynomials (the Hermite, Laguerre and Jacobi polynomials), namely, those belonging to the family of Askey-scheme of hypergeometric orthogonal polynomials, and the Askey-Wilson polynomial in \(q\)-analysis [9, 10, 11]. In fact, these deformed orthogonal polynomials arise in the problem [7] of describing the equilibrium positions of Ruijsenaars-Schneider-van Diejen (RSvD) type systems [12], which are the “discrete” counterparts of the celebrated exactly solvable multi-particle Calogero-Sutherland-Moser (CSM) systems [13], just as the Hermite, Laguerre and Jacobi polynomials describe the equilibrium positions of the CSM systems [14]. In this sense, the proposed deformed FP equations [5] can be considered as the “discrete” deformation of the usual FP equation.

In this paper, we would like to generalize the results in [5] for the one-particle cases to the many-body cases, and obtain the many-variable deformed FP equations associated with the RSvD systems. We first briefly review in Sect. II the connection between the ordinary multi-variable FP equation and the many-body Schrödinger equation. Sect. III summarizes the basics of the CSM systems associated with rational/trigonometric classical root systems. In Sect. IV we review the RSvD systems and derive the corresponding FP equations for the two rational types. The trigonometric type RSvD systems and the corresponding FP equations are discussed in Sect. V. Sect. VI concludes the paper.

II. CONNECTION BETWEEN FOKKER-PLANCK AND SCHRÖDINGER EQUATIONS

The general form of FP equation of the probability density \(P(x, t)\) with \(n\) variables \(x \equiv (x_1, x_2, \ldots, x_n)\) is [4]

\[
\frac{\partial P(x, t)}{\partial t} = L_{FP} P(x, t),
\]

\[
L_{FP} \equiv - \sum_j \frac{\partial}{\partial x_j} D_j(x) + \sum_{jk} \frac{\partial^2}{\partial x_j \partial x_k} D_{jk}(x).
\]

(1)

The functions \(D_j(x)\) and \(D_{jk}(x)\) in the FP operator \(L_{FP}\) are, respectively, the drift vector and the diffusion matrix (we consider only time-independent case). The drift vector represents the external forces acting on the particles, while the diffusion matrix accounts for the effects of fluctuation.

It is well known that in one-dimension, the FP equation is closely related to the Schrödinger equation. The correspondence between the two equations can be shown by transforming the FP equation into the corresponding
Schrödinger equation \( \lambda \), or vice versa \( \tilde{\lambda} \), based on a similarity transformation. The similarity transformation is related to the ground state of the Schrödinger equation, or equivalently, to the stationary state of the FP equation. One thus expects that similar connection is possible between a FP equation with multiple variables and a one-dimensional many-body Schrödinger equation. It turns out that such connection is possible only for special forms of \( D_j \) and \( D_{jk} \). Below we shall briefly review such connection.

Instead of transforming an assumed form of FP equation to a Schrödinger equation, as is usually done in the literature (see e.g. \( \tilde{\lambda} \), we think it is more direct to see what form of FP equation will emerge by directly transforming a known many-body Schrödinger equation. So let us consider a quantum mechanical Hamiltonian of \( n \) particles with equal masses (for simplicity, we adopt the unit system in which \( \hbar = 1 \), the speed of light \( c \) and normalized) eigenfunction \( \phi \) and normalized) eigenfunction \( \lambda \) can always be made zero, i.e. \( H\phi_0 = 0 \). Under these assumptions, the Hamiltonian \( H \) is completely determined by its ground state wave function \( \phi_0(x) \). By the well-known theorem of quantum mechanics, \( \phi_0(x) \) has no node and can be chosen real. Hence one can parametrize \( \phi_0 \) by a real prepotential \( W(x) \):

\[
\phi_0(x) = e^{-W(x)}. \quad \text{(3)}
\]

Then the condition \( H\phi_0 = 0 \) enables one to express \( V(x) \) in terms of \( W(x) \) as well:

\[
V(x) = \sum_{j=1}^{n} \left( \left( \frac{\partial W(x)}{\partial x_j} \right)^2 - \frac{\partial^2 W(x)}{\partial x_j^2} \right). \quad \text{(4)}
\]

Since only derivatives of \( W(x) \) appear in \( V(x) \), \( W(x) \) is defined only up to an additive constant \( W_0 \). We choose the constant \( W_0 \) in such a way as to normalize \( \phi_0(x) \) properly, \( \int \phi_0(x)^2 \, dx = 1 \). For simplicity of presentation, we consider the cases in which the ground state wave functions are square integrable, that is, the corresponding FP operators have the normalizable stationary distributions.

The Hamiltonian \( H \) can be expressed as a sum of factorized forms, \( H = \sum_j A_j^\dagger A_j \), where

\[
A_j = \frac{\partial}{\partial x_j} + \left( \frac{\partial W(x)}{\partial x_j} \right), \quad \text{(5)}
\]

\[
A_j^\dagger = -\frac{\partial}{\partial x_j} + \frac{\partial W(x)}{\partial x_j}.
\]

Since each term \( A_j^\dagger A_j \) in \( H \) is positive semi-definite, the condition \( A_j \phi_0 = 0 \) must hold in order to have \( H\phi_0 = 0 \). This latter condition is a sufficient and necessary condition for \( H\phi_0 = 0 \).

Now we define an operator from the \( H \) and \( \phi_0 \) by the similarity transformation

\[
L_{FP} \equiv -\phi_0 H\phi_0^{-1}, \quad \text{(6)}
\]

which guarantees the non-positivity of the eigenvalues of \( L_{FP} \). Using \( \phi_0 \) in \( \lambda \), one obtains

\[
L_{FP} = \sum_j \frac{\partial}{\partial x_j} \left( \frac{\partial (2W(x))}{\partial x_j} \right) + \sum_j \frac{\partial^2}{\partial x_j^2}. \quad \text{(7)}
\]

From \( \lambda \), it is seen that \( L_{FP} \) is just the corresponding FP operator with drift vector \( D_j = -2\partial W/\partial x_j \) and a diffusion matrix proportional to the unit matrix: \( D_{jk} = \delta_{jk} \) (the constant factor here equals unity which is a result of the choice of unit in \( H \)). This establishes the required forms of \( D_j \) and \( D_{jk} \) so that the multi-variable FP equation is related to a one-dimensional many-body Schrödinger equation. We see that \( D_j \) is in fact defined, as in the one variable FP equation, by a drift potential \( \Phi(x) = 2W(x) \): \( D_j = -\partial \Phi/\partial x_j \).

Both \( H \) and \( L_{FP} \) are determined by \( \phi_0 \). The relationship between their eigenfunctions is the same as that for the single particle case, except that special care is needed to ensure the reality of the eigenfunctions belonging to degenerate eigenvalues. The eigenfunction \( P_{m}(x) \) of \( L_{FP} \) corresponding to the eigenvalue \( -\lambda_m \) is related to the (real and normalized) eigenfunction \( \phi_m \) of \( H \) corresponding to \( \lambda_m \) by \( P_{m}(x) = \phi_0(x)\phi_m(x) \). Here \( m \) is, in general, a
certain multi-index. The stationary distribution is $P_0 = \phi_0^c = \exp(-2W)$, which is obviously non-negative, and is the zero mode of $\text{LFP}$: $\text{LFP}P_0 = 0$. Any positive definite initial probability density $P(x, 0)$ can be expanded as $P(x, 0) = \phi_0(x) \sum_m c_m \phi_m(x)$, with constant coefficients $c_m$

$$c_m = \int_{-\infty}^{\infty} \phi_m(x) \left( \phi_0^{-1}(x)P(x, 0) \right) \, dx.$$  

Then at any later time $t$, the solution of the FP equation is $P(x, t) = \phi_0(x) \sum_m c_m \phi_m(x) \exp(-\lambda_m t)$.

### III. CALOGERO-SUTHERLAND-MOSER SYSTEMS

One class of multi-variable FP equations having the special form discussed in the previous section which has been widely discussed in the literature is that related to the celebrated exactly solvable many-body quantum system, namely, the CSM models. Such FP equation has found applications in, e.g. random matrix theory [15] and others [16].

Here we consider four types of CSM systems associated with the classical root systems: two rational and two trigonometric types. As mentioned earlier, we consider only the cases with explicitly known ground state wave functions as well as the rational cases without the harmonic confining potential. The potential $V_{\text{CS}}(x)$ of the CSM systems can be written in terms of the prepotential $W(x)$ as in (4). The explicit forms of the four types of potentials $V_{\text{CS}}(x)$ and the corresponding prepotentials $W(x)$ are as follows:

(i) rational $A_{n-1}$ :

$$V_{\text{CS}}(x) = \sum_{j=1}^{n} x_j^2 + \sum_{j,k=1 \atop j \neq k}^{n} \frac{g(g-1)}{(x_j - x_k)^2} - n(1 + g(n-1)), \quad (9a)$$

$$W(x) = \frac{1}{2} \sum_{j=1}^{n} x_j^2 - g \sum_{1 \leq j < k \leq n} \log|x_j - x_k| + W_0, \quad (9b)$$

(ii) rational $BC_n$ :

$$V_{\text{CS}}(x) = \sum_{j=1}^{n} \left( x_j^2 + \frac{g_s(g_s - 1)}{x_j^2} \right) + g_M(g_M - 1) \sum_{j,k=1 \atop j \neq k}^{n} \left( \frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2} \right) - 2n \left( g_s + \frac{1}{2} + g_M(n-1) \right), \quad (10a)$$

$$W(x) = \frac{1}{2} \sum_{j=1}^{n} x_j^2 - g_M \sum_{1 \leq j < k \leq n} \left( \log|x_j - x_k| + \log|x_j + x_k| \right) - g_s \sum_{j=1}^{n} \log|x_j| + W_0, \quad (10b)$$

(iii) trigonometric $A_{n-1}$ :

$$V_{\text{CS}}(x) = \frac{\pi^2}{L^2} g(g-1) \sum_{j,k=1 \atop j \neq k}^{n} \frac{1}{\sin^2 \frac{\pi}{L}(x_j - x_k)} - \frac{\pi^2 g^2 n(n^2 - 1)}{3 L^2}, \quad (11a)$$

$$W(x) = -g \sum_{1 \leq j < k \leq n} \log|\sin \frac{\pi}{L}(x_j - x_k)| + W_0, \quad (11b)$$
(iv) trigonometric $BC_n$:

$$V_{CS}(x) = \frac{\pi^2}{L^2} \sum_{j=1}^{n} \left( \frac{(g_L + g_S)(g_L + g_S - 1)}{\sin^2 \frac{\pi}{L} x_j} + \frac{g_L(g_L - 1)}{\cos^2 \frac{\pi}{L} x_j} \right)$$

$$+ \frac{\pi^2 n}{L^2} \sum_{j,k=1}^{n} \left( \frac{g_M(g_M - 1)}{\sin^2 \frac{\pi}{L} (x_j - x_k)} + \frac{g_M(g_M - 1)}{\sin^2 \frac{\pi}{L} (x_j + x_k)} \right)$$

$$- \frac{\pi^2 n^2}{L^2} \left( (g_S + 2g_L + g_M(n - 1))^2 + \frac{g_M^2}{3} (n^2 - 1) \right),$$

$$W(x) = - \sum_{1 \leq j < k \leq n} g_M \left( \log |\sin \frac{\pi}{L} (x_j - x_k)| + \log |\sin \frac{\pi}{L} (x_j + x_k)| \right)$$

$$- \sum_{j=1}^{n} \left( g_S \log |\sin \frac{\pi}{L} x_j| + g_L \log |\sin \frac{\pi}{L} 2x_j \right) + W_0.$$  \hspace{1cm} (12a)

Here $W_0$ is the constant term necessary for the normalization of the ground state wave function $\phi_0(x)$. The constant terms in $V_{CS}(x)$ are the consequences of the expression (11) in terms of the prepotential. In the above formulas $g, g_S, g_M$, and $g_L$ are non-negative coupling constants, and $L$ is the circumference of the circle in which $x_j$ live for the trigonometric cases. In the rational potential cases, the coefficients of the harmonic confining potential $x_j^2$ are all equivalent. The trigonometric potentials in (12a) and (12b), the coefficient of the harmonic confining potential $x_j^2$ is usually written as $\frac{1}{2} m \omega^2$. Here we choose $2m = m \omega = 1$ for simplicity. The rational $D_n$ model is obtained from the rational $BC_n$ model by putting $g_S = 0$ in (10a) and (10b). The rational $B_n$, $C_n$ and $BC_n$ models are obtained from the trigonometric $BC_n$ model by putting $g_S = g_L = 0$ in (12a) and (12b), whereas the trigonometric $B_n$ and $C_n$ models are obtained by $g_L = 0$ and $g_S = 0$, respectively.

If we apply the transformation (6) to the Hamiltonian of the rational $A_{n-1}$ CSM system, the resulted FP equation is

$$L_{FP} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + 2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( x_j - g \sum_{k=1}^{n} \frac{1}{x_j - x_k} \right). \hspace{1cm} (13)$$

Multi-variable FP equation of this type was first employed by Dyson as a Brownian motion model of certain random matrix ensemble [12]. FP equations corresponding to the other three types of CSM systems can be obtained accordingly.

**IV. RUIJSENAARS-SCHNEIDER-VAN DIEJEN SYSTEMS**

Now we would like to derive new types of multi-variable FP equations corresponding to certain discrete deformations of the above CSM systems; namely, the Ruijsenaars-Schneider-van Diejen (RSvD) systems with the rational/trigonometric potentials.

The RSvD system is an integrable deformation of the CSM system. The Hamiltonian has the general form

$$H = \sum_{j=1}^{n} \sqrt{V_j(x)} e^{-i\partial_j} \sqrt{V_j^*(x)} + \sqrt{V_j^*(x)} e^{i\partial_j} \sqrt{V_j(x)} - V_j(x) - V_j^*(x). \hspace{1cm} (14)$$

Here $\partial_j = \partial/\partial x_j$, and $V_j(x)$ is given by

$$V_j(x) = w(x_j) \prod_{k=1 \atop k \neq j}^{n} v(x_j - x_k) \times \begin{cases} 1 & \text{for } A_{n-1}, \\ v(x_j + x_k) & \text{for } BC_n. \end{cases} \hspace{1cm} (15)$$

We use the conventional notation that $V_j^*(x)$ is the complex conjugate function of $V_j(x)$. (For an arbitrary function $f(x) = \sum_{n} a_n x^n$, $a_n \in \mathbb{C}$ we define $f^*(x) = \sum_{n} a_n^* x^n$. Here $c^*$ is the complex conjugation of a number $c \in \mathbb{C}$.) Since the operators $e^{\pm i\partial_j}$ cause finite shifts of the wave function in the imaginary direction ($e^{\pm i\partial_j} f(x) = f(x_1, \cdots, x_j \pm i, \cdots, x_n)$), these systems can thus be called “discrete” dynamical systems.

The basic potential functions $v(x)$ and $w(x)$ are as follows:
(i) rational $A_{n-1}$:
\[
v(x) = 1 - i \frac{q}{x}, \\
w(x) = \left( a_1 + ix \right) \left( a_2 + ix \right),
\]

(ii) rational $BC_n$:
\[
v(x) = 1 - i \frac{q}{x}, \\
w(x) = \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)(a_4 + ix)}{2ix(2ix + 1)},
\]

(iii) trigonometric $A_{n-1}$:
\[
v(x) = \frac{\sin \frac{\pi}{2}(x - i\theta_0)}{\sin \frac{\pi}{2}x}, \quad w(x) = 1,
\]

(iv) trigonometric $BC_n$:
\[
v(x) = \frac{\sin \frac{\pi}{2}(x - i\theta_0)}{\sin \frac{\pi}{2}x}, \\
w(x) = \frac{\sin \frac{\pi}{2}(x - i\theta_1) \sin \frac{\pi}{2}(x - \frac{1}{2} - i\theta_2) \cos \frac{\pi}{2}(x - \frac{1}{2})}{\sin \frac{\pi}{2}x \sin \frac{\pi}{2}(x - \frac{1}{2}) \cos \frac{\pi}{2}(x - \frac{1}{2})},
\]

Here the coupling constants $a_1, a_2, a_3, a_4, g, g_0, g_1, g_2, g'_1$ and $g'_2$ are assumed to be non-negative and one of $a_1, \ldots, a_4$ must be greater than $1/2$. Apparently, the deformed theories have usually more coupling constants than the original ones.

Note that the Hamiltonian \ref{eq:hamiltonian} can be expressed as a sum of factorized forms, $H = \sum_j A_j A_j^\dagger$, with
\[
A_j \equiv e^{-\frac{i}{2} \theta_j} \sqrt{V_j(x)} - e^{\frac{i}{2} \theta_j} \sqrt{V_j(x)}, \\
A_j^\dagger \equiv \sqrt{V_j(x)} e^{-\frac{i}{2} \theta_j} - \sqrt{V_j(x)} e^{\frac{i}{2} \theta_j}.
\]

It should be remarked that in the small momentum (the large $c$) limit, in which the shift operators $e^{\pm i \theta_j}$ can be approximated by differential operators $e^{\pm i \theta_j} \approx 1 \pm i \theta_j - \frac{1}{2} \theta_j^2$, the RsvD systems reduce to the CSM systems \cite{7, 8}.

We want now to derive new multi-variable FP equations associated with these RsvD systems, following the procedure described before. Let us note here that while the trigonometric cases can be treated exactly in the manner to be discussed below, it is more suitable to be treated as multiplicative shift systems instead of additive ones like the other two cases. For this reason, we will discuss the two rational cases first.

The ground state eigenfunctions $\phi_0(x)$ of \ref{eq:hamiltonian} for the rational cases are
\[
(i) : \phi_0(x) \propto \prod_{j=1}^{n} \frac{\Gamma(a_1 + ix_j) \Gamma(a_2 + ix_j)}{\Gamma(i(x_j - x_k))} \prod_{1 \leq j < k \leq n} \frac{\Gamma(g + i(x_j - x_k))}{\Gamma(i(x_j - x_k))},
\]

\[
(ii) : \phi_0(x) \propto \prod_{j=1}^{n} \prod_{a=1}^{q} \frac{\Gamma(a_1 + ix_j)}{\Gamma(2ix_j)} \prod_{1 \leq j < k \leq n} \prod_{\epsilon = \pm 1} \frac{\Gamma(g + i(x_j + \epsilon x_k))}{\Gamma(i(x_j + \epsilon x_k))},
\]

where $|f(z)| = \sqrt{f(z)*f(z)}$ for any complex function $f(z)$. These ground states have zero energy: $H \phi_0 = 0$. Since $H$ consists of a sum of positive semi-definite terms $A_j A_j^\dagger$, the condition of zero ground state energy implies $A_j \phi_0 = 0$ ($j = 1, \ldots, n$), or explicitly,
\[
\sqrt{V_j \left( x_1, \ldots, x_j - \frac{i}{2}, \ldots, x_n \right)} \phi_0 \left( x_1, \ldots, x_j - \frac{i}{2}, \ldots, x_n \right) = \sqrt{V_j \left( x_1, \ldots, x_j + \frac{i}{2}, \ldots, x_n \right)} \phi_0 \left( x_1, \ldots, x_j + \frac{i}{2}, \ldots, x_n \right).
\]

\[
\sqrt{V_j \left( x_1, \ldots, x_j - \frac{i}{2}, \ldots, x_n \right)} \phi_0 \left( x_1, \ldots, x_j - \frac{i}{2}, \ldots, x_n \right) = \sqrt{V_j \left( x_1, \ldots, x_j + \frac{i}{2}, \ldots, x_n \right)} \phi_0 \left( x_1, \ldots, x_j + \frac{i}{2}, \ldots, x_n \right).
\]
This can be verified explicitly using the forms of \( V(x) \) and \( \phi_0 \) for the two cases.

Now, we form the associated FP operator from (14) and \( \phi_0 \) in (22a) and (22b) according to the similarity transformation (6). With the help of (23), we find that

\[
L_{FP} = -\sum_{j=1}^{n} \left( e^{i\partial_j} V_j(x) + e^{-i\partial_j} V_j^*(x) - V_j(x) - V_j^*(x) \right). 
\]  

(24)

This is the general form of FP operator corresponding to the discrete Hamiltonian \( H \) in (14). One has \( L_{FP} \phi_0^2 = 0 \) as a consequence of \( H \phi_0 = 0 \). Thus \( \phi_0^2 \) is the stationary solution of the respective FP equation.

In the small momentum limit, or equivalently, the large \( c \) limit, all \( x \) and other parameters in \( V(x) \) are considered small quantities [7, 8]. Eq. (24) then reduces to the corresponding FP equation associated with the CSM system. Explicitly, keeping in the operators \( \exp(\pm i\partial_j) \) only up to the second order terms in the momentum, we get the FP operator

\[
L_{FP} = \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} \Re V_j(x) + \frac{\partial}{\partial x_j} \Im V_j(x) \right). 
\]  

(25)

Here in \( V_j \) only the lowest order terms in \( x \) and other parameters shall be retained.

As an illustration, let us take the case of rational \( A_{n-1} \) type. The potential is

\[
V_j(x) = (a_1 + ix_j)(a_2 + ix_j) \prod_{k=1, k \neq j}^{n} \left( 1 - i\frac{g}{x_j - x_k} \right). 
\]  

(26)

In the above-mentioned limit, we have

\[
\Re V_j(x) = a_1 a_2, \quad \Im V_j(x) = (a_1 + a_2) x_j - a_1 a_2 g \sum_{k=1, k \neq j}^{n} \frac{1}{x_j - x_k}, 
\]  

(27)

and hence the limiting FP operator is

\[
L_{FP} = a_1 a_2 \left( \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + 2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \frac{1}{a_1 + a_2} x_j - g \sum_{k=1, k \neq j}^{n} \frac{1}{x_j - x_k} \right) \right). 
\]  

(28)

This is just the corresponding FP operator for the \( A_{n-1} \) type CSM system in (13). The presence of the scale factors arises from the way the Hamiltonian and the potential of the RSvD systems are parameterized. If we take \( a_1 = a_2 = 2 \), and rescale \( H \to H/4 \), then the two formulas are the same.

V. MULTI-VARIABLE \( q \)-SHIFT FOKKER-PLANCK EQUATION

As mentioned before, the trigonometric cases can be treated more elegantly in terms of multiplicative shift type of Hamiltonian [1, 8]. Let us define \( z_j = \exp(2\pi i x_j/L) \), \( q = \exp(-2\pi /L) \) and \( a_0 = q^{a_0} \). For any function \( f(z) \) \( (z = \{z_1, \ldots, z_n\}) \) with real coefficients, we have \( f(z)^* = f(z^{-1}) \). Furthermore, we define \( D_j = q^{a_0^j} \). Then \( q^{D_j} \) is a multiplicative \( q \)-shift operator, i.e. \( q^{D_j} f(z) = f(z_1, \ldots, q z_j, \ldots, z_n) \). With these, the Hamiltonian (14) of the trigonometric case with (15a), (19a) and (19b) can be recast into (up to an overall constant factor)

\[
H = \sum_{j=1}^{n} \left( \sqrt{V_j(z)} q^{D_j} \sqrt{V_j(z^{-1})} + \sqrt{V_j^*(z)} q^{-D_j} \sqrt{V_j^*(z)} - V_j(z) - V_j(z^{-1}) \right), 
\]  

(29)

where the potential \( V_j(z) \) have the form

(iii) : \( V_j(z) = \prod_{k \neq j} \frac{1 - a_0 z_j / z_k}{1 - z_j / z_k} \),

(iv) : \( V_j(z) = \prod_{\alpha=1}^{4} (1 - a_\alpha z_j) \prod_{k \neq j} \prod_{\epsilon=\pm 1} \frac{1 - a_0 z_j z_k^\epsilon}{1 - z_j z_k^\epsilon} \).

(30a)
and the \( \{a_n\} \)'s are related to the parameters in (19a) and (19b) (see (4)):

\[
(a_1, a_2, a_3, a_4) = (e^{-2\pi g_1/L}, e^{-2\pi (g_2+1/2)/L}, -e^{-2\pi g_1'/L}, -e^{-2\pi (g_2'+1/2)/L}).
\]

The form of the Hamiltonian (29) is so chosen that it looks similar to that of the rational case (14) by the formal replacement \( e^{-i\partial_j} \rightarrow q^{D_j}, e^{i\partial_j} \rightarrow q^{-D_j} \). However, it should be emphasized that the potential \( V_j(z) \) in (29) corresponds to \( V_j'(x) \) in (14), which is due to the fact that \( 0 < q < 1 \) (\( q = e^{-2\pi/L} \)).

The ground state \( \phi_0 \) satisfying \( H\phi_0 = 0 \) is

\[
(iii) : \phi_0(z; g_0, q) \propto \prod_{1 \leq j < k \leq n} \frac{(z_j/z_k; q)^{\infty}}{(a_0 z_j/z_k; q)^{\infty}} ;
\]

\[
(iv) : \phi_0(z; \{a_n\}, g_0, q) \propto \prod_{j=1}^{n} \frac{(z_j^2; q)^{\infty}}{\prod_{\alpha=1}^{n} (a_\alpha z_j; q)^{\infty} \prod_{1 \leq j < k \leq n} (z_j z_k^*; q)^{\infty}}.
\]

Here we use the standard notation \( (a; q)^{\infty} = \prod_{n=0}^{\infty} (1 - aq^n) \).

Just like the rational cases, the Hamiltonian (29) is a sum of factorized forms, i.e. \( H = \sum_j A_j A_j^\dagger \), with

\[
A_j = q^{D_j} V_j(z^{-1}) - q^{D_j} V_j(z),
\]

\[
A_j^\dagger = \sqrt{V_j(z)} q^{D_j} - \sqrt{V_j(z^{-1})} q^{-D_j}.
\]

The ground state \( \phi_0 \) is annihilated by \( A_j \): \( A_j \phi_0(z) = 0 \) (\( j = 1, \ldots, n \)). Explicitly this equation reads

\[
\sum_{j=1}^{n} (q^{-D_j} V_j(z) + q^{D_j} V_j(z^{-1}) - V_j(z) - V_j(z^{-1})) \phi_0(z_1, \ldots, z_n).
\]

Using \( H, \phi_0, \) and (34), the similarity transformation (40) produces the FP operator

\[
L_{FP} = -\sum_{j=1}^{n} (q^{-D_j} V_j(z) + q^{D_j} V_j(z^{-1}) - V_j(z) - V_j(z^{-1})).
\]

This is the general discrete \( q \)-deformed FP operator corresponding to the Hamiltonian (29). Again, \( L_{FP} \) annihilates \( \phi_0^2 \).

VI. SUMMARY

In this paper we have proposed new types of multi-variable FP equations associated with the RSvD systems which are difference instead of differential equations. As the RSvD systems are the integrable “discrete” counterparts of the celebrated exactly solvable many-body CMS systems, the deformed FP equations presented here can be considered as “discrete” deformations of the ordinary multi-variable FP equations. They are also integrable. In the small momentum (large \( c \)) limit, our FP equations reduce to the usual FP equations associated with the respective CMS systems. For future study, it would be of interest to investigate solutions of these generalized FP equations with potentials \( V(x) \) and \( V(z) \) other than those exactly solvable ones discussed here.

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