Differential sandwich results for Wanas operator of analytic functions

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Abstract. In the present article, we determine some subordination and superordination results involving Wanas operator for certain normalized analytic functions defined in the unit disk $U$. These results are applied to establish sandwich results. Our results extend corresponding previously known results.

1. Introduction

Denote by $H = H(U)$ the collection of analytic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $H[a, n]$ be the subclass of $H$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \quad (a \in \mathbb{C}, \ n \in \mathbb{N} = \{1, 2, \ldots\}).$$

Also, let $A$ be the subclass of $H$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Now we recall the principal of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $U$, we say that the function $f$ is subordinate to $g$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 \ (z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) < g(z) \ (z \in U)$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalent (see [8]),

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\xi, h \in H$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $\xi$ and

$$\psi(\xi(z), z\xi'(z), z^2\xi''(z); z)$$

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are univalent functions in $U$ and if $\xi$ satisfies the second-order differential superordination

\begin{equation}
(2) \quad h(z) \prec \psi(\xi(z), z\xi'(z), z^2\xi''(z); z),
\end{equation}

then $\xi$ is called a solution of the differential superordination (2). (If $f$ is subordinate to $g$, then $g$ is superordinate to $f$). An analytic function $q$ is called a subordinant of (2), if $q \prec \xi$ for all $\xi$ satisfying (2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinates $q$ of (2) is called the best subordinant.

For $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, the Wanas operator $W_{\alpha,\beta}^{k,\delta} : \mathcal{A} \to \mathcal{A}$ (see [24]) is defined by

\begin{equation}
(3) \quad W_{\alpha,\beta}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n.
\end{equation}

**Remark 1.** It should be remarked that the operator $W_{\alpha,\beta}^{k,\delta}$ generalizes some known operators considered earlier:

1. For $k = 1$, the operator $W_{\alpha,\beta}^{1,\delta} \equiv I_{\alpha,\beta}^{\delta}$ was introduced and studied by Swamy [22],
2. For $k = \beta = 1$, $\delta = -\mu$, $Re(\mu) > 1$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W_{\alpha,1}^{1,-\mu} \equiv J_{\mu,\alpha}$ was investigated by Srivastava and Attiya [16]. The operator $J_{\mu,\alpha}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of the Srivastava-Attiya operator are found in [15, 17, 18, 19, 20] and in the references cited in each of these earlier works,
3. For $k = \beta = 1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,\delta} \equiv I_{\alpha}^{\delta}$ was investigated by Cho and Srivastava [6],
4. For $k = \alpha = \beta = 1$, the operator $W_{1,1}^{1,\delta} \equiv I^{\delta}$ was considered by Uralegaddi and Somanatha [23],
5. For $k = \alpha = \beta = 1$, $\delta = -\sigma$ and $\sigma > 0$, the operator $W_{1,1}^{1,-\sigma} \equiv I^{\sigma}$ was introduced by Jung et al. [7]. The operator $I^{\sigma}$ is the Jung-Kim-Srivastava integral operator,
6. For $k = \beta = 1$, $\delta = -1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,-1} \equiv L_\alpha$ was studied by Bernardi [4],
7. For $\alpha = 0$, $k = \beta = 1$ and $\delta = -1$, the operator $W_{0,1}^{1,-1} \equiv u$ was investigated by Alexander [1],
8. For $k = 1$, $\alpha = 1 - \beta$ and $\beta \geq 0$, the operator $W_{1-\beta,\beta}^{1,\delta} \equiv D_{\beta}^{\delta}$ was given by Al-Oboudi [2],
9. For $k = 1$, $\alpha = 0$ and $\beta = 1$, the operator $W_{0,1}^{1,\delta} \equiv S^{\delta}$ was considered by Sălăgean [13].
It is readily verified from (3) that
\[
(4) \quad z \left(W_{\alpha,\beta}^{k,\delta} f(z)\right)' = \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^{m} + 1 \right] W_{\alpha,\beta}^{k,\delta+1} f(z)
- \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^{m} \right] W_{\alpha,\beta}^{k,\delta} f(z).
\]

Very recently, Rahrovi [12], Attiya and Yassen [3], Seoudy [14], Wanas and Majeed [25] and Srivastava and Wanas [21] have obtained sandwich results for certain classes of analytic functions. Motivated by aforementioned works to investigate sufficient condition for \( f \) based on Wanas differential operator we define a new subclasses of normalized analytic functions satisfying the following:
\[
q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \prec q_2(z)
\]
and
\[
q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma} \prec q_2(z),
\]
where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

To establish our main results, we need the following definition and lemmas.

**Definition 1** ([8]). Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( U \setminus E(f) \), where
\[
E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}
\]
and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

**Lemma 1** ([8]). Let \( q \) be univalent in the unit disk \( U \) and let \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z)\phi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that
1. \( Q(z) \) is starlike univalent in \( U \),
2. \( \Re\left(z\frac{h'(z)}{Q(z)}\right) > 0 \) for \( z \in U \).

If \( \xi \) is analytic in \( U \), with \( \xi(0) = q(0) \), \( \xi(U) \subset D \) and
\[
(5) \quad \theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),
\]
then \( \xi \prec q \) and \( q \) is the best dominant of (5).

**Lemma 2** ([9]). Let \( q \) be a convex univalent function in \( U \) and let \( \mu \in \mathbb{C} \), \( \nu \in \mathbb{C} \setminus \{0\} \) with
\[
\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\nu}\right)\right\}.
\]
If $\xi$ is analytic in $U$ and
\begin{equation}
\mu \xi (z) + \nu z \xi' (z) \prec \mu q (z) + \nu z q' (z),
\end{equation}
then $\xi \prec q$ and $q$ is the best dominant of (6).

**Lemma 3** ([9]). Let $q$ be convex univalent in $U$ and let $\nu \in \mathbb{C}$. Further assume that $\Re (\nu) > 0$. If $\xi \in H [q (0), 1] \cap Q$ and $\xi (z) + \nu z \xi' (z)$ is univalent in $U$, then
\begin{equation}
q (z) + \nu z q' (z) \prec \xi (z) + \nu z \xi' (z),
\end{equation}
which implies that $q \prec \xi$ and $q$ is the best subordinant of (7).

**Lemma 4** ([5]). Let $q$ be convex univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q (U)$. Suppose that
\begin{enumerate}
\item $\Re \left( \frac{\theta' (q (z))}{\phi (q (z))} \right) > 0$ for $z \in U$,
\item $Q (z) = z q' (z \phi (q (z)))$ is starlike univalent in $U$.
\end{enumerate}
If $\xi \in H [q (0), 1] \cap Q$, with $\xi (\mathbb{U}) \subset D$, $\phi (\xi (z)) + z \xi' (z) \phi (\xi (z))$ is univalent in $U$ and
\begin{equation}
(\alpha \beta)^m + 1 \left( \frac{W_{k,\delta}f (z)}{z} \right)^\gamma \prec q (z) + \frac{\sigma}{\gamma} z q' (z),
\end{equation}
then $q \prec \xi$ and $q$ is the best subordinant of (8).

2. **Main Results**

**Theorem 1.** Let $q$ be convex univalent in $U$ with $q (0) = 1$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$ and suppose that $q$ satisfies
\begin{equation}
\Re \left( 1 + \frac{z q'' (z)}{q' (z)} \right) > \max \left\{0, -\Re \left( \frac{\gamma}{\sigma} \right) \right\}.
\end{equation}
If $f \in A$ satisfies the subordination
\begin{equation}
\left[ 1 - \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\sigma}{\beta} \right)^m + 1 \right] \left( \frac{W_{k,\delta}f (z)}{z} \right)^\gamma \prec q (z) + \frac{\sigma}{\gamma} z q' (z),
\end{equation}
then
\begin{equation}
\left( \frac{W_{k,\delta}f (z)}{z} \right)^\gamma \prec q (z)
\end{equation}
and $q$ is the best dominant of (10).
Proof. Define the function $\xi$ by
\begin{equation}
(12) \quad \xi(z) = \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^\gamma, \quad (z \in \mathbb{U}).
\end{equation}

Differentiating (12) logarithmically with respect to $z$, we get
\begin{equation*}
\frac{z\xi'(z)}{\xi(z)} = \gamma \left( z \left( \frac{W_{\alpha,\beta}^k f(z)'}{W_{\alpha,\beta}^k f(z)} - 1 \right) \right).
\end{equation*}

Now, in view of (4), we obtain the following subordination
\begin{equation*}
\frac{z\xi'(z)}{\xi(z)} = \sum_{m=1}^k \left( \begin{array}{c} k \\ m \end{array} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k+1} f(z)}{W_{\alpha,\beta}^k f(z)} - 1 \right).
\end{equation*}

Therefore,
\begin{equation*}
\frac{z\xi'(z)}{\gamma} = \sum_{m=1}^k \left( \begin{array}{c} k \\ m \end{array} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \times \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^\gamma \left( \frac{W_{\alpha,\beta}^{k+1} f(z)}{W_{\alpha,\beta}^k f(z)} - 1 \right).
\end{equation*}

The subordination (10) from the hypothesis becomes
\begin{equation*}
\xi(z) + \sigma \frac{z\xi'(z)}{\gamma} < q(z) + \sigma \frac{zq'(z)}{\gamma}.
\end{equation*}

Hence, an application of Lemma 2 with $\mu = 1$ and $\nu = \sigma$, we obtain (11). □

**Theorem 2.** Let $\eta, \tau \in \mathbb{C}$, $\gamma > 0$, $\lambda \in \mathbb{C}\{0\}$ and $q$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$, $q(z) \neq 0$ ($z \in \mathbb{U}$) and assume that $q$ satisfies
\begin{equation}
(13) \quad \Re \left( 1 + \frac{\tau}{\lambda} q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0.
\end{equation}

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$. If $f \in A$ satisfies
\begin{equation}
(14) \quad \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) < \eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)},
\end{equation}

where
\begin{equation}
(15) \quad \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) = \eta + \tau \left( \frac{W_{\alpha,\beta}^{k+1} f(z)}{W_{\alpha,\beta}^k f(z)} \right)^\gamma + \gamma \lambda \sum_{m=1}^k \left( \begin{array}{c} k \\ m \end{array} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k+2} f(z)}{W_{\alpha,\beta}^{k+1} f(z)} - \frac{W_{\alpha,\beta}^{k+1} f(z)}{W_{\alpha,\beta}^k f(z)} \right),
\end{equation}
then
\[
\left( \frac{W_{k,\delta+1}^{k,\delta} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} < q(z)
\]
and \( q \) is the best dominant of (14).

**Proof.** Define the function \( \xi \) by
\[
\xi(z) = \left( \frac{W_{k,\delta+1}^{k,\delta} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma}, \quad (z \in \mathbb{U}).
\]
By a straightforward computation and using (4), we have
\[
\eta + \tau \xi(z) + \lambda z \xi'(z) = \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z),
\]
where \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is given by (15). From (14) and (17), we obtain
\[
\eta + \tau \xi(z) + \lambda z \xi'(z) < \eta + \tau q(z) + \lambda z q'(z).
\]
By setting
\[
\theta (w) = \eta + \tau w \text{ and } \phi (w) = \frac{\lambda}{w}, \quad w \neq 0,
\]
we see that \( \theta (w) \) is analytic in \( \mathbb{C} \), \( \phi (w) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi (w) \neq 0, \ w \in \mathbb{C} \setminus \{0\} \). Also, we get
\[
Q(z) = z q'(z) \phi(q(z)) = \lambda \frac{z q'(z)}{q(z)}
\]
and
\[
h(z) = \theta(q(z)) + Q(z) = \eta + \tau q(z) + \lambda \frac{z q'(z)}{q(z)}.
\]
It is clear that \( Q(z) \) is starlike univalent in \( \mathbb{U} \),
\[
\Re \left( \frac{z h'(z)}{Q(z)} \right) = \Re \left( 1 + \frac{\tau}{\lambda} q(z) + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} \right) > 0.
\]
Thus, by Lemma 1, we get \( \xi(z) < q(z) \). By using (16), we obtain the desired result. \( \square \)

**Theorem 3.** Let \( q \) be convex univalent in \( \mathbb{U} \) with \( q(0) = 1 \), \( \gamma > 0 \) and \( \Re(\sigma) > 0 \). Let \( f \in A \) satisfies
\[
\left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \in H[q(0), 1] \cap Q
\]
and
\[
\left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \binom{\alpha}{\beta}^{m} + 1 \right) \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma}
\]


\begin{align*}
+ \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{m}{\beta} \right)^m + 1 \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f (z)}{W_{\alpha,\beta}^{k,\delta} f (z)} \right) \\
\end{align*}

be univalent in \( U \). If

\begin{equation}
q (z) + \frac{\sigma}{\gamma} z q' (z)
\end{equation}

\begin{align*}
\prec \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{m}{\beta} \right)^m + 1 \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma \\
+ \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{m}{\beta} \right)^m + 1 \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma \times \\
\times \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f (z)}{W_{\alpha,\beta}^{k,\delta} f (z)} \right) ,
\end{align*}

then

\begin{equation}
q (z) \prec \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma
\end{equation}

and \( q \) is the best subordinant of (18).

**Proof.** Let \( \xi \) be defined by (12), then differentiating \( \xi \) with respect to \( z \), we get

\begin{equation}
\frac{z \xi' (z)}{\xi (z)} = \gamma \left( \frac{z \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right)'}{\frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} - 1} \right).
\end{equation}

By using (4) for \( \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right)' \), in (20), we have

\begin{align*}
\left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{m}{\beta} \right)^m + 1 \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma \\
+ \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{m}{\beta} \right)^m + 1 \left( \frac{W_{\alpha,\beta}^{k,\delta} f (z)}{z} \right) \gamma \times \\
\times \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f (z)}{W_{\alpha,\beta}^{k,\delta} f (z)} \right) = \frac{1}{P \gamma} z \xi' (z).
\end{align*}

From (18) and (21), we get

\begin{equation}
q (z) + \frac{\sigma}{\gamma} z q' (z) \prec \xi (z) + \frac{\sigma}{\gamma} z \xi' (z).
\end{equation}

Hence, by using Lemma 3 with \( \mu = 1 \) and \( \nu = \frac{\sigma}{\gamma} \), we obtain (19). \( \square \)
Theorem 4. Let \( \eta \in \mathbb{C}, \gamma > 0, \lambda \in \mathbb{C}\setminus\{0\} \) and \( q \) be convex univalent in \( U \) with \( q(0) = 1, q(z) \neq 0 \ (z \in U) \) and assume that \( q \) satisfies

(22) \[ \Re \left( \frac{\tau}{\lambda} q(z) \right) > 0. \]

Suppose that \( zq'(z)/q(z) \) is starlike univalent in \( U \). If \( f \in \mathcal{A} \) satisfies

\[
\left( \frac{W^{k,\delta+1}_{\alpha,\beta} f(z)}{W^{k,\delta}_{\alpha,\beta} f(z)} \right)^{\gamma} \in H[q(0), 1] \cap Q
\]

and \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is univalent in \( U \), where \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is given by (15). If

(23) \[ \eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z), \]

then

\[ q(z) \prec \left( \frac{W^{k,\delta+1}_{\alpha,\beta} f(z)}{W^{k,\delta}_{\alpha,\beta} f(z)} \right)^{\gamma} \]

and \( q \) is the best subordinant of (23).

Proof. Assume that the function \( \xi \) be defined by (16). By a straightforward computation, we have

(24) \[ \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) = \eta + \tau \xi(z) + \lambda \frac{z\xi'(z)}{\xi(z)}, \]

where \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is given by (15). From (23) and (24), we obtain

\[ \eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \eta + \tau \xi(z) + \lambda \frac{z\xi'(z)}{\xi(z)}. \]

By setting \( \theta(w) = \eta + \tau w \) and \( \phi(w) = \frac{\lambda}{w} \), \( w \neq 0 \), we see that \( \theta(w) \) is analytic in \( \mathbb{C}, \phi(w) \) is analytic in \( \mathbb{C}\setminus\{0\} \) and that \( \phi(w) \neq 0 \), \( w \in \mathbb{C}\setminus\{0\} \). Also, we get

\[ Q(z) = zq'(z)\phi(q(z)) = \lambda \frac{zq'(z)}{q(z)}. \]

It is clear that \( Q(z) \) is starlike univalent in \( U \),

\[ \Re \left( \frac{\theta'(q(z))}{\phi(q(z))} \right) = \Re \left( \frac{\tau}{\lambda} q(z) \right) > 0. \]

Thus, by Lemma 4, we get \( q(z) \prec \xi(z) \). By using (16), we obtain the desired result. \( \square \)

Concluding the results of differential subordination and superordination, we state the following “sandwich results”.
**Theorem 5.** Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \) with \( q_1(0) = q_2(0) = 1 \). Suppose \( q_2 \) satisfies (9), \( \gamma > 0 \) and \( \Re(\sigma) > 0 \). Let \( f \in A \) satisfies

\[
\left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \in H[1,1] \cap Q
\]

and

\[
\left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^{m} \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^{m} \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)
\]

be univalent in \( U \). If

\[
q_1(z) + \frac{\sigma}{\gamma} z q_1'(z)
\]

\[
< \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^{m} \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^{m} \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)
\]

\[
< q_2(z) + \frac{\sigma}{\gamma} z q_2'(z)
\]

then

\[
q_1(z) < \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} < q_2(z)
\]

and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**Theorem 6.** Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \) with \( q_1(0) = q_2(0) = 1 \). Suppose \( q_1 \) satisfies (22) and \( q_2 \) satisfies (13). Let \( f \in A \) satisfies

\[
\left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} \in H[1,1] \cap Q
\]

and \( \Omega(\eta,\tau,\gamma,\lambda,k,\delta,\alpha,\beta;z) \) is univalent in \( U \), where \( \Omega(\eta,\tau,\gamma,\lambda,k,\delta,\alpha,\beta;z) \) is given by (15). If

\[
\eta + \tau q_1(z) + \lambda \frac{z q_1'(z)}{q_1(z)} < \Omega(\eta,\tau,\gamma,\lambda,k,\delta,\alpha,\beta;z)
\]

\[
< \eta + \tau q_2(z) + \lambda \frac{z q_2'(z)}{q_2(z)}
\]
then
\[ q_1(z) \prec \left( \frac{W_{\alpha,\beta}^{\gamma} f(z)}{W_{\alpha,\beta}^{\gamma} f(z)} \right) \prec q_2(z) \]

and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**Remark 2.** By selecting the particular values of \( \delta, k, \alpha \) and \( \beta \), we can derive a number of known results. Some of them are given below:

1. Taking \( \delta = 0 \) in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [10, Corollary 3.3],
2. Putting \( k = 1, \alpha = 1 - \beta \) and \( \beta \geq 0 \) in Theorems 1, 3 and 5, we get the results obtained by Răducanu and Nechita [11, Theorem 3.1, Theorem 3.6, Theorem 3.9],
3. Setting \( \alpha = 0 \) and \( k = \beta = 1 \) in Theorems 1, 3 and 5, we get the results obtained by Răducanu and Nechita [11, Corollary 3.3, Corollary 3.8, Corollary 3.11].

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