Stark resonances in 2-dimensional curved quantum waveguides.

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Abstract

In this paper we study the influence of an electric field on a two dimensional waveguide. We show that bound states that occur under a geometrical deformation of the guide turn into resonances when we apply an electric field of small intensity having a nonzero component on the longitudinal direction of the system.

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1 Introduction

The study of resonances occurring in a quantum system subjected to a constant electric field is now a well-known issue among the mathematical physics community. In a recent past a large amount of literature has been devoted to this problem (see e.g. \cite{14,16} and references therein). Mostly these works are concerned with quantum systems living in the whole space $\mathbb{R}^n$ as e.g. atomic systems \cite{6,11,15,17,25,26}. In the present paper we would like to address this question for an inhomogeneous quantum system consisting in a curved quantum waveguide in $\mathbb{R}^2$. It is known that

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bound states arise in curved guides \cite{1, 9} and the corresponding eigenfunctions are expected to be localized in space around the deformation. Therefore, based on these results the main question is what happens with these bound states when the electric field is switched on?

A first result is given in \cite{10} where the electric field is supposed to be orthogonal to the guide outside a bounded region. But in this situation there is no Stark resonance.

Here we are focusing on a strip $\Omega \subset \mathbb{R}^2$ of constant width curved within a compact region. The electric field is chosen with a strictly positive component both on the longitudinal direction of the left part and of the right part of the curved strip. Roughly speaking this situation is similar to the one of an atomic system interacting with an external electric field. Due to the field, an eigenstate of the curved waveguide at zero field turns into scattering state which is able to escape at infinity under the dynamics. It is then natural to expect spectral resonances for this system.

The resonances are defined as the complex poles in the second Riemann sheet of the meromorphic continuation of the resolvent associated to the Stark operator. We construct this extension using the distortion theory \cite{1, 18}. Our proof of existence of resonances borrows elements of strategy developed in \cite{6, 15}. It is mainly based on non-trapping estimates of \cite{6}. For the applicability of these techniques to our model, the difficulty we have to solve is that the system has a bounded transverse direction.

Let us briefly review the content of the paper. In section 2 we describe precisely the system and assumptions. The distortion and the definition of resonances are given respectively in section 3 and 4. In section 5 we prove the existence of resonances. Finally the section 6 is devoted to get an exponential estimate on the width of resonances. Actually we show that the imaginary part of resonances arising in this system follows a type of Oppenheimer’s law \cite{21} when the intensity of the field vanishes.

To end this section let us mention a still open question related to this problem and that we hope to solve in a future work. We claim that our regularity assumptions on the curvature imply that the corresponding Stark operator (see (2.4)) has no real eigenvalue \cite{5}. In that case the complex poles have a non zero imaginary part then they are resonances in the strict sense of the term \cite{23}.

**2 Setting**

Consider a curved strip $\Omega$ in $\mathbb{R}^2$ of a constant width $d$, around a smooth reference curve $\Gamma$, we suppose that $\Omega$ is not self-intersecting. The points $X = (x, y)$ of $\Omega$ are
described by the curvilinear coordinates \((s, u) \in \mathbb{R} \times (0, d)\),

\[
\begin{align*}
    x &= a(s) - ub'(s), \\
    y &= b(s) + ua'(s),
\end{align*}
\]  

(2.1)

where \(a, b\) are smooth functions defining the reference curve \(\Gamma = \{(a(s), b(s)), s \in \mathbb{R}\}\) in \(\mathbb{R}^2\). They are supposed to satisfy \(a'(s)^2 + b'(s)^2 = 1\).

Introduce the signed curvature \(\gamma(s)\) of \(\Gamma\),

\[
\gamma(s) = b'(s)a''(s) - a'(s)b''(s).
\]  

(2.2)

For a given curvature \(\gamma\), the functions \(a\) and \(b\) can be chosen as

\[
\begin{align*}
a(s) &= \int_0^s \cos \alpha(t) \, dt, \\
b(s) &= \int_0^s \sin \alpha(t) \, dt,
\end{align*}
\]  

(2.3)

where \(\alpha(s_1, s_2) = -\int_{s_1}^{s_2} \gamma(t) \, dt\) is the angle between the tangent vectors to \(\Gamma\) at the points \(s_1\) and \(s_2\) \([10]\). Set \(\alpha(s) = \alpha(s, 0), s \in \mathbb{R}\). We assume that \(\gamma\) has a compact support, \(\text{supp}(\gamma) = [0, s_0]\) for some \(s_0 > 0\) and

(h1) \(\gamma \in C^2(\mathbb{R})\),

(h2) \(d\|\gamma\|_\infty < 1\).

Set \(\alpha_0 = \alpha(s_0)\). Let \(F = F(\cos(\eta), \sin(\eta))\) be the electric field. Here \(|\eta| < \frac{\pi}{2}\) is fixed and satisfies

(h3) \(|\eta - \alpha_0| < \frac{\pi}{2}\).

See remark \([2.2]\) below for a discussion about assumptions on \(F\). We consider the Stark effect Hamiltonian on \(L^2(\Omega)\),

\[
H(F) = -\Delta_\Omega + F \cdot X, \quad F > 0,
\]  

(2.4)

with Dirichlet boundary condition on \(\partial \Omega\), the boundary of \(\Omega\). One can check using natural curvilinear coordinates that \(H(F)\) is unitarily equivalent to the Schrödinger operator defined by

\[
H(F) = H_0(F) + V_0, \quad H_0(F) = T_s + T_u + W(F)
\]  

(2.5)

on the Hilbert space \(L^2(\Omega)\), \(\Omega = \mathbb{R} \times (0, d)\) with Dirichlet boundary conditions on \(\partial \Omega = \mathbb{R} \times \{0, d\}\) \([10]\). Here

\[
T_s = -\partial_s g \partial_s, \quad g = g(s, u) = (1 + u\gamma(s))^{-2}, \quad T_u = -\partial_u^2
\]  

(2.6)

and \(W(F)\) is the operator multiplication by the function,

3
\[
W(F, s, u) = \begin{cases} 
F(\cos(\eta)s + \sin(\eta)u) & \text{if } s < 0 \\
F(\int_0^s \cos(\eta - \alpha(t)) \, dt + \sin(\eta - \alpha(s))u) & \text{if } 0 \leq s \leq s_0 \\
F(\cos(\eta - \alpha_0)(s - s_0) + A + \sin(\eta - \alpha_0)u) & \text{if } s > s_0 
\end{cases}
\] (2.7)

where \(A = \int_0^{s_0} \cos(\eta - \alpha(t)) \, dt\). 

\[
V_0(s, u) = -\frac{\gamma(s)^2}{4(1 + u\gamma(s))^2} + \frac{u\gamma''(s)}{2(1 + u\gamma(s))^3} - \frac{5}{4} \frac{u^2\gamma'(s)^2}{(1 + u\gamma(s))^4}.
\] (2.8)

Denote by \(H = H_0 + V_0\) where \(H_0 = T_s + T_u\). This is the Hamiltonian associated with the guide in absence of electric field.

We have

**Theorem 2.1.** Suppose that assumptions (h1) and (h2) hold, then for \(F > 0\),

(i) \(H(F)\) is an essentially self-adjoint operator on \(L^2(\Omega)\),

(ii) the spectrum of \(H(F)\), \(\sigma(H(F)) = \mathbb{R}\).

The proof of this theorem is given in the appendix of the paper.

**Remark 2.2.** The situation where \(\eta = \frac{\pi}{2}\) and \(\alpha_0 = 0\) has been considered in \([10]\), but in that case there is no Stark resonance. It is also true if we suppose \(|\eta| \geq \frac{\pi}{2}\) and \(|\eta - \alpha_0| < \frac{\pi}{2}\) since \(W(F)\) is now a confining potential. Note that the regime \(|\eta| > \frac{\pi}{2}\) and \(|\eta - \alpha_0| > \frac{\pi}{2}\) is a symmetric case of the one considered in this paper and can be studied in the same way. While for \(|\eta| < \frac{\pi}{2}\) and \(|\eta - \alpha_0| > \frac{\pi}{2}\), the situation is quite different since here the escape region is larger also including a neighbourhood of \(+\infty\). This will be studied in \([13]\).

### 3 The distortion

From now on we assume that (h1), (h2) and (h3) are satisfied. To give a sense to the construction below we need to consider electric fields of finite magnitude. Without loss of generality we may suppose that \(0 < F \leq 1\).

Introduce the distortion on \(\Omega\),

\[
S_\theta : (s, u) \mapsto (s + \theta f(s), u)
\] (3.9)

defined from the vector field \(f = -\frac{1}{F \cos(\eta)} \Phi\) where \(\Phi \in C^\infty(\mathbb{R})\) is as follow. Let \(E < 0\), be the reference energy, \(0 < \delta E < \frac{1}{2} \min\{1, |E|\}\), \(E_- = E - \delta E\) and
\( E_+ = E + \delta E \). Set \( \Phi(s) = \phi[F \cos(\eta)]s \) where \( \phi \in C^\infty(\mathbb{R}) \) is a non-increasing function such that

\[
\phi(t) = 1 \text{ if } t < E, \quad \phi(t) = 0 \text{ if } t > E_+.
\]

(3.10)

Note that for \( s < \frac{E}{F \cos(\eta)} \), \( S_\theta \) coincides with a translation w.r.t. the longitudinal variable \( s \).

Clearly for \( k \geq 1 \), \( \| \Phi^{(k)} \|_\infty \leq \frac{F \delta E}{k} \) and \( \| f^{(k)} \|_\infty \leq \frac{F^{k-1}}{(\delta E)^k} \). For \( \theta \in \mathbb{R} \), \( |\theta| < \delta E \), \( S_\theta \) implements a family of unitary operators on \( L^2(\Omega) \) by

\[
U_\theta \psi = (1 + \theta f')^{\frac{1}{2}} \psi \circ S_\theta.
\]

(3.11)

We note that

\[
H_\theta(F) = U_\theta H(F) U_\theta^{-1} = H_{0,0}(F) + V_0.
\]

(3.12)

where

\[
T_{s,\theta} = -(1 + \theta f')^{-\frac{1}{2}} g \partial_s (1 + \theta f')^{-\frac{1}{2}},
\]

(3.14)

\[
W_\theta(F) = W(F) \circ S_\theta.
\]

(3.15)

Set \( \theta_0 = \alpha \delta E \) where \( \alpha \) is some strictly positive constant which we fix in the proof of Theorems 3.15 and 6.11 below. In fact \( \theta_0 \) is the critical value of distortion parameter.

**Theorem 3.3.** There exists \( 0 < \alpha < 1/2 \) independent of \( E \) and \( F \) such that for \( 0 < F < \delta E \), \( \{ H_\theta(F), \Im \theta < \theta_0 \} \) is an self-adjoint analytic family of type A (see [19]).

**Proof.** An computation shows that

\[
T_{s,\theta} = -\partial_s (1 + \theta f')^{-2} g \partial_s + R_\theta,
\]

(3.16)

where \( R_\theta = \frac{\eta f'''}{2 (1 + \theta f')} - \frac{5 \sigma^2 f''^2}{4 (1 + \theta f')^2} \) is a bounded function. Let \( h(F) = H_0 + w(F) \) be the operator in \( L^2(\Omega) \) where \( w(F) \) is the multiplication operator by

\[
w(F, s) = \begin{cases} 
F \cos(\eta) s & \text{if } s < 0 \\
0 & \text{if } 0 \leq s \leq s_0 \\
F \cos(\eta - \alpha_0) s & \text{if } s > s_0.
\end{cases}
\]

(3.17)

Since \( h = h(F) \) differs from \( H(F) \) by adding a bounded symmetric operator, it is also a self-adjoint operator.
We have

\[ H_\theta(F) = h + \partial_s G_\theta \partial_s + R_\theta + W_\theta(F) - w(F) + V_0, \quad G_\theta = \left( \frac{2\theta f' + \theta^2 f'^2}{(1 + \theta f')^2} \right) g. \]  

(3.18)

Let us show that for |\theta| small enough then \( D(H_{0,\theta}(F)) = D(h) \). Through unitarity property we may suppose that \( R\theta = 0 \). In view of the perturbation theory \([19]\) and (3.18) it remains to show that \( \partial_s G_\theta \partial_s \) is \( h \)-bounded. By using the resolvent identity,

\[
\partial_s G_\theta \partial_s (h + i)^{-1} = \partial_s G_\theta \partial_s (H_0 + i)^{-1} - \partial_s G_\theta \partial_s (H_0 + i)^{-1} w(F)(h + i)^{-1}
\]

\[
= \partial_s G_\theta \partial_s (H_0 + i)^{-1} - \partial_s G_\theta \partial_s F_s (H_0 + i)^{-1} \frac{w(F)}{s}(h + i)^{-1}
\]

\[
- \partial_s G_\theta \partial_s (H_0 + i)^{-1}(\partial_s g + g \partial_s)(H_0 + i)^{-1} \frac{w(F)}{s}(h + i)^{-1}.
\]

We know that \( D(H_0) \subset \mathcal{H}_{0}^{2}(\Omega) \cap \mathcal{H}_{0}^{1}(\Omega), \) \([21, 8, 20]\) (Here we use the standard notation for Sobolev spaces). Let \( \chi \) be a characteristic function of supp\( (f') \). Then by the closed graph theorem \( \partial_s g(H_0 + i)^{-1}, g \partial_s(H_0 + i)^{-1}, \chi \partial_s g \partial_s(H_0 + i)^{-1} \) and \( \chi \partial_s g \partial_s F_s (H_0 + i)^{-1} \) are bounded operators. The multiplication operators \( \frac{w(F)}{s} \) and \( \frac{w(F)}{s} \) are also bounded. Hence this is true for the operator \( \partial_s G_\theta \partial_s (h + i)^{-1} \).

It is easy to check that under conditions on parameters \( \theta \) and \( F \),

\[
\| \partial_s G_\theta \partial_s (H_0 + i)^{-1} \| \leq \frac{3\alpha}{(1 - \alpha)^3} (\| \chi \partial_s g \partial_s (H_0 + i)^{-1} \| + \| g \partial_s(H_0 + i)^{-1} \|) \leq C \frac{3\alpha}{(1 - \alpha)^3}
\]

for some constant \( C > 0 \) independent of \( F \) and \( E \). Evidently \( \| \partial_s G_\theta \partial_s F_s (H_0 + i)^{-1} \| \) satisfies a similar estimate. Choosing \( \alpha \) so small such that \( \| \partial_s G_\theta \partial_s (h + i)^{-1} \| < 1 \), then \( \partial_s G_\theta \partial_s \) is relatively bounded to \( h \) with relative bound strictly smaller that one. Thus the statement follows.

The proof is complete if we can show that for \( \psi \in D(H_{\theta}(F)) \)

\[
\theta \in \{ \theta \in \mathbb{C}, |\theta| < \theta_0 \} \longmapsto (H_{\theta}(F)\psi, \psi)
\]

is an analytic function. But this last fact can be readily verified by using standard arguments and the explicit expression (3.18).

\[ \square \]

Remark 3.4. For \( \theta \in \mathbb{R}, \ |\theta| < \delta E \) consider the unitary transformation on \( L^2(\mathbb{R}) \)

\[ u_\theta \psi(s) = (1 + \theta f'(s))^\frac{1}{2} \psi(s + \theta f(s)), \ \psi \in L^2(\mathbb{R}). \]

There exists a dense subset of analytic vectors associated with \( u_\theta \) in \( |\theta| < \frac{\delta E}{\sqrt{2}} \) \([18, 22]\).

It is shown in \([15]\) that \( \mathcal{A}_1 \) is dense in \( L^2(\mathbb{R}) \) and universal w.r.t the distortion. Let \( \mathcal{A} \) be the linear subspace generated by vectors of the form \( \varphi \otimes \psi, \varphi \in \mathcal{A}_1, \psi \in L^2((0, d)) \).

Then \( \mathcal{A} \) is a dense subset of analytic vectors associated to the transformation \( U_\theta \) in \( |\theta| < \theta_0 \).
For further developments we need to introduce the following modified operator on $L^2(\Omega)$. Let $s_1 > s_0$, such that $\cos(\eta - \alpha_0)(s - s_0) + A + \sin(\eta - \alpha_0)u \geq 0$ for all $u \in (0, d)$. Set

$$\tilde{H}_0(F) = H_0 + \tilde{W}(F),$$

where $\tilde{W}(F)$ is a multiplication operator by

$$\tilde{W}(F, s, u) = \begin{cases} W(F, s, u) & \text{if } s < 0, \ s > s_1 \\ 0 & \text{if } 0 \leq s \leq s_1. \end{cases}$$

(3.20)

For $\theta \in \mathbb{R}$, $|\theta| < \theta_0$, let $\tilde{H}_{0,\theta}(F) = U_{\theta} \tilde{H}_0(F) U_{\theta}^{-1} = T_s \theta + T_u + \tilde{W}_\theta(F)$. Then we have

**Corollary 3.5.** For $0 < \theta < \delta E$, $\{\tilde{H}_{0,\theta}(F), \ |\Im\theta| < \theta_0\}$ is a type A self-adjoint analytic family of operators.

**Proof.** We have $H_\theta(F) - \tilde{H}_{0,\theta}(F) = V_\theta \tilde{H}_0(F) U_\theta^{-1} = T_s \theta + T_u + \tilde{W}_\theta(F)$, but $V_\theta$ as well as $W_\theta(F) - \tilde{W}_\theta(F)$ are bounded and $\theta$-independent so by the Theorem 3.3 the corollary follows. \qed

4 Meromorphic extension of the resolvent.

Let $\theta = i\beta$, we suppose that $0 < \beta < \theta_0$. Set

$$\mu_\theta = 1 + \theta f^z$$

(4.21)

with $f^z = \Phi - 1$ and $\Phi$ defined in the Section 3. Then $\mu_\theta$ defined a one to one map from $D(H_\theta(F))$ to $D(H_{0,\theta}(F))$. Let $\lambda_0 = \inf \sigma(T_u)$ be the first transversal mode and

$$\nu_\theta = \{z \in \mathbb{C}, \ \Im \mu_\theta^2(E_+ \lambda_0 - z) < \beta \delta E \frac{E}{2}\}. \quad (4.22)$$

$\nu_\theta$ denotes its complement in $\mathbb{C}$. It is easy to see that $\nu_\theta$ contains a $F$-independent complex neighbourhood of the semi axis $(-\infty, \lambda_0 + E - \frac{3\delta E}{4})$,

$$\tilde{\nu}_\theta = \{x \leq 0, \ y \geq -\frac{\beta \delta E}{2}\} \cup \{(x > 0, \ y \geq 2\beta x - \frac{\beta \delta E}{2}\} \quad (4.23)$$

where $x = \Re z - \lambda_0 - E_-$ and $y = \Im z$.

In this section our main result is the following. Let $F_0 = \alpha'(\delta E)^2 \min\{1, \frac{1}{d}\}$ where $\alpha'$ is a strictly positive constant independent of $E$ and $\beta$ which is determined in the proof of the Lemma 4.7. We have
Theorem 4.6. There exists \( \alpha' > 0 \) such that for all \( E < 0, 0 < F \leq F_0 \), the function
\[
z \in \mathbb{C}, \Im z > 0 \rightarrow f_\varphi(z) = \left((H(F) - z)^{-1}\varphi, \varphi\right), \varphi \in A
\]
has an meromorphic extension in \( \cup_{0 < \beta < \theta} \nu_\theta \).

We define the resonances of the pair \((H, H(F))\) as the set
\[
\cup_{\varphi \in A} \{\text{poles of } f_\varphi(z)\} \cap \mathbb{C}^-.
\]

The proof of this theorem is based on the two following results. For a given operator \( O \) on \( L^2(\Omega) \) we denote by \( \rho(O) \) its the resolvent set.

Lemma 4.7. There exists \( \alpha' > 0 \) such that for \( E < 0, 0 < F \leq F_0 \) and \( 0 < \beta < \theta_0 \).

(i) \( \nu_\theta \subset \rho(\tilde{H}_{0,\theta}(F)) \).

(ii) \( \forall z \in \nu_\theta, \| (\tilde{H}_{0,\theta}(F) - z)^{-1} \| \leq \text{dist}^{-1}(z, \nu_\theta^c) \).

Proof. By using a standard commutation relation we derive from (3.14),
\[
\mu_\theta T_{s,\theta} \mu_\theta = T_1(\theta) + iT_2(\theta) + \mu_\theta(T_{s,\theta} \mu_\theta)
\]
where \( T_1(\theta) = -\partial_s \Re \{\mu_\theta^2(1 + \theta f')^{-2}\} g \partial_s, \ T_2(\theta) = -\partial_s \Im \{\mu_\theta^2(1 + \theta f')^{-2}\} g \partial_s \). The operators \( T_1(\theta), T_2(\theta) \) are symmetric and \( T_2(\theta) \) is negative [6]. Moreover
\[
\Im \mu_\theta(T_{s,\theta} \mu_\theta) = O\left(\frac{\beta F^2}{(\delta E)^3}\right).
\]

In the other hand, let \( z \in \nu_\theta \), set \( \beta S = -\Im \mu_\theta^2(\tilde{W}_{\theta}(F) - E_-) - \Im \mu_\theta(T_{s,\theta} \mu_\theta) \) in fact
\[
S = (1 - \beta^2 f'^2)\Phi - 2f^2(\tilde{W}(F) - E_-) - \beta^{-1} \Im \mu_\theta(T_{s,\theta} \mu_\theta).
\]

On \( \text{supp}(f^2) = \text{supp}(\Phi - 1) \), we have \( \cos(\eta - \alpha_0)(s - s_0) + \sin(\eta - \alpha_0)u + A \geq 0 \) if \( s > s_1, F \cos(\eta)s - E_- \geq \delta E \) if \( s < 0 \) and then
\[
F \cos(\eta)s \chi_{\{s < 0\}} + F(\cos(\eta - \alpha_0) + \sin(\eta - \alpha_0)u + A) \chi_{\{s \geq s_1\}} - E_- \geq \delta E \chi_{\{s < 0\}} - E_- \chi_{\{s \geq 0\}} \geq \delta E.
\]

By using (4.25), we get for \( 0 < \beta < \theta_0 \)
\[
S \geq \frac{1}{2} \Phi + 2(1 - \Phi)(\delta E + Fu \sin \eta \chi_{\{s < 0\}}) + O\left(\frac{F^2}{(\delta E)^3}\right).
\]

Then we can choose \( \alpha' \) so small such that,
\[
S \geq \frac{1}{2} \min\left\{\frac{1}{2}, \delta E\right\} = \frac{\delta E}{2}.
\]
Further in the quadratic form sense on \(D(H_\theta(F)) \times D(H_\theta(F))\), we have
\[
\text{Im}\mu_\theta(\overline{H}_{0,\theta}(F) - z)\mu_\theta = T_2(\theta) - \beta S + \Re\mu_\theta^2 T_u + \text{Im}\mu_\theta^2 (E_- - z).
\]
(4.27)
Thus for \(0 < \beta < \theta_0\), \(0 < F \leq F_0\) and \(z \in \nu_\theta\), since \(\Re\mu_\theta^2 = 2\beta f^2 \leq 0\), we get
\[
\text{Im}\mu_\theta(\overline{H}_{0,\theta}(F) - z)\mu_\theta \leq -\beta \delta E + \Re\mu_\theta^2 (E_- + \lambda_0 - z) < 0.
\]
(4.28)
This last estimate with together some usual arguments (see e.g. [6]) complete the proof of the Lemma (4.7).

Introduce the following operator, let \(\theta \in \mathbb{C}, \; |\theta| < \theta_0\) and \(z \in \nu_\theta\)
\[
K_\theta(F, z) = (V_0 + W_\theta(F) - \overline{W}_\theta(F))(\overline{H}_{0,\theta}(F) - z)^{-1}.
\]
(4.29)

**Lemma 4.8.** In the same conditions as in the previous lemma.

(i) \(z \in \nu_\theta \rightarrow K_\theta(F, z)\) is an analytic compact operator valued function.

(ii) For \(z \in \nu_\theta\), \(\text{Im} z > 0\) large enough, \(\|K_\theta(F, z)\| < 1\).

**Proof.** By the Lemma 4.7 (i) follows if we show that \(K_\theta(F, z), z \in \nu_\theta\) are compact operators. Set \(V = V_0 + W_\theta(F) - \overline{W}_\theta(F)\). Notice that \(V\) has compact support in the longitudinal direction and it is a bounded operator.

Introduce the operator \(\tilde{h} = \tilde{h}(F) = H_0 + \tilde{w}(F)\) on \(L^2(\Omega)\) where \(\tilde{w}(F)\) is the multiplication operator by
\[
\tilde{w}(F, s) = \begin{cases} 
F \cos(\eta)s & \text{if } s < 0, \\
0 & \text{if } 0 \leq s \leq s_1 \\
F \cos(\eta - \alpha_0)s & \text{if } s > s_1.
\end{cases}
\]
(4.30)
Then
\[
\overline{H}_{0,\theta}(F) - \tilde{h} = \partial_\theta G_\theta \partial_\theta + R_\theta + \overline{W}_\theta(F) - \tilde{w}(F),
\]
(4.31)
where \(R_\theta, G_\theta\) and \(\overline{W}_\theta(F)\) are defined in the Section 3. Suppose \(|\theta| < \theta_0, 0 < F < \delta E\), this is satisfied under assumptions of the lemma. Then following step by step the proof of the Theorem 3.3 \(\overline{H}_{0,\theta}(F) - \tilde{h}\) is \(\tilde{h}\)-bounded with a relative bound smaller than one. Therefore, to prove (i) we are left to show that for \(z \in \nu_\theta, \Re z \neq 0\), \(V(\tilde{h} - z)^{-1}\) is compact.

Denote \(h_0 = -\partial_\theta^2 \otimes I + I \otimes T_u\) and \(G = g - 1\), we have
\[
V(\tilde{h} - z)^{-1} = V(h_0 - z)^{-1} + V(h_0 - z)^{-1}(\partial_\theta G \partial_\theta - \tilde{w}(F))(\tilde{h} - z)^{-1}
\]
(4.32)
Note that by using again the Herbst’s argument [15], the second term of the r.h.s of (4.32) can be written as
\[ V(h_0 - z)^{-1} \tilde{w}(F)(\hat{h} - z)^{-1} = V s(h_0 - z)^{-1} \frac{\tilde{w}(F)}{s}(\hat{h} - z)^{-1} + V(h_0 - z)^{-1} [s, h_0](h_0 - z)^{-1} \frac{\tilde{w}(F)}{s}(\hat{h} - z)^{-1}. \] (4.33)

In the one hand let \( \chi \) be a \( C^\infty \) characteristic function of \([0, s_1] \) then \( \chi(h_0 - z)^{-1} \) is a compact operator. Indeed,
\[ \chi(h_0 - z)^{-1} = \sum_{n \geq 0} \chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n \]
where \( \lambda_n, n \in \mathbb{N} \) are the eigenvalues of the operator \( T_u \) (transverse modes) and \( p_n, n \in \mathbb{N} \) the associated projectors. We know that \( \chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n \) is compact [22] and for large \( n \),
\[ \|\chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n\| \leq \|(\partial_s^2 - z)^{-1}\| = O\left(\frac{1}{n^2}\right). \] (4.34)
Thus \( \chi(h_0 - z)^{-1} \) is compact since it is a limit of a sequence of compact operators in the norm topology. This holds true for operators \( V(h_0 - z)^{-1} \) and \( V s(h_0 - z)^{-1} \).

On the other hand the function \( G \) has a bounded support in the longitudinal direction then the same arguments as in the proof of the Theorem 3.3 imply that the operator \( \partial_s G \partial_s(\hat{h} - z)^{-1} \) is bounded. By the closed graph theorem \( [s, h_0](h_0 - z)^{-1} = 2\partial_s(h_0 - z)^{-1} \) is also bounded.

Then by (4.32) and (4.33) the statement follows.

The assertion (ii) is a direct consequence of the Lemma 4.7 (ii) and the fact that \( V \) is a bounded operator.

\[ \square \]

**4.1 Proof of the Theorem 4.6.**

Let \( E < 0, |\theta| < \theta_0 \) and \( 0 < F \leq F_0 \). By Lemmas 4.7 and 4.8 then the operator \( \mathbb{I} + K_{\theta}(F, z) \) is invertible for all for \( z \in \nu_\theta \setminus \mathcal{R} \) where \( \mathcal{R} \) is a discrete set. In the bounded operator sense, we have
\[ (H_{\theta}(F) - z)^{-1} = (\hat{H}_{0, \theta}(F) - z)^{-1}(\mathbb{I} + K_{\theta}(F, z))^{-1}. \] (4.35)
This implies that \( \nu_\theta \setminus \mathcal{R} \subset \rho(H_{\theta}(F)) \).

Further let \( \mathcal{O} \) an open subset of \( \nu_\theta \setminus \mathcal{R} \). For \( \varphi \in \mathcal{A}, \) consider the function
\[ z \in \mathcal{O} \to f_{\varphi}(z) = ((H(F) - z)^{-1}\varphi, \varphi). \] (4.36)
For $\theta \in \mathbb{R}$, $|\theta| < \theta_0$, by using the identity $U_\theta^*U_\theta = \mathbb{I}$ in the scalar product of the r.h.s. of (4.36), we have $f_\varphi(z) = ((H_\theta(F) - z)^{-1}\varphi_\theta, \varphi_\theta)$, $\varphi_\theta = U_\theta \varphi$. Then together with the Theorem 3.3, it holds

$$f_\varphi(z) = ((H_\theta(F) - z)^{-1}\varphi_\theta, \varphi_\theta).$$

(4.37)

in the disk $\{\theta \in \mathbb{C}, |\theta| < \theta_0\}$.

Fix $\theta = i\beta$, $0 < |\beta| < \theta_0$ then $f_\varphi$ has an meromorphic extension in $\nu_\theta$ given by

$$f_\varphi(z) = ((\tilde{H}_{0,\theta}(F) - z)^{-1}(I + K_\theta(F, z))^{-1}\varphi_\theta, \varphi_\theta).$$

The poles of $f_\varphi$ are locally $\theta$-independent. From [18] and standard arguments, these poles are the set of $z \in \nu_\theta$ such that the equation $K_\theta(F, z)\psi = -\psi$ has non-zero solution in $L^2(\Omega)$. In view of (4.35) they are the discrete eigenvalues of the operator $H_\theta(F)$.

5 Resonances.

Theorem 5.9. Let $E_0$ be an eigenvalue of $H$ of multiplicity $j$. There exists $0 < F'_0 \leq F_0$ such that for $0 < F \leq F'_0$, the operator $H_\theta(F)$, $0 < |\theta| < \theta_0$ has $j$ eigenvalues near $E_0$ converging to $E_0$ as $F \to 0$.

We need first to show the following result. Let $\Im z \neq 0$, set $K(z) = V_0(H_0 - z)^{-1}$, they are compact operators. Note that formally $K(z) = K_\theta(F = 0, z)$. We have

Lemma 5.10. Let $E < 0$, $\theta = i\beta$, $0 < \beta < \theta_0$. Let $\kappa$ be a compact subset of $\tilde{\nu}_\theta \cap \rho(H_0)$, $\chi = \chi(s) \in C_0^\infty(\mathbb{R}^+)$. Then

(i) $\lim_{F \to 0} \| (\tilde{H}_{0,\theta}(F) - z)^{-1}\psi - (H_0 - z)^{-1}\psi \| = 0$, $\psi \in L^2(\Omega)$,

(ii) $\lim_{F \to 0} \| \chi(\tilde{H}_{0,\theta}(F) - z)^{-1} - \chi(H_0 - z)^{-1} \| = 0$,

(iii) $\lim_{F \to 0} \| K_\theta(F, z) - K(\zeta) \| = 0$,

uniformly in $z \in \kappa$.

Proof. By using the arguments of the appendix the operator $H_0 = T_s + T_u$ on $L^2(\Omega)$ has a core given by (7.51) i.e. for $z \in \rho(H_0)$, $\mathcal{C} = (H_0 - z)\mathcal{C}$ is dense in $L^2(\Omega)$. Let $0 < F \leq F_0$ and $z \in \kappa$. For all $\varphi \in \mathcal{C}$, set $\psi = (H_0 - z)\varphi$. The resolvent equation implies,

$$(\tilde{H}_{0,\theta}(F) - z)^{-1}\psi - (H_0 - z)^{-1}\psi = (\tilde{H}_{0,\theta}(F) - z)^{-1}(T_s - T_{s,\theta} - \tilde{W}_\theta(F))\varphi.$$  

(5.38)
Clearly \( \lim_{F \to 0} \| \tilde{W}_\theta(F) \varphi \| = 0 \). On the other hand we have
\[
\| (T_s - T_{s,\theta}) \varphi \| \leq \| \partial_s G_\theta \partial_s \varphi \| + \| R_\theta \varphi \|
\]
Where \( G_\theta \) and \( R_\theta \) are defined as in the Section 3. Evidently \( \lim_{F \to 0} \| R_\theta \varphi \| = 0 \).
Since \( \text{supp}(G_\theta) = \left[ \frac{E}{F \cos(\eta)} + \frac{E_s}{F \cos(\eta)} \right] \) then for such a \( \varphi \), \( \lim_{F \to 0} \| \partial_s G_\theta \partial_s \varphi \| = 0 \). So that \( \lim_{F \to 0} \| (T_s - T_{s,\theta}) \varphi \| = 0 \).
In view of the Lemma 4.7 \( (\tilde{H}_{0,\theta}(F) - z)^{-1} \) has a norm which is uniformly bounded w.r.t. \( F \). Thus (i) is proved on \( C' \), by standard arguments then the strong convergence follows.
Let us show (ii). For \( z \in \kappa \) then
\[
\chi(\tilde{H}_{0,\theta}(F) - z)^{-1} - \chi(H_0 - z)^{-1} = \chi(\tilde{H}_{0,\theta}(F) - z)^{-1} Q_\theta(F)
\] 
(5.39)
where \( Q_\theta(F) = (T_s - T_{s,\theta} - \tilde{W}_\theta(F))(H_0 - z)^{-1} \). On \( \text{supp}(\chi) \), \( f = 0 \) then the following resolvent identity holds,
\[
\chi(\tilde{H}_{0,\theta}(F) - z)^{-1} = (H_0 - z)^{-1} \chi + (H_0 - z)^{-1} ([T_s, \chi] - \chi \tilde{W}(F) - H_{0,\theta}(F) - z)^{-1}.
\] 
(5.40)
In view of (5.39) and (5.40) we have to consider two terms. First
\[
t_1(F) = (H_0 - z)^{-1} \chi Q_\theta(F) = (H_0 - z)^{-1} \chi \tilde{W}(F))(H_0 - z)^{-1}
\]
which clearly converges in the norm sense to \( 0_{B(L^2(\Omega))} \) as \( F \to 0 \) uniformly in \( z \in \kappa \) and
\[
t_2(F) = (H_0 - z)^{-1} ([T_s, \chi] - \chi \tilde{W}(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} Q_\theta(F).
\]
Let \( \chi \) be the characteristic function of \( \text{supp}(\chi) \). We know that the operator \( (H_0 - z)^{-1} \chi \) is compact (see e.g. the proof of the Lemma 4.8) then to prove that \( t_2(F) \) converges in the norm sense to \( 0_{B(L^2(\Omega))} \) as \( F \to 0 \) uniformly in \( z \in \kappa \), it is sufficient to show that \( ([T_s, \chi] - \chi \tilde{W}(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} Q_\theta(F) \) converges strongly to \( 0_{B(L^2(\Omega))} \) as \( F \to 0 \) uniformly in \( z \in \kappa \). But considering the proof of (i) it is then sufficient to prove that the operator \( ([T_s, \chi] - \chi \tilde{W}(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} \) is bounded operator and has a norm which is uniformly bounded w.r.t. \( F \) if \( F \) is small and \( z \in \kappa \).
Evidently by the Lemma 4.7 this is true for the operator \( \chi \tilde{W}(F)(\tilde{H}_{0,\theta}(F) - z)^{-1} \).
We have on \( L^2(\Omega) \),
\[
[T_s, \chi](\tilde{H}_{0,\theta}(F) - z)^{-1} = -(\chi' g \partial_s + \partial_s (g \chi')) (\tilde{H}_{0,\theta}(F) - z)^{-1} = -(2 \chi' g \partial_s + (g \chi'))(\tilde{H}_{0,\theta}(F) - z)^{-1}.
\]
Since the functions \( g \) and \( (g \chi')' \) are bounded and do not dependent on \( F \), we only have to consider the operator \( \chi' g^{1/2} \partial_s (\tilde{H}_{0,\theta}(F) - z)^{-1} \).
Let \( \varphi \in L^2(\Omega) \), \( \| \varphi \| = 1 \) set \( \psi = (\tilde{H}_{0,\theta}(F) - z)^{-1} \varphi \). Integrating by part, we have
\[
\| \chi' g^{1/2} \partial_s \psi \|^2 = (\partial_s (\chi')^2 g \partial_s \psi, \psi) \leq (\partial_s (\chi')^2 g \partial_s \psi, \psi) + (\chi'T_u \chi' \psi, \psi).
\]

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By using standard commutation relations, \(\partial_s(\chi')^2 g \partial_s = \frac{1}{2}((\chi')^2 \partial_s g \partial_s + \partial_s g \partial_s (\chi')^2 + \partial_s(g\partial_s(\chi')^2))\). Since the field \(f = 0\) on \(\text{supp}(\chi')\) we get,

\[
\|\chi'g^{1/2} \partial_s \psi\|^2 \leq |\mathcal{R}((\tilde{H}_{0,\theta}(F) - z)\psi, (\chi')^2 \psi)| \leq \|(\chi')^2\|_\infty \|(\tilde{H}_{0,\theta}(F) - z)^{-1}\| + \|(\partial_s(g\partial_s(\chi')^2))\|_\infty + \|(\chi')^2(\tilde{W}_\theta(F) - z)\| \|(\tilde{H}_{0,\theta}(F) - z)^{-1}\|^2. \tag{5.41}
\]

The Lemma 4.7 implies that the l.h.s. of the last inequality is bounded uniformly w.r.t. \(F\) if \(F\) is small and \(z \in \kappa\).

Note that once the strong convergence on \(\mathcal{C}'\) is proved, the strong convergence on \(L^2(\Omega)\) follows by using the fact that

\[
([T_s, \chi] - \chi \tilde{W}(F))(\tilde{H}_{0,\theta}(F) - z)^{-1}Q_\theta(F) = ([T_s, \chi] - \chi \tilde{W}(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} - (H_0 - z)^{-1}
\]

is uniformly bounded w.r.t \(F\) for small and \(z \in \kappa\). Hence the proof of (ii) is done.

We have

\[
K_\theta(F, z) - K(z) = (V_0 + W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} - (H_0 - z)^{-1}
\]

\[
= V_0((\tilde{H}_{0,\theta}(F) - z)^{-1} - (H_0 - z)^{-1}) - (W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{0,\theta}(F) - z)^{-1}.
\]

Clearly in the norm sense \((W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{0,\theta}(F) - z)^{-1} \to 0_{\mathcal{B}(L^2(\Omega))}\) as \(F \to 0\), uniformly w.r.t. \(z \in \kappa\). By applying (ii) this is also true for \(V_0((\tilde{H}_{0,\theta}(F) - z)^{-1} - (H_0 - z)^{-1})\) as \(F \to 0\). Then

\[
\lim_{F \to 0} \|K_\theta(F, z) - K(z)\| = 0.
\]

uniformly w.r.t. \(z \in \kappa\). \(\Box\)

5.1 Proof of the Theorem 5.9

Let \(E_0\) be an eigenvalue of the operator \(H\). Recall that \(\lambda_0 = \inf \sigma_{\text{ess}}(H)\). Choose the reference energy, \(E\) so that \(E_- = E_0 - \lambda_0 = E - \delta E\) and \(\delta E = \frac{|E|}{2}\).

Let \(0 < |\theta| < \theta_0, \exists \theta = \beta > 0\). Suppose \(R > 0\) is such that the complex disk, \(\mathcal{D} = \{z \in \mathbb{C}, |z - E_0| \leq R\} \subset \tilde{\nu}_\theta\) and \(\mathcal{D} \cap \sigma(H) = \{E_0\}\). First, we show that for \(F\) small enough, \(z \in \partial \mathcal{D}\), \((H_\theta(F) - z)^{-1}\) exists. Clearly \(H\) has no spectrum in \(\partial \mathcal{D}\) then in view of the identity

\[
(H - z)^{-1} = (H_0 - z)^{-1}(\mathbb{I} + K(z))^{-1}, \quad z \in \rho(H) \cap \rho(H_0),
\]

the operator \((\mathbb{I} + K(z))^{-1}\) is well defined on \(\partial \mathcal{D}\) and its norm is uniformly bounded w.r.t. \(z \in \partial \mathcal{D}\).
We have
\[ \mathbb{I} + K_\theta(F, z) = \left( \mathbb{I} + (K_\theta(F, z) - K(z))(\mathbb{I} + K(z))^{-1} \right)(\mathbb{I} + K(z)). \] (5.42)

Since by the Lemma 5.10 (iii), \( \|K_\theta(F, z) - K(z)\| \to 0 \) as \( F \to 0 \) uniformly for \( z \in \partial \mathcal{D} \), then for \( F \) small enough and \( z \in \partial \mathcal{D} \)
\[ \|(\mathbb{I} + K(z))^{-1}(K_\theta(F, z) - K(z))\| < 1. \]
and \( \mathbb{I} + (K_\theta(F, z) - K(z))(\mathbb{I} + K(z))^{-1} \) is invertible. Hence for \( F \) small enough \( \mathbb{I} + K_\theta(F, z) \) is invertible for \( z \in \partial \mathcal{D} \) and from (4.35), \( (H_\theta(F) - z)^{-1} \) is well defined on the contour \( \partial \mathcal{D} \). We define the spectral projector associated with \( H_\theta(F) \),
\[ P_\theta(F) = \frac{1}{2i\pi} \oint_{\partial \mathcal{D}} (H_\theta(F) - z)^{-1} dz. \] (5.43)
The algebraic multiplicity of the eigenvalues of \( H_\theta(F) \) inside \( \mathcal{D} \) is just the dimension of \( P_\theta(F) \). In the same way let
\[ P = \frac{1}{2i\pi} \oint_{\partial \mathcal{D}} (H - z)^{-1} dz \]
be the spectral projector associated with \( H \). Thus to prove the first part of the theorem, it is sufficient to show that for \( F \) small enough, \( \|P_\theta(F) - P\| < 1 \). We have
\[ (H_\theta(F) - z)^{-1} = (\tilde{H}_{0,\theta}(F) - z)^{-1}(\mathbb{I} + K_\theta(F, z))^{-1} \]
\[ = (\tilde{H}_{0,\theta}(F) - z)^{-1} - (\tilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F, z)(\mathbb{I} + K_\theta(F, z))^{-1} \] (5.44)
and similarly
\[ (H - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1}K(z)(\mathbb{I} + K(z))^{-1}. \]
By the Lemma 4.14, the operator \( \tilde{H}_{0,\theta}(F) \) has no spectrum inside \( \mathcal{D} \) this is also true for \( H_0 \) then \( \oint_{\partial \mathcal{D}} (H_0 - z)^{-1} dz = \oint_{\partial \mathcal{D}} (H_{0,\theta}(F) - z)^{-1} dz = 0 \). Hence, we get
\[ P_\theta(F) - P = \frac{1}{2i\pi} \oint_{\partial \mathcal{D}} ((H_0 - z)^{-1}K(z)(\mathbb{I} + K(z))^{-1} - \tilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F, z)(\mathbb{I} + K_\theta(F, z))^{-1} dz. \] (5.45)
Set \( \Delta K = K(z) - K_\theta(F, z) \), \( \Delta R = (H_0 - z)^{-1} - (\tilde{H}_{0,\theta}(F) - z)^{-1} \), we have the following identity,
\[ (H_0 - z)^{-1}K(z)(\mathbb{I} + K(z))^{-1} - (\tilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F, z)(\mathbb{I} + K_\theta(F, z))^{-1} = \Delta R K(z)(\mathbb{I} + K(z))^{-1} + (\tilde{H}_{0,\theta}(F) - z)^{-1}(\mathbb{I} + K_\theta(F, z))^{-1} \Delta K(\mathbb{I} + K(z))^{-1}. \]
By applying the Lemma \[5.10\] then in the norm operator sense $\Delta R K(z) \to 0_{B(L^2(\Omega))}$ and $\Delta K \to 0_{B(L^2(\Omega))}$ as $F \to 0$ uniformly in $z \in \partial \mathcal{D}$. Moreover the operators $(I + K(z))^{-1}, (I + K_\theta(F, z))^{-1}$ and $(\tilde{H}_{0, \theta}(F) - z)^{-1}$ are uniformly bounded w.r.t. $z \in \partial \mathcal{D}$ and $F$ for $F$ small. This implies

$$\lim_{F \to 0} \|P_\theta(F) - P\| = 0.$$  \hspace{1cm} (5.46)

The second part of the theorem follows from the fact that the radius of $\mathcal{D}$ can be chosen arbitrarily small, this shows that the eigenvalues of $H_\theta(F)$ inside $\mathcal{D}$ converge to $E_0$ as $F \to 0$.

\[\square\]

6 Exponential estimates

In this section we show that the width of resonances given in the Theorem \[5.9\] decays exponentially when the intensity of the field $F \to 0$. Let $E_0$ be an eigenvalue of $H$. For $0 < F \leq F_0''$, let $Z_0$ be an eigenvalue of the operator $H_\theta(F)$ in a small complex neighborhood of $E_0$ given by the Theorem \[5.9\]. Then

**Theorem 6.11.** Under conditions of the Theorem \[5.9\] there exists $0 < F_0'' \leq F_0'$ and two constants $0 < c_1, c_2$ such that for $0 < F \leq F_0''$,

$$|\Im Z_0| \leq c_1 e^{-c_2 F}$$

First we need to prove the following lemma.

**Lemma 6.12.** Let $\varphi_0$ be an eigenvector of $H$ associated with the eigenvalue $E_0$ i.e. $H\varphi_0 = E_0 \varphi_0$. Then there exist $a > 0$ such that $e^{i a s} \varphi_0 \in L^2(\Omega)$.

**Proof.** Here we use the standard Combes-Thomas argument (see e.g. \[23\]). Consider the following unitary transformation on $L^2(\Omega)$. Let $a \in \mathbb{R}$, for all $\varphi \in L^2(\Omega)$, set

$$W_a(\varphi)(s) = e^{-i a s} \varphi(s).$$

We have

$$H_a = W_a H W_a^{-1} = H - ia(\partial_s g + g \partial_s) + a^2.$$ 

The family of operators $\{H_a, a \in \mathbb{C}\}$ is an entire family of type A. Indeed it is easy to check that $D(H_a) = D(H)$, $\forall a \in \mathbb{C}$. This follows from the fact that $\forall z \in \mathbb{C}, \Im z \neq 0$,

$$\|g^{1/2} \partial_s (H - z)^{-1}\| \leq \|(H - z)^{-1}\| + (\|V_0\|_\infty + |z|)(H - z)^{-1}\|^2.$$ 

Thus, for a suitable choice of $z$, the r.h.s of this last inequality is arbitrarily small. This implies $\partial_s g + g \partial_s$ is $H$-bounded with zero relative bound.
Further let $\Re a = 0$. Denote by $H_{0,a} = H_0 - ia(\partial_s g + g\partial_s) + a^2$. For $\varphi \in D(H)$, we have
\[\Re(H_{0,a}\varphi, \varphi) = H_0 - (\Im a)^2 \geq \lambda_0 - (\Im a)^2.\] (6.47)
Then for $z \notin \Sigma_a = \{z \in \mathbb{C}, \Re z \geq \lambda_0 - (\Im a)^2\}$, $\|(H_{0,a} - z)^{-1}\| \leq \text{dist}^{-1}(z, \Sigma_a)$ [19].

Thus if we show that $V_0(H_{0,a} - z)^{-1}$ is compact, then by using usual arguments of the perturbation theory (see e.g. the proof of the Theorem [4,6]) the operator $H_a$ has only discrete spectrum in $\mathbb{C} \setminus \Sigma_a$ this will imply that the essential spectrum of $H_a$, $\sigma_{ess}(H_a) \subset \Sigma_a$.

Let $h_0 = -\partial^2_s \otimes I + I \otimes T_u$ be the operator introduced in the proof of the Lemma 4.8 and $G = g - 1$ we have
\[V_0(H_{0,a} - z)^{-1} = V_0(h_0 - z)^{-1} - V_0(h_0 - z)^{-1}(\partial_s G\partial_s + ia(\partial_s g + g\partial_s) - a^2)(H_{0,a} - z)^{-1}.\]

We know that $V_0(h_0 - z)^{-1}$ is compact (see the proof of the Lemma 4.8), so we are left to show that $(\partial_s G\partial_s + ia(\partial_s g + g\partial_s) - a^2)(H_{0,a} - z)^{-1}$ is a bounded operator. We have
\[\partial_s G\partial_s(H_{0,a} - z)^{-1} = \partial_s G\partial_s(H_0 - z)^{-1} + \partial_s G\partial_s(H_0 - z)^{-1}(ia(\partial_s g + g\partial_s) - a^2)(H_{0,a} - z)^{-1}.\]

since $D(H_0) \subset \mathcal{H}^2_{loc}(\bar{\Omega}) \cap \mathcal{H}^1(\Omega)$, by the closed graph theorem $\partial_s G\partial_s(H_0 - z)^{-1}$ is bounded. By using similar arguments as in the proof of the Lemma 5.10 $\text{(ii)}$, $(ia(\partial_s g + g\partial_s) - a^2)(H_{0,a} - z)^{-1}$ is also a bounded operator.

We now conclude the proof of the lemma by using usual arguments [23]. If $(\Im a)^2 < \lambda_0 - E_0$, $E_0$ remains an discrete eigenvalue of $H_a$ and $e^{3\Im a}\varphi \in L^2(\Omega)$. \[\square\]

### 6.1 Proof of the Theorem 6.11

Let $\theta = i\beta$, $0 < \beta < \theta_0$, $0 < F < F_0$. As in previous section $P; P_\theta = P_\theta(F)$ are the spectral projectors of $H, H_\theta$ associated respectively to the eigenvalue $E_0, Z_0$. Evidently $P_\varphi_0 = \varphi_0$. We have

\[(Z_0 - E_0)(P_\theta \varphi_0, P \varphi_0) = ((H_\theta - H)P_\theta \varphi_0, P \varphi_0) = ((\theta F \cos(\eta)f + T_s \theta - T_s)P_\theta \varphi_0, P \varphi_0).\] (6.48)

By using (5.46), for $F$ small enough, the l.h.s. of (6.48) is estimated as,
\[|(P_\theta \varphi_0, P \varphi_0)| \geq \frac{1}{2}.\]

From the Lemma 6.12, for $F$ small enough, the first term of the r.h.s. of (6.48) satisfies
\[|(\theta F \cos(\eta)fP_\theta \varphi_0, P \varphi_0)| \leq |\theta||\Phi \varphi_0|| = O(e^{-\tilde{F}})\]

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for some constant $c > 0$. Set $\Delta T = T_{s, \theta} - T_s$. Let $\chi$ be a characteristic function of $\text{supp}(f')$. Then (see e.g. (3.16) and (3.18)),

$$
|\langle \Delta T P_{\theta \varphi_0}, P_{\varphi_0} \rangle| = |\langle \Delta T P_{\theta \varphi_0}, \chi P_{\varphi_0} \rangle| \leq ||\chi \varphi_0|| ||\Delta T P_{\theta \varphi_0}||.
$$

Since $||\chi \varphi_0|| = O(e^{-\frac{r}{\gamma}})$ for $F$ small enough, to prove the theorem we need to show that for $F$ is small enough $||\Delta T P_{\theta \varphi_0}||$ and then by (5.43), that $||\Delta T(H_{\theta}(F) - z)^{-1}||$, $z \in \partial \mathcal{D}$ is uniformly bounded w.r.t. $F$.

Note that following the proof of the Theorem 5.9, (see e.g. (5.42) and (5.44)) then for $F$ small enough, the norm $||\Delta T(H_{\theta}(F) - z)^{-1}||$, $z \in \partial \mathcal{D}$ is uniformly bounded in $F$. Evidently this also holds for $||\Delta T(H - z)^{-1}||$, $z \in \partial \mathcal{D}$. The second resolvent equation implies for $F$ small and $z \in \partial \mathcal{D}$,

$$
\Delta T(H_{\theta}(F) - z)^{-1} = \Delta T(H - z)^{-1} - \Delta T(H - z)^{-1}(\Delta T + W_{\theta}(F))(H_{\theta}(F) - z)^{-1}.
$$

By the closed graph theorem the operator $\Delta T(H - z)^{-1}$, $z \in \mathcal{D} \subset \rho(H)$ is bounded and if $F$ is assumed small enough $||\Delta T(H - z)^{-1}|| < \frac{1}{2}$ uniformly in $z \in \partial \mathcal{D}$ (see e.g. the proof of the Theorem 3.3). In view of

$$
\Delta T(H - z)^{-1}W_{\theta}(F)(H_{\theta}(F) - z)^{-1} = \Delta T(Fs + i)(H - z)^{-1}\frac{W_{\theta}(F)}{F_{s+1}}(H_{\theta}(F) - z)^{-1} + F\Delta T(H - z)^{-1}(g\partial_s + \partial_s g)(H - z)^{-1}\frac{W_{\theta}(F)}{F_{s+1}}(H_{\theta}(F) - z)^{-1},
$$

the same arguments already used in the Section 3, then imply that there exists a constant $C > 0$ such that for $F$ small enough $||\Delta T(H - z)^{-1}W_{\theta}(F)(H_{\theta}(F) - z)^{-1}|| \leq C$ for $z \in \partial \mathcal{D}$. Therefore, by (6.49), we get for $z \in \partial \mathcal{D}$,

$$
||\Delta T(H_{\theta}(F) - z)^{-1}|| (1 - ||\Delta T(H - z)^{-1}||) \leq ||\Delta T(H - z)^{-1}|| + ||\Delta T(H - z)^{-1}W_{\theta}(F)(H_{\theta}(F) - z)^{-1}||,
$$

hence we get

$$
||\Delta T(H_{\theta}(F) - z)^{-1}|| \leq 1 + 2C.
$$

\hfill \Box

\section{Appendix: Self-adjointness}

In this section we prove the Theorem 2.1. Our proof is mainly based on the commutator theory \cite{22,24}. First we note that it is sufficient to show the theorem for the operator $h = h(F) = H_{\theta} + w(F)$ defined on $L^2(\Omega)$ where $w(F)$ is defined in
Choose \(a, b \in \mathbb{R}^+\) such that \(w(F, s) + as^2 + b > 1\) and consider the positive symmetric operator in \(L^2(\Omega)\),

\[
N = H_0 + w(F) + 2as^2 + b.
\]

Then \(N\) admits a self-adjoint extension since it is associated with a positive quadratic form, we denote its self-adjoint extension by the same symbol. Moreover \(N\) has compact resolvent and then only discrete spectrum (see section 7.1 below). So \(N\) is essentially self-adjoint on

\[
C = \{\varphi = \psi|_\Omega : \psi \in S(\mathbb{R}^2), \psi(s,0) = \psi(s,d) = 0 \text{ for all } s \in \mathbb{R}\}\quad (7.51)
\]

where \(S(\mathbb{R}^2)\) denotes the Schwartz class. In fact \(C\) contains a complete set of eigenvectors of \(N\). Indeed some standard arguments (see e.g. [3, 12, 23]) show that the corresponding eigenfunctions and their derivatives are smooth on \(\bar{\Omega}\) and super-exponentially decay in the longitudinal direction. From [22, XIII.12] we have to check that there exist \(c, d > 0\) such that for all \(\varphi \in C, \|\varphi\| = 1\),

\[
c\|N\varphi\| \geq \|h\varphi\| \quad (7.52)
\]

and

\[
d\|N^{\frac{1}{2}}\varphi\|^2 \geq |(h\varphi, N\varphi) - (N\varphi, h\varphi)|. \quad (7.53)
\]

In the quadratic forms sense on \(C\),

\[
N^2 = (h + b)^2 + 4asNs + [[h, s], s]. \quad (7.54)
\]

But in the form sense on \(C, [[h, s], s] = -2g\) and \(g\) is bounded function. Therefore,

\[
\|N\varphi\| + 2\|g\|_\infty \geq \|(h + b)\varphi\|
\]

and then since \(N \geq 1\) this last inequality implies \((7.52)\). Similarly,

\[
\pm i[h, N] = \pm i[h - N, N] = \pm i2a[s^2, T_s] = \mp 4a(\partial_s g + sg\partial_s),
\]

this gives that for all \(\varphi \in C, \|\varphi\| = 1\),

\[
|(h\varphi, N\varphi) - (N\varphi, h\varphi)| \leq 2a(\|g^{\frac{1}{2}}\partial_s \varphi\|^2 + \|sg^{\frac{1}{2}}\varphi\|^2). \quad (7.55)
\]

Clearly we have \(N \geq T_s + as^2\) on \(C\). Then from \((7.55)\) there exists a constant \(d > 0\) such that

\[
|(h\varphi, N\varphi) - (N\varphi, h\varphi)| \leq d(N\varphi, \varphi)
\]

proving \((7.53)\).
We now show (ii). Let \( E \in \mathbb{R} \). We denote by \( \tilde{E}_1 \) the first eigenvalue of the operator \( T_u + F \sin(\eta)u \) and \( \tilde{\chi}_1 \) the associated normalized eigenvector,
\[
(T_u + F \sin(\eta)u) \tilde{\chi}_1(u) = \tilde{E}_1 \tilde{\chi}_1(u).
\] (7.56)

Set \( \lambda = E - \tilde{E}_1 \) and \( \varphi \) be the solution of the Airy equation
\[
-\varphi''(s) + F \cos(\eta) \varphi(s) = \lambda \varphi(s) \quad \lambda \in \mathbb{R}.
\] (7.57)

It is known (see e.g. [1]) that \( \varphi(s) = (\lambda - F \cos(\eta)s)^{-1/4} e^{-i \frac{2}{3 F \cos(\eta)}} (\lambda - F \cos(\eta)s)^{3/2} + o((\lambda - F \cos(\eta)s)^{1/4}) \) as \( s \to -\infty \).

Let \( \xi \) be a \( C^\infty \) characteristic function of \((-1, 1)\) and \( s \in \mathbb{R} \to \xi_n(s) = \xi(\frac{s}{n^\alpha} + n), \) \( \frac{1}{2} < \alpha < 1, n \in \mathbb{N}^* \). Set
\[
\psi_n = \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|} \quad \text{where} \quad \tilde{\varphi}_n(s, u) = \tilde{\chi}_1(u) \varphi(s) \xi_n(s),
\]
then for \( n \) large enough, \( \|\tilde{\varphi}_n\| = \|\varphi_n\| \geq c n^{\alpha/2 - 1/4} \) for some constant \( c > 0 \). Since \( g = 1 \) if \( n \) is large, we have
\[
(H(F) - E)\psi_n = (-2 \tilde{\chi}_1(u) \varphi'(s) \xi'_n(s) - \chi_1(u) \varphi(s) \xi''_n(s)) \frac{1}{\|\psi_n\|}
\]
and then
\[
\|(H(F) - E)\psi_n\|_{L^2(\Omega)} \leq \frac{1}{\|\psi_n\|} \left( 2 \|\varphi'_n\|_{L^2(\mathbb{R})} + \|\varphi''_n\|_{L^2(\mathbb{R})} \right). \quad (7.58)
\]

For \( n \) large enough \( \|\varphi'_n\|_{L^2(\mathbb{R})}^2 = o(n^{-\alpha/2 + 1/4}) \) and \( \|\varphi''_n\|_{L^2(\mathbb{R})} = o(n^{-\alpha/2 - 1/4}) \).

Thus,
\[
\lim_{n \to \infty} \|(H(F) - E)\psi_n\|_{L^2(\Omega)} = 0.
\]

This completes the proof.

\[ \square \]

### 7.1 The operator \((N + 1)^{-1}\)

Consider first the positive self-adjoint operator on \( L^2(\Omega) \)
\[
N_0 = (-\partial^2_s + v(s)) \otimes \mathbb{I} + \mathbb{I} \otimes T_u
\]
where \( v(s) = w(F, s) + 2as^2 + b \) and \( w \) is defined in (3.17). It is known that the operator \(-\partial^2_s + v(s)\) is essentially self-adjoint on \( L^2(\mathbb{R}) \) and has a compact resolvent [22, 23]. By the min-max principle we can verify that the eigenvalues of this operator satisfy, there exists \( c_1, c_2 > 0 \) such that for large \( n \in \mathbb{N} \)
\[
c_1 n \leq \epsilon_n \leq c_2 n.
\]

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Then \((N_0 + 1)^{-1}\) is an Hilbert-Schmidt operator. By using the second resolvent equation we have

\[
(N + 1)^{-1} = (N_0 + 1)^{-1} + (N_0 + 1)^{-1} \partial_s G \partial_s (N + 1)^{-1}
\]

where \(G\) is defined in the proof of the Lemma 4.8. Therefore, the statement follows if we show that \(\partial_s G \partial_s (N + 1)^{-1}\) is a bounded operator.

We have

\[
\partial_s G \partial_s (N + 1)^{-1} = \partial_s G \partial_s (H_0 + 1)^{-1} - \partial_s G \partial_s (H_0 + 1)^{-1} v(N + 1)^{-1}.
\]

Since \(D(H_0) \subset H_{loc}^2(\Omega) \cap H_0^1(\Omega)\), by the closed graph theorem \(\partial_s G \partial_s (H_0 + 1)^{-1}\) and \(\partial_s G \partial_s (H_0 + 1)^{-1}\) are bounded. Standard commutation relations then imply,

\[
\partial_s G \partial_s (H_0 + 1)^{-1} v(N + 1)^{-1} = \partial_s G \partial_s (s + i)(H_0 + 1)^{-1} \frac{v}{s+i}(N + 1)^{-1} + \partial_s G \partial_s (H_0 + 1)^{-1} 2\partial_s (H_0 + 1)^{-1} \frac{v}{s+i}(N + 1)^{-1}.
\]

We know that the domain \(D(N) \subset D(|v|^{1/2})\) so \(\frac{v}{s+i}(N + 1)^{-1}\) is bounded, then it follows by using the same arguments as above that \(\partial_s G \partial_s (H_0 + 1)^{-1} v(N + 1)^{-1}\) is also bounded.

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