Symmetry as a sufficient condition for a finite flex

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Abstract
We show that if the joints of a bar and joint framework \((G, p)\) are positioned as ‘generically’ as possible subject to given symmetry constraints and \((G, p)\) possesses a ‘fully-symmetric’ infinitesimal flex (i.e., the velocity vectors of the infinitesimal flex remain unaltered under all symmetry operations of \((G, p)\)), then \((G, p)\) also possesses a finite flex which preserves the symmetry of \((G, p)\) throughout the path. This and other related results are obtained by symmetrizing techniques described by L. Asimov and B. Roth in their paper ‘The Rigidity Of Graphs’ from 1978 and by using the fact that the rigidity matrix of a symmetric framework can be transformed into a block-diagonalized form by means of group representation theory. The finite flexes that can be detected with these symmetry-based methods can in general not be found with the analogous non-symmetric methods.

1 Introduction
A bar and joint framework is said to be ‘rigid’ if, loosely speaking, it cannot be deformed continuously into another non-congruent framework while keeping the lengths of all bars fixed. Otherwise, the framework is said to be ‘flexible’ [1, 16, 17, 39]. The identification of flexes (sometimes also called ‘finite flexes’ or ‘mechanisms’) in frameworks is, in general, a very difficult problem. For frameworks whose joints lie in ‘generic’ positions, however, L. Asimov and B. Roth showed in 1978 that ‘rigidity’ is equivalent to ‘infinitesimal rigidity’ (see [1]). So, for ‘generic’ frameworks, finite flexes can easily be detected, since the infinitesimal rigidity properties of a framework are completely described by the linear algebra of its rigidity matrix [18, 38, 39].

While the result of L. Asimov and B. Roth applies to ‘almost all’ realizations of a given graph, it does, in general, not apply to frameworks that possess non-trivial symmetries, since the joints of a symmetric framework are typically forced to lie in special ‘non-generic’ positions [28, 29].

In this paper, we establish symmetric versions of the theorem of Asimov

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and Roth which allow us to detect finite flexes in symmetric frameworks that are as generic as possible subject to the given symmetry constraints. These results are based on the fact that the rigidity matrix of a symmetric framework \((G, p)\) can be transformed into a block-diagonalized form using techniques from group representation theory \([23, 27]\). In this block-diagonalization of the rigidity matrix of \((G, p)\), each submatrix block corresponds to an irreducible representation of the point group \(S\) of \((G, p)\), so that the (infinitesimal) rigidity analysis of \((G, p)\) can be broken up into independent subproblems, where each subproblem considers the relationship between external forces on the joints and resulting internal distortions in the bars of \((G, p)\) that share certain symmetry properties. In particular, the submatrix block that corresponds to the trivial irreducible representation of \(S\) describes the relationship between external displacement vectors on the joints and internal distortion vectors in the bars of \((G, p)\) that are ‘fully-symmetric’, i.e., unchanged under all symmetry operations in \(S\). By adapting the techniques described by L. Asimov and B. Roth in \([1]\) and applying them to the submatrix block corresponding to the trivial irreducible representation of \(S\), rather than the entire rigidity matrix, we show that the existence of a ‘fully-symmetric’ infinitesimal flex of \((G, p)\) also implies the existence of a finite flex of \((G, p)\) which preserves the symmetry of \((G, p)\) throughout the path, provided that \((G, p)\) is as generic as possible subject to the given symmetry constraints. As a corollary of this result, one obtains the Proposition 1 stated (but not proven) in \([19]\) (see also \([22]\)). By considering submatrix blocks that correspond to other irreducible representations of \(S\), finite flexes of \((G, p)\) that preserve some, but not all of the symmetries of \((G, p)\) can also be detected.

In order to apply our symmetry-adapted versions of the theorem of Asimov and Roth to a given framework \((G, p)\), we need to detect infinitesimal flexes that possess certain symmetry properties (in particular, ‘fully-symmetric’ infinitesimal flexes) in \((G, p)\). In many cases, this can be done with very little computational effort by means of the Fowler-Guest symmetry-extended version of Maxwell’s rule described in \([15]\) (see also \([10, 27, 28]\)). An alternate way of finding ‘fully-symmetric’ infinitesimal flexes, as well as ‘fully-symmetric’ self-stresses, in symmetric frameworks will be presented in \([31]\).

We note that there exist a number of famous and interesting examples of symmetric frameworks that can be shown to be flexible with our symmetry-based methods. These include the frameworks examined in \([4, 5, 8, 19, 28, 34]\), for example. For each of these classes of frameworks, our new approach provides a much simpler proof for the existence of a finite symmetry-preserving flex than previous methods. In the final section of this paper, we demonstrate the efficiency of our results by showing the flexibility of two of the three types of ‘Bricard octhedra’ \([5, 33]\). It is shown in \([31]\) that new finite flexes can also be detected with our methods.
2 Rigidity theoretic definitions and preliminaries

2.1 Rigidity

All graphs considered in this paper are finite graphs without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$.

A framework in $\mathbb{R}^d$ is a pair $(G, p)$, where $G$ is a graph and $p : V(G) \to \mathbb{R}^d$ is a map with the property that $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$. We also say that $(G, p)$ is a $d$-dimensional realization of the underlying graph $G$ [17, 18, 38, 39]. An ordered pair $(v, p(v))$, where $v \in V(G)$, is a joint of $(G, p)$, and an unordered pair $\{(u, p(u)), (v, p(v))\}$ of joints, where $\{u, v\} \in E(G)$, is a bar of $(G, p)$.

Given the vertex set $V(G) = \{v_1, \ldots, v_n\}$ of a graph $G$ and a map $p : V(G) \to \mathbb{R}^d$, it is often useful to identify $p$ with a vector in $\mathbb{R}^{dn}$ by using the order on $V(G)$. In this case we also refer to $p$ as a configuration of $n$ points in $\mathbb{R}^d$. For $p(v_i)$ we will frequently write $p_i$.

For a fixed ordering of the edges of a graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define the edge function $f_G : \mathbb{R}^{dn} \to \mathbb{R}^{E(G)}$ by

$$f_G(p_1, \ldots, p_n) = (\ldots, \|p_i - p_j\|^2, \ldots),$$

where $\{v_i, v_j\} \in E(G)$, $p_i \in \mathbb{R}^d$ for all $i = 1, \ldots, n$, and $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$ [1, 28].

If $(G, p)$ is a $d$-dimensional framework with $n$ vertices, then $f_G^{-1}(f_G(p))$ is the set of all configurations $q$ of $n$ points in $\mathbb{R}^d$ with the property that corresponding bars of the frameworks $(G, p)$ and $(G, q)$ have the same length. In particular, we clearly have $f_{K_n}^{-1}(f_{K_n}(p)) \subseteq f_G^{-1}(f_G(p))$, where $K_n$ is the complete graph on $V(G)$.

**Definition 2.1** [1, 26] Let $G$ be a graph with $n$ vertices and let $(G, p)$ be a framework in $\mathbb{R}^d$. A motion of $(G, p)$ is a differentiable path $x : [0, 1] \to \mathbb{R}^{dn}$ such that $x(0) = p$ and $x(t) \in f_G^{-1}(f_G(p))$ for all $t \in [0, 1]$.

A motion $x$ of $(G, p)$ is a rigid motion if $x(t) \in f_{K_n}^{-1}(f_{K_n}(p))$ for all $t \in [0, 1]$ and a flex of $(G, p)$ if $x(t) \notin f_{K_n}^{-1}(f_{K_n}(p))$ for all $t \in [0, 1]$.

$(G, p)$ is rigid if every motion of $(G, p)$ is a rigid motion, and flexible otherwise.

![Figure 1: A rigid (b) and a flexible (b) framework in the plane. The flex shown in (c) takes the framework in (b) to the framework in (d).](image)

There also exist some alternate definitions of a flexible framework all of which are equivalent to Definition 2.1, as shown in [1].
Theorem 2.1 Let \((G, p)\) be a framework in \(\mathbb{R}^d\) with \(n\) vertices. The following are equivalent:

(i) \((G, p)\) is flexible;

(ii) there exists a motion \(x : [0, 1] \to \mathbb{R}^d\) of \((G, p)\) such that \(x(t) \notin f_{K_n}^{-1}(f_{K_n}(p))\) for some \(t \in (0, 1)\);

(iii) for every neighborhood \(N_p\) of \(p \in \mathbb{R}^d\), we have \(f_{K_n}^{-1}(f_{K_n}(p)) \cap N_p \nsupseteq f_{G}^{-1}(f_{G}(p)) \cap N_p\).

Remark 2.1 In Definition 2.1 we may replace the term ‘differentiable path’ by the terms ‘continuous path’ or ‘analytic path’. The fact that all of these definitions are equivalent is a consequence of some basic results from algebraic geometry [1 26 39].

2.2 Infinitesimal rigidity

It is in general very difficult to determine whether a given framework \((G, p)\) is rigid or not since it requires solving a system of quadratic equations. It is therefore common to linearize this problem by considering the derivative of the edge function \(f_G\) of \(G\).

Definition 2.2 Let \(G\) be a graph with \(V(G) = \{v_1, v_2, \ldots, v_n\}\) and let \(p : V(G) \to \mathbb{R}^d\) be a map. The rigidity matrix of \((G, p)\) is the \(|E(G)| \times dn\) matrix

\[
R(G, p) = \begin{pmatrix}
0 & \ldots & 0 & p_i - p_j & 0 & \ldots & 0 & p_j - p_i & 0 & \ldots & 0 \\
\vdots & & & & & & & & & & \\
\end{pmatrix},
\]

that is, for each edge \(\{v_i, v_j\} \in E(G)\), \(R(G, p)\) has the row with \((p_i - p_j)_{\|d\|}, \ldots, (p_i - p_j)_{\|d\|}\) in the columns \(d(i-1) + 1, \ldots, d_i, (p_j - p_i)_{\|d\|}, \ldots, (p_j - p_i)_{\|d\|}\) in the columns \(d(j-1) + 1, \ldots, d_j, 0\) elsewhere.

Equivalently, \(R(G, p) = \frac{1}{d} df_G(p)\), where \(df_G(p)\) denotes the Jacobian matrix of the edge function \(f_G\) of \(G\), evaluated at the point \(p \in \mathbb{R}^d\).

Remark 2.2 The rigidity matrix is defined for arbitrary pairs \((G, p)\), where \(G\) is a graph and \(p : V(G) \to \mathbb{R}^d\) is a map. If \((G, p)\) is not a framework, then there exists a pair of adjacent vertices of \(G\) that are mapped to the same point in \(\mathbb{R}^d\) under \(p\) and every such edge of \(G\) gives rise to a zero-row in \(R(G, p)\).

Definition 2.3 An infinitesimal motion of a \(d\)-dimensional framework \((G, p)\) with \(V(G) = \{v_1, v_2, \ldots, v_n\}\) is a function \(u : V(G) \to \mathbb{R}^d\) such that

\[(p_i - p_j) : (u_i - u_j) = 0 \quad \text{for all} \quad \{v_i, v_j\} \in E(G),\]

where \(u_i = u(v_i)\). Equivalently, \(u\) is a vector in \(\mathbb{R}^{dn}\) that lies in the kernel of \(R(G, p)\).

An infinitesimal motion \(u\) of \((G, p)\) is an infinitesimal rigid motion if there
exists a skew-symmetric matrix $S$ (a rotation) and a vector $t$ (a translation) such that $u_i = Sp_i + t$ for all $i = 1, \ldots, n$; otherwise $u$ is an infinitesimal flex of $(G, p)$.

$(G, p)$ is infinitesimally rigid if every motion of $(G, p)$ is an infinitesimal rigid motion, and infinitesimally flexible otherwise.

Figure 2: The arrows indicate the non-zero displacement vectors of an infinitesimal rigid motion (a) and infinitesimal flexes (b, c) of frameworks in $\mathbb{R}^2$.

Physically, an infinitesimal motion of $(G, p)$ is a set of initial velocity vectors, one at each joint, that preserve the lengths of the bars of $(G, p)$ at first order (see also Figure 2).

It is well known that the infinitesimal rigid motions arising from $d$ translations and $\binom{d}{2}$ rotations of $\mathbb{R}^d$ form a basis for the space of infinitesimal rigid motions of $(G, p)$, provided that the points $p_1, \ldots, p_n$ span an affine subspace of $\mathbb{R}^d$ of dimension at least $d - 1$ [13,38]. Thus, for such a framework $(G, p)$, we have

$$\text{nullity } (\mathbf{R}(G, p)) \geq d + \binom{d}{2} = \frac{d(d+1)}{2}$$

and $(G, p)$ is infinitesimally rigid if and only if nullity $\left(\mathbf{R}(G, p)\right) = \frac{d(d+1)}{2}$ or equivalently, rank $\left(\mathbf{R}(G, p)\right) = d|V(G)| - \frac{d(d+1)}{2}$.

Theorem 2.2 [1,16] A framework $(G, p)$ in $\mathbb{R}^d$ is infinitesimally rigid if and only if either rank $\left(\mathbf{R}(G, p)\right) = d|V(G)| - \frac{d(d+1)}{2}$ or $G$ is a complete graph $K_n$ and the points $p(v)$, $v \in V(G)$, are affinely independent.

The following theorem gives the main connection between rigidity and infinitesimal rigidity. A proof of this result can be found in [1], [16] or [26], for example.

Theorem 2.3 If a framework $(G, p)$ is infinitesimally rigid, then $(G, p)$ is rigid.

2.3 Basic ‘generic’ results

In 1978, L. Asimov and B. Roth showed that for ‘almost all’ realizations of a given graph $G$, infinitesimal rigidity and rigidity are equivalent. We need the following definition.

Definition 2.4 [1,28] Let $G$ be a graph with $n$ vertices and let $d \geq 1$ be an integer. A point $p \in \mathbb{R}^{dn}$ is said to be a regular point of $G$ if there exists a neighborhood $N_p$ of $p$ in $\mathbb{R}^{dn}$ so that rank $\left(\mathbf{R}(G, p)\right) \geq \text{rank } \left(\mathbf{R}(G, q)\right)$ for all
A framework \((G, p)\) is said to be regular if \(p\) is a regular point of \(G\).

**Theorem 2.4 (Asimov, Roth, 1978)** \([1]\) Let \(G\) be a graph with \(n\) vertices and let \((G, p)\) be a \(d\)-dimensional framework. If \(p \in \mathbb{R}^{dn}\) is a regular point of \(G\), then \((G, p)\) is infinitesimally rigid if and only if \((G, p)\) is rigid.

Note that the set of all regular points of a graph \(G\) in \(\mathbb{R}^{dn}\) forms a dense open subset of \(\mathbb{R}^{dn}\). Moreover, all regular realizations of \(G\) share the same infinitesimal (and, by Theorem 2.4, also finite) rigidity properties. Regular frameworks are therefore sometimes also referred to as 'generic' frameworks \([18, 26]\). However, since in combinatorial (or generic) rigidity it is often useful to have a notion of 'generic' that is invariant under addition or deletion of edges in \(G\), a 'generic' framework is frequently also defined as follows.

**Definition 2.5** \([17, 18]\) Let \(G\) be a graph with \(n\) vertices, \(d \geq 1\) be an integer, and \(K_n\) be the complete graph on \(V(G)\). Further, let \(R(n, d)\) be the matrix that is obtained from the rigidity matrix \(R(K_n, p)\) of a \(d\)-dimensional realization \((K_n, p)\) by replacing each \((p_i)_j \in \mathbb{R}\) with the variable \((p'_i)_j\). Then we say that \(p \in \mathbb{R}^{dn}\) is generic if the determinant of any submatrix of \(R(K_n, p)\) is zero only if the determinant of the corresponding submatrix of \(R(n, d)\) is (identically) zero.

The framework \((G, p)\) is said to be generic if \(p\) is generic.

Like the set of all regular points of a graph \(G\), the set of all generic points is also a dense open subset of \(\mathbb{R}^{dn}\) \([18]\). Moreover, since a generic framework is clearly also regular, we immediately obtain the following corollary of Theorem 2.4.

**Corollary 2.5** If a framework \((G, p)\) is generic, then \((G, p)\) is infinitesimally rigid if and only if \((G, p)\) is rigid.

An easy but often useful observation concerning generic frameworks is that if a framework \((G, p)\) in \(\mathbb{R}^d\) is generic (in the sense of Definition 2.5), then the joints of \((G, p)\) are in general position, that is, for \(1 \leq m \leq d\), no \(m + 1\) joints of \((G, p)\) lie in an \((m - 1)\)-dimensional affine subspace of \(\mathbb{R}^d\) \([18]\).

For further information on generic frameworks, we refer the reader to \([17, 18, 37]\), for example.

**Definition 2.6** \([36, 38]\) A self-stress of a framework \((G, p)\) is a vector \(\omega \in \mathbb{R}^{|E(G)|}\) that satisfies \(R(G, p)^T \omega = 0\). If \((G, p)\) has a non-zero self-stress, then \((G, p)\) is said to be dependent (since in this case there exists a linear dependency among the row vectors of \(R(G, p)\)). Otherwise, \((G, p)\) is said to be independent.

An independent framework is clearly also regular. Therefore, the next result is also an immediate consequence of Theorem 2.4.

**Corollary 2.6** If a framework \((G, p)\) is independent, then \((G, p)\) is infinitesimally rigid if and only if \((G, p)\) is rigid.

We establish symmetric analogs to the Theorem of Asimov and Roth, as well as to Corollaries 2.5 and 2.6 in Section 4. These results will allow us to detect flexes in frameworks that are ‘generic’ with respect to certain symmetry constraints.
3 Symmetric frameworks

3.1 The set $\mathcal{R}(G,S,\Phi)$

Recall that an automorphism of a graph $G$ is a permutation $\alpha$ of $V(G)$ such that $\{u,v\} \in E(G)$ if and only if $\{\alpha(u),\alpha(v)\} \in E(G)$. The automorphisms of a graph $G$ form a group under composition which is denoted by $\text{Aut}(G)$.

A symmetry operation of a $d$-dimensional framework $(G,p)$ is an isometry $x$ of $\mathbb{R}^d$ such that for some $\alpha \in \text{Aut}(G)$, we have $x(p(v)) = p(\alpha(v))$ for all $v \in V(G)$ \cite{27, 28, 29}.

The set of all symmetry operations of a framework $(G,p)$ forms a group under composition, called the point group of $(G,p)$ \cite{3, 20, 28, 29}. Since translating a framework does not change its rigidity properties, we may assume wlog that the point group of any framework in this paper is a symmetry group, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^d)$.

We use the Schoenflies notation for the symmetry operations and symmetry groups considered in this paper, as this is one of the standard notations in the literature about symmetric structures (see \cite{10, 15, 19, 22, 27, 28, 29, 30}, for example). In this notation, the identity transformation is denoted by $Id$, a rotation about a $(d-2)$-dimensional subspace of $\mathbb{R}^d$ by an angle of $\frac{\pi}{m}$ is denoted by $C_m$, and a reflection in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ is denoted by $s$.

In this paper, we only consider three types of symmetry groups. In the Schoenflies notation, they are denoted by $C_s$, $C_m$, and $C_{mn}$: $C_s$ is a symmetry group consisting of the identity $Id$ and a single reflection $s$, $C_m$ is a cyclic group generated by a rotation $C_m$, and $C_{mn}$ is a dihedral group generated by a pair $\{C_m,s\}$. For further information about the Schoenflies notation we refer the reader to \cite{3, 20, 28}.

Given a symmetry group $S$ in dimension $d$ and a graph $G$, we let $\mathcal{R}(G,S)$ denote the set of all $d$-dimensional realizations of $G$ whose point group is either equal to $S$ or contains $S$ as a subgroup \cite{27, 28, 29}. In other words, the set $\mathcal{R}(G,S)$ consists of all realizations $(G,p)$ of $G$ for which there exists a map $\Phi : S \to \text{Aut}(G)$ so that

$$x(p(v)) = p(\Phi(x)(v)) \text{ for all } v \in V(G) \text{ and all } x \in S. \quad (1)$$

A framework $(G,p) \in \mathcal{R}(G,S)$ satisfying the equations in \eqref{eq:1} for the map $\Phi : S \to \text{Aut}(G)$ is said to be of type $\Phi$, and the set of all realizations in $\mathcal{R}(G,S)$ which are of type $\Phi$ is denoted by $\mathcal{R}(G,S,\Phi)$ (see again \cite{27, 28, 29} as well as Figure \ref{fig:Schoenflies}).

Remark 3.1 If the map $p$ of a framework $(G,p) \in \mathcal{R}(G,S)$ is non-injective, then $(G,p)$ can possibly be of more than just one type and a given type may not be a homomorphism. However, if $p$ is injective, then $(G,p) \in \mathcal{R}(G,S)$ is of a unique type $\Phi$ and $\Phi$ is necessarily also a homomorphism. See \cite{29} for further details.

3.2 The notion of $(S,\Phi)$-generic

Recall from equation \eqref{eq:1} in the previous section that for every framework $(G,p)$ in the set $\mathcal{R}(G,S,\Phi)$ with $V(G) = \{v_1, \ldots, v_n\}$, we have $x(p(v_i)) = p(\Phi(x)(v_i))$ for all $i = 1,2,\ldots,n$ and all $x \in S$. Since every element of $S$ is an orthogonal
Figure 3: 2-dimensional realizations of $K_{3,3}$ in $\mathcal{R}(K_{3,3},C_3)$ of different types: the framework in (a) is of type $\Phi_a$, where $\Phi_a : C_3 \rightarrow \text{Aut}(K_{3,3})$ is the homomorphism defined by $\Phi_a(s) = (v_1 v_2)(v_3 v_0)(v_4)$ and the framework in (b) is of type $\Phi_b$, where $\Phi_b : C_3 \rightarrow \text{Aut}(K_{3,3})$ is the homomorphism defined by $\Phi_b(s) = (v_1 v_4)(v_2 v_0)(v_3 v_0)$.

linear transformation, we may identify each $x \in S$ with its corresponding orthogonal matrix $M_x$ that represents $x$ with respect to the canonical basis of $\mathbb{R}^d$. Therefore, for each $x \in S$, the equations in (11) can be written as

$$M^{(x)} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = P_{\Phi(x)} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix},$$

where

$$M^{(x)} = \begin{pmatrix} M_x & 0 & \ldots & 0 \\ 0 & M_x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & M_x \end{pmatrix},$$

and $P_{\Phi(x)}$ is the $dn \times dn$ matrix which is obtained from the permutation matrix corresponding to $\Phi(x)$ by replacing each 1 by a $d \times d$ identity matrix and each 0 by a $d \times d$ zero matrix.

We denote $L_{x,\Phi} = \ker(M^{(x)} - P_{\Phi(x)})$ and $U = \bigcap_{x \in S} L_{x,\Phi}$. Then $U$ is a linear subspace of $\mathbb{R}^{dn}$ which may be interpreted as the space of all those (possibly non-injective) configurations of $n$ points in $\mathbb{R}^d$ that possess the symmetry imposed by $S$ and $\Phi$ \cite{28, 29}. In particular, if $(G,p)$ is a framework in $\mathcal{R}(G,S,\Phi)$, then the configuration $p$ is an element of $U$. Therefore, if we fix a basis $\mathcal{B}_U = \{u_1, u_2, \ldots, u_k\}$ of $U$, then every framework $(G,p) \in \mathcal{R}(G,S,\Phi)$ can be represented uniquely by the $k \times 1$ coordinate vector of $p$ relative to $\mathcal{B}_U$.

We let $R_{\mathcal{B}_U}(n,d)$ denote the matrix that is obtained from the ‘indeterminate’ rigidity matrix $R(n,d)$ (see Definition 25) by introducing a $k$-tuple $(t'_1, t'_2, \ldots, t'_k)$ of variables and replacing the $dn$ variables $(p'_j)$ of $R(n,d)$ as follows.

For each $i = 1, 2, \ldots, n$ and each $j = 1, \ldots, d$, we replace the variable $(p'_i)_j$ in $R(n,d)$ by the linear combination $t'_1(u_1)_i + t'_2(u_2)_i + \ldots + t'_k(u_k)_i$. 

8
Remark 3.2 [28, 29] Let \((G, p) \in \mathcal{R}_{(G, S, \Phi)}\) and \(\mathcal{B}_U = \{u_1, u_2, \ldots, u_k\}\) be a basis of \(U = \bigcap_{x \in S} L_{x, \Phi}\). Then

\[
\begin{pmatrix}
  p_1 \\
  p_2 \\
  \vdots \\
  p_n
\end{pmatrix} = t_1 u_1 + \ldots + t_k u_k, \text{ for some } t_1, \ldots, t_k \in \mathbb{R}.
\]

So, if for \(i = 1, \ldots, k\), the variable \(t_i'\) in \(R_{\mathcal{B}_U}(n, d)\) is replaced by \(t_i\) then we obtain the rigidity matrix \(R(K_n, p)\) of the framework \((K_n, p)\).

The following symmetry-adapted notion of ‘generic’ for the set \(\mathcal{R}_{(G, S, \Phi)}\) was introduced in [29] (see also [28, 30]).

Definition 3.1 [28, 29, 30] Let \(G\) be a graph with \(V(G) = \{v_1, v_2, \ldots, v_n\}\), \(K_n\) be the complete graph with \(V(K_n) = V(G)\), \(S\) be a symmetry group in dimension \(d\), \(\Phi\) be a map from \(S\) to \(\text{Aut}(G)\), and \(\mathcal{B}_U\) be a basis of \(U = \bigcap_{x \in S} L_{x, \Phi}\).

A point \(p \in \mathbb{R}^{d n}\) is called \((S, \Phi, \mathcal{B}_U)\)-generic if the determinant of any submatrix of \(R(K_n, p)\) is equal to zero only if the determinant of the corresponding submatrix of \(R_{\mathcal{B}_U}(n, d)\) is (identically) zero.

The point \(p\) is said to be \((S, \Phi)\)-generic if \(p\) is \((S, \Phi, \mathcal{B}_U)\)-generic for some basis \(\mathcal{B}_U\) of \(U\).

A framework \((G, p) \in \mathcal{R}_{(G, S, \Phi)}\) is \((S, \Phi, \mathcal{B}_U)\)-generic if \(p\) is \((S, \Phi, \mathcal{B}_U)\)-generic, and \((G, p)\) is \((S, \Phi)\)-generic if \((G, p)\) is \((S, \Phi, \mathcal{B}_U)\)-generic for some basis \(\mathcal{B}_U\) of \(U\).

Remark 3.3 It is shown in [29] that the definition of \((S, \Phi)\)-generic is independent of the choice of the basis of \(U\).

Intuitively, an \((S, \Phi)\)-generic realization of a graph \(G\) is obtained by placing the vertices of a set of representatives for the symmetry orbits \(Sv = \{\Phi(x)|v| x \in S\}\) into ‘generic’ positions. The positions for the remaining vertices of \(G\) are then uniquely determined by the symmetry constraints imposed by \(S\) and \(\Phi\) (see [28, 29], for further details).

As shown in [29], the set of \((S, \Phi)\)-generic realizations of a graph \(G\) is a dense open subset of the set \(\mathcal{R}_{(G, S, \Phi)}\) and the infinitesimal rigidity properties are the same for all \((S, \Phi)\)-generic realizations of \(G\).

As an example, consider the realizations of \(K_{3,3}\) with mirror symmetry depicted in Figure 3. All \((C_s, \Phi_n)\)-generic realizations of \(K_{3,3}\) are infinitesimally rigid, whereas all realizations in \(\mathcal{R}_{(K_{3,3}, C_s, \Phi_n)}\) are infinitesimally flexible (though rigid), since the joints of any realization in \(\mathcal{R}_{(K_{3,3}, C_s, \Phi_n)}\) are forced to lie on a conic section [29, 35].

3.3 The external and internal representation

Basic to our investigation of the rigidity and flexibility of symmetric frameworks in \(\mathcal{R}_{(G, S, \Phi)}\) are the ‘external’ and ‘internal’ representation of the group \(S\) which were first introduced in [15] and [23] by means of an example (see also [27, 28]). We need mathematically rigorous definitions of these representations.
Definition 3.2  Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$, $S$ be a symmetry group in dimension $d$, and $\Phi : S \rightarrow \text{Aut}(G)$ be a homomorphism.

The external representation of $S$ (with respect to $G$ and $\Phi$) is the matrix representation $H_e : S \rightarrow \text{GL}(dn, \mathbb{R})$ that sends $x \in S$ to the matrix $H_e(x)$ which is obtained from the transpose of the $n \times n$ permutation matrix corresponding to $\Phi(x)$ (with respect to the enumeration $V(G) = \{v_1, v_2, \ldots, v_n\}$) by replacing each 1 with the orthogonal $d \times d$ matrix $M_x$ which represents $x$ with respect to the canonical basis of $\mathbb{R}^d$ and each 0 with a $d \times d$ zero-matrix.

We let $H'_e : S \rightarrow \text{GL}(dn)$ be the corresponding linear representation of $S$ that sends $x \in S$ to the automorphism $H'_e(x)$ which is represented by the matrix $H_e(x)$ with respect to the canonical basis of the $\mathbb{R}$-vector space $\mathbb{R}^{dn}$.

The internal representation of $S$ (with respect to $G$ and $\Phi$) is the matrix representation $H_i : S \rightarrow \text{GL}(m, \mathbb{R})$ that sends $x \in S$ to the transpose of the permutation matrix corresponding to the permutation of $E(G)$ (with respect to the enumeration $E(G) = \{e_1, e_2, \ldots, e_m\}$) which is induced by $\Phi(x)$.

We let $H'_i : S \rightarrow \text{GL}(m)$ be the corresponding linear representation of $S$ that sends $x \in S$ to the automorphism $H'_i(x)$ which is represented by the matrix $H_i(x)$ with respect to the canonical basis of the $\mathbb{R}$-vector space $\mathbb{R}^m$.

Example 3.1  To illustrate the previous definition, let $K_3$ be the complete graph with $V(K_3) = \{v_1, v_2, v_3\}$ and $E(K_3) = \{e_1, e_2, e_3\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$ and $e_3 = \{v_2, v_3\}$. Further, let $C_s = \{I_d, s\}$ be the symmetry group in dimension 2 with

\[
M_{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and let $\Phi : C_s \rightarrow \text{Aut}(K_3)$ be the homomorphism defined by $\Phi(s) = (v_1 v_2)(v_3)$. (See also Figure 4.) Then we have

\[
H_e(\text{Id}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_e(s) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},
\]

\[
H_i(\text{Id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_i(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

For further examples, see [23] or [24].

From group representation theory we know that every finite group has, up to equivalency, only finitely many irreducible linear representations and that every linear representation of such a group can be written uniquely, up to equivalency of the direct summands, as a direct sum of the irreducible linear representations of this group [21, 32]. So, let $S$ have $r$ pairwise non-equivalent irreducible linear representations $I_1, I_2, \ldots, I_r$ and let

\[
H'_e = \lambda_1 I_1 \oplus \ldots \oplus \lambda_r I_r, \quad \text{where} \quad \lambda_1, \ldots, \lambda_r \in \mathbb{N} \cup \{0\}.
\]
Then, for each \( t = 1, \ldots, r \), there exist \( \lambda_t \) subspaces \( (V_e^{(I_t)})_1, \ldots, (V_e^{(I_t)})_{\lambda_t} \) of the \( \mathbb{R} \)-vector space \( \mathbb{R}^{d_n} \) which correspond to the \( \lambda_t \) direct summands in \( (3) \), so that

\[
\mathbb{R}^{d_n} = V_e^{(I_1)} \oplus \cdots \oplus V_e^{(I_r)},
\]

where

\[
V_e^{(I_t)} = (V_e^{(I_t)})_1 \oplus \cdots \oplus (V_e^{(I_t)})_{\lambda_t}.
\]

Let \( (B_e^{(I_t)})_1, \ldots, (B_e^{(I_t)})_{\lambda_t} \) be bases of the subspaces in \( (4) \). Then

\[
B_e^{(I_t)} = (B_e^{(I_t)})_1 \cup \cdots \cup (B_e^{(I_t)})_{\lambda_t}
\]

is a basis of \( V_e^{(I_t)} \) and

\[
B_e = B_e^{(I_1)} \cup \cdots \cup B_e^{(I_r)}
\]

is a basis of the \( \mathbb{R} \)-vector space \( \mathbb{R}^{d_n} \).

Similarly, let

\[
H'_t = \mu_1 I_1 \oplus \cdots \oplus \mu_r I_r,
\]

where \( \mu_1, \ldots, \mu_r \in \mathbb{N} \cup \{0\} \).

For each \( t = 1, \ldots, r \), there exist \( \mu_t \) subspaces \( (V_i^{(I_t)})_1, \ldots, (V_i^{(I_t)})_{\mu_t} \) of the \( \mathbb{R} \)-vector space \( \mathbb{R}^{m} \) which correspond to the \( \mu_t \) direct summands in \( (5) \), so that

\[
\mathbb{R}^{m} = V_i^{(I_1)} \oplus \cdots \oplus V_i^{(I_r)},
\]

where

\[
V_i^{(I_t)} = (V_i^{(I_t)})_1 \oplus \cdots \oplus (V_i^{(I_t)})_{\mu_t}.
\]

Let \( (B_i^{(I_t)})_1, \ldots, (B_i^{(I_t)})_{\mu_t} \) be bases of the subspaces in \( (6) \). Then

\[
B_i^{(I_t)} = (B_i^{(I_t)})_1 \cup \cdots \cup (B_i^{(I_t)})_{\mu_t}
\]

is a basis of \( V_i^{(I_t)} \) and

\[
B_i = B_i^{(I_1)} \cup \cdots \cup B_i^{(I_r)}
\]

is a basis of the \( \mathbb{R} \)-vector space \( \mathbb{R}^{m} \).

**Definition 3.3** [27] With the notation above, we say that a vector \( v \in \mathbb{R}^{d_n} \) is **symmetric with respect to the irreducible linear representation** \( I_t \) of \( S \) if \( v \in V_e^{(I_t)} \). Similarly, we say that a vector \( w \in \mathbb{R}^{m} \) is **symmetric with respect to** \( I_t \) if \( w \in V_i^{(I_t)} \).
Example 3.2 Let $K_3, C_s = \{\text{Id}, s\}$, and $\Phi$ be as in Example 3.1. The symmetry group $C_s$ has two non-equivalent irreducible linear representations, $I_1$ and $I_2$, both of which are of dimension 1 \[20\]. $I_1$ is the trivial representation which maps both $\text{Id}$ and $s$ to the identity transformation, whereas $I_2$ maps $\text{Id}$ to the identity transformation and $s$ to the linear transformation $I_2(s)$ which is defined by $I_2(s)(x) = -x$ for all $x \in \mathbb{R}$. It is easy to see that both of the $H'_e$-invariant subspaces $V_e^{(I_1)}$ and $V_e^{(I_2)}$ of $\mathbb{R}^6$ are of dimension 3. An element of each of these invariant subspaces is shown in Figure 5.

Similarly, the $H'_i$-invariant subspaces $V_i^{(I_1)}$ and $V_i^{(I_2)}$ of $\mathbb{R}^3$ are easily found to be of dimension 2 and 1, respectively.

Remark 3.4 While in Example 3.2 the $H'_e$- and $H'_i$-invariant subspaces can be found by inspection, this is of course generally not possible. There are, however, some standard methods and algorithms for finding the symmetry adapted bases $B_e$ and $B_i$ for any given symmetry group. Good sources for these methods are \[14\], for example.

The following fundamental theorem for analyzing the rigidity properties of a symmetric framework using group representation theory was established in \[27\].

**Theorem 3.1** \[27\] Let $G$ be a graph, $S$ be a symmetry group with pairwise non-equivalent irreducible linear representations $I_1, \ldots, I_r$, $\Phi$ be a homomorphism from $S$ to Aut($G$), and $p \in \bigcap_{x \in S} L_x, \Phi$. Then for every $t \in \{1, \ldots, r\}$, we have that if $R(G, p)u = z$ and $u$ is symmetric with respect to $I_t$, then $z$ is also symmetric with respect to $I_t$.

As an immediate consequence of Theorem 3.1 we obtain

**Corollary 3.2** \[23\] Let $G$ be a graph, $S$ be a symmetry group with pairwise non-equivalent irreducible linear representations $I_1, \ldots, I_r$, $\Phi$ be a homomorphism from $S$ to Aut($G$), and $p \in \bigcap_{x \in S} L_x, \Phi$. Further, let $T_e$ and $T_i$
be the matrices of the basis transformations from the canonical bases of the \( \mathbb{R} \)-
\( \mathbb{R}^{dn} \) and \( \mathbb{R}^{m} \) to the bases \( B_e \) and \( B_i \), respectively. Then the matrix 
\( \tilde{R}(G, p) = T_e^{-1} R(G, p) T_e \) is block-diagonalized in such a way that there exists
(at most) one submatrix block for each irreducible linear representation \( I_t \) of \( S \).

Note that in this block-diagonalization of the rigidity matrix \( R(G, p) \), the submatrix block corresponding to \( I_t \) is a matrix of the size \( \dim (V_i^{(I_t)}) \times \dim (V_e^{(I_t)}) \). In particular, a submatrix block can possibly be an ‘empty matrix’
which has rows but no columns or alternatively columns but no rows.

Consider the submatrix block \( \tilde{R}_t(G, p) \) of \( \tilde{R}(G, p) \) corresponding to the irreducible
representation \( I_t \) of \( S \). By definition, the kernel of \( \tilde{R}_t(G, p) \) is the space of all infinitesimal motions of \( (G, p) \) that are symmetric with respect to \( I_t \). Further, \( (G, p) \) has a non-zero self-stress that is symmetric with respect to \( I_t \)
if and only if there exists a linear dependency among the row vectors of \( \tilde{R}_t(G, p) \). So, the matrix \( \tilde{R}_t(G, p) \) comprises all the information about the infinitesimal motions and self-stresses of \( (G, p) \) that are symmetric with respect to \( I_t \).

Notice that knowledge of only the size of the submatrix block \( \tilde{R}_t(G, p) \) already allows us to gain significant insight into the ‘\( I_t \)-symmetric’ infinitesimal rigidity properties of \( (G, p) \): if
\[
\dim (V_i^{(I_t)}) < \dim (V_e^{(I_t)}) - \dim (W_e^{(I_t)}),
\]
where \( W_e^{(I_t)} \) denotes the space of all infinitesimal rigid motions in \( V_e^{(I_t)} \), then
\( (G, p) \) clearly possesses an infinitesimal flex that is symmetric with respect to \( I_t \). Similarly, if
\[
\dim (V_i^{(I_t)}) > \dim (V_e^{(I_t)}) - \dim (W_e^{(I_t)}),
\]
then \( (G, p) \) clearly possesses a non-zero self-stress which is symmetric with respect to \( I_t \).

So, for \( (G, p) \) to be isostatic (i.e., infinitesimally rigid and independent), we need to have equality in equation (10), for each \( t = 1, \ldots, r \). These are necessary, but not sufficient conditions for \( (G, p) \) to be isostatic, as they cannot predict the presence of paired ‘equi-symmetric’ infinitesimal flexes and self-stresses.

The symmetry-extended version of Maxwell’s rule described by P. Fowler and S. Guest in [15] combines all of these conditions into a single equation using some basic techniques from character theory. With this rule we can detect infinitesimal flexes and self-stresses of \( (G, p) \) that are symmetric with respect to \( I_t \) with very little computational effort. In particular, we do not need to determine the bases \( B_e \) and \( B_i \) in equations (5) and (9) explicitly. See also [10, 27, 28], for further details on this rule.

Note that since the dimensions of the subspaces \( V_i^{(I_t)} \), \( V_e^{(I_t)} \), and \( W_e^{(I_t)} \) do not depend on the map \( p \) of the framework \( (G, p) \), provided that the points \( p_1, \ldots, p_n \) span all of \( \mathbb{R}^d \), the symmetry-extended version of Maxwell’s rule provides information about the infinitesimal rigidity properties for all \( (S, \Phi) \)-generic realizations of \( G \) (see also [27, 28]).
4 Detection of symmetric flexes

Our main goal in this section is to find sufficient conditions for the existence of a ‘symmetry-preserving’ flex of a symmetric framework. As we will see, the existence of a flex that preserves some, but not all of the symmetries of a given framework can be predicted in an analogous way.

As usual, let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$, $S$ be a symmetry group in dimension $d$ with $r$ pairwise non-equivalent irreducible linear representations $I_1, \ldots, I_r$, $\Phi : S \to \text{Aut}(G)$ be a homomorphism, and $(G, p)$ be a framework in $\mathbb{R}(G,S,\Phi)$. In the following, $I_1$ will always denote the trivial irreducible linear representation of $S$, i.e., $I_1$ denotes the linear representation of dimension one with the property that $I_1(x)$ is the identity transformation for all $x \in S$.

Recall from Section 3.3 that $V_e^{(I_1)}$ denotes the $H'_e$-invariant subspace of $\mathbb{R}^{dn}$ which corresponds to $I_1$. So, $p \in V_e^{(I_1)}$ if and only if $H'_e(x)(p) = p$ for all $x \in S$. Further, recall from Section 3.2 that if $(G,p)$ is a framework in $\mathbb{R}(G,S,\Phi)$, then $p \in \mathbb{R}^{dn}$ is an element of the subspace $U = \bigcap_{x \in S} I_x \Phi$ of $\mathbb{R}^{dn}$.

Note that it follows immediately from the definitions of $U$ and $V_e^{(I_1)}$ that $U = V_e^{(I_1)}$, because $p \in U$ if and only if $M^{(x)} p = P_{\Phi(x)} p$ for all $x \in S$ if and only if $(P_{\Phi(x)})^T M^{(x)} p = p$ for all $x \in S$ if and only if $H_e(x) p = p$ for all $x \in S$ if and only if $p \in V_e^{(I_1)}$.

So, since we are interested in flexes of $(G,p) \in \mathbb{R}(G,S,\Phi)$ that preserve the symmetry of $(G,p)$, we need to restrict the edge functions $f_G$ of $G$ and $f_{K_n}$ of $K_n$ to the subspace $V_e^{(I_1)}$ of $\mathbb{R}^{dn}$. In the following, we let $\tilde{f}_G : V_e^{(I_1)} \to \mathbb{R}^{|E(G)|}$ denote the restriction of $f_G$ to $V_e^{(I_1)}$, and $\tilde{f}_{K_n} : V_e^{(I_1)} \to \mathbb{R}^{(3)}$ denote the restriction of $f_{K_n}$ to $V_e^{(I_1)}$. The Jacobian matrices of $\tilde{f}_G$ and $\tilde{f}_{K_n}$, evaluated at a point $p \in V_e^{(I_1)}$, are denoted by $d f_G(p)$ and $d \tilde{f}_{K_n}(p)$, respectively.

Definition 4.1 An element $p \in V_e^{(I_1)}$ is said to be a regular point of $G$ in $V_e^{(I_1)}$ if there exists a neighborhood $N_p$ of $p$ in $V_e^{(I_1)}$ so that rank $(d \tilde{f}_G(p)) = \max \{\text{rank} (d f_G(q)) \mid q \in N_p\}$. A regular point of $K_n$ in $V_e^{(I_1)}$ is defined analogously.

Definition 4.2 An $(S,\Phi)$-symmetry-preserving flex of a framework $(G,p) \in \mathbb{R}(G,S,\Phi)$ is a differentiable path $x : [0,1] \to V_e^{(I_1)}$ such that $x(0) = p$ and $x(t) \in \tilde{f}_G^{-1} (\tilde{f}_G(p)) \setminus \tilde{f}_{K_n}^{-1} (\tilde{f}_{K_n}(p))$ for all $t \in (0,1]$.

Lemma 4.1 Let $G$ be a graph, $S$ be a symmetry group, $\Phi : S \to \text{Aut}(G)$ be a homomorphism, and $(G,p)$ be a framework in $\mathbb{R}(G,S,\Phi)$. If $p$ is a regular point of $G$ in $V_e^{(I_1)}$, then there exists a neighborhood $N_p$ of $p$ in $V_e^{(I_1)}$ such that $\tilde{f}_G^{-1}(\tilde{f}_G(p)) \cap N_p$ is a smooth manifold of dimension $\dim \ (V_e^{(I_1)}) - \text{rank} (d \tilde{f}_G(p))$.

Proof. The result follows immediately from Proposition 2 (and subsequent remark) in [11]. □

Theorem 4.2 Let $G$ be a graph with $n$ vertices, $S$ be a symmetry group, $\Phi : S \to \text{Aut}(G)$ be a homomorphism, and $(G,p)$ be a framework in $\mathbb{R}(G,S,\Phi)$. If $p$ is a regular point of $G$ in $V_e^{(I_1)}$ and also a regular point of $K_n$ in $V_e^{(I_1)}$, then
(i) \( \text{rank} \left( d\tilde{f}_G(p) \right) = \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \) if and only if \((G,p)\) has no \((S,\Phi)\)-symmetry-preserving flex;

(ii) \( \text{rank} \left( d\tilde{f}_G(p) \right) < \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \) if and only if \((G,p)\) has an \((S,\Phi)\)-symmetry-preserving flex.

Proof. Since \( p \) is a regular point of both \( G \) and \( K_n \) in \( V^{(i_1)}_e \), it follows from Lemma 4.1 that there exist neighborhoods \( N_p \) and \( N'_p \) of \( p \) in \( V^{(i_1)}_e \) so that \( \tilde{f}^{-1}_G(\tilde{f}_G(p)) \cap N_p \) is a manifold of dimension \( \dim (V^{(i_1)}_e) - \text{rank} \left( d\tilde{f}_G(p) \right) \) and \( \tilde{f}^{-1}_{K_n}(\tilde{f}_{K_n}(p)) \cap N'_p \) is a manifold of dimension \( \dim (V^{(i_1)}_e) - \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \).

Since \( \tilde{f}^{-1}_{K_n}(\tilde{f}_{K_n}(p)) \cap N''_p \) is a submanifold of \( \tilde{f}^{-1}_G(\tilde{f}_G(p)) \cap N''_p \), where \( N''_p = N_p \cap N'_p \), it follows that

\[
\text{rank} \left( d\tilde{f}_G(p) \right) \leq \text{rank} \left( d\tilde{f}_{K_n}(p) \right).
\]

Clearly, \( \text{rank} \left( d\tilde{f}_G(p) \right) = \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \) if and only if there exists a neighborhood \( N''_p \) of \( p \) in \( V^{(i_1)}_e \) such that \( \tilde{f}^{-1}_{K_n}(\tilde{f}_{K_n}(p)) \cap N''_p = \tilde{f}^{-1}_G(\tilde{f}_G(p)) \cap N''_p \).

Therefore, if \( \text{rank} \left( d\tilde{f}_G(p) \right) = \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \), then there does not exist an \((S,\Phi)\)-symmetry-preserving flex of \((G,p)\).

If \( \text{rank} \left( d\tilde{f}_G(p) \right) < \text{rank} \left( d\tilde{f}_{K_n}(p) \right) \), then every neighborhood of \( p \) in \( V^{(i_1)}_e \) contains elements of \( \tilde{f}^{-1}_G(\tilde{f}_G(p)) \setminus \tilde{f}^{-1}_{K_n}(\tilde{f}_{K_n}(p)) \), and hence, by the proof of Proposition 1 in [4] (and references therein), there exists an \((S,\Phi)\)-symmetry-preserving flex of \((G,p)\). This completes the proof. \( \square \)

In order to make further use of Theorem 4.2, we need the following fundamental observations.

Recall from Section 3.3 that with respect to the bases \( B_e \) and \( B_i \), the rigidity matrix of a framework \((G,p)\) in \( \mathcal{R}(G,S,\Phi) \) has the block form

\[
\tilde{R}(G,p) =
\begin{pmatrix}
\tilde{R}_1(G,p) & 0 \\
0 & \ldots & \tilde{R}_r(G,p)
\end{pmatrix},
\]

where for \( t = 1, \ldots, r \), the block \( \tilde{R}_t(G,p) \) corresponds to the irreducible linear representation \( I_t \) of \( S \), and the size of the block \( \tilde{R}_t(G,p) \) depends on the dimensions of the subspaces \( V^{(i_1)}_e \) of \( \mathbb{R}^{d_n} \) and \( V^{(i_k)}_e \) of \( \mathbb{R}^{|E(G)|} \). (In particular, the block \( \tilde{R}_t(G,p) \) is an empty \( 0 \times 0 \) matrix if and only if both of the coefficients \( \lambda_t \) and \( \mu_t \) in equations (2) and (3) are equal to zero.)

Since with respect to the bases \( B_e \) and \( B_i \), the Jacobian matrix of \( f_G \), evaluated at \( p \), is (up to a constant) the matrix \( \tilde{R}(G,p) \), it follows that with respect to the bases \( B^{(i_1)}_e \) and \( B_i \), the Jacobian matrix of \( \tilde{f}_G \), evaluated at the point \( p \in V^{(i_1)}_e \), is (up to a constant) the matrix

\[
\begin{pmatrix}
\tilde{R}_1(G,p) & 0 \\
0 & \ldots & 0
\end{pmatrix}.
\]
Thus, we have
\[ \text{rank } (\tilde{R}_1(G, p)) = \text{rank } (d\tilde{f}_G(p)). \] (12)

Furthermore, note that if \( K_n \) is the complete graph on the vertex set \( V(G) \), then with respect to the bases \( B_i \) and \( \hat{B}_i \), where \( \hat{B}_i \) is an appropriate extension of the basis \( B_i \), the rigidity matrix of \((K_n, p)\) has a block form analogous to the one of \( \tilde{R}(G, p) \) in (11), namely
\[
\tilde{R}(K_n, p) = \begin{pmatrix}
\tilde{R}_1(K_n, p) & 0 \\
0 & \ddots \\
0 & \tilde{R}_r(K_n, p)
\end{pmatrix}.
\]

Clearly, \( \tilde{R}_t(G, p) \) is a submatrix of \( \tilde{R}_t(K_n, p) \) for all \( t = 1, \ldots, r \). Moreover, analogously to (12), we have
\[ \text{rank } (\tilde{R}_1(K_n, p)) = \text{rank } (d\tilde{f}_{K_n}(p)). \] (13)

Recall from Definition 3.3 that if \( u \in V_e^{(I_1)} \), then \( u \) is said to be symmetric with respect to \( I_1 \). So, if we think of the vector \( u \in \mathbb{R}^n \) as a set of displacement vectors with one vector at each joint of \((G, p) \in \mathcal{P}(G, S, \Phi)\), then \( u \) is symmetric with respect to \( I_1 \) if and only if all of the displacement vectors remain unchanged under all symmetry operations in \( S \). A vector \( u \in \mathbb{R}^n \) that is symmetric with respect to \( I_1 \) can therefore also be termed fully \((S, \Phi)\)-symmetric \([19, 22]\) (see also Figure 6).

Figure 6: Fully \((S, \Phi)\)-symmetric infinitesimal motions of frameworks: (a) a fully \((C_s, \Phi)\)-symmetric infinitesimal rigid motion of \((K_3, p) \in \mathcal{P}(K_3, C_s, \Phi)\); (b) a fully \((C_s, \Phi)\)-symmetric infinitesimal flex of \((K_{3,3}, p) \in \mathcal{P}(K_{3,3}, C_s, \Phi)\). Since each of the above frameworks is an injective realization, the type \( \Phi \) is uniquely determined in each case \([25, 22]\).

**Theorem 4.3** Let \( G \) be a graph, \( S \) be a symmetry group in dimension \( d \), \( \Phi : S \rightarrow \text{Aut}(G) \) be a homomorphism, and \((G, p)\) be a framework in \( \mathcal{P}(G, S, \Phi) \) with the property that the points \( p(v), v \in V(G) \), span all of \( \mathbb{R}^d \). If \( p \) is a regular point of \( G \) in \( V_e^{(I_1)} \) and also a regular point of \( K_n \) in \( V_e^{(I_1)} \) and there exists a fully \((S, \Phi)\)-symmetric infinitesimal flex of \((G, p)\), then there also exists an \((S, \Phi)\)-symmetry-preserving flex of \((G, p)\).
Proof. Let $K_n$ be the complete graph on $V(G)$. Since the points $p(v)$, $v \in V(G)$, span all of $\mathbb{R}^d$, the kernel of $R(K_n,p)$ is the space of all infinitesimal rigid motions of $(G,p)$ and the kernel of $\tilde{R}_1(K_n,p)$ is the space of all fully $(S,\Phi)$-symmetric infinitesimal rigid motions of $(G,p)$. So, since $(G,p)$ has an infinitesimal flex which is fully $(S,\Phi)$-symmetric, we have nullity $\tilde{R}_1(G,p) > \tilde{R}_1(K_n,p)$, and hence
\[ \text{rank} (\tilde{R}_1(G,p)) < \text{rank} (\tilde{R}_1(K_n,p)). \] (14)

Since, by (12), we have $\text{rank} (\tilde{R}_1(G,p)) = \text{rank} (d\tilde{f}_G(p))$ and, by (15), we have $\text{rank} (\tilde{R}_1(K_n,p)) = \text{rank} (d\tilde{f}_{K_n}(p))$, it follows from (14) that
\[ \text{rank} (d\tilde{f}_G(p)) < \text{rank} (d\tilde{f}_{K_n}(p)). \]

The result now follows from Theorem 4.2.

The above results concerning the subspace $V_e^{(I_1)}$ of $\mathbb{R}^{dn}$ may be transferred analogously to the affine subspaces of $\mathbb{R}^{dn}$ of the form $p + V_e^{(I_1)}$, where $t \neq 1$.

More precisely, if we define a point $q \in p + V_e^{(I_1)}$ to be a regular point of a graph $G$ in $p + V_e^{(I_1)}$, if there exists a neighborhood $N_q$ of $q$ in $p + V_e^{(I_1)}$ so that $\text{rank} (d\tilde{f}_G(q')) = \max \{ \text{rank} (d\tilde{f}_G(q)) \}$ for all $q' \in N_q$, where $f_G$ denotes the restriction of the edge function $f_G$ to $p + V_e^{(I_1)}$, then the following results can be proved completely analogously to the Theorems 4.2 and 4.3.

**Theorem 4.4** Let $G$ be a graph with $n$ vertices, $S$ be a symmetry group, $\Phi : S \rightarrow \text{Aut}(G)$ be a homomorphism, and $(G,p)$ be a framework in $\mathbb{R}(G,S,\Phi)$. If $p$ is a regular point of $G$ in $p + V_e^{(I_1)}$ and also a regular point of $K_n$ in $p + V_e^{(I_1)}$, then
\begin{enumerate}
\item[(i)] $\text{rank} (d\tilde{f}_G(p)) = \text{rank} (d\tilde{f}_{K_n}(p))$ if and only if $(G,p)$ does not have a flex $x$ with $x(t) \in p + V_e^{(I_1)}$ for all $t \in [0,1]$;
\item[(ii)] $\text{rank} (d\tilde{f}_G(p)) < \text{rank} (d\tilde{f}_{K_n}(p))$ if and only if $(G,p)$ has a flex $x$ with $x(t) \in p + V_e^{(I_1)}$ for all $t \in [0,1]$.
\end{enumerate}

**Theorem 4.5** Let $G$ be a graph, $S$ be a symmetry group in dimension $d$, $\Phi : S \rightarrow \text{Aut}(G)$ be a homomorphism, and $(G,p)$ be a framework in $\mathbb{R}(G,S,\Phi)$ with the property that the points $p(v)$, $v \in V(G)$, span all of $\mathbb{R}^d$. If $p$ is a regular point of $G$ in $p + V_e^{(I_1)}$ and also a regular point of $K_n$ in $p + V_e^{(I_1)}$ and there exists an infinitesimal flex $u$ of $(G,p)$ with $u \in V_e^{(I_1)}$, then there also exists a flex $x$ of $(G,p)$ with $x(t) \in p + V_e^{(I_1)}$ for all $t \in [0,1]$.

Note that if we define $\ker (I_1) = \{ x \in S | I_1(x) = id \}$, where $id$ is the identity transformation, then $\ker (I_1)$ is a normal subgroup of $S$ (see [21], for example). Therefore, Theorems 4.3 and 4.5 provide us with sufficient conditions for the existence of a flex of $(G,p)$ that preserves the sub-symmetry of $(G,p)$ given by $\ker (I_1)$ and $\Phi|_{\ker (I_1)}$.

An important property of the subspace $V_e^{(I_1)}$ which does not hold for the affine subspaces $p + V_e^{(I_1)}$, where $t \neq 1$, is that, by Corollary 5.2, for every
q ∈ V_{e}^{(I_{1})}, the rigidity matrix \( \tilde{R}(G, q) \) has the same block structure as the rigidity matrix \( \tilde{R}(G, p) \). Thus, \( p ∈ V_{e}^{(I_{1})} \) is a regular point of \( G \) in \( V_{e}^{(I_{1})} \) if and only if there exists a neighborhood \( N_{p} \) of \( p \) in \( V_{e}^{(I_{1})} \) so that \( \text{rank} (\tilde{R}_{1}(G, p)) \geq \text{rank} (\tilde{R}_{1}(G, q)) \) for all \( q ∈ N_{p} \).

Similarly, \( p ∈ V_{e}^{(I_{1})} \) is a regular point of \( K_{n} \) in \( V_{e}^{(I_{1})} \) if and only if there exists a neighborhood \( N_{p} \) of \( p \) in \( V_{e}^{(I_{1})} \) so that \( \text{rank} (\tilde{R}_{1}(K_{n}, p)) \geq \text{rank} (\tilde{R}_{1}(K_{n}, q)) \) for all \( q ∈ N_{p} \).

The fact that regular points of \( G \) and \( K_{n} \) in \( V_{e}^{(I_{1})} \) can be characterized in this way is essential to proving all the remaining results of this section. These results turn out to be very useful for practical applications of Theorem 4.6 as we will see in Section 5 (see also \[31\] as well as Section 6.3 in \[28\]).

**Theorem 4.6** Let \( G \) be a graph with \( n \) vertices, \( S \) be a symmetry group in dimension \( d \), \( \Phi : S → \text{Aut}(G) \) be a homomorphism, and \( (G, p) \) be a framework in \( \mathcal{A}_{(G, S, \Phi)} \). If the points \( p(v), v ∈ V(G) \), span all of \( \mathbb{R}^{d} \), then \( p \) is a regular point of \( K_{n} \) in \( V_{e}^{(I_{1})} \).

**Proof.** Since the points \( p(v), v ∈ V(G) \), span all of \( \mathbb{R}^{d} \), there exists a neighborhood \( N_{p} \) of \( p \) in \( V_{e}^{(I_{1})} \) so that for all \( q ∈ N_{p} \), the points \( q(v), v ∈ V(G) \), also span all of \( \mathbb{R}^{d} \). Therefore, for all \( q ∈ N_{p} \), the dimension of the subspace of \( \mathbb{R}^{dn} \) consisting of all fully \((S, \Phi)\)-symmetric infinitesimal rigid motions of \((G, p)\) is equal to the dimension of the subspace of \( \mathbb{R}^{dn} \) consisting of all fully \((S, \Phi)\)-symmetric infinitesimal rigid motions of \((G, q)\) (see \[27\] \[28\] for details). Therefore, we have \( \text{rank} (\tilde{R}_{1}(K_{n}, p)) = \text{rank} (\tilde{R}_{1}(K_{n}, q)) \) or equivalently, by \[13\], \( \text{rank} (d\tilde{f}_{K_{n}}(p)) = \text{rank} (d\tilde{f}_{K_{n}}(q)) \) for all \( q ∈ N_{p} \). Thus, \( p \) is a regular point of \( K_{n} \) in \( V_{e}^{(I_{1})} \). \( \square \)

By Theorem 4.6, the condition that \( p \) is a regular point of \( K_{n} \) in \( V_{e}^{(I_{1})} \) may be omitted in Theorem 4.4.

**Theorem 4.7** Let \( G \) be a graph, \( S \) be a symmetry group, \( \Phi : S → \text{Aut}(G) \) be a homomorphism, and \( (G, p) \) be a framework in \( \mathcal{A}_{(G, S, \Phi)} \). If \( p \) is \((S, \Phi)\)-generic, then \( p \) is a regular point of \( G \) in \( V_{e}^{(I_{1})} \).

**Proof.** Suppose \( G \) is a graph with \( n \) vertices and \( S \) is a symmetry group in dimension \( d \) with \( r \) pairwise non-equivalent irreducible representations \( I_{1}, \ldots, I_{r} \). Fix a basis \( U = \{v_{1}, \ldots, v_{k}\} \) of \( U = V_{e}^{(I_{1})} = \bigcap_{x ∈ S} L_{x, \Phi} \) and let \( p = t_{1}v_{1} + \ldots + t_{k}v_{k} \). Then the symmetry-adapted ‘indeterminate’ rigidity matrix \( R_{\mathcal{A}_{n}}(n, d) \) for \( \mathcal{A}_{(G, S, \Phi)} \) is a matrix in the variables \( t_{1}, \ldots, t_{k} \). More precisely, the entries of \( R_{\mathcal{A}_{n}}(n, d) \) are elements of the quotient field of the integral domain \( \mathbb{R}[t_{1}, \ldots, t_{k}] \). Over this field we can again do linear algebra. We let \( R_{\mathcal{A}_{n}}^{(G)}(n, d) \) denote the submatrix of \( R_{\mathcal{A}_{n}}(n, d) \) that corresponds to the submatrix \( R(G, p) \) of \( R(K_{n}, p) \), i.e., \( R_{\mathcal{A}_{n}}^{(G)}(n, d) \) is obtained from \( R_{\mathcal{A}_{n}}(n, d) \) by deleting those rows that do not correspond to edges of \( G \). If we replace each variable \( t_{i} \) in \( R_{\mathcal{A}_{n}}^{(G)}(n, d) \) with \( t_{i} \), then, by Remark 3.2 we obtain the rigidity matrix \( R(G, p) \). Therefore,

\[
\text{rank} (R(G, p)) \leq \text{rank} (R_{\mathcal{A}_{n}}^{(G)}(n, d)).
\]
Since \((G, p)\) is \((S, \Phi)\)-generic, we also have
\[
\text{rank } (R(G, p)) \geq \text{rank } (R_{\mathcal{B}_i}^{(G)}(n, d)),
\]
and hence
\[
\text{rank } (R(G, p)) = \text{rank } (R_{\mathcal{B}_i}^{(G)}(n, d)). \quad (15)
\]
Now, let \(T_e\) be the matrix of the basis transformation from the canonical basis of the \(\mathbb{R}\)-vector space \(\mathbb{R}^{dn}\) to the basis \(B_e\), and let \(T_i\) be the matrix of the basis transformation from the canonical basis of the \(\mathbb{R}\)-vector space \(\mathbb{R}^{|E(G)|}\) to the basis \(B_i\), so that the matrix \(\tilde{R}(G, p) = T_i^{-1}R(G, p)T_e\) is block-diagonalized as in (11). Then, by Corollary 4.2, the matrix \(\tilde{R}_{\mathcal{B}_i}^{(G)}(n, d) = T_i^{-1}R_{\mathcal{B}_i}^{(G)}(n, d)T_e\) has the same block form as \(\tilde{R}(G, p)\). For \(t = 1, \ldots, r\), let \(\tilde{R}_t^{(G)}(n, d)\) denote the block of \(\tilde{R}_{\mathcal{B}_i}^{(G)}(n, d)\) that corresponds to the block \(\tilde{R}_t(G, p)\) of \(\tilde{R}(G, p)\). Since the rank of a matrix is invariant under a basis transformation, and since the rank of a block-diagonalized matrix is equal to the sum of the ranks of its blocks, it follows from equation (15) that
\[
\sum_{t=1}^{r} \text{rank } (\tilde{R}_t(G, p)) = \text{rank } (\tilde{R}(G, p)) = \text{rank } (\tilde{R}_{\mathcal{B}_i}^{(G)}(n, d)) = \text{rank } (\tilde{R}_t^{(G)}(n, d)) = \sum_{t=1}^{r} \text{rank } (\tilde{R}_t^{(G)}(n, d)).
\]
Since we clearly have \(\text{rank } (\tilde{R}_t(G, p)) \leq \text{rank } (\tilde{R}_t^{(G)}(n, d))\) for each \(t\), it follows that \(\text{rank } (\tilde{R}_t(G, p)) = \text{rank } (\tilde{R}_t^{(G)}(n, d))\) for each \(t\). This gives the result. □

**Corollary 4.8** Let \(G\) be a graph, \(S\) be a symmetry group in dimension \(d\), \(\Phi : S \to \text{Aut}(G)\) be a homomorphism, and \((G, p)\) be a framework in \(\mathfrak{F}_{(G,S,\Phi)}\) with the property that the points \(p(v), v \in V(G)\), span all of \(\mathbb{R}^d\). If \((G, p)\) is \((S, \Phi)\)-generic and \((G, p)\) has a fully \((S, \Phi)\)-symmetric infinitesimal flex, then there also exists an \((S, \Phi)\)-symmetry-preserving flex of \((G, p)\).

**Proof.** The result follows immediately from Theorems 4.3, 4.6, and 4.7 □

In Section 5, we will use Corollary 4.3 to prove the existence of an \((S, \Phi)\)-symmetry-preserving flex for a variety of symmetric octahedral frameworks in 3-space (including two of the three types of ‘Bricard octahedra’ [5]). A number of other classes of frameworks, such as the ones examined in [4, 8, 13, 28, 34], can also be shown to be flexible with the help of Corollary 4.8.

Note that the framework in Figure 6(b) is not \((\mathcal{C}_s, \Phi)\)-generic (recall Section 3.2 and Figure 3), so that Corollary 4.8 does not apply to this framework. In fact, it can be verified that the framework in Figure 6(b) does not possess any flex, let alone a \((\mathcal{C}_s, \Phi)\)-symmetry-preserving flex.

Corollary 4.8 is a symmetrized version of Corollary 2.5 in Section 2.3. Next, we show that a symmetrized version of Corollary 2.6 can also be obtained from the previous results.
Corollary 4.9 Let $G$ be a graph, $S$ be a symmetry group in dimension $d$, $\Phi : S \to \text{Aut}(G)$ be a homomorphism, and $(G, p)$ be a framework in $\mathcal{R}(G, S, \Phi)$ with the property that the points $p(v), v \in V(G)$, span all of $\mathbb{R}^d$. If the block $\tilde{R}_1(G, p)$ of the block-diagonalized rigidity matrix $\tilde{R}(G, p)$ has linearly independent rows and $(G, p)$ has a fully $(S, \Phi)$-symmetric infinitesimal flex, then there also exists an $(S, \Phi)$-symmetry-preserving flex of $(G, p)$.

Proof. Since the block matrix $\tilde{R}_1(G, p)$ has linearly independent rows, $p$ is a regular point of $G$ in $V_i^{(1)}$. The result now follows from Theorems 4.3 and 4.6. □

Corollary 4.9 confirms the observation made by R. Kangwai and S. Guest in [22]. Note that the condition that the block matrix $\tilde{R}_1(G, p)$ has linearly independent rows is equivalent to the condition that the framework $(G, p)$ has no fully $(S, \Phi)$-symmetric non-zero self-stress, i.e., a non-zero self-stress in the subspace $V_i^{(1)}$ of $\mathbb{R} |E(G)|$. In particular, it follows that if $(G, p)$ is independent (i.e., $(G, p)$ does not possess any non-zero self-stress) and there exists a fully $(S, \Phi)$-symmetric infinitesimal flex of $(G, p)$, then there also exists an $(S, \Phi)$-symmetry-preserving flex of $(G, p)$.

In order to apply Corollary 4.9 to a given framework $(G, p)$, we need to compute the rank of the submatrix block $\tilde{R}_1(G, p)$. This can be done by finding the block-diagonalized rigidity matrix $\tilde{R}(G, p)$ with the methods and algorithms described in [14, 25], for example.

The rank of the submatrix block $\tilde{R}_1(G, p)$ can also be determined directly by finding the rank of an appropriate ‘orbit rigidity matrix’ whose columns and rows correspond to a set of representatives for the orbits of the group action from $S \times V(G)$ to $V(G)$ that sends $(x, v)$ to $\Phi(x)(v)$ and a set of representatives for the orbits of the group action from $S \times E(G)$ to $E(G)$ that sends $(x, e)$ to $\Phi(x)(e)$, respectively. The kernel of this matrix is the space of fully $(S, \Phi)$-symmetric infinitesimal motions of $(G, p)$ and the cokernel of this matrix is the space of fully $(S, \Phi)$-symmetric self-stresses of $(G, p)$. Further details on the ‘orbit rigidity matrix’ will be presented in [31].

5 Examples

In his famous paper from 1897, the French engineer R. Bricard proved that if an octahedron in 3-space with no self-intersecting faces is realized as a framework by placing bars along edges, and joints at vertices, then this framework must be rigid [5]. Moreover, he showed that there exist three distinct types of octahedra with self-intersecting faces whose realizations as frameworks are flexible. Two of these three types of octahedra possess non-trivial symmetries: Bricard octahedra of the first type have a half-turn symmetry and Bricard octahedra of the second type have a mirror symmetry. In the following, we consider both of these types of symmetric Bricard octahedra (as well as octahedra with dihedral symmetry) and use the results of Section 4 to not only show that they are flexible, but also that they possess a ‘symmetry-preserving’ flex.

Various other treatments of the Bricard octahedra can be found in [2] [33], for example. R. Connelly’s celebrated counterexample to Euler’s rigidity conjecture from 1776 (see [15]) is also based on a flexible Bricard octahedron (of
Let $G$ be the graph of the octahedron (see Figure 7), $C_2$ be a ‘half-turn’ symmetry group in dimension 3, and $\Phi_a : C_2 \to \text{Aut}(G)$ be the homomorphism defined by

\[
\Phi_a(Id) = id \\
\Phi_a(C_2) = (v_1 v_3)(v_2 v_4)(v_5 v_6).
\]

From the symmetry-extended version of Maxwell’s rule, applied to the $(C_2, \Phi_a)$-generic framework $(G, p)$ in Figure 7, we may deduce that $(G, p)$ has a fully $(C_2, \Phi_a)$-symmetric infinitesimal flex. It follows from Corollary 4.8 that $(G, p)$ also has a $(C_2, \Phi_a)$-symmetry-preserving flex.

Remark 5.1 Note that some of the configurations that lie on the path of the $(C_2, \Phi_a)$-symmetry-preserving flex of $(G, p)$ are not $(C_2, \Phi_a)$-generic.

For example, the $(C_2, \Phi_a)$-symmetry-preserving flex of $(G, p)$ passes through a configuration $q$ with the property that the four points $q_1, q_2, q_3,$ and $q_4$ are coplanar. The framework $(G, q)$ is therefore clearly not $(C_2, \Phi_a)$-generic. However, by computing the rank of $R_1(G, q)$ and showing that it is equal to the rank of $R_1(G, p)$, where $p$ is $(C_2, \Phi_a)$-generic, the configuration $q$ can be proven to be a regular point of $G$ in $V^I_{e}(I_1)$, where $I_1$ is the trivial irreducible representation of $C_2$. So, Theorem 4.3 can be used in this case to prove the existence of a $(C_2, \Phi_a)$-symmetry-preserving flex of $(G, q)$.

From the symmetry-extended version of Maxwell’s rule, applied to the framework $(G, p)$ in Figure 8, the symmetry group $C_s = \{Id, s\}$, and the homomorphism $\Phi_b : C_s \to \text{Aut}(G)$ defined by

\[
\Phi_b(Id) = id \\
\Phi_b(s) = (v_1 v_3)(v_2)(v_5 v_6),
\]
it follows that \((G, p)\), as well as any other \((\mathcal{C}_s, \Phi_b)\)-generic realization of \(G\), has a fully \((\mathcal{C}_s, \Phi_b)\)-symmetric infinitesimal flex \([15, 27, 28]\). Thus, by Corollary 4.8, any such realization of \(G\) also has a \((\mathcal{C}_s, \Phi_b)\)-symmetry-preserving flex.

Finally, consider the \((\mathcal{C}_{2v}, \Phi_c)\)-generic framework \((G, p)\) in Figure 8, where \(\Phi_c : \mathcal{C}_{2v} \to \text{Aut}(G)\) is the unique type determined by the injective realization of \(G\). Although \((G, p)\) is neither \((\mathcal{C}_2, \Phi_a)\)-generic nor \((\mathcal{C}_s, \Phi_b)\)-generic, we anticipate from the discussion above that \((G, p)\) possesses a flex that preserves both the \(C_2\) and the \(C_s\) symmetry defined in these examples.

The symmetry-extended version of Maxwell’s rule applied to \((G, p), C_{2v},\) and \(\Phi_c\) detects a fully \((\mathcal{C}_{2v}, \Phi_c)\)-symmetric flex \([15, 27, 28]\). Thus, by Corollary 4.8, the framework \((G, p)\), as well as any other \((\mathcal{C}_{2v}, \Phi_c)\)-generic realization of \(G\), indeed possesses a \((\mathcal{C}_{2v}, \Phi_c)\)-symmetry-preserving flex.

Remark 5.2 If \(G\) is the graph of the octahedron, \(\mathcal{C}_s\) is a symmetry group in
dimension 3, and $\Phi_d : \mathcal{C}_s \to \text{Aut}(G)$ is defined by

\[
\Phi_d(Id) = id \\
\Phi_d(s) = (v_2 v_4)(v_1)(v_3)(v_5)(v_6),
\]

then $G$ is $(\mathcal{C}_s, \Phi_d)$-generically isostatic. The framework $(G, p)$ in Figure 10, for example, is a realization of $G$ in $\mathcal{R}_{(G, \mathcal{C}_s, \Phi_d)}$ which is isostatic by Cauchy’s Theorem [6, 12].

![Figure 10: An isostatic octahedron in $\mathcal{R}_{(G, \mathcal{C}_s, \Phi_d)}$.](image)

**Remark 5.3** The above rigidity analyses of symmetric octahedra can directly be extended to analyses of symmetric frameworks that consist of an arbitrary $2n$-gon and two ‘cone-vertices’ that are linked to each of the joints of the $2n$-gon. These kinds of frameworks are also known as ‘double-suspensions’, and are studied in [8], for example.

A number of other interesting and famous examples of symmetric frameworks can also be proven to be flexible with the methods presented in this paper. These include Bottema’s famous mechanism in the plane (see [4], for example), ring structures and reticulated cylinder structures in 3-space like the ones examined in [19] and [34], for example, and various types of bipartite frameworks in 3-space (see [28]). Each of these structures possesses a ‘symmetry-preserving’ flex, and it is precisely this kind of flex that our symmetry-based methods detect in each case. While detection of these flexes is not new, our new approach allows a much simpler verification of the flexes than previous methods. New flexes can also be detected, and some will be presented in [31].

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