DeepStochLog: Neural Stochastic Logic Programming

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Abstract

Recent advances in neural symbolic learning, such as DeepProbLog, extend probabilistic logic programs with neural predicates. Like graphical models, these probabilistic logic programs define a probability distribution over possible worlds, for which inference is computationally hard. We propose DeepStochLog, an alternative neural symbolic framework based on stochastic definite clause grammars, a type of stochastic logic program, which defines a probability distribution over possible derivations. More specifically, we introduce neural grammar rules into stochastic definite clause grammars to create a framework that can be trained end-to-end. We show that inference and learning in neural stochastic logic programming scale much better than for neural probabilistic logic programs. Furthermore, the experimental evaluation shows that DeepStochLog achieves state-of-the-art results on challenging neural symbolic learning tasks.

1 Introduction

The integration of neural and symbolic learning methods is high on the research agenda. It is popular to use (variants of) logic programs to represent the available symbolic knowledge and use Prolog-like mechanisms to generate computation structures that can then be differentiated [23, 7, 31, 43, 5, 37, 36]. Several of these approaches also incorporate probability into these neural logic programming models, cf. [8]. There are several ways to incorporate probability into logic programs [9]. Most notably, one distinguishes probabilistic from stochastic logic programs (PLPs vs SLPs). The more popular PLPs are based on a possible worlds semantics (the so-called distribution semantics), which extends probabilistic graphical models, while the SLPs are based on stochastic grammars. The difference can also be described as a random graph vs a random walk model. So far, the emphasis in neurosymbolic computation has been on the PLP approach, especially [43, 23, 37, 40], with only Tensorlog [5] adopting the SLP semantics in an efficient but restricted Datalog or database setting that does not handle subsymbolic data such as images (see Section 7 for details).

To fill this gap, we introduce DeepStochLog, a neural stochastic logic programming approach. It incorporates ideas from DeepProbLog such as the neural predicate. The neural predicate encapsulates neural networks to cope, for instance, with subsymbolic data such as images. Without loss of generality, we base DeepStochLog on stochastic definite clause grammars (SDCGs) as this notation is not only easier to introduce and use, but also results in a sequence-based model. SDCGs are a kind of probabilistic unification-based grammar formalism [15]. However, SDCGs and SLPs are very closely related. SDCGs can be directly translated and executed as SLPs, and all the concepts we introduce for SDCGs can directly apply to SLPs as well. More specifically, the key contributions of this paper are: 1) the introduction of the neural stochastic logic programming framework DeepStochLog; 2) the introduction of inference and learning algorithms (through gradient

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Note that this paper will be extended in the future.
2 Stochastic DCGs

A context-free grammar (CFG) \( G \) is a 4-tuple \( (V, \Sigma, S, R) \), with \( V \) the set of non-terminals, \( \Sigma \) the set of terminals, \( S \in V \) the starting symbol and \( R \) a set of rewrite rules of the form \( N \rightarrow S_1, \ldots, S_k \) where \( N \) is a non-terminal, the \( S_i \) are either terminals or non-terminals. A probabilistic context-free grammar (PCFG) extends a CFG by adding probabilities to the rules, i.e., the rules take the form \( p_i : N \rightarrow S_1, \ldots, S_k \), where \( p_i \) is a probability. Furthermore, the sum of the probabilities of rules with the same non-terminal \( N \) on the left-hand side equals 1. We use list notation for sequences of terminals such as \([\text{cat}]\) and \([\text{the, cat}]\). Whereas CFGs define whether a sequence can be parsed, PCFGs define a probability distribution over possible parses. This allows for the most likely parse to be identified. An example PCFG, representing single digit additions, is shown on the left of Example 1.

Definite clause grammars (DCGs) are a well-known logic programming-based extension of CFGs \([28]\). DCGs can represent context-sensitive languages and are unification-based. They differ from CFGs in that logical atoms are used instead of the non-terminals. An atom \( a(t_1, \ldots, t_n) \) consists of a predicate \( a \) of arity \( n \) followed by \( n \) terms \( t_i \). Terms are either constants, logical variables or structured terms of the form \( f(t_1, \ldots, t_k) \) with \( f \) a functor and \( t_j \) terms. The production rules are called definite clause grammar rules because they can be directly translated to a set of definite clauses (i.e., Horn clauses with exactly one positive literal) and can be executed as a Prolog program using SLD-resolution. The right hand side of DCG rules are also allowed to contain queries to Prolog predicates \( q \) between curly brackets \( \{q_1(t_{1,1}, \ldots, t_{1,m_1}), \ldots, q_n(t_{n,1}, \ldots, t_{n,m_n})\} \) to impose further constraints and perform additional computations during the inference process. These are to be considered atoms as well. DCGs use substitutions \( \{V_1 = t_1, \ldots, V_n = t_k\} \), which are sets of variable/term pairs, to unify atoms with heads from the rules. Applying a substitution to an atom \( a \) yields the atom \( a \theta \) where all variables \( V_i \) have been replaced by their corresponding terms \( t_i \). \( \theta \) is a unifier of an atom \( s \) and an atom \( u \) if and only if \( s \theta = u \theta \). For more information, see standard textbooks on logic programming such as \([13, 38]\).

Stochastic definite clause grammars (SDCGs) extend DCGs by associating probabilities to the rules, just like how PCFGs extend CFGs \([15]\). As PCFGs, SDCGs require that the sum of the probabilities for the rules defining a single non-terminal predicate equals 1. SDCGs also correspond directly to stochastic logic programs (SLP) \([6, 25, 26]\) which are well-known in the probabilistic (logic) programming community \([9]\). An example SDCG is shown in Example 1 on the right.

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**Example 1.** A PCFG (left) and a similar SDCG (right) that constrains the result of the expression.

\[
\begin{align*}
0.5 &:: E \rightarrow N \\
0.5 &:: e(N) \rightarrow n(N) \\
0.5 &:: E \rightarrow E;[+]_N \\
0.5 &:: e(N) \rightarrow e(N1),[+],n(N2),\{N \text{ is N1 + N2}\} \\
0.1 &:: N \rightarrow [0] \quad \ldots \\
0.1 &:: N \rightarrow [9] \\
0.1 &:: n(0) \rightarrow [0] \quad \ldots \\
0.1 &:: n(9) \rightarrow [9]
\end{align*}
\]

The inference task in (S)DCGs consists of deriving a sequence of terminals from a goal (which often captures the starting symbol of the grammar). SLD-derivations are used for this. More formally, in an SLD-derivation for a DCG, a goal \( g_1, \ldots, g_n \) is a sequence where each \( g_i \) is either a logical atom (a non-terminal) or a list containing terminals and logical variables. An SLD-derivation is shown in Example 2 and uses several resolution steps. Applying resolution to a goal \( g_1, \ldots, g_n \) and a definite clause grammar rule \( n \rightarrow t_1, \ldots, t_k \) yields the goal \( g_1 \theta, \ldots, g_{n-1} \theta, t_1 \theta, \ldots, t_k \theta, g_{n+1} \theta, \ldots, g_n \theta \) provided that \( g_i \) is the leftmost atom in the goal (so \( g_1, \ldots, g_{n-1} \) are terminal symbols), \( \theta \) is the unifier of \( g_i \) and \( n \), i.e., \( g_i \theta = n \theta \). We write this as \( g_1, \ldots, g_n \vdash t_1 \theta, \ldots, t_k \theta, s_2 \theta, \ldots, s_n \theta \). A derivation

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\(^1\)When a Prolog query \( q \) is the first non-terminal to occur in a goal during the derivation process, the query is
$d(S)$ is then the repeated application $G \vdash G_1 \theta_1 \vdash G_2 \theta_2 \vdash \ldots \vdash G_n \theta_1 \theta_2 \ldots \theta_n$ of such resolution steps onto a goal G. We will write $G \vdash^* G_n$. Successful derivations of a goal end in a sequence $T$ that consists only of terminal symbols, see Example 1 for an example. We will write that $d(S) = T$ and also say that $\text{derives}(S \theta, T)$ is true, with $\theta = \theta_1 \theta_2 \ldots \theta_n$ the answer substitution. A successful derivation corresponds to a proof. The set of all possible proofs can be depicted using SLD-trees, see Figure 1a.

The probability $P(d(G))$ of a derivation $d(G)$ is the product of the probabilities $\prod p_i^{m_i}$ of the rules $i$ used in the derivation with $m_i$ the number of times the rule $i$ was used. An important difference between the probability of a parse in a PCFG and a derivation in an SDCG is that there can be a loss of probability mass in the latter whenever a derivation fails. Derivations can fail when there are non-terminals in the goal that do not unify with the heads of any of the rules. This is due to unification and is different from (P)CFGs, where non-terminals can always be resolved using rules for that non-terminal. Non-terminating derivations can also lead to loss of probability mass. Observe that every $G$ in this way induces a probability distribution $P_G$ over possible derivations $d(G)$. The goal can consist of one or more atoms, but in general for parsing, this will typically be the starting symbol or atom of the grammar. This in turn lets us define the probability $P_G(\text{derives}(G, T))$ of a terminal sequence $T$ relative to the goal $G$ as

$$\sum_{d_i(G\theta)=T} P_G(d_i(G\theta)), \text{i.e. the sum of the probabilities of all derivations } G\theta \text{ for } G \text{ that result in the terminal sequence } T \text{ with answer substitution } \theta.$$ 

In a similar way, this allows to define the probability of an answer substitution $\theta$ relative to a goal $G$ and sequence $t$ as $P_G(\text{derives}(G\theta, t\theta))$ where $t\theta$ could contain variables. For ease of notation, when the goal is clear, we shall omit the subscript $G$. Notice that if there are failing derivations, the total probability mass assigned to all sequences of terminals for a goal $G$ may be strictly less than 1. This is discussed at length by [6]. It is possible to obtain normalized probabilities by calculating the normalization constant, but this is computationally expensive. We avoid this normalization in the present paper because in practice, the goal is often to find the most likely derivation $d_{\text{max}}(G, T) = \arg \max_{d(G)=T} P_G(d(G))$, non-normalized probabilities usually suffice. Notice that a SDCG defines a parametric probability distribution, where the parameters $p$ of the distribution are the vector of probabilities of the rules. When we need to refer to these probabilities we write $P_G(\text{derives}(G, T); p)$.

**Example 2** (Derivations). Consider the following successful derivation using the SDCG in Example 1 for the goal $G = [e(X)]$, the answer substitution $\theta = \{X/2\}$ and the terminal sequence $T = [2, +, 0]$.

\[
\begin{align*}
\text{Example 2} \\
\text{Consider the following successful derivation using the SDCG in Example 1}} \\
\text{for the goal } G = [e(X)], \text{ the answer substitution } \theta = \{X/2\} \text{ and the terminal} \\
\text{sequence } T = [2, +, 0].
\end{align*}
\]

\[
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\text{Example 2} \\
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\text{for the goal } G = [e(X)], \text{ the answer substitution } \theta = \{X/2\} \text{ and the terminal} \\
\text{sequence } T = [2, +, 0].
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\[
\begin{align*}
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\text{for the goal } G = [e(X)], \text{ the answer substitution } \theta = \{X/2\} \text{ and the terminal} \\
\text{sequence } T = [2, +, 0].
\end{align*}
\]

\[
\begin{align*}
e(X) & \vdash e(N1), [+], n(N2), \{X \text{ is } N1 + N2\} & \theta_1 = {} & p = 0.5 \\
& \vdash n(N1), [+], n(N2), \{X \text{ is } N1 + N2\} & \theta_2 = {} & \times0.5 \\
& \vdash [2, +], n(N2), \{X \text{ is } 2 + N2\} & \theta_3 = \{N1/2\} & \times0.1 \\
& \vdash [2, +, 0], 2 \text{ is } 2 + 0 & \theta_4 = \{X/2, N2/0\} & \times0.1
\end{align*}
\]

### 3 DeepStochLog

DeepStochLog integrates neural networks and SDCGs by introducing neural definite clause grammars (NDCG). More formally, DeepStochLog allows for specifying an SDCG that additionally supports neural definite clause grammar rules, or neural rules for short. These are statements of the form:

\[nn(m; [I_1, ..., I_m], [O_1, ..., O_L], [D_1, ..., D_L]) :: nt \rightarrow g_1, ..., g_n\]

where $nt$ is an atom, $g_1, ..., g_n$ is a goal, and the $I_1...I_m$ and $O_1,...,O_L$ are variables occurring in $g_1,...,g_n$ and $nt$. The $D_i$ are unary predicates defining the domain of the output variables $O_i$. The $nn$ declaration states that $m$ is a neural network that takes the variables $I_1,...,I_m$ as input and

executed in Prolog possibly yielding an answer substitution $\theta$ such that $\phi\theta$ is true. For instance, in Example 2 there is the Prolog query $N \in N1 + N2$ which computes $N$ as the sum of $N1$ and $N2$. In this paper, we assume that when such a query is called there is at most one answer substitution that is true. If there were more such substitutions, we would have to introduce a probability for such substitutions in the SDCG case, which unnecessarily complicates the semantics.
outputs a probability distribution over output variables $O_1, ..., O_L$ (i.e. a probability distribution over the cross product of the domains specified by $D_i$). It thus maps an input substitution $\sigma$ for the variables $I_1, ..., I_m$ to a set of output substitutions $\theta_j$ with probability $p_j$. The neural rule serves as a template. For every input substitution $\sigma$, the template $(nt \rightarrow g_1, ..., g_n)\sigma$ defines the set of instantiated stochastic definite clause grammar rules $p_j : (nt \rightarrow g_1, ..., g_n)\sigma\theta_j$.

Example 3 (Neural definite clause grammar rules). Consider the SDCG in Example 1. We can substitute the $n(X) \rightarrow [X]$ rules with the following neural rule

$$nn(mnist, [Mnist], [N], [digit]) :: n(N) \rightarrow [Mnist].$$

Here, the neural network called $nn$ takes as input the $Mnist$ image and returns a probability for every possible number between 0 and 9, indicating how likely it is that every number is for the given MNIST image [20]. The predicate $digit$ is defined as $digit(0), digit(1), ..., digit(9)$. Given the neural network and the input substitution $\sigma = \{Mnist = \mathcal{O}\}$ (which could be obtained through unification with the terminal sequence), the neural network could generate the output substitutions $\theta_0 = \{N = 0\}; ...; \theta_9 = \{N = 9\}$, with probabilities 0.87; ...; 0.05. Thus, the neural rule with the input substitution $\sigma = \{Mnist = \mathcal{O}\}$ denotes the following set of grammar rules: 0.87 :: $n(0) \rightarrow \mathcal{O}$; ...; 0.05 :: $n(9) \rightarrow \mathcal{O}$

The neural rules are reminiscent of the neural predicates in DeepProbLog [23], which also encapsulate a neural network that outputs a distribution over a number of alternatives. It is worth analyzing how a neural rule behaves w.r.t the neural inputs (e.g. images). In fact, a neural rule defines a probability distribution over the values of the output variables given the neural inputs, whose distribution is not modeled in the program. This conditional setting is akin to conditional PCFGs [20, 39] and it is a common modeling strategy in discriminative parsing.

4 Inference in DeepStochLog

The goal of the inference is to compute the probability $P_G(derives(G, T))$ for a given goal $G$ and (possibly unknown) sequence $T$. This is divided into two steps, a logical and a probabilistic one, which we now explain.

Logical inference Given a DeepStochLog program, a goal $G$ and a (potentially unknown) sequence $T$, logical inference uses resolution to answer $derives(G, T)$. This corresponds to finding all the possible derivations for $G$ that result in a terminal sequence $T$. The resolution process is then turned into a compact AND-OR circuit, which represents all possible derivations and will be the input for the probabilistic inference. The logical inference procedure is illustrated in Figure 1 where the SLD resolution tree for the given goal is on the left and its corresponding AND-OR circuit on the right. The translation to the AND-OR circuit is straightforward. It has exactly the same structure as the SLD-tree. For every resolution step with a rule $p_i : r_i$, an AND node is added. Furthermore, for a normal SDCG rule, the corresponding probability $p_i$ is added, and for a neural grammar rule, there is a call to the neural network that returns the probability $p_m$. Whenever there are two (or more) branches in the SLD tree for a goal, an OR node is added. Notice that all the leaves are either probabilities given as parameters or the result of a neural call.

During SLD resolution, many identical intermediate goals may be proved multiple times which results in an explosion of inference time. To avoid proving the same goals, we use SLG resolution [4], which plays a similar role as the dynamic programming CYK algorithm for CFGs [16]. Tabling using SLG resolution is a standard logic programming technique that memoizes the answers of predicates by tabling the evaluations. This technique is incorporated in Prolog implementations such as XSB, SWI-Prolog and Prism [34]. The important difference with SLD resolution is that

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2DeepStochLog could also be used to define generative grammars on subsymbolic inputs (e.g. images) if provided with neural models that can provide a joint probability distribution of both outputs and images. This is also discussed in [23] but will not be further analyzed in the current paper.

3This is sometimes called phrase or sentence in actual Prolog implementations and it requires an automatic syntactical translation of a DCG into Prolog. We show an example in Appendix C.
the results are not a single derivation tree, but rather a forest, where certain parts are re-used for multiple derivations thanks to tabled evaluations. The effect of tabling is carried over to the creation of the AND-OR tree. Each time a derivation is reused from the table, its corresponding node is returned and linked to the new derivation. Thus, also the AND-OR circuit turns into a forest. When a goal admits a finite set of answers, we can resolve it only once and cache the corresponding AND-OR tree.

**Probabilistic inference.** With probabilistic inference, we refer to the task of calculating the probability $P(\text{derives}(G,T)) = \sum_{d(G\theta)=T} P(d(G\theta)) = \sum_{d(G\theta)=T} \prod_{i\in d(G\theta)} P_i^{\theta_i}$, i.e. the sum of the probabilities of all derivations for a given $G$ that result in a given terminal sequence $T$ and answer substitution $\theta$. Thanks to SLG resolution and tabling, the shared sub-structure of many derivations is explicit in the AND-OR circuit obtained from the logical inference. This dissipates the need for a specialized algorithm, like the inside algorithm used in the probabilistic extension of CYK. Computing the probability $P(\text{derives}(G\theta,T))$ is just a bottom-up evaluation of the AND-OR circuit where AND-nodes are substituted by multiplications and OR-nodes by summations, i.e. compiling the logical circuit to an arithmetic circuit using the $(+,\times)$ semiring \cite{DBLP:journals/ml/Deatom06}. Analogously, the most probable derivation for the goal $G$ is found with the $(\max,\times)$ semiring.

## 5 Learning in DeepStochLog

Learning in DeepStochLog is achieved by optimizing the parameters of the neural network and the parameters of the logic programs itself. Let us consider a dataset of triples $D = \{(G_i, \theta_i, T_i, t_i)\}$, where $G_i$ is a goal, $\theta_i$ a substitution for $G_i$, $T_i$ a sequence of terminals and $t_i$ a target probability. Let us also consider a DeepStochLog program parameterized by the vector $\theta$ of rule probabilities. Learning in DeepStochLog is defined as the following optimization problem, with $\mathcal{L}$ being any differentiable loss function:

$$\min_{\theta} \sum_{(G_i, \theta_i, T_i, t_i)\in D} \mathcal{L}
(P_G(\text{derives}(G_i\theta_i, T_i); p), t_i)$$

Representing the dynamic programming computation for the inside probability in terms of an arithmetic circuit has an important advantage. In fact, the corresponding computational graph is differentiable and the derivatives of the loss function $\mathcal{L}$ w.r.t. the probabilities $p$ can be carried out automatically using out-of-the-box differentiation frameworks. Moreover, when the probabilities $p$ are computed by a neural network as for a neural grammar rule, the gradients can be seamlessly backpropagated to the network to train its internal parameters. We solve the learning problem using standard gradient descent techniques from deep learning, e.g. the Adam optimizer \cite{Kingma2014Adam}. One interesting case is when the loss function $\mathcal{L}$ is the negative log-likelihood, as it brings DeepStochLog into the standard learning scenario for probabilistic grammars. Here, the optimization problem is usually carried out in the expectation-maximization (EM) framework. Given an inside algorithm, a correspondent outside algorithm is designed to extract the expected counts of the
various grammar rules from data (E-step) and then the counts are used to update the probabilities (M-step). Most of the developed inside-outside algorithms are tailored to a specific formalism. However, the gradient descent approach of DeepStochLog on the negative log-likelihood is actually equivalent to the EM approach but it does not require the explicit definition of the corresponding outside algorithm. In fact, the gradients obtained by the backward pass through the AND-OR circuit have been shown to actually compute the outside probabilities (E-step), while the gradient descent step is used to update the parameters (M-step) [33, 2, 13].

6 Evaluation

6.1 Research Questions

The goal of our experiments is to answer the following questions:

Q1 Does DeepStochLog reach state-of-the-art predictive performance on neural-symbolic tasks?

Q2 How does the inference time of DeepStochLog compare to other neural-symbolic frameworks and what is the role of tabling?

Q3 Can DeepStochLog handle larger-scale tasks?

Q4 Can DeepStochLog go beyond grammars and encode more general programs?

6.2 Tasks

We specify the tasks used in this paper. Complete details are specified in Appendix [13]

T1: MNIST Addition. In the MNIST Addition task [23], the model is given two sequences of length $N$ of MNIST images containing handwritten images [20], each representing an $N$-digit number. The task is to predict the sum of these numbers $37 + 25 = 62$. The training data only contains the two image sequences and the sum of the corresponding numbers, thus not providing the digit labels of the individual images. The datasets for each digit length use all 60K images of MNIST images exactly once.

T2: Handwritten Formulas. In the Handwritten Formulas (HWF) task, the goal is to solve mathematical expressions, where both digits and operators (addition, subtraction, multiplication and division) are images of handwritten characters, like $[337 \times 252]$. Like T1, the data of T2 only contains the outcome of the expression and the sequence of images. For this task, we use the Handwritten Formula (HWF) dataset, introduced in [21]. The dataset contains 10000 expressions of lengths 1, 3, 5 and 7. Unlike the original paper, we do not consider a curriculum learning setting here, and split the dataset into 4 separate parts by length.

T3: Well-formed Parentheses. We introduce the Well-formed Parentheses task, where the model is asked to recognize image sequences that represent well-formed parentheses. Well-formed parentheses is a classic context-free grammar language where $\Sigma = \{(, \})$, and $R = \{ s \rightarrow (()) | (ss) \}$, i.e. all open brackets are closed in the right order. As images, we use the zeros from MNIST as “(” and ones as “)”, and generate 1000 well-formed parenthesis sequences without labels as training data. The goal is to predict the most probable parse of the bracket sequence.

T4: Context-Sensitive Grammar. Since DCGs support context-sensitive grammars, we created a dataset of 2000 image sequences representing the canonical context-sensitive grammar $a^n b^k c^m$. Since each sequence of length $3n$ only has one valid parse, we increased the difficulty by allowing permutations such as $b^k a^n c^m$. We also generated 2000 negative examples, i.e. random sequences of the form $a^k b^l c^m$, and permutations like $c^k a^l b^m$ such that $k$, $l$, $m$ are all larger than 1, sum to a multiple of 3 and are not all the same number. The goal of the task is to recognize whether the input sequence belongs to the first grammar or the second.

T5: Semi-supervised classification in citation networks. Given a set of scientific papers represented as bag-of-words and their citation network, the goal is to assign the correct class to a large test set of documents by having access only to the true labels of a small training set. The intuition is that one can infer the class of a paper not only by the features of the document but also by the class of its neighbors. This task is interesting from a neural symbolic perspective because one
must be able to use both the features of the documents and the symbolic network. Two well-known datasets for this task are the Cora (2708 nodes and 5429 edges) and Citeseer (3327 nodes and 9228 edges) [35].

T6: Word Algebra Problems. In this task, a natural language text describes a word algebra problem (e.g., “Mark has 6 apples. He eats 2 and divides the remaining among his 2 friends. How many apples did each friend get?”). This dataset of this task contains 300 training instances and was introduced in [32]. Each text contains 3 numbers, and all numbers have to be used exactly once in a formula containing addition subtraction, multiplication and division. The task is to predict the right numerical answer to the expression implied by textual description.

6.3 Results
For all experiments and metrics, we report the mean accuracy (or the mean most likely parse accuracy where applicable, i.e. \( T1 \), \( T2 \) and \( T3 \)) and standard deviation over 5 runs. We report ‘timeout’ if a single of these 5 runs took more than 1 hour to execute.

Q1: Performance of DeepStochLog We first investigate whether DeepStochLog achieves state-of-the-art results compared to similar neural-symbolic frameworks. Table 1 shows the result for the MNIST addition task (T1), for training and testing on lengths 1 to 4. It shows that DeepStochLog performs similarly to the DeepProbLog [23] and NeurASP [43] frameworks but scales to larger sequences. Table 2 shows the result on the HWF task (T2). DeepStochLog performs similar to NGS and DeepProbLog for expressions of length 1 and 3. Starting from expression length 5, it becomes infeasible to train DeepProbLog. NGS [21] can still be trained, but for expression length 7, some runs fail to converge. DeepStochLog, however, performs well for all expression lengths. Table 3 shows the accuracy on task T3. Here we can see that both DeepStochLog and DeepProbLog achieve high accuracy, but DeepStochLog reaches a slightly higher accuracy for a longer length. For task T6, DeepStochLog and DeepProbLog achieve a similar accuracy of 94.8±1.1 and 94.2±1.4 respectively. We also compare to δ4 [3], but the authors only report the maximum accuracy reached. For all three frameworks, the maximum accuracy reached is 96.0. To conclude, DeepStochLog is able to achieve similar or better performance compared to other state-of-the-art neural-symbolic frameworks.

Q2: DeepStochLog scalability We now investigate whether DeepStochLog is more scalable than similar neural-symbolic frameworks. First, we observe that in the tasks T1, T2, T4 and T5, DeepStochLog scales to settings or datasets that are infeasible for the competitors. Next, in Table 7 we compare the execution times for inference in task T1. We selected 100 queries from the training

\[ \text{T1:} \] The test accuracy (%) on the MNIST addition (T1).

| Number of digits per number (N) | 1       | 2       | 3       | 4       |
|--------------------------------|---------|---------|---------|---------|
| NeurASP                        | 97.3 ± 0.3 | 93.9 ± 0.7 | timeout | timeout |
| DeepProbLog                    | 97.2 ± 0.5 | 95.2 ± 1.7 | timeout | timeout |
| DeepStochLog                   | 97.9 ± 0.1 | 96.4 ± 0.1 | 94.5 ± 1.1 | 92.7 ± 0.6 |

\[ \text{T2:} \] The accuracy (%) on the HWF dataset (T2).

| Expression length | 1       | 3       | 5       | 7       |
|-------------------|---------|---------|---------|---------|
| NGS               | 90.2 ± 1.6 | 85.7 ± 1.0 | 91.7 ± 1.3 | 20.4 ± 37.2 |
| DeepProbLog       | 90.8 ± 1.3 | 85.6 ± 1.1 | timeout | timeout |
| DeepStochLog      | 90.8 ± 1.0 | 86.3 ± 1.9 | 92.1 ± 1.4 | 94.8 ± 0.9 |

\[ \text{T6:} \] Word Algebra Problems. In this task, a natural language text describes a word algebra problem (e.g., "Mark has 6 apples. He eats 2 and divides the remaining among his 2 friends. How many apples did each friend get?"). This dataset of this task contains 300 training instances and was introduced in [32]. Each text contains 3 numbers, and all numbers have to be used exactly once in a formula containing addition subtraction, multiplication and division. The task is to predict the right numerical answer to the expression implied by textual description.

On the Cora dataset, another common task is link prediction, which is a purely symbolic task (i.e. there are no subsymbolic features on the documents).
| Method             | Maximum expression length |
|--------------------|----------------------------|
| DeepProbLog        | 100.0 ± 0.0                |
| DeepStochLog       | 100.0 ± 0.0                |

Table 4: The accuracy (%) on the $a^n b^n c^n$ dataset (T4).

| Method             | Expression length |
|--------------------|-------------------|
| DeepProbLog        | 99.8 ± 0.3        |
| DeepStochLog       | 99.4 ± 0.5        |

Q3: Larger scale relational datasets

The complexity of many of the previous experiments comes from the large number of derivations for a single goal, while the number of subsymbolic inputs (e.g. images) in a single relational example was quite limited. Here, we focus on task T5, i.e. semi-supervised classification in citation networks, where the complexity mainly comes from the large number of elements of the unique relational example, i.e. the citation network. This task is usually out of the scope of (neural) PLP approaches due to the fact that there is a unique large relational example and the possible world semantics is prohibitive in this scenario. We compare against the following baselines: label propagation (LP) [44], semi-supervised embedding (SemiEmb) [42], manifold regularization (ManiReg) [1], skip-gram based graph embeddings (DeepWalk) [29], ICA [22] and GCN [19]. All these baselines are specific to the semi-supervised classification task, while DeepStochLog is a much more general framework. We finally tried to compare with DeepProbLog, which, however, does not scale to the size of this problem due to the different probabilistic semantics. Results are reported in Table 5. DeepStochLog compares similarly or favorably to most of the other methods, even though it is the only one that has not been developed for the specific task. However, it still underperforms w.r.t. ICA and GCN. But these methods use extra knowledge as input to the classifier in the form of precomputed or learned features of the neighbors of a document, which is very useful for this task but not considered in the DeepStochLog experiment. Adding or learning relational features for input to the neural modules is, however, an interesting future direction.

Q4: General programs in DeepStochLog

Even though DeepStochLog naturally represents grammars for parsing sequences, NDCGs with Prolog goals are a powerful formalism to express more complex relational problems and programs. Actually, both task T5 and T6 have been solved with programs that depart from the pure grammar formalism and are more like general logic programs. We provide the complete models in Appendix B. The main ingredients are (neural) empty production rules, sometimes referred to as non-consuming or $\epsilon$-production rules. They allow to take probabilistic decisions, including also neural networks, without consuming any element of the sequence, as shown in Example 4. This also shows that DeepStochLog has the full power of stochastic logic programs.

Table 3: The parse accuracy (%) on the well-formed parentheses dataset (T3).
Table 5: Q3 Accuracy (%) of the classification on the test nodes on task T5

| Method       | Citeseer | Cora |
|--------------|----------|------|
| ManiReg      | 60.1     | 59.5 |
| SemiEmb      | 59.6     | 59.0 |
| LP           | 45.3     | 68.0 |
| DeepWalk     | 43.2     | 67.2 |
| ICA          | 69.1     | 75.1 |
| GCN          | 70.3     | 81.5 |
| DeepProbLog  | timeout  | timeout |
| DeepStochLog | 65.0     | 69.4 |

Table 6: Q4 Parsing time in seconds (T2). Comparison of the DeepStochLog with and without tabling (SLD vs SLG resolution).

| Lengths | # Answers | No Tabling | Tabling |
|---------|-----------|------------|---------|
| 1       | 10        | 0.067      | 0.060   |
| 3       | 95        | 0.081      | 0.096   |
| 5       | 1066      | 3.78       | 0.95    |
| 7       | 10386     | 30.42      | 10.95   |
| 9       | 68298     | 1494.23    | 132.26  |
| 11      | 416517    | timeout    | 1996.09 |

Table 7: Inference times in milliseconds for DeepStochLog, DeepProbLog and NeurASP on task T1 for variable number lengths.

| Numbers Length | 1     | 2     | 3     | 4     |
|----------------|------|------|------|------|
| DeepStochLog   | 1.3 ± 0.9 | 2.3 ± 0.4 | 4.0 ± 0.4 | 5.7 ± 1.8 |
| DeepProbLog    | 13.5 ± 3.0 | 36.0 ± 0.5 | 199.7 ± 14.0 | timeout |
| NeurASP        | 9.2 ± 1.4 | 85.7 ± 22.6 | 158.2 ± 47.7 | timeout |

Example 4 (Empty productions). We show a variant of the MNIST Addition problem using empty productions.

\[
nn([\text{mnist}, [X], [Y], [\text{digit}]] : \text{number}(X, Y) \rightarrow []).

\[
\text{addition}(X, Y, N) \rightarrow \text{number}(X, N1), \text{number}(Y, N2), N \equiv N1 + N2.
\]

This grammar will always produce the empty sequence but through the Prolog unification mechanism and the probabilistic modeling, we can express complex stochastic logic programs that include calls to neural networks.

7 Related Works

DeepStochLog is an expressive neural symbolic framework whose distinguishing features are: 1) it is based on the expressive stochastic logic programming paradigm, which can express probabilistic programs (as in T5-T6) as well as probabilistic unification based grammars (T1-T4); 2) it can work with both symbolic and subsymbolic data such as images (as shown in T1-T4); and 3) its inference and learning mechanism is based on SLG-resolution that naturally supports tabling, a form of dynamic programming (as shown in Q2).

There are several strands of related research. First, DeepStochLog is a neural logic programming language in the spirit of DeepProbLog [23], NeurASP [43], the neural theorem prover [31] and lifted relational neural networks (LRNNs) [37]. The first two systems are based on a probabilistic possible world semantics, while DeepStochLog is based on stochastic grammars, which—as we have shown—scales much better (in part also due to the use of tabling). The latter two approaches focus on Datalog (which cannot deal with function symbols) and use the logic to construct the neural network in a kind of knowledge based model construction approach. Furthermore, they are neither probabilistic nor do they deal with subsymbolic inputs such as images. Another related system is Tensorlog [5], which is based on stochastic logic programming. While sharing their roots in SLPs, it is less expressive than DeepStochLog, as it considers only Datalog and predicates of arity 2. While Tensorlog’s implementation is fast thanks to being in terms of tensors, it has only been applied to symbolic data.

Second, DeepStochLog can be viewed as a neural-based grammar, similarly to Neural Gram-
Neural Grammars have been introduced in the natural language community as an effective strategy to learn PCFGs. They are neural parameterizations of PCFG and it is possible to learn the structure of the grammar by enumerating a set of candidate rules and using neural networks to learn their probabilities. Differently from DeepStochLog, they are restricted to context-free grammars. Furthermore, Neural Grammars [12] do not consider subsymbolic inputs (as in all our tasks T1-T6). Different from the probabilistic interface of DeepStochLog, NGS uses backsearch, a greedy search that defines the backward feedback from the grammar to the neural nets. While this makes NGS very scalable, the backsearch must be defined per-program, while DeepStochLog backpropagates evidence automatically through any NDCG. Neural Attribute Grammars [27] integrate attribute grammars with neural networks. While this is also an expressive grammatical framework, they are quite different from DeepStochLog in their approaches and applications, and also are not applied to subsymbolic data.

Third, many systems in the neural symbolic community [11, 10, 37] obtain differentiable logics by relaxing logical programs or theories using fuzzy logic and t-norms. While the shift in semantics from probabilistic to fuzzy logic has known issues [41], fuzzy logic allows for more scalable systems as compared to probabilistic logic based on the possible world semantics. But by exploiting the stochastic grammars, DeepStochLog shows the same benefits as fuzzy logic in terms of computational complexity (i.e. no disjoint-sum problem required) by resorting to an alternative probabilistic semantics.

8 Conclusions

We have introduced a novel and very expressive neural symbolic model based on stochastic logic programming, that allows to integrate symbolic knowledge with subsymbolic representations, that scales well, and gives state-of-the-art results on various neural symbolic computation tasks.

There are several limitations of DeepStochLog that we want to explore in further research. First, DeepStochLog does not yet learn the structure of the rules, while the neural theorem prover [31], DiffLog [36] and the neural grammars [12] can all enumerate rules and then identify the most relevant ones. Second, DeepStochLog’s inference could be further optimised by parallelization of the circuit using ideas from TensorLog [5]. Third, SLPs and hence, DeepStochLog, may lose probability mass due to failing derivations. This can be addressed by normalizing and computing the partition function [6]. It would be interesting to approximate the partition function and also to further speed up the inference by sampling or by searching for the k-best derivations. Finally, it would be interesting to explore the use of DeepStochLog as a generative model to generate sequences.

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A Appendix

A.1 Data and Licenses

- For tasks T1, T3 and T4, we used the MNIST dataset from [20] to generate new datasets. The MNIST dataset itself was released under the Creative Commons Attribution-Share Alike 3.0 license.
- The Handwritten Formula Recognition (HWF) dataset (used in T2) originates from [21].
- The Cora and Citeseer datasets (T5) are from [35].
- The dataset of the Word Algebra Problem (T6) originates from [32].

A.2 Computational details

Inference time experiments are all executed on a MacBookPro 13 2020 (2.3 GHz Quad-Core Intel Core i7 and 16 GB 3733 MHz LPDDR4).

B Task Details

In this section we show the full DeepStochLog programs and additional experimental details for all tasks.

B.1 MNIST Digit Addition

For the MNIST addition problem, we trained the model for 25 epochs using the Adam optimizer with a learning rate of 0.001 and used 32 training terms in each batch for each digit length.

DeepStochLog program for single-digit numbers:

```prolog
1  digit(Y) :- member(Y,[0,1,2,3,4,5,6,7,8,9]).
2  nn(number, [X],[Y],[digit]):: number(Y) --> [X].
3  addition(N) --> number(N1), number(N2), {N is N1+N2}.
```

DeepStochLog program for \(L\)-long digits numbers:

```prolog
1  digit(Y) :- member(Y,[0,1,2,3,4,5,6,7,8,9]).
2  nn(number, [X],[Y],[digit]):: number(Y) --> [X].
3  addition(N) --> number(N1), number(N2), {N is N1+N2}.
4  0.5::multi_addition(N, 1) --> addition(N).
5  0.5::multi_addition(N, L) --> {L > 1, L2 is L - 1},
6        addition(N1), multi_addition(N2, L2),
7        {N is N1*(10**L2) + N2}.
```

B.2 Handwritten Mathematical Expressions

For the Handwritten Mathematical Expression problem, we trained the model for 20 epochs using the Adam optimizer with a learning rate of 0.003 and a batch size of 2. We used two separate, similar neural networks for recognising the numbers and the operators (see Table 8). The encoder of the neural networks has a convolutional layer with 1 input channel, 6 output channels, kernel size 3, stride 1, padding 1, a ReLu, max pooling with kernel size 2, a convolutional layer with 6 inputs, 16 outputs, kernel size 3, stride 1, padding 1, a ReLu, a max pooling with kernel size 2, and a 2d dropout layer with \(p = 0.4\). They then use two fully connected layers, one from 1936 to 128 with a ReLu, and one linear to the number of classes (10 and 4 respectively). This problem is modeled as follows in DeepStochLog:
B.3 Well-formed Parentheses

We ran the well-formed parenthesis problem for 1 epoch using the Adam optimizer with a learning rate of 0.001 and a batch size of 4. This problem is modelled as follows:

\[
\begin{align*}
\text{bracket_d(Y)} & : - \text{member(Y, ["(", "]")].} \\
\text{s_switch_d(Y)} & : - \text{member(Y, [0, 1, 2]).}
\end{align*}
\]

\[
\begin{align*}
\text{nn(bracket_nn, [X], [Y], [bracket_d])} & : \text{bracket(Y) --> [X].} \\
\text{nn(s_nn, [X], [Y], [s_switch_d])} & : \text{s --> s_switch(Y).} \\
0.33 & : \text{s_switch(0) --> s, s.} \\
0.33 & : \text{s_switch(1) --> bracket("(", s, bracket(")").} \\
0.33 & : \text{s_switch(2) --> bracket("(", bracket(")").}
\end{align*}
\]

B.4 Context-Sensitive Grammar

The model was trained on the canonical context-sensitive grammar \(a^nb^nc^n\) problem for 1 epoch using the Adam optimizer with a learning rate of 0.001 and batch size 4. This problem is modelled in DeepStochLog as follows:

\[
\begin{align*}
\text{rep_d(Y)} & : \text{domain(Y, [a, b, c]).} \\
0.5 & : \text{s(0) --> akblcm(K, L, M), \{K\=L; L\=M; M\=K\}.} \\
0.5 & : \text{s(1) --> akblcm(N, N, N).} \\
\text{akblcm(K, L, M)} & : \text{rep(K, A), rep(L, B), rep(M, C), \{A\=B, B\=C, C\=A\}.} \\
0.5 & : \text{rep(s(0), C) --> terminal(C).} \\
0.5 & : \text{rep(s(N), C) --> terminal(C), rep(N, C).} \\
\text{nn(mnist, [X], [C], [rep_d])} & : \text{terminal(C) --> [X].}
\end{align*}
\]

B.5 Semi-supervised classification in citation networks

The model was trained on the Cora and Citeseer datasets using the Adam optimizer with a learning rate of 0.01. We trained for 100 epochs and we selected the model corresponding to the epoch with
the maximum accuracy on the validation set. We can easily and concisely express this problem in DeepStochLog as follows:

```prolog
class(Y) :- member(Y, [class0, class1, ..., class6]).
nn(classifier, [X], [Y], [class]):: doc_neural(X, Y) --> [].
1/Na :: cite(a, b) --> [].
1/Na :: cite(a, c) --> [].
1/Nb :: cite(b, d) --> [].
...
0.5:: doc(X, Y) --> doc_neural(X, Y).
0.5:: doc(X, Y) --> cite(X, X1), doc(X1, Y).
...
s(X) --> doc(X, Y), [Y].
```

The program simply states that a document \( X \) is of class \( Y \) (and produce the terminal \( Y \)) either if a neural network classifies it so or if it is cited by one or more documents of the same class. Even if strange, many of these citation networks are actually cyclic. Therefore, we limited the depth of the derivations to a maximum value to avoid having a cyclical program. We also experimented with a variant of the program in which the probabilities of the rules expanding the \( \text{doc} \) predicate are trained separately for each class \( Y \).

### B.6 Word Algebra Problem

We trained the model on the Word Algebra Problem for 40 epochs using the Adam optimizer with a learning rate of 0.001 and a batch size of 32. The DeepStochLog program is then modelled as follows:

```prolog
permute_d(Y) :- member(Y, [0, 1, 2, 3, 4, 5]).
op(Y) :- member(Y, [plus, minus, times, div]).
swap_d(Y) :- member(Y, [no_swap, swap]).
nn(nn_permute, [X], [Y], [permute_d]):: nn_permute(X, Y) --> [].
nn(nn_op1, [X], [0], [op]):: nn_op1(X, 0) --> [].
nn(nn_swap, [X], [S], [swap_d]):: nn_swap(X, S) --> [].
nn(nn_op2, [X], [0], [op]):: nn_op2(X, 0) --> [].
0.1666:: permute(0, A, B, C, A, B, C) --> [].
0.1666:: permute(1, A, B, C, A, C, B) --> [].
0.1666:: permute(2, A, B, C, B, A, C) --> [].
0.1666:: permute(3, A, B, C, B, C, A) --> [].
0.1666:: permute(4, A, B, C, C, A, B) --> [].
0.1666:: permute(5, A, B, C, C, B, A) --> [].
0.5:: swap(no_swap, X, Y, X, Y) --> [].
0.5:: swap(swap, X, Y, X, Y) --> [].
0.25:: operator(plus, X, Y, Z) --> [], \{Z is X+Y\}.
0.25:: operator(minus, X, Y, Z) --> [], \{Z is X-Y\}.
0.25:: operator(times, X, Y, Z) --> [], \{Z is X\*Y\}.
0.25:: operator(div, X, Y, Z) --> [], \{Y > 0, 0 =:= X mod Y, Z is X/Y\}.
s(Out, X1, X2, X3) --> [String],
    nn_permute(String, Perm),
    nn_op1(String, Op1),
```
B.7 Neural network architectures

Table 8 summarizes the neural network architectures used in the experiment. $\text{Conv}(o, k)$ is a convolutional layer with $o$ output channels and kernel size $k$. $\text{Lin}(n)$ is a fully-connected layer of size $n$. $(\text{Bi})\text{GRU}(h)$ is a single-layer (bi-directional) GRU with a hidden size $h$. $\text{MaxPool}(k)$ is a max-pooling layer with kernel size $k$, and $\text{Dropout}(p)$ a dropout layer with probability $p$. A layer in bold means it is followed by a ReLU activation function. All neural networks end with a Softmax layer, unless otherwise specified.

Table 8: Overview of the neural network architectures used in the experiments.

| Task   | Network | Architecture                        |
|--------|---------|-------------------------------------|
| T1     | number  | MNISTConv, $\text{Lin}(120)$, $\text{Lin}(84)$, $\text{Lin}(10)$ |
| T2     | number  | $\text{Conv}(6, 3)$, $\text{MaxPool}(2)$, $\text{Conv}(16, 3)$, $\text{MaxPool}(2)$, $\text{Dropout}(0.4)$, $\text{Lin}(128)$, $\text{Lin}(10)$ |
|        | operator| $\text{Conv}(6, 3)$, $\text{MaxPool}(2)$, $\text{Conv}(16, 3)$, $\text{MaxPool}(2)$, $\text{Dropout}(0.4)$, $\text{Lin}(128)$, $\text{Lin}(4)$ |
| T3     | bracket nn | MNISTConv, $\text{Lin}(120)$, $\text{Lin}(84)$, $\text{Lin}(2)$ |
| T4     | mnist  | MNISTConv, $\text{Lin}(120)$, $\text{Lin}(84)$, $\text{Lin}(3)$ |
| T5     | classifier-Citeseer | $\text{Lin}(50)$, $\text{Lin}(6)$ |
|        | classifier-Cora  | $\text{Lin}(50)$, $\text{Lin}(7)$ |
| T6     | RNN     | Embedding(256), BiGRU(512), Dropout(0.5)* |
|        | nn_permute | $\text{Lin}(6)$ |
|        | nn_op1   | $\text{Lin}(4)$ |
|        | nn_swap   | $\text{Lin}(2)$ |
|        | nn_op2   | $\text{Lin}(4)$ |

* Does not end with a Softmax layer.

C Translation Example of Neural Definite Clause Grammar to Prolog

Consider the following DeepStochLog program.

```
1 digit(Y) :- member(Y, [0,1,2,3,4,5,6,7,8,9]).
2 op(Y) :- member(Y, [+,-]).
3 nn(mnist,[I],[N],[digit]) :: n(N) --> [I].
4 nn(operator,[I],[N],[op]) :: o(N) --> [I].
5 0.33::e(N) --> n(N).
6 0.33::e(S) --> e(E1), o(+), n(E2), \{S is E1 + E2\}.
7 0.33::e(S) --> e(E1), o(-), n(E2), \{S is E1 - E2\}.
```

It is translated into:
Both the calls to \texttt{nn} and \texttt{p} are considered always true during logical inference. During evaluation of the arithmetic circuit, \texttt{nn(mnist,[I],[O])} returns the probability of the output \(O\) when provided with the image \(I\) as input, while \texttt{p(x)} constantly returns the probability \(x\). It is easy to see that any derivation of the Prolog program always end with either a \texttt{nn} or a \texttt{p} call, which constitute the leaves of the correspondent arithmetic circuit.