T-duality, Gerbes and Loop Spaces

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Abstract

We revisit sigma models on target spaces given by a principal torus fibration $X \to M$, and show how treating the 2-form $B$ as a gerbe connection captures the gauging obstructions and the global constraints on the T-duality. We show that a gerbe connection on $X$, which is invariant with respect to the torus action, yields an affine double torus fibration $Y$ over the base space $M$ — the generalization of the correspondence space. We construct a symplectic form on the cotangent bundle to the loop space $LY$ and study the relation of its symmetries to T-duality. We find that geometric T-duality is possible if and only if the torus symmetry is generated by Hamiltonian vector fields. Put differently, the obstruction to T-duality is the non-Hamiltonian action of the symmetry group.

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1 Introduction

T-duality is a perturbative symmetry of string theory and has played an important role in a wide number of applications, ranging from the study of WZW models to flux compactifications. One curious aspect of this duality is that due to mixing of the metric and the $B$ field under its action, it may connect backgrounds which are very different not only geometrically but also topologically.

Conventional $T$-duality is defined for certain backgrounds with isometries in which there is an action of an $n$-dimensional torus on the target space $X$ preserving the metric and the 3-form $H$, which is locally given by the exterior derivative of the $B$-field. The equations of motion of the two-dimensional field theory then have global symmetries. The standard procedure for deriving the duality starts by gauging these isometries, i.e. by making the symmetries local by coupling to
world-sheet gauge fields. Then the target space of the model is enlarged to a space \( Y \) by adding fields which provide the \( n \) extra coordinates of \( Y \) and which couple as Lagrange multipliers. One can then either eliminate the extra coordinates to recover the original model with target \( X \), or integrate out the fibre coordinates of the torus fibration of \( X \) to obtain the \( T \)-dual theory, and the two \( T \)-dual models define the same quantum theory. Performing the calculations classically (locally) give the well-known formulae for the transformation of the target space metric and \( B \)-field. However some steps of the procedure outlined above can have global obstructions, which we revisit in this paper. See [1, 2] for the obstructions to gauging sigma models with WZ term, e.g. [3, 4, 5] for \( T \)-duality from gauging sigma-models, [6] for the obstructions to \( T \)-duality.

**Obstructions to gauging and \( T \)-duality.** A well studied case of geometric \( T \)-duality is that of a target space \( X \) that is a principal circle fibration over a base manifold \( M \). The enlarged space of the sigma model — the correspondence space \( Y \) — can be thought of as arising from the geometrization of the \( B \)-field and can be represented as two independent circle fibrations over \( M \) (see [7, 8, 9] for detailed discussions and generalizations). Two independent projections give the two dual geometries - the original \( X \) or the dual \( \tilde{X} \). The two manifolds are in general topologically distinct, since in the process of dualization we are exchanging the curvature of the original \( S^1 \) bundle with the integral of the 3-from \( H \) along the circle fibre. This picture can be extended to a higher dimensional case of principal torus fibre \( \mathbb{T}^n \), provided the right constraints on the \( B \)-field are imposed.

Consider a fibration \( \pi : X \to M \) with fibre \( \mathbb{T}^n \) and connection \( \Theta_I \) \((I = 1, \ldots, n)\) given by a globally well defined smooth 1-form on \( X \) with values in \( t := \text{Lie } \mathbb{T}^n \cong \mathbb{R}^n \). A closed 3-form \( H \) that is invariant with respect to the torus action can be written globally as

\[
H = \pi^* H_3 + \pi^* H_2^I \wedge \Theta_I + \frac{1}{2} \pi^* H_1^{IJ} \wedge \Theta_I \wedge \Theta_J + \frac{1}{6} \pi^* H_0^{JK} \Theta_I \wedge \Theta_J \wedge \Theta_K ,
\]

(1.1)

where \( H_j \in \Omega^j(M; \Lambda^{3-j} t) \) for \( j = 0, 1, 2, 3 \). Gauging the sigma model is not obstructed provided \( \Pi \)

\[
i(K^I) H = dv^I , \quad i(K^I) v^J + i(K^J) v^I = 0 ,
\]

(1.2)

where the vector fields \( K^I \) \((I, J = 1, \ldots, n)\) are the torus generators on \( X \) (the Lie derivative of \( \Theta \) with respect to \( K^I \) vanishes), \( v^I \) are globally well-defined one-forms and \( i(K^I) \) denotes the contraction with the vector \( K^I \). These conditions can be obtained by demanding that the gauged sigma-model action involves globally defined forms \( \Pi \) and are equivalent to requiring that the

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\(^1\)Our choice for the position of the indices \( I (I = 1, \ldots, n) \) is somewhat unconventional - they are placed down on objects that take values in \( \Lambda^* t \) and up - on objects in \( \Lambda^* t^* \). This way the formulae appear to be less cluttered with different indices. Whenever this will not cause a confusion, the index \( I \) will be suppressed.
3-form $H$ has an equivariant extension $\tilde{H}$ \cite{[11] \cite[12]} (This means $D\tilde{H} = 0$, where $D = d + \phi_I i(K^I)$ and $\phi_I$ are two-form generators of Lie $\mathbb{T}^n$. Imposing $\tilde{H}|_{\phi=0} = H$, allows us to write $\tilde{H} = H - \phi_I v^I$ iff \cite{[12]} holds.)

More recently it was shown that when the gauging and the addition of Lagrange multipliers is done together rather than in steps, these conditions can be weakened significantly \cite{[6], [19]}, to become

$$H^{IJK}_0 = 0, \quad H^{IJI} = dB^{IJ}_0,$$

(1.3)

where $B^{IJ}_0$ is globally defined. The basic picture of the correspondence space still holds — the geometrization of the $B$-field can still lead to a correspondence space $Y$ with a double-torus fibration over the base $M$. It becomes important to identify the correct connection that upon $T$-duality gets exchanged with the connection on the torus bundle. Its curvature is in the same de Rham cohomology class as the 2-form obtained from $H$ by a single contraction with a torus generator, $H_2^I = (K_I) \ H$ (and this class is no longer required to be trivial!).

One can see that, in the absence of $B_0^{IJ}$, the 2-form $H_2^I$ is closed and can be thought of as the curvature of a connection $\tilde{\Theta}$ on the dual principal torus fibre over $M$, $\tilde{\pi} : \tilde{X} \to M$, and we may indeed pass to the sigma model on the extended target space given by the fibrewise product $Y = X \times_M \tilde{X}$. $T$-duality acts to interchange $\Theta$ and $\tilde{\Theta}$.

**The generalized correspondence space.** Much of our understanding of $T$-duality is based on the relation with gauged sigma-models, so it is interesting to investigate further the cases in which gauging is not possible. We will focus here on one of the simplest obstructed cases, that in which $B_0^{IJ}$ is not globally defined but $B$ is invariant under the torus action. While being the simplest obstructed case, it is sufficiently nontrivial to illustrate some of the problems one encounters in attempting to perform an obstructed $T$-duality. To discuss $T$-duality in the more general case one has to specify how the torus group acts on the gerbe connection (the $B$-field), see \cite{[6]} and appendix \cite{[13]} of this paper for more details. The topological aspects of $T$-duality with nontrivial $B$-fields have been discussed in \cite{[10] \cite{[13] \cite{[14] \cite{[15]}.}

As when discussing the global aspects of WZ models \cite{[16] \cite{[17]}, it will be crucial for our discussion to treat the 2-form as a gerbe connection. As before, its geometrization leads to a new enlarged space $Y$. As we shall see, upon imposing certain conditions on the gerbe structure, $Y$ has two different descriptions. It can either be viewed as a principal torus fibration over the original torus fibration $X$ with a well defined connection form $\Theta_\#$, or as an affine $2n$-dimensional torus fibration over the base $M$. As we shall see the affine connection $(\Theta, \tilde{\Theta})$ has the following gluing conditions
on twofold overlaps \(M_{\alpha\beta}:\)

\[
\begin{pmatrix}
\tilde{\Theta}_\alpha \\
\Theta_\alpha
\end{pmatrix}
= \begin{pmatrix}
1 & m_{\alpha\beta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{\Theta}_\beta \\
\Theta_\beta
\end{pmatrix}
\] (1.4)

where \(m_{I\alpha\beta}\) are skewsymmetric integral valued matrices satisfying cocycle conditions on triple overlaps (see subsection 2.2 for details). They parameterize the non-triviality of the \(B\)-field. When \(m_{I\alpha\beta}\) can be set to zero, \(B_{0}^{I\beta}\) is a globally defined smooth function (and the \(T\)-duality is geometric).

The principal difference between the two connections \(\Theta\#\), \(\tilde{\Theta}\) becomes clear when the lifting of the original \(T^n\) action to \(Y\) is considered — even for well-defined \(B_{0}^{I\beta}\). When using the connection \(\Theta\#\), the torus group in general lifts only to the universal covering group \(\mathbb{R}^n\) (which will be the case even if \(B_{0}^{I\beta}\) is constant provided the matrix \(B_{0}^{I\beta}\) has irrational values at some points on \(M\), which will necessarily be the case if it is a non-constant function), while for \(\tilde{\Theta}\) it always lifts to \(T^n\). As mentioned, when \(B_{0}^{I\beta}\) is well defined, the two connections have curvatures which are in the same de Rham cohomology class. When it is not, this discrepancy becomes one of the principal difficulties. In this situation it is impossible to perform the \(T\)-duality in the standard way at the level of the sigma model. In such cases, it has been proposed that \(T\)-duality is nonetheless possible and gives a \(T\)-fold [18].

**T-duality as a symmetry of a loop space.** The phase space of the sigma model on the target \(X\), given by the cotangent bundle to the loop space \(LX\), has a natural symplectic structure with a closed 2-form

\[
\omega_X = \oint_{S^1} d\sigma \left[ \delta p + \iota(\partial_\sigma x)H \right],
\] (1.5)

where \(\delta\) is the differential on the loop space. We shall see how this structure extends upon enlarging the target space from the principal torus fibration \(X\) to the generalized correspondence space \(Y\). The new symplectic form on \(T^*LY\), \(\omega_Y\), has a natural \(O(n,n,\mathbb{Z})\) action. In the unobstructed case this action simply leads to a new derivation of the old results. There are two different torus actions with two different symplectic reductions leading either back to the original sigma model on \(X\) or the dual one on \(\tilde{X}\) with the exchange of the first Chern classes of the torus fibrations and the fibrewise integrals of \(h\) — the characteristic class of the gerbe connection. When \(B_{0}^{I\beta}\) is not globally defined, the situation changes radically. The natural extension of the construction for this case uses the globally well defined connection \(\Theta\#\) (in a way similar to the reduction of a centrally extended current algebra on a torus fibre as explained in the Appendix [D]). However, as already mentioned, when using \(\Theta\#\) the original torus \(T^n\) acts as \(\mathbb{R}^n\) in \(T^*LY\) and one has to deal with non-closed orbits in \(Y\). This problem manifests itself in the fact that the action of the torus \(T^n\) on \(\omega_Y\) is no longer hamiltonian. Not surprisingly, the obstruction is given by \(H_{1}^{I\beta}\). As
we shall see (Theorem 4.2) there exists a way of writing the symplectic form on $T^*LY$ using the
affine connection $\tilde{Θ}$ (see section 4 for details).

The structure of the paper is as follows. In section 2 we rederive the $T$-duality obstructions and
describe the construction of the enlarged target space. The way the original gerbe structure defined
the structure of this space is discussed there (and in Appendices A, B; Appendix C discusses the
toy example of a $U(1)$ bundle reduction). $T$-duality in sigma models is discussed in section 3. In
section 4 we discuss the construction of the phase space on the enlarged sigma model and the
action of the obstructed $T$-duality (with the current algebra being discussed in Appendix D).

2 Sigma-models on principal torus bundles

2.1 Review of the obstructions to $T$-duality

Principal torus bundle. Let $X$ be a principal torus bundle with fibre $\mathbb{T}^n$:

$$\mathbb{T}^n \hookrightarrow X \xrightarrow{\pi} M.$$  

A connection on $X$ is a globally well defined smooth 1-form $Θ$ on $X$ with values in $t := \text{Lie } \mathbb{T}^n \cong \mathbb{R}^n$. Let $K \in \Gamma(TX \otimes t^*)$ be a fundamental vector field — the generator of the $\mathbb{T}^n$-action on $X$. The connection $Θ$ is characterized by

$$\iota(K) Θ = 1 \in t^* \otimes t,$$

and the equivariance condition

$$\mathcal{L}(K) Θ = 0$$

where $\mathcal{L}(K)$ denotes the Lie derivative with respect to the vector field $K$. These two conditions imply that $dΘ = π^*F$ is a horizontal form, $F \in \Omega^2(M; t)$.

It is convenient to choose a basis on $t$, so that one can think of the connection $Θ$ as a collection of one-forms $\{Θ_I\}, I = 1, \ldots, n$. We denote the corresponding fundamental vector fields by $\{\frac{\partial}{\partial θ_I}\}$. Note that, given a connection $Θ$,

$$Θ_I \wedge \iota(\frac{\partial}{\partial θ_I}) \quad \text{and} \quad 1 - Θ_I \wedge \iota(\frac{\partial}{\partial θ_I})$$

are the projection operators onto the vertical and horizontal forms respectively. Similarly, we can decompose the differential $d$ into a horizontal differential $π^*d_M$ and the vertical one $d_{θ/∂θ}$:

$$d = [1 - Θ_I \wedge \iota(\frac{∂}{∂θ_I})]d + Θ_I \wedge \iota(\frac{∂}{∂θ_I})d = π^*d_M + d_{θ/∂θ}$$ (2.1a)
where \( d_M \) is the differential on \( M \). On the horizontal forms \( \omega_{\text{hor}} \), \( \iota(\frac{\partial}{\partial y_I})\omega_{\text{hor}} = 0 \), one has \( \iota(\frac{\partial}{\partial y_I})d\omega_{\text{hor}} = \mathcal{L}(\frac{\partial}{\partial y_I})\omega_{\text{hor}} \) with \( \mathcal{L}(\frac{\partial}{\partial y_I})\omega_{\text{hor}} \) also horizontal, and therefore
\[
d\omega_{\text{hor}} = (\pi^*d_M)\omega_{\text{hor}} + \Theta_I \wedge \mathcal{L}(\frac{\partial}{\partial y_I})\omega_{\text{hor}}.
\]
(2.1b)

The lift \( \pi^*d_M \) of the differential on \( M \) is not nilpotent: rather \( (\pi^*d_M)^2\omega_{\text{hor}} = -F_I \wedge \mathcal{L}(\frac{\partial}{\partial y_I})\omega_{\text{hor}}. \)

In the next section we will use a local description of the torus bundle. The following notation will be used. We choose an open cover \( \{M_\alpha\} \) of the base \( M \) by contractible open sets. We denote by \( \theta_{\alpha I} \) \((I = 1, \ldots, n)\), \( 0 \leq \theta_{\alpha I} < 1 \), coordinates in the torus fibre over the patch \( M_\alpha \) with the gluing condition on twofold overlaps \( \{M_{\alpha\beta}\} \)
\[
\theta_{\alpha}|_{M_{\alpha\beta}} - \theta_{\beta}|_{M_{\alpha\beta}} = -\lambda_{\alpha\beta}
\]
(2.2a)

where \{\( \lambda_{\alpha\beta}\)\} are functions on twofold overlaps with values in \( t \) satisfying the cocycle condition on threefold overlaps: \( \lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = 0 \). Then locally the connection \( \Theta \) can be written as
\[
\Theta|_{M_\alpha} = d\theta_\alpha + \pi^*A_\alpha \quad \text{and} \quad A_\alpha|_{M_{\alpha\beta}} - A_\beta|_{M_{\alpha\beta}} = d\lambda_{\alpha\beta}
\]
(2.2b)

where \( A_\alpha \) is a 1-form on \( M_\alpha \) with values in \( t \).

**Restrictions on the 3-form \( H \).** We are interested in sigma models on a target space \( X \), given by a principal torus fibration, and a Wess-Zumino term defined by a 2-form gauge field \( B \). To be more precise, \( B \) is a gerbe connection. The implications of this description are important and will be explained in the next subsection. For the moment, we are interested in the curvature of the gerbe connection — a globally well defined smooth closed 3-form \( H \in \Omega^3_2(X) \).

Since \( \pi : X \to M \) is a principal torus bundle we have a free torus action on \( X \). The Wess-Zumino term is invariant with respect to this torus action (more precisely, the holonomies\(^2\) of the gerbe connection \( B \) over 2-cycles in \( X \) are invariant with respect to the torus action) iff \( \iota(\frac{\partial}{\partial y_I})H \) is an exact form. This is a necessary condition for gauging the sigma model \([1]\). However the conditions for \( T \)-duality are less restrictive: \( \mathcal{L}(\frac{\partial}{\partial y_I})H = 0 \), i.e. \( \iota(\frac{\partial}{\partial y_I})H \) is a closed 2-form but not necessarily an exact one \([\ref{6}]\). Such a 3-form \( H \) can be written globally as
\[
H = \pi^*H_3 + \langle \pi^*H_2, \Theta \rangle + \frac{1}{2}\langle \pi^*H_1, \Theta \wedge \Theta \rangle + \frac{1}{6}\langle \pi^*H_0, \Theta \wedge \Theta \wedge \Theta \rangle,
\]
(2.3)

where \( H_j \in \Omega^j(M; \Lambda^{3-j}t) \) for \( j = 0, 1, 2, 3 \) and \( \langle \cdot, \cdot \rangle \) denotes the natural pairing \( t^* \otimes t \to \mathbb{R} \). We use the same notation for the linear extension of this pairing to antisymmetric powers of \( t \) and \( t^* \).

\(^2\)The holonomy of a 2-form gauge field \( B \) over a 2-cycle \( \Sigma \) is, roughly speaking, \( \exp(2\pi i \int_{\Sigma} B) \) and is defined in a way similar to the holonomy of a 1-form gauge field — see Appendix\([\ref{4}]\) for details. The holonomy of a gerbe connection is an exponential of a Wess-Zumino term: \( \text{Hol}(B, \Sigma) = \exp[2\pi i \text{WZ}(B, \Sigma)] \).
For example, \( \langle H_1, \Theta \wedge \Theta \rangle = H_1^{I J} \wedge \Theta_I \wedge \Theta_J \). The closure of \( H \) implies the following equations on \( \{ H_j \} \)

\[
dH_j + \langle H_{j-1}, F \rangle = 0, \quad (2.4a)
\]
or using the basis we have

\[
d_M H_3 + H_2^I \wedge F_I = 0, \quad d_M H_2^I + H_1^{IJ} \wedge F_J = 0, \\
d_M H_1^{IJ} + H_0^{IJK} F_K = 0, \quad d_M H_0^{IJK} = 0. \quad (2.4b)
\]

**Double fibration.** The contraction of the invariant 3-form \( H \) with the fundamental vector field \( K \) defines a closed 2-form \( F_\# \in \Omega^2_Z(M; t) \) on \( X \) with integral periods (provided that the fundamental vector field \( K \) is properly normalized, as we will now assume). Using the basis in \( t \) it can be written as

\[
F_\#^I := \iota(\partial/\partial \theta^I) H = H_2^I - H_1^{IJ} \wedge \Theta_J + \frac{1}{2} H_0^{IJK} \Theta_J \wedge \Theta_K. \quad (2.5)
\]

We would like now to geometrize this form, i.e. think of it as a curvature of a connection \( \Theta_\# \) on a principal torus bundle \( \mathbb{T}^n_\# \hookrightarrow Y \xrightarrow{p} X \) with \( \text{Lie} \mathbb{T}^n_\# = t^* \):

\[
p^*(F_\#^I) = d\Theta_\#^I. \quad (2.6)
\]

To this end one has to construct a 2-cocycle representing the first Chern class of the torus bundle, such that its image in the de Rham cohomology is \([F_\#], dR\).

**Torus actions on the double fibration.** The total space \( Y \) of the double fibration has a natural action of the torus \( \mathbb{T}^n_\# \). It is natural to ask whether the original torus \( \mathbb{T}^n \) acts on \( Y \). The connection \( \Theta_\# \) on \( Y \to X \) allows one to lift the action of \( \text{Lie} \mathbb{T}^n \): a fundamental vector field \( K \in \Gamma(TX \otimes t^*) \) can be lifted to \( Y \) as a horizontal vector field \( K_\text{hor} \in \Gamma(TY \otimes t^*) \). Note that these horizontal vector fields do not commute automatically. Indeed, the commutator of two such fields \( K_\text{hor} \) and \( K'_\text{hor} \) is given by their contraction with the curvature of \( \Theta_\# \):

\[
[K_\text{hor}, K'_\text{hor}] = [K, K'_\#]_{\text{hor}} + \iota(K_\text{hor})\iota(K'_\text{hor}) F_\#, \quad (2.7)
\]
or explicitly

\[
[(\partial/\partial \theta^I)_{\text{hor}}, (\partial/\partial \theta^J)_{\text{hor}}] = -H_0^{IJK} (\partial/\partial \theta^K_\#). \quad (2.7)
\]

\footnote{This means that \( \iota(K_\text{hor})\Theta_\# = 0 \).}
Thus the vanishing of $H_0 \in \Omega^0_\mathbb{R}(M; \Lambda^3 t)$ is the necessary condition for the action of Lie $\mathbb{T}^n$ to remain abelian after lifting to $Y$.

Having lifted the action of the Lie algebra we have not necessarily lifted the action of the Lie group as well. To lift the torus action $\mathbb{T}^n$ to $Y$ in addition to $H_0^{IJK} = 0$ we have to verify that the orbits of $\left( \frac{\partial}{\partial \theta_j} \right)_{\text{basis}}$ are closed for all $I = 1, \ldots, n$. If this is so, then we have an action of the double torus $\mathbb{T}^n \times \mathbb{T}^n_\#$ on $Y$. Otherwise, if $H_0^{IJK} = 0$ but not all orbits are closed we have the action of $\mathbb{R}^k \times \mathbb{T}^{n-k} \times \mathbb{T}^n_\#$ on $Y$ for some $k$ between 1 and $n$.

A free action of $\mathbb{T}^n \times \mathbb{T}^n_\#$ on $Y$ means that $Y$ itself is a double torus fibration over $M$. In particular,

$$[F^I_\#]_{dR} = [\pi^* H_2^I - \pi^* H_1^{IJ} \wedge \Theta_J]_{dR} \quad (2.8)$$

must be a pullback of some de Rham cohomology class on $M$. The first term in this expression is clearly a pullback. However the second one is not in general a pullback from $M$. Suppose that $H_1$ is exact, i.e. there exists a globally well defined smooth $B_0 \in \Omega^0(M; \Lambda^2 t^*)$ such that $H_1 = dB_0$. Then we can rewrite (2.8) as

$$[F^I_\#]_{dR} = \pi^*[H_2^I + B_0^{IJ}F_J]_{dR}.$$ 

Thus the necessary conditions for having a free action of the double torus on $Y$ are $H_0^{IJK} = 0$ and $H_1^{IJ}$ is exact. In this case $Y$ itself is a principal $\mathbb{T}^n \times \mathbb{T}^n_\#$ bundle over $M$. One can choose on $Y$ a connection $\tilde{\Theta}$ which respects the fact that $Y$ is principal double torus bundle over $M$:

$$\tilde{\Theta}^I = \Theta^I_\# - B_0^{IJ} \Theta_J. \quad (2.9)$$

It is this $\mathbb{T}^n \times \mathbb{T}^n_\#$ bundle over $M$ that is referred to as the doubled torus bundle in [18, 6].

We can summarize our discussion by the following

**Theorem 2.1.** The contraction of the invariant 3-form $H$ (2.3) with the fundamental vector field defines a closed 2-form $F_\#$ with integral periods on $X$. One can think of it as a curvature of a connection $\Theta_\#$ on a principal torus bundle $\mathbb{T}^n_\# \hookrightarrow Y \xrightarrow{p} X$: $p^* F_\# = d\Theta_\#$.

a) The action of Lie $\mathbb{T}^n$, the Lie algebra of the original torus, is abelian on $Y$ iff $H_0^{IJK} = 0$.

b) If in addition $H_1^{IJ} = dB_0^{IJ}$ is an exact form on $X$, then $Y$ is a principal double torus bundle on $M$ with connections $\Theta$ and $\tilde{\Theta}$ defined by (2.9).

**Comment.** In the next subsection we will show that if $H_1$ is not exact then $Y$ is an affine torus bundle over $M$ with very specific gluing functions.
2.2 Gerbes

The general sigma model includes a Wess-Zumino defined by a 2-form gauge field $B$. When discussing the global properties of the sigma model it is important to treat $B$ as a gerbe connection. In this subsection we first review the definition of a gerbe on a general manifold $X$, and then consider in detail what happens when $X$ is a principal torus bundle and the curvature of the gerbe connection is invariant with respect to the torus action. The geometrization of the gerbe for this case will be the key to the following discussion.

The main results of this subsection are Corollary 2.2 and Corollary 2.3 which state that an invariant gerbe connection on a principal torus bundle defines:

a) a principal torus bundle $p : Y → X$ with connection;

b) an affine (double) torus bundle over $M$ with an affine connection.

Gerbe. We use the formulation of a gerbe presented in section 1.2 in [20, 21]. Choose an open covering $\{X_\alpha\}$ of $X$. Note that these open sets do not need to be contractible. A gerbe is defined as the following structure: a line bundle $L_{\alpha\beta}$ on each twofold intersection $X_{\alpha\beta} = X_\alpha \cap X_\beta$; an isomorphism $L_{\alpha\beta} \cong L_{\beta\alpha}$; a trivialization $f_{\alpha\beta\gamma} : X_{\alpha\beta\gamma} → U(1)$ of the line bundle $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ on each threefold intersection $X_{\alpha\beta\gamma}$; $f_{\alpha\beta\gamma}$ is a cocycle, i.e. $δf_{\alpha\beta\gamma\delta} = f_{\alpha\beta\gamma}f_{\beta\gamma\delta}^{-1}f_{\gamma\delta\alpha}f_{\delta\alpha\beta}^{-1} = 1$ on each fourfold intersection $X_{\alpha\beta\gamma\delta}$.

A gerbe with connection is a gerbe plus a connection $A_{\alpha\beta}$ on the line bundle $L_{\alpha\beta}$ in each $X_{\alpha\beta}$ such that the section $f_{\alpha\beta\gamma}$ is covariantly constant with respect to the induced connection on $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$:

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = \frac{1}{2\pi i} f_{\alpha\beta\gamma}^{-1} df_{\alpha\beta\gamma},$$

(2.10)

and a two form (gerbe connection) $B_\alpha \in \Omega^2(X_\alpha)$ such that $B_\alpha - B_\beta = dA_{\alpha\beta}$ on $X_{\alpha\beta}$.

The gauge group of the gerbe is generated by a group of line bundles with connection. Given a line bundle $L$ with connection $A$ we shift $L_{\alpha\beta} \mapsto L|_{X_{\alpha\beta}} \otimes L_{\alpha\beta}$, $A_{\alpha\beta} \mapsto A|_{X_{\alpha\beta}} + A_{\alpha\beta}$ and $B_\alpha \mapsto B_\alpha + F|_{X_\alpha}$ where $F \in \Omega^2_Z(X)$ is the curvature of the connection on $L$. The gauge equivalence classes of gerbe connections form an abelian group — the Chiger-Simons cohomology $\tilde{H}^3(X)$ (for a pedagogical introduction to Chiger-Simons cohomology see section 2 in [22]).

Connections $A_{\alpha\beta}$ on line bundles $L_{\alpha\beta}$ should not be confused with the connection $A_\alpha$ on a principal torus bundle defined in subsection 2.1.
Gerbe on a principal torus bundle. As we have seen in section 2.1, a necessary condition for T-duality is the invariance of the curvature $H$ with respect to the torus action. Thus $H$ can be written as

$$H = \pi^*H_3 + \langle \pi^*H_2, \Theta \rangle + \frac{1}{2}\langle \pi^*H_1, \Theta \wedge \Theta \rangle + \frac{1}{6}\langle \pi^*H_0, \Theta \wedge \Theta \wedge \Theta \rangle \quad (2.11)$$

where $H_j$ for $j = 0, 1, 2, 3$ is a smooth $j$-form on $M$. The torus bundle $X$ can be covered by open sets $\{X_\alpha = \mathbb{T}^n \times M_\alpha\}$ where $\{M_\alpha\}$ is an open covering of the base manifold $M$.

Structure on a coordinate patch. In each coordinate patch $\mathbb{T}^n \times M_\alpha$ a gerbe connection $B_\alpha$ can be written as

$$B_\alpha = B_{2\alpha} + \langle B_{1\alpha}, \Theta \rangle + \frac{1}{2}\langle B_{0\alpha}, \Theta \wedge \Theta \rangle, \quad (2.12)$$

where $B_{2\alpha}$, $B_{1\alpha}$ and $B_{0\alpha}$ are horizontal 2-, 1- and 0-forms on $X_\alpha$. Note that there is no $\pi^*$ in front of these forms in the equation (2.12) since a priori they can depend on the torus coordinates. Locally the curvature $H_j|_{X_\alpha}$ can be written as

$$\pi^*H_3|_{X_\alpha} = (\pi^*d_M)B_{2\alpha} - \langle B_{1\alpha}, \pi^*F \rangle; \quad (2.13a)$$
$$\pi^*H_2|_{X_\alpha} = (\pi^*d_M)B_{1\alpha}^I + \mathcal{L}(\frac{\partial}{\partial\theta_I})B_{2\alpha} - B_{0\alpha}^{IJ} \wedge \pi^*F_J; \quad (2.13b)$$
$$\pi^*H_1^I|_{X_\alpha} = (\pi^*d_M)B_{0\alpha}^{IJ} - \mathcal{L}(\frac{\partial}{\partial\theta_I})B_{1\alpha}^J + \mathcal{L}(\frac{\partial}{\partial\theta_J})B_{1\alpha}^I; \quad (2.13c)$$
$$\pi^*H_0^{IK}|_{X_\alpha} = \frac{\partial}{\partial\theta_I}B_{0\alpha}^{JK} + \frac{\partial}{\partial\theta_J}B_{0\alpha}^{IK} \quad (2.13d)$$

where $\pi^*d_M$ is the horizontal exterior derivative defined in (2.1). Note that the left hand sides of the equations above do not depend on the torus coordinates, and thus the right hand sides should not depend on them either.

Structure on a twofold intersection. On twofold intersections $\{X_{\alpha\beta} = \mathbb{T}^n \times M_{\alpha\beta}\}$ the gerbe connections $\{B_\alpha\}$ are glued by 1-forms $\{A_{\alpha\beta}\}$ which can be written as

$$A_{\alpha\beta} = a_{\alpha\beta} + \langle h_{\alpha\beta}, \Theta \rangle, \quad (2.14)$$

where $a_{\alpha\beta}$ and $h_{\alpha\beta}$ are horizontal 1- and 0-forms on $X_\alpha$ respectively. There is no $\pi^*$ in this expression since a priori both $a_{\alpha\beta}$ and $h_{\alpha\beta}$ can depend on the torus coordinates. The gluing condition yields

$$B_{2\alpha} - B_{2\beta}|_{M_{\alpha\beta}} = (\pi^*d_M)a_{\alpha\beta} + \langle h_{\alpha\beta}, \pi^*F \rangle; \quad (2.15a)$$
$$B_{1\alpha}^I - B_{1\beta}^I|_{M_{\alpha\beta}} = (\pi^*d_M)h_{\alpha\beta}^I - \mathcal{L}(\frac{\partial}{\partial\theta_I})a_{\alpha\beta}; \quad (2.15b)$$
$$B_{0\alpha}^{IJ} - B_{0\beta}^{IJ}|_{M_{\alpha\beta}} = \frac{\partial}{\partial\theta_I}h_{\alpha\beta}^J - \frac{\partial}{\partial\theta_J}h_{\alpha\beta}^I. \quad (2.15c)$$
Structure on a threefold intersection. On threefold intersections \( \{X_{\alpha \beta \gamma} = \mathbb{T}^n \times M_{\alpha \beta \gamma}\} \) we are given sections \( f_{\alpha \beta \gamma} : X_{\alpha \beta \gamma} \to U(1) \) satisfying the cocycle condition on fourfold intersections. The connections \( \{A_{\alpha \beta}\} \) must be such that \( f_{\alpha \beta \gamma} \) is covariantly constant (2.10):

\[
a_{\alpha \beta} + a_{\beta \gamma} + a_{\gamma \alpha}\Big|_{M_{\alpha \beta \gamma}} = \frac{1}{2\pi i} (\pi^* d_M) \log f_{\alpha \beta \gamma} \quad \text{and} \quad h_{\alpha \beta}^I + h_{\beta \gamma}^I + h_{\gamma \alpha}^I\Big|_{M_{\alpha \beta \gamma}} = \frac{1}{2\pi i} \frac{\partial}{\partial \theta_I} \log f_{\alpha \beta \gamma}. \tag{2.16}
\]

T-duality constraints. Recall that the contraction of the fundamental vector field \( \frac{\partial}{\partial \theta_I} \) with the form \( H \) yields a closed 2-from \( F_\#^I \) on \( X \) with integral periods. In section 2.1 we interpreted this form as a curvature of a connection \( \Theta_\# \) on a principal torus bundle \( p : Y \to X \). To perform the T-duality one has to construct this torus bundle and connection on it explicitly. The torus bundle is defined by a 2-cocycle on \( X \). From equation (2.16) it follows that the information contained in \( \{h^I_{\alpha \beta}\} \) should be used to construct such a cocycle. Moreover locally \( H|_{X_\alpha} = dB_\alpha \) so it is natural to ask whether \( B \) alone defines the connection \( \Theta_\# \). From equation

\[
F_\#^I|_{X_\alpha} = \iota(\frac{\partial}{\partial \theta_I}) dB_\alpha = L(\frac{\partial}{\partial \theta_I}) B_\alpha - d \iota(\frac{\partial}{\partial \theta_I}) B_\alpha
\]

it follows that the necessary condition for this is the invariance of \( B_\alpha \) under the torus action:

\[L(\frac{\partial}{\partial \theta_I}) B_\alpha = 0 \text{ in all patches } X_\alpha.\]

In particular, this condition implies that \( H_0^{ijk} = 0 \). If \( L(\frac{\partial}{\partial \theta_I}) B_\alpha \neq 0 \) in some of the patches, then one has to introduce extra structure into the formulation; to simplify the discussion, we will restrict ourselves to the case in which \( B \) is invariant here.

The invariance of \( B_\alpha \) with respect to the torus action restricts the possible dependence of \( \{h^I_{\alpha \beta}\}, \{a_{\alpha \beta}\} \) and \( \{f_{\alpha \beta \gamma}\} \) on the torus coordinates: the right hand sides of (2.15) must be pullbacks from the base. The result can be summarized by the following

**Theorem 2.2.** The gluing conditions for the gerbe connection \( B_\alpha \) which are compatible with the \( \mathbb{T}^n \)-invariance \( L(\frac{\partial}{\partial \theta_I}) B_\alpha = 0 \) are

\[
B_{0 \alpha}^I - B_{0 \beta}^I = m_{\alpha \beta}^I, \tag{2.17a}
\]

\[
B_1^I - B_1^J = d_M \tilde{h}^I_{\alpha \beta} + m_{\alpha \beta}^I(A_\beta - \frac{1}{2} d_M \lambda_\alpha), \tag{2.17b}
\]

\[
B_2^I - B_2^J = [d_M a_{\alpha \beta} + \langle \tilde{h}^I_{\alpha \beta}, F \rangle] + \frac{1}{2} \langle m_{\alpha \beta}, (A_\beta - \frac{1}{2} d_M \lambda_\alpha) \wedge (A_\beta - \frac{1}{2} d_M \lambda_\alpha) \rangle, \tag{2.17c}
\]

where \( \{m_{\alpha \beta}^I\} \) are skewsymmetric integral valued matrices satisfying the cocycle condition on threefold overlaps, \( \{\tilde{h}^I_{\alpha \beta}\} \) are functions (skew-symmetric in \( \alpha, \beta \)) defined on twofold overlaps \( \{M_{\alpha \beta}\} \) and satisfying the following condition on threefold overlaps

\[
m_{\alpha \beta} + m_{\beta \gamma} + m_{\gamma \alpha} = 0 \quad \text{and} \quad \tilde{h}^I_{\alpha \beta} + \tilde{h}^I_{\beta \gamma} + \tilde{h}^I_{\gamma \alpha}\Big|_{M_{\alpha \beta \gamma}} = -\frac{1}{2} \left[m^I_{\alpha \beta} \lambda_{\beta \gamma} J - m^I_{\gamma \beta} \lambda_{\beta \alpha} J \right]. \tag{2.18}
\]
\{\tilde{a}_{\alpha\beta}\} \text{ are 1-forms defined on twofold overlaps and satisfying the following condition on threefold overlaps:}

\[\begin{align*}
\tilde{a}_{\alpha\beta} + \tilde{a}_{\beta\gamma} + \tilde{a}_{\gamma\alpha} & = \frac{1}{2\pi i} d_M \log \tilde{f}_{\alpha\beta\gamma} - \frac{1}{12} d_M \left[ \lambda_{\beta\alpha}(m_{\alpha\beta} + m_{\gamma\beta})\lambda_{\beta\gamma} \right] \\
& - \frac{1}{8} \left( \lambda_{\beta\alpha} m_{\gamma\beta} d_M \lambda_{\beta\alpha} + \lambda_{\beta\alpha} m_{\alpha\beta} d_M \lambda_{\beta\gamma} + \lambda_{\beta\gamma} m_{\beta\alpha} d_M \lambda_{\beta\gamma} + \lambda_{\beta\gamma} m_{\gamma\alpha} d_M \lambda_{\beta\alpha} \right) \quad (2.19)
\end{align*}\]

where \(\tilde{f}_{\alpha\beta\gamma} : M_{\alpha\beta\gamma} \to U(1)\) and it satisfies the following condition on fourfold overlaps:

\[\begin{align*}
\tilde{f}_{\alpha\beta\gamma} \tilde{f}_{\beta\gamma\delta} \tilde{f}_{\gamma\delta\alpha} \tilde{f}_{\delta\alpha\beta} & = \exp \left[ -\frac{2\pi i}{6} \left( \lambda_{\beta\gamma} m_{\delta\beta} \lambda_{\alpha\delta} - \lambda_{\beta\gamma} m_{\alpha\delta} \lambda_{\delta\gamma} + \lambda_{\beta\delta} m_{\gamma\delta} \lambda_{\delta\alpha} \right) \right]. \quad (2.20)
\end{align*}\]

Before proving the theorem let us discuss the implications of the result. The invariance of the gerbe connection with respect to the torus action, \(\mathcal{L}(\partial/\partial \theta) B_{\alpha} = 0\), while not being the most general case, allows for gluing functions that are sufficiently nontrivial. In particular, \(H^1_1\) can represent a nontrivial de Rham cohomology class. The corresponding integral cohomology class is represented by a cocycle \(\{m_{\alpha\beta}\}\). If \(\{m_{\alpha\beta}\}\) is a coboundary (so it can be set to zero) then \(B_0\) is a globally well defined smooth function, \(B_{1\alpha}\) has gluing functions \(\tilde{h}_{\alpha\beta}\) corresponding to a connection on a principal torus bundle:

**Corollary 2.1.** If \(\{m_{\alpha\beta}\}\) is a coboundary then \(Y\) is a principal double torus fibration. The gluing functions are \(\lambda_{\alpha\beta}\) and \(\tilde{h}_{\alpha\beta}\):

\[\theta_{\alpha} - \theta_{\beta} = -\lambda_{\alpha\beta} \quad \text{and} \quad \tilde{\theta}_{\alpha} - \tilde{\theta}_{\beta} = -\tilde{h}_{\alpha\beta}. \quad (2.21)\]

The connection one forms are \(\Theta_{I}\) and \(\tilde{\Theta}^I = d\tilde{h}_{\alpha}^I + B_{1\alpha}^I\). \(Y\) can also be thought of as a fibrewise product of two principal torus bundles, \(X\) and \(\tilde{X}\), defined by the gluing functions \(\lambda_{\alpha\beta}\) and \(\tilde{h}_{\alpha\beta}\) respectively.

In general, \(\{m_{\alpha\beta}\}\) is a nontrivial cocycle and \(Y\) is a double fibration — a principal torus bundle \(Y \to X\) over a principal torus bundle \(X \to M\):

**Corollary 2.2.** The following functions defined on twofold overlaps \(\{X_{\alpha\beta}\}\)

\[\lambda_{\#\alpha\beta}(\theta_{\beta}) = \tilde{h}_{\alpha\beta} - m_{\alpha\beta}(\theta_{\beta} + \frac{1}{2} \lambda_{\beta\alpha}) \quad (2.22)\]

satisfy the cocycle condition on threefold overlaps \(\{X_{\alpha\beta\gamma}\}\). This cocycle defines a principal torus bundle \(p : Y \to X\) by the gluing condition \(\theta_{\#\alpha} - \theta_{\#\beta} = -\lambda_{\#\alpha\beta}\). The connection \(\Theta_{\#}\) on \(Y\) can locally be written as

\[\Theta_{\#}^I |_{X_{\alpha}} = d\theta_{\#\alpha}^I + B_{1\alpha}^I - B_{0\alpha}^I \Theta_{I}. \quad (2.23)\]
The same space $Y$ can be represented in a slightly different geometrical form: $Y$ is an affine double torus bundle (for an introduction to affine torus bundles see section 4.1.1 in [23]) with very special gluing functions:

**Corollary 2.3.** The gerbe reduction (2.17) defines an affine $\mathbb{T}^m \times \mathbb{T}^m$-torus bundle over $M$: the gluing conditions for coordinates on twofold overlaps are

$$
\begin{pmatrix}
\bar{\theta}_\alpha + \frac{1}{2} \bar{h}_{\alpha\beta} \\
\theta_\alpha + \frac{1}{2} \lambda_{\alpha\beta}
\end{pmatrix} =
\begin{pmatrix}
I & m_{\alpha\beta} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\bar{\theta}_\beta + \frac{1}{2} \bar{h}_{\beta\alpha} \\
\theta_\beta + \frac{1}{2} \lambda_{\beta\alpha}
\end{pmatrix}.
$$

(2.24)

The corresponding affine connection has the form $\Theta_\alpha = d\theta_\alpha + A_\alpha$ and $\tilde{\Theta}_\alpha = d\tilde{\theta}_\alpha + B_{1\alpha}$ with the gluing condition on twofold overlaps

$$
\begin{pmatrix}
\bar{\theta}_\alpha \\
\theta_\alpha
\end{pmatrix} =
\begin{pmatrix}
I & m_{\alpha\beta} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\bar{\theta}_\beta \\
\theta_\beta
\end{pmatrix}.
$$

(2.25)

It also follows that $\Theta^I_{\alpha | Y^I_\alpha} = \tilde{\Theta}^I_\alpha - B_{0\alpha}^I \Theta^I_\alpha$ is globally well defined 1-form on the total space of the affine torus bundle.\(^6\)

**Proof of the theorem:** From the invariance of $B_\alpha$ on the torus coordinates it follows that the right hand sides of equations (2.15) do not depend on the torus coordinates. On the other hand the curvature $F_{\alpha\beta}$ of the connection $A_{\alpha\beta}$

$$
F_{\alpha\beta} = \cdots + \frac{1}{2} m_{\alpha\beta}^{IJ} \Theta_I \wedge \Theta_J \\
where \\
m_{\alpha\beta}^{IJ} = \frac{\partial}{\partial \theta_I} h_{\alpha\beta}^J - \frac{\partial}{\partial \theta_J} h_{\alpha\beta}^I
$$

must have integral periods. In particular, $m_{\alpha\beta}^{IJ}$ is an integral valued matrix. So locally (it means one has to cover the torus $\mathbb{T}^m$ by patches) we can write $h_{\alpha\beta}^I(\theta_\beta)$ as

$$
h_{\alpha\beta}^J(\theta_\beta) = \pi^* \bar{h}_{\alpha\beta}^J + \frac{1}{2} m_{\alpha\beta}^{IJ}(\theta_\beta + \frac{1}{2} \lambda_{\beta\alpha}).
$$

Note that $h_{\alpha\beta}(\theta_\beta) = -h_{\beta\alpha}(\theta_\alpha)$ provided $\bar{h}_{\alpha\beta}$ is skewsymmetric in $\alpha, \beta$. From the gluing condition for $B_{1\alpha}$ (2.15) we know that

$$
\pi^* B_{1\alpha}^I - \pi^* B_{1\beta}^I = \pi^*(d_M \bar{h}_{\alpha\beta}^I) + \frac{1}{2} m_{\alpha\beta}^{IJ}(\frac{1}{2} \lambda_{\beta\alpha} - A_{\beta})_J - \mathcal{L}(\frac{\partial}{\partial \theta_J}) a_{\alpha\beta}
$$

\(^6\)In fact, similar affinisation happens when one considers a simpler case of reduction of a $U(1)$ bundle on a principal torus bundle $X$. The basic steps are the same, but the derivation is much lighter and is presented in Appendix C.
should not depend on the torus coordinates. This means that $a_{\alpha\beta}$ is at most linear a function of the torus coordinates:

$$a_{\alpha\beta}(\theta_\beta) = \pi^*\tilde{a}_{\alpha\beta} + \langle\pi^*\rho_{\alpha\beta}, \theta_\beta + \frac{1}{2}\lambda_{3\alpha}\rangle$$

where $\tilde{a}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are smooth 1-forms on $M_{\alpha\beta}$. Again $a_{\alpha\beta}(\theta_\beta) = -a_{\beta\alpha}(\theta_\alpha)$ provided $\tilde{a}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are skewsymmetric in $\alpha, \beta$. From the gluing condition for $B_{2\alpha}$:

$$B_{2\alpha} - B_{2\beta} = \pi^*[d\tilde{a}_{\alpha\beta} + (\tilde{h}_{\alpha\beta}, F)] + (d\rho_{\alpha\beta} + \frac{1}{2}m_{IJ}^{\alpha\beta}F_J)(\theta_\beta + \frac{1}{2}\lambda_{3\alpha})_I + \langle\rho_{\alpha\beta}, A_{\beta} - \frac{1}{2}d\lambda_{3\alpha}\rangle.$$

one concludes that

$$\rho_{I\alpha\beta} = -\frac{1}{2}m_{IJ}^{\alpha\beta}(A_{\beta} - \frac{1}{2}d\lambda_{3\alpha})_J.$$

Combining the equations above we obtain the gluing conditions for the gerbe connection (2.17).

To obtain the cocycle conditions one has to study equations (2.16). From the first equation in (2.16) we learn that

$$f_{\alpha\beta\gamma}(\theta_\beta) = \exp\left[-\frac{2\pi i}{4}(\theta_\beta + \frac{1}{3}\lambda_{3\alpha} + \frac{1}{3}\lambda_{3\gamma})(m_{\alpha\beta}\lambda_{3\gamma} - m_{\gamma\beta}\lambda_{3\alpha})\right] \pi^*\tilde{f}_{\alpha\beta\gamma}$$

where $\tilde{f}_{\alpha\beta\gamma}$ satisfies the usual symmetric properties: $\tilde{f}_{\beta\alpha\gamma} = \tilde{f}_{\alpha\beta\gamma}^{-1}$ etc. Straightforward calculation yields the relation (2.20) on fourfold overlaps.

Comment. Note that the connection $A_{\alpha\beta}$ (2.14) for the invariant gerbe connection has the form

$$A_{\alpha\beta}(\theta_\beta) = \pi^*\tilde{a}_{\alpha\beta} + \langle\pi^*\tilde{h}_{\alpha\beta}, \Theta\rangle + \frac{1}{2}m_{IJ}^{\alpha\beta}(\theta_\beta + \frac{1}{2}\lambda_{3\alpha})_I (d\theta_\beta + \frac{1}{2}d\lambda_{3\alpha})_J. \quad (2.26)$$

3 T-duality in string sigma models

In this section we discuss $T$-duality for principal torus bundles with nontrivial $H$-flux (with vanishing $H^{IJK}_0$). First, we consider the case when $B^{IJ}_0$ is a smooth function on $M$ and present the standard derivation of the $T$-duality on the level of function integral. Second, we discuss the problems with the generalization for the case when $B_0$ is not globally defined.

3.1 Sigma model on a principal torus bundle

In this section we review the construction of the sigma model with a target space $X$ that is a principal torus bundle $\pi : X \to M$. The space $\text{Map}(\Sigma, X)$ of maps from $\Sigma$ to $X$ has itself a
structure of a fibre bundle\footnote{Strickly speaking, it is not a conventional fibre bundle but rather one that is defined for Fréchet manifolds (see e.g. \cite{16,17}).}:

\[ \Gamma(x^*X) \hookrightarrow \text{Map}(\Sigma, X) \rightarrow \text{Map}(\Sigma, M). \] (3.1)

This fibre bundle is defined as follows: given a map \( \chi \in \text{Map}(\Sigma, X) \) we define \( x = \pi \circ \chi \) as the composition of this map followed by the projection onto the base manifold \( M \). This defines the map \( x : \Sigma \rightarrow M \). Now we can restrict the principal torus bundle \( X \rightarrow M \) to the image of \( \Sigma \) and then pull it back to \( \Sigma \). The fibre in (3.1) is exactly the space of sections of the resulting torus bundle. It is convenient to write the sigma model functional integral in the following factorized form

\[
\mathcal{Z}(g) = \int_{\text{Map}(\Sigma, M)}[\mathcal{D}x^\mu(\sigma)] \exp \left[ -\pi \int_{\Sigma} g_{\mu\nu}(x) dx^\mu \wedge \ast_g dx^\nu + \frac{1}{4\pi} \int_{\Sigma} \text{vol}(g) \mathcal{R}(g) \Phi(x) \right] \Psi(x(\sigma)); \quad (3.2a)
\]

\[
\Psi(x) = \int_{\Gamma(x^*X)}[\mathcal{D}\theta_I(\sigma)] \exp \left[ -\pi \int_{\Sigma} h^{IJ}(x) \Theta_I \wedge \ast_g \Theta_J \right]
\times \text{Hol}(x^*B_2 + \langle x^*B_1, \Theta \rangle + \frac{1}{2}(x^*B_0, \Theta \wedge \Theta), \Sigma). \quad (3.2b)
\]

Here \( \Psi(x(\sigma)) \) is a function of the map \( x \) to the base space \( M \), \( g_{\mu\nu} \) is a metric on the base \( M \), \( h^{IJ}(x) \) is an invariant metric on the torus fibre over \( x \in M \), \( g \) is a metric on the worldsheet \( \Sigma \), \( \Phi(x) \) is the dilaton. \( \text{Hol}(B, \Sigma) \) denotes the holonomy of the gerbe connection \( B \) on \( \Sigma \) — the exponential of the Wess-Zumino term, see Appendix A. Note that the dilaton is only a function of the base coordinates. In string theory, one also has to calculate the integral over the space of 2d metrics, however we are not going to discuss this integral here.

### 3.2 T-duality

In this section we review the derivation of T-duality in a sigma model on a Riemann surface \( \Sigma \) with target space a principal torus bundle \( \pi : X \rightarrow M \).

The main result of this subsection can be summarized by the following

**Theorem 3.1.** If \( B_0^{IJ} \) is globally well defined then the functional (3.2b) can also be written as

\[
\Psi(x(\sigma)) = \left[ \frac{\det h^{IJ}(x)}{\det \tilde{h}_{IJ}(x)} \right]^{\chi(\Sigma)/2} \int_{\Gamma(x^*X)}[\mathcal{D}\tilde{\theta}_I(\sigma)] \exp \left[ -\pi \int_{\Sigma} \tilde{h}_{IJ}(x) \tilde{\Theta}^I \wedge \ast_g \tilde{\Theta}^J \right]
\times \text{Hol}(x^*\tilde{B}_2 - \langle \tilde{\Theta}, x^*A \rangle + \frac{1}{2}(\tilde{\Theta} \wedge \tilde{\Theta}, \tilde{B}_0), \Sigma) \quad (3.3)
\]
where $\tilde{X}$ is defined in Corollary 2.1. $\chi(\Sigma)$ is the Euler character of the surface $\Sigma$,

$$\tilde{h} = (h - B_0 h^{-1} B_0)^{-1} \quad \text{and} \quad \tilde{B}_0 = -\tilde{h} B_0 h^{-1}$$

are symmetric and antisymmetric parts of the matrix $(h + B_0)^{-1}$ respectively, and

$$\tilde{\Theta}^I = d\tilde{\theta}^I + x^* B_1 \quad \text{and} \quad \tilde{B}_2 = B_2 + \langle B_1, A \rangle.$$  

**Corollary 3.1.** Under classical $T$-duality the set $(F_I, H_3, \tilde{F}^I)$ maps to the set $(\tilde{F}^I, H_3, F_I)$ where $\tilde{F} = H_2 + B_0 F$ is the curvature of the connection $\tilde{\Theta}$, and

$$\Phi(x) \mapsto \Phi(x) + \frac{1}{2} \log \left[ \frac{\det h^{IJ}(x)}{\det h_{IJ}(x)} \right].$$

**Proof.** If $X$ were a product space then one could prove the theorem in the standard way: gauge the torus symmetry, add lagrange multipliers to impose the condition that the gauge-fields be pure gauge, change the order of integration and integrate out the original torus variables [3, 4]. However when the equivariant extension of $H$ does not exist, and equations (1.2) are not satisfied this approach does not work: it is impossible to gauge the sigma model. It was shown in [6] that although it is impossible to gauge the sigma model it makes sense to gauge and add the lagrange multiplier in one step. This result is explained in detail in Lemma 3.1. The extended sigma model is defined by the functional integral:

$$\Psi(x(\sigma)) = \int D\theta I D\Lambda I D\tilde{\theta}^I \exp \left[ -\pi \int_{\Sigma} h^{IJ}(x)(\Theta_I - \Lambda_I) \wedge * (\Theta_J - \Lambda_J) \right] \times \exp \left[ 2\pi i \int_{\Sigma} \left( x^* B_2 + \langle x^* B_1, \Theta - \Lambda \rangle + \frac{1}{2} \langle x^* B_0, (\Theta - \Lambda) \wedge (\Theta - \Lambda) \rangle \right) \right]. \quad (3.4)$$

where $\Lambda_I$ is a globally well defined smooth 1-form on $\Sigma$ and $\tilde{\theta}^I(\sigma)$ is a section of the pullback of the principal torus bundle with fibre $\tilde{T}^n$ as described in section 2. The gluing conditions on the twofold overlaps for $\tilde{\theta}$ are exactly those described in Corollary 2.1. The exponential in (3.4) is invariant with respect to the gauge transformations $\theta(\sigma) \mapsto \theta(\sigma) + \phi(\sigma)$ and $\Lambda \mapsto \Lambda + d_{\Sigma} \phi$. We can rewrite the last term in (3.4) in a slightly different way:

$$\Psi(x(\sigma)) = \int D\theta I D\Lambda I D\tilde{\theta}^I \exp \left[ -\pi \int_{\Sigma} h^{IJ}(x)(\Theta_I - \Lambda_I) \wedge * (\Theta_J - \Lambda_J) \right] \times \text{Hol}(B_2 + \langle B_1, A \rangle - \langle \tilde{\Theta}, A \rangle, \Sigma) \exp \left[ 2\pi i \int_{\Sigma} \left( \langle \tilde{\Theta}, \Theta - \Lambda \rangle + \frac{1}{2} \langle B_0, (\Theta - \Lambda) \wedge (\Theta - \Lambda) \rangle \right) \right].$$

In [6], the combination $x^*(A - \Lambda)$ is denoted $C$.  

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Note that the last line can be rewritten as an integral over a 3-disk $D$ with boundary $\Sigma$:

$$e^{2\pi i \int_{\Sigma} (\ldots)} = e^{2\pi i \int_D (x^*H_3 + (x^*H_2,\Theta - \Lambda) + \frac{1}{2}(H_1, (\Theta - \Lambda)(\Theta - \Lambda)) + (\tilde{\Theta} - B_0(\Theta - \Lambda), d\Lambda)).} \quad (3.5)$$

Note that the exponential on the right hand side contains only globally well defined quantities.

Imposing a gauge-fixing condition on $\theta_I$ and integrating over $\Lambda$ yields (3.3) (see Lecture 7 of E. Witten in [24] for details).

Comment.

1. Note that the component $B_2$ of the gerb connection (2.12) is not invariant under $T$-duality transformations (see also [25]). It transforms as in Theorem 3.1.

2. Theorem 3.1 can be generalized to cover the action of the whole $T$-duality group $O(n, n; \mathbb{Z})$. The pair $(F_I, \tilde{F}_I)$ transforms as a vector of $O(n, n; \mathbb{Z})$ while the matrix $h^{IJ} + B_0^{IJ}$ transforms by fractional linear transformations.

Lemma 3.1. Assuming $B_0$ is globally well defined the functional $\Psi(x)$ in (3.4) as a function of the gerbe connection $B$ descends to a well defined function of the gauge equivalent classes of gerbe connections (or in short $\Psi(x)$ is gauge invariant).

Proof. Suppose that the image of $\Sigma$, $x(\Sigma)$, lies in the patch $M_\alpha$ then we can write (3.4) in two different ways: using coordinates in the patch $M_\alpha$ or in the patch $M_\beta$. First notice that the Jacobian in change of measure between the patches $M_\alpha$ and $M_\beta$ is trivial, so

$$\mathcal{D}x \mathcal{D}\theta_\alpha \mathcal{D}\tilde{\theta}_\alpha |_{M_{\alpha\beta}} = \mathcal{D}x \mathcal{D}\theta_\beta \mathcal{D}\tilde{\theta}_\beta |_{M_{\alpha\beta}}.$$ 

The only nontrivial term we can obtain is from the second line in (3.4). So let us rewrite the second line of (3.4) written in patch $M_\alpha$ in terms of the quantities defined in the patch $M_\beta$:

$$\exp \left[ 2\pi i \int_{\Sigma} (x^*B_{2\alpha} + (x^*B_{1\alpha}, \Theta - \Lambda) + \frac{1}{2}(x^*B_{0\alpha}, (\Theta - \Lambda)(\Theta - \Lambda)) + (d\tilde{\theta}_\alpha, d\theta_\alpha - \Lambda)) \right] |_{M_{\alpha\beta}} = e^{2\pi i \int_{\Sigma} x^*a_{\alpha\beta}} \left[ 2\pi i \int_{\Sigma} (\tilde{\Theta}_{\beta} - \tilde{\Theta}_{\alpha}, \Lambda) \right] |_{M_{\alpha\beta}}$$

$$\exp \left[ 2\pi i \int_{\Sigma} d\Sigma x^*(\tilde{a}_{\alpha\beta} + (\tilde{h}_{\alpha\beta}, \Theta) + (d\tilde{\theta}_\beta + \frac{1}{2}d\tilde{h}_{\beta\alpha}, \lambda_{\alpha\beta}) - (\tilde{h}_{\alpha\beta}, d\theta_\beta + d\lambda_{\beta\alpha})) \right]$$

where $\tilde{\Theta}_\alpha = d\tilde{\theta}_\alpha + B_{1\alpha}$. To cancel the second exponent one has to require that $\tilde{\Theta}$ is a globally well defined 1-form which means that $\tilde{\theta}_\alpha$ is a coordinate on a principal torus bundle as in Corollary 2.2. The third exponent vanishes by itself since it is an integral of a total derivative over the compact closed surface $\Sigma$. □
3.3 \([H_1] dR \neq 0\)

In this subsection we rederive the result that it is impossible to construct the gauged sigma model with extended target space when \(H_1\) is not exact [1, 2].

The simplest way to see this is to consider the 3-dimensional form of the WZ term \(\int x^* H\), integrated over a 3-space whose boundary is the world-sheet, The first step is minimal coupling, i.e. the replacement \(x^* \Theta\) to \(x^* \Theta - \Lambda\) where \(\Lambda\) is a globally defined 1-form on \(\Sigma\). The lagrangian should be a closed 3-form. The minimally-coupled 3-form

\[
x^* H_3 + \langle x^* H_2, \Theta - \Lambda \rangle + \frac{1}{2} \langle x^* H_1, (\Theta - \Lambda)^2 \rangle
\]  

(3.7a)

is not closed. To make it closed we add two terms: one proportional to \(d \Lambda\) and another proportional to \(\Lambda \wedge d \Lambda\),

\[
\langle G + w \Lambda, d \Lambda \rangle
\]  

(3.7b)

where \(G\) is a 1-form on \(X\) and \(w^{IJ}\) is a function on \(X\). The closure of \((3.7)\) yields:

\[
dG^I = F^I_{\#}, \quad w^{IJ} = -w^{JI}, \quad H_1^{IJ} = dw^{IJ}, \quad L(\frac{\partial}{\partial \tau_I}) w = 0.
\]  

(3.8)

The invariance with respect to the shift \(\theta \rightarrow \theta + \phi(\sigma)\) requires \(G^I = G^I_1 - w^{IJ} \Theta_J\) where \(G_1\) is a pullback of 1-form from \(M\). To make \((3.7)\) globally defined requires \(w\) to be globally defined. So we conclude that \(H_1\) is an exact form. From the discussion in section 2 it follows that one can take \(w = B_0\) and \(G = \Theta_{\#}\).

If one continues the discussion of lemma 3.1 one obtains that \(\Psi(x)\) is not gauge invariant any more. It descends to a section of a non-trivial line bundle over the space of gauge equivalence classes of gerbe connections.

4 T-duality as a symmetry of a loop space

In this section we discuss \(T\)-duality in terms of the canonical quantization of the phase space of the sigma model. For an earlier treatment in which \(T\)-duality is understood as a canonical transformation, see [26]. A bosonic string sigma model on \(S^1 \times \mathbb{R}\) with the target space \(X\) has the configuration space \(LX\) — the loop space of \(X\) — and the phase space \(T^* LX\). We show that when \(B_0 = 0\) (or more generally when \(B_0\) is globally well defined), \(T\)-duality is a symmetry of a total space of a line bundle over the cotangent bundle to the loop space on \(Y\). The symplectic form \(\omega_Y\) has two different torus actions and the two corresponding hamiltonian reductions yield the two \(T\)-dual models.
When $B_0 \neq 0$ and is topologically nontrivial, there is still a symmetry but it is realized differently. There is a Hamiltonian action of one torus but the other has a non Hamiltonian action. The obstruction to having a Hamiltonian action is that $[H_1]_{dR} \neq 0$ (i.e. there can only be a Hamiltonian action if $[H_1]_{dR} = 0$).

In the following subsections we will review the construction of the sigma model phase space for a general smooth manifold $X$ and then restrict to the case in which $X$ is a principal torus bundle.

### 4.1 Phase space of string sigma model

Let $X$ be a compact smooth manifold. The phase space for the string sigma model on $S^1 \times \mathbb{R}$ is $T^*LX$ — the cotangent bundle to the loop space of $X$. $T^*LX$ is naturally a symplectic space: given a loop $x : S^1 \hookrightarrow X$, the symplectic form is

$$\omega = \oint_{S^1} d\sigma \delta p = \oint_{S^1} d\sigma \delta p_M(\sigma) \wedge \delta x^M(\sigma). \quad (4.1)$$

One can think of the momentum $p = p_M(\sigma)\delta x^M(\sigma)$ as of a section of the pullback of $T^*X$ to $S^1$. Here $\delta$ is the differential on the loop space and $x^M$ are coordinates on $X$. To quantize the theory we follow the standard procedure of geometrical quantization (see e.g. [27]) and specify a hermitian line bundle over the phase space with a connection that has curvature $\omega$. Since $\omega$ is exact one can take a trivial line bundle and choose a connection

$$\vartheta = \delta z + \oint_{S^1} d\sigma p \quad (4.2)$$

where $z \in \mathbb{C}$ is a coordinate on the fibre. The wave-functions are then sections of this bundle.

We are interested in sigma models which are twisted by a $B$-field. Mathematically, the $B$-field is a gerbe connection, and the relevance of this will become clear shortly. The gerbe connection has a curvature $H$ — a closed globally defined smooth 3-form on $X$ with integral periods. Using $H$ we can twist the symplectic form (4.3) to give

$$\omega_X = \oint_{S^1} d\sigma [\delta p + \imath(\partial_\sigma x)H]. \quad (4.3)$$

Here

$$\oint_{S^1} \imath(\partial_\sigma x)H = \frac{1}{2} \oint_{S^1} d\sigma \partial_\sigma x^M(\sigma) H_{MNP}(x(\sigma)) \delta x^N(\sigma) \wedge \delta x^P(\sigma) \quad (4.4)$$

is a 2-form on loop-space.

To quantize this phase space we specify a hermitian line bundle over $T^*LX$ with connection $\vartheta_X$ whose curvature is $\omega_X$ (The space of $L^2$ sections of this line bundle form a prequantum Hilbert space). Now the magic fact is that a gerbe connection on $X$ defines a principal circle bundle over $LX$ with connection whose curvature is exactly the second term in (4.3) [20]. Then the pullback of this circle bundle to $T^*LX$ can be taken as the required line bundle.
Connection and circle action. The connection on the line bundle over $T^*L_X$ can be written as (in the patch $X_\alpha$)
\[
\vartheta_X = \delta z_\alpha + \oint_{S^1} d\sigma [p - \iota(\partial_\sigma x)B_\alpha].
\]  
(4.5)

with $\iota(\partial_\sigma x)B = \partial_\sigma x^M B_{MN}\delta x^N(\sigma)$. Recall that $B$ is not a globally defined 2-form, rather $B_\alpha - B_\beta = dA_{\alpha\beta}$ on the twofold intersection $X_{\alpha\beta}$. The momentum $p$ is nevertheless globally well defined: it is a section of $x^*(T^*X)$. So one sees that $z_\alpha - z_\beta = \oint_{S^1} x^*A_{\alpha\beta}$.

The associated circle action is given by the group of line bundles with connection: a line bundle $L \rightarrow X$ with connection $A$ acts on the $B$-field by the shift $B_\alpha \mapsto B_\alpha + F |_{X_\alpha}$ where $F$ is the curvature of the connection on $L$. The pullback $x^*L$ of the line bundle $L$ to the loop $S^1$ is necessarily a trivial line bundle with a flat connection: the second Cheeger-Simons cohomology (essentially the space of connections modulo gauge transformations; see e.g. [22]) is $\tilde{H}^2(S^1) \cong U(1)$. In other words, a pullback of the gerbe to a loop is a principal homogeneous space for $U(1)$, so we have a principal circle bundle. From (4.5) it is easy to see that the coordinate $z$ shifts $z \mapsto z + \oint_{S^1} x^*A$.

**Sigma model on a principal torus bundle.** Let $X$ be the principal torus bundle $\pi: X \rightarrow M$ which we described in section 2.1. In order to define a sigma model on $X$ we have to specify a gerbe connection. We use coordinates $x^\mu$ on the base $M$ and fibre coordinates $\theta_I$, as before, so that $x^M = (x^\mu, \theta_I)$.

A connection (4.5) for a target $X$ which is a principal torus bundle has the following form
\[
\vartheta_X = \delta z + \oint_{S^1} d\sigma [p_\mu \delta x^\mu + \langle p, \Theta \rangle - \iota(\partial_\sigma x + \nabla_\sigma \theta)B]
\]  
(4.6)

where $x: S^1 \rightarrow M$ defines a loop on the base manifold $M$, and $\theta \in \Gamma(x^*X)$ is section of the pullback torus bundle; $\iota(\partial_\sigma x)$ stands for $\iota(\partial_\sigma x^\mu (\frac{\partial}{\partial x^\nu})_{hor})$ and
\[
\nabla_\sigma \theta_I = \partial_\sigma \theta_I + \iota(\partial_\sigma x)A_I
\]  
(4.7)

is the covariant derivative of $\theta$ with respect to the pullback connection. Explicitly, $\langle p, \Theta \rangle = p^I(\sigma)[\delta \theta_I(\sigma) + A_{I\mu}(x(\sigma))\delta x^\mu(\sigma)]$ etc.

**4.2 Symmetry of a loop space**

Recall that, under the assumptions of section 2.1, the generalized correspondence space $Y$ is a double torus bundle over $M$ (provided $B_0^{IJ}$ is globally well defined). We shall first discuss the case when $B_0^{IJ} = 0$.

\[9\text{Note that } \iota(\partial_\sigma x)\delta A = \mathcal{L}(\partial_\sigma x)(x^*A) - \delta(\iota(\partial_\sigma x)A) = \partial_\sigma[\iota(\partial_\sigma x)A] - \delta(\iota(\partial_\sigma x)A).\]
The cotangent bundle to the loop space of $Y$ is naturally a symplectic manifold with $\omega_Y = \delta \vartheta_Y$. Here $\vartheta_Y$ is a connection on the corresponding line bundle:

$$\vartheta_Y = \delta z_\alpha + \oint_{S^1} d\sigma \left[ p_\mu \delta x^\mu + \langle p, \Theta \rangle + \langle \tilde{\Theta}, \tilde{p} \rangle - \imath(\partial_\sigma x + \nabla_\sigma \theta)(B_\alpha - \langle \tilde{\Theta}, \Theta \rangle) \right].$$  

(4.8)

Note that there are two extra terms in this connection than were in (4.6): the meaning of the first one is obvious, while the second can be interpreted as the topologically trivial gerbe connection coming from the Poincaré line bundle (See e.g. [7] for an explanation of this and other relevant geometric structures.). The sympletic form is

$$\omega_Y = \oint_{S^1} d\sigma \left[ \delta p_\mu \wedge \delta x^\mu + \langle \delta p, \Theta \rangle + \langle p, F \rangle + \langle H_2, \tilde{p} \rangle - \langle \tilde{\Theta}, \delta \tilde{p} \rangle + \imath(\partial_\sigma x + \nabla_\sigma \theta)(H_3 + \langle \tilde{\Theta}, F \rangle) \right]$$  

(4.9)

where we have used $d\tilde{\Theta}^I = p^*H_2 = \bar{\pi}^*F$ (and we are assuming $B_0^{IJ} = 0$). The main result of this subsection is the following:

**Theorem 4.1.** The symplectic form $\omega_Y$ is invariant with respect to the $\tilde{T}^n \times T^n$ action generated by the fundamental vector fields $\delta_{\vartheta_I}$ and $\delta_{\tilde{\vartheta}^I}$. Moreover this action is hamiltonian:

$$\imath(\delta_{\vartheta_I})\omega_Y = \delta(-\tilde{p}_I + \nabla_\sigma \theta_I);$$  

(4.10a)

$$\imath(\delta_{\tilde{\vartheta}^I})\omega_Y = \delta(-p^I + \nabla_\sigma \tilde{\theta}^I).$$  

(4.10b)

The symplectic reduction with respect to $\tilde{T}^n$ or $T^n$ yields the symplectic forms $\omega_X$ or $\omega_{\tilde{X}}$ respectively where

$$\omega_X = \oint_{S^1} d\sigma \left[ \delta p_\mu \wedge \delta x^\mu + \langle \delta p, \Theta \rangle + \langle p, F \rangle + \imath(\partial_\sigma x + \nabla_\sigma \theta)(H_3 + \langle \tilde{\Theta}, F \rangle) \right];$$  

(4.11a)

$$\omega_{\tilde{X}} = \oint_{S^1} d\sigma \left[ \delta p_\mu \wedge \delta x^\mu - \langle \tilde{\Theta}, \delta \tilde{p} \rangle + \langle \tilde{F}, \tilde{p} \rangle + \imath(\partial_\sigma x + \nabla_\sigma \tilde{\theta})(H_3 + \langle \tilde{\Theta}, F \rangle) \right].$$  

(4.11b)

**Corollary 4.1.** Instead of doing symplectic reduction with respect to $\tilde{T}^n$ or $T^n$ one can reduce with respect to some sub-torus inside $\tilde{T}^n \times T^n$. The space of such sub-tori is an affine space with the group of translations being given by $O(n, n; \mathbb{Z})$. This space encompasses all $T$-dual backgrounds.

Clearly, the symmetry in (4.11) corresponds to the $T$-duality exchange that was discussed in section 3 (for $B_0^{IJ} = 0$).
4.3 Non-hamiltonian torus action

As was discussed in section 2, if $B_{0}^{I J}$ is non-zero the original torus action can in general be only lifted to an $\mathbb{R}^n$ action. However we have also seen that if $B_{0}^{I J}$ is globally well-defined then topologically $Y$ is a principal $2n$-torus bundle over $M$, and there exists another lift which defines the torus action. In other words, $B_{0}^{I J} \neq 0$ is a geometrical obstruction for $Y$ being a principal torus bundle with connection over $M$, but when it is well defined there exists a connection (2.9), $\tilde{\Theta}^{I} = \Theta^{I}_{\#} - B_{0}^{I J} \Theta^{J}$, which respects the double-torus fibered structure of $Y$. We may extend the construction of the previous subsection to the case when $B_{0}^{I J} \neq 0$ and is not necessarily globally defined.

Symplectic form on $Y$. A connection on a line bundle over the cotangent bundle to the loop space of $Y$ can be written in a form similar to (4.8):

$$\vartheta^{Y} = \delta z + \oint_{S^{1}} d\sigma \left[ p_{\mu} \delta x^{\mu} + \langle p, \Theta \rangle + \langle \Theta^{#}, p^{#} \rangle - \iota(\partial_{\sigma} x + \nabla_{\sigma} \theta)(B - \langle \Theta^{#}, \Theta \rangle) \right]$$

where $\nabla_{\sigma} \theta_{I}$ is as before but

$$\Theta^{I}_{\#} = \delta \theta^{I}_{\#} + B_{0}^{I J} \Theta^{J}. \quad (4.13)$$

Note that $\vartheta^{Y}$ is written in terms of globally well defined connection $\Theta^{#}$. One easily sees that if $B_{0} = 0$ the equation (4.12) reduces to (4.8). The sympletic form is

$$\omega^{Y} = \oint_{S^{1}} d\sigma \left[ \delta p_{\mu} \wedge \delta x^{\mu} + \langle \delta p, \Theta \rangle + \langle \delta p, F \rangle + \langle F^{#}, p^{#} \rangle - \langle \Theta^{#}, \delta p^{#} \rangle \right]$$

$$+ \iota(\partial_{\sigma} x + \nabla_{\sigma} \theta)H - \delta(\Theta^{#}, \nabla_{\sigma} \theta) \right]$$

where we have used $d\Theta^{I}_{\#} = p^{#} F^{I}_{\#}$. Recall that $F^{I}_{\#} = \iota(\frac{\partial}{\partial \theta^{I}}) \theta$ and thus it is globally well defined.

The action of $T^{n}_{\#}$ on the symplectic form $\omega^{Y}$ is still hamiltonian:

$$\iota\left( \frac{\delta}{\delta \theta^{I}} \right)_{\text{hor}} \omega^{Y} = \delta(-p_{#}^{I} + \nabla_{\sigma} \theta_{I}). \quad (4.15)$$

So the hamiltonian reduction by $T^{n}_{\#}$ yields a sigma model with the symplectic form $\omega_{X} = \delta \vartheta_{X}$ constructed from the connection (4.11). The action of $T^{n}$ however is no longer hamiltonian:

$$\iota\left( \frac{\delta}{\delta \theta^{I}} \right)_{\text{hor}} \omega^{Y} = \delta(-p_{I} + \nabla_{\sigma} \theta_{I}) + H^{I}_{J}(-p_{#}^{J} + \nabla_{\sigma} \theta_{J}), \quad (4.16)$$

where $(\frac{\delta}{\delta \theta^{I}})_{\text{hor}|M_{n}} = \frac{\delta}{\delta \theta^{I}} - B_{0}^{IJ} \delta_{\text{hor}^{#}_{\#}}$ denotes the horizontal lift of the vector field $\frac{\delta}{\delta \theta^{I}}$ via the connection $\Theta^{#}$. This is explained by the failure of the $T^{n}$ to lift to $Y$ and it is not surprising that $H^{I}_{J}$ defines the obstruction to the symplectic reduction.\textsuperscript{10}

\textsuperscript{10}Actually this is a weaker condition - having a constant irrational $B_{0}^{IJ}$ is sufficient for the failure of the torus action to lift. Once more, (4.16) holds regardless whether $B_{0}^{IJ}$ is globally defined or not.
When $B_0^{IJ}$ is well defined this situation can be fixed: the vector fields $\frac{\partial}{\partial \nu_I}$ and $\frac{\partial}{\partial \nu^I}$ are independently well defined and thus we can rewrite equation (4.16) as

$$\iota\left(\delta \frac{\partial}{\partial \nu_I}\right) \omega_Y - B_0^{IJ} \iota\left(\delta \frac{\partial}{\partial \nu^J}\right) \omega_Y = \delta(-p^I + \nabla_\sigma \theta^I) + \delta B_0^{IJ} (-p^J + \nabla_\sigma \theta_J)$$

$$\Rightarrow \iota\left(\delta \frac{\partial}{\partial \nu_I}\right) \omega_Y = \delta(-p^I - B_0^{IJ} p^J + \nabla_\sigma \theta^I + B_0^{IJ} \nabla_\sigma \theta_J). \quad (4.17)$$

Now on substituting $\Theta^I_\# = \tilde{\Theta}^I + B_0^{IJ} \Theta^J$ and redefining $(p^I - B_0^{IJ} p^J_\#, p^J_\#) \rightarrow (p^I, \tilde{p}^I)$ one finds that $\omega_Y$ can be written as

$$\omega_Y = \oint_{S^1} \ d\sigma \left[ \delta p_\mu \wedge \delta x^\mu + \langle \delta p, \Theta \rangle + \langle p, F \rangle + \langle \tilde{F}, \tilde{p} \rangle - \langle \tilde{\Theta}, \delta \tilde{p} \rangle \right.$$

$$\left. + \ i(\partial_\sigma x + \nabla_\sigma \theta)(H_3 + \langle \tilde{\Theta}, F \rangle) \right] \quad (4.18)$$

One can check that there are now two hamiltonian torus actions with respective reductions yielding symplectic forms $\omega_X$ and $\omega_{\tilde{X}}$ related via $(\Theta^I, p^I) \leftrightarrow (\tilde{\Theta}^I, \tilde{p}^I)$.

When $B_0^{IJ}$ is not globally well defined, similar steps can be made but instead of (4.18) one obtains:

**Theorem 4.2.** The symplectic form $\omega_Y$ on $T^*LY$ can be written as

$$\omega_Y = \oint_{S^1} \ d\sigma \left[ \delta p_\mu \wedge \delta x^\mu + \langle \delta p_\alpha, \Theta_\alpha \rangle + \langle p_\alpha, F_\alpha \rangle + \langle \tilde{F}_\alpha, \tilde{p}_\alpha \rangle - \langle \tilde{\Theta}_\alpha, \delta \tilde{p}_\alpha \rangle \right.$$

$$\left. + \ i(\partial_\sigma x + \nabla_\sigma \theta)(H_3 + \langle \tilde{\Theta}_\alpha, F_\alpha \rangle) \right] \quad (4.19)$$

where the both the momenta $\tilde{p}_\alpha, p_\alpha$ and connections $\tilde{\Theta}_\alpha, \Theta_\alpha$ are not globally defined. The gluing functions on twofold overlaps $M_{\alpha \beta}$ are

$$\begin{pmatrix} \tilde{p}_\alpha \\ p_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m_{\alpha \beta} & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_\beta \\ p_\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\Theta}_\alpha \\ \Theta_\alpha \end{pmatrix} = \begin{pmatrix} 1 & m_{\alpha \beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\Theta}_\beta \\ \Theta_\beta \end{pmatrix}. \quad (4.20)$$

The expressions (4.14) and (4.19) are the same. Note that although most of the terms in (4.19) are not well defined — their sum is well defined, and thus can be integrated.

Thus the string sigma model can be consistently quantized in the canonical approach with a phase space constructed from the generalized correspondence space $Y$, which an affine doubled torus fibration over a base manifold $M$ even in the case in which the $B_0^{IJ}$ component of $B$ is not globally well defined. This supports the view that passing to $Y$ is the correct way of dealing with sigma models in situations in which the gauging and T-duality is obstructed. The phase space on
$T^*LY$ with symplectic form on $\omega_Y$ has a natural $O(n,n)$ action and puts momentum and winding modes on an equal footing, so that gluing functions mixing the two can be incorporated easily. It seems that the phase space can then be defined in situations in which there is no well defined dual configuration space.

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A Wess-Zumino term and holonomy of the gerbe connection

In this appendix we review a definition of Wess-Zumino term or logarithm of a gerbe connection for topologically nontrivial $B$-field.

Holonomy of an abelian 1-form gauge field. Given a contractible cover $\{M_{\alpha}\}$ of a manifold $M$ a 1-form gauge field is specified by the following data

- a function $\lambda_{\alpha\beta}$ on each twofold intersection $M_{\alpha\beta}$ satisfying the cocycle condition that $\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = 0$ on threefold overlaps.
- a 1-form $A_\alpha$ on each $M_\alpha$ such that on twofold overlaps $M_{\alpha\beta}$: $A_\alpha - A_\beta|_{M_{\alpha\beta}} = d\lambda_{\alpha\beta}$.

A loop $\gamma : S^1 \to M$ does not necessarily lies within one patch. We break the loop into segments $\{\gamma_{\alpha} \subset M_{\alpha}\}$ and denote by $\gamma_{\alpha\beta} \in M_{\alpha\beta}$ a point where the segments $\gamma_\alpha$ and $\gamma_\beta$ intersect. Then the logarithm of the holonomy of the gauge field $A$ is defined as the following sum

$$\frac{1}{2\pi i} \log \text{Hol}(A, \gamma) = \sum_{\{\gamma_{\alpha}\}} \int_{\gamma_{\alpha}} A_\alpha + \sum_{\gamma_{\alpha\beta}} \lambda_{\alpha\beta}(\gamma_{\alpha\beta}).$$ (A.1)
One can easily verify that this sum does not depend on a particular choice of the partitioning \(\{\gamma_\alpha, \gamma_{\alpha\beta}\}\) of the loop \(\gamma\).

### Holonomy of a gerbe connection.

A gerbe connection is defined by a set of 2-forms \(B_\alpha\), 1-forms \(\{A_{\alpha\beta}\}\) on twofold intersections and functions \(f_{\alpha\beta\gamma} : M_{\alpha\beta\gamma} \to U(1)\) on threefold intersections (see section 2.2 for details). Given a 2-cycle \(\Sigma\) we can partition it as shown in the picture: \(\Sigma_\alpha\) are surfaces, \(\Sigma_{\alpha\beta}\) is the common boundary of \(\Sigma_\alpha\) and \(\Sigma_\beta\) and \(\Sigma_{\alpha\beta\gamma}\) is the intersection of segments \(\Sigma_{\alpha\beta}\), \(\Sigma_{\beta\gamma}\) and \(\Sigma_{\gamma\alpha}\). The holonomy of the gerbe connection is defined by the following sum (the orientation is important)

\[
\frac{1}{2\pi i} \log \text{Hol}(B, \Sigma) = \sum_{\{\Sigma_\alpha\}} \int_{\Sigma_\alpha} B_\alpha + \sum_{\{\Sigma_{\alpha\beta}\}} \int_{\Sigma_{\alpha\beta}} A_{\alpha\beta} + \sum_{\{\Sigma_{\alpha\beta\gamma}\}} \frac{1}{2\pi i} \log f_{\alpha\beta\gamma}(\Sigma_{\alpha\beta\gamma}).
\]

(A.2)

It requires a little bit more work to verify that this sum does not depend on a particular choice of partitioning of the 2-cycle \(\Sigma\).

### B Torus action on the gerbe connection

To discuss a group action on a sigma model one has to specify how it acts on the target space, metric and on any additional structure involved. In this paper we have assumed that the space \(X\) is a principal torus bundle (\(T^n\) acts freely on it) and that the metric is \(T^n\)-invariant. We now discuss how the torus group acts on a gerbe connection with \(T^n\)-invariant curvature.

In this appendix we choose a contractible covering \(\{U_\alpha\}\) of the target space \(X\). The main result can be summarized by the following

**Theorem B.1.** Let \(\{U_\alpha\}\) be a contractible covering of the space \(X\). The action of the torus group on a gerbe connection with \(T^n\)-invariant curvature is specified by a 1-forms \(\{w^I_\alpha\}\) in every path \(U_\alpha\), a function \(u^I_{\alpha\beta}\) in each twofold overlap \(U_{\alpha\beta}\) and a constant \(c^I_{\alpha\beta\gamma}\) in each threefold overlap \(U_{\alpha\beta\gamma}\) such that they satisfy the following conditions

\[
\mathcal{L}(\frac{\partial}{\partial \theta^I})B_\alpha = dw^I_\alpha; \quad (B.3a)
\]

\[
(u^I_\alpha - u^I_\beta)|_{U_{\alpha\beta}} = \mathcal{L}(\frac{\partial}{\partial \theta^I})A_{\alpha\beta} + du^I_{\alpha\beta}; \quad (B.3b)
\]

\[
(u^I_{\alpha\beta} + u^I_{\beta\gamma} + u^I_{\gamma\alpha})|_{U_{\alpha\beta\gamma}} = c^I_{\alpha\beta\gamma} - \frac{1}{2\pi i} \mathcal{L}(\frac{\partial}{\partial \theta^I}) \log f_{\alpha\beta\gamma}; \quad (B.3c)
\]

\[
(c^I_{\alpha\beta\gamma} - c^I_{\beta\gamma\delta} + c^I_{\gamma\delta\alpha} - c^I_{\delta\alpha\beta})|_{U_{\alpha\beta\gamma\delta}} = m^I_{\alpha\beta\gamma\delta} \in \mathbb{Z}. \quad (B.3d)
\]
Proof. The invariance of the curvature $H$ of the gerbe connection implies $\mathcal{L}(\frac{\partial}{\partial \theta})H = 0$. From this it follows that $\mathcal{L}(\frac{\partial}{\partial \theta})B_\alpha$ is a closed form. Since the patch $\mathcal{U}_\alpha$ is contractible this closed form is exact, so we denote it by $dw_\alpha$. From the gluing conditions for $B_\alpha$ we obtain (B.3b) for some $u_{\alpha\beta}^I$, from the gluing condition for $A_{\alpha\beta}$ one obtains (B.3c). Finally equation (B.3d) comes from the cocycle condition on $f_{\alpha\beta\gamma}$.

Note that THE simplest solution to these equations is that in which $B_\alpha$ is invariant with respect to the torus action in each patch.

Given a structure specified in the Theorem (B.1) we can construct a connection $\Theta_#$ and check whether or not we have a principal torus bundle. The curvature $F_# = d\Theta_#$ is

$$F_#|_{\mathcal{U}_\alpha} = d\{w_\alpha^I - \iota(\frac{\partial}{\partial \theta})B_\alpha\}.$$  \hspace{1cm} (B.4)

The gluing conditions for these 1-forms are

$$\{w_\alpha^I - \iota(\frac{\partial}{\partial \theta})B_\alpha\} - \{w_\beta^I - \iota(\frac{\partial}{\partial \theta})B_\beta\} = d\{\iota(\frac{\partial}{\partial \theta})A_{\alpha\beta} + u_{\alpha\beta}^I\}. \hspace{1cm} (B.5)$$

On threefold overlap the functions defined on the right hand side satisfy

$$\{\iota(\frac{\partial}{\partial \theta})A_{\alpha\beta} + u_{\alpha\beta}^I\}|_{\mathcal{U}_{\alpha\beta\gamma}} + \cdots = c_{\alpha\beta\gamma}. \hspace{1cm} (B.6)$$

Thus $\{c_{\alpha\beta\gamma}\}$ is an obstruction to the geometrization of $F_#'$: if it does not vanish then one does not have a principal torus bundle with connection over $X$ whose curvature is $F_#'$.

C Reduction of a $U(1)$ bundle

We consider here a toy example of a reduction of a principal circle bundle $L \xrightarrow{p} X$ onto the torus fibration $\mathbb{T}^n \hookrightarrow X \xrightarrow{\pi} M$, specified in subsection 2.1. This example captures the essential features of the affine bundle appearing in the gerbe reduction given in section 2.2, but is considerably simpler.

We denote by $\Theta_#$ a connection 1-form on the total space $L$ of the circle bundle. Locally it can be written as

$$\Theta_#|_{X_\alpha} = d\psi_\alpha + p^*B_\alpha \hspace{1cm} (C.1)$$

where $\psi_\alpha$ ($0 \leq \psi_\alpha I < 1$) is a coordinate on the circle in the patch $X_\alpha$ and $\{B_\alpha\}$ is a 1-form. We denote by $H \in \Omega^2_\mathbb{Z}(X)$ the curvature of this connection, and assume that both $H$ and $\{B_\alpha\}$ are invariant with respect to the torus action:

$$\mathcal{L}(\frac{\partial}{\partial \theta})H = 0 \quad \text{and} \quad \mathcal{L}(\frac{\partial}{\partial \theta})B_\alpha = 0.$$

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The gluing conditions on two-fold overlaps \( \{X_{\alpha\beta}\} \) are

\[
\psi_{\alpha}|_{X_{\alpha\beta}} - \psi_{\beta}|_{X_{\alpha\beta}} = -\sigma_{\alpha\beta}, \quad B_{\alpha}|_{X_{\alpha\beta}} - B_{\beta}|_{X_{\alpha\beta}} = d\sigma_{\alpha\beta}
\]

where \( \{\sigma_{\alpha\beta}\} \) are functions on twofold overlaps satisfying the cocycle condition on threefold overlaps \( X_{\alpha\beta\gamma} \): \( \sigma_{\alpha\beta} + \sigma_{\beta\gamma} + \sigma_{\gamma\alpha} = 0 \).

The invariant 2-form \( H \) can be decomposed in horizontal forms:

\[
H = \pi^*H_2 + \langle \pi^*H_1, \Theta \rangle + \frac{1}{2} \langle \pi^*H_0, \Theta \wedge \Theta \rangle,
\]

where \( H_j, j = 2, 1, 0 \), are \( j \)-forms on the base manifold \( M \). The assumption of the invariance of the 1-forms \( B_{\alpha} \) with respect to the torus action yields \( H_0^{IJ} = 0 \).

In each coordinate patch we can decompose the 1-form \( B_{\alpha} \) into vertical and horizontal forms:

\[
B_{\alpha} = B_{1\alpha} + \langle B_{0\alpha}, \Theta \rangle,
\]

where \( B_{1\alpha} \) and \( B_{0\alpha} \) are horizontal 1- and 0-forms respectively. The gluing conditions take the form:

\[
B_{1\alpha}|_{M_{\alpha\beta}} - B_{1\beta}|_{M_{\alpha\beta}} = (\pi^*d_M)\sigma_{\alpha\beta}, \quad B_{0\alpha}|_{M_{\alpha\beta}} - B_{0\beta}|_{M_{\alpha\beta}} = \mathcal{L}(\theta^\beta_\alpha)\sigma_{\alpha\beta}.
\]

The invariance of \( \{B_{\alpha}\} \) with respect to the torus action yields restrictions on a possible dependence of the gluing functions on the torus coordinates: the right hand side of (C.5) must be a pullback from the base manifold. The most general solution of this conditions is

\[
\sigma_{\alpha\beta}(\theta_\beta) = \pi^*\tilde{\sigma}_{\alpha\beta} + \langle m_{\alpha\beta}, \theta_\beta + \frac{1}{2} \lambda_{\beta\alpha} \rangle,
\]

where \( m_{\alpha\beta}^I \) is an integral valued vector and \( \tilde{\sigma}_{\alpha\beta} \) is a function on the twofold overlap \( M_{\alpha\beta} \) of the base manifold. On the threefold overlaps \( \{M_{\alpha\beta\gamma}\} \) they satisfy the following conditions:

\[
m_{\alpha\beta}^I + m_{\beta\gamma}^I + m_{\gamma\alpha}^I = 0, \quad \tilde{\sigma}_{\alpha\beta} + \tilde{\sigma}_{\beta\gamma} + \tilde{\sigma}_{\gamma\alpha} = \frac{1}{2} (\langle m_{\alpha\beta}, \lambda_{\beta\gamma} \rangle - \langle m_{\alpha\gamma}, \lambda_{\beta\alpha} \rangle).
\]
It is now not hard to see that the equation (C.6) and the first of (C.8) define an affine $S^1 \times \mathbb{T}^n$-torus bundle over $M$. On twofold overlaps $M_{\alpha \beta}$, the gluing conditions for coordinates $\psi_\alpha$ and $\theta_\alpha$ and the affine connection $\Theta_\alpha = d\theta_\alpha + A_\alpha$ and $\tilde{\Theta}_\alpha = d\psi_\alpha + B_1\alpha$ are given by:

\[
\begin{pmatrix}
\psi_\alpha + \frac{1}{2} \tilde{\sigma}_{\alpha \beta} \\
\theta_\alpha + \frac{1}{2} \lambda_{\alpha \beta}
\end{pmatrix} = \begin{pmatrix}
1 & m_{\alpha \beta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi_\beta + \frac{1}{2} \tilde{\sigma}_{\beta \alpha} \\
\theta_\beta + \frac{1}{2} \lambda_{\beta \alpha}
\end{pmatrix}, \quad \begin{pmatrix}
\check{\Theta}_\alpha \\
\Theta_\alpha
\end{pmatrix} = \begin{pmatrix}
1 & m_{\alpha \beta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\check{\Theta}_\beta \\
\Theta_\beta
\end{pmatrix}.
\]

(C.9)

Only when the gluing function $\sigma_{\alpha \beta}$ does not depend on torus coordinates, i.e. $m^{I}_{\alpha \beta} = 0$, the reduction of the $U(1)$ bundle yields a 1-form on $M$ and $n$ scalar fields. When the gluing function does not respect the torus action, the result of the reduction is given by an affine $S^1 \times \mathbb{T}^n$ fibration over $M$ and $n$ line bundles.

### D Reduction of the current algebra

**Current algebra.** Given a section $(v, \rho)$ of $TX \oplus T^*X$ one can construct a current

\[
J_\epsilon(v, \rho) = \oint_{S^1} d\sigma \epsilon(\sigma) [\iota(v)p + \iota(\partial_\sigma x) \rho]
\]

(D.1)

where $\epsilon(\sigma)$ is a smooth (test) function on the circle. From (4.11) it follows that the Poisson bracket of two such currents is [28, 29, 30]

\[
\{J_{\epsilon_1}(v_1, \rho_1), J_{\epsilon_2}(v_2, \rho_2)\} = J_{\epsilon_1 \epsilon_2}([[(v_1, \rho_1), (v_2, \rho_2)]_{H}] - \frac{1}{2} \oint_{S^1} d\sigma (\epsilon_1 \partial_\sigma \epsilon_2 - \epsilon_2 \partial_\sigma \epsilon_1) [\iota(v_1)\rho_2 + \iota(v_2)\rho_1]
\]

(D.2)

where $[,]_H$ is the twisted Courant bracket. The twisted Courant bracket is defined by

\[
[[v_1, \rho_1], (v_2, \rho_2)]_H = [v_1, v_2] + \{ \mathcal{L}(v_1)\rho_2 - \mathcal{L}(v_2)\rho_1 - \frac{1}{2} d(\iota(v_1)\rho_2 - \iota(v_2)\rho_1) + \iota(v_1)\iota(v_2)H \}
\]

(D.3)

where $[,]$ denotes the commutator of vector fields. Note that one can rewrite the Poisson bracket above in a slightly different form: as a twisted Courant bracket on the $T \oplus T^*$ bundle over $X \times S^1$ (see equation (30) in [28]).

**Reduction of the Courant bracket.** Taking $X$ to be a principal torus bundle, we can study the reduction of the twisted current algebra to the base $M$.

We start by decomposing the sections of $TX \oplus T^*X$ into horizontal and vertical components. Any vector $v$ and one-form $\rho$ can be written as

\[
v = v_M + \langle K, f \rangle
\]
\[
\rho = \rho_M + \langle \phi, \Theta \rangle
\]

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Demanding that both $L_K v = 0$ and $L_K \rho = 0$, implies in particular $f \in \Omega^0(M, t)$ and $\phi \in \Omega^0(M, t^*)$. In other words, a $\mathbb{T}^n$-invariant section of $TX$ can be written as an element $(v_M, f) \in TM \oplus t$, while a $\mathbb{T}^n$-invariant section of $TX^*$ can be written as $(\rho_M, \phi) \in T^*M \oplus t^*$. Given these elements, we can introduce some basic operations replacing the contractions, Lie brackets and Lie derivatives:

$$i((v_M, f))(\lambda_M, \omega) = i(v_M)\lambda_M + \omega(f)$$

$$d(\lambda_M, \omega) = (d\lambda_M + \omega, F), -d\omega$$

$$\mathcal{L}_{(v_M, f)}(\lambda_M, \omega) = (\mathcal{L}_{v_M} \lambda_M + \omega, i(v_M)F + df), \mathcal{L}_{v_M} \omega)$$

$$[(v_M, f), (w_M, g)] = [(v_M, w_M), i(v_M)i(w_M)F + \mathcal{L}_{v_M}g - \mathcal{L}_{w_M}f]$$

In this notation, a contraction of the element $(v_M, f)$ with a $p$-form in $\Omega^p_Z(X)$ can be thought of as a collection of forms in $\Omega^i_Z(M, \Lambda^{p-i-1})$ for $i = 0, \ldots, p - 1$. In particular, for $H \in \Omega^2_Z(X)$,

$$i((v_M, f))H = \left((i(v_M)H_3 + \langle H_2, f \rangle), (i(v_M)H_2 - \langle H_1, f \rangle), (i(v_M)H_1 + \langle H_0, f \rangle)\right)$$

We can now write down the reduction of the twisted Courant algebra to the base $M$ in a compact form.

**Theorem D.1.** The space of $\mathbb{T}^n$-invariant sections of $TX \oplus T^*X$ is isomorphic to $\Gamma(TM \oplus T^*M \oplus t \oplus t^*)$. The Courant bracket on $TX \oplus T^*X$ yields the following bracket on $\mathbb{T}^n$-invariant sections

$$[(v_M, f; \rho_M, \phi), (w_M, g; \lambda_M, \omega)]_H = \left([(v_M, f), (w_M, g)]; \mathcal{L}_{(v_M, f)}(\lambda_M, \omega) - \mathcal{L}_{(w_M, g)}(\rho_M, \phi) + \frac{1}{2}(i((v_M, f))(\lambda_M, \omega) - i((w_M, g))(\rho_M, \phi)) + i((v_M, f))i((w_M, g))H)\right).$$

The reduced Courant bracket in Theorem D.1 can be cast as

$$[(v_M, f; \rho_M, \phi), (w_M, g; \lambda_M, \omega)]_H = \left([(v_M; \rho_M), (w_M; \lambda_M)]_H + \left(0, \mathcal{L}_{v_M}g - \mathcal{L}_{w_M}f; \langle \omega, df \rangle - \langle \phi, dg \rangle + \frac{1}{2}d(\langle \omega, f \rangle - \langle \phi, g \rangle), \mathcal{L}_{v_M}\omega - \mathcal{L}_{w_M}\phi\right)\right) + \left(0, i(v_M)i(w_M)F; \langle \omega, i(v_M)F \rangle + \langle i(v_M)F_\#, g \rangle - \langle i(w_M)F_\#, f \rangle - \langle \phi, i(w_M)F \rangle, i(v_M)i(w_M)F_\#\right) - \left(0, 0; \langle H_1, [f, g] \rangle, \langle H_0, [f, g] \rangle\right).$$

The rhs of the first line is the Courant bracket on the base $M$ twisted by a 3-form $H_3$, which in general will not be closed. Adding the second line amounts to extending the bracket to $M \times \mathbb{T}^n$ (or $M \times \mathbb{R}^n$) [31]. On the third line we recover $F_\#_I = H_2^I + H_1^{I, J} \wedge \Theta_J + \frac{1}{2}H_0^{I, JK} \Theta_J \wedge \Theta_K$; in the absence of nontrivial $B_1$ and $B_0$ it displays an explicit $O(n, n, \mathbb{Z})$ symmetry, which exchanges the terms containing $F_I$ and $H_2^I$, reflecting the fact that there are two independent principal tori on $M$.
(and thus two choices to which of the two forms corresponds to the curvature of the connection, and which to twisting). It may appear strange that in the general case the natural $O(n, n, \mathbb{Z})$ action is on 2-forms $F^I_\#$ rather than $\tilde{F}^I$ preferred by the sigma model. However nontrivial $B^I_0$ and $H^IJK_0$ have contributions that spoil this symmetry of the Courant bracket. These are collected in the last line (with $[\cdot, \cdot]$ denoting antisymmetrization in $I, J$ indices).

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