On equations of motion on Siegel-Jacobi spaces
generated by linear Hamiltonians in the generators
of the Jacobi group

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Abstract. It is proved that the equations of classical motion and the quantum evolution
on the Siegel-Jacobi disk generated by a Hamiltonian linear in the generators of the Jacobi
group $G_j$ obtained by the Wei-Norman method and a method used in the context of Berezin’s
quantization are identical. In a certain set of variables the motion on the Siegel disk and $C$ are
decoupled. The geometric significance and the meaning in the context of coherent states of this
coordinates are emphasized.

1. Introduction
The Jacobi group $G_j$ - the semidirect product of the Heisenberg group $H_n$ with $\text{Sp}(n, \mathbb{R})$ - is
intensively studied in mathematics [1, 2, 3]. The Jacobi group $G_j$ is relevant in physics, in
particular in quantum mechanics [4], being responsible [5] for the squeezed states in quantum
optics [6, 7, 8]. To the Jacobi group [9] it is associated the so called Siegel-Jacobi ball
$D_j \approx \mathbb{C}^n \times D_n$, where the the Siegel ball $D_n = \text{Sp}(n, \mathbb{R})/U(n)$ admits a realization as a bounded domain [12]. To the
Jacobi group we have attached coherent states [13] based on the Siegel-Jacobi spaces [14, 9, 3, 15].
Different aspects of quantum mechanics via coherent states and geometry on the Siegel-Jacobi
disk $D_j$ are emphasized in [16]. The equations of motion on the Siegel-Jacobi ball and Siegel-
Jacobi upper half plane generated by linear Hamiltonians in the generators of the Jacobi group
are studied in [14, 15, 17] using a method developed in [18, 19, 17] in the context of Berezin’s
approach to quantization [20, 21] for coherent states [13]. On the other side, equations of motion on the Siegel-Jacobi disk are studied in [22] by the Wei-Norman method [23] applied to
the quasienergy operator [24] in the context of Lie systems of differential equations [25, 26]. The
Wei-Norman equations for $G_j$ were studied in [27, 28].

In this note I underline the equivalence between the equations of motion on the Siegel-Jacobi
disk $D_j$ (also called extended Poincaré disk, $\mathcal{D}$) generated by a hermitian Hamiltonian linear
in the generators of the Jacobi group $G_j$ obtained with apparently different methods - the Wei-
Norman method [22] and Berezin’s method [14, 17, 15, 29]. The equivalence of the equations
of motion on the Siegel-Jacobi disk is proved in a set of variables - called $FC$-variables - in
which the motion on $\mathbb{C}$ and $D_j$ are decoupled. The $FC$-transform has a special geometric
interpretation and also has significance in the context of coherent states. If $(z, w) \in \mathbb{C} \times D_j$ are

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local coordinates on the Siegel-Jacobi disk [14], it was proved in [17, 15] that under the FC-transform, $FC: (\eta, w) \rightarrow (z = \eta - w\bar{\eta}, w)$ the Kähler to form $\omega_{D_1^J}(z, w)$ becomes just the sum of $\omega_{C}(\eta)$ and $\omega_{D_1}(w)$, recalling the fundamental conjecture for homogeneous Kähler manifolds [30, 31]. The FC-transform expresses also the change of variables from the normalized to the unnormalized coherent state vectors [16]. The equations of motion on $\mathfrak{M}$ associated to a hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_1$ obtained in [22] are a particular case of equations of motion on the Siegel-Jacobi spaces attached to a hermitian Hamiltonian linear in the generators of $G^J_1$ [17].

The paper is laid out as follows. §2 recalls the method to obtain what we call Berezin’s equations of motion on a homogeneous Kähler manifold $M = G/H$ generated by a hermitian Hamiltonian linear in the generators of the Lie group $G$ if the Lie algebra $\mathfrak{g}$ admits a certain holomorphic differential representation [18, 19, 32, 17]. §3 briefly summarizes the Wei-Norman method [23] to solve linear differential equations attached to linear operators in the generators of a Lie algebra $\mathfrak{g}$. §4 recalls the commutation relations of the generators of the Jacobi algebra $\mathfrak{g}^J_1 := \mathfrak{h}_1 \ltimes \mathfrak{su}(1, 1)$ and the operators which appear in the unitary representation of the Jacobi group $G^J_1$.

§5 presents the quasienergy operator attached to the time-dependent Schrödinger equation on the Siegel-Jacobi disk $D_1^J$. A hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_1$ is considered. Using the results from [33, 34] and passing from the complex representation adopted in [14, 17] to the real representation of the Jacobi group $G^J_1$ used in [22], with the technique of §3, the equations of motion on $\mathfrak{M}$ from [22] are reobtained and shown to be identical with those obtained by Berezin’s method [17]. For completeness, the equations of motion on the Siegel-Jacobi ball [15] are briefly recalled in §5. The main contribution of the paper is contained in Proposition 2. More details are presented in [29].

Conventions. The Hilbert space $\mathfrak{H}$ considered in this paper is endowed with a scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first argument. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denotes the field of real, complex numbers, respectively the set of non-negative integers. We denote the imaginary unit $\sqrt{-1}$ by $i$, and the Real and Imaginary part of a complex number by $\Re$ and $\Im$ respectively, for $z \in \mathbb{C}$ we have $z = \Re z + i \Im z$, and $\bar{z} = \Re z - i \Im z$. Also we use the notation $cc(z) := \bar{z}$ for $z \in \mathbb{C}$ and $cc(A) = A^\dagger$ for an operator $A$. We denote by $M_n(\mathbb{F})$ the set of $n \times n$ matrices with entries in the field $\mathbb{F}$. If $A \in M_n(\mathbb{F})$, then $A^t (A^\dagger)$ denotes the transpose (respectively, the hermitian conjugate) of $A$. $I$ denotes the unit operator, while $\mathbb{I}_n$ denotes the unit matrix of $M_n(\mathbb{F})$. If $A \in M_n(\mathbb{F})$, we denote by $A^* := \frac{1}{2}(A + A^\dagger)$. We use Einstein convention that repeated indices are implicitly summed over. We denote the differential by $d$. If $\pi$ is an unitary irreducible representation of a Lie group $G$ with Lie algebra $\mathfrak{g}$ on a complex separable Hilbert space $\mathfrak{H}$, then we denote for the derived representation $X := d\pi(X), X \in \mathfrak{g}$. If $X, Y$ are elements in the lie algebra $\mathfrak{g}$, then $\text{ad}X(Y) = [X, Y]$.

2. Berezin’s equations of motion via coherent states

In the group theoretic approach to coherent states [13] it is considered the triplet $(G, \pi, \mathfrak{H})$, where $\pi$ is a continuous, unitary, irreducible representation of the Lie group $G$ on the separable complex Hilbert space $\mathfrak{H}$.

The normalized (unnormalized) vectors $e_x$ (respectively, $e_z$), based on the homogeneous manifold $M=G/H$, supposed to be Kählerian, are defined as

$$e_x = \exp(\sum_{\phi \in \Delta^+} x_\phi X^\dagger_\phi - \bar{x}_\phi X^-_\phi) e_0, \quad e_z = \exp(\sum_{\phi \in \Delta^+} z_\phi X^\dagger_\phi) e_0. \quad (1)$$

Above $e_0$ is the extremal weight vector of the representation $\pi$, $\Delta^+$ are the positive roots of the Lie algebra $\mathfrak{g}$ of $G$, and $X_\phi, \phi \in \Delta$, are the generators. $X^\dagger_\phi$ (respectively, $X^-_\phi$) corresponds to the positive (respectively, negative) generators [13, 32].
We denote by $FC$ [16] (from fundamental conjecture [30, 31]) the change of variables $x \rightarrow z$ in formula (1) such that

$$e_x = \tilde{e}_z, \quad \tilde{e}_z := < e_z, e_z >^{-\frac{1}{2}} e_z, \quad z = FC(x).$$

(2)

By a dequantization procedure, the motion on the classical phase space - a homogeneous Kähler manifold $M = G/H$ - can be described by the local equations of motion [20, 21]

$$\dot{z}_\alpha = i \{ \mathcal{H}, z_\alpha \}, \quad \alpha \in \Delta_+, \quad \mathcal{H} = < e_z, e_z >^{-1} < e_z, \mathcal{H} e_z >$$

(3)

attached to the quantum Hamiltonian $\mathcal{H}$, and the Poisson bracket $\{,\}$ is introduced using the inverse of the balanced metric [35] matrix $g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln < e_z, e_z >$. The time-dependent Schrödinger equation is expressed as

$$H(t)\psi(t) = i\hbar \frac{d\psi(t)}{dt}.$$  

(4)

A linear Hamiltonian in the generators $X_\lambda$ of the group of symmetry $G$ is considered

$$H = \sum_{\lambda \in \Delta} \epsilon_\lambda X_\lambda.$$  

(5)

We look for the solution of the Schrödinger equation of motion (4) generated by the Hamiltonian (5) as

$$\psi(t) = e^{i\phi} \tilde{e}_z.$$  

(6)

Let us suppose that the differential action corresponding to the operator $X_\lambda$ in (5) can be expressed in a local system of coordinates as a holomorphic first order differential operator with polynomial coefficients,

$$X_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda,\beta} \frac{\partial}{\partial z_\beta}, \quad \lambda \in \Delta.$$  

(7)

It can be proven [18, 19, 17] that:

**Proposition 1.** If the generators of the group of symmetry admit the representation (7), then the classical motion and quantum evolution on $M = G/H$ generated by the hermitian linear Hamiltonian (5) are solutions of the first order differential equations

$$i \dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda,\alpha}, \quad \alpha \in \Delta_+.$$  

(8)

3. Wei-Norman method

Let us consider a Lie algebra $\mathfrak{g}$ with the generators $\{X_i\}_{i=1,\ldots,n}$ and the linear operator

$$A(t) = \sum_{i=1}^n \epsilon_i(t) X_i,$$  

(9)

where $\epsilon_i(t)$ are scalar functions of $t$. Let us attach to the linear operators $U$ and $A$ the differential equation

$$\frac{dU(t)}{dt} = A(t)U(t), \quad U(0) = I.$$  

(10)
In the Wei-Norman method [23] the solution of the equation (10) is searched in the form of product of exponentials

\[ U(t) = \prod_{i=1}^{n} \exp(\xi_i(t)X_i). \]  

(11)

Then the functions \( \xi \) satisfy a first order differential equation which depends only on the Lie algebra \( \mathfrak{g} \) and the \( \epsilon(t) \)-s:

\[ \eta' = \epsilon, \]

(12)

where \( \eta \) is obtained by the formulae

\[ \left( \prod_{j=1}^{r} \exp(\xi_jX_j) \right) X_i \left( \prod_{j=1}^{r} \exp(-\xi_jX_j) \right) = \sum_{k=1}^{n} \eta_{ki}X_k, \quad i, r = 1, 2, \ldots, n. \]  

(13)

In our calculation, what we need is the following formula

\[ U^{-1}(t)U(t) = \sum_{i=1}^{n} \xi_i Y_i, \quad Y_i = \left( \prod_{k=n}^{i-1} \left[ \exp(-\xi_k \text{ad} X_k) \right] \right) X_i, \]  

(14)

obtained [29] with technique presented in [23] and the Baker-Hausdorff formula [36):

\[ e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{(\text{ad} X)^n}{n!} Y \]

\[ = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \cdots + \frac{1}{n!} \left[ \underbrace{X, [X, \ldots, [X, \ldots, [X, Y]]}_{n \text{ brackets}}, \ldots \right] + \ldots \]

4. The Jacobi Lie algebra \( \mathfrak{g}_1^J \) and the Lie Jacobi group \( G_1^J \)

The (6-dimensional) Jacobi algebra \( \mathfrak{g}_1^J \) is the semi-direct sum \( \mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1) \), where \( \mathfrak{h}_1 \) is the 3-dimensional Heisenberg algebra generated by the boson creation (respectively, annihilation) operators \( a^\dagger (a) \), \( \mathfrak{su}(1,1) \) has the generators \( K_{0,+,\pm} \), verifying the non-trivial commutation relations [14]

\[ [a, a^\dagger] = I, \]

\[ [K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0, \]  

(15a)

(15b)

(15c)

To the Heisenberg group \( H_1 \) it is attached the unitary displacement operator

\[ D(\alpha) := \exp(\alpha a^\dagger - \bar{\alpha} a) \]

\[ = \exp(-\frac{1}{2} |\alpha|^2) \exp(\alpha a^\dagger) \exp(-\bar{\alpha} a) \]

\[ = \exp(\frac{1}{2} |\alpha|^2) \exp(-\bar{\alpha} a) \exp(\alpha a^\dagger). \]

(16a)

(16b)

(16c)

We denote by \( S \) the unitary irreducible positive discrete series representation \( D_k^+ \) of the group \( SU(1,1) \) with Bargmann index \( k \) [37] and we introduce the notation \( \mathcal{S}(z) = S(w) \), where

\[ \mathcal{S}(z) := \exp(zK_+ - \bar{z}K_-), \quad z \in \mathbb{C}; \]  

\[ \mathcal{S}(w) = \exp(wK_+) \exp(\rho K_0) \exp(-\bar{w}K_-) \exp(-\rho K_0) \exp(wK_+), \quad |w| < 1; \]  

\[ w = \frac{z}{|z|} \tanh(|z|), \quad \rho = \ln(1 - w\bar{w}), \quad z \neq 0, \]  

(17a)

(17b)

(17c)

(17d)
and \( w = 0 \) for \( z = 0 \) in (17d). We attach to the Jacobi group \( G_1^J \) the unitary operator \([38]\) defined on the Siegel-Jacobi disk \( \mathcal{D}_1^J \) [14]:

\[
T(\xi) = D(\alpha)S(w), \quad \mathcal{D}_1^J \ni \xi = (\alpha, w) \in \mathbb{C} \times \mathcal{D}_1. \tag{18}
\]

In [14] we have introduced unnormalized Perelomov vectors \( e_\xi \) defined in the points \( \xi = (\alpha, w) \) of the Jacobi disk \( \mathcal{D}_1^J \).

The real base [2, 4] used in [22] and the base (15) of the Lie algebra \( \mathfrak{g}_1^J \) are related by the relations:

\[
\begin{align*}
N_1 &= a + a^\dagger; & N_2 &= i(a - a^\dagger); \\
K_1 &= \frac{1}{2}(K_+ + K_-); & K_2 &= \frac{1}{2i}(K_+ - K_-). \tag{19b}
\end{align*}
\]

In [22] it was considered instead of the representation (18) the equivalent representation

\[
\begin{align*}
T(\xi) &= D(x, y)S(u, v), \quad \xi = (u, v, x, y) \in \mathcal{M}, \quad \text{where} \\
D(x, y) &= \exp(iyN_1 + ixN_2), \quad S(u, v) = \exp(ik_1K_1 + ik_2K_2), \\
k_1 &= \frac{v}{2s}\ln\frac{1 + s}{1 - s}, \quad k_2 = \frac{u}{2s}\ln\frac{1 + s}{1 - s}, \quad s = (u^2 + v^2)^{\frac{1}{2}}, \quad |s| < 1. \tag{20c}
\end{align*}
\]

The parameters in the representations (20b) and (18) are related by the relations

\[
\alpha = x + i y; \quad w = u + i v. \tag{21}
\]

5. Equations of motion on the Siegel-Jacobi disk \( \mathcal{D}_1^J \)

To any linear operator \( A \) defined on \( \mathfrak{g}_1 \), we attach [4, 33, 34] the operator \( \hat{A}(\xi) \) based on the Siegel-Jacobi disk \( \mathcal{D}_1^J \)

\[
\hat{A}(\xi) := T^{-1}(\xi)AT(\xi), \quad \mathcal{D}_1^J \ni \xi = (\alpha, w) \in \mathbb{C} \times \mathcal{D}_1. \tag{22}
\]

As in [22], we consider the following family of unitary operators

\[
U(\xi, \varphi) := \exp(-i\varphi)T(\xi), \tag{23}
\]

where \( \xi \in \mathcal{D}_1^J \) and \( \varphi \) is a real phase. Let \( \tau := \frac{i}{\hbar} \). In [22] it was introduced the quasienergy operator \( E := i \frac{d}{d\tau} - \hat{H} \) [24]. With (22), we get for \( \hat{E}(\xi, \varphi) := U(-\xi, -\varphi)\hat{E}U(\xi, \varphi) \) the expression

\[
\hat{E}(\xi, \varphi) = \frac{d\varphi}{d\tau}I + iT(\xi)^{-1}\dot{T}(\xi) - \dot{\hat{H}}(\xi). \tag{24}
\]

We have the relations [33, 34]:

\[
\begin{align*}
\hat{a}(\alpha, w) &= r(a + wa^\dagger) + \alpha, \quad r = (1 - w\bar{w})^{-\frac{1}{2}}, \\
\hat{K}_0(\alpha, w) &= r^2[wK_- + (1 + |w|^2)K_0 + wK_+] + r\Re[\alpha(a^\dagger + \bar{w}a)] + \frac{1}{2}|\alpha|^2, \\
\hat{K}_-(\alpha, w) &= r^2[K_- + 2wK_0 + w^2K_+] + \alpha r(a + wa^\dagger) + \frac{1}{2}|\alpha|^2. \tag{27}
\end{align*}
\]

We express (18) as product of exponentials of the generators as in (11), where

\[
X_1 = I; \quad X_2 = a^\dagger; \quad X_3 = a; \quad X_4 = K_+; \quad X_5 = K_0; \quad X_6 = K_- . \tag{28}
\]
\[ \xi_1 = -\frac{1}{2}|z|^2; \xi_2 = z; \xi_3 = -\bar{z}; \xi_4 = w; \xi_5 = \ln(1 - w\bar{w}); \xi_6 = -\bar{w}. \]  

(29)

We apply (14) to the operator \( T(\xi) \) (18) in the variables \((z, w) \in \mathcal{D}_1^{\ell}\). With the notation:

\[
Y_2 = e^{-\xi_6 a}X_6 e^{-\xi_5 a}X_5 e^{-\xi_4 a}X_4 e^{-\xi_3 a}X_3 X_2, \\
Y_3 = e^{-\xi_6 a}X_6 e^{-\xi_5 a}X_5 e^{-\xi_4 a}X_4 X_3, \\
Y_4 = e^{-\xi_6 a}X_6 e^{-\xi_5 a}X_5 X_4, \\
Y_5 = e^{-\xi_6 a}X_6 X_5, \\
\]

we obtained the expressions:

\[
Y_1 = I; Y_2 = -\xi_3 + e^{\frac{\xi_3}{2}}(a^\dagger - \xi_6 a); Y_3 = (e^{\frac{\xi_3}{2}} - \xi_4 \xi_6 e^{-\frac{\xi_3}{2}}) a + \xi_4 e^{-\frac{\xi_3}{2}} a^\dagger; \\
Y_4 = e^{-\xi_3}(K_+ + 2\xi_4 K_0 + \xi_4^2 K_-); Y_5 = K_0 + \xi_4 K_-; Y_6 = K_-.
\]  

(30)

With (30) and (14) we obtain for \( T^{-1}\hat{T}(z, w) \) the value:

\[
T^{-1}\hat{T}(z, w) = i \Im(z\bar{\dot{z}}) + r[(\dot{z} - \bar{z}w)a^\dagger - cc] + [\bar{w}r^2K_+ - cc] + 2i \Im(\bar{w}w)r^2 K_0.
\]  

(31)

In the notation of [14, 15], we consider a hermitian Hamiltonian linear in the generators of the Jacobi group \( G_1^{\ell} \):

\[
H_0 = \epsilon_a a + \bar{\epsilon}_a a^\dagger + \epsilon_0 K_0 + \epsilon_+ K_+ + \epsilon_- K_-; \quad \bar{\epsilon}_+ = \epsilon_-; \quad \epsilon_0 = \bar{\epsilon}_0.
\]  

(32)

With equations (25)-(27), we calculate \( \hat{H}_0(\xi) \), \( \xi = (\alpha, w) \in \mathbb{C} \times \mathcal{D}_1^{\ell} \):

\[
\hat{H}_0(\alpha, w) = I_0 + C_1 a^\dagger + C_1 a + C_0 K_0 + C_+ K_+ + C_- K_-, \quad \text{where:}
\]

\[
I_0 = \epsilon_a \alpha + \bar{\epsilon}_a \bar{\alpha} + \frac{1}{2}(\epsilon_0 |\alpha|^2 + \epsilon_- |\bar{\alpha}|^2 + \epsilon_+ |\alpha|^2 + \epsilon_- |\bar{\alpha}|^2),
\]

\[
C_1 = \bar{\epsilon}_a + \epsilon_a w + \frac{\epsilon_0}{2}(\alpha + \bar{\alpha}w) + \epsilon_- \alpha w + \epsilon_+ \bar{\alpha},
\]

\[
C_0 = \epsilon_0 (1 + |w|^2) + 2(\epsilon_- w + \epsilon_+ \bar{w}),
\]

\[
C_+ = \epsilon_0 w + \epsilon_- \bar{w} + \epsilon_+.
\]

(33)

In [22] the Hamiltonian (32) was written down in real coordinates as:

\[
H_0 = 2\varepsilon_0 K_0 + 2\varepsilon_1 K_1 + 2\varepsilon_2 K_2 + 2\nu_1 N_1 + 2\nu_2 N_2.
\]  

(34)

The correspondence of the coefficients of the Hamiltonians (34) and (32) is

\[
\epsilon_a = \nu_1 + i \nu_2; \quad \epsilon_0 = 2\varepsilon_0; \quad \epsilon_+ = \varepsilon_1 - i \varepsilon_2.
\]  

(35)

Now we introduce (31) and (33) into (24), we make the change of variables (21) and operators...
Proposition 2. Let us consider \( \Phi \in \mathfrak{g} \) such that \( K_0 \Phi = k \Phi, \ k \in \mathbb{R} \). Then
\[
\Psi(\xi, \varphi) = U(\xi, \varphi)\Phi = e^{-i\varphi T(\xi)\Phi}, \ \xi = (u, v, x, y) \in \mathfrak{m},
\]
is a solution of (4) for the hermitian Hamiltonian (34) on the Siegel-Jacobi disk \( \mathcal{D}_1^r \). \( x, y \in \mathbb{R} \) verify (36), \( u, v \in \mathcal{D}_1 \) verify (37), while the phase \( \varphi \) in (23) verifies the differential equation:
\[
\dot{\varphi} = \nu_1 x - \nu_2 y + 2k(\varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v).
\]

In the complex variable \( w = u + iv \) the motion (37) on the Siegel disc \( \mathcal{D}_1 \) is described by the Riccati equation
\[
i \dot{w} = \epsilon_+ w + \epsilon_- w^2.
\]

The equations of motion (36) in the complex variable \( \eta = x + iy \) become
\[
i \dot{\eta} = \bar{\epsilon}_a + \epsilon_+ \eta + \frac{\epsilon_0}{2} \eta.
\]

In conclusion, we have proved that the Riccati equation (41) on \( \mathcal{D}_1 \) obtained with the Wei-Norman method coincides with Berezin’s equation of motion (4.8b) or (4.10b) in [15], with the difference of notation \( \epsilon_+ \leftrightarrow \epsilon_- = \bar{\epsilon}_+ \). The equation (42) for \( \eta \in \mathbb{C} \) obtained with the Wei-Norman method is just Berezin’s equation of motion (4.10a) in [15], with the correspondence

(19) and we get:

\[
\dot{E}(u, v, x, y) = G_0 \mathbf{I} + G_1 \mathbf{N}_1 + G_2 \mathbf{N}_2 + H_0 \mathbf{K}_0 + H_1 \mathbf{K}_1 + H_2 \mathbf{K}_2,
\]
where:
\[
G_0 = \dot{\varphi} + y \dot{x} - xy - 2(\nu_1 x - \nu_2 y) - \varepsilon_0(x^2 + y^2) - \varepsilon_1(x^2 - y^2) + 2\varepsilon_2 xy,
\]
\[
- \frac{G_1}{r} = (1 + u) \dot{y} - \dot{x} v + \nu_1 (1 + u) - \nu_2 v + \varepsilon_0 [x(1 + u) + yv]
\]
\[
+ \varepsilon_1 [x(1 + u) - yv] - \varepsilon_2 [y(1 + u) + xv],
\]
\[
\frac{G_2}{r} = -(1 - u) \dot{y} - \dot{x} + \nu_1 v + \nu_2 (u - 1) + \varepsilon_0 [y(1 - u) + xv]
\]
\[
+ \varepsilon_1 [xv + y(u - 1)] + \varepsilon_2 [x(u - 1) - yv],
\]
\[
- \frac{H_0}{2r^2} = \dot{v} u - \dot{u} v + \varepsilon_0 (1 + u^2 + v^2) + 2(\varepsilon_1 u - \varepsilon_2 v),
\]
\[
- \frac{H_1}{2r^2} = \dot{v} + \varepsilon_0 u + \varepsilon_1 (u^2 - v^2 + 1) - 2\varepsilon_2 uv,
\]
\[
\frac{H_2}{2r^2} = -\dot{u} + \varepsilon_0 v + 2\varepsilon_1 uv + \varepsilon_2 (u^2 - v^2 - 1).
\]

Identifying the coefficients of \( \mathbf{N}_1, \mathbf{N}_2 \), and respectively \( \mathbf{K}_1, \mathbf{K}_2 \), we get the equations of motion in the real coordinates \((u, v, x, y)\) and (38)

\[
\dot{x} = -\varepsilon_2 x + (\varepsilon_0 - \varepsilon_1)y - \nu_2,
\]
\[
\dot{y} = -(\varepsilon_0 + \varepsilon_1)x + \varepsilon_2 y - \nu_1;
\]
\[
\dot{u} = 2v(\varepsilon_1 u + \varepsilon_0) - \varepsilon_2 (1 - u^2 + v^2),
\]
\[
\dot{v} = 2u(\varepsilon_2 v - \varepsilon_0) - \varepsilon_1 (1 + u^2 - v^2);
\]
\[
- \frac{H_0}{2} = \varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v, \quad G_0 = \dot{\varphi} - (\nu_1 x - \nu_2 y).
\]
\( \epsilon_a \leftrightarrow \epsilon_a, \epsilon_+ \leftrightarrow \epsilon_- = \epsilon_+ \). The quantum and classical Berezin’s equations of motion on the Siegel-Jacobi disk determined by a hermitian Hamiltonian, linear in the generators of Jacobi group \( G_n^J \), expressed in the FC-coordinates, are the same as the equations obtained applying the Wei-Norman method. The equations of motion (41) on \( D_1 \) and (42) on \( C \) and (36) on \( \mathbb{R}^2 \), determined by the hermitian Hamiltonian (32) or (34) linear in the generators of the Jacobi group \( G_n^J \) obtained with the Wei-Norman method are a particular case of the Berezin’s equations of motion (45a) on \( D_n \), (46) on \( \mathbb{C}^n \) and respectively (48) on \( \mathbb{R}^{2n} \), determined by the hermitian Hamiltonian (43), linear in the generators of the Jacobi group \( G_n^J \). See [29] for a comparison of phases in the two approaches. In the present paper we have proved Proposition 2 by brute-force calculation in the case of the Jacobi group \( G_n^J = H_1 \times SU(1,1) \), but we believe that the identity of the results in the two methods is true in more general situations, for some semidirect products of Lie groups.

### Appendix: Berezin’s Equations of motion on the Siegel-Jacobi ball

In [9, 17] we have defined unnormalized coherent state vectors \( e_z,W \), where \( z \in \mathbb{C}^n \), and \( W \in M(n, \mathbb{C}) : W = W^\dagger, \text{Im} - W W^\dagger > 0 \) describes a point in the noncompact hermitian symmetric space \( D_n = \text{Sp}(n, \mathbb{R})/U(n) \), realized as a homogenous bounded domain.

We consider a hermitian Hamiltonian linear in the generators of the group \( G_n^J \) [17]

\[
H = \epsilon_i a_i + \epsilon_i a_i^\dagger + \epsilon_{ij} K_{ij}^0 + \epsilon_{ij} K_{ij}^- + \epsilon_{ij}^* K_{ij}^+,
\]

(43)

\[
\epsilon_0 = \epsilon_0; \quad \epsilon_- = \epsilon_-^*; \quad \epsilon_+ = \epsilon_+^*; \quad \epsilon_- = \epsilon_+.
\]

(44)

With the technique recalled in §2, we have proved in [17] that:

**Proposition 3.** a) The differential equations for \( W, z \in D_n^J \) are

\[
i W = \epsilon_- + (W\epsilon_0) W + W\epsilon_+ W, \quad W \in D_n,
\]

(45a)

\[
i \dot{z} = \epsilon + W\dot{\bar{\eta}} + \frac{1}{2} \epsilon_\eta \dot{z} + W\epsilon_+ z, \quad z \in \mathbb{C}^n,
\]

(45b)

b) Under the FC transform, \( z = \eta - W\bar{\eta} \), the differential equations in the variables \( \eta \in \mathbb{C}^n \) and \( W \in D_n \) become independent: \( W \) verifies (45a), while \( \eta \) verifies

\[
i \dot{\eta} = \epsilon + \epsilon_- \bar{\eta} + \frac{1}{2} \epsilon_0 \dot{\eta}, \quad \eta \in \mathbb{C}^n.
\]

(46)

c) The linear system of differential equations associated with matrix Riccati equation (45a) reads

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y}
\end{pmatrix} = h_c \begin{pmatrix}
X \\
Y
\end{pmatrix}, \quad h_c = \begin{pmatrix}
-i (\epsilon_0')^t & -i \epsilon_-^* \\
-i \epsilon_+ & i \epsilon_0^t
\end{pmatrix} \in \text{sp}(n, \mathbb{R})_C, \quad W = X/Y \in D_n.
\]

(47)

d) In (46) we introduce \( \eta = \xi - i \zeta, \xi, \zeta \in \mathbb{R}^n \) and we put \( \epsilon = b + i a, \epsilon_- = m + i n, \epsilon_0/2 = p + i q \) where \( a, b \in \mathbb{R}^n, m, n, p, q \in M(n, \mathbb{R}) \). The first order complex differential equation equation (46) is equivalent with a system of first order real differential equations with real coefficients, which we write as

\[
\dot{Z} = h_v Z + F, \quad Z = \begin{pmatrix}
\xi \\
\zeta
\end{pmatrix}, \quad F = \begin{pmatrix}
a \\
b
\end{pmatrix}, \quad h_v = \begin{pmatrix}
n + q & m - p \\
m + p & -n + q
\end{pmatrix} \in \text{sp}(n, \mathbb{R}).
\]

(48)

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