Evidence for mass zeros of the fermionic determinant in four-dimensional quantum electrodynamics

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Abstract

The Euclidean fermionic determinant in four-dimensional quantum electrodynamics is considered as a function of the fermionic mass for a class of $O(2) \times O(3)$ symmetric background gauge fields. These fields result in a determinant free of all cutoffs. Consider the one-loop effective action, the logarithm of the determinant, and subtract off the renormalization dependent second-order term. Suppose the small-mass behavior of this remainder is fully determined by the chiral anomaly. Then either the remainder vanishes at least once as the fermionic mass is varied in the interval $0 < m < \infty$ or it reduces to its fourth-order value in which case the new remainder, obtained after subtracting the fourth-order term, vanishes at least once. Which possibility is chosen depends on the sign of simple integrals involving the field strength tensor and its dual.

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I. INTRODUCTION

Within the standard model fermionic determinants are required for the calculation of every physical process. These determinants produce an effective functional measure for the gauge fields when the fermionic fields are integrated. They are the means by which virtual fermion loops are incorporated into a calculation. Without them color screening, quark fragmentation into hadrons and unitarity would be lost. In quantum electrodynamics the biggest barrier to understanding its nonperturbative structure is its fermionic determinant. These determinants are therefore fundamental.

They are also hard to calculate and physicists by and large lost interest in them during the 1980s. With the advent of large machines lattice QCD physicists are now starting to include the determinant in their calculations. Analytic results for QCD and QED determinants are very scarce, especially in four dimensions. Such results as they become available will serve as benchmarks for determinant algorithms, including the various lattice discretizations of the Dirac operator in use, and hence a means of reliably estimating computational error, a major problem in lattice QCD at present.

Most analytic nonperturbative results obtained so far deal with the dependence of the determinant on the coupling constant. Little attention has been given to their dependence on the fermion’s mass. One notable exception is the work of Dunne et al. [1], which gives a semi-analytic calculation of the QCD determinant’s mass dependence in an instanton background.

In two-dimensional Euclidean QED the author has shown that mass can have a profound effect on its determinant. Namely, for a large class of centrally symmetric, finite-range background gauge fields the growth of the determinant in the limit \( mR << 1 \) followed by \( |e\Phi| >> 1 \) is

\[
\ln \det \sim -\frac{|e\Phi|}{4\pi} \ln \left( \frac{|e\Phi|}{(mR)^2} \right),
\]

where \( \det \) denotes the determinant, \( R \) is the field strength’s range, and \( \Phi \) is the background field’s flux [2]. In the massless case, the Schwinger model, the determinant is quadratic in the field strength.

The second example of the nontrivial mass dependence of \( \det \) in Euclidean QED\(_2\) is the presence of mass zeros. Let \( \det \) be written as
\[ \ln \det = \Pi_2 + \ln \det_3, \] (2)

where \( \Pi_2 \) is the second-order vacuum polarization graph and \( \ln \det_3 \) is a technical term, defined in Sec. II, for the remainder after the conditionally convergent second-order term has been isolated and made gauge invariant by some regularization procedure. Then there is at least one real value of \( m \) at which \( \ln \det_3 = 0 \) when \( 0 < |e\Phi| < 2\pi \), subject to some mild restrictions on the field strength \( |e\Phi| \). There may be other mass zeros. Now recall Schwinger’s result \( |e\Phi| = 0 \) when \( \ln \det_3 = 0 \); for fields with \( \Phi = 0 \) then it is also true that

\[ \lim_{m=0} \ln \det_3 = 0; \] (3)

otherwise not. So the result is this: when \( 0 < |e\Phi| < 2\pi \) the zero in \( m \) of \( \ln \det_3 \) moves up from \( m = 0 \) to some finite value \( m > 0 \). Beyond \( |e\Phi| > 2\pi \) we can say nothing definite yet.

The obvious question to ask is whether there are mass zero(s) in the remainder term of \( \ln \det \) in QED_4, denoted by \( \ln \det_5 \). The background gauge fields \( A_\mu(x) \) considered in two dimensions have a slow \( 1/|x| \) falloff resulting in a nonvanishing chiral anomaly \( \Phi/2\pi \). Here we will consider a large class of \( O(2) \times O(3) \) symmetric background gauge fields that also have a \( 1/|x| \) falloff with a nonvanishing chiral anomaly. If the small-mass behavior of the remainder is fully determined by the chiral anomaly, as in two dimensions, then there are circumstances in which mass zeros are present in the remainder. The idea of the proof is extremely simple: show that for \( m \to 0 \) the remainder is negative and that as \( m \to \infty \) it becomes positive. The demonstration that the chiral anomaly determines the small-mass behavior of the remainder turns out to be nontrivial, and we are not able to settle this matter here. Evidence is presented that it does, but it is not conclusive.

At this point it may be asked why these mass zeros for a special class of background gauge fields are of interest. First and foremost they are a truly nonperturbative result for the exact QED_4 determinant. As such, they would serve as a benchmark result that lattice theorists could aim to reproduce. As discussed in Sec. II, once the second- and fourth-order contributions to \( \det_{\text{ren}} \) are isolated the remainder of \( \det_{\text{ren}} \) is determined by the distribution of its complex zeros in the coupling constant plane. Little is known about how these zeros distribute themselves. The presence of mass zeros in the remainder terms in \( \ln \det_{\text{ren}} \) must place a strong constraint on their distribution which future work could deal with.
In Sec. II det_{ren} is defined and some of its properties are reviewed. Section III introduces the background gauge fields used in the calculation. Section IV is an introduction to the zero-mass limit of the remainder and some of the subtleties involved. Section V establishes that all of the square-integrable zero modes of the Dirac operator $\partial$ have positive chirality. In addition it is necessary to know the scattering states and low-energy phase shifts associated with the background gauge field, and this is done in Sec. VI. Section VII gives an analysis of the low energy behavior of the exact negative chirality propagator and seeks to justify a particular approach to proving that the chiral anomaly is sufficient to describe the small-mass limit of the remainder. Section VIII demonstrates that the remainder can become positive as $m \to \infty$. Finally, Sec. IX summarizes our conclusions.

II. QED$_4$ DETERMINANT

We begin by reviewing some established results for the QED$_4$ determinant \cite{6, 7}. By fermionic determinant we mean the ratio of determinants of the interacting and free Euclidean Dirac operators, $\text{det}(\not{P} - e\not{A} + m) / \text{det}(\not{P} + m)$, defined by the renormalized determinant on $\mathbb{R}^4$, namely

$$\det_{\text{ren}} = \exp(\Pi_2 + \Pi_3 + \Pi_4)\text{det}_5(1 - eS\not{A}),$$

where

$$\ln \text{det}_5 = \text{Tr} \left[ \ln(1 - eS\not{A}) + \sum_{n=1}^{4} \frac{(eS\not{A})^n}{n} \right],$$

and $S = (\not{P} + m)^{-1}$; $\Pi_{2,3,4}$ are the second, third and fourth-order contributions to the one-loop effective action defined by some consistent regularization procedure together with a charge renormalization subtraction in $\Pi_2$. The regularization should also result in $\Pi_3 = 0$ by C-invariance, and it should give a gauge-invariant result for $\Pi_4$. The remainder, $\text{det}_5$, after these subtractions is gauge invariant and has a well-defined power series expansion without regularization. The remainder $\ln \text{det}_3$ in (2) is given by (5) with the restriction $n = 1, 2$.

The operator $S\not{A}$ is a bounded operator on the Hilbert space $L^2(\mathbb{R}^4, \sqrt{k^2 + m^2}d^4k)$ for $A_\mu \in \bigcap_{n>4} L^n(\mathbb{R}^4)$, in which case it belongs to the trace ideal $\mathcal{C}_n$ for $n > 4$ $[\mathcal{C}_n = \{ K | \text{Tr}(K^\dagger K)^{\frac{n}{2}} < \infty \}] \cite{6-9}$. This includes the case when $A_\mu(x)$ falls off as $1/|x|$.
as $|x| \to \infty$. As a result $\det_5$ is an entire function of the coupling $e$, and it can be represented in terms of the discrete complex eigenvalues $1/e_n$ of the non-Hermitian compact operator $SA$ [10] :

$$\det_5(1 - eSA) = \prod_n \left( 1 - \frac{e}{e_n} \right) \exp \left( \sum_{k=1}^{4} \frac{(e/e_n)^k}{k} \right).$$  \hfill (6)

By C-invariance and the reality of $\det_5$ these eigenvalues appear in quartets $\pm e_n, \pm \bar{e}_n$ or as imaginary pairs. Because $\det_{ren}$ has no zeros for real $e$ when $m \neq 0$ [11] and $\det_{ren}(e = 0) = 1$, it is positive for real $e$. Because $SA \in C_n, n > 4$, it is of order 4. This means that for suitable positive constants $A(\epsilon), K(\epsilon)$ and any complex value of $e$, $|\det_{ren}| < A(\epsilon) \exp \left( K(\epsilon)|e|^{4+\epsilon} \right)$ for any $\epsilon > 0$. The first paper to show that $\det_{ren}$ is of order 4 was that in [12].

In the coordinate space representation of the operator $S(P)A(X)$, the propagator is given by

$$S(x) = \frac{m}{4\pi^2} \left( i\partial_t + m \right) K_1(m|x|).$$  \hfill (7)

Here $S$ is an analytic function of $m$ throughout the complex $m$-plane cut along the negative real axis. A theorem of Gohberg and Krein [13] states that if $A(\mu) \in C_1$ and is analytic in $\mu$ in some region then so is $\det(1 - A(\mu))$. In our case $SA \in C_{4+\epsilon}$, requiring the four subtractions from the logarithm in (5). These subtractions are easily incorporated into Gohberg and Krein’s proof for $SA \in C_1$, provided use is made of the inequality [10, 14],

$$|\det_n(1 + A)| \leq e^{\Gamma_k||A||_n^2},$$  \hfill (8)

for $A \in C_n, ||A||_n^2 = \operatorname{Tr}|A^\dagger A|^\frac{n}{2}$, and $\Gamma_k$ is a constant. Therefore $\det_5(1 - SA)$ is infinitely differentiable in $m$ on the interval $(0, \infty)$.

The regularization procedure used here is Schwinger’s heat kernel representation [15]:

$$\ln \det_{ren} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \int d^4x \left\{ \operatorname{tr}(x)e^{-P^2t} - e^{-(D^2 + \frac{1}{2}\sigma F)t}|x\rangle + \frac{1}{24\pi^2} F_{\mu\nu}(x) \right\} e^{-tm^2}$$

$$= \frac{1}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} |\hat{F}_{\nu\mu}(k)|^2 \int_0^1 dzz(1 - z) \ln \left( \frac{z(1 - z)k^2 + m^2}{m^2} \right)$$

$$+ \Pi_4 + \ln \det_5(1 - SA).$$  \hfill (9)

Here $e$ has been absorbed into $A_\mu$, $D^2 = (P - A)^2$, $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i$, $\gamma_\mu^\dagger = -\gamma_\mu$, $\hat{F}_{\mu\nu}$ denotes the Fourier transform of $F_{\mu\nu}$, and $m$ is the fermionic mass. A second-order on-shell charge
renormalization subtraction has been incorporated. All terms appearing on the right-hand side of (9) follow from the heat kernel expression on the left-hand side. The requirement that $A_\mu \in \bigcap_{n>4} L_n(\mathbb{R}^4)$ and certain differentiability conditions on $A_\mu$ introduced later are sufficient to ensure that (9) makes mathematical sense.

In the representation

$$\gamma_0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

one gets

$$D^2 + \frac{1}{2}\sigma F = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

where

$$H_\pm = (P - A)^2 - \sigma \cdot (B \pm E).$$

(12)

Denote the remainder in (9) after removing the renormalization dependent second-order term by

$$\mathcal{R} = \Pi_4 + \ln \det_5(1 - S A).$$

(13)

It will be shown in Sec.V that all the zero modes of the Dirac operator are confined to the positive chirality sector for the class of $O(2) \times O(3)$ symmetric background fields used to calculate $\det_{\text{ren}}$. Differentiating (9) with respect to $m^2$ allows one to isolate $H_\pm$. After some rearrangement of terms there follows

$$m^2 \frac{\partial \mathcal{R}}{\partial m^2} = \frac{1}{2} m^2 \text{Tr} \left[ (H_+ + m^2)^{-1} - (H_- + m^2)^{-1} \right]$$

$$+ m^2 \int_0^\infty dt e^{-tm^2} \int d^4k \left\{ \text{tr}(k|e^{-tH_-} - e^{-tP^2}|k) \right.$$

$$- \frac{1}{128\pi^6} |\hat{F}_{\mu\nu}(k)|^2 \int_0^1 dzz(1-z)e^{-k^2z(1-z)t} \left. \right\}.$$  

(14)

where the spin traces are now over $2 \times 2$ matrices. Equation (14) expanded as a power series defines the conditionally convergent fourth-order term in $\mathcal{R}$. This requires iterating the second term in (14) four times using the operator identity

$$e^{-t(P^2 + V)} - e^{-tP^2} = - \int_0^t ds e^{-(t-s)(P^2 + V)} V e^{-sP^2},$$

(15)
with \( V = -AP - PA + A^2 + \sigma \cdot (E - B) \). The result is

\[
m^2 \frac{\partial R}{\partial m^2} = \frac{1}{2} m^2 \text{Tr} \left[ (H_+ + m^2)^{-1} - (H_- + m^2)^{-1} \right] - \frac{m^2}{16\pi^2} \int \frac{d^4k}{(2\pi)^4} \frac{\hat{E}(k) \cdot \hat{B}(-k) + \hat{E}(-k) \cdot \hat{B}(k)}{z(1 - z)k^2 + m^2} + m^2 \frac{\partial}{\partial m^2} \Pi_4^\sigma - \frac{m^2}{4} \text{Tr} \left[ \Delta_- V \Delta V \Delta V \Delta V \Delta \right.
\]

\[
- (\Delta A^2 \Delta V \Delta V \Delta V \Delta + \text{all perms. of } A^2, V^2) + \Delta A^2 \Delta A^2 \Delta A^2 \Delta \]

\[
(16)
\]

The second term in (16) is the remainder after adding the second-order contribution from the second term in (14) to the last term. This remainder would be canceled by the first term in (16) were it expanded in a power series. It, as well as \( \Pi_4^\sigma \), were calculated from the regulated expansion of the second term in (14) using (15).

The quantity \( \Pi_4^\sigma \) is obtained by adding all the fourth-order terms in the expansion. It is the contribution of \( H_- \) to the photon-photon scattering graph. Its structure is

\[
\Pi_4^\sigma = \Pi_4^{\text{scalar}} + \Pi_4^{\sigma \cdot (B - E)},
\]

(17)

where \( \Pi_4^{\text{scalar}} \) is the contribution to \( \Pi_4^\sigma \) neglecting the \( \sigma \cdot (B - E) \) term in \( V \). That is, \( \Pi_4^{\text{scalar}} \) is the contribution to \( \Pi_4 \) in scalar QED4 multiplied by 2. The factor 2, and not 4, is due to the factor of \( \frac{1}{2} \) in the definition (9) of the spinor determinant. The remainder in (17) is the
contribution to $\Pi_{\tau}$ from the $\sigma \cdot (\mathbf{B} - \mathbf{E})$ term in $V$. These two terms are

$$\Pi_{\tau}^{scalar} = -\frac{1}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \int_0^1 dz_1 z_1 \int_0^{1-z_1} dz_2 \int_0^{1-z_1-z_2} dz_3 \times \left[ z_1 k^2 + z_2 p^2 + z_3 q^2 - (z_1 k + z_2 p + z_3 q)^2 + m^2 \right]^{-2} \times \left\{ \left[ \frac{1}{4}(1 - 2z_1)^2(1 - 2z_2)^2 F_{\mu\nu}(p - k) \hat{F}_{\mu\nu}(k) \hat{F}_{\alpha\beta}(-q) \hat{F}_{\alpha\beta}(q - p) + \frac{1}{4}(1 - 2z_1 - 2z_2)^2(1 - 2z_2 - 2z_3)^2 \hat{F}_{\mu\nu}(p - k) \hat{F}_{\mu\nu}(q - p) \hat{F}_{\alpha\beta}(k) \hat{F}_{\alpha\beta}(q - p) + \frac{1}{4}(1 - 2z_2)^2(1 - 2z_1 - 2z_2)^2 \hat{F}_{\mu\nu}(p - k) \hat{F}_{\mu\nu}(q - p) \hat{F}_{\alpha\beta}(k) \hat{F}_{\alpha\beta}(q - p) + \frac{1}{4}(1 - 2z_1)(1 - 2z_2)(1 - 2z_3)(1 - 2z_2)(1 - 2z_2 - 2z_3)(1 - 2z_1 - 2z_2) \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(q - p) \hat{F}_{\gamma\delta}(q - p) \hat{F}_{\delta\alpha}(k) + \frac{1}{4}(1 - 2z_1)(1 - 2z_2)(1 - 2z_3)(1 - 2z_2)(1 - 2z_3)(1 - 2z_1 - 2z_2) \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(q - p) \hat{F}_{\gamma\delta}(q - p) \hat{F}_{\delta\alpha}(k) + \text{three additional field strength terms} \right\}. \quad (18)$$

and

$$\Pi_{\tau}^{(\mathbf{B} - \mathbf{E})} = -\frac{1}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \int_0^1 dz_1 z_1 \int_0^{1-z_1} dz_2 \int_0^{1-z_1-z_2} dz_3 \times \left[ z_1 k^2 + z_2 p^2 + z_3 q^2 - (z_1 k + z_2 p + z_3 q)^2 + m^2 \right]^{-2} \times \left\{ \left[ \hat{F}_{\mu\nu}(p - k) \hat{F}_{\mu\nu}(q - p) \right] [(1 - 2z_1 - 2z_2 - 2z_3)^2 + (1 - 2z_2)^2] + \frac{1}{2} [(1 - 2z_2)^2 + (1 - 2z_1)^2] \hat{F}_{\mu\nu}(p - k) \hat{F}_{\mu\nu}(k) \right\} \times \left\{ \hat{F}_{\mu\nu}(q - p) \cdot \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\alpha\beta}(q - p) \right\} + \frac{1}{2} [(1 - 2z_2)(1 - 2z_2)(1 - 2z_3)(1 - 2z_2 - 2z_3) \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(q - p) \hat{F}_{\gamma\delta}(q - p) \hat{F}_{\delta\alpha}(k) + \frac{1}{2} [(1 - 2z_1)(1 - 2z_2)(1 - 2z_3)(1 - 2z_2 - 2z_3)(1 - 2z_1 - 2z_2) \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(q - p) \hat{F}_{\gamma\delta}(q - p) \hat{F}_{\delta\alpha}(k) + \text{three additional field strength terms} \right\}. \quad (19)$$

In (18) the additional terms are gauge invariant expressions such as

$$\hat{F}_{\mu\nu}(q - p) q_\alpha \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(k) q_\gamma,$$

and in (19) they are of a similar form, such as

$$\hat{F}_{\mu\nu}(q - p) q_\alpha \hat{F}_{\alpha\beta}(p - k) \hat{F}_{\beta\gamma}(k) q_\gamma.$$
The full $\Pi_4$ graph has been calculated by Karplus and Neuman [16], although the above results do not appear in their paper. Results (18) and (19) require extensive, but straightforward, calculation. As a check on $\Pi_4^{\text{scalar}}$ we can specialize to the case of constant $B$ and $E$. Using $F_{\alpha\beta}F_{\beta\gamma}F_{\gamma\delta}F_{\delta\alpha} = 2(B^2 + E^2)^2 - 4(E \cdot B)^2$, $\Pi_4^{\text{scalar}}$ reduces to

$$\Pi_4^{\text{scalar}} = -\frac{V}{720\pi^4 m^4} \left( \frac{7}{4} (B^2 + E^2)^2 - (B \cdot E)^2 \right),$$

(20)

where $V$ is a Euclidean volume cutoff. This is Weisskopf’s [17] and Schwinger’s [15] constant field result for scalar QED$_4$’s fourth-order effective Lagrangian continued to Euclidean space modulo a factor of $-2$. The factor 2 was discussed above. The minus sign arises from the difference in statistics: we are calculating a contribution to the scalar QED$_4$ effective action from the spinor effective action.

Continuing with our discussion of (16), $\Delta_-$ in the last trace is the exact negative chirality propagator $\langle x| (H_- + m^2)^{-1} |y \rangle$ and $\Delta$ is the scalar propagator $\langle x| (P^2 + m^2)^{-1} |y \rangle$. The regulating exponentials have been removed as the terms in the trace are fifth order and higher, and so the implicit loop integral is unambiguous. We leave the discussion of $\Delta_-$ to Secs. VI and VII.

The $A^2$ insertions clutter up the trace in the sense that in a perturbative expansion of the trace they can be neglected. This is because they form part of gauge invariant expressions whose structure is already determined by the $AP + PA$ terms in $V$. We do not see any justification for neglecting such terms in a nonperturbative treatment of the trace, but nevertheless this remark should be kept in mind.

In order to discuss the $m = 0$ limit of (16) we must be more specific about the background gauge field.

**III. BACKGROUND GAUGE FIELDS**

QED determinants in constant field backgrounds have volume divergences and so are not defined on non-compact manifolds. Instead one considers the associated effective Lagrangians, which do make sense. In the simplest case of the Euclidean QED$_2$ determinant there is just a constant magnetic field, and the volume divergence arises from the degeneracy of the Landau levels. In constant-field QED$_4$, by making two rotations (a Lorentz boost plus a rotation in Minkowski space) the operator $(P - A)^2$ can be transformed into the sum of
two two-dimensional harmonic oscillator Hamiltonians, leading to a degeneracy factor that grows as a four-volume \[18\]. The lesson is that constant fields have too much degeneracy to define the determinant on a non-compact manifold.

We have found that \(O(2) \times O(3)\) symmetric background fields allow a satisfactory definition of the QED\(_4\) determinant and that they are sufficiently tractable to permit substantial analytic analysis. Such fields were first explicitly considered in QED\(_4\) by Adler \[12, 19\]. In this paper these fields take the form \[18–20\].

\[
A_\mu(x) = M_{\mu\nu} x_\nu a(r^2),
\]

(21)

where \(M_{\mu\nu}\) is chosen to be antiself-dual and is given by

\[
M_{\mu\nu} = \begin{pmatrix}
    -1 & \quad & \quad & \\
    1 & \quad & \quad & \\
    -1 & \quad & \quad & \\
    1 & \quad & \quad & 
\end{pmatrix}.
\]

(22)

This field has an \(O(2) \times O(3)\) invariance, subgroups present in the reduction of \(O(4)\) to \(O(3) \times O(3)\). It is further assumed that \(a(r^2)\) is smooth, well-behaved at the origin, and satisfies

\[
a(r^2) = \frac{\nu}{r^2}, \quad r > R,
\]

(23)

where \(\nu\) is a dimensionless constant. Without loss of generality assume \(\nu > 0\).

The orbital angular momentum operators of the first and second \(O(3)\) subgroups of \(O(4)\) are

\[
L^{(1)}_k = \frac{1}{2} i \left( x_0 \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_0} - \epsilon_{klm} x_l \frac{\partial}{\partial x_m} \right),
\]

\[
L^{(2)}_k = \frac{1}{2} i \left( x_k \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_k} - \epsilon_{klm} x_l \frac{\partial}{\partial x_m} \right),
\]

(24)

which satisfy

\[
\left[ L^{(p)}_i, L^{(q)}_j \right] = \delta_{pq} \epsilon_{ijk} L^{(p)}_k, \quad p, q, = 1, 2.
\]

(25)

The spin angular momentum operators in the representation (10) are

\[
S^{(1)}_k = \frac{1}{2} \begin{pmatrix}
    \sigma_k & 0 \\
    0 & 0
\end{pmatrix}, \quad S^{(2)}_k = \frac{1}{2} \begin{pmatrix}
    0 & 0 \\
    0 & \sigma_k
\end{pmatrix}.
\]

(26)
The total angular momentum operator relative to the second subgroup,
\[ J_k^{(2)} = L_k^{(2)} + S_k^{(2)} , \] (27)
commutes with \( \mathcal{A} \):
\[ [ J_k^{(2)}, \mathcal{A} ] = 0, \quad k = 1, 2, 3, \] (28)
while \( \mathcal{A} \) is invariant only with respect to rotations about the third axis of the first subgroup:
\[ [ J_3^{(1)}, \mathcal{A} ] = 0. \] (29)

We adopt the conventions of [22] for the four-dimensional rotation matrices \( D_{l_1m_2}^l \):
\[
L^{(1)} \cdot L^{(1)} D_{m_1m_2}^l (x) = l(l+1)D_{m_1m_2}^l (x),
\]
\[ L^{(1)}_3 D_{m_1m_2}^l (x) = m_1 D_{m_1m_2}^l (x), \]
\[ L^{(2)}_3 D_{m_1m_2}^l (x) = m_2 D_{m_1m_2}^l (x). \] (30)

The \( D_{m_1m_2}^l \) are normalized so that
\[
\int d\Omega_4 D_{m_1m_2}^{l_1*} (x) D_{m_3m_4}^{l_2} (x) = \frac{2\pi^2}{2l_1 + 1} \delta_{l_1l_2} \delta_{m_1m_3} \delta_{m_2m_4} (r^2)^{2l_1}, \] (31)
where \( \Omega_4 \) is the surface element in four dimensions. Some properties of these matrices appear in Appendix A.

Following [22] we construct eigenstates of \( J^{(1)} \cdot J^{(1)}, J_3^{(1)} \) (eigenvalues \( j \pm \frac{1}{2}, M \)) and \( J^{(2)} \cdot J_3^{(2)} \) (eigenvalues \( j, m \)). In the positive chirality sector these are
\[
\varphi_{j,m}^{\pm \frac{1}{2},M} (x) = \begin{pmatrix}
\mp (j \pm M + \frac{1}{2})^{\frac{1}{2}} D_{M-\frac{1}{2},m}^j (x) \\
(j \mp M + \frac{1}{2})^{\frac{1}{2}} D_{M+\frac{1}{2},m}^j (x) \\
0 \\
0
\end{pmatrix}, \tag{32}
\]
and in the negative chirality sector they are
\[
\psi_{j,m}^{\pm \frac{1}{2},M} (x) = \begin{pmatrix}
0 \\
0 \\
-(j - m + 1)^{\frac{1}{2}} D_{M,m-\frac{1}{2}}^{j+\frac{1}{2}} (x) \\
(j + m + 1)^{\frac{1}{2}} D_{M,m+\frac{1}{2}}^{j+\frac{1}{2}} (x)
\end{pmatrix}, \tag{33}
\]
\[ \psi^{j-\frac{1}{2},M}_{j,m}(x) = \begin{pmatrix} 0 \\ 0 \\ (j+m)\frac{1}{2}D_{M,m-\frac{1}{2}}(x) \\ (j-m)\frac{1}{2}D_{M,m+\frac{1}{2}}(x) \end{pmatrix}. \] (34)

Due to (28) and (29), eigenstates of \( \mathcal{D} = \mathcal{P} - \mathcal{A} \) are of the form 

\[ \psi^{\pm}_{EjMm}(x) = F(r^2)\varphi^{j-\frac{1}{2},M}_{j,m}(x) + G(r^2)\varphi^{j+\frac{1}{2},M}_{j,m}(x), \] (35)

\[ \psi^{-}_{EjMm}(x) = f(r^2)\varphi^{j-\frac{1}{2},M}_{j,m}(x) + g(r^2)\varphi^{j+\frac{1}{2},M}_{j,m}(x), \] (36)

where the superscripts on \( \psi^{\pm}_{EjMm} \) denote chirality and \( E \) is the energy eigenvalue. In the following we will write \( \psi^{\pm}_{EjMm} \) as two-component spinors.

From \( \ast F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \) and (21) it follows that

\[ \ast F_{\mu\nu}F_{\mu\nu} = -16a^2 - 16r^2aa', \] (37)

and

\[ F_{\mu\nu}F_{\mu\nu} = 8r^4a'^2 - \ast F_{\mu\nu}F_{\mu\nu}, \] (38)

where the prime denotes differentiation with respect to \( r^2 \). From (23) and (37) the chiral anomaly is

\[ -\frac{1}{16\pi^2} \int d^4x \ast F_{\mu\nu}F_{\mu\nu} = \frac{\nu^2}{2}, \] (39)

provided \( \lim_{r \to 0} r^2a = 0 \). Note, as expected, that \( F_{\mu\nu} \) is not square-integrable. But this does not matter as far as the remainder \( R \) in (13) is concerned. Recall that it is only required that \( A_{\mu} \in \bigcap_{n>4} L^n(\mathbb{R}^4) \), which it does here. Furthermore, because we have chosen on-shell charge renormalization the \( 1/k^2 \) behavior of \( \hat{F}_{\mu\nu} \) for small \( k \) in the first term on the right-hand side of (9) is regulated by the vanishing logarithm as \( k \to 0 \). So everything in (9) is finite.

If one wishes to deal with a negative anomaly then instead of choosing \( M_{\mu\nu} \) in (21) antiself-dual, require it to be self-dual. For example, let

\[ M_{\mu\nu} \to N_{\mu\nu} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}. \] (40)
IV. ZERO MASS LIMIT OF $\mathcal{R}$: PRELIMINARIES

We will now discuss in a preliminary way the limit of (16) as $m \to 0$. Consider the first term. A working definition of the chiral anomaly for $\mathcal{P}$ on non-compact manifolds is [23].

$$\lim_{m \to 0} m^2 \text{Tr} \left[ (H_+ + m^2)^{-1} - (H_- + m^2)^{-1} \right] = -\frac{1}{16\pi^2} \int d^4x \, *F_{\mu\nu}F_{\mu\nu}. \quad (41)$$

Because the manifold is a non-compact Euclidean one the right-hand side of (41) need not be the difference between numbers $n_+ - n_-$ of positive and negative chirality $L^2$ zero modes. The remainder, if any, is related to the zero-energy phase shifts associated with $H_{\pm}$ [23]. More will be said about this at the end of Sec. V. If the remaining terms in (16) vanish in the $m = 0$ limit then (39) and (41) indicate that $\mathcal{R}$ in (13) behaves as

$$\mathcal{R} \sim \frac{\nu^2}{4} \ln m^2 + \text{less singular in } m^2. \quad (42)$$

Thus, $\mathcal{R}$ would become negative as $m \to 0$.

A necessary condition for the vanishing of the remaining terms is that there be no $L^2$ zero modes in the negative chirality sector. It will be shown in Sec. V that this is true for our choice of gauge fields. Otherwise, $\Delta_-$ in (16) would develop a simple pole at $m = 0$ and (42) would contain more terms varying as $\ln m^2$ for $m \to 0$. But this is not a sufficient condition for the remaining terms in (16) to vanish at $m = 0$.

One can see already from the second term in (16) some of the subtleties involved. If $B(x)$ and $E(x)$ fall off as $1/r^2$, as our fields do, without any particular symmetry constraint then their Fourier transforms will be such that $\hat{B}(k), \hat{E}(k)$ behave as $1/k^2$ as $k \to 0$. In this case the integral will have an infrared divergence even when $m \neq 0$. But this does not happen due to the $O(2) \times O(3)$ symmetry of the gauge fields.

To see this define

$$\hat{F}_{\mu\nu}^<(k) = \int_{|x| > R} d^4xe^{-ikx}F_{\mu\nu}(x), \quad (43)$$

with

$$F_{\mu\nu} \big|_{|x| > R} = \frac{2\nu}{r^2} \left( M_{\mu\nu} + \frac{x_\mu M_{\nu\alpha}x_\alpha - x_\nu M_{\mu\alpha}x_\alpha}{r^2} \right). \quad (44)$$

Then

$$\hat{F}_{\mu\nu}^>(k) = \frac{8\pi^2\nu}{k^2} \left[ M_{\mu\nu}J_2(kR) + \frac{M_{\nu\alpha}k_\alpha k_\mu - M_{\mu\alpha}k_\alpha k_\nu}{k^2}(J_0(kR) + 2J_2(kR)) \right], \quad (45)$$
\[ \hat{B}^>(k) \cdot \hat{E}^>(-k) = -\frac{(8\pi^2\nu)^2}{k^4}(J_0(kR)J_2(kR) + J_2^2(kR)). \] (46)

Thus, \( \hat{B}^>(k) \cdot \hat{E}^>(-k) \) behaves as \( R^2/k^2 \) instead of \( 1/k^4 \) as \( k \to 0 \). For large \( k \), \( \hat{F}^{\mu\nu}(k) \), calculated from an integral like (43) but with \( |x| < R \), behaves as \( \sin(kR - 3\pi/4)/k^{5/2} \) for any reasonable behavior of \( a(r^2) \) near \( r = 0 \), such as \( a \sim Cr^\beta \) with \( \beta > -\frac{1}{2} \) or \( -\frac{1}{3} \) as required in Sec. VIII. Therefore, the integral in (16) is absolutely convergent in the ultraviolet and its small-mass limit varies as \( (\ln m^2)^2 \), allowing us to conclude that the second term in (16) vanishes in the limit \( m = 0 \).

Now consider the third term in (16), \( m^2 \partial \Pi^{-}_{\sigma} / \partial m^2 \). Referring to (17), (18), (19) and (45), simple power counting of momenta suggests that the integrals defining \( \Pi^\text{scalar}_4 \) and \( \Pi^\text{(B-E)}_4 \) have a logarithmic mass singularity of the form \( (\ln m^2)^n \) with \( n \geq 1 \). If so, then the \( m = 0 \) limit of \( m^2 \partial \Pi^{-}_{\sigma} / \partial m^2 \) would be nonvanishing, thereby falsifying (42). It is encouraging that there is no immediate infrared divergence for \( m = 0 \) that has to be canceled by the symmetry of \( F_{\mu\nu} \), as in the second-order term of (16). The confluence of singularities in \( \hat{E} \) and \( \hat{B} \) is no longer present in fourth order. Nor are they present in higher orders due to the result cited in Sec. II that \( \det_5 \) is well-defined for any \( A_\mu \in \cap_{n>4} L^n(\mathbb{R}^4) \), which includes our fields.

The fact that power counting does not ensure finiteness in the \( m = 0 \) limit of \( \Pi^{-}_4 \) indicates that the symmetry properties of \( \hat{F}_{\mu\nu}(k) \) will be required to give a finite limit to the individual terms in (18) and (19). Because of this reliance on symmetry, theorems on mass singularities of Feynman amplitudes known to the author are inapplicable here. The analysis required to give a definitive answer one way or another is beyond the scope of this paper. All we are able to do here is to present evidence for a finite limit of \( \Pi^{-}_4 \) as \( m \to 0 \). We note that Adler’s stereographic mapping to the surface of a 5-dimensional unit hypersphere [12], [19] cannot help here due to the slow \( 1/r \) falloff of the vector potential.

Consider, for example, the fourth term in (18). Let \( p \to p + k, q \to -q \) so that the chain
of field strengths is put into the form
\[ \hat{F}_{\alpha\beta}(p)\hat{F}_{\gamma\delta}(-k - p - q)\hat{F}_{\delta\alpha}(k) = \frac{1}{4} \hat{F}_{\alpha\beta}(p)\hat{F}_{\alpha\beta}(-k - p - q)\hat{F}_{\mu\nu}(q)\hat{F}_{\mu\nu}(k) + \frac{1}{4} \hat{F}_{\alpha\beta}(p)\hat{F}_{\alpha\beta}(k)\hat{F}_{\mu\nu}(-k - p - q)\hat{F}_{\mu\nu}(q) - \left[ \mathbf{B}(p) \cdot \mathbf{E}(q) + \mathbf{B}(q) \cdot \mathbf{E}(p) \right] \times \left[ \mathbf{B}(-k - p - q) \cdot \mathbf{E}(k) + \mathbf{B}(k) \cdot \mathbf{E}(-k - p - q) \right]. \] (47)

The \(1/k^2\) behavior of \(\hat{F}_{\mu\nu}(k)\) arises from the \(J_0(kR)\) term in (45). Fixing on the most singular terms, the first term on the right-hand side of (47) contributes
\[ \hat{F}_{\mu\nu}(q)\hat{F}_{\mu\nu}(k) \xrightarrow{k,q \to 0} \frac{128\pi^4\nu^2}{k^4q^4} \left[ (q \cdot k)^2 - (q_0k_3 - q_1k_2 + q_2k_1 - q_3k_0)^2 \right]. \] (48)

To isolate the leading singularity when \(k, q \to 0\) we neglect the denominator in the fourth term in (18) when integrating over the angles defining \(q_\mu\). Using
\[ \int d\Omega_q (k \cdot q)^2 = \frac{\pi^2}{2} k^2 q^2, \]
\[ \int d\Omega_q (q_0k_3 - q_1k_2 + q_2k_1 - q_3k_0)^2 = \frac{\pi^2}{2} k^2 q^2, \] (49)
we see that the leading singularity for small \(k\) and \(q\) cancels. The first term on the right-hand side of (47) also contains the term \(F_{\alpha\beta}(p)F_{\alpha\beta}(-k - p - q)\). The case \(k, p, q \to 0\) with \(p \ll k, q\) reduces to the case just considered when the angles defining \(p_\mu\) are integrated over. The same conclusions follow for the small \(k\) and \(p\) behavior of the second term in (47) as well as the case \(k, p, q \to 0\) with \(q \ll p, k\).

Finally, consider the third term on the right-hand side of (47). Referring to the result (46), the singularity in \(\mathbf{B}(-k - p - q) \cdot \mathbf{E}(k)\) and \(\mathbf{B}(k) \cdot \mathbf{E}(-k - p - q)\) is \(R^2/k^2\) at \(p, q = 0\) and not \(1/k^4\).

The gauge invariant expressions on the right-hand side of (47) occur in all the terms in (18) and (19), and so the above cancellations occur there too. There are three additional field strength terms in the integrands of (18) and (19) and these must also be considered.

Given that the second and fourth-order terms in (16) vanish at \(m = 0\) then presumably so will all higher order terms \(m^2 \partial \Pi_6 / \partial m^2, \ldots\) generated by expanding \(\Delta_-\) in (16). Since the zero modes reside in the positive chirality propagator \(\Delta_+\) this expansion may have some justification. However, the scattering states extend down to zero energy, and these may
result in nonperturbative mass singularities induced by $\Delta_-$. This will be examined in Secs. VI and VII.

V. ZERO MODES

In the representation (10) $\mathcal{D}$ has the supersymmetric structure

$$
\mathcal{D} = \begin{pmatrix} 0 & D \\ -D^\dagger & 0 \end{pmatrix},
$$

and hence positive chirality zero modes are square-integrable solutions of

$$
D^\dagger \psi^+ = 0,
$$

where all subscripts on $\psi^+$ have been dropped. From (32) and (35)

$$
\psi^+(x) = \begin{pmatrix} \left[(j - M + \frac{1}{2})^\frac{1}{2} F - (j + M + \frac{1}{2})^\frac{1}{2} G\right] D^j_{M-\frac{1}{2},m}(x) \\
\left[(j + M + \frac{1}{2})^\frac{1}{2} F + (j - M + \frac{1}{2})^\frac{1}{2} G\right] D^j_{M+\frac{1}{2},m}(x) \end{pmatrix},
$$

By (31), $\psi^+ \in L^2$ provided

$$
\int_0^\infty dr r^{4j+3}(F^2 + G^2) < \infty.
$$

Inserting (52) in (51) results in

$$
G' + \frac{a}{2j+1} \left(\sqrt{(j+\frac{1}{2})^2 - M^2} F - MG\right) = 0,
$$

$$
r^2 F' + (2j+1)F + \frac{ar^2}{2j+1} \left(MF + \sqrt{(j+\frac{1}{2})^2 - M^2} G\right) = 0.
$$

Here $j = 0, \frac{1}{2}, \ldots$ and $-j - \frac{1}{2} \leq M \leq j + \frac{1}{2}$. Equations (54) and (55) appear in [21, 22] in a different notation, although the authors are considering an entirely different problem. There are three cases to consider.

Case 1: $M = -j - \frac{1}{2}$. Then

$$
\psi^+ = \sqrt{2j+1} D^j_{-j,m}(x) G(r^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$
with

$$\frac{dG}{dr^2} = -\frac{a}{2} G,$$

and so

$$G(r^2) = G(r_0^2)e^{-\frac{1}{2}\int_{r_0^2}^{r^2} ds a(s)}.$$  

(58)

Since \(a = \nu/r^2\) for \(r > R\), \(\psi^+ \in L^2\) for \(j = 0, \frac{1}{2}, \ldots, j_{\text{max}}\), where \(j_{\text{max}}\) is the largest value of \(j\) for which \(\nu > 2j + 2\) is satisfied.

Case 2: \(M = j + \frac{1}{2}\). Then

$$\psi^+ = -\sqrt{2j + 1} D_{j,m}^j(x) G(r^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\frac{dG}{dr^2} = \frac{a}{2} G.$$  

(60)

Based on case 1 it is clear that \(G \notin L^2\) for any \(\nu > 0\).

Case 3: \(|M| < j + \frac{1}{2}\). We claim that there are no \(L^2\) zero modes in this case. To show this let \(z = r^2\),

$$\Delta = (2j + 1)F,$$

$$\Gamma = -2MF - 2\sqrt{(j + \frac{1}{2})^2 - M^2 G},$$

in (54) and (55). Then these become

$$z \frac{d\Delta}{dz} + (2j + 1) \Delta = \frac{1}{2} z a \Gamma,$$

$$z \frac{d\Gamma}{dz} = (2M + \frac{1}{2} za) \Delta.$$  

(64)

Assume \(a\) has a power series expansion about \(z = 0\) and let

$$a = \sum_{n=0} a_n z^n,$$

$$\Delta = \sum_{n=0} b_n z^n,$$

$$\Gamma = \sum_{n=0} c_n z^n.$$  

(65)
Then $\triangle$ and $\Gamma$ have the expansions

$$
\triangle = c_0 \left( \frac{a_0}{4(j+1)} z + \left( \frac{a_1}{2(2j+3)} + \frac{Ma_0^2}{4(j+1)(2j+3)} \right) z^2 + O(z^3) \right),
$$

$$
\Gamma = c_0 \left( 1 + \frac{Ma_0}{2(j+1)} z + \left( \frac{Ma_1}{2(2j+3)} + \frac{M^2a_0^2}{4(j+1)(2j+3)} + \frac{a_0^2}{16(j+1)} \right) z^2 + O(z^3) \right).
$$

It will now be shown that the solution $\triangle$, $\Gamma$ that is finite at $r = 0$ does not converge fast enough to make $\psi^+ \in L^2$ for $|M| < j + \frac{1}{2}$ with $\psi^+$ given by (52). Let $t = \ln z$ and

$$
\Gamma = \gamma e^{-(j + \frac{1}{2})t},
$$

$$
\triangle = \delta e^{-(j + \frac{1}{2})t},
$$

$$
a = \alpha e^{-t}.
$$

Then (63) and (64) become

$$
\frac{d\delta}{dt} + (j + \frac{1}{2})\delta = \frac{1}{2} \alpha \gamma,
$$

$$
\frac{d\gamma}{dt} - (j + \frac{1}{2})\gamma = (2M + \frac{1}{2}\alpha)\delta.
$$

These are the same equations appearing in Eq. (5.24) of [21]. Following their analysis, multiply (69) by $\delta$, (70) by $\gamma$ and subtract:

$$
\frac{1}{2} \frac{d}{dt}(\gamma^2 - \delta^2) = (j + \frac{1}{2})(\gamma^2 + \delta^2) + 2M\gamma\delta.
$$

Since $\gamma = r^{2j+1}\Gamma$, $\delta = r^{2j+1}\triangle$ and $\Gamma$, $\triangle$ are finite at $r = 0$, $\gamma$ and $\delta$ vanish at $r = 0$. From (53), if $\psi^+ \in L^2$ then $F$, $G \sim r^{-2j-2-\epsilon}$, $\epsilon > 0$ and hence $\gamma, \delta \sim r^{-1-\epsilon}$ for $r \to \infty$. Integrating (71) therefore gives

$$
\int_0^{\infty} \frac{dr}{r} \left[(j + \frac{1}{2})(\gamma^2 + \delta^2) + 2M\gamma\delta\right] = 0.
$$

Since $|M| < j + \frac{1}{2}$, (72) is impossible for real $\epsilon$. Hence the assumption that $\psi^+ \in L^2$ for $|M| < j + \frac{1}{2}$ is false.

We now turn to the negative chirality sector. From (33), (34) and (36),

$$
\psi^-(x) = \begin{pmatrix} (j + m)^{-\frac{1}{2}} D_{M,m+\frac{1}{2}}^{-\frac{1}{2}}(x) \left( f(r^2) + \begin{pmatrix} -(j - m + 1)^{-\frac{1}{2}} D_{M,m-\frac{1}{2}}^{j+\frac{1}{2}}(x) \\ (j + m + 1)^{-\frac{1}{2}} D_{M,m+\frac{1}{2}}^{j+\frac{1}{2}}(x) \end{pmatrix} g(r^2) \right) \end{pmatrix}.
$$
and $\psi^- \in L^2$ provided
\[ \int_0^\infty dr r^{4j+1} \left[ f^2 + (r^2 g)^2 \right] < \infty. \] (74)

From (50) negative chirality zero modes are $L^2$ solutions of
\[ D\psi^- = 0. \] (75)

Substitution of (73) in (75) results in
\[ 2\sqrt{j - M + \frac{1}{2} f'} - \sqrt{j + M + \frac{1}{2}} (2r^2 g' + 4(j + 1)g) + \sqrt{j - M + \frac{1}{2} a f} - \sqrt{j + M + \frac{1}{2} r^2 a g} = 0, \] (76)
\[ 2\sqrt{j + M + \frac{1}{2} f'} + \sqrt{j - M + \frac{1}{2}} (2r^2 g' + 4(j + 1)g) - \sqrt{j + M + \frac{1}{2} a f} - \sqrt{j - M + \frac{1}{2} r^2 a g} = 0. \] (77)

There are again three cases.

Case 1: $M = -j - \frac{1}{2}$ . From (73),
\[ \psi^-(x) = \begin{pmatrix} -(j - m + 1)\frac{1}{2} D^{j + \frac{1}{2}}_{-j - \frac{1}{2} - \frac{1}{2}, m - \frac{1}{2}}(x) \\ (j + m + 1)\frac{1}{2} D^{j + \frac{1}{2}}_{-j - \frac{1}{2}, m + \frac{1}{2}}(x) \end{pmatrix} g(r^2). \] (78)
From (77),
\[ 2r^2 g' + 4(j + 1)g - r^2 a g = 0, \] (79)
whose solution by inspection is
\[ g(r^2) = g(r_0^2) \left( \frac{r}{r_0} \right)^{-4j-4} e^{\frac{1}{2} f_{r_0^2}^{r_2} ds a(s)}. \] (80)

By (74) $\psi^- \in L^2$ only if
\[ \int_0^\infty dr r^{4j+5} g^2 < \infty, \] (81)
and therefore $g$ is too singular at $r = 0$ to be in $L^2$.

Case 2: $M = j + \frac{1}{2}$ . From (73),
\[ \psi^- = \begin{pmatrix} -(j - m + 1)\frac{1}{2} D^{j + \frac{1}{2}}_{j + \frac{1}{2}, m - \frac{1}{2}}(x) \\ (j + m + 1)\frac{1}{2} D^{j + \frac{1}{2}}_{j + \frac{1}{2}, m + \frac{1}{2}}(x) \end{pmatrix} g(r^2). \] (82)
and from (76),
\[ 2r^2 g' + 4(j + 1)g + r^2 a g = 0. \] (83)
As in case 1, \( g \) is too singular at \( r = 0 \) to be in \( L^2 \).

Case 3: \(|M| < j + \frac{1}{2}\). We will demonstrate that \( \psi \not\in L^2 \). Let \( z = r^2 \),

\[
\Gamma = \sqrt{(j + M + \frac{1}{2})(f + r^2 g) + (j - M + \frac{1}{2})(r^2 g - f)},
\]

\[
\triangle = \sqrt{(j + M + \frac{1}{2})(f - r^2 g) + (j - M + \frac{1}{2})(f + r^2 g)}.
\]

Then (76), (77) become

\[
z \frac{d\triangle}{dz} + \left( j + \frac{1}{2} - \sqrt{(j + \frac{1}{2})^2 - M^2} \right) \triangle = (M + \frac{1}{2} z a) \Gamma,
\]

\[
z \frac{d\Gamma}{dz} + \left( j + \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - M^2} \right) \Gamma = (M + \frac{1}{2} z a) \triangle.
\]

Making the expansions (65) gives for \( M \neq 0 \),

\[
\triangle = b_0 \left( \frac{2M^2 + j + \frac{1}{2} - \sqrt{(j + \frac{1}{2})^2 - M^2}}{4M(j + 1)} \right) a_0 z + O(z^2),
\]

\[
\Gamma = b_0 \left( \frac{j + \frac{1}{2} - \sqrt{(j + \frac{1}{2})^2 - M^2}}{2(j + 1)} + \frac{j + 1 - \sqrt{(j + \frac{1}{2})^2 - M^2}}{2(j + 1)} a_0 z + O(z^2) \right).
\]

For \( M = 0 \),

\[
\triangle = b_0 \left( 1 + \frac{a_0^2}{16(j + 1)} z^2 + O(z^3) \right),
\]

\[
\Gamma = b_0 \left( \frac{a_0}{4(j + 1)} z + \frac{a_1}{2(2j + 3)} z^2 + O(z^3) \right).
\]

Having established that there is a solution of (86) and (87) that is finite at \( r = 0 \) we now show that this solution is not square-integrable. There is also a solution that is too singular at the origin to satisfy (74); we therefore ignore it here. Let

\[
\triangle = z^{-j - \frac{1}{2}} \delta,
\]

\[
\Gamma = z^{-j - \frac{1}{2}} \gamma,
\]

\[
\lambda = \sqrt{(j + \frac{1}{2})^2 - M^2}.
\]
Then (86), (87) reduce to
\[ zd\delta - \lambda \delta = (M + \frac{1}{2}az)\gamma, \] (93)
\[ zd\gamma + \lambda \gamma = (M + \frac{1}{2}az)\delta. \] (94)

Multiply (93) by \( \delta \), (94) by \( \gamma \) and subtract:
\[ \frac{d}{dr}(\delta^2 - \gamma^2) = 4\lambda r(\gamma^2 + \delta^2), \] (95)
where
\[ \delta^2 - \gamma^2 = 4\left[\lambda(f^2 - r^4g^2) - 2Mr^2fg\right]r^{4j+2}, \] (96)
and
\[ \delta^2 + \gamma^2 = 2(2j + 1)\left[f^2 + (r^2g)^2\right]r^{4j+2}. \] (97)

From (74), if \( \psi^- \in L^2 \) then \( f, r^2g \sim r^{-2j-1-\epsilon} \), \( \epsilon > 0 \), in which case \( \lim_{r\to\infty}(\delta^2 - \gamma^2) = 0. \) Because \( \Delta, \Gamma \) are finite at \( r = 0 \), \( \gamma, \delta = O(r^{2j+1}) \) as \( r \to 0 \). Hence, integration of (95) using (97) gives
\[ \sqrt{(j + \frac{1}{2})^2 - M^2} \int_0^\infty dr r^{4j+1}\left[f^2 + (r^2g)^2\right] = 0. \] (98)

But this is impossible for \( |M| < j + \frac{1}{2} \). Therefore, the assumption that \( \psi^- \in L^2 \) is false.

Summarizing, it has been shown that all \( L^2 \) zero modes of \( \mathcal{D} \) have positive chirality and that these only occur when \( M = -j - \frac{1}{2} \) and for values of \( j \) satisfying \( \nu > 2j + 2. \)

These results raise an interesting problem. The main result of [23] is
\[ \frac{\nu^2}{2} = n_+ - n_- + \frac{1}{\pi} \sum_l \mu(l)\left[\delta_l^+(0) - \delta_l^-(0)\right], \] (99)
where \( n_\pm \) are the number of positive and negative chirality \( L^2 \) zero modes, \( \delta_l^\pm(0) \) are the zero-energy scattering phase shifts for \( H_\pm \) in (111), \( \mu(l) \) is a weight factor, and \( l \) are the quantum numbers required to specify the phase shifts discussed in Sec. VI. We have just shown that \( n_- = 0. \) Suppose \( \nu = 3. \) Due to the condition for a \( L^2 \) zero mode derived above only \( j = 0, M = -\frac{1}{2}, m = 0 \) are allowed. So \( n_+ = 1, \) and it must follow that
\[ \frac{9}{2} = 1 + \frac{1}{\pi} \sum_l \mu(l)\left[\delta_l^+(0) - \delta_l^-(0)\right]. \] (100)

Verification of this and (99) here would take us too far afield.
VI. SCATTERING STATES

Having established that there are no negative chirality zero modes it cannot be concluded that the \( m = 0 \) limit of the last term in (16) is zero. Equation (31) when combined with (39) and (99) caution against this. They demonstrate that a particular zero-mass limit receives contributions from the scattering states of \( H_- \) in (11). There seems to be no alternative to actually calculating the low-energy scattering states of \( H_- \) before deciding whether the \( m = 0 \) limit of the last term in (16) is zero.

Because \( \mathcal{D} \) is anti-Hermitian we look for eigenstates of the form

\[
\mathcal{D} \psi = i k \psi.
\]  

(101)

Decomposing \( \psi \) into its positive and negative chirality components and using (50) gives

\[
D D^\dagger \psi^+ = k^2 \psi^+,
\]

(102)

\[
D^\dagger D \psi^- = k^2 \psi^-.
\]

(103)

To get the scattering states \( \psi^- \) it is easier to calculate \( \psi^+ \) and then use \( D^\dagger \psi^+ = -ik \psi^- \). In the representation (10) the Zeeman term, \( \frac{1}{2} \sigma F \), is diagonal in the positive chirality sector, and so \( D D^\dagger = H_+ \) has the form

\[
D D^\dagger = \begin{pmatrix} H_{\frac{1}{2}} & 0 \\ 0 & H_{-\frac{1}{2}} \end{pmatrix},
\]

(104)

where the subscripts on \( H \) denote the eigenvalues of \( S_3^{(1)} \) in (26). In (52) let

\[
\sqrt{\frac{2j + 1}{2\pi^2}} r^{-2j - \frac{3}{2}} \rho_{\pm \frac{1}{2}} = (j \mp M + \frac{1}{2}) \frac{3}{4} F \mp (j \pm M + \frac{1}{2}) \frac{3}{4} G,
\]

(105)

and decompose \( \psi^+ \) into its upper and lower components:

\[
\psi_{\frac{1}{2}}^+ = \sqrt{\frac{2j + 1}{2\pi^2}} \begin{pmatrix} \rho_{\frac{1}{2}}(r) \\ 0 \end{pmatrix},
\]

(106)

\[
\psi_{-\frac{1}{2}}^+ = \sqrt{\frac{2j + 1}{2\pi^2}} \begin{pmatrix} 0 \\ \rho_{-\frac{1}{2}}(r) \end{pmatrix}.
\]

where \( \hat{x} \cdot \hat{x} = 1 \). Substituting Eqs. (106) in turn in (102) gives

\[
- \frac{d^2}{dr^2} + \frac{(2j + 1)^2 - \frac{1}{4}}{r^2} + (4M \pm 2)a + r^2 a^2 \pm r \frac{da}{dr} \rho_{\pm \frac{1}{2}} = k^2 \rho_{\pm \frac{1}{2}}.
\]

(107)
Equation (107) has to be supplemented by appropriate boundary conditions. For \( r > R \) according to (23) \( \rho = \nu/r^2 \). Let

\[
\rho_{\pm \frac{1}{2}} = r^{\frac{1}{2}} f_\pm.
\]

Then (107) becomes for \( r > R \)

\[
f''_\pm + \frac{1}{r} f'_\pm + \left( k^2 - \frac{(2j + 1)^2 + 4M\nu + \nu^2}{r^2} \right) f_\pm = 0,
\]

whose general solution is a superposition of Hankel functions

\[
f_\pm = \alpha_\pm H^{(1)}_\lambda(kr) + \beta_\pm H^{(2)}_\lambda(kr),
\]

with

\[
\lambda = \left[ (2j + 1)^2 + 4M\nu + \nu^2 \right]^{\frac{1}{2}}.
\]

Choosing \( \alpha_\pm, \beta_\pm \) so that

\[
\rho_{EjM,\pm \frac{1}{2}}(r) \sim \sqrt{\frac{1}{\pi k}} \cos \left( kr - \frac{\pi}{2} (2j + 1) + \delta_{jM,\pm \frac{1}{2}}^+(k) - \frac{\pi}{4} \right),
\]

gives for \( r > R \)

\[
\rho_{E\alpha}(r) = \sqrt{\frac{r}{8}} \left( e^{i \left( \frac{\pi}{2} - \frac{\pi}{4} (2j + 1) + \delta_{m,\pm \frac{1}{2}}^+(k) \right)} H^{(1)}_\lambda(kr)
\]

\[
+ e^{-i \left( \frac{\pi}{2} - \frac{\pi}{4} (2j + 1) + \delta_{m,\pm \frac{1}{2}}^+(k) \right)} H^{(2)}_\lambda(kr) \right),
\]

where \( E = k^2 \) and \( \alpha \) denotes \( j, M, \pm \frac{1}{2} \). The superscript on \( \delta^+ \) is a reminder that these are positive chirality phase shifts. The solutions (113) are to be joined to the solutions of (107) for \( r < R \). This will determine the phase shifts. Equation (113) fixes the normalization so that

\[
\int_0^\infty dr \rho_{E\alpha}(r) \rho_{E'\alpha}(r) = \delta(E - E').
\]

Then \( \psi^\pm_{\pm \frac{1}{2}} \) in (106) have the overall normalization

\[
(\psi^+_{E\beta}, \psi^+_{E'\beta'}) = \delta_{\beta \beta'} \delta(E - E'),
\]

where \( \beta \) represents \( j, M, m, \pm \frac{1}{2} \).

The calculation of the low-energy phase shifts is outlined in Appendix B. Define the energy-dependent part of \( \delta^+ \) by

\[
\Delta^+_{\alpha}(k) = \frac{\pi \lambda}{2} - \frac{\pi}{2} (2j + 1) + \delta^+_\alpha(k), \quad \text{mod } \pi,
\]

where \( \Delta^+_{\alpha}(k) \) is the dispersion relation for the phase shifts.

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and denote the expansion in powers of $k$ of the logarithmic derivative of the interior radial wave function at $r = R$ by

$$
\left( \frac{r \partial_r \rho_{Ea}}{\rho_{Ea}} \right)_{R} = \gamma_\alpha - (kR)^2 \Gamma_\alpha + O(kR)^4.
$$

(117)

The coefficients $\gamma_\alpha$, $\Gamma_\alpha$ are defined in Appendix B. Then for $|M| \neq j + \frac{1}{2}$

$$
\tan \Delta^+_\alpha = -\frac{\pi}{\lambda \Gamma^2(\lambda)} \gamma_\alpha - \lambda - \frac{1}{2} \left( \frac{kR}{2} \right)^{2\lambda} \left( 1 + O \left( (kR)^2, (kR)^{2\lambda} \right) \right),
$$

(118)

with $\lambda > 1$ for $\nu > 0$. There are several special cases to consider.

Case 1: $M = j + \frac{1}{2}$ and hence $\lambda = 2j + 1 + \nu$. From (106) the only phase shift in this case is $\delta^+_{j+j+\frac{1}{2}}(k)$ and

$$
\tan \Delta^+_j = -\frac{\pi}{\Gamma^2(1 + \lambda)} \left( \frac{1}{\lambda + 1} - 2\Gamma_{j+\frac{1}{2}} \right) \left( \frac{kR}{2} \right)^{2\lambda+2} \left( 1 + O \left( (kR)^2, (kR)^{2\lambda-2} \right) \right).
$$

(119)

Case 2: $M = -j - \frac{1}{2}$ and hence $\lambda = |2j + 1 - \nu|$. From (106) the only phase shift in this case is $\delta^+_{-j-\frac{1}{2}-\frac{1}{2}}(k)$.

Case 2.1: $2j < \nu < 2j + 1$ with $0 < \lambda < 1$,

$$
\tan \Delta^+_{j,j-\frac{1}{2},-\frac{1}{2}} = -\frac{\pi}{\lambda \Gamma^2(\lambda)} \left( \frac{1}{\lambda+1} - 2\Gamma_{j-j-\frac{1}{2}-\frac{1}{2}} \right) \left( \frac{kR}{2} \right)^{2\lambda+2} \left( 1 + O \left( (kR)^2, (kR)^{2\lambda} \right) \right).
$$

(120)

Case 2.2: $2j + 1 < \nu < 2j + 2$ with $0 < \lambda < 1$,

$$
\tan \Delta^+_{j,j-\frac{1}{2},-\frac{1}{2}} = \pi \lambda + O \left( (kR)^{2-2\lambda} \right).
$$

(121)

When $\lambda = \frac{1}{2} + \epsilon$, $|\epsilon| << 1$, (121) becomes

$$
\tan \Delta^+_{j,j-\frac{1}{2},-\frac{1}{2}} = -\frac{1 + 2\epsilon + O(kR)^2}{\pi \epsilon + 2(\Gamma_{j,j-\frac{1}{2},-\frac{1}{2}} - 1) \left( \frac{kR}{2} \right)^{1-2\epsilon} + O(\epsilon^2, \epsilon(kR)^{1-2\epsilon}, (kR)^{3-2\epsilon})}.
$$

(122)

Case 2.3: $0 < \nu < 2j$ with $\lambda > 1$,

$$
\tan \Delta^+_{j,j-\frac{1}{2},-\frac{1}{2}} = -\frac{\pi}{\lambda \Gamma^2(\lambda)} \left( \frac{1}{\lambda+1} - 2\Gamma_{j,j-\frac{1}{2},-\frac{1}{2}} \right) \left( \frac{kR}{2} \right)^{2\lambda+2} \left( 1 + O(kR)^2 \right).
$$

(123)

Case 2.4: $\nu > 2j + 2$ with $\lambda > 1$,

$$
\tan \Delta^+_{j,j-\frac{1}{2},-\frac{1}{2}} = \frac{\pi}{\lambda \Gamma^2(\lambda)} \left( \frac{2j + 1 - \nu}{2\Gamma_{j,j-\frac{1}{2},-\frac{1}{2}} + \frac{1}{\lambda-1}} \right) \left( \frac{kR}{2} \right)^{2\lambda-2} \left( 1 + O \left( (kR)^2, (kR)^{2\lambda-2} \right) \right).
$$

(124)
Case 2.5: $\nu = 2j + 1$ with $\lambda = 0$, 

$$
\tan \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^{+} = -\pi \left( 1 - 2\Gamma_{j,-j-\frac{1}{2},-\frac{1}{2}} \right) \left( \frac{kR}{2} \right)^2 \left( 1 + O \left[ (kR)^2 \ln(kR) \right] \right).
$$

(125)

Case 2.6: $\nu = 2j + 2$ with $\lambda = 1$, 

$$
\tan \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^{+} = \frac{\pi}{2} \left( 1 + O \left[ (kR)^2 \right] \right) \ln \left( \frac{kR}{2} \right) + \gamma_E - \Gamma_{j,-j-\frac{1}{2},-\frac{1}{2}} + O \left[ (kR)^2 \ln(kR) \right],
$$

(126)

where $\gamma_E$ is Euler's constant $0.577\ldots$

Case 2.7: $\nu = 2j$ with $\lambda = 1$, 

$$
\tan \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^{+} = -\pi \left( \frac{1}{2} - 2\Gamma_{j,-j-\frac{1}{2},-\frac{1}{2}} \right) \left( \frac{kR}{2} \right)^4 \left( 1 + O \left[ (kR)^2 \right] \right).
$$

(127)

Although it is not required for the analysis here there is a compact relation between the interior wave functions and the phase shifts that ought to be mentioned, namely for $|M| \neq j + \frac{1}{2}$

$$
2\pi \int_0^R \frac{ds}{s} \frac{d}{ds} (s^2 \alpha) \rho_{jM,\frac{1}{2}}(s) \rho_{jM,\frac{1}{2}}(s) = \sin \left( \delta_{jM,\frac{1}{2}}^{+}(k) - \delta_{jM,\frac{1}{2}}^{+}(k) \right).
$$

(128)

This is easily obtained by going back to (107) and noting that

$$
\left( \rho_{\frac{1}{2},\frac{1}{2}}(r) - \rho_{\frac{1}{2},\frac{1}{2}}(r) \right)' = \frac{2}{r} (r^2 a)' \rho_{\frac{1}{2},\frac{1}{2}}(r).
$$

(129)

For $r > R$ the right-hand side of (129) vanishes. The constant $\rho_{\frac{1}{2},\frac{1}{2}}(r) - \rho_{\frac{1}{2},\frac{1}{2}}(r)$ in the region $r > R$ can be calculated using (112). Then integrating (129) from 0 to $R$ gives (128). It holds for all energies.

We now proceed to get the negative chirality scattering states, in particular $f$ and $g$ in (36) by calculating $D^1 \psi^+_f = -ik\psi^-_f$. This results in two orthogonal states since

$$
\left( \psi_{\frac{3}{2}}^-, \psi_{\frac{3}{2}}^- \right) = \frac{1}{k^2} \left( \psi_{\frac{3}{2}}^-, D^1 D^1 \psi_{\frac{3}{2}}^- \right) = \frac{1}{k^2} \left( D\psi_{\frac{3}{2}}^-, D\psi_{\frac{3}{2}}^- \right) = \left( \psi_{\frac{3}{2}}^+, \psi_{\frac{3}{2}}^+ \right) = 0.
$$

(130)

The result is

$$
\psi_{EjMm,\frac{1}{2}}^-(x) = \frac{1}{\sqrt{2\pi^2 k r^2}} \sqrt{\frac{j - M + \frac{1}{2}}{2j + 1}} \left( \frac{j + m - \frac{1}{2}}{2j + 1} \right) D^{j-\frac{1}{2}}_{M,m,\frac{1}{2}}(\hat{x}) \left( \frac{d}{dr} - ar + \frac{2j + \frac{3}{2}}{r} \right) \rho_{EjM,\frac{1}{2}}(r)
$$

$$
- \frac{1}{\sqrt{2\pi^2 k r^2}} \sqrt{\frac{j + M + \frac{1}{2}}{2j + 1}} \left( \frac{j - m - \frac{1}{2}}{2j + 1} \right) D^{j+\frac{1}{2}}_{M,m,\frac{1}{2}}(\hat{x}) \left( \frac{d}{dr} - ar - \frac{2j + \frac{3}{2}}{r} \right) \rho_{EjM,\frac{1}{2}}(r),
$$

(131)
These states are normalized so that
\[
\langle \psi_E^{-\beta}, \psi_{E'}^{-\beta'} \rangle = \delta_{\beta \beta'} \delta(E - E'),
\]
where \( \beta \) represents \( j, M, m, \pm \frac{1}{2} \). Because there are no \( L^2 \) zero modes in the negative chirality sector we expect that the scattering states (131) and (132) form a complete set:
\[
\sum_{j=0}^{\infty} \sum_{M=-j}^{j} \int_{0}^{\infty} dE \left[ \psi_{EjMm}^{-\frac{1}{2}}(x) \psi_{E'jM'm}^{-\frac{1}{2}}(x') + \psi_{EjMm}^{-\frac{1}{2}}(x) \psi_{E'jM'm}^{-\frac{1}{2}}(x') \right] = \delta(x - x') I.
\]
(134)

VII. \( \Delta_- \) AT LOW ENERGY

The exact negative chirality propagator is
\[
\Delta_-(x, x') = \sum_{\alpha} \int_{0}^{\infty} dk^2 \frac{\psi_{E\alpha}^{-}(x) \psi_{E\alpha}^{+}(x')}{k^2 + m^2},
\]
with \( \psi_{E\alpha}^{-} \) given by (131), (132) and \( \alpha = jMm, \pm \frac{1}{2} \). Now suppose \( \Delta_-(x, x') \) is divided into its low and high energy parts by replacing the integral in (135) by \( \int_{0}^{\Lambda^2} + \int_{\Lambda^2}^{\infty} \), with \( \Lambda R << 1 \). Then our objective is to show that the low energy propagator has only minor deviations from the free propagator. This turns out to be the case except when \( \nu = 2j + 2 \) which results in a benign logarithmic mass singularity. The high energy propagator poses no obstacle to the \( m = 0 \) limit in (16) and is well-defined due to the assumed regularity of \( A_\mu \) at the origin.

In order to proceed we replace the differential equation (107) with the integral equation (107) with the integral equation
\[
\rho_{\pm}(r) = A_{\pm} \sqrt{\frac{r}{2}} J_{2j+1}(kr) + \frac{\pi}{2} \sqrt{r} \int_{0}^{r} dr' \sqrt{r'}
\times [J_{2j+1}(kr')Y_{2j+1}(kr) - J_{2j+1}(kr)Y_{2j+1}(kr')] V_{\pm}(r') \rho_{\pm}(r'),
\]
(136)
where $\rho_\pm$ represents $\rho_{EjM,\pm\frac{1}{2}}$, $A_\pm$ are constants to be determined and

$$V_\pm = (4M \pm 2)a + r^2 a^2 \pm r \frac{da}{dr}.$$  \hspace{1cm} (137)

By differentiating (136) it can be verified that (107) results. To fix $A_\pm$ require that $\rho_\pm$ join smoothly to the outgoing wave solution (113) at $r = R$ with $\delta_\alpha^+$ replaced by its energy dependent part defined in (116). Then

$$\rho_\pm(R) = \sqrt{\frac{R}{2}} \left( J_\alpha(kR) \cos \triangle_\alpha^+(k) - Y_\alpha(kR) \sin \triangle_\alpha^+(k) \right),$$  \hspace{1cm} (138)

together with (136) at $r = R$ determine $A_\pm$.

An upper bound on $\rho_\pm(r)$ for $0 \leq r \leq R$ will now be obtained. Starting with (2.60), (12.134) and (12.136a) in [24] deduce that for $z > 0$ and for fixed values of $j$

$$|J_{2j+1}(z)| \leq \frac{C_J(z^{2j+1}}{(2j+1)! (1+z)^{2j+\frac{\delta}{2}}},$$  \hspace{1cm} (139)

$$|H^{(1)}_{2j+1}(z)| \leq C_H \sqrt{\frac{2}{\pi z}} \left( \frac{1+z}{z} \right)^{2j+\frac{\delta}{2}},$$  \hspace{1cm} (140)

where the constants $C_J$, $C_H$ depend on $j$. From these results it follows that for $z \geq z' > 0$

$$\sqrt{zz'}|J_{2j+1}(z')Y_{2j+1}(z) - J_{2j+1}(z)Y_{2j+1}(z')| \leq C z \left( \frac{z}{z'} \right)^{2j+1},$$  \hspace{1cm} (141)

with $C$ of order one. Now iterate (136) and let

$$\rho_\pm(r) = \sum_0 \rho_\pm^{(n)}(r),$$

$$\rho_\pm^{(0)}(r) = A_\pm \sqrt{\frac{r}{2}} J_{2j+1}(kr),$$

$$|\rho_\pm^{(n)}(r)| = r \psi_\pm^{(n)}(r).$$  \hspace{1cm} (142)

From (136) and (141) the $n^{th}$ iterate satisfies

$$\psi_\pm^{(n)}(r) \leq \frac{\pi C}{2} \int_0^r \left| V_\pm(r') \right| \left( \frac{r}{r'} \right)^{2j+1} r' \psi_\pm^{(n-1)}(r').$$  \hspace{1cm} (143)

Since $0 < r < R$ and we assume $kR << 1$, then (142) gives

$$|\rho_\pm^{(0)}(r)| = r \psi_\pm^{(0)}(r) \leq \sqrt{\frac{r}{2}} \left( \frac{kr}{2} \right)^{2j+1} \frac{|A_\pm|}{(2j+1)!},$$  \hspace{1cm} (144)

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Thus (143) and (144) give
\[ \psi^{(n)}_{\pm} \leq \left( \frac{\pi C}{2} \right)^n \left( \frac{k r}{2} \right)^{2j+1} \frac{|A_{\pm}|}{(2j+1)!} n^n \int_0^r dr_{n} \ldots \int_0^{r_2} dr_{1} |V_{\pm}(r_1) \ldots V_{\pm}(r_n)| \]
\[ = \frac{|A_{\pm}|}{\sqrt{2r(2j+1)!}} \left( \frac{k r}{2} \right)^{2j+1} \left[ \frac{\pi C r}{2} \int_0^r ds |V_{\pm}(s)| \right]^n /n!. \] (145)

By (142)
\[ |\rho_{\pm}(r)| \leq \sqrt{r \frac{|A_{\pm}|}{2(2j+1)!}} \left( \frac{k r}{2} \right)^{2j+1} \exp \left( \frac{\pi C r}{2} \int_0^r ds |V_{\pm}(s)| \right), \] (146)
valid for \( 0 \leq r \leq R, kR << 1. \)

It remains to estimate the constants \( A_{\pm}. \) Suppose \(|M| \neq j + \frac{1}{2}. \) From (136), (138) and (118) with \( kR << 1 \) obtain
\[ A_{\pm} = \left( \frac{k R}{2} \right)^{\lambda-2j-1} \left[ 1 - \frac{\Gamma(2j+2)}{\Gamma(\lambda+1)} \frac{\gamma_{\pm} - \lambda - \frac{1}{2}}{\gamma_{\pm} + \lambda - \frac{1}{2}} + O(kR)^2 \right] \]
\[ - \frac{\Gamma(2j+1)}{\sqrt{2}} \int_0^R dr \sqrt{r} \left( 1 - \left( \frac{r}{R} \right)^{4j+2} + O(kR^2) \right) V_{\pm}(r)(\frac{1}{2}k r)^{-2j-1} \rho_{\pm}(r). \] (147)

Because \( k^2 \) is an analytic perturbation of \( \rho_{\pm} \) in (107) make the expansion
\[ \rho_{\pm}(r) = \left( \frac{k R}{2} \right) \left( \rho_{0\pm}(r) + \rho_{2\pm}(r)k^2 + O(k^4) \right), \] (148)
for \( 0 \leq r \leq R. \) Then to leading order in \( k \)
\[ A_{\pm} = \left( \frac{k R}{2} \right)^{\lambda-2j-1} \left[ 1 - \frac{\Gamma(2j+2)}{\Gamma(\lambda+1)} \frac{\gamma_{\pm} - \lambda - \frac{1}{2}}{\gamma_{\pm} + \lambda - \frac{1}{2}} \right] \]
\[ - \frac{1}{\sqrt{2}} \frac{1}{2j+1} \int_0^R dr \sqrt{r} \left( 1 - \left( \frac{r}{R} \right)^{4j+2} \right) V_{\pm}(r) \left( \frac{R}{r} \right)^{2j+1} \rho_{0\pm}(r). \] (149)

Substitution of (149) and (148) into (136) gives an integral equation for \( \rho_{0\pm}(r) \) whose solution we do not require here. The main conclusion is that for \( kR << 1, |M| \neq j + \frac{1}{2} \)
\[ A_{\pm} = \left( \frac{k R}{2} \right)^{\lambda-2j-1} N_{\pm}(j, M, \nu), \] (150)
where \( N_{\pm}(j, M, \nu) \) is \( k \)-independent. The centrifugal barrier term in (107) for large \( j \) (certainly \( j >> \nu \)) will cause \( \rho_{\pm} \) to approach the noninteracting solution \( \sqrt{\frac{\pi}{2} I_{2j+1}(rk)}. \)
Hence, for large $j$, $\lambda \to 2j + 1, \gamma \to 2j + \frac{3}{2}, N_\pm(j, M, \nu) \to 1$ and so $A_\pm \to 1$. Equations (146) and (150) give the explicit upper bound

$$|\rho_\pm(r)| \leq \frac{1}{(2j + 1)!} \sqrt{\frac{r}{R}} \frac{(r \Lambda_j + 1)}{(2j + 2)^\lambda} |N_\pm(j, M, \nu)| e^{\frac{1}{2} \pi C r_0 R \Delta V_\pm(s)},$$

(151)

for $0 \leq r \leq R, kR << 1, |M| \neq j + \frac{1}{2}$. The $k-$dependence of this bound is consistent with (128).

When $M = j + \frac{1}{2}$, case 1, Sec. VI, only $\rho_+$ is relevant and there is no change in its overall $k$ dependence. When $M = -j - \frac{1}{2}$ only $\rho_-$ is relevant, and the largest modification of (151) occurs in case 2.2 when $2j + 1 < \nu < 2j + 2$ with $0 < \lambda < 1$. Repeating the above analysis gives the same result as (151) except that the factor $(kR/2)^\lambda$ is replaced with $(kR/2)^{-\lambda}$ and $N_-$ is replaced with a new constant $\tilde{N}_-$. Thus, for case 2.2 (148) is replaced with

$$\rho_-(r) = \frac{(kR/2)^{-\lambda}}{(2j + 1)!} \left( \rho_{0-} + \rho_{2-}(r)k^2 + O(k^4) \right).$$

(152)

The remaining cases when $M = -j - \frac{1}{2}$ result in less singular $k-$factors than $(kR)^{-\lambda}$.

Now it is evident that the overall $k-$dependence of $\rho_\pm(r)$ is not changed by differentiating it with respect to $r$. Therefore, the leading small $k-$dependence of the radial wave functions in (131) and (132) remains $(kR)^\lambda, \lambda > 1$ for $M \neq -j - \frac{1}{2}$. Because of the factor $k^{-1}$ multiplying them the negative chirality wave functions $\psi^-_{E_j M, \pm, \frac{1}{2}}$ fall off as $(kR)^{\lambda - 1}$ as $k \to 0$ for $M \neq -j - \frac{1}{2}, 0 \leq r \leq R$. This statement was verified by deriving integral equations for the radial wave functions in (131) and (132) starting from (136) and proceeding as above in the derivation of the bounds on $\rho_\pm$.

The case $M = -j - \frac{1}{2}$ has to be handled with care because when $2j + 1 < \nu < 2j + 2$ we have noted that $\rho_-$ behaves as $(kR)^{-\lambda}$ as $k \to 0$ and hence one might naively conclude that $\psi^-_{E_j M, \pm, \frac{1}{2}}$ behaves as $(kR)^{-\lambda - 1}$ with $0 < \lambda < 1$ when $k \to 0$. This would induce a non-integrable singularity in the chiral propagator (135). This does not happen for the following reason. It may be explicitly checked that $\rho_-$ in (152) is given by

$$\rho_{0-}(r) = Cr^{2j + \frac{3}{2}} e^{-\int_{r_0}^r ds \sigma(s)};$$

(153)

that is, it is a regular solution of (107) when $M = -j - \frac{1}{2}$ and $k = 0$. Here $C$ and $r_0$ are arbitrary constants. This together with (152) show that the relevant radial wave function in (132) satisfies

$$\frac{1}{k} \left( \frac{d}{dr} + ar - \frac{2j + \frac{3}{2}}{r} \right) \rho_-(r) = O(kR)^{1-\lambda}.$$
Hence, $\psi_{EjMm, -\frac{1}{2}}$ continues to vanish as $k \to 0$ for $M = -j - \frac{1}{2}$ and $0 \leq r \leq R$. Another potential non-integrable singularity from $\rho_+$ in the second term in (131) is averted by the vanishing of the Clebsch-Gordon coefficient at $M = -j - \frac{1}{2}$. In view of the foregoing it is clear that the detailed analysis here is necessary.

It is now required to examine the low energy behavior of the radial wave functions in (131) and (132) when $r > R$. We choose to deal with the most troublesome cases in Sec.VI first, namely those arising when $M = -j - \frac{1}{2}$. We need only consider $\rho_-$ in this case. For cases 2.1 and 2.3 when $\lambda = 2j + 1 - \nu$ get from (23) and (113)

$$\frac{1}{k} \left( \frac{d}{dr} + ar - \frac{2j + \frac{3}{2}}{r} \right) \rho_-(r) = -\sqrt{\frac{r}{2}} \left( J_{2j+2-\nu}(kr) \cos \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^+ - Y_{2j+2-\nu}(kr) \sin \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^+ \right).$$

(155)

For cases 2.2 and 2.4 when $\lambda = \nu - 2j - 1$ then

$$\frac{1}{k} \left( \frac{d}{dr} + ar - \frac{2j + \frac{3}{2}}{r} \right) \rho_-(r) = \sqrt{\frac{r}{2}} \left( J_{\nu-2j-2}(kr) \cos \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^+ - Y_{\nu-2j-2}(kr) \sin \Delta_{j,-j-\frac{1}{2},-\frac{1}{2}}^+ \right).$$

(156)

Cases 2.5 and 2.7 are obtained by setting $\nu = 2j + 1$ and $2j$, respectively, in (155); case 2.6 is obtained from (156) by setting $\nu = 2j + 2$. Using (120)-(127) and $Y_\rho(z) \sim -\Gamma(\rho)(z/2)^{-\rho}/\pi$ for $z \to 0$ obtain the following results with $\alpha$ denoting $E, j, -j - \frac{1}{2}, m, -\frac{1}{2}$

- case 2.1: $\psi_\alpha^- = O(kR)^{\lambda+1}, 0 < \lambda < 1$,
- case 2.2: $\psi_\alpha^- = O(kR)^{1-\lambda}, 0 < \lambda < 1$,
- case 2.3: $\psi_\alpha^- = O(kR)^{\lambda+1}, \lambda > 1$,
- case 2.4: $\psi_\alpha^- = O(kR)^{\lambda-1}, \lambda > 1$,
- case 2.5: $\psi_\alpha^- = O(kR)$,
- case 2.6: $\psi_\alpha^- = O(1)$,
- case 2.7: $\psi_\alpha^- = O(kR)^2$.

(157)

For case 1, $M = j + \frac{1}{2}$, only $\psi_{Ej,j+\frac{1}{2},m,\frac{1}{2}}^-$ is relevant and

$$\frac{1}{k} \left( \frac{d}{dr} - ar - \frac{2j + \frac{3}{2}}{r} \right) \rho_+ = -\sqrt{\frac{r}{2}} \left( J_{2j+2+\nu}(kr) \cos \Delta_{j,j+\frac{1}{2},\frac{1}{2}}^+ - Y_{2j+2+\nu}(kr) \sin \Delta_{j,j+\frac{1}{2},\frac{1}{2}}^+ \right).$$

(158)
Thus $\psi_{Ej,j+\frac{1}{2},m,\frac{1}{2}} = O(kR)^{\lambda+1}$, $\lambda > 1$. Finally, when $|M| \neq j + \frac{1}{2}$ and therefore $\lambda > 1$, $\psi_{Ej,j+\frac{1}{2},m,\frac{1}{2}} = O(kR)^{\lambda-1}$ based on (113), (118) and the form of the radial wave functions appearing in (131) and (132). All of these cases are for $r > R$.

Now return to the last term in (16) and the interacting propagator (135). As noted earlier the centrifugal barrier term in (107) together with the regularity assumptions made on $a(r)$ will cause the large $j >> \nu$ contributions to $\Delta_-(x,x')$ to approach those of the noninteracting propagator. Therefore we need only consider a finite range of $j$ in the search for a possible mass singularity in $\Delta_-$ that would result in a non-vanishing remainder at $m = 0$.

For $r >> R$ the radial wave functions in (131) and (132) are seen from (112) to behave as

$$\frac{1}{k} \left( \frac{d}{dr} \pm ra + \frac{2j + \frac{1}{2}}{r} \right) \rho_\pm = -\sqrt{\frac{1}{\pi k}} \left[ \sin \left( kr - \frac{1}{2} \pi (2j + 1) + \delta^{+}_{jM,\pm\frac{1}{2}} - \frac{1}{4} \pi \right) ight. + O \left( \frac{\cos(kr)}{kr} \right)$$

(159)

with $k >> 1/r$, and similarly for the other group of wave functions. Therefore the leading large-distance behavior of these wave functions is the same as in the noninteracting case except for phase shifts.

We have shown that the low energy wave functions $\psi_{EjMm,\pm\frac{1}{2}} = O(kR)^{\lambda-1}$, or less, for $kR << 1$ in the region between $r = 0$ and $r > R$. Since $\lambda^2 = (2j + 1)^2 + 4M\nu + \nu^2$, then for $\nu > 0$ and $M \geq 0$, $\lambda > 2j + 1$ and $\psi_{EjMm,\pm\frac{1}{2}} = O(kR)^{2j+\epsilon}$, $\epsilon > 0$, which cannot lead to mass singularities in the last term in (16) that would result in a non-vanishing limit at $m = 0$. Hence we restrict our discussion to the case $M < 0$, in particular, the extreme case $M = -j - \frac{1}{2}$.

In (157) the largest deviation from the noninteracting case is case 2.6 when $\nu = 2j + 2$. Focus on this mode in $\Delta_-$. This mode first opens up when $\nu = 2$, the threshold value of $\nu$ for the formation of the first square-integrable zero mode in the positive chirality sector according to the discussion following (58). From (135), the second term in (132), (156) and (126) one obtains for the worst case $r, r' > R$

$$\Delta_{-}^{j=\frac{3}{2}\nu-1}(x,x') = \frac{M(\hat{x}, \hat{x}')}{rr'} \int_{0}^{\Lambda^2} \frac{dk^2}{k^2 + m^2} \left[ J_0(kr)J_0(kr') - \frac{\pi}{2} Y_0(kr)J_0(kr') + Y_0(kr')J_0(kr) \right] \ln \left( \frac{kR}{2} \right) + \frac{\pi^2}{4} Y_0(kr)Y_0(kr') \ln^2 \left( \frac{kR}{2} \right) + R_\Lambda$$

(160)
with $\Lambda R \ll 1$. The $k$--independent matrix $M$ is obtained from the second term in (132) in the calculation of $\psi_{E_0}^{-1}(x)\psi_{E_0}^{-1}(x')$, and $R_\Lambda$ is the contribution to $\Delta_-^l$ from the region $k > \Lambda$.

The most singular term in $m$ in (160) occurs in the first integral, written as

$$\int_0^\infty dk \frac{k J_0(kr) J_0(kr')}{k^2 + m^2} - \int_\Lambda^\infty dk \frac{k J_0(kr) J_0(kr')}{k^2 + m^2} = I_0(m r_<) K_0(m r_>) - \int_\Lambda^\infty (\cdot),$$

where $r_<(r_>)$ denotes the lesser (larger) of $r, r'$. The last integral in (161) can be put into the remainder $R_\Lambda$ in (160). Then for $m \to 0$ the most singular behavior in $m$ of $\Delta_-(x,x')$ occurs in the mode $j = \frac{1}{2} \nu - 1$ which has only a logarithmic mass singularity when $r, r' > R$

$$\Delta_-^{j=\frac{1}{2}\nu-1}(x,x') = -2 \frac{M(\hat{x}, \hat{x'})}{r r'} \ln(m r_>) + \text{less singular in } m. \quad (162)$$

In summary, a mode-by-mode analysis of the exact propagator in (135) uncovers only minor deviations from the free propagator in the low energy domain. If $\Pi_4^{-}$ is finite at $m = 0$ so that $\lim_{m=0} m^2 \partial \Pi_4^{-}/\partial m^2 = 0$ and if the role of the symmetry of $F_{\mu\nu}$ at large distances in reaching this conclusion is well-understood then it should be possible to generalize this fourth-order result to $m^2 \partial \Pi_6^{-}/\partial m^2$, etc., obtained by expanding $\Delta_-$ in (16) in a power series. We have shown in this section that its expansion is justified, considering that no nonperturbative singularities are induced in $\Delta_-$ by the scattering states that would cause the $m = 0$ limit of the last term in (16) to be nonvanishing. At this stage we do not see any other feasible way of demonstrating the vanishing of the last term in (16) at $m = 0$ since the available evidence relies on explicit gauge invariance and the long-distance symmetry of $F_{\mu\nu}$.

VIII. LARGE-MASS LIMIT OF $\mathcal{R}$

The leading term in the asymptotic expansion of $\mathcal{R}$ in (13) for large $m$ can be calculated from the effective Lagrangian density for QED$_4$ in a constant field background [15, 17, 25]. This is possible provided $F_{\mu\nu}$ is assumed to be smooth enough so that a meaningful derivative expansion of $\mathcal{R}$ can be carried out. Just how smooth will be made more precise below.

The photon-photon scattering graph in $\mathcal{R}$ has been thoroughly studied by Karplus and Neuman [16]. Using their results or the comprehensive review of the Heisenberg-Euler
In order for the remainder term in the asymptotic expansion in (163) to be finite it is necessary that $F_{\mu\nu}$ be twice differentiable. From (21) the most singular term in $F_{\mu\nu}$ contains terms like $x_{\nu} M_{\mu\alpha} x_{\alpha} a'$ and hence the most singular term in $\partial^2 F_{\mu\nu}$ is of the form $r^2 x_{\nu} M_{\mu\alpha} x_{\alpha} a''$. Thus, the finiteness of $\int F^2 F \partial^2 F$ requires

$$\left| \int_0^R dr r^7 \left( \frac{da}{dr} \right)^3 \frac{d^3 a}{dr^3} \right| < \infty,$$

and so $a(r^2)$ must be at least three times differentiable. For ease of analysis we assumed in Sec. V that $a(r^2)$ was regular at the origin, but this is not necessary. Condition (167) only
FIG. 1: Sketch of $a(r^2)$ versus radial distance for a class of gauge fields satisfying conditions (166) and (167).

requires $a \sim C r^\beta$ with $\beta > -\frac{1}{2}$. Of course requiring $A_\mu \in \bigcap_{n>4} L^n(\mathbb{R}^4)$ rules out $\beta < 0$. Any branches in $a(r^2)$ away from $r = 0$ of the form

$$a(r^2) \sim C(r^2 - r_0^2)^\alpha,$$

must have $\alpha > 5/4$ according to (167).

Now it may happen that a given $a(r^2)$ does not satisfy condition (166). This could mean that either there are no mass zeros in the remainder defined by (13) or that there are an even number of such zeros. This cannot be decided here. In our search for definite information we go back to (13) and deal only with $\ln \det_5$, treating the photon-photon graph as a subtraction like the second-order graph. If (42) is true then the $\ln m^2$ singularity is from $\ln \det_5$ alone. Then if the leading term in $\ln \det_5$’s asymptotic expansion in powers of $1/m$ is positive it certainly has at least one mass zero in the interval $0 < m < \infty$.

The leading term in the expansion of $\ln \det_5$ in powers of $1/m$ is the sixth-order graph given by $[15, 17, 25, 26]$

$$\ln \det_5 = \frac{1}{40320\pi^2 m^8} \int d^4 x \left[ 13( F_{\mu\nu} F_{\mu\nu})^2 - 8( F_{\mu\nu} F_{\mu\nu})^2 \right] F_{\alpha\beta} F_{\alpha\beta}$$

$$+ O \left( \frac{1}{m^{10}} \int d^4 x F^2 F_{\mu\nu} \partial^2 F_{\mu\nu}, \frac{1}{m^{10}} \int d^4 x F_{\mu_1\mu_2} \partial^2 F_{\mu_2\mu_3} \ldots F_{\mu_6\mu_1} \right),$$

and hence the positivity condition is

$$\int d^4 x \left[ 13( F_{\mu\nu} F_{\mu\nu})^2 - 8( F_{\mu\nu} F_{\mu\nu})^2 \right] F_{\alpha\beta} F_{\alpha\beta} > 0.$$
From (37) and (23) this becomes

$$\int_{r<R} d^4x \left[ 13\left( {}^*F_{\mu\nu}F_{\mu\nu}\right)^2 - 8\left( F_{\mu\nu}F_{\mu\nu}\right)^2 \right] F_{\alpha\beta}F_{\alpha\beta} > \frac{1024\pi^2\nu^6}{R^8}. \quad (171)$$

Further use of (37) and (23) results in the final positivity condition

$$\int_{r=0}^{R^2} dr^2 \left[ 2r^{14}a' a^6 + 12r^{12}a a' a^5 + 23r^{10}a^2 a' a^4 + 12r^8a^3 a' a^3 - 19r^6a^4 a' a^2 \right] < \frac{9\nu^6}{2R^8}. \quad (172)$$

The most singular terms in the remainder of the asymptotic expansion in (169) will arise from those containing $\partial^2 F$. Following the above discussion these will be finite provided

$$\left| \int_0^R drr^9 \left( \frac{da}{dr}\right)^5 \frac{d^3a}{dr^3} \right| < \infty, \quad (173)$$

This requires $a \sim C r^\beta$ with $\beta > -1/3$, at least, and any branch points in $a$ of the form (168) must have $\alpha > 7/6$.

A necessary condition for positivity can be easily derived from Hölder’s inequality, namely

$$\int d^4x |fg| \leq \left( \int d^4x |f|^p \right)^{\frac{1}{p}} \left( \int d^4x |g|^q \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q \geq 1. \quad (174)$$

Then with summation over indices understood, $f = F^2$, $g = ( {}^*FF)^2$, $p = 3$, $q = \frac{3}{2}$,

$$\int d^4x \left( {}^*FF\right)^2 F^2 \leq \left( \int d^4x (F^2)^3 \right)^{\frac{1}{3}} \left( \int d^4x |{}^*FF|^3 \right)^{\frac{2}{3}}, \quad (175)$$

and so

$$\int d^4x \left[ 13\left( {}^*FF\right)^2 F^2 - 8(F^2)^3 \right] \leq \left( \int d^4x (F^2)^3 \right)^{\frac{1}{3}} \left[ 13 \left( \int d^4x |{}^*FF|^3 \right)^{\frac{2}{3}} - 8 \left( \int d^4x (F^2)^3 \right)^{\frac{2}{3}} \right]. \quad (176)$$

Thus, it is necessary that

$$\int d^4x \left| {}^*F_{\mu\nu}F_{\mu\nu} \right|^3 > \left( \frac{8}{13} \right)^{\frac{2}{3}} \int d^4x (F_{\mu\nu}F_{\mu\nu})^3, \quad (177)$$

for (170) to be satisfied. It may be seen by inspection of (172) that one class of fields satisfying it are those with $a(0) \sim N\nu/R^2$, $N \geq 2$ and more or less monotonically decaying
to $\nu/R^2$ at $r = R$ as sketched in Fig. 2. Such fields will not satisfy the positivity condition (166).

To summarize, when (42), (166) and (167) are satisfied the remainder $R$ in (13) has at least one zero as $m$ varies over the interval $0 < m < \infty$. When (42), (172) and (173) are satisfied, $\ln \det_5$ has such a zero. In this case the entire function in (6) somehow manages to reduce to unity at the mass zero(s).

IX. CONCLUSION

By choosing $O(2) \times O(3)$ symmetric background gauge fields we were able to make some provisional nonperturbative statements about the behavior of the Euclidean fermionic determinant $\det_{\text{ren}}$ of QED$_4$ as a function of the fermionic mass. This determinant has the form

$$\ln \det_{\text{ren}} = \Pi_2 + \Pi_4 + \ln \det_5.$$  

The second-order term contains a charge renormalization subtraction. The remaining terms are denoted by $R$ in (13). It was assumed that for $r > R$ the radial profile function $a(r^2)$ in (21) takes the form $\nu/r^2$ for $r > R$, together with some mild regularity assumptions for $a(r^2)$.  

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for $r < R$. With these assumptions $\text{det}_{\text{ren}}$ is free of all cutoffs, including the second-order term if on-shell charge renormalization is used. Then we showed that if the mass singularity of $\mathcal{R}$ as $m \rightarrow 0$ is fully determined by the chiral anomaly, then $\mathcal{R}$ has at least one zero as $m$ varies in the interval $0 < m < \infty$, provided conditions (166) and (167) are satisfied. If not, then provided (172) and (173) are satisfied, $\ln \text{det}_5$ has at least one such zero at which the entire function in (6) becomes unity for any fixed coupling $\epsilon$. Then $\ln \text{det}_{\text{ren}}$ is dominated by $\Pi_4$ for $|\epsilon| >> 1$, which is consistent with $\text{det}_{\text{ren}}$ being an entire function of order four as discussed in Sec.II. If there is a mass zero such that $\mathcal{R}$ vanishes then $\ln \text{det}_{\text{ren}} = \Pi_2$ at this zero. If the number of mass zeros in $\mathcal{R}$ or $\ln \text{det}_5$ is even then they will not show up in the analysis here.

This raises an interesting possibility. If $\epsilon^2 << 1$ then $m$ does not have to be very large to make a meaningful $1/m$ asymptotic expansion. So, presumably, there are one or more “small” mass zeros in the weak coupling domain.

In plain language the result is this: set $\frac{e^2}{\hbar c} = \frac{1}{137} \ldots$. Select a gauge field that satisfies (166), (167) or (172), (173). Adjust $m$ until a mass zero appears. If $m$ is the physical fermion mass then it probably does not coincide with a mass zero. But if, for the selected gauge field, $m$ is near a zero then we would expect the remainder $\mathcal{R}$ or $\ln \text{det}_5$ to be anomalously small compared to the sum of the first few graphs in their expansion. By continuity there should be a class of gauge fields for which the physical coupling and mass coincide exactly with a mass zero.

In establishing these results we also demonstrated a vanishing theorem when the field strength tensor is not (anti-)self-dual, namely, that all of the square-integrable zero modes of the Dirac operator are of one chirality. This is a generalization of the vanishing theorem of Brown, Carlitz and Lee [27]. It would be useful to have a general vanishing theorem and to understand the physical principles underlying it.

In Sec. VIII it was assumed that the expansion of $\mathcal{R}$ in powers of $1/m$ is truly an asymptotic one so that the remainder after the series is truncated is of the order of the first neglected term. A proof is needed, but for the present it is an assumption physicists accept provided the background gauge field is smooth enough.

Most of this paper deals with the question of whether it is indeed true that the leading mass singularity of $\mathcal{R}$ in (13) is determined by the chiral anomaly. We have presented evidence that it is. It is true for the case of constant $B$ and $E$ [18], but this is a formal
result as the determinant has to be made finite by a volume cutoff. And it is also true for the QCD$_4$ determinant in the presence of an instanton background [28]. It is evident that the analytic, nonperturbative analysis of four-dimensional fermionic determinants is still at an early stage and may yet yield some surprises.
Here we list some properties of the four-dimensional rotation matrix. Several of the properties listed can be found in the Appendix of [22] from which our conventions and notations are taken.

Let \( \xi = x_0 + ix_3, \eta = x_2 + ix_1. \) Then explicitly

\[
D_{m_1m_2}^l(x) = [(l - m_1)!(l - m_2)!(l + m_1)!(l + m_2)!]^\frac{1}{4} \sum_{n_1 \ldots n_4} \frac{\xi^{n_1} \eta^{n_2} (-\bar{\eta})^{n_3} (\bar{\xi})^{n_4}}{n_1!n_2!n_3!n_4!}, \tag{A1}
\]

where \( n_i = 0, 1, \ldots \) and satisfy

\[
\begin{align*}
n_3 + n_4 &= l + m_1, \\
n_3 + n_1 &= l + m_2, \\
n_1 + n_2 + n_3 + n_4 &= 2l. \tag{A2}
\end{align*}
\]

The \( D_{m_1m_2}^l \) are normalized according to (31) where we set \( r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2. \) They satisfy

\[
D_{m_1m_2}^{ls} = (-1)^{m_1+m_2} D_{-m_1,-m_2}^l. \tag{A3}
\]

A useful relation is

\[
\sum_{m_2=-l}^l D_{m_1m_2}^{ls} (\hat{x}) D_{m_1m_2}^l (\hat{x}) = 1. \tag{A4}
\]

The \( D_{m_1m_2}^l \) satisfy an addition theorem

\[
\sum_{m_1,m_2=-l}^l D_{m_1m_2}^l (\hat{x}) D_{m_1m_2}^{ls} (\hat{y}) = C_{2l}^1 (\hat{x} \cdot \hat{y}), \tag{A5}
\]

where the Gegenbauer polynomial is

\[
C_{2l}^1 (\cos \varphi) = \frac{\sin(2l + 1) \varphi}{\sin \varphi}. \tag{A6}
\]

They also satisfy a completeness relation

\[
\frac{1}{2\pi^2} \sum_{l=0}^{\infty} (2l + 1) \sum_{m_1m_2=-l}^l D_{m_1m_2}^l (\hat{x}) D_{m_1m_2}^{ls} (\hat{y}) = \delta (\Omega_{\hat{x}} - \Omega_{\hat{y}}). \tag{A7}
\]
The raising and lowering operators of the $O(3)$ subgroups give

\[
\begin{align*}
L_+^{(1)} D_{m_1 m_2}^l &= -\sqrt{l(l+1) - m_1(m_1+1)} D_{m_1+1, m_2}^l, \\
L_-^{(1)} D_{m_1 m_2}^l &= -\sqrt{l(l+1) - m_1(m_1-1)} D_{m_1-1, m_2}^l, \\
L_+^{(2)} D_{m_1 m_2}^l &= \sqrt{l(l+1) - m_2(m_2+1)} D_{m_1, m_2+1}^l, \\
L_-^{(2)} D_{m_1 m_2}^l &= \sqrt{l(l+1) - m_2(m_2-1)} D_{m_1, m_2-1}^l.
\end{align*}
\] (A8)

Other relations we found useful are

\[
\begin{align*}
\sqrt{j + M + \frac{1}{2} \xi} D_{M+\frac{1}{2}, m}^j(x) + \sqrt{j - M + \frac{1}{2} \eta} D_{M-\frac{1}{2}, m}^j(x) &= \sqrt{j - m r^2} D_{M, m+\frac{1}{2}}^{j-\frac{1}{2}}(x), \\
\sqrt{j - M + \frac{1}{2} \xi} D_{M-\frac{1}{2}, m}^j(x) - \sqrt{j + M + \frac{1}{2} \eta} D_{M+\frac{1}{2}, m}^j(x) &= \sqrt{j + m r^2} D_{M, m-\frac{1}{2}}^{j+\frac{1}{2}}(x),
\end{align*}
\] (A9)

as well as

\[
\begin{align*}
\frac{\partial}{\partial \xi} D_{M-\frac{1}{2}, m}^j(x) &= (j - M + \frac{1}{2})^\frac{1}{2} (j + m)^\frac{1}{2} D_{M, m}^{j-\frac{1}{2}}(x), \\
\frac{\partial}{\partial \eta} D_{M-\frac{1}{2}, m}^j(x) &= (j - M + \frac{1}{2})^\frac{1}{2} (j - m)^\frac{1}{2} D_{M, m}^{j+\frac{1}{2}}(x), \\
\frac{\partial}{\partial \xi} D_{M+\frac{1}{2}, m}^j(x) &= (j + M + \frac{1}{2})^\frac{1}{2} (j - m)^\frac{1}{2} D_{M, m}^{j-\frac{1}{2}}(x), \\
\frac{\partial}{\partial \eta} D_{M+\frac{1}{2}, m}^j(x) &= -(j + M + \frac{1}{2})^\frac{1}{2} (j + m)^\frac{1}{2} D_{M, m}^{j+\frac{1}{2}}(x).
\end{align*}
\] (A10)
APPENDIX B

In order to calculate the low-energy phase shifts we follow the procedure of approximating the interior wave function by a small $k^2$ expansion about its zero-energy solution. Thus, (107) has the form

$$\left(\frac{d^2}{dr^2} + f_\pm(r)\right)\rho_{\pm \frac{1}{2}} = -k^2 \rho_{\pm \frac{1}{2}}. \quad (B1)$$

At a zero energy

$$\left(\frac{d^2}{dr^2} + f_\pm\right)\rho_{0,\pm \frac{1}{2}} = 0. \quad (B2)$$

From (B1) and (B2) get, after integrating,

$$\rho_{0,\pm \frac{1}{2}}(r) \sim \frac{1}{r} \rho_{\pm \frac{1}{2}}(r) - \rho_{\pm \frac{1}{2}}(r) \rho_{0,\pm \frac{1}{2}}(r) = -k^2 \int_0^r ds \rho_{\pm \frac{1}{2}}(r) \rho_{0,\pm \frac{1}{2}}(r), \quad (B3)$$

since $\rho_{\pm \frac{1}{2}} \sim r^{2j+\frac{3}{2}}$ provided $r^2 a \to 0$, which we have assumed. Hence

$$\left(\frac{r \partial_r \rho_{\pm \frac{1}{2}}}{\rho_{\pm \frac{1}{2}}}\right)_R - \left(\frac{r \partial_r \rho_{0,\pm \frac{1}{2}}}{\rho_{0,\pm \frac{1}{2}}}\right)_R = \frac{k^2 R \int_0^R dr \rho_{\pm \frac{1}{2}}(R) \rho_{0,\pm \frac{1}{2}}(R)}{\rho_{\pm \frac{1}{2}}(R) \rho_{0,\pm \frac{1}{2}}(R)}. \quad (B4)$$

Since $k^2$ is an analytic perturbation of $\rho_{\pm \frac{1}{2}}$ in (B1), make the expansion

$$\rho_{\pm \frac{1}{2}}(r) = \rho_{0,\pm \frac{1}{2}} + \rho_{2,\pm \frac{1}{2}}(r) k^2 + O(k^4), \quad (B5)$$

and substitute this in (B4) to get the interior logarithmic derivative

$$\left(\frac{r \partial_r \rho_{\pm \frac{1}{2}}}{\rho_{\pm \frac{1}{2}}}\right)_R = \gamma_{\pm \frac{1}{2}} - (kR)^2 \Gamma_{\pm \frac{1}{2}} + O(kR)^4. \quad (B6)$$

Here

$$\gamma_{\pm \frac{1}{2}} = \left(\frac{r \partial_r \rho_{0,\pm \frac{1}{2}}}{\rho_{0,\pm \frac{1}{2}}}\right)_R, \quad \Gamma_{\pm \frac{1}{2}} = \frac{\int_0^R dr \rho_{0,\pm \frac{1}{2}}^2}{\rho_{0,\pm \frac{1}{2}}(R)}, \quad (B7)$$

where $\gamma_{\pm \frac{1}{2}}$, $\Gamma_{\pm \frac{1}{2}}$ are denoted by $\gamma_\alpha$, $\Gamma_\alpha$, respectively, in Secs.VI and VII with $\alpha$ denoting $j$, $M$, $\pm \frac{1}{2}$. When $M = -j - \frac{1}{2}$, $\rho_{0,-\frac{1}{2}}$ is given by (153) and when $M = j + \frac{1}{2},$

$$\rho_{0,\frac{1}{2}} = C r^{2j+3/2} e^{i \gamma_\alpha} d^{\alpha} s a. \quad (B8)$$
Calculation of the low-energy phase shifts then proceeds by equating the interior derivative $B_6$ with the exterior derivative calculated from the small-$k$ expansion of

$$
\rho_{\pm \frac{1}{2}}(r) = \sqrt{\frac{r}{2}} J_\lambda(kr) \cos \Delta_{\pm \frac{1}{2}}^+(k) - \sqrt{\frac{r}{2}} Y_\lambda(kr) \sin \Delta_{\pm \frac{1}{2}}^+(k),
$$

(B9)

obtained from (113) and (116).
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