Operator-sum Representation for Bosonic Gaussian Channels

J. Solomon Ivan
Raman Research Institute, C. V. Raman Avenue, Sadashivanagar, Bangalore 560 080, India.

Krishnakumar Sabapathy† and R. Simon‡
Centre for Quantum Science, The Institute of Mathematical Sciences, C.I.T Campus, Tharamani, Chennai 600 113, India.

Operator-sum or Kraus representations for single-mode Bosonic Gaussian channels are developed, and several of their consequences explored. The fact that the two-mode metaplectic operators acting as unitary purification of these channels do not, in their canonical form, mix the position and momentum variables is exploited to present a procedure which applies uniformly to all families in the Holevo classification. In this procedure the Kraus operators of every quantum-limited Gaussian channel can be simply read off from the matrix elements of a corresponding metaplectic operator. Kraus operators are employed to bring out, in the Fock basis, the manner in which the antilinear, unphysical matrix transposition map when accompanied by injection of a threshold classical noise becomes a physical channel, denoted \( D(\kappa) \) in the Holevo classification. The matrix transposition channels \( D(\kappa), D(\kappa^{-1}) \) turn out to be a dual pair in the sense that their Kraus operators are related by the adjoint operation. The amplifier channel with amplification factor \( \kappa \) and the beamsplitter channel with attenuation factor \( \kappa^{-1} \) turn out to be mutually dual in the same sense. The action of the quantum-limited attenuator and amplifier channels as simply scaling maps on suitable quasi-probabilities in phase space is examined in the Kraus picture. Consideration of cumulants is used to examine the issue of fixed points. The semigroup property of the amplifier and attenuator families leads in both cases to a Zeno-like effect arising as a consequence of interrupted evolution. In the cases of entanglement-breaking channels a description in terms of rank one Kraus operators is shown to emerge quite simply. In contradistinction, it is shown that there is not even one finite rank operator in the entire linear span of Kraus operators of the quantum-limited amplifier or attenuator families, an assertion far stronger than the statement that these are not entanglement breaking channels. A characterization of extremality in terms of Kraus operators, originally due to Choi, is employed to show that all quantum-limited Gaussian channels are extremal. The fact that every noisy Gaussian channel can be realised as product of a pair of quantum-limited channels is used to construct a discrete set of linearly independent Kraus operators for noisy Gaussian channels, including the classical noise channel, and these Kraus operators have a particularly simple structure.

PACS numbers: 03.67-a, 03.67.Mn, 03.67.Hk, 42.50.Ex, 42.50.-p

I. INTRODUCTION

Gaussian states and Gaussian channels play a major role in quantum information processes, and this is primarily due to their potential experimental realization within current technologies [1–4]. Indeed, the basic protocols of quantum information processing including teleportation and dense coding have been implemented in the quantum optical domain [5, 6]. The feasibility of processing information using Gaussian channels was originally explored in [1, 7]. More recently, the problem of evaluating the classical capacity of Gaussian channels was addressed in [8–10], and the quantum capacities in [11–16]. In particular, the classical capacity of the attenuator channel was evaluated in [10], and the quantum capacity of a class of channels was studied in [12]. A systematic study of the structure of the family of all Gaussian channels has been carried out in [17–21]; single-mode Gaussian channels have been classified in [17, 18], and the case of multimodes in [19–21].

Phase space description in terms of quasi-probabilities or, equivalently, in terms of the associated characteristic functions underlies the very notions of Gaussian states and Gaussian channels. Gaussian states are those with Gaussian characteristic functions, and Gaussian channels are those trace-preserving CP maps which image every input Gaussian state into a Gaussian state at the output.
Gaussian states are fully specified by their first and second moments. Since the first moments play no significant role in our study, we may assume that they vanish (this can indeed be ensured using the unitary Weyl-Heisenberg displacement operators), so that a Gaussian state for our purpose is fully described by its covariance matrix \( \rho_2 \) and annihilation operators \([22, 24]\).

The symplectic group of real linear canonical transformations (acting through its unitary metaplectic representation) and the Weyl-Heisenberg group of phase space translations are the only unitary evolutions which preserve Gaussianity, and these groups are generated by hermitian Hamiltonians which are respectively quadratic and linear in the creation and annihilation operators \([22, 24]\). This suggests that more general Gaussian channels on system A may be realized as Gaussianity preserving unitaries on a suitably enlarged system:

\[
\rho_A \rightarrow \rho'_A = \text{Tr}_B \left( U_{AB} (\rho_A \otimes \rho_B) U_{AB}^\dagger \right).
\]

Here \( \rho_B \) is a Gaussian state of the ancilla B, and \( U_{AB} \) is a linear canonical transformation on the enlarged composite system consisting of the system of interest A and the ancilla B. That all Gaussian channels can indeed be realized in this manner has been shown by the work of Holevo and coauthors \([17, 18, 20, 21]\).

It is clear that the most general trace-preserving linear map \( \Omega \) which takes Gaussian characteristic functions to Gaussian, taking states with vanishing first moments to ones with vanishing first moments, are necessarily of the form \( \text{Tr}_S (U \chi_S \rho_S U^\dagger) \). Therefore in classifying Gaussian channels it is sufficient to classify these orbits or double cosets and, further, we may identify each orbit with the ‘simplest’ looking representative element of that orbit (the canonical form). Since \( \exp[-1/2 S_1^T S_2 S_2^T S_1] \) the task actually reduces to enumeration of the orbits of \((X, Y)\) under the transformation \((X, Y) \rightarrow (X', Y') = (S_2 X S_1, S_1^T Y S_1)\).

One final point before turning to the special case of single-mode Gaussian channels. The injection of an arbitrary amount of classical (Gaussian) noise into the state is obviously a Gaussian channel: \( \chi_X \rightarrow \chi_X \exp[-1/2 \xi^T S_1^T Y S_1 \xi], a > 0 \). It is called the classical noise channel. Now, given a Gaussian channel we may follow it up with a classical noise channel to obtain another Gaussian channel. A Gaussian channel will be said to be quantum-limited if it cannot be realized as another Gaussian channel followed by a classical noise channel. Conversely, the most general Gaussian channel is a quantum-limited Gaussian channel followed by a classical noise channel, and it follows that quantum-limited channels are the primary objects which need to be classified into orbits.

In the single-mode case where \( X, Y \) are \( 2 \times 2 \) matrices, \( S_1, S_2 \in Sp(2, R) \) can be so chosen that \( X' = \rho_1 \) equals a multiple of identity, a multiple of \( \sigma_3 \), or \( (I + \sigma_3)/2 \) while \( Y' \) equals a multiple of identity or \( (I + \sigma_3)/2 \). Thus the canonical form of a Gaussian channel \( X, Y \) is fully determined by the rank and determinant of \( X, Y \) and we have the following classification of quantum-limited bosonic Gaussian channels \([17, 18]\):

\[
\begin{align*}
D(\kappa; 0) : & \quad X = -\kappa \sigma_3, \quad Y_0 = (1 + \kappa^2) I, \quad \kappa > 0; \\
C_1(\kappa; 0) : & \quad X = \kappa I, \quad Y_0 = (1 - \kappa^2) I, \quad 0 < \kappa < 1; \\
C_2(\kappa; 0) : & \quad X = \kappa I, \quad Y_0 = \kappa^2 I, \quad \kappa > 1; \\
A_1(0) : & \quad X = 0, \quad Y_0 = I; \\
A_2(0) : & \quad X = (I + \sigma_3)/2, \quad Y_0 = I; \\
B_2(0) : & \quad X = I, \quad Y_0 = 0; \\
B_1(0) : & \quad X = I, \quad Y_0 = 0.
\end{align*}
\]
it is the injection of additional classical noise of magnitude (not less than) $1 + \kappa^2$, represented by $Y_0$, that mends it into a CP map.

The reason for the special emphasis on quantum-limited channels in our enumeration of the Holevo classification is this: every noisy Gaussian channel except $B_1(a)$ can be realized, as we shall see later, as the composite of a pair of quantum-limited channels. And this fact proves to be of much value to the study presented in this paper. In the original classification of Holevo the families $C_1$ and $C_2$, which correspond respectively to the attenuator (beamsplitter) and the amplifier (two-mode squeezing) channels, together constituted a single family $C$. From the point of view of the present study, however, these two families turn out to be qualitatively different from one another, hence we prefer to keep them as two distinct families.

It is well known that every trace-preserving completely positive map has an operator-sum representation of the form

$$\rho \rightarrow \rho' = \sum_{\alpha} W_\alpha \rho W_\alpha^\dagger, \quad \sum_{\alpha} W_\alpha^\dagger W_\alpha = \mathbb{1},$$

(1.4)

often called Kraus representation [28]. It may be noted, however, that this representation appears as Theorem 4 of a much earlier work of Sudarshan et al [29]. It has been presented also by Choi [30], apparently independently. Mathematicians seem to view it as a direct and immediate consequence of the dilation theorem of Stinespring [51]. In this paper we develop and present a systematic analysis of the operator-sum representation for single-mode bosonic Gaussian channels.

Knowledge of Kraus representation of a channel could prove useful for several purposes. For instance, since the set of channels for a given system is convex, it is of interest to know its extremals. And a theorem of Choi [31] gives a necessary and sufficient test for extremality of a channel in terms of Kraus operators. It is known that a channel is entanglement breaking if and only if it can be described in terms of a set of rank one Kraus operators [33, 34]. Further, the work of [38] and [39, 40] relate error correctability to the structure of the Kraus operators of a channel. Finally, there has been considerable recent interest in contrasting the Gaussian with nonGaussian states in respect of robustness and degradation of bipartite entanglement under one-sided and two-sided action by Gaussian channels [41, 43], and it is likely that Kraus representation could throw light on this problem.

The content of this paper is organized as follows. Section II presents a general scheme for computation of Kraus operators, and this scheme applies uniformly to all quantum-limited Gaussian channels. This scheme takes particular advantage of the fact that the symplectic two-mode transformation which realizes the channel in the sense of (1.1) does not couple, in the Holevo canonical form, the position variables with the momentum variables. With the ancilla mode assumed to be in its vacuum state initially, it turns out that the Kraus operators for each channel can be simply read off from the matrix elements of the appropriate two-mode metaplectic operator.

This scheme is applied in Sections III to VI to detail the Kraus operators of respectively the $D$, $C_1$, $C_2$, and $A_2$ families of quantum-limited channels. The Kraus operators in every case is found to have an extremely simple-looking sparse structure. In each case we ask if the channel has any fixed points (invariant states), and if there are sufficient number of rank one operators in the linear span of the Kraus operators, a question which is at the very root of the entanglement breaking property of the channel.

In the case of the phase conjugation or (matrix) transposition family $D(\kappa; 0)$ in Section III we explore how the threshold noise of magnitude $1 + \kappa^2$ renders the antilinear phase conjugation into a linear map and channel. We bring out a well-defined sense in which the channels $D(\kappa; 0)$ and $D(\kappa^{-1}; 0)$ are dual to one another. The case $D(1; 0)$ is self-dual and hence doubly stochastic, but it turns out that it is not a random unitary channel, a fact which is of relevance to the possibility or otherwise of extending the classical Birkhoff theorem to the quantum domain [52].

We examine in the Kraus picture the manner in which quantum-limited attenuator and amplifier families $C_1$, $C_2$ act as simple scaling maps respectively on the diagonal weight function and the Husimi $Q$-distribution. Comparing the Kraus operators of the $C_1$ family with those of the $C_2$ family, we show in Section V that these two families are dual to one another. The intersection between $C_1$, $C_2$ consists of just the identity channel, the only self-dual or doubly stochastic channel in the union of $C_1$, $C_2$. The manner in which the semigroup structures of the $C_1$ and $C_2$ families are reflected in their respective sets of Kraus operators is brought out in Sections IV and V, and this enables us to point to a Zeno-like effect [53], in both cases, arising as consequence of interrupted evolution. Finally, even though the single-quadrature classical noise channels $B_1(a)$, $a \neq 0$ [B_1(0) is the identity channel] are not quantum-limited, we deal with them briefly in Section VII just to bring out the fact that this case too is obedient to the general computational scheme presented in Section II.

In Section VIII wherein we use Choi’s theorem [30] to study if there are any extremals among Gaussian channels, we show that all quantum limited channels, and these alone, are extremal. That our concern upto this stage of the presentation is (almost) exclusively with the quantum-limited case gets justified by our demonstrations in Section IX that every noisy Gaussian channel except $B_1(a)$ can be realized as the composite of a pair of quantum-limited ones. This demonstration leads, in particular, to an operator-sum representation for all noisy channels, including the all
important classical noise channels $B_2(a)$ [but excluding $B_1(a)$, and only this case], in terms of a linearly independent discrete set of Kraus operators having very simple sparse structure.

The composition of pairs of quantum-limited channels studied in Section IX, and conveniently summarized in Table III there, assumes that both the constituent channels are simultaneously in their respective canonical forms. When this assumption is removed, the situation with the composition process gets much richer. The general case is fully classified and presented in Table III of the Appendix.

The final Section X contains a brief summary of the principal results and also some additional remarks.

II. KRAUS REPRESENTATION: SOME GENERAL CONSIDERATIONS

Given density operator $\rho^{(a)}$ describing the state of a single-mode radiation field, the action of a quantum-limited Gaussian channel takes it to 17, 18

$$\rho^{(a)} = \text{Tr}_b(U^{(ab)} \rho^{(a)} \otimes |0\rangle_b \langle 0|) U^{(ab)\dagger}. \quad (2.1)$$

Here $|0\rangle_b$ is the vacuum state of the ancilla mode $b$, and $U^{(ab)}$ is the unitary operator corresponding to a suitable two-mode linear canonical transformation. It is convenient to perform the partial trace in the Fock basis of mode $b$. We have

$$\rho^{(a)} = \sum_\ell b(\ell|U^{(ab)} \rho^{(a)} \otimes |0\rangle_b \langle 0|) U^{(ab)\dagger} |\ell\rangle_b$$

$$= \sum_\ell b(\ell|U^{(ab)}|0\rangle_b \rho^{(a)} b\langle 0| U^{(ab)\dagger} |\ell\rangle_b. \quad (2.2)$$

Clearly, $b(\ell|U^{(ab)}|0\rangle_b$ is an operator acting on the Hilbert space of mode $a$. The last expression thus leads us to the Kraus representation of the channel 28:

$$\rho \rightarrow \rho^{(a)} = \sum_\ell W_\ell \rho^{(a)} W_\ell^\dagger, \quad W_\ell = b(\ell|U^{(ab)}|0\rangle_b. \quad (2.3)$$

It follows that once the Fock basis matrix elements of $U^{(ab)}$ are known, the Kraus operators $W_\ell$ can be easily read off. Let $\langle m_1 m_2|U^{(ab)}|n_1 n_2 \rangle \equiv C_{n_1 n_2}^{m_1 m_2}$ be the matrix elements of $U^{(ab)}$ in the two-mode Fock basis. Since the ancilla mode $b$ is assumed to be in the vacuum state, the $W_\ell$’s are obtained by setting $n_2 = 0$ and $m_2 = \ell$:

$$W_\ell = \sum_{n_1, m_1 = 0}^{\infty} C_{n_1 0}^{m_1 \ell} |m_1 \rangle \langle n_1|. \quad (2.4)$$

Now, in evaluating $C_{n_1 n_2}^{m_1 m_2}$ it proves useful to employ a resolution of identity in the position basis 44:

$$C_{n_1 n_2}^{m_1 m_2} = \langle m_1 m_2|U^{(ab)}|n_1 n_2 \rangle$$

$$= \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2|x_1 x_2 \rangle \langle x_1 x_2|U^{(ab)}|n_1 n_2 \rangle. \quad (2.5)$$

Under conjugation by $U^{(ab)}$ the quadrature variables $q_j, p_j \ (j = 1, 2)$ undergo a linear canonical transformation $S \in Sp(4, R)$, of which $U^{(ab)}$ is the (metaplectic) unitary representation 24. Let us assume that this canonical transformation does not mix the position variables with the momentum variables. That is,

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow U^{(ab)\dagger} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} U^{(ab)} = \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow U^{(ab)\dagger} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} U^{(ab)} = \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad (2.6)$$

where $M$ is a real non-singular $2 \times 2$ matrix. This assumption that our $S \in Sp(4, R)$ has the direct sum structure
Written in detail, the phase space variables undergo, under the action of this channel, the transformation

\[ \psi_{m_1} \psi_{n_2} = \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1, m_2 | x_1 x_2 \rangle \langle x_1 x_2 | U^{(ab)} | n_1, n_2 \rangle \]

where \((x'_1, x'_2)\) is linearly related to \((x_1, x_2)\) through \(M\). These wavefunctions are the familiar Hermite functions, the Fock states in the position representation. The above integral may be evaluated using the generating function for Hermite polynomials \([44]\):

[\[
\psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} e^{-x^2} H_n(x) \\
= \frac{\pi^{-1/4}}{\sqrt{n!}} \frac{\partial^n}{\partial z^n} \exp \left( -\frac{1}{2} [(x - z\sqrt{2})^2 - z^2] \right) \bigg|_{z=0}.
\]\]

Inserting in Eq. (2.7) the generating function for each of the four wavefunctions we have

\[ C_{n_1 n_2}^{m_1 m_2} = \left. \frac{1}{\sqrt{n_1 n_2 m_1 m_2}} \frac{\partial^{m_1}}{\partial z_1^{m_1}} \frac{\partial^{m_2}}{\partial z_2^{m_2}} F(z_1, z_2, \eta_1, \eta_2) \right|_{z_1, z_2, \eta_1, \eta_2 = 0}, \]

where

\[ F(z_1, z_2, \eta_1, \eta_2) = \pi^{-1} \int_{-\infty}^{\infty} dx_1 dx_2 \exp \left\{ -\frac{1}{2} [(x_1 - \eta_1 \sqrt{2})^2 + (x_2 - \eta_2 \sqrt{2})^2] \right. \\
\left. + (x'_1 - z_1 \sqrt{2})^2 + (x'_2 - z_2 \sqrt{2})^2 - \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2 \right\}. \]  

The Gaussian integration over the variables \(x_1\) and \(x_2\) can be easily carried out to obtain \(F(z_1, z_2, \eta_1, \eta_2)\), and from \(F(z_1, z_2, \eta_1, \eta_2)\) we may readily obtain \(C_{n_1 n_2}^{m_1 m_2}\), and hence the Kraus operators. This is the general scheme we will employ in what follows to obtain Kraus representation for quantum-limited Gaussian channels of the various families.

**III. PHASE CONJUGATION OR TRANSPOSITION CHANNEL \(D(\kappa), \kappa \geq 0\)**

We now use the above scheme to evaluate a set of Kraus operators representing the phase conjugation channel. The metaplectic unitary operator \(U^{(ab)}\) appropriate for this case induces on the quadrature operators of the bipartite phase space a linear canonical transformation corresponding to the following \(S \in Sp(4, R)\) \([17]\):

[\[
S = \begin{pmatrix}
\text{sinh} \mu & 0 & \cosh \mu & 0 \\
0 & -\text{sinh} \mu & 0 & \cosh \mu \\
\cosh \mu & 0 & \text{sinh} \mu & 0 \\
0 & \cosh \mu & 0 & -\text{sinh} \mu
\end{pmatrix},
\]

Written in detail, the phase space variables undergo, under the action of this channel, the transformation

\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} q'_1 \\ q'_2 \\ p'_1 \\ p'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}, \]

where

\[ M = \begin{pmatrix} -\text{sinh} \mu & \cosh \mu \\ \cosh \mu & -\text{sinh} \mu \end{pmatrix}. \]

It is seen that the above \(S\) is indeed of the form \(S = M \oplus (M^{-1})^T \in Sp(4, R)\), and does not mix the position variables with the momentum variables, and so our general scheme above readily applies.
It is clear from the structure of \( S \) that the parameter \( \mu \) is related to \( \kappa \) in \( \mathcal{D}(\kappa) \) through \( \kappa = -\sinh \mu > 0 \), so that \( \cosh \mu = \sqrt{\kappa^2 + 1} \). Thus (2.10) translates, for the present case, to the following expression:
\[
F(z_1, z_2, \eta_1, \eta_2) = \pi^{-1} \int_{-\infty}^{\infty} dx_1 \, dx_2 \exp \left\{ -\frac{1}{2} \left[ (x_1 - \eta_1 \sqrt{2})^2 + (x_2 - \eta_2 \sqrt{2})^2 \right. \right. \\
\left. \left. + (-\kappa x_1 + \sqrt{1 + \kappa^2} x_2 - z_1 \sqrt{2})^2 + (\sqrt{1 + \kappa^2} x_1 - \kappa x_2 - z_2 \sqrt{2})^2 \right. \right. \\
\left. \left. - \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2 \right\} . \tag{3.3} \right.
\]

Performing the Gaussian integrals in \( x_1 \) and \( x_2 \) we obtain
\[
F(z_1, z_2, \eta_1, \eta_2) = (\sqrt{1 + \kappa^2} + 1)^{-1} \exp \left\{ (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 - (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \right. \\
\left. + \eta_2 [(\sqrt{1 + \kappa^2} - 1)^{-1} \eta_2 + (\sqrt{1 + \kappa^2} - 1)^{-1} z_2] \right\} . \tag{3.4} \right.
\]

Performing the \( z_2 \) and \( \eta_2 \) differentiations respectively \( n_2 \) and \( m_2 \) times on \( F(z_1, z_2, \eta_1, \eta_2) \), we obtain
\[
\left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 - (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \right]^{n_2} \left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 + (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \right]^{m_2} F = GF. \tag{3.6} \right.
\]

The remaining differentiations can be carried out using the Leibniz rule. Since we finally set \( z_1, z_2, \eta_1, \eta_2 = 0 \), and since \( F(0) = 1 \), the only terms that could possibly survive are necessarily of the form
\[
\frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 - (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \right]^{n_2} \left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 + (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \right]^{m_2}. \tag{3.7} \right.
\]

To evaluate the above expression we set \( x = (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 - (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \) and \( y = (\sqrt{1 + \kappa^2} - 1)^{-1} \eta_1 + (\sqrt{1 + \kappa^2} - 1)^{-1} z_1 \), and compute
\[
\left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \partial_x + (\sqrt{1 + \kappa^2} - 1)^{-1} \partial_y \right]^{n_2} \left[ (\sqrt{1 + \kappa^2} - 1)^{-1} \partial_x + (\sqrt{1 + \kappa^2} - 1)^{-1} \partial_y \right]^{m_2} \bigg|_{x,y=0}. \tag{3.8} \right.
\]
Straight forward algebra leads, in view of Eq. (2.9), to
\[
C_{n_1 n_2}^{m_1 m_2} = \frac{(\sqrt{1 + \kappa^2} - 1)^{-1}}{\sqrt{n_1! \, n_2! \, m_1! \, m_2!}} \sum_{j=0}^{n_1} \sum_{r=0}^{m_1} C_j^{m_1} C_r \left( -\frac{1}{\sqrt{1 + \kappa^2}} \right)^{(m_1 + j - r)} \left( \frac{1}{\sqrt{1 + \kappa^2}} \right)^{(n_1 - j + r)} \\
\times (-1)^{m_1 - r} \, n_2! \, m_2! \delta_{n_2, r+j} \delta_{m_2, n_1 - j + m_1 - r}. \tag{3.9} \right.
\]
The Kraus operators \( W_{\ell} \), denoted \( T_{\ell}(\kappa) \) in this case, are obtained from these matrix elements by setting \( n_2 = 0 \) and \( m_2 = \ell \). Since \( n_2 = 0 \Rightarrow r, j = 0 \), we have,
\[
T_{\ell}(\kappa) = (\sqrt{1 + \kappa^2} - 1)^{-1} \sum_{n_1, m_1 = 0}^{\infty} \frac{(\sqrt{1 + \kappa^2} - n_1) \, (-\sqrt{1 + \kappa^2} - m_1 \sqrt{\ell}) \, \delta_{\ell, n_1 + m_1} \, (-1)^{m_1}}{\sqrt{n_1! \, m_1!}} \left| m_1 \right\rangle \langle n_1 \left| . \tag{3.10} \right.
\]
We set \( n_1 + m_1 = \ell \) and denote \( n_1 = n \), leading to
\[
T_{\ell}(\kappa) = (\sqrt{1 + \kappa^2} - 1)^{-1} \sum_{n=0}^{\ell} \frac{(\sqrt{1 + \kappa^2} - n) \, (-\sqrt{1 + \kappa^2} - \ell - n) \, \sqrt{\ell} \, C_n \, \sqrt{\ell - n}}{\sqrt{\ell - n}} \left| m_1 \right\rangle \langle n \left| , \ \ell = 0, 1, 2, \cdots \tag{3.11} \right.
\] as our final form for the Kraus operators of the phase conjugation channel. We note that the \( T_{\ell}(\kappa)'s \) are real and manifestly trace-orthogonal: \( \text{tr}(T_{\ell}(\kappa) \, T_{\ell'}(\kappa)) = 0 \) if \( \ell \neq \ell' \).
**I. The dual of \( D(\kappa) \)**

As is well known (and also obvious), if a set of Kraus operators \( \{W_\ell\} \) describes the completely positive map \( \Phi : \rho \to \rho' = \sum_\ell W_\ell \rho W_\ell^\dagger \), then the dual map \( \tilde{\Phi} : \rho \to \rho'' = \sum_\ell W_\ell^\dagger \rho W_\ell \), described by the dual or adjoint set of operators \( \{W_\ell^\dagger\} \), is also completely positive. It is clear that the dual map \( \tilde{\Phi} \) is unital or trace-preserving according as \( \Phi \) is trace-preserving or unital.

For the present case of \( D(\kappa) \), it is readily verified that the Kraus operators \( \{T_\ell(\kappa)\} \) presented in (3.11) meet \( \sum_\ell T_\ell^\dagger(\kappa)T_\ell(\kappa) = 1 \), consistent with the expected trace-preserving nature of \( \rho \to \rho'' = \sum_\ell T_\ell(\kappa)\rho T_\ell^\dagger(\kappa) \). But the phase conjugation channel is not unital in general, for we have

\[
\sum_\ell T_\ell(\kappa)T_\ell^\dagger(\kappa) = \kappa^{-2} I. \tag{3.12}
\]

We may say that it is ‘almost unital’ to emphasise the minimal nature of the failure: the unit element is taken by the channel into a scalar multiple of itself. However, the scale factor \( \kappa^{-2} \) can not be transformed away by absorbing \( \kappa^{-1} \) into the Kraus operators, for the Kraus operators so modified would not then respect the trace-preserving property of the map.

It is thus of interest to understand the nature of the unital channel described by the set of Kraus operators \( \{T_\ell(\kappa)^\dagger\} \). We have

\[
T_\ell(\kappa)^\dagger = (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^\ell (\sqrt{1 + \kappa^2})^{-n} (\sqrt{1 + \kappa^2})^{-(\ell-n)} \sqrt{\kappa C_n} |\ell - n\rangle \langle n|.
\]

Thus the dual \( \{T_\ell(\kappa)^\dagger\} \) differs from the original \( \{T_\ell(\kappa)\} \) in two elementary aspects. The multiplicative factor \( \kappa^{-1} \) is the same for all Kraus operators, independent of \( \ell \). Thus the only significant difference is change in the argument of \( T_\ell \), from \( \kappa \) to \( \kappa^{-1} \). We conclude that the ‘dual’ channel whose Kraus operators are \( \kappa T_\ell(\kappa)^\dagger \) is the (trace-preserving) phase conjugation channel \( D(\kappa^{-1}) \). We have thus proved

**Theorem 1** While the Kraus operators \( \{T_\ell(\kappa)\} \) describe \( D(\kappa) \), the ‘dual’ channel described by Kraus operators \( \{\kappa T_\ell(\kappa)^\dagger\} \) is the trace-preserving phase conjugation channel \( D(\kappa^{-1}) \) with reciprocal scale parameter.

**II. Properties of the Kraus operators**

We now explore the properties of the Kraus operators of Eq. (3.11), connecting their explicit action in the Fock basis to the expected transformation of the characteristic function. The question of its fixed points is studied through the action of the channel on the cumulants. The action of the channel is illustrated with simple examples and, finally, the entanglement breaking nature of the channel is made transparent by obtaining a set of rank one Kraus operators describing the channel.

The expected or defining action of the phase conjugation channel on the characteristic function is (17):

\[
\chi_W(\xi) \to \chi_W'(\xi) = \chi_W(-\kappa \xi^* \exp[-(1 + \kappa^2)|\xi|^2/2]). \tag{3.14}
\]

It is of interest to understand how the ‘antilinear’ phase conjugation \( (\xi \to \xi^*) \) action of this channel on the characteristic function emerges from the linear action of the Kraus operators. To this end, it is sufficient to establish such an action on the ‘characteristic function’ corresponding to the operators \( |n\rangle \langle m| \), for arbitrary pairs of integers \( n, m \geq 0 \). The ‘characteristic function’ of \( |n\rangle \langle m| \) is given by (15)

\[
\chi_W(\langle n| \langle m| = m! D(\xi) |n\rangle \langle m| = \sqrt{\frac{n!}{m!}} (-\xi^*)^{-n-m} L_{m-n}(|\xi|^2) \exp[-|\xi|^2/2] \text{ for } n \geq m,
\]

\[
= \sqrt{\frac{n!}{m!}} (\xi)^{m-n} L_{n-m}(|\xi|^2) \exp[-|\xi|^2/2] \text{ for } n \leq m. \tag{3.15}
\]
Assuming $n \geq m$, the action of the phase conjugation channel on the operator $|n\rangle\langle m|$ is

$$
\sum_{\ell=0}^{\infty} T_{\ell}(\kappa)|n\rangle\langle m|T_{\ell}(\kappa)\\
= (1 + \kappa^2)^{-1} \sum_{\ell=n}^{\infty} \left( \sqrt{1 + \kappa^2} \right)^{-(n+m)} \left( \sqrt{1 + \kappa^{-2}} \right)^{-\left(2\ell-n-m\right)} \sqrt{C_\ell \ell C_m} |\ell-n\rangle\langle \ell-m|.
$$

(3.16)

Denoting $n = m + \delta$ and $\ell - n = \lambda$, we have

$$
\sum_{\ell=0}^{\infty} T_{\ell}(\kappa)|m + \delta\rangle\langle m|T_{\ell}(\kappa) = (1 + \kappa^2)^{-1} \left( \sqrt{1 + \kappa^2} \right)^{-(2m+\delta)} \left( \sqrt{1 + \kappa^{-2}} \right)^{-\delta} \times \sum_{\lambda=0}^{\infty} \frac{(\lambda + m + \delta)!}{\sqrt{(m+\delta)!m!\lambda!(\lambda + \delta)!}} (\lambda + \delta)!D(\xi)|\lambda\rangle\langle \lambda|.
$$

(3.17)

The manner in which $D(\kappa)$, matrix transposition accompanied by threshold Gaussian noise $\exp[-(1 + \kappa^2)|\xi|^2/2]$, acts as a channel may now be appreciated. Every operator $M$ can be written in the Kronecker delta basis $\{|j\rangle\langle \ell|\}$ as $M = \sum_{j,\ell} c_{j\ell}|j\rangle\langle \ell|$. The coefficient matrix $C$ associated with $|5\rangle\langle 3|$, for instance, is $c_{j,k} = \delta_{5j}\delta_{3\ell}$, with non-zero entry only at the lower-diagonal location (5,3) marked $\otimes$ in the matrix below.

$$
\begin{pmatrix}
0 & 0 & \times & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \times & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \otimes & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & \ldots \\
0 & 0 & \times & 0 & 0 & 0 & \times & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{pmatrix}
$$

On transposition this entry moves to the upper-diagonal location (3,5) marked $\oplus$, and the threshold noise then spreads it along the parallel upper diagonal $(3+r,5+r)$, $-3 \leq r < \infty$ marked $\times$.

Let the Weyl-ordered characteristic function $\text{tr}(D(\xi)|m + \delta\rangle\langle m|)$ where $D(\xi) = \exp[\xi a^\dagger - \xi^* a]$ is the displacement operator, be denoted $\chi_{W|m+\delta\rangle\langle m|}(\xi)$, and that of the output $\sum_{\ell=0}^{\infty} T_{\ell}(\kappa)|m + \delta\rangle\langle m|T_{\ell}(\kappa)^\dagger$ be denoted $\chi_{W|m+\delta\rangle\langle m|}(\xi)$. Then we have from Eq. (3.17)

$$
\chi_{W|m+\delta\rangle\langle m|}(\xi) = (1 + \kappa^2)^{-1} \left( \sqrt{1 + \kappa^2} \right)^{-(2m+\delta)} \left( \sqrt{1 + \kappa^{-2}} \right)^{-\delta} \times \sum_{\lambda=0}^{\infty} \frac{(\lambda + m + \delta)!}{\sqrt{(m+\delta)!m!\lambda!(\lambda + \delta)!}} (\lambda + \delta)!D(\xi)|\lambda\rangle\langle \lambda|.
$$

(3.18)

where we used (3.15), the Fock basis representation of the displacement operator. While no ‘phase conjugation’ is manifest as yet, we expect from Eq. (3.14) that the channel should take the characteristic function of $|m + \delta\rangle\langle m|$ to

$$
\chi_{W|m+\delta\rangle\langle m|}(\xi) = |m\rangle D(-\kappa^*\xi^*)|m + \delta\rangle \exp \left[ -\frac{1}{2} (1 + \kappa^2)|\xi|^2 \right] \frac{m!}{m + \delta!} \kappa^\delta \chi_{m+\delta\rangle\langle m|}(\xi) L^\delta_m (\kappa^2 |\xi|^2) \exp \left[ -\frac{1}{2} (1 + \kappa^2)|\xi|^2 \right].
$$

(3.19)
Thus the problem reduces to one of establishing equality of $\chi_W|_{m+\delta}\langle m|\xi\rangle$ in (3.18) and $\chi_W''|_{m+\delta}\langle m|\xi\rangle$ in (3.19). That is, it remains to prove

\[
\sqrt{m!/(m+\delta)!}(\kappa\xi)\delta L_m(\kappa^2|\xi|^2) \exp\left[-(1/2 + \kappa^2)|\xi|^2\right] \\
= \frac{(1 + \kappa^2)^{-1/2}e^{-|\xi|^2/2}}{\sqrt{(m + \delta)!m!}} \sum_{\lambda=0}^{\infty} (1 + \kappa^{-2})^{-\lambda} (\sqrt{1 + \kappa^2})^{-2m-\delta}(\sqrt{1 + \kappa^{-2}})^{-\delta} \\
\times \frac{(\lambda + m + \delta)!}{(\lambda + \delta)!} \xi^\delta L_\lambda^\delta(|\xi|^2),
\]
(3.20)

for all $m, \delta \geq 0$ [the case of $|m|\langle m+\delta|$ can be handled similarly].

Since the associated Laguerre functions form a complete orthonormal set, we may expand the LHS of Eq. (3.20) in the Laguerre basis. That is, we multiply both sides of Eq. (3.20) by $(\xi^*)^\delta L_\lambda^\delta(|\xi|^2) e^{-|\xi|^2/2}$ and evaluate the overlap integrals. We use the following two standard results: (i) orthogonality relation among Laguerres, and (ii) the overlap 'weight' function $\chi_A\{\}$.

Now, Eq. (3.14) written in terms of the normal-ordered characteristic function reads

\[
\sqrt{m!/(m+\delta)!}(\kappa\xi)\delta L_m(\kappa^2|\xi|^2) \exp\left[-(1 + \kappa^2)|\xi|^2/2\right] \\
= \frac{(1 + \kappa^2)^{-1/2}e^{-|\xi|^2/2}}{\sqrt{(m + \delta)!m!}} \sum_{\lambda=0}^{\infty} (1 + \kappa^{-2})^{-\lambda} (\sqrt{1 + \kappa^2})^{-2m-\delta}(\sqrt{1 + \kappa^{-2}})^{-\delta} \\
\times \frac{(\lambda + m + \delta)!}{(\lambda + \delta)!} \xi^\delta L_\lambda^\delta(|\xi|^2),
\]
(3.21)

Here $F[\cdot]$ is the hypergeometric function. In our case $t = \eta^2 + 1$, which implies that the last argument of $F[\cdot]$ in Eq. (3.21) is zero, and thereby $F[\cdot] = 1$. Performing the overlap integrals, we obtain for the left and right hand sides of (3.20)

\[
\text{LHS} = \frac{(m + \ell + \delta)!}{\ell!(m + \delta)!m!} \frac{\kappa^2\ell+\delta}{(1 + \kappa^2)^{m+\ell+\delta+1}}, \\
\text{RHS} = \frac{(m + \ell + \delta)!}{\ell!(m + \delta)!m!} \frac{(\sqrt{1 + \kappa^2})^{-2+2m+\delta}(\sqrt{1 + \kappa^{-2}})^{-2\ell+\delta}}{t^{m+\ell+\delta}},
\]
(3.22)

These two expressions obviously equal one another for all $\ell$. We have thus established Eq. (3.21), and the fact that the Kraus operators indeed effect the 'completely positive phase conjugation' operation, transforming the characteristic function as expected in (3.14).

**Theorem 2** The scaled phase conjugation transformation $\chi_W(\xi) \rightarrow \chi_W'\langle\xi\rangle = \chi_W(-\kappa \xi^*) \exp[-(1 + \kappa^2)|\xi|^2/2]$ is, in view of the threshold noise $\exp[-(1 + \kappa^2)|\xi|^2/2]$ a completely positive map, and is implemented linearly by the Kraus operators $\{T_\lambda(\kappa)\}$ in Eq. (3.11).

The phase conjugation channel has an interesting property in respect of classicality/nonclassicality of the output states. We may say a channel is nonclassicality breaking if the output of the channel is classical for every input state. That is, if the normal-ordered characteristic function $\chi_N(\xi)$ of the output, related to the Weyl-ordered characteristic function $\chi_W(\xi)$ of (3.14), through $\chi_N(\xi) = \chi_W(\xi) \exp[|\xi|^2/2]$, is such that its Fourier transform, called the diagonal 'weight' function $\phi(\alpha)$ [47], is a genuine probability density.

Now, Eq. (3.14) written in terms of the normal-ordered characteristic function reads

\[
\chi_N(\xi) \rightarrow \chi_N'(\xi) = \chi_W(-\kappa \xi^*) \exp[-\kappa^2|\xi^*|^2/2] \\
= \chi_A(-\kappa \xi^*),
\]
(3.23)

where $\chi_A(\xi) = \chi_N(\xi) \exp[-|\xi|^2]$ is the antinormal-ordered characteristic function corresponding to the Q or Husimi distribution.

Under Fourier transformation this important relation (3.22), namely $\chi_N(\xi) = \chi_A(-\kappa \xi^*)$, reads that the output diagonal weight function $\phi(\alpha)$ evaluated at $\alpha$ equals the input $Q(\alpha)$ evaluated at $\kappa^{-1}\alpha^*$. Thus $\phi(\alpha)$ is a genuine probability density for every input state, and we have

\[
\mathcal{D}(\kappa) : \phi_{\text{in}}(\alpha) \rightarrow \phi_{\text{out}}(\alpha) = \kappa^{-2}Q_{\text{in}}(\kappa^{-1}\alpha^*).
\]
(3.24)
Since the $Q$-distribution of a density operator is given by $Q(\alpha) = \langle \alpha | \rho | \alpha \rangle$, it is a genuine probability distribution for all states including nonclassical states. We have thus proved

**Theorem 3** The phase conjugation channel is a nonclassicality breaking channel.

### III. Fixed points

We now study the fixed points of the phase conjugation channel through consideration of cumulants. The characteristic function of the $s$-ordered quasiprobabilities differ from one and another just by a Gaussian factor, and it follows that the cumulants of order $> 2$ of the $s$-ordered quasi-probability are independent of the ordering parameter $s \ [48, 49]$, and are thus intrinsic to the state. Hence it is sufficient to work with a particular choice of $s$. We work with $s = 0$, the case of symmetric or Weyl ordering.

Given a symmetric ordered characteristic function $\chi_W(\xi)$, the corresponding cumulant generating function is defined as

$$\Gamma(\xi) = \log [\chi_W(\xi)].$$

(3.25)

With $\xi = \xi_1 + i\xi_2$, the cumulants are defined through

$$\gamma_{m_1, m_2} = \left. \frac{\partial^{m_1}}{\partial (i\xi_1)^{m_1}} \frac{\partial^{m_2}}{\partial (i\xi_2)^{m_2}} \Gamma(\xi) \right|_{\xi = 0}, \quad m_1, m_2 = 0, 1, 2, \ldots$$

(3.26)

From Eq. (3.17) we know that under the action of the phase conjugation channel

$$D(\kappa) : \Gamma(\xi) \rightarrow \Gamma'(\xi) = \log [\chi_{W'}(\xi)] = \log [\chi_{W(-\kappa\xi^*)}] - \frac{1}{2} \xi^2 (1 + \kappa^2).$$

(3.27)

Since the additional term on the right hand side is quadratic in $\xi$, the cumulants $\gamma_{m_1, m_2}$ of $\chi_{W'}(\xi)$ of order $m_1, m_2 \neq 2$ are

$$\gamma_{m_1, m_2} = \left. \left( \frac{\partial}{\partial (i\xi_1)} \right)^{m_1} \left( \frac{\partial}{\partial (i\xi_2)} \right)^{m_2} \log [\chi_{W(-\kappa\xi^*)}] \right|_{\xi = 0}.$$

(3.28)

Denoting $-\kappa\xi^* = t$ we have

$$\gamma_{m_1, m_2} = (-1)^{m_1} (\kappa)^{m_1 + m_2} \left( \frac{\partial}{\partial (it_1)} \right)^{m_1} \left( \frac{\partial}{\partial (it_2)} \right)^{m_2} \log [\chi_W(t)]$$

(3.29)

Now, for a state to be invariant all its cumulants need to remain invariant. By (3.28), none of the cumulants of order $m_1, m_2 > 2$ are preserved for $\kappa \neq 1$. Indeed, under repeated use of the channel the higher order cumulants monotonically increase or decrease depending on whether $\kappa$ is $> 1$ or $< 1$. In the case $\kappa = 1$, the cumulants with $m_1$ or $m_2 = 2$ are not preserved because of the last additional term on the right hand side of (3.27), showing that no non-Gaussian state can be a fixed point of $D(\kappa)$.

We are therefore left with the case $\kappa < 1$ to consider. It is clear that in this case all cumulants of order $\neq 2$ die out under repeated use of the channel, and any initial state is driven towards a fixed Gaussian (thermal state). A similar situation was discussed in [50], where linear devices were used to drive non-Gaussian pure states to Gaussian states.

We are thus led to look for fixed points among Gaussian states. The additional last term in Eq. (3.27) is proportional to $|\xi|^2$. Since there is no cross term involving the real and imaginary parts of $\xi$, it is sufficient to look for fixed point among the thermal states, given by

$$\rho_{th}(a_0) = \frac{2}{a_0 + 1} \sum_{n=0}^{\infty} \left( \frac{a_0 - 1}{a_0 + 1} \right)^n |n\rangle \langle n|$$

$$= (1 - x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n|,$$

(3.30)

where $x = (a_0 - 1)(a_0 + 1)^{-1}$, and the average photon number $\text{tr} (\rho_{th} a^\dagger a) = (a_0 - 1)/2$. By Eq. (3.17) the output of the channel is

$$\rho' = (1 + \kappa^2)^{-1} (1 - x) \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} n+j C_j (1 + \kappa^2)^{-j} (1 + \kappa^2)^{-n} x^j |n\rangle \langle n|.$$
Theorem 4 may summarize as follows:

Under repeated use of the channel. Channels $D$ to which all other states are attracted under repeated use of the channel. The second one corresponds to infinite temperature, and is uninteresting for this reason and for the fact that no state is attracted towards it.

As was to be expected from (3.34), the recursion relation in Eq. (3.35) is stable when $\kappa < 1$ and unstable for $\kappa \geq 1$. For a given $\kappa < 1$, any input state is driven towards this ‘stable’ thermal state under repeated use of the channel, and the thermal parameter of this attracting fixed state is $a_0 = (1 + \kappa^2)(1 - \kappa^2)^{-1}$. This is illustrated in Fig. 1. We may summarize as follows:

**Theorem 4** For the phase conjugation channel $D(\kappa)$, $\kappa < 1$, there is a unique thermal state $\rho_{th}(a_0)$ with parameter $a_0 = (1 + \kappa^2)(1 - \kappa^2)^{-1}$ which is left invariant by the channel. All other states are driven towards this thermal state under repeated use of the channel. Channels $D(\kappa)$ for which $\kappa \geq 1$ have no fixed points.

As a simple illustration of the action of the phase conjugation channel, assume the input to be a Fock state. By Eq. (3.17), we have

$$|n\rangle\langle n| \to \sum_{\ell=0}^{\infty} T_\ell(\kappa)|n\rangle\langle n|T_\ell(\kappa)^\dagger$$

$$= (1 + \kappa^2)^{-1}\sum_{\ell=n}^{\infty} \ell C_n(1 + \kappa^2)^{-n}(1 + \kappa^{-2})^{-\ell-n}|\ell - n\rangle\langle \ell - n|.$$  

FIG. 1: Showing the variation of the output thermal parameter $a'_0$ under repeated application of the quantum-limited phase conjugation channel $D(\kappa)$ with $\kappa = 0.8$ for various input thermal parameters. The stable thermal parameter is $a_0 = 4.56$.

With the use of the identity

$$\sum_{j=0}^{\infty} n j C_j (x(1 + \kappa^2)^{-1})^j = \frac{1}{(1 - (1 + \kappa^2)^{-1}x)^n + 1},$$  

the double summation in Eq. (3.31) reduces to a single sum, and we have

$$\rho_{th}(a_0) \to \rho_{th}(a'_0) = \frac{1 - x}{1 + \kappa^2 - x} \sum_{n=0}^{\infty} \left( \frac{\kappa^2}{1 + \kappa^2 - x} \right)^n |n\rangle\langle n|. \quad (3.33)$$

This is a thermal state, and we see that the effect of the channel is to change the thermal parameter $x$ as follows:

$$x \to x' = \frac{\kappa^2}{1 + \kappa^2 - x}. \quad (3.34)$$

Thus the fixed points are $\bar{x} = \kappa^2$ and $\bar{x} = 1$. The first, $\bar{x} = \kappa^2$, corresponds to a finite temperature state ($\kappa^2 < 1$), to which all other states are attracted under repeated use of the channel. The second one corresponds to infinite temperature, and is uninteresting for this reason and for the fact that no state is attracted towards it.

In terms of the parameter $a_0$, the channel action reads

$$D(\kappa) : \rho_{th}(a_0) \to \rho_{th}(a'_0), \quad a_0 \to a'_0 = \kappa^2 a_0 + 1 + \kappa^2. \quad (3.35)$$

As was to be expected from (3.34), the recursion relation in Eq. (3.35) is stable when $\kappa^2 < 1$ and unstable for $\kappa^2 \geq 1$. For a given $\kappa < 1$, any input state is driven towards this ‘stable’ thermal state under repeated use of the channel, and the thermal parameter of this attracting fixed state is $a_0 = (1 + \kappa^2)(1 - \kappa^2)^{-1}$. This is illustrated in Fig. 1. We may summarize as follows:

**Theorem 4** For the phase conjugation channel $D(\kappa)$, $\kappa < 1$, there is a unique thermal state $\rho_{th}(a_0)$ with parameter $a_0 = (1 + \kappa^2)(1 - \kappa^2)^{-1}$ which is left invariant by the channel. All other states are driven towards this thermal state under repeated use of the channel. Channels $D(\kappa)$ for which $\kappa \geq 1$ have no fixed points.

As a simple illustration of the action of the phase conjugation channel, assume the input to be a Fock state. By Eq. (3.17), we have

$$|n\rangle\langle n| \to \sum_{\ell=0}^{\infty} T_\ell(\kappa)|n\rangle\langle n|T_\ell(\kappa)^\dagger$$

$$= (1 + \kappa^2)^{-1}\sum_{\ell=n}^{\infty} \ell C_n(1 + \kappa^2)^{-n}(1 + \kappa^{-2})^{-\ell-n}|\ell - n\rangle\langle \ell - n|. \quad (3.36)$$
Setting \( \ell - n = j \), we have
\[
\sum_{\ell=0}^{\infty} T_\ell(\kappa)|n\rangle\langle T_\ell(\kappa)| = (1 + \kappa^2)^{-1} \sum_{j=0}^{\infty} n^+ j C_j (1 + \kappa^2)^{-\kappa} (1 + \kappa^{-2})^{-\kappa} |j\rangle\langle j|.
\] (3.37)

That is, a Fock state is taken to a convex sum of all Fock states. As an immediate consequence we have: if a density operator \( \rho \) has \( T_\ell(\kappa)|n\rangle\langle n| = 0 \) for some \( n \), then \( \rho \) cannot remain invariant under the action of the channel. This is true, in particular, of any state \( \rho \) which is in the support of a finite number of Fock states.

As another simple example, consider the phase averaged coherent state given by
\[
\rho = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} |j\rangle\langle j|.
\] (3.38)

This mixed state has Poissonian photon number distribution [PND]. Under the action of the channel, we have
\[
\rho \rightarrow \rho' = e^{-\lambda} (1 + \kappa^2)^{-1} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \sum_{n=0}^{\infty} n^+ j C_j (1 + \kappa^2)^{-\kappa} (1 + \kappa^{-2})^{-\kappa} |n\rangle\langle n|.
\] (3.39)

We cannot solve consistently for parameters \( \kappa \) and \( \lambda \) such that the output PND is also Poissonian. In view of Theorem 3, the output is a classical state, and hence is necessarily super-Poissonian [51]. To summarize, under the action of the channel \( D(\kappa) \) a thermal PND is taken to a thermal PND, whereas a Poissonian PND is taken to a super-Poissonian PND, all moving towards the fixed thermal PND if \( \kappa < 1 \).

IV. Entanglement breaking property

It is known that the phase conjugating channel is entanglement breaking [36, 37]. It is also known that every entanglement breaking channel has a description in terms of rank one Kraus operators [35]. We demonstrate these aspects using our Kraus operators \( \{T_\ell(\kappa)\} \).

The Kraus operators \( T_\ell(\kappa) \) presented in (3.11) are not of unit rank; indeed, rank \( T_\ell(\kappa) = \ell + 1, \ell = 0, 1, 2, \ldots \). We noted immediately following (3.11) that \( T_\ell(\kappa) \) are trace-orthogonal. In the generic case, trace-orthogonality requirement would render the Kraus operators unique, but this is not true with the present situation. The reason is that all these trace-orthogonal \( T_\ell(\kappa) \)'s have the same Frobenius norm: \( \text{tr} \left( T_\ell(\kappa)T_\ell(\kappa)\right)^\dagger \) is \( (1 + \kappa^2)^{-1} \), independent of \( \ell \). Thus the set \( \{T_\ell'\} \) defined through \( T_\ell'(\kappa) = \sum_\ell U_{\ell\ell'} T_\ell(\kappa) \), for any unitary matrix \( \{U_{\ell\ell'}\} \) will be a set of trace-orthogonal Kraus operators describing the same channel as the original trace-orthogonal set \( \{T_\ell(\kappa)\} \).

More generally, and independent of trace-orthogonality, the map \( \rho \rightarrow \rho' = \sum_\alpha T_\alpha(\kappa) \rho T_\alpha(\kappa) \) describes the same channel as \( \rho \rightarrow \rho' = \sum_\ell T_\ell(\kappa) \rho T_\ell(\kappa) \) if the matrix \( U \) connecting the sets \( \{T_\ell(\kappa)\} \) and \( \{T_\ell'(\kappa)\} \) is an isometry [34, 52]:
\[
T_\alpha'(\kappa) = \sum_\alpha U_{\ell\alpha} T_\ell(\kappa), \quad \sum_\alpha U_{\ell\alpha} U^*_{\ell'\alpha} = \delta_{\ell\ell'}
\Rightarrow \sum_\ell T_\ell(\kappa) \rho T_\ell(\kappa) = \sum_\alpha T_\alpha'(\kappa) \rho T_\alpha'(\kappa).
\] (3.40)

If the index set \( \alpha \) is continuous, as in the case below, then \( \sum_\alpha \) is to be understood, of course, as an integral. Now, the matrix elements between coherent states \(|\alpha\rangle\rangle \) and Fock states \(|k\rangle\rangle \) define such an isometry
\[
U_{\ell\alpha} = \langle \ell|\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right] \frac{\alpha^\ell}{\sqrt{\ell!}}.
\] (3.41)
The resulting new Kraus operators $T'_\alpha(\kappa)$ are

$$T'_\alpha(\kappa) = e^{-i|\alpha|^2} \sum_{\ell=0}^{\infty} \frac{\alpha^{\ell}}{\sqrt{\ell!}} T_\ell(\kappa)$$

$$= e^{-i|\alpha|^2} \sum_{\ell=0}^{\infty} \frac{\alpha^{\ell}}{\sqrt{\ell!}} (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^{\ell} \sqrt{\ell C_n} (\sqrt{1 + \kappa^2} - n) \langle \ell - n | n \rangle$$

$$= e^{-i|\alpha|^2} \sum_{\ell=0}^{\infty} (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^{\ell} [[(\sqrt{1 + \kappa^2})^{-1} \alpha]^{n} [((\sqrt{1 + \kappa^2})^{-1} \alpha]^{\ell-n} | \ell - n \rangle | n \rangle$$

$$= \frac{1}{\sqrt{1 + \kappa^2}} \langle \alpha / \sqrt{1 + \kappa^2} | \alpha^* / \sqrt{1 + \kappa^2} \rangle, \forall \alpha \in \mathcal{C}. \quad (3.42)$$

It is manifest that rank $T'_\alpha(\kappa) = 1$ for all $\alpha \in \mathcal{C}$, the complex plane, showing that the phase conjugation channel is indeed entanglement breaking. However $\{T'_\alpha(\kappa)\}$ are not trace-orthogonal even though $\{T_\ell(\kappa)\}$ from which the former are constructed were trace-orthogonal. This is due to the fact that the isometry $U$ defined in (3.41) is not an unitary, which in turn is a consequence of the overcompleteness of the coherent states.

This brings us to another aspect of $\mathcal{D}(\kappa)$. In terms of these new Kraus operators the phase conjugation channel $\mathcal{D}(\kappa)$ reads

$$\rho \rightarrow \rho' = \pi^{-1} \int d^2 \alpha T'_\alpha(\kappa) \rho T'^{-1}_\alpha(\kappa)$$

$$= \pi^{-1} (1 + \kappa^2)^{-1} \int d^2 \alpha \mathcal{Q}((\sqrt{1 + \kappa^2})^{-1} \alpha^*) | \alpha / \sqrt{1 + \kappa^2} \rangle | \alpha / \sqrt{1 + \kappa^2} \rangle. \quad (3.43)$$

Thus the diagonal weight function of the output state of the channel is the $Q$-distribution of the input state $\rho$: $\phi_{\text{out}} = \kappa^{-2} \mathcal{Q}_{\text{in}}(\kappa^{-1} \alpha^*)$. We may combine this result with the earlier one on rank one Kraus operators to state

**Theorem 5** The diagonal weight of the output of the quantum-limited phase conjugation channel is essentially the $Q$-distribution of the input state. The channel $\mathcal{D}(\kappa)$ is not only classicality breaking, but also entanglement breaking.

The diagonal weight of the output state at $\alpha$ is the $Q$-distribution of the input state evaluated at $\kappa^{-1} \alpha^*$. Since $\mathcal{Q}(\alpha) \geq 0$ for all $\alpha$ and for any $\rho$, the channel is nonclassicality breaking. The intimate relationship between this result and the earlier one on nonclassicality breaking may be noted. While the former followed directly from the behaviour of the characteristic function, the present one required consideration of the Kraus operators.

### IV. BEAMSLIPPER/ATTENUATOR CHANNEL $\mathcal{C}_1(\kappa)$, $0 < \kappa < 1$

The two-mode unitary operator corresponding to the beamsplitter channel induces the following symplectic transformation on the quadrature operators of the bipartite phase space $[17]$:

$$S = \begin{pmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (4.1)$$

Note that $S$ is a direct sum of identical two-dimensional rotations: as in the case of $\mathcal{D}(\kappa)$, the position and momentum operators are not mixed by this transformation. The position variables transform as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (4.2)$$

and, consequently, the momentum variables as

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \quad (4.3)$$
It is evident from $S$ that the parameter $\kappa$ in $C_1(\kappa)$ is related to $\theta$ through $\cos \theta = \kappa$, $\sin \theta = \sqrt{1-\kappa^2}$. The function $F(z_1, z_2, \eta_1, \eta_2)$ of (2.11) for the present case is given by

$$F(z_1, z_2, \eta_1, \eta_2) = \exp \left[ \eta_2 (\sqrt{1-\kappa^2} z_1 + \kappa z_2) + \eta_1 (\kappa z_1 - \sqrt{1-\kappa^2} z_2) \right]. \quad (4.4)$$

As in the previous case of $D(\kappa)$, the differentiation on $F(z_1, z_2, \eta_1, \eta_2)$ can be performed in a straightforward manner to obtain the matrix elements of the unitary operator $D$,[53] leading to

$$C_{n_1n_2}^{m_1m_2} = \frac{1}{\sqrt{n_1n_2m_1m_2}} \sum_{r=0}^{n_1} \sum_{j=0}^{n_2} n_1 C_r n_2 C_j (-1)^{n_2-j} \kappa^{n_1-r+j} (\sqrt{1-\kappa^2})^{r+n_2-j} \times m_1m_2 \delta_{m_2,r+j} \delta_{m_1,n_1+n_2-r-j}. \quad (4.5)$$

Now, to obtain the Kraus operators from these matrix elements we set, as in the case of $D(\kappa)$, $n_2 = 0$ and $m_2 = \ell$. Setting $n_2 = 0$ $\Rightarrow$ $j = 0$, and we have

$$B_\ell(\kappa) = \sum_{m=0}^{\infty} \sqrt{m+\ell} C_\ell (\sqrt{1-\kappa^2})^\ell \kappa^m |m\rangle \langle m| + \ell|, \quad \ell = 0, 1, 2, \ldots \quad (4.6)$$

as the Kraus operators of the beamsplitter or quantum-limited attenuator channel. It is easy to see that the Kraus operators are real and pairwise trace-orthogonal, as in the case of $D(\kappa)$.

I. Properties of the Kraus operators

We now explore the properties of the Kraus operators presented in Eq. (4.6), connecting the action of the channel on the Fock basis to that on the characteristic function. We firstly exhibit the fact that the beamsplitter channel simply affects a scaling on the weight function of the diagonal representation. We show that vacuum is the only fixed point of the channel. It is further shown that in any set of Kraus operators describing the channel $C_1(\kappa)$, there will be not even one operator of unit rank, thus demonstrating that $C_1(\kappa)$ is not an entanglement breaking channel. Finally, the manifestation of the semigroup structure of the family of channels $C_1(\kappa)$, $0 < \kappa < 1$ is brought out in the Kraus representation, as also an associated Zeno-like effect.

Recall that the beamsplitter channel induces the following transformation on the characteristic function $[17]$:

$$\chi_W(\xi) \rightarrow \chi_W(\kappa \xi) \exp[-(1-\kappa^2)|\xi|^2/2] = \chi_W(\kappa \xi) \exp[\kappa^2|\xi|^2/2] \exp[-|\xi|^2/2]. \quad (4.7)$$

Thus the normal ordered characteristic function $\chi_N(\xi)$ transforms as

$$\chi_N(\xi) = \chi_W(\xi) \exp(|\xi|^2/2) \rightarrow \chi_N'(\xi) = \chi_N(\kappa \xi). \quad (4.8)$$

Since $\chi_N(\xi)$ and the diagonal weight $\phi(\alpha)$ form a Fourier transform pair, it is immediately seen that $\phi(\alpha)$ gets simply scaled under the action of the $C_1(\kappa)$ channel: $\phi(\alpha) \rightarrow \phi'(\alpha) = \kappa^{-2}\phi(\kappa^{-1}\alpha)$ [54].

It is instructive to bring out this fact from the perspective of the Kraus operators. Since every state $\rho$ can be expressed through a diagonal ‘weight’ $\phi(\alpha)$ as $[17]$

$$\rho = \pi^{-1} \int d^2 \alpha \phi(\alpha) |\alpha\rangle \langle \alpha|, \quad (4.9)$$

to exhibit the action of the channel on an arbitrary state it is sufficient to consider its action on a generic coherent state. We have

$$|\alpha\rangle \langle \alpha| \rightarrow \sum_{\ell=0}^{\infty} B_\ell(\kappa) |\alpha\rangle \langle \alpha| B_\ell^\dagger(\kappa)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1-\kappa^2)|\alpha|^2)^\ell}{\ell!} (\kappa \alpha^*)^m (\kappa \alpha)^n \frac{e^{-|\alpha|^2}}{\sqrt{m!n!}} |m\rangle \langle n|, \quad (4.10)$$
where we used the fact that the operator

\[ |m\rangle\langle n| \rightarrow \sum_{\ell=0}^{\infty} B_\ell(\kappa)|m\rangle\langle n|B_\ell^\dagger(\kappa) \]

\[ = \sum_{\ell=0}^{\min\{m,n\}} \sqrt{mC_\ell} C_\ell (1 - \kappa^2)\ell \kappa^{m+n-2\ell}|m - \ell\rangle\langle n - \ell|. \]  

(4.11)

Carrying out the summations in Eq. (4.10), one finds \[56\]

\[ \sum_{\ell=0}^{\infty} B_\ell(\kappa)|\alpha\rangle\langle \alpha|B_\ell^\dagger(\kappa) = |\kappa\alpha\rangle\langle \kappa\alpha|. \]  

(4.12)

With this the action of the channel \( C_1(\kappa) \) reads

\[ \rho \rightarrow \rho' = \pi^{-1} \int d^2\alpha \phi(\alpha)|\kappa\alpha\rangle\langle \kappa\alpha| \]

\[ = \pi^{-1}\kappa^{-2} \int d^2\alpha \phi(\kappa^{-1}\alpha)|\alpha\rangle\langle \alpha|, \]  

(4.13)

which means

\[ C_1(\kappa) : \phi(\alpha) \rightarrow \kappa^{-2} \phi(\kappa^{-1}\alpha). \]  

(4.14)

We have thus proved in the Kraus representation

**Theorem 6** The scaling \( \phi_p(\alpha) \rightarrow \phi'_p(\alpha) = \kappa^{-2}\phi_p(\kappa^{-1}\alpha) \), \( 0 < \kappa < 1 \), is a completely positive map whose Kraus decomposition is given by \{\( B_\ell(\kappa) \)\} of (4.10).

As an immediate consequence we have

**Corollary 1** The beamsplitter channel cannot generate or destroy nonclassicality.

**Proof:** By definition a state is classical if and only if its diagonal weight function \( \phi(\alpha) \) is pointwise nonnegative everywhere in the complex plane \[47\]. Since a pointwise positive function goes to a pointwise positive function under the above scaling transformation, it follows that a classical state (and a classical state alone) is taken to a classical state under the action of the (quantum-limited) attenuator channel.

**II. Fixed Points**

We now examine, using cumulants, if there are any fixed points for this channel. In view of Eq. (4.17), the cumulant generating function transforms as follows under the action of this channel:

\[ C_1(\kappa) : \Gamma(\xi) \rightarrow \Gamma'(\xi) = \log [\chi_W(\kappa\xi)] - \frac{1}{2}(1 - \kappa^2)|\xi|^2 \]  

(4.15)

As in the previous case of \( D(\kappa) \), the cumulants of order \( > 2 \) of \( \chi'_W(\xi) \) are

\[ \gamma'_{m_1m_2} = (\kappa)^{m_1+m_2} \left( \frac{\partial}{\partial \ell(t_1)} \right)^{m_1} \left( \frac{\partial}{\partial \ell(t_2)} \right)^{m_2} \log [\chi(t)], \quad (\kappa\xi = t) \]

\[ = (\kappa)^{m_1+m_2}\gamma_{m_1m_2}, \]  

(4.16)

where \( \gamma'_{m_1m_2} \) and \( \gamma_{m_1m_2} \) are the cumulants of respectively the output and the input states. Thus, with the exception of the trivial case \( \kappa = 1 \), action of the channel \( C_1(\kappa) \) attains the higher order cumulants. Therefore any state that is preserved is necessarily Gaussian. Since the additional term \( \frac{1}{2}(1 - \kappa^2)|\xi|^2 \) in Eq. (4.15) does not involve a cross term between the real and imaginary parts of \( \xi \), in looking for fixed points it is sufficient to consider the action only on thermal states. Given an input thermal state \( \rho_\text{th} \) as in (3.30) the output, in view of Eq. (4.11), is

\[ \rho_{\text{out}} = \frac{(1 - x)}{1 - (1 - \kappa^2)x} \sum_{n=0}^{\infty} \left( \frac{\kappa^2x}{1 - (1 - \kappa^2)x} \right)^n |n\rangle\langle n|, \]  

(4.17)
where we have used the identity in Eq. \(\text{(3.32)}\). Since \(\rho_{\text{out}}\) is a thermal state, comparing Eqs. \(\text{(3.30)}\) and \(\text{(4.17)}\) we have this transformation law for the thermal parameter:

\[
C_1(\kappa) : x \rightarrow \frac{\kappa^2 x}{1 - (1 - \kappa^2)x} \quad \text{or} \quad a_0 \rightarrow \kappa^2 a_0 + (1 - \kappa^2).
\]

(4.18)

This means that the output thermal state is always strictly ‘cooler’ than the input thermal state. Thus any thermal state is driven towards the ground state under repeated use of the channel. Indeed, an arbitrary state is driven towards the ground state. We thus have

**Theorem 7** For the action of the beamsplitter channel \(C_1(\kappa)\), the ground state is the only fixed point.

As an illustration of the action of \(C_1(\kappa)\), consider the case of a Fock state as the input. By Eq. \(\text{(4.11)}\) we find

\[
\sum_{\ell} B_{\ell}(\kappa) |n\rangle \langle n| B_{\ell}^\dagger(\kappa) = \sum_{\ell=0}^{n} nC_{\ell}(1 - \kappa^2)\ell \kappa^{2n-2\ell} |n-\ell\rangle \langle n-\ell|.
\]

(4.19)

That is, the beamsplitter channel takes a Fock state \(|n\rangle \langle n|\) to a convex sum of all Fock states with photon number less than or equal to \(n\). It is thus clear that any input state which is diagonal in the Fock basis is taken to a Fock diagonal state at the output. We also note that for an arbitrary input \(\rho\), a Fock diagonal entry \(|m\rangle \langle m|\) at the output gets contribution only from the Fock diagonal entries \(|n\rangle \langle n|\) of the input with \(n \geq m\). Putting these facts together, it is easy to see that no state which is in the support of a finite number of Fock states is preserved under the beamsplitter channel, for the strength of the highest Fock state strictly decreases.

As a second example, consider the phase averaged coherent state \(\text{(3.38)}\) as the input. By Eq. \(\text{(4.12)}\), we have a phase averaged coherent state at the output: a Poissonian PND at the input is taken to another Poissonian PND at the output. Unlike the phase conjugation channel, both a thermal as well as a Poissonian PND at the input are taken at the output to respectively a thermal and Poissonian PND of strictly decreasing mean photon number. The fixed point should therefore be a Poissonian or thermal state of vanishing mean photon number; this is the vacuum state.

### III. The issue of Entanglement breaking

It is known that the beamsplitter channel is not entanglement breaking \(\text{(3.30)}\). It should thus be possible, as it is obligatory, to demonstrate that this channel cannot be represented using a set of rank one Kraus operators. We begin by noting that in the limiting case \(\kappa = 0\), all our Kraus operators \(B_{\ell}(0)\) are of rank one. Indeed, \((B_{\ell}(0))_{mn} = \delta_{m0}\delta_{n\ell}\). This singular limit corresponds to the quantum-limited \(A_1\) channel which is known to be entanglement breaking. We consider therefore the nontrivial case \(\kappa \neq 0\). It is manifestly clear that rank \(B_{\ell}(\kappa) = \infty\) for all \(\ell\) (for \(\kappa \neq 0\)). If we represent this channel using another set of Kraus operators \(\{B'_{\ell}(\kappa)\}\), then these new operators should necessarily be in the support of the set of operators \(\{B_{\ell}(\kappa)\}\). Thus a necessary condition that one is able to represent the channel \(\{B_{\ell}(\kappa)\}\) using rank one Kraus operators is that there be (sufficient number of) rank one operators in the support of \(\{B_{\ell}(\kappa)\}\). It turns out that there is not even one rank one operator in this support. Indeed, a much stronger result is true.

**Theorem 8 :** There exists no finite rank operator in the support of the set \(\{B_{\ell}(\kappa)\}\), \(\kappa \neq 0\).

Proof follows immediately from the structure of the \(B_{\ell}(\kappa)\)'s: \(B_0(\kappa)\) is diagonal, and the \(mn\)th entry of \(B_{\ell}(\kappa)\) is nonzero iff \(n = m + \ell\). Any matrix in the linear span of \(\{B_{\ell}(\kappa)\}\) is of the form \(M = \sum_{\ell} c_{\ell} B_{\ell}(\kappa)\), and is upper diagonal. Let \(N\) be the smallest \(\ell\) for which the \(c\)-number coefficient \(c_\ell \neq 0\). Let \(\tilde{M}\) be the matrix obtained from the upper-diagonal \(M\) by deleting the first \(N\) columns. Clearly, rank \(\tilde{M} = \text{rank} M\). Further, the diagonal entries of the upper triangular \(\tilde{M}\) are all nonzero, being the nonzero entries of \(B_N(\kappa)\). Now, the rank of an upper triangular matrix is not less than that of its diagonal part. Thus, rank \(\tilde{M}\) is not less than rank \(B_N(\kappa) = \infty\), thus completing the proof.
IV. Semigroup property

It is clear from (4.17) that successive actions of two beamsplitter channels with parameter values $\kappa_1, \kappa_2$ is a single beamsplitter channel whose parameter $\kappa$ equals the product $\kappa_1 \kappa_2$ of the individual channel parameters:

\[
C_1(\kappa_1) : \chi_W(\xi) \rightarrow \chi'_W(\xi) = \chi_W(\kappa_1 \xi) \exp\left[-(1 - \kappa_1^2)|\xi|^2/2\right], \\
C_1(\kappa_2) : \chi'_W(\xi) \rightarrow \chi''_W(\xi) = \chi''_W(\kappa_2 \xi) \exp\left[-(1 - \kappa_2^2)|\xi|^2/2\right] = \chi_W(\kappa_1 \kappa_2 \xi) \exp\left[-(1 - \kappa_1^2 \kappa_2^2)|\xi|^2/2\right].
\]  

(4.20)

It is instructive to see how this semigroup property emerges in the Kraus representation. Let $\{B_{\ell_1}(\kappa_1)\}$ and $\{B_{\ell_2}(\kappa_2)\}$ be the Kraus operators of the two channels. The product of two Kraus operators $B_{\ell_1}(\kappa_1), B_{\ell_2}(\kappa_2)$, one from each set, is

\[
B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2) = \sum_{n=0}^{\infty} \sqrt{\ell_1 + \ell_2} C_{\ell_1} \left(\sqrt{1 - \kappa_1^2}\right)^{\ell_1} \left(\sqrt{1 - \kappa_2^2}\right)^{\ell_2} \times \sqrt{m + \ell_1 + \ell_2} C_{\ell_1 + \ell_2} (\kappa_1 \kappa_2)^m m! |m\rangle \langle m + \ell_1 + \ell_2|.
\]

(4.21)

Thus the action of the product channel on the input operator $|r\rangle\langle r + \delta|$ is

\[
\sum_{\ell_1, \ell_2} B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2)|r\rangle\langle r + \delta|B_{\ell_2}(\kappa_2)^\dagger B_{\ell_1}(\kappa_1)^\dagger
\]

\[
= \sum_{\ell_1, \ell_2, m, n} \sqrt{\ell_1 + \ell_2} C_{\ell_1} \left(\sqrt{1 - \kappa_1^2}\right)^{\ell_1} \left(\sqrt{1 - \kappa_2^2}\right)^{\ell_2} \ times \sqrt{m + \ell_1 + \ell_2} C_{\ell_1 + \ell_2} (\kappa_1 \kappa_2)^m m! |m\rangle \langle m + \ell_1 + \ell_2|.
\]

(4.22)

Denoting $\ell_1 + \ell_2 = \ell$, the expression on the RHS becomes

\[
\text{RHS} = \sum_{\ell=0}^{r} \sum_{\ell_1=0}^{\ell} \sum_{m=0}^{\infty} \ell_1^{\ell_1} (1 - \kappa_1^2)^{\ell_1} (1 - \kappa_2^2)^{\ell_2} (\kappa_1 \kappa_2)^{m+n} \times \sqrt{\ell + n} C_{\ell} \delta_{r, \ell + \delta} \delta_{n, \ell} |m\rangle \langle n|.
\]

(4.23)

The sum over $\ell_1$ is the binomial expansion of $[(1 - \kappa_1^2) \kappa_2^2 + (1 - \kappa_2^2)]^\ell = (1 - \kappa_1^2 \kappa_2^2)^\ell$ and, in addition, we have the constraints $m + \ell = r$ and $n + \ell = r + \delta$. With this the expression (4.23) reduces to

\[
\text{RHS} = \sum_{\ell=0}^{r} (1 - \kappa_1^2 \kappa_2^2)^{\ell} \sqrt{\tau} C_{\ell} \delta_{r, \ell + \delta} C_{\ell} (\kappa_1 \kappa_2)^{2\ell - 2\ell + \delta} |r - \ell\rangle \langle r - \ell + \delta|.
\]

(4.24)

Comparing Eqs. (4.11) and (4.24) we find that the expression in (4.24) is precisely the action of a quantum-limited attenuator channel with parameter $\kappa_1 \kappa_2$:

\[
\sum_{\ell_1, \ell_2=0}^{\infty} B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2)|r\rangle\langle r + \delta|B_{\ell_2}(\kappa_2)^\dagger B_{\ell_1}(\kappa_1)^\dagger = \sum_{\ell=0}^{\infty} B_{\ell}(\kappa_1 \kappa_2)|r\rangle\langle r + \delta|B_{\ell}(\kappa_1 \kappa_2).
\]

(4.25)

An identical result can be similarly obtained for the behaviour of $|r + \delta\rangle\langle r|$, and thus we have proved the semigroup property

\[
C_1(\kappa_1) \circ C_1(\kappa_2) = C_1(\kappa_1 \kappa_2).
\]

(4.26)

Remark on interrupted evolution and Zeno-like effect:

As seen from (4.11) the parameter $\kappa$ specifying the channel $C_1(\kappa)$ equals $\cos \theta$, where $\theta$ is a measure of the two-mode rotation effected by the ‘beamsplitter’ coupling the system mode to an ancilla mode assumed to be in the vacuum state initially. The associated two-mode unitary operator $U(\theta) = \exp[-\theta(a^\dagger b - b^\dagger a)]$ may be viewed as effecting
evolution for ‘duration’ $\theta$ under the Hamiltonian $-i(a^\dagger b - b^\dagger a)$. It is clear that attenuation increases monotonically as $\theta$ varies from 0 to $\pi/2$, with total attenuation achieved at $\theta = \pi/2$.

Two-mode evolution for duration $N^{-1}\pi/2$ followed by tracing away of the ancilla mode results in the channel $C_1(\kappa_{N,1})$ where $\kappa_{N,1} \equiv \cos(N^{-1}\pi/2)$. Now suppose that we have interrupted evolution in the sense that this process leading to $C_1(\kappa_{N,1})$ is repeated $\ell$ times. By the semigroup property the net result is a quantum-limited attenuator $C_1(\kappa_{N,\ell})$ where $\kappa_{N,\ell} = (\kappa_{N,1})^\ell = (\cos(N^{-1}\pi/2))^\ell$. The behaviour of the attenuation factor $\kappa_{N,\ell}$ is depicted in Fig. 2. That the effect of interruption is to slow down attenuation is transparent. For large $N$ we have $\kappa_{N,\ell} \approx 1 - \pi \frac{\ell}{4N} \left(\frac{\ell}{N}\right)$ reminiscent of quantum Zeno effect [55].

V. AMPLIFIER CHANNEL $C_2(\kappa), \kappa \geq 1$

The two-mode metaplectic unitary operator describing a single-mode quantum-limited amplifier channel corresponds to the following symplectic transformation on the mode operators [17]:

$$S = \begin{pmatrix} \cosh \nu & 0 & \sinh \nu & 0 \\ 0 & \cosh \nu & 0 & -\sinh \nu \\ \sinh \nu & 0 & \cosh \nu & 0 \\ 0 & -\sinh \nu & 0 & \cosh \nu \end{pmatrix}. \tag{5.1}$$

As in the earlier two cases of $D(\kappa)$ and $C_1(\kappa)$, the position and momentum variables do not mix under the action of $C_2(\kappa)$. The position variables transform as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cosh \nu & -\sinh \nu \\ -\sinh \nu & \cosh \nu \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{5.2}$$
and the momentum variables transform according to \( M^{-1} \). Thus the parameter \( \kappa \) in \( C_2(\kappa) \) is related to the two-mode squeeze parameter \( \nu \) through \( \kappa = \cosh \nu \). The function \( F(z_1, z_2, \eta_1, \eta_2) \) in (2.10) is readily computed to be

\[
F(z_1, z_2, \eta_1, \eta_2) = \kappa^{-1} \exp \left\{ \kappa^{-1}(\eta_1 z_1 + \eta_2 z_2) + (\sqrt{1 - \kappa^{-2}}(\eta_1 \eta_2 - z_1 z_2)) \right\}. \tag{5.3}
\]

As in the earlier cases of \( D(\kappa) \) and \( C_1(\kappa) \), the differentiation on \( F(z_1, z_2, \eta_1, \eta_2) \) can be performed to obtain the matrix elements of the unitary operator corresponding to the symplectic \( S \) in (5.1). We obtain, after some algebra patterned after the earlier two cases,

\[
C_{n_1n_2}^{m_1m_2} = \frac{\kappa^{-1}}{\sqrt{n_1!n_2|m_1m_2|}} n_1!m_2!
\times \sum_{r=0}^{n_2} \sum_{j=0}^{m_1} n_2 C_r^{|m_1|} C_j (\kappa)^{-r}(\sqrt{1 - \kappa^{-2}})^{r+m_1-j}(\kappa^{-1})^{n_2+j-r} \delta_{n_1,r+j} \delta_{m_2,n_2+m_1-r-j}. \tag{5.4}
\]

The Kraus operators are obtained from \( C_{n_1n_2}^{m_1m_2} \) by setting \( n_2 = 0 \), and \( m_2 = \ell \). Setting \( n_2 = 0 \Rightarrow r = 0 \), and we have

\[
A_\ell(\kappa) = \kappa^{-1} \sum_{m=0}^{\infty} \sqrt{m+\ell} C_\ell \left( \sqrt{1 - \kappa^{-2}} \right)^{\ell} (\kappa^{-1})^{m+\ell} \langle m+\ell | m \rangle, \quad \ell = 0, 1, 2, \ldots \tag{5.5}
\]

as the Kraus operators of the quantum-limited amplifier channel \( C_2(\kappa), \kappa > 1 \).

**I. Duality between the attenuator family \( C_1(\cdot) \) and the amplifier family \( C_2(\cdot) \)**

The Kraus operators \( A_\ell(\kappa), \kappa > 1 \) of the amplifier channel \( C_2(\kappa) \) have an interesting dual relationship to the Kraus operators \( B_\ell(\kappa^{-1}), \kappa > 1 \) of the attenuator channel \( C_1(\kappa^{-1}) \). While \( \sum_{\ell=0}^{\infty} A_\ell(\kappa)A_\ell(\kappa) = \mathbb{1}, \kappa > 1 \) and \( \sum_{\ell=0}^{\infty} B_\ell(\kappa)B_\ell(\kappa) = \mathbb{1}, \kappa < 1 \), consistent with the trace-preserving property of \( C_2(\kappa) \) and \( C_1(\kappa) \), we have

\[
\sum_{\ell=0}^{\infty} A_\ell(\kappa)A_\ell(\kappa) = \kappa^{-2} \mathbb{1},
\]

\[
\sum_{\ell=0}^{\infty} B_\ell(\kappa)B_\ell(\kappa) = (\kappa')^{-2} \mathbb{1}. \tag{5.6}
\]

Thus the (trace-preserving) families \( C_1 \) and \( C_2 \) are not unital. But they are ‘almost unital’, for the failure to be unital is by just a scalar factor. This shows that each family \( \{ \kappa A_\ell(\kappa)^\dagger, \kappa > 1 \} \) and the family \( \{ \kappa' B_\ell(\kappa')^\dagger, \kappa' < 1 \} \) too describe trace-preserving CP maps, and we may ask what these ‘new’ channels stand for.

The meaning of these channels may be easily seen by considering the adjoints \( A_\ell(\kappa)^\dagger, \kappa > 1 \) of the Kraus operators of the amplifier channel:

\[
A_\ell(\kappa)^\dagger = \kappa^{-1} \sum_{m=0}^{\infty} \sqrt{m+\ell} C_\ell \left( \sqrt{1 - \kappa^{-2}} \right)^{\ell} \kappa^{-m} \langle m+\ell | m \rangle
= \kappa^{-1} B_\ell(\kappa^{-1}). \tag{5.7}
\]

Thus \( \{ \kappa A_\ell(\kappa)^\dagger \}, \kappa > 1 \) are the Kraus operators of the beamsplitter channel \( C_1(\kappa') \) with \( \kappa' = \kappa^{-1} < 1 \). Similarly it can be seen that \( \{ \kappa' B_\ell(\kappa')^\dagger \}, \kappa' < 1 \) represents the amplifier channel \( C_2(\kappa) \) with \( \kappa = (\kappa')^{-1} > 1 \). Thus we have

**Theorem 9** The amplifier family \( C_2(\kappa) \) and the attenuator family \( C_1(\kappa^{-1}) \), \( \kappa > 1 \) are mutually dual: their Kraus operators are connected through the adjoint operation.

**II. Properties of the Kraus operators**

We now explore the properties of the Kraus operators of \( C_2(\kappa) \) exhibited in Eq. (5.5), relating the action of these operators in the Fock basis to the defining or expected transformation property of the quasiprobability in phase
Thus, under the action of the channel $C_2(\kappa)$ the Weyl-ordered characteristic function transforms as follows, and this may be identified with the very definition of the channel:

$$\chi_W(\xi) \to \chi_W(\xi) = \chi_W(\kappa \xi) \exp\left[-(\kappa^2 - 1)|\xi|^2/2\right]. \quad (5.8)$$

Given a Weyl-ordered characteristic function $\chi_W(\xi)$, the corresponding antinormal ordered characteristic function corresponding to the $Q$-distribution is

$$\chi_A(\xi) = \chi_W(\xi) \exp\left[-|\xi|^2/2\right]. \quad (5.9)$$

Therefore the channel action Eq. (5.8), written in terms of $\chi_A(\xi)$, reads

$$\chi_A(\xi) \to \chi_A'(\xi) = \chi_A(\kappa \xi). \quad \tag{5.10}$$

That is, $\chi_A(\xi)$ simply scales under the action of the amplifier channel, a fact that should be profitably compared with the scaling behaviour for the attenuator channel. Since $\chi_A(\xi)$ and the $Q$-function form a Fourier transform pair, the action of the amplifier channel is fully described as a scaling transformation of the $Q$-function: $Q(\alpha) \to Q'(\alpha) = \kappa^{-2}Q(\kappa^{-1}\alpha)$, $\kappa > 1$.

It is instructive to see in some detail how our Kraus operators $A_\ell(\kappa)$ bring out this behaviour. Given a state

$$\rho = \sum_{n,m=0}^{\infty} |n\rangle\langle n| \rho |m\rangle\langle m| = \sum_{n,m=0}^{\infty} \rho_{nm} |n\rangle\langle m|, \quad (5.11)$$

its corresponding $Q$-function is

$$Q_\rho(\alpha) = \langle \alpha | \rho | \alpha \rangle = \exp\left[-|\alpha|^2\right] \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^n (\alpha)^m}{\sqrt{n!} \sqrt{m!}} \rho_{nm}. \quad (5.12)$$

To see the action of the linear map $C_2(\kappa)$ on an arbitrary $\rho$, it is sufficient to exhibit its action on the operators $|n\rangle\langle m|$, for all $n, m \geq 0$. We have

$$|n\rangle\langle m| \to \sum_{\ell=0}^{\infty} A_\ell(\kappa) |n\rangle\langle m| A_\ell^\dagger(\kappa) = \kappa^{-2} \left(\frac{-n+m}{\sqrt{n!} \sqrt{m!}} \sum_{\ell=0}^{\infty} \frac{(1 - \kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)! \sqrt{(m+\ell)!}} | n + \ell \rangle \langle m + \ell | \right). \quad (5.13)$$

Thus, under the action of the channel $C_2(\kappa)$, $\rho$ goes to

$$\rho' = \kappa^{-2} \sum_{n,m=0}^{\infty} \rho_{nm} \frac{\kappa^{-(n+m)}}{\sqrt{n!} \sqrt{m!}} \sum_{\ell=0}^{\infty} \frac{(1 - \kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)! \sqrt{(m+\ell)!}} | n + \ell \rangle \langle m + \ell |. \quad (5.14)$$

The $Q$-function of the resultant or output state $\rho'$ is

$$\langle \alpha | \rho' | \alpha \rangle = \kappa^{-2} \exp\left[-|\alpha|^2\right] \sum_{n,m=0}^{\infty} \rho_{nm} \frac{\kappa^{-(n+m)}}{\sqrt{n!} \sqrt{m!}} (\alpha^*)^n (\alpha)^m \left(\sum_{\ell=0}^{\infty} \frac{(1 - \kappa^{-2})^\ell}{\ell!} |\alpha|^2\right)$$

$$= \kappa^{-2} \exp\left[-|\kappa^{-1}\alpha|^2\right] \sum_{n,m=0}^{\infty} (\kappa^{-1} \alpha^*)^n (\kappa^{-1} \alpha)^m \frac{\rho_{nm}}{\sqrt{n!} \sqrt{m!}}, \quad (5.15)$$

We thus conclude
Theorem 10  The scaling \( Q_\rho(\alpha) \to Q_\rho'(\alpha) = \kappa^{-2}Q_\rho(\kappa^{-1}\alpha), 0 < \kappa^{-1} < 1, \) is a completely positive map whose Kraus decomposition is given by \( \{ A_k(\kappa) \} \).

This result may be compared with Theorem 6 for the \( C_k(\cdot) \) family of channels.

The amplifier channel has the following property in respect of nonclassicality of the output states:

**Corollary 2**  The amplifier channel cannot generate nonclassicality.

**Proof:**  By Eq. (5.8), the normal ordered characteristic function transforms as follows

\[
C_2(\kappa) : \chi_N(\xi) \to \chi_N'(\xi) = \chi_W(\kappa\xi) \exp[-(\kappa^2 - 2)|\xi|^2/2].
\]  \hspace{1cm} (5.16)

This may be rewritten in the suggestive form

\[
\chi_N(\xi) \to \chi_N'(\xi) = \chi_N(\kappa\xi) \exp[-(\kappa^2 - 1)|\xi|^2].
\]  \hspace{1cm} (5.17)

Fourier transforming, we see that the diagonal weight \( \phi(\alpha) \) of the output state is the convolution of the (scaled) input diagonal weight with a Gaussian (corresponding to the last factor), and hence it is pointwise nonnegative whenever the input diagonal weight \( \phi(\alpha) \) is pointwise nonnegative.

**Remark:**  We are not claiming that the amplifier channel cannot destroy nonclassicality [compare the structure of Corollary 2 with that of Corollary 1 following Theorem 6]. Indeed, it is easy to show that nonclassicality of every Gaussian state will be destroyed by any \( C_2(\kappa) \) with \( \kappa \geq \sqrt{2} \). \cite{41, 42, 57}. It is also easy to show that there are states whose nonclassicality will survive \( C_2(\kappa) \) even for arbitrarily large \( \kappa \). \cite{41, 42, 57}. To see this, note first of all, that any state \( \rho \) whose \( Q \)-function \( Q(\alpha) = \langle \alpha | \rho | \alpha \rangle \) vanishes for some \( \alpha \) is necessarily nonclassical. The assertion simply follows from the fact that under the scaling \( Q(\alpha) \to \kappa^{-2}Q(\kappa^{-1}\alpha) \) a zero \( \alpha_0 \) of \( Q(\alpha) \) goes to a zero at \( \kappa\alpha_0 \).

### III. Fixed points

By Eq. (5.8) we have, under the action of the channel \( C_2(\kappa) \), the following behaviour for the moment generating function:

\[
\Gamma(\xi) \to \Gamma'(\xi) = \log [\chi_W(\kappa\xi)] - \frac{1}{2}|\xi|^2(\kappa^2 - 1).
\]  \hspace{1cm} (5.18)

The cumulants of order \( > 2 \) of the output characteristic function are

\[
\gamma_{m_1m_2}' = (\kappa)^{m_1+m_2} \left( \frac{\partial}{\partial(it_1)} \right)^{m_1} \left( \frac{\partial}{\partial(it_2)} \right)^{m_2} \log [\chi_W(t)], \quad (\kappa\xi = t)
\]  \hspace{1cm} (5.19)

where \( \gamma_{m_1m_2} \) are the cumulants of the input state. Thus, for any non-Gaussian input state, the higher order cumulants grow monotonically with repeated use of the channel. Thus, leaving out the case \( \kappa = 1 \) which corresponds to the identity channel, there is no non-Gaussian state that is preserved. To see if there is any fixed point among the Gaussian states, it is sufficient to consider only thermal states as input. \cite{30}. By Eq. (5.18), the output state is

\[
\rho_{\text{out}} = (1-x)\kappa^{-2} \sum_{n=0}^{\infty} \kappa^{-2n}(\kappa^2 - 1 + x)^n |n\rangle \langle n|.
\]  \hspace{1cm} (5.20)

where we used the binomial expansion to perform a sum. Comparing Eqs. (5.19) and (5.20), we have

\[
C_2(\kappa) : x \to x' = \kappa^{-2}(\kappa^2 - 1 + x) \text{ or } a_0 \to a_0' = \kappa^2a_0 + \kappa^{-2} - 1.
\]  \hspace{1cm} (5.21)

It is clear that the output thermal parameter \( a_0' \) is strictly greater than \( a_0 \). Hence there is no thermal state that is left invariant under the action of the amplifier channel. Collecting these facts together, we conclude

**Theorem 11**  There exists no state which is a fixed point of the quantum-limited amplifier channel.
As a simple illustration, consider the action of the channel on a Fock state. We have by Eq. (5.13)

\[ C_2(\kappa) : |n\rangle \langle n| \rightarrow \kappa^{-2} \sum_{\ell=0}^{\infty} n^{+\ell} C_\ell (1 - \kappa^{-2})^{\ell} |n + \ell\rangle \langle n + \ell|. \] (5.22)

Thus an input Fock state |n\rangle \langle n| is taken to a convex combination of all Fock states with photon number greater than or equal to n. It may be noted that this behaviour is complementary to that of the beamsplitter channel where the output was a convex combination of all Fock states up to |n\rangle \langle n|. And we find, analogous to the beamsplitter case, that any state which is in the support of a finite number of Fock states is not preserved by the channel.

As a second example, consider the phase averaged coherent state \[|\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^n |n\rangle\] as the input. By Eq. (5.22) the output is

\[ \rho_{\text{out}} = \sum_{j,\ell} j^{+\ell} C_{j\ell} \chi_j e^{-\lambda} \frac{\lambda^{j+\ell}}{j!} (1 - \kappa^{-2})^{\ell} |j + \ell\rangle \langle j + \ell|. \] (5.23)

It is clear that we cannot solve consistently for parameters \(\kappa > 0\) and \(\lambda\) such that the output PND is also Poissonian. In view of Corollary 2 we may conclude that an input Poissonian PND generically results in a super-Poissonian PND.

**Remark on entanglement breaking** : It is well known that the quantum-limited amplifier channel is not entanglement breaking [35]. It may be pointed out in passing that this fact follows also from the structure of our Kraus operators \(\{A_\ell(\kappa)\}\). Since these operators coincide with the transpose of the beamsplitter channel Kraus operators \(\{B_\ell(\kappa^{-1})\}\), apart from a \(\ell\)-independent multiplicative factor, there exists no finite rank operator in the support of the set of operators \(\{A_\ell(\kappa)\}\). In particular, there are no rank one operators in the support of \(\{A_\ell(\kappa)\}\). Hence, \(C_2(\kappa)\) is not an entanglement breaking channel.

### IV. Semigroup property

It follows from the very definition of the amplifier channel that the composition of two quantum-limited amplifier channels with parameters \(\kappa_1\) and \(\kappa_2\) is also such an amplifier channel with parameter \(\kappa_1 \kappa_2 > 1\):

\[ C_2(\kappa_2) \circ C_2(\kappa_1) : \chi W(\xi) \rightarrow \chi W(\kappa_1 \kappa_2 \xi) \exp \left[ -(\kappa_1^2 \kappa_2^2 - 1) |\xi|^2 / 2 \right]. \] (5.24)

That is,

\[ C_2(\kappa_2) \circ C_2(\kappa_1) = C_2(\kappa_1 \kappa_2) = C_2(\kappa_1) \circ C_2(\kappa_2). \] (5.25)

It will be instructive to examine how this fact emerges from the structure of the Kraus operators. Let the set \(\{A_\ell(\kappa_1)\}\) be the Kraus operators of the first amplifier and let \(\{A_\ell(\kappa_2)\}\) be that of the second. Then the product of a pair of Kraus operators, one from each set, is

\[ A_{\ell_1}(\kappa_1) A_{\ell_2}(\kappa_2) = (\kappa_1 \kappa_2)^{-1} \sqrt{\ell_1 + \ell_2} C_{\ell_1} \sum_{n=0}^{\infty} \sqrt{n + \ell_1 + \ell_2} C_{\ell_1 + \ell_2} \left( \sqrt{1 - \kappa_1^{-2}} \right)^{\ell_1} \left( \sqrt{1 - \kappa_2^{-2}} \right)^{\ell_2} \times (\kappa_1 \kappa_2)^{-1} |n + \ell_1 + \ell_2\rangle \langle n|. \] (5.26)

Thus, under the successive action of these two amplifier channels the operator \(|j\rangle \langle j + \delta|\) goes to

\[ \sum_{\ell_1, \ell_2} A_{\ell_1}(\kappa_1) A_{\ell_2}(\kappa_2) |j\rangle \langle j + \delta| A_{\ell_2}(\kappa_2) \dagger A_{\ell_1}(\kappa_1) \dagger = (\kappa_1 \kappa_2)^{-2} \sum_{\ell_1, \ell_2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \ell_1 + \ell_2 C_{\ell_1} (1 - \kappa_1^{-2})^{\ell_1} (1 - \kappa_2^{-2})^{\ell_2} (\kappa_1 \kappa_2)^{-(n+m)} \kappa_1^{-2\ell_2} \times \sqrt{n + \ell_1 + \ell_2} m^\ell_1 + \ell_2 C_{\ell_1 + \ell_2} |n + \ell_1 + \ell_2\rangle \langle m + \ell_1 + \ell_2|. \] (5.27)

Denoting \(\ell_1 + \ell_2 = \ell\), the right hand side of the above expression reduces to

\[ (\kappa_1 \kappa_2)^{-2} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \ell C_{\ell} (1 - \kappa_1^{-2})^{\ell} (\kappa_1^{-2}(1 - \kappa_2^{-2}))^{(\ell - \ell_1)} (\kappa_1 \kappa_2)^{-(n+m)} \times \sqrt{n + \ell} m^\ell m^\ell \delta_{\ell, m + \delta} |n + \ell\rangle \langle n + \delta + \ell|. \] (5.28)
FIG. 3: Showing the evolution of the amplification factor κ as a function of the parameter μ, with the evolution interrupted every $2N^{-1}$ in μ. The broken curve represents uninterrupted evolution for $0 \leq μ \leq 2$. The other curves correspond to $N = 2, 3, 5$ and $10$, the total duration adding to 2 for each $N$. The Zeno tendency of κ, for large $N$, to become linear in μ with slope $\sim N^{-1}$ should be noted.

The summation over the index $\ell_1$ is a binomial expansion:

$$\sum_{\ell_1=0}^{\ell} C_{\ell_1} (1 - \kappa_1^{-2})^{\ell_1} (\kappa_1^{-2} (1 - \kappa_2^{-2}))^{(\ell-\ell_1)} = (1 - \kappa_1^{-2} \kappa_2^{-2})^\ell.$$

Thus the expression in (5.28) reduces to

$$(\kappa_1 \kappa_2)^{-2} \sum_{\ell=0}^{\infty} (1 - \kappa_1^{-2} \kappa_2^{-2})^\ell (\kappa_1 \kappa_2)^{-(j+j+\delta)} \sqrt{j+j+\delta} C_\ell |j+\ell\rangle \langle j+\ell+\delta|.$$  (5.29)

Comparing Eqs. (5.13) and (5.29), we see that the latter is the Kraus representation for a single quantum-limited amplifier channel. That is,

$$\sum_{\ell_1, \ell_2} A_{\ell_1} (\kappa_1) A_{\ell_2} (\kappa_2) |j\rangle \langle j+\delta| A_{\ell_2} (\kappa_2)^\dagger A_{\ell_1} (\kappa_1)^\dagger = \sum_{\ell} A_\ell (\kappa_1 \kappa_2) |j\rangle \langle j+\delta| A_\ell (\kappa_1 \kappa_2)^\dagger.$$  (5.30)

A similar behaviour holds for $|j+\delta\rangle \langle j|$ as well. And this is what we set out to demonstrate.

**Remark on interrupted evolution and Zeno-like effect:**

As seen from (5.1) the parameter κ of $C_2(\kappa)$ equals cosh μ, where μ is a measure of the two-mode squeezing effected with the help of an ancilla initially in the vacuum state. The relevant two-mode unitary squeeze operator $U(\mu) = \exp[-\mu(a^\dagger b^\dagger - ab)]$ can be viewed as evolution for a duration μ under the Hamiltonian $-i(a^\dagger b^\dagger - ab)$, the amplification factor $\kappa = \cosh \mu$ increasing monotonically with increasing μ.

For convenience let us fix the total duration in μ to some value, say 2. In place of this single evolution, let us consider a sequence of $N$ interrupted evolutions, by tracing away the ancilla after every $2N^{-1}$ duration in μ, the total duration adding up to 2. The semigroup property of these interrupted evolutions after a total duration of 2 in μ will
be a quantum-limited amplifier $C_2(\kappa)$, with $\kappa = [\cosh (2/N)]^N$, which should be compared with $\kappa = \cosh 2$. The effect of interruption in slowing down amplification is clear. The behaviour of this interrupted amplification is shown in Fig. 3. After $\ell$ such evolutions the amplification factor will be $\kappa = (\cosh 2/N)^\ell$, which for large $N$ has the behaviour $\kappa \approx 1 + \frac{2}{N} \left( \frac{2}{N} \right)$, reminiscent of quantum Zeno effect \[55\] as in the case of quantum-limited attenuation in Section IV.

VI. THE SINGULAR CASE $A_2$

We now consider briefly $A_2$, the last of the quantum limited Bosonic Gaussian channels. The two-mode metaplectic unitary operator representing $A_2$ produces a symplectic transformation on the quadrature variables which does not mix the position variables with the momentum variables \[17\] :

\[
\begin{pmatrix}
q_1 \\
p_1
\end{pmatrix} \rightarrow \begin{pmatrix}
q'_1 \\
p'_1
\end{pmatrix} = M \begin{pmatrix}
q_1 \\
p_1
\end{pmatrix},
\]

\[
\begin{pmatrix}
p_2 \\
p_2
\end{pmatrix} \rightarrow \begin{pmatrix}
p'_2 \\
p'_2
\end{pmatrix} = (M^{-1})^T \begin{pmatrix}
p_1 \\
p_2
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
0 & 1 \\
1 & -1
\end{pmatrix}.
\]

Therefore, our general scheme applies to this case as well. Unlike in the earlier cases of $D(\kappa)$, $C_1(\kappa)$, and $C_2(\kappa)$, in the present case it turns out to be more convenient to evaluate the matrix elements of $U^{(ab)}$ in a mixed basis:

\[
C_{n_1,n_2}^{m_1,q} = \langle m_1 | q U^{(ab)} | n_1 \rangle | n_2 \rangle.
\]

Here $|q\rangle$ labels the position basis of the ancilla mode. With this mixed choice, the Kraus operators are labelled by a continuous index `$q$', and are given by

\[
V_q = \langle q | U^{(ab)} | 0 \rangle = \sum_{m_1,n_1} C_{m_1,0}^{m_1,q} | m_1 \rangle \langle n_1 |,
\]

where

\[
C_{n_1,0}^{m_1,q} = \int dq_1 \langle m_1 | q_1 \rangle \langle q_1 | q U^{(ab)} | n_1 \rangle | 0 \rangle
\]

\[
= \int dq_1 \langle m_1 | q_1 \rangle \langle q, q_1 - q | n_1, 0 \rangle.
\]

Here we have used, as in the earlier cases, the action of the unitary operator in the position eigenstates of the two-mode system. Employing the position space wavefunctions of the Fock states, we have

\[
C_{n_1,0}^{m_1,q} = \frac{\pi^{-3/4}}{\sqrt{2^{n_1+m_1} n_1! m_1!}} H_{n_1} (q) e^{-q^2} \int dq_1 H_{m_1} (q_1) e^{-\frac{(q_1 - q)^2}{2} + q^2}.
\]

The above integral is easily evaluated \[58\], and we have

\[
C_{n_1,0}^{m_1,q} = \frac{\pi^{-1/4}}{\sqrt{2^{n_1+m_1} n_1! m_1!}} q^{m_1} H_{n_1} (q) \exp [-3q^2 / 4]
\]

\[
= \langle m_1 | q / \sqrt{2} \rangle \langle q | n_1 \rangle,
\]

where $|q / \sqrt{2}\rangle$ is the coherent state $|\alpha\rangle$ for $\alpha = q / \sqrt{2}$, and the purpose of the round bracket being to distinguish the same from the position eigenket $|q / \sqrt{2}\rangle$. With this notation the Kraus operators are

\[
V_q = | q / \sqrt{2} \rangle \langle q |
\]

That the trace-preserving condition on the Kraus operators is satisfied emerges from the fact that the position kets are complete: $\int dq V_q^* V_q = \int dq |q\rangle \langle q | = 1$. 

To connect these Kraus operators $V_q$ to the action of the channel in the phase space picture, we examine the behaviour of an arbitrary pure state $|\psi\rangle$ under passage through the channel. We have

$$A_2 : \rho = |\psi\rangle\langle\psi| \rightarrow \rho' = \int dq \frac{|q/\sqrt{2}\rangle\langle q/\sqrt{2}|}{|q/\sqrt{2}|\langle q/\sqrt{2}|}$$
$$= \int dq \frac{|\psi(q)|^2 |q/\sqrt{2}\rangle\langle q/\sqrt{2}|}{|q/\sqrt{2}|\langle q/\sqrt{2}|}$$
$$= \int dq dp \frac{|\psi(q)|^2 \delta(p) |q + ip/\sqrt{2}\rangle\langle q + ip/\sqrt{2}|}{|q + ip/\sqrt{2}|\langle q + ip/\sqrt{2}|}. \quad (6.8)$$

The last expression is already in the ‘diagonal’ form in the coherent states basis, with $|\psi(q)|^2 \delta(p) = (q + ip)/\sqrt{2}$ forming the diagonal weight function $\phi(\alpha)$. It follows by convexity that for an arbitrary input state $\rho$ the output of the channel is given by

$$\rho' = \pi^{-1} \int d^2 \alpha \phi(\alpha) |\alpha\rangle\langle\alpha|, \quad \phi(\alpha) = \langle q|\rho|q\rangle \delta(p). \quad (6.9)$$

It is seen that this transformation is the same as $\chi_\xi(\xi) \rightarrow \chi_\xi \left(\frac{1+i\alpha}{\sqrt{2}} \xi\right) \exp[-|\xi|^2/2]$, the expected behaviour of the characteristic function under passage through $A_2$. \[59\]

The above results can be alternatively understood through the action of the channel in the Fock basis. Under passage through the channel,

$$|n\rangle\langle m| \rightarrow \int dq V_q |n\rangle\langle m| V_q^\dagger$$
$$= \int dq \frac{|q/\sqrt{2}\rangle\langle q/\sqrt{2}|}{|q/\sqrt{2}|\langle q/\sqrt{2}|}$$
$$= \int dq \frac{\pi^{-1/2}}{\sqrt{2^{n+m}n!m!}} H_n(q) H_m(q) e^{-q^2/2} |q/\sqrt{2}\rangle\langle q/\sqrt{2}|, \quad (6.10)$$

for all $n, m$. The outcome for an arbitrary input state $\rho$ follows by linearity, and we have

**Theorem 12** The channel $A_2$ is both nonclassicality breaking and entanglement breaking.

**Proof**: We note from Eq. \[6.7\] that the Kraus operators are already in rank one form, thereby showing that the channel is entanglement breaking. And from Eq. \[6.9\] we see that the output of the channel, for every input state $\rho$, supports a diagonal representation with nonnegative weight $\langle q|\rho|q\rangle \delta(p) \geq 0$, for all $\alpha = (q + ip)/\sqrt{2}$, showing that the output is classical for all input states.

**Remark on fixed points**: It can be seen directly from the action of the channel on the characteristic function that the momentum variable of the output is set to zero, and then multiplied by a Gaussian in that variable. Thus any cumulant of order $> 2$ in this variable is set to zero by the channel action. Moreover it is easily seen that the second moments are not preserved, thus showing that there is no state that is invariant under the action of this channel.

**VII. SINGLE QUADRATURE CLASSICAL NOISE CHANNEL** $B_1(a)$, $a \geq 0$

The channel $B_1(a)$, whose action is to simply inject Gaussian noise of magnitude $a$ into one quadrature of the oscillator, is neither quantum-limited nor extremal. It can be realized in the form

$$B_1(a) : \rho \rightarrow \rho' = \frac{1}{\sqrt{\pi a}} \int dq \exp[-q^2/a] D(q/\sqrt{2}) \rho D(q/\sqrt{2})^\dagger, \quad (7.1)$$

where $D(\alpha)$’s are the unitary displacement operators. $B_1(a)$ is thus a case of the so-called random unitary channels \[60\], a convex sum of unitary channels. The continuum

$$Z_q \equiv (\pi a)^{-1/4} \exp[-q^2/2a] D(q/\sqrt{2}) \quad (7.2)$$

are the Kraus operators of this realization. The quantum-limited end of $B_1(a)$ is obviously the identity channel, corresponding to $a \rightarrow 0 \quad \lim_{a \rightarrow 0} \sqrt{\pi a}^{-1} \exp[-q^2/a] = \delta(q)$, and $Z_{q=0} = \text{identity}$. One may assume $a = 1$ without loss of generality.
The reason we present a brief treatment of this channel here is just to demonstrate that this case too subjects itself to our general scheme presented in Section II. A further reason why we treat this noisy channel here is this. In Section IX we shall treat every noisy channel as the composite of two quantum-limited channels, the case of $B_1(a)$ constituting the only exception wherein this cannot be done.

The two-mode metaplectic unitary operator representing $B_1$ produces a symplectic transformation on the quadrature variables which, as in the earlier cases of $D(\kappa)$, $C_1(\kappa)$, $C_2(\kappa)$, and $A_2$, does not mix the position variables with the momentum variables:  

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{7.3}$$

And $p_1, p_2$ transform according to $(M^{-1})^T$.

As in the immediate previous case $A_2$, the matrix elements of $U^{(ab)}$ are

$$C_{m,n_1n_2}^{m_1q} = \langle m_1 | q | U^{(ab)} | n_1 \rangle | n_2 \rangle, \tag{7.4}$$

where $|q\rangle$’s are the position eigenvectors. In view of this the Kraus operators are labelled by a continuous index ‘$q$’ and are given by

$$\langle q | U^{(ab)} | 0 \rangle = \sum_{m_1,n_1} C_{n_10}^{m_1q} | m_1 \rangle \langle n_1 |, \tag{7.5}$$

where

$$C_{n_10}^{m_1q} = \int dq_1 | m_1 \rangle | q_1 \rangle | q_1 | (U^{(ab)} | n_1 \rangle | 0 \rangle = \int dq_1 | m_1 \rangle | q_1 - q, q | n_1, 0 \rangle. \tag{7.6}$$

Here we made the two-mode metaplectic unitary operator act on the position basis. To evaluate the Kraus operator, it is sufficient to evaluate the matrix elements

$$C_{m_10}^{m_1q} = \frac{\pi^{-3/4}}{\sqrt{2^{m_1+m_2}n_1!m_1!}} e^{-\frac{q^2}{2}} \int dq_1 H_{n_1}(q_1 - q) H_{m_1}(q_1)e^{-\frac{q^2}{2}} e^{-\frac{(q_1-q)^2}{2}}. \tag{7.7}$$

The above integral can be readily performed \textsuperscript{[61]}, and we obtain

$$C_{m_10}^{m_1q} = \pi^{-1/4} e^{-\frac{q^2}{2}} \left[ e^{-\frac{q^2}{2}} \sqrt{\frac{m_1!}{n_1!}} \left( \frac{-q}{\sqrt{2}} \right)^{n_1-m_1} L_{m_1}^{n_1-m_1}(q^2/2) \right] \equiv \pi^{-1/4} e^{-\frac{q^2}{2}} (m_1 | D(q/\sqrt{2}) | n_1 \rangle = Z_q. \tag{7.8}$$

We have thus recovered \textsuperscript{[22]}, but staying entirely within our general scheme.

VIII. EXTREMAL GAUSSIAN CHANNELS

It follows from the very identification of channels with trace-preserving CP maps that channels form a convex set, and since a convex set is fully characterized by its extremal elements, it is of interest to know if there are any extremal elements among the quantum-limited Bosonic Gaussian channels. The present Section is devoted to this issue.

A convenient characterization of extremality in terms of Kraus representation is due to Choi \textsuperscript{[30]}: A trace-preserving CP map $\Omega$ is extremal if and only if $\Omega$ supports a Kraus representation $\rho \rightarrow \rho' = \sum_n W_n \rho W_n^\dagger$, with $m$ and $n$ independently running over their range, are linearly independent. While Choi’s original result was formulated in the finite dimensional case, it is to be expected that this result will generalize to the infinite-dimensional case with suitable technical regularization. That this is the case is indicated, for instance, by Theorem 2.4 of the work of Tsai \textsuperscript{[62]}. Since we are dealing with concrete physical situations, we may assume in what follows that such technical requirements are indeed satisfied.

With these preliminary remarks in place, we are now ready to test the quantum-limited Gaussian channels for extremality. We begin with the transpose or phase conjugation channel.
I. The phase conjugation channel $\mathcal{D}(\kappa)$

The Kraus operators $\{T_n(\kappa)\}$ for $\mathcal{D}(\kappa)$ have been presented in (3.11). We may readily express the products $\{T_m^\dagger(\kappa)T_n(\kappa)\}$, $0 \leq m, n < \infty$ in the convenient form

\[
T_{\ell}^\dagger(\kappa)T_{\ell+\delta}(\kappa) = \sum_{j=0}^{\infty} d(\delta)_{\ell j} |j+\delta\rangle\langle j|,
\]

\[
T_{\ell}^\dagger(\kappa)T_{\ell+\delta}(\kappa) = \sum_{j=0}^{\infty} d(\delta)_{\ell j} |j\rangle\langle j+\delta|,
\]

\[
d(\delta)_{\ell j} = \sqrt{\int C_{\ell}^j C_{\ell+\delta} C_{\ell+\delta} \left[ (1 + \kappa^2)^{-j-\delta/2-1} \right] \delta^{-j}} \text{ for } j \leq \ell = 0 \text{ for } j > \ell.
\]

The two cases involving $|\ell + \delta\rangle\langle \ell|$ and $|\ell\rangle\langle \ell + \delta|$, $\delta = 0, 1, 2, \cdots$ correspond respectively to $m - n \geq 0$ and $m - n \leq 0$. It is clear from (8.1) that the set $\{T_m^\dagger(\kappa)T_n(\kappa)\}$, $0 \leq m, n < \infty$ breaks into nonintersecting subsets labelled by $m-n = 0, \pm 1, \pm 2 \cdots$. The index $\ell = \min\{m, n\}$ labels the $T_m^\dagger(\kappa)T_n(\kappa)$’s within a subset. That subsets corresponding to two different values of $m - n$ are mutually orthogonal is manifest. Thus, we are left to examine only linear independence within each subset, and this is easily accomplished as follows.

For a given value of $m - n$ (i.e. fixed value of $\delta$) arrange the c-number coefficients $d(\delta)_{\ell j}$ into a matrix $d(\delta) = ((d(\delta)_{\ell j}))$. This matrix is lower diagonal, and its diagonals are nonzero [indeed, all the entries on and below the diagonal are nonzero]. Thus $d(\delta)$ is nonsingular. Therefore the linear independence of $T_{\ell}^\dagger(\kappa)T_0(\kappa)$, $T_{\ell+\delta}(\kappa)T_1(\kappa)$, $T_{\ell+\delta}(\kappa)T_2(\kappa)$, $\cdots$ follows from the linear independence of the mutually orthogonal $|\delta\rangle\langle 0|$, $|1\rangle\langle 1|$, $|2\rangle\langle 2|$, $\cdots$ and, similarly, the linear independence of $T_0^\dagger(\kappa)T_\delta(\kappa)$, $T_1^\dagger(\kappa)T_{\ell+\delta}(\kappa)$, $T_2^\dagger(\kappa)T_{\ell+\delta}(\kappa)$, $\cdots$ follows from the linear independence of $|\delta\rangle\langle 0|$, $|1\rangle\langle 1+\delta|$, $|2\rangle\langle 2+\delta|$, $\cdots$.

This completes proof of the linear independence of the set $\{T_m^\dagger(\kappa)T_n(\kappa)\}$, $0 \leq m, n < \infty$ and hence proves extremality of $\mathcal{D}(\kappa)$ in the sense of Choi, for all $\kappa \neq 1$.

Remark on the Doubly Stochastic Case $\mathcal{D}(1)$: The channel $\mathcal{D}(1)$ is exceptional within the phase conjugation family of quantum-limited channels in that it is both trace-preserving and unital, i.e. it is doubly stochastic. While all random unitary channels (convex sums of unitary channels) are manifestly doubly stochastic, the converse is not true. Early (finite-dimensional) counter examples to the converse can be found in Ref. [52, 64]. It is this phenomenon that underlies a conjecture of Winter et al [38]. The channel $\mathcal{D}(1)$ is a counter example from the infinite-dimensional Gaussian domain, and the only doubly stochastic case among quantum-limited Gaussian channels. That $\mathcal{D}(1)$ is not random unitary readily follows from its extremality which we have established above. Not every non-extremal doubly stochastic map is random unitary, but an extremal doubly stochastic map can have not even one unitary operator in the support of its Kraus operators, for if it had it will be a convex sum of that unitary channel and another doubly stochastic map. The noisy classical noise channels of families $\mathcal{B}_2$ and $\mathcal{B}_2$ are obviously doubly stochastic and obviously random unitary.

II. The attenuation/beam splitter channel $\mathcal{C}_1(\kappa)$

The beamsplitter channel $\mathcal{C}_1(\kappa)$ is described by the Kraus operators $\{B_m(\kappa)\}$ given in (1.6). As in the earlier case of $\mathcal{D}(\kappa)$ we can compute the products $\{B_m^\dagger(\kappa)B_n(\kappa)\}$ in the form

\[
B_{\ell+\delta}^\dagger(\kappa)B_{\ell}(\kappa) = \sum_{j=0}^{\infty} f(\delta)_{\ell j} |j\rangle\langle j+\delta|,
\]

\[
B_{\ell}^\dagger(\kappa)B_{\ell+\delta}(\kappa) = \sum_{j=0}^{\infty} f(\delta)_{\ell j} |j+\delta\rangle\langle j|,
\]

\[
f(\delta)_{\ell j} = \sqrt{\int C_{\ell}^{j+\delta} C_{\ell+\delta} \left[ 1 - \kappa^2 \right]^{j-\ell}} \kappa^{2(j-\ell)} \text{ for } j \geq \ell = 0 \text{ for } j < \ell.
\]

Again the two cases $|j+\delta\rangle\langle j|$ and $|j\rangle\langle j+\delta|$ correspond respectively to $m - n \geq 0$ and $m - n \leq 0$. 


The situation is similar to the earlier case of $D(\kappa)$. The set $\{B_m^\ell(\kappa)B_n(\kappa)\}$, $0 \leq m, n < \infty$ fibrates into nonintersecting subsets, and these are labelled by $m - n = 0, \pm 1, \cdots$. The index $\ell = \min\{m, n\}$ acts as the label within a given subset. The subsets being mutually orthogonal, it only remains to examine linear independence within each subset.

As in the case of $D(\kappa)$ we arrange the $c$-number coefficients $f(\delta)_{\ell j}$ into matrices $f(\delta) = (f(\delta)_{\ell j})$, one matrix for each subset $\delta$. These matrices are upper diagonal. None of the diagonal elements vanishes, and so the matrices are nonsingular, proving the linear independence of the sets $\{B_m^\ell(\kappa)B_n(\kappa)\}_\ell$ and $\{B_m^\ell(\kappa)B_{\ell n}(\kappa)\}_\ell$ and hence the extremality of $C_1(\kappa)$, for all $\kappa \geq 0$.

III. The amplifier channel $C_2(\kappa)$

The amplifier channel $C_2(\kappa)$ described by the Kraus operators $\{A_m(\kappa)\}$ presented in \[7.5\] turns out to be a little more subtle in respect of our present purpose. There is considerable similarity with the two earlier cases, though. As in the case of $D(\kappa)$ and $C_1(\kappa)$, let us begin by expressing the products $\{A_m^\dagger(\kappa)A_n(\kappa)\}$ in the form

$$A_m^\dagger(\kappa)A_n(\kappa) = \sum_{j=0}^{\infty} h(\delta)_{\ell j} |j\rangle\langle j+\delta|,$$

$$A_m^\dagger(\kappa)A_{n+\delta}(\kappa) = \sum_{j=0}^{\infty} h(\delta)_{\ell j} |j+\delta\rangle\langle j|,$$

$$h(\delta)_{\ell j} = \kappa^{-2} \sqrt{\ell+j+\delta}C_{\ell+\delta}^{\ell+j+\delta}C_{j+\delta}^{\ell+j+\delta} \sqrt{1-\kappa^{-2}}^{2\ell+\delta} \kappa^{-1(2j+\delta)}.$$

The two cases correspond respectively to $m - n \geq 0$ and $m - n \leq 0$. The set of operators $\{A_m(\kappa)A_n(\kappa)\}$ manifestly separate into nonintersecting subsets, determined by $m - n$; the index $\ell$ acts as a label within each subset; the subsets are mutually orthogonal; and it only remains to determine the linear independence within each subset. Up to this point the situation is similar to the earlier two cases.

As in the earlier two cases, let us arrange the $c$-number coefficients $h(\delta)_{\ell j}$ into matrices $h(\delta)$, one matrix for each $\ell$. And now arises the distinction: whereas the invertibility of $d(\delta)$ and $f(\delta)$ was manifest, being lower or upper diagonal matrices, this is not so in respect of the present case. We are therefore led to demonstrate the nonsingularity of $h(\delta)$ in a somewhat different manner.

The multiplicative scalar $(\kappa^{-1})^{2+\delta}(\sqrt{1-\kappa^{-2}})^{\delta}$ can be dropped from $h(\delta)$ for our present purpose. We note further that $h(\delta)$ can be simplified by left and right multiplication by diagonal matrices $L, M$:

$$h(\delta) = L \tilde{h} R$$

$$L_{rs} = \frac{(1-\kappa^{-2})^r}{\sqrt{r!(r+\delta)!}} \delta_{rs}, \quad R_{rs} = \frac{\kappa^{-2r}}{\sqrt{r!(r+\delta)!}} \delta_{rs}. \quad (8.4)$$

Since $L, R$ are nonsingular, invertibility of $h(\delta)$ is equivalent to that of $\tilde{h}(\delta)$. The new matrix $\tilde{h}$ is symmetric and has a simple structure: $\tilde{h}(\delta)_{rs} = (r + s + \delta)!$. To demonstrate the nonsingularity of $\tilde{h}(\delta)$, we begin by writing out its entries in detail:

$$\tilde{h}(\delta) = \begin{pmatrix}
\delta! & (\delta+1)! & (\delta+2)! & \cdots & (\delta+k)! & \cdots \\
(\delta+1)! & (\delta+2)! & (\delta+3)! & \cdots & \cdots & \cdots \\
(\delta+2)! & (\delta+3)! & (\delta+4)! & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
(\delta+\ell)! & \vdots & \vdots & \vdots & (\delta+k+\ell)! & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad (8.5)$$

Dividing the first row of $\tilde{h}(\delta)$ by $\delta!$, the second row by $(\delta+1)!$, the third row by $(\delta+2)!$, and so on, we obtain

$$\begin{pmatrix}
1 & (\delta+1) & (\delta+1)(\delta+2) & (\delta+1)(\delta+2)(\delta+3) & \cdots & \cdots \\
1 & (\delta+2) & (\delta+2)(\delta+3) & (\delta+2)(\delta+3)(\delta+4) & \cdots & \cdots \\
1 & (\delta+3) & (\delta+3)(\delta+4) & (\delta+3)(\delta+4)(\delta+5) & \cdots & \cdots
\end{pmatrix}. \quad (8.6)$$
Now performing the row transformations $R_1 \rightarrow R_1 - R_0$, $R_2 \rightarrow R_2 - R_1$, and so forth we obtain

\[
\begin{pmatrix}
1 & (\delta + 1) & (\delta + 1)(\delta + 2) & (\delta + 1)(\delta + 2)(\delta + 3) & \cdots & \\
0 & 1 & 2(\delta + 2) & 3(\delta + 2)(\delta + 3) & \cdots & \\
0 & 1 & 2(\delta + 3) & 3(\delta + 3)(\delta + 4) & \cdots & \\
0 & 1 & 2(\delta + 4) & 3(\delta + 4)(\delta + 5) & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\end{pmatrix},
\]

(8.7)

We can repeatedly do this kind of row transformations, starting with row $R_2$ in the next iteration and row $R_3$ in the subsequent iteration and so on. The matrix $h(\delta)$ gets finally transformed to the form

\[
\begin{pmatrix}
1 & \cdots & \cdots & \cdots & \\
0 & 1 & \cdots & \cdots & \\
0 & 0 & 2! & \cdots & \\
0 & 0 & 0 & 3! & \\
0 & 0 & 0 & 0 & 4! & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\end{pmatrix},
\]

(8.8)

which is upper triangular with non-vanishing diagonal entries, and thus has ‘full rank’. We may thus conclude that the coefficient matrix $h(\delta)$ is invertible, showing that the set $\{A_{\ell+\delta}^{\dagger}A_{\ell}\}$ as well as $\{A_{\ell+\delta}^{\dagger}A_{\ell}\}$ is linearly independent, for each $\delta$.

This completes proof of linear independence of the set $\{A_{m}^{\dagger}(\kappa)A_{n}(\kappa)\}$, $0 \leq m, n < \infty$. Extremality of the amplifier channel $C_2(\kappa)$ is thus established, for all $\kappa > 1$.

IV. The Singular case $A_2$

Computation of the product $V_{q}^{\dagger}V_{q}$ is nearly trivial in this case. We have from (6.7):

\[
V_{q}^{\dagger}V_{q} = (q'/\sqrt{2}|q/\sqrt{2})\langle q'|q\rangle.
\]

(8.9)

The inner product $(q'/\sqrt{2}|q/\sqrt{2}) = \exp[-q^2/4 - q'^2/4 + qq'/2]$ is nonzero for arbitrary $q, q'$. Since $|q\rangle\langle q|$ form a linearly independent set of operators, we conclude that the channel $A_2$ is extremal.

We have considered in this Section the quantum-limited channels $D(\kappa)$, $C_1(\kappa)$, $C_2(\kappa)$, and $A_2$ in that order. Since these are the only quantum-limited Bosonic Gaussian channels, the main conclusion of our study can be stated in the following concise form:

**Theorem 13** All quantum-limited Bosonic Gaussian channels are extremal.

**Remark on $A_1$**: We have not treated separately the quantum-limited channel $A_1$, for this may be viewed as a particular case of $C_1(\kappa)$, corresponding to $\cos(\theta) = 0$ in (4.11) or, equivalently, $\kappa = 0$. Thus, the assertion above covers $A_1$ as well.

IX. NOISY CHANNELS AS COMPOSITES OF QUANTUM-LIMITED CHANNELS

Our considerations so far have been in respect of quantum-limited channels. We turn our attention now to the case of noisy channels. It turns out that every noisy channel, except $B_1(a)$ which corresponds to injection of classical noise in just one quadrature, can be realised (in a non-unique way) as composition of two quantum-limited channels, so that the Kraus operators are products of those of the constituent quantum-limited channels.

We have noted in Sections IV and V that composition of two quantum limited attenuator (or amplifier) channels is again a quantum-limited attenuator (or amplifier) channel. This special semigroup property however does not obtain under composition for other quantum-limited channels. In general, composition of two quantum-limited channels results in a channel with additional classical noise. For this reason we restore in this Section the original notation $D(\kappa;0)$, $C_1(\kappa;0)$, and $C_2(\kappa;0)$ in order to make room for this additional classical noise.
Comparing this with (8.2), it is now clear that the actual numerical values of the nonzero coefficients. Hence the same argument should be expected to apply to these Kraus operators are linearly independent. may be seen as follows. Since assumed to be in their respective canonical forms simultaneously. The composition results, in several cases, in noisy channels thereby enabling description of noisy Gaussian channels, including the classical noise channel \( B_2(a) \), in terms of discrete sets of linearly independent Kraus operators.

### I. The composite \( C_2(\kappa_2; 0) \circ C_1(\kappa_1; 0) \)

It is clear from the very definition of these channels through their action on the characteristic function that the composite \( C_2(\kappa_2; 0) \circ C_1(\kappa_1; 0) \) is a noisy amplifier or attenuator depending on the numerical value of \( \kappa_2 \kappa_1 \): it equals \( C_1(\kappa_2 \kappa_1; 2(\kappa_2^2 - 1)) \) for \( \kappa_2 \kappa_1 \leq 1 \), and \( C_2(\kappa_2 \kappa_1; 2(1 - \kappa_2^2)) \) for \( \kappa_2 \kappa_1 \geq 1 \), as may be readily read off from Table I.

The Kraus operators for the composite is given by the set \( \{ A_m(2^z)B_n(\kappa_1) \} \) with \( m, n \) running independently over the range \( 0 \leq m, n < \infty \). That these Kraus operators are linearly independent may be seen as follows. Since \( A_m(\kappa_2) \) is the same as \( B_m(\kappa_2^{-1}) \) except for an \( m \)-independent multiplicative constant, linear independence of the set \( \{ A_m(\kappa_2)B_n(\kappa_1) \} \) is the same as linear independence of \( \{ B_m(\kappa_2^{-1})B_n(\kappa_1) \} \). It will be recalled that in proving linear independence of the set \( \{ B_m(\kappa)^\dagger B_n(\kappa) \} \) in Section V in the context of extremality of \( C_2(\kappa) \), we used only the structure of the \( B_\ell(\cdot) \)'s in respect of the expansion coefficients in the basis \( \{|m\rangle \langle n|\} \) being zero or nonzero, and not the actual numerical values of the nonzero coefficients. Hence the same argument should be expected to apply to proof of linear independence of \( \{ B_m(\kappa_2^{-1})B_n(\kappa_1) \} \sim \{ A_m(\kappa_2)B_n(\kappa_1) \} \) as well. That this is indeed the case is seen by computing the products \( A_m(\kappa_2)B_n(\kappa_1) \) in the form

\[
A_{\ell+\delta}(\kappa_2)B_{\ell}(\kappa_1) = \sum_{j=0}^{\infty} g_1(\delta)_{\ell j} |j+\delta\rangle \langle j|,
\]

\[
A_{\ell}(\kappa_2)B_{\ell+\delta}(\kappa_1) = \sum_{j=0}^{\infty} \hat{g}_1(\delta)_{\ell j} |j\rangle \langle j+\delta|,
\]

\[
g_1(\delta)_{\ell j} = \kappa_2^{-1} \sqrt{\beta^{\ell+\delta}C_{\ell+\delta}^3C_{\ell}} \left( \sqrt{1-\kappa_2^2} \right)^{\ell+\delta} (\kappa_2^{-1} \kappa_1)^{-\ell}, \text{ for } j \geq \ell,
\]

\[
= 0, \text{ for } j < \ell;
\]

\[
\hat{g}_1(\delta)_{\ell j} = \kappa_2^{-1} \sqrt{\beta^{\ell}C_{\ell+\delta}^3C_{\ell}} \left( \sqrt{1-\kappa_2^2} \right)^{\ell+\delta} (\kappa_2^{-1} \kappa_1)^{-\ell}, \text{ for } j \geq \ell,
\]

\[
= 0, \text{ for } j < \ell.
\]

Comparing this with (5.2), it is now clear that \( A_m(\kappa_2)B_n(\kappa_1) \)'s are linearly independent for exactly the same reason by which \( B_m(\kappa)^\dagger B_n(\kappa) \)'s were linearly independent rendering the quantum-limited attenuator \( C_1(\kappa; 0) \) extremal.
II. The composite $C_1(\kappa_2; 0) \circ C_2(\kappa_1; 0)$, $\kappa_2 \leq 1$, $\kappa_1 \geq 1$

Again the composite $C_1(\kappa_2; 0) \circ C_2(\kappa_1; 0)$ is a noisy amplifier or attenuator depending on the numerical value of $\kappa_2 \kappa_1$, and the details may be read off from Table III. The Kraus operators for the composite $C_1(\kappa_2; 0) \circ C_2(\kappa_1; 0)$ are given by $\{B_m(\kappa_2)A_n(\kappa_1)\}$, $0 \leq m, n < \infty$. To establish their linear independence we compute $B_m(\kappa_2)A_n(\kappa_1)$ in the form

$$B_{\ell+\delta}(\kappa_2)A_\ell(\kappa_1) = \sum_{j=0}^{\infty} g_2(\delta)_{\ell j} |j\rangle \langle j + \delta|,$$

$$B_{\ell}(\kappa_2)A_{\ell+\delta}(\kappa_1) = \sum_{j=0}^{\infty} \tilde{g}_2(\delta)_{\ell j} |j\rangle \langle j + \delta|,$$

$$g_2(\delta)_{\ell j} = \kappa_1^{-1} \sqrt{\sqrt{1+\kappa_1^2} \sqrt{1+\kappa_2^2}} C_\ell C_\ell^{\dagger} \kappa_1^{-\ell(j+\delta)} \left(\sqrt{1-\kappa_1^{-2}}\right)^{\ell} \kappa_2^{j} \left(\sqrt{1-\kappa_2^{-2}}\right)^{\ell+\delta},$$

$$\tilde{g}_2(\delta)_{\ell j} = \kappa_1^{-1} \sqrt{\sqrt{1+\kappa_1^2} \sqrt{1+\kappa_2^2}} C_\ell C_\ell^{\dagger} \kappa_1^{-\ell j} \left(\sqrt{1-\kappa_1^{-2}}\right)^{\ell+\delta} \kappa_2^{j+\delta} \left(\sqrt{1-\kappa_2^{-2}}\right)^{\ell}.$$

Comparing with (S.3) we see that the Kraus operators $B_m(\kappa_2)A_n(\kappa_1)$ of $C_1(\kappa_2; 0) \circ C_2(\kappa_1; 0)$ are linearly independent for the same reason by which the quantum-limited amplifier channel was extremal.

III. The composite $D(\kappa_2) \circ D(\kappa_1)$, $\kappa_2, \kappa_1 > 0$

Similar to the earlier two cases, the composite $D(\kappa_2; 0) \circ D(\kappa_1; 0)$ is a noisy amplifier or attenuator depending on the numerical value of $\kappa_2 \kappa_1$, and the details can be read off from Table III. It may be noted, again from Table III, that this case tends to be more noisy than the earlier two cases.

The Kraus operators for this composite are given by $\{T_m(\kappa_2)T_n(\kappa_1)\}$, $0 \leq m, n < \infty$. To exhibit the linear independence of these Kraus operators we compute the products $T_m(\kappa_2)T_n(\kappa_1)$ in the form

$$T_{\ell+\delta}(\kappa_2)T_\ell(\kappa_1) = \sum_{j=0}^{\infty} g_3(\delta)_{\ell j} |j\rangle \langle j + \delta|,$$

$$T_\ell(\kappa_2)T_{\ell+\delta}(\kappa_1) = \sum_{j=0}^{\infty} \tilde{g}_3(\delta)_{\ell j} |j\rangle \langle j + \delta|,$$

$$g_3(\delta)_{\ell j} = \left(\sqrt{1+\kappa_1^2}\right)^{-1} \left(\sqrt{1+\kappa_2^2}\right)^{-1} \sqrt{\sqrt{1+\kappa_1^2} \sqrt{1+\kappa_2^2}} C_\ell C_\ell^{\dagger} \left[\sqrt{(1+\kappa_2^2)(1+\kappa_1^{-2})}\right]^{-\ell-j} \times \left[\sqrt{(1+\kappa_1^2)(1+\kappa_2^{-2})}\right]^{-j} \left(\sqrt{1+\kappa_2^{-2}}\right)^{-\delta}, \text{ for } j \leq \ell,$$

$$= 0, \text{ for } j > \ell,$$

$$\tilde{g}_3(\delta)_{\ell j} = \left(\sqrt{1+\kappa_1^2}\right)^{-1} \left(\sqrt{1+\kappa_2^2}\right)^{-1} \sqrt{\sqrt{1+\kappa_1^2} \sqrt{1+\kappa_2^2}} \left[\sqrt{(1+\kappa_2^2)(1+\kappa_1^{-2})}\right]^{-\ell-j} \times \left[\sqrt{(1+\kappa_1^2)(1+\kappa_2^{-2})}\right]^{-j} \left(\sqrt{1+\kappa_1^{-2}}\right)^{-\delta}, \text{ for } j \leq \ell,$$

$$= 0, \text{ for } j > \ell.$$

Comparing with (S.4) it is readily seen that these Kraus operators are linearly independent for exactly the same reason by which the quantum-limited transpose channel was extremal.
IV. The composite $D(\kappa_2; 0) \circ C_1(\kappa_1; 0)$, $\kappa_2 > 0, 0 \leq \kappa_1 \leq 1$

Kraus operators of this composite, which always corresponds to a noisy transpose channel (see Table IV), are \{T_m(\kappa_2)B_n(\kappa_1)\}, $0 \leq m, n < \infty$. We have

$$T_m(\kappa_2)B_n(\kappa_1) = \sum_{j=0}^{\infty} \xi^j_{mn} |m-j\rangle \langle n+j|,$$

$$\xi^j_{mn} = \left( \sqrt{1 + \kappa_2^2} \right)^{-1} \sqrt{mC_j}^{n+j} C_j \left( \sqrt{1 + \kappa_2^{-2}} \right)^{-j} \left( \sqrt{1 + \kappa_2^{-2}} \right)^{-(m-j)}$$

$$\times \kappa_1^j \left( 1 - \kappa_1^{-2} \right)^n, \text{ for } j \leq m;$$

$$= 0, \text{ for } j > m. \quad (9.4)$$

It is immediately clear that $T_m(\kappa_2)B_n(\kappa_1)$ and $T_m(\kappa_2)B_n(\kappa_1)$ are (trace-)orthogonal unless $m + n = m' + n'$.

We may therefore divide the Kraus operators into orthogonal subsets determined by $m + n = \text{constant} \equiv N$. Then linear independence will have to be established just within each subset $\Omega_N = \{T_0(\kappa_2)B_N(\kappa_1), T_1(\kappa_2)B_{N-1}(\kappa_1), \cdots, T_N(\kappa_2)B_0(\kappa_1)\}$. It is seen from (9.4) that $T_0(\kappa_2)B_N(\kappa_1)$ is a multiple of $|0\rangle \langle N|$, $T_1(\kappa_2)B_{N-1}(\kappa_1)$ is a linear combination of $|0\rangle \langle N|$ and $|1\rangle \langle N-1|$, $T_2(\kappa_2)B_{N-2}(\kappa_1)$ is a linear combination of $|0\rangle \langle N|$, $|1\rangle \langle N-1|$, and $|2\rangle \langle N-2|$, and so on. Thus linear independence within $\Omega_N$ follows as an immediate consequence of the fact that $\xi^j_{mn} \neq 0 \forall m, n$.

V. The composite $C_1(\kappa_2; 0) \circ D(\kappa_1; 0)$, $\kappa_1 > 0, 0 \leq \kappa_2 \leq 1$

This composite channel corresponds to a quantum-limited transpose channel (see Table IV) which we have already considered in much detail in Section III. The Kraus operators \{B_m(\kappa_2)T_n(\kappa_1)\}, $0 \leq m, n < \infty$ (which as a set should be equivalent to \{T_\ell(\kappa_2\kappa_1)\}, $0 \leq \ell < \infty$), are

$$B_m(\kappa_2)T_n(\kappa_1) = \sum_{j=m}^{n} \xi^j_{mn} |j-m\rangle \langle n-j|,$$

$$\xi^j_{mn} = \sqrt{jC_m^{n}C_j} \left( \sqrt{1 - \kappa_1^2} \right)^m \kappa_2^{-m-j} \left( \sqrt{1 + \kappa_1^2} \right)^{-(n-j)}$$

$$\times \left( \sqrt{1 + \kappa_1^2} \right)^j, \text{ for } n \geq m;$$

$$= 0, \text{ for } n < m. \quad (9.5)$$

Thus $B_m(\kappa_2)T_n(\kappa_1) = 0$ if $m > n$. Further, all $B_m(\kappa_2)T_n(\kappa_1)$ with $m = n$ correspond to a multiple of the vacuum projector $|0\rangle \langle 0|$, showing that these Kraus operators are not linearly independent. Even so, this is also a valid representation of the quantum-limited channel $C_1(\kappa_2; 0) \circ D(\kappa_1; 0) = D(\kappa_2\kappa_1; 0)$.

VI. The composite $C_2(\kappa_2; 0) \circ D(\kappa_1; 0)$, $\kappa_2 \geq 1, \kappa_1 > 0$

This composite channel corresponds, for all $\kappa_1, \kappa_2$, to a noisy transpose channel, similar to the case of $D(\kappa_2; 0) \circ C_1(\kappa_1; 0)$ considered earlier. The Kraus operators \{A_m(\kappa_2)T_n(\kappa_1)\}, $0 \leq m, n < \infty$ have the form

$$A_m(\kappa_2)T_n(\kappa_1) = \sum_{j=0}^{n} \xi^j_{mn} |j+m\rangle \langle n-j|,$$

$$\xi^j_{mn} = \kappa_2^{-j} \left( \sqrt{1 + \kappa_1^2} \right)^{-1} \sqrt{j+m}C_m^{n}C_j \left( \sqrt{1 - \kappa_2^{-2}} \right)^m \kappa_2^{-j} \left( \sqrt{1 + \kappa_1^2} \right)^{-(n-j)}$$

$$\times \left( \sqrt{1 + \kappa_1^2} \right)^j, \text{ for } j \leq n;$$

$$= 0, \text{ for } j > n. \quad (9.6)$$
Linear independence of the Kraus operators can be established in a manner similar to the earlier case of \( D(\kappa_2; 0) \circ C_1(\kappa_1; 0) \).

### VII. The composite \( D(\kappa_2; 0) \circ C_2(\kappa_1; 0) \), \( \kappa_2 > 0, \kappa_1 \geq 1 \)

This composite is a quantum-limited transpose channel (see Table I), with Kraus operators \( \{ T_m(\kappa_2) A_n(\kappa_1) \} \), \( 0 \leq m, n < \infty \). Similar to the earlier quantum-limited case of \( C_1(\kappa_1; 0) \circ D(\kappa_2; 0) \), these Kraus operators too are not linearly independent; they do represent the quantum-limited channel \( D(\kappa_2\kappa_1; 0) \), though.

We conclude this Section with some further observations. It is seen from Table I that to realise the noisy attenuator channel \( C_1(\kappa; a) \), for the full parameter range \( 0 \leq \kappa \leq 1, a > 0 \), where \( a \) is a measure of the additional classical noise above the quantum limit, as the composite \( C_2(\kappa_2; 0) \circ C_1(\kappa_1; 0) \), we have to solve \( \kappa_2\kappa_1 = \kappa \) and \( 2(\kappa_2^2 - 1) = a \) for \( 0 \leq \kappa_1 \leq 1, \kappa_2 \geq 1 \), and we have as solution \( \kappa_2 = \sqrt{1 + a/2} > 1, \kappa_1 = \kappa/\kappa_2 < \kappa \leq 1 \). It should be appreciated that this realisation contains as special cases the noisy \( A_1(a) \) for \( \kappa = 0 \) (i.e., \( \kappa_1 = 0 \)), and the classical noise channel \( B_2(a) \) for \( \kappa = 1 \).

Similarly, the noisy amplifier \( C_2(\kappa; a) \), \( \kappa \geq 1, a > 0 \) can be realised through the same composite if we solve \( \kappa_2\kappa_1 = \kappa \) and \( 2\kappa_2^2(1 - \kappa_1^2) = a \) for \( 0 \leq \kappa_1 \leq 1, \kappa_2 \geq 1 \). We have the solution \( \kappa_2 = \sqrt{\kappa^2 + a/2} > 1 \) and \( \kappa_1 = \kappa/\kappa_2 < 1 \). Again the classical noise channel \( B_2(a) \) is contained as the special case \( \kappa = 1 \).

Finally, to realise the noisy transpose channel \( D(\kappa; a) \), \( \kappa \geq 0, a > 0 \) as the composite \( D(\kappa_2; 0) \circ C_1(\kappa_1; 0) \), \( 0 \leq \kappa_1 \leq 1, \kappa_2 > 0 \), of quantum-limited channels we solve \( \kappa = \kappa_2\kappa_1 \) and \( a = 2\kappa_2^2(1 - \kappa_1^2) \) to obtain \( \kappa_2 = \sqrt{\kappa^2 + a/2} > 1, \kappa_1 = \kappa/\kappa_2 < 1 \). The same can also be realised as the composite \( C_2(\kappa_2; 0) \circ D(\kappa_1; 0) \), \( \kappa_2 \geq 1, \kappa_1 > 0 \) by solving \( \kappa = \kappa_1\kappa_2 \) and \( a = 2(\kappa_2^2 - 1) \), and we have \( \kappa_2 = \sqrt{1 + a/2} > 1, \kappa_1 = \kappa/\kappa_2 \geq 0 \).

We may summarise some aspects of our consideration thus far in this Section in the following manner.

**Theorem 14** All the nonsingular noisy channels can be realised as the composition of a pair of quantum-limited channels. Equivalently, and as a consequence, each of the nonsingular Gaussian channel has an operator-sum representation in terms of a discrete set of linearly independent Kraus operators.

The above assertion includes in particular the case of the classical noise channel \( B_2(a) \). Further, the case \( A_1(a) \) is not exempted from our consideration above, for it is just a special case, corresponding to \( \kappa = 0 \) of the noisy attenuator \( C_1(\kappa; a) \).

It is seen from Table I that the noisy singular case \( A_2(a) \) can be realised as a composite of two quantum-limited channels: either following or preceding the quantum-limited \( A_2(0) \) by quantum limited \( C_1(\cdot; 0) \), \( C_2(\cdot; 0) \), or \( D(\cdot; 0) \). Consequently, the Kraus operators will be indexed by one discrete and one real variable. Thus, the single quadrature noise channel \( B_1(a) \) is the only singular case that does not submit itself to our consideration above, in the sense that there seems to be no way of realising it as composite of a pair of quantum limited channels.

While we have obtained in the Section Kraus representations for noisy channels with the aid of pairs of quantum-limited channels it is, of course, possible to obtain Kraus representation using unitary dilation of Section 2, with the ancilla in a thermal state rather than the vacuum state. But we believe our present approach has the advantage of leading to Kraus operators of extremely simple structure, in addition to the advantage of connecting the noisy case to the quantum-limited case.

We conclude this section with three remarks, two of them are in respect of Table I while the third one is in the context of error correctability.

**Remark 1:** We have already noted in Sections III and V that the quantum-limited family \( D(\cdot) \) is self dual \( D(\kappa) \sim D(\kappa^{-1}) \), whereas the families \( C_1(\cdot) \) and \( C_2(\cdot) \) are dual to one another : \( C_1(\kappa) \sim C_2(\kappa^{-1}) \). The reader will recall that these duality relations are a consequence of the failure to be unital of these (trace-preserving) maps by just a multiplicative scalar. Remnants of these duality relations may be readily observed in Table I. The composite \( D(\cdot) \circ C_1(\cdot) \) is quantum-limited, the dual fact being that \( C_2(\cdot) \circ D(\cdot) \) is quantum-limited. The fact that \( C_2(\cdot) \circ C_2(\cdot) \) is quantum-limited is dual to the fact that \( C_1(\cdot) \circ C_1(\cdot) \) is quantum-limited. The fact that \( D(\cdot) \circ C_1(\cdot) \) is a noisy conjugator has as its dual the fact that \( C_2(\cdot) \circ D(\cdot) \) is a noisy conjugator. Finally, \( D(\cdot) \circ D(\cdot) \), \( C_1(\cdot) \circ C_2(\cdot) \), and \( C_2(\cdot) \circ C_1(\cdot) \) are self duals. Since the failure of the quantum limited \( A_2 \) to be unital is nontrivial, \( A_2 \) does not figure in any such duality relation.

**Remark 2:** In Table I we have considered the composition of pairs of quantum-limited channels under the assumption that the two channels are simultaneously in their canonical forms. In order to help the reader appreciate this remark it should first be emphasised that a typical composite of this kind, say \( D(\kappa_2; 0) \circ C_1(\kappa_1; 0) \), should not stand for composition of two quantum-limited channels which are already in their respective canonical forms,
but rather to two channels picked one from either $Sp(2, R)$ orbit or double coset. That is $D(\kappa_2; 0) \circ C_1(\kappa_1; 0)$, for instance should stand for $((U(S_1) \circ D(\kappa_2; 0) \circ U(S_2)) \circ (U(S'_1) \circ C_1(\kappa_1; 0) \circ U(S'_2)))$, for arbitrary metaplectic unitaries corresponding to $S_1, S_2, S'_1, S'_2 \in Sp(2, R)$. The fact that two channels cannot in general be taken to their respective canonical forms simultaneously, i.e., with $S'_1 = S_1, S'_2 = S_2$, brings out the nontriviality of the assumption underlying Table 1. When this assumption is lifted, Table II gets much enriched into Table 2 as shown in the Appendix.

**Remark 3**: We have noted in Sections III to VI that the Kraus operators $\{W_\ell\}$ of quantum-limited Gaussian channels possess the property that the associated nonnegative operators $W_\ell^+ W_\ell$ are simultaneously diagonal (in the Fock basis for $D(\cdot), C_1(\cdot)$, and $C_2(\cdot)$ and in the position basis for $A_2$). In the present Section we presented for each nonsingular noisy channel a discrete set of Kraus operators, say $W_{mn}$, indexed by a pair of integer variables $m, n$ and it can be readily verified in each case that the associated nonnegative operators $W_{mn}^+ W_{mn}$ are simultaneously diagonal (in the Fock basis). For the noisy channel $A_2(a)$, it may be verified in the realization $D(\cdot) \circ A_2$ and $C_2(\cdot) \circ A_2$ [and not in $A_2 \circ D(\cdot), A_2 \circ C_1(\cdot)$, or $A_2 \circ C_2(\cdot)$] that the relevant nonnegative operators are simultaneously diagonal in the position basis. Finally, the single quadrature classical noise channel $B_1(a)$ being random unitary, the associated nonnegative operators are all multiples of unity. We may thus state

**Theorem 15**: For every Gaussian channel it is possible to obtain a Kraus representation such that the nonnegative operators associated with the Kraus operators are all simultaneously diagonal.

The above observation leads to the following remark on error correction.

**Remark on Error correction**: The fact that $W_\ell^+ W_\ell$’s are simultaneously diagonal in the Fock basis for all $\ell$ for the channels $D(\kappa; 0), C_1(\kappa; 0), C_2(\kappa; 0)$ and their composites, imply in the view of the work of [39] that the Fock states could be used to reliably transmit classical information through this channel. In such a case the channel is viewed as a generalised measurement. In other words, any classical information encoded in Fock states and passed through these channels can be reliably retrieved by a restoring channel.

**X. CONCLUSION**

We have obtained operator-sum representations for all single-mode bosonic Gaussian channels presented in their respective canonical forms. Evidently, the operator-sum representation of a channel not in the canonical form follows by adjoining of appropriate unitary Gaussian evolutions before and after the channel. The Kraus operators were obtained from the matrix elements of the two-mode metaplectic unitary operator which effects the channel action on a single mode. The two-mode symplectic transformation in each case did not mix the position and momentum variables and this fact proved valuable for our study. The Kraus operators for the quantum-limited channels except the singular case were found to have a simple and sparse structure in the Fock basis.

It was shown that the phase conjugation channels $D(\kappa)$ and $D(\kappa^{-1})$ are dual to one another, and the attenuator and the amplifier families $C_1(\kappa)$ and $C_2(\kappa^{-1})$, $\kappa < 1$ are mutually dual. The channels $D(\kappa), C_1(\kappa)$, and $C_2(\kappa)$ were found to be almost unital; in the sense that the unit operator was taken to a scalar times the unit operator. The channel $D(1)$ was found to be bistochastic but not random unitary. The unitary bistochastic channels being the classical noise channels $B_1$ and $B_2$ and the trivial identity channel $C_1(1) = C_2(1)$.

In the case of the phase conjugation channel, the action in phase space was brought out explicitly through the action of the Kraus operators on the Fock basis. The attenuator channel resulted in the scaling of the diagonal weight function $\phi(\alpha)$ and the amplifier channel resulted in the scaling of the Husimi $Q$-function as expected. Further, the output of the channel with respect to classicality/nonclassicality was studied. It was found that the phase conjugation channel $D(\kappa)$ and the singular channel $A_2$ are classicality breaking while the attenuator channel $C_1(\kappa)$ and the amplifier channel $C_2(\kappa)$ do not generate nonclassicality.

The action of the channel in the Fock basis gave an insight into the fixed points of the channel. The action in the Fock basis together with the action of the channel in phase space led us to conclude that there is a unique thermal state which is an invariant state for $D(\kappa)$, $\kappa < 1$. Further it was shown that the vacuum state is the only invariant state for the attenuator channel $C_1(\kappa)$, and that there is no finite energy state that is invariant for either the amplifier channel $C_2(\kappa)$ and the singular channel $A_2$.

Using Choi’s theorem, it was shown that all quantum-limited bosonic Gaussian channels are extremal. The Kraus operators of the phase conjugation channel was brought to a rank one form, thereby explicitly bringing out the entanglement breaking nature of the phase conjugation channel. It was further shown that there is no finite rank operator in the support of the Kraus operators of either the amplifier or the attenuator channel, and this explicitly demonstrates that the quantum-limited attenuator and the amplifier families of channels are not entanglement breaking. The Kraus
operators of the singular channel $A_2$ was also obtained in the rank one form thereby manifestly showing that this channel is entanglement breaking.

It was shown that every noisy Gaussian channel (except the singular case $B_1(a)$), can be obtained as the composition of a pair of quantum-limited channels as shown in Table [I]. This in turn implies that apart from those compositions that involve $A_2$, there is a discrete set of Kraus operators for all the noisy Gaussian channels. Further, the nonnegative operators $\{W_{\ell}^\dagger W_\ell\}$ were found to be simultaneously diagonal for all Gaussian channels. This throws light on the error correctability of these channels.

In bringing out the semigroup structure of the amplifier and the attenuator families of quantum-limited channels, it was shown that interrupted evolution slows down both amplification and attenuation in a manner characteristic of the quantum Zeno effect.

Acknowledgements: This work originated from an inspiring seminar on Gaussian channels which Raul Garcia-Patron Sanchez presented at the Institute of Mathematical Sciences, and the authors would like to thank him for stimulating their interest in the problems studied in this paper. They would like to thank K. R. Parthasarathy and V. S. Sunder for insightful comments, from mathematicians’ point of view, on the subtleties involved in the extension of Choi's theorem on extremality of unital [or trace-preserving] CP maps to the infinite dimensional case.

Appendix: Composition of a General pair of Quantum Limited Channels

Given two quantum-limited Gaussian channels whose $(X,Y)$ matrices in the sense of Section 1 are respectively $(X_1,Y_1), (X_2,Y_2)$, with $|\det X_j| = \kappa_j^2$, we have for the composite channel $(X,Y)$

$$X = X_1 X_2, \quad Y = X_1^T Y_1 X_2 + Y_2.$$  

If $(X_1^0,Y_1^0), (X_2^0,Y_2^0)$ are the canonical forms of $(X_1,Y_1), (X_2,Y_2)$ in the sense of [I], the most general $(X_1,Y_1), (X_2,Y_2)$ should necessarily have the form $(S_1 X_1^0 S_2, S_2^T Y_1^0 S_2), (S_3 X_2^0 S_4, S_4^T Y_2^0 S_4)$, with $S_1, S_2, S_3, S_4 \in Sp(2,R)$, so that

$$X = S_1 X_1^0 S_2 S_3 X_2^0 S_4,$$

$$Y = S_4^T X_2^0 S_3^T S_2^T Y_1^0 S_2 S_3 X_2^0 S_4 + S_4^T Y_2^0 S_4.$$  

Our problem now is to classify the orbits or cosets under the unitary equivalence $(X,Y) \sim (\tilde{S}_1 X \tilde{S}_2, \tilde{S}_2^T Y \tilde{S}_2)$, $\tilde{S}_1, \tilde{S}_2 \in Sp(2,R)$. Basically we have to determine the determinants of $X, Y$ in terms of the canonical parameters $\kappa_1, \kappa_2$ of the constituent channels. While $\det X$ is independent of $S_1, S_2, S_3, S_4$, $\det Y$ needs a careful consideration. It is this situation that $Y \geq 0$ is the sum of two (positive) terms that breaks our analysis into two distinct cases.

Let us first consider the case in which $X_2^0$ is nonsingular, so that both the terms of $Y$ are nonsingular: this case obviously corresponds to the first twelve entries of Table [I]. With the choice $\tilde{S}_2 = S_4^{-1}$ the second term of $Y$ becomes a multiple of the identity, and the first term of $Y$ becomes a multiple of $S_3^T S_2^T S_2 S_3 \in Sp(2,R)$, a positive symplectic matrix. We can now do a rotation $\tilde{S}_2 = R \in SO(2) \subset Sp(2,R)$ without affecting the second term in $Y$, so that the first term of $Y$ becomes a multiple of $\text{diag}(\lambda, \lambda^{-1})$, $\lambda, \lambda^{-1}$ being eigenvalues of $S_3^T S_2^T S_2 S_3$. Det $Y$ can now be easily evaluated. Removal of the mandatory quantum-limited noise, as dictated by the value of det $X$ [i.e., $|1 - \kappa^2|$, $1 + \kappa^2$, or 1 depending on det $X$ being positive, negative or zero, $\kappa^2$ equalling $\det |X|$], then yields the classical noise indicated in Table [I] by the second argument of the composite channel. This procedure is the one used for the first twelve entries of Table [I].

The case of singular $X_2^0$ is somewhat different. With the removal of $S_4$ with the choice $\tilde{S}_2 = S_4^{-1}$, as in the earlier case, in the first term of $Y$ the projection $X_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ picks out the 1,1 element of the matrix $S_2^T S_2 S_3 S_3 \in Sp(2,R)$. With the eigenvalues of this matrix again denoted $\lambda, \lambda^{-1}$ and assuming that $\theta$ is the rotation needed to diagonalize it, the 1,1 element equals $\lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta$ (a combination appearing in the last four entries of Table [I]). With this, the second term in $Y$ is a multiple of identity whereas the first term is a multiple of $\lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta$ times the projector $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: the determinant of $Y$ can be readily computed, and removal of the quantum-limited noise gives the numerical value of the second argument of the composite channel.

Finally, the two cases in the last entry of Table [I], the case of the composite $A_2(0) \circ A_2(0)$, correspond to the two possible situations that may arise with $X$ when both $X_1^0$ and $X_2^0$ are of rank one: $X$ could either be rank one or it could be a null matrix. Correspondingly, the composite $A_2(0) \circ A_2(0)$ should be viewed as a member of the $C_1(0; a) = A_1(a)$ family or the $A_2(a)$ family, the extra classical noise being ‘the same’ in both cases.
\[
\begin{align*}
C_2(k_2; 0) \circ C_1(k_1; 0) &= C_1\left(k_2k_1; \sqrt{(1 - k_2^2k_1^2)^2 + k_2^2(1 - k_1^2)(1 - k_2^2)(\lambda - \lambda^{-1})^2} - (1 - k_2^2k_1^2)\right), \\
C_2(k_2; 0) \circ C_2(k_1; 0) &= C_2\left(k_2k_1; \sqrt{(1 - k_2^2k_1^2 - 1)^2 + k_2^2(1 - k_1^2)(1 - k_2^2 - 1)(\lambda - \lambda^{-1})^2} - (k_2^2k_1^2 - 1)\right), \\
C_2(k_2; 0) \circ C_1(k_1; 0) &= C_1\left(k_2k_1; \sqrt{(2k_2^2 - k_1^2k_2^2 - 1)^2 + k_2^2(1 - k_1^2)(k_2^2 - 1)(\lambda - \lambda^{-1})^2} - (1 - k_2^2k_1^2)\right), \\
&\quad \text{for } k_2k_1 \leq 1; \\
C_2(k_2; 0) \circ C_1(k_1; 0) &= C_2\left(k_2k_1; \sqrt{(2k_2^2 - k_1^2k_2^2 - 1)^2 + k_2^2(1 - k_1^2)(k_2^2 - 1)(\lambda - \lambda^{-1})^2} - (k_2^2k_1^2 - 1)\right), \\
&\quad \text{for } k_2k_1 \geq 1; \\
C_1(k_2; 0) \circ C_2(k_1; 0) &= C_1\left(k_2k_1; \sqrt{(k_2^2k_1^2 - 2k_2^2 + 1)^2 + k_2^2(1 + k_1^2)(1 + k_2^2)(\lambda - \lambda^{-1})^2} - (1 - k_2^2k_1^2)\right), \\
&\quad \text{for } k_2k_1 \leq 1; \\
C_1(k_2; 0) \circ C_2(k_1; 0) &= C_2\left(k_2k_1; \sqrt{(k_2^2k_1^2 - 2k_2^2 + 1)^2 + k_2^2(1 + k_1^2)(1 + k_2^2)(\lambda - \lambda^{-1})^2} - (k_2^2k_1^2 - 1)\right), \\
&\quad \text{for } k_2k_1 \geq 1. \\
D(k_2; 0) \circ D(k_1; 0) &= D\left(k_2k_1; \sqrt{(2k_2^2 - k_1^2k_2^2 + 1)^2 + k_2^2(1 + k_1^2)(1 + k_2^2)(\lambda - \lambda^{-1})^2} - (1 + k_2^2k_1^2)\right), \\
&\quad \text{for } k_2k_1 \leq 1; \\
D(k_2; 0) \circ D(k_1; 0) &= D\left(k_2k_1; \sqrt{(2k_2^2 - k_1^2k_2^2 - 1)^2 + k_2^2(1 + k_1^2)(1 + k_2^2)(\lambda - \lambda^{-1})^2} - (1 + k_2^2k_1^2)\right), \\
&\quad \text{for } k_2k_1 \geq 1. \\
C_1(k_2) \circ A_2(0) &= A_2\left(1 + \sqrt{1 + k_2^2(\lambda - \lambda^{-1})^2 - 1}\right), \\
C_2(k_2) \circ A_2(0) &= A_2\left(\sqrt{(2k_2^2 - 1)^2 + k_2^2(\lambda - \lambda^{-1})^2} - 1\right), \\
D(k_2) \circ A_2(0) &= A_2\left(\sqrt{(2k_2^2 + 1)^2 + k_2^2(\lambda - \lambda^{-1})^2} - 1\right), \\
A_2(0) \circ C_1(k_1) &= A_2\left(\sqrt{1 + (\lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta)(1 - k_1^2)} - 1\right), \\
A_2(0) \circ C_2(k_1) &= A_2\left(\sqrt{1 + (\lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta)(k_1^2 - 1)} - 1\right), \\
A_2(0) \circ D(k_1) &= A_2\left(\sqrt{1 + (\lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta)(1 + k_1^2)} - 1\right), \\
A_2(0) \circ A_2(0) &= A_2\left(1 + \lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta - 1\right), \text{ or } \\
&\quad C_1\left(0; 1 + \lambda \cos^2 \theta + \lambda^{-1} \sin^2 \theta - 1\right).
\end{align*}
\]

TABLE II: Showing composition of two quantum-limited Gaussian channels which are not necessarily in their respective canonical forms. It may be seen that the case \( \lambda = 1 \) corresponds to Table III in the case of the first twelve entries the noise is always greater than what obtains in Table III. But in the remaining four cases the noise can be either more or less, depending on the value of \( \theta \).

We conclude with the following observations in respect of Table III. It is evident that in all cases \( \lambda = 1 \) corresponds to the situation in which both constituents of the composite are already in their canonical forms, and the reader can verify that Table III reduces to Table II in this case. And thus we see that \( \lambda \neq \lambda^{-1} \) results, in the first sixteen entries of Table III in classical noise which is always more in magnitude than the case \( \lambda = 1 \). The last four entries of Table III are distinguished in this regard: since the classical noise depends also on the choice of \( \theta \), it can be either more or less than what obtains in the \( \lambda = 1 \) case.

[1] C. M. Caves, P. D. Drummond, Rev. Mod. Phys. 66, 481 (1994).
[2] S. L. Braunstein, and P. van Loock, Rev. Mod. Phys. 77, 513–577 (2005); G. Adesso and F. Illuminati, J. Phys. A: Math. Theor. 40, 7821 (2007); X.-B. Wang, T. Hiroshima, A. Tomita, M. Hayashi, Phys. Rep. 448, 1 (2007); K. Hammerer A. S. Sørensen and E. S. Polzik, Rev. Mod. Phys. 82, 10411093 (2010).
[3] J. DiGuglielmo, B. Hage, A. Franzen, J. Fiuřák, and R. Schnabel, Phys. Rev. A 76, 012323 (2007); J. Laurat, G. Keller, J. A. O.-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso, and F. Illuminati, J. Opt. B: Quantum semiclass. Opt. 7, S577 (2005); V. D’Auria, S. Fornaro, A. Porzio, S. Solimeno, S. Olivares, and M. G. A. Paris, Phys. Rev. Lett. 102,
[56] Y. -X. Liu, Ş. K. Özdemir, A. Miranowicz, and N. Imoto, Phys. Rev. A 70, 042308 (2004); I. L. Chuang, D. W. Leung, and Y. Yamamoto, Phys. Rev. A 56, 1114 (1997).
[57] N. Lütkenhaus and S. M. Barnett, Phys. Rev. A 51, 3340 (1995).
[58] The integral 7.374 on p.796 of Ref.[46].
[59] V. Giovannetti, A. S. Holevo, S. Lloyd, and L. Maccone, J. Phys. A: Math. Theor. 43, 415305 (2010).
[60] K. Kraus, States, Effects, and Operations: Fundamental Notions in Quantum Theory, Lecture Notes in Physics vol. 190, Springer-Verlag, Berlin (1983).
[61] The integral 7.377 on p.797 of Ref.[46].
[62] S. -K. Tsui, Proc. Am. Math. Soc. 124, 437 (1996).
[63] R. Bhatia, *Matrix Analysis*, Springer (2004).
[64] S. L. Tregub, Sov. Math. 30, 105 (1986); B. Kummerer and H. Maassen, Commun. Math. Phys. 109, 1 (1987).