$q$-Fibonacci bicomplex quaternions

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Abstract
In the paper, we define the $q$-Fibonacci bicomplex quaternions and the $q$-Lucas bicomplex quaternions, respectively. Then, we give some algebraic properties of $q$-Fibonacci bicomplex quaternions and the $q$-Lucas bicomplex quaternions.

Keywords: Bicomplex number; $q$-integer; Fibonacci number; Bicomplex Fibonacci quaternion; $q$-quaternion; $q$-Fibonacci quaternion.

1. Introduction

The real quaternions were first described by Irish mathematician William Rowan Hamilton in 1843. The real quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors. Hamilton$^{[1]}$ introduced the set of quaternions which can be represented as

$$H = \left\{ q = q_0 + i q_1 + j q_2 + k q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \right\} \quad (1)$$

where

$$i^2 = j^2 = k^2 = -1, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j. \quad (2)$$

Horadam$^{[2,3]}$ defined complex Fibonacci and Lucas quaternions as follows

$$Q_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k \quad (3)$$

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and
\[ K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k \]  
(4)
where \( F_n \) and \( L_n \) denote the \( n \)-th Fibonacci and Lucas numbers, respectively. Also, the imaginary quaternion units \( i, j, k \) have the following rules
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

There are several studies on quaternions as the Fibonacci and Lucas quaternions and their generalizations, for example,[4, 5, 6, 8, 10, 11, 12, 13, 14].

Bicomplex numbers were introduced by Corrado Segre in 1892[15]. In 1991, G. Baley Price, the bicomplex numbers are given in his book based on multi-complex spaces and functions[16]. In recent years, fractal structures of these numbers have also been studied[17, 18]. The set of bicomplex numbers can be expressed by the basis \( \{1, i, j, ij\} \) (Table 1) as,
\[ C_2 = \{ q = q_1 + i q_2 + j q_3 + ij q_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R} \} \]  
(5)
where \( i, j \) and \( ij \) satisfy the conditions
\[ i^2 = -1, \quad j^2 = -1, \quad ij = ji. \]  
(6)

In 2018, the bicomplex Fibonacci quaternions defined by Aydn Torunbalci[19] as follows
\[ \mathbb{BF}_n = F_n + i F_{n+1} + j F_{n+2} + ij F_{n+3} \]
\[ = (F_n + i F_{n+1}) + (F_{n+2} + i F_{n+3})j \]  
(7)
where \( i, j \) and \( ij \) satisfy the conditions[6].

The theory of the quantum \( q \)-calculus has been extensively studied in many branches of mathematics as well as in other areas in biology, physics, electrochemistry, economics, probability theory, and statistics[20, 21]. For \( n \in \mathbb{N}_0 \), the \( q \)-integer \( [n]_q \) is defined as follows
\[ [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}. \]  
(8)

By (8), for all \( m, n \in \mathbb{Z} \), can be easily obtained \( [m + n]_q = [m]_q + q^m [n]_q \).

For more details related to the quantum \( q \)-calculus, we refer to[22, 23].
In 2019, \(q\)-Fibonacci hybrid and \(q\)-Lucas hybrid numbers defined by Kızılateş [24] as follows

\[
\mathcal{HF}_n(\alpha; q) = \alpha^{n-1}[n]_q + \alpha^n[n + 1]_q i + \alpha^{n+1}[n + 2]_q \varepsilon + \alpha^{n+2}[n + 3]_q h
\]

and

\[
\mathcal{HL}_n(\alpha; q) = \alpha^n[2n]_q + \alpha^{n+1}[2n + 2]_q i + \alpha^{n+2}[2n + 4]_q \varepsilon + \alpha^{n+3}[2n + 6]_q h
\]

where \(i, \varepsilon\) and \(h\) satisfy the conditions

\[
i^2 = -1, \quad \varepsilon^2 = 0, \quad h^2 = 1, \quad i h = h i = \varepsilon + i.
\]

Also, Kızılateş derived several interesting properties of these numbers such as Binet-Like formulas, exponential generating functions, summation formulas, Cassini-like identities, Catalan-like identities and d’Ocagne-like identities [24].

2. \(q\)-Fibonacci bicomplex quaternions

In this section, we define \(q\)-Fibonacci bicomplex quaternions and \(q\)-Lucas bicomplex quaternions by using the basis \(\{1, i, j, i j\}\), where \(i, j\) and \(i j\) satisfy the conditions (6) as follows

\[
\mathbb{BF}_n(\alpha; q) = \alpha^{n-1}[n]_q + \alpha^n[n + 1]_q i + \alpha^{n+1}[n + 2]_q j + \alpha^{n+2}[n + 3]_q i j
\]

\[
= \alpha^n \left( \frac{1-q^n}{\alpha-\alpha q} \right) i + \alpha^{n+1} \left( \frac{1-q^{n+1}}{\alpha-\alpha q} \right) j
\]

\[
+ \alpha^{n+2} \left( \frac{1-q^{n+2}}{\alpha-\alpha q} \right) i j + \alpha^{n+3} \left( \frac{1-q^{n+3}}{\alpha-\alpha q} \right) i j
\]

\[
= \frac{\alpha^n}{\alpha-(\alpha q)} \left[ 1 + \alpha i + \alpha^2 j + \alpha^3 i j \right]
\]

\[
- \frac{(\alpha q)^n}{\alpha-(\alpha q)} \left[ 1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j \right]
\]
bicomplex quaternions are defined by

\[ \mathbb{BF}_n(\alpha; q) = \alpha^n \left( \frac{[2n]_q}{[n]_q} + \alpha^n \frac{[2n+2]_q}{[n+1]_q} i + \alpha^{n+2} \frac{[2n+4]_q}{[n+2]_q} j + \alpha^{n+3} \frac{[2n+6]_q}{[n+3]_q} i j \right) \]

\[ = \alpha^{2n} \left( \frac{1-q^{2n}}{\alpha^n-(\alpha q)^n} \right) + \alpha^{2n+2} \left( \frac{1-q^{2n+2}}{\alpha^{n+1}-(\alpha q)^{n+1}} \right) i \\
+ \alpha^{2n+4} \left( \frac{1-q^{2n+4}}{\alpha^{n+2}-(\alpha q)^{n+2}} \right) j + \alpha^{2n+6} \left( \frac{1-q^{2n+6}}{\alpha^{n+3}-(\alpha q)^{n+3}} \right) i j \]

\[ = \alpha^n \left( 1 + \alpha i + \alpha^2 j + \alpha^3 i j \right) \\
- (\alpha q)^n \left( 1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j \right) \] (13)

For \( \alpha = \frac{1+\sqrt{5}}{2} \) and \( (\alpha q) = \frac{1}{\alpha} \), q-Fibonacci bicomplex quaternion \( \mathbb{BF}_n(\alpha; q) \) become the bicomplex Fibonacci quaternions \( \mathbb{BF}_n \).

The addition, subtraction and multiplication by real scalars of two q-Fibonacci bicomplex quaternions gives q-Fibonacci bicomplex quaternion. Then, the addition, subtraction and multiplication by scalar of q-Fibonacci bicomplex quaternions are defined by

\[ \mathbb{BF}_n(\alpha; q) \pm \mathbb{BF}_m(\alpha; q) = \left( \alpha^{n-1}[n]_q + \alpha^n[n+1]_q i + \alpha^{n+1}[n+2]_q j \right. \\
+ \alpha^{n+2}[n+3]_q i j) \\
\pm \left( \alpha^{m-1}[m]_q + \alpha^m[m+1]_q i + \alpha^{m+1}[m+2]_q j \right. \\
+ \alpha^{m+2}[m+3]_q i j) \\
= [\alpha^n \left( \frac{1-q^n}{\alpha-q} \right) \pm \alpha^m \left( \frac{1-q^m}{\alpha-q} \right)] \\
+ \left[ \alpha^{n+1} \left( \frac{1-q^{n+1}}{\alpha-q} \right) \pm \alpha^{m+1} \left( \frac{1-q^{m+1}}{\alpha-q} \right) \right] i \\
+ \left[ \alpha^{n+2} \left( \frac{1-q^{n+2}}{\alpha-q} \right) \pm \alpha^{m+2} \left( \frac{1-q^{m+2}}{\alpha-q} \right) \right] j \\
+ \left[ \alpha^{n+3} \left( \frac{1-q^{n+3}}{\alpha-q} \right) \pm \alpha^{m+3} \left( \frac{1-q^{m+3}}{\alpha-q} \right) \right] i j \]

\[ = \frac{1}{\alpha-q} \left\{ (\alpha^n \pm \alpha^m)(1 + \alpha i + \alpha^2 j + \alpha^3 i j) \\
+ ((\alpha q)^n \pm (\alpha q)^m)(1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j) \right\}. \] (14)

The multiplication of q-Fibonacci bicomplex quaternion by the real scalar \( \lambda \) is defined as

\[ \lambda \mathbb{BF}_n(\alpha; q) = \lambda \alpha^n(1 + \alpha i + \alpha^2 j + \alpha^3 i j) \\
+ \lambda (\alpha q)^n(1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j). \] (15)
Thus, the scalar and the vector part of $BF_n(\alpha; q)$ which is the $n$–th term of the $q$-Fibonacci bicomplex quaternion are denoted by

$$S_{BF_n(\alpha; q)} = \alpha^{n-1}[n]_q, \quad V_{BF_n(\alpha; q)} = \alpha^n[n+1]_q i + \alpha^{n+1}[n+2]_q j + \alpha^{n+2}[n+3]_q i j.$$  

(16)

Thus, the $q$-Fibonacci bicomplex quaternion $BF_n(\alpha; q)$ is given by

$$BF_n(\alpha; q) = S_{BF_n(\alpha; q)} + V_{BF_n(\alpha; q)}$$

The multiplication of two $q$-Fibonacci bicomplex quaternions is defined by

$$BF_n(\alpha; q) \times BF_m(\alpha; q) = \begin{array}{c}
(\alpha^{n-1}[n]_q + \alpha^n[n+1]_q i + \alpha^{n+1}[n+2]_q j \\
+ \alpha^{n+2}[n+3]_q i j)
\end{array} \times \begin{array}{c}
(\alpha^{m-1}[m]_q + \alpha^m[m+1]_q i + \alpha^{m+1}[m+2]_q j \\
+ \alpha^{m+2}[m+3]_q i j)
\end{array}$$

$$= \frac{1}{\alpha^m \alpha^n} (\alpha^{n+m}) \{ (1 - \alpha^2 - \alpha^4 + \alpha^6) \\
+ 2i (\alpha - \alpha^5) + 2j (\alpha^2 - \alpha^4) + 4ij (\alpha^3) \\
- q^m \{ (1 - \alpha(\alpha q) - \alpha^2(\alpha q)^2 + \alpha^3(\alpha q)^3) \\
+ i(\alpha + (\alpha q) - \alpha^2(\alpha q)^3 - \alpha^3(\alpha q)^2) \\
+ j(\alpha^2 + (\alpha q)^2 - \alpha(\alpha q)^3 - \alpha^3(\alpha q)) \\
+ i j (\alpha^3 + (\alpha q)^3 - \alpha(\alpha q)^2 - \alpha^2(\alpha q)) \\
- q^n \{ (1 - \alpha(\alpha q) - \alpha^2(\alpha q)^2 + \alpha^3(\alpha q)^3) \\
+ i(\alpha + (\alpha q) - \alpha^2(\alpha q)^3 - \alpha^3(\alpha q)^2) \\
+ j(\alpha^2 + (\alpha q)^2 - \alpha(\alpha q)^3 - \alpha^3(\alpha q)) \\
+ i j (\alpha^3 + (\alpha q)^3 + \alpha^2(\alpha q) + \alpha(\alpha q)^2) \\
+ q^{n+m} \{ (1 - (\alpha q)^2 - (\alpha q)^4 + (\alpha q)^6) \\
+ 2i ((\alpha q) - (\alpha q)^5) + 2j ((\alpha q)^2 - (\alpha q)^4) \\
+ 4i j ((\alpha q)^3) \}
\}$$

$$= BF_n(\alpha; q) \times BF_m(\alpha; q)$$

(17)

Here, quaternion multiplication is done using bicomplex quaternionic units (table 1), and this product is commutative.

Also, the $q$-Fibonacci bicomplex quaternion product may be obtained as follows
Table 1: Multiplication scheme of bicomplex units

| $x$ | 1  | $i$ | $j$ | $ij$ |
|-----|----|-----|-----|------|
| 1   | 1  | $i$ | $j$ | $ij$ |
| $i$ | $i$| $-1$| $ij$| $-j$ |
| $j$ | $j$| $ij$| $-1$| $-i$ |
| $ij$| $ij$| $-j$| $-i$| $1$  |

\[
\mathbb{BF}_n(\alpha; q) \times \mathbb{BF}_m(\alpha; q) = \begin{pmatrix}
\alpha^{n-1}[n]_q & -\alpha^n[n+1]_q & -\alpha^{n+1}[n+2]_q & \alpha^{n+2}[n+3]_q \\
\alpha^n[n+1]_q & -\alpha^{n-1}[n]_q & -\alpha^{n+2}[n+3]_q & -\alpha^{n+1}[n+2]_q \\
\alpha^{n+1}[n+2]_q & -\alpha^{n+2}[n+3]_q & \alpha^{n-1}[n]_q & -\alpha^n[n+1]_q \\
\alpha^{n+2}[n+3]_q & \alpha^{n+1}[n+2]_q & \alpha^n[n+1]_q & -\alpha^{n-1}[n]_q \\
\end{pmatrix} \begin{pmatrix}
\alpha^{m-1}[m]_q \\
\alpha^m[m+1]_q \\
\alpha^{m+1}[m+2]_q \\
\alpha^{m+2}[m+3]_q \\
\end{pmatrix}
\]

(18)

Three kinds of conjugation can be defined for bicomplex numbers \[17, 18\]. Therefore, conjugation of the $q$-Fibonacci bicomplex quaternion is defined in three different ways as follows

\[
(\mathbb{BF}_n(\alpha; q))^{*1} = (\alpha^{n-1}[n]_q - i \alpha^n[n+1]_q + j \alpha^{n+1}[n+2]_q - ij \alpha^{n+2}[n+3]_q),
\]

(19)

\[
(\mathbb{BF}_n(\alpha; q))^{*2} = (\alpha^{n-1}[n]_q + i \alpha^n[n+1]_q - j \alpha^{n+1}[n+2]_q - ij \alpha^{n+2}[n+3]_q),
\]

(20)

\[
(\mathbb{BF}_n(\alpha; q))^{*3} = (\alpha^{n-1}[n]_q - i \alpha^n[n+1]_q - j \alpha^{n+1}[n+2]_q + ij \alpha^{n+2}[n+3]_q).
\]

(21)

Therefore, the norm of the $q$-Fibonacci bicomplex quaternion $\mathbb{BF}_n(\alpha; q)$ is defined in three different ways as follows

\[
N(\mathbb{BF}_n(\alpha; q))^{*1} = \| (\mathbb{BF}_n(\alpha; q)) \times (\mathbb{BF}_n(\alpha; q))^{*1} \|^2,
\]

(22)

\[
N(\mathbb{BF}_n(\alpha; q))^{*2} = \| (\mathbb{BF}_n(\alpha; q)) \times (\mathbb{BF}_n(\alpha; q))^{*2} \|^2,
\]

(23)

\[
N(\mathbb{BF}_n(\alpha; q))^{*3} = \| (\mathbb{BF}_n(\alpha; q)) \times (\mathbb{BF}_n(\alpha; q))^{*3} \|^2.
\]

(24)
Theorem 2.1. (Binet’s Formula). Let $\mathbb{B}_n(\alpha; q)$ and $\mathbb{B}_n(\alpha; q)$ be the $q$-Fibonacci bicomplex quaternion and the $q$-Lucas bicomplex quaternion. For $n \geq 1$, Binet’s formula for these quaternions respectively, is as follows:

$$
\mathbb{B}_n(\alpha; q) = \frac{\alpha^n \hat{\gamma} - (\alpha q)^n \hat{\delta}}{\alpha - \alpha q}, \quad (25)
$$

and

$$
\mathbb{B}_n(\alpha; q) = \alpha^n \hat{\gamma} + (\alpha q)^n \hat{\delta}, \quad (26)
$$

where

$$
\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 i j, \quad \alpha = \frac{1 + \sqrt{5}}{2}
$$

and

$$
\hat{\delta} = 1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j, \quad \alpha q = -\frac{1}{\alpha}.
$$

Proof. (25): Using (8) and (12), we find that

$$
\mathbb{B}_n(\alpha; q) = \alpha^{n-1} [n]_q + \alpha^n [n + 1]_q i + \alpha^{n+1} [n + 2]_q j + \alpha^{n+2} [n + 3]_q i j
$$

$$
= \alpha^n \frac{1 - q^n}{\alpha - \alpha q} + \alpha^{n+1} \frac{1 - q^{n+1}}{\alpha - \alpha q} i + \alpha^{n+2} \frac{1 - q^{n+2}}{\alpha - \alpha q} j + \alpha^{n+3} \frac{1 - q^{n+3}}{\alpha - \alpha q} i j
$$

$$
= \frac{\alpha^n [1 + \alpha i + \alpha^2 j + \alpha^3 i j] - (\alpha q)^n [1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j]}{\alpha - (\alpha q)}
$$

$$
= \frac{\alpha^n \hat{\gamma} - (\alpha q)^n \hat{\delta}}{\alpha - \alpha q}
$$

In a similar way, equality (26) can be derived as follows

$$
\mathbb{B}_n(\alpha; q) = \frac{\alpha^n [2n]_q}{[n]_q} + \alpha^{n+1} \frac{[2n+2]_{[n]}_q}{[n+1]_q} i + \alpha^{n+2} \frac{[2n+4]_{[n+2]}_q}{[n+3]_q} j + \alpha^{n+3} \frac{[2n+6]_{[n+3]}_q}{[n+4]_q} i j
$$

$$
= \alpha^{2n} \frac{1 - q^{2n}}{(\alpha - (\alpha q)^n)} + \alpha^{2n+2} \left( \frac{1 - q^{2n+2}}{(\alpha^{n+1} - (\alpha q)^{n+1})} \right) i
$$

$$
+ \alpha^{2n+4} \left( \frac{1 - q^{2n+4}}{(\alpha^{n+2} - (\alpha q)^{n+2})} \right) i j + \alpha^{2n+6} \left( \frac{1 - q^{2n+6}}{(\alpha^{n+3} - (\alpha q)^{n+3})} \right) i j
$$

$$
= \frac{\alpha^{2n}}{\alpha^n} (1 + \alpha i + \alpha^2 j + \alpha^3 i j)
$$

$$
- \frac{(\alpha q)^{2n}}{\alpha^n} (1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 i j)
$$

$$
= \alpha^n \hat{\gamma} - (\alpha q)^n \hat{\delta}.
$$
where \( \hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k \), \( \hat{\delta} = 1 + (\alpha q) i + (\alpha q)^2 j + (\alpha q)^3 k \) and \( \hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma} \).

**Theorem 2.2. (Exponential generating function)**

Let \( BF_n(\alpha; q) \) be the \( q \)-Fibonacci bicomplex quaternion. For the exponential generating function for these quaternions is as follows:

\[
g_{BF_n(\alpha; q)} \left( \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} BF_n(\alpha; q) \frac{t^n}{n!} = \frac{\hat{\gamma} e^{\alpha t} - \hat{\delta} e^{(\alpha q) t}}{\alpha - \alpha q} \quad (27)
\]

**Proof.** Using the definition of exponential generating function, we obtain

\[
\sum_{n=0}^{\infty} BF_n(\alpha; q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{\alpha^n \hat{\gamma} - (\alpha q)^n \hat{\delta}}{\alpha - \alpha q} \right) \frac{t^n}{n!} = \frac{\hat{\gamma} e^{\alpha t} - \hat{\delta} e^{(\alpha q) t}}{\alpha - \alpha q} \quad (28)
\]

Thus, the proof is completed.

**Theorem 2.3. (Honsberger identity)**

For \( n, m \geq 0 \) the Honsberger identity for the \( q \)-Fibonacci bicomplex quaternions \( BF_n(\alpha; q) \) and \( BF_m(\alpha; q) \) is given by

\[
BF_n(\alpha; q) BF_m(\alpha; q) + BF_{n+1}(\alpha; q) BF_{m+1}(\alpha; q)
= \frac{\alpha^{n+m}}{\alpha - \alpha q} \left\{ (1 + \alpha^2 \hat{\gamma}) - \hat{\gamma} \delta (1 + \alpha (\alpha q)) (q^n + q^m) + (1 + (\alpha q)^2) q^{n+m} \hat{\delta}^2 \right\} .
\]

**Proof. (29):** By using (12) and (25) we get,

\[
BF_n(\alpha; q) BF_m(\alpha; q) + BF_{n+1}(\alpha; q) BF_{m+1}(\alpha; q)
= \left( \frac{\alpha^n \hat{\gamma} - (\alpha q)^n \hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^m \hat{\gamma} - (\alpha q)^m \hat{\delta}}{\alpha - \alpha q} \right) + \left( \frac{\alpha^{n+1} \hat{\gamma} - (\alpha q)^{n+1} \hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{m+1} \hat{\gamma} - (\alpha q)^{m+1} \hat{\delta}}{\alpha - \alpha q} \right)
= \frac{\alpha^{n+m}}{(\alpha - \alpha q)^2} \left\{ (\hat{\gamma} - q^n \hat{\delta})(\hat{\gamma} - q^m \hat{\delta}) \right\} + \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} \left\{ \hat{\gamma} - q^{n+1} \hat{\delta})(\hat{\gamma} - q^{m+1} \hat{\delta}) \right\}
= \frac{\alpha^{n+m}}{(\alpha - \alpha q)^2} \left\{ (1 + \alpha^2 \hat{\gamma}^2) - \hat{\gamma} \delta (1 + \alpha (\alpha q)) (q^n + q^m) + (1 + (\alpha q)^2) q^{n+m} \hat{\delta}^2 \right\}.
\]

where \( \hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma} \).
Theorem 2.4. (d’Ocagne’s identity)

For \( n, m \geq 0 \) the d’Ocagne’s identity for the \( q \)-Fibonacci bicomplex quaternions \( BF_n(\alpha; q) \) and \( BF_m(\alpha; q) \) is given by

\[
BF_m(\alpha; q) BF_{n+1}(\alpha; q) - BF_{m+1}(\alpha; q) BF_n(\alpha; q) = \frac{\alpha^{n+m-1}(q^n-q^m)\gamma\delta}{(1-q)}. \tag{30}
\]

**Proof.** (30): By using (12) and (25) we get,

\[
BF_m(\alpha; q) BF_{n+1}(\alpha; q) - BF_{m+1}(\alpha; q) BF_n(\alpha; q) = \frac{\alpha^m (\alpha q)^m\gamma}{(\alpha - \alpha q)} \left( \frac{\alpha^{n+1} (\alpha q)^{n+1}\gamma + \alpha^n (\alpha q)^n\delta}{(\alpha - \alpha q)} \right) - \frac{\alpha^{n+1} (\alpha q)^n\delta}{(\alpha - \alpha q)} \left( \frac{\alpha^n (\alpha q)^n\gamma}{(\alpha - \alpha q)} \right) = \frac{\alpha^{n+m+1} (q^n-q^m)\gamma\delta}{(1-q)}.
\]

Here, \( \hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma} \) is used.

Theorem 2.5. (Cassini Identity)

Let \( BF_n(\alpha; q) \) be the \( q \)-Fibonacci bicomplex quaternion. For \( n \geq 1 \), Cassini’s identity for \( BF_n(\alpha; q) \) is as follows:

\[
BF_{n+1}(\alpha; q) BF_{n-1}(\alpha; q) - BF_n(\alpha; q)^2 = \frac{\alpha^{2n-2} q^n (1 - q^{-1}) \gamma \delta}{(1-q)}. \tag{31}
\]

**Proof.** (31): By using (12) and (25) we get,

\[
BF_{n+1}(\alpha; q) BF_{n-1}(\alpha; q) - BF_n(\alpha; q)^2 = \left( \frac{\alpha^{n+1} (\alpha q)^{n+1}\gamma}{(\alpha - \alpha q)} \right) \left( \frac{\alpha^{n-1} (\alpha q)^{n-1}\gamma}{(\alpha - \alpha q)} \right) - \left( \frac{\alpha^n (\alpha q)^n\delta}{(\alpha - \alpha q)} \right)^2 = \alpha^{2n} q^n (1-q)(1-q^{-1}) \gamma \delta \left( \frac{(\alpha - \alpha q)^2}{(1-q)} \right) = \frac{\alpha^{2n-2} q^n (1-q^{-1}) \gamma \delta}{(1-q)}.
\]

Here, \( \hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma} \) is used.
Theorem 2.6. (Catalan’s Identity)
Let \( BF_n(\alpha; q) \) be the q-Fibonacci bicomplex quaternion. For \( n \geq 1 \), Catalan’s identity for \( BF_n(\alpha; q) \) is as follows:

\[
BF_{n+r}(\alpha; q) BF_{n-r}(\alpha; q) - BF_n(\alpha; q)^2 = \frac{\alpha^{2n-2} q^n (1 - q^r)(1 - q^{-r}) \hat{\gamma} \hat{\delta}}{(1 - q)^2}.
\] (32)

Proof. By using (12) and (25) we get

\[
BF_{n+r}(\alpha; q) BF_{n-r}(\alpha; q) - BF_n(\alpha; q)^2 = \left( \frac{\alpha^{n+r} \hat{\gamma} - (\alpha q)^{n+r} \hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{n-r} \hat{\gamma} - (\alpha q)^{n-r} \hat{\delta}}{\alpha - \alpha q} \right) - \left( \frac{(\alpha - \alpha q)^2}{\alpha q^{n-r} \hat{\gamma} \delta - \alpha^{2n} q^{n+r} \hat{\gamma} \hat{\delta} + 2 \alpha^{2n} q^n \hat{\gamma} \hat{\delta}} \right) = \frac{\alpha^{2n} q^n \hat{\gamma} \hat{\delta} (q^{-r} - 1) + (q^r - 1)}{(\alpha - \alpha q)^2} \frac{\alpha^{2n} q^n \hat{\gamma} \hat{\delta} (1 - q^{-r})(1 - q^r)}{(1 - q)^2}.
\]

Here, \( \hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma} \) is used.

3. Conclusion

In this paper, algebraic and analytic properties of the q-Fibonacci bicomplex quaternions are investigated.

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