ESSENTIAL DIMENSION OF FINITE GROUPS
IN PRIME CHARACTERISTIC

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ABSTRACT. Let \( F \) be a field of characteristic \( p > 0 \) and \( G \) be a smooth finite algebraic group over \( F \). We compute the essential dimension \( \text{ed}_F(G;p) \) of \( G \) at \( p \). That is, we show that
\[
\text{ed}_F(G;p) = \begin{cases} 
1, & \text{if } p \text{ divides } |G|, \\
0, & \text{otherwise.}
\end{cases}
\]

RéSUMÉ. Soit \( F \) un corps de caractéristique \( p > 0 \), et soit \( G \) un groupe algébrique fini étalesur \( F \). On calcule la dimension essentielle de \( G \) en \( p \), que l’on note \( \text{ed}_F(G;p) \). Plus précisément, on démontre que
\[
\text{ed}_F(G;p) = \begin{cases} 
1, & \text{si } p \text{ divise } |G|, \\
0, & \text{sinon.}
\end{cases}
\]

1. Introduction

Let \( F \) be a field and \( G \) be an algebraic group over \( F \). We begin by recalling the definition of the essential dimension of \( G \).

Let \( K \) be a field containing \( F \) and \( \tau: T \to \text{Spec}(K) \) be a \( G \)-torsor. We will say that \( \tau \) descends to an intermediate subfield \( F \subset K_0 \subset K \) if \( \tau \) is the pull-back of some \( G \)-torsor \( \tau_0: T_0 \to \text{Spec}(K_0) \), i.e., if there exists a Cartesian diagram of the form

\[
\begin{array}{ccc}
T & \longrightarrow & T_0 \\
\downarrow & & \downarrow \tau_0 \\
\text{Spec}(K) & \longrightarrow & \text{Spec}(K_0)
\end{array}
\]

The essential dimension of \( \tau \), denoted by \( \text{ed}_F(\tau) \), is the smallest value of the transcendence degree \( \text{trdeg}(K_0/F) \) such that \( \tau \) descends to \( K_0 \). The essential dimension of \( G \), denoted by \( \text{ed}_F(G) \), is the maximal value of \( \text{ed}_F(\tau) \), as \( K \) ranges over all fields containing \( F \) and \( \tau \) ranges over all \( G \)-torsors \( T \to \text{Spec}(K) \).

Now let \( p \) be a prime integer. A field \( K \) is called \( p \)-closed if the degree of every finite extension \( L/K \) is a power of \( p \). Equivalently, \( \text{Gal}(K^s/K) \) is a pro-\( p \)-group, where \( K^s \) is a

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separable closure of $K$. For example, the field of real numbers is $2$-closed. The essential dimension $\text{ed}_F(G; p)$ of $G$ at $p$ is the maximal value of $\text{ed}_F(\tau)$, where $K$ ranges over $p$-closed fields $K$ containing $F$, and $\tau$ ranges over the $G$-torsors $T \to \text{Spec}(K)$. For an overview of the theory of essential dimension, we refer the reader to the surveys [Rei10] and [Me13].

The case where $G$ is a finite group (viewed as a constant group over $F$) is of particular interest. A theorem of N. A. Karpenko and A. S. Merkurjev [KM08] asserts that in this case

$$\text{ed}_F(G; p) = \text{ed}_F(G_p; p) = \text{ed}_F(G_p) = \text{rdim}_F(G_p),$$

provided that $F$ contains a primitive $p$-th root of unity $\zeta_p$. Here $G_p$ is any Sylow $p$-subgroup of $G$, and $\text{rdim}_F(G_p)$ denotes the minimal dimension of a faithful representation of $G_p$ defined over $F$. For example, assuming that $\zeta_p \in F$, $\text{ed}_F(G) = \text{ed}(G; p) = r$ if $G = (\mathbb{Z}/p\mathbb{Z})^r$, and $\text{ed}(G) = \text{ed}(G; p) = p$ if $G$ is a non-abelian group of order $p^3$. Further examples can be found in [MR10].

Little is known about essential dimension of finite groups over a field $F$ of characteristic $p > 0$. A. Ledet [Le04] conjectured that

$$\text{ed}_F(\mathbb{Z}/p^r\mathbb{Z}) = r$$

for every $r \geq 1$. This conjecture remains open for every $r \geq 3$. In this paper we will prove the following surprising result.

**Theorem 1.** Let $F$ be a field of characteristic $p > 0$ and $G$ be a smooth finite algebraic group over $F$. Then

$$\text{ed}_F(G; p) = \begin{cases} 1, & \text{if } p \text{ divides } |G|, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, Ledet’s conjecture (2) fails dramatically if essential dimension is replaced by essential dimension at $p$. On the other hand, Theorem 1 fails if $\text{ed}(G; p)$ is replaced by $\text{ed}(G)$; see [Le07].

Before proceeding with the proof of Theorem 1 we remark that the condition that $G$ is smooth cannot be dropped. Indeed, it is well known that $\text{ed}_F(\mu^r_p; p) = r$ for any $r \geq 0$. More generally, if $G$ is a group scheme of finite type over a field $F$ of characteristic $p$ (not necessarily finite or smooth), then $\text{ed}_F(G; p) \geq \dim(\mathcal{G}) - \dim(G)$, where $\mathcal{G}$ is the Lie algebra of $G$; see [TV13, Theorem 1.2].

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2. Versality

Let $G$ be an algebraic group and $X$ be an irreducible $G$-variety (i.e., a variety with a $G$-action) over $F$. We will say that the $G$-variety $X$ is generically free if there exists a dense open subvariety $U$ of $X$ such that the scheme-theoretic stabilizer $G_u$ of every geometric point $u$ of $X$ is trivial. Equivalently, there exists a $G$-invariant dense open subvariety $U''$ of $X$, which is the total space of a $G$-torsor; see [Se03, Section 5].
Following [Se03, Section 5] and [DR15, Section 1], we will say that $X$ is weakly versal (respectively, weakly $p$-versal), if for every infinite field (respectively, every $p$-closed field) $E$, and every $G$-torsor $T \to \text{Spec}(E)$ there is a $G$-equivariant $F$-morphism $T \to X$. We will say that $X$ versal (respectively, $p$-versal), if every $G$-invariant dense open subvariety of $X$ is weakly versal (respectively, weakly $p$-versal).

It readily follows from these definitions that $\text{ed}(G)$ (respectively, $\text{ed}(G; p)$) is the minimal dimension $\dim(X) - \dim(G)$, where the minimum is taken over all versal (respectively $p$-versal) generically free $G$-varieties $X$; see [Se03, Section 5.7], [DR15, Remark 2.6 and Section 8]. Our proof of Theorem 1 will be based on the following facts.

(i) ([DR15, Proposition 2.2]) Every $G$-variety $X$ with a $G$-fixed $F$-point is weakly versal.

(ii) ([DR15, Theorem 8.3]) Let $X$ be a smooth geometrically irreducible $G$-variety. Then $X$ is weakly $p$-versal if and only if $X$ is $p$-versal.

Combining (i) and (ii), we obtain:

**Proposition 2.** ([DR15, Corollary 8.6(b)]) Let $G$ be a finite smooth algebraic group over $F$. If there exists a faithful geometrically irreducible $G$-variety $X$ with a smooth $G$-fixed $F$-point, then $\text{ed}(G; p) \leq \dim(X)$.

If we replace “$p$-versal” by “versal”, then (ii) fails: a weakly versal $G$-variety does not need to be versal. This is the underlying reason why both Proposition 2 and Theorem 1 fail if $\text{ed}(G; p)$ is replaced by $\text{ed}(G)$.

### 3. Proof of Theorem 1

In this section we will prove Theorem 1 assuming Lemmas 3 and 4 below. We will defer the proofs of these lemmas to sections 4 and 5 respectively.

By [MR10, Lemma 4.1], if $G' \subset G$ is a subgroup of index prime to $p$, then

$$\text{ed}_F(G; p) = \text{ed}_F(G'; p).$$

In particular, if $p$ does not divide $|G|$, then taking $G' = \{1\}$, we conclude that $\text{ed}_F(G; p) = 0$. On the other hand, if $p$ divides $|G|$, then $\text{ed}_F(G; p) \geq 1$; see [Mc09, Proposition 4.4] or [LMMR13, Lemma 10.1]. Our goal is thus to show that $\text{ed}_F(G; p) \leq 1$.

First let us consider the case where $G$ is a finite group, viewed as a constant algebraic group over $F$. After replacing $G$ by a Sylow $p$-subgroup, we may assume that $G$ is a $p$-group. Moreover, since $\mathbb{F}_p \subset F$, $\text{ed}_F(G; p) \leq \text{ed}_{\mathbb{F}_p}(G; p)$. Thus, for the purpose of proving the inequality $\text{ed}_F(G; p) \leq 1$, we may assume that $F = \mathbb{F}_p$. In view of Proposition 2 it suffices to prove the following.

**Lemma 3.** For every finite constant $p$-group $G$ there exists a faithful $G$-curve defined over $\mathbb{F}_p$ with a smooth $G$-fixed $\mathbb{F}_p$-point.

Now consider the general case, where $G$ is a smooth finite algebraic group over $F$. In other words, $G = \Gamma \rtimes \tau$, where $\Gamma$ is a constant finite group, $A = \text{Aut}_{\text{grp}}(\Gamma)$ is the group of automorphisms of $\Gamma$ and $\tau$ is a cocycle representing a class in $H^1(F, A)$.

**Lemma 4.** (a) $\text{ed}_F(G) \leq \text{ed}_F(\Gamma \rtimes A)$, (b) $\text{ed}_F(G; p) \leq \text{ed}(\Gamma \rtimes A; p)$.

The semidirect product $\Gamma \rtimes A$ is a constant finite group. Hence, as we showed above, $\text{ed}_F(\Gamma \rtimes A; p) \leq 1$. Theorem 1 now follows from Lemma 4(b).
4. Proof of Lemma 3

We will give two proofs: our original proof, extracted from the literature, and a self-contained proof suggested to us by the referee.

**Proof.** Recall that the Nottingham group $\text{Aut}_0(\mathbb{F}_p[[t]])$ is the group of automorphisms $\sigma$ of the algebra $\mathbb{F}_p[[t]]$ of formal power series such that $\sigma(t) = t + a_2t^2 + a_3t^3 + \ldots$, for some $a_2, a_3, \ldots \in \mathbb{F}_p$. By a theorem of of Leedham-Green and Weiss [C97, Theorem 3], every finite $p$-group $G$ embeds into $\text{Aut}_0(\mathbb{F}_p[[t]])$. Fix an embedding $\phi: G \hookrightarrow \text{Aut}_0(\mathbb{F}_p[[t]])$. By [Ka80, Theorem 1.4.1], there exists a smooth $G$-curve $X$ over $\mathbb{F}_p$, with an $\mathbb{F}_p$-point $x \in X$ fixed by $G$, such that the $G$-action in the formal neighborhood of $x$ is given by $\phi$; see also [Ha80, Section 2] and [BCPS17, Theorem 4.8]. Since $\phi$ is injective, the $G$-action on $X$ is faithful. ♠

**Alternative proof.** First consider the case, where $G = (\mathbb{Z}/p\mathbb{Z})^n$ is an elementary abelian $p$-group. Here we can construct $X$ as the cover of $\mathbb{P}^1$ (with function field $\mathbb{F}_p(s)$) given by the compositum of $n$ linearly disjoint Artin-Schreier extensions $\mathbb{F}_p(s, t_i)/\mathbb{F}_p(s)$ given by $t_i^p - t_i = f_i(s)$ (e.g., taking $f_i(s) = s^{p+1}$).

Now consider a general finite $p$-group $G$. Denote the Frattini subgroup of $G$ by $\Phi$ and the quotient $G/\Phi$ by $(\mathbb{Z}/p\mathbb{Z})^n$. Let $Y$ be the smooth curve and $Y \to \mathbb{P}^1$ be a $G/\Phi = (\mathbb{Z}/p\mathbb{Z})^n$-cover constructed in the previous paragraph, totally ramified at a point $y \in Y(\mathbb{F}_p)$ above $\in \mathbb{P}^1$. Let $E/\mathbb{F}_p(s)$ be the $(\mathbb{Z}/p\mathbb{Z})^n$-Galois extension associated to this cover. By [Se97, Proposition II.2.2.3], the cohomological dimension of $\mathbb{F}_p(s)$ at $p$ is $\leq 1$. Consequently by [Se97, Propositions I.3.4.16], $E/\mathbb{F}_p(s)$ lifts to a $G$-Galois extension $K/\mathbb{F}_p(s)$ such that $K^G = E$. Let $X$ be the smooth curve associated to $K$ and $x \in X(\mathbb{F}_p)$ is a point above $y$:

$$
x \in X \quad \text{and} \quad y \in Y \quad \text{and} \quad \in \mathbb{P}^1_{\mathbb{F}_p}.
$$

We claim that $x$ is fixed by $G$; in particular, this will imply that $x \in X(\mathbb{F}_p)$. Let $H$ be the stabilizer of $x$ in $G$. Since $\Phi$ acts transitively on the fiber above $y$ in $X$, we have $\Phi \cdot H = G$. By Frattini’s theorem (see, e.g., [Ro96, Theorem 5.2.12]), $\Phi$ is the set of non-generators of $G$. We conclude that $H = G$, as claimed. ♠

5. Proof of Lemma 4

We will make use of the following description of $\text{ed}_F(G)$ and $\text{ed}_F(G; p)$ in the case where $G$ is a finite algebraic group over $F$. Let $G \to \text{GL}(V)$ be a faithful representation. A compression (respectively, a $p$-compression) of $V$ is a dominant $G$-equivariant rational map $V \to X$ (respectively, a dominant $G$-equivariant correspondence $V \sim X$ of degree prime to $p$), where $G$ acts faithfully on $X$. Here by a correspondence we mean a $G$-equivariant subvariety $V'$ of $V \times X$ such that the $G$ transitively permutes the irreducible
components of $V'$, and the dimension of each component equals the dimension of $V$. The degree of this correspondence is defined as the degree of the projection $V' \to V$ to the first factor.

Recall that $\text{ed}_F(G)$ (respectively, $\text{ed}_F(G; p)$) equals the minimal value of $\dim(X)$ taken over all compressions $V \to X$ (respectively all $p$-compressions $V \simeq X$). In particular, these numbers depend only on $G$ and $F$ and not on the choice of the generically free representation $V$. For details, see [Rei10].

We are now ready to proceed with the proof of Lemma 4. To prove part (a), let $V$ be a generically free representation of $\Gamma \rtimes A$ and let $f : V \to X$ be a $\Gamma \rtimes A$-compression, with $X$ of minimal possible dimension. That is, $\dim_F(X) = \text{ed}_F(\Gamma \rtimes A)$. Twisting by $\tau$, we obtain a $G = \Gamma$-equivariant map $\tau f : \tau V \to \tau X$; see e.g., [PR17, Proposition 2.6(a)]. Now observe that by Hilbert’s Theorem 90, $\tau V$ is a vector space with a linear action of $G = \Gamma$ and $\tau f : \tau V \to \tau X$ is a compression. (To see that the $G$-action on $\tau V$ and $\tau X$ are faithful, we may pass to the algebraic closure $\overline{F}$ of $F$. Over $\overline{F}$, $\tau$ is split, so that $G = \Gamma$, $\tau V = V$, $\tau X = X$ and $\tau f = f$, and it becomes obvious that the $G$-actions on $\tau V$ and $\tau X$ are faithful.) We conclude that $\text{ed}_F(G) \leq \dim_F(\tau X) = \dim_F(X) = \text{ed}_F(\Gamma \rtimes A)$, as desired.

The proof of part (b) proceeds along the same lines. The starting point is a $p$-compression $f : V \to X$ with $X$ of minimal possible dimension, $\dim_F(X) = \text{ed}_F(\Gamma \rtimes A; p)$. We twist $f$ by $\tau$ to obtain a $p$-compression $\tau f : \tau V \simeq \tau X$ of the linear action of $G = \Gamma$ on $\tau V$. The rest of the argument is the same as in part (a). This completes the proof of Lemma 4 and thus of Theorem 1.

6. AN APPLICATION

In this section $G$ will denote a connected reductive linear algebraic group over a field $F$. It is shown in [CGR06, Theorem 1.1(c)] that there exists a finite $F$-subgroup $S \subset G$ such that every $G$-torsor over every field $K/F$ admits reduction of structure to $S$; see also [CGR08, Corollary 1.4]. In other words, the map $H^1(K, S) \to H^1(K, G)$ is surjective for every field $K$ containing $F$. If this happens, we will say that “$G$ admits reduction of structure to $S$”.

We will now use Theorem 1 to show that if $\text{char}(F) = p > 0$ and $p$ is a torsion prime for $G$, then $S$ cannot be smooth. For the definition of torsion primes, a discussion of their properties and further references, see [Sc00]. Note that by a theorem of A. Grothendieck [Gr58], if $G$ is not special (i.e., if $H^1(K, G) \neq \{1\}$ for some field $K$ containing $F$), then $G$ has at least one torsion prime; see also [Sc00, 1.5.1].

**Corollary 5.** Let $G$ be a connected reductive linear algebraic group over an algebraically closed field $F$ of characteristic $p > 0$.

(a) If $S$ is a smooth finite subgroup of $G$ defined over $F$, then the natural map

$$f_K : H^1(K, S) \to H^1(K, G)$$

is trivial for any $p$-closed field $K$ containing $F$. In other words, $f_K$ sends every $\alpha \in H^1(K, S)$ to $1 \in H^1(K, G)$.

(b) If $p$ is a torsion prime for $G$, then $G$ does not admit reduction of structure to any smooth finite subgroup.
Proof. (a) Let $\alpha \in H^1(K, S)$ and $\beta = f_K(\alpha) \in H^1(K, G)$. By Theorem [1], $\alpha$ descends to $\alpha_0 \in H^1(K_0, S)$ for some intermediate field $F \subset K_0 \subset K$, where $\text{trdeg}(K_0/F) \leq 1$. Since $F$ is algebraically closed, $\dim(K_0) \leq 1$; see [Se97, Sections II.3.1-3]. By Serre’s Conjecture I (proved by R. Steinberg [St65] for a perfect field $K_0$ and by A. Borel and T. A. Springer [BS68, §8.6] for an arbitrary $K_0$ of dimension $\leq 1$), $H^1(K_0, G) = \{1\}$. Tracing through the diagram

$H^1(K_0, S) \xrightarrow{f_{K_0}} H^1(K_0, G) = \{1\}$

we see that $\beta = 1$, as desired.

(b) If $p$ is a torsion prime for $G$, then $H^1(K, G) \neq \{1\}$ for some $p$-closed field $K$ containing $F$; see [Me09, Proposition 4.4]. In view of part (a), this implies that $f_K$ is not surjective.

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ESSENTIAL DIMENSION IN PRIME CHARACTERISTIC

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