Three-$\alpha$ model calculations of low-lying $^{12}$C states and the triple-$\alpha$ reaction

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Abstract. Properties of some $^{12}$C resonant states at low energies are studied in three-$\alpha$-particle model. Calculations for breakup reactions of $^{12}$C bound state leading to three-body continuum states are performed at energies up to a few MeV above the breakup threshold by applying the Faddeev three-body formalism in coordinate space. Calculated cross sections reveal some resonant peaks. Decay modes and the structure at interior region of these resonant states as well as their effects on the triple-alpha reaction rate will be reported.

1. Introduction
Low-lying resonant states of $^{12}$C nucleus are interesting objects to study, because some curious structures having a particular configuration of $\alpha$-particles, such as a linear chain, an equilateral triangle, or a bent-arm, have been predicted to exist [1, 2, 3, 4, 5]. Furthermore these states, especially the second $0^+$ state $^{12}$C$(0^+_2)$ (the Hoyle state) [6], are known to play an important role in the triple-alpha (3$\alpha$) process,

$$\alpha + \alpha + \alpha \rightarrow ^{12}\text{C} + \gamma,$$

(1)

by which three $\alpha$-particles (3$\alpha$) are fused into a $^{12}$C nucleus in stars.

In a recent paper [7], the present author studied an inverse reaction of (1), namely the E2-photodisintegration of $^{12}$C$(2^+_1)$ bound state leading to final 3$\alpha$ continuum states with total angular momentum $J = 0$,

$$^{12}\text{C}(2^+_1) + \gamma(\text{E2}) \rightarrow \alpha + \alpha + \alpha (J = 0),$$

(2)

and then calculated the 3$\alpha$ reaction rate for stellar temperatures of $T = 10^7$ K to $10^9$ K. It turned out that the calculated reaction rate for $T > 10^8$ K agrees with a standard rate by the Nuclear Astrophysics Compilation of Reaction Rates (NACRE) [8], and is about one order of magnitude larger than the NACRE rate at $T = 10^7$ K.

In Ref. [7], a wave function for the process (2) was defined and solved by applying the Faddeev three-body formalism in coordinate space. Calculations were performed for an energy range of the 3$\alpha$ system from 10 keV to 600 keV, which includes the energy of the two-$\alpha$ resonant state $^8\text{Be}(0^+_1)$ (92 keV) and that of the Hoyle state (380 keV). In the present work, aiming to calculate the 3$\alpha$ reaction rate at higher temperatures as $10^{10}$ K, the calculations are extended by increasing energies of the 3$\alpha$ system up to several MeV, where some additional resonant
states with higher angular momenta exist. In the following, after shortly reviewing a formalism to calculate the disintegration process in Sec. 2, interaction models among \( \alpha \)-particles used in this work, and then results for resonant energies, energy spectrum of decaying \( \alpha \)-particles, the structure of resonant states at interior region, and their effects on the thermal \( 3\alpha \) reaction rate will be presented in Sec. 3. Summary will be given in Sec. 4.

2. Formalism

Let us consider the photodisintegration process of the \( ^{12}\text{C}(J^P) \) bound state by the \( E \lambda \) photon of the energy \( E_\gamma \), leading to \( 3\alpha \) continuum states of the angular momentum \( J \). Once the cross section for the process \( \sigma_\gamma(E_\gamma) \) is obtained, the \( 3\alpha \) reaction rate \( \langle \alpha\alpha\alpha \rangle \) is calculated by

\[
\langle \alpha\alpha\alpha \rangle = \frac{(3)^{3/2} 8\pi c}{(m_\alpha c^2)^3} \frac{\hbar}{k_B T} \frac{\hbar}{k_B T} \sum_{J_A, J_B} (2J_A + 1) \int_0^\infty dE_\gamma E_\gamma^2 \sigma_\gamma(E_\gamma) e^{-(E_\gamma - |E_A|)/k_B T},
\]

where \( m_\alpha \) is the \( \alpha \) particle mass and \( E_A \) is the energy of the initial bound state.

Let us define a wave function for the disintegration process of a bound state \( |\Psi_A\rangle \) by the electromagnetic interaction \( H_\gamma \):

\[
\Psi(x, y) = \langle x, y \rvert \frac{1}{E + i\epsilon - H} H_\gamma |\Psi_A\rangle,
\]

where \( x \) is the relative coordinate of an \( \alpha \)-pair, \( y \) is the relative coordinate of the third (spectator) \( \alpha \)-particle with respect to the center of mass of the pair, \( E \) is the \( 3\alpha \) energy in the center of mass system, \( E = E_\gamma - |E_A| \), and \( H \) is the Hamiltonian of the \( 3\alpha \) system. The wave function \( \Psi \) will be obtained by applying the Faddeev three-body formalism [9] to solve integral equations in coordinate space with accommodating the long range Coulomb force effects. (See Refs. [7, 10] for the details of the calculations.) This procedure leads to a three-body breakup amplitude \( F^{(B)}(\hat{q}, \hat{p}, E_q) \), where \( \hat{q} \) is relative momentum of a \( \alpha\alpha \) pair in the final state, \( \hat{p} \) is the momentum of the spectator \( \alpha \) particle, and \( E_q \) is the relative \( \alpha\alpha \) energy. Here, the amplitude \( F^{(B)}(\hat{q}, \hat{p}, E_q) \) is the complex conjugate of an amplitude for the capture process (1), and from which the photodisintegration cross section is given by

\[
\sigma_\gamma(E_\gamma) = \frac{1}{8\pi} \frac{1}{2J_A + 1} \frac{1}{\hbar c K^3} \left( \frac{3}{4} \right)^2 \int d\hat{q} d\hat{p} dE_q q \bar{q} |F^{(B)}(\hat{q}, \hat{p}, E_q)|^2,
\]

where \( K = \sqrt{\frac{m_\alpha}{\hbar^2} E} \).

3. Results

3.1. Interactions

The mode D of the Ali-Bodmer potential [11] with the (point) Coulomb potential is used for the \( \alpha\alpha \) interaction. In addition, \( 3\alpha \) potentials (3\alphaP) are introduced to reproduce the properties of \( ^{12}\text{C} \), namely binding energies and resonance energies. We used two different functional forms: one is taken from Ref. [12]:

\[
W_\Delta = \sum_{J=0,2} \hat{P}_J W_J \exp \left( -\frac{r_{12}^2 + r_{23}^2 + r_{34}^2}{a^2} \right) + [c.p.],
\]

and the other is from Ref. [13]:

\[
W_V = \sum_{J=0,2} \hat{P}_J W_J \exp \left( -\frac{r_{12}^2 + r_{23}^2}{a^2} \right) + [c.p.],
\]
Table 1. The range and strength parameters of the 3\(\alpha\)P models used in this work, and results for the reaction (8) (See the text).

| Model | Eq. | \(a\) (fm) | \(W_{0}^{(b)}\) (MeV) | \(W_{0}^{(c)}\) (MeV) | \(W_{2}\) (MeV) | \(E_{\text{peak}}^{[12\text{C}(2_{2}^{+})]}\) (MeV) | \(B(E2,2_{2}^{+} \rightarrow 0_{1}^{+})\) (e\(^{2}\) fm\(^{4}\)) |
|-------|-----|-----------|---------------------|---------------------|----------------|----------------------|------------------|
| \(\Delta L\) | (6) | 3.39 | -29.2 | -30.95 | -15.3 | 2.4 | 2.1 |
| \(\Delta S\) | (6) | 2.61 | -99.0 | -101.22 | -52.7 | 2.5 | 0.92 |
| \(V_{L}\) | (7) | 3.33 | -14.3 | -12.21 | -7.7 | 1.9 | 3.6 |
| \(V_{S}\) | (7) | 2.50 | -42.8 | -39.64 | -22.4 | 2.2 | 1.8 |

where [c.p.] denotes the cyclic permutations and \(\hat{P}_{J}\) is the projection operator on a 3-\(\alpha\) state with the total angular momentum \(J\). We examined two different values for the range parameter \(a\) both for \(W_{\Delta}\) (6) and \(W_{V}\) (7). Calculations with these 3\(\alpha\)P models will be denoted as \(\Delta L\), \(\Delta S\), \(V_{L}\), and \(V_{S}\), in which the subscript \(L\) (\(S\)) indicates the choice of a long-range (short-range) parameter. Chosen values of the range and strength parameters will be described below (Table 1).

3.2. Photodisintegration cross section (Transition strength function)

We studied, in addition to the E2 photodisintegration of the \(^{12}\text{C}(2^{+}_{0})\) state (2), that of the \(^{12}\text{C}(0^{+}_{1})\) state, \(^{12}\text{C}(0^{+}_{1}) + \gamma(E2) \rightarrow \alpha + \alpha + \alpha (J = 2)\) (8).

In solving \(\Psi\) in (4), the maximum angular momentum of the \(\alpha-\alpha\) subsystem in 3-\(\alpha\) partial wave components is chosen to be 4, while was 2 in Ref. [7], which causes little change compared to the results in Ref. [7].

In calculating the reaction (2), the 3\(\alpha\)P strength parameter for \(J = 2\) state, \(W_{2}\), is determined to reproduce the binding energy of \(^{12}\text{C}(2^{+}_{0})\), and that for \(J = 0\) state, \(W_{0}^{(c)}\), is determined to reproduce the Hoyle state energy, as shown in Table 1.

In the following, the E2-strength function defined by

\[
\frac{dB(E\lambda)}{dE} = \frac{\lambda[(2\lambda + 1)!!]^2 \sigma_{\gamma}(E_{\gamma})}{(2\pi)^3(\lambda + 1) k_{\gamma}^{\lambda-1/2}},
\]

where \(\lambda = 2\) in the present case and \(k_{\gamma}\) is the momentum of the photon, will be presented instead of the photodisintegration cross section \(\sigma_{\gamma}(E_{\gamma})\) to accent peak structures corresponding to resonances.

Fig. 1 shows the calculated E2 strength functions for the reaction (2), in which we observe, in addition to sharp peak corresponding to \(^{12}\text{C}(0^{+}_{1})\) at \(E = 0.38\) MeV, two peaks: one is around \(E = 1\) MeV and the other is around \(E = 4\) MeV. The lower-energy peak corresponds to the resonant state \(^{12}\text{C}(0^{+}_{1})\) whose existence was predicted by complex scaling calculations [14, 15]. Such a two-peak structure of \(0^{+}\) \(^{12}\text{C}\) state is indicated experimentally in Ref. [16]. For higher-energy peak, which is assigned as \(^{12}\text{C}(0^{+}_{1})\), there is a rather large amount of model dependence in position, width, and height of the strength-peak. Therefore further experimental information about \(^{12}\text{C}(0^{+}_{1})\) state is useful for the improvement of 3-\(\alpha\) model interactions.

For the reaction (8), the 3\(\alpha\)P strength parameter for \(J = 0\) state had to be changed from \(W_{0}^{(c)}\) by at most 10 % to reproduce the \(^{12}\text{C}(0^{+}_{1})\) bound state energy (shown as \(W_{0}^{(b)}\) in Table 1). The construction of a 3\(\alpha\)P that reproduces both of bound and continuum states is left for a future work. For the \(J = 2\) strength of 3\(\alpha\)P, the same values are used as in previous ones.
Calculations of the E2-strength function for the process (8) are shown in Fig. 2. The results show a resonant structure corresponding to $^{12}\text{C}(2^+_2)$ state, which appeared in some theoretical calculations, (e.g., [2, 3, 4, 5, 14, 15]), and experimentally observed [16, 17, 18, 19]. Calculated peak energy, $E_{\text{peak}}[^{12}\text{C}(2^+_2)]$, and the E2 transition strength $B(2^+_2 \rightarrow 0^+_1)$ are summarized in Table 1. The calculated peak energies are subequal or small by a few hundred keV compared to the experimental results: 2.48(15) MeV [18] and 2.76(11) MeV [19]. Calculated values of the strength $B(2^+_2 \rightarrow 0^+_1)$ are larger than the recent result of $^{12}\text{C}(\gamma, \alpha)$ reaction, 0.73(13) $e^2\text{fm}^4$ [19], but not so much different from values obtained from $^{12}\text{C}(\alpha, \alpha')$ reaction, 1.83(9) $e^2\text{fm}^4$ or 1.6(2) $e^2\text{fm}^4$ [16]. These might suggest that we should examine to change the strength parameter $W_2$ as well as for $W_0$.

![Figure 1. E2 strength function for the process (2) as a function of the 3-$\alpha$ energy $E$. The (red) thick dashed and solid lines indicate results of the $\Delta_L$ and $\Delta_S$, respectively. The (blue) thin dashed and solid lines indicate results of the $V_L$ and $V_S$, respectively.](image1)

![Figure 2. E2 strength function for the process (8) as a function of the 3-$\alpha$ energy $E$. The (red) thick dashed and solid lines indicate results of the $\Delta_L$ and $\Delta_S$, respectively. The (blue) thin dashed and solid lines indicate results of the $V_L$ and $V_S$, respectively.](image2)

3.3. Spectrum of decaying $\alpha$ particles

Fig. 3 shows the spectrum of decaying $\alpha$-particles from the Hoyle state for the $V_S$ model, which is obtained by integrating $|F^{(B)}(\hat{q}, \hat{p}, E_q)|^2$ in Eq. (5) over $\hat{q}$ and $\hat{p}$. The spectrum has a peak at $E_q = E_q[^{8}\text{Be}(0^+_1)]$ corresponding to a sequential process, in which the Hoyle state first decays to $^{8}\text{Be}(0^+_1)$ and $\alpha$-particle and then $^{8}\text{Be}(0^+_1)$ to two $\alpha$-particles. A contribution of such sequential decay (SD) process is calculated by integrating the spectrum just around $E_q = E_q[^{8}\text{Be}(0^+_1)]$. More than 99 % of the total decay is coming from the SD process for the Hoyle state, which is consistent with experimental results of Refs. [20, 21, 22, 23], but not with that of Ref. [24]. While such large contributions of the SD channel are calculated for the decay processes of $^{12}\text{C}(0^+_1)$ and $^{12}\text{C}(2^+_2)$ states, a rather small (about 50% of the total) SD contribution is calculated for $^{12}\text{C}(0^+_1)$.

3.4. Structure of the Hoyle state

Although the function $\Psi(x, y)$ (4) is not the eigen-state of the 3-$\alpha$ Hamiltonian, its interior structure is expected to reflect that of the resonance state because of the 3-$\alpha$ Green’s function. Actually, $\Psi(x, y)$ at the resonance energy has a concentration of the amplitude at interior region.
In Fig. 4, the density distribution $\rho(x, y)$ of the disintegration wave functions,

$$\rho(x, y) = x^2 y^2 \int d\hat{x} d\hat{y} |\Psi(x, y)|^2,$$

(10)

calculated at the Hoyle resonance energy, is plotted. Here, the wave function $\Psi(x, y)$, which is not normalizable since it is continuum state function, is artificially normalized within the region: \{ $x < 12$ fm, $y \leq 12$ fm\}. The density has three distinct local peaks denoted by A, B, and C in the figure, which are located at $(x, y) \sim (2.6$ fm, $2.3$ fm), $(3.3$ fm, $4.4$ fm), and $(5.6$ fm, $1.8$ fm), respectively. Similar peak structure is appeared in calculations of a 3-$\alpha$ eigen-state performed in Refs. [25, 26].

The peak A corresponds to the configuration of an equilateral triangle of side length $2.6$ fm. The peaks B and C correspond to a bent-arm configuration, in which three $\alpha$-particles compose an isosceles triangle with two equal sides of length $3.3$ fm and the third side length being $5.6$ fm. Because of the symmetric property of the wave function, a bent-arm configuration appears at two points for the density distribution.

Since each peak is associated with wide slopes, three $\alpha$-particles may take each triangle configuration rather loosely. The probability to find 3-$\alpha$ taking the equilateral triangle configuration is estimated by integrating the density $\rho(x, y)$ for the domain of a square, 1.5 fm on a side, around the peak A. This gives about 10 %. Similar procedure for the peak B (C) gives about 20 % (10%). Therefore, probabilities for the equilateral triangle and the bent-arm configurations are roughly about 10 % and 30 %, respectively. The density distribution shows that the Hoyle state has a mixed configuration of the equilateral triangle and the bent-arm.

Figure 3. Energy distribution of decaying $\alpha$-particles from the Hoyle state for the $V_S$ model.

Figure 4. Contour plot of the density distribution $\rho(x, y)$ of the Hoyle state for the $V_S$ model. See the text for the meaning of the characters, 'A', 'B', and 'C' in the figure.
3.5. \(3\alpha\) reaction rate

The \(3\alpha\) reaction rates calculated by Eq. (3) including both of the processes (2) and (8) are shown as the ratio to the NACRE \(3\alpha\) rate [8] in Fig. 5. The dashed-dotted line demonstrates that effects of \(3\alpha\) states with \(J = 0\) are small on the \(3\alpha\) rate for \(T > 2 \times 10^9\) K, and the difference among the other four lines indicates an importance of detailed information on the \(^{12}\text{C}(2^+_2)\) resonant state.

The calculated rates at lower temperatures, as \(T \sim 10^7\) K, are large by an order at most than the NACRE rate, which is consistent with recent calculation by the imaginary time method [27], but not with those of Refs. [28, 25] that give a huge enhancement by many orders.

![Figure 5](image)

**Figure 5.** The ratio of the calculated \(3\alpha\) reaction rates to the NACRE \(3\alpha\) rate as a function of the temperature \(T_9 = T/10^9\) K. The (red) thick dashed and solid lines indicate results of the \(\Delta L\) and \(\Delta S\), respectively. The (blue) thin dashed and solid lines indicate results of the \(V_L\) and \(V_S\), respectively. The (black) dashed-dotted line denotes the rate calculated with only the \(J = 0\) \(3\alpha\) state (2) for the \(V_S\) model.

4. Summary

Low energy \(3\alpha\) continuum states are studied via the breakup reactions of the ground \(0^+_1\) state and the first \(2^+_1\) excited state of \(^{12}\text{C}\) by the E2 photon leading to \(2^+\) and \(0^+\) final states, respectively. The Ali-Bodmer \(\alpha\)-\(\alpha\) potential together with some forms of \(3\alpha\) potential are used for the \(3\alpha\) calculations. The calculated cross sections of the processes show some peaks corresponding to \(^{12}\text{C}(0^+_2), \(^{12}\text{C}(0^+_3), \(^{12}\text{C}(0^+_4),\) and \(^{12}\text{C}(2^+_2)\) resonant states. The sequential decay process dominates for \(3\alpha\) decay processes of \(^{12}\text{C}(0^+_2), \(^{12}\text{C}(0^+_3),\) and \(^{12}\text{C}(2^+_2)\), but not for \(^{12}\text{C}(0^+_4)\). The density distribution of the wave function for the process with the Hoyle state, \(^{12}\text{C}(0^+_2)\), has some peaks corresponding to the equilateral triangle configuration (10%) and the bent-arm configuration (30%). Contribution of the \(^{12}\text{C}(2^+_2)\) resonant state to the \(3\alpha\) reaction rate is significant for temperatures of \(2 \times 10^9\) K < \(T < 10^{10}\) K, although a rather large dependence on the choice of \(3\alpha\) potential requires further studies.

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