On the two-magnon bound states for the quantum Heisenberg chain with variable range exchange

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Abstract

The spectrum of finite-difference two-magnon operator is investigated for quantum $S=1/2$ chain with variable range exchange of the form $h(j-k) \propto \sinh^{-2} a(j-k)$. It is found that usual bound state appears for some values of the total pseudomomentum of two magnons as for the Heisenberg chain with nearest-neighbor spin interaction. Besides this state, a new type of bound state with oscillating wave function appears at larger values of the total pseudomomentum.

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The spectral theory of finite-difference operators associated with multimagnon states of one-dimensional ferromagnetic Heisenberg chains is far from being complete. From previous works, it is known that magnons not only scatter on each other, but can also create bound complexes with lower energy. A canonical example of this phenomenon was described first by Bethe [1] for the famous case of nearest-neighbor exchange. It was found that in this case just one bound state appears for all values of the total pseudomomentum $P$ in the interval $0 < P \leq \pi$. Much later, it was shown that for next-nearest-neighbor interaction the bound state is always present within this interval but can be destroyed at $P = \pi$ by second-neighbor interaction [2-4]. The situation for the third neighbor or more general nonlocal spin interaction has not been investigated till now.

In this letter, we shall investigate the two-magnon problem for the integrable ferromagnetic Heisenberg chain with variable range exchange defined by the Hamiltonian

$$H = -\frac{J}{2} \sum_{j \neq k; j, k = -\infty}^{\infty} h(j - k) \vec{\sigma}_j \vec{\sigma}_k - \frac{1}{2},$$  \hspace{1cm} (1)

where $\{\vec{\sigma}_j\}$ are Pauli matrices attached to the site $j$, $J > 0$ and

$$h(j - k) = \frac{1}{\sinh^2 a(j - k)}. $$ \hspace{1cm} (2)

The range of variation of the parameter $a$ is $0 < a < \infty$. It will be shown that for all values of $a$ in this interval there are two types of two-magnon bound states, the first is similar to the bound state for the Heisenberg chain with nearest-neighbor interaction which might be considered as a limit of (1-2) as $a \to \infty$ (with $J \propto \sinh^2 a$), and the second corresponds to the asymptotically decayed two-magnon wave functions with superimposed oscillations.

We shall consider two-magnon excitations over the ground state of the model in the form

$$|\psi > = \sum_{n_1 \neq n_2} \psi(n_1, n_2) S_{n_1}^- S_{n_2}^- |0 >, \quad S_{n}^- = \frac{1}{2}(\sigma_n^x - i\sigma_n^y),$$

where $|0 >$ is the ferromagnetic ground state with all spins aligned up and $\psi(n_1, n_2) = \psi(n_2, n_1)$. Acting with (1) on the state $|\psi >$, the equation $H|\psi > = \varepsilon|\psi >$ results in

$$H^{(2)} \psi(n_1, n_2) = \varepsilon \psi(n_1, n_2),$$  \hspace{1cm} (3)
where the action of the finite-difference operator $H^{(2)}$ is given by

$$H^{(2)}\psi(n_1, n_2) = -\left[ \sum_{k \neq n_1, n_2} (h(n_1 - k)\psi(k, n_2) + h(n_2 - k)\psi(n_1, k))
+ (2h(n_1 - n_2) - 2\varepsilon_0)\psi(n_1, n_2) \right],$$

(4)

where $\varepsilon_0 = \sum_{k \neq 0} h(k)$. A two-parametric solution to the eigenequation (3) with the exchange strength (2) reads as follows [5],

$$\psi(n_1, n_2) = e^{i(p_1 n_1 + p_2 n_2)}(\coth(\gamma(p_1, p_2) + \coth a(n_1 - n_2))) + e^{i(p_2 n_1 + p_1 n_2)}(\coth(\gamma(p_1, p_2) - \coth a(n_1 - n_2))),$$

(5)

where the phase factor $\gamma(p_1, p_2)$ is expressed via the magnon pseudomomenta $p_{1,2}$,

$$\coth \gamma(p_1, p_2) = \frac{f(p_1) - f(p_2)}{2a}.$$  

(6)

The formula for two-magnon energy reads

$$\varepsilon(p_1, p_2) = \varepsilon(p_1) + \varepsilon(p_2) = J \sum_{n \neq 0} \frac{2 - \cos(np_1) - \cos(np_2)}{\sinh^2(an)},$$

(7)

where the sum can be expressed via the Weierstrass functions. The structure of the function $f(p)$ will be crucial for our treatment. It is given by the expression

$$f(p) = \frac{p}{\pi} \frac{\zeta_1}{2a} \left( \frac{i\pi}{2a} \right) - \frac{\zeta_1}{2a} \left( \frac{ip}{2a} \right),$$

(8)

where $\zeta_1$ is the Weierstrass zeta function with quasiperiods 1 and $\omega = i\pi/a$. It admits also the representation [6]

$$f(p) = ia \cot \frac{p}{2} - a \sum_{n=1}^{\infty} \left[ \coth \left( \frac{ip}{2} + an \right) + \coth \left( \frac{ip}{2} - an \right) \right].$$

(9)

From these representations, one can easily find that $f$ is odd and double quasiperiodic in $p$,

$$f(p) = -f(-p),$$

(10)

$$f(p + 2\pi) = f(p),$$

(11)

$$f(p + 2ia) = f(p) + 2a.$$
The derivative $f'(p)$ is double periodic and due to (11), (12) determined by its restriction to the fundamental domain

$$D : 0 \leq \Re p < 2\pi, \quad -a < \Im p \leq a.$$ 

The representations of $f'(p)$ are

$$f'(p) = \frac{1}{\pi} \zeta_1 \left( \frac{i\pi}{2a} \right) + i \frac{p}{2a} \varphi_1 \left( \frac{ip}{2a} \right)$$

$$= -\frac{ia}{2\sin^2 \frac{q}{2}} + i \frac{a}{2} \sum_{n \neq 0} \frac{1}{\sinh^2 \left( \frac{ip}{2} + an \right)},$$

where $\varphi_1$ is the Weierstrass $\wp$-function with periods $(1, \omega)$.

If $p_1, p_2$ are real, the wave function (5) describes scattering of magnons. In the bound state, the wave function must vanish as $|n_1 - n_2| \to \infty$. It means that $p_1$ and $p_2$ should be complex with $P = p_1 + p_2$ real. The simplest possibility is given by the choice

$$p_1 = \frac{P}{2} + iq, \quad p_2 = \frac{P}{2} - iq,$$

where $q$ is real. One can always choose $q > 0$ for convenience. Then vanishing of $\psi(n_1, n_2)$ as $|n_1 - n_2| \to \infty$ is equivalent to the condition

$$\coth \gamma(p_1, p_2) = \frac{f(p_1) - f(p_2)}{2a} = 1.$$  

Moreover, one can choose $q \leq a$. At first sight, the interval $a < q \leq 2a$ should be also taken into consideration but one can see that in this case the rearrangement $p_1 \to p_2 + 2ia$, $p_2 \to p_1 - 2ia$, $q \to 2a - q$ returns the problem to the interval $0 \leq q < a$ due to quasiperiodicity property (12).

The equation (15) under the condition (14) can be rewritten in more detailed form,

$$F_P(q) = 1 - \frac{1}{2a} \left[ \frac{2iq}{\pi} \zeta_1 \left( \frac{i\pi}{2a} \right) - \zeta_1 \left( \frac{iP}{4a} - \frac{q}{2a} \right) + \zeta_1 \left( \frac{iP}{4a} + \frac{q}{2a} \right) \right] = 0.$$  

At fixed real $P$ and real $q$, the function $F_P(q)$ is real. In the limit $a \to \infty$ it reads

$$\tilde{F}_P(q) = 1 - \frac{i}{2} \left[ \cot \left( \frac{P}{4} + \frac{iq}{2} \right) - \cot \left( \frac{P}{4} - \frac{iq}{2} \right) \right] = \frac{e^{-q} - \cos P/2}{\cosh q - \cos P/2}.$$ 

One can see that for each $-\pi \leq P \leq \pi$ there is just one bound state with the solution $q = -\ln \cos \frac{P}{2}$ of the equation $\tilde{F}_P(q) = 0$. In the case of finite $a$, which is of our interest,
the situation becomes more involved. One has to investigate zeroes of transcendental function $F_p(q)$ (16). To this end, note that (10) and (12) imply the following properties of $F_p(q)$,

$$F_p(0) = 1,$$  \hfill (17)  

$$F_p(q) = -F_p(2a - q), \quad F_p(q) = -F_p(-q) + 2.$$  \hfill (18)  

This means that $F_p(a) = 0$, i.e. $q = a$ is a solution to (16) for all $P$. However, one can see by the direct substitution of (14) with $q = a$ into (5) that the wave function in this case vanishes identically, i.e. this zero is unphysical and physical solution must lie in the interval $0 < q < a$. Let us find the existence criterium for it.

Note first that due to (17),(18) $F_p(0) = 1$ and $F_p(a) = 0$. Then a nontrivial zero of $F_p(q)$ in the interval $0 < q < a$ must exist if the derivative of $F_p(q)$ at the point $q = a$ is positive,

$$F_p'(a) = -\frac{i}{\pi a} \zeta_1 \left( \frac{i\pi}{2a} \right) + \frac{1}{4a^2} \left[ \varphi_1 \left( \frac{iP}{4a} - \frac{1}{2} \right) + \varphi_1 \left( \frac{iP}{4a} + \frac{1}{2} \right) \right] > 0.$$  

Since $\varphi_1(x-1) = \varphi_1(x)$, this condition can be cast into the form

$$-if' \left( \frac{P}{2} + ia \right) = -\frac{i}{\pi} \zeta_1 \left( \frac{i\pi}{2a} \right) + \frac{1}{2a} \varphi_1 \left( \frac{iP}{4a} + \frac{1}{2} \right)$$

$$= \frac{a}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\sinh^2 \left( \frac{iP}{4} + a(n + 1/2) \right)} > 0.$$  \hfill (19)  

Let us prove that the inequality (19) has nontrivial solutions. Note first that the inequality definitely holds for $P = 0$. On the other hand, let us estimate the left-hand side at $P = \pi$,

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sinh^2 \left( \frac{i\pi}{4} + a(n + 1/2) \right)} = \sum_{n=0}^{\infty} \left[ \frac{1}{\sinh^2 \left( \frac{i\pi}{4} + a(n + 1/2) \right)} + \frac{1}{\sinh^2 \left( \frac{i\pi}{4} - a(n + 1/2) \right)} \right]$$

$$= -\sum_{n=0}^{\infty} \frac{4}{\cosh^2(2n + 1)a} < 0.$$  

It means that there should be some value $P_{cr}(a)$, $0 < P_{cr}(a) < \pi$, such that inequality (19) is satisfied for all values $0 \leq P < P_{cr}$, since the left-hand side of (19) is a continuous function of $P$. At these values of the total pseudomomentum there should be at least one bound state specified by (14),(15). At $P > P_{cr}$, the inequality (19) does not hold and
there are no bound states of the type I since the derivative of the left-hand side of (19) with respect to $P$ does not change sign in the interval $(0, \pi)$ (cf. the known zeroes of function $\varphi'_1$).

Let us consider another possibility for getting the bound state. To this end, note that

$$f(p + ia) = \frac{p}{\pi} \zeta_1 \left( \frac{ip}{2a} \right) + \frac{ia}{\pi} \zeta_1 \left( \frac{i\pi}{2a} \right) - \zeta_1 \left( \frac{ip}{2a} - \frac{1}{2} \right).$$

Taking into account the relations

$$\zeta_1 \left( \frac{ip}{2a} - \frac{1}{2} \right) = \zeta_1 \left( \frac{ip}{2a} \right) - \zeta_1 \left( \frac{1}{2} \right) + \frac{\varphi'_1 \left( \frac{ip}{2a} \right)}{2 \left( \varphi_1 \left( \frac{ip}{2a} \right) - \varphi_1 \left( \frac{1}{2} \right) \right)},$$

$$\frac{ia}{\pi} \zeta_1 \left( \frac{i\pi}{2a} \right) + \zeta_1 \left( \frac{1}{2} \right) = a,$$

one finds that

$$f(p + ia) = a + i\chi(p), \quad f(p - ia) = -a + i\chi(p), \quad (20)$$

where $\chi(p)$ is real for real $p$. Suppose now that

$$p_1 = \tilde{p}_1 + ia, \quad p_2 = \tilde{p}_2 - ia, \quad (21)$$

where both $\tilde{p}_1$ and $\tilde{p}_2$ are real, $\tilde{p}_1 \neq \tilde{p}_2$ (the case $\tilde{p}_1 = \tilde{p}_2$ corresponds to trivial solution to the equation (15) mentioned above). Substitution of (21) into (15) gives with the account of (20)

$$\chi(\tilde{p}_1) - \chi(\tilde{p}_2) = 0. \quad (22)$$

Note also that

$$\chi(0) = \chi(\pi) = 0,$$

$$\chi'(0) = \frac{a}{2} \sum_{n=\infty}^{\infty} \frac{1}{\sinh^2 a \left( n + \frac{1}{2} \right)} > 0, \quad (23)$$

$$\chi' \left( \frac{\pi}{2} \right) = \frac{a}{2} \sum_{n=\infty}^{\infty} \frac{1}{\sinh^2 \left( \frac{i\pi}{4} + a \left( n + \frac{1}{2} \right) \right)} < 0.$$

It follows from (23) that there should be some value $\tilde{p}_0$ at which $\chi(\tilde{p})$ has a maximum on the interval $[0, \pi]$ and the corresponding $\tilde{p}_0' = 2\pi - \tilde{p}_0$ at which $\chi(\tilde{p})$ has a minimum on the interval $[\pi, 2\pi]$. As a matter of fact, $\tilde{p}_0 = \frac{P_0}{2}$. Since $f'(p)$ is linearly related to the Weierstrass function $\varphi_1$, $\chi'(\tilde{p})$ can have only two zeroes in the fundamental domain,
i.e. there are no other extrema of $\chi(\tilde{p})$ on the interval $[0, \pi]$. The presence of a maximum means that the equation

$$\chi(\tilde{p}) = \chi_0$$

has two distinct real roots if $0 \leq \chi_0 < \chi \left(\frac{P_{cr}}{2}\right)$, $\frac{P_{cr}}{2} < \tilde{p}_1 \leq \pi$ and $0 \leq \tilde{p}_2 < \frac{P_{cr}}{2}$. These roots serve also as nontrivial solution to the equation (22) and thus give, via (21), the bound state of type II in which the wave function oscillates and decays exponentially as $|n_1 - n_2| \to \infty$. For $P_{cr} < P \leq \pi$ such a solution always exists. Similar solutions corresponding to $-\chi \left(\frac{P_{cr}}{2}\right) < \chi_0 < 0$ can be found with any $-\pi \leq P < -P_{cr}$. The methods used above do not allow us to say anything definite about the presence of bound state at $\pi < |P| \leq 2\pi$ different from the just described ones.

In conclusion, we have proved the existence of two types of bound states of two magnons in the model with nonlocal spin interaction (1-2). The states of type I appear for small values of the total pseudomomentum $P$, $0 < |P| < P_{cr} < \pi$. Due to oscillation of terms in the series (7), one can state that the energy for the type I states is lower than the continuum boundary only for sufficiently small values of $P$. The state of the type II appears if the total pseudomomentum exceeds the critical value, $P_{cr} < |P| \leq \pi$. Our treatment has been universal with respect to parameter $a$ in the interval $0 < a < \infty$. In the nearest-neighbor limit $a \to \infty$, the type II states with complex relative pseudomomentum and oscillating wave function disappear ($P_{cr} \to \pi$ as it can be seen from (19)) and the result coincides with the known one. An interesting open problem concerns the presence or absence of multimagnon bound states. The condition of their existence is given by the set of transcendental equations of the type (15). At present, we have not found the way of their analytical investigation.

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