Fractal to Nonfractal Phase Transition in the Dielectric Breakdown Model

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A fast method is presented for simulating the dielectric-breakdown model using iterated conformal mappings. Numerical results for the dimension and for corrections to scaling are in good agreement with the recent RG prediction of an upper critical \( \eta = 4 \), at which a transition occurs between branching fractal clusters and one-dimensional nonfractal clusters.

Diffusion-limited aggregation (DLA) is one of the most interesting and difficult fractal growth models, despite its simple rules. It models a wide range of processes in nature that give rise to branching structures, including viscous fingering, electrodeposition, and dendritic growth. Yet, despite the importance of the model and various theoretical approaches, a full, controlled understanding is lacking.

An extension of DLA is the dielectric-breakdown model (DBM), also known as the Eden model \( \eta = 0 \), the Eden model \( \eta = 1 \), to roughly 1.7 ( \( \eta = 1 \), DLA), to 1. Recently, it has been argued that \( d = 1 \) for \( \eta \geq \eta_c = 4 \), leading to a controlled renormalization group expansion for the DBM. Further, while continuum DLA has certain integrability properties, most other theoretical approaches do not rely on these and should apply to the DBM as well. Thus, numerical studies of the DBM provide a strong check on these and should apply to the DBM as well.

For the dielectric breakdown model with given \( \eta \), we need to chose the angle \( \theta_{n+1} \), and construct an elementary mapping \( f^{(n+1)} \) which produces a bump of linear size \( \sqrt{\lambda_{n+1}} \) at angle \( \theta_{n+1} \). Then, compose mappings to define \( F^{(n+1)}(z) = F^{(n)}(f^{(n+1)}(z)) \), which adds a bump to the cluster. To obtain bumps of fixed size in the physical plane, we chose

\[
\lambda_{n+1} = \frac{\lambda_0}{|F^{(n)}(e^{i\theta_{n+1}})|^2},
\]

where \( \lambda_0 \) is some constant. Various elementary functions are possible; we use the function \( f \) proposed in Ref. [10], which includes a parameter \( a \) describing the bump shape.

For the dielectric breakdown model with given \( \eta \), we use the function \( f \) proposed in Ref. [10], which includes a parameter \( a \) describing the bump shape. The conformal mapping method proceeds as follows: Define

\[
P = \left( \frac{|F^{(n)}(e^{i\theta})|}{|F^{(n)}(e^{i\theta_{old}})|} \right)^{1-\eta}.
\]

If \( P > 1 \), it replaces \( \theta_{old} \) with \( \theta \), “accepting” angle \( \theta \). If \( P < 1 \), it replaces \( \theta_{old} \) with \( \theta \) with probability \( P \). The program proceeds through a certain number \( T \) of these trials, and then takes \( \theta_{n+1} = \theta_{old} + 1 \) for the \( n \)th particle to the cluster. If \( T \) is large enough, this yields the desired distribution.

The time for the algorithm to produce a cluster of size \( N \) scales as \( N^2T \). We have used various means to determine the minimum value of \( T \) needed. In Fig. 1, we show a plot of \( F_1 \) against number of growth steps for single realizations with \( \eta = 4 \), \( N = 10000 \) and \( T = 50, 300, 400, 500 \). While the curve for \( T = 50 \) is clearly different, the other curves are all close, suggesting that \( T = 300 \) suffices. Determinations of the dimension using \( T = 300 \) are also indistinguishable from those using \( T = 500 \); however, to be careful, in all quantitative results given below, we have chosen \( T = 500 \), except for \( \eta = 2 \), where \( T = 200 \) was used. On an 850-MHz Pentium III, a 10000 particle cluster with \( T = 500 \) can be simulated in roughly 33 hours.
To provide an analytical estimate of the minimum $T$ needed we consider the worst case, a one-dimensional cluster, obtained by a suitably regularized version of the mapping $F(z) = F_0 z + F_1/z$. For $\eta > 2$, the probability distribution $|F^{(n)}(e^{i\theta})|^{1-\eta}$ diverges near $\theta = 0, \pi$. The appropriate regularization of the cluster at a physical length of order $\sqrt{\lambda}$ cuts off this divergence at an angle that scales as $\sqrt{\lambda}/F_0 \propto \sqrt{1/n}$. In order to have sufficient trials that the algorithm is able to find a point in this narrow region, one must have $T > \sqrt{\lambda}$. For clusters which are not one-dimensional, so that $F'(z)$ is less singular, a smaller value of $T$ may be used. A final check on the value of $T$ needed is to look at the number of times a trial angle is accepted, as shown in Table I.

In all simulations we used $\lambda_0 = 0.5$. We chose $a = 2/3$, suggested by Davidovitch et. al. [1] as minimizing numerical error due to regions in the fjords of the cluster in which the derivative of $F(z)$ varies rapidly. An improved technique for dealing with these regions involving an acceptance window was suggested by Stepanov and Levitov [12]. However, for $\eta > 1$, we automatically avoid regions with large $|F'(z)|$, so that we do not expect to have to worry about this problem; this justifies the very simple plotting technique employed to produce the images of the clusters below. We chose to plot only the images of certain points on the unit circle chosen to lie near singularities of the mapping, each point corresponding to a single growth step.

We used a parallel computer to generate many realizations of each cluster. However, generation of a single large cluster is a process that parallelizes very efficiently. Each node picks random angles and computes the Jacobian, returning the results to a central process, which does the acceptance calculation, and returns the chosen angle to the other nodes. This will increase cluster size, but at a cost in statistics.

Results — In Fig. 2, we show a large cluster with $\eta = 2, N = 50000, T = 30$ for purposes of illustration. In Figs. 3,4 we show clusters with $\eta = 3.5, 4.5$ and $N = 10000, T = 500$. A difference in the “fuzziness” of the two clusters is clear, with the cluster at $\eta = 3.5$ having many more small side branches. The single side branch in Fig. 4 is not the only possibility for this $\eta$. Other realizations show either a cluster with no side branches, or with a single branching near a growth tip.

For quantitative results, we have generated 13-14 realizations of clusters for each $\eta = 3, 3.5, 4, 4.5, 5$, with $N = 10000, T = 500$, and 8 realizations of clusters for $\eta = 2$ and $N = 15000, T = 200$. $F_1^{(n)}$ provides a measure of the linear size of the cluster; it may be shown that the radius of the cluster is at most 4 times $F_1$. We calculated the dimension $D$ by averaging $F_1$ over realizations of clusters and using linear regression on a log-log plot to fit

$$F_1^{(n)} \propto n^{1/D}. \quad (4)$$

In Fig. 5 we plot $F_1$ and the rms fluctuations in $F_1$ for $\eta = 3$ on a logarithmic scale; the rms fluctuations have a smaller slope, so the relative fluctuations in $F_1$ tend to zero, as found previously for $\eta = 1$ [1]. The use of $F_1$ to determine dimension avoids problems with finite size effects in box-counting that can lead to a spurious dimension less than 1, as found for some $\eta$ in previous studies [4].

While for $\eta = 3$, Fig. 5 is close to straight, for $\eta = 4$ there is a curvature, as shown in the solid line in Fig. 6 (the dashed line is a fit discussed below). To quantitatively measure this curvature, indicative of corrections to scaling, we performed the linear regression 4 times for each $\eta$, first on the full set, then on the last half, quarter, and eighth of the data set. The results are shown in Table II; we also show the rms fluctuations in $F_1$ after the 10000-th growth step. For $\eta < 4$, the corrections to scaling are small, and the dimension is relatively constant, sometimes increasing and sometimes decreasing in the last part of the data set; these dimensions are close to those found in previously [4, 12].

For $\eta \geq 4$, there is a clear trend for the dimension to approach unity at longer times, with the trend most clear for $\eta > 4$. We test the trend against the RG prediction for the corrections to scaling. The leading irrelevant variable in the RG is a tip splitting rate $g$, which is irrelevant for $\eta > 4$, with scaling dimension $(4-\eta)/2$. At $\eta = 4$ the tip-splitting is marginal, which may be shown to lead to $F_1 = c_1 n/(\log (n/c_2))^{(4-\eta)/2}$, where $c_2$ is a universal constant determined by the $\beta$-functions and $c_2$ is nonuniversal. The dashed line in Fig. 6 shows the fit at $\eta = 4$ with this scaling form; the fit is indistinguishable from the data over more than three decades, with only a slight difference on the first 5 data points out of 10000. (Within the RG, the $\beta$-functions are determined numerically, so we keep $c_3$ as a fitting parameter; the fit agrees with the numerical result from the RG). For $\eta > 4$ the leading correction to scaling is $F_1 = c_1 n(1 + (n/c_2)^{(4-\eta)/2})$. A fit with this form is shown in Fig. 7 for $\eta = 4.5$, with the fit distinguishable from the data over only roughly the first 30 data points out of 10000. (We checked the RG prediction for the scaling corrections by attempting to fit the numerical data with other forms; the results are much less accurate).

Conclusion — We have presented a fast algorithm for studying the dielectric breakdown model. Numerical results using this algorithm, using the RG corrections to scaling, support an upper critical $\eta_c = 4$ for which the DBM clusters become one-dimensional. The existence of finite size corrections for $\eta \geq 4$ makes it desirable to simulate even larger clusters, but, while we have greatly improved the cluster size over previous work, the results in this paper are close to the limit of what can presently be attained using the conformal mapping techniques.

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TABLE I. For various values of $\eta, N, T$, the total number of acceptances, and number of growth steps on which at least one acceptance occurred, for a typical run.

| $\eta$ | $N$   | $T$   | Total Acceptances | Growth Steps with Acceptance |
|--------|-------|-------|-------------------|-----------------------------|
| 2      | 15000 | 200   | 1291738           | 15000                       |
| 2      | 50000 | 30    | 580805            | 49902                       |
| 3      | 10000 | 500   | 837291            | 10000                       |
| 3.5    | 10000 | 500   | 442854            | 10000                       |
| 4      | 10000 | 500   | 226170            | 99900                       |
| 4.5    | 10000 | 500   | 143031            | 99500                       |
| 5      | 10000 | 500   | 99702             | 9826                        |

TABLE II. For various values of $\eta$, calculated dimension based on different parts of data set; also, rms fluctuations in $F_1$ after last step, normalized by $F_1$.

| $\eta$ | full set | last 1/2 | last 1/4 | last 1/8 | rms |
|--------|----------|----------|----------|----------|-----|
| 2      | 1.433    | 1.4256   | 1.435    | 1.4522   | .039|
| 3      | 1.263    | 1.264    | 1.262    | 1.243    | .056|
| 3.5    | 1.170    | 1.143    | 1.155    | 1.162    | .078|
| 4      | 1.128    | 1.090    | 1.078    | 1.071    | .072|
| 4.5    | 1.101    | 1.090    | 1.066    | 1.035    | .039|
| 5      | 1.068    | 1.030    | 1.025    | 1.009    | .046|
FIG. 1. $F_1$ as a function of $N$ for $\eta = 4$ and various values of $T$ (marked on the right-hand axis).

FIG. 2. Cluster with $\eta = 2$, $N = 50000$, $T = 30$.

FIG. 3. Cluster with $\eta = 3.5$, $N = 10000$, $T = 500$.

FIG. 4. Cluster with $\eta = 4.5$, $N = 10000$, $T = 500$.

FIG. 5. Average of $F_1$ (upper curve) and rms fluctuations in $F_1$ (lower curve) as a function of $n$ for $\eta = 3$.

FIG. 6. Plot of $F_1$ versus $n$ for $\eta = 4$. The barely visible dashed line is a fit including corrections to scaling.
FIG. 7. Plot of $F_1$ versus $n$ for $\eta = 4.5$. The dashed line is a fit including corrections to scaling.