GLOBAL EXACT CONTROLLABILITY TO THE TRAJECTORIES OF THE KURAMOTO-SIVASHINSKY EQUATION

PENG GAO∗

State Key Laboratory of Automotive Simulation and Control
Jilin University, Changchun 130012, China

and
School of Mathematics and Statistics
and Center for Mathematics and Interdisciplinary Sciences
Northeast Normal University, Changchun 130024, China

(Communicated by Andrei Fursikov)

Abstract. In this paper, we establish the global exact controllability to the trajectories of the Kuramoto-Sivashinsky equation by five controls: one control is the right member of the equation and is constant with respect to the space variable, the four others are the boundary controls.

1. Introduction. The Kuramoto-Sivashinsky (KS) equation reads as

$$y_t + ky_{xx} + y_{xxxx} + yy_x = 0,$$

where the real number $k > 0$ is called the “anti-diffusion” parameter. This equation was derived independently by Kuramoto et al. in [17, 18, 19] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky in [21] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. This nonlinear partial differential equation (PDE for short) describes incipient instabilities in a variety of physical and chemical systems (see, for instance, [8, 16]). This equation also arises in the modeling of the flow of a thin film of viscous liquid falling down on an inclined plane subject to an applied electric field [14]. The KS equation has been studied as a prototypical example for an infinite-dimensional dynamical system.

In this paper, we consider the following control system

$$\begin{align*}
    y_t + ky_{xx} + y_{xxxx} + yy_x &= u(t) & \text{in } Q, \\
    y(0, t) &= v_1(t), y(1, t) = v_3(t) & \text{in } (0, T), \\
    y_x(0, t) &= v_2(t), y_x(1, t) = v_4(t) & \text{in } (0, T), \\
    y(x, 0) &= y_0(x) & \text{in } I,
\end{align*}$$

(1)

where $I = (0, 1)$, $Q = I \times (0, T)$, the state is given by $y = y(x, t)$ and the time-dependent functions $u, v_1, v_2, v_3, v_4$ are controls. The main result in this paper is the global exact controllability to the trajectories of system (1):

2000 Mathematics Subject Classification. Primary: 93B05; Secondary: 35K55.

Key words and phrases. Global exact controllability, Kuramoto-Sivashinsky equation.

This work is supported by Foundation of State Key Laboratory of Automotive Simulation and Control and NSFC Grant (11601073).

∗ Corresponding author: Peng Gao.

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Theorem 1.1. For any $T > 0, y_0 \in L^2(I)$ and $\hat{y} \in L^\infty(0,T; H^2(I))$ which satisfies
\[
\begin{align*}
\hat{y}_t + k \hat{y}_{xx} + \hat{y}_{xxxx} + \hat{y}_{x} &= 0 & \text{in } Q, \\
\hat{y}(0, t) &= \hat{y}(1, t) & \text{in } (0, T), \\
\hat{y}_x(0, t) &= \hat{y}_x(1, t) & \text{in } (0, T).
\end{align*}
\]
There exist $v_1(t), v_2(t), v_3(t), v_4(t) \in L^2(0,T)$ and $u \in C^\infty([0,T])$ vanishing on a neighborhood of 0 and $T$ such that the solution $y$ of (1) satisfies
\[y(T,x) = \hat{y}(T,x).\]

Remark 1. The global exact controllability results to the trajectories of the Burgers equation and KdV equation have been considered in [6, 7]. To our best knowledge, Theorem 1.1 is the first result in the literature on the global exact controllability to the trajectories of the KS equation.

The controllability of the fourth order parabolic equation has been studied by many authors. The fourth order parabolic equation([1]), the KS equation([4, 9, 2, 3, 5, 13]), the Cahn-Hilliard equation([15], the Swift-Hohenberg equation([11]), the viscous Camassa-Holm equation([12]).

This article is organized as follows. In Section 2 we do some preliminaries. Theorem 1.1 is proved in Section 3.

2. Some preliminaries.

2.1. Basic assumptions. Through this paper, we make the following assumptions:
\(\diamond\) For $0 \leq s \leq 6$, set
\[
X_s = \{\varphi \in H^s(I) | \varphi \text{ satisfies the } s\text{-compatibility conditions}\},
\]
\[
Y_s(T) = C([0,T]; X_s) \cap L^2(0,T; H^{s+2}(I)).
\]
The $s$-compatibility conditions are following:
\[
\begin{align*}
\varphi(0) &= \varphi(1) = 0 & \text{when } & \frac{1}{2} < s \leq \frac{3}{2}, \\
\varphi(0) &= \varphi(1) = \varphi'(0) = \varphi'(1) = 0 & \text{when } & \frac{4}{3} < s \leq \frac{5}{2}, \\
\varphi(0) &= \varphi(1) = \varphi'(0) = \varphi'(1) = \varphi''(0) = \varphi''(1) = 0 & \text{when } & \frac{3}{2} < s.
\end{align*}
\]
The norms on $X_s$ and $Y_s(T)$ are defined by $\|\varphi\|_{X_s} \triangleq \|\varphi\|_{H^s(I)}$ and $\|y\|_{Y_s(T)} \triangleq \|y\|_{C([0,T]; H^s(I))} + \|y\|_{L^2(0,T; H^{s+2}(I))}$, for any $\varphi \in X_s$ and $y \in Y_s(T)$.
\(\diamond\) Throughout this paper, $C$ denotes various positive constants. Frequently, we will emphasize the fact that $C$ depends on $\nu$ by writing $C = C(\nu)$.

2.2. Local exact controllability to the trajectories of the KS equation. Let us recall the local exact controllability result in [4].

Proposition 1. (See [4, Theorem 1.1]) Let $T > 0$ and $u \in L^\infty(0,T; H^2_0(I))$ be a solution of following system
\[
\begin{align*}
u_t + ku_{xx} + u_{xxxx} + u_x &= 0 & \text{in } Q, \\
u(0, t) &= \nu(1, t) & \text{in } (0, T), \\
u_x(0, t) &= \nu_x(1, t) & \text{in } (0, T).
\end{align*}
\]
There exists $r_0 > 0$ such that for any $y_0 \in H^{-2}(I)$ with $\|y_0 - u(\cdot,0)\|_{H^{-2}(I)} < r_0$, there exist $h_1, h_2 \in L^2(0,T)$ and $y \in C([0,T]; H^{-2}(I)) \cap L^2(0,T; L^2(I))$ satisfying
It follows from the Gronwall inequality that

\[ y_t + ky_{xx} + y_{xxxx} + yy_x = 0 \quad \text{in } Q, \]
\[ y(0, t) = h_1(t), y(1, t) = 0 \quad \text{in } (0, T), \]
\[ y_x(0, t) = h_2(t), y_x(1, t) = 0 \quad \text{in } (0, T), \]
\[ y(x, 0) = y_0(x) \quad \text{in } I, \]

and

\[ y(t, T) = u(t, T). \]

2.3. **Smoothing effect for (1) without internal and boundary force.** In this section, we establish a regularity property for the KS equation that we shall use later. Let us consider the following system:

\[
\begin{aligned}
\begin{cases}
y_t + ky_{xx} + y_{xxxx} + yy_x = 0 & \text{in } Q, \\
y(0, t) = 0 = y(1, t) & \text{in } (0, T), \\
y_x(0, t) = 0 = y_x(1, t) & \text{in } (0, T), \\
y(x, 0) = y_0(x) & \text{in } I,
\end{cases}
\end{aligned}
\]

**Proposition 2.** The well-posedness of (2) is given in the following:

i) If \( y_0 \in X_0 \), (2) admits a unique solution \( y \in Y_0(T) \), which also satisfies

\[ \|y\|_{Y_0(T)} \leq \beta_0(\|y_0\|_{X_0}) \|y_0\|_{X_0}, \]  

where \( \beta_0 : R^+ \to R^+ \) is a nondecreasing continuous function. Moreover, the corresponding solution map is locally Lipschitz continuous: for any \( y_{10}, y_{20} \in X_0 \), the corresponding solutions \( y_1 \) and \( y_2 \) of (2) satisfy

\[ \|y_1 - y_2\|_{Y_0(T)} \leq \beta_0(\|y_{10}\|_{X_0} + \|y_{20}\|_{X_0}) \|y_{10} - y_{20}\|_{X_0}. \]

ii) If \( y_0 \in X_4 \), (2) admits a unique solution \( y \in Y_4(T) \) which also satisfies

\[ \|y\|_{Y_4(T)} \leq \beta_4(\|y_0\|_{X_0}) \|y_0\|_{X_4}, \]

where \( \beta_4 : R^+ \to R^+ \) is a nondecreasing continuous function.

iii) Let \( 0 \leq s \leq 4 \). If \( y_0 \in X_s \), (2) admits a unique solution \( y \in Y_s(T) \) which also satisfies

\[ \|y\|_{Y_s(T)} \leq \beta_s(\|y_0\|_{X_0}) \|y_0\|_{X_s}, \]

where \( \beta_s : R^+ \to R^+ \) is a nondecreasing continuous function.

iv) If \( y_0 \in X_6 \), (2) admits a unique solution \( y \in Y_6(T) \) which also satisfies

\[ \|y\|_{Y_6(T)} \leq C(\|y_0\|_{X_0}), \]

where \( C(\|y_0\|_{X_0}) \) is a constant dependants on \( y_0 \).

**Proof.** We proceed following the ideas in [20, Theorem 2.9].

i) The local existence and uniqueness of the solution of (2) can be obtained by the classical semigroup theory and Banach fixed point theorem. Next, we prove a priori estimate. Multiplying the equation in (2) by \( y \) and then performing integration by parts over \( I \) we get

\[ \frac{1}{2} \frac{d}{dt} \int_I y^2(x, t)dx + \int_I y^2_{xx}(x, t)dx = \int_I ky^2_x(x, t)dx. \]

Using Cauchy inequality and interpolation inequality, we can obtain

\[ \frac{d}{dt} \int_I y^2(x, t)dx + \int_I y^2_{xx}(x, t)dx \leq C(k) \int_I y^2(x, t)dx. \]

It follows from the Gronwall inequality that

\[ \|y\|_{X_0,T} \leq C(T, k) \|y_0\|_{L^2(I)}. \]
Set $z = y_1 - y_2$, we have
\[
\begin{aligned}
&z_t + kz_{xx} + z_{xxxx} + y_1z_x + y_2z = 0 \quad \text{in } Q, \\
&z(0, t) = 0 = z(1, t) \quad \text{in } (0, T), \\
&z_x(0, t) = z_x(1, t) \quad \text{in } (0, T), \\
&z(x, 0) = y_{10}(x) - y_{20}(x) \quad \text{in } I.
\end{aligned}
\]

Multiplying the equation in (4) by $z$ and then performing integration by parts over $I$ we get
\[
\frac{1}{2} \frac{d}{dt} \int_I z^2(x, t)dx + \int_I z_{xx}^2(x, t)dx = \int_I k z^2(x, t)dx - \int_I (y_1z_x + y_2z)zdx.
\]
Using Cauchy inequality, we can obtain
\[
\frac{d}{dt} \int_I z^2(x, t)dx + \int_I z_{xx}^2(x, t)dx \leq C(k)(\|y_1\|_{H^1(I)}^2 + \|y_2\|_{H^2(I)} + 1) \int_I z^2(x, t)dx.
\]
It follows from the Gronwall inequality that
\[
\|z\|_{X_0, T} \leq e^{C(T, k)(\|y_{10}\|_{L^2(I)}^2 + \|y_{20}\|_{L^2(I)}^2 + 1)}\|z_0\|_{L^2(I)}.
\]

ii) By i), (2) admits a unique solution $y \in Y_0(T)$. It is sufficient to show that $y \in Y_4(T)$. Let $u = y_1$, it is easy to know $u$ is a solution of
\[
\begin{aligned}
&u_t + ku_{xx} + u_{xxxx} + uy_x + yu_x = 0 \quad \text{in } Q, \\
&u(0, t) = 0 = u(1, t) \quad \text{in } (0, T), \\
&u_x(0, t) = 0 = u_x(1, t) \quad \text{in } (0, T), \\
&u(x, 0) = u_{10}(x) \quad \text{in } I
\end{aligned}
\]
with $u_0 = -ky_{0xx} - y_{0xxxx} - y_0y_{0x}$. Note that $u_0 \in X_0$ and an easy computation yields a constant $C = C(\|y_0\|_{X_0})$ such that
\[
\|u_0\|_{X_0} \leq C\|y_0\|_{X_4}.
\]
Multiplying the equation in (5) by $u$ and then performing integration by parts over $I$ we get
\[
\frac{1}{2} \frac{d}{dt} \int_I u^2(x, t)dx + \int_I u_{xx}^2(x, t)dx = \int_I ku_x^2(x, t)dx + \int_I yu_xudx.
\]
It follows from the interpolation inequality that
\[
\frac{1}{2} \frac{d}{dt} \int_I u^2(x, t)dx + \int_I u_{xx}^2(x, t)dx \\
\leq k \int_I u_x^2(x, t)dx + \|y\|_{L^\infty(I)}^2 \int_I u_{xx}^2(x, t)dx + \int_I u^2(x, t)dx \\
\leq \frac{1}{2} \int_I u_{xx}^2(x, t)dx + C(k)(\|y\|_{H^2(I)}^2 + 1) \int_I u^2(x, t)dx.
\]
Using Cauchy inequality, we can obtain
\[
\frac{d}{dt} \int_I u^2(x, t)dx + \int_I u_{xx}^2(x, t)dx \leq C(k)(\|y\|_{H^2(I)}^2 + 1) \int_I u^2(x, t)dx.
\]
It follows from the Gronwall inequality that
\[
\|u\|_{Y_4(T)} \leq e^{C(T, k)(\|y_0\|_{L^2(I)}^2 + 1)}\|u_0\|_{L^2(I)}.
\]
We can know that (5) admits a unique solution $u \in Y_0(T)$ according to Lemma 3.1 and the proof of i). Observe that $y_t = u$ and $u \in Y_0(T)$. 
Since \( y \in L^2(0, T; H^2(I)) \cap C([0, T]; L^2(I)) \), we have

\[
\|yy_x\|_{L^2(0,T;L^2(I))} = \left( \int_0^T \|yy_x\|_{L^2(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|y_x\|_{L^\infty(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|y\|_{H^1(I)}^2 \, dt \right)^{\frac{1}{2}} 
= \|y\|_{C([0,T];L^2(I))} \|y\|_{L^2(0,T;H^2(I))},
\]

namely, we have \( yy_x \in L^2(0,T;L^2(I)) \).

Note that

\[
y_{xxxx} = -y_t - ky_{xx} - yy_x,
\]

we have \( y_{xxxx} \in L^2(0,T;L^2(I)) \), thus we have \( y \in L^2(0, T; H^4(I)) \), hence \( y \in C([0,T]; H^3(I)) \).

Since \( y \in L^2(0, T; H^4(I)) \cap C([0, T]; H^3(I)) \) and \((yy_x)_x = y_x^2 + yy_{xx}\), we have

\[
\|y_x^2\|_{L^2(0,T;L^2(I))} = \left( \int_0^T \|y_xy_x\|_{L^2(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y_x\|_{L^2(I)}^2 \|y_x\|_{L^\infty(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y_x\|_{L^2(I)}^2 \|y\|_{H^2(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \|y\|_{C([0,T];H^1(I))} \left( \int_0^T \|y\|_{H^2(I)}^2 \, dt \right)^{\frac{1}{2}} 
= \|y\|_{C([0,T];H^1(I))} \|y\|_{L^2(0,T;H^4(I))},
\]

and

\[
\|yy_{xx}\|_{L^2(0,T;L^2(I))} = \left( \int_0^T \|yy_{xx}\|_{L^2(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|yy_{xx}\|_{L^\infty(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|y\|_{H^4(I)}^2 \, dt \right)^{\frac{1}{2}} 
\leq \|y\|_{C([0,T];L^2(I))} \left( \int_0^T \|y\|_{H^4(I)}^2 \, dt \right)^{\frac{1}{2}} 
= \|y\|_{C([0,T];L^2(I))} \|y\|_{L^2(0,T;H^4(I))},
\]

thus we have

\[
\|(yy_x)_x\|_{L^2(0,T;L^2(I))} \leq \|y\|_{C([0,T];H^3(I))} \|y\|_{L^2(0,T;H^4(I))},
\]

namely, we have \( yy_x \in L^2(0,T;H^4(I)) \).

Since \( y \in L^2(0, T; H^4(I)) \cap C([0, T]; H^3(I)) \) and \((yy_x)_{xx} = (y_x^2 + yy_{xx})_x = 3x_xyy_x + yy_{xxx}\), we have

\[
\|yy_{xx}\|_{L^2(0,T;L^2(I))} = \left( \int_0^T \|yy_{xx}\|_{L^2(I)}^2 \, dt \right)^{\frac{1}{2}}
\]
thus we have

\[
\int_0^T \|y_x\|_{L^2(I)}^2 \|y_{xx}\|_{L^\infty(I)}^2 \, dt \leq \left( \int_0^T \|y_x\|_{L^2(I)}^2 \|y_{xx}\|_{L^\infty(I)}^2 \, dt \right)^{1/2}
\]

and

\[
\|yy_{xx}\|_{L^2(0,T;L^2(I))} \leq \left( \int_0^T \|yy_{xx}\|_{L^2(I)}^2 \, dt \right)^{1/2}
\]

\[
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|yy_{xx}\|_{L^\infty(I)}^2 \, dt \right)^{1/2}
\]

\[
\leq \left( \int_0^T \|y\|_{L^2(I)}^2 \|y\|_{H^2(I)}^2 \, dt \right)^{1/2}
\]

\[
\leq \|y\|_{C([0,T];L^2(I))} \left( \int_0^T \|y\|_{H^2(I)}^2 \, dt \right)^{1/2}
\]

\[
= \|y\|_{C([0,T];L^2(I))} \|y\|_{L^2(0,T;H^2(I))},
\]

thus we have \( \|yy_{xx}\|_{L^2(0,T;L^2(I))} \leq \|y\|_{C([0,T];H^2(I))} \|y\|_{L^2(0,T;H^4(I))} \), namely, we have \( y_{xx} \in L^2(0,T;H^2(I)) \).

It can be deduced from (6) that \( y \in L^2(0,T;H^6(I)) \), hence \( y \in C([0,T];H^4(I)) \). Consequently, there exists a nondecreasing continuous function \( \beta_4 : R^+ \to R^+ \) such that

\[
\|y\|_{Y_4(T)} \leq \beta_4(\|y_0\|_{X_0}) \|y_0\|_{X_4}.
\]

iii) The cases \( s = 0 \) and \( s = 4 \) have been proved in i) and ii). The cases of \( 0 < s < 4 \) follow by the nonlinear interpolation theory in [20, Theorem 2.8], thus we can obtain iii).

iv) Let \( u \) be the same as in iii). Multiplying the equation in (5) by \( u_{xxx} \) and then performing integration by parts over \( I \) we get

\[
\frac{1}{2} \frac{d}{dt} \int_I u_{xx}^2(x,t) \, dx + \int_I u_{xxx}^2(x,t) \, dx
\]

\[
= -\int_I k u_{xx} u_{xxx} \, dx - \int_I y u_{xx} u_{xxx} \, dx - \int_I y_x u_{xxx} \, dx.
\]

Using Cauchy inequality, we can obtain

\[
\frac{1}{2} \frac{d}{dt} \int_I u_{xx}^2(x,t) \, dx + \int_I u_{xxx}^2(x,t) \, dx
\]

\[
= -\int_I k u_{xx} u_{xxx} \, dx - \int_I y u_{xx} u_{xxx} \, dx - \int_I y_x u_{xxx} \, dx
\]

\[
\leq \frac{1}{2} \int_I u_{xxx}^2(x,t) \, dx + C k^2 \int_I u_{xx}^2(x,t) \, dx
\]

\[
+ C \|y\|_{L^\infty(I)}^2 \int_I u_x^2(x,t) \, dx + C \|y_{xx}\|_{L^\infty(I)}^2 \int_I u^2(x,t) \, dx
\]

\[
\leq \frac{1}{2} \int_I u_{xxx}^2(x,t) \, dx + C (k^2 + \|y\|_{L^\infty(I)}^2 + \|y_{xx}\|_{L^\infty(I)}^2) \int_I u_{xx}^2(x,t) \, dx
\]
3. Proof of Theorem 1.1.

3.1. A global approximate controllability result for (1). In this subsection, we establish a global approximate controllability result for (1) with smooth initial and final states, this is important when proving Theorem 1.1.

Proposition 3. For any $M > 0$, there exist $K > 0$ and $\delta_1 > 0$ such that, for any $0 < \delta < \delta_1$ and any $z_0, z_1 \in C^4(\overline{I})$ which satisfy

$$
\|z_0\|_{C^4(\overline{I})} \leq M, \tag{7}
$$

$$
\|z_1\|_{C^4(\overline{I})} \leq M, \tag{8}
$$

there exist $h_1(t), h_2(t), h_3(t), h_4(t) \in L^2(0, \delta)$ and $u \in C^\infty([0, \delta])$ vanishing on a neighborhood of 0 and $\delta$ such that the solution $y \in Y_0(\delta)$ of

$$
\begin{align*}
&\frac{d}{dt} \int I u_{xx}^2(x,t)dx + \int I u_{xxxx}^2(x,t)dx \leq C(k)(\|y\|_{H^2(I)}^2 + 1) \int I u_{xx}^2(x,t)dx. \\
&\text{It follows from the Gronwall inequality that } \|u\|_{Y_0(\delta)} \leq C(\|y_0\|_{X_0}).
\end{align*}
$$

According to the equation $y_{xxxx} = -u - ky_{xx} - yy_x$ and the same method as in iii), we can obtain $y \in Y_0(T)$ and $\|y\|_{Y_0(\delta)} \leq C(\|y_0\|_{X_0}).$

The proof of Proposition 2 is complete. \qed

According to Proposition 2, we can obtain a strong smoothing property of system (2).

Corollary 1. For any $y_0 \in X_0$, the corresponding solution $y$ of (2) belongs to the space $C([\varepsilon, T]; H^6(I)) \cap L^2(\varepsilon, T; H^8(I))$ for any $\varepsilon > 0$.

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$$
\|z_0\|_{C^4(\overline{I})} \leq M, \tag{7}
$$

$$
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$$

there exist $h_1(t), h_2(t), h_3(t), h_4(t) \in L^2(0, \delta)$ and $u \in C^\infty([0, \delta])$ vanishing on a neighborhood of 0 and $\delta$ such that the solution $y \in Y_0(\delta)$ of

$$
\begin{align*}
y_t + ky_{xx} + y_{xxxx} + yy_x = u(t) & \text{ in } I \times (0, \delta), \\
y(0,t) = h_1(t), y(1,t) = h_3(t) & \text{ in } (0, \delta), \\
y_x(0,t) = h_2(t), y_x(1,t) = h_4(t) & \text{ in } (0, \delta), \\
y(x,0) = z_0(x), & \text{ in } I
\end{align*}
$$

satisfies

$$
\|y(\cdot, \delta) - z_1\|_{L^2(I)} \leq K\sqrt{\delta}.
$$

In order to prove Proposition 3, we need the following lemmas.

Lemma 3.1. There exists $C > 0$ such that, for every $M > 0$, there exists $\delta_0 > 0$ such that, for every $z_0, z_1 \in C^4(\overline{I})$ which satisfy (7) and (8) and for every $0 < \delta \leq \delta_0$, there exists $y^\delta \in C^4(\overline{I} \times [0, \delta])$ such that

$$
\begin{align*}
y_t^\delta + (a^\delta + y^\delta)y_x^\delta & = 0 \\
y^\delta(x,0) & = z_0(x) \\
y^\delta(x,\delta) & = z_1(x)
\end{align*}
$$

and

$$
\|y^\delta\|_{C([0, \delta]; C^4(\overline{I}))} \leq CM
$$

where

$$
a^\delta : [0, \delta] \to [0, +\infty),
$$

$$
t \to \frac{1}{\delta}a(t/\delta)
$$

and $a$ is defined as follows: $a \in C^\infty([0, T])$ such that $a(t) = 0$ on a neighborhood of 0 and $T$, such that $\int_0^T a(s)ds > 1$, for any $t \in [0, T]$, $a(t) = a(T - t)$. 

Remark 2. By the same methods as in [6, Lemma 5] and [7, Lemma 13] where they study the cases of $C^2$ and $C^4$-functions, we can prove Lemma 3.1.

Lemma 3.2. Let $M > 0$. Let $C, \delta_0$ and $y^\delta$ be as in Lemma 3.1. The problem

$$
\begin{align*}
\begin{cases}
R^\delta_t + kR^\delta_{xx} + R^\delta_{xxxx} + R^\delta y^\delta + (y^\delta + a^\delta)R^\delta_x + R^\delta y^\delta_x + y^\delta_{xxxx} = 0 \\
R^\delta(0, t) = 0, R^\delta(1, t) = 0, \\
R^\delta_x(0, t) = 0, R^\delta_x(1, t) = 0, \\
R^\delta(x, 0) = 0
\end{cases}
\end{align*}
$$

(10)

is well-posed in $Y_0(T)$. Moreover, there exists $K > 0$ and $0 < \delta_1 \leq \delta_0$ such that for every $z_0, z_1 \in C^4(\bar{T})$ which satisfy (7) and (8), for every $0 < \delta \leq \delta_1$, and for every $t \in (0, \delta)$,

$$
\|R^\delta(\cdot, t)\|_{L^2(T)} \leq K \sqrt{t}.
$$

(11)

Proof. The well-posedness of the initial value problem is classical, we omit it. We need to evaluate $R^\delta(\cdot, t)$. Multiplying the equation in (10) by $2R^\delta$ and taking into account the following estimates

$$
\begin{align*}
2 \int \int R^\delta_t R^\delta dx &= \frac{d}{dt} \int R^\delta dx, \\
2 \int \int R^\delta_{xxxx} R^\delta dx &= 2 \int \int R^\delta_{xx} dx, \\
2 \int \int R^\delta R^\delta_{xx} dx &\leq \|y^\delta\|_{C([0, T]; L^2(\bar{T}))}, \\
2 \int \int (k y^\delta_{xx} + y^\delta_{xxxx}) R^\delta dx &\leq \int \int R^\delta dx + C\|y^\delta\|_{C([0, T]; L^2(\bar{T}))}^2,
\end{align*}
$$

we can obtain that

$$
\begin{align*}
\frac{d}{dt} \int R^\delta dx &+ \int R^\delta_{xx} dx \\
\leq& C(1 + \|y^\delta\|_{C([0, T]; C^1(\bar{T}))}) \int R^\delta dx + 2k \int R^\delta_{xx} dx + C\|y^\delta\|_{C([0, T]; L^2(\bar{T}))}^2.
\end{align*}
$$

It follows from the interpolation inequality that

$$
\begin{align*}
\frac{d}{dt} \int R^\delta dx &+ \int R^\delta_{xx} dx \\
\leq& C(k)(1 + \|y^\delta\|_{C([0, T]; C^1(\bar{T}))}) \int R^\delta dx + C\|y^\delta\|_{C([0, T]; L^2(\bar{T}))}^2 \\
\leq& C(k)(1 + M) \int R^\delta dx + CM^2.
\end{align*}
$$

According to the Gronwall inequality, it holds that

$$
\int R^\delta(\cdot, t) dx \leq CM^2 e^{C(k)(1 + M)t} t \leq K^2 t,
$$

where $K^2 = CM^2 e^{C(k)(1 + M)T}$. \hfill \Box
Proof of Proposition 3. We define

\[ y := y^\delta + R^\delta + a^\delta \]
\[ u(t) := (a^\delta)'(t) \]
\[ h_1(t) := y^\delta(0, t) + a^\delta(t) \]
\[ h_2(t) := y^\delta(0, t) \]
\[ h_3(t) := y^\delta(1, t) + a^\delta(t) \]
\[ h_4(t) := y^\delta(1, t) \]

where \( y^\delta, R^\delta, a^\delta \) are the same as in Lemma 3.1 and Lemma 3.2. It is easy to see that \( y, h_1, h_2, h_3, h_4 \) solve Proposition 3.

\[ \square \]

3.2. **Proof of Theorem 1.1.** Arguing as in [6, 7], we take a sequence \( \{y_n\} \subseteq C^4(\mathcal{Q}) \) such that,

\[ y_n \to \hat{y} \text{ in } C([0,T];L^2(I)), \text{ as } n \to \infty. \]

One can find \( n_0 \geq 0 \) such that, for any \( t \in [0,T] \)

\[ ||y_n(\cdot, t) - \hat{y}(\cdot, t)||_{L^2(I)} < \frac{r_0}{2}. \]

Let \( y^1 \in Y_0(T) \) be the solution of (2), from Corollary 1, \( y^1 \in C([\frac{T}{4}, \frac{3T}{4}]; H^6(I)) \cap L^2((\frac{T}{4}, \frac{3T}{4}); H^8(I)). \)

We denote by

\[ M := ||y^1||_{C([\frac{T}{4}, \frac{3T}{4}]; C^4(\mathcal{Q}))} + ||y_{n_0}||_{C([0,T]; C^4(\mathcal{Q}))}, \]

\[ 0 < \delta < \min(\delta_1, \frac{r_0^2}{4K^2}, \frac{T}{2}), \]

where \( \delta_1, r_0, K \) are the same as in Proposition 1 and Proposition 3.

From Proposition 3, there exist \( u_2, v_2, v_3, v_4, v_5 \) such that the solution \( y^2 \) of

\[ \begin{cases} y_n + kyy_{xx} + yxxx + yy_{xx} = u_2 & \text{in } I \times (0, \delta), \\
y(0, t) = v_2(t), y(1, t) = v_2(t) & \text{in } (0, \delta), \\
y_x(0, t) = v_2(t), y_x(1, t) = v_2(t) & \text{in } (0, \delta), \\
y(x, 0) = y^1(\frac{3T}{4} - \delta, x) & \text{in } I \end{cases} \]

satisfies

\[ ||y^2(\cdot, \delta) - y_{n_0}(\cdot, \frac{3T}{4})||_{L^2(I)} \leq K\sqrt{\delta} < \frac{r_0}{2}. \]

Thus

\[ ||y^2(\cdot, \delta) - \hat{y}(\cdot, \frac{3T}{4})||_{L^2(I)} < r_0. \]

Applying Proposition 1, we can find \( v_3(t), v_4(t) \in L^2(0, T) \) and \( y^3 \) such that

\[ \begin{cases} y^3_t + kyy^3_{xx} + y^3xxx + y^3y^3_{xx} = 0 & \text{in } I \times (0, \frac{T}{4}), \\
y^3(0, t) = v_3(t), y^3(1, t) = 0 & \text{in } (0, \frac{T}{4}), \\
y^3(0, t) = v_4(t), y^3(1, t) = 0 & \text{in } (0, \frac{T}{4}), \\
y^3(x, 0) = y^2(x, \delta) & \text{in } I, \end{cases} \]

and

\[ y^3(\cdot, \frac{T}{4}) = \hat{y}(\cdot, T). \]
Then we define \( y, u, v_1, v_2, v_3 \) and \( v_4 \) by

\[
y = \begin{cases} 
y^1(x, t), & (x, t) \in I \times (0, \frac{3T}{4} - \delta), 
\end{cases}
\]

\[
y = \begin{cases} 
y^2(x, t - \frac{3T}{4} + \delta), & (x, t) \in I \times (\frac{3T}{4} - \delta, \frac{3T}{4}), 
\end{cases}
\]

\[
y^3(x, t - \frac{3T}{4}), & (x, t) \in I \times (\frac{3T}{4}, T), 
\end{cases}
\]

\[
u = \begin{cases} 
u_2(t - \frac{3T}{4} + \delta), t \in (\frac{3T}{4} - \delta, \frac{3T}{4}), 
0, & t \in (\frac{3T}{4}, T). 
\end{cases}
\]

\[
u_1 = \begin{cases} 
u_2^3(t - \frac{3T}{4} + \delta), t \in (\frac{3T}{4} - \delta, \frac{3T}{4}), 
\end{cases}
\]

\[
u_2 = \begin{cases} 
u_2^3(t - \frac{3T}{4} + \delta), t \in (\frac{3T}{4} - \delta, \frac{3T}{4}), 
\end{cases}
\]

\[
u_3 = \begin{cases} 
u_2^3(t - \frac{3T}{4} + \delta), t \in (\frac{3T}{4} - \delta, \frac{3T}{4}), 
0, & t \in (\frac{3T}{4}, T). 
\end{cases}
\]

\[
u_4 = \begin{cases} 
u_2^3(t - \frac{3T}{4} + \delta), t \in (\frac{3T}{4} - \delta, \frac{3T}{4}), 
0, & t \in (\frac{3T}{4}, T). 
\end{cases}
\]

It is easy to see that \( y, u, v_1, v_2, v_3 \) and \( v_4 \) solve Theorem 1.1.

**Acknowledgments.** I am grateful to the anonymous referees and the editor for their careful reading of the manuscript and numerous suggestions for its improvement. I sincerely thank Professor Yong Li for many useful suggestions and help.

**REFERENCES**

[1] N. Carreño and P. Guzmán, *On the cost of null controllability of a fourth order parabolic equation*, Journal of Differential Equations, **261** (2016), 6485–6520.

[2] E. Cerpa, A. Mercado and A. Pazoto, *Null controllability of the stabilized Kuramoto-Sivashinsky system with one distributed control*, SIAM J. Control Optim., **53** (2015), 1543–1568.

[3] E. Cerpa, *Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation*, Commun. Pure Appl. Anal., **9** (2010), 91–102.

[4] E. Cerpa and A. Mercado, *Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation*, J. Differential Equations, **250** (2011), 2024–2044.

[5] E. Cerpa, P. Guzmán and A. Mercado, *On the control of the linear Kuramoto-Sivashinsky equation*, ESAIM: Control, Optimisation and Calculus of Variations, **23** (2017), 165–194.

[6] M. Chapouly, *Global controllability of nonviscous and viscous Burgers type equations*, SIAM J. Control Optim., **48** (2009), 1567–1599.

[7] M. Chapouly, *Global controllability of a nonlinear Korteweg-de Vries equation*, Communications in Contemporary Mathematics, **11** (2009), 495–521.

[8] L. H. Chen and H. C. Chang, *Nonlinear waves on liquid film surfaces-II. Bifurcation analyses of the long-wave equation*, Chem. Eng. Sci., **41** (1986), 2477–2486.

[9] P. Gao, *A new global Carleman estimate for the one-dimensional Kuramoto-Sivashinsky equation and applications to exact controllability to the trajectories and an inverse problem*, Nonlinear Anal., **117** (2015), 133–147.

[10] P. Gao, *A new global Carleman estimate for Cahn-Hilliard type equation and its applications*, J. Differential Equations, **260** (2016), 427–444.

[11] P. Gao, *Local exact controllability to the trajectories of the Swift-Hohenberg equation*, Nonlinear Anal., **139** (2016), 169–195.
[12] P. Gao, Null controllability of the viscous Camassa-Holm equation with moving control, *Proc. Indian Acad. Sci. Math. Sci.*, **126** (2016), 99–108.

[13] P. Gao, Null controllability with constraints on the state for the 1-D Kuramoto-Sivashinsky equation, *Evol. Equ. Control Theory*, **4** (2015), 281–296.

[14] A. González and A. Castellanos, Nonlinear electrohydrodynamic waves on films falling down an inclined plane, *Phys. Rev. E.*, **53** (1996), 3573–3578.

[15] P. Guzmán, Local exact controllability to the trajectories of the Cahn-Hilliard equation, *Applied Mathematics & Optimization*, (2015), 1–28.

[16] A. P. Hooper and R. Grimshaw, Nonlinear instability at the interface between two viscous fluids, *Phys. Fluids*, **28** (1985), 37–45.

[17] Y. Kuramoto and T. Tsuzuki, On the formation of dissipative structures in reaction-diffusion systems, *Theor. Phys.*, **54** (1975), 687–699.

[18] Y. Kuramoto, Diffusion-induced chaos in reaction systems, *Suppl. Prog. Theor. Phys.*, **64** (1978), 346–367.

[19] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Prog. Theor. Phys.*, **55** (1976), 356–369.

[20] L. Rosier and B.-Y. Zhang, Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain, *SIAM Journal on Control and Optimization*, **45** (2006), 927–956.

[21] G. I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames-I Derivation of basic equations, *Acta Astronaut.*, **4** (1977), 1177–1206.

Received October 2018; revised March 2019.

E-mail address: gaopengjilindaxue@126.com