YETTER-DRINFELD MODULES
OVER WEAK MULTIPLIER BIALGEBRAS

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Abstract. We continue the study of the representation theory of a regular weak multiplier bialgebra with full comultiplication, started in [4, 2]. Yetter-Drinfeld modules are defined as modules and comodules, with compatibility conditions that are equivalent to a canonical object being (weakly) central in the category of modules, and equivalent also to another canonical object being (weakly) central in the category of comodules. Yetter-Drinfeld modules are shown to constitute a monoidal category via the (co)module tensor product over the base (co)algebra. Finite dimensional Yetter-Drinfeld modules over a regular weak multiplier Hopf algebra with full comultiplication are shown to possess duals in this monoidal category.

Introduction

Weak bialgebra (over a field) [15, 7] is a generalization of usual bialgebra in the following sense. It is a unital algebra and a counital coalgebra but the multiplication and the unit are not required to be coalgebra maps — symmetrically, the comultiplication and the counit are not required to be algebra maps. Instead, some weaker axioms are imposed which still ensure the expected behavior of the representation categories (see e.g. [3]). Namely, any (not necessarily finite dimensional) weak bialgebra $A$ has a distinguished unital subalgebra and counital quotient coalgebra $R$ (named the base (co)algebra). It is isomorphic to the base field if and only if $A$ is a usual bialgebra. The algebra and coalgebra structures of $R$ obey compatibility conditions that make it a separable Frobenius algebra [17, 18]. Consequently, the category of $R$-modules and the category of $R$-comodules are isomorphic and also the $R$-module tensor product of any two $R$-(co)modules is isomorphic to their $R$-comodule tensor product. In fact, any weak bialgebra $A$ contains not only $R$ but $R \otimes R^{\text{op}}$ both as a subalgebra and as a quotient coalgebra. Therefore any $A$-module and any $A$-comodule carries a natural $R$-bi(co)module structure. Via the $R$-(co)module tensor product, both the category $M_A$ of $A$-modules and the category $M^A$ of $A$-comodules are monoidal. That is to say, there are strict monoidal forgetful functors $U_A$ from $M_A$, and $U^A$ from $M^A$, to the category $R M_R$ of $R$-bi(co)modules. Similarly to the case of usual (i.e. non-weak) bialgebras studied in [13], the category $M^A$ can be recovered from the strict monoidal forgetful functor $U_A : M_A \to R M_R$ as a ‘commutant’ of the range of $U_A$ in $R M_R$. Symmetrically, the category $M_A$ can be re-covered from the strict monoidal forgetful functor $U^A : M^A \to R M_R$ as a ‘commutant’ of the range of $U^A$ in $R M_R$. The category of Yetter-Drinfeld modules over a weak bialgebra $A$ can be described as the weak monoidal center of $M_A$, equivalently, as the weak monoidal center of $M^A$, see [14, 8]. Thus it is again a monoidal category with a strict monoidal forgetful functor to $R M_R$.

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A weak Hopf algebra is a weak bialgebra equipped with an antipode in the appropriate sense [7]. If the antipode of a weak Hopf algebra $A$ is bijective, then finite dimensional Yetter-Drinfeld $A$-modules possess duals in the monoidal category of Yetter-Drinfeld $A$-modules [8].

Weak multiplier Hopf algebra was introduced in [20, 21]. It is a generalization of weak Hopf algebra in the same spirit as Van Daele’s multiplier Hopf algebra [19] extends a usual Hopf algebra. That is, the underlying algebra $A$ does not need to possess a unit. Instead, surjectivity and non-degeneracy of the multiplication is required. In this situation, there is a (largest possible) unital algebra $M(A)$ (known as the multiplier algebra of $A$) such that $A$ is a dense ideal in $M(A)$. A weak multiplier Hopf algebra is not a coalgebra either but it has a generalized comultiplication — which is a multiplicative map from $A$ to the multiplier algebra of the tensor product algebra $A \otimes A$ — and a generalized counit in the appropriate sense. Among the compatibility axioms, certain canonical maps $A \otimes A \rightarrow A \otimes A$ are required to be weakly (i.e. Von Neumann) invertible. The existence and the expected properties of the antipode map $A \rightarrow M(A)$ follow from the axioms. A weak multiplier Hopf algebra is said to be regular if the opposite algebra $A^{\text{op}}$ is a weak multiplier Hopf algebra too. In this case the antipode restricts to an isomorphism $A \rightarrow A$ [21].

Weak multiplier bialgebra was defined and studied in [4]. One expects that it should be ‘weak multiplier Hopf algebra without an antipode’ and this is almost the case. In fact, any weak multiplier bialgebra which possesses an antipode is a weak multiplier Hopf algebra. The converse is not true, however: one of the axioms of weak multiplier bialgebra (expressing some compatibility between the counit and the multiplication) may not hold in an arbitrary weak multiplier Hopf algebra (though it holds in a regular weak multiplier Hopf algebra).

As in the Hopf case, also a weak multiplier bialgebra $A$ is termed regular whenever $A^{\text{op}}$ is a weak multiplier bialgebra too. Whenever a regular weak multiplier bialgebra $A$ obeys a further property called fullness of the comultiplication (see [21]), the multiplier algebra $M(A)$ turns out to have a distinguished (non-unital) subalgebra $R$. This subalgebra is called the base (co)algebra for a classical analogy. Indeed, it was shown in [4] that $R$ possesses the structure of a coseparable coalgebra (that can be considered as a non-unital generalization of separable Frobenius algebra). As discussed in [6, 5], this implies that the category of firm $R$-modules and the category of $R$-comodules are isomorphic and also the $R$-module tensor product of any two firm $R$-modules (i.e. of $R$-comodules) is isomorphic to their $R$-comodule tensor product.

In the study of weak multiplier bialgebras, in the absence of an algebraic unit, it is a crucial question what is the appropriate notion of module. Working with any associative action seems to be too general: we cannot see a monoidal structure of the category of such modules. However, restricting to modules — over a regular weak multiplier bialgebra $A$ with a full comultiplication — whose action is surjective and non-degenerate, we can observe the expected behavior: As proven in [4], any such module can be equipped with the structure of a firm $R$-bimodule. What is more, they constitute a monoidal category $M(A)$ admitting a strict monoidal forgetful functor $U(A)$ to the category $R M_R$ of firm $R$-bimodules.

Since a weak multiplier bialgebra $A$ has no coalgebra structure, it is not obvious either what to mean by an $A$-comodule. This question was addressed in [2]. As in [22] in the case of a multiplier Hopf algebra, a right $A$-comodule $M$ is defined by a pair of compatible linear maps $\lambda, \varrho : M \otimes A \rightarrow M \otimes A$ (generalizing the left and right multiplication by the image of $m \in M$ under the usual coaction of a bialgebra $A$ on $M$). Coassociativity of such
a generalized comodule can be formulated as a pentagonal identity, see [22]. In the case of a weak multiplier bialgebra, it has to be supplemented by a normalization condition. The property which generalizes the counitality of a usual coaction, appears as the fullness of $\lambda$ and $g$. As proven in [2], full comodules over a regular weak multiplier algebra $A$ with a full comultiplication carry a firm $R$-bimodule structure. Moreover, they constitute a monoidal category $M^{(A)}$ admitting a strict monoidal forgetful functor $U^{(A)}$ to the category $RMM_{R}$ of firm $R$-bimodules.

As a next step, the aim of this paper is to define and study Yetter-Drinfeld modules over a regular weak multiplier bialgebra $A$ with full comultiplication. In Section 2 we study the ‘commutants’ $(M_{(A)})^{U(A)}$ and $U^{(A)}(M_{(A)})$ of the ranges of the functors $U_{(A)}$ and $U^{(A)}$ in $RMM_{R}$. Although — in contrast to usual (i.e. unital) weak bialgebras — they are not isomorphic to $M^{(A)}$ and $M_{(A)}$, respectively, there are full embeddings $M^{(A)} \rightarrow (M_{(A)})^{U(A)}$ and $M_{(A)} \rightarrow U^{(A)}(M_{(A)})$. Applying these embedding functors to a vector space $X$ carrying both a non-degenerate surjective $A$-action and a full $A$-coaction, we obtain canonical natural transformations $X \otimes_{R} U^{(A)}(-) \rightarrow U^{(A)}(-) \otimes_{R} X$ and $U_{(A)}(-) \otimes_{R} X \rightarrow X \otimes_{R} U_{(A)}(-)$. In Section 3 we prove that if the induced $R$-actions on $X$ coincide, then the first one of these natural transformations defines an object of the weak left center of $M^{(A)}$ if and only if the second one defines an object of the weak right center of $M_{(A)}$. We use these equivalent properties to define a Yetter-Drinfeld module over $A$. The resulting category of Yetter-Drinfeld $A$-modules is shown in Section 4 to be monoidal with strict monoidal forgetful functors to $P$ and identity map, meaning multiplication by the unit $1$ of the base field. For any vector spaces $P$ and $Q$, we denote by $\text{Lin}(P,Q)$ the vector space of linear maps $P \rightarrow Q$. We denote by $\text{tw}: P \otimes Q \rightarrow Q \otimes P$ the flip map $p \otimes q \mapsto q \otimes p$. For a linear map $f : P \otimes Q \rightarrow P' \otimes Q'$, we use the leg numbering notation $f^{21} := \text{tw}f\text{tw} : Q \otimes P \rightarrow Q' \otimes P'$. For any further vector space $R$, we also use the notation

\[
\begin{align*}
f^{13} & := (P' \otimes \text{tw})(f \otimes R)(P \otimes \text{tw}) : P \otimes R \otimes Q \rightarrow P' \otimes R \otimes Q', \\
f^{31} & := (Q' \otimes \text{tw})(f^{21} \otimes R)(Q \otimes \text{tw}) : Q \otimes R \otimes P \rightarrow Q' \otimes R \otimes P',
\end{align*}
\]

and its variants. For a subset $P_{0}$ of a vector space $P$, we denote by $\langle P_{0} \rangle$ the subspace of $P$ spanned by the elements of $P_{0}$.

1. Preliminaries

In this section we recall from [20], [21], [4] and [2] the most important information about weak multiplier bialgebras and weak multiplier Hopf algebras, their modules and comodules. No proofs that can be found in these papers are repeated. However, we prove here a few technical lemmata for later use. We also revisit the construction of commutants of strict monoidal functors in [13].

1.1. Notation. For any vector space $P$, we denote by $P$ also the identity map $P \rightarrow P$. On elements $p \in P$, also the notation $p \mapsto 1p$ or $p \mapsto p1$ is used for the action of the identity map, meaning multiplication by the unit $1$ of the base field. For any vector spaces $P$ and $Q$, we denote by $\text{Lin}(P,Q)$ the vector space of linear maps $P \rightarrow Q$. We denote by $\text{tw} : P \otimes Q \rightarrow Q \otimes P$ the flip map $p \otimes q \mapsto q \otimes p$. For a linear map $f : P \otimes Q \rightarrow P' \otimes Q'$, we use the leg numbering notation $f^{21} := \text{tw}f\text{tw} : Q \otimes P \rightarrow Q' \otimes P'$. For any further vector space $R$, we also use the notation

\[
\begin{align*}
f^{13} & := (P' \otimes \text{tw})(f \otimes R)(P \otimes \text{tw}) : P \otimes R \otimes Q \rightarrow P' \otimes R \otimes Q', \\
f^{31} & := (Q' \otimes \text{tw})(f^{21} \otimes R)(Q \otimes \text{tw}) : Q \otimes R \otimes P \rightarrow Q' \otimes R \otimes P',
\end{align*}
\]
1.2. Multiplier algebra. Consider an algebra $A$ over a field with an associative multiplication $\mu : A \otimes A \to A$ but possibly without a unit. On elements $a, b \in A$, multiplication is denoted by juxtaposition: $\mu(a \otimes b) = ab$. The algebra $A$ is called idempotent if $\mu$ is surjective; that is, any element of $A$ is a linear combination of elements of the form $ab$ for $a, b \in A$. The multiplication $\mu$ (or with an alternative terminology, the algebra $A$) is non-degenerate if both maps $A \to \text{Lin}(A, A)$,

$$a \mapsto \mu(a \otimes -) \quad \text{and} \quad a \mapsto \mu(- \otimes a)$$

are injective. If $A$ is an idempotent and non-degenerate algebra, then it was proven in [9] that there is a unique largest unital algebra $M(A)$ which contains $A$ as a dense ideal (i.e. such that if $aw = 0$ for all $a \in A$ or $wa = 0$ for all $a \in A$, then $M(A) \ni w = 0$). This algebra $M(A)$ is called the multiplier algebra of $A$ and it is of the following explicit form. An element $w$ of $M(A)$ is given by two linear maps $A \to A$, denoted by

$$a \mapsto wa \quad \text{and} \quad a \mapsto aw,$$

respectively, such that $a(wb) = (aw)b$ for all $a, b \in A$ (hence $w(-)$ is a right $A$-module map and $(-)w$ is a left $A$-module map). The multiplication — to be denoted by $\mu$ again, and written as juxtaposition of elements — is provided by the composition and opposite composition of these maps and the unit is the pair whose members are equal to the identity map. The embedding

$$A \to M(A), \quad a \mapsto (\mu(a \otimes -), \mu(- \otimes a))$$

makes $A$ a dense ideal, indeed.

For any idempotent non-degenerate algebra $A$, also the opposite algebra obtained from $A$ by reversing the order of the multiplication (that is, replacing $\mu$ by $\mu^{\text{op}} := \mu w$) is again an idempotent non-degenerate algebra. Its multiplier algebra obeys $M(A^{\text{op}}) \cong M(A)^{\text{op}}$. For idempotent non-degenerate algebras $A$ and $B$, the tensor product algebra $A \otimes B$ is an idempotent non-degenerate algebra again. For an element $w \in M(A \otimes B)$, we extend our leg numbering convention by putting for any labels $i, j$,

$$w^{ij}(-) = [w(-)]^{ij} \quad \text{and} \quad (-)w^{ij} = [(-)w]^{ij}.$$ 

The following theorem due to Van Daele and Wang is of crucial importance.

**Theorem 1.1.** [20] Proposition A.3 Let $A$ and $B$ be idempotent non-degenerate algebras. Let $\phi : A \to M(B)$ be a multiplicative map and $e$ be an idempotent element of $B$ (i.e. such that $e^2 = e$). If

$$\langle \phi(a) b ~|~ a \in A, b \in B \rangle = \langle eb ~|~ b \in B \rangle \quad \text{and} \quad \langle b \phi(a) ~|~ a \in A, b \in B \rangle = \langle be ~|~ b \in B \rangle$$

then there is a unique multiplicative map $\overline{\phi} : M(A) \to M(B)$ such that $\overline{\phi}(1) = e$ and for all elements $a \in A$, $\overline{\phi}(a) = \phi(a)$.

1.3. Weak multiplier bialgebra. A weak multiplier bialgebra $A$ over a field $k$ is given by the following data.

- an idempotent non-degenerate $k$-algebra $A$,
- linear maps $E_1, E_2, T_1, T_2 : A \otimes A \to A \otimes A$,
- a linear map $\epsilon : A \to k$.

They are subject to the axioms (i)-(ix) in [2] Definition 1.1]. An equivalent but more traditional definition can be found in [4] Definition 2.1].
By axioms (iii) and (iv) in [2] Definition 1.1, the maps $T_1$ and $T_2$ determine a multiplicative map $\Delta : A \rightarrow \mathbb{M}(A \otimes A)$ — called the comultiplication — via the prescriptions

$\Delta(a)(b \otimes c) = T_1(a \otimes c)(b \otimes 1)$ and $(b \otimes c)\Delta(a) = (1 \otimes c)T_2(b \otimes a), \quad \forall a, b, c \in A$

By axioms (i) and (ii) in [2, Definition 1.1], the maps $E_1$ and $E_2$ determine an idempotent element $E \in \mathbb{M}(A \otimes A)$ via

$E(a \otimes b) = E_1(a \otimes b)$ and $(a \otimes b)E = E_2(a \otimes b) \quad \forall a, b \in A$

By axiom (vii) in [2, Definition 1.1] and by Theorem 1.1, $\Delta$ extends to a multiplicative map $\Delta : \mathbb{M}(A) \rightarrow \mathbb{M}(A \otimes A)$ such that $\Delta(1) = E$.

By [4, Propositions 2.4 and 2.6], the idempotent element $E$ allows for the definition of linear maps

(1.1) $\nabla^R : A \rightarrow \mathbb{M}(A), \quad a \mapsto (\mathbb{M}(A) \otimes \epsilon)[E(1 \otimes a)]$

(1.2) $\nabla^L : A \rightarrow \mathbb{M}(A), \quad a \mapsto (\epsilon \otimes \mathbb{M}(A))[E(a \otimes 1)].$

Their properties are studied in Section 3 of [4].

A weak multiplier bialgebra $A$ is said to be regular if there is another weak multiplier bialgebra given by

- the idempotent non-degenerate $k$-algebra $A^{\text{op}}$,
- the same linear maps $E_2, E_1 : A \otimes A \rightarrow A \otimes A$ (but their roles interchanged) and the linear maps

$T_3 : A \otimes A \rightarrow A \otimes A \quad a \otimes b \mapsto (1 \otimes b)\Delta(a)$

$T_4 : A \otimes A \rightarrow A \otimes A \quad a \otimes b \mapsto \Delta(b)(a \otimes 1),$

- the same linear map $\epsilon : A \rightarrow k$.

This is equivalent to the existence of some linear maps $T_3$ and $T_4$ obeying axiom (x) in [2, Definition 1.1] (which axiom implies that the induced comultiplication on $A^{\text{op}}$ is equal to the comultiplication $\Delta$ on $A$). For a regular weak multiplier bialgebra $A$, there are further two linear maps

(1.3) $\nabla^R : A \rightarrow \mathbb{M}(A), \quad a \mapsto (\mathbb{M}(A) \otimes \epsilon)[(1 \otimes a)E]$

(1.4) $\nabla^L : A \rightarrow \mathbb{M}(A), \quad a \mapsto (\epsilon \otimes \mathbb{M}(A))[E(a \otimes 1)].$

The comultiplication of a weak multiplier bialgebra is said to be right full if the subspace

$\langle (A \otimes \omega)T_1(a \otimes b) \mid a, b \in A, \ \omega \in \text{Lin}(A, k) \rangle$

is equal to $A$. By [4, Theorem 3.13], the comultiplication of a regular weak multiplier bialgebra is right full if and only if the ranges of the maps $\nabla^R$ and $\nabla^R$ coincide. In this situation this coinciding range will be denoted by $R$. Symmetrically, the comultiplication of a weak multiplier bialgebra is left full if the subspace

$\langle (\omega \otimes A)T_2(a \otimes b) \mid a, b \in A, \ \omega \in \text{Lin}(A, k) \rangle$

is equal to $A$. By [4, Theorem 3.13], the comultiplication of a regular weak multiplier bialgebra is left full if and only if the ranges of the maps $\nabla^L$ and $\nabla^L$ coincide. In this situation this coinciding range will be denoted by $L$.

Consider a regular weak multiplier bialgebra $A$ over a field $k$ with a right full comultiplication. Then the subspace $R$ of $\mathbb{M}(A)$ above carries the following structure.

- By [4, Lemma 3.4], $R$ is a (non-unital) subalgebra of $\mathbb{M}(A)$.
• By [4] Theorem 4.4, $R$ is a coalgebra via the comultiplication and counit
  \[ \delta : R \to R \otimes R, \quad \nabla^R(ab) \mapsto (\nabla^R \otimes \nabla^R)T_2(a \otimes b) \quad \varepsilon : R \to k, \quad \nabla^R(a) \mapsto \varepsilon(a). \]

• By [4] Proposition 4.3 (3), the multiplication in $R$ is split by the comultiplication $\delta$.

• By [4] Proposition 4.3 (3), the multiplication in $R$ is a morphism of left and right $R$-comodules. Equivalently, the comultiplication $\delta$ is a morphism of left and right $R$-modules.

• By [4] Theorem 4.6 (2), the algebra $R$ has (idempotent) local units (so in particular it is a firm\(^1\) algebra).

All these amount to saying that $R$ is a coseparable coalgebra (and hence a firm Frobenius algebra, see [5]). Then we know from [6] Proposition 2.17 that the category of firm modules over the firm algebra $R$ is isomorphic to the category of comodules over the coalgebra $R$; and the $R$-module tensor product $V \otimes_R W$ of any firm $R$-modules $V$ and $W$ is isomorphic to their $R$-comodule tensor product. This implies that the canonical epimorphism

\[ \pi_{V,W} : V \otimes W \twoheadrightarrow V \otimes_R W \]

is split by the map

\[ \iota_{V,W} : V \otimes_R W \hookrightarrow V \otimes W, \quad v \cdot r \otimes_R w \mapsto (v \cdot (-) \otimes (-) \cdot w)\delta(r). \]

For $R$-bimodules $V, W$ and $Z$, we denote by $\pi_{V,W,Z}$ the epimorphism

\[ \pi_{V,\otimes_R W,Z}(\pi_{V,W} \otimes Z) = \pi_{V,W;\otimes_R Z}(V \otimes W \otimes Z) : V \otimes W \otimes Z \to V \otimes_R W \otimes_R Z \]

and we denote by $\iota_{V,W,Z}$ its section

\[ (\iota_{V,W} \otimes Z)\iota_{V,\otimes_R W,Z} = (V \otimes \iota_{W,Z})\iota_{V,W;\otimes_R Z} : V \otimes_R W \otimes_R Z \to V \otimes W \otimes Z. \]

If the comultiplication of a regular weak multiplier bialgebra $A$ is left full, then the subspace $L$ has analogous structures.

If the comultiplication is right full, then by [4] Lemma 4.8 there are anti-multiplicative linear maps

\[ \tau : R \to \nabla^L(A), \quad \nabla^R(a) \mapsto \nabla^L(a) \]

\[ (1.5) \]

\[ \bar{\tau} : R \to \nabla^L(A), \quad \nabla^R(a) \mapsto \nabla^L(a). \]

\[ (1.6) \]

If the comultiplication is left full, then by [4] Lemma 4.8 there are anti-multiplicative linear maps

\[ \sigma : L \to \nabla^R(A), \quad \nabla^L(a) \mapsto \nabla^R(a) \]

\[ (1.7) \]

\[ \bar{\sigma} : L \to \nabla^R(A), \quad \nabla^L(a) \mapsto \nabla^R(a). \]

\[ (1.8) \]

If the comultiplication is both left and right full, then by [4] Proposition 4.9 these are anti-coalgebra isomorphisms such that $\tau = \sigma^{-1}$ and $\bar{\tau} = \bar{\sigma}^{-1}$. The composite map $\vartheta := \sigma\bar{\sigma}^{-1}$ is the Nakayama automorphism of $R$ — meaning $\varepsilon(rs) = \varepsilon(\vartheta(s)r)$ for all $r, s \in R$. The map $\bar{\sigma}^{-1}\sigma$ is the Nakayama automorphism of $L$.

By [4] Proposition 4.3 (1) and Proposition 4.11], the formulae

\[ F(a \otimes b) := ((A \otimes \sigma)[E(a \otimes 1)])(1 \otimes b) \quad (a \otimes b)F := (1 \otimes b)((A \otimes \sigma)[(a \otimes 1)E]) \]

\[ (1.9) \]

\(^1\)A (say, right) module $P$ over a non-unital algebra $R$ is said to be firm if the $R$-action $P \otimes R \to P$ projects to a vector space isomorphism $P \otimes_R R \to P$. The algebra $R$ is firm if it is a firm left, equivalently, right $R$-module via the multiplication. This terminology is attributed to an unpublished preprint by D. Quillen in 1997.
define a multiplier $F$ on $A \otimes A$. By a symmetric form of [4, Proposition 4.11],
\[(1.10)\quad F^{21}(a \otimes b) := ((A \otimes \sigma)[E(a \otimes 1)])(1 \otimes b) \quad (a \otimes b)F^{21} := (1 \otimes b)((A \otimes \sigma)[(a \otimes 1)E]).\]

**Lemma 1.2.** For a regular weak multiplier bialgebra $A$ with left and right full comultiplication, and for any $a \in A$, the following expressions are equal.
- (a) $(R \otimes \cap R)[E(1 \otimes a)]$,
- (b) $(\cap R \otimes R)[(a \otimes 1)F]$,
- (c) $(\cap R \otimes R)[(a \otimes 1)E^{21}]$,
- (d) $(R \otimes \cap R)[F(1 \otimes a)]$.

**Proof.** (a)=(b) By [4, Proposition 2.5 (1)], $(a \otimes 1)E \in A \otimes L$ and by (2.3) in [4], $E(1 \otimes a) \in R \otimes A$. They obey
\[(\cap R \otimes L)[(a \otimes 1)E] \overset{(1.1)}{=} (R \otimes \epsilon \otimes L)[(E \otimes 1)(1 \otimes a \otimes 1)(1 \otimes E)] \overset{(1.2)}{=} (R \otimes \cap L)[E(1 \otimes a)].\]
Applying $R \otimes (\epsilon \otimes L)$ to both sides and using (1.3), we obtain (b)=(a).

(c)=(d) follows similarly using (1.3) and (1.4) together with (1.10).

(a)=(c) Using axiom (iii) in [2, Definition 1.1] in the second equality, (3.6) in [4] in the third equality and [4, Proposition 4.3 (1)] in the penultimate one, it follows for any $a, b, c \in A$ that
\[(c \otimes 1)((A \otimes \cap R)T_1(a \otimes b)) = (A \otimes \cap R)[(c \otimes 1)T_1(a \otimes b)]
= (A \otimes \cap R)[T_2(c \otimes a)(1 \otimes b)]
= (A \otimes \cap R)[((A \otimes \cap R)T_2(c \otimes a))(1 \otimes b)]
= (A \otimes \cap R)[(ca \otimes 1)F(1 \otimes b)]
= (c \otimes 1)((A \otimes \cap R)[(a \otimes 1)F(1 \otimes b)]).

Thus using the non-degeneracy of $A$ and simplifying by $c$, $(A \otimes \cap R)T_1(a \otimes b) = (A \otimes \cap R)[(a \otimes 1)F(1 \otimes b)]$. Symmetrically, using (1.6) in [2, 4, Lemma 3.2] and [4, Proposition 4.3 (1)], it follows that $(\cap R \otimes A)T_3^{21}(a \otimes b) = (\cap R \otimes A)[(a \otimes 1)F(1 \otimes b)]$. With these identities at hand,
\[(R \otimes \cap R)[E(1 \otimes ab)] = (\cap R \otimes \cap R)T_1(a \otimes b) = (\cap R \otimes \cap R)[(a \otimes 1)F(1 \otimes b)]
= (\cap R \otimes \cap R)T_3^{21}(a \otimes b) = (\cap R \otimes R)[(ab \otimes 1)E^{21}],\]
from which we conclude by the idempotency of $A$. In the first equality we used (2.3) in [4] and in the last one we used (3.4) in [4]. \hfill \Box

1.4. **Antipode.** The antipode on a regular weak multiplier bialgebra $A$ is a linear map $S : A \to M(A)$ satisfying the conditions in part (2) of [4, Theorem 6.8]. Then it follows that
\[(1.11)\quad \mu(S \otimes A)T_1 = \mu(\cap R \otimes A) \quad \mu(A \otimes S)T_2 = \mu(\cap R \otimes L) \quad \mu(A \otimes S)E_1 = \mu(S \otimes A) \quad \mu(A \otimes S)E_2 = \mu(A \otimes S),\]
see (6.14) in [4]. If the antipode exists then it is unique. Any weak multiplier bialgebra possessing an antipode is a weak multiplier Hopf algebra in the sense of [20] but not conversely.

By [4, Theorem 6.12] $S$ is anti-multiplicative. That is, for any $a, b \in A$, $S(ab) = S(b)S(a)$. By [4, Proposition 6.13] $S$ is non-degenerate. That is, any element of $A$ can be written as a linear combination of elements of the form $aS(b)$ and it can be written also as a linear combination of elements of the form $S(b)a$, in terms of elements $a, b \in A$.\]
These assertions together with Theorem 1.1 imply that \( S \) extends to a unital anti-algebra map \( \overline{S} : M(A) \to M(A) \). Assuming also that the comultiplication of \( A \) is left and right full, it follows by [4, Corollary 6.16] that \( S \) is anti-comultiplicative. That is, in terms of the opposite comultiplication \( \Delta^\text{op} : A \to M(A \otimes A) \) — defined by \( \Delta^\text{op}(a) = \Delta(a)^{21} \) for any \( a \in A \) — the equality \( \overline{\Delta}S = (S \otimes \overline{S})\Delta^\text{op} \) holds. By [4, Lemma 6.14], the restriction of \( \overline{S} \) to \( R \) is equal to \( \overline{\sigma}^{-1} \) and the restriction of \( \overline{S} \) to \( L \) is equal to \( \sigma \).

A regular weak multiplier Hopf algebra in \([20][21]\) is the same as a regular weak multiplier bialgebra \( A \) with left and right full comultiplication such that both \( A \) and its opposite \( A^\text{op} \) possess an antipode. The following is a variant of \([21, \text{Theorem 4.10}]\):

**Theorem 1.3.** For a regular weak multiplier bialgebra \( A \) with left and right full comultiplication possessing an antipode \( S \), the following are equivalent.

(a) \( A \) is a regular weak multiplier Hopf algebra.

(b) \( S \) factorizes through a vector space isomorphism \( A \to A \) (to be denoted by \( S \) too) via the embedding \( A \to M(A) \).

If assertion (b) in Theorem 1.3 holds then \( S^{-1} \) defines the antipode of \( A^\text{op} \).

1.5. Modules. Since there is an associative algebra underlying any weak multiplier bialgebra \( A \), there is an evident notion of right \( A \)-module \( V \) with an associative action \( \cdot : V \otimes A \to V \):

\[
v \cdot (ab) = (v \cdot a) \cdot b \quad \forall v \in V, \ a, b \in A.
\]

A morphism of \( A \)-modules is a linear map which commutes with the \( A \)-actions. Similarly to the properties of the multiplication, we say that \( V \) is an idempotent module if \( \cdot : V \otimes A \to V \) is surjective (i.e. any element of \( V \) is a linear combination of elements of the form \( v \cdot a \), in terms of \( v \in V \) and \( a \in A \)) and it is non-degenerate if the map

\[
V \to \text{Lin}(A, V), \quad v \mapsto v \cdot (-)
\]

is injective. We consider the category \( \text{M}(A) \) whose objects are the idempotent non-degenerate right \( A \)-modules and whose morphisms are the \( A \)-module maps.

If \( A \) is a regular weak multiplier bialgebra with right full comultiplication, then any idempotent non-degenerate right \( A \)-module \( V \) was equipped in [4, Proposition 5.2] with the structure of an \( R \)-bimodule with firm right and left actions:

\[
(v \cdot a) \triangleright r := v \cdot (ar) \quad r \triangleright (v \cdot a) := v \cdot (a\tau(r)) \quad \forall v \in V, \ a \in A, \ r \in R,
\]

where the map (1.5) is used. This defines a functor \( U(A) \) — acting on the morphisms as the identity map — from \( \text{M}(A) \) to the category \( \text{RM}_R \) of firm \( R \)-bimodules. Note that \( \text{RM}_R \) is a monoidal category via the \( R \)-module tensor product as the monoidal product and the regular \( R \)-bimodule as the neutral object. By [4, Theorem 5.6], \( \text{M}(A) \) admits a monoidal structure such that \( U(A) \) is strict monoidal.

1.6. Comodules. Applying the philosophy used in [22] for (non-weak) multiplier Hopf algebras, a comodule over a regular weak multiplier bialgebra \( A \) is a vector space \( M \) equipped with linear maps \( \lambda, \rho : M \otimes A \to M \otimes A \) such that

- \( (1 \otimes a)\lambda(m \otimes b) = \rho(m \otimes a)(1 \otimes b) \) for all \( m \in M, \ a, b \in A \),
- the same coassociativity condition \( (\lambda \otimes A)\lambda^{13}(M \otimes T_1) = (M \otimes T_1)(\lambda \otimes A) \) in [22] holds (for a few more equivalent forms see [2, Proposition 2.1]),
- together with the normalization condition \( (\lambda \otimes A)\lambda^{13}(M \otimes E_1) = (\lambda \otimes A)\lambda^{13} \) (some more equivalent forms can be found again in [2, Proposition 2.1]).
A morphism of $A$-comodules is a linear map $f : M \to M'$ such that $(f \otimes A)\lambda = \lambda'(f \otimes A)$, equivalently, $(f \otimes A)\varrho = g'(f \otimes A)$. The triple $(M, \lambda, \varrho)$ is a right comodule over a regular weak multiplier bialgebra $A$ if and only if $(M, \varrho, \lambda)$ is a right comodule over the opposite regular weak multiplier bialgebra $A^{\text{op}}$. Similarly to the comultiplication, a comodule $(M, \lambda, \varrho)$ is said to be full if any (hence by [2, Lemma 4.1] both) of the subspaces

$\langle (M \otimes \omega)\lambda(m \otimes a) | m \in M, a \in A, \omega \in \text{Lin}(A, k) \rangle$ and

$\langle (M \otimes \omega)\varrho(m \otimes a) | m \in M, a \in A, \omega \in \text{Lin}(A, k) \rangle$,

is equal to $M$. We consider the category $M(A)$ whose objects are the full right $A$-comodules and whose morphisms are the $A$-comodule maps.

If $A$ is a regular weak multiplier bialgebra with a right full comultiplication, then any full right $A$-comodule $M$ was equipped in [2, Theorem 4.5] with the structure of an $R$-bimodule with firm right and left actions:

$\begin{equation}
m \circ \Box^R (a) := (M \otimes \epsilon)\lambda(m \otimes a), \quad \Box^R (a) \circ m := (M \otimes \epsilon)\varrho(m \otimes a).
\end{equation}$

This defines a functor $U(A) = \text{RMod}_A$ acting on the morphisms as the identity map from $M(A)$ to the category $\mathcal{R}M_R$ of firm $R$-bimodules. By [2, Theorem 5.7], $M(A)$ admits a monoidal structure such that $U(A)$ is strict monoidal.

If $A$ is a regular weak multiplier bialgebra with a left and right full comultiplication and it possesses an antipode, then it was proven in [2, Theorem 6.7] that any finite dimensional object in $M(A)$ possesses a dual.

**Lemma 1.4.** Let $A$ be a regular weak multiplier bialgebra with right full comultiplication. Let $V$ be an idempotent non-degenerate right $A$-module regarded as a firm $R$-bimodule as in (1.12). Let $M$ be a full right $A$-comodule regarded as a firm $R$-bimodule as in (1.13).

1. The idempotent map $\iota_{M,V} \pi_{M,V} : M \otimes V \to M \otimes V$ satisfies

$\iota_{M,V} \pi_{M,V}(m \otimes v \cdot a) = (m \bullet (-) \otimes v \cdot (-))(1 \otimes a)E), \quad \forall m \in M, v \in V, a \in A$.

2. The idempotent map $\iota_{V,M} \pi_{V,M} : V \otimes M \to V \otimes M$ satisfies

$\iota_{V,M} \pi_{V,M}(v \cdot a \otimes m) = (v \cdot (-) \otimes (-) \bullet m)(a \otimes 1)F), \quad \forall m \in M, v \in V, a \in A$.

**Proof.** (1) For any $a \in A$, we have $(1 \otimes a)E \in R \otimes A$ by (3.4) in [4]. So the right hand side of the equality in the claim is meaningful. Since $A$ is idempotent and $M$ is a right $R$-module, $M$ is spanned by elements of the form $m \bullet \Box^R(cd)$, for $m \in M$ and $c, d \in A$. For such elements,

$\iota_{M,V} \pi_{M,V}(m \bullet \Box^R(cd) \otimes v \cdot a) = (m \bullet (-) \otimes v \cdot (-))(\Box^R \otimes \Box^R)(T_2(c \otimes d))$

$= (m \bullet (-) \otimes v \cdot (-))(1 \otimes a)(\Box^R \otimes \Box^L)(T_2(c \otimes d))$

$= (m \bullet (-) \otimes v \cdot (-))(\Box^R \otimes \Box^L)(T_2(c \otimes d))$

$= (m \bullet \Box^R(cd) \otimes v \cdot (-))(1 \otimes a)E)$

The first equality follows by (4.1) in [4], the penultimate one follows by [4, Proposition 2.5 (1)], and the last one does by [2, Lemma 1.3 (3)].

(2) For any $a \in A$, we have $(a \otimes 1)F \in A \otimes R$ by [4, Proposition 4.3 (1)]. So the right hand side of the equality in the claim is meaningful. By [4, Proposition 4.3 (3)], $\delta(r) = (r \otimes 1)F$ for any $r \in R$. Hence

$\iota_{V,M} \pi_{V,M}(v \cdot ar \otimes m) = (v \cdot (-) \otimes (-) \bullet m)\delta(r) = (v \cdot (-) \otimes (-) \bullet m)((ar \otimes 1)F)$.

By surjectivity of the right $R$-action on $A$ (cf. [4, Lemma 3.7 (4)]) this proves the claim. \hfill $\square$
1.7. Commutants of strict monoidal functors. The following notion due to Majid in [13] generalizes the weak center of a monoidal category in [16], which generalizes the center construction in [11].

**Definition 1.5.** For monoidal categories $(\mathcal{M}, \otimes, I)$ and $(\mathcal{M}', \otimes', I')$, and a strict monoidal functor $U : \mathcal{M} \to \mathcal{M}'$, the right commutant of $U$ is the following category $\mathcal{M}^U$: The objects are pairs consisting of an object $P$ of $\mathcal{M}'$ and a strict monoidal transformation $\sigma : U(-) \otimes P \to P \otimes' U(-)$ which is compatible with the monoidal structures in the sense of the following commutative diagrams.

\[
\begin{align*}
\xymatrix{& P \ar[r]^\cong & P \ar[d]^-\cong \ar[r] & (\mathcal{U} \otimes' \mathcal{W}) \otimes P \ar[r]^-\cong & P \otimes' (\mathcal{V} \otimes \mathcal{W}) \ar[d]^\cong \\
I' \otimes' P & P \otimes' I' & (\mathcal{U} \otimes' \mathcal{W}) \otimes P & & P \otimes' (\mathcal{V} \otimes \mathcal{W}) \\
& & (U \otimes' P) \otimes' UW & & P \otimes' (U \otimes' W) \\
\xymatrix{U \otimes' P \ar[r]^\cong_{\sigma_U} & P \otimes' U I} & \xymatrix{U(V \otimes \mathcal{W}) \otimes P \ar[r]^\cong_{\sigma_{U \otimes \mathcal{W}}} & P \otimes' (U \otimes \mathcal{W})}
\end{align*}
\]

Here the arrows carrying the label $\cong$ stand for Mac Lane’s coherence isomorphisms (not to be explicitly denoted later on). The morphisms $(P, \sigma) \to (P', \sigma')$ are morphisms $f : P \to P'$ in $\mathcal{M}'$ rendering commutative

\[
\begin{align*}
& (U(-) \otimes P) \xrightarrow{\sigma} P \otimes' U(-) \\
& U(-) \otimes f \quad \quad \quad f \otimes U(-) \\
& U(-) \otimes P' \xrightarrow{\sigma'} P' \otimes' U(-)
\end{align*}
\]

The left commutant $\mathcal{M}^U$ of $U$ is defined symmetrically. Its objects are pairs consisting of an object $Q$ of $\mathcal{M}'$ and a natural transformation $\vartheta : Q \otimes' U(-) \to U(-) \otimes' Q$ satisfying the evident compatibility conditions with the monoidal structures. The morphisms are morphisms in $\mathcal{M}'$ which commute with the structure morphisms $\vartheta$.

**Theorem 1.6.** [13] Theorem 3.3] For any monoidal categories $(\mathcal{M}, \otimes, I)$ and $(\mathcal{M}', \otimes', I')$, and a strict monoidal functor $U : \mathcal{M} \to \mathcal{M}'$, the left and right commutants of $U$ are monoidal categories admitting a strict monoidal forgetful functor to $\mathcal{M}'$.

For later reference, we recall that the monoidal product of two objects $(P, \sigma)$ and $(P', \sigma')$ in $\mathcal{M}^U$ is

\[
(P \otimes' P', U(-) \otimes P \otimes P' \xrightarrow{\sigma \otimes' \sigma'} P \otimes' U(-) \otimes P' \xrightarrow{P \otimes' \vartheta} P \otimes' P' \otimes' U(-))
\]

where we omitted the associativity constraints. The monoidal unit is $R'$ via the natural transformation built up from the unitor in $\mathcal{M}'$.

The (left or right) commutant of the identity functor on a monoidal category $\mathcal{M}$ is known as the (left or right) weak center [16]. The left weak center is denoted by $\mathcal{M}^M$ and the right weak center is denoted by $\mathcal{M}^M$. Any strict monoidal functor $U : \mathcal{M} \to \mathcal{M}'$ lifts to strict monoidal forgetful functors $\mathcal{M}^M \to \mathcal{M}^U$ and $\mathcal{M}^M \to \mathcal{M}^U$; to be denoted by $U$ again.
The \textit{center} of a monoidal category \([11]\) is the full subcategory of the left weak center for whose objects \((Q, \vartheta)\) the \(\vartheta\) is a natural isomorphism.

\section{Some embeddings of the module and comodule categories}

If \(A\) is a usual weak multiplier bialgebra possessing a unit as in \([15, 7]\), then both the category of (say, right) \(A\)-modules and the category of (say, right) \(A\)-comodules are monoidal, admitting strict monoidal forgetful functors to the bimodule category of the so-called base subalgebra (see e.g. \([3]\)). So one can look at the commutants (in the sense recalled in Section \([17]\)) of these forgetful functors. Generalizing the results in \([13]\), one can see that the left commutant of the forgetful functor from the category of right \(A\)-modules is isomorphic to the category of right \(A\)-modules. Symmetrically, the right commutant of the forgetful functor from the category of right \(A\)-modules is isomorphic to the category of right \(A\)-comodules.

Such isomorphisms do not seem to hold for a weak multiplier bialgebra \(A\). Instead, the aim of this section is to show that if \(A\) is regular and its comultiplication is right full, then there is a fully faithful functor from the category \(M_{(A)}\) of idempotent non-degenerate right \(A\)-modules in Section \([15]\) to the left commutant of the functor \(U_{(A)}\) in Section \([16]\) from the category \(M^{(A)}\) of full right \(A\)-comodules. Symmetrically, there is a fully faithful functor from the category \(M_{(A)}\) of full right \(A\)-comodules in Section \([15]\) to the right commutant of the functor \(U_{(A)}\) in Section \([16]\) from the category \(M_{(A)}\) of idempotent non-degenerate right \(A\)-modules.

\begin{example}
Let \(A\) be a regular weak multiplier bialgebra with right full comultiplication. Then by \([3]\) Theorem 5.6], the idempotent non-degenerate right \(A\)-modules constitute a monoidal category \(M_{(A)}\) admitting a strict monoidal functor \(U_{(A)}\) to the category \(RM_{R}\) of firm bimodules over the base algebra \(R := \cap R(A) = \Pi R(A)\). The right commutant of \(U_{(A)}\) is the following category \((M_{(A)})^{U_{(A)}\}}). An object is given by a firm \(R\)-bimodule \(P\) and an \(R\)-bimodule map \(\sigma_{V} : V \otimes_{R} P \to P \otimes_{R} V\) for any idempotent non-degenerate right \(A\)-module \(V\), such that the following hold.

\begin{itemize}
  \item Compatibility with the monoidal unit; that is,
  \[\sigma_{R}(r \otimes_{R} p \cdot s) = r \cdot p \otimes_{R} s\] for all \(p \in P, r, s \in R\).
  \item Compatibility with the monoidal product; that is,
  \[\sigma_{V \otimes_{R} W} = (\sigma_{V} \otimes_{R} W)(V \otimes_{R} \sigma_{W}),\] for all \(V, W \in M_{(A)}\).
  \item Naturality; that is,
  \[\sigma_{V'}(g \otimes_{R} P) = (P \otimes_{R} g)\sigma_{V},\] for all \(g \in \text{Hom}_{(A)}(V, V')\).
\end{itemize}

A morphism \((P, \sigma) \to (P', \sigma')\) is an \(R\)-bimodule map \(u : P \to P'\) such that
\[\sigma'_{V}(V \otimes_{R} u) = (u \otimes_{R} V)\sigma_{V},\] for all \(V \in M_{(A)}\).

\begin{example}
Let \(A\) be a regular weak multiplier bialgebra with right full comultiplication. Then by \([2]\) Theorem 5.7], the full right \(A\)-comodules constitute a monoidal category \(M^{(A)}\) admitting a strict monoidal functor \(U^{(A)}\) to the category \(RM_{R}\) of firm bimodules over the base algebra \(R := \cap R(A) = \Pi R(A)\). The left commutant of \(U^{(A)}\) is the following category \((M^{(A)})^{U^{(A)}\}}). An object is given by a firm \(R\)-bimodule \(Q\) and an \(R\)-bimodule map \(\vartheta_{M} : Q \otimes_{R} M \to M \otimes_{R} Q\) for any full right \(A\)-comodule \(M\), such that the following hold.

\begin{itemize}
  \item Compatibility with the monoidal unit; that is,
  \[\vartheta_{M}(s \cdot q \otimes_{R} r) = s \otimes_{R} q \cdot r,\] for all \(q \in Q, r, s \in R\).
\end{itemize}

• Compatibility with the monoidal product; that is,
\( \vartheta_{MN} = (M \otimes_R \vartheta_N)(\vartheta_M \otimes_R N) \), for all \( M, N \in \mathcal{M}(A) \).

• Naturality; that is,
\( \vartheta_M(Q \otimes_R f) = (f \otimes_R Q)\vartheta_M \), for all \( f \in \text{Hom}^A(M, M') \).

A morphism \((Q, \vartheta)\) is an \( R \)-bimodule map \( u : Q \to Q' \) such that
\[ \vartheta'_M(u \otimes_R M) = (M \otimes_R u)\vartheta_M, \quad \text{for all } M \in \mathcal{M}(A). \]

In order to give full embeddings of \( \mathcal{M}(A) \) into the category \( (\mathcal{M}(A))^{U(A)} \) in Example 2.1 and of \( \mathcal{M}(A) \) into the category \( \mathcal{U}(A)(\mathcal{M}(A)) \) in Example 2.2 some preparation is needed.

**Lemma 2.3.** Let \( A \) be a regular weak multiplier bialgebra with right full comultiplication. For any idempotent non-degenerate right \( A \)-module \((V, \cdot)\) and any full right \( A \)-comodule \((M, \xi, \varrho)\), there is an \( R \)-bimodule map
\[ \varphi_{V, M} : V \otimes M \to M \otimes V, \quad v \cdot a \otimes m \mapsto (M \otimes v \cdot (-))\varrho(m \otimes a) \]

obeying the following properties.

1. For the idempotent \( \iota_{M, V} \pi_{M, V} \) in Lemma 1.3 (1), \( \iota_{M, V} \pi_{M, V} \varphi_{V, M} = \varphi_{V, M} \).
2. It is \( R \)-balanced in the sense that \( \varphi_{V, M}(v \otimes r \cdot m) = \varphi_{V, M}(v \cdot r \cdot m) \), for any \( v \in V \), \( m \in M \) and \( r \in R \).
3. For any \( g \in \text{Hom}_{\mathcal{A}}(V, V') \), \( \varphi_{V', M}(g \otimes M) = (M \otimes g)\varphi_{V, M} \).
4. For any \( f \in \text{Hom}_{\mathcal{A}}(M, M') \), \( \varphi_{V, M'}(V \otimes f) = (f \otimes V)\varphi_{V, M} \).

**Proof.** In order to prove that \( \varphi_{V, M} \) is a well-defined linear map, we need to show that it takes a zero element to zero. So assume that \( v \cdot a \otimes m = 0 \). Then for any \( b \in A \)
\[ 0 = (M \otimes v \cdot a(-))\lambda(m \otimes b) = (M \otimes v \cdot (-))[(1 \otimes a)\lambda(m \otimes b)] \]
\[ = (M \otimes v \cdot (-))[\varrho(m \otimes a)(1 \otimes b)] = [(M \otimes v \cdot (-))\varrho(m \otimes a)](1 \otimes b), \]
where the third equality follows by the compatibility condition (2.10) in [2]. Since \( V \) is a non-degenerate right \( A \)-module, so is \( M \otimes V \) by [10] Lemma 1.11. So we can simplify by \( b \) and conclude that \( (M \otimes v \cdot (-))\varrho(m \otimes a) = 0 \) and hence \( \varphi_{V, M} \) is a well-defined linear map. It is a left and right \( R \)-module map by [2] Lemma 4.9 (4) and (6), respectively.

1. For any \( v \in V \), \( m \in M \) and \( a \in A \), using the implicit summation index notation \( \varrho(m \otimes a) =: m^\delta \otimes a^\epsilon \),
\[ \iota_{M, V} \pi_{M, V} \varphi_{V, M}(v \cdot a \otimes m) = (m^\delta \otimes (-) \otimes (1 \otimes a^\epsilon)E) \]
\[ = m^\delta \otimes v \cdot a^\epsilon = \varphi_{V, M}(v \cdot a \otimes m). \]
The first equality follows by Lemma 1.4 (1) and the second one follows by [2] Lemma 4.10 (3).

2. follows by [2] Lemma 4.9 (3).

3. For any \( g \in \text{Hom}_{\mathcal{A}}(V, V') \), \( v \in V, m \in M \) and \( a \in A \),
\[ \varphi_{V', M}(g \otimes M)(v \cdot a \otimes m) = \varphi_{V', M}(g(v) \cdot a \otimes m) = (M \otimes g(v) \cdot (-))\varrho(m \otimes a)) \]
\[ = (M \otimes g)((M \otimes v \cdot (-))\varrho(m \otimes a)) = (M \otimes g)\varphi_{V, M}(v \cdot a \otimes m). \]

4. For any \( f \in \text{Hom}_{\mathcal{A}}(M, M') \), \( v \in V, m \in M \) and \( a \in A \),
\[ \varphi_{V, M'}(V \otimes f)(v \cdot a \otimes m) = (M' \otimes v \cdot (-))(f \otimes A)(m \otimes a)) \]
\[ = (f \otimes v \cdot (-))\varrho(m \otimes a) = (f \otimes V)\varphi_{V, M}(v \cdot a \otimes m). \]

\[ \square \]
Example 2.4. For a regular weak multiplier bialgebra $A$ with right full comultiplication, for any full right $A$-comodule $(M, \lambda, \varrho)$ and the right $A$-module $A$ with action provided by the multiplication,

$$\varphi_{A,M}(ba \otimes m) = (1 \otimes b)\varrho(m \otimes a) = \varrho(m \otimes ba),$$

for any $m \in M$ and $a, b \in A$, where in the second equality we used that $\varrho$ is a left $A$-module map, cf. [2 Proposition 2.1 (1)]. Hence by the idempotency of $A$, $\varphi_{A,M}(a \otimes m) = \varrho(m \otimes a)$.

Example 2.5. For a regular weak multiplier bialgebra $A$ with right full comultiplication, for any idempotent non-degenerate right $A$-module $(V, \cdot)$ and the right $A$-comodule $(A, T_1, T_3),$

$$\varphi_{V,A}(v \cdot a \otimes b) = (A \otimes v \cdot (-))T_3(b \otimes a),$$

for any $v \in V$ and $a, b \in A$.

Under the assumptions and using the notation in Lemma 2.3 it follows from Lemma 2.3 (2) that $\varphi_{V,M}$ projects to an $R$-bimodule map

$$(2.1) \quad \hat{\varphi}_{V,M} : V \otimes_R M \rightarrow M \otimes_R V, \quad v \cdot a \otimes_R m \mapsto \pi_{M,V}(M \otimes v \cdot (-)) \varrho(m \otimes a) = m^e \otimes_R v \cdot a^e$$

where $\pi_{M,V} : M \otimes V \rightarrow M \otimes_R V$ is the canonical epimorphism and we used the notation $\varrho(m \otimes a) = : m^e \otimes a^e$ with implicit summation understood. In other words, $\hat{\varphi}_{V,M}$ is defined by the equality $\hat{\varphi}_{V,M}(\pi_{M,V} \varrho) = \pi_{M,V} \varphi_{V,M}$. Applying Lemma 2.3 (1) in the last equality, also

$$(2.2) \quad \iota_{M,V} \hat{\varphi}_{V,M} = \iota_{M,V} \varphi_{V,M} \pi_{M,V} \varphi_{V,M} = \iota_{M,V} \varphi_{V,M} \iota_{M,V} \varphi_{V,M} = \varphi_{V,M} \varphi_{V,M}.$$

Lemma 2.6. Let $A$ be a regular weak multiplier bialgebra with right full comultiplication. For any idempotent non-degenerate right $A$-module $(V, \cdot)$ and any full right $A$-comodule $(M, \lambda, \varrho)$, the map $\hat{\varphi}_{V,M}$ in (2.1) obeys the following properties.

1. $\hat{\varphi}_{R,M}(r \otimes_R m \cdot s) = r \cdot m \otimes_R s$.
2. $\hat{\varphi}_{V,R}(r^\circ v \otimes_R s) = r \otimes_R v \circ s$.
3. $(\hat{\varphi}_{V,M} \otimes_R W)(V \otimes_R \hat{\varphi}_{W,M}) = \hat{\varphi}_{V \otimes_R W,M}$.
4. $(M \otimes_R \hat{\varphi}_{V,N})(\hat{\varphi}_{V,M} \otimes_R N) = \hat{\varphi}_{V,M \otimes_R N}$.

Proof. (1) Since the $A$-action $r \otimes a \mapsto r \cdot a = \cap^R(ra)$ on $R$ is surjective by [4 Proposition 5.3], the claim follows by

$$\hat{\varphi}_{R,M}(r \cdot a \otimes_R m \cdot s) = (m \cdot s)^e \otimes_R \cap^R(ra)^e = m^e \otimes_R \cap^R(ra)^e \varrho$$

$$= m^e \otimes_R \cap^R((ra)^e)\varrho = \cap^R(ra) \cdot m \otimes_R s = (r \cdot a) \cdot m \otimes_R s,$$

for any $m \in M$, $a \in A$ and $s, r \in R$. In the first equality we applied the definition of $\hat{\varphi}_{R,M}$ via (2.1). The second equality follows by [2, Lemma 4.9 (6)]. In the third equality we used that $\varrho$ is a left $R$-module map being a left $A$-module map by [2 Proposition 2.1 (1)]. The fourth equality is a consequence of (3.8) in [4] and the penultimate one follows by [2 Lemma 4.11 (1)].

(2) Since the $A$-action on $V$ is surjective by assumption, the claim follows by

$$\hat{\varphi}_{V,R}(\cap^R(b)^\circ(v \cdot a) \otimes_R s) = \pi_{R,V}(R \otimes v \cdot (-))[1 \otimes a \cap^L(b)s]E$$

$$= \pi_{R,V}(\cap^R(b)(- \otimes v \cdot (-))[1 \otimes as]E$$

$$= \cap^R(b) \otimes_R (v \cdot a) \circ s,$$

for any $v \in V$, $a \in A$ and $r, s \in R$. The second equality follows by [4 Lemma 3.5] and [4 Lemma 3.9]. In the last equality we used Lemma 1.4 (1).
(3) Since the $R$-actions on $W$ and on $V \otimes_R W$ are surjective (by assumption, and by [4] Proposition 5.5, respectively), the claim follows by obtaining for any coassociativity condition (2.14) in [2] and the first and the last equalities follow by the form of the right $A$-action on $V \otimes_R W$ in [4 Proposition 5.5].

(4) Since the $A$-action on $V$ is surjective by assumption, the claim follows by

\[(M \otimes_R \hat{\varphi}_{V,N})(\hat{\varphi}_{V,M} \otimes_R N)(v \otimes_R m \otimes_R n) = \pi_{M,N\otimes V}(M \otimes N \otimes v \cdot (-))(M \otimes _N \otimes a)\]

\[= \pi_{M,N,V}(M \otimes N \otimes v \cdot (-))(M \otimes \otimes N \otimes A)(m \otimes n \otimes a)\]

\[= \pi_{M,N,V}(M \otimes N \otimes v \cdot (-))(\iota_{M,N} \otimes A)(\hat{\varphi}_{M \otimes_R N})(m \otimes n \otimes a)\]

\[= \pi_{M \otimes_R N,V}(M \otimes_R N \otimes v \cdot (-))(\hat{\varphi}_{M \otimes_R N})(m \otimes n \otimes a)\]

\[= \hat{\varphi}_{V,M \otimes_R N}(v \cdot a \otimes_R m \otimes_R n),\]

for any $v \in V$, $m \in M$, $n \in N$ and $a \in A$. The second equality follows by [2 Lemma 5.3 (8)] and the third one follows by the construction of $\hat{\varphi}_{M \otimes_R N}$ in [2 Proposition 5.4].

\[\square\]

Remark 2.7. Let $A$ be a regular weak multiplier Hopf algebra in the sense of [21] (that is, let it be a regular weak multiplier bialgebra with left and right full comultiplication such that both $A$ and $A^{\text{op}}$ possess an antipode). For any idempotent non-degenerate right $A$-module $(V, \cdot)$ and for any full right $A$-comodule $(M, \lambda, \varrho)$, $\hat{\varphi}_{V,M}$ in [21] is an isomorphism (of vector spaces).

Proof. Recall from [21] Proposition 4.3 that under the assumptions that we made on $A$, the antipode restricts to a vector space isomorphism $S : A \rightarrow A$. We use its inverse $S^{-1}$ to construct the to-be-inverse

\[(\hat{\varphi}_{V,M})^{-1} : M \otimes_R V \rightarrow V \otimes_R M, \quad m \otimes_R v \cdot a \mapsto v \cdot S^{-1}(\lambda(m \otimes a)) \otimes_R m^\lambda,\]

where the implicit summation index notation $\lambda(m \otimes a) = m^\lambda \otimes a^\lambda$ is used. Let us see first that $(\hat{\varphi}_{V,M})^{-1}$ is a well-defined linear map. Any element of $M \otimes_R V$ is a linear combination of elements of the form $m \otimes_R v \cdot a \mathcal{L}(bc) = m \otimes_R \iota^{R}(bc) \cdot (v \cdot a)$, for $m \in M$, $v \in V$ and $a, b, c \in A$. Assume that for some elements $m^i \in M$, $v^i \in V$ and $a^i, b^i, c^i \in A$, the finite sum $\sum_i m^i \otimes_R v^i \cdot a^i \mathcal{L}(b^i c^i)$ is equal to zero. Then applying the section $\iota_{M,V}$ of the canonical epimorphism $M \otimes V \rightarrow M \otimes R V$, using (4.5) in [4] and the implicit summation index notation $T_3(c \otimes b) =: c^3 \otimes b^3$, and omitting the summation symbol for brevity, we obtain

\[0 = m^i \otimes_R (c^3) \otimes R (b^3) \cdot (v^i \cdot a^i) = m^i \otimes_R (c^3) \otimes v^i \cdot a^i \mathcal{L}(b^3).\]
Hence using also the index notation \( g(v \otimes a) = v^g \otimes a^g \), where implicit summation is understood, it follows for any \( d \in A \) that

\[
\begin{align*}
0 &= v^i \cdot a^i \overline{\tau}(b^j)S^{-1}(S(d)^g) \otimes (m^i \circ \eta^R(c^3))^g \\
&= v^i \cdot a^i \overline{\tau}(b^j)S^{-1}(S(d)^g \cap R(c^3)) \otimes m^g \\
&= v^i \cdot a^i \overline{\tau}(b^j \overline{\tau}(c^3))S^{-1}(S(d)^g) \otimes m^g \\
&= v^i \cdot a^i \overline{\tau}(b^j \overline{\tau}(c^3))S^{-1}(S(d)^g) \otimes m^g \\
&= v^i \cdot a^i \overline{\tau}(b^j \overline{\tau}(c^3))S^{-1}(S(d)^g) \otimes m^g \\
&= v^i \cdot S^{-1}(S(d)^g S(a^i \overline{\tau}(b^j \overline{\tau}(c^3)))) \otimes m^g \\
&= v^i \cdot S^{-1}(S(d)S(a^i \overline{\tau}(b^j \overline{\tau}(c^3)))) \otimes m^g \\
&= v^i \cdot S^{-1}(S(a^i \overline{\tau}(b^j \overline{\tau}(c^3)))d \otimes m^g.
\end{align*}
\]

The second equality holds by \([2\text{ Lemma 4.9 (6)}]\), the third one holds by \([4\text{ Lemma 6.14}]\), the fourth one does by \([1\text{ Lemma 3.4}]\) and the fifth one holds by \([4\text{ Lemma 3.7 (1)}]\). The sixth and the last equalities follow by the anti-multiplicativity of \(S\), cf. \([4\text{ Theorem 6.12}]\). The penultimate equality follows by the compatibility condition (2.10) in \([2]\). Simplifying by \(d\), this proves that \(v^i \cdot S^{-1}(S(a^i \overline{\tau}(b^j \overline{\tau}(c^3)))d \otimes m^g\) is equal to zero. Thus applying the canonical epimorphism \(V \otimes M \rightarrow V \otimes_R M\),

\[
0 = v^i \cdot S^{-1}(S(a^i \overline{\tau}(b^j \overline{\tau}(c^3)))d \otimes m^g = (\tilde{\varphi}_{V,M})^{-1}(m^i \circ_R v^i \cdot a^i \overline{\tau}(b^j \overline{\tau}(c^3)))
\]

as needed. We turn to proving that \((\tilde{\varphi}_{V,M})^{-1}\) is indeed the inverse of \(\tilde{\varphi}_{V,M}\). By (2.3) in \([4]\), \(E(1 \otimes a) \in R \otimes A\) for any \(a \in A\). Hence there is a well-defined linear map

\[
E_{1}^{M,A} : M \otimes A \rightarrow M \otimes A, \quad m \otimes a \mapsto ((-) \cdot m \otimes A)[E(1 \otimes a)].
\]

By \([2\text{ Lemma 6.4 (2)}]\), for any \(m \in M\) and \(a, b \in A\) the identity \(m^{gA} \otimes S(a^e)^A b = E_{1}^{M,A}(m \otimes S(a)b)\) holds. So simplifying by \(b\),

\[
(2.3) \quad \lambda(M \otimes S)g = E_{1}^{M,A}(M \otimes S).
\]

Using \([2\text{3}]\) in the second equality,

\[
(\tilde{\varphi}_{V,M})^{-1} \tilde{\varphi}_{V,M}(v \cdot a \otimes_R m) = v \cdot S^{-1}(S(a^g)^A) \otimes_R m^{gA} = \pi_{V,M}(v \cdot (-) \otimes (-) \cdot m)(S^{-1} \otimes_R E_{21}(S(a) \otimes 1)] = \pi_{V,M}(v \cdot (-) \otimes (-) \cdot m)(a \otimes 1)F = v \cdot a \otimes_R m.
\]

The third equality holds since by \([4\text{ Lemma 6.14}]\) the restriction of the anti-multiplicative map \(\overline{\tau}\) to \(L\) is equal to \(\overline{\tau}\), hence it follows by \([1\text{10}]\) that \((R \otimes S^{-1})[E(1 \otimes S(a))] = (1 \otimes a)E^{21}\). The last equality follows by Lemma \([1\text{4}]\)(2). Symmetrically,

\[
\tilde{\varphi}_{V,M}(\tilde{\varphi}_{V,M})^{-1}(m \otimes_R v \cdot a) = m^{gA} \otimes_R v \cdot S^{-1}(S(a^g)^A)e = \pi_{M,V}(m \cdot (-) \otimes v \cdot (-))(1 \otimes a)E = m \otimes_R v \cdot a,
\]

where the second equality follows by applying \([2\text{3}]\) to the opposite weak multiplier Hopf algebra \(A^{op}\) and its right comodule \((M, g, \lambda)\); and the last equality follows by Lemma \([1\text{3}]\)(1). 

\[\square\]

**Proposition 2.8.** For any regular weak multiplier bialgebra \(A\) with right full comultiplication, the functor \(U^{(A)} : M^{(A)} \rightarrow _RM_R\) in \([2\text{ Theorem 5.7}]\) (see Section \([1\text{6}]\)) factorizes through an appropriate strict monoidal functor \(I^{(A)} : M^{(A)} \rightarrow (M^{(A)})^{U^{(A)}}\) (via the evident forgetful functor \((M^{(A)})^{U^{(A)}} \rightarrow _RM_R\)).
Proof. The object map of the stated functor $I^A$ takes a full right $A$-comodule $(M, \lambda, \varrho)$ to the firm $R$-bimodule $M$ (with the actions in (1.1.3)) and the family of $R$-bimodule maps $\hat{\varphi}_{V,M} : V \otimes_R M \rightarrow M \otimes_R V$ in (2.1), for any idempotent non-degenerate right $A$-module $(V, \cdot)$. Let us see that it obeys the conditions in Example 2.1. The first compatibility condition with the monoidal unit $R$ holds by Lemma 2.6 (1). The second compatibility condition with the monoidal product $\otimes_R$ follows by Lemma 2.6 (3). Naturality of $\hat{\varphi}_{V,M}$ in $V$ follows by Lemma 2.6 (3), proving that $(M, \hat{\varphi}_{-M})$ is an object of $(M(A(U^A))$. The stated functor $I^A$ acts on the morphisms as the identity map. Indeed, any morphism $f : M \rightarrow M'$ in $M(A)$ is a morphism $(M, \hat{\varphi}_{-M}) \rightarrow (M', \hat{\varphi}_{-M'})$ in $(M(A(U^A))$ by Lemma 2.3 (4). The functor $I^A$ is strict monoidal by Lemma 2.6 (2) and (4). □

Proposition 2.9. For any regular weak multiplier bialgebra $A$ with right full comultiplication, the functor $I^A : M(A) \rightarrow (M(A(U^A))$ in Proposition 2.8 is fully faithful.

Proof. Since $I^A$ acts on the morphisms as the identity map, it is evidently faithful. In order to see that is full as well, take a morphism $f : I^A(M, \lambda, \varrho) \rightarrow I^A(M', \lambda', \varrho')$ in $(M(A(U^A))$. This means that $f$ is an $R$-bimodule map satisfying

$$(f \otimes_R V)\hat{\varphi}_{V,M} = \hat{\varphi}_{V,M'}(V \otimes_R f),$$

for any object $(V, \cdot)$ in $M(A)$. Composing both sides of this equality by $\pi_{V,M}$ on the right and by $\iota_{M',V}$ on the left and applying Lemma 2.3 (1), we obtain

$$(f \otimes V)\varphi_{V,M} = \varphi_{V,M'}(V \otimes f),$$

for any object $(V, \cdot)$ in $M(A)$; so in particular for the right $A$-module $(A, \mu)$. Thus applying Example 2.1 we conclude that

$$(f \otimes A)\varrho = \varrho'(f \otimes A),$$

that is, that $f$ is a morphism of $A$-comodules. □

Summarizing Proposition 2.8 and Proposition 2.9 we proved the following.

Theorem 2.10. For any regular weak multiplier bialgebra $A$ with right full comultiplication, the functor $U^A : M(A) \rightarrow R M_R$ in [2. Theorem 5.7] (see Section 1.6) factorizes through the strict monoidal fully faithful functor $I^A : M(A) \rightarrow (M(A))^{U^A}$ in Proposition 2.8 (via the evident forgetful functor $(M(A(U^A)) \rightarrow R M_R$).

One can proceed symmetrically if interchanging the roles of modules and comodules.

Proposition 2.11. For any regular weak multiplier bialgebra $A$ with right full comultiplication, the functor $U(A) : M(A) \rightarrow R M_R$ in [3. Theorem 5.6] (see Section 1.3) factorizes through an appropriate strict monoidal functor $I(A) : M(A) \rightarrow U^A(M(A))$ (via the evident forgetful functor $U^A(M(A)) \rightarrow R M_R$).

Proof. The object map of the stated functor $I(A)$ takes an idempotent non-degenerate right $A$-module $(V, \cdot)$ to the firm $R$-bimodule $V$ (with the actions in (1.1.3)) and the family of $R$-bimodule maps $\hat{\varphi}_{V,M} : V \otimes_R M \rightarrow M \otimes_R V$ in (2.1), for any full right $A$-comodule $(M, \lambda, \varrho)$. Let us see that it obeys the conditions in Example 2.2. The first compatibility condition with the monoidal unit $R$ holds by Lemma 2.6 (2). The second compatibility condition with the monoidal product $\otimes_R$ follows by Lemma 2.6 (4). Naturality of $\hat{\varphi}_{V,M}$ in $M$ follows by Lemma 2.3 (4), proving that $(V, \hat{\varphi}_{V,-})$ is an object of $U^A(M(A))$. The stated functor $I(A)$ acts on the morphisms as the identity map. Indeed, any morphism
The functor $I_{(A)}$ for any regular weak multiplier bialgebra $A$ with right full comultiplication, the functor $I_{(A)} : M_{(A)} \rightarrow U^{(A)}(M_{(A)})$ in Proposition 2.11 is fully faithful.

Proof. Since $I_{(A)}$ acts on the morphisms as the identity map, it is evidently faithful. In order to see that it is full as well, take a morphism $g : I_{(A)}(V, \cdot) \rightarrow I_{(A)}(V', \cdot)$ in $U^{(A)}(M_{(A)})$. This means an $R$-bimodule map $g$ satisfying

$$(M \otimes_R g) \hat{\varphi}_{V,M} = \hat{\varphi}_{V,M}(g \otimes_R M)$$

for any object $(M, \lambda, \varrho)$ in $M_{(A)}$. Composing both sides of this equality by $\pi_{V,M}$ on the right and by $\iota_{M,V'}$ on the left and applying Lemma 2.3 (1), we obtain

$$(M \otimes g) \varphi_{V,M} = \varphi_{V',M}(g \otimes M)$$

for any object $(M, \lambda, \varrho)$ in $M_{(A)}$; so in particular for $(M, \lambda, \varrho) = (A, T_1, T_3)$. Composing both sides of the resulting equality with $\epsilon \otimes V'$ and using that by the counitality axiom of weak multiplier bialgebra, as appearing in (1.3) in [2], the map $\varphi_{V,A}$ in Example 2.5 obeys (2.4)

$$(\epsilon \otimes V) \varphi_{V,A}(v \otimes a) = v \cdot a, \quad \forall v \in V, \, a \in A,$$

we conclude that $g(v \cdot a) = g(v) \cdot a$ for all $v \in V$ and $a \in A$. That is, $g$ is a morphism of $A$-modules. \hfill $\square$

Summarizing Proposition 2.11 and Proposition 2.12, we proved the following.

Theorem 2.13. For any regular weak multiplier bialgebra $A$ with right full comultiplication, the functor $U_{(A)} : M_{(A)} \rightarrow R\mathcal{M}_R$ in [3, Theorem 5.6] (see Section 1.3) factorizes through the strict monoidal fully faithful functor $I_{(A)} : M_{(A)} \rightarrow U^{(A)}(M_{(A)})$ in Proposition 2.11 (via the evident forgetful functor $U^{(A)}(M_{(A)}) \rightarrow R\mathcal{M}_R$).

In contrast to usual, unital weak bialgebras in [7, 15], the functors in Theorem 2.10 and Theorem 2.13 do not seem to be equivalences.

3. Yetter-Drinfeld Modules

The aim of this section is to find the proper notion of Yetter-Drinfeld module over a regular weak multiplier bialgebra with right full comultiplication.

For a usual, unital (weak) bialgebra $A$, the isomorphism $I_A$ between the category of $A$-modules; and the left commutant of the forgetful functor from the category of $A$-comodules, induces an isomorphism $J_A$ between the category of Yetter-Drinfeld $A$-modules and the left weak center of the category of $A$-comodules. Symmetrically, the isomorphism $I^A$, between the category of $A$-comodules; and the right commutant of the forgetful functor from the category of $A$-modules, induces an isomorphism $J^A$ between the category of Yetter-Drinfeld $A$-modules and the right weak center of the category of $A$-modules. However, as we have seen in the previous section, for a weak multiplier bialgebra $A$, the analogous functors $I_{(A)}$ and $I^{(A)}$ are no longer isomorphisms but fully faithful embeddings. In this section we show that they induce fully faithful embeddings $J_{(A)}$ and $J^{(A)}$ of the category of appropriately defined Yetter-Drinfeld $A$-modules into the left weak center of the category of full $A$-comodules, and into the right weak center of the category of idempotent non-degenerate $A$-modules, respectively.
Let $A$ be a weak multiplier bialgebra and let $V$ and $W$ be idempotent non-degenerate right $A$-modules. Consider the map

$$E_{2}^{V,W} : V \otimes W \to V \otimes W, \quad v \cdot a \otimes w \cdot b \mapsto (v \cdot (\cdot -) \otimes w \cdot (\cdot -))[(a \otimes b)E]$$

in [4] Lemma 5.4. Regard $V \otimes W$ as a right $A$-module via the (so-called diagonal) action

$$(v \cdot a \otimes w \cdot b) \cdot c = (v \cdot (\cdot -) \otimes w \cdot (\cdot -))[(a \otimes b)\Delta(c)],$$

cf. [4] Proposition 5.5. Then for any $v \in V$, $w \in W$ and $c \in A$,

$$E_{2}^{V,W}((v \otimes w) \cdot c) = (v \otimes w) \cdot c = E_{2}^{V,W}(v \otimes w) \cdot c$$

so in particular $E_{2}^{V,W}$ is a right $A$-module map.

Throughout the section we use the notation

$$(v \otimes w)\Delta(a) := (v \otimes w) \cdot a \quad \text{and} \quad (v \otimes w)\Delta^{\text{op}}(a) := tw[(w \otimes v) \cdot a].$$

**Lemma 3.1.** Let $A$ be a regular weak multiplier bialgebra with right full comultiplication. Let $X$ be a vector space which carries the structure of an idempotent non-degenerate right $A$-module $(X, \cdot)$ and the structure of a full right $A$-comodule $(X, \lambda, \varpi)$. Denote the corresponding $R$-actions on $X$ in (1.12) by $\cdot$ and $\triangleright$, and those in (1.13) by $\triangleright$ and $\blacktriangleleft$.

1. The following assertions are equivalent.
   1.a) The right $R$-actions $\lhd$ and $\blacktriangleleft$ are equal. That is, for any $x \in X$ and $a, b \in A$,
   
   \[ x \cdot a^{R}R(b) = (X \otimes \epsilon)(x \cdot a \otimes b). \]
   
   1.b) $E_{2}^{X,A} \varpi = \varpi$ (where $E_{2}^{X,A}$ is as in (3.1)).

2. The following assertions are equivalent.
   2.a) The left $R$-actions $\triangleright$ and $\blacktriangleright$ are equal. That is, for any $x \in X$ and $a, b \in A$,
   
   \[ x \cdot a^{L}L(b) = (X \otimes \epsilon)\varpi(x \cdot a \otimes b). \]
   
   (2.b) $\varrho(E_{2}^{A,X})^{21} = \varrho$ (where $E_{2}^{A,X}$ is as in (3.1)).

3. If $\varrho[(x \otimes a)\Delta^{\text{op}}(b)] = \varrho(x \otimes a)\Delta(b)$ for all $x \in X$ and $a, b \in A$, then the assertions in part (1) and part (2) are equivalent also to each other.

**Proof.** (1) From (3.1) it follows that $E_{2}^{X,A}(x \otimes a) = (x \lhd(-) \otimes A)[(1 \otimes a)E]$. Hence assertion (1.a) implies (1.b) by [2] Lemma 4.10 (3)].

By (1.b) and [4] Lemma 3.9, for any $y \in X$ and $a, b \in A$,

\[ ((-) \cdot \varrho^{R}(a) \otimes A)\varrho(y \otimes b) = \varrho(y \otimes b)(1 \otimes \varrho^{L}(a)). \]

Using this identity in the first equality and [2] Lemma 4.9 (5)] in the second one, it follows for any $\omega \in \text{Lin}(A, k)$ that

\[ ((X \otimes \omega)\varrho(y \otimes b)) \varrho^{R}(a) = (X \otimes \omega)[\varrho(y \otimes b)(1 \otimes \varrho^{L}(a))] = ((X \otimes \omega)\varrho(y \otimes b)) \varrho^{R}(a). \]

Since $X$ is a full right $A$-comodule, any element of $X$ can be written as a linear combination of elements of the form $(X \otimes \omega)\varrho(y \otimes b)$, proving that (1.a) holds.

(2) For any $a, b, c \in A$, it follows by [4] Proposition 2.5 (1) that

\[ E_{2}^{21}(a \otimes bc) = (a \otimes 1)((\varrho^{R} \otimes A)T_{2}^{21}(c \otimes b)). \]

Using this identity in the second equality and introducing the implicit summation index notation $T_{2}(b \otimes c) = b^{2} \otimes c^{2}$, we see that (2.a) implies

\[ \varrho(E_{2}^{A,X})^{21}(x \cdot a \otimes bc) = \varrho(x \cdot (-) \otimes A)E_{2}^{21}(a \otimes bc) = \varrho(x \cdot a \varrho^{L}(c^{2}) \otimes b^{2}) \]

$$= \varrho((\varrho^{R}(c^{2}) \blacktriangleleft (x \cdot a \otimes b^{2}) = \varrho(x \cdot a \otimes b^{2} \varrho^{R}(c^{2})) = \varrho(x \cdot a \otimes bc);$$
that is, (2.b). The penultimate equality follows by [2, Lemma 4.9 (3)] and the last one follows by [4, Lemma 3.7 (4)].

If (2.b) holds then for any \( x \in X \) and \( a, b, c \in A \)
\[
ϕ(\nabla(c) ∘ (x ∙ a) ⊗ b) = ϕ((x ∙ a ⊗ c) ∘ b) = ϕ(x ∗ a ⊗ b ∘ c) = ϕ(\nabla(c) ∘ (x ∙ a) ⊗ b).
\]

In the second equality we used (2.b) together with [4, Lemma 3.9]. The last equality follows by [2, Lemma 4.9 (3)]. By surjectivity of the \( A \)-action on \( X \), this proves \( ϕ(\nabla(c) ∘ x ⊗ b) = ϕ(\nabla(c) ∘ x ⊗ b) \). Applying \( X ⊗ ε \) to both sides of this equality, we obtain
\[
\nabla(b) ∘ (\nabla(c) ∘ x) = \nabla(b) ∘ (\nabla(c) ∘ x), \quad \forall x ∈ X, b, c ∈ A.
\]

Since \((X, ⊘)\) is a firm left \( R \)-module by [2, Theorem 4.5] and \( R \) has local units by [4, Theorem 4.6 (2)], \((X, ⊘)\) is a non-degenerate left \( R \)-module. So we conclude that \( \nabla(c) ∘ x = \nabla(c) ∘ x \) for all \( x ∈ X \) and \( c ∈ A \); that is, (2.a) holds.

(3) Under the hypothesis in (3), for any \( x ∈ X \) and \( a, b, c ∈ A \)
\[
(3.3) \quad (ϕ(E_{2,A}X)^{21}(x ∙ a ⊗ b)) \Delta(c) = ϕ(((x ∙ (-)) ⊗ A)((a ⊗ b)E^{21})) \Delta^{op}(c) = ϕ(x ∙ (-) ⊗ A)((a ⊗ b)E^{21}) \Delta^{op}(c) = ϕ(x ∙ (-) ⊗ A)((a ⊗ b) \Delta^{op}(c)) = ϕ(x ∙ (-) ⊗ A)((a ⊗ b) \Delta^{op}(c)).
\]

In the second equality we used that for any \( x ∈ X \), \( A ⊗ x ∙ (-) : A ⊗ A → A ⊗ X \) is a morphism of right \( A \)-modules with respect to the diagonal actions in (3.2).

If (1.b) holds then the range of \( ϕ \) is contained in the range of \( E_{2,A}X \), which is a non-degenerate right \( A \)-module via the diagonal action (3.2) by [4, Lemma 5.4 and Proposition 5.5]. Thus we conclude from (3.3) that (2.b) holds.

By axiom (vii) of weak multiplier bialgebra in [2, Definition 1.1], for any given elements \( a, b ∈ A \) there exist finitely many elements \( p', q', r' \) such that \( (b ⊗ a)E = \sum_i (p' ⊗ q') \Delta(r') \). Then (omitting the summation symbol for brevity), it follows from (2.b) that for any \( x ∈ X \)
\[
ϕ(x ∙ a ⊗ b) = ϕ(E_{2,A}X)^{21}(x ∙ a ⊗ b) = ϕ((x ∙ q' ⊗ p') \Delta^{op}(r')) = ϕ(x ∙ q' ⊗ p') \Delta^{op}(r')
\]
so that (1.b) holds by \( E_2 ∆ = ∆ \). In the third equality we used our hypothesis in (3). □

**Theorem 3.2.** Consider a regular weak multiplier bialgebra \( A \) with right full comultiplication. Let \( X \) be a vector space which carries the structure of an idempotent non-degenerate right \( A \)-module \((X, *)\) and the structure of a full right \( A \)-comodule \((X, λ, ̺)\). If \( E_{2,A}X̺ = ̺ = E_{2,A}X^{21} \) holds for the maps \( E_{2,A}X \) and \( E_{2,A}X^{21} \) as in (3.1), then the following assertions are equivalent.

(a) For any full right \( A \)-comodule \( M \), \( ̺_{X,M} : X ⊗_R M → M ⊗_R X \) in (2.1) is a morphism of \( A \)-comodules.

(b) The datum \(((X, λ, ̺), ̺_{X,-})\) is an object in the left weak center of \( M^{(A)} \).

(c) For any idempotent non-degenerate right \( A \)-module \( V \), \( ̺_{V, X} : V ⊗_R X → X ⊗_R V \) is a morphism of \( A \)-modules.

(d) The datum \(((X, *), ̺_{-, X})\) is an object in the right weak center of \( M^{(A)} \).

(e) For any \( x ∈ X \) and \( a, b ∈ A \), \( ̺(x ∗ a) \Delta^{op}(b) = ̺(x ∗ a) \Delta(b) \).

**Proof.** The equivalences (a)⇔(b) and (c)⇔(d) are obvious.

(e)⇔(c) Regard \( X \) as a right \( R \)-module via the action \( * = ◦ \); cf. Lemma 3.1. For any object \( V \) of \( M^{(A)} \), it follows by the form of the \( A \)-action on \( X ⊗_R V \) in [4, Proposition 5.5]
that the canonical epimorphism $\pi_{X,V} : X \otimes V \to X \otimes_R V$ is a morphism of $A$-modules.

By [1] Lemma 5.4, the idempotent map $\iota_{X,V} \pi_{X,V}$ is equal to the $A$-module map $E_2^{X,V}$

proving that $\iota_{X,V} : X \otimes_R V \to X \otimes V$ is a morphism of $A$-modules too. Using this together

with Lemma 2.24 (1), we see that assertion (c) is equivalent to $\varphi_{V,X}$ being a right $A$-module map. That is, to

$$\Delta(b) = \Delta^{\text{op}}(b), \quad \forall v \in V, \; x \in X, \; a, b \in A.$$ 

This proves (e) $\Rightarrow$ (c). Conversely, applying (3.4) to the $A$-module $(V, \cdot) = (A, \mu)$ and using the non-degeneracy of $A$, we conclude that (c) $\Rightarrow$ (e).

(a) $\Rightarrow$ (e) Using (3.4) together with the construction of the $A$-comodules $X \otimes_R M$ and $M \otimes_R X$ in [1] Proposition 5.4 for any full right $A$-comodule $(M, \lambda_M, \varrho_M)$, (a) is equivalent to

$$(M \otimes \varrho)\theta_M^{13}((\iota_{X,M} \pi_X \otimes X \otimes A)(\varphi_{X,M} \otimes A)(X \otimes \varrho_M)\theta_3^{13}((\iota_{X,M} \pi_X \otimes A).$$

Thus by [1] Proposition 5.1 and Lemma 5.3 (8), also to

$$\Delta(b) = \Delta^{\text{op}}(b), \quad \forall v \in V, \; x \in X, \; a, b \in A.$$ 

proving (a) $\Rightarrow$ (e).

(e) $\Rightarrow$ (a) By axiom (vii) of weak multiplier bialgebra in [1] Definition 1.1, for any given elements $a, b \in A$ there exist finitely many elements $p^i, q^i, r^i \in A$ such that $(b \otimes a)E = \sum(p^i \otimes q^i)\Delta(r^i)$. In terms of these elements, for any full right $A$-comodule $(M, \lambda_M, \varrho_M)$, $m \in M$ and $x \in X$

$$\Delta(b) = \Delta^{\text{op}}(b), \quad \forall v \in V, \; x \in X, \; a, b \in A.$$ 

where we omitted the summation symbol for brevity; and used the implicit summation index notation $\varrho_M(m \otimes r) =: m^\varrho \otimes r^\varrho$ and $\varrho(x \otimes p) =: x^\varrho \otimes p^\varrho$. The first equality follows by the hypothesis $\varrho = \varrho(E_2^{A,X})^2$ and the penultimate equality follows by the coassociativity
condition [2] Proposition 2.1 (4.d)]. On the other hand, using the same notation,

\[(3.7) \ (M \otimes \varrho) \varrho_{13}^{13}(\varphi_{X,M} \otimes A)[x \cdot a \otimes m \otimes b] \]

\[= (M \otimes \varrho)(M \otimes x \cdot (-) \otimes A) \varrho_{13}^{13}(\varrho_{M} \otimes A)(m \otimes a \otimes b) \]

\[= (M \otimes \varrho)(M \otimes x \cdot (-) \otimes A) \varrho_{13}^{13}(\varrho_{M} \otimes A)[m \otimes (a \otimes b)E^{21}] \]

\[= (M \otimes \varrho)(M \otimes x \cdot (-) \otimes p^i(-))(M \otimes tw)(\varrho_{M} \otimes A) \varrho_{13}^{13}(M \otimes T_3)(m \otimes r^i \otimes q^i) \]

\[= (M \otimes \varrho)(M \otimes x \cdot (-) \otimes p^i(-))(M \otimes twT_3)(\varrho_{M} \otimes A)(m \otimes r^i \otimes q^i) \]

\[= m^e \otimes \varrho((x \cdot q^i \otimes p^i)\Delta^{op}(r^i\varrho)). \]

The second and the fourth equalities follow by part (2.b) and part (4.d) of [2, Proposition 2.1], respectively. The expressions in (3.6) and (3.7) are equal by assertion (e). This proves that (3.5) and hence assertion (a) holds. □

If the equivalent assertions in Theorem 3.2 hold, then we term X a (right-right) Yetter-Drinfeld A-module:

**Definition 3.3.** Consider a regular weak multiplier bialgebra A with right full comultiplication. A **right-right Yetter-Drinfeld module** over A is a vector space X which carries the structure of an idempotent non-degenerate right A-module (X, ·) and the structure of a full right A-comodule (X, λ, ρ) such that for all x ∈ X and a, b ∈ A

\[\varrho[(x \otimes a)\Delta^{op}(b)] = \varrho(x \otimes a)\Delta(b) \quad \text{and} \quad E^{X,A}_2 = \varrho = \varrho(E^{A,X}_2);\]

in terms of the maps \(E^{X,A}_2\) and \(E^{A,X}_2\) as in (3.1).

A morphism of right-right Yetter-Drinfeld modules is a linear map which is both a morphism of A-modules and a morphism of A-comodules. The category of right-right Yetter-Drinfeld A-modules will be denoted by \(\text{YD}(A)\).

**Remark 3.4.** If A is a regular weak multiplier Hopf algebra in the sense of [21] (that is, it is a regular weak multiplier bialgebra with left and right full comultiplication such that both A and \(A^{op}\) possess an antipode) and X is a vector space obeying the hypotheses of Theorem 3.2, then we conclude by Remark 2.7 that the equivalent assertions in Theorem 3.2 are equivalent also to the following.

(b’) The datum \(((X, \lambda, \varrho), \varphi_{X,-})\) is an object in the center of \(M(A)\).

(d’) The datum \(((X, \cdot), (\varphi_{-,X})^{-1})\) is an object in the center of \(M(A)\).

**Theorem 3.5.** For a regular weak multiplier bialgebra A with right full comultiplication, the following assertions hold.

1. There are fully faithful functors \(J^{(A)} : \text{YD}(A) \rightarrow (M(A))^{M(A)}\) and \(J^{(A)} : \text{YD}(A) \rightarrow M^{(A)}(M(A))\) rendering commutative the following diagram (in which the unlabelled arrows in the top row denote the evident forgetful functors).

\[
\begin{array}{l}
\text{M}(A) \\
\downarrow I^{(A)} \\
(M(A))^{U(A)} \quad \text{YD}(A) \\
\downarrow J^{(A)} \\
(M(A))^{M(A)} \quad \text{M}(A) \\
\end{array}
\]

\[
\begin{array}{l}
\downarrow J^{(A)} \\
M^{(A)}(M(A)) \quad U^{(A)}(M(A)) \\
\end{array}
\]

2. If \(I^{(A)}\) is an isomorphism (respectively, an equivalence) then so is \(J^{(A)}\).

3. If \(I^{(A)}\) is an isomorphism (respectively, an equivalence) then so is \(J^{(A)}\).
Proof. Recall that the morphisms \((V, \sigma) \to (V', \sigma')\) in the right weak center of \(M_{(A)}\) are the \(A\)-module maps \(V \to V'\) which are also morphisms \((U_{(A)} V, U_{(A)} \sigma) \to (U_{(A)} V', U_{(A)} \sigma')\) in \((M_{(A)})^{U_{(A)}}\); and there is a similar description of the morphisms in the left weak center of \(M_{(A)}\).

1. First we show that there is a functor \(J^{(A)} : \text{YD}(A) \to (M_{(A)})^{M_{(A)}}\), sending a Yetter-Drinfeld module \((X, \cdot, \lambda, \varrho)\) to \(((X, \cdot), \varphi_{-X})\) and acting on the morphisms as the identity map. By Theorem 3.2 (e) \((X, \cdot, \lambda, \varrho)\) is an object in the right weak center of \(M_{(A)}\). A morphism in \(\text{YD}(A)\) is a morphism of \(A\)-modules by definition and it is taken by \(I^{(A)}\) — acting as the identity map — to a morphism in \((M_{(A)})^{U_{(A)}}\). Hence it is a morphism in the right weak center of \(M_{(A)}\). Evidently, \(J^{(A)}\) is faithful and it renders commutative the left half of the diagram. In order to see that it is full, take any morphism \(J^{(A)}(X, \cdot, \lambda, \varrho) \to J^{(A)}(X', \cdot, \lambda', \varrho')\) in \((M_{(A)})^{M_{(A)}}\). It is by definition a morphism of \(A\)-modules and a morphism in \((M_{(A)})^{U_{(A)}}\). Thus by the fullness of \(I^{(A)}\) (see Proposition 2.9), it is a morphism of \(A\)-comodules hence a morphism in \(\text{YD}(A)\).

The existence and the stated properties of \(J^{(A)}\) are proven symmetrically.

2. Since \(J^{(A)}\) is fully faithful by part (1), we only need to show that if the object map of \(I^{(A)}\) is bijective (resp. essentially surjective) then so is the object map of \(J^{(A)}\). To this end, pick up any object \(((V, \cdot), \sigma)\) in the right weak center of \(M_{(A)}\), taken by the lifted functor \(U_{(A)}\) to the object \((V, \sigma)\) in \((M_{(A)})^{U_{(A)}}\). By the assumption about \(I^{(A)}\), there is a unique (resp. some) object \((M, \lambda, \varrho)\) in \(M_{(A)}\) such that \((M, \varphi_{-M})\) is equal (resp. isomorphic) to \((V, \sigma)\) in \((M_{(A)})^{U_{(A)}}\). Then we can use the \(R\)-bimodule isomorphism \(M \cong V\) to induce an idempotent non-degenerate right \(A\)-action \(\cdot\) on \(M\). By construction, the \(R\)-actions on \(M\), corresponding to its \(A\)-comodule structure and to its \(A\)-module structure, coincide. Hence assertions (1.b) and (2.b) in Lemma 3.1 hold. Moreover, since \(\sigma_{W}\) is an \(A\)-module map for any object \(W\) in \(M_{(A)}\), so is \(\varphi_{W,M}\). Thus by Theorem 3.2 (c) \((e) \Rightarrow (e), (M, \cdot, \lambda, \varrho)\) is a Yetter-Drinfeld module such that \(J^{(A)}(X, \cdot, \lambda, \varrho) = ((M, \cdot, \varphi_{-M})\) is equal (resp. isomorphic) to \(((V, \cdot), \sigma)\) in the right weak center of \(M_{(A)}\).

Part (3) is proven symmetrically.

4. The monoidal category of Yetter-Drinfeld modules

The aim of this section is to prove that the category of Yetter-Drinfeld modules, over a regular weak multiplier bialgebra \(A\) with right full comultiplication, carries a monoidal structure admitting strict monoidal forgetful functors to the category of idempotent non-degenerate \(A\)-modules and the category of full \(A\)-comodules. If \(A\) is in addition a regular weak multiplier Hopf algebra in the sense of [21] (that is, the comultiplication is both left and right full and both \(A\) and \(A^{op}\) possess an antipode), then finite dimensional Yetter-Drinfeld modules are shown to possess duals in this category.

Proposition 4.1. Consider a regular weak multiplier bialgebra \(A\) with right full comultiplication. For any right-right Yetter-Drinfeld \(A\)-modules \(X\) and \(Y\), the module tensor product \(X \otimes_{R} Y\) over the base algebra \(R\) of \(A\) is again a right-right Yetter-Drinfeld \(A\)-module via the module structure in [4] Proposition 5.5] and the comodule structure in [2] Proposition 5.4].

Proof. By [4] Proposition 5.5], \(X \otimes_{R} Y\) is an idempotent non-degenerate right \(A\)-module. By [2] Proposition 5.4 and Proposition 5.6], \(X \otimes_{R} Y\) is a full right \(A\)-comodule. So it remains to check the compatibility between them. Since the \(R\)-actions corresponding to
the $A$-module structure, and the $R$-actions corresponding to the $A$-comodule structure, coincide both on $X$ and $Y$, they evidently coincide on $X \otimes_R Y$ too. Since by Theorem 3.2 (e) $\rho$ and $\lambda$ both $\hat{\varphi}_{V,X}$ and $\hat{\varphi}_{V,Y}$ are morphisms of $A$-modules, for any object $V$ in $\text{M}_{(A)}$, so is $\hat{\varphi}_{V,X \otimes_R Y}$ by Lemma 2.6 (4). Thus $X \otimes_R Y$ is a Yetter-Drinfeld module by Theorem 3.2 (c) $\Rightarrow$ (e).

**Proposition 4.2.** Consider a regular weak multiplier bialgebra $A$ with right full comultiplication. Then the base algebra $R$ is a right-right Yetter-Drinfeld module via the action

$$\nabla^R(a) \cdot b = \nabla^R(\nabla^R(a)b) = \nabla^R(ab) \quad \text{(see [4, Proposition 5.3])}$$

and the comodule structure

$$(\lambda : r \otimes a \mapsto E(1 \otimes ra), \gamma : r \otimes a \mapsto (1 \otimes ar)E) \quad \text{(see [2, Example 2.5 and Example 4.4])}.$$

**Proof.** By [4, Proposition 5.3], $R$ is an idempotent non-degenerate right $A$-module via the stated action. By [2, Example 2.5 and Example 4.4] $R$ is a full right $A$-comodule via the stated maps. So we need to check the compatibility of these structures. Both the $R$-actions corresponding to the $A$-module structure, and the $R$-actions corresponding to the $A$-comodule structure, are given by the multiplication in $R$, see [4, Proposition 5.3] and [2, Example 4.8], respectively. Hence they are equal so that assertions (1.b) and (2.b) in Lemma 3.1 hold. By Lemma 2.6 (1), $((R, \lambda, \gamma), \hat{\varphi}_R)$ is the monoidal unit of $\text{M}_{(A)}(\text{M}_{(A)})$ and thus $R$ is a Yetter-Drinfeld module by Theorem 3.2 (b) $\Rightarrow$ (e). □

In view of [4, Theorem 5.6], idempotent non-degenerate right $A$-modules constitute a monoidal category via the $R$-module tensor product as a monoidal product. By [2, Theorem 5.7], also full right $A$-comodules constitute a monoidal category via the $R$-module tensor product as a monoidal product. Hence Proposition 4.1 and Proposition 4.2 give rise to the following.

**Theorem 4.3.** For a regular weak multiplier bialgebra $A$ with right full comultiplication, the category $\text{YD}(A)$ of right-right Yetter-Drinfeld $A$-modules is monoidal such that there is a commutative diagram of strict monoidal forgetful functors

$$\text{YD}(A) \longrightarrow \text{M}_{(A)}$$

$$\downarrow \quad \downarrow$$

$$\text{M}_{(A)} \longrightarrow R\text{M}_R.$$
Let us see that this $A$-action is surjective. Since the algebra $A$ has local units by [21, Proposition 4.9], since the $A$-action on $X$ is surjective and since $X$ is finite dimensional, there is an element $p \in A$ such that $x \cdot p = x$ for all $x \in X$. For this element $p$, any $\varphi \in X^*$ and $x \in X$,

$$(\varphi \cdot S(p))(x) = \varphi(x \cdot p) = \varphi(x)$$

so that $\varphi = \varphi \cdot S(p)$ and thus the $A$-action on $X^*$ is surjective. The $A$-action on $X^*$ is also non-degenerate. Indeed, if $\varphi \cdot a = 0$ for all $a \in A$, then

$$0 = (\varphi \cdot a)(x) = \varphi(x \cdot S^{-1}(a)) \quad \forall x \in X, \ a \in A,$$

so by the bijectivity of $S^{-1}$ and surjectivity of the $A$-action on $X$, $\varphi(x) = 0$ for all $x \in X$ proving that $\varphi = 0$.

Via the maps $\lambda^{sS}, \varrho^{sS} : X^* \otimes A \to X^* \otimes A$ in (6.11) of [2], $X^*$ is a full $A$-comodule, see [2 Proposition 6.2 and Proposition 6.5]. So it remains to check the Yetter-Drinfeld compatibility conditions. The $R$-actions on the above $A$-module $X^*$ are given by

$$(\varphi \cdot a \cdot \cap^R (b))(x \cdot c) = (\varphi \cdot a \cap^R (b))(x \cdot c) = \varphi(x \cdot c S^{-1}(a \cap^R (b)))$$

$$= \varphi(x \cdot c \cdot (-) \cap^L (b) S^{-1}(a)) = (\varphi \cdot a)(\cap^R (b) \Rightarrow (x \cdot c))$$

$$(\cap^R (b) \Rightarrow \varphi \cdot a)(x \cdot c) = (\varphi \cdot a \cap^R (b))(x \cdot c) = \varphi(x \cdot c S^{-1}(a \cap^R (b)))$$

$$= \varphi(x \cdot c (\vartheta^{-1} \cap^R (b) S^{-1}(a))) = (\varphi \cdot a)((x \cdot c) \cdot (\vartheta^{-1} \cap^R (b)))$$

for any $\varphi \in X^*$, $x \in X$ and $a, b, c \in A$, where $\vartheta$ is the Nakayama automorphism of the firm Frobenius algebra $R$, cf. [3 Theorem 4.6 (3)]. In the third equality of the first computation we used [4, Lemma 6.14] and in the third equality of the second computation we used the same lemma together with [4, Proposition 4.9]. Comparing this with [2 Proposition 6.6] and using that the $R$-actions (1.12) and (1.13) on $X$ coincide, we conclude that they coincide on $X^*$ too. For any $b, d \in A$, introduce the index notation $T_1(b \otimes d) := b^1 \otimes d^1$; for $x \in X$ and $d \in A$ introduce the index notation $\lambda(x \otimes d) := x^\lambda \otimes d^\lambda$ and $\rho(x \otimes d) := x^\rho \otimes d^\rho$; where in all cases implicit summation is understood. For any $\varphi \in X^*$ and $a, c \in A$,

$$\varrho^{sS}[(\varphi \otimes c S(d))(\Delta^{op} S(b))](1 \otimes S(a))$$

$$= (\varrho^{sS}(\varphi \cdot (-) \cap c(-))(S \otimes S) T_1(b \otimes d))(1 \otimes S(a))$$

$$= \varphi((-)^L \cdot b^1) \otimes cS(aa^1) = \varphi((-)^\rho \cdot b^1) \otimes cS(a^\rho d^1),$$

where the last equality follows by the compatibility condition (2.10) in [2]. Denoting also $T_2(a \otimes b) := a^2 \otimes b^2$ (with implicit summation understood),

$$\varrho^{sS}(\varphi \otimes c S(d))(\Delta(b))(1 \otimes S(a)) = \varrho^{sS}(\varphi \otimes c S(d))(S \otimes S) T_2(1 \otimes d)(a \otimes a)$$

$$= \varphi((-)^L \cdot S(b^2) \otimes cS(a^2 d^\lambda)$$

$$= \varphi((-)^\rho \cdot b^2)^\lambda \otimes cS(a^2 d^\lambda)$$

$$= \varphi((-)^\rho \cdot b^2)^\rho \otimes cS(a^\rho d^\rho)$$

where the last equality follows again by (2.10) in [2]. These expressions are equal since applying the Yetter-Drinfeld compatibility condition in the second equality,

$$x^\rho b^1 \otimes a^\rho d^1 = \varrho(x \otimes a) \Delta(b)(1 \otimes d) = \varrho((x \otimes a) \Delta^{op}(b))(1 \otimes d) = (x \cdot b^2)^\rho \otimes a^2 d^\rho$$

for any $x \in X$. By the bijectivity of $S$ and the non-degeneracy of $A$, this proves that $X^*$ is a Yetter-Drinfeld module.

For the comultiplication

$$\delta : R \to R \otimes R, \quad r \mapsto (r \otimes 1) F = F(1 \otimes r)$$
in [4] Proposition 4.3 (3)], introduce the implicit summation index notation \( \delta(r) =: r_1 \otimes r_2 \).

The evaluation map

\[
\text{ev} : X^* \otimes_R X \to R, \quad \varphi \otimes_R x \mapsto \varphi(x \cdot r_1) r_2
\]

in (6.13) in [2] and the coevaluation map

\[
\text{coev} : R \to X \otimes_R X^*, \quad r \mapsto \sum_i r \cdot x_i \otimes_R \xi_i
\]

in (6.14) in [2] — where \( \{x_i \in X\} \) and \( \{\xi_i \in X^*\} \) are finite dual bases — obey the triangular identities of duality and they are morphisms of \( A\)-comodules by [2] Theorem 6.7. So in order to complete the proof, we need to see that they are morphisms of \( A\)-modules too.

For any \( \varphi \in X^* \), \( x \in X \), \( a, b, c \in A \) and \( r, s \in R \),

\[
\text{ev}[(\varphi \otimes_R x \cdot ar) \cdot bs] = \text{ev} \pi_{X^*,X}(\varphi \cdot (\cdot) \otimes x \cdot (\cdot)) T_3(bs \otimes ar)
\]

\[
= \varphi(x \cdot \mu^{op}(S^{-1} \otimes A) T_3(bs \otimes ar)) s_2
\]

\[
= \varphi(x \cdot ar r_1(bs_1)) s_2 = \varphi(x \cdot ar r_1(bs_1)) s_2
\]

\[
= \cap^R(\varphi(x \cdot ar_1) r_2 bs) = \text{ev}(\varphi \cdot x \cdot ar) \cdot bs,
\]

so that the evaluation map is a morphism of \( A\)-modules. In the second equality we used twice that by [4, Lemma 3.3], \( T_3(bs \otimes a) = T_3(b \otimes a)(1 \otimes s) \). In the third equality we applied an identity in (6.14) in [4] to the opposite weak multiplier Hopf algebra \( A^{op} \). The fourth equality follows by [4, Lemma 3.4]. In the penultimate equality we used that by Lemma 1.2 (b) = (d),

\[
\cap^R(rbs_1) \otimes s_2 = (\cap^R \otimes R)[(rbs \otimes 1)F] = (R \otimes \cap^R)[F(1 \otimes rbs)] = r_1 \otimes \cap^R(r_2 bs).
\]

Recall from (6.12) in [2] the vector space isomorphism

\[
\kappa : X \otimes_R X^* \to \text{Hom}_R(X, X), \quad x \otimes_R \varphi \mapsto [y \mapsto \varphi(y \cdot r_1)x \cdot r_2].
\]

For any \( y \in X \), \( r \in R \) and \( a, b, c \in A \), omitting for brevity the summation symbol over \( i \),

\[
[\kappa((\text{coev} \cap^R(b)) \cdot a)(y \cdot r)] \cdot c = \left[(x_i \cdot \cap^L(b)(\cdot) \otimes \xi_i \cdot (\cdot))(y \cdot r_1)\right] T_4(r_2 c \otimes a)
\]

\[
= \left[x_i \cdot \cap^L(b)(\cdot) \otimes \xi_i(y \cdot r_1 S^{-1}(\cdot))\right] T_4(r_2 c \otimes a)
\]

\[
= \left[x_i \cdot (\cdot) \otimes \xi_i(y \cdot r_1 S^{-1}(\cdot))\right] T_4(r_2 c \otimes \cap^L(b)a)
\]

\[
= y \cdot r_1 (\mu^{op}(A \otimes S^{-1}) T_4(r_2 c \otimes \cap^L(b)a))
\]

\[
= y \cdot r_1(\cap^L(b)a) r_2 c = y \cdot r \cap^L(ba)c
\]

\[
= \left[\kappa \text{coev}(\cap^R(b) \cdot a)(y \cdot r)\right] \cdot c.
\]

Hence by the non-degeneracy of the right \( A\)-module \( X \), since \( \kappa \) is an isomorphism and by the surjectivity of the right \( R\)-action on \( X \), it follows that the coevaluation map is a morphism of \( A\)-modules. In the third equality we used that by [4, Lemma 3.3], \( (\cap^L(b) \otimes 1) T_4(c \otimes a) = T_4(c \otimes \cap^L(b)a) \), for any \( a, b, c \in A \). In the fifth equality we applied an identity in (6.14) in [4] to the opposite weak multiplier Hopf algebra \( A^{op} \). The penultimate equality follows by [4, Lemma 3.2], [4, Lemma 3.5], and the fact that \( \mu \delta(r) = r_1 r_2 = r \), see [4, Proposition 4.3 (1)].
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