Toda theory in AdS$_2$ and $\mathcal{W}A_n$-algebra structure of boundary correlators

Matteo Beccaria and Giulio Landolfi

Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, Lecce 73100, Italy

E-mail: matteo.beccaria@le.infn.it, giulio.landolfi@le.infn.it

Abstract: We consider the conformal $A_n$ Toda theory in AdS$_2$. Due to the bulk full Virasoro symmetry, this system provides an instance of a non-gravitational AdS$_2$/CFT$_1$ correspondence where the 1d boundary theory enjoys enhanced “$1/2$-Virasoro” symmetry. General boundary correlators are expected to be captured by the restriction of chiral correlators in a suitable $\mathcal{W}A_n$ Virasoro extension. At next-to-leading order in weak coupling expansion they have been conjectured to match the subleading terms in the large central charge expansion of the dual $\mathcal{W}A_n$ correlators. We explicitly test this conjecture on the boundary four point functions of the Toda scalar fields dual to $\mathcal{W}A_n$ generators with next-to-minimal spin 3 and 4. Our analysis is valid in the generic rank case and extends previous results for specific rank-2 Toda theories. On the AdS side, the extension is straightforward and requires the computation of a finite set of tree Witten diagrams. This is due to simple rank dependence and selection rules of cubic and quartic couplings. On the boundary, we exploit crossing symmetry and specific meromorphic properties of the $\mathcal{W}$-algebra correlators at large central charge. We present the required 4-point functions in closed form for any rank and verify the bulk-boundary correspondence in full details.

Keywords: AdS-CFT Correspondence, Conformal Field Theory

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1 Introduction

Recently, the old subject of quantum field theories in rigid AdS background [1] turned out to be conveniently seen as a non-gravitational instance of AdS/CFT, suggesting new ideas and methods. For example, flat space scattering amplitudes of a massive theory may be obtained at large curvature radius by studying the large scaling dimension regime of the boundary conformal correlators [2–4]. The specific AdS$_2$ case attracted much interest from the very beginning [5–7] and from the AdS/CFT perspective it has very special features, like the conjectured duality between a gravitational theory in AdS$_2$ and a chiral half of a 2d CFT [8].\footnote{Recently, the rigid AdS$_2$ background played a key role in the analysis of correlators of $N = 4$ SYM local operators inserted on a straight or circular Wilson line [9–16]. At strong coupling, the AdS$_2 \times S^5$ string action is expanded near the minimal surface associated with the 1d defect and leads to a 2d field theory action in AdS$_2$ background [14, 16].}

In the gravitational context, 2d diffeomorphisms are a gauge symmetry and
Virasoro symmetry may appear as an asymptotic symmetry whose boundary manifestation are 1d time reparametrizations [17–19], possibly spontaneously broken to SL(2, \mathbb{R}) [20–22]. Instead, for a rigid AdS_2 background, the natural counterpart of this setup is to consider a theory that is locally conformal in the bulk and to explore the occurrence of enhanced boundary conformal symmetry. As a first step in this direction, the analysis in [23] examined the case of Liouville theory [24–26] with curved space action

\[ S = \frac{1}{4\pi} \int d^2x \sqrt{g} \left( \partial^\mu \varphi \partial_\mu \varphi + \mu e^{2\beta \varphi} + Q R \varphi \right), \quad Q = \beta + \beta^{-1}. \] (1.1)

It is Weyl-covariant on a fixed curved 2d background with the central charge \( c = 1 + 6 Q^2 \). In particular, on Euclidean AdS_2 background with metric \( ds^2 = \frac{1}{\varphi^2}(dt^2 + dz^2) \) the Liouville field \( \varphi \) can be expanded near its constant vacuum expectation value and its fluctuations have classical mass \( m^2 = 2 \). Bulk properties of Liouville theory on AdS_2 have been discussed previously [6, 27, 28], while the recent study in [23] focused on boundary correlators. With Dirichlet boundary condition on the AdS boundary \( z = 0 \) the field \( \varphi \) has asymptotics \( \varphi(t, z) \big|_{z=0} = \varphi(t) + \cdots \), and is dual to the 1d CFT operator \( \Phi(t) \) with conformal dimension \( \Delta = 2 \) obeying the AdS_2 relation \( m^2 = \Delta(\Delta - 1) \). The associated boundary correlators are defined as usual by

\[ \langle \Phi(t_1) \cdots \Phi(t_n) \rangle \equiv \lim_{z_1, \ldots, z_n \to 0} z_1^{-2} \cdots z_n^{-2} \langle \varphi(t_1, z_1) \cdots \varphi(t_n, z_n) \rangle, \] (1.2)

and can be computed perturbatively in the weakly coupled Toda theory by expanding in Witten diagrams. Since we start from a 2d conformal theory in AdS_2, we can expect a correspondence between the boundary correlators and standard two-dimensional Virasoro correlators with the same central charge. Indeed, as noticed in [29], this is true at tree level, i.e. \( \beta \ll 1 \) or \( c \gg 1 \). The 2-, 3- and 4-point boundary correlators (1.2) match the correlators of the holomorphic stress tensor \( T(z) \) according to

\[ \langle \Phi(t_1) \cdots \Phi(t_n) \rangle = \kappa^n \left. \langle T(z_1) \cdots T(z_n) \rangle \right|_{z_i \to t_i}, \] (1.3)

where \( \kappa = \kappa(\beta) \) is a proportionality coefficient appearing in the formal identification \( \Phi(t) \to \kappa T(t) \) upon restriction of the 2d chiral stress tensor to the real boundary \( z_i = t_i + iy_i \to t_i \).\(^2\) In [23], the relation (1.3) has been tested beyond the leading tree level approximation by computing the one-loop corrections to various correlators \( \langle \Phi(t_1) \cdots \Phi(t_n) \rangle \). One of the outcomes of the analysis is the following proposal for the all-order expression of the intertwining coefficient \( \kappa(\beta) \)

\[ \kappa(\beta) = - \frac{4Q}{c} = - \frac{4\beta(1 + \beta^2)}{(3 + 2 \beta^2)(2 + 3 \beta^2)} = - \frac{2}{3} \beta + \frac{7}{9} \beta^3 + \cdots. \] (1.4)

A natural generalization of the Liouville correspondence (1.3) consists in its extension to conformal Toda theories of non-affine type [30–32] on the AdS_2 background. In the \( A_n \)

\(^2\)The identification can be explained at semiclassical level by identifying \( \Phi \) as the surviving piece in the boundary limit of the Toda stress tensor. This simple reasoning requires quantum refinements as discussed for the Liouville theory in [23].
case, expanding near the minimum of the Toda potential, one finds $n$ scalar fields $\varphi_\Delta$ with masses $m^2 = \Delta(\Delta - 1)$ corresponding to $\Delta = 2, \ldots, n + 1$ [29]. The expected generalization of the duality relation (1.3) reads then

$$\langle \Phi_{\Delta_1}(t_1) \cdots \Phi_{\Delta_n}(t_n) \rangle = \left( \prod_{i=1}^{n} \kappa_{\Delta_i} \right) \langle Q_{\Delta_1}(z_1) \cdots Q_{\Delta_n}(z_n) \rangle \bigg|_{z_i \rightarrow t_i}, \quad (1.5)$$

where $\langle \Phi_{\Delta_1}(t_1) \cdots \Phi_{\Delta_n}(t_n) \rangle = \lim_{z_1,\ldots,z_n \rightarrow 0} z_1^{-\Delta_1} \cdots z_n^{-\Delta_n} \langle \varphi_{\Delta_1}(t_1, z_1) \cdots \varphi_{\Delta_n}(t_n, z_n) \rangle$, $Q_\Delta = \{Q_2 \equiv T, Q_3, \ldots, Q_{n+1}\}$ are the generators of the chiral $W_{n+1}$ algebra replacing and extending the Virasoro symmetry and with the same central charge of the Toda theory. The coefficients $\kappa_{\Delta_i}$ are functions of the Toda coupling entering the correspondence $\Phi_\Delta \rightarrow \kappa_\Delta Q_\Delta$.

The relation (1.5) was noticed at tree level in [29] in a few sample 4-point functions involving the Toda theories associated to some rank-2 algebras with two scalar fields (one dual to the stress tensor $T$ and the other dual to a higher spin chiral field $Q_s$).

In this paper, we discuss the relation (1.5) for the four point functions involving the two fields with next-to-minimal higher ‘spin’ $\Delta = 3, 4$ in the general $A_n$ Toda theory. This analysis aims to exclude possible low-rank accidental good properties. Despite being a leading order analysis, not involving loops in AdS, the AdS/CFT matching of the full dependence on the rank $n$ proves to be a quite stringent constraint.

Technically, the $A_n$ case is feasible and rather straightforward on the AdS side due to some peculiar regularities of the cubic and quartic couplings with respect to the rank $n$. At leading non-trivial order, selection rules reduce the calculation to a finite sum of Witten diagrams that can be exactly computed. On the CFT side, the task is in principle harder and amounts to the calculation of 4-point correlators of spin-3 and spin-4 generators of the Casimir $W$-algebra $W_{n+1}$ [34]. The structure of such algebras depends non-trivially on the rank $n$ and the fusion structure constants are functions of $n$ and the central charge that are not known in general. Nevertheless, we shall be only interested in the leading and sub-leading correlators at large central charge. Known results about semiclassical Virasoro blocks together with a careful use of crossing symmetry and the meromorphic properties of the correlators will allow a simple determination of the desired correlators for generic rank. Our analysis confirms the validity of (1.5) for the considered 4-point functions in the $A_n$ Toda theory, at least at classical level. This lends further support to that relation and, in principle, allows to test higher loop AdS calculations by $W$-algebraic methods.

The structure of the paper is as follows. In section 2 we illustrate the tools needed to compute tree level boundary correlators in the $A_n$ Toda theory on AdS2 and present explicit results for the 4-point functions of scalars with $\Delta = 3, 4$. Section 3 presents the associated results for the dual CFT fields in the $W_n$ Virasoro extension. The correlators of generators with spin 3, 4 are computed for generic rank at subleading order in the large central charge regime.
charge expansion. Finally, section 4 concludes the analysis by checking agreement with the universal relation (1.5). Some technical tools are briefly collected in two appendices.

2 Tree-level 4-point functions in $A_n$ Toda theory in AdS$_2$

We shall consider the $A_n$ Toda theory in AdS$_2$ with classical action, see for instance [36],

$$S_n = \int d^2 x \sqrt{g} \left[ \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi + V_n(\phi) \right], \quad V_n(\phi) = \frac{1}{\beta^2} \sum_{i=1}^n q_i \epsilon^\beta \alpha_i \cdot \phi + \frac{1}{\beta} R \rho^\vee \cdot \phi, \quad (2.1)$$

where $\beta$ is a coupling and $V_n$ will be referred to as the potential. In the special case of $A_1$ the action reduces to the Liouville action, cf. (1.1). The field $\phi$ is an $n$-component multiplet of scalar fields, $\alpha_i$ are the simple roots of the Lie algebra $A_n$ and the Weyl vector $\rho^\vee$ satisfies $\alpha_i \cdot \rho^\vee = 1$ for all $i = 1, \ldots, n$. The numbers $q_i$ are taken to be the unique solution to the condition $\sum_{i=1}^n q_i \alpha_i \cdot \alpha_j = 2$, for $j = 1, \ldots, n$. The action (2.1) is Weyl invariant$^6$ and flat space integrability carries over to any conformally flat background. In particular we are interested in AdS$_2$ with unit radius and Poincaré coordinates

$$ds^2 = \frac{1}{z^2} (dz^2 + dt^2), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^+. \quad (2.2)$$

The kinetic part of the action (including mass terms) is diagonalized by going to a basis of normalized eigenvectors of the matrix $A_{ij} = q_k (A^{1/2})_{ki} (A^{1/2})_{kj}$, where $A$ is the (symmetric) Cartan matrix of the $A_n$ algebra

$$A_{ij} = \frac{2 \alpha_i \cdot \alpha_j}{|\alpha_i|^2} = \frac{1}{2} \alpha_i \cdot \alpha_j, \quad A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2
\end{pmatrix}. \quad (2.3)$$

The mass matrix $A_{ij}$ can be put in a simple tridiagonal form. To this aim, one writes the $A_n$ simple roots in the form $\alpha_1 = \sqrt{2} e_1, \alpha_2 = \beta_{21} e_1 + \beta_{22} e_2, \ldots, \alpha_p = \sum_{q=1}^p \beta_{pq} e_q$, where $e_1, \ldots, e_n$ are orthonormal vectors in $\mathbb{R}^n$. This gives the only non-zero elements

$$A_{p,p} = \frac{n+1}{2} + np - p^2, \quad A_{p+1,p+1} = A_{p+1,p} = \frac{1}{4} p (p+2) (p-n)^2. \quad (2.4)$$

In terms of the rotated fields $\phi \rightarrow \phi'$, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi' \cdot \partial_\mu \phi' + \frac{1}{2} \sum_{i=1}^n m_i^2 (\phi'_i)^2 + V_n(\phi'), \quad (2.4)$$

$^5$The numbers $q_i$ are not restricted to be integer since a shift of the scalar fields is understood and such that linear terms are removed from the action.

$^6$This requires a quantum shift $\frac{1}{2} \rightarrow \frac{1}{2} + \beta$ in the last term of the potential in (2.1). This correction will have no effects at the perturbative order of the calculations in this paper. Instead, it is an important ingredient in the loop corrections discussed in [23].
where the masses can be written \( m_i^2 = \Delta_i (\Delta_i - 1) \) with the simple pattern
\[
\{\Delta_i\} = 2, 3, 4, \ldots, n + 1.
\] (2.5)

According to the AdS/CFT dictionary, \( \Delta_i \) is the conformal dimensions of the dual boundary fields. Since they will turn out to be chiral fields, we shall often refer to \( \Delta_i \) as the spin quantum number. The non-polynomial potentials \( V_\beta(\phi) \) can be expanded in powers of \( \beta \) producing cubic, quartic and higher order couplings. It is convenient to relabel the fields by using their would-be conformal dimension \( \phi_i' \rightarrow \phi_i \). Besides, for the following discussion, it will be convenient to rescale \( \beta \rightarrow \sqrt{\frac{n(n+1)(n+2)}{6}} \beta \), depending on \( n \), in order to have the same \( \phi_i^3 \) coupling for all \( n \). With these conventions, in the first three cases \( n = 1, 2, 3 \), the explicit potentials read
\[
V_1(\phi_2) = \phi_2^2 + \frac{2}{3} \beta \phi_2^3 + \frac{1}{3} \beta^2 \phi_2^4 + \cdots,
\]
(2.6)
\[
V_2(\phi_2, \phi_3) = \phi_2^2 + 3 \phi_2^3 + \beta \left( \frac{2}{3} \phi_2^3 + 6 \phi_2^3 \phi_2 + \frac{1}{3} \beta^2 \phi_2^4 + \cdots \right) + \beta^2 \left( \frac{1}{3} \phi_2^4 + 6 \phi_2^3 \phi_2 + 3 \phi_2^4 \right) + \cdots,
\]
\[
V_3(\phi_2, \phi_3, \phi_4) = \phi_2^2 + 3 \phi_2^3 + 6 \phi_2^3 \phi_2 + \beta \left( \frac{2}{3} \phi_2^3 + 6 \phi_2^3 \phi_2 + 12 \phi_2^3 \phi_2 - 4 \phi_2^4 + 12 \phi_2^3 \phi_4 \right)
\]
\[+ \beta^2 \left( \frac{1}{3} \phi_2^4 + 6 \phi_2^3 \phi_2^2 + 12 \phi_2^4 \phi_2 - 8 \phi_2^4 \phi_2 + 24 \phi_2^3 \phi_2 \phi_2 + 5 \phi_2^4 + 14 \phi_2^3 + 24 \phi_2^3 \phi_4 \right) + \cdots.
\]

We want to discuss the 4-point function of the scalars dual to the first two higher spin fields, those with spin 3 and 4. The non-zero cases are the two diagonal and the one mixed 4-point functions
\[
\langle \phi_3 \phi_3 \phi_3 \phi_3 \rangle, \quad \langle \phi_3 \phi_3 \phi_4 \phi_4 \rangle, \quad \langle \phi_4 \phi_4 \phi_4 \phi_4 \rangle.
\] (2.7)

To compute them in the generic \( A_n \) Toda theory, we need some of the cubic and quartic couplings at generic \( n \). Not surprisingly, the non-zero couplings we shall need are only a finite set. Apart from the 3333, 3344, and 4444 contact Witten diagrams, we can have an exchange in one of three kinematical channels mediated by two cubic interactions. The 33s couplings are non-zero only for \( s = 2, 4 \) while the 44s couplings are non-zero only for \( s = 2, 4, 6 \). Finally, for the 3344 four-point function we also need the non-zero couplings 34s. Apart from \( s = 3 \), there is only \( s = 5 \). A systematic analysis of the potentials at increasing rank shows that we can write
\[
V_n(\phi) = \cdots + \beta \left( \frac{2}{3} \phi_2^3 + 6 \phi_2^3 \phi_2 + 12 \phi_2^4 \phi_2 + c_{3344} \phi_2^3 \phi_4 + c_{4444} \phi_4^2 \phi_2 + c_{3345} \phi_3 \phi_4 \phi_5 \right)
\]
\[+ \beta^2 \left( c_{3333} \phi_3^4 + c_{3344} \phi_4^2 \phi_4 + c_{4444} \phi_4^4 \right) + \cdots,
\] (2.8)

where the couplings are expressed by the following remarkably simple rational expressions\(^7\)
\[
(c_{334})^2 = \frac{1728}{7} \frac{(n+4)(n-2)}{(n-1)(n+3)}, \quad c_{3333} = \frac{15}{7} \frac{3n^2 + 6n - 17}{(n+3)(n-1)},
\]
\[
(c_{444})^2 = \frac{448(n^2 + 2n - 18)^2}{3(n-2)(n-1)(n+3)(n+4)}, \quad (c_{446})^2 = \frac{30000(n-4)(n-3)(n+5)(n+6)}{11(n-2)(n-1)(n+3)(n+4)}.
\]

\(^7\)The formulas for couplings involving the field \( \phi_k \) are valid if \( n \geq k - 1 \), otherwise that field is simply absent.
\[
c_{3344} = \frac{12(7n^2+14n-81)}{(n-1)(n+3)}, \quad c_{4444} = \frac{56(9n^4+36n^3-238n^2-548n+2316)}{11(n-2)(n-1)(n+3)(n+4)},
\]
\[
(c_{345})^2 = \frac{24000(n-3)(n+5)}{7(n-1)(n+3)}, \quad (c_{3333})^2 = \frac{24000(n-3)(n+5)}{7(n-1)(n+3)}.
\] (2.9)

Notice that in the case of affine Toda theories, the coupling structure is somewhat simpler and there are known selection rules for the cubic couplings, as well as (recursion) relations for their values [37–41]. Expressions (2.9) are valid for the finite Lie algebra Toda theories and are novel necessary ingredients.

To compute 4-point functions like (2.7) at leading order, we need the bulk-to-bulk propagator for a scalar field in AdS\(_2\) with mass term \(m^2 = \Delta(\Delta - 1)\). It is
\[
G_\Delta(t_1, z_1; t_2, z_2) = \mathcal{C}_\Delta u^{-\Delta} \binom{\Delta}{2} F_1 \left( \Delta, \Delta, 2\Delta; -\frac{1}{u} \right), \quad \mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2 \sqrt{\pi} \Gamma(\Delta + \frac{1}{2})},
\] (2.10)
where the chordal distance is \(u = \frac{(t_1-t_2)^2 + (z_1-z_2)^2}{z_1 z_2}\). The bulk-to-boundary propagator is, after suitable field rescaling,
\[
K_\Delta(t_0; t, z) = \mathcal{C}_\Delta \left[ \frac{z}{z^2 + (t-t_0)^2} \right]^\Delta,
\] (2.11)
and is associated with fields whose 2-point function has coefficient \(\mathcal{C}_\Delta\). Since we shall be interested in fields with unit normalized 2-point function, one factor \(\mathcal{C}_\Delta^{-1/2}\) will be attached to any boundary-to-bulk line. Some technical tools for the evaluation of AdS integrals are collected in appendix B.

### 2.1 Four point functions involving \(\Delta = 3, 4\) fields in the \(A_n\) Toda theory

We now compute the 4-point functions (2.7) by evaluating the associated Witten diagrams and using the general couplings in (2.9). The leading order will be given by two boundary-to-boundary propagators and is a disconnected contribution independent on \(\beta\). This part is almost trivial and will be discussed in the end. Instead, we focus on the non-trivial connected contribution. At leading order, this starts at quadratic order \(O(\beta^2)\).

#### 2.1.1 \(\Delta = 3\) boundary correlator \(\langle \varphi_3 \varphi_3 \varphi_3 \varphi_3 \rangle\)

From the couplings in (2.8), we have the following (Witten) Feynman rules for the relevant cubic vertices and quartic coupling
\[
3 \quad 2 = -12 \beta, \quad 3 \quad 4 = -2 \beta c_{334}, \quad 3 \times 3 \times 3 = -4! \beta^2 c_{3333},
\]
where the value of the couplings \(c_{334}\) and \(c_{3333}\) has been given in (2.9). The four-point function is then given by the sum of diagrams in figure 1. It can be computed in terms of
the $\mathcal{D}$ functions defined in appendix B. We find the connected contribution

$$
\mathcal{C}_3^{-2}\langle \varphi_3(t_1) \cdots \varphi_3(t_4) \rangle_{\text{conn}}
= (-12 \beta^2 (W^*_{3333:2} + W^d_{3333:2} + W^w_{3333:2})
+ (-2 \beta c_{334})^2 (W^*_{3333:4} + W^d_{3333:4} + W^w_{3333:4}) - 4! \beta^2 c_{3333} D_{3333}
= \frac{15 \pi \beta^2}{512} \left[ 7(c^2_{3333} + 36)D_{2,2,2,3,3,2} + 7(c^2_{3333} + 36)D_{2,2,3,2,3,2} + \frac{756 c_{3333} D_{3,3,3,3,3}}{\rho_3^h t_{24}^{+}} \right] + \frac{180 D_{1,1,1,3,3,3,3}}{t_{13}^{+} t_{24}^{+}}
+ \frac{180 D_{1,1,3,1,3,3,3}}{t_{12}^{+} t_{13}^{+} t_{24}^{+}} + \frac{180 D_{1,3,3,1,3,3}}{t_{13}^{+} t_{24}^{+}} \right].
$$

(2.12)

Using the explicit values of the relevant $\mathcal{D}$ functions, cf. (B.6), we may write

$$
\mathcal{C}_3^{-2}\langle \varphi_3(t_1) \cdots \varphi_3(t_4) \rangle_{\text{conn}} = \frac{\beta^2}{t_{12}^{+} t_{34}^{+}} G^{\text{AdS}}_{3333}(\chi), \quad \chi = \frac{t_{12}^{+} t_{34}^{+}}{t_{13}^{+} t_{24}^{+}},
$$

(2.13)

with

$$
G^{\text{AdS}}_{3333}(\chi) = -(c^2_{334} - 72 c_{3333} + 216) \frac{3 \pi \chi^6 (\chi^2 - \chi + 1)(2 \chi^2 - 7 \chi + 7)}{256 (1 - \chi)^3} \log \chi
- \frac{3}{256} \pi (c^2_{334} - 72 c_{3333} + 216) \chi (\chi^2 - \chi + 1)(2 \chi^2 + 3 \chi + 2) \log (1 - \chi)
- \frac{3 \pi \chi^2}{1024 (1 - \chi)^4} \left[ c^2_{334} (8 \chi^6 - 24 \chi^5 + 13 \chi^4 + 14 \chi^3 + 13 \chi^2 - 24 \chi + 8)
- 48 c_{3333} (12 \chi^6 - 36 \chi^5 + 37 \chi^4 - 14 \chi^3 + 37 \chi^2 - 36 \chi + 12)
- 36 (2 \chi^6 - 6 \chi^5 - 3 \chi^4 + 16 \chi^3 - 3 \chi^2 - 6 \chi + 2) \right].
$$

(2.14)

The logarithmic terms $\sim \log \chi, \log (1 - \chi)$ vanish using the explicit form of $c_{334}$ and $c_{3333}$, see (2.9). This non-trivial fact will have an explanation in terms of the boundary CFT and will be associated with the absence of anomalous dimensions of the various dual fields. The remaining expression for the 4-point function takes then the following compact form

$$
G^{\text{AdS}}_{3333}(\chi) = \frac{675 \pi}{256 (n - 1)(n + 3)} \frac{\chi^2}{(1 - \chi)^4} \left[ 2(n - 1)(n + 3)(1 - 3 \chi - 3 \chi^5 + \chi^6)
+ (9 n^2 + 18 n - 43) \chi^2 (1 + \chi^2) - 8 (n^2 + 2 n - 7) \chi^3 \right],
$$

(2.15)

\[ -7 \]
that obeys the correct crossing relations

\[ G_{3333}^{\text{AdS}}(\chi) = \frac{\chi^6}{(1-\chi)^6} G_{3333}^{\text{AdS}}(1-\chi), \quad G_{3333}^{\text{AdS}}(\chi) = G_{3333}^{\text{AdS}} \left( \frac{\chi}{\chi - 1} \right). \]  

(2.16)

### 2.1.2 \( \Delta = 4 \) boundary correlator \( \langle \varphi_4 \varphi_4 \varphi_4 \varphi_4 \rangle \)

In this case, again from (2.8), we have the following cubic vertices and quartic coupling

\[
\begin{align*}
4 \rightarrow 2 &= -2\beta, \\
4 \rightarrow 4 &= -3! \beta c_{444}, \\
4 \rightarrow 6 &= -2 \beta c_{446}, \\
4 \times 4 &= -4! \beta^2 c_{4444},
\end{align*}
\]

where the couplings \( c_{444}, c_{446}, \) and \( c_{4444} \) have been given in (2.9). The four-point (connected) function is then given by the diagrams in figure 2 and reads

\[ \mathcal{E}_4^{-2} \langle \varphi_4(t_1) \cdots \varphi_4(t_4) \rangle_{\text{conn}} \]

\[
= (-24\beta)^2(W_{4444,2}^s+W_{4444,6}^u) + (-3! \beta c_{444})^2(W_{4444,4}^s+W_{4444,4}^u) + (-2 \beta c_{446})^2(W_{4444,6}^s+W_{4444,6}^u)-4! \beta^2 c_{4444} D_{4444}
\]

\[
\begin{equation}
= \frac{\pi \beta^2}{6144} \left[ 315(9c_{444}^2+224)D_{2.2,4,4.4} + \frac{315(9c_{444}^2+224)}{t_{13}^4 t_{24}^4} + \frac{315(9c_{444}^2+224)}{t_{13}^4 t_{24}^4} \right]
\end{equation}

\[
+ \frac{385(c_{444}^2 c_{446}^2+144)D_{3.3,4,4}}{t_{13}^4 t_{24}^4} + \frac{385(c_{444}^2 c_{446}^2+144)}{t_{13}^4 t_{24}^4} + \frac{385(c_{444}^2 c_{446}^2+144)}{t_{13}^4 t_{24}^4}.
\]

(2.17)

This may be written

\[
\mathcal{E}_4^{-2} \langle \varphi_4(t_1) \cdots \varphi_4(t_4) \rangle_{\text{conn}} = \frac{\beta^2}{t_{12}^4 t_{34}^4} G_{4444}^{\text{AdS}}(\chi),
\]

(2.19)

where

\[ G_{4444}^{\text{AdS}}(\chi) = \frac{\pi \chi^8}{6144(1-\chi)^7} \left[ -5121c_{444}^2-110c_{446}^2+30888c_{4444}^2+350496 \\
+3(5121c_{444}^2+110c_{446}^2-30888c_{4444}^2+350496)\chi \right]. \]
strange couplings. This is due to the identities that obey the correct crossing relations
\begin{equation}
\begin{split}
(1-\chi) \log \chi + \frac{\pi \chi}{6444} \left[ -20(45 c_{444}^2 + 2 c_{446}^2 - 360 c_{4444} + 4320) \right] \log (1-\chi) \\
+ \frac{\pi \chi (1-\chi)^2}{36864 (1-\chi)^6} \left[ -120(45 c_{444}^2 + 2 c_{446}^2 - 360 c_{4444} + 400) \right] \\
+ 360(5 c_{444}^2 + 2 c_{446}^2 - 360 c_{4444} + 400) \chi + (-4851 c_{444}^2 + 140 c_{446}^2 + 44208 c_{4444} - 72576) \chi^2 \\
-2(8649 c_{444}^2 + 460 c_{446}^2 - 63792 c_{4444} + 400) \chi^3 + (-4851 c_{444}^2 - 140 c_{446}^2 + 44208 c_{4444} - 72576) \chi^4 \\
+ 360(5 c_{444}^2 + 2 c_{446}^2 - 360 c_{4444} + 400) \chi^5 - 120(45 c_{444}^2 + 2 c_{446}^2 - 360 c_{4444} + 400) \chi^6.
\end{split}
\end{equation}

Again, the logarithmic terms \( \sim \log \chi, \log (1-\chi) \) vanish using the explicit form of the couplings. This is due to the strange identities
\begin{equation}
(c_{444})^2 = \frac{8}{21} (-112 + 11 c_{4444}), \quad (c_{446})^2 = \frac{600}{7} (-14 + c_{4444}),
\end{equation}
that can be easily checked using (2.9). In conclusion, we find
\begin{equation}
G_{4444}^{AdS}(\chi) = \frac{245 \pi}{192 (n-1)(n-2)(n+3)(n+4)} \frac{\chi^2 (1-\chi)^2}{(1-\chi)^6} \left[ 10(n-2)(n-1)(n+3)(n+4) (1-3\chi-3\chi^5+\chi^6) \\
+ 9(2n^4 + 8n^3 - 39n^2 - 94n + 348) \chi^2 (1+\chi^2) + 2(7n^4 + 28n^3 + 176n^2 + 296n - 2532) \chi^3 \right],
\end{equation}
that obeys the correct crossing relations
\begin{equation}
G_{4444}^{AdS}(\chi) = \frac{\chi^8}{(1-\chi)^8} G_{4444}^{AdS}(1-\chi), \quad G_{4444}^{AdS}(\chi) = G_{4444}^{AdS} \left( \frac{\chi}{\chi - 1} \right).
\end{equation}

### 2.1.3 Mixed boundary correlator \( \langle \varphi_3 \varphi_3 \varphi_4 \varphi_4 \rangle \)

Finally, we can evaluate the mixed 4-point function \( \langle \varphi_3 \varphi_3 \varphi_4 \varphi_4 \rangle \) by the same methods. We do not repeat all the steps to get the final result, since they are completely similar to those for the diagonal 4-point functions. From the connected diagrams in figure 3 we can write
\begin{equation}
C_3^{-1} C_4^{-1} \langle \varphi_3(t_1) \varphi_3(t_2) \varphi_4(t_3) \varphi_4(t_4) \rangle_{\text{conn}} = \frac{\beta^2}{\hat{H}_{12} \hat{H}_{34}} G_{5444}^{AdS}(\chi),
\end{equation}

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Figure 3. Tree diagrams contributing the mixed boundary correlator $\langle \phi_3 \phi_3 \phi_4 \phi_4 \rangle$.

with

$$G_{3344}^{\text{AdS}}(\chi) = \frac{105 \pi}{128 (n-1)(n+3)} \frac{\chi^2}{(1-\chi)^4} \left[ 10(n-1)(n+3)(1-3\chi)+4(n-2)(n+4)\chi^5(-8+3\chi) 
+(-303+82n+41n^2)\chi^2-8(-57+8n+4n^2)\chi^3+(-469+86n+43n^2)\chi^4 \right],$$

(2.25)

obeying the single crossing relation

$$G_{3344}^{\text{AdS}}(\chi) = G_{3344}^{\text{AdS}}\left(\frac{\chi}{\chi-1}\right).$$

(2.26)

3 Chiral 4-point functions in $W_n$ Virasoro extensions

We now turn to the CFT side and discuss how to compute the relevant (chiral) 4-point functions in the $W_3$ and $W_4$ extended Virasoro algebras. In particular, we analyze the structure of their large $c$ expansion and show how to compute the subleading $O(c)$ contribution to the spin 3 and spin 4 correlators in the generic $W_n$ algebra.

3.1 General structure of 4-point functions in 2d CFT

Let us review some basic facts about the conformal block decomposition of 4-point functions of primary fields, see for instance [42–44]. Let us consider four chiral primaries with dimensions $\Delta_i$ and define $G(z)$ by

$$\langle \mathcal{O}_{\Delta_1}(\infty) \mathcal{O}_{\Delta_1}(1) \mathcal{O}_{\Delta_3}(z) \mathcal{O}_{\Delta_4}(0) \rangle = \lim_{w \to \infty} w^{2(\Delta_1+\Delta_2)} \langle \mathcal{O}_{\Delta_1}(w) \mathcal{O}_{\Delta_2}(1) \mathcal{O}_{\Delta_3}(z) \mathcal{O}_{\Delta_4}(0) \rangle = \frac{1}{z^{2(\Delta_3+\Delta_4)}} G(z).$$

(3.1)

The variable $z$ is again the cross ratio $\chi = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ for the configuration $z_i = (\infty, 1, z, 0)$. As we discussed in the Introduction, the reason why we consider chiral primaries is that we want to compare with the 1d boundary of AdS$_2$ that will be parametrized by $z$ restricted to be real $z = \overline{z}$.

The function $G(z)$ in (3.1) may be expanded in the $s$-channel by summing over the exchanged primaries $\mathcal{O}_p$,

$$G(z) = \sum_p C_{12p} C_{34p} \mathcal{F}(\Delta, \Delta_p; z),$$

(3.2)

where $C_{abc}$ are the three point function coefficients for primaries with unit normalization of 2-point function, and an obvious dependence on the central charge is understood.
Virasoro conformal block $\mathcal{F}(\Delta, \Delta_p; z)$ is fully determined by Virasoro symmetry. It is convenient to present it as a sum over contributions from the level $q$ quasi-primaries appearing in the Verma module $M(\Delta_p)$ built upon $O_p$. One has

$$\mathcal{F}(\Delta, \Delta_p; z) = z^{\Delta_p} \sum_{q=0}^{\infty} \chi_q (\Delta, \Delta_p) z^q 2F_1(\Delta_p+q+\Delta_{12}, \Delta_p+q+\Delta_{34}, 2(\Delta_p+q); z), \quad \Delta_{ij} = \Delta_i - \Delta_j,$$

where the expansion coefficients $\chi_q (\Delta, \Delta_p)$ are fully determined by the Virasoro algebra, i.e. they can be computed by summing over the explicit quasi-primaries that appear in the Virasoro Verma module built on $O_{-\Delta_p[0)}$. One important special case is the contribution from the identity $O_p = I$. In this case, we have $\Delta_p \to 0$ and the constraint $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$. Then,

$$\mathcal{F}(\Delta, 0; z) = \sum_{q=0}^{\infty} \chi_q (\Delta, 0) z^q 2F_1(q, q, 2q; z),$$

with

$$\chi_0 (\Delta, 0) = 1, \quad \chi_2 (\Delta, 0) = \frac{2 \Delta_1 \Delta_3}{c}, \quad \chi_4 (\Delta, 0) = \frac{10 (\Delta_1^2 + \Delta_3^2) (\Delta_3 + \Delta_4)}{c(5c+22)},$$

$$\chi_6 (\Delta, 0) = \frac{(14\Delta_1^2 + \Delta_1)(14\Delta_3^2 + \Delta_3)}{63c(70c+29)}$$

$$+ \frac{4\Delta_1 \Delta_3 [c(70\Delta_1^2 + 42\Delta_1 + 8) + 29\Delta_3^2 - 57\Delta_3 - 2] + c(70\Delta_3^2 + 42\Delta_3 + 8) + 29\Delta_3^2 - 57\Delta_3 - 2]}{3c(2c-1)(5c+22)(7c+68)(70c+29)},$$

and so forth. Remarkably, one has the general result

$$\lim_{c \to \infty} \mathcal{F}(\Delta, \Delta_p; z) = z^{\Delta_p} 2F_1(\Delta_p + \Delta_{12}, \Delta_p + \Delta_{34}, 2\Delta_p; z),$$

i.e. the Virasoro block reduces to the global block. This fact will have a role in the following discussion of the 4-point function analysis as a boundary 1d CFT, but we shall need some refinement, see later (3.23).

### 3.2 Virasoro extensions and $\mathcal{W}_n$

We are interested in CFT with extended Virasoro symmetry associated with additional chiral generator $Q_s$ with integer spin $s \geq 3$ [45]. This means that we have the singular operator expansions (OPE)

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{T'(0)}{z}, \quad T(z)Q_s(0) \sim \frac{s}{z^2} Q_s(0) + \frac{1}{z} Q'_s(0),$$

where $T$ is the stress-energy tensor, and $c$ the central charge. The conformal Ward identities read

$$\langle T(z_1)T(z_2) \cdots T(z_N) Q_{s_1}(w_1) \cdots Q_{s_M}(w_M) \rangle$$

$$= \sum_{i=2}^{N} \frac{c}{2(z_1 - z_i)^4} \langle T(z_2) \cdots T(z_{i-1}) T(z_{i+1}) \cdots T(z_N) Q_{s_1}(w_1) \cdots Q_{s_M}(w_M) \rangle$$

...
\[ + \left\{ \sum_{i=2}^{N} \left[ \frac{2}{(z_1 - z_i)^2} + \frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} \right] + \sum_{j=1}^{M} \left[ \frac{s_j}{(z_1 - w_j)^2} + \frac{1}{z_1 - w_j} \frac{\partial}{\partial w_j} \right] \right\} \times (T(z_2) \cdots T(z_N) Q_{s_1}(w_1) \cdots Q_{s_M}(w_M)). \tag{3.8} \]

Without higher spin fields, they imply the well known $T$ correlators
\[ \langle T(z_1)T(z_2) \rangle = \frac{c}{2 z_{12}^4}, \tag{3.9} \]
\[ \langle T(z_1)T(z_2)T(z_3) \rangle = \frac{3}{2} \left( \frac{2}{z_{12}^3 z_{13}^2 z_{23}^2} \right) \langle T(z_2)T(z_3) \rangle = \frac{c}{2 z_{12}^2 z_{13}^2 z_{23}^2}, \]
\[ \langle T(z_1)T(z_2)T(z_3)T(z_4) \rangle = \frac{c^2}{4} \left( \frac{1}{z_{12}^2 z_{13}^2 z_{14}^2} + \frac{1}{z_{13}^2 z_{14}^2 z_{23}^2} + \frac{1}{z_{14}^2 z_{23}^2 z_{12}^2} \right) + c \left( \frac{1}{z_{12}^2 z_{23}^2 z_{24}^2} + \frac{1}{z_{13}^2 z_{24}^2 z_{34}^2} + \frac{1}{z_{14}^2 z_{34}^2 z_{13}^2} \right). \tag{3.10} \]

The simplest case with higher spin fields is $\langle T(z_1)Q_s(z_2)Q_s(z_3) \rangle$. Assuming the standard (Zamolodchikov) normalization $\langle Q_s(z)Q_s(0) \rangle = \frac{c}{2 z^{2s-2}}$, we obtain, using (3.8), the expression, similar to (3.10),
\[ \langle T(z_1)Q_s(z_2)Q_s(z_3) \rangle = \frac{c}{2 z_{12}^2 z_{13}^2 z_{23}^2}. \tag{3.11} \]

For an extension with spins $s', s'', \ldots$, usually denoted by $W(2, s, s', \ldots)$, one can postulate a set of OPEs for all the generators (including the stress-tensor). To be consistent, they should be equivalent to associativity in correlators, or Jacobi identities. This is by far a non-trivial constraint, for a review of some basic constructions see for instance [34, 46].

Generally speaking, there are classes of solutions valid for generic central charges (apart from isolated special singular values) as well as specific solutions that are valid only at certain central charges. Here we consider the simplest case of the former type, i.e. the so-called quantum $W_n \equiv WA_{n-1}$ algebra that is the simplest example of a Casimir algebra [35, 47–49] and contains higher spin generators $Q_s$ with spin $s = 3, 4, \ldots, n$. As a preliminary step, we now discuss in some details the 4-point functions for spin 3 and spin 4 generators in $W_3$ and $W_4$. Then, we discuss their large $c$ limit, and its generalization to all $W_n$.

### 3.2.1 4-point functions in the $W_3$ algebra

The simplest Virasoro extension is $W_3 \equiv W(2, 3)$ first discussed in [45]. Denoting by $Q_3$ the spin-3 primary, we have the fusion data (singular OPE between local conformal families)
\[ Q_3 Q_3 = \frac{c}{3} [\mathbb{I}], \tag{3.12} \]

where $[\mathbb{I}]$ is the conformal family of the identity operator. Making explicit the descendents, this means the following singular OPE
\[ Q_3(z) Q_3(0) = \frac{c}{3 z^6} \left( \frac{2 T(0)}{z^4} + \frac{T''(0)}{z^3} + \frac{1}{z^2} \left[ \frac{3}{10} T''(0) + \frac{32}{22 + 5c} \Lambda(0) \right] + \frac{1}{z} \left[ \frac{1}{15} T'''(0) + \frac{16}{22 + 5c} \Lambda'(0) \right] \right) \cdots, \tag{3.13} \]
where $\Lambda(z)$ is the quasi-primary composite operator $\Lambda = (TT) - \frac{3}{10}T''$. The four point function $\langle Q_3 Q_3 Q_3 Q_3 \rangle$ can be computed by exploiting the fact that the OPE singularity in $Q_3 Q_3$ is local, i.e. a pole due to integer spin. Assuming the usual regularity condition at infinity for a spin $s$ field $Q_s(z) \sim z^{-2s}$ for $z \to \infty$, the four point function is fully captured by the poles predicted by the conformal Ward identity. This procedure is illustrated with examples in [45]. In particular, for the 4-point function $\langle Q_3 Q_3 Q_3 Q_3 \rangle$, we get the exact result for the $G$-function in (3.1)

$$G_{3333}(z) = \frac{c^2}{9} \left[ 1 + z^6 + \frac{z^6}{(1-z)^6} \right] + c \left[ 2z^4 + 2z^3 + \frac{9z^2}{5} + \frac{8z}{5} - \frac{96}{5(1-z)} + \frac{99}{5(1-z)^2} \right] - \frac{10}{(1-z)^3} + \frac{2}{(1-z)^4} + \frac{37}{5} + \frac{512c}{5(22 + 5c)} \frac{z^4}{(1-z)^2}. \tag{3.14}$$

This expression is fully equivalent to eq. (3.23) of [45]. As a further check, one can verify the correct crossing relations

$$G_{3333}(z) = G_{3333} \left( \frac{z}{z-1} \right) = \frac{z^6}{(1-z)^6} G_{3333}(1-z). \tag{3.15}$$

### 3.2.2 4-point functions in the $\mathcal{W}_4$ algebra

The $\mathcal{W}_4 \equiv \mathcal{W}(2, 3, 4)$ algebra fusion rules are discussed, e.g. in [50]. Denoting by $Q_3$ and $Q_4$ the spin-3 and 4 primaries, we have the OPEs

$$Q_3 Q_3 = \frac{c}{3} [I] + \gamma [Q_4],$$

$$Q_3 Q_4 = \frac{3}{4} \gamma [Q_3],$$

$$Q_4 Q_4 = \frac{c}{4} [I] + \mu [Q_4] + \lambda [\Phi_6], \tag{3.16}$$

where $\Phi_6 = (Q_3 Q_3) + \cdots$ is the dimension 6 (composite) primary appearing in (A.4). Up to automorphisms changing the sign of $\gamma$, the constants in (3.16) are

$$\gamma = \pm \frac{4}{3} \sqrt{\frac{3(7c+114)(c+2)}{(5c+22)(c+7)}}, \quad \mu = -12 \frac{c^2 + c + 218}{(5c+22)(c+7)\gamma}, \quad \lambda = \frac{45(5c+22)}{2(7c+114)(c+2)}. \tag{3.17}$$

The explicit form of $\Phi_6$ is

$$\Phi_6 = (Q_3 Q_3) + \frac{(5c+76)\sqrt{(c+2)(7c+114)(c+7)(5c+22)}}{9\sqrt{3(c+24)}} Q_4'' + \frac{88\sqrt{(c+2)(7c+114)(c+7)(5c+22)}}{3\sqrt{3(c+24)}} (T Q_4)$$

$$+ \frac{1504 - 2c(67c + 178)}{(2c - 1)(5c + 22)(7c + 68)} (T'' T) - \frac{c(225c + 1978) + 776}{2(2c - 1)(5c + 22)(7c + 68)} (T' T')$$

$$- \frac{16(191c + 22)}{3(2c - 1)(5c + 22)(7c + 68)} (TT T) - \frac{(c - 8)[5(c+12) + 4]}{6(2c - 1)(5c + 22)(7c + 68)} T^{(4)}. \tag{3.18}$$

*Here and later, we shall denote by $(\cdots)$ the (conformally) normal ordered composite operators, see for instance [34]. There should not be confusion with ordinary brackets.*
with squared norm

\[ \langle \Phi_6(z) \Phi_6(0) \rangle = \frac{4(c - 1)c(c + 2)(c + 13)(3c + 116)(7c + 114)}{27(c + 7)(c + 24)(2c - 1)(7c + 68)} \frac{1}{z^{12}}. \]  

(3.19)

We want to compute the \( G \)-functions associated with the correlators

\[ \langle Q_3 Q_3 Q_3 Q_3 \rangle, \quad \langle Q_3 Q_3 Q_4 \rangle, \quad \langle Q_4 Q_4 Q_4 \rangle. \]  

(3.20)

A tedious but straightforward calculation gives\(^9\)

\[
G_{3333}(z) = \frac{c^2}{9} [1 + z^6 + \frac{z^6}{(1-z)^6}] + c \left[ 2z^4 + 2z^3 + \frac{11}{2} z^2 + \frac{16}{3} z - \frac{80}{3(1-z)} + \frac{65}{3(1-z)^2} \right.
\]

\[
- \left. \frac{10}{(1-z)^3} + \frac{2}{(1-z)^4} + 13 \right] + \frac{100c}{3(c+7)} \frac{z^4}{(1-z)^2},
\]

\[
G_{3344}(z) = \frac{c^2}{12} + c \left[ \frac{7z^4}{5} + \frac{28z^3}{15} + 2z^2 + 2z + \frac{7}{5(1-z)^4} - \frac{112}{15(1-z)^3} + \frac{16}{(1-z)^2} - \frac{86}{5(1-z)} + \frac{109}{15} \right]
\]

\[
\left. + \frac{c}{15(7+c)(22+5c)} \frac{z^4}{(1-z)^4} \right] [2 (2844-5688z+3092z^2-248z^3+93z^4)
\]

\[
+ c (2484-4968z+4412z^2-1928z^3+723z^4)]
\]

\[
G_{4444}(z) = \frac{c^2}{16} \left[ 1 + z^8 + \frac{z^8}{(1-z)^8} \right] + c \left[ 2z^6 + 2z^5 + \frac{279z^4}{140} + \frac{139z^3}{70} + \frac{55z^2}{28} + \frac{27z}{14} \right.
\]

\[
\left. + \frac{2}{(1-z)^6} - \frac{14}{(1-z)^5} + \frac{5879}{140(1-z)^4} - \frac{4897}{70(1-z)^3} + \frac{9783}{140(1-z)^2} + \frac{585}{14(1-z)} + \frac{831}{70} \right]
\]

\[
\left. + \frac{9c}{140(2+c)(7+c)(22+5c)(114+7c)} \right] \frac{z^4(1-z+z^2)^2}{(1-z)^4},
\]  

(3.21)

These obey the exact crossing relations

\[
G_{3333}(z) = G_{3333} \left( \frac{z}{z-1} \right) = \frac{z^6}{(1-z)^6} G_{3333}(1-z),
\]

\[
G_{4444}(z) = G_{4444} \left( \frac{z}{z-1} \right) = \frac{z^8}{(1-z)^8} G_{4444}(1-z),
\]

\[
G_{3344}(z) = G_{3344} \left( \frac{z}{z-1} \right).
\]  

(3.22)

\subsection*{3.3 Large \( c \) analysis}

The \( G \)-functions in (3.14) and (3.21) may be expanded at large central charge

\[ G(z) = c^2 G_0(z) + c G_1(z) + \mathcal{O}(c^0). \]  

(3.23)

The \( \mathcal{O}(c^2) \) contributions are obvious, they come from disconnected contributions where two pairs of fields fuse into the identity. The next-to-leading \( \mathcal{O}(c) \) terms, i.e. \( G_1(z) \), display a

\(^9\)The package [51] is useful for such computations.
certain regularity and structural similarity. Finally, the NNLO contributions \( \mathcal{O}(e^0) \) appear to be more involved, but for our present purposes we are interested in the leading and next-to-leading terms.

It is useful to analyze the small \( z \) expansion of the \( G \) functions in terms of conformal blocks at large \( c \). The wealth of data we want to reproduce is summarized by the following expansions — we add an algebra suffix for better clarity —

\[
G^w_{3333}(z) = c^2 \left( \frac{1}{9} + \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + c \left( \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + c \left( \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + \mathcal{O}(e^0),
\]

\[
G^w_{3344}(z) = c^2 \left( \frac{1}{9} + \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + c \left( \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + \mathcal{O}(e^0),
\]

\[
G^w_{4444}(z) = c^2 \left( \frac{1}{16} + \frac{1}{8} z^8 + \cdots \right) + c \left( \frac{2}{9} z^6 + \frac{2}{3} z^7 + \frac{7}{3} z^8 + \cdots \right) + \mathcal{O}(e^0).
\]

In order to match (3.24) and the general representation (3.2), we need the large \( c \) expansion of Virasoro blocks. Let us focus on the case

\[
\Delta_1 = \Delta_2 = \Delta, \quad \Delta_3 = \Delta_4 = \Delta'.
\]

A systematic expansion of the Virasoro conformal block at large \( c \) and fixed dimensions \( \Delta, \Delta_p \) has been computed in [52–54]. The conformal block can be written

\[
\mathcal{F}(\Delta, \Delta', \Delta_p; z) = z^{\Delta_p} \left[ F_0(\Delta_p) + \frac{1}{c} F_1(\Delta, \Delta'; \Delta_p) + \cdots \right],
\]

\[
F_0(\Delta_p) = 2 F_1(\Delta_p, \Delta_p, 2 \Delta_p; z),
\]

\[
F_1(\Delta, \Delta'; \Delta_p) = 12 \left[ f_a(\Delta_p) \Delta \Delta' + f_b(\Delta_p) (\Delta + \Delta') + f_c(\Delta_p) \right],
\]

where the explicit functions \( f_a(\Delta_p) \) may be found in convenient form in [55]. In particular, for the vacuum block \( \Delta_p = 0 \) one has simply

\[
F_1(\Delta, \Delta'; 0) = -12 \Delta \Delta' \left[ 2 + \frac{2}{z} \log(1 - z) \right].
\]

Let us now discuss the various cases in (3.24) from this perspective and by means of these tools.

### 3.3.1 The \( W_3 \) case

We begin with a finite \( c \) analysis. The simple fusion algebra (3.12) implies that \( G^w_{3333}(z) \) starts with the vacuum block, cf. (3.2), and continues with other primary contributions that belong to the regular part of the OPE. So we expect

\[
G^w_{3333}(z) = \frac{c^2}{9} \mathcal{F}(\{3, 3, 3, 3\}, 0; z) + \text{other primary contributions}. \tag{3.28}
\]
The first primary is \( \Phi_6 = (Q_3 Q_3) + \cdots \) and has dimension 6, cf. (A.3). The explicit form of this primary, normalized in order to have unit 2-point function, is

\[
\Phi_6 = \frac{3}{2} \sum \frac{(2c - 1)(5c + 22)(7c + 68)}{c(c + 2)(c + 23)(5c - 4)(7c + 114)} \left[ (Q_3 Q_3) + \frac{1504 - 2c(67c + 178)}{(2c - 1)(5c + 22)(7c + 68)} (T'' T) - \frac{c(225c + 1978) + 776}{2(2c - 1)(5c + 22)(7c + 68)} (T' T') - \frac{16(191c + 22)}{3(2c - 1)(5c + 22)(7c + 68)} (T, T, T) \right]
\]

This is fully consistent with (3.2). Indeed, from (3.14) we can write

\[
G_{3333}^W(z) = \frac{c^2}{9} \left[ F_0(z) + \frac{18}{c} F_2(z) + \frac{4608}{5 c(22 + 5 c)} F_4(z) + \frac{9710 + 2189 c + 70 c^2}{7 c(22 + 5 c)} F_6(z) + O(z^8) \right],
\]

where \( F_q(z) = z^q \cdot F_1(q, q, 2q; z) \). Comparing the coefficients of the hypergeometric functions with (3.5) at \( \Delta_1 = \Delta_3 = 3 \) we see that we can continue (3.28) as

\[
G_{3333}^W(z) = \frac{c^2}{9} F([3, 3, 3, 3], 0; z) + \frac{4 c (c+2)(c+23)(5c-4)(7c+114)}{9(2c-1)(5c+22)(7c+68)} F([3, 3, 3, 3], 6; z) + O(z^8).
\]

The coefficients are in agreement with (3.2) taking into account that \( \langle Q_3(z_1) Q_3(z_2) \rangle = \frac{c^2}{9 z_1 z_2} \), and that the regular part of the OPE \( Q_3(z) Q_3(0) \) starts with \( (Q_3 Q_3) + \cdots \).

The large \( c \) limit of (3.31) may be computed by expanding both the coefficients and the conformal blocks. This gives

\[
G_{3333}^W(z) = \frac{c^2}{9} \left[ 1 + \frac{1}{c} F_1(3, 3; 0) + \cdots \right] + \left[ \frac{2 c^2}{9} + \frac{209 c}{35} + \cdots \right] z^6 \left[ F_0(6) + \frac{1}{c} F_1(3, 3, 6) + \cdots \right] + O(z^8).
\]

Using the explicit expressions

\[
F_1(3, 3; 0) = 108 \left[ -\frac{(2-z) \log(1-z)}{z} - 2 \right] = 18 z^2 + 18 z^3 + \frac{81 z^4}{5} + \frac{72 z^5}{5} + \frac{90 z^6}{7} + \frac{81 z^7}{7} + \frac{21 z^8}{2} + \cdots,
\]

\[
F_1(3, 3; 6) = \frac{997920}{z^{14}} (z-2)(z^4-28z^3+154z^2-252z+126) Li_2(z)
\]

one indeed checks that (3.32) reads

\[
G_{3333}^W(z) = c^2 \left[ \frac{1}{9} + \frac{2}{9} z^6 + \frac{2}{3} z^7 + O(z^8) \right] + c \left[ 2 c^2 + 2 z^2 + \frac{9}{5} z^4 + \frac{8}{5} z^5 + \frac{37}{5} z^6 + \frac{96}{5} z^7 + O(z^8) \right] + O(c^9),
\]
in agreement with (3.24). The $\mathcal{O}(c)$ contribution, has a very non-trivial origin. They depend on the $\mathcal{O}(c)$ term of the complicated coefficient in (3.31). Besides, looking at the $\mathcal{O}(z^5)$ term in $G(z)$ and in the representation limited to the two terms in (3.31) one sees that there are contributions associated with the dimension 8 primary built with $Q_3$ and $T$.

**Exploiting the analytic structure.** At this point, let us make a simple but useful remark. At order $\mathcal{O}(z^5)$, the full contribution to $G_{3333,1}^{W_3}(z)$ comes from, cf. (3.32),

$$
G_{3333,1}^{W_3}(z) = \frac{1}{9} F_1(3, 3; 0) + \mathcal{O}(z^6) = 2z^2 + 2z^3 + \frac{9}{5} z^4 + \frac{8}{5} z^5 + \mathcal{O}(z^6).
$$

On the other hand, the function $G_{3333,1}^{W_3}(z)$ is a rational function of $z$ with poles at $z = 1$. This means that we can find a unique polynomial $P\left(\frac{z}{z-1}\right)$ with no constant term such that $G_{3333,1}^{W_3}(z) = P\left(\frac{z}{z-1}\right) + \tilde{P}(z)$, where $\tilde{P}(z)$ is a polynomial. Notice that $P(z)$ is fully determined by the singular part of the Laurent expansion of $G_{3333,1}^{W_3}(z)$ around $z = 1$. Crossing invariance under $z \to \frac{z}{z-1}$ implies that the roles of $P(z)$ and $\tilde{P}(z)$ may be reversed. Thus, we can write in a unique way $G_{3333,1}^{W_3}(z) = P(z) + P\left(\frac{z}{z-1}\right) + c$ where $P(z)$ has no constant term and $c$ is a constant. This constant vanishes due to (3.35). The second crossing condition $G_{3333,1}^{W_3}(z) = \left(\frac{z}{z-1}\right)^6 G_{3333,1}^{W_3}(1-z)$ implies that $P(z)$ has degree 4. Matching with the four terms in the expansion (3.35) and imposing again the second crossing relation, we fully determine $P(z)$. In conclusion, we can write the exact representation

$$
G_{3333,1}^{W_3}(z) = P_{3333}^{W_1}(z) + P_{3333}^{W_2}\left(\frac{z}{z-1}\right), \quad P_{3333}^{W_3}(z) = 2z^4 + 2z^3 + \frac{9}{5} z^2 + \frac{8}{5} z.
$$

In summary, it has been possible to compute $G_{3333,1}^{W_3}(z)$ by just using the expression for $F_1(3, 3, 0)$ and some analytical constraint from the rational structure of the correlators. The representation (3.36) fully captures the exact $\mathcal{O}(c)$ term in (3.21).

### 3.3.2 The $W_4$ case

Let us begin with $G_{3333,1}^{W_4}$. Now, the starting point (3.35) is not enough because the fusion (3.16) implies a contribution to (3.2) from $Q_3 \times Q_3 \to Q_4$ at order $\mathcal{O}(z^4)$. However, this is governed by $\gamma^2 = \frac{112}{45} + \mathcal{O}(c^{-1})$ at large $c$. This means

$$
G_{3333,1}^{W_4}(z) = \frac{1}{9} F_1(3, 3; 0) + \frac{1}{4} \frac{112}{15} F_6(4) + \mathcal{O}(z^6) = 2z^2 + 2z^3 + \frac{11}{3} z^4 + \frac{16}{3} z^5 + \mathcal{O}(z^6).
$$

Matching this to the representation (3.36) and imposing crossing under $z \to 1-z$ is enough to completely determine

$$
G_{3333,1}^{W_4}(z) = P_{3333}^{W_3}(z) + P_{3333}^{W_4}\left(\frac{z}{z-1}\right), \quad P_{3333}^{W_4}(z) = 2z^4 + 2z^3 + \frac{11}{3} z^2 + \frac{16}{3} z.
$$

The same strategy may be applied to $\langle Q_3 Q_3 Q_4 Q_4 \rangle$. The identity exchange will require the correction

$$
F_1(3, 4, 0) = \frac{4}{3} F_1(3, 4, 0).
$$
Thus, we just need to determine $Q$ because we have less symmetry than in the previous case of 4-point functions with equal $\Delta$’s. Alternatively, one can combine the s-channel expansion (3.40) and the combination of (3.41) into account the normalization from the three point function $\langle Q_3 Q_3(Q_3 Q_3) \rangle$, this gives the improved version of (3.41)

$$G_{3344,1}^{W_4}(z) = \frac{1}{12} F_1(3, 4; 0) - \frac{1}{4} \frac{12}{5} F_0(4) + 2 \times \left(\frac{1}{3}\right)^2 \frac{225}{14} F_0(6) + O(z^8)$$

$$= 2z^2 + 2z^3 + \frac{6}{5} z^4 + \frac{2}{5} z^5 + \frac{10}{3} z^6 + 10z^7 + O(z^8),$$

and this is enough to obtain the representation

$$G_{3344,1}^{W_4}(z) = P_{3344}^{W_4}(z) + P_{3344}^{W_4} \left(\frac{z}{z-1}\right), \quad P_{3344}^{W_4}(z) = 2 z + 2 z^2 + \frac{28}{15} z^3 + \frac{7}{5} z^4.$$

Alternately, one can combine the s-channel expansion (3.41), with the t-channel expansion $3 \times 4 \rightarrow 3 + \cdots$. This gives\(^\text{10}\)

\[ z^3 G_{3344,1}^{W_4}(z^{-1}) = \left(\frac{3}{4}\right)^2 \frac{112}{5} \left(\frac{1}{3}\right) z^3 2 F_1(3 + 1, 3 - 1, 6, z) + O(z^5) = \frac{7}{5} z^3 + \frac{28}{15} z^4 + O(z^5), \]

and the combination of (3.41) and (3.44) fully determine the polynomial $P(z)$ and agrees with (3.43).

Finally, the $\langle Q_4 Q_4 Q_4(1) \rangle$ 4-point function is rather simple. Its polynomial representation obeying all crossing constraints is

$$G_{4444,1}^{W_4}(z) = P_{4444}^{W_4}(z) + P_{4444}^{W_4} \left(\frac{z}{z-1}\right),$$

$$P_{4444}^{W_4}(z) = 2 z^6 + 2 z^5 + k z^4 + 2(k - 1) z^3 + (5k - 8) z^2 + 2(5k - 9) z.$$  

Thus, we just need to determine $k$ that appears already at order $O(z^4)$

$$G_{4444,1}^{W_4}(z) = 2 z^2 + 2 z^3 + k z^4 + 2(k - 1) z^5 + (5k - 8) z^6 + O(z^7).$$

On the other hand, using $\mu^2 = \frac{27}{35} + O(c^{-1})$, we can certainly write

$$G_{4444,1}^{W_4}(z) = \frac{1}{16} F_1(4, 4; 0) + \frac{1}{4} \frac{27}{35} F_0(4) + O(z^5) = 2 z^2 + 2 z^3 + \frac{279}{140} z^4 + \cdots.$$

This fixes $k = \frac{279}{140}$ and determines

$$P_{4444}^{W_4}(z) = 2 z^6 + 2 z^5 + \frac{279}{140} z^4 + \frac{139}{70} z^3 + \frac{55}{28} z^2 + \frac{27}{14} z.$$

\(^{10}\)In (3.44), the factor $\frac{1}{2} \gamma$ is the $Q_3 Q_4 \rightarrow Q_4$ fusion coefficient, see (3.16). The factor $\frac{1}{2}$ is the inverse spin of the exchanged field, again due to normalization of the two point functions. Finally, the $\pm 1$ shift in the $2 F_1$ arguments are the conformal dimension difference, see (3.3).
3.4 Computing the 4-point functions in \( \mathcal{W}_n \)

We have analyzed the \( \mathcal{W}_3 \) and \( \mathcal{W}_4 \) cases to understand what is the origin of the \( c \to \infty \) subleading contribution to the 4-point functions of spin 3 and 4 generators. This is important to generalize the derivation to \( \mathcal{W}_n \). We have shown that the diagonal 4-point functions \( \langle Q_3 Q_4 Q_3 Q_3 \rangle \) and \( \langle Q_4 Q_4 Q_4 Q_4 \rangle \) may be computed at order \( \mathcal{O}(c) \) in terms of the large \( c \) expansion of the couplings \( \gamma, \mu \) in the \( Q_3 Q_3 \) and \( Q_4 Q_4 \) OPEs. Other primaries may be present in the \( Q_4 Q_4 \) OPE, but they do not enter our method of calculation. Instead, in the mixed 4-point function \( \langle Q_3 Q_3 Q_4 Q_4 \rangle \) we needed more information, and in particular the primary structure at dimension 6, including the coupling \( \lambda \). Nevertheless we have seen that by combining the conformal block expansions in the s- and t-channels, these problems can be overcome.

The above considerations are enough to compute the 4-point functions of spin 3 and 4 in the extended \( \mathcal{W}_n \) algebra. To this aim, we just require the \( n \)-dependent values of the couplings \( \gamma \to \gamma_n \) and \( \mu \to \mu_n \). These have been computed in \([56]\) (see also \([57, 58]\)) based on the free field representation derived in \([35]\). In our notation, we have the following couplings in \( \mathcal{W}_n \),

\[
(\gamma_n)^2 = 64 \frac{n - 3}{n - 2} \frac{c + 2}{5c + 22} \frac{c(n + 3) + 2(4n + 3)(n - 1)}{c(n + 2) + (3n + 2)(n - 1)},
\]
\[
\mu_n \gamma_n = 48 \frac{c^2(n^2 - 19) + 3c(6n^3 - 25n^2 + 15) + 2(n - 1)(6n^2 - 41n - 41)}{(5c + 22)(c(n + 2) + (3n + 2)(n - 1))}.
\]

(3.49)

In particular, expanding at large central charge,

\[
(\gamma_n)^2 = \frac{64}{5} \frac{n^2 - 9}{n^2 - 4} + \mathcal{O}(c^{-1}), \quad (\mu_n)^2 = \frac{36}{5} \frac{(n^2 - 19)^2}{(n^2 - 4)(n^2 - 9)} + \mathcal{O}(c^{-1}).
\]

(3.50)

The same calculation we did in the \( \mathcal{W}_4 \) case, cf. (3.37) and (3.47), gives now the general \( \langle Q_3 Q_3 Q_3 Q_3 \rangle \) 4-point function (at order \( \mathcal{O}(c) \)) in terms of the polynomial

\[
P_{3333}^{\mathcal{W}_n}(z) = 2z^4 + 2z^3 + \frac{5n^2 - 36}{(n - 2)(n + 2)}z^2 + \frac{8(n^2 - 8)}{(n - 2)(n + 2)}z.
\]

(3.51)

It reads

\[
G_{3333,1}^{\mathcal{W}_n}(z) = P_{3333}^{\mathcal{W}_n}(z) + P_{3333}^{\mathcal{W}_n} \left( \frac{z}{z - 1} \right) = \frac{1}{n^2 - 4} \frac{z^2}{(1 - z)^2} \left[ 2(n^2 - 4)(1 - 3z - 3z^5 + z^6) + (9n^2 - 52)z(1 + z^2) - 8(n^2 - 8)z^3 \right].
\]

(3.52)

The mixed 4-point function \( \langle Q_3 Q_3 Q_4 Q_4 \rangle \) is determined by the polynomial

\[
P_{\text{mix}}^{\mathcal{W}_n}(z) = \frac{12}{5} \frac{n^2 - 9}{n^2 - 4}z^4 + \frac{16}{5} \frac{n^2 - 9}{n^2 - 4}z^3 + \frac{7n^2 - 88}{n^2 - 4}z^2 + 12 \frac{n^2 - 14}{n^2 - 4}z.
\]

(3.53)

\[\text{In a more modern perspective, the couplings in (3.49) should be thought as a special limit of the structure constants of the quantum algebra } \mathcal{W}_n [\nu] \text{ when } \nu = n, \text{ see also } [59]. \text{ They are known to obey a remarkable triality symmetry with respect to the } \nu \text{ parameter [60].}\]
and reads
\[ G_{n444,1}^W(z) = P_{\text{mix}}^W(z) + P_{\text{mix}}^W \left( \frac{z}{z-1} \right) = \frac{1}{5(n^2-4)(1-z)^4} \]
\[ 10(n^2-4)(1-3z)+4(1n^2-344)z^2-8(4n^2-61)z^3+(43n^2-512)z^4+4(n^2-9)z^5(-8+3z) \].

The general \( \langle Q_4 Q_4 Q_4 Q_4 \rangle \) 4-point function turns out to be expressed in terms of the polynomial
\[ P_{4444}^W(z) = 2z^6 + 2z^5 + \frac{9(2n^4 - 51n^2 + 397)}{5(n^2 - 9)(n^2 - 4)} z^4 + \frac{2(13n^4 - 394n^2 + 3393)}{5(n^2 - 9)(n^2 - 4)} z^3 \]
\[ + \frac{5(2n^4 - 71n^2 + 657)}{(n^2 - 9)(n^2 - 4)} z^2 + \frac{18(n^2 - 19)^2}{(n^2 - 9)(n^2 - 4)} z \].

The general \( \langle Q_4 Q_4 Q_4 Q_4 \rangle \) 4-point function turns out to be expressed in terms of the polynomial
\[ G_{n4444,1}^W(z) = P_{n4444}^W(z) + P_{n4444}^W \left( \frac{z}{z-1} \right) = \frac{1}{5(n^2-4)(n^2-9)} \frac{z^2(1+z^2)^2}{(1-z)^6} \]
\[ 10(n^2-4)(n^2-9)(1-3z-3z^5+z^6) + 9(397-51n^2+2n^4) z^2(1+z^2) \]
\[ + 2(-2673+134n^2+7n^4) z^3 \].

4 Matching the two sides of the correspondence

We now have all the ingredients needed to compare the small \( \beta \) limit on AdS with the large \( c \) limit on the CFT side.

4.1 Field/generators normalization

The matching is based on the correspondence
\[ \varphi_s \rightarrow \kappa_s Q_s. \] (4.1)

The constant \( \kappa_2 \) is somewhat special since \( Q_2 \equiv T \), the stress-energy tensor. We can fix \( \kappa_2 \) as in [29] by considering the Liouville \( A_1 \) Toda theory. The Lagrangian for the \( \Delta = 2 \) field \( \varphi_2 \) is
\[ \mathcal{L} = \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 + \frac{2 \beta}{3} \varphi_2^3 + \frac{1}{3} \beta^2 \varphi_2^4 + \cdots. \] (4.2)

At leading order, the constant \( \kappa_2 \) is fixed by looking at the two point function. On AdS, we have — for our normalization of the bulk-to-boundary propagator —
\[ \langle \varphi_2(t) \varphi_2(0) \rangle = \frac{1 + \mathcal{O}(\beta^2)}{t^4}, \] (4.3)

\[ ^{12}\text{We remark that the final expressions (3.52), (3.54), (3.56) have a finite limit for } n \rightarrow \infty. \]
Figure 4. Tree diagrams contributing $\langle \varphi_2 \varphi_2 \varphi_2 \varphi_2 \rangle$. The external points and internal exchange are labeled by their $\Delta$.

and also

$$\langle \varphi_2(t) \varphi_2(0) \rangle = \kappa_2^2 \langle T(t) T(0) \rangle = \frac{\kappa_2^2 c}{2} \frac{1}{t^4}. \quad (4.4)$$

Hence (see later for the sign),

$$\kappa_2 = -\sqrt{\frac{2}{c}} [1 + O(\beta^2)]. \quad (4.5)$$

To find a relation connecting $\beta$ with $c$, we need connected diagrams. The associated (Witten) Feynman rules are (minus is from $e^{-S}$)

$$\begin{align*}
\begin{array}{c}
\text{vertex} \\
\end{array}
&= -4 \beta, \\
\begin{array}{c}
\text{cross} \\
\end{array}
&= -8 \beta^2.
\end{align*}$$

Using (B.2) we can compute the 3-point function

$$e_2^{-3/2} \langle \varphi_2(t_1) \varphi_2(t_2) \varphi_2(t_3) \rangle = (-4\beta) \times \frac{3\pi}{8} \frac{1}{t_1^{12} t_2^{13} t_3^{14}} [1 + O(\beta^2)]. \quad (4.6)$$

From (3.10) we have$^{13}$

$$-\frac{3\pi}{2} \beta [1 + O(\beta^2)] = \kappa^3 e_2^{-3/2} \rightarrow c = \frac{12\pi}{\beta^2} + O(\beta^0). \quad (4.7)$$

As a further check, we move to the connected 4-point function. This is given by the diagrams in figure 4. Their sum is

$$e_2^{-2} \langle \varphi_2(t_1) \cdots \varphi_2(t_4) \rangle_{\text{conn}} = 16 \beta^2 \left( W^{s}_{2222,2} + W^{t}_{2222,2} + W^{u}_{2222,2} - \frac{1}{2} D_{2222} \right)$$

$$= 16 \beta^2 \left( \frac{1}{4 \ell_{12}^2} D_{1122} + \frac{1}{4 \ell_{13}^2} D_{1212} + \frac{1}{4 \ell_{14}^2} D_{1221} - \frac{1}{2} D_{2222} \right)$$

$$= \frac{1}{\ell_{12}^4 \ell_{34}^4} \frac{3\pi \beta^2}{2} \chi^2 [D_{2222} + \chi^2 (D_{1122} + D_{1212} + D_{1221} - 5D_{2222})]$$

$$= \frac{1}{\ell_{12}^4 \ell_{34}^4} \frac{3\pi \beta^2}{2} \chi^2 (1 - \chi + \chi^2) \frac{2 (1 - \chi)^2}{(1 - \chi)^2}. \quad (4.8)$$

$^{13}$This is standard $6/\beta^2$ times $2\pi$ from the missing $1/(2\pi)$ in the action compared with standard Liouville action.
This can be written — let us add explicitly the higher order corrections —

\[ C_2^{-2} \langle \varphi_2(t_1) \cdots \varphi_2(t_4) \rangle_{\text{conn}} = \frac{3\pi \beta^2}{4} \left( \frac{1}{t_{12}^2 t_{23}^2 t_{13}^2 t_{14}^2} + \frac{1}{t_{13}^2 t_{24}^2 t_{14}^2 t_{23}^2} + \frac{1}{t_{12}^2 t_{23}^2 t_{14}^2 t_{13}^2} \right) + \mathcal{O}(\beta^4), \]

(4.9)

and is consistent with the above identifications because the coefficient of the \( \langle TTTT \rangle \) correlator is then predicted to be

\[ \kappa_2^{-4} C_2^{-2} \frac{3\pi \beta^2}{4} [1 + \mathcal{O}(\beta^2)] = c, \]

(4.10)
in agreement with (3.10).\footnote{We choose \( \kappa_2 < 0 \) in order to have \( \beta > 0 \). Notice also that the disconnected diagrams give the \( O(c^2) \) first term in (3.9).}

Of course, the relation found in (4.7) between the \( c \) and \( \beta \), i.e.

\[ c = \frac{12\pi}{\beta^2} + \mathcal{O}(\beta^0), \]

(4.11)

should be considered as the leading order term at small \( \beta \). The central charge of the \( A_n \) Toda theory is \( c_n = n[1 + (n + 1)(n + 2)(b + b^{-1})^2] \) where \( b \) is proportional to \( \beta \). With our conventions, i.e. requiring (4.11) to hold at leading order for all \( n \), this means that we could expect the following exact AdS/CFT map between the coupling \( \beta \) and the central charge \( c \)

\[ c = n + 12\pi \left[ \frac{1}{\beta} + \frac{n(n+1)(n+2)}{12\pi} \beta \right]^2 = \frac{12\pi}{\beta^2} + n(2n^2 + 6n + 5) + \frac{n^2(n+1)^2(n+2)^2}{12\pi} \beta^2 + \cdots. \]

(4.12)

As we reminded in the Introduction, the subleading \( \mathcal{O}(\beta^0) \) has been tested for the Liouville \( A_1 \) case in [23], see also [33] for the \( A_2 \) theory and other generalizations complementary to this analysis.

**Normalization of higher spin \( s \geq 3 \) duals.** If we assume that \( \varphi_s = \kappa_s Q_s \) where \( Q_s \) is a spin \( s \) generator in a certain Virasoro extension \( W(2, \ldots, s, \ldots) \), then the same analysis of the 2-point function and the relation \( \langle Q_s Q_s \rangle = \frac{c}{s^{12}} \), gives

\[ \kappa_s = -\sqrt{\frac{s}{c}} + \mathcal{O}(\beta^2). \]

(4.13)

This implies a constraint on the vertices of the form \( V = \beta g_{2ss} \varphi_2 \varphi_s^2 \). The associated Feynman rule is

\[ \begin{array}{c|c|c|c}
\text{s} & \text{s} & 2 \\
\hline
\end{array} \rightarrow -2 \beta g_{2ss}, \]

then from (B.2)

\[ C_2^{-1/2} C_s^{-1} \langle \varphi_2(t_1) \varphi_s(t_2) \varphi_s(t_3) \rangle = (2\beta g_{2ss}) \times \frac{\sqrt{\pi \Gamma(s + \frac{1}{2})}}{2(s-1) \Gamma(s)} \frac{1}{t_{12}^2 t_{13}^2 t_{23}^2} (1 + \mathcal{O}(\beta^2)). \]

(4.14)
On the other hand,
\[
\mathcal{C}_2^{-1/2} \mathcal{C}_s^{-1} \langle \varphi_2(t_1) \varphi_s(t_2) \varphi_s(t_3) \rangle = \mathcal{C}_2^{-1/2} \mathcal{C}_s^{-1} \kappa_2 \kappa_s^2 \langle T Q_s Q_s \rangle = \mathcal{C}_2^{-1/2} \mathcal{C}_s^{-1} \frac{\kappa_2 \kappa_s^2 c}{t_1^{7/2} t_2^{7/2} t_3^{7/2}}. 
\]

From (4.5) this gives for \( s > 2, \)
\[
\kappa_s^2 = \frac{g_{2ss} 2 \sqrt{\pi} \Gamma(s + \frac{1}{2})}{c(s - 1) \Gamma(s)} \mathcal{C}_s = \frac{g_{2ss}}{c} \left( \frac{1}{s - 1} (1 + O(\beta^2)) \right). 
\]

Using (4.13), we find
\[
g_{2ss} = s (s - 1), 
\]
consistently with the explicit values in (2.8), \( g_{233} = 6 \) and \( g_{244} = 12. \)

### 4.2 Matching the 4-point functions involving \( \Delta = 3, 4 \)

First of all, let us notice that the \( O(c^2) \) terms in the CFT results (3.21) immediately match the disconnected Witten diagrams where the four points on the boundary are connected with two boundary-to-boundary propagators. The next correction is \( O(c) \) on the CFT side and should match the \( O(\beta^2/\kappa^4) \) connected 4-point functions in AdS.\(^\text{16}\)

A comparison of \( G^{AdS}_{3333}(\chi) \) with the CFT result (3.52) valid for the \( W_n \) theory, shows that we have indeed
\[
\frac{\beta^2}{c^4} G^{AdS}_{3333}(\chi) = \frac{675 \pi \beta^2}{256 c} = \mathcal{C}_3^{-2} (\kappa_3)^4, 
\]
where we used (4.11) and (4.13), and identified \( z = \chi. \) Similarly, comparing \( G^{AdS}_{4444}(\chi) \) with the CFT result (3.56) we find
\[
\frac{\beta^2}{c^4} G^{AdS}_{4444}(\chi) = \frac{1225 \pi \beta^2}{192 c} = \mathcal{C}_4^{-2} (\kappa_4)^4, 
\]
where we used (4.11) and (4.13), and identified \( z = \chi. \) Finally, for the mixed correlator, we compare \( G^{AdS}_{3344}(\chi) \) with (3.54) and have again
\[
\frac{\beta^2}{c^4} G^{AdS}_{3344}(\chi) = \frac{525 \pi \beta^2}{128 c} = \mathcal{C}_3^{-1} \mathcal{C}_4^{-1} (\kappa_3 \kappa_4)^2. 
\]

These relations completes the proof that the four points functions of the \( \Delta = 3, 4 \) fields in the general \( A_n \) Toda theory obey the relation (1.5).

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\(^\text{15}\)The case \( s = 2 \) is special because of the extra permutation symmetry between the three \( \varphi_2 \) fields. In this case, the coefficient \( \kappa \) is that in (4.5).

\(^\text{16}\)This is correct using (4.11) and taking into account the \( \kappa^4 \sim 1/c^2 \) normalizations.
A Virasoro primary generating functions

Let us briefly recall how Virasoro primaries are easily counted in $W$-algebras by elementary character manipulations, see e.g. [61]. Acting on the vacuum with a bosonic primary with dimension $\Delta > 0$, i.e. starting from $\phi_- h |0\rangle$, we obtain the associated Virasoro character

$$\chi_{0,\Delta} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{\Delta+n}}.$$  \hspace{1cm} (A.1)

The full character of a CFT can be decomposed in highest weight representations of Virasoro separating out the identity module and the contributions from primaries with dimensions $\Delta_p$

$$\chi_{\text{full}} = \chi_{0,2} + \sum_{\Delta_p} d_{\Delta_p} q^{\Delta_p} \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \hspace{1cm} (A.2)$$

From this relation one can extract the generating function of primary fields $\sum_{\Delta_p} d_{\Delta_p} q^{\Delta_p}$. For example, to count the primaries in the $W_3$ algebra we simply evaluate

$$\sum_{\Delta_p} d_{\Delta_p} = \chi_{0,2} \chi_{0,3} - \chi_{0,3} = q^3 + q^4 + q^6 + q^8 + q^9 + q^{10} + 3q^{12} + 3q^{13} + 3q^{14} + \cdots. \hspace{1cm} (A.3)$$

The first term is the primary $Q_3$, the second one is a composite $\sim (Q_3Q_3) + \cdots$. In the similar case of the $W_4$ algebra we have

$$\sum_{\Delta_p} d_{\Delta_p} = \chi_{0,2} \chi_{0,4} - \chi_{0,4} = q^3 + q^4 + q^6 + q^7 + 3q^8 + 2q^9 + 4q^{10} + \cdots. \hspace{1cm} (A.4)$$

The first terms are the spin 3 and 4 generators, the third is a composite $\sim (Q_3Q_3) + \cdots$. Of course, this is not the same as the dimension 6 primary of $W_3$. In particular, it involves the spin 4 generator $Q_4$.

B Some AdS integrals

The basic $N$-point contact diagram connecting boundary points $t = (t_1, \ldots, t_N)$ to the bulk point $(t, z)$ is given by

$$W_\Delta(t) = \int_{AdS_2} \frac{dt \, dz}{z^2} \prod_{i=1}^{N} \left[ \frac{z}{z^2 + (t - t_i)^2} \right]^{\Delta_i}. \hspace{1cm} (B.1)$$

Special cases are the 3-point function ($t_{ij} = t_i - t_j$)

$$W_{\Delta_1, \Delta_2, \Delta_3}(t_1, t_2, t_3) = \sqrt{\pi} \Gamma \left( \frac{\Delta_1 + \Delta_2 - \Delta_3}{2} \right) \Gamma \left( \frac{\Delta_2 + \Delta_3 - \Delta_1}{2} \right) \Gamma \left( \frac{\Delta_1 + \Delta_3 - \Delta_2}{2} \right) \Gamma \left( \frac{\Delta_1 + \Delta_2 + \Delta_3 - 1}{2} \right) \frac{2^{\Delta_1}}{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3)} |t_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |t_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |t_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}, \hspace{1cm} (B.2)$$

and the 4-point function, cf. appendix B,

$$W_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(t_1, t_2, t_3, t_4) \overset{\text{def}}{=} D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(t_1, t_2, t_3, t_4). \hspace{1cm} (B.3)$$
Here, \( D \)-functions are discussed in general in [42, 43, 62]. \( D \)-functions are related to the more convenient \( \overline{D} \)-functions that depend only on the (unique) conformally invariant cross-ratio. They are defined by

\[
D_\Delta(t) = \frac{\sqrt{\pi} \Gamma\left(\frac{\Sigma - \frac{\Delta}{2}}{2}\right) t^{2(\Sigma - \Delta_1 - \Delta_4)} t^{2(\Sigma - \Delta_3 - \Delta_4)}}{2 \prod_n \Gamma(\Delta_n) t^{2(\Sigma - \Delta_4)} t^{2\Delta_2}} \overline{D}_\Delta(\chi), \tag{B.4}
\]

where \( \Sigma = \frac{1}{2} \sum_n \Delta_n \) and\(^{17}\)

\[
\chi = \frac{t_{12} t_{34}}{t_{13} t_{24}}, \quad t_{ij} = t_i - t_j. \tag{B.5}
\]

For integer \( \Delta_n \) we can evaluate all \( \overline{D} \) functions by the recursion identities of [63]. In particular, for \( 0 < \chi < 1 \), one has the special cases

\[
\overline{D}_{1133} = -\frac{2\chi^2 + 3\chi - 3}{15(1-\chi)^2} - \frac{2\chi^4 \log \chi}{15\chi} + \frac{2(\chi^2 + 3\chi + 6) \log(1-\chi)}{15\chi},
\]

\[
\overline{D}_{2233} = \frac{18\chi^4 - 29\chi^3 + 5\chi^2 + 48\chi - 24}{210(1-\chi)^3 \chi^2} + \frac{(9\chi^2 - 28\chi + 28) \chi^2 \log(\chi)}{105(1-\chi)^4} - \frac{(9\chi^3 + 8\chi^2 + 6\chi + 12) \log(1-\chi)}{105\chi^3},
\]

\[
\overline{D}_{3333} = -\frac{2(12\chi^6 - 36\chi^5 + 37\chi^4 - 14\chi^3 + 37\chi^2 - 3\chi + 12)}{315(1-\chi)^4 \chi^2} - \frac{4(\chi^2 - \chi + 1)(2\chi^2 - 3\chi + 7) \log(\chi)}{105(1-\chi)^5}
\]

\[
- \frac{4(\chi^2 - \chi + 1)(2\chi^2 + 3\chi + 2) \log(1-\chi)}{105\chi^5}, \tag{B.6}
\]

with the additional cases related to the above by crossing

\[
\overline{D}_{1313} = \frac{1}{(1-\chi)^2} \overline{D}_{1133} \left( \frac{1}{1-\chi} \right), \quad \overline{D}_{1331} = \frac{1}{(1-\chi)^3} \overline{D}_{1133}(1-\chi),
\]

\[
\overline{D}_{2323} = \frac{1}{(1-\chi)^4} \overline{D}_{2233} \left( \frac{1}{1-\chi} \right), \quad \overline{D}_{2332} = \frac{1}{(1-\chi)^2} \overline{D}_{2233} \left( \frac{1}{1-\chi} \right). \tag{B.7}
\]

Besides the contact diagram expressed by (B.3), a generic boundary 4-point correlation function \( \langle \Phi_{\Delta_1}(t_1) \cdots \Phi_{\Delta_4}(t_4) \rangle \) receives at tree level contributions from exchange diagrams mediated by two cubic interactions, see figure 5. In the s-channel and again with unit normalization, we have a simple formula valid when the exchanged field has conformal parameter \( \Delta_F \) with \( k = \frac{\Delta_1 + \Delta_2 - \Delta_f}{2} \in \mathbb{N}^+ \). In this case, the exchange diagram is given by [64]

\[
W_{\Delta,E}^a(t) = \sum_{\ell=1}^k \frac{\langle \Delta_1 \rangle_{-\ell} \langle \Delta_2 \rangle_{-\ell} \overline{D}_{\Delta_1-\ell,\Delta_2-\ell,\Delta_3,\Delta_4}(t)}{4(k)_{1-\ell} \langle \Delta_1 + \Delta_2 - 1 + \Delta_f \rangle_{1-\ell} \overline{D}_{12}^{2\Delta}} \frac{1}{|t_{12}|^{2\Delta}} \overline{D}_{\Delta_1-\ell,\Delta_2-\ell,\Delta_3,\Delta_4}(t), \tag{B.8}
\]

with similar expressions for the other channels.

\(^{17}\)Any ambiguity associated with odd exponents should be resolved by replacing \( t_{ij}^{2\chi} \rightarrow (t_{ij}^{2\chi})^{-1} \) and considering that \( \overline{D} \) is actually a function of \( \chi^2 \).
Figure 5. Tree diagram associated with the s-channel exchange of a field with conformal parameter $\Delta_E$ mediated by two cubic vertices. The dashed line is just a convenient graphical representation for the compactified boundary of AdS$_2$.

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