LG/CY CORRESPONDENCE:  
THE STATE SPACE ISOMORPHISM  
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Abstract. We prove the classical mirror symmetry conjecture for the mirror pairs constructed by Berglund, Hübsch, and Krawitz. Our main tool is a cohomological LG/CY correspondence which provides a degree-preserving isomorphism between the cohomology of finite quotients of Calabi–Yau hypersurfaces inside a weighted projective space and the Fan–Jarvis–Ruan–Witten state space of the associated Landau–Ginzburg singularity theory.

1. Introduction

Mirror symmetry has been one of the most inspirational problems arising from physics in the last twenty years. In the most common formulation, which we call classical mirror symmetry, it is a duality statement pairing two Calabi–Yau three-folds $X^3$ and $Y^3$ by interchanging $h^{1,1}$ and $h^{2,1}$. When the mirror symmetry was first proposed twenty years ago, only a few examples of Calabi–Yau three-folds were known. A major effort was launched to construct more examples. Soon, physicists constructed millions of examples which are (orbifolded) hypersurfaces and complete intersections lying inside weighted projective spaces or toric varieties. Since every three-dimensional Calabi–Yau orbifold admits a crepant resolution, we obtain millions of examples of smooth Calabi–Yau three-folds.

Among these millions of examples, an elementary and yet elegant mirror symmetry construction was proposed by the physicists Berglund and Hübsch [BH93], which will be the focus of our interest. In [BH93] a hypersurface $X_W$ in a weighted projective space $\mathbb{P}(w) = \mathbb{P}(w_1, \ldots, w_N)$ is considered: $X_W$ is defined by a quasihomogeneous polynomial $W$. Berglund and Hübsch describe a simple definition of the mirror of $X_W$.

The construction only involves cases when $W$ is “invertible”; i.e. $W$ is the sum of $N$ monomials, as many as the variables. In this case, one can transpose the exponents matrix and obtain another quasihomogeneous polynomial $W^T$ defining a hypersurface lying in another weighted projective space. The varieties $X_W$ and $X_{W^T}$ are not mirror pairs in general and...
a certain orbifolding construction must be involved. Berglund and Hübsch proposed a certain physical property for a correspondence between automorphism groups \( G \subset \text{Aut}(\{W = 0\}) \) and \( G^T \subset \text{Aut}(\{W^T = 0\}) \); in [BH93], the Calabi–Yau \( X_W/G \) is expected to be the mirror image of the Calabi–Yau \( X_{W^T}/G^T \). More precisely, the classical mirror symmetry conjecture should hold for these pairs: if we stick to Calabi–Yau three-folds, \( h^{1,1} \) and \( h^{2,1} \) should be interchanged. This group duality is precisely stated only in some cases, but already opens the way to several interesting tests: Kreuzer and Skarke checked thousands three-folds for which they computed the so-called “Landau–Ginzburg phase” [KS93]. Indeed, these invariants exhibit the classical mirror symmetry correspondence.

Unfortunately, this approach was mysteriously abandoned to favor a more geometric approach due to Batyrev and Borisov. Batyrev and Borisov considered the complete intersection of a Gorenstein toric variety. In this context, the mirror symmetry was interpreted as polar duality. A major theorem of the day was a solution of the classical mirror symmetry conjecture in this context. We should mention that Batyrev–Borisov imposed an important condition called Gorenstein in all their constructions. Indeed, Gorenstein conditions are also crucial on our recent investigation of Gromov-Witten theory [CIR]. It is interesting to consider it in the context of weighted projective spaces. The ambient weighted projective space \( \mathbb{P}(\mathbf{w}) \) is Gorenstein if and only if \( \sum_j w_j \) is a multiple of every weight \( w_j \); hence, with a Gorenstein ambient space we can reduce to the Calabi–Yau hypersurface defined by the Fermat polynomial of degree \( d = \sum_j w_j \); i.e. \( W(x_1, \ldots, x_N) = \sum_j x_j^{d/w_j} \). It was known that Fermat Calabi–Yau hypersurfaces only represents a small subclass of all Calabi–Yau hypersurfaces. It was a big surprise to us that a vast range of cases involved in the Berglund–Hübsch construction are not covered by Batyrev and Borisov (see Remark 5)! During the last two years, interest in this problem was revived by the introduction of a Gromov–Witten-type theory for singularities by Fan, Jarvis, and the second author. This fits within the framework of the Landau–Ginzburg model and is based on a proposal of Witten (FJRW theory). Recently, Krawitz [Kr] found a general construction for the dual group \( G^T \). Working on much more general grounds where \( X_W \) is not necessarily Calabi–Yau, Krawitz proved an “LG-to-LG” mirror symmetry theorem for all invertible polynomials \( W \) and all admissible groups \( G \). We should emphasize that the Berglund–Hübsch–Krawitz computations are purely in the Landau–Ginzburg setting. Whether \( X_W/G \) and \( X_{W^T}/G^T \) are a mirror pair of Calabi–Yau orbifolds is an open question. We shall give a firm answer in this article. To state our theorem, let us set up some notation.
The mirror symmetry setup. A hypersurface inside a weighted projective space is defined by a quasihomogeneous polynomial $W$ in the variables $x_1, \ldots, x_N$ of charges $q_1, \ldots, q_N \in \mathbb{Q}_{>0}$ such that
\begin{equation}
W(\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N) = \lambda W(x_1, \ldots, x_N).
\end{equation}
Write $q_1 = w_1/d, \ldots, q_N = w_N/d$ with common denominator so that we have $\gcd(w_1, \ldots, w_N, d) = 1$. Then, $X_W = \{W = 0\} \subset \mathbb{P}(w_1, \ldots, w_N)$ defines a degree-$d$ hypersurface. We always assume that $W$ has a unique singularity at zero; in other words, $X_W$ is a smooth Deligne–Mumford stack (an orbifold). Furthermore, $X_W$ is a Calabi–Yau orbifold if and only if $\sum_j q_j = 1$; we refer to this condition as the CY condition (see also Section 3.2). For three-dimensional Calabi–Yau orbifolds, the crepant resolution always exists and the Hodge numbers are equal to the Hodge numbers of the underlying Chen–Ruan orbifold cohomology. A wider range of Calabi–Yau orbifolds arises from quotients of $X_W$. Consider the group $\text{Aut}(W)$ of diagonal symmetries rescaling the coordinates and preserving $W$: $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^\times$ such that $W(\alpha_1 x_1, \ldots, \alpha_N x_N)$ equals $W(x_1, \ldots, x_N)$ for all $(x_1, \ldots, x_N) \in \mathbb{C}^N$. Clearly $J_W = (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_N))$ is contained in $\text{Aut}(W)$ and the action of $J_W$ on $X_W$ is trivial (see Section 3.2 for a discussion of group actions on these stacks). For any subgroup $G$ of diagonal symmetries containing $J_W$, let us consider the group $\tilde{G} = G/\langle J_W \rangle$ acting faithfully on $X_W$. The quotient is Calabi–Yau as long as $G \subseteq \text{SL}_N(\mathbb{C})$. Let $G \subset \text{Aut}(W)$ be such that $\langle J_W \rangle \subseteq G \subseteq \text{SL}_N(\mathbb{C})$. Then, there is a very natural construction associating $W^T$ and $G^T$ to $W$ and $G$ and preserving the following properties (see Section 3.2 (5) and 13, for a concise presentation of the construction of $W^T$ and $G^T$).

First, the polynomial $W^T$ — precisely as the polynomial $W$ — the polynomial $W^T: \mathbb{C}^N \to \mathbb{C}$ has a unique singularity at $0$ and the sum of its charges $q_1^T, \ldots, q_N^T$ equals $1$ (i.e. $X_W^T$ is Calabi–Yau). Second, the group $G^T$ — in perfect analogy with $\langle J_W \rangle \subseteq G \subseteq \text{SL}_N(\mathbb{C})$ — satisfies $\langle J_W^T \rangle \subseteq G^T \subseteq \text{SL}_N(\mathbb{C})$.

Our mirror symmetry theorem is

**Theorem 2**. The Calabi–Yau $[X_W/\tilde{G}]$ and the Calabi–Yau $[X_W^T/\tilde{G}^T]$ form a mirror pair; i.e. we have

$$H^p_{\text{CR}}([X_W/\tilde{G}]; \mathbb{C}) \cong H^{N-2-p,q}_{\text{CR}}([X_W^T/\tilde{G}^T]; \mathbb{C}),$$

where $H_{\text{CR}}(\ ; \mathbb{C})$ stands for Chen–Ruan orbifold cohomology.

The above theorem is precisely performing a “90 degrees rotation of the Hodge diamond” as predicted by the classical mirror symmetry conjecture in these cases.
Remark 1. Let us point out that one can find two different polynomials $W_1, W_2$ in the same family of degree-$d$ quasihomogeneous polynomials in the variables $x_1, \ldots, x_N$ with charges $q_1, \ldots, q_N$. Now, whereas $X_{W_1}$ may be regarded as a deformation of $X_{W_2}$, there is no apparent reason to claim that $W_1^T$ is related to $W_2^T$. Indeed the above statement implies that the cohomologies of the hypersurfaces defined by $W_1^T$ and $W_2^T$ are strictly related (in many cases, e.g. when $SL_W = \langle J_W \rangle$, they are isomorphic). This provides many examples of “multiple mirrors” which are not birational to each other—a rather interesting phenomenon which is certainly worth further investigation.

The LG/CY correspondence. The above mirror symmetry theorem is an outcome of our program to study so called Landau–Ginzburg (LG)/Calabi–Yau (CY) correspondence. In the early days of mirror symmetry, physicists noticed that regarding $W$ as a function on $\mathbb{C}^N$ leads to the Landau–Ginzburg (LG) singularity model. (In this correspondence, we place ourselves within a more general framework: we do not need to require that the number of variables equals the number of monomials.) The argument has been made on physical grounds that there should be a LG/CY correspondence connecting Calabi–Yau geometry to the LG singularity model [VW89], [Wi93]. In this context, CY manifolds are considered from the point of view of Gromov–Witten theory; this correspondence would therefore inevitably yield new predictions on Gromov–Witten invariants and is likely to greatly simplify their calculation (it is generally believed that the LG singularity model is relatively easy to compute). In a different context, the LG/CY correspondence led to identifying matrix factorization as the LG counterpart of the derived category of complexes of coherent sheaves [HW04], [Ko].

In [FJR1, FJR08, FJRb], a candidate quantum theory of singularities has been constructed by Fan, Jarvis, and Ruan. Using the Fan–Jarvis–Ruan–Witten theory as a candidate theory on the LG side, the authors have launched a program to solve LG/CY-correspondence for Calabi–Yau hypersurfaces inside weighted projective spaces. In [ChiR], the equivalence between FJRW theory and GW theory has been established in genus zero in the case of the famous quintic three-fold. The starting point of this equivalence is an isomorphism between the two cornerstones the two theories are built upon: the FJRW state space of the singularity and the cohomology of the hypersurface. This can be done explicitly in several examples, but it is rather intricate to prove it in full generality (see Section 4 for a case-by-case approach through elliptic curves, K3 surfaces and Calabi–Yau three-folds).

We will accomplish the isomorphism in full generality by building a common combinatorial model for both theories. Our model generalizes the combinatorial model of Boissière, Mann and Perroni [BMP09] for weighted
The main result is the following cohomological LG/CY correspondence where $H_{\text{CR}}^{p,q}(X_W/\tilde{G};\mathbb{C})$ denotes the Chen–Ruan orbifold cohomology while $H_{\text{FJRW}}^{p,q}(W,G;\mathbb{C})$ denotes the state space of Fan-Jarvis-Ruan-Witten theory (see Section 3 for the detailed definition).

**Theorem 1.** Let $W$ be a nondegenerate quasihomogeneous polynomial of degree $d$ in the variables $x_1,\ldots,x_N$ whose charges add up to 1 (CY condition). Then, for any group $G$ of diagonal symmetries containing $J_W$ we have a bidegree preserving isomorphism of vector spaces

$$H_{\text{CR}}^{p,q}(X_W/\tilde{G};\mathbb{C}) \cong H_{\text{FJRW}}^{p,q}(W,G;\mathbb{C}).$$

The mirror symmetry theorem is a direct consequence of our cohomological LG/CY correspondence and Krawitz LG-to-LG mirror symmetry theorem.

We point out, however, a most surprising aspect of our main theorem: not only does it hold for noninvertible polynomials, it also holds for $G \not\subseteq SL_W$ (e.g. $G$ equal to the group Aut($W$) itself). This goes beyond the LG/CY-correspondence stated in physics and yields several surprising consequences.

1.1. **Structure of the paper.** This article is organized as follows. In Section 2 we state precisely the mirror symmetry construction. In Section 3 we introduce the state spaces of both Gromov–Witten theory (CY side) and Fan–Jarvis–Ruan–Witten theory (LG side) and we state the cohomological Landau–Ginzburg(LG)/Calabi–Yau(CY) correspondence between them. In Section 4 we present several examples illustrating the correspondence, this prepares the ground to the combinatorics involved in the general proof. In Section 5 we prove the two theorems stated above. In Section 6 we review the examples introduced in Section 4 in the light of the combinatorial tools introduced in Section 5.

2. **The classical mirror symmetry construction**

Berglund and Hübsch [BH93] consider polynomials in $N$ variables having $N$ monomials

$$W(x_1,\ldots,x_N) = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{m_{i,j}}. \tag{2}$$

Note that each of the $N$ monomials has coefficient one; indeed, since the number of variables equals the number of monomials, even when we start from a polynomial of the form $\sum_{i=1}^{N} l_i \prod_{j=1}^{N} x_j^{m_{i,j}}$, it is possible to reduce
to the above form by conveniently rescaling the $N$ variables. In this way assigning a polynomial $W$ as above amounts to specifying a square matrix

$$M = (m_{i,j})_{1 \leq i,j \leq N}.$$ 

The polynomials studied in [BH93] are called “invertible”, because the matrix $M$ is an invertible $N \times N$ matrix. In fact, polynomials of this type may be regarded as quasihomogeneous polynomials in the variables $x_1, \ldots, x_N$ of charges $q_1, \ldots, q_N$. 

The geometrical meaning of this condition is the following: $X_W = \{W = 0\}$ is Calabi–Yau, or — more precisely — $\{W = 0\}$ is a degree-$d$ Calabi–Yau hypersurface in the weighted projective space $\mathbb{P}(dq)$, where $d$ is the least integer for which $dq \in \mathbb{Z}^N$. Let $G \subset \text{Aut}(W)$ be a group of diagonal symmetries satisfying $\langle J_W \rangle \subseteq G \subseteq SL_W$ (the fact that $J_W$ is contained in $SL_W$ follows from the Calabi–Yau condition).

In this context there is a natural way to associate to $W$ a polynomial $W^T$ and to $G$ a subgroup $G^T$ of the group of diagonal symmetries of the
polynomial $W^T$. The polynomial $W^T$ is defined by transposing the matrix $(m_{i,j})$:

\[
W^T(x_1, \ldots, x_N) = \sum_{i=1}^N \prod_{j=1}^N x_j^{m_{j,i}}.
\]

The group $G^T$ is defined by

\[
G^T = \left\{ \prod_{j=1}^N (\rho^T_i)^{a_i} \mid \text{if } \prod_{j=1}^N x_i^{a_i} \text{ is } G\text{-invariant} \right\},
\]

where $\rho^T_i$ is the diagonal symmetry corresponding to the $i$th column of $(M_W)^{-1}$ (note that, by construction, $M_W^T$ equals $(M_W)^T$).

Then we have the following properties:

- $W^T$ is nondegenerate and the sum of its charges $q^T_1, \ldots, q^T_N$ equals 1 (i.e. $X_{W^T}$ is Calabi–Yau).
- The group $G^T$ satisfies $\langle J_{W^T} \rangle \subseteq G^T \subseteq SL_{W^T}$.
- The quotients $[X_W/\tilde{G}]$ and $[X_{W^T}/\tilde{G}^T]$ form a mirror pair in the following sense.

Below, $M$, $W$, and $G$ satisfy these conditions: $M = (m_{i,j})$ is an invertible $N \times N$ matrix satisfying $\sum_{i,j} m_{i,j} = 1$ (CY condition, see (3) and (4)), the polynomial $W(x_1, \ldots, x_N) = \sum_i \prod_j x_j^{m_{i,j}}$ has a single isolated critical point at $0 \in \mathbb{C}^N$, $G$ is a group containing $J_W$ and contained in $SL_W$.

**Theorem 2.** Then, the Calabi–Yau $[X_W/\tilde{G}]$ and the Calabi–Yau $[X_{W^T}/\tilde{G}^T]$ form a mirror pair; i.e. we have

\[
H^{p,q}_{CR}([X_W/\tilde{G}]; \mathbb{C}) \cong H^{N-2-p,q}_{CR}([X_{W^T}/\tilde{G}^T]; \mathbb{C}),
\]

where $H^{p,q}_{CR}(\ ; \mathbb{C})$ stands for Chen–Ruan orbifold cohomology.

We prove this theorem in Section 5.

**Remark 3.** Let us mention that the fact that $W^T$ is nondegenerate follows from Kreuzer and Skarke [KS92] classification of invertible nondegenerate potentials. An invertible potential $W$ is nondegenerate if and only if it can be written as a sum of (decoupled) invertible potentials of one of the following three types, which we will refer to as atomic types:

- $W_{\text{Fermat}} = x^a$.
- $W_{\text{loop}} = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N} x_1$.
- $W_{\text{chain}} = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N}$.

If $W$ is a Fermat type polynomial, the ambient weighted projective stack is Gorenstein. However, if $W$ is of a loop or chain type, the ambient weighted projective stack is not Gorenstein in general.
Corollary 4. Assume that the quotient schemes $X_W/\tilde{G}$ and $X_W^T/\tilde{G}^T$ both admit crepant resolutions $Z$ and $Z^T$. Then the above statement yields a statement in ordinary cohomology:

$$h^{p,q}(Z; \mathbb{C}) = h^{N-2-p,q}(Z^T; \mathbb{C}).$$

We prove this corollary in Section 5.

Remark 5. In the case where $w_j$ divides $d$, Theorem 2 can be deduced from Batyrev’s construction of mirror pairs into toric geometry. The general case does not fit in this framework because the ambient space (unlike the space $X_W$) is not Gorenstein in general. The following example illustrates this well.

Example 6. In order to illustrate the above statement we provide an example straight-away and we refer to Section 6 for more. Consider the following quintic hypersurface in $\mathbb{P}^4$:

$$\{x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_5 + x_5^5 = 0\}.$$  

It is a chain-type Calabi–Yau variety $X$ whose Hodge diamond is clearly equal to that of the Fermat quintic and is well known: $h^{1,1} = 1$, $h^{0,3} = 1$, $h_{1,2} = 101$

\[
\begin{array}{cccc}
1 \\
0 & 0 \\
0 & 1 & 0 \\
1 & 101 & 101 & 1 \\
0 & 1 & 0 \\
0 & 0 \\
1 \\
\end{array}
\]  

(7)

The mirror Calabi–Yau is given by the vanishing of the polynomial

$$W^T(x_1, x_2, x_3, x_4, x_5) = x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3x_4^4 + x_4x_5^5 = 0,$$

which may be regarded as defining a degree-256 hypersurface $X^T$ inside $\mathbb{P}(64, 48, 52, 51, 41)$. This degree-256 hypersurface is Calabi–Yau (i.e. 256 is indeed the sum of the weights). But the ambient weighted projective stack is no longer Gorenstein. Note that the group $SL_W^T$ coincides with $\langle J_W^T \rangle$; therefore Corollary 2 reads

$$h^{p,q}(X; \mathbb{C}) = h^{3-p,q}(X^T; \mathbb{C}).$$
Indeed, the Hodge diamond satisfies $h^{1,1} = 101$, $h^{0,3} = 1$, $h_{1,2} = 1$.

\[
\begin{array}{ccc}
1 & & \\
0 & 0 & \\
0 & 101 & 0 \\
1 & 1 & 1 & 1 \\
0 & 101 & 0 & \\
0 & 0 & & \\
1 & & & \\
\end{array}
\]

3. The cohomological LG/CY correspondence

The geometrical Landau–Ginzburg/Calabi–Yau correspondence is a correspondence between two geometrical settings defined starting from the polynomial $W$ and the group $G$. With respect to the previous section we work in a more general setup.

3.1. The polynomial and its diagonal symmetries. We consider polynomials

\[
W(x_1, \ldots, x_N) = l_1 \prod_{j=1}^{N} x_j^{m_{1,j}} + \cdots + l_s \prod_{j=1}^{N} x_j^{m_{s,j}}.
\]

where $l_1, \ldots, l_s$ are nonzero complex numbers and $m_{i,j}$ (for $1 \leq i \leq N$ and $1 \leq j \leq s$) are nonnegative integers. We will always suppose that the summands of the above decomposition are distinct monomials; i.e. monomials with distinct exponents.

We assume that $W$ is quasihomogeneous; i.e. there exist positive integers $w_1, \ldots, w_N$, and $d$ satisfying

\[
W(\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \ldots, x_N) \quad \forall \lambda \in \mathbb{C},
\]

or, equivalently,

\[
W = \sum_{j=1}^{N} \frac{w_j}{d} x_j \partial_j W
\]

(we write $\partial_j$ for the partial derivative with respect to the $j$th variable). For $1 \leq j \leq N$, we say that the charge of the variable $x_j$ is $q_j = w_j/d$. As soon as $w_1, \ldots, w_N$ and $d$ are coprime, we say that the degree of $W$ is $d$ and that the weight of the variable $x_j$ is $w_j$. We assume that the origin is the only critical point of $W$; i.e. the only solution of

\[
\partial_j W(x_1, \ldots, x_N) = 0 \quad \text{for } j = 1, \ldots, N
\]

is $(x_1, \ldots, x_N) = (0, \ldots, 0)$. (By [11], if $(x_1, \ldots, x_N)$ satisfies [11], then $W(x_1, \ldots, x_N)$ is zero.)
Definition 7. We say that $W$ is a **nondegenerate quasihomogeneous polynomial** if it is a quasihomogeneous polynomial of degree $d$ in the variables $x_1, \ldots, x_N$ of charges $w_1/d, \ldots, w_N/d > 0$ and the following conditions are satisfied:

1. $W$ has a single critical point at the origin;
2. the charges are uniquely determined by $W$.

Remark 8. The second condition above may be regarded as saying that the $s \times N$ matrix $M_W = (m_{ij})$ defined by $W(x) = \sum_{i=1}^s \lambda_i \prod_{j=1}^N x_j^{m_{ij}}$ has rank $N$ (i.e. has a left inverse).

**CY condition.** The main result of this paper, the cohomological Landau–Ginzburg/Calabi–Yau correspondence, holds under the following condition:

\[ \sum_j q_j = 1. \]

The definition of $\text{Aut}(W)$ applies without changes to the polynomial $W$ in this context: $\text{Aut}(W)$ is the group of $(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^\times)^N$ satisfying $W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)$. Again $SL_W = SL(\mathbb{C}, N) \cap \text{Aut}(W)$ and $J_W := (\exp(2\pi i w_1/d), \ldots, \exp(2\pi i w_N/d))$ is in $SL_W$ and generates a cyclic subgroup of order $d$ as a consequence of the CY condition.

3.2. **The Calabi–Yau side.** On the Calabi–Yau side the picture is that of a hypersurface inside the weighted projective stack

\[ \mathbb{P}(w_1, \ldots, w_N) = [((\mathbb{C}^N \setminus \{0\})/\mathbb{C}^\times], \]

where $\mathbb{C}^\times$ acts as $\lambda(x_1, \ldots, x_N) = (\lambda^{w_1} x_N, \ldots, \lambda^{w_N} x_N)$ and $w_1, \ldots, w_N$ are the weights satisfying $q_j = w_j/d$. By the nondegeneracy condition, the equation $W = 0$ defines a smooth hypersurface inside $\mathbb{C}^N \setminus \{0\}$: the normal vector

\[ \vec{n}(x) = (\partial_j W(x))_{j=1}^N \]

never vanishes on $\mathbb{C}^N \setminus \{0\}$. By the quasihomogeneity condition the action of $\mathbb{C}^\times$ fixes the variety $\{W = 0\}$. We write $X_W$ for the quotient stack

\[ X_W := [\{W = 0\}_{\mathbb{C}^N \setminus \{0\}}/\mathbb{C}^\times] \subset \mathbb{P}(w_1, \ldots, w_N). \]

Remark 9. Note that the CY condition implies that $\omega_{X_W}$ is trivial, $X_W$ has canonical singularities, and $H^i(X_W, \mathcal{O}_{X_W}) = (0)$ for $i = 1, \ldots, n - 1$ (see [CG Lem. 1.12]). In other words $X_W$ is Calabi–Yau (see [Ba94, 4.1.8]). We point out that well-formedness conditions (see [IF00] and [CG, p.8]) are not needed here, see Remark 24 in Section 5.

\[ \text{From now on we will always stress the stack-theoretic nature of the above quotient, because this point of view is crucial here.} \]
Consider a group $G$ contained in $\text{Aut}(W)$ and containing $J_W$. The homomorphism mapping $\lambda \in \mathbb{C}^\times$ to $(\lambda^{w_1}, \ldots, \lambda^{w_N}) \in (\mathbb{C}^\times)^N$, is injective because $\cap_j \mu_j$ is trivial (the weights are coprime by definition). It is natural to identify $\mathbb{C}^\times$ with the image of the above injection: we write $\bar{\lambda}$ for the image of $\lambda \in \mathbb{C}^\times$, i.e.

$$\bar{\lambda} = (\lambda^{w_j})_{j=1}^N.$$  

Notice that we have

$$\mathbb{C}^\times \cap G = \langle J_W \rangle$$

as a straightforward consequence of the quasihomogeneity of $W$. The group $\tilde{G} = G/\langle J_W \rangle$ acts faithfully on the stack $X_W$. In fact, following Romagny’s treatment [Ro05] of actions on stacks we may consider the 2-stack $[X_W/\tilde{G}]$ which is equivalent to the quotient stack of $\{ W = 0 \} \mathbb{C}^N \backslash \{ 0 \}$ by the action of the product

$$G\mathbb{C}^\times = \{ g(\lambda^{w_1}, \ldots, \lambda^{w_N}) \mid g \in G \subset (\mathbb{C}^\times)^N, \lambda \in \mathbb{C}^\times \} \subseteq (\mathbb{C}^\times)^N$$

(this is a consequence of $G\mathbb{C}^\times / \mathbb{C}^\times = \tilde{G}$ and of [Ro05, Rem. 2.4]). In this way we may exhibit $[X_W/\tilde{G}]$ as a quotient and indeed a smooth stack of Deligne–Mumford type:

$$[X_W/\tilde{G}] = \left[ \{ W = 0 \} \mathbb{C}^N \backslash \{ 0 \} / G\mathbb{C}^\times \right]$$ (with $\tilde{G} = G/\langle J_W \rangle$).

Alternatively, one may take the above formula as a definition of the quotient $[X_W/\tilde{G}]$.

**Remark 10.** If $G \subseteq SL_W$, the $G$-action preserves the canonical form on $X_W$ and the quotient space $Y = X_W/\tilde{G}$ is still Calabi–Yau (see Remark [24] in Section 5). This motivates the hypothesis $G \subseteq SL_W$ in [BH93]; however, Theorem [14] holds for the orbifold $[X_W/\tilde{G}]$ even beyond $SL_W$. This happens because the theorem is phrased in terms of Chen–Ruan orbifold cohomology and applies to an orbifold which — in some sense — is Calabi–Yau (the CY condition insures that the canonical divisor $K$ of the stack $[X_W/G]$ has vanishing degree). Example [19] exhibits a situation where $G = \text{Aut}(W)$ is not contained in $SL_W$; there, the quotient space $X_W/\tilde{G}$ is not Calabi–Yau (it is a projective line) but the stack $[X_W/\tilde{G}]$ has canonical divisor of degree 0 and in fact there exists a tensor power of the canonical line bundle which is trivial (we have $\omega^{\otimes 4} \cong \mathcal{O}$). This is enough for Theorem [14] on LG/CY correspondence to hold at a stack-theoretic level even if there is no scheme-theoretic counterpart to this statement.

The main invariant on the Calabi–Yau side is the **Chen–Ruan orbifold cohomology**. For a smooth Deligne–Mumford quotient stack $\mathcal{X} = [U/G]$ it may be regarded essentially as follows. It is a direct sum over the elements
of the group $G$: the summands are ordinary cohomology groups $\mathcal{H}^\bullet(G, \mathbb{C})$ of the so-called sectors $X'_g = \{ u \in U \mid gu = u \}/G$. The sectors are algebraic stacks of Deligne–Mumford type; since the cohomology with complex coefficients can be identified with the cohomology of the coarse space, the summands can be expressed in terms of coarse spaces. We now detail this description for the quotient stack

$$[X_W/\tilde{G}] = \{(W = 0)_{\mathbb{C}^N \setminus \{0\}}/G\mathbb{C}^x \}. $$

For any $\gamma \in (\mathbb{C}^x)^N$, and in particular for $\gamma \in G\mathbb{C}^x$, we can define

$$\mathbb{C}^N_\gamma = \{ x \in \mathbb{C}^N \mid \gamma x = x \};$$

$$N_\gamma = \dim_{\mathbb{C}}(\mathbb{C}^N_\gamma);$$

$$W_\gamma = W|_{\mathbb{C}^N_\gamma}. $$

For $\gamma \in G\mathbb{C}^x$, we set the notation

$$\{W_\gamma = 0\}_\gamma := \{W_\gamma = 0\}_{\mathbb{C}^N \setminus \{0\}};$$

it is easy to show that $\{W_\gamma = 0\}$ defines a smooth hypersurface inside $\mathbb{C}^N \setminus \{0\}$. We illustrate this by distinguishing two cases: $\gamma \in G$ and $\gamma \not\in G$.

If $\gamma$ belongs to $G$, by [FJRa, Lem. 3.2.1], the condition $\bar{n}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \mathbb{C}^N_\gamma$ implies $\bar{n}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \mathbb{C}^N$; hence we have $\mathbf{x} = \mathbf{0}$. In other words the hypersurface $\{W_\gamma = 0\}$ inside $\mathbb{C}^N \setminus \{0\}$ is smooth.

On the other hand if $\gamma \not\in G$, then $\gamma = (g_1\lambda^{w_1}, \ldots, g_N\lambda^{w_N})$ with $\lambda \not\in \mathbf{\mu}_d$ and $(g_1, \ldots, g_N) \in G$ (see (14) and (15)). In this case $W_\gamma$ vanishes identically on $\mathbb{C}^N_\gamma$. Indeed suppose by way of contradiction that $x_1^{m_1} \cdots x_q^{m_q}$ is a nonzero monomial of $W$ involving only $\gamma$-fixed variables (i.e. $g_1\lambda^{w_1}x_1 = x_1$, ..., $g_q\lambda^{w_q}x_q = x_q$). Then $\lambda^d = 1$ because we have

$$x_1^{m_1} \cdots x_q^{m_q} = (g_1\lambda^{w_1}x_1)^{m_1} \cdots (g_q\lambda^{w_q}x_q)^{m_q}$$

$$= \lambda^{w_1 m_1 + \cdots + w_q m_q}((q_1 x_1)^{m_1} \cdots (q_q x_q)^{m_q}) = \lambda^d (x_1^{m_1} \cdots x_q^{m_q}).$$

A contradiction.

In this way, a sector is attached to each $\gamma \in G\mathbb{C}^x$ and its coarse space is always a quotient of a smooth variety

$$\{W_\gamma = 0\}_{\gamma/G\mathbb{C}^x} \subset (\mathbb{C}^N \setminus \{0\})/G\mathbb{C}^x \text{ if } \gamma \in G.$$

$$\{W_\gamma = 0\}_{\gamma/G\mathbb{C}^x} = (\mathbb{C}^N \setminus \{0\})/G\mathbb{C}^x \text{ if } \gamma \not\in G;$$

Remark 11. The second case of the above dichotomy corresponds to the situation where the intersection between $X_W$ and a twisted sector of the ambient space is not transverse. In fact, $X_W$ contains the twisted sector. This is the main difference between the Gorenstein and nonGorenstein cases,
see Example 18. For a while, we considered it to be a major obstacle for the LG/CY correspondence.

The action of $\gamma$ on a fixed point $x \in \{ W_\gamma = 0 \} \mathbb{C}^N \setminus \{ 0 \}$ on the tangent space $T_x(\{ W = 0 \})$ can be written (in a suitable basis) as a diagonal matrix

$$\text{Diag}(\exp(2\pi i a^1_1), \ldots, \exp(2\pi i a^N_{N-1}))$$

for $a^j_\gamma \in [0, 1]$. Note that the matrix above is $(N-1) \times (N-1)$ because $\{ W = 0 \}$ is a smooth hypersurface in $\mathbb{C}^N \setminus \{ 0 \}$. We can read from the above matrix the so-called age shift (22)

$$a(\gamma) = a(\text{Diag}(\exp(2\pi i a^1_1), \ldots, \exp(2\pi i a^N_{N-1}))) = \sum_{l=1}^{N-1} a^l_\gamma.$$

Note that here we regarded $\gamma$ inside $GL(T_x(\{ W = 0 \}), N-1)$, but in our situation $\gamma$ naturally operates also on the affine space $\mathbb{C}^N$; we refer to Lemma 22 in Section 5 for a formula expressing the age $a_x(\gamma)$ given above in terms of the age of $\gamma$ as an element of $GL(\mathbb{C}^N, N)$.

We finally define the bigraded Chen–Ruan cohomology as a direct sum of ordinary cohomology groups of twisted sectors

$$H^{p,q}_{\text{CR}}([X_W/\tilde{G}]; \mathbb{C}) = \bigoplus_{\gamma \in \mathbb{C}^N} H^{p-a(\gamma), q-a(\gamma)}([W = 0]_\gamma/G\mathbb{C}_x; \mathbb{C}),$$

where $\{ W = 0 \}_\gamma$ denotes the locus $\{ x \in \{ W = 0 \} \mathbb{C}^N \setminus \{ 0 \} | \gamma x = x \}$, and the quotients appearing on the right hand side are quotient schemes and will be referred to as sectors. The total degree $\deg_{\text{CR}}$ of a class $\alpha \in H^{p,q}_{\text{CR}}([X_W/\tilde{G}]; \mathbb{C})$ is $p + q$:

$$H^d_{\text{CR}}([X_W/\tilde{G}]; \mathbb{C}) = \bigoplus_{p+q=d} H^{p,q}_{\text{CR}}([X_W/\tilde{G}]; \mathbb{C}).$$

We do not discuss the Chen–Ruan orbifold product, because we only regard $H_{\text{CR}}$ as a bigraded vector space.

3.3. The Landau–Ginzburg side. On the Landau–Ginzburg side $W$ is regarded as a $G$-invariant function

$$W: \mathbb{C}^N \to \mathbb{C},$$

and the fibre over the origin is singular. We associate a nondegenerate bigraded vector space to this singularity: the Fan–Jarvis–Ruan–Witten state space. It will be the counterpart on the Landau–Ginzburg side of Chen–Ruan cohomology on the Calabi–Yau side.

For each $\gamma = (\exp(2\pi i \Theta^1_\gamma), \ldots, \exp(2\pi i \Theta^N_\gamma)) \in G$, with $\Theta^j_\gamma \in [0, 1]$; recall the notations $\mathbb{C}^N_\gamma$, $N_\gamma$, and $W_\gamma$ from (17-19). The only critical point of $W_\gamma$ is the origin (see [FJRa, Lem. 3.2.1]). Let $\mathcal{H}_\gamma$ be the $G$-invariant terms of the middle-dimensional relative cohomology of $\mathbb{C}^N_\gamma$

$$\mathcal{H}_\gamma = H^{N_\gamma}(\mathbb{C}^N_\gamma, W^{+\infty}_\gamma; \mathbb{C})^G,$$
where $W^{+\infty} = (ReW_\gamma)^{-1}(\]M, +\infty\])$ for $M \gg 0$. The Fan–Jarvis–Ruan–Witten state space is

$$H_{FJRW}(W, G; \mathbb{C}) = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma;$$

by analogy with Chen–Ruan cohomology, the summands will be often referred to as sectors. We point out a special sector: for $\gamma = J_W$ the term $\mathcal{H}_\gamma$ is 1-dimensional; indeed $N_{J_W} = 0$ and the relative cohomology has a single ($G$-invariant) generator $1_{J_W}$. This is a good spot to introduce the so called Neveu–Schwarz sectors:

**Definition 12.** A sector $\mathcal{H}_\gamma$ is a Neveu–Schwarz sector as soon as $N_\gamma$ vanishes. A Neveu–Schwarz sector $\mathcal{H}_\gamma$ has a single canonical generator $1_\gamma$. Following established practice we call the remaining sectors Ramond sectors (see [FJRa]).

Using the Hodge decomposition of $\mathcal{H}_\gamma$, we define a bigraded decomposition of $H_{FJRW}$. As in Chen–Ruan cohomology, the age shift (22) plays a role: for example the total degree $d_{FJRW}$ of the terms $\mathcal{H}_\gamma$ is equal to $N_\gamma - 2 + 2a(\gamma)$ rather than the ordinary relative cohomology degree $N_\gamma$. More precisely the decomposition of $\mathcal{H}_\gamma$ in terms of $\mathcal{H}^{p,q}_{\gamma}$ is as follows

$$\mathcal{H}_\gamma = \bigoplus_{p+q = N_\gamma - 2 + 2a(\gamma)} \mathcal{H}^{p,q}_{\gamma}.$$

The state space of FJRW theory is then equipped with a bigrading

$$H_{FJRW}^{p,q}(W, G; \mathbb{C}) = \bigoplus_{\gamma \in G} \mathcal{H}^{p,q}_{\gamma},$$

and the total degree $\deg_{FJRW}$ of a class in $H_{FJRW}^{p,q}(W, G; \mathbb{C})$ is $p + q$; note that, by construction, for any $\alpha \in \mathcal{H}_\gamma$ and $\beta \in \mathcal{H}_{\gamma-1}$ we have

$$\deg_W(\alpha) + \deg_W(\beta) = 2N - 4.$$

**Remark 13.** We make an observation which may be regarded as the LG analogue of Remark 9. The CY condition plays a crucial role here: the FJRW-degree of the canonical generator $1_{J_W}$ of $\mathcal{H}_{J_W}$ vanishes. We mention in passing that, when the product is introduced, $1_{J_W}$ may be regarded as a unit of $H_{FJRW}(W, G; \mathbb{C})$ (see [FJRa] and [Kr]).

Furthermore, in [FJRa] the above structure is defined beyond the case of the CY condition: it is important to notice that in order to extend the structure together with the property $\deg_{FJRW}(1_J) = 1$ the authors involve

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2 We refer to Example [16] and Step 3 of the Proof of Theorem [14] (Section [5]) for a geometric interpretation of these sectors on the Calabi–Yau side.
3.4. The isomorphism. The main theorem provides an isomorphism between the Landau–Ginzburg side and the Calabi–Yau side. As mentioned in the introduction this goes beyond the expected correspondence for $G$ satisfying $J_W \in G \subseteq SL_W$ (see Example 28 where $G \not\subseteq SL_W$).

**Theorem 14.** Let $W$ be a nondegenerate quasihomogeneous polynomial of degree $d$ in the variables $x_1, \ldots, x_N$ whose charges add up to 1 (CY condition). Then, for any group $G$ of diagonal symmetries containing $J_W$ we have a bidegree-preserving isomorphism of vector spaces

$$H^{p,q}_{CR}(X_W/\tilde{G}; \mathbb{C}) \cong H^{p,q}_{FJRW}(W; G; \mathbb{C}).$$

For a scheme-theoretic counterpart of the above theorem we should consider $G \subseteq SL_W$. Then we have the following statement.

**Corollary 15.** Let $G$ be a subgroup of $SL_W$. Assume that $X_W/\tilde{G}$ admits a crepant resolution $Z$. Then, we have $H^{p,q}(Z; \mathbb{C}) \cong H^{p,q}_{FJRW}(W; G; \mathbb{C})$.

See Section 5 for the proofs; we now discuss some examples.

4. A first approach

The following examples will provide a concrete introduction to CR orbifold cohomology and the FJRW state space. In each case we will establish by hand the isomorphism of Theorem 14 stated in the introduction. This illustrates how certain sectors of the FJRW state space on the Landau–Ginzburg side are interchanged with cohomology classes on the Calabi–Yau side. The exchange is nontrivial and provides some early motivation for the introduction of a bookkeeping device: the diagram introduced in Section 5. All the examples below will be examined in Section 6 using the diagram.

**Example 16** (homogeneous polynomials). Theorem 14 is rather straightforward for a degree-$d$ hypersurface in $\mathbb{P}^{d-1}$. Here $(w_1, \ldots, w_d)$ is the $d$-tuple $(1, \ldots, 1)$ and the CY condition is automatically satisfied, $d = \sum_j w_j$. This is the case of a cubic curve in $\mathbb{P}^2$, a K3 surface in $\mathbb{P}^3$ (degree 4), and a quintic three-fold in $\mathbb{P}^4$. The Lefschetz hyperplane theorem yields $N-1$ cohomology classes: $1 \cap X_d, h \cap X_d, \ldots, h^{d-2} \cap X_d$ of bidegrees $(0,0), (1,1), \ldots, (d-2, d-2)$. The remaining classes, the cokernel of $H^\bullet(\mathbb{P}^{d-1}; \mathbb{C}) \to H^\bullet(X_d; \mathbb{C})$, are the primitive cohomology classes of degree $d-2$: the $(p,q)$ primitive cohomology classes can be identified with the $J_W$-invariant $(p+1,q+1)$-classes of $H^d(\mathbb{C}^d, W^{\infty}; \mathbb{C})$. For the cubic curve we have $(h^{1,0}, h^{0,1}) = (1,1)$, for the K3 surface we have $(h^{2,0}, h^{1,1}, h^{0,2}) = (1,20,1)$, and for the quintic three-fold we have $(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1,101,101,1)$. The Hodge “diamond” for the
quintic polynomial \((W, \langle J_W \rangle)\) on the Calabi–Yau side (recall that \(\langle J_W \rangle/\langle J_W \rangle\) is the trivial group \(\langle J_W \rangle/\langle J_W \rangle\)).

If we switch to the Landau–Ginzburg side and we compute the FJRW state space for \((W, \langle J_W \rangle)\), we get

\[
H_{FJRW} = \bigoplus_{i=0}^{d-1} H_{J_i}.
\]

There are \(d-1\) sectors, \(H_{J_i}\) with \(i \neq 0\), for which \(N_{J_i}\) vanishes: these are \(J_W\)-invariant relative cohomology classes of bidegree \((0, 0)\) in \(H^{N_{J_i}}(\mathbb{C}^{N_{J_i}}, \mathbb{O}; \mathbb{C})\). In other words we have \(d-1\) Neveu–Schwarz generators \(1_{J_1}, 1_{J_2}, \ldots, 1_{J_{d-1}}\) of FJRW bidegree \((0, 0), (1, 1), \ldots, (d-2, d-2)\). The sector \(H_{J_0}\) is by definition the \(J_W\)-invariant part of \(H^d(\mathbb{C}^d, \mathbb{C}; \mathbb{C})\); therefore we get the same Hodge diamond as on the Calabi–Yau side; i.e. for the quintic three-fold we get \((7)\).

We can further test Theorem 14 by choosing a larger group \(G \supseteq \langle J_W \rangle\).

There is only one observation that we wish to retain from this example: the Neveu–Schwarz sectors on the LG side are interchanged with the hyperplane sections on the CY side. Note also that their degrees match.

**Example 17** (quasihomogeneous polynomials inside a Gorenstein \(\mathbb{P}(w)\)). Let us consider \(W(x_1, x_2, x_3, x_4) = x_1^6 + x_2^3 + x_3^4 + x_4^3\), which is quasihomogeneous of degree 12 in four variables of weight 2, 3, 3, 4. On the Calabi–Yau side, we are interpreting this datum as a K3 surface \(S\) inside the Gorenstein weighted projective stack \(\mathbb{P}(2, 3, 3, 4)\) (all weights divide the sum of the weights 12).

We point out that the surface \(S\) has only two types of stack-theoretic points with nontrivial stabilizers: the 3 intersections of \(\{W = 0\}\) with \(\{x_2 = x_3 = 0\}\), which have stabilizer \(\mu_2\), and the 4 intersections of \(\{W = 0\}\) with \(\{x_1 = x_4 = 0\}\), which have stabilizer \(\mu_3\). These points contribute to the twisted sectors: on the one hand a point \(p\) with stabilizer \(\mu_2\) yields the pair (point, automorphism)\(= (p, 1)\) in the “untwisted” sector \(S_1\) and the pair \((p, -1)\) in the twisted sector \(S_{-1}\), on the other hand a point \(p\) with stabilizer \(\mu_3\) yields \((p, 1)\) in the “untwisted” sector \(S_1\) and \((p, \xi_3)\) in the twisted sector \(S_{\xi_3}\), and \((p, \xi_2^3)\) in the twisted sector \(S_{\xi_2^3}\). In this way the “twisted” sectors \((S, \gamma \neq 1)\) consist of \(4 + 4 + 3 = 11\) points. It is straightforward to see that all these points have age 1: therefore they contribute to an 11-dimensional subspace of \(H^{1,1}\) in CR orbifold cohomology. The remaining CR cohomology generators come from the sector \(S_1\), whose Hodge numbers are \((h^{2,0}, h^{1,1}, h^{0,2}) = (1, 9, 1)\). Putting everything together, we get the K3
surface Hodge diamond

\[
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
1 & 20 & 1 & 0 \\
1 & 0 & 0 & 1
\end{array}
\]

On the LG side we compute the FJRW state space. There are 12 sectors

\[ J^W_0 \quad J^W_1 \quad J^W_2 \quad J^W_3 \quad J^W_4 \quad J^W_5 \quad J^W_6 \quad J^W_7 \quad J^W_8 \quad J^W_9 \quad J^W_{10} \quad J^W_{11} \]

where the entry \( m \) for a coordinate stands for a coordinate \( \exp(2\pi i m) \) of the power of \( J^W \) which we are considering. (We have put no entries where there is no invariant element.) Putting everything together we recover the same Hodge diamond \((25)\).

We can test this further with the degree-60 three-fold \( \{x_1^{20} + x_2^6 + x_3^5 + x_4^4 + x_5^3 = 0\} \) contained in \( \mathbb{P}(3, 10, 12, 15, 20) \). We leave to the reader this interesting case, see Figure 9 at the end of the paper. The main point we wish to observe at this stage is that we find again the correspondence between Neveu–Schwarz sectors and hyperplane generated cohomology classes. This is less obvious than in the previous example because hyperplane generated classes occur also in the twisted sector: for instance the sector \( S_{-1} \) has 3-dimensional and corresponds to \( H_{J^W} \) for the primitive part and to one of the Neveu–Schwarz sectors for the nonprimitive part.

Example 18 (a nonGorenstein ambient space \( \mathbb{P}(w) \)). We now consider the polynomial \( W = x_1^4 x_2 + x_2^3 x_3 + x_3^3 x_4 + x_4^3 \) of degree 27 and weights 5, 7, 6, 9. On the CY side we have a K3 surface \( S \) inside the nonGorenstein weighted projective stack \( \mathbb{P}(5, 7, 6, 9) \). The study of the special points whose stabilizer is nontrivial is rather subtle. The ambient weighted projective stack has a
point with stabilizer $\mu_9$ and a point with stabilizer $\mu_5$. These two fixed loci behave differently with respect to $\{W = 0\}$ and illustrate the dichotomy (21): the first one $\{x_2 = x_3 = x_4 = 0\}$ is intersected transversely (i.e. the intersection is empty because $\{x_2 = x_3 = x_4 = 0\}$ is a point), the second one $\{x_1 = x_2 = x_3 = 0\}$ is intersected nontransversely (i.e. it is contained in $\{W = 0\}$). In Lemma 22 we show that this happens because the first stabilizer is an element of $\langle J \rangle$ whereas the second stabilizer is not. This phenomenon is the crucial point of this example and may be phrased as follows.

The stabilizers $\mu_7$, $\mu_6$, and $\mu_5$ arise as subgroups of $\mathbb{C}^\times$ generated by $\xi_7, \xi_6, \xi_5$ acting as $\lambda(x_1, \ldots, x_4) = (\lambda^5 x_1, \lambda^7 x_2, \lambda^6 x_3, \lambda^9 x_4)$. These elements are not contained in the group generated by $J_W = (\xi_{27}, \xi_{27}^2, \xi_{27}^3, \xi_{27}^6)$. These special group elements should be treated in a special way both on the CY side and the LG side. This happens whenever the ambient space is not special and will require the study of extra group elements (beyond $\langle J_W \rangle$) (see Example 27 and Figure 4 illustrating the present example).

We continue the computation, which yields the Hodge diagram for K3 surfaces (25). Indeed, the untwisted sector has one $(0,0)$-class, one $(2,2)$-class and the following decomposition in degree two, $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 3, 1)$. On the other hand there are four special points with stabilizers of order 5, 7, 6, and 3: namely $\{x_2 = x_3 = x_4 = 0\}$ (order 5), $\{x_1 = x_3 = x_4 = 0\}$ (order 7), $\{x_1 = x_2 = x_4 = 0\}$ (order 6) and $\{x_1 = x_2 = x_3^2 + x_4^2 = 0\}$ (order 3). These contribute to the twisted sectors with $(5-1) + (7-1) + (6-1) + (3-1) = 17$ points representing $(1,1)$-classes due to the age shift (which is again 1). This matches (25).

On the LG side we only can run a simple check for brevity. The CY side shows 16 sectors, as many as the elements of $\mu_7 \cup \mu_6 \cup \mu_3 \cup \mu_5$, which contribute with 18 hyperplane sections (because the untwisted sector is two-dimensional and yields $1, h, h^2$). We find 20 corresponding Neveu–Schwarz sectors on the LG side: $J_W^h$ for $h$ prime to $\text{deg}(W) = 27$.

Example 19 (group quotients). We conclude this first study of the claim of Theorem 14 with an example where $G \supseteq \langle J_W \rangle$. As the previous section already shows, a detailed analysis of the twisted sectors on the CY side may be very delicate. Fortunately, the theory of elliptic curves provides a very well known and illuminating example. We mention that this provides an example where the Landau–Ginzburg/Calabi–Yau correspondence holds beyond $SL(3, \mathbb{C})$.

Let $W(x_1, x_2, x_3) = x_1^2 x_2 + x_2^2 x_3 + x_3^3$ and set $G$ equal to the maximal group $\text{Aut}(W)$, which is cyclic of order 12 and is generated by the element $(\exp(2\pi i/12), \exp(2\pi i/12), \exp(2\pi i/12))$. The hypersurface defined by $W = 0$ is a cubic curve in $\mathbb{P}^2$. The group $G = G/\langle J_W \rangle$ is cyclic of order
4 and the action fixes the point represented by $e_0 := \{x_2 = x_3 = 0\}$ (over this coordinate subspace the polynomial $W$ vanishes). We may regard $E = \{W = 0\}$ as a genus-1 curve with a marking $e_0 \in E$: an elliptic curve $(E, e_0)$. Since there is only one elliptic curve with automorphism group of order 4 ($j$-invariant 1728), we know that $(E, e_0)$ is isomorphic to

$$(\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), [0] \in \mathbb{C})$$

and the automorphism may be regarded as the complex multiplication by $i$. There are only three special points which do not consist of four distinct points: the one-point orbit $\{e_0 = [0]\}$ (with stabilizer $\tilde{G}$), the one-point orbit $\{1/2 + i/2\}$ (with stabilizer $\tilde{G}$), and the two-point orbit containing $1/2$ and $i/2$ (both with stabilizer of order 2). Therefore the stack-theoretic quotient $[E/\tilde{G}]$ has only three special (i.e. nonrepresentable) points with stabilizers of order $m_0 = 4$, $m_1 = 4$, and $m_2 = 2$ (the coarse space is actually a projective line $E/\tilde{G} \cong \mathbb{P}^1$). It is now easy to visualize the sectors: apart from the “untwisted” sector, we find $\sum_i (m_i - 1) = 7$ “twisted” sectors corresponding to points paired with their nontrivial automorphism. We expect a 9-dimensional CR cohomology vector space $H^\bullet_{CR}([E/\tilde{G}]; \mathbb{C})$ with a 2-dimensional contribution from the “untwisted” sector ($H^\bullet(\mathbb{P}^1) \cong \mathbb{C} \oplus \mathbb{hC}$) and seven twisted 1-dimensional contributions mentioned above (graded by twice the age). The picture is illustrated in Figure 1 where the Hodge numbers are also listed.
We finally check that the above computation matches the LG side. By $\gamma$, we denote the order-12 generator of $G$.

| $\gamma^h$ | $x_1$ | $x_2$ | $x_3$ | $\deg_{\text{FJRW}}$ | $(h^p.q \mid p + q = \deg_{\text{FJRW}})$ |
|------------|-------|-------|-------|-----------------|----------------------------------|
| $\gamma^0$ | 0     | 0     | 0     |                 |                                  |
| $\gamma^1$ | 1     | 10    | 4     | 0.5             | $h^{1/4}.1/4 = 1$                |
| $\gamma^2$ | 2     | 8     | 8     | 1               | $h^{1/2}.1/2 = 1$                |
| $\gamma^3$ | 3     | 6     | 0     |                 |                                  |
| $\gamma^4$ | 4     | 4     | 4     | 0               | $h^{0}.0 = 1$                    |
| $\gamma^5$ | 5     | 2     | 8     | 0.5             | $h^{1/4}.1/4 = 1$                |
| $\gamma^6$ | 6     | 0     | 0     | 1               | $h^{1/2}.1/2 = 1$                |
| $\gamma^7$ | 7     | 10    | 4     | 3/2             | $h^{3/4}.3/4 = 1$                |
| $\gamma^8$ | 8     | 8     | 8     | 2               | $h^{1}.1 = 1$                    |
| $\gamma^9$ | 9     | 6     | 0     |                 |                                  |
| $\gamma^{10}$ | 10   | 4     | 4     | 1               | $h^{1/2}.1/2 = 1$                |
| $\gamma^{11}$ | 11   | 2     | 8     | 1               | $h^{3/4}.3/4 = 1$                |

Once again we put no entries where there is no invariant element. The Hodge numbers match those listed in Figure [1]

5. Proof of the main result: a combinatorial model

The proof is structured in five steps as follows. On the Calabi–Yau side, we further detail the decomposition of the CR cohomology (Step 1). Then, we do the same for the FJRW state space on the Landau–Ginzburg side (Step 2). We provide a diagram which schematizes and assembles into one picture the sectors on the two sides (Step 3). We prove a lemma which allows us to read off $\deg_{\text{CR}}$ and $\deg_{\text{FJRW}}$ on the diagram (Step 4). We establish an isomorphism using the combinatorial model (Step 5).

Step 1: Calabi–Yau side. Consider the decomposition (23) of $H_{\text{CR}}$ as a sum over $G\mathbb{C}^\times$. The complex dimension of $H_{\text{CR}}$ is finite although this is not evident from (23). Indeed, we can decompose $G\mathbb{C}^\times$ modulo $\mathbb{C}^\times$ into $M = |G|/d$ cosets. Let us choose $M$ distinct cosets $g^{(1)}\mathbb{C}^\times, \ldots, g^{(M)}\mathbb{C}^\times$ so that $g^{(1)}, \ldots, g^{(M)} \in G$ and the set $\sqcup_{i=1}^{M} g^{(i)} \mathbb{C}^\times$ equals the set $G\mathbb{C}^\times$. Now, we describe the direct sum

$$\bigoplus_{\gamma \in g\mathbb{C}^\times} H^\bullet(\{W_\gamma = 0\}/G\mathbb{C}^\times; \mathbb{C})$$

where $g$ is any of the elements $\{g^{(1)}, \ldots, g^{(M)}\}$. By construction $H_{\text{CR}}$ is the direct sum of the expressions above for $g$ ranging over $\{g^{(1)}, \ldots, g^{(M)}\}$.

Now we exhibit a finite number of terms of $g\mathbb{C}^\times$, outside which the summand of (26) vanishes. Regard an element $g \in G$ as an $N$-tuple of elements
of \( \mathbb{C}^\times \),

\[ g = (g_j)_{j=1}^N. \]

Notice that specifying \( \gamma \) in \( g\mathbb{C}^\times \) is equivalent to choosing \( \lambda \in \mathbb{C}^\times \) so that \( \gamma = g\lambda = (g_j)_{j=1}^N(\lambda^{w_j})_{j=1}^N. \) Since \( g\lambda \) acts by multiplication on the coordinates, the fixed locus is nonempty if and only if \( \lambda \) is contained in the finite set \( \bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \} \). In this way (26) can be rewritten as a direct sum of a finite number of finite dimensional vector spaces

\[ \bigoplus_{\lambda \in \bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \}} H^\bullet(\{ W_{g\lambda} = 0 \}_{g\lambda}/G\mathbb{C}^\times; \mathbb{C}), \]

where the notation \( \lambda \) of (14) has been used.

The quotient scheme \( \{ W_{g\lambda} = 0 \}_{g\lambda}/G\mathbb{C}^\times \) may be regarded as the quotient scheme by \( G\mathbb{C}^\times /\mathbb{C}^\times = \tilde{G} \) of the hypersurface \( \{ W_{g\lambda} = 0 \} \) inside the weighted projective space \( \mathbb{P}(w_{\lambda}) \) where \( w_{\lambda} \) is the multi-index

\[ w_{\lambda} = \{ w_j \mid \lambda^{-w_j} = g_j \}. \]

In this way we have

\[ H^\bullet(\{ W_{g\lambda} = 0 \}_{g\lambda}/G\mathbb{C}^\times; \mathbb{C}) = H^\bullet(\{ W_{g\lambda} = 0 \}_{\mathbb{P}(w_{\lambda})}; \mathbb{C})^{\tilde{G}}. \]

Notice that the number of entries of \( w_{\lambda} \) equals \( N_{g\lambda}. \)

The cohomology \( H^\bullet \) of a hypersurface \( S \) inside a weighted projective stack splits into two summands. The first summand is generated by the self-intersections of the hyperplane sections: \( 1_S, h \cap S, h^2 \cap S, \ldots \). In the case of \( \{ W_{g\lambda} = 0 \}_{\mathbb{P}(w_{\lambda})} \), this summand of \( H^\bullet(\{ W_{g\lambda} = 0 \}_{\mathbb{P}(w_{\lambda})}; \mathbb{C}) \) is \((N_{g\lambda}-1)\)-dimensional. We point out that all these terms are \( \tilde{G} \)-invariant. The second summand is the primitive cohomology and is concentrated in degree \( \delta = \dim_{\mathbb{C}}(S) \) (if \( \dim_{\mathbb{C}}(S) \) is odd this summand is the entire cohomology group \( H^\delta(\cdot; \mathbb{C}) \), otherwise the rank of this summand equals the Betti number \( b_\delta = \dim H^\delta \) minus 1). By the theory of the Milnor fibre \[ \text{[St77] [Do82] [Di92] we may express the primitive cohomology as} \]

\[ H^\bullet_N(\mathbb{C}^\gamma, W_{\gamma}^+; \mathbb{C})^{(J_W)}. \]

This happens because the \( J_W \)-action is the monodromy action on the Milnor fibre of

\[ W_{\gamma}: \mathbb{C}^{N_{\gamma}} \to \mathbb{C}. \]

In this way the \( \tilde{G} \)-invariant part of the primitive cohomology of the hypersurface \( \{ W_{g\lambda} = 0 \}_{\mathbb{P}(w_{\lambda})} \) is isomorphic to \( H^\bullet_N(\mathbb{C}^\gamma, W_{\gamma}^+; \mathbb{C})^{(J_W)} \) (the isomorphism identifies \((p, q)\)-classes in \( H^\bullet_N(\{ W_{g\lambda} = 0 \}_{g\lambda}/G\mathbb{C}^\times; \mathbb{C}) \) with \((p+1, q+1)\)-classes in \( H^\bullet_N(\mathbb{C}^\gamma, W_{\gamma}^+; \mathbb{C})^{(J_W)} \). In this way the group \( H^\bullet(\{ W_{g\lambda} = 0 \}_{g\lambda}/G\mathbb{C}^\times; \mathbb{C}) \)

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can be decomposed as

\[(29) \quad H^{N_{g\bar{\lambda}}} \left( \mathbb{C}^{N_{g\bar{\lambda}}}_g, W^{+\infty}_g \right) \mathbb{C}^G \oplus \bigoplus_{i=0}^{N_{g\bar{\lambda}}-2} \left[ \mathfrak{h}^i \cap \{ W_{g\lambda} = 0 \} \mathbb{P}(w_{\lambda}) \right] \mathbb{C}.\]

**Remark 20.** The summands on the right hand side contain \((i, i)\)-classes corresponding to cohomology classes in \(H^{2i}(\{ W_{g\lambda} = 0 \} g_{\bar{\lambda}}/GC^\times)\); whereas the first summand consists of \((p + 1, q + 1)\)-classes (with \(p, q \geq 0\)) of degree \(N_{\gamma}\) which represent \((p, q)\)-classes in the primitive cohomology of \(\{ W_{g\lambda} = 0 \} g_{\bar{\lambda}}/GC^\times\).

By summing the above expression over all \(\lambda \in \bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \}\) we get the entire finite-dimensional contribution to \(H^\bullet_{\text{CR}}\) coming from the coset \(g_{\bar{\lambda}}C\).

**Step 2: Landau–Ginzburg side.** We analyze the FJRW state space in a similar way:

\[H^\bullet_{\text{FJRW}}(W, G; \mathbb{C}) = \bigoplus_{\gamma \in G} \mathcal{H}_{\gamma}.\]

For \(J = J_W\), we decompose \(G\) into \(M = |G|/d\) distinct cosets \(g(1)\langle J \rangle, \ldots, g(M)\langle J \rangle\) (we choose the same \(g(1), \ldots, g(M)\) as in the previous step). Therefore the FJRW state space is a direct sum of the terms

\[\bigoplus_{i=0}^{d-1} H^{N_{g, J_i}} \left( \mathbb{C}^{N_{g, J_i}}_g, W^{+\infty}_{g, J_i} \right) \mathbb{C}^G\]

for \(g\) ranging in \(\{ g(1), \ldots, g(M) \}\) (we are just making the definition of \(\mathcal{H}_{g, J_i}\) explicit).

Write \(g = (g_j)_{j=1}^N\) as usual. We point out that if \(\xi_d^i\) does not belong to \(\bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \}\), then \(N_{g, J_i} = 0\). In other words \(\mathcal{H}_{g, J_i}\) is of Neveu–Schwarz type. We finally express the entire contribution to \(H^\bullet_{\text{FJRW}}\) coming from the coset \(g\langle J \rangle:\)

\[\bigoplus_{\lambda \in \mu_d \cap \bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \}} H^{N_{g\bar{\lambda}}} \left( \mathbb{C}^{N_{g\bar{\lambda}}}_g, W^{+\infty}_{g\bar{\lambda}} \right) \mathbb{C}^G \oplus \bigoplus_{\lambda \in \mu_d \setminus \bigcup_{j=1}^N \{ \lambda \mid \lambda^{-w_j} = g_j \}} 1_{g\bar{\lambda}} \mathbb{C},\]

where we used the notation \([14]\), and we identified the terms of \(\langle J \rangle\) as \(\bar{\lambda}\) for \(\lambda \in \mu_d\) (e.g. \(J = \xi_d\)).

**Step 3: the diagram.** In the previous two steps we split the state spaces into \(M\) summands corresponding to a set of \(M\) elements \(g(1), \ldots, g(M)\) in \(G\). Each summand is efficiently represented by a diagram, which may be regarded as a generalization of Boissière, Mann, and Perroni’s model [BMP09].
Again, let us choose one of the above elements $g^{(1)}, \ldots, g^{(M)}$ and denote it by $g$; we describe the corresponding diagram. It consists of half-lines (rays) stemming from the origin in the complex plane and points lying on them (dots). The dots will correspond to (sets of) generators in CR cohomology, whereas the rays will represent sectors of the FJRW state space. Draw a ray
\[
\{ \rho \nu \in \mathbb{C} \mid \rho \in \mathbb{R}^+ \} \subset \mathbb{C}
\]
whenever we have
\[
(30) \quad \nu \in \mu_d \cup \bigcup_{j=1}^{N} \{ \alpha \in \mathbb{C} \mid \alpha^{w_j} = g_j \}.
\]
Mark a dot
\[
j \nu \in \mathbb{C}
\]
whenever $\nu^{w_j} = g_j$ for some $j$; in other words, whenever $\nu$ and $j$ satisfy
\[
\nu \in \{ \alpha \in \mathbb{C} \mid \alpha^{w_j} = g_j \}.
\]
Mark further dots
\[
(N+1) \nu \quad \text{whenever}
\]
\[
\nu \in \left( \bigcup_{j=1}^{N} \{ \alpha \in \mathbb{C} \mid \alpha^{w_j} = g_j \} \right) \setminus \mu_d.
\]
For a nontrivial but low-dimensional example we refer the reader to Figure 4 where the diagram is drawn for the above-mentioned K3 surface $\{x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2\} \subset \mathbb{P}(5, 7, 6, 9)$.

This model can be related to the sectors of the two CR and FJRW spaces. The coset determined by $h$ with $h = 1 \in G$ is the case treated in [BMP09] and, for the sake of clarity, we discuss it first. This corresponds to assuming $G = \langle J \rangle$ and looking at the hypersurface $\{W = 0\} \subset \mathbb{P}(w)$ (if $G = \langle J \rangle$, then $\tilde{G} = 1$). Since $g_j = 1$ for all $j$, following (30), we find that the rays correspond to the elements of $\mu_d \cup \mu_{w_1} \cup \cdots \cup \mu_{w_N}$. The rays that carry some dots are in one-to-one correspondence with the sectors associated to the hypersurface $\{W = 0\}$ inside $\mathbb{P}(w_1, \ldots, w_N)$. If we write a ray as $\{ \rho \nu \mid \rho \in \mathbb{R}^+ \}$ with $|\nu| = 1$ then the corresponding sector is the hypersurface $\{W_\lambda = 0\}_{\mathbb{P}(w_\lambda)}$ for $\lambda = \nu^{-1}$. Simply by unraveling the definitions, the authors of [BMP09] make the following useful observation: a ray carries as many dots as the quasihomogeneous coordinates of the corresponding weighted projective subspace $\mathbb{P}(w_\lambda)$. Building upon this, one can derive a combinatorial model for the cohomology of the sector $S = \{W_\lambda = 0\}_{\mathbb{P}(w_\lambda)}$: namely, we let the first $N_\lambda - 1$ dots represent the hyperplane sections $1S, h \cap S, h^2 \cap S, \ldots, h^{N_\lambda-2} \cap S$ and the $N_\lambda$th dots represent the primitive
cohomology. In this way all the dots are attached to a summand of the CR cohomology of $X_W$. On the Landau–Ginzburg side, we can use the diagram as follows: the rays with angular coordinate $2\pi l/d$ can be associated to the summand $H_{j-l}$ of the FJRW state space of $(W, G = \langle J \rangle)$. The number of dots on one of these rays corresponds to the index $N_{j-l}$.

The general procedure for a coset represented by $h$ is as follows. Similarly to the case $h = 1$, the rays whose angular coordinate is $2\pi l/d$ represent the sector of the FJRW space $H_{hJ-l}$. We point out that, by construction, a sector is of Neveu–Schwarz type if and only if it is empty; i.e. it does not carry any dot. The dots always lie on some ray by construction: consider the dot $m_\nu$ (with $m \in \mathbb{N}$ and $\mu \in \{z \mid |z| = 1\}$) lying on the ray $\{\rho_\nu \mid \rho \in \mathbb{R}^+\}$. We say that it is an extremal dot if there is no other dot with higher polar coordinate and is an internal dot otherwise. An extremal dot $m_\nu$ corresponds to the primitive cohomology of $H^\bullet(\{W_{g\lambda} = 0\}_{g\lambda} / G \times \mathbb{C})$ for $\lambda = \nu^{-1}$. The internal dots $m_1\nu, m_2\nu, m_3\nu, \ldots$ lying on $\{\rho_\nu \mid \rho \in \mathbb{R}\}$ can be ordered with respect to their polar coordinates and represent hyperplane sections in Chen–Ruan cohomology of the sector $\{W_{g\lambda} = 0\}_{g\lambda} / G \times \mathbb{C}$ for $\lambda = \nu^{-1}$; the first dot corresponds to the fundamental class of $\{W_{g\lambda} = 0\}_{g\lambda} / G \times \mathbb{C}$, the next corresponds to the intersection with $h$, and so on.

We refer to Example 28 for a simple, and nevertheless interesting, demonstration of the above procedure (we wrote it in such a way that the reader can skip directly there for a detailed description of the diagram attached to a coset).

Now, we define two functions $D$ and $R$ on the union of the sets of rays and of dots. They essentially count dots and rays and they can be efficiently used in order to express the quantities $\deg_{\text{CR}}$ and $\deg_{\text{FJRW}}$ for the corresponding classes. Notice that dots and rays are naturally ordered: the rays can be arranged according to the angular coordinate ranging over $[0, 1]$ whereas the dots can be arranged in lexicographic order $\preceq$ (recall that for $\vartheta, \vartheta' \in [0, 1]$ we write $\rho \exp(2\pi i \vartheta) \preceq \rho' \exp(2\pi i \vartheta')$ if and only if we have $\vartheta \leq \vartheta'$ or, for $\vartheta = \vartheta'$, we have $\rho \leq \rho'$). We can actually order the set given by the union of dots and rays: for this, we require that a ray precedes all dots lying on it and on the following rays (to this effect a ray $\{\rho_\nu \mid \rho \in \mathbb{R}^+\}$ may be treated as the point $(1/2)\nu$ and arranged according to $\preceq$). Now we define the functions $R$ and $D$. The function $R$ is naturally defined on all rays and takes values in the natural numbers ranging from 0 to the size of the set $\bigcup_{j=1}^N \{\alpha \in \mathbb{C} \mid \alpha^{w_j} = g_j\}$ minus one. It is defined by simply counting the rays in the sense of the angular coordinate (i.e. anticlockwise). The function $D$ is naturally defined on the set of dots and takes values in the natural numbers ranging from 0 to the number of dots minus 1. It is defined by counting the dots in lexicographic order. We may naturally extend the
function $D$ to the set of rays: simply assign to a ray the value $D$ of the first preceding dot (if the ray precedes all dots we set $D = -1$). We naturally extend $R$ to the set of dots: a dot takes the value $R$ assigned to the ray on which it lies.

Remark 21. The functions $R$ and $D$ range over the same finite set of numbers. This happens because the number of rays is clearly $d$ plus the number of elements of $\bigcup_j \mu_{w_j} \setminus \mu_d$. On the other hand the number of dots can be computed as follows. The number of dots $jv$ with $|jv| \leq N$ is $\sum_j w_j$ because each equation $\nu_{w_j} = g_j$ has $w_j$ solutions. The remaining dots are precisely as many as the elements of $\bigcup_j \mu_{w_j} \setminus \mu_d$ by construction. The two counts match by the CY condition: $d = \sum_j w_j$.

Step 4: the degrees $\deg_{\text{CR}}$ and $\deg_{\text{FJRW}}$. Let $x \in \mathbb{C}^N$ be a point in
\[(\mathbb{C}^N \setminus \{0\}) \cap \{W = 0\} = \{W = 0\}_{\gamma}.
\]
(i.e. $\gamma x = x$ and $W(x) = 0$). By (21), if $\gamma \not\in G$, then the intersection is not transversal and $\mathbb{C}^N$ lies inside $\{W = 0\}$; otherwise, if $\gamma \in G$, the intersection is transversal and the intersection locus is again a smooth variety. Indeed, one can see directly that if $\gamma \in G$ the normal vector $\vec{n}(x)$ to $x \in \{W = 0\}$ lies in $\mathbb{C}^N$: hence the whole line
\[
\{y = x + \rho \vec{n}(x) \in \mathbb{C}^N \mid \rho \in \mathbb{R}\}
\]
is fixed (lies inside $\mathbb{C}^N$).

The explicit argument is as follows: let us arrange the coordinates so that $x_1, \ldots, x_q$ are all the $\gamma$-fixed coordinates: i.e. if $\gamma = (g_1, \ldots, g_N)$ we have $g_1 = \cdots = g_q = 1$. Then, for any $j > q$ we have $g_j \neq 1$. We conclude that $\partial_j W(x) = 0$. This happens because $x \in \mathbb{C}^N$ is of the form $x = (x_1, \ldots, x_q, 0, \ldots, 0)$ and $\partial_j W(x) \neq 0$ only if there is a monomial of $W$ of the form $x_1^{m_1} \cdots x_q^{m_q} x_j$, which contradicts $g_j \neq 1$ because
\[
x_1^{m_1} \cdots x_q^{m_q} x_j = (g_1 x_1)^{m_1} \cdots (g_q x_q)^{m_q} (g_j x_j) = g_j (x_1^{m_1} \cdots x_q^{m_q} x_j).
\]

In the case $\gamma \not\in G$ we know that the normal line passing through $x$ with vector $\vec{n}(x)$ has only one fixed point: $x$. The following lemma describes this action precisely and embodies the previous observation that $\gamma$ acts trivially on $x$ for $\gamma \in G$.

Lemma 22. For any $\gamma = g \lambda \in G \mathbb{C}^N$, let $x \in \mathbb{C}^N \setminus \{0\}$ be a point of the hypersurface $\{W = 0\}$, which is fixed by $\gamma$; i.e. $x$ belongs to $(\mathbb{C}^N_{g \lambda} \setminus \{0\}) \cap \{W = 0\}$. Then $g \lambda$ acts on the normal line $\{y = x + \rho \vec{n}(x) \in \mathbb{C}^N \mid \rho \in \mathbb{R}\}$ by multiplication by $\lambda^d$ as follows
\[
g \lambda (x + \rho \vec{n}(x)) = x + \lambda^d \rho \vec{n}(x).
\]
In particular, the age $\alpha$ of $g\bar{\lambda}$ in $GL(\mathbb{C}, N)$ and the age $a_\star(\gamma)$ of $g\bar{\lambda}$ acting on the $(N - 1)$-dimensional tangent space $T_\star(\{W = 0\})$ are related as follows:

$$a_\star(g\bar{\lambda}) = \alpha - \langle sd \rangle \quad \text{if } \lambda = \exp(2\pi is) \quad \text{and} \quad s \in [0, 1[,$$

where $\langle sd \rangle$ denotes the fractional part of $sd$ (i.e. $sd - \lfloor sd \rfloor$).

As a consequence, on the diagram attached to $g = (g_1, \ldots, g_N) \in G$, the degree $\deg_{FJRW}$ of a class represented by an empty ray and the degree $\deg_{CR}$ of a class represented by an internal dot can be expressed as

$$2 \left( \sum_{j=1}^{N} s_j + D - R \right),$$

where $g_j = \exp((2\pi is_j))$ with $s_j \in [0, 1[.$

Proof. The first part is well known: the normal bundle to the hypersurface is a $\mathbb{C}^\times$-linearized line bundle $\mathcal{O}(d)$ with character $\lambda \mapsto \lambda^d$. We detail the argument by choosing a nonvanishing coordinate $\partial_{j_0}W(x)$ of $\bar{n}(x)$ and by proving that multiplying it by $g_{j_0}\lambda^w_{j_0}$ is the same as rescaling it by $\lambda^d$. To begin with, notice that the fact that $\partial_{j_0}W(x)$ does not vanish guarantees the existence of a monomial of $W$ with exponents $m_1, \ldots, m_N$ only involving the $j_0$th coordinate and coordinates for which $g_{j_0}\lambda^w_{j_0} = 1$. In other words, for $j \neq j_0$ we have $(g_j\lambda^w_j)^{m_j} = 1$, because either $m_j$ vanishes or $g_j\lambda^w_j$ equals 1.

Then there are two possibilities. First, if $g_{j_0}\lambda^w_{j_0} = 1$, then $\lambda^d = 1$,

$$\lambda^d = \lambda^{m_1w_1 + \cdots + m_Nw_N} = g_1^{m_1}\lambda^{m_1w_1} \cdots g_N^{m_N}\lambda^{m_Nw_N} = (g_{j_0}\lambda^w_{j_0})^{m_{j_0}} = 1.$$ 

Otherwise $g_{j_0}\lambda^w_{j_0} \neq 1$ and the $x_{j_0}$ coordinate is not $\gamma$-fixed. In this case $\partial_{j_0}W(x) \neq 0$ implies that $m_{j_0}$ is necessarily equal to 1: we have

$$g_{j_0}\lambda^w_{j_0} = g_{j_0}\lambda^w_{j_0} \prod_{j \neq j_0} (g_j\lambda^w_j)^{m_j} = \prod_{j} g_j^{m_j} \prod_{j} \lambda^{m_jw_j} = \lambda^d.$$ 

This completes the proof of the first part of the claim.

The formula immediately implies the expression for $a_\star(g\bar{\lambda})$ in terms of $\alpha$ and $\lambda$ in the statement. Indeed, we make that expression more explicit by assuming that $g$ equals $\exp((2\pi is_j))$ for $j = 1, \ldots, N$ and by writing $\lambda$ as $\exp(-2\pi it)$. 

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Then we have

\[ a_{x}(g\bar{\lambda}) = \sum_{j=1}^{N} \langle s_j - tw_j \rangle - \langle -td \rangle \]

\[ = \sum_{j=1}^{N} (s_j - tw_j) - \sum_{j=1}^{N} [s_j - tw_j] - (-td) - (-[td]) \]

\[ = \sum_{j=1}^{N} s_j + \left( - \sum_{j=1}^{N} [s_j - tw_j] \right) - (-[td]), \]

where the CY condition has been used in the last equality. The last part of the statement follows from relating the last two summands to the function \( D - R \) evaluated on an empty ray and internal dots.

The functions \( D \) and \( R \) introduced above have particularly convenient properties, which will be evident in the next step; however, in order to match the above expression we need to define two slightly different functions \( \tilde{D} \) and \( \tilde{R} \). The functions \( \tilde{D} \) and \( \tilde{R} \) only count (and are defined on) a special kind of dots and rays: the rays are those with angular coordinate within \((2\pi/d)N\) and the dots are those whose polar coordinate is (strictly) smaller than \(N + 1\) (i.e. \(|\cdot| \leq N\)). The union of these dots and rays is naturally ordered by the lexicographic order \( \preceq \) and the prescription that a ray precedes all dots lying on it and on the following rays. The function \( \tilde{R} \) is naturally defined on the considered rays by the angular coordinate times \(d/2\pi\) and takes values in \(\{0, 1, \ldots, d - 1\}\). The definition extends immediately to dots lying on the above-mentioned rays and also to a dot which does not lie on the considered rays: we assign to it the value \( \tilde{R} \) of the next ray (and we assign \(d\) if there is no next ray). On the other hand, the function \( \tilde{D} \) is defined by counting in lexicographic order the dots with \(|\cdot| \leq N\). Again, we may naturally extend the function \( \tilde{D} \) to the set of rays: simply assign to a ray the value \( \tilde{D} \) of the first preceding dot (if the ray precedes all dots we set the value of the function here to \(-1\)). We point out that \( D - R \) coincides with \( \tilde{D} - \tilde{R} \) on internal dots and on empty rays.

The claim follows. An empty ray necessarily has angular coordinate \((2\pi)l/d\) and corresponds to the sector \(\mathcal{H}_{gJ-l}\). Since \(-\sum_{j}[s_j - tw]\) equals

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3This is straightforward apart from the case of an internal dot whose angular coordinate is not in \((2\pi/d)N\), where it holds because, there, \( \tilde{R} \) has been defined as the value of the next ray.
On the other hand, for internal dots, the only interesting check concerns the first dot of one ray \( \{ \rho \exp(2\pi i t) \mid \rho \in \mathbb{R}^+ \} \). There, the identities \( \tilde{R} = -[-td] \) and \( \tilde{D} = -\sum_j [s_j - tw_j] \) hold. Therefore the degree \((1/2) \deg_{CR}\) of the fundamental class of \( \{ W_{g\lambda} = 0 \}/G \mathbb{C}^\times \) for \( \lambda = \exp(-t) \) equals

\[
a(g\tilde{\lambda}) = \sum_{j=1}^{N} s_j + \left( -\sum_{j=1}^{N} \left[ s_j - tw_j \right] \right) - \left( -[-td] \right) = \sum_{j} s_j + \tilde{D} - \tilde{R} = \sum_{j} s_j + D - R.
\]

Step 5: The correspondence. We finally establish the bidegree preserving isomorphism. We will be guided by the above diagram which highlights sets of generators of \( H_{FJRW} \) (the rays) and sets of generators of \( H_{CR} \) (the dots). They correspond to each other in a degree-preserving way.

Let us first remark that the subspaces corresponding to extremal dots in the CR-cohomology are isomorphic to the subspaces corresponding to the non-empty rays in the FJRW-state space. First, if the angular coordinate of the ray is not contained in \((2\pi/d)\mathbb{N}\), then no sector of \( H_{FJRW} \) is attached to this ray. On the other hand the primitive cohomology corresponding to the extremal point on this ray is \( \{ 0 \} \) because the sector is the quotient of a weighted projective stack by a finite group action, see (21). Let us focus on a ray \( \{ \rho \nu \mid \rho \in \mathbb{R}^+ \} \) with \( \nu \in \mu_d \). In this case, the extremal dot is the primitive cohomology of the quotient of a hypersurface inside a weighted projective stack; this has already been expressed in terms of \( G \)-invariant cohomology classes in relative cohomology. Remark 20 yields the required bidegree-preserving isomorphism.

We finally need to match the internal dots with the empty rays. As remarked above, these objects correspond to \((p, p)\)-classes in the respective \( H_{CR} \) and \( H_{FJRW} \) spaces (hyperplane sections and Neveu–Schwarz sectors). By Lemma 22, we only need to provide an involution exchanging internal dots and empty rays and preserving \( D - R \). This is constructed in the next lemma.
Lemma 23. There exists a 1-to-1 correspondence between internal dots and empty rays that preserves $F = D - R$.

Proof. The domain formed by all rays and dots introduced in Step 3 is totally ordered. The last element is a dot and the first is the real-axis ray $R^+$. Using this order, for any element $n$ different from the last dot $n + 1$ will denote the next element, whereas for any element $n$ different from the real-axis ray $R^+$ we will write $n - 1$ for the preceding element.

On the one hand, $n$ is a ray if and only if $F(n - 1) = F(n) + 1$ or $n = R$. On the other hand, $n$ is a dot if and only if $F(n - 1) = F(n) - 1$. In other words $F$ is decreasing when it reaches a marking and is increasing when it reaches a ray. It never varies by more than 1. Furthermore the CY condition ensures that $F$ vanishes on the last value of its domain (in other words the number of dots equals the number of rays). It follows that $F$ may be regarded as a function defined on a set of elements forming a circuit where the last dot is followed by the first real-axis ray $R^+$. Now notice that if $F$ attains a given value at a given number of internal markings (going down) it must attains the same value at the same number of empty rays (going up). Notice that extremal dots and nonempty rays are the relative maxima and minima of $F$. □

This completes the proof of Theorem 14. □

Proof of Theorem 2. By Krawitz’s main theorem we have $h^{p,q}(W, G; \mathbb{C}) = h^{N-2-p,q}(W^T, G^T; \mathbb{C})$ (see [Kr, §2.4] and use the fact that $\hat{c} = N - 2$). In this way Theorem 14 yields the claim. □

Remark 24. For $G \subseteq SL_W$, the action of $G\mathbb{C}^x$ on $\{W = 0\} \subset \mathbb{C}^N$ satisfies the following property. Consider the point $x$ in $\{W = 0\}$ and any element $\gamma = g\lambda$ of $G\mathbb{C}^x$ fixing $x$; then, the $(N - 1)$-dimensional representation $\gamma$ in $GL(T_x\{W = 0\})$ has determinant 1. This happens because $\gamma$ acts on the line through $x$ orthogonal to $T_x\{W = 0\}$ as $z \mapsto \lambda^d z$. Therefore we have $\det(\gamma \in GL(T_x\{W = 0\}) \lambda^d = \prod_{j=1}^{N} (g_j \lambda^{w_j})$; by the CY condition and $G \subseteq SL_W$, we obtain

$$\det(\gamma \in GL(T_x\{W = 0\})) = \lambda^{-d} \prod_{j=1}^{N} (g_j \lambda^{w_j}) = \lambda^{\sum_j w_j - d} \prod_j g_j = 1.$$ 

As a consequence the quotient stack $[X_W/\tilde{G}]$ has no nontrivial orbifold behaviour in codimension 1. Therefore, we can relate the ordinary cohomology of the coarse space to the Chen–Ruan orbifold cohomology of the stack. Let us assume that the coarse space of $[X_W/\tilde{G}]$, the scheme-theoretic quotient
Figure 2. Diagram of the Fermat quintic in \( \mathbb{P}^4 \).

\( X_W/\tilde{G} \), admits a crepant resolution \( Z \). Then, there is a bidegree preserving isomorphism between the cohomology of \( Z \) and the orbifold Chen–Ruan cohomology of \( X_W/\tilde{G} \). In this way Corollaries 4 and 15 follow.

6. Examples

We now recover the examples treated in Section 4 and see how they fit in the diagram illustrated in the course of the proof.

Example 25. Let us consider the case of a degree-\( d \) hypersurface in \( \mathbb{P}^{d-2} \) (Example 16). In general, the diagram has \( d - 1 \) empty rays and \( d - 1 \) dots on the real-axis ray. The diagram for the quintic polynomial in five variables looks as in Figure 2. The four internal points are the hyperplane sections of the quintic hypersurface whereas the four empty rays are the Neveu–Schwarz sectors of the FJRW state space. They correspond to each other and the degrees match (they can be computed following the definition or evaluating the function \( D - R \) as in Lemma 22 using the diagram).

Example 26. Here we illustrate the model in the case of a K3 surface inside a Gorenstein weighted projective stack. We take the same polynomial as in Example 17 and we get the diagram found by Boissière, Mann and Perroni without modifications. In fact, in [BMP09], this diagram is used to describe the sectors of the weighted projective stack \( \mathbb{P}(2, 3, 3, 4) \); indeed, the dotted rays correspond to the sectors, and the number of dots lying on one ray corresponds to the dimension of the cohomology of the corresponding sector (which, in turn, is a weighted projective stack). If we consider the hypersurface where \( W(x_1, \ldots, x_4) = x_1^6 + x_2^4 + x_3^4 + x_4^3 \) vanishes we can use the diagram as described in Step 3 of the proof. The sectors should be regarded
as hypersurfaces lying inside the sectors of the ambient weighted projective stack. In the surface above we actually have six dotted rays corresponding to the sectors of the ambient projective stack. When the ray carries a single dot, the hypersurface is empty. When the ray carries two dots the hypersurface is 0-dimensional. Hence, in the example there are only four nonempty sectors corresponding to \( J^0 = 1, J^{-4}, J^{-6}, \) and \( J^{-8} \). In general \( n \) dots on one ray correspond to an \((n-2)\)-dimensional hypersurface: the first \( n - 1 \) dots counting from the origin are the classes cut out by \( 1, h, \ldots, h^{n-2} \), whereas the extremal dot corresponds to the contribution from primitive cohomology. Beside each dot we mark the value of \( D - R \); the reader may check that this coincides with half \( \text{deg}_{CR} \) of the corresponding class in Chen–Ruan orbifold cohomology (see Example 17).

We leave to the reader the three-fold \( x_1^{20} + x_2^6 + x_3^5 + x_4^4 + x_5^3 \) inside the Gorenstein weighted projective stacks; we only provide the combinatorial diagram (see Figure 9 at the end).

**Example 27.** We now illustrate by means of the diagram the case where the hypersurface is embedded in a nonGorenstein weighted projective stack.
Consider the K3 surface of Example 18. We illustrate the corresponding diagram (Figure 4).

Two groups should be considered. On the one hand the union of the roots of unity of order 5, 7, 6, and 9 (the weights): \( H_1 = \mu_5 \cup \mu_7 \cup \mu_6 \cup \mu_9 \). On the other hand the roots of unity of order \( d = 27 \) (the degree): \( H_2 = \mu_{27} \). The nonGorenstein case is characterized by the following feature: \( H_2 \not\subseteq H_1 \).

Let us now go through the definition. We draw a ray for every element of \( H_1 \cup H_2 \). In this way we have 40 rays (13 of them are special because they correspond to elements of \( H_2 \setminus H_1 \)). We mark dots on the four circles corresponding to the four coordinates: 5 dots on the first, 7 dots on the second, 6 on the third, and 9 on the fourth. Following the construction of Step 3 of the proof, we mark 13 further dots with polar coordinate \( N + 1 \).

The presence of rays whose angular coordinate is not in \( 2\pi \{0, \frac{1}{27}, \ldots, \frac{26}{27}\} \) corresponds to the fact that there are sectors that do not intersect transversely \( \{W = 0\} \). The correspondence still holds because the presence of extra rays is balanced by the presence of extra dots.
Example 28. This example is meant to illustrate the setup of the proof in the more delicate cases where nontrivial \( \langle J \rangle \)-cosets are involved. We consider the cubic equation already studied in Example 19, i.e. \( x^2_1 x_2 + x^2_2 x_3 + x^3_1 = 0 \), and the order-12 cyclic group \( G = \text{Aut}(W) \).

As in the proof, we proceed coset by coset. Note that \( \gamma^4 = J \), therefore the natural choices corresponding to \( g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)} \) in the proof are \( \gamma^0, \gamma^1, \gamma^2, \gamma^3 \).

We start from the coset attached to \( g = \gamma^0 = (1, 1, 1) \) and we apply the previous construction. The terms \( (g_1, \ldots, g_N) \) are the \( N \) coordinates of \( g \in (\mathbb{C}^\times)^N \): in this case they are all equal to 1. We have \( \{ \alpha \mid \alpha^w = g_j \} = \{ 1 \} \) because the weights are all equal to 1. We have

\[
\alpha \cup \bigcup_{j=1}^N \{ \alpha \mid \alpha^w = g_j \} = \mu_d,
\]

hence there are three rays (as many as \( d \), which equals 3). Similarly there are three dots, as many as the solutions (in the variables \( \nu \) and \( j \)) of \( \nu^w = 1 \): \( (\nu, j) \) is necessarily \( (1, 1), (1, 2), \) or \( (1, 3) \). Note that the further dots mentioned in the construction of the model do not occur in this coset because \( \bigcup_{j=1}^N \{ \alpha \mid \alpha^w = g_j \} \) is contained in \( \mu_d \). The picture is that of Figure 5.

\[ \text{Figure 5. Diagram attached to (1, 1, 1).} \]

We can move on to the coset corresponding to \( g = \gamma \). This time the three coordinates differ \( g_1 = \exp(2\pi i 1/12) \): there is a single solution to \( \alpha^{w_1} = g_1 \) which is \( \alpha = \exp(2\pi i 1/12) \). Similarly there is a single solution to \( \alpha^{w_2} = g_2 \), i.e. \( \alpha \) equal to \( \exp(2\pi i 10/12) \), and there is a single solution to \( \alpha^{w_3} = g_3 \), i.e. \( \alpha \) equal to \( \exp(2\pi i 14/12) \). We have

\[
\mu_d \cup \bigcup_{j=1}^N \{ \alpha \mid \alpha^w = g_j \} = \mu_3 \cup \{ \exp(2\pi i 1/12), \exp(2\pi i 10/12) \}.
\]
Therefore we draw five rays (whose angular coordinates range among those of the above set). Following the rules of Section 5 we draw five dots:

1 exp(2πi 1/12), 2 exp(2πi 10/12), 3 exp(2πi 4/12), 4 exp(2πi 1/12), 4 exp(2πi 10/12),

where the last two dots correspond to the set \( \bigcup_{j=1}^{N} \{ \alpha \mid \alpha^{w_j} = g_j \} \setminus \mu_d \)

which consists of two elements: exp(2πi1/12) and exp(2πi10/12).

**Figure 6.** Diagram for exp(2πi(1/12, 10/12, 4/12)).

The analysis of the third and fourth cosets is completely analogous to that we just carried out and yields Figures 7 and 8.

**Figure 7.** Diagram for exp(2πi(2/12, 8/12, 8/12)).

This setting allows one to check that there is a degree-preserving isomorphism. We can focus on the eight empty rays (on the FJRW side) and compare them to the eight internal points. Using Lemma 22 we get the degrees on the four diagrams. On Figure 5 there are two internal dots on the real axis for which \( \deg_{CR} \) is 0 and 1 (if we read in lexicographic order), and — correspondingly — two empty rays for which \( \deg_{FJRW} \) is 1 and 0.
Figure 8. Diagram for $\exp(2\pi i(3/12,6/12,0))$.

(reading in the sense of the angular coordinate). It is an interesting exercise to verify that all the internal dots and empty rays appearing in Figure 6 have degree $1/2$ (twice $a(h) + D - R$), all internal dots on Figure 7 have degree 1, and, finally, all internal dots on Figure 8 have degree $3/2$. This matches the orbifold curve, Figure 1.

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Figure 9. The model for the Calabi–Yau three-fold \( \{x_1^{20} + x_2^6 + x_3^5 + x_4^4 + x_5^3 = 0\} \) contained in \( \mathbb{P}(3, 10, 12, 15, 20) \).

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