UNITARY CONNECTIONS ON BRATTELI DIAGRAMS

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1. Introduction

Abstract. In this paper, we extend Ocneanu’s theory of connections on graphs to define a 2-category whose 0-cells are tracial Bratteli diagrams, and whose 1-cells are generalizations of unitary connections. We show that this 2-category admits an embedding into the 2-category of hyperfinite von Neumann algebras, generalizing fundamental results from subfactor theory to a 2-categorical setting.

Jones seminal results on the index for subfactors gave rise to the modern theory of subfactors [J83]. Popa proved that amenable finite index subfactors of the hyperfinite II$_1$ factor are completely classified by their standard invariant [P94], which are axiomatized in general by standard $\lambda$-lattices [P95] or planar algebras [J99]. This has led to remarkable progress in the classification of finite index hyperfinite subfactors, by transforming a large part this fundamentally analytic problem to the (essentially) algebraic problem of classifying abstract standard invariants [JMS14], [AMP15].

In the finite depth setting, Ocneanu introduced and established the theory of biunitary connections on 4-partite graph as an essential tool for constructing hyperfinite subfactors. Biunitary connections feature in his paragroup axiomatization of finite depth standard invariants ([O88], [EK98]) but can also be used to construct infinite depth hyperfinite subfactors from finite graphs [S90]. While the other approaches to standard invariants are now more common, the theory of biunitary connections remain an important ingredient in the construction and classification of hyperfinite subfactors [EK98], [JMS14]. Many features of subfactor theory now have a clear higher-categorical interpretation ([M03], [CPJP22], [JMS14]), and while there is some work investigating biunitary connections from a categorical viewpoint ([C20]), the general theory of biunitary connections and particularly their role in hyperfinite subfactor construction has remained mysterious from the categorical viewpoint.

In this paper, we shed some light on this problem by showing that graphs and biunitary connections can be viewed naturally as part of a larger $\text{W}^*$ 2-category $\text{UC}_{tr}$ (see Section 4). We then build a 2-functor to the 2-category of tracial von Neumann algebras, which puts the hyperfinite subfactor construction from biunitary connections into a larger categorical context. The 0-cells (or objects) of the 2-category $\text{UC}_{tr}$ are Bratteli diagrams equipped with tracial weighting data. These generalize the Bratteli diagrams arising from taking the tower of relative commutants of a finite index subfactor. 1-cells in our 2-category are unitary connections between Bratteli diagrams which are compatible with the tracial data. These naturally generalize Ocneanu’s biunitary connections from subfactor theory to our Bratteli diagram setting. Finally, the 2-cells of our category are built as certain fixed points under a UCP map, strongly resembling a noncommutative Poisson boundary as in [104], [NY17].

Recall $\text{vNA}_{\text{Alg}}$ denotes the 2-category of von Neumann algebras, bimodules, and intertwiners. The following is the main theorem of the paper.
Theorem 1.1. There is a $W^*$ 2-functor $\mathcal{PB} : \mathbf{UC}^{\text{tr}} \to \mathbf{vNAlg}$ which is fully faithful at the level of 2-cells.

We note that the von Neumann algebras in the image of $\mathcal{PB}$ are always hyperfinite by construction. To see how the usual subfactor theory construction fits into this story, from a 4-partite graph and a biunitary connection we build a pair of tracial Bratteli diagrams by repeatedly reflecting the “vertical” bipartite graphs, and taking the Markov trace as data. The horizontal graphs and the biunitary connection assemble into a 1-morphism in $\mathbf{UC}^{\text{tr}}$ from this pair of tracial Bratteli diagrams. By carefully choosing the initial vertex data, we can build a unital inclusion of hyperfinite von Neumann algebras from this data, which we will see is just a special case of our construction Section 7.1. One way of looking at our result is that we are generalizing connections to be 1-morphisms between graphs that can be composed. Our main result is that a “compositional” version of the subfactor construction holds, and many of the results from the subfactor setting are true for bimodules as well. For example, it is well known that the relative commutants of the subfactor constructed as above can be computed as the “flat part” of the initial biunitary connection. We prove a generalization of this Proposition 6.6.

To motivate our definitions in $\mathbf{UC}^{\text{tr}}$, we first consider a purely algebraic category $\mathbf{UC}$ consisting of Bratteli diagrams (without tracial data), unitary connections between them, and natural intertwiners between connections which we call flat sequences Section 3. This 2-category is essentially equivalent to the 2-category studied in [CPJ22] in the context of fusion category actions on AF-C*-algebras, with only minor differences at the level of 0-cells and 2-only. As in [CPJ22], from a 0-cell we define an AF-algebra. We see that the 1-morphisms in $\mathbf{UC}$ are precisely the data we need to define inductive limit bimodules between the AF-algebras built from the 0-cells, and the 2-cells in $\mathbf{UC}$ are precisely the intertwiners between the resulting bimodules. Then picking a tracial state on the AF-algebras, we ask which 1-cell bimodules extend to the corresponding von Neumann completion, and if they do, what are the morphisms between them? This consideration leads us precisely to our definitions of 1-cell and 2-cell in $\mathbf{UC}^{\text{tr}}$, which answers this question and proves our main theorem simultaneously.

There are at least two natural questions arising from our investigations. First, which bimodules between hyperfinite von Neumann algebras can be realized by this construction? This is deeply related to the question about possible values of the index for irreducible hyperfinite subfactors and the recent work of Popa [P21]. Second, how is the story modified if we pick arbitrary states on our Bratteli diagrams instead of tracial ones? This could have interesting applications for the study of defects and categorical symmetries in 1-D spin chains.

The outline of the paper is as follows. In Section 2, we record some categorical preliminaries and introduce notation. In Section 3, we detail the 2-category $\mathbf{UC}$, and its realization as a category of bimodules over AF-algebras. In Section 4, we introduce $\mathbf{UC}^{\text{tr}}$ and prove our main theorem in Section 5. In Section 6, we investigate flatness and in Section 7 we consider some examples, including the relationship between our work and classical subfactor constructions, as well as the work of Izumi [I04].

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1 we are slightly abusing terminology: by AF-algebra we mean inductive limit of finite dimensional C*-algebras in the category of *-algebras, so we do not complete in norm.
2. Preliminaries and notations

In this section, we set up several notations and state well-known facts which will be used in the latter part of this article. Although, we have not used any pictures in this section, we urge the reader to translate the expressions in terms of pictures using standard graphical calculus of morphisms and see what pictorial moves are given by the equations and maps.

2.1. Categorical trace. Let $\mathcal{M}$ be a semisimple C*-category and $V$ be a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}$. For all $v \in V$, $x \in \text{ob}(\mathcal{M})$, consider the inner product $\langle \cdot, \cdot \rangle_{v,x}$ on $\mathcal{M}(v, x)$ defined by $\tau^*\sigma = \langle \sigma, \tau \rangle_{v,x} 1_v$. An orthonormal basis for such spaces is basically a maximal orthogonal family of isometries in $\mathcal{M}(v, x)$.

**Convention.** If a statement is independent of the choice of orthonormal basis for $\mathcal{M}(v, x)$, then we denote it by $\text{ONB}(v, x)$. For instance, $1_x = \sum_{v \in V} \sum_{\sigma \in \text{ONB}(v, x)} \sigma^* \sigma$.

Given a map $\mu : V \to (0, \infty)$ (referred as a weight function on $\mathcal{M}$), consider the linear functional
\[
\text{End}(x) \ni \alpha \mapsto \sum_{v \in V} \sum_{\sigma \in \text{ONB}(v, x)} \mu_v \langle \alpha \sigma, \sigma \rangle_{v,x} \in \mathbb{C} .
\]

Clearly, $\text{Tr}_x$ is a faithful, positive functional. Moreover, $\text{Tr} = (\text{Tr}_x)_{x \in \text{ob}(\mathcal{M})}$ is a ‘categorical’ trace, namely it satisfies $\text{Tr}_x(\alpha \beta) = \text{Tr}_y(\beta \alpha)$ for all $\alpha \in \mathcal{M}(y, x)$, $\beta \in \mathcal{M}(x, y)$. We refer $\text{Tr}$ as the categorical trace associated to the weight function $\mu$.

2.2. Graphs and functors. Let $\Gamma = (V_\pm, E)$ (also denoted by $V_\pm \xrightarrow{\Gamma} V_\pm$) be a bipartite graph with vertex sets $V_\pm$ and edge sets $E_{v_+, v_-}$ for $(v_+, v_-) \in V_+ \times V_-$, such that the set of edges attached to any vertex is non-empty and finite. Consider the semisimple C*-category $\mathcal{M}_\pm$ whose objects consists of finitely supported $V_\pm$-graded finite dimensional Hilbert spaces. Note that $\Gamma$ induces the following pair of faithful functors
\[
\mathcal{M}_- \ni (H_{v_-})_{v_- \in V_-} \xrightarrow{F_-} \left( \bigoplus_{v_- \in V_-} H_{v_-} \otimes \ell^2(E_{v_+, v_-}) \right)_{v_+ \in V_+} \in \mathcal{M}_+
\]
\[
\mathcal{M}_+ \ni (H_{v_+})_{v_+ \in V_+} \xrightarrow{F_+} \left( \bigoplus_{v_+ \in V_+} H_{v_+} \otimes \ell^2(E_{v_+, v_-}) \right)_{v_- \in V_-} \in \mathcal{M}_-
\]

where the action of each of the functors on a morphism is obtained by distributing it over the direct sum and tensor product keeping the edge vectors (in $\ell^2(E_{v_+, v_-})$’s) fixed. One can easily show that such $F_\pm$ is *-linear, bi-faithful (that is, both itself and it adjoint are faithful), and $F_+$ and $F_-$ are adjoints of each other. Conversely, every adjoint pair of *-linear faithful functors $F_\pm : \mathcal{M}_\pm \to \mathcal{M}_\pm$ between semisimple C*-categories $\mathcal{M}_\pm$, gives rise to such a bipartite graph by setting the vertex set $V_\pm$ as a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}_\pm$, and edge set $E_{v_+, v_-}$ as a choice of orthonormal basis in $\mathcal{M}_+(v_+, F_+ v_-)$ with respect to $\langle \cdot, \cdot \rangle_{v_+, F_+ v_-}$ (defined in Section 2.1).

For $F_\pm, \mathcal{M}_\pm, V_\pm$ as before, we will try to characterize the set of solutions to conjugate equations implementing the duality of $F_\pm$. At this point, it will be useful for us to introduce some pictorial notation for morphisms and natural transformations which are quite standard in articles appearing in category theory.
Pictorial notation.  (i) A morphism \( f : C \to D \) will be denoted by \( \begin{array}{c} D \\ \downarrow f \\ C \end{array} \), and composition of two morphisms will be represented by two vertically stacked labelled boxes.

(ii) Let \( C \) and \( D \) be two categories and \( F, G : C \to D \) be two functors. Then a natural transformation \( \eta : F \to G \) will be denoted by \( \begin{array}{c} G \\ \downarrow \eta \\ F \end{array} \). For an object \( x \) in \( C \), the morphism \( \eta_x : Fx \to Gx \) will be denoted by \( \begin{array}{c} G \\ \downarrow \eta_x \\ F \end{array} \).

(iii) For a \( \ast \)-linear functor \( F : C \to D \) between two semisimple \( C^\ast \)-categories, categories, we will denote a solution to the conjugate equation by

\[
\rho = F \begin{array}{c} \rightarrow \end{array} F' : id_D \to FF' \quad \text{and} \quad \rho' = F' \begin{array}{c} \rightarrow \end{array} F : id_C \to F'F
\]

where \( F' : D \to C \) is an adjoint functor of \( F \).

We will extend the above dictionary (between things appearing in the category world and pictures) in an obvious way. For instance, composition of morphisms and natural transformations will be pictorially represented by stacking the boxes vertically whereas tensor product (resp., composition) of objects (resp., functors) by parallel vertical strings. For simplicity, sometimes we will not label all of the strings (with any object or functor) emanating from a box (labelled with a morphism or a natural transformation) when it can be read off from the context. We urge the reader to get used to the various picture moves which are induced by relations satisfied by operations, such as, composition, tensor product, etc. between objects, morphisms, functors and natural transformations. In fact, the main purpose of introducing this graphical language is because of the ease of working with these moves instead of long equations.

**Fact 2.1.** If \( \rho^\pm : id_{\mathcal{M}^\pm} \to F_\pm F_\pm^\ast \) is a solution to the conjugate equation for \( F_\pm \), then for each \( (v_+, v_-) \in V_+ \times V_- \), there exists an orthonormal basis \( E_{v_+, v_-} \) of \( \mathcal{M}_+(v_+, F_+ v_-) \) and a ‘weight’ function \( \kappa_{v_+, v_-} : E_{v_+, v_-} \to (0, \infty) \) and satisfying the following:

\[
(2.1) \quad (\rho_{v_-}^\ast) F_-(\sigma \tau^\ast) \rho_{v_-} = \delta_{\sigma=\tau} \kappa_{v_+, v_-}(\sigma) 1_{v_-} \quad \text{for all} \quad \sigma, \tau \in E_{v_+, v_-}, \ v_\pm \in V_\pm.
\]

Conversely, to every such family of orthonormal bases and weight functions, one can associate a solution to the conjugate equations implementing the duality of \( F_\pm \) satisfying Equation (2.1).

The above easily follows from the spectral decomposition of the faithful positive functional \( \left((\rho_{v_-}^\ast) F_-(\bullet) \rho_{v_-}\right) : \text{End}(F_+ v_-) \to \text{End}(v_-) = \mathbb{C}1_{v_-} \). Thus, from the adjoint pair of \( \ast \)-linear faithful functors, we not only get a bipartite graph, the solution to the conjugate equations puts a positive scalar weight on each edge. Further, the set

\[
E_{v_-, v_+} := \left\{ \left(\kappa_{v_+, v_-}(\sigma)\right)^{-\frac{1}{2}} [F_- \sigma^\ast] \rho_{v_-} : \sigma \in E_{v_+, v_-} \right\}
\]

turns out to be an orthonormal
basis of $\mathcal{M}_-(v_-, F_-v_+)$ and satisfies an equation analogous to Equation (2.1) with weight function $\kappa_{v_-, v_+} := \frac{1}{\kappa_{v_-, v_-}}$.

The solution $\rho^\pm$ will be called ‘tracial’ if the weight function is constant on edges for every fixed pair of vertices. Indeed, for tracial solution $\rho^\pm$, Equation (2.1) becomes

$$
(2.2) \quad \begin{cases}
(\rho_{v_+}^\pm)^* \ F_- (\sigma \tau^*) \rho_{v_-}^- = \kappa_{v_+, v_-} \langle \sigma, \tau \rangle_{v_+, F_-v_-} \ 1_{v_-} \quad \text{for all } \sigma, \tau \in \mathcal{M}_+(v_+, F_+v_-) \\
(\rho_{v_+}^\pm)^* \ F_+ (\sigma \tau^*) \rho_{v_-}^+ = \kappa_{v_-, v_+} \langle \sigma, \tau \rangle_{v_-, F_+v_+} \ 1_{v_+} \quad \text{for all } \sigma, \tau \in \mathcal{M}_-(v_-, F_-v_+)
\end{cases}
$$

and the map $[(\rho_{v_+}^-)^* F_- (\bullet) \rho_{v_-}^-]$ is tracial and so is $[(\rho_{v_+}^+)^* F_+ (\bullet) \rho_{v_-}^+]$. We also get a conjugate linear unitaries

$$
\mathcal{M}_+(v_+, F_+v_-) \ni \sigma \quad \mapsto \quad \sqrt{\kappa_{v_-, v_+}} \ 1_{v_-} \quad \text{for } \sigma \in \mathcal{M}_+(v_+, F_+v_-), \\
\mathcal{M}_-(v_-, F_-v_+) \ni \sigma \quad \mapsto \quad \sqrt{\kappa_{v_-, v_+}} \ 1_{v_+} \quad \text{for } \sigma \in \mathcal{M}_-(v_-, F_-v_+).
$$

The two ‘loops’ are given by:

$$
(2.3) \quad (\rho_{v_+}^+)^* \circ \rho_{v_-}^- = \left\{ \sum_{v_+ \in V_+} N_{v_+, v_-} \kappa_{v_-, v_+} \right\} \ 1_{v_+} \quad \in \text{End}(\text{id}_{\mathcal{M}_+}),
$$

$$
(2.4) \quad (\rho_{v_+}^-)^* \circ \rho_{v_-}^+ = \left\{ \sum_{v_+ \in V_+} N_{v_+, v_-} \kappa_{v_-, v_+} \right\} \ 1_{v_-} \quad \in \text{End}(\text{id}_{\mathcal{M}_-}),
$$

where $N_{v_+, v_-} := \dim C(\mathcal{M}_+(v_+, F_+v_-)) = \dim C(\mathcal{M}_-(v_-, F_-v_+))$ (that is, the number of edges between $v_+$ and $v_-$ in the bipartite graph). (Note that a natural linear transformation between *-linear functors from one semisimple C*-category to another, is captured fully by its components corresponding to the simple objects.)

### 2.3. Trace on natural transformations.

In this article, we will be working with the 2-category of weighted semisimple C*-categories, denoted by WSSC*Cat, whose 0-cells are finite semisimple C*-categories along with a weight function on it (that is, a positive real valued map from the isomorphism classes of simple objects, as considered in Section 2.1), 1-cells are *-linear bi-faithful functors and 2-cells are natural linear transformations. Further, for the duality of the adjoint pair of 1-cells $\left(\mathcal{M}_-, \mu^- \right) \xrightarrow{\mathcal{F}_-} \left(\mathcal{M}_+, \mu^+ \right)$, we will consider tracial solution $\rho^\pm : \text{id}_{\mathcal{M}_\pm} \to F_\pm F_\mp$ to the conjugate equations associated to the constant weight on edges given by $\kappa_{v_+, v_-} = \mu_{\pm} v_+ v_- \mu_{v_-}^{-1}$ for $(v_+, v_-) \in V_+ \times V_-;$ we refer such a solution to be commensurate with the weight functions (on the simple objects) $(\mu^-, \mu^+)$. Using the categorical trace $\text{Tr}$ associated to the weight function $\mu^\pm$, one may obtain the relation:

$$
(2.5) \quad \text{Tr}_x \left((\rho_x^\pm)^* F_\pm (\alpha) \rho_x^\pm\right) = \text{Tr}_{F_\pm(x)}(\alpha) \quad \text{for all } x \in \text{ob}(\mathcal{M}_\pm), \alpha \in \text{End}(F_\pm(x)).
$$

We will now exhibit a similar categorial trace on the endomorphism space of every 1-cell between two ‘finite’ 0-cells (that is, there is finitely many isomorphism classes of simple objects in the semisimple C*-category of the 0-cell); further, this trace will be compatible with the tracial solution commensurate with the weight function in the 0-cells.
Proposition 2.2. Let $\langle M, \mu \rangle$, $\langle N, \nu \rangle$, $\langle Q, \pi \rangle$ be finite 0-cells in WSSC*Cat, and $\Lambda : M \to N$, $\Sigma : N \to Q$ be $\ast$-linear bi-faithful functors. Suppose $V_M, V_N, V_Q$ are maximal sets of mutually non-isomorphic simple objects in $M, N, Q$ respectively.

(a) The map

$$\text{End}(\Lambda) \ni \eta \mapsto \sum_{u \in V_M} \mu_u \text{Tr}_{\Lambda(u)}(\eta_u) \in \mathbb{C}$$

is a positive faithful trace. We will refer $\text{Tr}^\Lambda$ as the 'trace on $\text{End}(\Lambda)$ commensurate with $\langle \mu, \nu \rangle$'.

(b) If $\left( id_M \xrightarrow{\rho} \Lambda x, \ x \xrightarrow{\sigma} \Lambda x \right)$ (resp., $\left( id_N \xrightarrow{\beta} \Sigma x, \ x \xrightarrow{\alpha} \Sigma x \right)$) is a solution to conjugate equations for the duality of $\Lambda$ (resp., $\Sigma$) commensurate with $\langle \mu, \nu \rangle$ (resp., $\langle \mu, \pi \rangle$), then

$$\text{Tr}^\Lambda (\beta_u \Sigma(\eta) \beta_u) = \text{Tr}^\Sigma (\Sigma(\eta) \eta \Sigma(\nu)) \text{ for } \eta \in \text{End}(\Sigma \Lambda) .$$

Proof. (a) To each $\alpha \in \text{End}(\Lambda x)$ for $x \in \text{Ob}(M)$, we associate the natural transformation

$$[\alpha] := \left( \sum_{u \in V_M} \sum_{\sigma \in \text{ONB}(u, x)} \Lambda(\tau \sigma^*) \alpha \Lambda(\sigma \tau^*) \right)_{y \in \text{Ob}(M)} .$$

In terms of this association, we may express $\text{End}(\Lambda)$ as a direct sum of full matrix algebras indexed by $V_M \times V_N$, and a system of matrix units of the summand corresponding to $(u, v) \in V_M \times V_N$ is given by $\{ [\sigma \tau^*] : \sigma, \tau \in \text{ONB}(v, \Lambda u) \}$. Note that $\text{Tr}^\Lambda ([\sigma \tau^*]) = \delta_{\sigma=\tau} \mu_u \nu_v$ which is positive and independent of the choice of $\sigma$ and $\tau$.

(b) From the definition, the left side turns out to be

$$\sum_{u \in V_M} \mu_u \text{Tr}_{\Lambda(u)}(\beta_u \Sigma(\eta_u) \beta_u)$$

which is equal to (applying Equation (2.3))

$$\sum_{u \in V_M} \mu_u \text{Tr}_{\Sigma \Lambda(u)}(\eta_u) = \text{Tr}^\Sigma \Lambda(\eta) .$$

Pictorially the right side can be expressed as

$$\sum_{v \in V_N} \nu_v \text{Tr}_{\Sigma \Lambda(u)}(\eta \Sigma(\tau \tau^*)) = \text{Tr}^\Sigma \Lambda(\eta) .$$

$\Box$

Remark 2.3. The trace in Proposition 2.2 (a), is ‘categorical’, that is, $\tilde{\Lambda} : M \to N$ is another functor with the same PF vectors as that of $\Lambda$, then $\text{Tr}^\Lambda (\gamma \eta) = \text{Tr}^{\tilde{\Lambda}} (\eta \gamma)$ for $\eta \in \text{NT}(\Lambda, \tilde{\Lambda}), \gamma \in \text{NT}(\tilde{\Lambda}, \Lambda)$.
3. UNITARY CONNECTIONS AND RIGHT CORRESPONDENCES

Bratteli diagrams are incredibly useful tools for studying inductive limits of semisimple algebras (also called locally semisimple algebras). In this section we introduce a combinatorial 2-category whose objects are Bratteli diagrams and 1-cells are generalizations of Ocneanu’s connections. Our perspective is that our 1-cells can naturally be viewed as “Bratteli diagrams for bimodules” between locally semisimple algebras. Thus as we describe our 2-category \( \text{UC} \), we will explain its relationship to algebras and bimodules.

As a consequence, we build a fully faithful 2-functor \( \text{UC} \) into the 2-category of algebras, bimodules, and intertwiners.

3.1. The 0-cells.

These are sequences consisting of finite bipartite graphs

\[
V_0 \xrightarrow{\Gamma_1} V_1 \xrightarrow{\Gamma_2} V_2 \xrightarrow{\Gamma_3} V_3 \cdots
\]

(where \( V_j \)'s are the vertex sets) such that none of the vertices is isolated. As described in the Section 2.2 given such a data, we will often work with the corresponding \(*\)-linear, bi-faithful functor \( \Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k \) (where \( \mathcal{M}_k \) is a semisimple C*-category whose isomorphism classes of the simple objects are indexed by the vertex set \( V_k \)). We will denote such a 0-cell by \( \{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \}_{k \geq 1} \) or sometimes simply \( \Gamma_* \).

Given such a 0-cell, we fix an object \( m_0 := \bigoplus_{v \in V_0} v \in \text{ob}(\mathcal{M}_0) \). Consider the sequence of finite dimensional C*-algebras

\[
\{ A_k := \text{End}(\Gamma_k \cdots \Gamma_1 m_0) \}_{k \geq 0}
\]

(assuming \( A_0 = \text{End}(m_0) \)) along with the unital \(*\)-algebra inclusions given by \( A_{k-1} \ni \alpha \mapsto \Gamma_k \alpha \in A_k \). Indeed, the Bratteli diagram of \( A_{k-1} \) inside \( A_k \) is given by the graph \( \Gamma_k \). To the 0-cell \( \Gamma_* \), we associate the \(*\)-algebra \( A_\infty := \bigcup_{k \geq 0} A_k \).

3.2. The 1-cells.

**Definition 3.1.** A 1-cell from the 0-cell \( \{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \}_{k \geq 1} \) to the 0-cell \( \{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \}_{k \geq 1} \) consists of a sequence of \(*\)-linear bi-faithful functors \( \{ \Lambda_k : \mathcal{M}_k \to \mathcal{N}_k \}_{k \geq 0} \) and natural unitaries \( W_k : \Delta_k \Lambda_{k-1} \to \Lambda_k \Gamma_k \) for \( k \geq 1 \). Such a 1-cell will be denoted by \( (\Lambda_* , W_*) \) or simply by \( \Lambda_* \), and \( W_* \) will be referred as a \textit{unitary connection associated to} \( \Lambda_* \). Denote the set of 1-cells from \( \Gamma_* \) to \( \Delta_* \) by \( \text{UC}_1 (\Gamma_*, \Delta_*) \).

We will abuse the notation \( \Lambda_k \) to denote the functor \( \Lambda_k : \mathcal{M}_k \to \mathcal{N}_k \) as well as its associated adjacency matrix \((V_{\mathcal{N}_k} \times V_{\mathcal{M}_k})\), and the same will be done for \( \Gamma_k \)'s and \( \Delta_k \)'s. From the context, it will be clear whether we are using it as a functor or a matrix. Pictorially, the natural unitary \( W_k \) appearing in the 1-cell will be represented by

\[
W_k^* \xrightarrow{\Delta_k} \Lambda_{k-1} \xrightarrow{\Lambda_k} \Gamma_k
\]

To each such 1-cell \( (\Lambda_*, W_*) \), we will associate an \( A_\infty \)-\( B_\infty \) right correspondence where \( n_0 \) and \( B_k \)'s are related to \( \{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \}_{k \geq 1} \) exactly the way \( m_0 \) and \( A_k \)'s are related to...
We have an obvious $A_k$-$B_k$-bimodule structure on $H_k$ in the following way:

$$A_k \times H_k \times B_k \ni (\alpha, \xi, \beta) \mapsto \Lambda_k(\alpha) \circ \xi \circ \beta \in H_k .$$

Again, there is a $B_k$-valued inner product on $H_k$ given by

$$H_k \times H_k \ni (\xi, \zeta) \mapsto B_k \langle \xi, \zeta \rangle_B := \zeta^* \circ \xi \in B_k .$$

Next, observe that $H_k$ sits inside $H_{k+1}$ via the map

$$H_k \ni \xi \mapsto [(W_{k+1})_{\Gamma_k \cdots \Gamma_m} \circ [\Delta_{k+1}] \circ \Delta_{k+1} \beta \in H_{k+1} .$$

**Lemma 3.2.** The inclusions $H_k \hookrightarrow H_{k+1}$, $A_k \hookrightarrow A_{k+1}$, $B_k \hookrightarrow B_{k+1}$ and the corresponding actions are compatible in the obvious sense.

**Proof.** Naturality of $W$ implies

$$\Lambda_{k+1} \Gamma_{k+1} \alpha \circ [(W_{k+1})_{\Gamma_k \cdots \Gamma_m} \circ [\Delta_{k+1}] \circ \Delta_{k+1} \beta = [(W_{k+1})_{\Gamma_k \cdots \Gamma_m} \circ \Delta_{k+1} (\Lambda_k \alpha \circ \xi \circ \beta)$$

for all $\xi \in H_k$, $\alpha \in A_k$, $\beta \in B_k$.

Set $H_\infty := \bigcup_{k \geq 0} H_k$ which clearly becomes an $A_\infty$-$B_\infty$ right correspondence. Further, we will exhibit a Pimsner-Popa (PP) basis of the right-$B_\infty$-module $H_\infty$ with respect to the $B_\infty$-valued inner product.

**Lemma 3.3.** There exists a subset $\mathcal{I}$ of $H_0$ such that $\sum_{\sigma \in \mathcal{I}} \sigma \circ \sigma^* = 1_{\Lambda_0 m_0}$; moreover, any such $\mathcal{I}$ is a PP-basis for the right $B_\infty$-module $H_\infty$.

**Proof.** Since $n_0$ contains every simple object of $\mathcal{N}_0$ as a subobject, therefore expressing the identity of $\text{End}(\Lambda_0 m_0)$ as a sum of minimal projections, we have a resolution of identity $1_{\Lambda_0 m_0}$ factoring through $n_0$, that is, there exists a subset $\mathcal{I}$ of $\mathcal{N}(n_0, \Lambda_0 m_0) = H_0$ satisfying:

(i) $\sigma^* \sigma$ is a minimal projection of $\text{End}(n_0)$, and

(ii) $\sum_{\sigma \in \mathcal{I}} \sigma \sigma^* = 1_{\Lambda_0 m_0}$.

Condition (ii) and the definition of $B_\infty$-valued inner product directly imply that $\mathcal{I}$ is indeed a PP-basis for the right $B_\infty$-module $H_\infty$.

3.3. The 2-cells.

Let $\Lambda_\bullet$ and $\Omega_\bullet$ be two 1-cells from the 0-cell $\left\{ \mathcal{M}_{k_1} \xrightarrow{k} \mathcal{M}_k \right\}_{k \geq 1}$ to $\left\{ \mathcal{N}_{k-1} \xrightarrow{k} \mathcal{N}_k \right\}_{k \geq 1}$.

The natural way to define a 2-cell will be considering a sequence of natural linear transformations from $\Lambda_k$ to $\Omega_k$ which are compatible with the natural unitaries $W_k^T$ and $W_k^\Omega$ for $k \geq 1$. We define such compatibility in the following way.
Definition 3.4. A pair \((\eta, \kappa) \in \text{NT}(\Lambda_k, \Omega_k) \times \text{NT}(\Lambda_{k+1}, \Omega_{k+1})\) is said to satisfy exchange relation if the condition
\[
\begin{array}{c}
\Omega_{k+1} \\
\downarrow \eta \\
\Omega_k \\
\downarrow \kappa \\
\Lambda_{k+1} \\
\downarrow \gamma \\
\Lambda_k
\end{array}
\]
holds.

Remark 3.5. The exchange relation pair is unique separately in each variable, that is, if \((\eta_1, \kappa_1)\) and \((\eta_2, \kappa_2)\) (resp., \((\eta_1, \kappa)\) and \((\eta_2, \kappa)\)) both satisfy exchange relation, then \(\kappa_1 = \kappa_2\) (resp., \(\eta_1 = \eta_2\)); this is because the connections are unitary and the functors \(\Gamma_k\) and \(\Delta_k\) are bi-faithful.

We only require that the 2-cells satisfy this exchange relation eventually. To make this precise, we let
\[
\text{Ex}(\Lambda_*, \Omega_*) := \frac{\text{Ex}(\Lambda_*, \Omega_*)}{\text{Ex}_0(\Lambda_*, \Omega_*)}
\]
denote the space of sequences \(\{\eta^{(k)}(\in \text{NT}(\Lambda_k, \Omega_k))\}_{k \geq 0}\) such that there exists an \(N\) such that \((\eta_k, \eta_{k+1})\) satisfies the exchange relation for all \(k \geq N\). Consider the subspace

\[
\text{Ex}_0(\Lambda_*, \Omega_*) := \{\{\eta_k\}_{k \geq 0} \in \text{Ex}(\Lambda_*, \Omega_*) : \eta_k = 0 \text{ for all } k \geq N \text{ for some } N \in \mathbb{N}\}
\]

Definition 3.6. Let \(\Lambda_*, \Omega_* \in \text{UC}_1(\Gamma_*, \Delta_*)\). We define the space of 2-cells

\[
\text{UC}_2(\Lambda_*, \Omega_*) := \frac{\text{Ex}(\Lambda_*, \Omega_*)}{\text{Ex}_0(\Lambda_*, \Omega_*)}
\]

For notational convenience, instead of denoting a 2-cell by an equivalence class of sequences, we simply use a sequence in the class and truncate upto a level after which the exchange relation holds for every consecutive pair, namely, \(\{\eta^{(k)}\}_{k \geq N} \in \text{UC}_2(\Lambda_*, \Omega_*)\) where \((\eta_k, \eta_{k+1})\) satisfies the exchange relation for all \(k \geq N\).

If \(\eta = \{\eta^{(k)}\}_{k \geq K} \in \text{UC}_2(\Lambda_*, \Omega_*)\) and \(\kappa = \{\kappa^{(k)}\}_{k \geq L} \in \text{UC}_2(\Omega_*, \Xi_*),\) then define the ‘vertical’ composition of 2-cells by \(\kappa \cdot \eta := \{\eta^{(k)} \circ \kappa^{(k)}\}_{k \geq \max\{K, L\}}\). It is easy to check that \(\kappa \cdot \eta \in \text{UC}_2(\Lambda_*, \Xi_*)\) is well defined and the composition is associative.

Given two 0-cells \(\Gamma_*\) and \(\Delta_*\), we have obtained a category whose object space consists of 1-cells \(\Lambda_*\), and morphisms are given by 2-cells. We call this the category of unitary connections from \(\Gamma_*\) to \(\Delta_*\) and denote by \(\text{UC}_{\Gamma_*, \Delta_*}\).

Following with the structure in the previous subsections, we will see that 2-cells uniquely define bimodule intertwiners between the bimodules associated to the 1-cells. We will borrow the notations \(H_k, H_\infty, \mathcal{J},\) etc. (arising out of \(\Lambda_*\)) from previous subsections, and for those arising out of \(\Omega_*\), we will use the notation \(G_k, G_\infty, \mathcal{J},\) etc. and we will also work with the pictures as before. For each \(k \geq 0\), we define \(\mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \ni \gamma \xrightarrow{\Phi} \Phi_\gamma \in L(H_\infty, G_\infty)\) (the space of adjointable operators with respect to the \(B_\infty\)-valued inner product) in the following way

\[
\begin{array}{c}
\Omega_{k+1} \\
\downarrow \eta \\
\Omega_k \\
\downarrow \kappa \\
\Lambda_{k+1} \\
\downarrow \gamma \\
\Lambda_k
\end{array}
\]

for \(l \geq 0\). It is easy to check that \(\Phi_\gamma\) is well-defined and adjointable. We list a few basic properties of \(\Phi\) in the following lemma.
Lemma 3.7. For all \( k \geq 0 \) and \( \gamma \in \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \), the following conditions hold

(i) \( \Phi_{\gamma^*} = (\Phi_{\gamma})^* \),

(ii) \( \Phi_{\gamma}(H_l) \subset G_1 \) for all \( l \geq k \),

(iii) the map \( \gamma \mapsto \Phi_{\gamma}|_{H_k} \) is one-to-one, and

(iv) \( \Phi_{\gamma} \in \mathcal{L}_{B_{\infty}}(H_\infty, G_\infty) \).

Proof. The only nontrivial part is to prove (iii). This easily follows from the equality

\[
\gamma = \sum_{\sigma \in \mathcal{Y}} \frac{\gamma_{\sigma}}{m_0} \quad \text{In fact, we have deduced a stronger statement, namely, } \gamma \text{ is nonzero if and only if } \Phi_{\gamma}|_{H_0} \text{ is nonzero.} \]

Lemma 3.8. For each \( k \geq 0 \), the space \( \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \) gets an \( A_k \)-\( A_k \)-bimodule structure via

\[
(\alpha_1, \gamma, \alpha_2) \in A_k \times \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \times A_k \]

\[
\Omega_k \alpha_1 \circ \gamma \circ \Lambda_k \alpha_2 \in \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0)
\]

and the space \( \text{NT}(\Lambda_k, \Omega_k) \) of natural linear transformations is isomorphic to the space of \( A_k \)-central vectors in \( \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \) via \( \eta \mapsto \eta \Gamma_k \cdots \Gamma_1 m_0 \).

Proof. The map

\[
\gamma \in \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \]

\[
\left( \sum_{v \in V_{\mathcal{M}_k}} \frac{\text{dim}_C(\mathcal{M}_k(v, \Gamma_k \cdots \Gamma_1 m_0))}{\text{dim}_{\mathcal{M}_k}(v, \Gamma_k \cdots \Gamma_1 m_0)} \right) \sum_{\sigma \in \text{ONB}(v, x)} \Omega_k(\sigma \tau^*) \circ \gamma \circ \Lambda_k(\tau \sigma^*) \in \text{NT}(\Lambda_k, \Omega_k)
\]

when restricted to the \( A_k \)-central vectors, turns out to be the inverse of the map in the statement of the lemma (since \( m_0 \) contains every simple as a subobject and \( \Gamma_j \)'s are bi-faithful).

Lemma 3.9. The pair \( (\eta \cdot \kappa) \in \text{NT}(\Lambda_k, \Omega_k) \times \text{NT}(\Lambda_{k+1}, \Omega_{k+1}) \) satisfies exchange relation if and only if \( \Phi_{\eta \cdot \kappa} = \Phi_{\kappa_{1} \cdots \Gamma_1 m_0} \).

Proof. The ‘only if’ part direct follows from the definitions.
For the ‘if’ part, let \( ^{\times} \eta \) and \( \kappa_{\times} \) denote the left and the right sides of the exchange relation equation. Applying Lemma 3.7(iii) on the equation in our hypothesis, we deduce \( ^{\times} \eta |_{\Gamma_k \cdots \Gamma_1 m_0} = (\kappa_{\times}) |_{\Gamma_k \cdots \Gamma_1 m_0} \).
Now, by bi-faithfulness, \( \Gamma_k \cdots \Gamma_1 m_0 \) contains all the simples of \( \mathcal{M}_k \) as subobjects, and thereby \( ^{\times} \eta = \kappa_{\times} \).
Theorem 3.10. Starting from a 2-cell \( \{ \eta^{(k)} \in NT(\Lambda_k, \Omega_k) \}_{k \geq K} \), we have an intertwiner \( \Phi_{\eta^{(k)}_{k \cdot \Gamma_1 m_0}} \in A_\infty L_{B_\infty}(H_\infty, G_\infty) \) which is independent of \( k \geq K \).

Conversely, for every \( T \in A_\infty L_{B_\infty}(H_\infty, G_\infty) (= \text{the space of } A_\infty B_\infty \text{-linear adjointable operator}) \) and for all \( k \geq K_T := \min \{ t : TH_0 \subset G_t \} \), there exists unique \( \eta^{(k)} \in NT(\Lambda_k, \Omega_k) \) such that \( T = \Phi_{\eta^{(k)}_{k \cdot \Gamma_1 m_0}} \). Further, \( (\eta^{(k)}, \eta^{(k+1)}) \) satisfies exchange relation for all \( k \geq K_T \).

Proof. The forward direction trivially follows from Lemma 3.9 and the \( A_\infty B_\infty \)-linearity is obvious. For the converse, set

\[
\zeta_k := \sum_{\sigma \in \mathcal{S}} T_{\sigma} \in N(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \quad \text{for } k \geq K_T
\]

where \( T_{\sigma} \) is treated as an element of \( G_k \) and \( \mathcal{S} (\subset H_0) \) is a PP-basis for the right \( B_\infty \)-module \( H_\infty \). Using the right \( B_\infty \)-valued inner product, the PP-basis and right \( B_k \)-linearity of \( T \), one can easily conclude \( T \xi = \Phi_{\zeta_k} \xi \) for all \( \xi \in H_k \); moreover, this equation uniquely determines \( \zeta_k \) by Lemma 3.7 (iii). Further, the left side of the equation is \( A_k \)-linear; then so is the right side. Again by Lemma 3.7 (iii), \( \zeta_k \) becomes \( A_k \)-central. Applying Lemma 3.8 we get a unique \( \eta^{(k)} \in NT(\Lambda_k, \Omega_k) \) satisfying \( \zeta_k = \eta^{(k)}_{k \cdot \Gamma_1 m_0} \). The rest of the proof is straightforward. \( \square \)

Remark 3.11. If \( C_{B_\infty, A_\infty} \) denotes the category of \( A_\infty B_\infty \) right correspondences where \( A_\infty \) and \( B_\infty \) are the unital filtration of finite dimensional C*-algebras associated to the 0-cells \( \Gamma \) and \( \Delta \), respectively, then combining Theorem 3.10 Definition 3.6 and the definition of the vertical composition of 2-cells, we have a fully faithful \( * \)-linear functor from

\[
\Psi_{\Gamma_*, \Delta_*} : UC_{\Gamma_*, \Delta_*} \longrightarrow C_{B_\infty, A_\infty}.
\]

3.4. The horizontal structure.

This is the final step of constructing a \( * \)-linear 2-category of unitary connections, denoted by \( UC \) whose 0-, 1- and 2-cells are already defined in Sections 3.1, 3.2 and 3.3 respectively. For 0-cells \( \Gamma_*, \Delta_* \), we will define a bifunctor

\[
\boxtimes : UC_{\Delta_*, \Sigma_*} \times UC_{\Gamma_*, \Delta_*} \longrightarrow UC_{\Gamma_*, \Sigma_*}
\]

in such a way that it corresponds to the reverse relative tensor product of the associated right correspondences. For \( \Omega_* \in UC_1(\Delta_*, \Sigma_*) \) and \( \Lambda_* \in UC_1(\Gamma_*, \Delta_*) \), define

\[
(3.2) \quad \Omega_* \otimes \Lambda_* := \left\{ \Omega_k \Lambda_k \}_{k \geq 0}, \left( \begin{array}{c}
\Omega_k \\
\Lambda_k \\
\Sigma_k
\end{array} \right) \right\}_{k \geq 1}.
\]

Proposition 3.12. The bimodule \( \Psi_{\Gamma_*, \Sigma_*}(\Omega_\otimes \Lambda_\ast) \) is isomorphic to the relative tensor product of the right correspondences \( \Psi_{\Gamma_*, \Delta_*}(\Lambda_\ast) \) and \( \Psi_{\Delta_*, \Sigma_*}(\Omega_\ast) \).
Proof. We first consider the following notations:
\[
\begin{align*}
\mathcal{M}_{k-1} &\xrightarrow{\Gamma_k} \mathcal{M}_k \quad \text{for } k \geq 1, \\
\mathcal{N}_{k-1} &\xrightarrow{\Delta_k} \mathcal{N}_k \quad \text{for } k \geq 1, \\
\mathcal{Q}_{k-1} &\xrightarrow{\Sigma_k} \mathcal{Q}_k \quad \text{for } k \geq 1,
\end{align*}
\]
where \( \mathcal{M}_k \), \( \mathcal{N}_k \), and \( \mathcal{Q}_k \) are defined as follows:

- \( \mathcal{M}_k \) is the set of all \( k \)-dim 2-cells \( \eta \) satisfying the exchange relation, which is a requirement for being a 2-cell.

We consider the following notations:
\[
\begin{align*}
\mathcal{M}_k &\xrightarrow{\Gamma_k} \mathcal{M}_k \quad \text{for } k \geq 1, \\
\mathcal{N}_k &\xrightarrow{\Delta_k} \mathcal{N}_k \quad \text{for } k \geq 1, \\
\mathcal{Q}_k &\xrightarrow{\Sigma_k} \mathcal{Q}_k \quad \text{for } k \geq 1,
\end{align*}
\]

Consider the linear map
\[
\mathcal{Q}_k \xrightarrow{\Sigma_k} \mathcal{Q}_k \quad \text{for } k \geq 1,
\]

and the inclusions
\[
\mathcal{M}_k \xrightarrow{\Gamma_k} \mathcal{M}_k \quad \text{for } k \geq 1.
\]

It follows directly from the definition that the above map is \( \mathcal{A}_k \)-linear and compatible with the inclusions \( H_k \hookrightarrow H_{k+1} \), \( G_k \hookrightarrow G_{k+1} \) and \( F_k \hookrightarrow F_{k+1} \); as a result, it extends to a \( \mathcal{A}_\infty \)-linear map \( f : H_\infty \otimes G_\infty \rightarrow F_\infty \). Consider the \( \mathcal{C}_\infty \)-valued sesquilinear form on \( H_\infty \otimes G_\infty \) defined by
\[
(\langle (\xi_1, \xi_2)_B \otimes (\zeta_1, \zeta_2)_B \rangle_{\mathcal{C}_\infty} \in \mathcal{C}_\infty)
\]

which after applying \( f \), clearly goes to the desired one on the right correspondence \( G_\infty \). Moreover, kernel of \( f \) matches exactly with the null space of the form (using non-degeneracy of the \( \mathcal{C}_\infty \)-valued inner product on \( F_\infty \)). Thus \( f \) factors through the relative tensor product and induces an injective \( \mathcal{A}_\infty \)-linear map.

Finally, we need to show that it is surjective as well. For this, consider the PP-basis \( \mathcal{F} \) (resp., \( \mathcal{F}_\infty \)) sitting inside \( H_0 \) (resp., \( G_0 \)) for the right \( B_\infty \)- (resp., \( C_\infty \))-module \( H_\infty \) (resp., \( G_\infty \)) considered in Lemma 3.3. Note that \( \sum \sigma \in \mathcal{F} \sum \Omega_0(\sigma) \circ \tau = 1_{\mathcal{G}_0} \). Thus by Lemma 3.3 \( \{\Omega_0(\sigma) \circ \tau\}_{(\sigma, \tau) \in \mathcal{F} \times \mathcal{F}} \) turns out to be a PP-basis for the right \( \mathcal{C}_\infty \)-module \( F_\infty \).

We next proceed towards defining \( \mathcal{X} \) at the level of 2-cells.

**Definition 3.13.** For \( \Omega_k \in \mathcal{U} \mathcal{C}_1(\Delta_k, \Sigma_k) \) and \( \Lambda_k \in \mathcal{U} \mathcal{C}_1(\Gamma_k, \Delta_k) \) where \( i = 1, 2 \), and 2-cells \( \eta = \{\eta(k)\}_{k \geq K} \in \mathcal{U} \mathcal{C}_2(\Lambda_k, \Lambda_k^2) \) and \( \kappa = \{\kappa(k)\}_{k \geq L} \in \mathcal{U} \mathcal{C}_2(\Omega_k^1, \Omega_k^2) \), define \( \mathcal{K} \boxtimes \mathcal{Y} \in \mathcal{U} \mathcal{C}_2(\Omega_k^1 \boxtimes \Lambda_k^1, \Omega_k^2 \boxtimes \Lambda_k^2) \) by
\[
(\mathcal{K} \boxtimes \mathcal{Y})_k := \Omega_k^2(\eta(k)) \circ \kappa(k) = \kappa(k) \circ \Omega_k^1(\eta(k)) = \left[ \begin{array}{c}
\Omega_k^2 \\
\kappa(k) \\
\eta(k)
\end{array} \right] \begin{array}{c}
\Lambda_k^1 \\
\Lambda_k^2 \\
\Lambda_k
\end{array}
\]
for \( k \geq \max\{K, L\} \). (It is easy to check that every pair of consecutive terms in \( \mathcal{K} \boxtimes \mathcal{Y} \) satisfies the exchange relation, which is a requirement for being a 2-cell.)
The compatibility of the vertical and horizontal compositions - and \(\boxtimes\) between 2-cells follows easily from the pictures.

**Proposition 3.14.** Continuing with the above set up, \(\Psi_{\Gamma \cdot \Delta \cdot}(\eta) \otimes \Psi_{\Delta \cdot \Sigma \cdot}(\kappa)\) corresponds to the operator \(\Psi_{\Gamma \cdot \Delta \cdot}(\eta) \otimes \Psi_{\Delta \cdot \Sigma \cdot}(\kappa)\) via the isomorphism of bimodules in Proposition 3.12.

**Proof.** Let \(H_i^1, G_i^1\) and \(F_i^1\) denote the \(A_i \otimes B_i\), \(B_i \otimes C_i\) and \(A_i \otimes C_i\)-right correspondences \(\Psi_{\Gamma \cdot \Delta \cdot}(\Lambda_i)\), \(\Psi_{\Delta \cdot \Sigma \cdot}(\Omega_i)\) and \(\Psi_{\Gamma \cdot \Delta \cdot}(\Omega_i \boxtimes \Lambda_i)\) for \(i = 1, 2\) respectively.

Set \(T := \Psi_{\Gamma \cdot \Delta \cdot}(\eta) \in A_i \otimes B_i(H_i^1, H_i^2)\) and \(S := \Psi_{\Delta \cdot \Sigma \cdot}(\kappa) \in B_i \otimes C_i(G_i^1, G_i^2)\). Suppose \(X\) denote the intertwiner in \(A_i \otimes C_i(F_i^1, F_i^2)\) induced by \(T \otimes S\) under the isomorphism in Proposition 3.12. For \(k \geq \max\{K, L\}\) and \(\xi \in H_k^1, \zeta \in G_k^1\), applying \(X\) on the element corresponding to the basic tensor \(\xi \otimes \zeta\) (via Proposition 3.12), we get

\[
X(\Omega^1_k(\xi) \circ \zeta) = \Omega^2_k(\Phi(\eta^1_k, \Gamma = m_0) \circ \Phi(\kappa^1_k, \Delta = m_0) \circ \zeta) = \Psi_{\Gamma \cdot \Delta \cdot}(\eta \boxtimes \kappa)(\Omega^1_k(\xi) \circ \zeta).
\]

We summarize the above finding in the following theorem.

**Theorem 3.15.** \(\Psi\) is a \(*\)-linear, fully faithful, tensor-reversing 2-functor from the 2-category of unitary connections \(\text{UC} = \{\text{UC}_{\Gamma \cdot \Delta \cdot} : \Gamma, \Delta, \Sigma\) are 0-cells\} to the 2-category of right correspondence over pairs of \(\text{AFD pre-C*}\)-algebras.

**Remark 3.16.** The \(*\)-algebras \(A_i, B_i\) associated to 0-cells \(\Gamma, \Delta, \Sigma\) can be completed using their unique C*-norm, and obtain the C*-algebras \(A, B\) respectively. Then, the \(A\)-\(B\) right correspondence associated to the 1-cell \(\Lambda_i\) will be the completion \(H_i\) of the space \(\Lambda_i\) with respect to the norm \(\|\xi\|_{\text{c*}} = \sqrt{\|\langle \xi, \xi \rangle_B\|}\). The PP-basis \(\mathcal{S}\) for the right \(B_i\)-module \(H_i\) continue to be so for the right \(B\)-module \(H\). As a result, \(H\) as a \(B\)-module becomes isomorphic to \(q[C \otimes B]\) where the right \(B\)-action on the latter module is the diagonal one and \(q\) is the projection \(\sum_{\sigma_1, \sigma_2 \in \mathcal{S}} E_{\sigma_1, \sigma_2} \otimes (\sigma_2, \sigma_1) B\) in the C*-algebra \(M_{\mathcal{S} \times \mathcal{S}} \otimes B\). The left \(A\)-action on \(H\) will translate into \(*\)-homomorphism \(\Pi : A \rightarrow q[M_{\mathcal{S} \times \mathcal{S}} \otimes B]q\) giving rise to an \(A\)-action on \(q[C \otimes B]\). Now, at the level of 2-cells from the 1-cell \(\Lambda_i\) to \(\Omega_i\), the obvious candidates that come up are the adjointable \(A\)-\(B\) intertwiners; these are in one-to-one correspondence with elements in \(s[M_{\mathcal{S} \times \mathcal{S}} \otimes B]q\) which intertwines \(\Pi(a)\) and \(\Pi(a)\) for \(a \in A\) (where \(\mathcal{S}, s, \Pi\) are related to the 1-cell \(\Omega_i\) in the same way as \(\mathcal{S}, q, \Pi\) are related to \(\Lambda_i\) respectively). However, there is no apparent interpretation of such 2-cells in terms of natural transformations between \(\Lambda_k\)’s and \(\Omega_k\)’s compatible with the \(W_k\)’s as shown in Theorem 3.10.
4. The tracial case

Locally semisimple algebras equipped with a tracial state, extend to finite von Neumann algebras. Hyperfinite subfactor reconstruction works by passing from the algebraic category described in the previous section to von Neumann algebras, and showing that for special cases arising in finite index subfactor theory, it is fully faithful. A natural question is to figure out what happens in our more general setting.

To make this question precise, we consider modifications of the 2-category $\mathbf{UC}$ at every level. At the level of 0-cells, we will be considering Bratteli diagrams with extra tracial data, and make necessary adjustments to the 1- and 2-cells in order to “preserve” this extra structure. Our particular choice of adjustments admittedly appears ad hoc, but it is the condition that was required to make our functor work in the forthcoming proofs. We define the 2-category $\mathbf{UC}^\text{tr}$ as follows:

- **0-cells.** The 0-cells are given by pairs $(\Gamma_\bullet, \mu^\bullet)$ where $\Gamma_\bullet$ is a 0-cell $\{M_{k-1} \xrightarrow{\Gamma_k} M_k\}_{k \geq 0}$ in $\mathbf{UC}$, and $\mu^\bullet$ denotes weight vectors $\mu^k = (\mu^k_v)_{v \in V_{M_k}}$ with positive entries satisfying $\sum_{v \in V_{M_0}} \mu^0_v = 1$ and $(\Gamma_k)' \mu^k = \mu^{k-1}$ all $k \geq 1$ (where we use the same symbol for the functor and its adjacency matrix). In other words (recasting in terms of semisimple categories and functors), the data of a 0-cell in $\mathbf{UC}^\text{tr}$ is a sequence of weighted finitely semisimple $C^*$-categories $\{ (\mathcal{M}_k, \mu^k) \}_{k \geq 0}$ along with a sequence of *-linear, bi-faithful functors $\Gamma_k : \mathcal{M}_{k-1} \rightarrow \mathcal{M}_k$ such that the tracial solution (say $(\rho_k : \text{id}_{\mathcal{M}_k} \rightarrow \Gamma_k \Gamma_k' \rho_k') : \text{id}_{\mathcal{M}_{k-1}} \rightarrow \Gamma_k' \Gamma_k$) where $\Gamma_k'$ is adjoint to $\Gamma_k$ to the conjugate equations commensurate with the weight functions $(\mu^{k-1}, \mu^k)$ satisfies

$$\text{(4.1)} \quad (\rho_k')^* \circ (\rho_k') = 1,$$

which is equivalent to the matrix equation $(\Gamma_k)' \mu^k = \mu^{k-1}$ (via Equation (2.3)). The purpose of the equation $\sum_{v \in V_{M_0}} \mu^0_v = 1$ is to normalize scaling. For simplicity, we will denote the 0-cell of $\mathbf{UC}^\text{tr}$ by $\Gamma_\bullet$.

- **1-cells.** A 1-cell in $\mathbf{UC}^\text{tr}$ from the 0-cell $\{ (\mathcal{M}_{k-1}, \mu^{k-1}) \xrightarrow{\Gamma_k} (\mathcal{M}_k, \mu^k) \}_{k \geq 0}$ to

$$\{ (\Delta_{k-1}, \nu^{k-1}) \xrightarrow{\Lambda_k} (\Delta_k, \nu^k) \}_{k \geq 0}$$

is given by a 1-cell $\Lambda_\bullet$ in $\mathbf{UC}_1 (\Gamma_\bullet, \Delta_\bullet)$ such that there exists $\epsilon, M > 0$ satisfying the boundedness condition:

$$\text{(4.2)} \quad \epsilon \mu^k_v \leq \left[ \Lambda_k' \nu^k \right]_v \leq M \mu^k_v \text{ for all } k \geq 0, \ v \in V_{M_k}.$$

- **Tensor of 1-cells.** For 0-cells $\Gamma_\bullet, \Delta_\bullet, \Sigma_\bullet$, we will define a map

$$\boxtimes : \mathbf{UC}_1^\text{tr} (\Delta_\bullet, \Sigma_\bullet) \times \mathbf{UC}_1^\text{tr} (\Gamma_\bullet, \Delta_\bullet) \rightarrow \mathbf{UC}_1^\text{tr} (\Gamma_\bullet, \Sigma_\bullet)$$

exactly the same as that for $\mathbf{UC}_1$ given by Equation (3.2); however, we have to check whether Equation (4.2) is satisfied by $\Omega_\bullet \boxtimes \Lambda_\bullet$, where $\Omega_\bullet \in \mathbf{UC}_1^\text{tr} (\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \mathbf{UC}_1^\text{tr} (\Gamma_\bullet, \Delta_\bullet)$. Suppose we have,

$$\epsilon \mu^k_v \leq [\Lambda_k' \nu^k]_u \leq M \mu^k_v \quad \text{and} \quad \delta \nu^k_v \leq [\Omega_k' \Pi^k]_v \leq N \nu^k_v \text{ for each } k \geq 0, u \in V_{M_k}, v \in V_{\Delta_k}.$$
Applying $\Lambda'_k$ on the second set of inequalities, we get $\epsilon \delta k^u \leq [\Lambda'_k, \Omega'_u] u \leq MN \mu^k$ for each $k \geq 0, u \in V_{M_k}$. Thus, Equation (4.12) is satisfied for $\Omega_* \otimes \Lambda_*$. 

- **2-cells.** Consider two 1-cells $\Lambda_*, \Omega* \in \text{UC}^\text{tr}_1((\Gamma_*, \mu^*), (\Delta^*, \nu^*))$. The 2-cells in $\text{UC}_2(\Lambda_*, \Omega_*)$, that is, the sequences eventually satisfying the exchange relation at every level, do not use the extra data of $\mu^*$ and $\nu^*$. We introduce the following exchange relation.

**Definition 4.1.** The loop operator from $\Lambda_*$ to $\Omega_*$ is the sequence of linear maps $\{S_k : \text{NT}(\Lambda_k, \Omega_k) \to \text{NT}(\Lambda_{k-1}, \Omega_{k-1})\}_{k \geq 1}$ defined by

$$\text{NT}(\Lambda_k, \Omega_k) \ni \eta \xrightarrow{S_k} \frac{\Delta_k}{\Lambda_k} \frac{\Omega_k}{\Omega_{k-1}} \Gamma_k \in \text{NT}(\Lambda_{k-1}, \Omega_{k-1}) \ .$$

where the cap and the cup come from tracial solution to the conjugate equation for the duality of $\Delta_k : \mathcal{N}_{k-1} \to \mathcal{N}_k$ commensurate with $(\nu^{k-1}, \nu^k)$.

We will encounter equations and inequalities involving multiple loop operators all of which might not have the same source 1-cell or the same target 1-cell in $\text{UC}_1^\text{tr}$; for notational convenience, we will simply use $S_*$, and from the context, it will be clear what the source and the targets are.

**Remark 4.2.** The loop operator satisfies the following properties which are easy to derive:

(i) $S_k$ is unital when $\Lambda_* = \Omega_*$, (which follows from Equation (4.11)),

(ii) $S_k \eta^* = (S_k \eta)^*$,

(iii) $S_k \eta^* \circ S_k \eta \leq S_k (\eta^* \eta)$ in the $C^*$-algebra $\text{NT}(\Lambda_{k-1}, \Lambda_{k-1})$ and the loop operator is a contraction.

**Definition 4.3.** A sequence $\eta = \{ \eta^{(k)} \in \text{NT}(\Lambda_k, \Omega_k) \}_{k \geq 0}$ will be referred as:

(a) quasi-flat sequence from $\Lambda_*$ to $\Omega_*$ if it satisfies $S_{k+1} \eta^{(k+1)} = \eta^{(k)}$ for all $k \geq 0$,

(b) flat sequence from $\Lambda_*$ to $\Omega_*$ if it is quasi-flat and there exists $K \in \mathbb{N}$ such that $(\eta^{(k)}, \eta^{(k+1)})$ satisfies the exchange relation (Definition 3.3) for every $k \geq K$.

**Remark 4.4.** There is a one-to-one correspondence between flat sequences from $\Lambda_*$ to $\Omega_*$, and the 2-cells in $\text{UC}_2(\Lambda_*, \Omega_*)$. Note that the exchange relation

$$\begin{align*}
\eta^{(k)} \quad &\xrightarrow{S_k} \quad \frac{\eta^{(k+1)}}{\Lambda_{k+1}} \\
\Lambda_k \quad &\xrightarrow{\pi_k} \quad \Lambda_k
\end{align*}$$

unitarity of the connection and Equation (4.1) yield the equation $S_{k+1} \eta^{(k+1)} = \eta^{(k)}$. Thus every 2-cell $\{ \eta^{(k)} \}_{k \geq K} \in \text{UC}_2(\Lambda_*, \Omega_*)$ extends to a unique quasi-flat sequence from $\Lambda_*$ to $\Omega_*$ by setting $\eta^{(k)} := S_{k+1} \cdots S_K \eta^{(K)}$ for $k < K$. Further, a flat sequence is bounded in $C^*$-norm.

**Definition 4.5.** A 2-cell in $\text{UC}^\text{tr}_2(\Lambda_*, \Omega_*)$ is given by a bounded (in $C^*$-norm) quasi-flat sequence (abbreviated as ‘BQFS’) from $\Lambda_*$ to $\Omega_*$.
• Horizontal and vertical compositions of 2-cells.

**Definition 4.6.** (a) The **composition of the 2-cells** \( \kappa \in \text{UC}_{2}^{\text{tr}}(\Omega_{*}, \Xi_{*}) \) and \( \eta \in \text{UC}_{2}^{\text{tr}}(\Lambda_{*}, \Psi_{*}) \) is defined as

\[
(\kappa \cdot \eta) := \left\{ (\kappa \cdot \eta)^{(k)} := \lim_{l \to \infty} S_{k+1} \cdots S_{k+l} (\kappa^{(k+l)} \circ \eta^{(k+l)}) \right\}_{k \geq 0} \in \text{UC}_{2}^{\text{tr}}(\Lambda_{*}, \Psi_{*}) .
\]

(b) Let \( \Lambda_{*} \in \text{UC}_{1}^{\text{tr}}(\Gamma_{*}, \Delta_{*}) \) and \( \Omega_{*} \in \text{UC}_{1}^{\text{tr}}(\Delta_{*}, \Sigma_{*}) \) for \( i = 1, 2 \). Then, the **horizontal composition** (or the tensor product) of the 2-cells \( \eta = \{ \eta^{(k)} \}_{k \geq 0} \in \text{UC}_{2}^{\text{tr}}(\Omega_{1}^{*}, \Omega_{2}^{*}) \) is given by

\[
\kappa \boxtimes \eta := \left\{ (\kappa \boxtimes \eta)^{(k)} := \lim_{l \to \infty} S_{k+1} \cdots S_{k+l} \left( \Omega_{k+1}^{j}(\eta^{(k+l)}) \circ \kappa^{(k+l)}_{\Lambda_{k+l}} \right) \right\}_{k \geq 0} \in \text{UC}_{2}^{\text{tr}}(\Omega_{1}^{*} \boxtimes \Lambda_{1}^{*}, \Omega_{2}^{*} \boxtimes \Lambda_{2}^{*}) .
\]

**Remark 4.7.** A natural question to ask is why the limits in Definition 4.6 exists and even if they all exist, why the sequences built by these limit will yield a 2-cell in \( \text{UC}^{\text{tr}} \). One way to settle this issue is by viewing the loop operator \( S \) as a UCP operator and express the compositions as a certain Choix-Effros product (along the lines of Izumi’s Poisson boundary approach in [10,14]). However, we will not take this route. Instead we will make \( \text{UC}^{\text{tr}} \) sit inside the 2-category of von Neumann algebras, bimodules and intertwiners in a fully faithful way. The benefit of this approach is that the details which are left out in defining \( \text{UC}^{\text{tr}} \) as a \( \text{W}^{*} \)-2-category, namely, unit object, associativity of the two types of compositions, etc. will be automatically verified.

**Remark 4.8.** In the passage from \( \text{UC} \) to \( \text{UC}^{\text{tr}} \), we are imposing restriction at the level of 1-cells but the 2-cell spaces have been generalized. So, on the nose, neither we have a forgetful functor nor one turns out as a subcategory of the other. However, we do have a subcategory of \( \text{UC}^{\text{tr}} \) which we call its **flat part** and denote by \( \text{UC}^{\text{flat}} \) where everything is the same as that of \( \text{UC}^{\text{tr}} \) at the level of 0- and 1-cells but the 2-cells in \( \text{UC}^{\text{flat}} \) are only flat sequences (and not all BQFS). Indeed compositions of the 2-cells in \( \text{UC}^{\text{flat}} \) correspond to exactly to those in \( \text{UC} \); this easily follows from Remark 4.4.

## 5. A Concret Realization of \( \text{UC}^{\text{tr}} \)

The goal of this section is to build a fully faithful 2-functor \( \text{PB} : \text{UC}^{\text{tr}} \to \text{vNAlg} \) where \( \text{vNAlg} \) is the 2-category of von Neumann algebras, bimodules and intertwiners (as stated in Theorem 1.1). Our starting point will be the pre-C*-algebras and right correspondences produced from 0- and 1-cells in \( \text{UC}^{\text{tr}} \) viewed as those in \( \text{UC} \) as described in Section 3 and then take their appropriate completions. At this point, it might seem it is enough to build the functor starting from the flat part \( \text{UC}^{\text{flat}} \); however, in that case, the functor may not be fully faithful at the level of 2-cells (which are only flat sequences). In this section, we will be analyzing “the kernel” of the 2-functor from \( \text{UC}^{\text{flat}} \); as a consequence, we justify the need of generalizing the 2-cells in \( \text{UC}^{\text{flat}} \) to those in \( \text{UC}^{\text{tr}} \).

### 5.1. \( \text{PB} \) on 0-cells.

Given a 0-cell \( (\Gamma_{*}, \mu_{*}) \) in \( \text{UC}^{\text{tr}} \), we consider \( m_{0}, A_{k} \)'s and their inclusions as in the non-tracial case Section 3. Using the categorical trace \( \text{Tr} = (\text{Tr}_{x})_{x \in \text{ob}(A_{k})} \) associated to the weight vector \( \mu^{k} \), we define \( \text{Tr}_{A_{k}} := \text{Tr}_{\Gamma \cup \Gamma_{m_{0}}} : A_{k} \to \mathbb{C} \) which turns out to be a faithful tracial state which by Equation (2.5), turns out to be compatible with the inclusion. Thus, we have a faithful tracial state \( \text{Tr}_{A_{\infty}} \) on the *-algebra \( A_{\infty} \). Note that the action of an element of \( A_{\infty} \) on the GNS Hilbert \( L^{2}(A_{\infty}, \text{Tr}_{A_{\infty}}) \) is bounded. Let \( A \)
denote the type $H_1$ von Neumann algebra obtained by taking the WOT closure of $A_\infty$ acting on $L^2(A_\infty, \Tr_{A_\infty})$.

**Definition 5.1.** We define $\mathcal{PB}(\Gamma_\bullet, \mu^\bullet) := A = A'' \subseteq \mathcal{L}(L^2(A, \Tr(A_\infty)))$.

### 5.2. $\mathcal{PB}$ on 1-cells.

Let $\Lambda_\bullet \in \mathcal{UC}^1((\Gamma_\bullet, \mu^\bullet), (\Delta_\bullet, \nu^\bullet))$. Consider the $A_\infty \cdot B_\infty$ right correspondence $H_\infty$ associated to the 1-cell $\Lambda_\bullet$ treated as a 1-cell in $\mathcal{UC}$. Let $H$ be the completion of $H_\infty$ with respect to the scalar inner product $\langle \xi, \zeta \rangle \overset{\text{def}}{=} \Tr_{B_\infty}(\langle \xi, \zeta \rangle_{B_\infty})$ for $\xi, \zeta \in H_\infty$. $A_\infty, B_\infty$ being locally semisimple $*$-algebras, must have the action of their elements on $H_\infty$ bounded, and hence extend to action on $H$.

To obtain a right $B$-action on $H$, we work with the Pimsner-Popa basis $\mathcal{S}$ for the right-$B_\infty$-module $H_\infty$ with respect to the $B_\infty$-valued inner product obtained in Lemma 3.3. Observe that the map

$$H \ni H_\infty \ni \xi \mapsto \sum_{\sigma \in \mathcal{S}} \sigma \otimes \langle \xi, \sigma \rangle_{B_\infty} \in q [\ell^2(\mathcal{S}) \otimes L^2(B_\infty, \Tr_{B_\infty})] =: K$$

extends to an isometric isomorphism preserving the right $B_\infty$-action where $q$ is the projection $\sum_{\sigma, \tau \in \mathcal{S}} E_{\sigma, \tau} \otimes \langle \tau, \sigma \rangle_{B_\infty}$. Clearly, the $B_\infty$ action on $K$ extends to a normal action of $B$ and hence, the same holds for the Hilbert space $H$.

In order to extend the $A_\infty$-action on $H$ (which is clearly bounded) to a normal action of $A$, we first analyse the commutant of $B$ in $\mathcal{L}(H)$. For $k \geq 0$, define $C_k := \text{End}(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0)$. The $*$-homomorphism $\mathcal{L}(H) \ni \gamma \mapsto \Phi_{\gamma} |_{H_k} = \gamma \circ \bullet \in \mathcal{L}(C_k)$ is faithful by Lemma 3.7 and hence an isometry. Thus, $\Phi_{\gamma}$ extends to the whole of $H$ as a bounded operator commuting with the right action of $B_\infty$ (and thereby $B$). Consider the unital $*$-algebra inclusion

$$C_k \ni \gamma \mapsto (W_k)_{\Gamma_k \cdots \Gamma_1 m_0} \circ \Delta_{k+1} \gamma \circ (W_k)^\ast_{\Gamma_k \cdots \Gamma_1 m_0} = \Lambda_{k+1} \cdot \begin{pmatrix} \gamma \\ \cdots \\ \gamma \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ \cdots \\ m_0 \end{pmatrix} \in C_{k+1}$$

Note that $\Phi_{\gamma}$ is compatible with the above inclusion. Indeed, $C_k \ni \gamma \mapsto \Phi_{\gamma} \in \mathcal{L}_B(H)$ becomes a unital faithful $*$-algebra homomorphism where $C_\infty := \bigcup_{k \geq 0} C_k$.

**Proposition 5.2.** $\mathcal{L}_B(H) = \{ \Phi_{\gamma} : \gamma \in C_\infty \}''$.

*Proof.* Consider the projection $p_k \in \mathcal{L}(H)$ such that $\text{Range}(p_k) = H_k$. Since $A_k \cdot H_k \cdot B_k = H_k$, therefore $p_k$ must be $A_k \cdot B_k$-linear.

Let $T \in \mathcal{L}_B(H)$. Set $\zeta_k := \sum_{\sigma \in \mathcal{S}} \begin{pmatrix} m_0 \\ \cdots \\ m_0 \end{pmatrix}$ in $C_k$ where $\mathcal{S} (\subset H_0)$ is a PP-basis of the right $B$-module $H$ as constructed in Lemma 3.3. Since $T$ (resp. $p_k$) is right $B$-(resp. $B_{k\cdot}$) linear, one may deduce the relation $p_k T p_k = \Phi_{\zeta_k} p_k$. Clearly, $p_k$ converges to
id_\mathcal{H} in SOT as \( k \) goes to \( \infty \), and \( \{ \Phi_k \}_{k \geq 0} \) is a norm bounded subset of \( \mathcal{L}(\mathcal{H}) \). Hence, 
\[
T \in \{ \Phi_\gamma : \gamma \in C_\infty \}^{SOT} = \{ \Phi_\gamma : \gamma \in C_\infty \}''.
\]

**Remark 5.3.** Using Equation (4.1), we may represent the projection \( p_k \) in the following way:

\[
H \supset H_{k+l} \ni \xi \xrightarrow{p_k} \cdots \Lambda_k \Lambda_{k+l} \cdots \xi \in H_k \subset H
\]

where the local maxima and minima are given by the natural transformation appearing in the tracial solution to conjugate equation for the duality of the functors \( \Delta_i \)'s associated to the positive weights \( \nu_i^{-1} \) and \( \nu_i \) on the vertices (that is, the simple objects of \( \mathcal{N}_{i-1} \) and \( \mathcal{N}_i \)). Clearly, \( p_k \) is \( A_k \)-linear.

Consider the unital \(*\)-algebra inclusion \( A_k \ni \alpha \xrightarrow{\Lambda_k} A_k \alpha \in C_k \). Again, this inclusion is compatible with \( C_k \hookrightarrow C_{k+1} \) and \( A_k \hookrightarrow A_{k+1} \); thus, \( A_\infty \) sits as a unital \(*\)-subalgebra inside \( C_\infty \). Observe that if \( \gamma \in C_k \) comes from \( A_k \), that is, \( \gamma = \Lambda_k \alpha \) for some \( \alpha \in A_k \), then \( \Phi_\gamma \) matches exactly with the action of \( \alpha \) on \( H_\infty \). Now, the functional

\[
\text{Tr}' := [d_B(\mathcal{H})]^{-1} \sum_{\sigma \in \mathcal{S}} \langle \bullet, \sigma \rangle : \mathcal{L}_B(\mathcal{H}) \to \mathbb{C}
\]

is a faithful normal tracial state where \( \mathcal{S} \) is a PP-basis for the module \( H_B \) and \( d_B(\mathcal{H}) := \sum_{\sigma \in \mathcal{S}} \|\sigma\|^2 \); however, its restriction on \( A_\infty \) may not match with that of \( \text{Tr}_{A_\infty} \).

**Proposition 5.4.** The above inclusion of \( A_\infty \) inside \( C_\infty \) extends to a normal inclusion of \( A \) inside \( \mathcal{L}_B(\mathcal{H}) \), and thereby \( H \) becomes a ‘von Neumann’ \( A \)-\( B \)-bimodule.

**Proof.** By construction, \( A \) is the von Neumann algebra obtained from the GNS of \( A_\infty \) with respect to \( \text{Tr}_{A_\infty} \). Let \( A_\infty'' \) denote the double commutant of \( A_\infty \) sitting inside \( \mathcal{L}(\mathcal{H}) \) via the inclusions \( A_\infty \ni \alpha \xrightarrow{\Lambda_k} A_k \alpha \in C_k \) and \( C_k \xrightarrow{\Phi} \mathcal{L}(\mathcal{H}) \). It is enough to produce a central positive invertible element \( T \) in \( A_\infty'' \) satisfying \( \text{Tr}'(\Phi_{A_k} \alpha T) = \text{Tr}_{A_\infty}(\alpha) \) for \( \alpha \in A_\infty \) (that is, \( \text{Tr}_{A_\infty} \) extends to a faithful normal trace on \( A_\infty'' \)).

Consider the natural transformation \( \theta^k := \left( \frac{\mu^k}{[A_k, \nu^k]_v} \right)_{v \in \mathcal{V}_{M_k}} \in \text{End}(\text{id}_{M_k}) \). Set

\[
T_k := \Phi_{A_k}(\theta^k_{r_1 \cdots r_{m_0}}) \in A_\infty'' \quad \text{and} \quad \psi := \sum_{\sigma \in \mathcal{S}} \langle \bullet, \sigma \rangle = d_B(\mathcal{H}) \text{Tr}'.
\]

**Assertion:** \( \psi(\Phi_{A_k}(\bullet) T_k) = \text{Tr}_{A_k} \) for all \( k \geq 0 \).
Proof of the assertion. Let $\alpha \in A_k$. Then, \( \psi (\Phi_{\Lambda k}(\alpha) T_k) = \sum_{\sigma \in \mathcal{S}} \left\langle \Phi_{\Lambda k}(\alpha \theta^k_{\Gamma k \cdots \Gamma_m 0}) \sigma, \sigma \right\rangle \) for the duality of the functor $\Lambda_k$ by Equation (2.5) where the red cap and cup correspond to tracial solution to conjugate thereby.

Using the property of the categorical trace, the equation in Lemma 3.3 satisfied by the set $\mathcal{S}$ and the natural unitaries (namely, the crossings), we may rewrite the last expression as
\[
\text{Tr}_{\Lambda_k \Gamma_k \cdots \Gamma_{1m0}} \left( \begin{array}{c} \Lambda_k \theta^k \alpha \\ \cdots \\ \sigma \\ \cdots \\ \eta_m \end{array} \right) = \text{Tr}_{\Gamma_k \cdots \Gamma_{1m0}} \left( \begin{array}{c} \Lambda_k \theta^k \alpha \\ \cdots \\ \sigma \\ \cdots \\ \eta_m \end{array} \right)
\]

by Equation (2.5) where the red cap and cup correspond to tracial solution to conjugate for the duality of the functor $\Lambda_k$ with respect to weight vectors $\mu^k$ and $\nu^k$ on the vertex sets $V_{\Lambda_k}$ and $V_{\Lambda_k}$ respectively. Now, it is a matter of routine verification that the red loop appearing above is indeed the inverse of $\theta^k$ in the algebra $\text{End}(id_{\Lambda_k})$. Cancelling the two, we get $\text{Tr}_{\Lambda_k}(\alpha)$.

Equation (1.2) implies that $C^*$-norm of $\theta^k$ is uniformly bounded by $\epsilon^{-1}$ for $k \geq 0$, and thereby $\{T_k\}_{k \geq 0}$ is norm-bounded sequence in $A'_\infty \subset \mathcal{L}(H)$. By compactness, there exists a subsequence $\{T_{k_l}\}_l$ which converges in WOT to $T_0 \in A'_\infty$ (say). Clearly, $\psi(\Phi_{\Lambda_k}(\alpha) T_0) = \text{Tr}_{\Lambda_k}(\alpha)$ for all $\alpha \in A_k$. Observe that $T_k$ commutes with $\Phi_{\Lambda_k}$ for all $\alpha \in A_k$, $k \geq 0$; this implies $T_0$ must be central in $A'_\infty$. Again, $T_k$ is a positive element in $A'_\infty$ satisfying $T_k \geq M^{-1}$ (using Equation (1.2)); thus, the subsequential WOT-limit $T_0$ also satisfies the same.

\[\square\]

Definition 5.5. Define
\[\text{UC}_1^\mu \left( (\Gamma_\ast, \mu_\ast), (\Delta_\ast, \nu_\ast) \right) \ni \Lambda_\ast \xrightarrow{\mathcal{P}\mathcal{B}} \mathcal{P}\mathcal{B}(\Lambda_\ast) := A_{HB} \in \text{vNalg}_1 \left( \mathcal{P}\mathcal{B}(\Delta_\ast, \nu_\ast), \mathcal{P}\mathcal{B}(\Gamma_\ast, \mu_\ast) \right).
\]

5.3. $\mathcal{P}\mathcal{B}$ on 2-cells.

Let $\Lambda_\ast, \Omega_\ast \in \text{UC}_1^\mu \left( (\Gamma_\ast, \mu_\ast), (\Delta_\ast, \nu_\ast) \right)$ and $\mathcal{P}\mathcal{B}(\Gamma_\ast, \mu_\ast) = A, \mathcal{P}\mathcal{B}(\Delta_\ast, \nu_\ast) = B$. We will borrow the notations $H_k, H, p_k, \mathcal{F}$, etc. (arising out of $\Lambda_\ast$) from previous subsections, and for those arising out of $\Omega_\ast$, we will use $G_k, G, q_k, \mathcal{F}$, etc. respectively, and we will also work with the pictures as before. For $\gamma \in \mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_{1m0}, \Omega_k \Gamma_k \cdots \Gamma_{1m0})$, we will consider the unique bounded extension of $\Phi_{\gamma} \in \mathcal{L}_B(H_\infty, G_\infty)$ (defined in Equation (3.1)) and denote it with the same symbol $\Phi_{\gamma} \in \mathcal{L}_B(H, G)$.

Proposition 5.6. (a) $q_{k-1} \Phi_{\eta \Gamma_k \cdots \Gamma_{1m0}} p_{k-1} = \Phi_{(s_k \eta) \Gamma_k \cdots \Gamma_{1m0}} p_{k-1}$ for all $\eta \in \text{NT}(\Lambda_k, \Omega_k)$ and $k \geq 0$.

(b) If $\eta = \{\eta^{(k)}\}_{k \geq 0} \in \text{UC}_1^\mu(\Lambda_\ast, \Omega_\ast)$ (that is, a BQFS from $\Lambda_\ast$ to $\Omega_\ast$), then it gives rise to a unique bounded operator $T \in A_{\mathcal{L}_B(H, G)}$ satisfying
\[q_k T p_k = \Phi_{\eta^{(k)} \Gamma_k \cdots \Gamma_{1m0}} p_k \quad \text{for all} \quad k \geq 0.
\]
Proof. Part (a) directly follows from Remark 5.3. 

For (b), set \( T_k := \Phi_{\eta_{k}^{(k)}}^{\eta_{k} \cdot R_{1}m_{0}} \) for each \( k \geq 0 \); it is easy to see that \( T_k \in A_k \mathcal{L}_B(H, G) \) and \( \| T_k \| = \| \eta(k) \| \). For all \( \gamma \in H_k \), using part (a) followed by quasi-flat condition of \( \eta \), we have
\[
q_k T_k + \gamma = \Phi_{\eta_{k}^{(k+1)}}^{\eta_{k} \cdot R_{1}m_{0}} \gamma = \Phi_{\eta_{k}^{(k)}}^{\eta_{k} \cdot R_{1}m_{0}} \gamma = T_k \gamma.
\]
In other words, \( q_k T_k + \gamma = T_k p_k \). Applying this iteratively, we get \( q_k T_k + p_k = T_k p_k = q_k T_k p_k \) for all \( k, l \geq 0 \). This again implies
\[
\| T_{l+m} - T_l \gamma \| = \| T_{l+m} \gamma \| - \| T_l \gamma \| \quad \text{for all } k, l, \gamma \in H_k.
\]

Now, fix \( \gamma \in H_k \). Equation (5.2) tells us that the sequence \( \{ \| T_l \gamma \| \}_{l \geq k} \) must be increasing; also, it is bounded by \( \sup_{m \geq 0} \| \eta^{(m)} \| \) and hence convergent. Letting \( l \) tend towards \( \infty \) in Equation (5.2), we find that \( \{ T_k \}_{k \geq 0} \) converges pointwise on \( H_\infty \) (because of completeness of \( G \)). Since \( H_\infty \) is dense in \( H \) and \( \{ T_k \}_{k \geq 0} \) is norm-bounded by \( \sup_{k \geq 0} \| \eta^{(k)} \| \), we may conclude that \( \{ T_k \}_{k \geq 0} \) converges in \( \text{SOT} \) to some \( T \in \mathcal{L}(H, G) \).

To prove Equation (5.1), consider \( q_k T p_k = \text{SOT-lim}_{l \to \infty} q_k T_{k+l} p_k = q_k T p_k = \Phi_{\eta_{k}^{(k)}}^{\eta_{k} \cdot R_{1}m_{0}} p_k \). Since the right side of condition (b) is \( A_k \)-linear, so is the other side namely, \( q_k T p_k \). Since \( \{ q_k T p_k \}_{k \geq 0} \) converges in \( \text{SOT} \) to \( T \), therefore, \( T \) must be \( A_k \)-linear, and thereby \( A_\infty \)-linear, and finally \( A_\infty \)-linear.

If \( T_1 \) is any other operator satisfying Equation (5.1), then \( q_k (T - T_1) p_k = 0 \). Now, \( p_k \) and \( q_k \) increase to \( id_H \) and \( id_G \) respectively. This forces \( T_1 \) and \( T \) to be identical. \( \square \)

Definition 5.7. For \( \overline{\eta} = \{ \eta^{(k)} \}_{k \geq 0} \in \text{UC}^2_\infty (\Lambda \bullet, \Omega \bullet) \), define
\[
\mathcal{P}B (\overline{\eta}) := \text{SOT-lim}_{k \to \infty} \Phi_{\eta_{k}^{(k)}}^{\eta_{k} \cdot R_{1}m_{0}} = \mathcal{L}_B(H, G) = \text{vNAAlg}_2 (\mathcal{P}B (\Lambda \bullet), \mathcal{P}B (\Omega \bullet)) .
\]

Proposition 5.8. For every \( T \in A_\mathcal{L}_B(H, G) \) and \( k \geq 0 \), there exists unique \( \eta^{(k)} \in \text{NT}(\Lambda_k, \Omega_k) \) such that \( q_k T p_k = \Phi_{\eta_{k}^{(k)}}^{\eta_{k} \cdot R_{1}m_{0}} p_k \) (which is the same as Equation (5.1)).

Proof. We will use a modified version of a trick which we have already seen twice before, namely, in the proofs of Lemma 3.7 and Theorem 3.10. Set \( \zeta_k := \sum_{\sigma \in \mathcal{A}} \Omega_{k}^{m_0} q_k (T \sigma) \in \mathcal{N}_k (\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \). With similar reasoning as before, one can easily conclude \( q_k T p_k = \Phi_{\zeta_k} p_k \); moreover, this equation uniquely determines \( \zeta_k \) by Lemma 3.7 (iii). Further, the left side of the equation is \( A_k \)-linear; then so is the right side. Again by Lemma 3.7 (iii), \( \zeta_k \) becomes an \( A_k \)-central vector of \( \mathcal{N}_k (\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Omega_k \Gamma_k \cdots \Gamma_1 m_0) \). Applying Lemma 3.8, we get a unique \( \eta^{(k)} \in \text{NT}(\Lambda_k, \Omega_k) \) satisfying \( \zeta_k = \eta_{\Gamma_k}^{(k)} \). This completes the proof. \( \square \)

Theorem 5.9. The following is an isomorphism
\[
\text{UC}^2_\infty (\Lambda \bullet, \Omega \bullet) \ni \overline{\eta} \mapsto \mathcal{P}B (\overline{\eta}) \in \text{vNAAlg}_2 (\mathcal{P}B (\Lambda \bullet), \mathcal{P}B (\Omega \bullet)) .
\]
(This will ultimately imply that the 2-functor \( \mathcal{P}B \) is fully faithful.)
Proof. Suppose $\mathcal{PB}(\eta) = 0$. Then, by Equation (5.1), we have $\Phi_{\eta_{k_1 \cdots k_m}} = 0$ which (by Lemma 3.7 (iii)) implies $\eta_{k_1 \cdots k_m} = 0$. Now, Lemma 3.8 ensures that $\eta^{(k)}$ must be zero for all $k$.

For surjectivity, pick $T \in A\mathcal{L}_B(H, G)$. We only need to show that the unique sequence \( \{\eta^{(k)} \in NT(\Lambda_k, \Omega_k)\}_{k \geq 0} \) associated to $T$ obtained in Proposition 5.8 is quasi-flat and bounded in $C^*$-norm. Note that for all $\gamma \in H_k = \mathcal{N}_k(\Delta_k \cdots \Delta_1 n_0, \Lambda_k \Gamma_k \cdots \Gamma_1 m_0)$, we apply Equation (5.1) twice and obtain
\[
\Phi_{\eta_{k_1 \cdots k_m}} \gamma = q_k T p_k \gamma = q_k q_{k+1} T p_{k+1} \gamma = q_k \Phi_{\eta_{k_1 \cdots k_m}} \gamma = \Phi_{\eta_{k_1 \cdots k_m}} \gamma
\]
where the last equality follows from Proposition 5.6 (a). By Lemma 3.7 (iii), we must have $\eta_{k_1 \cdots k_m} = [S \eta^{(k+1)}]_{\Gamma_{k_1 \cdots k_m}}$ which via the isomorphism in Lemma 3.8 implies $\eta^{(k)} = S^{k+1} \eta^{(k+1)}$. For boundedness, we apply the norm on both sides of Equation (5.1); note that the map in Lemma 3.7 (iii) is actually an isometry (with respect to the $C^*$-norms) which yields the inequality $\|T\| \geq \|\eta_{k_1 \cdots k_m}\| = \|\eta^{(k)}\|$ where the last equality holds because $\Gamma_{k_1 \cdots k_m}$ contains every simple of $\mathcal{M}_k$ as a subobject.

5.4. $\mathcal{PB}$ preserves tensor product of 1-cells and compositions of 2-cells.

Our goal here is clear from the title of this section. As a by product of achieving this goal, we will prove the existence of the limits appearing in

Proposition 5.10. For $\eta = \{\eta^{(k)}\}_{k \geq 0} \in \text{UC}_2^{tr}(\Lambda_\bullet, \Omega_\bullet)$ and $\kappa = \{\kappa^{(k)}\}_{k \geq 0} \in \text{UC}_2^{tr}(\Omega_\bullet, \Xi_\bullet)$,

(a) the sequence $\{S_{k+1} \cdots S_{l+k} \left( \kappa^{(l+1)} \circ \eta^{(k)} \right) \}_{l \geq 0}$ converges (in $NT(\Lambda_k, \Xi_k)$) for every $k \geq 0$, and

(b) $\mathcal{PB}(\kappa \cdot \eta) = \mathcal{PB}(\kappa) \circ \mathcal{PB}(\eta)$.

Proof. We continue using the previous notations and let us denote the Hilbert spaces and the projection corresponding to $\Xi$ by $F_k$, $F$ and $s_k$; the intertwiners corresponding to $\eta$ and $\kappa$ will be denoted by $X \in A\mathcal{L}_B(H, G)$ and $Y \in A\mathcal{L}_B(G, F)$ respectively. Set $Z := XY \in A\mathcal{L}_B(H, F)$ whose corresponding BQFS will be $\{\psi^{(k)}\}_{k \geq 0}$.

For fixed $k \geq 0$, using Proposition 5.8 and Proposition 5.10 (a), we obtain
\[
\Phi_{\eta_{k_1 \cdots k_m}} p_k = s_k Y X p_k = \text{SOT-\,lim}_{l \to \infty} s_k Y q_{k+l} X p_k = \text{SOT-\,lim}_{l \to \infty} s_k \Phi_{\eta_{k_1 \cdots k_m}} p_k = \text{SOT-\,lim}_{l \to \infty} \Phi_{[S_{k+1} \cdots S_{l+k} \left( \kappa^{(l+1)} \circ \eta^{(k)} \right)]_{\Gamma_{k_1 \cdots k_m}}} p_k
\]

Since $\mathcal{N}_k(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Xi_k \Gamma_k \cdots \Gamma_1 m_0)$ is finite dimensional, by the isometry in Lemma 3.7 (iii), we may conclude that $[S_{k+1} \cdots S_{l+k} \left( \kappa^{(l+1)} \circ \eta^{(k)} \right)]_{\Gamma_{k_1 \cdots k_m}}$ converges as $l$ approaches $\infty$ which again implies convergence of $\{S_{k+1} \cdots S_{l+k} \left( \kappa^{(l+1)} \circ \eta^{(k)} \right)\}_{l \geq 0}$ via Lemma 3.8.

Next, we deal with tensor product of 1-cells. We will show that $\mathcal{PB}$ preserves it in the reverse order.

Proposition 5.11. For 0-cells $\Gamma_\bullet, \Delta_\bullet, \Sigma_\bullet$ in $\text{UC}_0^{tr}$, and $\Omega_\bullet \in \text{UC}_1^{tr}(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \text{UC}_1^{tr}(\Gamma_\bullet, \Delta_\bullet)$, the bimodule $\mathcal{PB}(\Omega_\bullet \boxtimes \Lambda_\bullet)$ is isomorphic to the Connes fusion $\mathcal{PB}(\Lambda_\bullet) \otimes \mathcal{PB}(\Omega_\bullet)$.
Proof. We borrow the notations in Proposition 5.12 where we have already obtained an $A_{\infty}$-$C_{\infty}$-linear isomorphism from the relative tensor product of the dense subspace of $\mathcal{P}B(\Lambda, \Omega)$ to that of $\mathcal{P}B(\Omega \otimes \Lambda)$. Moreover, this isomorphism preserves the right $C_{\infty}$-valued inner product; composing with $\text{Tr}_{C_{\infty}}$, it preserves the scalar inner product as well, and hence extends to a $A$-$C$-linear unitary. \hfill \Box

We next proceed towards the horizontal composition of 2-cells.

Proposition 5.12. Let $\Lambda_i \in \text{UC}_1(\Gamma, \Delta)$ and $\Omega_j \in \text{UC}_1(\Delta, \Sigma)$ for $i = 1, 2$, and $\eta = \{(\eta^{(k)})\}_{k \geq 0} \in \text{UC}_2(\Lambda_1, \Lambda_2)$ and $\kappa = \{(\kappa^{(l)})\}_{l \geq 0} \in \text{UC}_2(\Omega_1, \Omega_2)$. Then,

(a) for each $k \geq 0$, the sequence $\left\{ S_{k+1} \cdots S_{k+l} \left( \Omega_k^{(k+l)} \circ \kappa_k^{(k+l)} \right) \right\}_{l \geq 0}$ converges in $NT(\Omega_k^{(k)} \Lambda_k, \Omega_k^{(k)} \Lambda_k)$ where $S$ is the loop operator from $\Omega_k^{(k)} \otimes \Lambda_k$ to $\Omega_k^{(k)} \otimes \Lambda_k$, and indeed the sequence $\kappa \otimes \eta := \{(\kappa \otimes \eta)^{(k)}\}_{k \geq 0} = \lim_{k \to \infty} S_{k+1} \cdots S_{k+l} \left( \Omega_k^{(k+l)} \circ \kappa_k^{(k+l)} \right)$ is a BQFS from $\Omega_k \otimes \Lambda_k$ to $\Omega_k \otimes \Lambda_k$.

(b) $\mathcal{P}B(\kappa \otimes \eta)$ corresponds to the operator $\mathcal{P}B(\eta) \otimes \mathcal{P}B(\kappa)$ via the isomorphism of bimodules in Proposition 5.11.

Proof. Continuing with the notations used in Proposition 3.12, we set $\mathcal{A}_i := \mathcal{P}B(\Lambda_i)$, $\mathcal{B}_i := \mathcal{P}B(\Omega_i)$, $\mathcal{A}_C := \mathcal{P}B(\Omega_1, \Omega_2)$ for $i = 1, 2$.

Set $T := \mathcal{P}B(\eta) \in \mathcal{A}_C \mathcal{L}_B(\mathcal{H}_1, \mathcal{H}_2)$ and $T' := \mathcal{P}B(\kappa) \in \mathcal{B}_C(\mathcal{G}_1, \mathcal{G}_2)$. Suppose $X$ denote the intertwiner in $\mathcal{A}_C \mathcal{L}_C(\mathcal{F}_1, \mathcal{F}_2)$ induced by $T \otimes T'$ under the isomorphism in Proposition 5.11. We need to prove that $\kappa \otimes \eta$ is the unique BQFS which gets mapped to $X$ under the functor $\mathcal{P}B$. For $\xi_i \in \mathcal{H}_i$, $\zeta_i \in \mathcal{G}_i$, $i = 1, 2, k \geq 0$, applying Proposition 5.8 and using the isomorphism of bimodules in Proposition 5.11 (in fact, Proposition 3.12), we get

$$I := \left\langle \phi^{(k)}_{(\xi, \zeta)}(\Omega_k^{(k)} \circ \zeta), \Omega_k^{(k)} \circ \zeta \right\rangle_{F_k} = \left\langle X \left( \Omega_k^{(k)} \circ \zeta \right), \Omega_k^{(k)} \circ \zeta \right\rangle_{F_k} = \left\langle \langle T \xi_1, \xi_2 \rangle_B, T' \xi_1, \xi_2 \rangle \right\rangle_{G_2}.$$

We will now express $\langle T \xi_1, \xi_2 \rangle_B$ as a limit. Consider the sequence

\begin{align*}
\left\{ 
\begin{array}{l}
\xi_2 \quad \cdots \quad \xi_0 \\
\eta_{k+l} \quad \cdots \quad \eta_0 \\
\end{array}
\right.
\end{align*}

Observe that $\langle b_l b', b'' \rangle_{L_2(B)}$ eventually becomes $\langle \langle T \xi_1, \xi_2 \rangle_B, b', b'' \rangle_{L_2(B)}$ as $l$ grows bigger for $b', b'' \in B_\infty$. Since $\{b_l\}_{l \geq 0}$ is bounded, it converges ultraweakly to $\langle T \xi_1, \xi_2 \rangle_B$. Thus,

$$I = \lim_{l \to \infty} \langle T'(b_l \xi_1), \xi_2 \rangle_{G_2} = \lim_{l \to \infty} \left\langle \phi^{(k+l)}_{(\xi, \zeta)}(b_l \xi_1), \xi_2 \right\rangle_{G_2}.$$
We have seen that a BQFS depends solely on the loop operator. In order to understand when a BQFS turns out to be flat, analyzing the loop operator becomes crucial. We take on this job next.

Let $\Lambda \star$ and $\Omega \star$ be two 1-cells from the 0-cell $\Gamma \star$ to $\Delta \star$ in $\text{UC}^{tr}$. We will work with the adjoints of the functors $\Gamma_k$, $\Delta_k$, $\Lambda_k$, and solution to conjugate equations commensurate with the given weight functions associated to the objects in $\text{WSSC}^\star\text{Cat}$ (defined in Section 2.3).

**Proposition 6.1.** (a) If the spaces $\text{NT}(\Lambda_k, \Omega_k)$ and $\text{NT}(\Lambda_{k-1}, \Omega_{k-1})$ are equipped with the inner product induced by the categorical traces $\text{Tr}^{\Lambda_k}$ and $\text{Tr}^{\Lambda_{k-1}}$ (as defined in Proposition 2.2(a) ) respectively, then the adjoint of the loop operator $S_k : \text{NT}(\Lambda_k, \Omega_k) \longrightarrow \text{NT}(\Lambda_{k-1}, \Omega_{k-1})$ is given by

\[
\text{NT}(\Lambda_{k-1}, \Omega_{k-1}) \ni \kappa \xrightarrow{S_k^*} \Delta_k \xrightarrow{\Delta_{k-1}} \Omega_k \Omega_{k-1} \in \text{NT}(\Lambda_k, \Omega_k).
\]

(b) For $\eta \in \text{NT}(\Lambda_k, \Omega_k)$, the pair $(S_k \eta, \eta)$ satisfy the exchange relation (as in Definition 3.4) if and only if $S_k^* S_k \eta = \eta$ where $\Gamma^\prime_k$ is an adjoint of $\Gamma_k$ and the
loop is the natural transformation from $\text{id}_{M_k}$ to $\text{id}_{M_k}$ coming from the solution to the conjugate equation commensurate with $(\mu_k^{-1}, \mu_k)$. [cf. [99] Theorem 2.11.8]

**Proof.** (a) Using Proposition 2.2 multiple times, the inner product $\langle S_k \eta, \kappa \rangle$ turns out to be

$$\text{Tr}^{\Delta_{k-1}}_{\Lambda_{k-1}} (\eta) = \text{Tr}^{\Lambda_k \Gamma_k} (\eta) = \text{Tr}^{\Lambda_k} (\eta) = \langle \eta, S_k^* \kappa \rangle$$

where $\eta \in \text{NT} (\Lambda_k, \Omega_k)$ and $\kappa \in \text{NT} (\Lambda_{k-1}, \Omega_{k-1})$.

(b) The ‘only if’ part easily follows from the pictorial relations.

**if part:** Consider the maps

$$\text{NT} (\Lambda_k, \Omega_k) \ni \sigma \xrightarrow{\text{f}} \in \text{NT} (\Delta_k \Lambda_{k-1}, \Delta_k \Omega_{k-1})$$

and the subspace $Q$ of $\text{NT} (\Delta_k \Lambda_{k-1}, \Delta_k \Omega_{k-1})$ generated by the ranges of $f$ and $g$. Let $\eta$ satisfy the hypothesis. We need to establish the equation $f(\eta) = g(S_k \eta)$. It is enough to show that $\langle f(\eta), \chi \rangle = \langle g(S_k \eta), \chi \rangle$ for all $\chi \in Q$ where the inner product is induced by $\text{Tr}^{\Delta_k \Lambda_{k-1}}$.

For $\sigma \in \text{NT} (\Lambda_k, \Omega_k)$, we get (from the ‘categorical trace’ property) $\langle f(\eta), f(\sigma) \rangle = \text{Tr}^{\Lambda_k \Gamma_k} ([\sigma^* \eta]_{\Gamma_k})$ which by Proposition 2.2(b) becomes

$$\text{Tr}^{\Lambda_k} (\sigma^* S_k S_k (\eta)) = \text{Tr}^{\Lambda_k (\sigma^* S_k S_k (\eta))} = \text{Tr}^{\Lambda_k} (\sigma^* S_k S_k (\eta))$$

where the last equality follows from part (a). Applying Proposition 2.2(b) and categorical trace property again on the the last expression, we get $\text{Tr}^{\Delta_k \Lambda_{k-1}} (f(\sigma)\ g(S_k(\eta))) = \langle g(S_k(\eta)), f(\sigma) \rangle$.

For $\tau \in \text{NT} (\Lambda_{k-1}, \Omega_{k-1})$, we use Proposition 2.2(b) and deduce

$$\langle f(\eta), g(\tau) \rangle = \text{Tr}^{\Delta_k \Lambda_k} ([f(\tau)]^* f(\eta)) = \text{Tr}^{\Lambda_k} (\tau^* S_k(\eta))$$

which by Equation 4.11 along with Proposition 2.2(b) turns out to be $\langle g(S_k(\eta)), g(\tau) \rangle$.

**Remark 6.2.** Similar to Proposition 6.1, one can prove that for $\kappa \in \text{NT} (\Lambda_{k-1}, \Omega_{k-1})$,

$$\left( \kappa, S_k^* \kappa \odot \left[ \Gamma_k \bigotimes \Gamma_k' \right]^{-1} \right)$$

satisfy exchange relation if and only if

$$S_k \left( S_k^* \kappa \odot \left[ \Gamma_k \bigotimes \Gamma_k' \right]^{-1} \right) = \kappa$$

where $\odot$ stands for tensor product of natural transformations.
Remark 6.3. If the 0-cell \((\Gamma_*, \mu^*)\) satisfy an extra condition that \(\Gamma_k \bigcirc \Gamma'_k\) is trivial (that is, \(\Gamma_k \mu^{k-1} = d_k \mu^k\) for some \(d_k > 0\)) eventually for all \(k\), then a BQFS \(\{\eta^{(k)}\}_{k \geq 0}\) is flat if and only if \(\eta^{(k)}\) is an eigenvector of \(S_k^2 S_k\) with respect to the eigenvalue \(d_k\) eventually for all \(k\).

6.1. Periodic case.

In this section, we focus on a particular case where the 0- and the 1-cells are periodic in nature, and investigate whether the 2-cells are flat. To do that, we need a fact in linear algebra which could very well be standard; we give a proof nevertheless.

Definition 6.4. For \(X \in M_n(\mathbb{C})\), a sequence \(\{x^{(k)}\}_{k \geq 0}\) in \(\mathbb{C}^n\) is called \(X\)-harmonic if \(X x^{(k+1)} = x^{(k)}\) for all \(k \geq 0\).

Proposition 6.5. If spectral radius of \(X \in M_n(\mathbb{C})\) is at most 1, then the space of bounded \(X\)-harmonic sequences are spanned by elements of the form \(\{\lambda^{-k}x\}_{k \geq 0}\) where \(\lambda\) is an eigenvalue of \(X\) and \(x\) is a corresponding eigenvector such that \(|\lambda| = 1\).

Proof. Let \(X = TY T^{-1}\) where \(Y\) is in Jordan canonical form. Note that \(\{y^{(k)}\}_{k \geq 0}\) is bounded \(Y\)-harmonic if and only if \(\{T Y^{(k)}\}_{k \geq 0}\) is bounded \(X\)-harmonic. Suppose \(\{p_s : 1 \leq s \leq t\}\) be projections in \(M_n\) such that \(I_n = \sum_{1 \leq s \leq t} p_s\), and for each \(s\), \(p_s Y = Y p_s\) has exactly one nonzero Jordan block corresponding an eigen value, say, \(\lambda_s\). As a result, any bounded \(Y\)-harmonic sequence \(\{y^{(k)}\}_{k \geq 0}\) splits into the sum of \(\{p_s y^{(k)}\}_{k \geq 0}\) which is bounded \(Y\)-harmonic as well as \((p_s Y)\)-harmonic. So, it becomes essential to find bounded harmonic sequences for a Jordan block.

Suppose \(J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{m \times m}\)

the only \(J\)-harmonic sequence would be the zero sequence. So, let us assume \(\lambda \neq 0\). Any nonzero \(J\)-harmonic sequence is of the form \(\{J^{-k} x\}_{k \geq 0}\) for some nonzero vector \(x \in \mathbb{C}^m\); however, it may not always be bounded as \(k\) varies. Now, \(J^{-k} = \lambda^{-k} \sum_{l=0}^{m-1} \left(\frac{k+l-1}{l}\right) (-\lambda^{-1} N)^l\) for \(k \geq 1\). Fix nonzero \(x \in \mathbb{C}^m\). Set \(t := \max\{l : x_l \neq 0\}\) and \(C = \max\{|x_l| : 1 \leq l \leq m\}\). Hence

\[
[J^{-k} x]_1 = \lambda^{-k} \sum_{l=0}^{t-1} \left(\frac{k+l-1}{l}\right) (-\lambda^{-1} N)^l x_{t+1}.
\]

Since \(0 < |\lambda| \leq 1\) and \((\frac{k+l-1}{l})\) increase as \(l\) increases, we have the following inequality if \(t > 1\)

\[
|[J^{-k} x]_1| \geq |\lambda|^{-k} \left[ \left(\frac{k+t-2}{t-1}\right) |\lambda|^{-(t-1)} |x_t| - \left(\frac{k+t-3}{t-2}\right) |\lambda|^{-(t-2)} Ct \right].
\]

If \(t > 1\), then

\[
|[J^{-k} x]_1| \geq |\lambda|^{-(k+t-1)} \left(\frac{k+t-3}{t-2}\right) \left[ \frac{k+t-2}{t-1} |x_t| - |\lambda| Ct \right] \rightarrow \infty \text{ as } k \rightarrow \infty.
\]
Thus, in order to have a bounded $J$-harmonic sequence, $x$ must be in the unique one-dimensional eigen space of $J$, that is, $\mathbb{C}\xi_t$. On the other hand, if $|\lambda| < 1$ and $t = 1$, then
\[
\left|\frac{\xi}{J - k\xi}\right| = |\lambda|^{-k}|x_1| \to \infty \text{ as } k \to \infty.
\]
So, a nonzero bounded $J$-harmonic sequence exists only when $|\lambda| = 1$, and then it is a scalar multiple of $\{\lambda^{-k}\xi\}_{k \geq 0}$.

Getting back to the matrix $Y$ (which is in Jordan canonical form) and applying the above result, we may conclude that all bounded $Y$-harmonic sequences are linear combination of sequences of the form
\[
\{\lambda^{-k}y\}_{k \geq 0}
\]
where $\lambda$ is an eigenvalue with absolute value 1 and $y$ is a corresponding eigenvector. By similarity, the same result holds for $X$ too. □

**Proposition 6.6.** Let $(\Lambda_\bullet, W_\Lambda^\bullet)$ and $(\Omega_\bullet, W_\Omega^\bullet)$ be two 1-cells from the 0-cell $(\Gamma_\bullet, \mu^\bullet)$ to $(\Delta_\bullet, \nu^\bullet)$ in UC$^{tr}$ such that there exists a ‘period’ $K \in \mathbb{N}$ and a ‘Perron-Frobenius (PF) value’ $d > 0$ satisfying:

(i) the periodic condition: $\Gamma_k$’s, $\Delta_k$’s, $\Lambda_k$’s, $\Omega_k$’s, $W_\Lambda^A$’s and $W_\Omega^A$’s repeat with a periodicity $K$ eventually for all $k$;

(ii) the PF condition:
\[
d^{-1} \Gamma_{k+K} \cdots \Gamma_{k+1} \mu^k = \mu = d \mu^{k+K}
\]
\[
d^{-1} \Delta_{k+K} \cdots \Delta_{k+1} \nu^k = \nu = d \nu^{k+K}
\]

eventually for all $k$.

Then, all BQFS from $(\Lambda_\bullet, W_\Lambda^\bullet)$ to $(\Omega_\bullet, W_\Omega^\bullet)$ are flat.

**Proof.** Choose a level $L \in \mathbb{N}$ large enough after which the ingredients in (i) keep repeating with periodicity $K$, and the equations in (ii) hold. Set
\[
\mathcal{M} := \mathcal{M}_L = \mathcal{M}_{L + nK},
\]
\[
\mathcal{N} := \mathcal{N}_L = \mathcal{N}_{L + nK},
\]
\[
\Gamma := \Gamma_{L+K} \cdots \Gamma_{L+1} = \Gamma_{L+(n+1)K} \cdots \Gamma_{L+nK+1} : \mathcal{M} \to \mathcal{M},
\]
\[
\Delta := \Delta_{L+K} \cdots \Delta_{L+1} = \Delta_{L+(n+1)K} \cdots \Delta_{L+nK+1} : \mathcal{N} \to \mathcal{N},
\]
\[
\Lambda := \Lambda_L = \Lambda_{L+nK} : \mathcal{M} \to \mathcal{N},
\]
\[
\Omega := \Omega_L = \Omega_{L+nK} : \mathcal{M} \to \mathcal{N},
\]
\[
W_\Lambda^A := \begin{array}{c}
\Lambda_{L+(n+1)K} \\
\Delta_{L+(n+1)K}
\end{array} \begin{array}{c}
\Gamma_{L+K} \\
\Delta_{L+K}
\end{array} \cdots \begin{array}{c}
\Gamma_{L+nK+1} \\
\Delta_{L+nK+1}
\end{array}
\]
and also denoted by
\[
\begin{array}{c}
\Lambda \\
\Delta
\end{array} \begin{array}{c}
\Gamma \\
\Delta
\end{array}
\]

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\[ W^\Omega := \begin{array}{c} \Omega_{L+(n+1)K} \\ \Delta_{L+(n+1)K} \end{array} \cdots \begin{array}{c} \Gamma_{L+nK+1} \\ \Delta_{L+nK+1} \end{array} \] and also denoted by \[ \begin{array}{c} \Omega \\ \Delta \end{array} \begin{array}{c} \Gamma \\ \Omega \end{array} \]

\[ \mu := \mu^L = d^n \mu^{L+nK} \]
\[ \nu := \nu^L = d^n \nu^{L+nK} \]
for any \( n \geq 0 \). Now condition (ii) and Equation 4.1 imply the following relations:

\[ \Gamma \mu = d \mu = \Gamma' \mu \quad \text{and} \quad \Delta \nu = d \nu = \Delta' \nu . \]

Consider the loop operators given by

\[ S := d^{-1} \begin{array}{c} \Delta' \\ \Omega \\ \Gamma \end{array} \quad \text{and} \quad S^* := d^{-1} \begin{array}{c} \Omega \\ \Gamma \end{array} : NT(\Lambda, \Omega) \rightarrow NT(\Lambda, \Omega) \]

where we use tracial solution to the conjugate equation for \( \Gamma \in \text{End}(\mathcal{M}) \) (resp., \( \Delta \in \text{End}(\mathcal{N}) \)) commensurate with the weight function \( \mu \) on \( \mathcal{M} \) (resp., \( \nu \) on \( \mathcal{N} \)) for both source and target, and the crossings are given by \( W^\Omega, W^\Lambda, W^\Omega, W^\Omega \).

Observe that \( S = S_{L+1} \cdots S_{L+K} = S_{L+nK+1} \cdots S_{L+(n+1)K} \) for all \( n \geq 0 \). To see this, note that an \( S \) in composition \([S_{L+1} \cdots S_{L+K}]\) is defined using tracial solution to conjugate equation for \( \Delta_k \) commensurate with \( \nu^{k-1} \) and \( \nu^k \). So, for the composition \([S_{L+1} \cdots S_{L+K}]\), we are effectively using tracial solution to the conjugate equation for \( \Delta_{L+K} \cdots \Delta_{L+1} = \Delta \) commensurate with \( \nu^L = \nu \) and \( \nu^{L+K} = d^{-1} \nu \); let us denote this solution by \((\text{id}_\mathcal{N}, \frac{\rho}{\nu}, \frac{\rho'}{\Delta' \Delta})\). The solution to the conjugate equation for \( \Delta \) commensurate with \( \nu \) for both source and target, is given by \((d^{-1} \frac{\rho}{\nu}, d^2 \rho')\). Only \( d^2 \rho' \) is used while defining \( S \). Replacing the cap and the cup by \([d^2 \rho']^* \) and \( d \rho' \), we get the desired equation.

We next prove a one-to-one correspondence between bounded \( S \)-harmonic sequences and BQFS’s. Let \( \{ \eta^{(k)} \}_{k \geq 0} \) be a BQFS. Clearly, \( \{ \eta^{(L+nK)} \}_{n \in \mathbb{N}} \) becomes and \( S \)-harmonic sequence. Equip \( NT(\Lambda, \Omega) \) with the inner product induced by the trace \( \text{Tr}^\Lambda \) commensurate with \( (\mu, \nu) \). Finite dimensionality of \( NT(\Lambda, \Omega) \) implies that boundedness of a subset in \( C^* \)-norm is equivalent to that of the 2-norm.

Conversely, let \( \{ \kappa_n \}_{n \in \mathbb{N}} \) be a bounded \( S \)-harmonic sequence. Set \( \eta^{(k)} := S_{k+1} \cdots S_{L+nK}(\kappa_n) \) for any \( n \) such that \( L+nK > k \). Indeed \( \eta^{(k)} \) is well-defined and by construction \( \{ \eta^{(k)} \}_{k \geq 0} \) is quasi-flat. Again by finite dimensionality of \( NT(\Lambda, \Omega) \), \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is bounded in \( C^* \)-norm, and by Remark 4.2(iii), \( \eta^{(k)} \)'s become uniformly bounded as well.

In order to apply Proposition 6.5 on \( S \), it is enough to show that operator norm of \( S \) (acting on the finite dimensional Hilbert space \( NT(\Lambda, \Omega) \)) is at most 1. Note that
$d^{-1} \Delta' \cap \Delta$ is a projection in $\text{End} (\Delta' \Delta)$ and hence less than $1_{\Delta' \Delta}$. Using this and applying Equation (2.2) multiple times, we have

$$
\|S\kappa\|_2^2 = \text{Tr}^\Lambda ( (S\kappa)^* S\kappa) \leq d^{-1} \text{Tr}^\Lambda \Lambda^\kappa \kappa \Lambda \Gamma = \|\kappa\|_2^2.
$$

Thus, by Proposition 6.5, every bounded $S$-harmonic sequence turns out to be a linear combination of sequences of the form $\{\lambda^{-n}\kappa\}_{n \geq 0}$ where $|\lambda| = 1$ and $\kappa$ is an eigenvector of $S$ with respect to the eigenvalue $\lambda$. We will call such sequences 

**elementary.**

We will also borrow the notion of flatness for sequences in $\text{NT} (\Lambda, \Omega)$ from previous sections when every consecutive pair satisfy the exchange relation with respect to $\Gamma, \Delta, \Lambda, \Omega, W^\Lambda$ and $W^\Omega$. Since flat sequences form a vector space, we may conclude every bounded $S$-harmonic sequence will be flat if all elementary ones are so. Consider the above elementary $S$-harmonic sequence given by $\lambda$ and $\kappa$. It is enough to show

$$
\begin{align*}
\begin{array}{c}
\Lambda \\
\Omega
\end{array}
\begin{array}{c}
\Delta \\
\kappa
\end{array}
\end{align*} = \lambda
\begin{align*}
\begin{array}{c}
\Omega \\
\kappa
\end{array}
\begin{array}{c}
\Delta \\
\Lambda
\end{array}
\end{align*}
$$

Note that both sides of the above equation belongs to the space $\text{NT} (\Delta \Lambda, \Delta \Omega)$. We equip this space with inner product induced by $\text{Tr}^\Delta$ commensurate with $(\mu, \nu)$. Consider the subspace $\Delta (\text{NT} (\Lambda, \Omega)) \subset \text{NT} (\Delta \Lambda, \Delta \Omega)$. It is routine to check that the orthogonal projection onto this subspace is given by

$$
E := d^{-1} \Delta' \circ \Delta : \text{NT} (\Delta \Lambda, \Delta \Omega) \to \Delta (\text{NT} (\Lambda, \Omega))
$$

Now,

$$
E \begin{align*}
\begin{array}{c}
\Delta \\
\Omega
\end{array}
\begin{array}{c}
\kappa \\
\Lambda
\end{array}
\end{align*} = \Delta (S\kappa) = \lambda \Delta \kappa \quad \text{and} \quad \left\| \begin{align*}
\begin{array}{c}
\Delta \\
\Omega
\end{array}
\begin{array}{c}
\kappa \\
\Lambda
\end{array}
\end{align*} \right\|_2^2 = \|\lambda \Delta \kappa\|_2^2.
$$

So, the above equation must hold.

From correspondence between bounded $S$-harmonic sequences and BQFS’s and the flatness of the former, we may conclude that in a BQFS $\{\eta^{(k)}\}_{k \geq 0}$, two terms which are $K$ steps apart, must satisfy exchange relation starting from level $L$ (namely, $\eta^{(L)}, \eta^{(L+K)}, \eta^{(L+2K)}, \ldots$). Now, we have the freedom of choosing higher $L$'s; as a result, we obtain exchange relation of any two terms which are $K$ steps apart after level $L$. To establish exchange relation for consecutive terms, pick a $k > L$ and recall the maps $f$ and $g$ defined in the proof of
Proposition \(6.1\)(b). By quasi-flat property, we get

\[
f (\eta^{(k)}) = f \left( S_{k+1} \cdots S_{k+K-1} (\eta^{(k+K-1)}) \right) = \ldots \cdot \eta^{(k+K-1)} \ldots
\]

which (by Equation \([4.1]\), unitarity of the connection and the \(K\)-step exchange relation) turns out to be \(g (\eta^{(k-1)})\). Hence, \((\eta^{(k-1)}, \eta^{(k)})\) satisfies the exchange relation. This ends the proof. \(\square\)

7. Examples

7.1. Subfactors.

Subfactors, more specifically, their standard invariants constitute the initial source of examples generalizing which we arrived at our objects of interest, namely, \(\text{UC}\) and \(\text{UC}^{\text{tr}}\). Basically, we associate a 1-cell in \(\text{UC}^{\text{tr}}\) to the subfactor which captures all the information of the associated planar algebra.

Let \(C_{X_D}\) be an extremal bifinite bimodule over \(\mathcal{II}_1\) factors \(C, D\). For any bifinite bimodule \(A_{Y_B}\), let \(\mathcal{M}_0 := \mathcal{H}_{\text{fd}}\) (the category of finite dimensional Hilbert spaces), and for \(k \geq 0\), let

\[
\mathcal{M}_{2k+1} := \left\langle \left( X \otimes X \right)^{\otimes k}_{\overline{\mathcal{D}}} \right\rangle_{\overline{\mathcal{D}}} \text{ and } \mathcal{M}_{2k+2} := \left\langle \left( X \otimes X \right)^{\otimes k}_{\overline{\mathcal{D}}} \otimes X \right\rangle_{\overline{\mathcal{D}}}
\]

and functors \(\Gamma_k : \mathcal{M}_{k-1} \rightarrow \mathcal{M}_k\) be defined by

\[
\text{ob}(\mathcal{M}_0) \ni C \overset{\Gamma_1}{\longrightarrow} D \overset{L^2(D) \otimes D}{\longrightarrow} \text{ob}(\mathcal{M}_1), \quad \Gamma_{2k+2} := \cdot \otimes X \bigg|_{\mathcal{M}_{2k+1}} \text{ and } \Gamma_{2k+3} := \cdot \otimes X \bigg|_{\mathcal{M}_{2k+2}}.
\]

The sequence \(\Gamma_*\) will serve as the source 0-cell in \(\text{UC}\). For the target 0-cell in \(\text{UC}\), define

\[
\mathcal{N}_{2k} := \left\langle \left( X \otimes X \right)^{\otimes k}_{\overline{\mathcal{C}}} \right\rangle_{\overline{\mathcal{C}}} \text{ and } \mathcal{N}_{2k+1} := \left\langle \left( X \otimes X \right)^{\otimes k}_{\overline{\mathcal{C}}} \otimes X \right\rangle_{\overline{\mathcal{C}}}
\]

and \(\Delta_{2k+1} := \cdot \otimes X \bigg|_{\mathcal{N}_{2k}} \text{ and } \Delta_{2k+2} := \cdot \otimes X \bigg|_{\mathcal{N}_{2k+1}}\). Next, consider the 1-cell \(\Lambda_*\) in \(\text{UC}\) given by

\[
\text{ob}(\mathcal{M}_0) \ni C \overset{\Lambda_0}{\longrightarrow} C \overset{L^2(C) \otimes C}{\longrightarrow} \text{ob}(\mathcal{N}_0) \text{ and } \Lambda_k := X \otimes X \bigg|_{\mathcal{M}_k}
\]

where the unitary connection for the squares

\[
\begin{array}{c}
\mathcal{N}_{2k-1} \overset{\overline{\mathcal{A}_{2k-1}}}{\underset{\overline{\mathcal{A}_{2k}}}{\longrightarrow}} \mathcal{N}_{2k} \overset{\overline{\mathcal{A}_{2k+1}}}{\underset{\overline{\mathcal{A}_{2k+2}}}{\longrightarrow}} \mathcal{N}_{2k+1} \\
\begin{array}{c}
\mathcal{M}_{2k-1} \overset{\overline{\mathcal{B}_{2k-1}}}{\underset{\overline{\mathcal{B}_{2k}}}{\longrightarrow}} \mathcal{M}_{2k} \overset{\overline{\mathcal{B}_{2k+1}}}{\underset{\overline{\mathcal{B}_{2k+2}}}{\longrightarrow}} \mathcal{M}_{2k+1} \\
\end{array}
\end{array}
\]

is induced by the associativity constraint of the bimodules.

In order to turn the \(\text{UC}-0\)-cells \(\Gamma_*\) and \(\Delta_*\) into \(\text{UC}^{\text{tr}}\) ones, we work with the statistical dimension (same as the square root of the index) of an extremal bimodule \(A_{Y_B}\), denoted by
Set $\delta := d(X)$. Let $V_{M_k}$ and $V_{N_k}$ denote maximal sets of mutually non-isomorphic family of irreducible bimodules in $M_k$ and $N_k$ respectively.

Now we define the weight functions $\mu^k$ and $\nu^k$ on $V_{M_k}$ and $V_{N_k}$ respectively as follows:

$$\mu_Y^k := \delta^{-(k-1)}d(Y), \quad \nu_Z^k := \delta^{-k}d(Z) \quad \text{for} \quad Y \in V_{M_k}, \quad Z \in V_{N_k}$$

Since the dimension function is a linear homomorphism with respect to direct sum and Connes fusion of bimodules, we get that $\mu^k$ and $\nu^k$ satisfy Equation (4.2). For the same reason, the boundedness condition Equation (4.1) holds with the inequalities replaced by equality where both the bounds are 1.

7.1.1. Planar algebraic view of the associated bimodule.

Let $A := \mathcal{PB}(\Gamma_\ast)$ and $B := \mathcal{PB}(\Delta_\ast)$ be the AFD's and $H := \mathcal{PB}(\Lambda_\ast)$ be the $A$-$B$-bimodule where $\Gamma_\ast, \Delta_\ast$, and $\Lambda_\ast$ and 0- and 1-cells in $UC^\text{tr}$ associated to the extremal bimodule $C^\ast X_D$. Denote the planar algebra associated to $C^\ast X_D$ by $P = \{P_{\pm k}\}_{k \geq 0}$ where the vector spaces are given by

$$P_{+k} = \text{End}\left( X \otimes \cdots \otimes_k \text{tensor components} \right) \quad \text{and} \quad P_{-k} = \text{End}\left( \cdots \otimes_k \otimes \cdots \right).$$

Immediately from the definitions, we get the following.

(a) $A_{k+1} = P_{-k}$ and $B_k = P_{+k} = H_k$.

(b) Both the inclusions $B_k \hookrightarrow B_{k+1}$ and $H_k \hookrightarrow H_{k+1}$ are the same as $P_{+k} \hookrightarrow P_{+(k+1)}$, and $A_{k} \hookrightarrow A_{k+1}$ is same as $P_{-(k-1)} \hookrightarrow P_{-k}$, induced by the action of inclusion tangle by a string on the right.

(c) Action of $B_k$ on $H_k$ is given by right multiplication of $P_{+k}$ on itself whereas that of $A_k$ on $H_k$ is given by the left multiplication of the left inclusion $P_{-(k-1)} \hookrightarrow P_{+k}$ induced the action of inclusion tangle by a string on the left.

(d) The trace on $A_k$ and $B_k$ turns out to be the normalized picture trace on $P_{-(k-1)}$ and $P_{+k}$ respectively.

Remark 7.1. Let $P_{\pm \infty}$ be the union $\bigcup_{k \geq 0} P_{\pm k}$ of the filtered unital algebras, and $P_{\pm}$ be the von Neumann algebra generated by it acting on the GNS with respect to the canonical normalized picture trace $tr_\pm$. Finally, the bimodule $A_H B$ turns out to be the same as $p_\ast L^2(P_{\pm \infty}, tr_\pm) P_i$ where the $P_-$-action on left extends from treating $P_{\pm \infty}$ as a leftmodule over the subalgebra $LI(P_{\pm \infty})$. As a result, the BQFS’s from $\Lambda_\ast$ to $\Lambda_\ast$ are given by intertwiners in $P_- L_{P_\ast}(L^2(p_\ast, tr_\pm)) = [LI(P_-)]' \cap P_+$ via Theorem 5.9 and by Remark 4.2, the flat sequences correspond to elements in $[LI(P_{- \infty})]' \cap P_{+ \infty}$.

7.1.2. Loop operators and Izumi’s Markov operator.

We provide a description of loop operators $\{S_k : \text{End}(\Lambda_k) \to \text{End}(\Lambda_{k-1})\}_{k \geq 1}$ in terms of maps between intertwiner spaces. We continue to employ the graphical calculus pertaining to bimodules and intertwiners as well. Let us analyze the odd ones first. For
$Y \in V_{M_{2k}}$ and $\eta \in \text{End}(\Lambda_{2k+1})$,

$$
[S_{2k+1}\eta]_Y = \sum_{Y_1 \in V_{M_{2k+1}}} \sum_{\alpha \in \text{ONB}(Y, \Gamma_{2k+1}Y)} \eta_{Y_1} \alpha^\ast \alpha Y
$$

The last expression is in terms of the functors $\Gamma_n$'s, $\Delta_n$'s and $\Lambda_n$'s; to express it purely using bimodules and intertwiners, we prove the following relation.

**Lemma 7.2.** For $Z_1, Z_2 \in \text{ob}(\mathcal{N}_{2k})$, $\gamma \in \mathcal{N}_{2k+1}(\Delta_{2k+1}Z_1, \Delta_{2k+1}Z_2)$ we have

$$
\begin{pmatrix}
\gamma \\
Z_2
\end{pmatrix}_{Z_1} = \delta^{-1} \begin{pmatrix}
Z_2 \\
X
\end{pmatrix}_{X}
$$

where the cap and the cup on the right (resp. left) side come from balanced spherical solution (resp. solution) to the conjugate equations for the duality of $X$ (resp. $\Delta_{2k+1}$ commensurate with $(\nu_{2k+2}, \nu_{2k})$).

**Proof.** Without loss of generality, we may assume $Z_1 = Z_2 = Z$ (say) is irreducible and

$$
\begin{pmatrix}
\Delta_{2k+1}Z \\
\alpha
\end{pmatrix} Y = \begin{pmatrix}
Z \\
X
\end{pmatrix} Y
$$

where $Y \in V_{\mathcal{N}_{2k+1}}$. So, the right side of the equation in the statement becomes

$$
\delta^{-1} [d(Z)]^{-1} \begin{pmatrix}
Z \\
X
\end{pmatrix}_{\alpha} 1_Z = \frac{d(Y)}{\delta d(Z)} \langle \alpha, \beta \rangle 1_Z = \frac{\nu_{2k}^2}{\nu_{2k+1}^2} \langle \alpha, \beta \rangle 1_Z
$$

where the first equality follows from traciality of spherical solutions. The last term by Equation (2.2), is same as the left side. \qed
Coming back to the loop operators, we apply the lemma on the blue box below and obtain

\[
[S_{2k+1} \eta]_Y = \sum_{Y_1 \in V_{M_{2k+1}}} \sum_{\alpha \in \text{ONB}(Y_1, \Gamma_{2k+1} Y)} \alpha \begin{array}{c}
\eta \nu_Y \\
\alpha \end{array} Y_1 = \delta^{-1} \sum_{Y_1 \in V_{M_{2k+1}}} \sum_{\alpha \in \text{ONB}(Y_1, Y \otimes \chi)} \alpha \begin{array}{c}
\eta \nu_Y \\
\alpha \end{array} Y_1 X.
\]

**Proposition 7.3.** For all \(\eta \in \text{End}(\Lambda_k)\) and \(Y \in V_{M_{k-1}}\), the following equation holds

\[
[S_k \eta]_Y = \sum_{Y_1 \in V_{M_k}} \frac{d(Y_1)}{\delta} \begin{array}{c}
\beta^* \\
\beta
\end{array} Y_1 X
\]

where \(X_k\) is \(X\) or \(\chi\) according as \(k\) is even or odd.

**Proof.** In Equation (7.1), substituting \(\beta := \left(\frac{d(Y)}{d(Y_1)}\right) \begin{array}{c}
\beta^* \\
\beta
\end{array} Y_1 X\) (which yeilds an orthonormal basis of \(\mathcal{M}_{2k+1} \left( Y, Y_1 \otimes \chi \right) \)) as \(\alpha\) varies over \(\text{ONB}(Y_1, Y \otimes \chi)\), we get the desired equation for the odd case. The proof of the even case is exactly similar. \(\square\)

We now recall the Markov operator (that is, a UCP map) associated to an extremal finite index subactor / bifinite bimodule defined by Izumi in [I04]. Consider the finite dimensional C*-algebra \(D_k := \text{End}(\Lambda_k) \cong \bigoplus_{Y \in V_{M_k}} c\mathcal{L}(X \otimes Y) \) or \(\bigoplus_{Y \in V_{M_k}} c\mathcal{L}(X \otimes Y) \) according as \(k\) is even or odd. Define the von Neumann algebra \(D := \bigoplus_{k \geq 0} D_k\). Then, Izumi’s Markov operator \(P : D \longrightarrow D\) is defined as

\[
D \ni \eta = (\eta^{(k)})_{k \geq 0} \xrightarrow{P} P\eta := (S_{k+1} \eta^{(k+1)})_{k \geq 0} \in D.
\]

By [Lemma 3.2, [I04]], the space of \(P\)-harmonic elements \(H^\infty(D, P)\) (that is, the fixed points of \(P\)) is precisely the space of bounded quasi-flat sequences corresponding to our loop operators \(\{S_k\}_{k \geq 0}\).

**7.1.3. Temperley-Lieb - TL\(_\delta\) for \(\delta > 2\).**

Continuing with the same set up, let us further assume \(X\) is symmetrically self-dual and tensor-generates the Temperley-Lieb category for a generic modulus \(\delta > 2\). This example had already been investigated extensively, in particular, by Izumi in [I04] in our
context. Here, we address the question whether every UC-endomorphism of $\Lambda_\bullet$ extends to a UC\(_{\text{tr}}\)-one.

**Proposition 7.4.** The 1-cell in UC\(_{\text{tr}}\) corresponding to the TL-bimodule $X$ possesses a BQFS to itself which is not flat.

**Proof.** In [IoM], Izumi showed that $H^\infty(D,P)$ (and hence $\text{End}_{\text{UC}_{\text{tr}}}((\Lambda_\bullet))$) is 2-dimesional. So, it is enough to show that $\text{End}_{\text{UC}_{\text{tr}}}((\Lambda_\bullet))$ is the one-dimensional space generated by the identity in it. Again, by Remark 7.1 and Remark 4.4, this boils down to showing that $[\text{LI } (P_\infty)]' \cap P_\infty$ is 1-dimensional where $P = \{P_k\}_{k \geq 0}$ denotes the unshaded planar algebra associated to the symmetrically self-dual bimodule $X$.

Let $x \in [\text{LI } (P_{-\infty})]' \cap P_{+\infty}$. Then there exists some $k \geq 0$ such that $x \in [\text{LI } (P_{-\infty})]' \cap P_{+k}$, equivalently

$$\begin{array}{c}
\begin{array}{ccc}
& k & \\
& \cdots & \\
& y & \\
\end{array}
\end{array} = \begin{array}{ccc}
\begin{array}{ccc}
& k+l & \\
& \cdots & \\
& x & \\
\end{array}
\end{array}$$

for all $y \in P_{-(k+l)}$ and $l \geq 0$.

Using Equation (7.2) we get $x = \delta^{-k} \begin{array}{ccc}
\begin{array}{c}
\cdots
\end{array}
\end{array} = \delta^{-k} \begin{array}{c}
\cdots
\end{array} \in P_1$ where the thick line denotes $k$ many parallel strings. Since $P_1$ is one-dimensional, $x$ must be a scalar multiple of identity. \(\square\)

### 7.2. Directed graphs.

We will discuss an example arising out of directed graphs (where we allow multiple edges from one vertex to the other). Further, we assume the directed graphs are 'strongly connected', that is, for $v, w$ in the vertex set, there exists a path from $v$ to $w$. As a result, the corresponding adjacency matrices are irreducible and thereby, each possesses a Perron-Frobenius (PF) eigenvalue and PF eigenvectors. In terms of category and functor, it is equivalent to consider a finite semisimple category $M$ and a $*$-linear functor $\Gamma \in \text{End } (M)$ such that for simple $v, w \in \text{ob}(M)$, there exists $k \in \mathbb{N}$ satisfying $M(v, \Gamma^k w) \neq \{0\}$. From such a $\Gamma$, we build the 0-cell $\{M_{k-1} \xrightarrow{\Gamma_k} M_k\}_{k \geq 1}$ in UC\(_{\text{tr}}\) where $M_k = M$ and $\Gamma_k = \Gamma$ for all $k$, and the weight $\mu^k$ on $M_k$ is given by $d^{-k} \mu$ where $d$ is the PF eigenvalue of the adjacency matrix of $\Gamma'$ and $\mu$ is PF eigenvector whose sum of the coordinates is 1.

Consider the 1-cell $\Lambda_\bullet$ in UC\(_{\text{tr}}\)($\Gamma_\bullet$, $\Gamma_\bullet$) by setting $\Lambda_k := \Gamma$ for $k \geq 0$, with unitary connection $W^k := 1_{FP}$. Note that the loop operator $S_k : \text{End } (\Lambda_k) \to \text{End } (\Lambda_{k-1})$ is independent of $k$ because (although the weight on $M_k$ varies as $k$ varies) our solution to conjugate equation for the duality of $\Gamma_k$ : $M_{k-1} \to M_k$ is independent ; let us rename it as $S : \text{End } (\Gamma) \to \text{End } (\Gamma)$. More explicitly, $S\eta = d^{-1} \begin{array}{c}
\begin{array}{c}
\Omega
\end{array}
\end{array}$ $\Gamma$ for all $\eta \in \text{End } (\Gamma)$ where we use tracial solution to conjugate equation for $\Gamma$ commensurate with $(\mu, \mu)$. Clearly, the range of $S$ is contained in $\{\xi \otimes 1_\Gamma : \xi \in \text{End } (\text{id}_{M})\}$. Then an $S$-harmonic sequence $\{\xi_k \otimes 1_{\Gamma} \}_{k \geq 0}$ is completely captured by a sequence $\{\xi_k\}_{k \geq 0}$ in the
finite dimensional abelian C*-algebra $\text{End} \,(\text{id}_M)$ satisfying $d^{-1} \Gamma'(\zeta_k) \Gamma = \xi_{k-1}$ for all $k \geq 1$. The operator $X := d^{-1} \Gamma'(\bigotimes) \Gamma : \text{End} \,(\text{id}_M) \to \text{End} \,(\text{id}_M)$ is UCP and $\{\xi_k\}_k$ is $X$-harmonic. Using the categorical trace on natural transformations, the operator $X$ has norm at most 1 and so is the spectral radius. Applying Proposition 1, the bounded $X$-harmonic sequences are linear span of elementary ones, namely, $\{c^{-k} \xi\}_{k \geq 0}$ where $\xi$ is an eigenvector of $X$ for the eigenvalue $c$ such that $|c| = 1$. However, it is unclear whether such an elementary $X$-harmonic sequence contribute towards a flat sequence from $\Lambda_\bullet$ to $\Lambda_\bullet$: a necessary condition for this is $c \Gamma \odot \xi = d \xi \odot 1\Gamma$. A straightforward deduction from this condition will tell us that flat sequences are simple scalar multiples of the identity.

In the above example, if we would have started with a finite connected undirected graph $\Gamma$, then by Proposition 6.6 all BQFS from $\Lambda_\bullet$ to $\Lambda_\bullet$ would have been vertex.

### 7.3. Vertex models.

Let $\mathcal{M}$ be the category of finite dimensional Hilbert spaces, and $\Gamma := \text{id}_\mathcal{M} \otimes \ell^2(X) \in \text{End} \,(\mathcal{M})$, $\Lambda := \text{id}_\mathcal{M} \otimes \ell^2(Y) \in \text{End} \,(\mathcal{M})$ be two functors where $X, Y$ are some nonempty finite sets. Consider the 0-cell $\Gamma_\bullet$ in $\text{UC}^{\text{tr}}$ defined by $\mathcal{M}_k := \mathcal{M}$ and $\Gamma_k = \Gamma$ for all $k$ where the weight of $\mathcal{C}$ in $\mathcal{M}_k$ is $|X|^{-k}$. For a 1-cell in $\text{UC}^{\text{tr}}(\Gamma_\bullet, \Gamma_\bullet)$, we consider $\{\Lambda_k := \Lambda\}_{k \geq 0}$ with the unitary connections $W^k := \text{id}_\bullet \otimes UF$ where $U : \ell^2(X) \otimes \ell^2(Y) \to \ell^2(X) \otimes \ell^2(Y)$ is a unitary and $F : \ell^2(Y) \otimes \ell^2(X) \to \ell^2(X) \otimes \ell^2(Y)$ is canonical flip map. Note that $\text{End}(\Lambda) \cong M_Y(\mathbb{C})$ and the loop operators are independent of $k$; let us denote it by $S : M_Y(\mathbb{C}) \to M_Y(\mathbb{C})$. One can deduce the following two formula,

$$(S\eta)_{y,y'} = |X|^{-1} \sum_{x_1, x_2 \in X \atop y_1, y_2 \in Y} U_{x_2 y_2}^{x_2 y_1} \eta_{y_2 y_1} U_{x_1 y_1}^{x_2 y_1} \text{ and } (S^*\eta)_{y,y'} = |X|^{-1} \sum_{x_1, x_2 \in X \atop y_1, y_2 \in Y} U_{x_2 y_2}^{x_1 y_1} \eta_{y_2 y_1} U_{x_1 y_1}^{x_2 y_1}$$

for all $\eta \in M_Y(\mathbb{C})$ and $y, y' \in Y$. By Proposition 6.6 every BQFS from $\Lambda_\bullet$ to $\Lambda_\bullet$ becomes flat.

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