Pseudo-linear regression identification based on
generalized orthonormal transfer functions:
Convergence conditions and bias distribution
analysis

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Abstract
In this paper we generalize three identification recursive algorithms belonging to the pseudo-linear class, by introducing a predictor established on a generalized orthonormal function basis. Contrary to the existing identification schemes that use such functions, no constraint on the model poles is imposed. Not only this predictor parameterization offers the opportunity to relax the convergence conditions of the associated recursive schemes, but it entails a modification of the bias distribution linked to the basis poles. This result is specific to pseudo-linear regression properties, and cannot be transposed to most of prediction error method algorithms. A detailed bias distribution is provided, using the concept of equivalent prediction error, which reveals strong analogies between the three proposed schemes, corresponding to ARMAX, Output Error and a generalization of ARX models. That leads to introduce an indicator of the basis poles location effect on the bias distribution in the frequency domain. As shown by the simulations, the said basis poles play the role of tuning parameters, allowing to manage the model fit in the frequency domain, and allowing efficient identification of fast sampled or stiff discrete-time systems.

1 Introduction
Algorithms dedicated to discrete-time identification are generally subdivided in three classes [12]: Prediction error methods (PEM), Instrumental variable methods (IV), and pseudo-linear regression methods (PLR). This third category presents a specific interest, especially in the perspective of recursive (online) identification [13]. For example, the extended recursive least-squares [22].

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or the recursive output error algorithm [6], that belong to this class are celebrated schemes that have been widely used in adaptive control [9]. A little more recently, in the nineties, several closed-loop identification structures belonging to the pseudo-linear regression class appeared [7], [8]. It has been emphasized that the choice of the sampling frequency is crucial in discrete-time identification [1], and that discrete-time identification algorithms are generally not robust in fast sampling situations [11] (chap. 13). For example, some specialists of pseudo-linear regression algorithms recommend that the sampling frequency be not higher than 25-times the system expected bandwidth (for open-loop identification), and they systematically represent Bode Diagrams on a frequency linear scale [10]. Generally speaking, models obtained with PLR schemes are even less reliable than others at low frequency, which prevents from using them in a fast sampling situation. As a result, the identification of systems having modes with frequencies separated from several decades (stiff systems) is intractable with these structures. The reason for these limitations has been pointed out recently in [17]: It is a consequence of the specific PLR schemes bias distribution over frequency, which differs from the bias distribution of the corresponding PEM algorithms for a given predictor model. For example, the open-loop PLR output error and the ARMAX limit models are both weighted (for the definition of the limit model see [12], chap. 8), exactly as the least squares algorithm limit model, for which it is well-known that the model misfit in low frequency is poorly minimized in the criterion, see [12], pp. 268-269. In order to overcome the above limitations, this paper presents a parameterization for the regressor of the predictor developed on the orthonormal transfer function bases introduced by Heuberger et al. [3], which are at the origin of the Hambo transform [4], [5]. In the literature, identification schemes using series expansion of orthonormal transfer functions (for example, in the case of Laguerre transfer function see [20]), are made of predictors fed only by the system input [4], thus they can be considered as a generalization of finite impulse response systems, with a specification of the model poles. Here we do not impose any poles to the estimated model, the use we make of the orthonormal transfer function can be interpreted as -roughly speaking- a generalization of infinite impulse response systems, i.e. the predictor is fed not only with the system input, but depends also on the measured (or estimated) system output. In the context of PLR, the parametrization we propose has a clear impact on the convergence conditions of the identification algorithm, and the basis poles can be used as tuning parameters in order to relax the convergence of classical PLR schemes. The hereafter convergence conditions, based on [18], generalize those of classical recursive identification structures, established on hyperstability theory [9]. Furthermore, we show that the basis poles have a crucial impact on the bias distribution -contrary to what would happen if the same parameterization were employed in the context of PEM-. The bias distribution analysis is carried out with the recently developed concept of equivalent prediction error [17], which corresponds to the signal whose variance is effectively minimized in the PLR scheme. We demonstrate that, regarding the deterministic part, the weighting functions of the limit models are the same for the output-error, AR-
MAX, and a generalized version of ARX predictor. The optimization problem can be expressed in the Hambo frequency domain, in which it has always the same structure. Since this Hambo frequency scale is distorted compared with the classical frequency scale, a measure of this distortion from the logarithmic frequency scale to the Hambo frequency scale, in function of the basis poles, is provided. We show that it can be interpreted as an indicator of the basis poles effect on the bias distribution over frequency. The simulations show that the basis poles play the role of tuning parameters, impacting the bias distribution, and making it possible to identify accurately discrete-time fast sampled or stiff systems. The identification of stiff systems is an emerging area and is reputed to be a quite challenging subject in identification, see [2]. This paper is the first to propose a methodology dedicated to discrete-time identification of such systems.

2 Definitions related to generalized orthonormal functions

In this section we recall very briefly some definitions related to orthonormal transfer functions from a balanced realization of an all-pass function, as proposed in [3]. The reader interested in all theoretical aspects of these functions can refer to [4], and [5]. Let us consider the Blashke product $G_b(z^{-1})$, with $G_b(z^{-1})G_b(z) = 1$, such that

$$G_b(z^{-1}) = \prod_{k=0}^{\eta_p-1} \frac{p_k - z^{-1}}{1 - p_k z^{-1}}$$

where $p_k$ are the basis poles, and $\eta_p$, the poles number.

This transfer function can be represented by means of a balanced state-space realization

$$G_b(z) = D_b + C_b (z I - A_b)^{-1} B_b,$$

which satisfies

$$\begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}^* \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} = I$$

(2)

The orthonormal functions basis proposed by Heuberger and al. [3] corresponds to the vectors $V_k$ with size $(\eta_p, 1)$, such that

$$V_1(z) = (z I - A_b)^{-1} B_b$$

$$V_k(z) = (z I - A_b)^{-1} B_b G_b^{k-1}(z)$$

These functions form a Hilbert basis of strictly proper stable transfer functions in $H_2$. The orthonormality holds because of the orthonormal state space expression of $G_b(z)$. Particular configurations of $\eta_p$ and $p_k$ correspond to well known cases: $\eta_p = 1, p_0 = 0$ is the classical $z^{-1}, z^{-2}, \cdots$ basis, and
\[ p_1 = 1, |p_0| < 1 \] corresponds to the Laguerre basis.
The so-called signal and operator Hambo transforms stem from these orthogonal transfer function bases. They are not detailed here.

### 3 Optimal predictors expressed on orthonormal functions bases

In the sequel, we use the following notations:

- \( \theta_0 \) is the parameters vector of the true system,
- \( \theta \) is the parameter vector of the predictor,
- \( \hat{\theta}(t) \) is the estimated parameter vector,
- \( \hat{\theta}^* \) is the limit estimated parameter vector,
- \( \phi(t) \) is the regressor of the predictor.

Let us consider \( \{u(t)\}, \{y(t)\}\) the monovariable LTI system input and output, \( \{e(t)\}\) a centered gaussian white noise, and \( \{v(t)\}\) a centered noise uncorrelated with \( \{u(t)\}\). According to Landau [9], we distinguish two classes of stochastic models. The equation error model:

\[
A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t)
\]  

(4)

where \( A(q^{-1}) \) is a monic polynomial in \( q^{-1} \), the case \( C(q^{-1}) = 1 \) corresponding to the ARX model, and the case where \( C(q^{-1}) \) is a monic polynomial in \( q^{-1} \), corresponding to the ARMAX model. On the other hand the output error model is given by (\( v(t) \) being a disturbance uncorrelated with respect to \( u(t) \))

\[
A(q^{-1})y(t) = B(q^{-1})u(t) + A(q^{-1})v(t)
\]  

(5)

Let \( \hat{y}(t) \) be the predicted output, and \( \varepsilon(t) = y(t) - \hat{y}(t) \), the prediction error. The optimal predictor of the equation error model is classically given by (see [9])

\[
\hat{A} (q^{-1}) \hat{y}(t) = \hat{B} (q^{-1}) u(t) + \left( \hat{C}(q^{-1}) - \hat{A}(q^{-1}) \right) \varepsilon(t)
\]  

(6)

where \( \hat{A} (q^{-1}) , \hat{B} (q^{-1}) , \hat{C} (q^{-1}) \) are the estimations of polynomials \( A (q^{-1}) , B (q^{-1}) , C (q^{-1}) \). On the other hand, the optimal predicted output of the output error model is

\[
\hat{A} (q^{-1}) \hat{y}(t) = \hat{B} (q^{-1}) u(t)
\]  

(7)

In the context of PLR identification, whatever the predictor structure is, the predicted output at time \( t + 1 \) is written as:

\[
\hat{y}(t + 1) = \hat{\theta}^T (t + 1) \phi(t, \hat{\theta}(t))
\]  

(8)

where \( \hat{\theta}(t) \) is the estimated parameter vector, and \( \phi(t, \hat{\theta}(t)) \), the regressor depending on past inputs and (system and/or predictor) outputs. The basic
philosophy of pseudo-linear class consists in neglecting the regressor dependence with respect to \( \hat{\theta} \) in the computation of the estimated parameter vector. The purpose of this paper is to study identification algorithms belonging to the pseudo-linear class, when the regressor of the predictor is expressed not in function of the \( \{q^{-1}, q^{-2}, \cdots\} \) basis, but on the orthonormal function basis \( \{V_1(q^{-1}), V_2(q^{-2}), \cdots\} \) basis, as defined in the previous section. That leads to consider the following expressions of the predicted output, according to the various stochastic models:

- **Generalized-ARX predictor:**

  \[
  \hat{y}(t+1) = -\sum_{k=1}^{\eta_a} \hat{m}_k^T V_k(q^{-1}) y(t+1) + \cdots \]

  \[
  \cdots + \sum_{k=1}^{\eta_a} \hat{n}_k^T V_k(q^{-1}) u(t+1) \quad (9)
  \]

- **Generalized-ARMAX predictor:**

  \[
  \hat{y}(t+1) = -\sum_{k=1}^{\eta_a} \hat{m}_k^T V_k(q^{-1}) y(t+1) + \cdots \]

  \[
  \cdots + \sum_{k=1}^{\eta_a} \hat{n}_k^T V_k(q^{-1}) u(t+1) + \sum_{k=1}^{\eta_a} \hat{l}_k^T V_k(q^{-1}) \varepsilon(t+1) \quad (10)
  \]

- **Generalized-output error predictor:**

  \[
  \hat{y}(t+1) = -\sum_{k=1}^{\eta_a} \hat{m}_k^T V_k(q^{-1}) \hat{y}(t+1) + \cdots \]

  \[
  \cdots + \sum_{k=1}^{\eta_a} \hat{n}_k^T V_k(q^{-1}) u(t+1) \quad (11)
  \]

where \( \eta_a \) is the predictor order, and we assume that it is a multiple of \( \eta_p \), \( \hat{m}_k, \hat{n}_k, \hat{l}_k \) the estimated parameter vector (size \( \eta_p, 1 \)). As the orthonormal transfer function vectors \( V_k(q^{-1}) \) are strictly proper, there is no algebraical loop in expressions (9), (10), and (11).

Set

\[
A_o(q^{-1}) = \prod_{k=0}^{\eta_a-1} \left( 1 - p_k(q^{-1}) \right) \frac{\eta_a}{\eta_p} \quad (12)
\]

and consider \( \hat{G}(q^{-1}) = \frac{\sum_{k=1}^{\eta_a} \hat{n}_k^T V_k(q^{-1})}{1 + \sum_{k=1}^{\eta_a} \hat{m}_k^T V_k(q^{-1})} \)

It is clear from (11) that \( V_{\frac{\eta_a}{\eta_p}}(q^{-1}) \) has a characteristic polynomial equal to
Thus for the generalized ARX model we have $\hat{G}(q^{-1}) = \frac{\sum_{k=1}^{n_u} \hat{b}_k q^{-k}}{1 + \sum_{k=1}^{n_a} a_k q^{-k}}$

which agrees with the classical output error model $y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) + v(t)$, where $B(q^{-1}) = \sum_{k=1}^{n_u} b_k q^{-k}$ and $A(q^{-1}) = 1 + \sum_{k=1}^{n_a} a_k q^{-k}$.

Similarly, the stochastic part of the equation error model entails

$$\hat{W}(q^{-1}) = \frac{\sum_{k=1}^{n_u} \hat{p}_k T V_k(q^{-1})}{1 + \sum_{k=1}^{n_a} \hat{m}_k T V_k(q^{-1})} = \frac{1 + \sum_{k=1}^{n_u} \hat{c}_k q^{-k}}{1 + \sum_{k=1}^{n_a} \hat{a}_k q^{-k}}$$

Therefore the generalized ARMAX predictor agrees with the classical ARMAX model $y(t) = G(q^{-1}) u(t) + W(q^{-1}) \epsilon(t)$, with $G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}$ and $W(q^{-1}) = C(q^{-1})$, where $C(q^{-1}) = 1 + \sum_{k=1}^{n_e} \hat{c}_k q^{-k}$.

The generalized ARX predictor corresponds to $l_k = 0$ in the expression of $\hat{W}$. Thus for the generalized ARX model we have $\hat{W}(q^{-1}) = \frac{A_o(q^{-1})}{1 + \sum_{k=1}^{n_a} \hat{a}_k q^{-k}}$, and this predictor agrees with the model $y(t) = G(q^{-1}) u(t) + W(q^{-1}) \epsilon(t)$, where $W(q^{-1}) = \frac{A_o(q^{-1})}{A(q^{-1})}$.

### 4 Algorithms and their convergence conditions

In the context of PLR, the predicted output is expressed from a linear combination of the estimated parameter vector $\hat{\theta}(t)$ and a regressor $\phi(t)$ following (8). Let: $\epsilon(t + 1) = y(t + 1) - \hat{y}(t + 1)$ be the prediction error. The estimation of model parameters is, most of the time, computed recursively, with the so-called parameter adaptation algorithm (PAA) [9]

$$\begin{align*}
\hat{\theta}(t + 1) &= \hat{\theta}(t) + F(t) \phi(t) \epsilon(t + 1) \\
F^{-1}(t + 1) &= \lambda_1 F^{-1}(t) + \lambda_2 \phi(t) \phi^T(t)
\end{align*}$$

(13a) (13b)

Where $F(t)$ is the adaptation gain (positive definite matrix), and $0 < \lambda_1 \leq 1, 0 \leq \lambda_2 < 2$ the forgetting factors.

Each predictor is linked to an algorithm presented below. The generalized ARX is included in what we call the H-Recursive Least Square (H-RLS), (H stands for the Hambo transform which is associated with the bases used in this article). The generalized ARMAX predictor is associated with the H-Recursive Extended Least Squares (H-ERLS) algorithm, and the generalized Output Error predictor
is the one used in what we call the H-Open-Loop Output Error algorithm (H-OLOE) in the sequel.

It is well known that the algorithm convergence depends upon the strict real positiveness of a transfer transfer function appearing in the expression of the prediction error [9], (chap. 3 and 4). For each algorithm we now present these convergence conditions, that partially differ from the convergence conditions of the classical algorithms established with the basis $q^{-1}, q^{-2}, \ldots$. Furthermore, we make use of recent results regarding Parameter Adaptation Algorithm (PAA) convergence [18].

4.1 Generalized ARX predictor, and H-RLS algorithm

From (4), (8), and (9), we obtain immediately

$$\varepsilon(t + 1) = (\theta_0 - \theta)^T \phi(t) + e(t + 1)$$

with:

$$\phi^T(t) = \left[ -V_1^T(q^{-1})y(t+1) - V_2^T(q^{-1})y(t+1) \cdots \right.$$  

$$\left. \cdots V_1^T(q^{-1})u(t+1) V_2^T(q^{-1})u(t+1) \cdots \right]$$

and:

$$\theta_0^T = [m_1^T m_2^T \cdots n_1^T n_2^T \cdots]$$

Exactly as for the classical recursive least-square there is no convergence condition.

4.2 Generalized ARMAX predictor, and H-ERLS algorithm

From (4), (8), and (10) in a deterministic context we have again

$$\varepsilon(t + 1) = (\theta_0 - \theta)^T \phi(t),$$

$$\phi^T(t) = \left[ -V_1^T(q^{-1})y(t+1) - V_2^T(q^{-1})y(t+1) \cdots \right.$$  

$$\left. \cdots V_1^T(q^{-1})u(t+1) V_2^T(q^{-1})u(t+1) \cdots \right.$$  

$$\left. \cdots V_1^T(q^{-1})\varepsilon(t+1) V_2^T(q^{-1})\varepsilon(t+1) \cdots \right]$$

$$\theta_0^T = [m_1^T m_2^T \cdots n_1^T n_2^T \cdots]$$

Therefore there is no convergence condition in a deterministic context.

In a stochastic context, from (4), (8), and (10), we get easily

$$C(q^{-1})\varepsilon(t + 1) = A_o (\theta_0 - \theta)^T \phi(t) + C(q^{-1})e(t + 1)$$

(14)
Notice that this expression generalizes the expression of the classical prediction error expression of extended-least squares algorithms corresponding to the case \( A_0(q^{-1}) = 1 \).

The convergence analysis of the H-ERLS algorithm in a stochastic context can be carried out by means of the martingale theory, and the recent results of [18], generalizing those of theorem 4.2 in [9]. For this purpose, notice that \( \{e(t)\} \) is a martingale difference sequence as defined in [9] p. 135, with

\[
E[e(t+1)|\mathcal{F}_t] = 0
\]  

(15)

\[
\lim_{N \to \infty} \sup_{1 \leq t \leq N} e^2(t) < \infty
\]  

(16)

Where \( \mathcal{F}_t \) is the \( \sigma \)-algebra of all observations generated up to \( t \).

Preliminary remark: The theorems [1] [2] [3] result directly of [18]. However in this reference the true parameters is denoted \( \theta \) contrary to the notation we adapt here \( \theta_0 \).

**Theorem 1.** Consider the H-ERLS algorithm associated with the generalized ARMAX predictor in a stochastic context, and a prediction error as in (14). Assume that the following assumptions hold

a) The true system is in the model set,

b) There exists \( \delta > 0 \) such that for any \( t \geq n_\phi \), the matrix \( \sum_{j=t+1}^{t+n_\phi} \phi(t)^T \delta I \) is positive definite, where \( n_\phi \) is the length of vector \( \phi(t) \),

c) The transfer function

\[
\frac{A_0(\sqrt{2} - \lambda_1 z^{-1})}{C(\sqrt{2} - \lambda_1 z^{-1})} - \frac{\lambda_2}{2}
\]

is strictly positive real (SPR).

Then for any \( \nu > 0 \) one has

1. \( \lim_{N \to \infty} \frac{1}{N+\nu} \sum_{t=1}^{N} e(t) - e(t)^2 = 0 \) a.s.

2. \( \lim_{N \to \infty} \frac{1}{N+\nu} \sum_{t=1}^{N} e^2(t) = \lim_{N \to \infty} \frac{1}{N+\nu} \sum_{t=1}^{N} e^2(t) \) a.s.

3. \( \lim_{N \to \infty} \frac{1}{N+\nu} \sum_{t=1}^{N} \left[ \left( \hat{\theta} - \theta_0 \right)^T \phi(t-1) \right]^2 = 0 \) a.s.

Moreover if \( \lambda_1 = 1 \) then \( \nu \) can be taken equal to 0, condition b) is not necessary and if \( \lim_{t \to \infty} F^{-1}(t) > 0 \) a.s. then

\[
\lim_{t \to \infty} \hat{\theta}(t) = \theta_0 \quad \text{a.s.}
\]

8
Proof. The results are directly derived from theorem 2 of [18]. Condition b) of theorem 2 in [18] ($\phi(t)$ is $F_t$ measurable) is obvious, and condition c) of the same theorem ($\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^T(t)\phi(t) < \infty$) is obtained from lemma 4.1 of [9]. The case $\lambda = 1$ is treated in theorem 4.2 of [9].

Remark that condition b) of this theorem imposes a sufficiently rich excitation signal. Additionally the choice of the poles basis, $A_o(q^{-1})$ is depending on, can be used to relax the convergence condition of the H-ERLS algorithm in a stochastic context.

4.3 Generalized Output Error predictor, and H-OLOE algorithm

From (5), (8), and (10) we have

$$A(q^{-1})\varepsilon(t+1) = A_o(q^{-1}) (\theta_0 - \theta)^T \phi(t) + A(q^{-1})v(t+1)$$

where

$$\phi^T(t) = \begin{bmatrix} -V_1^T(q^{-1})\hat{y}(t+1) & -V_2^T(q^{-1})\hat{y}(t+1) & \cdots & -V_1^T(q^{-1})u(t+1) & V_2^T(q^{-1})u(t+1) \end{bmatrix}$$

$$\theta_0^T = [m_1^T m_2^T \cdots n_1^T n_2^T \cdots]$$

In a deterministic context, $v(t+1)$ is taken equal to 0. The expression (17) generalizes the prediction error expression of the classical recursive output error algorithm corresponding to the case $A_o(q^{-1}) = 1$.

Hence the following theorem is obtained from [18]:

**Theorem 2.** Consider the H-OLOE algorithm associated with the generalized output error in a deterministic context, the prediction error being provided by (17) where $\phi(t)$ is a non necessarily bounded vector sequence. Assume that the true system is in the model set. Then if:

$$\frac{A_0 \left( \sqrt{2} - \lambda_1 z^{-1} \right)}{A \left( \sqrt{2} - \lambda_1 z^{-1} \right)} - \frac{\lambda_2}{2}$$

is SPR, one has

- $\lim_{t \to \infty} \varepsilon(t+1) = 0$
- $\lim_{t \to \infty} \left[ \theta_0 - \hat{\theta}(t+1) \right]^T \phi(t) = 0$
- $\left[ \hat{\theta}(t) - \theta_0 \right]^T F^{-1}(t) \left[ \hat{\theta}(t) - \theta_0 \right] < C < \infty$
Proof. It is a direct consequence of theorem 1 in [18] that states that if there exists \( \rho \geq 1, 0 < \lambda \leq (2 - \rho^2) \), if \( \varepsilon(t + 1) = H(q^{-1})\hat{(\theta)}^T \phi(t) \), and if the transfer function \( H(\rho z^{-1}) - \frac{\lambda_2}{2} \) is SPR, then one has: \( \lim_{t \to \infty} \rho^t \varepsilon(t + 1) = 0 \), \( \lim_{t \to \infty} \left[ \theta_0 - \hat{\theta}(t + 1) \right] \phi(t) \rho^t = 0 \), and
\[
\left[ \hat{\theta}(t) - \theta_0 \right]^T F^{-1}(t) \left[ \hat{\theta}(t) - \theta_0 \right] \rho^{2t} < C < \infty.
\]
The result is obtained by taking \( \lambda_1 = 2 - \rho^2 \).

The convergence conditions of theorem 2 are less restrictive than the usual conditions (see theorem 3.1, of [9]), since whenever \( A_0(\sqrt{2-\lambda_1 z^{-1}}) - \frac{\lambda_2}{2} \) is SPR with \( \lambda_1 = 1 \), it is so with \( 0 < \lambda_1 < 1 \) too.

In a stochastic context, if \( v(t) = e(t) \) (meaning that the output noise is a white noise and therefore a martingale difference sequence), one can use the theorem 2 of [18], and we have the following result:

Theorem 3. Consider the H-OLOE algorithm and the associated generalized output error predictor in a stochastic context, and its prediction error given by (17) where \( \{v(t)\} \) is a white noise \( (v(t) = e(t)) \). Assume that the following assumptions hold

a) The true system is in the model set,

b) There exists \( \delta > 0 \) such that for any \( t \geq n_\phi \), \( \sum_{j=t+1}^{t+n_\phi} \phi(t)^T - \delta I \) is positive definite, where \( n_\phi \) is the length of vector \( \phi(t) \),

c) The transfer function
\[
\frac{A_0(\sqrt{2-\lambda_1 z^{-1}})}{A(\sqrt{2-\lambda_1 z^{-1}})} - \frac{\lambda_2}{2}
\]
is SPR.

Then for any \( \nu > 0 \) one has

1. \( \lim_{N \to \infty} \frac{1}{N1^{+\nu}} \sum_{t=1}^N \left[ \varepsilon(t) - e(t) \right]^2 = 0 \) a.s.

2. \( \lim_{N \to \infty} \frac{1}{N1^{+\nu}} \sum_{t=1}^N \varepsilon(t) = \lim_{N \to \infty} \frac{1}{N1^{+\nu}} \sum_{t=1}^N e(t) \) a.s.

3. \( \lim_{N \to \infty} \frac{1}{N1^{+\nu}} \sum_{t=1}^N \left[ \hat{\theta} - \theta_0 \right]^T \phi(t^{-1}) \right]^2 = 0 \) a.s.

Moreover if \( \lambda_1 = 1 \) then \( \nu \) can be taken equal to 0, condition a) is not necessary and if \( \lim_{t \to \infty} F^{-1}(t) > 0 \) a.s. then
\[
\lim_{t \to \infty} \hat{\theta}(t) = \theta_0 \text{ a.s.}
\]
Proof. Once again, the results are inferred from theorem 2 of [18]. The case \(\lambda = 1\) is treated in theorem 4.2 of [9]. \(\square\)

If \(v(t)\) is not a white noise, the convergence of the algorithm can be proved for \(\lambda = 1\), provided the transfer function \(\frac{A_o(z^{-1})}{A(z^{-1})} - \frac{\lambda_2}{2}\) is SPR with theorem 4.1 of [9].

Remark that the choice of the poles basis, \(A_o(q^{-1})\) is depending on, can be used to relax the convergence condition of the H-OLOE algorithm, both in a deterministic or stochastic context.

5 Bias distribution analysis

5.1 Limit models expressions

The stationary condition of the parameter adaptation algorithm is

\[
E[\varepsilon(t+1)\phi(t, \theta)] = 0 \quad (18)
\]

This limit exists for a strictly decreasing adaptation gain \(F(t)\), i.e. for \(\lambda_1 = 1\). We assume in this section that \(\lambda_1 = 1\). Except the case of least squares algorithm, the regressor \(\phi(t, \theta)\) depends on the estimated parameters. As shown in [17], condition (18) does not imply in general the minimization of \(E[\varepsilon^2(t)]\) (particularly if the system is not in the model set). This is the major difference with prediction error methods (PEM) that aim directly at minimizing this latter expression. Thus it is important to determine the signal whose variance is effectively minimized if the condition (18) is satisfied, in order to infer the effective bias distribution in the frequency domain. As in [17], let us denote by \(\varepsilon_E(t+1, \theta)\) the equivalent prediction error signal (in general non measurable) such that the optimal estimated parameters vector \(\hat{\theta}^*\) of PLR algorithms is given by

\[
\hat{\theta}^* = \text{Argmin}_{\theta} E[\varepsilon_E^2(t+1, \theta)] \quad (19)
\]

It is shown in [17], that for the equation error model one has

\[
\varepsilon_E(t+1, \theta) = Q(q^{-1}, \theta)\varepsilon(t+1, \theta) + (1 - Q(q^{-1}, \theta))e(t+1) \quad (20)
\]

and for the output error model

\[
\varepsilon_E(t+1, \theta) = Q(q^{-1}, \theta)\varepsilon(t+1, \theta) + (1 - Q(q^{-1}, \theta))v(t+1) \quad (21)
\]

Where \(Q(q^{-1}, \theta)\frac{\partial \varepsilon(t+1, \theta)}{\partial \theta} = -\phi(t, \theta)\).

Consequently we infer the two following theorems:

**Theorem 4.** The equivalent prediction error signal for the H-ERLS algorithm associated with the generalized-ARMAX predictor is given by

\[
\varepsilon_E(t) = \hat{A} \left[ (G - \hat{G}) u(t) + \left( W - \frac{\hat{C}}{\hat{A}} \right) e(t) \right] + e(t) \quad (22)
\]
The results in table 1, lead to some remarks:

\[ Q_{\text{uu}} \]

where \( \Phi \)dictors expressed with generalized orthonormal transfer functions

Table 1: Limit model expressions for open-loop PLR algorithms including pre-

Additionally, we check immediately that for the H-RLS algorithm corresponding

ciated with the generalized-output error predictor is given by

Theorem 5. The equivalent prediction error of the H-OLOE algorithm, asso-

For the ARMAX predictor, one has

For the output error predictor, we have

Proof. For the ARMAX predictor, one has \( Q(q^{-1}, \theta) \frac{\partial \phi(t+1)}{\partial \theta} = -\phi(t) \) with

\[ Q(q^{-1}, \theta) = 1 + \sum_{k=1}^{m_p} i_k T V_k(q^{-1}) = \frac{\hat{C}}{\hat{A}_0}, \]

and owing to theorem 1 of [17], \( \varepsilon_E(t+1) = Q(q^{-1}, \theta) \varepsilon(t+1) + (1 - Q(q^{-1}, \theta)) \varepsilon(t+1) \), that yields the result.

Additionally, we check immediately that for the H-RLS algorithm corresponding to the generalized-ARX predictor, since the regressor \( \phi(t) \) is independent of \( \theta(t) \), the prediction error and the equivalent prediction error are equal and

\[ \varepsilon_E(t) = \varepsilon(t) = \frac{\hat{A}}{\hat{A}_0} \left( (G - \hat{G}) u(t) + \left( W - \frac{A_o}{\hat{A}} \right) e(t) \right) + e(t) \]

From (22), (23), (24), we can infer the limit models expressed in Table 1.

| ALGORITHM | \( \theta^* \) | \( \hat{A}(e^{i\omega}) \) | \( A_o(e^{i\omega}) \) | \( A_o(e^{i\omega}) \) | \( A_o(e^{i\omega}) \) |
|---|---|---|---|---|
| H-RLS (generalized ARX predictor) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) |
| H-ERLS (Generalized ARMAX predictor) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) |
| H-OLOE (Generalized OUTPUT ERROR predictor) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) | \( \text{Argmin} \int_{-\pi}^{\pi} \left| \frac{\hat{A}(e^{i\omega})}{A_o(e^{i\omega})} \right|^2 \left| G(e^{i\omega}) - \hat{G}(e^{i\omega}) \right|^2 \Phi_{uu}(\omega) \) |

Table 1: Limit model expressions for open-loop PLR algorithms including predictors expressed with generalized orthonormal transfer functions

\[ \Phi_{uu}(\omega), \Phi_{ee}(\omega), \] are the spectral density associated with respectively \( \{u(t)\} \) and \( \{e(t)\} \).

The results in table lead to some remarks:
• The bias distribution of algorithms parameterized with generalized orthonormal functions differ from standard PLR algorithms.

• The bias distribution depends on the basis poles. Therefore these poles can be considered as tuning parameters in order to adjust the model fit over the frequency domain. This dependence is a direct consequence of the results of [17], and are analyzed with the concept of equivalent prediction error. Note that this dependence is specific to PLR algorithms and would not apply to PEM schemes, for which a parameterization modification has no effect on the identified model (see [12] p. 437).

• The limit expressions in Table I depend all on the same weighting function: 

\[ \left| \frac{\hat{A}(e^{i\omega})}{A(e^{i\omega})} \right|^2, \]

given there is a homogeneity in the effect due to the basis poles, independently of the predictor structure.

• For Output-Error and ARMAX predictors based schemes, the noise model is not affected by the the parameterization (contrary to a classical prediction error filtering applied on standard schemes, that modify the noise model, cf. [21], [12]).

5.2 Effect of the basis poles on the bias distribution

The remarks of the latter subsection lead to consider more thoroughly the bias distribution over the frequency domain, for PLR algorithms with the above predictor parameterizations. According to [17], if the system is in the model set, the equivalent prediction error can -for the equation error models- be expressed as

\[ \varepsilon_E(t + 1, \theta) = [\theta_0 - \theta]^T \phi_E(t) + e(t + 1) \] (25)

and for the output error model

\[ \varepsilon_E(t + 1, \theta) = [\theta_0 - \theta]^T \phi_E(t) + v(t + 1) \] (26)

The equivalent regressor \( \phi_E \), independent of \( \theta \), is by definition (see [17]):

\[ \phi_E(t) = -\frac{\partial \varepsilon_E(t+1, \theta)}{\partial \theta}. \]

In general, the system to be identified has an order \( \eta_g \) greater than that of the predictor, and this remarks concerns both its deterministic part

\[ G(q^{-1}) = \frac{\sum_{k=1}^{\eta_g} \hat{n}_k^T V_k(q^{-1})}{1 + \sum_{k=1}^{\eta_g} \hat{n}_k^T V_k(q^{-1})} \]

and its stochastic part (both of them are assumed to have order \( \leq \frac{\eta_g}{\eta_p} \)):

\[ W(q^{-1}) = \frac{1 + \sum_{k=1}^{\eta_g} \hat{i}_k^T V_k(q^{-1})}{1 + \sum_{k=1}^{\eta_g} \hat{n}_k^T V_k(q^{-1})} \]
It follows that the equivalent prediction error of the prediction error models, can be put under the generic form (where $\phi_E$ is the equivalent regressor)

$$\varepsilon_E(t + 1, \theta) = [\theta_0 - \theta]T \phi_E(t) + \theta_0 \phi_E(t) + e(t + 1)$$  \hspace{1cm} (27)

In (27), $\theta$ is the parameter vectors with indexes up to $\frac{n_\omega}{\eta_p}$, $\hat{\theta}$, the corresponding estimated parameter vector, and the vector $\theta_0$ includes the other system parameters with indexes from $\frac{n_\omega}{\eta_p}$ up to $\frac{n_\omega}{n_p}$. A similar equation holds for the output error model

$$\varepsilon_E(t + 1, \theta) = [\theta_0 - \theta]T \phi_E(t) + \theta_0 \phi_E(t) + v(t + 1)$$  \hspace{1cm} (28)

Owing to (22), (23), (24), the equivalent regressors, $\phi_E$, have the following expressions

| MODEL              | EQUIVALENT REGRESSOR $\phi_E^n(t)$ |
|--------------------|-----------------------------------|
| Generalized-AR     | $-V_1^T(q^{-1})y(t + 1)$, $-V_1^T(q^{-1})u(t + 1)$, ... |
|                    | $-V_2^T(q^{-1})y(t + 1)$, $-V_2^T(q^{-1})u(t + 1)$, ... |
| Generalized-ARMAX  | $-V_1^T(q^{-1})y(t + 1)$, $-V_1^T(q^{-1})u(t + 1)$, ... |
|                    | $-V_2^T(q^{-1})y(t + 1)$, $-V_2^T(q^{-1})u(t + 1)$, ... |
| Generalized-Output error | $-V_1^T(q^{-1})\tilde{y}(t + 1)$, $-V_1^T(q^{-1})\tilde{u}(t + 1)$, ... |
|                    | $-V_2^T(q^{-1})\tilde{y}(t + 1)$, $-V_2^T(q^{-1})\tilde{u}(t + 1)$, ... |

Table 2: Equivalent regressor in function of the predictor structure

where $y_d(t) = G(q^{-1})u(t)$. The term $\phi_E(t)$ has the same structure as $\phi_E$, but with index $j$ of $V_j(q^{-1})$, belonging to $\{\frac{n_\omega}{\eta_p} + 1, \ldots, \frac{n_\omega}{n_p}\}$

In (27) and (28), the expression $\theta_0 \phi_E(t)$ can be considered as the "tail" of the system to be identified, as in [4], chap. 4 pp. 78-79.

The criterion to be minimized is $J = \mathbf{E}[\varepsilon_E^n(t, \theta)]$, by definition of the equivalent prediction error.

For the purpose of the following theorem, let us introduce the Hambo operator $\lambda$ such that $\lambda^{-1} = G_b(z)$, where $G_b(z)$ is given by (1), and the Hambo frequency $\omega_\lambda \in [-\pi \eta_p, +\pi \eta_p]$ such that $\lambda = e^{i \omega_\lambda}$. The relation between $\omega$ and $\omega_\lambda$ has been first studied in [14], with the introduction of the phase function called the $\beta$ function (see [4], p.222) and $\omega_\lambda = \beta(\omega)$, which is a one to one strictly increasing function. In particular one has $d \omega_\lambda = \beta'(\omega)d\omega$ and as shown in the same reference

$$\beta'(\omega) = V_1^T(e^{i\omega})V_1(e^{-i\omega})$$  \hspace{1cm} (29)

Consider two signals $\{x_1(t)\}, \{x_2(t)\}$ and their associated interspectral density in the $\omega$ domain $\Phi_{x_1,x_2}(\omega)$. Define

$$\tilde{\Phi}_{x_1,x_2}(\omega_\lambda) = \Phi_{x_1,x_2}(\omega)\left|\frac{V_1(e^{i\omega})V_1^T(e^{-i\omega})}{V_1^T(e^{i\omega})V_1(e^{-i\omega})}\right|_{\omega = \beta^{-1}(\omega_\lambda)}$$  \hspace{1cm} (30)
\( \tilde{\Phi}_{x_1,x_2}(\omega_\lambda) \) can be interpreted as an interspectral density associated with the Hambo frequency \( \omega_\lambda \). Remark however that this expression differs from the interspectral density as defined in chap. 3 of [4], since it is expressed over \( \omega_\lambda \in [-\pi \eta_p, +\pi \eta_p] \). Our goal here is to preserve the one-to-one relation from \( \Phi_{x_1,x_2}(\omega) \) to \( \tilde{\Phi}_{x_1,x_2}(\omega_\lambda) \), for reasons that will become clear in the following theorem. From proposition 6.8 of [5], one has

\[
\| \tilde{\Phi}_{x_1,x_2}(\omega_\lambda) \|_2 = |\Phi_{x_1,x_2}(\omega)|_{\omega=\beta^{-1}(\omega_\lambda)} \tag{31}
\]

**Theorem 6.** Set \( \Delta \theta = \theta_0 - \theta \).
The minimization of \( J = E[\varepsilon_E^2(t,\theta)] \) is equivalent to the minimization of

\[
\frac{1}{2\pi \eta_p} \int_{-\eta_p}^{\eta_p} \left\{ \Delta \theta \left[ \begin{array}{cccc}
1 & e^{-i\omega_\lambda \eta_p} & \cdots \\
e^{i\omega_\lambda \eta_p} & 1 & \cdots \\
\vdots & \vdots & \ddots \\
\end{array} \right] \otimes \tilde{\Phi}(\omega_\lambda) \Delta \theta + \\
\Delta \theta \left[ \begin{array}{cccc}
1 & e^{-i\omega_\lambda \eta_p} & \cdots \\
e^{i\omega_\lambda \eta_p} & 1 & \cdots \\
\vdots & \vdots & \ddots \\
\end{array} \right] \otimes \tilde{\Phi}(\omega_\lambda) \theta_2 \right\} d\omega_\lambda \tag{32}
\]

Where \( \otimes \) is the Kronecker product, and \( \tilde{\Phi}(\omega_\lambda) \) is equal to

- \[
\begin{bmatrix}
\tilde{\Phi}_{yy}(\omega_\lambda) & -\tilde{\Phi}_{yu}(\omega_\lambda) \\
-\tilde{\Phi}_{uy}(\omega_\lambda) & \tilde{\Phi}_{uu}(\omega_\lambda)
\end{bmatrix}
\]

for the generalized ARX model,

- \[
\begin{bmatrix}
\tilde{\Phi}_{yy}(\omega_\lambda) & -\tilde{\Phi}_{yu}(\omega_\lambda) & -\tilde{\Phi}_{ye}(\omega_\lambda) \\
-\tilde{\Phi}_{uy}(\omega_\lambda) & \tilde{\Phi}_{uu}(\omega_\lambda) & \tilde{\Phi}_{ue}(\omega_\lambda) \\
-\tilde{\Phi}_{ye}(\omega_\lambda) & \tilde{\Phi}_{ue}(\omega_\lambda) & \tilde{\Phi}_{ee}(\omega_\lambda)
\end{bmatrix}
\]

for the generalized ARMAX model,

- \[
\begin{bmatrix}
\tilde{\Phi}_{ydu}(\omega_\lambda) & -\tilde{\Phi}_{ydu}(\omega_\lambda) \\
-\tilde{\Phi}_{udy}(\omega_\lambda) & \tilde{\Phi}_{udu}(\omega_\lambda)
\end{bmatrix}
\]

for the generalized Output Error model.

**Proof.** The minimization of \( J \) is equivalent to that of

\[
E[\Delta \theta^T \phi_E(t) \phi_E^T(t) \Delta \theta] + E[\Delta \theta^T \phi_E(t) \phi_{E_2}^T(t)] \theta_2
\]

where \( \Delta \theta = \theta_0 - \theta \).

Let us consider \( R_E = E[\phi_E(t) \phi_E^T(t)] \), with:

\[
\begin{pmatrix}
R_{Ey} & -R_{Ey} \\
-R_{Ey} & R_{Euu}
\end{pmatrix}
\]

for the generalized ARX model,
\begin{align*}
\begin{pmatrix}
R_{EyEy} & -R_{EyEy} \\
-R_{EyEy} & R_{EyuEyu}
\end{pmatrix}
\end{align*}
for the generalized Output Error model,
\begin{align*}
\begin{pmatrix}
R_{EyEy} & -R_{EyEy} & -R_{EyEy} \\
-R_{EyEy} & R_{EyuEyu} & R_{EyuEyu} \\
-R_{EyEy} & R_{EyuEyu} & R_{EyuEyu}
\end{pmatrix}
\end{align*}
for the generalized ARMAX model.

The matrix $R_{Ex1x2}$ has a Toeplitz-like structure:
\begin{align*}
R_{Ex1x2} &= 
\begin{bmatrix}
R_{Ex1x2}(0) & R_{Ex1x2}(1) & R_{Ex1x2}(2) \\
R_{Ex2x1}(1) & R_{Ex1x2}(0) & R_{Ex1x2}(2) \\
R_{Ex2x1}(2) & R_{Ex2x1}(1) & R_{Ex1x2}(0) \\
\cdots & \cdots & \cdots \\
\end{bmatrix}
\end{align*}
where
\begin{align*}
R_{Ex1x2}(k) &= E[V_1(q^{-1})x_1, V_{k+1}(q^{-1})x_2]
\end{align*}
and in the frequency domain
\begin{align*}
R_{Ex1x2}(k) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Phi_{x_1,x_2}(\omega) V_1(e^{i\omega}) V_{k+1}^T(e^{-i\omega}) d\omega
\end{align*}
From (29) and (30)
\begin{align*}
R_{Ex1x2}(k) &= \frac{1}{2\pi \eta} \int_{-\eta \pi}^{+\eta \pi} \tilde{\Phi}_{x_1,x_2}(\omega) e^{-i \omega k} d\omega
\end{align*}
hence the result.

For a given basis pole number, and because of (32) and (31), the problem to be solved is always the same in the $\omega_\lambda$ frequency domain (from $-\eta \pi$ to $+\eta \pi$), whatever the basis poles values are ($\theta_2$ depends on the basis poles but is independent of the frequency, and $\Phi(\omega_\lambda)$ is independent of the estimated parameters). The relation from $\omega$ to $\omega_\lambda$ being non-linear (except in the case where $p_k = 0$), the $\omega_\lambda$ frequency scale is distorted compared to the $\omega$ scale. This frequency distortion is well known for the Laguerre basis, see [16]. It depends on the basis poles values (since $\beta(\omega)$ is a function of these poles). As an example, we provide the plot of the $\beta$ function for a 2 poles basis ($p_0 = 0.7$ and $p_1 = 0.9$), without giving the expression of $\beta(\omega)$ that can be found in [4] p. 222.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{plot.png}
\caption{Example of $\omega_\lambda = \beta(\omega)$ for a two poles basis ($p_0 = 0.7, p_1 = 0.9$)}
\end{figure}
Since the minimization problem is structurally always the same, on the $\omega_\lambda$ distorted frequency scale (whose distortion depends on the basis poles), the analysis of this frequency distortion provides insights about the bias distribution.

5.3 A heuristic method for evaluating the effect of the basis poles on the bias distribution

If we consider (32), and the relation from $\omega$ to $\omega_\lambda$ by means of the $\beta(\omega)$ function, the frequencies $\omega$ for which the distortion (or dilatation) rate from $\omega$ scale to $\omega_\lambda$ scale is maximum, are over-penalized in the criterion minimization (we can expect better model fit around these frequencies), whereas the frequencies corresponding to a low dilatation are under-weighted (inducing a worse model approximation). Then the frequency distortion analysis from $\omega$ to $\omega_\lambda$ scales, gives an useful indication about the effect of the basis poles on the model fit quality.

However as most of linear systems are represented in Bode diagrams with a logarithmic scale such that $\bar{\omega} = \log(\omega)$, it is more interesting to study the dilatation (or distortion) rate from $\bar{\omega}$ to $\omega_\lambda$. The relation between measures of integration is
\[
d\omega_\lambda = e^{\bar{\omega}} \beta'(e^{\bar{\omega}}) d\bar{\omega}
\] (33)

According to [15], and [4] p. 222, one has
\[
\beta'(\omega) = \sum_{k=0}^{\eta-1} \beta_k(\omega)
\] (34)
\[
\beta_k'(\omega) = \frac{1 - |p_k|^2}{|1 - p_k e^{i\omega}|^2}
\] (35)

Remark that $\beta'(\omega)$ is nothing but a particular expression of the reproducing Kernel of the associated orthogonal transfer function basis, see [22] (chap.4).

Equation (33) leads to define the distortion rate function $\chi(\omega)$ from $\bar{\omega}$ scale to $\omega_\lambda$ scale, such that
\[
\chi(\omega) = \frac{1}{\pi} \omega \beta'(\omega) = \frac{1}{\pi} e^{\bar{\omega}} \beta'(e^{\bar{\omega}})
\] (36)

The following property 1, corresponds to a conservation principle of $\chi(\omega)$

Property 1. One has
\[
\int_{-\infty}^{\log(\pi)} \chi(e^{\bar{\omega}}) d\bar{\omega} = 1
\] (37)

Proof. One has $\beta'(\omega) = V_1^T(e^{i\omega}) V_1(e^{i\omega})$, and because of the orthonormality of $V_1(e^{i\omega})$ we have the result immediately, see [4], p.88. \qed
Consider $\chi_k(\omega) = \frac{1}{\pi} \omega \beta_k'(\omega)$ and define the k-th basis pole from its proper frequency $\omega_{pk}$ and its damping $\zeta_k$, such that $p_k = p_k e^{i\sigma_k}$ with $p_k = e^{-\zeta_k \omega_{pk}}$ and $\sigma_k = \sqrt{1 - \zeta_k^2 \omega_{pk}}$. Let $0 < p_k < 1$. This function $\chi_k(\omega)$ has nice properties presented in the following theorems:

**Theorem 7.** Assume $\zeta_k^2 \geq 1 - \frac{\pi^2}{4\omega_{pk}}$. One has the following results

1. If $\cosh(\zeta_k \omega_{pk}) - \sqrt{1 - \zeta_k^2 \omega_{pk}} \geq \frac{\pi}{2}$, $\chi_k(\omega)$ is an increasing function on $[0, \pi]$, and has its maximum at $\omega = \pi$.

2. If $\cosh(\zeta_k \omega_{pk}) - \sqrt{1 - \zeta_k^2 \omega_{pk}} < \frac{\pi}{2}$, $\chi_k(\omega)$ has a unique maximum on $[0, \pi]$. Additionnally if:

   $\cosh(\zeta_k \omega_{pk}) + \cos(\sqrt{1 - \zeta_k^2 \omega_{pk}}) - \pi \sin(\sqrt{1 - \zeta_k^2 \omega_{pk}}) > 0$, $\chi(\omega)$ has necessarily a minimum.

3. If $p_k$ is real ($\zeta_k = 1$), and if $\frac{\pi - \sqrt{\pi^2 - 4}}{2} < p_k < 1$, $\chi_k(\omega)$ has a unique maximum on $[0, \pi]$, and a unique minimum. If $p_k \leq \frac{\pi - \sqrt{\pi^2 - 4}}{2}$, $\chi_k(\omega)$ is an increasing function on $[0, \pi]$, and has its maximum at $\omega = \pi$.

**Proof.** One has

$$\chi_k(\omega) = \frac{1}{\pi} \frac{1 - |p_k|^2}{|1 - p_k e^{i\omega}|^2} \omega = \frac{1}{\pi} \frac{1 - p_k^2}{1 + \frac{1}{2} p_k^2 - 2 p_k \cos(\omega - \sigma_k)} \omega$$

and

$$\frac{\partial \chi_k(\omega)}{\partial \omega} = \frac{1}{\pi} \left(1 - \frac{2 p_k}{2} - \cos(\omega - \sigma_k) - \omega \sin(\omega - \sigma_k) \right)$$

The sign of $\frac{\partial \chi_k(\omega)}{\partial \omega}$ depends upon the sign of

$$g(\omega) = \frac{1 + p_k^2}{2 p_k} - \cos(\omega - \sigma_k) - \omega \sin(\omega - \sigma_k).$$

One has

$$\frac{\partial g(\omega)}{\partial \omega} = -\omega \cos(\omega - \sigma_k) = -\omega \cos(\omega - \sqrt{1 - \zeta_k^2 \omega_{pk}}), \quad \frac{\partial g(\omega)}{\partial \omega} \leq 0 \text{ if and only if } \omega \leq \frac{\pi}{2} + \sqrt{1 - \zeta_k^2 \omega_{pk}}, \text{ and } \frac{\partial g(\omega)}{\partial \omega} > 0 \text{ otherwise.}$$

Set $\omega$ the frequency for which $g(\omega)$ is minimum. One has

$$g(\omega) = \cos(\zeta_k \omega_{pk}) - \sqrt{1 - \zeta_k^2 \omega_{pk}} - \frac{\pi}{2}.$$  

This quantity is strictly negative if and only if

$$\cosh(\zeta_k \omega_{pk}) - \sqrt{1 - \zeta_k^2 \omega_{pk}} < \frac{\pi}{2} \quad (38)$$

Additionally $g(0) = \cosh(\zeta_k \omega_{pk}) - \cos(\sqrt{1 - \zeta_k^2 \omega_{pk}}) > 0$ for any $\omega_{pk} > 0$, $g(\omega)$ is a positive decreasing function for $\omega$ close to 0, and has a minimum at $\omega = \frac{\pi}{2} + \sqrt{1 - \zeta_k^2 \omega_{pk}}$ only if $\omega_{pk} < \frac{\pi}{2 \sqrt{1 - \zeta_k^2}}$, i.e.

$$\zeta_k^2 \geq 1 - \frac{\pi^2}{4\omega_{pk}^2} \quad (39)$$
Since $g(\pi) = \cosh(\zeta_k \omega_k) + \cos(\sigma_k) - \pi \sin(\sigma_k)$, therefore $g(\pi) > 0$ is equivalent to

$$\cosh(\zeta \omega_k) + \cos(\sqrt{1 - \zeta_k^2 \omega_k}) - \pi \sin(\sqrt{1 - \zeta_k^2 \omega_k}) > 0$$

(40)

Therefore, if we assume that (39) and (38) are satisfied, $\chi(\omega)$ has a unique maximum on $[0, \pi]$. Furthermore if (40) is satisfied $\chi(\omega)$ has a unique minimum.

If condition (39) is satisfied and (38) is not, $\chi(\omega)$ is an increasing function on this interval and admits a unique maximum at $\omega = \pi$.

If the pole $p_k$ is real condition (40) is necessarily fulfilled, and (38) reduces to:\n
$$1 + p_k^2 - \pi p_k < 0$$

Since we consider only stable poles, this is equivalent to $p_k > \frac{\pi - \sqrt{\pi^2 - 4}}{2}$.

**Theorem 8.** Set $\omega_{\text{max}}$ the frequency for which $\chi_k(\omega)$ is maximum. If $\omega_{\text{ok}} \to 0$, one has

$$\omega_{\text{max}} = \omega_{\text{ok}} + o(|\omega_{\text{ok}}|)$$

(41)

**Proof.** According to theorem 7, $\omega_{\text{max}}$ is the smallest frequency such that $g(\omega) = 0$. This frequency is such that

$$h(\omega_{\text{ok}}, \omega) = \cosh(\zeta \omega_{\text{ok}}) - \cos \left( \omega - \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right) \cdots$$

$$\cdots - \omega \cos \left( \omega - \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right).$$

Let us consider $\omega_{\text{ok}}$ as the function variable and $\omega$ as a parameter. One has

$$h(\omega_{\text{ok}}, \omega) = -\left( \cos(\omega) \cos \left( \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right) + \sin(\omega) \sin \left( \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right) \right) \cdots$$

$$-\omega \left( \sin(\omega) \cos \left( \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right) - \cos(\omega) \sin \left( \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right) \right) \cdots$$

$$+ \cosh(\omega_{\text{ok}}) \cdot \omega(\omega_{\text{ok}}, \omega).$$

A first order Taylor-Young approximation yields

$$h(\omega_{\text{ok}}, \omega) = 1 - \omega \cos(\omega) \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} - \omega \sin(\omega) + o(\omega_{\text{ok}}).$$

This quantity can be null only if $1 - \cos(\omega) - \omega \sin(\omega) = 0$, implying $\omega = 0$.

If we perform a second order Taylor-Young expansion near $0$, we get

$$h(\omega_{\text{ok}}, \omega) = 1 + \frac{\zeta_k^2 \omega_{\text{ok}}^2}{2} - \frac{1 - \omega^2}{2} \left( 1 - \frac{1 - \zeta_k^2 \omega_{\text{ok}}^2}{2} \right) - \omega \sqrt{1 - \zeta_k^2 \omega_{\text{ok}} - \omega \left( \omega - \sqrt{1 - \zeta_k^2 \omega_{\text{ok}}} \right)} + o(\omega_{\text{ok}}) \cdot \omega(\omega_{\text{ok}}, \omega).$$

$$= \frac{1}{2} \left( \omega^2 - \omega_{\text{ok}}^2 \right) + o^2(\omega_{\text{ok}}) \cdot \omega(\omega_{\text{ok}}, \omega).$$

Consequently $h(\omega_{\text{ok}}, \omega) = o^2(\omega_{\text{ok}})$ if and only if $\omega_0^2 = \omega_{\text{ok}}^2$.

The relation $h(\omega_{\text{ok}}, \omega) = 0$ entails an implicit function $\omega_{\text{max}}(\omega_{\text{ok}})$, and one has:

$$h \left( \omega_{\text{ok}}, \Omega(\omega_{\text{ok}}) \right) = o^2(\omega_{\text{ok}}).$$

Hence the result.
For a one pole basis (Laguerre basis), if \( p_0 \) is sufficiently close to 1, the maximum of \( \chi(\omega) \) corresponds to a frequency \( \omega_{\text{max}} \approx \omega_{a0} \), and we can expect that, according to the above remarks, the model fit is enhanced around this frequency. If \( p_0 = 0 \), corresponding to the classical basis \( z^{-1}, z^{-2} \cdots \), one has \( \chi(\omega) = e^{\omega} \), showing that \( \chi(\omega) \) is maximum at the Nyquist frequency, and insignificant at low \( \omega \); thus the model misfit at those frequencies plays a quasi negligible role in the minimization problem (22). This is the reason why classical PLR algorithms with basis \( z^{-1}, z^{-2} \cdots \) generally exhibit important bias at low frequency and are absolutely not suited for fast sampled systems identification, hence the quite stringent rules regarding the sample period choice [10] that have been introduced for a long time. Likewise some specialists of PLR identification prefer to represent linear systems on Bode plots with a linear scale [9], [10]. Figure 2 displays the frequency distortion rate \( \chi(\omega, p_0) \) corresponding to Laguerre bases for many values of the Laguerre poles. One can observe the conservation principle of property 1.

![Frequency distortion rate \( \chi(\omega) \) for Laguerre bases](image)

Figure 2: Frequency distortion rate \( \chi(\omega) \) for Laguerre bases

Figure 3 shows three examples of \( \chi(\omega) \), corresponding to 1) one pole basis \( p_0 = 0.99, 2) \) two poles basis with \( p_0 = 0.9, p_1 = 0.999, 3) \) three poles basis with \( p_0 = 0.9, p_1 = 0.99, p_2 = 0.999 \). This function \( \chi \) provides a tool to assess qualitatively the effect of the basis pole on the model approximation in the frequency domain, and the said poles can therefore be considered as tuning parameters to specify, for a given experiment where to enhance the model fit in the frequency domain.
6 Simulation results

6.1 Identification of a reduced order system with a Laguerre basis

In these simulations, the system to be identified consists of two clusters of two resonant and two antiresonant modes, separated from 3 decades, which corresponds clearly to a stiff system. The overall system has order equal to 9, and is disturbed by a white output noise (signal/noise ratio: 22 dB). We identify it by means of the H-ERLS algorithm (corresponding to an ARMAX model), and we choose a predictor parameterized with a one pole basis ($\eta_p = 1$), corresponding to a Laguerre basis. We look for a reduced order model $\eta_a = 6$.

The first simulation (figure 4) shows how the initial model is approximated if the Laguerre pole is chosen such that the frequency distortion rate maximum is near the high frequency modes ($p_0 = 0.6$), the system being excited by a PRBS (11 registers, length 2047 samples, no decimation). These high frequency modes are well captured, whereas the low frequency ones are sheerly ignored. On the contrary, if we set the Laguerre pole such that the frequency distortion rate is maximum at a frequency close to the low frequency modes ($p_0 = 0.9996$), we obtain a good model approximation at those frequencies as shown in figure 5; the system is excited by a 20 register PRBS, length $2^{20} - 1$, without decimation, corresponding roughly to 44 periods of the lowest mode period (a lower noise level would allow for a lower test duration). This example shows that the frequency distortion rate function can be viewed as a tool allowing to appreciate the effect of the predictor parameterization on the bias distribution.
Figure 4: Reduced order identification, recursive extended least squares, predictor with one pole basis, Laguerre pole $p_o = 0.6$.

Figure 5: Reduced order identification, recursive extended least squares, predictor with one pole basis, Laguerre pole $p_o = 0.9996$. 
6.2 Identification of a stiff system with a two poles basis

Finally, we carry out an identification of the system aiming at capturing both low and frequency modes. This is made possible by selecting a 2 poles basis, and choosing a system order equal to 10 i.e. $\eta_p = 2$, and $\eta_a = 10$. The frequency distortion rate has now two maximal values, and we choose their frequencies in order to correspond roughly to those of low and high frequency modes clusters. The resulting identified model is displayed in Figure 6; this figure shows that a good fit has been obtained over all the spectrum, and that the model cannot be distinguished from the system to identify. The noise level is the same as in the previous subsection, and the excitation signal is a 20 register PRBS, (lenght $2^{20} - 1$, without decimation).

![Bode diagram](image)

Figure 6: Identification with a 10th order model, and a two poles basis ($p_0 = 0.6, p_1 = 0.9996$)

7 Conclusion

In this paper, we have proposed a predictor parameterization of identification schemes belonging to the pseudo-linear regression class. This parameterization is established on an orthonormal transfer function basis, and it addresses Output Error, ARMAX and a generalization of ARX models. We have shown that the choice of the basis poles has a clear influence on the convergence conditions of recursive pseudo-linear algorithms. Moreover these poles modify the bias distribution of the estimated model. A method for assessing the basis poles
effect on the bias distribution is presented; it is established on the analysis of
the distortion between the classical frequency scale and the Hambo frequency
one. Successful simulations of identifications performed on a stiff system show
the interest of this analysis.

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