Solving A Class of Fredholm Integral Equations of the First Kind via Wasserstein Gradient Flows

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Abstract

Solving Fredholm equations of the first kind is crucial in many areas of the applied sciences. In this work we consider integral equations featuring kernels which may be expressed as scalar multiples of conservative (i.e. Markov) kernels and we adopt a variational point of view by considering a minimization problem in the space of probability measures with an entropic regularization. Contrary to classical approaches which discretize the domain of the solutions, we introduce an algorithm to asymptotically sample from the unique solution of the regularized minimization problem. As a result our estimators do not depend on any underlying grid and have better scalability properties than most existing methods. Our algorithm is based on a particle approximation of the solution of a McKean–Vlasov stochastic differential equation associated with the Wasserstein gradient flow of our variational formulation. We prove the convergence towards a minimizer and provide practical guidelines for its numerical implementation. Finally, our method is compared with other approaches on several examples including density deconvolution and epidemiology.

1 Introduction

Fredholm integral equations of the first kind are ubiquitous in applied sciences and engineering, they model density deconvolution [36, 69, 86, 106], image reconstruction [7, 28, 107, 95], inverse boundary problems for partial differential equations [98, 29] and find applications in epidemiology [45, 46, 71] and statistics [51, 76]. Fredholm integral equations of the first kind are typically defined by

\[ g(y) = \int_{\mathbb{R}^d} f(x)k(x,y)dx \]

where the kernel \( k : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R} \) is known, and \( g : \mathbb{R}^p \to \mathbb{R} \) is known but indirectly observed, and the objective is to identify the unknown function \( f : \mathbb{R}^d \to \mathbb{R} \). If we interpret this as a Lebesgue integral with \( dx \) denoting the Lebesgue measure then we may identify the functions with measures \( \mu(dy) := g(y)dy, \pi(dx) := f(x)dx \) and \( k \) with an integral operator \( K(x,dy) := k(x,y)dy \) which leads to the representation that we will consider throughout this paper:

\[ \mu(A) = \pi K(A) := \int_{\mathbb{R}^d} K(x,A)d\pi(x), \quad (1) \]

for any Borel set \( A \in \mathcal{B}(\mathbb{R}^p) \), with \( \pi \) an unknown measure on \( \mathcal{B}(\mathbb{R}^d) \), \( \mu \) an observed measure on \( \mathcal{B}(\mathbb{R}^p) \) and \( K : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^p) \to \mathbb{R} \) an integral operator. These equations model the task of reconstructing the signal \( \pi \) from a distorted observed version \( \mu \). Since many applications of Fredholm integral equations are concerned with the reconstruction of signals that are a priori known to be non-negative [28], we consider the case in which both \( \pi \in \mathcal{P}(\mathbb{R}^d) \) and \( \mu \in \mathcal{P}(\mathbb{R}^p) \) are probability measures on \( \mathbb{R}^d \) and \( \mathbb{R}^p \), respectively,

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and $K$ is a Markov kernel. A great many Fredholm integral equations, particularly those for which the kernel is conservative up to a scalar multiplier in the sense that $\int_{\mathbb{R}^d} K(x, dy) = \int_{\mathbb{R}^d} K(x', dy)$ for every $x, x' \in \mathbb{R}^d$, can be recast in this framework with appropriate translation and normalization [25, Section 6].

In a probabilistic and variational approach, we seek a solution of (1) by minimizing the Kullback-Leibler divergence between $\mu$ and $\pi K$ [81]:

$$\pi^* = \arg\min_{\pi} \{ KL(\mu|\pi K) : \pi \in \mathcal{P}(\mathbb{R}^d) \},$$

(2)

where $KL(\mu|\pi K) = \int_{\mathbb{R}^d} \log(\frac{d\mu}{d\pi K})(y) \, d\mu(y)$ if $\mu$ admits a density w.r.t. $\pi K$ and $+\infty$ otherwise. The probability measure $\pi^*$ corresponds to the maximum likelihood estimator (MLE) for $\pi$. Using this formulation, it is possible to define approximate solutions of Fredholm integral equations of the first kind as solutions of a minimization problem on the space of measures. However, in most interesting cases, the integral equation (1) is ill-posed, i.e. there might exist more than one solution $\pi$ to (1), and, even when the solution is unique, it is unstable with respect to changes in $\mu$. This lack of stability is the primary concern when attempting to solve Fredholm integral equations of the first kind: in practical applications we often only have access to (noisy) observations from $\mu$ and not to its analytic form, and the instability of (1) means that small errors in $\mu$ do not necessarily correspond to small errors in the recovered solution $\pi$ [65, Chapter 15]. This issue reflects onto the minimization problem (2), which often does not admit a unique minimizer [66].

In order to circumvent this issue, a regularization term is often added to the variational formulation (2) [19, 48]. In a similar fashion to the Tikhonov regularization [49], we consider a regularized version of (2) in which a cross-entropic penalty with respect to a reference measure $\pi$ [19, 48]. In particular, [43] considers a cross-entropy regularization in a least-square setting, [57] studies (3) in the discrete setting, i.e. when $\pi$ and $\mu$ are vectors in the simplex of $\mathbb{R}^d, \mathbb{R}^p$ respectively, and [89] studies (3) when $\pi$ and $\mu$ are integrable densities. In the discrete setting, [57] establishes uniqueness of the minimizers of a discrete version of (3) and studies the behaviour of the minimizers as $\alpha \to 0$; in the continuous setting, [89] shows existence and uniqueness of a minimizer of (3) under the assumption that (1) admits a maximum entropy solution.

The use of cross-entropy regularization is not new in the literature on Fredholm integral equations [43, 57, 89]. In particular, [43] considers a cross-entropy regularization in a least-square setting, [57] studies (3) in the discrete setting, i.e. when $\pi$ and $\mu$ are vectors in the simplex of $\mathbb{R}^d, \mathbb{R}^p$ respectively, and [89] studies (3) when $\pi$ and $\mu$ are integrable densities. In the discrete setting, [57] establishes uniqueness of the minimizers of a discrete version of (3) and studies the behaviour of the minimizers as $\alpha \to 0$; in the continuous setting, [89] shows existence and uniqueness of a minimizer of (3) under the assumption that (1) admits a maximum entropy solution.

In related works, functionals involving the Kullback-Leibler divergence $KL(\mu|\pi K)$ are minimized iteratively by discretizing the domain of $\pi$, $\mu$ [48, 25, 17, 106, 95], while maximum entropy estimators are usually obtained approximating $\pi$ through a set of basis functions [64, 56, 59, 73]. However, when $\pi, \mu$ are probability measures it is natural to consider Monte Carlo strategies, and approximate $\pi$ using particles [31]. This is particularly convenient when the only information on $\mu$ is given by a sample drawn from it, as is the case in most applications [36, 69, 86, 106, 45, 46, 71, 51, 76]. Recently, [31] introduced a sequential Monte Carlo method (SMC-EMS) to approximately solve (1), this algorithm provides an adaptive stochastic discretization of the expectation maximization smoothing algorithm first studied in [94], which aims at obtaining solutions of (1) which achieve a good trade-off between solving (2) and having enough smoothness.

The functional (3) can be seen as a probabilistic counterpart of the classic Tikhonov regularization setting (e.g [49]); [44] propose an interacting particle system based on a Wasserstein gradient flow construction to approximately solve the Tikhonov-regularized problem for a wide class of (nonlinear) inverse problems. While the particle system introduced in [44] is similar in spirit to the approach taken here, the former requires an a priori discretization of the support of $\pi$, which makes it impractical when we deal with Fredholm integral equations, since the object of interest (i.e. $\pi$) is infinite dimensional. By focussing on the case of linear integral equations, we can avoid the discretization step required by the particle system of [44], and work directly with the measures $\pi, \mu$.

In this work, we adopt a probabilistic approach and present a novel particle algorithm to solve Fredholm integral equations of the first kind. More precisely, we derive a particle approximation to the
Stochastic Differential Equation (SDE) associated with the Wasserstein gradient flow \([2, 91]\) minimizing surrogates of \((3)\). Wasserstein gradient flows have been widely used to solve a variety of optimization problems in the space of probability measures \([5, 60, 68, 41]\), including finding solutions to finite dimensional inverse problems \([44]\), but, to the best of our knowledge, not for infinite dimensional inverse problems like the integral equation \((1)\). The SDE associated with this gradient flow is non-linear in the sense that it depends not only upon the position of the solution but also on its marginal distribution. Such SDEs are called McKean–Vlasov SDEs (MKVSDE), and have been actively studied during the past decades \([97, 96, 74, 75, 84, 72, 22, 47]\). In this work, we derive a particular MKVSDE whose solution converges towards a minimizer of \((3)\). However, this MKVSDE cannot be solved analytically, and we introduce a numerical approximation using an Euler–Maruyama discretization and a particle system.

We perform an extensive simulation study in which we compare our method with SMC-EMS and problem-specific algorithms and show that it achieves comparable performances. The results obtained with our method and SMC-EMS are particularly similar, since both methods aim at minimizing a regularized Kullback–Leibler divergence. However, while in this work we explicitly have a minimization problem for the regularized functional \((3)\), the SMC-EMS algorithm does not minimize a particular functional but corresponds to a smoothed version of the EM algorithm minimizing the Kullback–Leibler divergence \((2)\). When \(\alpha\) and the smoothness parameter controlling the amount of regularization in SMC-EMS are both small, the two algorithms will therefore give very similar reconstructions. Nevertheless, the method proposed in this work comes with a number of additional theoretical guarantees and practical improvements. From the theoretical point of view, we show that \((3)\) admits a unique minimizer for any \(\alpha\) and that this minimizer can be seen as the probabilistic counterpart of the classical solution to the Tikhonov regularized least-squares problem. On the other hand, uniqueness of the fixed point of the EMS recursion approximated by SMC-EMS has not been established. From the practical point of view, minimizing \((3)\) allows us to introduce additional information about the solution through \(\pi_0\), this proves particularly beneficial when the kernel \(K\) leads to model misspecification (see Section 5.2). When compared on higher dimensional problems, the proposed method outperforms both standard discretization-based methods and SMC-EMS (Section 5.4), opening the door to solving Fredholm integral equations for high-dimensional problems.

In addition, the value of the regularization parameter \(\alpha\) can be chosen through common approaches for selecting the regularization parameter in Tikhonov regularization settings (e.g. cross–validation \([102, 1]\)), while in the case of SMC-EMS the link between the particle system and the functional which is minimized is less clear, and cannot be exploited as straightforwardly.

To summarize, our main contributions are as follows:

(a) First, we establish that the functional in \((3)\) is continuous and admits a unique minimizer under mild conditions on \(\mu, K\). In addition, we show that a modification of \((3)\) is stable with respect to perturbations of the observation \(\mu\) and study the limit behaviour of \(\pi^\star\) as \(\alpha \to 0\).

(b) Second, we derive the Wasserstein gradient flow used to minimize a surrogate of \((3)\) and show that the associated MKVSDE converges towards a unique minimizer using recent results from optimal transport \([53]\).

(c) Third, we present and study a discrete-time particle approximation of this MKVSDE. In particular, we establish geometric ergodicity of the particle system under regularity conditions.

(d) Finally, we illustrate the efficiency of our method to approximately solve Fredholm integral equations of the first kind on a range of different numerical examples and show that it achieves superior or comparable performances to those of problem-specific algorithms and SMC-EMS, outperforming the latter for high dimensional problems. In addition, we show on a task from epidemiology that, when the kernel \(K\) does not correctly model the problem at hand, the presence of a reference measure \(\pi_0\) allows us to outperform even problem-specific estimators.

The rest of the manuscript is organized as follows. In Section 2 we establish that the functional in \((3)\) is well-founded and study its key properties. The link between gradient flows and MKVSDE in the context of Fredholm integral equations of the first kind is studied in Section 3. In Section 4 we describe a numerical scheme which approximates the continuous time dynamics of the particle system and provide error bounds. Finally, in Section 5, we test the algorithm on a number of applications of Fredholm integral equations from statistics, epidemiology and image processing, and explore the scaling properties of our method through a comparison with the SMC-EMS algorithm recently proposed in \([31]\).
Notation

We endow \( \mathbb{R}^d \) with the Borel \( \sigma \)-field \( \mathcal{B}(\mathbb{R}^d) \) with respect to the Euclidean norm \( \| \cdot \| \). For a matrix \( A \) we denote \( \| A \| = (\sum_{ij} a_{ij}^2)^{1/2} \) the Hilbert-Schmidt norm. We denote by \( C(\mathbb{R}^d) \) the set of continuous functions defined over \( \mathbb{R}^d \) and by \( C^n(\mathbb{R}^d) \) the set of \( n \)-times differentiable functions defined over \( \mathbb{R}^d \) for any \( n \in \mathbb{N}^* \), where \( \mathbb{N}^* \) denotes the non-zero natural numbers. For all \( f \in C^1(\mathbb{R}^d) \), we denote by \( \nabla f \) its gradient. Furthermore, if \( f \in C^2(\mathbb{R}^d) \) we denote by \( \nabla^2 f \) its Hessian and by \( \Delta f \) its Laplacian. For all \( f \in C^1(\mathbb{R}^d \times \cdots \times \mathbb{R}^m) \) with \( m \in \mathbb{N}^* \), we denote by \( \nabla, f \) its gradient w.r.t. component \( i \). We say that a function \( f : \mathbb{X} \to \mathbb{R} \) where \( \mathbb{X} \) is a metric space is coercive if for any \( t \in \mathbb{R} \), \( f^{-1}((-\infty, t]) \) is relatively compact. This definition can be extended to the case where \( \mathbb{X} \) is only a topological space, see [33, Definition 1.12]. A family of functions \( \{ f_{\alpha} : \alpha \in A \} \) such that for any \( \alpha \in A \), \( f_{\alpha} : \mathbb{X} \to \mathbb{R} \), is said to be equicoercive if there exists \( g : \mathbb{X} \to \mathbb{R} \) such that for any \( \alpha \in A \), \( f_{\alpha} \geq g \) and \( g \) is coercive. For any \( p \in \mathbb{N} \) we denote by \( \mathcal{P}_p(\mathbb{R}^d) = \{ \pi \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \| x \|^p \mu(x) < +\infty \} \) the set of probability measures over \( \mathbb{R}^d \) with finite \( p \)-th moment. For ease of notation, we define \( \mathcal{P}(\mathbb{R}^d) = \mathcal{P}_0(\mathbb{R}^d) \) the set of probability measures over \( \mathcal{B}(\mathbb{R}^d) \) and endow this space with the topology of weak convergence. For any \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \) we define the \( p \)-Wasserstein distance \( W_p(\mu, \nu) \) between \( \mu \) and \( \nu \) by

\[
W_p(\mu, \nu) = \left( \inf_{\gamma \in \mathcal{T}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \| x - y \|^p \mu(dx) \right)^{1/p}
\]

where \( \mathcal{T}(\mu, \nu) = \{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \gamma(A \times \mathbb{R}^d) = \mu(A), \gamma(\mathbb{R}^d \times A) = \nu(A) \forall A \in \mathcal{B}(\mathbb{R}^d) \} \) denotes the set of all transport plans between \( \mu \) and \( \nu \). In the following, we metrize \( \mathcal{P}_p(\mathbb{R}^d) \) with \( W_p \). Assume that \( \mu \ll \nu \) and denote by \( \frac{d\mu}{d\nu} \) its Radon-Nikodym derivative. The Kullback–Leibler divergence, \( \text{KL}(\mu|\nu) \), between \( \mu \) and \( \nu \) is defined by

\[
\text{KL}(\mu|\nu) = \int_{\mathbb{R}^d} \log \left( \frac{d\mu(x)}{d\nu(x)} \right) d\mu(x).
\]

We say that \( K : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^p) \to [0, +\infty) \) is a Markov kernel if for any \( x \in \mathbb{R}^d \), \( K(x, \cdot) \) is a probability measure over \( \mathcal{B}(\mathbb{R}^p) \) and for any \( A \in \mathcal{B}(\mathbb{R}^p) \), \( K(\cdot, A) \) is a Borel-measurable function over \( \mathbb{R}^d \).

2 A minimization problem for Fredholm integral equations

In this section, we study the properties of the regularized functional \( \mathcal{F}_\alpha : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup +\infty \) given for any \( \alpha \geq 0 \) and \( \pi \in \mathcal{P}(\mathbb{R}^d) \) by

\[
\mathcal{F}_\alpha(\pi) = \text{KL}(\mu|\pi K) + \alpha \text{KL}(\pi|\pi_0),
\]

where we recall that \( \mu \in \mathcal{P}(\mathbb{R}^p) \), \( \pi_0 \in \mathcal{P}(\mathbb{R}^d) \) and the Markov kernel \( K : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^p) \to [0, 1] \) are given. For ease of notation we denote \( \mathcal{F} = \mathcal{F}_0 \).

In Section 2.1 we introduce a surrogate functional which enjoys better stability properties than \( \mathcal{F}_\alpha \). We prove that under mild assumption on \( \mu \) and \( K \), a minimizer of the surrogate exists, is unique and continuous w.r.t. the parameters of the functional. In Section 2.2, we investigate the links between our approach and other regularization methods for Fredholm integration equations of the first kind.

2.1 A surrogate functional: stability and regularization

First, we introduce a surrogate functional \( \mathcal{G}_\alpha \) whose minimizers are the same as \( \mathcal{F}_\alpha \) if \( \mu \) is absolutely continuous w.r.t. the Lebesgue measure but which is also easier to study when \( \mu \) is approximated by an empirical distribution. Assume that there exists \( k : \mathbb{R}^d \times \mathbb{R}^p \to [0, +\infty) \) such that for any \( x \in \mathbb{R}^d \) and \( A \in \mathcal{B}(\mathbb{R}^p) \), \( K(x, A) = \int_A k(x, y) dy \). Then, using the Fubini–Tonelli theorem we have that for any \( A \in \mathcal{B}(\mathbb{R}^p) \) and \( \pi \in \mathcal{P}(\mathbb{R}^d) \),

\[
\pi K(A) = \int_{\mathbb{R}_d} K(x, A) d\pi(x) = \int_A \left( \int_{\mathbb{R}_d} k(x, y) d\pi(x) \right) dy.
\]

Therefore \( \pi K \) admits a density w.r.t. the Lebesgue measure given by \( y \mapsto \pi[k(\cdot, y)] \). In addition, assume that \( \mu \) admits a density w.r.t. the Lebesgue measure \( d\mu(y) = \mu(y) dy \) such that \( \int_{\mathbb{R}_p} |\log(\mu(y))| d\mu(y) < +\infty \). Then for any \( \pi \in \mathcal{P}(\mathbb{R}^d) \)

\[
\text{KL}(\mu|\pi K) = -\text{H}(\mu) - \int_{\mathbb{R}_p} \log(\pi[k(\cdot, y)]) d\mu(y),
\]

where \( \text{H}(\mu) := -\int_{\mathbb{R}_p} \log(\mu(y)) d\mu(y) \) is the entropy of \( \mu \).
Since \( \int_{\mathbb{R}^d} |\log(p_k(\cdot, y))| \, \text{d} \mu(y) < +\infty \) if \( \text{KL}(\mu | \pi K) < +\infty \), minimizing \( \mathcal{F}_\alpha \) on the set \( \{ \pi \in \mathcal{P}(\mathbb{R}^d) : \text{KL}(\mu | \pi K) < +\infty \} \) for any \( \alpha \geq 0 \) is equivalent to minimizing \( \mathcal{G}_\eta \) on the same set, where for any \( \pi \in \mathcal{P}(\mathbb{R}^d) \)

\[
\mathcal{G}_\eta(\pi) = -\int_{\mathbb{R}^d} \log(p_k(\cdot, y)) \, \text{d} \mu(y) + \alpha \text{KL}(\pi | \pi_0),
\]

with \( \mathcal{G} = \mathcal{G}_0 \) for ease of notation.

The functional (4) is not defined if \( \int_{\mathbb{R}^d} \log(p_k(\cdot, y)) \, \text{d} \mu(y) = +\infty \) or \( \text{KL}(\pi | \pi_0) = +\infty \). We consider the following assumption on the kernel \( K \) so that the functional is well-defined for any \( \pi \in \mathcal{P}(\mathbb{R}^d) \)

Assume \( \alpha, \eta > 0 \) with respect to the parameters \( (\alpha, \eta) \), see Proposition 1. The following stability result guarantees that if we substitute \( \pi \) with \( \nu \) in \( \text{KL}(\mu | \pi K) \), we have \( \text{KL}(\mu | \nu K) = \text{KL}(\mu | \pi K) \) for ease of notation.

In this section, we could restrict ourselves to bounded second derivatives. The condition \( \| \partial_{i,j}^2 k(x,y) \| \leq M \) is used only in Proposition 10, in which we establish a convergence result for the time discretization of the MKVSDE. Under \( A_1 \), the density \( k \) of the Markov kernel \( K \) is Lipschitz continuous for every \( y \in \mathbb{R}^d \).

To address stability issues we also consider the following regularized functional: for any \( \alpha, \eta \geq 0 \) and \( \pi \in \mathcal{P}(\mathbb{R}^d) \)

\[
\mathcal{G}^\eta_\alpha(\pi) = -\int_{\mathbb{R}^d} \log(p_k(\cdot, y)) + \eta \, \text{d} \mu(y) + \alpha \text{KL}(\pi | \pi_0),
\]

where \( \eta \geq 0 \) is a hyperparameter. Using \( \mathcal{G}^\eta_\alpha \) with \( \eta > 0 \) will be crucial in order to establish the Lipschitz continuity of the drift of the MKVSDE in Section 3, and hence the stability of the proposed procedure. For clarity, we write \( \mathcal{G}_\eta = \mathcal{G}^\eta_0, \mathcal{G}^\eta = \mathcal{G}^\eta_0^n \) and \( \mathcal{G} = \mathcal{G}^\eta_0^n \). In the following proposition, we derive regularity properties for the family of functionals \( \{ \mathcal{G}^\eta : \eta \geq 0 \} \).

**Proposition 1.** Assume \( A_1 \), then the following hold:

(a) For any \( \eta \geq 0 \), \( \mathcal{G}^\eta \) is lower bounded, lower-semi continuous, convex and is not coercive.

(b) For any \( \eta > 0 \), \( \mathcal{G}^\eta \) is proper and \( \mathcal{G}^\eta \in C(\mathcal{P}(\mathbb{R}^d), \mathbb{R}) \).

**Proof.** See Appendix A.2.

Proposition 1 shows that \( \mathcal{G}^\eta \) enjoys better regularity properties for \( \eta > 0 \). In Proposition 3, we will also show that if \( \eta, \alpha > 0 \) then the minimizers of \( \mathcal{G}^\eta_\alpha \) are stable w.r.t. \( \mu \). In addition, note that for any \( \eta > 0 \), \( \mathcal{G}^\eta \) is not coercive. As a result, the uniqueness of the minimizer of \( \mathcal{G}^\eta \) is difficult to establish. In what follows we study the function \( \mathcal{G}^\eta_\alpha \) for \( \alpha, \eta > 0 \) and show that \( \mathcal{G}^\eta_\alpha \) is coercive in this case. Hence, \( \mathcal{G}^\eta_\alpha \) admits a unique minimizer \( \pi^\eta_\alpha \), and we show that the family of minimizers \( \{ \pi^\eta_\alpha : \alpha, \eta > 0 \} \) is smooth with respect to the parameters \( \alpha \) and \( \eta \). In Proposition 4 we will characterize the limit of \( \pi^\eta_\alpha \) when both \( \alpha, \eta \to 0 \).

**Proposition 2.** Assume \( A_1 \). Then the following hold:

(a) For any \( \alpha, \eta > 0 \), \( \mathcal{G}^\eta_\alpha \) is proper, strictly convex, coercive and lower semi-continuous. In particular, \( \mathcal{G}^\eta_\alpha \) admits a unique minimizer \( \pi^\eta_\alpha \in \mathcal{P}(\mathbb{R}^d) \).

(b) \( \pi : (\alpha, \eta) \mapsto \pi^\eta_\alpha \in C(0, +\infty)^2, \mathcal{P}(\mathbb{R}^d) \) and \( \mathbf{d} : (\alpha, \eta) \mapsto \inf_{\mathcal{P}(\mathbb{R}^d)} \mathcal{G}^\eta_\alpha \in C((0, +\infty)^2, \mathbb{R}) \).

**Proof.** See Appendix A.3.

In most applications, \( \mu \) is unknown but we have access to samples \( y^{1:M} = \{ y^{k:M} \}_{k=1}^M \in (\mathbb{R}^p)^M \) drawn from \( \mu \), see [36, 69, 86, 106, 71, 51, 76]. The following stability result guarantees that if we substitute \( \mu \) with the empirical measure \( \mu^M = (1/M) \sum_{k=1}^M \delta_{y^{k:M}} \) in \( \mathcal{G}^\eta_\alpha \) we obtain a minimization problem whose solution converges to the minimizer we would obtain if we had full knowledge of \( \mu \) as the number of samples \( M \) increases.

**Proposition 3.** Assume \( A_1 \). Let \( \alpha, \eta > 0 \) and for any \( \nu \in \mathcal{P}(\mathbb{R}^d) \) denote \( \pi^\nu_{\alpha, \eta} \) the unique minimizer of (5) with \( \mu \leftarrow \nu \). Let \( (\mu_n) \in \mathcal{P}(\mathbb{R}^d) \) such that \( \lim_{n \to +\infty} \mathbf{W}_1(\mu_n, \mu) = 0 \) with \( \mu \in \mathcal{P}(\mathbb{R}^d) \). Then, we have \( \lim_{n \to +\infty} \mathbf{W}_1(\pi^\nu_{\alpha, \eta} \mu_n, \pi^\nu_{\alpha, \eta} \mu) = 0 \).

**Proof.** See Appendix A.4.

We emphasize that in the proof of Proposition 3, the fact that \( \eta > 0 \) plays a crucial role. Indeed, if \( \eta > 0 \) and \( A_1 \) holds we can exploit the fact that \( y \mapsto \log(p_k(\cdot, y)) + \eta \) is Lipschitz continuous.


2.2 Variants and connections with other methods

2.2.1 Maximum entropy methods

The functional $\mathcal{F}_\alpha$ can be seen as the Lagrangian associated with the following primal problem

$$\arg\min \{ KL(\pi|\pi_0) : \pi \in \mathcal{P}(\mathbb{R}^d), \ KL(\mu|\pi K) = 0 \}.$$ 

The latter problem is a maximum entropy problem in the sense of [32] (where the entropy functional is replaced by the Kullback–Leibler divergence). Closely related to (3) is the functional $\tilde{\mathcal{F}}(\pi)$ given (when it is defined) by

$$\tilde{\mathcal{F}}(\pi) = KL(\mu|\pi K) - \alpha H(\pi).$$  

The choice of an entropic penalty does not require specifying a reference measure $\pi_0$ and connects (3) with the classical maximum entropy methods [58]. Indeed, (6) is the Lagrangian associated with the following primal problem

$$\arg\max \{ H(\pi) : \pi \in \mathcal{P}(\mathbb{R}^d), \ KL(\mu|\pi K) = 0 \},$$

where $\mathcal{P}(\mathbb{R}^d)$ is the set of probability distributions with finite entropy. However, the functional $\tilde{\mathcal{F}}(\pi)$ is not lower bounded and the corresponding minimization problem is not well-defined and therefore is not considered here. On the contrary, $\mathcal{F}_\alpha$ is always non-negative making the optimization problem well-defined. A number of maximum entropy approximations of solutions of Fredholm integral equations have been proposed in the literature, most of which maximize the entropy subject to moment constraints obtained by integrating $\mu$ and $\pi K$ w.r.t. a (possibly infinite) set of basis functions [64, 56, 59, 73]. These approaches work particularly well in the one-dimensional case, as in this case the maximum entropy solution can be written analytically (see, e.g., [56, Proposition 3.1]). For higher dimensional problems, the maximum entropy solution is usually approximated by discretizing the support of $\pi$ (see, e.g., [78]).

2.2.2 Generalized Bayesian inference

As mentioned in the introduction, minimizing $\mathcal{F}$ results in the maximum likelihood estimator (2). Given a set of observations $y_1, \ldots, y_M$ from $\mu$, the log-likelihood for the infinite-dimensional parameter $\pi$ is [24]

$$(1/M) \sum_{j=1}^M \log \pi(k(\cdot, y_j)) = \int_{\mathbb{R}^d} \log \pi(k(\cdot, y)) d\mu^M(y),$$ (7)

with $\mu^M = (1/M) \sum_{j=1}^M \delta_{y_j}$. For any $\alpha > 0$, minimizing the functional $\mathcal{G}_\alpha$ in (4) is equivalent to minimizing

$$-\frac{1}{\alpha} \int_{\mathbb{R}^d} \log(\pi(k(\cdot, y))) d\mu(y) + KL(\pi|\pi_0),$$

which can be interpreted as generalized Bayesian inference in the sense of adjusting the weight (or learning rate) of the likelihood (7) w.r.t. the prior $\pi_0$ (see, e.g., [12, Section 3] and [50]). In particular, if we assume that the target measure $\pi$ is a discrete distribution with a fixed and finite number of components (a straightforward case which is not considered here), i.e. $\pi = \sum_{i=1}^N \delta_{x^i} p_i$, with $\sum_{i=1}^N p_i = 1$ and $\{x^i\}_{i=1}^N \in (\mathbb{R}^d)^N$, then (1) is a finite mixture model and minimizing (4) corresponds to posterior inference for the locations $\{x^i\}_{i=1}^N$ and the weights $\{p_i\}_{i=1}^N$ with prior $\pi_0$ and learning rate $\alpha^{-1}$.

Choosing the appropriate $\alpha$ then amounts to the calibration problem in generalized Bayesian inference with standard Bayesian inference obtained when $\alpha = 1$ [50, 12]. Contrary to the standard setting, in which values of $0 < \alpha^{-1} < 1$ are preferred since the model is believed to be misspecified, we are in the setting in which we want to reduce the influence of the prior $\pi_0$ which is used as a regularizer in (4) and favour values of $\alpha^{-1} > 1$.

2.2.3 Tikhonov regularization

We now move onto considering the link between $\mathcal{G}_\alpha$ and Tikhonov regularization. In fact, minimising $\mathcal{G}_\alpha$ can be seen as the probabilistic counterpart to the classical Tikhonov regularization setting [49] which we recall briefly.

Let $A \in \mathbb{R}^{p \times d}$ and $y \in \mathbb{R}^p$ (note that this discussion is still valid in more general Hilbert spaces but we restrict ourselves to the finite-dimensional case for the sake of clarity). Define $g : \mathbb{R}^d \rightarrow [0, +\infty)$ for any $x \in \mathbb{R}^d$ by $g(x) = ||Ax - y||^2$. Denote by $A^\dagger$ the Moore-Penrose inverse of $A$, see [80] for a definition. $A^\dagger y \in \arg\min_{\mathbb{R}^d} g$ and moreover $A^\dagger y \in \arg\min_{\mathbb{R}^d} (x \mapsto ||x|| : x \in \arg\min_{\mathbb{R}^d} g)$, i.e. $A^\dagger y$ is the solution of the minimization of $g$ with least norm. For any $\alpha \geq 0$, the Tikhonov regularization with level $\alpha$ of $g$ is
given by \( g_\alpha : \mathbb{R}^d \to [0, +\infty) \) such that \( g_\alpha(x) = g(x) + \alpha \|x\|^2 \) for any \( x \in \mathbb{R}^d \). There exists a unique \( x_\alpha \in \mathbb{R}^d \) such that \( x_\alpha \in \arg\min_{x \in \mathbb{R}^d} g_\alpha \). It is known that \( \lim_{\alpha \to 0} x_\alpha = A^\dagger y \), i.e. the solution of the Tikhonov regularization converges towards the solution of \( g \) with minimal norm \([49]\).

In Proposition 4, we show a similar property in the probabilistic setting. In this case the norm function is replaced by the Kullback–Leibler divergence with respect to some reference measure \( \pi \mapsto \text{KL}(\pi|\pi_0) \). Given the functional \( \mathcal{G} : \mathcal{P}(\mathbb{R}^d) \to (-\infty, \infty] \) in (5), we obtain for any \( \alpha, \eta \geq 0 \) a regularized functional \( \mathcal{G}_\alpha^n(\pi) = \mathcal{G}(\pi) + \alpha \text{KL}(\pi|\pi_0) \). Similarly to the Tikhonov regularization, there exists a unique minimizer to \( \mathcal{G}_\alpha^n, \pi_{\alpha, \eta} \in \mathcal{P}(\mathbb{R}^d) \), see Proposition 2. In Proposition 4, we show that any limiting point of the sequences \((\pi_{\alpha, \eta_n})_{n \in \mathbb{N}} \) with \( \lim_{n \to +\infty} \alpha_n = 0 \) and \( \lim_{n \to +\infty} \eta_n = 0 \), is a minimizer of \( \mathcal{G} \) with least Kullback–Leibler divergence with respect to \( \pi_0 \).

To control the behaviour of the family of minima \( \{\inf_{\pi \in \mathcal{P}(\mathbb{R}^d)} \mathcal{G}_\alpha^n(\pi) : \alpha, \eta > 0\} \) when \( \alpha \) is close to 0, we make use of the following assumption which controls the tail behaviour of \( \pi_0 \) and the tail behaviour of the density \( \pi \).

**A2. The following hold:**

(a) \( \pi_0 \) admits a density w.r.t. the Lebesgue measure, \( d\pi_0(x) = \pi_0(x)dx \), with \( \pi_0(x) \propto \exp[-U(x)] \), where, \( U : \mathbb{R}^d \to \mathbb{R} \) is such that there exist \( \tau, C_1 > 0 \) satisfying for any \( x \in \mathbb{R}^d \)

\[-C_1 - \tau \|x\|^2 \leq U(x) \leq C_1 + \tau \|x\|^2.\]

(b) \( \mu \in \mathcal{P}_2(\mathbb{R}^p) \) and there exists \( C_2 \geq 0 \) such that for any \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^p \)

\[k(x, y) \geq C_2^{-1} \exp(-C_2(1 + \|x\|^2 + \|y\|^2)).\]

Under this assumption we have the following result.

**Proposition 4. Assume A1 and A2. Then, for any \( \alpha > 0, \eta \geq 0, \pi_{\alpha, \eta}^* \in \mathcal{P}_2(\mathbb{R}^d) \) admits a density w.r.t the Lebesgue measure. If there exist \( \pi^* \in \mathcal{P}_2(\mathbb{R}^d), (\alpha_n)_{n \in \mathbb{N}} (0, +\infty)^{\mathbb{N}}, (\eta_n)_{n \in \mathbb{N}} [0, +\infty)^{\mathbb{N}} \) such that \( \lim_{n \to +\infty} \alpha_n = 0 \), \( \lim_{n \to +\infty} \eta_n = 0 \) and \( \lim_{n \to +\infty} \mathcal{W}_2(\pi_{\alpha_n, \eta_n}^*, \pi^*) = 0 \), then \( \pi^* \in \arg\min_{\pi \in \mathcal{P}_2(\mathbb{R}^d)} \{\text{KL}(\pi|\pi_0) : \pi \in \arg\min_{\pi \in \mathcal{P}_2(\mathbb{R}^d)} \mathcal{G}\}. \]

**Proof.** See Appendix A.5. \( \square \)

### 3 McKean-Vlasov Stochastic Differential Equation and minimization of functionals

In most interesting cases the direct minimization of (5) is not tractable. In this work, we propose to follow a Wasserstein gradient flow approach in order to approximate these minimizers. Our methodology stems from the connection between minimization of functionals in the space of probability measures and partial differential equations (PDEs) pointed out in [60, 85]. In particular, we draw links between the minimization of \( \mathcal{G}_\alpha^n \) and a SDE whose invariant measure is the minimizer of \( \mathcal{G}_\alpha^n \). An informal presentation of the bridges between Wasserstein gradient flows and a certain class of SDEs, namely McKean-Vlasov SDEs (MKVSDEs), is presented in Section 3.1. We investigate the long-time behaviour of these MKVSDEs in the context of the minimization of \( \mathcal{G}_\alpha^n \) in Section 3.2. Finally, we show that there exists a system of interacting SDEs which approximates the obtained MKVSDE in Section 3.3.

#### 3.1 Minimization and gradient flows

In this section, we set \( \alpha, \eta > 0 \) and informally derive the continuous-time dynamics we study in the rest of this paper by drawing a link with gradient flows; formal results are given in Appendix B.1. We assume that all the probability measures we consider admit densities w.r.t the Lebesgue measure and do not distinguish between the distribution and its density. Let \( (\pi_t)_{t \geq 0} \) be a family of probability measures satisfying (in a weak sense)

\[\partial_t \pi_t = -\text{div}(\pi_t \nabla_{\pi_t}),\]

where \( v_t \) belongs to the Wasserstein subdifferential of a functional \( \mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty], \) see [2, Definition 10.1.1] for a definition. Then \( (\pi_t)_{t \geq 0} \) is called a Wasserstein gradient flow associated with
\[ F \] [2, Definition 11.1.1] and it has been shown in numerous settings that such gradient flows converge towards the minimizers of \( F \) when they exist [2, 91]. In Proposition 19 in Appendix B.1 we show that

\[ x \mapsto -\int_{\mathcal{R}^p} (\eta + \pi[k(x,y)])^{-1} \nabla k(x,y) d\mu(y) + \alpha \{ \nabla U(x) + \nabla \log \pi(x) \}, \]

belongs to the subdifferential of \( \mathcal{G}^\alpha \) at \( \pi \) under mild conditions and assuming that \( \pi_0(x) \propto \exp[-U(x)] \).

We also derive the corresponding gradient flow equation

\[ \partial_t \pi_t = -\text{div} \left[ \int_{\mathcal{R}^p} \pi_t[k(\cdot, y)] + \eta^{-1} \nabla k(x,y) d\mu(y) - \alpha \nabla U(x) \right] + \alpha \Delta \pi_t. \tag{9} \]

For strongly geodesically convex (i.e. convex along geodesics) functionals the gradient flow (8) converges geometrically towards the unique minimizer. However, in our setting \( \mathcal{G}^\alpha \) is not geodesically convex but only convex. Using recent results from [53] we establish the convergence of the Wasserstein gradient flow in Section 3.2 but without quantitative convergence rates.

We use the gradient flow equation (9) to draw a link between the minimization of \( \mathcal{G}^\alpha \) and MKVSDEs [72], a class of SDEs in which the drift and diffusion coefficients depend not only on the current position and its derivative, but also on the distribution of the current position. In particular, we show that under certain conditions, the MKVSDE in (10) admits a unique strong solution and is ergodic provided that the gradient of the potential \( U \) is Lipschitz continuous. We present here several assumptions on the regularity of \( U \) which we use throughout the remainder of this paper. However, the results presented in this section only rely on A3-(a) and A3-(b).

A3. The following hold:

(a) There exists \( L \geq 0 \) such that \( \| \nabla U(x_1) - \nabla U(x_2) \| \leq L \| x_1 - x_2 \| \), for any \( x_1, x_2 \in \mathbb{R}^d \).

(b) There exist \( m, c > 0 \) such that for any \( x_1, x_2 \in \mathbb{R}^d \), \( \langle \nabla U(x_1) - \nabla U(x_2), x_1 - x_2 \rangle \geq m \| x_1 - x_2 \|^2 - c \).

(c) There exists \( L_2 \geq 0 \) such that for any \( x_1, x_2 \in \mathbb{R}^d \), \( \| \nabla^2 U(x_1) - \nabla^2 U(x_2) \| \leq L_2 \| x_1 - x_2 \|^2 \).

Under these conditions, if \( \eta > 0 \) the drift given in (11) is Lipschitz continuous and we can use standard tools for McKean-Vlasov processes to establish existence and uniqueness.

**Proposition 5.** Assume A1 and A3-(a). Then for any \( \alpha, \eta > 0 \) there exists a unique strong solution to (10) for any initial condition \( X^*_t \in \mathcal{P}(\mathbb{R}^d) \).

**Proof.** See Appendix B.2.

The previous proposition is limited to the case where \( \eta > 0 \). Indeed, if \( \eta = 0 \) the drift is not Lipschitz continuous and the SDE (10) might be unstable, with solutions existing up to a (possibly small) explosion time. Having shown that (10) admits a unique strong solution when \( \eta > 0 \), we verify that the distribution \( \lambda^*_t \) converges to the unique minimizer \( \pi^*_\alpha, \eta \) of \( \mathcal{G}^\alpha \) when \( t \to +\infty \).

**Proposition 6.** Assume A1, A3-(a) and A3-(b). Then for any \( \alpha, \eta > 0 \), we have \( \lim_{t \to +\infty} \mathcal{W}_2(\lambda^*_t, \pi^*_\alpha, \eta) = 0 \).

**Proof.** See Appendix B.3.

There exists a rich literature on the exponential ergodicity of MKVSDE, see [18, 42, 70, 13] for instance. However, these results require the interaction term \( b_\eta \) to be small when compared to \( \alpha \nabla U \). This is not the case in our applications where we study the behaviour of the McKean-Vlasov process for small values of \( \alpha > 0 \). The original approach of [53] differs from these previous works and relies on the fact the MKVSDE under consideration stems from a Wasserstein gradient flow. However, [53, Theorem 2.11] does not establish quantitative bounds contrary to [18, 42, 70, 13] which derive explicit convergence rates. The result above can be strengthened to show that, if \( \alpha_t \to 0 \) at the appropriate rate, then \( \mathcal{G}^\alpha(\lambda^*_t) \) converges towards the minimum of the unregularized functional (see [27, Theorem 4.1] and [82, Theorem 1]).
3.3 Particle system

The MKVSDE (10) has several shortcomings from a methodological point of view. First, since the drift of the SDE depends not only on the current position of the solution but also on its distribution, it is usually difficult to sample from continuous-time processes. We circumvent the first issue by introducing a time discretization scheme for this particle system SDE in order to obtain a particle system Markov chain which can be implemented.

We introduce the particle system $(X^i_t^{1:N})_{t \geq 0}$ which satisfies the following SDE: for any $k \in \{1, \ldots, N\}$, $X_0^{k,N} \in \mathbb{R}^d$ and

$$X_t^{k,N} = X_0^{k,N} + \int_0^t b(X_s^{k,N}, \lambda_s^{N}(X_0^{1:N}))ds + \sqrt{2\sigma}B_t^k,$$

where for any $x \in \mathbb{R}^d$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ we have

$$b(x, \nu) = \int_{\mathbb{R}^d} \tilde{b}_i(x, \nu, y)d\mu(y) - \alpha \nabla U(x),$$

$(\{B_t^k\}_{t \geq 0} : k \in \mathbb{N})$ is a family of independent Brownian motions and $\lambda_t^N(X_0^{1:N})$ is the empirical measure associated with $(X_t^{k,N})_{t \geq 0}^N$, i.e. we have for any $N \in \mathbb{N}^+$

$$\lambda_t^N(X_0^{1:N}) = (1/N) \sum_{k=1}^{N} \delta_{X_t^{k,N}}.$$  

In the following proposition we show using classical propagation of chaos tools that there exists a unique solution to (12) and that this solution approximates (10) for any finite time horizon.

**Proposition 7.** Assume A1 and A3-(a). Then, for any $\alpha, \eta > 0$ and $N \in \mathbb{N}^*$ there exists a unique strong solution to (12) for any initial condition $X_0^{1:N}$ such that $\mathcal{L}(X_0^{1:N}) \in \mathcal{P}_1((\mathbb{R}^d)^N)$ and $X_0^{k,N}$ is exchangeable. In addition, if $X_t^{k,N} = \tilde{X}_t^k$, for any $T \geq 0$ there exists $C_T \geq 0$ such that for any $N \in \mathbb{N}^*$ and $\ell \in \{1, \ldots, N\}$

$$\mathbb{E}^{1/2}[\sup_{t \in [0,T]} \|X_t^\ell - \tilde{X}_t^\ell\|^2] \leq C_T N^{-1/2}.$$  

**Proof.** See Appendix B.4. 

We now turn to the exponential ergodicity of the particle system. Using recent advances in the theory of convergence of Markov chains [35] we derive quantitative bounds. However, these bounds are not uniform w.r.t. to the number of particles and therefore we cannot use this result to conclude that the McKean-Vlasov process itself is exponentially ergodic by letting $N \to +\infty$.

**Proposition 8.** Assume A1, A3-(a) and A3-(b). Then, for any $\alpha, \eta > 0$ and $N \in \mathbb{N}^*$ there exist $C_N \geq 0$ and $\rho_N \in [0,1)$ such that for any $x_1^{1:N}, x_2^{1:N} \in (\mathbb{R}^d)^N$ and $t \geq 0$

$$W_1(\lambda_t^N(x_1^{1:N}), \lambda_t^N(x_2^{1:N})) \leq C_N \rho_N \|x_1^{1:N} - x_2^{1:N}\|,$$

where for any $x^{1:N} \in (\mathbb{R}^d)^N$, $\lambda_t^N(x^{1:N})$ is the distribution of $(X_t^{k,N})_{t \geq 0}$ with initial condition $x^{1:N}$. In particular, (12) admits a unique invariant probability measure denoted $\pi^N \in \mathcal{P}_p((\mathbb{R}^d)^N)$ and for any $x^{1:N} \in (\mathbb{R}^d)^N$ and $t \geq 0$

$$W_1(\lambda_t^N(x^{1:N}), \pi^N) \leq C_N \rho_N (\|x^{1:N}\| + \int_{\mathbb{R}^d} \|\tilde{x}\|d\pi^N(\tilde{x})).$$

**Proof.** See Appendix B.5. 

In the two previous propositions, the exponential ergodicity and the approximation between the particle system and the McKean-Vlasov process are only valid for a finite number of particles or a finite time-horizon, respectively. In particular, we cannot directly conclude that the projection of the invariant distribution of the particle system onto its first component on $\mathcal{P}(\mathbb{R}^d)$, denoted by $\pi_t^{1:N}$, is close to $\pi_{\alpha, \eta}^*$. This is the topic of the next proposition.

**Proposition 9.** Assume A1, A3-(a) and A3-(b). Then, for any $\alpha, \eta > 0$, $\lim_{N \to +\infty} W_1(\pi^{1:N}, \pi_{\alpha, \eta}^*) = 0$. In addition, $\pi_{\alpha, \eta}^*$ is the unique invariant probability measure of (10).

**Proof.** See Appendix B.6.
4 Numerical Implementation

In the previous section, we introduced a continuous time Markov process \((X_t^{1,N})_{t \geq 0}\) such that the distribution \(X_t\) approximates \(\pi_{0,\eta}\) for large values of \(t \geq 0\) and \(N \in \mathbb{N}\). In this section, we derive an algorithm approximating this particle system. 

In order to obtain a numerical approximation of (12) we need two consider factors: first, as in the case of classical SDEs, the continuous time process (12) cannot be simulated directly, and we need to introduce time discretization, secondly, in practice the integral w.r.t. \(\mu\) in (13) is intractable.

First, in Section 4.1 we consider an Euler–Maruyama discretization of the particle system. For stability issues we consider a tamed version of the classical Euler–Maruyama discretization. To address the second issue, given that in applications we will often be in the situation in which the Lipschitz constant can be large. In practice, we observe that the use of a tamed Euler–Maruyama discretization scheme prevents some numerical stability issues appearing for small values of \(\eta > 0\).

The tamed Euler–Maruyama discretization (and its extension to the Milstein scheme) for classical SDEs has been investigated in [54, 55, 104, 15]. More recently, strong approximation results have been established for McKean–Vlasov SDEs in [8]. In the following proposition, we use the results of [8, Theorem 2.5] to derive strong approximation results between \((X_{n}^{k,N})_{n \in \mathbb{N}}\) and \((X_t^{k,N})_{t \geq 0}\) for any \(k \in \{1, \ldots, N\}\); since the diffusion coefficient of (12) is constant, the tamed scheme (14) coincides with a Milstein scheme, allowing us to show strong convergence of order 1.

**Proposition 10.** Assume A1 and A3. Then for any \(\eta, \alpha > 0\) and any \(T \geq 0\) there exists \(C_T \geq 0\) such that for any \(N \in \mathbb{N}^\star\), \(\ell \in \{1, \ldots, N\}\) and \(\gamma > 0\)

\[ \mathbb{E}[^{\sup_{n \in \{0, \ldots, n_T\}} \|X_{n_T}^{\ell,N} - X_n^{\ell,N}\|} \leq C_T \gamma \]

where \(n_T = \lfloor T/\gamma \rfloor\).

**Proof.** See Appendix C.2.

A similar result holds for a continuous-time interpolation of \((X_{n}^{\ell,N})_{n \in \mathbb{N}}\), see [8, Theorem 2.5] for more details. For small values of \(\gamma > 0\) and large values of \(n_T\) and \(N \in \mathbb{N}\) we get that \((1/N) \sum_{k=1}^{N} \delta_{X_{n}^{k,N}}\) is an approximation of \(\pi_{n,\eta}\).

4.2 Stability of the particle system w.r.t the observed measure

Proposition 3 shows that if \(\mu\) is approximated by the empirical measure \(\mu^M\) obtained from a sample \(y_{1,M} = (y_{1,1,M})_{k=1}^{M}\) then the minimizer of \(\varphi_{0}\) obtained using \(\mu^M\) instead of \(\mu\) converges to the minimizer of \(\varphi_{0}\) obtained using \(\mu\) as \(M \to +\infty\). We now consider the discrete-time alternative particle system \((\{X_{n}^{k,N,M}\}_{n \in \mathbb{N}})_{k=1}^{N}\) which evolves according to (14) with \(b\) replaced by its empirical approximation obtained by approximating \(\mu\) with the empirical measure of the \(M\) samples \(y_1, \ldots, y_M\):

\[ X_{n+1}^{k,N,M} = X_{n}^{k,N,M} + \gamma b^M(X_{n}^{k,N,M}, \lambda_{n}^{M})/(1 + \gamma \|b^M(X_{n}^{k,N,M}, \lambda_{n}^{M})\|) + \sqrt{2\alpha} Z_{n+1}^{k} \]
where \( \{Z^k_{n,k,n} \}_{k,n \in \mathbb{N}} \) is a family of independent Gaussian random variables, \( \gamma > 0 \) is a stepsize, \( \lambda^N,M = (1/N) \sum_{k=1}^N \delta_{X^N_{k,n},M} \) for any \( n \in \mathbb{N} \) and for any \( x \in \mathbb{R}^d \) and \( \nu \in \mathcal{P}(\mathbb{R}^d) \) we have
\[
 b^M(x, \nu) = \int_{\mathbb{R}^d} \tilde{b}_\nu(x, \nu, y) d\mu(y) - \alpha \nabla U(x) = (1/M) \sum_{j=1}^M \tilde{b}_\nu(x, \nu, y_j) - \alpha \nabla U(x). \tag{16}
\]
In the next proposition, we show that the two particle systems (14) and (15) are close when \( M \) is large.

**Proposition 11.** Assume A1, A3-(a) and A3-(b). Let \( \eta, \alpha > 0 \). For any \( N, M \in \mathbb{N}^* \), let \((X^N_{1:n,k,n})_{k \geq 0} \) and \((X^{1:N}_{1:n,k,n})_{k \geq 0} \) be obtained with (14) and (15) respectively, driven by the same underlying family of independent Gaussian random variables \( \{Z^k_{n,k,n} \}_{k,n \in \mathbb{N}} \), with initial condition \( X^N_{1:n,0} \in \mathcal{P}(\mathbb{R}^d)^N \). \( \{X^N_{1:n,k,n} \}_{k=1}^N \) is exchangeable and \( \{X^{1:N}_{1:n,k,n} \}_{k=1}^N = \{X^{0,k,n,M} \}_{k=1}^N \). Then, for any \( T \geq 0 \) there exists \( C_T \geq 0 \) such that for any \( \ell \in \{1, \ldots, N \} \) and \( \gamma > 0 \)
\[
 E[\sup_{n \in (0, \ldots, n_T]} \|X^{\ell,N}_{n} - X^{\ell,N,M}_{n}\|] \leq C_T \gamma M^{-1/2}.
\]

**Proof.** See Appendix C.3. \( \square \)

An equivalent result showing convergence of the continuous process
\[
 X^N_{t,k,n} = X^{k,n} + \int_0^t b^M(X^N_{s,k,n}, \lambda^N(X^{1:N}_{t,k,n})) ds + \sqrt{2\alpha} b_t,
\]
with \( b^M \) as in (16), to (12) could also be obtained. The argument follows the same line of that used to prove Proposition 7 and exploits the convergence of \( \mu^M \) to \( \tilde{b} \) established in the proof of Proposition 11. The resulting rate is \( M^{-1/2} \) (see Appendix C.5).

As we discuss in the next section, in some cases it might be beneficial to resample from the empirical measure \( \mu^M \) at each time step. In the following proposition we show that this resampling operation does not change the error bound obtained in Proposition 11.

**Proposition 12.** Assume A1, A3-(a) and A3-(b). Let \( \eta, \alpha > 0 \). For any \( N, M \in \mathbb{N}^* \), let \((X^N_{1:n,k,n})_{k \geq 0} \) and \((X^{1:N}_{1:n,k,n})_{k \geq 0} \) be obtained with (14) and (15) respectively, driven by the same underlying family of independent Gaussian random variables \( \{Z^k_{n,k,n} \}_{k,n \in \mathbb{N}} \), with initial condition \( X^N_{1:n,0} \in \mathcal{P}(\mathbb{R}^d)^N \). \( \{X^N_{1:n,k,n} \}_{k=1}^N \) is exchangeable and \( \{X^{1:N}_{1:n,k,n} \}_{k=1}^N = \{X^{0,k,n,m} \}_{k=1}^N \). Then, for any \( T \geq 0 \) there exists \( C_T \geq 0 \) such that for any \( \ell \in \{1, \ldots, N \} \) and \( \gamma > 0 \)
\[
 E[\sup_{n \in (0, \ldots, n_T]} \|X^{\ell,N}_{n} - X^{\ell,N,m}_{n}\|] \leq C_T \gamma m^{-1/2}.
\]

**Proof.** See Appendix C.4. \( \square \)

The empirical measure \( \lambda^N,M \) provides an approximation of \( \pi^\alpha,\eta \). In low dimensional settings, for the purpose of visualization we derive a smooth approximation of \( \pi^\alpha,\eta \) using standard kernel density estimation tools and define \( \tilde{\pi}^N,M : \mathbb{R}^d \to \mathbb{R}^d \) such that for any \( x \in \mathbb{R}^d \)
\[
 \tilde{\pi}^N,M(x) = (1/N) \sum_{k=1}^N \det(\mathbf{H})^{-1/2} \varphi(\mathbf{H}^{-1/2}(x - X_{k,n,M}^{N,M})), \tag{17}
\]
where \( \varphi \) is the density of a \( d \)-dimensional Gaussian distribution with zero mean and identity covariance matrix and \( \mathbf{H} \) is a positive definite bandwidth matrix, see, e.g., [93]. Combining the results in Proposition 7 and Proposition 10 with standard arguments from the kernel density estimation literature, the estimator (17) can be shown to converge to \( \pi^\alpha,\eta \) as \( N \to \infty \) and \( \gamma \to 0 \) (e.g., [4, Theorem 3.1] and [14, Theorem 2.2]). Our final algorithm is summarized in Algorithm 1.

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**Algorithm 1** Solving Fredholm integral equations with Wasserstein gradient flows (FE-WGF)

**Require:** \( N, M, n_T \in \mathbb{N}, \alpha, \eta, \gamma > 0, H \in \mathbb{R}^{d \times d}, \mu, \pi_0, \pi_{\text{init}} \in \mathcal{P}(\mathbb{R}^d) \).

**Draw** \( \{X^{0,k,n,M}_{t,k,n} \}_{k=1}^N \) from \( \pi_{\text{init}} \)  

**for** \( n = 0 : n_T \) **do**

**Draw** \( \{y_{k,n}^{M} \}_{k=1}^M \) from \( \mu^M \)  

**Compute** \( b^M(X_{n,k,n,M}, \lambda_{n,M}) \) as in (16)  

**Update** \( X_{n+1,k,n}^{M} = X_{n,k,n,M} + \gamma b^M(X_{n,k,n,M}, \lambda_{n,M}) + (1 + \gamma\|b^M(X_{n,k,n,M}, \lambda_{n,M})\|) + \sqrt{2\alpha} Z_{n+1} \) as in (14)  

**end for**

**return** \( \tilde{\pi}^N,M(x) \) as in (17)
4.3 Implementation guidelines

Obtaining the estimator (17) of the minimizer of $\mathcal{G}_n^\alpha$ requires specification of a number of parameters: the reference measure $\pi_0$ and the regularization parameter $\alpha$ control the properties of the measure $\pi$ we want to reconstruct (e.g. its smoothness), while the parameter $\eta$, the number of particles $N, M$ and the time discretization step $\gamma$ control accuracy and stability of the numerical implementation.

Choice of $\pi_0$ There are of course, many possible choices for the reference measure $\pi_0$ to which we want the regularized solution to be close depending on the constraints we want to impose on the reconstructed solution. The choice of an improper reference measure $\pi_0 \propto C$ with $C > 0$ results in a McKean-Vlasov SDE in which there is no dependence on $\pi_0$

$$dX^*_t = \left\{ \int_{\mathbb{R}^d} b(\hat{X}^*_t, \hat{X}^*_t; y) d\mu(y) \right\} dt + \sqrt{2} \sigma dB_t, \quad \hat{X}^*_0 \in \mathbb{R}^d,$$

where $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion and $\hat{X}^*_t$ is the distribution of $(X^*_t)_{t \geq 0}$. This scheme corresponds to what we would obtain by applying the gradient flow procedure to the functional (6). Under additional assumptions on the tail behaviour of the kernel $k$, it is possible to show that this functional is proper, lower semi-continuous and coercive in an appropriate sense; [30, Chapter 7] shows empirically that the corresponding particle system is stable. Nevertheless, in many scenarios some knowledge of the target will be embedded in the problem at hand. In deconvolution problems, $\mu$ is the distribution of noisy samples from the target distribution $\pi$; in this case it is sensible to expect that $\pi$ itself will be close to $\mu$ and therefore choose $\pi_0$ to be reasonably close to $\mu$. More generally, for problems in which $k(x, y) = k(y - x)$ is a deconvolution kernel, the distribution $\mu$ will be the convolution of $\pi$ and $k$ and therefore encapsulates information on $\pi$. In these problems we choose $\pi_0$ to be a Gaussian distribution with mean $\mu$ and variance given by the empirical mean and variance of the sample from $\mu$. For the epidemiology examples in Section 5.2, we set $\pi_0$ to be $\mu$ shifted back by a number of days equal to the mean [45] or the mode [103] of the infection-to-confirmed delay distribution $k$. For image reconstruction, $\pi_0$ could be selected to enforce particular characteristics in the reconstructed image, e.g. smoothness [79] or sparsity [108, 67]. Alternatively, if a large data set of previously reconstructed images is available one could use score matching priors [62] to obtain a reference measure $\pi_0$ encapsulating all the relevant features of the image to reconstruct. We study the influence of the reference measure $\pi_0$ on a toy example in Appendix D.1 and find that $\pi_0$ influences the shape of the minimizer of $\mathcal{G}_n^\alpha$ and its minimum but not the speed at which convergence to the minimum occurs.

Choice of $\alpha$ As the parameter $\alpha > 0$ controls the amount of regularization introduced by the cross-entropy penalty, its value should be chosen to give a good trade-off between the distance from the data distribution $\mu$, $\text{KL}(\mu|\pi_0 k)$, and that from $\pi_0$, $\text{KL}(\pi|\pi_0 k)$. A common approach for selecting the regularization parameter in Tikhonov regularization is cross-validation [102, 1]. Since the case in which $\mu$ is not known but a sample drawn from it is available is the most likely in applications, we propose the following approach to cross-validation: to estimate the value of $\alpha$ we divide the sample from $\mu$ into $L$ subsets and find the value of $\alpha$ which minimizes

$$\text{CV}(\alpha) = (1/L) \sum_{j=1}^L \mathcal{G}_n^\alpha(\hat{\pi}^j),$$

where $\hat{\pi}^j$ is obtained using Algorithm 1 with the samples from $\mu$ which are not in group $j \in \{1, \ldots, L\}$ and $\mathcal{G}_n^\alpha$ is an approximation of $\mathcal{G}_n^\alpha$

$$\mathcal{G}_n^\alpha(\hat{\pi}^j) := -\frac{1}{M} \sum_{\ell=1}^M \log \left( \frac{1}{N} \sum_{k=1}^N k(X^{k,N,M}_n, y_j) \right) + \frac{\alpha}{N} \sum_{k=1}^N \log \left( \frac{\hat{\pi}^j(X^{k,N,M}_n)}{\pi_0(\hat{\pi}^j)} \right).$$

(18)

If some prior information on the smoothness of the solution $\pi$ is known (e.g. its variance), one could chose $\alpha$ so that the smoothness of (17), matches the expected smoothness of $\pi$. We show how the value of $\alpha$ influences the reconstructions of $\pi$ on simple problems in Appendix D.2.

Choice of $\eta$ The parameter $\eta$ has been introduced in (5) to deal with the possible instability of the functional $\mathcal{G}_n^\alpha$; we did not find performances to be significantly influenced by this parameter as long as its value is sufficiently small. In practice, in the experiments in Section 5 we set $\eta \equiv 0$. This choice might seem counter-intuitive, since most of our theoretical results are obtained with $\eta > 0$. On one hand, this suggests that for regular enough $k$, the functional $\mathcal{G}_n^\alpha$ is well-behaved; on the other hand, we observe that the introduction of the tamed Euler–Maruyama scheme (14) prevents most of the numerical instability issues arising when discretizing (12).
Choice of \( N, \gamma \) and \( m \) The values of the number of particles \( N \), the time discretization step \( \gamma \) and the number of samples \( m \) from \( \mu \) to use at each iteration control the quality of the numerical approximation of (10), their choice is therefore largely application dependent. However, the results in Proposition 7, 10 and 12 give the following global error estimate: for any \( T \geq 0 \) we have

\[
E[\sup_{t \in [0, nT]} ||X^\mu_{N,t} - X^\mu_{N,\gamma,m}||] \leq C_T(N^{-1/2} + \gamma + \gamma m^{-1/2}).\]  

(19)

Choosing \( N \) amounts to the classical task of selecting an appropriate sample size for Monte Carlo approximations, while the choice of \( \gamma \) corresponds to the specification of a timescale on which to discretize a continuous time (stochastic) process; hence, one can exploit the vast literature on Monte Carlo methods and discrete time approximations of SDEs to select these values [63]. In practice, we found that the choice of \( N \) largely depends on the dimensionality of the problem; for one-dimensional examples values between \( N = 200 \) and \( N = 1000 \) achieve high enough accuracy to compete with specialized algorithms (see Sections 5.1, 5.2); as the dimension \( d \) increases larger values of \( N \) should be considered (see Section 5.3). Similar considerations apply to the choice of the time discretization step \( \gamma \). In particular, this parameter should be chosen taking into account the order of magnitude of the gradient, to give a good trade-off between the Monte Carlo error and the time discretization error. In practice, we observed good results with \( \gamma \) between \( 10^{-3} \) and \( 10^{-1} \).

The number of samples \( m \) from \( \mu \) to employ at each iteration is again problem dependent. In many cases, we are given a number \( M \) of observations from \( \mu \), if \( M \) is very large, the resulting algorithm might become computationally too expensive. In this setting, we propose the following batch strategy to approximate \( \mu \): fix \( N \) and at each time step approximate the integral with respect to \( \mu \) using \( m \) samples where \( m \) is the smallest between the number of particles \( N \) chosen for the particle system (12) and the total number of samples available from \( \mu \); if \( N \) is larger than the total number of observed samples from \( \mu \) then the whole sample is used at each iteration, if \( N \) is smaller we resample without replacement \( m \) times from the empirical measure of the sample. This amounts to a maximum cost of \( \mathcal{O}(N^2) \) per time step.

In cases in which we have access to a sampling mechanism from \( \mu \), and we can therefore draw \( m \) samples at each iteration, we simply set \( m = N \).

Choice of \( n_T \) It is straightforward to choose the number of time steps \( n_T \), adaptively. The number of steps necessary to give convergence of (12) to its stationary distribution is estimated by approximating the value of \( \mathcal{G}_n \) through numerical integration as in (18) and once the value of \( \mathcal{G}_n \) stops decreasing, a minimizer has apparently been reached and the iteration in Algorithm 1 can be stopped.

5 Experiments

We test FE-WGF on a number of examples of Fredholm integral equations (1) and compare our density estimator (17) with the reconstructions given by state-of-the-art methods for each experiment.

First, in Section 5.1 we consider a classical problem in statistics, density deconvolution. In this case, \( k(x, y) = k(y - x) \) and (1) models the task of reconstructing the density of a random variable \( X \) from noisy measurements \( Y \). In Section 5.2 we test FE-WGF on a problem drawn from epidemiology. More precisely, we aim at reconstructing the incidence profile of a disease from the observed number of cases. This is a particular instance of the deconvolution problem in which a smoothness constraint on the solution \( \pi \) of (1) is necessary to make the estimator robust to noise in the observations \( Y \) [77]. In Section 5.3, we consider an application to computed tomography (CT), in which a cross-sectional image of the lungs needs to be recovered from the radial measurements provided by the CT scanner. Finally, in Section 5.4, we study how the performances of the proposed method scale with the dimensionality of the support of \( \pi \) on a toy model.

In most examples, \( \mu \) is known through a sample and at each iteration we resample \( m \) times from its empirical distribution. For image restoration problems we consider the observed distorted image as the empirical distribution of a sample from \( \mu \) and use it to draw \( m \) samples at each iteration in Algorithm 1. In order to implement FE-WGF we follow the guidelines provided in Section 4.3. Julia code to reproduce all examples is available online at https://github.com/FrancescaCrucinio/FE_WGF.

5.1 Density Deconvolution

The focus of this section is on deconvolution problems, i.e. those in which \( \mu \) and \( \pi \) satisfy (1) with \( k(x, y) = k(y - x) \). In particular, we consider the case in which we want to recover the density of a
Figure 1: Average accuracy and runtime for FE-WGF, SMC-EMS and DKDE with number of particles $N$ ranging between $10^2$ and $10^4$. The shaded regions represent an interval of two standard deviations over 100 repetitions centred at the average ISE.

random variable $X$ from observations of $Y = X + Z$ where $Z$ is a zero mean error variable independent of $X$. The deconvolution problem is widely studied, and recently a number of approaches based on normalizing flows [39], deep learning [3], and generative adversarial networks [34] have been explored.

We consider the Gaussian mixture model of [69] (see also [31]) where, with a slight abuse of notation, we denote both a measure and its density w.r.t. the Lebesgue measure with the same symbol:

$$\pi(x) = (1/3)\mathcal{N}(x; 0.3, 0.015^2) + (2/3)\mathcal{N}(x; 0.5, 0.043^2),$$

$$k(x, y) = \mathcal{N}(y; x, 0.045^2),$$

$$\mu(y) = (1/3)\mathcal{N}(y; 0.3, 0.045^2 + 0.015^2) + (2/3)\mathcal{N}(y; 0.5, 0.045^2 + 0.043^2),$$

and compare our method with the deconvolution kernel density estimators with plug-in bandwidth (DKDEpi) [37], a class of estimators for deconvolution problems which given a sample from $\mu$ produce a kernel density estimator for $\pi$, and SMC-EMS, a particle method to solve (1) which achieves state-of-the-art performance.

We check the accuracy of the reconstructions through the integrated square error

$$\text{ISE}(\hat{\pi}) = \int \{\pi(x) - \hat{\pi}(x)\}^2 dx,$$  \hspace{1cm} (20)

with $\hat{\pi}$ an estimator of $\pi$. Even if the analytic form of $\mu$ is known, for SMC-EMS and (17) we draw a sample of size $M = 10^3$ and draw without replacement $m = \min(N, M)$ samples at each iteration; for DKDE we draw a sample of size $N$ for each $N$.

Both SMC-EMS and the estimator (17) require specification of a number of parameters: we fix the number of time steps for both algorithms to $n_T = 100$ as we observed that convergence occurs in less than 100 iterations, the initial distribution is that of the samples from $Y$, since we expect the distribution of $Y$ to be close to that of $X$. The regularization parameter $\alpha$ for FE-WGF and the smoothing parameter for SMC-EMS are chosen through cross validation, using (5) as target functional for the former and the Kullback–Leibler divergence (2) for the latter. To implement FE-WGF we also need to specify the reference measure $\pi_0$, a Gaussian with mean and variance given by the empirical mean and variance of $Y$, and the time step $\gamma = 10^{-3}$.

We consider different particle sizes (from $10^2$ to $10^4$) and compare reconstruction accuracy and total runtime of SMC-EMS, FE-WGF and DKDE (Figure 1). In particular, we compare the cost per iteration since we run both algorithms for a fixed number of steps, however, we found that SMC-EMS and FE-WGF converge in a similar number of steps for this example. The computational cost could be reduced by considering stopping criteria using approximations of the functional $\mathcal{G}_n^\alpha$. The performances of FE-WGF and SMC-EMS are similar both in terms of runtime and in terms of accuracy, with FE-WGF generally giving more accurate reconstructions for the same computational cost.
Figure 2: Distribution of MSE as a function of runtime (in log seconds) for FE-WGF, SMC-EMS and DKDE. The number of particles \( N \) ranges between 100 and 10,000.

The DKDE-pi estimator has runtime comparable to that of SMC-EMS and FE-WGF, but poorer accuracy. Despite being better than SMC-EMS on average, the accuracy of FE-WGF has a slightly higher variance (about twice that of SMC-EMS), while DKDE-pi has still higher variance (about three times that of SMC-EMS).

As both SMC-EMS and FE-WGF are regularized versions of the inconsistent maximum likelihood estimator for \( \pi \), it is natural to compare the smoothness of the reconstructions. To characterize smoothness, we take 100 points \( x_c \) in the support of \( \pi \) and approximate (with 100 replicates) the mean squared error

\[
\text{MSE}(x_c) = \mathbb{E} \left[ (\pi(x_c) - \hat{\pi}(x_c))^2 \right].
\]

The MSE measures locally the fit of the reconstructions to the solution \( \pi \), which is known to be smooth, providing information on the (relative) smoothness of the reconstructions.

The distribution of the MSE over the 100 points (Figure 2) shows that SMC-EMS and FE-WGF can achieve considerably lower MSE than DKDE-pi, and therefore that the smoothness of the reconstructions provided by these two methods is closer to that of the true density \( \pi \).

5.2 Epidemiology

In epidemiology, Fredholm integral equations link the unknown infection incidence curve \( \pi \) to the number of reported cases, deaths or hospitalizations. This is a particular instance of the deconvolution problem, in which the kernel \( k(x, y) = k(y - x) \) describes the delay distribution between time of infection and time of death or hospitalization. Deconvolution techniques have been used to infer \( \pi \) in the case of HIV [9] and influenza [45] and have been recently applied to estimate the incidence curve of COVID-19 [71, 103, 26, 77].

As an example, we take the spread of the pandemic influenza in the city of Philadelphia between the end of September and the beginning of October 1918 [45]. The count of daily deaths and the distribution of delay between infection and death are available through the \texttt{R} package \texttt{incidental} [77].

To obtain a parametric form for \( k \) we fit a mixture of Gaussians to the delay data using the expectation maximization algorithm (\texttt{normalmixEM} function in \texttt{R}; [10])

\[
k(x, y) = 0.595\mathcal{N}(y - x; 8.63, 2.56^2) + 0.405\mathcal{N}(y - x; 15.24, 5.39^2).
\]

Although this choice assigns \( \approx 10^{-3} \) mass to the negative reals, we found that a mixture of Gaussians fits the observed delay distribution better than other commonly used distributions (e.g. Gamma, log-normal, see [83]) and also benefits from a bounded derivative and therefore satisfies \textit{A1} and \textit{A2}.
Figure 3: Example fit of the reconstructions of the synthetic incidence curve (21) and of the corresponding reconstruction of the number of cases.

Table 1: Comparison of reconstructions of the synthetic incidence curve (21), the average reconstruction accuracy and fit to data over 100 repetitions are reported.
by convolution of the estimates of $\pi$ with $k$

$$\mu^{rec}(y) = \int k(y - x)d\hat{\pi}(x).$$

The RL algorithm gives the best fit the observed death counts and has the lowest runtime (< 1 second), however, the reconstruction of the incidence curve is not smooth and sensitive to noise [77]. RIDE, SMC-EMS and FE-WGF address the lack of smoothness introducing regularization and provide smoother reconstructions (Figure 3) which are considerably closer to the incidence curve both in the well-specified and in the misspecified setting (Table 1) with FE-WGF achieving the best fit to $\pi$. The runtime of RIDE and FE-WGF are comparable, while SMC-EMS is cheaper as convergence is achieved in only 100 steps, and the reconstructions do not change considerably after that. This is not unexpected, as the reconstructions provided by RIDE, SMC-EMS and FE-WGF are a regularized version of the measure $\pi$ minimizing the Kullback–Leibler divergence (2), while that given by RL minimizes (2) with no regularization and inherits the well-known inconsistency of the maximum likelihood estimator in the infinite dimensional setting [66].

Comparing SMC-EMS and FE-WGF, we find that the latter always achieves better accuracy, but the increase in accuracy is more pronounced for the misspecified model. In fact, even if SMC-EMS and FE-WGF both aim at minimizing a regularized Kullback–Leibler divergence (2), the latter minimizes (3) which allows us to impose that the reconstructions are close (in the Kullback–Leibler sense) to the reference measure $\pi_0$, which for this example is particularly informative, since the corresponding term in the drift (13) pushes the particles towards the mode of the distribution of the number of cases shifted back by 9 days. On the contrary, SMC-EMS does not allow us to exploit this information, and $\pi_0$ is only used to initialize the particle system for SMC-EMS.

### 5.3 Computed Tomography

Fredholm integral equations of the first kind find wide application in medical imaging [105]; in this context they model the reconstruction of cross-section images of the organ of interest from the noisy measurements provided by positron emission tomography (PET) and CT scanners.

CT scanners provide noisy measurements by mapping each point of the organ’s cross-section $\pi$ onto its radial projection $\mu$ defined for angles $\phi$ between 0 and $2\pi$ and depths $\xi \in \mathbb{R}$ with 0 corresponding to the centre of the scanner. The distribution of the radial projections $\mu$ is linked to the cross-section image of the organ of interest $\pi$ through the Radon transform [87], i.e. for any $\phi \in [0, 2\pi]$ and $\xi \in \mathbb{R}$

$$\mu(\phi, \xi) = \int_{\mathbb{R}} \pi(\xi \cos \phi - t \sin \phi, \xi \sin \phi + t \cos \phi) dt,$$  

(22)

(with a slight abuse of notation we denote both a measure and its density w.r.t. the Lebesgue measure with the same symbol) where the right hand side is the line integral along the line with equation $x \cos \phi + y \sin \phi = \xi$. We model the alignment between the projections onto $(\phi, \xi)$ and the corresponding location $(x, y)$ in the reference image in (22) using a Gaussian distribution with small variance, renormalized to integrate to 1 for fixed $(x, y) \in \mathbb{R}^2$ and obtain the following Fredholm integral equation (1)

$$\mu(\phi, \xi) = \int_{\mathbb{R}^2} (2\pi \sigma^2)^{-1} \exp\left(\frac{|x \cos \phi + y \sin \phi - \xi|^2}{2\sigma^2}\right) \pi(x, y) dxdy,$$
We test FE-WGF on a 512 × 512 pixels lung CT scan from the LIDC-IDRI database [6] (in particular, scan LIDC_IDRI_0683_L_048). The data image is obtained using the ASTRA toolbox [100] by projecting the CT scan at 512 equally spaced angles \( \phi \) in \([0, 2\pi]\) and at 729 depths, and then corrupting the obtained projections with Poisson noise. We compare the reconstructions given by FE-WGF with those obtained by filtered back projection (FBP), one of the most common methods for analytic image reconstruction [99], available in the ASTRA toolbox [100].

The reference measure is a Gaussian distribution with mean corresponding to the centre of the image and variance \( s^2 \)Id with \( s > 0 \) large enough to approach a uniform distribution over the image (in our case \( s^2 = 0.35^2 \)), this guarantees that \( \mathcal{A}^3 \) is satisfied. The number of particles \( N = 10^4 \) and the time step \( \gamma = 10^{-3} \) are chosen to achieve high enough resolution, the regularization parameter \( \alpha = 7 \times 10^{-3} \) is selected by cross validation over \( L = 5 \) replicates. Convergence is measured by numerically approximating \( \mathcal{G}^\alpha \) and occurs in 200 steps.

The reconstructions provided by FBP are less robust to noise in the data, which reflects in a higher ISE, on the contrary FE-WGF is more robust to noise and achieves lower ISE. However, the reconstructions obtained with FE-WGF rarely result in sharp edges due to the inherent features of the estimator (17) (i.e. the diffusive behaviour of (12) and the kernel density estimator).

5.4 Scaling with dimension

To explore the scaling with the dimension \( d \) of the support of \( \pi \) we take the generalization of the Gaussian mixture model in Section 5.1 described in [31] (with a slight abuse of notation we denote both a measure and its density with the same symbol)

\[
\begin{align*}
\pi(x) &= (1/3)\mathcal{N}(x; 0.3 \cdot 1_d, 0.07^2 \text{Id}_d) + (2/3)\mathcal{N}(x; 0.7 \cdot 1_d, 0.1^2 \text{Id}_d), \\
\kappa(x, y) &= \mathcal{N}(y; x, 0.15^2 \text{Id}_d), \\
\mu(y) &= (1/3)\mathcal{N}(y; 0.3 \cdot 1_p, (0.07^2 + 0.15^2) \text{Id}_p) + (2/3)\mathcal{N}(y; 0.7 \cdot 1_p, (0.1^2 + 0.15^2) \text{Id}_p),
\end{align*}
\]

where \( p = d \) and \( 1_d, \text{Id}_d \) denote the unit function in \( \mathbb{R}^d \) and the \( d \times d \) identity matrix, respectively.

We compare the reconstructions given by FE-WGF with those obtained with the one-step-late expectation maximization (OSL-EM) algorithm [48], an iterative algorithm for maximum penalized likelihood estimation, and SMC-EMS [31]. First, we focus on the comparison with OSL-EM since this algorithm can be implemented to minimize \( \mathcal{G}^\alpha \) in (5) as does FE-WGF. In particular, FE-WGF is a stochastic approach to the problem of minimizing (3), while OSL-EM relies on a deterministic discretization of \( \pi \). Secondly, we compare the proposed approach with SMC-EMS [31].

5.4.1 Comparison with OSL-EM

As briefly discussed in Section 2.2, the first term in the definition of the functional \( \mathcal{G}^\alpha \) in (4) (with \( \mu \) replaced by \( \mu^M \)) is the likelihood associated with the incomplete-data \( \{y^k, M^k\}_{k=1}^B \). Minimizing \( \mathcal{G}^\alpha \) is equivalent to maximizing a penalised likelihood with penalty \( \text{KL}(\pi|\pi_0) \). The one-step-late expectation maximization (OSL-EM) algorithm [48] is a variant of the well-known EM algorithm [38] to maximize any penalized likelihood.

In order to implement OSL-EM we need to discretize the support of \( \pi \), which essentially requires \( \pi \) to have compact support. This is not the case for the measure \( \pi \) considered in this example, however, \( \int_{[0,1]^d} |\pi(x)dx - 1| \ll 10^{-2} \) for \( d \leq 10 \); therefore we define a discretization grid for OSL-EM over the \( d \)-dimensional hypercube \([0,1]^d \) (similar considerations apply to \( \mu \)). We consider \( B \in \mathbb{N} \) equally spaced bins for each dimension and obtain the discretizations of \( \pi \) and \( \mu \), \( \{\pi_{b}^k\}_{k=1}^B, \{\mu_{b}^k\}_{k=1}^B \), setting \( \pi_{b} = \pi(x_b) \), where \( x_b \) is the centre of the \( b \)-th bin. The same discretization mechanism is used for \( k \) to obtain \( \{k_{b,c}\}_{b,c=1}^B \), where \( k_{b,c} = k(x_b, x_c) \). The one-dimensional OSL-EM iteration is given for any \( n \in \mathbb{N} \) by

\[
\pi_{b}^{(n+1)} = \pi_{b}^{(n)}/(1 + \alpha(1 + \log \pi_{b}^{(n)} - \log \pi_{0,b})) \sum_{c=1}^B \mu_{c} k_{b,c} / \sum_{d=1}^B \pi_{d}^{(n)} k_{d,c},
\]

where \( \pi_{0,b} = \pi_0(x_b) \), for \( b \in \{1, \ldots, B^d\} \). Contrary to the estimator provided by FE-WGF, the reconstructions provided by OSL-EM are deterministic and might not converge to the unique minimizer of \( \mathcal{G}^\alpha \). For the mixture model in (23), we observe empirical convergence to a unique fixed point for reasonable values of \( \alpha \), e.g. \( \alpha < 1 \).

To compare OSL-EM and FE-WGF we fix \( \alpha = 0.01 \) (as both algorithms aim at minimizing \( \mathcal{G}^\alpha \) the value of \( \alpha = 0.01 \) is chosen to compare them in a regime of practical interest, with the aim of obtaining good
Figure 5: Distribution of ISE ratios (ISE of OSL-EM divided by ISE of FE-WGF) over 100 repetitions. The runtime of FE-WGF is 1.3 times that of OSL-EM on average, while the average gain ranges from 1.3 for \( d = 1 \) to \( \approx 30 \) for \( d = 3, 4 \).

approximations of \( \pi \), set the reference measure \( \pi_0 \) to be a Gaussian distribution with mean \((0.5, \ldots, 0.5)\) and covariance matrix \( \sigma_0^2 \mathbb{I}_d \) with \( \sigma_0^2 = 0.25^2 \) and use \( \pi_0 \) as initial condition for FE-WGF. With this choice of parameters, convergence (assessed empirically by approximating the value of \( G(\eta) \)) occurs within 50 steps for both algorithms and \( d \in \{1, \ldots, 4\} \). The number of particles is set to \( N = 10^4 \), and we obtain the number of bins per dimension, \( B \), so that \( B^d \approx N \), and the two algorithms have similar runtime. This corresponds to coarser discretizations as \( d \) increases which give low resolution reconstructions, for instance, for \( d = 4 \) each dimension is discretized into \( B = 10 \) bins while for \( d = 1 \) we have \( B = N = 10^4 \) bins; however, such fine discretization schemes are impractical for \( d \gg 1 \) since the number of points at which \( \pi \) needs to be approximated grows exponentially with \( d \).

To assess the quality of the reconstructions we use the integrated squared error (20) of the first marginal, \( \text{ISE}(\hat{\pi}_1) \), where \( \hat{\pi} \) is the estimated density. Since OSL-EM is a deterministic algorithm we take its ISE as reference and investigate the gains obtained by using FE-WGF for dimension \( d = 1, 2, 3, 4 \), see Figure 5. For \( d = 1, 2 \) the gain with respect to OSL-EM is moderate, but for \( d = 3, 4 \) the reconstructions obtained with FE-WGF are at least 10 times more accurate than those obtained by OSL-EM with an average runtime which is only 1.3 times that of OSL-EM.

5.4.2 Comparison with SMC-EMS

We now turn to comparing the reconstructions obtained with FE-WGF with those given by SMC-EMS [31]. Both methods approximate \( \pi \) through a family of interacting particles from which smooth estimates can be obtained via standard kernel density estimation as in (17). Since typical applications of Fredholm integral equations are low-dimensional, mostly one to three dimensional with some notable exceptions [92], we focus on \( d \) up to 10.

To compare the two algorithms we consider \( N = 10^3 \) and we assume that \( \mu \) is not known but a sample of size \( M = 10^6 \) is available and resample without replacement \( m = N = 10^3 \) times from it at each iteration of both SMC-EMS and FE-WGF. SMC-EMS converges quickly to an approximation of \( \pi \) [31], for the model in (23) we observe convergence in less than 50 steps (measured by numerically approximating the Kullback–Leibler divergence (2) using the samples from \( \mu \) and \( \hat{\pi} \), the estimated distribution). For FE-WGF we set \( \gamma = 10^{-2} \) and select the reference measure \( \pi_0 \) to be a Gaussian distribution with mean \((0.5, \ldots, 0.5)\) and covariance matrix \( \sigma_0^2 \mathbb{I}_d \) with \( \sigma_0^2 = 0.25^2 \); with this choice of parameters convergence occurs in less than 50 iterations also for FE-WGF. To specify the smoothing parameter for SMC-EMS and \( \alpha \) for FE-WGF we use the cross-validation approach in Section 4.3 with \( L = 10 \).

Since kernel density estimators do not perform well for \( d \gg 1 \) [93], we do not build the kernel density estimator (17) and compare the reconstructions obtained by SMC-EMS and FE-WGF using the particle population given by each method as an approximation for \( \pi \) (as justified by Proposition 9), and compute the Wasserstein-1 distance between the first marginal \( \pi_1 \) and its particle approximation (Figure 6).
Figure 6: Reconstruction accuracy over 100 of SMC-EMS and FE-WGF for the Gaussian mixture model in (23). The left panel shows the distribution of the Wasserstein-1 distance between the first marginal of $\pi$ and its particle approximations for $d = 2, \ldots, 10$, the solid lines highlight the average accuracy. The right panel reports average runtime.

SMC-EMS and FE-WGF give similar results for all $d$, with the average runtime of SMC-EMS being slightly lower than that of FE-WGF for $d = 2, 3, 4$ and about 15% lower for $d \geq 5$. The increase in runtime for $d \geq 5$ is caused by the computation of the norm of the drift for the tamed Euler scheme (15) and is likely due to the increased algebraic complexity of this operation with dimension. In terms of accuracy, FE-WGF performs slightly better than SMC-EMS with $W_1$ distance 5% smaller on average for FE-WGF.

6 Discussion

We consider a probabilistic approach to the solution of Fredholm integral equations of the first kind (1) by introducing the minimization problem (3) and its surrogate (5). Under mild smoothness assumptions on the kernel $K$ and the reference measure $\pi_0$, we show that the surrogate functional admits a unique minimizer which is stable w.r.t. $\mu$. Leveraging a Wasserstein gradient flow construction, we introduce an interacting particle system (12) which approximates the minimizer of the surrogate functional and is ergodic for any finite $N$. Combining standard propagation of chaos results and strong convergence of tamed Euler–Maruyama schemes we obtain the bound (19) which provides a practical guideline for the selection of the time discretization step $\gamma$, the number of particles $N$ and the number of samples $m$ from $\mu$ to use at each iteration.

In Section 5 we compare FE-WGF with both problem specific and state-of-the-art methods to solve (1); specifically, we compare our algorithm with SMC-EMS, a particle method to solve (1) which also aims at minimizing a regularized Kullback–Leibler divergence. This particle method has been shown to achieve state-of-the-art performances in a number of settings and to scale with the dimension of the state space $d$ better than standard discretization based algorithms [31]. Our results show that FE-WGF has always comparable or superior performances to problem specific algorithms and outperforms SMC-EMS in high dimensional setting. Moreover, FE-WGF allows the introduction of a reference measure $\pi_0$, which can be particularly beneficial when some features of the measure $\pi$ to be reconstructed are known (as, for instance, in epidemiology applications as that in Section 5.2).

From a theoretical perspective, Proposition 2 and the convergence properties in Section 3 guarantee that the particle system (12) approximates the unique minimizer of (5); an equivalent uniqueness property has not yet been established for SMC-EMS. However, the results presented in Section 5 also provide some additional confidence in the use of SMC-EMS, demonstrating empirically that it recovers comparable solutions to the provably convergent method developed in this paper in a range of settings.

Moreover, while both algorithms minimize a regularized Kullback–Leibler divergence, and for small values of the regularization parameters the reconstruction they provide are similar, the methods through which they approximate the minimizer present some notable differences. FE-WGF is based on a numerical approximation of the MKVSDE (10) and therefore the associated particle system has a diffusive behaviour, while SMC-EMS uses a resampling mechanism to eliminate the particles which are not in the support of $\pi$ and replicate those close to high-probability regions. The effect of this different algorithmic approach can be observed by considering different initial distributions for $\{X_{0,N}^k\}_{k=1}^N$: SMC-EMS benefits from overly
diffused initial distributions, since the resampling mechanism allows to quickly move the particles towards high probability regions, for concentrated distributions (e.g. Dirac delta) convergence still occurs but in a larger number of iterations [31, Appendix E.1]. The convergence speed of FE-WGF is less influenced by the initial state \( \{X_{k,N}^{t} \}_{k=1}^{N} \) as shown in Appendix D. In contrast with SMC-EMS, FE-WGF requires specification of a reference measure \( \pi_0 \). While this might seem a limitation of our method, we found that in practice we often have sufficient knowledge about the problem at hand to build \( \pi_0 \). For instance, in image processing applications usually a large number of images needs to be processed and it is possible to build reference measures using previously processed images (e.g. [62]), while in epidemiology, one can exploit previous studies on similar diseases, since they often present incidence curves with the same characteristics (e.g. [45]). In fact, it is in this latter setting that we find that FE-WGF outperforms SMC-EMS in terms of accuracy (Section 5.2).

FE-WGF describes one possible numerical approximation of the particle system (12), however, we anticipate that more sophisticated time discretization schemes will prove beneficial in some settings. For instance, in our experiments we observed that in the first few iterations the drift component in (12) has larger values, and pushes the particles towards high \( \pi \)-probability regions; after the first iterations, the magnitude of the drift component decreases and the shape of the approximated solution is refined. In terms of the functional \( \mathcal{F}^\alpha_{\eta} \), the first few iterations correspond to a steepest decrease in \( \mathcal{F}^\alpha_{\eta} \), which then stabilizes around the minimum (see e.g., Appendix D). To speed up convergence to the minimizer, one could consider having a larger time step \( \gamma \) for earlier iterations, and then gradually reduce \( \gamma \) to refine the approximation. More sophisticated adaptive strategies for MKVSDE have been recently studied in [88].

In conclusion, FE-WGF offers a general recipe to obtain regularized solutions to Fredholm integral equations of the first kind, which can be refined exploiting specific knowledge of the problem at hand; for example, choosing informative reference measures \( \pi_0 \) or adapting the time discretization step \( \Delta t \) to the magnitude of the drift (13).

Acknowledgments

FRC, VDB, AD and AMJ acknowledge support from the EPSRC (grant # EP/R034710/1). AMJ acknowledges further support from the EPSRC (grant # EP/T004134/1) and the Lloyd’s Register Foundation Programme on Data-Centric Engineering at the Alan Turing Institute. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising from this submission.

Data access statement: No new data was created during this research.

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A Proofs of Section 2

We start this section by recalling some basic facts on Γ-convergence which will be used in the proof of Proposition 2 and Proposition 3 below. We refer to [33] for a more complete introduction to Γ-convergence.

A.1 Basics on Γ-convergence

First, we recall that a function \( f : X \to \mathbb{R} \) (where \( X \) is a metric space) is coercive if for any \( t \in \mathbb{R} \), \( f^{-1}((-\infty, t]) \) is relatively compact. This definition can be extended to the case where \( X \) is only a topological space, see [33, Definition 1.12].

Let \( F = \{ f_\alpha : \alpha \in A \} \) where \( A \) is a topological space and for any \( \alpha \in A \), \( f_\alpha : X \to \mathbb{R} \cup \{ +\infty \} \) with \( X \) a metric space. We say that \( F \) is a \( \Gamma \)-system if the following hold:

(a) For any \( x \in X \), \( \alpha^* \in A \) with \( \lim_{n \to +\infty} \alpha_n = \alpha^* \) and \( (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \) with \( \lim_{n \to +\infty} x_n = x \) we have \( \liminf_{n \to +\infty} f_{\alpha_n}(x_n) \geq f_{\alpha^*}(x) \).

(b) For any \( x \in X \), \( \alpha^* \in A \), \( (\alpha_n)_{n \in \mathbb{N}} \in A^\mathbb{N} \) with \( \lim_{n \to +\infty} \alpha_n = \alpha^* \) there exists \( (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \) with \( \lim_{n \to +\infty} x_n = x \) such that \( \lim_{n \to +\infty} f_{\alpha_n}(x_n) = f_{\alpha^*}(x) \).

Under some additional coercivity assumption (see Section 1 for a definition), we have the following result, see [33, Corollary 7.24] for a proof.

**Proposition 13.** Let \( F = \{ f_\alpha : \alpha \in A \} \) where \( A \) is a topological space and for any \( \alpha \in A \), \( f_\alpha : X \to \mathbb{R} \cup \{ +\infty \} \) with \( X \) a metric space. Assume that \( F \) is a \( \Gamma \)-system, is equicoercive and is such that for any \( \alpha \in A \), \( f_\alpha \) admits a unique minimizer. Then for any \( (\alpha_n)_{n \in \mathbb{N}} \in A^\mathbb{N} \) and \( (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \) such that for any \( n \in \mathbb{N} \), \( x_n \) is the minimizer of \( f_{\alpha_n} \) and \( \lim_{n \to +\infty} \alpha_n = \alpha^* \), we have that \( \lim_{n \to +\infty} x_n = x^* \) the minimizer of \( f_{\alpha^*} \).

The equicoercivity condition can be removed if we assume that the sequence \( (x_n)_{n \in \mathbb{N}} \) converges, see [33, Corollary 7.20].

**Proposition 14.** Let \( F = \{ f_\alpha : \alpha \in A \} \) where \( A \) is a topological space and for any \( \alpha \in A \), \( f_\alpha : X \to \mathbb{R} \cup \{ +\infty \} \) with \( X \) a metric space. Assume that \( F \) is a \( \Gamma \)-system and is such that for any \( \alpha \in A \), \( f_\alpha \) admits a unique minimizer. Then for any \( (\alpha_n)_{n \in \mathbb{N}} \in A^\mathbb{N} \) and \( (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \) such that for any \( n \in \mathbb{N} \), \( x_n \) is the minimizer of \( f_{\alpha_n} \) and \( \lim_{n \to +\infty} \alpha_n = \alpha^* \), if there exists \( x^* \in X \) such that \( \lim_{n \to +\infty} x_n = x^* \) then \( x^* \) is a minimizer of \( f_{\alpha^*} \).

A.2 Proof of Proposition 1

We recall that for any \( \eta \geq 0 \) and \( \pi \in \mathcal{P}(\mathbb{R}^d) \)

\[
\mathcal{G}^\eta(\pi) = -\int_{\mathbb{R}^d} \log(\pi[k(\cdot, y)] + \eta)d\mu(y).
\]

We divide the proof into two parts.

(a) Let \( \eta \geq 0 \). We first show that \( \mathcal{G}^\eta \) is lower bounded. For any \( \pi \in \mathcal{P}(\mathbb{R}^d) \) we have using A1

\[
\int_{\mathbb{R}^d} \mathbb{I}_{[1, +\infty)}(\pi[k(\cdot, y)] + \eta) \log(\pi[k(\cdot, y)] + \eta)d\mu(y) \leq \max(\log(\mathbb{M} + \eta), 0).
\]

Therefore, \( \mathcal{G}^\eta : \mathcal{P}(\mathbb{R}^d) \to [-\max(\log(\mathbb{M} + \eta), 0), +\infty) \). \( \mathcal{G}^\eta \) is convex since \( t \mapsto -\log(t + \eta) \) is convex and \( \pi \mapsto \pi[k(\cdot, y)] \) is a linear function of \( \pi \) (and therefore convex too). We now show that \( \mathcal{G}^\eta \) is lower semi-continuous. Let \( (\pi_n)_{n \in \mathbb{N}} \) be such that \( \lim_{n \to +\infty} \pi_n = \pi \in \mathcal{P}(\mathbb{R}^d) \). Then, since for any \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) with \( k(x, y) \in [0, \mathbb{M}] \) and \( k \in C(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}) \) we have for any \( y \in \mathbb{R}^d \), \( \lim_{n \to +\infty} \pi_n[k(\cdot, y)] = \pi[k(\cdot, y)] \). For any \( n \in \mathbb{N} \) and \( y \in \mathbb{R}^d \) we have

\[
\mathbb{I}_{[1, +\infty)}(\pi_n[k(\cdot, y)] + \eta) \log(\pi_n[k(\cdot, y)] + \eta) \leq \max(\log(\mathbb{M} + \eta), 0).
\]

Therefore, using the reverse Fatou’s lemma and the fact that \( t \mapsto \mathbb{I}_{[1, +\infty)}(t) \) is upper semi-continuous on \([0, +\infty)\) we obtain that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^d} \mathbb{I}_{[1, +\infty)}(\pi_n[k(\cdot, y)] + \eta) \log(\pi_n[k(\cdot, y)] + \eta)d\mu(y)
\]

\[
\leq \int_{\mathbb{R}^d} \mathbb{I}_{[1, +\infty)}(\pi[k(\cdot, y)] + \eta) \log(\pi[k(\cdot, y)] + \eta)d\mu(y).
\]

Using Fatou’s lemma and the fact that \( t \mapsto -\mathbb{I}_{[0, 1]}(t) \) is lower semi-continuous on \([0, +\infty)\), we have

\[
\liminf_{n \to +\infty} -\int_{\mathbb{R}^d} \mathbb{I}_{[0, 1]}(\pi_n[k(\cdot, y)]) \log(\pi_n[k(\cdot, y)])d\mu(y)
\]

\[
\geq -\int_{\mathbb{R}^d} \mathbb{I}_{[0, 1]}(\pi[k(\cdot, y)]) \log(\pi[k(\cdot, y)])d\mu(y).
\]
Hence, combining (24), (25), we get that lim inf $\mathcal{G}^n(\pi_n) \geq \mathcal{G}^n(\pi)$ and $\mathcal{G}^n$ is lower semi-continuous.

We now show that $\mathcal{G}^n$ is not coercive. Let $\beta = \log(2) - \int_{\mathcal{P}} \log(\kappa(0, y))d\mu(y)$ and $\Pi = \{\pi_\infty = (\delta_0 + \delta_R)/2 \in \mathcal{P}(\mathbb{R}^d) : R > 0\}$. Then, for any $R \geq 0$, $\pi_\infty(\mathbb{B}(0, R/2)^c) = 1/2$, where $\mathbb{B}(0, R/2)$ denotes the ball of radius $R/2$ centred at $0$. Hence $\Pi$ is not relatively compact in $\mathcal{P}(\mathbb{R}^d)$. However, for any $R \geq 0$, we have

$$
\mathcal{G}^n(\pi_\infty) = -\int_{\mathcal{P}} \log((\kappa(0, y) + k(R, y))/2 + \eta)d\mu(y) \leq \log(2) - \int_{\mathcal{P}} \log(\kappa(0, y))d\mu(y) \leq \beta.
$$

Hence, $\Pi \subset (\mathcal{G}^n)^{-1}((-\infty, \beta])$ and $(\mathcal{G}^n)^{-1}((-\infty, \beta])$ is not relatively compact which implies that $\mathcal{G}^n$ is not coercive.

(b) Second, let $\eta > 0$. Then, we have for any $y \in \mathbb{R}^p$ and $\pi \in \mathcal{P}(\mathbb{R}^d)$

$$
|\log(\pi[k(\cdot, y)])| \leq \max(|\log(\eta)|, |\log(M + \eta)|).
$$

Hence, $\mathcal{G}^n$ is proper. Let $(\pi_n) \in (\mathcal{P}(\mathbb{R}^d))^\mathbb{N}$ be such that $\lim_{n \to +\infty} \pi_n = \pi \in \mathcal{P}(\mathbb{R}^d)$. Since for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^p$, $|k(x, y)| \leq M$ and $k \in C(\mathbb{R}^d \times \mathbb{R}^p, \mathbb{R})$, we have that $\lim_{n \to +\infty} \pi_n[k(\cdot, y)] = \pi[k(\cdot, y)]$. Combining this result, (26) and the dominated convergence theorem we get that $\lim_{n \to +\infty} \mathcal{G}^n(\pi_n) = \mathcal{G}^n(\pi)$. Therefore, we have that $\mathcal{G}^n \in C(\mathcal{P}(\mathbb{R}^d), \mathbb{R})$.

### A.3 Proof of Proposition 2

We recall that for any $\alpha, \eta \geq 0$ and $\pi \in \mathcal{P}(\mathbb{R}^d)$

$$
\mathcal{G}^n_\alpha(\pi) = -\int_{\mathcal{P}} \log(\pi[k(\cdot, y)]) + \eta)d\mu(y) + \alpha\text{KL}(\pi[\pi_0]).
$$

Since, for $\eta > 0$, $\mathcal{G}^n$ is proper as shown in Proposition 1–(b) and $\text{KL}(\pi[\pi_0]) \geq 0$ with equality for $\pi = \pi_0$, $\mathcal{G}_\alpha^0$ is proper too. [40, Lemma 1.4.3-(b)] guarantees that $\text{KL}(\pi[\pi_0])$ is strictly convex and lower semi-continuous. Combining this with the results on $\mathcal{G}^n$ in Proposition 1 we find that $\mathcal{G}^n_\alpha$ is strictly convex and lower semi-continuous.

To see that $\mathcal{G}^n_\alpha$ is coercive observe that $\mathcal{G}^n_\alpha$ is the sum of the lower bounded lower semi-continuous functional $\mathcal{G}^n$ and of the coercive functional $[40, \text{Lemma } 1.4.3-(c)] \text{KL}(\pi[\pi_0])$, then for any $\beta \in \mathbb{R}$

$$
S := \left\{ \pi \in \mathcal{P}(\mathbb{R}^d) : \mathcal{G}^n(\pi) + \alpha\text{KL}(\pi[\pi_0]) \leq \beta \right\}
$$

$$
\subseteq \left\{ \pi \in \mathcal{P}(\mathbb{R}^d) : \alpha\text{KL}(\pi[\pi_0]) \leq \beta - \max(|\log(\eta)|, |\log(M + \eta)|) \right\} := \tilde{S},
$$

since $|\mathcal{G}^n(\pi)| \leq \max(|\log(\eta)|, |\log(M + \eta)|)$. $\tilde{S}$ is relatively compact since $\text{KL}(\pi[\pi_0])$ is coercive and thus $S$ is also relatively compact, showing that $\mathcal{G}^n_\alpha$ is coercive. Hence, for any $\alpha, \eta > 0$, $\mathcal{G}^n_\alpha$ admits a unique minimizer denoted $\pi_{\alpha, \eta}$.

For any $\tilde{\alpha}, \tilde{\eta} > 0$, $\pi \in \mathcal{P}(\mathbb{R}^d)$, let $(\alpha_n)_{n \in \mathbb{N}} \in (0, +\infty)^\mathbb{N}$ be such that $\lim_{n \to +\infty} \alpha_n = \tilde{\alpha}$ and $(\eta_n)_{n \in \mathbb{N}} \in (0, +\infty)^\mathbb{N}$ such that $\lim_{n \to +\infty} \eta_n = \tilde{\eta}$, then we have $\lim_{n \to +\infty} \mathcal{G}^n_{\alpha_n}(\pi) = \mathcal{G}^n_{\tilde{\alpha}}(\pi)$ using the dominated convergence theorem. In addition, using the dominated convergence theorem and [40, Lemma 1.4.3] we get that for any $(\pi_n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R}^d)^\mathbb{N}$ such that $\lim_{n \to +\infty} \pi_n = \pi \in \mathcal{P}(\mathbb{R}^d)$ we have $\liminf_{n \to +\infty} \mathcal{G}^n_{\alpha_n}(\pi_n) \geq \mathcal{G}^n_{\tilde{\alpha}}(\pi)$. Therefore, $(\mathcal{G}^n_{\tilde{\alpha}} : \alpha, \eta > 0)$ is a $\Gamma$-system. In addition, for any $\tilde{\alpha} > 0$ and $\eta > 0$ we have that for any $\alpha \in (\tilde{\alpha}/2, +\infty)$, $\mathcal{G}^n_{\alpha} \geq \mathcal{G}^n_{\alpha/2}$. Hence the $\Gamma$-system $(\mathcal{G}^n_{\tilde{\alpha}} : \alpha > \tilde{\alpha}/2, \eta > 0)$ is equicoercive and using Proposition 13 we have that $\mathcal{G}^n_\alpha$ is equicoercive for any $\alpha, \eta > 0$.

### A.4 Proof of Proposition 3

Let $\alpha, \eta > 0$, let $(\mu_n)_{n \in \mathbb{N}} \in (\mathcal{P}(\mathbb{R}^d))^\mathbb{N}$ be such that $\lim_{n \to +\infty} \mathcal{W}_1(\mu_n, \mu) = 0$ with $\mu \in \mathcal{P}(\mathbb{R}^d)$, and $(\pi_n)_{n \in \mathbb{N}} \in (\mathcal{P}(\mathbb{R}^d))^\mathbb{N}$ such that $\lim_{n \to +\infty} \mathcal{W}_1(\pi_n, \pi) = 0$ with $\pi \in \mathcal{P}(\mathbb{R}^d)$. For any $\pi \in \mathcal{P}(\mathbb{R}^d)$ denote $f_\pi(y) = \log(\pi[k(\cdot, y)]) + \eta)$. For any $\nu \in \mathcal{P}(\mathbb{R}^p)$ define $\mathcal{G}^n_{\alpha}(\cdot, \nu)$ as

$$
\mathcal{G}^n_{\alpha}(\pi, \nu) := -\int_{\mathcal{P}} \log(\pi[k(\cdot, y)]) + \eta)d\nu(x) + \alpha\text{KL}(\pi[\pi_0])
$$

for any $\pi \in \mathcal{P}(\mathbb{R}^d)$. Using the dominated convergence theorem, for any $\pi \in \mathcal{P}(\mathbb{R}^d)$, $f_\pi \in C(\mathbb{R}^d, \mathbb{R})$ is differentiable and we have that for any $y \in \mathbb{R}^p$

$$
||\nabla f_\pi(y)|| \leq \pi[||\nabla k(\cdot, y)||]/(\pi[k(\cdot, y)]) + \eta) \leq \mathcal{M}/\eta.
$$

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Therefore, for any \( \pi \in \mathcal{P}(\mathbb{R}^d) \) we have for any \( y_1, y_2 \in \mathbb{R}^p \)

\[
|\log(\pi[k(\cdot, y_1)] + \eta) - \log(\pi[k(\cdot, y_2)] + \eta)| \leq (\Theta/\eta) \|y_1 - y_2\|.
\]

Using this result we have that

\[
|\mathcal{A}_n^\alpha(\pi_n, \mu_n) - \mathcal{A}_n^\alpha(\pi, \mu)| \leq |\mathcal{A}_n^\alpha(\pi_n, \mu_n) - \mathcal{A}_n^\alpha(\pi_n, \mu)| + |\mathcal{A}_n^\alpha(\pi_n, \mu) - \mathcal{A}_n^\alpha(\pi, \mu)|
\]

\[
\leq (\Theta/\eta) \mathcal{W}_1(\mu_n, \mu)
\]

\[
+ \int_{\pi_n} |\log(\pi_n[k(\cdot, y)] + \eta) - \log(\pi[k(\cdot, y)] + \eta)| \, d\mu(y)
\]

\[
+ \alpha \left|\text{KL}(\pi_n|\pi_0) - \text{KL}(\pi|\pi_0)\right|,
\]

where the second inequality follows using the dual representation of \( \mathcal{W}_1 \).

Combining the result above, the dominated convergence theorem, the lower semi-continuity of the KL [40, Lemma 1.4.3-(b)] and the fact that \( \lim_{n \to +\infty} \mathcal{W}_1(\mu_n, \mu) = 0 \), we get that \( \liminf_{n \to +\infty} \mathcal{A}_n^\alpha(\pi_n, \mu_n) \geq \mathcal{A}_n^\alpha(\pi, \mu) \).

Thus, \( \{\mathcal{A}_n^\alpha(\cdot, \mu_n) : n \in \mathbb{N}\} \) is a \( \Gamma \)-system. Since we have that for any \( \pi \in \mathcal{P}(\mathbb{R}^d) \) and \( n \in \mathbb{N} \), \( \mathcal{A}_n^\alpha(\pi, \mu_n) \geq -\max(|\log(\eta)|, |\log(\pi + \eta)|) + \alpha \text{KL}(\pi|\pi_0) \), we have that \( \{\mathcal{A}_n^\alpha(\cdot, \mu_n) : n \in \mathbb{N}\} \) is equicoercive. We conclude using Proposition 13.

### A.5 Proof of Proposition 4

Take \( \alpha > 0 \) and \( \eta \geq 0 \). Since by Proposition 2, \( \mathcal{A}_n^\alpha \) is proper, \( \mathcal{A}_n^\alpha(\pi_n, \eta_0) < +\infty \) and therefore, \( \text{KL}(\pi_n^\alpha|\pi_0) < +\infty \). Hence, \( \pi_n^\alpha \) admits a density w.r.t. \( \pi_0 \). Since \( \pi_0 \) is equivalent to the Lebesgue measure, we get that \( \pi_n^\alpha \) admits a density with respect to the Lebesgue measure. In addition, let \( \pi_1 \in \mathcal{P}(\mathbb{R}^d) \) be such that for any \( x \in \mathbb{R}^d \) we have for some \( \tau > 0 \)

\[
\left( d\pi_1/d\pi_0 \right)(x) = f(x)/\pi_0[f], \quad f(x) = \exp((\tau/2) \|x\|^2).
\]

Since \( \text{KL}(\pi_n^\alpha|\pi_0) < +\infty \) we have

\[
\int_{\mathbb{R}^d} \log((d\pi_n^\alpha/d\pi_0)(x)) \|x\|^2 \, d\pi_n^\alpha(x) < +\infty
\]

and

\[
\int_{\mathbb{R}^d} \log((d\pi_n^\alpha/d\pi_0)(x)) \|x\|^2 \, d\pi_n^\alpha(x) = \int_{\mathbb{R}^d} \left\{ \log((d\pi_n^\alpha/d\pi_0)(x)) \right\} \frac{1}{\|x\|^2 + \log(\pi_0[f])}
\]

\[
\times \frac{(\tau/2) \|x\|^2}{\|x\|^2 + \log(\pi_0[f])}
\]

\[
\leq \int_{\mathbb{R}^d} \left\{ \log((d\pi_n^\alpha/d\pi_0)(x)) \right\} \frac{1}{\|x\|^2 + \log(\pi_0[f])}
\]

\[
\times \frac{(\tau/2) \|x\|^2}{\|x\|^2 + \log(\pi_0[f])}
\]

\[
< +\infty.
\]

Since \( t \mapsto t \log(t) \) is bounded on \([0, 1]\) we have that

\[
-\int_{\mathbb{R}^d} \log((d\pi_n^\alpha/d\pi_0)(x)) \|x\|^2 \, d\pi_n^\alpha(x) < +\infty.
\]

Combining (28) and (27) we get that \( \text{KL}(\pi_n^\alpha|\pi_1) < +\infty \). Therefore, using [32, Equation 2.6], we get that

\[
((\tau/2) \int_{\mathbb{R}^d} \|x\|^2 \, d\pi_n^\alpha(x) - \log(\pi_0[f])
\]

\[
= \int_{\mathbb{R}^d} \log((d\pi_1/d\pi_0)(x)) \, d\pi_n^\alpha(x) = \text{KL}(\pi_n^\alpha|\pi_0) - \text{KL}(\pi_n^\alpha|\pi_1) < +\infty.
\]

Therefore, \( \pi_n^\alpha \in \mathcal{P}_2(\mathbb{R}^d) \).

Let \( (\alpha_n)_{n \in \mathbb{N}} \in (0, +\infty)^\mathbb{N} \), \( (\eta_n)_{n \in \mathbb{N}} \in [0, +\infty)^\mathbb{N} \) such that \( \lim_{n \to +\infty} \alpha_n = 0 \), \( \lim_{n \to +\infty} \eta_n = 0 \) and \( \lim_{n \to +\infty} \pi_n^\alpha, \eta_n = \pi^* \) with \( \pi^* \in \mathcal{P}_2(\mathbb{R}^d) \). For any \( (\pi_n)_{n \in \mathbb{N}} \in \mathcal{P}_2(\mathbb{R}^d)^\mathbb{N} \) such that \( \lim_{n \to +\infty} \pi_n = \pi \) we have using the monotonicity in \( \alpha \) of \( \mathcal{A}_n^\alpha \) and Fatou’s lemma

\[
\liminf_{n \to +\infty} \mathcal{A}_n^\alpha_n(\pi_n) \geq \liminf_{n \to +\infty} \mathcal{A}_n^\alpha_n(\pi_n) \geq \mathcal{A}(\pi).
\]

Let \( \pi \in \mathcal{P}_2(\mathbb{R}^d) \) and for any \( n \in \mathbb{N} \) define \( \pi_n \in \mathcal{P}_2(\mathbb{R}^d) \) with density w.r.t. the Lebesgue measure given for any \( x \in \mathbb{R}^d \) by

\[
\pi_n(x) = \int_{\mathbb{R}^d} (2\pi_n)^{-d/2} \exp[-\|x - \tilde{x}\|^2/(2\pi_n)] \, d\pi(\tilde{x}).
\]

Let \( X \) be a random variable with distribution \( \pi \) and for any \( n \in \mathbb{N} \), let \( X_n = X + \alpha_n^{1/2}Z \) where \( Z \) is a Gaussian random variable with zero mean and identity covariance matrix. For any \( n \in \mathbb{N} \), \( X_n \) has distribution \( \pi_n \). Therefore, we have \( \lim_{n \to +\infty} \mathcal{W}_2(\pi_n, \pi)^2 \leq \lim_{n \to +\infty} \mathbb{E}[\|X - X_n\|^2] = 0 \).
In what follows, we show that \( \lim_{n \to +\infty} \mathcal{G}^n(\pi_n) = \mathcal{G}(\pi) \). Define \( h : [0, +\infty) \to \mathbb{R} \) as \( h(t) = t \log(t) \). For any \( n \in \mathbb{N} \), using Jensen’s inequality and the Faber–Tukey theorem we have

\[
\int_{\mathbb{R}^d} h(\pi_n(x)) \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h((2\alpha_n x)^{-d/2} \exp[-\|x - \tilde{x}\|^2/(2\alpha_n)]) \, d\pi(\tilde{x}) \, dx \\
\leq -(d/2) \log(2\alpha_n) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (2\alpha_n x)^{-d/2} \exp[-\|x - \tilde{x}\|^2/(2\alpha_n)] \, d\pi(\tilde{x}) \, dx \\
\leq -(d/2) \log(2\alpha_n). \tag{29}
\]

Using A2 there exists \( C_1 \geq 0 \) and \( \tau > 0 \) such that for any \( x \in \mathbb{R}^d \), \( |U(x)| \leq C_1 + \tau \|x\|^2 \). Using this result, the Faber–Tukey theorem, the fact that for any \( \tilde{x} \in \mathbb{R}^d \), \( \|x\|^2 \leq 2\|x - \tilde{x}\|^2 + 2\|\tilde{x}\|^2 \) and that 

\[
\int_{\mathbb{R}^d} |U(x)| \, d\pi_n(x) \leq C_1 + \tau \int_{\mathbb{R}^d} \|x\|^2 (2\alpha_n x)^{-d/2} \exp[-\|x - \tilde{x}\|^2/(2\alpha_n)] \, d\tilde{x} \, dx \\
\leq C_1 + 2\tau \sup_{n \in \mathbb{N}} \alpha_n + 2\tau \int_{\mathbb{R}^d} \|x\|^2 \, d\pi(\tilde{x}) < +\infty. \tag{30}
\]

Combining (29) and (30) we have \( \lim_{n \to +\infty} \alpha_n KL(\pi_n | \pi_0) = 0 \).

For any \( n \in \mathbb{N} \) and \( y \in \mathbb{R}^p \), \( \log(\pi_n[k(\cdot, y)] + \eta_n) \leq \log(M + \sup_{n \in \mathbb{N}} \eta_n) \). Define \( \Phi : \mathbb{R}^d \to \mathbb{R}, \Phi : x \mapsto \|x\|^2 \) and \( \Psi : \mathbb{R}^p \to \mathbb{R}, \Psi : y \mapsto \|y\|^2 \). Using Jensen’s inequality and the fact that under A2 there exists \( C_2 \geq 0 \) such that for any \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^p \), \( -\Phi(x) - \Psi(y)/\|k(x, y)\| \leq C_2 \exp(C_2) \), we have for any \( y \in \mathbb{R}^p \) and

\[
\log(\pi_n[k(\cdot, y)] + \eta_n) \geq -\Psi(y) - \sup_{n \in \mathbb{N}} \pi_0 [\Phi - C_2 - \log C_2].
\]

Since, \( (\pi_n)_{n \in \mathbb{N}} \) is relatively compact in \( \mathcal{P}_2(\mathbb{R}^d) \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), there exists \( C \geq 0 \) such that for any \( n \in \mathbb{N}, \pi_n[\Phi] \leq C \), see [101, Definition 6.8] and \( \mu[\Psi] < +\infty \). Hence, there exists \( M \geq 0 \) such that for any \( n \in \mathbb{N} \) and \( y \in \mathbb{R}^p \) such that \( |\log(\pi_n[k(\cdot, y)] + \eta_n)| \leq M \). Combining this result and the dominated convergence theorem, we get that \( \lim_{n \to +\infty} \mathcal{G}^n(\pi_n) = \mathcal{G}(\pi) \).

Therefore, we have that \( \mathcal{G}(\pi) = \lim_{n \to +\infty} \mathcal{G}^n(\pi_n) \), and \( \{\mathcal{G}^n : \alpha, \eta \geq 0\} \) is a \( \Gamma \)-system. Using Proposition 14, we get that \( \pi^* \) is a minimizer of \( \mathcal{G} \).

Assume that there exists \( \pi^1 \in \arg\min_{\mathcal{P}(\mathbb{R}^d)} \mathcal{G} \) such that \( KL(\pi^1 | \pi_0) < KL(\pi^* | \pi_0) \). In what follows we assume that \( KL(\pi^1 | \pi_0) < +\infty \), the case where \( KL(\pi^1 | \pi_0) = +\infty \) is similar and omitted. There exists \( \varepsilon > 0 \) such that \( KL(\pi^1 | \pi_0) = KL(\pi^* | \pi_0) - \varepsilon \). Since \( \lim_{n \to +\infty} KL(\pi_{\alpha_n, \eta_n} | \pi_0) \geq KL(\pi^* | \pi_0) \), there exists \( n_0 \in \mathbb{N} \) such that for any \( n \in \mathbb{N} \) with \( n \geq n_0 \) we have

\[
KL(\pi_{\alpha_n, \eta_n} | \pi_0) \geq KL(\pi^* | \pi_0) - \varepsilon/2 \geq KL(\pi^1 | \pi_0) + \varepsilon/2.
\]

Therefore we have that for any \( n \geq n_0 \)

\[
\mathcal{G}^n(\pi_{\alpha_n, \eta_n}) = \mathcal{G}^n(\pi^1_{\alpha_n, \eta_n}) + \alpha_n KL(\pi_{\alpha_n, \eta_n} | \pi_0) \\
\geq \mathcal{G}^n(\pi^1) + \alpha_n KL(\pi^1 | \pi_0) + \alpha_n \varepsilon/2 > \mathcal{G}^n(\pi^1),
\]

which is a contradiction since for any \( n \in \mathbb{N} \), \( \pi_{\alpha_n, \eta_n} \) is a minimizer of \( \mathcal{G}^n(\pi_n) \). Hence, for any \( \pi \in \arg\min_{\mathcal{P}(\mathbb{R}^d)} \mathcal{G} \), \( KL(\pi^* | \pi_0) \leq KL(\pi | \pi_0) \), which concludes the proof.

B Proofs of Section 3

B.1 Basics on Wasserstein gradient flows and subdifferential of \( \mathcal{G}^n \)

In this section we recall basic results on gradient flows in Wasserstein spaces and provide a slight extension of the classical variational formula. We refer to [2] for further details. We denote by \( L^2(\mathbb{R}^d, \nu) \) the set of square integrable functions w.r.t. a measure \( \nu \), by \( C^\infty(\mathbb{R}^d) \) the set of infinitely many times differentiable functions with compact support and by \( W^{1,1}(\mathbb{R}^d) \) the set of functions which are in \( W^{1,1}_{\text{loc}}(\mathbb{X}) \) for all compact sets \( \mathbb{X} \subset \mathbb{R}^d \), where \( W^{1,1}_{\text{loc}}(\mathbb{X}) \) denotes the Sobolev space of functions whose zero-th and first derivative are bounded in \( L^1 \). For any \( \nu \in \mathcal{P}(\mathbb{R}^d) \) and \( t : \mathbb{R}^d \to \mathbb{R}^d \) measurable, we denote by \( t_{\nu, \nu}(A) = \nu(t^{-1}(A)) \) for all \( A \in \mathcal{B}(\mathbb{R}^d) \). We start by recalling the definition of the subdifferentiability in Wasserstein spaces, see [2, Definition 10.1.1]. We denote \( \mathcal{P}^1_2(\mathbb{R}^d) \) the space of probability distributions in \( \mathcal{P}(\mathbb{R}^d) \) which are absolutely continuous w.r.t. the Lebesgue measure.

**Definition 15** (Fréchet subdifferential). Let \( \Phi : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty) \). Let \( \pi \in \mathcal{P}^1_2(\mathbb{R}^d) \), then \( \xi \in L^2(\mathbb{R}^d, \pi) \) belongs to the strong Fréchet subdifferential \( \partial \Phi(\pi) \) of \( \Phi \) at \( \pi \) if for any \( t \in L^2(\mathbb{R}^d, \pi) \)

\[
\lim_{t \to \pi} \frac{\Phi(\pi + t) - \Phi(\pi) - \int_{\mathbb{R}^d} \langle \xi(x), t(x) - x \rangle \, d\pi(x)}{\|t - 1\|_{L^2(\mathbb{R}^d, \pi)}} = 0.
\]

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The following lemma is a specific case of the chain-rule formula.

Lemma 16. Let $g \in C^1(\mathbb{R}, \mathbb{R})$ be Lipschitz, convex and lower-bounded. For any $y \in \mathbb{R}^p$, let $\mathcal{L} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^p \to \mathbb{R}$, $\pi \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu \in \mathcal{P}_2(\mathbb{R}^p)$ and $\xi \in L^1(\mathbb{R}^d \times \mathbb{R}^p, \pi \otimes \mu)$ be such that for any $y \in \mathbb{R}^p$ and $\pi \in \mathcal{P}_2(\mathbb{R}^d)$, $\xi(\pi, y) \in \partial \mathcal{L}(\pi, y)$ and $-\xi(\pi, y) \in \partial (-\mathcal{L})(\pi, y)$. In addition, assume that there exists $h \in L^1(\mathbb{R}^p, \mu)$ such that for any $\pi_1, \pi_2 \in \mathcal{P}_2(\mathbb{R}^d)$ and $y \in \mathbb{R}^p$ we have

$$|\mathcal{L}(\pi_1, y) - \mathcal{L}(\pi_2, y)| \leq h(y)W_2(\pi_1, \pi_2).$$

Let $\mathcal{M}$ be given for any $\pi \in \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{M}(\pi) = \int_{\mathbb{R}^p} g(\mathcal{L}(\pi, y))d\mu(y).$$

Then, we have that $\partial_{\pi} \mathcal{M}(\pi) \neq \emptyset$ and $\{x \mapsto \int_{\mathbb{R}^p} g'(\mathcal{L}(\pi, y))\xi(x, y)d\mu(y)\} \in \partial_{\pi} \mathcal{M}(\pi)$.

Proof. Let $\pi \in \mathcal{P}(\mathbb{R}^d)$, $t \in L^2(\mathbb{R}^d, \pi)$ and $y \in \mathbb{R}^d$. Since $g$ is convex we have

$$g(\mathcal{L}(t\#\pi, y)) \leq g(\mathcal{L}(\pi, y)) + g'(\mathcal{L}(\pi, y))(\mathcal{L}(t\#\pi, y) - \mathcal{L}(\pi, y)).$$

We have

$$g(\mathcal{L}(t\#\pi, y)) - g(\mathcal{L}(\pi, y)) - \int_{\mathbb{R}^d} \xi(x, y, t(x) - x)d\pi(x) \geq \int_{\mathbb{R}^d} \xi(x, y, t(x) - x)d\pi(x).$$

In addition, using that $\xi(\pi, y) \in \partial \mathcal{L}(\pi, y)$ and $-\xi(\pi, y) \in \partial (-\mathcal{L})(\pi, y)$ we have

$$\lim_{t \to 0} \mathcal{L}(t\#\pi, y) - \mathcal{L}(\pi, y) = \int_{\mathbb{R}^d} \xi(x, y, t(x) - x)d\pi(x) \to t \to 0.$$

Hence, we have

$$\lim_{t \to 0} \mathcal{L}(t\#\pi, y) - \mathcal{L}(\pi, y) - \int_{\mathbb{R}^d} \xi(x, y, t(x) - x)d\pi(x) / ||t - \text{Id}||_2 = 0.$$
Lemma 18 (Strong subdifferential of KL$(\pi|\pi_0)$). Assume that $\pi_0$ admits a density, also denoted $\pi_0$, w.r.t. the Lebesgue measure such that for any $x \in \mathbb{R}^d$, $\pi_0(x) \propto \exp[-U(x)]$ for some convex $U : \mathbb{R}^d \to \mathbb{R}$. Let $\pi \in \mathcal{P}_2^\#(\mathbb{R}^d)$ such that $\mathcal{H}(\pi) < +\infty$ and $(d\pi/d\pi_0) \in W^{1,1}_p(\mathbb{R}^d, (0, +\infty))$. Then,

$$\partial_s \mathcal{H}(\pi) = \{x \mapsto \nabla \log(d\pi/d\pi_0)(x)\}.$$ 

Combining these two lemmas we are now ready to derive the subdifferential of $\mathcal{G}^\alpha_0$ given by (5) as well as one associated Wasserstein gradient flow.

Proposition 19 (Gradient flow). Assume A1 and that $\pi_0$ admits a density, also denoted $\pi_0$, w.r.t. the Lebesgue measure such that for any $x \in \mathbb{R}^d$, $\pi_0(x) \propto \exp[-U(x)]$ for some convex $U : \mathbb{R}^d \to \mathbb{R}$. Let $\pi \in \mathcal{P}_2^\#(\mathbb{R}^d)$ be such that KL$(\pi|\pi_0) < +\infty$ and $(d\pi/d\pi_0) \in W^{1,1}_p(\mathbb{R}^d, (0, +\infty))$. Then for any $\alpha, \eta > 0$

$$\{x \mapsto -\int_{\mathbb{R}^d} (\eta - \pi[k(x,y)])^{-1} \nabla k(x,y) d\mu(y) + \alpha \nabla \log(d\pi/d\pi_0)(x)\} \in \partial_s \mathcal{G}^\alpha_0(\pi).$$

In addition, if $U \in C^1(\mathbb{R}^d, \mathbb{R})$ and there exists $(\nu_t)_{t \geq 0}$ such that for any $t \in (0, +\infty)$, $\nu_t$ admits a density w.r.t. the Lebesgue measure also denoted (with a slight abuse of notation) by $\nu_t \in C^1(\mathbb{R}^d, \mathbb{R})$, and for any $\varphi \in C^\infty_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x)d\nu_t(x) - \int_{\mathbb{R}^d} \varphi(x)d\nu_0(x) = \int_0^t \int_{\mathbb{R}^d} \{\alpha (\varphi(x) - \varphi(y)) / (\int_{\mathbb{R}^d} k(x,y) d\nu_0(x))\} d\mu(y) + \alpha (\nabla U(x), \nabla \varphi(x)),$$

then $(\nu_t)_{t \geq 0}$ is a Wasserstein gradient flow associated with $\mathcal{G}^\alpha_0$.

Proof. The first part of the proof is a straightforward application of Lemma 17 and Lemma 18. The second part of the proof follows from the definition of a Wasserstein gradient flow [2, Definition 11.1.1].

B.2 Proof of Proposition 5

Let $\alpha, \eta > 0$. First, we show that $\hat{b}_\alpha$ is Lipschitz continuous. Let $f(t) = (\eta + t)^{-1}$. Using that for any $t \geq 0$, $|f'(t)| \leq \eta^{-2}$ for any $x_1, x_2 \in \mathbb{R}^d$, $y \in \mathbb{R}^p$ and $\pi_1, \pi_2 \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|\hat{b}_\alpha(x_1, \pi_1, y) - \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| + \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\|,$$

where we have used the fact that, under A1, $k$ is Lipschitz continuous. The rest of the proof is classical, see for instance [96, Theorem 1.1], but is given for completeness. Define $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ such that for any $x \in \mathbb{R}^d$ and $\pi \in \mathcal{P}_1(\mathbb{R}^d)$

$$b(x, \pi) = \int_{\mathbb{R}^d} \hat{b}_\alpha(x, \pi, y) d\mu(y) - \alpha \nabla U(x).$$

Using A3-(a) and (34) we have that for any $x_1, x_2 \in \mathbb{R}^d$ and $\pi_1, \pi_2 \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|b(x_1, \pi_1) - b(x_2, \pi_2)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\|.$$ 

Combining this result and A1, we get that for any $x_1, x_2 \in \mathbb{R}^d$ and $\pi_1, \pi_2 \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|b(x_1, \pi_1) - b(x_2, \pi_2)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\| \leq \|\nabla \hat{b}_\alpha(x_1, \pi_1, y) - \nabla \hat{b}_\alpha(x_2, \pi_2, y)\|.$$ 

where the inequality follows using the dual representation of $\mathbf{W}_1$. For any $T \geq 0$ and any $\nu \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))$ denote by $(X_\nu^\nu)_{t \in [0, T]}$ the unique strong solution to (10) with initial condition $\nu \in \mathcal{P}_1(\mathbb{R}^d)$ (see [61, Chapter 5, Theorem 2.9 and 2.5]) given for any $t \in [0, T]$ by

$$X_\nu^\nu = X_0^\nu + \int_0^t b(X_\nu^\nu, \nu_s) ds + \sqrt{2\alpha} \mathbf{B}_s,$$ 

(37)
where \((B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion with \(X_0^\ast = X_0 \in \mathbb{R}^d\). Using \(A1\) and \((36)\), there exists \(C \geq 0\) such that for any \(x \in \mathbb{R}^d\) and \(\pi \in \mathcal{P}_1(\mathbb{R}^d)\)
\[
\|b(x, \pi)\| \leq C \left(1 + \|x\|\right).
\]

Denote \((\mathcal{F}_t)_{t \geq 0}\) the filtration associated with \((B_t)_{t \geq 0}\). Using Jensen’s inequality we have that \(\|\|B_t\|\|_{\mathcal{F}_t} \geq 0\) is a \((\mathcal{F}_t)_{t \geq 0}\)-supermartingale. Therefore, using \([90],\) Chapter 2, Corollary 1.6\), we get that for any \(t \geq 0\),
\[
\mathbb{E}[\sup_{s \in [0,t]} \|B_s\|] \leq 2\mathbb{E}^{1/2}[\|B_t\|^2] \leq 2(dt)^{1/2}.
\]

Using this result and \((38)\) we get that for any \(t \geq 0\)
\[
\mathbb{E}[\sup_{s \in [0,t]} \|X^\ast_s\|] 
\leq \mathbb{E}[\|X^\ast_{t}\|] + 2(2\alpha d)^{1/2} + (C + 2(2\alpha d)^{1/2}) t + C \int_0^t \mathbb{E}[\sup_{u \in [0,s]} \|X^\ast_u\|]\,ds.
\]

Using Grönwall’s lemma we get that \(\mathbb{E}[\sup_{s \in [0,T]} \|X^\ast_s\|] < +\infty\). In particular, for any \(t \in [0,T]\), \(X^\ast_t \in \mathcal{P}_1(\mathbb{R}^d)\), where \(X^\ast\) is the distribution of \((X^\ast_t)_{t \in [0,T]}\). Similarly there exists \(C \geq 0\) such that for any \(t, s \in [0,T]\) we have
\[
W_t(X^\ast_t, X^\ast_s) \leq \mathbb{E}[\|X^\ast_t - X^\ast_s\|] \leq C(t - s).
\]

Therefore, \(X^\ast \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))\). In addition, using \((36)\) we get that for any \(\nu_1, \nu_2 \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))\) and \(t \in [0,T]\)
\[
\mathbb{E}[\|X^\ast_t - X^\ast_{2}\|] \leq (\mathbb{E}[\|X^\ast_{t}\|] + \mathbb{E}[\|X^\ast_{2}\|]) \sup_{s \in [0,T]} W_t(\nu_1, s), \nu_2, s\right) dt\].

By Grönwall’s lemma, there exists \(C \geq 0\) such that for any \(\nu_1, \nu_2 \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))\) and \(t \geq 0\)
\[
\mathbb{E}[\|X^\ast_{t} - X^\ast_s\|] \leq T((\mathbb{E}[\|X^\ast_{t}\|] + \mathbb{E}[\|X^\ast_{s}\|]) \sup_{s \in [0,T]} W_t(\nu_1, s), \nu_2, s\right) dt\].

Let \(F_T : \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d)) \rightarrow \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))\) denote the map which associates to \(\nu\) to law of \((X^\ast_t)_{t \in [0,T]}\), \(X^\ast_t \in \mathcal{P}_1(\mathbb{R}^d)\). Then there exists \(T_0 \geq 0\) such that for any \(T_0 \in [0,T]\) and \(\nu_1, \nu_2 \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))\)
\[
\sup_{t \in [0,T_0]} W_t(F_T(\nu_1), t), F_T(\nu_2), t) \leq (1/2) \sup_{t \in [0,T_0]} W_t(\nu_1, t), \nu_2, t).
\]

Since \(\mathcal{P}_1(\mathbb{R}^d)\) is complete, see \([101],\) Theorem 6.18, \(F_T\) admits a unique fixed point using Picard’s theorem (see, e.g., \([21],\) Theorem 1.7). Denote \(X^\ast\) this fixed point and \((X^\ast_t)_{t \in [0,T]}\) the solution of \((37)\) with \(\nu \leftarrow X^\ast\). Then \((X^\ast_t)_{t \in [0,T]}\) is a solution of \((10)\) up to time \(T \geq 0\). By recursion, we obtain a solution \((X^\ast_t)_{t \geq 0}\).

Assume that there exist two solutions \((X^\ast_{1,t})_{t \geq 0}\) and \((X^\ast_{2,t})_{t \geq 0}\) and denote \(\lambda_1\) and \(\lambda_2\) their associated distribution. Since there exists a unique fixed point to \(F_T\) we get that \(\lambda_1 = \lambda_2\). Since there exists a unique strong solution to \((37)\), we get that \((X^\ast_{1,t})_{t \geq 0} = (X^\ast_{2,t})_{t \geq 0}\), which concludes the proof.

**B.3 Proof of Proposition 6**

Under \(A1\), Proposition 1 ensures that the functional \(U\) is convex and lower bounded. In addition, \(A2\) guarantees that \(U\) is sufficiently regular to satisfy \([53],\) Assumption 2.2. To see this, we can use \(A3\)-(b) and the Cauchy-Schwarz inequality to obtain
\[
\langle \nabla U(x), x \rangle = \langle \nabla U(x) - \nabla U(0), x \rangle + \langle \nabla U(0), x \rangle \geq \mathbb{E}[\|x\|^2 - c - \|x\| \parallel \nabla U(0)\parallel] .
\]

Then Young’s inequality yields, \(\alpha \beta \leq \alpha^2/(2\epsilon) + \beta^2 \epsilon/2\) for all \(\epsilon > 0\), and thus that:
\[
-\|x\| \|\nabla U(0)\| \geq -\|x\|^2/(2\epsilon) - \epsilon \|\nabla U(0)\|^2/2.
\]

It follows that for all \(\epsilon > 1/(2\alpha)\) we have, for any \(x \in \mathbb{R}^d\)
\[
\langle \nabla U(x), x \rangle \geq \alpha \|x\|^2 + b, \text{ where } a := a - 1/(2\epsilon) > 0, b := -c - \epsilon \|\nabla U(0)\|^2/2.
\]

As shown in \((36)\), the drift of the MKVSDE is Lipschitz continuous.

In addition, using the fact that, under \(A1\), \(k \in C^\infty(\mathbb{R} \times \mathbb{R}^d)\) and Leibniz integral rule for differentiation under the integral sign (e.g., \([11],\) Theorem 16.8), we have that \(b(x, \nu)\) in \((13)\) is \(C^\infty(\mathbb{R}^d \times \mathbb{R}^d)\).
for all fixed $\nu \in \mathcal{P}(\mathbb{R}^d)$. Finally, we need to show that $\nabla b(x, \nu)$ is jointly continuous in $(x, \nu)$. Similarly to (35), we have for any $x_1, x_2 \in \mathbb{R}^d$ and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$

$$
\|\nabla b(x_1, \nu_1) - \nabla b(x_2, \nu_2)\| \leq \| \int_{\mathbb{R}^d} \nabla^2 k(x_1, y)/(\nu_1[k(\cdot, y)] + \eta) - \int_{\mathbb{R}^d} \nabla^2 k(x_2, y)/(\nu_2[k(\cdot, y)] + \eta)\| + \alpha \|\nabla^2 U(x_1) - \nabla^2 U(x_2)\| 
$$

which gives continuity. Then, the result follows directly from [53, Theorem 2.11].

**B.4 Proof of Proposition 7**

Let $N \in \mathbb{N}$. The existence and strong uniqueness of a solution to (12) is a straightforward consequence of (35), [61, Chapter 5, Theorem 2.9 and 2.5] and the fact that for any $\{x^k_{1,N}\}_{k=1}, \{x^k_{2,N}\}_{k=1} \in (\mathbb{R}^d)^N$ and $\ell \in \{1, \ldots, N\}$ we have

$$
\int_{\mathbb{R}^d} \| \partial_k \frac{\delta}{\partial \nu} (x^{\ell,N}_1, (1/N) \sum_{k=1}^N \delta_{x^k_{1,N}, y} - \delta_{x^k_{2,N}, y}) \| \, d\mu(y) 
\leq \frac{(\|\mathbb{H}\|(1 + (1/\theta)) + \alpha L)}{\mathbb{N}^2} \|x^{\ell,N}_1 - x^{\ell,N}_2\|. 
$$

We now turn to the quantitative functional propagation of chaos result. For any $x \in \mathbb{R}^d$ and $\pi \in \mathcal{P}_1(\mathbb{R}^d)$ denote $b(x, \pi) = \int_{\mathbb{R}^d} b_\pi(x, \pi(y)) \, d\mu(y) - \alpha \nabla U(x)$. We recall that using (35) there exists $C_0 \geq 0$ such that for any $x_1, x_2 \in \mathbb{R}^d$ and $\pi_1, \pi_2 \in \mathcal{P}(\mathbb{R}^d)$ we have

$$
\|b(x_1, \pi_1) - b(x_2, \pi_2)\| \leq C_0 \big\|\|x_1 - x_2\| + \int_{\mathbb{R}^d} |\pi_1[k(\cdot, y)] - \pi_2[k(\cdot, y)]| \, d\mu(y)\big\|.
$$

Using this result, we have for any $t \geq 0$

$$
\mathbb{E}[\sup_{s \in [0,t]} \|X_t^* - X_t^{1,N}\|] \leq \int_0^t \mathbb{E}[\|b(X_s^*, \mathcal{A}^*_s) - b(X_t^{1,N}, \mathcal{A}_t^N)\|] \, ds 
\leq C \int_0^t \mathbb{E}[\sup_{u \in [0,s]} \|X_u^* - X_u^{1,N}\|] \, ds 
+ C \int_0^t \mathbb{E}[\|1/N \sum_{k=1}^N k(X_s^k, y) - \mathcal{A}_s^* [k(\cdot, y)]\|] \, d\mu(y) \, ds. 
$$

Now, consider $N$ independent copies of the nonlinear process $X_t^*$, $\{X_t^{k,N}\}_{k=1}^N$. We can bound the second term in the above with

$$
\mathbb{E}[\|1/N \sum_{k=1}^N k(X_s^k, y) - \mathcal{A}_s^* [k(\cdot, y)]\|] \leq (1/N) \mathbb{E}[\|\sum_{k=1}^N \{k(X_s^k, y) - k(X_s^{k,N}, y)\}\|] 
\leq (1/N) \mathbb{E}[\|\sum_{k=1}^N \{k(X_s^{k,N}, y) - k(X_s^{k,N}, y)\}\|] 
\leq \mathbb{E}[\|\sup_{u \in [0,t]} \|X_u^* - X_u^{1,N}\|\|] + (1/N) \mathbb{E}[\|\sum_{k=1}^N \{k(X_s^{k,N}, y) - \mathcal{A}_s^* [k(\cdot, y)]\}\|], 
$$

where we used the Lipschitz continuity of $k$ and the fact that $\{(X_t^{k,N})_{t \geq 0}\}_{k=1}^N$ is exchangeable to obtain the last inequality. Plugging the above into (40) and using Jensen’s inequality we obtain

$$
\mathbb{E}[\sup_{s \in [0,t]} \|X_s^* - X_s^{1,N}\|] \leq C_0 \int_0^t \mathbb{E}[\sup_{u \in [0,s]} \|X_u^* - X_u^{1,N}\|] \, ds 
+ C \int_0^t \mathbb{E}[\|1/N \sum_{k=1}^N \{k(X_s^{k,N}, y) - \mathcal{A}_s^* [k(\cdot, y)]\}\|] \, d\mu(y) \, ds 
\leq C_0 \mathbb{E}[\|X_t^* - X_t^{1,N}\|] + C \mathbb{E}[\|1/N \sum_{k=1}^N \{k(X_s^{k,N}, y) - \mathcal{A}_s^* [k(\cdot, y)]\}\|] + \mathbb{E}[\|X_t^* - X_t^{1,N}\|].
$$

Using Grönwall’s lemma we get that for any $T \geq 0$ there exists $C_T \geq 0$ such that for any $N \in \mathbb{N}$

$$
\mathbb{E}[\sup_{t \in [0,T]} \|X_t^* - X_t^{1,N}\|] \leq C_T N^{-1/2},
$$

which concludes the proof.
B.5 Proof of Proposition 8

First, we recall that \( b \) and \( \tilde{b} \) are given, for any \( x \in \mathbb{R}^d, \nu, \eta \in \mathcal{P}(\mathbb{R}^d) \) and \( y \in \mathbb{R}^p \), by
\[
 b(x, \nu) = \int_{\mathbb{R}^d} \tilde{b}_0(x, \nu, y) d\mu(y) - \alpha \nabla U(x), \quad \tilde{b}_0(x, \nu, y) = -\nabla_1 k(x, y)/(\nu[k(\cdot, y)] + \eta).
\]

Using A1 and (35), there exist \( C_0, C_1 \geq 0 \) such that for any \( x_1, x_2 \in \mathbb{R}^d, \pi_1, \pi_2 \in \mathcal{P}(\mathbb{R}^d) \) and \( y \in \mathbb{R}^p \),
\[
 \| \tilde{b}_0(x_1, \pi_1, y) \| \leq C_0, \quad \| b(x_1, \pi_1) - b(x_2, \pi_2) \| \leq C_1 (\| x_1 - x_2 \| + \mathcal{W}_1(\pi_1, \pi_2)). \tag{41}
\]

In addition, note that for any \( x_1^{1:N}, x_2^{1:N} \in (\mathbb{R}^d)^N \) we have for any \( i \in \{1, 2\} \)
\[
 \mathcal{W}_1(\mu_i^N, \mu_i^N) \leq (1/N) \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|, \quad \mu_i^N = (1/N) \sum_{k=1}^N \delta_{x_i^{k,N}}. \tag{42}
\]

Let \( N \in \mathbb{N} \) and denote \( B_N : (\mathbb{R}^d)^N \rightarrow (\mathbb{R}^d)^N \) given for any \( x^{1:N} \in (\mathbb{R}^d)^N \) by
\[
 B_N(x^{1:N}) = (b(x^{k,N}, \mu_i^N))_{k=1}^N, \quad \mu_i^N = (1/N) \sum_{k=1}^N \delta_{x_i^{k,N}}.
\]

Using (41), (42) and the Cauchy-Schwarz inequality we have for any \( x_1^{1:N}, x_2^{1:N} \in (\mathbb{R}^d)^N \)
\[
 \left\| B_N(x_1^{1:N}) - B_N(x_2^{1:N}) \right\| \leq C_1 \left( \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|^2 \right)^{1/2} + N \mathcal{W}_1 \left( \frac{1}{N} \sum_{k=1}^N \delta_{x_1^{k,N}}, \frac{1}{N} \sum_{k=1}^N \delta_{x_2^{k,N}} \right)^2
\]
\[
 \leq C_1 \left( \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|^2 \right)^{1/2} + \frac{1}{N} \left( \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|^2 \right)
\]
\[
 \leq 2C_1 \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|
\]
\[
 \leq 2C_1 N^{1/2} \left\| x_1^{1:N} - x_2^{1:N} \right\|.
\]

Let \( R = \max(4C_0N^{1/2}/(\alpha m), (2cN/m)^{1/2}) \) with \( m \) and \( c \) as in A3-(b). Using (41) and the Cauchy-Schwarz inequality we have for any \( x_1^{1:N}, x_2^{1:N} \in (\mathbb{R}^d)^N \)
\[
 \langle B_N(x_1^{1:N}), x_1^{1:N} - x_2^{1:N} \rangle \leq -\alpha m \| x_1^{1:N} - x_2^{1:N} \|^2 + \alpha c N + 2C_0 \sum_{k=1}^N \| x_1^{k,N} - x_2^{k,N} \|
\]
\[
 \leq -\alpha m \| x_1^{1:N} - x_2^{1:N} \|^2 + \alpha c N + 2C_0 N^{1/2} \left\| x_1^{1:N} - x_2^{1:N} \right\|.
\]

Then for any \( x_1^{1:N}, x_2^{1:N} \in (\mathbb{R}^d)^N \) with \( \| x_1^{1:N} - x_2^{1:N} \| \geq R \) we further have
\[
 \langle B_N(x_1^{1:N}) - B_N(x_2^{1:N}), x_1^{1:N} - x_2^{1:N} \rangle \leq -\left( \alpha m/2 \right) \left\| x_1^{1:N} - x_2^{1:N} \right\|^2. \tag{44}
\]

We conclude upon combining (43), (44) and [35, Corollary 2].

B.6 Proof of Proposition 9

First, we start with the following lemma.

Lemma 20. Assume A1, A3-(a) and A3-(b). Let \( \nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R}^d) \). Denote \( (X^*_t, X^*_n)_{t \geq 0} \) the stochastic process such that \( (X^*_0, X^*_2, \nu_1) \) is the optimal coupling between \( \nu_1 \) and \( \nu_2 \) w.r.t. to \( \mathcal{W}_1 \). Then for any \( T \geq 0 \), there exists \( C_T \geq 0 \) such that for any \( t \in [0, T] \)
\[
 \mathcal{W}_1(\lambda_t^*(\nu_1), \lambda_t^*(\nu_2)) \leq C_T \mathcal{W}_1(\nu_1, \nu_2),
\]
where for any \( \nu \in \mathcal{P}_1(\mathbb{R}^d) \), \( \lambda^*(\nu) \) is the distribution of \( (X^*_t)_{t \geq 0} \) with initial distribution \( \nu \).
Proof. Let $T \geq 0$. Using (35), we have for any $t \in [0, T]$

$$E[|X^1_{t,t} - X^2_{t,t}|] \leq W_1(\nu_1, \nu_2) + C \int_0^t (E[|X^1_{s,s} - X^2_{s,s}|] + W_1(\lambda^*_s(\nu_1), \lambda^*_s(\nu_2)))ds,$$

where $C = (\delta / \eta)(1 + (1 / \eta))(1 + \alpha L)(1 + \delta)$. Using Grönwall’s lemma we have for any $t \in [0, T]$

$$E[|X^1_{t,t} - X^2_{t,t}|] \leq (1 + T)C(1 + T)\{W_1(\nu_1, \nu_2) + C \int_0^t W_1(\lambda^*_s(\nu_1), \lambda^*_s(\nu_2))ds\}.$$

Using that for any $t \geq 0$, $W_1(\lambda^*_s(\nu_1), \lambda^*_s(\nu_2)) \leq E[|X^1_{t,t} - X^2_{t,t}|]$ and Grönwall’s lemma we get that there exists $C_T \geq 0$ such that for any $t \in [0, T]$

$$W_1(\lambda^*_s(\nu_1), \lambda^*_s(\nu_2)) \leq C_T W_1(\nu_1, \nu_2),$$

which concludes the proof.

Proof of Proposition 9. Let $\lambda, \eta > 0$. First we show that $\{\pi^{1,N} : N \in \mathbb{N}\}$ is relatively compact in $\mathcal{P}_1(\mathbb{R}^d)$. Let $N \in \mathbb{N}$ and assume that $X^1_{0,N} = 0$. Let us define for any $t \geq 0$

$$M^N_t = \frac{1}{2} \|X^1_{t,N}\|^2 - \int_0^t \{(X^1_{u,N}, b(X^1_{u,N}, \lambda^*_u)) + \alpha d\}ds,$$

where $\{X^1_{t,N}\}_{t \geq 0}$ is given in (12). Using Itô’s formula we have that

$$\frac{1}{2} E[\|X^1_{t,N}\|^2] - \frac{1}{2} E[\|X^1_{0,N}\|^2] = E \left[ \int_0^t \left\{ \langle X^1_{u,N}, b(X^1_{u,N}, \lambda^*_u) \rangle + \alpha d\right\} du \right]$$

$$= E \left[ \int_0^t \left\{ \langle X^1_{u,N}, \int_{\mathbb{R}^d} b(y)(X^1_{u,N}, \lambda^*_u, y) d\mu(y) \rangle \rangle du \right\right]$$

$$- E \left[ \int_0^t \left\{ \langle X^1_{u,N}, \alpha \nabla U(X^1_{u,N}) \rangle - \alpha du \right\right] ,$$

where we used the definition of $b$ in (13). Note that using A3-(b) with $x_1 = x$ and $x_2 = 0$, we have for any $x \in \mathbb{R}^d$

$$\langle \nabla U(x), x \rangle \geq \langle \nabla U(0), x \rangle + \alpha \|x\|^2 - \alpha.$$

Therefore, using this result, (13) and the Cauchy-Schwarz inequality we obtain that for any $t \geq 0$

$$\frac{1}{2} E[\|X^1_{t,N}\|^2] - \frac{1}{2} E[\|X^1_{0,N}\|^2] \leq \int_0^t \int_{\mathbb{R}^d} E[|b(y)(X^1_{u,N}, \lambda^*_u, y)| \|X^1_{u,N}\|] d\mu(y) du$$

$$+ \int_0^t \left\{ -\alpha m E[\|X^1_{u,N}\|^2] + \alpha \|\nabla U(0)\| E[\|X^1_{u,N}\|] + \alpha \|X^1_{u,N}\| + \alpha \alpha + \alpha \right\} du ,$$

where the last line follows from Tonelli’s Theorem (e.g. [11, Theorem 18.3]) since all integrated functions are positive (or always negative, in which case we can consider minus the integral itself).

Let $\gamma^N_t = E[\|X^1_{t,N}\|^2]$. Appealing to the fundamental theorem of calculus we get that

$$\frac{1}{2} d\gamma^N_t/dt \leq E \left[ \int_{\mathbb{R}^d} |b(\hat{\nu}_{\eta}(X^1_{u,N}, \lambda^*_u, y))| \|X^1_{u,N}\| d\mu(y) \right]$$

$$- \alpha m E[\|X^1_{t,N}\|^2] + \alpha \|\nabla U(0)\| E[\|X^1_{t,N}\|] + \alpha \|X^1_{t,N}\| + \alpha \alpha + \alpha \right\} ,$$

where

$$\frac{1}{2} d\gamma^N_t/dt \leq M/\eta (\gamma^N_t)^{1/2} - \alpha m \gamma^N_t + \alpha \|\nabla U(0)\| (\gamma^N_t)^{1/2} + \alpha \alpha + \alpha \right\} + \alpha \eta \gamma^N_t.$$
Noting that for any $a, b$ such that $ab \geq 1/2$, for any $x \geq 0$ we have $\sqrt{x} \leq a + bx$, and setting $a = (b/\eta + \alpha \|\nabla U(0)\|)/\alpha$ and $b = 1/2\alpha$ we have:

$$\frac{1}{2}d\gamma_t^N/dt \leq \frac{(b/\eta + \alpha \|\nabla U(0)\|)^2}{\alpha} + \alpha c + a d - \frac{1}{2}\alpha \alpha m \gamma_t^N$$

Hence, for any $t \geq 0$ and any $N \in \mathbb{N}$ we get that $\gamma_t^N \leq C$ with

$$C = 2 \left( \frac{(b/\eta + \alpha \|\nabla U(0)\|)^2}{\alpha} + \alpha c + a d \right) / \alpha m$$

Therefore, letting $t \to +\infty$ we get that for any $N \in \mathbb{N}$, $\int_{\mathbb{R}^d} \|\tilde{\xi}\|^2 d\pi^{1,N}(\tilde{\xi}) \leq C$. Hence, $\{\pi^{1,N} : N \in \mathbb{N}\}$ is relatively compact in $\mathcal{P}_2(\mathbb{R}^d)$ using [2, Proposition 7.1.5].

Let $\pi^*$ be a cluster point of $\{\pi^{1,N} : N \in \mathbb{N}\}$. Let $(N_k)_{k \in \mathbb{N}}$ be an increasing sequence such that $\lim_{k \to +\infty} W_1(\pi^{1,N_k}, \pi^*) = 0$. We have that for any $t \geq 0$

$$W_1(\pi^*, \pi_{\alpha, \eta}^*) \leq W_1(\pi^*, \pi_{1,N_k}^*)$$

$$+ \frac{W_1(\pi_{1,N_k}, \lambda_{1,N_k}^*(\pi_{N_k}^*))}{\lambda_{1,N_k}^*(\pi_{N_k}^*)} + \frac{W_1(\lambda_{1,N_k}^*(\pi_{N_k}^*), \lambda_{1,N_k}^*(\pi_{N_k}^*))}{\lambda_{1,N_k}^*(\pi_{N_k}^*)}$$

$$+ \frac{W_1(\lambda_{1,N_k}^*(\pi_{N_k}^*), \lambda_{1,N_k}^*(\pi_{N_k}^*))}{\lambda_{1,N_k}^*(\pi_{N_k}^*)}$$

We now control each of these terms. Let $\varepsilon > 0$ and set $t \geq 0$ such that, using Proposition 6, $W_1(\lambda_{1,N_k}^*(\pi^*), \pi_{\alpha, \eta}^*) \leq \varepsilon$. Using Lemma 20, there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we have $W_1(\lambda_{1,N_k}^*(\pi^*), \pi_{N_k}^*) \leq \varepsilon$. Using Proposition 7, there exists $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$ we have that $W_1(\lambda_{1,N_k}^*(\pi_{N_k}^*), \lambda_{1,N_k}^*(\pi_{N_k}^*)) \leq \varepsilon$. Since $\pi_{N_k}$ is invariant for $(X_{1,N_k})_{t \geq 0}$ we get that $W_1(\pi_{1,N_k}, \lambda_{1,N_k}^*(\pi_{N_k}^*)) = 0$. Finally, there exists $k_2 \in \mathbb{N}$ such that for any $k \geq k_2$ we have $W_1(\pi^*, \pi_{1,N_k}^*) \leq \varepsilon$. Combining these results in (45), we get that $W_1(\pi^*, \pi_{\alpha, \eta}^*) \leq 4\varepsilon$. Therefore, since $\varepsilon > 0$ is arbitrary, we have that $\pi^* = \pi_{\alpha, \eta}^*$, which concludes the proof.

C Proofs of Section 4

C.1 Basics on Lions derivatives

We briefly recall here the definition of Lions derivative of functionals defined over $\mathcal{P}_2(\mathbb{R}^d)$. In this section, we fix an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued, $\mathcal{F}$-measurable random variables with finite second moment. We refer to [23, 20, 52, 16, 8] for a thorough exposition of Lions derivatives with applications to mean-field games. The following definitions can be found in [8, 52]. We start by introducing the notion of $L$-differentiability.

**Definition 21.** Let $\mathcal{M} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\overline{\mathcal{M}} : L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ such that for any $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, $\overline{\mathcal{M}}(X) = \mathcal{M}(\mathcal{L}(X))$, where $\mathcal{L}(X)$ is the distribution of $X$. $\mathcal{M}$ is called a lifted version of $\mathcal{M}$. We say that $\mathcal{M}$ is $L$-differentiable at $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists a random variable $X_0$ such that $\mathcal{L}(X_0) = \nu_0$ and $\mathcal{M}$ is Fréchet differentiable at $X_0$, i.e. there exists $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that

$$\lim_{||X-X_0||_2 \to 0} \frac{||\overline{\mathcal{M}}(X) - \overline{\mathcal{M}}(X_0) - \langle Y, X-X_0 \rangle||}{||X-X_0||_2} = 0.$$
Example 23. Consider \( \mathcal{M} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^p \to \mathbb{R} \) given by \( \mathcal{M}(\nu, y) = \frac{1}{\nu(k(y)) + \eta} \) with \( k \) satisfying A1 and \( \eta > 0 \). We have \( \mathcal{M}(X, y) = \frac{1}{\mathcal{M}(X, y)} \) with \( X \sim \nu \) and we have for any \( H \in L^2(\Omega, \mathbb{R}^d) \)

\[
\mathcal{M}(X + H) - \mathcal{M}(X) = \left\langle \nabla_1 k(X, y), H \right\rangle - \left( \frac{1}{\mathbb{E} [k(X + H, y)] + \eta} - \frac{1}{\mathbb{E} [k(X, y)] + \eta} \right) \left( \nabla_1 k(X, y) \right).
\]

Now, since \( \mathbb{E} [k(X + H, y)] + \eta \geq \mathbb{E} [k(X, y)] + \eta \geq 0 \) and \( \mathbb{E} [k(X, y)] + \eta \geq 0 \) and, as shown in [8, Example 2] the L-derivative of \( \mathcal{Q}(\nu, y) = \nu(k(y)) \) is \( D^L \mathcal{Q}(\nu, y)(x) = \nabla_1 k(\cdot, y)(x) \), we have

\[
D^L \mathcal{M}(\nu, y)(x) = -\frac{\nabla_1 k(x, y)}{\nu(k(y)) + \eta}.
\]

Similarly to Definition 22, we introduce the higher-order Lions derivatives (see, e.g., [8, Appendix A] or [52, Appendix A.2]).

Definition 24. Let \( \mathcal{M} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) be L-differentiable. For any \( x \in \mathbb{R}^d \) and \( i \in \{1, \ldots, d\} \), define \( g^i_\nu : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \) such that for any \( x \in \mathbb{R}^d \), \( i \in \{1, \ldots, d\} \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) we have \( g^i_\nu(\mu) = D_{\nu}^L \mathcal{M}(\mu)(x)_i \). If for any \( x \in \mathbb{R}^d \) and \( i \in \{1, \ldots, d\} \), \( g^i_\nu \) is L-differentiable then we say that \( \mathcal{M} \) is twice L-differentiable and we define \( (D^L)^2 \mathcal{M} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) given for any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( x_0, x_1 \in \mathbb{R}^d \) by

\[
(D^L)^2 \mathcal{M}(\mu)(x_0, x_1) = (D^L(g^i_\nu))(\mu)(x_1))_{i \in \{1, \ldots, d\}}.
\]

Finally, we define the class \( C^{2,(2,1)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \mathbb{R}^p) \) used to establish the main strong approximation results in [8]. An equivalent definition for \( \mathbb{R} \)-valued functionals \( \mathcal{M} \) is given in [8, Definition A.2].

Definition 25. Let \( \mathcal{M} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^p \). We say that \( \mathcal{M} \in C^{2,(2,1)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \mathbb{R}^p) \) if the following hold:

\( (a) \) For any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), \( \mathcal{M}(\cdot, \mu) \) is twice differentiable.

\( (b) \) For any \( x \in \mathbb{R}^d \), \( \mathcal{M}(x, \cdot) \) is L-differentiable and \( (x, \mu, y) \mapsto D^L \mathcal{M}(x, \mu)(y) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^p) \).

\( (c) \) For any \( x \in \mathbb{R}^d \), \( \mathcal{M}(x, \cdot) \) is twice L-differentiable and we have that \( (x, \mu, y_0, y_1) \mapsto (D^L)^2 \mathcal{M}(x, \mu)(y_0, y_1) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d) \).

\( (d) \) For any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( y \in \mathbb{R}^d \), \( x \mapsto D^L \mathcal{M}(x, \mu)(y) \) is differentiable and \( (x, \mu, y) \mapsto d_x D^L \mathcal{M}(x, \mu)(y) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d) \).

C.2 Proof of Proposition 10

In this section, we let \( \alpha, \eta > 0 \) and verify that the drift in (10) satisfies the conditions of [8, Theorem 2.2]. For completeness we report the conditions of [8, Theorem 2.2] below. We denote by \( b(x, \nu) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \) and \( \sigma(x, \nu) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \) the drift and diffusion coefficient of the MKVSDE, respectively. In the following, \( ||s|| \) denotes the standard Euclidean norm if \( s \) is a vector and the Hilbert-Schmidt norm \( ||s|| = (\sum_{i,j} s_{ij}^2)^{1/2} \) if \( s \) is a matrix.

Conditions on the drift coefficient

\( (A_2^2) \) There exists constants \( L_1^2, \alpha_1 > 0 \) such that

\[
||b(x_1, \nu) - b(x_2, \nu) || \leq L_1^2 (1 + ||x_1||^{\alpha_1} + ||x_2||^{\alpha_1}) ||x_1 - x_2||.
\]

\[
\langle x_1 - x_2, b(x_1, \nu) - b(x_2, \nu) \rangle \leq L_1^2 ||x_1 - x_2||^2.
\]

\[
||b(x, \nu_1) - b(x, \nu_2)|| \leq L_1^2 W_2(\nu_1, \nu_2).
\]

\( (A_2^2) \) Let \( b_i \in C^{2,(2,1)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \mathbb{R}^d) \), where \( b_i \) denotes the \( i \)-th component of \( b \). There exists constants \( L_2^2, \alpha_2 > 0 \) such that for any \( i = 1, \ldots, d \)

\[
||\nabla_i b_i(x, \nu)(z)|| \sqrt{||D^L \{D^L b_i(x, \nu)(\cdot)\}(z)||} \sqrt{||D^L b_i(x, \nu)(z, z)||} \leq L_2^2 \left( 1 + ||x||^{1+\alpha_2} + ||z||^{1+\alpha_2} + \nu(|\cdot|^2)^{(1+\alpha_2)/2} \right).
\]
(A₃) There exists constants $L^3, \alpha_3 > 0$ such that
\[
\| \nabla b(x_1, \nu_1) - \nabla b(x_2, \nu_2) \| \leq L^3 (\| x_1 - x_2 \| + W_2(\nu_1, \nu_2)) \\
\cdot (1 + \| x_1 \|^{\alpha_3} + \| x_2 \|^{\alpha_3} + \nu_1(1 \cdot | \cdot |^{3/2})^{\alpha_3/2} + \nu_2(1 \cdot | \cdot |^{3/2})^{\alpha_3/2})
\]
\[
\| D^2 b(x_1, \nu_1)(z_1) - D^2 b(x_2, \nu_2)(z_2) \| \leq L^3 (\| x_1 - z_1 \| + \| x_2 - z_2 \| + W_2(\nu_1, \nu_2)) \\
\cdot (1 + \| x_1 \|^{\alpha_3} + \| x_2 \|^{\alpha_3} + \| z_1 \|^{\alpha_3} + \| z_2 \|^{\alpha_3} + \nu_1(1 \cdot | \cdot |^{3/2})^{\alpha_3/2} + \nu_2(1 \cdot | \cdot |^{3/2})^{\alpha_3/2})
\]

(A₄) There exists a constant $L_b^4 > 0$ such that
\[
\| b(0, \nu) \| \leq L_b^4.
\]

Conditions on the diffusion coefficient
(A₅) There exists a constant $L_\sigma^0 > 0$ such that
\[
\| \sigma(x_1, \nu_1) - \sigma(x_2, \nu_2) \| \leq L_\sigma^0 (\| x_1 - x_2 \| + W_2(\nu_1, \nu_2))
\]

(A₆) Let $\sigma_{ij} \in C^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \mathbb{R})$, where $\sigma_{ij}$ denotes the $(i,j)$-th component of $\sigma$. There exists a constant $L_\sigma^{3} > 0$ such that for any $i,j = 1, \ldots, d$
\[
\| D^2 \sigma_{ij}(x, \nu)(z) \| + \| \nabla(D^2 \sigma_{ij}(x, \nu)(\cdot)(z)) \| + \| (D^2)^2 \sigma_{ij}(x, \nu)(z, z) \| \leq L_\sigma^2
\]

(A₇) There exists a constant $L_\sigma^{3} > 0$ such that
\[
\| \nabla \sigma(x_1, \nu_1) - \nabla \sigma(x_2, \nu_2) \| \leq L_\sigma^3 (\| x_1 - x_2 \| + W_2(\nu_1, \nu_2))
\]
\[
\| D^2 \sigma(x_1, \nu_1)(z_1) - D^2 \sigma(x_2, \nu_2)(z_2) \| \leq L_\sigma^3 (\| x_1 - z_1 \| + \| x_2 - z_2 \| + W_2(\nu_1, \nu_2))
\]

(A₈) There exists a constant $L_\sigma^{3} > 0$ such that
\[
\| \sigma(0, \nu) \| + \| D^2 \sigma(x_1, \nu)(x_2, \nu) \sigma(x_2, \nu) \| \leq L_\sigma^4.
\]

We start by computing the $L$-derivatives of $b$. We recall that $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is given for any $x \in \mathbb{R}^d$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ by
\[
b(x, \nu) = \int_{\mathcal{P}} \tilde{b}_\eta(x, \nu, y) \, d\mu(y) - \alpha \nabla U(x), \quad \tilde{b}_\eta(x, \nu, y) = \nabla_x k(x, y)/\nu[k(\cdot, y)] + \eta.
\]
In all of our results, $\mu$ can be replaced by $\mu^M$.

Lemma 26. Let $\alpha, \eta > 0$ and assume $A_1$ and $A_3$. Then $b \in C^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \mathbb{R}^d)$. In addition, we have for any $\nu \in \mathcal{P}_2(\mathbb{R}^d), x_0, x_1, x_2 \in \mathbb{R}^d$

1. $D^2 b(x_0, \nu)(x_1) = \int_{\mathcal{P}} \nabla_x k(x_0, y) \odot \nabla_x k(x_1, y)/\nu[k(\cdot, y)] + \eta \, d\mu(y)$
2. $D^2 b(x_0, \nu)(x_1) = \int_{\mathcal{P}} \nabla_x k(x_1, y) \odot \nabla_x k(x_0, y)/\nu[k(\cdot, y)] + \eta \, d\mu(y)$
3. $(D^2)^2 b(x_0, \nu)(x_1, x_2) = \int_{\mathcal{P}} \nabla_x \nabla_x k(x_0, y) \odot \nabla_x \nabla_x k(x_1, y) \odot \nabla_x \nabla_x k(x_2, y)/\nu[k(\cdot, y)] + \eta \, d\mu(y)$
4. $D^2 b(x_0, \nu)(x_1) = \int_{\mathcal{P}} \nabla_x k(x_0, y) \odot \nabla_x k(x_1, y) \odot \nabla_x k(x_2, y)/\nu[k(\cdot, y)] + \eta \, d\mu(y)$

where $\odot$ denotes the outer product.

Proof. We prove each result separately.

1. Recall that $b_i(x, \nu) = \int_{\mathcal{P}} \tilde{b}_{i,\eta}(x, \nu, y) \, d\mu(y) - \alpha \partial_i U(x)$ where $\partial_i$ denotes the derivative w.r.t. the $i$-th component and $\tilde{b}_{i,\eta}(x, \nu, y) = \partial_i k(x, y)/\nu[k(\cdot, y)] + \eta$. We have that for any $x \in \mathbb{R}^d, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$
\[
\partial_i \tilde{b}_{i,\eta}(x, y) = \partial_i \tilde{b}_{i,\eta}(x, y)/\nu[k(\cdot, y)] + \eta.
\]

We have that for any $x \in \mathbb{R}^d, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, $\| \partial_i \tilde{b}_{i,\eta}(x, y) \| \leq M/\eta$ and $\| \tilde{b}_{i,\eta}(x, y) \| \leq M$. Hence, using the dominated convergence theorem, $b \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and we have that for any $x_0 \in \mathbb{R}^d$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$
\[
\partial^2 b(x_0, \nu) = \int_{\mathcal{P}} \partial^2 \tilde{b}(x_0, y) \odot \nabla_x k(x_0, y)/\nu[k(\cdot, y)] + \eta \, d\mu(y) - \alpha \partial^2 \tilde{b}(x_0, y)\partial_i U(x_0).
\]

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2. Define $\tilde{b}: \mathbb{R}^d \times L^2(\Omega, \mathbb{R}^d) \to \mathbb{R}^d$, for any $x \in \mathbb{R}^d$ and $X \in L^2(\Omega, \mathbb{R}^d)$, via

$$\tilde{b}(x, X) = \int_{\mathcal{P}_R} \nabla_1 k(x, y)/(E[k(X, y)] + \eta)\,d\mu(y) - \alpha \nabla U(x),$$

so that for any $X \in L^2(\Omega, \mathbb{R}^d)$, $\tilde{b}(\cdot, X) = b(\cdot, \mathcal{L}(X))$. In addition, using Example 23, the fact that $D^L, \mathcal{M}$ therein is bounded under $A_1$ and the dominated convergence theorem we have that for any $H \in L^2(\Omega, \mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$\lim_{\|H\| \to 0} \left| \|\tilde{b}(x, X + H) - \tilde{b}(x, X)\| \right| = \int_{\mathcal{P}_R} \frac{\nabla_1 k(x, y)/(E[k(X, y)] + \eta)\,d\mu(y)}{E[k(X, y)] + \eta} \|\tilde{b}(x, X)\| \to 0.$$

Therefore, for any $x_0, x_1 \in \mathbb{R}^d$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$

$$D^Lb(x_0, \nu)(x_1) = \int_{\mathcal{P}_R} \nabla_1 k(x_0, y) \otimes \nabla_1 k(x_1, y)/(E[k(X, y)] + \eta)\,d\mu(y).$$

Using the boundedness of $\nabla_1 k$ ensured by $A_1$ and the dominated convergence theorem, we have $(x_0, \nu, x_1) \mapsto D^Lb(x_0, \nu)(x_1) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d, \mathbb{R}^{d \times d})$.

3. For any $x_1 \in \mathbb{R}^d$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $x_0 \mapsto D^Lb(x_0, \nu)(x_1) \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and we have that for any $x_0 \in \mathbb{R}^d$

$$\partial_x D^L b(x_0, \nu)(x_1) = \int_{\mathcal{P}_R} \partial_x \partial_x \partial_x k(x_0, y) \otimes \nabla_1 k(x_1, y)/(E[k(X, y)] + \eta)\,d\mu(y).$$

4. Finally, we define $\tilde{g}: \mathbb{R}^d \times L^2(\Omega, \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ such that for any $x_0, x_1 \in \mathbb{R}^d$ and $X \in L^2(\Omega, \mathbb{R}^d)$

$$\tilde{g}(x_0, X, x_1) = \int_{\mathcal{P}_R} \nabla_1 k(x_0, y) \otimes \nabla_1 k(x_1, y)/(E[k(X, y)] + \eta)\,d\mu(y).$$

For any $X \in L^2(\Omega, \mathbb{R}^d)$, $\tilde{g}(\cdot, X) = D^L(\cdot, \mathcal{L}(X), \cdot)$. In addition, using the boundedness ensured by $A_1$ and dominated convergence theorem we have that for any $H \in L^2(\Omega, \mathbb{R}^d)$ and $x_0, x_1 \in \mathbb{R}^d$

$$\frac{1}{\|H\|} \left\| \tilde{g}(x_0, X + H, x_1) - \tilde{g}(x_0, X, x_1) - \int_{\mathcal{P}_R} \nabla_1 k(x_0, y) \otimes \nabla_1 k(x_1, y)/(E[k(X, y)] + \eta)\,d\mu(y) \right\| \to 0.$$

as $\|H\| \to 0$. Therefore, for any $x_0, x_1, x_2 \in \mathbb{R}^d$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$

$$\begin{align*}
(D^L)^2 b(x_0, \nu)(x_1, x_2) & = \int_{\mathcal{P}_R} \nabla_1 k(x_0, y) \otimes \nabla_1 k(x_1, y) \otimes \nabla_1 k(x_2, y)/(E[k(X, y)] + \eta)\,d\mu(y). \\

\end{align*}$$

We conclude using again the boundedness of $\nabla_1 k$ ensured by $A_1$ and the dominated convergence theorem to show that $(x_0, \mu, x_1, x_2) \mapsto (D^L)^2 b(x_0, \nu)(x_1, x_2) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{d \times d \times d}).$

Proof of Proposition 10. Using Lemma 26 there exists $K \geq 0$ such that for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $x_0, x_1, x_2 \in \mathbb{R}^d$ we have

$$\|b(0, \nu)\| + \|\nabla b(0, \nu)\| + \|D^Lb(x_0, \nu)(x_1)\|$$

$$+ \|D^Lb(x_0, \nu)(x_1, x_2)\| + \|(D^L)^2 b(x_0, \nu)(x_1, x_2)\| \leq K.$$

In addition, using (35), for any $x_1, x_2 \in \mathbb{R}^d$ and $x_1, x_2 \in \mathbb{R}^d$ we have

$$\|b(x_1, x_2) - b(x_1, x_2)\| \leq (\Phi/\eta)(1 + (1/\eta))(1 + \alpha L)(1 + \Phi)\|x_1 - x_2\| + W_2(\nu_1, \nu_2).$$

Hence, $(A^2_k)$ in [8] is satisfied. Using (46) we have that $(A^2_k)$ and $(A^2_k)$ in [8] are satisfied. Similarly to (35), we have for any $x_1, x_2 \in \mathbb{R}^d$ and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$

$$\|D^Lb(x_1, \nu_1)(x_2) - D^Lb(x_2, \nu_2)(x_3)\|$$

$$\leq \|\int_{\mathcal{P}_R} \nabla_1 k(x_1, y) \otimes \nabla_1 k(x_2, y)/(\nu_1[k(X, y)] + \eta)\,d\mu(y)$$

$$- \int_{\mathcal{P}_R} \nabla_1 k(x_2, y) \otimes \nabla_1 k(x_1, y)/(\nu_2[k(X, y)] + \eta)\,d\mu(y)\|$$

$$\leq (\Phi^2/\eta)(1 + (1/\eta))(1 + \Phi)\{|x_1 - x_2\| + \|x_2 - x_3\| + W_2(\nu_1, \nu_2).$$

Combining (39) and (47), we get that $(A^3_k)$ in [8] is satisfied. In addition, note that $(A^3_k)$, $(A^3_k)$, $(A^3_k)$ and $(A^3_k)$ are immediately satisfied since $\sigma = \sqrt{2\alpha}$ in our case. We conclude the proof upon using [8, Theorem 2.5].
C.3 Proof of Proposition 11

Let \( M, N \in \mathbb{N}^* \) and \( \alpha, \eta > 0 \). Using (35), we have for any \( x_1, x_2 \in \mathbb{R}^d \) and \( \nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R}^d) \)

\[
\|b(x_1, \nu_1) - b^M(x_2, \nu_2)\|
\leq ((\mathbb{M}/\eta)(1 + (1/\eta)) + L)(\|x_1 - x_2\| + \int_{\mathbb{R}^d} |\nu_1[k(\cdot, y)] - \nu_2[k(\cdot, y)]| \, d\mu(y))
+ \|(1/M) \sum_{j=1}^M \{b_0(x_2, \nu_2, y^k, M) - \int_{\mathbb{R}^d} b_0(x_2, \nu_2, y) \, d\mu(y) \}\|.
\]  

(48)

For any \( i \in \{1, 2\} \) and \( x_i \in (\mathbb{R}^d)^N \), let \( \nu_i = (1/N) \sum_{k=1}^N \delta_{x_i} \). Using (48), we have that for any \( x_1 \in (\mathbb{R}^d)^N \)

\[
\|b(x_1, \nu_1) - b^M(x_2, \nu_2)\|
\leq ((\mathbb{M}/\eta)(1 + (1/\eta)) + L)(1 + M)
\times \left(\|x_1 - x_2\|^2 + (1/N) \sum_{k=1}^N \|x_{1,k} - x_{2,k}\|\right)
+ \|(1/M) \sum_{j=1}^M \{b_0(x_2, \nu_2, y^k, M) - \int_{\mathbb{R}^d} b_0(x_2, \nu_2, y) \, d\mu(y) \}\|.
\]

(49)

Let \( C \geq 0 \) and \( \{A_k\}_{k=1}^M \) be a family of \( d \)-dimensional random variables such that for any \( i, j \in \{1, \ldots, M\} \) with \( i \neq j \)

\[ E[\|A_i\|^2] \leq C, \quad E[\langle A_i, A_j \rangle] = 0. \]  

(50)

Then, we have that

\[ E[\|(1/M) \sum_{k=1}^M A_k\|] \leq (C/M)^{1/2}. \]  

(51)

Let \( A_k := b_0(X^{1,N,M}_n, \lambda^{1,N,M}_n, y^{k,M}) - \int_{\mathbb{R}^d} b_0(X^{1,N,M}_n, \lambda^{1,N,M}_n, y) \, d\mu(y) \) for any \( k \in \{1, \ldots, M\} \). Then \( \{A_k\}_{k=1}^M \) satisfies (50) with \( C = \mathbb{M}/\eta \). Combining this result, (49), (51) and that for any \( n \in \mathbb{N} \), \( \{X^{1,N}_n - X^{k,N,M}_n\}_{k=1}^\infty \) is exchangeable, we have

\[
E[\|b(X^{1,N}_n, \lambda^{1}_n) - b^M(X^{1,N,M}_n, \lambda^{N,M}_n)\|]
\leq ((\mathbb{M}/\eta)(1 + (1/\eta)) + L)(1 + M)
\times \left(\|X^{1,N}_n - X^{1,N,M}_n\|^2 + (1/N) \sum_{k=1}^N \|X^{1,N}_n - X^{k,N,M}_n\|\right)
+ \|(1/M) \sum_{k=1}^M \{b_0(X^{1,N,M}_n, \lambda^{1,N,M}_n, y^{k,M}) - \int_{\mathbb{R}^d} b_0(X^{1,N,M}_n, \lambda^{1,N,M}, y) \, d\mu(y) \}\|.
\]

(52)

Using this result, we have for any \( n \in \mathbb{N} \)

\[
E[\|X^{1,N}_{n+1} - X^{1,N,M}_{n+1}\|] \leq E[\|X^{1,N}_n - X^{1,N,M}_n\|]
+ \gamma E \left[ \frac{\gamma \|b(X^{1,N}_n, \lambda^{1}_n)\|}{(1 + \gamma \|b(X^{1,N}_n, \lambda^{1}_n)\|)} \right] \left(1 + \gamma \|b^M(X^{1,N,M}_n, \lambda^{N,M}_n)\|\right)
+ \gamma E \left[ \frac{\|b(X^{1,N}_n, \lambda^{1}_n) - b^M(X^{1,N,M}_n, \lambda^{N,M}_n)\|}{(1 + \gamma \|b^M(X^{1,N,M}_n, \lambda^{N,M}_n)\|)} \right]
\leq E[\|X^{1,N}_n - X^{1,N,M}_n\|] + 2\gamma E \left[ \|b(X^{1,N}_n, \lambda^{1}_n) - b^M(X^{1,N,M}_n, \lambda^{N,M}_n)\| \right]
\leq C \left(1 + 2\gamma\right) E[\|X^{1,N}_n - X^{1,N,M}_n\|] + 2\gamma M^{-1/2} .
\]

(53)

Proceeding recursively and recalling that \( X^{1,N}_0 = X^{0,N,M}_0 \) the result follows.

C.4 Proof of Proposition 12

The proof follows the same structure of that of Proposition 11. Let \( M, N, m \in \mathbb{N}^* \) with \( m \leq M \) and \( \alpha, \eta > 0 \). Proceeding as above we can write a result equivalent to (52)

\[ E[\|b(X^{1,N}_n, \lambda^{1}_n) - b^M(X^{1,N,m}_n, \lambda^{N,m}_n)\|]
\leq ((\mathbb{M}/\eta)(1 + (1/\eta)) + L)(1 + M)
\times \left(\|X^{1,N}_n - X^{1,N,m}_n\|^2 + (1/N) \sum_{k=1}^N E[\|X^{k,N}_n - X^{k,N,m}_n\|]\right)
+ \|(1/M) \sum_{k=1}^M \{b_0(X^{1,N,m}_n, \lambda^{1,N,m}_n, y^{k,M}) - \int_{\mathbb{R}^d} b_0(X^{1,N,m}_n, \lambda^{1,N,m}, y) \, d\mu(y) \}\|. \]  

(54)
where \( A_k := \tilde{b}_\eta(X_{1,k}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,m}) - \int_{\mathbb{R}^p} \tilde{b}_\eta(X_{1,k}^{1,m}, \lambda_\eta^{1,m}, y) d\mu(y) \) and \( \{y_1, \ldots, y_m\} \) is a sample from \( \mu^M \). Consider the last expectation and decompose

\[
E \left[ \left( 1/m \right) \sum_{j=1}^m \left\{ \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,m}) - \int_{\mathbb{R}^p} \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y) d\mu(y) \right\} \right]
\]

\[
\leq E \left[ \left( 1/m \right) \sum_{j=1}^m \left\{ \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,m}) - (1/M) \sum_{k=1}^M \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,M}) \right\} \right]
\]

\[
+ E \left[ \left( 1/M \right) \sum_{k=1}^M \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,M}) - \int_{\mathbb{R}^p} \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y) d\mu(y) \right] \right].
\]

As shown in the proof of Proposition 11, the second term satisfies

\[
E \left[ \left( 1/M \right) \sum_{k=1}^M \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,M}) - \int_{\mathbb{R}^p} \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y) d\mu(y) \right] \right] \leq \left( \Theta/(M\eta) \right)^{1/2}.
\]

Let us denote the \( \sigma \)-field generated by \( \mu^M \) by \( \mathcal{S}^M := \sigma \{ (y_j : j = 1, \ldots, M) \} \). For any \( k \in \{1, \ldots, m\} \), \( x_1 \in \mathbb{R}^d \) let \( A_k = \tilde{b}_\eta(x_1, \nu_1, y_j^{1,m}) - (1/M) \sum_{k=1}^M \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,M}) \). Conditionally on \( \mathcal{S}^M \), \( \{A_k\}_{k=1} \) satisfies (50) with \( C = \Theta/\eta \). Hence,

\[
E[\| (1/m) \sum_{j=1}^m \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,m}) - (1/M) \sum_{k=1}^M \tilde{b}_\eta(X_{1,n}^{1,m}, \lambda_\eta^{1,m}, y_j^{1,M}) \|] \leq \left( \Theta/(m\eta) \right)^{1/2}.
\]

Combining the two expectations we obtain the following bound for (54)

\[
E[\| b(X_{1,n}^{1,N}, \lambda_\eta^{1,N}) - b^M(X_{1,n}^{1,N}, \lambda_\eta^{1,N}) \|] \leq \left( \Theta/(\eta) + \Theta/(M\eta) \right) \left( 1 + \Theta \right)
\]

\[
+ \left( \Theta/(m\eta) \right)^{1/2} + \left( \Theta/(M\eta) \right)^{1/2}
\]

\[
+ (1/N) \sum_{k=1}^N E[\| X_{1,n}^{1,N} - X_{1,k,n}^{1,N} \|] + (1/N) \sum_{k=1}^N E[\| X_{1,k,n}^{1,N} - X_{1,M,n}^{1,N} \|] + 2(\Theta/(m\eta))^{1/2},
\]

where we used the fact that \( m \leq M \). The above allows us to obtain an equivalent result to (53). Proceeding recursively and recalling that \( X_{1,n}^{1,N} = X_0^{1,N} \) we have the result.

### C.5 Stability of continuous time process

**Proposition 27.** Assume \( A1, A3 \). Let \( \eta, \alpha > 0 \). For any \( N, M \in \mathbb{N}^* \), let \( \{X_{1,n}^{1,N}\}_{n \geq 0} \) and \( \{X_{1,n}^{1,N,M}\}_{n \geq 0} \) be the solution of \( (12) \) and \( (15) \) respectively, with initial condition \( X_{1,0}^{1,N} \in \mathcal{P}_1((\mathbb{R}^d)^N) \) such that \( \{X_{1,n}^{1,N}\}_{n \geq 0} \) is exchangeable and \( \{X_{1,n}^{1,N,M}\}_{n \geq 0} \). Then for any \( T \geq 0 \) there exists \( C_T \geq 0 \) such that for any \( M, N \in \mathbb{N}^* \) and \( t \in \{1, \ldots, N\} \)

\[
E[\sup_{t \in [0, T]} \| X_{1,n}^{1,N} - X_{1,t}^{1,N,M} \|] \leq C_T M^{-1/2}.
\]

**Proof.** The proof is identical to that of Proposition 11 with the discrete recursion replaced by an application of Grönwall’s lemma. Let \( M, N \in \mathbb{N}^* \) and \( \eta, \alpha > 0 \). For any \( k \in \{1, \ldots, M\} \), \( x_1 \in \mathbb{R}^d \) and \( v_1 \in \mathcal{P}_1(\mathbb{R}^d) \) let \( A_k = \tilde{b}_\eta(x_1, \nu_1, y_j^{1,M}) - \int_{\mathbb{R}^p} \tilde{b}_\eta(x_1, \nu_1, y) d\mu(y) \). Then \( \{A_k\}_{k=1}^M \) satisfies (50) with \( C = \Theta/\eta \).

Combining this result, (51), (49) and that for any \( t \geq 0 \), \( \{X_{1,n}^{1,N} - X_{1,k,n}^{1,N,M}\}_{k=1}^M \) is exchangeable, we have that for any \( t \geq 0 \),

\[
E[\| b(X_{1,n}^{1,N}, \lambda_\eta^{1,N}) - b^M(X_{1,n}^{1,N,M}, \lambda_\eta^{1,N,M}) \|] \leq (\Theta/\eta) + \Theta/(M\eta) \left( 1 + \Theta \right)
\]

\[
+ \left( \Theta/(m\eta) \right)^{1/2} + \left( \Theta/(M\eta) \right)^{1/2}
\]

\[
+ (1/N) \sum_{k=1}^N E[\| X_{1,n}^{1,N} - X_{1,k,n}^{1,N} \|] + (1/N) \sum_{k=1}^N E[\| X_{1,k,n}^{1,N} - X_{1,M,n}^{1,N} \|] + 2(\Theta/(m\eta))^{1/2},
\]

where we used the fact that \( m \leq M \). The above allows us to obtain an equivalent result to (53). Proceeding recursively and recalling that \( X_{1,n}^{1,N} = X_0^{1,N} \) we have the result.
Using this result, we have for any $t \geq 0$
\[
E[\sup_{s \in [0,t]} \|X_{s,N}^1 - X_{s,M}^N\|] 
\leq C(\delta(t) + \epsilon(t) + L)\int_0^t E[\sup_{s \in [0,t]} \|X_{s,n}^1 - X_{s,M}^N\|]ds + t(\delta/M)\epsilon(t)^{1/2}.
\]
We conclude upon using Grönwall’s lemma.

\section*{D Additional Experiments}

\subsection*{D.1 Choice of initial distribution and reference measure}

To study the influence of the reference measure $\pi_0$ on $\mathcal{G}_n^\alpha$ and of the initial condition $X_0^{1,N}$ on the speed of convergence we consider the following toy Fredholm integral equation
\[
\mathcal{N}(y;0,\sigma_y^2 + \sigma_x^2) = \int_{\mathbb{R}} \mathcal{N}(y;x,\sigma_x^2)dx,
\]
where $\mathcal{N}(x;m,\sigma^2)$ is a Gaussian distribution with mean $m$ and variance $\sigma^2$, with $\sigma_y^2 = 0.43^2$ and $\sigma_x^2 = 0.45^2$. We assume that $\mu$ is known through a sample of size $M = 10^4$, we consider four reference measures: a uniform distribution, a zero-mean Gaussian distribution with variance given by the empirical variance of the sample from $\mu$, a Student’s $t$-distribution with 100 degrees of freedom and a zero-mean Laplace distribution with variance given by the empirical variance of the sample from $\mu$; and three initial conditions: the solution $\pi$, a uniform over the interval $[-2,2]$ which contains more than 99% of the mass and a highly peaked zero-mean Gaussian distribution. As an example we set $N = m = 500$, $\gamma = 10^{-2}$, $\alpha = 0.02$ and iterate for 300 time steps. The behaviour for other values of $N, m$ is similar.

Figure 7 shows that the reference measure does not influence the speed of convergence but influences the shape of the reconstructions, which are more peaked when the reference measure is Laplace and flatter when the reference measure is a Gaussian distribution; this corresponds to slightly different minima for $\mathcal{G}_n^\alpha$. In particular, the order of the minima is not influenced by the initial condition, with the uniform reference measure always giving the smallest minima. Indeed, if we compare the values of $\mathcal{G}_n^\alpha$ at the solution $\pi$ we find that when the reference measure is Uniform, $\text{KL}(\pi|\pi_0)$ is not defined and
\[
\mathcal{G}_n^\alpha(t) \geq \mathcal{G}_n^\alpha(\text{Gaussian}) \simeq \mathcal{G}_n^\alpha(\text{Laplace}).
\]

The initial condition $X_{0,n}^{1,N}$ affects the speed of convergence with light tail distributions giving faster convergence as already observed in [14, 4], but does not affect the shape of the reconstructions. This is not surprising as Proposition 7 guarantees the existence of a unique solution for any initial condition $X_{0,n}^{1,N}$.

\subsection*{D.2 Choice of $\alpha$}

\subsubsection*{D.2.1 Toy problem}

Consider again the toy Fredholm integral equation in Appendix D.1 and consider $\mathcal{G}_n$ with the reference measure $\pi_0$ given by a Gaussian distribution with mean 0 and variance $\sigma_0^2$. If we restrict the class of distributions on which we look for a minimizer to that of Gaussian distributions with mean 0 and variance $\beta$, i.e. $\pi(x) = \mathcal{N}(x;0,\beta)$, the functional $\mathcal{G}_n$ can be computed exactly
\[
\mathcal{G}_n(\pi) = -\int_{\mathbb{R}} \log(\pi(k,y))d\mu(y) + \alpha\text{KL}(\pi|\pi_0) 
= (1/2)\log(2(\beta + \sigma_0^2)) + \sigma_0^2/(2(\beta + \sigma_0^2)) + \alpha/2(\log\sigma_0^2/\beta + \beta/\sigma_0^2 - 1).
\]
Finding the optimal $\beta$ for given $\alpha$ amounts to solving the following cubic equation
\[
\alpha\beta^3 + \beta^2(2\alpha\sigma_0^2 + (1-\alpha)\sigma_0^2) + \beta\left(\alpha\sigma_0^4 - \sigma_0^4\sigma_0^2 + \sigma_0^4\sigma_0^2 - 2\alpha\sigma_0^4\right) - \alpha\sigma_0^2\sigma_0^4 = 0,
\]
clearly when $\alpha = 0$ (no cross-entropy constraint), $\beta = \sigma_0^2$, while for all $\alpha > 0$ (at least) one optimal $\beta$ exists. However, as $\alpha$ increases the value of $\beta$ stops increasing and stabilizes around a fixed value which depends on $\sigma_0^2$. Figure 8 shows the result for $\alpha \in [0,1]$ and $\sigma_0^2 = 0.1^2$.

As discussed in Section 2.2, taking $\alpha = 1$ amounts to Bayesian inference for a model with prior $\pi_0$ and data given by the available sample from $\mu$. Using the cubic equation above we obtain that the corresponding optimal $\beta$ is 0.44; comparing the minimizer obtained with $\alpha = 1$ with the solution $\pi$ and the minimizer corresponding to the value of $\alpha = 4 \cdot 10^{-5}$ obtained by cross-validation we find that $\alpha = 1$ results in significantly over-smoothed reconstructions (Figure 9) while the value chosen by cross-validation provides reconstructions which are indistinguishable from the solution $\pi$. 

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Figure 7: Influence of reference measure and initial condition, reconstructions (left panel) and approximate value of $G_{\eta}$ (right panel).

Figure 8: Example of dependence of the variance of the minimizer $\pi^\star_{\alpha}$ (left) and the functional $G_{\alpha}$ (right) on the parameter $\alpha$. The red lines denote the values obtained with $\alpha = 0$, i.e. no regularization.
Figure 9: Effect of regularizing parameter $\alpha$ on reconstructions. We compare $\alpha = 1$ which corresponds to Bayesian inference for the incomplete data model with prior $\pi_0$ and $\alpha = \text{selected by cross-validation}$. 