Vortex with Fractional Quantum Numbers in Chiral \( p \)-Wave Superconductor

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We show that a vortex in a chiral \( p \)-wave superconductor, which has the \( p_x + ip_y \)-wave pairing state and breaks \( U(1) \), parity and time reversal symmetry simultaneously, has fractional charge \( -\frac{n\pi}{\lambda} \) and fractional angular momentum \( -\frac{n\pi}{\lambda} \) (\( n \); vorticity). This suggests that the vortex could be anyon and could obey fractional statistics. Electromagnetic property of the vortex is also discussed and we find that an electric field is induced near the vortex core.

I. INTRODUCTION

Recently, the superconductivity of \( \mathrm{Sr}_2\mathrm{RuO}_4 \) was discovered by Maeno et al. This superconductivity seems to exhibit chiral \( p \)-wave order, in which the symmetry of the wave function of Cooper pair is \( p_x + ip_y \) and parity (P) and time reversal symmetry (T) are violated. In this paper, we show that a vortex in chiral \( p \)-wave superconductor has fractional charge and fractional angular momentum as the consequence of P- and T-violation.

\( \mathrm{Sr}_2\mathrm{RuO}_4 \) has a layered perovskite crystal structure without cupper and has low transition temperature \( T_c \approx 1.5[K] \). \( \mathrm{Sr}_2\mathrm{RuO}_4 \) is a strongly correlated 2-dimensional Fermi liquid and is partially similar to \( ^3\mathrm{He} \), which shows \( p \)-wave superfluidity. The evidence for unconventional (i.e., non-\( s \)-wave) superconductivity was shown by the absence of a Habel-Slichter peak in \( 1/T_1T \) of NQR-measurements and \( T_c \)-suppression by non-magnetic impurities. Rice and Sigrist proposed that pairing symmetry of orbital part is chiral \( p \)-wave, which has the same orbital symmetry of superfluid \( ^3\mathrm{He} \).

The proposal is consistent with the discovery of internal magnetic field in the superconducting phase by \( \mu\mathrm{SR} \) measurement and the experiment of \( ^{17}\mathrm{O}-\mathrm{NMR} \) Knight shift. In the chiral \( p \)-wave superconductor, Cooper pair has orbital angular momentum \( l_z = 1 \), where \( z \) is perpendicular to the superconducting plane, i.e., the same direction as the \( c \) axis of the crystal. Therefore, local \( U(1) \) gauge symmetry, \( P \) and \( T \) are spontaneously broken by the same order parameter.

The above situation is analogous to the quantum Hall system (QHS), which is the 2-dimensional electron system in an external magnetic field. In QHS the external magnetic field violates \( P \) and \( T \), but the explicit local \( U(1) \) gauge symmetry is preserved. It was shown that the Chern-Simons term is induced in the effective action of the gauge field when Fermion is integrated out. The Chern-Simons term is a P- and T-odd bi-linear form of the gauge fields and has one derivative. Therefore, this term plays important roles in low energy and long distance physics and causes P- and T-violating electromagnetic phenomena. The coefficient of the induced Chern-Simons term is quantized exactly as \( (integer) \times \frac{\pi^2}{16n} \). \( U(1) \) gauge invariance guarantees the exact quantization. Since the coefficient of the term becomes the Hall conductance, the integer quantum Hall effect was explained from gauge invariance.

It was shown that the Chern-Simons term is also induced in the Ginzburg-Landau action of chiral \( p \)-wave superconductors. In this case, the Chern-Simons term does not have exactly quantized coefficient because of the spontaneous breaking of local \( U(1) \) gauge symmetry. The coefficient is approximately equal to the fine structure constant.

In this paper, we discuss a vortex in chiral \( p \)-wave superconductor starting from the Ginzburg-Landau Lagrangian with the Chern-Simons term. Such a vortex could obey unusual statistics by the existence of the Chern-Simons term. The above possibility has been pointed out in Chern-Simons-Higgs model by several authors. In this model, vortex solutions have been found and they have fractional angular momentum and fractional charge. The Chern-Simons vortex is a candidate of anyon that obeys the fractional statistics due to fractional angular momentum. In the present paper we find a vortex solution in chiral \( p \)-wave superconductor and show that it has fractional charge \( -\frac{n\pi}{\lambda} \) and fractional angular momentum \( -\frac{n\pi}{\lambda} \) with integrally quantized magnetic flux \( \frac{n\pi}{16} \) \((n = 0, \pm 1, \pm 2 \cdots \); vorticity\). Hence the vortex could be anyon. The starting Ginzburg-Landau Lagrangian with the Chern-Simons term is an extended version of the Chern-Simons-Higgs model, so to say, the non-relativistic Maxwell-Chern-Simons-vector Higgs model. We also discuss electromagnetic property of the vortex. In the region \( r < \lambda \) (\( r \) is the distance from the vortex core, and \( \lambda \) is London penetration depth), the vortex has an electric field from the Chern-Simons term and it would be expected that non-trivial electromagnetic phenomena occur. \( \mathrm{Sr}_2\mathrm{RuO}_4 \) is known as a type II superconductor and therefore it has a stable vortex. It is interesting if these exotic properties of the vortex are observed experimentally in the superconductivity of \( \mathrm{Sr}_2\mathrm{RuO}_4 \).

The paper is organized in the following manner. In section II, we review the derivation of the Ginzburg-Landau Lagrangian with the Chern-Simons term in chiral \( p \)-wave superconductor. In section III, we study charge and angular momentum of a vortex. In section IV, we discuss electromagnetic property of the vortex and find that an electric field is induced near the vortex core. Summary is
II. DERIVATION OF THE CHERN-SIMONS TERM AND THE GINZBURG-LANDAU LAGRANGIAN OF CHIRAL P-WAVE SUPERCONDUCTOR

Following Ref. [24], we review how the Chern-Simons term is induced in the Ginzburg-Landau Lagrangian of chiral p-wave superconductor such as Sr$_2$RuO$_4$ in this section. At first, we consider one superconducting layer, with cylindrical Fermi surface, Fermi energy given in section V. We use the natural unit ($\hbar = c = 1$) in the present paper.

It is known that the massive particle gives no higher order corrections to the Chern-Simons term, but the massless particle may give a correction. To find out a higher order correction, the authors of Ref. [16] considered a correction to the induced Chern-Simons term by $U(1)$ Goldstone mode. It was shown that $U(1)$ Goldstone mode does not give a correction to the induced Chern-Simons term and its degree of freedom corresponds to the redundant gauge degree of freedom of the system.

From Eq. (6), we can obtain Fermion propagator and calculate Fermion loop. Before doing that, to simplify notations, we introduce a vector $\vec{g}(\vec{p})$ as

$$\vec{g}(\vec{p}) = \begin{pmatrix} \text{Re} \Delta(\vec{p}) \\ -\text{Im} \Delta(\vec{p}) \end{pmatrix} = \begin{pmatrix} \eta_0 \eta_1 & i \sigma_3 \sigma_2 \\ -\eta_0 \eta_2 & i \sigma_3 \sigma_2 \end{pmatrix}.$$  \hspace{1cm} (6)

By using $\vec{g}(\vec{p})$, Eq. (1) is rewritten as

$$\begin{align*}
\mathcal{L} &= \Psi^\dagger (i\partial_0 - eA_0 \tau_3) \Psi \\
&= \Psi^\dagger \begin{pmatrix} \epsilon(\vec{p} + eA) & \Delta(\vec{p}) \\ \Delta^\dagger(\vec{p}) & -\epsilon(\vec{p} - eA) \end{pmatrix} \Psi,
\end{align*}$$

where $\Psi_\alpha = \psi_1(\psi_\psi)$ is Fermion field in Nambu-Bogoliubov representation with “isospin” $\alpha$, and $\psi_\alpha$ is the usual representation for Fermion field with spin index $s$. $(\tau)^{\alpha\beta}$ is Pauli matrix with isospin indices, and $\epsilon(\vec{p})$ is the kinetic energy of Fermion and is measured from Fermi energy $\epsilon_F$. For simplicity, we choose 1 band model with cylindrical Fermi surface therefore, $\epsilon(\vec{p})$ is written as $\epsilon(\vec{p}) = \frac{\vec{p}^2}{2m_e} - \epsilon_F$ with electron effective mass $m_e$. $\Delta(\vec{p}) \sim V_{pair} \psi \psi^\dagger > 0$ is the gap function ($\cdots > 0$ denotes the expectation value of operator in the ground state and $V_{pair}$ shows electron-electron interaction by which pairing state arises) and it is written as

$$\Delta(\vec{p}) = i \sigma_3 \vec{d}(\vec{p}),$$  \hspace{1cm} (2)

in the p-wave (spin-triplet) pairing state. $\vec{d}_{\alpha\nu'}$ is Pauli’s spin matrix and $\vec{d}$ is so-called $d$-vector. In the superconductivity of Sr$_2$RuO$_4$, proposed $d$-vector is written as

$$\begin{align*}
\vec{d}(\vec{p}) &= \vec{\eta} \cdot \vec{p}, \\
\eta(\vec{x}) &= (\eta_x(\vec{x}), \eta_y(\vec{x})),
\end{align*}$$

where $\eta$ is a vector order parameter field and it has an expectation value written as

$$\langle \eta \rangle = |\eta_0|(1, i),$$  \hspace{1cm} (4)

which shows chiral $p$-wave pairing state. For a while, we neglect the spatial dependence of $\eta$ and put $\eta = \langle \eta \rangle$.

Gauge transformation of $\Psi$ and $A_\mu$ is written as

$$\begin{align*}
\Psi &\rightarrow e^{-i\sigma_3 \eta \cdot \vec{x}} \Psi, \\
A_\mu &\rightarrow A_\mu + \partial_\mu \eta.
\end{align*}$$

(5)

For convenience, we take $\partial_0 A_0 - e_s^2 \partial_1 A_1 = 0$ gauge, where $e_s$ is the velocity of $U(1)$ Goldstone mode, and fix the redundant gauge degree of freedom which corresponds to the degree of freedom of $U(1)$ Goldstone mode.
and a contribution from the diamagnetic current in the same order is
\[ \pi^{(d)}_{ij}(q) = \left(\frac{F.T.}{\delta A^i(x')} - j^{(d)}_i(x) \right) \]
\[ = - \frac{e^2}{m_e} \delta_{ij}, \tag{13} \]
where \( x \) denotes the coordinate in (2+1)-dimensional spacetime, \((F.T.)\langle \cdot \cdot \cdot \rangle\) means that we make the Fourier transformation of following terms, \( T_r \) means the trace about isospin and real spin indices, and \( \rho_e = \frac{m_e}{\pi} \) is Fermion number density. Since we are interested in low energy and long distance region, we neglect \( O(q^2) \) terms. The constant \( \sigma_{xy} \) will be the coefficient of the Chern-Simons term, and it is written as
\[ \sigma_{xy} = \frac{i}{2!} \frac{e^{ij}}{\pi} \frac{\partial}{\partial q} \pi_0(q)|_{q=0} \]
\[ = \frac{e^2}{8} \int \frac{d^2 p}{(2\pi)^2} \frac{\text{tr} \left[ \tilde{g} \cdot (\tilde{g} \times \partial \tilde{g}) - g_3 (\tilde{g} \times \partial \tilde{g})_3 \right]}{\text{tr} \left[ \tilde{g} \cdot \tilde{g} \cdot \frac{1}{2} \right]}. \tag{14} \]
here \( \text{tr} \) means trace about real spin indices and \( \partial = \frac{\partial}{\partial q} \). The first term in Eq. \((14)\) is a topological invariant. Volovik argued this type of topological invariant. The second term is a contribution from the diamagnetic current in the r.h.s. of Eq. \((16)\), \( n_p \) is a number of the superconducting plane per unit length along the \( z \) axis (the \( c \) axis), and \( A, K_i, \) and \( \beta_i \), are phenomenological constants with appropriate dimensions.

By substituting Eq. \((8)\) into Eq. \((14)\), we obtain the value
\[ \sigma_{xy} = \frac{e^2}{4\pi}, \tag{15} \]
which coincides with the fine structure constant. In this case, the non-topological term in Eq. \((14)\) vanishes accidentally. In general, the non-topological term does not vanish. Actually, if we calculate \( \sigma_{xy} \) in the tight binding scheme, the non-topological term has negligibly small value and \( \sigma_{xy} \) is approximately equal to the fine structure constant \( \frac{e^2}{4\pi} \).

From Eqs. \((8), (11), (12), \) and \((13)\), the low energy effective Lagrangian in the case that the vector order parameter field is constant \( \langle i.e., \eta(x) = \eta = |\eta| (1, i) \rangle \) is written as
\[ \mathcal{L}_{eff.} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left( \frac{m^2}{c_s^2} - m^2 A^2 \right) + \frac{\sigma_{xy}}{2} \epsilon_{0ij} (A_0 \partial_i A_j + A_i \partial_j A_0) + O(e^3). \tag{16} \]
The second term in the r.h.s. of Eq. \((16)\) contributes Thomas-Fermi screening, and the third term contributes Meissner effect. The parameters \( m \) and \( c_s \) are determined as \( m = (\frac{m_e c_s}{v_F})^{1/2} \), \( c_s \approx (\frac{m_e}{\gamma v_F})^{1/2} = (\frac{2\pi}{\gamma})^{1/2} \), where \( m_e \), \( \rho_e \) and \( v_F \) are mass, number density and Fermi velocity of electrons, respectively. \( \lambda = m^{-1} \) is the penetration depth for the magnetic field in the usual case \( i.e. \sigma_{xy} = 0 \). The combination of the forth term and the fifth term in the r.h.s. of Eq. \((16)\) is the Chern-Simons term in \( \partial_0 A_0 - c_s^2 \partial_i A_i = 0 \) gauge.

Next we take into account spatial dependence of order parameter fields. Actual superconductor is spatially 3-dimensional object which consists of layer of the superconducting plane. Therefore, we extend above discussion to layered system. In low energy and long distance region, it is sufficient to consider lower mass dimensional terms. Except for the Chern-Simons term, the quite general phenomenological Ginzburg-Landau free energy of the chiral p-wave superconductor with tetragonal symmetry which corresponds to the layered perovskite structure of the crystal lattice has been proposed by Sigrist and Ueda. We combine both of them. Lagrangian would have the following form:
\[ \mathcal{L}_{G-L} = n_p \sigma_{xy} \frac{e^2}{2} \epsilon_{ij3} (A_0 \partial_i A_j + A_i \partial_j A_0) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]
\[ + |D_0 \eta|^2 - K_1 (|D_z \eta|^2 + |D_y \eta|^2)^2 \]
\[ - K_2 (|D_y \eta|^2 + |D_x \eta|^2)^2 \]
\[ - K_3 \{ (D_x \eta_1 \eta_2^* + (D_x \eta_2 \eta_1^*) + c.c \} \]
\[ - K_4 \{ (D_x \eta_3)^2 + c.c \} - K_5 |D_z \eta|^2 \]
\[ - A |\eta|^2 - \beta_1 |\eta|^4 - \beta_2 |\eta^* \times \eta|^2 \]
\[ - \beta_3 |\eta|^2 |\eta|^2 + C, \tag{17} \]
where \( D_{\mu} = \partial_{\mu} + 2ieA_{\mu}, n_p \) is a number of the superconducting plane per unit length along the \( z \) axis (the \( c \) axis), and \( A, K_i, \) and \( \beta_i \), are phenomenological constants with appropriate dimensions.

To simplify the calculations, we use
\[ \begin{cases} K_1 = K_2 = c_s^2, \quad K_3 = -K_4 = c_s^2 \gamma, \\ \beta_1 = \frac{\gamma}{3} + b, \quad \beta_2 = -b, \quad \beta_3 = 0, \\ A = -\frac{a |\eta|^2}{2}, \quad C = \frac{a |\eta|^4}{4}, \quad (0 < a, b). \end{cases} \tag{18} \]
These assignments of the phenomenological parameters in Eq. \((18)\) seems artificial, however, charge and angular momentum of a vortex do not depend on these phenomenological parameters as will become clear in the next section. Eq. \((17)\) reduces its form with cylindrical symmetry to
\[ \mathcal{L}_{G-L} = n_p \sigma_{xy} \frac{e^2}{2} \epsilon_{ij3} (A_0 \partial_i A_j + A_i \partial_j A_0) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]
\[ + |D_0 \eta|^2 - c_s^2 |D_{/\mu} \eta|^2 - c_s^2 \gamma \epsilon_{ij3} \eta_k^* (D_i \eta_\kappa) D_j \eta_\kappa \]
\[ - K_5 |D_z \eta|^2 \]
\[ - \frac{a}{4} (|\eta|^2 - |\eta|^2 - b |\eta|^4 - |\eta^* \times \eta|^2), \tag{19} \]
where \( D_{/\mu} = (D_x, D_y) \). The Ginzburg-Landau Lagrangian Eq. \((19)\) is an extended version of the Chern-Simons-Higgs model, so to say, a non-relativistic Maxwell-Chern-Simons-vector Higgs model.
III. VORTEX WITH FRACTIONAL CHARGE AND FRACTIONAL ANGULAR MOMENTUM IN CHIRAL P-WAVE SUPERCONDUCTOR

In this section, we study charge and angular momentum of a vortex in chiral p-wave superconductor. As we mentioned before, charge and angular momentum take fractional values. From these facts the vortex in the present case could be anyon.

Suppose the magnetic field is directed to the z-axis. We consider static and cylinder-symmetric solution around the z axis and we neglect time and z-dependence. Eqs. of motion in $\nabla \cdot A = 0$ gauge are written as

$$\nabla \cdot E = -8e^2|\eta|^2 A_0 + n_p \sigma_{xy} B,$$

$$(\nabla \times B)_i = -8e^2 c_s^2 |\eta|^2 A_i + 2iec_s^2 (\eta^* \partial_i \eta - \eta \partial_i \eta^*)$$

$$+ 2iec_s^2 \gamma \epsilon_{ij3} \partial_j (\eta^* \times \eta) + n_p \sigma_{xy} \epsilon_{ij3} \partial_j A_0,$$

where $\xi = c_s \sqrt{|\eta|}$ and $\lambda = (8e^2 c_s^2 |\eta|^2)^{-1/2}$ for gauge fields and another is $\xi = c_s \sqrt{|\eta|}$ for $\eta$. $\lambda$ coincides with so-called London penetration depth of the magnetic field in the case $\sigma_{xy} = 0$. We call $\lambda$ also “penetration depth” in this paper. $\xi$ is the coherent length. The Ginzburg-Landau parameter $\kappa$ is defined as $\kappa = \lambda/\xi$. It distinguishes type I superconductor ($\kappa < \frac{1}{\sqrt{2}}$) and type II superconductor ($\kappa > \frac{1}{\sqrt{2}}$). Type II superconductor has a stable vortex solution but type I does not. The chiral p-wave superconductor $Sr_2RuO_4$ has values $\lambda \simeq 1800A$ and $\kappa = 1.2$ for the magnetic field directed to the c axis of the crystal, which corresponds to z axis in this paper. Therefore, $Sr_2RuO_4$ is a type II superconductor and it has a stable vortex.

We use cylindrical coordinate and consider an Ansatz written as,

$$\{ \begin{array}{c}
\eta = \frac{\eta_0}{r} (\rho_1 (r), i \rho_2 (r)) e^{i \theta}, \\
A_0 = \frac{\eta_0}{r} a_0 (r), \\
A_\theta = \frac{\eta_0}{r} (n - a_0 (r)),
\end{array} \}$$

which shows a situation that a vortex with vorticity $n$ is located at the origin. For the total energy of the system to be finite, boundary condition at infinity is written as

$$\{ \begin{array}{c}
\rho_1 (\infty) = 1 ; i = 1, 2, \\
\rho_2 (\infty) = 0, \\
a_0 (\infty) = 0.
\end{array} \}$$

are required to exclude singularities of the fields.

First, we consider the asymptotic behavior of Ansatz Eq. (21). As we mentioned before, we see from Eq. (21) that a typical length scale of $\eta$ is $\xi$. Hence we regard $\rho_1 \simeq 1$ when $r$ is sufficiently larger than $\xi$ and behavior of $a_0$ and $a_\theta$ are written as

$$a_0 (r) \simeq c_+ K_0 \left( \frac{r}{c_s \lambda} \right) + c_- n_p \sigma_{xy} \lambda c_s^2 K_0 \left( \frac{r}{\lambda} \right),$$

$$a_\theta (r) \simeq -\frac{r}{\lambda} \left( C_+ c_s n_p \sigma_{xy} \lambda K_1 \left( \frac{r}{c_s \lambda} \right) - C_- K_1 \left( \frac{r}{\lambda} \right) \right).$$

where $K_i (r)$ is the l-th order modified Bessel function and $C_+$ and $C_-$ are numerical factors. The power behavior of the fields near the origin is written as

$$a_0 \sim a_0 + n_p \sigma_{xy} \lambda a_0 (r/\lambda)^2 + O((r/\lambda)^4),$$

$$a_\theta \sim n + a_\theta (r/\lambda)^2 + O((r/\lambda)^4),$$

$$\rho_1 \sim \rho_1^{(n)} (r/\lambda)^{|n|} + O((r/\lambda)^{|n|+2}),$$

where $\alpha_0, \alpha_\theta^{(n)}, \rho_1^{(n)}$ are numerical constants.

By using asymptotic behavior of the fields, we calculate the charge and the angular momentum of the vortex. In the rest of this section, we consider pure 2-dimensional space to avoid complexity in the arguments. It corresponds to putting $n_p = 1$ in our calculation.

The vortex charge is obtained by integrating charge density of the matter, such as

$$Q = \int d^2 x j_0^{\text{Matter}} = \int d^2 x (-8e^2 A_0 |\eta|^2)$$

$$= \int d^2 x \{ \nabla \cdot E - \sigma_{xy} B \}$$

$$= -2\pi \sigma_{xy} (r A_\theta)|_{r=\infty}^{r=0}$$

$$= -\sigma_{xy} \frac{2\pi n}{2e} = -\frac{ne}{4},$$

by using Eqs. (21), (22), (23) and (24). We can see the flux $\frac{2e}{4}$ is attached to the charge, i.e. the vortex is an object such as flux-charge composite. Volovik has mentioned the vortex charge caused by the Chern-Simons term in superfluid $^3$He-A film. His result has half value comparing with our result Eq. (24) because the factor $2$ was missing when the vortex charge was derived from the Chern-Simons term.

It was argued by many authors that the change in the density of electrons in the vortex core due to the spatial dependence of the orderparameter also gives the charge of the vortex and we call it “the regular charge”. Recently, Volovik pointed out that the occupation of the zero-energy bound states of electrons, which exist in the vortex core of the chiral p-wave superconductor could
contributes the charge of the vortex. We call it “zero-mode charge”. It is important how to distinguish these two charges and the fractional charge Eq. always comes from the Chern-Simons term. It can be done, even in the case that the regular charge or the zero-mode charge are of order $e$, by comparing the vortex charges of vorticities $n = 1$ and $n = -1$ because the fractional charge Eq. changes its sign but other two charges do not depend on the sign of vorticity.

The definition of the vortex angular momentum is
\[ J = \int d^2 x r x \cdot \mathcal{P}. \] (27)
\[ \mathcal{P} \] is the momentum density, which is the generator of the translation. It is defined to be gauge invariant such as
\[ \mathcal{P}_i = \frac{\partial \mathcal{L}}{\partial (\partial_\eta A_j)} D_i \eta + (h.c.) + \frac{\partial \mathcal{L}}{\partial (\partial_\eta A_j)} F_{ij} \]
\[ = (D_\eta)^* D_i \eta + D_\eta (D_i)^* + \epsilon_{ij} E_j B. \] (28)
By using Eqs.(28), (21), (24) and (25), angular momentum Eq. (27) is calculated as follows;
\[ J = \int d^2 x r p_y \]
\[ \quad = \int d^2 x \left\{ 8 e^2 A_0 (A_\theta - \frac{n}{2e r^2}) |\eta|^2 - E_x B \right\} \]
\[ = -2 \pi \sigma_{xy} \int_0^\infty dr \frac{r}{2} \partial_r \{ n - a_\theta (r) \}^2 \]
\[ = -\frac{n^2}{16}. \] (29)
We see the vortex in our system Eq. (14) has fractional charge and fractional angular momentum same as the vortex in Chern-Simons-Higgs model. Since the charge and the angular momentum are proportional to $\sigma_{xy}$, we see that the Chern-Simons term creates these non-zero fractional values. These values are topological and depends only on asymptotic behavior of the gauge fields, although there is Maxwell term $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ and the Higgs field is vector-like.

The results Eq. (26) and Eq. (29) would be general and would not depend on the form of Ansatz and phenomenological parameters. Actually, in unconventional superconductors there is another Ansatz which is different from Eq. (21), and there are plausible phenomenological parameters for $Sr_2 RuO_4$ studied by Agterberg and they are different from Eq. (18). The results Eq. (24) and Eq. (28) would not be changed even if we use such a different Ansatz and phenomenological parameters. The reasons are written as follows;

1. The definition of charge density and angular momentum density do not depend on the phenomenological parameters explicitly and always become total derivative, since
2. charge density of matter $j^Matter_0$ in Eq. (26) always satisfies the Maxwell equation $j^Matter_0 = \nabla \cdot E - \sigma_{xy} B$, and
3. the momentum density $\mathcal{P}_i$ defined by Eq. (28) depends on the form of time derivative terms in Lagrangian Eq. (19) and these terms do not contain the phenomenological parameters.

2. Asymptotic behavior of the fields is investigated without the Chern-Simons term in Ref. by using Ansatz written in Ref. and the phenomenological parameters written in Ref. The essential property of the asymptotic behavior is the same with Eq. (24) and Eq. (23), and it would not be changed even if the Chern-Simons term is taken into account.

Therefore, the results Eq. (24) and Eq. (29) would be valid for realistic systems such as superconducting $Sr_2 RuO_4$. Evidence that the vortex we consider is a candidate of anyon can be seen in the fact that vortex has fractional angular momentum. It is expected that transmutation of the statistics occurs. The further discussion is needed to see the fractional statistics of these vortices experimentally.

IV. ELECTROMAGNETIC PROPERTY OF VORTEX WITH FRACTIONAL QUANTUM NUMBERS

In this section, we show numerical solution of Eq. (20) by using Ansatz Eq. (21) and discuss electromagnetic property of the vortex. We consider layered system again. In this case, $n_p \approx 10^{-1} A^{-1}$. For simplicity, we assume $\gamma$ in Eq. (19) is zero. In this case, there is an exchange symmetry such as $\rho_1 \leftrightarrow \rho_2$ in the equation which is obtained by substituting Eq. (21) into Eq. (20). We introduce a real field $\phi$ written as
\[ \phi(r) = \rho_1(r) = \rho_2(r). \] (30)
The magnetic field, the electric field and $\phi$ with parameters $\kappa = 1.2$ and $c_s \simeq (\frac{e^2}{\hbar})^{1/2} \approx 10^{-4}$ in the case $n = 1$ are shown in FIG. 1. Contrary to the usual vortex (i.e. the vortex in the case $\sigma_{xy} = 0$), we found that an electric field in the radial direction is induced in the region $r < \lambda$ and its magnitude is about 1 Volt/meter.

Let $\Delta E$ stands for energy difference between our vortex $E(\sigma_{xy} = \frac{e^2}{\hbar \pi})$ and the usual one $E(\sigma_{xy} = 0)$. There is a relation between them written as
\[ \frac{\Delta E}{E(\sigma_{xy} = 0)} = \frac{E(\sigma_{xy} = \frac{e^2}{\hbar \pi})}{E(\sigma_{xy} = 0)} - 1 \]
\[ \sim 10^{-8}. \] (31)
Therefore, energy increase caused by the existence of the Chern-Simons term is extremely small and the vortex
with fractional quantum numbers could exist in realistic systems. The magnitude of the vortex energy and also that of the electric field depend on the form of Ansatz and phenomenological parameters in Eq. (17), however, these calculation would have validity as an order estimation. It is interesting if some phenomena caused by the induced electric field are detected by experiments.

V. CONCLUSION

We have discussed a vortex in chiral p-wave superconductor such as $Sr_2RuO_4$, where $U(1)$, P- and T-symmetry are broken simultaneously. We have investigated the vortex based on the Ginzburg-Landau Lagrangian. The Ginzburg-Landau Lagrangian of the system contains the Chern-Simons term, and it corresponds to an extended model of the Chern-Simons-Higgs model. We have found that the vortex in chiral p-wave superconductor has fractional charge $\frac{\pi}{\alpha}$ and fractional angular momentum $-\frac{\alpha^2}{16}$ with integrally quantized magnetic flux $\frac{\alpha^2}{\alpha^2}$. These values are topologically stable and do not depend on the form of Ansatz and the phenomenological parameters in the Ginzburg-Landau Lagrangian. Following the discussion in Ref. 17, the vortex could obey the fractional statistics. We have also investigated the electromagnetic property of the vortex. We have found that the electric field is induced by the Chern-Simons term near the vortex core. Energy increase caused by the existence of the Chern-Simons term is so small that the vortex with fractional quantum numbers could exist in realistic systems. It is interesting if these exotic feature is observed experimentally.

Our discussion is valid for other chiral superconductors, such as “anisotropic chiral p-wave” superconductor in which the symmetry of the wave function of Cooper pair is $(\sin p_x, \sin p_y)$-wave and $d_{x^2-y^2}+id_{xy}$-wave superconductor since it has been shown that the Chern-Simons term is also induced in the Ginzburg-Landau Lagrangian of these superconductors.

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FIGURE CAPTION

FIG. 1. \( r \) dependence of (a) the order parameter \( \phi \), (b) the magnetic field directed to the \( z \)-axis, and (c) the radial electric field.

FIGURES

FIG. 1(a).
FIG 1(b).
FIG. 1(c).