Topology of Conic Bundles - II

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Abstract
For conic bundles on a smooth variety (over a field of characteristic $\neq 2$) which degenerate into pairs of distinct lines over geometric points of a smooth divisor, we prove a theorem which relates the Brauer class of the non-degenerate conic on the complement of the divisor to the covering class (Kummer class) of the 2-sheeted cover of the divisor defined by the degenerate conic, via the Gysin homomorphism in etale cohomology. This theorem is the algebro-geometric analogue of a topological result proved earlier.

1 Introduction

Let $X$ be a smooth scheme over a field $F$ of characteristic $\neq 2$, and let $C \to X$ be a conic bundle on $X$, whose discriminant defines a smooth divisor $Y \subset X$ with multiplicity $\tau$ (which, if $Y$ is not irreducible, will consist of a positive integer $\tau_i$ for each component $Y_i$ of $Y$). Let $\beta \in H^2(X, \mu_2)$ be the Brauer class of the restriction of $C$ to $X-Y$, which is a $P^1$ fibration which is etale locally trivial (all cohomologies are with respect to etale topology unless otherwise indicated). Suppose that over each geometric point of $Y$, the fiber of $C$ consists of two distinct projective lines meeting at a point. Therefore, the relative Hilbert scheme of lines in $C|Y \to Y$ is a two sheeted finite etale cover $\tilde{Y} \to Y$. Let $\alpha \in H^1(Y, \mu_2)$ be its covering class ('Kummer class'). By smoothness, we have a Gysin homomorphism $H^2(X-Y, \mu_2) \to H^1(Y, \mu_2)$. We prove here the following

Theorem 1.1 Under the Gysin homomorphism $H^2(X-Y, \mu_2) \to H^1(Y, \mu_2)$, the Brauer class $\beta$ of the $P^1$ fibration on $X-Y$ maps to $\tau \alpha$, where $\tau$ is the vanishing multiplicity of the discriminant and $\alpha$ is the cohomology class of the two sheeted finite etale cover $\tilde{Y} \to Y$.

A topological version of this result was proved in [Ni] for topological conic bundles on manifolds, which is equivalent for complex algebraic conic bundles to the above
result. This is because there is a natural isomorphism between etale cohomology with finite constant coefficients (in this case, coefficients $\mathbb{Z}/(2)$) and the corresponding singular cohomology, which commutes with the two Gysins.

The theorem is proved below in two steps. In section 2, we reduce it to proving a purely algebraic lemma (Lemma 2.1 below, which we call as the ‘main lemma’) over a discrete valuation ring. In section 3, we prove the main lemma.

**Remarks 1.2**

1. The main lemma is more transparent than its topological counterpart in [Ni].
2. After proving the main lemma, enquiries with algebraist colleagues revealed that a more general lemma has already been proved by Colliot-Thélène and Ojanguren (see proposition 1.3 in [C-O]). Our proof is more geometric but less general.
3. When the total space of $C$ is nonsingular, it is known that (see [H-N] Proposition 1.4 or [Ne] Theorem 2) $\beta$ is zero only if $\alpha$ is zero. This follows from theorem 1.1 by taking $\tau = 1$, though theorem 1.1 makes a stronger statement even in this case.

## 2 Reduction to Main Lemma

For basic definitions about conic bundles, see for example Newstead [Ne].

It is clearly enough to prove the theorem for each connected component of $X$, so we will assume $X$ to be connected. If $Y = \bigcup_i Y_i$ are the connected components of $Y$, then by replacing $(X, Y)$ by $(X \setminus \bigcup_{j \neq i} Y_i, Y_i)$ we are reduced to the case where $Y$ also is connected. Hence we can assume that both $X$ and $Y$ are irreducible.

Let the conic bundle $C \to X$ be defined via a rank 3 vector bundle $E$ on $X$, together with quadratic form $q$ on $E$ with values in some line bundle $L$ on $X$. The quadratic form $q$ is of rank 3 on $X \setminus Y$ because we have a non-degenerate conic over $X \setminus Y$, and $q$ has rank 2 on $Y$ as we have pairs of distinct lines over geometric points of $Y$.

Let $U = \text{Spec} R$ be an affine open subscheme of $X$ which intersects $Y$, such that $E$ and $L$ are trivial on $U$, and $Y \cap U$ is defined by a principal ideal $(\pi)$ in $R$. By injectivity of $H^1(Y, \mu_2) \to H^1(Y \cap U, \mu_2)$, it is enough to prove the theorem for $(U, Y \cap U)$ in place of $(X, Y)$. Hence we can assume that the conic bundle is defined by an explicit quadratic form $x^2 - ay^2 - bz^2$ on the ring $R$, where $a$ is a unit in $R$, while $b \in (\pi)$ vanishes over $Y$.

Let $\eta$ be the generic point of $Y$, and let $A$ be the discrete valuation ring $\mathcal{O}_{X, \eta}$. Let $k$ be the function field of $Y$. The morphism $\text{Spec}(k) \to Y$ induces an injective homomorphism $H^1(Y, \mu_2) \to H^1(k, \mu_2)$. Hence the theorem follows from the following lemma.

**Lemma 2.1 (Main Lemma)** Let $F$ be a field of characteristic $\neq 2$. Let $A$ be a discrete valuation ring which is the local ring at the generic point of a smooth divisor in a smooth $F$-variety. Let $K$ be the quotient field of $A$, let $k$ be the residue field, and let $\nu : A - \{0\} \to \mathbb{Z}$ be the discrete valuation. Let $x^2 - ay^2 - bz^2$ be
a quadratic form on $A$, with $a$ a unit in $A$, and $b \neq 0$. Let $(a, b) \in H^2(K, \mu_2)$ be the Brauer class (Hilbert symbol) of the quadratic form on $K$. Let $\pi \in k - \{0\}$ be the residue class of $a$, and let $\chi(\pi) \in H^1(k, \mu_2)$ be the class of the two sheeted etale cover $k(\pi^{1/2})/k$ (Kummer character). Then under the Gysin homomorphism $H^2(K, \mu_2) \to H^1(k, \mu_2)$, we have (in additive notation)

$$(a, b) \mapsto \nu(b)\chi(\pi)$$

3 Proof of the Main Lemma

In the course of the proof below, we use the following elementary facts which can be found for example in the textbook of Milne [M]. If $S$ is a field or more generally a Henselian local ring, then $Br(S) \to H^2(B, GL_1)$ is an isomorphism, and provided the characteristic of the residue field is $\neq 2$, the homomorphism $H^2(S, \mu_2) \to H^2(B, GL_1)$ is injective. Moreover the etale cohomology of $S$ with coefficients in a smooth representable sheaf (for example $\mu_2$ or $PGL_2$) is isomorphic by restriction to the corresponding etale cohomology of the residue field of $S$.

We need two more fact, which are contained in the following two remarks.

Remark 3.1 Let $A$ be a henselian local ring and $B/A$ be a 2-sheeted finite etale cover. If the image of $\gamma \in H^2(A, \mu_2)$ is zero under the composite

$$H^2(A, \mu_2) \to H^2(B, \mu_2) \to H^2(L, \mu_2)$$

where $L$ is the quotient field of $B$, then $\gamma$ lies in the image of the canonical (connecting) set map $H^1(A, \underline{PGL_2}) \to H^2(A, \mu_2)$ for the following reason. Any generic section of a Brauer-Severi variety on $Spec(B)$ extends to a global section by the valuative criterion of properness and so the map $H^2(B, \mu_2) \to H^2(L, \mu_2)$ is injective, hence $\gamma$ maps to zero under $H^2(A, \mu_2) \to H^2(B, \mu_2)$. Hence $\gamma$ is represented by an element (factor set) of the group cohomology $H^2(Gal(B/A), \mu_2)$. As $Gal(B/A)$ is of order 2, the factor set $\gamma$ defines an Azumaya algebra of rank 2, showing $\gamma$ comes from $H^1(R, \underline{PGL_2})$.

Remark 3.2 Let $K$ be a field of characteristic $\neq 2$, let $a, b \in K - \{0\}$ such that $a$ is not a square in $K$, and let $L = K[t]/(t^2 - a)$. As the conic in $P^2_K$ defined by $x^2 - ay^2 - bz^2 = 0$ has an $L$-rational point, its Brauer class $(a, b)$ is an element of the group cohomology set $H^1(Gal(L/K), \underline{PGL_2}(L))$. Let $Gal(L/K) = \{1, \sigma\}$. A 1-cocycle for $Gal(L/K)$ with coefficients $\underline{PGL_2}(L)$ therefore consists of an element $g \in \underline{PGL_2}(L)$ such that $g\sigma(g) = I$ in $PGL_2$. (The corresponding ‘crossed homomorphism’ $Gal(L/K) \to PGL_2(L)$ is defined by $\sigma \mapsto g$.) If $g' \in GL_2(L)$ is an arbitrary lift of $g$, then there must exist some $c \neq 0$ in $L$ such that $g'\sigma(g') = cI$ in $GL_2(L)$ (which implies $c \in K$). In particular it can be seen by using stereographic projection from $(\sqrt{a}, 1, 0) \in P^2_K(L)$ that $(a, b)$ is represented in $H^1(Gal(L/K), \underline{PGL_2}(L))$ by
the 1-cocycle defined by

$$h = \begin{pmatrix} 1 & b \\ \end{pmatrix}$$

We now prove the main lemma.

**Proof** If \(b\) is a unit in \(A\) (that is, \(\nu(b) = 0\)), then the \(P^1\) bundle given by \(x^2 - ay^2 - bz^2\) is defined over all of \(\text{Spec}(A)\), and so by exactness of the Gysin sequence

$$H^2(A, \mu_2) \to H^2(K, \mu_2) \to H^1(k, \mu_2)$$

the image of \((a, b) \in H^2(K, \mu_2)\) in \(H^1(k, \mu_2)\) is also zero. So we now assume \(\nu(b) = 0\) (which is anyway the case which interests us). If \(\nu(b) = 2n\) is even then as \(\chi(\pi)\) is 2-torsion we have \(\nu(b)\chi(\pi) = 0\). Making the change of variable \(\nu' = \pi^{n}z\) over \(K\), we see that \((a, b) = (a, b') \in H^2(K, \mu_2)\) where \(b' = b/\pi^{2n}\). As \(\nu(b') = 0\), the argument above now completes the proof when \(\nu(b)\) is even. When \(\nu(b) = 2n + 1\) is odd, the same change of variables enables us to reduce to the case where \(\nu(b) = 1\). Hence we assume from now on that \(\nu(b) = 1\).

Note that passing to the completion of \(A\) does not affect the residue field and gives a commutative square of the Gysins. Hence it is enough to show the conclusion of the lemma assuming \(A\) is complete. If \(\pi \in k\) is a square then (by the Henselian property of a complete local ring) \(a\) will be a square in \(A\) and hence in \(K\), and conversely (this uses the discrete valuation) if \(a \in A\) is a square in \(K\) then it is a square in \(A\) and hence \(\pi\) is a square in \(k\). The conic defined in \(P^2_K\) by \(x^2 - ay^2 - bz^2\) will have a \(K\) rational point \((a_1, 1, 0)\) whenever \(a_1\) is a squareroot of \(a\) in \(K\), showing its Brauer class over \(K\) is zero. Hence if \(\chi(\pi) = 0\) then \((a, b) = 0\), and so the lemma holds. Hence we now assume that \(\pi\) is not a square in \(k\), and therefore \(a\) is not a square in \(K\) or \(A\). (As moreover \(\nu(b) = 1\), it follows by remark 1.3 (3) that \((a, b) \neq 0\), though this will also follow from the argument below.)

Let \(B = A[t]/(t - a^2)\) which is a 2-sheeted finite etale cover of \(A\), and let \(L\) and \(l\) be respectively the quotient field and the residue field of \(B\). Consider the commutative diagram where the horizontal maps are the two Gysins.

$$
\begin{array}{ccc}
H^2(K, \mu_2) & \to & H^1(k, \mu_2) \\
\downarrow & & \downarrow \\
H^2(L, \mu_2) & \to & H^1(l, \mu_2)
\end{array}
$$

The kernel of \(H^1(k, \mu_2) \to H^1(l, \mu_2)\) is of order two, generated by \(\chi(\pi)\). As \(a\) has a squareroot in \(L\), \((a, b) \in Br(K)\) maps to zero in \(Br(L)\). Hence to show that \((a, b) \mapsto \chi(\pi)\), it is enough to show that the image of \((a, b)\) in \(H^1(k, \mu_2)\) is nonzero. By exactness of the Gysin sequence, this will follow if we show that \((a, b)\) does not lie in the image of \(H^2(A, \mu_2) \to H^2(K, \mu_2)\).

Suppose \((a, b)\) was the image of \(\gamma \in H^2(A, \mu_2)\). As \((a, b)\) restricts to zero over \(L\), it follows from remark 3.1 that \(\gamma\) lies in the image of some \(\gamma' \in H^1(A, \mathbf{PGL}_2)\) under \(H^1(A, \mathbf{PGL}_2) \to H^2(A, \mu_2)\). As \(\gamma'\) splits over \(B\), it can be regarded as an element
of the group cohomology set $H^1(Gal(B/A), PGL_2(B))$, and hence as in remark 3.2, its 1-cocycle is represented by an element $g \in GL_2(B)$ such that

$$g\sigma(g) = eI$$

where $Gal(B/A) = \{1, \sigma\}$ and $e$ is a unit in $A$.

On the other hand, the group cohomology class of $(a, b) \in H^1(Gal(L/K), PGL_2(L))$ can be represented by the matrix $h$ given by remark 3.2. If the cohomology class of $(g)$ were to map to the cohomology class of $(h)$ in $H^1(Gal(L/K), PGL_2(L))$, then by definition of group cohomology there would exist elements $M \in GL_2(L)$ and $0 \neq c \in L$ such that

$$h = cMg\sigma(M^{-1})$$

Applying $\sigma$, this would give

$$h = \sigma(h) = \sigma(c)\sigma(M)\sigma(g)M^{-1}$$

Multiplying the two equations and using $h^2 = bI$ and $g\sigma(g) = eI$, we would get

$$b/e = \text{Norm}(c)$$

But this is impossible as on one hand $e \in A$ is a unit while $b \in A$ has valuation $\nu(b) = 1$ so $\nu(b/e) = 1$, and on the other hand $\nu$ takes only even values on norms. Hence the image of $(a, b)$ in $H^1(k, \mu_2)$ is nonzero. This completes the proof of the main lemma and hence that of the theorem.

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