Planar Superconductivity via Kosterlitz-Thouless mechanism

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Abstract

The phase structure of a (2 + 1) - dimensional model of relativistic fermions with a four fermi interaction is analyzed in the strong coupling regime using the large N perturbation theory. It is shown that, this model exhibits a low temperature superconducting phase due to the vortex-anti vortex binding via Kosterlitz-Thouless mechanism. Above a critical temperature, vortices unbind and superconductivity is destroyed; at a still higher temperature the vacuum expectation value of a neutral order parameter vanishes. The ground state respects parity and time reversal symmetries.
1. Introduction
Superconductivity in planar field theoretical models have aroused considerable interest in recent times because of their relevance to condensed matter systems.\textsuperscript{1} A thorough analyses of the phase structure of these models is necessitated due to the fact that the conventional explanation based on the spontaneous breaking of a $U(1)$ symmetry may not be applicable to the lower dimensional world. In $2+1$ dimensions the vacuum expectation value (VEV) of a charged order parameter vanishes for non-zero temperatures due to infrared divergences, much akin to the celebrated Coleman-Mermin-Wagner\textsuperscript{2} theorem, which states that spontaneous breaking of a continuous symmetry does not take place in $1+1$ dimensions. In light of the above constraint, one usually invokes a small interplanar coupling or takes recourse in unconventional mechanisms e.g. parity and time reversal violating anyon superconductivity.\textsuperscript{3} This note briefly describes an alternate model\textsuperscript{4}, involving relativistic fermions with a four-fermi interaction, which exhibits superconductivity due to the vortex confinement mechanism of Kosterlitz and Thouless(KT).\textsuperscript{5} Relativistic fermions appear as the relevant low energy degrees of freedom in the Hubbard and related models,\textsuperscript{6} widely believed to be of relevance to high $T_c$ superconductors and four-fermi couplings also arise naturally in these models in the presence of doping.\textsuperscript{7} Without a precise knowledge about these couplings, we consider a four-fermi term of BCS type and study the various phases, taking advantage of the model’s large $N$ renormalizability.\textsuperscript{8}

The paper is organized as follows. Section 2 is devoted to the computation of the low energy effective action in parallel to the derivation of the Landau-Ginzburg effective action in the BCS theory. It is shown that a neutral order parameter can have a nonvanishing VEV up to a critical temperature $T_0$, although the VEV of a charged order parameter vanishes for $T \neq 0$. In Sec.3, we study the occurrence of superconductivity in this model, after taking into account the dynamics of the phase degrees of freedom. We conclude in Sec.4 with some comments.
2. Low Energy Effective Action

Calculation of the low energy effective action which encapsulates the relevant dynamical degrees of freedom, has been a common approach since the early days of superconductivity. Physically, there is a gap in the fermion spectrum below the critical temperature, because of which long wavelength excitations of the condensate are energetically cheaper to create than fermionic excitations. The effective action exhibits the spontaneous breaking of the $U(1)$ symmetry responsible for superconductivity. Below, we will outline the derivation of the effective action for the BCS Lagrangian

$$L_{BCS} = \psi_\uparrow(i\partial_t - \epsilon(p))\psi_\uparrow + \psi_\downarrow(i\partial_t - \epsilon(p))\psi_\downarrow - \lambda \psi_\uparrow \psi_\uparrow \psi_\downarrow \psi_\downarrow, \quad (1)$$

and follow the same procedure for the model presented here. To get the effective theory, one introduces auxiliary fields to rewrite the $L_{BCS}$ as,

$$L_1 = \psi_\uparrow(i\partial_t - \epsilon(p))\psi_\uparrow + \psi_\downarrow(i\partial_t - \epsilon(p))\psi_\downarrow + \sqrt{\lambda g} (\phi^* \psi_\downarrow \psi_\uparrow + \bar{\phi} \psi^\dagger \psi^\dagger) + g \phi^* \phi. \quad (2)$$

In the path integral formalism, the generating functional of $(2)$ is,

$$Z = \int D\phi^* D\phi e^{iS_{eff}(\phi^*, \phi)}.$$

Performing the fermion functional integral:

$$Z = \int D\phi^* D\phi e^{iS_{eff}(\phi^*, \phi)}.$$

where

$$e^{iS_{eff}[\phi^*, \phi]} = \int D\psi^\dagger D\psi e^{iS_1}$$

and

$$S_{eff} = -iT_{\text{Tr}} \log \left( \begin{pmatrix} p_0 + \epsilon(p) & -\sqrt{\lambda} g \phi \\ \sqrt{\lambda} g \phi^* & p_0 - \epsilon(p) \end{pmatrix} \right) + \int d^4x g \phi^* \phi.$$  

Here the matrix, if sandwiched between $\psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger)$ and $\psi = (\psi_\uparrow, \psi_\downarrow)^T$, gives the fermion-dependent part of $(4)$. The term without derivatives, is the effective potential for the condensate and at finite temperature is given by,
\[ V_{\text{eff}} \sim a \log \frac{T}{T_c} \phi^* \phi + b(\phi^* \phi)^2 \, , \]  

(7)

where \(a, b\) are positive parameters. It is easily seen that \(\phi\) attains a nonzero VEV below \(T_c\), which results in the spontaneous breaking of the electromagnetic \(U(1)\) symmetry and hence superconductivity. In the following we will proceed along similar lines, keeping in mind the non-trivial vortex excitations on the plane.

The relevant Lagrangian is,

\[ \mathcal{L} = \bar{\psi}_\alpha (i \partial - e A) \psi_\alpha - \frac{1}{4\lambda N} \bar{\psi}_\alpha \psi^c_\alpha \bar{\psi}^c_\beta \psi_\beta \, . \]

(8)

Here \(\alpha, \beta\) range from 1 to \(N\), \(N\) being the parameter of the large \(N\) expansion. Although the coupling constant is dimensionful, this model is renormalizable in large \(N\) perturbation theory.

Introducing the auxiliary fields \(\phi\) and \(\phi^*\) to decouple the four fermi term, one gets,

\[ \mathcal{L} = \bar{\psi}_\alpha (i \partial - e A) \psi_\alpha + \frac{1}{2} (\phi^* \bar{\psi}^c_\alpha \psi_\alpha - \phi \bar{\psi}_\alpha \psi^c_\alpha) - \lambda \phi^* \phi \, . \]

(9)

Proceeding along the same lines as in the BCS case, a derivative expansion leads to

\[ V_{\text{eff}} = V^{(0)} + iN \int \frac{d^3 k}{(2\pi)^3} \log \left( 1 - \frac{\phi^* \phi}{k^2} \right) \, . \]

(10)

The gap equation involving \(v_0 \equiv <\phi>\),

\[ \frac{1}{N} \frac{\delta V_{\text{eff}}}{\delta v_0} = 0 = 2\lambda v_0 - \frac{v_0}{2\pi}(\sqrt{\Lambda^2 + v_0^2} - v_0) \, , \]

(11)

yields a nontrivial VEV for \(v_0\) if \(\lambda\) is less than a critical value \(\lambda_c \equiv \frac{A}{4\pi}\). Notice that the four-fermi coupling is \(\sim 1/\lambda\) and hence \(\lambda < \lambda_c\) is a strong coupling regime. The effective potential finally is

\[ V_{\text{eff}} = N(\lambda - \lambda_c) \phi^* \phi + \frac{N}{6 \pi}(\phi^* \phi)^{3/2} \, . \]

(12)

At finite temperature the potential takes the form

\[ \frac{1}{N} V_{\text{eff}}(\phi, \phi^*; T) = \lambda \phi^* \phi - \frac{1}{\beta} \sum_{-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \log \left( 1 + \frac{\phi \phi^*}{k^2 + k_3^2 \left( n + \frac{1}{2} \right)^2} \right) \, . \]

(13)
We analyze the gap equation for a nontrivial constant solution \( v_T \equiv \langle |\phi| \rangle \):

\[
\lambda v_T - \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + v_T^2 + \frac{4\pi^2}{\beta} \left( n + \frac{1}{2} \right)^2} = 0 .
\]  

(14)

Sum over frequencies yields,

\[
v_T \left( \lambda - \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\sqrt{k^2 + v_T^2}} \frac{\sinh(\beta \sqrt{k^2 + v_T^2})}{\cosh(\beta \sqrt{k^2 + v_T^2}) + 1} \right) = 0 .
\]  

(15)

From above, defining the critical temperature \( T_0 \) to be the point where \( v_T = 0 \) one gets

\[
T_0 = \frac{v_0}{2 \log 2} .
\]  

(16)

Hence \( T_0 \) is the temperature where the neutral order parameter vanishes. Incorporating electromagnetism, the zero temperature effective Lagrangian can be computed in a straightforward manner and is given by

\[
\mathcal{L}_{\text{eff}} = \frac{N}{16\pi v_0} |(\partial_\mu + 2ieA_\mu)\phi|^2 - \frac{N}{192\pi v_0^3} (\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*) (\phi^* \partial^\mu \phi + \phi \partial^\mu \phi^*) - V_{\text{eff}} .
\]  

(17)

It is worth noting that the effective action does not contain any parity or time reversal violating terms, e.g. it can be shown that Chern-Simons type terms will never be generated.

Now the expectation value of a charged order parameter can be easily computed at finite temperatures. Writing \( \phi = \rho e^{i\theta(x)} \) and neglecting fluctuations of \( \rho \), one gets

\[
\langle \phi(x) \rangle = \langle \rho e^{i\theta(x)} \rangle \simeq v_T e^{-\frac{1}{2} \langle \theta^2(x) \rangle} .
\]

\( \langle \theta^2(x) \rangle \) can be computed in a straightforward manner using the effective action, yielding finally

\[
\langle \phi \rangle = \text{const.} (\beta \eta)^\chi .
\]  

(18)

Here \( \eta \) is an infrared cutoff and \( \chi = 2/N(\beta v_T \tanh(\beta v_T/2)) \). We see that in the limit \( \eta \to 0, \langle \phi \rangle \to 0 \), since \( \chi \) is positive. This proves that \( \theta \), the massless excitation associated with the Goldstone mode, is responsible for the vanishing of the VEV of the charged order
parameter $\phi$ at nonzero temperatures.

Next-to-leading order corrections in the large $N$ perturbation theory shows that the quantum fluctuations do not destabilize the vacuum either due to ultraviolet or infrared effects.

3. **Superconductivity due to Vortex binding**

We study occurrence of superconductivity in this model using a duality transformation to take into account the non-trivial topological excitations on the plane. The phase of the complex order parameter is multivalued and gives rise to vortex excitations. The duality transformation, allows the effective separation of the single and multivalued components of the complex field. At low temperature, it will be shown that vortices and anti-vortices are tightly bound, and the model exhibits superconductivity. At a critical temperature $T_c$ the celebrated Kosterlitz-Thouless transition takes place and the vortices become free. This is precisely the temperature where superconductivity is destroyed.

To capture the physics of vortex excitations, we write $\phi = \rho e^{i\theta}$ with $\rho$ fixed at $v_T$. This is the long wavelength London limit which neglects fluctuations of $\phi$. Replacing $\partial_\mu \theta \to \partial_\mu \theta - i\varphi^* \partial_\mu \varphi$ with $\varphi^* \varphi = 1$, we see that $\theta$ describes a single valued field and $\phi$ accounts for the vortex dynamics. The identically conserved vortex current is,

$$J_\mu^{\text{vort}} = (2\pi i)^{-1} \epsilon_{\mu\nu\lambda} \partial^\nu (\varphi^* \partial^\lambda \varphi) \ .$$  \hspace{1cm} (19)

At finite temperature the dynamics is effectively described by

$$\mathcal{L}_{\text{eff}} = \frac{N \tanh(\beta v_T/2)}{16\pi v_T} \nu_T^2 (\partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu)^2 \ .$$  \hspace{1cm} (20)

It is useful to rewrite this in terms of the auxiliary current $J_\mu = \partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu$:

$$\frac{1}{N} \mathcal{L}_{\text{eff}} = -\frac{1}{2K} J_\mu J^\mu - J_\mu (\partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu)$$  \hspace{1cm} (21)

where $K = \frac{v_T \tanh(\beta v_T/2)}{8\pi}$. The single valued field $\theta$ can now be integrated out, giving the constraint, $\partial^\mu J_\mu = 0$. This is readily solved by

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu a^\lambda \ .$$ \hspace{1cm} (22)
In terms of this auxiliary gauge field $a_\mu$ the new Lagrangian after integration by parts becomes,

$$\frac{1}{N} \mathcal{L}_{\text{eff}} = -\frac{1}{4K} f_{\mu\nu}^2 - 2 e \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu A^\lambda - 2\pi a_\mu J^\text{vort}_\mu, \quad (23)$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The integration over $a_\mu$ can now be performed, and the result is

$$\mathcal{L}_{\text{eff}} = -2\pi^2 N K J^\text{vort}_\mu(x) \frac{g^\mu\nu}{\partial^2} J^\text{vort}_\nu(x)$$

$$- 4\pi e N K J^\text{vort}_\mu \frac{g^\mu\nu}{\partial^2} \epsilon_{\nu\lambda\rho} \partial^\lambda A^\rho - e^2 N K F^\mu\nu \frac{1}{\partial^2} F_{\mu\nu}. \quad (24)$$

Neglecting for the moment the second term which describes the interaction of the electromagnetic field with the vortex, the current-current correlation function is

$$\delta^2 S_{\text{eff}}/\delta A_\mu(x) \delta A_\nu(y) = \langle j^\mu(x) j^\nu(y) \rangle = -4e^2 N K \left( \frac{\partial^\mu \partial^\nu - g^\mu\nu}{\partial^2} \right) \delta^3(x - y). \quad (24)$$

The above expression clearly indicates a pole at zero momentum and hence superconductivity. In the presence of a Maxwell term, one can show that the photon propagator, reveals a pole at non-zero momenta which indicates Meissner effect.

As has been seen above, the pole in the current-current correlator arises because of the $F^\mu\nu \partial^{-2} F_{\mu\nu}$ term in the effective action. At low temperature a slowly varying magnetic field sees pairs of tightly bound vortices and hence the net contribution from each pair to the above term vanishes. However above $KT$ phase transition temperature the vortex contribution to this term exactly cancels the contribution from the single-valued part, thereby destroying superconductivity.

This can be shown more explicitly, by looking at only the static vortex configurations ($J^\text{vort}_i$ and $\partial_0 J^\text{vort}_0 = 0$) and computing the vortex contribution to the current-current correlation function. Writing

$$J^\text{vort}_0 \equiv \rho^\text{vort}(x) = \sum_a m_a \delta(x - x_a(t)) \quad (25)$$

and using the Green’s function properties in two dimensions, the effective action reduces to

$$S_{\text{eff}}/N = \pi K \beta \sum_{a,a'} m_a m_{a'} \log |x_a - x_{a'}|$$
Here $m_a$ is the integer valued vorticity and $x_a$ is the position of the $a^{th}$ vortex. Without the interaction term, the above is the action of the familiar XY model. In the static limit, the contribution of the vortices to the current-current correlation function is

$$< J^i(q) j^i(-q) >^{vort} = \frac{\delta^2 S_{eff}}{\delta A^i(q) \delta A^j(-q)} = \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right) q^2 < \rho^{vort}(q) \rho^{vort}(-q) > .$$

In momentum space,

$$\rho^{vort}(q) = \sum_a m_a e^{i q x_a} .$$

and hence in the confined phase,

$$< \rho^{vort}(q) \rho^{vort}(-q) > = \sum_{a,a'} e^{i q (x_a - x_{a'})} < m(x_a) m(x_{a'}) > .$$

Using the results of the XY model,

$$< m(x_a) m(x_{a'}) > \sim \frac{1}{|x_a - x_{a'}|^2 P}$$

and

$$< \rho^{vort}(q) \rho^{vort}(-q) > \sim \frac{1}{q^{2-2P}} ;$$

where $P = \pi NK \beta$. Since in the confined phase $P > 2$, the zero momentum pole in $< J^i(q) j^i(-q) >$ does not get a contribution from the confined vortices and hence the Meissner effect is not affected. In contrast, when the vortex deconfinement, one can directly integrate out the $J_\mu$ variable and arrive at

$$L^{vort}_{eff} = NK e^2 F^{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} .$$

This is exactly of opposite sign to that of the contribution of the single-valued sector. This proves, that in the deconfined phase, superconductivity is destroyed, and $T_c = T_{KT}$. $T_{KT}$ in this model can be obtained by solving the following equation
\[
\frac{\beta v_T}{2} \tanh \frac{\beta v_T}{2} = \frac{8}{N}.
\] 

This can be shown to be lower than the temperature where the neutral order parameter vanishes.

4. Conclusion

To conclude the relativistic four-fermi model presented here, exhibits superconductivity due to the KT mechanism of vortex-binding. Once the vortices unbind, the pole in the current-current correlator disappears and superconductivity is destroyed. This result was anticipated, but not proven, in some recent works in the literature.\(^\text{11}\) It would be interesting to see if in three dimensions, this mechanism generates superconductivity.

This paper is dedicated to the memory of Prof. H. Umezawa, who has contributed significantly to the understanding of various aspects of superconductivity and in general to the understanding of the structure of field theories, both at zero and finite temperatures.

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