Hindman’s finite sums theorem and
its application to topologizations of algebras

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Abstract

The first part of the paper is a brief overview of Hindman’s finite sums theorem, its
prehistory and a few of its further generalizations, and a modern technique used in proving
these and similar results, which is based on idempotent ultrafilters in ultrafilter extensions of
semigroups. The second, main part of the paper is devoted to the topologizability problem
of a wide class of algebraic structures called polyrings; this class includes Abelian groups,
rings, modules, algebras over a ring, differential rings, and others. We show that the Zariski
topology on such an algebra is always non-discrete. Actually, a much stronger fact holds: if
$K$ is an infinite polyring, $n$ a natural number, and a map $F$ of $K^n$ into $K$ is defined by a term
in $n$ variables, then $F$ is a closed nowhere dense subset of the space $K^{n+1}$ with its Zariski
topology. In particular, $K^n$ is a closed nowhere dense subset of $K^{n+1}$. The proof essentially
uses a multidimensional version of Hindman’s finite sums theorem established by Bergelson
and Hindman. The third part of the paper lists some problems concerning topologization of
various algebraic structures, their Zariski topologies, and some related questions.

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1 Ramsey theory of finite sums

In this section we shortly recall some basic things related to the famous Hindman finite sums
theorem, including some historic information, a modern technique used in proving results of such
kind, which is based on idempotent ultrafilters in ultrafilter extensions of semigroups. The we
formulate a couple of generalizations of the theorem, one of which will be essential for the proof
of our main result in the next section of this paper.

1.1 Algebraic Ramsey theory

As well-known, Ramsey theory is a vast area having various aspects including purely combinatorial
and set-theoretic; for general information we refer the reader to the classical textbooks [1] and
(from a more set-theoretic perspective) [2]. Informally speaking, a statement can be considered
as Ramsey-theoretic iff it has the form “Any small partition of a large structure has a large part.”

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In this paper, we shall be interested in infinite Ramsey theory, where the weakest meaning of “small” naturally is “finite”. An easy observation (see e.g. [3], Theorem 5.7) is that in this case, “large” can be always understood as “ultrafilter large”:

Theorem 1.1.1. Let $X$ be a set and $A$ a family of its subsets. The following are equivalent:

(i) any finite partition of $X$ has a part $S$ such that there is $A \in A$ with $A \subseteq S$;
(ii) there exists an ultrafilter $u$ over $X$ such that for any $S \in u$ there is $A \in A$ with $A \subseteq S$.

Furthermore, we shall discuss here an algebraic aspect of infinite Ramsey theory, where (again informally speaking) “large” means “having a rich algebraic structure”. More specifically, for our purposes, “large” will mean “having many finite sums (or products)”. The answer to what is an ultrafilter counterpart in this case, will be given a bit later, in Section 1.3.

1.2 Finite sums

As usually, $\mathbb{N}$ is the set of natural numbers, which are understood as finite ordinals, and $\omega = \mathbb{N}$ is the first infinite ordinal.

Notation 1.2.1. 1. Given a (finite or infinite) sequence $x_0, x_1, \ldots, x_i, \ldots$ in $\mathbb{N}$, let

$$\text{FS} (x_i)_i$$

denote the set of finite sums $x_{i_0} + x_{i_1} + \ldots + x_{i_n}$ for all $n$ and $i_0 < i_1 < \ldots < i_n$ in $\mathbb{N}$.

2. If $X$ is a (multiplicatively written) semigroup, for a sequence $x_0, x_1, \ldots, x_i, \ldots$ in $X$, let

$$\text{FP} (x_i)_i$$

denote the set of finite products $x_{i_0} \cdot x_{i_1} \cdot \ldots \cdot x_{i_n}$ for all $n$ in $\mathbb{N}$ and $i_0 < i_1 < \ldots < i_n$ in $X$.

Let us point out that if the sequence $(x_i)_i$ was injective, then all summands in $\text{FS} (x_i)_i$ (respectively, factors in $\text{FP} (x_i)_i$) involve distinct elements. Also notice that if the semigroup $X$ is non-commutative, then the increasing ordering of $i_0, i_1, \ldots, i_n$ is essential.

Example 1.2.2. $\text{FS} (1, 3, 5) = \{1, 3, 4, 5, 6, 8, 9\}$.

The historically first Ramsey-theoretic result was due to Hilbert [4] (much before Ramsey’s [5], which gave the name of the theory) and related to finite sums:

Theorem 1.2.3 (Hilbert, 1892). For any finite partition of $\mathbb{N}$ and any $n \in \mathbb{N}$ there exist a part $A$, a sequence $(x_i)_{i<n}$ in $\mathbb{N}$, and an infinite $B \subseteq \mathbb{N}$ such that

$$\bigcup_{b \in B} (b + \text{FS} (x_i)_{i<n}) \subseteq A.$$

Here $b + C$ denotes $\{b + c : c \in C\}$, the shift of the set $C$ by $b$.

Another early result, Schur’s theorem [6], involved only sums with two summands, but without shifts:

Theorem 1.2.4 (Schur, 1916). For any finite partition of $\mathbb{N}$ there exist a part $A$ and $x, y \in \mathbb{N}$ such that

$$\{x, y, x + y\} \subseteq A.$$
Remark 1.2.5. Interesting enough, Schur used this result to prove that $x^n + y^n \equiv z^n \pmod{p}$ has solutions for all sufficiently large prime $p$.

A natural question is whether two summands $x, y$ can be improved to three summands $x, y, z$, or more. What is known now under the name of Folkman’s theorem (see [1]) gives an affirmative answer even for any finite number:

**Theorem 1.2.6** (Folkman’s theorem, Sanders, 1968, Rado, 1969). For any finite partition of $\mathbb{N}$ and any $n \in \mathbb{N}$ there exist a part $A$ and an $n$-tuple $x_0, \ldots, x_{n-1}$ such that

$$FS(x_i)_{i<n} \subseteq A.$$  

Clearly, then some part $A$ should include even sets $FS(x_i)_{i<n}$ for some (distinct) sequences $(x_i)_{i<n}$ for all $n \in \mathbb{N}$. Finally, one can ask whether it is possible to have a single infinite sequence whose finite sums lie in some part. This was known as the Graham–Rothschild conjecture until it was proved by Hindman [7]:

**Theorem 1.2.7** (The Finite Sums Theorem, Hindman, 1974). For any finite partition of $\mathbb{N}$ there exist a part $A$ and an infinite sequence $x_0, \ldots, x_i, \ldots$ such that

$$FS(x_i)_{i<\omega} \subseteq A.$$  

Hindman’s original proof in [7] was purely combinatorial and rather complicated. In the same year Baumgartner provided a much shorter combinatorial proof [8]. Although both proofs remain interesting (in particular, in point of view of reverse mathematics, see Remark 1.3.5), only the third proof, based on the idea to use idempotent ultrafilters, put beginning of a new era in algebraic Ramsey theory.

### 1.3 Idempotent ultrafilters

The idea to use such type of ultrafilters for the proof of the Graham–Rothschild conjecture was proposed (but not published) by Galvin around 1970. In [9], Hindman shown that under CH (Continuum Hypothesis), the conjecture is equivalent to the existence of such ultrafilters. Finally, in 1975 Glazer observed that their existence follows from some topological facts, which have been already known; this proof was first published in [10].

Recall that a groupoid is a set $X$ with an arbitrary binary operation on it (e.g. a semigroup is just an associative groupoid). A (multiplicativele written) groupoid $(X, \cdot)$ is right topological iff $X$ is endowed with a topology in which all its right shifts, i.e. the maps $x \mapsto xa$ for all $a \in X$, are continuous.

The first of the topological facts is the following statement, due in its final form to Ellis [11]:

**Theorem 1.3.1** (Ellis, 1969). Every compact Hausdorff right topological semigroup has an idempotent.

The second fact is that $(\mathbb{N}, +)$, the additive semigroup of natural numbers, extends (in a canonical way) to $(\beta \mathbb{N}, +)$, a compact Hausdorff extremally disconnected semigroup of ultrafilters over $\mathbb{N}$ which right topological; moreover, all its left shifts by principal ultrafilters are continuous.

Combining these two facts, we see that the semigroup $(\beta \mathbb{N}, +)$ has an idempotent. Moreover, it has a free ultrafilter which is idempotent; to see, take rather $\mathbb{N}\{0\}$. It remains to use Galvin’s

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1 In 2008 the author hear from Hindman an amusing phrase: “I never understood my proof”.
observation: If \( u \) is an idempotent in \((\beta \mathbb{N}, +)\), then for any \( S \subseteq u \) there is an infinite \((x_i)_{i<\omega} \) in \( \mathbb{N} \) such that \( FS(x_i)_{i<\omega} \subseteq S \). Now Hindman’s Finite Sums Theorem (Theorem 1.2.7) is immediate: whenever \( u \) is an idempotent ultrafilter then one part of any finite partition does belong to it.

A real value of these observations, however, is that they have a very broad character; in fact, they lead to a general version of Theorem 1.2.7, as was (according to [10]) independently pointed out by Glazer and Hindman around 1975.

Recall that \( \beta X \), the set of ultrafilters over \( X \), carries the standard topology generated by open sets of form \( A = \{ u \in \beta X : A \subseteq u \} \) for all \( A \subseteq X \), which is compact, Hausdorff, and extremally disconnected (the latter means that the closure of any open set is open).

Given a groupoid \((X, \cdot)\), it canonically extends to the groupoid \((\beta X, \cdot)\) whose operation is defined by letting \( u \cdot v = \{ A \subseteq X : \{ x \in X : y \in X : x \cdot y \in A \} \subseteq v \} \subseteq u \) for all \( u, v \in \beta X \), the ultrafilter extension of \((X, \cdot)\). (Here the word “extension” relates to the usual identification of elements of \( X \) with the principal ultrafilters over \( X \), under which one lets \( X \subseteq \beta X \).) Topologically the extension is going as follows: first we continuously extend all left shifts of the initial operation, and then all right shifts of the obtained partially extended operation (see [12] or [13] for details and explaining of canonicity of the construction).

**Lemma 1.3.2.** Every discrete semigroup \((X, \cdot)\) canonically extends to the semigroup \((\beta X, \cdot)\). The latter is, w.r.t. the standard compact Hausdorff extremally disconnected topology on \( \beta X \), a right topological semigroup with continuous left shifts by principal ultrafilters.

A groupoid is weakly left cancellative iff for any its elements \( a, b \) the equation \( a \cdot x = b \) has only finitely many solutions. It can be verified that whenever \( X \) is weakly left cancellative then the free ultrafilters form a closed subgroupoid of \( \beta X \); therefore, Ellis’ theorem (Theorem 1.3.1) is applicable to it. Thus combining 1.3.1 and 1.3.2 we obtain:

**Corollary 1.3.3.** For every semigroup \((X, \cdot)\), the semigroup \((\beta X, \cdot)\) has an idempotent. Moreover, if \((X, \cdot)\) is infinite and either without idempotents or weakly left cancellative, then \((\beta X, \cdot)\) has a free idempotent.

**Lemma 1.3.4.** For every groupoid \((X, \cdot)\), if \( u \) is a free idempotent in \((\beta X, \cdot)\), then for any \( S \in u \) there is an infinite \((x_i)_{i<\omega} \) in \( S \) such that \( FP(x_i)_{i<\omega} \subseteq S \).

**Remark 1.3.5.** The statement converse to Lemma 1.3.4 is also true: any such \( S \) belongs to some idempotent ultrafilter. Thus ultrafilters each element of which includes some \( FP(x_i)_{i<\omega} \) are exactly those that belong to \( cl_{\beta X}\{ u \in \beta X : u \cdot u = u \} \), the closure (in \( \beta X \)) of the set of idempotents ultrafilters. (The latter set is not closed, see [3].)

Let us point out also that the existence of ultrafilters \( u \in \beta \mathbb{N} \) such that any \( S \in u \) includes a set of form \( FS(x_i)_{i<\omega} \) which itself belongs to \( u \), called strongly summable ultrafilters, is independent of ZFC (the Zermelo–Fraenkel set theory); see [3], Chapter 12.

Now 1.3.3 and 1.3.4 together immediately give the general version of Hindman’s theorem:

**Theorem 1.3.6** (The Finite Products Theorem, 1975). Let \((X, \cdot)\) be a semigroup either without idempotents or weakly left cancellative. For any finite partition of \( X \) there exist a part \( A \) and an infinite sequence \( x_0, \ldots, x_i, \ldots \) such that

\[
FP(x_i)_{i<\omega} \subseteq A.
\]

**Example 1.3.7.** The multiplicative semigroup \((\mathbb{N}, \cdot)\) of natural numbers.
1.4 Generalizations

After these initial steps, algebra of ultrafilters quickly became an advanced area cultivated by many prominent authors (Bergelson, Blass, van Douwen, Hindman, Leader, Protasov, Strauss, Zelenyuk among others). Earlier results, including classical Ramsey’s theorem and van der Waerden’s arithmetic progressions theorem as well as newer Hales–Jewett’s theorem and Furstenberg’s multiple recurrence theorem, were restated by means of this technique. This provided a better understanding of algebraic Ramsey theory and led to new deep results and applications to number theory, algebra, topological dynamics, and ergodic theory; most of them have no known elementary proofs.

Here we mention only two immediate generalizations of Hindman’s theorem, both obtained by Hindman and Bergelson. The first provides a simultaneous additive and multiplicative version of the theorem (see [3], Corollary 5.22):

**Theorem 1.4.1** (Bergelson–Hindman, 1993). For any finite partition of \( \mathbb{N} \) there exist a part \( A \) and two infinite sequences \( x_0, \ldots, x_i, \ldots \) and \( y_0, \ldots, y_i, \ldots \) such that

\[
\text{FS}(x_i)_{i<\omega} \cup \text{FP}(y_i)_{i<\omega} \subseteq A.
\]

The proof is not difficult modulo the above observations; it suffices first to show that the set \( \text{cl}_{\mathbb{B}\mathbb{N}}(\{u \in \mathbb{B}\mathbb{N} : u = u + u\}) \) (the closure of the set of additive idempotents) forms a left ideal of the multiplicative semigroup \( (\mathbb{B}\mathbb{N}, \cdot) \); then apply Theorem 1.3.1 to pick some \( v = v \cdot v \) in it.

**Remark 1.4.2.** This result cannot be improved by showing the existence of a single sequence \( (x_i)_{i<\omega} \) such that \( \text{FS}(x_i)_{i<\omega} \cup \text{FP}(x_i)_{i<\omega} \) is included into a part of a given partition. In fact, this is impossible even for sums and products of all pairs of distinct elements in \( (x_i)_{i<\omega} \); see [3], Theorem 17.16.

Another generalization of Hindman’s theorem we want to formulate in this overview, is an its multidimensional (more precisely, finite-dimensional) version (see [3], Theorem 18.11):

**Theorem 1.4.3** (Bergelson–Hindman, 1996). For any \( m, n \in \mathbb{N} \), infinite semigroups \( X_0, \ldots, X_n \) each of which is either without idempotents or weakly left cancellative, and finite partition of the Cartesian product \( \prod_{i \leq n} X_i \) there exist a part \( A \), \( m \)-sequences \( (x_{i,k})_{k<m} \) in each \( X_i \), \( i < n \), and an infinite sequence \( (x_{n,k})_{k<\omega} \) in \( X_n \) such that

\[
(\prod_{i<n} \text{FP}(x_{i,k})_{k<m}) \times \text{FP}(x_{n,k})_{k<\omega} \subseteq A.
\]

**Remark 1.4.4.** This result cannot be improved by showing the existence of two infinite sequences, say, \( (x_{n-1,k})_{k<m} \) and \( (x_{n,k})_{k<\omega} \), generating such sets of finite products; see [3].

Theorem 1.4.3 will be crucial in our application to topologization of certain universal algebras discussed in Section 2.

For further information about the topic, we refer the reader to the literature. In particular, [3] is a comprehensive treatise on algebra of ultrafilters, with an historic information and a vast list of references; [14] provides a clear introduction to this area; [15] is the classical textbook on the general theory of ultrafilters.
Remark 1.4.5. There are investigations of Hindman’s theorem and related assertions in point of view of their proof-theoretic and set-theoretic strength.

An arithmetic version of Hindman’s theorem was studied in reverse mathematics. It was shown in [16] that its proof-theoretic strength (over RCA\(_0\)) lies between ACA\(_0\) and ACA\(_0^+\). For more recent results see [17, 18, 19] and the literature mentioned there.

Very recently it was shown [20] that Ellis’ theorem (Theorem 1.3.1 here) follows from the Boolean Prime Ideal Theorem, thus showing it is weaker than the full AC (the Axiom of Choice). The article [20] contains also a series of other relevant results.

Remark 1.4.6. In [21], Theorems 1.3.1, 1.3.6, 1.4.1 were generalized to certain non-associative algebras. Further generalizations of Hindman’s theorem involving larger sets of finite products (taken rather along partially ordered sets than sequences) were studied in [22].

The above extension of semigroups by ultrafilters is a partial case of a certain canonical procedure of ultrafilter extension of arbitrary first-order models, which was defined in [12, 13].

2 Finite sums and topologizations

In this section, we apply the multidimensional generalization of the Finite Sums Theorem, which was formulated above (Theorem 1.4.3), to the problem of topologizability of universal algebras.

First we recall the origin of the problem and results obtained earlier; then we define a certain class of algebras, called here “polyrings”, which includes various classical algebras like rings, differential rings, algebras over rings, and others, and the Zariski topology of these algebras and their finite powers.

After this, we formulate the main result of this section (Theorem 2.4.1), which states that for any infinite polyring \(K\), any map of \(K^n\) into \(K\) defined by a term in \(n\) variables is a closed nowhere dense subset of \(K^{n+1}\) in its Zariski topology. In particular, \(K^n\) is closed nowhere dense in \(K^{n+1}\), and a fortiori, all the Zariski spaces \(K^n\) are non-discrete. Then we state that all countable polyrings are Hausdorff topologizable, and conclude the section with an outline of a proof of Theorem 2.4.1.

2.1 Topologizations of algebras

Recall some facts concerning the topologization problem.

Definition 2.1.1. 1. A universal algebra is a topological algebra iff it is endowed with a topology in which all its main operations are continuous.

2. A universal algebra is \(T_1\)-topologizable iff it admits a non-discrete \(T_1\)-topology which turns it into a topological algebra.

Topological universal algebras were defined by Maltsev in [23]. A topologizability problem was first posed by Markov (Jr.) in [24–26], who asked whether any group admits a non-discrete Hausdorff topology in which its multiplication and inversion become continuous. He (implicitly) defined a \(T_1\)-topology on a group, called now its Zariski topology, and proved that any countable group is \(T_2\)-topologizable iff its Zariski topology is non-discrete.

Later it was proved that the answer is affirmative for certain classes of groups, as Abelian [27] and free [28], and negative in general; for uncountable groups this was proved under CH in [29] and without CH in [30], for countable groups in [31] (based on Adian’s construction):
Theorem 2.1.2.
1 (Kertész and Szele, 1953, Zelenyuk, 2011). All Abelian groups as well as and all free groups are $T_2$-topologizable.
2 (Shelah, 1976, Hesse, 1979, Olshanski, 1980). There exist groups of any infinite cardinality that are not $T_2$-topologizable.

There are known other classes of groups in which all countable groups are $T_2$-topologizable (see [32], Corollary 3). Other examples of non-topologizable groups (e.g. non-topologizable torsion-free groups) can found in [33]. For further studies of topologizability of groups and other related problems posed by Markov in [24]–[26] we refer the reader to [34]–[36] and the literature there.

Remark 2.1.3. Zelenyuk proved [37] that any infinite group admits a non-discrete zero-dimensional $T_3$-topology in which all its left and right shifts and inversion are continuous. Cf. also Section 9.2 in [3], which discusses left topologizability of groups obtained by idempotent ultrafilters over them.

The situation with topologization of rings slightly differs from the case of groups. It is still possible to prove that any countable ring is $T_2$-topologizable iff its Zariski topology is non-discrete. In 1970, Arnautov obtained the following principal results: the Zariski topology of every infinite ring is non-discrete [38] and so countable rings admit Hausdorff topologies, the same holds for all commutative rings [39], but not in general [40]:

Theorem 2.1.4 (Arnautov, 1970).
1. All infinite rings have non-discrete Zariski topologies.
2. All countable rings as well as and all commutative rings are $T_2$-topologizable.
3. There exist rings of an uncountable cardinality that are not $T_2$-topologizable.

A survey on topologizability of rings and modules can be found in [41], Chapter 5.

The topologizability problem was studied for other algebras. Prior the negative solution was obtained for groups, it was done for groupoids [42] and semigroups [43]. For universal algebras this was studied in [44]. That countable algebras admit Hausdorff topologizations iff their Zariski topologies are non-discrete, was shown for unoids (i.e. algebras with arbitrary families of unary operations) in [45]. In [46] this fact was announced for all universal algebras; the proof was provided in [32] and completed in [47]. Moreover, the latter paper proves this fact for all first-order models (requiring that all their relations should be closed), and provides a sufficient condition for $T_2$-topologizability of models of any cardinality; we shall use this condition below (Theorem 2.4.3). Let us mention also that the $T_1$-topologizability generally does not imply the $T_2$-topologizability (see e.g. [48]).

2.2 Polyrings

In [14] Protasov gave an elegant proof of the non-discreteness of the Zariski topologies of rings by using Hindman’s Finite Sums Theorem. Following close ideas, but replacing this theorem with its stronger multidimensional version (Theorem 1.4.3), we shall state stronger facts, from which will follow the non-discreteness of the Zariski topologies for universal algebras of a much wider class, called here “polyrings”. Moreover, it will follow that all finite powers of these algebras have

\footnote{2 It is worth to note that the original proof in [40] used van der Waerden’s Arithmetic Progression Theorem, another standard result having a short proof via ultrafilter algebra (which can be found in [3] or [14]).}
non-discrete Zariski topologies, and even that for all finite $n$, the $n$th power is closed nowhere dense in the $(n + 1)$th power.

**Definition 2.2.1.** A universal algebra $(K, 0, +, \Omega)$ is a polyring iff $(K, 0, +)$ is an Abelian group and any operation $F \in \Omega$ (of arbitrary arity) is distributive w.r.t. the addition, i.e. the additive shifts

$$x \mapsto F(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$$

are endomorphisms of $(K, 0, +)$, for all $i < n$ and $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1} \in K$.

**Example 2.2.2.** Various classical algebras can be considered as special instances of polyrings: Abelian groups with operators, modules, Lie algebras, rings, differential rings, algebras over a ring, Boolean algebras with operators (in sense of [49]), etc.

On the other hand, every polyring is an Abelian $\Omega$-group, but not conversely. (Recall that $\Omega$-groups are groups, not necessarily Abelian, with operations $F \in \Omega$ satisfying $F(0, \ldots, 0) = 0$.)

**Lemma 2.2.3.** There are two standard expansion procedures:

1. For every Abelian group $(K, 0, +)$ there exists the most expanded polyring $(K, 0, +, \Omega)$, i.e. such that for any polyring $(K, 0, +, \Omega')$ we have $\Omega' \subseteq \Omega$.

2. For every universal algebra $(K', \Omega')$ there exists a polyring $(K, 0, +, \Omega)$ such that $(K', \Omega')$ is embedded into $(K, \Omega)$.

Clause 2 of Lemma 2.2.3 generalizes the standard construction of a groupoid ring, which provides the ring of formal sums with integer coefficients whose multiplicative groupoid extends a given groupoid; in the general case for each $n$-ary operation $F' \in \Omega'$ we define the $n$-ary operation $F \in \Omega$ by using products of $n$ integers.

### 2.3 Zariski topologies

As said above, the Zariski topology on groups was implicitly considered by Markov in [24]–[26]. The first explicit description of it, under the name of “verbal topology”, was given in [50]. The name “Zariski topology” became standard after the paper [51], in which the authors developed algebraic geometry over abstract groups; in analogy with classical algebraic geometry over fields they defined the Zariski topologies on finite powers $G^n$ of a group $G$ by using solution sets of $n$ variables equations (the case $n = 1$ gives the verbal topology of [50]). Later this approach, together with close ideas in [52]–[53] resulted in universal algebraic geometry; see [54]–[55] and the references there. The Zariski topologies of universal algebras were considered in [44], [46], later in [32], and generalized to arbitrary first-order models in [47]. We mention also [56] providing an abstract, model-theoretic approach to Zariski topologies of classical algebraic geometry.

For simplicity, here we define only the Zariski topologies of polyrings, although the general definition is not much more complicated (it involves the solutions of all atomic formulas).

**Definition 2.3.1.** Let $K$ be a polyring and $n \in \mathbb{N}$.

1. If $F \in K[x_1, \ldots, x_n]$ is a term in $n$ variables, let

$$S_F = \{(a_1, \ldots, a_n) \in K^n : F(a_1, \ldots, a_n) = 0\}$$

denote the set of its roots, i.e. solutions of the equation $F(x_1, \ldots, x_n) = 0$ in $K$. Finite unions of sets of form $S_F$ are called algebraic.

2. A set $S \subseteq K^n$ is closed in the Zariski topology on $K^n$ iff $S$ is an intersection of algebraic sets. In other words, sets of roots of equations in $n$ variables form a closed subbase of the Zariski topology on $K^n$, and the resulting algebraic sets a closed base of the topology.
As easy to see, the Zariski topology on $K$ is a $T_1$- (but not necessarily $T_2$-) topology in which all shifts are continuous. Moreover:

**Lemma 2.3.2.** Let $K$ be any polyring and $n \in \mathbb{N}$. Then:

(i) the Zariski topology on $K^n$ is a $T_1$- topology in which all maps of $K^n$ into $K$ defined by terms in $n$ variables are continuous;

(ii) the Zariski topology on $K^{n+1}$ includes the product of the Zariski topologies on $K^n$ and $K$, and can be stronger (e.g. if $K$ is an infinite field);

(iii) the space $K^n$ is homeomorphic to $K^n \times \{0\} \subseteq K^{n+1}$ (and will be identified with it below);

(iv) the space $K^n$ is homogeneous.

**Remark 2.3.3.** In Lemma 2.3.2, only clause (iv) uses the group structure on $K^n$ (autohomeomorphisms which connect given points can be realized as additive shifts). It is easy to see that (iv) remains true for any structure with the transitive group of invertible shifts (cf. Lemma 6 in [32]).Clauses (i)–(iii) are of a general character.

For further information on Zariski topologies we refer the reader to the above literature.

### 2.4 The main result

Now we are ready to formulate the main result of this section:

**Theorem 2.4.1** (The Main Theorem). Let $K$ be an infinite polyring. For any term $F \in K[x_1, \ldots, x_n]$ the mapping of $K^n$ into $K$ defined by $F$ is closed and nowhere dense in $K^{n+1}$ (where $F$ is a subspace of the space $K^{n+1}$ with its Zariski topology). In particular, $K^n$ is closed and nowhere dense in $K^{n+1}$.

Loosely speaking, this shows that such spaces, although can be even not Hausdorff, allow a reasonable notion of topological dimension (this remark will be refined in Section 3.3). Certainly, this is much stronger fact than the non-discreteness of these space, which easily follows:

**Corollary 2.4.2.** If $K$ is an infinite polyring and $0 < n < \omega$, then $K^n$ with its Zariski topology has no isolated points.

**Remark 2.4.3.** The following observations are immediate from Lemma 2.3.2:

1. If $\Omega \subseteq \Omega'$ then the Zariski topology of $(K, 0, +, \Omega')$ is stronger than one of $(K, 0, +, \Omega)$. Since there exists the most expanded polyring with a given $(K, 0, +)$, Theorem 2.4.1 provides the best possible result in this direction.

2. The procedure which embeds a given universal algebra $(K', \Omega')$ into a universal algebra $(K, \Omega)$ expanded to a polyring $(K, 0, +, \Omega)$ of formal sums, provides also a natural extension of the Zariski space of any given algebra, which can be discrete, to a non-discrete Zariski space of a larger algebra.

Recall that for a $T_1$-space $X$, its *pseudocharacter* at a point $x \in X$ is the least cardinality of a family of open sets whose intersection is $\{x\}$, and the pseudocharacter of the whole space $X$ is the supremum of these cardinals for all its points. Clearly, the pseudocharacter of $X$ does not exceed

---

1 This theorem was established around 2010 but not published until [57] though presented at several conferences, e.g. Colloquium Logicum 2012 in Paderborn, and included in author’s course lectured at the Steklov Mathematical Institute in 2014.
its cardinality \(|X|\), and is 1 if \(X\) is discrete, and infinite otherwise. Thus Corollary 2.4.2 shows that whenever \(K\) is an infinite polyring endowed with its Zariski topology then its pseudocharacter is also infinite.

Applying the sufficient condition for topologizability of arbitrary structures given in [47] to the Main Theorem, we obtain:

**Theorem 2.4.4.** Let \(K\) be an infinite polyring, \(|K| = \kappa \geq \omega\).
1. If \(\kappa = \omega\), then \(K\) is topologizable by some Hausdorff topology without isolated points.
2. In general, if the pseudocharacter of \(K\) with its Zariski topology is \(\cf \kappa\), then \(K\) is topologizable by some Hausdorff topology of the same pseudocharacter \(\cf \kappa\).

**Example 2.4.5.** 1. Let \(K\) be a non-topologizable ring constructed by Arnautov in [40]. In the Zariski space of \(K\), the pseudocharacter is infinite (like for any polyring) but not equal to \(\cf |K|\).
2. Let \(K\) be \((\mathbb{F}_2)^\omega\), the countable direct power of the two-element field \(\mathbb{F}_2\). Clearly, \(K\) is a ring, and its Zariski topology coincides with the usual topology of the Cantor set.

To see, note that for any \(a = (a_i)_{i < \omega} \in K\), some \(b = (b_i)_{i < \omega} \in K\) is a solution of the equation \(ax = 0\) iff \(b_i = 0\) whenever \(a_i = 1\), and a solution of the equation \(ax + a = 0\) iff \(b_i = 1\) whenever \(a_i = 1\) (so we have \(S_{ax + a} = K \setminus S_{ax}\)). It easily follows that any basic clopen set in the Cantor space is algebraic.

As \(K\) has cardinality \(2^\omega\) and pseudocharacter \(\omega < \cf 2^\omega\), we see that the sufficient condition from [47] is not necessary.

### 2.5 Sketch of proof

Here we outline the proof of the Main Theorem. Let \(K\) be a polyring.

**Notation 2.5.1.** To simplify the reading, we adopt the following conventions:

1. Given \(n < \omega\), we write \(\bar{x}, \bar{0}, \bar{K}\) instead of \((x_1, \ldots, x_n), (0, \ldots, 0), K^n\), respectively.
2. The pointwise addition in \(\bar{K}\) is denoted also by +, so \(\bar{x} + \bar{y}\) denotes \((x_1 + y_1, \ldots, x_n + y_n)\).
3. \(K[x_1, \ldots, x_n]\) denotes the set of all terms over \(K\) in the variables \(x_1, \ldots, x_n\).

**Definition 2.5.2.** A term is a *monomial* iff it does not contain +, and a *polynomial* iff it is a sum of monomials.

**Lemma 2.5.3.** In polyrings, any term is represented by a polynomial.

*Proof.* By distributivity.

**Definition 2.5.4.** For a term \(F\) over a polyring \(K\), the *degree of \(F\)* w.r.t. a given set of variables is defined by induction on the construction of \(F\): it is equal to

1. the number of occurrences of these variables in \(F\) if \(F\) is a monomial;
2. the maximum of degrees of monomials in \(F\) if \(F\) is a polynomial;
3. the minimal degree of a polynomial representing \(F\).

The *degree of \(F\)* is its degree w.r.t. all variables.

The following lemma states that in polyrings, any mapping defined by a term is “almost an endomorphism”, namely, an endomorphism up to a mapping defined by a term of a lesser degree:
Lemma 2.5.5. Let \( \vec{x} \) and \( \vec{y} \) have the same length. For every \( F \in K[\vec{x}] \) of a nonzero degree there exists \( G \in K[\vec{x}, \vec{y}] \) of a lesser degree w.r.t. \( \vec{x} \) such that

\[
F(\vec{x} + \vec{y}) = F(\vec{x}) + F(\vec{y}) + G(\vec{x}, \vec{y}).
\]

Proof. Induction on the degree of \( F \).

Remark 2.5.6. Here is the only place in the proof using the assumption that the groups \((K, 0, +)\) in polyrings are Abelian. Recall, however, that the Zariski topology of non-Abelian groups can be discrete; thus the assumption is essential.

Corollary 2.5.7. For every \( F \in K[\vec{x}] \) of a nonzero degree and any \( \vec{a} \in \vec{K} \) there exists \( H \in K[\vec{x}] \) of a lesser degree such that

\[
F(\vec{x} + \vec{a}) = F(\vec{x}) + H(\vec{x}).
\]

The next lemma connects roots of terms and sets of finite sums; this crucial fact will allow us to apply Hindman-type results to our purposes:

Lemma 2.5.8 (The Key Lemma). Let \( F \in K[\vec{x}] \) have the degree \( n \) and \( (\vec{a}_i)_{i \leq n} \in \vec{K}^{n+1} \). If \( F(\vec{b}) = 0 \) for all \( \vec{b} \in \text{FS}(\vec{a}_i)_{i \leq n} \), then \( F(\vec{0}) = 0 \).

Proof. Induction on \( n \) using Corollary 2.5.7.

Thus \( \text{FS}(\vec{a}_i)_{i \leq n} \subseteq S_F \) implies \( \vec{0} \in S_F \). In terms of the Zariski topology, we obtain:

Corollary 2.5.9. For any \( (\vec{a}_i)_{i < \omega} \) in \( \vec{K} \), the closure of the set \( \text{FS}(\vec{a}_i)_{i < \omega} \) in the Zariski topology of \( \vec{K} \) has the element \( \vec{0} \).

Remark 2.5.10. Recall that sets \( S \supseteq \text{FS}(\vec{a}_i)_{i < \omega} \) are just elements of idempotent ultrafilters in the semigroup \((\beta \vec{K}, +)\).

Definition 2.5.11. A set \( A \subseteq \prod_{i < n} X_i \) is finite-valued iff there is \( j < n \) such that all sections in \( X_j \) are finite, i.e. for all \( (a_i)_{i < j} \in \prod_{i < j} X_i \) and \( (a_i)_{j < i < n} \in \prod_{j < i < n} X_i \) the sets

\[
\{a_j \in X_j : (a_i)_{i < n} \in A\}
\]

are finite. (Such an \( A \) can be regarded as a partial finite-valued map of \( \prod_{i \in \mathbb{N} \setminus \{j\}} X_i \) into \( X_j \), which explains the name.)

The following theorem immediately leads to the main result but is also interesting in its own right:

Theorem 2.5.12 (The Key Theorem). For every infinite polyring \( K \), if \( A \subseteq \vec{K} \) is finite-valued then \( A \) has the empty interior in the Zariski topology of \( \vec{K} \).

Proof. Notice first that if a set \( A \) is finite-valued then so is any \( B \subseteq A \). Therefore, it suffices to show that \( A \) is not open.

Furthermore, notice that we can w.l.g. suppose that \( \vec{0} \in A \) (otherwise additively shift the set by using that the space is homogeneous) and, assuming \( \vec{K} \) denotes \( K^{n+1} \), that \( A \) is finite-valued in the \( n \)-th coordinate (otherwise rename the coordinates).

Toward a contradiction, assume that \( A \) is open. Then \( \vec{K} \setminus A \) is closed so is the intersection of sets of form \( S_{F_0} \cup \ldots \cup S_{F_j} \) for some terms \( F_0, \ldots, F_j \). Pick any of these sets and show that it has \( \vec{0} \). It will follow that \( \vec{0} \in \vec{K} \setminus A \), thus reaching a contradiction.
Let $m$ be maximum of the degrees of the terms $F_0, \ldots, F_j$. The sets $A, S_{F_0}, \ldots, S_{F_j}$ cover the whole space $\vec{K} = K^{n+1}$:

$$A \cup S_{F_0} \cup \ldots \cup S_{F_j} = K^{n+1}. $$

Therefore, by Theorem 1.4.3 (the multidimensional generalization of the Finite Sums Theorem), some of these sets includes the set

$$P = \left( \prod_{i<n} \text{FS} (a_{i,k})_{k\leq m} \right) \times \text{FS} (a_{n,k})_{k<\omega}$$

for some sequences $(a_{i,k})_{k\leq m}$ and $(a_{n,k})_{k<\omega}$ consisting of distinct elements of $K$.

For $A$, however, this is impossible: $P \not\subseteq A$ since $A$ is finite-valued in the $n$th coordinate. Hence, $P \subseteq S_{F}$ for some $F \in \{F_0, \ldots, F_j\}$. But then $\vec{0} \in S_{F}$ follows from the Key Lemma (Lemma 2.5.8).

This completes the proof of Theorem 2.5.12.

**Corollary 2.5.13.** For every infinite polyring $K$, if $A \subseteq \vec{K}$ is finite-valued and closed in the Zariski topology of $\vec{K}$, then $A$ is nowhere dense.

**Proof.** For closed sets, to have empty interior is the same that to be nowhere dense. □

**Remark 2.5.14.** The assumption that $A$ is closed cannot be omitted even for single-valued maps; e.g., the set $A = \{(a, a^2) : a \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the additive group of integers, is everywhere dense in $\mathbb{Z}^2$.

Now the Main Theorem follows since $F \subseteq K^{n+1}$ is single-valued and closed in $K^{n+1}$ as $F = S_G$ for the map $G$ defined by letting

$$G(x_0, \ldots, x_{n-1}, x_n) = F(x_0, \ldots, x_{n-1}) - x_n.$$ 

The proof is complete.

### 3 Problems

In this section, we provide several problems and tasks related to subjects discussed above.

### 3.1 Discreteness

**Problem 3.1.1.** Characterize groups whose Zariski topology is non-discrete. Is the class of such groups first-order axiomatizable? at least, second-order axiomatizable?

The same questions for other algebraic structures.

**Problem 3.1.2.** Is the class of $T_2$-topologizable groups first-order (or at least, second-order) axiomatizable?

The same questions for other algebraic structures.

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1. Let us point out also that Section 8 of [36] and Section 13 in [28] both provide lists of open problems concerning topologies on groups and related questions.
Problem 3.1.3. If the Zariski topology of a group is non-discrete, does it remain non-discrete under adding an automorphism (endomorphism) as a new operation? all under adding all automorphisms (endomorphisms)?

The same questions for other algebraic structures.

Problem 3.1.4. If the Zariski topology of a group $K$ is non-discrete, is so the Zariski topology of $K^2$ of $K^n$? Furthermore, is then $K$ nowhere dense in $K^2$? $K^n$ nowhere dense in $K^{n+1}$ for all $n \in \mathbb{N}$?

The same questions for skew (i.e. non-commutative) fields, rings, polyrings, other algebraic structures.

Notice that Problems 3.1.1–3.1.4 are open only for non-Abelian groups; for Abelian case all four answers are affirmative by results of Section 2.

Let us say a polyring $(K, +, 0, \Omega)$ is commutative iff for all $F \in \Omega$ and permutations $\sigma$ of $n$, where $n$ is the arity of $F$, we have $F(x_1, \ldots, x_n) = F(x_{\sigma 1}, \ldots, x_{\sigma n})$ for all $x_1, \ldots, x_n$ in $K$.

Problem 3.1.5. Are all commutative polyrings $T_2$-topologizable?

By [39], the answer are affirmative for commutative rings (see Theorem 2.1.4).

Problem 3.1.6. Is there a variety (or at least, a quasivariety) of universal algebras, e.g. groupoids, having an infinite free algebra whose Zariski topology is discrete?

3.2 Connectedness

Problem 3.2.1. Characterize groups whose Zariski topology is connected. Is the class of such groups first-order axiomatizable? second-order axiomatizable?

The same questions for other algebraic structures (Abelian groups, rings, skew fields, etc.).

Problem 3.2.2. If the Zariski topology of a group is connected, does it remain connected under adding an automorphism (endomorphism) as a new operation? all under adding all automorphisms (endomorphisms)?

The same questions for other algebraic structures.

Problem 3.2.3. Are all Abelian torsion-free groups connected?

Recall that [33] provides examples of (non-Abelian) torsion-free groups with discrete Zariski topologies.

Problem 3.2.4. If an Abelian group is disconnected, can it be decomposed in a direct product in which a finite (of size $> 1$) factor? What about rings?

Problem 3.2.5. Is any free (non-Abelian) group connected? The same question for algebraically closed groups, torsion-free groups.

Problem 3.2.6. Is a ring connected if its additive group is free? algebraically closed? torsion-free?

Problem 3.2.7. Is any ring with non-trivial zero divisors disconnected?

Problem 3.2.8. Is any skew field connected? If yes, what about quasifields (near-fields, rings without non-trivial zero divisors, etc.)?
The connectedness may fail for some rings. E.g. the Zariski topology of \((\mathbb{Z}^\omega, +, \cdot)\) coincides with the usual topology on the Cantor set, and thus is disconnected. Its subring \((\mathbb{Z}_{<\omega}^\omega, +, \cdot)\) consisting of all eventually zero sequences is homeomorphic to the space of rationals.

**Problem 3.2.9.** Let \(K\) be an infinite skew field and \(F \in K[x]\) a polynomial such that not all elements of \(K\) are roots of \(F\). Is there a sequence \((a_i)_{i \in \mathbb{N}}\) of distinct elements of \(K\) such that all the elements of the set \(FS(a_i)_{i \in \mathbb{N}}\) are not roots of \(F\)?

What about near-fields (rings without non-trivial zero divisors, etc.)?

Let us point out (without a proof) that an affirmative answer to Problem 3.2.9 implies an affirmative answer to Problem 3.2.8.

### 3.3 Dimension

Let \(\text{ind}(X) \geq -1\) for all topological spaces \(X\), and let: \(\text{ind}(X) = -1\) iff \(X\) is empty, and whenever spaces \(Y\) with \(\text{ind}(Y) \leq n\) have already been defined then \(\text{ind}(X) \leq n + 1\) iff there exists an open base \(\Gamma\) of \(X\) such that \(\text{ind}(\partial O) \leq n\) for all \(O \in \Gamma\). (Here \(\partial S\) denotes the boundary of the set \(S\).)

Let us mention (without a proof) that Theorem 2.4.1 gives the inequality \(\text{ind}(K^n) \geq (\text{ind} K) + n - 1\) for all polyrings \(K\) and \(n \in \mathbb{N}\).

**Problem 3.3.1.** Is it true that for any infinite field \(K\) and all \(n \in \mathbb{N}\) we have \(\text{ind}(K^n) = n\)? (It is not difficult to verify that this is the case for \(n \leq 2\), and that \(\text{ind}(K^n) \geq n\) for \(n > 1\).)

Let us point out (without a proof) that an affirmative answer to Problem 3.3.1 below implies an affirmative answer to Problem 3.3.1.

**Problem 3.3.2.** Calculate \(\text{ind}(\mathbb{H}^n)\) for all \(n \in \mathbb{N}\). Is it true \(\text{ind}(\mathbb{H}) = 2\)? (Here \(\mathbb{H}\) is the skew field of quaternions.) An analogous question about other Cayley–Dickson algebras.

**Problem 3.3.3.** For which algebras (in particular, polyrings) the equality \(\text{ind}(K^n) = (\text{ind} K) \cdot n\) does hold for all \(n \in \mathbb{N}\)?

**Problem 3.3.4.** What can be values of the function \(\text{ind}\) for various algebras (skew fields, Abelian groups, groups, rings, polyrings, etc.)? How does \(\text{ind} K^n\) depend on \(\text{ind} K\)?

**Problem 3.3.5.** Assume \(K\) is a topologizable polyring. Is then \(K\) topologizable by a (Hausdorff) topology such that \(K^n\) is closed nowhere dense subspace of \(K^{n+1}\) in the usual product topology on \(K^{n+1}\)?

### 3.4 Other properties

**Problem 3.4.1.** Characterize groups (Abelian groups, rings, skew fields, etc.) whose Zariski topology is:

(i) compact (locally compact, Lindelöf, paracompact, metrizable, etc.);
(ii) Hausdorff (Tychonoff, normal, etc.);
(iii) zero-dimensional (totally disconnected, extremally disconnected);
(iv) a base of a filter (a filter, an ultrafilter, a filter of a special kind, etc.) plus the empty set.

Which of these classes of algebras are first-order axiomatizable? second-order axiomatizable?
Some similar questions were discussed in the literature. For any universal algebra \( A \) and \( n \in \mathbb{N} \), the Zariski spaces of \( A^n \) are Noetherian iff each system of equations in \( n \) variables has the same set of solutions that some its finite subsystem, see [54] (in fact, this reformulation is easy since Noetherian spaces can be characterized as the spaces in which all subspaces are compact). \( \Omega \)-groups in which all Zariski closed sets are algebraic (thus any intersection of algebraic sets is algebraic) are characterized in [58].

**Problem 3.4.2.** Establish properties of Zariski topologies that are preserved under:

(i) homomorphic images (pre-images);
(ii) subalgebras (normal subgroups, ideals);
(iii) direct products (reduced products, ultraproducts);
(iv) existential closure.

More generally, what is an interplay between the properties of a given algebra and algebras obtained from it by these (or other) model-theoretic operations?

An interplay between some separability axioms and some model-theoretic operations on topological algebras is discussed in [59].

**Problem 3.4.3.** Which properties of their Zariski topologies are shared by \( A^n \) and \( B^n \) if \( A \) and \( B \) are groups (divisible groups, rings, skew fields, etc.) having the same equational theory? elementary equivalent?

**Problem 3.4.4.** Characterize polyrings (and other algebras) \( K \) such that the Zariski space \( K^n \) coincides with the product space of \( n \) copies of the Zariski space \( K \).

**Problem 3.4.5.** Characterize skew fields (rings, Abelian groups, groups) in which any polynomial in one variable has only finitely many roots. (This is in fact Problem 3.4.1(iv) for the Fréchet filter.) Do such skew fields (algebraically closed skew fields) coincide with fields?

**Problem 3.4.6.** Are all countable polyrings topologizable by a metrizable topology?

Recall that by Theorem 2.4.4, all countable polyrings are \( T_2 \)-topologizable. Proposition 4.2 in [47] states that all countable first-order models in a relational language are topologizable by a metrizable zero-dimensional topology.

### 3.5 Topological classification

Let us say that a topological space \( X \) is *locally homeomorphic* to a space \( Y \) iff any open \( U \subseteq X \) includes some open \( V \subseteq U \) homeomorphic to \( Y \). The following questions are formulated for fields as even in this simplest case answers may appear to be unclear.

**Problem 3.5.1.** Let \( K \) be an infinite field. Provide a topological classification of connected definable subspaces of \( K^n \) up to: (i) homeomorphism; (ii) local homeomorphism. (E.g., do \( K^2 \) and \( K^2 \setminus A \) homeomorphic if \( A \) is a singleton? a proper closed subset of \( K^2 \)?)

The same task for skew fields (rings, Abelian groups, groups, etc.).

**Problem 3.5.2.** Is it true that for any polynomial \( F \in K[x_1, \ldots, x_n] \) over a field \( K \) the set of its roots (as a subspace of \( K^n \)) is locally homeomorphic to the space \( K^m \) for some \( m \)? (This is clear for \( n \leq 2 \); for \( n = 3 \) the simple instance which seems unclear is a 2-dimensional sphere in \( K^3 \).)

If the answer is affirmative, what about skew fields (rings, Abelian groups, groups)?
Problem 3.5.3. Let $K$ be a field. Is any space locally homeomorphic to the space $K^n$ embeddable into the space $K^m$ for some $m \in \mathbb{N}$ (e.g. into $K^{2n+1}$)?

If yes, what about skew fields (rings, Abelian groups, groups)?

Problem 3.5.4. Is the skew field $\mathbb{H}$ (with the Zariski topology given by polynomials in one variable) homeomorphic to the complex plane $\mathbb{C}^2$ (with the Zariski topology given by polynomials in two variables)? An analogous question about $\mathbb{H}^n$, other Cayley–Dickson algebras.

3.6 Miscellaneous

Problem 3.6.1. Characterize topological spaces $X$ which are the Zariski spaces of groups.

An analogous question about quasigroups, fields, rings, etc. If $X$ is the Zariski space of a quasigroup, is it the Zariski space of a group?

Problem 3.6.2. Characterize Hausdorff topological spaces $X$ which are spaces of groups, i.e. such that there is a group structure on $X$ with continuous group operations in this topology.

An analogous question about quasigroups, fields, rings, etc. If a Hausdorff space $X$ is a space of a quasigroup, is it the space of a group?

Problem 3.6.3. For the ultrafilter extension of the additive semigroup $(\mathbb{N}, +)$, study the smallest topology on $\beta \mathbb{N}$ which includes its standard topology and: (i) makes all left shifts continuous; (ii) includes the Zariski topology of $(\beta \mathbb{N}, +)$. (Notice that in case (i) the addition of ultrafilters becomes separately topological, thus turning $(\beta \mathbb{N}, +)$ into a quasi-topological semigroup, and that this topology is included into one defined in case (ii).)

Analogous questions about $(\mathbb{N}, +, \cdot)$, about ultrafilter extensions of other groups, rings, etc.

Problem 3.6.4. Determine the proof-theoretic strength of (arithmetic versions of) the Finite Products Theorem for $(\mathbb{N}, \cdot)$, the simultaneous Finite Sums and Products Theorem for $(\mathbb{N}, +, \cdot)$ (Theorem 1.4.1 here), their multidimensional generalizations (variants of Theorem 1.3.6).

Problem 3.6.5. Study restricted Zariski topologies given by a set of terms, e.g. the set of terms of degrees $\leq n$. Characterize algebras having such topologies non-discrete for a given $n$.

For the case of groups, such topologies were studied in [34].

Problem 3.6.6. Can some of results on the Zariski topology of polyrings in Section 2 be reproved for Abelian $\Omega$-groups?

Problem 3.6.7. With regard to Lemma 2.2.3, what is an interplay between properties of:

(i) a given Abelian group and the most expanded polycring over it?
(ii) a given universal algebra and the polycring of formal sums generated by it?

Problem 3.6.8. Generalize (non-commutative) ring theory to polyrings.
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