PRIMITIVE EQUATIONS WITH HALF HORIZONTAL VISCOSITY

MARTIN SAAL

Abstract. We consider the 3D primitive equations and show, that one does need less than horizontal viscosity to obtain a well-posedness result in Sobolev spaces. Furthermore, we will also investigate the primitive equations with horizontal viscosity and show that these equations are well-posed without imposing any boundary condition for the horizontal velocity components on the vertical boundary.

1. Introduction

The primitive equations are one of the fundamental models for geophysical flows and they are used to describe oceanic and atmospheric dynamics. They are derived from the Navier-Stokes equations in domains where the vertical scale is much smaller than the horizontal scale by performing the formal small aspect ratio limit. They describe the velocity $u$ of a fluid and the pressure $p$. Putting $u = (v, w)$, where $v = (v_1, v_2)$ denotes the horizontal components and $w$ stands for the vertical one, the equations read with full viscosity

\[
\begin{align*}
\partial_t v + v \cdot \nabla_H v + w \partial_z v - \nu_1 \Delta_H v - \nu_2 \partial_{zz} v + \nabla_H p &= 0, & \text{in } \Omega \times (0, T), \\
\partial_z p &= 0, & \text{in } \Omega \times (0, T), \\
\text{div}_H v + \partial_z w &= 0, & \text{in } \Omega \times (0, T), \\
v(t = 0) &= v_0, & \text{in } \Omega.
\end{align*}
\]

Here $\Omega := G \times (-h, h) \subset \mathbb{R}^3$ for $h > 0$, $G \subset \mathbb{R}^2$; $\nu_1$ stands for the horizontal viscosity and $\nu_2$ for the vertical one; $\nabla_H$, $\text{div}_H$ and $\Delta_H$ denote the horizontal gradient, divergence and Laplacian, respectively and $v \cdot \nabla_H = v_1 \partial_x + v_2 \partial_y$. Throughout this work we take $G = (-1, 1)^2$. A rigorous justification of the small aspect ratio limit of the Navier-Stokes equations to the primitive equations is given in [11]. For simplicity we formulated the equations without the Coriolis force, but being a zero order term it does not change the qualitative results obtained by us.

Note, that the vertical velocity $w$ is determined by the divergence free condition and boundary conditions for $w$ on the bottom of the domain, so it has less regularity than $v$ making the nonlinear term $w \partial_z v$ stronger compared to the nonlinearity of the Navier-Stokes equation.

The mathematical analysis of the primitive equations has been started by Lions, Temam and Wang [12,14], and in difference to the 3D Navier-Stokes equations the primitive equations are known to be time-global well-posed for initial data in $H^1(\Omega)$. This break through result has been proven by Cao and Titi [3] and launched a lot of

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activity in the analysis of those equations. For more information on previous results on the primitive equations we refer to the works of Washington and Parkinson [22], Pedlosky [17], Majda [15] and Vallis [21]; see also the survey by Li and Titi [10] for recent results and further references.

In the case $\nu_1 = \nu_2 = 0$, i.e. when there is no viscosity at all, one obtains the 3D primitive Euler equations, which are also called the hydrostatic Euler equations. The only existence result for this system is due to Kukavica, Temam, Vicol and Ziane [8]. They show that real-analytic data leads to a real-analytic local in time solution. For the 2D hydrostatic Euler equations Han-Kwan and Nguyen have shown in [7], that this set of equations is ill-posed in the Sobolev space setting, in the sense that - without any additional condition on the data - the solution map cannot be Hölder continuous. Such an additional condition was first used by Brenier [1] and later by Masmoudi and Wong [16] to show the local in time well-posedness of the 2D hydrostatic Euler (see also [9] for a generalization). They assume, that the initial velocity has a convex profile in the vertical direction which means, that one has a Rayleigh condition of the form $\partial_{zz} v \neq 0$ in $\Omega$. Regarding the question of global well-posedness, Cao et al [2] and Wong [23] have shown, that smooth solutions to the primitive Euler equations blow-up in finite time. In contrast to this situation, for the system with horizontal viscosity, i.e $\nu_1 > 0, \nu_2 = 0$ Cao, Li and Titi [4] prove not only local but even global well-posedness for initial data in $H^2(\Omega)$.

So the question is natural, how much anisotropic viscosity is needed to obtain a local well-posedness result in Sobolev spaces. We will show, that one can take out at least "half" the horizontal viscosity by considering the following system.

$$\begin{align*}
\partial_t v + v \cdot \nabla_H v + w \partial_z v - A_\perp v + \nabla_H p = 0 & \quad \text{in} \ (0, T) \times \Omega, \\
\partial_z p = 0 & \quad \text{in} \ (0, T) \times \Omega, \\
\operatorname{div} H v + \partial_z w = 0 & \quad \text{in} \ (0, T) \times \Omega, \\
v(t = 0) = v_0 & \quad \text{in} \ \Omega
\end{align*}$$

(1.1)

with periodic boundary conditions imposed on $v$ and $p$ in the horizontal directions, $w(z = \pm h) = 0$ and

$$A_\perp = \begin{pmatrix}
\partial_{yy} & 0 \\
0 & \partial_{xx}
\end{pmatrix}.$$

Note, that we do not have any boundary condition for $v$ on the vertical boundary. The main difficulty arises due to the nonlinear term $w \partial_z v$, because of the lack of regularity of $w$. In [16] Masmoudi and Wong make use of a special cancellation property related to this term when considering the 2D hydrostatic Euler equations to deduce bounds for $v_z$ under the condition $\partial_{zz} v \neq 0$. While in the 2D case this is sufficient to control also $v$, in the 3D setting this is not immediately possible. To work around that problem we combine their method with the approach Cao and Titi used in [3] to split the function $v$ into two parts, the vertical average $\overline{v}$ (also called the baroclinic mode) and an average free remainder $\tilde{v}$ (the barotropic modes). Then $\overline{v}$ is the solution to a 2D equation containing the pressure and $\tilde{v}$ can be controlled by $v_z = \tilde{v}_z$. Our result than reads as follows.

**Theorem 1.1.** Let $s \geq 3$. Then for any horizontally periodic $v_0 = (v_{01}, v_{02}) \in H^s(\Omega)$ with $\partial_z v_0 \in H^s(\Omega)$, $\int_{-h}^h \operatorname{div}_H v_0(x, y, \xi) \, d\xi = 0$ and $\partial_{zz} v_{0i} \neq 0$ in $\Omega$ for
there exists a time $T > 0$ and a unique strong solution $v$ to (1.1) with
\begin{align*}
v &\in L^\infty((0, T), H^s(\Omega)) \cap C^0([0, T], H^{s-\kappa}(\Omega)), \\
\partial_x v_2, \partial_y v_1 &\in L^2((0, T), H^s(\Omega))
\end{align*}
for all $\kappa \in (0, 1)$, and $\partial_z v$ has the same regularity as $v$.

Furthermore, we show a second possibility to take out half the horizontal viscosity by replacing $A_\perp$ in (1.1) with $A_\parallel = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{yy} \end{pmatrix}$.

This case is easier than the previous one, because the operator $A_\parallel$ controls the horizontal divergence of $v$ and thus also the function $w$ directly. We will prove the following well-posedness result for which no Rayleigh condition is needed.

**Theorem 1.2.** Let $s \geq 3$. Then for any horizontally periodic $v_0 \in H^s(\Omega)$ with $\int^h_{-h} \operatorname{div} H v_0(x, y, \xi) \, d\xi = 0$ there exists a time $T > 0$ and a unique strong solution $v$ to (1.1) with $A_\perp$ replaced by $A_\parallel$ and we have
\begin{align*}
v &\in L^\infty((0, T), H^s(\Omega)) \cap C^0([0, T], H^{s-\kappa}(\Omega)), \\
\partial_x v_1, \partial_y v_2 &\in L^2((0, T), H^s(\Omega))
\end{align*}
for all $\kappa \in (0, 1)$.

We will need the well-posedness of the primitive equations with only horizontal viscosity in the proofs of the Theorems (1.1) and (1.2), but that system is also of interest on its own. Due to turbulent mixing in the horizontal plane even in the case of full viscosities, the horizontal viscosity $\nu_1$ is much stronger than the vertical $\nu_2$. In the limiting case this means, that we have no vertical viscosity and we will consider that system,
\begin{align}
\partial_t v + v \cdot \nabla_H v + w \partial_z v - \Delta_H v + \nabla_H p &= 0 \quad \text{in } (0, T) \times \Omega, \\
\partial_z p &= 0 \quad \text{in } (0, T) \times \Omega, \\
\operatorname{div} H v + \partial_z w &= 0 \quad \text{in } (0, T) \times \Omega, \\
v(t = 0) &= v_0 \quad \text{in } \Omega
\end{align}
(1.2)
with periodic boundary conditions imposed on $v$ and $p$ in the horizontal directions and $w(z = \pm h) = 0$. Here we set $\nu_1 = 1$ for simplicity. The first results for the primitive equations with only horizontal viscosity were obtained by Cao, Li and Titi [4] and further investigated by them in [5]. In [4] they show the global well-posedness for initial data in $H^2(\Omega)$ additionally assuming homogeneous Neumann boundary conditions for $v$ on the top $G \times \{h\}$ and the bottom $G \times \{-h\}$ of the domain. Their approach is to consider the case of full viscosity for which the global existence of solutions is known and to derive a priori bounds independent of the vertical viscosity $\nu_2$. For this boundary conditions there is no formation of a boundary layer and by letting $\nu_2$ tend to zero they obtain a solution to the primitive equations with only horizontal viscosity. However, not all solutions of (1.2) can be found by that method. We show, that the well-posedness result holds also without the Neumann conditions for $v$ on the top and on the bottom. Our approach is to interpret the term $w \partial_z v$ as a transport term with a non-constant coefficient $w$, which vanishes on the boundary. So there is no transport through
the boundary and thus we do not need any condition on \( v \) there and we obtain the following result.

**Theorem 1.3.** Let \( s \geq 2 \). Then for any horizontally periodic \( v_0 \in H^s(\Omega) \) with \( \int_{-h}^h \text{div} v_0(x, y, \xi) \, d\xi = 0 \) there exists a time \( T > 0 \) and a unique strong solution \( v \) to (1.2) with

\[
v \in L^\infty((0, T), H^s(\Omega)) \cap C^0([0, T], H^{s - \kappa}(\Omega))
\]

\[
\partial_x v, \partial_y v \in L^2((0, T), H^s(\Omega))
\]

for all \( \kappa \in (0, 1) \). For \( s = 2 \) this solution extends globally in time.

We will prove the local existence result in detail, the global existence then follows from the estimates proven in [4] for initial data in \( H^2(\Omega) \). Although no boundary conditions for \( v \) on the top and the bottom are not needed for the well-posedness, homogeneous Neumann boundary conditions are preserved in time by the equations if they hold for the initial value.

This paper is organized as follows. In section 2 we list the most important definitions and notations and we reformulate the equation (1.2) for \( v \) into a system of equations for the mean value \( \bar{v} \) and the remainder \( \tilde{v} \). In section 3 we show the well-posedness of a linearized version of the primitive equations with horizontal viscosity by a Galerkin-approach, and in section 4 we use that result to prove Theorem (1.3). In section 5 we turn to the case of half horizontal viscosity and give the proofs for the Theorems (1.1) and (1.2).

## 2. Notations and basic Lemmas

By \( L^2(\Omega), L^2(G) \) we denote the standard Lebesgue spaces with the scalar products

\[
\langle f, g \rangle_\Omega := \int_\Omega g(x, y, z) h(x, y, z) \, d(x, y, z)
\]

and \( \langle f, g \rangle_G \) defined analogously, by \( \| f \|_{L^2(\Omega)} \) and \( \| f \|_{L^2(G)} \) we denote the induced norm. We drop \( \Omega \) and \( G \) in the notation if the dependence is obvious.

For a function \( f \in L^\infty((0, T), L^\infty(\Omega)) \) we use the abbreviation

\[
\| f \|_\infty := \sup_{t \in (0, T)} \| f(t) \|_{L^\infty}.
\]

We write \( v(t = 0) \) for the function \( v|_{t=0} \) and \( v(z = \pm h) \) as a short form of \( v|_{z=\pm h} \).

For \( s \in \mathbb{N} \) the space \( H^s(\Omega) \) consists of \( f \in L^2(\Omega) \) such that \( \nabla^\alpha f \in L^2(\Omega) \) for \( |\alpha| \leq s \) endowed with the norm

\[
\| f \|_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \| \nabla^\alpha f \|_{L^2(\Omega)}.
\]

Here we used the multi-index notation \( \nabla^\alpha = \partial_{x}^\alpha \partial_{y}^\beta \partial_{z}^\gamma \) for \( \alpha \in \mathbb{N}_0^3 \). The spaces \( H^s(G) \) are defined analogously, and we will again just write \( \| f \|_{H^s} \) if there is no ambiguity. If \( s \notin \mathbb{N} \) the spaces \( H^s(\Omega), H^s(G) \) are defined by complex interpolation, see [20] for details.

To handle the periodic boundary conditions in the horizontal variables we set

\[
H^s_{\text{per}}(\Omega) := \{ f \in H^s(\Omega) | f \text{ is periodic of order } s - 1 \text{ on } \partial G \times (-h, h) \}.
\]
and

\[ C^\infty_{\text{per}}(\Omega) := \{ f \in C^\infty(\Omega) \mid f \text{ is periodic of arbitrary order on } \partial G \times (-h, h) \}. \]

It is easy to see that \( H^s_{\text{per}}(\Omega) \) equipped with the \( H^s \)-norm is a Banach space and that \( C^\infty_{\text{per}}(\Omega) \) is a dense subset.

When investigating the case of half horizontal viscosity we need furthermore a subset of \( H^s(\Omega) \), which reflects the Rayleigh condition mentioned in the introduction. For \( s \geq 3 \) and \( \eta > 1 \) we define

\[ H^s_{\text{per}, \eta}(\Omega) := \left\{ f \in H^s_{\text{per}}(\Omega) \mid \frac{1}{\eta} \leq \frac{1}{|\partial_z f(x, y, z)|} \right\} \]

with

\[ \|f\|^2_{H^s_{\text{per}, \eta}} = \|f\|^2_{H^{s-1}} + \|\partial_z^s f\|^2_{L^2} + \sum_{|\alpha| = s, \alpha_3 = 0} \left\| \frac{\nabla^\alpha f}{\sqrt{|\partial_z f|}} \right\|^2_{L^2}. \]

For \( f \in H^s_{\text{per}, \eta}(\Omega) \) this expression is equivalent to the \( H^s(\Omega) \)-norm.

The following inequalities will be helpful when we show a-priori estimates for the solutions to the primitive equations.

**Lemma 2.1.**

a) Let \( f, h, \nabla_h h \in L^2(\Omega) \) and \( g \in H^1(\Omega) \), then

\[ |(f g, h)| \leq c \|f\|_{L^2(\Omega)} \|\nabla_h h\|_{L^2(\Omega)}^{1/2} \|h\|_{L^2(\Omega)}^{1/2} \|g\|_{H^1(\Omega)}. \]

b) Let \( f \in H^2(\Omega) \), then

\[ \|f\|_{L^\infty(\Omega)} \leq c \|f\|_{H^2(\Omega)}^{3/4} \|f\|_{L^2(\Omega)}^{1/4}. \]

**Proof.**

a) Consider

\[ |(f g, h)| \leq \int_{-h}^h \|f(z) g(z) h(z)\|_{L^2(G)} \, dz \]
\[ \leq \|g\|_{L^\infty((-h,h), L^4(G))} \int_{-h}^h \|f(z)\|_{L^2(G)} \|h(z)\|_{L^4(G)} \, dz \]
\[ \leq \|g\|_{L^\infty((-h,h), L^4(G))} \|f\|_{L^2(\Omega)} \|h\|_{L^2((-h,h), L^4(G))}. \]

Ladyzhenskaya’s inequality \( \|h(z)\|^2_{L^4(G)} \leq c \|\nabla_h h(z)\|_{L^2(G)} \|h(z)\|_{L^2(G)} \) gives

\[ \|h\|^2_{L^2((-h,h), L^4(G))} \leq \int_{-h}^h \|\nabla_h h(z)\|_{L^2(G)} \|h(z)\|_{L^2(G)} \, dz \leq c \|\nabla_h h\|_{L^2(\Omega)} \|h\|_{L^2(\Omega)}. \]

In \([4, \text{Lemma 2.3}]\) it has been shown that

\[ \|g\|_{L^\infty((-h,h), L^4(G))} \leq c \left( \|g\|_{L^2(\Omega)} + \|\partial_z g\|_{L^2(\Omega)} \right)^{1/2} \left( \|g\|_{L^2(\Omega)} + \|\nabla_h g\|_{L^2(\Omega)} \right)^{1/2} \]

and thus

\[ |(f g, h)| \leq c \|g\|_{H^1(\Omega)} \|f\|_{L^2(\Omega)} \|\nabla_h h\|_{L^2(\Omega)}^{1/2} \|h\|_{L^2(\Omega)}. \]

b) This is the well-known Agmon’s inequality.

The following Aubin-Lions Lemma is shown in \([18, \text{Corollary 4}]\).

**Lemma 2.2.** Let \( T > 0 \) and \( X, Y \) and \( Z \) be Banach spaces such that \( X \) is compactly embedded in \( Y \), and \( Y \) is continuously embedded in \( Z \).
(i) If \((f_n)_n \subset L^2((0, T), X)\) is bounded and \((\partial_t f_n)_n\) is bounded in \(L^2((0, T), Z)\) then there exists an in \(L^2((0, T), Y)\) convergent subsequence.

(ii) If \((f_n)_n \subset L^\infty((0, T), X)\) is bounded and \((\partial_t f_n)_n\) is bounded in \(L^2((0, T), Z)\) then there exists an in \(C^0([0, T], Y)\) convergent subsequence.

Now let us reformulate the primitive equations (1.2) where we replace the operator \(\Delta H\) by any constant coefficient operator \(A\) acting only in the horizontal directions. It is easy to see that the divergence free condition \(\partial_z w + \text{div}_H v = 0\) and the boundary condition \(w(z = \pm h) = 0\) are equivalent to

\[
w(t, x, y, z) = -\text{div}_H \int_{-h}^h v(t, x, y, \xi) \, d\xi \quad \text{and} \quad \text{div}_H \int_{-h}^h v(t, x, y, \xi) \, d\xi = 0.
\]

This means, that the mean value of \(v\) in the vertical direction

\[
\overline{v}(t, x, y) := \frac{1}{2h} \int_{-h}^h v(t, x, y, \xi) \, d\xi
\]

is divergence free. The pressure is constant in the vertical direction, and thus \(\overline{p} = p\). For \(\overline{v}, \overline{p}\) and the remainder

\[
\tilde{v} = v - \overline{v}
\]

we obtain the system of coupled equations

\[
\begin{align*}
\partial_t \overline{v} + \overline{v} \cdot \nabla_H \overline{v} - A\overline{v} + \nabla_H p &= -K(\tilde{v}), \\
\text{div}_H \overline{v} &= 0, \\
\overline{v}(t = 0) &= \overline{v}_0
\end{align*}
\]

(2.1)

and

\[
\begin{align*}
\partial_t \tilde{v} + \tilde{v} \cdot \nabla_H \tilde{v} + \overline{v} \cdot \nabla_H \tilde{v} + \tilde{v} \cdot \nabla_H \overline{v} + w\partial_z \tilde{v} - A\tilde{v} &= K(\tilde{v}), \\
w(t, x, y, z) &= -\text{div}_H \int_{-h}^h \tilde{v}(t, x, y, \xi) \, d\xi, \\
\tilde{v}(t = 0) &= \tilde{v}_0
\end{align*}
\]

(2.2)

both with periodic boundary conditions in the horizontal directions and where the coupling term \(K(\tilde{v})\) is given by

\[
K(\tilde{v})(t, x, y) = \frac{1}{2h} \int_{-h}^h \tilde{v}(t, x, y, \xi) \cdot \nabla_H \tilde{v}(t, x, y, \xi) + \tilde{v}(t, x, y, \xi) \, \text{div}_H \tilde{v}(t, x, y, \xi) \, d\xi.
\]

(2.3)

Therefore, the well-posedness of the primitive equations (1.2) with periodic boundary conditions in the horizontal directions and \(w(z = \pm h) = 0\) is equivalent to the well-posedness of (2.1)-(2.3) with periodic boundary conditions in the horizontal directions, and the same holds for the cases \(A = A_\perp\) and \(A = A_{||}\).

3. A linearized equation

For given functions \(w, a, b\) and \(f\) and initial data \(v_0\) we consider the equation

\[
\partial_t v + av + b \cdot \nabla_H v + w\partial_z v - \Delta_H v = f \quad \text{in} \quad (0, T) \times \Omega,
\]

\[
v(t = 0) = v_0 \quad \text{in} \quad \Omega
\]

(3.1)
with periodic boundary conditions in the horizontal directions and show the existence of a solution by a Galerkin-approach. Note that if \( w(z = \pm h) = 0 \) we do not need any boundary condition for \( v \) in the vertical direction.

This is a linearized version of (2.2) for \( A = \Delta_{H} \), and based on its well-posedness we will show the local existence of solutions to the primitive equations with horizontal viscosity.

We work in the spaces
\[
H := \{ v \in L^{2}(\Omega) \mid \partial_{z}v \in L^{2}(\Omega) \} = H^{1}( (-h, h), L^{2}(G))
\]
equipped with the scalar product \( \langle u, v \rangle_{H} := \langle u, v \rangle_{\Omega} + \langle \partial_{z}u, \partial_{z}v \rangle_{\Omega} \) and
\[
V := \{ v \in H \mid \nabla_{H}v \in H \} \cap H_{per}^{1}(\Omega) = H^{1}( (-h, h), H^{1}(G)) \cap H_{per}^{1}(\Omega)
\]
with the scalar product \( \langle u, v \rangle_{V} := \langle u, v \rangle_{H} + \langle \nabla_{H}u, \nabla_{H}v \rangle_{H} \). By \( V' \) we denote the dual space of \( V \) with respect to the norm in \( H \), i.e.,
\[
V' = H^{1}( (-h, h), H^{-1}(G)).
\]
For \( f, g \in L^{2}((0, T), L^{2}(\Omega)) \) we denote by
\[
\langle g, h \rangle_{T} := \int_{0}^{T} \int_{\Omega} g(t, x, y, z)h(t, x, y, z) \, dx \, dy \, dz \, dt
\]
the scalar product in space and time. We will also denote the dual pairings \( L^{2}((0, T), V') \times L^{2}((0, T), V) \) and \( V \times V' \) by \( \langle \cdot, \cdot \rangle_{T} \) and \( \langle \cdot, \cdot \rangle_{H} \) to keep the notation simple.

The solution we obtain in the first step will be a weak solution, where weak means “weak with respect to \( x, y \)”, i.e., that for all \( \varphi \in C_{c}^{\infty}( (0, T), L^{2}( (-h, h), H_{per}(G)) ) \)
\[
- \langle v, \partial_{t}\varphi \rangle_{T} + \langle av, \varphi \rangle_{T} + \langle b \cdot \nabla_{H}v, \varphi \rangle_{T} + \langle w\partial_{z}v, \varphi \rangle_{T} + \langle \nabla_{H}v, \nabla_{H}\varphi \rangle_{T} = \langle f, \varphi \rangle_{T}.
\]
With this notion of solution we then have the following existence result.

**Theorem 3.1.** Let \( v_{0} \in H, w, w_{z}, a, a_{z}, b = (b_{1}, b_{2}), b_{z} \in L^{\infty}( (0, T) \times \Omega) \) with \( w(z = \pm h) = 0 \) and \( f \in L^{2}( (0, T), V') \). Then there is a unique weak solution
\[
v \in L^{2}( (0, T), V) \cap H^{1}( (0, T), V') \cap C^{0}( [0, T], H)
\]
to (3.1) with
\[
\|v\|_{L^{2}( (0, T), H)}^{2} + \|v\|_{L^{2}( (0, T), V)}^{2} + \|v_{t}\|_{L^{2}( (0, T), V')}^{2}
\leq \left( \|v_{0}\|_{H}^{2} + 2\|f\|_{L^{2}( (0, T), V')}^{2} \right) e^{\left( \frac{1}{2}\|w_{z}\|_{\infty} + 1\|a, a_{z}\|_{\infty} + 2\|b, b_{z}\|_{\infty} \right) T}.
\]

_Proof._ Uniqueness: Let \( v_{1}, v_{2} \) be weak solutions to the same initial data. Then the difference \( v := v_{1} - v_{2} \) solves
\[
\partial_{t}v + av + b \cdot \nabla_{H}v + w\partial_{z}v - \Delta_{H}v = 0,
\]
\[
v(t = 0) = 0.
\]
v is regular enough to test this equation with itself in \( L^{2}(\Omega) \), which yields
\[
\frac{1}{2}\partial_{t}\|v\|_{L^{2}}^{2} + \|\nabla_{H}v\|_{L^{2}}^{2} = -\langle av, v \rangle_{\Omega} - \langle b \cdot \nabla_{H}v, v \rangle_{\Omega} + \frac{1}{2}\langle w_{z} \cdot v, v \rangle_{\Omega}
\leq \|a\|_{\infty}\|v\|_{L^{2}}^{2} + \|b\|_{\infty}\|\nabla_{H}v\|_{L^{2}}\|v\|_{L^{2}} + \frac{1}{2}\|w_{z}\|_{\infty}\|v\|_{L^{2}}^{2}
\leq \left( \|a\|_{\infty} + \frac{1}{2}\|b\|_{\infty}^{2} + \frac{1}{2}\|w_{z}\|_{\infty} \right) \|v\|_{L^{2}}^{2} + \frac{1}{2}\|\nabla_{H}v\|_{L^{2}}^{2}.
\]
Gronwall’s Lemma now implies \( v = 0 \).

Existence: Let \((\Phi_n)_n \subset V\) be orthonormal in \( H\) with \( \partial_z \Phi_n \in L^2(\Omega)\) and \( \text{span}\{\Phi_n|n \in \mathbb{N}\}\) dense in \( H\). We set \( V_n := \text{span}\{\Phi_j|1 \leq j \leq n\}\) and denote by \( P_n : H \to V_n\) the orthogonal projection onto it.

We project the equation onto the finite dimensional subspace \( V_n\) and we are looking for a solution \( v_n : [0, T] \to V_n\) of the system of ordinary differential equations

\[
(3.2) \quad \langle \partial_t v_n, \Phi_i \rangle_H + \langle b \cdot \nabla_H v_n, \Phi_i \rangle_H + \langle av_n, \Phi_i \rangle_H + \langle w \partial_z v_n, \Phi_i \rangle_H - \langle \Delta_H v_n, \Phi_i \rangle_H = \langle f, \Phi_i \rangle_H, \quad (1 \leq i \leq n)
\]

\( v_n(0) = P_n v_0 \).

We have \( v_n(t) = \sum_{j=1}^{n} g_{nj}(t) \Phi_j \) for some \( g_{nj} : [0, T] \to \mathbb{R} \) and this yields

\[
\sum_{j=1}^{n} \frac{d}{dt} g_{nj}(t) \langle \Phi_j, \Phi_i \rangle_H + g_{nj}(t) \langle b \cdot \nabla_H \Phi_j, \Phi_i \rangle_H + g_{nj}(t) \langle a \Phi_j, \Phi_i \rangle_H
\]

\[
+ g_{nj}(t) \langle w \partial_z \Phi_j, \Phi_i \rangle_H - g_{nj}(t) \langle \Delta_H \Phi_j, \Phi_i \rangle_H = \langle f, \Phi_i \rangle_H,
\]

\[
\sum_{j=1}^{n} g_{nj}(0) \Phi_j = P_n v_0
\]

for \( 1 \leq i \leq n\). Denoting

\[
g_n(t) := (g_{1n}(t), \ldots, g_{nn}(t)), \quad f_n(t) := (\langle f, \Phi_1 \rangle_H, \ldots, \langle f, \Phi_n \rangle_H),
\]

\[
a_n(t) := (\langle a(t) \Phi_j, \Phi_i \rangle_H)_{1 \leq i, j \leq n}, \quad b_n(t) := (\langle (b(t) \cdot \nabla_H) \Phi_j, \Phi_i \rangle_H)_{1 \leq i, j \leq n},
\]

\[
w_n(t) := (\langle w(t) \partial_z \Phi_j, \Phi_i \rangle_H)_{1 \leq i, j \leq n}, \quad D_n := (\langle \nabla_H \Phi_j, \nabla_H \Phi_i \rangle_H)_{1 \leq i, j \leq n}
\]

this system has the form

\[
\frac{d}{dt} g_n(t) + [a_n(t) + b_n(t) + w_n(t)] g_n(t) + D_n g_n(t) = f_n(t),
\]

\[
\sum_{j=1}^{n} g_{nj}(0) \Phi_j = P_n v_0.
\]

By standard theory for ordinary differential equations there is a solution \( g_n \in L^2((0, T), \mathbb{R})\). Multiplying \( \langle \Phi_n \rangle \) by \( g_{ni} \) and summing over \( i \) yields

\[
\langle \partial_t v_n, v_n \rangle_H + \langle av_n, v_n \rangle_H + \langle b \cdot \nabla_H v_n, v_n \rangle_H + \langle w \partial_z v_n, v_n \rangle_H - \langle \Delta_H v_n, v_n \rangle_H = \langle f, v_n \rangle_H.
\]

From the periodic boundary conditions we obtain

\[
-\langle \Delta_H v_n, v_n \rangle_H = \|\partial_z v_n\|_H^2 + \|\partial_y v_n\|_H^2,
\]

and \( w(z = \pm h) = 0 \) gives

\[
\|w \partial_z v_n, v_n\|_H = \left| \frac{1}{2} \langle w_z \cdot v_n, v_n \rangle_\Omega + \frac{1}{2} \langle w_z \cdot \partial_z v_n, \partial_z v_n \rangle_\Omega \right| \leq \frac{1}{2} \|w_z\|_\infty \|v_n\|_H^2.
\]
With
\[ |\langle b \cdot \nabla_H v_n, v_n \rangle_H | = |\langle b \cdot \nabla_H v_n, v_n \rangle_{\Omega} + \langle b \cdot \nabla_H \partial_z v_n + (b_z \cdot \nabla_H )v_n, \partial_{z} v_n \rangle_{\Omega}| \]
\[ \leq \|b\|_\infty \|\nabla_H v_n\|_{L^2} \|v_n\|_{L^2} + \|b\|_\infty \cdot \|\nabla_H \partial_z v_n\|_{L^2} \|\partial_z v_n\|_{L^2} \]
\[ + \|b_z\|_\infty \|\nabla_H v_n\|_{L^2} \|\partial_z v_n\|_{L^2} \]
\[ \leq (\|b\|_\infty^2 + \|b_z\|_\infty^2) \|v_n\|_{H}^2 + \frac{1}{2} \|\nabla_H v_n\|_{H}^2 \]
and
\[ |\langle a v_n, v_n \rangle_H | \leq \|a\|_\infty \cdot \|v_n\|_{L^2}^2 + \|a\|_\infty \cdot \|\partial_z v_n\|_{L^2}^2 + \|a_z\|_\infty \cdot \|v_n\|_{L^2} \|\partial_z v_n\|_{L^2} \]
\[ = \|a\|_\infty \cdot \|v_n\|_{H}^2 + \frac{1}{2} \|a_z\|_\infty \|v_n\|_{H}^2 \]
we get
\[ \frac{d}{dt} \|v_n\|_{H}^2 + 2 \|\nabla_H v_n\|_{H}^2 \]
\[ = (2\|a\|_\infty + \|a_z\|_\infty) \cdot \|v_n\|_{H}^2 + 2(\|b\|_\infty^2 + \|b_z\|_\infty^2) \cdot \|v_n\|_{H}^2 \]
\[ + \|w_z\|_\infty \cdot \|v_n\|_{H}^2 + \|\nabla_H v_n\|_{H}^2 + \frac{1}{2} \|v_n\|_{H}^2 + \frac{1}{2} \|\nabla_H v_n\|_{H}^2 + 2 \|f\|_{V}^2, \]
and thus
\[ \frac{d}{dt} \|v_n\|_{H}^2 + \frac{1}{2} \|\nabla_H v_n\|_{H}^2 \]
\[ \leq \left( \frac{1}{2} + \|w_z\|_\infty + 2\|a, a_z\|_\infty + 2\|b, b_z\|_\infty^2 \right) \cdot \|v_n\|_{H}^2 + 2 \|f\|_{V}^2. \]
Integration in time
\[ \|v_n(t)\|_{H}^2 + \int_0^t \|\nabla_H v_n(r)\|_{H}^2 \, dr \leq \|v_0\|_{H}^2 + \int_0^t \|f(r)\|_{V}^2, \, dr \]
\[ + \left( \frac{1}{2} + \|w_z\|_\infty + 2\|a, a_z\|_\infty + 2\|b, b_z\|_\infty^2 \right) \cdot \int_0^t \|v_n(r)\|_{H}^2 \, dr \]
and Gronwall’s inequality give
\[ \|v_n(t)\|_{H}^2 + \int_0^t \|\nabla_H v_n(r)\|_{H}^2 \, dr \]
\[ \leq \left( \|v_0\|_{H}^2 + 2 \int_0^t \|f(r)\|_{V}^2, \, dr \right) e^\left( \frac{1}{2} + \|w_z\|_\infty + 2\|a, a_z\|_\infty + 2\|b, b_z\|_\infty^2 \right) T. \]
This estimate yields the boundedness of the sequence \( (v_n)_n \) in \( C^0([0, T], H) \cap L^2((0, T), V) \). Analogously the strong convergence in \( C^0([0, T], H) \cap L^2((0, T), V) \) follows by performing the above estimates for \( v_m - v_n \).

Let now \( \varphi \in C_0^\infty([0, T], V) \) be of the form \( \varphi(t) = \sum_{i=1}^{k} h_i(t) \Phi_i \) with \( k \in \mathbb{N} \) and \( h_i \in C_0^\infty((0, T], \mathbb{R}) \) \( (1 \leq i \leq k) \). From [3,2] we get with
\[ \langle w \partial_z v_n, \varphi \rangle_{H} = \langle w \partial_z v_n, \varphi \rangle_{\Omega} - \langle w \partial_z v_n, \partial_z \varphi \rangle_{\Omega} = \langle w \partial_z v_n, (1 - \partial_z \varphi) \rangle_{\Omega} \]
that
\[- \int_0^T \langle v_n, \partial_t \varphi \rangle_H + \langle av_n, \varphi \rangle_H + (b \cdot \nabla_H v_n, \varphi)_H \]
\[+ \langle w \partial_z v_n, (1 - \partial_{zz}) \varphi \rangle_{\Omega} + \langle \nabla_H v_n, \nabla_H \varphi \rangle_H \, dt = \int_0^T \langle f, \varphi \rangle_H \, dt \]
and passing to the limit gives
\[- \int_0^T \langle v, \partial_t \varphi \rangle_H + \langle av, \varphi \rangle_H + (b \cdot \nabla_H v, \varphi)_H \]
\[+ \langle w \partial_z v, (1 - \partial_{zz}) \varphi \rangle_{\Omega} + \langle \nabla_H v, \nabla_H \varphi \rangle_H \, dt = \int_0^T \langle f, \varphi \rangle_H \, dt . \]
To show that \( v \) is a weak solution we need to have this equality with scalar products in \( L^2(\Omega) \) instead of \( H \), and without the term \( (1 - \partial_{zz}) \varphi \). By density arguments the above equality holds for all \( \varphi \in C_c^\infty([0,T], V) \) with \( \partial_{zz} \varphi \in C_c^\infty([0,T], L^2(\Omega)) \), and taking such a \( \varphi \) with \( \partial_z \varphi \in C_c^\infty([0,T], V) \) and \( \partial_z \varphi(z = \pm h) = 0 \) we obtain
\[- \langle v, \partial_t (1 - \partial_{zz}) \varphi \rangle_T + \langle av, (1 - \partial_{zz}) \varphi \rangle_T + (b \cdot \nabla_H v, (1 - \partial_{zz}) \varphi)_{\Omega} \]
\[+ \langle w \partial_z v, (1 - \partial_{zz}) \varphi \rangle_T + \langle \nabla_H v, \nabla_H (1 - \partial_{zz}) \varphi \rangle_T = \langle f, (1 - \partial_{zz}) \varphi \rangle_T. \]
The set
\[\{(1 - \partial_{zz}) \varphi \mid \varphi, \partial_z \varphi \in C_c^\infty([0,T], V), \partial_z \varphi(z = \pm h) = 0\}\]
is dense in \( C_c^\infty((0,T), L^2((-h,h), H_{per}^1(G))) \), and so we have
\[\langle v, \partial_t \varphi \rangle_T + \langle av, \varphi \rangle_T + (b \cdot \nabla_H v, \varphi)_{\Omega} + \langle w \partial_z v, \varphi \rangle_T + \langle \nabla_H v, \nabla_H \varphi \rangle_T = \langle f, \varphi \rangle_T\]
for all \( \varphi \in C_c^\infty((0,T), L^2((-h,h), H_{per}^1(G))) \).
Thus, \( v \in H^1((0,T), V') \cap C^0([0,T], H) \cap L^2((0,T), V) \) is a weak solution. \( \square \)

A direct consequence is the following result on \( C^\infty \)-data.

**Corollary 3.2.** Let \( a, b, w, f \in C^\infty([0,T], C^\infty_{per}(\Omega)) \) with \( w(z = \pm h) = 0 \) and \( v_0 \in C^\infty_{per}(\Omega) \). Then we have for the solution \( v \) to (3.1) obtained in Theorem 3.1 that
\[v \in C^\infty([0,T], C^\infty_{per}(\Omega)).\]

4. **Primitive equations with horizontal viscosity**

In this section we show that the equation (1.2) is well-posed. First we prove the local in time well-posedness part of Theorem 1.3.

**Theorem 4.1.** Let \( s \geq 2 \). Then for any \( v_0 \in H^s_{per}(\Omega) \) with \( \text{div}_H v_0 = 0 \) there exists a time \( T > 0 \) and a unique strong solution \( v \) to (1.2) with
\[v \in L^\infty((0,T), H^s_{per}(\Omega)) \cap C^0([0,T], H^{s-\kappa}_{per}(\Omega)) \]
\[\partial_z v, \partial_y v \in L^2((0,T), H^s_{per}(\Omega)) \]
for all \( \kappa \in (0,1) \).
Proof. Let \((v_{0,n})_n \subset C^\infty_v(\Omega)\) with \(v_{0,n} \rightarrow v_0\) in \(H^s(\Omega)\) and \(\|v_n\|_{H^s} \leq \|v\|_{H^s}\).

For \(n \in \mathbb{N}\) and \(\tilde{v}_{n-1}, w_{n-1} \in C^\infty([0,T], C^\infty_v)\) given let \(\overline{v}_n = \overline{v}_n(t, x, y)\) and \(p_n = p_n(t, x, y)\) be the solution to
\[
\begin{aligned}
\partial_t \overline{v}_n + \overline{v}_{n-1} \cdot \nabla_H \overline{v}_n - \Delta_H \overline{v}_n + \nabla_H p_n &= -K(\tilde{v}_{n-1}), \\
\text{div}_H \overline{v}_n &= 0, \\
\overline{v}_n(t = 0) &= \frac{1}{2h} \int_{-h}^h v_{0,n}(0, x, y, \xi) \, d\xi,
\end{aligned}
\]
where \(K\) is defined as in (2.3), and \(\tilde{v}_n = \tilde{v}_n(t, x, y, z)\) be the solution to
\[
\begin{aligned}
\partial_t \tilde{v}_n + (\tilde{v}_{n-1} + \overline{v}_{n-1}) \cdot \nabla_H \tilde{v}_n + \tilde{v}_{n-1} \cdot \nabla_H \overline{v}_n + w_{n-1} \partial_z \tilde{v}_n - \Delta_H \tilde{v}_n &= K(\tilde{v}_{n-1}), \\
\tilde{v}_n(t = 0) &= v_{0,n} - \overline{v}_{n,n},
\end{aligned}
\]
both equations are complemented by periodic boundary conditions in the horizontal directions. We define
\[
w_n(t, x, y, z) = -\int_{-h}^h \text{div}_H \tilde{v}_n(t, x, y, \xi) \, d\xi + \frac{z + h}{2h} \int_{-h}^h \text{div}_H \tilde{v}_n(t, x, y, \xi) \, d\xi.
\]
Starting with \(\overline{v}_0 = \tilde{v}_0 = w_0 = 0\) the sequence is well defined by Corollary 3.2 and known results for the 2D Stokes equation.

Note, that this set of equations looks similar to (2.1)-(2.3), but \(\tilde{v}_n\) is not average free in the vertical direction and therefore we need the correction term in the equation for \(w_n\) to guarantee that \(w_n(z = \pm h) = 0\). However, after passing to the limit the resulting function \(\tilde{v}\) will be average free and thus the correction term vanishes.

For \(\overline{v}_n\) we obtain the inequality
\[
\frac{1}{2} \frac{d}{dt} \|\overline{v}_n\|^2_{H^s} + \|\nabla_H \overline{v}_n\|^2_{H^s} 
\leq c \|\overline{v}_{n-1}\|_{H^s} \|\overline{v}_n\|_{H^s} \|\nabla_H \overline{v}_n\|_{H^s} + c \|\tilde{v}_{n-1}\|_{H^s} \|\nabla_H \tilde{v}_{n-1}\|_{H^s} \|\overline{v}_n\|_{H^s},
\]
where the last term is due to the coupling \(K(\tilde{v})\). By applying \(\text{div}_H\) to the equation for \(\overline{v}_n\) we get that
\[
-\Delta_H p_n = \text{div}_H (K(\tilde{v}_{n-1}) + \overline{v}_{n-1} \cdot \nabla_H \overline{v}_n),
\]
and so we have for the pressure
\[
(4.1) \quad \|\nabla_H p_n\|_{H^s} \leq c \|\tilde{v}_{n-1}\|_{H^s} \|\nabla_H \tilde{v}_{n-1}\|_{H^s} + c \|\overline{v}_{n-1}\|_{H^s} \|\nabla_H \overline{v}_n\|_{H^s}.
\]
Next we show an estimate for \(v_n := \overline{v}_n + \tilde{v}_n\). It fulfills the equation
\[
\begin{aligned}
\partial_t v_n + v_{n-1} \cdot \nabla_H v_n + w_{n-1} \partial_z v_n - \Delta_H v_n + \nabla_H p_n &= 0, \\
\partial_z p_n &= 0, \\
\text{div}_H v_n &= 0, \\
v_n(t = 0) &= v_{0,n}.
\end{aligned}
\]
Applying \(\nabla^\alpha\) and multiplying with \(\nabla^\alpha v_n\) we obtain
\[
\begin{aligned}
\frac{1}{2} \partial_t \|\nabla^\alpha v_n\|^2_{L^2} + \|\nabla_H \nabla^\alpha v_n\|^2_{L^2} 
&= -\langle \nabla^\alpha (v_{n-1} \cdot \nabla_H v_n + w_{n-1} \partial_z v_n + \nabla_H p_n), \nabla^\alpha v_n \rangle_{\Omega}.
\end{aligned}
\]
For the pressure term we only have \( \langle \nabla^\alpha \nabla_H p, \nabla^\alpha v_n \rangle_\Omega = 0 \) if \( \nabla^\alpha \) contains a derivative in the \( z \) direction, because of the correction term in the definition of \( w_{n-1} \), but with (E.1) and \( \bar{v}_n = v_n - \varpi_n \) it follows for \( \nabla^\alpha = (\partial_x, \partial_y)^\alpha \) that

\[
\left| \langle \nabla^\alpha \nabla_H p, \nabla^\alpha v_n \rangle_\Omega \right| \\
\leq c (\|v_{n-1}\|_{H^s} \|\nabla_H v_{n-1}\|_{H^s} + \|v_{n-1}\|_{H^s} \|\nabla_H \nabla_{\Omega} v_{n-1}\|_{H^s} + \|\nabla v_{n-1}\|_{H^s} \|\nabla v_{n-1}\|_{H^s} \\
+ \|\nabla v_{n-1}\|_{H^s} \|\nabla_H v_{n-1}\|_{H^s} + \|\nabla v_{n-1}\|_{H^s} \|\nabla_H v_{n-1}\|_{H^s} \|\nabla^\alpha v_n \|_{L^2})
\]

The first part of the nonlinearity can be estimated directly

\[
\left| \langle \nabla^\alpha (v_{n-1} \cdot \nabla_H v_n), \nabla^\alpha v_n \rangle_\Omega \right| \\
\leq c \|v_{n-1}\|_{H^s} \|\nabla_H v_n\|_{H^s} \|\nabla^\alpha v_n \|_{L^2}.
\]

The other one we write as

\[
\langle \nabla^\alpha (w_{n-1} \partial_z v_n), \nabla^\alpha v_n \rangle_\Omega = \langle \nabla^\alpha w_{n-1} \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega + \langle w_{n-1} \nabla^\alpha \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega \\
+ \sum_{0 < \alpha' < \alpha} \langle \nabla^\alpha w_{n-1} \nabla^\alpha \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega.
\]

By \( w_{n-1}(z = \pm h) = 0 \) and because of \( \|\partial_z w_{n-1}\|_{L^\infty} \leq \|\text{div}_H v_n\|_{H^2} \) we obtain

\[
\left| \langle w_{n-1} \nabla^\alpha \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega \right| \leq c \|\text{div}_H v_n\|_{H^2} \|\nabla^\alpha v_n \|_{L^2}^2.
\]

Let us recall that by Lemma 2.1

\[
\left| \langle f, g \rangle_\Omega \right| \leq c \|f\|_{L^2} \|g\|_{H^s} \|\nabla_H k\|_{L^2}^{1/2} \|h\|_{L^2}^{1/2}
\]

holds. This implies

\[
\left| \langle \nabla^\alpha w_{n-1} \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega \right| \\
\leq c \|\nabla^\alpha w_{n-1}\|_{L^2} \|\partial_z v_n\|_{H^s} \|\nabla_H \nabla^\alpha v_n \|_{L^2} \|\nabla^\alpha v_n \|_{L^2} \\
\leq c \|\text{div}_H v_n\|_{H^2} \|v_n\|_{H^s} \|\nabla_H v_n\|_{H^s} \|\nabla^\alpha v_n \|_{L^2}.
\]

Here we have two terms which contain third order derivatives, but they come with a power strictly less then two, so they also can be absorbed by the horizontal Laplacian in the end. For the sum we proceed similarly and get

\[
\sum_{0 < \alpha' < \alpha} \left| \langle \nabla^\alpha w_{n-1} \nabla^\alpha \partial_z v_n, \nabla^\alpha v_n \rangle_\Omega \right| \\
\leq c \sum_{0 < \alpha' < \alpha} \|\nabla^\alpha w_{n-1}\|_{H^s} \|\nabla^\alpha \partial_z v_n\|_{L^2} \|\nabla_H \nabla^\alpha v_n \|_{L^2} \|\nabla^\alpha v_n \|_{L^2} \\
\leq c \|\nabla_H v_{n-1}\|_{H^s} \|v_n\|_{H^s} \|\nabla_H v_n\|_{H^s} \|\nabla^\alpha v_n \|_{L^2}.
\]

With Young’s inequality it follows that

\[
\partial_t \|v_n\|_{H^s}^2 + \|\nabla_H v_n\|_{H^s}^2 \\
\leq c \left( \|v_{n-1}\|_{H^s}^2 + \|\nabla v_{n-1}\|_{H^s}^2 + \|\nabla v_{n-1}\|_{H^s}^2 \right) \|v_n\|_{H^s}^2 \\
+ \|\nabla_H \nabla v_n\|_{H^s}^2 + \frac{1}{2} \|\nabla_H v_{n-1}\|_{H^s}^2 + \frac{1}{2} \|\nabla_H v_{n-1}\|_{H^s}^2.
\]

Combined with the estimate for \( \varpi \) we have

\[
\partial_t (\|v_n\|_{H^s}^2 + \|\nabla v_n\|_{H^s}^2 + \|\nabla_H \nabla v_n\|_{H^s}^2) \\
\leq c \left( \|v_{n-1}\|_{H^s}^2 + \|\nabla v_{n-1}\|_{H^s}^2 + \|\nabla v_{n-1}\|_{H^s}^2 + \|\nabla_H v_{n-1}\|_{H^s}^2 \|v_n\|_{H^s}^2 \\
+ \|\nabla_H v_{n-1}\|_{H^s}^2 \right) \|\nabla v_n\|_{H^s}^2 + \frac{1}{2} \|\nabla_H \nabla v_{n-1}\|_{H^s}^2 + \frac{1}{2} \|\nabla_H v_{n-1}\|_{H^s}^2.
\]
and Gronwall’s inequality yields
\[ \|v_n\|_{H^s}^2 + \|\overline{v}_n\|_{H^s}^2 + \int_0^t \|\nabla_H v_n(r)\|_{H^s}^2 + \|\nabla_H \overline{v}_n(r)\|_{H^s} \, dr \]
\[ \leq \left( \|v(0)\|_{H^s}^2 + \|\overline{v}(0)\|_{H^s}^2 + \frac{1}{2} \int_0^t \|\nabla_H v_{n-1}(r)\|_{H^s}^2 + \|\nabla_H \overline{v}_{n-1}(r)\|_{H^s} \, dr \right) e^{f_n-1(t)}, \]
where
\[ f_n-1(t) = ct \left( \|v_{n-1}\|_{L^\infty((0,T),H^s)} + \|\overline{v}_{n-1}\|_{L^\infty((0,T),H^s)} + \|v_{n-1}\|_{L^\infty((0,T),H^s)}^4 \right) \]
\[ + c \int_0^t \|\nabla_H v_{n-1}(r)\|_{H^s} + \|\nabla_H \overline{v}_{n-1}(r)\|_{H^s}^{4/3} \, dr \]
\[ \leq ct \left( \|v_{n-1}\|_{L^\infty((0,T),H^s)} + \|\overline{v}_{n-1}\|_{L^\infty((0,T),H^s)} + \|v_{n-1}\|_{L^\infty((0,T),H^s)}^4 \right) \]
\[ + c t^{1/3} \|\nabla_H v_{n-1}\|_{L^2((0,T),H^s)}^{4/3} + ct^{1/2} \|\nabla_H v_{n-1}\|_{L^2((0,T),H^s)}. \]

For \( T \) sufficiently small this implies that \( \|v_n\|_{L^\infty((0,T),H^s)}, \|\overline{v}_n\|_{L^\infty((0,T),H^s)}, \|
abla_H v_n\|_{L^2((0,T),H^s)} \) and \( \nabla_H \overline{v}_n \) are uniformly bounded, and thus also \( \tilde{v}_n \) and \( \nabla_H p_n \) are bounded in these norms.

From the equation for \( v \) we then get a uniform bound for \( \|\partial_t \nabla_H v_n\|_{L^2((0,T),H^{s-2})} \) and \( \|\partial_t v_n\|_{L^\infty((0,T),H^{s-2})} \). By Lemma 2.2 we find a \( v_n \) such that
\[ v_n \to v \quad \text{in} \quad C^0([0,T], H^{s-1}(\Omega)) \]
\[ \nabla_H v_n \to \nabla_H v \quad \text{in} \quad L^2((0,T), H^{s-1}(\Omega)), \]
where we have not renamed the subsequences. By the energy inequality we obtain
\[ v \in L^\infty((0,T), H_{\text{per}}^{s-1}(\Omega)), \quad \nabla_H v \in L^2((0,T), H_{\text{per}}^s(\Omega)) \]
\[ \partial_t v \in L^\infty((0,T), H_{\text{per}}^{s-2}(\Omega)), \quad \partial_t \nabla_H v \in L^2((0,T), H_{\text{per}}^{s-2}(\Omega)). \]

Additionally, we have for \( \kappa \in (0,1) \)
\[ \|v - v_n\|_{H^{s-1}} \leq C \|v\|_{H^s} + \|v_n\|_{H^s} \|v - v_n\|_{H^{s-1}} \to 0 \quad (n \to \infty), \]
and thus \( v \in C^0([0,T], H_{\text{per}}^{s-1}(\Omega)) \). By the same arguments \( (\overline{v}_n)_n \) and \( (\tilde{v}_n)_n \) converge to some \( \overline{v} \) and \( \tilde{v} \) with the same regularities as \( v \) and the equation for \( p_n \) yields the convergence of \( (p_n)_n \) to some \( p \in L^\infty((0,T), H_{\text{per}}^s(G)) \cap L^2((0,T), H_{\text{per}}^{s+1}(G)) \) (which is uniquely determined up to a constant) with
\[ \partial_t \overline{v} + \nabla_H \overline{v} - \Delta_H \overline{v} + \nabla_H p = -K(\overline{v}), \]
\[ \text{div}_H \overline{v} = 0, \]
\[ \overline{v}(t) = \frac{1}{2h} \int_{-h}^{h} v_0(0,x,y) \, d\xi \]
and
\[ \partial_t \tilde{v} + \nabla_H \tilde{v} + \nabla_H \tilde{v} + \tilde{v} \cdot \nabla_H \overline{v} + \tilde{v} \cdot \nabla_H \overline{v} + w \partial_z \tilde{v} - \Delta_H \tilde{v} = K(\tilde{v}), \]
\[ w(t,x,y,z) = - \int_{-h}^{z} \text{div}_H \tilde{v}(t,x,y,\xi) \, d\xi + \frac{z + h}{2h} \int_{-h}^{h} \text{div}_H \tilde{v}(t,x,y,\xi) \, d\xi, \]
\[ \tilde{v}(t) = v_0 - \overline{v}(t) = 0. \]
What is left is to show that $\tilde{v}$ is average free in the $z$ direction. We set $u(t, x, y) := \frac{1}{2h} \int_{-h}^{h} \tilde{v}(t, x, y, \xi) \, d\xi$. From the above equation it follows that

\[ \partial_t u + \nabla_H u + u \cdot \nabla_H \nabla_H u = 0, \]

\[ \tilde{u}(t) = 0. \]

Multiplication with $u$ in $L^2(G)$ gives

\[ \frac{1}{2} \partial_t \| u \|_{L^2}^2 + \| \nabla_H u \|_{L^2}^2 \leq c \| \nabla \|_{L^\infty} \| u \|_{L^2} \| \nabla_H u \|_{L^2}, \]

and from this we get $\| u(t) \|_{L^2}^2 = 0$. Thus, we eventually have

\[ w(t, x, y, z) = -\int_{-h}^{z} \text{div}_H \tilde{v}(t, x, y, \xi) \, d\xi, \]

and so $\nabla$ and $\tilde{v}$ solve $(2.1)-(2.3)$ with $A = \Delta_H$, which implies that $v$ is a solution to $(1.2)$.

The uniqueness and continuous dependence on the data of that solution is a direct consequence of the energy inequality shown above.

It follows immediately from the above proof, that if in addition $\partial_z v_0 \in H^s(\Omega)$ then also the regularity of the solution in the vertical directions is increased. We will need this when investigating the equations with half horizontal viscosity.

**Corollary 4.2.** Assume that under the conditions of Theorem 4.1 additionally $\partial_z v_0 \in H^s_{\text{per}}(\Omega)$. Then we have for the solution $v$ to (1.2) obtained in Theorem 4.1 that additionally

\[ \partial_z v \in L^\infty((0, T), H^s_{\text{per}}(\Omega)) \cap C^0([0, T], H^{s-\kappa}_{\text{per}}(\Omega)), \]

\[ \partial_x \partial_z v, \partial_y \partial_z v \in L^2((0, T), H^s_{\text{per}}(\Omega)) \]

for some $T > 0$ and all $\kappa \in (0, 1)$.

Under the additional boundary condition $\partial_z v(z = \pm h) = 0$ Cao, Li and Titi showed in [4] that for $s = 2$ the solution to (1.2) exists global in time. They construct the solution by approximating the system with only horizontal viscosity by the system with full viscosity, showing bounds on the solution which are independent of the vertical viscosity and then letting this vertical tend to 0. The uniform bounds proved by them can be carried over directly to our equation, the additional boundary condition (which is preserved by the equation, see Proposition 4.4) is only needed for estimates on the $\partial_z v$ term and therefore we obtain literally the same estimates for our problem as they do in the limiting case. This implies that the local solutions obtained in Theorem 4.1 can be extended globally in time, which yields the global in time part of Theorem 1.3.

**Theorem 4.3.** For any $v_0 \in H^2_{\text{per}}(\Omega)$ with $\text{div}_H \nabla v_0 = 0$ and any $T > 0$ there exists a unique strong solution $v$ to (1.2) with

\[ v \in L^\infty((0, T), H^2_{\text{per}}(\Omega)) \cap C^0([0, T], H^{2-\kappa}_{\text{per}}(\Omega)) \]

\[ \partial_x v, \partial_y v \in L^2((0, T), H^2_{\text{per}}(\Omega)) \]

for all $\kappa \in (0, 1)$.
If we additionally assume a homogeneous Neumann boundary condition for the initial data in the vertical direction, then this boundary condition is preserved in time.

**Proposition 4.4.** Assume that under the conditions of Theorem 4.1 additionally \( \partial_z v_0(z = h) = 0 \). Then we have also for the solution \( v \) to (1.2) that \( \partial_z v(z = h) = 0 \), and the same holds for \( z = -h \).

**Proof.** Taking the derivative in the vertical direction of (1.2) gives
\[
\partial_t \partial_z v + v \cdot \nabla_H \partial_z v + \partial_z v \cdot \nabla_H v + \partial_z w \partial_z v + w \partial_z z v - \Delta_H \partial_z v = 0,
\]
and for \( z = h \) we obtain for \( u(t, x, y) := \partial_z v(z = h) \)
\[
\partial_t u + v(z = h) \cdot \nabla_H u + u \cdot \nabla_H v(z = h) - \text{div}_H v(z = h) u - \Delta_H u = 0,
\]
with \( u(t = 0) = 0 \). Multiplication with \( u \) in \( L^2(G) \) gives
\[
\frac{1}{2} \partial_t |u|^2_{L^2} + \|\nabla_H u\|^2_{L^2} \leq c \|v\|_{L^\infty} \|u\|_{L^2} \|\nabla_H u\|_{L^2},
\]
and this implies \( \|u(t)\|_{L^2} \equiv 0 \). For \( z = -h \) we proceed analogously. \( \square \)

5. **Half horizontal viscosity**

Here we show that the equation at least is locally well-posed, when we only have half horizontal viscosity. We first turn to the more involved case (1.1) and prove Theorem 1.1.

5.1. **Proof of Theorem 1.1.** Let us briefly describe the strategy of the proof. We will assume initial data \( v_0, \partial_z v_0 \in H^{s\per}_{\text{per}} \) with \( s \geq 3 \) for which additionally a Rayleigh condition
\[
\frac{1}{\eta} \leq \frac{1}{|\partial_z v_1(t = 0)|} \leq \eta, \quad \frac{1}{\eta} \leq \frac{1}{|\partial_z v_2(t = 0)|} \leq \eta
\]
holds for some \( \eta > 1 \), i.e. \( \partial_z v_0 \in H^s_{\text{per}, \eta}(\Omega) \). We have \( v_{zz}(t = 0) \in C^0(\Omega) \) because of \( s \geq 3 \), so the above point-wise condition makes sense.

We consider for \( \varepsilon > 0 \) the system (1.1) with \( A_{\text{per}} \) replaced by
\[
A_\varepsilon = \begin{pmatrix} \varepsilon \partial_{xx} + \partial_{yy} & 0 \\ 0 & \partial_{xx} + \varepsilon \partial_{yy} \end{pmatrix}.
\]
Corollary 4.2 guarantees the existence of a solution for \( \varepsilon > 0 \) and implies the continuity of \( \partial_z v \), so we have a Rayleigh condition \( \frac{1}{2\eta} \leq \frac{1}{|\partial_z v_1|} \leq 2\eta \) also in some initial time interval. Using this we show an \( \varepsilon \)-independent estimate for the \( H^s \)-norm of the corresponding solutions and perform the limit \( \varepsilon \to 0 \).

The main difficulty will be to control the highest derivatives in the horizontal directions \( \| (\partial_z, \partial_y)^\alpha v(t) \|_{L^2(\Omega)} \) for \( |\alpha| = s \). To obtain an estimate for this norms we follow the idea by Masmoudi and Wong for the 2D primitive Euler equation and use the Rayleigh condition to obtain bounds on \( \| \partial_z v(t) \|_{H^s(\Omega)} \), but in difference to the 2D Euler case these bounds cannot be carried over to bounds on \( v \) directly.

To work around that problem we combine this approach with the idea of Cao and Titi to split \( v \) via \( v = \overline{\nu} + \tilde{v} \) and consider the set of coupled equations (2.1) for \( A = A_\varepsilon \). We can deduce bounds for the mean value \( \overline{\nu} \) from the 2D Navier-Stokes equation (2.1), and by Poincaré’s inequality it suffices to have a bound for \( \partial_z v \) to control \( \tilde{v} \). We divide this proof into three steps, the estimates on the baroclinic
mode, the estimates on the barotropic modes and the convergence of the solutions for \(\varepsilon > 0\) to the solution of (1.1) when \(\varepsilon\) tends to 0.

**Estimates for the baroclinic mode.** The estimates for \(\nabla\) are the easier part, only the coupling-term has to be handled with some care. We obtain the following result.

**Lemma 5.1.** Let \(s \geq 3\), \(\varepsilon > 0\), \(v_0 \in H^s_{\text{per}}(\Omega)\) with \(\nabla v_0 = 0\), \(\partial_z v_0 \in H^s_{\text{per}}(\Omega)\) and \(v = \nabla + \dot{v}\) be the solution to (2.1)-(2.3) for \(A = A_\varepsilon\) according to Corollary 4.3. Then for any \(\delta > 0\)

\[
\frac{1}{2} \frac{d}{dt} \|\nabla\|_{H^s(G)}^2 + \|\partial_t v_1\|_{H^s(G)}^2 + \|\partial_t v_2\|_{H^s(G)}^2 + \|\partial_x \nabla v_1\|_{H^s(G)}^2 + \varepsilon \|\partial_y \nabla v_2\|_{H^s(G)}^2 \\
\leq c \|\nabla\|_{H^s(G)}^3 + \frac{c}{\delta} \|\partial_x \dot{v}\|_{H^s(G)}^2 + \varepsilon \|\nabla \partial_y \nabla v_1\|_{L^2}^2 + \varepsilon \|\nabla \partial_y \nabla v_2\|_{L^2}^2
\]

holds.

**Proof.** Applying \(\nabla^\alpha\) to (2.1) and multiplying it in \(L^2(G)\) by \(\nabla^\alpha \nabla\) for \(|\alpha| \leq s\) yields

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^\alpha \nabla\|_{L^2(G)}^2 + \|\nabla^\alpha \partial_x \nabla v_1\|_{L^2(G)}^2 + \|\nabla^\alpha \partial_x \nabla v_2\|_{L^2(G)}^2 + \varepsilon \|\nabla^\alpha \partial_y \nabla v_1\|_{L^2(G)}^2 + \varepsilon \|\nabla^\alpha \partial_y \nabla v_2\|_{L^2(G)}^2
\]

Due to the divergence free condition, the periodic boundary conditions and \(s \geq 3\) we get

\[
\|\nabla^\alpha \nabla H p, \nabla^\alpha \nabla v\|_{L^2(G)} = 0 \quad \text{and} \quad \|\nabla^\alpha (\nabla \cdot \nabla v), \nabla^\alpha \nabla v\|_{L^2(G)} \leq c \|\nabla\|_{H^s}^3.
\]

For the coupling-term we have

\[
2 h \langle \nabla^\alpha K(\dot{v}), \nabla^\alpha v\rangle_G = \int_{-h}^h \langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi)) \hat{v}_1(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_1\rangle_G d\xi
\]

and thus

\[
\langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi)) \hat{v}_1(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_1\rangle_G
\]

and similarly for the second integrand

\[
\langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi))) + \partial_y (\hat{v}_2(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_2\rangle_G
\]

Analogously it follows for the second integrand

\[
\langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi))) + \partial_y (\hat{v}_2(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_2\rangle_G
\]

and similarly for the second integrand

\[
\langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi))) + \partial_y (\hat{v}_2(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_2\rangle_G
\]

and similarly for the second integrand

\[
\langle \nabla^\alpha (\partial_x (\hat{v}_1(\cdot, \cdot, \xi)) \partial_y (\hat{v}_2(\cdot, \cdot, \xi))) + \partial_y (\hat{v}_2(\cdot, \cdot, \xi)), \nabla^\alpha \nabla v_2\rangle_G
\]
Using
\[ \int_{-h}^{h} \| \tilde{v}_1 (\cdot, \cdot, \xi) \|_{H^s(\Omega)} \| \partial_y \tilde{v}_1 (\cdot, \cdot, \xi) \|_{H^r(\Omega)} \frac{d\xi}{H^s(\Omega)} \leq \frac{1}{2} \| \tilde{v}_1 \|_{L^2 (\Omega)} \| \tilde{v}_2 \|_{H^s(\Omega)} + \| \tilde{v}_2 \|_{H^s(\Omega)} \| \tilde{v}_1 \|_{H^r(\Omega)} \]
we get
\[ \| (\nabla^\alpha K (\tilde{v})), \nabla^\alpha \varphi ) \|_{L^2 (\Omega)} \leq c \| \tilde{v}_1 \|_{H^s(\Omega)} \| \tilde{v}_2 \|_{H^s(\Omega)} + \| \tilde{v}_2 \|_{H^s(\Omega)} \| \tilde{v}_1 \|_{H^r(\Omega)} \]
for any \( \delta > 0 \). This leads after summing over \( \alpha \) and with Poincaré’s inequality for \( \tilde{v} \) the stated estimate. \( \square \)

Estimates for the barotropic modes. Here we prove estimates for the vertical derivative \( \partial_z v \), which is given by the equation
\[ \partial_t \partial_z v + v \cdot \nabla_H \partial_z v + \partial_z v \cdot \nabla_H v + \partial_z w \partial_z v + w \partial_z z v - A_z \partial_z v = 0. \]
The straightforward part is to estimate the lower derivatives and those which contain at least one derivative in the vertical direction (because \( w \) and \( w_z \) have the same regularity with respect to \( x \) and \( y \)). We multiply the equation by \( \nabla^\alpha \partial_z v \),

\[ \frac{d}{dt} \partial_z v + \nabla^\alpha \partial_z v \Omega - (A_z \nabla^\alpha \partial_z v, \nabla^\alpha \partial_z v) \Omega = -\nabla^\alpha (w \partial_z v + v \cdot \nabla_H \partial_z v - \partial_z v \cdot \nabla_H v + \partial_z v \nabla_H v, \nabla^\alpha \partial_z v) \Omega. \]

The following lemma is a direct consequence of the fact that for \( s \geq 3 \) the first order derivatives of \( v \) and \( \partial_z v \) are in \( L^\infty (\Omega) \).

Lemma 5.2. Let \( s \geq 3, \epsilon > 0 \), \( v_0 \in H^{s}_p(\Omega) \) with \( \text{div}_H \varphi = 0 \), \( \partial_z v_0 \in H^{s}_p(\Omega) \) and \( v = \tilde{v} + \tilde{v} \) be the solution to (2.1)-(2.2) for \( A = A_z \) according to Corollary 4.2. Then we have for \( |\alpha'| < s \) or \( \nabla^\alpha = \partial_z \nabla^\alpha \) with \( |\alpha'| = s - 1 \)

\[ \frac{1}{2} \frac{d}{dt} \| \nabla^\alpha \partial_z v \|_{L^2}^2 + \| \nabla^\alpha \partial_z \partial_y v_1 \|_{L^2}^2 + \| \nabla^\alpha \partial_z \partial_x v_2 \|_{L^2}^2 + \epsilon \| \nabla^\alpha \partial_z \partial_y v_1 \|_{L^2}^2 + \epsilon \| \nabla^\alpha \partial_z \partial_x v_2 \|_{L^2}^2 \leq c \left( \| v \|_{H^s} + \| \partial_z v \|_{H^s} \right) \| \partial_z v \|_{H^r}^2. \]

We replace the multiplier \( \nabla^\alpha \partial_z v_i \) in (5.2) by \( \nabla^\alpha \partial_z v_i \) to get an estimate if \( \nabla^\alpha = (\partial_z, \partial_y)^\alpha \) with \( |\alpha| = s \). In the next lemma we give the estimates for the different terms, this is the key step in the proof of our local well-posedness result. We write here \( A_{z,1} = \epsilon \partial_{zx} + \partial_{yy} \) and \( A_{z,2} = \epsilon \partial_{xx} + \epsilon \partial_{yy} \).

Lemma 5.3. Let \( s \geq 3, \eta > 1, \epsilon > 0 \), \( v_0 \in H^{s}_p(\Omega) \) with \( \text{div}_H \varphi = 0 \), \( \partial_z v_0 \in H^{s}_p(\Omega) \) and \( v = \tilde{v} + \tilde{v} \) be the solution to (2.1)-(2.2) for \( A = A_z \) according to Corollary 4.2. Then there exists a time \( T \) such that \( \partial_z v (t) \in H^{s}_p(\Omega) \) (\( t \leq T \))
and for $\nabla^\alpha = (\partial_x, \partial_y)^\alpha$ with $|\alpha| = s$ the following identities and estimates hold:

a) 
\[
\langle \nabla^\alpha A_{x,1} \partial_x v_1, \nabla^\alpha \partial_{zz} v_1 \rangle \Omega \leq -\varepsilon \frac{1}{2\eta} \| \nabla^\alpha \partial_x \partial_z v_1 \|_{L^2}^2 - \frac{1}{2\eta} \| \nabla^\alpha \partial_y \partial_z v_1 \|_{L^2}^2 + c\eta^2 \| \partial_z v_1 \|^{5/4}_H (\varepsilon \| \partial_x \partial_z v_1 \|^{7/4}_H + \| \partial_y \partial_z v_1 \|^{7/4}_H)
\]

and
\[
\langle \nabla^\alpha A_{x,2} \partial_x v_2, \nabla^\alpha \partial_{zz} v_2 \rangle \Omega \leq -\varepsilon \frac{1}{2\eta} \| \nabla^\alpha \partial_x \partial_z v_2 \|_{L^2}^2 - \frac{1}{2\eta} \| \nabla^\alpha \partial_y \partial_z v_2 \|_{L^2}^2 + c\eta^2 \| \partial_z v_2 \|^{5/4}_H (\| \partial_x \partial_z v_2 \|^{7/4}_H + \| \partial_y \partial_z v_2 \|^{7/4}_H).
\]

b) 
\[
\langle \nabla^\alpha (v \cdot \nabla_H \partial_x v_1), \nabla^\alpha \partial_{zz} v_1 \rangle \Omega = \frac{1}{2} \langle v \cdot \nabla_H \partial_{zz} v_1, \nabla^\alpha \partial_{zz} v_1 \rangle \Omega + \sum_{\alpha' < \alpha} \langle \nabla^\alpha v \cdot \nabla_H \nabla^\alpha \partial_x v_1, \nabla^\alpha \partial_{zz} v_1 \rangle \Omega
\]

with
\[
\left| \sum_{\alpha' < \alpha} \langle \nabla^\alpha v \cdot \nabla_H \nabla^\alpha \partial_x v_1, \nabla^\alpha \partial_{zz} v_1 \rangle \Omega \right| \leq c\eta \| v \|_H \| \partial_x v \|_H \| \partial_z v_1 \|_H.
\]

c) 
\[
\langle \nabla^\alpha (w \partial_{zz} v_i), \nabla^\alpha \partial_{zz} v_i \rangle \Omega = \frac{1}{2} \langle w \partial_{zz} v_i, \nabla^\alpha \partial_{zz} v_i \rangle \Omega + \langle \nabla^\alpha (\partial_x v_1 + \partial_y v_2), \nabla^\alpha v_i \rangle \Omega
\]

with
\[
\left| \sum_{\alpha' < \alpha} \langle \nabla^\alpha w \nabla^\alpha \partial_x v_1, \nabla^\alpha \partial_{zz} v_1 \rangle \Omega \right| \leq c\eta \| v \|_H \| \partial_z v_1 \|_H \| \partial_z v_1 \|_H.
\]

d) 
\[
\langle \nabla^\alpha (\partial_z v \cdot \nabla_H v_1 + \partial_z w \partial_x v_1), \nabla^\alpha \partial_{zz} v_1 \rangle \Omega \leq c\eta \| v \|_H \| \partial_z v \|_H^2,
\]
\[
+ c\eta \| \partial_z v \|_H (\| v \|_H + \| \partial_z v \|_H) \| \partial_x v_1 \|_H^2 + c\eta^2 \| v \|_H \| \partial_z v \|_H \| \partial_y v_1 \|_H
\]

and
\[
\langle \nabla^\alpha (\partial_z v \cdot \nabla_H v_2 + \partial_z w \partial_x v_2), \nabla^\alpha \partial_{zz} v_2 \rangle \Omega \leq c\eta \| v \|_H \| \partial_z v \|_H^2,
\]
\[
+ c\eta \| \partial_z v \|_H (\| v \|_H + \| \partial_z v \|_H) \| \partial_x v_2 \|_H^2 + c\eta^2 \| v \|_H \| \partial_z v \|_H \| \partial_z v_2 \|_H.
\]
Proof. We have $\partial_x v(t = 0) \in H^s_{\text{per}, \partial_x}(\Omega)$ and $\partial_x v$ is continuous, so there exists a $T > 0$ such that $\partial_x v(t) \in H^s_{\text{per}, 2\alpha}(\Omega)$ for $t \leq T$. We assume in the following for simplicity $\partial_{zz} v_i > 0$.

a) By integration by parts we obtain

$$\langle \nabla^\alpha \partial_{xx} \partial_{zz} v_i, \nabla^\alpha \partial_{zz} v_i \rangle_\Omega = -\frac{1}{2} \left\langle \nabla^\alpha \partial_{xx} \partial_{zz} v_i, \nabla^\alpha \partial_{zz} v_i \right\rangle_{L^2} \Rightarrow$$

$$\leq -\frac{1}{2\eta} \left\langle \nabla^\alpha \partial_{xx} \partial_{zz} v_i \right\rangle^2_L + \eta^2 \left\| \nabla^\alpha \partial_{x} \partial_{zz} v_i \right\|_{L^2} \left\| \nabla^\alpha \partial_{zz} v_i \right\|_{L^\infty} \nabla^\alpha \partial_{zz} v_i \|_{L^2}$$

where we used Lemma 2.1 to estimate $\|\partial_{x} \partial_{zz} v_i\|_{L^\infty}$. In the same way we get this estimate for the $y$-derivatives and adding them up for $i = 1, 2$ we obtain the assertion.

b) For

$$\langle \nabla^\alpha (v \cdot \nabla_H \partial_{zz} v_i), \nabla^\alpha \partial_{zz} v_i \rangle_\Omega = \langle v \cdot \nabla_H \nabla^\alpha \partial_{zz} v_i, \nabla^\alpha \partial_{zz} v_i \rangle_\Omega$$

$$+ \sum_{\alpha' < \alpha} \langle \nabla^{\alpha'-\alpha} v \cdot \nabla_H \nabla^\alpha \partial_{zz} v_i, \nabla^\alpha \partial_{zz} v_i \rangle_\Omega$$

with

$$\langle \langle \nabla^\alpha \partial_{zz} v_i \rangle, A_{\varepsilon, 1} \partial_{zz} v_i \rangle \rangle \leq c \eta^2 \left\| \partial_{zz} v_i \right\|_{H^s}^5 \left( \varepsilon \left\| \partial_x \partial_{zz} v_i \right\|_{H^s}^{7/4} + \left\| \partial_y \partial_{zz} v_i \right\|_{H^s}^{7/4} \right)$$

$$+ c \eta^3 \left\| \partial_y \partial_{zz} v_i \right\|_{H^s}^{3/2} \left( \varepsilon \left\| \partial_x \partial_{zz} v_i \right\|_{H^s}^{3/2} + \left\| \partial_y \partial_{zz} v_i \right\|_{H^s}^{3/2} \right),$$

and

$$\left\| \partial_{zz} v \cdot \nabla_H v_i + \partial_{zz} w \partial_{zz} v_i + 2 \partial_v v \cdot \nabla_H \partial_{zz} v_i + 2 \partial_v w \partial_{zz} v_i \right\|_{L^\infty} \leq c \eta^2 \left\| \partial_{zz} v \right\|_{H^s}^2 + \left\| \partial_z v \right\|_{H^s} \left\| v \right\|_{H^s}.$$
we have
\[
\langle v \cdot \nabla H \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega = -\frac{1}{2} \left( \frac{\partial v_1}{\partial_z v_i} - \frac{v_1 \partial_z \partial_z v_i}{(\partial_z v_i)^2}, \langle \nabla^\alpha \partial_z v_i \rangle_\Omega \right)^2 \Omega
\]
\[
- \frac{1}{2} \left( \frac{\partial v_2}{\partial_z v_i} - \frac{v_2 \partial_z \partial_z v_i}{(\partial_z v_i)^2}, \langle \nabla^\alpha \partial_z v_i \rangle_\Omega \right)^2 \Omega
\]
\[
- \frac{1}{2} \left( \frac{\text{div}_H v}{\partial_z v_i} - \frac{v \cdot \nabla H \partial_z v_i}{(\partial_z v_i)^2}, \langle \nabla^\alpha \partial_z v_i \rangle_\Omega \right)^2 \Omega.
\]
and all the terms in the sum contain only derivatives of order less or equal \(s\), so
\[
|\langle \nabla^\alpha v \cdot \nabla H \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega| \leq c\eta \|v\|_{H^s} \|\partial_z v\|_{H^s} \|\nabla^\alpha \partial_z v_i\|_{L^2}.
\]
c) This is shown similarly to b), but due to the bad regularity of \(w\) we have to separate two terms from the sum,
\[
\langle \nabla^\alpha (w \partial_z v_i), \nabla^\alpha \partial_z v_i \rangle_\Omega = \langle \nabla^\alpha w \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega + \langle w \partial_z \nabla^\alpha v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega
\]
\[
+ \sum_{\alpha' < \alpha} \langle \nabla^\alpha \partial_z v_i \rangle_\Omega.
\]
Only the first term is new compared to b), we get
\[
\langle \nabla^\alpha w \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega = \langle \nabla^\alpha w, \nabla^\alpha \partial_z v_i \rangle_\Omega = \langle \nabla^\alpha (\partial_z v_1 + \partial_y v_2), \nabla^\alpha v_i \rangle_\Omega
\]
and this is also the term for which forces us to use the more complicated multiplier.
d) Here we also split the expression into the highest order derivatives and a sum,
\[
\langle \nabla^\alpha (\partial_z v \cdot \nabla H v_i + \partial_z w \partial_z v_i), \nabla^\alpha \partial_z v_i \rangle_\Omega
\]
\[
= \langle \partial_z v \cdot \nabla H \nabla^\alpha v_i - \nabla^\alpha \text{div}_H v \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega
\]
\[
+ \sum_{\alpha' < \alpha} \langle \nabla^\alpha \nabla^\alpha \partial_z v \cdot \nabla H \nabla^\alpha v_i - \nabla^\alpha \text{div}_H v \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega.
\]
The terms in the sum are once again easy to estimate,
\[
|\langle \nabla^\alpha \nabla^\alpha \partial_z v \cdot \nabla H \nabla^\alpha v_i - \nabla^\alpha \text{div}_H v \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i \rangle_\Omega|
\]
\[
\leq c\eta \|v\|_{H^s} \|\partial_z v\|_{H^s} \|\nabla^\alpha \partial_z v_i\|_{L^2}.
\]
For the other term we use a cancellation, for \(i = 1\) we obtain
\[
\langle \partial_z v \cdot \nabla H \nabla^\alpha v_1 - \nabla^\alpha \text{div}_H v \partial_z v_1, \nabla^\alpha \partial_z v_1 \rangle_\Omega
\]
\[
= \langle \partial_z v_2 \cdot \partial_y \nabla^\alpha v_1 - \nabla^\alpha \partial_y v_2 \cdot \partial_z v_1, \nabla^\alpha \partial_z v_1 \rangle_\Omega.
\]
The first term can be estimated directly
\[
|\langle \partial_z v_2 \cdot \partial_y \nabla^\alpha v_1, \nabla^\alpha \partial_z v_1 \rangle_\Omega| \leq c\eta \|\partial_z v_2\|_{H^2} \|\nabla^\alpha \partial_y v_1\|_{L^2} \|\nabla^\alpha \partial_z v_1\|_{L^2}.
\]
and for the second one we integrate by parts
\[
(\nabla^\alpha \partial_z v_1, \frac{\nabla^\alpha \partial_y v_2}{\partial_z v_1})_\Omega = - \langle \nabla^\alpha \partial_z v_1, \partial_y \partial_z v_1 + \partial_z v_1 \nabla^\alpha \partial_y \partial_z v_1, \frac{\nabla^\alpha v_2}{\partial_z v_1} \rangle_\Omega \\
+ \langle \nabla^\alpha \partial_z v_1, \partial_z \partial_y \partial_z v_1 \frac{\nabla^\alpha v_2}{(\partial_z v_1)^2} \rangle_\Omega.
\]
It follows
\[
|\langle \nabla^\alpha \partial_z v_1, \frac{\nabla^\alpha \partial_y v_2}{\partial_z v_1} \rangle_\Omega| \leq \eta \|v_2\|_{H^\alpha} \|\partial_z v_1\|_{H^3} (\|\partial_z v_1\|_{H^\alpha} + \|\partial_y \partial_z v_1\|_{H^\alpha}) \\
+ \eta^2 \|v_2\|_{H^\alpha} \|\partial_z v_1\|_{H^2} \|\partial_z \partial_y \partial_z v_1\|_{H^3} \|\partial_z v_1\|_{H^\alpha}.
\]
So we have
\[
|\langle \partial_z v \cdot \nabla_H \nabla^\alpha v_1 - \nabla^\alpha \text{div}_H v \partial_z v_1, \frac{\nabla^\alpha \partial_y v_2}{\partial_z v_1} \rangle_\Omega| \leq \eta \|v\|_{H^\alpha} \|\partial_z v\|_{H^3} \\
+ \eta^2 \|v\|_{H^\alpha} \|\partial_z v_1\|_{H^2} \|\partial_z \partial_y \partial_z v_1\|_{H^3} \|\partial_z v_1\|_{H^\alpha}.
\]
For \(i = 2\) we proceed in the same way.

e) Last we have to handle the time derivative for this multiplier,
\[
\langle (\nabla^\alpha \frac{d}{dt} \partial_z v_1, \frac{\nabla^\alpha \partial_y v_2}{\partial_z v_1})_\Omega \rangle = \frac{1}{2} \frac{d}{dt} \left( \frac{\nabla^\alpha \partial_z v_1}{|\partial_z v_1|^2} \right)_{L^2} - \frac{1}{2} \langle (\nabla^\alpha \partial_z v_1)^2, \frac{d}{dt} \frac{1}{\partial_z v_1} \rangle_\Omega
\]
with
\[
\frac{d}{dt} \frac{1}{\partial_z v_1} = \frac{v \cdot \partial_z \nabla_H v_1 + \partial_z v \cdot \nabla_H v_1}{(\partial_z v_1)^2} - \frac{A_{x,i} \partial_z v_i}{(\partial_z v_1)^2} \\
+ \frac{\partial_z w \partial_z v_1 + \partial_z w \partial_z v_i}{(\partial_z v_1)^2} + 2 \partial_z v \cdot \nabla_H \partial_z v_1 + 2 \partial_z w \partial_z v_1.
\]
The first fraction will cancel with terms obtained in b) and c), and the lengthy expression inherits only functions which are in \(L^\infty(\Omega)\). This yields
\[
\left| \frac{\partial_z v \cdot \nabla_H v_1 + \partial_z w \partial_z v_1 + 2 \partial_z v \cdot \nabla_H \partial_z v_1 + 2 \partial_z w \partial_z v_1}{(\partial_z v_1)^2} \right|_{L^\infty} \\
\leq \eta \|\partial_z v\|_{H^3} (\|v\|_{H^3} + \|\partial_z v\|_{H^3}).
\]
Finally, for \(\langle (\nabla^\alpha \partial_z v_1)^2, \frac{A_{x,i} \partial_z v_i}{(\partial_z v_1)^2} \rangle_\Omega\) we obtain
\[
\langle (\nabla^\alpha \partial_z v_1)^2, \frac{\partial_x \partial_z v_i}{(\partial_z v_1)^2} \rangle_\Omega \\
= -\langle 2(\nabla^\alpha \partial_z v_1)(\partial_x \nabla^\alpha \partial_z v_1), \frac{\partial_x \partial_z v_i}{(\partial_z v_1)^2} \rangle_\Omega + \langle (\nabla^\alpha \partial_z v_1)^2, \frac{\partial_x \partial_z v_i}{(\partial_z v_1)^3} \rangle_\Omega.
\]
As in a) we get
\[
|\langle 2(\nabla^\alpha \partial_z v_1)(\partial_x \nabla^\alpha \partial_z v_1), \frac{\partial_x \partial_z v_i}{(\partial_z v_1)^2} \rangle_\Omega| \leq \eta \|\nabla^\alpha \partial_z v_1\|_{L^2} \|\partial_x \partial_z v_i\|_{H^\alpha} \|\partial_z v_i\|_{H^\alpha}^{1/4}
\]
and from Lemma 2.11 follows
\[
|\langle (\nabla^\alpha \partial_z v_1)^2, \frac{\partial_x \partial_z v_i}{(\partial_z v_1)^3} \rangle_\Omega| \leq \eta^3 \|\nabla^\alpha \partial_z v_1\|_{L^2} \|\partial_x \partial_z v_i\|_{H^\alpha} \|\partial_z v_i\|_{L^\infty}^{3/2} \|\partial_z v_i\|_{H^\alpha}^{1/2}.
\]
Combining the two estimates yields

\[
\left| \left( \nabla^\alpha \partial_z v_i \right)^2, \frac{\partial_x \partial_z v_i}{(\partial_z v_i)^2} \right| \leq c \eta^2 \left( \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \nabla_x \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \partial_x \partial_z v_i \right)^{3/4} \left( \partial_z v_i \right)^{1/4} \\
+ c \eta^3 \left( \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \partial_x \partial_z v_i \right)^{3/2} \left( \partial_z v_i \right)^{1/2} 
\]

and similarly we get

\[
\left| \left( \nabla^\alpha \partial_z v_i \right)^2, \frac{\partial_y \partial_z v_i}{(\partial_z v_i)^2} \right| \leq c \eta^2 \left( \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \partial_y \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \partial_y \partial_z v_i \right)^{3/4} \left( \partial_z v_i \right)^{1/4} \\
+ c \eta^3 \left( \nabla^\alpha \partial_z v_i \right)_{L^2} \left( \partial_y \partial_z v_i \right)^{3/2} \left( \partial_z v_i \right)^{1/2}. 
\]

Now we can give the estimate for the barotropic mode.

**Lemma 5.4.** Let \( s \geq 3, \eta > 1, \varepsilon > 0, v_0 \in H^s_{per}(\Omega) \) with \( \text{div}_H \varphi_0 = 0 \), \( \partial_z v_0 \in H^s_{per}(\Omega) \) and \( \varphi = \varphi_0 + \tilde{v} \) be the solution to (2.1), (2.2) for \( A = A_{\varepsilon} \) according to Corollary 4.2. Then the following estimate holds in \([0, T]\) for some \( T > 0 \)

\[
\frac{d}{dt} \eta \left( \nabla^\alpha \partial_z v_i \right)^2_{H_2} + \frac{1}{2} \left( \partial_y \partial_z v_i \right)^2_{H^s} + \frac{1}{2} \left( \partial_x \partial_z v_i \right)^2_{H^s} + \frac{\varepsilon}{2} \left( \partial_x \partial_z v_i \right)^2_{H^s} + \frac{\varepsilon}{2} \left( \partial_y \partial_z v_i \right)^2_{H^s} \leq c(\eta) \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} + c(\eta) \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} + c(\eta) \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} + c(\eta) \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} \\
+ \frac{1}{2} \left( \partial_y \partial_z v_i \right)^2_{H^s} + \frac{1}{2} \left( \partial_y \partial_z v_i \right)^2_{H^s}. 
\]

**Proof.** For \( \nabla^\alpha = (\partial_x, \partial_y)^\alpha \) we multiply (5.1) by \( \nabla^\alpha \partial_z v_i |_{\partial_z v_i} \).

\[
(\partial_t \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i |_{\partial_z v_i})_\Omega - \langle A_{\varepsilon,i} \nabla^\alpha \partial_z v_i, \nabla^\alpha \partial_z v_i |_{\partial_z v_i} \rangle_\Omega \\
= -\langle \nabla^\alpha (\partial_z v \cdot \nabla H v_i + \partial_z w \partial_z v_i + v \cdot \nabla_H \partial_z v_i + w \partial_z v_i), \nabla^\alpha \partial_z v_i |_{\partial_z v_i} \rangle_\Omega. 
\]

With

\[
\left| \langle \nabla^\alpha (\partial_z v_i + \partial_y v_i), \partial^\alpha v_i \rangle \right| \leq \left( \nabla^\alpha v_2 \right)_{L^2} \left( \nabla^\alpha \partial_z v_i \right)_{L^2} 
\]

follows from Lemma 5.5 for \( i = 1 \)

\[
\left. \sum_{|\alpha|=s} \frac{1}{2} \frac{d}{dt} \left( \nabla^\alpha \partial_z v_i |_{\partial_z v_i} \right)^2_{H^s} + \frac{1}{2} \left( \nabla^\alpha \partial_z v_i \right)^2_{L^2} + \frac{1}{2} \left( \partial_y \partial_z v_i \right)^2_{H^s} \right. \\
\left. \leq \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \nabla^\alpha \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} \\
\right. \\
\left. + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} \\
\right. \\
\left. + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} + c \eta \left( \partial_y \partial_z v_i \right)^{10}_{H^s} \right). 
\]

We use \( v = \tilde{v} + \varphi \), where \( \tilde{v} \) is average free in the vertical direction, and Poincaré’s inequality to get

\[
\left( \partial_y v \right)_{H^s(\Omega)} \leq \left( \partial_y \tilde{v} \right)_{H^s(\Omega)} + \left( \partial_y \varphi \right)_{H^s(\Omega)} \leq c \left( \partial_y \partial_z \tilde{v} \right)_{H^s(\Omega)} + \sqrt{2} \left( \partial_y \varphi \right)_{H^s(\Omega)} 
\]

By this and by applying Young’s inequality a couple of times follows
\[
\sum_{|\alpha|=s} \eta \frac{d}{dt} \left[ \left\| \nabla^\alpha \partial_t v_1 \right\|_{L^2}^2 + \varepsilon \left\| \nabla^\alpha \partial_x \partial_z v_1 \right\|_{L^2}^2 + \left\| \nabla^\alpha \partial_y \partial_z v_1 \right\|_{L^2}^2 \right] + c(\eta) \left( \left\| \nabla v \right\|_{H^s}^2 + \left\| \partial_x v \right\|_{H^s}^2 + \left\| \partial_x v \right\|_{H^s}^2 \right) + c(\eta)(1 + \varepsilon) \left( \left\| \partial_x v \right\|_{H^{s+1}}^{10} + \left\| \partial_x v \right\|_{H^{s+1}}^{10} \right) + \frac{1}{2} \left\| \partial_y \nabla v \right\|_{H^s}^2 + \frac{1}{2} \left\| \partial_y \partial_x v \right\|_{H^s}^2 + \varepsilon \left\| \partial_x \partial_z v \right\|_{H^s}^2.
\]
For \( i = 2 \) we can proceed in the same way. Together with Lemma 5.2 and after applying Young’s inequality some more times we obtain the stated estimate. \( \square \)

**Local existence.** We finally have everything in place to give the proof of Theorem 1.1 i.e. to show the local in time well-posedness of (1.1). Let us restate the Theorem using the \( H^s_{per, \eta} \)-spaces.

**Theorem 5.5.** Let \( s \geq 3 \) and \( \eta > 0 \). Then for any \( v_0 \in H^s_{per, \eta}(\Omega) \) with \( \text{div}_H \tau_0 = 0 \) and \( \partial_z v_0 \in H^s_{per, \eta}(\Omega) \) there exists a \( T > 0 \) and a unique strong solution \( v \) to (1.1) with \( v \in L^\infty((0, T), H^s_{per, \eta}(\Omega)) \cap C^0((0, T), H^{s-\kappa}_{per, \kappa}(\Omega)) \), \( \partial_x v_2, \partial_y v_1 \in L^2((0, T), H^s_{per, \eta}(\Omega)) \) for all \( \kappa \in (0, 1) \) and \( \partial_z v \) has the same regularity as \( v \) with \( \partial_z v(t) \in H^s_{per, 2\eta}(\Omega) \).

**Proof.** Let \( n \in \mathbb{N} \) and \( v^{(n)} = \tau^{(n)} + \tilde{v}^{(n)} \) be the solution to (2.1)–(2.3) for \( A = A_\varepsilon \) with \( \varepsilon = \frac{1}{n} \) according to Corollary 4.2. Following Lemma 5.3 and 5.4 there exists some \( T_n > 0 \) such that
\[
\frac{d}{dt} \left( \eta \left\| \partial_z v^{(n)} \right\|_{H^2_{per, \eta}} + \left\| \tau^{(n)} \right\|_{H^s_{per, \eta}} \right) + \frac{1}{4} \left( \left\| \partial_y \tau^{(n)}_1 \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \partial_z v^{(n)}_2 \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_y \tau^{(n)}_2 \right\|_{H^s_{per, \eta}}^2 \right) + \frac{1}{n} \left( \frac{3}{2} \left\| \partial_x \partial_z v^{(n)}_1 \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_y \partial_z v^{(n)}_2 \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \tau^{(n)}_1 \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \tau^{(n)}_2 \right\|_{H^s_{per, \eta}}^2 \right) \leq c(\eta) \left( \left\| \tau^{(n)} \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_z v^{(n)} \right\|_{H^s_{per, \eta}}^2 + \left( \left\| \partial_z v^{(n)} \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x v^{(n)} \right\|_{H^s_{per, \eta}}^2 \right)^5 \right)
\]
holds for \( t \leq T_n \). Integration in time and Gronwall’s inequality yield now
\[
\left( \left\| \tau^{(n)}(t) \right\|_{H^s_{per, \eta}}^2 + \eta \left\| \partial_z v^{(n)}(t) \right\|_{H^2_{per, \eta}}^2 \right) + \frac{1}{4} \int_0^t \left( \left\| \partial_y \tau^{(n)}_1(r) \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \partial_z v^{(n)}_2(r) \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_y \partial_z v^{(n)}_1(r) \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \tau^{(n)}_2(r) \right\|_{H^s_{per, \eta}}^2 \right) \, dr \leq \left[ \left( 1 + \left\| \tau_0 \right\|_{H^s_{per, \eta}}^2 + \eta \left\| \partial_z v_0 \right\|_{H^2_{per, \eta}}^2 \right) e^{-c(\eta)t} - 1 \right]^{-1/4}
\]
and so there exists some \( T_n \) independent of \( n \) such that
\[
\left( \left\| \tau^{(n)}(t) \right\|_{H^s_{per, \eta}}^2 + \eta \left\| \partial_z v^{(n)}(t) \right\|_{H^2_{per, \eta}}^2 + \frac{1}{4} \int_0^t \left( \left\| \partial_y \tau^{(n)}_1(r) \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_y \partial_z v^{(n)}_1(r) \right\|_{H^s_{per, \eta}}^2 \right) + \left\| \partial_x \tau^{(n)}_2(r) \right\|_{H^s_{per, \eta}}^2 + \left\| \partial_x \partial_z v^{(n)}_2(r) \right\|_{H^s_{per, \eta}}^2 \right) \, dr \leq 2 \left( \left\| \tau_0 \right\|_{H^s_{per, \eta}}^2 + \eta \left\| \partial_z v_0 \right\|_{H^2_{per, \eta}}^2 \right)
\]
for \( t \leq \min\{T_n, T\} \). Additionally, we have to show that the existence time \( T_n \) does not tend to 0. The uniform bound for the norm is not sufficient, because of the dependence of \( T_n \) on the condition \( \frac{1}{2\eta} \leq \frac{1}{|v^{(n)}_z(t)|} \leq 2\eta \), we need to control
\[
\left\| \partial_{zz}v^{(n)}(t) - \partial_{zz}v^{(n)}(0) \right\|_{L^\infty(\Omega)}.
\]
Using \( v_{zz} \in L^\infty((0, T_n), H^2(\Omega)) \cap W^{1,\infty}((0, T_n), L^2(\Omega)) \) we obtain first that
\[
\partial_{zz}v \in L^q((0, T_n), H^2(\Omega)) \cap H^{1,q}((0, T_n), L^2(\Omega))
\]
for all \( 1 \leq q \leq \infty \), where \( H^{1,q} \) denotes the Bessel potential space. This implies (see [19, Theorem 2.4.1] and [6, Lemma 2.61])
\[
\partial_{zz}v \in H^{\kappa,q}(0, T_n), H^{2(1-\kappa)}(\Omega)
\]
for \( \kappa \in [0, 1] \) and both embeddings are continuous, thus
\[
\left\| \partial_{zz}v^{(n)} \right\|_{H^{\kappa,q}(0, T_n), H^{2(1-\kappa)}} \leq cT_n^{1/q} \left( \left\| \partial_{zz}v^{(n)} \right\|_{L^\infty((0, T), H^2)} + \left\| \partial_{zz}v^{(n)} \right\|_{L^\infty((0, T), H^3)} \left\| v^{(n)} \right\|_{L^\infty((0, T), H^3)} \right).
\]
Choosing now \( \kappa = \frac{1}{8} \) (any \( \kappa < \frac{1}{4} \) is possible) and \( q = \frac{4}{\kappa} = 16 \) we get
\[
H^{\kappa,q}(0, T), H^{2(1-\kappa)}(\Omega) \hookrightarrow C^{0,\alpha}([0, T], C^0(\Omega))
\]
by Sobolev embedding with \( \alpha = \frac{1}{8} \) and
\[
\left\| \partial_{zz}v^{(n)} \right\|_{C^{0,\alpha}([0, T], L^\infty(\Omega))} \leq cT_n^{1/q} \left( 1 + \left\| v^{(n)} \right\|_{L^\infty((0, T), H^3)} \right) \left\| \partial_{zz}v^{(n)} \right\|_{L^\infty((0, T), H^3)}.
\]
Therefore,
\[
\left\| \partial_{zz}v^{(n)}(t) - \partial_{zz}v^{(n)}(0) \right\|_{L^\infty(\Omega)} \leq cT_n^{1/q} \left( 1 + \left\| v_0 \right\|_{H^s}^2 + \eta \left\| \partial_{zz}v_0 \right\|_{H^s}^2 \right) \left( \left\| v_0 \right\|_{H^s}^2 + \left\| \partial_{zz}v_0 \right\|_{H^s}^2 \right)
\]
and so there exists a \( T_0 \), independent of \( n \) such that \( \partial_{zz}v(t) \in H^{2\eta}(\Omega) \) for \( t \leq T_0 \).

Especially (5.3) holds for \( t \leq \min\{T_n, T_0\} \) and the convergence to a solution follows now as in the proof of Theorem 4.1 by using Lemma 2.2. \( \square \)

**Remark 5.6.** Masmoudi and Wong handled \( \left\| \partial_{zz}v(t) - \partial_{zz}v(0) \right\|_{L^\infty} \) for the primitive Euler equation in 2D by assuming enough regularity to have \( \frac{d}{dt} \partial_{zz}v(t) \in L^\infty \) with a uniform estimate against the initial data, which gives
\[
\left\| \partial_{zz}v(t) - \partial_{zz}v(0) \right\|_{L^\infty} \leq c \left\| \partial_{zz}v_0 \right\|_{H^s}.
\]
This approach would have forced us to take initial values \( v_0, \partial_{zz}v_0 \) at least in \( H^4(\Omega) \). Applying our method in their proof allows to lower their regularity assumptions on the data to \( \partial_{zz}v_0 \in H^3((-1, 1) \times (-h, h)) \) instead of \( \partial_{zz}v_0 \in H^4((-1, 1) \times (-h, h)) \).
5.2. Proof of Theorem 1.2

Proof. Here we also consider for $n \in \mathbb{N}$ an equation with full, but anisotropic horizontal viscosity. However, deducing the energy estimate is much simpler than in the proof of Theorem 1.1. Let $v_n$ be the solution to

$$
\partial_t v_n + v_n \cdot \nabla_H v_n + w_n \partial_z v_n - A_n' v_n + \nabla_H p_n = 0 \quad \text{in } (0, T) \times \Omega,
$$

$$
\partial_z p_n = 0 \quad \text{in } (0, T) \times \Omega,
$$

$$
\text{div}_H v_n + \partial_z w_n = 0 \quad \text{in } (0, T) \times \Omega,
$$

$$
v(t = 0) = v_0 \quad \text{in } \Omega
$$

with periodic boundary conditions imposed on $v_n$ and $p_n$ in the horizontal directions, $w_n(z = \pm h) = 0$ and $A_n' = \begin{pmatrix} \partial_{xx} + \frac{1}{n} \partial_{yy} & 0 \\ 0 & \partial_{yy} + \frac{1}{n} \partial_{xx} \end{pmatrix}$.

Theorem 4.1 yields the existence of such a $v_n$ and multiplication of the equation in $H^s(\Omega)$ with $v_n$ gives because of $s \geq 3$

$$
\frac{1}{2} \frac{d}{dt} \left\| v_n \right\|_{H^s}^2 + \left\| \partial_x v_n \right\|_{H^s}^2 + \frac{1}{n} \left\| \partial_y v_n \right\|_{H^s}^2 + \frac{1}{n} \left\| \partial_y v_n \right\|_{H^s}^2 + \left\| \partial_z v_n \right\|_{H^s}^2
\leq c \left\| v_n \right\|_{H^s}^2 + c \left\| v_n \right\|_{H^s}^2
\leq c \left\| v_n \right\|_{H^s}^2 + c \left\| v_n \right\|_{H^s}^2
\leq c \left\| v_n \right\|_{H^s}^2 + c \left\| v_n \right\|_{H^s}^2
\leq c \left\| v_n \right\|_{H^s}^2 + c \left\| v_n \right\|_{H^s}^2 + \frac{1}{2} \left\| \partial_x v_n \right\|_{H^s}^2 + \frac{1}{2} \left\| \partial_y v_n \right\|_{H^s}^2.
$$

This already implies for $T$ sufficiently small that we have a uniform bound for $\left\| v_n \right\|_{L^\infty((0,T),H^s)}$, $\left\| \partial_x v_n \right\|_{L^2((0,T),H^s)}$ and $\left\| \partial_y v_n \right\|_{L^2((0,T),H^s)}$, which yields again the convergence to a solution by using Lemma 2.2. □

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