Annular embeddings of permutations for arbitrary genus

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Abstract

In the symmetric group on a set of size 2n, let $P_{2n}$ denote the conjugacy class of involutions with no fixed points (equivalently, we refer to these as “pairings”, since each disjoint cycle has length 2). Harer and Zagier explicitly determined the distribution of the number of disjoint cycles in the product of a fixed cycle of length 2n and the elements of $P_{2n}$. Their famous result has been reproved many times, primarily because it can be interpreted as the genus distribution for 2-cell embeddings in an orientable surface, of a graph with a single vertex attached to n loops. In this paper we give a new formula for the cycle distribution when a fixed permutation with two cycles (say the lengths are $p, q$, where $p + q = 2n$) is multiplied by the elements of $P_{2n}$. It can be interpreted as the genus distribution for 2-cell embeddings in an orientable surface, of a graph with two vertices, of degrees $p$ and $q$. In terms of these graphs, the formula involves a parameter that allows us to specify, separately, the number of edges between the two vertices and the number of loops at each of the vertices. The proof is combinatorial, and uses a new algorithm that we introduce to create all rooted forests containing a given rooted forest.

1 Introduction

Let $[p] = \{1, \ldots, p\}$, and $S_p$ be the set of permutations of $[p]$, for $p \geq 0$. When $p \geq 0$ is even, let $P_p$ be the set of pairings on $[p]$, which are partitions of the set $[p]$ into disjoint pairs (subsets of size 2). We refer to the single element of $P_0$ as the empty pairing. Where the context is appropriate, we shall also regard $P_p$ as the conjugacy class of involutions with no fixed points in $S_p$. In this latter context, each pair becomes a disjoint cycle consisting of that pair of elements. Of course, the number of pairings in $P_p$ is $(p-1)!! = \prod_{j=1}^{p/2} (2j-1)$, with the empty product convention that $(-1)!! = 1$.

Now, for $p > 0$ and even, let $\gamma_p = (1 \ldots p)$, in disjoint cycle notation, and let $A_p = \{\mu \gamma_p^{-1} : \mu \in P_p\}$. Let $a_{p,k}$ be the number of permutations in $A_p$ with $k$ cycles in the disjoint cycle representation, for $k \geq 1$. The generating series for these numbers are given by $A_p(x) = \sum_{k \geq 1} a_{p,k} x^k$. Harer and Zagier [4] obtained the following result.

Theorem 1.1. (Harer and Zagier [4]) For a positive, even integer $p$, with $n = \frac{1}{2}p$,

$$A_p(x) = (2n-1)!! \sum_{k \geq 1} 2^{k-1} \binom{n}{k-1} \binom{x}{k}.$$ 

Other proofs of Theorem 1.1 have been given by Itzykson and Zuber [5], Jackson [6], Kerov [7], Kontsevich [8], Lass [10], Penner [12] and Zagier [15] (see also the survey by Zvonkin [16], Section 3.2.7 of Lando and Zvonkin [9] and the discussion in Section 4 of the paper by Haagerup and Thorbjørnsen [2]). Recently, Goulden and Nica [8] gave a direct bijective proof of Theorem 1.1. In the present paper, we consider a similar bijective approach to extend this important result of Harer and Zagier to the case in which the permutation $\gamma_p$ is replaced by a fixed permutation with two cycles in its disjoint cycle representation. Some additional notation is required.

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Let \([q]' = \{1', \ldots, q'\}\), and let \(S_{p,q}\) be the set of permutations of \([p] \cup [q]'\), for \(p, q \geq 0\). Let \(P_{p,q}\) be the set of pairings on \([p] \cup [q]'\), for \(p, q \geq 0\), where \(p + q\) is even (we refer to the single element of \(P_{0,0}\) as the empty pairing). A pair in a pairing is called mixed if it consists of one element from \([p]\) and one element from \([q]'\). Where the context is appropriate, we shall also regard \(P_{p,q}\) as the conjugacy class of involutions with no fixed points in \(S_{p,q}\). For \(p, q \geq 1\), we consider the permutation \(\gamma_{p,q} = (1 2 \ldots p)(1' 2' \ldots q')\), and let 
\[A_{p,q}^{(s)}(s) = \{\mu \gamma_{p,q} : \mu \in P_{p,q} \text{ has } s \text{ mixed pairs}\},\]
and \(a_{p,q,k}^{(s)}\) be the number of permutations in \(A_{p,q}^{(s)}\) with \(k\) cycles in the disjoint cycle representation, for \(k \geq 1\). Consider the generating series 
\[A_{p,q}^{(s)}(x) = \sum_{k \geq 1} a_{p,q,k}^{(s)} x^k.\]
The main result of this paper is the following expression for \(A_{p,q}^{(s)}(x)\).

**Theorem 1.2.** For \(p, q, s \geq 1\), with \(p, q, s\) of the same odd-even parity and \(n = \frac{1}{2}(p + q)\), we have 
\[A_{p,q}^{(s)}(x) = plq! \sum_{k=1}^{n+1} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \sum_{j=0}^{n+j} \frac{(n - s - j)!}{j!} \Delta_{k,p,q}.\]
where 
\[\Delta_{k,p,q} = \left( k - 1 \right) \left( \frac{1}{2} (p - s) - i \right) \left( \frac{1}{2} (q - s) - j \right) - \left( k - 1 \right) \left( \frac{1}{2} (p + s) - i \right) \left( \frac{1}{2} (q + s) - j \right).\]

Note that Theorem 1.2 gives a summation of nonnegative terms, since for all choices of summation indices \(k, i, j\) with \(k - 1 \leq n - i - j\) (so that \(\binom{n - i - j}{k - 1}\) is nonzero), the difference \(\Delta_{k,p,q}^{(s)}\) is nonnegative. The proof of Theorem 1.2 is based on a combinatorial model that is developed in Section 2. As a consequence, it is sufficient to enumerate a particular graphical object that we call a paired array. We then give two combinatorial reductions, in Sections 3 and 5, in terms of a simpler class of paired arrays called vertical paired arrays. These are explicitly enumerated in Section 6, which allows us to complete the proof of Theorem 1.2. One of the combinatorial conditions on paired arrays is that two graphs associated with them must be acyclic. Because of this, a key component of Sections 5 and 6 is the enumeration of rooted forests which contain a given forest as a subgraph. Thus in Section 4 we give a new bijection for this fundamental combinatorial problem. However, before we turn to our combinatorial model and subsequent reductions, we consider some consequences of Theorem 1.2 and give some comparisons to results in the existing literature.

A major reason that Harer and Zagier’s result (Theorem 1.1) is important (as evidenced by so many published proofs) is that it can be restated as an equivalent geometric problem in terms of maps. A map is an embedding of a connected graph (with loops and multiple edges allowed) in an orientable surface, partitioning the surface into disjoint regions (called the faces of the map) that are homeomorphic to discs (this is called a two-cell embedding). A rooted map is a map with a distinguished edge and incident vertex (so, the map is “rooted” at that end of the distinguished edge). The well-known embedding theorem allows us to consider this as equivalent to a pair of permutations and their product (see, e.g., Tutte [11], where the terminology “rotation system” is used to describe this triple of permutations). From this point of view, the \(k\)th coefficient \(a_{p,k}\) in the generating series \(A_p(x)\) evaluated in Theorem 1.1 is equal to the number of rooted maps with 1 vertex, \(n\) edges and \(k\) faces (where \(n = \frac{1}{2} p\), as in Theorem 1.1). Denoting the genus of the surface in which such a map is embedded by \(g\), then the Euler-Poincaré Theorem implies that \(1 - n + k = 2 - 2g\), or that \(k = n - 2g + 1\).

Similarly, Theorem 1.2 has a geometric interpretation. Let \(C_{p,q}\) be the conjugacy class of \(S_{p+q}\) in which there are two disjoint cycles, of lengths \(p\) and \(q\). Then the coefficient \(a_{p,q,k}^{(s)}\) in the generating series \(A_{p,q}^{(s)}(x)\) is equal to \(2(n - 1)!/[C_{p,q}]\) times the number of rooted maps with 2 vertices (of degrees \(p\) and \(q\)), \(n\) edges (exactly \(s\) of which join the two vertices together, plus \(\frac{1}{2}(p - s)\) that are loops at the vertex of degree \(p\), plus \(\frac{1}{2}(q - s)\) that are loops at the vertex of degree \(q\)), and \(k\) faces (where \(n = \frac{1}{2}(p + q)\), as in Theorem 1.2). In this case, if we denote the genus of the surface in which such a map is embedded by \(g\), then we obtain \(k = n - 2g\).

Of course, since genus is a nonnegative integer, we must have \(a_{p,q,n+1}^{(s)} = 0\), and indeed the coefficient of \(x^{n+1}\) in the summation for \(A_{p,q}^{(s)}(x)\) given in Theorem 1.2 is zero, since the summand corresponding to
Theorem 1.3. Our notation (by applying the proportionality constant \(2^{n} s^{T}\) to compare this result to our main result, we must sum over

\[ \sum_{k=0}^{n} A_{p,q}^{(s)}(x) \]

in which the total number of edges is specified, but not the exact number joining the two vertices together.

This explains the term “genus” in the title; the term “annular” is adapted from its usage in Mingo and Nica [11]. It refers to an equivalent embedding for a map with two vertices, in an annulus. The ends of the edges incident with one of the vertices (say the one of degree \(p\)) are identified with \(p\) points arranged around the disc on the exterior of the annulus, and the ends incident with the other vertex are identified with \(q\) points arranged around the disc on the interior of the annulus. The points corresponding to the two ends of an edge are joined by an arc in the interior of the annulus.

We have been able to find one relevant enumerative result (Jackson [6]) in the literature about such maps, in which the total number of edges is specified, but not the exact number joining the two vertices together. To compare this result to our main result, we must sum over \(s \geq 1\) (since the underlying graph must be connected, then \(s\), the number of edges joining the two vertices together, must be positive), and thus define

\[ A_{p,q}(x) = \sum_{s \geq 1} A_{p,q}^{(s)}(x). \]

Then Jackson [6] has considered the case \(p = q = n\), and obtained the following result, restated in terms of our notation (by applying the proportionality constant \((2n-1)!/|C_{n,n}| = n\).

**Theorem 1.3. (Jackson [6])** For \(n \geq 1\),

\[ A_{n,n}(x) = n! \sum_{j=0}^{1} \frac{1}{2^{(n-1)}} \sum_{i=0}^{n-2j-1} \frac{1}{2^{(n-2j-1)}} \sum_{k=0}^{} 4^{-k} \binom{n}{2k} \binom{2j}{n-2j-1} \binom{n-j-1}{i} \binom{x+j+i}{n}. \]

By slightly modifying Jackson’s [6] integration argument we are able to obtain the following expression for \(A_{p,q}(x)\), with arbitrary \(p,q\) of the same parity.

**Theorem 1.4.** For \(1 \leq p \leq q\), with \(p+q\) even, and \(n = \frac{1}{2}(p+q)\),

\[ A_{p,q}(x) = p!q! \sum_{j=0}^{1} \frac{1}{2^{(p-1)}} \sum_{i=0}^{n-2j-1} \frac{1}{2^{(p-2j-1)}} \sum_{k=0}^{2^{n-p+2k}k!(p-2k)!(n-p+k)!} \binom{2j}{n-2j-1} \binom{x+j+i}{n}. \]

We have checked computationally, with the help of Maple, that Theorems [1.3] and [1.4] agree with Theorem [1.2] summed over \(s \geq 1\), for a wide range of values of \(p,q\). However, we have been unable to prove this for all \(p,q\), since we have not been able to show that the sum over \(s \geq 1\) of the result of Theorem [1.2] is equal to the result of Theorem [1.4]. Note that the summation in Theorem [1.4] can be made symmetrical in \(p,q\) (so the ordering \(p \leq q\) is not required) by changing the summation variable \(k\) to \(m = p - 2k\).

The method employed in Jackson [6] for Theorem [1.3] and in many of the papers listed above that give proofs of Theorem [1.1] is matrix integration. However, we do not see how to adapt the matrix integration methodology to prove our main result, Theorem [1.2], since it doesn’t seem possible to specify that there are exactly \(s\) edges joining the two vertices together in the matrix method. The simplicity of our result seems to suggest that an extended theory of matrix integration to allow a specified number of edges between particular vertices might be possible, and worth investigating. The simplicity of the result also suggests that there should be a more direct combinatorial proof than the one presented in this paper.
2 The combinatorial model

2.1 Paired surjections

The combinatorial model for our proof of Theorem 1.2 is based on a paired surjection, which has the following definition.

Definition 2.1. For \( p, q, s, k \geq 1 \), with \( p, q, s \) of the same odd-even parity, let \( B^{(s)}_{p,q,k} \) be the set of ordered pairs \((\mu, \phi)\), where \( \mu \in \mathcal{P}_{p,q} \) has \( s \) mixed pairs, and \( \phi \) is a surjection from \([p] \cup [q]'\) onto \([k]\), satisfying the condition

\[
\phi(\mu(i)) = \phi(\gamma_{p,q}(i)) \quad \text{for all} \quad i \in [p] \cup [q]' .
\]

Such an ordered pair \((\mu, \phi)\) is called a paired surjection. Let \( b^{(s)}_{p,q,k} = |B^{(s)}_{p,q,k}| \).

In the following result, the generating series \( A^{(s)}_{p,q}(x) \) evaluated in Theorem 1.2 is expressed in terms of the numbers \( b^{(s)}_{p,q,k} \) of paired surjections. Paired surjections are closely related to shift-symmetric partitions, that arose in Goulden and Nica [3]. Indeed, the proof of the following result is identical to the proof of Proposition 1.3 in Goulden and Nica [3], and is hence omitted.

Proposition 2.2. For \( p, q, s \geq 1 \), with \( p, q, s \) of the same odd-even parity, we have

\[
A^{(s)}_{p,q}(x) = \sum_{k \geq 1} b^{(s)}_{p,q,k} \left( \frac{x}{k} \right) .
\]

We consider \((\mu, \phi) \in B^{(s)}_{p,q,k}\) and construct various objects associated with \((\mu, \phi)\). First let \( C_i = \phi^{-1}(i) \cap [p] \) and \( C'_i = \phi^{-1}(i) \cap [q]' \), for \( i \in [k] \). Let \( D = \{i : |C_i| \geq 1\} \), and \( D' = \{i : |C'_i| \geq 1\} \), and let \( m_i = \max C_i \), \( i \in D \), and \( m'_i = \max C'_i \), \( i \in D' \). Suppose that 1 is contained in \( C_a \), and that 1' is contained in \( C'_a \). Define \( \psi : D \setminus \{a\} \rightarrow D \) by \( \psi(i) = j \) when \( \phi(\mu(m_i)) = j \), and \( \psi' : D' \setminus \{b\} \rightarrow D' \) by \( \psi'(i) = j \) when \( \phi(\mu(m'_i)) = j \).

Now, if \( \psi(i) = j \), then (interpreting 1 as \( p+1 \) condition) means that \( m_i + 1 \in C_j \), so we have \( m_i < m_j \). This implies that the functional digraph of \( \psi \) (the directed graph on vertex-set \( D \) with an arc directed from \( i \) to \( \psi(i) \) for each \( i \in D \setminus \{a\} \)) is actually a tree, in which all arcs are directed towards vertex \( a \) (which we consider as the root of this tree). We denote this rooted tree by \( T \). Similarly, the functional digraph of \( \psi' \), on vertex-set \( D' \), is also a tree, with all arcs directed towards vertex \( b \) (which we consider as the root of this tree). We denote this rooted tree by \( T' \).

One condition that the paired surjection \((\mu, \phi)\) satisfies is that the number of mixed pairs containing an element of \( C_i \) is equal to the number of mixed pairs containing an element of \( C'_i \) for all \( i \in [k] \). (For the reason that this necessary condition arises, see the discussion of “unique label recovery” in the next section.) We call this the balance condition for \((\mu, \phi)\). The fact that \( \phi \) is a surjection is equivalent to \(|C_i| + |C'_i| \geq 1\), for \( i \in [k] \), and we call this the nonempty condition for \((\mu, \phi)\). The fact, established above, that the graphs of \( \psi \) and \( \psi' \) are trees is called the tree condition for \((\mu, \phi)\).

2.2 A graphical model

Now we consider a graphical representation for the paired surjection \((\mu, \phi)\), called its labelled paired array. This is an array of cells, arranged in \( k \) columns, indexed 1, \( \ldots, k \) from left to right, and two rows. In column \( i \) of row 1, place an ordered list of \(|C_i|\) vertices, labelled by the elements of \( C_i \) from left to right; in column \( i \) of row 2, place an ordered list of \(|C'_i|\) vertices, labelled by the elements of \( C'_i \) from left to right. For each pair of \( \mu \) draw an edge between the vertices whose labels are given by the pair.

For example, when \( p = 11, q = 9, s = 5, k = 4 \), consider \((\mu, \phi) \in B^{(s)}_{p,q,k}\), given by \( \mu = \{1, 9\}, \{5, 8\}, \{6, 7\}, \{2, 3\}, \{7, 8\}, \{2, 4\}, \{3, 1\}, \{4, 9\}, \{10, 6\}, \{11, 5\}\), and \( \phi^{-1}(1) = \{3, 6, 8, 2, 4\} \), \( \phi^{-1}(2) = \{3', 8', 9\} \), \( \phi^{-1}(3) = \{1, 2, 5, 9, 10, 5', 7', 9'\} \), \( \phi^{-1}(4) = \{4, 7, 11, 1', 6'\} \). The corresponding labelled paired array is given in Figure 1 and the trees \( T \) and \( T' \) are given in Figure 2.

Now suppose that we mark the cells in column a of row 1 and in column b of row 2 (by placing a small box in the top righthand corner of the marked cell in row 1, and in the bottom righthand corner of the marked cell in row 2), and remove the labels from all vertices -- call the resulting object the paired array of \((\mu, \phi)\).
The ordered list of vertices in each cell is now to be interpreted as a generic totally ordered set, with the given left to right order, and the pairing $\mu$ now acts on these ordered sets in the obvious way. For example, the paired array determined from the labelled paired array displayed in Figure 1 is given in Figure 3.

What information have we lost when the labels are removed? The answer, perhaps surprisingly, is that no information is lost, since we have unique label recovery by applying condition (1) iteratively, as follows: for the first row, place label 1 on the leftmost vertex in the marked cell of row 1; for each $i$ from 2 to $p$, place label $i$ on the leftmost unlabelled vertex in column $\varphi(\mu(i - 1))$ of row 1. The same process applied to the second row will place labels $1'$ to $q'$ on the vertices in row 2. (The reader can apply this to the paired array in Figure 3 to check that indeed the labelled paired array in Figure 1 is recovered in this way.) The proof that this process always works for a paired array satisfying the balance, nonempty and tree conditions (and the proof that these conditions are necessary for this process to work) requires only a slight modification of the results in Section 3 of [3], and is not given here (note that neither the functions $\psi$ and $\psi'$, nor the trees $T$ and $T'$, depend on the labels of the vertices, and the number of mixed pairs incident with the vertices in...
each cell of the paired array also does not depend on the labels, so the balance, nonempty and tree conditions can be checked on the paired array alone).

2.3 Paired arrays

This motivates us to define a \textit{paired array} in the abstract (and not as obtained by removing the labels from a labelled paired array), and in fact to extend it to a more general class of objects, in the following definition.

\textbf{Definition 2.3.} For \( p, q, s, k \geq 1 \), with \( p, q, s \) of the same odd-even parity, we define \( \mathcal{PA}^{(s)}_{p,q,k} \) to be the set of arrays of cells, arranged in \( k \) columns and 2 rows, subject to the following conditions:

- Each cell contains an ordered list of vertices, so that there is a total of \( p \) vertices in the first row, and \( q \) vertices in the second row. The vertices are paired (in the language of graph theory, there is a perfect matching on the vertices), so that \( s \) pairs join a vertex in the first row to a vertex in the second row (these are the mixed pairs). The number of mixed pairs containing a vertex in column \( i \) of row 1 is equal to the number of mixed pairs containing a vertex in column \( i \) of row 2, for all \( i = 1, \ldots, k \) (this is called the balance condition).

- There is at least one marked (with a small box) cell in row 1, and we denote the set of such columns by \( R \). There is at least one marked (with a small box) cell in row 2, and we denote the set of such columns by \( R' \). There is at least one vertex in every column that is not contained in \( R \cup R' \) (this is called the nonempty condition).

- Denote the set of columns in which there is at least one vertex in row 1 by \( D \), and the set of columns in which there is at least one vertex in row 2 by \( D' \). Define the function \( \psi : D \setminus R \rightarrow D \) as follows: if the rightmost vertex in column \( i \) of row 1 is paired with a vertex in column \( j \), then \( \psi(i) = j \). Similarly, define \( \psi' : D' \setminus R' \rightarrow D' \) as follows: if the rightmost vertex in column \( i \) of row 2 is paired with a vertex in column \( j \), then \( \psi'(i) = j \). The functional digraph of \( \psi \) is a forest with \( |R| \) components (called the rightmost forest for row 1); each component is a tree in which all edges are directed towards an element of \( R \) (and this is called the root of that tree). The functional digraph of \( \psi' \) is a forest with \( |R'| \) components (called the rightmost forest for row 2); each component is a tree in which all edges are directed towards an element of \( R' \) (and this is called the root of that tree). Together, these specify the forest condition.

The elements of \( \mathcal{PA}^{(s)}_{p,q,k} \) are called paired arrays. A paired array is defined to be canonical if \( |R| = |R'| = 1 \). Define \( c^{(s)}_{p,q,k} \) to be the set of canonical paired arrays in \( \mathcal{PA}^{(s)}_{p,q,k} \), and \( c^{(s)}_{p,q,k} = |c^{(s)}_{p,q,k}| \).

The uniqueness of label recovery described in the previous section proves that there is a bijection (via labelled paired arrays) between the set \( \mathcal{B}^{(s)}_{p,q,k} \) of paired surjections and the set \( \mathcal{C}^{(s)}_{p,q,k} \) of canonical paired arrays, so we have

\[ b^{(s)}_{p,q,k} = c^{(s)}_{p,q,k}, \]

(It is straightforward to verify that the conditions for canonical paired arrays imply that every column is nonempty.) In this paper we shall determine \( b^{(s)}_{p,q,k} \), and hence the generating series \( A^{(s)}_{p,q}(x) \) via Proposition 2.2 by giving a combinatorial reduction for canonical paired arrays, thus directly determining \( c^{(s)}_{p,q,k} \).

3 Removing redundant pairs and minimal paired arrays

A \textit{redundant} pair in a paired array is a vertex pair that is not mixed, and does not contain a vertex that is rightmost in an unmarked cell. A \textit{minimal} paired array is a paired array without redundant pairs. We define \( \mathcal{M}^{(s)}_{p,q,k} \) to be the set of minimal, canonical paired arrays in \( \mathcal{PA}^{(s)}_{p,q,k} \), and \( m^{(s)}_{p,q,k} = |\mathcal{M}^{(s)}_{p,q,k}| \). In our next result, we remove redundant pairs from a canonical paired array, and thus show that the enumeration of canonical paired arrays can be reduced to the enumeration of minimal, canonical paired arrays.
Theorem 3.1. For \( p, q, s, k \geq 1 \), with \( p, q, s \) of the same odd-even parity, we have

\[
C_{p,q,k}^{(s)} = \sum_{i,j \geq 0} \binom{p}{2i} (2i - 1)!! \binom{q}{2j} (2j - 1)!! m_{p-2i,q-2j,k}^{(s)}.
\]

Proof. For the proof, it is convenient to introduce some notation. A partial pairing on \([p]\) is a pairing on a set \( \alpha \subseteq [p] \) of even cardinality. If \(|\alpha| = 2i\), then we also call it an \( i \)-partial pairing. For each of these partial pairings \( \mu \), we call \( \alpha \) the support, and denote this by \( \text{supp}(\mu) = \alpha \). Let \( R_{p,i} \) be the set of \( i \)-partial pairings on \([p]\). Similarly, let \( R_{q,i} \) be the set of \( i \)-partial pairings on \([q]\).

Consider an arbitrary \( \alpha \in C_{p,q,k}^{(s)} \). We now describe a construction for three objects, \( \mu_1, \mu_2 \) and \( \beta \), obtained from \( \alpha \). We begin by attaching the numbers \( 1, \ldots, p+1 \) to the vertices and the small box in row 1 of \( \alpha \), from left to right (under the interpretation that all vertices in column \( i \) are to the left of all vertices in column \( j \) for \( i < j \), and that the small box representing a marking is rightmost in its cell). Let \( \mu_1 \) be the partial pairing consisting of pairs of numbers attached to the redundant pairs in row 1 of \( \alpha \). We follow the analogous procedure for row 2: we attach primed numbers \( 1', \ldots, (q + 1)' \) to the vertices and small box in row 2, and let \( \mu_2 \) be the partial pairing consisting of the pairs of (primed) numbers attached to the redundant pairs in row 2 of \( \alpha \). Third, we remove all redundant pairs (both vertices and edges) from \( \alpha \), to get the paired array \( \beta \), with the same marked cells as \( \alpha \). The vertices in each cell of \( \beta \) have the same relative order as they did in \( \alpha \). For example, if \( \alpha \) is the paired array in Figure 3, then we have \( \mu_1 = \{\{2,11\},\{4,7\}\} \), \( \mu_2 = \{\{1',3'\}\}, \) and \( \beta \) is given in Figure 4.

![Figure 4: A minimal paired array.](image-url)

Now, the only vertex that can be numbered \( p + 1 \) in row 1 is the rightmost vertex of the rightmost nonempty cell in row 1 (if this cell is not marked), but this vertex cannot appear in a redundant pair, since it is rightmost in an unmarked cell. This implies that the numbers on redundant pairs in row 1 all fall in the range \( 1, \ldots, p \), and so \( \mu_1 \) is a partial pairing on \([p]\). Similarly, \( \mu_2 \) is a partial pairing on \([q]\). Also, since the redundant pairs that were removed in the construction do not involve the rightmost vertex in any nonempty cell, \( \beta \) has the same rightmost functions \( \psi \) and \( \psi' \) as \( \alpha \), and the same mixed pairs as \( \alpha \), so it must satisfy the balance, nonempty and tree conditions, which implies that \( \beta \) is a minimal paired array. Thus we have a mapping

\[
\xi : C_{p,q,k}^{(s)} \rightarrow \bigcup_{i,j \geq 0} R_{p,i} \times R_{q,j} \times M_{p-2i,q-2j,k}^{(s)} : \alpha \mapsto (\mu_1, \mu_2, \beta).
\]

We now prove that \( \xi \) is a bijection. It is sufficient to describe the inverse mapping, so that we can uniquely recover \( \alpha \) from an arbitrary triple \( (\mu_1, \mu_2, \beta) \in \bigcup_{i,j \geq 0} R_{p,i} \times R_{q,j} \times M_{p-2i,q-2j,k}^{(s)} \). Given \((\mu_1, \mu_2, \beta)\), let \( \sigma_i = \text{supp}(\mu_i), i = 1, 2, \) and \( \rho_1 = [p+1] \setminus \sigma_1, \rho_2 = [q+1] \setminus \sigma_2 \). Number the vertices and small box in row 1 of \( \beta \) with the elements of \( \rho_1 \). Then insert vertices numbered with the elements of \( \sigma_1 \), so that the numbers on all vertices and the small box in row 1 increase from left to right, and so that the vertex numbered with \( l \in \sigma_1 \) is placed in the same cell as either the vertex or small box numbered with \( \min\{i \in \rho_1 : i > l\} \). Inserting vertices numbered from \( \sigma_2 \) in row 2 using an analogous process, pairing the inserted vertices with \( \mu_1 \) and \( \mu_2 \), and removing the numbers, we arrive at a paired array \( \alpha \). It is straightforward to check that \( \alpha \) satisfies the balance, nonempty, and tree conditions, and that the process described above reverses the numbering scheme used in
4 A bijection for rooted forests

In this section, we detour to consider the basic combinatorial question of how many rooted forests with a given set of root vertices contain a given rooted forest. We give a bijection for this that differs from the standard ones in the literature, like the Cycle Lemma (see, e.g., [13], p. 67) or the Prüfer Code (see, e.g., [13], p. 25), because it is more convenient for our constructions involving paired arrays.

Suppose we have a rooted forest $F$ (all edges directed towards a root vertex in each component) on vertex-set $[k]$, whose components are the rooted trees $T_1, \ldots, T_{m+n}$, $m, n \geq 1$. Suppose the root vertex of $T_j$ is $r_j$, $j = 1, \ldots, m$, and the root vertex of $T_{m+j}$ is $s_j$, $j = 1, \ldots, n$. For convenience, we order the trees so that $r_1 < \cdots < r_m$, $s_1 < \cdots < s_n$, and we let $S$ denote the union of the sets of vertices in the trees $T_{m+1}, \ldots, T_{m+n}$.

**Theorem 4.1 (Forest Completion Theorem).** There is a bijection between $[k]^{m-1} \times S$ and the set of rooted forests on vertex-set $[k]$ with root vertices $s_1, \ldots, s_n$ that contain $F$ as a subforest.

**Proof.** We describe such a mapping, which we call the “Forest Completion Algorithm” (FCA). Consider the $m$-tuple $a = (a_1, \ldots, a_m) \in [k]^{m-1} \times S$. We construct the forest corresponding to $a$ iteratively, in $m+1$ stages $0, 1, \ldots, m$. At each stage, we have a forest $G$ containing $F$ as a subforest, a permutation $\pi$ of $[m]$, and a sequence $b = (b_1, \ldots, b_m)$ in $[k]^m$. Initially, at stage 0, we have $G = F$, $\pi$ is the identity permutation and $b = a$. Then, for $i = 1, \ldots, m$:

- if $b_i$ is in a different component of $G$ from $r_i$, then add an arc directed from $r_i$ to $b_i$ in $G$, and leave $\pi$ and $b$ unchanged;
- otherwise (so $b_i$ is in the same component of $G$ as $r_i$), add an arc directed from $r_i$ to $b_m$ in $G$ (to obtain the new $G$), switch $\pi(i)$ and $\pi(m)$ in $\pi$, and switch $b_i$ and $b_m$ in $b$.

The forest corresponding to the $m$-tuple $a$ is the terminating forest $G$. We call the terminating permutation $\pi$ the “Forest Completion Permutation” (FCP). The significance of the FCP is that it identifies precisely the arcs that are added to $F$ — they are $(r_i, a_{\pi(i)})$, $i = 1, \ldots, m$. In our examples throughout the paper, we shall specify the second line in the two line representation of $\pi$ — the list of images $(\pi(1), \ldots, \pi(m))$.

In Figure 5, we give an example of the FCA with $k = 9$, $m = 3$, $n = 2$. The trees $T_1, T_2, T_3$, with $r_1 = 2$, $r_2 = 4$, $r_3 = 7$, are given in the box at the top left; the trees $T_4, T_5$, with $s_1 = 6$, $s_2 = 8$, are given in the box at the top right. Then, corresponding to the triple $a = (9, 2, 3)$, we construct the forest at the bottom of Figure 5. The corresponding FCP is $(3, 2, 1)$.

In analyzing this mapping, it is convenient to use the term “safe” to describe a vertex in a component of a forest rooted at one of the vertices $s_1, \ldots, s_n$. Thus, initially, $b_m$ is safe. It is trivial to prove by induction that, after stage $i$, for $i = 1, \ldots, m-1$, $G$ is a forest with root vertices $r_{i+1}, \ldots, r_m, s_1, \ldots, s_n$, and that $b_m$ is safe for $G$ (which implies that $r_{i+1}, \ldots, r_m$ are in different components of $G$ from $b_m$). Thus, at stage $m$, $b_m$ is indeed in a different component of $G$ from $r_m$, so we successfully add the final arc from $r_m$ to $b_m$, to obtain a terminating forest $G$ rooted at $s_1, \ldots, s_n$ (this explains our use of “safe” — for extending our forest, it is always safe to add the new arc directed to $b_m$). This proves that the FCA does indeed produce a rooted forest on vertex-set $[k]$ with root vertices $s_1, \ldots, s_n$, that contains $F$ as a subforest.

To prove that the FCA is a bijection, we describe its inverse. Suppose that we are given a rooted forest $H$ on vertex-set $[k]$, with root vertices $s_1, \ldots, s_n$, and that we wish to remove the arcs directed from $r_i$ to $c_i$, $i = 1, \ldots, m$, where $r_1, \ldots, r_m$ are distinct non-root vertices, with the convention that $r_1 < \cdots < r_m$. We proceed iteratively, through stages $m, \ldots, 0$. At every stage we have a subforest $G$ of $H$, a permutation $\sigma$ of $[m]$, and a sequence $b = (b_1, \ldots, b_m)$ in $[k]^m$. Initially, at stage $m$, we have $G = H$, $\sigma$ is the identity permutation, and $b = c$. Then, for $i = m-1, \ldots, 0$, remove the arc from $r_{i+1}$ to $b_{i+1}$ in $G$ (to get the new $G$), and:

- if $b_m$ is safe for (the new) $G$, leave $\sigma$ and $b$ unchanged;
Figure 5: A rooted forest and subforest.

- otherwise (so \( b_m \) is not safe for \( G \)), switch \( \sigma_{i+1} \) and \( \sigma_m \) in \( \sigma \), and switch \( b_{i+1} \) and \( b_m \) in \( b \).

We claim that the \( m \)-tuple corresponding to the forest \( H \) is the terminating \( m \)-tuple \( b \), so that this mapping uniquely reverses the FCA. In fact, it is easy to establish that this mapping uniquely reverses the FCA by stage, since it is trivial to prove by induction that the values of \( b \) and \( G \) after stage \( i \) of the above mapping are exactly the same as \( b \) and \( G \) after stage \( i \) of the FCA. It is also easy to prove that the FCP is given by \( \sigma^{-1} \) for the terminating \( \sigma \).

The result follows, since the FCA is a bijection between the required sets. \( \square \)

Of course, it is an immediate enumerative consequence of Theorem 4.1 that there are \( k^{m-1}|S| \) rooted forests on vertex-set \([k]\) with root vertices \( s_1, \ldots, s_n \) that contain \( F \) as a subforest. In the special case \( m + n = k \) (so that \( F \) has no edges) this gives the classical result that there are \( k^{k-n-1}n \) rooted forests on vertex-set \([k]\), with a prescribed set of \( n \) root vertices (see, e.g., [13], p. 25).

5 Removing non-mixed pairs and vertical paired arrays

A vertical paired array is a paired array in which all pairs are mixed. We define \( \mathcal{V}_{k,i,j}^{(s)} \) to be the set of vertical paired arrays in \( \mathcal{P}\mathcal{A}^{(s)}_{p,s,k} \), in which there are \( i + 1 \) marked cells in row 1 and \( j + 1 \) marked cells in row 2, for \( i, j \geq 0 \), and let \( v_{k,i,j}^{(s)} = |\mathcal{V}_{k,i,j}^{(s)}| \).

In our next result, we remove non-mixed pairs from a minimal, canonical paired array, and thus show that the enumeration of minimal, canonical paired arrays can be reduced to the enumeration of vertical paired arrays. We use the following notation. For a finite set \( X \), let \( \mathcal{L}_{X,i} \) denote the set of \( i \)-tuples consisting of \( i \) distinct elements of \( X \). Thus \( |\mathcal{L}_{X,i}| = (x)_i \), where \( x = |X| \) and \( (x)_i \) is the falling factorial: for positive integers \( i \), \( (x)_i = x(x-1) \cdots (x-i+1) \); for \( i = 0 \), \( (x)_i = 1 \); otherwise \( (x)_i = 0 \).

Theorem 5.1. For \( p, q, s \geq 1 \) of the same odd-even parity, let \( i = \frac{1}{2}(p-s) \) and \( j = \frac{1}{2}(q-s) \). Then

\[
m_{p,q,k}^{(s)} = (p)_i (q)_j v_{k,i,j}^{(s)}.
\]

Proof. Note that every element of \( \mathcal{M}_{p,q,k}^{(s)} \) has exactly \( i \) and \( j \) non-mixed pairs in the top and bottom rows, respectively. Taking an arbitrary \( \alpha \in \mathcal{M}_{p,q,k}^{(s)} \), we now describe a mapping that is initially identical to that used in the proof of Theorem 3.1. We attach the numbers 1, \ldots, \( p+1 \) to the vertices and the small box in row 1, using the same left to right convention as in the proof of Theorem 3.1. Let the pairs of numbers
attached to the non-mixed pairs in row 1 be denoted by \((u_1, v_1), \ldots, (u_i, v_i)\), where \(u_1, \ldots, u_i\) are attached to the rightmost vertices in these pairs (with \(u_1 < \cdots < u_i\)), and \(v_1, \ldots, v_i\) are attached to the other (not rightmost in their cell) vertices in these pairs.

Suppose that the marked cell in row 1 is in column \(m\), and that the rightmost tree for row 1 of \(\alpha\) is \(T\), so \(T\) is rooted at vertex \(m\). Now run the inverse of the FCA on \(T\), to remove the arcs directed from \(c(u_\ell)\) to \(c(v_\ell)\), \(\ell = 1, \ldots, i\), (here, \(c(\ell)\) denotes the column in which the number \(\ell\) appears) and let \(\rho\) be the corresponding FCP. Let \(\kappa_1 = (v_{\rho - 1(1)}, \ldots, v_{\rho - 1(i)})\).

We follow the analogous procedure for row 2: we attach primed numbers \(1', \ldots, (q + 1)'\) to the vertices and small box in row 2. Let the pairs of numbers attached to the non-mixed pairs in row 1 be denoted by \((x_1', y_1'), \ldots, (x_j', y_j')\), where \(x_1' < \cdots < x_j'\), and \(y_1', \ldots, y_j'\) are attached to the other (not rightmost in their cell) vertices in these pairs.

Suppose that the marked cell in row 2 is in column \(n\), and that the rightmost tree for row 2 of \(\alpha\) is \(T'\), so \(T'\) is rooted at vertex \(n\). Now run the inverse of the FCA on \(T'\), to remove the arcs directed from \(c(x_\ell')\) to \(c(y_\ell')\), \(\ell = 1, \ldots, i\), and let \(\tau\) be the corresponding FCP. Let \(\kappa_2 = (y_{\tau - 1(1)}, \ldots, y_{\tau - 1(j)})\).

Finally, we mark the cells in row 1 containing \(v_1, \ldots, v_i\) (in addition to the existing marked cell in column \(m\)), and the cells in row 2 containing \(y_1', \ldots, y_j'\) (in addition to the existing marked cell in column \(n\)), and then remove all non-mixed pairs (both vertices and edges) from \(\alpha\), to get the vertical paired array \(\beta\). The vertices in each cell of \(\beta\) have the same relative order as they did in \(\alpha\).

![Figure 6: A minimal paired array.](image6)

![Figure 7: The rightmost trees for Figure 6](image7)

For example, suppose that \(\alpha\) is the paired array in Figure 6. Then we have \(i = 3\), with \(u_1 = 2, u_2 = 5, u_3 = 10, v_1 = 4, v_2 = 8, v_3 = 3\), and \(j = 2\), with \(x_1' = 3, x_2 = 8', y_1' = 4', y_2' = 1'\). We have \(m = 3\), \(n = 4\), and the trees \(T\) and \(T'\) are given in Figure 7. When we run the inverse of the FCA on \(T\) to remove the arcs \((1, 2), (2, 4)\) and \((5, 2)\) (for which we use dashed lines in Figure 7, we obtain \((1, 3, 2)\) as the FCP, which gives \(\kappa_1 = (4, 3, 8)\). When we run the inverse of the FCA on \(T'\) to remove the arcs \((1, 3)\) and \((5, 1)\) (for which we use dashed lines in Figure 7), we obtain \((2, 1)\) as the FCP, which gives \(\kappa_2 = (1', 4')\). The vertical paired array \(\beta\) in this example is given in Figure 8.
Now, for the same reasons as in the proof of Theorem 3.1, we have \( 1 \leq v_{\ell} \leq p \) for \( \ell = 1, \ldots, i \), so \( \kappa_1 \in \mathcal{L}_{[p],i} \), and similarly, \( \kappa_2 \in \mathcal{L}_{[q]',j} \). Thus we have a mapping

\[
\zeta : \mathcal{M}_{p,q,k}^{(s)} \to \mathcal{L}_{[p],i} \times \mathcal{L}_{[q]',j} \times \mathcal{V}^{(s)}_{k,i,j} : \alpha \mapsto (\kappa_1, \kappa_2, \beta).
\]

In fact, \( \zeta \) is a bijection. Before proving this, for \( \zeta(\alpha) = (\kappa_1, \kappa_2, \beta) \), we first note the key dependencies between \( \alpha \) and \( (\kappa_1, \kappa_2, \beta) \) that follow from the FCA: let \( F \) and \( F' \) denote the rightmost forests of \( \beta \) for rows 1 and 2, respectively. Then, in \( \alpha \) with numbers attached as in the construction above, the column containing the vertex whose number is the last entry of \( \kappa_1 \) is contained in the component of \( F \) rooted at \( m \). Similarly, the column containing the vertex whose number is the last entry of \( \kappa_2 \) is contained in the component of \( F' \) rooted at \( n \). For example, for \( \alpha, \kappa_1, \kappa_2, \beta \) given in Figures 6 and 7, the last entries of \( \kappa_1 \) and \( \kappa_2 \) are 8 and 4, respectively, corresponding to vertices of \( \alpha \) in columns 4 and 3, respectively. Now, the forests \( F \) and \( F' \) in this case are obtained from the trees \( T \) and \( T' \), respectively, by removing the dashed edges in Figure 7. Then, indeed, vertex 4 is contained in the component of \( F \) rooted at \( m = 3 \), and vertex 3 is contained in the component of \( F' \) rooted at \( n = 4 \).

We now prove that \( \zeta \) is a bijection by describing the inverse mapping, so that we can uniquely recover \( \alpha \) from an arbitrary triple \( (\kappa_1, \kappa_2, \beta) \in \mathcal{L}_{[p],i} \times \mathcal{L}_{[q]',j} \times \mathcal{V}^{(s)}_{k,i,j} \). Let \( F \) denote the rightmost forest of \( \beta \) for row 1. Let \( \{\kappa_1\} \) denote the set consisting of the entries in \( \kappa_1 \), and let \( p = s + 2i, \delta_1 = [p + 1] \setminus \{\kappa_1\} \). Next (as a generalization of the procedure for the inverse of \( \xi \) described in the proof of Theorem 3.1), we number the vertices and small boxes in row 1 of \( \beta \) with the elements of \( \delta_1 \). Then insert vertices numbered with the elements of \( \{\kappa_1\} \), so that the numbers on all vertices and small boxes increase from left to right (with any small box regarded as the rightmost object in its cell) and so that the vertex numbered with \( l \in \{\kappa_1\} \) is placed in the same cell as the object numbered with \( \min\{t \in \delta_1 : t > l\} \). Now suppose that \( \kappa_1 = (w_1, \ldots, w_l) \), and use the key dependency noted above: let the column containing the vertex numbered \( w_i \) be contained in the component of \( F \) rooted at vertex \( m \). Let \( u_1 < \cdots < u_i \) denote the numbers attached to the small boxes that are not in column \( m \) (the columns containing \( u_1, \ldots, u_i \) are the root vertices for the components of \( F \) not rooted at \( m \)). Now, apply the FCA on \( i \)-tuple \( (c(w_1), \ldots, c(w_l)) \), to give the tree \( T \) rooted at \( m = c(w_1) \), that contains the forest \( F \) as a subforest. Let \( \rho \) be the FCP. Finally, replace the small boxes numbered \( u_1, \ldots, u_i \) by rightmost vertices (in the same cells) numbered \( u_1, \ldots, u_i \), pair the vertex numbered \( u_\ell \) with the vertex numbered \( w_{\rho(\ell)}, \ell = 1, \ldots, i \), and remove the numbers from row 1. Repeat the analogous process for row 2, and we arrive at a minimal paired array \( \alpha \). It is straightforward to check that the process described above reverses \( \zeta \). Thus, we have described \( \zeta^{-1} \), and our proof that \( \zeta \) is a bijection is complete. The result follows immediately.

6 Enumeration of vertical paired arrays

For every column of a vertical paired array, the cells in rows 1 and 2 have the same number of vertices, because of the balance condition. A full, vertical paired array is a vertical paired array with a positive number of
vertices in every column. Let \( F_{k,i,j}^{(s)} \) be the set of full, vertical paired arrays in \( \mathcal{V}_{k,i,j}^{(s)} \), and \( f_{k,i,j}^{(s)} = |F_{k,i,j}^{(s)}| \).

In Theorem 6.1 we shall give an explicit construction for the elements of \( F_{k,i,j}^{(s)} \), and thus obtain an explicit formula for \( f_{k,i,j}^{(s)} \). To help in the proof of this result, we first introduce some terminology and notation associated with an arbitrary \( \alpha \in F_{k,i,j}^{(s)} \). A vertex is said to be dependent if it is paired with the rightmost vertex of an unmarked cell (in the other row). If the rightmost vertex of an unmarked cell in row 1, column \( i \) is paired with the rightmost vertex of an unmarked cell in row 2, column \( j \), then we call this a shared pair of \( \alpha \). In this case, the rightmost forest \( F \) for row 1 of \( \alpha \) contains arc \((i,j)\) and the rightmost forest \( F' \) for row 2 of \( \alpha \) contains arc \((j,i)\), and we call each of these a shared arc.

Now, canonically number the vertices in row 1 of \( \alpha \) \( 1, \ldots, s \), from left to right, and number the vertices in row 2 of \( \alpha \) \( 1', \ldots, s' \), from left to right. Let \( E \) be the subforest of \( F \) with only the shared arcs of \( F \). Suppose that \( F \) has \( n \) non-shared arcs, corresponding to pairs \((x_1,y_1'), \ldots, (x_n,y'_n)\), where \( x_1 < \cdots < x_n \), and \( x_1, \ldots, x_n \) are rightmost vertices in their (unmarked) cells. Run the inverse of the FCA on the forest \( F \), to obtain the subforest \( E \) by removing the non-shared arcs directed from \( c(x_\ell) \) to \( c(y'_\ell) \), \( \ell = 1, \ldots, n \), and let \( \tau \) be the corresponding FCP. Define \( \alpha' = y_\ell' - 1(m) \). If all arcs of \( F \) are shared, then let \( \alpha' \) be the vertex in row 2 that is paired with the rightmost non-dependent vertex in row 1 (we call this the non-FCA option). In both cases, define \( A = c(\alpha') \), and let \( \rho_0 = A, \rho_1, \ldots, \rho_l \) be the vertices on the unique directed path in \( E \) from vertex \( A \) to the root vertex \( \rho_\ell \) of the component of \( E \) containing \( A \). Thus \( l \geq 0 \), and the cell in row 2, column \( \rho_0 \) is marked, and the cell in row 2, column \( \rho_l \) is not marked, \( \ell = 1, \ldots, l \). Also, the cell in row 1, column \( \rho_0 \) is not marked, \( \ell = 0, \ldots, l - 1 \). Now define \( E' \) to be the subforest of \( F' \) containing the arcs \((\rho_\ell, \rho_{\ell+1})\), \( \ell = 0, \ldots, l - 1 \). Suppose that \( F' \) has \( m \) arcs that are not in \( E' \), corresponding to pairs \((w_1', z_1), \ldots, (w_m', z_m)\), where \( w_1 < \cdots < w_m \), and \( w_1', \ldots, w_m' \) are rightmost vertices in their (unmarked) cells. Run the inverse of the FCA on the forest \( F' \), to obtain the subforest \( E' \) by removing the arcs directed from \( c(w_\ell') \) to \( c(z_\ell) \), \( \ell = 1, \ldots, m \), and let \( \kappa \) be the corresponding FCP. Define \( b = \sum_{\ell=1}^m (z_\ell - 1(m)) \). If \( F' = E' \), let \( b \) be the vertex in row 1 that is paired with the rightmost non-dependent vertex in row 2 (again, we call this the non-FCA option). Let \( \rho = (\rho_0, \rho_1, \ldots, \rho_l) \), which we call the tail of \( \alpha \). The tail length is \( l \). We say that \( b \) is in the tail when the column containing vertex \( b \) is one of \( \rho_0, \rho_1, \ldots, \rho_l \). The type of \( \alpha \) is given by \((l, \rho, \alpha', b)\). If the cells in rows 1 and 2 of column \( \ell \) in \( \alpha \) have \( \lambda_\ell \) vertices, \( \ell = 1, \ldots, k \), then we say that \( \alpha \) has shape \( \lambda = (\lambda_1, \ldots, \lambda_k) \). Note that \( \lambda_1 + \cdots + \lambda_k = s \), and that \( \lambda_\ell \) is positive for all \( \ell = 1, \ldots, k \), so \( \lambda \) is a composition of \( s \) with \( k \) parts.

For example, suppose that \( \alpha \) is the full, vertical paired array given at the top of Figure 5 with \( s = 10 \), \( k = 7 \), \( i = 1 \), \( j = 0 \), shape \((1, 2, 1, 1, 2, 2, 1)\), and rightmost forests \( F \) and \( F' \) given at the bottom of Figure 5. The lines joining the pairs in \( \alpha \) are of various types (and the same type of line is used in the rightmost forests when the pair corresponds to an edge in one or other of these forests): a thick solid line indicates a shared pair in the tail, a thick dashed line a shared pair not in the tail, a thick dashed and dotted line a pair contributing to \( F \) only, a thin solid line a pair contributing to \( F' \) only, and a thin dashed line a pair that contributes to neither of \( F, F' \). When we run the inverse of the FCA to remove the arcs \((6, 2)\) and \((7, 6)\) from \( F \), we obtain the ordered pair \((6, 2)\), and hence obtain \( \alpha' = 2' \) as indicated in Figure 5 contained in column 2. Thus the tail is \( \rho = (2, 1) \), of length \( l = 1 \). When we run the inverse of the FCA to remove the arcs \((2, 3), (3, 4), (4, 5), (5, 6)\) and \((7, 5)\) from \( F' \), we obtain the 5-tuple \((3, 4, 5, 5, 6)\), and hence obtain \( b = 8 \) as indicated in Figure 6 contained in column 6. Thus we conclude that \( \alpha \) has type \((1, (2, 1), 2', 8)\).

**Theorem 6.1.** For \( i, j \geq 0 \), \( k, s \geq 1 \), we have

\[
f_{k,i,j}^{(s)} = s! \sum_{l=0}^{k-1} \binom{s - 1 - l}{k - 1 - l} \binom{k - 1 - l}{i} \binom{k - 1 - l}{j}.
\]

**Proof.** Each paired array in \( F_{k,i,j}^{(s)} \) has a unique type and shape, and we can uniquely construct those of given type and shape as follows. For the given shape, we begin with a 2 by \( k \) array, with each cell containing an ordered set of vertices of prescribed size. Then we pair these vertices (all are mixed pairs) in all possible ways for the given type in four stages. First, we pair the rightmost vertices as prescribed by the tail. Second, we use the FCA to pair the rightmost vertices in row 2 (note that \( b \) is safe, by construction). Third, we use the FCA to pair the unpaired rightmost vertices in row 1 (note that \( \alpha' \) is safe, by construction). Fourth, we pair the remaining vertices arbitrarily. (In addition, along the way, we have to choose the marked cells in a
consistent fashion.) In this way, for each composition \( \lambda \) of \( s \) with \( k \) parts and \( l \geq 0 \), we enumerate elements of \( F^{(s)}_{k,i,j} \) with shape \( \lambda \) and tail of length \( l \). There are three cases:

**Case 1 (Vertex \( a' \) is not rightmost in its cell):** There are \( s - k \) choices for \( a' \), which then fixes \( \rho_0 \). There are then \( (k - 1)l \) choices for \( \rho_1, \ldots, \rho_l \), and then \( l \) of the pairs are determined. This leaves \( s - l \) choices for \( b \) (any vertex not yet paired in row 1). Now mark the cell in row 1, column \( \rho_l \). Also, if \( b \) is in the tail, mark the cell in row 2, column \( \rho_l \), or if \( b \) is not in the tail, mark the cell in row 2 of the column that contains \( b \) (\( b \) is in row 1). Choose, from the \( k - l - 1 \) cells not in the tail or already marked, \( i \) cells to mark in the top row, and \( j \) cells to mark in the bottom row. Now pair the unpaired rightmost vertices of unmarked cells in row 2 with vertices in row 1 to satisfy the forest condition, using vertex \( b \) as the safe position. There are \( (s - l - 1)_{k-j-l-2} \) possible choices for this, from the FCA. Suppose that there are \( n \) unmarked cells in row 1 whose rightmost vertices are not yet paired. Then pair these with vertices in row 2 to satisfy the forest condition, using vertex \( a' \) as the safe position. There are \( (s - k + j - n + 1)! \) possible choices for this, from the FCA. Finally, there are \( (s - k + j - n + 1)! \) ways to pair the remaining vertices, arbitrarily. But \( (s - k + j)_{n-1} \cdot (s - k + j - n + 1)! = (s - k + j)! \), and we conclude that the number of elements in \( F^{(s)}_{k,i,j} \) with shape \( \lambda \) and tail of length \( l \) in this case is

\[
(s - k)(k - 1)_i(s - l)(s - l - 1)_{k-j-l-2}(s - k + j)! \binom{k - l - 1}{i} \binom{k - l - 1}{j}.
\]

(3)

(It is straightforward the check that these cardinalities are correct when we use the non-FCA options also.)

**Case 2 (Vertex \( a' \) is rightmost in its cell, and \( b \) is in the tail):** The number of choices for \( b \) and \( \rho \) is \( s(s - 1)_{l-1}(l + 1)! - (l(k))_{l+1} \). Then \( a' \), and \( l \) of the pairs, are uniquely determined. The cell in row 1, column \( \rho_l \) is marked. The cell in row 2, column \( \rho_0 \) is marked for two reasons – because \( b \) is in the tail, and because the cell contains \( a' \), rightmost, so that \( a' \) will be paired with a non-dependent vertex in row 1. The rest of the enumeration proceeds as in Case 1, and we conclude that the number of elements in \( F^{(s)}_{k,i,j} \) with shape \( \lambda \)
and tail of length \( l \) in this case is
\[
\binom{k-1}{l} \binom{l+1}{i} (s-l-1)_{k+j-l-2} (s-k+j)! \binom{k-l-1}{i} \binom{k-l-2}{j}.
\] (4)

**Case 3 (Vertex \( a' \) is rightmost in its cell, and \( b \) is not in the tail):** The number of choices for \( b \) and \( \rho \) is \( s(k-1)_{l+1} \), and then \( a' \) together with \( l \) of the pairs are uniquely determined. In this case three different cells must now be marked: in row 1, column proceeds as in Cases 1 and 2. We conclude that the number of elements in \( F_{s,i,j}^{(k)} \) with shape \( \lambda \) and tail of length \( l \) in this case is
\[
\binom{k-1}{l} (s-l-1)_{k+j-l-2} (s-k+j)! \binom{k-l-1}{i} \binom{k-l-2}{j}.
\] (5)

Adding (3), (4), and (5), and simplifying, we obtain that the total number of elements in \( F_{s,i,j}^{(k)} \) with shape \( \lambda \) and tail of length \( l \) is
\[
s(k-1)_l (s-l-1)! \binom{k-l-1}{i} \binom{k-l-2}{j},
\]
and the result follows, by summing over \( l \geq 0 \) and multiplying by \( \binom{k-1}{l} \), the number of choices for \( \lambda \).

In the next result, we give an explicit enumeration for vertical paired arrays, by applying Theorem 6.1.

**Theorem 6.2.** For \( i, j \geq 0 \), \( k, s \geq 1 \), we have
\[
v_{k,i,j}^{(s)} = \frac{(s+i)! (s+j)!}{(s+i+j)!} \sum_{k=1}^{s+i+j} \binom{k-1}{i} \binom{k-1}{j} - \binom{k-1}{s+i} \binom{k-1}{s+j}
\]

**Proof.** If a column in a vertical paired array has no vertices, then at least one of the cells in rows 1 and 2 of that column must be marked, from the nonempty condition. Thus, suppose that a vertical paired array with \( s \) mixed pairs and \( k \) columns, has \( k-m \) columns with no vertices and \( m \) with a positive number of vertices (the same number in both rows of such columns). Of the \( k-m \) columns with no vertices, suppose that \( a \) are marked in row 1 only, \( b \) are marked in row 2 only, and that \( c-m \) are marked in both row 1 and 2 (we use this parameterization for convenience in determining the summations below). Then we have
\[
v_{k,i,j}^{(s)} = \sum_{a,b,c \geq 0}^{k!} \frac{k!}{a!b!c!} S_{a,b,c}
\]
where
\[
S_{a,b,c} = \sum_{m=0}^{s} \binom{c}{m} f_{m,i-a-c+m,j-b-c+m}^{(s)}
\]
But, from Theorem 6.1, we have
\[
S_{a,b,c} = \sum_{s \geq m - l \geq 0} \binom{c}{m} \frac{m-1}{m-l} \frac{m-l}{a-c-i-l} \frac{m-1}{b-c-j-l}
\]
\[
= \sum_{s \geq m - l \geq 0} \binom{c}{m} \frac{m-1}{m-l} \frac{m-l}{a-c-i-l} \frac{m-1}{b-c-j-l}
\]
where we use the notation \([A]B\) to denote the coefficient of \( A \) in the expansion of \( B \). Now
\[
\binom{c}{m} = \frac{(-m-1)}{c-m} (-1)^{c-m} = [x^c] x^m (1-x)^{-m-1},
\]

which gives

\[ S_{a,b,c} = s! [x^c] y^a z^b ] G(x, y, z), \tag{7} \]

where, summing over \( m \) by the binomial theorem, we have

\[
G(x, y, z) = \frac{x y^{i+1} z^{j+1}}{(1 - x)^2} \sum_{l=0}^{s-1} \left( \frac{xyz}{1 - x} \right)^l \left( 1 + \frac{x(1+y)(1+z)}{1 - x} \right)^{s-l-1}
\]

\[
= \frac{x y^{i+1} z^{j+1}}{(1 - x)^2} \left( 1 + \frac{x(1+y)(1+z)}{1 - x} \right)^{s-1} \left( \frac{xyz}{1 - x} \right)^s
\]

\[
= \frac{x y^{i+1} z^{j+1}}{(1 - x)^{s+1}} \left( 1 + x(y + z) \right)^s \left( \frac{xyz}{1 - x} \right)^s.
\]

But, changing variables in \((7)\), we have

\[
S_{a,b,c} = s! [x^0 y^a z^b ] x^k \frac{G(x, y, z)}{x^k},
\]

so from \((6)\) we obtain

\[
t_{k,i,j}^{(s)} = s! [x^0 y^k z^k ] x^k (1 + y + z)^k \frac{G(x, y, z)}{x^k}
\]

\[
= s! [x^0 y^k z^k ] x^k \left( \frac{1}{(1 - x)^{s+1}} \right) (x(1 + y + z) + yz)^s - (yz)^s
\]

\[
= R_1 - R_2,
\]

where

\[
R_2 = s! \sum_{m \geq 0} \binom{s}{m} \left( \frac{s + i + j + m - k + 1}{m} \right) \left( \frac{1}{(1 - x)^{s+1}} \right) (x(1 + y + z) + yz)^s
\]

\[
= s! \sum_{m \geq 0} \binom{s}{m} \left( \frac{s + i + j + m - k + 1}{m} \right) \left( \frac{1}{(1 - x)^{s+1}} \right) (x(1 + y + z) + yz)^s
\]

Now, for this latter sum over \( m \geq 0 \), we observe that the ratio of the \( m + 1 \)st term to the \( m \)th term is a rational function of \( m \), which implies that it is a hypergeometric sum. In particular, using the standard notation for hypergeometric series, we have

\[
R_1 = \frac{(s + k - 1)!}{(k - i - 1)!(k - j - 1)!(i + j - k + 1)!} \binom{s + i + j}{s} \binom{s + k - 1}{i} \binom{s}{j} \binom{3F_2}{i + 1 - k, j + 1 - k, -s}{i + j - k + 2, 1 - s - k, 1}
\]

where the second last equality follows from the Pfaff-Saalschütz Theorem for \( 3F_2 \) hypergeometric summations (see, e.g., Theorem 2.2.6 on page 69 of \([5]\)). The result follows immediately.
We finish with the proof of our main result, now that we have completed all the intermediate results in our reduction.

**Proof of Theorem 1.2.** From Proposition 2.2, (2) and Theorems 3.1, 5.1, we obtain

\[ A^{(s)}_{p,q}(x) = \sum_{k \geq 1} \binom{x}{k} \sum_{i,j \geq 0} \binom{p}{2i}(2i-1)!! \binom{q}{2j}(2j-1)!! (p-2i)_{\frac{1}{2}(p-s-2i)} (q-2j)_{\frac{1}{2}(q-s-2j)} (p-2i)_{\frac{1}{2}(p-s-2i)} (q-2j)_{\frac{1}{2}(q-s-2j)}. \]

But, simplifying, we obtain

\[ \binom{p}{2i}(2i-1)!! \binom{q}{2j}(2j-1)!! (p-2i)_{\frac{1}{2}(p-s-2i)} (q-2j)_{\frac{1}{2}(q-s-2j)} = \frac{p!q!}{2^{i+j} i!j! (\frac{1}{2}(p+s)-i)! (\frac{1}{2}(q+s)-j)!}, \]

and the result follows from Theorem 6.2.

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