Support $\tau_n$-tilting pairs

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Abstract

We introduce the higher version of the notion of Adachi-Iyama-Reiten’s support $\tau$-tilting pairs, which is a generalization of maximal $\tau_n$-rigid pairs in the sense of Jacobsen-Jørgensen. Let $\mathcal{C}$ be an $(n+2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and an Opperman-Thomas cluster tilting object. We show that relative $n$-rigid objects in $\mathcal{C}$ are in bijection with $\tau_n$-rigid pairs in the $n$-abelian category $\mathcal{C}/\text{add}\Sigma^n T$, and relative maximal $n$-rigid objects in $\mathcal{C}$ are in bijection with support $\tau_n$-tilting pairs. We also show that relative $n$-self-perpendicular objects are in bijection with maximal $\tau_n$-rigid pairs. These results generalise the work for $\mathcal{C}$ being $2n$-Calabi-Yau by Jacobsen-Jørgensen and the work for $n = 1$ by Yang-Zhu.

Key words: $(n+2)$-angulated categories; cluster tilting objects; support $\tau_n$-tilting pairs; maximal $\tau_n$-rigid pairs; maximal $n$-rigid objects; $n$-self-perpendicular objects.

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1 Introduction

Higher homological algebra emerged from the higher Auslander-Reiten theory by Iyama in [I1, I2]. It has gained more and more attentions since the introduction of $(n+2)$-angulated categories by Geiss-Keller-Oppermann [GKO], and the introduction of $n$-abelian categories by Jasso [Ja]. The $n$-versions of triangulated or abelian categories appear as an attempt to get a better understanding of cluster tilting subcategories: some $n$-cluster tilting subcategories in triangulated categories are $(n+2)$-angulated categories and $n$-cluster tilting subcategories in abelian categories are $n$-abelian categories.

Cluster tilting objects (or subcategories) and cluster categories provide insight into cluster algebras and their related combinatorics. They have also been used to define a new kind of tilting theory, known as cluster tilting theory, which generalizes APR-tilting for hereditary algebras. Recall that the notion of cluster tilting object and related objects from [BMRRT] [KR] [KZ] [IY].

Definition 1.1. Let $\mathcal{C}$ be a triangulated category with a shift functor $\Sigma$.

(1) An object $T \in \mathcal{C}$ is called rigid if $\text{Hom}_\mathcal{C}(T, \Sigma T) = 0$.

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(2) An object \( T \in C \) is called **maximal rigid** if it is rigid and maximal with respect to the property:
\[
\text{add} \, T = \{ M \in C \mid \text{Hom}_C(T \oplus M, \Sigma(T \oplus M)) = 0 \}.
\]

(3) An object \( T \in C \) is called **cluster tilting** if
\[
\text{add} \, T = \{ M \in C \mid \text{Hom}_C(T, \Sigma M) = 0 \} = \{ M \in C \mid \text{Hom}_C(M, \Sigma T) = 0 \}.
\]

In fact, Koenig and Zhu [KZ, Lemma 3.2] show that \( T \) is cluster tilting if and only if \( \text{Hom}_C(T, \Sigma T) = 0 \) and for any object \( C \in C \), there exists a triangle \( T_0 \to T_1 \to C \to \Sigma T_0 \) where \( T_0, T_1 \in \text{add} \, T \).

Cluster tilting theory permitted to construct abelian categories from some triangulated categories. By Buan-Marsh-Reiten [BMR, Theorem 2.2] in cluster categories, by Keller-Reiten [KR, Proposition 2.1] in the 2-Calabi-Yau case, then by Koenig-Zhu [KZ, Theorem 3.3] and Iyama-Yoshino [IY, Corollary 6.5] in the general case, one can pass from triangulated categories to abelian categories by factoring out cluster tilting subcategories. This permits to link cluster tilting objects in triangulated categories with tilting modules in the abelian quotient categories.

In fact, Adachi, Iyama and Reiten [AIR] introduced the \( \tau \)-tilting theory for finite dimensional algebras. It is a generalization of classical tilting theory. They proved that for a 2-Calabi-Yau triangulated category \( C \) with a cluster tilting object \( T \), there exists a bijection between the basic cluster tilting objects in \( C \) and the basic support \( \tau \)-tilting pairs in \( \text{modEnd}_C(T) \). Note that each cluster tilting object is maximal rigid in a 2-Calabi-Yau triangulated category, but the converse is not true in general. Zhou and Zhu [ZhZ, Theorem 2.6] proved that if \( C \) is a 2-Calabi-Yau triangulated category with a cluster tilting object, then every maximal rigid object is cluster tilting. The bijection above in a 2-Calabi-Yau triangulated category was generalized to any triangulated category by Yang and Zhu [YZ]. They introduced the notion of relative cluster tilting objects in a triangulated category \( C \) with a cluster tilting object, which are a generalization of cluster tilting objects. Let \( C \) be a triangulated category with a cluster tilting object \( T \). They established a one-to-one correspondence between the basic relative cluster tilting objects in \( C \) and the basic support \( \tau \)-tilting pairs in \( \text{modEnd}_C(T) \). For more works on this line, please see [CZZ, FGL, IJY, LX, YZZ, ZZ2].

Now we recall some higher versions of the notions and results above.

In [GKO], Geiss, Keller and Oppermann introduced \((n + 2)\)-angulated categories. These are a higher dimensional analogue of triangulated categories, in the sense that triangles are replaced by \((n + 2)\)-angles, that is, morphism sequences of length \((n + 2)\). Thus a 1-angulated category is precisely a triangulated category. They appear for example as certain cluster tilting subcategories of triangulated categories.

The notion of cluster tilting objects can be generalised to \((n + 2)\)-angulated categories, it due to Oppermann and Thomas.

**Definition 1.2.** [OT, Definition 5.3] Let \( C \) be an \((n + 2)\)-angulated category with an \( n \)-suspension functor \( \Sigma^n \). An object \( T \in C \) is called **cluster tilting** if

1. \( \text{Hom}_C(T, \Sigma^n T) = 0 \).
(2) For any object $C \in \mathcal{C}$, there exists an $(n+2)$-angle

$$T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{n-1} \rightarrow T_n \rightarrow C \rightarrow \Sigma^n T_0$$

where $T_0, T_1, \cdots, T_n \in \text{add} T$.

Jacobsen and Jørgensen introduced the notions of maximal $n$-rigid objects and $n$-self-perpendicular objects.

**Definition 1.3.** [JJ2, Definitions 0.2 and 0.3] Let $\mathcal{C}$ be an $(n+2)$-angulated category with an $n$-suspension functor $\Sigma^n$.

1. An object $T \in \mathcal{C}$ is called $n$-rigid if $\text{Hom}_\mathcal{C}(T, \Sigma^n T) = 0$.
2. An object $T \in \mathcal{C}$ is called maximal $n$-rigid if it is rigid and maximal with respect to the property: $\text{add} T = \{ M \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(T \oplus M, \Sigma^n (T \oplus M)) = 0 \}$.
3. An object $T \in \mathcal{C}$ is called $n$-self-perpendicular object if

$$\text{add} T = \{ M \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(T, \Sigma^n M) = 0 \} = \{ M \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(M, \Sigma^n T) = 0 \}.$$

There are also an $n$-version of the result that the quotient categories of triangulated categories by cluster tilting objects are abelian.

**Theorem 1.4.** [JJ1, Theorem 0.5] and [ZZ1, Theorem 3.8] Let $\mathcal{C}$ be an $(n+2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and $T$ an Opperman-Thomas cluster tilting object with endomorphism algebra $\Lambda = \text{End}_\mathcal{C}(T)$. Consider the essential image $\mathcal{D}$ of the functor $C(T, -) : \mathcal{C} \rightarrow \text{mod} \Lambda$.

Then $\mathcal{D}$ is an $n$-cluster tilting subcategory of $\text{mod} \Lambda$. There exists a commutative diagram, as shown below, where the vertical arrow is the quotient functor and the diagonal arrow is an equivalence of categories:

$$\begin{array}{c}
\mathcal{C} \xrightarrow{C(T, -)} \mathcal{D} \\
\downarrow \vee \downarrow \\
\mathcal{C}/\text{add} \Sigma^n T.
\end{array}$$

The category $\mathcal{D}$ is an $n$-abelian category in the sense of Jasso by [Ja, Theorem 3.16]. By [I1, Theorem 2.3.1], we know that $\mathcal{D}$ has an $n$-Auslander-Reiten translation $\tau_n$, which is a higher analogue of the classic Auslander-Reiten translation $\tau$. It is natural to ask if $\mathcal{D}$ permits a higher analogue of the $\tau$-tilting theory of [AIR] for any $(n+2)$-angulated category.

In this article, we give an affirmative answer to this question based on the work of Jacobsen and Jørgensen [JJ2].

We introduce the notion of support $\tau_n$-tilting pairs which are a generalization of maximal $\tau_n$-rigid pairs. At the same time, we also introduce the concepts of relative $n$-rigid objects,
relative maximal $n$-rigid objects, relative $n$-self-perpendicular objects which can be regarded as the generalization of $n$-rigid objects, maximal $n$-rigid objects, $n$-self-perpendicular objects, respectively.

Our main result is the following.

**Theorem 1.5.** (see Theorem 3.7 for details) Let $\mathcal{C}$ be an $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and an Opperman-Thomas cluster tilting object. Then there are three bijections

\[
\begin{align*}
\text{isomorphism classes of} & \quad \text{isomorphism classes of} \\
\text{relative $n$-rigid objects in } \mathcal{C} & \quad \tau_n\text{-rigid pairs in } \mathcal{D} \\
\text{relative maximal $n$-rigid objects in } \mathcal{C} & \quad \text{support } \tau_n\text{-tilting pairs in } \mathcal{D} \\
\text{relative $n$-self-perpendicular objects in } \mathcal{C} & \quad \text{maximal } \tau_n\text{-rigid pairs in } \mathcal{D}.
\end{align*}
\]

When $n = 1$, this theorem covers [YZ, Theorem 1.2] and [YZZ, Theorem 1.1]. When $\mathcal{C}$ is an $2n$-Calabi-Yau $(n + 2)$-angulated category, we show the following consequence which is a completion and generalisation of a recent result of Jacobsen and Jørgensen [JJ2].

**Corollary 1.6.** Let $\mathcal{C}$ be an $2n$-Calabi-Yau $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and an Opperman-Thomas cluster tilting object. Then there are three bijections

\[
\begin{align*}
\text{isomorphism classes of} & \quad \text{isomorphism classes of} \\
n\text{-rigid objects in } \mathcal{C} & \quad \tau_n\text{-rigid pairs in } \mathcal{D} \\
\text{maximal } n\text{-rigid objects in } \mathcal{C} & \quad \text{support } \tau_n\text{-tilting pairs in } \mathcal{D} \\
n\text{-self-perpendicular objects in } \mathcal{C} & \quad \text{maximal } \tau_n\text{-rigid pairs in } \mathcal{D}.
\end{align*}
\]

This corollary is a generalisation of a recent result of Jacobsen and Jørgensen: See Theorem 4.1 in [AIR].

**Theorem 1.7.** [JJ2, Theorem B and Theorem 1.1] Let $\mathcal{C}$ be a $2n$-Calabi-Yau $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and an Opperman-Thomas cluster tilting object. If each indecomposable object of $\mathcal{C}$ is $n$-rigid, then

1. an object $X$ is $n$-self-perpendicular if and only if $X$ is maximal $n$-rigid;
2. there exists a bijection

\[
\text{isomorphism classes of} \quad \text{isomorphism classes of} \\
\text{maximal } n\text{-rigid objects in } \mathcal{C} \quad \text{maximal } \tau_n\text{-rigid pairs in } \mathcal{D}.
\]

This article is organized as follows. In Section 2, we review some elementary definitions and facts about $\tau$-tilting theory and $(n + 2)$-angulated categories. In Section 3, we prove our main result and give some applications.
2 Preliminaries

Let $\Lambda$ be a finite dimensional $k$-algebra and $\tau$ the Auslander-Reiten translation. We denote by $\text{proj} \Lambda$ is the category of finitely generated projective right $\Lambda$-modules. $|M|$ denotes the number of non-isomorphic indecomposable direct summands of $M$. Support $\tau$-tilting modules were introduced by Adachi, Iyama and Reiten \cite{AIR}, which can be regarded as a generalization of tilting modules.

**Definition 2.1.** Let $(M, P)$ be a pair with $M \in \text{mod} \Lambda$ and $P \in \text{proj} \Lambda$.

1. $M$ is called $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$.
2. $(M, P)$ is called a $\tau$-rigid pair if $M$ is $\tau$-rigid and $\text{Hom}_\Lambda(P, M) = 0$.
3. $(M, P)$ is called a support $\tau$-tilting pair if it is a $\tau$-rigid pair and $|M| + |P| = |\Lambda|$. In this case, $M$ is called a support $\tau$-tilting module.

Adachi, Iyama and Reiten gave some equivalent characterizations of support $\tau$-tilting pairs.

**Remark 2.2.** \cite[Corollary 2.13]{AIR} Let $(M, P)$ be a $\tau$-rigid pair. Then the following statements are equivalent:

- (a) $(M, P)$ is a support $\tau$-tilting pair.
- (b) If $(M \oplus N, P)$ is a $\tau$-rigid pair for any $N \in \text{mod} \Lambda$, then $N \in \text{add} M$.
- (c) If $\text{Hom}_\Lambda(M, \tau N) = 0, \text{Hom}_\Lambda(N, \tau M) = 0$ and $\text{Hom}_\Lambda(P, N) = 0$, then $N \in \text{add} M$.

We give the fact which will be used later.

**Lemma 2.3.** \cite[Proposition 2.2]{JK} Let $D$ be an $n$-cluster tilting subcategory of $\text{mod} \Lambda$.

- (a) If $0 \to L \to M_1 \to M_2 \to \cdots \to M_n$ is an exact sequence in $\text{mod} \Lambda$ whose terms all lie in $D$, then, for any $X \in D$ there exists the following exact sequence in $\text{mod} \Lambda$

  $$0 \to \text{Hom}_\Lambda(X, L) \to \text{Hom}_\Lambda(X, M_1) \to \text{Hom}_\Lambda(X, M_2) \to \cdots \to \text{Hom}_\Lambda(X, M_n).$$

- (b) If $M_1 \to M_2 \to \cdots \to M_n \to N \to 0$ is an exact sequence in $\text{mod} \Lambda$ whose terms all lie in $D$, then, for any $Y \in D$ there exists the following exact sequence in $\text{mod} \Lambda$

  $$0 \to \text{Hom}_\Lambda(N, Y) \to \text{Hom}_\Lambda(M_n, N) \to \text{Hom}_\Lambda(M_{n-1}, N) \to \cdots \to \text{Hom}_\Lambda(M_1, N).$$

Jasso \cite{Ja} introduced $n$-abelian categories which are categories inhabited by certain exact sequences with $n + 2$ terms, called $n$-exact sequences. The case $n = 1$ corresponds to the classical concepts of abelian categories. An important source of examples of $n$-abelian categories are $n$-cluster tilting subcategories of abelian categories. Hence it deals with $n$-cluster tilting subcategories of abelian categories. For example, Jørgensen introduced the notion of torsion classes in $n$-abelian categories, and established the bijection between intermediate aisles and
torsion classes in $n$-abelian categories associated to $n$-representation finite algebras, see [J] Theorem 8.5].

Recall that an $n$-abelian category in the sense of Jasso [Ja, Definition 3.1] is projectively generated if for every objects $M \in \mathcal{M}$ there exists a projective object $P \in \mathcal{M}$ and an epimorphism $f : P \to M$. Building on work of Jasso [Ja, Theorem 3.20], Kvamme [K, Theorem 1.3] proved that any projectively generated $n$-abelian category is equivalent to a $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$ with enough projectives.

We define a higher analogue of the $\tau$-tilting theory of [AIR] for any $n$-abelian category.

**Definition 2.4.** Let $\mathcal{M}$ be a projectively generated $n$-abelian category with an $n$-Auslander-Reiten translation $\tau_n$. By the discussion above, $\mathcal{M}$ is equivalent to a $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$ with enough projectives. We denote by $\mathcal{P}$ the full subcategory of projective objects in $\mathcal{A}$. Assume that $(\mathcal{M}, \mathcal{P})$ is a pair with $M \in \mathcal{M}$ and $P \in \mathcal{P}$.

1. An object $M \in \mathcal{M}$ is called $\tau_n$-rigid if $\text{Hom}_\mathcal{A}(M, \tau_n M) = 0$.
2. $(M, P)$ is called a $\tau_n$-rigid pair in $\mathcal{A}$ if $M$ is $\tau_n$-rigid and $\text{Hom}_\mathcal{A}(P, M) = 0$.
3. $(M, P)$ is called a maximal $\tau_n$-rigid pair in $\mathcal{A}$ if it satisfies:
   (i) If $N \in \mathcal{M}$, then
   $$N \in \text{add}M \iff \begin{cases} 
   \text{Hom}_\mathcal{A}(M, \tau_n N) = 0, \\
   \text{Hom}_\mathcal{A}(N, \tau_n M) = 0, \\
   \text{Hom}_\mathcal{A}(P, N) = 0.
   \end{cases}$$
   (ii) If $Q \in \mathcal{P}$, then
   $$Q \in \text{add}P \iff \text{Hom}_\mathcal{A}(Q, M) = 0.$$
4. A $\tau_n$-rigid pair $(M, P)$ is called support $\tau_n$-tilting pair if it satisfies
   (i) If $(M \oplus N, P)$ is a $\tau_n$-rigid pair for any $N \in \mathcal{M}$, then $N \in \text{add}M$.
   (ii) If $Q \in \mathcal{P}$, then
   $$Q \in \text{add}P \iff \text{Hom}_\mathcal{A}(Q, M) = 0.$$

**Remark 2.5.** It is obvious that a maximal $\tau_n$-rigid pair is a support $\tau_n$-tilting pair, and a support $\tau_n$-tilting pair is a $\tau_n$-rigid pair. It will be seen that all the reverse statements are not true in general, see explanation at the end of this article.

Let $k$ be a field and $n$ a positive integer. In what follows, we assume that $\mathcal{C}$ is a $k$-linear Hom-finite $(n + 2)$-angulated category with split idempotents. The $n$-suspension functor of $\mathcal{C}$ is denoted by $\Sigma^n$. We let $T$ be an Oppermann-Thomas cluster tilting object in $\mathcal{C}$ with endomorphism algebra $\Lambda = \text{End}_\mathcal{C}(T)$. We denote by $\mathcal{D}$ the essential image of the functor $\mathcal{C}(T, -) : \mathcal{C} \to \text{mod}\Lambda$, where $\text{mod}\Lambda$ is the category of finite dimensional right $\Lambda$-modules. For any object $M \in \mathcal{C}$, we denote by $\text{add}M$ the full subcategory of $\mathcal{C}$ consisting of direct summands of direct sum of finitely many copies of $M$.

**Remark 2.6.** Jacobsen-Jørgensen [JJ1, Theorem 0.5] and Zhou-Zhu [ZZ1, Theorem 3.8] show that $\mathcal{D}$ is a projectively generated $n$-abelian category with an $n$-Auslander-Reiten translation.
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Moreover, it is equivalent to a \( n \)-cluster tilting subcategory of a module category \( \text{mod} \Lambda \). In this case, the items (1), (2) and (3) in Definition 2.4 are precisely defined by Jacobsen-Jørgensen [JJ2, Definition 0.6 and Definition 0.7]. Note that when \( n = 1 \), by [AIR, Proposition 2.3] and Remark 2.2, we know that \((M, P)\) is a maximal \( \tau_1 \)-rigid pair if and only if it is a support \( \tau \)-tilting pair, and \((M, P)\) is a support \( \tau_1 \)-rigid pair if and only if it is a support \( \tau \)-tilting pair. Hence support \( \tau_n \)-tilting pairs can be viewed as higher support \( \tau \)-tilting pairs.

Remark 2.7. [JJ1, Lemma 2.1 and Lemma 2.2]

(1) The functor \( C(T, -) \) restricts to an equivalence \( \text{add} T \to \text{proj} \Lambda \).

(2) \( C(T', X) \cong \text{Hom}_\Lambda(C(T, T'), C(T, X)) \) for any \( T' \in \text{add} T \) and \( X \in C \).

The following observation is useful in the sequel.

Lemma 2.8. Let \( M, N \in D \) and

\[
P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0
\]

be a minimal projective resolution of \( M \). Then \( \text{Hom}_\Lambda(M, \tau_n N) = 0 \) if and only if the map

\[
\text{Hom}_\Lambda(P_{n-1}, M) \xrightarrow{\text{Hom}_\Lambda(d_n, N)} \text{Hom}_\Lambda(P_n, M)
\]

is surjective. In particular, \( M \) is \( \tau_n \)-rigid if and only if the map

\[
\text{Hom}_\Lambda(P_{n-1}, M) \xrightarrow{\text{Hom}_\Lambda(d_n, M)} \text{Hom}_\Lambda(P_n, M)
\]

is surjective.

Proof. Since \( P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0 \) is a minimal projective resolution of \( M \), by definition, there exists a complex

\[
0 \to \tau_n M \to D(P_n, \Lambda) \to D(P_{n-1}, \Lambda) \to \cdots \to D(P_1, \Lambda) \to D(P_0, \Lambda),
\]

where we omitted \( \text{Hom}_\Lambda \) because of lack of space. We claim that it is exact. Indeed, for all \( i \in \{1, 2, \cdots, n-1\} \) the cohomology of the above complex at \( D\text{Hom}_\Lambda(P^i, \Lambda) \) is isomorphic to \( D\text{Ext}_\Lambda^i(M, \Lambda) \) and hence vanishes since \( M \) and \( \Lambda \) belong to \( D \).

By Lemma 2.3 we have a commutative diagram of exact sequences.

\[
\begin{array}{ccccccccccc}
0 & \to & (M, \tau_n N) & \to & (N, D(P^n, \Lambda)) & \to & (N, D(P^{n-1}, \Lambda)) & \to & \cdots & \to & (N, D(P^0, \Lambda)) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & D(P_n, N) & \xrightarrow{D(d_n, N)} & D(P^{n-1}, N) & \to & \cdots & \to & D(P^0, N).
\end{array}
\]

Thus the assertion follows.

Lemma 2.9. [JJ2, Lemma 2.2] If \( M \in C \) has no non-zero direct summands in \( \text{add} \Sigma^n T \), then there exists an \( (n + 2) \)-angle

\[
T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} M \xrightarrow{d_{n+1}} \Sigma^n T_0
\]
where $T_0, T_1, \cdots, T_n \in \text{add} T$ and for each $i \in \{1, 2, \cdots, n - 1\}$ the morphism $d_i : T_i \rightarrow T_{i+1}$ is in the radical of $C$. Moreover, applying the functor $C(T, -)$ gives a complex

$$C(T, T_0) \rightarrow C(T, T_1) \rightarrow C(T, T_2) \rightarrow \cdots \rightarrow C(T, T_n) \rightarrow C(T, M) \rightarrow 0$$

which is the start of the minimal projective resolution of $C(T, M)$.

3 Relative maximal $n$-rigid objects and support $\tau_n$-tilting pairs

We first introduce the notion of relative maximal $n$-rigid objects in $C$, which are a generalization of maximal $n$-rigid objects. For an object $M$ in $C$, we use $[M](X, Y)$ to denote the subgroup of $\text{Hom}_C(X, Y)$ consisting of the morphisms from $X$ to $Y$ factoring through $\text{add} M$.

**Definition 3.1.** Let $C$ be an $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and an Opperman-Thomas cluster tilting object.

(i) An object $X$ in $C$ is called relative $n$-rigid if there exists an Opperman-Thomas cluster tilting object $T$ such that $[\Sigma^n T](X, \Sigma^n X) = 0$. In this case, $X$ is also called $\Sigma^n T$-n-rigid.

(ii) An object $X$ in $C$ is called relative maximal $n$-rigid if there exists an Opperman-Thomas cluster tilting object $T$ such that $X$ is $\Sigma^n T$-n-rigid and $[\Sigma^n T](X \oplus M, \Sigma^d (X \oplus M)) = 0$ implies $M \in \text{add} X$.

In this case, $X$ is also called maximal $\Sigma^n T$-n-rigid.

(iii) An object $X \in C$ is called relative $n$-self-perpendicular object if there exists an Opperman-Thomas cluster tilting object $T$ such that

$$\text{add} X = \{ M \in C \mid [\Sigma^n T](X, \Sigma^n M) = 0 = [\Sigma^n T](M, \Sigma^n X) \}.$$

In this case, $X$ is also called $\Sigma^n T$-n-self-perpendicular.

**Remark 3.2.** From the definition, we can immediately conclude that the following implications:

$$\Sigma^n T \text{-n-self-perpendicular objects} \downarrow \Sigma^n T \text{-n-rigid objects} \downarrow \text{maximal } \Sigma^n T \text{-n-rigid objects}$$

The implications cannot be reversed in general. See explanation at the end of this article.

**Remark 3.3.** Any $n$-rigid object in $C$ is relative $n$-rigid.

The following result indicates that $\Sigma^n T$-n-rigid objects and $n$-rigid objects coincide in some cases. We say that an $(n + 2)$-angulated category $C$ is $2n$-Calabi-Yau if

$$C(X, \Sigma^n Y) \simeq DC(Y, \Sigma^n X)$$
naturally in $X, Y \in \mathcal{C}$ where $D$ is the duality functor $\text{Hom}_k(-, k)$.

**Proposition 3.4.** If $\mathcal{C}$ is a $2n$-Calabi-Yau and $T$ is an Oppermann-Thomas cluster tilting object in $\mathcal{C}$, then $X$ is $\Sigma^n T$-$n$-rigid if and only if $X$ is $n$-rigid. In particular, $X$ is maximal $\Sigma^n T$-$n$-rigid if and only if $X$ is maximal n-rigid. $X$ is $\Sigma^n T$-$n$-self-perpendicular if and only if $X$ is $n$-self-perpendicular.

**Proof.** It is obvious that any $n$-rigid object is $\Sigma^n T$-$n$-rigid.

Now we assume that $X$ is $\Sigma^n T$-$n$-rigid. Since $T$ is an Oppermann-Thomas cluster tilting object, there exists $(n+2)$-angle

$$T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} X \xrightarrow{d_{n+1}} \Sigma^n T_0$$

where $T_0, T_1, \ldots, T_n \in \text{add} T$. Applying the functor $\text{Hom}_\mathcal{C}(-, \Sigma^n X)$ to the above $(n+2)$-angle, we have the following exact sequence

$$\cdots \rightarrow \text{Hom}_\mathcal{C}(\Sigma^n T_0, \Sigma^n X) \xrightarrow{(d_{n+1}, \Sigma^n X)} \text{Hom}_\mathcal{C}(X, \Sigma^n X) \xrightarrow{(d_n, \Sigma^n X)} \text{Hom}_\mathcal{C}(T_n, \Sigma^n X) \rightarrow \cdots .$$

Since $X$ is $\Sigma^n T$-$n$-rigid, we obtain that $\text{Hom}_\mathcal{C}(d_n, \Sigma^n X)$ is an injective. It follows that the morphism

$$D \text{Hom}_\mathcal{C}(d_n, \Sigma^n X) : D \text{Hom}_\mathcal{C}(T_n, \Sigma^n X) \rightarrow D \text{Hom}_\mathcal{C}(X, \Sigma^n X)$$

is surjective. By the $2n$-Calabi-Yau property, we have the following commutative diagram

$$\begin{array}{ccc}
D \text{Hom}_\mathcal{C}(T_n, \Sigma^n X) & \xrightarrow{D \text{Hom}_\mathcal{C}(d_n, \Sigma^n X)} & D \text{Hom}_\mathcal{C}(X, \Sigma^n X) \\
\downarrow & & \downarrow \\
\text{Hom}_\mathcal{C}(X, \Sigma^n T_n) & \xrightarrow{\text{Hom}_\mathcal{C}(X, \Sigma^n d_n)} & \text{Hom}_\mathcal{C}(X, \Sigma^n X).
\end{array}$$

Thus we get that $\text{Hom}_\mathcal{C}(X, \Sigma^n d_n) : \text{Hom}_\mathcal{C}(X, \Sigma^n T_n) \rightarrow \text{Hom}_\mathcal{C}(X, \Sigma^n X)$ is surjective. Since $X$ is $\Sigma^n T$-$n$-rigid, we have that the morphism $\text{Hom}_\mathcal{C}(X, \Sigma^n d_n)$ is zero. Hence $\text{Hom}_\mathcal{C}(X, \Sigma^n X) = 0$. This shows that $X$ is $n$-rigid. \hfill $\Box$

**Lemma 3.5.** If $M, N \in \mathcal{C}$ has no non-zero direct summands in $\text{add} \Sigma^n T$ and

$$\text{Hom}_\mathcal{A}(\mathcal{C}(T, M), \tau_n \mathcal{C}(T, N)) = 0 = \text{Hom}_\mathcal{A}(\mathcal{C}(T, N), \tau_n \mathcal{C}(T, M)),$$

then $[\Sigma^n T](M, \Sigma^n N) = 0 = [\Sigma^n T](N, \Sigma^n M)$.

**Proof.** Since $M \in \mathcal{C}$ has no non-zero direct summands in $\text{add} \Sigma^n T$, by Lemma 2.9 there exists an $(n+2)$-angle

$$T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} M \xrightarrow{d_{n+1}} \Sigma^n T_0$$

(3.1)

where $T_0, T_1, \ldots, T_n \in \text{add} T$ and for each $i \in \{1, 2, \cdots, n-1\}$ the morphism $d_i : T_i \rightarrow T_{i+1}$ is in the radical of $\mathcal{C}$. Moreover, applying the functor $\mathcal{C}(T, -)$ gives a complex

$$\mathcal{C}(T, T_0) \xrightarrow{\mathcal{C}(T, d_0)} \mathcal{C}(T, T_1) \rightarrow \mathcal{C}(T, T_2) \rightarrow \cdots \rightarrow \mathcal{C}(T, T_n) \rightarrow \mathcal{C}(T, M) \rightarrow 0$$
Lemma 3.6. Assume that

which is the start of the minimal projective resolution of \( C(T, M) \). By Lemma 2.8, we have that \( \text{Hom}_A(C(T, d_0), C(T, N)) \) is surjective since \( \text{Hom}_A(C(T, M), \tau_n C(T, N)) = 0 \).

Applying the functor \( C(-, \Sigma^n N) \) to the \((n + 2)\)-angle \( \text{3.3} \), we get the following exact sequence:

\[
C(\Sigma^0 T_1, \Sigma^n N) \xrightarrow{C(\Sigma^0 d_0, \Sigma^n N)} C(\Sigma^0 T_0, \Sigma^n N) \rightarrow C(M, \Sigma^n N) \xrightarrow{C(d_n, \Sigma^n N)} C(T_n, \Sigma^n N).
\]

Thus we have

\[
\text{Ker } C(d_n, \Sigma^n N) \approx \text{Coker } C(\Sigma^n d_0, \Sigma^n N).
\]

(3.2)

According to Remark 2.7, we have the following commutative diagram

\[
\begin{array}{ccc}
C(T_1, N) & \xrightarrow{\cong} & \text{Hom}_A(C(T, T_1), C(T, N)) \\
\downarrow C(d_0, N) & & \downarrow \text{Hom}_A(C(T, d_0), C(T, N)) \\
C(T_0, N) & \xrightarrow{\cong} & \text{Hom}_A(C(T, T_0), C(T, N))
\end{array}
\]

It follows that \( C(d_0, N) \) is surjective and then \( C(\Sigma^n d_0, \Sigma^n N) \) is also surjective. Combining with \( \text{3.2} \), we obtain \( \text{Ker } C(\Sigma^n d_0, \Sigma^n N) = 0 \).

Now we show that \([\Sigma^n T](M, \Sigma^n N) = 0\). For any morphism \( u \in [\Sigma^n T](M, \Sigma^n N) \), since \( T \) is \( n \)-rigid, we have \( u \) implies \( u \in \text{Ker } C(d_n, \Sigma^n M) = 0 \). Hence \( u = 0 \). This shows that \([\Sigma^n T](M, \Sigma^n N) = 0\).

Similarly, we can show \([\Sigma^n T](M, \Sigma^n N) = 0\). \( \square \)

**Lemma 3.6.** Assume that \( M, N \in \mathcal{C} \) has no non-zero direct summands in \( \text{add} \Sigma^n T \). If

\[
[\Sigma^n T](M, \Sigma^n N) = 0 = [\Sigma^n T](N, \Sigma^n M)
\]

then

\[
\text{Hom}_A(C(T, M), \tau_n C(T, N)) = 0 = \text{Hom}_A(C(T, N), \tau_n C(T, M)).
\]

**Proof.** Since \( M \in \mathcal{C} \) has no non-zero direct summands in \( \text{add} \Sigma^n T \), by Lemma 2.9, there exists an \((n + 2)\)-angle

\[
T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} M \xrightarrow{d_{n+1}} \Sigma^n T_0
\]

(3.3)

where \( T_0, T_1, \cdots, T_n \in \text{add } \mathcal{T} \) and for each \( i \in \{1, 2, \cdots, n - 1\} \) the morphism \( d_i: T_i \rightarrow T_{i+1} \) is in the radical of \( \mathcal{C} \). Moreover, applying the functor \( C(T, -) \) gives a complex

\[
C(T, T_0) \xrightarrow{C(T, d_0)} C(T, T_1) \rightarrow C(T, T_2) \rightarrow \cdots \rightarrow C(T, T_n) \rightarrow C(T, M) \rightarrow 0
\]

which is the start of the minimal projective resolution of \( C(T, M) \). Combining this with Lemma 2.8, we only need to show that

\[
\text{Hom}_A(C(T, d_0), C(T, N)) : \text{Hom}_A(C(T, T_1), C(T, N)) \rightarrow \text{Hom}_A(C(T, T_0), C(T, N))
\]

is surjective.
According to Remark 2.7, we have the following commutative diagram

\[
\begin{array}{ccc}
C(T_1, N) & \xrightarrow{\cong} & \text{Hom}_\Lambda (C(T, T_1), C(T, N)) \\
\downarrow C(d_0, N) & & \downarrow \text{Hom}_\Lambda (C(T, d_0), C(T, N)) \\
C(T_0, N) & \xrightarrow{\cong} & \text{Hom}_\Lambda (C(T, T_0), C(T, N))
\end{array}
\]

Thus it suffices for us to prove that \( C(d_0, N) \) is surjective. Indeed, for any morphism \( v : T_0 \to N \), we have that \( \Sigma v \circ d_{n+1} \in [\Sigma^n T](M, \Sigma^n N) = 0 \). So there exists a morphism \( w : T_1 \to N \) such that \( \Sigma v = \Sigma w \circ \Sigma d_0 \) and then \( v = w d_0 \). This shows that \( C(d_0, N) \) is surjective.

Similarly, we can show \( \text{Hom}_\Lambda (C(T, N), \tau_n C(T, M)) = 0 \).

Now we state and prove our main result.

**Theorem 3.7.** Let \( C \) be an \((n+2)\)-angulated category with an \( n \)-suspension functor \( \Sigma^n \) and \( T \) an Oppermann-Thomas cluster tilting object with endomorphism algebra \( \Lambda = \text{End}_C(T) \). Let \( D \) be the essential image of the functor \( C(T, -) : C \to \text{mod} \Lambda \), which is \( n \)-abelian and equivalent to the quotient category \( C/\text{add} \Sigma^n T \). Then we have

(a) Let \( M \) be an object in \( C \) satisfying that \( \text{add} M \cap \text{add} \Sigma^n T = \{0\} \). Then \( M \) is \( \Sigma^n T \)-rigid in \( C \) if and only if \( C(T, M) \) is \( \tau_n \)-rigid in \( D \).

(b) Decompose any object \( M \) in \( C \) as \( M = M_0 \oplus \Sigma^n T_M \) where \( \Sigma^n T_M \) is a maximal direct summand of \( M \) which belongs to \( \text{add} \Sigma^n T \). Then the correspondence

\[
M \mapsto (C(T_0, M_0), C(T, T_M))
\]

gives a bijection between the set of isomorphism classes of \( \Sigma^n T \)-rigid objects in \( C \) and the set of isomorphism classes of \( \tau_n \)-rigid pairs in \( D \).

(c) The functor \( C(T, -) \) induces a bijection between the set of isomorphism classes of maximal \( \Sigma^n T \)-rigid objects in \( C \) and the set of isomorphism classes of support \( \tau_n \)-tilting pairs in \( D \).

(d) The functor \( C(T, -) \) induces a bijection between the set of isomorphism classes of \( \Sigma^n T \)-\( n \)-self-perpendicular in \( C \) and the set of isomorphism classes of maximal \( \tau_n \)-rigid pairs in \( D \).

**Proof.** (a) This follows from Lemma 3.5 and Lemma 3.6.
(b) We decompose any object $M$ in $C$ as $M = M_0 \oplus \Sigma^n T_M$ where $\Sigma^n T_M$ is a maximal direct summand of $M$ which belongs to $\text{add} \Sigma^n T$. Put

$$F(M) := (C(T, M_0), C(T, T_M)).$$

If $M$ is $\Sigma^n T$-n-rigid, according to (a), we get that $C(T, M_0)$ is a $\tau_n$-rigid. Since $M$ is $\Sigma^n T$-n-rigid, we have $[\Sigma^n T](\Sigma^n T_M, \Sigma^n M_0) = 0$ and then $C(T, M_0) = 0$. By Remark 2.7, we obtain that $\text{Hom}_A(C(T, T_M), C(T, M_0)) = 0$. Thus $F(M)$ is a $\tau_n$-rigid pair of $A$-modules. On the other hand, for any $\tau_n$-rigid pair $(M, P)$ of $A$-modules, then $M \simeq C(T, M')$ where $M'$ has no non-zero direct summands in $\text{add} \Sigma^n T$ and $P \simeq C(T, T')$ where $T' \in \text{add} T$. By definition, we have $F(M' \oplus \Sigma^n T') = (M, P)$. It suffices to show that $M' \oplus \Sigma^n T'$ is $\Sigma^n T$-n-rigid, which is a consequence of (a), Remark 2.7 and the fact that $T$ is $n$-rigid.

(c) Let $M = M_0 \oplus \Sigma^n T_M$ be a maximal $\Sigma^n T$-n-rigid object in $C$, where $T_M \in \text{add} T$ and $M_0$ has no non-zero direct summands in $\text{add} \Sigma^n T$. We claim that

$$F(M) := (C(T, M_0), C(T, T_M)).$$

is a support $\tau_n$-tilting pair. Since $M$ is a maximal $\Sigma^n T$-n-rigid object, by (b), we know that $F(M)$ is a $\tau_n$-rigid pair. It remains to show that $F(M)$ is a support $\tau_n$-tilting pair.

(i) We assume that $(C(T, M_0) \oplus N, C(T, T_M))$ is a $\tau_n$-rigid pair for any $N \in D$. Since $N \in D$, we have $N \simeq C(T, N')$ where $N'$ has no non-zero direct summands in $\text{add} \Sigma^n T$. That is to say, $(C(T, M_0 \oplus N'), C(T, T_M))$ is a $\tau_n$-rigid pair. By (b), we know that $M_0 \oplus N' \oplus \Sigma^n T_M = M \oplus N'$ is $\Sigma^n T$-n-rigid. Since $M$ is maximal $\Sigma^n T$-n-rigid, we have $N' \in \text{add} M$. Note that $N'$ has no non-zero direct summands in $\text{add} \Sigma^n T$, we get that $N' \in \text{add} M_0$ which implies $N \in \text{add} C(T, M_0)$.

(ii) If $Q \in \text{proj} A$, then there exists $Q \simeq C(T, T')$ where $T' \in \text{add} T$. It follows that

$$Q \in \text{add} C(T, T_M) \implies \text{Hom}_A(Q, C(T, M_0)) = 0$$

since $\text{Hom}_A(C(T, T_M), C(T, M_0)) = 0$. Conversely, if $\text{Hom}_A(Q, C(T, M_0)) = 0$, by Remark 2.7, we have that $C(T', M_0) = 0$. Note that $T$ is $n$-rigid, it is straightforward to verify that $M \oplus \Sigma^n T' = M_0 \oplus \Sigma^n T_M \oplus \Sigma^n T'$ is $\Sigma^n T$-n-rigid. Since $M$ is maximal $\Sigma^n T$-n-rigid, we have $\Sigma^n T' \in \text{add} M$. Note that $M_0$ has no non-zero direct summands in $\text{add} \Sigma^n T$, we get that $\Sigma^n T' \in \text{add} \Sigma^n T_M$ which implies $T' \in \text{add} T_M$. Thus $Q \in \text{add} C(T, M_0)$.

This shows that $F(M)$ is support $\tau_n$-tilting pair.

On the other hand, now assume that $(M, P)$ is a support $\tau_n$-tilting pair in $D$. Then $M \simeq C(T, M_0)$ where $M_0$ has no non-zero direct summands in $\text{add} \Sigma^n T$ and $P \simeq C(T, T')$ where $T' \in \text{add} T$. By (b), we have $N := M_0 \oplus \Sigma^n T'$ is $\Sigma^n T$-n-rigid. We will show that $N$ is maximal. By definition, we need to show that if $X$ is an object such that $[\Sigma^n T](N \oplus X, \Sigma^n (N \oplus X)) = 0$, then $X \in \text{add} N$. Without loss of generality, we assume that $X$ is indecomposable.

If $X \notin \text{add} \Sigma^n T$, then $M \oplus C(T, X)$ is a $\tau_n$-rigid by (a). Since $[\Sigma^n T](N \oplus X, \Sigma^n (N \oplus X)) = 0$, we have $C(T', X) = 0$. By Remark 2.7, we have that $\text{Hom}_A(P, C(T, X)) \simeq C(T', X) = 0$. Thus $(M \oplus C(T, X), P)$ is a $\tau_n$-rigid pair. By hypothesis, $(M, P)$ is a support $\tau_n$-tilting pair, we infer
that \( C(T, X) \in \text{add} M \) and then \( X \in \text{add} M_0 \subseteq \text{add} N \).

If \( X \in \text{add} \Sigma^n T \), then \( \Sigma^{-n} X \in \text{add} T \) implies \( C(T, \Sigma^{-n} X) \in \text{proj} \Lambda \). Since \( \Sigma^n T \mid (N \oplus X, \Sigma^n (N \oplus X)) = 0 \), we have \( C(X, \Sigma^n M_0) = 0 \) and then \( C(\Sigma^{-n} X, M_0) \). By Remark 2.7 we obtain \( \text{Hom}_\Lambda(C(T, \Sigma^{-n} X), C(T, M_0)) \simeq C(\Sigma^{-n} X, M_0) = 0 \). By hypothesis, \((M, P)\) is a support \( \tau_n \)-tilting pair, we conclude that \( C(T, \Sigma^{-n} X) \in \text{add} C(T, T') \). Hence \( \Sigma^{-n} X \in \text{add} T' \) implies \( X \in \text{add} \Sigma^n T' \subseteq \text{add} N \). This completes the proof of (c).

(d) Assume that \( M = M_0 \oplus \Sigma^n T_M \) is \( \Sigma^n T \)-n-self-perpendicular in \( C \), where \( T_M \in \text{add} T \) and \( M_0 \) has no non-zero direct summands in \( \text{add} \Sigma^n T \). We claim that

\[
F(M) := (C(T, M_0), C(T, T_M))
\]

is a maximal \( \tau_n \)-rigid pair.

(i) Suppose that \( N \in D \) satisfies

\[
\begin{cases}
\text{Hom}_\Lambda(C(T, M_0), \tau_n N) = 0, \\
\text{Hom}_\Lambda(N, \tau_n C(T, M_0)) = 0, \\
\text{Hom}_\Lambda(C(T, T_M), N) = 0.
\end{cases}
\]

For any \( N \in D \), we have \( N \simeq C(T, N_0) \) where \( N_0 \) has no non-zero direct summands in \( \text{add} \Sigma^n T \). By Lemma 3.5 we get that \( [\Sigma^n T](M_0, \Sigma^n N_0) = 0 = [\Sigma^n T](N_0, \Sigma^n M_0) \).

Since \( \text{Hom}_\Lambda(C(T, T_M), N) = 0 \), by Remark 2.7, we have \( \text{Hom}_\Lambda(C(T, M_0), N_0) = 0 \) and then \( [\Sigma^n T](\Sigma^n T_M, N_0) = 0 \). We infer that \( [\Sigma^n T](M, \Sigma^n N_0) = [\Sigma^n T](M_0 \oplus \Sigma^n T_M, \Sigma^n N_0) = 0 \).

Note that \( [\Sigma^n T](N_0, \Sigma^n (\Sigma^n T_M)) = 0 \) since \( T \) is \( n \)-rigid. Thus we obtain

\[
[\Sigma^n T](N_0, \Sigma^n M) = [\Sigma^n T](N_0, \Sigma^n (M_0 \oplus \Sigma^n T_M)) = 0.
\]

Since \( M \) is \( \Sigma^n T \)-n-self-perpendicular, we get \( N_0 \in \text{add} M \) implies \( N_0 \in \text{add} M_0 \) since \( N_0 \) has no non-zero direct summands in \( \text{add} \Sigma^n T \). It follows that \( N \in \text{add} C(T, M_0) \). The opposite direction is obvious.

(ii) Since \( M \) is \( \Sigma^n T \)-n-self-perpendicular, thus \( M \) is \( \Sigma^n T \)-n-rigid. By (b), we know that \( FM \) is a \( \tau_n \)-rigid pair. Hence \( \text{Hom}_\Lambda(C(T, T_M), C(T, M_0)) = 0 \).

If \( Q \in \text{proj} \Lambda \), then there exists \( Q \simeq C(T, T') \) where \( T' \in \text{add} T \). It follows that

\[
Q \in \text{add} C(T, T_M) \implies \text{Hom}_\Lambda(Q, C(T, M_0)) = 0,
\]

since \( \text{Hom}_\Lambda(C(T, T_M), C(T, M_0)) = 0 \).

Conversely, if \( \text{Hom}_\Lambda(Q, C(T, M_0)) = 0 \), by Remark 2.7 we have that \( C(T', M_0) = 0 \). Note that \( T \) is \( n \)-rigid, we have that \( [\Sigma^n T](M, \Sigma^n (\Sigma^n T')) = 0 \) and it is straightforward to verify that \( M \oplus \Sigma^n T' = M_0 \oplus \Sigma^n T_M \oplus \Sigma^n T' \) is \( \Sigma^n T \)-n-rigid. Thus we have \( [\Sigma^n T](\Sigma^n T', \Sigma^n M) = 0 \).

Since \( M \) is \( \Sigma^n T \)-n-self-perpendicular, we obtain \( \Sigma^n T' \in \text{add} M \). Note that \( M_0 \) has no non-zero direct summands in \( \text{add} \Sigma^n T \), we get that \( \Sigma^n T' \in \text{add} \Sigma^n T_M \) which implies \( T' \in \text{add} T_M \). Thus \( Q \in \text{add} C(T, M_0) \). This shows that \( F(M) := (C(T, M_0), C(T, T_M)) \) is a maximal \( \tau_n \)-rigid pair.

On the other hand, now assume that \((M, P)\) is a maximal \( \tau_n \)-rigid pair. Then \( M \simeq C(T, M_0) \) where \( M_0 \) has no non-zero direct summands in \( \text{add} \Sigma^n T \) and \( P \simeq C(T, T') \) where \( T' \in \text{add} T \).
By (b), we have $N := M_0 \oplus \Sigma^n T'$ is $\Sigma^n T$-$n$-rigid.

We claim that $N$ is $\Sigma^n T$-$n$-self-perpendicular in $\mathcal{C}$. Indeed, by definition, it suffices to show that if $X \in \mathcal{C}$ is an object such that

$$[\Sigma^n T](N, \Sigma^n X) = 0 = [\Sigma^n T](X, \Sigma^n N), \quad (3.4)$$

then $X \in \text{add} N$.

Now we decompose $X = X_0 \oplus \Sigma^n T X$ where $T X \in \text{add} T$ and $X_0$ has no non-zero direct summands in $\text{add} \Sigma^n T$. By the equality (3.4), we have

$$[\Sigma^n T](M_0, \Sigma^n X_0) = 0 = [\Sigma^n T](X_0, \Sigma^n M_0).$$

By Lemma 3.6, we get then

$$\text{Hom}_\Lambda(\mathcal{C}(T, M_0), \tau_n \mathcal{C}(T, X_0)) = 0 = \text{Hom}_\Lambda(\mathcal{C}(T, X_0), \tau_n \mathcal{C}(T, M_0)).$$

Since $[\Sigma^n T](N, \Sigma^n X) = 0$, we have $[\Sigma^n T](\Sigma^n T', \Sigma^n X_0) = 0$ implies $\mathcal{C}(T', X_0) = 0$. By Remark 2.7, we obtain

$$\text{Hom}_\Lambda(\mathcal{C}(T, T'), \tau_n \mathcal{C}(T, X_0)) \simeq \text{Hom}_\mathcal{C}(T', X_0) = 0.$$

Since $(M, P)$ is a maximal $\tau_n$-rigid pair, we have $C(T, X_0) \in \text{add} C(T, M_0)$ implies $X_0 \in \text{add} M_0$.

Since $[\Sigma^n T](X, \Sigma^n N) = 0$, we have $[\Sigma^n T](\Sigma^n T X, \Sigma^n M_0) = 0$ implies $C(T X, M_0) = 0$. By Remark 2.7, we obtain

$$\text{Hom}_\Lambda(\mathcal{C}(T, T X), \tau_n \mathcal{C}(T, M_0)) \simeq \text{Hom}_\mathcal{C}(T X, M_0) = 0.$$

Since $(M, P)$ is a maximal $\tau_n$-rigid pair, we have $C(T, T X) \in \text{add} C(T, T')$ implies $T X \in \text{add} T'$. We also have $\Sigma^n T X \in \Sigma^n \text{add} T'$.

Hence $X = X_0 \oplus \Sigma^n T X \in \text{add}(M_0 \oplus \Sigma^n T') = \text{add} N$ since $X_0 \in \text{add} M_0$ and $\Sigma^n T X \in \Sigma^n \text{add} T'$. This completes the proof of (d).

This theorem immediately yields the following conclusion.

**Corollary 3.8.** Let $\mathcal{C}$ be a $2n$-Calabi-Yau $(n + 2)$-angulated category with an Oppermann-Thomas cluster tilting object. Then there exists a bijection between the set of $n$-rigid objects of $\mathcal{C}$ and the set of $\tau_n$-rigid pairs, which induces two one-to-one correspondences to the set of maximal $n$-rigid objects and the set of support $\tau_n$-tilting pairs, the set of $n$-self-perpendicular objects and the set of maximal $\tau_n$-rigid pairs.

**Proof.** This follows from Theorem 3.7 and Proposition 3.4.

Zhou and Zhu [ZhZ, Theorem 2.6] proved that: if $\mathcal{C}$ is a 2-Calabi-Yau triangulated category with a cluster tilting object, then every maximal rigid object is cluster tilting. Thus we get the following.

**Remark 3.9.** In Corollary 3.8 if $n = 1$, it is the just the Theorem 4.1 in [AIR].
Remark 3.10. In Theorem 3.7 if \( n = 1 \), it covers [YZ, Theorem 1.2] and [YZZ, Theorem 1.1].

The following result shows that maximal \( \tau_n \)-rigid pairs and support \( \tau_n \)-tilting pairs are the same under suitable conditions.

**Lemma 3.11.** If each indecomposable object of \( \mathcal{C} \) is \( n \)-rigid, then any maximal \( \tau_n \)-rigid pair is precisely support \( \tau_n \)-tilting pair in \( \mathcal{D} \).

**Proof.** By definition, we know that any maximal \( \tau_n \)-rigid pair is a support \( \tau_n \)-tilting pair. Conversely, we assume that \((M, P)\) is a support \( \tau_n \)-tilting pair. If \( N \in \mathcal{D} \) satisfies

\[
\begin{align*}
\text{Hom}_\Lambda(M, \tau_n N) &= 0, \\
\text{Hom}_\Lambda(N, \tau_n M) &= 0, \\
\text{Hom}_\Lambda(P, N) &= 0.
\end{align*}
\]

Without loss of generality, we assume that \( N \) is indecomposable. Thus \( N \simeq \mathcal{C}(T, N') \) where \( N' \) has no non-zero direct summands in \( \text{add}\Sigma^n T \) and \( N' \) is indecomposable. By hypothesis, \( N' \) is \( n \)-rigid. By Lemma 3.6, \( N \) is \( \tau_n \)-rigid. Hence \((M \oplus N, P)\) is a \( \tau_n \)-rigid pair. Since \((M, P)\) is a support \( \tau_n \)-tilting pair, we get that \( N \in \text{add} M \). The remaining conditions are clearly satisfied. This shows that \((M, P)\) is a maximal \( \tau_n \)-rigid pair. \( \square \)

As an application of Theorem 3.7, we have the following.

**Corollary 3.12.** In Corollary 3.8 if each indecomposable object in \( \mathcal{C} \) is \( n \)-rigid, then Corollary 3.8 is just the Theorem B in [JJ2].

**Proof.** This follows from Corollary 3.8, Proposition 3.4 and Lemma 3.11. \( \square \)

We use the following diagram to explain our main result.

Let \( \mathcal{C} \) be an \((n + 2)\)-angulated category. Jacobsen and Jørgensen [JJ2, Theorem 1.1] gave the following implications.

\[
\begin{align*}
n\text{-self-perpendicular objects in } \mathcal{C} &\quad \Downarrow \\
\text{maximal } n\text{-rigid objects in } \mathcal{C} &\quad \Downarrow \\
n\text{-rigid objects in } \mathcal{C}
\end{align*}
\]

But the implications cannot be reversed in general, see [JJ2, Remark 1.2] and a concrete example in [JJ2, Section 4].

If \( \mathcal{C} \) is an \((n + 2)\)-angulated category with an Opperman-Thomas cluster tilting object. We give the following implications.

\[
\begin{align*}
\text{relative } n\text{-self-perpendicular objects in } \mathcal{C} &\quad \Downarrow \\
\text{relative maximal } n\text{-rigid objects in } \mathcal{C} &\quad \Downarrow \\
\text{relative } n\text{-rigid objects in } \mathcal{C}
\end{align*}
\]
Moveover, we get the following three bijections.

\[
\begin{align*}
\text{relative } n\text{-self-perpendicular objects in } \mathcal{C} & \leftrightarrow \text{maximal } \tau_n\text{-rigid pairs} \\
\downarrow & \downarrow \\
\text{relative maximal } n\text{-rigid objects in } \mathcal{C} & \leftrightarrow \text{support } \tau_n\text{-tilting pairs} \\
\downarrow & \downarrow \\
\text{relative } n\text{-rigid objects in } \mathcal{C} & \leftrightarrow \tau_n\text{-rigid pairs}
\end{align*}
\]

When \( n = 1 \), the first two bijections are the same, then the bijections are the corresponding ones in [YZ, YZZ]. When \( \mathcal{C} \) is \( 2n \)-Calabi-Yau \( (n+2) \)-angulated category with an Oppermann-Thomas cluster tilting object, we obtain the following three bijections.

\[
\begin{align*}
\text{n-self-perpendicular objects in } \mathcal{C} & \leftrightarrow \text{maximal } \tau_n\text{-rigid pairs} \\
\downarrow & \downarrow \\
\text{maximal } n\text{-rigid objects in } \mathcal{C} & \leftrightarrow \text{support } \tau_n\text{-tilting pairs} \\
\downarrow & \downarrow \\
\text{n-rigid objects in } \mathcal{C} & \leftrightarrow \tau_n\text{-rigid pairs}
\end{align*}
\]

Hence we also know that left and right implications cannot be reversed in general.

**References**

[AIR] T. Adachi, O. Iyama, I. Reiten. \( \tau \)-tilting theory. Compos. Math. 150(3): 415-452, 2014.

[BMR] A. B. Buan, R. Marsh, I. Reiten. Cluster-tilted algebra. Trans. Amer. Math. Soc. 359(1): 323-332, 2007.

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov. Tilting theory and cluster combinatorics. Adv. Math. 204(2): 572-618, 2006.

[CZZ] W. Chang, J. Zhang and B. Zhu. On support \( \tau \)-tilting modules over endomorphism algebras of rigid objects, Acta Math. Sin. (Engl. Ser.), 31(9): 1508-1516, 2015.

[FGL] C. Fu, S. Geng and P. Liu. Relative rigid objects in triangulated categories. J. Algebra, 520: 171-185, 2019.

[GKO] C. Geiss, B. Keller and S. Oppermann. \( n \)-angulated categories. J. Reine Angew. Math. 675: 101-120, 2013.

[I1] O. Iyama. Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories. Adv. Math. 210(1): 22-50, 2007.

[I2] O. Iyama. Auslander correspondence. Adv. Math. 210(1): 51-82, 2007.

[IJY] O. Iyama, P. Jørgensen and D. Yang. Intermediate co-\( t \)-structures, two-term silting objects, \( \tau \)-tilting modules, and torsion classes, Algebra and Number Theory, 8(10): 2413-2431, 2014.

[IY] O. Iyama and Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. 172(1): 117-168, 2008.
[Ja] G. Jasso. $n$-abelian and $n$-exact categories. Math. Z. 283(3-4): 703-759, 2016.

[J] P. Jørgensen. Torsion classes and $t$-structures in higher homological algebra. Int. Math. Res. Not. IMRN 13: 3880-3905, 2016.

[JJ1] K. Jacobsen, P. Jørgensen. $d$-abelian quotients of $(d+2)$-angulated categories. J. Algebra, 521: 114-136, 2019.

[JJ2] K. Jacobsen, P. Jørgensen. Maximal $\tau_d$-rigid pairs. J. Algebra, 546: 119-134, 2020.

[JK] G. Jasso, S. Kvaamme. An introduction to higher Auslander-Reiten theory. Bull. Lond. Math. Soc. 51(1): 1-24, 2019.

[K] S. Kvaamme. Projectively generated $d$-abelian categories are $d$-cluster tilting. arXiv: 1608.07985, 2016.

[KR] B. Keller, I. Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211: 123-151, 2007.

[KZ] S. Koenig, B. Zhu. From triangulated categories to abelian categories: cluster tilting in a general framework. Math. Z. 258: 143-160, 2008.

[LX] P. Liu and Y. Xie. On the relation between maximal rigid objects and $\tau$-tilting modules. Colloq. Math. 142(2): 169-178, 2016.

[OT] S. Oppermann, H. Thomas. Higher-dimensional cluster combinatorics and representation theory. J. Eur. Math. Soc. 14(6): 1679-1737, 2012.

[YZ] W. Yang, B. Zhu. Relative cluster tilting objects in triangulated categories. Trans. Amer. Math. Soc. 371(1): 387-412, 2019.

[YZZ] W. Yang, P. Zhou, B. Zhu. Triangulated categories with cluster-tilting subcategories. Pacific J. Math. 301(2): 703-740, 2019.

[ZZ1] P. Zhou, B. Zhu. $n$-Abelian quotient categories. J. Algebra 527: 264-279, 2019.

[ZZ2] P. Zhou, B. Zhu. Two-term relative cluster tilting subcategories, $\tau$-tilting modules and silting subcategories. J. Pure Appl. Algebra 224, no. 9, 106-365, 2020.

[ZhZ] Y. Zhou, B. Zhu. Maximal rigid subcategories in 2-Calabi-Yau triangulated categories. J. Algebra 348: 49-60, 2011.

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