Are there two sterile neutrinos cooscillating with $\nu_e$ and $\nu_\mu$?  

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Abstract

The existence of two sterile neutrinos $\nu_s$ and $\nu'_s$ (blind to all Standard–Model interactions) is shown to be implied by a model of fermion ”texture” that we develop since some time. They may mix nearly maximally with two of three conventional neutrinos, say $\nu_e$ and $\nu_\mu$, thus leading to neutrino oscillations, say $\nu_e \rightarrow \nu_s$ and $\nu_\mu \rightarrow \nu'_s$, with nearly maximal amplitudes. Then, they can be responsible for the observed deficits of solar $\nu_e$’s and atmospheric $\nu_\mu$’s, respectively, but by themselves do not help to explain the LSND results for $\nu_\mu \rightarrow \nu_e$ oscillations. On the other hand, they are consistent with the CHOOZ negative result. At the moment, the experiment cannot decide, whether the deficit of atmospheric $\nu_\mu$’s, confirmed by the recent Super–Kamiokande findings, has to be related to the oscillations $\nu_\mu \rightarrow \nu_\tau$ or $\nu_\mu \rightarrow \nu'_s$. In the last Section of the paper, a new notion of ”non–Abelian spin–1/2 fermions” is presented in the context of a composite option for fermion families.

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*Dedicated to Jan Lopuszański in honour of his 75th birthday.
1. Introduction

The hypothetic sterile neutrinos, by definition interacting only gravitationally, are blind to all Standard–Model interactions, in contrast to the conventional neutrinos (or, rather, their lefthanded parts) which participate first of all in the weak sector of Standard–Model interactions. Such Standard–Model–inactive fermions are invoked from time to time by theorists, who want to explain [1] through neutrino oscillations not only the observed deficits of solar and atmospheric neutrinos, but also the results of LSND experiment. The sterile neutrinos may also form a Standard–Model–inactive fraction of the dark matter.

In the present paper, we demonstrate how two different sterile neutrinos are implied by a model of fermion ”texture” [2,3] that we develop since some time. As shown previously, this model justifies [2] the existence of three and only three families of conventional leptons and quarks \((\nu_e, e^-, u, d)\), \((\nu_\mu, \mu^-, c, s)\), \((\nu_\tau, \tau^-, t, b)\) and, moreover, describes reasonably [3] the masses and mixing parameters of quarks and charged leptons, making also some useful suggestions as to neutrinos. Note that in this model all neutrinos are Dirac particles having both lefthanded and righthanded parts.

In order to make our presentation fairly comprehensible, we will first recapitulate briefly the basic features of the model in its part concerning the existence of fundamental–particle families [2]. Then, we shall discuss the existence of two sterile neutrinos and the related neutrino oscillations.

2. Dirac’s generalized square root

The starting point of our model is the conjecture that all kinds of matter’s fundamental particles existing in Nature can be deduced from Dirac’s square–root procedure \(\sqrt{p^2} \rightarrow \Gamma \cdot p\).

As is easy to observe, this procedure leads in general to the sequence \(N = 1, 2, 3, \ldots\) of different (generally reducible) representations

\[
\Gamma^\mu \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu
\]

of the Dirac algebra.
\[ \{ \Gamma^\mu, \Gamma^\nu \} = 2g^{\mu\nu}, \]  

(2)

carried out with the use of the sequence \( N = 1, 2, 3, \ldots \) of Clifford algebras

\[ \{ \gamma^\mu_i, \gamma^\nu_j \} = 2\delta_{ij}g^{\mu\nu} (i, j = 1, 2, \ldots, N). \]  

(3)

Then, the sequence \( N = 1, 2, 3, \ldots \) of Dirac–type equations follows,

\[ \{ \Gamma \cdot [p - gA(x)] - M \} \psi(x) = 0, \]  

(4)

where \( g\Gamma \cdot A(x) \) may symbolize the minimal coupling of \( \psi(x) \) to the Standard–Model gauge fields \( A_\mu(x) \) including all \( SU(3) \times SU_L(2) \times U(1) \) coupling matrices: \( \lambda \)'s, \( \tau \)'s, \( Y \) and \( \Gamma^5 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 \).

In Eqs. (4) the matrices (1) can be presented in the reduced forms

\[ \Gamma^\mu = \gamma^\mu \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \]  

(5)

with \( \gamma^\mu \) and \( \mathbf{1} \) denoting the usual \( 4 \times 4 \) Dirac matrices. Then, the Dirac–type equations (4) can be rewritten as

\[ \{ \gamma \cdot [p - gA(x)] - M \}_{\alpha_1\beta_1} \psi_{\beta_1\alpha_2\cdots\alpha_N}(x) = 0 \]  

(6)

with \( \psi(x) = (\psi_{\alpha_1\alpha_2\cdots\alpha_N}(x)) \), where \( \alpha_1, \alpha_2, \ldots, \alpha_N \) stand for \( N \) Dirac bispinor indices: \( \alpha_i = 1, 2, 3, 4 \) for \( i = 1, 2, \ldots, N \). Here, the chiral representations are used to define all \( \alpha_i \) (\( i = 1, 2, \ldots, N \)). This means that \( \alpha_i = 1, 2, 3, 4 \) correspond to four different pairs (1,1), (1,-1), (-1,1), (-1,-1) of eigenvalues of the matrices

\[ \Gamma^5_i \equiv i\Gamma^0_i\Gamma^1_i\Gamma^2_i\Gamma^3_i, \quad \Sigma^3_i \equiv i\Gamma^5_i\Gamma^0_i\Gamma^3_i, \]  

(7)

simultaneously diagonal for all \( i \), which choice is allowed because all \( \Gamma^5_i \) and \( \Sigma^3_i \) commute both for equal and different \( i \). The \( \Gamma^\mu_i \) matrices \( (i = 1, 2, \ldots, N) \) appearing in Eqs. (7) are defined as \( N \) (properly normalized) Jacobi combinations of \( \gamma^\mu_i \) matrices \( (i = 1, 2, \ldots, N) \), where in particular \( \Gamma^\mu_i \equiv \Gamma^\mu \) is given as in Eq. (1). Then, \( \{ \Gamma^\mu_i, \Gamma^\nu_j \} = 2\delta_{ij}g^{\mu\nu} (i, j = 1, 2, \ldots, N) \) due to Eqs. (3), and also \( \{ \Gamma^\mu_i, \Gamma^5_j \} = 0 \), but \( [\Gamma^5_i, \Gamma^5_j] = 0 \), where particularly
\[ \Gamma_1^5 \equiv \Gamma^5 \]. Note that in the one–body Dirac–type equations (4) there appear only the “centre–of–mass” \( \Gamma_1^\mu \) matrices, while all ”relative” matrices \( \Gamma_2^\mu, \ldots, \Gamma_N^\mu \) are absent. In spite of this, all \( \alpha_1, \alpha_2, \ldots, \alpha_N \) are present in Eqs. (6): both the ”centre–of–mass” Dirac bispinor index \( \alpha_1 \) as well as the ”relative” Dirac bispinor indices \( \alpha_2, \ldots, \alpha_N \), the latter are decoupled, however, even in the presence of Standard–Model coupling \( g \Gamma_i \cdot A(x) \).

For \( N = 1 \) Eq. (6) is obviously the usual Dirac equation, for \( N = 2 \) it is known as the Dirac form [4] of the Kähler equation [5], whilst for \( N = 3, 4, 5, \ldots \) we obtain new Dirac–type equations [2].

If the Standard–Model coupling \( g \Gamma \cdot A(x) \) is really present in Eqs. (6), then the Dirac bispinor index \( \alpha_1 \), which is the only \( \alpha_i \) affected by the gauge fields \( A_\mu(x) \), is distinguished by its correlation with the set of all diagonal Standard–Model charges ascribed to any particle of the fields \( \psi_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \) (a label \( f \) of this set is here suppressed). The remaining Dirac bispinor indices \( \alpha_2, \ldots, \alpha_N \) are all decoupled and so, physically unobservable in the gauge fields \( A_\mu(x) \). It is natural to conjecture that they are physically undistinguishable and, therefore, are formal objects obeying Fermi statistics along with Pauli principle. This implies that \( \psi_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \) is fully antisymmetric with respect to \( \alpha_2, \ldots, \alpha_N \).

The above conjecture, together with the probabilistic interpretation of wave functions \( \psi_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \) and the requirement of their relativistic covariance applied to all bispinor indices \( \alpha_1, \alpha_2, \ldots, \alpha_N \), leads to the conclusion that there are three (and only three) families \( N = 1, 3, 5 \) of leptons and quarks [2], and two (and only two) families \( N = 2, 4 \) of some, not yet observed, fundamental scalars [6]. They correspond to the wave functions

\[
\begin{align*}
\psi^{(1)}_{\alpha_1} &\equiv \psi_{\alpha_1}, \\
\psi^{(3)}_{\alpha_1} &\equiv \frac{1}{4} \left( C^{-1} \gamma^5 \right)_{\alpha_2\alpha_3} \psi_{\alpha_1\alpha_2\alpha_3} = \psi_{\alpha_1 12} = \psi_{\alpha_1 34}, \\
\psi^{(5)}_{\alpha_1} &\equiv \frac{1}{24} \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5} \psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \psi_{\alpha_1 1234}
\end{align*}
\]

and

\[
\begin{align*}
\phi^{(2)} &\equiv \frac{1}{2\sqrt{2}} \left( C^{-1} \gamma^5 \right)_{\alpha_1\alpha_2} \psi_{\alpha_1\alpha_2} = \frac{1}{\sqrt{2}} (\psi_{12} - \psi_{21}) = \frac{1}{\sqrt{2}} (\psi_{34} - \psi_{43}), \\
\phi^{(4)} &\equiv \frac{1}{6\sqrt{4}} \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} \psi_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{\sqrt{4}} (\psi_{1234} - \psi_{2134} + \psi_{3412} - \psi_{4312})
\end{align*}
\]
respectively. Each of these wave functions carries the (here suppressed) Standard–Model label \( f = \nu, e, u, d \) denoting four sorts of fundamental particles corresponding to the signature of conventional neutrinos \( \nu \) and charged leptons \( e \) as well as up quarks \( u \) and down quarks \( d \), all four following from the Standard Model (though the existence of three and two fundamental–particle families does not follow from it). In the case of fundamental fermions, the three families are, of course, \( (\nu_e, e^-, u, d), (\nu_\mu, \mu^-, c, s), (\nu_\tau, \tau^-, t, b) \), while in the case of fundamental scalars one of (a priori) possible options may be that the two families correspond to the first and second fermion family \([6]\).

Now, in contrast, if the Standard–Model coupling \( g \Gamma \cdot A(x) \) is absent from Eqs. (6), then only physically undistinguishable \( i.e., \) antisymmetric bispinor indices \( \alpha_i \) \( (i = 1, 2, \ldots, N) \) can appear at the wave functions \( \psi_{\alpha_1\alpha_2\ldots\alpha_N}(x) \). In this case, the argument similar to the used before shows that on the fundamental level there are two (and only two) Standard–Model–inactive spin–1/2 fermions \( N = 1, 3 \) corresponding to the wave functions

\[
\psi^{(1)}_{\alpha_1} \equiv \psi_{\alpha_1}, \\
\psi^{(3)}_{\alpha_1} \equiv \frac{1}{6} \left( C^{-1} \gamma^5 \right)_{\alpha_1\alpha_2} \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5} \psi_{\alpha_3\alpha_4\alpha_5}
\]

(with no suppressed \( f \) label). They can be identified with two sterile neutrinos denoted in this paper by \( \nu_s \) and \( \nu'_s \), respectively. Analogically, on the fundamental level there should exist also two (and only two) Standard–Model–inactive spin–0 bosons \( N = 2, 4 \) that may be called sterile scalars, \( \phi^{(2)} \) and \( \phi^{(4)} \) (with no suppressed \( f \) label).

3. Neutrino oscillations involving \( \nu_s \) and \( \nu'_s \)

Let us conjecture tentatively that the sterile neutrinos \( \nu_s \) and \( \nu'_s \) are compelled to mix nearly maximally with the conventional neutrinos \( \nu_e \) and \( \nu_\mu \), respectively, in order to form four related neutrino mass states \( \nu_1 \) or \( \nu_4 \) and \( \nu_2 \) or \( \nu_5 \). Other neutrino mixings are assumed not to appear at all or to be negligible. In particular, the third conventional neutrino \( \nu_\tau \) is left not mixed and so, \( \nu_3 = \nu_\tau \) is a neutrino mass state. Evidently, the mixings of \( \nu_e \) with \( \nu_s \) and \( \nu_\mu \) with \( \nu'_s \) would be forbidden, if the electroweak \( SU_L(2) \times U(1) \) symmetry were not spontaneously broken. Thus, we can say that neutrino oscillations, being a consequence of these mixings, are caused in fact by the spontaneous breaking of
electroweak symmetry (if, of course, sterile neutrinos exist).

Under the above conjecture, the unitary transformation $\nu_I = \sum \alpha V_{I \alpha} \nu_\alpha$ between neutrino mass states $\nu_I = \nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ and neutrino flavor states $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau, \nu_s, \nu'_s$ is given as

$$
\begin{align*}
\nu_1 &= V_{11} \nu_e + V_{14} \nu_s, \\
\nu_4 &= V_{41} \nu_e + V_{44} \nu_s, \\
\nu_2 &= V_{22} \nu_\mu + V_{25} \nu'_s, \\
\nu_5 &= V_{52} \nu_\mu + V_{55} \nu'_s, \\
\nu_3 &= \nu_\tau,
\end{align*}
$$

(11)

where the nonzero coefficients are

$$
\begin{align*}
V_{11} &= V_{44} = \frac{1}{\sqrt{1 + Y^2}}, & V_{14} &= -V_{41}^* = -\frac{Y}{\sqrt{1 + Y^2}} e^{i\varphi}, \\
V_{22} &= V_{55} = \frac{1}{\sqrt{1 + X^2}}, & V_{25} &= -V_{52}^* = -\frac{X}{\sqrt{1 + X^2}} e^{i\varphi'}, \\
V_{33} &= 1
\end{align*}
$$

(12)

(in Eqs. (11) and (12), for notation convenience, we write $V_{IJ}$ in place of $V_{I \alpha}$, where $I, J = 1, 2, 3, 4, 5$). The magnitudes of these coefficients are determined by the parameters

$$
\begin{align*}
Y &= \frac{M_{11} - m_{\nu_1}}{|M_{14}|} = -\frac{M_{44} - m_{\nu_4}}{|M_{14}|}, \\
X &= \frac{M_{22} - m_{\nu_2}}{|M_{25}|} = -\frac{M_{55} - m_{\nu_5}}{|M_{25}|}
\end{align*}
$$

(13)

involving neutrino masses

$$
\begin{align*}
m_{\nu_1, \nu_4} &= \frac{M_{11} + M_{44}}{2} \pm \sqrt{\left(\frac{M_{11} - M_{44}}{2}\right)^2 + |M_{14}|^2}, \\
m_{\nu_2, \nu_5} &= \frac{M_{22} + M_{55}}{2} \pm \sqrt{\left(\frac{M_{22} - M_{55}}{2}\right)^2 + |M_{25}|^2}.
\end{align*}
$$

(14)

On the other hand $m_{\nu_3} = M_{33}$. Here, $(M_{IJ})$ $(I, J = 1, 2, 3, 4, 5)$ is a $5 \times 5$ neutrino mass matrix with $M_{14} = M_{41}^* = |M_{14}| \exp i \varphi$ and $M_{25} = M_{52}^* = |M_{25}| \exp i \varphi'$ as the only off-diagonal elements. Then, $(V_{IJ})$ $(I, J = 1, 2, 3, 4, 5)$ is a $5 \times 5$ lepton counterpart of
the familiar Cabibbo—Kobayashi—Maskawa matrix for quarks, where now $V_{14} = -V_{11}^*$ and $V_{25} = -V_{52}^*$ are the only nonzero off-diagonal elements.

Some (here neglected) small corrections to the neutrino mixings (11) may be caused by possible small deviations of the charged–lepton mass matrix from a diagonal form [3]. In fact, these deviations produce small deviations of the related diagonalizing unitary matrix from the unit matrix. In turn, such a charged–lepton diagonalizing matrix contributes multiplicatively to the lepton Cabibbo—Kobayashi—Maskawa matrix [3], changing a little its leading form (12) (in particular, nearly all zero elements of its leading form become nonzero but small).

Now, making use of Eqs. (12), we can calculate the probabilities of neutrino oscillations $\nu_e \rightarrow \nu_s$ and $\nu_\mu \rightarrow \nu'_s$ (in the vacuum) from the general formula

$$P(\nu_\alpha \rightarrow \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{K,L} V_{L\beta} V_{L\alpha}^* V_{K\beta}^* V_{K\alpha} \exp \left(i \frac{\Delta m_{LK}^2}{2|\vec{p}|} t \right),$$

where $\Delta m_{LK}^2 = m_{\nu_L}^2 - m_{\nu_K}^2$ (on the rhs of Eq. (15), for notation convenience, we will replace $\alpha, \beta$ by $I, J = 1, 2, 3, 4, 5$). Here, $\nu_\alpha(0) = \nu_\alpha$, $\langle \nu_\beta | = \langle 0 | \nu_\beta$, $\langle \nu_\beta | \nu_\alpha \rangle = \delta_{\beta \alpha}$ and, as usual, $t/|\vec{p}| = L/E$ ($c = 1 = \hbar$), what is equal to $4 \times 1.2663L/E$ if $\Delta m_{LK}^2$, $L$ and $E$ are measured in $eV^2$, $m$ and $MeV$, respectively ($L$ is, of course, the source–detector distance). Further on, we will denote

$$x_{LK} = 1.2663 \frac{\Delta m_{LK}^2 L}{E}$$

and use the identity $\cos 2x_{LK} = 1 - 2 \sin^2 x_{LK}$.

In such a way, we derive the following formulae for probabilities of neutrino oscillations $\nu_e \rightarrow \nu_s$ and $\nu_\mu \rightarrow \nu'_s$ (in the vacuum):

$$P(\nu_e \rightarrow \nu_s) = 4 \frac{Y^2}{(1 + Y^2)^2} \sin^2 x_{41},$$

$$P(\nu_\mu \rightarrow \nu'_s) = 4 \frac{X^2}{(1 + X^2)^2} \sin^2 x_{52},$$

while all other $P(\nu_\alpha \rightarrow \nu_\beta)$ with $\alpha \neq \beta$ vanish [except, of course, for $P(\nu_s \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_s)$ and $P(\nu'_s \rightarrow \nu_\mu) = P(\nu_\mu \rightarrow \nu'_s)$]. Thus, the neutrino–oscillation formulae (in the vacuum) for survival probabilities of $\nu_e$ and $\nu_\mu$ are
\[ P(\nu_e \to \nu_e) = 1 - 4 \frac{Y^2}{(1 + Y^2)^2} \sin^2 x_{41}, \]
\[ P(\nu_\mu \to \nu_\mu) = 1 - 4 \frac{X^2}{(1 + X^2)^2} \sin^2 x_{52}. \] (18)

In the case of solar neutrinos, the observed deficit of $\nu_e$'s can be explained through the neutrino oscillations (in the vacuum), when using the two–flavor formula for survival probability of $\nu_e$,
\[ P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta_{\text{solar}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{solar}} L}{E}\right), \] (19)
with the parameters [7]
\[ \sin^2 2\theta_{\text{solar}} \sim 0.65 \text{ to } 1, \quad \Delta m^2_{\text{solar}} \sim (5 \text{ to } 8) \times 10^{-11} \text{ eV}^2. \] (20)

These give the so called vacuum fit, in contrast to two other known fits based on the resonant MSW mechanism [8] in the Sun matter. In our model of neutrino oscillations (where $\nu_e \to \nu_\tau$ oscillations are responsible for the deficit of solar $\nu_e$'s), this fit leads to
\[ \frac{4Y^2}{(1 + Y^2)^2} \sim 0.65 \text{ to } 1, \quad \Delta m^2_{41} \sim (5 \text{ to } 8) \times 10^{-11} \text{ eV}^2 \] (21)
(as $m^2_{\nu_4} > m^2_{\nu_1}$). Hence, $Y \sim 0.507$ to 1 and so, we get a large mixing of $\nu_e$ with $\nu_\tau$: $V_{11} = V_{44} \sim 0.892$ to $1/\sqrt{2}$ and $V_{14} = -V_{41}^{\ast} \sim -(0.452 \text{ to } 1/\sqrt{2}) \exp i\varphi$ (the phase $\varphi$ remains not determined).

In the case of atmospheric neutrinos, the recent findings of the Super–Kamiokande experiment [9] show that the observed deficit of $\nu_\mu$'s can be explained also through the neutrino oscillations (in the vacuum), when making use of the two–flavor formula for survival probability of $\nu_\mu$,
\[ P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta_{\text{atm}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{atm}} L}{E}\right), \] (22)
with the parameters
\[ \sin^2 2\theta_{\text{atm}} \sim 0.82 \text{ to } 1, \quad \Delta m^2_{\text{atm}} \sim (0.5 \text{ to } 6) \times 10^{-3} \text{ eV}^2. \] (23)
In our model of neutrino oscillations (where $\nu_\mu \rightarrow \nu'_s$ oscillations are responsible for the deficit of atmospheric $\nu'_s$'s), this implies

$$\frac{4X^2}{(1 + X^2)^2} \sim 0.82 \to 1 \ , \ \Delta m^2_{52} \sim (0.5 \to 6) \times 10^{-3} \text{eV}^2$$  \ (24)

(as $m^2_{\nu_5} > m^2_{\nu_2}$). Hence, $X \sim 0.636$ to 1 and thus, we obtain a large mixing of $\nu_\mu$ with $\nu'_s$:

$$V_{22} = V_{55} \sim 0.844 \ \text{to} \ \frac{1}{\sqrt{2}} \ \text{and} \ V_{25} = -V^*_{52} \sim -(0.537 \ \text{to} \ \frac{1}{\sqrt{2}}) \text{exp} \ i\phi' \ (\text{the phase } \phi' \ \text{remains not determined}).$$

On the other hand, the CHOOZ experiment [10] found no evidence for neutrino–oscillation modes of $\bar{\nu}_e$ in a parameter region overlapping the range (23) of $\sin^2 2\theta_{\text{atm}}$ and $\Delta m^2_{\text{atm}}$: what shows that within this parameter range there are no mixings of $\nu_e$ with $\nu_\mu$, $\nu_\tau$, $\nu_s$, $\nu'_s$. In particular for $\nu_\mu$, this is consistent with the assumed dominance of mixing between $\nu_\mu$ and $\nu'_s$ over mixing between $\nu_\mu$ and $\nu_e$ within the range (23) of $\sin^2 2\theta_{\text{atm}}$ and $\Delta m^2_{\text{atm}}$ (at the moment, however, it cannot be decided experimentally [9], whether the mixing of $\nu_\mu$ with $\nu'_s$ or the here neglected mixing of $\nu_\mu$ with $\nu_\tau$ is responsible for the deficit of atmospheric $\nu_\mu$'s). For $\nu_s$, this requires that the assumed strong mixing of $\nu_e$ with $\nu_s$ must correspond to parameters $\sin^2 2\theta_{\text{solar}}$ and $\Delta m^2_{\text{solar}}$ belonging to a range very different from (23) [in fact, they can lie in the range (20)]. Finally for $\nu_\tau$, the lack of mixing between $\nu_\tau$ and $\nu_e$ is one of necessary and sufficient conditions for the assumed identity $\nu_3 = \nu_\tau$ (another is the lack of mixing between $\nu_\tau$ and $\nu_\mu$, if this really is true).

Of course, the sterile neutrinos $\nu_s$ and $\nu'_s$ by themselves cannot help to explain the results of LSND experiment [11] which gave evidence for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations corresponding to a much larger $\Delta m^2_{\text{LSND}}$ than both $\Delta m^2_{\text{solar}}$ and $\Delta m^2_{\text{atm}}$. These oscillations, if eventually confirmed, would require a considerable mixing of $\nu_\mu$ with $\nu_e$, corresponding to parameters $\sin^2 2\theta_{\text{LSND}}$ and $\Delta m^2_{\text{LSND}}$ lying in a range very different from (23). This mixing should be stronger than that induced by the (mentioned before) nondiagonal charged–lepton corrections appearing in our model of fermion “texture” [3].

The comparison of mass squared differences $\Delta m^2_{41}$ and $\Delta m^2_{52}$ as estimated in Eqs. (21) and (24) suggests that $m^2_{\nu_1}$ and $m^2_{\nu_4}$ are possibly much smaller than $m^2_{\nu_2}$ and $m^2_{\nu_5}$ (alternatively, $m^2_{\nu_1}$ and $m^2_{\nu_4}$ may be much more degenerate than $m^2_{\nu_2}$ and $m^2_{\nu_5}$).

4. **Outlook: Non–Abelian spin–1/2 fermions**
When the Dirac–type equations (4) are considered, one may ask a (perhaps) profound question, as to whether these one–body equations could be understood physically as point–like limits of some \( N \)–body equations for tight bound states of \( N \) spin–1/2 preons with equal masses. If it was so, the four–positions of such subelementary constituents (called here preons, as usual) should tend practically (within the bound states) to their centre–of–mass four–position,

\[
x_i = X + \delta x_i \to X, \quad X \equiv \frac{1}{N} \sum_{i=1}^{N} x_i, \quad \sum_{i=1}^{N} \delta x_i \equiv 0,
\]

while then \( \delta p_i \), defined by their four–momenta

\[
p_i = P + \delta p_i, \quad P \equiv \sum_{i=1}^{N} p_i, \quad \sum_{i=1}^{N} \delta p_i \equiv 0,
\]

should vanish in action on the wave functions [here, \( x_i = (t_i, \vec{x}_i) \), \( \delta x_i = (\delta t_i, \delta \vec{x}_i) \) and \( X = (t, \vec{X}) \)].

Of course, the physical mechanism for realization of such practically point–like limits in \( N \)–body systems would be provided by an unknown, very strong and shortrange attraction between their \( N \) constituents (described, for convenience, in the equal–time formalism, where \( \delta t_i \equiv 0 \) and \( \delta p_i^0 \) vanish in action on the wave functions). The (necessarily) non–Standard–Model nature of such an attraction would be certainly the most obscure aspect of the compound option for the Dirac–type equations (4).

Let us denote by \( P_i \) and \( X_i \) \((i = 1, 2, \ldots, N)\), with \( P_1 \equiv P \) and \( X_1 \equiv X \), the (properly normalized) Jacobi combinations of four–momenta \( p_i \) and four–positions \( x_i \) \((i = 1, 2, \ldots, N)\), respectively, for \( N \) particles. Then,

\[
\left[ P_i^\mu, X_j^\nu \right] = i \delta_{ij} g^{\mu\nu}.
\]

Making use of this notation, we can write the identities

\[
\sum_{i=1}^{N} (\gamma_i \cdot p_i - m_i) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \Gamma_i \cdot P_i - \sqrt{N} m_i \right),
\]

where \( \Gamma_i^\mu \) \((i = 1, 2, \ldots, N)\), with \( \Gamma_1^\mu \equiv \Gamma^\mu \), stand for the (properly normalized) Jacobi combinations of \( \gamma_i^\mu \) matrices \((i = 1, 2, \ldots, N)\) for \( N \) particles [the \( \Gamma_i^\mu \) matrices were already
introduced in Eqs. (7), though only in reference to the one–body Dirac–type equations (4). Then,

\[ \{ \Gamma^\mu_i, \Gamma^\nu_j \} = 2\delta_{ij}g^{\mu\nu} \quad (i, j = 1, 2, \ldots, N), \]

(29)
as follows from Eqs. (3). Here, in particular, \( \Gamma^\mu_1 \equiv \Gamma^\mu \) is given as in Eq. (1).

In this notation, the natural candidates for the hypothetic \( N \)–body equations would be

\[ \left\{ \Gamma_1 \cdot [P_1 - gA(X_1)] + \sum_{i=2}^N \Gamma_i \cdot P_i - \sqrt{N} \left( \sum_{i=1}^N m_i + I \right) \right\} \psi(X_1, X_2, \ldots, X_N) = 0, \]

(30)
where \( I(X_2, \ldots, X_N) \) would symbolize the unknown non–Standard–Model attraction between \( N \) constituents. In Eqs. (30), the Standard–Model gauge fields \( A_\mu(x) \) are coupled to the hypothetic \( N \)–body systems at four–points describing their centre–of–mass four–positions \( X_1 \equiv X \). This is approximately true, when \( A_\mu(X + \delta x_i) \) are only weakly dependent on \( \delta x_i \).

In the point–like limits, where the relative four–positions \( X_2, \ldots, X_N \) (i.e., also all \( \delta x_i \)) tend to zero and then the relative four–momenta \( P_2, \ldots, P_N \) (i.e., also all \( \delta p_i \)) vanish in action on the wave functions, Eqs. (30) are really reduced to the Dirac–type equations (4) with \( p \equiv P \equiv P_1, X \equiv X \equiv X_1, \Gamma \equiv \Gamma_1 \) and \( M \equiv \sqrt{N}(Nm + I_{X_i \to 0}) \) \((m_i \equiv m)\). Note that \( M \) grows with \( N \) faster than linearly.

In the equal–time formalism, where the relative times \( t_2, \ldots, t_N \) (i.e., also all \( \delta t_i \)) are zero and the relative energies \( P^0_2, \ldots, P^0_N \) (i.e., also all \( \delta p^0_i \)) vanish in action on the wave functions, Eqs. (30) assume the forms

\[ P^0_1 \psi(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_N, t) = \left\{ \Gamma^0_1 \Gamma_1 \cdot [\vec{P}_1 - g\vec{A}(\vec{X}_1, t)] + gA^0(\vec{X}_1, t) + \sum_{i=2}^N \Gamma^0_1 \Gamma_1 \cdot \vec{P}_i \right. \\
+ \left. \Gamma^0_1 \left( \sqrt{N} \sum_{i=1}^N m_i + I_{X^p_i=0} \right) \right\} \psi(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_N, t), \]

(31)
where \( P^0_1 \equiv P^0 = i\partial/\partial t \) and \( I_{X^p_i=0} = I(\vec{X}_2, \ldots, \vec{X}_N) \).

In the point–like limits, Eqs. (31) are reduced to the equations.
\[ p^0 \psi(\vec{x}, t) = \left\{ \Gamma_0 \vec{\Gamma} \cdot \left[ \vec{p} - g \vec{A}(\vec{x}, t) \right] + g A^0(\vec{x}, t) + \Gamma^0 M \right\} \psi(\vec{x}, t) \]  \hspace{1cm} (32)

with \( p \equiv P \equiv P_1 \), \( x \equiv X \equiv X_1 \), \( \Gamma \equiv \Gamma_1 \) and \( M \equiv \sqrt{N} (Nm + I_{X_{i \rightarrow 0}}) \) \((m_i \equiv m)\). Of course, \( p^0 \equiv P^0 = i\partial / \partial t \) and \( I_{X_{i \rightarrow 0}} \) stands for a reasonably defined point–like limit of \( I \).

Note that the eigenvalues \((P^0_{1\text{kin}})_{\pm}\) of the kinetic part of the hamiltonian appearing on the rhs of the state equation (31) get for any \( N \) the form
\[
\pm \sqrt{N} \left[ p^2_1 + \ldots + p^2_N + N(Nm)^2 \right]^{1/2}
\]

as if our \( N \)-body system were a single Dirac particle with the mass \( Nm \) in a \((3N+1)\)-dimensional spacetime (notice, however, the additional factor \( \sqrt{N} \)).

A fundamental feature of Eqs. (30) is that, via \( \Gamma_{\mu}^i \) \((i = 1, 2, \ldots, N)\), they contain \( N \) Dirac nonconventional \( \gamma_5^i \) matrices \((i = 1, 2, \ldots, N)\) which do not commute for different particles, in contrast to Dirac conventional gammas commuting for different particles [in fact, the nonconventional \( \gamma_5^i \) \((i = 1, 2, \ldots, N)\) anticommute for different \( i \), as is seen from Eqs. (3)]. The spin–1/2 fermions \( i = 1, 2, \ldots, N \) described within an \( N \)-body system with the use of such nonconventional \( \gamma_5^i \) matrices \((i = 1, 2, \ldots, N)\), anticommuting for different particles, might be called non–Abelian spin–1/2 fermions [12]. In contrast, in the familiar case of Dirac conventional gammas, commuting for different particles, one could use the term Abelian spin–1/2 fermions.

Now, let us observe that the form of spin tensors for spin–1/2 fermions \( i = 1, 2, \ldots, N \) is identical in the non–Abelian and Abelian case:

\[
\sigma_{\mu \nu}^i \equiv \frac{i}{2} [\gamma_{\mu}^i, \gamma_{\nu}^i] = \begin{cases} 
\alpha_{\mu \nu}^i & \text{for } \mu = 0, \nu = l \\
\epsilon^{kln} \sigma_{\mu}^m & \text{for } \mu = k, \nu = l
\end{cases},
\]

(33)

where \( \alpha_{\mu \nu}^i \equiv \gamma_{0}^i \gamma_{\mu}^i \) and \( \sigma_{\mu}^m \equiv \gamma_{5}^i \gamma_{0}^i \gamma_{\mu}^i \equiv \gamma_{5}^i \alpha_{\mu}^m \) with \( \gamma_{5}^i \equiv i \gamma_{0}^i \gamma_{1}^i \gamma_{2}^i \gamma_{3}^i \). In fact, for each \( i \) the components \( \frac{1}{2} \sigma_{\mu \nu}^i \) satisfy in both cases the usual Lorentz–group commutation relations, while for different \( i \) they commute in both cases as being bilinear in \( \gamma_5^i \). Also \( \gamma_5^i \) commute for different \( i \) in both cases. So, the total generators of Lorentz group for a system of \( N \) spin–1/2 fermions are given in both cases by the operators

\[
J^\mu = \sum_{i=1}^{N} \left( x_i^\mu p_i^\nu - x_i^\nu p_i^\mu + \frac{1}{2} \sigma_{\mu \nu}^i \right).
\]

(34)

Let us note, by the way, the following identity valid for the total spin tensor in both cases.
where

\[ \Sigma_{\mu\nu} \equiv \frac{i}{2} [\Gamma_{\mu}, \Gamma_{\nu}] = \begin{cases} i A_i^l \varepsilon_{klm} \Sigma_{m} & \text{for } \mu = 0, \nu = l \\
\varepsilon_{klm} \Sigma_{m} & \text{for } \mu = k, \nu = l \end{cases} \]

(36)

with \(A_i^l \equiv \Gamma_i^0 \Gamma_i^l\) and \(\Sigma_i^m \equiv \Gamma_5^{\mu} \Gamma_4^0 \Gamma_i^m \equiv \Gamma_5 A_i^m\). Evidently, \(\Sigma_{\mu\nu} \equiv \Sigma^\mu_{\nu}\) with \(\Sigma_{\mu\nu} \equiv \frac{i}{2} [\Gamma^\mu, \Gamma^\nu]\) is the centre-of-mass spin tensor for the system of \(N\) spin–1/2 fermions, while \(\Sigma_{\mu\nu}^2, \ldots, \Sigma_{\mu\nu}^N\) are its relative spin tensors. All spin tensors \(\Sigma_{\mu\nu}^i\), being bilinear in \(\Gamma_i^\mu\), commute for different \(i\) in both cases.

Thus, in this Section, we can draw the important conclusion that for a system of \(N\) spin–1/2 fermions the Lorentz-group commutation relations get two (and only two) realizations: either with the use of Dirac conventional gammas commuting for different particles, or with the use of Dirac nonconventional gammas anticommuting for different particles. Such an intriguing statement seems to support the logical consistency and unexpected naturalness of the notion of non–Abelian spin–1/2 fermions. They may provide an unconventional alternative for familiar Abelian spin–1/2 fermions in the potential structure of particle theory. In this Section, their role as hypothetic preons was underlined.

Finally, we should like to emphasize some unconventional features of the quantization procedure which would work in the case of non–Abelian spin–1/2 fermions. It is not difficult to observe that in the case of spin–1/2 fermions, only the Fock–space states related to Dirac conventional gammas (commuting for different particles) can be constructed by means of the familiar second–quantization procedure based on Fermi creation and annihilation operators for single particles. This is so, because the repetition of single–particle creation operators can lead to Fock–space states of particles with commuting Dirac gammas only. In order to construct the Fock–space states related to Dirac nonconventional gammas (anticommuting for different particles), new Fermi or Bose operators creating and annihilating at once whole \(N\)–particle configurations with odd or even \(N = 1, 2, 3, \ldots\), respectively, must be introduced. Of course, these \(N\) particles are then non–Abelian spin–1/2 fermions. Such a new procedure might be called the third quantization [12].

\[
\sum_{i=1}^{N} \sigma_{\mu\nu}^i = \sum_{i=1}^{N} \Sigma_{\mu\nu}^i, \quad (35)
\]
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