Quantum Ising model in transverse and longitudinal fields: chaotic wave functions

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Abstract

The construction of a statistical model for eigenfunctions of the Ising model in transverse and longitudinal fields is discussed in detail for the chaotic case. When the number of spins is large, each wave function coefficient has the Gaussian distribution with zero mean and variance calculated from the first two moments of the Hamiltonian. The main part of the paper is devoted to the discussion of various corrections to the asymptotic result. One type of correction is related to higher order moments of the Hamiltonian, and can be taken into account by Gibbs-like formulae. Other corrections are due to symmetry contributions, which manifest as different numbers of non-zero real and complex coefficients. The statistical model with these corrections included agrees well with numerical calculations of wave function moments.

Keywords: quantum spin chain, quantum many body, chaotic wave functions

(Some figures may appear in colour only in the online journal)

1. Introduction

With the ever increasing power of modern computers, it has become possible to perform full numerical calculations for non-integrable quantum many-body systems containing few tens of particles. Direct numerical calculation is, however, a challenging problem of modern computational physics. In these systems, the size of the Hilbert space generally grows exponentially with the number of particles, and the development of approximate statistical description remains thus of great importance. In recent years, there has been an increasing interest in this field, mostly due to progress made on the experimental side and the realization of simple models of quantum many body physics in the laboratory. This in turn has stimulated theoretical studies, and enabled meaningful formulations of questions about fundamental principles of quantum thermodynamics and foundations of quantum statistical physics. In particular,
the questions of knowing how/which quantum systems thermalize or what corrections to the thermodynamic limit are appropriate in finite systems are still under investigation.

For macroscopic bodies, the number of interacting particles \(N\) is so large that only the thermodynamic limit corresponding to \(N \to \infty\) is meaningful. The behaviour of a small subsystem \(\mathcal{S}\) is described by its reduced density matrix—constructed from the knowledge of the eigenvalues \(E_p\) and the associated eigenfunctions \(\psi(p)\)—when the interaction with exterior particles (not belonging to \(\mathcal{S}\)) is switched off. According to general properties of quantum thermodynamics (see e.g. [1]), the mean value of an observable \(A\) of the sub-system is calculated as follows

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \Psi(t) | \hat{A} | \Psi(t) \rangle dt = \frac{1}{Z} \sum_p e^{-\beta E_p} \langle \psi(p) | \hat{A} | \psi(p) \rangle, \quad Z = \sum_p e^{-\beta E_p}
\]

and the only information of the existence of a larger system is the inverse temperature \(\beta\), which is determined by the mean energy of the initial state.

This text-book approach has proved to be so successful that there is no doubt, at least in physical literature, that it is the correct approach to the calculation of the thermodynamic limit in generic systems. The existence of powerful perturbation expansion around equilibrium states with zero and non-zero temperature [2], and non-equilibrium states [3] contributes considerably to the success of quantum thermodynamics. The important particularity of these constructions is that effectively infinite number of particles never appears explicitly at any step of the construction. The formalism is built in such a way that only finite quantities like temperature and chemical potential give information of the existence of the external world.

Let \(E_\alpha\) and \(\Psi(\alpha)\) be exact eigenvalues and eigenfunctions of the full \(N\) particles Hamiltonian. For finite \(N\), instead of canonical Gibbs measure (1), one has to use the microcanonical average

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \Psi(t) | \hat{A} | \Psi(t) \rangle dt = \frac{1}{N_{E_0,E_0}} \sum_{|E_\alpha-E_0|<\Delta E_0} \langle \Psi(\alpha) | \hat{A} | \Psi(\alpha) \rangle
\]

where \(E_0\) is the mean energy of the initial state, \(\Delta E_0\) is an energy window assumed to be small with respect to \(E_0\) but much larger than the distance between nearest levels

\[
1/\rho(E_0) \ll \Delta E_0 \ll E_0
\]

where \(\rho(E)\) is the mean spectral density at energy \(E\). \(N_{E_0,E_0}\) is the number of states in this window. In what follows, we consider only the so-called chaotic systems, where there is no integral of motion except classical ones. The important difference between equations (1) and (2) is that in the former \(\psi(n)\) are the eigenfunctions of a sub-system whereas in the latter \(\Psi(\alpha)\) are eigenfunctions of the full Hamiltonian.

In this approach, the central object of investigation is the \(N\)-particle eigenfunction \(\Psi(\alpha)\). There exist different scenarios about the emergence of thermodynamic behaviour (see e.g. [4]). One of the most accepted is the so-called eigenstate thermalization conjecture (ETH) [5–7] which states that relation (2) is valid for (almost) all individual eigenstates. A similar conjecture is known in low-dimensional quantum chaos (see e.g. [8]), where it is explicitly stated that for a smooth classical observable \(A(p,q)\) almost all diagonal matrix elements of the corresponding quantum counterpart tend to the microcanonical semiclassical limit when \(\hbar \to 0\)

\[
\langle \Psi(\alpha) | \hat{A} | \Psi(\alpha) \rangle \xrightarrow[\hbar \to 0]{} \frac{1}{Z} \int \delta(E - H(p,q))A(p,q) dp dq, \quad Z = \int \delta(E - H(p,q)) dp dq.
\]
The smallness of off-diagonal matrix elements \( \langle \Psi_\alpha | \hat{A} | \Psi_\gamma \rangle \ll \langle \Psi_\alpha | \hat{A} | \Psi_\alpha \rangle \), when \( \gamma \neq \alpha \), also appears naturally in quantum chaos (see e.g. [9]), as only periodic orbits give contribution to this quantity.

These and many other arguments strongly suggest that for systems where quantum thermodynamics can be applied, the majority of eigenfunctions are universal, depending only on a few parameters. It has even been proved that (in a certain sense) typical functions in quantum mechanics lead automatically to canonical averaging [13, 14]. A well known example of such a phenomenon is given by Berry’s conjecture [10–12], which postulates that wave functions of low-dimensional chaotic systems are Gaussian random functions with variance determined from semiclassical microcanonical average, as in equation (4). A similar construction has been applied to various problems ranging from nuclear and atomic physics to other many-body models [4, 16–30].

The purpose of this paper is to construct and carefully check a statistical model for eigenfunctions of the quantum Ising model in transverse and longitudinal fields determined by the Hamiltonian

\[
\mathcal{H} = -\sum_{l=1}^{N} \sigma^x_l \sigma^x_{l+1} - \lambda \sum_{l=1}^{N} \sigma^z_l - \alpha \sum_{l=1}^{N} \sigma^y_l,
\]

where \( \sigma^{x,y,z} \) are the usual Pauli matrices, and parameters \( \lambda \) and \( \alpha \) fix values of transverse and longitudinal fields.

This model is a prototypical example of one-dimensional spin chains with nearest-neighbour interactions. When \( \alpha = 0 \) it reduces to the well known quantum Ising model in transverse field, which is integrable by the Jordan–Wigner transformation [31, 32] and for \( \lambda = 1 \) becomes critical and serves as the paradigmatic model of a quantum critical phenomenon [34]. For non-zero \( \alpha \) (and \( \lambda \neq 0 \)) the model is considered as non-integrable, but at the critical value of transverse field, \( \lambda = 1 \), and specially fine-tuned weak longitudinal field, \( \alpha \to 0 \), it is integrable but not conformal [33].

In [35] it was shown that the ground state wave function of the Ising model (as well as of practically all one-dimensional spin-chain models) is multifractal in the initial spin basis. So it does not fit the standard thermodynamic scheme. For highly excited states, the situation is different. The spectral density for this model has been discussed in [36]. For finite number of spins \( N \), in the bulk of the spectrum, when \( E \sim \sqrt{N} \), there exist two different regimes. When both fields \( \lambda \) and \( \alpha \) are of the order of 1 (i.e. of order of the hopping term), the spectral density is well approximated by a simple Gaussian function whose parameters are calculated from the knowledge of the first moments of the Hamiltonian. If \( \lambda \) is small or large the spectral density of the Ising model in two fields for large but finite \( N \) has many peaks well described by a sum of Gaussian functions with parameters calculated directly from the moment of the Hamiltonian without full diagonalization.

In parallel with two regimes of spectral density of the Ising model, there are two regimes for eigenfunctions in the bulk of the spectrum. Here we consider in detail the case when all coupling constants are of the same order. The multi-peaks case will be discussed elsewhere.

The plan of the paper is the following. The construction of chaotic wave functions for the Ising model is discussed in section (2). To check the accuracy of such statistical models it is convenient to calculate moments of wave functions and compare them with statistical approximations. In section (3) this is done for the participation ratio for the Ising model in two fields. The main part of this section is devoted to the calculation of various corrections to the lowest order approximation. It is demonstrated that corrected statistical modeling of eigenfunctions
agrees very well with the results of direct numerical calculations. The first moments of the full Ising model Hamiltonian are determined in appendix.

2. Statistical description of wave functions

The construction of chaotic wave functions for different problems follow approximately the same steps as in [10]. An eigenfunction of the Hamiltonian (5) with energy \( E_\alpha \) by definition obeys

\[
\mathcal{H} |\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle
\]

and can be represented as a finite series of basis set functions

\[
|\Psi_\alpha\rangle = \sum_\vec{n} C_{\vec{n}}(\alpha) |\vec{n}\rangle.
\]

For \( N \) spins the full dimension of the Hilbert space is \( \mathcal{N} = 2^N \). Here a state is indicated by symbol \( \vec{n} = (n_1, n_2, \ldots, n_N) \) with the convention that \( n_k = -1 \) corresponds to spin down at site \( k \) and \( n_k = 1 \) shows that site \( k \) is occupied by spin up.

All information about the wave function is contained in the coefficients \( C_{\vec{n}}(\alpha) \). Let us consider the collection of coefficients \( C_{\vec{n}}(\alpha) \) with fixed symbol \( \vec{n} \) and with energies in a small interval \( I = [E < E_\alpha < E + \Delta E] \). The energy window \( \Delta E \) is assumed to be small with respect to \( E \), but much larger than the distance between nearest levels as in (3). The principal assumption is that for chaotic systems coefficients \( C_{\vec{n}}(\alpha) \) in such intervals are such irregular functions of eigen-energies \( E_\alpha \) that their exact values are irrelevant for most purposes, and one has to develop and rely on a statistical description of chaotic wave functions. For spin-chain models without random parameters, such as that given by equation (5), we are not aware of any strict definition of chaoticity. The usual lore is that non-integrability of systems with large number of degrees of freedom is (almost always) synonymous with chaoticity. For our purposes this notion is sufficient.

As an example, we present in figure 1(a) one particular coefficient with fixed symbol \( \vec{n} \), and for all eigen-energies of the Ising model with \( \alpha = 1 \) and \( \lambda = 1 \) for 17 spins with periodic boundary conditions obtained by direct numerical diagonalization. This type of picture strongly suggests the following conjecture.

**Conjecture 1.** The irregular behaviour of coefficients \( C_{\vec{n}}(\alpha) \) in a small energy window can be mimicked by the assumption that they are random functions of energies with a certain distribution.

For strongly chaotic systems there are two widely used \textit{a priori} assumptions inspired from random matrix theory [40]. If coefficients are real, they should well be described by the Gaussian distribution. This means that at fixed symbol \( \vec{n} \), the distribution density of coefficients is

\[
P(C_{\vec{n}} = x) = \frac{1}{\sqrt{2\pi \Sigma_{\vec{n}}^2}} \exp \left( -\frac{x^2}{2\Sigma_{\vec{n}}^2} \right).
\]

If coefficients are complex, one assumes that they have Gaussian distribution for both real and imaginary parts with zero mean and the same variance

\[
P(\text{Re } C_{\vec{n}} = x, \text{ Im } C_{\vec{n}} = y) = \frac{1}{\pi \Sigma_{\vec{n}}^2} \exp \left( -\frac{x^2 + y^2}{\Sigma_{\vec{n}}^2} \right).
\]
In the both expressions \( \Sigma_2^{\vec{n}} \) indicates the mean value of modulus square of coefficient \( C_{\vec{n}} \)

\[
\Sigma_2^{\vec{n}} = \langle |C_{\vec{n}}|^2 \rangle.
\]  
(10)

For illustration, in figure 1(b) we compare distribution of the same coefficient as in figure 1(a) with the best Gaussian fit, and a good agreement is clearly seen—hence supporting the first conjecture.

Accepting the above conjecture means, in particular, that moments of variables \( C_{\vec{n}} \) averaged over a small energy interval \( I \) should be well approximated (at least for large \( N \)) by the moments of the corresponding Gaussian distributions

\[
\langle |C_{\vec{n}}|^2q \rangle_I \equiv \sum_{E, \alpha}^{E \in I} |C_{\vec{n}}(\alpha)|^2q \delta(E - E_\alpha) = \mathcal{R}_q \Sigma_2^{\vec{n}}
\]  
(11)

where for complex coefficients \( \mathcal{R}_q = \mathcal{R}_q^{(\text{complex})} \) and for real ones \( \mathcal{R}_q = \mathcal{R}_q^{(\text{real})} \) with

\[
\mathcal{R}_q^{(\text{complex})} = \Gamma(q + 1), \quad \mathcal{R}_q^{(\text{real})} = \frac{2^q \Gamma \left(q + \frac{1}{2} \right)}{\sqrt{\pi}}.
\]  
(12)

In particular, \( \mathcal{R}_2^{(\text{complex})} = 2 \) and \( \mathcal{R}_2^{(\text{real})} = 3 \). Under the validity of the conjecture all average moments of \( C_{\vec{n}}(\alpha) \) (with fixed symbol \( \vec{n} \)) are determined by one quantity, the variance \( \Sigma_2^{\vec{n}} = \Sigma_2^{\vec{n}}(E) \) which depends on symbol \( \vec{n} \) and the center of energy window \( E \).

To calculate this variance, it is convenient to consider the so-called strength function (or the local density of states)

\[
P_{\vec{n}}(E) = \sum_{\alpha} |C_{\vec{n}}(\alpha)|^2 \delta(E - E_\alpha).
\]  
(13)

\[\text{Figure 1. (a) Coefficient corresponding to symbol } |000 010 001 101 110 01\rangle \text{ for the Ising model with } \alpha = 1 \text{ and } \lambda = 1 \text{ for } N = 17 \text{ spins in the sector with zero translational momentum. Red vertical lines indicate energy windows (i)–(iv) used in the calculation of distributions of this coefficient in figure 1(b). Each window contains approximately 2000 levels. (b) Distribution of the coefficient as in figure 1(a) in four energy windows indicated in that figure (red histograms). Blue solid lines are the best Gaussian fits to these distributions. Inserts: The difference between numerical histograms and the Gaussian fits.}\]
Due to the pseudo-random character of $C_{\vec{n}}(\alpha)$ and $E_{\alpha}$, the strength function itself can be considered as a pseudo-random or random function of energy $E$. The mean value of a certain function $f(E)$ is defined as in (2)

$$\left\langle P_{\vec{n}}(E) f(E) \right\rangle = \frac{1}{N_{E^2, \Delta E}} \sum_{|E - E_{\alpha}| < \Delta E} |C_{\vec{n}}(\alpha)|^2 f(E_{\alpha})$$  \hspace{0.5cm} (14)

where the width of the energy window $\Delta E$, obeys (3).

The knowledge of the strength function makes it possible to find $|C_{\vec{n}}(\alpha)|^2$ averaged over a small energy window (which can be used as the definition of the variance)

$$\Sigma_{\vec{n}}^2 = \frac{\left\langle P_{\vec{n}}(E) \right\rangle}{\rho(E)}$$  \hspace{0.5cm} (15)

where $\rho(E)$ is the average density of states with energy $E$ i.e. the number of states in an interval $E < E_{\alpha} < E + \Delta E$. The advantage of the strength function is that for this quantity there exist exact sum rules [15]

$$\int P_{\vec{n}}(E) E^k dE \equiv \langle E^k \rangle = \langle \vec{n} | \hat{H}^k | \vec{n} \rangle.$$  \hspace{0.5cm} (16)

In other words, it means that these moments can be calculated directly from the Hamiltonian without solving the full problem. The exact calculation of the strength function is, of course, equivalent to finding the full solution of the problem. In order to obtain a simple approximation, one has to assume that the strength function can be well approximated from the knowledge of its first moments. One can argue that for $N$-body systems with short-range interactions, the first two moments reproduce higher order moments well when $N \to \infty$. This then leads to the second conjecture.

**Conjecture 2.** The functional dependence of $P_{\vec{n}}(E)$ on $E$ is well approximated when $N \to \infty$ by a Gaussian

$$P_{\vec{n}}(E) = \frac{1}{\sqrt{2\pi \sigma_{\vec{n}}^2}} \exp \left( -\frac{(E - E_{\vec{n}})^2}{2\sigma_{\vec{n}}^2} \right)$$  \hspace{0.5cm} (17)

where $E_{\vec{n}}$ and $\sigma_{\vec{n}}^2$ are the first two moments of the Hamiltonian

$$E_{\vec{n}} = \langle \vec{n} | \hat{H} | \vec{n} \rangle, \quad \sigma_{\vec{n}}^2 = \langle \vec{n} | (\hat{H} - E_{\vec{n}})^2 | \vec{n} \rangle.$$  \hspace{0.5cm} (18)

For the case of an integrable system, this functional form is nothing else but the central limit theorem, as proven in [36]. Under this conjecture the variance of coefficient distribution given by equation (15) is

$$\Sigma_{\vec{n}}^2 \approx \frac{1}{\rho(E) \sqrt{2\pi \sigma_{\vec{n}}^2}} \exp \left( -\frac{(E - E_{\vec{n}})^2}{2\sigma_{\vec{n}}^2} \right)$$  \hspace{0.5cm} (19)

with the mean spectral density calculated by the formula (to ensure the normalization $\sum_{\vec{n}} |C_{\vec{n}}(\alpha)|^2 = 1$)

$$\rho(E) = \sum_{\vec{n}} \frac{1}{\sqrt{2\pi \sigma_{\vec{n}}^2}} \exp \left( -\frac{(E - E_{\vec{n}})^2}{2\sigma_{\vec{n}}^2} \right).$$  \hspace{0.5cm} (20)
Assuming that coefficients have Gaussian distribution (8), the values of the $2q$–th moments of the wave function are

$$
M_q = \left\langle \sum_\vec{n} |C_\vec{n}(\alpha)|^{2q} \right\rangle = \frac{R_q}{\rho(E)} \sum_\vec{n} \frac{1}{(2\pi\sigma^2)^{(q/2)}} \exp \left( -\frac{(E - E_\vec{n})^2}{2\sigma^2} \right).
$$

(21)

The above two conjectures are the basis ingredients for the construction of statistical models of chaotic wave functions in different problems [4], [16–30]. Although they were not proved in full generality (see [37, 38]), they are simply enough checked and compared to exact results for a given particular system. One can even reverse the arguments and make the following statement: the wave functions of an $N$-body model with short range interactions are called chaotic if and only if they obey and satisfy the above conjectures.

Another line of reasoning one can use is the maximum-entropy principle (see e.g. [39]), according to which 'the best' statistical distribution ($P_{\vec{n}}(E)$ in our case) is given by the one which maximises the entropy with certain restrictions (as in (16))

$$
S = - \int P_{\vec{n}}(E) \ln P_{\vec{n}}(E) dE.
$$

(22)

Assuming, for instance, that a finite number of first moments in (16) are specified, the maximum-entropy principle predicts that the corresponding probability density takes the Gibbsian form

$$
P_{\vec{n}}(E) = \frac{e^{-\sum_j \mu_j E_j}}{Z(\vec{\mu})}, \quad Z(\vec{\mu}) = \int e^{-\sum_j \mu_j E_j} dE
$$

(23)

with Lagrangian multipliers $\mu_j = \mu_j(\vec{n})$ calculated from the partition function using the knowledge of the first moments

$$
\langle \vec{n}|H_j|\vec{n} \rangle = -\frac{\partial}{\partial \mu_j} \ln Z(\vec{\mu}).
$$

(24)

When only two moments are taken into account one gets the Gaussian distribution (17). In theory, knowing all the moments of the Hamiltonian, or equivalently the Lagrangian multipliers, makes it possible to reconstruct the exact strength function. However, formula (23) requires numerical calculations of Lagrangian multipliers, which complicates the separation of different sources of corrections. To increase the accuracy of approximation for the strength function, we shall include the third and forth moments, assuming that they are much smaller than the first two moments, by using Gram-Charlier A series in Hermite polynomials

$$
P_{\vec{n}}(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(E - E_\vec{n})^2}{2\sigma^2} \right) \left[ 1 + \frac{k_3}{3\sigma^3} H_3 \left( \frac{E - E_\vec{n}}{\sigma} \right) + \frac{k_4}{4!\sigma^4} H_4 \left( \frac{E - E_\vec{n}}{\sigma} \right) \right].
$$

(25)

Here $k_3$ and $k_4$ are the third and the forth cumulants for the Ising model given by equations (A.18) and (A.24) from appendix, $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$.

3. Participation ratio for the Ising model in two fields

For the Hamiltonian (5) simple calculations give

$$
E_n = \lambda(N - 2n), \quad \sigma^2_n = \sigma^2 = N(1 + \alpha^2)
$$

(26)
where \( n \) is the number of spins up. Each state with fixed \( n \) is \( C_n^N \) degenerated. Notice that the second moment is independent of the number of spins up. The strength function can thus be written

\[
P_n(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(E - E_n)^2}{2\sigma^2}\right).
\]

(27)

We have written \( n \), and not \( \vec{n} \), to stress that in this approximation all quantities depend only on the number of spins up. The full density, normalized to 1 is

\[
\rho(E) = 2^{-N} \sum_{n=0}^{N} C_n^N P_n(E).
\]

(28)

In figure 2 this equation is compared with the spectral density of the Ising model (5) with \( \alpha = 1 \) and \( \lambda = 1 \) for \( N = 17 \) spins calculated numerically. The agreement is good but a small difference is still visible. When higher moments are taken into account as in (25) the result is practically indistinguishable from numerics.

To approximate moments of wave functions accurately, one has to know the first moments of wave function coefficients but also if they are real or complex. Usually, the answer is simple. If the Hamiltonian is real symmetric, wave functions (and their coefficients) are real. If the Hamiltonian is complex Hermitian, coefficients are also complex. For the considered Ising model in two fields (5) the situation is more tricky, due to the presence of symmetries in the problem.

The model with periodic boundary conditions has the translational invariance

\[
\tilde{T}: n \rightarrow n + 1, \quad n = 1, 2, \ldots, N
\]

(29)

so if \( \Psi \) is a wave function with energy \( E \), with the following basis expansion \( \Psi = \sum_{n_1, n_2, \ldots, n_N = \pm 1} C_{n_1, n_2, \ldots, n_N} |n_1, n_2, \ldots, n_N \rangle \) then \( \tilde{T}\Psi = \sum_{n_1, n_2, \ldots, n_N = \pm 1} C_{n_1, n_2, \ldots, n_N} |n_N, n_1, \ldots, n_{N-1} \rangle \) is also an eigenfunction with the same energy, which imposes certain relations on the coefficients.

As \( \tilde{T}^N = 1 \), its eigenvalues are \( e^{2\pi i k/N} \) and the eigenfunctions can be classified by the quantum number \( k = 0, 1, \ldots, N \) which we call translational momentum for simplicity.
\[ \hat{T}\Psi_k = e^{2\pi i k/N} \Psi_k. \] (30)

The numerical calculations presented below are performed in the basis with fixed translational momentum \( k \). To construct this basis explicitly, one fixes one element, \( |\vec{n}\rangle, \vec{n} = (n_1, n_2, \ldots, n_N) \) and constructs all its translations with corresponding phases. For prime \( N \) one can choose

\[ |\vec{n}\rangle_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i kj/N} \hat{T}^j |\vec{n}\rangle. \] (31)

Here \( \hat{T}^j |\vec{n}\rangle \) indicates the shift on \( j \) elements of the initial sequence

\[ \hat{T}^j |n_1, n_2, \ldots, n_N\rangle \equiv |n_{N-j+1}, \ldots, n_N, n_1, n_2, \ldots, n_N-j\rangle. \] (32)

After performing one step, one takes an arbitrary element which is not in the sum (31), and repeats the construction till all elements are exhausted. Two states \( |\vec{n}\rangle \) and \( |\vec{m}\rangle \) are said to be equivalent or in the same equivalence class if they can be obtained from one another by a finite number of translations. For composite \( N \), there exist elements with primitive periods \( t \) (the size of the equivalence class) equal to a divisor of \( N \). The number of equivalence classes of size \( t \) is given by

\[ r(t) = \frac{1}{t} \sum_{d|t} 2^{\ell/d} \mu(d) \] (33)

where \( \mu(n) \) is the Möbius function: \( \mu(n) = (-1)^p \) if \( n \) is square-free number with \( n_p \) prime divisors, and \( \mu(n) = 0 \) if \( n \) is divisible on a square of a prime.

The total dimension of the basis with \( k = 0 \) is

\[ \mathcal{N}_{\text{tot}} = \sum_{n|N} r(n) = \frac{1}{N} \sum_{d|N} 2^{N/d} \phi(d) \] (34)

where \( \phi(n) \) is Euler’s totient function and equals the number of positive integers smaller than \( n \) and co-prime with it. For non-zero \( k \) the contribution of basis states with primitive period \( t < N \) (with \( t|N \)), is non-zero only when \( k \equiv 0 \mod N/t \). The simplest approximation

\[ \mathcal{N}_{\text{tot}} \approx \frac{2^N}{N} \] (35)

is sufficient in many cases when \( N \) is large. The total number of states with fixed momentum and fixed number of spins up \( \nu_{\text{tot}}(n) \) can also be approximated in a similar manner

\[ \nu_{\text{tot}}(n) \approx \frac{1}{N} C_n^2. \] (36)

After rewriting the Hamiltonian in the basis of states with fixed translational momentum, \( |\vec{n}\rangle_k \), it becomes complex and eigenfunctions

\[ \Psi_k = \sum_{\vec{n}} C_{\vec{n}} |\vec{n}\rangle_k \] (37)

are, in general, also complex.

Nevertheless, one cannot conclude that in the chaotic regime eigenstates and eigenfunctions are distributed as for a Gaussian unitary ensemble of random matrices (GUE). The reason is that for the model considered there exists another discrete symmetry, namely the geometric inversion.
Hamiltonian (5) is invariant under this transformation. Therefore, if \( \Psi \) is an eigenfunction \( \hat{S} \Psi \) is also an eigenfunction with the same energy. By definition the initial basis \( |\vec{n}\rangle \) transforms under \( \hat{S} \) as follows
\[
\hat{S}|n_1, n_2, \ldots, n_N\rangle = |n_N, n_{N-1}, \ldots, n_1\rangle.
\]
Functions with zero translational momentum transform into themselves under this inversion
\[
\hat{S}\Psi_0 = \epsilon \Psi_0
\]
where \( \epsilon = \pm 1 \) is the parity under the inversion. But functions with \( k \neq 0 \) are transformed up to a phase into functions with opposite momentum
\[
\hat{S}\Psi_k = \Psi_{N-k}.
\]
States \( |\vec{n}\rangle_k \) with fixed momentum \( k \) (see equation (31)) split into two groups under the inversion symmetry. The first group includes all states which under this inversion transform into themselves but with opposite momentum (up to a certain non-dynamical phase which can be included in the definition of basis states)
\[
\hat{S}|\vec{n}\rangle_k = |\vec{n}\rangle_{N-k}.
\]
In other words, the inversion of one element of this group is equivalent to a certain shift of this element. Such elements are called invariant under the inversion.

Elements of the second group are organised in different pairs transformed one to another by the inversion
\[
\hat{S}|\vec{n}\rangle_k = |\vec{n}'\rangle_{N-k}, \quad \vec{n}' \neq \vec{n}.
\]
We call them non-invariant elements.

The above arguments show that eigenfunctions with \( k \neq 0, N/2 \) can be written in the form
\[
\Psi_k = \sum_{\text{non-invariant}} \left( C^{(\text{non-in})}_{\vec{n}} |\vec{n}\rangle_k + \epsilon C^{(\text{non-in})\ast}_{\vec{n}} |\vec{n}'\rangle_k \right) + \sum_{\text{invariant}} C^{\text{in}}_{\vec{n}} |\vec{n}\rangle_k
\]
where \( \vec{n} \) and \( \vec{n}' \) are pairs of elements connected by the inversion. Taking real and imaginary parts of this expression demonstrates that all coefficients in the expansion can be chosen real, which indicates that the spectral statistics of this model is well described by GOE statistics even though the Hamiltonian is complex (see [41]).

For \( k = 0 \), eigenfunctions have a particular parity under the inversion, and all coefficients can be chosen real but invariant elements are identically zero for states with negative parity
\[
\Psi^{(\epsilon)}_0 = \sum_{\text{non-invariant}} \left( C^{(\text{non-in})}_{\vec{n}} |\vec{n}\rangle_0 + \epsilon C^{(\text{non-in})\ast}_{\vec{n}} |\vec{n}'\rangle_0 \right) + \frac{1 + \epsilon}{2} \sum_{\text{invariant}} C^{\text{in}}_{\vec{n}} |\vec{n}\rangle_0
\]
with \( \epsilon = \pm 1 \).

It is plain that the majority of elements at large \( N \) are non-invariant. Nevertheless, corrections from invariant elements, though small, are noticeable for \( N \) accessible in numerical calculations. The number of invariant elements is especially simple to find for prime \( N \) when all elements have the same primitive period. As \( N = 2K + 1 \) is odd, all invariant elements in the periodic cycle appear in pairs under the inversion except one element, which has to be invariant under the inversion. This element can be constructed by arbitrarily choosing the first
$K$ elements and reflecting them into another part of the cycle. The number of possibilities is $2^K$, and one has 2 choices for the central element; therefore, the total number of invariant elements is

$$N_{\text{inv}} = 2^{K+1} = 2^{[N/2]+1}$$

where $[x]$ is the integer part of $x$. The number of invariant elements with fixed total number of spins up is

$$\nu_{\text{inv}}(n) = C_{[n/2]}^{\lfloor n/2 \rfloor}.$$  \hspace{1cm} (46)

A similar expression may be obtained for even $N$.

According to the above statistical conjectures, coefficients $C_{\vec{n}_k}^{(n-in)}$ with $k \neq 0$ are distributed as complex Gaussian variables with zero mean and variance $\Sigma_{\vec{n}}^2$ calculated from $P_n(E)$ as in equation (15)

$$\Sigma_{\vec{n}}^2 = \frac{P_n(E)}{\rho(E)}, \quad \rho(E) = \sum_{n=0}^{N} \nu_{\text{tot}}(n) P_n(E)$$

with $P_n(E)$ given by equations (27) or (25).

Similarly, coefficients of invariant terms, $C_{\vec{n}_k}^{(in)}$, have real Gaussian distribution with the same variance. Therefore, when moments are calculated, one has to take into account corrections from different types of distributions (see equations (12)). For large $N$ one can neglect the existence of invariant elements and consider that for non-zero $k$ functions are complex, and for zero momentum functions are real.

As a typical example, we consider the participation ratio determined by

$$Pr(E_{\alpha}) = \left( \sum_{\vec{n}} |C_{\vec{n}}(E_{\alpha})|^4 \right)^{-1}.$$  \hspace{1cm} (49)

The simple Gaussian approximation of this quantity takes the form

$$Pr(E) \approx N_{\text{tot}} \frac{\left( 2^{-N} \sum_{n=0}^{N} C_n P_n(E) \right)^2}{\mathcal{R}_2 \left( 2^{-N} \sum_{n=0}^{N} C_n^2 P_n(E)^2 \right)}$$

where $P_n(E)$ is given by equation (27). Here we have used approximation (36) for the number of total terms with fixed $n$. $N_{\text{tot}} = \nu_{\text{tot}}(n) = 2^{N-n} / N$, $\mathcal{R}_2 = 2$ for states with non-zero momentum and $\mathcal{R}_2 = 3$ for zero momentum states.

In figures 3(a) and (b) we present the numerical results for this quantity for the Ising model with $\alpha = \lambda = 1$ and $N = 17$ spins for states with translational momentum $k = 2$ and $k = 0$ respectively. Notice that the left-hand side of the participation ratio (with $E < -10$) fluctuates strongly, and the use of statistical description at energies close to the ground state is questionable. In the same figures the above Gaussian approximation is also indicated. In general the agreement is good, but there exist noticeable differences from numerical calculations.

There are two different types of correction. The first is related with higher moments of the Hamiltonian, and these can be taken into account by modifications indicated in equation (25). This formula differs from the Gibbs-like formulae (23). The later usually gives better results, but as they require numerical calculations to find necessary Lagrangian multipliers, their use leads to non-transparent expressions where the relative importance of different terms is hidden. To get clear separation of different contributions, we prefer to use series expansion, as in
though this leads to big errors at large arguments. As in all cases the advocated statistical approach can, strictly speaking, be applied only in the bulk of the spectrum; it is not an important restriction. Another correction is related to the fact that in the expansion of eigenfunctions, there exist basis sets which are invariant under inversion (see equations (44) and (45)) and have different properties as compared to non-invariant terms.

For the moments of wave function coefficients with non-zero momentum, one gets the following formula

\[
\langle \sum_{\vec{n}} |\Psi_{\vec{n}}(E)|^{2q} \rangle = \left( R_q^{\text{complex}} + [R_q^{\text{real}} - R_q^{\text{complex}}]\delta_q(E) \right) \frac{\sum_{n=0}^{N} \nu_{\text{tot}}(n) P_n(E)}{\left( \sum_{n=0}^{N} \nu_{\text{tot}}(n) P_n(E) \right)^{q}}
\]

where

\[
\delta_q(E) = \frac{\sum_{n=0}^{N} \nu_{\text{inv}}(n) P_n(E)}{\sum_{n=0}^{N} \nu_{\text{tot}}(n) P_n(E)} \approx \frac{N_{\text{inv}}}{N_{\text{tot}}}.
\]

The approximative expression of \(\delta_q \approx \delta = N_{\text{inv}}/N_{\text{tot}}\) corresponds to the uniform approximation where the summation over the invariant set is proportional to the summation over all elements. When the invariant elements are taken into account in equation (50) for the participation ratio, one has instead of \(R_2 = 2\) to use \(R_2 = 2 + \delta\) for states with non-zero translational momentum.

For \(k = 0\) all eigenfunctions are either symmetric or anti-symmetric with respect to inversion. The full dimensions of symmetric and anti-symmetric sub-spaces (i.e. the number of independent real coefficients) are (see (45))

\[
N_{\pm} = \frac{1}{2} (1 \pm \delta) N_{\text{tot}}.
\]
Eigenvalues corresponding to the two sub-spaces are statistically independent, so the local densities of these eigenvalues are proportional to the above numbers,

$$\rho_\pm(E) = \frac{1}{2}(1 \pm \delta)\rho(E).$$  \hfill (54)

Correspondingly, the strength function (13) has contributions from both the symmetric and anti-symmetric states. According to the above conjecture, each term in equation (45) has real Gaussian distribution with the same variance. But these variances depend on the number of independent components. In the uniform approximation variances in the different sub-spaces are

$$\tilde{\Sigma}^2_{\vec{n}}(\pm) = \frac{2}{1 \pm \delta} \Sigma^2_{\vec{n}}.$$  \hfill (55)

For positive parity states the variance of \( C_{\vec{n}}^{(m)} \) equals \( \tilde{\Sigma}^2_{\vec{n}}(+) \), but variance of non-invariant coefficients, \( C_{\vec{n}}^{(n-m)} \) is, according to our definition (45), equal to \( \frac{1}{2} \tilde{\Sigma}^2_{\vec{n}}(+) \). Calculating the \( 2q^{th} \) moments of eigen-functions in the initial spin basis gives

$$M_q^{(+)} = \frac{1 - \delta + 2^q\delta}{(1 + \delta)^q} M_q, \quad M_q^{(-)} = \frac{1}{(1 - \delta)^q} M_q.$$  \hfill (56)

As we do not separate states with definite parity in the numerical calculation, the mean value of moments is (see (54))
The last equation is valid in the first order on $\delta$. One can perform exact summation with the substitution $\delta \rightarrow \delta q$, and obtain a formula with the same accuracy. In particular, the participation ratio is given by equation (50) but instead of $R_2 = 3$ one has to use $R_2 = 3(1 + \delta)$.

Numerically calculated inverse moments for the Ising model with the same values of parameters as above, and for zero momentum sector, are plotted in figure 4 for different values of $q$ and compared with equation (57). Participation ratios for different values of $\alpha$, $\lambda$, and $N$ are shown in figure 5. In all cases, the statistical model with corrections gives good agreement across the bulk of the spectrum.

4. Conclusion

Exact determination of wave functions in many-body problems is rarely possible. For the overwhelming majority of models numerical calculations remain the only way of getting information about wave functions. In certain models (called chaotic) the complexity of wave
functions is so high that, with a good precision, they can be considered as random, and the development of statistical methods becomes a valuable alternative to numerics. This in turn enables meaningful formulations of questions about the foundations of quantum thermodynamics, which is a work in progress in modern physics.

In this paper, we have discussed the construction of statistical models for eigenfunctions in a particular example of one-dimensional models of \( N \) interacting spin-\( \frac{1}{2} \), namely the Ising model in transverse and longitudinal fields. The model is simple enough to demonstrate general phenomena without unnecessary complications.

The investigation of the model is restricted to the chaotic regime when all coupling constants are of the same order. It is attested that for large number of spins, eigenfunction coefficients in the bulk of the spectrum are well approximated by Gaussian functions with zero mean and variance determined analytically from the Hamiltonian. Such asymptotic results are supposed to give a good description of wave functions only in the thermodynamic limit, when the number of spins tends to infinity. For numbers of spins accessible in numerical calculations there exist small but noticeable deviations from asymptotic formulae, and a large part of the paper is devoted to calculations of different types of correction.

The first type of correction is common. It appears every time one calculates corrections to the central limit theorem by taking into account higher order moments. The best way to incorporate them is to use Gibbs-like formulae, which requires numerical calculations to find necessary parameters. When the number of spins is large, contributions of higher order cumulants are small, and one can incorporate these corrections by using simple series in Hermite polynomials.

The second type of correction appears for models with periodic boundary conditions, and is related to the conservation of translational momentum and the parity transformation. Due to symmetry considerations, some coefficients of the wave function expansion are real and others complex. In addition, for certain values of translational momentum the symmetries impose that a subset of coefficients is zero.

When all corrections are included, the resulting statistical expressions agree well with the results of numerical calculations of wave function moments, which clearly confirms that wave functions are fully extended in the bulk of the spectrum.

All constructions used in the paper can be applied only in the bulk of the spectrum, where eigen-energies are of the order of the square root of the number of spins. The structure of eigenfunctions close to the ground state and to the highest energy state are more complicated. In particular, the ground state eigenfunctions for practically all spin chains have non-trivial multifractal dimensions [35]. The existence or not of a sharp transition between multifractal behaviour of boundary states and the extended states in the bulk and more careful determination of the nature of these states require additional investigations. The behaviour of wave functions for small or large values of coupling constants in the bulk is different from that investigated so far. The existence in such cases of perturbation series for large but finite number of spins leads to multiple peaks in both the average spectral density [36] and the moments of wave functions. These questions will be discussed elsewhere.

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Appendix. Calculations of first moments for the Ising model

Hamiltonian of the Ising model in two fields (5) can be rewritten in the form

$$
\mathcal{H} = \mathcal{H}_x + \mathcal{H}_z
$$

where

$$
\mathcal{H}_x = -\sum_n h_n, \quad h_n = \sigma_n^x \sigma_{n+1}^x + \alpha \sigma_n^z, \quad \mathcal{H}_z = -\lambda \sum_n \sigma_n^z.
$$

The purpose of this Appendix is to calculate moments of the Hamiltonian $\mu_k(n) \equiv \langle \vec{n} | \mathcal{H}^k | \vec{n} \rangle$ taken between basis states with definite projection of spins in each point.

As $\mathcal{H}_x | \vec{n} \rangle = E_n | \vec{n} \rangle$ with $E_n = \lambda (N - 2n)$ where $n$ is the number of spins up, one has $\langle \vec{n} | \mathcal{H}_x^k | \vec{n} \rangle = E_n^k$. Therefore the first two moments of the Ising model Hamiltonian are

$$
\mu_1(n) = E_n, \quad \mu_2(n) = N(1 + \alpha^2) + E_n^2
$$

and the variance (i.e. the second cumulant)

$$
\sigma_n^2 = \langle \vec{n} | \mathcal{H}_x^2 | \vec{n} \rangle - \langle \vec{n} | \mathcal{H}_x | \vec{n} \rangle^2 = N(1 + \alpha^2).
$$

Hamiltonian $\mathcal{H}_x$ flips spins so simple counting gives

$$
\langle \vec{n} | \mathcal{H}_x | \vec{n} \rangle = 0, \quad \langle \vec{n} | \mathcal{H}_x^2 | \vec{n} \rangle = N(1 + \alpha^2), \quad \langle \vec{n} | \mathcal{H}_x^3 | \vec{n} \rangle = -6N\alpha^3.
$$

The forth power of $\mathcal{H}_x$ is

$$
\mathcal{H}_x^4 = \sum_n h_n^4 + 4 \sum_{m \neq n} h_m^2 h_n^2 + 24 \sum_{m < n \neq k < r} h_m h_n h_r h_k.
$$

By inspection one can check that only the even powers of $h_n$ give contribution to $\langle \vec{n} | \mathcal{H}_x^4 | \vec{n} \rangle$ and

$$
\langle \vec{n} | \mathcal{H}_x^4 | \vec{n} \rangle = 3N^2(1 + \alpha^2)^2 + N(24\alpha^2 - 2 - 2\alpha^4).
$$

As it does not depend on $n$, it coincides with $1/2^N \text{Tr} \mathcal{H}_x^4$ [36].

The third power of the Hamiltonian is

$$
\mathcal{H}^3 = \mathcal{H}_x^3 + \mathcal{H}_z^3 + \mathcal{H}_x \mathcal{H}_z \mathcal{H}_x + \mathcal{H}_x \mathcal{H}_z^2 + \mathcal{H}_z \mathcal{H}_x^2 + \mathcal{H}_z \mathcal{H}_x \mathcal{H}_z + \mathcal{H}_x^2 \mathcal{H}_z + \mathcal{H}_z^2 \mathcal{H}_x.
$$

Due to equation (A.5), one can find all contributions to $\langle \vec{n} | \mathcal{H}^3 | \vec{n} \rangle$ except $\mathcal{H}_x^3 \mathcal{H}_z$. One gets

$$
\langle \vec{n} | \mathcal{H}^3 | \vec{n} \rangle = \langle \vec{n} | \mathcal{H}_x^3 | \vec{n} \rangle + 2E_n \langle \vec{n} | \mathcal{H}_x^2 | \vec{n} \rangle + 3E_n^2 \langle \vec{n} | \mathcal{H}_x | \vec{n} \rangle + E_n^3 + \langle \vec{n} | \mathcal{H}_x \mathcal{H}_z \mathcal{H}_x | \vec{n} \rangle.
$$

As has been discussed above, $\mathcal{H}_z$ acting on a product state $| \vec{n} \rangle$ changes the number of spins which can be symbolically written as follows

$$
\mathcal{H}_z \overset{n \to n-1}{\longrightarrow} -1, \ 2k \text{ times} \quad (A.10)
$$

$$
\mathcal{H}_z \overset{n \to n+1}{\longrightarrow} -\alpha, \ (N - n) \text{ times} \quad (A.11)
$$

$$
\mathcal{H}_z \overset{n \to n-1}{\longrightarrow} \overset{x \to x-1}{\longrightarrow} -\alpha, \ n \text{ times} \quad (A.12)
$$
\[ \mathcal{H}_s \longrightarrow -1, \ (N - n - k) \text{ times} \]  
\[ \mathcal{H}_s \longrightarrow -1, \ n - k \text{ times} \]

where \( k \) is the number of groups of spins in the same direction.

One gets
\[ \mathcal{H}_s \mathcal{H}_s |n\rangle = -2kE_n |n\rangle - (N - n - k)E_{n+2}|n + 2\rangle - (n - k)E_{n-2}|n - 2\rangle - \alpha(N - n)E_{n+1}|n + 1\rangle - \alpha nE_{n-1}|n - 1\rangle. \tag{A.15} \]

Here \( |n\rangle \) stays for a state with \( n \) spins up.

Finally the action \( \mathcal{H}_s \) will select the corresponding term, and one obtains
\[ \langle \vec{m}|\mathcal{H}_s^2|\vec{m}\rangle = \frac{1}{2}kE_n + (N - n - k)E_{n+2} + (n - k)E_{n-2} + \alpha^2[(N - n)E_{n+1} + nE_{n-1}] \]
\[ = (N - n)E_{n+2} + nE_{n-2} + \alpha^2[(N - n)E_{n+1} + nE_{n-1}] \]
\[ = [N(1 + \alpha^2) - 4 - 2\alpha^2]E_n. \tag{A.16} \]

Combining all terms together gives that for \( N \geq 4 \)
\[ \mu_3(n) \equiv \langle \vec{m}|\mathcal{H}_s^3|\vec{m}\rangle = E_n^3 + 2(N + k)E_n + (N - n - k)E_{n+2} + (n - k)E_{n-2} + \alpha^2[2kE_n + (N - n)E_{n+1} + nE_{n-1} - 6N] \]
\[ = E_n^3 + [3(1 + \alpha^2)N - 4 - 2\alpha^2]E_n - 6N\alpha^2. \tag{A.17} \]

Notice that the dependence on \( k \) disappears.

Finally, the third cumulant, \( k_3 = 3\mu_3 - 3\mu_2\mu_1 + 2\mu_3^2 \), takes the following value
\[ k_3 = -6N\alpha^2 - 2E_n(\alpha^2 + 2). \tag{A.18} \]

The next step is the calculation of the fourth moment of Hamiltonian (A.1)
\[ \mathcal{H}_s^4 = \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4 + \mathcal{H}_s^4. \tag{A.19} \]

When calculating \( \langle \vec{m}|\mathcal{H}_s^4|\vec{m}\rangle \) many terms are known from the above expressions
\[ \langle \vec{m}|\mathcal{H}_s^4|\vec{m}\rangle = \langle \vec{m}|\mathcal{H}_s^2|\vec{m}\rangle + 2\langle \vec{m}|\mathcal{H}_s^2|\vec{m}\rangle E_n + 3\langle \vec{m}|\mathcal{H}_s^2|\vec{m}\rangle E_n^2 + 4\langle \vec{m}|\mathcal{H}_s^2|\vec{m}\rangle E_{n+1} \]
\[ + 2\langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s|\vec{m}\rangle E_n + E_n^3 + \langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s|\vec{m}\rangle \]
\[ + \langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s^2|\vec{m}\rangle + \langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s^2|\vec{m}\rangle. \tag{A.20} \]

Only three last quantities require separate calculation.

Term \( \langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s^2|\vec{m}\rangle \) can be performed as above and the result is the same as equation (A.16) but with substitution \( E_{n+1} \rightarrow E_{n+1}^2 \)
\[ \langle \vec{m}|\mathcal{H}_s^2|\mathcal{H}_s^2|\vec{m}\rangle = 2kE_n^2 + (N - n - k)E_{n+2}^2 + (n - k)E_{n-2}^2 + \alpha^2((N - n)E_{n+1}^2 + \alpha^2nE_{n-1}^2 \]
\[ = -32k\lambda^2 + (N - 8)E_1^2 + 16N\lambda^2 + \alpha^2[(N - 4)E_2^2 + 4N\lambda^2]. \tag{A.21} \]

Operator \( \mathcal{H}_s \) flips one or two nearby spins, therefore only the following contributions are non-zero
\begin{align}
\langle \vec{n} | \hat{H}_n^2 \hat{H}_z | \vec{n} \rangle &= -2\alpha^2 \langle \vec{n} | \left( \sum_n \sigma^x_n \sigma^x_{n+1} \right) \left( \sum_{n_1} \sigma^x_{n_1} \right) \hat{H}_z \left( \sum_n \sigma^x_n \right) | \vec{n} \rangle \\
&\quad - \alpha^2 \langle \vec{n} | \left( \sum_{n_1} \sigma^x_{n_1} \right) \left( \sum_{n_2} \sigma^x_{n_2} \right) \hat{H}_z \left( \sum_n \sigma^x_n \sigma^x_{n+1} \right) | \vec{n} \rangle. 
\end{align}
\tag{A.22}

Consider the first term. Operator \( \sum_n \sigma^x_n \) acting on \( | \vec{n} \rangle \) produces \( N - n \) states with \( n + 1 \) spins up and \( n \) states with \( n - 1 \) spins down (see (A.11) and (A.10)). Operator \( \hat{H}_z \) multiplies them by \( E_{n+1} \) and \( E_{n-1} \) correspondingly. Other terms should combine to produce the initial state. In the end one gets

\begin{align}
\langle \vec{n} | \hat{H}_n^2 \hat{H}_z | \vec{n} \rangle &= -4\alpha^2 [ (N - n) E_{n+1} + n E_{n-1} ] \\
&\quad - 2\alpha^2 [ 2k E_n + (N - n - k) E_{n+2} + (n - k) E_{n-2} ] = -2\alpha^2 (3N - 8) E_n. 
\end{align}
\tag{A.23}

Taking into account all terms, we obtain that for \( N \geq 5 \)

\[ \mu_4(n,k) \equiv \langle \vec{n} | \hat{H}_4 | \vec{n} \rangle = E_n^4 + E_n^2 (6N(1 + \alpha^2) - 16 - 8\alpha^2] + 8E_n (4 - 3N) \alpha^2 \\
+ 3N^2 (1 + \alpha^2)^2 - 32k \lambda^2 + N(24\alpha^2 - 2 - 2\alpha^4 + 16 \lambda^2 + 4 \lambda^4 \alpha^2). \]
\tag{A.24}

The forth cumulant by definition is the following combination, \( k_4 \equiv 4\mu_3 \mu_1 - 3\mu_2^2 + 12\mu_2 \mu_1^2 - 6\mu_1^4 \). Simple calculations give

\[ k_4 = 32\alpha^2 E_n - 32k \lambda^2 + N(24\alpha^2 - 2 - 2\alpha^4 + 16 \lambda^2 + 4 \lambda^4 \alpha^2). \]
\tag{A.25}

In these calculations we used, instead of \( k \), its mean value in all states with fixed \( n \), \( k = n(N - n)/(N - 1) \) [36].

All the above calculations were performed without selecting a particular translational momentum. If instead of basis set \( | \vec{n} \rangle \) one uses states which are eigenstates of the translational operator as in (30), moments may be different. We check that for lower order moments the difference is small, and does not noticeably change the results. For simplicity we ignore such corrections.

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