An algorithm for hiding and recovering data using matrices

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We present an algorithm for the recovery of a matrix \( M \) (non-singular \( \in \mathbb{C}^{N \times N} \)) by only being aware of two of its powers, \( M_{k_1} := M^{k_1} \) and \( M_{k_2} := M^{k_2} \) \((k_1 > k_2)\) whose exponents are positive coprime numbers. The knowledge of the exponents is the key to retrieve matrix \( M \) out from the two matrices \( M_{k_1} \). The procedure combines products and inversions of matrices, and a few computational steps are needed to get \( M \), almost independently of the exponents magnitudes. Guessing the matrix \( M \) from the two matrices \( M_{k_1} \), without the knowledge of \( k_1 \) and \( k_2 \), is comparatively highly consuming in terms of number of operations. If a private message, contained in \( M \), has to be conveyed, the exponents can be encrypted and then distributed through a public key method as, for instance, the DF (Diffie-Hellman), the RSA (Rivest-Shamir-Adleman), or any other.

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I. INTRODUCTION

The sender of a message \( M \) can choose to convey it to a receiver as a single continuous string of numbers, in a chosen base as, binary, decimal, hexadecimal, etc... or he also can choose to use the matrix architecture, where each entry conceals part of the message. He can also split a string, or a matrix, in several parts to be sent separately. Besides being easy to handle, the structure and properties of matrices allow to accommodate not only one but several messages if one considers that each entry, or each row (or column), corresponds to an instruction or a sentence. Now, if the message must be private, or confidential, then, as it was already practiced in the antiquity (for instance, by Julius Cesar) the message must be encrypted and hopping that only the receiver could get access and knows the procedure to decrypt it. To our knowledge the use of matrices to encode messages was initiated by L. S. Hill [1]. The Hill cipher is a polygraphic substitution that makes use of matrix multiplications in order to change a plaintext letters into a ciphertext. One of the basic components of classical ciphers is the substitution cipher: a ciphertext matrix \( C \) is obtained by multiplication of a plaintext matrix \( P \) by a key matrix \( K \), \( C = KP \mod (q) \), i.e., by a linear transformation to be followed by an arithmetic modular operation, \( q \) being the number of digits. So a square matrix \( C \), of order \( N \) can host, for instance, \( 2N^2 \) messages to be conveyed by the sender \( A \) to the receiver \( B \). More recently, a report about the usefulness of matrices in public-key cryptography was published [2].

When a sender \( A \) (aka Alice, in the jargon) wants to remit a message contained in a matrix \( A \) to a receiver \( B \) (aka Bob), such that it could not be known by any third person \( E \) (aka the eavesdropper Eve, that is also a cryptanalyst), it is necessary to encode \( A \mapsto M \) in some special form such to be hard to be decoded by Eve, conceding that she has access to \( M \). That this matrix is difficult to decrypt means that even if Eve succeeds to decode \( M \) and gets the message, the time she consumed for the task is long enough so that the acquired data becomes already obsolete. The standard procedure to encrypt \( A \) into \( M \) consists in producing of a secret key owned only by Bob that he uses to get \( M \) in a very short time compared to the one Eve needs to arrive at the same result.

If higher confidentiality of the message is needed, then a second layer of encryption can be implemented. For instance, a matrix \( M \), already subjected to a first encryption, is subjected to a second one, becoming matrix \( Z \). An interesting procedure could be the calculation of \( Z := M^k \) the \( k \)-th \((k \text{ an integer})\) power of a non-singular square matrix defined in the field of complex numbers \( M \in \mathbb{C}^{N \times N} \). The inverse process to extract the unknown \( M \) from a know \( Z \) consists in seeking a solution \( X \) (or more than one) of the equation \( X = Z^{1/k} \) and expect that \( X = M \). The direct procedure consists in diagonalizing \( Z \) and to each eigenvalue \( z_i \) solve the equation \( x^k - z_i = 0 \) and the zeros \( x_{i,l} \), \( l = 1,...,k \) will result; any zero can be one eigenvalue of matrix \( M \). Here lies the ambiguity! An eavesdropper may have access to the matrix \( Z \) but Alice and Bob have to keep \( k \) as they secret key. Several algorithms are proposed in Ref. [3] that make use of iterative approaches, which however lead to approximate solutions.

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Nevertheless, one could envisage another method still based on powers of a matrix $M$, making the unraveling of $Z$ less time consuming than to construct it, since $k-1$ matrix multiplications are necessary to construct $M^k$, without storage of intermediary results. We propose an algorithm to reconstruct a matrix $M \in \mathbb{C}^{N \times N}$ to be conveyed from Alice to Bob, having arbitrary eigenvalue multiplicity (degeneracy). The only requirement is that $M$ be non-singular [13]. The procedure goes as follows: Alice compute two powers of $M$, with exponents that are two positive coprime (relatively prime) numbers, $k_1$ and $k_2$, such that $M_{k_1} := M^{k_1}$ and $M_{k_2} := M^{k_2}$, and sends the matrices $M_{k_1}$ and $M_{k_2}$ to Bob, publicly or privately. Bob’s task consists in recovering $M_1 := M$ from those two matrices in a quite short time (the minimal number of matrix operations), before it could be unraveled by a third party. The knowledge of the exponents, $k_1$ and $k_2$, is the key element to invert the process in order to get $M_1$. The exponents can be either shared by Alice and Bob previously, or they can be encoded, creating an enciphering key, by using, for instance, the Diffie-Hellman proposal [4], the RSA method [5] or the quantum BB84 one [6], and then distributed by Bob (indeed, it is Bob that could determine the values $k_1$ and $k_2$ that Alice should use), one key can be made public (a unique key for many several senders) or private (one key for each message sender) while the other one is Bob’s own key (or keys) that he keeps secretly.

Regarding the computational time consumption, there are efficient algorithms that reduce the necessary time to calculate the product of two square $N \times N$ matrices to $O(N^{2.376})$ [7], instead of the direct method that is $O(N^3)$. The inversion of an invertible matrix consumes the same amount of time [7]. For Alice, the time consumed to calculate the powers of a matrix $M^k$ can be reduced from $(k-1)$ to approximately $2\sqrt{k}$ for $k \gg 1$, if storage of lower powers ($< k/2$) matrices is possible. Regarding our algorithm the inverse operation, extracting $M$ from the matrices $M_{k_1}$ and $M_{k_2}$, turns out to be much less time consuming (to Bob) than to produce it (by Alice). In the following sections we present the algorithm that can be used to encrypt messages to be sent from Alice to Bob and still keeping a degree of confidentiality against a skillful eavesdropper. To illustrate the procedures of Alice and Bob we present a simple example.

Example 1. Alice wants to send a message to Bob that is encoded in the non-singular matrix $M \in \mathbb{C}^{2 \times 2}$. Alice computes two powers of $M$, choosing, for instance, the exponents $k_1 = 17$ and $k_2 = 11$ (or receives them from Bob), and she constructs the matrices $C := M_{17}$ and $D := M_{11}$,

$$C = \begin{pmatrix}
-8229303833 + i9607837931 & 159011972369 + i193528618710 \\
221214366306 + i117483350975 & 600042299893 - i104462213832
\end{pmatrix}, \quad (1)
$$

and

$$D = \begin{pmatrix}
-717135 - i728137 & -17385865 - i6190286 \\
-18429370 + i972761 & -31527077 + i28463112
\end{pmatrix} \quad (2)
$$

respectively, and sends them to Bob, through some channel, openly or privately. Alice then produces an enciphering key, $K_E$, for the numbers $k_1$ and $k_2$ (or she does not need to send anything else if she received $k_1$ and $k_2$ from Bob) that she sends to Bob through another channel. Bob uses the deciphering key $K_D$, that only he knows, to retrieve $k_1$ and $k_2$ and he then makes use of the algorithm (to be explained below), that consists in doing the matrix operations $C^2D^{-3}$ that results in

$$M = \begin{pmatrix}
1 + 4i & 3 - 2i \\
2 - 3i & -1 - 5i
\end{pmatrix}. \quad (3)
$$

The algorithm dispenses the need of calculating eigenvectors, eigenvalues or to perform any decomposition of $C$ and $D$ and do not depend on the possible eigenvalue multiplicities of matrices (1) and (2). Once the matrices $C$ and $D$ are stored, the number of matrix operations Bob needs to perform are 4: one matrix inversion ($D^{-1}$) (stored), one product $C D^{-1}$ (stored), two products ($C^{-1}$) ($D^{-1}$) ($D^{-1}$), whereas in order to construct $C$ and $D$ Alice has to calculate 16 matrix multiplications without storage of powers of $M$ or 7 multiplications considering storage, as to be explained in details below. For a different pair of numbers, $k_1 = 38$ and $k_2 = 23$, for instance, as exponents, the sequence of operations needed to retrieve $M$ is a different one, being $C^{-3}D^5$.

The pair of exponents (-3,5) is unique for each pair of coprime integers $k_1$ and $k_2$, as shown in Theorem 3 in Appendix A. As observed by D. Knuth [3], working numerically with integers or with fractions, as entries of matrix $M$, instead of using floating point numbers, avoids accumulated rounding errors, so the accuracy of the calculations is absolute. In the case where an error occurs during the transmission of the key conveyed by Alice to Bob (or vice versa), for instance instead of receiving $k_2 = 11$ Bob receives the number 12, then, according to the proposed Algorithm [4] below, he will calculate a different sequence of operations involving matrices $C$ and $D$, namely $C^5 D^{-7}$, and he will get a fully different matrix,

$$M' = \begin{pmatrix}
5319 + i84141 & -1142 + i119291 \\
44827 + i110554 & 288728 + i196979
\end{pmatrix}, \quad (4)$$
instead of the sought matrix $[3]$. Therefore, the sequence of operations to get the matrix $M$, from matrices $C$ and $D$, is unique for each pair of coprime numbers $k_1$ and $k_2$. If an error occurs by changing $k_1$ or $k_2$, even by one unit, the sequence of operations changes, resulting in a wrong matrix as in Eq. $4$.

II. ALGORITHM AND EXAMPLES

We consider a variant of Euclid’s algorithm, see Lemma $1$ (in Appendix A), namely, instead of looking for the gcd (greatest common divisor) of a pair of arbitrary coprime integers – gcd($k_1$, $k_2$) = 1 – we are essentially interested in determining the sequence of quotients that are inherent to the algorithm. For one unit, the sequence of operations changes, resulting in a wrong matrix as in Eq. (4), and $D$ instead of the sought matrix (3). Therefore, the sequence of operations to get the matrix of matrix $M$.

Algorithm 1.

Since $k_1 = q_2 k_2 + k_3$ we write the decomposition $M_{k_1} = M_{k_2} ((M_{k_2})^{-1})^{q_2}$ to get the matrix $M_{k_2}$, again, the numbers ($k_2$, $k_3$) are also coprime which permits us to write the decomposition $M_{k_3} = M_{k_2} ((M_{k_3})^{-1})^{q_3}$ to get the matrix $M_{k_3}$. Continuing this procedure by reducing the powers of matrix $M_{k_1}$, one arrives at the original matrix $M_1$ (the seed), $M_1 = M_{k_{r-1}} ((M_{k_r})^{-1})^{q_r} = M_{k_{r-1}} ((M_{k_{r-1}}^{-1})^{q_r}$. All the $r-1$ pairs ($k_1$, $k_2$), ($k_2$, $k_3$), ..., ($k_{r-1}$, $k_r$) are coprime numbers. The choice by Alice (or by Bob) of ($k_1$, $k_2$) being coprime is an important feature in order that the last equation of the sequence (5) be $k_{r-1} \mod k_r = 1$; the value $k_{r+1} = 1$ is necessary to retrieve the seed matrix $M$. The set of quotients ($q_2, q_3, ..., q_r$) constitutes the essential element to construct the sequence of operations:

\[
M_{k_1} = M_{k_2} ((M_{k_2})^{-1})^{q_2}, \\
M_{k_3} = M_{k_2} ((M_{k_3})^{-1})^{q_3}, \\
\vdots \\
M_{k_{r+1}} = M_1 = M_{k_{r-1}} ((M_{k_r})^{-1})^{q_r}.
\]

From another point of view, as to be shown below, any sequence of integers ($q_2, q_3, ..., q_r, q_{r+1} = k_r$) expressed as a continued fraction leads to the ratio $k_1/k_2$ where $k_1$ and $k_2$ are coprime. Because of the one-to-one correspondence, Alice (or Bob) can choose either a pair ($k_1$, $k_2$) or a sequence of quotients ($q_2, q_3, ..., q_r$) to construct the matrices $C$ and $D$. However, for not facilitating the task of decryption by Eve, it is more convenient to choose judiciously the sequence of integers that will produce the pair of exponents. The number of operations in the form of products of matrices (without storage) and inversions is

\[
N_{op,r} = \sum_{i=2}^{r} (q_i + 1).
\]

For any pair of coprime numbers ($k_1, k_2$) the uniqueness of the sequence of quotients is proved in Theorem $3$.

Below we illustrate the use of the Algorithm $1$ by two examples

Example 2. For the prime numbers $k_1 = 1019$ and $k_2 = 239$, the sequence of the modular calculation goes as shown in Table $4$.  

---
\[
\begin{array}{|c|c|c|c|c|}
\hline
i+1 & k_{i+1} & k_i \mod k_{i+1} = k_{i+2} & q_{i+1}k_{i+1} + k_{i+2} = k_i & q_{i+1} \\
\hline
2 & 239 & 1019 \mod 239 = 63 & 4 \times 239 + 63 = 1019 & 4 \\
3 & 63 & 239 \mod 63 = 50 & 3 \times 63 + 50 = 239 & 3 \\
4 & 50 & 63 \mod 50 = 13 & 1 \times 50 + 13 = 63 & 1 \\
5 & 13 & 50 \mod 13 = 11 & 3 \times 13 + 11 = 50 & 3 \\
6 & 11 & 13 \mod 11 = 2 & 1 \times 11 + 2 = 13 & 1 \\
7 & 2 & 11 \mod 2 = 1 & 5 \times 2 + 1 = 11 & 5 \\
8 & 1 & 2 \mod 1 = 0 & 2 \times 1 + 0 = 2 & 2 \\
\hline
\end{array}
\]

TABLE I: Sequence of the modular calculation with remainders and quotients

where \((q_2, q_3, ..., q_8) = (4, 3, 1, 3, 1, 5, 2), r = 7, \text{ and all the pairs } (1019, 239), (239, 63), (63, 50), (50, 13), (13, 11), (11, 2), (2, 1) \text{ are coprime. From these results the six sequential operations to be done by Bob are:}

\[
\begin{align*}
M_{63} &= M_{1019} \left( (M_{239})^{-1} \right)^4, \\
M_{50} &= M_{239} \left( (M_{63})^{-1} \right)^3, \\
M_{13} &= M_{63} (M_{50})^{-1}, \\
M_{11} &= M_{50} \left( (M_{13})^{-1} \right)^3, \\
M_{2} &= M_{13} (M_{11})^{-1}, \\
M_{1} &= M_{11} (M_{2})^{-1},
\end{align*}
\]

Each step is calculated as \( M_{k_{i+2}} = M_{k_i} \left( (M_{k_{i+1}})^{-1} \right)^{q_{i+1}} \), and in each row, in parenthesis, we give the number of operations (matrix multiplications plus inversions) which is \((q_i + 1)\). From Table I we observe that the total number of operations calculated from Eq. (9) is \(N_{op,7} = 23\). Calling \(M_{1019} = C\) and \(M_{239} = D\), we then rewrite each row in the set of Eqs. (9) as

\[
\begin{align*}
M_{63} &= C (D^{-1})^4, \\
M_{50} &= (C^{-1})^3 D^{13}, \\
M_{13} &= C^4 (D^{-1})^7, \\
M_{11} &= (C^{-1})^{15} D^{64}, \\
M_{2} &= C^{19} (D^{-1})^{81}, \\
M_{1} &= (C^{-1})^{110} D^{469},
\end{align*}
\]

and the necessary number of operations with storage to be performed in the RHS of Eq. (10) is: 6 matrix inversions and 15 multiplications \((3 + 3 + 1 + 3 + 1 + 4 = 15)\). So the total number of matrix operations reduces to 21.

We anticipate the result of Theorem 3 that says: in the RHS of each equation in the set (10) the exponents in \(C^n D^l\) are related through the equation

\[
p_k t_1 + t_2 k_2 = k_{i+2}, \quad l = 1, ..., 6,
\]

where \(p_k\) and \(t_l\) are two integers such that \(p_k t_1 < 0\), whose dependence on the quotients is shown in Table I. and the last line, \(p_k t_1 + t_2 k_2 = 1\), is known as Bezout identity. The moduli of the coefficient \([p_k]\) and \([t_l]\) can be used for calculating the number of necessary operations (matrix products) to retrieve the matrices \(M_k\) out of the matrices \(C\) and \(D\). In fact, instead of sending to Bob the coprime numbers \(k_1\) and \(k_2\) Alice could choose to send the coefficients of the Bezout relation, \(p_k = -110\) and \(t_6 = 469\), that are not necessarily coprime numbers, thus sparing time for Bob to do any further calculations. However this approach has a drawback, it asks a lot of memory for storage since huge matrices \((C, D)\) have to be exponentiated, \(C^n D^l\), with quite large exponents, whereas considering the sequence of matrices of the kind \(\left( (M_{k_i})^{-1} \right)^{q_i}\) where the exponents \((the\ quotients \ q_2, q_3, ..., q_r, q_{r+1})\) are quite small compared with the coefficient in the Bezout identity - so sparing the use of large amount of bytes and computational time - that turns out to be advantageous to Bob if his computational capabilities are limited.
Example 3. For the coprime numbers \( k_1 = 1001 \) and \( k_2 = 213 \), the sequence of modular calculations is given in Table III with the quotients \((4, 1, 2, 3, 21)\). From Eq. \((12)\), the number of operations (without storage) is

\[
N_{\text{op}}, 5 = 14 \text{ and the sequence of calculation steps is}
\]

\[
M_{149} = M_{1001} \left( (M_{213})^{-1} \right)^4 = C \left( D^{-1} \right)^4, \quad (12a)
\]

\[
M_{64} = M_{213} \left( M_{149}^{-1} \right) = C^{-1}D^5, \quad (12b)
\]

\[
M_{21} = M_{149} \left( (M_{64})^{-1} \right)^2 = C^3 \left( D^{-1} \right)^{14}, \quad (12c)
\]

\[
M_1 = M_{64} \left( (M_{21})^{-1} \right)^3 = (C^{-1})^{10} D^{47}. \quad (12d)
\]

The relations \( p_t k_1 + t_t k_2 = k_{t+2} \) \((p_t t_t < 0)\) are verified in Table IV.

The necessary times needed to encrypt and to decrypt a message are unbalanced. If Alice has a high performance computer but Bob does not or he reckons on a short time to decode the encryption, then the algorithm works well for him as long as he knows the numbers \( k_1 \) and \( k_2 \), whereas an eavesdropper should consume a time that can be comparatively much larger to break the code and get \( M_1 \), as to be discussed below. In order to retrieve \( M_1 \) from, for instance, the matrices of example \((13)\), \( C = M_{1001} \) and \( D = M_{213} \), Bob has to perform 13 operations: the factor \((C^{-1})^{10}\) needs 4 and \( D^{47}\) needs 8, thus to get \( M_1 = (C^{-1})^{10} D^{47}\), 13 operations with storage of partial products are necessary. However, to produce the matrices \( M_{1001} \) and \( M_{213} \) Alice has to perform 14 plus 3 matrix products (with storage) respectively, so 17 in total.

### III. THEOREMS: CONTINUED FRACTIONS AND QUOTIENTS

We now explore the relation between the exponent \( k_1 \) and \( k_2 \) and a continued fraction involving the quotients, and show its usefulness for Alice to choose the numbers \( k_1 \) and \( k_2 \). We make use of a theorem presented in \((9)\) \((10)\):

**Theorem 1.** There is a one-to-one correspondence between the coprime numbers \( k_1 \) and \( k_2 \) and the sequence of quotients that makes the continued fraction \( \lfloor / q_2, q_3, ..., q_r, q_{r+1} / \rfloor \) (it is customary to denote the sequence of numbers in a continued fraction between two double slashes \( \lfloor / ... / \rfloor \)). Alternatively, given a sequence of chosen
positive integers \((q_2, q_3, \ldots, q_r, q_{r+1})\), the calculation of the finite continued fraction, involving the \(q_i\)'s, results in the ratio of coprime numbers \(k_1/k_2\).

**Proof.** We write the equations that are on the left side in the first and second lines of the set (5) as

\[
\begin{align*}
k_1/k_2 &= q_2 + k_3/k_2, \\
k_2/k_3 &= q_3 + k_4/k_3,
\end{align*}
\]  

(13) \quad (14)

we then substitute \((k_2/k_3)^{-1}\) from Eq. (14) into Eq. (13) to obtain

\[
\frac{k_1}{k_2} = q_2 + \frac{1}{q_3 + \frac{1}{k_3/k_4}}.
\]  

(15)

One repeats this operation for the finite continued fraction for all the \(q_i\)'s and the result is the ratio

\[
\frac{k_1}{k_2} = q_2 + \frac{1}{q_3 + \frac{1}{q_4 + \frac{1}{\ddots + \frac{1}{q_{r+1}}}}}
\]  

(16)

where \(q_{r+1} = k_r\), because \(k_r = q_r + 1, k_{r+1} = 0 = q_{r+1} \times 1 + 0\).

**Example 4.** For the coprime numbers \((k_1, k_2) = (1019, 239)\), the sequence of modular calculations in Table I permits us to write the finite continued fraction (16) as

\[
/4, 3, 1, 3, 1, 5, 2/ = 4 + \frac{1}{3 + \frac{1}{1 + \frac{1}{5 + \frac{1}{2}}}} = \frac{1019}{239},
\]  

(17)

which is the ratio \(k_1/k_2\). See also [10]

Another theorem related to Theorem 1 and useful for our method follows:

**Theorem 2.** A one-to-one correspondence between the coprime numbers \((k_1, k_2)\) and the sequence of quotients \((q_2, q_3, \ldots, q_r, q_{r+1})\) of Eq. (5) exists and is given as the inverse of the product of a finite number of \(2 \times 2\) matrices,

\[
\mathbb{B}_r := \prod_{j=2}^{r} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix}^{-1},
\]  

(18)

where

\[
\mathbb{B}_r \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_r \\ 1 \end{pmatrix},
\]  

(19)

and reminding that \(k_r = q_{r+1}\).

**Proof.** We consider the equations on the right side of the set (5) and write the first two equations as

\[
\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} q_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_2 \\ k_3 \end{pmatrix};
\]  

(20)

proceeding in the same fashion for the second and third equations,

\[
\begin{pmatrix} k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} q_3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_3 \\ k_4 \end{pmatrix}
\]  

(21)
and substituting Eq. (21) into Eq. (20) we get

\[
\begin{pmatrix}
  k_1 \\
  k_2 
\end{pmatrix} = \begin{pmatrix}
  q_2 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  q_3 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  q_4 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  k_3 \\
  k_4 
\end{pmatrix}.
\]

By repeating the process of substitution, with the last \( k_{r+1} = 1 \), and then inverting the relation one gets Eq. (19).

**Example 5.** For the prime numbers \( k_1 = 1019 \) and \( k_2 = 239 \), the sequence of quotients is \((q_2, q_3, ..., q_8) = (4, 3, 1, 3, 1, 5, 2)\), with \( r = 7 \) and \( k_7 = 2 \); so

\[
\begin{pmatrix}
  1019 \\
  239 
\end{pmatrix} = \left[ \begin{pmatrix}
  4 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  3 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  1 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  3 & 1 \\
  1 & 0 
\end{pmatrix} \right. \\
\left. \begin{pmatrix}
  1 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  5 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  2 \\
  1 
\end{pmatrix} \right].
\]

(23)

**Example 6.** The relations between the continued fraction (16) and the matrix (19) with \( k \) and the ratio \( (19) \).

The coefficients are

\[
\begin{array}{c}
\begin{matrix}
    p_1 & t_1 \\
    p_2 & t_2 \\
    p_3 & t_3
\end{matrix}
\end{array}
\]

for \( p_1 \) and \( t_1 \) are obtained by inverting Eqs. (20) and (22),

\[
\begin{pmatrix}
  q_2 & 1 \\
  1 & 0 
\end{pmatrix}^{-1} \begin{pmatrix}
  k_1 \\
  k_2 
\end{pmatrix} = \begin{pmatrix}
  k_2 \\
  k_3 
\end{pmatrix} = \begin{pmatrix}
  k_2 \\
  k_3 
\end{pmatrix}.
\]

(26)

\[
\begin{pmatrix}
  q_2 & 1 \\
  1 & 0 
\end{pmatrix} \begin{pmatrix}
  q_3 & 1 \\
  1 & 0 
\end{pmatrix}^{-1} \begin{pmatrix}
  k_1 \\
  k_2 
\end{pmatrix} = \begin{pmatrix}
  k_3 - k_2 q_2 \\
  k_2 (q_2 q_3 + 1) - k_1 q_3 
\end{pmatrix} = \begin{pmatrix}
  k_3 \\
  k_4 
\end{pmatrix}.
\]

(27)

and with the matrix

\[
\mathbb{B}_4 := \left[ \prod_{i=2}^{4} \begin{pmatrix}
  q_i & 1 \\
  1 & 0 
\end{pmatrix} \right]^{-1}
\]

we get the relation,

\[
\mathbb{B}_4 \begin{pmatrix}
  k_1 \\
  k_2 
\end{pmatrix} = \begin{pmatrix}
  k_2 (q_2 q_3 + 1) - k_1 q_3 \\
  k_1 (q_2 q_3 + 1) - k_2 (q_2 + q_4 + q_2 q_3 q_4) 
\end{pmatrix} = \begin{pmatrix}
  k_4 \\
  k_5 
\end{pmatrix}.
\]

(29)

The coefficients are

\[
\mathbb{B}_3 = \begin{pmatrix}
  p_1 & t_1 \\
  p_2 & t_2 \\
  p_3 & t_3
\end{pmatrix} = \begin{pmatrix}
  1 & -q_2 \\
  -q_3 & q_2 q_3 + 1
\end{pmatrix},
\]

\[
\mathbb{B}_4 = \begin{pmatrix}
  p_2 & t_2 \\
  p_3 & t_3
\end{pmatrix} = \begin{pmatrix}
  -q_3 & q_2 q_3 + 1 \\
  q_3 q_4 + 1 - q_2 - q_4 - q_2 q_3 q_4 
\end{pmatrix}.
\]
thus the sequence of equations

\[
\begin{align*}
    k_1 - q_2k_2 &= k_3 \\
    -q_3k_1 + (q_2q_3 + 1)k_2 &= k_4 \\
    k_1(q_3q_4 + 1) - (q_2 + q_4 + 2q_3q_4)k_2 &= k_5
\end{align*}
\]

represents the relations \( p_1k_1 + t_1k_2 = k_{l+2} \) with \( l = 1, \ldots, r \). Thus we can write

\[
B_r \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} p_{r-1} & t_{r-1} \\ p_r & t_r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_{r+1} \\ k_{r+2} \end{pmatrix}
\]

and the coefficients \( p_l \) and \( t_l \), see Eqs. (11) and (IV), depend only on the quotients, as shown in Table V and we can confront these expression with those calculated in Example 3, for instance, \((q_2, \ldots, q_6) = (4, 1, 2, 3, 21)\).

| \( l \) | \( p_l \) | \( t_l \) |
|---|---|---|
| 1 | 1 | \(-q_2\) |
| 2 | \(-q_3\) | \(q_2q_3 + 1\) |
| 3 | \(-q_3(1 + q_4q_5) - q_4q_6 + q_5q_6 + q_2q_3q_4q_5 + 1\) |

TABLE V: The coefficients in terms of the quotients.

IV. TWO STRATEGIES FOR A MORE SECURE ALICE-BOB COMMUNICATION

A. Choosing the exponents \( k_1 \) and \( k_2 \)

For the construction of the matrices \( M_{k_1} \) and \( M_{k_2} \), we consider that the best strategy for Alice (or Bob) consists in choosing a certain sequence of positive integers \((q_2, q_3, \ldots, q_r, q_{r+1})\) and then to calculate the continued fraction \(//q_2, q_3, \ldots, q_r, q_{r+1}//\), from which she (or he) obtains the coprime numbers \( k_1 \) and \( k_2 \) to be used as exponents. We envisage this procedure as a good strategy because according to a theorem exposed in [9], in the sequence \( (q_2, q_3, \ldots, q_8) \) for the quotients in Euclid algorithm, the approximate probability for one integer \( q_j \) to take the value \( a \) is

\[
p(a) = \log_2 \left( 1 + \frac{1}{a} \right) - \log_2 \left( 1 + \frac{1}{a+1} \right) = \log_2 \left( 1 + \frac{(a + 1)^2}{(a + 1)^2 - 1} \right),
\]

thus \( p(1) = 0.41505 \), \( p(2) = 0.16993 \), \( p(3) = 0.09311 \), \( p(4) = 0.05889 \), etc. So Alice (or Bob) should choose quotients that contradict that pattern in order to fool an eavesdropper, turning the decryption a hard task, i.e., more computational time consuming. The statistics of appearance of the numbers 1 to 4 in a sequence of quotients \((q_2, q_3, \ldots, q_r, q_{r+1})\), that comes from a fraction \(k_1/k_2\), with \( k_1 \) and \( k_2 \) being coprime numbers picked at random, has probability 74%. Therefore, if Alice (or Bob) believes that to guess the matrix \( M \) the eavesdropper Eve is going to use mostly the numbers 1 to 4, thus Alice (or Bob) have the option to construct \( k_1 \) and \( k_2 \) using most of the quotients \( q_j \) higher than 4.

B. Disguising the ratio \( k_1/k_2 \)

The matrix \( B \) has determinant 19 + 4i and trace \(-i\); raising it to the two prime numbers \( k_1 = 19 \) and \( k_2 = 13 \), for instance, results in

\[
C = \begin{pmatrix} 513852777463 - i2298638980668 & -2931007733267 - i5289191614510 \\
-473985067058 - i3755020054445 & -14804380518615 - i191643568579 \end{pmatrix},
\]

and

\[
D = \begin{pmatrix} 14127153 + i166148948 & 378280035 + i217438622 \\
432811810 + i55220253 & 881816207 - i535190869 \end{pmatrix}.
\]
Utilizing the Algorithm \[1\], one retrieves the original matrix \((C^{-1})^2 D^3 = M\). However, there is a weak point that makes the task easier to a cryptanalyst: he has a direct access to the ratio \(k_1/k_2\) since \((\ln(\det C))/\ln(\det D)) = k_1/k_2\), and in the current numerical example the result is
\[
\frac{\ln(\det C)}{\ln(\det D)} \approx 1.4502 - i 0.1622, \tag{35}
\]
which is a strong clue to guess the exponents since the real part is quite close to 19/13 \(\approx 1.4615\). Another relation relies on the difference \(\ln(\det C) - \ln(\det D) = (k_1 - k_2) \ln(\det M)\); but the presence of the unknown \(\det M\) does not introduce an additional way to guess \((k_1 - k_2)\).

In order to blur the relation \((\ln(\det C))/\ln(\det D)) = k_1/k_2\), Alice can adopt a simple additional procedure, without affecting sensibly the sequence of steps of the Algorithm \[1\] making it part of the protocol. Instead of placing publicly the matrices \(C\) and \(D\), she will place other two associated matrices as to be explained below, although without ruling out that other simple procedures can be invented and be more efficient. Firstly she chooses two positive integer numbers, \(m\) and \(l\), and calculates the inverse modulo \(m\) equation for \(k_1\) and \(k_2\), i.e. \((k_i x_i) \mod m = 1\), \(i = 1, 2\), so that she gets the class of equivalent solutions, \(x_i(l) = ml + x_i(0)\), \(l = 0, 1, 2, \ldots\). Thus the inverse modulo of \(k_i\) is \([k_i^{-1}] = x_i(l)\) and any value \(l\) can be picked by Alice (or Bob), \(k_i := x_i(l)\), and she defines the complex number \(z = k_1 + ik_2\). A set of \(L\) matrices is constructed and displayed publicly, \(M = \{Z(z, z^*)\}\). These matrices have the same dimension of matrix \(M\), they are different from each other, they are enumerable (as \(p_1, \ldots, p_r, \ldots, p_L\)) and the argument \(z\) is not defined numerically. Alice then picks two matrices of the set \(M\) that she calls \(Z_C(z, z^*)\) and \(Z_D(z, z^*)\), where they now depend on the numerical values \(k_1, k_2\). In sequence Alice calculates the matrices \(\tilde{C} = Z_C C\) and \(\tilde{D} = Z_D D\) that she places publicly. Now the key Alice has to share with Bob, in order to enable him to get \(M\), is constituted by the 6-tuple \((k_1, k_2, m, l, p_r, p_s) := K\), that he uses to calculate \((Z_C^{-1} C)^{-1} (Z_D^{-1} D)^3 = M\). If a cryptanalyst calculates the ratio \((\ln(\det \tilde{C}))/\ln(\det \tilde{D}))\) she/he will not extract an obvious information about the ratio \(k_1/k_2\), as before, due to the presence of \(\ln(\det Z_C)\) and \(\ln(\det Z_D)\).

\[
\frac{\ln(\det \tilde{C})}{\ln(\det \tilde{D})} = \frac{\ln det(Z_C) + k_1 \ln det M}{\ln det(Z_D) + k_2 \ln det M}. \tag{36}
\]

Example 7. Alice adopts \(m = 11\) and \(l = 2\): \(k_1 := x_1(2) = 11 \times 2 + 7 = 29\) and \(k_2 := x_2(2) = 11 \times 2 + 6 = 28\); she defines \(z = k_1 + ik_2 = 29 + i28\) and chooses, for instance, from the set \(M\) the matrices,
\[
Z_C(z, z^*) = \begin{pmatrix} z & 1 \\ -1 & z^* \end{pmatrix}, \quad Z_D(z, z^*) = \begin{pmatrix} i & z \\ z^* & -i \end{pmatrix}, \tag{37}
\]
then the ratio \[(36)\] becomes
\[
\frac{\ln(\det \tilde{C})}{\ln(\det \tilde{D})} = \frac{\ln \left(\frac{|z|^2 + 1}{-|z|^2 + 1}\right) + k_1 \ln det M}{\ln \left(\frac{|z|^2 + 1}{-|z|^2 + 1}\right) + k_2 \ln det M} \approx 1.0456 - i 5.696 \times 10^{-2}, \tag{38}
\]
whose real part does not give a hint about the ratio \(k_1/k_2\) \(\approx 1.4615\), so that the case where Alice does not uses in her protocol the matrices \(Z_C\) and \(Z_D\). This kind of disguising is not very costly: the multiplications of \(C\) and \(D\) add only two additional operations for Alice, while for Bob, the task to retrieve \(M\) needs the use of \(Z_C^{-1}\) and \(Z_D^{-1}\), that introduces four additional operations: two inversions and two multiplications. Worth to note that the 6-tuple key \(K\) can also be split into two (or more) keys \(K_1\) and \(K_2\) that complement each other.

V. UPPER BOUND FOR THE NUMBER OPERATIONS TO CONSTRUCT A MATRIX \(M^k\)

The number of operations, \(N_{op}(k)\), Alice has to perform (the product of \(k\) matrices, namely \(M^k\)) is \(k - 1\) without storage of partial products (in the example \[2\] there should be 1000 of such products). Now, if Alice stores the already multiplied matrices, namely \(M^2, M^3, \ldots, M^{k-1}\) and uses them for further operations, the necessary number of operations to calculate \(M^k\) is significantly reduced, we estimate that \(2\sqrt{k}\) is an upper bound for \(k \gg 1\), as we presume that other approaches could reduce this number. To prove this result we assume that for a given \(M\) one calculates \(M^2\) and store it, so the available matrices are \(S_2 = \{M, M^2\}\) and the number of multiplications is reduced from \(k - 1\) to
\[
N_{op}(k|S_2) = 1 + \left\lfloor \frac{k - 1}{2} \right\rfloor, \tag{39}
\]
for $k > 2$, where the square brackets $[x]$ means the integer part of $x$. Thus, for $k = 113$, for instance, the number of multiplications is reduced from 112 without storage to 57 with $S_2$ storage, as shown in Table VI. For $S_3 = \{M, M^2, M^3\}$ we have to perform two operations to produce $M^2$ and $M^3$ to be then stored. Thus, for $k > 3$ the number of operations is reduced from 57 (storage using the set $S_2$) to 39 when one uses the set $S_3$, as shown in Table VI.

$$\mathcal{N}_{op}(k|S_3) = 2 + \left\lfloor \frac{k-1}{3} \right\rfloor.$$ (40)

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... | 113 | ... |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{N}_{op}(k|S_2)$ | 2 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 6 | ... | 57 | ... |
| $\mathcal{N}_{op}(k|S_3)$ | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | ... | 39 | ... |

TABLE VI: Number of operations for calculating $M^k$ with $S_2$ and $S_3$ storages.

In general, for any number of stored matrix products $x_s$ the number of multiplications to construct $M^k$ is

$$\mathcal{N}_{op}(k|S_{x_s}) = (x_s - 1) + \left\lfloor \frac{k-1}{x_s} \right\rfloor,$$ (41)

where the term in parenthesis stands for the number of multiplication necessary to construct the set $S_{x_s}$. For each $k$ we can find the optimum number of products to be stored in order to minimize the number of matrix multiplications by looking for the point of minimum of Eq. (41)

$$\frac{d\mathcal{N}_{op}(k|S_{x_s})}{dx_s} = 1 - \left\lfloor \frac{k-1}{x_s^2} \right\rfloor = 0 \implies [x_{s,\text{min}}] := \bar{k} = \left\lfloor \sqrt{k-1} \right\rfloor$$ (42)

and

$$\mathcal{N}_{op}(\bar{k}) := \mathcal{N}_{op}(k|S_{\bar{k}}) = \left(2\sqrt{\bar{k}} - 1 - 1\right),$$ (43)

where $\bar{k}$ is the minimum number of powers of $M$ necessary to construct the set $S_{\bar{k}} = \{M, M^2, ..., M^{\bar{k}}\}$. This is exemplified in Table VII. For $k \gg 1$ we have $\bar{k} \approx \sqrt{k}$ and $\mathcal{N}_{op}(k) \approx 2\sqrt{k}$. In Fig. 1 we plot the graphs

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... | 113 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\bar{k}$ | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | ... | 11 |
| $\mathcal{N}_{op}(\bar{k})$ | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | ... | 20 |

TABLE VII: Minimizing the number of operations for calculating $M^k$ with storage $S_{\bar{k}}$ with $\bar{k} = \left\lfloor \sqrt{k-1} \right\rfloor$.

FIG. 1: Eq. (41) for $k = 53, 113, 159$ (red solid line, black dashed and circles in green)
for $k = 53, 113, 159$ and we find $k = 7, 10, 12$, that imply the following lowest numbers of necessary operations $N_{\text{op}}(k) \approx 13, 20, 24$.

In Fig. 2 we plot the number of $k$ operations Alice has to perform without (dashed line) and with (solid line) storage. Indeed, the result, that we consider being only an upperbound, is quite remarkable in terms of reducing the number of operations. Nonetheless, we did not considered here the necessary computational time for storing and calling back a matrix.

VI. SOME STRATEGIES OF ATTACK THAT A CRYPTANALYST MAY CHOOSE TO RECOVER $M$

A pertinent question that can be asked is: how a cryptanalyst could get the matrix $M$ by only knowing the matrices $C$ and $D$? To answer this question, some considerations are advanced, as discussed in the Algorithm $\text{IV.B}$.

(a) Eve has the matrices $C$ and $D$ of Example 1, for instance, but she is in the dark about their meaning, or whether they contain any message. In this case she has no clue to follow in order to identify some pattern, the only obvious property is that they commute. She shall try to find some kind of correlation between the entries of the matrices, between its inverses, or between the eigenvalues and eigenvectors. Otherwise she waits for another two matrices, $C'$ and $D'$, to be conveyed by Alice and then tries to find some pattern.

(b) Eve have access to the matrices $C$ and $D$, she knows that they originate from a single matrix $M$, of dimension $N$, raised to the powers $k_1$ and $k_2$ that may be the carrier of some message, but $k_1$ and $k_2$ are unknown to her. She can embrace the strategy of the direct approach by diagonalizing the matrices, such to get access to the eigenvalues and eigenvectors. The eigenvalues \{\lambda_i^C\} and \{\lambda_i^D\} ($i = 1, ..., N$) are related to the eigenvalues of matrix $M$, \{\lambda_i^M\}, namely $\lambda_i^C = (\lambda_i^M)^{k_1}$ and $\lambda_i^D = (\lambda_i^M)^{k_2}$, then Eve should try many integer values $p$ and $t$ to solve $(\lambda_i^C)^{1/p}$ and $(\lambda_i^D)^{1/t}$ ($p, t = K, K + 1, ..., K + L$, where $K$ and $L$ are integer numbers chosen arbitrarily by Eve) in order to find a common eigenvalue. However, Eve have to deal with the multiplicity of roots, $\mu_{i,r}^C$ and $\mu_{i,s}^D$ with $i, j$ standing for the eigenvalues and $r = 1, ..., p$ and $s = 1, ..., t$ are the indices that classify the multiplicity of the roots for each eigenvalue. The number of eigenvalues to be analyzed is $N^2 \times (L - K + 1)^2$, which is time and, as well as, memory consuming, compared to Bob’s task to get $M$ once he receives the numbers $k_1$ and $k_2$.

(c) Eve knows that each matrix, $C$ and $D$, originates from the exponents $k_1$ and $k_2$ of a single matrix $M$ that is the carrier of a message, but $k_1$ and $k_2$ are unknown to her. Then another strategy consists in solving the equations $C - \Sigma_i^C = 0$ and $D - \Sigma_i^D = 0$, where many positive integers $l_C$ and $l_D$ should be tried for the unknown matrices $\Sigma_C$ and $\Sigma_D$. If the matrices $C$ and $D$ belong to $C^{N \times N}$, then $2N^2$ (the factor 2 is present because the entries belong to the field of complex numbers) polynomial algebraic equations, having degrees $l_C$ and $l_D$ respectively, must be solved. Each one of them has the same number, $2N^2$, of unknown variables. Each matrix equation accepts $L_i$ ($i = C, D$) different sets of solutions \{\Sigma_{C,r}\} and \{\Sigma_{D,r}\}, $r = 1, ..., L_i$ if $C$ and $D$ do not show any kind of symmetry or constraints that could reduce the number of independent equations. For each pair $(l_C,l_D)$ Eve has to verify whether one solution, of the

\[ \mathcal{N}_{\text{op}}(k) \]

FIG. 2: The solid line stands for the number of operations with storage ($\lfloor 2\sqrt{k-1} \rfloor$) versus the number of operations without storage ($k - 1$), dashed line.
2N² possible ones, of each set of equations is common to both; if that happens then Eve has successfully uncovered the matrix $\bar{M}$, once she concluded that $(\bar{I}_C, \bar{I}_D) = (k_1, k_2)$, such that $C = M^{k_1}$ and $D = M^{k_2}$. Worth to mention that Bob can use that very same method to get the right matrix because he knows beforehand the values of $k_1$ and $k_2$, and he does not need, as Eve needs, to browse through many pairs of numbers to be used as exponents. However, this approach is more time consuming than the proposition of our algorithm needs and depends on high computational capacity.

(d) Eve knows that $C = M^{k_1}$ and $D = M^{k_2}$ and she is acquainted with the algorithm, but she ignores the values of the exponents $k_1$ and $k_2$ ($k_1 > k_2$). Knowing the algorithm she decides to calculate the product of matrices $C(D^{-1})^{m_1} \equiv \bar{M}_{k_1}(m_1)$, and as she doesn’t know which is the right exponent she tries the values $m_1 = 1, ..., L$. She repeats the process described in Eq. (6) sequentially for each value $m_1$, as it is explicitly presented in Table VIII. She then computes and analyzes the matrices, stopping at some value $r > 2$ when she succeeds in finding the seed matrix in the last group of $L^{-1}$, after having checked $\sum_{r=1}^{L-1} L^r = (L (L^{-1} - 1)) / (L - 1)$ matrices. The analysis: Eve considers the matrices $M_k(m_1, m_2)$, with $l = 3, ..., r+1$, and raises each one to powers that begin at some positive integer $K_1$ and stops at a value $K_2$, $k = K_1, K_1 + 1, ..., K_2$, chosen arbitrarily ($(M_k(m_1, m_2))^{k_2}$). She then begins to quest whether an obtained matrix coincides with matrix $C$ or with $D$. In the case she succeeds, she then acknowledges that she got the seed matrix $M$. However, this procedure is of exponential complexity since for $L = 9$ and $r = 10$ she has to analyze 435,848,049 $\approx 4.4 \times 10^{6}$ matrices. For the same $L$ and for $r = 15$ the number of matrices goes to 25,736,391,511,830 $\approx 2.6 \times 10^{12}$; this number grows exponentially with $r$. Besides, if she is sure that she did not find the meaningful matrix, she can try another possibility, namely, she changes $C \rightarrow D^{-1}$ and $D^{-1} \rightarrow C$ and repeats the same procedure.

We can now analyze this procedure in terms of probabilities as a function of time. What is the probability that Eve achieves the identification of the right matrix $M$ at the $s$-th attempt? i.e., the first $s-1$ computed matrices are the wrong ones. If she assumes that $M$ is one out of the $N = (L (L^{-1} - 1)) / (L - 1)$ matrices, and all being equally likely, then she attributes the success probability $p = N^{-1}$. The probability to find $M$ at the $s$-th trial is $q^{-1} p$ with $q = (1 - p) < 1$, for the probability of not being the right matrix. Normalizing, the probability is given as $P_s = q^{-1} p / (1 - q^N)$, such that $\sum_{s=1}^{N} P_s = 1$. The estimated average time for Eve to discover the right matrix then is

$$T(N) = \tau_0 \frac{\sum_{s=1}^{N} s P_s}{\sum_{s=1}^{N} P_s} \approx \tau_0 N$$  \hspace{1cm} (44)$$

a time that grows exponentially with $r$, where $\tau_0$ is an arbitrary unit of time. We do not rule out the possibility of creation of ingenuous strategies, more efficient than those we proposed here, to extract the seed matrix $M$ from matrices $C$ and $D$, in a comparatively quite shorter time.

If Alice extends her protocol by blurring the matrices $C$ and $D$ as proposed in subsection IV B we have not been able to devise a numerical procedure for a cryptanalyst to extract matrices $Z$ from matrices $C$ and $D$.

### VII. PUBLIC KEY DISTRIBUTION

Here we present two methods for the public key distribution, whose discussion is included because the encryption and then decryption are essential for the distribution of the 6-tuple key $K$, as discussed in subsection IV B.
A. Diffie-Hellman proposal

In Ref. [4], W. Diffie and M. E. Hellman (DF) reported a method by which two, or even \( n \) persons could exchange information through a secure connection created over a public channel and using a publicly divulged method only. In fact, it seems that the idea was already proposed and used by the British intelligence agencies, but it was not publicized [14]. The proposal was later improved and turned operational by R. L. Rivest, A. Shamir and L. Adleman, as reported in Ref. [5], after getting a patent for the method. As recognized by these authors, “Their [DH] article motivated our research, since they presented the concept but not any practical implementation of such a system.” Indeed, although the RSA method was a new encryption approach, the principle of the public key distribution was first publicized by Diffie and Hellman for civilian use purposes, although it was recently reported that it fails in practice [12]. With this in mind we present below the principle of the public key distribution was first publicized by Diffie and Hellman for civilian use purposes, containing a plaintext, by raising it to two powers, and the calculated matrices are the recipient of an encrypted publicly exposed message, and how a confidential plaintext can be shared by the parts. This procedure can be used in our proposal of encrypting a matrix \( M \), containing a plaintext, by raising it to two powers, \( k_1 \) and \( k_2 \), and complementing it with the protocol exposed in subsection IV B. The calculated matrices are the recipient of an encrypted publicly exposed message, and the 6-tuple key \( K \) is essential for the encryption and decryption, so to get the seed matrix \( M \), it was recently reported that it fails in practice [12].

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FIG. 3: Schematic picture of the key distribution between Alice and Bob. The solid line boxes represent the public codes and dashed boxes are for the personal calculations and the secret keys.

$K_{i\rightarrow j} := K_{j,i}$. By its turn, using his secret key, $P_j$ composes a number $k_{a,j} = q^{a_j} \mod p$ and sends it to $P_i$ that does the symmetric operation by calculating $(k_{a,j})^{a_i} \mod p = K_{j\rightarrow i} := K_{i,j}$. As $(k_{a,j})^{a_i} \mod p = (k_{a,j})^{a_i} \mod p$, therefore $P_i$ and $P_j$ share the same number $K_{j,i} = K_{i,j}$.

B. RSA method and Algorithm 1

The method of encrypting a message invented by Rivest, Shamir and Adleman [5] has its security based, in part, on the difficulty of factoring large numbers, dispensing the use of a courier to carry secret keys. We present below we illustrate the RSA method to be employed in conjunction with the Algorithm 1.

Example 9. Bob chooses two “many digits” prime numbers, $p$ and $q$, and computes their product $n = pq$ (n is called biprime number), he then considers the Euler totient function $\varphi(n)$ (that gives the number of positive integers relatively prime to but less than $n$); in this case $\varphi(n) = \varphi(p)\varphi(q) = (p - 1)(q - 1)$, see [13] for instance.

Then Bob chooses an integer number $d$ that is relatively prime to $\varphi(n)$, $\gcd(d, \varphi(n)) = 1$, and also to $n$, $\gcd(d, n) = 1$. Thus $d$ has an inverse modulo $\varphi(n)$, $[d^{-1}] := e$, such that $e \cdot d \equiv 1 \pmod{\varphi(n)}$, or $e \equiv d^{-1} \pmod{\varphi(n)}$. The rôle of $e$ within the method is to encrypt the message $M$ (shuffling the digits). Thereafter Bob makes the pair of numbers $(e, n)$ public.

After reading $(e, n)$ Alice encrypts her message $M$ as $C = M^e \mod n$ and makes it public, then Bob uses his secret decryption key $C^d \mod n = M$ to retrieve $M$ from $C$. The RSA method relies essentially on the difficulty that a cryptanalyst will have to decompose a large – some 200 digits – biprime number $n$ into its two factors, as long as he does not know $d$. Below we illustrate the RSA method to be employed in conjunction with the reversal operation discussed in subsection IV B and the Algorithm 1.

Example 11. Bob chooses the numbers $p = 37$ and $q = 53$, such that $n = p \cdot q = 1961$, $\varphi(n) = (q - 1)(p - 1) = 1872$ and turns $n$ public. He then picks $d = 163$ as his secret decryption key, checks that $\gcd(\varphi(n), d) = 1$ is satisfied and calculates the encryption key $e = d \mod \varphi(n) = 379$ that he turns public. Alice wants to send to Bob $K_1 = 361$ and $K_2 = 079$ without worrying whether they will be intercepted by an eavesdropper. Once she has free access to $e$ and $n$ she encrypts them as $Y_1 = ((K_1)^e \mod n) = 324$ and $Y_2 = ((K_2)^e \mod n) = 1253$, and makes these numbers public together with the matrices $C$ and $D$.

In order to get a swift access to the matrix $M$ using the reversal operation in subsection IV B and the Algorithm 1, Bob uses his secret key $d$ to get $K_1 = (Y_1)^d \mod n = 361$ and $K_2 = (Y_2)^d \mod n = 079$. In this way Alice was able to send to Bob the necessary keys that he uses to get access to matrix $M$. It is worth noting that, otherwise,
Alice could have sent a single number 361079 instead of two if there was an understanding that the first half of the digits (beginning from the left) corresponds to $K_1 = 361$ and the other half to $K_2 = 079$.

VIII. SUMMARY AND CONCLUSIONS

We have here proposed a method to encrypt a matrix $M$ and to decrypt it that is based on the calculation consisting in raising $M$ to powers whose exponents are positive coprime numbers $k_1$ and $k_2$, producing the matrices $C$ and $D$, respectively. The knowledge of these numbers, that can be be considered as a key, makes the decryption (extracting $C$ and $D$) quite immediate in terms of a number of a certain number of operations, according to the Algorithm [1]. In the case the key is unknown, we estimated that a third party could spend a much larger time to get $M$ by the method of trial and error. Our algorithm follows the standard approach to cryptography: easy to encrypt and also to decrypt to those that have the key, but quite hard to a third party that has to guess it. The keys for the decryption, $k_1$ and $k_2$ can be shared between the parts by using either the DH method or the RSA, as described above.

Our proposal for encrypting a message is one possible application of the discussed algorithm, nevertheless we believe that it can be useful in other instances. For example, in retrodictim a seed matrix $M$ in a discrete $n$-step Markov process. If one only knows two matrices $M_{k_1}$ and $M_{k_2}$ (entries belonging to $\mathbb{R}$ in the closed interval $[0, 1]$) describing the evolution at two times (known), $k_1$ and $k_2$ ($k_1 > k_2$), once $M_{k_1}$ is retrieved we are able to calculate the powers $M_{k_2}^k$ for any integer $k \in [1, \infty)$, being thus able to run all the history of the changes of the matrix, so establishing a kind of determinism, although the entries are conditional probabilities.

Appendix A: Lemma and Theorem

Lemma 1. Euclid’s algorithm for coprime numbers: given two positive integers, $k_1$ and $k_2$, with $k_1 > k_2$, being coprime, $(\gcd (k_1, k_2) = 1)$, in modular arithmetic we have $k_1 \mod k_2 = k_3$, or

$$k_1 = q_1k_2 + k_3, \quad (A1)$$

$q_1$ being the quotient and $k_3 (< k_2)$ is the remainder, then it follows that the numbers $k_2$ and $k_3$ are also coprime.

Proof. If we assume that $k_2$ and $k_3$ are not coprime, then they share a common factor $b > 1$, and we can write $k_2 = bk_2' \text{ and } k_3 = bk_3'$, where now

$$k_1 = b(q_1k_2' + k_3') \equiv bk_1', \quad (A2)$$

therefore one gets for the pair of numbers $(k_1, k_2) = (bk_1', bk_2')$, thus $k_1$ and $k_2$ are not coprime, which contradicts our initial assumption, therefore $k_2$ and $k_3$ are necessarily coprime, i.e. $b = 1$. $\Box$

Here we give a proof of the uniqueness of the sequence [5] of two arbitrary coprime integers, $k_1$ and $k_2$, $k_1 > k_2$ and $r > 2$. Namely, any other pair of coprime numbers leads to a different sequence.

Theorem 3. For a pair of coprime numbers $(k_1, k_2)$ the sequence of the $(r - 1)$-tuple $(n_2, n_3, ..., n_r)$ is unique for $r > 2$. Two pairs of coprime numbers, $(k_1, k_2)$ and $(k_1', k_2')$, cannot lead to the same sequence of integers $(n_2, n_3, ..., n_r)$, for $k_1' = k_3$ and $k_2' = k_4$, (according to the sequence [5]) .

Proof. We construct another pair of coprime numbers $(k_1', k_2')$ as $k_1' = m_1k_1 + t_1$ and $k_2' = m_2k_2 + t_2$ (a dilation and a shift for $k_1$ and $k_2$) with $m_1$, $m_2$ chosen as positive integers and $t_1$, $t_2$ chosen as non-negative integers, and $k_1' > k_2'$. As $k_1 = n_2k_2 + k_3$ we can write

$$k_1' = n_2'k_2' + k_3' \implies m_1k_1 + t_1 = n_2'(m_2k_2 + t_2) + k_3', \quad (A3)$$

or

$$m_1(n_2k_2 + k_3) + t_1 = n_2'(m_2k_2 + t_2) + k_3'. \quad (A4)$$

Imposing $k_3 = k'_3$ we get the relation

$$n_2' = \frac{m_1k_2}{m_2k_2 + t_2}n_2 + \frac{(m_1 - 1)k_3 + t_1}{m_2k_2 + t_2} \quad (A5)$$

and setting $n'_2 = n_2$ we obtain a specific relation on $m_1$, $m_2$, $t_1$ and $t_2$,

$$n_2 = \frac{(m_1 - 1)k_3 + t_1}{(m_2 - m_1)k_2 + t_2}, \quad (A6)$$
that can be a positive integer, due to the arbitrariness of $m_1$, $m_2$, $t_1$ and $t_2$. The second step of the demonstration consists in considering the already imposed conditions $k_3 = k'_3$ and $n'_2 = n_2$, and use them in the second row in the sequence \([5]\),

\[
\begin{align*}
k_2 &= n_3k_3 + k_4, \quad \text{and} \quad k'_2 &= n'_3k'_3 + k'_4 \\
\implies m_2k_2 + t_2 &= n'_3k'_3 + k'_4 \\
\implies m_2(n_3k_3 + k_4) + t_2 &= n'_3k'_3 + k'_4.
\end{align*}
\tag{A7}
\]

For the additional condition $k_4 = k'_4$, such that

\[
m_2(n_3k_3 + k_4) + t_2 = n'_3k'_3 + k_4
\tag{A8}
\]

we obtain

\[
n'_3 = m_2n_3 = \frac{(m_2 - 1)k_4 + t_2}{k_3},
\tag{A9}
\]

and initially considering $m_2 = 1$ we get the relation

\[
n'_3 = n_3 + \frac{t_2}{k_3},
\tag{A10}
\]

As so, we may have $n'_3 = n_3$ as long as $t_2 = 0$ and $m_2 = 1$. Introducing these conditions in Eq. \([A6]\), it becomes

\[
n_2 = -\frac{(m_1 - 1)k_3 + t_1}{k_2(m_1 - 1)} = -\frac{k_3}{k_2} + \frac{t_1}{k_2(m_1 - 1)},
\tag{A11}
\]

which is a negative number! Thus $n'_3 = n_3$ is incompatible with $n'_2 = n_2$ positive. Now we consider the case $m_2 > 1$: imposing $n'_3 = n_3$ in Eq. \([A9]\) we have

\[
n_3 = -\frac{k_3}{k_2} + \frac{t_2}{k_3(m_2 - 1)},
\tag{A12}
\]

which is a negative number! Thus, although $n'_2 = n_2$ is possible for $m_2 > 1$, $n'_3 = n_3$ is not, so, here too, necessarily $n'_3 \neq n_3$. Summarizing, for two pairs of coprime numbers $(k_1, k_2)$ and $(k'_1, k'_2)$ that differ under a dilation and shift transformation, we can have both conditions $k_3 = k'_3$ and $k_4 = k'_4$ but not $n'_2 = n_2$, (see Eq. \([A6]\), and $n'_3 = n_3$, see (Eq. \([A10]\)), fulfilled simultaneously. The equalities $k'_1 = k_4$ and $n'_3 = n_3$ happen iff $m_2 = 1$ and $t_2 = 0$, but $n_2 = n'_2$ is possible only for a negative integer; when $m_2 > 1$ it is $n'_3 = n_3$ that becomes a negative number. Thence the conditions $k_3 = k'_3$, $n'_2 = n_2$, $k'_4 = k_4$ and $n'_3 = n_3$ cannot be fulfilled simultaneously. \(\square\)

**Example 10.** For instance, for $k_1 = 17$ and $k_2 = 11$, $n_2 = 1$ and $k_3 = 6$. Now we choose $t_1 = 6$ and $t_2 = 4$, such that $k'_1 = 23$, $k'_2 = 15$ with the condition $n'_2 = n_2 = 1$, we will get

\[
k'_3 = t_1 + \frac{(k_2 + t_2)k_3 - t_2k_1}{k_2} = 8,
\tag{A13}
\]

therefore $k'_3 \neq k_3$.

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