Method of generalized Reynolds operators and Pauli’s theorem in Clifford algebras

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(February 26, 2016)

We consider real and complex Clifford algebras of arbitrary even and odd dimensions and prove generalizations of Pauli’s theorem for two sets of Clifford algebra elements that satisfy the main anticommutative conditions. In our proof we use some special operators - generalized Reynolds operators. This method allows us to obtain an algorithm to compute elements that connect two different sets of Clifford algebra elements.

Keywords: Clifford algebra; Reynolds operator; Pauli’s theorem; Salingaros’ vee group

AMS Subject Classification: 15A66

1. Introduction

In the brief report\textsuperscript{23} we present generalization of Pauli’s theorem\textsuperscript{20} on the case of Clifford algebras (without a proof). In the present paper we present complete proof of this theorem using generalized Reynolds operators - special operators in Clifford algebras (see Theorems 4.2 - 4.5). The proof of generalization of Pauli’s theorem may be not very elegant, but with the use of generalized Reynolds operators we obtain not only existence of one or another relation between 2 different sets of Clifford algebra elements but also we obtain explicit algorithm to find elements $T$, that relate these 2 sets. We do not use abstract theorems of representation theory but our method is closely related to representation theory.

In\textsuperscript{24} we consider the following operators acting on Clifford algebras

\[ F_S(U) = \frac{1}{|S|} \sum_{A \in S} (e^A)^{-1} U e^A, \]  

where $e^A = e^{a_1} e^{a_2} \cdots e^{a_k}$, $A = a_1 a_2 \cdots a_k$, $a_1 < a_2 < \cdots < a_k$, are basis elements generated by an orthonormal basis in vector space $V$. Here $S \subseteq \mathcal{I}$ is a subset of the set of all ordered multi-indices $A$ of the length from 0 to $n$. We denote

\footnote{We use notation $e^A$ from\textsuperscript{8}. Note that $e^A$ is not exponent, $A$ is a multi-index.}
the number of elements in $S$ by $|S|$. Note that not for every subset $S \subseteq I$ in (1), the set $\{e^A | A \in S\}$ is a group.

Reynolds operator (see, for example, [8]) acts on a Clifford algebra element $U \in \mathcal{A}(p,q)$

$$R_G(U) = \frac{1}{|G|} \sum_{g \in G} g^{-1}Ug,$$  

(2)

where $|G|$ is the number of elements in a finite subgroup $G \subset \mathcal{A}(p,q)^\times$. We denote the group of all invertible Clifford algebra elements by $\mathcal{A}(p,q)^\times$. These operators “average” an action of group $G$ on Clifford algebra $\mathcal{A}(p,q)$.

We can take Salingaros’ vee group $G = \{\pm e^A, A \in I\}$ (see [1], [2], [3]), where $e^A$ are basis elements of Clifford algebra $\mathcal{A}(p,q)$. Note that Salingaros’ vee group is a finite subgroup of spin groups $\text{Pin}(p,q), \text{Spin}(p,q)$ (see [2], [13], [26], [27]).

We can write in this case

$$\frac{1}{|G|} \sum_{g \in G} g^{-1}Ug = \frac{1}{2n+1} \sum_{A \in I} ((e^A)^{-1}Ue^A + (-e^A)^{-1}U(-e^A)) = \frac{1}{2n} \sum_{A \in I} (e^A)^{-1}Ue^A.$$

Note that operators (2) are used in representation theory of finite groups (see [22], [10], [4]). We use these operators in Clifford algebras to obtain some new properties. In [24] we present a relation between these operators and projection operators onto fixed subspaces of Clifford algebras and present solutions of the system of commutator equations in Clifford algebras.

In this paper we consider some new operators (we call them generalized Reynolds operators) that generalize operators (1):

$$W_S(U) = \frac{1}{|S|} \sum_{A \in S} (\gamma^A)^{-1}U\beta^A,$$  

(3)

where elements $\gamma^A$ and $\beta^A$ are generated by 2 different sets of Clifford algebra elements $\gamma^a$ and $\beta^a$, $a = 1, \ldots, n$, that satisfy the main anticommutative conditions.

Clifford algebra was invented by W.K. Clifford in 1878 [7]. In his research he combined Hamilton’s quaternions (1843, [12]) and Grassmann’s exterior algebra (1844, [11]). Further development of Clifford algebra theory associates with a number of famous mathematicians and physicists - R.Lipschitz [15], T.Vahlen, E.Cartan, E.Witt, C.Chevalley [6], M.Riesz and others. Dirac equation (1928, [9]) had a great influence on the development of Clifford algebra. This equation contains $\gamma$-matrices that satisfy the anticommutative relations of Clifford algebra.

Clifford algebra is one of the possible generalizations of real numbers, complex numbers and quaternions. Clifford algebra is used in different branches of modern mathematics and physics. For example, Clifford algebra has applications in field theory [14], [17], robotics, signal processing, computer vision, chemistry, celestial mechanics, electrodynamics, etc.

In 1936 Pauli published [20] his fundamental theorem for Dirac gamma-matrices $\gamma^a, a = 1, 2, 3, 4$. He was interested in relation between 2 different sets of $\gamma$-matrices.
Theorem 1.1 (Pauli, [20]) Let two sets of square complex matrices \( \gamma^a, \beta^a, a = 1, 2, 3, 4 \) of order 4 satisfy the relations
\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab} 1, \quad \beta^a \beta^b + \beta^b \beta^a = 2 \eta^{ab} 1,
\]
where \( \eta = |\eta^{ab}| = \text{diag}(1, -1, -1, -1) \) is a diagonal matrix.

Then there exists an unique matrix \( T \) (up to a multiplicative complex constant) such that \( \gamma^a = T^{-1} \beta^a T, a = 1, 2, 3, 4 \). Note that this theorem follows from the Skolem-Noether theorem [28] proved in 1927. This theorem states that any automorphism of a central simple algebra is an inner automorphism.

In various publications (see, for example, [30]) you can find generalizations of Theorem 1.1 for arbitrary even number \( n = 2k \) complex square matrices \( \gamma^a \) of order \( 2^k \) that satisfy (4) with diagonal matrix \( \eta = |\eta^{ab}| = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) with \( p \) elements equal to 1 and \( q \) elements equal to \( -1 \), \( p + q = n \).

Also you can find in literature some discussions about odd case (\( n \) - odd): that in different cases there exists such element \( T_+ \), that \( \gamma^a = T_+^{-1} \beta^a T_+ \), or there exists such element \( T_- \), that \( \gamma^a = -T_-^{-1} \beta^a T_- \).

In this paper we investigate this question. We formulate and prove generalizations of Pauli’s theorem on the cases of real and complex Clifford algebras. We consider the more general question (not always reducing to studying representations). We show that in odd case there are 6 different variants (not 2) for relations between 2 different sets of elements satisfying Clifford algebra anticommutation relations. Also we present an algorithm to compute the element \( T \) that connects two sets of elements. We use generalized Reynolds operators to do it.

2. Clifford algebras over the field of real and complex numbers

There are several different (equivalent) definitions of Clifford algebras. We consider Clifford algebras [18], [16] with the fixed basis - enumerated by the ordered multi-indices. Note that generators and basis elements are fixed.

Let \( E \) be a vector space over the field \( F \) of real \( \mathbb{R} \) or complex \( \mathbb{C} \) numbers. Dimension of \( E \) equals \( 2^n \), where \( n \) is a natural number. Let we have a basis in \( E \)
\[
e, e^a, e^{a_1 a_2}, \ldots, e^{1 \ldots n}, \quad \text{where } a_1 < a_2 < \cdots, \quad (2^n \text{ elements})
\]
enumerated\(^2\) by the ordered multi-indices of length from 0 to \( n \). Indices \( a, a_1, a_2, \ldots \) take the values from 1 to \( n \). Note that vector space \( E \) contains subspace of dimension \( n \) with basis \( e^a, a = 1, \ldots, n \).

Let \( p \) and \( q \) be nonnegative integer numbers such that \( p + q = n, n \geq 1 \). Consider the diagonal matrix \( \eta = |\eta^{ab}| = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \), whose diagonal contains \( p \) elements equal to +1 and \( q \) elements equal to −1.

We introduce the operation of Clifford multiplication \( U, V \rightarrow UV \) on \( E \) such that

\(^2\)We use notation from [3] (see, also [12]). Note that there exists another notation instead of \( e^a \) - with lower indices. But we use upper indices because we take into account relation with differential forms. Note that \( e^a \) is not exponent.
we have the properties of distributivity, associativity, $e$ is identity element and
\[ e^{a_1} \cdots e^{a_k} = e^{a_1 \cdots a_k}, \quad 1 \leq a_1 < \cdots < a_k \leq n, \quad e^a e^b + e^b e^a = 2\eta^{ab} e, \quad \forall a, b = 1, \ldots, n. \]

Then, introduced in this way algebra is called real (complex) Clifford algebra and it is denoted by $\mathcal{C}^F(p, q)$ or $\mathcal{C}^C(p, q)$. When results are true for both cases, we write $\mathcal{C}^F(p, q)$.

Any Clifford algebra element $U \in \mathcal{C}^F(p, q)$ can be written in the form
\[ U = ue + u_1 e^a + \sum_{a_1, a_2} u_{a_1, a_2} e^{a_1, a_2} + \cdots + u_{1 \cdots n} e^{1 \cdots n}, \quad u, u_a, u_{a_1 a_2}, \ldots, u_{1 \cdots n} \in F. \] (6)

### 2.1. Notions of ranks and parity

We denote the vector subspaces spanned by the elements $e^{a_1 \cdots a_k}$, enumerated by the ordered multi-indices of length $k$, by $\mathcal{C}^F_k(p, q)$. The elements of the subspace $\mathcal{C}^F_k(p, q)$ are called elements of rank $k$. We have
\[ \mathcal{C}^F(p, q) = \bigoplus_{k=0}^n \mathcal{C}^F_k(p, q). \]

Clifford algebra $\mathcal{C}^F(p, q)$ is a superalgebra. It is represented as the direct sum of even and odd subspaces $\mathcal{C}^F(p, q) = \mathcal{C}^F_{\text{Even}}(p, q) \oplus \mathcal{C}^F_{\text{Odd}}(p, q)$, where
\[ \mathcal{C}^F_{\text{Even}}(p, q) = \bigoplus_{k-\text{even}} \mathcal{C}^F_k(p, q), \quad \mathcal{C}^F_{\text{Odd}}(p, q) = \bigoplus_{k-\text{odd}} \mathcal{C}^F_k(p, q). \]

### 2.2. Projection operators

Suppose $U \in \mathcal{C}^F(p, q)$ is written in the form (6). Then denote
\[ \langle U \rangle_k = \sum_{a_1, \ldots, a_k} u_{a_1 \cdots a_k} e^{a_1 \cdots a_k} \in \mathcal{C}^F_k(p, q). \]

Using the projection operator to the 1-dimensional vector space $\mathcal{C}^F_0(p, q)$ we define the trace of an element $U \in \mathcal{C}^F(p, q)$
\[ \text{Tr}(U) = \langle U \rangle_0 |_{e \rightarrow 1} = u. \]

The main property of the trace is $\text{Tr}(UV) = \text{Tr}(VU)$.

Also consider projection operator to the vector space $\mathcal{C}^F_n(p, q)$
\[ \pi(U) = \langle U \rangle_n |_{e^{1 \cdots n} \rightarrow 1} = u_{1 \cdots n}. \]

3 All complex Clifford algebras of the same dimension $n$ are isomorphic as algebras (see Theorem 2.3). That’s why we can consider only signature $(n, 0)$ in complex case. But in physics it is convenient to use complex Clifford algebras of signatures $(p, q)$, $q \neq 0$ too. For example, when we consider Dirac equation we use $\gamma$-matrices for signature $(1, 3)$.

4 We use Einstein summation convention (there is a sum over index $a$ in $u_a e^a$).

5 There is a difference in notation in literature. We use term “rank” and notation $\mathcal{C}^F_k(p, q)$ because we take into account relation with differential forms, see [17].
2.3. Some properties of Clifford algebras

We have the following well-known statement about center \( \text{Cen}(\mathcal{O}^F(p,q)) = \{ U \in \mathcal{O}^F(p,q) | UV = VU \ \forall V \in \mathcal{O}^F(p,q) \} \) of Clifford algebra.

**Theorem 2.1** \([16]\) The center \( \text{Cen}(\mathcal{O}^F(p,q)) \) of Clifford algebra \( \mathcal{O}^F(p,q) \) of dimension \( n = p + q \) is subspace \( \mathcal{O}^F_0(p,q) \) in the case of even \( n \) and subspace \( \mathcal{O}^F_0(p,q) \oplus \mathcal{O}^F_0(p,q) \) in the case of odd \( n \).

All real and complex Clifford algebras are isomorphic as algebras to matrix algebras.

**Theorem 2.2** \([16]\) We have the following algebra isomorphisms

\[
\mathcal{O}^F(p,q) \simeq \begin{cases} 
\text{Mat}(2^{\frac{p+q}{2}}, \mathbb{R}), & \text{if } p - q \equiv 0 \text{ mod 8}; \\
\text{Mat}(2^{\frac{p-q}{2}}, \mathbb{R}) \oplus \text{Mat}(2^{\frac{p-q}{2}}, \mathbb{R}), & \text{if } p - q \equiv 1 \text{ mod 8}; \\
\text{Mat}(2^{\frac{p+q}{2}}, \mathbb{C}), & \text{if } p - q \equiv 3; 7 \text{ mod 8}; \\
\text{Mat}(2^{\frac{p-q}{2}}, \mathbb{H}), & \text{if } p - q \equiv 4; 6 \text{ mod 8}; \\
\text{Mat}(2^{\frac{p+q}{2}}, \mathbb{H}) \oplus \text{Mat}(2^{\frac{p-q}{2}}, \mathbb{H}), & \text{if } p - q \equiv 5 \text{ mod 8}.
\end{cases}
\]

**Theorem 2.3** \([16]\) We have the following algebra isomorphisms

\[
\mathcal{O}^C(p,q) \simeq \begin{cases} 
\text{Mat}(2^{\frac{p+q}{2}}, \mathbb{C}), & \text{if } n \text{ is even}, \\
\text{Mat}(2^{\frac{p-q}{2}}, \mathbb{C}) \oplus \text{Mat}(2^{\frac{p-q}{2}}, \mathbb{C}), & \text{if } n \text{ is odd}.
\end{cases}
\]

3. Sets of anticommutative elements in Clifford algebras

Let we have a set of Clifford algebra elements

\[
\{ \gamma^a, \ a = 1, \ldots, n \} \in \mathcal{O}^F(p,q), \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} e. \tag{7}
\]

Let denote multi-index of arbitrary length by \( A \) and denote its length by \( |A| \). Expression \( \gamma^{a_1} \cdots \gamma^{a_k} \) is denoted by \( \gamma^{a_1 \cdots a_k} \) for \( a_1 < \cdots < a_k \). We have the following notations for sets of multi-indices

\[
\mathcal{I} = \{ \emptyset, \ 1, \ \ldots, \ n, \ 12, \ 13, \ \ldots, \ 1 \ldots n \}, \tag{8}
\]

\[
\mathcal{I}_{\text{Even}} = \{ A \in \mathcal{I}, \ |A| - \text{even} \}, \quad \mathcal{I}_{\text{Odd}} = \{ A \in \mathcal{I}, \ |A| - \text{odd} \}. \tag{9}
\]

We consider the following set

\[
\mathcal{B} = \{ \gamma^A, A \in \mathcal{I} \} = \{ e, \gamma^a, \gamma^{ab}, \gamma^{abc}, \ldots, \gamma^{1\ldots n} \}, \quad a < b < c < \cdots \tag{10}
\]

**Theorem 3.1** Consider real or complex Clifford algebra \( \mathcal{O}^F(p,q) \) and the set \( \{A\} \).

(1) If \( n = p + q \) - even, then \( \{A\} \) is a basis of \( \mathcal{O}^F(p,q) \).

(2) If \( n = p + q \) - odd, then either

- we have \( \gamma^{1\ldots n} = \pm e^{1\ldots n} \) and \( \{A\} \) is a basis, or
- we have \( \gamma^{1\ldots n} = \pm e \) and \( \{A\} \) is not a basis (this is possible only in the case \( p - q \equiv 1 \text{ mod 4} \), or
• we have $\gamma^{1...n} = \pm i e$ and (11) is not a basis (this is possible only in complex Clifford algebra and only in the case $p - q \equiv 3 \mod 4$).

Proof. We omit proof of this theorem, You can find proof of similar statements, for example, in [29] (pp. 289-290) and [21] (pp. 127-128).

THEOREM 3.2 Consider real or complex Clifford algebra $\mathcal{C}^{F}(p,q)$ of dimension $n$ and the set $\{\gamma^{1}, \ldots, \gamma^{n}\}$. Then

(1) if $n = p + q$ - even, then $\text{Tr}(\gamma^{a_{1}...a_{k}}) = 0$, $k = 1, \ldots, n$.
(2) if $n = p + q$ - odd, then $\text{Tr}(\gamma^{a_{1}...a_{k}}) = 0$, $k = 1, \ldots, n - 1$ and

$$\text{Tr}(\gamma^{1...n}) = \begin{cases} 0, & \text{if (10) is s basis;} \\ \pm 1, \pm i & \text{if (10) is not a basis.} \end{cases}$$

(where $\pm i$ are possible in the case of complex Clifford algebra).

Proof. For any element $\gamma^{A}$ (except $\gamma^{1...n}$ in the case of odd $n$ and except $e$ in the case of any $n$) there exists such element $\gamma^{a}$, that $\gamma^{A}$ anticommutes with $\gamma^{a}$ (if $|A|$ - even, then we take $a \in A$; if $|A|$ - odd, then we take $a \notin A$). We obtain $\text{Tr}(\gamma^{A}) = \text{Tr}(-\gamma^{a}\gamma^{A}(\gamma^{a})^{-1}) = -\text{Tr}(\gamma^{A})$, and so, $\text{Tr}(\gamma^{A}) = 0$.

THEOREM 3.3 Consider real or complex Clifford algebra $\mathcal{C}^{F}(p,q)$ of odd dimension $n$ and the set $\{\gamma^{1}, \ldots, \gamma^{n}\}$. Then $\pi(\gamma^{a_{1}...a_{k}}) = 0$, $k = 1, \ldots, n - 1$ and

$$\pi(\gamma^{1...n}) = \begin{cases} \pm 1, & \text{if (10) is a basis;} \\ 0, & \text{if (10) is not a basis.} \end{cases}$$

Proof. The theorem can be proved similarly to Theorem 3.2 using the following property: for any two elements $U, V$ of Clifford algebra $\mathcal{C}^{F}(p,q)$ of odd dimension $n$ holds $\pi(UV) = \pi(VU)$.

THEOREM 3.4 Consider real or complex Clifford algebra $\mathcal{C}^{F}(p,q)$ of odd dimension $n$ such that $p - q \equiv 1 \mod 4$ and the set $\{\gamma^{1}, \ldots, \gamma^{n}\}$. Then elements

$$\sigma^{a} = e^{1...n}\beta^{a}, \quad a = 1, \ldots, n \quad (11)$$

satisfy the following conditions $\sigma^{a}\sigma^{b} + \sigma^{b}\sigma^{a} = 2\eta^{ab} e$.

If $\{\gamma^{A}, A \in I\}$ is a basis, then $\{\sigma^{A}, A \in I\}$ is not a basis. If $\{\gamma^{A}, A \in I\}$ is not a basis, then $\{\sigma^{A}, A \in I\}$ is a basis.

Proof. Using (11) we obtain $\sigma^{a}\sigma^{b} + \sigma^{b}\sigma^{a} = e^{1...n}\beta^{a}e^{1...n}\beta^{b} + e^{1...n}\beta^{b}e^{1...n}\beta^{a} = 2\eta^{ab} e$, because $e^{1...n} \in \text{Cen}(\mathcal{C}^{F}(p,q))$ and $(e^{1...n})^{2} = (-1)^{\frac{n+1}{2}}(-1)^{9}e = e$ in the case $p - q \equiv 1 \mod 4$.

If $\beta^{1...n} = \pm e^{1...n}$, then $\sigma^{1...n} = \pm (e^{1...n})^{n}\beta^{1...n} = \pm e^{1...n}e^{1...n} = \pm e$, and if $\beta^{1...n} = \pm e$, then $\sigma^{1...n} = \pm (e^{1...n})^{n}\beta^{1...n} = \pm e^{1...n}e = \pm e^{1...n}$. Further we use Theorem 3.1.

THEOREM 3.5 Consider complex Clifford algebra $\mathcal{C}^{F}(p,q)$ of odd dimension $n$ such that $p - q \equiv 3 \mod 4$ and the set $\{\gamma^{1}, \ldots, \gamma^{n}\}$. Then elements

$$\sigma^{a} = ie^{1...n}\beta^{a}, \quad a = 1, \ldots, n \quad (12)$$
satisfy the following conditions \( \sigma^a \sigma^b + \sigma^b \sigma^a = 2 \eta^{ab} e. \)

If \( \{\gamma^A, A \in \mathcal{I}\} \) is a basis, then \( \{\sigma^A, A \in \mathcal{I}\} \) is not a basis. If \( \{\gamma^A, A \in \mathcal{I}\} \) is not a basis, then \( \{\sigma^A, A \in \mathcal{I}\} \) is a basis.

**Proof.** The proof is analogous to the proof of Theorem 3.4. We have \( \sigma^a \sigma^b + \sigma^b \sigma^a = ie^{1...n} \beta^{i} e^{1...n} \beta^{b} + ie^{1...n} \beta^{b} e^{1...n} \beta^{a} = 2 \eta^{ab} e, \) because \( e^{1...n} \in \text{Cen}(\mathcal{O}(p, q)) \) and \((e^{1...n})^2 = (-1)^\frac{n}{2}(-1)^{q}e = -e\) in the case \( p - q \equiv 3 \mod 4 \).

If \( \beta^{1...n} = \pm e^{1...n}, \) then \( \sigma^{1...n} = \pm (ie^{1...n})^n \beta^{1...n} = \pm ie^{1...n}e^{1...n} = \pm ie, \) and if \( \beta^{1...n} = \pm ie, \) then \( \sigma^{1...n} = \pm (ie^{1...n})^n \beta^{1...n} = \pm ie^{1...n}ie = \pm e^{1...n}. \)

**Theorem 3.6** Consider real or complex Clifford algebra \( \mathcal{O}(p, q), p + q = n \) and the set of elements \( \mathcal{B} = \{\gamma^A, A \in \mathcal{I}\} \) with the property [7]. Then each element of this set (if it is neither e nor \( \gamma^{1...n} \)) commutes with 2\( n - 2 \) even elements of the set \( \mathcal{B}, \) commutes with 2\( n - 2 \) odd elements of the set \( \mathcal{B}, \) anticommutes with 2\( n - 2 \) even elements of the set \( \mathcal{B} \) and anticommutes with 2\( n - 2 \) elements of the set \( \mathcal{B}. \) Element e commutes with all elements of the set \( \mathcal{B}. \)

(1) if \( n - \text{even} \), then \( \gamma^{1...n} \) commutes with all 2\( n - 1 \) even elements of the set \( \mathcal{B} \) and anticommutes with all 2\( n - 1 \) odd elements of the set \( \mathcal{B}; \)

(2) if \( n - \text{odd} \), then \( \gamma^{1...n} \) commutes with all 2\( n \) elements of the set \( \mathcal{B}. \)

**Proof.** Let us fix one multi-index \( A \) of the length \( k. \) The cases \( k = 0 \) and \( k = n \) are trivial (see Theorem 2.2).

In the other cases there are \( C^i_kC^{m-i}_{n-k} \) different multi-indices of the fix length \( m \) that have fixed number \( i \) coincident indices with multi-index \( A. \) Here \( C^i_n = \binom{n}{i} = \frac{n!}{i!(n-i)!} \) is binomial coefficient (we have \( C^i_n = 0 \) for \( k > n \)). Note, that

\[
\sum_{i=0}^{n} C^i_kC^{m-i}_{n-k} = C^m_n \quad \text{(Vandermonde's convolution)} - \text{the full number of ordered multi-indices of the length } m.
\]

When we swap basis element with multi-index \( A \) of the length \( k \) with another basis element with multi-index of the length \( m, \) then we obtain coefficient \((-1)^{km-i}, \) where \( i \) is the number of coincident indices in these 2 multi-indices, i.e. \( \gamma^{a_1...a_k} \gamma^{b_1...b_m} = (-1)^{km-i} \gamma^{b_1...b_m} \gamma^{a_1...a_k}. \)

If \( k \) is even and does not equal to 0 and \( n, \) then the number of even and odd elements \( \gamma^{b_1...b_m} \) that commute (in this case coefficient \( km - i \) must be even, and so \( i \) is even) with fixed \( \gamma^{a_1...a_k} \) respectively equals

\[
\sum_{m-even} \sum_{i-even} C^i_kC^{m-i}_{n-k} = 2^{n-2}, \quad \sum_{m-odd} \sum_{i-even} C^i_kC^{m-i}_{n-k} = 2^{n-2}.
\]

If \( k \) is odd and does not equal to \( n, \) then the number of even and odd elements \( \gamma^{b_1...b_m} \) that anticommute (in this case \( km - i \) is odd, and so \( m - i \) is even) with \( \gamma^{a_1...a_k} \) respectively equals

\[
\sum_{m-even} \sum_{i-even} C^i_kC^{m-i}_{n-k} = 2^{n-2}, \quad \sum_{m-odd} \sum_{i-odd} C^i_kC^{m-i}_{n-k} = 2^{n-2}.
\]

We can prove the last 4 identities if we regroup summands. For example, we have

\[
\sum_{m-even} \sum_{i-even} C^i_kC^{m-i}_{n-k} = \left( \sum_{j-even} C^j_k \right) \left( \sum_{l-odd} C^l_{n-k} \right) = 2^{k-1}2^{n-k-1} = 2^{n-2}.
\]
4. Generalized Reynolds operators in Clifford algebras

Let us consider the following operator\[ F(U) = \frac{1}{2^n} \gamma_A U \gamma^A, \] where we have a sum over multi-index \( A \in \mathcal{I} \). Note, that if \( \{\gamma^A, A \in \mathcal{I}\} \) is a basis of Clifford algebra \( C^F(p, q) \), then \( F \) is the Reynolds operator of the Salingaros’ vee group, see [24]. We have\[ \gamma_A = \gamma_{a_1 \ldots a_k} = (\gamma^A)^{-1}, \quad \gamma_a = \eta_{ab} \gamma^b = (\gamma^a)^{-1}. \] (13)

**Theorem 4.1** [24] If \( \{\gamma^A, A \in \mathcal{I}\} \) is a basis of Clifford algebra \( C^F(p, q) \), then we have
\[
F(U) = \frac{1}{2^n} \gamma_A U \gamma^A = \begin{cases} 
\text{Tr}(U)e, & \text{if } n \text{ is even}; \\
\text{Tr}(U)e + \pi(U)e^{1 \ldots n}, & \text{if } n \text{ is odd},
\end{cases}
\] (14)

Operator \( F \) is a projector \( F^2 = F \) (on the center of Clifford algebra).

**Proof.** We have
\[
(\gamma^a)^{-1} F(U) \gamma^a = \sum_A (\gamma^A \gamma^a)^{-1} U (\gamma^A \gamma^a) = \sum_B (\gamma^B)^{-1} U \gamma^B = F(U).
\]

So, \( F(U) \) is in the center of Clifford algebra (see Theorem 2.1). For elements \( U \) of ranks \( k = 1, \ldots, n-1 \) (and \( k = n \) in the case of even \( n \)) we have \( F(U) = 0 \). In other particular cases we have \( \gamma_A e^{1 \ldots n} \gamma^A = 2^n e^{1 \ldots n} \).

It is also easy to verify that \( F^2 = F \).

Note that \( \text{Cen}(\mathcal{O}(p, q)) \) is the “ring of invariants” (in the language of [8]) of Salingaros’ vee group.

Consider 2 different sets of Clifford algebra elements
\[
\gamma^a, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}e, \quad a, b = 1, \ldots, n, \] (15)
\[
\beta^a, \quad \beta^a \beta^b + \beta^b \beta^a = 2\eta^{ab}e \quad a, b = 1, \ldots, n \] (16)

and generalized Reynolds operators
\[
H(U) = \frac{1}{2^n} \beta^A U \gamma_A, \quad P(V) = \frac{1}{2^n} \gamma^A V \beta_A, \quad U, V \in \mathcal{O}^F(p, q),
\] (17)

where we have sum over multi-index \( A \) of \( 2^n \) elements.

---

6We use Einstein summation convention for multi-indices too.
7We want to deal only with ordered multi-indices. So, multi-index \( a_1 \ldots a_k \) is indivisible object in our consideration. It is not a set of indices \( a_1, a_2, \ldots, a_k \). That’s why \( \gamma_{a_1 \ldots a_k} \neq \gamma_{a_1} \cdots \gamma_{a_k} \) in our notation.
Theorem 4.2 We have
\[ \beta^B H(U) = H(U)\gamma^B, \quad \gamma^B P(V) = P(V)\beta^B, \quad \forall B. \] (18)

Proof. \[ \beta^B H(U)(\gamma^B)^{-1} = \sum_A (\beta^B \beta^A) U(\gamma^B \gamma^A)^{-1} = \sum_A (\beta^A) U(\gamma^A)^{-1} = H(U). \]
In the other case the proof is similar. \[ \square \]

Theorem 4.3 (1) For Clifford algebra \(\mathbb{O}^{p,q}(p,q)\) of even dimension \(n = p + q\), bases \(\{\gamma^A, A \in \mathcal{I}\}, \{\beta^A, A \in \mathcal{I}\}\) and elements \(17\) we have
\[ P(V)H(U) = H(U)P(V) = \text{Tr}(VH(U))e, \] (19)
where \(\text{Tr}(VH(U)) = \text{Tr}(H(U)V)\).

(2) For Clifford algebra \(\mathbb{O}^{p,q}(p,q)\) of odd dimension \(n = p + q\), bases \(\{\gamma^A, A \in \mathcal{I}\}, \{\beta^A, A \in \mathcal{I}\}\) and elements \(17\) we have
\[ P(V)H(U) = H(U)P(V) = \text{Tr}(VH(U))e + \pi(VH(U))e^{1..n}, \] (20)
where \(\text{Tr}(VH(U)) = \text{Tr}(H(U)V)\) and \(\pi(VH(U)) = \pi(H(U)V)\).

Proof. Using \(17\), \(18\) and \(14\), we obtain
\[ P(V)H(U) = \frac{1}{2^n} \beta^A H(U) = \frac{1}{2^n} \gamma^A VH(U)\gamma_A = \]
\[ \begin{cases} \text{Tr}(VH(U))e, & n \text{ - even;} \\ \text{Tr}(VH(U))e + \pi(VH(U))e^{1..n}, & n \text{ - odd.} \end{cases} \]

\[ \square \]

Theorem 4.4 Consider Clifford algebra \(\mathbb{O}^{p,q}(p,q)\) of dimension \(n = p + q\) and 2 sets \(15\), \(10\). Then
\[ \begin{aligned} 
\sum_{A} \sum_{B \in \mathcal{I}_{\text{even}}} \beta^A \gamma^B \gamma_A \gamma_B &= \begin{cases} 2^{n-1}(e + \beta^{1..n} \gamma_{1..n}), & n \text{ - even;} \\ 2^{n-1}(e + \beta^{1..n} \gamma_{1..n}), & n \text{ - odd,} \end{cases} \tag{21} \\
\sum_{A} \sum_{B \in \mathcal{I}_{\text{odd}}} \beta^A \gamma^B \gamma_A \gamma_B &= \begin{cases} 2^{n-1}(e - \beta^{1..n} \gamma_{1..n}), & n \text{ - even;} \\ 2^{n-1}(e + \beta^{1..n} \gamma_{1..n}), & n \text{ - odd,} \end{cases} \tag{22} 
\end{aligned} \]

Proof. Consider the following expressions in the case of even \(n\):
\[ \begin{align*} 
\beta^A e \gamma_A &= e + \beta^1 \gamma_1 + \cdots + \beta^{1..n} \gamma_{1..n}, \\
\beta^A_1 \gamma_A &= (e + \beta^1 \gamma_1 - \cdots - \beta^{1..n} \gamma_{1..n}) \gamma^1, \\
\beta^A_2 \gamma_A &= (e - \beta^1 \gamma_1 + \cdots - \beta^{1..n} \gamma_{1..n}) \gamma^2, \\
& \vdots \\
\beta^A_1^{1..n} \gamma_A &= (e - \beta^1 \gamma_1 - \cdots + \beta^{1..n} \gamma_{1..n}) \gamma^{1..n}. 
\end{align*} \]
Then multiply each equation on the right by \(\gamma_B\) and sum equations. From Theorem 3.6 we obtain the statement of this theorem. In the case of odd \(n\) the proof is similar. \[ \square \]
Theorem 4.5 Consider Clifford algebra $\mathcal{C}^F(p, q)$ of dimension $n = p + q$, 2 sets (15), (16) and element $H(U)$ (17).

1. If $n$ - even, then there exists such element $U$ in the set $\{\gamma^A, A \in \mathcal{I}\}$ that $H(U)$ is nonzero. Moreover, $U$ is among elements $\{\gamma_A, A \in \mathcal{I}_{\text{Even}}\}$ if $\beta_1^{1..n} \neq -\gamma_1^{1..n}$ and it is among elements $\{\gamma_A, A \in \mathcal{I}_{\text{Odd}}\}$ if $\beta_1^{1..n} \neq \gamma_1^{1..n}$.

2. If $n$ - odd and $\beta_1^{1..n} \neq -\gamma_1^{1..n}$, then there exists such element $U$ in the set $\{\gamma^A, A \in \mathcal{I}\}$ that $H(U)$ is nonzero. Moreover, there exists $U$ from the set $\{\gamma^A, A \in \mathcal{I}_{\text{Even}}\}$ and there exists $U$ from the set $\{\gamma^A, A \in \mathcal{I}_{\text{Odd}}\}$ at the same time.

Proof. Let $n$ is even. Suppose that for all elements $U$ element $H(U) = \frac{1}{2^n} \beta^A U \gamma_A$ equals zero. By Theorem 4.4 we obtain $2^n = (\beta^A \gamma_B \gamma_A) = 0$ and obtain a contradiction. Using (21), (22) we can similarly prove statement in other cases.

5. Using generalized Reynolds operators to prove generalized Pauli’s theorem in Clifford algebras

Now we are ready to prove the following generalizations of Pauli’s theorem using Theorems from Sections 3 and 4.

Theorem 5.1 Consider real (or, respectively complex) Clifford algebra $\mathcal{C}^F(p, q)$ of even dimension $n = p + q$. Let two sets of Clifford algebra elements

$$\gamma^a, \quad \beta^a, \quad a = 1, 2, \ldots, n$$

satisfy conditions

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab} e, \quad \beta^a \beta^b + \beta^b \beta^a = 2 \eta^{ab} e. \quad (24)$$

Then both sets generate bases of Clifford algebra and there exists an unique (up to multiplication by a real (respectively, complex) number) element $T \in \mathcal{C}^F(p, q)$ such that

$$\gamma^a = T^{-1} \beta^a T, \quad \forall a = 1, \ldots, n. \quad (25)$$

Moreover, $T = H(U) = \frac{1}{2^n} \beta^A U \gamma_A$, where $U$ is such element

- of the set $\{\gamma^A, A \in \mathcal{I}_{\text{Even}}\}$ if $\beta_1^{1..n} \neq -\gamma_1^{1..n}$,
- of the set $\{\gamma^A, A \in \mathcal{I}_{\text{Odd}}\}$ if $\beta_1^{1..n} \neq \gamma_1^{1..n}$.

that $H(U) \neq 0$.

Proof. Statement about bases follows from Theorem 3.1 For two arbitrary elements $F, G \in \mathcal{C}^F(p, q)$ and elements (17) we have (18) and (19). There exists such $U$ that $H(U)$ is nonzero (see Theorem 4.5). Further we take such element $V$ that $\text{Tr}(V H(U)) \neq 0$ (we can take $V$ from the set of basis elements $\{e^A\}$). So, from (19) we obtain that $H(U)$ is invertible. From (18) we obtain (25).

Let prove that $T$ is unique up to multiplication by a constant. Suppose that we have two elements $T_1, T_2$ that satisfy (25). Then for any $a = 1, \ldots, n$ we have
\[ T_1^{-1} \beta^a T_1 = T_2^{-1} \beta^a T_2. \] Let multiply this equation on the left by \( T_1 \) and on the right by \((T_2)^{-1}\). We obtain \([T_1 T_2^{-1}, \beta^a] = 0\) for \(a = 1, \ldots, n\). Using Theorem 2.1 we obtain \(T_1 = \mu T_2\), where \(\mu \neq 0\).

**Theorem 5.2** Let \(\mathcal{O}^{\mathbb{R}}(p, q)\) be the real Clifford algebra of odd dimension \(n = p + q\). Suppose that 2 sets of Clifford algebra elements

\[ \gamma^a, \beta^a, \quad a = 1, 2, \ldots, n \]

satisfy the conditions

\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} e, \quad \beta^a \beta^b + \beta^b \beta^a = 2\eta^{ab} e. \]

Then, for the Clifford algebra \(\mathcal{O}^{\mathbb{R}}(p, q)\) of signature \(p - q \equiv 1 \mod 4\), the elements \(\gamma^1 \ldots n\) and \(\beta^1 \ldots n\) either take the values \(\pm e^1 \ldots n\) and the corresponding sets generate bases of Clifford algebra or take the values \(\pm e\) and then the sets do not generate bases. In this situation, we have cases 1-4 below.

For the Clifford algebra \(\mathcal{O}^{\mathbb{R}}(p, q)\) of signature \(p - q \equiv 3 \mod 4\), the elements \(\gamma^1 \ldots n\) and \(\beta^1 \ldots n\) always take the values \(\pm e^1 \ldots n\) and the corresponding sets always generate bases of the Clifford algebra. In this situation, cases 1 and 2 only hold.

There exists a unique element \(T\) of the Clifford algebra (up to multiplication by an invertible element of the center of the Clifford algebra) such that

1. \(\gamma^a = T^{-1} \beta^a T, \quad \forall a = 1, \ldots, n \quad \iff \beta^1 \ldots n = \gamma^1 \ldots n;\)
2. \(\gamma^a = -T^{-1} \beta^a T, \quad \forall a = 1, \ldots, n \quad \iff \beta^1 \ldots n = -\gamma^1 \ldots n;\)
3. \(\gamma^a = e^1 \ldots n T^{-1} \beta^a T, \quad \forall a = 1, \ldots, n \quad \iff \beta^1 \ldots n = e^1 \ldots n \gamma^1 \ldots n;\)
4. \(\gamma^a = -e^1 \ldots n T^{-1} \beta^a T, \quad \forall a = 1, \ldots, n \quad \iff \beta^1 \ldots n = -e^1 \ldots n \gamma^1 \ldots n.\)

Note that all 4 cases have the unified notation \(\gamma^a = (\beta^1 \ldots n \gamma^1 \ldots n) T^{-1} \gamma^a T\).

Additionally, in the case of real Clifford algebra of signature \(p - q \equiv 1 \mod 4\), the element \(T\), whose existence is stated in all 4 cases of the theorem, equals

\[ T = H_{\text{Even}}(U) = \frac{1}{2^{n-1}} \sum_{A \in \mathcal{I}_{\text{Even}}} \beta^A U \gamma_A, \quad (26) \]

where \(U\) is an element of the set \(\{\gamma^A + \gamma^B, A, B \in \mathcal{I}_{\text{Even}}\}\).

In the case of real Clifford algebra of signature \(p - q \equiv 3 \mod 4\), the element \(T\), whose existence is stated in 1-2 cases of the theorem, equals \(T = H_{\text{Even}}(U)\), where \(U\) is such element of the set \(\{\gamma^A, A \in \mathcal{I}_{\text{Even}}\}\), that \(H_{\text{Even}}(U) \neq 0\).

**Theorem 5.3** Let \(\mathcal{O}^{\mathbb{C}}(p, q)\) be the complex Clifford algebra of odd dimension \(n = p + q\). Suppose that 2 sets of Clifford algebra elements

\[ \gamma^a, \beta^a, \quad a = 1, 2, \ldots, n \]

satisfy the conditions

\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} e, \quad \beta^a \beta^b + \beta^b \beta^a = 2\eta^{ab} e. \]

\[ \text{Note: } \text{See footnote 3 on the page 4.} \]
Then, for the Clifford algebra $\mathcal{C}(p, q)$ of the signature $p - q \equiv 1 \pmod{4}$ the elements $\gamma^1\ldots^n$ and $\beta^1\ldots^n$ either take the values $\pm e^{1\ldots n}$ and the corresponding sets generate bases of the Clifford algebra or take the values $\pm e$ and then the sets do not generate bases. In this situation, we have cases 1-4 below.

For the Clifford algebra $\mathcal{C}(p, q)$ of signature $p - q \equiv 3 \pmod{4}$ the elements $\gamma^1\ldots^n$ and $\beta^1\ldots^n$ either take the values $\pm e^{1\ldots n}$ and the corresponding sets generate bases of the Clifford algebra or take the values $\pm ie$ and then the sets do not generate bases. In this situation, we have cases 1, 2, 5 and 6 below.

There exists a unique element $T$ of the Clifford algebra (up to multiplication by an invertible element of the center of the Clifford algebra) such that

1. $\gamma^a = T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = \gamma^1\ldots^n$;
2. $\gamma^a = -T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = -\gamma^1\ldots^n$;
3. $\gamma^a = e^{1\ldots n}T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = e^{1\ldots n}\gamma^1\ldots^n$;
4. $\gamma^a = -e^{1\ldots n}T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = -e^{1\ldots n}\gamma^1\ldots^n$;
5. $\gamma^a = ie^{1\ldots n}T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = ie^{1\ldots n}\gamma^1\ldots^n$;
6. $\gamma^a = -ie^{1\ldots n}T^{-1}\beta^a T, \quad \forall a = 1, \ldots, n \iff \beta^1\ldots^n = -ie^{1\ldots n}\gamma^1\ldots^n$.

Note that all 6 cases have the unified notation $\gamma^a = (\beta^1\ldots^n\gamma_1\ldots^n)T^{-1}\gamma^a T$.

Additionally, the element $T$, whose existence is stated in all 6 cases of the theorem, equals

$$T = H_{\text{Even}}(U) = \frac{1}{2^{n-1}} \sum_{A \in \mathcal{T}_{\text{Even}}} \beta^A U \gamma_A,$$

(27)

where $U$ is an element of the set $\{\gamma^A + \gamma^B, A, B \in \mathcal{T}_{\text{Even}}\}$.

Proof. (of Theorems 5.2 and 5.3) Statements about sets that generate bases (or not) follow from Theorem 3.1. So we have 4 cases in real Clifford algebra $\mathcal{C}(p, q)$ and 6 cases in complex Clifford algebra $\mathcal{C}(p, q)$.

Cases 3, 4, 5 and 6 are reduced to the cases 1 and 2 by Theorem 3.4 and Theorem 3.5. To do this, we must change one of the given sets by the set $\sigma^a$. Case 2 (when we have $\beta^1\ldots^n = -\gamma^1\ldots^n$) is reduced to case 1. We must consider the set $\sigma^a = -\beta^a$ for $a = 1, \ldots, n$. For this set we have $\sigma^1\ldots^n = (-1)^n\beta^1\ldots^n = -\beta^1\ldots^n = \gamma^1\ldots^n$ and obtain $\gamma^a = T^{-1}\sigma^a T = T^{-1}\beta^a T$.

Thus we will consider and prove only Case 1 (when $\beta^1\ldots^n = \gamma^1\ldots^n$). We will consider only case, when sets $\beta^a, \gamma^a$ generate bases i.e. $\beta^1\ldots^n = \gamma^1\ldots^n = \pm e^{1\ldots n}$ (another case is reduced to this case by Theorem 3.4 and Theorem 3.5).

Consider arbitrary elements $U, V \in \mathcal{O}(p, q)$ and expressions (17). Then we have (18) and (20). We must prove that there exist such elements $H(U)$ and $V$ that $\text{Tr}(VH(U))e + \pi(VH(U))e^{1\ldots n}$ is invertible. Then from (20) we will obtain that $T = H(U)$ is invertible and from (18) we will obtain $\gamma^a = T^{-1}\beta^a T$.

We have

$$(\text{Tr}(VH(U))e + \pi(VH(U))e^{1\ldots n})(\text{Tr}(VH(U))e - \pi(VH(U))e^{1\ldots n}) =$$

$$= (\text{Tr}^2(VH(U)) - \pi^2(VH(U)))(-1)^{\frac{p(q-1)}{2}}(-1)^q)e = (\text{Tr}^2(VH(U)) \pm \pi^2(VH(U)))e,$$

where sign “+” is in the case $p - q \equiv 3 \pmod{4}$ and sign “−” is in the case $p - q \equiv 1$. 

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mod 4.

I) Let consider the case \( p - q \equiv 1 \mod 4 \). We must choose such elements \( H(U) \) and \( V \) that \( \text{Tr}(H(U)V) \neq \pm \pi H(U)V \). By Theorem 4.5 there exists such element \( F \) from the set \( \gamma \) that \( T \neq 0 \). Since \( \lambda \) is a basis, \( H(U) \) can be written in the form \( H(U) = h_A \gamma^A \).

If there exists such index \( C \) that \( h_C \neq 0 \) and \( h_{\bar{C}} \neq \pm h_C \) (where \( \bar{C} \) is such index that \( \bar{C} \) and \( C \) together constitute the full multi-index \( 1 \ldots n \)), we call \( \bar{C} \) and \( C \) adjacent multi-indices, see [24], then we can take \( V = \gamma_C \). Using Theorems 3.2 and 3.3 we obtain \( \text{Tr}(H(U)V) = h_C \neq \pm h_{\bar{C}} = \pm \pi H(U)V) \).

If there is no such index \( C \), then \( H(U) \) can be represented in the form

\[
H(U) = \sum_{j=1}^{k} \lambda_j \gamma^A_j (e + e^{1 \ldots n}) + \sum_{j=1}^{l} \mu_j \gamma^B_j (e - e^{1 \ldots n}), \quad \lambda_j, \mu_j \neq 0,
\]

where all multi-indices \( A_j \) and \( B_j \) are different and any 2 of them do not constitute the full multi-index \( 1 \ldots n \).

Using Theorem 4.3 we have \( \beta^A \gamma^B \gamma^C = 2^n + 1 e \) (element of the rank 0). So we have at least one such \( U \in \{ \gamma^B \} \) that \( k \neq 0 \) and at least one such \( U \in \{ \gamma^B \} \) that \( l \neq 0 \). So, there exists such element \( U \) of the set \( \{ \gamma^A + \gamma^B, A \neq B \} \), that \( k, l \neq 0 \) in (28). Then we will take the following element \( V \)

\[
V = \sum_{j=1}^{k} \frac{1}{\lambda_j} \gamma^A_j + \sum_{j=1}^{l} \frac{1}{\mu_j} \gamma^B_j.
\]

We have

\[
\text{Tr}(H(U)V) = \left( \sum_{j=1}^{k} \lambda_j \gamma^A_j (e + e^{1 \ldots n}) + \sum_{j=1}^{l} \mu_j \gamma^B_j (e - e^{1 \ldots n}) \right) \left( \sum_{j=1}^{k} \frac{1}{\lambda_j} \gamma^A_j + \sum_{j=1}^{l} \frac{1}{\mu_j} \gamma^B_j \right) =
\]

\[
= k(e + e^{1 \ldots n}) + \left( \sum_{i,j=1, i \neq j}^{k} \frac{\lambda_i}{\lambda_j} \gamma^A_i \gamma^A_j \right) + \sum_{i=1}^{k} \lambda_i \gamma^A_i(e + e^{1 \ldots n}) +
\]

\[
= l(e - e^{1 \ldots n}) + \left( \sum_{i,j=1, i \neq j}^{l} \frac{\mu_i}{\mu_j} \gamma^B_i \gamma^B_j \right) + \sum_{i=1}^{l} \mu_i \gamma^B_i(e - e^{1 \ldots n}),
\]

so, \( k + l = \text{Tr}(H(U)V) \neq \pm \pi (H(U)V) = \pm (k - l) \).

II) Now consider the case \( p - q \equiv 3 \mod 4 \). We must choose such elements \( H(U) \) and \( V \), that \( \text{Tr}^2(V H(U)) + \pi^2 (V H(U)) \neq 0 \). We always have such element \( U \) of the set \( \gamma \) that \( H(U) \neq 0 \) (by Theorem 4.5). And we can always take such element \( V \) (from the basis \( \{ e^A \} \) that \( \text{Tr}(V H(U)) \neq 0 \) or \( \pi(V H(U)) \neq 0 \). In the case of real Clifford algebra the theorem is proved.

In the case of complex Clifford algebra we must choose such elements \( H(U) \) and \( V \) that \( \text{Tr}(H(U)V) \neq \pm i \pi (H(U)V) \). Further proof is similar to the proof of the case \( p - q \equiv 1 \mod 4 \), but we consider instead of the elements (28) elements

\[
H(U) = \sum_{j=1}^{k} \lambda_j \gamma^A_j (e + i e^{1 \ldots n}) + \sum_{j=1}^{l} \mu_j \gamma^B_j (e - i e^{1 \ldots n}), \quad \lambda_j, \mu_j \neq 0.
\]
Proof of uniqueness of element $T = H(U)$ up to multiplication on an invertible element of Clifford algebra element is similar to the proof of uniqueness in Theorem 5.1.

According to the proof above, we can find element $T$ in different cases in the following form (up to multiplication by nonzero constant):

1. $\beta^{1...n} = \gamma^{1...n} \Rightarrow T = \beta^{A} U \gamma^{A}$,
2. $\beta^{1...n} = -\gamma^{1...n} \Rightarrow T = (-1)^{|A|} \beta^{A} U \gamma^{A}$,
3. $\beta^{1...n} = e^{1...n} \gamma^{1...n} \Rightarrow T = \sum_{A \in I_{\text{Even}}} \beta^{A} U \gamma^{A} + e^{1...n} \sum_{A \in I_{\text{Odd}}} \beta^{A} U \gamma^{A}$,
4. $\beta^{1...n} = -e^{1...n} \gamma^{1...n} \Rightarrow T = \sum_{A \in I_{\text{Even}}} \beta^{A} U \gamma^{A} - e^{1...n} \sum_{A \in I_{\text{Odd}}} \beta^{A} U \gamma^{A}$,
5. $\beta^{1...n} = ie^{1...n} \gamma^{1...n} \Rightarrow T = \sum_{A \in I_{\text{Even}}} \beta^{A} U \gamma^{A} + ie^{1...n} \sum_{A \in I_{\text{Odd}}} \beta^{A} U \gamma^{A}$,
6. $\beta^{1...n} = -ie^{1...n} \gamma^{1...n} \Rightarrow T = \sum_{A \in I_{\text{Even}}} \beta^{A} U \gamma^{A} - ie^{1...n} \sum_{A \in I_{\text{Odd}}} \beta^{A} U \gamma^{A}$.

It is not difficult to prove that all these elements $T$ equal (27) up to multiplication by nonzero constant. For example, if $\beta^{1...n} = \gamma^{1...n}$, then

$$\beta^{a_1...a_k} U (\gamma^{a_1...a_k})^{-1} = \beta^{a_1...a_k} \beta^{1...n} U (\gamma^{a_1...a_k} \gamma^{1...n})^{-1} = \beta^{a_{k+1}...a_n} U \gamma^{a_{k+1}...a_n}^{-1}$$

and $\beta^{A} U \gamma^{A} = 2 \sum_{A \in I_{\text{Even}}} \beta^{A} U \gamma^{A}$. Another cases are similar.

Note that Theorems 5.1, 5.2 and 5.3 can be reformulated in matrix formalism using Theorems 2.2 and 2.3.

Generalized Pauli’s theorem (Theorems 5.1, 5.2 and 5.3) are useful in different applications of mathematical physics, for example, when we consider Weyl, Majorana and Weyl-Majorana spinors (see [25]). Also these theorems can be useful in the proof of various algebraic properties of Clifford algebras. In [19] we present a local variant of these theorems.

These theorems are used in the proof of theorem about double covers of orthogonal groups by spin groups (see [26]). Generalized Pauli’s theorem give us an algorithm to compute elements that connect two different sets of Clifford algebra elements. This algorithm allows us to compute elements of spin groups that correspond to given elements of orthogonal groups (see [27]).

Acknowledgements

The author is grateful to N.G.Marchuk for fruitful discussions.

This work was supported by Russian Science Foundation (project RSF 14-11-00687, Steklov Mathematical Institute).

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