Finite-dimensional Lie algebras of order $F$

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Abstract

$F$-Lie algebras are natural generalisations of Lie algebras ($F = 1$) and Lie superalgebras ($F = 2$). When $F > 2$ not many finite-dimensional examples are known. In this paper we construct finite-dimensional $F$-Lie algebras $F > 2$ by an inductive process starting from Lie algebras and Lie superalgebras. Matrix realisations of $F$-Lie algebras constructed in this way from $su(n)$, $sp(2n)$, $so(n)$ and $sl(n|m)$, $osp(2|m)$ are given. We obtain non-trivial extensions of the Poincaré algebra by İnönü-Wigner contraction of certain $F$-Lie algebras with $F > 2$.

1 Introduction

The classification of algebraic objects satisfying certain axioms may be considered a fundamental objective on purely mathematical grounds. If in addition, these objects turn out to be relevant for the description of the possible symmetries of a physical system, such a classification takes on a whole new meaning. The main question is, of course, what are the mathematical structures which are useful in describing the laws of physics. Simple complex finite-dimensional Lie algebras were classified at the end of the 19th century by W. Killing and E. Cartan well before any physical applications were known. Since then, Lie algebras have become essential for the description of space-time symmetries and fundamental interactions. On the other hand, it was the discovery of supersymmetry in relativistic quantum field theory or as a possible extension of Poincaré invariance [1] which gave rise to the concept of Lie superalgebras and their subsequent classification [2, 3].

It is generally accepted that because of the theorems of Coleman & Mandula [4] and Haag, Lopuszanski & Sohnius [5], one cannot go beyond supersymmetry. However, if one weakens the hypotheses of these two theorems, one can imagine symmetries which go beyond supersymmetry [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28], the idea being that then the generators of the Poincaré algebra can be obtained as an appropriate product of more than two fundamental additional symmetries.

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These new generators are in a representation of the Lorentz algebra which is neither bosonic nor fermionic. Two kinds of representations are generally taken: parafermionic representations \([29]\), or infinite-dimensional representations (Verma module) \([30]\).

Fractional supersymmetry (FSUSY) \([1, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]\) is among the possible extensions of supersymmetry which have been studied in the literature. Basically, in such extensions, the generators of the Poincaré algebra are obtained as \(F\)-fold \((F \in \mathbb{N}^*)\) symmetric products of more fundamental generators. A natural generalisation of Lie (super)algebras which is relevant for the algebraic description of FSUSY was defined in \([24, 28]\) and called an \(F\)-Lie algebra. An \(F\)-Lie algebra admits a \(\mathbb{Z}_F\)-gradation, the zero-graded part being a Lie algebra. An \(F\)-fold symmetric product (playing the role of the anticommutator in the case \(F = 2\)) expresses the zero graded part in terms of the non-zero graded part.

The purpose of this paper is to show how one can construct many examples of finite-dimensional \(F\)-Lie algebras by an inductive process starting from Lie algebras and Lie superalgebras. Some preliminary results is this direction were given in \([28]\). Two types of finite-dimensional \(F\)-Lie algebras will be constructed. The first family of examples, which we call trivial, are obtained by taking the direct sum of a Lie (super)algebra with the trivial representation. The second family is more interesting: by an inductive procedure we show how one can give the underlying vector space of any Lie algebra or any classical Lie superalgebra the structure of an \(F\)-Lie algebra. This procedure involves Casimir operators in the case of Lie algebras and invariant symmetric forms on the odd part of the algebra in the case of Lie superalgebras.

The paper is organized as follows. In section 2 we recall the definition of an \(F\)-Lie algebra and show how one can construct an \(F\)-Lie algebra of order \(F_1 + F_2\) from an \(F\)-Lie algebra of order \(F_1 \geq 2\) and an invariant symmetric form of order \(F_2\) on its non-zero graded part (c.f. Theorem 2.6). In section 3 we introduce the notion of a graded 1-Lie algebra in order to prove a version of theorem 2.6 when \(F_1 = 1\) (theorem 3.6), and give some explicit examples of \(F\)-Lie algebras associated to Lie algebras. In section 4 we give explicit examples of \(F\)-Lie algebras associated to Lie superalgebras. In section 5 we obtain FSUSY extensions of the Poincaré algebra by Inönü-Wigner contraction of certain \(F\)-Lie algebras constructed in the two previous sections. In section 6 we define a notion of simplicity for \(F\)-Lie algebras and give examples of simple and non-simple \(F\)-Lie algebras. Finally, in section 7 we give finite-dimensional matrix realisations of the \(F\)-Lie algebras of section 4 induced from \(\mathfrak{su}(m|n)\) and \(\mathfrak{osp}(2|2n)\) and a quadratic form. Using finite-dimensional matrices, we also show that the underlying vector spaces of the graded 1-Lie algebras \(\mathfrak{su}(n) \oplus \mathfrak{su}(n), \mathfrak{so}(n) \oplus \mathfrak{so}(n)\) and \(\mathfrak{sp}(2n) \oplus \mathfrak{sp}(2n)\) can be given \(F\)-Lie algebra structures which cannot be obtained by our inductive process.

## 2 \(F\)-Lie algebras

### 2.1 Definition of \(F\)-Lie algebras

In this section, we recall briefly the definition of \(F\)-Lie algebras given in \([24, 28]\). Let \(F\) be a positive integer and let \(q = e^{\frac{2\pi i}{F}}\). We consider \(S\) a complex vector space and \(\varepsilon\) an automorphism of \(S\) satisfying \(\varepsilon^F = 1\). We set \(A_k = S_{q^k}, 1 \leq k \leq F - 1\) and \(B = S_1\) \((S_{q^k}\) is the eigenspace corresponding to the eigenvalue \(q^k\) of \(\varepsilon\)). Then we have \(S = B \oplus_{k=1}^{F-1} A_k\).

**Definition 2.1** \(S = B \oplus_{k=1}^{F-1} A_k\) is called a (complex) \(F\)-Lie algebra if it is endowed with the following structure:
1. \( \mathcal{B} \) is a (complex) Lie algebra and \( A_k, 1 \leq k \leq F - 1 \) are representations of \( \mathcal{B} \). If \([,]\) denotes the bracket on \( \mathcal{B} \) and the action of \( \mathcal{B} \) on \( S \) it is clear that \( \forall b \in \mathcal{B}, \forall s \in S, [\varepsilon(b), \varepsilon(s)] = \varepsilon([b, s]) \).

2. There exist multilinear \( \mathcal{B} \)-equivariant maps \( \{ , \cdots , \} : S^F(A_k) \to \mathcal{B} \), where it reduces to \( \mathcal{B} \)-equivariant maps \( \{ , \cdots , \} : S^F(D) \to \mathcal{B} \), where \( S^F(D) \) denotes the \( F \)-fold symmetric product of \( D \). It is easy to see that \( \{\varepsilon(a_1), \cdots, \varepsilon(a_F)\} = \varepsilon(\{a_1, \cdots, a_F\}), \forall a_1, \cdots, a_F \in A_k \).

3. For \( b_i \in \mathcal{B} \) and \( a_j \in A_k \) the following “Jacobi identities” hold:

\[
\sum_{i=1}^{F+1} [a_i, \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_F\}] = 0. \tag{J_4}
\]

Note that the three first identities are automatic but the fourth, which we will refer to as \( J_4 \), is an extra constraint.

**Remark 2.2** An \( F \)-Lie algebra is more than a Lie algebra \( g_0 \), a representation \( g_1 \) of \( g_0 \) and a \( g_0 \)-valued \( g_0 \)-equivariant symmetric form on \( g_1 \). Indeed, although the three first Jacobi identities are manifest in this situation, the fourth is not necessarily true. As an example, consider \( g_0 = sl(2, \mathbb{C}) \) and \( g_1 = S_{2k+1}, k \in \mathbb{N} \) (the irreducible representation of dimension \( 2k + 2 \)). From the decomposition \( S^2(S_{2k+1}) = S_{4k+2} \oplus S_{4k-2} \oplus \cdots \oplus S_2 \) one has an \( sl(2, \mathbb{C}) \)-equivariant mapping from \( S^2(S_{2k+1}) \to S_2 \to sl(2, \mathbb{C}) \). But \( g = sl(2, \mathbb{C}) \oplus S_{2k+1} \) is not a Lie superalgebra (the fourth Jacobi identity is not satisfied) except when \( k = 0 \) where it reduces to \( osp(1|2) \).

**Remark 2.3** A 1–Lie algebra is a Lie algebra, and a 2–Lie algebra is a Lie superalgebra. We will also refer to these objects as \( F \)-Lie algebras of order one and two respectively.

**Remark 2.4** Notice that \( \{a_1, \cdots, a_F\} \) is only defined if the \( a_i \) are in the same \( A_k \) and that \( \forall k = 1, \cdots, F - 1 \), the spaces \( S_k = \mathcal{B} \oplus A_k \) are \( F \)-Lie algebras.

From now on, we consider only \( F \)-Lie algebras \( S = \mathcal{B} \oplus A \) such that \( A \) is an eigenspace of \( \varepsilon \).

**Remark 2.5** If \( \mathfrak{h} \subset \mathcal{B} \) is a Cartan subalgebra and \( F_{\lambda_1}, \cdots, F_{\lambda_F} \in F \) are respectively of weight \( \lambda_1, \cdots, \lambda_F \), then \( \{F_{\lambda_1}, \cdots, F_{\lambda_F}\} \in \mathcal{B} \) is of weight \( \lambda_1 + \cdots + \lambda_F \). In particular, if \( \lambda_1 + \cdots + \lambda_F \neq 0 \) is not a root of \( \mathcal{B} \) this bracket is zero.

This structure can be seen as a possible generalisation of Lie algebras (\( F = 1 \)) or Lie superalgebras (\( F = 2 \)) and can be compared, in some sense, to the ternary algebras (\( F = 3 \)) considered in [33], and to the \( n \)-ary algebras (\( F = n \)) introduced in [33] but in a different context. We have shown [24, 28] that all examples of FSUSY considered in the literature can be described within the framework of \( F \)-Lie algebras.
2.2 An inductive construction of $F$–Lie algebras

Let $\mathfrak{g}$ be a complex Lie algebra and let $\tau, \tau'$ be representations of $\mathfrak{g}$ such that there is a $\mathfrak{g}$–equivariant map $\mu_F : S^F(\tau) \to \tau'$. We set:

$$S = \mathcal{B} \oplus \mathcal{A}_1 = (\mathfrak{g} \oplus \tau') \oplus \tau.$$ 

Then, $\mathcal{B} = \mathfrak{g} \oplus \tau'$ is a Lie algebra as the semi-direct product of $\mathfrak{g}$ and $\tau'$ (the latter with the trivial bracket). We can extend the action of $\mathfrak{g}$ on $\mathfrak{g}$ to an action of $\mathcal{B}$ on $\mathfrak{g}$ by letting $\tau'$ act trivially on $\mathfrak{g}$. This defines the bracket $[\cdot, \cdot]$ on $S$. For the map $\{\cdots\}$ we take $\mu_F$. The first three Jacobi identities are clearly satisfied, and the fourth is also satisfied as each term in the expression on the L.H.S. of $J_4$ vanishes. Hence $S$ is an $F$–Lie algebra. There are two essentially opposite ways of giving explicit examples of $F$–Lie algebras of this type. One can either start from $\mathfrak{g}$ and $\tau'$ and extract an “$F$–root” of $\tau'$, or one can decompose $S^F(\tau)$ into irreducible summands and project onto one of them [24]. The first approach is the more difficult since, in general, it involves infinite-dimensional representation theory. For example if one starts with $\tau' = \mathcal{D}_{\mu_1}$, the vector representation of $\mathfrak{so}(1, d - 1)$ of highest weight $\mu_1$, the representation $\tau = \mathcal{D}_{\mu_2}$ of highest weight $\mu_2$, is not exponentialisable (see e.g. [33]) and does not define a representation of the Lie group $SO(1, d - 1)$, except when $d = 3$ where such representations describe relativistic anyons [34]. The second approach on the other hand will always give finite-dimensional $F$–Lie algebras if one starts from finite-dimensional representations.

The following theorem gives an inductive procedure for constructing finite-dimensional $F$–Lie algebras.

**Theorem 2.6** Let $\mathfrak{g}_0$ be a Lie algebra and $\mathfrak{g}_1$ a representation of $\mathfrak{g}_0$ such that

(i) $S_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is an $F$–Lie algebra of order $F_1 > 1$;

(ii) $\mathfrak{g}_1$ admits a $\mathfrak{g}_0$–equivariant symmetric form $\mu_2$ of order $F_2 \geq 1$.

Then $S = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ admits an $F$–Lie algebra structure of order $F_1 + F_2$, which we call the $F$–Lie algebra induced from $S_1$ and $\mu_2$.

**Proof:** By hypothesis, there exist $\mathfrak{g}_0$–equivariant maps $\mu_1 : S^{F_1}(\mathfrak{g}_1) \to \mathfrak{g}_0$ and $\mu_2 : S^{F_2}(\mathfrak{g}_1) \to \mathbb{C}$. Now, consider $\mu : S^{F_1+F_2}(\mathfrak{g}_1) \to \mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{g}_0$ defined by

$$\mu(f_1, \cdots, f_{F_1+F_2}) = \frac{1}{F_1! F_2!} \sum_{\sigma \in S_{F_1+F_2}} \mu_1(f_{\sigma(1)}, \cdots, f_{\sigma(F_1)}) \otimes \mu_2(f_{\sigma(F_1+1)}, \cdots, f_{\sigma(F_1+F_2)}),$$

where $f_1, \cdots, f_{F_1+F_2} \in \mathfrak{g}_1$ and $S_{F_1+F_2}$ is the group of permutations on $F_1 + F_2$ elements. By construction, this is a $\mathfrak{g}_0$–equivariant map from $S^{F_1+F_2}(\mathfrak{g}_1) \to \mathfrak{g}_0$, thus the three first Jacobi identities are satisfied. The last Jacobi identity $J_4$ is more difficult to check and is a consequence of $J_4$ for the $F$–Lie algebra $S_1$ and a factorisation property. Indeed, setting $F = F_1 + F_2$, the identity $J_4$ for the terms in

$$\sum_{i=0}^{F} \left[ f_i, \mu(f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_{F}) \right],$$

of the form $\mu_1(f_{\sigma(1)}, \cdots, f_{\sigma(F_1)}) \otimes \mu_2(f_{\sigma(F_1+1)}, \cdots, f_{\sigma(F_1+F_2)})$ with $\sigma \in S_{F_1+F_2+1}$, reduces to
\[
\sum_{i=0}^{F_1} \left[ f_{\sigma(i)}, \mu_1 \left( f_{\sigma(1)}, \cdots, f_{\sigma(i-1)}, f_{\sigma(i+1)}, \cdots, f_{\sigma(F_1)} \right) \right] \otimes \mu_2 (f_{\sigma(F_1+1)}, \cdots, f_{\sigma(F_1+F_2)}) = 0,
\]

using \(\mu_2 (f_{\sigma(F_1+1)}, \cdots, f_{\sigma(F_1+F_2)}) \in \mathbb{C}\). But the L.H.S. vanishes by \(J_4\) for the \(F\)-Lie algebra \(S_1\). A similar argument works for the other terms and hence \(J_4\) is satisfied and \(S\) is an \(F\)-Lie algebra of order \(F_1 + F_2\).

QED

Remark 2.7 Theorem 2.6 is equivalent to the fact that the product of two \(g_0\)-equivariant symmetric forms satisfying \(J_4\) also satisfies \(J_4\) if one of them is scalar-valued.

3 Finite-dimensional \(F\)-Lie algebras associated to Lie algebras

In this section we first introduce the notion of a graded 1–Lie algebra in order to have a version of 2.6 when \(F_1 = 1\).

3.1 Graded 1–Lie algebras

Definition 3.1 A graded 1–Lie algebra is a \(\mathbb{Z}_2\)-graded vector space \(S = \mathcal{B} \oplus \mathcal{F}\) such that:

1. \(\mathcal{B}\) is a Lie algebra;
2. \(\mathcal{F}\) is a representation of \(\mathcal{B}\);
3. there is a \(\mathcal{B}\)-equivariant map \(\mu : \mathcal{F} \to \mathcal{B}\);
4. \([\mu (f_1), f_2] + [\mu (f_2), f_1] = 0, \forall f_1, f_2 \in \mathcal{F}\).

Example 3.2 Let \(\mathfrak{g}\) be a Lie algebra. Set \(\mathcal{B} = \mathfrak{g}\), \(\mathcal{F} = \text{ad} \ \mathfrak{g}\) and \(S = \mathcal{B} \oplus \mathcal{F}\). If \(\mu : \mathcal{F} \to \mathcal{B}\) is the identity then \((S, \mu)\) is a graded 1–Lie algebra.

Remark 3.3 A graded 1–Lie algebra is not a priori a Lie algebra but it easy to see that, in fact, it has a natural graded Lie algebra structure.

Remark 3.4 \(\text{Ker} \mu\) is a \(\mathcal{B}\)-invariant subspace of \(\mathcal{F}\) and \(\text{Im} \mu\) is a \(\mathcal{B}\)-invariant subspace of \(\mathcal{B}\). In particular, if \(\mathcal{B}\) is simple, \(\mathcal{F}\) irreducible and \(\mu\) non-trivial, then \(\mu\) defines a \(\mathcal{B}\)-equivariant isomorphism between \(\mathcal{F}\) and \(\mathcal{B}\).

A graded 1–Lie algebra is a graded Lie algebra in the usual sense. In general, however, a graded Lie algebra is not a graded 1–Lie algebra since there is no preferred map from the odd to the even part.

Proposition 3.5 Let \(\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-\) be a graded Lie algebra, and let \(\mu : \mathfrak{g}_+ \to \mathfrak{g}\) be an odd \(\mathfrak{g}_+\)-equivariant map of \(\mathfrak{g}\) such that \(\mu\) is injective on \([\mathfrak{g}_+, \mathfrak{g}_-]\). Then \((\mathfrak{g}, \mu)\) is a graded 1–Lie algebra.
Proof: One only has to check (3.1)(4). One has \( \forall f_1, f_2 \in g_-, \mu ([\mu(f_1), f_2] + [\mu(f_2), f_1]) = [\mu(f_1), \mu(f_2)] + [\mu(f_2), \mu(f_1)] = 0 \). Since \( \mu \) is injective on \([g_+, g_-]\) this implies (3.1)(4). QED

**Theorem 3.6 (2.6-bis)** Let \( g_0 \) be a Lie algebra and \( g_1 \) a representation of \( g_0 \) such that

(i) \( S_1 = g_0 \oplus g_1 \) is an graded 1–Lie algebra;

(ii) \( g_1 \) admits a \( g_0 \)-equivariant symmetric \( \mu_2 \) form of order \( F_2 \geq 1 \).

Then \( S = g_0 \oplus g_1 \) admits an \( F \)-Lie algebra structure of order \( 1 + F_2 \) which we call the \( F \)-Lie algebra induced from \( S_1 \) and \( \mu_2 \).

**Proof:** Analogous to 2.6. QED

### 3.2 Trivial and induced \( F \)-Lie algebras

Consider the graded 1–Lie algebra \( S = g_0 \oplus g_1 \) where \( g_0 \) is a Lie algebra, \( g_1 \) is the adjoint representation of \( g_0 \) and \( \mu : g_1 \rightarrow g_0 \) is the identity. Let \( J_1, \ldots, J_{\dim g_0} \) be a basis of \( g_0 \), and \( A_1, \ldots, A_{\dim g_0} \) the corresponding basis of \( g_1 \). The graded 1–Lie algebra structure on \( S \) is then:

\[
[J_a, J_b] = f_{ab}^c J_c, \quad [J_a, A_b] = f_{ab}^c A_c, \quad \mu(A_a) = J_a,
\]

where \( f_{ab}^c \) are the structure constant of \( g_0 \). Two types of \( F \)-Lie algebras associated to \( S \) will be defined.

The first type of \( F \)-Lie algebras associated to \( S \), will be called trivial and are constructed as follows:

**Theorem 3.7** Let \( g_0 \) be a Lie algebra and let \( F \geq 1 \) be an integer. Then \( S = g_0 \oplus (g_1 \oplus \mathbb{C}) \) can be given the structure of an \( F \)-Lie algebra (graded 1–Lie algebra if \( F = 1 \)) where \( g_1 \) is the adjoint representation of \( g_0 \) and \( \mathbb{C} \) is the trivial representation.

**Proof:** The map \( \mu : S^F(g_1 \oplus \mathbb{C}) \rightarrow g_0 \) is given by projection on \( g_1 \) in the decomposition \( S^F(g_1 \oplus \mathbb{C}) = S^F(g_1) \oplus S^{F-1}(g_1) \oplus \cdots \oplus S^1(g_1) \oplus g_1 \oplus \mathbb{C} \), followed by the identification of \( g_1 \) with \( g_0 \).

With the notations of (3.1) the brackets are:

\[
\{\lambda, \cdots, \lambda\} = 0
\]
\[
\{\lambda, \cdots, \lambda, A_a, \} = J_a
\]
\[
\vdots
\]
\[
\{\lambda, \cdots, \lambda, A_{a_1}, \cdots, A_{a_k}\} = 0, \quad 1 < k \leq F
\]
\[
\vdots
\]
\[
\{A_{a_1}, \cdots, A_{a_F}\} = 0.
\]

with \( A_a, \in g_1, \lambda \in \mathbb{C}, J_a \in g_0 \).

It is easy to check that the four Jacobi identities are satisfied. QED
Theorem 3.8 Let $g_0$ be a simple (complex) Lie algebra and $g_1$ be the adjoint representation of $g_0$. Then a Casimir operator of $g_0$ of order $m$ induces the structure of an $F$–Lie algebra of order $m+1$ on $S_{m+1} = g_0 \oplus g_1$.

Proof: By example 3.2 $g_0 \oplus g_1$ is a graded 1–Lie algebra and the result follows from 3.6. QED

Remark 3.9 One can give explicit formulae for the bracket of these $F$–Lie algebras as follows. Let $J_a, a = 1, \cdots, \dim(g_0)$ and let $A_a, a = 1, \cdots, \dim(g_0)$ be bases as at the beginning of this section. Let $h_{a_1\cdots a_m}$ be a Casimir operator of order $m$ (for $m = 2$, the Killing form $g_{ab} = \text{Tr}(A_aA_b)$ is a primitive Casimir of order two). Then, the $F$–bracket of the $F$–Lie algebra is

$$\{A_{a_1}, A_{a_2}, \cdots, A_{a_{m+1}}\} = \sum_{\ell=1}^{m+1} h_{a_1\cdots a_{\ell-1}a_{\ell+1}\cdots a_{m+1}} J_{a_\ell}$$

(3.3)

For the Killing form this gives

$$\{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a.$$  

(3.4)

If $g_0 = \mathfrak{sl}(2)$, the $F$–Lie algebra of order three induced from the Killing form is the $F$–Lie algebra of $[36]$.

4 Finite-dimensional $F$–Lie algebras associated to Lie superalgebras

In this section we will consider some $F$–Lie algebras which can be associated to Lie superalgebras using Theorem 2.6.

4.1 Lie superalgebras

We first recall some basic results on simple complex Lie superalgebras (for more details see [37, 38]). Simple Lie superalgebras can be divided into two types: classical and the Cartan-type. Classical Lie superalgebras can be further divided into two families: basic and strange. A basic Lie superalgebra $g$ is a graded Lie algebra of order two. Then, the Killing form on $g$ induces the structure of an $F$–Lie algebra as follows. Let $h_{a_1\cdots a_m}$ be a primitive Casimir of $g$ (for $m = 2$, the Killing form $g_{ab} = \text{Tr}(A_aA_b)$ is a primitive Casimir of order two). Then, the $F$–bracket of the $F$–Lie algebra is

$$\{A_{a_1}, A_{a_2}, \cdots, A_{a_{m+1}}\} = \sum_{\ell=1}^{m+1} h_{a_1\cdots a_{\ell-1}a_{\ell+1}\cdots a_{m+1}} J_{a_\ell}$$

(3.3)

For the Killing form this gives

$$\{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a.$$  

(3.4)

If $g_0 = \mathfrak{sl}(2)$, the $F$–Lie algebra of order three induced from the Killing form is the $F$–Lie algebra of $[36]$.
2. (Basic of type II)

(i) \(B(m, n) : m \geq 0, n \geq 1, \mathfrak{g}_0 = \mathfrak{so}(2m + 1) \oplus \mathfrak{sp}(2n), \mathfrak{g}_1 = (2m + 1, 2n)\):

(ii) \(D(m, n) : m \geq 2, n \geq 1, m \neq n + 1, \mathfrak{g}_0 = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n), \mathfrak{g}_1 = (2m, 2n)\):

(iii) \(D(n + 1, n) : \mathfrak{g}_0 = \mathfrak{so}(2(n + 1)) \oplus \mathfrak{sp}(2n), \mathfrak{g}_1 = (2(n + 1), 2n)\):

(iv) \(D(2, 1; \alpha) : \alpha \in \mathbb{C} - \{0, -1\}, \mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{g}_1 = (2, 2, 2)\):

(v) for \(F(4) : \mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{so}(7), \mathfrak{g}_1 = (2, 8)\):

(vi) for \(G(3) : \mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{g}_2, \mathfrak{g}_1 = (2, 7)\).

3. (Strange)

(i) \(Q(n) : n > 1 \mathfrak{g}_0 = \mathfrak{sl}(n), \mathfrak{g}_1 = \text{ad}(\mathfrak{sl}(n)), \) with ad the adjoint representation;

(ii) \(P(n) : n > 1 \mathfrak{g}_0 = \mathfrak{sl}(n), \mathfrak{g}_1 = [2] \oplus [1^{n-2}], \) where \([2]\) denotes \(S^2(\mathbb{C}^n)\) the two-fold symmetric representation and \([1^{n-2}]\) denotes \(\Lambda^{n-2}(\mathbb{C}^n)\) the \((n-2)\)-fold antisymmetric representation.

(The superscript in 1(i) and 1(iii) indicates the \(\mathfrak{gl}(1)\) charge).

4.2 Symmetric invariant forms

By Theorem 2.6 one can construct an \(F\)–Lie algebra from a Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) and a \(\mathfrak{g}_0\)–invariant symmetric form on \(\mathfrak{g}_1\). In general determining all invariant symmetric forms on a given representation of a given Lie algebra is very difficult. However, for the Lie superalgebras given in the above list we will show how one can construct many invariant symmetric forms. The key observation is that for each basic Lie superalgebra in the list, the odd part \(\mathfrak{g}_1\) is either a tensor product (type II) or a sum of two dual tensor products (type I) as a representation of \(\mathfrak{g}_0\). Thus, to find \(\mathfrak{g}_0\)–invariant symmetric forms on \(\mathfrak{g}_1\) one can use the following well known isomorphisms of representations of \(GL(A) \times GL(B)\) \([39]\):

\[
S^p(A \oplus B) = \sum_{k=0}^{p} S^k(A) \otimes S^{p-k}(B) \tag{4.1}
\]

\[
S^p(A \otimes B) = \sum_{\Gamma} S^\Gamma(A) \otimes S^{\Gamma}(B), \tag{4.2}
\]

where the second sum is taken over all Young diagrams \(\Gamma\) of length \(p\) and \(S^\Gamma(A)\) denotes the irreducible representation of \(GL(A)\) corresponding to the Young symmetriser of \(\Gamma\).

Type I

We consider the Lie superalgebra \(A(m, n)\). The case of the other basic type I Lie superalgebras is similar. Then \(\mathfrak{g}_0 = \mathfrak{sl}(n + 1) \oplus \mathfrak{sl}(n + 1) \oplus \mathfrak{gl}(1)\) and \(\mathfrak{g}_1 = (\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}) \oplus (\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C})^*\).

Using the formulae (4.1) and (4.2), one sees that \(S^\Gamma(\mathfrak{g}_1)\) is a direct sum of terms of the form:

\[
S^\Gamma(\mathbb{C}^{m+1}) \otimes S^\Gamma(\mathbb{C}^{n+1}) \otimes S^\Gamma(\mathbb{C}^{n+1}) \otimes S^{\Gamma'}(\mathbb{C}^{m+1}) \otimes \mathbb{C}^{[\Gamma] - [\Gamma']}, \tag{4.3}
\]

where \(|\Gamma|\) is the length of \(\Gamma\) and \(|\Gamma| + |\Gamma'| = p\). If this term contains the trivial representation then \(n\) must be even and \(|\Gamma| = |\Gamma'|\). Furthermore the dimension of the vector space of \(\mathfrak{g}_0\) invariants is then
$I_{\Gamma, \Gamma'} = \dim \text{Hom}_{\mathfrak{sl}(m+1)} \left( \mathfrak{g}^{\Gamma'}(\mathbb{C}^{m+1}), \mathfrak{g}^{\Gamma}(\mathbb{C}^{m+1}) \right) \times \dim \text{Hom}_{\mathfrak{sl}(n+1)} \left( \mathfrak{g}^{\Gamma'}(\mathbb{C}^{n+1}), \mathfrak{g}^{\Gamma}(\mathbb{C}^{n+1}) \right), \quad (4.4)

where $\text{Hom}_{\mathfrak{sl}(m+1)}$ denotes homomorphisms which are $\mathfrak{sl}(n+1)$ equivariant. One can calculate the dimensions of these spaces using well known results [39]. If $\Gamma = \Gamma'$ then $I_{\Gamma, \Gamma'} \geq 1$; if $\Gamma = \Gamma'$ and $|\Gamma| = |\Gamma'| = 1$ then $I_{\Gamma, \Gamma'} = 1$ and the invariant quadratic form corresponds to the tautological metric on $\mathfrak{g}_1$. In [28] $F$–Lie algebras were constructed using this symmetric form.

**Type II**

All basic type II Lie superalgebras except (iv) have $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathfrak{g}_0''$ and $\mathfrak{g}_1 = \mathcal{D}' \otimes \mathcal{D}''$, where $\mathcal{D}'$ and $\mathcal{D}''$ are irreducible self-dual representations of respectively $\mathfrak{g}_0'$ and $\mathfrak{g}_0''$. Therefore $\mathcal{S}^p(\mathfrak{g}_1)$ is the direct sum of terms of the form:

$$\mathcal{S}^\Gamma (\mathcal{D}') \otimes \mathcal{S}^\Gamma (\mathcal{D}'')$$

where $|\Gamma| = p$. The dimension of the vector space of $\mathfrak{g}_0$ invariants is

$$I_\Gamma = \dim \mathcal{S}^\Gamma (\mathcal{D}') \mathfrak{g}_0' \times \dim \mathcal{S}^\Gamma (\mathcal{D}'') \mathfrak{g}_0'', \quad (4.6)$$

where $\mathcal{S}^\Gamma (\mathcal{D}') \mathfrak{g}_0'$ denotes the space of $\mathfrak{g}_0'$ invariant vectors in $\mathcal{S}^\Gamma (\mathcal{D}')$. Although the respective factors in the product $\left(4.5\right)$ are irreducible for $GL(\mathcal{D}')$ and $GL(\mathcal{D}'')$, they may become reducible for $\mathfrak{g}_0'$, $\mathfrak{g}_0''$. For example the representations associated to the Young diagram are reducible for both $\mathfrak{g}_0' = \mathfrak{so}(m)$ and $\mathfrak{g}_0'' = \mathfrak{sp}(2n)$.

**The strange superalgebra $Q(n)$**

Up to duality $\mathcal{S}^\ast (\mathfrak{g}_1)$ (the symmetric algebra on $\mathfrak{g}_1$) is generated by the Casimir operators of $\mathfrak{sl}(n)$ (see section 3.).

**The strange superalgebra $P(n)$**

In this case $\mathcal{S}^\ast (\mathfrak{g}_1)$ is a direct sum of terms of the form

$$\mathcal{S}^k \left( \mathcal{S}^2 (\mathbb{C}^n) \right) \otimes \mathcal{S}^{p-k} \left( \Lambda^{n-2} (\mathbb{C}^n) \right) \quad (4.7)$$

This representation is in general reducible but we do not know of a simple general formula for the dimension of $\mathfrak{sl}(n)$ invariants.

**4.3 Trivial and induced $F$–Lie algebras**

In this section $F$–Lie algebras associated to Lie superalgebras will be constructed explicitly. To fix our notations, consider $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a classical Lie superalgebra. Let $J_\alpha, 1 \leq \alpha \leq \dim \mathfrak{g}_0$ be a basis of $\mathfrak{g}_0$ and $F_\alpha, 1 \leq \alpha \leq \dim \mathfrak{g}_1$ be a basis of $\mathfrak{g}_1$. The structure constants of $\mathfrak{g}$ are given by
\[ [J_a, J_b] = f_{ab}^c J_c \]
\[ [J_a, F_\alpha] = (R_\alpha)_\beta^\alpha F_\beta, \]
\[ \{ F_\alpha, F_\beta \} = E_{\alpha\beta} = S_{\alpha\beta} J_a \]  
(4.8)

The structure constants are given e. g. in [33] for particular choices of bases.

The first type of \( F \text{--} \) Lie algebras associated to \( g \) will be called trivial and are constructed as follows:

**Theorem 4.2** Let \( g = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra and let \( F \geq 1 \) be an integer. Then \( S = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \oplus \mathbb{C}) \) (with \( \mathbb{C} \) the trivial representation of \( \mathfrak{g}_0 \)) can be given the structure of an \( F \text{--} \) Lie algebra.

**Proof:** The proof is analogous to the proof of [33] QED

The second type of \( F \text{--} \) Lie algebras associated to \( g \) are those induced from \( g \) and symmetric forms on \( \mathfrak{g}_1 \). Let \( g = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be one of the classical Lie superalgebras in the statement of [4.1] and let \( g \) be a \( \mathfrak{g}_0 \) invariant symmetric form of order \( m \) on \( \mathfrak{g}_1 \). The bracket of the associated \( F \text{--} \) Lie algebra of order \( m+2 \) in the above basis is given by [2.1]

\[ \{ F_{\alpha_1}, \cdots , F_{\alpha_{m+2}} \} = \frac{1}{m!} \sum_{i<j} g_{\alpha_1 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha_{m+2}} E_{\alpha_{i} \alpha_{j}} \]  
(4.9)

**Example 4.3** We denote by \( S \) the \( F \text{--} \) Lie algebra of order 4 induced from the Lie superalgebra

\[ A(m-1, n-1) = (\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(1)) \oplus (\mathbb{C}^m \otimes \mathbb{C}^n^* \otimes \mathbb{C}) \oplus (\mathbb{C}^m \otimes \mathbb{C}^n^* \otimes \mathbb{C})^*, \]

and the tautological quadratic form on \( (\mathbb{C}^m \otimes \mathbb{C}^n^* \otimes \mathbb{C}) \oplus (\mathbb{C}^m \otimes \mathbb{C}^n^* \otimes \mathbb{C})^* \). Let \( \{ E_{IJ} \} \) \( 1 \leq i \leq m \) and \( m+1 \leq i \leq m+n \) be the standard bases of \( \mathfrak{gl}(m) \) and \( \mathfrak{gl}(n) \) respectively. Let \( \{ F_{IJ} \} \) \( 1 \leq i \leq m \) and \( m+1 \leq i \leq m+n \) be bases of \( (\mathfrak{m}, \mathfrak{n})^+, \) and \( (\mathfrak{m}, \mathfrak{n})^- \) respectively.

Then the four brackets of \( S \) have the following simple form:

\[ \{ F_{I_1 J_1}, F_{I_2 J_2}, F_{I_3 J_3}, F_{I_4 J_4} \} = \delta_{I_1 I_2} \delta_{J_1 J_2} \left( \delta_{I_3 I_4} E_{J_3 J_4} + \delta_{J_3 I_4} E_{I_3 J_4} \right) + \delta_{I_1 I_3} \delta_{J_1 J_3} \left( \delta_{I_2 I_4} E_{J_2 J_4} + \delta_{J_2 I_4} E_{I_2 J_4} \right) + \delta_{I_1 I_4} \delta_{J_1 J_4} \left( \delta_{I_2 I_3} E_{J_2 J_3} + \delta_{J_2 I_3} E_{I_2 J_3} \right) + \delta_{I_2 I_4} \delta_{J_2 J_4} \left( \delta_{I_1 I_3} E_{J_1 J_3} + \delta_{J_1 I_3} E_{I_1 J_3} \right) + \delta_{I_2 I_3} \delta_{J_2 J_3} \left( \delta_{I_1 I_4} E_{J_1 J_4} + \delta_{J_1 I_4} E_{I_1 J_4} \right) + \delta_{I_3 I_4} \delta_{J_3 J_4} \left( \delta_{I_1 I_2} E_{J_1 J_2} + \delta_{J_1 I_2} E_{I_1 J_2} \right). \]  
(4.10)

The fact that the R.H.S is in \( \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(1) \) is a consequence of theorem [2.6]
Example 4.4 We denote by $S$ the $F$–Lie algebra of order 4 induced from the Lie superalgebra

$$\mathfrak{osp}(2|2m) = (\mathfrak{so}(2) \oplus \mathfrak{sp}(2m)) \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2m},$$

and the quadratic form $g = \varepsilon \otimes \Omega$, where $\varepsilon$ is the invariant symplectic form on $\mathbb{C}^2$ and $\Omega$ the invariant symplectic form on $\mathbb{C}^{2m}$. Let $\{|S_\alpha \beta = S_\beta \alpha\} 1 \leq \alpha \leq 2m$ be a basis of $\mathfrak{sp}(2m)$ and $\{h\}$ be a basis of $\mathfrak{so}(2)$.

Let $\{F_{q\alpha}\}_{q = \pm 1 \ \ 1 \leq \alpha \leq 2m}$ be a basis of $\mathbb{C}^2 \otimes \mathbb{C}^{2m}$. Then the four brackets of $S$ take the following form

$$\{F_{q\alpha_1}, F_{q_2 \alpha_2}, F_{q_3 \alpha_3}, F_{q_4 \alpha_4}\} = \varepsilon_{q_1 q_2} \Omega_{\alpha_1 \alpha_2} (\delta_{q_3 + q_4} S_{\alpha_3 \alpha_4} + \varepsilon_{q_3 + q_4} \Omega_{\alpha_3 \alpha_4} h) + \varepsilon_{q_1 q_3} \Omega_{\alpha_1 \alpha_3} (\delta_{q_2 + q_4} S_{\alpha_2 \alpha_4} + \varepsilon_{q_2 + q_4} \Omega_{\alpha_2 \alpha_4} h)$$

$$+ \varepsilon_{q_1 q_4} \Omega_{\alpha_1 \alpha_4} (\delta_{q_2 + q_3} S_{\alpha_2 \alpha_3} + \varepsilon_{q_2 + q_3} \Omega_{\alpha_2 \alpha_3} h) + \varepsilon_{q_2 q_3} \Omega_{\alpha_2 \alpha_3} (\delta_{q_1 + q_4} S_{\alpha_1 \alpha_4} + \varepsilon_{q_1 + q_4} \Omega_{\alpha_1 \alpha_4} h)$$

$$+ \varepsilon_{q_2 q_4} \Omega_{\alpha_2 \alpha_4} (\delta_{q_1 + q_3} S_{\alpha_1 \alpha_3} + \varepsilon_{q_1 + q_3} \Omega_{\alpha_1 \alpha_3} h) + \varepsilon_{q_3 q_4} \Omega_{\alpha_3 \alpha_4} (\delta_{q_1 + q_2} S_{\alpha_1 \alpha_2} + \varepsilon_{q_1 + q_2} \Omega_{\alpha_1 \alpha_2} h).$$

Remark 4.5 By repeated application of theorem 2.6 one construct $F$–Lie algebras of higher and higher order.

5 Finite-dimensional FSUSY extensions of the Poincaré algebra

It is well known that supersymmetric extensions of the Poincaré algebra can be obtained by Inönü-Wigner contraction of certain Lie superalgebras. In fact, one can also obtain FSUSY extensions of the Poincaré algebra by Inönü-Wigner contraction of certain $F$–Lie algebras as we now show with two examples.

For the first example, we let $S_3 = \mathfrak{sp}(4) \oplus \text{ad} \mathfrak{sp}(4)$ be the real $F$–Lie algebra of order three (see Remark 3.9) induced from the real graded $1$–Lie algebra $S_1 = \mathfrak{sp}(4) \oplus \text{ad} \mathfrak{sp}(4)$ (see Example 3.2) and the Killing form on $\text{ad} \mathfrak{sp}(4)$. Using vector indices of $\mathfrak{so}(1, 3)$ coming from the inclusion $\mathfrak{so}(1, 3) \subset \mathfrak{so}(2, 3) \cong \mathfrak{sp}(4)$, the bosonic part of $S_3$ is generated by $M_{\mu \nu}, M_{\mu \lambda}$, with $\mu, \nu = 0, 1, 2, 3$ and the graded part by $J_{\mu \nu}, J_{4\mu}$. Letting $\lambda \to 0$ after the Inönü-Wigner contraction,

$$M_{\mu \nu} \to L_{\mu \nu}, \quad M_{\mu \lambda} \to \frac{1}{\sqrt{3}} P_{\mu},$$

$$J_{\mu \nu} \to \frac{1}{\sqrt{3}} Q_{\mu \nu}, \quad J_{4\mu} \to \frac{1}{\sqrt{3}} Q_{\mu},$$

one sees that $L_{\mu \nu}$ and $P_{\mu}$ generate the $(1 + 3)D$ Poincaré algebra and that $Q_{\mu \nu}, Q_{\mu}$ are the fractional supercharges in respectively the adjoint and vector representations of $\mathfrak{so}(1, 3)$. This $F$–Lie algebra of order three is therefore a non-trivial extension of the Poincaré algebra where translations are cubes of more fundamental generators. The subspace generated by $L_{\mu \nu}, P_{\mu}, Q_{\mu}$ is also an $F$–Lie algebra of order three extending the Poincaré algebra in which the trilinear symmetric brackets have the simple form:

$$\{Q_{\mu}, Q_{\nu}, Q_{\rho}\} = \varepsilon_{\mu \nu \rho} P_{\rho} + \varepsilon_{\mu \rho \nu} P_{\nu} + \varepsilon_{\nu \mu \rho} P_{\rho},$$

(5.2)
where $\eta_{\mu\nu}$ is the Minkowski metric. This algebra should be compared to the algebra recently obtained in a different context, where a “trilinear” extension of the Poincaré algebra involving “supercharges” in the vector representation was constructed [41].

For the second example, we let $S_4 = (\mathfrak{so}(2) \oplus \mathfrak{sp}(4)) \oplus 2 \otimes 4$ be the real $F$–Lie algebra of order four induced from $\mathfrak{osp}(2|4)$ and the symmetric form $\varepsilon \otimes \Omega$, where $\Omega$ is the symplectic form on 4 and $\varepsilon$ the antisymmetric two-form on 2. Using spinor indices coming from $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,3) \subset \mathfrak{so}(2,3)$ the bosonic part is generated by $E_{\alpha\beta}, E_{\dot{\alpha}\dot{\beta}}, E_{\dot{\alpha}\beta}$ and the fermionic part by $F_{\alpha}^\pm, \tilde{F}_{\dot{\alpha}}^\pm, \alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}$. Letting $\lambda \to 0$ after the Inonü-Wigner contraction

$$
E_{\alpha\beta} \to L_{\alpha\beta} \quad E_{\dot{\alpha}\dot{\beta}} \to L_{\dot{\alpha}\dot{\beta}} \quad E_{\dot{\alpha}\beta} \to \frac{1}{\chi} P_{\alpha\dot{\alpha}} \quad \h \to \frac{1}{\chi} Z
$$

one sees that $L_{\alpha\beta}, L_{\dot{\alpha}\dot{\beta}}$ and $P_{\alpha\dot{\alpha}}$ generate the $(1 + 3)D$ Poincaré algebra, that $Z$ is central and that $Q_{\alpha}^\pm, \bar{Q}_{\dot{\alpha}}^\pm$ are the fractional-supercharges in the spinor representations of $\mathfrak{so}(1,3)$. This $F$–Lie algebra of order four is therefore a non-trivial extension of the Poincaré algebra where translations are fourth powers of more fundamental generators. The four bracket can be expressed simply if we introduce the following notation: $\sigma^\mu_{\alpha\dot{\alpha}}, \bar{\sigma}^\mu_{\dot{\alpha}\alpha}$ are the Dirac matrices, $\sigma^\mu_{\alpha\beta}, \bar{\sigma}^\mu_{\dot{\alpha}\dot{\beta}}$ and $P^\mu$ are the Poincaré generators (for details e.g. [42]). One then has:

$$
\{Q^{q_1}_{\alpha_1}, Q^{q_2}_{\alpha_2}, Q^{q_3}_{\alpha_3}, Q_{\alpha_4}^{q_4}\} = 2\varepsilon^{q_1 q_2} \varepsilon^{q_3 q_4} \varepsilon^{\alpha_{12}} \varepsilon_{\alpha_{34}2} \varepsilon_{\alpha_{23}4} Z + 2\varepsilon^{q_1 q_4} \varepsilon^{q_2 q_3} \varepsilon^{\alpha_{14}} \varepsilon_{\alpha_{23}2} \varepsilon_{\alpha_{23}4} Z + 2\varepsilon^{q_1 q_3} \varepsilon^{q_2 q_4} \varepsilon^{\alpha_{13}} \varepsilon_{\alpha_{24}2} \varepsilon_{\alpha_{24}3} Z
$$

(5.4)

$$
\{Q^{q_1}_{\alpha_1}, Q^{q_2}_{\dot{\alpha}_2}, Q^{q_3}_{\dot{\alpha}_3}, \bar{Q}_{\dot{\alpha}_4}^{q_4}\} = \delta^{q_1+q_4} \varepsilon^{q_2 q_3} \varepsilon_{\alpha_{23}} \sigma^\mu_{\alpha_{14}} P^\mu + \delta^{q_2+q_4} \varepsilon^{q_1 q_3} \varepsilon_{\alpha_{13}} \sigma^\mu_{\alpha_{24}} P^\mu + \delta^{q_3+q_4} \varepsilon^{q_1 q_2} \varepsilon_{\alpha_{12}} \sigma^\mu_{\alpha_{34}} P^\mu
$$

$$
\{Q^{q_1}_{\alpha_1}, Q^{q_2}_{\dot{\alpha}_2}, \bar{Q}_{\dot{\alpha}_3}^{q_3}, \bar{Q}_{\dot{\alpha}_4}^{q_4}\} = 0,
$$

together with similar relations involving $\{Q^{q_1}_{\alpha_1}, \bar{Q}^{q_2}_{\dot{\alpha}_2}, \bar{Q}^{q_3}_{\dot{\alpha}_3}, \bar{Q}^{q_4}_{\dot{\alpha}_4}\}$ and $\{\bar{Q}^{q_1}_{\dot{\alpha}_1}, \bar{Q}^{q_2}_{\dot{\alpha}_2}, \bar{Q}^{q_3}_{\dot{\alpha}_3}, \bar{Q}^{q_4}_{\dot{\alpha}_4}\}$.

Analogous constructions lead to FSUSY extensions of the Poincaré algebra in any space-time dimensions.

### 6 Simple $F$–Lie algebras

By analogy with the case of Lie (super)algebras we define ideals and the notion of simplicity for $F$–Lie algebras.

**Definition 6.1** Let $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$ be an $F$–Lie algebra, or a graded $1$–Lie algebra. Then $\mathcal{I} = \mathcal{B}' \oplus \mathcal{F}'$ is an ideal of $\mathcal{S}$ if and only if

1. $\forall f \in \mathcal{F}' \setminus \{f\}, \forall f \in \mathcal{F} : \{f_1, f_2, \ldots, f_F\} \in \mathcal{B}'$;
2. $\mathcal{B}'$ is an ideal of $\mathcal{B}$ ($\forall b \in \mathcal{B}', \forall b \in \mathcal{B}, [b, b'] \in \mathcal{B}')$;
(iii) \( \forall b \in \mathcal{B}, \forall f' \in \mathcal{F}' [b, f'] \in \mathcal{F}' \);
(iv) \( \forall b' \in \mathcal{B}', \forall f \in \mathcal{F} [b', f] \in \mathcal{F} \).

Remark 6.2 For a graded 1–Lie algebra \( S = \mathcal{B} \oplus \mathcal{F} \), denoting \( \mu \) the map from \( \mathcal{F} \) to \( \mathcal{B} \), the property (i) of \( \mathcal{F} \) becomes \( \text{Im} \mu \subset \mathcal{B}' \).

Remark 6.3 By \( \mathcal{B} \) \( \text{Im} \mu \oplus \mathcal{F} \) is an ideal of \( S \) (\( \mu \) denotes the \( \mathcal{B} \)–equivariant map from \( S^F(\mathcal{F}) \to \mathcal{B} \)).

Remark 6.4 In the case of Lie algebras and Lie superalgebras, this is the usual definition. In the case of a graded 1–Lie algebra \( S = \mathcal{B} \oplus \mathcal{F}, S' = \mathcal{B}' \oplus \mathcal{F}' \) is an ideal if and only if it is a \( \mathbb{Z}_2 \)–graded ideal for the natural Lie bracket on \( S \) (c.f. 3.3).

Definition 6.5 An \( F \)–Lie algebra \( S \) is said to be simple if and only if its only ideals are \( S \) and \( \{0\} \), and \( \mu : S^F(\mathcal{F}) \to \mathcal{B} \) is non-zero.

Remark 6.6 Let \( S = \mathcal{B} \oplus \mathcal{F} \) be a graded 1–Lie algebra such that \( \mu : \mathcal{F} \to \mathcal{B} \) is non-zero. Then, \( S \) is simple if and only if \( \mathcal{B} \) is a simple Lie algebra and \( \mathcal{F} \) is an irreducible representation of \( \mathcal{B} \).

Remark 6.7 If \( \mathfrak{g} \) is a simple Lie algebra, and \( S = \mathfrak{g} \oplus \text{ad} \mathfrak{g} \) is the graded 1–Lie algebra of Example 3.2, then \( S \) is simple as a graded 1–Lie algebra but is not simple as a Lie algebra, with respect to the natural Lie bracket \( \mathfrak{g} \).

Proposition 6.8 Let \( S = \mathcal{B} \oplus \mathcal{F} \) be an \( F \)–Lie algebra such that (i) \( \mathcal{B} \) is semi-simple, (ii) the map \( \mu : S^F(\mathcal{F}) \to \mathcal{B} \) is a surjection and (iii) no non-zero ideal of \( \mathcal{B} \) has non-zero fixed points in \( \mathcal{F} \). Then:

(a) \( S \) is simple.

(b) The \( F \)–Lie algebra of order \( (F + 2) \) induced from a \( \mathcal{B} \)–equivariant non-degenerate quadratic form on \( \mathcal{F} \) (see 2.3) also satisfies (i) and (ii).

Proof: Let \( \mathcal{F} = \mathcal{B}' \oplus \mathcal{F}' \) be a non-trivial ideal of \( S \). Then \( \mathcal{B}' \) is an ideal of \( \mathcal{B} \) and \([\mathcal{B}', \mathcal{F}'] \subset \mathcal{F}' \). But if \( \mathcal{F} = \mathcal{F}' \oplus \mathcal{F}'' \) as \( \mathcal{B}' \)–modules then \([\mathcal{B}', \mathcal{F}''] = 0 \) and therefore, \( \mathcal{F}' = \{0\} \) since by hypothesis \( \mathcal{B}' \) does not admit non-zero fixed points. This proves (a).

To prove (b), it is enough to prove that the induced \( (F + 2) \)–bracket is surjective. Since the \( F \)–bracket \( \mu : S^F(\mathcal{F}) \to \mathcal{B} \) is surjective, by diagonalising the quadratic form, it is easy to see that the \( (F + 2) \)–bracket \( (2.1) \) is also surjective. QED

Remark 6.9 If \( \mathfrak{g} \) is a simple Lie algebra, the graded 1–Lie algebras \( \mathfrak{g} \oplus \text{ad} \mathfrak{g} \) satisfies (i), (ii) and (iii) above. As one can check, the Lie superalgebras in the list \( 3.1 \) also satisfy (i), (ii) and (iii). Thus the induced \( F \)–Lie algebras associated to non-degenerate quadratic forms and these graded 1–Lie algebras or Lie superalgebras are always simple.

The trivial \( F \)–Lie algebras associated to graded 1–Lie algebras or Lie superalgebras \( 3.7, 4.2 \) are not simple since in both cases \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is an ideal of \( S \). In particular, when \( F = 2 \), the trivial Lie superalgebras associated to graded 1–Lie algebras are not simple. The direct sum of two simple \( F \)–Lie algebras of the same order is clearly not simple. These two kinds of examples of non-simple \( F \)–Lie algebras indicate that probably, as for Lie superalgebras, there are different inequivalent ways to define semi-simple Lie \( F \)–Lie algebras.
7 Representations

Definition 7.1 A representation of an $F$–Lie algebra $S$ is a linear map $\rho : S \to \text{End}(H)$, and a automorphism $\hat{\epsilon}$ such that $\hat{\epsilon}^F = 1$ which satisfy

\begin{align}
\text{a) } & \rho ([x, y]) = \rho (x)\rho (y) - \rho (y)\rho (x) \\
\text{b) } & \rho \{a_1, \cdots , a_F\} = \sum_{\sigma \in S_F} \rho (a_{\sigma(1)}) \cdots \rho (a_{\sigma(F)}) \\
\text{d) } & \hat{\epsilon}\rho (s)\hat{\epsilon}^{-1} = \rho (\hat{\epsilon} (s))
\end{align}

(\text{S}_F\text{ being the group of permutations of } F\text{ elements}).

As a consequence of these properties, since the eigenvalues of $\hat{\epsilon}$ are $F^{\text{th}}$ – roots of unity, we have the following decomposition

$$H = \bigoplus_{k=0}^{F-1} H_k,$$

where $H_k = \{ |h\rangle \in H : \hat{\epsilon}|h\rangle = q^k |h\rangle \}$ where $q$ is any linear map whose minimal polynomial is $\chi = \ldots - \lambda^3 + 1$. Since $\hat{\epsilon}\rho (b) = \rho (b)\hat{\epsilon}$, $\forall b \in B$ each $H_k$ provides a representation of the Lie algebra $B$. Furthermore, for $a \in A$, $\hat{\epsilon}\rho (a) = q^\ell \rho (a)\hat{\epsilon}$ and so we have $\rho (a).H_k \subseteq H_{k+\ell (\text{mod } F)}$.

Example 7.2 Let $X, Y, Z$ be $n \times n$ (resp. $2n \times 2n$) matrices in $\mathfrak{so}(n)$ (resp. $\mathfrak{sp}(2n)$). Then, it is easy to see that $\{X,Y,Z\}$ is also in $\mathfrak{so}(n)$ (resp. $\mathfrak{sp}(2n)$). Consequently, $S = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$ (resp. $S = \mathfrak{sp}(2n) \oplus \mathfrak{sp}(2n)$), is an $F$–Lie algebra of order 3 (the only non-trivial point to be checked is the Jacobi identity (J4) in Definition 2.1). A similar property is true for any odd number of matrices. We will calculate the structure constants in the case of $\mathfrak{so}(n)$, the calculation for $\mathfrak{sp}(2n)$ being analogous. If $X_a, 1 \leq a \leq \dim \mathfrak{so}(n)$ is a basis of $\mathfrak{so}(n)$, then the 3–bracket of $S$ is given by

$$\{X_a, X_b, X_c\} = k_{abc}^d X_d. \quad (7.2)$$

Writing $\{X_a, X_b, X_c, X_d\} = \{X_a, X_b, X_c\} X_d + \{X_a, X_b, X_d\} X_c + \{X_a, X_c, X_d\} X_b + \{X_b, X_c, X_d\} X_a$ and taking the trace using (7.2), we get $4k_{abc}^d \text{tr}(X_d X_e) = \text{Tr}(\{X_a, X_b, X_c, X_e\})$. Since the trace defines a metric on $\mathfrak{so}(n)$ this gives $k_{abc}^d = \frac{1}{4} \text{Tr} \{X_a, X_b, X_c, X_d\} g^{de}$.

This $F$–Lie algebra of order three is not induced from the graded 1–Lie algebra $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$ and the Killing form: if this where the case we would have $\{X_a, X_b, X_c\} = \text{Tr}(X_a X_b) X_c + \text{Tr}(X_a X_c) X_b + \text{Tr}(X_b X_c) X_a$ which is clearly false if $a = b = c$. However, by proposition 5.8 $S$ is simple.

We can construct a representation of $S$ in $\mathbb{C}^n \otimes \mathbb{C}^3$ as follows: define $\rho : S \to \text{End} (\mathbb{C}^n \otimes \mathbb{C}^3)$ by

$$\rho (X) = \begin{cases} X \otimes \text{Id} & \text{if } X\text{ is in the first } \mathfrak{so}(n) \\ X \otimes Q & \text{if } X\text{ is in the second } \mathfrak{so}(n), \end{cases} \quad (7.3)$$

where $Q : \mathbb{C}^3 \to \mathbb{C}^3$ is any linear map whose minimal polynomial is $\lambda^3 - 1$ (i.e., $Q^3 = \text{Id}$ and $Q$ has three distinct eigenvalues).

Related results were obtained for $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ in [43].
Example 7.3 Let $X, Y, Z$ be three $n \times n$ matrices in $\mathfrak{u}(n)$. Then, it is easy to see that $\{X, Y, Z\}$ is also in $\mathfrak{u}(n)$. As in the previous example, this simple observation enables us to give $\mathfrak{u}(n) \oplus \mathfrak{u}(n)$ or $\mathfrak{u}(n) \oplus \mathfrak{su}(n)$ the structure of an $F$--Lie algebra of order 3.

Example 7.4 Let $A(m - 1, n - 1), n \neq m$ be the Lie superalgebra of $(n + m) \times (n + m)$ matrices $[\mathfrak{b} \mathfrak{b} \mathfrak{b}, \mathfrak{b}]$,

$M = \begin{pmatrix} E_{mm} & F_{mn} \\ F_{nm} & E_{nn} \end{pmatrix},$

of supertrace zero (i.e., $s\text{Tr}M = \text{tr}E_{mm} - \text{tr}E_{nn} = 0$).

If $J_{i_1}, \cdots, J_{i_2p}$ are arbitrary matrices then

$$\{J_{i_1}, \cdots, J_{i_2p}\} = \sum_{a < b = 1}^F \left\{ \{J_{i_a}, J_{i_b}\}, \{J_{i_a}, \hat{J}_{i_b}, J_{i_1}, \cdots, J_{i_2p}\} \right\}. \quad (7.4)$$

Applying this formula to $2F$ odd matrices in $A(m - 1, n - 1)$ one sees by an induction that the supertrace of the $2F$--bracket $(7.4)$ vanishes. Using the $\mathbb{Z}_2$ gradation of $A(m - 1, n - 1)$ one sees that this bracket belongs to the even part of the algebra and hence defines the structure of an $F$--Lie algebra of order $2F$ on the underlying vector space of $A(m - 1, n - 1)$. For $F = 4$ this is just the the $F$--Lie algebra of order 4 induced by the tautological quadratic form of Example 7.3. Indeed, let $V = \mathbb{C}^{*n} \otimes \mathbb{C}^m \otimes \mathbb{C}$ and let $\mathfrak{g}_0 = \mathfrak{sl}(n) \oplus \mathfrak{sl}(m) \oplus \mathfrak{gl}(1)$. Then, comparing $\mathfrak{gl}(1)$ charges, we have $\text{Hom}_{\mathfrak{g}_0}(\mathfrak{S}^4(V \oplus V^*), \mathfrak{g}_0) = \text{Hom}_{\mathfrak{g}_0}(\mathfrak{S}^2(V) \otimes \mathfrak{S}^2(V^*), \mathfrak{g}_0)$. Since,

$$\mathfrak{S}^2(V) \otimes \mathfrak{S}^2(V^*) \cong \left( \mathfrak{S}^2(\mathbb{C}^{*n}) \otimes \mathfrak{S}^2(\mathbb{C}^m) \otimes \Lambda^2(\mathbb{C}^{*n}) \otimes \Lambda^2(\mathbb{C}^m) \right) \otimes \left( \mathfrak{S}^2(\mathbb{C}^n) \otimes \mathfrak{S}^2(\mathbb{C}^{*n}) \otimes \Lambda^2(\mathbb{C}^n) \otimes \Lambda^2(\mathbb{C}^{*n}) \right)$$

and since the representations $\mathbf{1}$ and $\mathfrak{sl}(n)$ occur exactly once in $\mathfrak{S}^2(\mathbb{C}^n) \otimes \mathfrak{S}^2(\mathbb{C}^{*n})$ and not at all in $\Lambda^2(\mathbb{C}^n) \otimes \Lambda^2(\mathbb{C}^{*n})$, we deduce that $\text{Hom}_{\mathfrak{g}_0}(\mathfrak{S}^4(V \oplus V^*), \mathfrak{g}_0)$ is of dimension one.

By definition, the fundamental $(n + m) \times (n + m)$ matrix representation of the Lie superalgebra $A(m - 1, n - 1)$ is also a representation of the $F$--Lie algebra of order $2F$ constructed above. In general, this is not true: for instance if $m = 2, n = 1$, one can check that the $6$--dimensional representation of $A(2, 1)$ is not a representation of the associated $F$--Lie algebra of order 4.

Example 7.5 Let $S$ be the set of all matrices of the form

$$M = \begin{pmatrix} q & 0 & F_+ \\ 0 & -q & F_- \\ -\Omega F^t_+ & -i\Omega F^t_- & S \end{pmatrix}, \quad (7.5)$$

where $q$ is a complex number, $F_{\pm}$ are two $1 \times 2n$ matrices, $\Omega$ is the standard $2n \times 2n$ symplectic form on $\mathbb{C}^{2n}$ and $S$ is a $2n \times 2n$ matrix in $\mathfrak{sp}(2n)$, i.e., $S^t = \Omega S \Omega$. Let $\mathfrak{B} = \left\{ \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, q \in \mathbb{C}, S \in \mathfrak{sp}(2n) \right\}$.

$\mathfrak{so}(2) \oplus \mathfrak{sp}(2n)$ and let $\mathfrak{T} = \left\{ \begin{pmatrix} 0 & F_+ \\ 0 & F_- \\ -\Omega F^t_+ & -i\Omega F^t_- \end{pmatrix}, F_{\pm} \in \mathcal{M}_{1,2n}(\mathbb{C}) \right\}$. If one now takes
This shows that \( \mathcal{F} \) is not closed under the superbracket. From the formula \( \{ \mathcal{F}_a, \mathcal{F}_b \} = \begin{pmatrix} \alpha_{ab} & 0 & 0 \\ 0 & -i\alpha_{ab} & 0 \\ 0 & 0 & A_{ab} \end{pmatrix} \), where \( A_{ab} = -\Omega F^t_{a-} F_{b+} - i\Omega F^t_{a+} F_{b-} - \Omega F^t_{b-} F_{a+} - i\Omega F^t_{b+} F_{a-} \) and where \( \alpha_{ab} = -F^a_{a} F^b_{b} - F^a_{b} F^b_{a} \). This shows that \( \mathcal{B} \oplus \mathcal{F} \) is not closed under the superbracket.

From the formula \( \{ \mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d \} = \{(\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d)\} + \{(\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c)\} + \{(\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_d)\} + \{(\mathcal{F}_a, \mathcal{F}_c, \mathcal{F}_d)\} \), observing that \( \{\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d\} = 0 \) the four bracket \( \{\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d\} = 0 \) if \( q_1 + q_2 + q_3 + q_4 \neq 0 \). We then calculate 4-brackets for \( q_1 = q_2 = -q_3 = -q_4 = 1 \) and obtain

\[
\{\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d\} = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & S \end{pmatrix},
\]

\[
q = 2 \left( F^a_{a} + \Omega F^t_{c-} \right) \left( F^b_{b} + \Omega F^t_{d-} \right) + 2 \left( F^a_{a} + \Omega F^t_{c-} \right) \left( F^b_{b} + \Omega F^t_{d-} \right)
\]

\[
S = F^a_{a} + \Omega F^t_{c-} \left( \Omega F^t_{c-} F^b_{b} + \Omega F^t_{b+} F^t_{c-} \right) + F^a_{a} + \Omega F^t_{c-} \left( \Omega F^t_{d-} F^b_{b} + \Omega F^t_{b+} F^t_{d-} \right) + F^b_{b} + \Omega F^t_{d-} \left( \Omega F^t_{c-} F^a_{a} + \Omega F^t_{a+} F^t_{c-} \right) + F^b_{b} + \Omega F^t_{d-} \left( \Omega F^t_{c-} F^a_{a} + \Omega F^t_{a+} F^t_{d-} \right).
\]

This shows that \( \mathcal{B} \oplus \mathcal{F} \) is an \( F \)-Lie algebra of order 4 since \( S^t = \Omega \sum \Omega \).

In fact, the matrices of \( \mathcal{B} \oplus \mathcal{F} \) define a representation of the \( F \)-Lie algebra of order 4 induced from \( \mathfrak{osp}(2|2m) \) and \( \varepsilon \otimes \Omega \) (see [4.]). Indeed setting

\[
\mathcal{F}_a = \begin{pmatrix} 0 & 0 & F^a_{a+} \\ 0 & 0 & 0 \\ -\Omega F^t_{a+} & 0 \end{pmatrix}, \quad \mathcal{F}_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F^a_{a-} \\ -\Omega F^t_{a-} & 0 & 0 \end{pmatrix},
\]

we see that \( \mathcal{B} \oplus \mathcal{F} \cong \mathfrak{osp}(2|2m) \) and that \( \{\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d\} = \{\mathcal{F}_a, \mathcal{F}_c\} < \{\mathcal{F}_b, \mathcal{F}_d\} < \{\mathcal{F}_a, \mathcal{F}_d\} < \{\mathcal{F}_a, \mathcal{F}_c\} \).

where \( < \mathcal{F}_a, \mathcal{F}_c \) \) denotes the \( \varepsilon \otimes \Omega \) invariant form.

Given an \( F \)-Lie algebra \( S = \mathcal{B} \oplus \mathcal{F} \) one can define the universal enveloping algebra \( \mathcal{U}(S) \) by taking the quotient of the tensor algebra \( \mathcal{T}(S) \) by the two-sided ideal generated by (see definition [2.1])

\[
\sum_{\sigma \in \Sigma_F} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(F)} - \{a_1, \cdots, a_F\}, \quad \text{(7.7)}
\]

with \( a_1, \cdots, a_F \in \mathcal{A}_1, b_1, b_2 \in \mathcal{B} \). It is not necessary to impose the Jacobi identity (I4) since it is true in \( \mathcal{T}(S) \).

The natural filtration of \( \mathcal{T}(S) \) factors to a filtration of \( \mathcal{U}(S) \) and, denoting the associated graded algebra by \( \text{gr}(\mathcal{U}(S)) \), we conjecture the following:
(1) \( \text{gr}(\mathcal{U}(S)) \) is isomorphic to \( \mathcal{T}(S)/\bar{I} \), where \( \bar{I} \) is the two-sided ideal generated by

\[
\begin{align*}
\sum_{\sigma \in \Sigma_F} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(F)}, \\
b_1 \otimes b_2 - b_2 \otimes b_1, \\
b_1 \otimes a_2 - a_2 \otimes b_1.
\end{align*}
\]

(This would then imply that \( \text{gr}(\mathcal{U}(S)) \cong S(\mathcal{B}) \otimes \Lambda_F(\mathcal{F}) \), where \( S(\mathcal{B}) \) is the symmetric algebra on \( \mathcal{B} \) and \( \Lambda_F(\mathcal{F}) \) is the \( F \)-exterior algebra on \( \mathcal{F} \) [34]).

(2) The natural map \( \pi : \mathcal{U}(S) \to \text{gr}(\mathcal{U}(S)) \) is a linear isomorphism. (This would be an analogue of the Poincaré-Birkhoff-Witt theorem).

In the usual way, the representations of \( S \) are in bijective correspondence with the representations of the associative algebra \( \mathcal{U}(S) \). Consequently, if \( \mathcal{J} \subset \mathcal{U}(S) \) is a two-sided ideal, then the quotient \( \mathcal{U}(S)/\mathcal{J} \) gives a representation of \( S \). It would be very convenient to have a theory of “Cartan sub-algebras”, “roots” and “weights” for \( S \). However, even for simple Lie superalgebras this kind of theory only works well for basic Lie superalgebras [38]. One might expect \( F \)-Lie algebras induced from basic Lie superalgebras to be amenable to this approach. This seems not to be the case. Indeed, recall that if \( S \) is a basic Lie superalgebra with Borel decomposition \( S = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \) and \( \lambda \in \mathfrak{h}^* \) is a dominant weight, then \( \mathcal{V}_\lambda = \mathcal{U}/\mathcal{J}_\mu \) (where \( \mathcal{J}_\lambda \) is the ideal corresponding to \( \lambda \)) is (i) generated by the action of \( \mathfrak{n}_+ \) on the vacuum and (ii) has a unique quotient \( \mathcal{D}_\lambda \) on which the action of \( \mathfrak{n}_+ \) is nilpotent and which is therefore finite-dimensional. However, if \( \mathcal{S}_g \) is the \( F \)-Lie algebra induced from \( S \) and a symmetric form \( g \), the quotient \( \mathcal{V}'_\lambda = \mathcal{U}(\mathcal{S}_g)/\mathcal{J}'_\lambda \) is (i) not generated by the action of \( \mathfrak{n}_+ \) on the vacuum and (ii) the nilpotence of the action of \( \mathfrak{n}_+ \) in a quotient does not guarantee finite-dimensionality. This means that in finite-dimensional representations of \( S \), as in the examples of section 7, the elements of \( \mathfrak{n}_+ \) are not only nilpotent but also satisfy additional relations.

8 Conclusion

The mathematical structure underlying supersymmetry is that of a Lie superalgebra. Given the classification of Lie superalgebras, one can list the possible supersymmetric extensions of the Poincaré algebra. These extensions have had a wide range of applications in physics.

Fractional supersymmetries were first studied in the early 1990’s in relation with low dimensional physics (\( D \leq 3 \)) where fields which are neither bosonic nor fermionic [34] do exist. It was understood a few years later that FSUSY can be considered in arbitrary dimensions and the definition of an \( F \)-Lie algebra, the underlying mathematical structure, was given [24]. However when \( F > 2 \), most of the examples of \( F \)-Lie algebras which have been found since then are of infinite-dimensions. In this paper, we show how one can construct many finite-dimensional \( F \)-Lie algebras starting from Lie algebras or Lie superalgebras equipped with appropriate symmetric forms. We define a notion of simplicity in this context and show that some of our examples are simple. Furthermore, we construct the first finite-dimensional FSUSY extensions of the Poincaré algebra by Inönü-Wigner contraction of certain \( F \)-Lie algebras.

These results can be seen as a first step in classifying \( F \)-Lie algebras.
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