ON THE RANDOM VERSION OF THE ERDŐS MATCHING CONJECTURE

MEYSAM ALISHAHI AND ALI TAHERKHANI

ABSTRACT. The Kneser hypergraph $KG_{n,k}^r$ is an $r$-uniform hypergraph with vertex set consisting of all $k$-subsets of $\{1, \ldots, n\}$ and any collection of $r$ vertices forms an edge if their corresponding $k$-sets are pairwise disjoint. The random Kneser hypergraph $KG_{n,k}^r(p)$ is a spanning subhypergraph of $KG_{n,k}^r$ in which each edge of $KG_{n,k}^r$ is retained independently of each other with probability $p$. The independence number of random subgraphs of $KG_{n,k}^2$ was recently addressed in a series of works by Bollobás, Narayanan, and Raigorodskii (2016), Balogh, Bollobás, and Narayanan (2015), Das and Tran (2016), and Devlin and Kahn (2016). It was proved that the random counterpart of the Erdős-Ko-Rado theorem continues to be valid even for very small values of $p$. In this paper, generalizing this result, we will investigate the independence number of random Kneser hypergraphs $KG_{n,k}^r(p)$. Broadly speaking, when $k$ is much smaller than $n$, we will prove that the random analogue of the Erdős matching conjecture is true even for extremely small values of $p$.

1. Motivations and Main Results

Let $n, k$ and $r$ be three positive integers such that $n \geq 2k$ and $r \geq 2$. Throughout the paper, the two symbols $[n]$ and $\binom{[n]}{k}$ respectively stand for the sets $\{1, \ldots, n\}$ and $\{A \subseteq [n] : |A| = k\}$. The Kneser hypergraph $KG_{n,k}^r$ is an $r$-uniform hypergraph whose vertex set is $\binom{[n]}{k}$ and its edge set consists of all pairwise disjoint $r$-tuples of elements in $\binom{[n]}{k}$, i.e.,

$$E(KG_{n,k}^r) = \left\{ \{A_1, \ldots, A_r\} : A_1, \ldots, A_r \in \binom{[n]}{k}, \text{ are pairwise disjoint} \right\}.$$ 

For each $x \in [n]$, the set $S_x = \left\{ A \in \binom{[n]}{k} : x \in A \right\}$ is called a star. It is clear that any star is an independent set of $KG_{n,k}^2$, that is, a set of vertices containing no edge. We remind the reader that the maximum size of an independent set in a hypergraph $\mathcal{H}$ is called the independence number of $\mathcal{H}$, denoted by $\alpha(\mathcal{H})$. The seminal Erdős-Ko-Rado theorem states that for $n \geq 2k$, the independence number of $KG_{n,k}^2$ is $\binom{n-1}{k-1}$; furthermore if $n > 2k$, the only independent sets of this size are the stars. As an extension of the Erdős-Ko-Rado theorem, Erdős [10] conjectured that $\alpha(KG_{n,k}^r) = \max \left\{ \binom{rk-1}{k}, \binom{n}{k} - \binom{n-r+1}{k} \right\}$ provided that $n \geq r(k-1)$. Easy computation shows that for $n \geq r(k + \frac{1}{2})$, the aforementioned maximum is $\binom{n}{k} - \binom{n-r+1}{k}$. In recent years, this conjecture has received significant attention and several papers were devoted to the study of this conjecture; see, e.g., [5] [10] [11] [12] [13] [17] [19] [24]. Regarding this conjecture, the best known result is proved by Frankl [12]. Provided $n \geq (2r-1)k - r + 1$, he proved that $\alpha(KG_{n,k}^r) = \binom{n}{k} - \binom{n-r+1}{k}$; furthermore, any independent set of this size is formed by the union of some $r-1$ distinct stars which confirms the conjecture in this range.

Key words and phrases. Erdős matching conjecture, random Kneser hypergraphs, independence number.
For more recent results concerning this conjecture, one can refer to [15, 16]. It is worth noting that there is another interesting extension of the Erdős-Ko-Rado theorem due to Hilton and Milner [18] asserting that for $n > 2k$, any independent set of Kneser graph $KG_{n,k}$ which is contained in no star has cardinality at most $(n-1)\binom{n-k-1}{k-1} + 1$. For recent results, one can see [14, 23].

Let $KG_{n,k}(r)$ be the random subhypergraph of $KG_{n,k}$ whose vertex set is the same as $KG_{n,k}$ and each edge of $KG_{n,k}$ is retained independently of each other with probability $p$. Throughout the paper, when $r = 2$, we shall drop the super-index $r$ and write $KG_{n,k}$ and $KG_{n,k}(p)$ instead of $KG_{n,k}^r$ and $KG_{n,k}^r(p)$, respectively. Also, we say an event occurs with high probability or likely happens if it can be made as close as desired to 1 by making $n$ large enough.

As a fast growing branch of hypergraph theory, many articles are recently devoted to investigating the properties of random Kneser hypergraphs $KG_{n,k}(p)$; see [1, 2, 3, 4, 6, 8, 9, 20, 21, 22, 25, 26, 27, 28]. Extending some results in [3, 4], Bollobás, Narayanan and Raigorodskii [6] studied the independence number of random Kneser graphs $KG_{n,k}(p)$. They tried to answer the question that for which $p$, the Erdős-Ko-Rado theorem is likely valid in $KG_{n,k}(p)$. Surprisingly, when $k$ is much smaller than $n$, they proved that an analogue of the Erdős-Ko-Rado theorem continues to hold even after deleting practically all the edges of the Kneser graphs.

**Theorem A.** [6 Theorem 1.2] Fix a real number $\varepsilon > 0$ and let $k = k(n)$ be a natural number such that $2 \leq k = o(n^{1/3})$. Then as $n \to \infty$,

$$
P\left(\alpha(KG_{n,k}(p)) = \binom{n-1}{k-1} \right) \to \begin{cases} 1 & p \geq \left(1 + \varepsilon\right)\frac{(r+1)\ln n - r\ln r}{(n-k-1)} \\ 0 & p \leq \left(1 - \varepsilon\right)\frac{(r+1)\ln n - r\ln r}{(n-k-1)} \end{cases}.
$$

Furthermore, when $p \geq \left(1 + \varepsilon\right)\frac{(r+1)\ln n - r\ln r}{(n-k-1)}$, with high probability, the only independent sets of size $\binom{n-1}{k-1}$ in $KG_{n,k}(p)$ are the stars.

In addition, they conjectured that a similar result should hold for $k = o(n)$ which first was partially answered by Balogh, Bollobás and Narayanan [2]. Then, a significantly sharper result was proved by Das and Tran [8]. They extended the Bollobás-Narayanan-Raigorodskii theorem to $k$ as large as linear in $n$ subsuming the earlier results. Finally, Delvin and Kahn [9] extended this theorem to general $k$ with $n \geq 2k + 2$. Also, for $n = 2k + 1$, they proved that there is a fixed $p < 1$ such that, with high probability, $\alpha(KG_{2k+1,k}(p)) = \binom{2k}{k-1}$ and the stars are the only maximum independent sets. It is worth mentioning that some other kinds of generalizations of Theorem [A] can be found in [25, 26, 27, 28].

Seeing the Erdős matching conjecture as a generalization of the Erdős-Ko-Rado theorem to the case of Kneser hypergraphs, one may naturally ask for which $p > 0$ the Erdős matching conjecture continues likely to hold in $KG_{n,k}(p)$. Mainly motivated by this question, in this paper, we shall investigate the size and structure of maximum independent sets in random Kneser hypergraphs. We will show that the random counterpart of the Erdős matching conjecture continues to hold when $k$ is very small in comparison to $n$. More precisely, when $r \geq 2$, we shall prove a hypergraph version of Theorem [A] which in part implies a slightly weaker version of this theorem. It should be mentioned that our technique in the
proof of this result is different from that of Theorem \(\mathcal{A}\) in [9]. A natural candidate for the probability threshold could be obtained by seeking for a threshold \(p_c\) such that for each positive constant \(\varepsilon\), if \(p \leq (1 - \varepsilon)p_c\), then the expected number of independent sets \(\mathcal{A}\) in KG\(_{n,k}^r\) of size \((\binom{n}{r}) - (\binom{n-r+1}{k}) + 1\) which contain some \(r - 1\) distinct stars goes to zero as \(n\) tends to infinity. Since for any such family \(\mathcal{A}\), we have \(|E(\text{KG}_{n,k}^r[\mathcal{A}])| = \prod_{i=1}^{r-1} \left(\binom{n-r+1}{k} - (\binom{n-r+i}{k})\right)\), the expected number of such independent sets would be

\[
\left(\frac{n}{r-1}\right)\left(\frac{n-r+1}{k}\right)(1-p)^{r-1}\prod_{i=1}^{r-1} \left(\binom{n-r+i}{k} - (\binom{n-r+i}{k})\right)
\]

which clearly suggests \(p_c = \frac{\ln\left(\frac{\binom{n-r+1}{k}}{\prod_{i=1}^{r-1} \left(\binom{n-r+i}{k} - (\binom{n-r+i}{k})\right)}\right)}{\prod_{i=1}^{r-1} \left(\binom{n-r+i}{k} - (\binom{n-r+i}{k})\right)}\). Our main result is the following theorem.

**Theorem 1.** Let \(n, k\) and \(r\) be positive integers such that \(k = k(n) \geq 2\), \(r \geq 2\), and \(n \geq r(k + \frac{1}{2})\).

**I:** There are positive constants \(\zeta = \zeta(r)\) and \(C = C(r)\) such that

\[
P\left(\alpha(\text{KG}_{n,k}^r(p)) = \left(\frac{n}{k}\right) - \left(\frac{n-r+1}{k}\right)\right) \rightarrow 1
\]

provided \(p > \zeta p_c\) and \(k \leq Cn^\ast\) (\(k = o(n^\ast)\) for \(r = 2, 3\)).

Furthermore, with high probability, the only independent sets of size \((n\binom{n-r+1}{k}) - (n-r+1\binom{k}{k})\) are the trivial ones, namely the union of \(r - 1\) distinct stars.

**II:** For each \(\varepsilon \in (0, 1]\), we have

\[
P\left(\alpha(\text{KG}_{n,k}^r(p)) = \left(\frac{n}{k}\right) - \left(\frac{n-r+1}{k}\right)\right) \rightarrow 0
\]

provided \(p \leq (1 - \varepsilon)p_c\).

This theorem generalizes Theorem \(\mathcal{A}\) to the case of Kneser hypergraphs. As stated above (see the discussion after Theorem \(\mathcal{A}\)), owing to the works [2, 8, 9], Theorem \(\mathcal{A}\) has been extended to \(k\) as large as \(\frac{n}{2} - 2\). We believe that the condition on \(k\) in Theorem \(\mathcal{I}\) is superfluous as well. By the way, we conjecture that the same formula for the critical threshold continues to work for \(r \geq 3\) and \(n > r(k + \frac{1}{2})\), but we are unable to prove this presently. Also, for \(1 \leq n-rk \leq \frac{r}{2}\), it is interesting to study the behavior of \(\alpha(\text{KG}_{n,k}^r(p))\). Note that the case \(r = 2\) is already addressed by the aforementioned result by Delvin and Kahn [9]. Indeed, for \(1 \leq n-rk \leq \frac{r}{2}\), we surmise that there is a constant \(p < 1\) such that, with high probability, \(\alpha(\text{KG}_{n,k}^r(p))\) is equal to \max\{\(\binom{n-r+1}{k}\), \(\binom{n}{k}\)\} and the only maximum independent sets are the trivial ones.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem \(\mathcal{I}\) which is divided into three subsections. In the first subsection, we set up some notations, then the proof of the first and the second parts of the theorem will be discussed separately.
2. Proof of Theorem 1

2.1. Notation. For two functions $f(n)$ and $g(n)$, we write $f \sim g$ and $f = o(g)$ whenever
\[
\lim_{n \to \infty} \frac{f}{g} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{f}{g} = 0, \text{ respectively. For simplicity of notation, we set } V = \binom{n}{k},
\]
\[
M = \prod_{i=1}^{r-1} \binom{n-ik-r+i}{k-1}, \quad N = \binom{n}{k} - \binom{n-r+1}{k}, \quad N_i = \binom{n-i}{k-1}, \quad \text{and } H = \binom{n-1}{k-1} - \binom{n-k-1}{k-2}.
\]
Note that $N = N_1 + \cdots + N_{r-1},$ $H \leq k\binom{r-2}{k-2},$ and
\[
(r-1)\binom{n-r+1}{k-1} \leq N \leq (r-1)\binom{n-1}{k-1}.
\]
Let us remind that $r \geq 2$ is a fixed positive integer and $k \leq Cn^{\frac{1}{r}} \left( k = o(n^{\frac{1}{r}}) \right)$ for $r = 2, 3).$ Accordingly, we have $N \sim (r-1)N_i$ and $H = o\left(\frac{n}{k}\right)$ for each $i \in [r-1]$. Moreover, $M \sim \frac{N^{r-1}}{(r-1)^{r-1}}$ which implies
\[
p_c = \frac{\ln \left( \binom{n}{r-1} \binom{n-r+1}{k} \right)}{\prod_{i=1}^{r-1} \binom{n-ik-r+i}{k-1}} \sim \frac{(r-1)^{r-1} \ln \left( \binom{n}{r-1} \binom{n-r+1}{k} \right)}{N^{r-1}}.
\]

2.2. Proof of Theorem 1 Part I. For the ease of reading and without loss of generality, we can suppose that $p > \zeta \ln\left(\frac{n}{N+r-1}\right) \binom{n-r+1}{k}$ and \( k \leq Cn^{\frac{1}{r}} \left( k = o(n^{\frac{1}{r}}) \right) \) for $r = 2, 3$) for some suitable fixed $\zeta$ and $C$ which will be determined during the proof. Set
\[
\mathcal{C} = \left\{ \mathcal{A} \subseteq \binom{[n]}{k} : |\mathcal{A}| = N \text{ and } \mathcal{A} \text{ is not the union of any } r-1 \text{ stars} \right\}.
\]
Suppose that $\mathcal{A}$ is an independent set of $\text{KG}_{n,k}^r(p)$ with size $N+1$. Since there is an $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{A}' \in \mathcal{C}$ and $|\mathcal{A}'| = N$, the event that $\alpha(\text{KG}_{n,k}^r(p)) \geq N+1$ is a subset of the event that some member of $\mathcal{C}$ is an independent set of $\text{KG}_{n,k}^r(p)$. Therefore, to prove the first part of Theorem 1 it suffices to show that with high probability no member of $\mathcal{C}$ is an independent set of $\text{KG}_{n,k}^r(p)$ which will be clearly done if we prove
\[
(1) \quad \sum_{\mathcal{A} \in \mathcal{C}} \mathbb{P} \left( \mathcal{A} \text{ is an independent set of } \text{KG}_{n,k}^r(p) \right) = o(1).
\]
Let $\mathcal{T}_{n,k}^r(p)$ denote the collection of independent sets of $\text{KG}_{n,k}^r(p)$. For each $\mathcal{A} \in \mathcal{C}$ and $x \in [n]$, define $\mathcal{A}_x = \mathcal{A} \cap \mathcal{S}_x.$ Moreover, consider fixed (with respect to $\mathcal{A}$) distinct elements $x_1, \ldots, x_{n} \in [n]$ such that
\[
|\mathcal{A}_{x_1}| \geq \cdots \geq |\mathcal{A}_{x_n}|.
\]
Throughout the paper, we will refer to these $x_i$'s several times. For an $\mathcal{A}$, if there is more than one choice for $(x_1, \ldots, x_n)$, we choose one of them arbitrarily and fix it for the rest of the paper. Now, for each $i \in [r-1]$, set $z_i = N_i - |\mathcal{A}_{x_i}| \setminus \bigcup_{j=1}^{r-1} \mathcal{A}_{x_j}|. \) Note that $\sum_{i=1}^{r-1} z_i = |\mathcal{A} \setminus \bigcup_{i=1}^{r-1} \mathcal{A}_{x_i}|.$ Define
\[
\mathcal{C}_1 = \left\{ \mathcal{A} \in \mathcal{C} : |\mathcal{A}_{x_{r-1}}| < \frac{1}{2r^2k}N \right\}, \quad \mathcal{C}_2 = \left\{ \mathcal{A} \in \mathcal{C} \setminus \mathcal{C}_1 : \sum_{i=1}^{r-1} z_i \geq \frac{N}{4r^2} \right\}.
\]
\[ C_3 = \left\{ A \in \mathcal{C} \setminus \mathcal{C}_1 : \sum_{i=1}^{r-1} z_i < \frac{N}{4r^2} \right\}. \]

To prove Equation (1), we will show that for each \( \ell \in \{1, 2, 3\} \),
\[
(2) \sum_{A \in \mathcal{C}_\ell} \mathbb{P} \left( A \in I_{r,n,k}(p) \right) = o(1).
\]

The rest of our discussion in this subsection is devoted to the proof of Equation (2), which will be done separately for each \( \ell \in \{1, 2, 3\} \).

**Proof of Equation 2 when \( \ell = 1 \).** We here first need to estimate the minimum number of edges of KG\(_{n,k}^r[A]\) when \( A \in \mathcal{C}_1 \).

**Lemma 1.** There is a constant \( \eta_1 = \eta_1(r) \) such that for any \( A \in \mathcal{C}_1 \),
\[ |E(KG_{n,k}^r[A])| \geq \eta_1 N^r. \]

**Proof.** Let \( A \in \mathcal{C}_1 \). According to the definition of \( \mathcal{C}_1 \), we have \( |A_{x_{r-1}}| < \frac{N}{2r^2} \). Set \( A' = A \setminus \bigcup_{j=1}^{r-2} A_{x_j} \). Note that \( |A'| \geq N_{r-1} = (\frac{1}{r} - o(1))N \); moreover, each \( A \in A' \) intersects at most \( k \frac{N}{2r^2} \) elements in \( A' \). This observation concludes in
\[
|E(KG_{n,k}^r[A'])| \geq \frac{1}{r!} \prod_{i=0}^{r-1} \left( |A'| - ik \frac{N}{2r^2} \right)
\geq \frac{1}{r!} \left( |A'| - \frac{N}{2r} \right)^r
\geq \frac{1}{r!} \left( \frac{1}{r} - o(1) \right)^r N^r
\geq \eta_1 N^r,
\]
for some appropriate \( \eta_1 \), as desired. \( \square \)

By using Lemma 1, we thus have
\[
\sum_{A \in \mathcal{C}_1} \mathbb{P} \left( A \in I_{r,n,k}(p) \right) \leq |\mathcal{C}_1|(1 - p)^{\eta_1 N^r}
\leq \left( \frac{V}{N} \right) e^{-\eta_1 N^r}
\leq \left( \frac{V}{(r-1)N_1} \right) e^{-\eta_1 N^r}
\leq \exp \left\{ -\eta_1 N^r + (r - 1)N_1 \ln \frac{ne}{(r-1)k} \right\}
\leq \exp \left\{ \left( -\zeta \eta_1 \ln \left( \left( \frac{n}{r-1} \right)^{r-1} \right) + (1 + o(1)) \ln \frac{ne}{(r-1)k} \right) N \right\} \to 0
\]
provided that \( \zeta > \frac{1}{\eta_1} \), which completes the proof of Equation 2 for \( \ell = 1 \). \( \square \)
Proof of Equation 2 when $\ell = 2$. The minimum possible number of edges of $KG_{n,k}^{r}[A]$ when the size of $A$ is given was studied by Das, Gan, and Sudakov in [7]. To state their result precisely, we first need to recall some definitions. We consider $\binom{[n]}{k}$ as a poset equipped with the lexicographical ordering: $A < B$ if $\min(A \Delta B) \in A$. In other words, in the lexicographical ordering, we prefer sets with smaller elements. Define $L_{n,k}(s)$ to be the set of $s$ first sets in $\binom{[n]}{k}$ according to the lexicographical ordering.

Theorem B. [7, Theorems 1.6 and 1.7] If $n > 108k^2(l + k)$ and $1 \leq s \leq \binom{n}{k} - \binom{n-l}{k}$, then $L_{n,k}(s)$ minimizes the number of edges of $KG_{n,k}^{r}[A]$ among all sets $A$ of $s$ sets in $\binom{[n]}{k}$.

Also, for $q \geq 3$, there is a positive constant $\eta$ such that if $n > \eta l 2k^5(l^2 + k^2)e^{3q}$ and $1 \leq s \leq \binom{n}{k} - \binom{n-l}{k}$, then $L_{n,k}(s)$ minimizes the number of edges of $KG_{n,k}^{q}[A]$ among all sets $A$ of $s$ sets in $\binom{[n]}{k}$.

Although, the next corollary is a simple consequence of this theorem, for the sake of completeness, we prove it here.

Corollary 1. Let $q \geq 2$ be a fixed positive integer. There are positive constants $\alpha$ and $\beta$ such that for $n \geq \alpha k^7$ (for $q = 2$, $n \geq \alpha k^3$), we have

$$|E(KG_{n,k}^{q}[A])| \geq \beta m|A|^{q-1}$$

provided that $|A| = N_1 + \cdots + N_{q-1} + m$, where $1 \leq m \leq N_q$.

Proof. Set $s = N_1 + \cdots + N_{q-1} + m$. In view of Theorem B, since $|E(KG_{n,k}[A])|$ will be minimized when $A$ is the set of $s$ first sets in $\binom{[n]}{k}$ according to the lexicographical ordering, we may assume that $A = S_1 \cup \cdots \cup S_{q-1} \cup T$ for some $T \subseteq S_q \setminus \bigcup_{i=1}^{q-1} S_i$ with $|T| = m$. In conclusion, one can verify that

$$|E(KG_{n,k}^{q}[A])| \geq m \prod_{i=1}^{q-1} \left( \binom{n-i-k+q+i}{k-1} \right)$$

$$= m(1 - o(1))N_1^{q-1}$$

$$\geq m(1 - o(1)) \left( \frac{1}{q} \sum_{i=1}^{q} N_i \right)^{q-1}$$

$$\geq \beta m|A|^{q-1}$$

for an appropriate positive constant $\beta$. \hfill $\square$

Using this corollary, by the following lemma, we will prove that $KG_{n,k}^{r}[A]$ has many edges whenever $A \in C_2$.

Lemma 2. There is a positive constant $\eta_2 = \eta_2(r)$ such that for each $A \in C_2$, we have

$$|E(KG_{n,k}^{r}[A])| \geq \eta_2 \frac{N^r}{k}.$$ 

Proof. Consider distinct elements $x_1, \ldots, x_n \in [n]$ (as is defined fixedly above) such that $|A_{x_1}| \geq \cdots \geq |A_{x_n}|$. 

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Since \( \mathcal{A} \in \mathcal{C}_2 \), we have \( |\mathcal{A}_{x_i}| \geq \cdots \geq |\mathcal{A}_{x_{r-1}}| \geq \frac{N}{2r^2k} \). Let \( a \in [r-1] \) be the largest index for which \( |\mathcal{A}_{x_a}| \geq \frac{N}{2r^2k} \) (if there is no such an index, then set \( a = 0 \)). Note that \( |\mathcal{A}_x \cap \mathcal{A}_y| \leq |S_x \cap S_y| = \binom{n-2}{k-2} = o\left(\frac{N}{k}\right) \) for each \( x \neq y \in [n] \). Accordingly, for each \( i \leq a \),

\[
\left| \mathcal{A}_{x_i} \setminus \bigcup_{j \in [r-1]\setminus \{i\}} \mathcal{A}_{x_j} \right| \geq |\mathcal{A}_{x_i}| - (r-2) \binom{n-2}{k-2} \geq \frac{N}{r^2} - o\left(\frac{N}{k}\right)
\]

and for each \( a + 1 \leq i \leq r-1 \),

\[
\left| \mathcal{A}_{x_i} \setminus \bigcup_{j \in [r-1]\setminus \{i\}} \mathcal{A}_{x_j} \right| \geq |\mathcal{A}_{x_i}| - (r-2) \binom{n-2}{k-2} \geq \frac{N}{2r^2k} - o\left(\frac{N}{k}\right).
\]

Note that each \( A \not\in S_x \) is disjoint from all but \( H \) elements in \( S_x \). Consequently, if \( a \geq r-2 \), then

\[
E(KG_{n,k}^r[\mathcal{A}]) \geq \left| \mathcal{A} \setminus \bigcup_{i=1}^{r-1} \mathcal{A}_{x_i} \right| \times \prod_{i=1}^{r-1} \left( \left| \mathcal{A}_{x_i} \setminus \bigcup_{j \in [r-1]\setminus \{i\}} \mathcal{A}_{x_j} \right| - iH \right)
\]

\[
\geq (z_1 + \cdots + z_{r-1}) \left( \frac{N}{r^2} - o\left(\frac{N}{k}\right) \right)^{r-2} \left( \frac{N}{2r^2k} - o\left(\frac{N}{k}\right) \right)
\]

\[
\geq \frac{N}{4r^2} \left( \frac{N}{r^2} - o\left(\frac{N}{k}\right) \right)^{r-2} \left( \frac{N}{2r^2k} - o\left(\frac{N}{k}\right) \right)
\]

\[
\geq \beta' \frac{N}{k}
\]

for some positive constant \( \beta' \) (note that \( H = o\left(\frac{N}{k}\right) \)). Henceforth, we assume that \( a < r-2 \).

Set \( \mathcal{A}' = \mathcal{A} \setminus \bigcup_{i=1}^{a+1} \mathcal{A}_{x_i} \) and \( m = N_{r-1} - \frac{N}{r^2} \). Note that

\[
|\mathcal{A}'| \geq N_{a+1} + \cdots + N_{r-2} + m \geq \left( \frac{1}{r} - o(1) \right) N
\]

and

\[
m = N_{r-1} - \frac{N}{r^2} = \left( \frac{1}{r - 1} - o(1) \right) N - \frac{N}{r^2} \geq \left( \frac{1}{r} - o(1) \right) N.
\]

Without loss of generality, we assume that \( |\mathcal{A}'| = N_{a+1} + \cdots + N_{r-2} + m \). Consequently, in view of Corollary \([\square]\) there is a constant \( \beta \) for which

\[
|E(KG_{n,k}^{r-a-1}[\mathcal{A}'])| \geq \beta \left( \frac{1}{r} - o(1) \right) N |\mathcal{A}'|^{r-a-2}
\]

\[
\geq \beta \left( \frac{1}{r} - o(1) \right) N \left( \left( \frac{1}{r} - o(1) \right) N \right)^{r-a-2}
\]

\[
= \beta \left( \frac{1}{r^{r-a-1}} - o(1) \right) N^{r-a-1}.
\]
Since $H = o\left(\frac{N}{k}\right)$,

$$|E(KG_{n,k}^r[A])| \geq |E(KG_{n,k}^{r-a-1}[\mathcal{A}'])| \times \prod_{i=1}^{a+1} \left(\left|A_{x_i} \setminus \bigcup_{j \in [a+1] \setminus \{i\}} A_{x_j}\right| - (r - a - i)H\right)$$

$$\geq \beta\left(\frac{1}{r-a-1} - o(1)\right)N^{r-a-1} \left(\frac{N}{r^2} - o\left(\frac{N}{k}\right)\right)^a \left(\frac{N}{2^{r-2}k} - o\left(\frac{N}{k}\right)\right)$$

$$\geq \beta' \frac{N^r}{k}$$

for some positive constant $\beta''$. Setting $\eta_2 = \min\{\beta', \beta''\}$ completes the proof of lemma. □

Now, by use of this lemma, we have

$$\sum_{\mathcal{A} \in \mathcal{C}_2} \mathbb{P}\left(\mathcal{A} \in \mathcal{T}_{n,k}(p)\right) \leq |\mathcal{C}_2| (1 - p)^{\eta_2 \frac{N^r}{k}}$$

$$\leq \left(\frac{V}{N}\right) \exp\{-\eta_2 p \frac{N^r}{k}\}$$

$$\leq \left(\frac{V}{(r-1)N_1}\right) \exp\{-\eta_2 p \frac{N^r}{k}\}$$

$$\leq \exp\left\{-p\eta_2 \frac{N^r}{k} + (r - 1)N_1 \ln \frac{n}{(r-1)k}\right\}$$

$$\leq \exp\left\{\left(-\zeta \eta_2 k \ln \left(\binom{n}{r-1} (n-r+1)\right) + (1 + o(1)) \ln \frac{n}{(r-1)k}\right) N\right\}$$

$$\leq \exp\left\{\left(-\zeta \eta_2 \ln \left(\frac{n-r+1}{k}\right) + (1 + o(1)) \ln \frac{n}{(r-1)k}\right) N\right\} \to 0$$

provided that $\zeta > \frac{1}{\eta_2}$. □

**Proof of Equation 2 when $\ell = 3$.** For each $\mathcal{A} \in \mathcal{C}_3$ and each $i \in [r-1]$, we clearly have

$$\left|A_{x_i} \setminus \bigcup_{j=1}^{i-1} A_{x_j}\right| = N_i - z_i \geq \left(\frac{1}{r - 1} - o(1)\right) N - z_i.$$ 

Consequently,

$$\left|A_{x_i} \setminus \bigcup_{j \in [r-1] \setminus \{i\}} A_{x_j}\right| \geq |A_{x_i} \setminus \bigcup_{j=1}^{i-1} A_{x_j}| - \sum_{j=i+1}^{r-1} |A_{x_i} \cap A_{x_j}|$$

$$\geq N_i - z_i - (r - 2) \binom{n-2}{k-2}$$

$$\geq \left(\frac{1}{r - 1} - o(1)\right) N - z_i.$$
Accordingly, since $H = o(N)$, for large enough $n$, we have

$$E(KG_{n,k,[A]}^r) \geq \left| A \backslash \bigcup_{i=1}^{r-1} A_{x_i} \right| \times \prod_{i=1}^{r-1} \left( \left| A_{x_i} \backslash \bigcup_{j \in [r-1]\backslash \{i\}} A_{x_j} \right| - iH \right)$$

$$= (z_1 + \cdots + z_{r-1}) \prod_{i=1}^{r-1} \left( \left| A_{x_i} \backslash \bigcup_{j \in [r-1]\backslash \{i\}} A_{x_j} \right| - iH \right)$$

$$\geq (z_1 + \cdots + z_{r-1}) \prod_{i=1}^{r-1} \left( \frac{N}{r} - z_i \right).$$

Hence, if we define $f(z_1, \ldots, z_{r-1}) = (z_1 + \cdots + z_{r-1}) \prod_{i=1}^{r-1} \left( \frac{N}{r} - z_i \right)$, then, for large enough $n$,

$$\sum_{A \in C_3} P(A \in T_{n,k}(p)) \leq \sum_{1 \leq z_1 + \cdots + z_{r-1} \leq cN} \binom{n}{r-1} \binom{N_1}{z_1} \cdots \binom{N_{r-1}}{z_{r-1}} \left( \frac{V}{z_{r-1}} \right)^{V} (1 - p)^{f(z_1, \ldots, z_{r-1})}$$

$$\leq \sum_{1 \leq z_1 + \cdots + z_{r-1} \leq cN} \binom{n}{r-1} \binom{N_1}{z_1} \cdots \binom{N_{r-1}}{z_{r-1}} \left( \frac{V}{z_{r-1}} \right)^{V} e^{-pf(z_1, \ldots, z_{r-1})}.$$

Now, we set

$$g(z_1, \ldots, z_{r-1}) = \binom{n}{r-1} \binom{N_1}{z_1} \cdots \binom{N_{r-1}}{z_{r-1}} \left( \frac{V}{z_{r-1}} \right)^{V} e^{-pf(z_1, \ldots, z_{r-1})}.$$ 

It is simple to check that there is a constant $\zeta_0$ such that for $\zeta > \zeta_0$, if $\sum_{i=1}^{r-1} z_i \geq 2$, then for each $z_i \geq 1$,

$$\frac{g(z_1, \ldots, z_i, \ldots, z_{r-1})}{g(z_1, \ldots, z_i - 1, \ldots, z_{r-1})} = \frac{\binom{N_i}{z_i} \left( \frac{V}{z_{i-1}} \right)^{V}}{\binom{N_{i-1}}{z_{i-1}} \left( \frac{V}{z_{i-1}} \right)^{V}} e^{-p \left( f(z_1, \ldots, z_{r-1}) - f(z_1, \ldots, z_i - 1, \ldots, z_{r-1}) \right)} = o(1).$$

Therefore, for sufficiently large $n$, we have

$$\frac{g(z_1, \ldots, z_i, \ldots, z_{r-1})}{g(z_1, \ldots, z_i - 1, \ldots, z_{r-1})} < 1$$

which clearly concludes in

$$g(z_1, \ldots, z_{r-1}) \leq g(1, 0, \ldots, 0).$$
This implies that there is a constant \( c = c(r) \) for which
\[
\sum_{A \in C_3} \Pr(A \in \mathcal{I}_{n,k}(p)) \leq \sum_{1 \leq z_1 + \cdots + z_{r-1} \leq \frac{N}{2r}} g(1, 0, \ldots, 0)
\]
\[
\leq \sum_{1 \leq z_1 + \cdots + z_{r-1} \leq \frac{N}{2r}} \binom{n}{r-1} N_1 V e^{-pcN^{r-1}}
\]
\[
\leq \binom{n}{r-1} N^r V e^{-pcN^{r-1}}
\]
\[
= \exp \{-pcN^{r-1} + \ln((\binom{n}{r-1})^{N^r V})\}
\]
\[
\leq \exp \{-\zeta c \ln((\binom{n}{r-1})^{(n-r+1)}) + \ln((\binom{n}{r-1})^{N^r V})\}
\]
\[
= \frac{(\binom{n}{r-1})^{N^r V}}{((\binom{n}{r-1})^{(n-r+1)})^{\zeta c}} \to 0
\]
provided that \( \zeta > \frac{r+1}{c} \).

We are now ready to finish the proof of Theorem 1: Part I.

Completing the proof of Theorem 1: Part I. In conclusion, if we set \( \zeta > \max\{\zeta_0, \frac{1}{\eta_1}, \frac{1}{\eta_2}, \frac{r+1}{c}\} \), then for all \( \ell \in \{1, 2, 3\} \), we simultaneously have
\[
\sum_{A \in C_\ell} \Pr(A \in \mathcal{I}_{n,k}(p)) = o(1)
\]
finishing the proof.

2.3. Proof of Theorem 1: Part II. It should be noticed that our proof is similar to that of the second part of Theorem A in [6]. Let \( p \leq (1 - \varepsilon)p_c \) for some constant \( \varepsilon \in (0, 1] \). Here we prove that
\[
\Pr(\alpha(KG_{n,k}(p)) \leq \binom{n}{k} - \binom{n-r+1}{k}) = o(1).
\]

Let \( Y \) denote the random variable counting the number of pairs \((A, Q)\) such that \( Q \in ([n]) \setminus \bigcup_{x \in Q} S_x \), and \( E(KG_{n,k}(p)[S_Q \cup \{A\}]) = \emptyset \). Clearly, to prove the desired assertion, it suffices to show that \( \Pr(Y > 0) \) goes to 1 as \( n \) tends to infinity. Let us remind that
\[
M = \prod_{i=1}^{r-1} \binom{n-ik-1}{k-1}.
\]
It is easy to check that
\[
\mathbb{E}[Y] = \binom{n}{r-1}^{(n-r+1)}(1 - p)^M
\]
\[
\geq \binom{n}{r-1}^{(n-r+1)} \exp(-(p + p^2)M)
\]
\[
\geq \binom{n}{r-1}^{(n-r+1)} \exp \{- (1 + p)(1 - \varepsilon) \ln((\binom{n}{r-1})^{(n-r+1)})\}
\]
\[
\geq (\binom{n}{r-1})^{(n-r+1)} \varepsilon - p + \varepsilon p.
\]
Therefore, $E[Y] \to \infty$ when $p \leq (1 - \varepsilon)p_c$. Hence, by using the classical second moment technique, to prove that $\mathbb{P}(Y > 0) \to 1$, it is sufficient to show that $\text{Var}[Y] = o(E[Y]^2)$. Let $Y'$ denote the random variable counting the number of 4-tuples $(A, B, Q, T)$ with $Q, T \in \binom{[n]}{r-1}$, $A \in \binom{[n]}{k} \setminus S_Q$ and $B \in \binom{[n]}{k} \setminus S_Q$ such that $(A, Q) \neq (B, T)$ and

$$E(KG_{n,k}^r(p)[S_Q \cup \{A\}]) = E(KG_{n,k}^r(p)[S_T \cup \{B\}]) = \emptyset.$$ 

Clearly,

$$E[Y'] = \sum \mathbb{P}(S_Q \cup \{A\}, S_T \cup \{B\} \in T_{n,k}),$$

where the summation is taken over all ordered 4-tuples $(A, B, Q, T)$ with $Q, T \in \binom{[n]}{r-1}$, $A \in \binom{[n]}{k} \setminus S_Q$, $B \in \binom{[n]}{k} \setminus S_Q$, and $(A, Q) \neq (B, T)$. Now, one can verified that

$$\sum_{Q \neq T} \mathbb{P}(S_Q \cup \{A\}, S_T \cup \{B\} \in T_{n,k}) \leq \left(\binom{n}{r-1}\right)^2 \left(1 - p\right)^{2M - O(N^{r-2})}$$

$$= (1 + o(1))E[Y]^2$$

and

$$\sum_{Q = T, A \neq B} \mathbb{P}(S_Q \cup \{A\}, S_T \cup \{B\} \in T_{n,k}) \leq \left(\binom{n}{r-1}\right)^2 \left(1 - p\right)^{2M}$$

$$= o(E[Y]^2).$$

Note that

$$\text{Var}[Y] = E[Y^2] - E[Y]^2 = E[Y] + E[Y'] - E[Y]^2.$$ 

Hence, $\text{Var}[Y] = E[Y] + o(E[Y^2]) = o(E[Y^2])$, as desired. □

ACKNOWLEDGMENTS

The research of Meysam Alishahi was in part supported by a grant from IPM (No. 96050013).

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M. Alishahi, Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran & School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
*E-mail address: meysam_alishahi@shahroodut.ac.ir*

A. Taherkhani, Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran
*E-mail address: ali.taherkhani@iasbs.ac.ir*