CYCLIC DUALITY FOR SLICE
AND ORBIT 2-CATEGORIES

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Abstract. The self-duality of the paracyclic category is extended
to a certain class of homotopy categories of (2,1)-categories. These
generalise the orbit category of a group and are associated to cer-
tain self-dual preorders equipped with a presheaf of groups and
a cosieve. Slice 2-categories of equidimensional submanifolds of a
compact manifold without boundary form a particular case, and
for $S^1$, one recovers cyclic duality. This provides in particular a
visualisation of the results of Böhm and Ştefan on the topic.

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1. Introduction

1.1. Slice 2-categories. This article is about embeddings of subobjects, their deformations, and complements. Our basic setting is:

**Assumption 1.** $C$ is a (2,1)-category all of whose 1-cells are monic.

Recall that the *slice category* of $C$ over an object $T \in C_0$ is the preorder of all 1-cells in $C$ with codomain $T$ and the preorder relation $x \leq y :\iff \exists f \in C_1 : x = yf$.

By definition, a *subobject* of $T$ is an isomorphism class $[x]$ of an object in this preorder, see e.g. [ML98, Section V.7] or [Lur09, Section 6.1.6].

The *slice 2-category* $C/T$ records further information about the subobjects of $T$: its objects are 1-cells with codomain $T$, and its 1-cells $x \to y$ are 2-cells $\phi: x \Rightarrow z$ in $C$ with $z \leq y$; think of a deformation of a subobject $[x]$ to a subobject $[z]$ contained in $[y]$. Its 2-cells are 2-cells in $C$ that deform the target object $[z]$ inside $[y]$ (see Definition 2.2.1), so in the *homotopy category* $\text{ho}(C/T)$ (Definition 2.3.1) such final perturbations of the target $[z]$ get identified.

1.2. Orbit 2-categories. For many $C$, the ordinary slice category over $T$ is self-dual, with the dual $x^\circ$ of an object representing some form of complement of $[x]$ in $T$. The question we are interested in is:

**Question.** When does a self-duality of $(C/T_0, \leq)$ lift to $\text{ho}(C/T)$?

This was triggered by the following example that we will return to in Section 1.3 below, where we provide details and definitions:

**Example 1.2.1.** In the (2,1)-category $\text{Mfld}^1$ of embeddings of compact 1-dimensional manifolds (Definition 2.4.1), all ordinary slice categories are self-dual. The homotopy category $\text{ho}(\text{Mfld}^1/[0,1])$ is a model of the simplicial category hence is not self-dual. In contrast, $\text{ho}(\text{Mfld}^1/S^1)$ is a model of the paracyclic category which is self-dual.

The answer we give assumes that the self-duality of $(C/T_0, \leq)$ is equivariant with respect to the natural action of the group $\text{Aut}(T)$ of invertible 1-cells $T \to T$:

**Assumption 2.** $\circ : C/T_0 \to C/T_0$, $x \mapsto x^\circ$ is a map such that $[x^\circ] = [x]$, $[(gx)^\circ] = [g(x^\circ)]$, $x \leq y \iff y^\circ \leq x^\circ$ holds for all $x,y \in C/T_0$ and $g \in \text{Aut}(T)$. 

Such a self-duality gives rise to a subrelation $\ll$ of $\leq$ which is a $\text{Aut}(T)$-cosieve in $(\mathcal{C}/T_0, \leq)$, i.e. which is closed under the $\text{Aut}(T)$-action and under postcomposition (Definition 4.1.1, Proposition 4.1.2). In $\mathcal{C} = \text{Mfld}^d$, $x \ll y$ means that $[x]$ is contained in the interior of $[y]$.

Our main result states that if all $x \in \mathcal{C}/T_0$ satisfy a strong form of the homotopy extension property (Assumptions 3 and 4 in Section 4, where this is discussed in full detail) and admit an abstract version of tubular neighbourhoods (Assumption 5 therein), then the answer to our question is affirmative. More precisely, we prove the following theorem, where expressions such as $\gamma y$ and $y \xi$ denote the horizontal composition of the 2-cell $\text{id}_y$ with 2-cells $\gamma$ respectively $\xi$, and where

$$G := \bigcup_{g \in \text{Aut}(T)} C_2(\text{id}_T, g).$$

**Theorem.** If $\ll$ is an $\text{Aut}(T)$-cosieve in $(\mathcal{C}/T_0, \leq)$ with

$$\text{id}_T \ll \text{id}_T,$$

and if for all $f, h: X \to Y$, $y: Y \to T$, and $\phi: yf \Rightarrow yh$, we have

$$\exists \xi: f \Rightarrow h: \phi = y\xi \quad \text{and} \quad (\forall u \ll y^\ast \exists \gamma \in G: \gamma u = u, \gamma yf = \phi)$$

and for all $u \ll y^\ast, v \ll y^\ast$ there exists $\tau: \text{id}_T \Rightarrow t$ in $G$ and $r \ll y$ with

$$\tau u = u, \tau v = v, [tr] = [y],$$

then $\circ$ lifts to an $\text{Aut}(T)$-equivariant self-duality on $\text{ho}(\mathcal{C}/T)$.

This applies in particular to $\text{Mfld}^d/T$ if $T$ is a manifold with empty boundary (Examples 3.4.2, 3.5.3, 3.7.3, and 4.2.4).

We present the above theorem as a special case of a more general self-duality result: let $\mathcal{G}$ be a (strict) 2-group, $A$ be the group of its 1-cells, and $G$ be its source group. Then we associate a $\mathcal{G}$-category $\mathcal{I}_s$ to any $A$-preorder $(S, \subseteq)$ equipped with an $A$-equivariant presheaf $s: x \mapsto G_x$ of subgroups $G_x \subseteq G$, (Proposition 3.3.2). When $S$ is the poset of all subgroups of $A = G$ itself and $s$ is the identity, then $\mathcal{I}_s$ is the dual of the orbit category of $G$ (Example 3.3.4, see [tD87] for more information). When the $A$-preorder is self-dual and $\subseteq$ is an $A$-cosieve satisfying (1.1), then $\mathcal{I}_s$ gets upgraded to a (2,1)-category with a self-dual homotopy category (Proposition 3.6.1, Corollary 3.7.2). The remaining assumptions of our theorem are there to imply $\text{ho}(\mathcal{C}/T) \cong \text{ho}(\mathcal{I}_{s/T})$ as $\text{Aut}(T)$-categories, where $\mathcal{G} = \text{Aut}(T)$ is the automorphism 2-group of $T$ (Corollary 3.7.2, Example 3.7.3).

We are not aware of a reference that considers this exact type of (2,1)-category, and we only develop the part of the theory required to prove the above theorem. Studying other examples and applications

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might therefore be an interesting topic for future research, as there are many applications of classical orbit categories in equivariant algebraic topology, see e.g. [Wan82, Brö71, BGPT97] and in particular [Elm83], or the more algebraically motivated articles [PYn14, LL15, Web08, Ber17, So01], as well as [MP15].

1.3. Cyclic duality. The motivation for this article lies in homological and homotopical algebra. The categories of chain complexes and of simplicial objects are not self-dual – applying contravariant functors yields cochain complexes respectively cosimplicial objects. So by construction, the category of chain and cochain complexes (same graded module, no compatibility between boundary and coboundary map assumed) is self-dual. Building on the seminal work of Connes ([Con83], see also [Con94, Appendix 3.A]), Dwyer and Kan [DK85] extended the Dold-Kan correspondence to this setting and called the corresponding homotopical objects and the governing index category $K$ duplicial. The self-duality of $K$ also descends to Connes’ cyclic category $\Lambda$ which is a quotient, and extends to the paracyclic category $\Lambda^c$ which is a localisation:

**Definition 1.3.1.** The categories $\Delta \subset K \subset \Lambda^c$ are defined as follows:

- The objects are the natural numbers 0, 1, 2, 3, ….
- The morphisms $f: n \to m$ are the maps $\mathbb{Z} \to \mathbb{Z}$ satisfying
  1. $i \leq j \Rightarrow f(i) \leq f(j)$ for all $i, j$,
  2. $f(j + n + 1) = f(j) + m + 1$ for all $j$,
  3. $f(0) \geq 0$ (in $K$ and $\Delta$),
  4. $f(n) \leq m$ (in $\Delta$).

The cyclic dual of $f: n \to m$ is the morphism $f^c: m \to n$ given by

$$f^c(i) := \max\{j \mid -f(-j) \leq i\}.$$  

See [Con83, DK85, GJ93, FT87, Elm93] for some original references for these definitions, and [NS18, Appendix B] for some recent reflections. Note that the above self-duality is not the most studied one on $\Lambda^c$, but one that restricts to $K$, see e.g. [DK85, p585] and [KK11, Section 4.2] – this also addresses the warning on [NS18, p381].

Böhm and Ţefan [BS12] explored the self-duality of $\Lambda^c$ further from the perspective of the bar construction. Our focus is different: we describe $\Lambda^c$ and $\Delta$ in a unified way in which we can point exactly at the reason why the one is self-dual and the other is not: both are (skeletal subcategories of) $\text{ho}(\mathcal{C}/T)$ for suitable $\mathcal{C}$ and $T$. For $\Delta$ we are looking at $\text{Mfld}^1/[0, 1]$, while for $\Lambda^c$ it is $\text{Mfld}^1/S^1$, and the latter is self-dual as $S^1$ has empty boundary.
We also find the resulting visualisation of $\Lambda^c$ clarifying in several ways. In the standard description, the object $n$ of $\Lambda^c$ gets visualised as $n + 1$ points on $S^1$. We replace these by tubular neighbourhoods, which is anyway natural in many settings, e.g. the study of the cyclic homology of DG algebras. Furthermore, the fact that the objects of $\Lambda^c$ are self-dual, $n^c = n$, is seen to be in a sense coincidental – the complement of $n + 1$ intervals in $S^1$ happens to be again $n + 1$ intervals. These are isotopic to the original ones, but in higher-dimensional manifolds $T$, $[x]$ and the closure $[x^c]$ of the complement of $\text{im} x$ are in general not diffeomorphic.

Most importantly to us, this provides a spatial view on the results of Böhm and Ştefan. Their main aim was to conceptually explain cyclic duality in the setting of Hopf-cyclic (co)homology \cite{ HKRS04, Kay11, CM, Cra02}. They showed (see \cite[Theorem 4.7]{BS12}) that the simplicial object resulting from the bar construction associated to a comonad $S_l$ and coefficients that they denote by $\Box, \triangle$ becomes para-cyclic in the presence of a second comonad $S_r$, a comonad distributive law $\Psi: S_lS_r \Rightarrow S_rS_l$, and $\Psi$-(op)coalgebra structures $i, w$ on the coefficients. In our visualisation, this corresponds to connecting the end points of the interval $[0, 1]$ in which the simplicial object is realised using a second interval to obtain a circle $S^1$, and the second comonad $S_r$ lives on the dark side of the moon.

The string diagrams in \cite{BS12} can now be seen as planar projections of our depiction of morphisms in $\Lambda^c$ in which we draw the track of a point in $S^1$ under an isotopy on a cylinder. In particular, Figure 1 depicts the cyclic operator $f: \mathbb{Z} \rightarrow \mathbb{Z}$, $f(j) = j + 1$, acting on the
object 2 ∈ \( \mathcal{X} \) (which is denoted \( t_2 \) in [BS12]). The natural transformation \( w \) corresponds to the track of an interval passing on the right end from the front to the back of the cylinder, the distributive law \( \Psi \) corresponds to two tracks crossing in the planar projection, while in the spatial resolution one of them runs down the front of the cylinder and the other one the back, and finally the natural transformation \( i \) is where the track from the back reappears on the left end of the cylinder.

The remainder of this article is divided into three sections. In the first, we provide some definitions from the theory of slice 2-categories, and discuss the example \( \text{Mfld}^d \) of embeddings of compact \( d \)-dimensional manifolds. In the second, we develop parts of a general theory of orbit 2-categories associated to certain presheaves of groups on a preorder. Ordinary slice categories are used throughout as a guiding example, but we also discuss a few group-theoretic examples in order to demonstrate the scope of the concepts. In the final section we focus again on \( \mathcal{C}/T \) and discuss the assumptions of our main theorem in detail.

Throughout the paper, we suppress all set-theoretic problems, so we tacitly assume all categories to be as small as required. Readers who are concerned about the application to manifolds should restrict to submanifolds of \( \mathbb{R}^{2d} \). Similarly, we focus on strict 2-categories.

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2. **Slice 2-categories**

This section contains background material on (2,1)-categories (see e.g. [JY21] for further information). We recall in particular the definition of the homotopy category \( \text{ho}(\mathcal{C}/T) \) of a slice 2-category and discuss in some detail the example of the (2,1)-category \( \text{Mfld}^d \) of embeddings of compact \( d \)-dimensional manifolds with 2-cells given by isotopies.

2.1. **(2,1)-categories.** Throughout, \( \mathcal{C} \) is a strict (2,1)-category, that is, the composition of 1-cells as well as both the vertical and the horizontal composition of 2-cells is strictly associative, and all 2-cells are invertible. In addition, we assume that all 1-cells are monic. The set of 1-cells between objects \( X, Y \in \mathcal{C}_0 \) is denoted by \( \mathcal{C}_1(X, Y) \); 1-cells are denoted by lower case Roman letters and their composition is written as concatenation. The set of 2-cells between \( f, g \in \mathcal{C}_1(X, Y) \) is denoted by \( \mathcal{C}_2(f, g) \); 2-cells are denoted by lower case Greek letters, and their
vertical respectively horizontal compositions by $\alpha \circ \beta$ respectively $\alpha \beta$. The identity 1-cell in $C_1(X, X)$ is denoted by $\text{id}_X$; analogously, the identity 2-cell in $C_2(f, f)$ is denoted by $\text{id}_f$. We denote the source and target maps $C_2 \to C_1$ and $C_1 \to C_0$ both by $s$ respectively $t$. Horizontal composition of 2-cells with identity 1-cells is called left- respectively right-whiskering, and we write $f \xi := \text{id}_f \xi$ respectively $\xi g := \xi \text{id}_g$, if no confusion arises.

**Remark 2.1.1.** Let $\alpha : f \Rightarrow g$ be a 2-cell and $\alpha^* : g \Rightarrow f$ be its inverse, so $\alpha \circ \alpha^* = \text{id}_g$ and $\alpha^* \circ \alpha = \text{id}_f$. If in addition $f, g : X \to Y$ are invertible as 1-cells, then a straightforward computation shows (see e.g. [KR21, Lemma 7]) that $\alpha$ also has a horizontal inverse given by

$$\alpha^{-1} := g^{-1} \alpha^* f^{-1} : f^{-1} \Rightarrow g^{-1}$$

and the vertical inverse of $\alpha^{-1}$ is $(\alpha^{-1})^* := g^{-1} \alpha f^{-1} : g^{-1} \Rightarrow f^{-1}$. If furthermore $\delta : g \Rightarrow h$ is another 2-cell, then we have in this case

$$\begin{align*}
\delta \circ \alpha & = (\delta \circ \alpha) \text{id}_{\text{id}_X} \\
& = (\delta \circ \alpha) ((\alpha^{-1})^* \circ \alpha^{-1}) \text{id}_f \\
& = ((\delta (\alpha^{-1})^*) \circ (\alpha \alpha^{-1})) \text{id}_f \\
& = ((\delta (\alpha^{-1})^*) \circ \text{id}_{\text{id}_Y}) \text{id}_f \\
& = \delta (\alpha^{-1})^* \text{id}_f = \delta \text{id}_{g^{-1}} \alpha.
\end{align*}$$

(2.1)

An alternative way to carry out such computations is offered by the graphical calculus of string diagrams:

2.2. **Slice categories.** In ordinary category theory, the slice category $\mathcal{C}/T$ over some object $T$ of a category $\mathcal{C}$ has as objects all morphisms
$x: X \rightarrow T$, and as morphisms commutative triangles
\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow x \downarrow \downarrow y \\
T
\end{array}
\]

For (2,1)-categories $\mathcal{C}$, there is the following generalisation in which the equality $x = yf$ is replaced with a 2-cell $x \Rightarrow yf$:

**Definition 2.2.1.** The *slice 2-category* over an object $T \in \mathcal{C}_0$ is the (2,1)-category $\mathcal{C}\{T\}$ with the following data:

- The objects are 1-cells $x: X \rightarrow T$ of $\mathcal{C}$,
- 1-cells between objects $x: X \rightarrow T$ and $y: Y \rightarrow T$ are pairs $(f, \phi)$ consisting of a 1-cell $f: X \rightarrow Y$ and a 2-cell $\phi: x \Rightarrow yf$ that we will usually depict as
\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow x \Downarrow \phi \downarrow y \\
T
\end{array}
\]
- The composition of 1-cells $\phi: x \Rightarrow yf$ and $\psi: y \Rightarrow zg$ is given by $(g, \psi)(f, \phi) := (gf, \psi f \circ \phi)$.
- 2-cells between $\phi: x \Rightarrow yf$ and $\psi: x \Rightarrow yg$ are 2-cells $\xi: f \Rightarrow g$ such that
\[
\psi = y\xi \circ \phi
\]
- Vertical and horizontal composition of 2-cells in $\mathcal{C}/T$ is defined as the vertical (respectively horizontal) composition in $\mathcal{C}$.

See [JY21, Definition 7.1.1(3)] for a diagrammatic depiction of the *ice cream cone condition* (2.3).

**Remark 2.2.2.** As we focus on (2,1)-categories in which all 1-cells $x: X \rightarrow T$ are monic, the 2-cell $\phi$ which is part of a 1-cell $(f, \phi): x \rightarrow y$ in $\mathcal{C}/T$ uniquely determines the 1-cell $f$. We denote the latter by $f_\phi$ and simply write $\phi$ for $(f_\phi, \phi)$.

2.3. **Homotopy categories.** Instead of just forgetting the 2-cells, one can construct an ordinary category out of a (2,1)-category $\mathcal{C}$ by identifying 1-cells if they are related by a 2-cell:

**Definition 2.3.1.** We call $f, g \in \mathcal{C}_1$ *homotopy equivalent* and write $f \sim_h g$ if there exists a 2-cell $\alpha: f \Rightarrow g$. The *homotopy category* $\text{ho}(\mathcal{C})$ of $\mathcal{C}$ is the category with objects $\text{ho}(\mathcal{C})_0 := \mathcal{C}_0$ and morphisms
\[
\text{ho}(\mathcal{C})_1(X, Y) := \mathcal{C}_1(X, Y)/ \sim_h.
\]
Note that [JY21] calls $ho(C)$ the classifying category of $C$, see Example 2.1.27 therein.

Remark 2.3.2. As we assume all 1-cells in $C$ to be monic, the objects of $C/T$ represent the subobjects of $T \in C_0$. By definition, these are equivalence classes $[x]$ with $x: X \to T, y: Y \to T$ being equivalent if and only if there is an invertible 1-cell $f: X \to Y$ such that $x = yf$. A homotopy equivalence between objects $X, Y \in C_0$ is by definition a pair of 1-cells $f: X \to Y, g: Y \to X$ whose classes in $ho(C)$ are inverses of each other, $fg \sim_h id_Y, gf \sim_h id_X$. Thus the isomorphism classes of the objects in $C/T$ are the homotopy classes of subobjects of $T$.

2.4. Embeddings of manifolds. We now define the motivating example for this paper:

Definition 2.4.1. By the (2,1)-category $\text{Mfld}^d$ of embeddings of compact $d$-dimensional manifolds we mean the following:

- The objects of $\text{Mfld}^d$ are compact smooth $d$-dimensional manifolds $X$ with (possibly empty) boundary $\partial X$, together with the empty manifold $\emptyset$.
- A 1-cell in $\text{Mfld}^d$ is an embedding, by which we mean a smooth injective immersion.
- The composition of 1-cells is the ordinary composition of maps.
- A 2-cell $f \Rightarrow g$ between embeddings $f, g: X \to Y$ is an isotopy class $[\phi]$ of isotopies $\phi$ from $f$ to $g$, that is, of smooth maps $\phi: [0, 1] \times X \to Y$ such that the restrictions $\phi(t, -): X \to Y$ are embeddings and for some $\epsilon > 0$, we have
  $$\phi(t, -) = f, \quad \phi(1 - t, -) = g \quad \forall t \in [0, \epsilon].$$
- The horizontal composition of 2-cells is induced by the level-wise composition of isotopies,
  $$(\alpha \beta)(t, p) := \alpha(t, \beta(t, p))$$
  while the vertical composition is induced by the concatenation of the path $\beta(-, p)$ followed by the path $\alpha(-, \beta(1, p))$. The vertical inverse of a 2-cell is taken by inverting the orientation of a path.

See e.g. [Hir94, p. 111] for further details. Note that we do not make any additional assumptions on the behaviour of embeddings on $\partial X$; in particular, we do not assume it embeds $X$ as a neat submanifold in the sense of [Hir94, p. 30]. Note further that the vertical composition of isotopies themselves is not strictly associative; however, since we define 2-cells to be isotopy classes of isotopies, $\text{Mfld}^d$ is indeed a strict 2-category.
2.5. Submanifolds. The slice 2-category $\text{Mfld}^d/T$ describes embeddings of manifolds into an ambient manifold $T$ of the same dimension $d$.

For this entire Section 2.5, we fix embeddings $x: X \to T$ and $y: Y \to T$.

A 1-cell in $(\text{Mfld}^d/T)_1(x, y)$ is represented by an isotopy
\[ \phi: x \Rightarrow y f_{\phi} \]
in $\text{Mfld}^d$, where $f_{\phi}: X \to Y$ is the unique embedding such that
\[ \phi(1, -) = y f_{\phi}. \]

Note that we are in the situation of Remark 2.2.2.

To visualise such 1-cells, it is convenient to introduce their track:

**Definition 2.5.1.** The *track* of an isotopy $\phi$ is the smooth map
\[ \text{track}(\phi): [0,1] \times X \to [0,1] \times T, \quad (t, p) \mapsto (t, \phi(t, p)). \]

**Example 2.5.2.** Figure 2 depicts the track of an isotopy $\phi$ which represents a 1-cell $[\phi] \in (\text{Mfld}^1/S^1)_1(x, y)$. Recall that we work with isotopy classes of isotopies as 2-cells in $\text{Mfld}^d$ rather than isotopies themselves. The isotopies in the class $[\phi]$ share the embeddings $x$ and $y$ of $X$ respectively $Y$ into $S^1$ at the top ($t = 0$) respectively bottom ($t = 1$) of the cylinder. If $\psi \in [\phi]$ is another representative, then track($\psi$) differs for $0 < t < 1$ from track($\phi$) by an isotopy
\[\omega: [0,1] \times [0,1] \times S^1 \to [0,1] \times S^1,\]
so we have
\[\text{track}(\phi)(t, p) = (t, \omega(0, t, p)), \quad \text{track}(\psi)(t, p) = (t, \omega(1, t, p)),\]
as well as
\[\omega(s, 0, p) = x(p), \quad \omega(s, 1, p) = y(p).\]

The vertical composition of 1-cells in $\text{Mfld}^1/S^1$ can be visualised as stacking such cylinders on top of each other.

Assume now that $[\phi], [\psi]: x \to y$ are two 1-cells in $\text{Mfld}^d/T$, and let $f_{\phi}, f_{\psi}$ be the underlying embeddings of $X$ into $Y$. A 2-cell $[\xi]$ in $(\text{Mfld}^d/T)_2([\phi], [\psi])$ is by definition a 2-cell $[\xi]: f_{\phi} \Rightarrow f_{\psi}$ in $\text{Mfld}^d$, so the representative $\xi: [0,1] \times X \to Y$ is an isotopy from $f_{\phi}$ to $f_{\psi}$ satisfying $\psi = (y \xi) \circ \phi$.

**Example 2.5.3.** As in Example 2.5.2 above, we consider $d = 1$ and $T = S^1$. Then the action of a 2-cell $\xi$ can be pictured as in Figure 3. We stress that the action of 2-cells is given by the vertical composition with 1-cells that are not arbitrary but have to be of the form $y \xi$. In Figure 3 this means that for all possible choices of $\xi$, the (grey) track of $y \xi$ will stay within $\text{im} y \subset S^1$, it can not freely use all of $S^1$. 
Figure 2. The track of an isotopy representing a 1-cell $[\phi]: x \to y$ of $\text{Mfld}^1/S^1$. $X = s(x)$ consists of three copies of the interval $[0,1]$, $Y = s(y)$ of two. The thick lines at the top and bottom mark the subsets $\{0\} \times \text{im} \, x$ and $\{1\} \times \text{im} \, y$ of $[0,1] \times S^1$.

Figure 3. The isotopy $\xi: [0,1] \times X \to Y$ between $f_\phi$ and $f_\psi$ represents a 2-cell in $(\text{Mfld}^1/S^1)_2([\phi],[\psi])$.

One observes by direct inspection that the paracyclic category $\mathcal{A}^\circ$ (Definition 1.3.1) with an initial and a terminal object added can be realised as a skeletal subcategory of $\text{ho}(\text{Mfld}^1/S^1)$: the object $n$ of $\mathcal{A}^\circ$
can be identified with any embedding of \( n + 1 \) intervals into \( S^1 \), say
\[
x_n : \bigcup_{j=0}^{n} [j, j + 1/2] \to S^1, \quad t \mapsto \exp \left( \frac{2\pi it}{n + 1} \right).
\]

An isotopy \( \phi \) that represents a 1-cell \( x_n \to x_m \) in \( \text{Mfld}^1/S^1 \) defines unique smooth maps \( \phi_j : [0, 1] \to \mathbb{R} \) with
\[
\phi(t, x_n(j)) = \exp(2\pi i \phi_j(t)), \quad \phi_j(0) = \frac{j}{n + 1}, \quad j = 0, \ldots, n.
\]
Now \( \phi(1, -) \leq x_m \) implies that there is a unique morphism \( f : n \to m \) in \( \Lambda^2 \) such that
\[
\phi_j(1) \in \left[ \frac{f(j)}{m + 1}, \frac{f(j) + 1/2}{m + 1} \right],
\]
and the assignment \( \phi \mapsto f \) induces an isomorphism between the full subcategory of \( \text{ho}(\text{Mfld}^1/S^1) \) consisting of all \( x_n, 0 \leq n < \infty \), and \( \Lambda^2 \).

**Example 2.5.4.** Analogously, together with the empty embedding and \( \text{id}_{[0, 1]} \), the simplicial category \( \Delta \) can be realised as a skeletal subcategory of \( \text{ho}(\text{Mfld}^1/[0, 1]) \).

**Example 2.5.5.** When \( d = 3 \) and \( T = S^3 \), then embeddings of a solid 3-torus are knots, and the 1-cells between them are given by isotopies.

**Example 2.5.6.** Consider the 2-dimensional manifolds
\[
X := \{(a, b) \in \mathbb{R}^2 \mid \sqrt{a^2 + b^2} \leq 1/5\},
\]
\[
Y := \{(a, b) \in \mathbb{R}^2 \mid 1/2 \leq \sqrt{a^2 + b^2} \leq 1\},
\]
\[
T := \{(a, b) \in \mathbb{R}^2 \mid \sqrt{a^2 + b^2} \leq 1\}
\]
and the embeddings
\[
f : X \to Y, \quad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ b + 3/4 \end{pmatrix},
\]
\[
y : Y \to T, \quad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix},
\]
\[
x := yf : X \to T.
\]

The identity 2-cell \( \text{id}_x : x \Rightarrow x \) in \( \text{Mfld}^2 \) yields a 1-cell \( (f, \text{id}_x) : x \to y \) in the slice 2-category \( \text{Mfld}^2/T \) and there is a nontrivial 2-cell \( \xi : f \Rightarrow f \) in \( \text{Mfld}^2 \) which moves the small disc (resp. its embedding) once round the annulus without rotating the disc itself,
\[
\xi(t, -) := R_t f R_{-t} : X \to Y, \quad R_t \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) := \begin{pmatrix} \cos(2\pi t)a + \sin(2\pi t)b \\ -\sin(2\pi t)a + \cos(2\pi t)b \end{pmatrix}.
\]
As $y^\xi$ is (isotopic to) $1d_x$ in $\text{Mfld}^2$, $\xi$ satisfies (2.3) (with $\psi = \phi = 1d_x$). It thus defines a nontrivial 2-cell in $(\text{Mfld}^2/T)_2([f, 1d_x], [f, 1d_x])$. This example also shows that even when $y$ is monic, whiskering on the left with $y$ is not necessarily injective.

3. CYCLIC DUALITY FOR ORBIT 2-CATEGORIES

We now study a certain class of 2-thin (2,1)-categories that generalise the orbit category of a group, and sufficient conditions under which their homotopy categories are self-dual. In the subsequent section, we will realise homotopy categories of suitable slice 2-categories in this way and thus prove our main theorem.

3.1. 2-Groups and Crossed Modules. Throughout this section, $G$ denotes a (strict) 2-group (see e.g. [BL04] for an excellent account on the concept), that is, a (2,1)-category with a single object $T$ in which all 1-cells are invertible. Recall that by the Brown-Spencer theorem [BS76], such a 2-group can be equivalently described as follows:

**Definition 3.1.1.** A crossed module is a pair of group homomorphisms
\[ \mathfrak{t}: G \to A, \quad \mathfrak{a}: A \to \text{Aut}(G) \]
such that for all $h \in A$ and $\gamma, \alpha \in G$, we have
\[ \mathfrak{t}(\mathfrak{a}(h)(\gamma)) = h\mathfrak{t}(\gamma)h^{-1}, \]
that is, $\mathfrak{t}$ is $A$-equivariant with respect to the action $\mathfrak{a}$ on $G$ and the adjoint action on itself, and the Peiffer identity
\[ (\gamma \alpha)^{-1} = \mathfrak{t}(\gamma)\mathfrak{a}t(\gamma)^{-1}. \]

The crossed module that is associated to (and describes) $G$ has
\[ A := G_1(T, T), \]
the group of 1-cells, and
\[ G := \bigcup_{g \in A} G_2(\text{id}_T, g), \]
the so-called source group of \( G \) (with horizontal composition as group structure, cf. Remark 2.1.1). The group homomorphism \( t \) is given by the target map in \( G \), and \( a(h) \) is given by left and right whiskering with \( h \) respectively \( h^{-1} \),
\[ a(h)(\gamma) := h\gamma h^{-1}. \]

**Example 3.1.2.** Every group \( A \) acts on itself by conjugation, and taking \( t := \text{id}_A \) yields a crossed module. The corresponding 2-group has the group \( A \) as its 1-cells and for any \( g, h \in A \) a unique 2-cell \( g \Rightarrow h \).

### 3.2. \( A \)-preorders and \( G \)-categories.

Recall that a preorder is a thin category, i.e. one with at most one morphism between any two objects. We denote such a category \( S \) by \( (S, \subseteq) \), where \( S := S_0 \) is the set of objects and \( \subseteq \) is the reflexive and transitive binary relation on \( S \) that a morphism \( x \to y \) exists. We will write
\[ x \sim y \Leftrightarrow (x \subseteq y \text{ and } y \subseteq x) \]
if \( x, y \in S \) are isomorphic in \( S \).

**Definition 3.2.1.** An \( A \)-preorder is a preorder \( (S, \subseteq) \) with an action of \( A \) on \( S \) such that \( x \subseteq y \Rightarrow gx \subseteq gy \) holds for all \( x, y \in S \) and \( g \in A \).

**Example 3.2.2.** The set \( U_G \) of all subgroups of \( G \) carries a natural action of \( A \) and hence defines an \( A \)-preorder \( (U_G, \subseteq) \).

**Definition 3.2.3.** By a \( G \)-presheaf on an \( A \)-preorder \( (S, \subseteq) \) we shall mean an \( A \)-equivariant map of preorders
\[ s : (S, \subseteq) \to (U_G, \subseteq), \quad x \mapsto G_x, \]
that is, a map such that for all \( x, y \in S \) and all \( g \in A \), we have
\[ G_{gx} = a(g)(G_x), \quad x \subseteq y \Rightarrow G_y \subseteq G_x. \]

Here is a toy example:

**Example 3.2.4.** Let \( A = G \) be any group acting on itself by conjugation (Example 3.1.2), let \( S \) be any \( A \)-set, and
\[ A_x := \{ g \in A \mid gx = x \} \]
be the isotropy group of \( x \in S \). Then \( x \subseteq y \Leftrightarrow A_y \subseteq A_x \) turns \( S \) into a preorder, and the \( \{ A_x \} \) form a \( G \)-presheaf on it.

This generalises as follows to any action of a 2-groups on a category:
Example 3.2.5. Assume that a 2-group \( \mathcal{G} \) acts on a category \( \mathcal{S} \), that is, \( g \in A \) acts by a functor \( \mathcal{S} \to \mathcal{S} \), and \( \gamma : h \Rightarrow g \) in \( \mathcal{G}_2 \) by a natural transformation between the functors given by \( g, h \) (more abstractly, we are given a 2-functor \( \mathcal{G} \to \text{Cat} \) which sends the unique object \( T \) of \( \mathcal{G} \) to \( \mathcal{S} \)). If we denote the components of these natural transformations by \( \gamma_x : hx \Rightarrow gx \), then

\[
G_x := \{ \gamma \in G \mid \gamma_x = \text{id}_x \}, \quad x \leq y \iff G_y \subseteq G_x
\]

turns \( S := \mathcal{S}_0 \) into an \( A \)-preorder equipped with a \( \mathcal{G} \)-presheaf.

In these examples, \( x \leq y \) is defined as \( G_y \subseteq G_x \), but in general, it is a subrelation. The example that we will use to prove our main result arises in this way as a subobject of one of the above type:

Example 3.2.6. Let \( \mathcal{C} \) be a \((2,1)\)-category, \( T \in \mathcal{C}_0 \), and \( \text{Aut}(T) \) be the automorphism 2-group of \( T \), that is, the 2-group of all invertible 1-cells \( T \to T \) and of all 2-cells between these. As a special case of the preceding Example 3.2.5, \( \text{Aut}(T) \) acts on the category underlying the slice 2-category \( \mathcal{C}/T \), with the action \( gx \) of \( g \in A = \text{Aut}(T)_1 \) on \( x \in \mathcal{C}/T_0 \) given by composition, and with \( \gamma_x := \gamma x \) (\( \gamma \) whiskered on the right by \( x \)). Note that this \( \text{Aut}(T) \)-action descends to \( \text{ho}(\mathcal{C}/T) \).

As above, we set \( G_x := \{ \gamma \in G \mid \gamma x = x \} \). In this way, we obtain an \( \text{Aut}(T) \)-presheaf

\[
s_{\mathcal{C}/T} : (\mathcal{C}/T_0, \leq) \to (U_G, \supseteq)
\]

on the ordinary slice category of \( \mathcal{C} \) over \( T \), so here

\[
x \leq y \iff \exists f \in \mathcal{C}_1 : x = yf.
\]

We will show that in good cases, \( \text{ho}(\mathcal{C}/T) \) only depends on \( s_{\mathcal{C}/T} \).

Remark 3.2.7. To paint a more complete picture, one could define a 2-category \( A\text{-Pro} \) of \( A \)-preorders and a 2-category \( \mathcal{G}\text{-Psh} = A\text{-Pro}/U_G \) of \( \mathcal{G} \)-presheaves. Example 3.2.5 then becomes part of the definition of a 2-functor \( \mathcal{G}\text{-Cat} \to \mathcal{G}\text{-Psh} \), where \( \mathcal{G}\text{-Cat} \) is the 2-category of all categories with an action of \( \mathcal{G} \). Note also that \( \mathcal{G}\text{-Psh} \) does not make use of the target map \( t \) of \( \mathcal{G} \), the construction works for all actions of a group \( A \) on another group \( G \). As this is not directly relevant for our main problem, we decided not to work this out here though.

3.3. Orbit categories. Let \( s : (S, \subseteq) \to (U_G, \supseteq) \) be a \( \mathcal{G} \)-presheaf.

Remark 3.3.1. The group \( G \) acts via the target map \( t : G \to A \) on \( S \), and to avoid confusion, we denote this action by

\[
\gamma \triangleright x := t(\gamma)x, \quad \gamma \in G, x \in S.
\]
Note that the Peiffer identity (3.1) implies
\[ G_{\gamma \triangleright x} = G_{t(\gamma) x} = a(t(\gamma))(G_x) = \gamma G_x \gamma^{-1}. \]
Here and later, we freely use that the product
\[ LR := \{ \alpha \beta \in G \mid \alpha \in L, \beta \in R \} \]
turns the power set \( P(G) \) of \( G \) into a monoid with unit element \( \{1\} \). Note that subgroups of \( G \) are idempotent elements of \( P(G) \). Expressions such as \( \gamma G_x \gamma^{-1} \) are a shorthand notation for \( \gamma \gamma G_x \gamma^{-1} \).

As we will explain in Example 3.3.4 below, the following construction generalises the (dual of the) orbit category of a group (see e.g. [tD87, Section I.10]). We will denote the composition of morphisms using \( \circ \) to distinguish it from the product of elements or subsets of \( G \) which we continue to denote simply by concatenation. Note also that \( G/G_x \) refers to the set of cosets \( \gamma G_x \) of the subgroup \( G_x \), not a slice 2-category.

**Proposition 3.3.2.** The following defines a \( G \)-category \( \mathcal{I}_s \):

- the set of objects is \( \mathcal{I}_{s,0} := S \),
- the morphism sets are \( \mathcal{I}_{s,1}(x, y) := \{ \gamma G_x \in G/G_x \mid \gamma \triangleright x \subseteq y \} \),
- the composition is induced by the product in \( G \); that is, if \( \gamma G_x : x \to y, \delta G_y : y \to z \) are morphisms, then
  \[ (\delta G_y) \circ (\gamma G_x) := \delta \gamma G_x, \]
- the functor given by \( h \in A \) acts on objects via the original action on \( \mathcal{I}_{s,0} = S \) and on morphisms by
  \[ \mathcal{I}_{s,1}(x, y) \to \mathcal{I}_{s,1}(hx, hy), \quad \gamma G_x \mapsto (h \gamma h^{-1})G_{hx}, \]
- and the natural transformation assigned to \( \gamma \in G \) has components \( \gamma_x := \gamma G_x : x \to \gamma \triangleright x \).

**Proof.** Note first that the composition is equal to the multiplication of subsets of \( G \): \( \gamma \triangleright x \subseteq y \) implies \( G_y \subseteq G_{\gamma \triangleright x} = \gamma G_x \gamma^{-1} \) so that
\[ \delta G_y \gamma G_x \subseteq \delta \gamma G_x \gamma^{-1} \gamma G_x = \delta \gamma G_x G_x = \delta \gamma G_x = (\delta G_y) \circ (\gamma G_x). \]
The reverse inclusion holds as \( 1 \in G_y \), so \( \delta \gamma G_x \subseteq \delta G_y \gamma G_x \).

It follows that the definition of \( \delta G_y \circ \gamma G_x \) is independent of the choice of representatives \( \delta, \gamma \) and also that \( \circ \) is associative.

Furthermore, \( \delta \triangleright y \subseteq z \) and \( \gamma \triangleright x \subseteq y \) together imply \( (\delta \gamma) \triangleright x \subseteq \delta \triangleright y \), hence \( (\delta \gamma) \triangleright x \subseteq \delta \triangleright y \subseteq z \). Thus \( \delta \gamma G_x \) is a morphism \( x \to z \) in \( \mathcal{I}_s \).

That the \( \mathcal{G} \)-action is well-defined is verified straightforwardly – for the naturality of the transformations \( \gamma_x \) use the Peiffer identity (3.1); note also that \( \gamma_x \) is an isomorphism with inverse \( \gamma^{-1} G_{\gamma \triangleright x} \). \( \square \)

As an immediate consequence of the definition, we have:
Corollary 3.3.3. All $\gamma G_x \in \mathcal{I}_{s,1}(x, y)$ are monic.

Example 3.3.4. Consider $G = A$ as in Example 3.1.2 and $s = \text{id}_{U_A} : U_A \to U_A, \ x \mapsto A_x = x$.

In this case, $\mathcal{I}_{\text{id}_{U_A}}$ has the subgroups $x \subseteq A$ as objects, and

$$\mathcal{I}_{s,1}(x, y) = \{gx \in A/x \mid y \subseteq gxg^{-1}\}.$$  

Note $gx$ is a coset of the subgroup $x$. The orbit category (see e.g. [tD87, Section I.10]) of $A$ is the category whose objects are the coset spaces $A/x, x \in U_A$, and whose morphisms $g : A/y \to A/x$ are $A$-equivariant maps. Such a map $g$ is uniquely determined by its value on $y = 1y \in A/y$, and an element $gx \in A/x$ occurs as such a value if and only if $y \subseteq gxg^{-1}$. Thus $\mathcal{I}_s$ is (isomorphic to) the dual of the orbit category.

Thus we define in full generality:

Definition 3.3.5. We call $\mathcal{I}_s^*$ the orbit category of $s$.

We will show later that under the assumptions in our main theorem, the category underlying $\mathcal{C}/T$ is isomorphic to $\mathcal{I}_{s_{\mathcal{C}/T}}$ (recall Example 3.2.6). This is why we focus on $\mathcal{I}_s$ rather than its dual.

Remark 3.3.6. Expanding Remark 3.2.7, the assignment $s \mapsto \mathcal{I}_s$ can be made part of a 2-functor $\mathcal{G}\text{-Psh} \to \mathcal{G}\text{-Cat}$. The construction from Example 3.2.5 almost recovers $s$ from $\mathcal{I}_s$, only the relation $\subseteq$ is replaced by the potentially weaker one meaning $G_y \subseteq G_x$. So there is a natural transformation from the identity 2-functor on $\mathcal{G}\text{-Psh}$ to the composition of the two constructions. As far as we can see, this is in general not part of a 2-adjunction: starting with any $\mathcal{G}$-category $\mathcal{S}$ and defining $s$ as in Example 3.2.5, the morphisms in the resulting $\mathcal{G}$-category $\mathcal{I}_s$ are generated by the $\gamma_x$ together with virtual embeddings $x \hookrightarrow y$ whenever $G_y \supseteq G_x$. These might not correspond to any actual morphisms in $\mathcal{S}$, and, conversely, there might by morphisms in $\mathcal{S}$ that are entirely unrelated to the $\mathcal{G}$-action (e.g. when $\mathcal{G}$ is nontrivial but acts trivially on $\mathcal{S}$). However, our main result roughly says that for $\mathcal{S} = \mathcal{C}/T$ with the necessary transitivity conditions added in the form of the homotopy extension property discussed below, $\mathcal{S}$ can be reconstructed from $\mathcal{I}_s$. So we believe there is a class of well-behaved $\mathcal{G}$-categories for which the two 2-functors form a split adjunction of suitable 2-categories.

3.4. Self-dual $A$-preorders. The aim of the remainder of Section 3 is to upgrade $\mathcal{I}_s$ to a (2,1)-category with a self-dual homotopy category. In order to do so, we need to assume the presence of two additional structures on the underlying $A$-preorders. The first one is an $A$-equivariant self-duality:
Definition 3.4.1. An $A$-self-duality on an $A$-preorder $(S, \preceq)$ is a map $S \rightarrow S$, $x \mapsto x^\circ$ such that for all $x, y \in S$ and $g \in A$, we have
\begin{equation}
 x \sim x^\circ, \quad x \preceq y \iff y^\circ \subseteq x^\circ, \quad (gx)^\circ \sim g(x^\circ).
\end{equation}

Here is the main example that we have in mind:

Example 3.4.2. Consider $s\mathcal{C}/T$ (Example 3.2.6) with $\mathcal{C} = \text{Mfld}^d$, so $(\mathcal{C}/T)_0$ consists of all embeddings $x : X \rightarrow T$ of a manifold $X$ into a compact manifold $T$ of the same dimension $d$. The preorder relation $x \preceq y \iff \exists f : x = yf$ means that $\text{im} x \subseteq \text{im} y$, and if $T$ has empty boundary, then the inclusion $x^\circ$ of the closure of the complement $T \setminus \text{im} x$ into $T$ is a $\text{Diff}(T)$-self-duality.

Bear in mind though that our setting is quite general. In particular, $x^\circ$ is in general not a complement in most of the standard meanings of the word. Here is a somewhat fake example which shows amongst other things that $x$ and $x^\circ$ do not need to be jointly epic:

Example 3.4.3. Let $\mathcal{C}$ be the (2,1)-category whose objects are the intervals of the form $(-\infty, t]$ and $[s, \infty)$, $s, t \in \mathbb{R}$, plus $\emptyset$ and $T = \mathbb{R}$, whose 1-cells are inclusions (so $\mathcal{C}$ is a preorder and in fact a poset), and all of whose 2-cells are identities (so $G = A$ is trivial). Then
\begin{align*}
 [s, \infty)^\circ & := (-\infty, s - 1], \quad (-\infty, t]^\circ := [t + 1, \infty) \\
 [s, \infty)^* & := (-\infty, s + 1], \quad (-\infty, t]^* := [t - 1, \infty)
\end{align*}
both define self-dualities on the preorder $(\mathcal{C}/T_0, \preceq)$, but we have
\begin{equation}
 \text{im} x \cup \text{im} x^\circ \neq \mathbb{R}, \quad \text{im} x \cap \text{im} x^* \neq \emptyset.
\end{equation}

Finally, here are two examples of a very different nature:

Example 3.4.4. Let $G = A$ be any group, viewed as a 2-group as in Example 3.1.2, and let $S$ be any $A$-set. If $H \lhd G$ is a normal subgroup, setting $G_x := H$ for all $x \in S$ and $x \sim y$ for all $x, y$ defines a $G$-presheaf, and $x^\circ := x$ is an $A$-self-duality.

Example 3.4.5. Let again $G = A$ be any group, $H, K$ be normal subgroups, and $S := \langle H, K \rangle$ with trivial $G$-action, and set $G_x := x$, $x \preceq y \iff G_x \supseteq G_y$. Then $H^\circ := K, K^\circ := H$ defines an $A$-self-duality.

In this example, $G_x^\circ$ is not necessarily contained in the centraliser $Z_G(G_x)$, but note that we always have:
Lemma 3.4.6. If $s$ is a $G$-presheaf and $\circ$ is a self-duality on the underlying $A$-preorder, then we have

$$N_G(G_x) = N_G(G_{x^e}).$$

In particular, $G_x \subseteq N_G(G_{x^e}).$

Proof. By (3.4) and (3.3), $G_x = G_{\gamma \circ x}$ implies

$$G_{x^e} = G_{(\gamma \circ x)^e} = G_{\gamma \circ x^e}.$$  \[ \Box \]

Corollary 3.4.7. $G_x G_{x^e} = G_x G_x$ is a subgroup of $G$. If, in addition, we have $G_x \cap G_{x^e} = \{1\}$, then $G_x G_{x^e} \cong G_x \times G_{x^e}$.

3.5. $A$-cosieves. The second ingredient we use for the (2,1)-upgrade of $I_s$ is a cosieve in the $A$-preorder underlying $s$ (recall that a cosieve in a category is just a set of morphisms closed under postcomposition with any morphism). This provides an abstract concept of an “interior” of a subobject; like the self-duality it should be compatible the $A$-action:

Definition 3.5.1. A binary relation $\subseteq$ is an $A$-cosieve in the $A$-preorder $(S, \subseteq)$ if for all $x, y, z \in S$ with $x \subseteq y$ and all $g \in A$, we have

$$x \subseteq y, \quad gx \subseteq gy, \quad y \subseteq z \Rightarrow x \subseteq z.$$  

Note that this implies:

Lemma 3.5.2. If $y \sim z$, then $x \subseteq y \leftrightarrow x \subseteq z$.

Again, we first consider the example that motivates the definition:

Example 3.5.3. Consider $s_{\text{Mfld}^d/T}$ (Example 3.2.6, Example 3.4.2). Then the relation $x \ll y$ ($x : X \to T, y : Y \to T$ embeddings of manifolds) that im $x$ is contained in the interior of im $y$ (i.e. the boundary of im $x$ does not intersect the boundary of im $y$) is a Diff$(T)$-cosieve.

Example 3.5.4. Any self-duality defines a cosieve

$$x \subseteq y :\iff (x \subseteq y \text{ and } \langle G_{y^e} \cup G_x \rangle = G),$$

where the right hand side denotes the fact that $G_{y^e}$ and $G_x$ together generate $G$ as a group. In the preceding Example 3.5.3 that we are most interested in, this is, however, a stricter relation than $\ll$: if im $x$ is not properly contained in the interior of im $y$, then it has a nontrivial intersection with im $y^e$, the closure of the complement of im $y$ in $T$. Thus this intersection contains at least one point $p$ which is fixed by all elements of $\langle G_{y^e} \cup G_x \rangle$ and in particular by their codomains $g : T \to T$. However, for each point in a manifold there is some diffeomorphism that is isotopic to id$_T$ and that moves this point, so $\langle G_{y^e} \cup G_x \rangle \subseteq G$. Therefore, we have $x \subseteq y \Rightarrow x \ll y$, but the converse does not hold in general. In
particular, if im \( x \subset T = S^1 \) is a nonempty proper submanifold, then \( G_x \) consists of isotopy classes of isotopies \( \gamma: \text{id}_{S^1} \Rightarrow g \in \text{Diff}(S^1) \) with \( \gamma x = \text{id}_x \), so \( g \) belongs to the subgroup \( \text{Diff}(S^1)|_x \) of diffeomorphisms that restrict to the identity on im \( x \). Since the complement of im \( x \) is contractible and we consider isotopy classes of isotopies rather than isotopies themselves, such \( \gamma \) are uniquely determined by \( g \). That is, \( t: G_x \to \text{Diff}(S^1)|_x \) is a group isomorphism. So if \( x \preceq y \) for im \( y \subsetneq S^1 \), then \( \langle G_y \cap G_x \rangle \) does not contain the 2-cell \( \text{id}_{S^1} \Rightarrow \text{id}_{S^1} \) that is represented by the isotopy

\[
(3.5) \quad [0,1] \times S^1 \to S^1, \quad (t,e^{2\pi is}) \mapsto e^{2\pi i(s+t)}
\]

that rotates the entire circle once, so we do not have \( x \preceq y \).

Here is an abstract algebraic example that illustrates that the behaviour of \( \preceq \) can be very different from what one might expect:

**Example 3.5.5.** Let \( A = G \) be a group viewed as a 2-group (Example 3.1.2) that acts trivially on a set \( S \). If \( \{A_x\}_{x \in S} \) is a family of normal subgroups with \( A_x \subseteq A_y \Leftrightarrow x = y \), then we obtain an \( A \)-preorder with a \( G \)-presheaf by setting \( x \preceq y \Leftrightarrow x = y \), and an \( A \)-self-duality by setting \( x^\circ := x \) for all \( x \). Assume furthermore that for any \( x \neq y \), \( \langle A_x \cup A_y \rangle = A \). As a concrete example, we can take \( A = \mathbb{Z} \), \( S = \) the set of prime numbers, and \( A_x = \langle x \rangle \) the group of all integers divisible by \( x \). Then there are no \( x, y \in S \) with \( x \preceq y \) at all.

Before we move on to the construction of the \((2,1)\)-category \( \mathcal{I}_s \), we briefly discuss for the example \( (\mathcal{C}/T_0, \preceq) \) the close relation between Aut\((T)\)-cosieves in the Aut\((T)\)-preorder and cosieves in \( \mathcal{C} \) itself:

**Proposition 3.5.6.** If \( \preceq \) is an Aut\((T)\)-cosieve in \( (\mathcal{C}/T_0, \preceq) \), then

\[
S_\preceq := \{f \in \mathcal{C}_1 \mid \forall y \in \mathcal{C}_1 : s(y) = t(f) \Rightarrow yf \preceq y\};
\]

\[
S^\preceq := \{hf \mid h: Y \to Z, f: X \to Y, \text{ and } \exists y: Y \to T : yf \preceq y\}.
\]

are cosieves \( S_\preceq \subseteq S^\preceq \) in \( \mathcal{C} \). Conversely, if \( S \) is any cosieve in \( \mathcal{C} \), then

\[
x \preceq_S y :\Leftrightarrow \exists f \in S : x = yf
\]

is an Aut\((T)\)-cosieve, and we have \( S_\preceq_S = S^\preceq_S = S \) as well as

\[
x \preceq_{S_\preceq} y \Rightarrow x \preceq y \Rightarrow x \preceq_{S^\preceq} y.
\]

**Proof.** Let \( \preceq \) be an Aut\((T)\)-cosieve. Given \( f: X \to Y \) in \( S_\preceq \) and \( h: Y \to Z \), we need to show \( hf \in S_\preceq \), that is, that for all \( z: Z \to T \) we have \( zhf \preceq z \). To see this, set \( y := zh, x := yf = zhf \). Then on the one hand, we have \( x = yf \preceq y \) by the definition of \( S_\preceq \), and on the other hand \( y = zh \preceq z \) by the definition of \( \preceq \). Since \( \preceq \) is a cosieve,
this implies $x \ll z$ as required. That $\bar{S}_{\ll}$ is a cosieve is immediate (it is the cosieve generated by all $f$ with $yf \ll y$ for some $y$).

Conversely, if $S$ is a cosieve and $x \ll_S y$, then evidently $x \leq y$ and $gx \ll_S gy$ (as $gx = gyf \iff x = yf$ for all $g \in \Aut(T)$). Also if $y \leq z$ with $y = zh$ for some $h : Y \to Z$, then $x = yf = zhf$ and as $hf \in S$ ($S$ is a cosieve), $x \ll_S z$. So $\ll_S$ is an $\Aut(T)$-cosieve.

We have

$$S_{\ll_S} = \{ f : X \to Y \mid \forall y : Y \to T \exists \bar{f} \in S : yf = y\bar{f} \}$$

and as all 1-cells are monic, we evidently have $S_{\ll_S} = S$. Similarly, $\bar{S}_{\ll_S} = S$. Conversely, $x \ll_{\bar{S}_{\ll_S}} y \Rightarrow x \ll y$ follows immediately from

$$x \ll_{\bar{S}_{\ll_S}} y \iff \exists f : (x = yf \ and \ \forall \bar{y} : Y \to T : \bar{y}f \ll \bar{y}).$$

The implication $x \ll y \Rightarrow x \ll_{\bar{S}_{\ll_S}} y$ follows analogously from

$$x \ll_{\bar{S}_{\ll_S}} y \iff \exists f : (x = yf \ and \ \exists \bar{y}, h, g : \bar{y}g \ll \bar{y}, f = hg),$$

just take $g = f, \bar{y} = y, h = \id_{S(y)}$. \hfill $\square$

Here is an example of an $A$-cosieve that is not of the form $\ll_S$:

**Example 3.5.7.** Let $C$ be the $(2,1)$-category of all sets with injective but not surjective maps plus identities as 1-cells and all 2-cells being identities. If $T$ is any set, 1-cells in $C\{T\}$ are given by the relation $x \ll y \iff \im x \subseteq \im y$; the 2-cells are all identities. The group $\Aut(T)$ is trivial, all 1-cells in $C$ are monic, and any $t \in T$ defines a cosieve

$$x \ll y :\iff x \leq y \text{ and } t \in \im y \setminus \im x,$$

but $S_{\ll}$ (and hence $\ll_{S_{\ll}}$) is empty, while $\bar{S}_{\ll}$ consists of all 1-cells that are not identities.

Note, however, that in our main example, $\ll$ is of the form $\ll_S$:

**Example 3.5.8.** If $C = \text{Mfld}^d$, then the set of all embeddings $X \to Y$ of a manifold $X$ into the interior of $Y$ is a cosieve, hence $\ll$ from Example 3.5.3 is a $\Diff(T)$-cosieve given by a cosieve in $C$.

3.6. **Orbit 2-categories.** We now show that the choice of an $A$-cosieve $\sqsubseteq$ and of an $A$-self-duality $\circ$ upgrades $\mathcal{I}_s$ to a $(2,1)$-category which is $2$-thin (contains at most one 2-cell between any two 1-cells):

**Proposition 3.6.1.** Let $s : (S, \sqsubseteq) \to (U_G, \equiv)$ be a $G$-presheaf, $\circ$ be an $A$-self-duality, and $\sqsubseteq$ be an $A$-cosieve. Then we have:

1. The relation

$$\gamma_{G_x} \equiv \varepsilon_{G_x} :\iff (\forall u \in S : u \sqsubseteq y \Rightarrow G_u \cap \varepsilon G_x \gamma^{-1} \neq \emptyset)$$

is an equivalence relation on the morphism set $\mathcal{I}_s(x, y)$. 
(2) Interpreting \( \equiv \) as a 2-cell turns \( \mathcal{I}_s \) into a (2,1)-category.
(3) The \( \mathcal{G} \)-action on \( \mathcal{I}_s \) induces an \( \mathcal{G} \)-action on \( \text{ho}(\mathcal{I}_s) \).

**Proof.** For all \( u \in S \), we have \( 1 \in G_u \cap \gamma G_x \gamma^{-1} \), hence
\[
\gamma G_x \equiv \gamma G_x.
\]
Next, if \( \alpha \in G_u \cap \varepsilon G_x \gamma^{-1} \), then \( \alpha^{-1} \in G_u \cap \gamma G_x \varepsilon^{-1} \), so
\[
\gamma G_x \equiv \varepsilon G_x \Rightarrow \varepsilon G_x \equiv \gamma G_x.
\]
Finally, if \( \alpha \in G_u \cap \varepsilon G_x \gamma^{-1} \) and \( \beta \in G_u \cap \rho G_x \varepsilon^{-1} \), then we obtain
\[
\beta \alpha \in G_u \cap \rho G_x \gamma^{-1},
\]
so
\[
\gamma G_x \equiv \varepsilon G_x, \varepsilon G_x \equiv \rho G_x \Rightarrow \gamma G_x \equiv \rho G_x.
\]
So \( \equiv \) is an equivalence relation and if we interpret it as a 2-cell, there is a (necessarily unique and associative) vertical composition of 2-cells (which in a sense is induced by the product in \( G \)), there is an identity 2-cell for each 1-cell \( \gamma G_x \) (represented by 1), and all 2-cells are invertible.

Up to here no properties of \( \equiv \) have been used. However, they are required to establish a horizontal composition of 2-cells \( \gamma G_x \equiv \varepsilon G_x \) and \( \delta G_y \equiv \lambda G_y \) for two other 1-cells \( \delta G_y, \lambda G_y : y \to z \).

We have to show \( \delta \gamma G_x \equiv \lambda \varepsilon G_x \). To do so, assume \( u \in S \) with \( u \equiv z^\circ \). Then as \( \delta G_y \equiv \lambda G_y \), there exists an element
\[
\mu \in G_u \cap \lambda G_y \delta^{-1}.
\]
Since \( \varepsilon G_x \) is a 1-cell \( x \to y \), we have \( \varepsilon \cdot x \equiv y \Rightarrow G_y \subseteq \varepsilon G_x \varepsilon^{-1} \), so we also have
\[
(3.6) \quad \mu \in G_u \cap \lambda \varepsilon G_x \varepsilon^{-1} \delta^{-1}.
\]
Furthermore, as \( \delta G_y \) is a 1-cell \( y \to z \), we have \( \delta \cdot y \equiv z \Rightarrow z^\circ \subseteq \delta \cdot y^\circ \).

Thus \( u \equiv z^\circ \) implies \( u \subseteq \delta \cdot y^\circ \Rightarrow \delta^{-1} \cdot u \subseteq y^\circ \) and as \( \gamma G_x \equiv \varepsilon G_x \), there exists some
\[
\alpha \in G_{\delta^{-1} \cdot y \circ} \cap \varepsilon G_x \gamma^{-1},
\]
which means
\[
\delta \alpha \delta^{-1} \in G_u \cap \delta \varepsilon G_x \gamma^{-1} \delta^{-1}.
\]
In combination with (3.6) we conclude
\[
\mu \delta \alpha \delta^{-1} \in G_u \cap \lambda \varepsilon G_x \gamma^{-1} \delta^{-1},
\]
so this set is not empty as we had to show in order to establish the horizontal composition \( \delta \gamma G_x \Rightarrow \lambda \varepsilon G_x \) of \( \gamma G_x \Rightarrow \varepsilon G_x \) with \( \delta G_y \Rightarrow \lambda G_y \).

Due to the uniqueness of 2-cells, the horizontal composition is automatically associative and satisfies the exchange law.
Last but not least, the action of $A$ on $\mathcal{I}_s$ defined in Proposition 3.3.2 descends to $\text{ho}(\mathcal{I}_s)$, since for all $h \in A$, we have
\[
\gamma G_x \equiv \varepsilon G_x \Leftrightarrow (h \gamma h^{-1} G_h) x \equiv (h \varepsilon h^{-1}) G_h x,
\]
simply conjugate all of $G_u \cap \varepsilon G_x \gamma^{-1} \neq \emptyset$ by $h$. Similarly, the natural transformations $\gamma_x = \gamma G_x$ descend to $\text{ho}(\mathcal{I}_s)$.

\textbf{Example 3.6.2.} We keep extending Examples 3.4.2 and 3.5.3 with $\mathcal{C} = \text{Mfld}^d$, $T$ a compact $d$-dimensional manifold without boundary. Two 1-cells $\gamma G_x, \varepsilon G_x$: $x \to y$ in $\mathcal{I}_{s\mathcal{C}/T}$ are represented by isotopy classes of isotopies $\gamma: \text{id}_T \Rightarrow g, \varepsilon: \text{id}_T \Rightarrow e$ with $gx \subseteq im y, \text{im } ex \subseteq im y$. To distinguish this from the generic case, we write here $\bullet$ instead of $\circ$; recall that $\gamma \bullet x = gx, \varepsilon \bullet x = ex, y \leq z \Leftrightarrow \text{im } y \subseteq \text{im } z$. The group $G_x$ contains the isotopies whose restriction $\gamma x$ to im $x$ is (isotopic to) the constant isotopy with value $\text{id}_{im x}$. So we can identify $\gamma G_x$ via the assignment
\[
(3.7) \quad \gamma G_x \mapsto \gamma x: x \Rightarrow gx
\]
with the restriction of $\gamma$: $\text{id}_T \Rightarrow g$ to im $x$. The coset $\gamma G_x$ defines a 1-cell $\gamma G_x$: $x \to y$ whenever $im gx \subseteq im y$. Now $\gamma G_x \equiv \varepsilon G_x$ means that for all submanifolds im $u \subseteq T$ that are disjoint from im $y$ (i.e. $u \ll y$) there exists a 2-cell $\alpha: \text{id}_T \Rightarrow a$ for some diffeomorphism $a: T \to T$ which is on the one hand in $G_u$, that is, the restriction of $\alpha$ to im $u$ is constantly equal to the identity,

$$
\alpha(t, p) = p \quad \forall t \in [0, 1], p \in \text{im } u \subseteq T \setminus \text{im } y.
$$

At the same time, $\alpha \in \varepsilon G_x \gamma^{-1} = \varepsilon \gamma^{-1} G_{x^\circ}$ means that $\alpha x$ is a 2-cell that composes with $\gamma x$ to $\varepsilon x$. The upshot is that $\gamma G_x \equiv \varepsilon G_x$ means that for any choice of submanifold im $u$ that is disjoint from im $y$, we find an isotopy that fixes im $u$ pointwise and deforms the embedding $gx: X \to T$ inside the complement of im $u$ to $ex: X \to T$.

\textbf{3.7. Lifting of self-dualities.} Let $s: (S, \sqsubseteq) \to (U_G, \vDash)$ be a $G$-presheaf and $(S, \sqsubseteq)$ be self-dual. If $\gamma G_x: x \to y$ is a morphism in $\mathcal{I}_s$, then $\gamma \circ x \sqsubseteq y$. As $\circ$ is a self-duality, $y^\circ \sqsubseteq (\gamma \circ x)^\circ \gamma^{-1} G_{x^\circ}$, hence $\gamma^{-1} \circ y^\circ \sqsubseteq x^\circ$ and there is a morphism $\gamma^{-1} G_{y^\circ}: y^\circ \to x^\circ$. However, in general this does not lead to a self-duality of $\mathcal{I}_s$ itself; $\gamma G_x \to \gamma^{-1} G_{y^\circ}$ is not well-defined unless $G_x \sqsubseteq G_{y^\circ}$. However, on $\text{ho}(\mathcal{I}_s)$ we obtain a somewhat satisfactory resolution.

One easily verifies that $\gamma G_x = \varepsilon G_x$ implies $\gamma^{-1} G_{y^\circ} \equiv \varepsilon^{-1} G_{y^\circ}$, but in general, $\gamma G_x \equiv \varepsilon G_x$ does not necessarily imply $\gamma^{-1} G_{y^\circ} \equiv \varepsilon^{-1} G_{y^\circ}$. We now formulate a technical condition that ensures it is; we will explain in Example 3.7.3 that this is an abstract replacement of the existence of a tubular neighbourhood of a submanifold.
Proposition 3.7.1. Assume that $\circ$ is an $A$-self-duality, $\mathcal{E}$ is an $A$-cosieve, and that for all $b, c, d \in S$ we have

$$(c \in b^\circ \text{ and } d \in b^\circ)$$

$$\Rightarrow \exists \rho \in G_c \cap G_d, a \in S : (\rho \triangleright a \sim b \text{ and } a \in b).$$

If $\gamma G_x \equiv \varepsilon G_x : x \to y$ in $\mathcal{I}_a$, then we have $\gamma^{-1}G_{y^\circ} \equiv \varepsilon^{-1}G_{y^\circ} : y^\circ \to x^\circ$.

Proof. We need to show

$$\gamma^{-1}G_{y^\circ} \equiv \varepsilon^{-1}G_{y^\circ} \iff \forall v \in x : G_v \cap \varepsilon^{-1}G_{y^\circ} \gamma \neq \emptyset$$

$$\iff \forall v \in x : \varepsilon G_v \gamma^{-1} \cap G_{y^\circ} \neq \emptyset.$$

We are going to apply (3.8) with

$$b = y^\circ, \quad c = \gamma \triangleright v, \quad d = \varepsilon \triangleright v,$$

so we need to show $(\gamma \triangleright v) \equiv y^\circ$ and $(\varepsilon \triangleright v) \equiv y^\circ$. To do so, recall once more that $\gamma G_x, \varepsilon G_x : x \to y$ are 1-cells, so $(\gamma \triangleright x) \equiv y, (\varepsilon \triangleright x) \equiv y$. Together with $v \in x \Rightarrow \gamma \triangleright v \in \gamma \triangleright x$ and $v \in x \Rightarrow \varepsilon \triangleright v \in \varepsilon \triangleright x$ we conclude $\gamma \triangleright v \in y, \varepsilon \triangleright v \in y$, and now we can use $y^\circ \sim y$ and Lemma 3.5.2.

So by (3.8), there are $\rho \in G, a \in S$ satisfying

$$\rho \in G_{\gamma \triangleright v} \cap G_{\varepsilon \triangleright v}, \quad \rho \triangleright y^\circ \sim a, \quad a \in y^\circ.$$

Now we use $\gamma G_x \equiv \varepsilon G_x$ with $u := a$. This shows that

$$G_a \cap \varepsilon G_x \gamma^{-1} = G_{\rho \triangleright y^\circ} \cap \varepsilon G_x \gamma \neq \emptyset$$

which implies (conjugate with $\rho^{-1}$ and use $v \in x \Rightarrow v \in x \Rightarrow G_x \subseteq G_v$)

$$G_{y^\circ} \cap \rho^{-1} \varepsilon G_v \gamma^{-1} \rho \neq \emptyset.$$

As $\rho$ and hence $\rho^{-1}$ is both in $G_{\gamma \triangleright v}$ and $G_{\varepsilon \triangleright v}$ we finally have

$$\rho^{-1} \varepsilon G_v \gamma^{-1} \rho = \rho^{-1} G_{\varepsilon \triangleright x} \varepsilon \gamma^{-1} \rho$$

$$= G_{\varepsilon \triangleright v} \varepsilon \gamma^{-1} \rho$$

$$= \varepsilon \gamma^{-1} G_{\gamma \triangleright v} \rho$$

$$= \varepsilon \gamma^{-1} G_{\varepsilon \triangleright v}$$

$$= \varepsilon G_v \gamma^{-1}. \quad \square$$

Corollary 3.7.2. Under the assumptions of Proposition 3.7.1, $\text{ho}(\mathcal{I}_a)$ is a self-dual category, with the dual of $[\gamma G_x] : x \to y$ given by

$$[\gamma G_x]^\circ := [\gamma^{-1}G_{y^\circ}] : y^\circ \to x^\circ.$$
Example 3.7.3. For \( s_{\text{Mfld}^d/T} \), (Example 3.6.2), \( b, c, d \) correspond to submanifolds \( B := \text{im} \, b, C := \text{im} \, c, D := \text{im} \, d \subseteq T \) with

\[
B \cap C = B \cap D = \emptyset,
\]
so \( C, D \) are both contained in the interior of \( T \setminus B \) (as \( c \in b^\circ, d \in b^\circ \)). Condition (3.8) asserts the existence of an isotopy \( \rho: \text{id}_T \Rightarrow r \) for some diffeomorphism \( r: T \to T \) such that \( \rho c \) is constantly \( \text{id}_C \) and \( \rho d \) is constantly \( \text{id}_D \) while \( \rho \) shrinks \( B \) to a manifold \( A = \text{im} \, a \) contained in the interior of \( B \) which is however diffeomorphic to \( B \) via \( r \). To obtain such an isotopy, choose a Riemannian metric on \( T \). Extend the outward normal vectors of length 1 on \( B \subset T \) to a vector field on all of \( T \) that is only supported on a small neighbourhood of \( \partial B \) disjoint from \( C \) and \( D \) (using e.g. a partition of 1 and bump functions). Following the inverse flow of this vector field for times in a sufficiently small closed time interval yields \( \rho \) (or rather \( \rho^\circ \)).

4. Application to slice 2-categories

Our main theorem follows more or less immediately from the results above, but we discuss its assumptions and our main example in more detail, as well as the canonical choice of the relation \( \ll \). Throughout, we make Assumptions 1 and 2, and \( A = \text{Aut} (T), G, G_x, \triangleright, \leq \) and \( s_{\mathcal{C}/T} \) are as in Examples 3.2.6 and 3.6.2.

4.1. The interior of a subobject. The theory developed in Section 3 crucially relies on the choice of an \( \text{Aut} (T) \)-cosieve with certain properties. This is an auxiliary structure though, neither the category \( \text{ho}(\mathcal{C}/T) \) nor the resulting self-duality on it depends on this choice.

As we will discuss in the next subsection, the central assumption of our theorem is a strong form of the homotopy extension property well-studied in algebraic and differential topology. We now define an \( \text{Aut} (T) \)-cosieve \( \ll \) that is adapted to the formulation of this assumption:

Definition 4.1.1. If \( u \leq v \), then we set

\[
\ll u \ll v \Leftrightarrow \forall \xi: x \Rightarrow z \exists \gamma \in G_u: t(x) = s(v^\circ) \Rightarrow \gamma v^\circ x = v^\circ \xi.
\]

As we will explain in Example 4.2.4 below, this is consistent with the notation introduced for the special case \( \text{Mfld}^d/T \) in Example 3.5.3 above. The interpretation derived from this example is that a subobject \([u]\) of \( T \) that is contained in a subobject \([v]\) is in the interior of \([v]\) if and only if any 2-cell \( \xi \) which only acts in the complement \([v^\circ]\) of \([v]\) can be extended to a 2-cell \( \gamma: \text{id}_T \Rightarrow g \) for which \( g \in \text{Aut} (T) \) and \( \gamma \) is constantly the identity on \([u]\), \( \gamma u = u \).
Let us verify that Definition 4.1.1 indeed defines an $\text{Aut}(T)$-cosieve:

**Proposition 4.1.2.** $\ll$ is an $\text{Aut}(T)$-cosieve in $(\mathcal{C}/T_0, \cdot, \leq)$.

**Proof.** Assume $u \ll v$. Then $u \leq v$ holds by definition, say $u = vb$. If $d \in \text{Aut}(T)$, then we have $du \leq dv$ as $du = d(vb)$. Also, the domain of $(dv)^\circ = dv^\circ$ agrees with the domain of $v^\circ$. Hence if $\xi: x \Rightarrow z$ is any $2$-cell in $\mathcal{C}$ between $1$-cells $x, z$ whose codomain is $t(x) = t((dv)^\circ = t(v^\circ)$, then by assumption, there exists $\gamma \in G$ with $\gamma u = u$ and $\gamma v^\circ x = v^\circ \xi$. Then $\eta := d\gamma d^{-1}$ satisfies

$$\eta(du) = d\gamma d^{-1} du = d\gamma u = du$$

and

$$\eta(dv)^\circ x = \eta(dv^\circ) x = d\gamma d^{-1} dv^\circ x = d\gamma v^\circ x = dv^\circ \xi = (dv)^\circ \xi.$$ 

Thus $du \ll dv$. Similarly, if $v \leq w$, say $v = wc$, then we have $u = vb = wcb \leq w$ and $w^\circ \leq v^\circ$, say $w^\circ = v^\circ l$. Furthermore, if $\psi: m \rightarrow n$ is a $2$-cell between $1$-cells $m, n$ with codomain

$$t(m) = s(w^\circ) = s(l),$$

then $\xi := l\psi: x \Rightarrow z$, $x := lm, z := ln$, is a $2$-cell and the target $t(x) = s(v^\circ)$, so there exists $\gamma \in G_u$ with

$$\gamma w^\circ m = \gamma v^\circ lm = \gamma v^\circ x = v^\circ \xi = v^\circ l\psi = w^\circ \psi.$$ 

So $u \ll w$ as we had to show. \hfill \Box

So far, this subsection has not made any assumptions on $\mathcal{C}$ and $T$, but what we need to demand is that the converse of Definition 4.1.1 holds, that is, that we can characterise the image of the maps $\mathcal{C}_2 \rightarrow \mathcal{C}_2, \xi \mapsto e\xi$ in terms of $\ll$:

**Assumption 3.** If $x, z: X \rightarrow E$, $e: E \rightarrow T$, and $\phi: ex \Rightarrow ez$, then

$$(\exists \xi: x \Rightarrow z : \phi = e\xi) \iff (\forall u \ll e^\circ \exists \gamma \in G_u : \gamma ex = \phi).$$

To be clear: $\Rightarrow$ holds by definition of $\ll$, what we assume is $\Leftarrow$.

Assumption 3 enters the proof of our theorem in the following way:

**Proposition 4.1.3.** There exists a $2$-cell $\xi$ between $1$-cells $\gamma x, \varepsilon x$ in $\mathcal{C}/T$ with $x \in \mathcal{C}/T_0, \gamma, \varepsilon \in G$ if and only if $\gamma G_x \equiv \varepsilon G_x$. 

Proof. If \( \gamma : \text{id}_T \Rightarrow g \), then we have
\[
\varepsilon x = y \xi \circ \gamma x \Leftrightarrow \varepsilon x \circ (\gamma x)^* = y \xi \\
\Leftrightarrow \varepsilon x \circ \gamma^* x = y \xi \\
\Leftrightarrow (\varepsilon \circ \gamma^*) x = y \xi \\
\Leftrightarrow (\varepsilon \circ g\gamma^{-1}) x = y \xi \\
\Leftrightarrow \varepsilon g\gamma^{-1} x = y \xi \\
\Leftrightarrow \varepsilon \gamma^{-1} g x = y \xi.
\]
Now apply Assumption 3 with \( \phi = \varepsilon \gamma^{-1} g x \).
\[\square\]

4.2. The homotopy extension property. A priori, the \( \text{Aut}(T) \)-cosieve \( \ll \) could be empty. As we will show now, the backbone of our main theorem is that the objects in \( \mathcal{C}/T_0 \) are cofibrations, which in the language we have developed means that all objects have a nonempty interior: as \( \text{id}_T \) is terminal in \( \mathcal{C}/T_0, \leq \), \( \text{id}_T \) is initial, that is, we have \( \text{id}_T \leq v \) for all \( v \in \mathcal{C}/T_0 \), so the following implies \( \text{id}_T \ll v \) for all \( v \).

Assumption 4. \( \text{id}_T \ll \text{id}_T \).

So explicitly, we assume that for all \( x, z \in \mathcal{C}/T_0 \), we have
\[
(4.1) \quad \forall \xi : x \Rightarrow z \exists \gamma \in G_{\text{id}_T} : \gamma x = \xi.
\]
This implies:

**Proposition 4.2.1.** If \( g \sim_h \text{id}_T \), then \( g \) is invertible.

*Proof. If \( \xi : \text{id}_T \Rightarrow g \) is a 2-cell, then by Assumption 4, \( \xi \in G_{\text{id}_T} \subseteq G \), so its codomain is invertible. \[\square\]

More importantly, if \( \gamma G_x : x \to y \) is a morphism in \( \mathcal{I}_{\mathcal{C}/T} \), then by definition, we have \( \gamma \bullet x \leq y \), so there exists a unique (all 1-cells are monic) \( f : x \to y \) with \( \gamma \bullet x = y f \), and with \( \phi := \gamma x \) we obtain a morphism \( x \to y \) in \( \mathcal{C}/T \), viewed as an ordinary category. Thus we obtain a functor
\[
I_{\mathcal{C}/T} : \mathcal{I}_{\mathcal{C}/T} \to \mathcal{C}/T, \quad (\gamma G_x : x \to y) \mapsto ((f, \gamma x) : x \to y)
\]
which is the identity map on objects and is easily seen to be compatible with the \( \text{Aut}(T) \)-actions. It is by the definition of \( G_x \) faithful, and by (4.1) and Proposition 4.1.3, we in fact finally obtain:

**Proposition 4.2.2.** The functor \( I_{\mathcal{C}/T} \) is an isomorphism and induces an isomorphism of \( \text{Aut}(T) \)-categories \( \text{ho(\mathcal{C}/T)} \cong \text{ho(\mathcal{I}_{\mathcal{C}/T})} \).
More precisely, “⇒” in Assumption 3 implies that $I_{C/T}^{-1}$ extends to a 2-functor $\mathcal{C}/T \to \mathcal{I}_{sC/T}$ and if in addition “⇐” holds, this induces an isomorphism $\text{ho}(\mathcal{C}/T) \cong \text{ho}(\mathcal{I}_{sC/T})$.

**Remark 4.2.3.** A 1-cell $x: X \to T$ in a 2-category is a cofibration (see e.g. [LR20]) if for all $\phi: sx \Rightarrow v, s: T \to V$, $v: X \to V$ there exists $w: T \to V$ with $v = wx$ and $\rho: s \Rightarrow w$ with $\phi = \rho x$. As we assume all 1-cells are monic, $\mathcal{C}/T$ remains unchanged if we discard all objects $V$ in $\mathcal{C}$ without a 1-cell $V \to T$, and then Assumption 4 simply says that all 1-cells in $\mathcal{C}$ are cofibrations. In many examples, there is a path space object $V^I$ in $\mathcal{C}$ that comes equipped with source and target 1-cells $s, t: V^I \to V$ (think of a space of paths $[0,1] \to V$ in a space $V$), and 2-cells $\phi: r \Rightarrow v$ between 1-cells $r, v: X \to V$ correspond to 1-cells $p: X \to V^I$ with $r = sp, v = tp$. Then the cofibration property becomes depicted by the following standard diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p} & V^I \\
\downarrow x & & \downarrow \exists ! \\
T & \xrightarrow{s} & V.
\end{array}
$$

**Example 4.2.4.** In $\mathcal{C} = \text{Mfld}^d$, $\text{id}_T^v$ is the empty embedding of the empty set, and Assumptions 3 and 4 hold if and only if $T$ has no boundary. We refer e.g. [Hir94, Theorem 8.1.6] for the proof of the homotopy extension property; see also the recent post [Goo18] by Goodwillie who discusses the uniqueness of the extensions. Reformulated in our language, he therein points out that $G_x$ and hence $\gamma G_x$ is not just a path-connected, but a contractible topological space. In this sense, the extension $\gamma$ of a given $\phi$ to all of $T$ is from a homotopy-theoretic point of view unique. The non-uniqueness of the extension was the reason why we introduced the auxiliary tool of the orbit 2-category $\mathcal{I}_{sC/T}$. Its 1-cells are the cosets $\gamma G_x$ rather than the representatives $\gamma$ themselves, and that is why $I_{C/T}$ is faithful by definition.

Once it is established that 2-cells between $x, z: X \to T$ extend to all of $T$, it is easily seen that Assumption 3 holds and that $u \ll v$ as defined in Definition 4.1.1 means that $\text{im } u$ is contained in the interior of $\text{im } v$ (Example 3.5.3): indeed, if we have

$$
\text{im } x, \text{im } z \subseteq E \subseteq T, \quad U = \text{im } u \subseteq V = T \setminus E,
$$

then there is a tubular neighbourhood of $\partial E$ that is disjoint from $U$. Using a partition of 1 we obtain a smooth bump function $b \in C^\infty(T)$
with value 1 on $E$ and value 0 on $U$; now any extension $\eta \in G$ of an isotopy $\phi: x \Rightarrow z$ can be replaced by $\gamma \in G_u$ given by

$$\gamma(t, p) := \eta(tb(t), p), \quad t \in [0, 1], p \in T$$

with $\gamma(t, p) = p$ for all $t$ and $p \in U$ (as in Example 3.6.2). Conversely, if such an extension of $\phi$ exists, then $\phi$ only acts inside $T \setminus U$. So if $U$ can be chosen arbitrarily in $T \setminus E$, $\phi$ is of the form $e \xi$.

4.3. Tubular neighbourhoods. To complete the proof of our theorem, we need to show that $\circ$ extends to $I_{s_C/T}$, and here we simply assume outright the required condition (3.8):

**Assumption 5.** For all $b, c, d \in C/T_0$ with $c \ll b^*$ and $d \ll b^*$, there are $\rho \in G_c \cap G_d$ and $a \in C/T_0$ with $\rho \triangleright a \sim b$ and $a \ll b$.

The picture we have in mind has already been discussed in Example 3.7.3: we are given submanifolds $B \subseteq T$ and $C, D \subseteq T \setminus B$ and then use a tubular neighbourhood of $\partial B$ to replace $B$ by a slightly smaller but diffeomorphic sub manifold $A \subset B$. So Assumption 5 is about an abstract form of tubular neighbourhoods and deformation retracts.

Nowhere in this paper we have assumed that the preorders studied are lattices, i.e. that one can take some form of unions or intersections of subobjects, and in fact in the case of $C = \text{Mfld}^d$, one can not. However, the union of submanifolds $C, D$ contained in the interior of a submanifold is obviously contained in a slightly larger submanifold $E$, and if $C/T$ has this property, then we can use the homotopy extension property to formulate Assumption 5 internally in $B$:

**Example 4.3.1.** Assume that in $C/T$, there exists for all $c \ll v, d \ll v$ an $e \ll v$ with $c \preceq v, d \preceq v$. Then Assumption 5 can be reduced to the assumption that for all $b: B \to T$ there exists an invertible 1-cell $i: B \to B$ with $a := bi \ll b$ and $i \sim_h \text{id}_B$, as by Assumption 3 a 2-cell $\iota: i \Rightarrow \text{id}_B$ gives rise to $bu: a \Rightarrow b$ which can be extended to a 2-cell $\rho \in G_e \subseteq G_c \cap G_d$ as in Assumption 5.

To sum up: the assumptions made in the present section are those from our main theorem, which follows from Proposition 4.2.2 (Assumptions 3 and 4 imply $\text{ho}(C/T) \cong \text{ho}(s_C/T)$) in combination with Corollary 3.7.2 (Assumption 5 implies that $\text{ho}(s_C/T)$ is self-dual).

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