A finite speed of propagation approximation for the incompressible Navier-Stokes equations

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Abstract

In this paper, we introduce a finite propagation speed perturbation of the incompressible Navier-Stokes equations (\textit{NS}). The model we consider is inspired by a hyperbolic perturbation of the heat equation due to Cattaneo in [4] and by an equation that Višik and Fursikov investigated in [5] in order to find statistical solutions to (\textit{NS}). We prove that the solutions to the perturbed Navier-Stokes equation approximate those to (\textit{NS}).

We use refined energy methods involving fractional Sobolev spaces and precise estimates on the nonlinear term due to the dyadic Littlewood-Paley decomposition.

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1 Introduction

The purpose of this paper is to approximate the solutions to the incompressible Navier-Stokes equations with quasi-critical regularity initial datum by solutions to a nonlinear wave equation with a finite speed of propagation which is obtained by penalizing the incompressibility constraint. First, let us recall the Navier-Stokes equations which govern the motion of an incompressible, viscous and homogeneous Newtonian fluid whose velocity and pressure are denoted by \( v \) and \( p \) respectively.

\[
(\text{NS}) \quad \partial_t v(t, x) - v \Delta v(t, x) + (v, \nabla)v(t, x) = -\nabla p(t, x), \quad \text{div} \, v(t, x) = 0,
\]
where \( t > 0 \) and \( x \in \mathbb{R}^n \) for \( n = 2, 3 \). The velocity is a \( \mathbb{R}^n \) valued vector field and the pressure is scalar. The coefficient of the Laplacian is the viscosity and, without loss of generality, is assumed to be 1 in the following.

Applying the Leray projector \( \mathbb{P} \) which maps \( L^2 \) into \( L^2_\sigma := \{ u \in L^2 : \text{div} \, u = 0 \} \) to \((\text{NS})\), we obtain the equations

\[
(\text{NS}) \quad \partial_t v - \Delta v + \mathbb{P}(v, \nabla)v = 0, \quad \text{div} \, v = 0
\]
from which we can recover the pressure \( p \).

A first hyperbolic perturbation of \((\text{NS})\) has been obtained after relaxation of the Euler equations and rescaling variables (see [3] and references therein):

\[
(\text{HNS}^\varepsilon) \quad \varepsilon \partial_t u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon + \mathbb{P}(u^\varepsilon, \nabla)u^\varepsilon = 0, \quad \text{div} \, u^\varepsilon = 0.
\]

In [4], Cattaneo introduced this equation (without the nonlinear term) as a perturbation of the heat equation. In [3] and [9], the authors approximate the solutions to \((\text{NS})\) by solutions to \((\text{HNS}^\varepsilon)\) under some assumptions on the size and the regularity of the initial data. In [6], we improve the results of [3] and [9]. Under weaker assumptions on the initial data size, we prove the convergence of solutions to \((\text{HNS}^\varepsilon)\) towards solutions to \((\text{NS})\) in the critical Sobolev space norm, that is \( H^{\frac{n}{3}}(\mathbb{R}^n) \), as \( \varepsilon \) goes to 0. More precisely, the theorem is:

**Theorem 1.** Let \( n = 2 \) or \( 3 \) and \( 0 < s, \delta < 1 \). Let \( v_0 \in H^{\frac{n}{2} + s}(\mathbb{R}^n) \) be a divergence-free vector field and \((u_0^0, u_1^0) \in H^{\frac{n}{2} + \delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2} + \delta}(\mathbb{R}^n)^n\) be a sequence of initial data for problem \((\text{HNS}^\varepsilon)\). Assume

\[
\begin{align*}
\|u_0^0 - v_0\|_{H^{\frac{n}{2} - 1}} + \varepsilon \|u_1^0\|_{H^{\frac{n}{2} - 1}} + \varepsilon^{\frac{1}{2}} \|u_0^0\|_{H^{\frac{n}{2} + s}} + \varepsilon^{\frac{1}{2}} \|u_0^0\|_{H^{\frac{n}{2} - 1 + s}} &= O(\varepsilon^{\frac{s}{2}}), \\
\varepsilon^{1 + \delta} \|u_1^0\|_{H^{\frac{n}{2} - 1 + s}} &= o(1).
\end{align*}
\]

Moreover, if \( n = 3 \), assume that \( \|u_0^0\|_{H^{\frac{n}{2}}(\mathbb{R}^3)^3} < \frac{1}{16} \).

Then, for \( \varepsilon \) small enough, there exists a global solution \( u^\varepsilon \) to system \((\text{HNS}^\varepsilon)\) that converges, when \( \varepsilon \) goes to 0, in the \( L_\infty^\infty(\mathbb{R}^+: H^{\frac{n}{2} - 1}(\mathbb{R}^n)^n) \) norm, towards the unique solution \( v \) to \((\text{NS})\), with \( v_0 \) as initial datum. Moreover, for all positive \( T \), there exists a constant \( C_T \), depending only on \( T \) and \( v \), such that

\[
\sup_{t \in [0, T]} \|u^\varepsilon - v\|_{H^{\frac{n}{2} - 1}(\mathbb{R}^n)^n} dx \leq C_T \varepsilon^{\frac{(s)}{2}}.
\]

**Remark.** As a consequence of the assumptions \( \|u_0^0\|_{H^{\frac{n}{2}}(\mathbb{R}^3)^3} < \frac{1}{16} \) and \( \|u_0^0 - v_0\|_{H^{\frac{n}{2}}(\mathbb{R}^3)} = O(\varepsilon^{\frac{s}{2}}) \), we obtain the smallness of \( \|v_0\|_{H^{\frac{n}{2}}} \), which is a necessary condition to the existence of a global strong solution to the Navier-Stokes equations in \( \mathbb{R}^3 \).
In this paper, we shall consider another hyperbolic perturbation of \((NS)\) which is inspired by Cattaneo’s perturbation \((HNS)\) and by Višik and Fursikov’s weakly compressible equations in [5]. The point is that we try to get around the difficulties which come from the Leray projector \(\mathbb{P}\) (or, equivalently, from the pressure) on the one hand and, on the other hand, from the infinite propagation speed due to the heat kernel. The equation we introduce in this paper is:

\[
(HNS^{\varepsilon,\alpha}) \quad \varepsilon \partial_t u^{\varepsilon,\alpha} + \partial_i u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} + \frac{1}{\alpha} \nabla (\text{div } u^{\varepsilon,\alpha}).
\]

Under the same assumptions as in Theorem 1, we prove that the solutions to \((HNS^{\varepsilon,\alpha})\) approximate those to \((NS)\). Notice that we do not need any more restrictions involving \(\alpha\) on the initial data.

**Theorem 2.** Let \(n = 2\) or \(3\) and \(0 < s, \delta < 1\). Let \(v_0 \in H^{2-1+s}(\mathbb{R}^n)\) be a divergence-free vector field and \((u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha}) = (u_0^0, u_1^0) \in H^{1+\delta}(\mathbb{R}^n) \times H^{1+\delta}(\mathbb{R}^n)\) be a sequence of divergence-free initial data for problem \((HNS^{\varepsilon,\alpha})\), independent of \(\alpha\). Assume

\[
\begin{aligned}
\|u_0^{\varepsilon,\alpha} - v_0\|_{H^{1/2-1}} + \varepsilon \|u_1^{\varepsilon,\alpha}\|_{H^{1/2-1}} + \varepsilon^{1/2}\|u_0^{\varepsilon,\alpha}\|_{H^{1/2-1}} &= O(\varepsilon^{1/2}) \\
\varepsilon^{-1/2}\|u_0^{\varepsilon,\alpha}\|_{H^{1/2-1}} + \varepsilon^{1/2}\|u_0^{\varepsilon,\alpha}\|_{H^{1/2-1}} &= O(\varepsilon^{1/2}) \\
\varepsilon^{1/2}\|u_1^{\varepsilon,\alpha}\|_{H^{1/2-1}} &= o(1).
\end{aligned}
\]

Moreover, if \(n = 3\), assume that \(\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{1/2}(\mathbb{R}^3)^3} < \frac{1}{36K_2}\), where \(K_2\) is the constant such that

\[
\|f\|_{L^\infty(\mathbb{R}^3)} \leq K_2\|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}.
\]

Then, for \(\varepsilon, \alpha\) small enough and for all positive \(T\), the global solutions to \((HNS^{\varepsilon,\alpha})\) approximate those to \((NS)\) in the \(L_T^\infty \dot{H}^{1/2-1}(\mathbb{R}^n)\) norm.

**Remark.** A possible choice of initial data is \(\tilde{u}_0^\varepsilon(\xi) = \tilde{v}_0(\xi)1_{\{\sqrt{\xi} | \xi| < 1\}}\) and \(u_1^\varepsilon = 0\).

In this paper, we shall prove that the solutions to \((HNS^{\varepsilon,\alpha})\) with initial data \((u_0^0, u_1^0)\) converge, as \(\alpha\) goes to 0, towards the solutions to \((HNS)\) with the same initial data. Then we conclude according to Theorem 1.

The next section is dedicated to the introduction of the model. In section 3, we shall prove that \((HNS^{\varepsilon,\alpha})\) has a finite speed of propagation using the results of section 4, where we prove local existence. Then, we focus on the 2D case in section 5.1. First, we recall subsection 5.1 some important estimates on the solutions \(u^\varepsilon\) to \((HNS)\) then we prove global existence for \((HNS^{\varepsilon,\alpha})\) in part 5.2 and we show that its solutions approximate those to \((HNS)\) in subsection 5.3. Finally, section 6 is devoted to the 3D case and follows the same plan as section 5. In subsection 6.1, we recall the useful regularity results on \(u^\varepsilon\) then we prove that the local solutions to \((HNS^{\varepsilon,\alpha})\) are global in subsection 6.2 and that the global solutions approximate those to \((HNS)\) in subsection 6.3. Finally, Theorem 1 allows to conclude the proof of Theorem 2. Some important estimates coming from the framework of Littlewood-Paley theory are recalled in appendix A.

### 2 Introducing the model

In this section, we shall introduce the finite speed of propagation equation that we will study in the next sections. We will work in the setting of \(\mathbb{R}^2\). The 3D case is similar.

First, let us perturb the Navier-Stokes system \((NS)\) into the damped nonlinear wave equation \((HNS)\) which we recall:

\[
(HNS^\varepsilon) \quad \varepsilon \partial_t u^\varepsilon + \partial_i u^\varepsilon - \Delta u^\varepsilon = -(u^\varepsilon \cdot \nabla) u^\varepsilon - \nabla p^\varepsilon, \quad \text{div } u^\varepsilon = 0.
\]

Then, applying the \(\text{div} = \sum_{i=1}^n \partial_i(.)\) operator to this equation, we obtain

\[
0 = \text{div } (u^\varepsilon) - \Delta p^\varepsilon, \quad (2)
\]
where \( f(u) = - (u, \nabla) u \). Let us now consider the stationary problem

\[
- \Delta u^\varepsilon = f - \nabla p^\varepsilon, \quad \text{div} u^\varepsilon = 0
\]

(3)

which is related to (2). Indeed, applying \( \text{div} \) to (3), we obtain (2). Given \( f \in \dot{H}^{-1} \), we look for a solution \( u^\varepsilon \in \dot{H}^{1} \) to (3), i.e. such that

\[
J^\varepsilon(u^\varepsilon) = \min \left\{ J^\varepsilon(v) : v \in \dot{H}^{1}, \text{div} v = 0 \right\},
\]

where \( J^\varepsilon(v) = \int \left( \frac{1}{2} |\nabla v|^2 - f \cdot v \right) \, dx \). Now, using a penalization method, we change the problem into minimizing

\[
J^{\varepsilon, \alpha}(v) = J^\varepsilon(v) + \frac{1}{2\alpha} \int |\text{div} v|^2 \, dx
\]

in the space \( \dot{H}^{1} \) so that the constraint \( \text{div} u^\varepsilon = 0 \) is integrated to the functional \( J^{\varepsilon, \alpha} \) to minimize. Let us call the minimizer \( u^{\varepsilon, \alpha} \). Letting \( \alpha \) go to zero in \( u^{\varepsilon, \alpha} \), we obtain the desired solution \( u^\varepsilon \). Since \( u^{\varepsilon, \alpha} \) minimizes \( J^{\varepsilon, \alpha} \), we know that it solves

\[
- \Delta u^{\varepsilon, \alpha} = f + \frac{1}{\alpha} \nabla (\text{div} u^{\varepsilon, \alpha}) = 0.
\]

(4)

Now, recall that \( f \in \dot{H}^{-1} \) and write

\[
\int_{\mathbb{R}^2} \left( |\nabla u^{\varepsilon, \alpha}|^2 + \frac{1}{\alpha} |\text{div} u^{\varepsilon, \alpha}|^2 \right) \, dx = \int_{\mathbb{R}^2} f \cdot u^{\varepsilon, \alpha} \, dx \leq \|f\|_{\dot{H}^{-1}} \|u^{\varepsilon, \alpha}\|_{\dot{H}^{1}} \leq \frac{1}{2} \|f\|_{\dot{H}^{-1}}^2 + \frac{1}{2} \|u^{\varepsilon, \alpha}\|_{\dot{H}^{1}}^2.
\]

(5)

So we immediately deduce that

\[
\|\text{div} u^{\varepsilon, \alpha}\|_{L^2}^2 \leq C\alpha \|f\|_{\dot{H}^{-1}}^2 = O(\alpha).
\]

On this basis, we shall consider the equation

\[
(\text{HNS}^{\varepsilon, \alpha}) \quad \varepsilon \partial_t u^{\varepsilon, \alpha} + \partial_x u^{\varepsilon, \alpha} - \Delta u^{\varepsilon, \alpha} = -(u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha} + \frac{1}{\alpha} \nabla (\text{div} u^{\varepsilon, \alpha}).
\]

Due to (5), we say that \( (\text{HNS}^{\varepsilon, \alpha}) \) is weakly compressible. Let us point out here that this model reminds of the one studied by Višik and Fursikov in [5] and, later, by Basson [2] and Lelièvre [7] in order to study statistical solutions to the Navier-Stokes equations.

In the next section, we shall prove that \( (\text{HNS}^{\varepsilon, \alpha}) \) has a finite propagation speed through Picard’s fixed point theorem and energy inequalities.

### 3 Finite speed of propagation

In order to prove that the equation

\[
(\text{HNS}^{\varepsilon, \alpha}) \quad \varepsilon \partial_t u^{\varepsilon, \alpha} + \partial_x u^{\varepsilon, \alpha} - \Delta u^{\varepsilon, \alpha} = -(u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha} + \frac{1}{\alpha} \nabla (\text{div} u^{\varepsilon, \alpha})
\]

has a finite propagation speed, let us consider the Helmholtz-Hodge decomposition of \( u^{\varepsilon, \alpha} \):

\[
u^{\varepsilon, \alpha} = u^{\varepsilon, \alpha} + z^{\varepsilon, \alpha},
\]

where \( w^{\varepsilon, \alpha} = \mathbb{P} u^{\varepsilon, \alpha} := \frac{1}{\alpha} \nabla \text{div} u^{\varepsilon, \alpha} \) is irrotational and \( z^{\varepsilon, \alpha} = \mathbb{Q} u^{\varepsilon, \alpha} := u^{\varepsilon, \alpha} - w^{\varepsilon, \alpha} \) is divergence-free. We obtain the system

\[
\begin{cases}
\varepsilon \partial_t z^{\varepsilon, \alpha} + \partial_x z^{\varepsilon, \alpha} - \Delta z^{\varepsilon, \alpha} = -\mathbb{P} ((w^{\varepsilon, \alpha} + z^{\varepsilon, \alpha}) \cdot \nabla) (w^{\varepsilon, \alpha} + z^{\varepsilon, \alpha}) \\
\varepsilon \partial_t w^{\varepsilon, \alpha} + \partial_x w^{\varepsilon, \alpha} - \frac{\alpha + 1}{\alpha} \Delta w^{\varepsilon, \alpha} = -\mathbb{Q} ((w^{\varepsilon, \alpha} + z^{\varepsilon, \alpha}) \cdot \nabla) (w^{\varepsilon, \alpha} + z^{\varepsilon, \alpha})
\end{cases}
\]

(6)
From now on, we assume $u$ cannot prove that the Duhamel’s formula related to system (\(\mathcal{P}\)). Let us recall the standard energies associated to the wave equations in $(\mathcal{S})$.

Remark. As we shall see in the following, this way of writing the equation $(\mathcal{HNS}^{c,\alpha})$ is convenient for the proof of finite speed of propagation but, unless we assume that $u$ and $\tilde{u}$ are smooth, we cannot prove directly that the Duhamel’s formula related to system $(\mathcal{S})$ is locally contractive (due to the term $\nabla\tilde{u}$). Besides, we cannot prove that $(\mathcal{HNS}^{c,\alpha})$ has a finite speed of propagation through the Helmholtz-Hodge decomposition since the operators $\mathcal{P}$ and $\mathcal{Q}$ are non-local.

Moreover, applying $\nabla$ to (6), we obtain the equation

$$
\varepsilon\partial_{tt}\tilde{u}^{j+1} - \frac{\alpha+1}{\alpha}\Delta\tilde{u}^{j+1} = -\partial_t\tilde{u}^j - \nabla_ju^j.
$$

Then we have the coupled system of equations

$$
(S')\;\begin{cases}
\varepsilon\partial_{tt}\tilde{u}^{j+1} - \frac{\alpha+1}{\alpha}\Delta\tilde{u}^{j+1} = -\partial_t\tilde{u}^j - \nabla_ju^j \\
\varepsilon\partial_{tt}w^{j+1} - \frac{\alpha+1}{\alpha}\Delta w^{j+1} = -\partial_t w^j - \nabla_jw^j + \frac{1}{\alpha}\nabla\tilde{u}^j.
\end{cases}
$$

Now, let us consider the problem

$$(P')\;\begin{cases}
(S')\;\begin{cases}
w^{j+1}|_{t=0} = u_0 & , \; \partial_tw^{j+1}|_{t=0} = u_1 \\
\tilde{u}^{j+1}|_{t=0} = \text{div}u_0 & , \; \partial_{tt}\tilde{u}^{j+1}|_{t=0} = \text{div}u_1 \\
u^0 & , \; v^0 \equiv 0
\end{cases}
\end{cases}
$$

From now on, we assume $u_0$ and $u_1$ to be supported in a ball $B(0,R)$ and we set

$$
c_1 = \sqrt{\frac{\alpha+1}{\alpha\varepsilon}} \; , \; c_2 = \frac{1}{\sqrt{\varepsilon}}.
$$

Let us recall the standard energies associated to the wave equations in $(\mathcal{S})$:

$$
E_{c_1}(\tilde{u})(t) = \int_{\Gamma_{c_1,t,x}} e_{c_1}(\tilde{u})(t,y) \, dy \; , \; E_{c_2}(u)(t) = \int_{\Gamma_{c_2,t,x}} e_{c_2}(u)(t,y) \, dy,
$$

$$
E_{c_3}(\tilde{u})(t) = \int_{\Gamma_{c_3,t,x}} e_{c_3}(\tilde{u})(t,y) \, dy.
$$
where the densities are

\[ e_{c_1}(\tilde{u})(t, y) = \frac{\varepsilon}{2} (\partial_t \tilde{u}(t,y))^2 + \frac{\alpha + 1}{2\alpha} |\nabla \tilde{u}(t,y)|^2, \]

\[ e_{c_2}(u)(t, y) = \frac{\varepsilon}{2} (\partial_t u(t,y))^2 + \frac{1}{2} |\nabla u(t,y)|^2, \]

and the cones \( \Gamma_{c_i,t,x} \) are defined by

\[ \Gamma_{c_i,t,x} = \{ (s, y) : 0 \leq s \leq \frac{t}{c_i}, |y - x| \leq t - cs \}. \]

Notice that, since \( c_1 > c_2 \), we have \( \Gamma_{c_1,t,x} \subseteq \Gamma_{c_2,t,x} \). Now, assume that we know that \( u^j, \partial_t u^j, \tilde{u}^j, \partial_t \tilde{u}^j = 0 \)

on the cone \( \Gamma_{c_1,t,x} \). Then, multiplying the first equation in \( (S^j) \) by \( \partial_t \tilde{u}^{j+1} \) and integrating on \( \Gamma_{c_1,t,x} \), we obtain:

\[ 0 = \int_{\Gamma_{c_1,t,x}} \left( \varepsilon \partial_t \tilde{u}^{j+1} + \partial_t \tilde{u}^j - \frac{\alpha + 1}{\alpha} \Delta \tilde{u}^{j+1} + \text{div} (u^j, \nabla) u^j \right) \partial_t \tilde{u}^{j+1} \, dt \, dy \]

\[ = \int_{\Gamma_{c_1,t,x}} \left( \varepsilon \partial_t \tilde{u}^{j+1} - \frac{\alpha + 1}{\alpha} \Delta \tilde{u}^{j+1} \right) \partial_t \tilde{u}^{j+1} \, dt \, dy \]

\[ = \int_{\Gamma_{c_1,t,x}} \frac{\varepsilon}{2} \partial_t |\partial_t \tilde{u}^{j+1}|^2 - \frac{\alpha + 1}{\alpha} \sum_{i=1}^{n} \partial_i \partial_t \tilde{u}^{j+1} \partial_i \tilde{u}^{j+1} \, dt \, dy \]

\[ = \int_{\Gamma_{c_1,t,x}} \frac{\varepsilon}{2} \partial_t |\partial_t \tilde{u}^{j+1}|^2 - \frac{\alpha + 1}{\alpha} \sum_{i=1}^{n} \left[ \partial_i \left( \partial_t \tilde{u}^{j+1} \partial_i \tilde{u}^{j+1} \right) - \partial_i \partial_t \tilde{u}^{j+1} \partial_i \tilde{u}^{j+1} \right] \, dt \, dy \]

\[ = \int_{\Gamma_{c_1,t,x}} \partial_t \left( \frac{\varepsilon}{2} |\partial_t \tilde{u}^{j+1}|^2 + \frac{\alpha + 1}{2\alpha} |\nabla \tilde{u}^{j+1}|^2 \right) - \frac{\alpha + 1}{\alpha} \sum_{i=1}^{n} \partial_i \left( \partial_t \tilde{u}^{j+1} \partial_i \tilde{u}^{j+1} \right) \, dt \, dy. \]

In order to apply the divergence theorem, let us compute the unit outgoing normal \( \nu \) to the cone \( \Gamma_{c_i,t,x} \). We have

\[ \nu(s, y) = \begin{cases} 
(+1, 0) & \text{if } s = t \\
(-1, 0) & \text{if } s = 0 \\
\frac{1}{\sqrt{1 + c_i^2}} \left( c_i, \frac{y - x}{|y - x|} \right) & \text{if } 0 < s < t.
\end{cases} \]

The divergence theorem yields:

\[ E_{c_1}(\tilde{u}^{j+1})(0) = E_{c_1}(\tilde{u}^{j+1})(t) + \frac{1}{\sqrt{1 + c_1^2}} \int_0^{\frac{t}{c_1}} \int_{\setminus_{c_1} \Gamma_{c_1,t,x}} \frac{\varepsilon}{2} |\partial_t \tilde{u}^{j+1}|^2 + \frac{\alpha + 1}{2\alpha} |\nabla \tilde{u}^{j+1}|^2 - \frac{\alpha + 1}{\alpha} \sum_{i=1}^{n} \left( \partial_t \tilde{u}^{j+1} \partial_i \tilde{u}^{j+1} \right) \, dt \, dy. \]
By the Cauchy-Schwarz inequality followed by Young, we obtain:
\[
\int \sum_{i=1}^{n} \frac{(y-x)_i}{|y-x|} \partial_i \tilde{u}^{j+1} \cdot \partial_i \tilde{u}^{j+1} \leq \sum_{i=1}^{n} \| \partial_i \tilde{u}^{j+1} \|_{L^2} \| \partial_i \tilde{u}^{j+1} \|_{L^2} \leq \frac{1}{2\lambda} \| \nabla \tilde{u}^{j+1} \|^2_{L^2} + \frac{\lambda}{2} \| \partial_t \tilde{u}^{j+1} \|^2_{L^2},
\]
where \( \lambda = c_1^{-1} \). Since \( \tilde{u}^{j+1}_{|t=0} = \text{div} u_0 \) and \( \partial_t \tilde{u}^{j+1}_{|t=0} = \text{div} u_1 \), we know that
\[
\tilde{u}^{j+1}, \partial_t \tilde{u}^{j+1} = 0 \text{ on } B(x,t).
\]
So, finally, we have
\[
E_{c_1}(\tilde{u}^{j+1})(t) \leq E_{c_1}(\tilde{u}^{j+1})(0) = 0
\]
and we deduce that \( \tilde{u}^{j+1}, \partial_t \tilde{u}^{j+1} = 0 \) on \( \Gamma_{c_1,t,x} \).

Now, in order to handle the second equation in (S1), we cover the cone \( \Gamma_{c_1,x,t} \) with cones of the type \( \Gamma_{c_2,t,x_i} \) and integrate the second equation in (S1) multiplied by \( \partial_t w^{j+1} \) on these cones. Doing so, we obtain that

\[
\int_{\Gamma_{c_1,t,x}} \varepsilon \partial_t u^{j+1} + \partial_t w^{j+1} - \Delta w^{j+1} = 0 \]

\[
|u^{j+1}| \leq \frac{c_1}{\varepsilon}
\]

\[
\text{and}
\]

\[
\text{as } \alpha \text{ goes to } 0.
\]

4 Local existence for equation \( (HNS^{\varepsilon,\alpha}) \)

4.1 Introduction

Let us consider the Cauchy problem
\[
\left\{
\begin{array}{l}
(HNS^{\varepsilon,\alpha}) \\
\varepsilon \partial_t u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} - \frac{1}{\alpha} \nabla (\text{div } u^{\varepsilon,\alpha}) = f(u^{\varepsilon,\alpha}) \\
u^{\varepsilon,\alpha}_{|t=0} = u^{\varepsilon,\alpha}_0 \in H^{2+\delta}_R(\mathbb{R}^n), \quad \partial_t u^{\varepsilon,\alpha}_{|t=0} = u^{\varepsilon,\alpha}_1 \in H^{2+\delta-1}_R(\mathbb{R}^n)
\end{array}
\right.
\]

where \( n = 2,3 \) and \( f(u) = -(u, \nabla u) \). First, let us assume that \( f = 0 \) and split the solution \( u^{\varepsilon,\alpha} \) to equation \( (HNS^{\varepsilon,\alpha}) \) into its irrotational part \( w^{\varepsilon,\alpha} \) and its divergence-free part \( z^{\varepsilon,\alpha} \):
\[
u^{\varepsilon,\alpha}(t,x) = w^{\varepsilon,\alpha}(t,x) + z^{\varepsilon,\alpha}(t,x).
\]

More precisely, \( w^{\varepsilon,\alpha} \) and \( z^{\varepsilon,\alpha} \) are defined as follows:
\[
w^{\varepsilon,\alpha} = \mathbb{Q} u^{\varepsilon,\alpha} := \frac{1}{\Delta} \nabla (\text{div } u^{\varepsilon,\alpha})
\]
\[
z^{\varepsilon,\alpha} = \mathbb{P} u^{\varepsilon,\alpha} := (1 - \mathbb{Q}) u^{\varepsilon,\alpha} = u^{\varepsilon,\alpha} - w^{\varepsilon,\alpha}.
\]

Then \( w^{\varepsilon,\alpha} \) and \( z^{\varepsilon,\alpha} \) solve the following equations:
\[
\left\{
\begin{array}{l}
\varepsilon \partial_t w^{\varepsilon,\alpha} + \partial_t w^{\varepsilon,\alpha} - (1 + \frac{1}{\alpha}) \Delta w^{\varepsilon,\alpha} = 0 \\
w^{\varepsilon,\alpha}_{|t=0} = w^{\varepsilon,\alpha}_0 := \mathbb{Q} u^{\varepsilon,\alpha}_0, \quad \partial_t w^{\varepsilon,\alpha}_{|t=0} = w^{\varepsilon,\alpha}_1 := \mathbb{Q} u^{\varepsilon,\alpha}_1
\end{array}
\right.
\]

Figure 2: Covering the cone \( \Gamma_{c_1,x,t} \) with cones of the type \( \Gamma_{c_2,t,x_i} \).
\[
\begin{cases}
\varepsilon \partial_t z^{\varepsilon,\alpha} + \partial_z z^{\varepsilon,\alpha} - \Delta z^{\varepsilon,\alpha} = 0 \\
z^{\varepsilon,\alpha}|_{t=0} = z_0^{\varepsilon,\alpha} := P u_0^{\varepsilon,\alpha}, \quad \partial_z z^{\varepsilon,\alpha}|_{t=0} = z_1^{\varepsilon,\alpha} := P u_1^{\varepsilon,\alpha}.
\end{cases}
\]  

(9)

Since (9) is a wave equation, we know that

\[ w^{\varepsilon,\alpha}(t,x) = A_Q(t) w_0^{\varepsilon,\alpha}(x) + B_Q(t) w_1^{\varepsilon,\alpha}(x) - \int_0^t B_Q(t-s) \partial_t w^{\varepsilon,\alpha}(s) \, ds, \]

where \( A_Q \) and \( B_Q \) are defined as follows.

\[ A_Q(t) = \cos \left( \sqrt{\frac{\alpha + 1}{\varepsilon}} t \Lambda \right), \quad B_Q(t) = \frac{\sin \left( \sqrt{\frac{\alpha + 1}{\varepsilon}} t \Lambda \right)}{\sqrt{\frac{\alpha + 1}{\varepsilon}} \Lambda}. \]

Similarly, (7) is a wave equation and we have

\[ z^{\varepsilon,\alpha}(t,x) = A_P(t) z_0^{\varepsilon,\alpha}(x) + B_P(t) z_1^{\varepsilon,\alpha}(x) - \int_0^t B_P(t-s) \partial_z z^{\varepsilon,\alpha}(s) \, ds, \]

where \( A_P \) and \( B_P \) are

\[ A_P(t) = \cos \left( \frac{t \Lambda}{\sqrt{\varepsilon}} \right), \quad B_P(t) = \sqrt{\varepsilon} \frac{\sin \left( \frac{t \Lambda}{\sqrt{\varepsilon}} \right)}{\Lambda}. \]

Now, setting \( A(t) = A_Q(t) Q + A_P(t) P \) and \( B(t) = B_Q(t) Q + B_P(t) P \), we can write Duhamel’s formula for the Cauchy problem (7) with \( f = 0 \):

\[ \phi(u^{\varepsilon,\alpha})(t) = A(t) u_0^{\varepsilon,\alpha} + B(t) u_1^{\varepsilon,\alpha} - \int_0^t B(t-s) \partial_t u^{\varepsilon,\alpha}(s) \, ds. \]

Adding the source term \( f(u^{\varepsilon,\alpha}) = -(u^{\varepsilon,\alpha}, \nabla) u^{\varepsilon,\alpha} \), the formula becomes

\[ \phi(u^{\varepsilon,\alpha})(t) = A(t) u_0^{\varepsilon,\alpha} + B(t) u_1^{\varepsilon,\alpha} + \int_0^t B(t-s) (f(u^{\varepsilon,\alpha}) - \partial_t u^{\varepsilon,\alpha})(s) \, ds. \]

### 4.2 Contraction argument

We shall show local existence for (7) in the complete metric space

\[ X_T(u) = \left\{ (u, \partial_t u) \in \dot{H}^{\frac{\delta}{\varepsilon}} \cap \dot{H}^{\frac{\delta}{\varepsilon} - 1}(\mathbb{R}^n) \times \dot{H}^{\frac{\delta}{\varepsilon} + 1}(\mathbb{R}^n) : \right. \]

\[ \left. \|u\|_{X_T} := \|u\|_{L^\infty \dot{H}^{\frac{\delta}{\varepsilon} + \delta}} + \|u\|_{L^\infty \dot{H}^{\frac{\delta}{\varepsilon} + \delta - 1}} + \|\partial_t u\|_{L^\infty \dot{H}^{\frac{\delta}{\varepsilon} - 1}} \leq a \right\}, \]
with $\alpha > 0$ and $0 < T < 1$ to be chosen later. Let us estimate $\|\phi(u)\|_{X_T}$. First, we have

$$
\|\phi(u)(t)\|_{\dot{H}^{s+\delta}} \leq \|A_Q u_0\|_{\dot{H}^{s+\delta}} + \|A_P u_0\|_{\dot{H}^{s+\delta}} + \|B_Q u_1\|_{\dot{H}^{s+\delta}} + \|B_P u_1\|_{\dot{H}^{s+\delta}}
$$

$$
+ \int_0^t \|B_Q(t-s)Q(f(u)-\partial_t u)\|_{\dot{H}^{s+\delta}} \, dt
$$

$$
+ \int_0^t \|B_P(t-s)P(f(u)-\partial_t u)\|_{\dot{H}^{s+\delta}} \, dt
$$

$$
\leq 2\|u_0\|_{\dot{H}^{s+\delta}} + \left(1 + \sqrt[\alpha+1]{\frac{\alpha}{\alpha+1}}\right) \sqrt{\varepsilon}\|u_1\|_{\dot{H}^{s+1}}
$$

$$
+ \sqrt{\varepsilon} \int_0^t \left\| \frac{\sin \left(\frac{\alpha(t-s)}{\alpha+1}\right) \Lambda}{\Lambda} Q[(u,\nabla)u - \partial_t u](s) \right\|_{\dot{H}^{s+\delta}} \, ds
$$

$$
+ \sqrt{\varepsilon} \int_0^t \left\| \frac{\sin \left(\frac{\alpha(t-s)}{\alpha+1}\right) \Lambda}{\Lambda} P[(u,\nabla)u - \partial_t u](s) \right\|_{\dot{H}^{s+\delta}} \, ds + I + II.
$$

In order to estimate $I$, recall that $\text{div} u \neq 0$, so that we have

$$
(u,\nabla)u = \sum_{i=1}^n \partial_i (u_i u) - \text{div} u u.
$$

Using that $\dot{H}^{s+\delta} \cap \dot{H}^{s+1} \cap (\mathbb{R}^n)$ is an algebra, we obtain the inequality

$$
\|\partial_i (u_i u)\|_{\dot{H}^{s+\delta}} \leq \|u_i u\|_{\dot{H}^{s+\delta}} \leq 2C\|u\|_{\dot{H}^{s+\delta}} \|u\|_{\dot{H}^{s+1}}.
$$

Now, the following Lemma allows to estimate the remaining term.

**Lemma 1.** Let $u \in \dot{H}^{s+\delta} \cap \dot{H}^{s+1}(\mathbb{R}^n)$ and $\text{div} u \neq 0$. Then we have

$$
\|\text{div} u u\|_{\dot{H}^{s+\delta}} \leq 2C\|u\|_{\dot{H}^{s+\delta}} \|u\|_{L^\infty}.
$$

**Proof.** Applying the paraproduct decomposition (see Lemma 7 in the appendix), we can write

$$
\|\partial_i u u\|_{\dot{H}^{s+\delta}} \leq \sum_p 2^{p(\frac{s+\delta}{2})-1} \|\Delta_p \partial_i u S_{p+1} u\|_{L^2} + \sum_q 2^{q(\frac{s+\delta}{2})-1} \|\Delta_q u S_{p+1} \partial_i u\|_{L^2}
$$

$$
\leq \sum_p 2^{p(\frac{s+\delta}{2})-1} \|\Delta_p \partial_i u\|_{L^2} \|S_{p+1} u\|_{L^\infty} + \sum_q 2^{q(\frac{s+\delta}{2})-1} \|\partial_i u\|_{L^2} \|S_{p+1} u\|_{L^\infty}
$$

$$
\leq \sum_p 2^{p(\frac{s+\delta}{2})} \|\Delta_p u\|_{L^2} \|u\|_{L^\infty} + \sum_q 2^{q(\frac{s+\delta}{2})} \|\Delta_q u\|_{L^2} \|u\|_{L^\infty}
$$

$$
\leq 2C\|u\|_{\dot{H}^{s+\delta}} \|u\|_{L^\infty}.
$$

The Lemma is thereby proved.

Now, using the interpolation estimate $\|u\|_{L^\infty} \leq C\|u\|_{\dot{H}^{s+\delta}}^{\delta} \|u\|_{\dot{H}^{s+\delta}}^{1-\delta}$ followed by a Young inequality, we obtain that

$$
\|\text{div} u u\|_{\dot{H}^{s+\delta}} \leq 2C\|u\|_{\dot{H}^{s+1}} \|u\|_{\dot{H}^{s+\delta}} + \|u\|_{\dot{H}^{s+\delta}}^2.
$$

Finally, we have

$$
\|(u,\nabla)u\|_{\dot{H}^{s+\delta-1}} \leq 4C\|u\|_{\dot{H}^{s+\delta} \cap \dot{H}^{s+\delta-1}}^2.
$$
and the integral $I$ estimates

$$
I \leq \sqrt{\frac{\varepsilon}{\alpha + 1}} \int_0^t (\|Q(u,\nabla)u(s)\|_{\dot{H}^{\frac{\alpha}{2}} + \delta} + \|Q\partial_t u(s)\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1}) \, ds
$$

Similarly, we obtain

$$
II \leq \sqrt{\varepsilon} T(4Ca^2 + a).
$$

So, finally, we have the inequality

$$
\|\phi(u)\|_{L_\infty^\infty \dot{H}^{\frac{\alpha}{2}} + \delta} \leq 2\|u_0\|_{\dot{H}^{\frac{\alpha}{2}} + \delta} + \left(1 + \sqrt{\frac{\alpha}{\alpha + 1}}\right) \sqrt{\varepsilon}\|u_1\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + \sqrt{\varepsilon}(1 + \sqrt{\frac{\alpha}{\alpha + 1}}) aT(1 + 4Ca).
$$

Analogous estimates using that $|\frac{\sin \frac{\pi x}{a}}{\pi x}| \leq 1$ give

$$
\|\phi(u)\|_{L_\infty^\infty \dot{H}^{\frac{\alpha}{2}} + \delta - 1} \leq 2\|u_0\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + 2T\|u_1\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + 2aT^2(1 + 4Ca).
$$

Then, let us compute the time derivative of $\phi(v)$.

$$
\partial_t \phi(v)(t) = -\sqrt{\frac{\alpha + 1}{\alpha \varepsilon}} \Lambda \sin \left(\sqrt{\frac{\alpha + 1}{\alpha \varepsilon}} t \Lambda\right) Qv_0 - \frac{\Lambda}{\sqrt{\varepsilon}} \sin \left(\frac{t}{\sqrt{\varepsilon}} \Lambda\right) P v_0 + \cos \left(\sqrt{\frac{\alpha + 1}{\alpha \varepsilon}} t \Lambda\right) Qv_1 + \cos \left(\frac{t}{\sqrt{\varepsilon}} \Lambda\right) P v_1 + \int_0^t \cos \left(\sqrt{\frac{\alpha + 1}{\alpha \varepsilon}} (t-s) \Lambda\right) Q(v,\nabla v - \partial_t v)(s) \, ds + \int_0^t \cos \left(\frac{t-s}{\sqrt{\varepsilon}} \Lambda\right) P (v,\nabla v - \partial_t v)(s) \, ds.
$$

We have

$$
\|\partial_t \phi(u)\|_{L_\infty^\infty \dot{H}^{\frac{\alpha}{2}} + \delta - 1} \leq \frac{1}{\sqrt{\varepsilon}} \left(1 + \sqrt{\frac{\alpha + 1}{\alpha}}\right) \|u_0\|_{\dot{H}^{\frac{\alpha}{2}} + \delta} + 2\|u_1\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + 2aT(1 + 4Ca).
$$

Finally, we obtain

$$
\|\phi(u)\|_{X_T} \leq \left(2 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\frac{\alpha + 1}{\alpha \varepsilon}}\right) \|u_0\|_{\dot{H}^{\frac{\alpha}{2}} + \delta} + 2\|u_0\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + \left(2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha \varepsilon}{\alpha + 1}}\right) \|u_1\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + 2T\|u_1\|_{\dot{H}^{\frac{\alpha}{2}} + \delta - 1} + \left(2 + 2T + \sqrt{\varepsilon} + \sqrt{\frac{\alpha \varepsilon}{\alpha + 1}}\right) aT(1 + 4Ca).
$$
Let us set
\[ \frac{a}{2} = \left( 2 + \frac{1}{\sqrt{\varepsilon}} + \frac{\alpha + 1}{\alpha \varepsilon} \right) \|u_0\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta}} + 2\|u_0\|_{\mathcal{H}^{\frac{\alpha}{2}+\theta}} + \left( 2 + \sqrt{\varepsilon} + \frac{\alpha \varepsilon}{\alpha + 1} \right) \|u_1\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta-1}}. \]

So we have
\[ \|\phi(u)\|_{\mathcal{X}_T} \leq \frac{a}{2} + \frac{aT}{2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha \varepsilon}{\alpha + 1}}} + \left( 2 + 2T + \sqrt{\varepsilon} + \sqrt{\frac{\alpha \varepsilon}{\alpha + 1}} \right) aT(1 + 4C). \]

Finally, we have the following bound on the local existence time:
\[ T \leq \frac{C}{1 + \left[ \left(C_{\varepsilon} + \frac{\alpha + 1}{\alpha \varepsilon}\right) \|u_0^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta}} + 2\|u_0^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\theta-1}} + \left(C_{\varepsilon} + \frac{\alpha \varepsilon}{\alpha + 1}\right) \|u_1^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta-1}} \right]. \]

In order to prove that the solutions obtained in this section are global, we shall prove that the denominator
\[ \left(C_{\varepsilon} + \frac{\alpha + 1}{\alpha \varepsilon}\right) \|u_0^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta}} + 2\|u_0^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\theta-1}} + \left(C_{\varepsilon} + \frac{\alpha \varepsilon}{\alpha + 1}\right) \|u_1^{\varepsilon,\alpha}\|_{\mathcal{H}^{\frac{\alpha}{2}+\delta-1}} \]
remains bounded for all fixed \( \alpha \) and \( \varepsilon \).

In subsections 5.2 and 6.2, we will globalize the solutions obtained in this section and then, in subsections 5.3 and 6.3, we will prove that they converge towards the solutions to \((HNS^c)\) as \( \alpha \) goes to 0.

## 5 The 2D case

### 5.1 Preliminary estimates

In this part, we shall recall the regularity results we have on the solution \( u^\varepsilon \) to \((HNS^c)\).

**Lemma 2.** Let \( T > 0 \) and \( u^\varepsilon \in L^\infty_t(\mathcal{H}^{1+\delta} \cap \hat{\mathcal{H}}^\delta)(\mathbb{R}^2)^2 \) be the global solution to \((HNS^c)\) with initial data \((u_0^\varepsilon, u_0^1) \in H^{1+\delta} \times H^\delta(\mathbb{R}^2)^2\). Then we have
\[ u^\varepsilon \in L^\infty_T \mathcal{H}^{1+\delta} \cap L^2_T \hat{\mathcal{H}}^{1+\delta} \cap L^2_T L^2 \cap L^\infty_T \mathcal{H}^\delta \]
and
\[ \partial_t u^\varepsilon \in L^2_T L^2. \]

We sketch here the proof of this lemma since all the details can be found in [6]. In the following, we shall denote by \( C \) all the constants, even those depending on \( T \).

**Proof.** First, let us introduce the energy
\[ E^\varepsilon(t) := \frac{1}{2} \|u^\varepsilon + \varepsilon \partial_t u^\varepsilon\|_{\mathcal{H}^{\delta}}^2 + \frac{\varepsilon^2}{2} \|\partial_t u^\varepsilon\|^2_{\mathcal{H}^{\varepsilon}} + \varepsilon \|\nabla u^\varepsilon\|^2_{\mathcal{H}^{\varepsilon}}, \]
for non-negative \( \delta \). Then, according to [6], we know that
\[ \exists C > 0 : \forall t \geq 0, \ E^\varepsilon(t) \leq C\varepsilon^{-\delta}. \]

From this inequality, we immediately deduce that
\[ \varepsilon \|u^\varepsilon\|_{L^\infty_T \mathcal{H}^{1+\delta}} + \varepsilon \|u^\varepsilon\|_{L^2_T \hat{\mathcal{H}}^{1+\delta}} \leq C\varepsilon^{-\delta} \text{ and } \|u^\varepsilon\|_{L^\infty_T L^\infty} = o \left( \frac{1}{\sqrt{\varepsilon}} \right), \quad (10) \]
Now, let us compute the time derivative of $E_{\varepsilon}^0$. We have
\[
\frac{d}{dt}E_{\varepsilon}^0(t) + \varepsilon \int_{\mathbb{R}^2} |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \, dx + \int_{\mathbb{R}^2} (|\nabla u^\varepsilon|^2 - \varepsilon|\nabla(u^\varepsilon \otimes u^\varepsilon)|^2) \, dx = 0.
\]
Due to the control of the norm $\|u^\varepsilon\|_{L^2_T L^\infty}$ by $\frac{1}{C\varepsilon}$ for any $C$ provided that $\varepsilon$ is small enough, the last term in the left hand side is lower bounded by $\int_{\mathbb{R}^2} \frac{1}{2} |\nabla u^\varepsilon|^2 \, dx$. Now, integrating in time, we have
\[
E_{\varepsilon}^0(T) + \int_0^T \int_{\mathbb{R}^2} \varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u^\varepsilon|^2 \, dx \, dt \leq E_{\varepsilon}^0(0) \leq C.
\]
So, we have that $\varepsilon \|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_T L^2}^2 \leq C$ and
\[
\frac{1}{2} \|u^\varepsilon\|_{L^2 T H^1}^2 \leq C.
\]
The later yields
\[
\varepsilon \|\nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_T L^2}^2 \leq C \varepsilon \|u^\varepsilon\|_{L^2_T L^\infty}^2 \|u^\varepsilon\|_{L^2_T H^1}^2 \leq C.
\]
Consequently, we have
\[
\|\sqrt{\varepsilon} \partial_t u^\varepsilon\|_{L^2_T L^2} \leq 2\varepsilon \|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_T L^2} + 2\varepsilon \|\nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_T L^2} \leq C.
\]
We have thereby proven that $\sqrt{\varepsilon} \partial_t u^\varepsilon \in L^2_T L^2$ uniformly in $\varepsilon$.

Finally, notice that $\|u^\varepsilon\|_{L^2_T} \leq 2E_{\varepsilon}^0(t) \leq 2C_0$, so that
\[
u^\varepsilon \in L^2_T L^2.
\]

### 5.2 Globalization

As in [6], let us define the energy
\[
E_{\varepsilon, \alpha}^\delta(t) = \int_{\mathbb{R}^2} \frac{1}{2} |\Lambda^\delta (u^\varepsilon \otimes \alpha + \varepsilon \partial_t u^\varepsilon \otimes \alpha)|^2 + \frac{\varepsilon}{2} |\Lambda^\delta \partial_t u^\varepsilon \otimes \alpha|^2 + \varepsilon |\Lambda^\delta \nabla u^\varepsilon \otimes \alpha|^2 + \varepsilon |\Lambda^\delta \text{div } u^\varepsilon \otimes \alpha|^2.
\]
In view of the dependence of the local time existence on the initial data, we know that proving that $E_{\varepsilon, \alpha}^\delta$ is bounded yields the global existence for $(HNS)^{\varepsilon, \alpha}$. We shall first show that it is true on a time interval $[0, T)$ then prove that $T = +\infty$.

First, let us point out that $\dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$ is an algebra and that the product estimate
\[
\|fg\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} \leq C_1 (\|f\|_{\dot{H}^{1+\delta}} \|g\|_{\infty} + \|g\|_{\dot{H}^{1+\delta}} \|f\|_{\infty}).
\]
holds (see Proposition [7] in the appendix or [11]) for all functions $f, g \in \dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$. Moreover, we know that the homogeneous Besov space $B_{2,1}^1(\mathbb{R}^2)$ embeds into $L^\infty(\mathbb{R}^2)$ and, interpolating, we obtain
\[
\|f\|_{\infty} \leq \tilde{C} \|f\|_{B_{2,1}^1} \leq C_2 \|f\|_{\dot{H}^\delta} \|f\|_{\dot{H}^{1-\delta}}.
\]
Finally, notice that $\text{div } u^\varepsilon \otimes \alpha \neq 0$, so that we have
\[
(u^\varepsilon \otimes \alpha \cdot \nabla) u^\varepsilon \otimes \alpha = \sum_{i=1}^2 u^\varepsilon \otimes \alpha_i \partial_i u^\varepsilon \otimes \alpha = \sum_{i=1}^2 \partial_i (u^\varepsilon \otimes \alpha \cdot u^\varepsilon \otimes \alpha) - \text{div } u^\varepsilon \otimes \alpha \times u^\varepsilon \otimes \alpha.
\]

\(\text{For definitions and properties of the Besov spaces, see the book by P.-G. Lemarié-Rieusset [5].}\)
but we still can prove the estimate

\[ \|(u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha}\|_{\dot{H}^{1+\delta}} \leq C_3 \|u^{\varepsilon, \alpha}\|_{L^\infty} \|u^{\varepsilon, \alpha}\|_{\dot{H}^{1+\delta}} \]  \tag{17} \]

using that \( \dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2) \) is an algebra for the first term in \[16\] and due to Lemma \[1\] for the second term.

Now, let us define \( T^{\max} \) the maximal existence time of \((HNS^{\varepsilon, \alpha})\) and prove the following lemma.

**Lemma 3.** Assume the following, when \( \varepsilon \) goes to zero :

\[
(H) \left\{ \begin{array}{ll}
i) & \varepsilon \frac{1+\delta}{2} \|u_0^{\varepsilon, \alpha}\|_{H^{1+\delta}} + \varepsilon^\frac{\delta}{2} \|u_0^{\varepsilon, \alpha}\|_{H^{\delta}} = o(1) \\
ii) & \varepsilon^\frac{\delta}{2} \|u_0^{\varepsilon, \alpha}\|_{H^1} + \varepsilon \|u_1^{\varepsilon, \alpha}\|_{L^2} = o(1).
\end{array} \right.
\]

Let us define \( 0 \leq T \leq T^{\max} \) by

\[
T = \sup \left\{ 0 \leq \tau \leq T^{\max} : \forall t \in [0, \tau), \|u^{\varepsilon, \alpha}(t)\|_{L^\infty} < \frac{1}{2C_3\sqrt{\varepsilon}} \right\}. \tag{18}
\]

Then, for \( \varepsilon \) small enough, there exists a large number \( N \), depending only on \( \delta \) and \( \|u_0^{\varepsilon, \alpha}\|_{L^2} \) (which is arbitrary), such that, for all \( 0 \leq t < T \),

\[
E_{\varepsilon, \alpha}^{\delta}(t) \leq E_{\varepsilon, \alpha}^{\delta}(0) \left(2\|u_0^{\varepsilon, \alpha}\|_{L^2}^2 + 1 \right)^N. \tag{19}
\]

**Proof.** Let us compute the time derivative of \( E_{\varepsilon, \alpha}^{\delta} \):

\[
\frac{d}{dt} E_{\varepsilon, \alpha}^{\delta}(t) = \varepsilon \|(u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha}\|_{H^\delta}^2 - \|\nabla u^{\varepsilon, \alpha}\|_{H^\delta}^2 - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon, \alpha} \Lambda^\delta (u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha} \, dx - \\
- \varepsilon \|\partial_t u^{\varepsilon, \alpha} + (u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha}\|_{H^\delta}^2 - \frac{1}{\alpha} \|\text{div}\, u^{\varepsilon, \alpha}\|_{H^\delta}^2
\]

\[
\leq \varepsilon \|(u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha}\|_{H^\delta}^2 - \|\nabla u^{\varepsilon, \alpha}\|_{H^\delta}^2 - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon, \alpha} \Lambda^\delta (u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha} \, dx.
\]

Using \[17\], the derivative of the energy estimates as follows:

\[
\frac{d}{dt} E_{\varepsilon, \alpha}^{\delta}(t) \leq \left( C_3^2 \varepsilon \|u^{\varepsilon, \alpha}\|_{L^\infty}^2 - 1 \right) \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^2 - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon, \alpha} \Lambda^\delta (u^{\varepsilon, \alpha} \cdot \nabla) u^{\varepsilon, \alpha} \, dx.
\]

Since we assume i) and using inequality \[15\], we can write, for \( \varepsilon \) small enough,

\[
\|u_0^{\varepsilon, \alpha}\|_{L^\infty} \leq \frac{1}{2C_3\sqrt{\varepsilon}}. \tag{20}
\]

Now, by continuity of the (local) solution \( u^{\varepsilon, \alpha} \) with respect to \( t \), we deduce that \( T > 0 \) and that the inequality

\[
\frac{d}{dt} E_{\varepsilon, \alpha}^{\delta}(t) \leq - \frac{1}{4} \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^2 + C \|u^{\varepsilon, \alpha}\|_{H^{\delta}} \|u^{\varepsilon, \alpha}\|_{L^\infty} \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}
\]

holds on \([0, T]\). Then, using the interpolation inequalities

\[
\|u^{\varepsilon, \alpha}\|_{H^\delta} \leq C_4 \|u^{\varepsilon, \alpha}\|_{L^2}^{1-\delta} \|u^{\varepsilon, \alpha}\|_{H^{1}}^{\delta} \tag{21}
\]

and \[15\], we obtain the estimate

\[
\frac{d}{dt} E_{\varepsilon, \alpha}^{\delta}(t) \leq - \frac{1}{4} \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^2 + C \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^{1-\delta} \left( \|u^{\varepsilon, \alpha}\|_{H^{1}} \|u^{\varepsilon, \alpha}\|_{H^{\delta}} \right)^{\delta} \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^{2-\delta}.
\]

Finally, a Young inequality yields

\[
\frac{d}{dt} E_{\varepsilon, \alpha}^{\delta}(t) \leq C \|u^{\varepsilon, \alpha}\|_{H^1}^{2-\delta} \|u^{\varepsilon, \alpha}\|_{H^{1+\delta}}^2 E_{\varepsilon, \alpha}^{\delta}(t). \tag{22}
\]
In order to show that $E^0_{\epsilon, \alpha}$ is bounded, we shall use the decay of $E^0_{\epsilon, \alpha}$. So, let us estimate its time derivative:
\[
\frac{d}{dt}E^0_{\epsilon, \alpha}(t) \leq \epsilon \|(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha}\|_{L^2}^2 - \|u^{\epsilon, \alpha}\|_{L^2}^2 - \int_{\mathbb{R}^2} u^{\epsilon, \alpha}(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha} \, dx - \frac{1}{\alpha} \|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}^2.
\]
First, by Lemma 1, we immediately have
\[
\|(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha}\|_{L^2} \leq C_3\|u^{\epsilon, \alpha}\|_{L^\infty} \|u^{\epsilon, \alpha}\|_{H^1}.
\]
Besides, by integrations by parts, we obtain
\[
\int_{\mathbb{R}^2} u^{\epsilon, \alpha}(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha} \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} u^{\epsilon, \alpha}, \text{div} \, u^{\epsilon, \alpha} u^{\epsilon, \alpha} = -\frac{1}{2} \int_{\mathbb{R}^2} \text{div} \, u^{\epsilon, \alpha} |u^{\epsilon, \alpha}|^2.
\]
Recall that $\text{div} \, u^{\epsilon, \alpha} \in L^2(\mathbb{R}^2)$ and $u^{\epsilon, \alpha} \in L^2 \cap \dot{H}^1 \subset \dot{H}^{\frac{1}{2}} \subset L^4(\mathbb{R}^2)$. Thus the integral estimates
\[
\left| \int_{\mathbb{R}^2} u^{\epsilon, \alpha}(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha} \, dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}^2} \text{div} \, u^{\epsilon, \alpha} |u^{\epsilon, \alpha}|^2 \right| \\
\leq \frac{1}{2}\|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}\|u^{\epsilon, \alpha}\|_{L^2}^2 \\
\leq \frac{1}{2}\|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}\|u^{\epsilon, \alpha}\|_{L^4}^2 \\
\leq \frac{K}{2}\|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}\|u^{\epsilon, \alpha}\|_{L^2}\|u^{\epsilon, \alpha}\|_{H^1} \\
\leq \frac{K^2}{8}\|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}^2\|u^{\epsilon, \alpha}\|_{L^2}^2 + \frac{1}{2}\|u^{\epsilon, \alpha}\|_{H^1}^2,
\]
where we have used the Gagliardo-Nirenberg inequality
\[
\|f\|_{L^4}^2 \leq K\|f\|_{L^2}\|f\|_{H^1}
\]
for $f \in H^1(\mathbb{R}^2)$. We therefore obtain the following estimate on $E^0_{\epsilon, \alpha}$ on $[0, T)$ for $\epsilon$ small enough:
\[
\frac{d}{dt}E^0_{\epsilon, \alpha}(t) \leq \epsilon \|(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha}\|_{L^2}^2 - \|u^{\epsilon, \alpha}\|_{L^2}^2 - \int_{\mathbb{R}^2} u^{\epsilon, \alpha}(u^{\epsilon, \alpha} \cdot \nabla)u^{\epsilon, \alpha} \, dx - \frac{1}{\alpha} \|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}^2 \\
\leq \left( C_3^2 \epsilon \|u^{\epsilon, \alpha}\|_{L^\infty}^2 - \frac{1}{2}\|u^{\epsilon, \alpha}\|_{L^2}^2 \right) + \left( \frac{K^2}{8} \|u^{\epsilon, \alpha}\|_{L^2}^2 - \frac{1}{\alpha}\right) \|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}^2 \\
\leq -\frac{1}{4}\|u^{\epsilon, \alpha}\|_{H^1}^2 + \left( \frac{K^2}{8} \|u^{\epsilon, \alpha}\|_{L^2}^2 - \frac{1}{\alpha}\right) \|\text{div} \, u^{\epsilon, \alpha}\|_{L^2}^2.
\]
Moreover, assuming that $\alpha \leq \frac{2}{K^2\|u_0^{\epsilon, \alpha}\|_{L^2}^2}$, we have
\[
\frac{d}{dt}E^0_{\epsilon, \alpha}(0) \leq -\frac{1}{4}\|u_0^{\epsilon, \alpha}\|_{H^1}^2.
\]
Besides, the smallness assumptions $ii)$ on the initial data yield
\[
\|u^{\epsilon, \alpha}(t)\|_{L^2}^2 \leq 2E^0_{\epsilon, \alpha}(t) \leq 2E^0_{\epsilon, \alpha}(0) \leq 4\|u_0^{\epsilon, \alpha}\|_{L^2}^2
\]
and $E^0_{\epsilon, \alpha}$ is therefore decreasing on $[0, T)$.

Let us now define the functional $E^\delta_{\epsilon, \alpha} := E^\delta_{\epsilon, \alpha}(1 + E^0_{\epsilon, \alpha})^N$. One can easily check that
\[
\frac{d}{dt}E^\delta_{\epsilon, \alpha} \leq E^\delta_{\epsilon, \alpha}(1 + E^0_{\epsilon, \alpha})^{N-1}\|u^{\epsilon, \alpha}\|_{H^1}^2 \left( C\|u_0^{\epsilon, \alpha}\|_{L^2}^2 \frac{1}{\epsilon^2} \left( 1 + 2\|u_0^{\epsilon, \alpha}\|_{L^2}^2 \right) - \frac{N}{4}\right).
\]
So, if the positive integer $N = N(\delta, \|u_0^{\epsilon, \alpha}\|_{L^2})$ is large enough, $E^\delta_{\epsilon, \alpha}$ decreases and $E^\delta_{\epsilon, \alpha}(t)$ satisfies the estimate (11) for all $t \in [0, T)$.

The aim of the following lemma is to ensure the control of $\|u^{\epsilon, \alpha}(t)\|_{\infty}$ throughout the time, so that we can reiterate the reasoning.
Lemma 4. Assume that
\[ ε^{1+\frac{2}{3}}\|u_{1}^{ε,α}\|_{H^δ} \rightarrow 0 , \ \ ε \rightarrow 0 \] (23)
in addition to the assumptions (H) in Lemma 3. Then the inequality
\[ E_{ε,α}^{δ}(0) < C_{δ}(2\|u_{0}^{ε,α}\|_{L^2} + 1)^{-N}. \] (24)
holds for ε small enough. Moreover, for all \( t \in [0,T) \),
\[ \|u^{ε,α}\|_{L^∞} \leq \frac{1}{4C_{δ}\sqrt{ε}}. \] (25)

We skip the proof of this lemma since it can be found in [6].

Remark. Notice that, under the assumptions (1) in Theorem 2, the conditions (H) in Lemma 3 and (24) in Lemma 2 are fulfilled.

As a consequence of Lemmas 3 and 4 we obtain that \( E_{ε}^{δ} \) satisfies inequality (28) on the whole existence interval \( [0,T^{max}) \). Therefore \( (HNS^{ε,α}) \) has a global solution.

5.3 Convergence
Let \( u^{ε,α} \) and \( u^{ε} \) be the solutions to
\[
(HNS^{ε,α}) \quad \begin{array}{c}
ε\partial_{t}u^{ε,α} + \partial_{i}u^{ε,α} - \Delta u^{ε,α} = -(u^{ε,α}.\nabla)u^{ε,α} + \frac{1}{α}\nabla \text{div} u^{ε,α} \\
(HNS^{ε}) \quad \begin{array}{c}
ε\partial_{t}u^{ε} + \partial_{i}u^{ε} - \Delta u^{ε} = -(u^{ε}.\nabla)u^{ε} + \nabla p^{ε}, \\
\text{div} u^{ε} = 0
\end{array}
\end{array}
\]
with the same initial data \( u_{0}^{ε,α} = u_{0}^{ε} \in H^{1+δ}(R^{2})^{2} \) and \( u_{1}^{ε,α} = u_{1}^{ε} \in H^{δ}(R^{2})^{2} \).

Let us now define the following modulated energy
\[
E_{ε,α,u^{ε}} = \frac{1}{2}\|u^{ε,α} - u^{ε}\|_{L^{2}}^{2} + \frac{ε^{2}}{2}\|\partial_{t}(u^{ε,α} - u^{ε})\|_{L^{2}}^{2} + \frac{ε}{2}\|u^{ε,α} - u^{ε}\|_{H^{1}}^{2} + \frac{ε}{α}\|\text{div} u^{ε,α}\|_{L^{2}}^{2}
\]
which is inspired by the Dafermos modulated energy (see [3] [4]). Notice that
\[ E_{ε,α,u^{ε}}(u^{ε,α}) = E_{ε,α}(u^{ε,α} - u^{ε}). \]

Through a Gronwall estimate on this energy, we shall prove that \( u^{ε,α} \) converges to \( u^{ε} \) in the \( L^{∞}_{T}L^{2}(R^{2}) \) norm, as \( α \) goes to 0. To this end, let us estimate the time derivative of the modulated energy \( E_{ε,α,u^{ε}} \) using the equations \( (HNS^{ε}) \) and \( (HNS^{ε,α}) \). We have
\[
\frac{d}{dt}E_{ε,α,u^{ε}} = \int_{R^{2}} \left( \partial_{t}(u^{ε,α} - u^{ε}) + ε\partial_{t}(u^{ε,α} - u^{ε}) \right)(u^{ε,α} - u^{ε} + ε\partial_{t}(u^{ε,α} - u^{ε}))
+ ε^{2}\partial_{t}(u^{ε,α} - u^{ε})\partial_{t}(u^{ε,α} - u^{ε}) + 2ε\partial_{i}(u^{ε,α} - u^{ε})\n\begin{array}{c}
\partial_{i}(u^{ε,α} - u^{ε})
\end{array}
\] dx
\[
= \int_{R^{2}} \left( \partial_{t}(u^{ε,α} - u^{ε}) + ε\partial_{t}(u^{ε,α} - u^{ε}) \right)(u^{ε,α} - u^{ε} + 2ε\partial_{i}(u^{ε,α} - u^{ε}))
- ε\partial_{i}(u^{ε,α} - u^{ε})|^{2} - 2ε\partial_{t}(u^{ε,α} - u^{ε})\Delta(u^{ε,α} - u^{ε}) - \frac{2ε}{α}\partial_{t}u^{ε,α}\n\begin{array}{c}
\nabla \text{div} u^{ε,α}
\end{array}
\] dx
\[
= \int_{R^{2}} \left( \begin{array}{c}
\nabla(u^{ε,α} - u^{ε})\nabla(u^{ε,α} - u^{ε}) + (u^{ε,α}\nabla)u^{ε} - (u^{ε,α}\nabla)u^{ε,α}\n\begin{array}{c}
\nabla \text{div} u^{ε,α}
\end{array}
\end{array}
\right) \partial_{t}(u^{ε,α} - u^{ε}) - ε\partial_{t}(u^{ε,α} - u^{ε})|^{2}
+ 2ε\partial_{t}(u^{ε,α} - u^{ε}).(u^{ε,α}\nabla)u^{ε} - (u^{ε,α}\nabla)u^{ε,α} + \frac{1}{α}\nabla(u^{ε,α})\partial_{t}(u^{ε,α} - u^{ε})
+ \frac{2ε}{α}\nabla(u^{ε,α})\partial_{t}(u^{ε,α}) - \partial_{i}(u^{ε,α}) - \nabla p^{ε} - \partial_{i}(u^{ε,α} - u^{ε}) - 2ε\n\begin{array}{c}
p^{ε}
\end{array}
\] dx.
Now, using that \( \text{div} \ u_\varepsilon = 0 \) and the inequality
\[
-\varepsilon |\partial_t (u^{\varepsilon,\alpha} - u_\varepsilon)|^2 + 2\varepsilon \partial_t (u^{\varepsilon,\alpha} - u_\varepsilon). (u^{\varepsilon,\alpha} \nabla u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}) \leq \varepsilon |(u^{\varepsilon,\alpha} \nabla) u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}|^2
\]
and integrating in time, we finally obtain the estimate
\[
E_{\varepsilon,\alpha,\omega} (T) \leq E_{\varepsilon,\alpha,\omega} (0) - \|u^{\varepsilon,\alpha} - u_\varepsilon\|_{L^2_T H^1}^2 + \int_0^T \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u_\varepsilon). (u^{\varepsilon,\alpha} \nabla) u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}) \ dx \ dt + \\
+ \varepsilon \|u^{\varepsilon,\alpha} \nabla u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}\|_{L^2_T L^2}^2 - \int_0^T \int_{\mathbb{R}^2} \nabla p^{\varepsilon,\alpha} + 2\varepsilon \partial_t u^{\varepsilon,\alpha} \ dx \ dt
\]
In the following subsections, we shall estimate the terms in the right hand side.

**Notation:** We shall write \( f = \mathcal{O}(1) \) if there exists a constant \( C = C(\varepsilon, u_\varepsilon) \) such that \( f \leq C \). Similarly, \( f = \mathcal{O}(\sqrt{\alpha}) \) means that \( f \leq C(\varepsilon, u_\varepsilon)\sqrt{\alpha} \).

### 5.3.1 Estimate on \( \int_0^T \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u_\varepsilon). (u^{\varepsilon,\alpha} \nabla) u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}) \ dx \ dt \)

From the estimates on the energies \( E_{\varepsilon,\alpha} \) and \( E_{\varepsilon} \), we know that
\[
u^{\varepsilon,\alpha}, u_\varepsilon \in L^\infty_T L^2 (\mathbb{R}^2) \cap L^2_T H^1 (\mathbb{R}^2)
\]
and the boundedness of \( E_{\varepsilon,\alpha} \) yields
\[\|\text{div} u^{\varepsilon,\alpha}\|_{L^2_T L^2} = \mathcal{O}(\sqrt{\alpha}).\]
Besides, let us recall Ladyzhenskaya’s inequality:
\[
\|f\|_{L^2}^2 \leq c \|f\|_{L^2} \|f\|_{H^1}
\]
which holds for all \( f \in H^1 (\mathbb{R}^2) \). Hence we can estimate the integral
\[
I = \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u_\varepsilon). ((u^{\varepsilon,\alpha} \nabla) u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}) \ dx
\]
as follows:
\[
I = \int_{\mathbb{R}^2} [(u^{\varepsilon,\alpha} - u_\varepsilon). ((u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}) \nabla) u_\varepsilon + (u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}) \nabla (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}] \ dx
\]
\[
= \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}) . ((u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}) \nabla) u_\varepsilon \ dx + \frac{1}{2} \int_{\mathbb{R}^2} \text{div} u^{\varepsilon,\alpha} |u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}|^2 \ dx
\]
\[
\leq \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{L^2} \|\nabla u_\varepsilon\|_{L^2} + \frac{1}{2} \|\text{div} u^{\varepsilon,\alpha}\|_{L^2} \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{L^2}
\]
\[
= C \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{L^2} \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|_{H^1} (\|u^{\varepsilon,\alpha}\|_{H^1} + \|\text{div} u^{\varepsilon,\alpha}\|_{L^2})
\]
\[
\leq C \frac{\eta}{\eta} (\|u^{\varepsilon,\alpha}\|^2_{H^1} + \eta \|\text{div} u^{\varepsilon,\alpha}\|^2_{L^2})^2 \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{H^1} + \frac{\eta}{\eta} \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{H^1},
\]
where \( \eta > 0 \) is a small number to be chosen in the conclusion.

### 5.3.2 Estimate on \( \varepsilon \|u^{\varepsilon,\alpha} \nabla u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}\|^2_{L^2_T L^2} \)

Let us set
\[
A^{\varepsilon,\alpha} (t) = \varepsilon \|u^{\varepsilon,\alpha} \nabla u_\varepsilon - (u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha}\|_{L^2_T L^2}^2 - \frac{1}{2} \|u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\|^2_{L^2_T H^1}
\]
and write
\[
(u^{\varepsilon,\alpha} \nabla) u^{\varepsilon,\alpha} - (u^{\varepsilon,\alpha} \nabla) u_\varepsilon = (u^{\varepsilon,\alpha} \nabla) (u^{\varepsilon,\alpha} - u_\varepsilon) + ((u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}) \nabla) u_\varepsilon
\]
then, using Lemma 1 estimate

\[ \|(u^\varepsilon, \nabla u^\varepsilon - (u^\varepsilon, \alpha, \nabla) u^\varepsilon, \alpha\|_{L^2} \leq \|u^\varepsilon, \alpha\|_{L^\infty} \|u^\varepsilon, \alpha - u^\varepsilon\|_{H^1} + \|(u^\varepsilon, \alpha - u^\varepsilon), \nabla) u^\varepsilon\|_{L^2}. \]

The second term on the right hand side estimates using the Sobolev embeddings

\[ \dot{H}^s \subset L^\frac{2}{s+2}(\mathbb{R}^2) \]

as follows. Notice that \( \nabla u^\varepsilon \in L^\infty_T \dot{H}^\delta \subset L_x^\infty \dot{H}^{\frac{3}{2}} \) and

\[ u^\varepsilon, \alpha - u^\varepsilon \in \dot{L}_x^2 \dot{H}^1 \subset \dot{L}_x^\frac{3}{2} \dot{H}^{1-\delta} \subset \dot{L}_x^2 L^{\frac{3}{2}}. \]

So we have

\[ \|((u^\varepsilon, \alpha - u^\varepsilon), \nabla) u^\varepsilon\|_{L^2} \leq \|u^\varepsilon, \alpha - u^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\frac{2}{s+2}} \leq C \|u^\varepsilon, \alpha - u^\varepsilon\|_{L^2} \|u^\varepsilon, \alpha - u^\varepsilon\|_{H^1} \|\nabla u^\varepsilon\|_{H^1}. \]

Now, integrating in time the squared norm and performing a Young inequality, we obtain

\[ \|((u^\varepsilon, \alpha - u^\varepsilon), \nabla) u^\varepsilon\|_{L^2}^2 \leq \frac{C_s}{\eta} \int_0^T \|u^\varepsilon\|_{H^{1+\delta}}^2 E_{\varepsilon, \alpha, u^\varepsilon}(t) \, dt + \frac{C_s}{\eta} \int_0^T \|u^\varepsilon\|_{H^{1+\delta}}^2 E_{\varepsilon, \alpha, u^\varepsilon}(t) \, dt. \]

Finally, recalling that \( \varepsilon \left( \|u^\varepsilon\|_{L^2_x \dot{H}^{1+\delta}}^2 + \|u^\varepsilon\|_{L^2_x \dot{H}^{1+\delta}}^2 \right) = \mathcal{O}(1) \), we obtain the inequality

\[ \int_0^T A^\varepsilon, \alpha \, dt \leq \left( C \varepsilon \|u^\varepsilon, \alpha\|_{L^\infty}^2 + \frac{C_s}{\eta} \int_0^T \|u^\varepsilon\|_{H^{1+\delta}}^2 E_{\varepsilon, \alpha, u^\varepsilon}(t) \, dt \right) \leq \frac{C_s}{\eta} \int_0^T \|u^\varepsilon\|_{H^{1+\delta}}^2 E_{\varepsilon, \alpha, u^\varepsilon}(t) \, dt \]

if \( \varepsilon \) and \( \eta \) are small enough.

**5.3.3 Estimate on \( \int_0^T \int_{\mathbb{R}^2} \nabla p^\varepsilon, u^\varepsilon, \alpha \, dx \, dt \)**

First, notice that, applying the div operator to equation \((HNS^\varepsilon)\), we obtain the identity

\[ \Delta p^\varepsilon = \text{div} (u^\varepsilon, \nabla) u^\varepsilon = \text{div} \nabla : u^\varepsilon \otimes u^\varepsilon \]

(27)

from which we deduce that \( p^\varepsilon \) has the same regularity as \( u^\varepsilon \otimes u^\varepsilon \) and

\[ \|p^\varepsilon\|_{L^2_x L^2} \leq C \|u^\varepsilon \otimes u^\varepsilon\|_{L^2_x L^2}. \]

Since \( u^\varepsilon \in L_x^\frac{3}{2} \dot{H}^1 \cap \dot{L}_x^\infty \dot{L}^2 \subset \dot{L}_x^\frac{3}{2} \dot{L}^4(\mathbb{R}^2)^2 \), we immediately conclude that

\[ \int_0^T \int_{\mathbb{R}^2} \nabla p^\varepsilon, u^\varepsilon, \alpha \, dx \, dt = - \int_0^T \int_{\mathbb{R}^2} p^\varepsilon \cdot \text{div} u^\varepsilon, \alpha \, dx \, dt \leq \|p^\varepsilon\|_{L^2_x L^2} \|\text{div} u^\varepsilon, \alpha\|_{L^2_x L^2} \leq C \sqrt{\alpha} \|p^\varepsilon\|_{L^2_x L^2} \leq C \sqrt{\alpha} \|u^\varepsilon \otimes u^\varepsilon\|_{L^2_x L^2} \leq C \sqrt{\alpha} \|u^\varepsilon\|_{L^2_x L^4}^2 = \mathcal{O}(\sqrt{\alpha}). \]
5.3.4 Estimate on $2\varepsilon \int_0^T \int_{\mathbb{R}^2} \nabla p^\varepsilon \cdot \partial_t u^{\varepsilon,\alpha} \, dx \, dt$

From identity (27), we know that

$$\partial_t p^\varepsilon = \sum_{i,j=1}^2 \partial_i \partial_j \frac{1}{\varepsilon} (u^\varepsilon_i u^\varepsilon_j).$$

Besides, due to the control of the energies $E_0^\varepsilon$ and $E_\delta^\varepsilon$, we have

$$\partial_t u^\varepsilon \in L^2_\varepsilon L^2 \quad u^\varepsilon \in L^\infty_T H^{1+\delta} \subset L^\infty_T L^\infty.$$

After two integrations by parts (one in each variable), we obtain an integral which easily estimates:

$$\int_0^T \int_{\mathbb{R}^2} \nabla p^\varepsilon \cdot \partial_t u^{\varepsilon,\alpha} \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} \partial_t p^\varepsilon \cdot \text{div} u^{\varepsilon,\alpha} \, dx \, dt \leq \|\partial_t p^\varepsilon\|_{L^2_\varepsilon L^2} \|\text{div} u^{\varepsilon,\alpha}\|_{L^2_\varepsilon L^2} \leq C\sqrt{\alpha} \|\partial_t u^\varepsilon\|_{L^2_\varepsilon L^2} \|u^\varepsilon\|_{L^\infty_T L^\infty} = O(\sqrt{\alpha}).$$

5.3.5 Conclusion

First, notice that since we take the same initial data for $(HNS^\varepsilon,\alpha)$ and $(HNS_\varepsilon)$, we have in particular

$$\text{div} u^{\varepsilon,\alpha} = 0 \quad \text{so} \quad E_{\varepsilon,\alpha,u^\varepsilon}(0) = 0.$$ 

Now, gathering the estimates in the previous subsections, we obtain that

$$E_{\varepsilon,\alpha,u^\varepsilon}(T) \leq O(\sqrt{\alpha}) + C_\delta,\eta \int_0^T \left( \|u^\varepsilon\|_{H^{1+\delta}}^2 + \varepsilon \|u^\varepsilon\|_{H^{1+\delta}}^2 + O(\alpha) \right) E_{\varepsilon,\alpha,u^\varepsilon}(t) \, dt + \left( \eta + (1 - \delta)\varepsilon \|u^\varepsilon\|_{H^{1+\delta}}^2 \right) \frac{1}{2} \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^2_\varepsilon H^{1+\delta}}^2.$$

So, choosing $\eta$ small enough, Gronwall’s lemma yields, for all positive $T$,

$$E_{\varepsilon,\alpha,u^\varepsilon}(T) = O(\sqrt{\alpha}).$$

Now, recall that Theorem 1 tells that $u^\varepsilon$ converges towards the solution $v$ to $(NS)$. Theorem 2 is now proved.

6 The 3D case

6.1 Preliminary estimates

As in subsection 5.1, we start by recalling the regularity results we have on $u^\varepsilon$. All the proofs are in [6].

**Lemma 5.** Let $u^\varepsilon \in L^\infty_T (H^{1/2+\delta} \cap H^{1/2+\delta})(\mathbb{R}^3)^3$ be the global solution to $(HNS^\varepsilon)$ with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1/2+\delta} \times H^{1/2+\delta}(\mathbb{R}^3)^3$. Then we have

$$u^\varepsilon \in L^\infty_T H^{1/2+\delta} \cap L^2_T H^{1/2+\delta} \cap L^\infty_T H^{1/2} \cap L^\infty_T H^{1/2}$$

and

$$\partial_t u^\varepsilon \in L^2_T H^{1/2} \cap L^\infty_T H^{1/2+\delta}.$$

**Remark.** In this paper, we are not interested in the dependence of the norms on $\varepsilon$. 

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Proof. Here is a sketch of the proof. First, let us define the 3D energy

\[ E_\varepsilon^{1+\delta}(t) = \frac{1}{2}||u^\varepsilon + \varepsilon \partial_t u^\varepsilon||_{H^{\frac{1}{2} + \delta}}^2 + \frac{\varepsilon^2}{2}||\partial_t u^\varepsilon||_{H^{\frac{1}{2} + \delta}}^2 + \varepsilon||\nabla u^\varepsilon||_{H^{\frac{1}{2} + \delta}}^2 \]

for non-negative \( \delta \). In order to obtain that the solutions \( u^\varepsilon \) to \((HNS_2^\varepsilon)\) are global, we proved in [6] that

\[ \exists C > 0 : \forall t \geq 0, \quad E_\varepsilon^{1+\delta}(t) \leq C \varepsilon^{-\delta}. \] (28)

Directly from the expression of \( E_\varepsilon^{\delta} \) and from (28), we deduce that

\[ \partial_t u^\varepsilon \in L_T^\infty \dot{H}^{1+\delta}, \quad u^\varepsilon \in L_T^\infty \dot{H}^{\frac{1}{2} + \delta} \cap L_T^2 \dot{H}^{\frac{3}{2} + \delta}. \]

Besides, let us consider the time derivative of \( E_\varepsilon^{\frac{1}{2}} \). According to [6], we have

\[ E_\varepsilon^{\frac{1}{2}}(T) + \varepsilon||\partial_t u^\varepsilon + (u^\varepsilon, \nabla) u^\varepsilon||_{L_T^2 \dot{H}^{\frac{1}{2}}}^2 + ||u^\varepsilon||_{L_T^2 \dot{H}^{\frac{3}{2}}}^2 - \varepsilon||((u^\varepsilon, \nabla) u^\varepsilon)||_{L_T^2 H^{\frac{1}{2}}}^2 = E_\varepsilon^{\frac{1}{2}}(0) + I(\varepsilon, T), \]

where

\[ I(\varepsilon, T) := \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}((u^\varepsilon, \nabla) u^\varepsilon, \Lambda^{\frac{1}{2}} u^\varepsilon) \, dx \, dt \leq \|u^\varepsilon\|_{L_T^\infty \dot{H}^{1+\delta}} \|u^\varepsilon\|_{L_T^2 \dot{H}^{\frac{3}{2}}}. \]

The \( \Lambda \) above is the Fourier multiplier defined by \( \hat{\Lambda}f(\xi) = |\xi|\hat{f}(\xi) \).

So, since \( E_\varepsilon^{\frac{1}{2}}(0) \leq C \) and \( ||u^\varepsilon||_{L_T^\infty \dot{H}^{1+\delta}} \leq 2||u_0^\varepsilon||_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{8} \) by the decay of \( E_\varepsilon^{\frac{1}{2}} \), we obtain the inequality

\[ E_\varepsilon^{\frac{1}{2}}(T) + \varepsilon||\partial_t u^\varepsilon + (u^\varepsilon, \nabla) u^\varepsilon||_{L_T^2 \dot{H}^{\frac{1}{2}}}^2 + \frac{1}{2}||u^\varepsilon||_{L_T^2 \dot{H}^{\frac{3}{2}}}^2 \leq C \]

if \( \varepsilon \) is small enough. Now, we complete the reasoning as in the 2D case and deduce that

\[ \partial_t u^\varepsilon \in L_T^2 \dot{H}^{\frac{1}{2}}, \quad u^\varepsilon \in L_T^2 \dot{H}^{\frac{3}{2}}. \]

Finally, since \( E_\varepsilon^{\frac{1}{2}} \) is bounded, we immediately have

\[ u^\varepsilon \in L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^\infty \dot{H}^{\frac{3}{2}}. \]

### 6.2 Globalization

As in the 2D case, we shall prove that the energy

\[ E_{\varepsilon, \alpha}^{1+\delta}(t) = \frac{1}{2}||u^{\varepsilon, \alpha} + \varepsilon \partial_t u^{\varepsilon, \alpha}||_{H^{\frac{1}{2} + \delta}}^2 + \frac{\varepsilon^2}{2}||\partial_t u^{\varepsilon, \alpha}||_{H^{\frac{1}{2} + \delta}}^2 + \varepsilon||u^{\varepsilon, \alpha}||_{H^{\frac{1}{2} + \delta}}^2 + \frac{\varepsilon}{\alpha}||\text{div} u^{\varepsilon, \alpha}||_{H^{\frac{1}{2} + \delta}}^2 \]

is bounded in order to show that the local solution obtained by the contraction argument is global. First, let us define \( 0 \leq T \leq T^{\text{max}} \) by

\[ T = \sup \left\{ 0 \leq \tau \leq T^{\text{max}} : \forall t \in [0, \tau), \quad ||u^{\varepsilon, \alpha}(t)||_{L^\infty} < \frac{1}{2K_1 \sqrt{\varepsilon}} \right\}, \]

where \( K_1 = \max(K, \tilde{K}) \) such that

\[ ||(u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha}||_{H^{\frac{1}{2}}} \leq K ||u^{\varepsilon, \alpha}||_{L^\infty} ||u^{\varepsilon, \alpha}||_{\dot{H}^{\frac{1}{2}}}, \]

(30)

\[ ||(u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha}||_{H^{\frac{1}{2} + \delta}} \leq \tilde{K} ||u^{\varepsilon, \alpha}||_{L^\infty} ||u^{\varepsilon, \alpha}||_{\dot{H}^{\frac{1}{2} + \delta}}. \]

(31)
We shall start by proving that the energy $E^{\frac{1}{2},\alpha}_{\varepsilon}$ decreases on $[0, T)$. To this end, we estimate its time derivative:
\[
\frac{dE^{\frac{1}{2},\alpha}_{\varepsilon}}{dt} \leq \varepsilon \| (u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2 - \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2 - \int_{\mathbb{R}^3} \Lambda^\frac{1}{2}(u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} \cdot \Lambda^\frac{1}{2} u^{\varepsilon,\alpha} dx \\
\leq (K_1^2 \varepsilon \| u^{\varepsilon,\alpha} \|_{L^\infty}^2 - 1) \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2 - \int_{\mathbb{R}^3} \Lambda^\frac{1}{2}(u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} \cdot \Lambda^\frac{1}{2} u^{\varepsilon,\alpha} dx.
\]
So, for $\varepsilon$ small enough, we have
\[
\frac{dE^{\frac{1}{2},\alpha}_{\varepsilon}}{dt} \leq -\frac{3}{4} \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2 + \int_{\mathbb{R}^3} (u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} \cdot \Lambda u^{\varepsilon,\alpha} dx
\]
on $[0, T)$. Then, recalling that $\text{div} u^{\varepsilon,\alpha} \neq 0$, we have
\[
\int_{\mathbb{R}^3} (u^{\varepsilon,\alpha} \cdot \nabla) u^{\varepsilon,\alpha} \cdot \Lambda u^{\varepsilon,\alpha} dx = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u^{\varepsilon,\alpha}_i \partial_i u^{\varepsilon,\alpha}_j \partial_j u^{\varepsilon,\alpha}_i dx \\
= -\int_{\mathbb{R}^3} \left( \sum_{i,j=1}^3 \partial_j u^{\varepsilon,\alpha}_i \partial_i u^{\varepsilon,\alpha}_j u^{\varepsilon,\alpha}_i + \sum_{i=1}^3 (u^{\varepsilon,\alpha}_i)^2 \partial_i \text{div} u^{\varepsilon,\alpha} \right) dx \\
= \sum_{i=1}^3 \int_{\mathbb{R}^3} u^{\varepsilon,\alpha}_i \partial_i u^{\varepsilon,\alpha}_i \text{div} u^{\varepsilon,\alpha} dx \\
\leq \sum_{i=1}^3 \| u^{\varepsilon,\alpha}_i \|_{L^3} \| \partial_i u^{\varepsilon,\alpha} \|_{L^3} \| \text{div} u^{\varepsilon,\alpha} \|_{L^3} \\
\leq 3K_2^3 \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}} \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}} \| \text{div} u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}} \\
\leq 9K_2^3 \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2 \| u^{\varepsilon,\alpha} \|_{H^\frac{1}{2}}^2,
\]
where $K_2$ is the constant such that
\[
\| f \|_{L^3(\mathbb{R}^3)} \leq K_2 \| f \|_{H^\frac{1}{2}(\mathbb{R}^3)}.
\]
Now, assume that
\[
\| u^{\varepsilon,\alpha}_0 \|_{H^\frac{1}{2}} \leq \frac{1}{36K_2^3}.
\tag{32}
\]
Then we obtain
\[
\frac{d}{dt} E^{\frac{1}{2},\alpha}_{\varepsilon}(0) \leq \left( -\frac{3}{4} + 9K_2^3 \| u^{\varepsilon,\alpha}_0 \|_{H^\frac{1}{2}} \right) \| u^{\varepsilon,\alpha}_0 \|_{H^\frac{1}{2}}^2 < -\frac{1}{4} \| u^{\varepsilon,\alpha}_0 \|_{H^\frac{1}{2}}^2
\]
and $E^{\frac{1}{2},\alpha}_{\varepsilon}$ therefore decreases on an interval $[0, \tau)$. Set
\[
\tau = \sup \left\{ t < T : E^{\frac{1}{2},\alpha}_{\varepsilon} \text{ decreases on } [0, t] \right\}.
\]
Since $E^{\frac{1}{2},\alpha}_{\varepsilon}$ decreases on $[0, \tau]$, we have
\[
\| u^{\varepsilon,\alpha}(\tau) \|_{H^\frac{1}{2}}^2 \leq 2E^{\frac{1}{2},\alpha}_{\varepsilon}(0) \leq 4 \| u^{\varepsilon,\alpha}_0 \|_{H^\frac{1}{2}}^2 \leq \left( \frac{1}{18K_2^3} \right)^2
\]
so that
\[
\frac{d}{dt} E^{\frac{1}{2},\alpha}_{\varepsilon}(\tau) \leq -\frac{1}{4} \| u^{\varepsilon,\alpha}(\tau) \|_{H^\frac{1}{2}}^2.
\]
Thus $E^{\frac{1}{2},\alpha}_{\varepsilon}$ decreases on $[0, \tau]$, where $\tau' > \tau$. We have proved by contradiction that the energy decays on the whole interval $[0, T)$.

Now, we can prove the following lemma:
Lemma 6. Assume that \( \|u_{0}^{\varepsilon,\alpha}\|_{H^{2}} \leq \frac{1}{36K^2} \) and
\[
(H') \quad i) \ \varepsilon^{\frac{1}{3}+\delta} \|u_{0}^{\varepsilon,\alpha}\|_{H^{2+\delta}} = o(1), \quad ii) \ \varepsilon^{\frac{3}{2}} \|u_{0}^{\varepsilon,\alpha}\|_{H^{2+\delta}} = o(1).
\] (33)
when \( \varepsilon \) goes to zero.
Now, recall that \( 0 \leq T \leq T_{max} \) is defined by
\[
T = \sup \left\{ 0 \leq \tau \leq T_{max} : \forall \ t \in [0, \tau), \ ||u^{\varepsilon,\alpha}(t)||_{L^{\infty}} < \frac{1}{2K_{1}/\sqrt{\varepsilon}} \right\}.
\]
Then \( T > 0 \) and there exists a large number \( N \), depending only on \( \delta \), and a constant \( C > 1 \) such that
\[
E_{\varepsilon}^{\frac{3}{2}+\delta}(t) \leq C^{N} E_{\varepsilon}^{\frac{3}{2}+\delta}(0)
\] (34)
for all \( t \in [0, T) \) and \( \varepsilon \) small enough.

Proof. First, let us compute the time derivative of this energy:
\[
\frac{d}{dt} E_{\varepsilon,\alpha}(t) = \varepsilon \|u^{\varepsilon,\alpha,\alpha} \|_{H^{2+\delta}}^{2} - \varepsilon \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} - \int_{\mathbb{R}^{3}} \Lambda^{3+\delta}(u^{\varepsilon,\alpha,\alpha}) u^{\varepsilon,\alpha} \Lambda^{3+\delta} u^{\varepsilon,\alpha} \ dx - \frac{1}{\alpha} \|\text{div} \ u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \leq \varepsilon \|u^{\varepsilon,\alpha,\alpha} \|_{H^{2+\delta}}^{2} - \varepsilon \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} + \int_{\mathbb{R}^{3}} \Lambda^{3+\delta}(u^{\varepsilon,\alpha,\alpha}) u^{\varepsilon,\alpha} \Lambda^{3+\delta} u^{\varepsilon,\alpha} \ dx.
\]
On the time interval \([0, T)\), we have \( K_{1}^{2} \varepsilon \|u^{\varepsilon,\alpha} \|_{L^{\infty}}^{2} - 1 \leq -\frac{1}{2} \) and we can estimate the integral on the right hand side by
\[
\int_{\mathbb{R}^{3}} \Lambda^{3+\delta}(u^{\varepsilon,\alpha,\alpha}) u^{\varepsilon,\alpha} \Lambda^{3+\delta} u^{\varepsilon,\alpha} \ dx \leq \|u^{\varepsilon,\alpha,\alpha} \|_{H^{2+\delta}} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}} \leq K_{1} \|u^{\varepsilon,\alpha} \|_{L^{\infty}} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}
\]
due to Lemma 1. Using that followed by \( (13) \), we obtain
\[
\frac{d}{dt} E_{\varepsilon,\alpha}(t) \leq -\frac{1}{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} - K_{1} \|u^{\varepsilon,\alpha} \|_{L^{\infty}} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}} \leq -\frac{1}{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} - C \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \leq -\frac{1}{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} - C \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \leq C_{6} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} E_{\varepsilon,\alpha}(t),
\]
where we have used standard interpolations and a Young inequality. Now, since we do not know if the energy \( E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \) decays, we will use that \( E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \) satisfies the inequality
\[
\frac{d}{dt} E_{\varepsilon,\alpha}(t) \leq -\frac{1}{4} \|u^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2}
\]
on the interval \([0, T)\).
Now, as in the 2D case, let us define the functional \( E_{\varepsilon,\alpha, N} := E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right)^{N} \). Then we have
\[
\frac{d}{dt} E_{\varepsilon,\alpha, N}(t) = \frac{d}{dt} E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right)^{N} + N E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \frac{d}{dt} E_{\varepsilon,\alpha}^{\frac{3}{2}} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right)^{N-1}
\]
\[
\leq E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right)^{N-1} \|u^{\varepsilon,\alpha}(t)\|_{H^{2+\delta}}^{2} \left[ C_{6} \|u^{\varepsilon,\alpha}(t)\|_{H^{2+\delta}}^{2} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right) - \frac{N}{4} \right]
\]
\[
\leq E_{\varepsilon,\alpha}^{\frac{3}{2}+\delta} \left( 1 + E_{\varepsilon,\alpha}^{\frac{3}{2}} \right)^{N-1} \|u^{\varepsilon,\alpha}(t)\|_{H^{2+\delta}}^{2} \left[ C_{6} u_{0}^{\varepsilon,\alpha}(t) \|u_{0}^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \left( 1 + 2 \|u_{0}^{\varepsilon,\alpha} \|_{H^{2+\delta}}^{2} \right) - \frac{N}{4} \right].
\]
Taking \( N = N(\delta) \) large enough, we obtain that \( \mathcal{E}^{\frac{1}{2}}_{\varepsilon, \alpha, N} \) decays and we have

\[
E^{\frac{1}{2} + \delta}_{\varepsilon, \alpha} (t) \leq E^{\frac{1}{2} + \delta}_{\varepsilon, \alpha} (0) \left( 1 + E^{\frac{1}{2}}_{\varepsilon, \alpha} (0) \right)^N.
\]

Finally, notice that

\[
E^{\frac{1}{2}}_{\varepsilon, \alpha} (0) \leq 2 \| u^\varepsilon_0 \|_{H^3}^2,
\]

so the proof of the lemma is finished.

As in the 2D case, the smallness assumptions on the initial data yield the boundedness of \( E^{\frac{1}{2} + \delta}_{\varepsilon, \alpha} \) for \( \alpha \) and \( \varepsilon \) small enough. From this, we deduce that the solutions to \((HNS^{\varepsilon, \alpha})\) are global.

### 6.3 Convergence

Let \( u^{\varepsilon, \alpha} \) and \( u^\varepsilon \) be the solutions to \((HNS^{\varepsilon, \alpha})\) and \((HNS^\varepsilon)\) respectively with the same initial data

\[
(u^\varepsilon_0, u^\varepsilon_1) = (u^{\varepsilon, \alpha}_0, u^{\varepsilon, \alpha}_1) \in H^{\frac{1}{2} + \delta} (\mathbb{R}^3)^3 \times H^{\frac{1}{2} + \delta} (\mathbb{R}^3)^3.
\]

In order to prove that \( u^{\varepsilon, \alpha} \) converges to \( u^\varepsilon \) in the \( L^2_T H^{\frac{1}{2}} \) norm, as \( \alpha \) goes to 0, we shall use, as in the 2D case, a variant of the Dafermos modulated energy:

\[
\mathcal{E}_{\varepsilon, \alpha, u^\varepsilon} (t) := \frac{1}{2} \| u^{\varepsilon, \alpha} - u^\varepsilon + \varepsilon \partial_t (u^{\varepsilon, \alpha} - u^\varepsilon) \|_{H^\frac{1}{2}}^2 + \frac{\varepsilon^2}{2} \| \partial_t (u^{\varepsilon, \alpha} - u^\varepsilon) \|_{H^\frac{1}{2}}^2 + \varepsilon \| u^{\varepsilon, \alpha} - u^\varepsilon \|_{H^\frac{1}{2}}^2 + \frac{\varepsilon}{\alpha} \| \text{div } u^{\varepsilon, \alpha} \|_{H^\frac{1}{2}}^2.
\]

As done in section 5.3, we shall compute the time derivative of \( \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon} \) and use equations \((HNS^{\varepsilon, \alpha})\) and \((HNS^\varepsilon)\). Doing so, we obtain the following estimate:

\[
\mathcal{E}_{\varepsilon, \alpha, u^\varepsilon} (T) \leq \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon} (0) + \varepsilon \| (u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha} - (u^\varepsilon, \nabla) u^\varepsilon \|_{L^3_T H^{\frac{1}{2}}}^2 - \| u^{\varepsilon, \alpha} - u^\varepsilon \|_{L^3_T H^{\frac{1}{2}}}^2
\]

\[
- \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} p^\varepsilon \cdot \Lambda^{\frac{1}{2}} \text{div } u^{\varepsilon, \alpha} \ dx dt + 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \partial_t p^\varepsilon \cdot \Lambda^{\frac{1}{2}} \text{div } u^{\varepsilon, \alpha} \ dx dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^3} [(u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha} - (u^\varepsilon, \nabla) u^\varepsilon] \cdot (\Lambda (u^{\varepsilon, \alpha} - u^\varepsilon)) \ dx dt - \frac{1}{\alpha} \| \text{div } u^{\varepsilon, \alpha} \|_{L^3_T H^{\frac{1}{2}}}^2.
\]

#### 6.3.1 Estimate on \( \varepsilon \| (u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha} - (u^\varepsilon, \nabla) u^\varepsilon \|_{L^3_T H^{\frac{1}{2}}}^2 \)

First, \( A := (u^{\varepsilon, \alpha}, \nabla) u^{\varepsilon, \alpha} - (u^\varepsilon, \nabla) u^\varepsilon = ((u^{\varepsilon, \alpha} - u^\varepsilon), \nabla) u^\varepsilon + (u^{\varepsilon, \alpha}, \nabla) (u^{\varepsilon, \alpha} - u^\varepsilon) \).

Let us estimate the RHS using the dyadic Littlewood-Paley decomposition. First, by Lemma 7 in the appendix, we have

\[
\| (u^{\varepsilon, \alpha} - u^\varepsilon), \nabla u^\varepsilon \|_{H^{\frac{1}{2}}}^2 \leq \sum_{i,j=1}^3 \sum_{p \in \mathbb{Z}} 2^p \| \Delta_p (u^{\varepsilon, \alpha}_i - u^\varepsilon_i) \cdot S_{p+1} (\partial_i u^\varepsilon_j) \|_{L^2}^2 + \sum_{i,j=1}^3 \sum_{q \in \mathbb{Z}} 2^q \| \Delta_q (\partial_i u^\varepsilon_j) \cdot S_q (u^{\varepsilon, \alpha}_i - u^\varepsilon_i) \|_{L^2}^2.
\]

The first term estimates easily:

\[
\sum_p 2^p \| \Delta_p (u^{\varepsilon, \alpha}_i - u^\varepsilon_i) \cdot S_{p+1} (\partial_i u^\varepsilon_j) \|_{L^2}^2 \leq \sum_p 2^p \| \Delta_p (u^{\varepsilon, \alpha}_i - u^\varepsilon_i) \|_{L^\infty}^2 \| S_{p+1} (\partial_i u^\varepsilon_j) \|_{L^\infty}^2 \leq \| u^\varepsilon \|_{L^\infty}^2 \| u^{\varepsilon, \alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}}^2.
\]
due to property \((36)\) in the appendix. Now, notice that Sobolev embeddings imply that \(u^\varepsilon \in \dot{H}^{\frac{2}{3}+\delta} \subset W^{\frac{2}{3},\alpha}(\mathbb{R}^3)\), where \(\alpha = \frac{6}{3 - 2\varepsilon}\). Then, if \(\bar{\alpha} = \frac{3}{\varepsilon}\), we have

\[
\sum_q 2^q \| \Delta_q (\partial_i u_j^\varepsilon) \cdot S_q (u_i^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \leq \sum_q 2^q \| \Delta_q (\partial_i u_j^\varepsilon) \|_{L^\infty} \| S_q (u_i^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \\
\leq \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{L^\infty}^2 \sum_q 2^{q+3q(1-\frac{2}{3})} \| \Delta_q (\partial_i u_j^\varepsilon) \|_{L^2}^2 \\
\leq C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}-\delta}}^2 \sum_q 2^{3q+3q(1-\frac{2}{3})} \| \Delta_q u_j^\varepsilon \|_{L^2}^2 \\
= C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}-\delta}}^2 \sum_q 2^{6q-\frac{6q}{3}} \| \Delta_q u_j^\varepsilon \|_{L^2}^2 \\
= C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}-\delta}}^2 \sum_q 2^{2q(\frac{1}{3}+\delta)} \| \Delta_q u_j^\varepsilon \|_{L^2}^2 \\
\leq C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}-\delta}}^2 \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}}^{1-\delta} \| u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}+\delta}} \\
\leq C_\delta \left( \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}} + \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}} \right) \| u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}+\delta}}.
\]

Now, let us estimate

\[
\| u_i^{\varepsilon,\alpha} \nabla (u_i^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{\dot{H}^{\frac{2}{3}}} \leq \sum_{i,j=1}^3 \sum_{p \in \mathbb{Z}} 2^p \| \Delta_p u_i^{\varepsilon,\alpha} \cdot S_{p+1} \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \\
+ \sum_{i,j=1}^3 \sum_{q \in \mathbb{Z}} 2^q \| \Delta_q \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \cdot S_q u_i^{\varepsilon,\alpha} \|_{L^2}^2.
\]

We know that \(u_i^{\varepsilon,\alpha} \in \dot{H}^{\frac{2}{3}+\delta} \subset W^{\frac{2}{3},\alpha}(\mathbb{R}^3)\), where \(\alpha = \frac{6}{3 - 2\varepsilon}\). Let \(\bar{\alpha} = \frac{3}{\varepsilon}\). Then

\[
\sum_p 2^p \| \Delta_p u_i^{\varepsilon,\alpha} \cdot S_{p+1} \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \leq \sum_p 2^p \| \Delta_p u_i^{\varepsilon,\alpha} \|_{L^\infty} \| S_{p+1} \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \\
\leq \sum_p 2^{3p} \| \Delta_p u_i^{\varepsilon,\alpha} \|_{L^2}^2 \| S_{p+1} (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \\
\leq \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{L^2}^2 \sum_p 2^{3p} \| \Delta_p u_i^{\varepsilon,\alpha} \|_{L^2}^2 \\
\leq \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{L^2}^2 \sum_p 2^{3p+6p(\frac{1}{3}+\delta)} \| \Delta_p u_i^{\varepsilon,\alpha} \|_{L^2}^2 \\
= \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{L^2}^2 \sum_p 2^{2p(\frac{1}{3}+\delta)} \| \Delta_p u_i^{\varepsilon,\alpha} \|_{L^2}^2 \\
\leq C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}} \| u_i^{\varepsilon,\alpha} \|_{\dot{H}^{\frac{2}{3}+\delta}}
\]

since \(\dot{H}^{\frac{2}{3}-\delta} \subset L^\infty(\mathbb{R}^3)\). So, by interpolation and a Young inequality, we get

\[
\sum_p 2^p \| \Delta_p u_i^{\varepsilon,\alpha} \cdot S_{p+1} \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^2}^2 \leq C_\delta \| u_i^{\varepsilon,\alpha} \|_{\dot{H}^{\frac{2}{3}+\delta}} \left( \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}} + \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}} \dot{H}^{\frac{2}{3}+\delta}} \right).
\]

Immediately, the remaining term estimates

\[
\sum_q 2^q \| \Delta_q \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \cdot S_q u_i^{\varepsilon,\alpha} \|_{L^2}^2 \leq C \| u_i^{\varepsilon,\alpha} - u_j^\varepsilon \|_{\dot{H}^{\frac{2}{3}}} \| u_i^{\varepsilon,\alpha} \|_{L^\infty}.
\]
Finally, we obtain
\[
\varepsilon \|A\|_{L^2_T H^{\frac{1}{2}}}^2 \leq C_{\delta, \eta} \varepsilon \int_0^T \left( \|u^{\varepsilon, \alpha}\|_{H^{\frac{3}{2}+\delta}}^2 + \|u^{\varepsilon}\|_{H^{\frac{3}{2}+\delta}}^2 \right) \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon(t)} \, dt + \\
+ C \varepsilon \left( \|u^{\varepsilon, \alpha}\|_{L^\infty}^2 + \|u^\varepsilon\|_{L^\infty}^2 \right) \|u^{\varepsilon, \alpha} - u^\varepsilon\|_{L^2_T H^{\frac{3}{2}}}^2 + \\
+ C\delta \eta \varepsilon \left( \|u^{\varepsilon, \alpha}\|_{L^\infty_T H^{\frac{3}{2}+\delta}}^2 + \|u^\varepsilon\|_{L^\infty_T H^{\frac{3}{2}+\delta}}^2 \right) \|u^{\varepsilon, \alpha} - u^\varepsilon\|_{L^2_T H^{\frac{3}{2}}},
\]
where we use that \(\varepsilon \|u^{\varepsilon, \alpha}\|_{L^\infty_T H^{\frac{3}{2}+\delta}}^2 = O(1)\) (see subsection 6.1).

6.3.2 Estimate on \(\int_0^T \int_{\mathbb{R}^3} \Lambda^\frac{1}{2} \nabla p^\varepsilon. \Lambda^\frac{1}{2} u^{\varepsilon, \alpha} \, dx \, dt\)

Applying the div operator to \((HNS^\varepsilon)\), we obtain the identity
\[
p^\varepsilon = -\frac{1}{\Delta} \text{div} (u^\varepsilon \nabla) u^\varepsilon.
\]
Then, recall that \(\text{div} u^{\varepsilon, \alpha} \in L^2_T \dot{H}^{\frac{1}{2}}\) and \(\|\text{div} u^{\varepsilon, \alpha}\|_{L^2_T \dot{H}^{\frac{1}{2}}} = O(\sqrt{\alpha})\). Now, we easily estimate the integral as follows:
\[
\left| \int_0^T \int_{\mathbb{R}^3} \Lambda^\frac{1}{2} \nabla p^\varepsilon. \Lambda^\frac{1}{2} u^{\varepsilon, \alpha} \, dx \, dt \right| = \left| \int_0^T \int_{\mathbb{R}^3} \Lambda p^\varepsilon . \text{div} u^{\varepsilon, \alpha} \, dx \, dt \right| \\
\leq \|\Lambda p^\varepsilon\|_{L^2_T L^2} \|\text{div} u^{\varepsilon, \alpha}\|_{L^2_T L^3} \\
\leq C \sum_{i,j,k=1}^3 \|\partial_i u^\varepsilon_j\|_{L^2_T L^2} \|\text{div} u^{\varepsilon, \alpha}\|_{L^2_T L^3} \\
\leq C \sum_{i,j,k=1}^3 \|\partial_i u^\varepsilon_j\|_{L^2_T L^3} \|u^\varepsilon_j\|_{L^\infty_T L^3} \|\text{div} u^{\varepsilon, \alpha}\|_{L^2_T L^3} \\
\leq C \|u^\varepsilon\|_{L^2_T \dot{H}^{\frac{1}{2}}} \|u^\varepsilon\|_{L^\infty_T \dot{H}^{\frac{1}{2}}} \|\text{div} u^{\varepsilon, \alpha}\|_{L^2_T \dot{H}^{\frac{1}{2}}} \\
= O(\sqrt{\alpha}).
\]

6.3.3 Estimate on \(2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^\frac{1}{2} \nabla p^\varepsilon. \Lambda^\frac{1}{2} \partial_t u^{\varepsilon, \alpha} \, dx \, dt\)

First, two integrations by parts (one in space and another in time) give
\[
I := 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^\frac{1}{2} \nabla p^\varepsilon. \Lambda^\frac{1}{2} \partial_t u^{\varepsilon, \alpha} \, dx \, dt = 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^\frac{1}{2} \partial_t p^\varepsilon. \Lambda^\frac{1}{2} \text{div} u^{\varepsilon, \alpha} \, dx \, dt
\]
which is easier to estimate.
Notice that \(\Lambda^\frac{1}{2} \text{div} u^{\varepsilon, \alpha} \in L^2_T L^2(\mathbb{R}^3)\). Now, recall that
\[
\Lambda^\frac{1}{2} \partial_t p^\varepsilon = \Lambda^\frac{1}{2} \partial_t \frac{1}{\Delta} \text{div} (u^\varepsilon . \nabla) u^\varepsilon.
\]
So we have
\[
\|\Lambda^\frac{1}{2} \partial_t p^\varepsilon\|_{L^2_T L^2} \leq C \|\partial_t (u^\varepsilon . \nabla) u^\varepsilon\|_{L^2_T \dot{H}^{-\frac{1}{2}}} = C \|\partial_t (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_T \dot{H}^{\frac{1}{2}}} \leq C \|\partial_t u^\varepsilon \otimes u^\varepsilon\|_{L^2_T \dot{H}^{\frac{1}{2}}}.
\]
We shall estimate this term by a dyadic Littlewood-Paley decomposition. Using Lemma [7] in the appendix, we have
\[
\|u^\varepsilon \otimes \partial_t u^\varepsilon\|_{L^2}^2 \leq \sum_{i,j} \sum_{p \in \mathbb{Z}} 2^p \|\Delta_i u^\varepsilon_j S_{p+1} \partial_t u^\varepsilon_j\|_{L^2}^2 + \sum_{i,j} \sum_{q \in \mathbb{Z}} 2^q \|\Delta_i \partial_t u^\varepsilon_j S_q u^\varepsilon_i\|_{L^2}^2.
\]
Then we estimate each term separately, using Bernstein inequalities. First, we have

\[ \sum_p 2^p \| \Delta_p u_t^\varepsilon S_{p+1} \partial_t u_j^\varepsilon \|_{L^2}^2 \leq \sum_p 2^p \| \Delta_p u_t^\varepsilon \|^2_{L^2} \| S_{p+1} \partial_t u_j^\varepsilon \|^2_{L^\infty} \]

\[ \leq C \| u_t^\varepsilon \|^2_{H^{\frac{3}{2}}} \sum_p 2^{-2p} \left( \sum_{-1 \leq k \leq p} \| \Delta_k \partial_t u_j^\varepsilon \|^2_{L^\infty} \right) \]

\[ \leq C \| u_t^\varepsilon \|^2_{H^{\frac{3}{2}}} \sum_p 2^{-2p} \sum_{-1 \leq k \leq p} 2^{4k} \| \Delta_k \partial_t u_j^\varepsilon \|^2_{L^2} \]

\[ \leq C \| u_t^\varepsilon \|^2_{H^{\frac{3}{2}}} \| \partial_t u_j^\varepsilon \|^2_{H^{\frac{3}{2}}}. \]

Similarly, we obtain that

\[ \sum_q 2^q \| \Delta_q \partial_t u_j^\varepsilon S_q u_t^\varepsilon \|_{L^2}^2 \leq \sum_q 2^q \| \Delta_q \partial_t u_j^\varepsilon \|^2_{L^2} \| S_q u_t^\varepsilon \|^2_{L^\infty} \leq C \| \partial_t u_t^\varepsilon \|^2_{H^{\frac{3}{2}} \dot{H}^{\frac{1}{2}}} \| u_t^\varepsilon \|^2_{H^{\frac{3}{2}} \dot{H}^{\frac{1}{2}}}. \]

So finally, we have

\[ I \leq C \varepsilon \left( \| u_t^\varepsilon \|_{L^p \dot{H}^{\frac{3}{2}}} \| \partial_t u_j^\varepsilon \|_{L^\infty \dot{H}^{\frac{1}{2}}} + \| \partial_t u_t^\varepsilon \|_{L^p \dot{H}^{\frac{3}{2}} \dot{H}^{\frac{1}{2}}} \| u_t^\varepsilon \|_{L^\infty \dot{H}^{\frac{3}{2}}} \right) \| \text{div } u^{\varepsilon, \alpha} \|_{L^\infty \dot{H}^{\frac{1}{2}}} \]

\[ = O (\sqrt{\varepsilon}). \]

**6.3.4 Estimate on \( \int_0^T \int_{\mathbb{R}^3} \Lambda(u^{\varepsilon, \alpha} - u^\varepsilon), (u^\varepsilon, \nabla u^\varepsilon - u^{\varepsilon, \alpha}, \nabla u^{\varepsilon, \alpha}) \) dx dt**

Let us write \( u^\varepsilon, \nabla u^\varepsilon - u^{\varepsilon, \alpha}, \nabla u^{\varepsilon, \alpha} = (u^\varepsilon - u^{\varepsilon, \alpha}), \nabla u^\varepsilon + u^{\varepsilon, \alpha}, \nabla (u^\varepsilon - u^{\varepsilon, \alpha}) \). First, using the Sobolev embeddings \( H^1 \subset L^6(\mathbb{R}^3) \) and \( \dot{H}^{\frac{1}{2}} \subset L^3(\mathbb{R}^3) \), we estimate the integral

\[ \tilde{A} := \int_0^T \int_{\mathbb{R}^3} \Lambda(u^{\varepsilon, \alpha} - u^\varepsilon), (u^\varepsilon - u^{\varepsilon, \alpha}), \nabla u^\varepsilon \) \]

as follows:

\[ \tilde{A} \leq \int_0^T \| \Lambda(u^{\varepsilon, \alpha} - u^\varepsilon) \|_{L^\infty} \| u^\varepsilon - u^{\varepsilon, \alpha} \|_{L^\infty} \| \nabla u^\varepsilon \|_{L^3} dt \]

\[ \leq C \int_0^T \| u^\varepsilon - u^\varepsilon \|^2_{H^1} \| u^\varepsilon \|^2_{\dot{H}^{\frac{1}{2}}} dt \]

\[ \leq C \int_0^T \| u^\varepsilon - u^\varepsilon \|^2_{\dot{H}^{\frac{1}{2}}} \| u^{\varepsilon, \alpha} - u^\varepsilon \|^2_{H^{\frac{1}{2}}} \| u^\varepsilon \|^2_{H^{\frac{1}{2}}} dt \]

\[ \leq \frac{C}{\eta} \int_0^T \| u^\varepsilon \|^2_{\dot{H}^{\frac{1}{2}}} \| u^{\varepsilon, \alpha} - u^\varepsilon \|^2_{L^\infty \dot{H}^{\frac{1}{2}}} dt \]

Now, we are left with the term:

\[ \Lambda(u^{\varepsilon, \alpha} - u^\varepsilon), u^{\varepsilon, \alpha}, \nabla (u^\varepsilon - u^{\varepsilon, \alpha}) = \sum_{i,j=1}^3 \partial_j (u_i^{\varepsilon, \alpha} - u_i^\varepsilon) (u_i^{\varepsilon, \alpha} - u_i^\varepsilon) \partial_i (u_j^{\varepsilon, \alpha} - u_j^\varepsilon) + \]

\[ + \sum_{i,j=1}^3 \partial_j (u_i^{\varepsilon, \alpha} - u_i^\varepsilon) u_i^\varepsilon \partial_i (u_j^{\varepsilon, \alpha} - u_j^\varepsilon) = I_1 + I_2. \]
We estimate these three terms as follows:

\[
\int_{\mathbb{R}^3} I_1 \, dx = \frac{1}{2} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i (u_i^{\varepsilon,\alpha} - u_i^\varepsilon)^2 \text{div} \, u^{\varepsilon,\alpha} \, dx
\]

\[
\leq C \sum_{i=1}^{3} \| u_i^{\varepsilon,\alpha} - u_i^\varepsilon \|_{L^3} \| \partial_i (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \|_{L^3} \| \text{div} \, u^{\varepsilon,\alpha} \|_{L^3}
\]

\[
\leq C \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
\leq \frac{C \alpha}{\eta} \sum_{i,j=1}^{3} \partial_i (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \sum_{i,j=1}^{3} \partial_j (u_j^{\varepsilon,\alpha} - u_j^\varepsilon)
\]

where \( \eta \) is small. Besides, by integrations by parts, we obtain for the second term

\[
\int_{\mathbb{R}^3} I_2 \, dx = - \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \partial_j u_j^\varepsilon \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \, dx - \sum_{i=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) u_i \partial_i \text{div} \, u^{\varepsilon,\alpha} \, dx
\]

\[
= - \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \partial_j u_j^\varepsilon \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \, dx
\]

\[
+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) u_i^\varepsilon \, \text{div} \, u^{\varepsilon,\alpha} \, dx
\]

\[
+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \partial_i u_i^\varepsilon \, \text{div} \, u^{\varepsilon,\alpha} \, dx
\]

\[
= I + II + III
\]

We estimate these three terms as follows:

\[
I \leq \sum_{i,j=1}^{3} \| u_i^{\varepsilon,\alpha} - u_i^\varepsilon \|_{L^3} \| \partial_j u_j^\varepsilon \|_{L^3} \| \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon) \|_{L^3}
\]

\[
\leq C \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| u^\varepsilon \|_{H^{\frac{1}{2}}} \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}}
\]

\[
\leq \frac{C}{\eta} \| u^\varepsilon \|_{H^{\frac{3}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
II \leq \sum_{i=1}^{3} \| \partial_i (u_i^{\varepsilon,\alpha} - u_i^\varepsilon) \|_{L^3} \| u_i^\varepsilon \|_{L^3} \| \text{div} \, u^{\varepsilon,\alpha} \|_{L^3}
\]

\[
\leq C \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| u^\varepsilon \|_{H^{\frac{1}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
\leq C \| u^\varepsilon \|_{H^{\frac{3}{2}}} \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
\leq \eta \| u^{\varepsilon,\alpha} - u^\varepsilon \|_{H^{\frac{1}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
III \leq \sum_{i=1}^{3} \| u_i^\varepsilon \|_{L^3} \| \delta_i u_i^\varepsilon \|_{L^3} \| \text{div} \, u^{\varepsilon,\alpha} \|_{L^3}
\]

\[
\leq C \| u^\varepsilon \|_{H^{\frac{3}{2}}} \| u^\varepsilon \|_{H^{\frac{1}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]

\[
\leq \frac{C}{\eta} \| u^\varepsilon \|_{H^{\frac{3}{2}}} \| \text{div} \, u^{\varepsilon,\alpha} \|_{H^{\frac{1}{2}}}
\]
Now, we shall approximate the functions \( u^\varepsilon - u^\varepsilon \). (\( u^\varepsilon \cdot \nabla u^\varepsilon - u^\varepsilon \cdot \nabla u^\varepsilon \)) dx dt \leq \int_0^T \left( C_\eta \| u^\varepsilon \|_{\dot{H}^{3/2}+\delta} + C_\eta \right) \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon}(t) dt + \frac{\| u^\varepsilon - u^\varepsilon \|_{L^2_{\varepsilon} H^{1/2}+\delta} + O(\varepsilon)}{\| u^\varepsilon - u^\varepsilon \|_{L^2_{\varepsilon} H^{1/2}+\delta} + O(\varepsilon)}.

### 6.3.5 Conclusion

Since \( \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon}(0) = 0 \), if we choose \( \eta \) and \( \varepsilon \) small enough, we obtain the estimate

\[ \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon}(T) \leq O(\sqrt{\alpha}) + \int_0^T \left[ C_{\delta, \eta} \| u^\varepsilon \|_{\dot{H}^{3/2}+\delta} + \| u^\varepsilon \|_{\dot{H}^{3/2}+\delta} + C_\eta \| u^\varepsilon \|_{\dot{H}^{3/2}+\delta} + C_\eta \right] \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon}(t) dt. \]

Now, notice that \( u^\varepsilon \in L^2_T \dot{H}^{3/2} \cap L^2_T \dot{H}^{3/2+\delta} \) and that \( \varepsilon \| u^\varepsilon \|_{L^2_{\varepsilon} H^{1/2}+\delta} = O(1) \) then apply the Gronwall’s lemma and obtain that, for all positive \( T \),

\[ \mathcal{E}_{\varepsilon, \alpha, u^\varepsilon}(T) = O(\sqrt{\alpha}). \]

As in the 2D case, Theorem 1 concludes the proof of Theorem 2.

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### Appendix A Littlewood-Paley theory

One of the main tools we use in this paper is the dyadic Littlewood-Paley decomposition. In this subsection, we briefly recall some important results. Our main references for the subject are the books by Alinhac and Gérard [1] and by Lemarié-Rieusset [8].

First, recall that the homogeneous Sobolev norm \( \dot{H}^{s} \) writes

\[ \| u \|^2_{\dot{H}^{s}} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \sum_{p \in \mathbb{Z}} 2^{ps} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi. \]

In particular, we can estimate this norm in terms of the \( L^2 \) norm of \( u|_{2^p \leq |\xi| \leq 2^{p+1}} \) as follows:

\[ \sum_{p \in \mathbb{Z}} 2^{ps} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\hat{u}(\xi)|^2 d\xi \leq \| u \|^2_{\dot{H}^{s}} \leq 2^{2s} \sum_{p \in \mathbb{Z}} 2^{2ps} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\hat{u}(\xi)|^2 d\xi. \]

Now, we shall approximate the functions \( 1_{2^p \leq |\xi| \leq 2^{p+1}} \) by smooth ones. So let us consider a nonnegative function \( \varphi \in C_0^\infty(\mathbb{R}^d) \) such that \( \varphi(\xi) = 1 \) if \( |\xi| \leq 1 \) and \( \varphi(\xi) = 0 \) if \( |\xi| \geq 1 + \varepsilon \). Then define the function

\[ \psi : \xi \mapsto \varphi \left( \frac{\xi}{2} \right) - \varphi(\xi) \]

which is supported in the annulus \( \{ 1 \leq |\xi| \leq 2(1 + \varepsilon) \} \).

For all tempered distribution \( u \), that is \( u \in \mathcal{S}'(\mathbb{R}^d) \), we define \( \Delta_p u \) the \( p \)th dyadic block of \( u \) by

\[ \Delta_p u(\xi) = \psi(2^{-p}\xi) \hat{u}(\xi) \]

and we denote the partial sums by

\[ S_p u = \sum_{q < p} \Delta_q u. \]
for all \( p \in \mathbb{Z} \).

Now, notice that \( \forall \xi \neq 0 \ , \ \sum_{p \in \mathbb{Z}} \psi(2^{-p} \xi) = 1 \). So, in particular for all the functions \( u \) considered in this work, we have

\[
\forall \xi \neq 0 \ , \ \hat{u}(\xi) = \sum_{p \in \mathbb{Z}} \hat{\Delta}_p u(\xi).
\]

**Remark.** Every dyadic block is in \( H^{1,\infty} \) once \( u \in H^s \).

We recall here some important properties of this dyadic decomposition. The results are stated for homogeneous Sobolev spaces but hold also for classical Sobolev spaces.

- **(Almost-orthogonality)** For all \( u \in L^2 \),

\[
\sum_{p \in \mathbb{Z}} \| \Delta_p u \|_{L^2}^2 \leq \| u \|_{L^2}^2 \leq 2 \sum_{p \in \mathbb{Z}} \| \Delta_p u \|_{L^2}^2.
\]

- There exists a constant \( C \) such that for all \( s \in \mathbb{R} \) and \( p \in \mathbb{Z} \),

\[
\frac{1}{C} \sum_{p \in \mathbb{Z}} 2^{2ps} \| \Delta_p u \|_{L^2}^2 \leq \| u \|_{\dot{H}^s}^2 \leq C \sum_{p \in \mathbb{Z}} 2^{2ps} \| \Delta_p u \|_{L^2}^2.
\]

We will use this property several times in this work.

- There exists a constant \( C \) such that for all \( \alpha \in \mathbb{N}^d \) and \( p \in \mathbb{Z} \),

\[
\| \partial^\alpha \Delta_p u \|_{L^2} \leq C 2^{p|\alpha|} \| u \|_{L^2} \ , \ \| \partial^\alpha S_p u \|_{L^2} \leq C 2^{p|\alpha|} \| u \|_{L^2},
\]

\[
\| \partial^\alpha \Delta_p u \|_{L^\infty} \leq C 2^{p|\alpha|} \| u \|_{L^\infty} \ , \ \| \partial^\alpha S_p u \|_{L^\infty} \leq C 2^{p|\alpha|} \| u \|_{L^\infty}.
\]

Using this theory, we can prove some important product estimates. The following lemma tells how to write a product of tempered distributions:

**Lemma 7** (Paraproduct). Given two tempered distributions \( u, v \in \mathcal{S}'(\mathbb{R}^d) \), we can write their product, if it exists, as follows:

\[
uv = \sum_{p \in \mathbb{Z}} \Delta_p u S_{p+1} v + \sum_{q \in \mathbb{Z}} \Delta_q v S_q u.
\]

Then, the following proposition is very useful to estimate some products.

**Proposition 1.** Let \( s > 0 \) and \( u, v \in L^\infty \cap \dot{H}^s \). Then the product \( uv \) is also in \( L^\infty \cap \dot{H}^s \) and

\[
\| uv \|_{L^\infty \cap \dot{H}^s} \leq C \left( \| u \|_{L^\infty} \| v \|_{\dot{H}^s} + \| u \|_{H^s} \| v \|_{L^\infty} \right).
\]

For a detailed proof of this proposition, see the book by Alinhac and Gérard. In this paper, we also use Bernstein inequalities:

**Proposition 2.** There exists a positive constant \( C_0 \) such that, for all \( u \in \mathcal{S}'(\mathbb{R}^d) \), we have

\[
\begin{cases}
\forall k \geq 0, \ \forall p \geq 1, \ \ 2^{nk} C_0^{-k} \| \Delta^k u \|_{L^p} \leq \sup_{|\alpha|=k} \| \partial^\alpha \Delta^k u \|_{L^p} \leq 2^{nk} C_0^k \| \Delta^k u \|_{L^p} \\
\forall p' \geq p \geq 1, \ \ |\Delta^k u|_{L^{p'}} \leq C_0 2^{k} k^{n/d} \left( 1 + \frac{d}{p'} \right) \| \Delta^k u |_{L^{p}}
\end{cases}
\]

and, for the partial sums:

\[
\begin{cases}
\forall k \geq 0, \ \forall p \geq 1, \ \ \sup_{|\alpha|=k} \| \partial^\alpha S_k u \|_{L^p} \leq 2^{nk} C_0^k \| S_k u \|_{L^p} \\
\forall p' \geq p \geq 1, \ \ |S_k u|_{L^{p'}} \leq C_0 2^{kp} \left( 1 + \frac{d}{p'} \right) \| S_k u |_{L^{p}}
\end{cases}
\]

Using the dyadic blocks, we can easily define the homogeneous Besov spaces \( \dot{B}_{p,r}^s(\mathbb{R}^d) \), where \( s \in \mathbb{R} \), \( p, r \geq 1 \) and \( d \geq 1 \),

\[
\dot{B}_{p,r}^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \| u \|_{\dot{B}_{p,r}^s} := \left( \sum_{j \geq -1} 2^{jsr} \| \Delta^j u \|_{L^p}^r \right)^{\frac{1}{r}} \right\}.
\]

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