THE $\alpha$-INVARIANTS ON TORIC FANO MANIFOLDS

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§1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian\cite{14}, Tian and Yau\cite{13} is closely related to the existence of Kähler-Einstein metrics. In his solution to the Calabi conjecture, Yau\cite{19} proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with nonpositive first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there exist known obstructions such as the Futaki invariant. For a compact Kähler manifold $M$ with positive first Chern class, Tian\cite{14} proved that $M$ admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $\mathbb{C}P^2 \# 1 \mathbb{C}P^2$ and $\mathbb{C}P^2 \# 2 \mathbb{C}P^2$\cite{16}.

There have been many nice results on the classification of toric Fano manifolds. Mabuchi discovered that if a toric Fano manifold is Kähler-Einstein then the barycenter of the polyhedron defined by its anticanonical divisor is at the origin. V. Batyrev and E. Selivanova\cite{2} estimate the lower bound of $\alpha$-invariant of symmetric toric Fano manifolds which is sufficient to show the existence of Kähler-Einstein metric.

In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman kernel on polarized Kähler metrics to approximate plurisubharmonic functions and obtain a formula to calculate the $\alpha_G$-invariants of all toric Fano manifolds precisely. This gives a generalization of the result by V. Batyrev and E. Selivanova\cite{2} and also this formula confirms the earlier result\cite{12} on the estimates of $\alpha$ invariants on $\mathbb{C}P^2 \# 1 \mathbb{C}P^2$ and $\mathbb{C}P^2 \# 2 \mathbb{C}P^2$.

Our main theorems are

**Theorem 1.1** If $X$ is a toric Fano manifold of complex dimension $n$ then
(a) $\alpha_G(X) = 1$ if $X$ is symmetric, otherwise
(b) $\alpha_G(X) = \frac{\min_{0 \neq v \in S} \frac{|wv|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|wv|}{|v|}} \leq \frac{1}{2}$. 

1
Corollary 1.1  If $X$ is a toric Fano manifold, then $X$ is symmetric if and only if $\alpha_G(X) = 1$.

Theorem 1.2  If $X$ is a toric Fano manifold, then $\{\alpha_{m,G}(X)\}_{m \geq 1}$ is stationary. More precisely, $\alpha_{m,G}(X) = \alpha_G(X)$ if $m \geq m_0$, where $m_0$ is the least positive integer such that $m_0v$ is an integral point and $v$ is the minimizer of $\min_{0 \neq v \in S} \frac{|w_v|}{|v|}$.

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§2. Notations

Let us first recall the definition of an invariant $\alpha_G(X)$ introduced by Tian. Let $X$ be an $n$-dimensional compact complex manifold with positive first Chern class $c_1(X)$ and $G$ a compact subgroup of $Aut(X)$. Choose a $G$-invariant Kähler metric $g = g^\omega_G$ on $X$ such that $\omega_g = \frac{i}{2\pi} \sum g_{ij} dz_i \wedge d\bar{z}_j \in c_1(X)$.

Definition  Let $P_G(X, g)$ be the set of all $C^2$-smooth $G$-invariant real-valued functions $\varphi$ such that $\sup_X \varphi = 0$ and $\omega_g + \frac{i}{2\pi} \partial \bar{\partial} \varphi > 0$. The $\alpha_G(X)$ invariant is defined as superemum of all $\lambda > 0$ such that

$$\int_X e^{-\lambda \varphi} \omega^n \leq C(\alpha)$$

for all $\varphi \in P_G(X, g)$, where $C(\alpha)$ is a positive constant depending only on $\alpha, g$ and $X$.

Let $N$ be a lattice of rank $n$, $M = Hom(N, Z)$ the dual lattice. $M_R = M \otimes Z \mathbb{R}$, $N_R = N \otimes Z \mathbb{R}$. Let $X = X_\Sigma$ be a smooth projective toric $n$-fold defined by a complete fan $\Delta$ of regular cones $\Delta \subset M_R$ and denote $\Delta(i)$ the $i$-dimensional cone of $\Delta$. We put $T = C^* = \{(t_1, t_2, ..., t_n) | t_i \in \mathbb{C}^* \}$. For $a \in M$ and $b \in N$, we define $\langle a, b \rangle \in \mathbb{Z}$, $\chi^a \in Hom_{algpo}(T, C^*)$ by

$$\langle a, b \rangle = \sum_{i=1}^n a_i b_i,$$

$$\chi^a((t_1, ..., t_n)) = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}.$$
For each $\rho \in \Delta(1)$, let $b_\rho$ denote the unique fundamental generator of $\rho$. We now consider the divisor $K = -\sum_{\rho \in \Delta(1)} D(\rho)$ on $X = X_\Delta$. The following theorem is due to Demazure\[4\].

**Theorem 2.3** $K$ is a canonical divisor of $X_\Delta$, and the following are equivalent:

(a) $X_\Delta$ is a toric fano manifold.
(b) $-K$ is ample.
(c) $-K$ is very ample.
(d) $\Sigma_{-K} = \{ a \in M_R | <a, b_\rho> \leq 1 \text{ for all } \rho \in \Delta(1) \}$ is an $n$-dimensional compact convex polyhedron whose vertices are exactly $\{a_\tau | \tau \in \Delta(n)\}$, where each $a_\tau$ denotes the unique element of $M$ such that $<a_\tau, b> = 1$ for all fundamental generators $b$ of $\tau$.

The maximal torus $T \subset Aut(X)$ acting on $X$ has an open dense orbit $U \subset X$, so the normalizer $N(T) \subset Aut(X)$ of $T$ has a natural action on $U$. Let $W(X) = N(T)/T$ and we identify the maximal torus $T \subset Aut(X)$ with an open dense orbit $U$ in $X$ by choosing an arbitrary point $x_0 \in U$, then we have the following splitting short exact sequence

$$1 \to T \to N(T) \to W(X) \to 1,$$

i.e., an embedding $W(X) \hookrightarrow N(T)$. Denote by $K(T) = (S^1)^n$ the maximal compact subgroup in $T$. We choose $G$ to be the maximal compact subgroup in $N(T)$ generated by $W(X)$ and $K(T)$, so that we have the short exact sequence

$$1 \to K(T) \to G \to W(X) \to 1.$$

**Proposition 2.1** Let $X = X_\Delta$ be a smooth projective toric $n$-fold defined by a complete regular polyhedral fan $\Delta$. Then the group $W(X)$ is isomorphic to the finite group of all symmetries of $\Delta$, i.e., $W(X)$ is isomorphic to a subgroup of $GL(N)(\simeq GL(n, Z))$ consisting of all elements $\gamma \subset GL(N)$ such that $\gamma(\Delta) = \Delta$.

**Remark:** $W(X)$ is as well isomorphic to a subgroup of $GL(M)(\simeq GL(n, Z))$ consisting of all elements $\gamma \subset GL(M)$ such that $\gamma(\Sigma) = \Sigma$.

**Definition** A toric n-fold $X$ is symmetric if the trivial character is the only $W(X)$-invariant algebraic character of $T$, i.e. $N^{W(X)} = \{ \chi \in N | g\chi = \chi \text{ for all } g \in W(X) \} = \{0\}$.

**Definition** Let $S = \{ v \in \partial \Sigma | gv = v \text{ for all } g \in W(X) \}$ be the stable points of $W(X)$ on the boundary of $\Sigma$. If $S \neq \{0\}$ then for any $0 \neq v \in S$, we define $w_v$ related with $v$ by $w_v = \partial \Sigma \cap \{-tv | t \geq 0\}$.
Remark: It’s easy to see $X$ is symmetric if and only if $S = 0$.

§3. Holomorphic approximation of PSH

In this section, we will employ the technique in [15, 20] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [20].

**Theorem 3.4** Let $M$ be a compact complex manifold of dimension $n$ and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h)$. For each $m \in \mathbb{N}$, $h$ induces a Hermitian metric $h_m$ on $L^m$. Let $\{S^m_0, S^m_1, \ldots, S^m_{d_m-1}\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!} \omega^n_g$ is the volume form of $g$. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} \|S^m_i(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \ldots$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$:

$$\left\| \sum_{i=0}^{d_m-1} \|S^m_i(x)\|_{h_m}^2 - \sum_{j<R} a_j(x)m^{n-j} \right\|_{C^k} \leq C_{R,k} m^{n-R}$$

where $C_{R,k}$ depends on $R, k$ and the manifold $M$.

Let

$$\tilde{\omega}_g = \omega_g + \sqrt{-1} \partial\bar{\partial}\phi > 0 \quad \tilde{h} = he^{-\phi}$$

Let $\tilde{h}_m$ be the induced Hermitian metric of $\tilde{h}$ on $L^m$, $\{\tilde{S}^m_0, \tilde{S}^m_1, \ldots, \tilde{S}^m_{d_m-1}\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x))dV_g.$$
By Theorem 3.4, we have
\[ d_m - 1 \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||_{h_m}^2 = \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||_{h_m}^2 \right) e^{-m\phi}. \]
Thus
\[ \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||_{h_m}^2 \right) = -\frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||_{h_m}^2 \right) e^{-m\phi}. \]
As \( m \to +\infty \), we obtain for any positive integer \( R \)
\[ \frac{1}{m} \log \left( \sum_{j<R} \tilde{a}_j(x)m^{n-j} \right) \]
\[ = \frac{1}{m} \log m^n \left( \sum_{j<R} \tilde{a}_j(x)m^{-j} \right) \]
\[ = \frac{n}{m} \log m + \frac{1}{m} \log(1 + O(\frac{1}{m})) \to 0 \]
Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

**Corollary 3.2**
\[ \left\| \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||_{h_m}^2 \right) \right\|_{C^k} \to 0, \text{ as } m \to +\infty. \]
In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of \( L^m \).

**§4. Proof of The Main Theorem**
Suppose \( X_\Delta \) is Fano, then one obtains a convex \( W(X) \)-invariant polyhedron \( \Sigma \) in \( M_R \)
defined by \( \Sigma = \{ \ a \in M_R \ | \ <a, b_{\rho}> \leq 1, \text{ for all } \rho \in \Delta(1) \} \) where \( b_{\rho} \) is the fundamental generator of \( \rho \). Let \( L(\Sigma) = \{ v_0, v_1, ..., v_k \} = M \cap \Sigma \). Then \( v_0, v_1, ..., v_k \) determine algebraic characters \( \chi_i : T \to C^* \) of \( T(i=0, 1, ..., k) \). Moreover, we have
\[ |\chi_i(x)|^2 = e^{<v_i, y>}, i = 0, ..., k, \]
where \( y \) is the image of \( x \) under the canonical projection \( \pi : T \to M_R \). Let us define \( u : U \to R \) as follows:
\[ u = \log(\sum_{i=0}^{k} |\chi_i(x)|^2), x \subset U \simeq T. \]
Since $u$ is $K(T)$-invariant, $u$ descends to a function $\tilde{u} : M_R \to R$ defined as

$$\tilde{u} = \log(\sum_{i=0}^{k} e^{<v_i, y>}), y \subset M_R.$$ 

Consider the $G$-invariant hermitian metric $g = g_{\mathcal{I}}$ on $X$ such that the restriction of the corresponding to $g$ differential 2-form on $U$ is defined by

$$\omega_g = \partial \bar{\partial} u.$$ 

The metric $g$ is exactly the pull-back of Fubini-Study metric from $P^m$ with respect to the anticanonical embedding $X \hookrightarrow P^m$ defined by the algebraic characters $\chi_0, \chi_1, ..., \chi_k$.

Let $\Sigma^{(m)} = \{ a \in M_R | <a, b_a> \leq m \}$ and $L(\Sigma^{(m)}) = \{v_0, ..., v_{km}\} = M \cap \Sigma^{(m)}$, where $k_m + 1 = \dim H^0(X, O((-K)^m))$ and $\chi^\mu : T \to C^*$ defined by $|\chi^\mu(x)|^2 = e^{<\mu, y>}$.

We have the following lemma (see [7] p66)

**Lemma 4.1** $H^0(X, O((-K)^m)) = \oplus_{\mu \in L(\Sigma^{(m)})} C \cdot \chi^\mu$.

**Proposition 4.2** $\{\chi^\mu\}_{\mu \in L(\Sigma^{(m)})}$ is an orthogonal basis of $H^0(X, O((-K)^m))$ with respect to the inner product $<, >_{h^m}$, where $h^m = \frac{1}{(\sum_{i=0}^{k} |\chi_i(x)|^2)^m}$.

**Proof**

$$\int_X \chi^\mu \cdot \chi^{\nu} = \int_T \frac{(z_1^{\mu_1} ... z_n^{\mu_n})(\bar{z}_1^{\nu_1} ... \bar{z}_n^{\nu_n})}{(\sum_{i=0}^{k} |z_i|^m)} \cdot \omega^n = \int_T |z_1|^{\mu_1+\nu_1} ... |z_n|^{\mu_n+\nu_n} e^{i(\mu_1-\nu_1)\theta_1} ... e^{i(\mu_n-\nu_n)\theta_n} \frac{(\sum_{i=0}^{k} |z_i|^m)}{\omega^n}.$$ 

which is 0 if $\mu \neq \nu$.

For any $\varphi \in P_G(X, \omega)$, by Corollary 3.3

$$\varphi(x) = \lim_{m \to \infty} \frac{1}{m} \log \frac{\sum_{\mu \in L(\Sigma^{(m)})} a^{(m)}_{\mu}|\chi^\mu(x)|^2}{(\sum_{i=0}^{k} |\chi_i(x)|^2)^k}$$

**Lemma 4.2** There exists $\epsilon > 0$ such that for $\varphi \in P_G(X, \omega)$ and $\tilde{m} > 0$ there exist $m > \tilde{m}$ and $\mu \in L(\Sigma^{(m)})$ with $(a^{(m)}_{\mu})^{\frac{1}{m}} > \epsilon$.  

6
Proof Otherwise, for any $\epsilon > 0$ there exists $\varphi_\epsilon$ and $\tilde{m}$ such that for any $m > \tilde{m}$ and $\mu \in L(\Sigma^{(m)})$ we have $(a^{(m)}_\mu)^{\frac{1}{m}} < \epsilon$. By choosing $m$ large enough we have

\[
\varphi_\epsilon(x) \leq \frac{1}{m} \log \frac{\sum_{\mu \in L(\Sigma^{(m)})} |\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} + \log \epsilon \\
= \frac{1}{m} \log \left( \sum_{\mu \in L(\Sigma^{(m)})} \frac{|\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} \right) + \log \epsilon \\
\leq \frac{1}{m} \log \left( \sum_{\mu \in L(\Sigma^{(m)})} 1 \right) + \log \epsilon \\
\leq \text{Const} + \log \epsilon.
\]

Since $\epsilon$ can be chosen arbitrarily small, the above inequality implies that $\varphi_\epsilon \to -\infty$ uniformly as $\epsilon$ goes to 0, which contradicts the fact that $\sup_X \varphi = 0$.

For any $\varphi \in P_G(X, \omega)$, by Lemma 4.1 we have

\[
\varphi(x) = \lim_{m \to \infty} \frac{1}{m} \log \frac{\sum_{\mu \in L(\Sigma^{(m)})} a^{(m)}_\mu |\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} \\
\geq \frac{1}{m} \log \frac{\sum_{g \in W(X)} |\chi^g(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} - C_1 \\
\geq \log \frac{|\chi(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)} - C_1
\]

Put $v = \frac{\sum_{g \in W(X)} g\mu}{|W(X)|K}$, then we have $\bar{\varphi}(y) \geq \log \frac{e^{v \cdot y}}{\sum_{i=0}^m e^{v \cdot y}}$

Put $y_i = \log |t_i|^2 t_i = e^{\frac{y_i}{2}} + \sqrt{-1} \theta_i$, then

\[
\frac{dt_i}{t_i} = \frac{1}{2} dy_i + \sqrt{-1} d\theta_i \\
\frac{d\bar{t}_i}{\bar{t}_i} = \frac{1}{2} dy_i + \sqrt{-1} d\theta_i \\
\frac{dt_i \wedge d\bar{t}_i}{|t_i|^2} = -\sqrt{-1} dy_i \wedge d\theta_i \\
\partial \bar{\partial} u = \sum_{i,j} \frac{\partial^2 u}{\partial y_i \partial y_j} \frac{dt_i \wedge d\bar{t}_j}{t_i \bar{t}_j} \\
(\partial \bar{\partial} u)^n = \det \left( \frac{\partial^2 u}{\partial y_i \partial y_j} \right) dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n
\]
Lemma 4.3 Let $\tilde{F} = e^u \det \frac{\partial^2 u}{\partial y_i \partial y_j}$, then $0 < c \leq \tilde{F} \leq C$.

**Proof** $e^{-u} \frac{dt_1 \wedge dt_1 \wedge ... \wedge dt_n \wedge d\tilde{u}}{|t_1|^2 |t_n|^2} = e^{-\tilde{u}} dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n$ can be extended to a non-vanishing volume form on $X$. Also

$$(\hat{\partial u})^n = \det \frac{\partial^2 u}{\partial t_i \partial t_j} dt_1 \wedge dt_1 \wedge ... \wedge dt_n \wedge d\tilde{u}$$

is a non-vanishing volume form, so the quotient of these two volume forms must be positive and bounded. Which proves the lemma.

Now we can prove the Theorem 1.1.

$$\int_X e^{-\alpha \phi} \omega^n = \int_X e^{-\alpha \phi} (\hat{\partial u})^n$$

$$= \int_{R^n} e^{-\alpha \phi} \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} dy_1 ... dy_n$$

$$\leq \int_{R^n} e^{-\alpha \phi - \tilde{u}} dy_1 ... dy_n$$

$$\leq \int_{R^n} e^{-\alpha \log \sum e^{<v,y>} - \log(\sum e^{<v,y>})} dy_1 ... dy_n$$

$$= \int_{R^n} \frac{e^{-\alpha \sum e^{<v,y>}}}{(\sum e^{<v,y>})^{1-\alpha}} dy_1 ... dy_n$$

If the stable points $S = \{0\}$, then $X$ is symmetric so that $v = \frac{\sum_{\mu \in W(X)} g \mu}{m|W(X)|} = 0$ for all $\mu \in L(\Sigma^{(m)})$. Therefore for all $\alpha < 1$ the integral

$$\int_X e^{-\alpha \phi} \omega^n = \int_{R^n} \frac{1}{(\sum e^{<v,y>})^{1-\alpha}} dy_1 ... dy_n$$

is finite since every $n$-dimensional cone $\sigma_j \in \Delta(j = 1, ..., l)$ is generated by a basis of the lattice $N$ and the fact that $N_R = \sigma_1 \cup ... \cup \sigma_l$. This implies $\alpha_G(X) \geq 1$ so that by Tian’s theorem[14] $X$ admits Kahler-Einstein metric. This is a result by V. Batyrev and E.N. Selivanova[2].

If $S \neq \{0\}$ then for any $0 \neq v \in S$, we have $w_v \in \partial \Sigma$ related with $v$ satisfying

$$<w_v,v> = -|w_v||v|.$$
The integral
\[ \int_{\mathbb{R}^n} \left( \sum e^{-\alpha v, y} \right) \frac{1}{1 - \alpha} dy_1 \ldots dy_n = \int_{\mathbb{R}^n} \left( \sum e^{-\alpha v, y} \right) 1 - \alpha dy_1 \ldots dy_n \]
is finite if
\[ -\frac{\alpha}{1 - \alpha} v \in \text{int}(\Sigma), \]
i.e.
\[ < -\frac{\alpha}{1 - \alpha} v, w_v > = \frac{\alpha}{1 - \alpha} |v||w_v| \leq |w_v|^2 \]

Then for all \( \alpha < \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}} \) the integral \( \int_X e^{-\alpha \varphi} \omega^n \) is finite. Therefore
\[ \alpha_G(X) \geq \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}}. \]

In order to estimate the upper bound of \( \alpha_G(X) \) we will construct a sequence of PSH functions. Suppose \( S \neq \{0\} \), then for all \( \alpha \) with \( 1 > \alpha > \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}} \) we choose \( \tilde{\varphi}_\epsilon = \log(\sum e^{<\epsilon \varphi >} \frac{|w_v|}{|v|}) \) which is increasing and uniformly bounded from above where \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \) is achieved at \( \tilde{v} \in S \). Then by Fatou lemma we have
\[ \lim_{\epsilon \to 0} \int_X e^{-\alpha \varphi} \omega^n = \int_X e^{-\alpha \varphi_0} \omega^n = \infty \]
which implies \( \alpha_G(X) \leq \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}}. \) Combined the above estimates together, we have proved the Theorem 1.1.

Also it’s obvious to see that \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \leq 1 \) for non-symmetric toric Fano manifold \( X \) thus \( \alpha_G(X) \leq \frac{1}{2} \). This shows that there doesn’t exist any non-symmetric toric Fano manifold such that its \( \alpha_G \)-invariant is greater than \( \frac{n}{m+1} \) which is a sufficient condition for the existence of Kähler-Einstein metrics.

Now we prove Theorem 1.2 which is a direct corollary of the proof of Theorem 1.1. Define \( \mathcal{P}_{m,a}(X) = \{ \varphi \in C^\infty(X, R) | \sup_X \varphi = 0, \varphi \text{ is } G \text{-invariant and there exists a basis } \{ S_i^m \}_{1 \leq i \leq N_m} \text{ of } H^0(X, K_X^{-m}) \text{ such that } \omega_g + \overline{\partial} \overline{\partial} \varphi = \frac{1}{m} \overline{\partial} \overline{\partial} \log(\sum_{i=0}^{N_m} |S_i^m|^2) \}, \) where \( N_m + 1 = \dim H^0(X, K_X^{-m}) \) and \( m \) is large.

We also define for \( m \) large, \( \alpha_{m,a}(X) = \sup \{ \alpha | \text{there exists } C > 0 \text{ such that for all } \varphi \in \mathcal{P}_{m,a}(X), \int_X e^{-\alpha \varphi} dV \leq \infty \}. \)
It’s easy to see $\alpha_{m,G}(X)$ is decreasing as $m$ goes to the infinity. By the argument to give the upper bound for the $\alpha_G$-invariant, we can directly have the following corollary which answers the question proposed by Tian\cite{15} in the special case of toric Fano manifolds.

**Corollary 4.3** If $X$ is a toric Fano manifold, then $\{\alpha_{m,G}(X)\}_m$ is decreasing and stationary. More precisely, $\alpha_{m,G}(X) = \alpha_G(X)$ if $m \geq m_0$, where $m_0$ is the least positive integer such that $m_0 v \in M$ and $v$ is the minimizer of $\min_{0 \neq v \in S} \frac{|w_v|}{|v|}$.

§5. Multiplier ideal sheaf

In this section we relate the $\alpha$-invariant on toric Fano manifolds with the method of the multiplier ideal sheaf employed by Nadel\cite{9}. Here we follow the lines in \cite{3}.

**Theorem 5.5** (Nadel) Let $X$ be a Fano manifold of dimension $n$ and $G$ be a compact subgroup of the group of complex automorphisms of $X$. Then $X$ admits a $G$-invariant Kähler-Einstein metric, unless $K_X^{-1}$ possesses a $G$-invariant singular hermitian metric $h = h_0 e^{-\varphi}$ ($h_0$ is a smooth $G$-invariant metric and $\varphi$ is a $G$-invariant function in $L^1_{\text{loc}}(X)$), such that the following properties occur.

1. $h$ has a semipositive curvature current

\[ \Theta_h = -\frac{i}{2\pi} \partial \bar{\partial} \log h = \Theta_{h_0} + \frac{i}{2\pi} \partial \bar{\partial} \varphi \leq 0. \]

2. For every $\gamma \in (\frac{n}{n+1}, 1)$, the multiplier ideal sheaf $\mathcal{J}(\gamma \varphi)$ is nontrivial, (i.e. $0 \neq \mathcal{J}(\gamma \varphi) \neq O_X$).

**Theorem 5.6** (Nadel) Let $(X, \omega)$ be a Kähler manifold and let $L$ be a holomorphic line bundle over $X$ equipped with a singular hermitian metric $h$ of weight $\phi$ with respect to a smooth metric $h_0$ (i.e. $h = h_0 e^{-\phi}$). Assume that the curvature form $\Theta_h(L)$ is positive definite in the sense of currents, i.e. $\Theta_h(L) \geq \epsilon \omega$ for some $\epsilon > 0$. Then we have $H^q(X, K_X \otimes L \otimes \mathcal{J}(\phi)) = 0$ for all $q \geq 1$.

**Corollary 5.4** (Nadel) Let $X$, $G$, $h$ and $\varphi$ be in theorem 4.1. then for all $\gamma \in (\frac{n}{n+1}, 1)$,

1. The multiplier ideal sheaf $\mathcal{J}(\gamma \varphi)$ satisfies $H^q(X, \mathcal{J}(\gamma \varphi)) = 0$ for all $q \geq 1$.

2. The associated subscheme $V_\gamma$ of structure sheaf $O_{V_\gamma} = O_X / \mathcal{J}(\gamma \varphi)$ is nonempty, distinct from $X$, $G$-invariant and satisfies $H^q(V_\gamma, O_{V_\gamma}) = C$ for $q=0$ and vanishes for $q \geq 1$. 
In order to construct Kahler-Einstein metrics it’s sufficient to rule out the existence of any $G$-invariant subscheme with the property (2) in the corollary.

In the case of toric Fano manifolds, we have the following theorem.

**Theorem 5.7** Let $X$ be a toric Fano manifold. If $X$ is not symmetric then there always exists a $G$-invariant subscheme with the property (2) in the corollary.

**Proof** If $X$ is not symmetric then $\alpha_G(X) \leq \frac{1}{2}$ and we can construct $G$-invariant $\varphi \in L^1_{loc}(X)$ such that for all $\gamma \in (\alpha_G(X), 1)$

$$\int_X e^{-\gamma \varphi} \omega^n = +\infty$$

therefore $J(\gamma \varphi)$ is nontrivial and there exist subschemes $V$, satisfying property (2) of the corollary.

§6. Examples

In this sections we will calculate the $\alpha$ invariants for 2-dimensional toric Fano manifolds.

Here (1) (2) (3) (4) are corresponding to $CP^2$ and $CP^2$ blow-up at 1, 2 and 3 points and (5) (6) (7) (8) are the corresponding polyhedrons.

$CP^2$ and $CP^2$ blow-up at 3 points are symmetric thus the their $\alpha_G$-invariants are both equal to 1.

For $CP^2\#1CP^2$ its stable points of $G$ on the boundary of the polyhedron in (6) is $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ and $\|\frac{1}{2}, \frac{1}{2}\| = 1$ then it is easy to see $\alpha_G(CP^2\#1CP^2) = \frac{1}{2}$.

For $CP^2\#2CP^2$ its stable points of $G$ on the boundary of the polyhedron in (7) is $(\frac{1}{2}, \frac{1}{2})$ and $(-1, -1)$ and $\|\frac{1}{2}, \frac{1}{2}\| = 1$ then it is easy to see $\alpha_G(CP^2\#2CP^2) = \frac{1}{3}$.

The above calculation confirms our earlier results[12].
References

[1] Abdesselem, B. A., Équations de Monge-Ampère d’origine géométrique sur certaines variétés algébriques, J.Funct. Anal. 149 (1997), 102-134 Math. Soc. (3), 50 (1985), 1-26.

[2] Batyrev, V. and Selivanova, E., Einstein-Kähler metrics on symmetric toric Fano manifolds, J. reine angew. Math. 512 (1999), 225-236.

[3] Demailly, J.-P. and Kollár, J., Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.

[4] Demazure, M., sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. École Norm. Sup. 3 (1970), 507-588.

[5] Danilov, V.I., Geometry of toric varieties, Russ. Math. Surv. 33, No.2 (1978), 97-154.

[6] Ding, W and Tian, G., The generalized Moser-Trudinger Inequality, Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.

[7] Fulton, W., Introduction to Toric Varieties, Ann. Math. Stud. 131, Princeton Univ. Press, 1993.

[8] Mabuchi, T., Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, Osaka J. Math. 24 (1987), 705-737.

[9] Nadel, A.M., Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. (2) 132 (1990), no. 3, 549–596.

[10] Phong, D. H. and Sturm, J., Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions, Ann. of Math. (2) 152 (2000), no. 1, 277–329.

[11] Siu, Y.T., The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. of Math. (2) 127 (1988), no. 3, 585–627.

[12] Song, J., The α-Invariant on $CP^2#2\overline{CP}^2$, preprint.
[13] Tian, G. and Yau, S.T., \textit{K"ahler-Einstein metrics on complex surfaces with }$C_1 > 0$, Comm. Math. Phys. 112 (1987), no. 1, 175–203.

[14] Tian, G., \textit{On K"ahler-Einstein metrics on certain K"ahler manifolds with }$C_1(M) > 0$, Invent. Math. 89 (1987), no. 2, 225–246.

[15] Tian, G., \textit{On a set of polarized K"ahler metrics on algebraic manifolds}, J. Differential Geometry 32 (1990) 99-130.

[16] Tian, G., \textit{On Calabi’s conjecture for complex surfaces with positive first Chern class}, Invent. Math. 101 (1990), no. 1, 101–172.

[17] Tian, G., \textit{K"ahler-Einstein metrics with positive scalar curvature}, Invent. Math. 130 (1997), no. 1, 1–37.

[18] Tian, G. and Zhu, X.H., \textit{A nonlinear inequality of Moser-Trudinger type}, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 349–354.

[19] Yau, S.T., \textit{On the Ricci curvature of a compact K"ahler manifold and the complex Monge-Ampère equation I}, Comm. Pure Appl. Math. 31 (1978) 339-411.

[20] Zelditch, S., \textit{Szego Kernels and a Theorem of Tian}, IMRN1998, No. 6 317-331.