On the Fundamental Limit of Private Information Retrieval for Coded Distributed Storage

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Abstract—We consider private information retrieval (PIR) for distributed storage systems (DSSs) with noncolluding nodes where data is stored using a non maximum distance separable (MDS) linear code. It was recently shown that if data is stored using a particular class of non-MDS linear codes, the MDS-PIR capacity, i.e., the maximum possible PIR rate for MDS-coded DSSs, can be achieved. For this class of codes, we prove that the PIR capacity is indeed equal to the MDS-PIR capacity, giving the first family of non-MDS codes for which the PIR capacity is known. For other codes, we provide asymmetric PIR protocols that achieve a strictly larger PIR rate compared to existing symmetric PIR protocols.

I. INTRODUCTION

The concept of private information retrieval (PIR) was first introduced by Chor et al. [1]. A PIR protocol allows a user to privately retrieve an arbitrary data item stored in multiple servers (referred to as nodes in the sequel) without disclosing any information of the requested item to the nodes. The efficiency of a PIR protocol is measured in terms of the total communication cost between the user and the nodes, which is equal to the sum of the upload and download costs. In distributed storage systems (DSSs), data is encoded by an \( [n, k] \) linear code and then stored on \( n \) nodes in a distributed manner. Such DSSs are referred to as coded DSSs [2], [3].

One of the primary aims in PIR is the design of efficient PIR protocols from an information-theoretic perspective. Since the upload cost does not scale with the file size, the download cost dominates the total communication cost [3], [4]. Thus, the efficiency of a PIR protocol is commonly measured by the amount of information retrieved per downloaded symbol, referred to as the PIR rate. Recently, Sun and Jafar derived the maximum achievable PIR rate for an arbitrarily coded DSS. To this purpose, we mainly consider non-MDS-PIR capacity-achieving codes. Most of the earlier works focus on designing symmetric PIR protocols and it was shown in [5], [7], [11] that any PIR scheme can be made symmetric for MDS-coded DSSs. However, this is in general not the case for non-MDS codes. Specifically, we propose an asymmetric PIR protocol, Protocol A, that allows asymmetry in the responses from the storage nodes. For non-MDS-PIR capacity-achieving codes, Protocol A achieves improved PIR rates compared to the PIR rates of existing symmetric PIR protocols. Furthermore, we present an asymmetric PIR protocol, named Protocol B, that applies to non-MDS-PIR capacity-achieving codes that can be written as a direct sum of MDS-PIR capacity-achieving codes. Finally, we give an example showing that it is possible to construct an improved (compared to Protocol A) asymmetric PIR protocol. However, the protocol is code-dependent and strongly relies on finding good punctured MDS-PIR capacity-achieving subcodes of the non-MDS-PIR capacity-achieving code.

II. PRELIMINARIES AND SYSTEM MODEL

A. Notation and Definitions

We denote by \( \mathbb{N} \) the set of all positive integers and by \( \mathbb{N}_a \triangleq \{1, 2, \ldots, a\} \). Vectors are denoted by lower case bold letters, matrices by upper case bold letters, and sets by calligraphic upper case letters, e.g., \( x \), \( X \), and \( \mathcal{X} \) denote a vector, a matrix, and a set, respectively. In addition, \( \mathcal{X}^c \) denotes the complement of a set \( \mathcal{X} \) in a universe set. For a given index set \( S \), we also write \( X_S^m \) and \( Y_S \) to represent \( \{X^{(m)}: m \in S\} \) and \( \{Y_l: l \in S\} \), respectively. The fonts of random and deterministic quantities are not distinguished.
typographically since it should be clear from the context. We denote a submatrix of $X$ that is restricted in columns by the set $\mathcal{I}$ by $X|_{\mathcal{I}}$. The function $\text{LCM}(n_1, n_2, \ldots, n_a)$ computes the lowest common multiple of $a$ positive integers $n_1, n_2, \ldots, n_a$. The function $H(\cdot)$ represents the entropy of its argument and $I(\cdot; \cdot)$ denotes the mutual information of the first argument with respect to the second argument. $(\cdot)^T$ denotes the transpose of its argument. We use the customary code parameters $[n, k]$ to denote a code $C$ over the finite field $\text{GF}(q)$ of blocklength $n$ and dimension $k$. A generator matrix of $C$ is denoted by $G^C$, while $G^{\mathcal{I}}$ represents the corresponding code generated by $G$.

The function $\chi(x)$ denotes the support of a vector $x$, while the support of a code $C$ is defined as the set of coordinates where not all codewords are zero. A set of coordinates of $C_k$ of size $k$ is said to be an information set if and only if $G^{\mathcal{I}}|_{\mathcal{I}}$ is invertible. The $s$-th generalized Hamming weight of an $[n, k]$ code $C$, denoted by $d^C_s$, $s \in \mathbb{N}_k$, is defined as the cardinality of the smallest support of an $s$-dimensional subcode of $C$.

B. System Model

We consider a DSS that stores $f$ files $X^{(1)}, \ldots, X^{(f)}$, where each file $X^{(m)} = (x_i^{(m)})$, $m \in \mathbb{N}_f$, can be seen as a random $\beta \times k$ matrix over $\text{GF}(q)$ with $\beta, k \in \mathbb{N}$. Assume that each entry $x_i^{(m)}$ of $X^{(m)}$ is chosen independently and uniformly at random from $\text{GF}(q)$, $m \in \mathbb{N}_f$. Thus,

$$H(X^{(m)}) = L, \forall m \in \mathbb{N}_f,$$

$$H(X^{(1)}, \ldots, X^{(f)}) = fL \ (\text{in } q\text{-ary units}),$$

where $L \triangleq \beta \cdot k$. Each file is encoded using a linear code as follows. Let $x_i^{(m)} = (x_{i,1}^{(m)}, \ldots, x_{i,k}^{(m)})$, $i \in \mathbb{N}_\beta$, be a message vector corresponding to the $i$-th row of $X^{(m)}$. Each $x_i^{(m)}$ is encoded by an $[n, k]$ code $C$ over $\text{GF}(q)$ into a length-$n$ codeword $c_i^{(m)} = (c_{i,1}^{(m)}, \ldots, c_{i,n}^{(m)})$. The $\beta f$ generated codewords $c_i^{(m)}$ are then arranged in the array $C = ((C^{(1)}_{i,1})^T) \ldots (C^{(f)}_{i,1})^T$ of dimensions $\beta f \times n$, where $C^{(m)} = ((c_{i,1}^{(m)})^T) \ldots (c_{i,n}^{(m)})^T$. The code symbols $c_{i,1}^{(m)}, \ldots, c_{i,n}^{(m)}$, $m \in \mathbb{N}_f$, for all $f$ files are stored on the $l$-th storage node, $l \in \mathbb{N}_n$.

C. Privacy Model

To retrieve file $X^{(m)}$ from the DSS, the user sends a random query $Q_l^{(m)}$ to the $l$-th node for all $l \in \mathbb{N}_n$. In response to the received query, node $l$ sends the response $A_l^{(m)}$ back to the user. $A_l^{(m)}$ is a deterministic function of $Q_l^{(m)}$ and the code symbols stored in the node.

Definition 1. Consider a DSS with $n$ noncolluding nodes storing $f$ files. A user who wishes to retrieve the $m$-th file sends the queries $Q_l^{(m)}$, $l \in \mathbb{N}_n$, to the storage nodes, which return the responses $A_l^{(m)}$. This scheme achieves perfect information-theoretic PIR if and only if

Privacy:

$$I(m; Q_l^{(m)}) = 0, \forall l \in \mathbb{N}_n, \quad (1a)$$

Recovery:

$$H(X^{(m)} | A_l^{(m)}, \ldots, A_{l-1}^{(m)}, Q_1^{(m)}, \ldots, Q_{l-1}^{(m)}) = 0. \quad (1b)$$

D. PIR Rate and Capacity

Definition 2. The PIR rate of a PIR protocol, denoted by $R$, is the amount of information retrieved per downloaded symbol, i.e., $R \triangleq \frac{\mu}{\nu}$, where $D$ is the total number of downloaded symbols for the retrieval of a single file.

We will write $R(C)$ to highlight that the PIR rate depends on the underlying storage code $C$. It was shown in [7] that for the noncolluding case and for a given number of files $f$ stored using an $[n, k]$ MDS code, the MDS-PIR capacity is

$$C_f^{[n,k]} \triangleq \left(\frac{n-k}{n}\right)\left(1 - \left(\frac{k}{n}\right)^f\right)^{-1}, \quad (2)$$

where superscript “[n, k]” indicates the code parameters of the underlying MDS storage code. When the number of files $f$ tends to infinity, (2) reduces to

$$C_f^{[n,k]} \triangleq \lim_{f \to \infty} C_f^{[n,k]} = \frac{n-k}{n},$$

which we refer to as the asymptotic MDS-PIR capacity. Note that for the case of non-MDS linear codes, the PIR capacity is unknown.

E. MDS-PIR Capacity-Achieving Codes

In [9], two symmetric PIR protocols for coded DSSs, named Protocol 1 and Protocol 2, were proposed and shown to achieve the MDS-PIR capacity for certain important classes of non-MDS codes. Their PIR rates depend on the following property of the underlying storage code $C$.

Definition 3. Let $C$ be an arbitrary $[n, k]$ code. A $\nu \times n$ binary matrix $\Lambda_{\kappa, \nu}(C)$ is said to be a PIR achievable rate matrix for $C$ if the following conditions are satisfied.

1) The Hamming weight of each column of $\Lambda_{\kappa, \nu}$ is $\kappa$, and
2) for each row matrix $\lambda_i$, $i \in \mathbb{N}_\nu$, $\chi(\lambda_i)$ always contains an information set.

The following theorem gives the achievable PIR rate of Protocol 1 from [9, Thm. 1].

Theorem 1. Consider a DSS that uses an $[n, k]$ code $C$ to store $f$ files. If a PIR achievable rate matrix $\Lambda_{\kappa, \nu}(C)$ exists, then the PIR rate

$$R_{f, S}(C) \triangleq \left(\frac{\nu - \kappa}{\kappa n}\right)\left(1 - \left(\frac{\kappa}{\nu}\right)^f\right)^{-1} \quad (3)$$

is achievable.

In (3), we use subscript $S$ to indicate that this PIR rate is achievable by the symmetric Protocol 1 in [9]. Define $R_{\infty, S}(C)$ as the limit of $R_{f, S}(C)$ as the number of files $f$ tends to infinity, i.e., $R_{\infty, S}(C) \triangleq \lim_{f \to \infty} R_{f, S}(C) = \frac{(\nu - \kappa)}{\kappa n}$. The asymptotic PIR rate $R_{\infty, S}(C)$ is also achieved by the file-independent Protocol 2 from [9].

Corollary 1. If a PIR achievable rate matrix $\Lambda_{\kappa, \nu}(C)$ with $\frac{\nu}{\kappa} = \frac{\kappa}{\nu}$ exists for an $[n, k]$ code $C$, then the MDS-PIR capacity (2) is achievable.
Definition 4. A PIR achievable rate matrix $\Lambda_{k,\nu}(C)$ with $\frac{n}{\nu} = \frac{k}{n}$ for an $[n,k]$ code $C$ is called an MDS-PIR capacity-achieving matrix, and $C$ is referred to as an MDS-PIR capacity-achieving code.

In the following, we briefly state a main result for Protocol 1 and Protocol 2 from [9] and compare the required number of stripes and download cost of these protocols.

Theorem 2. If an MDS-PIR capacity-achieving matrix exists for an $[n,k]$ code $C$ with $\frac{n}{\nu} = \frac{k}{n}$, then the PIR rates $C_f^{[n,k]}$ and $C_{\infty}^{[n,k]}$ are achievable by Protocol 1 and Protocol 2 from [9], respectively, using the corresponding required $\beta$ and $D$. From Definition 2, we have

$$\frac{nD}{\beta} = \begin{cases} k\left(C_f^{[n,k]}\right)^{-1} & \text{for Protocol 1,} \\ k\left(C_{\infty}^{[n,k]}\right)^{-1} & \text{for Protocol 2.} \end{cases}$$

Furthermore, the smallest number of stripes $\beta$ of Protocol 1 and Protocol 2 is equal to $\nu_1$ and $\frac{k\Lambda_{NM}(k,n-k)}{k}$, respectively.

The following theorem from [9, Thm. 3] provides a necessary condition for the existence of an MDS-PIR capacity-achieving matrix.

Theorem 3. If an MDS-PIR capacity-achieving matrix exists for an $[n,k]$ code $C$, then $d_f^{[n,k]} \geq \frac{1}{\nu_1} s$, $\forall s \in \mathbb{N}_k$.

III. PIR CAPACITY FOR MDS-PIR CAPACITY-ACHIEVING CODES

In this section, we prove that the PIR capacity of MDS-PIR capacity-achieving codes is equal to the MDS-PIR capacity.

Theorem 4. Consider a DSS that uses an $[n,k]$ MDS-PIR capacity-achieving code $C$ to store $f$ files. Then, the maximum achievable PIR rate over all possible PIR protocols, i.e., the PIR capacity, is equal to the MDS-PIR capacity $C_f^{[n,k]}$ in (2).

Proof: See Appendix A.

Theorem 4 provides an expression for the PIR capacity for the family of MDS-PIR capacity-achieving codes (i.e., (2)). Moreover, for any finite number of files $f$ and in the asymptotic case where $f$ tends to infinity, the PIR capacity can be achieved using Protocols 1 and 2 from [9], respectively.

IV. ASYMMETRY HELPS: IMPROVED PIR PROTOCOLS

In this section, we present three asymmetric PIR protocols for non-MDS-PIR capacity-achieving codes, illustrating that asymmetry helps to improve the PIR rate. By asymmetry we simply mean that the number of symbols downloaded from the different nodes is not the same, i.e., for any fixed $m \in \mathbb{N}_f$, the entropies $H(A_i^{(m)})$, $l \in \mathbb{N}_m$, may be different. This is in contrast to the case of MDS codes, where any asymmetric protocol can be made symmetric while preserving its PIR rate [5], [7], [11]. We start with a simple motivating example showing that the PIR rate of Protocol 1 from [9] can be improved for some underlying storage codes.

A. Protocol 1 from [9] is Not Optimal in General

Example 1. Consider the $[5,3]$ code $C$ with generator matrix $G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$.

The smallest possible value of $\frac{n}{\nu}$ for which a PIR achievable rate matrix exists is $\frac{2}{3}$ and a corresponding PIR achievable rate matrix is $\Lambda_{2,3} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

It is easy to verify that $\Lambda_{2,3}$ above is a PIR achievable rate matrix for code $C$. Thus, the largest PIR rate for $f = 2$ files with Protocol 1 from [9] is $R_{2,5} = \frac{3}{5} \frac{1}{3} = \frac{27}{50}$. In Table I (taken from [9, Sec. IV]), we list the downloaded sums of code symbols when retrieving file $X^{(1)}$ and $f = 2$ files are stored. In the table, for each $m \in \mathbb{N}_2$ and $\beta = \nu_1 = 3^2$, the interleaved code array $Y^{(m)}$ with row vectors $y_i^{(m)} = c_i^{(m)}$, $i \in \mathbb{N}_2$, is generated (according to Protocol 1 from [9]) by a randomly selected permutation function $\pi(\cdot)$.

Observe that since $\{2,3,4\} \subset \chi(\nu_1) = \{2,3,4,5\}$ is an information set of $C$, the five sums of $\{y_3^{(1)} \cdot 2, y_2^{(1)} \cdot 3, y_4^{(2)} \cdot 3, y_3^{(2)} \cdot 3, y_2^{(2)} \cdot 3\}$ are not necessarily required to recover $X^{(1)}$. For privacy concerns, notice that the remaining sums of code symbols from the 5-th node would be $\{y_3^{(1)} \cdot 2, y_2^{(1)} \cdot 3, y_4^{(2)} \cdot 3, y_3^{(2)} \cdot 3, y_2^{(2)} \cdot 3\}$.

This ensures the privacy condition, since for every combination of files, the user downloads the same number of linear sums. This shows that by allowing asymmetry in the responses from the storage nodes, the PIR rate can be improved to $\frac{27}{50} = \frac{27}{50} = \frac{3}{5},$ which is much closer to the MDS-PIR capacity $C_{f}^{[n,k]} = \frac{1}{\nu_1} = \frac{5}{2}$.

Example 1 indicates that for a coded DSS using a non-MDS-PIR capacity-achieving code, there may exist an asymmetric PIR scheme that improves the PIR rate of the symmetric Protocol 1 from [9].

B. Protocol A: A General Asymmetric PIR Protocol

In this subsection, we show that for non-MDS-PIR capacity-achieving codes, by discarding the redundant coordinates that are not required to form an information set within $\chi(\nu_1)$, i.e., $i \in \mathbb{N}_\nu$, it is always possible to obtain a larger PIR rate compared to that of Protocol 1 from [9].

Theorem 5. Consider a DSS that uses an $[n,k]$ code $C$ to store $f$ files. If a PIR achievable rate matrix $\Lambda_{k,\nu}(C)$ exists, then the PIR rate

$$R_{f,\Lambda}(C) = \left(1 - \frac{k}{\nu}\right)^{-1} \left(1 - \frac{n}{\nu}\right)^{-1}$$

is achievable.

Proof: See Appendix B.
We will make use of the following lemma from [9, Lem. 2].

**Lemma 1.** If a matrix $A_{\nu,k}(C)$ exists for an $[n,k]$ code $C$, then we have

$$\frac{k}{\nu} \leq \frac{k}{n},$$

where equality holds if $\chi(\lambda_i), i \in \mathbb{N}_\nu$, are all information sets.

Proposition 1 can be easily verified using Lemma 1.

**Proposition 1.** Consider a DSS that uses an $[n,k]$ code $C$ to store $f$ files. Then, $R_{f,S}(C) \leq R_{f,A}(C) \leq C_{[n,k]}^{(\nu,k)}$ with equality if and only if $C$ is an MDS-PIR capacity-achieving code.

**Proof:** The result follows since

$$R_{f,S}(C) = \left(\frac{\nu - k}{\nu n}\right) \left[1 - \left(\frac{k}{\nu}\right)^f\right]^{-1} \leq \left(\frac{\nu - k}{\nu n - (\nu n - \nu k)}\right) \left[1 - \left(\frac{k}{\nu}\right)^f\right]^{-1} = \left[1 + \frac{k}{\nu} + \ldots + \frac{k}{\nu} \left(\frac{k}{\nu}\right)^{-1}\right]^{-1} = C_{[n,k]}^{(\nu,k)},$$

where (6) and (7) hold since $\frac{\nu}{\nu} \leq \frac{k}{n}$.

In the following, we refer to the asymmetric PIR protocol that achieves the PIR rate in Theorem 5 as Protocol A (thus the subscript A in $R_{f,A}(C)$ in (5)). Similar to Theorem 1, there also exists an asymmetric PIR protocol that achieves the asymptotic PIR rate $R_{\infty,A}(C) \triangleq \lim_{f \to \infty} R_{f,A}(C) = 1 - \frac{1}{\beta}$ and we simply refer to this protocol as the file-independent Protocol A. $A_{\nu,k}(C)$ can be used for both the file-dependent Protocol A and the file-independent Protocol A.

1As for Protocol 1 and Protocol 2 from [9, Remark 2]
Example 2. Continuing with Example 1, by elementary matrix operations, the generator matrix of the $[5,3]$ code of Example 1 is equivalent to the generator matrix
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix} = \begin{pmatrix} G_1 & G_2 \end{pmatrix}.
\]

It can easily be verified that both $C^{G_1}$ and $C^{G_2}$ are MDS-PIR capacity-achieving codes. Hence, from Theorem 6, the asymptotic PIR rate
\[
R_{\infty,B} = \left( \frac{2 \cdot 1 + 1}{3 \cdot 1 - \frac{2}{3}} \right) = 3
\]
is achievable. $R_{\infty,B} = \frac{3}{8}$ is strictly larger than both $R_{\infty,S} = \frac{3}{16}$ and $R_{\infty,A} = \frac{3}{16}$.

D. Protocol C: Code-Dependent Asymmetric PIR Protocol

In this subsection, we provide a code-dependent, but file-independent asymmetric PIR protocol for non-MDS-PIR capacity-achieving codes that cannot be decomposed into a direct sum of MDS-PIR capacity-achieving codes as in (8). The protocol is tailor-made for each class of storage codes. The main principle of the protocol is to further reduce the number of downloaded symbols by looking at punctured MDS-PIR capacity-achieving subcodes. Compared to Protocol A, which is simpler and allows for a closed-form expression for its PIR rate, Protocol C gives larger PIR rates.

The file-independent Protocol 2 from [9] utilizes interference symbols. An interference symbol can be defined through a summation as [9]
\[
\sum_{m=1}^{f} \sum_{j=(m-1)\beta+1}^{m\beta} u_{h,j} x^{(m)}_{j-(m-1)\beta, h'},
\]
where $h, h' \in \mathbb{N}_k$ and the symbols $u_{h,j}$ are chosen independently and uniformly at random from the same field as the code symbols.

Example 3. Consider a $[9,5]$ code $C$ with generator matrix
\[
G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
It has $d_g^C = 3 < \frac{n}{2} = 2$, thus it is not MDS-PIR capacity-achieving (see Theorem 3). Note that this code cannot be decomposed into a direct sum of MDS-PIR capacity-achieving codes as in (8).

The smallest $\xi$ for which a PIR achievable rate matrix exists for this code is $\frac{1}{3}$, and a corresponding PIR achievable rate matrix is
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The idea of the file-independent Protocol 2 from [9] is to use the information sets $I_1 = \{2, 6, 7, 8, 9\}$ and $I_2 = \{1, 3, 4, 5\}$ to recover the $\beta k = 1 \cdot 5$ requested file symbols that are located in $I_3 = \{1, 2, 3, 4, 5\}$. Specifically, we use the information sets $I_1$ to reconstruct the required code symbols located in $\chi(c_1)^{\leq} = \{1, 3, 4, 5\}$ and $I_2 \subseteq \chi(c_2) = \{1, 3, 4, 5, 6, 7, 8, 9\}$ to reconstruct the required code symbol located in $\chi(c_2)^{\leq} = \{2\}$. Since the code coordinates $\{1, 2, 4, 5, 9\}$ form an $[n', k'] = [5, 4]$ punctured MDS-PIR capacity-achieving subcode $C^{G'}$ with generator matrix
\[
G' = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]
it can be seen that the code coordinates $\{1, 4, 5, 9\}$ are sufficient to correct the erasure located in $\chi(c_2)^{\leq}$. Therefore, compared to Protocol A, we can further reduce the required number of downloaded symbols. The responses from the nodes when retrieving file $X^{(m)}$ are listed in Table II. The PIR rate of Protocol C is then equal to
\[
R_{\infty,C} = \frac{1 \cdot 5}{n + n'} = \frac{5}{14},
\]
which is strictly larger than $R_{\infty,A} = \frac{1}{1}$. Notice that it can readily be seen from Table II that the privacy condition in (1a) is ensured.

Finally, we remark that, using the same principle as outlined above, other punctured MDS-PIR capacity-achieving subcodes can be used to construct a valid protocol, giving the same PIR rate. For instance, we could pick the two punctured subcodes $C^{G_1}$ and $C^{G_2}$ with generator matrices
\[
G_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix},
\]
respectively.

Example 3 above illustrates the main working principle of Protocol C and how the redundant set of code coordinates is taken into account. Its general description will be given in a forthcoming extended version. However, some numerical results are given below, showing that it can attain larger PIR rates than Protocol A.
In this appendix, we only provide the converse proof of different storage codes. We proposed three asymmetric codes in (8). For all presented codes except $C_3$, Protocol $C$ achieves strictly larger PIR rate than Protocol $A$, although smaller than the MDS-PIR capacity.

### VI. Numerical Results

In Table III, we compare the PIR rates for different storage codes using several binary linear codes. The second column gives the smallest fraction $\frac{\alpha}{\nu}$ for which a PIR achievable rate matrix exists. In the table, code $C_1$ is from Example 1, code $C_2$ is from Example 3, $C_3$ is a [7, 4] code with generator matrix $(1, 2, 4, 8, 14, 5)$ (in decimal form, e.g., $(1, 0, 1, 1)^T$ is represented by $13$) and $d_{\nu}^{C_3} = 5 < \frac{1}{3} \cdot 3$, and $C_4$ is an [11, 6] code with generator matrix $(1, 2, 4, 8, 16, 32, 48, 40, 24, 56, 55)$ and $d_{\nu}^{C_4} = 4 < \frac{11}{6} \cdot 3$. Note that $C_2$, $C_3$, and $C_4$ cannot be decomposed into a direct sum of MDS-PIR capacity-achieving codes as in (8). For all presented codes except $C_3$, Protocol $C$ achieves strictly larger PIR rate than Protocol $A$, although smaller than the MDS-PIR capacity.

### Appendix A

**Proof of Theorem 4**

Achievability is by Theorem 1 and Corollary 1. Hence, in this appendix, we only provide the converse proof of Theorem 4.

Before we proceed with the converse proof, we give some general results that hold for any PIR protocol.

1) Given a query $Q_{l_i}^{(m)}$ sent to the $l$-th node, $m \in \mathbb{N}_f$, the response $A_{l_i}^{(m)}$ received by the user is a function of $Q_{l_i}^{(m)}$ and the $f$ coded chunks (denoted by $c_i \triangleq (c_{i,1}, \ldots, c_{i,f})$) that are stored in the $l$-th node. It follows that

$$H(A_{l_i}^{(m)} | Q_{l_i}^{(m)}, X^{n}) = H(A_{l_i}^{(m)} | Q_{l_i}^{(m)}, c_i) = 0.$$  

2) From the condition of privacy, the $l$-th node should not be able to differentiate between the responses $A_{l_i}^{(m)}$ and $A_{l_i}^{(m')}$, when the user requests $X^{(m)}$, $m \neq m'$. Hence,

$$H(A_{l_i}^{(m)} | Q, X^{(m)}) = H(A_{l_i}^{(m')} | Q, X^{(m)}),$$  

where $Q \triangleq \{Q_{l_i}^{(m)} : m \in \mathbb{N}_f, l \in \mathbb{N}_n\}$ denotes the set of all possible queries made by the user. Although this seems to be intuitively true, a proof of this property is still required and can be found in [13, Lem. 3].

3) Consider a PIR protocol for a coded DSS that uses an $(n, k)$ code $C$ to store $f$ files. For any subset of files $M \subseteq \mathbb{N}_f$ and for any information set $I \subseteq C$, we have

$$H(A_{I}^{(m)} | X^M, Q) = \sum_{i \in I} H(A_{i}^{(m)} | X^M, Q).$$

The proof uses the linear independence of the columns of a generator matrix of $C$ corresponding to an information set, and can be seen as a simple extension of [7, Lem. 2] or [13, Lem. 4].

Next, we state Shearer’s Lemma, which represents a very useful entropy method for combinatorial problems.

**Lemma 2** (Shearer’s Lemma [14]). Let $\mathcal{S}$ be a collection of subsets of $\mathbb{N}_n$, with each $l \in \mathbb{N}_n$ included in at least $k$ members of $\mathcal{S}$. For random variables $Z_1, \ldots, Z_n$, we have

$$\sum_{S \in \mathcal{S}} H(Z_S) \geq \kappa H(Z_1, \ldots, Z_n).$$

Now, we are ready for the converse proof. By Lemma 1, since the code $C$ is MDS-PIR capacity-achieving, there exist $\nu$ information sets $I_1, \ldots, I_\nu$ such that each coordinate $l \in \mathbb{N}_n$ is included in exactly $k$ members of $\mathcal{S} = \{I_1, \ldots, I_\nu\}$ with $\frac{\alpha}{\nu} = \frac{k}{n}$.

Applying the chain rule of entropy we have

$$H(A_{I_i}^{(m)} | X^M, Q) \geq \sum_{i=1}^\nu H(A_{I_i}^{(m)} | X^M, Q)$$

$$= \sum_{i=1}^\nu \left( \sum_{l \in I_i} H(A_{l}^{(m)} | X^M, Q) \right)$$

$$= \sum_{i=1}^\nu \left( \sum_{l \in I_i} H(A_{l}^{(m)} | X^M, Q) \right)$$

$$= \sum_{i=1}^\nu H(A_{I_i}^{(m)} | X^M, Q)$$

$$\geq \kappa H(A_{I_i}^{(m)} | X^M, Q)$$

$$= \kappa [H(A_{I_i}^{(m)}, X^{(m)} | X^M, Q)] - \sum_{i=1}^\nu H(A_{I_i}^{(m)} | X^M, Q)$$

$$\geq \kappa H(X^{(m)} | X^M, Q)$$

where (14) and (16) follow from (13), (15) is because of (12), (17) is due to Shearer’s Lemma, (18) is from the fact that the $m'$-th file $X^{(m')}$ is determined by the responses $A_{I_i}^{(m')}$ and

$$\hat{\nu} H(A_{I_i}^{(m')} | X^M, Q)$$

$$= \sum_{i=1}^\nu H(A_{I_i}^{(m')} | X^M, Q)$$

$$= \nu H(A_{I_i}^{(m')} | X^M, Q)$$

$$= \nu H(A_{I_i}^{(m')} | X^M, Q)$$

$$\geq \kappa H(X^{(m')} | X^M, Q) + H(A_{I_i}^{(m')} | X^M, X^{(m')}, Q)$$

Table III

| Code      | $\alpha$ | $\beta_{\infty}$ | $\beta_{\infty}$ | $\beta_{\infty}$ | $\beta_{\infty}$ |
|-----------|----------|-------------------|-------------------|-------------------|-------------------|
| $C_1 : [5, 3]$ | 2/3 | 0.3 | 0.3333 | 0.375 | 0.375 | 0.4 |
| $C_2 : [9, 5]$ | 2/3 | 0.2778 | 0.3333 | – | 0.3571 | 0.4444 |
| $C_3 : [7, 4]$ | 3/5 | 0.3810 | 0.4 | – | 0.4 | 0.4286 |
| $C_4 : [11, 6]$ | 3/4 | 0.1818 | 0.25 | – | 0.2824 | 0.4545 |
the queries \( Q \), and finally, (19) follows from the independence between the queries and the files. Therefore, we can conclude that

\[
\begin{aligned}
H(A^{(m)}_{\nu_n} | X^M, Q) &
\geq \sum_{\nu} \frac{\kappa}{n} H(X^{(m)} | X^M) + \sum_{\nu} \frac{\kappa}{n} H(A^{(m)}_{\nu_n} | X^M, X^{(m')}, Q) \\
&= \sum_{\nu} \frac{k}{n} H(X^{(m')} | X^M) + \sum_{\nu} \frac{k}{n} H(A^{(m')}_{\nu_n} | X^M, X^{(m')}, Q),
\end{aligned}
\]

where we have used Definition 4 to obtain (20).

Since there are in total \( f \) files, we can recursively use (20) \( f - 1 \) times to obtain

\[
H(A^{(1)}_{\nu_n} | X^{(1)}, Q) \geq \sum_{m=1}^{f-1} \left( \frac{k}{n} \right)^m H(X^{(m+1)} | X^{(m)}) + \left( \frac{k}{n} \right)^{f-1} H(A^{(f)}_{\nu_n} | X^{(f)}, Q).
\]

(21)

(22)

where (21) follows from (11), (22) holds since \( H(X^{(m)} | X^{(m-1)}) = H(X^{(m)}) = L \).

Now,

\[
\begin{aligned}
L &= H(X^{(1)}) \\
&= H(X^{(1)} | Q) - H(X^{(1)} | A^{(1)}_{\nu_n} | Q) \\
&= I(X^{(1)} ; A^{(1)}_{\nu_n} | Q) \\
&= H(A^{(1)}_{\nu_n} | Q) - H(A^{(1)}_{\nu_n} | X^{(1)}, Q) \\
&\leq H(A^{(1)}_{\nu_n} | Q) - \sum_{m=1}^{f-1} \left( \frac{k}{n} \right)^m L,
\end{aligned}
\]

(23)

(24)

where (23) follows since any file is independent of the queries \( Q \), and knowing the responses \( A^{(1)}_{\nu_n} \) and the queries \( Q \), one can determine \( X^{(1)} \). Inequality (24) holds because of (22).

Finally, the converse proof is completed by showing that

\[
R = \frac{L}{\sum_{i=1}^{n} H(A^{(1)}_{l_i})} \leq \frac{L}{H(A^{(1)}_{\nu_n})} \leq \frac{L}{H(A^{(1)}_{\nu_n} | Q)} \leq \frac{1}{1 + \sum_{m=1}^{f-1} \left( \frac{k}{n} \right)^m} = C_f^{[n,k]},
\]

(25)

(26)

(27)

where (25) holds because of the chain rule of entropy, (26) is due to the fact that conditioning reduces entropy, and we apply (24) to obtain (27).

**APPENDIX B**

**PROOF OF THEOREM 5**

The theorem is proved by showing that some downloaded symbols in Protocol 1 from [9] are not really necessary both from the recovery and the privacy perspective. The resulting protocol is named Protocol A, and the proof is based on the fact that for a PIR achievable rate matrix \( A_{\kappa,\nu}(C) \) of a code \( C \), to recover a file of size \( \beta \times k \), exactly \( v(k) \) code coordinates of the \( \nu \) information sets \( \{ \chi(\lambda_i) \}_{i \in \mathbb{N}} \) are required to be exploited in Protocol 1. In order to illustrate the achievability proof, we have to review the steps and proof of Protocol 1 in [9, Sec. IV and App. B], and we refer the reader to [9] for the details. In particular, Protocol 1 in [9] is constructed from two matrices as defined below.

**Definition 5.** For a given \( \nu \times n \) PIR achievable rate matrix \( A_{\kappa,\nu}(C) = (\lambda_{u,l},) \), we define the PIR interference matrices \( A_{\kappa} = (a_{i,l}) \) and \( B_{(\nu-k)\times n} = (b_{i,l}) \) for the code \( C \) with

\[
a_{i,l} \triangleq u \text{ if } \lambda_{u,l} = 1, \forall l \in \mathbb{N}_n, i \in \mathbb{N}_\nu, \quad b_{i,l} \triangleq u \text{ if } \lambda_{u,l} = 0, \forall l \in \mathbb{N}_n, i \in \mathbb{N}_{\nu-k}, u \in \mathbb{N}_\nu.
\]

Note that in Definition 5, for each \( l \in \mathbb{N}_n \), distinct values of \( u \in \mathbb{N}_\nu \) should be assigned for all \( i \). Thus, the assignment is not unique in the sense that the order of the entries of each column of \( A \) and \( B \) can be permuted. Further, by \( S(a|A_{\kappa}) \) we denote the set of column coordinates of matrix \( A_{\kappa} = (a_{i,l}) \) in which at least one of its entries is equal to \( a \), i.e.,

\[
S(a|A_{\kappa}) \triangleq \{ l \in \mathbb{N}_n : \exists a_{i,l} = a, i \in \mathbb{N}_\kappa \}.
\]

Thus, Definition 5 leads to the following claim.

**Claim 1 ([9, Claim 1]).** \( S(a|A_{\kappa}) \) contains an information set of code \( C \), \( \forall a \in \mathbb{N}_\nu \). Moreover, for an arbitrary entry \( b_{i,l} \) of \( B_{(\nu-k)\times n} \), \( S(b_{i,l}|A_{\kappa}) = S(a|A_{\kappa}) \subseteq \mathbb{N}_n \setminus \{ l \} \) if \( b_{i,l} = a \).

From Definition 5 we see that there are in total \( \kappa n \) entries in \( A \) and each entry \( a_{i,l} \) is related to a coordinate within \( \chi(\lambda_i) \), \( i \in \mathbb{N}_\nu \), \( l \in \mathbb{N}_n \). In Protocol 1 the user downloads the needed symbols in a total of \( \kappa \) repetitions and in the \( i \)-th repetition, \( i \in \mathbb{N}_\kappa \), the user downloads the required symbols in a total of \( f \) rounds. Two types of symbols are downloaded by the user, desired symbols, which are directly related to the requested file (say \( X^{(1)} \)), and undesired symbols, which are not related to the requested file, but are exploited to decode the requested file from the desired symbols.

Consider a fixed \( i \in \mathbb{N}_\kappa \) and denote by \( D(a_{i,l}) \) the total download cost of Protocol 1 resulting from a particular entry \( a_{i,l} \), \( l \in \mathbb{N}_n \). First, we focus on the undesired symbols downloaded in Step 2 of Protocol 1. In each repetition the user downloads

\[
\kappa \left( \frac{f - 1}{\ell} \right) \left[ U(\ell) - 1 - U(\ell - 1) + 1 \right] = \left( \frac{f - 1}{\ell} \right)^{\kappa - (\ell + 1)(\nu - k) - 1}
\]

where \( U(\ell) \) denotes the utility function.
undesired symbols resulting from a particular \( a_{i,l} \) in the \( \ell \)-th round, \( \ell \in \mathbb{N}_{f-1} \), where \( \mathfrak{U}(\ell) \triangleq \sum_{h=1}^{\ell} \kappa^{f-(h+1)} (\nu - \kappa)^{\ell-1} \). Hence, for the undesired symbols associated with \( a_{i,l} \), in total

\[
\kappa \binom{f-1}{\ell} \kappa^{f-(\ell+1)} (\nu - \kappa)^{\ell-1}
= \binom{f-1}{\ell} \kappa^{f-\ell} (\nu - \kappa)^{\ell-1}
\tag{28}
\]

symbols are downloaded in every \( \ell \)-th round of all \( \kappa \) repetitions.

Secondly, for a particular entry \( a_{i,l} \) in the \( i \)-th repetition, the user downloads \( \kappa f^{\ell-1} \) desired symbols from the \( \ell \)-th node in round \( \ell = 1 \), and

\[
W(\ell) - 1 - W(\ell - 1) + 1 = \binom{f-1}{\ell} \kappa^{f-(\ell+1)} (\nu - \kappa)^{\ell}
\tag{29}
\]

extra desired symbols in the \((\ell + 1)\)-th round, \( \ell \in \mathbb{N}_{f-1} \), where \( W(\ell) \) is defined as

\[
W(\ell) \triangleq \kappa f^{\ell-1} + \sum_{h=1}^{\ell} \binom{f-1}{h} \kappa^{f-(h+1)} (\nu - \kappa)^h.
\]

In summary, using (28) and (29), the download cost associated to entry \( a_{i,l} \) is obtained as

\[
\mathcal{D}(a_{i,l}) = \sum_{\ell=1}^{f-1} \binom{f-1}{\ell} \kappa^{f-\ell} (\nu - \kappa)^{\ell-1}
+ \sum_{\ell=0}^{f-1} \binom{f-1}{\ell} \kappa^{f-(\ell+1)} (\nu - \kappa)^{\ell}
= \frac{\nu f - \kappa f}{\nu - \kappa}.
\]

In the part of Step 2 of Protocol 1 that exploits side information, we only require \( \nu \) information sets induced by the matrix \( \mathbf{A} \) to reconstruct code symbols induced by \( \mathbf{B} \). Moreover, from [9, App. B], after Step 2 of Protocol 1, \( \beta = \nu f \) rows of code symbols of length \( n \) have been downloaded, and again the information sets induced by the matrix \( \mathbf{A} \) are enough to recover all length-\( k \) stripes of the requested file. In other words, \( \kappa n - \nu k \) entries of \( \mathbf{A} \) are redundant for the reconstruction of all \( \beta = \nu f \) stripes of the requested file. Thus, the improved PIR rate becomes

\[
\beta k = \frac{\nu f k}{\mathcal{D}(a_{i,l})}
= \frac{\nu f k}{\nu f k} \left[ \frac{\kappa n - \nu k}{\nu - \kappa} (\nu f - \kappa f) - \frac{\kappa n - \nu k}{\nu - \kappa} (\nu f - \kappa f) \right]
= \frac{\nu f k}{\nu - \kappa} \left[ \nu f - \kappa f \right] \left[ 1 - \left( \frac{\kappa n}{\nu f} - 1 \right) \right]^{-1}.
\]

Finally, we would like to emphasize that by removing the redundant downloaded sums of code symbols in Protocol 1, it can be shown that within each storage node in each round \( \ell \in \mathbb{N}_{f} \) of all repetitions, file symmetry still remains. This follows from a similar argumentation as in the privacy part of the proof of Protocol 1 in [9, App. B]. In the following, we briefly explain that in each round \( \ell \in \mathbb{N}_{f} \) of all repetitions, for each particular entry \( a_{i,l} \) and for every combination of files \( \mathcal{M} \subseteq \mathbb{N}_{f} \) with \( |\mathcal{M}| = \ell \), the user requests the same number of every possible combination of files in \( \mathcal{D}(a_{i,l}) \).

- In the first round (\( \ell = 1 \)) of all \( \kappa \) repetitions, it follows from (28) that, for each \( m' \in \mathbb{N}_{2,f} \), the number of downloaded undesired symbols resulting from a particular entry \( a_{i,l} \) is \( \kappa f^{\ell-1} \), the same as the number of downloaded desired symbols resulting from \( a_{i,l} \).
- In the \((\ell + 1)\)-th round of all \( \kappa \) repetitions, \( \ell \in \mathbb{N}_{f-2} \), arbitrarily choose a combination of files \( \mathcal{M} \subseteq \mathbb{N}_{2,f} \), where \( |\mathcal{M}| = \ell \). For a particular entry \( a_{i,l} \), it follows from (29) that the total number of downloaded desired symbols for files pertaining to \( \mathcal{M} \) is equal to \( \kappa f^{\ell-1} (\nu - \kappa)\ell \). On the other hand, for the undesired symbols resulting from a particular \( a_{i,l} \), it follows from (28) that in the \((\ell + 1)\)-th round the user downloads \( \kappa f^{\ell-1} (\nu - \kappa)\ell \) linear sums for a combination of files \( \mathcal{M} \subseteq \mathbb{N}_{2,f} \), \( |\mathcal{M}| = \ell + 1 \). Thus, in rounds \( \mathbb{N}_{f-1} \setminus \{1\} \), an equal number of linear sums for all combinations of files \( \mathcal{M} \subseteq \mathbb{N}_{f} \) are downloaded.
- In the \( f \)-th round, only desired symbols are downloaded. Since each desired symbol is a linear combination of code symbols from all \( f \) files, an equal number of linear sums is again downloaded from each file.

In summary, in response to each particular \( a_{i,l} \), the user downloads the same number of linear sums for every possible combination of files. As illustrated above, this is inherent from Protocol 1, and hence the privacy condition of (1a) is still satisfied.

**APPENDIX C**

**PROOF OF THEOREM 6**

The result follows by treating Protocol 1 and Protocol 2 from [9] as subprotocols for each punctured MDS-PIR capacity-achieving subcode \( C_{G_{P}} \) and \( \beta_{P}, p \in \mathbb{N}_{P} \). If Protocol 1 is used as a subprotocol, then we obtain the file-dependent Protocol B and the PIR rate in (9), while if Protocol 2 is used as a subprotocol, then we obtain the file-independent Protocol B and the PIR rate in (10).

For the asymmetric Protocol B, we require \( \beta = \text{LCM}(\beta_{1}, \ldots, \beta_{P}) \) stripes, where \( \beta_{p}, p \in \mathbb{N}_{P} \), is the smallest number of stripes of either Protocol 1 or Protocol 2 for a DSS that uses only the punctured MDS-PIR capacity-achieving subcode \( C_{G_{P}} \) to store \( f \) files (see Theorem 2). Note that for Protocol 1 the index preparation\(^2\) should be made for all \( \beta \) stripes. Since \( \sum_{p=1}^{P} k_{p} = k \) and \( \sum_{p=1}^{P} n_{p} = n \), to privately retrieve the entire requested file consisting of \( k \) symbols in each stripe, we have to privately recover all \( P \) substripes of all \( \beta \) stripes, where the \( p \)-th substripe is of length \( k_{p} \) by processing the subprotocol (either Protocol 1 or Protocol 2) for every punctured subcode \( C_{G_{P}} \). In particular, for each punctured subcode \( C_{G_{P}} \) we repeat the subprotocol \( \beta_{P}/\beta_{p} \) times to recover all the length-\( k_{p} \) requested substripes. This can be done since both Protocol 1 and Protocol 2 recover \( \beta_{p} \) stripes of length \( k_{p} \).

\(^2\)This terminology was introduced in Step 1 of Protocol 1 from [9], i.e., the indices of the rows for each file are interleaved randomly and independently of each other.
while repeating it $\beta/\beta_p$ times enables the recovery of $\beta$ length-$k_p$ substripes. Note that privacy is ensured since the storage nodes of each punctured subcode are disjoint and within the nodes associated with each punctured subcode $C^{G_p}$ the subprotocol (Protocol 1 or Protocol 2) yields privacy against each server [9].

Denote by $D_p$ the total download cost for each node for the punctured subcode $C^{G_p}$ using the subprotocol, $p \in \mathbb{N}_P$. The PIR rates of the file-dependent and file-independent Protocol B are given by

$$\frac{\beta k}{D} = \frac{\beta k}{\sum_{p=1}^{P} \frac{\beta}{\beta_p} n_p D_p} \tag{30}$$

$$= \left( \sum_{p=1}^{P} \frac{1}{k} \frac{n_p D_p}{\beta_p} \right)^{-1}$$

$$= \begin{cases} 
\left( \sum_{p=1}^{P} \frac{1}{k} \frac{k_p \left(C_{n_p,k_p}^f\right)^{-1}}{\beta_p} \right)^{-1} & \text{if Protocol 1 is used as subprotocol,} \\
\left( \sum_{p=1}^{P} \frac{1}{k} \frac{k_p \left(C_{n_p,k_p}^\infty\right)^{-1}}{\beta_p} \right)^{-1} & \text{if Protocol 2 is used as subprotocol,} 
\end{cases} \tag{31}$$

where (30) holds since within each punctured subcode, the subprotocol is required to be repeated $\frac{\beta}{\beta_p}$ times and (31) follows from (4).

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