SHARP BOUNDS FOR HARMONIC NUMBERS

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Abstract. In the paper, we first survey some results on inequalities for bounding harmonic numbers or Euler-Mascheroni constant, and then we establish a new sharp double inequality for bounding harmonic numbers as follows: For \( n \in \mathbb{N} \), the double inequality

\[
\frac{1}{12n^2 + 2(7 - 12\gamma)/(2\gamma - 1)} \leq H(n) - \ln n - \frac{1}{2n} - \gamma < \frac{1}{12n^2 + 6/5}
\]

is valid, with equality in the left-hand side only when \( n = 1 \), where the scalars \( \frac{2(7 - 12\gamma)}{2\gamma - 1} \) and \( \frac{6}{5} \) are the best possible.

1. Introduction

The series

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots
\]

(1)

is called harmonic series. The \( n \)-th harmonic number \( H(n) \) for \( n \in \mathbb{N} \), the sum of the first \( n \) terms of the harmonic series, may be given analytically by

\[
H(n) = \sum_{i=1}^{n} \frac{1}{i} = \gamma + \psi(n + 1),
\]

(2)

see [1, p. 258, 6.3.2], where \( \gamma = 0.57721566 \cdots \) is Euler-Mascheroni constant and \( \psi(x) \) denotes the psi function, the logarithmic derivative \( \Gamma'(x)/\Gamma(x) \) of the classical Euler gamma function \( \Gamma(x) \) which may be defined by

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt, \quad x > 0.
\]

(3)

In [17], the so-called Franel’s inequality in literature was given by

\[
\frac{1}{2n} - \frac{1}{8n^2} < H(n) - \ln n - \gamma < \frac{1}{2n}, \quad n \in \mathbb{N}.
\]

(4)

In [11, pp. 105–106], by considering

\[
I_n = \int_{1/n}^{1} \left( \frac{1}{x} - \left[ \frac{1}{x} \right] \right) \, dx = \ln n - H(n)
\]

(5)
and $0 < I_n < \frac{1}{2}$, where $[t]$ denotes the largest integer less than or equal to $t$, it was established that

$$\frac{1}{2} < H(n) - \ln n < 1, \quad n \in \mathbb{N}. \tag{6}$$

In [11, pp. 128–129, Problem 65], it was verified that

$$\frac{1}{2} \ln(2n + 1) < \sum_{k=1}^{n} \frac{1}{2k-1} < 1 + \frac{1}{2} \ln(2n - 1), \quad n \in \mathbb{N}. \tag{7}$$

In [27], it was obtained that

$$\frac{1}{2(n+1)} < H(n) - \ln n - \gamma < \frac{1}{2n}, \quad n \in \mathbb{N}. \tag{8}$$

In [8], it was proved that

$$\frac{1}{24(n+1)^2} < H(n) - \ln \left( n + \frac{1}{2} \right) - \gamma < \frac{1}{24n^2}, \quad n \in \mathbb{N}. \tag{9}$$

In [21], the following problems were proposed:

1. Prove that for every positive integer $n$ we have

$$\frac{1}{2n + 2/5} < H(n) - \ln n - \gamma < \frac{1}{2n + 1/3}. \tag{10}$$

2. Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In [10], by using

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4}, \tag{11}$$

for $0 < \varepsilon_n < 1$, these problems were answered affirmatively. The editorial comment in [10] said that the number $\frac{2}{5}$ in (10) can be replaced by $\frac{2\gamma - 1}{\pi^2}$ and equality holds only when $n = 1$. This means that

$$\frac{1}{2n + 1/\gamma - 2} \leq H(n) - \ln n - \gamma < \frac{1}{2n + 1/3}, \quad n \in \mathbb{N}. \tag{12}$$

This double inequality was recovered and sharpened in [6, 7] and [18, Theorem 2].

In [26], basing on an improved Euler-Maclaurin summation formula, some general inequalities for the $n$-th harmonic number $H(n)$ are established, including recovery of the inequality (10).

In [25], the problems above-mentioned was solved once again by employing

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{2} \sum_{i=1}^{q-1} \frac{B_{2i}}{in^{2i}} - \int_{n}^{\infty} \frac{B_{2q}(x)}{x^{2q}} \, dx \tag{13}$$

and

$$\int_{n}^{\infty} \frac{B_{2q-1}(x)}{x^{2q}} \, dx < \frac{(-1)^q B_{2q}}{2qn^{2q}}, \tag{14}$$

where $n$ and $q$ are positive integers, $B_i(x)$ are Bernoulli polynomials and $B_{2i} = B_{2i}(0)$ denote Bernoulli numbers for $i \in \mathbb{N}$. For definitions of $B_i(x)$ and $B_{2i}$, please refer to [1, p. 804].

In [23], the inequality (10) was verified again by calculus.
In [13], by utilizing Euler-Maclaurin summation formula, the following general result was obtained:

\[
H(n) = \ln n + \gamma + \frac{1}{2n^2} + \frac{1}{2} \sum_{i=3}^{m} \frac{B_{2(i-1)}}{(i-1)n^{2(i-1)}} + O\left(\frac{1}{n^{2m}}\right),
\]

(15)

See also [15, p. 77]. In fact, this is equivalent to the formula in [1, p. 259, 6.3.18].

In [22], by considering the decreasing monotonicity of the sequence

\[
x_n = \frac{1}{\left|\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k}\right|} - 2n,
\]

(16)

it was shown that the best constants \(a\) and \(b\) such that

\[
\frac{1}{2n+a} \leq \left|\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k}\right| < \frac{1}{2n+b}
\]

(17)

for \(n \geq 1\) are \(a = 1\) and \(b = 1\).

In [4, Theorem 2.8] and [19], alternative sharp bounds for \(H(n)\) were presented: For \(n \in \mathbb{N}\),

\[
1 + \ln(\sqrt{e} - 1) - \ln(e^{1/(n+1)} - 1) \leq H(n) < \gamma - \ln(e^{1/(n+1)} - 1).
\]

(18)

The constants \(1 + \ln(\sqrt{e} - 1)\) and \(\gamma\) in (18) are the best possible. This improves the result in [3, pp. 386–387].

In [20], it was established that

\[
\ln \left( n + \frac{1}{2} \right) + \gamma < H(n) \leq \ln \left( n + e^{1-\gamma} - 1 \right) + \gamma, \quad n \in \mathbb{N}. \tag{19}
\]

In [5], it was obtained that

\[
\frac{1}{24\{n + 1/2\sqrt{6(1 - \gamma - \ln(3/2))}\}^2} \leq H(n) - \ln \left( n + \frac{1}{2} \right) - \gamma < \frac{1}{24(n + 1/2)^2} \tag{20}
\]

for \(n \in \mathbb{N}\), where the constants

\[
\frac{1}{2\sqrt{6(1 - \gamma - \ln(3/2))}}
\]

and \(\frac{1}{3}\) are the best possible.

For more information on estimates of harmonic numbers \(H(n)\), please refer to [9, 24], [14, pp. 68–86], [15, pp. 75–79] and closely-related references therein.

The aim of this paper is to establish a double inequality for bounding harmonic numbers, which is sharp and refines those inequalities above-mentioned.

**Theorem 1.** For \(n \in \mathbb{N}\), the double inequality

\[
-\frac{1}{12n^2 + 2(7 - 12\gamma)/(2\gamma - 1)} \leq H(n) - \ln n - \frac{1}{2n} - \gamma < -\frac{1}{12n^2 + 6/5} \tag{21}
\]

is valid, with equality in the left-hand side of (21) only when \(n = 1\), where the scalars \(\frac{2(7 - 12\gamma)}{2\gamma - 1}\) and \(\frac{6}{5}\) in (21) are the best possible.

**Remark 1.** When \(n \geq 2\), the double inequality (21) refines (20) and those mentioned before it.
2. Proof of Theorem 1

We now are in a position to prove Theorem 1. Let
\[
f(x) = \frac{1}{\ln x + \frac{1}{2}x - \psi(x + 1)} - 12x^2
\]  
(22)
for \(x \in (0, \infty)\). An easy computation gives
\[
f'(x) = \frac{4x^2\psi'(x + 1) - 4x + 2}{[2x \ln x - 2x\psi(x + 1) + 1]^2} - 24x = \frac{4x^2g(x)}{[2x \ln x - 2x\psi(x + 1) + 1]^2},
\]
where
\[
g(x) = \psi'(x + 1) - \frac{1}{x} + \frac{1}{2x^2} - 24x\left[\psi(x + 1) - \ln x - \frac{1}{2x}\right]^2.
\]  
(23)
In [2, Theorem 8], the functions
\[
F_n(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}
\]  
(24)
and
\[
G_n(x) = -\ln \Gamma(x) + \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}
\]  
(25)
for \(n \geq 0\) were proved to be completely monotonic on \((0, \infty)\). This generalizes [16, Theorem 1] which states that the functions \(F_n(x)\) and \(G_n(x)\) are convex on \((0, \infty)\). The complete monotonicity of \(F_n(x)\) and \(G_n(x)\) was proved in [12, Theorem 2] once again. In particular, the functions
\[
F_2(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi)
\]  
(26)
and
\[
G_1(x) = -\ln \Gamma(x) + \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi)
\]  
(27)
are completely monotonic on \((0, \infty)\). Therefore, we have
\[
\ln x + \frac{1260x^5 + 210x^4 - 21x^2 + 10}{2520x^6} \psi(x)
\]  
(28)
and
\[
\frac{1260x^5 + 210x^4 + 35x^6 - 7x^4 + 5x^2 - 7}{210x^9} \psi'(x)
\]  
(29)
on \((0, \infty)\). From this, it follows that
\[
\ln x + \frac{1}{2x} - \psi(x + 1) = \ln x - \frac{1}{2x} - \psi(x) < \frac{1260x^3 + 210x^4 - 21x^2 + 10}{2520x^6} - \frac{1}{2x} = \frac{10 - 21x^2 + 210x^4}{2520x^6},
\]

and
\[
g(x) > \psi'(x) - \frac{1}{x^2} + \frac{1}{2x^2} - \frac{(10 - 21x^2 + 210x^4)^2}{264600x^{11}} > \frac{210x^8 + 105x^7 + 35x^6 - 7x^4 + 5x^2 - 7}{210x^9} - \frac{1}{x^2} - \frac{1}{x} + \frac{1}{2x^2} - \frac{(10 - 21x^2 + 210x^4)^2}{264600x^{11}} = \frac{1659x^4 - 8400x^2 - 100}{264600x^{11}} = \frac{1659(x - 3)^4 + 19908(x - 3)^3 + 81186(x - 3)^2 + 128772(x - 3) + 58679}{264600x^{11}}.
\]

Hence, the function \( g(x) \) is positive on \([3, \infty)\). So the derivative \( f'(x) > 0 \) on \([3, \infty)\), that is, the function \( f(x) \) is strictly increasing on \([3, \infty)\).

It is easy to obtain
\[
\begin{align*}
f(1) &= \frac{2(7 - 12\gamma)}{2\gamma - 1} = 0.9507 \ldots, \\
f(2) &= \frac{4(48\gamma + 48\ln 2 - 61)}{5 - 4\gamma - 4\ln 2} = 1.109 \ldots, \\
f(3) &= \frac{3(108\gamma + 108\ln 3 - 181)}{5 - 3\gamma - 3\ln 3} = 1.1549 \ldots.
\end{align*}
\]

This means that the sequence \( f(n) \) for \( n \in \mathbb{N} \) is strictly increasing.

Employing the inequality (30) yields
\[
f(x) > \frac{2520x^6}{10 - 21x^2 + 210x^4} - 12x^2 = \frac{12x^2(21x^2 - 10)}{10 - 21x^2 + 210x^4} \to \frac{6}{5}
\]
as \( x \to \infty \). Utilizing the right-hand side inequality in (28) leads to
\[
f(x) = \frac{1}{\ln x - 1/2x - \psi(x)} - 12x^2 < \frac{1}{(2520x^7 + 420x^6 - 42x^4 + 20x^2 - 21)/5040x^8 - 1/2x} - 12x^2
\]
\[
= \frac{12x^2(42x^4 - 20x^2 + 21)}{420x^6 - 42x^4 + 20x^2 - 21} \to \frac{6}{5}
\]
as \( x \to \infty \). As a result, it follows that \( \lim_{x \to \infty} f(x) = \frac{6}{5} \). Therefore, it is derived that \( f(1) \leq f(n) < \frac{6}{5} \) for \( n \in \mathbb{N} \), equivalently,
\[
\frac{2(7 - 12\gamma)}{2\gamma - 1} \leq \frac{1}{\ln n + 1/2n - \psi(n + 1)} - 12n^2 < \frac{6}{5}
\]
which can be rearranged as
\[
\frac{1}{12n^2 + 2(7 - 12\gamma)/(2\gamma - 1)} \geq \ln n + \frac{1}{2n} - \psi(n + 1) > \frac{1}{12n^2 + 6/5}.
\]
Combining this with (2) yields (21). The proof of Theorem 1 is proved.

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