On Polynomial Remainder Codes

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Abstract

Polynomial remainder codes are a large class of codes derived from the Chinese remainder theorem that includes Reed-Solomon codes as a special case. In this paper, we revisit these codes and study them more carefully than in previous work. We explicitly allow the code symbols to be polynomials of different degrees, which leads to two different notions of weight and distance.

Algebraic decoding is studied in detail. If the moduli are not irreducible, the notion of an error locator polynomial is replaced by an error factor polynomial. We then obtain a collection of gcd-based decoding algorithms, some of which are not quite standard even when specialized to Reed-Solomon codes.

Index Terms—Chinese remainder theorem, redundant residue codes, polynomial remainder codes, Reed-Solomon codes, polynomial interpolation.

1 Introduction

Polynomial remainder codes are a large class of codes derived from the Chinese remainder theorem. Such codes were proposed by Stone [2], who also pointed out that these codes include Reed-Solomon codes [3] as a special case. Variations of Stone’s codes were studied in [4–6]. In [2] and [4], the focus is on codes with a fixed symbol size, i.e., the moduli are relatively prime polynomials of the same degree. A generalization of such codes was proposed by Mandelbaum [5], who also pointed out that using moduli of different degrees can be advantageous for burst error correction [6].

Although the codes in [2,4–6] can, in principle, correct many random errors, no efficient decoding algorithm for random errors was proposed in these papers. In 1988, Shiozaki [7] proposed an efficient decoding algorithm for Stone’s codes [2] using Euclid’s algorithm, and he also adapted this algorithm to decode Reed-Solomon codes. However, the algorithm of [7] is restricted to codes with a fixed symbol size, i.e., fixed-degree moduli. Moreover,
the argument given in \[7\] seems to assume that all the moduli are irreducible although this assumption is not stated explicitly.

In \[8\], Mandelbaum made the interesting observation that polynomial remainder codes (generalized as in \[5\]) contain Goppa codes \[9\] as a special case. By means of this observation, generalized versions of Goppa codes such as in \[10\] may also be viewed as polynomial remainder codes. In subsequent work \[11,12\], Mandelbaum actually used the term “generalized Goppa codes” for (generalized) polynomial remainder codes. He also proposed a decoding algorithm for such codes using a continued-fractions approach \[11,12\]. However, this connection between (generalized) polynomial remainder codes and Goppa codes will not be further pursued in this paper.

There is also a body of work on Chinese remainder codes over integers, cf. \[13, 14\]. However, the results of the present paper are not directly related to that work.

In this paper, we revisit polynomial remainder codes as in \[2\]. We explicitly allow moduli of different degrees (i.e., variable symbol sizes) within a codeword. In this way, we can, e.g., lengthen a Reed-Solomon code by adding some higher-degree symbols without increasing the size of the underlying field. In consequence, we obtain two different notions of distance—Hamming distance and degree-weighted distance—and the corresponding minimum-distance decoding rules. Algebraic decoding as in \[7\] is studied in detail. If the moduli are not irreducible, the notion of an error locator polynomial is replaced by an error factor polynomial. We then obtain a collection of gcd-based decoding algorithms, some of which are not quite standard even when specialized to Reed-Solomon codes.

This paper is organized as follows. In Section 2, we recall the Chinese remainder theorem and the definition of Chinese remainder codes over integers and polynomials. We also discuss erasures-only decoding, i.e., the recovery of a codeword from a subset of its symbols, for which we propose a method that appears to be new. In Section 3, we focus on polynomial remainder codes and their minimum-distance decoding, both for Hamming distance and degree-weighted distance. In Section 4, we introduce error locator polynomials and error factor polynomials and a key equation for the latter. In Section 5, we derive gcd-based decoding algorithms. A synopsis of these algorithms is given in Section 5.4 and their relation to prior work is discussed in Section 5.5. Section 6 concludes the paper.

The cardinality of a set \(S\) will be denoted by \(|S|\) and the absolute value of an integer \(n\) will be denoted by \(|n|\). In Section 2.2, this same symbol will also be used for the degree of a polynomial, i.e., \(|a(x)| \triangleq \deg a(x)\).

### 2 Chinese Remainder Codes

#### 2.1 Chinese Remainder Theorem and Codes

Let \(R = \mathbb{Z}\) or \(R = F[x]\) for some field \(F\). (Later on, we will focus on \(R = F[x]\).) For \(R = \mathbb{Z}\), for any positive \(m \in \mathbb{Z}\), let \(R_m\) denote the ring \{0, 1, …, \(m - 1\}\) with addition and multiplication modulo \(m\); for \(R = F[x]\), for any monic polynomial \(m(x) \in F[x]\), let
$R_m$ denote the ring of polynomials over $F$ of degree less than $\deg m(x)$ with addition and multiplication modulo $m(x)$. For $R = \mathbb{Z}$, $\gcd(a, b)$ denotes the greatest common divisor of $a, b \in \mathbb{Z}$, not both zero; for $R = F[x]$, $\gcd(a, b)$ denotes the monic polynomial of largest degree that divides both $a, b \in F[x]$, not both zero.

We will need the Chinese remainder theorem [2] in the following form.

**Theorem 1 (Chinese Remainder Theorem).** For some integer $n > 1$, let $m_0, m_1, \ldots, m_{n-1} \in R$ be relatively prime (i.e., $\gcd(m_i, m_j) = 1$ for $i \neq j$) and let $M_n \triangleq \prod_{i=0}^{n-1} m_i$. Then the mapping

$$\psi : R_{M_n} \to R_{m_0} \times \cdots \times R_{m_{n-1}} : a \mapsto (\psi_0(a), \ldots, \psi_{n-1}(a))$$

(1)

with $\psi_i(a) \triangleq a \mod m_i$ is a ring isomorphism.

The inverse of the mapping (1) is

$$\psi^{-1} : R_{m_0} \times \cdots \times R_{m_{n-1}} \to R_{M_n} : (c_0, \ldots, c_{n-1}) \mapsto \sum_{i=0}^{n-1} c_i \beta_i \mod M_n$$

(2)

with coefficients

$$\beta_i = \frac{M_n}{m_i} \cdot \left( \frac{M_n}{m_i} \right)^{-1} \mod m_i$$

(3)

where $(b)^{-1} \mod m_i$ denotes the inverse of $b$ in $R_{m_i}$.

**Definition 1.** A Chinese remainder code (CRT Code) over $R$ is a set of the form

$$C \triangleq \{(c_0, \ldots, c_{n-1}) : c_i = a \mod m_i \text{ for some } a \in R_{M_k}\}$$

(4)

where $n$ and $k$ are integers satisfying $1 \leq k \leq n$, where $m_0, m_1, \ldots, m_{n-1} \in R$ are relatively prime, and where $M_k \triangleq \prod_{i=0}^{k-1} m_i$.

In other words, a CRT code consists of the images $\psi(a)$, with $\psi$ as in (1), of all $a \in R_{M_k}$. For $R = F[x]$, CRT codes are linear (i.e., vector spaces) over $F$; for $R = \mathbb{Z}$, however, CRT codes are not linear since the pre-image of the sum of two codewords may exceed the range of $M_k$.

The components $c_i = \psi_i(a)$ in (1) and (4) will be called symbols. Note that each symbol is from a different ring $R_{m_i}$; these rings need not have the same number of elements. We will often (but not always) assume that the moduli $m_i$ in Definition 1 satisfy the condition

$$|R_{m_0}| \leq |R_{m_1}| \leq \cdots \leq |R_{m_{n-1}}|.$$  

(5)

We will refer to (5) as the Ordered-Symbol-Size Condition.
2.2 Interpolation

Consider the problem of reconstructing a codeword \( c = (c_0, \ldots, c_{n-1}) \) from a subset of its symbols. Specifically, let \( C \) be a CRT code as in Definition 1 and let \( S \) be a subset of \( \{0, 1, 2, \ldots, n-1\} \) with cardinality \( |S| > 0 \). Let \( c = (c_0, \ldots, c_{n-1}) = \psi(a) \in C \) be the codeword corresponding to some \( a \in \mathbb{R}^{M_k} \) by (4). Suppose we are given \( \tilde{c} = (\tilde{c}_0, \ldots, \tilde{c}_{n-1}) \) with
\[
\tilde{c}_i = c_i \quad \text{for } i \in S
\] (and with arbitrary \( \tilde{c}_i \in \mathbb{R}^{m_i} \) for \( i \notin S \)) and we wish to reconstruct \( a = \psi^{-1}(c) \) from \( \tilde{c} \).

This problem arises, for example, when the channel erases some symbols (and lets the receiver know the erased positions) but delivers the other symbols unchanged. However, this problem also arises as the last step in the decoding procedures that will be discussed later in the paper.

This interpolation problem can certainly be solved if \( S \) is sufficiently large. A first solution follows immediately from the CRT (Theorem 1). Specifically, with \( M_S \triangleq \prod_{i \in S} m_i \), Theorem 1 can be applied as follows: if
\[
|M_S| \geq |M_k|
\] then
\[
a = \sum_{i=0}^{n-1} \tilde{c}_i \tilde{\beta}_i \mod M_S
\] with
\[
\tilde{\beta}_i \triangleq \begin{cases} 
\frac{M_S}{m_i} \cdot \left( \frac{M_S}{m_i} \right)^{-1} \mod m_i, & i \in S \\
0, & i \notin S.
\end{cases}
\] (9)

Obviously, the coefficients \( \tilde{\beta}_i \) in (9) depend on the support set \( S \). Interestingly, there is a second solution to the interpolation problem that avoids the computation of these coefficients: the following theorem shows how \( a = \psi^{-1}(c) \) can be computed from \( \psi^{-1}(\tilde{c}) \), which in turn may be computed using the fixed coefficients (3).

**Theorem 2 (Fixed-Transform Interpolation).** If
\[
|M_S| \geq |M_k|
\] then
\[
\psi^{-1}(c) = Z/M_{\overline{\Sigma}}
\] (11)
where \( M_{\overline{\Sigma}} \triangleq M_n/M_S \) and where
\[
Z \triangleq (M_{\overline{\Sigma}} \cdot \psi^{-1}(\tilde{c})) \mod M_n
\] (12)
is a multiple of \( M_{\overline{\Sigma}} \). □
This theorem does not appear in standard expositions of the CRT; perhaps it is new. Its application to coding, even to Reed-Solomon codes (cf. Section 3.3), also appears to be new.

**Proof of Theorem 2**

Let \( \bar{c} \triangleq c - \tilde{c} \), let \( \bar{a} \triangleq \psi^{-1}(\bar{c}) \), and note that \( \psi^{-1}(\tilde{c}) = (a - \bar{a}) \mod M_n \). Note also that \( |\bar{a}| < |M_n| \) because of (10). Then

\[
Z = (M_{\Sigma} \cdot (a - \bar{a})) \mod M_n
\]

\[
= M_{\Sigma} \cdot a - (M_{\Sigma} \cdot \bar{a}) \mod M_n
\]

\[
= M_{\Sigma} \cdot a
\]

where the last step follows from

\[
\psi(M_{\Sigma} \cdot \bar{a}) = \psi(M_{\Sigma})\psi(\bar{a})
\]

\[
= 0.
\]

### 2.3 Hamming Distance and Singleton Bound

For any \( a \in R_{M_n} \), the Hamming weight of \( \psi(a) \) (i.e., the number of nonzero symbols \( \psi_i(a), 0 \leq i \leq n - 1 \)) will be denoted by \( w_H(\psi(a)) \). For any \( a, b \in R_{M_n} \), the Hamming distance between \( \psi(a) \) and \( \psi(b) \) will be denoted by \( d_H(\psi(a), \psi(b)) = w_H(\psi(a) - \psi(b)) \).

**Theorem 3.** Let \( C \) be a CRT code as in Definition 1 satisfying the Ordered-Symbol-Size Condition (5). Then the Hamming weight of any nonzero codeword \( \psi(a) \) (\( a \in R_{M_k}, a \neq 0 \)) satisfies

\[
w_H(\psi(a)) \geq n - k + 1
\]

and

\[
d_{\text{min}}(C) = n - k + 1.
\]

**Proof:** For any nonzero \( a \in R_{M_n} \), assume that the image \( \psi(a) \) has Hamming weight \( w_H(\psi(a)) \leq n - k \); i.e., the number of zero symbols of \( \psi(a) \) is at least \( k \). For \( R = \mathbb{Z} \), this implies \( a \geq M_k \); for \( R = F[x] \), this implies \( \deg a \geq \deg M_k \). In both cases, \( a \not\in R_{M_k} \), which proves (18).

As for (19), consider \( d_H(\psi(a), \psi(b)) \) for any \( a, b \in R_{M_k}, a \neq b \). For \( R = F[x], a - b \in R_{M_k} \) and thus

\[
d_H(\psi(a), \psi(b)) = w_H(\psi(a) - \psi(b))
\]

\[
= w_H(\psi(a - b))
\]

\[
\geq n - k + 1
\]

by (18). For \( R = \mathbb{Z} \), either \( a - b \in R_{M_k} \) or \( b - a \in R_{M_k} \) and the same argument applies. It follows that \( d_{\text{min}}(C) \geq n - k + 1 \). Finally, the equality in (19) follows from the Singleton bound below. \( \square \)
In the following theorem, we will use the following notation. For any subset $S \subset \{0, 1, \ldots, n-1\}$, let $\overline{S} \triangleq \{0, 1, \ldots, n-1\} \setminus S$ and let

$$R_S \triangleq \bigotimes_{i \in S} R_{m_i},$$

the direct product of all rings $R_{m_i}$ with $i \in S$.

**Theorem 4 (Singleton Bound for Hamming Distance).** Let $C$ be a code in $R_{\{0, \ldots, n-1\}}$ (i.e., a nonempty subset of $R_{m_0} \times \cdots \times R_{m_{n-1}}$) with minimum Hamming distance $d_{\text{minH}}$. Then

$$|C| \leq \min_{S \subset \{0, 1, \ldots, n-1\}} \{|R_S| : |S| > n - d_{\text{minH}}\}.$$  \hfill (24)

Note that this theorem does not require the Ordered-Symbol-Size Condition (5).

**Proof:** Let $\overline{S}$ be a subset of $\{0, 1, \ldots, n-1\}$ with $|\overline{S}| < d_{\text{minH}}$. For every word $c \in C$, erase its components in $\overline{S}$. The resulting set of shortened words, which are elements of $R_S$, has still $|C|$ elements.

For CRT codes satisfying the Ordered-Symbol-Size Condition (5), we have $|C| = |R_{M_k}|$; on the other hand, the right-hand side of (24) becomes

$$|R_{\{0, \ldots, n-d_{\text{minH}}\}}| = |R_{M_{n-d_{\text{minH}}+1}}|$$

where $M_{n-d_{\text{minH}}+1} \triangleq \prod_{i=0}^{n-d_{\text{minH}}} m_i$. It then follows from (24) that $|R_{M_k}| \leq |R_{M_{n-d_{\text{minH}}+1}}|$ and thus

$$k \leq n - d_{\text{minH}} + 1.$$ \hfill (26)

3 Polynomial Remainder Codes

From now on, we will focus on the case $R = F[x]$ for some finite field $F$.

3.1 Definition and Some Examples

**Definition 2.** A polynomial remainder code is a CRT code over $R = F[x]$ with monic moduli $m_i(x)$, i.e., a set of the form

$$C = \{(c_0, \ldots, c_{n-1}) : c_i = a(x) \mod m_i(x) \text{ for some } a(x) \in R_{M_k}\}.$$ \hfill (27)

A polynomial remainder code is irreducible if the polynomials $m_0(x), \ldots, m_{n-1}(x)$ are all irreducible. \hfill (1)
For such codes, the Ordered-Symbol-Size Condition (5) may be written as
\[
\deg m_0(x) \leq \deg m_1(x) \leq \ldots \leq \deg m_{n-1}(x),
\]
which we will call the Ordered-Degree Condition.

Example 1 (Binary Irreducible Polynomial Remainder Codes). Let \( F = \mathrm{GF}(2) \) be the finite field with two elements and let \( m_0(x), \ldots, m_{n-1}(x) \) be different irreducible binary polynomials.

The number of irreducible binary polynomials of degree up to 16 is given in Appendix A. For example, by using only irreducible moduli of degree 16, we can obtain a code with \( \deg M_n(x) = 4080 \); by using irreducible moduli of degree up to 16, we can achieve \( \deg M_n(x) = 130'486 \).

Example 2 (Polynomial Evaluation Codes and Reed-Solomon Codes). Let \( \beta_0, \beta_1, \ldots, \beta_{n-1} \) be distinct elements of some finite field \( F \) (which implies \( n \leq |F| \)). A polynomial evaluation code over \( F \) is a code of the form
\[
C \triangleq \{(c_0, \ldots, c_{n-1}) : c_i = a(\beta_i) \text{ for some } a(x) \in F[x] \text{ of } \deg a(x) < k\}.
\]
A Reed-Solomon code is a polynomial evaluation code with \( \beta_i = \alpha^i \), where \( \alpha \) is a primitive \( n \)-th root of unity in \( F \). With
\[
m_i(x) \triangleq x - \beta_i,
\]
a polynomial evaluation code may be viewed as a polynomial remainder code since
\[
c_i = a(\beta_i) = a(x) \mathrm{mod} m_i(x).
\]
For Reed-Solomon codes (as defined above), we then have
\[
M_n(x) = x^n - 1.
\]

Example 3 (Polynomial Extensions of Reed-Solomon Codes). When Reed-Solomon codes are viewed as polynomial remainder codes as in Example 2, the code symbols are constants, i.e., polynomials of degree at most zero. Reed-Solomon codes can be extended with additional symbols in \( F[x] \) by adding some moduli \( m_i(x) \) of degree two (or higher).

3.2 Degree-weighted Distance

Let
\[
N \triangleq \deg M_n(x) = \sum_{i=0}^{n-1} \deg m_i(x)
\]
and
\[
K \triangleq \deg M_k(x) = \sum_{i=0}^{k-1} \deg m_i(x).
\]
Note that \( K \) is the dimension of the code as a subspace of \( F^N \).
Definition 3. The degree weight of a set $S \subset \{0,1,\ldots,n-1\}$ is

$$w_D(S) \triangleq \sum_{i \in S} \deg m_i(x).$$

(35)

For any $a(x) \in R_{M_n}$, the degree weight of $\psi(a) = (\psi_0(a), \ldots, \psi_{n-1}(a))$ is

$$w_D(\psi(a)) \triangleq \sum_{i: \psi_i(a) \neq 0} \deg m_i,$$

(36)

and for any $a(x), b(x) \in R_{M_n}$, the degree-weighted distance between $\psi(a)$ and $\psi(b)$ is

$$d_D(\psi(a), \psi(b)) \triangleq w_D(\psi(a) - \psi(b)).$$

(37)

Note that the degree-weighted distance satisfies the triangle inequality:

$$d_D(\psi(a), \psi(b)) \leq d_D(\psi(a), \psi(c)) + d_D(\psi(b), \psi(c))$$

(38)

for all $a(x), b(x), c(x) \in R_{M_n}$.

Let $d_{\min D}(C)$ denote the minimum degree-weighted distance of a polynomial remainder code $C$, i.e.,

$$d_{\min D}(C) \triangleq \min_{c,c' \in C: c \neq c'} d_D(c,c'),$$

(39)

and let

$$w_{\min D}(C) \triangleq \min_{c \in C: c \neq 0} w_D(c)$$

(40)

be the minimum degree weight of any nonzero codeword. We then have the following analog of Theorem 3:

Theorem 5 (Minimum Degree-Weighted Distance). Let $C$ be a code as in Definition 2. Then

$$d_{\min D}(C) = w_{\min D}(C)$$

(41)

$$= \min_{S \subset \{0,\ldots,n-1\}} \left\{ w_D(S) : w_D(S) > N-K \right\}$$

(42)

$$> N-K.$$

(43)

If all moduli $m_i(x)$ have degree one, then the right-hand side of (42) equals $N-K+1$.

Note also that unlike Theorem 3, Theorem 5 does not require the Ordered-Degree Condition (28).
Let $d$ be the right-hand side of (42). For any nonzero $a(x) \in R_{M_k}$, assume that the image $\psi(a)$ has degree weight $w_D(\psi(a)) \leq N - K$, i.e., the sum of $\deg m_i(x)$ over the zero symbols of $\psi(a)$ is at least $K$. Then $\deg a(x) \geq K = \deg M_k(x)$, which is impossible since $a(x) \in R_{M_k}$. We thus have $w_D(\psi(a)) > N - K$. It then follows from Definition 3 that $w_D(\psi(a)) \geq d$ and thus $w_{\min D}(C) \geq d$.

Conversely, let $S$ be a subset of $\{0, 1, \ldots, n - 1\}$ such that $w_D(S) = d$. Then there exists some nonzero $a(x) \in R_{M_k}$ such that $\psi_i(a) \neq 0$ for each $i \in S$ but $\psi_j(a) = 0$ for each $j \in \{0, 1, \ldots, n - 1\} \setminus S$. Thus $w_D(\psi(a)) = w_D(S) = d$, which implies $w_{\min D}(C) \leq d$. 

Theorem 6 (Singleton Bound for Degree-weighted Distance). Let $C$ be a nonempty subset of $R_{m_0} \times \cdots \times R_{m_{n-1}}$ with minimum degree-weighted distance $d_{\min D}$ and with $N$ as in (33). Then

$$\log_F |C| \leq \min_{S \subset \{0, \ldots, n-1\}} \{w_D(S) : w_D(S) > N - d_{\min D}\}. \quad (44)$$

Proof: Recall the notation $\overline{S}$ and $R_S$ as in (23). Let $\overline{S}$ be a subset of $\{0, 1, \ldots, n - 1\}$ with $w_D(\overline{S}) < d_{\min D}$. For every word $c \in C$, erase its components in $\overline{S}$. The resulting set of shortened words, which are elements of $R_S$, has still $|C|$ elements. Thus $|C| \leq |R_S| = |F|^{w_D(S)}$, and (44) follows.

For polynomial remainder codes, we have $\log_F |C| = K$ and (44) holds with equality. To see this, we first write (44) as

$$K \leq \min_{S \subset \{0, \ldots, n-1\}} \{w_D(S) : w_D(S) > N - d_{\min D}\}. \quad (45)$$

On the other hand, for $S = \{0, \ldots, k-1\}$, we have $w_D(S) = K$, and using (43), we obtain

$$\min_{S \subset \{0, \ldots, n-1\}} \{w_D(S) : w_D(S) > N - d_{\min D}\} \leq K. \quad (46)$$

We thus have equality in (45) and (46), and therefore also in (44).

In the special case where all the moduli $m_0(x), \ldots, m_{n-1}(x)$ have the same degree, the two Singleton bounds (44) and (24) are equivalent.

3.3 Interpolation and Erasures Decoding

We now return to the subject of Section 2.2 and specialize it to polynomial remainder codes. Let $C$ be a code as in Definition 2. Let $c = (c_0, \ldots, c_{n-1}) = \psi(a(x)) \in C$ be the codeword corresponding to some polynomial $a(x) \in R_{M_k}$. Let $S$ be a set of positions $i \in \{0, \ldots, n - 1\}$ where $c_i$ is known. Let $\tilde{c} = (\tilde{c}_0, \ldots, \tilde{c}_{n-1})$ satisfy $\tilde{c}_i = c_i$ for $i \in S$ with arbitrary $\tilde{c}_i \in R_{m_i}$ for $i \notin S$. Suppose we wish to reconstruct $a(x)$ from $\tilde{c}$ and $S$.

Let $\overline{S} = \{0, \ldots, n - 1\} \setminus S$ be the indices of the unknown components of $c$ and let $M_{\overline{S}}(x) = \prod_{i \in \overline{S}} m_i(x)$ as in Section 2.2. Recall that $w_D(\overline{S})$ denotes the degree weight of the unknown (erased) components of $c$. Then Theorem 2 can be restated as follows:
**Theorem 7 (Fixed-Transform Interpolation for Polynomial Remainder Codes).**
If
\[ w_D(S) \leq N - K, \]  
then
\[ a(x) = Z(x)/M_S(x) \]  
with
\[ Z(x) \overset{\triangle}{=} M_S(x)\psi^{-1}(\tilde{c}) \mod M_n(x). \]

The equivalence of (47) and (10) follows from noting that the left-hand side of (10) is \(|M_S| = N - w_D(S)\) and the right-hand side of (10) is \(|M_k| = K\).

Since \(S\) contains the support set of \(\tilde{c} - c\), the polynomial \(M_S(x)\) is a multiple of an error locator polynomial (as will be defined in Section 4).

In contrast to most other statements in this paper, Theorem 7 appears to be new even when specialized to Reed-Solomon codes (as in Example 2), where \(M_n(x) = x^n - 1\) and the modulo operation in (49) is computationally trivial.

### 3.4 Minimum-Distance Decoding

Let \(C\) be a code as in Definition 2. The receiver sees \(y = c + e\), where \(c \in C\) is the transmitted codeword and \(e\) is an error pattern. A **minimum Hamming distance decoder** is a decoder that produces
\[ \hat{c} = \arg\min_{c \in C} d_H(c,y). \]  

A **minimum degree-weighted distance decoder** is a decoder that produces
\[ \hat{c} = \arg\min_{c \in C} d_D(c,y). \]

In general, the decoding rules (50) and (51) produce different estimates \(\hat{c}\) as will be illustrated by the examples below.

**Theorem 8 (Basic Error Correction Bounds).** If \(d_H(c,y) < d_{\min H}(C)/2\), then the rule (50) produces \(\hat{c} = c\). If \(d_D(c,y) < d_{\min D}(C)/2\), then the rule (51) produces \(\hat{c} = c\).

**Proof:** The proof follows the standard pattern; we prove only the second part. Assume \(\hat{c} \neq c\), which implies \(d_D(\hat{c},y) \leq d_D(c,y)\). Using the triangle inequality (38), we obtain
\[ d_{\min D}(C) \leq d_D(\hat{c},c) \leq d_D(\hat{c},y) + d_D(c,y) \leq 2d_D(c,y). \]

The second part of Theorem 8 can also be formulated as follows: if
\[ w_D(e) \leq t_D \overset{\triangle}{=} \left\lfloor \frac{N - K}{2} \right\rfloor, \]  
(52)
then the rule (51) produces \( \hat{c} = c \). If the Ordered-Degree Condition (28) is satisfied, then the first part of Theorem 8 implies the following: if

\[
  w_H(e) \leq t_H = \left\lfloor \frac{n - k}{2} \right\rfloor,
\]

then the rule (50) produces \( \hat{c} = c \).

Depending on the degrees \( \deg m_i(x) \), it is possible that the condition \( w_H(e) \leq t_H \) implies \( w_D(e) \leq t_D \) (see Example 5 below). In general, however, none of the two decoding rules (50) and (51) is uniformly stronger than the other.

**Example 4.** Let \( k = 3 \) and \( n = 5 \), and let \( \deg m_i(x) = i \) for \( i = 1, 2, \ldots, 5 \). We then have \( t_H = 1, K = 6, N = 15, \) and \( t_D = 4 \). Consider the following two decoders: Decoder A corrects all errors with \( w_H(e) \leq t_H \) and Decoder B corrects all errors with \( w_D(e) \leq t_D \). We then observe:

- Decoder A corrects all single symbol errors in any position.
- Decoder B corrects all single symbol errors in the first 4 symbols (but not in position 5), and it corrects two symbol errors in positions 1 and 2, or in positions 1 and 3.

**Example 5.** Let \( k = 3 \) and \( n = 5 \), and let \( \deg m_1(x) = \deg m_2(x) = \deg m_3(x) = 1 \) and \( \deg m_4(x) = \deg m_5(x) = 2 \). We then have \( t_H = 1, K = 3, N = 7, \) and \( t_D = 2 \). Considering the same decoders as in Example 4, we observe:

- Decoder A corrects all single symbol errors in any position.
- Decoder B also corrects all single symbol errors, and in addition, it corrects any two symbol errors in the first 3 symbols.

### 3.5 Summary of Code Parameters

Let us summarize the key parameters of a polynomial remainder code \( C \) both in terms of Hamming distance and in terms of degree-weighted distance. For the latter, the code parameters are \((N, K, d_{\text{minD}})\) with \( N, K \), and \( d_{\text{minD}} \) defined as in (33), (34), (39) and with \( d_{\text{minD}} \) as in (42). By the rate of the code, we mean the quantity

\[
  \frac{1}{N} \log_{|F|}|C| = \frac{K}{N},
\]

where \( F \) is the underlying field.

With respect to Hamming distance, we have the parameters \((n, k, d_{\text{minH}})\) and the symbol rate \( k/n \). If the code \( C \) satisfies the Ordered-Degree Condition (28), we have \( d_{\text{minH}} = n - k + 1 \).

In the special case where all the moduli \( m_0(x), \ldots, m_{n-1}(x) \) have the same degree, the two triples \((N, K, d_{\text{minD}})\) and \((n, k, d_{\text{minH}})\) are equal up to a scale factor and the rate (54) equals the symbol rate \( k/n \).
4 Error Factor Polynomial

Decoding Reed-Solomon codes can be reduced to solving a key equation that involves an error locator polynomial [15]. We are going to propose such an approach for polynomial remainder codes. As it turns out, in general (i.e., beyond irreducible remainder codes), we will need a slight generalization of an error locator polynomial.

Let \( C \) be a polynomial remainder code of the form (27). For the received \( y = c + e \), where \( c = (c_0, \ldots, c_{n-1}) \in C \) is a transmitted codeword, and where \( e = (e_0, \ldots, e_{n-1}) \) is an error pattern, let \( Y(x) = a(x) + E(x) \) denote the pre-image \( \psi^{-1}(y) \) of \( y \) with \( \psi^{-1} \) as in [2], where \( a(x) = \psi^{-1}(c) \) is the transmitted-message polynomial, and where \( E(x) \) denotes the pre-image \( \psi^{-1}(e) \) of the error \( e \).

4.1 Error Factor Polynomial, Key Equation, and Interpolation

Definition 4. An error factor polynomial is a nonzero polynomial \( \Lambda(x) \in F[x] \) such that

\[
\Lambda(x)E(x) \mod M_n(x) = 0.
\]

(55)

Clearly, the polynomial

\[
\Lambda_f(x) \triangleq \frac{M_n(x)}{\gcd(E(x), M_n(x))}
\]

(56)

is the unique monic polynomial of the smallest degree that satisfies (55).

A closely related notion is the error locator polynomial

\[
\Lambda_e(x) \triangleq \prod_{i: e_i \neq 0} m_i(x),
\]

(57)

which is of degree \( \deg \Lambda_e(x) = w_D(e) \). Note that \( \Lambda_e(x) \) qualifies as an error factor polynomial. In the special case where all the moduli \( m_i(x), 0 \leq i \leq n - 1, \) are irreducible (e.g., for irreducible polynomial remainder codes), we have

\[
\gcd(E(x), M_n(x)) = \prod_{i: e_i = 0} m_i(x)
\]

(58)

and thus \( \Lambda_f(x) = \Lambda_e(x) \).

In any case, every error factor polynomial \( \Lambda(x) \) is a multiple of \( \Lambda_f(x) \). This applies, in particular, to \( \Lambda_e(x) \) and thus

\[
\deg \Lambda_f(x) \leq \deg \Lambda_e(x) = w_D(e).
\]

(59)

The following theorem is then obvious:
Theorem 9 (Key Equation). The error factor polynomial (56) satisfies

\[ A(x) M_n(x) = \Lambda_f(x) E(x) \]  

for some polynomial \( A(x) \in F[x] \) of degree smaller than \( \deg \Lambda_f(x) \). Conversely, if some monic polynomial \( G(x) \in F[x] \) satisfies

\[ A(x) M_n(x) = G(x) E(x) \]  

for some \( A(x) \in F[x] \), then \( G(x) \) is a multiple of \( \Lambda_f(x) \).

For irreducible polynomial remainder codes, \( \Lambda_f(x) \) in Theorem 9 can be replaced everywhere by \( \Lambda_e(x) \) because, in this case, \( \Lambda_f(x) = \Lambda_e(x) \).

The following theorem is a slight generalization of Theorem 7.

Theorem 10 (Error Factor-based Interpolation). If \( G(x) \) is a multiple of \( \Lambda_f(x) \) with

\[ \deg G(x) \leq N - K, \]  

then

\[ a(x) = \frac{G(x) Y(x) \mod M_n(x)}{G(x)} \]  

Proof: With \( Y(x) = a(x) + E(x) \) and with \( G(x) \) satisfying (62), we have

\[ G(x) Y(x) \mod M_n(x) = G(x) (a(x) + E(x)) \mod M_n(x) = G(x) a(x) + \tilde{E}(x) \]  

with

\[ \tilde{E}(x) \triangleq G(x) E(x) \mod M_n(x). \]  

If \( G(x) \) is a multiple of \( \Lambda_f(x) \), then \( \tilde{E}(x) = 0 \) by Theorem 9 and (63) follows.

For irreducible polynomial remainder codes, \( \Lambda_f(x) \) in Theorem 10 can be replaced by \( \Lambda_e(x) \) and Theorem 10 reduces to Theorem 7. For non-irreducible codes, however, Theorem 10 is more general than Theorem 7 because error patterns with \( w_D(e) > N-K \) but \( \deg \Lambda_f(x) \leq N-K \) can exist.

4.2 Error Factor Test and Error Locator Test

Recall \( t_D \triangleq \left\lfloor \frac{N-K}{2} \right\rfloor \) from (52) and \( t_H \triangleq \left\lfloor \frac{n-k}{2} \right\rfloor \) from (53).

Theorem 11 (Error Factor Test). Let \( y = \psi(a) + e \) as above, let \( G(x) \) be a nonzero polynomial, and let

\[ Z(x) \triangleq G(x) Y(x) \mod M_n(x). \]

Assume that the following conditions are satisfied:
1. \( \deg \Lambda_f(x) \leq t_D \)
2. \( \deg G(x) \leq t_D \)
3. \( G(x) \) divides \( Z(x) \)
4. \( \deg Z(x) - \deg G(x) < K \).

Then \( G(x) \) is a multiple of \( \Lambda_f(x) \) and \( Z(x) = G(x)a(x) \).

Note that the conditions in the theorem are satisfied for \( G(x) = \Lambda_f(x) \). Note also that for non-irreducible polynomial remainder codes, there may exist error patterns such that \( w_D(e) > t_D \) but \( \deg \Lambda_f(x) \leq t_D \). For irreducible polynomial remainder codes, Condition 1 in Theorem 11 is equivalent to \( \deg \Lambda_e(x) = w_D(e) \leq t_D \), and \( \Lambda_f(x) \) in Theorem 11 can be replaced everywhere by \( \Lambda_e(x) \).

Proof of Theorem 11: Assume that Conditions 1–4 are satisfied. Note that Condition 2 implies (62), and thus (64) and (65). From (64) and Condition 3, we have

\[
\tilde{E}(x) = G(x)Q(x)
\]

for some polynomial \( Q(x) \) and (64) can be written as

\[
Z(x) = G(x)(a(x) + Q(x)).
\]

From Condition 4, we then have

\[
\deg Q(x) < K.
\]

Furthermore, from (65) and (66), we have \( G(x)E(x) = b(x)M_n(x) + G(x)Q(x) \) for some polynomial \( b(x) \) and thus

\[
G(x)(E(x) - Q(x)) = b(x)M_n(x).
\]

Let

\[
\Lambda_f(x) \triangleq M_n(x)/\Lambda_f(x) = \gcd(E(x), M_n(x)).
\]

Since \( \deg \Lambda_f(x) \leq t_D \), we have \( \deg \Lambda_f(x) \geq N - t_D \). Taking (69) modulo \( \Lambda_f(x) \) yields

\[
G(x)Q(x) \mod \Lambda_f(x) = 0
\]

since \( E(x) \mod \Lambda_f(x) = 0 \). From (66), we have either \( Q(x) = 0 \) or \( \deg Q(x) \geq \deg \Lambda_f(x) - \deg G(x) \geq N - 2t_D \geq K \) since \( \deg G(x) \leq t_D \). From (68), we then conclude \( Q(x) = 0 \). Thus \( \tilde{E}(x) = 0 \) from (66) and \( Z(x) = G(x)a(x) \) from (64). Finally, from (65) (with \( \tilde{E}(x) = 0 \)) and the converse part of Theorem 9, it follows that \( G(x) \) is a multiple of \( \Lambda_f(x) \).

If the code \( C \) further satisfies the Ordered-Degree Condition (28), we have the following analog of Theorem 11. Let \( N_{\text{zero}}(G) \) denote the number of indices \( j \in \{0, \ldots, n-1\} \) such that \( G(x) \mod m_j(x) = 0 \). Note that \( N_{\text{zero}}(\Lambda_e) = w_H(e) \).
Theorem 12 (Error Locator Test). Let \( C \) be a polynomial remainder code that satisfies the Ordered-Degree Condition and let \( y = \psi(a) + e \) as above. For some set \( S \subset \{0, 1, \ldots, n - 1\} \) of indices, let \( G(x) = \prod_{i \in S} m_i(x) \neq 0 \) and let

\[
Z(x) = G(x)Y(x) \mod M_n(x).
\]

Assume that the following conditions are satisfied:

1. \( w_H(e) \leq t_H \)
2. \( N_{\text{zero}}(G) \leq t_H \) and \( \deg G(x) \leq \sum_{i=n-t_H}^{n-1} \deg m_i(x) \)
3. \( G(x) \) divides \( Z(x) \)
4. \( \deg Z(x) - \deg G(x) < K. \)

Then, \( G(x) \) is a multiple of \( \Lambda_e(x) \) and \( Z(x) = G(x)a(x) \).

Proof: Note that Condition 2 implies (62) and Conditions 3 and 4 are the same as the two corresponding conditions in Theorem 11. Assume now that Conditions 1–4 are satisfied. It is easily verified that we then have both (64)–(65) and (66)–(69) for some polynomial \( Q(x) \). Let \( S_{\text{zero}} \) denote the set of indices \( i \in \{0, 1, \ldots, n - 1\} \) such that \( E(x) \mod m_i(x) = 0 \). Equation (69) implies that, for each \( i \in S_{\text{zero}} \), we have

\[
G(x)Q(x) \mod m_i(x) = 0
\]

and thus \( N_{\text{zero}}(Q) \geq |S_{\text{zero}}| - N_{\text{zero}}(G) \). Since \( N_{\text{zero}}(G) \leq t_H \) and \( |S_{\text{zero}}| = n - w_H(e) \geq n - t_H \), we have \( N_{\text{zero}}(Q) \geq n - 2t_H \). It follows that \( N_{\text{zero}}(Q) \geq K \), which implies either \( \deg Q(x) \geq K \) or \( Q(x) = 0 \). It then follows from (68) that \( Q(x) = 0 \).

We then have \( \tilde{E}(x) = 0 \) from (66) and thus \( Z(x) = G(x)a(x) \) from (64). Finally, from (65) (with \( \tilde{E}(x) = 0 \)) and the converse part of Theorem 9, it follows that \( G(x) (= \prod_{i \in S} m_i(x)) \) is a multiple of \( \Lambda_e(x) \).

\[ \square \]

5 Decoding by the Extended GCD Algorithm

For Reed-Solomon codes, the use of the extended gcd algorithm to compute an error locator polynomial is standard [15, 16]. Gcd-based decoding of polynomial remainder codes was proposed by Shiozaki [7]. However, the assumptions in [7] do not cover all codes considered in the present paper. In particular, in [7], the moduli \( m_i(x) \) are assumed to have the same degree and they are implicitly assumed to be irreducible, as will be discussed in Section 5.5. In order to properly address these issues, we need to develop gcd-based decoding accordingly. We then obtain several versions of gcd-based decoding (summarized in Section 5.4), some of which are not quite standard even when specialized to Reed-Solomon codes.
5.1 An Extended GCD Algorithm

As in Section 4, let $c$ be the transmitted codeword, let $e$ be the error pattern, and let $y = c + e$ be the corrupted codeword that the receiver gets to see. Let $a(x), E(x) = \sum_{\ell=0}^{N-1} E_\ell x^\ell$, and $Y(x) = \sum_{\ell=0}^{N-1} Y_\ell x^\ell$ be the pre-images of these quantities with respect to $\psi$. The general idea of gcd decoding is to compute $\gcd(M_n(x), E(x))$ despite the fact that $E(x)$ is not fully known. We begin by stating the extended gcd algorithm in the following (not quite standard) form, where we assume for the moment that $E(x)$ is fully known.

Extended GCD Algorithm

Input: $M_n(x)$ and $E(x)$ with $\deg M_n(x) > \deg E(x)$.
Output: polynomials $\hat{r}(x), s(x), t(x) \in F[x]$ where $\hat{r}(x) = \gamma \gcd(M_n(x), E(x))$ for some nonzero $\gamma \in F$ and where $s(x)$ and $t(x)$ satisfy $s(x) \cdot M_n(x) + t(x) \cdot E(x) = 0$.

1. if $E(x) = 0$ begin
2. \[ \hat{r}(x) := M_n(x), \quad s(x) := 0, \quad t(x) := 1 \]
3. return $\hat{r}(x), s(x), t(x)$
4. end
5. $r(x) := M_n(x)$
6. $\tilde{r}(x) := E(x)$
7. $s(x) := 1$
8. $t(x) := 0$
9. $\tilde{s}(x) := 0$
10. $\tilde{t}(x) := 1$
11. loop begin
12. \[ i := \deg r(x) \]
13. \[ j := \deg \tilde{r}(x) \]
14. while $i \geq j$ begin
15. \[ q(x) := \frac{r(x)}{\tilde{r}(x)} x^{-j} \]
16. \[ r(x) := r(x) - q(x) \cdot \tilde{r}(x) \]
17. \[ s(x) := s(x) - q(x) \cdot \tilde{s}(x) \]
18. \[ t(x) := t(x) - q(x) \cdot \tilde{t}(x) \]
19. \[ i := \deg r(x) \]
20. end
21. if $r(x) = 0$ begin
22. return $\hat{r}(x), s(x), t(x)$
23. end
24. \[ (r(x), \hat{r}(x)) := (\hat{r}(x), r(x)) \]
25. \[ (s(x), \tilde{s}(x)) := (\tilde{s}(x), s(x)) \]
26. \[ (t(x), \tilde{t}(x)) := (\tilde{t}(x), t(x)) \]
27. end
The inner loop between lines 14 and 20 essentially computes the division of \( r(x) \) by \( \tilde{r}(x) \). In line 15, \( r_i \) denotes the coefficient of \( x^i \) in \( r(x) \) and \( \tilde{r}_j \) denotes the coefficient of \( x^j \) in \( \tilde{r}(x) \). For polynomials over \( F = \text{GF}(2) \), the scalar division \( r_i/\tilde{r}_j \) in line 15 disappears.

**Theorem 13 (GCD Loop Invariants).** The condition

\[
gcd\left(M_n(x), E(x)\right) = gcd\left(r(x), \tilde{r}(x)\right)
\]  
(73)

holds everywhere after line 6. The condition

\[
r(x) = s(x) \cdot M_n(x) + t(x) \cdot E(x)
\]  
(74)

holds both between lines 13 and 14 and between lines 20 and 21. The condition

\[
\deg M_n(x) = \deg \tilde{r}(x) + \deg t(x)
\]  
(75)

holds between lines 20 and 21.

Equations (73) and (74) are the standard loop invariants of extended gcd algorithms, cf. e.g. [15]. The proof of Theorem 13 is given in Appendix B.

**Theorem 14 (GCD Output).** When the algorithm terminates, we have both

\[
\tilde{r}(x) = \gamma \gcd\left(M_n(x), E(x)\right)
\]  
(76)

\[
= \gamma \frac{M_n(x)}{\Lambda_f(x)}
\]  
(77)

for some nonzero \( \gamma \in F \) and

\[
t(x) = \tilde{\gamma} \Lambda_f(x)
\]  
(78)

for some nonzero \( \tilde{\gamma} \in F \). Moreover, the returned \( s(x) \) and \( t(x) \) satisfy

\[
s(x) \cdot M_n(x) + t(x) \cdot E(x) = 0.
\]  
(79)

**Proof:** If \( E(x) = 0 \), the algorithm terminates at line 3 and (76)–(79) are easily verified. We now prove the case where \( E(x) \neq 0 \). Equation (76) follows from (73) and (77) follows from (74). It remains to prove (78) and (79). With \( r(x) = 0 \) and from (74), Equation (79) follows. We then conclude from the second part of Theorem 9 that \( t(x) \) is a multiple of \( \Lambda_f(x) \). Finally, it follows from (75) and (77) that \( t(x) \) and \( \Lambda_f(x) \) have the same degree.

From (78), we see that the gcd algorithm computes the error factor polynomial \( \Lambda_f \) (up to a scale factor). The main idea of gcd decoding (discovered by Sugiyama [16]) is that this still works even if \( E(x) \) is only partially known.
5.2 Modifications for Partially Known \( E(x) \)

Recall that \( Y(x) = a(x) + E(x) \) where \( E(x) = \sum_{\ell=0}^{N-1} E_\ell x^\ell \) is the pre-image of \( e \). Since \( \deg a(x) < K \), the receiver knows the coefficients \( E_K, E_{K+1}, \ldots, E_{N-1} \) of \( E(x) \), but not \( E_0, \ldots, E_{K-1} \). With the following modifications, the Extended GCD Algorithm of Section 5.1 can still be used to compute (78).

Partial GCD Algorithm I

Input: \( M_n(x) \) and \( Y(x) \) with \( \deg M_n(x) > \deg Y(x) \).

Output: \( r(x) \), \( s(x) \) and \( t(x) \), cf. Theorem 15 below.

The algorithm is the same as the Extended GCD Algorithm of Section 5.1 except for the following changes:

- Line 1: if \( \deg Y(x) < K \) begin
- Line 2: \( r(x) := Y(x) \), \( s(x) := 0 \), \( t(x) := 1 \)
- Line 6: \( \tilde{r}(x) := Y(x) \)
- Line 21: if \( \deg r(x) < \deg t(x) + K \) begin

or alternatively

if \( \deg r(x) < (N + K)/2 \) begin

\[ 80 \]

\[ 81 \]

Theorem 15. If

\[ \deg \Lambda f(x) \leq (N - K)/2, \]

then the Partial GCD Algorithm I (with either \[ 80 \] or \[ 81 \]) returns the same polynomials \( s(x) \) and \( t(x) \) (after the same number of iterations) as the Extended GCD Algorithm of Section 5.1. Moreover, the returned \( r(x) \) is such that

\[ r(x) = t(x)a(x). \]

The proof is given in Appendix B. Note that \( a(x) \) can be recovered directly from (83).

5.3 Alternative Modifications for Partially Known \( E(x) \)

The Partial GCD Algorithm I of the previous section involves a lot of computations with the unknown lower parts of \( E(x) \). These computations are avoided in the following algorithm, which works only with the known part of \( E(x) \) as follows. Let

\[ E_U(x) \triangleq \sum_{\ell=0}^{N-K-1} E_{K+\ell} x^\ell = \sum_{\ell=0}^{N-K-1} Y_{K+\ell} x^\ell, \]
which is the known upper part of \( E(x) = \sum_{\ell=0}^{N-1} E_\ell x^\ell \), and let
\[
M_U(x) = \sum_{\ell=0}^{N-K} (M_n)_K x^\ell
\]
(85)
be the corresponding upper part of \( M_n(x) = \sum_{\ell=0}^{N} (M_n)_\ell x^\ell \).

Partial GCD Algorithm II
Input: \( M_U(x) \) and \( E_U(x) \) with \( \deg M_U(x) > \deg E_U(x) \).
Output: \( s(x) \) and \( t(x) \), cf. Theorem 16 below.

The algorithm is the same as the Extended GCD Algorithm of Section 5.1 except for the following changes:

- Line 1: \( \text{if } E_U(x) = 0 \text{ begin} \)
- Line 2: \( s(x) := 0, t(x) := 1 \)
- Line 5: \( r(x) := M_U(x) \)
- Line 6: \( \tilde{r}(x) := E_U(x) \)
- Line 21: \( \text{if } \deg r(x) < \deg t(x) \text{ begin} \)
  \begin{equation}
  \text{(86)}
  \end{equation}
  or alternatively
  \begin{equation}
  \text{(87)}
  \end{equation}

\textbf{Theorem 16.} If the condition (82) is satisfied, then the Partial GCD Algorithm II (with either (86) or (87)) returns the same polynomials \( s(x) \) and \( t(x) \) (after the same number of iterations) as the Extended GCD Algorithm of Section 5.1.

The proof is given in Appendix C. Note, however, that this algorithm does not compute \( r(x) \) as in (83).

5.4 Summary of Decoding
We can now put together several decoding algorithms that consist of the following three steps. The relation of all these decoding algorithms to the prior literature is discussed in Section 5.5.

1. **Transform**: Compute \( Y(x) = \psi^{-1}(y) \). If \( \deg Y(x) < K \), we conclude \( E(x) = 0 \) and \( a(x) = Y(x) \), and the following two steps can be skipped.
2. **Partial GCD**: If $\deg Y(x) \geq K$, run either the Partial GCD Algorithm I (Section 5.2) or the Partial GCD Algorithm II (Section 5.3). Either algorithm yields the polynomial $t(x) = \tilde{\gamma} \Lambda_f(x)$ (for some scalar $\tilde{\gamma} \in F$) provided that $\deg \Lambda_f(x) \leq (N - K)/2$.

If $\deg t(x) > (N - K)/2$, we declare a decoding failure.

Depending on Step 3 (below), the computation of the polynomials $s(x)$ and $\tilde{s}(x)$ may be unnecessary. In this case, lines 7, 9, 17, and 25 of the gcd algorithm can be deleted.

3. **Recovery**: Recover $a(x)$ by any of the following methods:

   (a) From (63), we have
   
   $$a(x) = \frac{t(x)Y(x) \mod M_n(x)}{t(x)}$$
   
   (88)

   (If the numerator of (88) is not a multiple of $t(x)$ or if $\deg a(x) \geq K$, then decoding failed due to some uncorrectable error.)

   (b) When using the Partial GCD Algorithm I in the Step 2, we can compute $a(x) = r(x)/t(x)$ according to (83).

   (If $t(x)$ does not divide $r(x)$ or if $\deg a(x) \geq K$, we declare a decoding failure.)

   (c) Alternatively, from (79), we can compute
   
   $$E(x) = \frac{-s(x) \cdot M_n(x)}{t(x)}$$
   
   (89)

   and then obtain $a(x) = Y(x) - E(x)$.

   (If the numerator of (89) is not a multiple of $t(x)$ or if $\deg a(x) \geq K$, we declare a decoding failure.)

The computation can be simplified as follows. Let $E_L(x) \triangleq E(x) - x^K E_U(x)$ denote the unknown part of $E(x)$. Then

$$E_L(x) = \frac{-s(x) \cdot M_n(x) - x^K t(x) E_U(x)}{t(x)}$$

(90)

and $a(x)$ can be recovered by

$$a(x) = \sum_{\ell=0}^{K-1} Y_\ell x^\ell - E_L(x).$$

As stated, the described decoding algorithms are guaranteed to correct all errors $e$ with $\deg \Lambda_f(x) \leq t_D$, which by (59) implies that they also correct all errors $e$ with $w_D(e) \leq t_D$ (52). If the code satisfies the Ordered-Degree Condition (28) as well as the additional condition

$$\deg m_k(x) = \cdots = \deg m_{n-1}(x),$$

(91)

then the algorithm is guaranteed to correct also all errors $e$ with $w_H(e) \leq t_H$ (53) since in this case, from (57), $w_H(e) \leq t_H$ implies $w_D(e) \leq t_D$. 

20
An Extension

Assume that the code satisfies the Ordered-Degree Condition (28) but not the additional condition (91). In this case, we can still correct all errors $e$ with $w_H(e) \leq t_H$ (in addition to all errors with $w_D(e) \leq t_D$) by the following procedure, which, however, is practical only in special cases.

Decoder with List of Special Error Positions

First, run the gcd decoder of the previous section. If it succeeds, stop. Otherwise, let $S_A$ be a precomputed list of candidate error locator polynomials $G(x)$ with $N_{\text{zero}}(G) \leq t_H$ and $\deg G(x) > (N - K) / 2$. Check if any $G(x) \in S_A$ satisfies all conditions of Theorem 12. If such a polynomial $G(x)$ exists, we conclude that it is a multiple of the error locator polynomial and we compute $a(x)$ from (63).

5.5 Relation to Prior Work

The idea of gcd-based decoding is due to Sugiyama [16] and its application to polynomial remainder codes is due to Shiozaki [7]. As it turns out, most (and perhaps all) gcd-based decoding algorithms in the literature, both for Reed-Solomon codes and for polynomial residue codes, are essentially identical to one of the algorithms of Section 5.4. However, even when specialized to Reed-Solomon codes, no single paper (not even [18, 19]) seems to cover all these algorithms. In particular, recovering $a(x)$ by (88) does not seem to have appeared in the literature. For Reed-Solomon codes, the work by Gao [17] appears to be the most pertinent, see also [18, 19]. As for polynomial remainder codes, our algorithms overcome the limitations of Shiozaki’s algorithm [7] as will be discussed below.

Relation to Gao’s Decoding Algorithms for Reed-Solomon Codes

In the same paper [17] from 2003, Gao proposed two algorithms for decoding Reed-Solomon codes. Each algorithm comprises three steps, and the first step of each algorithm is essentially Step 1 (“Transform”) of Section 5.4.

Gao’s first algorithm: Step 2 of this algorithm is essentially the Partial GCD Algorithm I of Section 5.2 with (81) as the stopping condition. Step 3 is identical to Step 3.b in Section 5.4.

As pointed out in [19], this algorithm is actually identical to Shiozaki’s 1988 algorithm for decoding Reed-Solomon codes [7].

Gao’s second algorithm: The stopping condition of the gcd-algorithm (Step 2) as stated in [17] is not quite correct: it should be changed from $\deg g(x) < (d + 1) / 2$ to $\deg g(x) < (d - 1) / 2$ where $d \triangleq n - k + 1$ is the minimum Hamming distance of the code.

With this correction, Step 2 of this algorithm is identical to the Partial GCD Algorithm II of Section 5.2 with (87) as the stopping condition. Step 3 of the algorithm
turns out to be equivalent to the first part of 3.c in Section 5.4, i.e., computing $a(x) = Y(x) - E(x)$ with $E(x)$ as in (89).

**Relation to Shiozaki’s Decoding Algorithms**

In [7], Shiozaki proposed a new version of gcd-based decoding for Reed-Solomon codes, which he also extended to polynomial remainder codes. (For Reed-Solomon codes, Shiozaki’s algorithm is equivalent to Gao’s first decoding algorithm, as noted above.)

Shiozaki’s algorithm also consists of three steps: the first step agrees with Step 1 in Section 5.4, the second step is equivalent to the Partial GCD Algorithm I with (81) as the stopping condition, and the third step is identical to Step 3.b of Section 5.4.

However, the assumptions in [7] do not cover all codes considered in the present paper. First, it is assumed in [7] that all the moduli $m_i(x), 0 \leq i \leq n - 1$, have the same degree.

Second, the argument given in [7] seems to assume that all the moduli are irreducible although this assumption is not stated explicitly. Specifically, Shiozaki derived a congruence (see (37) in [7]) involving an error locator polynomial as defined in (57), and then used the gcd-based decoding algorithm to solve the congruence. However, if the moduli are not irreducible, then the gcd-based decoding algorithm will find an error factor polynomial (56) (as shown in our Theorems 14 and 15) rather than an error locator polynomial.

**6 Conclusion**

We considered polynomial remainder codes and their decoding more carefully than in previous work. We explicitly allowed the code symbols to be polynomials of different degrees, which leads to two different notions of weight and distance and, correspondingly, to two different Singleton bounds.

Our discussion of algebraic decoding revolved around the notion of an error factor polynomial, which is a generalization of an error locator polynomial. From a correct error factor polynomial, the transmitted codeword can be recovered in various ways, including a new method for erasures-only decoding of general Chinese remainder codes.

Error factor polynomials can be computed by a suitably adapted partial gcd algorithm. We obtained several versions of such decoding algorithms, which generalize previous work and which include the published gcd-based decoders of Reed-Solomon codes as special cases.
Appendix A: The Number of Monic Irreducible Polynomials

The number of monic irreducible polynomials of any degree over any finite field can be expressed in closed form [15]. However, this closed-form expression is not easy to evaluate. Therefore, for the convenience of the reader, we tabulate some of these numbers.

The first table gives the number $N_i$ of binary irreducible polynomials of degree $i$:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| $N_i$ | 2 | 1 | 2 | 3 | 6 | 9 | 18 | 30 | 56 | 99 | 186 | 335 |
| $S_i$ | 1 | 4 | 10 | 22 | 52 | 106 | 232 | 472 | 976 | 1966 | 4012 | 8032 |

The table also gives the number $S_i \triangleq \sum_{\ell=1}^i \ell N_\ell$, which is the maximum degree of $M_n(x)$ of a polynomial remainder code that uses only irreducible moduli of degree at most $i$.

The second table gives the number $N_i$ of monic irreducible polynomials over GF($2^j$) of degree $i$:

| $i$ | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|
| $N_i$ | 630 | 1161 | 2182 | 4080 |
| $S_i$ | 16222 | 32476 | 65206 | 130486 |

E.g., over GF($2^8$), there are 256 monic irreducible polynomials of degree 1 and 32640 polynomials of degree 2.

Appendix B: Proof of Theorem [15]

In this section, we first prove the loop invariant properties of the Extended GCD Algorithm in Section 5.1 and the Partial GCD Algorithm I in Section 5.2 and then proceed to prove Theorem [15].

We begin with the Extended GCD Algorithm of Section 5.1. In order to prove Theorem [13], we first recall that, for $R = \mathbb{Z}$ or $R = F[x]$ for some field $F$,

$$\gcd(a, b) = \gcd(a + qb, b)$$

(92)

for all $a, b, q \in R$, provided that $a$ and $b$ are not both zero. It follows that (73) holds everywhere after line 6.

The other claims of Theorem [13] are covered by the following lemma.
Lemma 1 (GCD Loop Invariant). For the Extended GCD Algorithm in Section 5.1, the condition

\[ r(x) = s(x) \cdot M_n(x) + t(x) \cdot E(x) \]  

holds both between lines 13 and 14 and between lines 20 and 21. For the Partial GCD Algorithm I in Section 5.2, the condition

\[ r(x) = s(x) \cdot M_n(x) + t(x) \cdot Y(x) \]  

also holds both between lines 13 and 14 and between lines 20 and 21.

For both algorithms, the conditions

\[ \deg r(x) < \deg \tilde{r}(x) \] \hspace{1cm} (95)
\[ \deg t(x) > \deg \tilde{t}(x) \] \hspace{1cm} (96)
\[ \deg M_n(x) = \deg r(x) + \deg t(x) \] \hspace{1cm} (97)

hold between lines 20 and 21.

Specifically, let \( \delta_\ell \) denote the degree of \( q(x) \) (line 15) in the first iteration of the while block (lines 14–20) of the \( \ell \)-th loop iteration. Then, for the respective algorithms,

\[ \deg t(x) = \deg \tilde{t}(x) + \delta_\ell = \sum_{v=1}^{\ell} \delta_v \] \hspace{1cm} (98)

holds between lines 20 and 21 in the \( \ell \)-th loop iteration. \( \square \)

Proof: Conditions (93) and (94) are loop invariants (of the respective algorithms), as is easily verified. Inequality (95) is obvious. It remains to prove (96)–(98). For both algorithms, assume the conditions

\[ \deg r(x) > \deg \tilde{r}(x) \] \hspace{1cm} (99)
\[ \deg t(x) < \deg \tilde{t}(x) \] \hspace{1cm} (100)
\[ \deg M_n(x) = \deg r(x) + \deg \tilde{t}(x) \] \hspace{1cm} (101)

hold between lines 13 and 14 in the \( \ell \)-th loop iteration. Note that \( r(x), \tilde{r}(x), t(x), \) and \( \tilde{t}(x) \) are initialized to \( M_n(x), E(x) \) or \( Y(x), 0, \) and 1, respectively; thus (99)–(101) obviously hold between lines 13 and 14 in the first iteration. In the following, we begin with \( \ell = 1 \) and then complete the proof by induction.

For both algorithms, let \( d_\ell = \deg r(x) \) denote the degree of \( r(x) \) between lines 13 and 14 in the \( \ell \)-th loop iteration, and let \( \delta_\ell \) denote the degree of \( q(x) \) (line 15) in the first iteration of the while block (lines 14–20) of the \( \ell \)-th loop iteration. Note that \( \delta_\ell = d_\ell - \deg \tilde{r}(x) > 0 \) and from (101)

\[ \deg M_n(x) = d_\ell + \deg \tilde{t}(x). \] \hspace{1cm} (102)

Recall that, from (100), \( \deg t(x) < \deg \tilde{t}(x) \) holds before entering the while block, and recall the update rule for \( t(x) \) in line 18. Clearly, in the first execution of line 18, the
Algorithm starts with the algorithm of Section 5.1 when we run both algorithms simultaneously. Clearly, the algorithm is "probably unmatched" with the corresponding algorithm of either the Partial GCD Algorithm I in Section 5.2, let \( g \) denote the largest integer such that the coefficient of \( x^g \) of either \( r(x) \) or of \( \tilde{r}(x) \) is unknown, or alternatively let \( g \) denote the largest integer such that the coefficient of \( x^g \) of either \( r(x) \) or of \( \tilde{r}(x) \) is "probably unmatched" with the corresponding \( r(x) \) or the corresponding \( \tilde{r}(x) \) in the Extended GCD Algorithm of Section 5.1 when we run both algorithms simultaneously. Clearly, the algorithm starts with \( g = K - 1 \), since the coefficients \( E_0, E_1, \ldots, E_{K-1} \) of \( \tilde{r}(x) := Y(x) \) (line 6) are unknown. Moreover, let \( h \triangleq \max\{\deg r(x), \deg \tilde{r}(x)\} \). Clearly, the algorithm starts with \( h = \deg M_n(x) = N \).

**Lemma 2.** For the Partial GCD Algorithm I of Section 5.2, let \( \delta_\ell \) denote the degree of \( q(x) \) in the first iteration of the while block (lines 14–20) of the \( \ell \)-th loop iteration. If \( h - g > 2\delta_1 \) holds between lines 13 and 14, then the value of \( q(x) \) (line 15) throughout the while block in the \( \ell \)-th loop iteration is exactly the same as the corresponding one of the Extended GCD Algorithm of Section 5.1 in the same loop iteration. In addition, \( g = (K - 1) + \sum_{v=1}^{\ell} \delta_v \) and \( h = N - \sum_{v=1}^{\ell} \delta_v \) both hold between lines 20 and 21 in the \( \ell \)-th loop iteration.

**Proof:** We will prove this theorem by induction. Recall that the update rule for \( r(x) \) in line 16 is

\[
r(x) := r(x) - q(x) \cdot \tilde{r}(x).
\]

In the first loop iteration, \( h = \deg r(x) = N \) and \( g = K - 1 \) clearly hold between lines 13 and 14, and \( g \) is the largest integer such that the coefficient of \( x^g \) of \( \tilde{r}(x) \) is unknown. If \( h - g > 2\delta_1 \) holds between lines 13 and 14, then the first execution of (105) in the while block increases \( g \) by \( \delta_1 \); afterwards, further iterations in the same block will not change \( g \) since \( \deg q(x) \) decreases in each iteration. Moreover, after executing the while block,
$h = \deg \tilde{r}(x) = N - \delta_1$ holds between lines 20 and 21. It is also easily seen that throughout the while block, the value of $q(x)$ in line 15 is exactly identical to the corresponding one of the Extended GCD Algorithm.

Note that the increased $g$, i.e., after the first execution of (105), will become to denote the largest integer such that the coefficient of $x^g$ of $r(x)$ is unknown. It follows after the swap of $r(x)$ and $\tilde{r}(x)$ in line 24 that the increased $g$ will again become to denote the largest integer such that the coefficient of $x^g$ of $\tilde{r}(x)$ is unknown between lines 13 and 14 for subsequent loop iteration, and the decreased $h$ will again become to denote $\deg r(x)$ between lines 13 and 14 for subsequent loop iteration. The proof is then completed by induction.

Since $h - g = N - K + 1$ holds between lines 13 and 14 in the first loop iteration, it follows from Lemma 2 that if

$$2 \sum_{v=1}^{\ell} \delta_v < N - K + 1,$$

then, from the first to the $\ell$-th loop iteration, $q(x)$ and thus $s(x)$ and $t(x)$ are exactly the same as in the Extended GCD Algorithm. Moreover from Lemma 1, $\deg t(x) = \sum_{v=1}^{\ell} \delta_v$ holds between lines 20 and 21. In order to obtain (78), which implies that $\deg t(x) = \deg \Lambda f(x)$, it turns out from (106) that if

$$2 \deg \Lambda f(x) \leq N - K,$$

which agrees with (82), then the algorithm maintains exactly the same $s(x)$ and $t(x)$ as the Extended GCD Algorithm of Section 5.1 until $\deg t(x) = \deg \Lambda f(x)$.

It remains to argue the validity of (80) and (81) (i.e., line 21 in the Partial GCD Algorithm I) as appropriate terminating conditions. Assume now that (82) is satisfied and suppose the Extended GCD Algorithm (in Section 5.1) terminates (at line 22) in the $\mu$-th loop iteration. We will show in the following that the Partial GCD Algorithm I also terminates (at line 22) in the $\mu$-th loop iteration.

As shown above, since both the gcd algorithms maintain exactly the same $s(x)$ and $t(x)$ until $\deg t(x) = \deg \Lambda f(x)$, clearly, before the $\mu$-th loop iteration,

$$\deg t(x) < \deg \Lambda f(x) \leq (N - K)/2$$

holds between lines 20 and 21. Moreover, by (97) of Lemma 1

$$\deg \tilde{r}(x) = \deg M_n(x) - \deg t(x)$$

$$> (N + K)/2$$

$$> \deg t(x) + K$$

also holds between lines 20 and 21. Further, from (96), $\deg t(x) > \deg \tilde{t}(x)$ holds as well between lines 20 and 21. Therefore,

$$\deg \tilde{r}(x) > (N + K)/2 > \deg t(x) + K > \deg \tilde{t}(x) + K$$
holds between lines 20 and 21 in every but before the \( \mu \)-th loop iteration. It then follows after swapping all auxiliary polynomials in lines 24–26 that
\[
\deg r(x) > (N + K)/2 > \deg \tilde{r}(x) + K > \deg t(x) + K \tag{113}
\]
holds between lines 13 and 14 for each subsequent loop iteration. Then, after executing the \texttt{while} block in the \( \mu \)-th loop iteration, the Extended GCD Algorithm in Section 5.1 terminates with \( r(x) = 0 \), and (79) holds; meanwhile, for the Partial GCD Algorithm I, we obtain the desired \( t(x) \) (with \( \deg t(x) = \deg \Lambda_f(x) \)) and \( s(x) \), and we have from (94)
\[
\begin{align*}
  r(x) & = s(x)M_n(x) + t(x)Y(x) \\
         & = s(x)M_n(x) + t(x)E(x) + t(x)a(x) \\
         & = t(x)a(x)
\end{align*}
\tag{114}
\tag{115}
\tag{116}
\]
of \( \deg r(x) = \deg t(x) + \deg a(x) < \deg t(x) + K \), where (115) to (116) follows from (79). Finally, since from (113) \( \deg r(x) > \deg t(x) + K \) holds between lines 13 and 14 but from (116) \( \deg r(x) < \deg t(x) + K \) holds between lines 20 and 21, thus the correctness of (80) as a terminating condition is guaranteed; meanwhile from (116) we obtain (83). As for (81), since from (113) \( \deg r(x) > (N + K)/2 \) holds between lines 13 and 14 but (from (116) and then (82)) \( \deg r(x) < \deg t(x) + K = \deg \Lambda_f(x) + K \leq (N + K)/2 \) holds between lines 20 and 21, we thus conclude that (81) can serve as an alternative terminating condition.

Appendix C: Proof of Theorem \textbf{16}

In this section, we prove Theorem \textbf{16} in an analogous way as proving Theorem \textbf{15}. The following theorem is an analog of Lemma \textbf{1}.

**Lemma 3 (GCD Loop Invariant).** For the Partial GCD Algorithm II in Section 5.3 the condition
\[
r(x) = s(x)M_n(x) + t(x)Y(x)
\tag{117}
\]
holds both between lines 13 and 14 and between lines 20 and 21. moreover, the conditions
\[
\begin{align*}
  \deg r(x) & < \deg \tilde{r}(x) \tag{118} \\
  \deg t(x) & > \deg \tilde{t}(x) \tag{119} \\
  \deg M_U(x) & = \deg \tilde{r}(x) + \deg t(x) \tag{120}
\end{align*}
\]
hold between lines 20 and 21.

Specifically, let \( \delta_\ell \) denote the degree of \( q(x) \) (line 15) in the first iteration of the \texttt{while} block (lines 14–20) of the \( \ell \)-th loop iteration. Then, \( \deg t(x) = \deg \tilde{t}(x) + \delta_\ell = \sum_{\ell=1}^{\ell} \delta_\ell \) holds between lines 20 and 21 in the \( \ell \)-th loop iteration.

The proof of Lemma 3 is the same as the proof of Lemma 1, except for replacing the \( M_n(x) \) in the proof of Lemma 1 by \( M_U(x) \), and is thus omitted.
We now start to prove Theorem 16. If \( E(x) = 0 \), which implies \( E_U(x) = 0 \), Theorem 16 holds obviously; we thus prove in the following only the case where \( E(x) \neq 0 \). For the Partial GCD Algorithm II of Section 5.3, let \( g \) denote the largest integer such that \( x^g \) of either \( r(x) \) or of \( \tilde{r}(x) \) is unknown. Clearly, with \( M_U(x) \) and \( E_U(x) \) as inputs, the algorithm starts with \( g = -1 \). Moreover, let \( h = \max\{\deg r(x), \deg \tilde{r}(x)\} \). Clearly, the algorithm starts with \( h = \deg M_U(x) = N - K \).

**Lemma 4.** For the Partial GCD Algorithm II in Section 5.3, let \( \delta_\ell \) denote the degree of \( q(x) \) in the first iteration of the while block (lines 14–20) of the \( \ell \)-th loop iteration. If \( h - g > 2\delta_\ell \) holds between lines 13 and 14, then the value of \( q(x) \) (line 15) throughout the while block in the \( \ell \)-th loop iteration is exactly the same as the corresponding one of the Extended GCD Algorithm of Section 5.1 in the same loop iteration. In addition, \( g = -1 + \sum_{v=1}^\ell \delta_v \) and \( h = N - K - \sum_{v=1}^\ell \delta_v \) both hold between lines 20 and 21 in the \( \ell \)-th loop iteration. \( \square \)

The proof is similar to that of Lemma 2 and is thus omitted. Since \( h - g = N - K + 1 \) holds between lines 13 and 14 in the first loop iteration, it follows from Lemma 4 that if \( 2 \sum_{v=1}^\ell \delta_v < N - K + 1 \), then, from the first to the \( \ell \)-th loop iteration, \( q(x) \) and thus \( s(x) \) and \( t(x) \) are exactly the same as in the Extended GCD Algorithm. Moreover, from Lemma 3, \( \deg t(x) = \sum_{v=1}^\ell \delta_v \) holds between lines 20 and 21. In order to obtain (78), which implies that \( \deg t(x) = \deg \Lambda_f(x) \), it turns out that if

\[
2 \deg \Lambda_f(x) \leq N - K, \tag{121}
\]

which agrees with (82), then the algorithm maintains exactly the same \( s(x) \) and \( t(x) \) as the Extended GCD Algorithm of Section 5.1 until \( \deg t(x) = \deg \Lambda_f(x) \).

It remains to argue the validity of (86) and (87) as appropriate terminating conditions. Assume that (82) is satisfied and suppose the Extended GCD Algorithm (in Section 5.1) terminates (at line 22) in the \( \mu \)-th loop iteration. As shown above, it has been clear that the Extended GCD Algorithm in Section 5.1 and the Partial GCD Algorithm II maintain exactly the same \( s(x) \) and \( t(x) \) until \( \deg t(x) = \deg \Lambda_f(x) \). Thus, before the \( \mu \)-th loop iteration

\[
\deg t(x) < \deg \Lambda_f(x) \leq (N - K)/2 \tag{122}
\]

holds between lines 20 and 21, moreover, by (120) of Lemma 3

\[
\deg \tilde{r}(x) = \deg M_U(x) - \deg t(x) \tag{123}
\]

\[
> (N - K)/2 \tag{124}
\]

\[
> \deg t(x) \tag{125}
\]

also holds between lines 20 and 21 for the Partial GCD Algorithm II. Further, from (119), \( \deg t(x) > \deg \tilde{t}(x) \) holds as well between lines 20 and 21. Therefore, for the Partial GCD Algorithm II,

\[
\deg \tilde{r}(x) > (N - K)/2 > \deg t(x) > \deg \tilde{t}(x) \tag{126}
\]
holds between lines 20 and 21 in every but before the \( \mu \)-th loop iteration. It then follows after swapping all auxiliary polynomials in lines 24–26 that

\[
\deg r(x) > (N - K)/2 > \deg \tilde{t}(x) > \deg t(x)
\]

(127)

holds between lines 13 and 14 for each subsequent loop iteration. Then, after executing the while block in the \( \mu \)-th loop iteration, we obtain the desired \( t(x) \) (with \( \deg t(x) = \deg \Lambda_f(x) \)) and \( s(x) \) that coincide with the corresponding ones of the Extended GCD Algorithm in Section 5.1, thus \( t(x) \) and \( s(x) \) (in the Partial GCD Algorithm II) at this moment satisfy both (117) and (79). From (79), we have

\[
-s(x)M_n(x) = t(x)E(x)
\]

(128)

with \( \deg s(x) < \deg t(x) \). Note that (128) can also be written as

\[
-s(x)(x^K M_U(x) + M_L(x)) = t(x)(x^K E_U(x) + E_L(x)),
\]

(129)

where \( M_U(x) \) and \( E_U(x) \) are defined in Section 5.3 and \( M_L(x) = M_n(x) - x^K M_U(x) \) and \( E_L(x) = E(x) - x^K E_U(x) \). Further, let \( V(x) \overset{\Delta}{=} -s(x)M_L(x) - t(x)E_L(x) = \sum_{\ell=0}V_{\ell} x^\ell \), which is of degree \( \deg V(x) \leq (K - 1) + \deg t(x) \) because \( \deg s(x) < \deg t(x) \). Equation (129) can then be written as

\[
x^K (s(x)M_U(x) + t(x)E_U(x)) = V(x).
\]

(130)

Observing the left hand side of (130), we know that all the terms on the right hand side of (130) of degree less than \( K \) will vanish. Thus, we have the following equivalent expression for (130):

\[
s(x)M_U(x) + t(x)E_U(x) = V_U(x)
\]

(131)

where \( V_U(x) \overset{\Delta}{=} \sum_{\ell=0} V_{K+\ell} x^\ell \) has degree

\[
\deg V_U(x) \leq \deg V(x) - K \leq (K - 1) + \deg t(x) - K < \deg t(x).
\]

(132)

Comparing (131) with (117) and from (132), clearly, \( \deg r(x) = \deg V_U(x) < \deg t(x) \), which coincides with (86), holds between lines 20 and 21 in the \( \mu \)-th loop iteration. Thus, the correctness of (86) as a terminating condition is guaranteed (because from (127) \( \deg r(x) > \deg t(x) \) holds between lines 13 and 14). On the other hand, since from (127) \( \deg r(x) > (N - K)/2 \) holds between lines 13 and 14 but \( \deg r(x) < \deg t(x) = \deg \Lambda_f(x) \leq (N - K)/2 \) holds between lines 20 and 21, we thus conclude that (87) can serve as an alternative terminating condition.
References

[1] J.-H. Yu and H.-A. Loeliger, “On irreducible polynomial remainder codes,” *IEEE Int. Symp. on Information Theory*, Saint Petersburg, Russia, July 31 – Aug. 5, 2011.

[2] J. J. Stone, “Multiple-burst error correction with the Chinese Remainder Theorem,” *J. SIAM*, vol. 11, pp. 74–81, Mar. 1963.

[3] I. S. Reed and G. Solomon, “Polynomial codes over certain finite fields,” *J. SIAM*, vol. 8, pp. 300–304, Oct. 1962.

[4] D. C. Bossen and S. S. Yau, “Redundant residue polynomial codes,” *Information and Control*, vol. 13, pp. 597–618, 1968.

[5] D. Mandelbaum, “A method of coding for multiple errors,” *IEEE Trans. Information Theory*, vol. 14, pp. 518–621, May 1968.

[6] D. Mandelbaum, “On efficient burst correcting residue polynomial codes,” *Information and Control*, vol. 16, pp. 319–330, 1970.

[7] A. Shiozaki, “Decoding of redundant residue polynomial codes using Euclid’s algorithm,” *IEEE Trans. Information Theory*, vol. 34, pp. 1351–1354, Sep. 1988.

[8] D. Mandelbaum, “On the derivation of Goppa codes,” *IEEE Trans. Information Theory*, vol. 21, pp. 110–101, Jan. 1975.

[9] V. D. Goppa, “A new class of linear error-correction codes,” *Probl. Peredach. Inform.*, vol. 6, pp. 24–30, Sept. 1970.

[10] S. V. Bezzateev and N. A. Shekhunova, “One generalization of Goppa codes,” *Proc. 1997 IEEE Int. Symp. on Information Theory*, Ulm, Germany, June 29 – July 4, 1997, p. 299.

[11] D. Mandelbaum, “A method for decoding of generalized Goppa codes,” *IEEE Trans. Information Theory*, vol. 23, pp. 137–140, Jan. 1977.

[12] D. Mandelbaum, “Addition to ‘A method for decoding of generalized Goppa codes’,” *IEEE Trans. Information Theory*, vol. 24, p. 268, Jan. 1978.

[13] O. Goldreich, D. Ron, and M. Sudan, “Chinese remaindering with errors,” *IEEE Trans. Information Theory*, vol. 46, pp. 1330–1338, July 2000.

[14] V. Guruswami, A. Sahai, and M. Sudan, “Soft-decision decoding of Chinese remainder codes,” *Proc. 41st IEEE Symp. Foundations Computer Science*, Redondo Beach, CA, 2000, pp. 159–168.

[15] R. M. Roth, *Introduction to Coding Theory*. New York: Cambridge University Press, 2006.
[16] Y. Sugiyama, M. Kasahara, S. Hirasawa, and T. Namekawa, “A method for solving key equation for decoding Goppa codes,” *Information and Control*, vol. 27, pp. 87–99, 1975.

[17] S. Gao, “A new algorithm for decoding Reed-Solomon codes,” in *Communications, Information and Network Security*, V. Bhargava, H. V. Poor, V. Tarokh, and S. Yoon, Eds. Norwell, MA: Kluwer, 2003, vol. 712, pp. 55-68.

[18] S. V. Fedorenko, “A simple algorithm for decoding Reed-Solomon codes and its relation to the Welch-Berlekamp algorithm,” *IEEE Trans. Information Theory*, vol. IT-51, pp. 1196-11198, Sep. 2005.

[19] S. V. Fedorenko, “Correction to ‘A simple algorithm for decoding Ree-Solomon codes and its relation to the Welch-Berlekamp algorithm’,” *IEEE Trans. Information Theory*, vol. IT-52, pp. 1278, Mar. 2006.

[20] L. Welch and B. R. Berlekamp, “Error correction for algebraic block codes,” US. Patent 4 633 740, Sep. 27, 1983.