Tunneling and energy splitting
in an asymmetric double-well potential

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Abstract

An asymmetric double-well potential is considered, assuming that the minima of
the wells are quadratic with a frequency $\omega$ and the difference of the minima is close
to a multiple of $\hbar \omega$. A WKB wave function is constructed on both sides of the local
maximum between the wells, by matching the WKB function to the exact wave
functions near the classical turning points. The continuities of the wave function
and its first derivative at the local maximum then give the energy-level splitting
formula, which not only reproduces the instanton result for a symmetric potential,
but also elucidates the appearance of resonances of tunneling in the asymmetric
potential.

Key words: quantum tunneling, energy splitting, double-well potential
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1 Introduction

The quantum tunneling in a double-well potential appears in a variety of
physical cases. Well-known examples include inversion of ammonia molecule
[1], for which the double-well is symmetric, and proton tunneling in hydrogen
bonds, for which the two wells could be unsymmetrical [2]. For a symmetric
potential, instanton method developed in [3], has been elaborated and applied
to calculate energy splitting [4]. The tunneling splitting calculated from the
instanton method exactly agrees with the WKB result when the quadratic
connection formula is adopted, and it has been confirmed that the result is
very accurate for large separation between the two wells [5,6,7].
Quantum tunneling in asymmetric double-well potentials has also long been considered \[8,9\], and one of the intriguing properties in these cases is the appearance of resonances. For example, the wave function appropriate for the false vacuum of the potential

\[
V_D(x) = \begin{cases} 
\frac{m\omega^2}{2}[(x + \alpha)^2 + (\beta^2 - \alpha^2)] & \text{for } x < 0 \\
\frac{m\omega^2}{2}(x - \beta)^2 & \text{for } x \geq 0,
\end{cases}
\]

\[(\beta > \alpha > 0)\] can significantly tunnel to the side of \(x \geq 0\) only when \(\beta\) is tuned to satisfy the condition

\[
\frac{m\omega^2}{2}(\beta^2 - \alpha^2) \approx n\hbar\omega, \quad n = 0, 1, 2, 3 \cdots.
\]

Through various numerical calculations, it has been known that the appearance of these resonances is not limited to \(V_D(x)\), but is a general property of tunneling in asymmetric potentials \[8\]. In this paper, in order to elucidate the analytic structures of the resonances, we construct the WKB wave functions for a class of asymmetric double-well potentials.

Specifically, we consider a smooth double-well potential \(V(x)\), assuming that \(V(x)\) has minima at \(x = b\) and at \(x = -a\) \((a, b > 0)\), and a local maximum at \(x = 0\). The minima are taken to be quadratic with a frequency \(\omega\), \(V(b) = 0\), and \(V(-a) = (n + \epsilon)\hbar\omega\) (See Fig. 1). As in the instanton method, we are interested in the large separation between the two wells, and consider the ground and low lying excited states of energy eigenvalue \(E\) satisfying \(V(0) \gg E > n\hbar\omega\). We also assume that the potential is still quadratic near the classical turning points between the wells.

Around the minima, exact solutions to the Schrödinger equation are described by the parabolic cylinder functions \[10\]. On both sides of \(x = 0\), a WKB wave function is constructed by matching the WKB function to the asymptotic forms of the exact solutions near the classical turning points. The continuities of the wave function and its first derivative at the local maximum then give the energy-level splitting formula. Though our method of requiring continuities is very different from the instanton or WKB method in \[4\], the splitting formula reduces to the known one in the symmetric case \[4,11,12\].

In the symmetric potential, for a given energy, an approximate solution to the Schrödinger equation localized in left(right) well implies, by the inversion symmetry, another solution localized in the right(left) well, and this fact has been conveniently used to evaluate energy splittings \[13,11\]. In this paper, we also show that tunneling in the asymmetric potential of \(\epsilon = 0\) can be explored by assuming the degenerate approximate solutions to the Schrödinger equation
Fig. 1. An asymmetric double-well potential $V(x)$: $V(b) = 0$, $V(-a) = (n + \epsilon)\hbar\omega$, $(n = 0, 1, 2, \cdots)$. We assume that, for a given energy $E$, $V(x)$ is quadratic in the regions of classical motions with the frequency $\omega$, and concentrate on the case of $\epsilon \ll 1$. 
\(\psi_R(x)\) and \(\psi_L(x)\) which are localized in the right and left wells, respectively. By explicitly constructing \(\psi_R(x)\) and \(\psi_L(x)\), the splitting formula is found from these wave functions. Indeed it turns out that the WKB wave functions satisfying the continuities could be written as linear combinations of \(\psi_R(x)\) and \(\psi_L(x)\), while the linear combinations lift the degeneracy to make the splitting. Therefore, a linear combination of the time-dependent WKB wave functions gives a system which shuttles back and forth between \(\psi_R(x)\) and \(\psi_L(x)\), to clearly elucidate the resonance structure of the tunneling in the asymmetric potential of \(\epsilon = 0\).

This paper is organized as follows. In Section 2, we construct the WKB wave function on both sides of the local maximum and, by requiring the continuities at the maximum, we evaluate the energy splitting. In Section 3, we find the appropriate localized wave functions \(\psi_R(x)\) and \(\psi_L(x)\), to re-obtain the energy splitting formula. We also establish a time-dependent WKB wave function of a system shuttling back and forth. In Section 4, we give some concluding remarks. Finally in Appendix A we give exact solutions for the system of \(V_D(x)\) in the limit of large separation between the two wells.

2 WKB method with continuity requirements

As the results could be easily modified to incorporate the small non-zero \(\epsilon\), we start with \(\epsilon = 0\).

2.1 WKB wave function for \(x \geq 0\)

For the wave function \(\psi_I(x)\) of the energy eigenvalue \(E = (\nu+n+\frac{1}{2})\hbar\omega\) around the right minimum, the Schrödinger equation is written as

\[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_I + \frac{m\omega^2}{2} (x-b)^2 \psi_I = \hbar\omega(\nu+n+\frac{1}{2}) \psi_I. \tag{3}\]

By introducing

\[l_{ho} = \sqrt{\frac{\hbar}{m\omega}}, \tag{4}\]

and \(z_R = \sqrt{2}(x-b)/l_{ho}\), we rewrite the equation as

\[\frac{d^2 \psi_I}{dz_R^2} + (\nu+n+\frac{1}{2} - \frac{z_R^2}{4} - \frac{z_R^2}{4}) \psi_I = 0,\]
to obtain
\[ \psi_I(x) = C_R D_{\nu+n}(z_R) = \frac{C_R D_{\nu+n}(\sqrt{2}(x-b))}{l_{ho}}, \]
(5)

where \( C_R \) is a constant and \( D_{\nu+n} \) denotes the parabolic cylinder function [10].

Bearing in mind that we wish to construct a normalizable wave function, we choose the solution in (5) so that \( \int_b^{\infty} |\psi_I(x)|^2 dx \) is finite if the expression of \( \psi_I(x) \) is valid for \( x > b \). Asymptotic expansions of the parabolic cylinder function are well-known. For large and negative \( z (z \ll -1 \text{ and } z \ll -|k|) \), we have

\[ D_k(z) \sim e^{-\frac{z^2}{2}} z^k \left[ 1 - \frac{k(k-1)}{2z^2} + \cdots \right] \]
\[-\frac{\sqrt{2\pi}}{\Gamma(-k)} e^{k\pi i} e^{\frac{z^2}{2}} z^{-k-1} \left[ 1 + \frac{(k+1)(k+2)}{2z^2} + \cdots \right]. \]
(6)

For \( x > 0 \), a classical turning point may be written as
\[ x = b_\nu = b - \sqrt{2\nu + 2n + 1} \ l_{ho}. \]
(7)

In the classically forbidden region of \( b_\nu - x \gg l_{ho} \), within the WKB approximation, a solution to the Schrödinger equation is
\[ \psi_{II}(x) = \frac{A_R}{\sqrt{p(x)}} e^{\int_{b}^{x} \frac{\mu(y)}{\hbar^2} dy} + \frac{B_R}{\sqrt{p(x)}} e^{-\int_{b}^{x} \frac{\mu(y)}{\hbar^2} dy}, \]
(8)

where \( p(x) \) is defined as
\[ p(x) = \sqrt{2m [V(x) - E]}, \]
(9)

and \( A_R, B_R \) are constants. In the region of quadratic potential satisfying \( b - x \gg b - b_\nu \), by introducing
\[ \Phi_R(x) = -\int_x^{b_\nu} \frac{p(y)}{\hbar} dy = -\int_x^{b_\nu} \frac{[(b-y)^2 - (b-b_\nu)^2]^{1/2}}{l_{ho}^2} dy, \]
(10)
we have [14,11]
\[ \Phi_R(x) = -\frac{(b-x)^2}{2l_{ho}^2} + \frac{1}{4} (2\nu + 2n + 1) \]
\[ + \frac{1}{2}(2\nu + 2n + 1) \ln \left( \frac{2(b - x)}{b - b_\nu} \right) + O \left( \frac{(b - b_\nu)^2}{b - x} \right), \quad (11) \]

and thus

\[
\psi_{II}(x) \approx \frac{A_R}{\sqrt{m\omega(b - x)}} \left( \frac{2\sqrt{e}(b - x)}{b - b_\nu} \right)^{\nu + \frac{1}{2}} \exp \left[ \int_0^{b_\nu} \frac{p(y)}{\hbar} dy - \frac{(b - x)^2}{2l_{ho}^2} \right] \\
+ \frac{B_R}{\sqrt{m\omega(b - x)}} \left( \frac{b - b_\nu}{2\sqrt{e}(b - x)} \right)^{\nu + \frac{1}{2}} \exp \left[ - \int_0^{b_\nu} \frac{p(y)}{\hbar} dy + \frac{(b - x)^2}{2l_{ho}^2} \right]. \quad (12)
\]

As we are interested in the limit of large separation between the two wells where the energy splitting is small, we introduce \( \delta_l \) with an integer \( l = \lceil \nu \rceil \) as

\[ \delta_l = \nu - l, \quad (13) \]

so that

\[ |\delta_l| \ll 1. \quad (14) \]

In the region of the quadratic potential the wave function is also described by \( \psi_I(x) \). Making use of the asymptotic form in (6), for \( b - x \gg l_{ho} \), in the leading orders we obtain

\[
\psi_I(x) \sim C_R e^{-\frac{(x-b)^2}{2l_{ho}^2}} \left( \frac{\sqrt{2(x - b)}}{l_{ho}} \right)^l e^{\frac{(x-b)^2}{2l_{ho}^2}} \left( \frac{2\sqrt{\pi}(l + n)!}{\sqrt{2l_{ho}^{-1}(x - b)}} \right)^{l+n+1}. \quad (15)
\]

By matching the asymptotic form of \( \psi_I(x) \) onto that of \( \psi_{II}(x) \) in this overlap region, we have

\[
A_R = (-1)^{l+n} C_R \sqrt{\frac{h(l + n)!}{2\sqrt{\pi}l_{ho}}} e^{-\int_0^{b_\nu} \frac{p(y)}{\hbar} dy}, \quad (16)
\]

\[
B_R = (-1)^{l+n+1} C_R \delta_l \sqrt{\frac{2\pi^3/2h(l + n)!}{l_{ho}g_{l+n}}} e^{\int_0^{b_\nu} \frac{p(y)}{\hbar} dy}, \quad (17)
\]

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where
\[ g_k = \frac{\sqrt{2\pi}}{k!} \left( k + \frac{1}{2} \right)^{k+\frac{1}{2}} e^{-(k+1/2)}. \] (18)

2.2 WKB wave function for \( x \leq 0 \)

Since \( V(x) = m\omega^2(x + a)^2/2 + n\hbar\omega \) near the minimum of the left well, a classical turning point is given as
\[ x = -a_v = -a + \sqrt{2\nu + 1} \hbar. \] (19)

In the classically forbidden region of \( a_v + x \gg \hbar \), a WKB solution may be written as
\[ \psi_{III}(x) = \frac{A_L}{\sqrt{p(x)}} e^{-\int_x^0 \frac{p(y)}{\hbar} dy} + \frac{B_L}{\sqrt{p(x)}} e^{\int_x^0 \frac{p(y)}{\hbar} dy}, \] (20)

where \( A_L, B_L \) are constants. In the region of the quadratic potential satisfying \( a + x \gg a - a_v \), from the fact that
\[ \Phi_L(x) = -\int_{-a_v}^x \frac{p(y)}{\hbar} dy = -\int_{-a_v}^x \left[ (a + y)^2 - (a - a_v)^2 \right]^{1/2} \frac{dy}{\hbar^2}, \] (21)

we obtain
\[ \psi_{III}(x) \approx \frac{A_L}{\sqrt{m\omega(a + x)}} \left( \frac{a - a_v}{2\sqrt{c(a + x)}} \right)^{t_2 + \frac{1}{2}} \exp \left[ -\int_{-a_v}^0 \frac{p(y)}{\hbar} dy + \frac{(a + x)^2}{2\hbar^2} \right] + \frac{B_L}{\sqrt{m\omega(a + x)}} \left( \frac{2\sqrt{c(a + x)}}{a - a_v} \right)^{t_2 + \frac{1}{2}} \exp \left[ \int_{-a_v}^0 \frac{p(y)}{\hbar} dy - \frac{(a + x)^2}{2\hbar^2} \right]. \] (22)

With \( z_L = \sqrt{2(x + a)}/\hbar \), the Schrödinger equation around the left minimum is written as
\[ \frac{d^2\psi_{IV}}{dz_L^2} + (\nu + 1/2 - \frac{z_L^2}{4})\psi_{IV} = 0. \]
and thus
\[ \psi_{IV}(x) = C_L D_\nu(-z_L) = C_L D_\nu\left(-\frac{\sqrt{2}(x+a)}{l_{ho}}\right), \tag{23} \]

with a constant $C_L$. The solution in (23) is chosen, so that $\int_{-\infty}^{-a} |\psi_{IV}(x)|^2 dx$ is finite if the expression of $\psi_{IV}(x)$ is valid for $x < -a$. Around the left minimum of $a + x \gg l_{ho}$, from (14) and the asymptotic form in (6), in the leading orders we have

\[ \psi_{IV}(x) \sim (-1)^l C_L e^{-\frac{(x+a)^2}{2l_{ho}}} \left( \frac{\sqrt{2}(x+a)}{l_{ho}} \right)^l - \delta_l e^{\frac{(x+a)^2}{2l_{ho}}} \frac{\sqrt{2\pi} l!}{\left(\sqrt{2l_{ho}}(x+a)\right)^{l+1}} \tag{24} \]

By matching $\psi_{III}(x)$ to $\psi_{IV}(x)$ in the overlap region, we obtain

\[ A_L = (-1)^{l+1} C_L \delta_l \sqrt{\frac{2\pi^{3/2}\hbar!}{l_{ho}^l}} e^{\int_{-a_l}^{0} \frac{p(y)}{\hbar} dy}, \tag{25} \]
\[ B_L = (-1)^l C_L \sqrt{\frac{\hbar!}{2\sqrt{\pi}l_{ho}^l}} e^{-\int_{-a_l}^{0} \frac{p(y)}{\hbar} dy}. \tag{26} \]

### 2.3 Continuity and energy splitting

For a smooth potential, a wave function and its first derivative must be continuous at $x = 0$, which gives the relations

\[ A_L = A_R, \quad B_L = B_R. \tag{27} \]

There are three unknowns $C_L$, $C_R$, $\delta_l$ in these two equations, while another equation may come from the normalization of the wave function. From $A_L/B_L = A_R/B_R$, making use of Eqs. (16,17,25,26), we obtain

\[ \delta_l^2 = \frac{g_l g_{l+n}}{(2\pi)^2} \exp \left[ -2 \int_{-a_l}^{b_l} \frac{p(y)}{\hbar} dy \right], \tag{28} \]

which indicates that the splitting of the energy level $\Delta_l$ is given by

\[ \Delta_l = \sqrt{\frac{g_l g_{l+n}}{\pi}} \hbar \omega \exp \left[ -2 \int_{-a_l}^{b_l} \frac{p(y)}{\hbar} dy \right]. \tag{29} \]
For a symmetric potential of \( n = 0 \), (29) exactly agrees with the result in [11,12].

Expression in (29) is not easy to use, because the integrand in the exponential is close to a singularity near the limits. By introducing

\[
I_a = \int_{-a}^{0} \sqrt{2m[V(y) - nh\omega]} dy, \quad I_b = \int_{0}^{b} \sqrt{2mV(y)} dy, \quad (30)
\]

and

\[
\gamma_a = \int_{0}^{a} \left( \frac{\sqrt{m\omega^2}}{\sqrt{2[V(y-a) - nh\omega]}} - \frac{1}{y} \right) dy, \\
\gamma_b = \int_{0}^{b} \left( \frac{\sqrt{m\omega^2}}{\sqrt{2V(b-y)}} - \frac{1}{y} \right) dy, \quad (31)
\]

the splitting is written for a general potential as

\[
\Delta_l = \hbar \omega \sqrt{2e^{-(I_a+I_b)/\hbar}} \left( \frac{\sqrt{2} ae^{\gamma_a}}{l_{ho}} \right)^{l+1/2} \left( \frac{\sqrt{2} be^{\gamma_b}}{l_{ho}} \right)^{l+n+1/2}. \quad (32)
\]

The expression in (32) can be conveniently used to find that our formula reduces to the known one in [12] for a symmetric potential.

### 2.4 For a non-zero \( \epsilon \)

The above formalism can be modified to include non-zero \( \epsilon \), as far as \( \delta_l - \epsilon \ll 1 \). In this case, without a change in \( \psi_I(x) \) and \( \psi_{II}(x) \), the modifications of \( \psi_{III}(x) \) and \( \psi_{IV}(x) \) are obtained by replacing \( \delta_l \) with \( \delta_l - \epsilon \) (or, \( \nu \) with \( \nu - \epsilon \)). Due to the changes in Eqs. (24,25), the continuity requirements then give the relation

\[
\delta_l(\delta_l - \epsilon) = \left( \frac{\Delta_l}{2\hbar\omega} \right)^2. \quad (33)
\]

If \( \Delta_l^{\epsilon} \) denotes the energy splitting in the presence of \( \epsilon \), (33) implies

\[
\Delta_l^{\epsilon} = \sqrt{\Delta_l^2 + (\hbar\omega\epsilon)^2}. \quad (34)
\]
3 An alternative method with localized wave functions

For $\epsilon = 0$, if we assume two states of the normalized real wave functions $\psi_R(x)$ and $\psi_L(x)$ with energy $E_0$ as mentioned in Section 1, the Hamiltonian in this two-state subspace may be given by

$$H = \begin{pmatrix} E_0 & \Delta/2 \\ \Delta/2 & E_0 \end{pmatrix},$$

(35)

where the small tunneling splitting $\Delta$ is written as

$$\Delta = 2 \int_{-\infty}^{\infty} \psi_L(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi_R(x) dx.$$

(36)

The two eigenstates of the Hamiltonian are given by $(\psi_R(x) \pm \psi_L(x))/\sqrt{2}$.

From the Schrödinger equations for these eigenstates, and the equations

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi_i(x) = E_0 \psi_i(x) \quad (i = R, L),$$

we find

$$\Delta \approx \frac{\hbar^2}{m} \left| \psi_L(0) \psi_R'(0) - \psi_R(0) \psi_L'(0) \right|,$$

(38)

with $\psi'_i(x) = d\psi_i(x)/dx$ ($i = R, L$), which is a generalization of the method used for the symmetric case [11,13].

In the classically forbidden region, we may write

$$\psi_R(x) = \frac{N_R}{\sqrt{p(x)}} e^{\int_0^x \frac{\mu(y)}{\hbar} dy}, \quad \psi_L(x) = \frac{N_L}{\sqrt{p(x)}} e^{-\int_0^x \frac{\mu(y)}{\hbar} dy},$$

(39)
where \( N_R \) and \( N_L \) are constants. From these expressions of \( \psi_R(x) \) and \( \psi_L(x) \), by assuming that the validity condition for the WKB approximation

\[
\left| \frac{d \frac{h}{p}}{dx} \right| \ll 1
\]  

(40)

is satisfied at \( x = 0 \), we obtain

\[
\Delta \approx 2\frac{\hbar}{m} |N_L N_R| .
\]  

(41)

For \( E_0 = (l + n + \frac{1}{2})\hbar \omega \), near the right-hand well, \( \psi_R(x) \) would be accurately described by the \((l+n)\)th harmonic oscillator eigenfunction. This description holds well into the forbidden region, and, for \((b-x)/\hbar \omega \gg 1\), we may write

\[
\psi_R(x) \approx \exp \left[ -\frac{(x-b)^2}{2\hbar^2} \right] \left( \frac{\sqrt{2}(x-b)}{\hbar \omega} \right)^{l+n} .
\]  

(42)

On the other hand, making use of (11), we can find the asymptotic expansion form of \( \psi_R(x) \) of (39) in the overlap region. By matching the asymptotic form onto the expression in (42), we obtain

\[
N_R = (-1)^{l+n} \frac{\sqrt{\hbar} \gamma_{l+n}}{\sqrt{2\pi} \hbar \omega} \exp \left[ -\frac{b_\nu}{\hbar} \int_0^y \frac{p(y)}{\hbar} dy \right] .
\]  

(43)

Similarly, by matching the asymptotic form of \( \psi_L(x) \) onto that of the \( l \)th excited harmonic oscillator state near the left-hand well, we have

\[
N_L = \frac{\sqrt{\hbar} \gamma_l}{\sqrt{2\pi} \hbar \omega} \exp \left[ -\int_{-b_\nu}^0 \frac{p(y)}{\hbar} dy \right] .
\]  

(44)

By plugging these explicit forms of \( N_R \) and \( N_L \) into (41), for \( E_0 = (l+n+\frac{1}{2})\hbar \omega \), we confirm that \( \Delta \) reduces to \( \Delta_l \).

If (27) is satisfied, after some algebra, we find that \( \psi_{II}(x) \) and \( \psi_{III}(x) \) of \( \epsilon = 0 \) can be merged, in the classically forbidden region, into

\[
\psi_{WKB}^\pm(x) = \frac{\sqrt{2\pi} A_R \hbar \omega}{\sqrt{\hbar g_{l+n}}} \exp \left[ \int_0^{b_\nu} \frac{p(y)}{\hbar} dy \right] \left[ (-1)^{l+n} \psi_R(x) \mp \psi_L(x) \right] ,
\]  

(45)
namely, $\psi_{WKB}^\pm(x)$ becomes $\psi_{II}(x)$ for $x \geq 0$, and $\psi_{III}(x)$ for $x \leq 0$, where $\psi_{WKB}^+(x)$ ($\psi_{WKB}^-(x)$) is given when we choose $\delta_l > 0$ ($\delta_l < 0$) in (28). Thus this alternative method is in fact equivalent to the WKB method of requiring the continuities. We also note that $\psi_{WKB}^-(x)$ has a node in the classically forbidden region between the wells, while $\psi_{WKB}^+(x)$ has no node in the same region.

A (unnormalized) time-dependent WKB solution is given as

$$
\psi(x, t) = e^{-i\omega(n+l+1/2+|\delta_l|)t} \psi_{WKB}^+(x) + e^{-i\omega(n+l-1/2-|\delta_l|)t} \psi_{WKB}^-(x)
$$

$$
= 2\sqrt{2\pi A_R l_ha} \exp \left[ -i\omega(n + l + \frac{1}{2})t + \int_0^{by} \frac{p(y)}{\hbar} dy \right]
$$

$$
\times (-1)^{l+n} \cos \left( \frac{\Delta_l}{2\hbar} t \right) \psi_R(x) - i \sin \left( \frac{\Delta_l}{2\hbar} t \right) \psi_L(x). \quad (46)
$$

This last form shows clearly that the system shuttles back and forth between $\psi_R(x)$ and $\psi_L(x)$ with the frequency $\Delta_l/\hbar$.

In order to include $\epsilon$, let us consider a slight modification of the potential around the left well so that $V(-a)$ changes from $n\hbar\omega$ to $(n+\epsilon)\hbar\omega$. While $\psi_R(x)$ would still be an appropriate solution of the new system with energy $E_0$, we introduce $\psi_L^\epsilon(x)$ as an approximate solution with energy $E_0 + \epsilon\hbar\omega$ localized in the left well. If we confine our attention on the two state subspace described by $\psi_L^\epsilon(x)$ and $\psi_R(x)$, the Hamiltonian of the new system is analogous to that of a particle in a magnetic field [15]. We note that this analogy can be used to derive (34) within the approximation that $\psi_L^\epsilon(x)$ is the same with $\psi_L(x)$. If $\psi(x, t)$ is a solution in this subspace with $\psi(x, 0) = \psi_R(x)$, the maximum of the probability of $| \int_0^\infty (\psi_L^\epsilon(x))^* \psi^\epsilon(x, t) |^2$ during the time-evolution is $\Delta_l^2/(\Delta_l^2 + \epsilon^2)$, which indicates that the resonance peaks in tunneling have the Lorentzian shape.

4 Concluding remarks

Energy splitting formula has been obtained for the asymmetric double-well potential, by assuming that the potential is quadratic near the minima. As has been well known for the symmetric case, we expect that the splitting formula given here would be very accurate for the large separation between wells, which needs to be confirmed through numerical calculations. If we could add a linear term $sx$ to the potential $V(x)$ of $\epsilon = 0$ with a controllable constant $s$, $V(x) + sx$ has two minima at $x = b - s/m\omega^2$ and at $x = -a - s/m\omega^2$. Since the difference of the minima is given as $n\hbar\omega - s(a+b)$, in the light of numerical results [8], the tunneling would be significant only if $s$ is close to a multiple of

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\( \hbar \omega / (a + b) \). It would be of great interest to realize the asymmetric system with controllable constants. As a final remark, for \( n = 0 \) and \( \epsilon = 0 \), we note that 

\[ \Delta_{l} / \Delta_{0} = \left( 2abe^{\gamma_{a}+\gamma_{b}} / l_{ho}^{2l} \right) / (2l)! \]

which shows the quasi-Weierstrassian nature of the tunneling spectrum [12]. In this asymmetric case, thus, the tunneling behavior of an initially squeezed wave packet is erratic, and the trajectory of the expected position of the wave packet has a fractal structure.

**Appendix A**

For the potential \( V_{D}(x) \) in (1), WKB method may not be applicable, since the potential is not differentiable at \( x = 0 \). In this case, however, exact wave functions could be written in terms of the parabolic cylinder functions on both sides of \( x = 0 \). From the continuities of the wave function and its first derivative at \( x = 0 \), assuming

\[ m\omega^{2}(\beta^{2} - \alpha^{2}) = (n + \epsilon)\hbar \omega \],

we find that the eigenstate of an energy eigenvalue \((\nu + n + \epsilon + \frac{1}{2})\hbar \omega \) exists if the condition

\[ D_{\nu} \left( -\frac{\sqrt{2} \alpha}{l_{ho}} \right) D'_{\nu+n+\epsilon} \left( -\frac{\sqrt{2} \alpha}{l_{ho}} \right) = -D'_{\nu} \left( -\frac{\sqrt{2} \alpha}{l_{ho}} \right) D_{\nu+n+\epsilon} \left( -\frac{\sqrt{2} \beta}{l_{ho}} \right) \]  
(47)

is satisfied. (47) can be solved in the limit of \( \alpha, \beta \gg l_{ho} \) and \( \epsilon \ll 1 \). Making use of the asymptotic expansion of (6), in this limit we obtain

\[ \delta_{l}^{2} + (r(R_{l} - L_{l}) + \epsilon) \delta_{l} - R_{l}L_{l} - \epsilon rL_{l} = 0, \]  
(48)

where

\[ R_{l} = \frac{(\sqrt{2} \beta / l_{ho})^{2(n+l)+1}}{\sqrt{2\pi(1+l)!}} e^{-\beta^{2}/l_{ho}^{2}}, \quad L_{l} = \frac{(\sqrt{2} \alpha / l_{ho})^{2l+1}}{\sqrt{2\pi l!}} e^{-\alpha^{2}/l_{ho}^{2}}, \]

\[ r = \frac{\beta - \alpha}{\alpha + \beta}. \]  
(49)

(48) implies that the energy splitting is given as

\[ \hbar \omega \sqrt{4R_{l}L_{l} + \epsilon^{2} + 2\epsilon r(R_{l} + L_{l}) + r^{2}(R_{l} - L_{l})^{2}}. \]  
(50)

If we formally use the formulas in Eqs. (30,31) by replacing \( V(x) \) with \( V_{D}(x) \), \( \Delta_{l} \) coincides with the splitting of (50) in the symmetric case \((n = 0, \epsilon = 0)\).
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