A computationally efficient non-iterative four-parameter sine fitting method

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Abstract
A computationally efficient four-parameter least squares (LS) sine fitting method in the time domain is presented here. Unlike the most widespread procedure defined in the relevant IEEE standard, the proposed fitting is non-iterative. This is achieved by the second-order approximation of the cost function (CF) around the actual frequency of the sinusoidal excitation. The approximation reduces the four-parameter non-linear fitting problem to a defined set of three-parameter linear fitting problems. Therefore, the computational demand can be predicted precisely, which is an essential aspect of real-life applications. Furthermore, the proposed method is shown to have increased numerical stability. Finally, measurements and computer simulations are carried out to demonstrate the reduced computational demand, while preserving the accuracy compared with the algorithm proposed in the IEEE standard.

1 | INTRODUCTION

Parameter estimation of a sinusoidal waveform is a general signal processing task. Sinusoidal excitation is applied in a wide variety of application areas since these functions are eigenfunctions of linear systems: a sine wave with arbitrary amplitude and phase, and with a given frequency becomes a sine wave with the same frequency at the output of the system, but the amplitude and the phase are altered. In power systems, parameters of synchronphasors (complex sinusoidal functions) are estimated [1, 2]. In the field of biomedical engineering, sinusoidal excitation can be used to measure bioimpedances both for single- and multi-tone measurements [3, 4]. Furthermore, it can be applied for analogue-to-digital converter (ADC) testing, where quantised signals are to be treated [5, 6].

A sinusoidal waveform can be described with the following signal model:

\[ y_k = R \cdot \cos(2\pi ft_k + \Phi) + C, \]  

where \( y_k \) is the \( k \)th sample of the waveform, \( R, \Phi \) and \( C \) are the amplitude, initial phase and dc component of the signal, respectively, \( f \) is the frequency and \( t_k \) is the \( k \)th sampling time.

The above form is non-linear in the frequency and initial phase parameters. Thus, mostly a modified form is used in sine fitting applications:

\[ y_k = A \cdot \cos(2\pi ft_k) + B \cdot \sin(2\pi ft_k) + C, \]  

where parameters \( A \) and \( B \) are the amplitudes of the cosinusoidal and sinusoidal component. They can be expressed in the following form:

\[ A = R \cdot \cos(\Phi), \quad B = -R \cdot \sin(\Phi). \]  

In real world applications, the sampling of the signal is mostly equidistant, that is, an \( f_s \) sampling frequency is applied. Introducing notation

\[ \vartheta = 2\pi f, \]  

for the relative angular frequency of the signal, Equation (2) can be re-written as follows:
\[ y_k = A \cdot \cos(k\theta) + B \cdot \sin(k\theta) + C. \] (5)

Consequently, an equidistantly sampled sine wave can be uniquely characterised by \( A, B, C \) and \( \theta \). It is usually assumed that the sampled signal can be described as the sum of a pure sine wave and an additive noise:

\[ x = y + e, \] (6)

where \( x \) is a vector containing the sampled sine wave, \( y \) is the fitted sine wave, and \( e \) contains the samples of the additive noise. Vector \( e \) is mostly assumed to contain independent Gaussian noise samples, though this model is not appropriate if the sine wave is quantised by an ADC [7]. The most widely used method fits \( y \) on the measured data set, so that the sum of squared differences is minimal. That is, an LS fitting is performed. This fitting is prescribed in IEEE Standard 1241-2010, which describes methods for ADC testing [8]. Among time-domain LS sine wave fitting methods, two cases are distinguished in the standard: in the first case, the frequency of the signal is assumed to be known, while in the second one, the frequency of the signal is estimated, as well.

Prior knowledge on the signal frequency makes the fitting fast and robust. The signal model is linear in the other three parameters. However, in this case, the accuracy of the frequency estimate influences the result of the fitting significantly. If no prior knowledge is present, an initial frequency estimate should be provided, and it is fine-tuned by the four-parameter sine fitting that also estimates the frequency. In this case, the problem becomes non-linear, and its solution is more involved compared with the three-parameter problem [8].

Former research investigated the issue, whether the frequency estimation can be separated from the other three parameters [9, 10]. If a precise frequency estimate can be provided in advance, the robust linear three-parameter fitting can be performed. Performance comparison between the time-domain LS four-parameter fitting and the three-parameter fitting with separated frequency estimation is detailed in [11]. Certainly, if the frequency estimate is accurate enough, the three-parameter fitting can be preferred. However, it is hard to determine whether the frequency estimate needs to be refined. In [12], it is suggested that the accurate initial estimator is further fine-tuned by the four-parameter method.

The aim of the paper is to improve the robustness and evaluation time of the time-domain four-parameter LS method. It will be shown that we can apply the linear three-parameter method without needing to decide whether the initial frequency estimate is accurate enough for the evaluation. In the vicinity of the optimum, the LS cost function is of quadratic nature with respect to the frequency. The core of the idea is such that starting from an initial frequency estimate close to the optimum, the bottom of the cost function is mapped through three-parameter fittings. By this means, the parameters of the sine wave can be obtained.

The estimation of sinusoidal parameters is even a recently investigated area. Methods beyond [8] are suggested to precisely determine the frequency or the amplitude parameters of the sinusoidal signal, see [13, 14], respectively. This paper intends to implement the widely used time-domain four-parameter LS method, improving the weak points of the standardised procedure described in [8]. By this means, the evaluation of ADC testing can be improved, and the proposed method may be adequate to support the prospective revision of the IEEE Standard 1241-2010, but the method can be applied in any area where sinusoidal parameter estimation is needed.

The paper is organised as follows: In Section 2, both the three- and four-parameter time-domain LS sine fitting methods will be described in order to see their advantages and drawbacks when compared with each other. In Section 3, the proposed non-iterative method will be introduced for the estimation of all the four parameters. In Section 4, the accuracy of the proposed method will be compared with the existing four-parameter method through simulation results. In Section 5, the computational demand of the existing and proposed methods will be compared, while in Section 6, the significant improvement in this quantity will be demonstrated through microcontroller implementation. Finally, the paper is concluded in Section 7.

2 | TIME-DOMAIN LEAST SQUARES SINE FITTING METHODS

2.1 | Sine fitting with known frequency

The signal model in Equation (5) is linear in \( A, B \) and \( C \). Consequently, if the frequency of the signal is assumed to be known accurately, these parameters can be estimated in LS sense without iteration, introducing the following matrix:

\[
D_0 = \begin{bmatrix} \cos(\theta_0) & \sin(\theta_0) & 1 \\ \cos(2\theta_0) & \sin(2\theta_0) & 1 \\ \vdots & \vdots & \vdots \\ \cos(N\theta_0) & \sin(N\theta_0) & 1 \end{bmatrix}, \] (7)

where \( \theta_0 \) is the applied estimate of the actual relative angular frequency. The LS solution is

\[
[A_0 \ B_0 \ C_0]^T = D_0^T x, \] (8)

where \( (\cdot)^\dagger \) denotes the Moore-Penrose pseudoinverse operator [8]. Furthermore, \( A_0, B_0 \) and \( C_0 \) are estimates of signal parameters \( A, B \) and \( C \), respectively. This is the three-parameter LS fitting method (LS3p).

In [15], it has been pointed out that the calculation of the parameters applying

\[
[A_0 \ B_0 \ C_0]^T = (D_0^T D_0)^{-1} D_0^T x \] (9)
does not introduce numerical problems. \((D_k^T D_k)\) was proven to be well conditioned.

If the frequency estimate cannot be considered accurate, the problem becomes iterative, as it will be shown in the following section.

2.2 Sine fitting with unknown frequency

In the IEEE Standard 1240-2010, an iterative solution is suggested to estimate all the four parameters simultaneously [8]. In the procedure, the frequency of the signal is fine-tuned in every step. The system matrix is extended with the derivative of the fitted signal with respect to \(\dot{\theta}\):

\[
D_i = \begin{bmatrix}
\cos(\theta_{i-1}) & \sin(\theta_{i-1}) & 1 & D_{i,1,4} \\
\cos(2\theta_{i-1}) & \sin(2\theta_{i-1}) & 1 & D_{i,2,4} \\
\vdots & \vdots & \vdots & \vdots \\
\cos(N\theta_{i-1}) & \sin(N\theta_{i-1}) & 1 & D_{i,N,4}
\end{bmatrix}
\]

(10)

where subscript \(i\) denotes the actual iteration step. The \(D_{i,k,4}\) elements of the matrix can be calculated as:

\[
D_{i,k,4} = -A_{i-1} \cdot k \cdot \sin(k\theta_{i-1}) + B_{i-1} \cdot k \cdot \cos(k\theta_{i-1}).
\]

(11)

Note that in order to evaluate the first iteration, parameters \(A_0\) and \(B_0\) and \(\delta_0\) are needed. These parameters are obtained from the previously shown LS3p method. The resulting parameter vector is

\[
\begin{bmatrix}
A_i & B_i & C_i & (\Delta \delta)_{i1}^T
\end{bmatrix}^T = (D_i^T D_i)^{-1} (D_i^T x).
\]

(12)

After the evaluation of the parameters, the frequency estimate is updated by \(\theta_i = \theta_{i-1} + (\Delta \delta)_i\).

It should be noted that contrary to the LS3p method, the evaluation of Equation (12) may become numerically unstable due to the ill-conditioning of \(D_i\) [15]. To overcome this problem, [8] suggests the application of numerically stable methods for the calculation of the pseudoinverse. Since it would further increase the evaluation time of the fitting, in the following, the calculation method of Equation (12) will be utilised in the paper.

The method is called four-parameter LS fitting (LS4p). To have satisfactorily accurate results, this algorithm needs a good initial frequency estimator [12]. In the following section, some frequency estimation methods will be described.

2.3 Frequency estimation methods

The simplest frequency estimation method is based on the Fast Fourier Transform (FFT): the frequency estimator of this method is the frequency corresponding to the maximum element of the FFT. The worst-case error of \(f/f\) equals to

\[
1/2N. \text{ Though this approach is straightforward, in some applications, a more accurate estimator is needed.}
\]

Further investigation reveals that frequency estimation in itself is an involved signal processing task. Even recently, different estimators have been introduced to have both accurate and computationally efficient solutions [16, 17]. A common approach is to apply a specific window function, and launch interpolated FFT (IpFFT) methods. These methods are connected to the type of window that is applied at sine fitting. If no specific window is used, that is, if the signal is windowed with a rectangular window, a frequency estimator is derived in [18]. However, this window type is very sensitive to harmonic distortions. To overcome this problem, the shape of the window can be altered, resulting in different IpFFT estimators [19, 20]. Another possibility is that instead of the discrete frequency domain, the frequency estimation is performed in the continuous frequency domain, applying discrete-time Fourier-transform (DTFT) [21, 22].

In this paper, the Hanning window-based IpFFT method will be applied. The reason for the choice is its simplicity and ability to provide an initial estimator for the frequency, which is accurate enough to ensure the convergence of the four-parameter sine fitting method [23]. The time-domain expression of the window is given in the following [24]:

\[
\omega_k = 0.5 - 0.5 \cos\left(\frac{2\pi k}{N}\right),
\]

(13)

where \(\omega_k\) and \(N\) are the \(k\)th sample and the total number of samples, respectively. Once the measured sine wave is windowed, an FFT can be performed, and the absolute value of the result is calculated:

\[
z = |\text{FFT}\{y_k \cdot \omega_k\}|.
\]

(14)

Let \(l\) be the index of the largest spectral component, and \(\delta\) be the required correction for the frequency. The latter can be determined using the following formula [25]:

\[
\delta = \begin{cases} 
\frac{2z_{l+1} - z_l}{z_{l+1} + z_l} & \text{if } z_{l+1} \geq z_{l-1} \\
\frac{z_l - 2z_{l-1}}{z_{l-1} + z_l} & \text{if } z_{l+1} < z_{l-1}
\end{cases}
\]

(15)

Finally, the initial estimator of \(\theta\) can be calculated as:

\[
\theta_0 = 2\pi \cdot \frac{l + \delta}{N}.
\]

(16)

3 | DESCRIPTION OF THE METHOD

In this section, an improved four-parameter LS sine fitting method is proposed. During the procedure, we make use of the fact that when close to the optimum, the CF is approximately of second-order. However, if this holds, the parameters could
be obtained in one step, applying the Newton-Raphson method [26]:

$$\Delta p = -\frac{\partial CF(p)}{\partial p} \cdot \frac{\partial CF(p)}{\partial \Delta p}.$$ (17)

Though this calculation is non-iterative, it has two main drawbacks. First, it is not certain that after the fine-tuning of the parameters, we step closer to the optimum since the matrix containing the second derivative is not assuredly positive semi-definite. On the other hand, the calculation of the first and second derivatives is rather time consuming. We can overcome the first problem if a precise initial frequency estimator can be obtained. This is the case when IpFFT with Hanning window is applied. However, the computational burden of the derivatives makes the method impractical to be applied.

In the following, a parabola fitting approach will be introduced that is non-iterative and contains easy-to-implement operations, contrary to the Newton-Raphson method. We make use of the assumption that when close to the optimum, the CF is approximately of second order. This is a higher-dimensional paraboloid with input parameters $A, B, C,$ and $\theta$. With the proposed approach, we pick a cross-section of this paraboloid along the frequency axis. This cross-section is a parabola, which has its minimum at the actual frequency of the signal. To gain this parabola, we choose three frequency values that are close to the optimum. A natural choice is an IpFFT estimator. This frequency value is given as input for the LS3p method. After the other three parameters are determined according to Equation (9), the CF is evaluated. We repeat this procedure with the other two frequency values that are selected symmetrically to this estimate, so that we remain close to the optimum. After the evaluations, we have three CF values at three frequencies. Recall that these are samples from the cross-section of the higher dimensional paraboloid, that is, a parabola. The three points determine the parabola exactly, so that we can collect the information on the signal frequency. The procedure is depicted in Figure 1.

The only question is how the frequencies should be chosen. On the one hand, they have to be close to the optimum, so the deviation from the IpFFT estimator should be small. On the other hand, if the deviation is too small, the minimisation will not be effective, either. In this case, the algorithm will be very sensitive to the roundoff errors of the CF evaluation [27]. Thus, we have to choose frequencies that are far enough from the IpFFT estimate, not to be disturbed by numerical accuracy problems, but not too far to be able to use the assumption that the CF is of second order. The two other frequency points are chosen symmetrically, so that the deviation from the IpFFT estimate $\theta_{IpFFT}$ equals to $\Delta \theta = \pm 0.01$. This formula expresses that the longer a record, the more accurate the IpFFT frequency estimate. Using these three CF points, a parabola fit can be executed. The abscissa of the parabola is the deviation from the IpFFT frequency estimator, that is, $\Delta \theta = \theta - \theta_{IpFFT}$. To fit the parabola to the evaluated points, the following system of equations is to be solved:

$$
\begin{bmatrix}
(-\Delta \theta)^2 & -\Delta \theta & 1 \\
0 & 0 & 1 \\
(\Delta \theta)^2 & \Delta \theta & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
CF_{-\Delta \theta} \\
CF_{\theta_{IpFFT}} \\
CF_{\Delta \theta}
\end{bmatrix}.
$$ (18)

where $a, b$ and $c$ are the parameters of the parabola. The index of the CF denotes the abscissa position. For instance, $CF_{\Delta \theta}$ is the cost function value at $\theta_{IpFFT} + \Delta \theta$. Since $\Delta \theta$ is rather small and the first and second power of it is contained in the matrix of the system of equations, the matrix may become ill-conditioned. For instance, if $\Delta \theta = 10^{-6}$, the condition number equals to $2.1 \cdot 10^{12}$. To overcome this issue, $(\Delta \theta)^2$ from the first column and $\Delta \theta$ from the second column can be factored out, so that we obtain

$$
\begin{bmatrix}
1 & -1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
a \cdot (\Delta \theta)^2 \\
b \cdot \Delta \theta \\
c
\end{bmatrix}
= 
\begin{bmatrix}
CF_{-\Delta \theta} \\
CF_{\theta_{IpFFT}} \\
CF_{\Delta \theta}
\end{bmatrix}.
$$ (19)

Besides better conditioning, the advantage of the modified parabola fit is that the inverse calculation of the system matrix becomes independent of $\Delta \theta$:

$$
\begin{bmatrix}
d' \\
b' \\
c'
\end{bmatrix}
= 
\begin{bmatrix}
1 & -1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}^{-1}
\cdot
\begin{bmatrix}
CF_{-\Delta \theta} \\
CF_{\theta_{IpFFT}} \\
CF_{\Delta \theta}
\end{bmatrix}.
$$ (20)

**FIGURE 1** Illustration on the sequence of the proposed method: CF evaluation at three frequencies, parabola fitting, CF evaluation at the optimum. CF, cost function.
With the parameters of the parabola, the minimum can be expressed as:

\[
\Delta \theta_{opt} = \frac{b}{2d} = \frac{b'}{2d'} (\Delta \theta_p)^2 = -\frac{b'}{2d'} \Delta \theta_p.
\]  \hspace{1cm} (21)

After setting the angular frequency to the optimal value, a further LS3p fit is to be executed to find the LS estimates of \(A, B,\) and \(C\). Hence, a total number of four LS3p fittings are needed to evaluate the LS4p fitting. MATLAB and C implementations of the proposed method are available in [28].

It should be noted that the parabola approach for frequency estimation has been exploited in [29]. However, that approach was applied in the frequency domain, where periodogram samples were considered. In this paper, not the FFT, but the CF itself is optimised with respect to the frequency.

### 4 | SIMULATION RESULTS

The goal of this section is to compare the statistical properties of the proposed parabolic method with the original standard sine fitting method suggested by [8]. The statistical properties are determined through the analysis of the estimation errors for each parameter of the sine wave. This requires the exact knowledge of the true parameters of the input sine, thus we decided to perform simulations. In these tests, the sine fitting was executed using both the proposed and the standard method on the same input, then the estimation errors were determined for each procedure. To reproduce real-life circumstances, the generated sine waves were influenced by the following disturbances:

- non-ideal quantisation,
- harmonic distortion,
- additive random noise.

To model the effect of the non-uniform quantisation, an artificial 14-bit quantiser was used. Its integral and differential non-linearity (INL and DNL) can be seen in Figure 2.

In addition to the quantisation, a harmonic component was also added to the signal on the double frequency of the excitation. The amplitude of the harmonic component is \(10^{-3}\) where FS is the full scale of the ADC. The additive random noise is of Gaussian distribution with standard deviation of \(\sigma = 10^{-3}\). First, the random noise and the harmonic component was added to the generated sine, and then the resulting vector was quantised:

\[
y_q = Q(y + h + n),
\]  \hspace{1cm} (22)

where \(Q(\cdot)\) is the quantisation operator, \(y_q\) is the quantised, noisy input on which the estimation methods are performed, \(y\) is the original pure sine wave, \(h\) denotes the harmonic component and \(n\) is the additive Gaussian noise.

The general form of the pure sine wave (y) is described by Equation (1). During the simulations, the offset and amplitude parameters are constant, \(C = 0\) and \(R = FS\), while the angular frequency and the initial phase are random variables, uniformly distributed in domains \([2\pi/N_1, 2\pi/N_2]\) and \([0, 2\pi]\), respectively. The selection of the bounds of the angular frequency is in accordance with the Nyquist theorem and with the IEEE 1241-2010 standard [8], which recommends at least five measured periods in the measured signal. In addition, due to the complex conjugate symmetry of real-valued signals, the sine must not be closer to the Nyquist frequency (bin \(N/2\) in the FFT) by five periods. By this means, the selection fully covers the available frequency range. A total number of 10,000 random sine wave inputs were generated, each of \(N = 10^5\) samples. The signal parameters were estimated with both estimators. The error of the estimations was determined, and finally the mean value and the standard deviation of the error were calculated for every parameter. Histograms of errors for different estimates are depicted in Figures 3–6, while the statistical properties of the estimators are summarised in Table 1 and Table 2.

Figures 3–6 show that the error of the two estimators behaves similarly for every parameter of the sine wave. The most important consequence of the tests is that despite the reduced computational burden (which will be demonstrated in Section 5) and the parabolic approximation, the quality of the two estimators is almost exactly the same, statistically significant difference could not be demonstrated. The statistical properties show that the estimation is biased for the amplitude and offset parameters for both methods (the mean value and the standard deviation are in the same order of magnitude). This could be explained by the non-ideal quantiser characteristics, which mainly influence these two parameters. Results show that the estimators of the initial phase and relative angular frequency can be regarded as unbiased.

The statistical properties of the estimators were also compared in the case when the number of measured periods is less than five. In the following simulations, the number of periods was uniformly distributed in [2, 5]. The other parameters of the signal and measurement conditions were not modified. Results are summarised in Table 3 and 4. Results
show that the estimators still provide similar statistical values, but the mean value of the frequency estimate and the standard deviation of each parameter is increased for both methods. This behaviour can be explained by the proximity of the second-order harmonic component, which is not taken into account in the model of the sine wave described in Equation (1).

For records consisting of less than two periods, the proposed algorithm is likely to diverge. This is caused by the imprecise angular frequency estimation of the IpFFT routine. In this case, the effect of the symmetric window component (which is not modelled in Equation (15)) is the main source of the error. Consequently, the usage of such a procedure for signals with less than two periods is not recommended. Note that imprecise initial estimation of the angular frequency also reduces the precision of the standard method. Consequently, this behaviour is not related to the sine fitting method itself.

Finally, we compared the proposed method to the Hanning window IpFFT procedure in terms of angular frequency estimation. The experimental environment was the same as in the comparison of the proposed and the standard estimator.
Results for 10,000 cases can be seen in Figure 7. Results show that the estimation error of the proposed method is significantly lower when compared with that of the IpFFT. Thus, the algorithm is able to find a better solution for the angular frequency parameter. The standard deviation of the error of the IpFFT method was \( \sigma_{\text{IpFFT}} = 1.4827 \cdot 10^{-10} \), while for the proposed method, it was \( \sigma_{\text{parab}} = 2.4873 \cdot 10^{-11} \). It should be noted that in real measurements, the accuracy of frequency estimation is unknown. The required accuracy should be determined by the user. If the accuracy of the IpFFT frequency estimate is sufficient, the LS3p method can be utilised [11]. Nevertheless, the IEEE Standard 1241-2010 suggests the IpFFT estimate as an initial estimator for the LS4p method [8]. Since this paper intends to implement this method in a computationally efficient way, the demonstration of the improvement in frequency estimation is reasonable.

5 | CALCULATION OF THE COMPUTATIONAL DEMAND

In this section, the proposed LS4p method will be compared with the standard one suggested by [8] from the point of computational demand. Since the proposed method is based on the LS3p fitting, this procedure will be analysed. The computational burden of the proposed method will be determined based on these results. The calculations will be performed for the following characteristic parts of the algorithm:

- Calculation of system matrix \( \mathbf{D}_0 \) or \( \mathbf{D}_1 \)
- Parameter estimation
- Sine wave fitting
- CF evaluation

5.1 | Calculation of system matrix \( \mathbf{D}_0 \) or \( \mathbf{D}_1 \)

To determine system matrix \( \mathbf{D}_0 \), \( N \) instantaneous phases has to be calculated, see Equation (7). After calculating \( \vartheta_0 \), it can be performed with \( N \) multiplications. Furthermore, to obtain the columns of the system matrix, \( 2N \) trigonometric evaluations are to be performed. It should be noted that for large \( N \) (\( N > 1000 \)), the computational demand of the calculation of \( \vartheta_0 \) can be neglected besides that of the instantaneous phases.

5.2 | Parameter estimation

To evaluate the parameter estimation of the proposed method, the LS3p fitting has to be evaluated four times. To solve (9), the following operations have to be carried out. \( \mathbf{D}_0^\top \mathbf{D}_0 \) contains nine elements, and \( \mathbf{D}_0^\top \mathbf{y} \) contains three elements. To obtain each element, \( N \) multiplications and \( N+1 \) additions are needed. The latter will be approximated by \( N \). For instance,

\[
(\mathbf{D}_0^\top \mathbf{D}_0)_{12} = \sum_{k=1}^{N} \cos(k\vartheta_0) \cdot \sin(k\vartheta_0). \tag{23}
\]

Compared with these operations, the computational demand of the remaining steps of Equation (9), that is, the multiplication of the resulting matrices and matrix inversion, is negligible if \( N \) is large (\( N > 1000 \)). The computational demand is therefore approximately \( 12N \) multiplications and \( 12N \) additions.

It can be shown similarly that in order to evaluate Equation (12), approximately \( 20N \) additions and \( 20N \) multiplications are needed in each iteration cycle, if \( N \) is large.

5.3 | Sine wave fitting

When the parameter estimation is finished, the estimated sine wave can be determined as

\[
\mathbf{y} = \mathbf{D}_0 \cdot [A_0 \ B_0 \ C_0]^\top. \tag{24}
\]
In this subsection, the computational demands calculated in subsections 5.1 and 5.4 are evaluated. The cost function is the sum of squared errors, that is

$$\text{CF}_{\theta} = e^T e = \sum_{k=1}^{N} (x_k - y_k)^2.$$ (25)

To evaluate this expression, the elements of the error can be obtained with $N$ multiplications (subtractions). Each element is to be squared. It can be achieved with $N$ multiplications. Finally, the summation can be performed with $N - 1$ additions, which will be approximated by $N$. To sum up, the number of needed operations at this stage is $2N$ additions and $N$ multiplications. This holds for both the three-parameter and the four-parameter fittings.

### 5.4 CF evaluation

In case of LS fittings, the cost function is the sum of squared errors, that is

$$\text{CF}_{\theta} = e^T e = \sum_{k=1}^{N} (x_k - y_k)^2.$$ (25)

To evaluate this expression, the elements of the error can be obtained with $N$ additions (subtractions). Each element is to be squared. It can be achieved with $N$ multiplications. Finally, the summation can be performed with $N - 1$ additions, which will be approximated by $N$. To sum up, the number of needed operations at this stage is $2N$ additions and $N$ multiplications. This holds for both the three-parameter and the four-parameter fittings.

### 5.5 Comparison of the computational demands

In this subsection, the computational demands calculated in the previous parts will be summarised. In the proposed method, the LS3p fitting is evaluated four times. Since the method is based on the evaluation of the CF at different frequencies, every step of the LS3p method needs to be performed four times. Contrarily, for the LS4p fitting, the actual fitting and the CF evaluation only need to be evaluated once, after the parameter set is determined. However, the system matrix evaluation and the parameter estimation steps are performed as many times as the number of iterations. The number of operations is summarised in Table 5.

The analysis shows that even for four iterations, the computational demand of the proposed method is smaller than that of the standard LS4p procedure. However, the number of needed iteration cycles is only vaguely defined in [8]. The standard suggests that six iteration cycles are enough to get accurate LS estimators. Thus, only an upper bound can be determined for the number of operations. The advantage of the proposed method is that it is non-iterative. Hence, its computational demand can be predicted precisely, contrarily to the standard method. Besides, since the proposed method is based on LS3p fittings, it is proven to be well conditioned, see [15]. Therefore, numerical stability is guaranteed, which does not hold for the standard LS4p method. For a special but common case, the computational demand of Equation (9) can be significantly reduced by applying methods proposed in [15].

### Table 5

| Multiplications | Additions | Trigonometric Evaluations |
|-----------------|----------|--------------------------|
| Calculation of the system matrix | $4N/5N \cdot i$ | $0/N \cdot i$ | $8N/2N \cdot i$ |
| Parameter estimation | $48 N/20N \cdot i$ | $48 N/20N \cdot i$ | $0/0$ |
| Sine fitting | $12 N/3N$ | $8N/2N$ | $0/0$ |
| CF evaluation | $4N/N$ | $8N/2N$ | $0/0$ |
| Sum | $68 N/(25N \cdot i + 4N)$ | $64 N/(20N \cdot i + 4N)$ | $8N/2N \cdot i$ |

Abbreviation: CF, cost function.

For this aim, $3N$ multiplications and $2N$ additions are needed. Note that $D_0$ has already been evaluated in Subsection 5.1. After the parameters are determined, the same operations have to be performed in the case of the LS4p fitting. In that case, the parameter vector contains $A_i$, $B_i$, and $C_i$ respectively.

### 6 REAL MEASUREMENT RESULTS

The aim of this section is to present the suggested method’s ability to provide accurate results with less computational demand on a microcontroller device. For this purpose, the original and the proposed methods were implemented on an
STM32 Nucleo F446RE board in C language. This board contains an ARM Cortex M4 32-bit microcontroller unit (MCU) with up to 180 MHz frequency, 512 kByte Flash memory, and 128 kByte RAM. DSP instructions are supported by the hardware. During the implementation of the algorithms, we used the standard CMSIS library that provides optimised C language implementations for the computations with vectors and matrices, and also for the FFT [30]. We used the following 32-bit floating-point functions in the implementations:

- `arm_mult_f32()` for multiplying vectors,
- `arm_cfft_f32()` for the calculation of the FFT,
- `arm_emplx_mag_f32()` for the calculation of the absolute value of the result of the FFT,
- `arm_max_f32()` for finding the maximum element of a vector,
- `arm_mat_init_f32()` for initialisation of matrices,
- `arm_mat_trans_f32()` for matrix transpose,
- `arm_mat_mult_f32()` for matrix multiplication,
- `arm_mat_inverse_f32()` for matrix inversion.

Using the above functions, the standard four-parameter sine wave fit was implemented along with the proposed non-iterative method. The performance was compared by means of execution times. A sine wave was measured using the ADC of the MCU, and the parameters were estimated. Four and six iterations were used in the standard LS4p algorithm. Results can be seen in Figure 8. Results show that the proposed method’s computational burden was 85% of the original method’s execution time for four iterations. In case of six iterations, the parabolic method is about twice as fast as the original estimator (the computational costs were lower by 44%). The outcome is in accordance with the theoretical results of Table 5.

7 | CONCLUSION

In this paper, a novel time-domain sine wave fitting method was presented. It was highlighted that the four-parameter method defined in the IEEE Standard 1241-2010 is iterative due to the estimation of frequency. By exploiting the fact that when near to the optimum, the LS cost function is approximately of second-order, it was proposed that the frequency of the signal should be evaluated separately from the other three parameters. This can be achieved by performing three-parameter estimations and fitting a parabola to the results. To have a well-conditioned parabola fitting, its parameters were slightly modified. After the frequency estimator was obtained, the other three parameters were determined by applying one last three-parameter fitting. Simulation results showed that the proposed method has the same accuracy as the one defined in the IEEE Standard 1241-2010 [8]. However, its computational demand is be much lower if at least four iterations are needed in the latter case. Contrary to the method defined in the quoted standard, the computational burden of the proposed method can be predicted precisely, which is an important aspect for real-life applications. Furthermore, the proposed method is guaranteed to be well conditioned. Therefore, it outperforms the standard procedure from the point of numerical stability as well. Considering these advantages, the algorithm may be adequate to support the prospective revision of the IEEE Standard 1241-2010. MATLAB and C implementations of the proposed method are available at [28].

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