The Trajectory-Coherent Approximation and the System of Moments for the Hartree Type Equation

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Abstract

The general construction of quasi-classically concentrated solutions to the Hartree type equation, based on the complex WKB–Maslov method, is presented. The formal solutions of the Cauchy problem for this equation, asymptotic in small parameter $\hbar (\hbar \to 0)$, are constructed with a power accuracy of $O(\hbar^{N/2})$, where $N$ is any natural number. In constructing the quasi-classically concentrated solutions, a set of Hamilton–Ehrenfest equations (equations for centered moments) is essentially used. The nonlinear superposition principle has been formulated for the class of quasi-classically concentrated solutions of Hartree type equations. The results obtained are exemplified by a one-dimensional Hartree type equation with a Gaussian potential.

Introduction

The nonlinear Schrödinger equation

$$\{-i\partial_t + \hat{H}(t, |\Psi|^2)\} \Psi = 0,$$

(0.1)

where $\hat{H}(t, |\Psi|^2)$ is a nonlinear operator, arises in describing a broad spectrum of physical phenomena. In statistical physics and quantum field theory, the generalized model of the evolution of bosons is described in terms of the second quantization formalism by the Schrödinger equation \[\] which, in Hartree’s approximation, leads to the classical multidimensional Schrödinger equation with a nonlocal nonlinearity for one-particle functions, i.e., a Hartree type equation.

The quantum effects associated with the propagation of an optical pulse in a nonlinear medium are also described in the second quantization formalism by the one-dimensional Schrödinger equation with a delta-shaped interaction potential. In this case, the Hartree approximation results in the classical nonlinear Schrödinger equation \[\] which is integrated by the Inverse Scattering Transform (IST) method and has soliton solutions \[\]. Solitons are localized wave packets propagating without distortion and interacting elastically in mutual collisions. The soliton theory has found wide application in various fields of nonlinear physics \[\].

Investigations of the statistical properties of optical fields have led to the concept of compressed states of a field in which quantum fluctuations are minimized and the highest possible accuracy of optical measurements is achieved. The important problem of the correspondence between the stressed states describing the quantum properties of a radiation and the optical solitons is analyzed in \[\].

The Hartree type equation is nonintegrable by the IST method. Nevertheless, approximate solutions showing some properties characteristic of solitons can be constructed. Solutions of this type are referred to as solitary waves or “quasi-solitons” to differentiate them from the solitons (in the strict sense) arising in IST integrable models.
An efficient method for constructing solutions of this type is offered by the technique of quasi-classical asymptotics. Thus, for nonlinear operators of the self-consistent field type, the theory of canonical operators with a real phase has been constructed for the Cauchy problem in 3, 10 and for spectral problems, including those with singular potentials, in 11, 12 (see also 13, 15, 16). Solitonlike solutions of the Hartree type equation and some types of interaction potentials have also been constructed in 17.

In this paper, localized solutions asymptotical in small parameter \( h (h \to 0) \) for the (nonlinear) Hartree type equation are constructed using the so-called WKB method or the Maslov complex germ theory 18, 19, 20. The constructed solutions are a generalization of the well-known quantum mechanical coherent and compressed states for linear equations 21, 22 for the case of nonlinear Hartree type equations with variable coefficients. We refer to the corresponding asymptotical solutions, like in the linear case 21, as quasi-classically concentrated solutions (or states).

Most typical of solitary waves (“quasi-solitons”) is that they show some properties characteristic of particles. For the “quasi-solitons” being quasi-classically concentrated states of a Hartree type equation these properties are represented by a dynamic set of ordinary differential equations for the “quantum” means \( \bar{X}(t, h) \) and \( \bar{P}(t, h) \) of the operators of coordinates \( \hat{x} \) and momenta \( \hat{p} \) and for the centered higher-order moments. In the limit of \( h \to 0 \), the centroid of such a quasi-soliton moves in the phase space along the trajectory of this dynamic system: at each point in time, the quasi-classically concentrated state is efficiently concentrated in the neighborhood of the point \( \bar{X}(t, 0) \) (in the \( x \) representation) and in the neighborhood of the point \( \bar{P}(t, 0) \) (in the \( p \) representation). Note that a similar set of equations in quantum means has been obtained in 19, 21 for the linear case (Schrödinger equation) and in 22 for a more general case. It has been shown 22, 23 that these equations are Poisson equations with respect to the (degenerate) nonlinear Dirac bracket. Therefore, we call the equations in quantum means for the Hartree type equation, like in the linear case 21, Hamilton–Ehrenfest equations. The Hamiltonian character of these equations is the subject of a special study. Nevertheless, it should be noted that, as distinct from the linear case, the construction of the quasi-classically concentrated states for the Hartree type equation essentially uses the solutions of the Hamilton–Ehrenfest equations.

The specificity of the Hartree type equation, where nonlinear terms are only under the integral sign, is that it shows some properties inherent in linear equations. In particular, it has been demonstrated that for the class of quasi-classically concentrated solutions of this type of equation (with a given accuracy \( h, h \to 0 \)), the nonlinear superposition principle is valid.

In terms of the approach under consideration, the formal asymptotical solutions of the Cauchy problem for this equation and the evolution operator have been constructed in the class of trajectory-concentrated functions, allowing any accuracy in small parameter \( h, h \to 0 \).

It should be stressed that throughout the paper we deal with the construction of the formal asymptotical solutions to the Hartree type equation with the residual whose norm has a small estimate in parameter \( h, h \to 0 \). To substantiate these asymptotics for finite times \( t \in [0, T], T = \text{const} \), is a special nontrivial mathematical problem. This problem is concerned with obtaining a priori estimates for the solution of nonlinear equation 1, 2, which are uniform in parameter \( h \in [0, 1] \), and is beyond the scope of the present work. Note that, in view of the heuristic considerations given in 3, it seems that the difference between the exact and the constructed formal asymptotical solution can be found with the use of the method developed in 1, 2.

This paper is arranged as follows: The first section gives principal notions and definitions. In the second section, a class of trajectory-concentrated functions is specified and the simplest properties of these functions are considered. In the third section, Hamilton–Ehrenfest equations are constructed which describe the “particle-like” properties of the quasi-classically concentrated solutions of the Hartree type equation. In the fourth section, the Hartree type equation is linearized for the solutions of the Hamilton–Ehrenfest equations, and a set of associated linear equations which determine the asymptotical solution of the starting problem is obtained. In the fifth section, we construct, accurate to \( O(h^{3/2}) \), quasi-classical coherent solutions to the Hartree type equation. In the sixth section, the principal term of the quasi-classical asymptotic of this equation is obtained in a class of quasi-classically concentrated functions. The quasi-classically concentrated solutions to the Hartree type equation are constructed with an arbitrary accuracy in \( \sqrt{h} \) in Sec. 7, while the kernel of the evolution operator (Green function) of the Hartree type equation is constructed in Sec. 8. Herein, the nonlinear superposition principle is substantiated for the class of quasi-classically concentrated solutions. In the ninth section, the Hartree type equation with a Gaussian potential is considered as an example. Appendix A presents the properties of the solutions to a set of equations in variations necessary to construct the asymptotical solutions and the approximate evolution operator to the Hartree type equation.
The operators

\[
\hat{H}(t) = \mathcal{H}(\hat{z}, t),
\]

\[
\hat{V}(t, \Psi) = \int_{\mathbb{R}^n} d\hat{y} \Psi^*(\hat{y}, t)V(\hat{z}, \hat{w}, t)\Psi(\hat{y}, t)
\]

are functions of the noncommuting operators

\[
\hat{z} = (-i\hbar \frac{\partial}{\partial \xi}, \xi), \quad \hat{w} = (-i\hbar \frac{\partial}{\partial \eta}, \eta), \quad \xi, \eta \in \mathbb{R}^n,
\]

the function \(\Psi^*\) is complex conjugate to \(\Psi\), \(\alpha\) is a real parameter, and \(\hbar\) is a “small parameter”, \(\hbar \in [0, 1]\). For the operators \(\hat{z}\) and \(\hat{w}\), the following commutative relations are valid:

\[
[\hat{z}_k, \hat{z}_j]_\text{\(\alpha\)} = [\hat{w}_k, \hat{w}_j]_\text{\(\alpha\)} = i\hbar J_{kj}, \quad k, j = 1, 2n,
\]

\[
[\hat{z}_k, \hat{w}_j]_\text{\(\alpha\)} = 0,
\]

where \(J = ||J_{kj}||_{2n \times 2n}\) is a unit symplectic matrix

\[
J = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}_{2n \times 2n}.
\]

For the functions of noncommuting variables, we use the Weyl ordering \([27, 28]\). In this case, we can write, for instance, for the operator \(\mathcal{H}\):

\[
\mathcal{H}(t)\Psi(\hat{x}, t, \hbar) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} d\hat{p} d\hat{\delta} \exp \left(\frac{i}{\hbar} (\langle \hat{p}, \hat{x} \rangle - \hat{\delta}) \right) \mathcal{H}(\hat{p}, \frac{\hat{x} + \hat{\delta}}{2}, t) \Psi(\hat{p}, \hat{\delta}, t, \hbar), \quad (1.6)
\]

where \(\mathcal{H}(z, t) = \mathcal{H}(\hat{p}, \xi, t)\) is the Weyl symbol of the operator \(\mathcal{H}(t)\) and \(\langle , , \rangle\) is the Euclidean scalar product of the vectors

\[
\langle \hat{p}, \hat{x} \rangle = \sum_{j=1}^{n} p_j x_j, \quad \hat{p}, \hat{x} \in \mathbb{R}^n, \quad \langle z, w \rangle = \sum_{j=1}^{2n} z_j w_j, \quad z, w \in \mathbb{R}^{2n}.
\]

We here are interested in localized solutions of equation (1.1) for each fixed \(\hbar \in [0, 1]\) and \(t \in \mathbb{R}\), belonging to the Schwartz space with respect to the variable \(\hat{x} \in \mathbb{R}^n\). For the operators \(\mathcal{H}(t)\) and \(V(t, \Psi)\) to be at work in this space, it is sufficient that their Weyl symbols \(\mathcal{H}(z, t)\) and \(V(z, w, t)\) be smooth functions\(^1\) and grow, together with their derivatives, with \(|z| \to \infty\) and \(|w| \to \infty\) no more rapidly as the polynomial and uniformly in \(t \in \mathbb{R}\). Therefore, we believe that the following conditions for the functions \(\mathcal{H}(z)\) and \(V(z, w, t)\) are satisfied:

**Supposition 1** For any multi-indices \(\alpha, \beta, \mu, \) and \(\nu\) there exist constant \(C_{\beta}^\alpha(T)\) and \(C_{\beta \mu}^{\alpha}(T)\), such that the inequalities

\[
|z^\alpha \frac{\partial |\beta| \mathcal{H}(z, t)}{\partial z^\beta}| \leq C_{\beta}^\alpha(T),
\]

\[
|z^\alpha w^\mu \frac{\partial |\beta + \nu| V(z, w, t)}{\partial z^\beta \partial w^\nu}| \leq C_{\beta \mu}^{\alpha}(T),
\]

are fulfilled.

\(^1\)In what follows we assume that for all the operators under consideration, \(\hat{A} = A(\hat{z}, t)\), their Weyl symbols satisfy Supposition 1.
Here, \( \alpha, \beta, \mu, \nu \) are multi-indices \((\alpha, \beta, \mu, \nu \in \mathbb{Z}_2^n)\) defined as
\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n}), \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_{2n},
\]
\[
\frac{\partial^{|\alpha|} V(z)}{\partial z^\alpha} = \frac{\partial^{\alpha_1} \partial^{\alpha_2} \ldots \partial^{\alpha_{2n}} V(z)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \ldots \partial z_n^{\alpha_{2n}}}, \quad \alpha_j = 0, \infty, \quad j = 1, 2n.
\]

We are coming now to the description of the class of functions for which we shall find asymptotical solutions to equation (1.1).

2 The class of trajectory-concentrated functions

Let us introduce a class of functions singularly depending on a small parameter \( \hbar \), which is a generalization of the notion of a solitary wave. It appears that asymptotical solutions to equation (1.1) can be constructed based on functions of this class, which depend on the phase trajectory \( z = Z(t, \hbar) \), the real function \( S(t, \hbar) \) (analogous to the classical action at \( \varkappa = 0 \) in the linear case), and the parameter \( \hbar \). For \( \hbar \to 0 \) the functions of this class are concentrated in the neighborhood of a point moving along a given phase curve \( z = Z(t, 0) \). Functions of this type are well known in quantum mechanics. In particular, among these are coherent and “compressed” states of quantum systems with a quadric Hamiltonian \([24, 25, 29, 30, 31, 32, 33, 34, 35, 36]\). Note that the soliton solution localized only with respect to spatial (but not momentum) variables does not belong to this class.

Let us denote this class of functions as \( \mathcal{P}_h^l(Z(t, \hbar), S(t, \hbar)) \) and define it as
\[
\mathcal{P}_h^l = \mathcal{P}_h^l(Z(t, \hbar), S(t, \hbar)) = \left\{ \Phi : \Phi(\vec{x}, t, \hbar) = \varphi \left( \frac{\Delta \vec{x}}{\sqrt{\hbar}} , t \right) \exp \left[ \frac{i}{\hbar} \left( S(t, \hbar) + \langle \vec{P}(t, \hbar), \Delta \vec{x} \rangle \right) \right] \right\},
\]
(2.1)

where the function \( \varphi(\vec{z}, t, \hbar) \) belongs to the Schwartz space \( \mathcal{S} \) in variable \( \vec{z} \in \mathbb{R}^n \) and depends smoothly on \( t \) and regularly on \( \sqrt{\hbar} \) for \( \hbar \to 0 \). Here, \( \Delta \vec{x} = \vec{x} - \vec{X}(t, \hbar) \), and the real function \( S(t, \hbar) \) and the \( 2n \)-dimensional vector function \( Z(t, \hbar) = (\vec{P}(t, \hbar), \vec{X}(t, \hbar)) \), which characterize the class \( \mathcal{P}_h^l(Z(t, \hbar), S(t, \hbar)) \), depend regularly on \( \sqrt{\hbar} \) in the neighborhood of \( \hbar = 0 \) and are to be determined. In the cases where this does not give rise to ambiguity, we use a shorthand symbol of \( \mathcal{P}_h^l \) for \( \mathcal{P}_h^l(Z(t, \hbar), S(t, \hbar)) \).

The functions of the class \( \mathcal{P}_h^l \) are normalized to
\[
\| \Phi(t) \|^2 = \langle \Phi(t) | \Phi(t) \rangle
\]
in the space \( L_2(\mathbb{R}^n) \) with the scalar product
\[
\langle \Psi(t) | \Phi(t) \rangle = \int_{\mathbb{R}^n} d\vec{x} \Psi^* (\vec{x}, t, \hbar) \Phi(\vec{x}, t, \hbar).
\]

In the subsequent manipulation, the argument \( t \) in the expression for the norm may be omitted: \( \| \Phi(t) \|^2 = \| \Phi \|^2 \).

In constructing asymptotical solutions, it is useful to define, along with the class of functions \( \mathcal{P}_h^l(Z(t, \hbar), S(t, \hbar)) \), the following class of functions
\[
\mathcal{C}_h^l(Z(t, \hbar), S(t, \hbar)) = \left\{ \Phi : \Phi(\vec{x}, t, \hbar) = \varphi \left( \frac{\Delta \vec{x}}{\sqrt{\hbar}} , t \right) \exp \left[ \frac{i}{\hbar} \left( S(t, \hbar) + \langle \vec{P}(t, \hbar), \Delta \vec{x} \rangle \right) \right] \right\},
\]
(2.2)

where the functions \( \varphi \), as distinct from (2.1), are independent of \( \hbar \).

At any fixed point in time \( t \in \mathbb{R}^1 \), the functions belonging to the class \( \mathcal{P}_h^l \) are concentrated, in the limit of \( \hbar \to 0 \), in the neighborhood of a point lying on the phase curve \( z = Z(t, 0) \), \( t \in \mathbb{R}^1 \) (the sense of this property is established exactly in theorems 2.1, 2.2 below). Therefore, it is natural to refer to the functions of the class \( \mathcal{P}_h^l \) as trajectory-concentrated functions. The definition of the class of trajectory-concentrated functions
includes the phase trajectory $Z(t, \hbar)$ and the scalar function $S(t, \hbar)$ as free “parameters”. It appears that these “parameters” are determined unambiguously from the Hamilton–Ehrenfest equations (see Sec. 3) fitting the nonlinear $(x \neq 0)$ Hamiltonian of equation (1.6). Note that for a linear Schrödinger equation, in the limiting case of $x = 0$, the principal term of the series in $\hbar \to 0$ determines the phase trajectory of the Hamilton system with the Hamiltonian $\mathcal{H}(\vec{p}, \vec{x}, t)$, and the function $S(t, 0)$ is the classical action along this trajectory. In particular, in this case, the class $P^1_\hbar$ includes the well-known dynamic (compressed) coherent states of quantum systems with quadric Hamiltonians when the amplitude of $\varphi$ in (2.4) is taken as a Gaussian exponential:

$$
\varphi(\vec{\xi}, t) = \exp \left[ \frac{i}{2} (\vec{\xi}, Q(t) \vec{\xi}) \right] f(t),
$$

where $Q(t)$ is a complex symmetrical matrix with a positive imaginary part, and the time factor is given by

$$
f(t) = \sqrt{\text{Im} \ Q(t)} \exp \left[ - \frac{i}{2} \int_0^t \text{Im} \ Q(\tau) \, d\tau \right]
$$

(see for details [21]).

Let us consider the principal properties of the functions of the class $P^1_\hbar(Z(t, \hbar), S(t, \hbar))$, which are also valid for those of the class $C^0_\hbar(Z(t, \hbar), S(t, \hbar))$.

**Theorem 2.1** For the functions of the class $P^1_\hbar(Z(t, \hbar), S(t, \hbar))$, the following asymptotical estimates are valid for centered moments $\Delta_\alpha(t, \hbar)$ of order $|\alpha|$, $\alpha \in \mathbb{Z}_+^n$:

$$
\Delta_\alpha(t, \hbar) = \frac{\langle \Phi| (\Delta \hat{\zeta})^\alpha |\Phi \rangle}{||\Phi||^2} = O(|\hbar|^{n/2}), \quad \hbar \to 0.
$$

(2.3)

Here, $\{\Delta \hat{\zeta}\}^\alpha$ denotes the operator with the Weyl symbol $(\Delta \zeta)^\alpha$,

$$
\Delta \zeta = z - Z(t, \hbar) = (\Delta \vec{p}, \Delta \vec{x}), \quad \Delta \vec{p} = \vec{p} - \vec{P}(t, \hbar), \quad \Delta \vec{x} = \vec{x} - \vec{X}(t, \hbar).
$$

**Proof.** The operator symbol $\{\Delta \hat{\zeta}\}^\alpha$ can be written as

$$
(\Delta \zeta)^\alpha = (\Delta \vec{p})^{\alpha_\eta} (\Delta \vec{x})^{\alpha_x}, \quad (\alpha_\eta, \alpha_x) = \alpha,
$$

and, hence, according to the definition of Weyl-ordered pseudodifferential operators (1.6), we have for the mean value $\sigma_\alpha(t, \hbar)$ of the operator $\{\Delta \hat{\zeta}\}^\alpha$:

$$
\sigma_\alpha(t, \hbar) = \langle \Phi| (\Delta \hat{\zeta})^\alpha |\Phi \rangle = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} d\vec{y} d\vec{\eta} d\vec{p} \Phi^*(\vec{x}, t, \hbar) \times \exp \left( \frac{i}{\hbar} ((\vec{x} - \vec{y}), \vec{p}) \right) [\Delta \vec{p}]^{\alpha_\eta} \left( \frac{\Delta \vec{x} + \Delta \vec{y}}{2} \right)^{\alpha_x} \Phi(\vec{y}, t, \hbar).
$$

Here, we have denoted

$$
\Delta \vec{y} = \vec{y} - \vec{X}(t, \hbar).
$$

After the change of variables

$$
\Delta \vec{x} = \sqrt{\hbar} \vec{\xi}, \quad \Delta \vec{y} = \sqrt{\hbar} \vec{\zeta}, \quad \Delta \vec{p} = \sqrt{\hbar} \vec{\omega}
$$

and taking into consideration the implicit form of the functions

$$
\Phi(\vec{x}, t, \hbar) = \exp \{i/\hbar(S(t, \hbar) + \langle \vec{P}(t, \hbar), \Delta \vec{x} \rangle) \} \varphi \left( \frac{\Delta \vec{x}}{\sqrt{\hbar}}, t, \hbar \right),
$$

belonging to the class $P^1_\hbar(Z(t, \hbar), S(t, \hbar))$, we find

$$
\sigma_\alpha(t, \hbar) = \frac{1}{(2\pi \hbar)^n} \hbar^{3n/2} h^{n|\alpha|/2 - |\alpha_\eta|} \int_{\mathbb{R}^n} d\vec{\xi} d\vec{\zeta} d\vec{\omega} \varphi^*(\vec{\xi}, t, \hbar) \times \exp \{i(\vec{\xi} - \vec{\zeta}, \vec{\omega}) \} \vec{\omega}^{\alpha_\eta} \vec{\xi}^{\alpha_x} \varphi(\vec{\zeta}, t, \hbar) = \hbar^{(n + |\alpha|)/2} M_\alpha(t, \hbar),
$$

$$
||\Phi||^2 = \hbar^{n/2} \int_{\mathbb{R}^n} d\vec{\xi} \varphi^*(\vec{\xi}, t, \hbar) \varphi(\vec{\xi}, t, \hbar) = \hbar^{n/2} M_0(t, \hbar).
$$
Since $\varphi(\xi, t, h)$ depends on $\sqrt{h}$ regularly and $M_0(t, h) > 0$, we get
\[
\Delta_{\alpha}(t, h) = \frac{\sigma_{\alpha}(t, h)}{\|\Phi\|^2} = h^{\alpha/2} \frac{M_0(t, h)}{M_0(t, h)} \leq h^{\alpha/2} \max_{t \in [0, T]} \frac{M_0(t, h)}{M_0(t, h)} = O(h^{\alpha/2}),
\]
and thus the theorem is proved.

Let us denote by the symbol $\hat{O}(h^\alpha)$ an operator $\hat{F}$, such that for any function $\Phi$ belonging to the space $\mathcal{P}_h^k(Z(t, h), S(t, h))$ the asymptotical estimate
\[
\frac{\|\hat{F}\Phi\|}{\|\Phi\|} = O(h^\alpha), \quad h \to 0,
\]
is valid.

**Theorem 2.2** For the functions belonging to $\mathcal{P}_h^k(Z(t, h), S(t, h))$, the following asymptotical estimates are valid:
\[
\{\Delta_{\alpha}\}^\alpha = \hat{O}(h^{\alpha/2}), \quad \alpha \in \mathbb{Z}_+^n, \quad h \to 0. \tag{2.5}
\]

**Proof** is similar to that of relation (2.3).

**Corollary 2.2.1** For the functions belonging to $\mathcal{P}_h^k(Z(t, h), S(t, h))$, the following asymptotical estimates are valid:
\[
\{\Delta_{\alpha}\}^\alpha = \hat{O}(h^{\alpha/2}), \quad \alpha \in \mathbb{Z}_+^n, \quad h \to 0. \tag{2.6}
\]

Let us consider an arbitrary function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^k(Z(t, h), S(t, h))$ and from the estimates (2.3).

**Theorem 2.3** For any function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^k(Z(t, h), S(t, h))$, the limiting relations
\[
\lim_{h \to 0} \frac{1}{\|\Phi\|^2} |\Phi(\vec{x}, t, h)|^2 = \delta(\vec{x} - \vec{X}(t, 0)), \tag{2.8}
\]
\[
\lim_{h \to 0} \frac{1}{\|\Phi\|^2} |\Phi(\vec{p}, t, h)|^2 = \delta(\vec{p} - \vec{P}(t, 0)), \tag{2.9}
\]
where $\Phi(\vec{p}, t, h) = F_{h, \vec{x} \to \vec{p}}\Phi(\vec{x}, t, h)$, $F_{h, \vec{x} \to \vec{p}}$ is the $h^{-1}$ Fourier transform $[18]$, are valid.

**Proof.** Let us consider an arbitrary function $\phi(\vec{x}) \in S$. Then for any function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^k$ the integral
\[
\langle \frac{|\Phi(t, h)|^2}{\|\Phi(t, h)\|^2} |\phi \rangle = \frac{1}{\|\Phi(t, h)\|^2} \int_{\mathbb{R}^n} \phi(\vec{x}) |\Phi(\vec{x}, t, h)|^2 \, d\vec{x} = \frac{1}{\|\varphi(t, h)\|^2} \int_{\mathbb{R}^n} \phi(\vec{x}) |\varphi(\Delta_{\vec{\xi}}/\sqrt{h}, t)|^2 \, d\vec{x}
\]
after the change of variables $\vec{\xi} = \Delta_{\vec{\xi}}/\sqrt{h}$, becomes
\[
\langle |\Phi(t, h)|^2 |\phi \rangle = \frac{h^{n/2}}{\|\varphi(t, h)\|^2} \int_{\mathbb{R}^n} \phi(\vec{X}(t, h) + \sqrt{h}\vec{\xi}) |\varphi(\vec{\xi}, t, h)|^2 \, d\vec{\xi}.
\]
Let us pass in the last equality to the limit of $h \to 0$, and, in view of
\[
\|\varphi(t, h)\|^2 = h^{n/2} \int_{\mathbb{R}^n} |\varphi(\vec{\xi}, t, h)|^2 \, d\vec{\xi},
\]

\[6\]
and a regular dependence of the function $\varphi(\vec{z}, t, h)$ on $\sqrt{h}$, we arrive at the required statement.

The proof of relation (2.9) is similar to the previous one if we notice that the Fourier transform of the function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^t$ can be represented as

$$\hat{\Phi}(\vec{p}, t, h) = \exp\left\{ \frac{i}{\hbar} [S(t, h) - \langle \vec{p}, \vec{X}(t, h) \rangle] \right\} \varphi\left( \frac{\vec{p} - \vec{P}(t, h)}{\sqrt{\hbar}}, t, h \right),$$

where

$$\varphi(\vec{z}, t, h) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \vec{z}, \vec{\xi} \rangle} \varphi(\vec{z}, t, h) d\vec{\xi}.$$ 

Denote by $\langle \hat{L}(t) \rangle$ the mean value of the operator $\hat{L}(t)$, $t \in \mathbb{R}$, self-conjugate in $L^2(\mathbb{R}^n)$, calculated from the function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^t$. Then the following corollary is valid:

**Corollary 2.3.2** For any function $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^t(Z(t, h), S(t, h))$ and any operator $\hat{A}(t, h)$ whose Weyl symbol $A(z, t, h)$ satisfies Supposition 1, the equality

$$\lim_{\hbar \to 0} \langle \hat{A}(t, h) \rangle = \lim_{\hbar \to 0} \frac{1}{|\Phi|^2} \langle \Phi(\vec{x}, t, h) | \hat{A}(t, h) | \Phi(\vec{x}, t, h) \rangle = A(Z(t, 0), t, 0)$$

is valid.

**Proof** is similar to that of relations (2.8) and (2.9).

Following [21], we introduce

**Definition 2.1** We refer to the solution $\Phi(\vec{x}, t, h) \in \mathcal{P}_h^t$ of equation (1.3) as quasi-classically concentrated on the phase trajectory $Z(t, h)$ for $h \to 0$, provided that the conditions (2.8) and (2.9) are fulfilled.

**Remark 2.1** The above estimates (2.5) of operators $\{\Delta z\}^\alpha$ allow a consistent expansion of the functions of the class $\mathcal{P}_h^t(Z(t, h), S(t, h))$ and the operator of equation (1.3) in a power series for $\sqrt{\hbar}$. This expansion gives rise to a set of recurrent equations which determine the sought-for asymptotical solution of equation (1.3).

For any function $\Phi \in \mathcal{P}_h^t(Z(t, h), S(t, h))$, the representation

$$\Phi(\vec{x}, t, h) = \sum_{k=0}^N \hbar^{k/2} \Phi^{(k)}(\vec{x}, t, h) + O(h^{(N+1)/2}),$$

where $\Phi^{(k)}(\vec{x}, t, h) \in \mathcal{C}_h^t(Z(t, h), S(t, h))$, is valid. Representation (2.11) naturally induces the expansion of the space $\mathcal{P}_h^t(Z(t, h), S(t, h))$ in a direct sum of subspaces

$$\mathcal{P}_h^t(Z(t, h), S(t, h)) = \bigoplus_{l=0}^\infty \mathcal{P}_h^t(Z(t, h), S(t, h), l).$$

Here, the functions $\Phi \in \mathcal{P}_h^t(Z(t, h), S(t, h), l) \subset \mathcal{P}_h^t(Z(t, h), S(t, h))$, according to (2.2), have estimates by the norm

$$\frac{1}{h^{n/2}} \|\Phi\|_{L^2(\mathbb{R}^n)} = h^{1/2} \mu(t),$$

where the function $\mu(t)$ is independent of $\hbar$ and continuously differentiable with respect to $t$.

Similar to the proof of the estimates (2.5) and (2.6), it can be shown that the operators

$\{\Delta z\}^\alpha, \{-i\hbar \partial_t - \hat{S}(t, h) + \langle \hat{P}(t, h), \hat{\vec{X}}(t, h) \rangle + \langle \hat{Z}(t, h), J \Delta \hat{z} \rangle \}$

do not disrupt the structure of the expansion (2.11), (2.12), and

$$\{\Delta z\}^\alpha : \mathcal{P}_h^t(Z(t, h), S(t, h), l) \to \mathcal{P}_h^t(Z(t, h), S(t, h), l + |\alpha|),$$

$$\{-i\hbar \partial_t - \hat{S}(t, h) + \langle \hat{P}(t, h), \hat{\vec{X}}(t, h) \rangle + \langle \hat{Z}(t, h), J \Delta \hat{z} \rangle \} :$$

$$: \mathcal{P}_h^t(Z(t, h), S(t, h), l) \to \mathcal{P}_h^t(Z(t, h), S(t, h), l + 2).$$
Remark 2.2 From Corollary 2.3.3 it follows that the solution \( \Psi(\vec{x}, t, \hbar) \) of equation (1.1), belonging to the class \( P_\hbar^h \), is quasi-classically concentrated.

The limiting character of the conditions (2.3) and (2.4) and the asymptotical character of the estimates (2.3)–(2.6) valid for the class of trajectory-concentrated functions make it possible to construct quasi-classically concentrated solutions to the Hartree type equation not exactly, but approximately. In this case, the \( L_2 \) norm of the error has an order of \( \hbar^\alpha, \alpha > 1 \) for \( \hbar \to 0 \) on any finite time interval \([0, T]\). Denote such an approximate solution as \( \Psi_{as} = \Psi_{as}(\vec{x}, t, \hbar) \). This solution satisfies the following problem:

\[
\begin{align*}
&\left[-i\hbar \frac{\partial}{\partial t} + \hat{H}(t) + z\hat{V}(t, \Psi_{as})\right] \Psi_{as} = O(\hbar^\alpha), \\
&\Psi_{as} \in P_\hbar^h(Z(t, \hbar), S(t, \hbar), \hbar), \quad t \in [0, T],
\end{align*}
\]

where \( O(\hbar^\alpha) \) denotes the function \( g^{(\alpha)}(\vec{x}, t, h) \), the “residual” of equation (1.1). For the residual, the following estimate is valid:

\[
\max_{0 \leq t \leq T} \|g^{(\alpha)}(\vec{x}, t, \hbar)\| = O(\hbar^\alpha), \quad \hbar \to 0.
\]

Below we refer to the function \( \Psi_{as}(\vec{x}, t, \hbar) \) satisfying the problem (2.13)–(2.17) as a quasi-classically concentrated solution (mod \( \hbar^\alpha \), \( \hbar \to 0 \)) of the Hartree type equation (1.1).

The main goal of this work is to construct quasi-classically concentrated solutions to the Hartree type equation (1.1) with any degree of accuracy in small parameter \( \sqrt{\hbar} \), \( \hbar \to 0 \), i.e., functions \( \Psi_{as}(\vec{x}, t, \hbar) = \Psi^{(N)}(\vec{x}, t, \hbar) \) satisfying the problem (2.18)–(2.17) in mod(\( \hbar^{(N+1)/2} \)), where \( N \geq 2 \) is any natural number.

Thus, the quasi-classically concentrated solutions \( \Psi^{(N)}(\vec{x}, t, \hbar) \) of the Hartree type equation describe approximately the evolution of the initial state \( \Psi_0(\vec{x}, \hbar) \) if the latter has been taken from a class of trajectory-concentrated functions \( P_\hbar^0 \). The operators \( \hat{H}(t) \) and \( \hat{V}(t, \Psi) \) entering in the Hartree type equation (1.1) leave the class \( P_\hbar^h \) invariant on a finite time interval \( 0 \leq t \leq T \) since their symbols satisfy Supposition I. Therefore, in constructing quasi-classically concentrated solutions to the Cauchy problem, the initial conditions can be taken in the form

\[
\Psi(\vec{x}, t, \hbar)|_{t=0} = \Psi_0(\vec{x}, \hbar), \quad \Psi_0 \in P_\hbar^0(z_0, S_0).
\]

The functions from the class \( P_\hbar^0 \) have the following form:

\[
\Psi_0(\vec{x}, \hbar) = \exp\left\{ \frac{i}{\hbar} \bar{S}(0, \hbar) + \langle \bar{F}_0(h), (\vec{x} - \vec{X}_0(h)) \rangle \right\} \varphi_0\left( \frac{\vec{x} - \vec{X}_0(h)}{\sqrt{\hbar}} \right), \quad \varphi_0(\vec{\xi}, \hbar) \in S(\mathbb{R}^2),
\]

where \( \bar{F}_0(h), \vec{X}_0(h) \) is an arbitrary point of the phase space \( \mathbb{R}^{2n} \), and the constant \( S_0(h) \) can be put equal to zero without loss of generality.

Important particular cases of the initial conditions of type (2.19) are

1) \( \varphi_0(\vec{\xi}) = e^{-|A\vec{\xi}|^2/2} \),

where the real \( n \times n \) matrix \( A \) is positive definite and symmetrical. Then relationship (2.19) defines the Gaussian packet;

2) \( \varphi_0(\vec{\xi}) = e^{i\bar{\xi}Q\vec{\xi}/2} H_\nu(\text{Im} \ Q\vec{\xi}), \)

where the complex \( n \times n \) matrix \( Q \) is symmetrical and has a positive definite imaginary part \( \text{Im} \ Q \) and \( H_\nu(\vec{y}) \) and \( \vec{y} \in \mathbb{R}^n \) are multidimensional Hermite polynomials of multi-index \( \nu = (\nu_1, \ldots, \nu_n) \) \([12]\). In this case, relationship (2.19) defines the Fock states of a multidimensional oscillator.

The solution of the Cauchy problem (1.1), (2.18) leads in turn to a set of Hamilton–Ehrenfest equations to the study of which we are coming.

3 The set of Hamilton–Ehrenfest equations

In view of Supposition I for the symbols \( \hat{H}(z, t) \) and \( \hat{V}(z, w, t) \), the operator \( \hat{H}(\zeta, t) \) (1.2) is self-conjugate to the scalar product \( \langle \Psi | \Phi \rangle \) in the space \( L_2(\mathbb{R}^2) \) and the operator \( \hat{V}(\zeta, \bar{w}, t) \) (1.3) is self-conjugate to the scalar
product $L_2(\mathbb{R}^{2n}_x)$:

$$\langle \Psi(t)|\Phi(t)\rangle_{\mathbb{R}^{2n}} = \int_{\mathbb{R}^{2n}} d\vec{x} d\vec{y} \Psi^\ast(\vec{x}, \vec{y}, t, h) \Phi(\vec{x}, \vec{y}, t, h).$$

Therefore, for the exact solutions of equation (1.1) we have

$$\|\Psi(t)\|^2 = \|\Psi(0)\|^2,$$

and for the mean values of the operator $\hat{A}(t) = A(\vec{z}, t)$, calculated for these solutions, the equality

$$\frac{d}{dt} \langle \hat{A}(t) \rangle = \left( \frac{\partial \hat{A}(t)}{\partial t} \right) + \frac{i}{\hbar} \langle [\hat{H}, \hat{A}(t)] \rangle + \frac{i\kappa}{\hbar} \int d\vec{y} \Psi^\ast(\vec{y}, t, h) [\hat{A}(t), V(\vec{z}, \vec{w}, t)] \Psi(\vec{y}, t, h), \tag{3.1}$$

where $[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of the operators $\hat{A}$ and $\hat{B}$, is valid. We refer to equation (3.1) as the Ehrenfest equation for the operator $\hat{A}$ and function $\Psi(\vec{x}, t, h)$. This term was chosen in view of the fact that in the linear case ($\kappa = 0$) equation (1.1) goes into the quantum mechanical Schrödinger equation and relationship (2.1) into the Ehrenfest equation (2.6).

Let us make the following notations:

$$\vec{z} = (\vec{p}, \vec{x}), \quad Z(t, h) = (F(t, h), \vec{X}(t, h)), \quad \Delta \vec{z} = \vec{z} - Z(t, h). \tag{3.2}$$

Using the rules of composition for Weyl symbols [27], we find for the symbol of the operator $\hat{C} = \hat{A}\hat{B}$

$$C(z) = A\left( \frac{\partial^2}{\partial \vec{z}^2} + \frac{i\hbar}{2} J^\beta \right) B(z) = B\left( \frac{\partial^2}{\partial \vec{z}^2} - \frac{i\hbar}{2} J^\beta \right) A(z). \tag{3.3}$$

Here, the index over an operator symbol specifies the turn of its action. We suppose that for the Hartree type equation (1.1), exact solutions (or solutions differing from exact ones by a quantity $O(\hbar^\infty)$) exist in the class of trajectory-concentrated functions. Let us write Ehrenfest equations (3.1) for the mean values of the operators $\bar{z}_j, \{\Delta \bar{z}\}^\alpha$ calculated from such (trajectory-coherent) solutions of equation (1.1). After cumbersome, but not complicated calculations similar to the calculations that were performed for the linear case with $\kappa = 0$ (see for details [21]), we then obtain, restricting ourselves to the moments of order $N$, the following set of ordinary differential equations:

$$\dot{\vec{z}} = \sum_{|\mu|=0}^N \frac{1}{\mu!} J^\mu \mathcal{H}_{z\mu}(z, t) \Delta_\mu + \vec{z} \sum_{|\nu|=0}^N \frac{1}{\nu!} V_{z\nu}(z, t) \Delta_\nu, \tag{3.4}$$

$$\dot{\Delta}_\alpha = \sum_{|\mu+\gamma|=0}^N \left(-i\hbar\right)^{|\gamma|-1} \frac{[(-1)^{|\gamma|} - (-1)^{|\gamma|}] \alpha! \beta!}{\gamma!(\alpha - \gamma)!(\beta - \gamma)! \mu!} \times \left( \mathcal{H}_{\mu}(z, t) + \vec{z} \sum_{|\nu|=0}^N \frac{1}{\nu!} V_{\nu}(z, t) \Delta_\nu \right) \Delta_{\alpha - \gamma + J\beta - J\gamma - k} - \sum_{k=1}^{2n} \dot{\lambda}_k \alpha_k \Delta_{\alpha(k)}$$

with initial conditions

$$z|_{t=0} = z_0 = (\Psi_0|\bar{z}|\Psi_0), \quad \Delta_\alpha|_{t=0} = (\Psi_0|\{ \bar{z} - z_0 \}^\alpha|\Psi_0), \tag{3.5}$$

$$\alpha \in \mathbb{Z}^{2n}_+, \quad |\alpha| \leq N.$$

Here, $\bar{z} = \kappa \|\Psi_0(\bar{x}, h)\|^2$ and $\Psi_0(\bar{x}, h)$ is the initial function from (2.18),

$$\mathcal{H}_{\mu}(z, t) = \frac{\partial^{|\mu|}}{\partial \bar{z}^\mu} \mathcal{H}(z, t), \quad V_{\mu
u}(z, t) = \frac{\partial^{|\mu+\nu|} V(\bar{z}, \bar{w}, t)}{\partial \bar{z}^\mu \partial \bar{w}^\nu} \bigg|_{\bar{w}=z}, \tag{3.6}$$

$$\theta(\alpha - \beta) = \prod_{k=1}^{2n} \theta(\alpha_k - \beta_k), \quad \alpha(k) = (\alpha_1 - \delta_{1,k}, \ldots, \alpha_{2n} - \delta_{2n,k}).$$
By analogy with the linear theory \((x = 0)\) [21], we refer to equations (3.4) as **Hamilton–Ehrenfest equations** of order \(N\). In view of the estimates (2.3), these equations are equivalent, for the class \(\mathcal{P}_h^t\), to the nonlinear Hartree type equation (1.1) accurate to \(O(h^{(N+1)/2})\).

For the case of \(N = 2\), the Hamilton–Ehrenfest equations take the form

\[
\begin{align*}
\dot{z} &= J\partial_z \left(1 + \frac{1}{2} (\partial_z, \Delta_2 \partial_z) + \frac{1}{2} (\partial_\omega, \Delta_2 \partial_\omega) \right) (H(z, t) + \dot{z}V(z, \omega, t))|_{\omega = z}, \\
\dot{\Delta}_2 &= J M \Delta_2 - \Delta_2 M J,
\end{align*}
\]

(3.7)

where

\[
M = [H_{zz}(z, t) + \dot{z}V_{zz}(z, \omega, t)]|_{\omega = z}.
\]

Equations (3.7) can be written in the equivalent form if in the second equation we put

\[
\Delta_2(t) = A(t) \Delta_2(0) A^+(t),
\]

and then it becomes

\[
\dot{A} = J M A, \quad A(0) = I.
\]

(3.8)

### 4 Linearization of the Hartree type equation

Let us now construct a quasi-classically concentrated (for \(\hbar \to 0\)) solution to equation (1.1), satisfying the initial condition (2.18).

Designate by

\[
y^{(N)}(t, \hbar) = (Z_j, \Delta_j^{(2)}, \Delta_j^{(3)}, \ldots) = (Z(t, \hbar), \Delta_\alpha(t, \hbar)), \quad |\alpha| \leq N
\]

(4.1)

the solution of the Hamilton–Ehrenfest equations of order \(N\) (3.4), with the initial data \(y^{(N)}(0, \hbar)\) (3.5) determined by the initial function \(\Psi_0(\hat{x}, \hbar)\) (2.18), i.e., the mean values \(Z(t, \hbar)\) and \(\Delta_\alpha(0, \hbar)\) are calculated from the function \(\Psi_0(\hat{x}, \hbar)\). Let us expand the “kernel” of the operator \(\hat{V}(t, \Psi)\) in a Taylor power series for the operators \(\Delta \hat{w} = \hat{w} - Z(t, \hbar)\):

\[
V(\hat{z}, \hat{w}, t) = \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha!} \left. \frac{\partial^{(|\alpha|)}}{\partial \hat{w}^{\alpha}} V(\hat{z}, \hat{w}, t) \right|_{\hat{w} = Z(t, \hbar)} \{\Delta \hat{w}\}^{\alpha}.
\]

(4.2)

Substituting this series into equation (1.3), we obtain for the functions \(\Psi \in \mathcal{P}_h^t\)

\[
\left[-i\hbar \partial_t + H(\hat{z}, t) + \dot{z} \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha!} \left. \frac{\partial^{(|\alpha|)}}{\partial \hat{w}^{\alpha}} V(\hat{z}, \hat{w}, t) \right|_{\hat{w} = Z(t, \hbar)} \Delta_\alpha(t, \hbar) \right] \Psi = O(\hbar^{(N+1)/2}), \quad \Psi|_{t = 0} = \Psi_0.
\]

(4.3)

where

\[
Z(t, \hbar) = \frac{1}{\|\Psi(t, \hbar)\|^2} (\Psi(t, \hbar)|\hat{z}|\Psi(t, \hbar));
\]

\[
\Delta_\alpha(t, \hbar) = \frac{1}{\|\Psi(t, \hbar)\|^2} (\Psi(t, \hbar)|\{\Delta \hat{z}\}^{\alpha}|\Psi(t, \hbar)).
\]

(4.4)

In view of the asymptotical estimates (2.3), the functions \(z(t, \hbar)\) and \(\Delta_\alpha(t, \hbar)\) can be determined with any degree of accuracy from the Hamilton–Ehrenfest equations (1.3) as

\[
z(t, \hbar) = z(t, h, N) + O(h^{(N+1)/2});
\]

\[
\Delta_\alpha(t, \hbar) = \Delta_\alpha(t, h, N) + O(h^{(N+1)/2}) \quad |\alpha| \leq N.
\]

(4.5)

where \(z(t, h, N)\) and \(\Delta_\alpha(t, h, N)\) are solutions of the Hamilton–Ehrenfest equations of order \(N\), which are completely determined by the initial condition of the Cauchy problem for the Hartree type equation, \(\Psi_0(\hat{x}, t, \hbar)\),
and do not use the explicit form of the solution \( \Psi(\vec{x}, t, h) \). Thus, the change of the mean values of the operators for the solutions of the Hamilton–Ehrenfest equations of order \( N \) linearizes the Hartree type equation (4.3) accurate to \( O(h^{N+1}) \). So, to find an asymptotical solution to the Hartree type equation (4.3), we should consider the linear Schrödinger type equation:

\[
\hat{L}^{(N)}(t, \Psi_0) \Phi = O(h^{N+1/2}), \quad \Phi|_{t=0} = \Phi_0;
\]

\[
\hat{L}^{(N)}(t, \Psi_0) = -ih\partial_t + \mathcal{H} (\hat{z}, t) + \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} \partial^{[|\alpha|]} V(\hat{z}, w, t) \bigg|_{w=Z(t, h, N)} \Delta_{\alpha}(t, h, N).
\]

**Definition 4.2** We call an equation of type (4.4) with a given \( \Psi_0 \) a Hartree equation in the trajectory-coherent approximation or a linear associated Schrödinger equation of order \( N \) for the Hartree type equation (4.3).

The following statement is valid:

**Statement 4.1** If the function \( \Phi^{(N)}(\vec{x}, t, h, \Psi_0) \in \mathcal{P}_h^N \) is an asymptotical (accurate to \( O(h^{N+1/2}) \), \( h \to 0 \)) solution of equation (4.3), satisfying the initial condition \( \Phi|_{t=0} = \Psi_0 \), the function

\[
\Psi^{(N)}(\vec{x}, t, h) = \Phi^{(N)}(\vec{x}, t, h, \Psi_0)
\]

is an asymptotical (accurate to \( O(h^{N+1/2}) \), \( h \to 0 \)) solution of the Hartree type equation (4.3).

Now we expand the operators

\[
\mathcal{H}(\hat{z}, t), \quad \frac{\partial^{[|\alpha|]} V(\hat{z}, w, t)}{\partial w^\alpha} \bigg|_{w=Z(t, h, N)}
\]

in a Taylor power series for the operator \( \Delta \hat{z} \) and present the operator \( -ih\partial_t \) in the form

\[
-ih\partial_t = \{-\hat{P}(t, h, N), \hat{X}(t, h, N)\} + \hat{S}(t, h) + \{\hat{S}(t, h), \mathcal{H}(\hat{z}, t, h)\}
\]

Here, the group of terms in braces containing \( -ih\partial_t \), in view of (2.14), has an order of \( \hat{O}(h) \). Other terms can be estimated, in view of (2.3), by the parameter \( h \). Substitute the obtained expansions into (4.4). Take (accurate to \( O(h^{N+1/2}) \)) the real function \( S(t, h) \) entering in the definition of the class \( \mathcal{P}_h^N(Z(t, h), S(t, h)) \) in the form

\[
S(t, h) = S^{(N)}(t, h) = \int_0^t \{\hat{P}(t, h, N), \hat{X}(t, h, N)\} - \mathcal{H}(Z(t, h, N), t) - \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} \partial^{[|\alpha|]} V(Z(t, h, N), w, t) \bigg|_{w=Z(t, h, N)} \Delta_{\alpha}(t, h, N) dt.
\]

As a result, equation (4.4) will not contain operators of multiplication by functions depending only on \( t \) and \( h \).

In view of the estimates (2.3) and (2.6) valid for the class \( \mathcal{P}_h^N(Z(t, h), S(t, h)) \), we obtain for (4.3)

\[
\left\{ -ih\partial_t + \hat{S}_0(t, \Psi_0) + h\hat{S}^{(N)}(t, \Psi_0) \right\} \Phi = O(h^{N+1/2}),
\]

with the following notations:

\[
\hat{S}_0(t, \Psi_0) = \sum_{k=1}^{N} h^{k/2} \hat{S}_k(t, \Psi_0),
\]

\[
\hat{S}_0(t, \Psi_0) = -\hat{S}(t, h) + (\hat{P}(t, h), \hat{X}(t, h) + (\hat{Z}(t, h), J\Delta \hat{z}) + \frac{1}{2} (\Delta \hat{z}, \hat{S}_{zz}(t, \Psi_0) \Delta \hat{z}),
\]

\[
\hat{S}_{zz}(t, \Psi_0) = \left[ \mathcal{H}_{zz}(z, t) + \hat{z} V_{zz}(z, w, t) \right]_{z=Z(t, h, N)},
\]
\[ h^{k+2}/2 \tilde{\phi}_k(t, \Psi_0) = -\langle \Delta_t(0), J \Delta \hat{\psi} \rangle + \sum_{|\alpha|=k+2} \frac{1}{\alpha!} \left. \frac{\partial^{|\alpha|} H(z, t)}{\partial z^{\alpha}} \right|_{z=Z(t, h, N)} \{ \Delta \hat{\psi} \}^\alpha + \]
\[ + \frac{\kappa}{\hbar} \sum_{|\alpha+\beta|=k+2} \frac{1}{\alpha! \beta!} \left. \frac{\partial^{|\alpha+\beta|} V(z, w, t)}{\partial z^{\alpha} \partial w^{\beta}} \right|_{z=w=Z(t, h, N)} \{ \Delta \hat{\psi} \}^\beta \Delta_{\alpha}(t, h, N). \]

(4.11)

Here, \( k = 1, N \) and the functions \( Z(k)(t) \) are the coefficients of the expansion of the projection \( Z(t, h) \) of the solution \( y^{(N)}(t, h) \) of the Hamilton–Ehrenfest equations on the phase space \( \mathbb{R}^{2n} \) in a power series of the regular perturbation theory for \( \sqrt{h} \):

\[ Z(t, h) = Z(t, h, N) = Z(t, 0) + \sum_{k=2}^N \hbar^{k/2} Z(k)(t). \]

From the Hamilton–Ehrenfest equations, in view of the fact that the first-order moments are zero (\( \Delta_{\alpha}(t, h, N) = 0 \) for \( |\alpha| = 1 \)), it follows that the coefficient \( \tilde{\phi}_{(1)}(t) \) is equal to zero.

**Remark 4.1** The solutions of the set of Hamilton–Ehrenfest equations depend on the index \( N \) that denotes the highest order of the centered moments \( \Delta_{\alpha}, \alpha \in \mathbb{Z}^{2n} \). We shall omit the index \( N \) if this does not give rise to ambiguity.

The operators \( \hat{\phi}_0(t) \) (4.10) and \( \hat{\phi}_k(t) \) (4.13) depend on the mean \( Z(t, h) \) and moments \( \Delta_{\alpha}(t, h) \), i.e., on the solution \( y^{(N)}(t, h) \) of the Hamilton–Ehrenfest equations (3.4). The solutions of equation (4.8) in turn depend implicitly on \( y^{(N)}(t, h) \):

\[ \Phi(\vec{x}, t, h) = \Phi(\vec{x}, t, h, y^{(N)}(t, h)). \]

Below the function arguments \( y^{(N)}(t, h) \) or \( \Psi_0 \) can be omitted if this does not give rise to ambiguity. For example, we may put \( \hat{\phi}_0(t) = \hat{\phi}_0(t, \Psi_0) \).

In accordance with the expansion (2.12) and (2.14), the solution of equation (4.8) can be represented in the form

\[ \Phi(\vec{x}, t, h, \Psi_0) = \sum_{k=0}^N h^{k/2} \Phi^{(k)}(\vec{x}, t, h, \Psi_0) + O(h^{(N+1)/2}), \]

(4.12)

where

\[ \Phi^{(k)}(\vec{x}, t, h, \Psi_0) \in C^1_h(Z(t, h), S(t, h)). \]

In view of (2.14), for the operators \( \{-i\hbar \partial_t + \hat{\phi}_0(t, \Psi_0)\} \) (4.10) and \( h^{(k+2)/2} \tilde{\phi}_k(t, \Psi_0) \), \( k = 1, N \) (4.11) the following is valid:

\[ h^{(k+2)/2} \tilde{\phi}_k(t, \Psi_0) : \mathcal{P}_h^1(Z(t, h), S(t, h), l) \rightarrow \mathcal{P}_h^1(Z(t, h), S(t, h), l + k + 2), \]

\[ \{-i\hbar \partial_t + \hat{\phi}_0(t, \Psi_0)\} : \mathcal{P}_h^1(Z(t, h), S(t, h), l) \rightarrow \mathcal{P}_h^1(Z(t, h), S(t, h), l + 2). \]

(4.13)

Substitute (4.12) into (4.8) and equate the terms having the same order in \( h^{1/2}, h \to 0 \) in the sense of (4.13). As a result we obtain a set of recurrent associated linear equations of order \( k \) to determine the functions \( \Phi^{(k)}(\vec{x}, t, h, \Psi_0) \):

\[ h^1 \{ -i\hbar \partial_t + \hat{\phi}_0(t, \Psi_0) \} \Phi^{(0)} = 0, \]

(4.14)

\[ h^{3/2} \{-i\hbar \partial_t + \hat{\phi}_0(t, \Psi_0)\} \Phi^{(1)} + h \tilde{\phi}_1(t, \Psi_0) \Phi^{(0)} = 0, \]

(4.15)

\[ h^2 \{-i\hbar \partial_t + \hat{\phi}_0(t, \Psi_0)\} \Phi^{(2)} + h \tilde{\phi}_1(t, \Psi_0) \Phi^{(1)} + h^{3/2} \tilde{\phi}_2(t, \Psi_0) \Phi^{(0)} = 0, \]

(4.16)

It is natural to call equation (4.14) for the principal term of the asymptotical solution as the Hartree type equation in the trajectory-coherent approximation in \( \mod h^{3/2} \). This equation is the Schrödinger equation with the Hamiltonian quadric with respect to the operators \( \vec{p} \) and \( \vec{x} \).
5 The trajectory-coherent solutions of the Hartree type equation

The solution of the Schrödinger equation with a quadric Hamiltonian is well known \[24, 25\]. For our purposes, it is convenient to take quasi-classical trajectory-coherent states (TCS’s) \[21\] as a basis of solutions to equation (4.14). We shall refer to the solution of the nonlinear Hartree type equation, which coincides with the TCS at the time zero, as a trajectory-coherent solution of the Hartree type equation. Now we pass to constructing solutions like this.

Let us write the symmetry operators \( \hat{a}(t, \Psi_0) \) of equation (4.14), linear with respect to the operators \( \Delta \hat{\varepsilon} \), in the form

\[
\hat{a}(t, \Psi_0) = N_a \langle b(t, \Psi_0), \Delta \hat{\varepsilon} \rangle,
\]

where \( N_a \) is a constant and \( b(t) \) is a 2n-space vector. From the equation

\[
- i\hbar \frac{\partial \hat{a}(t)}{\partial t} + [\hat{H}_0(t, \Psi_0), \hat{a}(t)] = 0,
\]

which determines the operators \( \hat{a}(t) \), in view of the explicit form of the operator \( \hat{H}_0(t, \Psi_0) \), we obtain

\[
- i\hbar \langle \hat{b}(t), \Delta \hat{\varepsilon} \rangle + i\hbar \langle \hat{b}(t), \hat{Z}(t, \hbar) \rangle +
\]

\[
+ \left\{ \langle -\hat{S}(t, \hbar) \rangle + \langle \hat{P}(t, \hbar), \hat{X}(t, \hbar) \rangle \right\} + \langle \hat{J}(t, \hbar), \Delta \hat{\varepsilon} \rangle + \frac{1}{2} \langle \Delta \hat{\varepsilon}, \hat{H}_{zz}(t, \Psi_0) \Delta \hat{\varepsilon} \rangle = 0.
\]

Taking into account the commutative relations

\[
[\Delta \hat{\varepsilon}_j, \Delta \hat{\varepsilon}_k] = i\hbar J_{jk}, \quad j, k = 1, 2n,
\]

which follow from (1.4) and (1.5), we find

\[
- i\hbar \langle \hat{b}(t), \Delta \hat{\varepsilon} \rangle + i\hbar \langle \Delta \hat{\varepsilon}, \hat{H}_{zz}(t, \Psi_0) \rangle \hat{b}(t) = 0.
\]

Hence, we have

\[
\hat{b} = \hat{H}_{zz}(t, \Psi_0) \hat{b}.
\]

Denote \( \hat{b}(t) = -Ja(t) \). Then we obtain for the determination of the 2n-space vector \( a(t) \) from (5.3)

\[
\hat{a} = J \hat{H}_{zz}(t, \Psi_0) a.,
\]

We call the set of equations (5.4), by analogy with the linear case \[18\], a set of equations in variations.

Thus, the operator

\[
\hat{a}(t) = \hat{a}(t, \Psi_0) = N_a \langle b(t, \Psi_0), \Delta \hat{\varepsilon} \rangle = N_a \langle a(t), J \Delta \hat{\varepsilon} \rangle
\]

is a symmetry operator for equation (4.14) if the vector \( a(t) = a(t, \Psi_0) \) is a solution of the equations in variations (1.4).

For each given solution \( Z(t, \hbar) \) of the Hamilton–Ehrenfest equations (5.4), we can find 2n linearly independent solutions \( a_k(t) \in C^{2n} \) to the equations in variations (5.4). Since to each 2n-space vector \( a_k(t) \) corresponds an operator \( \hat{a}_k(t, \Psi_0) \), we obtain 2n operators \( n \) of which commute with one another and form a complete set of symmetry operators for equation (4.14).

Now we turn to constructing the basis of solutions to equation (4.14) with the help of the operators \( \hat{a}_k(t, \Psi_0) \). Equation (4.14) is a (linear) Schrödinger equation with a quadric Hamiltonian and admits solutions in the form of Gaussian wave packets

\[
\Phi(\vec{x}, t, \Psi_0) = N_{\hbar} \exp \left\{ \frac{i}{\hbar} \left[ S(t, \hbar) + i\phi_0(t) + i\hbar \phi_1(t) + \langle \hat{P}(t, \hbar), \Delta \vec{x} \rangle + \frac{1}{2} \langle \Delta \vec{x}, Q(t) \Delta \vec{x} \rangle \right] \right\},
\]

where the real phase \( S(t, \hbar) \) is defined in (4.7), \( N_{\hbar} \) is a normalized constant, and the real functions \( \phi_0(t) \) and \( \phi_1(t) \) and the complex \( n \times n \) matrix \( Q(t) \) are to be determined.

**Remark 5.1** Asymptotical solutions in the form of Gaussian packets (5.6) for equations with an integral nonlinearity of more general form than (1.1) were constructed in [14]. In this case, the Hamilton–Ehrenfest equations depend substantially on the initial condition for the starting nonlinear equation.
Substitution of (5.6) into (4.14) yields
\[
\Phi \left\{ \tilde{S}(t, h) + i\phi_0(t) + i\hbar \phi_1(t) + \langle \tilde{P}(t, h), \Delta \vec{x} \rangle - \langle \tilde{P}(t, h), \tilde{X}(t, h) \rangle + \frac{1}{2} (\Delta \vec{x}, \hat{Q}(t) \Delta \vec{x}) - 
\right.
\]
\[
- \langle \Delta \vec{x}, t \rangle \hat{X}(t, h) \rangle - \tilde{S}(t, h) + \langle \tilde{P}(t, h), \tilde{X}(t, h) \rangle + \langle \tilde{X}(t, h), Q(t) \Delta \vec{x} \rangle - \langle \tilde{P}(t, h), \Delta \vec{x} \rangle + 
\]
\[
+ \frac{1}{2} \left\{ (\Delta \vec{x}, \hat{S}_{xx}(t, \Psi_0) \Delta \vec{x}) + (\Delta \vec{x}, \hat{S}_{pp}(t, \Psi_0) t Q(t) \Delta \vec{x}) + 
\right. 
\]
\[
+ \langle - i\hbar \nabla + Q(t) \Delta \vec{x} \rangle, \hat{S}_{px}(t, \Psi_0) \Delta \vec{x} \rangle + \langle - i\hbar \nabla + Q(t) \Delta \vec{x} \rangle, \hat{S}_{pp}(t, \Psi_0) t \rangle 0 \rangle = 0. 
\]
Equating the coefficients at the terms with the same powers of the parameter \( \hbar \) and the operator \( \Delta \vec{x} \), we obtain
\[
(\Delta \vec{x})^0 h^0 : \quad i\phi_0(t) = 0; 
\]
\[
(\Delta \vec{x})^0 h^1 : \quad i\phi_1(t) + \frac{-i}{2} \text{Sp} [\hat{S}_{px}(t, \Psi_0) + \hat{S}_{pp}(t, \Psi_0) t Q(t)] = 0; 
\]
\[
(\Delta \vec{x})^1 h^0 : \quad (\Delta \vec{x}, 0) = 0; 
\]
\[
(\Delta \vec{x})^2 h^0 : \quad (\Delta \vec{x}, [Q(t) + \hat{S}_{xx}(t, \Psi_0) + \hat{S}_{xp}(t, \Psi_0) t Q(t) + Q(t) \hat{S}_{pp}(t, \Psi_0) + 
\]
\[
+ Q(t) \hat{S}_{pp}(t, \Psi_0) t Q(t)] \Delta \vec{x}) = 0. 
\]
As a result we have
\[
\phi_0(t) = 0, \quad (5.7) 
\]
\[
\phi_1(t) = \frac{1}{2} \int_0^t \text{Sp} [\hat{S}_{px}(t) + \hat{S}_{pp}(t) t Q(t)] dt. \quad (5.8) 
\]
The matrix \( Q(t) \) is determined from the Riccati type equation
\[
Q(t) + \hat{S}_{xx}(t) + Q(t) \hat{S}_{px}(t) + \hat{S}_{xp}(t) Q(t) + Q(t) \hat{S}_{pp}(t) Q(t) = 0. \quad (5.9) 
\]
Thus, the construction of a solution to equation (4.14) in the form of the Gaussian packet (5.8) is reduced to solving the set of ordinary differential equations (5.9).

Let us now construct the Fock basis of solutions to the (linear) Hartree equation in the trajectory-coherent approximation (4.14). This is the first step in constructing the solution to recurrent equations (4.14)–(4.16).

Consider the properties of the symmetry operators \( \hat{a}_k(t) \) (5.3) of the zero-order associated Schrödinger equation (4.14), which are necessary to construct the Fock basis.

**Statement 5.1** Let \( a_1(t) \) and \( a_2(t) \) be two solutions of the equations in variations and \( \hat{a}_1(t) \) and \( \hat{a}_2(t) \) be the respective symmetry operators of equation (4.14), defined in (5.5). Then the equality
\[
[\hat{a}_1(t), \hat{a}_2(t)] = i\hbar N_1 N_2 \{ a_1(t), a_2(t) \} = i\hbar N_1 N_2 \{ a_1(0), a_2(0) \} \quad (5.10) 
\]
is valid.

Actually, upon direct checking we are convinced that
\[
[\hat{a}_1(t), \hat{a}_2(t)] = N_1 N_2 \{ a_1(t), J \Delta \hat{\phi}, a_2(t), J \Delta \hat{\phi} \} = 
\]
\[
i\hbar N_1 N_2 \{ J a_1(t), J a_2(t) \} = 
\]
\[
i\hbar N_1 N_2 \{ a_1(t), J J a_2(t) \} = i\hbar N_1 N_2 \{ a_1(t), J a_2(t) \} = 
\]
\[
i\hbar N_1 N_2 \{ a_1(t), a_2(t) \}. 
\]
Here, we have used the rules of commutation for the operators \( \Delta \hat{\phi} \). The skew scalar product holds and, hence, the statement is proved.
Remark 5.2 If the initial conditions for the equations in variations are taken such that

\[ \{a_j(0), a_k(0)\} = \{a_j^*(0), a_k^*(0)\} = 0, \quad \{a_j(0), a_k^*(0)\} = id_k \delta_{kj}, \quad (5.11) \]

and \( N_k = 1/\sqrt{d_k} \), then the following canonical commutation relations for the boson operators of “creation” \((\hat{a}_k^+(t))\) and “annihilation” \((\hat{a}_k(t))\) are valid:

\[ [\hat{a}_k(t), \hat{a}_j(t)] = [\hat{a}_k^+(t), \hat{a}_j^+(t)], \quad [\hat{a}_k(t), \hat{a}_j^+(t)] = \delta_{kj}. \quad (5.12) \]

The simplest example of initial data satisfying the conditions \((5.11)\) is

\[ a_1(0) = (b_1, 0, \ldots, 0, 1, 0, \ldots); \]
\[ a_2(0) = (0, b_2, \ldots, 0, 0, 1, \ldots); \quad (5.13) \]

Here, we have \( d_k = 2 \text{Im } b_k > 0, \quad k = 1, n \).

Theorem 5.1 The function

\[ |0, t\rangle = |0, t, \Psi_0\rangle = \frac{N_h}{\text{det } C(t)} \times \]
\[ \times \exp \left\{ \frac{i}{\hbar} \left[ S(t, h) + \langle \hat{P}(t, h), \Delta \hat{x} \rangle + \frac{1}{2} \langle \Delta \hat{x}, Q(t) \Delta \hat{x} \rangle \right] \right\}. \quad (5.14) \]

where \( N_h = [(\pi \hbar)^{-n} \text{det } D_0]^{1/4} \) is a “vacuum” state for the operators \( \hat{a}_j(t) \), such that

\[ \hat{a}_j(t)|0, t\rangle = 0, \quad j = 1, n. \quad (5.15) \]

Proof. Actually, substituting \((5.5)\) and \((5.14)\) into \((5.13)\), we get

\[ |0, t\rangle [\hat{Z}_j(t), Q(t) \Delta \hat{x}] - [\hat{W}_j(t), \Delta \hat{x}] = 0, \]

since we have

\[ Q(t) \hat{Z}_j(t) = B(t) C^{-1}(t) \hat{Z}_j(t) = \hat{W}_j(t). \]

Recollect that from the fact that the matrix \( D_0 \) is positive definite and diagonal follows \( \text{det } C(t) \neq 0 \), and so the matrix \( \text{Im } Q(t) \) is positive definite as well (see Appendix A).

Let us define the denumerable set of states \( |\nu, t\rangle \) as a result of the action of the “creation” operators upon the “vacuum” state \(|0, t\rangle\):

\[ |\nu, t\rangle = |\nu, t, \Psi_0\rangle = \frac{1}{\nu^!} (\hat{a}_k^+(t, \Psi_0))^\nu |0, t, \Psi_0\rangle = \prod_{k=1}^n \frac{1}{\nu_k!} (\hat{a}_k^+(t, \Psi_0))^\nu |0, t, \Psi_0\rangle. \quad (5.16) \]

By analogy with the linear theory \((\kappa = 0)\), we call the functions \(|\nu, t\rangle\) \((5.16)\) quasi-classical trajectory-coherent states and consider their simplest properties.

Statement 5.2 The relationships

\[ \hat{a}_k |\nu, t\rangle = \sqrt{\nu_k} |\nu_k^{(-)}, t\rangle, \]
\[ \hat{a}_k^+ |\nu, t\rangle = \sqrt{\nu_k + 1} |\nu_k^{(+)}, t\rangle, \]
\[ \nu_k^{(\pm)} = (\nu_1 \pm \delta_{1,k}, \nu_2 \pm \delta_{2,k}, \ldots, \nu_n \pm \delta_{n,k}) \quad (5.17) \]

are valid.
Actually, we have

\[ [\hat{a}_j, (\hat{a}_k^+)^{\nu_k}] = \nu_k (\hat{a}_k^+)^{\nu_k-1} \delta_{j,k}. \]

It follows that

\[ \hat{a}_j |\nu, t\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu_k}} [\hat{a}_j, (\hat{a}_k^+)^{\nu_k}] |0, t\rangle = \prod_{k=1}^{n} \frac{\nu_j}{\sqrt{\nu_k^2}} (\hat{a}_k^+)^{\nu_k-\delta_{k,j}} |0, t\rangle = \sqrt{\nu_j} |\nu, t\rangle; \]

\[ \hat{a}_j^+ |\nu, t\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu_k^2}} (\hat{a}_k^+)^{\nu_k+\delta_{k,j}} |0, t\rangle = \prod_{k=1}^{n} \frac{\nu_j + 1}{\sqrt{\nu_k^2}} (\hat{a}_k^+)^{\nu_k+\delta_{k,j}} |0, t\rangle = \sqrt{\nu_j + 1} |\nu, t\rangle, \]

and thus the statement is proved.

**Statement 5.3** The states \(|\nu, t, \Psi_0\rangle\) with \(t \in \mathbb{R}\) and \(\Psi_0 \in \mathcal{P}_h^0\) form a set of orthonormal functions:

\[ \langle \Psi_0, t, \nu|\nu, t, \Psi_0\rangle = \delta_{\nu,\nu'}, \quad \nu, \nu' \in \mathbb{Z}_+^n. \]

(5.18)

Let us consider the expression

\[ \langle \Psi_0, t, \nu'|\nu, t, \Psi_0\rangle = \frac{1}{\sqrt{\nu'!\nu!}} \langle \Psi_0, t, 0|\hat{a}^{\nu'} (t, \Psi_0)|\hat{a}^+\rangle|\nu (t, \Psi_0)\rangle |0, t, \Psi_0\rangle. \]

Commuting the operators of “creation” and “annihilation” in view of commutation relations (5.12) and using relationship (5.15), we obtain

\[ \langle t, \nu'|\nu, t \rangle = \langle t, 0|0, t \rangle \delta_{\nu,\nu'}. \]

(5.19)

Then we calculate

\[ \langle t, 0|0, t \rangle = \frac{N_h^2}{\det C(t)} \int \exp\left[-\frac{2}{\hbar} \text{Im} S(\bar{x}, t)\right] d\bar{x}. \]

(5.20)

In view of (A.8) and the explicit form of the complex phase in (5.14), we have

\[ \text{Im} S(\bar{x}, t) = \frac{1}{2} \langle \Delta \bar{x}, \text{Im} Q(t) \Delta \bar{x} \rangle. \]

The matrix \(\text{Im} Q(t)\) is real and positive definite; hence, the matrix \(\sqrt{\text{Im} Q(t)}\) does exist, such that

\[ \text{det} \sqrt{\text{Im} Q(t)} = \frac{\sqrt{\text{det} D_0}}{|\det C(t)|}. \]

Let us perform in the integral of (5.20) the change

\[ \bar{\xi} = \frac{1}{\hbar} \sqrt{\text{Im} Q(t)} \Delta \bar{x}, \]

and then obtain

\[ \langle t, 0|0, t \rangle = \frac{N_h^2}{\sqrt{\text{det} D_0}} \hbar^{n/2} \int e^{-\bar{\xi}^2} d\bar{\xi} = \frac{(\pi \hbar)^{n/2} N_h^2}{\sqrt{\text{det} D_0}} = 1 \]

since \(\text{det} D_0 = \prod_{k=1}^{n} \text{Im} b_k\). Thus, the functions \(|\nu, t, \Psi_0\rangle\) (5.16) form the Fock basis of solutions to equation (1.14), Q.E.D.

**Theorem 5.2** Let the symbols of the operators \(\hat{H}(t)\) and \(\hat{V}(t, \Psi)\) satisfy the conditions of Supposition 1. Then for any \(\nu \in \mathbb{Z}_+^n\) the function

\[ \Psi_{\nu}(x, t, h) = |\nu, t\rangle, \]

(5.21)

where the functions \(|\nu, t\rangle\) are defined by formula (5.16), is an asymptotical (accurate to \(O(h^{3/2})\), \(h \to 0\)) solution to the Hartree type equation (1.1) with the initial conditions

\[ \Psi_{\nu}(x, t, h)|_{t=0} = |\nu, t\rangle|_{t=0}. \]

(5.22)
6 The principal term of the quasi-classical asymptotic of the Hartree type equation

The solution of the Cauchy problem (1.1), (5.22) is a special case of the quasi-classically concentrated solutions of equation (1.1). However, in the case of arbitrary initial conditions (2.18) belonging to the class \( \mathcal{P}_h \), the functions \( \nu, t \) are not asymptotical solutions of the Hartree type equation (1.1). This is a fundamental difference between the complex germ method for the Hartree type equation (1.1), being developed here, and a similar method developed for linear equations [18, 21]. The coefficients of the Hartree type equation in the trajectory-coherent approximation (4.14) depend on the initial condition (2.18) since they are determined by the solutions of the set of Hamilton–Ehrenfest equations. It follows that among the whole set of solutions to equation (4.14) only one (satisfying the condition \( \Psi(\vec{x}, t, h)|_{t=0} = \Psi_0(\vec{x}, h) \)) will be an asymptotical (accurate to \( O(h^{3/2}) \)) solution to equation (1.1). However, the Fock basis (TCS's) \( \nu, t \) makes it possible to construct an asymptotical solution to equation (1.1) with any degree of accuracy in \( h^{1/2}, h \to 0 \) in an explicit form which would satisfy the initial condition (2.18).

Let us illustrate in more detail the relation of the solutions of the associated linear Schrödinger equation to the solution of the Hartree type equation. To do this, we construct the Green function of the Cauchy problem for the zero-order associated Schrödinger equation. Although the Green function \( G^{(0)}(\vec{x}, \vec{y}, t, s) \) for quadratic quantum systems is well known [22, 23, 24, 25], we give for completeness its explicit form, as convenient to us. This function will allow us to demonstrate explicitly the nontrivial dependence of the evolution operator of the associated linear equation on the initial conditions for the starting Hartree type equation.

By definition we have

\[
\begin{align*}
-\imath \hbar \partial_t + \hat{\mathcal{G}}_0(t, \Psi_0)G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0) &= 0, \\
\lim_{t \to s} G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0) &= \delta(\vec{x} - \vec{y}),
\end{align*}
\]

where the operator \( \hat{\mathcal{G}} \) is defined in (4.10). We make use of the simplifying assumption that

\[
\det \mathcal{S}_{pp}(s) \neq 0, \quad \det \left| \frac{\partial p_k(t, z_0)}{\partial p_{0j}} \right| \neq 0.
\]

If the condition (6.2) is not valid, the solution of the problem can be found following the work [39, 40]. For the problem under consideration, exact solutions of the Schrödinger equation (4.14) are known: these are the functions \( \nu, t, \Psi_0 \) (5.16) that form a complete set of functions. Thus we have

\[
G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0) = \sum_{|\nu|=0}^{\infty} \Phi_\nu(\vec{x}, t, h)\Phi_\nu^*(\vec{y}, s, h),
\]

where

\[
\Phi_\nu(\vec{x}, t, h) = |\nu, t, \Psi_0\rangle.
\]

Details of similar calculations can be found, for instance, in [23]. However, for our purposes the following approach seems to be convenient.

Let us carry out an \( h^{-1} \) Fourier transform in equation (6.1). For the Fourier transform of the Green function

\[
\hat{G}^{(0)}(\vec{p}, \vec{y}, t, s, \Psi_0) = \int_{\mathbb{R}^n} \frac{d\vec{x}}{(2\pi\imath h)^{n/2}} G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0) \exp \left\{ -\frac{i}{\hbar} \langle \vec{x}, \vec{p} \rangle \right\}
\]

we obtain

\[
\begin{align*}
-\imath \hbar \partial_t + \hat{\mathcal{S}}_0(\vec{p}, \vec{x}, t, \Psi_0)\hat{G}^{(0)}(\vec{p}, \vec{y}, t, s, \Psi_0) &= 0, \\
\lim_{t \to s} \hat{G}(\vec{p}, \vec{y}, t, s, \Psi_0) &= \frac{1}{(2\pi\imath h)^{n/2}} \exp \left\{ -\frac{i}{\hbar} \langle \vec{p}, \vec{y} \rangle \right\}.
\end{align*}
\]

Here, \( \vec{p} = \vec{p} \) and \( \vec{x} = \imath \hbar \frac{\partial}{\partial \vec{p}} \) and the symbols of the operators of equations (6.1) and (6.5) coincide:

\[
\hat{\mathcal{S}}_0(\vec{p}, \vec{x}, t) = \mathcal{S}_0(\vec{p}, \vec{x}, t).
\]
Equation (6.5) coincides to notations with (4.14) and, hence, admits solutions of type (5.6)
\[
\tilde{G}^{(0)}(\vec{p}, \vec{g}, t, s, \Psi_0) = \exp \left\{ -\frac{i}{\hbar} \left[ S_0(t, s, \vec{y}) + \langle \tilde{G}(t, s, \vec{y}), \Delta \vec{p} \rangle + \frac{1}{2} \langle \Delta \vec{p}, \tilde{Q}(t, s, \vec{y}) \Delta \vec{p} \rangle \right] \right\},
\] (6.6)
where \( \Delta \vec{p} = \vec{p} - \vec{P}(t, \hbar) \). Here, the functions \( S_0(t, s, \vec{y}) = S_0(t), \tilde{G}(t, s, \vec{y}) = \tilde{G}(t) \) and \( \tilde{Q}(t, s, \vec{y}) = \tilde{Q}(t) \) are to be determined and, according to \((6.3)\), satisfy the initial conditions
\[
\lim_{t \to s} \tilde{Q}(t, s, \vec{y}) = 0, \quad \lim_{t \to s} \langle \tilde{G}(t, s, \vec{y}) \rangle = \vec{y}, \quad \lim_{t \to s} S_0(t, s, \vec{y}) = \langle \vec{p}_0, \vec{y} \rangle.
\] (6.7)
Substituting \((6.6)\) into \((6.5)\), we write
\[
\tilde{G}^{(0)}(\vec{p}, \vec{g}, t, s, \Psi_0) \left\{ -S_0(t) - \langle \tilde{G}(t), \Delta \vec{p}(t) \rangle + \langle \tilde{G}(t), \tilde{P}(t, \hbar) \rangle - \frac{1}{2} \langle \Delta \vec{p}, \tilde{Q}(t) \Delta \vec{p} \rangle + \langle \tilde{P}(t, \hbar), \tilde{Q}(t) \Delta \vec{p} \rangle - \langle \tilde{P}(t, \hbar), \langle \tilde{G}(t) + \tilde{Q}(t) \Delta \vec{p} - \tilde{P}(t, \hbar) \rangle \rangle + \langle \tilde{X}(t, \hbar), \Delta \vec{p} \rangle + \frac{1}{2} \langle (\tilde{G}(t) + \tilde{Q}(t) \Delta \vec{p} - \tilde{P}(t, \hbar)), \partial_{xx}(t) \tilde{G}(t) + \tilde{Q}(t) \Delta \vec{p} - \tilde{P}(t, \hbar) \rangle \rangle + \langle (\tilde{G}(t) + \tilde{Q}(t) \Delta \vec{p} - \tilde{P}(t, \hbar)), \partial_{xp}(t) \rangle \Delta \vec{p} + \langle (\Delta \vec{p}, \partial_{xp}(t) \tilde{G}(t) + \tilde{Q}(t) \Delta \vec{p} - \tilde{P}(t, \hbar)) \rangle + \frac{i\hbar}{2} \text{Sp} [\partial_{xx}(t) \tilde{G}(t) + \partial_{xp}(t)] \right\} = 0.
\]
Equating the terms with the same powers of \( \Delta \vec{p} \), we obtain the following set of equations:
\[
\dot{\tilde{Q}} = \partial_{xx}(t) \tilde{Q} + \partial_{xp}(t) \tilde{Q} + \partial_{xx}(t) \tilde{Q} + \partial_{xp}(t) = 0,
\] (6.8)
\[
\dot{\tilde{G}} - \dot{\tilde{P}}(t, \hbar) + \tilde{Q}(t) \partial_{xx}(t) (\tilde{G} - \tilde{P}(t, \hbar)) + \partial_{xp}(t) (\tilde{G} - \tilde{P}(t, \hbar)) = 0,
\] (6.9)
\[
\dot{S}_0 + \langle \tilde{X}(t, \hbar), \tilde{P}(t, \hbar) \rangle - \tilde{S}(t, \hbar) + \frac{i\hbar}{2} \text{Sp} [\partial_{xx}(t) \tilde{Q}(t) + \partial_{xp}(t) = 0
\]
\[
+ \frac{1}{2} \langle (\tilde{G}(t) - \tilde{P}(t, \hbar)), \partial_{xx}(t) (\tilde{G}(t) - \tilde{P}(t, \hbar)) \rangle = 0
\] (6.10)
with the initial conditions \((6.7)\).

Let \( \tilde{B}(t) \) and \( \tilde{C}(t) \) be solutions of equations in variations \((A.2)\) with the initial conditions
\[
\tilde{B}(t)|_{t=s} = B_0(s), \quad \tilde{C}(t)|_{t=s} = 0, \quad B_0^t(s) = B_0(s)
\] (6.11)
and the matrix \( \text{Im} \ B_0(s) \) be positive definite.

In view of \((6.2)\), the solution to the Cauchy problem \((A.2)\), \((6.11)\) will then have the form
\[
\tilde{B}(t) = \lambda_4(\Delta t) B_0(s), \quad \tilde{C}(t) = -\lambda_3(\Delta t) B_0(s), \quad \Delta t = t - s,
\] (6.12)
where the matrices \( \lambda_3(t) \) and \( \lambda_4(t) \) are defined in \((A.16)\). The matrix
\[
\tilde{Q}(t) = \tilde{C}(t) \tilde{B}^{-1}(t) = -\lambda_3(\Delta t) (\lambda_4^{-1}(\Delta t))^t
\] (6.13)
will then satisfy equation \((6.8)\) with the initial conditions \((6.7)\).

Provided that \((6.2)\) is valid, from \((A.12)\) and \((A.10)\) follows
\[
\tilde{G}(t) = (\tilde{B}^{-1}(t))^t B_0^t(s) (\vec{y} - \vec{x}_0) + \tilde{X}(t, \hbar) = \lambda_4^{-1}(\Delta t) (\vec{y} - \vec{x}_0) + \tilde{X}(t, \hbar).
\] (6.14)
In a similar manner, we obtain for \( S_0 \)
\[
S_0(t, s, h) = S(t, h) - S(s, h) + \frac{i\hbar}{2} \int_s^t d\tau \text{Sp} [\partial_{xx}(\tau) + \partial_{xx}(\tau) \tilde{Q}(\tau)] + \frac{1}{2} \int_s^t d\tau \langle (\tilde{G}(\tau) - \tilde{X}(\tau, \hbar)), \partial_{xx}(\tau) (\tilde{G}(\tau) - \tilde{X}(\tau, \hbar)) \rangle + (\vec{p}_0, \vec{y}).
\] (6.15)
In view of (A.12) and Liouville’s lemma (A.3), we obtain
\[
\frac{1}{2} \int_{s}^{t} d\tau \text{Sp} \left[ \hat{\Sigma}_{xp}(\tau) + \hat{\Sigma}_{xx}(\tau)\hat{Q}(\tau) \right] = \frac{1}{2} \ln \det \hat{B}^{-1}(\tau) = - \frac{1}{2} \ln \det \lambda_{4}(\Delta t).
\]
(6.16)

To calculate the last integral in (6.16), we use relationship (A.15) and, in view of (6.15), we get
\[
\frac{1}{2} \int_{s}^{t} d\tau \langle (\hat{G}(\tau) - \bar{X}(\tau, h))\hat{\Sigma}_{xx}(\tau)(\hat{G}(\tau) - \bar{X}(\tau, h)) \rangle = \frac{1}{2} \langle (\vec{y} - \vec{x}_{0})\lambda_{2}(\Delta t)\lambda_{4}^{-1}(\Delta t)(\vec{y} - \vec{x}_{0}) \rangle,
\]
(6.17)

where the matrix \(\lambda_{4}(t)\) is defined in (A.16). Hence, we have
\[
S_{0}(t, s, h) = S(t, h) - S(s, h) - \frac{i\hbar}{2} \ln(\det \lambda_{4}(\Delta t)) + \frac{1}{2} \langle (\vec{y} - \vec{x}_{0})\lambda_{2}(\Delta t)\lambda_{4}^{-1}(\Delta t)(\vec{y} - \vec{x}_{0}) + (\vec{p}_{0}, \vec{y}) \rangle.
\]
(6.18)

Substituting (6.13), (6.14), and (6.19) into (6.4), we obtain the well-known expression (see, e.g., [3])
\[
G^{(0)}(\vec{p}, \vec{y}, t, s, \Psi_{0}) = \frac{1}{(2\pi\hbar)^{n/2}} \frac{1}{\sqrt{\det \lambda_{4}(\Delta t)}} \exp \left\{ - \frac{i}{\hbar}(S(t, h) - S(s, h))- \frac{i}{2\hbar} \langle (\vec{y} - \vec{x}_{0})\lambda_{2}(\Delta t)\lambda_{4}^{-1}(\Delta t)(\vec{y} - \vec{x}_{0}) - \frac{i}{\hbar}\langle \vec{p}_{0}, \vec{y} \rangle - \frac{i}{\hbar} \langle \Delta \vec{p}, \bar{X}(t, h) \rangle - \frac{i}{\hbar} \langle \Delta \vec{p}, \lambda_{4}^{-1}(\Delta t)\lambda_{3}(\Delta t)\Delta \vec{p} \rangle \right\}.
\]
(6.19)

Now we substitute (6.19) into (6.4) and make use of the relationship
\[
\int_{\mathbb{R}^{n}} d\vec{x} \exp \left\{ - \frac{1}{2} \langle \vec{x}, \Gamma \vec{x} \rangle + \langle \vec{b}, \vec{x} \rangle \right\} = \sqrt{(2\pi\hbar)^{n} \det \Gamma^{-1}} \exp \left\{ \frac{\vec{b} \cdot \Gamma^{-1} \vec{b}}{2} \right\},
\]
(6.20)
in which we put \(\Gamma = -(i/\hbar)\lambda_{4}^{-1}(\Delta t)\lambda_{3}(\Delta t), \bar{b} = -(i/\hbar)[\bar{X}(t, h) - \vec{x} + \lambda_{4}^{-1}(\Delta t)(\vec{y} - \vec{x}_{0})].\) We then obtain
\[
G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_{0}) = \frac{1}{\sqrt{\det(-i2\pi\hbar\lambda_{3}(\Delta t))}} \exp \left\{ i \left\{ \frac{1}{\hbar}(S(t, h) - S(s, h)) + \langle \vec{P}(t, h), \Delta \vec{x} \rangle - \frac{i}{\hbar}\langle \vec{p}_{0}, (\vec{y} - \vec{x}_{0}) \rangle - \frac{1}{2} \langle (\vec{y} - \vec{x}_{0})\lambda_{1}(\Delta t)\lambda_{3}^{-1}(\Delta t)(\vec{y} - \vec{x}_{0}) - \frac{i}{\hbar} \langle \Delta \vec{x}, \lambda_{3}^{-1}(\Delta t)\lambda_{4}(\Delta t)\Delta \vec{x} \rangle \right\}.
\]
(6.21)

Here, we used the relationships
\[
\lambda_{1}^{(t)}(t)\lambda_{4}(t) - \lambda_{1}^{(t)}(t)\lambda_{2}(t) = \mathbb{I}_{n \times n}, \quad \lambda_{3}(t)\lambda_{4}^{(t)}(t) - \lambda_{4}(t)\lambda_{3}^{(t)}(t) = 0,
\]
that follow immediately from (A.5), (A.19), and from the definition of matriciant (A.16).

Let us consider the limit of expression (6.21) for \(\Delta t = t - s \to 0.\) We obtain
\[
\lambda_{1}(\Delta t) = \mathbb{I}_{n \times n} + O(\Delta t), \quad \lambda_{1}^{(t)}(t)\Delta t - \mathbb{I}_{n \times n} + O((\Delta t)^{2}),
\]
\[
\lambda_{3}^{-1}(\Delta t) = - \frac{1}{\Delta t} \delta_{pp}(s)\Delta t + O((\Delta t)^{2}), \quad \lambda_{4}(\Delta t) = \mathbb{I}_{n \times n} + O(\Delta t), \quad \lambda_{2}(\Delta t) = O(\Delta t).
\]
It follows that for short times we have (see, e.g., [41])

\[
\lim_{\Delta t \to 0} G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0) = \frac{1}{\sqrt{\text{det}(-i2\pi \hbar \Delta t \delta_{pp}(s))}} \times \\
\exp \left\{ \frac{i}{2\hbar \Delta t} \left( (\vec{x} - \vec{y}), \delta_{pp}^{-1}(s)(\vec{x} - \vec{y}) \right) + O(\Delta t^0) \right\}.
\]

(6.22)

Thus we have proved

**Theorem 6.1** Let the symbols of the operators \( \hat{H}(t) \) and \( \hat{V}(t, \Psi) \) satisfy the conditions of Supposition 4. Then the function

\[
\Psi^{(0)}(\vec{x}, t, \hbar) = \hat{U}^{(0)}(t, 0, \Psi_0) \Psi_0,
\]

(6.23)

where \( \hat{U}^{(0)}(t, 0, \Psi_0) \) is the evolution operator of the zero-order associated Schrödinger equation (6.14) with the kernel \( G^{(0)}(\vec{x}, \vec{y}, t, 0, \Psi_0) \), is an asymptotical (accurate to \( O(\hbar^{3/2}) \), \( \hbar \to 0 \)) solution of the Hartree type equation (1.1) and satisfies the initial condition

\[
\Psi^{(0)}(\vec{x}, t, \hbar)|_{t=0} = \Psi_0.
\]

**Remark 6.1** The principal term of the quasi-classical asymptotic \( \Psi^{(0)}(\vec{x}, t, \hbar) \) will not change (accurate to \( O(\hbar^{3/2}) \), \( \hbar \to 0 \)) if the phase function \( S^{(N)}(t, \hbar) \) in the operator \( \hat{S}_0(t, \Psi_0) \) is substituted by its value \( S^2(t, \hbar) \) for \( N = 2 \) and we restrict ourselves to the first terms in \( \hbar \to 0 \) in the phase trajectory \( Z^{(2)}(t, \hbar) \), and in the other expressions \( Z^{(N)}(t, \hbar) \) is changed by \( Z^0(t, \hbar) \).

## 7 Quasi-classically concentrated solutions of the Hartree type equation

Now we construct asymptotical solutions to the Hartree type equation (1.1) with an arbitrary accuracy in powers of \( \sqrt{\hbar} \). To do this, we find asymptotical solutions to the associated linear Schrödinger equation (4.6) with an arbitrary accuracy in powers of \( \sqrt{\hbar} \). Let us present an arbitrary initial condition \( \Phi_0(\vec{x}, \hbar) \in \mathcal{P}_h^0 \)

\[
\Phi_0(\vec{x}, \hbar) = \sum_{k=0}^{N} \hbar^{k/2} \Phi_0^{(k)}(\vec{x}, \hbar),
\]

(7.1)

where

\[
\Phi_0^{(k)}(\vec{x}, \hbar) \in C_\delta^k(z_0, S_0).
\]

Then for the recurrent associated linear equations (4.14)–(4.16) we arrive at a Cauchy problem with initial data:

\[
\Phi^{(k)}|_{t=0} = \Phi_0^{(k)}(\vec{x}, \hbar) \quad k = 0, N.
\]

The solution to these recurrent equations can readily be constructed as its expansion over the complete set of orthonormalized Fock functions \( \{\nu, t\} (7.15) \). As a result we obtain

\[
\Phi^{(0)}(\vec{x}, t, \hbar) = \sum_{|\nu|=0}^{\infty} \langle \nu, t, \Psi_0 | \Psi_0, 0, \nu | \Phi_0^{(0)}(\vec{x}, \hbar) \rangle,
\]

(7.2)

\[
\Phi^{(1)}(\vec{x}, t, \hbar) = \sum_{|\nu|=0}^{\infty} \langle \nu, t, \Psi_0 | \Psi_0, 0, \nu | \Phi_0^{(1)}(\vec{x}, \hbar) \rangle -
\]

\[
-\frac{i}{\hbar} \sum_{|\nu|=0}^{\infty} \langle \nu, t, \Psi_0 | \Psi_0, \tau, \nu | \hat{S}_1(t, \Psi_0) \Phi^{(0)}(\vec{x}, \tau, \hbar) \rangle,
\]

(7.3)

\[
\Phi^{(2)}(\vec{x}, t, \hbar) = \sum_{|\nu|=0}^{\infty} \langle \nu, t, \Psi_0 | \Psi_0, 0, \nu | \Phi_0^{(2)}(\vec{x}, \hbar) \rangle -
\]

(7.3)
where for the associated linear Schrödinger equation (4.8) has the form

\[ -\frac{i}{\hbar} \sum_{|\nu|=0}^{\infty} \nu \int_0^t d\tau \langle \Psi_0, \tau, \nu | \hat{\mathcal{S}}_1(t, \Psi_0) \Phi(t, \tau, h) \rangle - \]

\[ -\frac{i}{\hbar} \sum_{|\nu|=0}^{\infty} \nu \int_0^t d\tau \langle \Psi_0, \tau, \nu | \hat{\mathcal{S}}_2(t, \Psi_0) \Phi(t, \tau, h) \rangle, \]

(7.4)

Theorem 7.1 Let the symbols of the operators \( \hat{\mathcal{H}}(t) \) and \( \hat{V}(t, \Psi) \) satisfy the conditions of Supposition (4). Then the function

\[ \Psi^{(N)}(x, t, \hbar) = \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ -\frac{i}{\hbar} \hat{\mathcal{F}}^{(N)}(t, \Psi_0) \right\}^k \hat{U}_0(t, 0, \Psi_0) \Psi_0(x, \hbar), \]

(7.7)

where \( N \geq 2 \), is an asymptotical, accurate to \( O(\hbar^{N+1/2}) \), solution of equation (1.1) and satisfies the initial condition (2.8).

8 The Green function and the nonlinear superposition principle

Let us show that in the class of trajectory-concentrated functions for the Hartree type equation (1.1) we can construct, with any given accuracy in \( \hbar^{1/2} \), the kernel of the evolution operator or the Green function of the Cauchy problem for equation (1.1). The explicit form of the quasi-classical asymptotics \( \Psi^{(N)}(x, t, \hbar) \) (7.7) makes it possible to obtain an expression for the Green function \( G^{(N)}(x, y, t, s, \Psi_0) \) valid on finite time intervals \( t \in [0, T] \). Actually, according to (7.7), for any function \( \varphi(x, \hbar) \in \mathcal{P}_h \), the solution of the Cauchy problem with the initial condition

\[ \Phi(x, t, \hbar)|_{t=0} = \varphi(x, \hbar) \]

(8.1)

for the associated linear Schrödinger equation (4.8) has the form

\[ \Phi^{(N)}(x, t, \hbar) = \]

\[ = \hat{\mathcal{F}}^{(N)}(t, \Psi_0) \int_{\mathbb{R}^n} d\tilde{y} G^{(0)}(x, \tilde{y}, t, 0, \Psi_0) \varphi(\tilde{y}, \hbar) + O(\hbar^{N+1/2}) = \]

\[ = \int_{\mathbb{R}^n} d\tilde{y} G^{(N)}(x, \tilde{y}, t, 0, \Psi_0) \varphi(\tilde{y}, \hbar) + O(\hbar^{N+1/2}), \]

where

\[ \hat{\mathcal{F}}^{(N)}(t, \Psi_0) = \sum_{k=0}^{N} \frac{1}{k!} \left\{ -\frac{i}{\hbar} \hat{\mathcal{F}}^{(N)}(t, \Psi_0) \right\}^k, \]

(8.2)

and the function \( G^{(0)}(x, \tilde{y}, t, s, \Psi_0) \) is defined in (6.21).

It follows that

\[ G^{(N)}(x, \tilde{y}, t, 0, \Psi_0) = \hat{\mathcal{F}}^{(N)}(t, \Psi_0) G^{(0)}(x, \tilde{y}, t, 0, \Psi_0). \]
Since $\hat{R}^{(N)}(0, \Psi_0) = 1$, we have for an arbitrary $s < t$

$$G^{(N)}(\vec{x}, \vec{y}, t, s, \Psi_0) = \hat{R}^{(N)}(t, \Psi_0)G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0)(\hat{R}^{(N)}(s, \Psi_0))^+$$  \hspace{1cm} (8.3)

being the Green function of the Cauchy problem $\hat{S}_I^+(t)$ with $s \neq 0$. Obviously, for the functions $G^{(N)}(\vec{x}, \vec{y}, t, s, \Psi_0)$ the following composition rule is valid:

$$\int d\vec{u}G^{(N)}(\vec{x}, \vec{u}, t, \tau, \Psi_0)G^{(N)}(\vec{u}, \vec{y}, \tau, s, \Psi_0) = G^{(N)}(\vec{x}, \vec{y}, t, s, \Psi_0) + O(\hbar^{(N+1)/2}).$$  \hspace{1cm} (8.4)

Denoting by $\hat{U}^{(N)}(t, 0, \Psi_0)$ the approximate evolution operator of the linear equation (4.8)

$$\hat{U}^{(N)}(t, 0, \Psi_0)\varphi(\vec{x}, \hbar) = \int d\vec{y}G^{(N)}(\vec{x}, \vec{y}, t, 0, \Psi_0)\varphi(\vec{y}, \hbar),$$

we obtain it from (8.3) in the form of the T-ordered Dyson expansion

$$\hat{U}^{(N)}(t, 0, \Psi_0) = \sum_{k=0}^N \left( -\frac{i}{\hbar} \right)^k \int d^k\tau \hat{S}_1(\tau_1, t, \Psi_0) \cdots \hat{S}_1(\tau_k, t, \Psi_0) \hat{U}_0(t, 0, \Psi_0).$$  \hspace{1cm} (8.5)

Here, we have used the following notations $[11]$: the domain of integration is an open hypertriangle

$$\Delta_2^\tau \equiv \{ \tau \in [0, t^k] | t > \tau_1 > \tau_2 > \cdots > \tau_N > s \};$$

the operator $\hat{S}_1(\tau, t, \Psi_0)$ is a perturbation operator in the representation of the interaction

$$\hat{S}_1(\tau, t) = \hat{U}_0(t, \tau, \Psi_0)\hat{S}_1^{(N)}(\tau, \Psi_0)\hat{U}_0^+(\tau, \Psi_0),$$  \hspace{1cm} (8.6)

where $\hat{S}_1^{(N)}(t, \Psi_0)$ has been defined in (4.4), and $\hat{U}_0(t, s, \Psi_0)$ is the evolution operator of the associated linear Schrödinger equation with the kernel $G^{(0)}(\vec{x}, \vec{y}, t, s, \Psi_0)$.  \hspace{1cm} (8.7)

In view of Statement 4.1, the action of operator $\hat{U}_0(t, s, \Psi_0)$ on the function $\varphi = \Psi_0(\vec{x}, \hbar)$ determines the asymptotical solution of the Cauchy problem (4.4)–(2.8) for the Hartree type equation (1.1)

$$\Psi^{(N)}(\vec{x}, t, \hbar) = \hat{U}^{(N)}(t, 0, \Psi_0)\Psi_0(\vec{x}, \hbar), \quad \Psi_0(\vec{x}, \hbar) \in \mathcal{P}^0_\hbar.$$  \hspace{1cm} (8.7)

It follows that operator $\hat{S}_1^{(N)}(t, \Psi_0)$ is an approximate evolution operator for the Hartree type equation (1.1) in the class of trajectory-concentrated functions.

For the constructed asymptotical solutions, from expression (8.7) immediately follows (15)–(16).

**Theorem 8.1 (nonlinear superposition principle)** Let $\Psi_1(\vec{x}, t, \hbar, y_{11}^{(N)}(t, \hbar))$ be an asymptotical, accurate to $O(\hbar^{(N+1)/2})$, solution of the Cauchy problem for the Hartree type equation (1.1) with the initial condition $\Psi_{01}(\vec{x}, \hbar)$ and the function $\Psi_2(\vec{x}, t, \hbar, y_{02}^{(N)}(t, \hbar))$ is a solution of the same problem with the initial condition $\Psi_{02}(\vec{x}, \hbar)$. Then the solution of the Cauchy problem with the initial condition

$$\Psi_{03}(\vec{x}, \hbar) = c_1\Psi_{01}(\vec{x}, \hbar) + c_2\Psi_{02}(\vec{x}, \hbar), \quad c_1, c_2 = \text{const},$$

has the form

$$\Psi_3(\vec{x}, t, \hbar, y_{33}^{(N)}(t, \hbar)) = \hat{U}^{(N)}(t, 0, \Psi_{03})\Psi_{03}(\vec{x}) =$$

$$= c_1\hat{U}^{(N)}(t, 0, \Psi_{03})\Psi_{01}(\vec{x}) + c_2\hat{U}^{(N)}(t, 0, \Psi_{03})\Psi_{02}(\vec{x}) =$$

$$= c_1\Psi_1(\vec{x}, t, \hbar, y_{33}^{(N)}(t, \hbar)) + c_2\Psi_2(\vec{x}, t, \hbar, y_{33}^{(N)}(t, \hbar)).$$

Here, $y_{k}^{(N)}(t, \hbar)$ denotes the solution of the Hamilton–Ehrenfest equations of order $N$, $N \geq 2$ (3.4) with an initial condition which is determined from the functions $\Psi_{0k}(\vec{x}, \hbar)$, $k = 1, 3$, respectively.
9 The one-dimensional Hartree type equation with a Gaussian potential

Let us illustrate the above scheme for constructing asymptotical solutions by the example of a nonlinear interaction with a Gaussian potential [17]. By this example we shall show in an explicit form how the procedure of constructing quasi-classically concentrated solutions to the Hartree type equation necessarily leads to Hamilton–Ehrenfest equations. Moreover, it becomes possible to elucidate the “nonlinearity” of the generalized superposition principle for the Hartree type equation.

We write equation (1.1) with a Gaussian potential for the one-dimensional case as

\[
\left\{-i\hbar \frac{\partial}{\partial t} + \frac{(\hat{p})^2}{2m} + \kappa V_0 \int_{-\infty}^{+\infty} dy \exp \left[ \frac{(x-y)^2}{2\gamma^2} \right] |\Psi(y,t)|^2 \right\} \Psi = 0. \tag{9.1}
\]

In this case, for the class of functions \(P_h^i(S(t, \hbar), Z(t))\) in which we shall find solutions to equation (9.1), in accordance with (2.4), we find

\[
P_h^i(S(t, \hbar), Z(t)) = \left\{ \Psi : \Psi(x,t,\hbar) = \varphi \left( \frac{\Delta x}{\sqrt{\hbar}} t, \hbar \right) \exp \left[ \frac{i}{\hbar} (S(t) + Pt \Delta x) \right] \right\}. \tag{9.2}
\]

Here, the function \(\varphi(\xi, t, \hbar) \in \mathcal{S}\) (Schwartz space) with respect to the variable \(\xi = \frac{\Delta x}{\sqrt{\hbar}}\) and depends regularly on \(\hbar\) with \(\Delta x = x - x(t)\). The functions \(S(t, \hbar)\) and \(Z(t) = (P(t), X(t))\) are real, depend regularly on \(\hbar\), and are to be determined.

Let us expand the exponential in equation (9.1) in a Taylor series for \(\Delta x = x - X(t)\), \(\Delta y = y - X(t)\) and restrict ourselves to the terms of the order of not above four in \(\Delta x\) and \(\Delta y\). In view of the estimates (2.7), equation (9.1) will then take the form

\[
\left\{-i\hbar \frac{\partial}{\partial t} + \frac{P^2(t)}{2m} + \frac{P(t)\Delta \hat{p}}{m} + \frac{\Delta p^2}{2m} + \right.
\]

\[+ \tilde{\kappa} V_0 \left[ 1 - \frac{1}{2\gamma^2} (\Delta x^2 - 2\Delta x \alpha_\psi^{(1)}(t, \hbar) + \alpha_\psi^{(2)}(t, \hbar)) + \right.\]

\[+ \frac{1}{8\gamma^2} (\Delta x^4 - 4\Delta x^3 \alpha_\psi^{(1)}(t, \hbar) + 6\Delta x^2 \alpha_\psi^{(2)}(t, \hbar) - \right.\]

\[\left. - 4\Delta x \alpha_\psi^{(3)}(t, \hbar) + \alpha_\psi^{(4)}(t, \hbar) \right]\right\} \Psi = O(h^{5/2}), \tag{9.3}
\]

where \(\Delta \hat{p} = \hat{p} - P(t), \tilde{\kappa} = \kappa ||\Psi||^2\), and

\[
\alpha_\psi^{(k)}(t, \hbar) = \frac{1}{||\Psi||^2} \int_{-\infty}^{+\infty} (\Delta y)^k |\Psi(y,t)|^2 dy, \quad k = 0, \infty
\]

are the \(k\)-order moments centered about \(X(t)\). Equation (9.3) can be simplified if we make the change

\[
\Psi(x,t,\hbar) = \exp \left\{ -\frac{i}{\hbar} \int_0^t \left[ \frac{P^2(t)}{2m} - \tilde{\kappa} V_0 + \frac{\tilde{\kappa}}{2\gamma^2} V_0 \sigma_{xx}(t, \hbar) \right] dt \right\} \Phi(x, t, \hbar), \tag{9.4}
\]

where

\[
\sigma_{xx}(t, \hbar) = \alpha_\psi^{(2)}(t, \hbar) = \frac{1}{||\Psi||^2} \int_{-\infty}^{+\infty} \Delta y^2 |\Psi(y,t)|^2 dy
\]
is the variance. The function $\Phi(x, t, \hbar)$ belongs to the class $\mathcal{P}_h^k(S(t, \hbar), Z(t))$, where

$$\tilde{S}(t, \hbar) = S(t, \hbar) - \int_0^t \left[ \frac{P^2(t)}{2m} + \tilde{z} V_0 - \frac{\tilde{z}}{2\gamma^2} V_0 \sigma_{xx}(t, \hbar) \right] dt,$$

and satisfies the equation

$$\left\{ -i\hbar \partial_t + \frac{P(t) \Delta \tilde{p}}{m} + \frac{\Delta \tilde{p}^2}{2m} + \tilde{z} V_0 \left[ -\frac{1}{2\gamma^2} \Delta x^2 - 2\Delta x^2 \alpha^{(1)}_\Psi(t, \hbar) + \frac{1}{8\gamma^4} (\Delta x^3 - 4\Delta x^3 \alpha^{(1)}_\Psi(t, \hbar) + 6\Delta x^2 \alpha^{(2)}_\Psi(t, \hbar) - 4\Delta x \alpha^{(3)}_\Psi(t, \hbar) + \alpha^{(4)}_\Psi(t, \hbar)) \right] \right\} \Phi = O(\hbar^{5/2}). \tag{9.5}$$

Here, we have made use of

$$\alpha^{(k)}_\Psi(t, \hbar) = \alpha^{(k)}_\Psi(t, \hbar), \quad k = 1, N.$$  

We shall seek the approximate (mod $\hbar^{5/2}$) solution $\Phi(x, t, \hbar)$ to equation (9.5) in the form

$$\Phi(x, t, \hbar) = \Phi^{(0)}(x, t) + \sqrt{\hbar} \Phi^{(1)}(x, t) + \hbar \Phi^{(2)}(x, t) + \ldots,$$

where $\Phi^{(k)}(x, t) \in \mathcal{C}_h^k(S(t, \hbar), Z(t))$. In equation (9.5) we equate the terms having the same estimate in $\sqrt{\hbar}$ in the sense of (2.7). Denote by $\hat{L}_0$ the operator

$$\hat{L}_0 = -i\hbar \partial_t + \frac{1}{m} P(t) \Delta \tilde{p} + \frac{1}{2m} \Delta \tilde{p}^2 - \frac{\tilde{z}}{2\gamma^2} \Delta x^2;$$

Earlier we have shown that $\hat{L}_0 = \hat{O}(\hbar)$. We then have

$$h^1 : \ (\hat{L}_0 + \frac{\tilde{z}}{2\gamma^2} \Delta x \alpha^{(1)}_\Psi(0)) \Phi^{(0)} = 0; \tag{9.6}$$

$$h^{3/2} : \ \sqrt{\hbar} (\hat{L}_0 + \frac{\tilde{z}}{2\gamma^2} \Delta x \alpha^{(1)}_\Psi(0)) \Phi^{(1)} = -2\sqrt{\hbar} \frac{\tilde{z} V_0}{\gamma^2} \Delta x \operatorname{Re} \langle \Phi^{(0)} | \Delta x | \Phi^{(1)} \rangle \Phi^{(0)},$$

$$h^2 : \ h (\hat{L}_0 + \frac{\tilde{z}}{2\gamma^2} \Delta x \alpha^{(1)}_\Psi(0)) \Phi^{(2)} = -2h \frac{\tilde{z} V_0}{\gamma^2} \Delta x \left\{ \left[ 2 \operatorname{Re} \langle \Phi^{(0)} | \Delta x | \Phi^{(2)} \rangle + \langle \Phi^{(1)} | \Delta x | \Phi^{(1)} \rangle \right] \Phi^{(0)} + 2 \operatorname{Re} \langle \Phi^{(0)} | \Delta x | \Phi^{(1)} \rangle \Phi^{(1)} \right\} - \hat{\gamma}(t) \Phi^{(0)};$$

The function

$$\Phi^{(0)}(x, t) = \Phi^{(0)}(x, t) = \left( \frac{1}{|C(t)|} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left( \frac{v^2 t}{m} + v \Delta x + \frac{1}{2} \Delta x^2 m \frac{C(t)}{C(t)} \right) \right\} =$$

$$= \left( \frac{1}{|C(t)|} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left( \frac{v(x - x_0)}{m} + \frac{1}{2} \Delta x^2 m \frac{C(t)}{C(t)} \right) \right\}$$

is a solution of equation (9.4). Here, we have used the fact that $X(t)$ and $P(t)$ are solutions of the ordinary differential equations

$$\left\{ \begin{array}{l}
\dot{p} = 0 \\
\dot{x} = \frac{p}{m}, \\
p(0) = p_0, x(0) = x_0,
\end{array} \right. \tag{9.7}$$

and $C(t)$ denotes the complex function satisfying the equations

$$\left\{ \begin{array}{l}
\dot{B} = \frac{\tilde{z} V_0}{\gamma^2} C \\
\dot{C} = \frac{B(t)}{m}.
\end{array} \right. \tag{9.8}$$
Equations (9.7) are Hamiltonian equations with the Hamiltonian \( H(p, x, t) = \frac{p^2}{2m} \) and their solution is
\[
P(t) = mv = p_0, \quad X(t) = \frac{p_0}{m} t + x_0.
\]
Similarly, equations (9.7) are Hamiltonian equations for a harmonic oscillator with frequency
\[\Omega = \sqrt{\frac{\hbar |V_0|}{m \gamma}},\]
and their solution is
\[
\begin{cases}
C(t) = c_1 \exp \left(-\sqrt{\frac{\hbar |V_0|}{m \gamma^2}} t\right) + c_2 \exp \left(\sqrt{\frac{\hbar |V_0|}{m \gamma^2}} t\right) \\
\dot{B}(t) = m \dot{C}(t).
\end{cases}
\]
For the initial conditions (5.13)
\[C(0) = 1, \quad \dot{B}(0) = b, \quad \text{Im} \ b > 0\]
two cases are possible:
1) \( \hbar V_0 > 0 : \quad C(t) = \cosh(\Omega t) + \frac{b}{\Omega} \sinh(\Omega t) \)
2) \( \hbar V_0 < 0 : \quad C(t) = \cos(\Omega t) + \frac{b}{\Omega} \sin(\Omega t). \)

In the linear case (\( \hbar = 0 \)), the frequency \( \Omega = 0 \) and equations (9.8) become equations in variations for equations (9.7). In view of (9.8), we find the variance of the coordinate \( x \) in an explicit form:
\[
\sigma_{xx}(t, \hbar) = \sqrt{\frac{m \text{Im} \ b}{\pi \hbar}} \cdot \int_{-\infty}^{+\infty} \frac{\Delta x^2}{|C(t)|} \exp \left[-\frac{m}{\hbar} \Delta x^2 \frac{\text{Im} \ b}{|C(t)|^2}\right] dx = \frac{|C(t)|^2 \hbar}{2m \cdot \text{Im} \ b}.
\]

Then we get
\[
\Psi_0^{(0)}(x, t, \hbar) = \exp \left[-i \frac{\hbar}{2m} \left(\frac{v^2}{2m} + \hbar V_0 t + \hbar V_0 \frac{\hbar}{2m \cdot \text{Im} \ b} \int \frac{t}{0} |C(t)|^2 dt\right)\right] \Phi_0^{(0)}(x, t, \hbar).
\]
It can readily be noticed that \( \alpha_{\Phi_0^{(0)}}(t, \hbar) = \alpha_{\Phi_0^{(0)}}(t, \hbar) \). Hence, from (13.5) and (9.14) it can be inferred that for \( \hbar V_0 < 0 \) the variance \( \alpha_{\Phi_0^{(0)}}(t, \hbar) \) is limited in \( t \), i.e., we have \( |\sigma_{xx}(t)| \leq M, \ M = \text{const} \), and for \( \hbar V_0 > 0 \) it increases exponentially. In the limit of \( \gamma \to 0 \) and with \( V_0 = (2\pi \gamma)^{-1/2} \), equation (13.3) becomes a nonlinear Schrödinger equation, while in the case where \( \hbar V_0 < 0 \) (\( \hbar V_0 > 0 \)) it corresponds to the condition of existence (nonexistence) of solitons. Note that if \( \alpha_{\Phi_0^{(0)}}(t, \hbar) = 0 \), the equation for the function \( \Phi^{(0)} \) takes the form
\[
\hat{L}_0 \cdot \Phi^{(0)} = 0
\]
becoming a Schrödinger equation with a quadric Hamiltonian. We shall find the solution to equation (9.11) satisfying an additional condition \( \alpha_{\Phi_0^{(0)}}(t, \hbar) = 0 \). To do this, we denote
\[
\dot{a}(t) = N_0(C(t) \Delta \dot{p} - B(t) \Delta x).
\]
If \( C(t) \) and \( B(t) \) are solutions of equations (9.8), the operator \( \dot{a}(t) \) commutes with the operator \( \hat{L}_0 \). So the function
\[
\Phi_k^{(0)} = \frac{1}{k!} (\hat{a}^+)^k \Phi_0^{(0)}
\]
we get

$$H_{\nu}(\xi)$$ are Hermite polynomials. Determining \( N_{\alpha} \) from the condition \([\hat{a}(t), \hat{a}^+(t)] = 1\) and representing the solution of the equations in variations as

$$C(t) = |C(t)| \exp \{- i \text{Arg}[C(t)]\},$$

we get

$$\Phi_{\nu}^{(0)}(x, t) = \frac{1}{k!} (-i)^k \exp \{- i k \text{Arg}[C(t)]\} \left( \frac{1}{\sqrt{2}} \right)^k H_k \left( \frac{\sqrt{m} \text{Im} b}{|C(t)| \sqrt{\hbar}} \right)^k \Phi_{\nu}^{(0)}(x, t). \quad (9.12)$$

Using the properties of Hermite polynomials, we can readily be convinced that the mean \( \alpha_{\Phi_{\nu}^{(0)}}(t, \hbar) = 0, k = \infty \). Then we have

$$\Psi_{\nu}^{(0)}(x, t, \hbar) = \exp \left\{ - \frac{i}{\hbar} \left[ \left( \frac{\nu^2}{2m} + \nu V_0 \right)t + \nu V_0 \langle \Phi_{\nu}^{(0)} | \Delta x^2 | \Phi_{\nu}^{(0)} \rangle \right] \right\} \Phi_{\nu}^{(0)}(x, t, \hbar).$$

Similarly, we find

$$\langle \Phi_{\nu}^{(0)} | \Delta x^2 | \Phi_{\nu}^{(0)} \rangle = \frac{1}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \Delta x^2 | \Phi_{\nu}^{(0)}(x, t) |^2 dx = \frac{\hbar |C(t)|^2 (2k + 1)}{2m \text{Im} b}$$

and for functions \( \Psi_{\nu}^{(0)}(t, \hbar) \) obtain

$$\Psi_{\nu}^{(0)}(x, t, \hbar) = \exp \left[ - \frac{i}{\hbar} \left( \frac{\nu^2}{2m} + \nu V_0 \right)t + \nu V_0 \frac{h(2k + 1)}{2m \text{Im} b} \int_0^t |C(\tau)|^2 d\tau \right] \Phi_{\nu}^{(0)}(x, t, \hbar). \quad (9.13)$$

The functions \((9.13)\) are approximate, accurate to \(O(\hbar^{3/2})\), solutions of the Hartree type equation \((9.1)\). However, since for the linear combination

$$\Phi(x, t) = c_1 \Phi_{\nu}^{(0)}(x, t) + c_2 \Phi_{\nu}^{(0)}(x, t) \quad (9.14)$$

the condition \( \alpha_{\Phi}^{(1)}(t, \hbar) = 0 \) is not fulfilled, \( \Phi(x, t) \) is not a solution of equation \((9.1)\) and, hence, the linear superposition principle is invalid for the functions \((9.13)\) even in the class of asymptotical solutions \(P_{\nu}^{(0)}(S(t, \hbar), P(t), X(t))\) accurate to \(O(\hbar^{3/2})\). Thus, the presence of the term \( \alpha_{\Phi_{\nu}^{(0)}}^{(1)}(t, \hbar) \) in equation \((9.6)\) violates the linear superposition principle \((9.14)\).

We seek the solution to equation \((9.1)\) in the class \(P_{\nu}^{(0)}(S(t, \hbar), Z(t, \hbar))\), i.e., localize the solution asymptotically in the neighborhood of the trajectory \(z = Z(t, \hbar)\) depending explicitly on parameter \(\hbar\). With that, the estimates \((2.7)\) remain valid. Let us take the dependence of \(Z(t, \hbar)\) on the parameter \(\hbar \to 0\) such that the equation for the function \(\Phi^{(0)}(x, t, \hbar)\) be linear. For doing this, we subject the functions \(X(t, \hbar)\) and \(P(t, \hbar)\) to the equations

$$\begin{align*}
\dot{p} &= \frac{2\nu V_0}{\nu^2} \alpha_{\Phi_{\nu}^{(0)}}^{(1)}(t, \hbar) + \frac{1}{\gamma^2} \alpha_{\Phi_{\nu}^{(0)}}^{(3)}(t, \hbar), \\
\dot{x} &= p/m,
\end{align*} \quad (9.15)$$
and the functions \( C(t) \) and \( B(t) \) to the equations

\[
\begin{aligned}
\dot{B} &= \frac{2V_0}{\gamma^2} C + \frac{3}{4\gamma^2} a_{\Phi(0)}^{(2)}(t, \hbar) C, \\
\dot{C} &= B/m.
\end{aligned}
\]  

(9.16)

The function \( \Phi^{(0)}(x, t, \hbar) \) will then satisfy the equation

\[
\hat{L}_0 \Phi^{(0)}(0) = 0.
\]  

(9.17)

Unlike equations (9.6)–(9.8), equations (9.15)–(9.17) are dependent. Note that, within the accuracy under consideration, the principal term of the asymptotic will not change if equations (9.15) and (9.16) are solved accurate to \( \mathcal{O}(\hbar^{3/2}) \) and \( \mathcal{O}(\hbar) \), respectively. Then equations (9.15) become

\[
\begin{aligned}
\dot{p} &= 2\tilde{\kappa} V_0 \gamma^2 \Phi(0)(t, \hbar) \\
\dot{x} &= p/m.
\end{aligned}
\]  

(9.18)

and equations (9.16) coincide with equations (9.8) and their solution has the form (9.9). Equation (9.17) is linear and its general solution can be represented as an expansion over a complete set of orthonormalized functions \( \Phi^{(0)}_k(x, t, \hbar) \):

\[
\Phi^{(0)}(x, t, \hbar) = \sum_{k=0}^{\infty} c_k \Phi^{(0)}_k(x, t, \hbar).
\]  

(9.19)

Here, \( \Phi^{(0)}_k(x, t) \) is determined by expression (9.12) where \( X(t) \) and \( P(t) \) ought to be replaced by \( X(t, \hbar) \) and \( P(t, \hbar) \), respectively. Substitute (9.19) into (9.9). In view of the properties of Hermite polynomials

\[
\int_{-\infty}^{+\infty} H_n(\xi) H_l(\xi) e^{-\xi^2} d\xi =
\]

\[
= \int_{-\infty}^{+\infty} \left[ \frac{1}{2} H_{n+1}(\xi) + nH_{n-1}(\xi) \right] H_l(\xi) e^{-\xi^2} d\xi = \frac{1}{2} \delta_{n+1,l} + n\delta_{n-1,l},
\]

we obtain

\[
a_{\Phi(0)}^{(1)}(t, \hbar) = \frac{\sqrt{\hbar} C(t)}{m \Im b} \sum_{n=0}^{\infty} \left( \frac{1}{2} \delta_{n+1,l} + n\delta_{n-1,l} \right) c_n c_l^* =
\]

\[
= \frac{\sqrt{\hbar} C(t)}{m \Im b} \sum_{n=0}^{\infty} \left( \frac{1}{2} c_{n+1}^* + nc_{n-1}^* \right) c_n.
\]

Equations (9.18) will then take the form

\[
\begin{aligned}
\dot{p} &= \Theta_1 \sqrt{\hbar} \cdot |C(t)| \\
\dot{x} &= \frac{p}{m},
\end{aligned}
\]  

where

\[
\Theta_1 = \frac{2\tilde{\kappa} V_0}{m \gamma^2 \Im b} \sum_{n=0}^{\infty} \left( \frac{1}{2} c_{n+1}^* + nc_{n-1}^* \right) c_n.
\]  

(9.20)

Integration of the obtained equations yields

\[
\begin{aligned}
P(t, \hbar) &= m \dot{X}(t, \hbar), \\
X(t, \hbar) &= \frac{\sqrt{\hbar} \Theta_1}{m} \int_0^t dt' \int_0^{t'} |C(s)| ds + \frac{p_0}{m} t + x_0.
\end{aligned}
\]
As a result, the principal term of the asymptotic can be represented in the form

$$\Psi^{(0)}(x, t, h) = \exp \left\{ - \frac{i}{\hbar} \int_0^t \left( P(\tau, h) \dot{X}(\tau, h) - \frac{P^2(\tau, h)}{2m} + h \Theta_2 |C(\tau)|^2 \right) d\tau \right\} \Phi^{(0)}(x, t, h), \quad (9.21)$$

where

$$\Theta_2 = \frac{\tilde{z}V_0}{2m \text{Im} \eta} \sum_{n=0}^{\infty} \left[ \frac{1}{4} c_{n+2}^* + (n + \frac{1}{2}) c_n^* + \left( n^2 - n \right) c_{n-2}^* \right] c_n. \quad (9.22)$$

It follows that function (9.21) depends on $\Theta_1$ and $\Theta_2$ as on parameters:

$$\Psi^{(0)}(x, t, h) = \Psi^{(0)}(x, t, h, \Theta_1, \Theta_2).$$

Here, $\Theta_1$ and $\Theta_2$ are determined by the sets of equations (9.20) and (9.22), respectively.

Let us consider the Cauchy problem for equation (1.1):

$$\begin{align*}
\Psi_1 \big|_{t=0} &= \Psi_{10}(x); & \Psi_2 \big|_{t=0} &= \Psi_{20}(x), \\
\Psi_3 \big|_{t=0} &= \Psi_{30}(x) = G_1 \Psi_{10}(x) + G_2 \Psi_{20}(x), \\
\Psi_k(x) &\in \mathcal{P}_h^k.
\end{align*} \quad (9.23)$$

where $G_k = \text{const}$. Denote by $\Psi_k(x, t, h, \Theta_1^k, \Theta_2^k)$ the principal term of the asymptotic solution to equation (1.1), satisfying the initial conditions (1.23). Then from the explicit form of function (9.13) follows

$$\Psi_3(x, t, h, \Theta_1^3, \Theta_2^3) = G_1 \Psi_2(x, t, h, \Theta_1^2, \Theta_2^2) + G_2 \Psi_2(x, t, h, \Theta_1^2, \Theta_2^2). \quad (9.24)$$

Relationship (9.24) represents the nonlinear superposition principle for the asymptotical solutions of equation (1.1) in the class $\mathcal{P}_h^k(S(t, h), Z(t, h))$.

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Appendix A. The set of equations in variations

We already mentioned that to construct solutions to equation (4.14) in the class $\mathcal{P}_h^k$, it is necessary to find solutions to the equations in variations (2.4) and to the type Riccati matrix equation (2.3). Let us show that the solutions of the Riccati type matrix equation can be completely expressed through the solutions of the equations in variations $a(t)$.

Let us present the $2n$-space vector $a(t)$ in the form

$$a(t, \Psi_0) = (\hat{W}(t, \Psi_0), \hat{Z}(t, \Psi_0)),$$

where the $n$-space vector $\hat{W}(t) = \hat{W}(t, \Psi_0)$ is the “momentum” part and $\hat{Z}(t) = \hat{Z}(t, \Psi_0)$ is the “coordinate” part of the solution of the equations in variations. Thus we can write the latter as

$$\begin{align*}
\dot{\hat{W}} &= -S_{xp}(t, \Psi_0)\hat{W} - S_{xx}(t, \Psi_0)\hat{Z}, \\
\dot{\hat{Z}} &= S_{pp}(t, \Psi_0)\hat{W} + S_{px}(t, \Psi_0)\hat{Z}.
\end{align*} \quad (A.1)$$
The set of equations (A.1) is called a set of equations in variations in vector form. Denote by $B(t)$ and $C(t)$ the $n \times n$ matrices composed of the “momentum” and “coordinate” parts of the solution of the equations in variations:

$$B(t) = (\vec{W}_1(t), \vec{W}_2(t), \ldots, \vec{W}_n(t)), \quad C(t) = (\vec{Z}_1(t), \vec{Z}_2(t), \ldots, \vec{Z}_n(t)).$$

The matrices $B(t)$ and $C(t)$ satisfy the set of equations

$$\begin{cases}
\dot{B} = -\delta_{xp}(t, \Psi_0)B - \delta_{zz}(t, \Psi_0)C, \\
\dot{C} = \delta_{pp}(t, \Psi_0)B + \delta_{px}(t)C,
\end{cases} \tag{A.2}$$

which is called a set of equations in variations (5.4) in matrix form.

Let us consider some properties of the solutions of this set of equations, which determine the explicit form of the asymptotical solution of the Hartree type equation and its approximate evolution operator.

**Remark A.1** The set of equations in variations (5.4) is a set of linear Hamiltonian equations with the Hamiltonian function

$$H(a, t) = \frac{1}{2} \langle a, \delta_{zz}(t)a \rangle, \quad a \in \mathbb{C}^{2n}.$$ 

The complex number $\{a_1, a_2\} = \langle a_1, Ja_2 \rangle$ is called a skew-scalar product of the vectors $a_1$ and $a_2, a_k \in \mathbb{C}^{2n}$. Obviously, the skew-scalar product is antisymmetric:

$$\{a_1, a_2\} = -\{a_2, a_1\}.$$ 

**Statement A.1** The skew-scalar product $\{a_1(t), a_2(t)\}$ of the two solutions, $a_1(t)$ and $a_2(t)$, of equations in variations (5.4) is invariable in time, i.e.,

$$\begin{align*}
\{a_1(t), a_2(t)\} & = \{a_1(0), a_2(0)\} = \text{const}, \quad (A.3) \\
\{a_1(t), a^*_2(t)\} & = \{a_1(0), a^*_2(0)\} = \text{const}. \quad (A.4)
\end{align*}$$

This statement can be checked immediately by differentiating the skew-scalar product $\{a_1(t), a_2(t)\}$ with respect to $t$:

\[
\frac{d}{dt}\{a_1(t), a_2(t)\} = \langle \dot{a}_1(t), Ja_2(t) \rangle + \langle a_1(t), J\dot{a}_2(t) \rangle = \langle J\delta_{zz}(t)a_1(t), Ja_2(t) \rangle + \langle a_1(t), JJ\delta_{zz}(t)a_2(t) \rangle = \langle a_1(t), J\delta_{zz}(t)a_2(t) \rangle - \langle a_2(t), J\delta_{zz}(t)a_2(t) \rangle = 0.
\]

Here, we have made use of the fact that $J^2 = -2n \times 2n$ and $J^t = -J$. Relationship (A.4) follows from (A.4) since $a^*_2(t)$ is also a solution of the equations in variations in view of the fact that these equations are real and linear.

For the set of equations in variations in matrix form, Statement A.1 becomes

**Statement A.2** The matrices

$$D_0 = \frac{1}{2i} [C(t)B(t) - B^+(t)C(t)], \quad (A.5)$$

$$\dot{D}_0 = C^t(t)B(t) - B^t(t)C(t), \quad (A.6)$$

where the matrices $B(t)$ and $C(t)$ are arbitrary solutions of the set of equations in variations (A.2), are invariable in time, and so we have

$$D_0 = \frac{1}{2i} [C^+(0)B(0) - B^+(0)C(0)] / (2i), \quad \dot{D}_0 = C^t(0)B(0) - B^t(0)C(0).$$

The relation of the matrices $B(t)$ and $C(t)$ to the matrix $Q(t)$ and, in view of (5.6), to the function $\phi_1(t)$ yields
Actually, in view of
\[ \dot{C}^{-1}(t) = -C^{-1}(t)\dot{C}(t)C^{-1}(t) \]
and since from \( C^{-1}(t)C(t) = I \) follows \( C^{-1}(t)C(t) + C^{-1}(t)\dot{C}(t) = 0 \), we have
\[ \dot{Q} = B(t)C^{-1}(t) + B(t)C^{-1}(t) = B(t)C^{-1}(t) - Q(t)\dot{C}(t)C^{-1}(t) = \]
\[ = -\delta_{xp}(t)B - \delta_{xx}(t)\dot{C}C^{-1}(t) - Q[\delta_{pp}(t)B + \delta_{px}(t)C]C^{-1} = \]
\[ = -\delta_{xp}(t)Q - \delta_{xx}(t) - Q\epsilon_{pp}(t)Q - Q\delta_{px}(t), \]
Q.E.D.
A similar property is also valid for the matrix \( Q^{-1}(t) \):
\[ -\dot{Q}^{-1} + Q^{-1}\delta_{xx}(t)Q^{-1} + \delta_{px}(t)Q^{-1} + Q^{-1}\delta_{xp}(t) + \delta_{pp}(t) = 0. \tag{A.7} \]

Statement A.4 If at the time zero the matrix \( Q(t) \) is symmetrical \([Q(0) = Q^t(0) at t = 0]\), it is symmetrical at any time \( t \in [0, T] \) [i.e., \( Q(t) = Q^t(t) \)]. Here, \( A^t \) denotes the transpose to the matrix \( A \).

Actually, from \((5.9)\) follows
\[ \dot{Q}^t + \delta_{xx}^t(t) + \delta_{pp}^t(t)Q^t + Q^t\delta_{xp}^t(t) + Q^t\delta_{pp}^t(t)Q^t = 0, \]
since we have
\[ \delta_{xx} = \delta_{xx}^t, \quad \delta_{pp} = \delta_{pp}^t, \quad \delta_{px} = \delta_{xp}^t. \]
Hence, the matrix \( Q^t(t) \) satisfies equation \((5.9)\) with the same initial conditions as the matrix \( Q(t) \) since, as agreed, the matrix \( Q(0) \) is symmetrical. The validity of the statement follows from the uniqueness of the solution to the Cauchy problem.

Statement A.5 The imaginary parts of the matrices \( Q(t) \) and \( Q^{-1}(t) \) can be presented in the form
\[ \text{Im } Q(t) = (C^t(t))^{-1}D_0C^{-1}(t), \tag{A.8} \]
\[ \text{Im } Q^{-1}(t) = -(B^{-1}(t))^tD_0B^{-1}(t). \tag{A.9} \]

Here, the matrix \( D_0 \) is defined by relationship \((A.5)\).

Actually, by definition we have
\[ \text{Im } Q(t) = \frac{i}{2}[Q^t(t) - Q(t)] = \frac{i}{2}[[B(t)C^{-1}(t)]^t - B(t)C^{-1}(t)] = \]
\[ = \frac{i}{2}[C^t(t)]^{-1}[B^t(t)C(t) - C^t(t)B(t)]C^{-1}(t) = [C^t(t)]^{-1}D_0C^{-1}(t), \]
Q.E.D. Similarly, relationship \((A.9)\) can be proved.

Statement A.6 Let the matrix \( Q(t) \) be definite and symmetrical and the components of the vector \( \vec{y}_j(t), j = 1, n \) be the row elements of the matrix \( C^{-1}(t) \) \((A.24)\). Then the vectors \( \vec{y}_j(t) \) satisfy the set of equations
\[ \dot{\vec{y}} = -\mathcal{H}_{xp}(t)\vec{y} - Q(t)\mathcal{H}_{pp}(t)\vec{y}. \tag{A.10} \]

Actually, we have
\[ \dot{C}^{-1} = -C^{-1}\dot{C}C^{-1}. \]
and hence
\[ \dot{C}^{-1} = -C^{-1}[\mathcal{H}_{pp}(t)Q(t) + \mathcal{H}_{px}(t)]. \tag{A.11} \]
Transposing relationship \((A.11)\) for the vectors \( \vec{y}(t) \) \((A.24)\), we obtain equation \((A.10)\), Q.E.D.
Remark A.2 If the matrix

\[ Q^{-1}(t) = C(t)B^{-1}(t) \]

is definite and symmetrical, then the matrix \( B^{-1} \) satisfies the equation

\[ \dot{B}^{-1} = B^{-1}[H_{xx}(t)Q^{-1}(t) + H_{xp}(t)]. \]  
(A.12)

The proof is similar to that of Statement A.6.

Statement A.7 If the matrix \( D_0 \) is positive definite, the relation

\[ 2iB^{-1}(t)H_{xx}(t)(B^{-1}(t))^{\dagger} = \frac{d}{dt}[D_0^{-1}B^{+}(t)(B^{\dagger}(t))^{-1}] \]  
(A.13)

is valid.

Actually, from (A.9) follows

\[ B^{-1}(t)H_{xx}(t)(B^{-1}(t))^{\dagger} = \]
\[ = \frac{i}{2}D_0^{-1}B^+(t)[Q^{-1}(t) - (Q^*(t))^{-1}]H_{xx}(t)(B^{-1}(t))^{\dagger} = \]
\[ = \frac{i}{2}D_0^{-1}[B^{-1}(t)(H_{xx}(t)C(t)B^{-1}(t) - H_{xx}(t)C^+(t)(B^{-1}(t)))^* + \]
\[ + H_{px}(t)B(t)B^{-1}(t) - H_{px}(t)B^*(t)(B^{-1}(t))^*](2i \text{Im } Q^{-1}(t))B^+(t)D_0^{-1}]^{\dagger} = \]
\[ = \frac{i}{2}D_0^{-1}[B^{-1}(t)\dot{B}(t)B^{-1}(t)B^+(t) - B^{-1}(t)\dot{B}^+(t)]^{\dagger} = \]
\[ = \frac{i}{2}D_0^{-1}\left[\frac{d}{dt}B^{-1}(t)B^+(t)\right]^{\dagger}, \]

Q.E.D.

Statement A.8 If for equations in variations (A.2) the Cauchy problem is formulated as

\[ \dot{B}(t)|_{t=s} = B_0, \quad \dot{C}(t)|_{t=s} = 0, \quad B_0^t = B_0, \]  
(A.14)

then the relation

\[ \int_s^t \dot{B}^{-1}(\tau)H_{xx}(t)(\dot{B}^{-1}(\tau))^{\dagger}d\tau = (B_0^{-1})^{\dagger}\lambda_2(\Delta t, \Psi_0)\lambda_4^\dagger(\Delta t, \Psi_0)(B_0^{-1})^{\dagger}, \]  
(A.15)

is valid. Here, \( \lambda_k(\Delta t, \Psi_0), \ k = 1,4 \) denote the \( n \times n \) matrices being blocks of the matriciant of the set of equations in variations (A.4).

\[ A(t, \Psi_0) = \begin{pmatrix} \lambda_1^\dagger(t, \Psi_0) & \lambda_2^\dagger(t, \Psi_0) \\ \lambda_3^\dagger(t, \Psi_0) & \lambda_4^\dagger(t, \Psi_0) \end{pmatrix}, \quad A(0, \Psi_0) = \mathbb{I}_{2n \times 2n}. \]  
(A.16)

Let us consider an auxiliary Cauchy problem formulated as

\[ B(t, \epsilon)|_{t=s} = B_0, \quad C(t, \epsilon)|_{t=s} = \epsilon \text{I}, \quad \text{I} = \|\delta_{k,j}\|_{n \times n}. \]  
(A.17)

Obviously, we have

\[ \lim_{\epsilon \to 0} B(t, \epsilon) = \dot{B}(t), \quad \lim_{\epsilon \to 0} C(t, \epsilon) = \dot{C}(t) \]

and

\[ D_0(\epsilon) = \frac{\epsilon}{2t}(B_0 - B_0^\dagger). \]
We assume that the matrix $D_0(\epsilon)$ is symmetrical and positive definite for $\epsilon \neq 0$. Hence, we may use relationship (A.13) and then obtain

\[
\int_s^t B^{-1}(\tau, \epsilon)H_{xx}(\tau)(B^{-1}(\tau, \epsilon))^t d\tau = -\frac{1}{\epsilon}(B_0 - B_0^*)^{-1}B^+(\tau, \epsilon)(B^{-1}(\tau, \epsilon))^t|_s^t.
\]

(A.18)

In view of (A.16), we have

\[
B(t, \epsilon) = \lambda_4^1(\Delta t)B_0 - \epsilon\lambda_2^1(\Delta t),
\]

and, hence,

\[
B^{-1}(t, \epsilon) = (1 + \epsilon B_0^{-1}(\lambda_4^{-1}(\Delta t))^t\lambda_2^1(\Delta t))B_0^{-1}(\lambda_4^{-1}(\Delta t))^t + O(\epsilon^2).
\]

Then we obtain

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} B^+(\tau, \epsilon)|B^{-1}(t, \epsilon)|^t = \lim_{\epsilon \to 0} \left[ \lim_{\epsilon \to 0} [B^{-1}(t, \epsilon)B^+(t, \epsilon) - B_0^{-1}B_0^*] \right]^t = -\epsilon B_0^{-1}(\lambda_2^1(\Delta t)\lambda_4^{-1}(\Delta t))(B_0^{-1})^t.
\]

Substitution of the obtained expression into (A.18) yields (A.15).

**Statement A.9** If the matrix $D_0$ (A.6) is positive definite and symmetrical and the matrix $\tilde{D}_0$ (A.6) is zero, the following relationships are valid:

\[
C^*(t)D_0^{-1}B^t(t) - C(t)D_0^{-1}B^+(t) = B(t)D_0^{-1}C^+(t) - B^*(t)D_0^{-1}C(t) = 2iI_{n \times n},
\]

(A.19)

\[
C^*(t)D_0^{-1}C^t(t) - C(t)D_0^{-1}C^+(t) = B(t)D_0^{-1}B^+(t) - B^*(t)D_0^{-1}B^t(t) = 0_{n \times n}.
\]

(A.20)

Let us consider an auxiliary matrix $T(t)$ of dimension $2n \times 2n$

\[
T(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} D_0^{-1/2}C^t(t) & -D_0^{-1/2}B^t(t) \\ D_0^{-1/2}C^+(t) & -D_0^{-1/2}B^+(t) \end{pmatrix}
\]

(A.21)

and find its inverse matrix. Direct checking makes us convinced that

\[
T^{-1}(t) = -\frac{i}{\sqrt{2}} \begin{pmatrix} (B(t)D_0^{-1/2})^* & B(t)D_0^{-1/2} \\ (C(t)D_0^{-1/2})^* & C(t)D_0^{-1/2} \end{pmatrix}.
\]

(A.22)

Actually, we have

\[
T(t)T^{-1}(t) = -\frac{i}{2} \begin{pmatrix} C^t(t)B^t(t) & -B^+(t)C^+(t) \\ -C^+(t)B^+(t) & B^+(t)C^+(t) \end{pmatrix}(D_0^{-1})^* = \begin{pmatrix} -D_0^{-1/2}(2iD_0)^* & D_0^{-1/2}D_0^{-1/2} \end{pmatrix} = I_{2n \times 2n}.
\]

From the uniqueness of the inverse matrix follows

\[
T(t)T^{-1}(t) = T^{-1}(t)T(t) = I_{2n \times 2n},
\]

i.e.,

\[
T(t)T^{-1}(t) =
\]

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\[-\frac{i}{2} \left( - (BD_0^{-1})^* C^t + BD_0^{-1} C^+ \left( BD_0^{-1}\right)^* B^t + BD_0^{-1} B^+ \right) \]
\[-\frac{i}{2} \left( - (CD_0^{-1})^* C^t + CD_0^{-1} C^+ \left( CD_0^{-1}\right)^* B^t + CD_0^{-1} B^+ \right) = \mathbb{I}_{2n \times 2n}. \] (A.23)

However, as assigned, we have \(D_0^t = D_0\), and from Definition (A.5) follows
\[D_0^+ = -\frac{1}{2i} (C^+ B - B^+ C)^+ = -\frac{1}{2i} (B^+ C - C^+ B) = D_0.\]

We then have \(D_0^t = D_0\) and, hence, \((D_0^{-1})^* = D_0^{-1}\). Then from (A.23) we obtain (A.19) and (A.27).

The following properties of the solutions to the set of equations in variations are dramatically important for the construction of quasi-classical asymptotics in the class of functions \(\mathcal{P}_h^t(Z(t, h), S(t, h))\).

**Lemma A.1** Let the matrix \(D_0\) be diagonal and positive definite and \(\det C(t) \neq 0\). The matrix \(\text{Im} Q(t)\) will then be positive definite as well.

**Proof.** Let we have \(D_0 = \text{diag} (\alpha_1, \ldots, \alpha_n), \alpha_j > 0, j = 1, n.\) Denote by \(\bar{y}_j\) the rows of the matrix \(C^{-1}(t)\):
\[C^{-1}(t) = \begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \\ \cdots \\ \bar{y}_n(t) \end{pmatrix}. \] (A.24)

Then for an arbitrary complex vector \(|\vec{p}\rangle \neq 0\) we obtain
\[ (\vec{p})^+ \text{Im} Q(t) \vec{p} = \sum_{j=1}^n (\bar{p}, \bar{y}_j(t))^+ \alpha_j (\bar{y}_j(t), \bar{p}) = \sum_{j=1}^n \alpha_j |(\bar{p}, \bar{y}_j(t))|^2 > 0. \] (A.25)

The above inequality is true since \(|\bar{y}_j(t)\rangle \neq 0, \alpha_j > 0, j = 1, n.\) From this inequality, in view of the arbitrariness of the vector \(\bar{p} \in \mathbb{C}^n, |\bar{p}\rangle \neq 0\), follows the validity of the lemma statement.

**Lemma A.2** The matrix \(C(t)\) is nondegenerate \(\det C(t) \neq 0\) if the matrix \(D_0 = (2i)^{-1} (C^+ (0) B (0) - B^+ (0) C(0))\) is positive definite.

**Proof.** Let, for some \(t_1\), \(\det C(t_1) = 0\) be valid. Then a vector \(\vec{k}, |\vec{k}\rangle \neq 0\) exists, such that
\[C(t_1) \cdot \vec{k} = 0, \quad (\vec{k}^t C^+ (t_1) = 0).\]

Since relationship (A.6) is valid for any \(t\), then
\[\vec{k}^t D_0 \vec{k} = \vec{k}^t \left( \frac{i}{2} [B^+(t_1) C(t_1) - C^+(t_1) B(t_1)] \right) \vec{k} = 0.\]

As agreed, the matrix \(D_0\) is positive definite, and, hence, the above equality holds only for \(|\vec{k}| = 0\). The obtained contradiction proves the lemma.

**Lemma A.3** (Liouville’s lemma) If the matrix \(Q(t)\) is continuous, the relation
\[ \exp \left\{ -\frac{1}{2} \int_0^t \text{Sp} \left[ \mathcal{S}_{pp}(t) Q(t) + \mathcal{S}_{px}(t) \right] dt \right\} = \sqrt{\frac{\det C(0)}{\det C(t)}} \] (A.26)
is valid.

**Proof.** From (A.2) follows
\[ \dot{C} = [\mathcal{S}_{pp}(t) Q(t) + \mathcal{S}_{px}(t)] C, \] (A.27)
where the matrix \(Q(t)\) is a solution of equation (A.9) and the matrices \(\mathcal{S}_{pp}(t)\) and \(\mathcal{S}_{px}(t)\) are continuous. Then for the matrix \(C(t)\) the Jacobi identity
\[ \det C(t) = [\det C(0)] \exp \int_0^t \text{Sp} \left[ \mathcal{S}_{pp}(t) Q(t) + \mathcal{S}_{px}(t) \right] dt \]
is valid. Raising the left and right parts of the equality to the power \(-1/2\) yields (A.26), Q.E.D.
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