I. INTRODUCTION

The field of cuprate superconductors has been considered for more than forty years now as the Rosetta stone for condensed matter physics[13, 115, 121, 144]. It is the subject in which the highest number of papers have been published in the condensed matter field. The reason for this intense activity might elude a physicist outside the field. When giving a first look at the phase diagram of the cuprates, however, we start to see very unusual features of this material. This family of copper oxides has the highest superconducting (SC) temperature (without external pressure applied) and, although the phenomenon of superconductivity expels magnetic flux, the SC phase is situated in the vicinity of an antiferromagnetic (AF) phase with a transition temperature three times higher than the SC one. More surprisingly, it is situated also at the vicinity of a Mott insulating phase. This is not the only mystery hidden in this phase diagram. A pseudogap (PG) phase exists in the underdoped region of the hole-doped compounds, which remains still a mystery eluding understanding for forty years[11, 177]. On the right-hand side of the phase diagram, a metallic phase exists which obeys the standard laws of the Fermi liquid theory. But again around optimal doping, in between the PG and Fermi liquid phases, we find what resembles one of the most strongly entangled fixed point of condensed matter theory[182]. Here, experiments show very unusual electric and thermal transport properties, which defy the standard theories of transport valid within the Fermi liquid paradigm (see, e.g., [59, 101, 102]). Indeed, this phase denoted by “strange metal phase” shows linear in $T$ resistivity up to a thousand of Kelvins, whereas the Wiedemann-Franz law is satisfied.

Recently, the field has been revived by the experimental discovery of charge order (CO) in the underdoped regime of cuprates. It started with the observation by STM that modulations exist inside vortices when a magnetic field is applied[96]. These modulations were soon enough seen by NMR experiment[178–180], which also was able to confirm that the signal is non-magnetic, suggesting CO as the main source. Meanwhile, a very important quantum oscillation (QO) experiment showed that the Fermi surface is reconstructed in the underdoped region of the phase diagram, and that small electron pockets are present[70, 71, 159]. The origin of the reconstruction was soon linked to the presence of CO, which enables the creation of small electron pockets detectable by QO. X-ray experiments and resonant x-rays on Cu site confirmed the presence of CO in this system [27, 29, 30, 38, 42, 43, 55–58, 82, 117] and also explored, concommitantly with ultrasound experiments[44, 119], the $B-T$ phase diagram when a magnetic field is applied. In this paper, we will also focus on a few other experimental probes prominent in the study of the underdoped region, like Raman scattering[23, 24, 124], angle-resolved photoemission[63, 170–173], neutron and muon scattering (elastic and inelastic)[17, 31, 62, 77, 83, 93–95, 125–128], ultrasound and x-ray observation of phonons[28, 116], and probes of the electric and thermal transport[6, 7, 20, 33, 46, 61, 64, 88, 91, 92, 97, 98, 100, 101, 103, 110, 111, 123, 153–155, 165, 166, 175].

This very mysterious phase diagram resisted theoretical approaches for a very long time, but also led experimentalists and theoreticians to propose new ideas and unconventional scenarios[3–5, 8–10, 12–16, 20–22, 25, 26,
34–37, 39, 41, 45, 47–49, 52, 54, 60, 67–69, 72, 76, 79–81, 83, 91, 99, 104, 107, 110, 112–115, 120, 121, 140, 143–145, 147, 149, 152, 156, 160, 161, 164, 169, 174, 182, 183].

We would like to highlight here three main types of ideas that have been influential over the years for the study of the cuprate phase diagram. First of all, when we look at the phase diagram of the cuprates, we see that we are close to a Mott transition. This metal-insulator transition implies a large Coulomb repulsion on each site, which is of the order of 1eV. At half filling, the electrons are totally localized on the different sites of the lattice, forming an insulator. This formation of insulator is accompanied by an antiferromagnetic order that persists until a temperature of about 700 K. The presence of strong Coulomb energy, which is the largest energy scale of the system, induces strong correlations on the electrons, to the point of giving the name of strongly correlated electron system to this part of solid state physics. In this first approach, the formation of the PG is due to strong Coulomb interactions, which typically induce a "fractionalization" of the electron into spinon and holon. In typical theories, the process is described by an emergent gauge symmetry, typically U(1) or SU(2), and the PG is attributed to spinon pairing above $T_c$. This type of theories invoking a form of spin liquid or spinon pairing in the PG phase of the cuprates, was first introduced intuitively by P. W. Anderson under the name of "resonating valence bond" state[13, 18, 41, 53, 79, 105, 113, 114, 147, 161–163, 168], which involves entangled pairs of spins. The second striking aspect when looking at the phase diagram is that, starting from large enough oxygen doping, we find the phenomenology of the Fermi liquid, which describes conventional metals at low temperatures. This has given rise to the idea that a quantum critical point (QCP), or a zero-temperature phase transition, is hidden under the superconducting dome and that the PG phase is related to a "broken" symmetry of the system corresponding to this QCP. This situation where a QCP is under the superconducting dome is quite common in quantum materials physics[2, 3, 5, 51, 133, 174, 176, 181, 183]. It can be found, for example, in iron pnictides, and in heavy fermion compounds. The last angle of approach is to consider that the intermediate oxygen doping phase is a phase in which many fluctuations are present. There are several reasons for this. On the one hand, the system is very anisotropic, quasi-bidimensional which induces quantum fluctuations. On the other hand, the PG phase is close to a metal-insulating transition (the Mott transition), and during a localization transition, the respective phases of the different particles fluctuate. This line of thought has been pursued by considering the fluctuations of the phase of superconducting pairs[67, 68, 73–75, 80, 87, 91, 106, 107, 139, 184–187]. While it gives a solid phenomenology, in particular, for the closure of the PG observed by angle-resolved photoemission spectroscopy, it is now consensual to think that the phase of

![FIG. 1. Phase diagram of the cuprate superconductors from Ref.[151]. We note the Mott insulator on the left-hand side, the Fermi liquid region on the right-hand side and the PG phase is in the underdoped part of the phase diagram. The superconducting pairs alone does not extend to $T^*$, but only a few tens of degrees above $T_c$.](image-url)

This paper gives an alternative view on the PG phase and the strange metal phase of the cuprates, through a set of original ideas with the potential to explain a large body of experimental data.

II. EMERGENT SU(2) SYMMETRY

A. Effective field theory

In order to account for the the experimental observations, and the presence of CO inside the PG phase, we introduced the concept of emergent SU(2) symmetry. This concept is not new: It amounts to considering rotations from the SC pairing state to another state with different symmetry. For example, one can rotate the SC state into the AF state within the SO(5) symmetry group[67, 68, 185, 186]. This was an obvious choice, since the AF phases and the SC phases are the most visible physical states in the phase diagram of these compounds. This led the authors of this theory to produce an original explanation for the magnetic resonance seen in inelastic neutron scattering (INS) around the phase diagram. Indeed, it has been shown experimentally that such a resonance is critical inside the AF phase, and progressively gets massive when the hole doping is increased[50, 68]. A criticism for this idea came from the observation that the resonance of the SO(5) theory forms an antibonding state and thus is situated at an energy too high to account for the experimental data. The deep reason why this idea although brilliant seem to be failing to account
for the data, is that between the AF and SC phase in the phase diagram, there is a Mott transition. Namely, the two states are not quasi-degenerate to allow for the rotation between them to be really possible[167].

Another symmetry known to be present in the phase diagram of the cuprates is the SU(2) symmetry which rotates between the particle-particle wave function and particle-hole wave function. This symmetry has been extensively studied close to half-filling where it is exact, especially within strong coupling techniques that include slave-bosons methods. Our idea was to build on this exact symmetry and consider a rotation from the SC state to the observed CO[184]. In real space the rotation involves two η-operators, \[ \eta = \frac{1}{2} \sum_\sigma \left( c_\sigma^\dagger c_{\sigma-\sigma}^\dagger + c_\sigma^\dagger c_{\sigma-\sigma}^\dagger \right) \exp(\i Q_0 \cdot \mathbf{R}_{ij}) \] going from \[ |\text{SC}\rangle = \sum_\sigma c_\sigma c_{\sigma-\sigma} |0\rangle \] to \[ |\text{CO}\rangle = \frac{1}{2} \sum_\sigma \left( c_\sigma^\dagger c_{\sigma} + c_\sigma^\dagger c_{\sigma} \right) \exp(\i Q_0 \cdot \mathbf{R}_{ij}), \] where the indices \( i \) and \( j \) refer typically to first nearest neighbors, \( Q_0 \) is the modulation wave vector, and \( \mathbf{R}_{ij} = (r_i + r_j)/2 \). We have \( \eta |\text{SC}\rangle = |\text{CO}\rangle \). The rotation operators between the two states form a SU(2) group with a triplet representation. At half filling for the Hubbard model, this symmetry is exact with \( Q_0 = (\pi, \pi) \).

Our idea was to generalize this emergent symmetry concept away from half filling with \( Q_0 \) being the ordering wave vector observed in the underdoped region of the phase diagram.

With K. B. Efetov, one of us found a model where the emergent SU(2) symmetry is exact [72]. It consists of eight hot spots situated at the antiferromagnetic (AF) zone boundary of the Brillouin zone, as depicted in Fig. 3[131, 132]. Within this restriction of the Fermi surface, we could work out the model exactly, and found that the two kinds of pairings \( \Delta_{pp} = \langle c_{\alpha}^\dagger c_{\beta}^\dagger c_{-\alpha} c_{-\beta} \rangle \) and

\[ \Delta_{ph} = \langle c_{\alpha}^\dagger c_{-\beta} \rangle \]

in the particle-particle and particle-hole channels, respectively, are fully degenerate. The effective field theory is a non-linear \( \sigma \) model with the effective action

\[ F = \lambda \int d^2r \text{Tr} \left[ \partial u^\dagger \partial u + k^2 u^\dagger \tau_3 u \tau_3 \right], \]

with \( u = \begin{bmatrix} \Delta_{ph} & \Delta_{pp} \\ \Delta_{pp}^* & -\Delta_{ph} \end{bmatrix} \) and \( u^\dagger u = 1 \).

The model in Eq. (2) has properties of chirality: the constraint has to be understood as a local constraint. This has several experimental consequences: First, the constraint \( u^\dagger u = 1 \) has the potential to open a PG in the Fermi surface when coupled to fermions. This situation will be treated in detail later in this paper, when we study the special case of Bi2201[84, 86]. Additionally, typically chiral models like Eq. (2) lead to phase separation. We have treated this idea in a set of works where the concept of “droplets” was introduced to account for the fluctuations below the PG temperature \( T^* \) [130, 134, 135].

**B. Modulations in the vortex core**

It was maybe the first experiment that showed unambiguously the presence of charge order in the underdoped regime of the cuprates[96]. STM groups have done measurements of Bi2212 around 12% of oxygen doping, using an applied field of 8 T and also at zero field[89, 90](Fig.
When the two set of data is subtracted form one another, the vortex lattice becomes visible, but also, inside the vortex core, some modulations are observed with a period around 3 to 4 lattice sites. The presence of modulations inside the vortex is a very striking feature which is not necessarily present for systems with coexistence of CDW and SC states. In the example of NiSe$_2$, where a charge order with ordering temperature five times as big as the SC temperature is observed, no modulations are observed inside the vertices. The chiral theory of Eq. (2) gives a very elegant explanation for this feature, which identifies the charge modulations inside the vortex core to a skyrmion. In order to understand this, we notice that within the framework of the non-linear $\sigma$ model, a space of pseudospins is created which mimics the spin operators. The correspondence goes as

$$S_x \to \Delta, \quad S_y \to \Delta^*,$$

and vector rotations operate between the three components. Away from the vortex core, the vector is aligned with the SC state, say, in the $x$-direction. Going inside the vortex core, the vector turns progressively toward the $z$-direction, reaching precisely the $z$-direction inside the core. The result is a skyrmion in the space of pseudospins. Note that this feature that a chiral model produces skyrmions is general, and was one of the predictions of the theory of emergent symmetries in the past. For example, within the SO(5) theory, the competitor state being AF order, it was predicted that AF order would be seen inside the vortex core. Similarly, within the SU(2) gauge theory of Ref. [121], a state with loop currents was predicted inside the vortex core. The SU(2) theory which rotates from a SC state to a CO state elegantly satisfies the prediction that charge modulations would be seen inside the vortex core.

So the theory passes this experimental test, but note that the test is not decisive, since it has been argued that a simple, but strong enough Ginzburg-Landau (GL) interaction between the two modes is enough to create charge modulations inside the vortex core.

**C. B-T phase diagram**

We now know that SC and CO orders in underdoped cuprates interact strongly with each other, but what is the form of their interaction? They could do it through a simple GL coupling $\lambda |\Delta| |\chi|$, where $\Delta_i$ and $\chi_i$ are, respectively, the SC and CO order parameters, but they could as well be "entangled" and interact through a local constraint $|\Delta_i|^2 + |\chi_i|^2 = 1$. In both cases, the two orders are in competition, but in the second case, the constraint produces an entangled ground state with the two orders. The issue here is whether the effective model describing the PG phase is chiral or not chiral. In order to examine the situation, we turn to a very powerful experiment where the phase diagram in the $B$-$T$ plane was determined. The compound studied through ultrasound and spectroscopy is YBCO at 12% of oxygen doping. The phase diagram shows a SC phase at $B = 0$ up to a temperature $T_c \approx 45$ K. Above a magnetic field of $B_c \sim 17$ T, the CO sets in abruptly, with a very flat phase transition and a transition temperature of the same magnitude as $T_c$, with $T_{CO} \approx T_c$. The flatness in temperature of this phase transition is one of the most striking features of the $B$-$T$ phase diagram. It is reminiscent of a spin-flop transition in magnetic structures when the magnetic field is applied perpendicular to the...
orientation of the spin: nothing happens for a while when the magnitude of the magnetic field is increased, but suddenly all the spins “flip” to get aligned to the magnetic field, producing a very abrupt and flat phase transition.

We have investigated the $B$-$T$ phase diagram in a series of two papers[40, 129]. As could be expected from the arguments above, the chiral models describe naturally the spin flop transition whereas the simple GL theory has to be heavily fine-tuned to account for the observed features. At low coupling between the two modes, the GL theory cannot produce a flat transition: the coupling has to be “strong”, typically larger than one in renormalized units to exhibit the flat transition. Now, if the coupling is larger than one, the two orders repel each other fully and it is impossible to get a coexisting phase. This is the difference with the “entanglement” between competitive orders given by chiral models. The constraint of chiral models in Eq. (2) naturally produces a flat “spin-flop” transition, but in the space of pseudospin. In order to account for the coexistence, the exact SU(2) symmetry has to be slightly broken, by adding for example a mass term in Eq. (2), but there are many other ways to break the exact SU(2) symmetry, as we will see a bit later. The conclusion of our study of the $B$-$T$ phase diagram is that the chiral model accounts for the experimental data one order of magnitude better than the GL theory.

D. Raman scattering and charge gap

A question very much debated with the observation of charge order in underdoped cuprates is whether it is a key stone for our understanding of the remaining mysterious phases, like the PG, or whether it is an epiphenomenon, which comes as a decoration of the phase diagram but is not central to it. Then came a very intriguing experiment from Raman scattering, which claimed to be able to extract the amplitude of the charge gap from the $B_{2g}$ resonance[124]. As a reminder, the Raman spectroscopy is able to scan parts of the Brillouin zone (BZ) by selecting different symmetries. The $A_{1g}$ channel is uniform around the BZ and thus does not select any symmetry. The $B_{1g}$ channel scans the antinodal (AN) region of the BZ, whereas the $B_{2g}$ channel scans the nodal region roughly up to the AF zone boundary. When entering the SC phase, a peak is seen in Raman scattering at $\omega = 2\Delta$, both in the $B_{1g}$ and $B_{2g}$ channels. In the $B_{1g}$ channel, however, we have an additional feature visible as a “peak-dip-hump” (the same feature is also visible in momentum resolved ARPES around the same part of the BZ), which is identified as a signature of the PG line. The novelty of Raman scattering studies, is that a “new” peak was observed in the $B_{2g}$ channel, separate from the SC peak. This was identified with the CO gap in the underdoped region. This charge gap has two remarkable features: First, it is of the same order of magnitude as the SC gap, which provides an answer to the question of whether it is an epiphenomenon or not. Second, the gap varies with doping, but does not follow the CO transition temperature $T_{co}$, but rather follows the PG temperature $T^*$.

Those two ingredients are part of the SU(2) emergent symmetry scenario, since in this scenario, the two gaps (the charge gap $\chi$ and the SC gap $\Delta$) behave essentially in the same way, i.e., they have the same order of magnitude and the same variation with doping. One consequence of the Raman experiment is that at low doping the charge sector has a “preformed gap” in the same manner as the SC sector. Namely, the bare value of the gap, both in the charge and SC sectors, is larger than the ordering temperature $T_{co}$. There is a fluctuation regime where the gap is “pre-formed”, but not condensed in both the charge and the SC sectors[85].
out the CDW potential which highlights the dip-hump CDW.

Our next goal is to map out the doping dependence of the CDW and the ones at T\text{\textsubscript{\textsc{c}}} = 250K. The left panel ('' is the reference between the SC Raman response at T\text{\textsubscript{\textsc{c}}} = 12K, but at a finite energy inside the SC phase, has to retain a certain amount of coherence in the SC state in the optimally and overdoped cuprate superconductors."

FIG. 2. (Color online). (a) Temperature dependence of the B\textsubscript{1}g (nodal) Raman responses of HgBa\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{3}O\textsubscript{8+}\textsubscript{x}. Correspondence and request for materials should be addressed to A.S. (alain.sacuto@univ-paris-diderot.fr).

E. Inelastic Neutron Spectroscopy (INS)

The idea of emergent SU(2) symmetry was criticized from a few standpoints. The most important criticism is whether the symmetry survives the full Fermi surface. Indeed, so far the SU(2) symmetry was found to be exact on the eight-hot-spot model which consists of eight points situated at intersection of the AF zone boundary and the Fermi surface. But what happens when the whole Fermi surface is involved? Will the symmetry survive, even for a point of the Fermi surface not far way from the eight hot spots? In order to address this criticism, we first considered a model of "droplets" made with charge modulations involving a series of wave vectors around the original CO one Q\textsubscript{0} [108, 136]. This idea pushed us to consider "hot regions" of the Fermi surface, situated in the AN part of the BZ. The INS experiment tells us that there is a "butterfly shape" of the neutron scattering resonance situated at vector Q = (\pi, \pi), but at a finite energy inside the SC phase. The same resonance changes to a "Y" shape between T\textsubscript{c} and T\textsuperscript{*}[94, 95]. The standard interpretation of the butterfly resonance below T\textsubscript{c} is that the upper part is the dispersion of AF spin density waves. It is gapless at zero doping and gradually becomes gapped at finite doping [76]. The origin of the lower part has led to many controversial ideas, with the most credibility given to the scenario of magnetic excitons showing below T\textsubscript{c}[141, 142, 167]. Within the exciton scenario the shape of the lower part of the "butterfly" is driven by the shape of the d-wave dispersion of the gap around the node, a fact that is well-reproduced experimentally. Although the "butterfly" shape is rather well-understood theoretically, the change into a "Y" shape above T\textsubscript{c} resists interpretation. Within our scenario of "hot regions" of the Fermi surface, we interpret this result with the idea of a loss of coherence around the AN region of the Fermi surface when the T\textsubscript{c} line is crossed. With this idea, we were able to reproduce the entire phase diagram for the INS resonance [137], as depicted in Fig. 8.

FIG. 7. Raman B\textsubscript{1}g and B\textsubscript{2}g channels scan, respectively, the antinodal and the nodal parts of the Brillouin zone. The charge gap is extracted from the B\textsubscript{2}g channel. We observe that the charge gap is of the same order of magnitude as the SC gap, and varies like T\textsuperscript{*} (and not like T\textsubscript{\textsc{c}}) with doping.

FIG. 8. Schematic phase diagram showing the various type of neutron resonances in the SC phase and PG phases of the cuprate superconductors.

III. FRACTIONALIZED PAIR DENSITY WAVE

A. The concept

We turned then to a more drastic solution to the criticism, with a new chiral model which encompasses the same constraint as in Eq. (2) but which sees an emergent U(1) gauge field be responsible for the formation of the PG. The main context is the formation of hard
core bosons at high energy (of the order of 0.5 eV) in the phase diagram of the cuprates. This is our starting hypothesis. These bosons may have different symmetries, singlet, triplet, charge zero or charge two, and due to their hopping from site to site a continuum of wave vectors is explored. Of course, this idea is difficult to prove theoretically, but we will see later that some experimental evidence inclines towards it.

Out of this boson “soup” which interact with fermions, some bosons will condense at low energy, and the various charge orderings seen in the underdoped regime, but also at some places in the overdoped regime in the phase diagram could be the result of such a condensation [39, 85]. On the other hand, charge two particle-particle bosons with finite center-of-mass momentum, also called pair-density waves (PDW) might be unstable when the temperature is lowered and “fractionalize” into their elementary symmetry components \( \hat{\Delta}, \hat{\chi} \), where \( \hat{\Delta} = \langle \sum \sigma c_{i\sigma}^\dagger c_{j-\sigma} \rangle \) is a charge-two, \( Q = 0 \) particle-particle pair and \( \hat{\chi} = \langle \sum \sigma c_{i\sigma}^\dagger c_{j\sigma} e^{i\Phi(R_{ij})} \rangle \) is the particle-particle order breaking the translation symmetry, with \( R_{ij} = (r_i + r_j)/2 \). The system has the same constraint as Eq. (2) with \( |\Delta|^2 + |\chi|^2 = (E^*)^2 \), where \( E^* \) is a constant (note that both \( \Delta \) and \( \chi \) have dimension of energy). The effective model is now another chiral model, which can be written in the form of a quantum rotor model

\[
S = \int d^2 x \sum_{a,b=1}^2 |\omega_{ab}|,
\]

with \( \omega_{ab} = z_a \partial_\mu z_b - z_b \partial_\mu z_a \), and

\[
\sum_{a=1}^2 |z_a|^2 = 1,
\]

with \( z_1 = \Delta/E^* \), \( z_2 = \chi/E^* \), \( z_1^* = \Delta^*/E^* \), \( z_2^* = \chi^*/E^* \). The model has a natural \( U(1) \) gauge symmetry with \( z_a \to z_a e^{i\theta} \) (\( z_1^* \to z_1^* e^{-i\theta} \)). The model in Eq. (5) is formally equivalent to the CP\(^1\) model

\[
S = \int d^2 x |D_\mu \psi|^2,
\]

with \( D_\mu = \partial_\mu - i\alpha_\mu \), and

\[
\alpha_\mu = \frac{1}{2} \sum_a z_a^* \partial_\mu z_a - z_a \partial_\mu z_a^*.
\]

with \( \psi = (z_1, z_2)^T \). The model in Eq. (6) is in turn “almost” the same as the non-linear \( \sigma \) model of Eq. (2), but with an additional gauge field \( \alpha \) taking care of the intrinsic \( U(1) \) gauge symmetry.

An advantage of this new formulation is that the constraint in Eq. (4) is now very robust because it is projected by an emergent gauge field. Within this new formalism, the PG of the phase diagram of the cuprates reads in the following way. At high temperature, PDW bosons are present in the “boson soup”. When the temperature is decreased like a deconfining transition at \( T^* \) followed by a reconfining transition at lower temperature when each of the elementary field (\( \Delta \) and \( \chi \)) is condensed.

### B. Angle-Resolved Photoemission Spectroscopy (ARPES)

One of the first studies in these materials that now can be made for the case of the emergent SU(2) symmetry is related to ARPES in order to determine, in particular, how the Fermi surface is affected by CO and SC gaps, away from the eight hot-spots[84, 86]. The first check that we have done is to see that the constraint in Eq. (4) when coupled back to the fermions in a phenomenological model, opens a gap in the AN region of the Fermi surface, precisely in the same region that was identified by ARPES as the PG region. This leads to the formation of the so-called “Fermi arcs” around the nodal region of the Fermi surface. The particularity of the opening of the gap on part of the Fermi surface is due to the wave vector of the CO gap, which affects only part of the conduction electrons, typically in the AN region of the Fermi surface. Moreover, the scenario belongs to the category of “fluctuation” scenarios, which were intensively studied in the past in the context of the ARPES data, considering the fluctuations of the phase of the pairing \( \Delta \). In our scenario, both the phase and the amplitude of the two order parameters \( \Delta \) and \( \chi \) vary, only related by the constraint in Eq. (4). One very nice feature of the fluctuation scenarios compared to the other ones is that the PG gap is naturally closing with temperature at \( T^* \), the Fermi “arcs” getting smaller and smaller as the temperature is increased, and finally vanishing at \( T^* \), leading to a large Fermi surface observed in experiments. It should be mentioned that among the various scenarios for the formation of the PG, the fluctuation scenario is the only one which reproduces the very gradual closing of the Fermi surface with the temperature.

We turn now to the case study of Bi2201, from which extensive ARPES experiments have been performed. The study of this compound has got some visibility because of two observations in the opening of the PG. First, the opening of the PG in the AN of the Fermi surface occurs at specific wave vectors \( k_G \), which are a bit different from the Fermi wave vector \( k_F \) identified above \( T^* \) when the PG closes. Then there is a back-bending of the electronic dispersion precisely at the point \( k_G \) where the gap opens, as can be seen in Fig. 9. These two observations actually restrict very strongly the realm of possibilities for the PG, since the back-bending indicates that a particle-hole transformation has to be present, whereas
the finite wave vector different from \( k_F \) is calling for a breaking of translational invariance. Note that the observed wave vector \( k_Q \) corresponds well to a charge ordering of wave vector \( Q_0 \simeq 0.3 \). All these observations led a group of researchers to propose a fluctuating PDW as a candidate for the PG [120].

C. Phase locking viewed from STM

A very striking experiment, as already mentioned in the introduction, is the study of the phase slips inside the CO and SC phases. Recent STM experiments were carried out and were able to extract the phase \( \theta \) in the modulations amplitudes \( \chi = |\chi_0 \cos (Q_0 \cdot r + \theta)\) [39]. As can be seen in Fig. 10, at zero field, the histogram distribution of \( \theta \) is totally random, spreading on all values. The surprise comes when the values at \( B = 8.5 \) T is subtracted from the values at \( B = 0 \) T. One sees clearly the vortices with the modulations inside, but then the phase slip \( \theta \) of the modulations is locked at a unique value (say, \( \theta = 0 \)). The phase locking is long-ranged, of the size of the sample, whereas the charge order itself is short-ranged, spreading typically about four to five lattice sites. We are thus in a situation where the phase locking in the charge sector is much longer than the correlations of the amplitude of the order parameter. This very original situation is unique to the cuprates. Within the PDW fractionalization scenario, we address this situation by noticing that the phase of the CDW and the one of the SC order parameters are related by a gauge phase, and lock together below \( T_c \). This has striking consequences, both in terms of the symmetry of the charge order, which behaves typically like a PDW below \( T_\text{c} \), and in particular reacts to an electromagnetic field (note that a reconfining transition occurs below \( T_\text{c} \)). A second consequence is that the phase \( \theta_r \) of the charge order is locked to the SC phase and acquires long-range correlations across the whole sample.

D. Phonon softening

In the same line of ideas, we considered the experiment of softening of the phonon line due to the presence of a CDW order[28, 116]. This experimental result is depicted in Fig. 11. The standard Peierls theory of CO stipulates that below the CO transition temperature \( T_{\text{CO}} \), and at the charge ordering modulation wave vector \( Q_0 \), the phonon line is softened, namely, the phonon dispersion \( \varepsilon_q \to 0 \) at \( q = Q_0 \) [122]. The theory actually works for one di-
E. Cascade of phase transitions, loop currents

One of our early discoveries was that a cascade of phase transitions could be produced around an AF quantum critical point (QCP), thus masking the QCP itself [130]. This can be one interpretation of the experimental phase diagram of the cuprates, where spin-glass, charge ordering, time-reversal (TR) symmetry breaking, inversion-symmetry (IS) breaking, and loop currents have been reported in the vicinity of the AF QCP. Out of this complexity, one can already notice a body of phase transitions involving discrete symmetries and not breaking translational symmetries. Since they do not break Galilean invariance, they cannot be directly responsible for the opening of the PG in the antinodal (AN) region, but experiments show that in the underdoped region they “accompany” the formation of the PG.

We focus here on the formation of the loop currents (LC) and see how they can integrate into our theoretical framework [9, 65, 66]. Loops currents are a structure observed by elastic neutron scattering, namely, at \( Q = 0 \) wave vector. They do not break translational invariance and thus are notoriously difficult to measure in neutron scattering. They break both TR and IS but not the product of the two. There are two approaches to the issue of LC order in the cuprates. In the first approach, the LC order comes directly from the three-band Hubbard model as an exotic, but very important primary order. The second way is to consider that the LC order is a “vestigial” order, namely, a discrete order that comes as a precursor of a continuous phase transition. One then needs to check if the discrete order is compatible with the main continuous order as far as symmetries are concerned. A notorious candidate for the precursor order is the pair-density wave (PDW) order, which allows for the formation of a particle-particle pairs with finite center-of-mass momentum [9]. In Ref. [157], we discuss the idea of LC as a precursor order in the context of a fractionalized PDW scenario. Our conclusion is that the symmetry of fractionalized PDW allows for LC as a precursor order.

The LC order being a \( Q = 0 \) order, the experimental evidence for its presence through elastic neutron scattering is difficult and has been controversial since a lot of subtractions have to be made to the signal to finally extract the response corresponding to the LC [31, 32, 78]. Recent developments, nevertheless, reveal the presence of a signal at finite wave vector \((0, \pm \pi)\) and \((\pm \pi, 0)\). The presence of LC at a finite wave vector being a clear signature of the signal gives hope for a resolution of the experimental controversy in the near future.

A question that is often debated with the idea of LC is that of the nature of the transition at \( T^* \). It is important to note that the LC are not a static “order”, but rather a dynamical phenomenon. For one thing, they are not visible by NMR, which shall be the case for a static mag-

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mensional systems; for 2D and 3D systems a softening is still observed but does not go exactly to zero. For the cuprates superconductors, with doping around 12%, a clean softening of the phonon dispersion line is observed, but surprisingly instead of taking place below \( T_{co} \), it occurs below \( T_c \). It is very unusual since there is no specific reason where the CDW modulations should be sensitive to the superconducting ordering temperature \( T_c \), rather than \( T_{co} \). In this theoretical scenario, we account for the sensitivity of the phonon line to \( T_c \) by considering that the phase freezing of the CDW and SC modes both occur below the minimum of the two temperatures \( T_c \) and \( T_{co} \), which here is \( T_c \). Above \( T_c \), the phases of the particle-particle and particle-hole order parameters unlock and fluctuate, which wash out the softening of the line [158].

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FIG. 11. (Upper panel) Experimental observation of the phonon softening below \( T_c \). (Lower panel) The schematic depiction of the phenomenon [28].
netic order since the typical timescale of NMR is $10^{-6}$ s. For comparison, the timescale of neutron scattering is $10^{-9}$ s while the one for muon scattering resonance is ($\mu$SR) $10^{-12}$ s. A signal corresponding to LC has been reported in $\mu$SR experiments, supporting the dynamical nature of this phenomenon. The transition at $T^*$ even from the point of view of a discrete order like LC is a dynamic phenomenon which is closer to a crossover rather than a phase transition.

Another concept highly debated around the LC order and discrete phase transitions is whether the phase diagram has a quantum critical point (QCP) around optimal doping, precisely at the point where the LC order and other discrete phase transition line seem to terminate at $T = 0$. Here the experiments are a bit contradictory with both ARPES spectroscopy and Raman experiments, which seem to show an abrupt end of the PG line around optimal doping [24]. The number of charge carriers was studied by Hall conductivity by one experimental group in YBCO, Nd-LSCO and LSCO. They find a transition around optimal doping for a number of carriers going from $p$ to $1 + p$, where $p$ is the oxygen doping, suggesting a QCP at a critical doping $p^*$ [138, 150]. Another experimental group studied the same quantity in Ti2201 and Bi2201 and, although they converged in finding that the number of carriers goes from $p$ to $1 + p$, the transition between the two regimes looks more like a wide region around the critical doping where “another species” seem to coexist with electronic carriers. We will come back to these different viewpoints in our study of transport in the strange metal phase of those compounds.

IV. STRANGE METAL

A. Longitudinal conductivity and Hall conductivity

The strange metal phase of the cuprates is one of the greatest mysteries of condensed matter physics. Around the optimal doping, the longitudinal resistivity is linear in temperature over several temperature decades ($\sigma_{xx} \sim T$), while the Hall cotangent which is the ratio of the linear resistivity to the transverse resistivity varies as the square of the temperature $\cot \theta_H = \sigma_{xx}/\sigma_{xy} \sim T^2$ (see, e.g., [101, 102] and references therein, and Fig. 13). In a theory where only one species of charge carrier is present, these two quantities should be inversely proportional to the transport lifetime of the charge carrier, and thus the same power law should be observed. Moreover, it has been shown that the optical conductivity has a Drude form as a function of energy, with the Drude width being of the order of $k_BT$. This set of observations is notoriously difficult to account for theoretically. Recently, theories have emerged which consider the special case of optimal doping to be one of the most strongly entangled fixed points that nature has ever produced. Models using Sachdev-Ye-Kitaev (SYK) Hamiltonians are studied to describe such a strongly-correlated fixed point [92].

B. The scenario of entangled bosons

In our scenario, we take a new approach based on recent transport experiments that suggest the presence of a new carrier species in a doping region around the optimal doping. This species would participate to the linear resistivity in temperature but would not contribute to the Hall conductivity. Moreover, a scaling relation in $H/T$, where $H$ is the applied magnetic field has been observed. The picture that emerges (and is depicted in Fig. 14) for the cuprates is that at "high energy" with $0.5 < \omega < 1$ eV, many bosons form due to strong coupling interactions. Many symmetry channels are represented, spin 1, spin 0, zero charge or charge two bosons, and they form at different wave stretches. Under the effect of the strong coupling, these bosons have a “hard core”, the pairing energy being approximately equal to the Coulomb energy. When the temperature decreases, some bosons are unstable, like the charge two bosons of finite wave vector (or PDW). These bosons "fractionalize" and some other bosons condense around the optimal doping, like the particle-hole bosons of zero charge and finite wave vector, producing charge density waves. A recent experiment that supports this picture is the MEEELS experiment, at optimal doping for the cuprates. We see that the compressibility as a function of frequency admits a plateau at low frequency, whose width is of the order of 0.7 eV. Scalings relations are also shown experimentally. These data suggest that we are dealing here with a "jamming" transition, which appears naturally if we have interacting bosons near the optimal doping [1].

In a series of papers [19, 146], we calculated the conductivity and thermal conductivity of the boson-fermion “soup” with charge two bosons. In this picture, the bosons scatter through the fermions and acquire Landau damping given by

$$D_{q,\omega}^{-1} = |\omega| + q^2 + \mu(T),$$  \hspace{1cm} (7)
why are high temperature superconductors interesting?  

The superconductivity emerges out of a highly correlated insulating state. Their superconducting order parameter is unconventional \( \sigma^d \) phase diagram of the cuprates within the fractionalized scenario. At high energy, due to strong coupling, incoherent bosons are formed. At \( T^* \), some bosons like the PDW do not “survive” and fractionalize. At lower temperatures, particle-hole pairs can condense, forming the charge orders observed in the underdoped regime of the phase diagram.

with \( \mu (T) \) being the boson mass. Besides, \( \mu (T) \sim g T + \mu_0 \), where the term proportional to the temperature is due to the bosons interacting together and \( \mu_0 = 0 \) when the bosons are critical. Since the Landau damping is particle-hole symmetric, bosons with propagators described in Eq. (7) do not contribute to the transverse conductivity and thus do not affect the Hall conductivity. We are thus oriented on a global picture where two types of carriers are present. Bosons contribute to the temperature linear resistivity but not to the Hall conductivity. On the other hand, fermions will contribute to the Hall conductivity and, in a model with “hot” and “cold” regions, a phenomenological study has shown that fermions contribute to the Hall conductivity as \( \sigma_{xy} \sim T^{-3} \). Assuming that the bosons short-circuit the fermions and contribute to the longitudinal conductivity as \( \sigma_{xx} \sim T^{-1} \), we get the correct scaling for the cotangent Hall \( \cot \theta_H \sim T^2 \).

We have adopted a Kubo formulation of the transport for the bosons, and also a more hydrodynamic formulation using the memory-matrix approach, and found that indeed such bosons contribute to the linear in \( T \) resistivity in \( d = 2 \). To obtain this result, we assumed Umklapp scattering as the mechanism for total momentum relaxation in the theory.
V. DISCUSSION

In this paper, we have given an overview of the PG and strange metal phases of the cuprate superconductors. We focused on the study of the charge order in the underdoped part of the phase diagram. The main idea is that the charge order is in competition with superconductivity, but also, in a surprising and unprecedented way in condensed matter, the two orders turn out to be entangled. These mixed effects of entanglement and competition are described by chiral models, in which the square of the amplitudes of the two modes are linked by a local constraint. The first chiral model introduced to describe the PG is the non-linear $O(3)$ model. In this context, it corresponds to the idea of an emergent SU(2) symmetry, where $\eta$-modes describe a rotation from the superconducting state to the charge order state. An exact realization of the SU(2) symmetry was found and treated in the eight hot-spot model. The constraint linking the two orders form a pseudospin space. This type of model allows to explain the opening of the PG in the antinodal part of the Brillouin zone and the formation of Fermi arcs. We obtain a good description of the PG opening observed by ARPES in Bi2201, which takes into account the back-bending of the electronic dispersion around a wave vector $Q_0/2 > k_F$, where $k_F$ is the Fermi wave vector. The phase diagram as a function of magnetic field and temperature is also very well described. In particular, we hold that the flat transition at $B = 18$ T to a three-dimensional charge order cannot be described by a Ginzburg-Landau model but requires a constraint. This transition is well-described by the nonlinear $\sigma$ model, which captures the flatness of the transition, interpreted here as a pseudo spin-flop transition by analogy with the “spin -flop” transitions observed in spin models. A recent Raman diffusion experiment also goes in the same direction with the observation of the “charge gap” in the $B_{1g}$ channel. Strikingly, the charge gap is of the same order of magnitude as the observed SC gap in the $B_{1g}$ channel and, moreover, it follows $T^*$ with the oxygen doping, rather than following the charge ordering temperature $T_{co}$. In this type of interpretation, the occurrence of charge modulations inside the vortex core upon applying an external magnetic field is a topological object: a skyrmion in the pseudospin space. The presence of $\eta$-modes in the system can also be related to an x-ray experiment showing some optical modes around $Q_0$, interacting with a phonon.

In the second part, we introduced the concept of fractionalization of an order of density of Cooper pairs. The idea of fractionalization is not new, and was introduced at the beginning of the study of the cuprate superconductors with the original paper of P. W. Anderson’s resonating valence bond. This idea was implemented in large part in gauge theories for which the electron is fractionalized into its spinon and holon elementary components. Here we take up the same idea of fractionalization, but on a particle much less stable than an electron, a Cooper pair density wave (PDW). The PDW fractionalizes at $T^*$ into a particle-particle field breaking the particle-hole invariance and a modulated particle-hole field, breaking the translational invariance. As before, the PG state is described by these two fields which are both intertwined and in competition. The cuprate phase diagram is now interpreted as a deconfinement transition at $T^*$ with decreasing temperature and the quantum rotor model describes the physics under $T^*$. This approach has given us a way to think about the strange metal phase observed at optimal doping in the cuprates. The idea is that at a high energy scale, of the order of 0.7 eV and under the effect of the very strong Coulomb interaction, hard core bosons form with different symmetries. Some have a charge two, some have charge zero, others are spin singlet, etc. These bosons, coupled to the lattice could jump from site to site and thus have a dispersion. When the temperature is lowered, some bosons will be unstable and will fractionalize while others will be stable and will condense, giving rise to charge orders at different locations in the phase diagram. If we consider that bosons of charge two are present in the strange metal phase, we have shown that they will contribute with a $T$-linear resistivity, thus shedding potentially new light on this big mystery of quantum material physics.

VI. ACKNOWLEDGMENT

This paper is dedicated to late Konstantin B. Efetov with gratitude for the very creative research undertaken together and with nostalgia for the wonderful time spent working in various places of the world. Kostya was a rare physicist, independent, inventive and brilliant. The community is missing him sorely.

We would like to thank the many young physicists who participated in this project at various stages of its development: Anurag Banerjee, Corentin Morice, Debmalaya Chakraborty, Emile Pangburn, Evrard Kandelaki, Hendrik Meier, Louis Haurie, Matthias Einenken, Maxence Grandadam, Saheli Sarkar, Thomas Kloss, Vanyuldo S. de Carvalho and Xavier Montiel. We acknowledge the International Institute of Physics (IIP) in Natal (Brazil), where part of the collaboration with Kostya was happening. A special thanks to A. Ferraz, its director, for his kind hospitality. Discussions with experimentalists were invaluable for the ideas exposed in this work. We are especially grateful to Y. Sidis for numerous and often heated discussions about the experiments. We also acknowledge very useful interactions with H. Aloulou, V. Baledent, D. Bounoua, P. Bourges, D. Colson, S.C. Davis, Y. Gallais, G. Grissonnanche, M. Hamidian, N. Hussey, M-H. Julien, D. Leboeuf, B. Leridon, M. Le...
Tacon, M-A. Méasson, C. Proust, B. Ramshaw, A. Sacuto, S.E. Sebastian, L. Tiffefer. H.F. also acknowledges funding from CNPq under Grant No. 311428/2021-5.

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