Affine Anosov diffeomorphims of affine manifolds.

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Abstract.
In this paper, we show that a compact affine manifold endowed with an affine Anosov transformation map is finitely covered by a complete affine nilmanifold. This answers a conjecture of Franks for affine manifolds.

0. Introduction.

An $n$—affine manifold $(M, \nabla_M)$ is an $n$—differentiable manifold $M$ endowed with a locally flat connection $\nabla_M$, that is a connection $\nabla_M$ whose curvature and torsion forms vanish identically. The connection $\nabla_M$ defines on $M$ an atlas (affine) whose transition functions are affine maps. The connection $\nabla_M$ pulls-back on the universal cover $\hat{M}$ of $M$, and defines on it a locally flat connection $\nabla_{\hat{M}}$. The affine structure of $\hat{M}$ is defined by a local diffeomorphism $D_M : \hat{M} \rightarrow \mathbb{R}^n$ called the developing map. The developing map gives rise to a representation $h_M : \pi_1(M) \rightarrow Aff(\mathbb{R}^n)$ called the holonomy. The linear part $L(h_M)$ of $h_M$ is the linear holonomy. The affine manifold $(M, \nabla_M)$ is complete if and only if $D_M$ is a diffeomorphism. This means also that the connection $\nabla_M$ is complete.

A diffeomorphism $f$ of $M$ is an Anosov diffeomorphism if and only if there is a norm $\| \|$ associated to a riemannian metric $<,>$, a real number $0 < \lambda < 1$ such that the tangent bundle $TM$ of $M$ is the direct summand of two bundles $TM^s$ and $TM^u$ called respectively the stable and the unstable bundle such that:

$$\| df^m(v) \| \leq c \lambda^m \| v \|, v \in TM^s$$

$$\| df^m(w) \| \geq c \lambda^{-m} \| w \|, w \in TM^u;$$

where $d$ is the usual differential, $c$ a positive real number, and $m$ a positive integer.

The stable distribution $TM^s$ (resp. the unstable distribution $TM^u$) is tangent to a topological foliation $\mathcal{F}^s$ (resp. $\mathcal{F}^u$).

The property for a diffeomorphism to be Anosov is independent of the choice of the riemannian metric if $M$ is compact. In this case we can suppose that $c$ is 1.

A Franks’s conjecture asserts that an Anosov diffeomorphism $f$ of a compact manifold $M$ is $C^0$—conjugated to an hyperbolic infranilautomorphism. This conjecture is proved in [1] with the assumptions that $f$ is topologically transitive
it’s stable and unstable foliations are $C^\infty$, it preserves a symplectic form or a connection.

The goal of this paper is to characterize compact affine manifolds endowed with affine Anosov transformations. More precisely, we show:

**Theorem 1.** Let $(M, \nabla_M)$ be a compact affine manifold, and $f$ an affine Anosov transformation of $M$, then $(M, \nabla_M)$ is finitely covered by a complete affine nilmanifold.

1. The Proof of the main theorem.

The main goal of this part is to show theorem 1. In the sequel, $(M, \nabla_M)$ will be an $n$–compact affine manifold endowed with an affine Anosov diffeomorphism $f$. The stable foliation $\mathcal{F}^s$ (resp. the unstable foliation $\mathcal{F}^u$) pulls-back on the universal cover $\hat{M}$ to a foliation $\hat{\mathcal{F}}^s$, (resp. $\hat{\mathcal{F}}^u$).

Let $(U, \phi)$ be an affine chart of $M$, the restriction of a riemannian metric of $M$ to $U$ will be said an euclidean metric adapted to the affine structure of $U$ if its restriction to $U$ is the pulls-back by $\phi$ on $U$, of the restriction to $\phi(U)$ of an euclidean metric of $\mathbb{R}^n$.

**Proposition 1.1.** The stable and the unstable distributions of $f$ define on $(M, \nabla_M)$ foliations whose leaves are immersed affine submanifolds.

**Proof.** Let $x$ be an element of $M$, $|| \cdot ||$ a norm associated to a riemannian metric $<,>$ of $M$. Let $v$ be an element of $T_xM^s$ the subspace of $T_xM$ tangent to $\mathcal{F}^s$, we have $|| d_f^m(v) || \leq \lambda^m || v ||$ where $0 < \lambda < 1$, and $m \in \mathbb{N}$. Let $(U, \phi)$ be an affine chart which contains a point of accumulation of the sequence $(f^p(x))_{p \in \mathbb{N}}$. We can suppose that the restriction of $<,>$ to $U$ is euclidean and adapted to the affine structure. Let $p > p'$ such that $f^p(x)$ and $f^{p'}(x)$ are elements of $U$, and $v \in T_{f^{p'}(x)}M^s$, we have $|| df^{p-p'}(v) || \leq \lambda^{p-p'} || v ||$. This implies that for every element $y = f^{p'}(x) + w, w \in T_{f^{p'}(x)}M^s$ and $f^{p-p'}(y) \in U$ is an element of the stable leaf of $f^{p'}(x)$ since the distance between $f^{n_q}(x)$ and $f^{n_q}(y)$ goes to zero for a subsequence $n_q > p$ such that $f^{n_q}(x)$ is an element of $U$. We deduce that this leaf is an immersed submanifold. The result for the unstable foliation is deduced considering $f^{-1}$.

**Proposition 1.2.** The leaves of $\mathcal{F}^s$ are geodesically complete for the affine structure.

**Proof.** Let $x$ be an element of $M$, as $M$ is supposed to be compact, the sequence $(f^m(x))_{m \in \mathbb{N}}$ has a point of accumulation $y$. Let $U_y$ be an open set containing $y$ such that there is a strictly positive number $r$, such that for every $z \in U_y, v \in T_zM$ whose norm is less than $r$ for a given riemannian metric, a (affine) geodesic from $x$ whose derivative at 0 is $v$ is defined at 1. Let $w$ be an element of $T_xM^s$ such that $|| df^p(w) || \leq r$. Without loss generality, we can suppose that $f^p(x) \in U_y$, the (affine) geodesic from $f^p(x)$ whose derivative
is $df^p(w)$ at 0 is defined at 1. This implies that the geodesic from $x$ which derivative at 0 is $w$ is defined at 1, since $f^{-1}$ is an affine map. This shows the result.

It is a well-known fact that an Anosov’s diffeomorphism of a compact manifold as a periodic fixed point. We will assume up to change $f$ by an iterated that $f$ has a fixed point $x$. There exists also a map $F: M \to M$ over $f$ which fixed the element $\hat{x}$ over $x$.

**Proposition 1.3.** Let $\hat{y}$ and $\hat{z}$ be two elements of $\hat{F}^y_i$, where $t$ is an element of $\hat{M}$, then the images of $\hat{F}^s_i$ and $\hat{F}^u_i$ by the developing map are parallel affine subspaces.

**Proof.** Let $\hat{y}$ be an element of $\hat{F}^u_i$, it is sufficient to show that $D(\hat{F}^s_i)$ and $D(\hat{F}^u_i)$ are parallel affine subspaces.

We know that the tangent bundle of a simply connected affine manifold is trivial. We have $TM = M \times T_1M$. Let $<,>$ be a riemannian metric on $M$ which pulls back on $\hat{M}$ is $\langle,\rangle$. The map $F$ is Anosov relatively to $\langle,\rangle$. Without loss generality, we can assume that the distributions tangent to $\hat{F}^s$ and $\hat{F}^u$ are orthogonal.

Let $w$ be a vector of $T\hat{M}_t$ tangent to $\hat{F}^s_i$. We can write $w = s + u$ where $s$ is a vector of $T\hat{M}_t$ tangent to $\hat{F}^s_i$ and $u$ is a vector of $T\hat{M}_t$ tangent to $\hat{F}^u_i$. The vector $u$ is also an element of $T\hat{M}_t$ tangent to $\hat{F}^u_i$ since the unstable foliation is affine (we can identify the projection on the second factor of $TM \times T\hat{M}_t$ of vectors tangent to $\hat{t}$ and $\hat{y}$ since $T\hat{M}$ is trivial). This implies that $||DF^u_{\hat{y}}(u)|| \geq \lambda^{-n}||u||$ for $0 < \lambda < 1$. But on the other hand we have $||DF^u_{\hat{y}}(s + u)|| \leq \lambda^n ||(s + u)||$, which implies that the limit of $||DF^u_{\hat{y}}(s + u)||$ is zero when $n$ goes to infinity.

We deduce that $u = 0$ since we have supposed that the distributions tangent to $\hat{F}^s$ and $\hat{F}^u$ are orthogonal, so $||DF^s_{\hat{y}}(s + u)|| = ||DF^u_{\hat{y}}(s)|| + ||DF^u_{\hat{y}}(u)||$. This implies the result.

The images by $D_M$ of the leaf $\hat{F}^s_i$ of $\hat{F}^s$ and $\hat{F}^u_i$ of $\hat{F}^u$ are affine subspaces of $\mathbb{R}^n$ whose direction are direct summand of $\mathbb{R}^n$. Since for every element $z$ of $\hat{F}^s_i$ the leaf of $\hat{F}^u$ passing by $z$ is complete, we deduce that the developing map is surjective.

**Proposition 1.4.** The affine manifold $(M, \nabla_M)$ is complete.

**Proof.** Let $\hat{t}$ be an element of $\hat{M}$, and $\hat{E}_t$ the set of elements $y$ of $\hat{M}$ such that there is and element $z$ in $\hat{F}^u_i$ such that $y$ is an element of $\hat{F}^u_i$. The image of $\hat{E}_t$ by $D_M$ is $\mathbb{R}^n$, and the restriction of $D_M$ to $\hat{E}_t$ is injective. The set $\{\hat{E}_t, \hat{t} \in \hat{M}\}$ is a partition of $\hat{M}$ by disjoint open sets. It has only one element since $\hat{M}$ is connected. We deduce that $(M, \nabla_M)$ is complete.

**Remark.**
The existence of an affine Anosov transformation \( \hat{f} \) on the universal cover of a compact affine manifold \((M, \nabla_M)\) which pulls forward onto a diffeomorphism \( f \) of \( M \) does not imply that \( f \) is an Anosov diffeomorphism as shows the following example:

Let \( Hopf(n) \) be the quotient of \( \mathbb{R}^n/\{0\} \) by an homothetie \( h_\lambda \) whose ratio \( \lambda \) is such that \( 0 < \lambda < 1 \). It is a compact affine manifold. Every homothetie \( h_c \) which ratio \( c \) is positive and different from 1 and \( \lambda \) is an Anosov diffeomorphism of \( \mathbb{R}^n \) endowed with an euclidean metric. But the pulls forward of \( h_c \) is an isometry of \( Hopf(n) \) endowed with the pulls forward of the riemannian metric of \( \mathbb{R}^n/\{0\} \) defined as follows:

\[
\frac{1}{||x||^2} < u, v >
\]

where \( x \) is an element of \( \mathbb{R}^n/\{0\} \), and \( u, v \) are elements of its tangent space.

**Proof of theorem 1.**

First we show that the module of the eigenvalues of the elements of the linear holonomy of \((M, \nabla_M)\) is 1.

Let \((A, a)\) be an element of \( \pi_1(M) \) such that \( A \) has an eigenvector \( u \) associated to an eigenvalue \( b \) (which may be a complex number) whose norm is different from 1.

It is a well-known fact that an Anosov diffeomorphism of a compact manifold has a fixed periodic point. Without loss of generality, we will assume that up to change \( f \) by an iterated that \( f \) has a fixed point \( x \). This implies that up to a change of coordinates, there exists on \( \mathbb{R}^n \) a linear map \( F \) over \( f \).

Put \( x = p(0) \) where \( p \) is the covering map and consider a riemannian metric \( <,> \) on \( M \) whose restriction on an affine neighborhood \( N \) of \( x \) is euclidean adapted to the affine structure. Expressing on \( N \) the fact that \( F \) is an Anosov diffeomorphism using the metric \( <,> \), one obtains that \( \mathbb{R}^n = U \oplus V \), where \( U \) and \( V \) are two subvectors spaces such that there exists a number \( 0 < \lambda < 1 \) such that

\[
|| F^n(u) || \leq \lambda^n || u ||, u \in U,
\]

\[
|| F^n(v) || \geq \lambda^{-n} || u ||, v \in V
\]

where \( u \), and \( v \) are respectively elements of \( U \) and \( V \) and \( || || \) is a norm associated to an euclidean metric \(<,>\) of \( \mathbb{R}^n \). The subvectors spaces \( U \) and \( V \) pulls forward respectively onto \( T_xM^e \) and \( T_xM^u \). We will assume that they are orthogonal with respect to \(<,>\). The vectors spaces \( U \) and \( V \) are stable by the linear holonomy (see proposition 1.3)

Put \( u = u_1 + u_2 \), where \( u_1 \) and \( u_2 \) are respectively elements of \( U' \) and \( V' \) the complexified vectors spaces respectively associated to \( U \) and \( V \).

Without restrict the generality, one can assume that after eventually having changed \( \gamma \) by \( \gamma^{-1} \) and (or) \( f \) by \( f^{-1} \) that \( u_1 \) is not zero and that the norm of \( b \) is strictly superior to 1.

Let \( q \) be a positive integer, consider the smallest positive integer \( n_q \) such that \( F^{n_q} \circ \gamma^q \) has a fixed point \( \hat{m}_q \). The integer \( n_q \) exists since \( A \) preserves \( U \).
and $V$. Let $y$ be a point of accumulation of the sequence $p(\hat{m}_q)$. Up to replace $(\hat{m}_q)$ by a subsequence and replace $m_q$ by another element over $m_q$, we can suppose that the sequence $(\hat{m}_q)$ converges to $\hat{y}$ over $y$. The linear part of the elements over $f^n$ which fix $m_q$ have the same eigenvalues since they are conjugated. Consider a riemannian metric of $M$ which pulls back on $\mathbb{R}^n$ coincides on a neighborhood of $\hat{y}$ with an euclidean metric.

1. The sequence $(n_q)$ is bounded.

The restriction of the linear part of the element of $f^n$ conjugated to $F^n \circ \gamma^q$ which fixe $\hat{m}_q$ at $U$ is contractant (for the last euclidean metric) with ratio $\lambda^n_q$ ($0 < \lambda < 1$). This is not possible since the restriction of the linear part of this element to $U'$ has the same eigenvalues than the restriction of $F^n \circ A^q$ to $U'$, and the limit when $q$ goes to infinity of the norm of $F^n \circ A^q(u_1)$ for the hermitian metric associated to the last euclidean metric is infinity, since the sequence $n_q$ is bounded and the norm of $b > 1$.

2. The sequence $(n_q)$ is not bounded.

Up to change $(n_q)$ by a subsequence, we can suppose that $(n_q)$ goes to infinity. The linear map $F^{n_q-1} \circ \gamma^q$ does not have a fixed point. Its linear part has the eigenvalue 1 associated to the eigenvector $v_q$. Write $v_q = v_{1q} + v_{2q}$ where $v_{1q}$ and $v_{2q}$ are respectively element of $U$ and $V$. If $v_{1q}$ is not zero, then $\| F^{n_q} \circ A^q(v_{1q}) \| = \| F(v_{1q}) \|$ (the norm considered is the precedent euclidean norm); the restriction of $F^{n_q} \circ A^q$ cannot be contractant with ratio $\lambda^{n_q}$ for $q$ enough big since the sequence $(n_q)$ goes to infinity.

If $v_{2q}$ is not zero, then $\| F^{n_q} \circ A^q(v_{2q}) \| = \| F(v_{2q}) \|$, the restriction of $F^{n_q} \circ A^q$ to $V$ cannot be dilatant (for the last euclidean metric) with ratio $\lambda^{-n_q}$ for $q$ enough big since $n_q$ goes to infinity. There is a contradiction since the eigenvalues of the restriction of $F^{n_q} \circ A^q$ to $U'$ (resp to $V'$) coincide with the eigenvalues of the restriction to $U'$ of the linear part of the conjugated to $F^n \circ \gamma^q$ which fixes $\hat{m}_q$ (resp. the eigenvalues of the restriction to $V'$ of the linear part of the element conjugated to $F^n \circ \gamma^q$ which fixes $\hat{m}_q$). We deduce that the eigenvalues of $A$ are roots of unity.

We deduce from the first paragraph of [4] p.6 that $\pi_1(M)$ has a subgroup of finite index $G$ such that the eigenvalues of the linear part of the elements of $G$ are 1. We deduce from [3] that the quotient of $\mathbb{R}^n$ by $G$ is a complete affine nilmanifold.

**Remark.**

Let $(M', \nabla')$ be the finite cover of $(M, \nabla)$ which fundamental group is $G$, and $f'$ the pulls-back of $f$ to $M'$. The map $\text{Hom}(M, \mathbb{R}) \to \text{Hom}(M, \mathbb{R})$, $\alpha \to (f'^{\sharp})^\ast \alpha$ is the identity since $f'^{\sharp}$ is a diffeomorphism which preserves the orientation. Let $\omega$ be the parallel form of $(M', \nabla')$, we deduce that $(f'^{\sharp})$ preserves the parallel volume form of $(M', \nabla')$; then a well-known result of Anosov implies that $f'^{\sharp}$ is ergodic and its periodic points are dense.

**Bibliography.**
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