Toward quantization of Galois theory

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February 10, 2015

Abstract

In this note, we explore an unknown land of quantized Galois theory. We know Hopf-Galois theory for linear equations or Picard-Vessiot theory in terms of Hopf algebra [2] that is a general Galois theory of linear equations with a set of non-commutative operators. The Hopf algebras in this theory are, however, essentially assumed to be co-commutative. In other words, they are interested in only commutative rings with operators. Consequently their Galois groups are linear algebraic groups. In other words, the Galois theory is not quantized.

Heiderich [7] discovered that we can combine the Hopf Galois theory for linear equations and our general Galois theory for non-linear equations. We apply this theory to some concrete examples and show that the quantization of Galois group happens in the first part.

In fact, quantization occurs even for linear equations. In the second part, we analyze, one particular example of linear difference-differential equation to show the unique existence of the non-commutative Picard-Vessiot ring and asymmetric Tannaka theory.

Starting from this example and other similar examples of linear equations, Akira Masuoka [11] generalized this example to any Hopf linear equations over a constant field.
Part I
Quantization of non-linear $q$-SI $\sigma$-differential equations

1 Introduction

The pursuit of $q$-analogue of hypergeometric functions goes back to the 19th century. Galois group of a $q$-hypergeometric function is not a quantum group but it is a linear algebraic group. This shows that if we consider a $q$-deformations of the hypergeometric equation, Galois theory is not quantized. In fact, generally we know that the Galois group of a linear difference equation is a linear algebraic group.

Y. André [3] was the first who studied linear difference-differential equations in the framework of non-commutative geometry. He encountered only linear algebraic groups treating linear difference-differential equations. Hardouin [5] also studied Picard-Vessiot theory of $q$-skew iterative $\sigma$-differential field extensions but in this theory, the Galois group is a linear algebraic group. We clarified the situation in [21]. So far as they studied linear difference-differential equations, however twisted or non-commutative the ring of difference and differential operators might be, Galois group, according to general Hopf Galois theory, is a linear algebraic group.

We believed for a long time that it was impossible to quantize Picard-Vessiot theory, Galois theory for linear difference or differential equations. Namely, there was no Galois theory for linear difference-differential equations, of which the Galois group is a quantum group that is, in general, neither commutative nor co-commutative. This is not correct as we see in this note. Our mistake came from a misunderstanding of preceding works of Hardouin [5] and of Masuoka and Yanagawa [12].

The correct understanding of the picture seems that despite they considered a set of non-commutative operators, as they assumed that the rings of functions on which the set of non-commutative operators act were commutative, they did not arrive at a quantization of Galois theory. In fact, in their Picard-Vessiot theory [5], [12], a Picard-Vessiot extension is a difference-differential field extension.

With this misbelief, it was natural to wonder how about considering non-linear difference-differential equations. We proposed to study the $q$-Painlevé equations in [21]. We elaborated and we can answer this question in the following way. As we observe in the first part, that quantization of Galois group happens for much simpler equations than the $q$-Painlevé equations (Sections 4, 5 and 6). Moreover the First Example reduces to a pair of linear difference-differential equations breaking our wrong belief. In the first part, after a brief review of our framework, we analyze three examples of difference-differential field extensions. In these examples, however, the Galois hulls or the normalizations are not commutative rings yielding quantum Galois group that are neither commutative nor co-commutative Hopf algebras.

Among these three examples the first one is given by a pair of linear difference-differential equations. In the second part, we analyze this example throughly. We show
that the Picard-Vessiot ring exists uniquely and the asymmetric Tannaka theory holds for this particular example. Looking at this and further examples found in Section 12, Masuoka has established a general quantum Picard-Vessiot theory over a constant field \[11\]. See Introduction to the second part.

We work over a field \(C\) of characteristic 0. We consider \(C\)-algebras. Except for Lie algebras, all the rings or algebras are associative \(C\)-algebras and contain the unit element. So the field \(C\) is in the center of the algebras. Morphisms between them are unitary \(C\)-morphisms. For a commutative algebra \(A\), we denote by \((\text{Alg}/A)\) the category of commutative \(A\)-algebras, which we sometimes denote by \((\text{CAlg}/A)\) to emphasize that we are dealing with commutative \(A\)-algebras. In fact, to study quantum groups, we have to also consider non-commutative \(A\)-algebras. We denote by \((\text{NCAlg}/A)\) the category of not necessarily commutative \(A\)-algebras \(B\) such that \(A\) (or to be more logic, the image of \(A\) in \(B\)) is contained in the center of \(B\).

We thank Professors Akira Masuoka and Katsutoshi Amano for for teaching us their Galois theory and for valuable discussions.

\section{Foundation of a general Galois theory \cite{17, 19, 20}}

\subsection{Notation}

Let us recall basic notation. Let \((R, \delta)\) be a differential ring so that \(\delta : R \to R\) is a derivation of a commutative ring \(R\) of characteristic 0. When there is no danger of confusion of the derivation \(\delta\), we simply say the differential ring \(R\) without referring to the derivation \(\delta\). We often have to talk, however, about the abstract ring \(R\) that we denote by \(R^\circ\). For a commutative ring \(S\) of characteristic 0, the power series ring \(S[[X]]\) with derivation \(d/dX\) gives us an example of differential ring.

\subsection{General Galois theory of differential field extensions}

Let us start by recalling our general Galois theory of differential field extensions.

\subsubsection{Universal Taylor morphism}

Let \((R, \delta)\) be a differential algebra so that \(S\) is a commutative \(C\)-algebra and \(\delta : R \to R\) is a \(C\)-derivation:

\begin{enumerate}
\item \(\delta : R \to R\) is a \(C\)-linear map.
\item \(\delta(ab) = \delta(a)b + a\delta(b)\) for all \(a, b \in R\).
\end{enumerate}

For the differential algebra \((R, \delta)\) and a commutative \(C\)-algebra \(S\), a Taylor morphism is a differential morphism

\[ (R, \delta) \to (S[[X]], d/dX). \]
Given a differential ring \((R, \delta)\), among the Taylor morphisms (1), there exists the universal one. In fact, for an element \(a \in R\), we define the power series
\[
\iota(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a) X^n \in R^\mathbb{Z}[[X]].
\]
Then the map
\[
\iota: (R, \delta) \rightarrow (R^\mathbb{Z}[[X]], d/dX)
\]
is the universal Taylor morphism.

### 2.2.2 Galois hull \(L/K\) for a differential field extension \(L/k\)

Let \((L, \delta)/(k, \delta)\) be a differential field extension such that the abstract field \(L^k\) is finitely generated over the abstract base field \(k^k\). We constructed the Galois hull \(L/K\) in the following manner.

We take a mutually commutative basis
\[
\{D_1, D_2, \cdots, D_d\}
\]
of the \(L^k\)-vector space \(\text{Der}(L^k/k^k)\) of \(k^k\)-derivations of the abstract field \(L^k\). So we have
\[
[D_i, D_j] = D_iD_j - D_jD_i = 0 \quad \text{for } 1 \leq i, j \leq d.
\]

Now we introduce a partial differential structure on the abstract field \(L^k\) using the derivations \(\{D_1, D_2, \cdots, D_d\}\). Namely we set
\[
L^k := (L^k, \{D_1, D_2, \cdots, D_d\})
\]
that is a partial differential field. Similarly we define a differential structure on the power series ring \(L^k[[X]]\) with coefficients in \(L^k\) by considering the derivations
\[
\{D_1, D_2, \cdots, D_d\}
\]
that operate on the coefficients of the power series. In other words, we work with the differential ring \(L^k[[X]]\). So the power series ring \(L^k[[X]]\) has differential structure defined by the differentiation \(d/dX\) with respect to the variable \(X\) and the set
\[
\{D_1, D_2, \cdots, D_d\}
\]
of derivations. Since there is no danger of confusion of the choice of the differential operator \(d/dX\), we denote this differential ring by
\[
L^k[[X]].
\]
We have the universal Taylor morphism
\[
\iota: L \rightarrow L^k[[X]]
\]
that is a differential morphism. We added further the \(\{D_1, D_2, \cdots, D_d\}\)-differential structure on \(L^d[[X]]\) or we replace the target space \(L^d[[X]]\) of the universal Taylor morphism \(\mathfrak{t}\) by \(L^d[[X]]\) so that we have

\[ \mathfrak{t} : L \to L^d[[X]]. \]

In Definition 2.1 below, we work in the differential ring \(L^d[[X]]\) with differential operators \(d/dX\) and \(\{D_1, D_2, \cdots, D_d\}\). We identify the differential field \(L^d\) with the set of power series consisting only of constant terms. Namely,

\[ L^d = \left\{ \sum_{n=0}^{\infty} a_n X^n \in L^d[[X]] \mid \text{The coefficients } a_n = 0 \text{ for every } n \geq 1 \right\}. \]

Therefore \(L^d\) is a differential sub-field of the differential ring \(L^d[[X]]\). The differential operator \(d/dX\) kills \(L^d\). Similarly, we set

\[ k^d := \left\{ \sum_{n=0}^{\infty} a_n X^n \in L^d[[X]] \mid \text{The coefficients } a_0 \in k \text{ and } a_n = 0 \text{ for every } n \geq 1 \right\}. \]

So all the differential operators \(d/dX, D_1, D_2, \cdots, D_d\) act trivially on \(k^d\) and so \(k^d\) is a differential sub-field of \(L^d\) and hence of the differential algebra \(L^d[[X]]\).

**Definition 2.1.** The Galois hull \(\mathcal{L}/\mathcal{K}\) is the differential sub-algebra of \(L^d[[X]]\), where \(\mathcal{L}\) is the differential sub-algebra generated by the image \(\mathfrak{t}(L)\) and \(L^d\) and \(\mathcal{K}\) is the sub-algebra generated by the image \(\mathfrak{t}(k)\) and \(L^d\). So \(\mathcal{L}/\mathcal{K}\) is a differential algebra extension with differential operators \(d/dX\) and \(\{D_1, D_2, \cdots, D_d\}\).

### 2.2.3 Universal Taylor morphism for a partial differential ring

The universal Taylor morphism has a generalization for partial differential ring. Let

\[ (R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \]

be a partial differential ring. So \(R\) is a commutative ring of characteristic 0 and \(\partial_i : R \to R\) are mutually commutative derivations. For a ring \(S\), the power series ring

\[ (S[[X_1, X_2, \cdots, X_d]], \left\{ \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d} \right\}) \]

gives us an example of partial differential ring.

A Taylor morphism is a differential morphism

\[ (R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \to (S[[X_1, X_2, \cdots, X_d]], \left\{ \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d} \right\}). \quad (4) \]

For a differential algebra \((R, \{\partial_1, \partial_2, \cdots, \partial_d\})\), among Taylor morphisms \((4)\), there exists the universal one \(\mathfrak{t}_R\) given below.
Definition 2.2. The universal Taylor morphism is a differential morphism
\[ \iota_R: (R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \rightarrow (R^d[[X_1, X_2, \cdots, X_d]], \{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\}) \] (5)
defined by the formal power series expansion
\[ \iota_R(a) = \sum_{n \in \mathbb{N}^d} \frac{1}{n!} \partial^n(a) X^n \]
for an element \( a \in R \), where we use the standard notation for multi-index. Namely, for \( n = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d \),
\[ |n| = \sum_{i=1}^{d} n_i, \]
\[ \partial^n = \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_d^{n_d} \]
\[ n! = n_1! n_2! \cdots n_d! \]
and
\[ X^n = X_1^{n_1} X_2^{n_2} \cdots X_d^{n_d}. \]

See Proposition (1.4) in Umemura [17].

2.2.4 The functor \( \mathcal{F}_{L/k} \) of infinitesimal deformations for a differential field extension

For the partial differential field \( L^s \), we have the universal Taylor morphism
\[ \iota_{L^s}: L^s \rightarrow L^s[[W_1, W_2, \cdots, W_d]] = L^s[[W]], \] (6)
where we replaced the variables \( X \)'s in (5) by the variables \( W \)'s for a notational reason. The universal Taylor morphism (6) gives a differential morphism
\[ L^s[[X]] \rightarrow L^s[[W_1, W_2, \cdots, W_d]][[X]]. \] (7)

Restricting the morphism (7) to the differential sub-algebra \( \mathcal{L} \) of \( L^s[[X]] \), we get a differential morphism \( \mathcal{L} \rightarrow L^s[[W_1, W_2, \cdots, W_d]][[X]] \) that we denote by \( \iota \). So we have the differential morphism
\[ \iota: \mathcal{L} \rightarrow L^s[[W_1, W_2, \cdots, W_d]][[X]]. \] (8)

Similarly for every commutative \( L^s \)-algebra \( A \), thanks to the differential morphism
\[ L^s[[W]] \rightarrow A[[W]] \]

arising from the structural morphism \( L^s \rightarrow A \), we have the canonical differential morphism
\[ \iota: \mathcal{L} \rightarrow A[[W_1, W_2, \cdots, W_d]][[X]]. \] (9)
We define the functor
\[ F_{L/k} : (\text{Alg}/L^\natural) \to (\text{Set}) \]
from the category \((\text{Alg}/L^\natural)\) of commutative \(L^\natural\)-algebras to the category \((\text{Set})\) of sets, by associating to an \(L^\natural\)-algebra \(A\), the set of infinitesimal deformations of the canonical morphism \(\mathbf{S}\). So
\[ F_{L/k}(A) = \{ f : L \to A[[W_1, W_2, \cdots, W_d]][[X]] \mid f \text{ is a differential morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted on the sub-algebra } K \}. \]

### 2.2.5 Group functor \(\text{Inf-gal}(L/k)\) of infinitesimal automorphisms for a differential field extension

The Galois group in our Galois theory is the group functor
\[ \text{Inf-gal}(L/k) : (\text{Alg}/L^\natural) \to (\text{Grp}) \]
defined by
\[ \text{Inf-gal}(L/k)(A) = \{ f : L^\natural \otimes L^\natural A[[W]] \to L^\natural \otimes L^\natural A[[W]] \mid f \text{ is a differential } \mathcal{K}^\natural \otimes L^\natural A[[W]]\text{-automorphism continuous with respect to the } W\text{-adic topology and congruent to the identity modulo nilpotent elements} \} \]
for a commutative \(L^\natural\)-algebra \(A\). Here the completion is taken with respect to the \(W\)-adic topology. See Definition 2.19 in [13].

Then the group functor \(\text{Inf-gal}(L/k)\) operates on the functor \(F_{L/k}\) in such a way that the operation \((\text{Inf-gal}(L/k), F_{L/k})\) is a principal homogeneous space (Theorem (5.11), [17]).

### 2.2.6 Origin of the group structure

For the differential equations, the Galois group is a group functor. We are going to generalize differential Galois theory in such a way that the Galois group is a quantum group. Quantum group is a generalization of affine algebraic group. We can not, however, regard a quantum group as a group functor. Therefore, we have to understand the origin of the group functor \(\text{Inf-gal}\). We illustrate it by an example.

**Example 2.3.** Let us consider a differential field extension
\[ L/k := (\mathbb{C}(y), \delta)/\mathbb{C} \]
such that \(y\) is transcendental over the field \(\mathbb{C}\) and
\[ \delta(y) = y \quad \text{and} \quad \delta(\mathbb{C}) = 0 \]
so that \(k = \mathbb{C}\) is the field of constants of \(L\).
The universal Taylor morphism

$$\iota : L \to L^b[[X]]$$

maps $y \in L$ to

$$Y := y \exp X \in L^b[[X]].$$

Since the field extension $L^b/k^b = \mathbb{C}(y)/\mathbb{C}$, taking $d/dy \in \text{Der}(L^b/k^b)$ as a basis of 1-dimensional $L^b$-vector space $\text{Der}(L^b/k^b)$, we get $L^b := (L^b, d/dy)$. As we have relations

$$\frac{\partial Y}{\partial X} = Y, \quad y \frac{\partial Y}{\partial y} = Y \quad (11)$$

in the power series ring $L^b[[X]]$ so that the Galois hull $L/K$ is

$$L = K.\mathbb{C}(\exp X), \quad K = L^b \subset L^b[[X]] \quad (12)$$

by definition of the Galois hull.

Now let us look at the infinitesimal deformation functor $\mathcal{F}_{L/k}$. To this end, we Taylor expand the coefficients of the power series in $L^b[[X]]$ to get

$$\iota : L \to L^b[[X]] \to L^b[[W]][[X]] = L^b[[W, X]]$$

so that

$$\iota(y) = (y + W) \exp X \in L^b[[W, X]].$$

We identify $L^b[[X]]$ with its image in $L^b[[W]][[X]] = L^b[[W, X]]$. In particular we identify $Y = y \exp X \in L^b[[X]]$ with $Y(W, X) = (y + W) \exp X \in L^b[[W, X]]$. Equalities (11) become in $L^b[[W, X]]$

$$\frac{\partial Y(W, X)}{\partial X} = Y(W, X), \quad (y + W) \frac{\partial Y(W, X)}{\partial W} = Y \quad (13)$$

It follows from (13), for a commutative $L^b$-algebra $A$, an infinitesimal deformation $\varphi \in \mathcal{F}_{L/k}(A)$ is determined by the image

$$\varphi(Y(W, X)) = cY(W, X) \in A[[W, X]], \quad (14)$$

where $c \in A$. Conversely any invertible element $c \in A$ infinitesimally close to 1 defines an infinitesimal deformation so that we conclude

$$\mathcal{F}_{L/k}(A) = \{ c \in A \mid c - 1 \text{ is nilpotent} \} \quad (15)$$

Where does the group structure come from?

There are two ways of answering to this question, which are closely related.

(1) Algebraic answer.

By (14), we have

$$\varphi(y) = c(y + W) \exp X \in A[[W, X]],$$

where $c - 1 \in A$ is a nilpotent element. Consequently we have

$$\varphi(y) = Y((c - 1)y + cW, X). \quad (16)$$
In other words \( \varphi(y) \) coincides with

\[
Y(W, X) \big|_{W=(c-1)y+cW}.
\]

Equivalently \( \varphi(y) \) is obtained by substituting \((c-1)y+cW\) for \(W\) in \(Y(W, X)\). This is quite natural in view of differential equations (14). We only have to look at the initial condition at \(X = 0\) of the solutions \(Y(W, X)\) and \(\varphi(y) = c(y+W)Y(W, X)\) of the differential equation \(\partial Y/\partial X = Y\). The transformation

\[
W \mapsto (c-1)y+cW \quad \text{where} \quad c \in A \quad \text{and} \quad c-1 \text{ is nilpotent},
\]

is an infinitesimal coordinate transformation of the initial condition and the multiplicative structure of \(c\) is nothing but the composite of coordinate transformations (17).

(II) Geometric answer.

To see this geometrically, we have to look at the dynamical system defined by the differential equation (10). Geometrically the differential equation (10) gives us a dynamical system on the line \( \mathbb{C} \).

\[
y \mapsto Y = y \exp X
\]

describes the dynamical system. Observing the dynamical system through algebraic differential equations, is equivalent to considering the deformations of the Galois hull. So the (infinitesimal) deformation functor measures the ambiguity of the observation. In other words, the result due to our method is (15). In terms of the initial condition, it looks as

\[
y \mapsto cY \big|_{X=0} = cy \exp X \big|_{X=0} = cy.
\]

Namely,

\[
y \mapsto cy.
\]

If we have two transformations (18)

\[
y \mapsto cy, \quad y \mapsto c'y
\]

the composite transformation corresponds to the product

\[
y \mapsto cc'y.
\]

Our generalization depends on the first answer (I). See Section 7.

### 2.3 Difference Galois theory

If we replace the universal Taylor morphism by the universal Euler morphism, we can construct a general Galois theory of difference equations ([13], [14]).
### 2.3.1 Universal Euler morphism

Let \((R, \sigma)\) be a \(C\)-difference algebra so that \(\sigma : R \to R\) is a \(C\)-algebra automorphism of a commutative \(C\)-algebra \(R\). See Remark 3.7. When there is no danger of confusion of the automorphism \(\sigma\), we simply say the \(C\)-difference algebra \(R\) without referring to the automorphism \(\sigma\). We often have to talk however about the abstract ring \(R\) that we denote by \(R^\natural\). For a commutative ring \(S\), we denote by \(F(Z, S)\) the ring of functions on the set of integers \(Z\) taking values in the ring \(R\). For a function \(f \in F(Z, S)\), we define the shifted function \(\Sigma f \in F(Z, S)\) by

\[
(\Sigma f)(n) = f(n + 1) \quad \text{for every } n \in \mathbb{Z}.
\]

Hence the shift operator

\[
\Sigma : F(Z, S) \to F(Z, S)
\]

is an automorphism of the ring \(F(Z, S)\) so that \((F(Z, S), \Sigma)\) is a difference ring.

**Remark 2.4.** In this paragraph 2.3.1 and the next 2.3.2, in particular for the existence of the universal Euler morphism, we do not need the commutativity assumption of the underlying ring.

Let \((R, \sigma)\) be a difference ring and \(S\) a ring. An Euler morphism is a difference morphism

\[
(R, \sigma) \to (F(Z, S), \Sigma).
\]  

(19)

Given a difference ring \((R, \sigma)\), among the Euler morphisms (19), there exists the universal one. In fact, for an element \(a \in R\), we define the function \(u[a] \in F(Z, R^\natural)\) by

\[
u[a](n) = \sigma^n(a) \quad \text{for } n \in \mathbb{Z}.
\]

Then the map

\[
\iota : (R, \sigma) \to (F(Z, R^\natural), \Sigma) \quad a \mapsto u[a]
\]  

(20)

is the universal Euler morphism (Proposition 2.5, [13]).

### 2.3.2 Galois hull \(L/K\) for a difference field extension \(L/k\)

Let \((L, \sigma)/(k, \sigma)\) be a difference field extension such that the abstract field \(L^\natural\) is finitely generated over the abstract base field \(k^\natural\). We constructed the Galois hull \(L/K\) as in the differential case. Namely, we take a mutually commutative basis

\[
\{D_1, D_2, \cdots, D_d\}
\]

of the \(L^\natural\)-vector space \(\text{Der}(L^\natural/k^\natural)\) of \(k^\natural\)-derivations of the abstract field \(L^\natural\). We introduce the partial differential field

\[
L^\natural := (L^\natural, \{D_1, D_2, \cdots, D_d\}).
\]

Similarly we define a differential structure on the ring \(F(Z, L^\natural)\) of functions taking values in \(L^\natural\) by considering the derivations

\[
\{D_1, D_2, \cdots, D_d\}.
\]
In other words, we work with the differential ring $F(\mathbb{Z}, L^\sharp)$. So the ring $F(\mathbb{Z}, L^\sharp)$ has a difference-differential structure defined by the shift operator $\Sigma$ and the set

$$\{D_1, D_2, \cdots, D_d\}$$

of derivations. Since there is no danger of confusion of the choice of the difference operator $\Sigma$, we denote this difference-differential ring by

$$F(\mathbb{Z}, L^\sharp).$$

We have the universal Euler morphism

$$\iota: L \to F(\mathbb{Z}, L^\sharp)$$  \hspace{1cm} (21)

that is a difference morphism. We added further the $\{D_1, D_2, \cdots, D_d\}$-differential structure on $F(\mathbb{Z}, L^\sharp)$ or we replace the target space $F(\mathbb{Z}, L^\sharp)$ of the universal Euler morphism (21) by $F(\mathbb{Z}, L^\sharp)$ so that we have

$$\iota: L \to F(\mathbb{Z}, L^\sharp).$$

In Definition 2.5 below, we work in the difference-differential ring $F(\mathbb{Z}, L^\sharp)$ with difference operator $\Sigma$ and differential operators $\{D_1, D_2, \cdots, D_d\}$. We identify with $L^\sharp$ the set of constant functions on $\mathbb{Z}$. Namely,

$$L^\sharp = \{f \in F(\mathbb{Z}, L^\sharp) \mid f(0) = f(\pm 1) = f(\pm 2) = \cdots \in L^\sharp\}.$$

Therefore $L^\sharp$ is a difference-differential sub-field of the difference-differential ring $F(\mathbb{Z}, L^\sharp)$. The action of the shift operator on $L^\sharp$ being trivial, the notation is adequate. Similarly, we set

$$k^\sharp := \{f \in F(\mathbb{Z}, L^\sharp) \mid f(0) = f(\pm 1) = f(\pm 2) = \cdots \in k \subset L^\sharp\}.$$

So both the shift operator and the derivations act trivially on $k^\sharp$ and so $k^\sharp$ is a difference-differential sub-field of $L^\sharp$ and hence of the difference-differential algebra $F(\mathbb{Z}, L^\sharp)$.

**Definition 2.5.** The Galois hull $L/K$ is a difference-differential sub-algebra extension of $F(\mathbb{Z}, L^\sharp)$, where $L$ is the difference-differential sub-algebra generated by the image $\iota(L)$ and $L^\sharp$ and $K$ is the sub-algebra generated by the image $\iota(k)$ and $L^\sharp$. So $L/K$ is a difference-differential algebra extension with difference operator $\Sigma$ and derivations $\{D_1, D_2, \cdots, D_d\}$.

### 2.3.3 The functor $F_{L/k}$ of infinitesimal deformations for a difference field extension

For the partial differential field $L^\sharp$, we have the universal Taylor morphism

$$\iota_{L^\sharp}: L^\sharp \to L^\sharp[[W_1, W_2, \cdots, W_d]] = L^\sharp[[W]].$$  \hspace{1cm} (22)

The universal Taylor morphism (22) gives a difference-differential morphism

$$F(\mathbb{Z}, L^\sharp) \to F(\mathbb{Z}, L^\sharp[[W_1, W_2, \cdots, W_d]]).$$  \hspace{1cm} (23)
Restricting the morphism \((23)\) to the difference-differential sub-algebra \(\mathcal{L}\) of \(F(\mathbb{Z}, L^2)\), we get a difference-differential morphism \(\mathcal{L} \to F(\mathbb{Z}, L^2[[W_1, W_2, \cdots, W_d]])\) that we denote by \(\iota\). So we have the difference-differential morphism

\[
\iota: \mathcal{L} \to F(\mathbb{Z}, L^2[[W_1, W_2, \cdots, W_d]]).
\]  

(24)

Similarly for every commutative \(L^2\)-algebra \(A\), thanks to the differential morphism \(L^2[[W]] \to A[[W]]\), arising from the structural morphism \(L^2 \to A\), we have the canonical difference-differential morphism

\[
\iota: \mathcal{L} \to F(\mathbb{Z}, A[[W_1, W_2, \cdots, W_d]]).
\]  

(25)

We define the functor

\[
\mathcal{F}_{L/k}: (\text{Alg}/L^2) \to (\text{Set})
\]

from the category \((\text{Alg}/L^2)\) of commutative \(L^2\)-algebras to the category \((\text{Set})\) of sets, by associating to a commutative \(L^2\)-algebra \(A\), the set of infinitesimal deformations of the canonical morphism \((24)\). So

\[
\mathcal{F}_{L/k}(A) = \{ f: \mathcal{L} \to F(\mathbb{Z}, A[[W_1, W_2, \cdots, W_d]]) \mid f \text{ is a difference-differential morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted on the sub-algebra } K \}.
\]

See Definition 2.13 in [13], for a rigorous definition.

### 2.3.4 Group functor \(\text{Inf-gal}(L/k)\) of infinitesimal automorphisms for a difference field extension

The Galois group in our Galois theory is the group functor

\[
\text{Inf-gal}(L/k): (\text{Alg}/L^2) \to (\text{Grp})
\]

defined by

\[
\text{Inf-gal}(L/k)(A) = \{ f: \mathcal{L} \hat{\otimes}_{L^2} A[[W]] \to \mathcal{L} \hat{\otimes}_{L^2} A[[W]] \mid f \text{ is a difference-differential } K \hat{\otimes}_{L^2} A[[W]] -\text{automorphism continuous with respect to the } W\text{-adic topology and congruent to the identity modulo nilpotent elements} \}
\]

for a commutative \(L^2\)-algebra \(A\). Here the completion is taken with respect to the \(W\)-adic topology. See Definition 2.19 in [13].

Then the group functor \(\text{Inf-gal}(L/k)\) operates on the functor \(\mathcal{F}_{L/k}\) in such a way that the operation \((\text{Inf-gal}(L/k), \mathcal{F}_{L/k})\) is a principal homogeneous space (Theorem 2.20, [13]).

The group functor \(\text{Inf-gal}(L/k)\) arises from the same origin as in the differential case, namely from the automorphism of the initial conditions as we explained in 2.2.6. In the quantum case too, where in Hopf Galois theory, the Galois hull \(\mathcal{L}\) is non-commutative.

We we are going to see that we can apply this principle to define the Galois group that is a quantum group, in the quantum case. See Section 4. The First Example, Section 5. The Second Example and Section 6. The Third Example.
2.4 Introduction of more precise notations

So far, we explained general differential Galois theory and general difference Galois theory. To go further, we have to make our notations more precise.

For example, we defined the Galois hull for a differential field extension in Definition 2.1 and the Galois hull for a difference field extension in Definition 2.5. Since they are defined by the same principle, we denoted both of them by $L/K$. We have to, however, distinguish them.

**Definition 2.6.** We denote the Galois hull for a differential field extension by $L_δ/K_δ$ and we use the symbol $L_σ/K_σ$ for the Galois hull of a difference field extension.

We also have to distinguish the functors $F_{L/k}$ and $\text{Inf-gal}(L/k)$ in the differential case and in the difference case: we add the suffix $δ$ for the differential case and the suffix $σ$ for the difference case:

1. We use $F_{δL/k}$ and $\text{Inf-gal}_δ(L/k)$, when we deal with differential algebras.
2. We use $F_{σL/k}$ and $\text{Inf-gal}_σ(L/k)$ for difference algebras.

We denoted, according to our convention, for a commutative algebra $A$ the category of commutative $A$-algebras by $(\text{Alg}/A)$. As we are going to consider the category of not necessarily commutative $A$-algebras. This notation is confusing. So we clarify the notation.

3 Hopf Galois theory

Picard-Vessiot theory is a Galois theory of linear differential or difference equations. The idea of introducing Hopf algebra in Picard-Vessiot theory is traced back to Sweedler [16]. Specialists in Hopf algebra succeeded in unifying Picard-Vessiot theories for differential equations and difference equations [2]. They further succeeded in generalizing the Picard-Vessiot theory for difference-differential equations, where the operators are not necessarily commutative. Heiderich [7] combined the idea of Picard-Vessiot theory via Hopf algebra with our general Galois theory for non-linear equations [17], [13]. This is a wonderful idea. After our Examples, it becomes, however, apparent that his result requires a certain modification in the non-co-commutative case. His general theory includes a wide class of difference and differential algebras.

There are two major advantages in his theory.

1. Unified study of differential equations and difference equations in non-linear case.
2. Generalization of universal Euler morphism and Taylor morphism.

$C$ being the field, for $C$-vector spaces $M, N$, we denote by $C\text{M}(M,N)$ the set of $C$-linear maps from $M$ to $N$.
Example 3.1. Let $\mathcal{H} := \mathbb{C}[\mathbb{G}_{ac}] = \mathbb{C}[t]$ be the $\mathbb{C}$-Hopf algebra of the coordinate ring of the additive group scheme $\mathbb{G}_{ac}$ over the field $\mathbb{C}$. Let $A$ be a commutative $\mathbb{C}$-algebra and

$$\Psi \in \mathbb{C}M(A \otimes_\mathbb{C} \mathcal{H}, A) = \mathbb{C}M(A, \mathbb{C}M(\mathcal{H}, A))$$

so that $\Psi$ defines two $\mathbb{C}$-linear maps

1. $\Psi_1: A \otimes_\mathbb{C} \mathcal{H} \to A$,
2. $\Psi_2: A \to \mathbb{C}M(\mathcal{H}, A)$.

Definition 3.2. We keep the notation of Example 3.1. We say that $(A, \Psi)$ is an $\mathcal{H}$-module algebra if the following equivalent conditions are satisfied.

1. The $\mathbb{C}$-linear map $\Psi_1: A \otimes_\mathbb{C} \mathcal{H} \to A$ makes $A$ into a left $\mathcal{H}$-module in such a way that we have in the algebra $A$,

$$h(ab) = \sum (h(1)a)(h(2)b) \in A,$$

for every element $h \in \mathcal{H}$ and $a, b \in A$, where we use the sigma notation so that

$$\Delta(h) = \sum h(1) \otimes h(2),$$

$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ being the co-multiplication of the Hopf algebra $\mathcal{H}$.

2. The $\mathbb{C}$-linear map

$$\Psi_2 : A \to \mathbb{C}M(\mathcal{H}, A)$$

is a $\mathbb{C}$-algebra morphism, the dual $\mathbb{C}M(\mathcal{H}, A)$ of co-algebra $\mathcal{H}$ being a $\mathbb{C}$-algebra.

cf. p.153 of Sweedler [10].

Concretely the dual algebra $\mathbb{C}M(\mathcal{H}, A)$ is the formal power series ring $A[[X]]$.

It is a comfortable exercise to examine that $(A, \Psi)$ is an $\mathcal{H}$-module algebra if and only if $A$ is a differential algebra with derivation $\delta$ such that $\delta(\mathbb{C}) = 0$. When the equivalent conditions are satisfied, for every element $a$ in the algebra $A$, $\Psi(a \otimes t) = \delta(a)$ and the $\mathbb{C}$-algebra morphism

$$\Psi_2 : A \to \mathbb{C}M(\mathcal{H}, A) = A[[X]]$$

is the universal Taylor morphism. So

$$\Psi_2(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(a)X^n \in A[[X]]$$

for every $a \in A$. See Heiderich [7], 2.3.4.

In Example 3.1 we explained the differential case. If we take the Hopf algebra $\mathbb{C}[\mathbb{G}_{mc}]$ of the coordinate ring of the multiplicative group $\mathbb{G}_{mc}$ for $\mathcal{H}$, we get difference structure
and the universal Euler morphism. See [7, 2.3.1. More generally we can take any Hopf algebra $H$ to get an algebra $A$ with operation of the algebra $H$ and a morphism

$$\Psi_2 : A \rightarrow \mathcal{CM}(H, A)$$

generalizing the universal Taylor morphism and Euler morphism. So we can define the Galois hull $L/K$ and develop a general Galois theory for a field extension $L/k$ with operation of the algebra $H$. In the differential case as well as in the difference case, the corresponding Hopf algebra $H$ is co-commutative so that the dual algebra $\mathcal{CM}(H, A)$ is a commutative algebra. Consequently the Galois hull $L/K$ that are sub-algebras in the commutative algebra $\mathcal{CM}(H, A)$. In these cases, the Galois hull is an algebraic counterpart of the geometric object, algebraic Lie groupoid. See Malgrange [9]. Therefore the most fascinating question is

**Question 3.3.** Let us consider a non-co-commutative bi-algebra $H$ and assume that the Galois hull $L/K$ that is a sub-algebra of the dual algebra $\mathcal{CM}(H, A)$, is not a commutative algebra. Does the Galois hull $L/K$ quantize the algebraic Lie groupoid?

We answer affirmatively the question by analyzing examples in $q$-SI $\sigma$-differential field extensions.

**Remark 3.4.** Looking at the works of Hardouin [5] and Masuoka and Yanagawa [12], even if we consider a twisted Hopf algebra $H$, so far as we consider linear difference-differential equations, the Galois hull $L$ often happens to be a commutative sub-algebra of the non-commutative algebra $\mathcal{M}(H, A)$ and the Galois group is a linear algebraic group. See also [21]. We show by examples that quantization of Galois theory really occurs for non-linear equations. We prove further that the first of our Examples reduces to a linear equation giving us the First Example of linear equation where quantization of Galois theory takes place.

Let $q$ an element of the field $C$. We use a standard notation of $q$-binomial coefficients. To this end, let $Q$ be a variable over the field $C$.

We set $[n]_Q = \sum_{i=0}^{n-1} Q^i \in C[Q]$ for positive integer $n$. We need also $q$-factorial

$$[n]_Q! := \prod_{i=1}^{n} [i]_Q \quad \text{for a positive integer } n \quad \text{and} \quad [0]_Q! := 1.$$  

So $[n]_Q \in C[Q]$. The $Q$-binomial coefficient is defined for $m, n \in \mathbb{N}$ by

$$\left( \begin{array}{c} m \\ n \end{array} \right)_Q = \begin{cases} \frac{[m]_Q!}{[m-n]_Q! [n]_Q!} & \text{if } m \geq n, \\
0 & \text{if } m < n. \end{cases}$$

Then we can show that the rational function

$$\left( \begin{array}{c} m \\ n \end{array} \right)_Q \in C(Q)$$

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is in fact a polynomial or
\[ \binom{m}{n} Q = C[Q]. \]

We have a ring morphism
\[ C[Q] \to C[q], \quad Q \mapsto q \]  
over \( C \) and we denote the image of the polynomial
\[ \binom{m}{n} Q \]
under morphism (26) by
\[ \binom{m}{n} q. \]

### 3.1 \( q \)-skew iterative \( \sigma \)-differential algebra \([5], [6]\)

The first non-trivial example of a Hopf Galois theory dependent on a non-co-commutative Hopf algebra is Galois theory of \( q \)-skew iterative \( \sigma \)-differential field extensions, abbreviated as \( q \)-SI \( \sigma \)-differential field extensions.

#### 3.1.1 Definition of \( q \)-SI \( \sigma \)-differential algebra

**Definition 3.5.** Let \( q \neq 0 \) an element of the field \( C \). A \( q \)-skew iterative \( \sigma \)-differential algebra \((A, \sigma, \sigma^{-1}, \theta^*) = (A, \sigma, \{\theta^{(i)}\}_{i \in \mathbb{N}})\), a \( q \)-SI \( \sigma \)-differential algebra for short, consists of a \( C \)-algebra \( A \) that is eventually non-commutative, a \( C \)-automorphism \( \sigma : A \to A \) of the \( C \)-algebra \( A \) and a family
\[ \theta^{(i)} : A \to A \quad \text{for } i \in \mathbb{N} \]
of \( C \)-linear maps, called derivations, satisfying the following conditions.

1. \( \theta^{(0)} = \text{Id}_A \),
2. \( \theta^{(i)} \sigma = q^i \sigma \theta^{(i)} \) for every \( i \in \mathbb{N} \),
3. \( \theta^{(i)}(ab) = \sum_{l+m=i, l,m \geq 0} \sigma^m(\theta^{(i)}(a))\theta^{(m)}(b) \) for every \( i \in \mathbb{N} \) and \( a, b \in A \),
4. \( \theta^{(i)} \circ \theta^{(j)} = \binom{i+j}{i} q^{i+j} \theta^{(i+j)} \) for every \( i, j \in \mathbb{N} \).

We say that an element \( a \) of the \( q \)-SI \( \sigma \)-differential algebra \( A \) is a constant if \( \sigma(a) = a \) and \( \theta^{(i)}(a) = 0 \) for every \( i \geq 1 \).

A morphism of \( q \)-SI \( \sigma \)-differential \( C \)-algebras is a \( C \)-algebra morphism compatible with the automorphisms \( \sigma \) and the derivations \( \theta^* \).

Both differential algebras and difference algebras are \( q \)-SI \( \sigma \)-differential algebras as we see below.

**Remark 3.6.** There is also a weaker version of \( q \)-SI \( \sigma \)-differential differential algebra, in which we do not require that \( \sigma \) is a \( C \)-linear automorphism of \( A \).
3.1.2 Difference algebra and a $q$-$SI\sigma$-differential algebra

Let $A$ be a commutative $C$-algebra and $\sigma : A \to A$ be a $C$-automorphism of the ring $A$. So $(A, \sigma)$ is a difference algebra. If we set $\theta(0) = \text{Id}_A$ and

$$\theta^{(i)}(a) = 0 \text{ for every element } a \in A \text{ and for } i = 1, 2, 3, \ldots.$$ 

Then $(A, \sigma, \sigma^{-1}, \theta^*)$ is a $q$-$SI\sigma$-differential algebra.

Namely we have a functor of the category $(\text{Diff}'\text{ceAlg}/C)$ of $C$-difference algebras to the category $(q$-$SI\sigma$-$\text{diff}'\text{ialAlg}/C)$ of $q$-$SI\sigma$-differential algebras over $C$:

$$(\text{Diff}'\text{ceAlg}/C) \to (q$-$SI\sigma$-$\text{diff}'\text{ialAlg}/C).$$

Let $t$ be a variable over the field $C$ and let us now assume $q^n \neq 1$ for every positive integer $n$. (27)

We denote by $\sigma : C(t) \to C(t)$ the $C$-automorphism of the rational function field $C(t)$ sending the variable $t$ to $qt$. We consider a difference algebra extension $(A, \sigma)/(C(t), \sigma)$.

If we set

$$\theta^{(1)}(a) = \frac{\sigma(a) - a}{(q - 1)t} \text{ for every element } a \in A$$

and

$$\theta^{(i)} = \frac{1}{[i]_q!} \theta^{(1)i} \text{ for } i = 2, 3, \ldots.$$ 

Then $(A, \sigma, \theta^*)$ is a $q$-$SI\sigma$-differential algebra. Therefore if $q \in C$ satisfies (27), then we have a functor

$$(\text{Diff}'\text{ceAlg}/(C(t), \sigma)) \to (q$-$SI\sigma$-$\text{diff}'\text{ialAlg}).$$

(28)

**Remark 3.7.** In coherence with Remark 3.6, when we speak of difference $C$-algebra $(A, \sigma)$, we assume that $\sigma : A \to A$ is a $C$-linear automorphism.

3.1.3 Differential algebra and $q$-$SI\sigma$-differential algebra

Let $(A, \theta)$ be a $C$-differential algebra such that the derivation $\theta : A \to A$ is $C$-linear. We set

$$\theta^{(0)} = \text{Id}_A,$$

$$\theta^{(i)} = \frac{1}{i!} \theta^i \text{ for } i = 1, 2, 3, \ldots.$$ 

Then $(A, \text{Id}_A, \theta^*)$ is a $q$-$SI\sigma$-differential algebra for $q = 1$. In other words, we have a functor

$$(\text{Diff}'\text{ialAlg}/C) \to (q$-$SI\sigma$-$\text{diff}'\text{ialAlg}/C)$$

of the category of (commutative) differential $C$-algebras to the category of $q$-$SI\sigma$-differential algebras over $C$. We have shown that both difference algebras and differential algebras are particular instances of $q$-$SI\sigma$-differential algebra.
3.1.4 Example of $q$-SI $\sigma$-differential algebra \[7\]

We are going to see that $q$-SI $\sigma$-differential algebras live on the border between commutative algebras and non-commutative algebras. The example below seems to suggest that it looks natural to seek $q$-SI $\sigma$-differential algebras in the category of non-commutative algebras.

An example of $q$-SI $\sigma$-differential algebra arises from a commutative $C$-difference algebra $(S, \sigma)$. We need, however, a non-commutative ring, the twisted power series ring $(S, \sigma)[[X]]$ over the difference ring $(S, \sigma)$ that has a natural $q$-SI $\sigma$-differential algebra structure.

Namely, let $(S, \sigma)$ be the $C$-difference ring so that $\sigma : S \to S$ is a $C$-algebra automorphism of the commutative ring $S$. We introduce the following twisted formal power series ring $(S, \sigma)[[X]]$ with coefficients in $S$ that is the formal power series ring $S[[X]]$ as an additive group with the following commutation relation

$$aX = X\sigma(a) \quad \text{and} \quad Xa = \sigma^{-1}(a)X \quad \text{for every } a \in S.$$ 

So more generally

$$aX^n = X^n\sigma^n(a) \quad \text{and} \quad X^n a = \sigma^{-n}(a)X^n \quad \text{(29)}$$

for every $n \in \mathbb{N}$. The multiplication of two formal power series is defined by extending (29) by linearity. Therefore the twisted formal power series ring $(S, \sigma)[[X]]$ is non-commutative in general. By commutation relation (29), we can identify

$$(S, \sigma)[[X]] = \left\{ \sum_{i=0}^{\infty} X^i a_i \mid a_i \in S \text{ for every } i \in \mathbb{N} \right\}$$

as additive groups.

We are going to see that the twisted formal power series ring has a natural $q$-SI $\sigma$-differential structure. We define first a ring automorphism

$$\hat{\Sigma} : (S, \sigma)[[X]] \to (S, \sigma)[[X]]$$

by setting

$$\hat{\Sigma}\left(\sum_{i=0}^{\infty} X^i a_i \right) = \sum_{i=0}^{\infty} X^i q^i \sigma(a_i) \quad \text{for every } i \in \mathbb{N}, \quad \text{(30)}$$

for every element

$$\sum_{i=0}^{\infty} X^i a_i \in (S, \sigma)[[X]].$$

As we assume that $\sigma : A \to A$ is an isomorphism, the $C$-linear map,

$$\hat{\Sigma} : (A, \sigma)[[X]] \to (A, \sigma)[[X]]$$

is an automorphism of the $C$-linear space. The operators $\Theta^* = \{\Theta^{(l)}\}_{l \in \mathbb{N}}$ are defined by

$$\Theta^{(l)}\left(\sum_{i=0}^{\infty} X^i a_i \right) = \sum_{i=0}^{\infty} X^i \left(\begin{array}{c} i + l \\ l \end{array}\right)_q a_{i+l} \quad \text{for every } l \in \mathbb{N}. \quad \text{(31)}$$
Hence the twisted formal power series ring \((S, \sigma)[[X]], \hat{\Sigma}, \Theta^*\) is a non-commutative \(q\)-SI \(\sigma\)-differential ring. We denote this \(q\)-SI \(\sigma\)-differential ring simply by \((S, \sigma)[[X]]\). See [7], 2.3. In particular, if we take as the coefficient difference ring \(S\) the difference ring \((F(Z, A), \Sigma)\) where \(\Sigma : F(Z, A) \to F(Z, A)\) is the shift operator, we obtain the \(q\)-SI \(\sigma\)-differential ring \((F(Z, A), [X]], \hat{\Sigma}, \Theta^*)\). (32)

Remark 3.8. We assumed that the coefficient difference ring \((S, \sigma)\) is commutative. The commutativity assumption on the ring \(S\) is not necessary. Consequently we can use non-commutative ring \(A\) in (32).

3.1.5 Hopf algebra for \(q\)-SI \(\sigma\)-differential structures

As we explained for differential algebras in Definition 3.2, a \(q\)-SI \(\sigma\)-differential structure is nothing but a \(H_q\)-module algebra structure for a Hopf algebra \(H_q\).

Definition 3.9. Let \(q \neq 0\) be an element of the field \(C\). Let \(H_q\) is a \(C\)-algebra generated over the field \(C\) by \(s, s^{-1}\) and the \(t_i\)'s for \(i \in \mathbb{N}\) subject to the relations

\[
t_1 = 1, \quad ss^{-1} = s^{-1}s = 1, \quad t_is = q^ist_i, \quad q^it_is^{-1} = s^{-1}t_i, \quad t_it_j = \binom{i + j}{i}_q t_{i+j}
\]

for every \(i, j \in \mathbb{N}\). We define a co-algebra structure \(\Delta : H_q \to H_q \otimes_C H_q\) by

\[
\Delta(s) = s \otimes s, \quad \Delta(s^{-1}) = s^{-1} \otimes s^{-1}, \quad \Delta(t_i) = \sum_{l=0}^{i} s^l t_{i-l} \otimes t_i
\]

for every \(i \in \mathbb{N}\). In fact \(H_q\) is a Hopf algebra with co-unit \(\epsilon : H_q \to C\) defined by

\[
\epsilon(s) = \epsilon(s^{-1}) = 1, \quad \epsilon(t_i) = 0
\]

for every \(i \in \mathbb{N}\). Antipode is an anti-automorphism \(S : H_q \to H_q\) of the \(C\)-algebra \(H_q\) given by

\[
S(s) = s^{-1}, \quad S(s^{-1}) = s, \quad S(t_i) = (-1)^i q^{i(i+1)/2} t_is^{-i}
\]

for every \(i \in \mathbb{N}\).

Proposition 3.10. For a not necessarily commutative \(C\)-algebra \(A\), there exists a 1 : 1 correspondence between the elements of the following two sets.

1. The set of \(q\)-SI \(\sigma\)-differential algebra structures on the \(C\)-algebra \(A\).
2. The set if \(H_q\)-module algebra structures on the \(C\)-algebra \(A\).

This result is well-known. See Heiderich [7]. We recall for a \(q\)-SI \(\sigma\)-differential algebra \(A\), the corresponding left \(H_q\)-module structure is given by

\[
s \mapsto \sigma, \quad s^{-1} \mapsto \sigma^{-1}, \quad t_i \mapsto \theta^{(i)}\text{ for every } i \in \mathbb{N}.
\]
3.1.6 Universal Hopf morphism for a q-SI σ-differential algebra

We introduced in [2.3.1] the difference ring of functions \((F(Z, A), \Sigma)\) on the set \(Z\) taking values in a ring \(A\). It is useful to denote the function \(f\) by a matrix

\[
\begin{bmatrix}
\cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & f(-2) & f(-1) & f(0) & f(1) & f(2) & \cdots \\
\end{bmatrix}.
\]

For an element \(b\) of a difference algebra \((R, \sigma)\) or a q-SI σ-differential algebra \((R, \sigma, \theta^*)\), we denote by \(u[b]\) a function on \(Z\) taking values in the abstract ring \(R^{\natural}\) such that

\[u[b](n) = \sigma^n(b) \quad \text{for every } n \in Z\]

so that

\[u[b] = \begin{bmatrix}
\cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & \sigma^{-2}(b) & \sigma^{-1}(b) & b & \sigma^1(b) & \sigma^2(b) & \cdots \\
\end{bmatrix}.
\]

Therefore \(u[b] \in F(Z, R^{\natural})\).

**Proposition 3.11** (Proposition 2.3.17, Heiderich [7]). For a q-SI σ-differential algebra \((R, \sigma, \theta^*)\), there exists a canonical morphism, which we call the universal Hopf morphism

\[
i : (R, \sigma, \theta^*) \rightarrow \left((F(Z, R^{\natural}), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^*\right), \quad a \mapsto \sum_{i=0}^{\infty} X^i u[\theta^i(a)]\tag{33}
\]

of q-SI σ-differential algebras.

We can also characterize the universal Hopf morphism as the solution of a universal mapping property.

When \(q = 1\) and \(\sigma = \text{Id}_R\) and \(R\) is commutative so that the q-SI σ-differential ring \((R, \text{Id}_R, \theta^*)\) is simply a differential algebra as we have seen in [3.1.3] the universal Hopf morphism (33) is the universal Taylor morphism in [2]. Similarly a commutative difference ring is a q-SI σ-differential algebra with trivial derivations as we noticed in [3.1.2]. In this case the universal Hopf morphism (33) is nothing but the universal Euler morphism (20). Therefore the universal Hopf morphism unifies the universal Taylor morphism and the Universal Euler morphism.

Let us recall the following fact.

**Lemma 3.12.** Let \((R, \sigma, \theta^*)\) be a q-SI σ-differential domain. If the endomorphism \(\sigma : R \rightarrow R\) is an automorphism, then the field \(Q(R)\) of fractions of \(R\) has the unique structure of q-SI σ-differential field extending that of \(R\).

**Proof.** See for example, Proposition 2.5 of [6].

We can interpret the Example in [3.1.4] from another view point. We constructed there from a difference ring \((S, \sigma)\) a q-SI σ-differential algebra \(((S, \sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^*)\). We notice that this procedure is a particular case of Proposition 3.11. In fact, given a difference
ring \((S, \sigma)\). So as in 3.1.2 by adding the trivial derivations, we get the \(q\)-SI \(\sigma\)-differential algebra \((S, \sigma, \theta^*)\), where

\[
\begin{align*}
\theta^{(0)} &= \text{Id}_S, \\
\theta^{(i)} &= 0 \quad \text{for } i \geq 1
\end{align*}
\]

Therefore we have the universal Hopf morphism

\[
(S, \sigma, \theta^*) \to (F(Z, S^\natural)[[X]], \hat{\Sigma}, \hat{\Theta}^*)
\]

by Proposition 3.11. So we obtained the \(q\)-SI \(\sigma\)-differential algebra \((F(Z, S^\natural)[[X]], \hat{\Sigma}, \hat{\Theta}^*)\) as a result of composite of two functors. Namely,

1. The functor: (Category of Difference algebras) \(\to\) (Category of \(q\)-SI \(\sigma\)-differential algebras) of adding trivial derivations
2. The functor: (Category of \(q\)-SI \(\sigma\)-differential algebras) \(\to\) (Category of \(q\)-SI \(\sigma\)-differential algebras), \(A \mapsto B\) if there exists the universal Hopf morphism \(A \to B\).

### 3.1.7 Galois hull \(L/K\) for a \(q\)-SI \(\sigma\)-differential field extension

We can develop a general Galois theory for \(q\)-SI \(\sigma\)-differential field extensions analogous to our theories in [18], [19] and [20] thanks to the universal Hopf morphism. Let \(L/k\) be an extension of \(q\)-SI \(\sigma\)-differential fields such that the abstract field \(L^\natural\) is finitely generated over the abstract field \(k^\natural\). Let us assume that we are in characteristic 0. General theory in [7] works, however, also in characteristic \(p \geq 0\). We have by Proposition 3.11 the universal Hopf morphism

\[
\iota: (L, \sigma, \theta^*) \to (F(Z, L^\natural)[[X]], \hat{\Sigma}, \hat{\Theta}^*)
\]

so that the image \(\iota(L)\) is a copy of the \(q\)-SI \(\sigma\)-differential field \(L\). We have another copy of \(L^\natural\). The set

\[
\{ f = \sum_{i=0}^{\infty} X^i a_i \in F(Z, L^\natural)[[X]] \mid a_0 = 0 \text{ for every } i \geq 1 \text{ and } \Sigma(a_0) = a_0 \}
\]

forms the sub-ring of constants in the \(q\)-SI \(\sigma\)-differential algebra of the twisted power series

\[
\left( (F(Z, L^\natural), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right).
\]

We identify \(L^\natural\) with the ring of constants through the following morphism. For an element \(a \in L^\natural\), we denote the constant function \(f_a\) on \(Z\) taking the value \(a \in L^\natural\) so that

\[
L^\natural \to \left( (F(Z, L^\natural), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right), \quad a \mapsto f_a
\]

is an injective ring morphism. We may denote the sub-ring in (35) by \(L^\natural\). In fact, as an abstract ring it is isomorphic to the abstract field \(L^\natural\) and the endomorphism \(\hat{\Sigma}\) and the derivations \(\Theta^{(i)}\), \(i \geq 1\) operate trivially on the sub-ring.
We are now exactly in the same situation as in 2.2.2 of the differential case and in 2.3.2 of the difference case. We choose a mutually commutative basis \{D_1, D_2, \ldots, D_d\} of the \(L^2\)-vector space \(\text{Der}(L^2/k^2)\) of \(k^2\)-derivations. So \(L^2 := (L^k, \{D_1, D_2, \ldots, D_d\})\) is a differential field.

So we introduce derivations \(D_1, D_2, \ldots, D_d\) operating on the coefficient ring \(F(Z, L^2)\). In other words, we replace the target space \(F(Z, L^2)[[X]]\) by \(F(Z, L^2)[[X]]\). Hence the universal Hopf morphism in Proposition 3.11 becomes \(\iota : L \to F(Z, L^2)[[X]]\).

In the twisted formal power series ring \((F(Z, L^2)[[X]], \hat{\Sigma}, \hat{\Theta}^*)\), we add differential operators \(D_1, D_2, \ldots, D_d\).

So we have a set \(\mathcal{D}\) of the following operators on the ring \((F(Z, L^2), \Sigma)[[X]]\).

1. The endomorphism \(\hat{\Sigma}\).
\[
\hat{\Sigma}(\sum_{i=0}^{\infty} X^i a_i) = \sum_{i=0}^{\infty} X^i q^i (\Sigma(a_i)),
\]

\(\Sigma : F(Z, L^2) \to F(Z, L^2)\) being the shift operator of the ring of functions on \(Z\).

2. The \(q\)-skew \(\hat{\Sigma}\)-derivations \(\hat{\Theta}^{(i)}\)'s in (31).
\[
\hat{\Theta}^{(l)}(\sum_{i=0}^{\infty} X^i a_i) = \sum_{i=0}^{\infty} X^i \binom{l+i}{l}_q a_{i+l} \quad \text{for every } l \in \mathbb{N}.
\]

3. The derivations \(D_1, D_2, \ldots, D_d\) operating through the coefficient ring \(F(Z, L^2)\) as in (33).

Hence we may write \((F(Z, L^2), \mathcal{D})\), where \(\mathcal{D} = \{\hat{\Sigma}, D_1, D_2, \ldots, D_d, \hat{\Theta}^*\}\) and \(\hat{\Theta}^* = \{\hat{\Theta}^{(i)}\}_{i \in \mathbb{N}}\).

We identify using inclusion (36)
\[
L^2 \to F(Z, L^2)[[X]].
\]

We sometimes denote the image \(f_a\) of an element \(a \in L^2\) by \(a^2\).

We are ready to define Galois hull as in Definition 2.1.

**Definition 3.13.** The Galois hull \(L/K\) is a \(\mathcal{D}\)-invariant sub-algebra of \(F(Z, L^2)[[X]]\), where \(L\) is the \(\mathcal{D}\)-invariant sub-algebra generated by the image \(\iota(L)\) and \(L^2\) and \(K\) is the \(\mathcal{D}\)-invariant sub-algebra generated by the image \(\iota(k)\) and \(L^2\). So \(L/K\) is a \(\mathcal{D}\)-algebra extension.

As in 2.4, if we have to emphasize that we deal with \(q\)-SI \(\sigma\)-differential algebras, we denote the Galois hull by \(L_{\sigma \theta}/K_{\sigma \theta}\).
We notice that we are now in a totally new situation. In the differential case, the universal Taylor morphism maps the given fields to the commutative algebra of the formal power series ring so that the Galois hull is an extension of commutative algebras. Similarly for the universal Euler morphism of a difference rings. The commutativity of the Galois hull comes from the fact in the differential and the difference case, the theory depends on the co-commutative Hopf algebras. When we treat the q-SI σ-differential algebras, the Hopf algebra $H$ is not co-commutative so that the Galois hull $L/K$ that is an algebra extension in the non-commutative algebra of twisted formal power series algebra, the dual algebra of $H$. So even if we start from a (commutative) field extension $L/k$, the Galois hull can be non-commutative. See the Examples in sections 4, 5 and 6. We also notice that when $L/k$ is a Picard-Vessiot extension fields in q-SI σ-differential algebra, the Galois hull is commutative [21].

As the Galois hull is a non-commutative, if we limit ourselves to the category of commutative $L^\flat$-algebras ($\text{Alg}/L^\flat$), we can not detect non-commutative nature of the q-SI σ-differential field extension. So it is quite natural to extend the functors over the category of not necessarily commutative algebras.

3.1.8 Infinitesimal deformation functor $F_{L/k}$ for a q-SI σ-differential field extension.

We pass to the task of defining the infinitesimal deformation functor $F_{L/k}$ and the Galois group functor. The latter is a subtle object and we postpone discussing it until Section 7. Instead we define naively the infinitesimal automorphism functor $\text{Inf-gal}(L/k)$, which does not seem useful in general.

We have the universal Taylor morphism

$$\iota_L: L^\sharp \to (L^\sharp[[W_1, W_2, \ldots, W_d]], \{ \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d} \})$$

as in [6]. So by (37), we have the canonical morphism

$$(F(Z, L^\sharp)[[X]], D) \to (F(Z, L^\sharp[[W]])[[X]], D),$$

where in the target space

$$D = \{ \Sigma, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d}, \Theta^* \}$$

by abuse of notation.

For an $L^\sharp$-algebra $A$, the structure morphism $L^\sharp \to A$ induces the canonical morphism

$$(F(Z, L^\sharp[[W]])[[X]], D) \to (F(Z, A[[W]])[[X]], D).$$

The composite of the $D$-morphisms (38) and (39) gives us the canonical morphism

$$(F(Z, L^\sharp[[X]], D) \to (F(Z, A[[W]])[[X]], D).$$

The restriction of the morphism (40) to the $D$-invariant sub-algebra $L$ gives us the canonical morphism

$$\iota: (L, D) \to (F(Z, A[[W]])[[X]], D).$$
We can define the functors exactly as in paragraphs 2.2.4 for the differential case and 2.3.3 for the difference case.

**Definition 3.14** (Introductory definition). We define the functor

\[ \mathcal{F}_{L/k} : (\text{Alg}/L^2) \to (\text{Set}) \]

from the category \((\text{Alg}/L^2)\) of commutative \(L^2\)-algebras to the category \((\text{Set})\) of sets, by associating to an \(L^2\)-algebra \(A\), the set of infinitesimal deformations of the canonical morphism \((40)\).

Hence

\[ \mathcal{F}_{L/k}(A) = \{ f : (\mathcal{L}, \mathcal{D}) \to (F(Z, A[[W_1, W_2, \cdots, W_d]][[X]], \mathcal{D}) \mid f \text{ is an algebra morphism compatible with } \mathcal{D}, \text{ congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted to the sub-algebra } \mathcal{K} \}. \]

The introductory definition 3.14 is exact, analogous to Definitions in 2.2.4 and 2.3.3, and easy to understand. As we explained in 3.1.7, we, however, have to consider also deformations over non-commutative algebras, the notation is confusing.

We have to treat both the category of commutative \(L^2\)-algebras and that of not necessarily commutative \(L^2\)-algebras.

**Definition 3.15.** All the associative algebras that we consider are unitary and the morphisms between them are assumed to be unitary. For a commutative algebra \(R\), we denote by \((C\text{Alg}/R)\) the category of associative commutative \(R\)-algebras. We consider also the category \((N\text{CAlg}/R)\) of not necessarily commutative \(R\)-algebras \(A\) such that (the image in \(A\) of) \(R\) is in the center of \(A\). When there is no danger of confusion the category of commutative algebras is denoted simply by \((\text{Alg}/R)\).

Let us come back to the \(q\)-SI \(\sigma\)-differential field extension \(L/k\). We can now give the infinitesimal deformation functors in an appropriate language.

**Definition 3.16.** The functor \(\mathcal{F}_{L/k}\) defined in 3.14 will be denoted by \(\mathcal{C}\mathcal{F}_{L/k}\). So we have

\[ \mathcal{C}\mathcal{F}_{L/k} : (C\text{Alg}/L^2) \to (\text{Set}). \]

We extend formally the functor \(\mathcal{C}\mathcal{F}_{L/k}\) in 3.14 from the category \((C\text{Alg}/L^2)\) to the category \((N\text{CAlg}/L^2)\). Namely, we define the functor

\[ \mathcal{N}\mathcal{C}\mathcal{F}_{L/k} : (N\text{CAlg}/L^2) \to (\text{Set}) \]

by setting

\[ \mathcal{F}_{L/k}(A) = \{ f : (\mathcal{L}, \mathcal{D}) \to (F(Z, A[[W_1, W_2, \cdots, W_d]][[X]], \mathcal{D}) \mid f \text{ is an algebra morphism compatible with } \mathcal{D}, \text{ congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted to the sub-algebra } \mathcal{K} \} \]

for \(A \in \text{Ob}(N\text{CAlg}_{L/k})\).
In the examples, we consider $q$-SI $\sigma$-differential structure, differential structure and difference structure of a given field extension $L/k$ and we study Galois groups with respect to the structures. So we have to clarify which structure is in question. For this reason, when we treat $q$-SI $\sigma$-differential structure, we sometimes add suffix $\sigma\theta$ to indicate that we treat the $q$-SI $\sigma$-differential structure as in $[2,3]$. For example $N\mathcal{CF}_{\sigma\theta L/k}$.

### 3.1.9 Definition of commutative Galois group functor $\text{CInf-gal}(L/k)$

Similarly to the Galois group functor $\text{Inf-gal}(L/k)$ in the differential and the difference cases, we may introduce the group functor $\text{CInf-gal}(L/k)$ called commutative Galois group functor, on the category $(\text{CAlg}/L^\natural)$. 

**Definition 3.17.** In the differential case and in the difference case, the Galois group in our Galois theory is the group functor

$$\text{CInf-gal}(L/k): \text{(CAlg}/L^\natural) \rightarrow \text{(Grp)}$$

defined by

$$\text{CInf-gal}(L/k)(A) = \{ f : L\hat{\otimes}_{L^\sharp}A[[W]] \rightarrow L\hat{\otimes}_{L^\sharp}A[[W]] \mid f \text{ is a } K \otimes_{L^\sharp} A[[W]]\text{-automorphism compatible with } D, \text{ continuous with respect to the } W\text{-adic topology and congruent to the identity modulo nilpotent elements } \}$$

for a commutative $L^\natural$-algebra $A$. See Definition 2.19 in $[13]$.

Then the group functor $\text{CInf-gal}(L/k)$ would operates on the functor $\mathcal{CF}_{L/k}$ in such a way that the operation $(\text{CInf-gal}(L/k), \mathcal{F}_{L/k})$ is a principal homogeneous space.

**Remark 3.18.** For a $q$-SI $\sigma$-differential field extension $L/k$, the Galois hull $L/K$ is, in general, a non-commutative algebra extension so that the commutative Galois group functor $\text{CInf-gal}(L/k)$ on the category $(\text{CAlg}/L^\natural)$ is not adequate for the following two reasons.

1. If we measure the extension $L/K$ over the category $(\text{CAlg}/L^\natural)$ by the commutative Galois group functor $\text{CInf-gal}(L/k)$, the non-commutative data of the extension $L/K$ are lost.

2. We hope to get a quantum group as a Galois group. A quantum group is, however, in any sense not a group functor on the category $(\text{NCAlg}/L^\natural)$ of non-commutative $L^\natural$-algebras.

In the three coming sections, we settle these points for three concrete Examples. Looking at these Examples, we are led to a general Definition in Section $[7]$. The idea is to look at the coordinate transformations of initial conditions. As it is easier to understand it with examples, we explain the definition there. See Questions $[7\mathbb{A}]$. 

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4 The First Example, the field extension \( \mathbb{C}(t)/\mathbb{C} \)

From now on, we assume \( \mathbb{C} = \mathbb{C} \). The arguments below work for an algebraic closed field \( \mathbb{C} \) of characteristic 0. So \( q \) is a non-zero complex number.

4.1 Analysis of the example

Let \( t \) be a variable over \( \mathbb{C} \). The field \( \mathbb{C}(t) \) of rational functions has various structures: the differential field structure, the \( q \)-difference field structure and the \( q \)-SI \( \sigma \)-field structure that we are going to define. We are interested in the Galois group of the field extension \( \mathbb{C}(t)/\mathbb{C} \) with respect to these structures. Let \( \sigma : \mathbb{C}(t) \to \mathbb{C}(t) \) be the \( \mathbb{C} \)-automorphism of the rational function field \( \mathbb{C}(t) \) sending \( t \) to \( qt \). So \( (\mathbb{C}(t), \sigma) \) is a difference field. We assume \( q^n \neq 1 \) fore every positive integer \( n \). We define a \( \mathbb{C} \)-linear map \( \theta^{(1)} : \mathbb{C}(t) \to \mathbb{C}(t) \) by

\[
\theta^{(1)}(f(t)) := \frac{\sigma(f) - f}{\sigma(t) - t} = \frac{f(qt) - f(t)}{(q - 1)t} \quad \text{for } f(t) \in \mathbb{C}(t).
\]

For an integer \( n \geq 2 \), we set

\[
\theta^{(n)} := \frac{1}{[n]_q!} (\theta^{(1)})^n.
\]

It is convenient to define

\[
\theta^{(0)} = \text{Id}_{\mathbb{C}(t)}.
\]

It is well-known and easy to check that \( (\mathbb{C}(t), \sigma, \theta^*) = (\mathbb{C}(t), \sigma, \{\theta^{(i)}\}_{i \in \mathbb{N}}) \) is a \( q \)-SI \( \sigma \)-differential algebra.

We have to clarify a notation. For an algebra \( R \), a sub-algebra \( S \) of \( R \) and a sub-set \( T \) of \( R \), we denote by \( S(T)_{\text{alg}} \) the sub-algebra of \( R \) generated over \( S \) by \( T \).

**Lemma 4.1.** The difference field extension \( (\mathbb{C}(t), \sigma)/(\mathbb{C}, \text{Id}_\mathbb{C}) \) is a Picard-Vessiot extension. Its Galois group is the multiplicative group \( G_m \).

**Proof.** Since \( t \) satisfies the linear difference equation \( \sigma(t) = qt \) over \( \mathbb{C} \) and the field \( \mathbb{C}(t) \) of constant of \( \mathbb{C}(t) \) is \( \mathbb{C} \), the extension \( (\mathbb{C}(t), \sigma)/(\mathbb{C}, \text{Id}_\mathbb{C}) \) is a difference Picard-Vessiot extension. The result follows from the definition of the Galois group. \( \square \)

When \( q \to 1 \), the limit of the \( q \)-SI \( \sigma \)-differential ring \( (\mathbb{C}(t), \sigma, \theta^*) \) is the differential algebra \( (\mathbb{C}(t), d/dt) \). We denote by \( AF_{1k} \), the algebraic group of affine transformations of the affine line so that

\[
AF_{1k} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \bigg| a, b \in \mathbb{C}, a \neq 0 \right\}.
\]

Then

\[
AF_{1\mathbb{C}} \simeq G_{m \mathbb{C}} \ltimes G_{a \mathbb{C}},
\]

where

\[
G_{m \mathbb{C}} \simeq \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in AF_{1 \mathbb{C}} \bigg| a \in \mathbb{C}^* \right\},
\]

\[
G_{a \mathbb{C}} \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in AF_{1 \mathbb{C}} \bigg| b \in \mathbb{C} \right\}.
\]
Lemma 4.2. The Galois group of differential Picard-Vessiot extension \((\mathbb{C}(t), d/dt)/\mathbb{C}\) is \(\mathbb{G}_a\mathbb{C}\).

Proof. We consider the linear differential equation
\[
Y' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y,
\]
where \(Y\) is a 2 \times 2-matrix with entries in a differential extension field of \(\mathbb{C}\). Then \(\mathbb{C}(t)/\mathbb{C}\) is the Picard-Vessiot extension for \((42)\),
\[
Y = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}
\]
being a fundamental solution of \((42)\). The result is well-known and follows from, the definition of Galois group. \(\square\)

The \(q\)-SI \(\sigma\)-differential field extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is not a Picard-Vessiot extension in the sense if Hardouin [5] and Masuoka and Yanagawa [11] so that we can not treat it in the framework of Picard-Vessiot theory. We can apply, however, Hopf Galois theory of Heiderich [7].

Proposition 4.3. The commutative Galois group \(C_{\text{Inf-gal}}((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C})\) of the extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is isomorphic to the formal completion \(\mathbb{G}_m\mathbb{C}\) of the multiplicative group \(\mathbb{G}_m\mathbb{C}\).

Before we start the proof, we explain the behavior of the Galois group under specializations. Theory of Umemura [17] and Heiderich [7] single out only the Lie algebra. Proposition 4.3 should be understood in the following manner. We have two specializations of the \(q\)-SI \(\sigma\)-differential field extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\).

(i) \(q \to 1\) giving the differential field extension \((\mathbb{C}(t), d/dt)/\mathbb{C}\). See 2.2.2

(ii) Forgetting \(\theta^*\), or equivalently specializing
\[
\theta^{(i)} \to 0 \quad \text{for } i \geq 1,
\]
we get the difference field extension \((\mathbb{C}(t), \sigma)/\mathbb{C}\). See 3.1.2

We can summarize the behavior of the Galois group under the specializations.

(1) Proposition 4.3 says that the commutative Galois group
\[C_{\text{Inf-gal}}(L/k)\]

of \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is the formal completion of the multiplicative group \(\mathbb{G}_m\mathbb{C}\). This describes the Galois group at the generic point.

(2) By Lemma 4.1, the Galois group of the specialization (i) is the formal completion of the multiplicative group
(3) The Galois group of the specialization (ii) is the additive group \( \mathbb{G}_a \) by Lemma 4.2.

**Proof of Proposition 4.3.** Let us set \( L = (\mathbb{C}(t), \sigma, \theta^*) \) and \( k = (\mathbb{C}, \sigma, \theta^*) \). By definition of the universal Hopf morphism (33),

\[
\iota \colon (L, \sigma, \theta^*) \rightarrow \left( F(\mathbb{Z}, L^3)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right), \quad \iota(t) = tQ + X \in F(\mathbb{Z}, L^3)[[X]],
\]

where

\[ Q \in F(\mathbb{Z}, L^3) \]

is a function on \( \mathbb{Z} \) taking values in \( \mathbb{C} \subset L^3 \) such that

\[ Q(n) = q^n \quad \text{for } n \in \mathbb{Z}. \]

We denote the function \( Q \) by the matrix

\[
Q = \begin{bmatrix}
\cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & q^{-2} & q^{-1} & 1 & q & q^2 & \cdots 
\end{bmatrix}
\]

according to the convention. We take the derivation \( \partial/\partial t \in \text{Der}(L^3/k^3) \) as a basis of the 1-dimensional \( L^3 \)-vector space \( \text{Der}(L^3/k^3) \) of \( k^3 \)-derivations of \( L^3 \). So \( (\partial/\partial t)(\iota(t)) = Q \) is an element of the Galois hull \( \mathcal{L} \). Therefore

\[
\mathcal{L} \supset \mathcal{L}^o := L^3\langle X, Q \rangle_{\text{alg}},
\]

which is the \( L^3 \)-sub-algebra of \( F(\mathbb{Z}, L^3)[[X]] \) generated by \( X \) and \( Q \). So the algebra \( \mathcal{L}^o \) is invariant under \( \hat{\Sigma}, \hat{\Theta}^*, \partial/\partial t \). Since \( QX = qXQ \), the Galois hull \( \mathcal{L} \) is a non-commutative \( L^3 \)-algebra. Now we consider the universal Taylor expansion

\[
(L^3, \partial/\partial t) \rightarrow L^3[[W]]
\]

and consequently we get the canonical morphism

\[
\iota : \mathcal{L} \rightarrow F(\mathbb{Z}, L^3)[[X]] \rightarrow F(\mathbb{Z}, L^3[[W]] [[X]]).
\]

We study infinitesimal deformations of \( \iota \) in (43) over the category \( (C\text{Alg}/L^3) \) of commutative \( L^3 \)-algebras. Let \( A \) be a commutative \( L^3 \)-algebra and

\[
\varphi : \mathcal{L} \rightarrow F(\mathbb{Z}, A[[W]] [[X]])
\]

be an infinitesimal deformation of the canonical morphism

\[
\iota : \mathcal{L} \rightarrow F(\mathbb{Z}, A[[W]]) [[X]].
\]

**Sublemma 4.4.** We keep the notation above.

1. There exists a nilpotent element \( n \in A \) such that \( \varphi(Q) = (1 + n)Q \) and \( \varphi(X) = X \).

2. The commutative infinitesimal deformation \( \varphi \) is determined by the nilpotent element \( n \) such that \( \varphi(Q) = (1 + n)Q \).
(3) Conversely, for every nilpotent element \( n \in A \), there exists a unique commutative infinitesimal deformation \( \varphi_e \in \mathcal{F}_{L/k}(A) \) such that \( \varphi_e(Q) = eQ \), where we set \( e = 1 + n \).

Sublemma proves Proposition 4.3.

Proof of Sublemma. The elements \( X, Q \in \mathcal{L} \) satisfy the following equation.

\[
\frac{\partial X}{\partial W} = \frac{\partial Q}{\partial W} = 0, \\
\hat{\Sigma}(X) = qX, \quad \hat{\Sigma}(Q) = qQ, \\
\hat{\Theta}^{(1)}(X) = 1, \quad \hat{\Theta}^{(i)}(X) = 0 \quad \text{for } i \geq 2, \\
\hat{\Theta}^{(i)}(Q) = 0 \quad \text{for } i \geq 1.
\]

So \( \varphi(X), \varphi(Q) \) satisfy the same equations as above, which shows

\[
\varphi(X) = X + fQ \in F(Z, A[[W]])[[X]], \\
\varphi(Q) = eQ \in F(Z, A[[W]])[[X]],
\]

where \( f, e \in A \). Since \( \varphi \) is an infinitesimal deformation of \( \iota \), \( f \) and \( e - 1 \) are nilpotent elements in \( A \). We show first to show \( f = 0 \). In fact, it follows from the equation

\[ QX = qXQ \]

that

\[ \varphi(Q)\varphi(X) = q\varphi(X)\varphi(Q) \]

or

\[ eQ(X + fQ) = q(X + fQ)eQ. \]

So we have

\[ eQfQ = qfQeQ \]

and so

\[ eQ^2 = qfQ^2. \]

Therefore

\[ ef = qfe. \]

Since \( e \) is a unit, \( e - 1 \) being nilpotent in \( A \),

\[ f - qf = 0, \]

so that

\[ (1 - q)f = 0. \]

As \( 1 - q \) is a non-zero complex number, \( f = 0 \). So we proved (1). In other words, we determined the restriction of \( \varphi \) to the sub-algebra \( \mathcal{L}^0 = L^2 < X, Q >_{alg} \subset \mathcal{L} \). To prove
(2), we have to show that \( \varphi \) is determined by its restriction on \( \mathcal{L}^o \). To this end, we take two commutative infinitesimal deformations \( \varphi, \psi \in \mathcal{F}_{L/k}(A) \) such that

\[
\varphi(Q) = eQ \quad \text{and} \quad \psi(Q) = eQ,
\]

where \( n \) is a nilpotent element in \( \in A \) and we set \( e = 1 + n \). Since

\[
\mathcal{L} = L^i \iota(\mathbb{C}(t)) < X, Q > \text{alg} = L^j < X, Q, \iota((t + c)^{-1}) >_{c \in \mathbb{C}\text{alg}},
\]

and since \( \varphi \) is a \( \mathcal{K} = L^j \)-morphism, it is sufficient to show that

\[
\varphi((t + c)^{-1}) = \psi((t + c)^{-1})
\]

for every complex number \( c \in \mathbb{C} \). Since \( \iota(t + c) \in \mathcal{L}^o \), \( \varphi(t + c) = \psi(t + c) \) and so

\[
\varphi((t + c)^{-1}) = \varphi(t + c)^{-1} = \psi(t + c)^{-1} = \psi((t + c)^{-1}).
\]

This is what we had to show.

Now we prove (3). We introduce another sub-algebra

\[
\tilde{L} := \{ \sum_{n=0}^{\infty} X^n a_n \in F(\mathbb{Z}, L^j)[[X]] \mid a_n \in L^j(Q) \text{ for every } n \in \mathbb{N} \}
\]

so that, by commutation relation (29), \( \tilde{L} \) is a sub-algebra of \( F(\mathbb{Z}, L^j)[[X]] \) invariant under \( \hat{\Sigma}, \hat{\Theta}^* \) and the derivation \( \partial/\partial t \). We show \( \mathcal{L} \subset \tilde{L} \). Since the sub-algebra \( \mathcal{L} \) is generated by \( \iota(L) \) and \( L^j \) along with operators \( \hat{\Sigma}, \hat{\Theta}^* \) and \( \partial/\partial t \). So it is sufficient to notice \( L^j \) and \( \iota(L) \) are sub-algebras of \( \tilde{L} \). The first inclusion \( L \subset e\tilde{L} \) being trivial, it remains to show the second inclusion:

\[
\iota(\mathbb{C}(t)) \subset \tilde{L}.
\]

We have to show that (i) \( \iota(t) \in \tilde{L} \), and (ii) \( \iota(t + c)^{-1} \in \tilde{L} \) for every complex number \( c \in \mathbb{C} \). The first assertion (i) follows from the equality \( \iota(t) = tQ + X \). As for the assertion (ii), we notice

\[
\iota((t + c)^{-1}) = \iota(t + c)^{-1}
\]

\[
= (tQ + X + c)^{-1}
\]

\[
= (tQ + c)^{-1}(1 + (tQ + c)^{-1}X)^{-1}
\]

\[
= (tQ + c)^{-1}(1 - A)^{-1}
\]

\[
= (tQ + c)^{-1}\sum_{n=0}^{\infty} A^n,
\]

where we set \( A = -(tQ + c)^{-1}X \). Upon writing \( a(Q) := -(tQ + c)^{-1} \), we have

\[
A = Xa(qQ), \quad A^2 = X^2a(qQ)a(q^2Q), \ldots, \quad A^n = X^n \prod_{i=1}^{n} a(q^iQ), \ldots
\]

by commutation relation (29). Hence, by (44) \( \iota(t^c)^{-1} \in \tilde{L} \). Thus we proved the inclusion \( \mathcal{L}^o \subset \tilde{L} \).

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To complete the proof of (3), a nilpotent element \( n \) of the algebra \( A \) being given, we set \( e = 1 + n \). As we have \( qXeQ = eQX \), by the commutation relation (29), there exists an infinitesimal deformation

\[
\psi_e : \tilde{L} \rightarrow F(\mathbb{Z}, A[[W]])[[X]]
\]

such that \( \psi_e(X) = X \) and \( \psi_e(Q) = eQ \) and continuous with respect to the \( X \)-adic topology. Therefore to be more concrete \( \psi_e \) maps an element of the algebra \( \tilde{L} \)

\[
\sum_{n=0}^{\infty} X^n a_n(Q) \text{ with } a_n(Q) \in L^5(Q) \text{ for every } n \in \mathbb{N}
\]

to an element

\[
\sum_{n=0}^{\infty} X^n a_n(eQ) \in F(\mathbb{Z}, A[[W]])[[X]].
\]

If we denote the restriction \( \psi_e | \mathcal{L} \) to \( \mathcal{L} \) by \( \varphi_e \), then \( \varphi_e \) satisfies all the required conditions except for the uniqueness. The uniqueness follows from (2) that we have already proved above.

We have shown that the functor

\[
\mathcal{F}_{L/k} : \text{Alg}/L^5 \rightarrow (\text{Set})
\]

is a principal homogeneous space of the group functor \( \hat{G}_{m, \mathbb{C}} \). For origin of the group structure, see paragraph 2.2.6 as well as paragraph 1.3.1 below.

In the course of the proof of Proposition 4.43 we have proved the following

**Proposition 4.5.** The Galois hull \( \mathcal{L} \) coincides with the sub-algebra

\[
L^5\langle X, Q, (c + tQ + X)^{-1} | c \in \mathbb{C}_{\text{alg}} \rangle
\]

of \( F(\mathbb{Z}, L^5)[[X]] \) generated by \( L^5, X, Q \) and the set \( \{(c + tQ + X)^{-1} | c \in \mathbb{C}\} \). The commutation relation of \( X \) and \( Q \) is

\[
QX = qXQ.
\]

In particular, if \( q \neq 1 \), then the Galois hull is non-commutative.

### 4.2 Non-commutative deformation functor \( \mathcal{NCF}_{\sigma \theta} \) for \( \mathbb{C}(t)/\mathbb{C} \)

We are ready to describe the non-commutative deformations. Let \( A \in \text{Ob}(\text{NCAlg}/L^5) \).

**Lemma 4.6.** If \( q \neq 1 \), we have

\[
\mathcal{NCF}_{L/k}(A) = \{ (e, f) \in A^2 | qfe = ef \text{ and } e - 1, f \text{ are nilpotent } \}
\]

for every \( A \in \text{Ob}(\text{NCAlg}/L^5) \).
Proof. Since $q \neq 1$, it follows from the argument of the proof of Sublemma 4.4 that if we take

$$\varphi \in \mathcal{NCF}_{L/k\sigma}(A)$$

for $A \in \text{Ob}(\mathcal{NCAlg}/L^2)$, then $\varphi(X) = X + f$ and $\varphi(Q) = eQ$, $f, e \in A$.

Since $\varphi$ is an infinitesimal deformation of $\iota$, $f$ and $e - 1$ are nilpotent.

It follows from $QX = qXQ$ that

$$eQ(X + f) = q(X + f)eQ$$

so $ef = qfe$.

Suppose conversely that elements $e, f \in A$ such that $e - 1, f$ are nilpotent and such that $ef = qfe$ are given. Then the argument of the proof of Sublemma 4.4 allows us to show the unique existence of the infinitesimal deformation $\varphi \in \mathcal{NCF}_{\sigma\theta^*}(A)$ such that

$$\varphi(X) = X + fQ, \varphi(Q) = eQ.$$ 

We are going to see in 4.3.3 that theoretically, we can identify

$$\mathcal{NCF}_L(A) = \left\{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \mid e \in A, f \in A, qfe = ef \text{ and } e - 1, f \text{ are nilpotent} \right\}.$$ (45)

Corollary 4.7 (Corollary to the proof of Lemma 4.6). When $q = 1$ that is the case excluded in our general study, we consider the $q$-SI $\sigma$-differential differential field

$$(\mathbb{C}(t), \text{Id}, \theta^*)$$
as in 3.1.3. So $\theta^*$ is the iterative derivation;

$$\theta^{(0)} = \text{Id},$$

$$\theta^{(i)} = \frac{1}{i!} \frac{d^i}{dt^i} \quad \text{for } i \geq 1.$$ Then we have

$$\mathcal{L}_{\sigma\theta^*} \simeq \mathcal{L}_{d/dt},$$

$$\mathcal{NCF}_{((\mathbb{C}(t), \text{Id}, \theta^*), \mathbb{C}}(A) = \{ f \in A \mid f \text{ is a nilpotent element} \}.$$ (47)

for $A \in \text{Ob}(\mathcal{NCAlg}/L^2)$.

Proof. In fact, if $q = 1$, then

$$Q = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \end{bmatrix} = 1 \in \mathbb{C}. $$

So $\mathcal{L}_{q, \sigma}$ is generated by $X$ over $\mathcal{K}$. Therefore $\mathcal{L}_{\sigma\theta^*} \simeq \mathcal{L}_{d/dt}$. Since $Q = 1 \in \mathcal{K}$, $\varphi(Q) = Q$ for an infinitesimal deformation

$$\varphi \in \mathcal{NCF}_{\sigma\theta^*}(A)$$

and we get (47).
4.2.1 Quantum group enters

To understand Lemma 4.6, it is convenient to introduce a quantum group.

**Definition 4.8.** We work in the category \((\text{NCAlg}/\mathcal{C})\). Let \(A\) be a not necessarily commutative \(\mathbb{C}\)-algebra. We say that two sub-sets \(S, T\) of \(A\) are mutually commutative if for every \(s \in S, t \in T\), we have \([s, t] = st - ts = 0\).

For \(A \in \text{Ob}(\text{NCAlg}/L^\times)\), we set
\[
H_q(A) = \left\{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \mid e, f \in A, \ e \text{ is invertible in } A, \ ef = qfe \right\}.
\]

**Lemma 4.9.** For two matrices
\[
Z_1 = \begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix} \in H_q(A),
\]
if \(\{e_1, f_1\}\) and \(\{e_2, f_2\}\) are mutually commutative, then the product matrix
\[
Z_1Z_2 \in H_q(A).
\]

**Proof.** Since
\[
Z_1Z_2 = \begin{bmatrix} e_1e_2 & e_1f_2 + f_1 \\ 0 & 1 \end{bmatrix},
\]
we have to prove
\[
qe_1e_2(e_1f_2 + f_1) = (e_1f_2 + f_1)e_1e_2.
\]
This follows from the mutual commutativity of \(\{e_1, f_1\}\) and \(\{e_2, f_2\}\), and the conditions \(e_1f_1 = qf_1e_1\), \(e_2f_2 = qf_2e_2\). \(\square\)

**Lemma 4.10.** For a matrix
\[
Z = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \in H_q(A),
\]
if we set
\[
\tilde{Z} = \begin{bmatrix} e^{-1} & -e^{-1}f \\ 0 & 1 \end{bmatrix} \in M_2,
\]
then
\[
\tilde{Z} \in H_{q^{-1}}(A) \text{ and } \tilde{Z}Z = ZZ = I_2.
\]

**Proof.** We can check it by a simple calculation. See also Remark 4.11, where the first assertion is proved. \(\square\)

**Remark 4.11.** If \(q^2 \neq 1\), for \(f \neq 0\), \(\tilde{Z} \not\in H_q(A)\). In fact let us set
\[
\tilde{Z} = \begin{bmatrix} \tilde{e} & f \\ 0 & 1 \end{bmatrix}.
\]
so that, \( \tilde{e} = e^{-1}, \tilde{f} = -e^{-1}f \). Then \( \tilde{e}\tilde{f} = e^{-1}(-e^{-1}f) = -e^{-2}f \) and \( \tilde{f}\tilde{e} = -e^{-1}fe^{-1} = -qe^{-2f} \). So

\[
\tilde{e}\tilde{f} = -e^{-2}f = q - 1\tilde{f}\tilde{e}
\]  

(48)

showing

\( \tilde{Z} \in H_{q^{-1}}(A) \).

Now we assume to the contrary that \( \tilde{Z} \in H_q(A) \). We show that it would lead us to a contradiction. The assumption would imply that we have

\[
\tilde{e}\tilde{f} = q\tilde{f}\tilde{e}.
\]  

(49)

It follows from (48) and (49)

\[
q\tilde{f}\tilde{e} = q^{-1}\tilde{f}\tilde{e}.
\]  

(50)

so that we would have

\[
q^2\tilde{f}\tilde{e} = \tilde{f}\tilde{e}.
\]  

(51)

Since \( \tilde{e} = e^{-1} \) is invertible in \( A \),

\[
(q^2 - 1)\tilde{f} = 0.
\]

As the algebra \( A \) is a \( \mathbb{C} \)-vector space and \( \tilde{f} \neq 0 \), the complex number \( q^2 - 1 = 0 \) which is a contradiction.

**Lemma 4.12.** Let \( u \) and \( v \) be symbols over \( \mathbb{C} \). We have shown that we find a \( \mathbb{C} \)-Hopf algebra

\[
\mathcal{H}_q = \mathbb{C}\langle u, u^{-1}, v \rangle_{\text{alg}}/(uv - qvu)
\]  

(52)

as an algebra so that

\[
u u^{-1} = u^{-1}u = 1, \quad u^{-1}v = q^{-1}vu^{-1}.
\]

**Definition of the algebra** \( \mathcal{H}_q \) **determines the multiplication**

\[
m: \mathcal{H}_q \otimes_{\mathbb{C}} \mathcal{H}_q \to \mathcal{H}_q,
\]

the co-unit

\[
\eta: \mathbb{C} \to \mathcal{H}_q,
\]

that is the composition of natural morphisms

\[
\mathbb{C} \to \mathbb{C}\langle u, u^{-1}, v \rangle_{\text{alg}}
\]

and

\[
\mathbb{C}\langle u, u^{-1}, v \rangle_{\text{alg}} \to \mathbb{C}\langle u, u^{-1}, v \rangle_{\text{alg}}/(uv - qvu) = \mathcal{H}_q.
\]

**The product of matrices gives the co-multiplication**

\[
\Delta: \mathcal{H}_q \to \mathcal{H}_q \otimes_{\mathbb{C}} \mathcal{H}_q,
\]

that is a \( \mathbb{C} \)-algebra morphism defined by

\[
\Delta(u) = u \otimes u, \quad \Delta(u^{-1}) = u^{-1} \otimes u^{-1}, \quad \Delta(v) = u \otimes v + v \otimes 1,
\]
for the generators $u, u^{-1}, v$ of the algebra $\mathcal{H}_q$, the co-unit is a $\mathbb{C}$-algebra morphism
\[
\epsilon: \mathcal{H}_q \to \mathbb{C}, \quad \epsilon(u) = \epsilon(u^{-1}) = 1, \epsilon(v) = 0
\]
for the generators $u, u^{-1}, v$ of the algebra $\mathcal{H}_q$. The antipode
\[
S: \mathcal{H}_q \to \mathcal{H}_q
\]
is a $\mathbb{C}$-anti-algebra morphism given by
\[
S(u) = u^{-1}, \quad S(u^{-1}) = u, \quad S(v) = -u^{-1}v.
\]
Let us set
\[
\mathcal{H}_{qL^3} := \mathcal{H}_q \otimes_{\mathbb{C}} L^3
\]
so that $\mathcal{H}_{qL^3}$ is an $L^3$-Hopf algebra. We notice that for an $L^3$-algebra $A$
\[
\mathcal{H}_{qL^3}(A) := \text{Hom}_{L^3, \text{algebra}}(\mathcal{H}_{qL^3}, A) = \left\{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \mid e, f \in A, ef = qfe, e \text{ is invertible} \right\}.
\]

Remark 4.13. We know by general theory that the antipode $i: H \to H$ that is a linear
map making a few diagrams commutative, is necessarily an anti-endomorphism of the
algebra $H$ so that
\[
S(ab) = S(b)S(a) \text{ for all elements } a, b \in H \text{ and } S(1) = 1.
\]
See Manin [10], section 1, 2.

The Hopf algebra $\mathcal{H}_q$ is a $q$-deformation of the affine algebraic group $AF_1\mathbb{C}$ of affine
transformations of the affine line.

Anyhow, we notice that the quantum group appears in this very simple example
showing that quantum groups are indispensable for a Galois theory of $q$-$SI$ $\sigma$-differential
field extensions.

4.3 Observations on the Galois structures of the field extension
$\mathbb{C}(t)/\mathbb{C}$

4.3.1 Where does quantum group structure come from?

Let us now examine that the group structure in [2.2.6] arising from the variation of initial
conditions coincides with the quantum group structure defined in [4.2.1].

To see this, we have to clearly understand the initial condition of a formal series
\[
f(W, X) = \sum_{i=0}^{\infty} X^i a_i(W) \in F(\mathbb{Z}, A[[W]])[[X]]
\]
so that the coefficients $a_i$’s, which are elements of $F(\mathbb{Z}, A[[W]])$, are functions on $\mathbb{Z}$ taking
values in the formal power series ring $A[[W]]$. The initial condition of $f(W, X)$ is the value
of the function $f(W, 0) = a_0(W) \in F(\mathbb{Z}, A)$ at $n = 0$ which we may denote by
\[
f(W, 0)|_{n=0} \in A[[W]].
\]
As in Example 2.3, we set
\[ T(W, X) := \iota(t) = (t + W)Q + X \in F(Z, L^3[[W]][[X]]). \]
For \( A \in \text{Ob}(NCAlg/L^3) \), we take an infinitesimal deformation \( \varphi \in NC\mathcal{F}_{L/k}(A) \) so that the morphism \( \varphi : \mathcal{L} \to F(Z, A[[W]])([[X]]) \) is determined by the image
\[ \tilde{T}(W, X) := \varphi(t) \in F(Z, A[[W]])([[X]]) \]
of \( t \in L \subset \mathcal{L} \), the \( q \)-SI \( \sigma \)-differential field \( L \) being a sub-algebra of \( \mathcal{L} \) by the universal Hopf morphism. It follows from Lemma 4.6 that there exist \( e, f \in A \) such that \( ef = qfe \), the elements \( e - 1, f \) are nilpotent and such that
\[ \varphi(t) = (e(t + W) + f)Q + X. \] (53)
Therefore,
\[ \tilde{T}(W, X) = T((t(e - 1) + f)Q + eW, X). \]
Since \( T(W, X) \) and \( \tilde{T}(W, X) \) satisfy
\[ \hat{\Sigma}(T) = qT \text{ and } \hat{\Theta}^{(1)}(T) = 1, \]
their difference is measured at the initial conditions. The initial condition of \( T(W, X) \) is \( t + W \) and that of \( \tilde{T}(W, X) \) is \( et + f + W \). Namely, the infinitesimal deformation \( \varphi \) arises from the coordinate transformation
\[ t + W \mapsto et + f + eW \]
or equivalently
\[ W \mapsto t(e - 1) + f + eW. \]
We answer the question above in Observation 10.2.

4.3.2 Quantum Galois group \( \text{NCInf} \)-gal \( \sigma_\theta, \ast(C(t)/\mathbb{C}) \)
The Hopf algebra \( \mathcal{H}_q \) in 4.2.1 defines a functor
\[ \hat{\mathcal{H}}_{qL^3} : (NCAlg/L^3) \to (Set) \]
such that
\[ \hat{\mathcal{H}}_{qL^3}(A) = \{ \psi : \mathcal{H}_q \otimes_{\mathbb{C}} L^3 \to A \mid \psi \text{ is a } L^3\text{-algebra morphism such that } \psi(u) - 1, \eta(v) \text{ are nilpotent} \} \]
for every \( A \in (NCAlg/L^3) \). In other words \( \hat{\mathcal{H}}_{qL^3} \) is the formal completion of the quantum group \( \mathcal{H}_q \otimes_{\mathbb{C}} L^3 = \mathcal{H}_q L^3 \). We can summarize our results in the following form.
Theorem 4.14. The quantum formal group \( \hat{\mathcal{H}}_{qL} \) operates on the functor \( \mathcal{NCF}_{L/k} \) in such a way that there exists a functorial isomorphism
\[
\mathcal{NCF}_{L/k} \simeq \hat{\mathcal{H}}_{qL}.
\] (54)

The restriction of the functor \( \mathcal{NCF}_{L/k} \) on the sub-category \((\text{CAlg}/L^3)\) gives the functorial isomorphism
\[
\mathcal{NCF}_{L/k} \mid (\text{CAlg}/L^3) \simeq \hat{\mathbb{G}}_{mL^3}.
\]

Or equivalently,

(1) We have not only isomorphism (54) of functors on the category \( \text{NCAlg}/L^3 \) taking values in the category \((\text{Set})\) of sets, but also we can identify, by this isomorphism, the co-product of the quantum formal group \( \hat{\mathcal{H}}_{qL} \) arising from the multiplication of triangular matrices in 4.2.1 with composition of coordinate transformations of the initial condition in 4.3.1. For these two reasons, we say that the quantum infinitesimal Galois group of the \( q \)-SI \( \sigma \)-differential field extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \) is the quantum formal group \( \hat{\mathcal{H}}_{qL} \). Namely,
\[
\text{NCinf-gal}\((\mathbb{C}(t), \sigma \theta^*)/\mathbb{C}\) \simeq \hat{\mathcal{H}}_{qL}.
\]

(2) The commutative Galois group functor \( \text{CInf-gal}_{\sigma \theta^*}(L/k) \) of the \( q \)-SI \( \sigma \)-differential extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \) on the category \((\text{Alg}/L^3)\) of commutative \( L^3 \)-algebras is isomorphic to the formal group \( \hat{\mathbb{G}}_{m} \).

The operation of quantum formal group requires a precision.

Remark 4.15. We should be careful about the operation of quantum formal group. To be more precise, for \( \varphi \in \mathcal{F}_{L/k}(A) \) and \( \psi \in \hat{\mathcal{H}}_{qL^3}(A) \) so that we have
\[
\varphi(t) = (e(t + W) + f)Q + X \in F(\mathbb{Z}, A[[W]])([X])
\]
with \( e, f \in A \) and we imagine the matrix
\[
\begin{bmatrix}
\psi(u) & \psi(v) \\
0 & 1
\end{bmatrix} \in M_2(A)
\]
corresponding to \( \psi \). If the sub-sets of the algebra \( A \), \( \{\psi(u), \psi(v)\} \) and \( \{e, f\} \) are commutative, the product
\[
\psi \cdot \varphi = \varpi \in \mathcal{F}_{L/k}(A)
\]
is defined to be
\[
\varpi(t) = (\psi(u)e(t + W) + (\psi(u)f + \psi(v))Q + X \in F(\mathbb{Z}, A[[W]])([X]).
\]
4.3.3 Non-commutative Picard-Vessiot ring

So far we analyzed the First Example, which is a non-linear $q$-SI $\sigma$-differential equation, according to general principle of Hopf Galois theory. We finally arrived at Theorem 4.14 that shows a quantum formal group appears as a Galois group. Our experiences of dealing Picard-Vessiot theory in our general framework done in our previous works [21], [17], teach us that we discovered here a new phenomenon, a non-commutative Picard-Vessiot extension.

We work in the $q$-SI $\sigma$-differential ring $(F(Z, C(t))[X], \hat{\Sigma}, \hat{\Theta}^*)$. We are delighted to assert that a non-commutative $q$-SI $\sigma$-differential ring extension

\[(C(Q, Q^{-1}, X)_{alg}, \hat{\Sigma}, \hat{\Theta}^*)/C) \quad (55)\]

is a non-commutative Picard Vessiot ring with quantum Galois group $H_q$. We consider the fundamental system

\[Y := \begin{bmatrix} Q & X \\ 0 & 1 \end{bmatrix} \in M_2(C(Q, Q^{-1}, X)_{alg})\]

so that the homogeneous linear $q$-SI $\sigma$-differential equations is

\[\hat{\Sigma}(Y) = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y, \quad \hat{\Theta}^{(1)}(Y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y. \quad (56)\]

In fact, we can check the first equation in (56):

\[\hat{\Sigma}(Y) = \begin{bmatrix} \hat{\Sigma}(Q) & \hat{\Sigma}(X) \\ \hat{\Sigma}(0) & \hat{\Sigma}(1) \end{bmatrix} = \begin{bmatrix} qQ & qX \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & X \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y.\]

The second equality of (56) is also checked easily.

Leaving heuristic reasoning totally aside, we study the Picard-Vessiot extension (55) in detail in Sections 10 and 11.

5 The Second Example, the $q$-SI $\sigma$-differential field extension $(C(t, t^\alpha), \sigma, \theta^*)/C$

5.1 Commutative deformations

As in the previous section, let $t$ be a variable over $C$ and we assume that the complex number $q$ is not a root of unity if we do not mention other assumptions on $q$. Sometimes we write the condition that $q$ is not a root of unity, simply to recall it. We work under the condition that $\alpha$ is an irrational complex number so that $t$ and $t^\alpha$ are algebraically independent over $C$. Therefore the field $C(t, t^\alpha)$ is isomorphic to the rational function field of two variables over $C$. We denote by $\sigma$ the $C$-automorphism of the field $C(t, t^\alpha)$ such that

\[
\sigma(t) = qt \quad \text{and} \quad \sigma(t^\alpha) = q^\alpha t^\alpha.
\]
Let us set \( \theta^{(0)} := \text{Id}_{\mathbb{C}(t, t^\alpha)} \), the map

\[
\theta^{(1)} := \frac{\sigma - \text{Id}}{(q - 1)t} : \mathbb{C}(t, t^\alpha) \to \mathbb{C}(t, t^\alpha)
\]

and

\[
\theta^{(n)} = \frac{1}{[n]_q!} (\theta^{(1)})^n \quad \text{for} \quad n = 2, 3, \ldots.
\]

So the \( \theta^{(i)} \)'s are \( \mathbb{C} \)-linear operators on \( \mathbb{C}(t, t^\alpha) \) and

\[ L := (\mathbb{C}(t, t^\alpha), \sigma, \theta^*) \]

is a \( q \)-SI \( \sigma \)-differential field. The restriction of \( \sigma \) and \( \theta^* \) to the subfield \( \mathbb{C} \) is trivial. We denote the \( q \)-SI \( \sigma \)-differential field extension \( \mathbb{C}(t, t^\alpha)/\mathbb{C} \) by \( L/k \). We denote \( t^\alpha \) by \( y \) so that as we mentioned above, the abstract field \( \mathbb{C}(t, t^\alpha) = \mathbb{C}(t, y) \) is isomorphic to the rational function field of 2 variables over \( \mathbb{C} \). We take the derivations \( \partial/\partial t \) and \( \partial/\partial y \) as a basis of the \( L^2 \)-vector space \( \text{Der}(L^2/k^2) \) of \( k^2 \)-derivations of \( L^2 \). Hence \( L^2 = (L^2, \{\partial/\partial t, \partial/\partial y\}) \) as in [21].

Let us list the fundamental equations.

\[
\sigma(t) = qt, \quad \sigma(y) = q^\alpha y, \quad (57)
\]

\[
\theta^{(1)}(t) = 1, \quad \theta^{(1)}(y) = [\alpha]_q y \quad (58)
\]

We explain below the notation \([\alpha]_q\). We are going to determine the Galois group

\[ \text{NCInf-gal}(L/k). \]

Before we start, we notice that by Proposition 1.3 the Galois hull of the extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \) is not a commutative algebra and since \( \mathbb{C}(t) \) is a sub-field of \( \mathbb{C}(t, t^\alpha) \), the Galois hull of the \( q \)-SI \( \sigma \)-differential field extension \( (\mathbb{C}(t, t^\alpha), \sigma, \theta^*)/\mathbb{C} \) is not a commutative algebra either. Consequently the \( q \)-SI \( \sigma \)-differential field extension \( \mathbb{C}(t, t^\alpha)/\mathbb{C} \) is not a Picard-Vessiot extension (See [5], [12], [21]). So we have to go beyond the general theory of Heiderich [7], Umemura [21] for the definition of the Galois group NCInf-gal (\( L/k \)).

It follows from general definition that the universal Hopf morphism

\[ \iota : L \to F(\mathbb{Z}, L^2)[[X]] \]

is given by

\[
\iota(a) = \sum_{n=0}^{\infty} X^n u[\theta^{(n)}(a)] \in F(\mathbb{N}, L^2)[[X]]
\]

for \( a \in L \). Here for \( b \in L \), we denote by \( u[b] \) the element

\[
u[b] = \begin{bmatrix} \cdots & -2 & -2(b) & \cdots \\
\cdots & \sigma^{-2}(b) & \sigma^{-1}(b) & \cdots \\
\cdots & 1 & \sigma(b) & \cdots \\
\cdots & 2 & \sigma^2(b) & \cdots 
\end{bmatrix} \in F(\mathbb{N}, L^2).
\]

It follows from the definition above of the universal Hopf morphism \( \iota \),

\[
\iota(y) = \sum_{n=0}^{\infty} X^n \begin{pmatrix} \alpha \\ n \end{pmatrix}_q t^{-n} q^{\alpha-n} y,
\]

39
where we use the following notations. For a complex number \( \beta \in \alpha + \mathbb{Z} \),

\[
[\beta]_q = \frac{q^\beta - 1}{q - 1}
\]

and

\[
\binom{\alpha}{n}_q = \frac{[\alpha][\alpha - 1]_q \cdots [\alpha - n + 1]_q}{[n]_q!}
\]

\[
Q = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & q^{-2} & q^{-1} & q & q^2 & \cdots \end{bmatrix}
\]

and

\[
Q^\alpha = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & q^{-2\alpha} & q^{-\alpha} & 1 & q^n & q^{2\alpha} & \cdots \end{bmatrix}
\]

We set

\[
Y_0 := \sum_{n=0}^{\infty} X^n \left( \frac{\alpha}{n} \right)_q t^{-n} Q^{\alpha - n}
\]

so that

\[
\iota(y) = Y_0 y \quad \text{in } F(\mathbb{N}, L)[[X]].
\]

(59)

Considering \( k \)-derivations \( \partial/\partial t, \partial/\partial y \) in \( L \) and therefore in \( F(\mathbb{N}, L^2) \) or in \( F(\mathbb{N}, L^2)[[X]] \), we generate the Galois hull \( L \) by \( \iota(L) \) and \( L^2 \) so that \( L \subset F(\mathbb{N}, L^2)[[X]] \) is invariant under \( \hat{\Sigma}, \Theta^* \)'s and \( \{ \partial/\partial t, \partial/\partial y \} \). We may thus consider

\[
L \hookrightarrow F(\mathbb{N}, L^2)[[X]].
\]

By the universal Taylor morphism

\[
L^2 = (L^2, \{ \partial/\partial t, \partial/\partial y \}) \rightarrow L^2[[W_1, W_2]],
\]

we identify \( L \) by the canonical morphism

\[
\iota: L \rightarrow F(\mathbb{Z}, L^2)[[X]] \rightarrow F(\mathbb{Z}, L^2[[W_1, W_2]])[[X]].
\]

We study first the infinitesimal deformations \( C\mathcal{F}_{L/k} \) of \( \iota \) on the category \( (C\text{Alg}/L^2) \) of commutative \( L^2 \)-algebras and then generalize the argument to the category \( (N\text{CAlg}/L^2) \) of not necessarily commutative \( L^2 \)-algebras.

For a commutative \( L^2 \)-algebra \( A \), let \( \varphi: \mathcal{L} \rightarrow F(\mathbb{Z}, A[[W_1, W_2]])[[X]] \) be an infinitesimal deformation of the canonical morphism \( \iota: \mathcal{L} \rightarrow F(\mathbb{Z}, L^2[[W_1, W_2]])[[X]] \) so that both \( \iota \) and \( \varphi \) are compatible with operators \( \{ \Sigma, \Theta^*, \partial/\partial W_1, \partial/\partial W_2 \} \).

**Lemma 5.1.** The infinitesimal deformation \( \varphi \) is determined by the images \( \varphi(Y_0), \varphi(Q) \) and \( \varphi(X) \).

**Proof.** The Galois hull \( \mathcal{L}/\mathcal{K} \) is generated over \( \mathcal{K} = L^2 \) by \( \iota(t) = tQ + X \) and \( \iota(y) = Y_0 y \) with operators \( \Theta^*, \Sigma \) and \( \partial/\partial t, \partial/\partial y \) along with localizations. This proves the Lemma.

Let us set \( Z_0 := \varphi(Y_0) \in F(\mathbb{Z}, A[[W_1, W_2]])[[X]] \) and expand it into a formal power series in \( X \):

\[
Z_0 = \sum_{n=0}^{\infty} X^n a_n, \quad \text{with } a_n \in F(\mathbb{Z}, A[[W_1, W_2]]) \quad \text{for every } n \in \mathbb{N}.
\]
It follows from (57) and (59)

\[ \hat{\Sigma}(Z_0) = q^\alpha Z_0 \]

so that

\[ \sum_{n=0}^{\infty} X^n q^n \hat{\Sigma}(a_n) = q^\alpha \sum_{n=0}^{\infty} X^n a_n. \]  \hspace{1cm} (60)

Comparing the coefficient of the \( X^n \)'s in (60) we get

\[ \hat{\Sigma}(a_n) = q^{\alpha - n} a_n \text{ for } n \in \mathbb{N}. \]

So \( a_n = b_n Q^{\alpha - n} \) with \( b_n \in A[[W_1, W_2]] \) for \( n \in \mathbb{N} \). Namely we have

\[ Z_0 = \sum_{n=0}^{\infty} X^n b_n Q^{\alpha - n} \text{ with } b_n \in A[[W_1, W_2]]. \]  \hspace{1cm} (61)

It follows from (58),

\[ \sigma(y) - y = \theta^{(1)}(y)(q - 1)t \]

and so by (57)

\[ (q^\alpha - 1)y = \theta^{(1)}(y)(q - 1)t. \]

Applying the canonical morphism \( \iota \) and the deformation \( \varphi \), we get

\[ (q^\alpha - 1)Y_0 = \Theta^{(1)}(Y_0)(q - 1)(tQ + X) \]  \hspace{1cm} (62)

as well as

\[ (q^\alpha - 1)Z_0 = \Theta^{(1)}(Z_0)(q - 1)(teQ + X). \]  \hspace{1cm} (63)

Substituting (61) into (63), we get a recurrence relation among the \( b_m \)'s;

\[ b_{m+1} = \left[ \frac{e(t + W_1)}{m + 1} \right] q^m b_m. \]

Hence

\[ b_m = \left( \frac{\alpha}{m} \right)_q (e(t + W_1))^{-m} b_0 \text{ for every } m \in \mathbb{N}, \]  \hspace{1cm} (64)

where \( b_0 \in A[[W_1, W_2]] \) and every coefficient of the power series \( b_0 - 1 \) are nilpotent.

Since

\[ \frac{\partial Y_0}{\partial y} = \frac{\partial}{\partial W_2} \left( \sum_{n=0}^{\infty} X^n \left( \begin{array}{c} \alpha \\ n \end{array} \right)_q (t + W_1)^{-n} Q^{\alpha - n} \right) = 0, \]

we must have

\[ 0 = \varphi \left( \frac{\partial Y_0}{\partial y} \right) = \frac{\partial \varphi(Y_0)}{\partial W_2} = \frac{\partial Z_0}{\partial W_2} \]

and consequently

\[ \frac{\partial b_0}{\partial W_2} = 0 \]

so that

\[ b_0 \in A[[W_1]]. \]
by (61). Therefore, we have determined the image

\[ Z_0 = \varphi(Y_0) = \sum_{n=0}^{\infty} X^n \left( \frac{\alpha}{n} \right)_q (e(t + W_1))^{-n} Q^{\alpha-n} b_0 \]  

by (64), where all the coefficients of the power series \( b_0(W_1) - 1 \) are nilpotent.

### 5.2 The functor \( \mathcal{CF}_{L/k} \) of commutative deformations

In the Second Example, when we deal with the \( q \)-SI \( \sigma \)-differential field extension \( L/k \), the Galois hull \( \mathcal{L}/\mathcal{K} \) is a non-commutative algebra extension, so we have to consider the functor \( \mathcal{NCF}_{L/k} \) on the category \( \mathcal{NCAlg}/L \) of not necessarily commutative \( L \)-algebras. It is, however, easier to understand first the commutative deformation functor \( \mathcal{CF}_{L/k} \) that is the restriction on the sub-category \( (\mathcal{CAlg}/L) \) of the functor \( \mathcal{NCF}_{L/k} \). We using the notation of Lemma 5.1, it follows from (65) the following Proposition.

**Proposition 5.2.** We set

\[ Y_1(W_1, W_2; X) := (t + W_1)Q + X, \]  

\[ Y_2(W_1, W_2; X) := \sum_{n=0}^{\infty} X^n \left( \frac{\alpha}{n} \right)_q (t + W_1)^{-n} Q^{\alpha-n} (y + W_2). \]

Then we have

\[ \iota(t) = Y_1(W_1, W_2; X), \]  

\[ \iota(y) = Y_2(W_1, W_2; X) \]

and

\[ \varphi(Y_1(W_1, W_2; X)) := Y_1((e - 1)t + eW_1, [b_0(W_1) - 1]y + b_0(W_1)W_2; X), \]  

\[ \varphi(Y_2(W_1, W_2; X)) := Y_2((e - 1)t + eW_1, [b_0(W_1) - 1]y + b_0(W_1)W_2; X). \]

In other words the infinitesimal deformation \( \varphi \) is given by the coordinate transformation of the initial conditions

\[ (W_1, W_2) \mapsto (\varphi_1(W_1, W_2), \varphi_2(W_1, W_2)), \]

where

\[ \varphi_1(W_1, W_2) = (e - 1)t + eW_1, \]  

\[ \varphi_2(W_1, W_2) = [b_0(W_1) - 1]y + b_0(W_1)W_2. \]

The set of transformations in the form of (70), (71) forms a group.

**Lemma 5.3.** For a commutative \( L^\sharp \)-algebra \( A \), we set

\[ \hat{G}_{II}(A) := \{(e - 1)t + eW_1, [b(W_1) - 1]y + b(W_1)W_2) \in A[[W_1, W_2]] \times A[[W_1, W_2]] | \]  

\[ e \in A, b(W_1) \in A[[W_1]], \text{ all the coefficients of } b(W_1) - 1 \text{ and } e - 1 \text{ are nilpotent} \}. \]  

Then the set \( \hat{G}_{II}(A) \) is a group, the group law being the composition of coordinate transformations.
Proof. We have shown in Umemura [17] that the set of coordinate transformations of \( n \)-variables with coefficients in a commutative ring that are congruent to the identity modulo nilpotent elements forms a group under the composite of transformations. So it is sufficient to show:

1. The set \( \hat{G}_{II}(A) \) is closed under the composition.
2. The identity is in \( \hat{G}_{II}(A) \).
3. The inverse of every element in \( \hat{G}_{II}(A) \) is in \( \hat{G}_{II}(A) \).

In fact, let

\[
((e - 1)t + eW_1, [b(W_1) - 1]y + b(W_1)W_2), \quad ((f - 1)t + fW_1, [c(W_1) - 1]y + c(W_1)W_2)
\]

be two elements of \( \hat{G}_{II}(A) \). We mean by their composite

\[
((ef - 1)t + efW_1, [b((f - 1)t + fW_1)c(W_1) - 1]y + b((f - 1)t + fW_1)c(W_1)W_2)
\]

that is an element of \( \hat{G}_{II}(A) \). Certainly the identity \((W_1, W_2)\) is expressed for \( e = 1 \) and \( b(W_1) = 1 \). As for the inverse

\[
((e - 1)t + eW_1, [b(W_1) - 1]y + b(W_1)W_2)^{-1} = ((e^{-1} - 1)t + e^{-1}W_1, [c(W_1) - 1]y + c(W_1)W_2),
\]

where

\[
c(W_1) = \frac{1}{b((e^{-1} - 1)t + e^{-1}W_1)}.
\]

We can summarize what we have proved as follows.

**Proposition 5.4.** There exists a functorial inclusion on the category \((CAlg/L^3)\) of commutative \( L^3 \)-algebras

\[
\mathcal{CF}_{L/k}(A) := \mathcal{NCF}_{L/k}(A) \to \hat{G}_{II}(A)
\]

that sends infinitesimal deformation \( \varphi \) to

\[
((e - 1)t + eW_1, [b_0(W_1) - 1]y + b_0(W_1)W_2) \in \hat{G}_{II}(A)
\]

for every commutative \( L^3 \)-algebra \( A \).

In the Definition of the group functor \( \hat{G}_{II} \) in Lemma 5.3, we can eliminate the variable \( W_2 \).

**Lemma 5.5.** We introduce a group functor

\[
\hat{G}_2 : (CAlg/L^3) \to (Grp),
\]
setting

\[ \hat{G}_2(A) = \{ (e, b(W_1)) \in A \times A[[W_1]] \mid \]

All the coefficients of \( b(W_1) - 1 \) and \( e - 1 \) are nilpotent\}

for every \( A \in \text{Ob}(C\text{Alg}/L^3) \). The group law, the identity and the inverse are given as below.

For two elements \( (e, b(W_1)), (f, c(W_1)) \), their product is by definition

\[ (ef, b((f - 1)t + fW_1)c(W_1)). \]  \hspace{1cm} (76)

The identity is \( (1, 1) \) and the inverse

\[ (e, b(W_1))^{-1} = \left( \frac{1}{e}, \frac{1}{b((e^{-1} - 1)t + e^{-1}W_1)} \right). \]

Then there exists an isomorphism of group functors.

\[ \hat{G}_{II} \simeq \hat{G}_2. \]

Proof. In fact, for a every commutative algebra \( A \in \text{Ob}(C\text{Alg}/L^3) \), the map

\[ \hat{G}_{II}(A) \rightarrow \hat{G}_2(A), \]

\[ ((e - 1)t + eW_1, (b(W_1) - 1)y + b(W_1)W_2) \rightarrow (e, b(W_1)) \]  \hspace{1cm} (78)

gives an isomorphism of group functors. \( \square \)

Remark 5.6. In the composition laws for \( \hat{G}_{II} \) \hspace{1cm} (75) and for \( \hat{G}_2 \) \hspace{1cm} (76), we substitute in the variable \( W_1 \) the linear polynomial \( (e - 1)t + W_1 \) in the power series \( c(W_1) \) to get \( c((e - 1)t + eW_1) \). Since \( c(W_1) \) is a power series, in order that the substitution has sense, we can not avoid the condition that \( e - 1 \) is nilpotent. We can neither define the global group functors \( G_{II} \) nor \( G_2 \) whose completions are \( \hat{G}_{II}, \hat{G}_2 \) respectively.

It is natural to wonder what is the image of the inclusion map in Proposition 5.4.

Conjecture 5.7. If \( q \) is not a root of unity, the inclusion in Proposition 5.4 is the equality.

Proposition 5.8. Origin of the group structure teaches us that if the Conjecture 5.7 is true, then the group functor

\[ \hat{G}_{II} : (C\text{Alg}/L^3) \rightarrow (\text{Grp}) \]

operates on the functor

\[ \mathcal{CF}_{L/k} : (C\text{Alg}/L^3) \rightarrow (\text{Set}) \]

through the transformations of the initial conditions \( (W_1, W_2) \), in such a way that

\[ (\hat{G}_{II}, \mathcal{CF}_{L/k}) \]

is a torsor. So we may say that the Galois group functor

\[ \text{CInf-gal} \left( (\mathbb{C}(t, t^\alpha, \theta^*)/\mathbb{C}) \right) \simeq \hat{G}_{II}. \]
Remark 5.9. We explain a background of Conjecture 5.7.

Lemma 5.10. The Galois hull $\mathcal{L}$ is a localization of the following ring

$$L^\sharp\langle Q, X, \frac{1}{tQ + X}\rangle_{alg}\langle \frac{\partial^j}{\partial t^j}Y_0\rangle_{alg, j \in \mathbb{N}}.$$ 

Proof. Since $\iota(t) = tQ + X$, as we have seen in the First Example,

$$L^\sharp\langle Q, X\rangle_{alg}\langle \frac{\partial^j}{\partial t^j}Y_0\rangle_{alg, j \in \mathbb{N}} \subseteq \mathcal{L}.$$ 

We show that the ring

$$L^\sharp\langle Q, X\rangle_{alg}\langle \frac{\partial^j}{\partial t^j}Y_0\rangle_{alg, j \in \mathbb{N}}$$

is closed under the operations $\hat{\Sigma}$, $\Theta^{(1)}$, $\partial/\partial t$ and $\partial/\partial y$ of $F(\mathbb{Z}, L^\sharp)[[X]]$. Evidently the ring is closed under the last two operators. Since the operators $\hat{\Sigma}$ and $\partial^n/\partial t^n$ operating on $F(\mathbb{Z}, L^\sharp)[[X]]$ mutually commute, it follows from (57)

$$\hat{\Sigma}\left(\frac{\partial^nY_0}{\partial t^n}\right) = \frac{\partial^n}{\partial t^n}\hat{\Sigma}(Y_0) = \frac{\partial^n}{\partial t^n}(q^nY_0) = q^n\frac{\partial^nY_0}{\partial t^n}.$$ 

So the ring is closed under $\hat{\Sigma}$. Similarly since the operators $\Theta^{(1)}$ and $\partial^n/\partial t^n$ mutually commute on $F(\mathbb{Z}, L^\sharp)[[X]]$,

$$\Theta^{(1)}\left(\frac{\partial^nY_0}{\partial t^n}\right) = \frac{\partial^n}{\partial t^n}\Theta^{(1)}(Y_0)$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\Theta^{(1)}(Y_0y)$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\Theta^{(1)}(\iota(y))$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\iota(\theta^{(1)}(y))$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\left(\sigma(y) - \frac{y}{q - 1}t\right)$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\left(\frac{q^nY_0y - Y_0y}{(q - 1)(tQ + X)}\right)$$

$$= \frac{1}{y}\frac{\partial^n}{\partial t^n}\left(\frac{q^nY_0 - Y_0}{(q - 1)(tQ + X)}\right),$$

which is an element of the ring. 

Conjecture 5.7 arises from experience that if $q$, is not a root of unity, we could not find any non-trivial algebraic relations among the partial derivatives

$$\frac{\partial^nY_0}{\partial t^n} \quad \text{for } n \in \mathbb{N}$$

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over $L^2$ so that we could guess that there would be none.

In fact, assume that we could prove our guess. Let $\varphi: L \to F(Z, A[[W_1, W_2]][[X]])$ be an infinitesimal deformation of $\iota$. So as we have seen

$$Z_0 = \varphi(Y_0) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} (et)^{-n} Q^{n-n} b(W_1)$$

with $b(W_1) \in A[[W_1]]$. There would be no constraints among the partial derivatives $\partial^n b(W_1)/\partial W_1^n$, $n \in \mathbb{N}$ and hence we could choose any power series $b(W_1) \in A[[W_1]]$ such that every coefficient of the power series $b(W_1) - 1$ is nilpotent.

5.3 The functor $\mathcal{NCF}_{L/k}$ of non-commutative deformations

We study the functor $\mathcal{NCF}_{L/k}(A)$ of non-commutative deformations

$$\mathcal{NCF}_{L/k}: (\text{Alg}/L^2) \to (\text{Set}).$$

For a not necessarily commutative $L^2$-algebra $A \in \text{Ob}(\text{NCAlg}/L^2)$, let $\varphi: L \to F(Z, A[[W_1, W_2]][[X]])$ be an infinitesimal deformation of the canonical morphism $\iota: L \to F(Z, A[[W_1, W_2]][[X]])$.

Both $t$ and $y$ are elements of the field $\mathbb{C}(t, t^\alpha) = \mathbb{C}(t, y)$ so that $[t, y] = yt - ty = 0$. So for the deformation $\varphi \in \mathcal{NCF}_{L/k}(A)$ we must have

$$[\varphi(t), \varphi(y)] = \varphi(t)\varphi(y) - \varphi(y)\varphi(t) = 0. \quad (80)$$

When we consider the non-commutative deformations, the commutativity $(80)$ gives a constraint for the deformation. To see this, we need a Lemma.

**Lemma 5.11.** For every $l \in \mathbb{N}$, we have

$$q^l \binom{\alpha}{l} q + \binom{\alpha}{l-1} q = \binom{\alpha}{l} q + q^{\alpha-l+1} \binom{\alpha}{l-1} q.$$

**Proof.** This follows from the definition of $q$-binomial coefficient. \qed

**Lemma 5.12.** Let $A$ be a not necessarily commutative $L^2$-algebra in $\text{Ob}(\text{NCAlg}/L^2)$. Let $e, f \in A$ such that $e - 1$ and $f$ are nilpotent. We set

$$A := (e(t + W_1) + f)Q + X$$

and for a power series $b(W_1) \in A[[W_1]]$, we also set

$$Z := \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} q (e(t + W_1) + f)^{-n} Q^{n-n} b(W_1)$$

so that $A$ and $Z$ are elements of $F(Z, A[[W_1]][[X]])$. The following conditions are equivalent.
(1) \([A, Z] := AZ - ZA = 0\).

(2) \([e(t + W_1) + f, b(W_1)] = 0\).

**Proof.** We formulate condition (1) in terms of coefficients of the power series in \(X\). Assume condition (1) holds so that we have
\[
((e(t + W_1) + f)Q + X) \left( \sum_{n=0}^{\infty} X^n \left( \begin{array}{c} \alpha \\ n \end{array} \right)_q (e(f + W_1) + f)^{-n} Q^{a-n} b(W_1) \right)
\]
\[
= \sum_{n=0}^{\infty} X^n \left( \begin{array}{c} \alpha \\ n \end{array} \right)_q (e(f + W_1) + f)^{-n} Q^{a-n} b(W_1) ((e(t + W_1) + f)Q + X).
\]

Comparing degree \(l\) terms in \(X\) of (81), we find condition (1) is equivalent to
\[
q^l \left( \begin{array}{c} \alpha \\ l \end{array} \right)_q (e(t + W_1) + f)^{-l+1} Q^{a-l+1} b(W_1)
\]
\[
+ \left( \begin{array}{c} \alpha \\ l-1 \end{array} \right)_q (e(t + W_1) + f)^{-l+1} Q^{a-l+1} b(W_1)
\]
\[
= \left( \begin{array}{c} \alpha \\ l \end{array} \right)_q (e(t + W_1) + f)^{-l} b(W_1) (e(t + W_1) + f) Q^{a-l+1}
\]
\[
+ \left( \begin{array}{c} \alpha \\ l-1 \end{array} \right)_q q^{a-l+1} (e(t + W_1) + f)^{-l+1} Q^{a-l+1} b(W_1).
\]

So the condition (1) is equivalent to
\[
q^l \left( \begin{array}{c} \alpha \\ l \end{array} \right)_q (e(t + W_1) + f) b(W_1)
\]
\[
+ \left( \begin{array}{c} \alpha \\ l-1 \end{array} \right)_q (e(t + W_1) + f) b(W_1)
\]
\[
= \left( \begin{array}{c} \alpha \\ l \end{array} \right)_q b(W_1) (e(t + W_1) + f)
\]
\[
+ \left( \begin{array}{c} \alpha \\ l-1 \end{array} \right)_q q^{a-l+1} (e(t + W_1) + f) b(W_1)
\]
for every \(l \in \mathbb{N}\). Condition (83) for \(l = 0\) is condition (2). Hence condition (1) implies condition (2). Conversely condition (1) follows from (2) in view of (83) and Lemma 5.11. \(\square\)

Now let us come back to the infinitesimal deformation (79) of the canonical morphism \(\iota\). The argument in Section 4 allows us to determine the restriction \(\varphi\) on the sub-algebra generated by \(\iota(t) = tQ + X\) over \(L^1\) invariant under the \(\Theta^{(1)}\)’s, \(\Sigma\) and \(\{\partial/\partial t, \partial/\partial y\}\) in \(F(Z, L^1)[[X]]\). So there exist \(e, f \in A\) such that \(ef = qfe, e - 1, f\) are nilpotent and such that
\[
\varphi(Q) = eQ \quad \text{and} \quad \varphi(X) = X + fQ.
\]
that are equations in $F(\mathbb{Z}, A[[W_1, W_2]])[[X]]$. In particular
\[ \varphi(t) = \varphi(tQ + X) = (et + f)Q + X = (e(t + W_1) + f)Q + X, \]
where we naturally identify rings
\[ F(\mathbb{Z}, L^2)[[X]] \to F(\mathbb{Z}, L^2[[W_1, W_2]])[[X]] \to F(\mathbb{Z}, A[[W_1, W_2]])[[X]] \]
through the canonical maps.

Then the argument in the commutative case allows us to show that there exists a power series $b_0(W_1) \in A[[W_1]]$ such that
\[ \varphi(Y_0) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} (e(t + W_1) + f)^{-n}Q^{\alpha - n}b_0(W_1). \]
such that all the coefficients of the power series $b_0(W_1) - 1$ are nilpotent. As we deal with the not necessarily commutative algebra $A$, the commutation relation in $L$ gives a constraint. Namely since $\iota(y) = yY_0$ and $ty = yt$ in $L$ so that $\iota(t)\iota(y) = \iota(y)\iota(t)$, we get $\iota(t)(yY_0) = (yY_0)\iota(t)$ in $L$ and $\varphi(tQ + X)\varphi(Y_0) = \varphi(Y_0)\varphi(tQ + X)$. So we consequently have
\[ AZ_0 = Z_0A \quad \text{in } F(\mathbb{Z}, A[[W_1, W_2]])[[X]], \quad (84) \]
setting
\[ A := (e(t + W_1) + f)Q + X, \quad Z_0 := \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} (e(t + W_1) + f)^{-n}Q^{\alpha - n}b_0(W_1). \]

**Lemma 5.13.** We have
\[ [e(t + W_1) + f, b_0(W_1)] = 0. \]

**Proof.** This follows from (84) and Lemma 5.12. \hfill \Box

**Definition 5.14.** We define a functor
\[ QG_{2q} : (NCAAlg/L^2) \to (Set) \]
by putting
\[ QG_{2q}(A) = \{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}, b(W_1) \in M_2(A) \times A[[W_1]] \mid e, f \in A, ef = qfe, \]
\[ e \text{ is invertible in } A, b(W_1) \in A[[W_1]], [e(t + W_1) + f, b(W_1)] = 0 \} \]
for $A \in Ob(NCAAlg/L^2)$.

The functor $QG_{2q}$ is almost a quantum group in usual sense of the word. See Remark 5.6. We also need the formal completion $\hat{QG}_{2q}$ of the quantum group functor $QG_{2q}$ so that
\[ \hat{QG}_{2q} : (NCAAlg/L^2) \to (Set) \]

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is given by

\[ \hat{Q}G_{2q}(A) = \{ \left( \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right), b(W_1) \in QG_{2q}(A) \mid e - 1, f \text{ and all the coefficients of } b(W_1) - 1 \text{ are nilpotent} \} \]

for \( A \in \text{Ob}(NC\text{Alg}/L^2) \).

Studying commutative deformations of the Galois hull \( L/K \) of \((C(t, t^\alpha), \sigma, \theta^*)/C \), we introduced in Lemma 5.3 the functor \( \hat{G}_1 \) and in Lemma 5.5 the functor \( \hat{G}_2 \). They are isomorphic. The former involves the variable \( W_2 \) but the latter does not. The functor \( \hat{Q}G_{2q} \) does not involve the variable \( W_2 \). As you imagine, we also have another functor \( \hat{Q}G_{11} \) equivalent to the functor \( \hat{Q}G_{2q} \) and involving the variable \( W_2 \). Using Definition 5.14, we can express what we have shown.

**Proposition 5.15.** There exists a functorial inclusion

\[ NCF_{L/k}(A) \hookrightarrow \hat{Q}G_{2q}(A) \]

sending \( \varphi \in NCF_{L/k}(A) \) to

\[ \left( \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right), b_0(W_1) \in \hat{Q}G_{2q}(A). \]

We show that \( \hat{Q}G_{2q} \) is a quantum formal group over \( L^2 \). In fact, we take two elements

\[ (G, \xi(W_1)) = \left( \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right), \xi(W_1), \quad (H, \eta(W_1)) = \left( \begin{array}{cc} g & h \\ 0 & 1 \end{array} \right), \eta(W_1) \]

of \( \hat{Q}G_{2q}(A) \) so that \( e, f, g, h \in A \) satisfying

\[ ef = qfe, \quad gh = qhg, \]

the elements \( e - 1, g - 1 \) and \( f, h \) are nilpotent and such that

\[ [e(t + W_1) + 1, \xi(W_1)] = 0, \quad [g(t + W_1) + h, \eta(W_1)] = 0. \quad (85) \]

When the following two sub-sets of the ring \( A \)

\[ \{e, f\} \cup (\text{the sub-set consisting of all the coefficients of the power series } \xi(W_1)), \quad (86) \]

\[ \{g, h\} \cup (\text{the sub-set consisting of all the coefficients of the power series } \eta(W_1)), \quad (87) \]

are mutually commutative, we define the product of \((G, \xi(W_1))\) and \((H, \eta(W_1))\) by

\[ (G, \xi(W_1)) \ast (H, \eta(W_1)) := (GH, \xi(gW_1 + (g - 1)t + h)\eta(W_1)). \]

**Lemma 5.16.** The product \((GH, \xi((g - 1)t + h + gW_1)\eta(W_1))\) is indeed an element of \( \hat{Q}G_{2q}(A) \).
Proof. First of all, we notice that the constant term \((g-1)t+h\) of the linear polynomial in \(W_1\)
\[(g-1)t+h+gW_1\] (88)
is nilpotent so that we can substitute (88) into the power series \(\xi(W_1)\). Therefore
\[
\xi((g-1)t+h+gW_1)\eta(W_1)
\]
is a well-determined element of the power series ring \(A[[W_1]]\). We have seen in Section 4 that if \(\{e, f\}\) and \(\{g, h\}\) are mutually commutative, then the product \(GH\) of matrices \(G, H \in \mathcal{H}_{L^L}(A)\) is in \(\mathcal{H}_{L^L}(A)\). Since
\[
GH = \begin{bmatrix} eg & eh + f \\ 0 & 1 \end{bmatrix},
\]
it remains to show
\[
[eg(t+W_1) + eh + f, \xi(gW_1 + (g-1)t+h)\eta(W_1)] = 0. \tag{89}
\]
The proof of (89) is done in several steps.
First we show
\[
[\xi((g-1)t+h+gW_1), \eta(W_1)] = 0. \tag{90}
\]
This follows, in fact, from the mutual commutativity of the sub-sets (86) and (87) above, and the second equation of (85).
Second, we show
\[
[eg(t+W_1) + eh + f, \xi(gW_1 + (g-1)t+h)] = 0. \tag{91}
\]
To this end, we notice
\[
eg g(t+W_1) + eh + f = e(gW_1 + (g-1)t+h) + et + f. \tag{92}
\]
So we have to show
\[
[e(gW_1 + (g-1)t+h) + et + f, \xi(gW_1 + (g-1)t+h)] = 0. \tag{93}
\]
This follows from the first equation of (85) and the mutual commutativity of the sub-sets (86) and (87).
We prove third
\[
[eg(t+W_1) + eh + f, \eta(W_1)] = 0. \tag{94}
\]
In fact, by mutual commutativity of sub-sets (86) and (87),
\[
[e(t + gW_1) + eh + f, \eta(W_1)] = [e(t + gW_1) + eh, \eta(W_1)]
= [e(t + gW_1 + h), \eta(W_1)],
\]
which is equal to 0 thanks to mutual commutativity of the sub-sets (86) and (87) and the second equality of (85). \qed
One can check associativity for the multiplication by a direct calculation. The co-unit element is given by 

\[(I_2, 1) \in \hat{QG}_{2q}(L^2) .\]

The antipode is given by the formula below. For an element 

\[(G, b(W_1)) = \left( \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right), (b(W_1)) \in \hat{QG}_{2q}(A),\]

we set 

\[ (G, b(W_1))^{-1} := \left( \begin{array}{cc} e^{-1} & -e^{-1}f \\ 0 & 1 \end{array} \right), b(e^{-1}W_1) + (e^{-1}t - e^{-1}f)^{-1} \in \hat{QG}_{2q^{-1}}(A),\]

denote

\[ (G, b(W_1))^{-1} \ast (G, b(W_1)) = (G, b(W_1)) \ast (G, b(W_1))^{-1} = (I_2, 1).\]

**Conjecture 5.17.** If \( q \) is not a root of unity, the injection in Proposition 5.15 is bijective for every \( A \in \text{Ob}(NCAlg/L^2). \)

**Proposition 5.18.** Conjecture 5.17 implies Conjecture 5.7.

**Proof.** Let us assume Conjecture 5.17. Take an element \((e, \xi(W_1)) \in \hat{G}_{II}(A)\) for \(A \in \text{Ob}(Alg/L^2).\) Since \(A\) is commutative, the commutation relation in Lemma 5.13 imposes no condition on \(\xi(W_1), (e, \xi(W_1)) \in \hat{QG}_{2q}(A).\) Conjecture 5.17 says that if \( q \) is not a root of unity, \((e, \xi(W_1))\) arise from an infinitesimal deformation 

\[ t: \mathcal{L} \rightarrow F(\mathbb{Z}, A[[W_1, W_2]] [[X]]) .\]

\[ \square \]

Conjecture 5.17 says that we can identify the functor \( \mathcal{NCF}_{L/k} \) with the quantum formal group \( \hat{QG}_{II,q} \). To be more precise, the argument in the first Example studied in 4 allows us to define a formal \( \mathbb{C} \)-Hopf algebra \( \hat{C}_q \) and hence

\[ \hat{C}_{q,L^2} := \hat{C}_q \hat{\otimes} \mathbb{C}L^2, \]

which is a functor on the category \((NCAlg/L^2)\) so that we have a functorial isomorphism

\[ \hat{C}_{q,L^2}(A) \simeq \hat{QG}_{2q}(A) \quad \text{for every } L^2 \text{-algebra } A \in \text{Ob}(NCAlg/L^2). \]

**Definition 5.19.** We define a functor

\[ \hat{QG}_{II,q} : (NCAlg/L^2) \rightarrow \text{(Set)} \]

by setting

\[ \hat{QG}_{II,q}(A) := \{(e - 1)t + f + eW_1, (b(W_1) - 1)y + b(W_1)W_2 \in A[[W_1, W_2]] \times A[[W_1, W_2]] | e, f \in A \text{ and } b(W_1) \in A[[W_1]], \]

\[ [(e - 1)t + f + eW_1, (b(W_1) - 1)y + b(W_1)W_2] = 0, \]

\[ e - 1, f \text{ and all the coefficients of the power series } b(W_1) - 1 \text{ are nilpotent}\} \]

for every \( A \in \text{Ob}(NCAlg/L^2). \)
Lemma 5.20. The functor $\hat{QG}_{II_q}$ is a quantum formal group. Namely, for two elements

$u := (u_1, u_2) := ((e - 1)t + f + eW_1, (b(W_1) - 1)y + b(W_1)W_2)$,

$v := (v_1, v_2) := ((g - 1)t + h + gW_1, (c(W_1) - 1)y + c(W_1)W_2)$

of $\hat{QG}_{II_q}(A)$, we consider the following two sub-sets of the ring $A$:

1. The sub-set $S_u$ of the coefficients of the two power series $u_1, u_2$ of $u$ and
2. the sub-set $S_v$ of the coefficients of the two power series $v_1, v_2$ in $v$.

If the sets $S_1$ and $S_2$ are mutually commutative, we define their product $u * v$ by

$((eg - 1)t + eh + f + egW_1, (b(gW_1 + (g - 1)t + h)c(W_1) - 1)y + b(gW_1 + (g - 1)t + h)c(W_1)W_2)$.

that is the composite of coordinate transformations

$(W_1, W_2) \mapsto ((e - 1)t + f + eW_1, (b(W_1) - 1)y + b(W_1)W_2)$ and

$(W_1, W_2) \mapsto ((g - 1)t + h + gW_1, (c(W_1) - 1)y + c(W_1)W_2)$,

then the product $u * v$ is an element of $\hat{QG}_{II_q}$. The co-unit is given by the identity transformation of $(W_1, W_2)$.

The quantum formal group $\hat{QG}_{II_q}$ arises as symmetry of the initial conditions of $q$-SI $\sigma$-differential equations.

$\sigma(t) = qt, \quad \sigma(t^\alpha) = q^\alpha t^\alpha,$

$\theta^{(1)}(t) = 1, \quad \theta^{(1)}(t^\alpha) = [\alpha]_q t^\alpha.$

Proposition 5.21. For every algebra $A \in (NCAlg/L^\bullet)$, we have a functorial isomorphism of quantum formal group

$\hat{QG}_{2q}(A) \to \hat{QG}_{II_q}(A)$

sending an element

$\left( \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right), b(W_1) \in \hat{QG}_{2q}(A)$ to $(eW_1 + (e - 1)t + f, b(W_1)W_2 + (b(W_1) - 1)y) \in \hat{QG}_{II_q}(A)$.

Thanks to Propositions 5.15, 5.21 and Conjecture 5.17, we are in the similar situation as in the commutative deformations in 5.2.

Theorem 5.22. We have an inclusion

$NCF_{L/k} \hookrightarrow \hat{QG}_{II_q}$

of functors on the category $(NCAlg/L^\bullet)$ taking values in the category of sets, where

$L/k = (C(t, t^\alpha), \sigma, \theta^\ast)/C.$ (95)
Let us assume Conjecture 5.17. Then the inclusion \((95)\) is bijection so that we can identify the functors
\[ \mathcal{NCF}_{L/k} \simeq \hat{\mathcal{Q}}_G_{IIq}. \]
The quantum formal group \(\hat{\mathcal{Q}}_G_{IIq}\) operates on the functor \(\mathcal{NCF}_{L/k}\) in an appropriate sense, through the initial conditions. (cf The commutativity condition in Lemma 5.20.) So we may say that the quantum formal Galois group
\[ \text{NCInf-gal}(L/k) \simeq \hat{\mathcal{Q}}_G_{IIq}. \]

5.4 Summary on the Galois structures of the field extension \(\mathbb{C}(t, t^\alpha)/\mathbb{C}\)

Let us summarize our results on the field extension \((\mathbb{C}(t, t^\alpha)/\mathbb{C})\).

1. Difference field extension \((\mathbb{C}(t, t^\alpha), \sigma)/\mathbb{C}\). This is a Picard-Vessiot extension with Galois group \(\mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}\).

2. Differential field extension \((\mathbb{C}(t, t^\alpha), d/dt)/\mathbb{C}\). This is not a Picard-Vessiot extension. The Galois group
\[ \text{Inf-gal}(L/k) : (\text{CAlg}/L^\natural) \to (\text{Grp}) \]
is isomorphic to \(\hat{\mathbb{G}}_{m, L^\natural} \times L^\natural \hat{\mathbb{G}}_{a, L^\natural}\), where \(\hat{\mathbb{G}}_{m, L^\natural}\) and \(\hat{\mathbb{G}}_{a, L^\natural}\) are formal completion of the multiplicative group and the additive group. So as a group functor on the category \((\text{CAlg}/L^\natural)\), we have
\[ \hat{\mathbb{G}}_{m, L^\natural}(A) = \{ b \in A | b - 1 \text{ is nilpotent} \}, \]
the group law being the multiplication and
\[ \hat{\mathbb{G}}_{a, L^\natural}(A) = \{ b \in A | b \text{ is nilpotent} \} \]
is the additive group for a commutative \(L^\natural\)-algebra \(A\).

3. Commutative deformation of \(q\)-SI \(\sigma\)-differential extension \((\mathbb{C}(t, t^\alpha), \sigma, \theta^\ast)/\mathbb{C}\). If \(q\) is not a root of unity, \(\text{Inf-gal}(L/k)\) is an infinite dimensional formal group such that we have
\[ 0 \to A[[W_1]]^* \to \text{Inf-gal}(L/k)(A) \to \hat{\mathbb{G}}_m(A) \to 0, \]
where \(A[[W_1]]^*\) denotes the multiplicative group
\[ \{ a \in A[[W_1]] | \text{all the coefficients of power series } a - 1 \text{ are nilpotent} \} \]
modulo Conjecture 5.17.
(4) Non-commutative Galois group. If \( q \) is not a root of unity, the Galois group \( \text{NCInf-gal}(L/k) \) is isomorphic to the quantum formal group \( \hat{QG}_{IIq} \):

\[
\text{NCInf-gal}(L/k) \cong \hat{QG}_{IIq}
\]

modulo Conjecture 5.17.

We should be careful about the group law. Quantum formal group structure in \( \hat{QG}_{IIq} \) coincides with the group structure defined from the initial condition \( s \) as in Remark 4.15.

(5) Let us assume \( q \) is not a root of unity. If we have a \( q \)-difference field extension \((L, \sigma)/(k, \sigma)\) such that \( t \in L \) with \( \sigma(t) = qt \), then we can define the operator \( \theta^{(1)} : L \to L \) by setting

\[
\theta^{(1)}(a) := \frac{\sigma(a) - a}{qt - t}
\]

We also assume the field \( k \) is \( \theta^{(1)} \) invariant. Defining the operator \( \theta^{(n)} : L \to L \) by

\[
\theta^{(0)} = \text{Id} \quad (96)
\]

\[
\theta^{(n)} = \frac{1}{[n]_q!}(\theta^{(1)})^n \quad (97)
\]

for every positive integer \( n \) so that we have a \( q \)-SI \( \sigma \)-differential field extension \((L, \sigma, \theta^*)/(k, \sigma, \theta^*)\).

Here arises a natural question of comparing the Galois groups of the difference field extension \((L, \sigma)/(k, \sigma)\) and \( q \)-SI \( \sigma \)-differential field extension \((L, \sigma, \theta^*)/(k, \sigma, \theta^*)\).

As the \( q \)-SI \( \sigma \)-differential field extension is constructed from the difference field extension in a more or less trivial way, one might imagine that they coincide or they are not much different.

This contradicts Conjecture 5.17. Let us take our example \( \mathbb{C}(t, \log t)/\mathbb{C} \). Assume Conjecture 5.17 is true. Then the Galois group for the \( q \)-SI \( \sigma \)-differential extension is \( \hat{QG}_{IIqL} \) that is infinite dimensional, whereas the Galois group is of the difference field extension is of dimension 2.

6 The Third Example, the field extension \( \mathbb{C}(t, \log t)/\mathbb{C} \)

We assume that \( q \) is a complex number not equal to 0. Let us study the field extension \( L/k := \mathbb{C}(t, \log t)/\mathbb{C} \) from various view points as in Sections 4 and 5.

6.1 \( q \)-difference field extension \( \mathbb{C}(t, \log t)/\mathbb{C} \)

We consider \( q \)-difference operator \( \sigma : L \to L \) such that \( \sigma \) is the \( \mathbb{C} \)-automorphism of the field \( L \) satisfying

\[
\sigma(t) = qt \quad \text{and} \quad \sigma(\log t) = \log t + \log q. \quad (98)
\]
It follows from (98) that if \( q \) is not a root of unity, then the field of constants of the difference field \((\mathbb{C}(t, \log t), \sigma)\) is \( \mathbb{C} \) and hence \((\mathbb{C}(t, \log t), \sigma) / \mathbb{C} \) is a Picard-Vessiot extension with Galois group \( \mathbb{G}_{m \mathbb{C}} \times \mathbb{G}_{a \mathbb{C}} \).

### 6.2 Differential field extension \((\mathbb{C}(t, \log t), d/dt) / \mathbb{C}\)

As we have
\[
\frac{dt}{dt} = 1 \quad \text{and} \quad \frac{d \log t}{dt} = \frac{1}{t},
\]
both differential field extensions \(\mathbb{C}(t, \log t) / \mathbb{C}(t)\) and \(\mathbb{C}(t) / \mathbb{C}\) are Picard-Vessiot extensions with Galois group \( \mathbb{G}_{a \mathbb{C}} \). The differential extension \(\mathbb{C}(t, \log t) / \mathbb{C}\) is not, however, a Picard-Vessiot extension. Therefore, we need general differential Galois theory \[17\] to speak of the Galois group of the differential field extension \(\mathbb{C}(t, \log t) / \mathbb{C}\).

The universal Taylor morphism
\[
\iota: L \to L^\flat[[X]]
\]
sends
\[
\iota(t) = t + X, \quad (99)
\]
\[
\iota(\log t) = \log t + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t} \right)^n \in L^\flat[[X]]. \quad (100)
\]
Writing \( \log t \) by \( y \), we take \( \partial / \partial t, \partial / \partial y \) as a basis of \( L^\flat = \mathbb{C}(t, y)^\flat \)-vector space \( \text{Der}(L^\flat / k^\flat) \) of \( k^\flat \)-derivations of \( L^\flat \). It follows from (99), (100) that
\[
\mathcal{L} = \text{a localization of the algebra } L^\flat \left[ t + X, \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1} \right)^n \right] \subset L^\flat[[X]].
\]

We argue as in \[4.3.1\] and Section 5. For a commutative algebra \( A \in \text{Ob}(\mathcal{C}\text{Alg}/L^\flat) \) and \( \varphi \in \mathcal{F}_{L/k}(A) \), there exist nilpotent elements \( a, b \in A \) such that
\[
\varphi(t + X) = t + W_1 + X + a,
\]
\[
\varphi \left( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1} \right)^n \right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1 + a} \right)^n + b.
\]

Therefore we arrived at the dynamical system
\[
\begin{cases}
  t, \\ y,
\end{cases} \mapsto \begin{cases}
  \phi(t) = t + X + W_1 + a, \\ \phi(y) = y + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1 + a} \right)^n + b.
\end{cases} \quad (101)
\]

In terms of initial conditions, dynamical system (101) reads
\[
^t(t, y) \mapsto ^t(t + a, y + b),
\]
where \( a, b \) are nilpotent elements of \( A \). So we conclude
\[
\text{Inf-gal } (L/k)(A) = \hat{\mathbb{G}}_a(A) \times \hat{\mathbb{G}}_a(A)
\]
for every commutative \( L^\flat \)-algebra \( A \). Consequently we get
\[
\text{Inf-gal } (L/k) \simeq (\hat{\mathbb{G}}_{a, \mathbb{C}} \times \hat{\mathbb{G}}_{a, \mathbb{C}}) \otimes_{\mathbb{C}} L^\flat.
\]
6.3 \( q\)-SI \( \sigma\)-differential field extension \((C(t, \log t), \sigma, \sigma^{-1}, \theta^*)/\mathbb{C}\)

For the automorphism \(\sigma: C(t, \log t) \rightarrow C(t, \log t)\) in Subsection 6.1 we set

\[
\theta^{(0)} = \text{Id}_{C(t, \log t)} \quad \text{and} \quad \theta^{(1)} = \frac{\sigma - \text{Id}_{C(t, \log t)}}{(q - 1)t}
\]

so that \(\theta^{(1)}: C(t, \log t) \rightarrow C(t, \log t)\) is a \(\mathbb{C}\)-linear map. We further introduce

\[
\theta^{(i)} := \frac{1}{[i]_q!} (\theta^{(1)})^i: C(t, \log t) \rightarrow C(t, \log t)
\]

that is a \(\mathbb{C}\)-linear map for \(i = 1, 2, 3, \cdots\). Hence if we denote the set \(\{\theta^{(i)}\}_{i \in \mathbb{N}}\) by \(\theta^*\), then \((C(t, \log t), \sigma, \sigma^{-1}, \theta^*)\) is a \(q\)-SI \(\sigma\)-differential ring.

The universal Hopf morphism

\[\iota: C(t, \log t) \rightarrow F(\mathbb{Z}, L^\sharp)[[X]]\]

sends, by Proposition 3.11, \(t\) and \(y\) respectively to

\[
\iota(t) = tQ + X,
\]

\[
\iota(y) = y + (\log q)Z + \frac{\log q}{q - 1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (tQ)^{-n}
\]

that we identify with

\[
y + W_2 + (\log q)Z + \frac{\log q}{q - 1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (t + W_1)^{-n} Q^{-n}
\]

that is an element of \(F(\mathbb{Z}, L^\sharp[[W_1, W_2]])[[X]]\), where we set

\[
Z := \begin{bmatrix} \cdots & -1 & 0 & 1 & 2 & \cdots \\ \cdots & -1 & 0 & 1 & 2 & \cdots \end{bmatrix} \in F(\mathbb{Z}, \mathbb{Z}).
\]

In particular we have

\[
\frac{\partial \iota(y)}{\partial W_2} = 1.
\] (102)

We identify further \(t \in L^\sharp\) with \(t + W_1 \in L^\sharp[[W_1, W_2]]\) and hence

\[
\sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (tQ)^{-n} \in F(\mathbb{Z}, L^\sharp)[[X]]
\]

with

\[
\sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (t + W_1)^{-n} Q^{-n} \in F(\mathbb{Z}, L^\sharp[[W_1, W_2]])[[X]].
\]
6.3.1 Commutative deformations $F_{L/k}$ for $(C(t, \log t), \sigma, \theta^*)/\mathbb{C}$

Now the argument of Section 5 allows us to describe infinitesimal deformations on the category of commutative $L^\natural$-algebras $(\text{CAlg}/L^\natural)$. Let $\varphi: L \to F(Z, A[[W_1, W_2]][[X]])$ be an infinitesimal deformation of the canonical morphism $\iota: L \to F(Z, A[[W_1, W_2]][[X]])$ for $A \in \text{Ob}(\text{CAlg}/L^\natural)$. Then there exist $e \in A$ such that $e - 1$ is nilpotent and such that

$$ \varphi((t + W_1)Q + X) = e(t + W_1)Q + X, $$

as we learned in the First Example. To determine the image $Z := \varphi(y)$, we argue as in the Second Example. We have

$$ \sigma(y) = y + \log q, \quad (103) $$

$$ \theta^{(1)}(y) = \frac{\log q}{(q - 1)t}. \quad (104) $$

Since the deformation $\varphi$ is $q$-SI $\sigma$-differential morphism, the two equations above give us relations

$$ \hat{\Sigma}(Z) = Z + \log q, \quad (105) $$

$$ \Theta^{(1)}(Z) = \frac{\log q}{(q - 1)(t + W_1)eQ + X}. \quad (106) $$

We determine the expansion of the element $Z$:

$$ Z = \sum_{n=0}^{\infty} X^n a_n \in F(cZ, A[[W_1, W_2]][[X]]) $$

so that

$$ a_n \in F(Z, A[[W_1, W_2]]) \text{ for every } n \in \mathbb{N}. $$

It follows from (102) and (105)

$$ a_0 = y + W_2 + b(W_1) + (\log q)N \in F(Z, A[[W_1, W_2]]), $$

where $b(W_1)$ is an element of $A[[W_1]]$ such that all the coefficients of the power series $b(W_1)$ are nilpotent. On the other hand (103) tells us

$$ a_1 = \frac{\log q}{q - 1} \frac{1}{(t + W_1)eQ}, \quad (107) $$

$$ a_{n+1} = -\left[n\right]_q \frac{1}{(n + 1)q(t + W_1)eQq^n} a_n \quad \text{for } n \geq 1. \quad (108) $$

Hence

$$ a_n = (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (t + W_1)^{-n} (eQ)^{-n} \quad \text{for } n \geq 1. \quad (109) $$

So we get

$$ Z = y + W_2 + b(W_1) + (\log q)N $$

$$ + \frac{\log q}{q - 1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q q^{n(n-1)/2}} (t + W_1)^{-n} (eQ)^{-n}, \quad (110) $$

which is an element of $F(Z, A[[W_1, W_2]][[X]])$.  

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Proposition 6.1. For every commutative $L^3$-algebra $A \in \text{Ob}(CAlg/L^3)$, we have a functorial injection
\[
\mathcal{CF}_{L/k}(A) \to \hat{G}_3(A) := \{(e, b(W_1)) \in A \times A[[W_1]] \mid e-1 \text{ and all the coefficients of } b(W_1) \text{ are nilpotent}\}
\]
sending an element \( \varphi \in \mathcal{CF}_{L/k}(A) \) to \((e, b(W_1)) \in \hat{G}_3(A)\).

Conjecture 6.2. If \( q \) is not a root of unity, then the injection in Proposition 6.1 is a bijection.

\( \hat{G}_3 \) is a group functor on \((CAlg/L^3)\). In fact, for \( A \in \text{Ob}(Alg/L^3) \), we define the product of two elements
\[
(e, b(W_1)), (g, c(W_1)) \in \hat{G}_3(A)
\]
by
\[
(e, b(W_1)) \ast (g, c(W_1)) := (eg, b(gW_1 + (g-1)t) + c(W_1)).
\]
Then, the product is, in fact, an element of \( \hat{G}_3(A) \), the product is associative, the unit element of the group law is \((I_2, 0) \in \hat{G}_3(A)\) and the inverse \((e, b(W_1))^{-1} = (e^{-1}, -b(e^{-1}W_1 + (e^{-1}-1)t))\).

So if Conjecture 6.2 is true, we have a splitting exact sequence
\[
0 \to A[[W_1]]_+ \to \text{Inf-gal } (L/k)(A) \to \hat{G}_{mL^3}(A) \to 1,
\]
where \( A[[W_1]]_+ \) denote the additive group of the power series in \( A[[W_1]] \) whose coefficients are nilpotent element.

6.3.2 Non-commutative deformations \( NCF_{L/k} \) for \((C(t, \log t), \sigma, \sigma^{-1}, \theta^*)/\mathbb{C}\)

The arguments in Section 5 allows us to prove analogous results on the non-commutative deformations for the \( q \)-SI \( \sigma \)-differential field extension \((C(t, \log t), \sigma, \sigma^{-1}, \theta^*)/\mathbb{C}\). We write assertions without giving detailed proofs. For, since the proofs are same, it is easy to find complete proofs.

As in the Second Example, doing calculations \((107), ..., (110)\) in the non-commutative case, we can determine the set \( NCF_{(C(t, \log t), \sigma, \theta^*)/\mathbb{C}}(A) \).

Proposition 6.3. For a not necessarily commutative $L^3$-algebra $A \in \text{Ob}(NCAlg/L^3)$, we can describe an infinitesimal deformation
\[
\varphi \in NCF_{(C(t, \log t), \sigma, \theta^*)/\mathbb{C}}(A).
\]
Namely putting \( y := \log t \), we have
\[
\varphi(t) = (e(t + W_1) + f)Q + X,
\]
\[
\varphi(y) = y + W_2 + b(W_1) + (\log q)Z
\]
\[
+ \frac{1}{q-1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n] q^n [n-1]/2} [e(t + W_1) + f]^{-n} Q^{-n}
\]
that are elements of \( F(\mathbb{Z}, A[[W_1, W_2]])([X]) \), where \( e, f \in A \) and \( b(W_1) \in A[[W_1]] \) satisfying the following conditions.
(1) ef = qfe.
(2) e − 1 and f are nilpotent elements of A.
(3) All the coefficients of the power series b(W_1) are nilpotent.
(4) [b(W_1), e(t + W_1) + f] = 0.

The commutativity condition (3) comes from the commutativity relation between the elements t and y = log t in the field L.

**Definition 6.4.** We introduce a functor

\[ \hat{Q}G_{3q} : (NCAlg/L^3) \to (Set) \]

by setting

\[ \hat{Q}G_{3q}(A) := \{ (G, \xi(W_1)) \in \hat{H}_{\Delta L^3}(A) \times A[[W_1]] \mid \text{(1) } G = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \in \hat{H}_{\Delta}(A) \text{ so that } ef = qfe, \ e - 1, f \in A \text{ are nilpotent. (2) All the coefficients of } \xi(W_1) \text{ are nilpotent. (3) } [e(t + W_1) + f, \xi(W_1)] = 0. \} \]

\[ \hat{Q}G_{3q} \] is a quantum formal group. Namely, for

\[ (G, \xi(W_1)), (H, \eta(W_1)) \in \hat{Q}G_{3q}(A) \]

such that the two sub-sets

\[ \{ \text{all the entries of matrix } G, \text{ all the coefficients of the power series } \xi(W_1) \}, \]

\[ \{ \text{all the entries of matrix } H, \text{ all the coefficients of the power series } \eta(W_1) \} \]

of A are mutually commutative, we define their product by

\[ (G, \xi(W_1)) \ast (H, \eta(W_1)) := (GH, \xi(gW_1 + (g - 1)t + h) + \eta(W_1)), \]

where

\[ H = \begin{bmatrix} g & h \\ 0 & 1 \end{bmatrix}. \]

Then, the argument of Lemma 5.10 shows that the product of two elements is, in fact, an element in the set \( \hat{Q}G_{3q}(A) \) and the product is associative. The co-unit element is\( (I_2, 0) \in \hat{Q}G_{3q}(A) \). The inverse

\[ (G, \xi(W_1))^{-1} = (G^{-1}, -\xi(e^{-1}W_1 + (e^{-1} - 1)t - e^{-1}f) \in \hat{Q}G_{3q^{-1}}(A), \]

where

\[ G = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}. \]
Proposition 6.5. We have a functorial injection
\[
\mathcal{NF}_{L/k}(A) \to \hat{QG}_{3q}(A)
\]
that sends \( \varphi \in \mathcal{NF}_{L/k}(A) \) to \( \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \) \( b(W_1) \). Here
\[
\varphi((t + W_1)Q + X) = (e(t + W_1) + f)Q + X, \tag{111}
\]
\[
\varphi(\iota(y)) = \varphi(y + W_2 + \log qN + \frac{\log q}{q - 1} \\
\times \sum_{n=1}^{\infty} X^n((-1)^{n+1} \frac{1}{n! q^{n(n-1)/2}} ((t + W_1)^{-n} Q^n))) \tag{112}
\]
\[
= y + W_2 + b(W_1) + \log qN + \frac{\log q}{q - 1} \\
\times \sum_{n=1}^{\infty} X^n((-1)^{n+1} \frac{1}{n! q^{n(n-1)/2}} (e(t + W_1) + f)^{-n} Q^{-n}). \tag{113}
\]
We also have a Conjecture.

Conjecture 6.6. If q is not a root of unity, then the injection in Proposition 6.5 is a bijection. So
\[
\mathcal{NF}_{L/k} \simeq \hat{QG}_{3q}.
\]

Remark 6.7. The argument in 5.3 allows us to prove that Conjecture 6.6 implies Conjecture 5.17.

We can also define the quantum formal group \( \hat{QG}_{3q} \) in terms of non-commutative coordinate transformations as in the Second Example.

Definition 6.8. We define a functor
\[
\hat{QG}_{IIIq} : (NCalg/L^2) \to (Set)
\]
by setting
\[
\hat{QG}_{IIIq}(A) := \{ (eW_1 + (e-1)t + f, W_2 + b(W_1)) \in A[[W_1, W_2]] \times A[[W_1, W_2]] \\
| e - 1, f, and all the coefficients of the power series b(W_1) are nilpotent satisfying ef = qfe, [eW_1 + (e-1)t + f, W_2 + b(W_1)] = 0 \}. 
\]

We regard an element
\[
\varphi = (\varphi_1(W_1, W_2), \varphi_2(W_1, W_2)) \in \hat{QG}_{IIIq}(A)
\]
as an infinitesimal coordinate transformation \( \Phi \)
\[
(W_1, W_2) \mapsto (\varphi_1(W), \varphi_2(W))
\]
with non-commutative coefficients. The product in the quantum formal group \( \hat{QG}_{IIIq} \) is the composition of coordinate transformations if they satisfy a commutation relation so that the product is defined. To be more concrete, let

\[
(eW_1 + (e - 1)t + f, W_2 + b(W_1)) \text{ and } (gW_1 + (g - 1)t + h, W_2 + c(W_1))
\]

be two elements of \( \hat{QG}_{IIIq}(A) \) such that the following two sub-sets of the ring \( A \) is mutually commutative:

1. \( \{e, f\} \cup \text{the set of coefficients of the power series } b(W_1) \),
2. \( \{e, f\} \cup \text{the set of coefficients of the power series } bc(W_1) \),

then the product is

\[
(eW_1 + (e - 1)t + f, W_2 + b(W_1)) \ast (gW_1 + (g - 1)t + h, W_2 + c(W_1)) = (egW_1 + (eg - 1)t + eh + f, W_2 + b(gW_1 + (g - 1)t + h) + c(W_1))
\]

which is certainly an element of \( \hat{QG}_{IIIq}(A) \).

Though we reversed the procedure, the quantum formal group \( \hat{QG}_{IIIq} \) arises from \( \hat{QG}_{IIIq} \) and we arrived at the last object as a natural extension of Lie-Ritt functor in \([17]\) of coordinate transformations in the space of initial conditions.

**Proposition 6.9.** For every algebra \( A \in (NCAlg/L^\natural) \), we have a functorial isomorphism of quantum formal groups

\[
\hat{QG}_{3q}(A) \rightarrow \hat{QG}_{IIIq}(A)
\]

sending an element

\[
\left[\begin{array}{c} e \\ 0 \\ 1 \end{array}\right], b(W_1)) \in \hat{QG}_{3q}(A) \text{ to } (eW_1 + (e - 1)t + f, W_2 + b(W_1)) \in \hat{QG}_{IIIq}(A).
\]

Looking at Propositions 6.5, 6.9 and Conjecture 6.6, we find that we are in the same situation as in 5.3, where we studied non-commutative deformations of the Second Example.

**Theorem 6.10.** We have an inclusion

\[
NCF_{L/k} \hookrightarrow \hat{QG}_{IIIq}
\]

of functors on the category \((NCAlg/L^\natural)\) taking values in the category of sets, where

\[
L/k = (\mathbb{C}(t, \log t), \sigma, \theta^*)/\mathbb{C}.
\]

If we assume Conjecture 6.6, then the inclusion \([114]\) is bijection so that we can identify the functors

\[
NCF_{L/k} \simeq \hat{QG}_{IIIq}.
\]

The quantum formal group \( \hat{QG}_{IIIq} \) operates on the functor \( NCF_{L/k} \) in an appropriate sense, through the initial conditions. (cf The commutativity condition in Definition 6.8.)

So we may say that the quantum formal Galois group

\[
\text{NCInf-gal}(L/k) \simeq \hat{QG}_{IIIq}.
\]
6.4 Summary on the Galois structures of the field extension \( \mathbb{C}(t, \log t)/\mathbb{C} \)

Let us summarize our results on the field extension \( \mathbb{C}(t, \log t)/\mathbb{C} \).

1. Difference field extension \( \mathbb{C}(t, \log t), \sigma)/\mathbb{C} \). This is a Picard-Vessiot extension with Galois group \( \mathbb{G}_{a \mathbb{C}} \times \mathbb{G}_{a \mathbb{C}} \).

2. Differential field extension \( \mathbb{C}(t, \log t), d/dt)/\mathbb{C} \). This is not a Picard-Vessiot extension. The Galois group

\[
\text{Inf-gal} (L/k): (\mathcal{CAlg}/L^2) \to (\text{Grp})
\]

is isomorphic to \( \hat{\mathbb{G}}_{aL^1} \times \hat{\mathbb{G}}_{aL^1} \), where \( \hat{\mathbb{G}}_{aL^1} \) is the formal completion of the additive group. So as a group functor on the category \( (\mathcal{CAlg}/L^2) \), we have

\[
\hat{\mathbb{G}}_{aL^1}(A) = \{ b \in A | b \text{ is nilpotent} \},
\]

the group law being the addition and hence

\[
\text{Inf-gal} (L/k)(A) = \{(a, b) | a, b \text{ are nilpotent elements of } A \}
\]

for a commutative \( L^2 \)-algebra \( A \).

3. Commutative deformations of \( q \)-SI \( \sigma \)-differential extension \( \mathbb{C}(t, \log t), \sigma, \sigma^{-1}, \theta^\ast)/\mathbb{C} \). If \( q \) is not a root of unity, \( \text{Inf-gal} (L/C) \) is an infinite dimensional formal group such that we have a splitting sequence

\[
0 \to A[[W_1]]_+ \to \text{Inf-gal} (L/k)(A) \to \hat{\mathbb{G}}_m(A) \to 0,
\]

where \( A[[W_1]]_+ \) denotes the additive group

\[
\{ a \in A[[W_1]] | \text{ all the coefficients of power series } a \text{ are nilpotent } \}
\]

modulo Conjecture 5.17.

4. Non-commutative Galois group. If \( q \) is not a root of unity, the Quantum Galois group \( \text{NCInf-gal} (L/k) \) is isomorphic to a quantum formal group \( \overline{QG}_{IIIq} \):

\[
\text{NCInf-gal} (L/k) \simeq \overline{QG}_{IIIq}.
\]

modulo Conjecture 6.6.

We should be careful about the group law. Quantum formal group structure in \( \overline{QG}_{IIIq} \) coincides with the group structure defined from the initial conditions as in Proposition 6.9.
7 General scope of quantized Galois theory for $q$-SI $\sigma$-differential field extensions

After we worked with three examples of $q$-SI $\sigma$-differential field extensions

$$C(t)/C, \quad C(t, t^\alpha)/C \quad \text{and} \quad C(t, \log t)/C,$$

there arises naturally, in our mind, the idea of formulating general quantized Galois theory for $q$-SI $\sigma$-differential field extensions. The simplest differential Example 2.3 is also very inspiring. As we are going to see, it seems to work.

7.1 Outline of the theory

Let $L/k$ be a $q$-SI $\sigma$-differential field extension such that the abstract field extension $L^\natural/k^\natural$ is of finite type. Galois theory for $q$-SI $\sigma$-differential filed extensions is a particular case of Hopf Galois theory in Section 3. So as we learned in 3.1.6 we have the universal Hopf morphism

$$\iota : L \rightarrow F(Z, L^\natural)[[X]].$$

We choose a basis

$$\{D_1, D_2, \ldots, D_d\}$$

of mutually commutative derivations of the $L^\natural$-vector space $\text{Der}(L^\natural/k^\natural)$ of $k^\natural$-derivations of $L^\natural$. We constructed the Galois hull $\mathcal{L}/\mathcal{K}$ in Definition 3.13. So we have the canonical morphism

$$\iota : \mathcal{L} \rightarrow F(Z, L^\natural[[W_1, W_2, \ldots, W_d]])[[X]]. \quad (115)$$

The rings $\mathcal{L}$ and $\mathcal{K}$ are invariant under the set of operators

$$\mathcal{D} := \{\hat{\Sigma}, \Theta^*, \frac{\partial}{\partial W_i} (1 \leq i \leq d)\} \quad (116)$$

on $F(Z, L^\natural[[W]])[[X]].$

In general, the Galois hull $\mathcal{L}/\mathcal{K}$ is not commutative. So we measure it by infinitesimal deformations of the canonical morphism (115) over the category $(\text{NCAlg}/L^\natural)$ of not necessarily commutative $L^\natural$-algebras. We set in Definition 3.15

$$\mathcal{NCF}_{L/k}(A) = \{\varphi : \mathcal{L} \rightarrow F(Z, A[[W]])[[X]] \mid \varphi \text{ is an infinitesimal deformation } /\mathcal{K} \text{ compatible with } \mathcal{D} \text{ of canonical morphism (115)}\}$$

so that we got the functor

$$\mathcal{NCF}_{L/k} : (\text{NCAlg}/L^\natural) \rightarrow (\text{Set}).$$

Now we compare the differential case and $q$-SI $\sigma$-differential case. to understand their similarity and difference.

(1) Differential case
(a) The Galois hull $L/K$ is an extension of commutative algebras.
(b) It is sufficient to consider commutative deformation functor $F_{L/k}$ of the Galois hull $L/K$ over the category $(\text{CA}lg/L^2)$ of commutative $L^2$-algebras.
(c) The Galois group $\text{Inf-gal}(L/k)$ is a kind of generalization of algebraic group. In fact, it is at least a group functor on the category $(\text{Ca}lg/L^2)$.
(d) Indeed the group functor $\text{Inf-gal}(L/k)$ is given as the functor of automorphisms of the Galois hull $L/K$.

(2) $q$-SI $\sigma$-differential case

(a) Galois hull $L/K$ is not always an extension of commutative algebras.
(b) We have to consider the non-commutative deformation functor $\mathcal{NC}F_{L/k}$ over the category $(\text{NC}A\text{lg}/L^2)$ of not necessarily commutative $L^2$-algebras.
(c) The Galois group should be a quantum group that we cannot interpret in terms of group functor.

The comparison above shows that we have to find a counterpart of (d) in the $q$-SI $\sigma$-differential case. The three examples suggest the following solution.

**Solution that we propose.** Let $y_1, y_2, \cdots, y_d$ be a transcendence basis of the abstract field extension $L^2/k^2$. We set by

$$\iota(y_i) = Y_i(W_1, W_2, \cdots, W_d; X) \in F(Z, L^2[[W]])[[X]] \text{ for every } 1 \leq i \leq d.$$ 

**Questions 7.1.** (1) For an $L^2$-algebra $A \in \text{Ob}(\text{NC}A\text{lg}/L^2)$, let

$$f : L \to F(Z, A[[W]])[[X]]$$

be an infinitesimal deformation of the canonical morphism $\iota$. Then there exist an infinitesimal coordinate transformation

$$\Phi = (\varphi_1(W), \varphi_2(W), \cdots, \varphi_d(W)) \in A[[W_1, W_2, \cdots, W_d]]^d$$

with coefficients in the not necessarily commutative algebra $A$ such that

$$f(Y_i) = Y_i(\Phi(W); X) \text{ for every } 1 \leq i \leq d.$$ 

(2) Assume that Question (1) is affirmatively answered. Then we have a functorial morphism

$$\mathcal{NC}F_{L/k}(A) \to$$

$$\{ \Phi \in A[[W]]^d | W \mapsto \Phi(W) \text{ is an infinitesimal coordinate transformation} \} \quad (117)$$

sending $f$ to $\Phi$ using the notation of (1). We set

$$\text{QInf-gal}(L/k)(A) := \text{the image of map (117)}$$

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so that
\[ \text{QInf-gal}(L/k) : (NCA_{L/k}) \to (\text{Set}) \]
is a functor. Our second question is if the functor \( \text{QInf-gal}(L/k) \) is a quantum formal group.

(3) Assume that Question (1) has an affirmative answer. Since the identity transformation is in \( \text{QInf-gal}(L/k) \), Question (2) reduces to the following concrete question. Let \( f, g \) be elements of \( NCF_{L/k}(A) \) and let \( \Phi \) and \( \Psi \) be the corresponding coordinate transformations to \( f \) and \( g \) respectively. If the set of the coefficients of \( \Phi \) and the set of the coefficients of \( \Psi \) is mutually commutative, then does the composite coordinate transformation \( \Phi \circ \Psi \) arise from an infinitesimal deformation \( h \in NCF_{L/k}(A) \)?

In view of Corollary 10.9, the universal deformation or the universal coaction seems to solve the Questions. It seems that we are very close to the solutions.

Part II
Quantization of Picard-Vessiot theory

8 Introduction to the second part

Keeping the notation of the first part, we denote by \( C \) a field of characteristic 0.

We believed for a long time that it was impossible to quantize Picard-Vessiot theory, Galois theory for linear difference or differential equations. Namely, there was no Galois theory for linear difference-differential equations, of which the Galois group is a quantum group that is, in general, neither commutative nor co-commutative. Our mistake came from a misunderstanding of preceding works, Hardouin [5] and Masuoka and Yanagawa [12]. They studied linear \( q \)-\( SI \) \( \sigma \)-differential equations, \( qsi \) equations for short, under two assumptions on \( qsi \) base field \( K \) and \( qsi \) module \( M \):

(1) The base field \( K \) contains \( C(t) \).
(2) On the \( K[\sigma, \sigma^{-1}, \theta^*] \)-module \( M \) the equality
\[ \theta^{(1)} = \frac{1}{(q-1)t}(\sigma - \text{Id}_M). \]

holds. Under these conditions, their Picard-Vessiot extension is realized in the category of commutative \( qsi \) algebras. The second assumption seems too restrictive as clearly explained in [12]. If we drop one of these conditions, there are many linear \( qsi \) equations whose Picard-Vessiot ring is not commutative and the Galois group is a quantum group that is neither commutative nor co-commutative.

We analyze only one favorite example [122] over the base field \( C \) in detail, which is equivalent to the non-linear equation in Section 4. We add three more example in Section
Looking at these examples, the reader’s imagination would go far away, as Cartier and Masuoka did it.

In the favorite example, we have a Picard-Vessiot ring $R$ that is non-commutative, simple $qsi$ ring (Observation 10.3 and Lemma 10.4). The Picard-Vessiot ring $R$ is a torsor of a quantum group (Observation 10.6). We have the Galois correspondence (Observation 10.14) and non-commutative Tannaka theory (Observation 10.13). We prove the uniqueness of the Picard-Vessiot ring in Section 11.

We are grateful to A. Masuoka and K. Amano for teaching us their Picard-Vessiot theory and clarifying our idea.

9 Field extension $C(t)/C$ from classical and quantum viewpoints

In Section 4, we studied a non-linear $q$-SI $\sigma$-differential equation, which we call $qsi$ equation for short,

$$\theta^{(1)}(y) = 1, \quad \sigma(y) = qy, \quad \sigma^{-1}(y) = q^{-1}y,$$

(118)

where $q$ is an element of the field $C$ not equal to 0 nor 1. Let $t$ be a variable over the complex number field $C$. We assume to simplify the situation that $q$ is not a root of unity. We denote by $\sigma : C(t) \to C(t)$ the $C$-automorphism of the field $C(t)$ of rational functions sending $t$ to $qt$. We introduce the $C$-linear operator $\theta^{(1)} : C(t) \to C(t)$ by

$$\theta^{(1)} (f(t)) := \frac{f(qt) - f(t)}{(q - 1)t} \quad \text{for every } f(t) \in C(t).$$

We set

$$\theta^{(m)} := \begin{cases} 1 \text{d}_{C(t)} & \text{for } m = 0 \\ \frac{1}{|m|!} \left(\theta^{(1)}\right)^m & \text{for } m = 1, 2, \cdots. \end{cases}$$

As we assume that $q$ is not a root of unity, the number $|m| q$ in the formula is not equal to 0 and hence the formula determines the family $\theta^* = \{\theta^{(i)} | i \in \mathbb{N}\}$ of operators. So $(C(t), \sigma, \sigma^{-1}, \theta^*)$ is a $qsi$ field. See Section 4 and $y = t$ is a solution for system (118).

The system (118) is non-linear in the sense that for two solutions $y_1, y_2$ of (118), a $C$-linear combination $c_1y_1 + c_2y_2$ ($c_1, c_2 \in C$) is not a solution of the system in general.

However, the system is very close to a linear system. To illustrate this, let us look at the differential field extension $(C(t), \partial_t)/(C, \partial_t)$, where we denote the derivation $d/dt$ by $\partial_t$. The variable $t \in C(t)$ satisfies a non-linear differential equation

$$\partial_t t - 1 = 0.$$

(119)

The differential field extension $(C(t), \partial_t)/(C, \partial_t)$ is, however, the Picard-Vessiot extension for the linear differential equation

$$\partial_t^2 t = 0.$$

(120)

To understand the relation between (119) and (120), we introduce the 2-dimensional $C$-vector space

$$E := Ct \oplus C \subset C[t].$$

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The vector space $E$ is closed under the action of the derivation $\partial_t$ so that $E$ is a $C[\partial_t]$-module. Solving the differential equation associated with the $C[\partial_t]$-module $E$ is to find a differential algebra $(L, \partial_t)/C$ and a $C[\partial_t]$-module morphism

$$\varphi: E \to L.$$ 

Writing $\varphi(t) = f_1, \varphi(1) = f_2$ that are elements of $L$, we have

$$\begin{bmatrix} \partial_t f_1 \\ \partial_t f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$ 

Since $\partial_t 1 = 1$, $\partial_t 0 = 0$, in the differential field $(C(t), \partial_t)/C$, we find two solutions $^t(1, 1)$ and $^t(1, 0)$ that are two column vectors in $C(t)^2$ satisfying

$$\partial_t \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} \neq 0.$$ 

Namely, $C(t)/C$ is the Picard-Vessiot extension for linear differential equation \((121)\).

We can argue similarly for the $qsi$ field extension $(C(t), \sigma, \sigma^{-1}, \theta^*)/C$. You will find a subtle difference between the differential case and the $qsi$ case. Quantization of Galois group arises from here.

Let us set

$$M = Ct \oplus C \subset C[t]$$

that is a $C[\sigma, \sigma^{-1}, \theta^*]$-module. Maybe to avoid the confusion that you might have in Remark \(9.3\) below, writing $m_1 = t$ and $m_2 = 1$, we had better define formally

$$M = Cm_1 \oplus Cm_2$$

as a $C$-vector space on which $\sigma$ and $\theta^{(1)}$ operate by

$$\begin{bmatrix} \sigma(m_1) \\ \sigma(m_2) \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad \begin{bmatrix} \sigma^{-1}(m_1) \\ \sigma^{-1}(m_2) \end{bmatrix} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad \begin{bmatrix} \theta^{(1)}(m_1) \\ \theta^{(1)}(m_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}. \quad(122)$$

Since in $(122)$ the first equation is equivalent to the second, we consider the first and third equations. Solving $C[\sigma, \sigma^{-1}, \theta^*]$-module $M$ is equivalent to find elements $f_1, f_2$ in a $qsi$ algebra $(A, \sigma, \sigma^{-1}, \theta^*)$ satisfying the system of linear difference-differential equation

$$\begin{bmatrix} \sigma(f_1) \\ \sigma(f_2) \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \begin{bmatrix} \theta^{(1)}(f_1) \\ \theta^{(1)}(f_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad(123)$$

in the $qsi$ algebra $A$.

**Lemma 9.1.** Let $(L, \sigma, \sigma^{-1}, \theta^*)/C$ be a $qsi$ field extension. If a $2 \times 2$ matrix $Y = (y_{ij}) \in M_2(L)$ satisfies a system of difference-differential equations

$$\sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \quad \text{and} \quad \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y, \quad(124)$$

then $\det Y = 0$. 

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Proof. It follows from (124)\[ \sigma(y_{11}) = qy_{11}, \sigma(y_{12}) = qy_{12}, \sigma(y_{21}) = y_{21}, \sigma(y_{22}) = y_{22} \] and \[ \theta^{(1)}(y_{11}) = y_{21}, \theta^{(1)}(y_{12}) = y_{22}, \theta^{(1)}(y_{21}) = 0, \theta^{(1)}(y_{22}) = 0. \] It follows from (125) and (126)\[ \theta^{(1)}(y_{11}y_{12}) = \theta^{(1)}(y_{11})y_{12} + \sigma(y_{11})\theta^{(1)}(y_{12}) = y_{21}y_{12} + qy_{11}y_{22} \] and similarly\[ \theta^{(1)}(y_{12}y_{11}) = y_{22}y_{11} + qy_{12}y_{21}. \] As \( y_{11}y_{12} = y_{12}y_{11} \), equating (127) and (128), we get\[ (q - 1)(y_{11}y_{22} - y_{12}y_{21}) = 0 \] so that \( \det Y = 0. \)

Corollary 9.2. Let \((K, \sigma, \sigma^{-1}, \theta^*)\) be a qsi field over \(C\). Then the qsi linear equation\[ \sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \text{ and } \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y \] has no qsi Picard-Vessiot extension \(L/K\) in the following sense. There exists a solution \( Y \in \text{GL}_2(L) \) to (129) such that the abstract field \(L\) is generated by the entries of the matrix \(Y\) over \(K\). The field of constants of the qsi over-field \(L\) coincides with the field of constants of the base field \(K\).

Proof. This is a consequence of Lemma 9.1.

Remark 9.3. Masuoka pointed out that Corollary 9.2 is compatible with Remark 4.4 and Theorem 4.7 of Hardouin [5]. See also Masuoka and Yanagawa [13]. They assure the existence of Picard-Vessiot extension for a \(K[\sigma, \sigma^{-1}, \theta^*]\)-module \(N\) if the following two conditions are satisfied;

(1) The qsi base field \(K\) contains \((C(t), \sigma, \sigma^{-1}, \theta^*)\),

(2) The operation of \(\sigma\) and \(\theta^{(1)}\) on the module \(N\) as well as on the base field \(K\), satisfy the relation\[ \theta^{(1)} = \frac{1}{(q - 1)t}(\sigma - \text{Id}_N). \]

In fact, even if the base field \(K\) contains \((C(t), \sigma, \sigma^{-1}, \theta^*)\), in \(K \otimes_C M\), we have by definition of the \(C[\sigma, \sigma^{-1}, \theta^*]\)-module \(M\),\[ \theta^{(1)}(m_1) = m_2 \neq \frac{1}{t}m_1 = \frac{1}{(q - 1)t}(\sigma(m_1) - m_1). \]

So \(K \otimes_C M\) does not satisfy the second condition above.
10 Quantum normalization of \( (C(t), \sigma, \sigma^{-1}, \theta^*)/C \)

We started from the \( qsi \) field extension \( C(t)/C \). The column vector \( \i(t, 1) \in C(t)^2 \) is a solution to the system of equations (122), i.e. we have

\[
\begin{bmatrix}
\sigma(t) \\
\sigma(1)
\end{bmatrix} = \begin{bmatrix}
q & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix},
\begin{bmatrix}
\theta^{(1)}(t) \\
\theta^{(1)}(1)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}.
\]

By applying to the \( qsi \) field extension \( (C(t), \sigma, \sigma^{-1}, \theta^*)/C \), the general procedure of [17], [7] that is believed to lead us to the normalization, we arrived at the Galois hull \( L = C(t)[Q, Q^{-1}, X]_{alg} \) modulo localization. This suggests an appropriate model of the non-commutative \( qsi \) ring extension \( C(t)[Q, Q^{-1}]_{alg}/C \) is a (maybe the) \( qsi \) Picard-Vessiot ring of the system of equations (122). More precisely, \( Q \) is a variable over \( C(t) \) satisfying the commutation relation

\[ Qt = qtQ. \]

We understand \( R = C(t, Q, Q^{-1})_{alg} \) as a sub-ring of

\[ S = C[[t, Q]][t^{-1}, Q^{-1}]. \]

We know that the usage of \( \langle \rangle_{alg} \) is more logical than \( [\ ] \) but as it is too heavy, we do not adopt it. The ring \( S \) is a non-commutative \( qsi \) algebra by setting

\[ \sigma(Q) = qQ, \theta^{(1)}(Q) = 0 \text{ and } \sigma(t) = qt, \theta^{(1)}(t) = 1 \]

and \( R = C(t, Q, Q^{-1})_{alg} \) is a \( qsi \) sub-algebra. Thus we get a \( qsi \) ring extension

\[ (R, \sigma, \sigma^{-1}, \theta^*)/C = (C(t, Q, Q^{-1})_{alg}, \sigma, \sigma^{-1}, \theta^*)/C. \]

We examine that \( (R, \sigma, \sigma^{-1}, \theta^*)/C \) is a non-commutative Picard-Vessiot ring for the systems of equations (122).

**Observation 10.1.** The \( C[\sigma, \sigma^{-1}, \theta^*] \)-module \( M \) has two solutions in the \( qsi \) ring \( R \) linearly independent over \( C \). In fact, setting

\[ Y := \begin{bmatrix} Q & t \\ 0 & 1 \end{bmatrix} \in M_2(R), \]

we have

\[ \sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \text{ and } \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y. \]

So the column vectors \( ^t(Q, 0), ^t(t, 1) \in R^2 \) are \( C \)-linearly independent solution of the system of equations (122).

**Observation 10.2.** The ring \( R = C(t, Q, Q^{-1})_{alg} \) has no zero-divisors. We can consider the ring \( K \) of total fractions of \( R = C(t, Q, Q^{-1})_{alg} \).

**Proof.** In fact, we have \( R \subset C[[t, Q]][t^{-1}, Q^{-1}] \). In the latter ring every non-zero element is invertible. □
**Observation 10.3.** The ring of qsi constants $C_K$ coincide with $C$. The ring of $\theta^*$ constants of $C[[t, Q]][t^{-1}, Q^{-1}]$ is $C(Q)$. Moreover as we assume that $q$ is not a root of unity, the ring of $\sigma$-constants of $C(Q)$ is equal to $C$.

**Lemma 10.4.** The non-commutative qsi algebra $R$ is simple. There is no qsi bilateral ideal of $R$ except for the zero-ideal and $R$.

*Proof.* Let $I$ be a non-zero qsi bilateral ideal of $R$. We take an element

$$0 \neq f := a_0 + ta_1 + \cdots + t^n a_n \in I,$$

where $a_i \in C[Q, Q^{-1}]$ for $0 \leq i \leq n$. We may assume $a_n \neq 0$. Applying $\theta^{(n)}$ to the element $f$, we conclude that $0 \neq a_n \in C[Q, Q^{-1}]$ is in the ideal $I$. Multiplying a monomial $bQ^i$ with $b \in C$, we find a polynomial $1 + b_1 Q + \cdots + b_s Q^s \in C[Q]$ with $b_s \neq 0$ is in the ideal $I$. We show that $1$ is in $I$ by induction on $s$. If $s = 0$, then there is nothing to prove. Assume that the assertion is proved for $s \leq m$. We have to show the assertion for $s = m + 1$. Then, since $Q^i$ is an eigenvector of the operator $\sigma$ with eigenvalue $q^i$ for $i \in \mathbb{N}$,

$$\frac{1}{q^{m+1}}(q^{m+1} = \sigma(h)) = 1 + c_1 Q + \cdots + c_m Q^m \in C[Q]$$

is an element of $I$ and by induction hypothesis $1$ is in the ideal $I$. \qed

**Observation 10.5.** The extension $R/C$ trivializes the $C[\sigma, \sigma^{-1}, \theta^*]$-module $M$. Namely, there exist constants $c_1, c_2 \in R \otimes C M$ such that

$$R \otimes C M \simeq Rc_1 \oplus Rc_2.$$ 

*Proof.* In fact, it is sufficient to set

$$c_1 := Q^{-1} m_1 - Q^{-1} m_2, \quad c_2 := m_2.$$

Then

$$\sigma(c_1) = c_1, \quad \sigma(c_2) = c_2, \quad \theta^{(1)}(c_2) = 0$$

and

$$\theta^{(1)}(c_1) = q^{-1} Q^{-1} \theta^{(1)}(m_1) - q^{-1} Q^{-1} m_2 = q^{-1} Q^{-1} m_2 - q^{-1} Q^{-1} m_2 = 0.$$

So we have an $(R, \sigma, \sigma^{-1}, \theta^*)$-module isomorphism $R \otimes C M \simeq Rc_1 \oplus Rc_2$. \qed

**Observation 10.6.** The Hopf algebra $\mathcal{H}_q = C(u, u^{-1}, v)$ with $uv = q vu$ co-acts from right on the non-commutative algebra $R$. Namely, we have an algebra morphism

$$R \to R \otimes C \mathcal{H}_q$$

sending

$$t \mapsto t \otimes 1 + Q \otimes v, \quad Q \mapsto Q \otimes u, \quad Q^{-1} \mapsto Q^{-1} \otimes u^{-1}.$$

Morphism (132) is compatible with $C[\sigma, \sigma^{-1}, \theta^{(1)}]$-module structures, where $\sigma, \sigma^{-1}$ and $\theta^{(1)}$ operate on the Hopf algebra $\mathcal{H}_q$ trivially.
We can prove the assertion of Observation 10.6 by a simple direct calculation, which is very much unsatisfactory. For, we are eager to know where the Hopf algebra $\mathcal{H}_q$ comes from. We answer this question in two steps:

1. Characterization of the non-commutative algebra $\mathcal{H}_q$.
2. Origin of the co-multiplication structure on the Hopf algebra $\mathcal{H}_q$.

We answer question (1) in Corollary 10.9 and question (2) in Observation 10.12. To this end, we admit the algebra structure of $\mathcal{H}_q$ and characterize it.

Proof of Observation. To show that we have the morphism $R \to R \otimes_C \mathcal{H}_q$ in the Observation, we had better notice Corollary 10.8 below and apply it to $T = R \otimes_C \mathcal{H}_q$ and the inclusion morphism

$$\varphi_0 : R \to R \otimes_C \mathcal{H}_q, \quad a \mapsto a \otimes 1.$$ 

Let us first fix some notation. For a not necessarily commutative $C$-qsi algebra $T$ and for a morphism $\varphi : R \to T$ of qsi algebras over $C$, we set

$$\varphi(Y) = \begin{bmatrix} \varphi(Q) & \varphi(t) \\ 0 & 1 \end{bmatrix}.$$ 

So $\varphi(Y)$ is an invertible element in the matrix ring $M_2(T)$, the inverse being given by

$$\varphi(Y)^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix},$$

where we set $a = \varphi(Q)$ and $b = \varphi(t)$ so that we have

$$\varphi(Y) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$ 

We have seen the following Lemma in Section 10.

**Lemma 10.7.** For a not necessarily commutative $C$-qsi algebra $T$, there exists a $C$-qsi algebra morphism $\varphi : R \to T$ such that

$$\varphi(Y) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

if and only if the following two conditions are satisfied:

1. We have a commutation relation

$$ab = qba,$$

2. The elements $a, b$ satisfies difference differential equations

$$\sigma(a) = qa, \quad \theta^{(1)}(a) = 0, \quad \sigma(b) = qb, \quad \theta^{(1)}(b) = 1,$$
(3) the element \( a \) is invertible in the ring \( T \) or equivalently the matrix
\[
\begin{bmatrix}
a & b \\
0 & 1
\end{bmatrix}
\]
is invertible in the ring \( M_2(T) \).

**Corollary 10.8.** Let \( \varphi : R \to T \) be a qsi algebra morphism over \( C \). Using the notation above, let
\[
H' = \begin{bmatrix}
e' & f' \\
0 & 1
\end{bmatrix} \in M_2(C_T)
\]
be an invertible element in the matrix ring \( M_2(C_T) \) satisfying the following two conditions.

1. \( e'f' = qf'e' \) and the element \( e' \) is invertible in the ring \( C_T \) of constants of \( T \).
2. The set \( \{ e', f' \} \) and the set of entries of the matrix \( \varphi(Y) \) are mutually commutative.

Then, there exists a qsi algebra morphism \( \psi \in \text{Hom}_{qsi}(R, T) \) over \( C \) such that
\[\psi(Y) = \varphi(Y)H'.\]

**Proof.** By Lemma [10.7], the matrix
\[
\varphi(Y) = \begin{bmatrix}
a & b \\
0 & 1
\end{bmatrix}
\]
satisfies conditions of Lemma [10.7]. This together with the assumption (1) and (2) in this Corollary implies that the matrix
\[\varphi(Y)H' \]
satisfies the conditions of Lemma [10.7]. Now the assertion follows from Lemma [10.7].

In particular if we take \( T = R \otimes_C \mathcal{H}_q \) and
\[
H' = \begin{bmatrix}
e & f \\
0 & 1
\end{bmatrix}
\]
and
\[
\varphi_0(Y) = Y = \begin{bmatrix}
Q & t \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
Q \otimes 1 & t \otimes 1 \\
0 & 1
\end{bmatrix},
\]
then the conditions of Corollary are satisfied and we get the morphism \( \mathcal{R} \to \mathcal{R} \otimes_C \mathcal{H}_q \) in the Observation [10.6].

The following Corollary describes in the quantum case, that with respect to the right co-action of the Hopf algebra \( \mathcal{H}_q \), the Picard-Vessiot ring \( R \) is a principal homogeneous space.

It also characterizes the algebra \( \mathcal{H}_q \). Namely, if we consider a functor
\[F : (NCAlg/C) \to (Set)\]
on the category of not necessarily commutative \( C \)-algebras defined by
\[F(S) = \text{Hom}_{qsi}(R, R \otimes_C S) \text{ for } S \in \text{ob}(NCAlg/C),\]
then the functor \( F \) is representable by the algebra \( \mathcal{H}_q \).
Corollary 10.9. For an object $S$ of the category $(\text{NCAlg}/C)$, we have

$$\text{Hom}_{\text{qsi}}(R, R \otimes_C S) \simeq \text{Hom}_{\text{alg}}(\mathcal{H}_q, S),$$

where the left hand side denotes the set of qsi algebra morphisms over $C$ and the right hand side is the set of $C$-algebra morphisms.

Proof. If we notice $C_R \otimes_C S = S$ and take as $\varphi : R \to R \otimes_C S$ the canonical inclusion

$$\varphi_0 : R \to R \otimes_C R, \quad a \mapsto a \otimes 1,$$

it follows from Corollary 10.8 that we have a map

$$\text{Hom}_{\text{alg}}(\mathcal{H}_q, S) \to \text{Hom}_{\text{qsi}}(R, R \otimes_C S),$$

that sends $\pi \in \text{Hom}_{\text{alg}}(\mathcal{H}_q, S)$ to $\psi \in \text{Hom}_{\text{qsi}}(R, R \otimes_C S)$ such that

$$\psi(Y) = \varphi_0(Y) \begin{bmatrix} \pi(e) & \pi(f) \\ 0 & 1 \end{bmatrix}.$$

To get the mapping of the other direction, let $\psi : R \to R \otimes_C \mathcal{H}_q$ be a qsi morphism over $C$. Then using the morphism $\varphi_0$ above, since both $\varphi_0(Y)$ and $\psi(Y)$ are solutions to the linear qsi equations (10.12), an easy calculation shows that the entries of the matrix

$$H' := \varphi(Y)^{-1}\psi(Y) \in M_2(R \otimes_C \mathcal{H}_q)$$

are constants so that

$$H' \in M_2(S) \subset M_2(R \otimes_C S).$$

We single out a Sub-lemma because we later use the same argument.

Sublemma 10.10. We have the commutation relation

$$e'f' = qf'e'$$

among the entries of the matrix

$$H' := \begin{bmatrix} e' & f' \\ 0 & 1 \end{bmatrix}.$$

Proof of Sub-lemma. Let us set

$$\varphi_0(Y) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \psi(Y) = \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix}.$$

So we have

$$ab = qba \quad \quad a'b' = qb'a' \quad \quad (133)$$

$$a' = ae' \quad \quad b' = af' + b. \quad \quad (134)$$

Since the set $\{e', f'\} \subset S$ and $\{a, b\} \subset R$ are mutually commutative in $\mathcal{R} \otimes_C S$, substituting equations (134) into the second equation of (133) and then using the first equation of (133), Sublemma follows.
By Sub-lemma, we get a morphism $\pi : \mathcal{H}_q \to S$ sending $e$ to $e'$ and $f$ to $f'$. So $\psi \mapsto \pi \psi$ gives the mapping of the other direction.

\textbf{Remark 10.11.} As we are in the non-commutative case, the converse of the Corollary is false. In fact, for two qsi morphisms $\varphi, \psi : R \to T$ over $C$, let us set

$$\varphi(Y) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad \psi(Y) = \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}.$$ 

It follows from difference differential equations

$$\sigma\left( \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad \theta^{(1)}\left( \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

that the entries of the matrix

$$\begin{bmatrix} e' & f' \\ 0 & 1 \end{bmatrix} := \varphi(Y)^{-1} \psi(Y) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^{-1}c & a^{-1}d - a^{-1}b \\ 0 & 1 \end{bmatrix}$$

are constants. Namely

$$\begin{bmatrix} e' & f' \\ 0 & 1 \end{bmatrix} \in M_2(C_T).$$

So equivalently

$$\psi(Y) = \varphi(Y) \begin{bmatrix} e' & f' \\ 0 & 1 \end{bmatrix}.$$ 

The entries of the matrices do not necessarily satisfy the commutation relations in the Corollary.

For algebras $A, S \in \text{ob}(NCAlg/C)$, we set

$$A(S) := \text{Hom}_{alg}(A, S)$$

that is the set of $C$-algebra morphisms.

\textbf{Observation 10.12} (Origin of co-multiplication of the Hopf algebra $\mathcal{H}_q$). The co-multiplication $\Delta : \mathcal{H}_q \to \mathcal{H}_q \otimes_C \mathcal{H}_q$ comes from the multiplications of matrices. More precisely, to construct an algebra morphism $\Delta : \mathcal{H}_q \to \mathcal{H}_q \otimes_C \mathcal{H}_q$, it is sufficient to give a functorial morphism

$$\mathcal{H}_q \otimes_C \mathcal{H}_q(S) \to \mathcal{H}_q(S) \quad \text{for } S \in \text{ob}(NCAlg/C). \quad (135)$$

An element of $\mathcal{H}_q \otimes_C \mathcal{H}_q(S)$ being given, it determines a pair $(\pi_1, \pi_2)$ of morphisms $\pi_1, \pi_2 : \mathcal{H}_q \to S$ such that the images $\pi_1(\mathcal{H}_q)$ and $\pi_2(\mathcal{H}_q)$ are mutually commutative. This condition is equivalent to mutually commutativity of the set of the entries $\{e'_1, f'_1\}$ and $\{e'_2, f'_2\}$ of the matrices

$$H'_1 := \begin{bmatrix} e'_1 & f'_1 \\ 0 & 1 \end{bmatrix} := \begin{bmatrix} \pi_1(e) & \pi_1(f) \\ 0 & 1 \end{bmatrix}, \quad H'_2 := \begin{bmatrix} e'_2 & f'_2 \\ 0 & 1 \end{bmatrix} := \begin{bmatrix} \pi_2(e) & \pi_2(f) \\ 0 & 1 \end{bmatrix}.\]
We show that there exists a morphism \( \pi_3 : \mathcal{S}_q \to S \) such that
\[
H_1' H_2' = \begin{bmatrix}
\pi_3(e) & \pi_3(f) \\
0 & 1
\end{bmatrix}.
\]
In fact, by Corollary 10.8, there exists a morphism \( \psi_1 : R \to R \otimes_C S \) such that
\[
\psi_1(Y) = \varphi_0(Y) H_1'.
\]
Then since the entries of \( H_2' \) and the union
\[
(\text{the entries of } \varphi_0(Y)) \cup (\text{the entries of } H_1')
\]
are mutually commutative and consequently the entries of \( H_1' \) and the entries of the product \( \varphi_0(Y) H_1' \) are mutually commutative, by Corollary 10.8 there exists a morphism \( \psi_2 : R \to R \otimes_C S \) such that
\[
\psi_2(Y) = (\varphi_0(Y) H_1') H_2' = \varphi_0(Y) (H_1' H_2').
\]
So if we note that the entries of \( H_1' H_2' \) and the entries of the matrix \( \varphi_0(Y) \) are mutually commutative, writing
\[
H_1' H_2' = \begin{bmatrix}
e'_3 & f'_3 \\
e'_3 & 1
\end{bmatrix},
\]
the argument of the proof of Sub-lemma 10.11 shows us that, we have \( e'_3 f'_3 = qf'_3 e'_3 \). Hence there exists a morphism \( \pi_3 : \mathcal{S}_q \to S \) sending \( e \) to \( e'_3 \) and \( f \) to \( f'_3 \). Now the mapping \((\pi_1, \pi 2) \mapsto \pi_3 \) defines the morphism \( \{\pi_3\} \).

We study category \( C(C[\sigma, \sigma^{-1}, \theta^*]) \) of left \( C[\sigma, \sigma^{-1}, \theta^*] \)-modules that are finite dimensional as \( C \)-vector spaces. We notice first the internal homomorphism
\[
\text{Hom}_C((M_1, \sigma_1, \theta_1^*), (M_2, \sigma_2, \theta_2^*)) \in \text{Ob}(C(C[\sigma, \sigma^{-1}, \theta^*]))
\]
events for two objects \((M_1, \sigma_1, \theta_1^*), (M_2, \sigma_2, \theta_2^*) \in \text{Ob}(C(C[\sigma, \sigma^{-1}, \theta^*]))\). In fact, let \( N := \text{Hom}(M_1, M_2) \) be the set of \( C \)-linear maps from \( M_1 \) to \( M_2 \). It sufficient to consider two \( C \)-linear maps
\[
\sigma_h : N \to N \text{ and } \theta_h^{(1)} : N \to N
\]
given by
\[
\sigma_h(f) := \sigma_2 \circ f \circ \sigma_1^{-1} \text{ and } \theta_h^{(1)}(f) := -(\sigma_2f) \circ \theta_1^{(1)} + \theta_1^{(1)} \circ f.
\]
So we have \( q \sigma_h \circ \theta_h^{(1)} = \theta_h^{(1)} \circ \sigma \). Since \( q \) is not a root of unity, we define \( \theta_h^{(m)} \) in an evident manner
\[
\theta_h^{(m)} = \begin{cases} 
\text{Id}_N, & \text{for } m = 0, \\
\frac{1}{[m]} (\theta_h^{(1)})^m, & \text{for } m \geq 1.
\end{cases}
\]
Since \( C[\sigma, \sigma^{-1}, \theta^*] \) is a Hopf algebra, for two objects \( M_1, M_2 \in \text{Ob}(C(C[\sigma, \sigma^{-1}, \theta^*])) \) the tensor product \( M_1 \otimes_C M_2 \) is defined as an object of \( C(C[\sigma, \sigma^{-1}, \theta^*]) \). However, as \( C[\sigma \theta^*] \) is not co-commutative, we do not have, in general, \( M_1 \otimes_C M_2 \simeq M_2 \otimes_C M_1 \). Taking the forgetful functor
\[
\omega : C(C[\sigma, \sigma^{-1}, \theta^*]) \to \text{category of } C \text{-vector spaces},
\]
we get a non-commutative Tannaka category.
Observation 10.13. The non-commutative Tannaka category \{\{M\}\} generated by the \(C[\sigma, \sigma^{-1}, \theta^*]\)-module \(M\) is equivalent to the category \(\mathcal{C}(\mathfrak{H}_q)\) of right \(\mathfrak{H}_q\)-co-modules that are finite dimensional as \(C\)-vector space.

Proof. We owe this proof to Masuoka and Amano. Our Picard-Vessiot ring \(R\) is not commutative. However, by Observations 10.3, 10.5 and Lemma 10.4, we can apply the arguments of the classical differential Picard-Vessiot theory according to Amano, Masuoka and Takeuchi [2], [1]. We first show that every \(C[\sigma, \sigma^{-1}, \theta^*]\)-module \(N \in \text{Ob}(\{\{M\}\})\) is trivialized over \(R\). Then, the functor

\[
\phi: \{\{M\}\} \to \mathcal{C}(\mathfrak{H}_q)
\]

is given by

\[
\phi(N) = \text{Constants of } C[\sigma, \sigma^{-1}, \theta^*]\text{-module } R \otimes_C N \text{ for } N \in \text{Ob}(\{\{M\}\}).
\]

In fact, the Hopf algebra \(\mathfrak{H}_q\) co-acts on \(R\) and so on the trivial \(R\)-module \(R \otimes_C N\) and consequently on the vector space of constants of \(R \otimes_C N\).

Observation 10.14. We have an imperfect Galois correspondence between the elements of the two sets.

1. The set of quotient \(C\)-Hopf algebras of \(\mathfrak{H}_q\):

\(\mathfrak{H}_q, \mathfrak{H}_q/J, C\)

with the sequences of the quotient morphisms

\(\mathfrak{H}_q \to \mathfrak{H}_q/J \to C\),

where \(J\) is the bilateral ideal of the Hopf algebra \(\mathfrak{H}_q\) generated by \(v\).

2. The sub-set of intermediate qsi division rings of \(K/C\):

\(C, C(t), K\)

with inclusions

\(C \subset C(t) \subset K\).

The intermediate qsi division rings \(C(Q)\) is not written as the ring of constants of a quotient Hopf algebra so that our Galois correspondence is imperfect.

The extensions

\(K/C\) and \(K/C(t)\)

are qsi Picard-Vessiot extensions with Galois groups

\(\text{Gal}(K/C) \simeq \mathfrak{H}_q, \quad \text{Gal}(K/C(t)) \simeq C[\mathbb{G}_a C]\)

Here we denote by \(C[G]\) the Hopf algebra of the coordinate ring of an affine group scheme \(G\) over \(C\).
11 On the uniqueness of the Picard-Vessiot ring

We show that our Picard-Vessiot ring $R/C$ is unique. Let us start with a Lemma on the $R$-module $R^n$ of column vectors for not necessarily commutative $C$-algebra $R$. The Lemma is trivial if the ring is commutative. We give a proof of the Lemma so that the reader could understand the logical structure of the whole argument.

Lemma 11.1. Let $Y = (y_1, y_2, \ldots, y_n) \in M_n(R)$ be an $n \times n$-matrix with entries in the ring $R$ so that the $y_i$'s are column vectors for $1 \leq i \leq n$. The following conditions (1), (2) and (3) on the matrix $Y$ are equivalent.

(1) (1.1) The column vectors $y_i$'s ($1 \leq i \leq n$) generate the right $R$-module $R^n$.
   (1.2) The column vectors $y_i$'s ($1 \leq i \leq n$) are right $R$-linearly independent or they are linearly independent elements in the right $R$-module $R^n$.

(2) We have the direct sum decomposition of the right $R$-module

$$R^n = \bigoplus_{i=1}^{n} y_i R.$$  \hspace{1cm} (138)

(3) The matrix $Y$ is invertible in the ring $M_n(R)$.

Proof. The equivalence of (1) and (2) is evident. We prove that (1) implies (3). In fact, let us set

$$e_1 := t(1, 0, \ldots, 0), e_2 := t(0, 1, 0, \ldots, 0), \ldots, e_n := t(0, 0, \ldots, 0, 1) \in R^n.$$  \hspace{1cm} (139)

If we assume (1), since the vectors $e_i$'s that are elements of $R^n$ are right $R$-linear combinations of the column vectors $y_j$'s, there exists a matrix $Z \in M_n(R)$ such that

$$YZ = I_n$$ \hspace{1cm} (138)

or the matrix $Y$ has a right inverse in $M_n(R)$. Multiplying $Y$ on (138) from left, we get

$$YZY = Y.$$  \hspace{1cm} \Box

So we have

$$Y(ZY - I_n) = 0.$$ \hspace{1cm} (139)

We notice here that (1.2) implies that if we have

$$Yu = 0 \text{ for } u = (u_1, u_2, \ldots, u_n) \in R^n,$$

then $u = 0$. Therefore (139) implies $ZY - I_n = 0$ and consequently $ZY = I_n$. So $Z$ is the inverse of $Y$ and the condition (3) is satisfied. We now assume the condition (3). Then for every element $v \in R^n$ a linear equation

$$Yx = u,$$

has the unique solution $x = Y^{-1}v \in R^n$ so that (2) is satisfied.
Dually we can prove the following result for the left $R$-module $^tR^n$ of row vectors.

**Corollary 11.2.** Let $Y = ^t(y_1, y_2, \ldots, y_n) \in M_n(R)$ be an $n \times n$-matrix with entries in the ring $R$ so that the $y_i$’s are column vectors. The following conditions (1), (2) and (3) on the matrix $Y$ are equivalent.

1. (1.1) The row vectors $^t y_i$’s ($1 \leq i \leq n$) generate the left $R$-module $^tR^n$.
2. (1.2) The row vectors $^t y_i$’s ($1 \leq i \leq n$) are left $R$-linearly independent.
3. We have the direct sum decomposition of the left $R$-module

$$^tR^n = \bigoplus_{i=1}^{n} R^t y_i.$$

(3) The matrix $Y$ is invertible in the ring $M_n(R)$.

Let $M$ be a left $C[\sigma, \sigma^{-1}, \theta^*]$-module that is of finite dimension $n$ as a $C$-vector space. Let $\{m_1, m_2, \ldots, m_n\}$ be a basis of the $C$-vector space $M$. Setting $m = ^t(m_1, m_2, \ldots, m_n)$, there exist matrices $A \in \text{GL}_n(C)$, $B \in M_n(C)$ satisfying

$$\sigma(m) = Am, \quad \text{and} \quad \theta^{(1)}(m) = Bm. \quad (140)$$

As we have seen in Section 10, the left $C[\sigma, \sigma^{-1}, \theta^*]$-module $M$ defines a system of $q$-SI $\sigma$-differential equation

$$\sigma(y) = Ay, \quad \theta^{(1)}(y) = By \quad \text{for an unknown column vector } y \text{ of length } n. \quad (141)$$

We are interested in solutions $y \in R^n$ for a $q$-SI $\sigma$-differential algebra $R$ over $C$.

**Definition 11.3.** For $qsi$ algebra $R$ over $C$, we say that a set $\{y_1, y_2, \ldots, y_n\}$ of solutions to (141) so that $y_i \in R^n$ for $1 \leq i \leq n$, is a fundamental system of solutions to (141) if the matrix $Y = (y_1, y_2, \ldots, y_n) \in M_n(R)$ satisfies the equivalent conditions of Lemma 11.1.

The dual to a fundamental system is a trivializing matrix of the $q$-SI $\sigma$-differential module $M$.

**Definition 11.4.** We assume that the $q$-SI $\sigma$-differential module $R \otimes_C M$ is trivialized over a $C$-$q$-SI $\sigma$-differential ring $R$. Namely, there exist elements

$$c_1, c_2, \ldots, c_n \in R \otimes_C M$$

such that

$$\sigma(c_i) = c_i, \quad \theta^{(1)}(c_i) = 0 \quad \text{for every } 1 \leq i \leq n$$

and such that we have $R$-module decomposition

$$R \otimes_C M = \bigoplus_{i=1}^{n} Rc_i.$$
So writing the elements \( c_i \)'s as a left \( \mathcal{R} \)-linear combination of the basis 
\[
\{ m_1, m_2, \ldots, m_n \},
\]
we get a matrix \( Y \in M_n(\mathcal{R}) \) such that 
\[
^t(c_1, c_2, \ldots, c_n) = Y^t(m_1, m_2, \ldots, m_n).
\]
We call the matrix \( Y \in M_n(\mathcal{R}) \) a trivializing matrix of \( q \)-SI \( \sigma \)-differential module \( M \) over \( \mathcal{R} \).

**Lemma 11.5.** A trivializing matrix over \( \mathcal{R} \) is invertible in the matrix ring \( M_n(\mathcal{R}) \).

**Proof.** It is sufficient to follow the argument of the proof of Corollary 11.2. \( \square \)

Now we make clear the relation between fundamental system and trivializing matrix.

**Proposition 11.6.** The following four conditions on an invertible matrix \( Y \in M_n(\mathcal{R}) \) are equivalent. We denote \( Y^{-1} \) by \( Z \) or \( Y = Z^{-1} \).

1. The matrix \( Y \) satisfies \( q \)-SI \( \sigma \)-differential equations

\[
\sigma(Y) = AY, \quad \theta^{(1)}(Y) = BY,
\]

\( A, B \) being the matrices in (140).

2. The matrix \( Y \) is a fundamental system of solutions of \( M \).

3. The matrix \( Z \) satisfies \( q \)-SI \( \sigma \)-differential equations

\[
\sigma(Z) = ZA^{-1}, \quad \theta^{(1)}(Z) = -ZA^{-1}B
\]

4. The matrix \( Z \) is a trivializing matrix for \( M \) over \( \mathcal{R} \).

**Proof.** The equivalence of conditions (1) and (2) follows from Lemma 11.1 and Definition 11.3. To prove the equivalence of (3) and (4), we set
\[
^t(c_1, c_2, \ldots, c_n) := Z^t(m_1, m_2, \ldots, m_n),
\]
where the \( m_i \)'s are the basis of \( M \) chosen above, so that
\[
c_i = \sum_{i=1}^n z_{il} m_l \text{ for every } 1 \leq i \leq n.
\]

It is convenient to introduce
\[
c := ^t(c_1, c_2, \ldots, c_n), \text{ and } m := ^t(m_1, m_2, \ldots, m_n).
\]

So we have
\[
c = Zm.
\]
Now we assume Condition (3) and show Condition (4). To this end, we prove that the $c_i$’s that are elements of $\mathcal{R} \otimes_{\mathbb{C}} M$, are constants. In fact, if we apply $\sigma$ to (144), it follows from the first equation in (142),

$$\sigma(c) = \sigma(Z)\sigma(m)$$

$$= (ZA^{-1})(Am)$$

$$= Zm$$

$$= c.$$  

Namely $\sigma(c) = c$. Now we apply $\theta^{(1)}$ to (143) to get

$$\theta^{(1)}(c) = \theta^{(1)}(Z)m + \sigma(Z)\theta^{(1)}(m)$$

$$= (-ZA^{-1}B)m + (ZA^{-1})Bm$$

$$= 0.$$  

So $\theta^{(1)}(c) = 0$ and $c$ is a constant. Hence $Z$ is a trivializing matrix by Definition 11.4 and the argument in the proof of Corollary 11.2. Conversely, we start from Condition (4). If we recall

$$c := Zm,$$  

then, as we assume Condition (4), $c$ is a constant. Applying $\sigma$ and $\theta^{(1)}$ to (145), we get Condition (3).

It remains to show the equivalence of (1) and (3). Let us assume (1) and show (3). If we apply the automorphism $\sigma$ to the equality $ZY = I_n$, the first equality in (142) implies the first equality of (143). On the other hand, applying $\theta^{(1)}$ to the equality $ZY = I_n$, we get

$$\theta^{(1)}(Z)Y + \sigma(Z)\theta^{(1)}(Y) = 0.$$  

It follows from equation (142)

$$\theta^{(1)}(Z)Y + ZA^{-1}BY = 0.$$  

Since the matrix $Y$ is invertible, we conclude

$$\theta^{(1)}Z = -ZA^{-1}B.$$  

So the matrix $Z$ satisfies Condition (3). The proof of the converse that Condition (3) implies (1) is similar. Applying first $\sigma$ and then $\theta^{(1)}$ to $ZY = I_n$, we immediately get Condition (1).

We are ready to characterize the Picard-Vessiot ring $\mathcal{R}/C$. Besides the properties we mentioned above, we have a $C$-morphism or a $C$-valued point of the abstract ring $\mathcal{R}$

$$\mathcal{R} \to C \text{ sending } Q^{\pm 1} \mapsto 1, \quad X \mapsto 0.$$  

We sometimes call it a $C$-rational point.

**Lemma 11.7.** Let $\mathcal{R}$ be a simple $q$-$SI$ $\sigma$-differential algebra over $C$. If the abstract algebra $\mathcal{R}$ has a $C$-valued point, then the ring of constants of $\mathcal{R}$ coincides with $C$.  

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Proof. Assume to the contrary. Then there would be a constant \( f \in \mathcal{R} \) that is not an element of \( C \). Let 
\[
\varphi : \mathcal{R} \rightarrow C
\]
be the \( C \)-valued point. We set \( c := \varphi(f) \) that is an element of \( C \). So the element \( f - c \neq 0 \) is a constant of \( q \)-SI \( \sigma \)-differential algebra \( \mathcal{R} \). Therefore the bilateral ideal \( I \) generated by \( f - c \) is a \( q \)-SI \( \sigma \)-differential bilateral ideal of \( \mathcal{R} \) because the ideal \( I \) is generated by the constant \( f - c \). As the ideal \( I \) contains \( f - c \neq 0 \), the simplicity of \( \mathcal{R} \) implies \( I = \mathcal{R} \). So there would be a positive integer \( n \) and elements \( a_i, b_i \in \mathcal{R} \) for \( 0 \leq i \leq n \) such that 
\[
\sum_{i=1}^{n} a_i (f - c) b_i = 1. \tag{148}
\]
Applying the morphism \( \varphi \), we would have \( 0 = 1 \) in \( C \) by \( \varphi(f) = c \) which is a contradiction. \( \Box \)

So far in this section, we studied general \( C[\sigma, \sigma^{-1}, \theta] \)-module \( M \). From now on, we come back to the \( C[\sigma, \sigma^{-1}, \theta] \)-module \( M \) in Section 10 so that writing \( m = t(m_1, m_2) \),
\[
M = Cm_1 \oplus Cm_2, \tag{149}
\]
\[
\sigma(m) = Am, \quad \theta^{(1)}(m) = Bm, \tag{150}
\]
where 
\[
A = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Theorem 11.8. Using the notation above, we can characterize the Picard-Vessiot ring \( R/C \) for \( M \) constructed in Section 10 in the following way.

Let \( \mathcal{R}/C \) be a \( q \)-SI \( \sigma \)-differential extension satisfying the following conditions.

1. There exists a fundamental system of solutions \( \mathbf{Y} \in M_2(\mathcal{R}) \) for \( M \) such that 
\[
\mathcal{R} = C[\mathbf{Y}, \mathbf{Y}^{-1}]_{\text{alg}}.
\]

2. The \( q \)-SI \( \sigma \)-differential algebra \( \mathcal{R} \) is simple.

3. There exists a \( C \)-rational point of the abstract \( C \)-algebra \( \mathcal{R}^\sharp \).

Then the \( q \)-SI \( \sigma \)-differential algebra \( \mathcal{R} \) is \( C \)-isomorphic to the Picard-Vessiot ring \( R \).

Proof. Let us express \( \mathbf{Y} \) in the matrix form:
\[
\mathbf{Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathcal{R}).
\]
Hence by \( \theta^{(1)} \), the matrix \( \mathbf{Y} \) satisfies
\[
\begin{bmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} \theta^{(1)}(a) & \theta^{(1)}(b) \\ \theta^{(1)}(c) & \theta^{(1)}(d) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]
or to be more concrete

\[
\begin{align*}
\sigma(a) &= qa, & \theta^{(1)}(a) &= c, & \sigma(c) &= c, & \theta^{(1)}(c) &= 0, \\
\sigma(b) &= qb, & \theta^{(1)}(b) &= d, & \sigma(d) &= d, & \theta^{(1)}(d) &= 0.
\end{align*}
\]

(151) (152)

It follows from (151) and (152) that \(c, d\) are constants of \(R\). By Lemma 11.7 and assumption (2) on \(R\), the ring \(C_R\) of constants of \(R\) coincides with \(C\). So \(c, d\) are complex numbers and hence by replacing the column vectors of the matrix \(Y\) by their appropriate \(C\)-linear combinations if necessary, we may assume that \(c = 0\) and \(d = 1\) so that

\[
Y = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.
\]

Consequently the set of equations (151) and (152) reduces to

\[
\begin{align*}
\sigma(a) &= qa, & \theta^{(1)}(a) &= 0, & \sigma(b) &= qb, & \theta^{(1)}(b) &= 1.
\end{align*}
\]

(153)

Since the matrix \(Y\) is invertible, \(a\) is an invertible element of the ring \(R\). We show that \(f := qa^{-1}b - ba^{-1} \in R\) is a constant. In fact, since the complex number \(q\) is in the center of \(R\), it follows from (153) that

\[
\begin{align*}
\sigma(f) &= q\sigma(a^{-1})\sigma(b) - \sigma(b)\sigma(a^{-1}) \\
&= qa^{-1}q^{-1}qb - qba^{-1}q^{-1} \\
&= qa^{-1}b - ba^{-1} \\
&= f
\end{align*}
\]

and

\[
\begin{align*}
\theta^{(1)}(f) &= q\sigma(a^{-1})\theta^{(1)}(b) - \theta^{(1)}(b)a^{-1} \\
&= qa^{-1}q^{-1}1 - 1a^{-1} \\
&= 0.
\end{align*}
\]

Therefore \(f\) is a complex number. Now we denote by \(g\) the complex number

\[
\frac{f}{1 - q}.
\]

and set

\[
b' := b + ga.
\]

Then

\[
Y' := \begin{bmatrix} a & b' \\ 0 & 1 \end{bmatrix}
\]

is a fundamental system of solutions so that

\[
\begin{align*}
\sigma(a) &= qa, & \sigma(a^{-1}) &= q^{-1}a^{-1}, & \sigma(b') &= qb', & \theta^{(1)}(b') &= 1
\end{align*}
\]

(154)
and we have
\[ R = C\langle Y, Y^{-1}\rangle_{alg} = C\langle Y', Y'^{-1}\rangle_{alg} = C\langle a, b', a^{-1}\rangle_{alg} \] (155)

and moreover we have
\[ ab' = qb'a. \] (156)

We have seen in Section 10 that \( R = \langle Q, Q^{-1}, t \rangle_{alg} \) and the relations among the generators \( Q, Q^{-1}, t \) are reduced to
\[ QQ^{-1} = Q^{-1}Q = 1, \quad qtQ -Qt = 0, \quad C \text{ commutes with } Q, Q^{-1} \text{ and } t. \]

Thus, there exists a \( C \)-morphism \( \varphi : R \rightarrow R \) of abstract \( C \)-algebras by (156). It follows from (154) and difference differential equations for \( Q, t \), the morphism \( \varphi \) is \( q \)-SI \( \sigma \)-differential morphism. By (155), the morphism \( \varphi \) is surjective. Since \( R \) is simple \( qsi \) algebra, the kernel of the \( qsi \) morphism \( \varphi \) is 0 and the morphism is injective. Therefore the \( qsi \) morphism \( \varphi \) is an isomorphism.

\[ \square \]

12 Further examples and generalizations

Looking at the Example above, analogy with theory of linear differential equations with constant coefficients lead Pierre Cartier [4] to discover that one can generalize the results to every \( qsi \) linear equations over \( C \). Recently being intrigued by our further Examples below, Akira Masuoka brought a Hopf algebraic view point and succeeded in theoreti-
cally simplifying our results and generalizing them to every Hopf algebra \( H \) and a finite
dimensional left \( H \)-module. [11].

Example 12.1. Let us consider two \( 3 \times 3 \) matrices
\[ A = \begin{bmatrix} q & 1 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

so that \( AB = qBA \). As in the previous section, we consider
\[ \sigma Y = AY \quad \text{and} \quad \theta^{(1)} Y = BY \] (157)
over \( C \), where \( Y \) is a \( 3 \times 3 \) unknown matrix.

The linear \( qsi \) equation is equivalent to considering a 3-dimensional vector space \( V \) equipped with \( q \) \( qsi \)-module structure defined by the \( C \)-algebra morphism
\[ C[\sigma, \sigma^{-1}, \theta^*] \rightarrow M_2(C) = \text{End}(V), \quad \sigma \mapsto tA^{\pm 1}, \quad \theta^{(1)} \mapsto tB. \]

The first task is to solve linear \( qsi \) equation (157) in \( qsi \) algebra \( F(Z, C)[[t]] \). To this end, we set
\[ Y := \sum_{i=0}^{\infty} t^i A_i \in M_3(F(Z, C)[[t]]) = M_3(F(Z, C))[[t]] \] (158)
so that $A_i \in M_3(F(Z, C))$ for every $i \in \mathbb{N}$. We may also identify

$$M_3(F(Z, C)) = F(\mathbb{Z}, M_3(C)).$$

Therefore $A_i$ is a function on the set $Z$ taking values in the set $M_3(C)$ of matrices. So

$$A_i = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & a^{(i)}_2 & a^{(i)}_1 & a^{(i)}_0 & a^{(i)}_1 & a^{(i)}_2 & \cdots \end{bmatrix}$$

with $a^{(i)}_j \in M_3(C)$ for every $i \in \mathbb{N}, j \in Z$. Substituting (158) into (157) and comparing coefficients of $t^i$, we get recurrence relations among the $A_i$'s

$$\sigma(A_i) = q^{-i}AA_i \quad \theta^{(1)}(A_{i+1}) = \frac{1}{[i+1]_q}BA_i$$

If we solve recurrence relations (159) with the initial condition $a^{(0)}_0 = I_3$, since $B^2 = 0$, $A_i = 0$ for $i \geq 2$ and

$$A_0 = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & A^{-2} & A^{-1} & I_3 & A & A^2 & \cdots \end{bmatrix} = \begin{bmatrix} Q & q^{-1}ZQ & 0 \\
0 & Q & 0 \\
0 & 0 & 1 \end{bmatrix}, \quad A_1 = BA_0 = \begin{bmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \end{bmatrix}.$$

So

$$Y = A_0 + tBA_0 = \begin{bmatrix} Q & q^{-1}ZQ & t \\
0 & Q & 0 \\
0 & 0 & 1 \end{bmatrix},$$

where $Z$ is an element of the ring of functions $F(Z, C)$ taking the value $n$ at $n \in \mathbb{N}$ so that

$$Z = \begin{bmatrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & -2 & -1 & 0 & 1 & 2 & \cdots \end{bmatrix}$$

The solution $Y$ is an invertible element in the matrix ring $M_3(F(Z, C)[[t]])$. We introduce a qsi $C$-algebra $R$ generated by the entries of the matrices $Y$ and $Y^{-1}$ in the qsi $C$-algebra $F(Z, C)[[t]]$. To be more concrete

$$R := C\langle Q, Q^{-1}, Z, t \rangle_{alg}.$$

The commutation relations among the generators are

$$QQ^{-1} = Q^{-1}Q = 1, \quad qtQ = Qt, \quad t(Z + 1) = Zt \quad ZQ = QZ.$$ (160)

and the operators act as

$$\sigma(t) = qt, \quad \theta^{(1)}(t) = 1,$$ (161)

$$\sigma Q = qQ, \quad \theta^{(1)}(Q) = 0,$$ (162)

$$\sigma(Q^{-1}) = q^{-1}Q^{-1}, \quad \theta^{(1)}(Q^{-1}) = 0,$$ (163)

$$\sigma Z = Z + 1, \quad \theta^{(1)}(Z) = 0.$$ (164)
Then the arguments in the previous Example shows that the ring $R$ trivializes the qsi module defined by the matrices $A$ and $B$, the qsi ring $R$ is simple and that the ring of constants $C_R = C$. The abstract $C$-algebra has a $C$-algebra morphism $R^g \to C$. So we may call it the Picard-Vessiot ring of the qsi module. Of course we can pro the uniqueness. Now we can speak of the Galois group of qsi equation (157). The argument of the previous section, (160) and the actions of the operators (161), (162), (163) and (164) allow us to prove the following result.

**Lemma 12.2.** The following conditions for a $C$-algebra $T$ and four elements $e, e', f, g \in T$ are equivalent.

1. There exists a $C$-qsi morphism $\varphi : R \to R \otimes_C T$ such that
   \[
   \varphi(Q) = eQ, \quad \varphi(Q^{-1}) = e'^{-1}Q^{-1}, \quad \varphi(Z) = Z + f, \quad \varphi(t) = t + gQ.
   \]

2. The four elements satisfy the following relations.
   \[
   ee' = e'e = 1, \quad eg = qfg, \quad ef = fe, \quad fg - gf = g. \tag{165}
   \]

Lemma 12.2 tells us the universal co-action. To see the co-algebra structure, let $\varphi_1 : R \to R \otimes_C T$ be the $C$-qsi morphism determined by four elements $e_1, e'_1, f_1, g_1 \in T$ satisfying relations (165). We take another $C$-qsi algebra morphism $\varphi_2 : R \to R \otimes_C T$ defined by four elements $e_2, e'_2, f_2, g_2 \in T$ satisfying relations (165). We assume that the subsets \{ $e_1, e'_1, f_1, g_1$ \} and \{ $e_2, e'_2, f_2, g_2$ \} of $T$ are mutually commutative. Let us compose $\varphi_1$ and $\varphi_2$.

\begin{align*}
Q & \mapsto e_1Q \\
Q^{-1} & \mapsto e_1^{-1}Q^{-1} \\
Z & \mapsto Z + f_1 \\
t & \mapsto t + g_1Q
\end{align*}

\begin{align*}
\mapsto & \quad e_2(e_1Q) = (e_1e_2)Q, \\
\mapsto & \quad e_2^{-1}(e_1^{-1}Q^{-1}) = (e_1^{-1}e_2^{-1})Q, \\
\mapsto & \quad (Z + f_1) + f_2 = Z + (f_1 + f_2), \\
\mapsto & \quad (t + g_1Q) + g_2e_1Q = t + (e_1g_2 + g_1)Q.
\end{align*}

Let us now set

\[ A := C\langle e, e', f, g \rangle_{\text{alg}}, \]

where we assume that the elements $e, e', f, g$ satisfy only relations (165) so that we have an isomorphism

\[ R^g \simeq A, \quad Q \mapsto e, Q^{-1} \mapsto e', Z \mapsto f, t \mapsto g \]

as abstract $C$-algebras. This remark is due to A. Masuoka. It follows from the result above of the composition of $\varphi_1$ and $\varphi_2$ that

\[ \Delta : A \to A \otimes_C A \]

with

\[ \Delta(e) = e \otimes e, \quad \Delta(e') = e' \otimes e', \quad \Delta(f) = f \otimes 1 + 1 \otimes f, \quad \Delta(g) = g \otimes 1 + e \otimes g \]
defines a $C$-algebra morphism and together with a $C$-algebra morphism
$$\epsilon : A \to C, \text{ with } \epsilon(e) = \epsilon(e') = 1, \epsilon(f) = \epsilon(g) = 0$$
makes $A$ a Hopf algebra over $C$.

The Galois group of the rank 3 qsi module is the Hopf algebra $A$.

We add another example.

**Example 12.3.** We consider matrices
$$A = \begin{bmatrix} lq & 0 \\ 0 & l \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(C),$$
where $l$ is an element of the field $C$. Since $AB = qBA$, the $C$-algebra morphism
$$C[\sigma, \sigma^{-1}, \theta^*] \to M_2(C) = \text{End}(V), \quad \sigma^{\pm 1} \mapsto t A^{\pm 1}, \theta(1) \mapsto t B$$
defines on a 2-dimensional $C$-vector space $V$ a 2-dimensional qsi module structure. We assume that $q$ and $l$ are linearly independent over $\mathbb{Q}$.

We do not give details here as it is useless to repeat the arguments.

1. The solution matrix in $M_2(F(\mathbb{Z}, C)[[t]])$ is
$$LQ \begin{bmatrix} t \\ 0 \end{bmatrix},$$
where
$$L = \begin{bmatrix} \cdots & -1 & 0 & 1 & 2 & \cdots \\ \cdots & l^{-1} & 1 & l & l^2 & \cdots \end{bmatrix} \in F(\mathbb{Z}, C).$$

2. The Picard-Vessiot ring is
$$C(Q, Q^{-1}, L, L^{-1}, t)_{alg}$$
with commutation relations
$$QQ^{-1} = Q^{-1}Q = 1, \quad LL^{-1} = L^{-1}L = 1, \quadQL = LQ, \quad Qt = qtQ, \quadLt = ltL.$$  

Actions of operators:
$$\sigma Q = qQ, \quad \sigma(Q^{-1}) = q^{-1}Q^{-1}, \quad \sigma L = lL, \quad \sigma(L^{-1}) = l^{-1}L^{-1}, \quad \sigma(t) = qt $$
$$\theta^{(1)} Q = 0, \quad \theta^{(1)}(Q^{-1}) = 0, \quad \theta^{(1)}(L) = 0, \quad \theta^{(1)}(L^{-1}) = 0, \quad \theta^{(1)}(t) = 1.$$

3. The Galois group is the Hopf algebra
$$\mathcal{H}_q := C\langle e, e^{-1}, g, h, h^{-1} \rangle_{alg},$$
satisfying commutation relations
$$ee^{-1} = e^{-1}e = 1, \quad hh^{-1} = h^{-1}h = 1, \quad eg = qge, \quad hg = lgh.$$  

Co-algebra structure $\Delta : \mathcal{H}_q \to \mathcal{H}_q \otimes_C \mathcal{H}_q$:
$$\Delta(e^{\pm 1}) = e^{\pm 1} \otimes e^{\pm 1}, \quad \Delta(h^{\pm 1}) = h^{\pm 1} \otimes h^{\pm 1}, \quad \Delta(g) = g \otimes 1 + e \otimes g.$$  

The co-unit $\epsilon : \mathcal{H}_q \to C$ is given by
$$\epsilon(e) = \epsilon(e^{-1}) = \epsilon(h) = \epsilon(h^{-1}) = 1, \quad \epsilon(g) = 0.$$
The last example is inspired of work of Masatoshi Noumi [15] on the quantization of hypergeometric functions. His idea is that q-hypergeometric functions should live on the quantized Grassmannians. Namely, he quantizes the framework of Gelfand of defining general hypergeometric functions.

Example 12.4. Let \( V \) be the natural 2-dimensional representation of \( U_q(sl_2) \) over \( C \). Hence \( V \) is a left \( U_q(sl_2) \)-module. So we can speak of the Picard-Vessiot extension \( R/C \) attached to the left \( U_q(sl_2) \)-module \( V \). The argument in the Examples so far studied allows us to guess that \( R \) is given by

\[
R := C\langle a, b, c, d \rangle_{\text{alg}},
\]

with relations

\[
ab = qba, \quad bd = qdb, \quad ac = qca, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q + q^{-1})bc, \quad ad - qbc = 1.
\]

Imagine a matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

and on the space of matrices, the quantum group or Hopf algebra \( U_q(sl_2) \) operates from right.

Let us recall the definitions. The Hopf algebra \( U_q(sl_2) = C\langle a^\pm H/2, X, Y \rangle \) is generated by four elements \( q^H, q^{-H}, X, Y \) over \( C \) satisfying the commutation relations

\[
q^H q^{-H} = q^{-H} q^H = 1, \quad q^H X q^{-H} = q^2 X, \quad q^H Y q^{-H} = q^{-2} Y, \quad [X, Y] = \frac{q^2 H - q^{-2}}{q - q^{-1}}.
\]

The co-algebra structure \( \Delta : U_q(sl_2) \to U_q(sl_2) \otimes_C U_q(sl_2) \) is given by

\[
\Delta(q^H) = q^H \otimes q^H, \quad \Delta(X) = X \otimes 1 + q^H \otimes X, \quad \Delta(Y) = Y \otimes q^{-H} + 1 \otimes Y.
\]

We define the co-unit \( \epsilon : U_q(sl_2) \to C \) by

\[
\epsilon(q^H) = 1, \quad \epsilon(X) = \epsilon(Y) = 0.
\]

See S. Majid [8], 3.2, for example.

The \( C \)-algebra \( R \) is a \( U_q(sl_2) \)-module algebra by the action of \( U_q(sl_2) \) on \( R \) defined by

\[
q^H, \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} q^{\pm 1}a & q^{\mp 1}b \\ q^{\pm 1}c & q^{\mp 1}d \end{bmatrix}, \quad X, \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}, \quad Y, \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}.
\]

We have not exactly examined but we believe

(1) The algebra extension \( R/C \) is the Picard-Vessiot extension for the \( U_q(sl_2) \)-module \( V \).

(2) The Galois group is the Hopf algebra on the abstract \( C \)-algebra \( R \) with adjunction of the co-algebra structure defined by

\[
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(c) = c \otimes a + d \otimes b, \quad \Delta(d) = c \otimes b + d \otimes d
\]

and the co-unit \( \epsilon : R \to C \) with

\[
\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0.
\]

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References

[1] Katsutoshi Amano and Akira Masuoka. Picard-Vessiot extensions of Artinian simple algebras. *J. of Algebra*, 285:713–767, 2005.

[2] Katsutoshi Amano, Akira Masuoka, and Mitsuhiro Takeuchi. *Hopf algebra approach to Picard-Vessiot thoery*, volume 6, pages 127–171. Elsvier/North-Holland, Amsterdam, 2009.

[3] Yves André. Différentielles non commutatives et théorie de Galois différentielle ou aux différences. (French) [Noncommutative differentials and Galois theory for differential or difference equations]. *Ann. Sci. École Norm. Sup. (4)*, 34:685–739, 2001.

[4] Pierre Cartier. Personal correspondence. November 2013.

[5] Charlotte Hardouin. Iterative difference Galois theory. *Journal Reine Angew. Math.*, 644:101–144, 2010.

[6] Heidi Haynal. PI degree parity in $q$-skew polynomial rings. *J. Algebra*, 319:4199–4221, 2008.

[7] Florian Heiderich. *Galois Theory of Module Fields*. PhD thesis, Barcelona University, 2010.

[8] Shahn Majid. *Foundation of quantum group theory*. Cambridge University press, Cambridge, 1995.

[9] Bernard Malgrange. Le groupoïde de Galois d’un feuilletage. (French) [The Galois groupoid of a foliation]. In *Essays on geometry and related topics, Vol. 1, 2*, Monogr. Enseign. Math., pages 465–501. Enseignement Math., Geneva, 2001.

[10] Yu. I. Manin. *Quantum groups and noncommutative geometry*. Centre de Recherches Mathématiques de l’université de Montréal, Montréal, 1988.

[11] Akira Masuoka. On the non-commutative Picard-Vessiot theory. Note in Japanese, December 4, 2014.

[12] Akira Masuoka and Makoto Yanagawa. $\times_R$-bialgebras associated with iterative $q$-difference rings. *Internat. J. Math.*, 24, 2013.

[13] Shuji Morikawa. On a general difference Galois theory. I. *Ann. Inst. Fourier (Grenoble)*, 59:2709–2732, 2009.

[14] Shuji Morikawa and Hiroshi Umemura. On a general difference Galois theory. II. *Ann. Inst. Fourier (Grenoble)*, 59:2733–2771, 2009.

[15] Masatoshi Noumi. Quantum Grassmannians and $q$-hypergeometric functions. *CWI Quarterly*, 5:293–307, 1992.
[16] Moss Sweedler. *Hopf Algebras*. Mathematics Lecture Series. Benjamin, New York, 1969.

[17] Hiroshi Umemura. Differential Galois theory of infinite dimension. *Nagoya Math. J.*, 144:59–135, 1996.

[18] Hiroshi Umemura. Galois theory of algebraic and differential equations. *Nagoya Math. J.*, 144:1–58, 1996.

[19] Hiroshi Umemura. Galois theory and Painlevé equations. In *Théories asymptotiques et équations de Painlevé*, volume 14 of *Sémin. Congr.*. pages 299–339. Soc. Math. France, Paris, 2006.

[20] Hiroshi Umemura. Invitation to Galois theory. In *Differential equations and quantum groups*, volume 9 of *IRMA Lect. Math. Theor. Phys.*, pages 269–289. Eur. Math. Soc., Zürich, 2007.

[21] Hiroshi Umemura. *Picard-Vessiot theory in general Galois theory*, volume 94, pages 263–293. Banach Center Publ., Warsaw, 2011.