THE FACTORIZATION METHOD FOR SCATTERERS WITH DIFFERENT PHYSICAL PROPERTIES

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Abstract. The scattering of time-harmonic acoustic plane waves by a mixed type scatterer is considered. Such a scatterer is given as the union of several components with different physical properties. Some of them are impenetrable obstacles with Dirichlet or impedance boundary conditions, while the others are penetrable inhomogeneous media with compact support. This paper is concerned with modifications of the factorization method for the following two basic cases, one is the scattering by a priori separated sound-soft and sound-hard obstacles, the other one is the scattering by a scatterer with impenetrable (Dirichlet) and penetrable components. The other cases can be dealt with similarly. Finally, some numerical experiments are presented to demonstrate the feasibility and effectiveness of the modified factorization methods.

1. Introduction. The factorization method is a sampling method for solving certain kinds of inverse problems where the shape and location of a domain have to be reconstructed. It has first been introduced by Kirsch in [6] for inverse acoustic scattering by impenetrable obstacles with either Dirichlet or Neumann boundary conditions. The case of scattering by obstacles with impedance boundary conditions has been treated in [4, 10, 13]. In [7] and, more satisfactorily, in [2, 8, 11] it has been extended to the scattering by an inhomogeneous medium. Recently, the factorization method has also been applied to reconstruction of penetrable obstacles with buried objects [1, 18] and to reconstruction of impenetrable cavities with interior measurements [15]. In this paper, we consider the factorization method for a scatterer with different physical properties. Such a scatterer is given as the union of several components with different physical properties. Some of them are impenetrable obstacles with Dirichlet or impedance boundary conditions, while the others are penetrable inhomogeneous media with compact support. For simplicity, we consider the following two cases: one is the scattering by a scatterer with a-priori Dirichlet and Neumann parts, the other one is the scattering by a scatterer with impenetrable (Dirichlet) and penetrable parts.

Let \( k > 0 \) be the wave number and

\[ u^i(x, d) = e^{ikx \cdot d}, \quad x \in \mathbb{R}^n (n = 2, 3) \]

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be the incident plane wave with direction \( d \in S^{n-1} \). Here, \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) denotes the unit sphere in \( \mathbb{R}^n \). Let \( D \subset \mathbb{R}^n \) be an open and bounded domain such that the exterior \( \mathbb{R}^n \setminus D \) of \( D \) is connected. Assume that \( D \) consists of two open and bounded Lipschitz domains, i.e., \( D = D_1 \cup D_2 \), such that \( D_1, D_2 \subset \mathbb{R}^n \) satisfy \( \partial D_1 \cap \partial D_2 = \emptyset \). Note that each \( D_i \) may consist of several connected components. \( D_1 \) is always assumed to be the impenetrable obstacles with Dirichlet boundary condition.

We will consider the following two cases according to the physical property of \( D_2 \).

**The first case.** \( D_2 \) is an impenetrable obstacle with Neumann boundary condition. In this case, the scattered field \( u^s \) satisfies the following exterior boundary value problem with \( f = -u^i \) and \( q = -\partial u^i/\partial \nu \):

Given \( f \in H^{1/2}(\partial D) \) and \( g \in H^{-1/2}(\partial D) \) find \( u^s \in H^1_{\text{loc}}(\mathbb{R}^n \setminus D) \) such that

\[
\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^n \setminus D, \\
\left. u^s \right|_{\partial D} = f \quad \text{on} \quad \partial D_1, \\
\left. \partial u^s / \partial \nu \right|_{\partial D} = g \quad \text{on} \quad \partial D_2, \\
\lim_{r \to \infty} r^{n-1} \left( \frac{\partial u^s}{\partial r} -iku^s \right) = 0, \\
\]

(4)

where \( r = |x| \) and (4) is the Sommerfeld radiation condition which holds uniformly in all directions \( \tilde{x} := x/|x| \) and physically implies that energy is transported to infinity.

Here and throughout this paper, \( \nu = \nu(x) \) denotes the unit normal vector at \( x \in \partial D \) directed into the exterior of \( D \). We note that the solution of (1) is understood in the variational sense. We refer to Theorem 7.15 in [16] for the existence and uniqueness of solutions to the exterior boundary value problem (1)-(4).

**The second case.** \( D_2 \) is a penetrable inhomogeneity. In this configuration, the inhomogeneous medium is characterized by a “contrast function” \( q \in L^\infty(D_2) \). The scattered field \( u^s \) solves the following problem with boundary data \( f_1 = u^i \) on \( \partial D_1 \) and source term \( f_2 = \sqrt{|q|}u^i \) in \( D_2 \):

Given \( f_1 \in H^{1/2}(\partial D_1) \) and \( f_2 \in L^2(D_2) \) find \( u^s \in H^1_{\text{loc}}(\mathbb{R}^n \setminus D) \) such that

\[
\Delta u^s + k^2(1 + q)u^s = -k^2 \frac{q}{\sqrt{|q|}} f_2 \quad \text{in} \quad \mathbb{R}^n \setminus D_1, \\
\left. u^s \right|_{\partial D} = -f_1 \quad \text{on} \quad \partial D_1, \\
\lim_{r \to \infty} r^{n-1} \left( \frac{\partial u^s}{\partial r} -iku^s \right) = 0 \quad \text{uniformly in all directions} \quad x/r. \\
\]

(5) (6) (7)

Also the solution of (5) is understood in the variational sense. Here, and in the following, we do not distinguish between a function defined in \( D_2 \) and its extension by zero to all of \( \mathbb{R}^n \setminus D_1 \). The well-posedness of the problem (5)-(7) has been established in [12].

It is well known that \( u^s(x) \) has the following asymptotic behavior

\[
u^s(x) = \gamma_n \frac{e^{ikr}}{r^{n-1}} \left\{ u^\infty(\tilde{x}) + O \left( \frac{1}{r} \right) \right\} \quad \text{as} \quad r \to \infty, \\
\]

(8)

uniformly for all directions \( \tilde{x} := x/|x| \), where

\[
\gamma_n = \begin{cases} 
\frac{1}{4\pi}, & n = 3, \\
e^{\pi/4}, & n = 2, \\
\sqrt{8k}, & n = 2
\end{cases}
\]
depends on the dimension $n$ and the function $u^\infty(\hat{x})$ defined on the unit sphere $S^{n-1}$ is known as the far field pattern with $\hat{x}$ denoting the observation direction. Let $u^\infty(\hat{x},d)$ be the far field pattern corresponding to the observation direction $\hat{x}$ and the incident direction $d$. Then, the inverse problem we consider in this paper is, given the wave number $k$ and the far field pattern $u^\infty(\hat{x},d)$ for all $\hat{x}, d \in S^{n-1}$, to determine the location and shape of $D$.

The far field patterns $u^\infty(\hat{x},d)$, $\hat{x}, d \in S^{n-1}$, define the far field operator $F : L^2(S^{n-1}) \to L^2(S^{n-1})$ by

$$(Fg)(\hat{x}) = \int_{S^{n-1}} u^\infty(\hat{x},d)g(d)ds(d) \text{ for } \hat{x} \in S^{n-1}. \quad (9)$$

From the definition of the far field operator we note that $Fg$ is the far field pattern corresponding to the incident field given as a Herglotz wave function $v_g$, i.e., a function of the form

$$v_g(x) = \int_{S^{n-1}} e^{ikx \cdot d}g(d)ds(d), \quad x \in \mathbb{R}^n, \quad (10)$$

with density $g \in L^2(S^{n-1})$.

The essential idea of the Factorization method is to decide for any $z \in \mathbb{R}^n$ whether or not the equation

$$\tilde{F}g = \phi_z$$

is solvable in $L^2(S^{n-1})$. Here, $\tilde{F}$ is a self-adjoint operator which can be explicitly computed from $F$, and $\phi_z \in L^2(S^{n-1})$ is given by the exponential function

$$\phi_z(\hat{x}) = e^{-ikz \cdot \hat{x}}, \quad \hat{x} \in S^{n-1}. \quad (11)$$

For both cases, it is still an open problem whether or not the factorization method can be justified without any additional a priori information. Following the approach of [5], in Section 3.3 of [10], Kirsch and Grinberg provide a modification such that the factorization method works under the following Assumption 1.1. Assume further that their exists $t \in (\pi, 2\pi)$ and $c > 0$ such that

$$\Re \left( \frac{e^{itq}}{|q|} \right) \geq c \quad \text{almost everywhere in } D_2. \quad (12)$$

Kirsch and Liu [12] provide a new modification of the factorization method for the second case. One feature of the modification given in [12] is that it avoids studying the boundary value problems with $L^2$-boundary data. We refer to [13] where a similar modification has been proposed for impenetrable obstacles such that the factorization method is independent of the interior eigenvalues. The first contribution of this paper is, following the idea of [12], to give a new modification of the factorization method for the first case, i.e., the scattering by obstacles with a priori separated Dirichlet and Neumann parts.

**Assumption 1.1.** Assume that we a priori know open and bounded domains $\Omega_i$ such that $\bar{D}_i \subset \Omega_i$ for $i = 1, 2$. Here the closures of the domains $\Omega_i$, $i = 1, 2$, are disjoint and their complements are connected. We further assume that $k^2$ is not a Dirichlet/Neumann eigenvalue of $-\Delta$ in $\Omega_2/\Omega_1$.

Without Assumption 1.1, but assume that their exists $c_0 > 0$ such that

$$\Re \left( \frac{q}{|q|} \right) \leq -c_0 \quad \text{a.e. in } D_2, \quad (13)$$
Kirsch and Liu [12] proved that the factorization method works for the second case. The second contribution of this paper is to weaken this assumption to

\[ \Re \left( \frac{e^{itq}}{|q|} \right) \leq -c_0 \quad \text{a.e. in } D_2 \tag{14} \]

for some \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( c_0 > 0 \). Clearly, (13) is a special case of (14) with \( t = 0 \).

This paper is organized as follows. In the next section, we will study new modification of the factorization method for the first case under Assumption 1.1. Section 3 is devoted to a study of the factorization method for the second case under the new assumption (14) on \( q \). Some numerical simulations in two dimensions will be presented in Section 4 to justify the validity of our method. Finally, we draw some conclusions in Section 5.

2. A new modification of the factorization method for obstacles with a priori separated Dirichlet and Neumann parts. We start with the following functional analysis result behind the Factorization Method. The proof is completely analogous to Theorem 2.15 in [10] and Theorem 2.1 in [14]. We refer to [17] for a proof.

**Theorem 2.1.** Let \( X \subset U \subset X^* \) be a Gelfand triple with a Hilbert space \( U \) and a reflexive Banach space \( X \) such that the imbedding is dense. Furthermore, let \( Y \) be a second Hilbert space and let \( A : Y \to Y, B : Y \to X, \) and \( C : X \to X^* \) be linear bounded operators such that

\[ A = B^* C B. \]

We make the following assumptions:

1. \( B \) is compact and injective.
2. There exists \( t \in [0, 2\pi] \) such that \( \Re[e^{itC}] \) has the form \( \Re[e^{itC}] = \tilde{C} + K \) with some compact operator \( K \) and some self-adjoint and coercive operator \( \tilde{C} : X \to X^* \).
3. \( \Im C \) is non-negative or non-positive on \( X \), i.e., \( \Im(C\phi, \phi) \geq 0 \) or \( \Im(C\phi, \phi) \leq 0 \) for all \( \phi \in X \).
4. (a) \( C \) is injective and the number \( t \) from 2. does not belong to \( \{\pi/2, 3\pi/2\} \) or
   (b) \( \Im C \) is positive or negative on the closure \( \overline{R(B)} \) of \( R(B) \).

Then the operator \( A_2 = |\Re[e^{itA}]| + |\Im A| \) is positive, and the ranges of \( B : Y \to X \) and \( A_2^{1/2} : Y \to Y \) coincide.

We want to remark that, in this paper, the real and the imaginary parts of an operator \( T \) are self-adjoint operators given by

\[ \Re(T) = \frac{T + T^*}{2} \quad \text{and} \quad \Im(T) = \frac{T - T^*}{2i}. \]

For the proof of the factorization method we define the following two operators

\[ H_1 : L^2(S^{n-1}) \to H^{1/2}(\partial D_1) \text{ and } H_2 : L^2(S^{n-1}) \to H^{-1/2}(\partial D_2) \]

by

\[ H_1 g = v_g|_{\partial D_1}, \quad H_2 g = \frac{\partial v_g}{\partial \nu} \bigg|_{\partial D_2}. \]
where \( v_g \) is the Herglotz wave function given in (10). Recall that the fundamental solution of the Helmholtz equation is given by

\[
\Phi(x, y) = \begin{cases} 
\frac{e^{ik|x-y|}}{4\pi|x-y|} & \text{for } x, y \in \mathbb{R}^3, x \neq y, \\
\frac{i}{4} H_0^{(1)}(k|x-y|) & \text{for } x, y \in \mathbb{R}^2, x \neq y,
\end{cases}
\]

(15)

where \( H_0^{(1)} \) is the Hankel function of the first kind of order zero. Also, for \( i = 1, 2 \), and any \( \varphi \in H^{-1/2}(\partial D_i), \psi \in H^{1/2}(\partial D_i) \), define the boundary integral operators

\[
\begin{align*}
(S_i \varphi)(x) &= \int_{\partial D_i} \varphi(y) \Phi(x, y) \, ds(y), & x \in \partial D_i, \\
(K \psi)(x) &= \int_{\partial D_i} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \, ds(y), & x \in \partial D_i, \\
(K_i' \varphi)(x) &= \int_{\partial D_i} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} \, ds(y), & x \in \partial D_i, \\
(N_i \psi)(x) &= -\frac{\partial}{\partial n(x)} \int_{\partial D_i} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \, ds(y), & x \in \partial D_i.
\end{align*}
\]

Furthermore, the operators \( \tilde{S}_i, \tilde{K}_i, \tilde{K}_i' \) and \( \tilde{N}_i \) are analogously defined by interchanging the roles of \( \partial D_i \) and \( \partial \Omega_i \). We refer to [3, 10, 16] for properties of these operators. Here we just mention that the operators \( \Im(\tilde{S}_2) \) and \( \Im(\tilde{N}_1) \) are strictly positive, i.e.,

\[
\Im(\tilde{S}_2 \eta_2, \eta_2) > 0, \quad \text{for all } \eta_2 \in H^{-1/2}(\partial \Omega_2) \text{ with } \eta_2 \neq 0 \tag{16}
\]

and

\[
\Im(\tilde{N}_1 \eta_1, \eta_1) > 0, \quad \text{for all } \eta_1 \in H^{1/2}(\partial \Omega_1) \text{ with } \eta_1 \neq 0. \tag{17}
\]

For a proof we refer to Lemma 1.14 and Theorem 1.26 in [10]. For the modification of the factorization later we define \( \tilde{H}_1 : L^2(S^{n-1}) \to H^{-1/2}(\partial \Omega_1) \) and \( \tilde{H}_2 : L^2(S^{n-1}) \to H^{1/2}(\partial \Omega_2) \) by

\[
\tilde{H}_1 g = \frac{\partial v_g}{\partial n_{\partial \Omega_1}}, \quad \tilde{H}_2 g = v_g|_{\partial \Omega_2},
\]

where again \( v_g \) is the Herglotz wave function given in (10). Using (3.26) in [10], the far field operator \( F : L^2(S^{n-1}) \to L^2(S^{n-1}) \) defined in (9) can be factorized as

\[
F = -\left( \begin{array}{c} H_1 \\ H_2 \end{array} \right)^* M_{old}^{-1} \left( \begin{array}{c} H_1 \\ H_2 \end{array} \right) \tag{18}
\]

with \( M_{old} : H^{-1/2}(\partial D_1) \times H^{1/2}(\partial D_2) \to H^{1/2}(\partial D_1) \times H^{-1/2}(\partial D_2) \) of the form

\[
M_{old} = \begin{pmatrix} S_1 & K_2 \\ K'_1 & N_2 \end{pmatrix}.
\]

In the following we collect some properties for the operator \( M_{old} \) in the factorization (18). The proof can be found in Theorem 3.4 in [10].

**Theorem 2.2.** Assume that \( k^2 \) is neither a Dirichlet eigenvalue of \( -\Delta \) in \( D_1 \) nor a Neumann eigenvalue of \( -\Delta \) in \( D_2 \).
(a) The middle operator $M_{\text{old}}^{-1}$ in the factorization (18) has the decomposition of the form

$$M_{\text{old}}^{-1} := \begin{pmatrix} S_1 & K_2 \\ K_1' & N_2 \end{pmatrix}^{-1} = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & N_2^{-1} \end{pmatrix} + K_{\text{old}}$$  \hfill (19)

with some compact operator $K_{\text{old}}$.

(b) The operator $\mathfrak{M}(M_{\text{old}})$ is strictly positive, i.e., for any $\psi \neq 0$,

$$\mathfrak{M}(M_{\text{old}}\psi, \psi) > 0.$$  

In the following, we assume that $k^2$ is neither a Dirichlet eigenvalue of $-\Delta$ in $\Omega_2$ nor a Neumann eigenvalue of $-\Delta$ in $\Omega_1$. Note that this requirement is not essential since we have the freedom to choose $\Omega_1$ and $\Omega_2$. For any parameter $\rho > 0$ we introduce two modified data operators $F_D$ and $F_N$ by

$$F_D = F - \rho \tilde{H}_2 S_2^{-1} \tilde{H}_2, \quad F_N = F - \rho \tilde{H}_1 S_1^{-1} \tilde{H}_1.$$  \hfill (20)

We want to remark that $-\tilde{H}_2 S_2^{-1} \tilde{H}_2$ is the factorization of the far field operator corresponding to inverse scattering of plane wave by the sound-soft obstacle $\Omega_2$ (see Remark in page 19 of [10]), while $-\tilde{H}_1 S_1^{-1} \tilde{H}_1$ is that by the sound-hard obstacle $\Omega_1$. Then we can show:

**Theorem 2.3.** Assume that $k^2$ is neither a Dirichlet eigenvalue of $-\Delta$ in $D_1, \Omega_2$ nor a Neumann eigenvalue of $-\Delta$ in $D_2, \Omega_1$.

(a) The operators $F_D$ and $F_N$ from (20) have the factorizations

$$F_D = -\begin{pmatrix} H_1' & H_2' \\ H_1 & H_2 \end{pmatrix}^* \begin{pmatrix} H_1' & H_2' \\ H_1 & H_2 \end{pmatrix}, \quad F_N = -\begin{pmatrix} H_1' & H_2' \\ H_1 & H_2 \end{pmatrix}^* \begin{pmatrix} H_1' & H_2' \\ H_1 & H_2 \end{pmatrix}$$  \hfill (21)

with some Fredholm operators $T_D : H^{1/2}(\partial D_1) \times H^{1/2}(\partial \Omega_2) \to H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial \Omega_2)$ and $T_N : H^{-1/2}(\partial \Omega_1) \times H^{-1/2}(\partial D_2) \to H^{1/2}(\partial \Omega_1) \times H^{1/2}(\partial D_2)$.

Here, $T_D$ and $T_N$ have the following forms:

$$T_D = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & \rho S_2^{-1} \end{pmatrix} + K_D, \quad T_N = \begin{pmatrix} \rho \tilde{N}_1^{-1} & 0 \\ 0 & N_2^{-1} \end{pmatrix} + K_N$$  \hfill (22)

with some compact operators $K_D$ and $K_N$.

(b) The operators $\mathfrak{M}(T_D)$ and $\mathfrak{M}(T_N)$ are strictly negative, i.e., for any $\phi \neq 0$,

$$\mathfrak{M}(T_D\phi, \phi) < 0, \quad \mathfrak{M}(T_N\phi, \phi) < 0.$$  

(c) The real parts $\Re(T_D)$ and $\Re(T_N)$ have the decompositions in the form

$$\Re(T_D) = C_D + \tilde{K}_D, \quad \Re(T_N) = C_N + \tilde{K}_N$$

with self-adjoint and coercive operators $C_D, C_N$ and compact operators $\tilde{K}_D, \tilde{K}_N$.

**Proof.** (a) We first note that $H_2 = R_2 \tilde{H}_2$, where $R_2 : H^{1/2}(\partial \Omega_2) \to H^{-1/2}(\partial D_2)$ is defined by

$$(R_2 f)(x) = \frac{\partial v(x)}{\partial \nu}, \quad x \in \partial D_2,$$  \hfill (23)

where $v \in H^1(\Omega_2)$ solves the interior Dirichlet boundary value problem

$$\Delta v + k^2 v = 0 \text{ in } \Omega_2, \quad v = f \text{ on } \partial \Omega_2.$$
Here we have used the assumption that \(k^2\) is not a Dirichlet eigenvalue of \(-\Delta\) in \(\Omega_2\). The operator \(R_2\) is compact because of interior regularity. \(R_2\) is clearly also injective. Indeed, the relation \(R_2 f = 0\) implies that \(v = 0\) in \(D_2\) since \(k^2\) is not a Neumann eigenvalue of \(-\Delta\) in \(D_2\). Analytic continuation yields \(v = 0\) in \(\Omega_2\) and therefore \(f = 0\).

We write the factorization (18) in the form
\[
F = - \left( \frac{H_1}{\widetilde{H}_2} \right)^* \left( \begin{array}{cc} I & 0 \\ 0 & R_2^2 \end{array} \right) M_{\text{old}}^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) \left( \frac{H_1}{\widetilde{H}_2} \right)
\]
and thus
\[
F_D = - \left( \frac{H_1}{\widetilde{H}_2} \right)^* \left[ \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) M_{\text{old}}^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \rho \widetilde{S}_2^{-1} \end{array} \right) \right] \left( \frac{H_1}{\widetilde{H}_2} \right). \tag{24}
\]
Using (19) the operator \(M_{\text{old}}^{-1}\) can be written as
\[
M_{\text{old}}^{-1} = \left( \frac{S_1^{-1} 0}{0 \ N_2^{-1}} \right) - M_{\text{old}}^{-1} K_{\text{old}} \left( \begin{array}{cc} S_1^{-1} & 0 \\ 0 & N_2^{-1} \end{array} \right). \tag{25}
\]
Substituting this into (24) yields
\[
F_D = - \left( \frac{H_1}{\widetilde{H}_2} \right)^* \left[ \left( \begin{array}{cc} S_1^{-1} & 0 \\ 0 & \rho \widetilde{S}_2^{-1} \end{array} \right) + K_D \right] \left( \frac{H_1}{\widetilde{H}_2} \right) \tag{26}
\]
where we have collected all compact parts in the operator \(K_D\). This proves the factorization (21) for \(F_D\).

For \(F_N\) we proceed analogously. It is \(H_1 = R_1 \widetilde{H}_1\) where \(R_1 : H^{-1/2}(\partial \Omega_1) \to H^{1/2}(\partial D_1)\) is given by
\[
(R_1 g)(x) = w(x), \quad x \in \partial D_1, \tag{27}
\]
where \(w \in H^1(\Omega_1)\) solves the interior Dirichlet boundary value problem
\[
\Delta w + k^2 w = 0 \text{ in } \Omega_1, \quad \frac{\partial w}{\partial \nu} = g \text{ on } \partial \Omega_1.
\]
Here we have used the assumption that \(k^2\) is not a Neumann eigenvalue of \(-\Delta\) in \(\Omega_1\). Then we follow the lines of the proof for \(F_D\).

(b) From (24) we observe that \(T_D\) has the form
\[
T_D = \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) M_{\text{old}}^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \rho \widetilde{S}_2^{-1} \end{array} \right).
\]
For any \(\varphi = (\varphi_1, \varphi_2)^T \in H^{1/2}(\partial D_1) \times H^{1/2}(\partial \Omega_2)\) with \(\varphi \neq 0\), define \(\psi := M_{\text{old}}^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) \varphi\) and \(\tilde{\varphi}_2 = \widetilde{S}_2^{-1} \varphi_2\). By the injectivity of \(M_{\text{old}}\), \(R_2\) and \(\widetilde{S}_2^{-1}\), we conclude that \(\psi \neq 0\) and \(\tilde{\varphi}_2 \neq 0\). Then
\[
\Im(T_D \varphi, \varphi) = \Im(\psi, M_{\text{old}} \psi) + \rho \Im(\tilde{\varphi}_2, \widetilde{S}_2 \tilde{\varphi}_2) < 0,
\]
where we have used (16) and the fact that \(\Im(M_{\text{old}})\) is strictly positive by Theorem 2.2. The proof for \(T_N\) can be carried out analogously.

(c) By Lemma 1.14 in [10], both \(\Re(S_1)\) and \(\Re(S_2)\) are the sum of a positively coercive operator and a compact operator. Thus the statement for \(\Re(T_D)\) follows from the first decomposition in (22). Similar arguments hold for \(\Re(T_N)\). \(\square\)
Theorem 2.4. Assume that $k^2$ is neither a Dirichlet eigenvalue of $-\Delta$ in $D_1, \Omega_2$ nor a Neumann eigenvalue of $-\Delta$ in $D_2, \Omega_1$. For any $z \in \mathbb{R}^n$ define again $\phi_z \in L^2(\Sigma^{n-1})$ by (11).

(a) A point $z \in \mathbb{R}^n \backslash \overline{\Omega}_2$ belongs to $D_1$ if, and only if, the function $\phi_z$ belongs to the range of the operator $(\hat{H}_1^*, \hat{H}_2^*) : H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial \Omega_2) \to L^2(\Sigma^{n-1})$.

(b) A point $z \in \mathbb{R}^n \backslash \overline{\Omega}_1$ belongs to $D_2$ if, and only if, the function $\phi_z$ belongs to the range of the operator $(\hat{H}_1^*, \hat{H}_2^*) : H^{1/2}(\partial \Omega_1) \times H^{1/2}(\partial D_2) \to L^2(\Sigma^{n-1})$.

Proof. (a) Let first $z \in D_1$. From Theorem 1.12 in [10] and the invertibility of the single layer boundary operator $S_1$ we conclude that $\phi_z \in \mathcal{R}(H_1^*)$, i.e., there exists $\phi_1 \in H^{-1/2}(\partial D_1)$ such that $\phi_z = H_1^*\phi_1$. Setting $\phi_2 = 0$ yields that $\phi_z = H_1^*\phi_1 + \hat{H}_2^*\phi_2$, i.e., $\phi_z$ belongs to the range of the operator $(H_1^*, \hat{H}_2^*)$.

Second, let $z \notin D_1$ and assume on the contrary that $\phi_z = H_1^*\phi_1 + \hat{H}_2^*\phi_2$ for some $\phi_1 \in H^{-1/2}(\partial D_1)$ and $\phi_2 \in H^{-1/2}(\partial \Omega_2)$. Noting that both sides are far field patterns, then by Rellich’s lemma and unique continuation, we have

$$\Phi(x, z) = \int_{\partial D_1} \Phi(x, y)\phi_1(y) dy + \int_{\partial \Omega_2} \Phi(x, y)\phi_2(y) dy, \quad x \in \mathbb{R}^n \backslash \{D_1 \cup \overline{\Omega}_2 \cup \{z\}\}.$$ 

This leads to a contradiction because the left hand side has a singularity at $x = z$ while the right hand side is continuous in $\mathbb{R}^n$.

(b) This can be proved analogously. \hfill \Box

Application of Theorem 2.1 and combination with Theorems 2.3 and 2.4 yield the first main result of this paper.

Theorem 2.5. In addition to Assumption 1.1 we assume that $k^2$ is neither a Dirichlet eigenvalue of $-\Delta$ in $D_1, \Omega_2$ nor a Neumann eigenvalue of $-\Delta$ in $D_2, \Omega_1$. For any $z \in \mathbb{R}^n$ define again $\phi_z \in L^2(\Sigma^{n-1})$ by (11), i.e.,

$$\phi_z(\hat{x}) = e^{-ikz \cdot \hat{x}}, \quad \hat{x} \in \Sigma^{n-1}.$$ 

Then the following characterizations hold:

(a) For any point $z \notin \overline{\Omega}_2$ we have

$$z \in D_1 \iff \phi_z \in \mathcal{R}(F_{D_2}^{1/2})$$

where $F_{D_2} := |\Re(F_D)| + |\Im(F_D)|$ with $F_D$ given by (20). Consequently

$$z \in D_1 \iff \sum_{j=1}^{\infty} \left| \langle \phi_z, \psi_j \rangle_{L^2(\Sigma^{n-1})} \right|^2 < \infty. \quad (28)$$

where $(\lambda_j, \psi_j)$ is the eigensystem of the operator $F_{D_2} : L^2(\Sigma^{n-1}) \to L^2(\Sigma^{n-1})$.

(b) For any point $z \notin \overline{\Omega}_1$ we have

$$z \in D_2 \iff \phi_z \in \mathcal{R}(F_{N_2}^{1/2})$$

where $F_{N_2} = |\Re(F_N)| + |\Im(F_N)|$ and $F_N$ is given in (20) and consequently

$$z \in D_2 \iff \sum_{j=1}^{\infty} \left| \langle \phi_z, \psi_j \rangle_{L^2(\Sigma^{n-1})} \right|^2 < \infty. \quad (29)$$

where $(\lambda_j, \psi_j)$ is the eigensystem of the operator $F_{N_2} : L^2(\Sigma^{n-1}) \to L^2(\Sigma^{n-1})$. 
3. The factorization method for obstacles with impenetrable and penetrable components. We begin this section with a general assumption on the contrast function \( q \).

**Assumption 3.1.** Assume that the contrast function \( q \in L^\infty(D_2) \) satisfies

1. \( \Im q \geq 0 \) in \( D_2 \);
2. There exists \( c_0 \in (0, 1) \) such that \( 1 + \Re q \geq c_0 \) in \( D_2 \);
3. \( |q| \) is locally bounded below, i.e., for every compact subset \( M \subset D_2 \) there exists \( c_1 \in (0, 1 - c_0) \) (depending on \( M \)) such that \( |q| \geq c_1 \) in \( M \). Note that this is satisfied for a continuous contrast \( q \) which vanishes at most on \( \partial D_2 \).

We extend \( q \) by zero into \( \mathbb{R}^n \setminus D_2 \).

In [12] (Theorem 3.2) the following result has been shown.

**Theorem 3.2.** Let Assumption 3.1 hold. Then we have:

(a) The far field operator \( F : L^2(S^{n-1}) \rightarrow L^2(S^{n-1}) \) can be factorized as

\[
F = GM^*G^*.
\]

(b) The factorization method for obstacles with impenetrable and penetrable components.

Here, the data-to-pattern operator \( G : H^{1/2}(\partial D_1) \times L^2(D_2) \rightarrow L^2(S^{n-1}) \) is given by

\[
G \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = v^\infty
\]

where \( v^\infty \) is the far field pattern of the solution to (5)–(7) with \( f_1 \in H^{1/2}(\partial D_1) \) and \( f_2 \in L^2(D_2) \). The middle operator \( M : H^{-1/2}(\partial D_1) \times L^2(D_2) \rightarrow H^{1/2}(\partial D_1) \times L^2(D_2) \) has the form

\[
M = M_0 + M_1 := \begin{pmatrix} -S_{1i} & 0 \\ 0 & \frac{|q|}{k^2 q} \end{pmatrix} + M_1
\]

with some compact operator \( M_1 \). Here, \( S_{1i} \) denotes the single layer operator \( S_1 \) with the wave number \( k = i \).

(c) \( G \) is compact with dense range in \( L^2(S^{n-1}) \). For any \( z \in \mathbb{R}^n \), define \( \phi_z \in L^2(S^{n-1}) \) as in (11), then \( z \in D \) if, and only if, \( \phi_z \) belongs to the range \( \mathcal{R}(G) \) of \( G \).

For any \( (\phi, \psi)^T \in H^{-1/2}(\partial D_1) \times L^2(D_2) \) define

\[
w(x) = \int_{\partial D_1} \phi(y)\Phi(x, y)ds(y) + \int_{D_2} \psi(y)\Phi(x, y)\sqrt{|q(y)|}dy, \quad x \in \mathbb{R}^n.
\]

We set

\[
f_1 := -w|_{\partial D_1} = -\int_{D_2} \psi(y)\Phi(x, y)\sqrt{|q(y)|}dy|_{\partial D_1} - S_1\phi
\]

and

\[
f_2 := -\sqrt{|q|}w + \frac{|q|}{k^2 q}\psi.
\]

Then \( w \) is a radiating solution (in the weak sense) of

\[
\Delta w + k^2(1 + q)w = -k^2 \frac{q}{\sqrt{|q|}}f_2 \quad \text{in } \mathbb{R}^n \setminus \overline{D_1}, \quad \text{(33)}
\]

\[
w = -f_1 \quad \text{on } \partial D_1. \quad \text{(34)}
\]
Recall that the operator \( M : H^{-\frac{1}{2}}(\partial D_1) \times L^2(D_2) \to H^{\frac{1}{2}}(\partial D_1) \times L^2(D_2) \) in the factorization (30) is given by

\[
M\left( \begin{array}{c} \phi \\ \psi \end{array} \right) := \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right).
\]

We claim that the operator \( M \) is injective under the condition that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( D_1 \). Indeed, assume that

\[
M\left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
\]

for some \( \phi \in H^{-\frac{1}{2}}(\partial D_1) \) and \( \psi \in L^2(D_2) \) which means

\[
f_1 = -\int_{D_2} \psi(y) \Phi(x,y) \sqrt{q(y)} dy \bigg|_{\partial D_1} - S_1 \phi = 0 \quad \text{on} \quad \partial D_1,
\]

\[
f_2 = \frac{|q|}{k^2 q} \psi - \sqrt{|q|} v = 0 \quad \text{in} \quad D_2.
\]

The uniqueness result of the direct problem (see Theorem 2.2 in [12]) implies that \( v = 0 \) in \( \mathbb{R}^n \setminus \overline{D_1} \) and therefore \( \psi = 0 \) in \( D_2 \) and \( S\phi = 0 \) on \( \partial D_1 \). The assumption that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( D_1 \) implies \( S \) is an isomorphism (see Lemma 1.14 in [10]) and thus \( \phi = 0 \) on \( \partial D_1 \).

Recall the assumption (14), i.e., their exists \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( c_0 > 0 \) such that

\[
\Re\left( \frac{e^{it}}{|q|} \right) \leq -c_0 \quad \text{a.e. in} \quad D_2.
\]

We note that \( \Re(e^{it}) = \cos(t) > 0 \) since \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). It is known that the operator \( S_i \) is self-adjoint and coercive, i.e., there exists \( c_1 > 0 \) with

\[
\langle \phi, S_i \phi \rangle \geq c_1 \| \phi \|^2_{H^{-1/2}(\partial D_1)}
\]

for all \( \phi \in H^{-1/2}(\partial D_1) \) (see Lemma 1.14 in [10]) and this implies

\[
\Re(\langle \phi, e^{-it} S_i \phi \rangle) \geq c_0 \| \phi \|^2_{H^{-1/2}(\partial D_1)}
\]

for all \( \phi \in H^{-1/2}(\partial D_1) \). For all \( \psi \in L^2(D_2) \), we have

\[
-\Re\left( \psi, e^{-it} \frac{|q|}{k^2 |q|} \psi \right) = -\int_{D_2} \Re\left( \frac{e^{it} q}{k^2 |q|} \right) |\psi|^2 dx \geq \frac{c_0}{k^2} \| \psi \|^2_{L^2(D_2)}.
\]

Thus, for those contrasts \( q \) satisfying (14), the operator \( -\Re(e^{-it} M_0) \) is coercive, i.e., there exists \( c > 0 \) with

\[
-\Re\left( \begin{array}{c} \phi \\ \psi \end{array} \right), e^{-it} M_0 \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right) \geq c(\| \phi \|^2_{H^{-1/2}(\partial D_1)} + \| \psi \|^2_{L^2(D_2)})
\]

for all \( \phi \in H^{-1/2}(\partial D_1) \) and \( \psi \in L^2(D_2) \).

We summarize these results in the following theorem.

**Theorem 3.3.** Let \( M : H^{-\frac{1}{2}}(\partial D_1) \times L^2(D_2) \to H^{\frac{1}{2}}(\partial D_1) \times L^2(D_2) \) be defined as (35).

(a) Assume that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( D_1 \). Then \( M \) is injective.

(b) In addition to Assumption 3.1, let \( q \in L^\infty(D_2) \) satisfy (14). Then \( -\Re(e^{-it} M_0) \) is coercive.

Combining Theorems 2.1, 3.2 and 3.3, we obtain the second main result of this paper.
Theorem 3.4. In addition to Assumption 3.1 we further assume that \( q \) satisfies (14) and \( k^2 \) is not a Dirichlet eigenvalue of \(-\Delta\) in \( D_1 \). For any \( z \in \mathbb{R}^n \) define again \( \phi_z \in L^2(S^{n-1}) \) by (11), i.e.,

\[
\phi_z(\hat{x}) = e^{-ikz \cdot \hat{x}}, \quad \hat{x} \in S^{n-1}.
\]

Then

\[
z \in D \iff \phi_z \in \mathcal{R}(F_z^{1/2})
\]

and consequently

\[
z \in D \iff \sum_{j=1}^{\infty} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(S^{n-1})}|^2}{|\lambda_j|} < \infty,
\]

where \((\lambda_j, \psi_j)\) is the eigensystem of the operator \( F_z : L^2(S^{n-1}) \to L^2(S^{n-1}) \) given by \( F_z := |\Re(e^{-it}F)| + |\Im(F)| \). In other words, the sign of the function

\[
W(z) = \left[ \sum_{j=1}^{\infty} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(S^{n-1})}|^2}{|\lambda_j|} \right]^{-1}
\]

is just the characteristic function of \( D \).

4. Numerical tests. In this section, we present some numerical examples in two dimensions to illustrate the applicability and effectiveness of the factorization method. The scatterer \( D \) under concern is given as the union of two disjoint components \( D = D_1 \cup D_2 \), where \( D_1 \) located in the south-west corner of Figure 1 is an ellipse with axes 1, 0.7, and center \((-4, -4)\) while \( D_2 \) is a kite-shaped domain with boundary described by the parametric representation

\[
x(t) = (3.5 + \cos t + 0.65\cos 2t - 0.65, 3.5 + 1.5\sin t), \quad t \in [0, 2\pi].
\]

The sampling region \( G \) is deliberately chosen as \([-8, 8]^2\) and a 321 \times 321 uniform mesh is used to cover the sampling region \( G \).

![Image of the original domain](image)

**Figure 1.** The original domain.

There are totally four groups of numerical tests to be considered, and they are respectively referred to as **Hard**, **Medium1**, **Medium2** and **Medium3**. In all examples, we imposed Dirichlet boundary condition on \( \partial D_1 \) and took two wave numbers \( k = 2 \) and \( k = 4 \). To generate the measured far field data \( F \) and the artificial far field data \( F_D \) and \( F_N \), we used the boundary integral equation method.
For the numerical treatment of the integral equations, the Nyström method, with 64 quadrature points on each boundary and each interface, has been applied. The following pictures show the values of the function

$$W(z) = \left[ \sum_{j=1}^{64} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(S^{n-1})}|^2}{|\lambda_j|} \right]^{-1},$$

(41)

where $(\lambda_j, \psi_j)$ is the eigensystem of the operators defined in Theorems 2.5 and 3.4.

**Example Hard.** We imposed the Neumann boundary condition on $\partial D_2$. We assume that the ellipse is contained in a disk $\Omega_1$ of radius 2.5 with center $(-4, -4)$ while the “kite” is contained in a disk $\Omega_2$ of radius 2.5 with center $(3.5, 3.5)$. Because we are only interested in the sampling points located in $\Omega_1 \cup \Omega_2$, the values of the function $W$ are set to be 0 deliberately for those points lying outside of $\Omega_1 \cup \Omega_2$. For the sampling points located in $\Omega_1$, $W$ is computed by using the eigensystem of the operator $F_{D_2} : L^2(S^{n-1}) \to L^2(S^{n-1})$, while for those located in $\Omega_2$, it is computed by eigensystem of the operator $F_{N_2} : L^2(S^{n-1}) \to L^2(S^{n-1})$. Pictures 2 and 3 show the reconstruction for different wave numbers $k$ and parameters $\rho$.

**Figure 2.** Reconstructions for the Example Hard with $k = 2$ and $\rho = 0$ (left), $\rho = 1$ (right).

**Figure 3.** Reconstructions for the Example Hard with $k = 4$ and $\rho = 0$ (left), $\rho = 1$ (right).
We observe that the reconstruction is independent of the choice of the parameter \( \rho \). In particular, we note that the factorization method works for \( \rho = 0 \). However, a rigorous proof for this case is still not established.

**Example Mediumq1.** In the second example, \( q_1 = -0.5 + 0.5i \) in \( D_2 \) which satisfies the relation (14) with \( t = 0 \). Picture 4 shows the reconstruction.

![Figure 4](image)

**Figure 4.** Reconstructions for the Example Medium1 with \( k = 2 \) (left) and \( k = 4 \) (right).

**Example Mediumq2.** In the third example, \( q_2 = 0.5 + 0.5i \) in \( D_2 \) which satisfies the relation (14) with \( t = \pi/3 \). Figure 5 shows the reconstruction. Note that \( q_2 \) does not satisfy the relation (14) with \( t = 0 \). Numerical experiment (see Figure 6) indicates that the factorization method with \( t = 0 \) still works but the theory analysis is not established.

![Figure 5](image)

**Figure 5.** Reconstructions for the Example Mediumq2 with \( t = \pi/3 \) and two wave numbers \( k = 2 \) (left) and \( k = 4 \) (right).

**Example Mediumq3.** From the previous section, we observe that the reconstructions are independent of the choice of \( t \). In this example we took \( q_2 = 0.5 \) in \( D_2 \) which even does not satisfy the relation (14) for any \( t \). However, numerical experiment (see Figure 7) indicates that the factorization method still works.

5. **Conclusions.** In this paper, we provide some modification of the factorization method for scatterers with different physical properties. It has been shown that the
factorization method works numerically without any additional assumption, i.e.,
one does not need to know the type of obstacle - penetrable or impenetrable - in
advance. Some additional assumptions are imposed to justify the validity of the
modified factorization method. Numerical results are satisfactory and thus verify
our theoretical analysis.

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