Abstract
In the paper, the question whether truth values can be assigned to the propositions representing
 categorical properties that a physical system possesses prior to their measurement is discussed.
 To answer this question, a notion of a propositionally noncontextual theory that can provide a
 map linking each element of a bounded lattice to a truth value in order to explain the outcomes
 of the experimental propositions associated with the state of the system is introduced. The pa-
 per demonstrates that no model based on such a theory can be consistent with the occurrence
 of a non-vanishing “two-path” quantum interference term and the quantum collapse postulate.

Keywords: Contextuality, Propositional logic, Truth-value assignment, Many-valued logic,
Pairwise decidability, Higher-order interference.

1 Introduction
As it is known, the presence of contextuality in quantum theory makes it impossible to view a mea-
 surement as merely revealing pre-existing properties of a quantum system [1]. More specifically,
 no hidden-variable model in quantum theory can assign \{0, 1\}-valued outcomes to the projectors
 of the measurement in a way that depends only on the projector and not the context in which it
 appeared, even though the Born probabilities associated with those projectors are independent of
 the context.\footnote{For an overview of different aspects of contextuality in quantum theory and beyond, see [2, 3]. Also, see a review
 of the framework of ontological models in [4, 5].}

On the other hand, thanks to a trivial probability function mapping true propositions to prob-
 ability 1 and false propositions to probability 0, the sum of all the Born probabilities after the
 measurement can be presented as a disjunction of all the corresponding propositions where exactly
 one proposition is true while the others are false. This naturally raises the question whether one
 can assign truth values to the propositions about properties of a state of a physical system before
 the measurement. In more general terms, can the assignment of pre-existing truth values to lattice
 elements, particularly, elements of a bounded lattice, be always made available in quantum theory?
In this paper, we introduce a notion of a propositionally noncontextual theory, which can provide a map linking each element of a bounded lattice to a truth value so as to explain the outcomes of experimental propositions associated with the state of the system. Using a quantum version of the double-slit experiment as an example, this paper demonstrates that no model based on such a theory can agree with the occurrence of a non-vanishing “two-path” quantum interference term and the quantum collapse postulate.

2 Pairwise decidability in a double-slit interference set-up

Let us consider a double-slit quantum interference set-up in which detectors placed just behind slits 1 and 2 register a particle passing through a corresponding slit. Let \( \{X_1\} \) denote the proposition of a click of the detector behind slit 1 such that \( \{X_1\} = 1 \) (“true”) if the detector 1 clicks (registering in this way that the particle has indeed passed through slit 1) and \( \{X_1\} = 0 \) (“false”) if this detector does not click (thus registering that the particle has in fact not passed through slit 1); in this manner, after the registration the truth value of the proposition \( \{X_1\} \) can be regarded as a random variable whose sample space \( \Omega_1 \) is given by \( \{0, 1\} \). Let \( \{X_2\} \) analogously denote the proposition of the second detector’s click.

Let us introduce the proposition:

\[
\{X_1, X_2\} \equiv \{X_1\} \lor \{X_2\},
\]

where the symbol \( \lor \) stands for the associative and commutative operation of exclusive disjunction that outputs 1 when one of its inputs is 1 and the other is 0. This proposition corresponds to the assertion that the particle passes through exactly one slit – either 1 or 2. Provided the truth value of \( \{X_1, X_2\} \) may be denoted as a bivariate random variable, the proposition \( \{X_1, X_2\} \) asserts that after the which-slit information has been registered (i.e., after the detectors have registered the particle’s passage through either slit), the sample space \( \Omega_{12} \) of this variable is given by \( \{X_1, X_2\} = \{(1, 0), (0, 1)\} \).

To keep things general, let us express the proposition \( \{X_1, X_2\} \) in the lattice-theoretic terms as follows

\[
x_{12} \equiv \left( \bigvee_i x_i \right) \land \left( \bigwedge_i x_i \right),
\]

where \( x_{12} \) and \( x_i \) are elements of a bounded lattice \( L = (L, \land, \lor, 1, 0) \) identified with the propositions \( \{X_1, X_2\} \) and \( \{X_i\} \), respectively, such that each \( y \in L \) satisfies the condition \( 0 \leq y \leq 1 \) in which 0 is the lattice’s bottom and 1 is the lattice’s top; both the lattice join \( \lor \) and the lattice meet \( \land \) are order-preserving operations over \( L \) defined correspondingly as a greatest lower bound and a least upper bound of pair of elements \( y, z \in L \), namely, \( y \land z = g.l.b.(y, z) \) and \( y \lor z = l.u.b.(y, z) \).
In addition to the binary operations $\land$ and $\lor$, the lattice $\mathcal{L}$ contains a unary operation $-$ defined such that $-0 \equiv 1$ as well as $-1 \equiv 0$ (which causes $-0$ to be a complement of the lattice’s bottom and $-1$ to be a complement of the lattice’s top).

Let us introduce the following definition: Let $\Upsilon$ denote a non-empty set of truth-values having a range with 1 and 0, $Y$ stand for an experimental proposition associated with a state of a physical system and $y \in L$ be a lattice element identified with the proposition $Y$. Then, a theory will be defined as propositionally noncontextual if to explain (predict) the registration outcomes (i.e., the truth values of the proposition $Y$) it provides a map $\mathcal{M} : L \to \Upsilon$, specifically, $Y = \mathcal{M}(y)$. Correspondingly, a theory, which is not propositionally noncontextual, will be defined as propositionally contextual.²

Consider the following valuational axioms: (a) $Y \lor Z = \mathcal{M}(y \lor z) = \max\{\mathcal{M}(y), \mathcal{M}(z)\}$; (b) $Y \land Z = \mathcal{M}(y \land z) = \min\{\mathcal{M}(y), \mathcal{M}(z)\}$; (c) $-Y = \mathcal{M}(-y) = 1 - \mathcal{M}(y)$. Using them, one can valuate the proposition $\{\{X_1, X_2\}\}$ as follows

$$\{\{X_1, X_2\}\} = \begin{cases} \min\{\mathcal{M}(x_2), 1 - \mathcal{M}(x_1)\}, & \mathcal{M}(x_1) \leq \mathcal{M}(x_2) \\ \min\{\mathcal{M}(x_1), 1 - \mathcal{M}(x_2)\}, & \mathcal{M}(x_2) \leq \mathcal{M}(x_1) \end{cases}. \quad (3)$$

Let us analyze the following assumption: Even before the registration, the lattice elements $x_i$ in the formula (2) can be assigned truth values. Otherwise stated, it is conceivable that the propositions $\{\{X_i\}\}$ are in possession of pre-existing (i.e., existing before the detectors’ clicks) truth values which are either merely revealed or somehow transformed by the registration. And so, it is possible to predict the registration outcomes using a propositionally noncontextual quantum theory.

In agreement with this assumption, let us suppose that before the registration the proposition $\{\{X_1, X_2\}\}$ is already true, that is, $\mathcal{M}(x_{12}) = 1$. As a consequence, one gets $\mathcal{M}(\vee^2 x_i) = 1$ and $\mathcal{M}(\wedge^2 x_i) = 0$.

Let $P[\cdot]$ be a probability function mapping proposition $A, B, \ldots$ to the real interval $[0, 1]$ such that $P[A] = 1$ if $A = 1$, $P[A] = 0$ if $A = 0$, and $P[A \lor B] = P[A] + P[B] - P[A \land B]$. Along these lines, the probability function $P[\cdot]$ can be considered as the degree of belief that the corresponding proposition is true (or it is expecting to be true).³

As $\vee^2(\{X_i\}) = \mathcal{M}(\vee^2 x_i) = 1$ while $\wedge^2(\{X_i\}) = \mathcal{M}(\wedge^2 x_i) = 0$, the probability function mapping the proposition $\{\{X_1, X_2\}\}$ to the interval $[0, 1]$ can be written by the sum of the probabilities $P(\{\{X_1\}\}) + P(\{\{X_2\}\}) = 1$. So, were the pre-existing truth values of the propositions $\{\{X_1\}\}$ and $\{\{X_2\}\}$ to be such that $\mathcal{M}(x_{12}) = 1$, the interference pattern $p_{12}$ in the two-slit set-up with none of the detectors registering the particle’s path would be the sum of one-slit patterns $p_1$ and $p_2$, namely, $p_{12} = p_1 + p_2$, and thus the second-order interference term $I_{12} \equiv p_{12} - p_1 - p_2$ would be absent.⁴

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²This definition is motivated by one introduced in the paper [6].

³This approach to the generalization of the notion of a probability function allows to accommodate variation in the background logic of the account while maintaining the core of standard probability theory [7].

⁴The hierarchy of interference terms and corresponding sum-rules is given in accordance with [8].
By contrast, let us suppose that before the registration \( M(x_{12}) = 0 \). But then, in the case \( M(\sqrt{2}x_i) = 1 \), one would find – in contradiction to the quantum collapse postulate – that \( \wedge_i^2(\{X_i\}) = 1 \), that is, it is not true that only one detector will click if the particle’s passage through the slits is observed.

As a result, if the propositions \( \{\{X_1\}\} \) and \( \{\{X_2\}\} \) are in possession of pre-existing truth values, then these values must obey the following two conditions: \( M(x_{12}) \neq 1 \) and \( M(x_{12}) \neq 0 \).

Clearly, for such inequalities to be concurrently in force, the set of the pre-existing truth values must contain more than 1 and 0. Specifically, the pre-existing truth values of \( \{\{X_1\}\} \) and \( \{\{X_2\}\} \) must be some middle elements of the totally ordered more-than-two-element set \( \{0, \ldots, m_i, \ldots, 1\} \) where \( m_i \) are ‘middle’ truth values ordered by an order \( 0 < \cdots < m_i < m_{i+1} \cdots < 1 \) based on the degree of truth (showing that the value \( m_{i+1} \) is more true than the value \( m_i \)).

The set \( \{0, \ldots, m_i, \ldots, 1\} \) can be the truth values of Kleene (strong) logic \( K_3 \) (where the only middle element \( m_1 \) stands for the "undefined" or "indeterminate" truth value which should be treated ontologically and not confused with the macroscopic epistemological unknown) or Gödel logics \( G_N \) or a finitely-valued family of Łukasiewicz logics \( L_N \) [9, 10, 5].

In this manner, the assignment of the pre-existing truth values to the lattice elements \( y \in L \) must be a map \( M : L \to \Upsilon \) in which \( \Upsilon \) represents a set \( \{0, \ldots, m_i, \ldots, 1\} \) whose cardinality is greater than or equal to 3.

Then again, because after the registration, the truth values of the propositions \( \{\{X_1\}\} \) are only members of the two-element set \( \{0, 1\} \), during the registration the middle truth values are expected to be transformed (converged) to either 0 or 1. This means that a propositionally noncontextual theory must provide an order-preserving map, which can be abbreviated by:

\[
Y = M(y) = M_a(M_b(y))
\]

where \( M_b \) denotes a truth-function that associates \( y \) with truth values before the registration, while \( M_a \) is a function that during the registration maps the elements of \( \{0, \ldots, m_i, \ldots, 1\} \) to the elements of \( \{0, 1\} \) with respect to the order of \( m_i \).

Along with the given presupposition, assume that before the registration, the middle truth values of the propositions \( \{\{X_1\}\} \) and \( \{\{X_2\}\} \) are both \( m_i \), namely, \( M_b(x_i) = m_i \). This implies that the pre-existing truth values of \( \{\{X_1, X_2\}\} \) will be neither true nor false:

\[
M_b(x_{12}) = \min\{m_i, 1-m_i\}
\]

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5The foregoing discussion is in line with the arguments asserting that many-valued logic is a proper tool for description of not-yet-measured (i.e. pre-existing) properties of quantum objects, see, for example, [11] [12] [13].
According to this expression, for the proposition \(\{X_1, X_2\}\) to turn into true during the registration, i.e., to get \(M_a(M_b(x_{12})) = 1\), the middle values \(m_i\) and \(1 - m_i\) must be converged to 1. However, this would cause a contradiction. Let’s take the case \(m_i \leq (1 - m_i)\), it would imply \(\{X_1, X_2\} = M_a(m_i) = 1\) and also \(\{X_1\} = \{X_2\} = M_a(m_i) = 1\); but then it would be \(1 \lor 1 = 0\), meaning that during the registration the proposition \(\{X_1, X_2\}\) would become both true and false.

Consequently, it is impossible to assign middle truth values \(m_i\) to pairs of \(\{X_1\}\) and \(\{X_2\}\) such that at some point in the registration they will always noncontradictory become elements of the ordered set \(\{0, 1\}\). One may have no other alternative here but to conclude that the propositions \(\{X_1\}\) and \(\{X_2\}\) in \(\{X_1, X_2\}\) cannot be in possession of pre-existing truth values. This could mean that there is no algorithm telling us how to compute the value of the truth-function \(M_b(x_{12})\) given the elements \(x_i\) of the lattice \(L\) (i.e., the truth-function \(M_b(x_{12})\) is undecidable).\(^6\)

However, if they are registered independently of each other, the propositions \(\{X_1\}\) can be considered having pre-existing truth values without bringing about a contradiction.

To show this, let us consider double-slit set-up where only one slit is opened while the other is made unobtainable (for example, by using a mask). Provided that masking a slit \(l \in \{1, 2\}\) is equivalent of identifying the corresponding proposition \(\{X_l\}\) with the lattice’s bottom 0, the element \(x_l = 0\) will correspond to the always-false proposition, i.e., \(\{X_l\} = M(x_l) = 0\) (which is another way to say that the detector behind the blocked slit \(l\) will never click).

By blocking slit \(l \in \{1, 2\}\), one gets that the pre-existing truth values of the proposition \(\{X_1, X_2\}\) in the double-slit set-up with a single slit opened are determined with

\[
M_b \left( \bigvee_{l=1}^2 x_{12x_l=0} \right) = \begin{cases} 
M_b(x_1), & x_2 = 0 \\
M_b(x_2), & x_1 = 0 
\end{cases}.
\]

(6)

Now suppose \(M_a(M_b(\cdot)) = 1\): It would imply either \(\{X_1\} = M_a(M_b(x_1)) = 1\) and \(\{X_2\} = 0\) or \(\{X_1\} = 0\) and \(\{X_2\} = M_a(M_b(x_2)) = 1\), which would produce \(\{X_1\} \lor \{X_2\} = 1\), meaning that during the registration the proposition \(\{X_1, X_2\}\) would become true noncontradictory.

Consequently, the proposition \(\{X_1, X_2\}\), which has no pre-existing truth values in the set-up with both slits opened, cannot be presented as logical disjunction \(\{X_1, 0\} \lor \{0, X_2\}\) on pre-existing truth values of the propositions \(\{X_1\}\) and \(\{X_2\}\) in the set-ups with a single slit opened. This can be expressed by the following inequality:

\[
M_b(x_{12}) \neq M_b \left( \bigvee_{l=1}^2 x_{12x_l=0} \right). \quad (7)
\]

\(^6\)In accordance with the definition of a decidable logical system (see, for example [13]), the truth-function \(M_a(x_{12})\) cannot be decidable since the truth-table method cannot be used to determine whether the propositional formula for \(\{X_1, X_2\}\) is logically valid before the registration.
As $P(\{X_1, 0\}) \lor (\{0, X_2\}) = P(\{X_1\}) + P(\{X_2\})$, following the above inequality one must get a deviation of the prior probability $P(\{X_1, X_2\})$ from the classical (Kolmogorovian) sum rule, namely, $P(\{X_1, X_2\}) \neq P(\{X_1, 0\}) \lor (\{0, X_2\})$, which entails a presence of the interference term when the particle’s passage through the slits is not observed. So therefore, two-path quantum interference can be written off as the inequality (7) (on a par with the violation of the second-order sum-rule $I_{12} \neq 0$).

The fact that the propositions (\{X_1\}) and (\{X_2\}) if taken independently may well be in possession of pre-existing truth values, whereas the prediction map (4) for (\{X_1, X_2\}) cannot be provided, means that the truth values of the propositions (\{X_1\}) and (\{X_2\}) are not pairwise decidable and so a propositionally noncontextual quantum theory is not possible.

3 Absence of high-order quantum interference

The chain of two operations of exclusive disjunction, namely, $\bigvee_i^3 (\{X_i\}) = (\{X_j, X_k\}) \lor (\{X_i\}) \lor (\{X_i\}) \lor (\{X_i\})$, can be written in terms of the propositions of exactly one true outcome in the double-slit set-ups (\{X_j, X_k\}) and the corresponding propositions in the single-slit set-ups (\{X_i\}) as follows:

$$\bigvee_i^3 (\{X_i\}) \equiv \bigvee_{i \neq j < k} (\{X_j, X_k\}) \lor (\{X_i\}) \quad .$$

After the registration, the propositional formula (8) is true in the case that only one of (\{X_i\}) has truth value 1 or all of them have truth value 1. Consequently, the proposition of exactly one true outcome in a triple-slit set-up (\{X_1, X_2, X_3\}) should be presented by the propositional expression

$$\langle X_1, X_2, X_3 \rangle \equiv (\bigvee_i^3 (\{X_i\})) \land (\bigwedge_i^3 (\{X_i\})) \quad .$$

In view of that, the proposition (\{X_1, X_2, X_3\}) can be expressed in the lattice-theoretic terms as follows

$$x_{123} \equiv \left( \bigvee_{i \neq j < k} (x_{jk} \lor x_i) \land (x_{jk} \land x_i) \right) \land \left( \bigwedge_i^3 x_i \right) \quad ,$$

where $x_{123}$ is the element of the lattice $\mathcal{L}$ identified with the proposition (\{X_1, X_2, X_3\}) and the following abbreviation is used: $x_{jk} \equiv (x_j \lor x_k) \land (x_j \land x_k)$.

The proposition (\{X_1, X_2, X_3\}) cannot be in possession of pre-existing truth values. Indeed, the truth-function $M_6(x_{123})$ is determined with the expression

$$M_6(x_{123}) = \min\left\{ M_6(y), 1 - \min_i \{M_6(x_i)\} \right\} \quad ,$$

6
where $M_b(y)$ stands for

$$M_b(y) \equiv \max_{i \neq j < k} \left\{ \min \{ \max\{M_b(x_{jk}), M_b(x_i)\}, 1 - \min\{M_b(x_{jk}), M_b(x_i)\} \right\}.$$  \hspace{1cm} (12)

Thus, if $(\{X_1, X_2, X_3\})$ were to possess pre-existing truth values, the truth-functions $M_b(x_{jk})$ would exist in direct contradiction to the conclusion reached in the previous section.

Suppose that the lattice element $x_l$ turns to zero because of the blocking of slit $l \in \{1, 2, 3\}$. Putting zero in place of any $x$ in (11) and (12) gives that in the triple-slit set-up with only two slits opened the truth-function $M_b(x_{123})$ takes the following form

$$M_b \left( \bigvee_{l=1}^{3} x_{123x_l=0} \right) = \begin{cases} M_b(x_{12}), & x_3 = 0 \\ M_b(x_{13}), & x_2 = 0 \\ M_b(x_{23}), & x_1 = 0 \end{cases}.$$  \hspace{1cm} (13)

Consistent with the inequality (7), it reasonable to imply that in order for a non-vanishing “three-path quantum interference” term to be present, the truth-function $M_b(x_{123})$ would have been obliged to be differentiated from the truth-function (13). However, since those functions cannot be set apart as different – both functions do not exist and for this reason cannot be distinguished from each other – there is no point of difference between the proposition $(\{X_1, X_2, X_3\})$ and the logical disjunction $(\{X_1, X_2, 0\}) \lor (\{X_1, 0, X_3\}) \lor (\{0, X_2, X_3\})$ prior to the registration. This implies the equivalence of the prior probabilities $P[(\{X_1, X_2, X_3\})] = P[(\{X_1, X_2, 0\}) \lor (\{X_1, 0, X_3\}) \lor (\{0, X_2, X_3\})]$ and, therefore, the fact that the interference pattern in the triple-slit set-up $p_{123}$ can be computed from contributions of pairs of slits $p_{jk}$ only, that is, $p_{123} = p_{12} + p_{13} + p_{23} - \sum_i p_i$ (in other words, the third-order sum-rule $I_{123} = 0$ holds).

It must be noticed that the sort of contextuality described above does not arise in quantum logic. The reason is as follows. In an interference set-up with any number of spatially arranged slits, the information about the slit the particle has traversed is encoded by commuting projections (representing outcomes in a von Neumann measurement of the particle’s position). Therefore, the truth values of the propositions associated with such projections must be simultaneously (or jointly) decidable in accordance with quantum logic [15].

### 4 Concluding remarks

Suppose a double-slit quantum interference experiment is described in the following manner: After the registration, the proposition that the particle passes through exactly one slit is true. Now, let us ask the question, is this a complete description of the quantum interference experiment?

The first answer is no: In a complete description, the particle passes through either slit regardless of the registration (i.e., the context). Any specific proposition about the properties of the state of the particle can be either true or false as well as neither true nor false no matter whether the
detector registers this proposition. Accordingly, in the complete description, the elements of the partially ordered set $L(\mathcal{H})$ of all closed subspaces of a separable, infinite dimensional Hilbert space $\mathcal{H}$ represent categorical properties that a system possesses independently of whether the system is observed or not.

The second answer is *yes*: Prior to the detectors’ clicks, the particle is by no means has passed through either slit. If both slits are opened, the passage through the slit only comes about when the corresponding detector records it. Consequently, the proposition that the particle passes through exactly one slit can be true only after the detectors have registered the particle’s passage through either slit.$^7$

It is clear that the assumption of pre-existing truth values of the propositions $(\{X_1\})$ and $(\{X_2\})$ in the formula for $(\{X_1, X_2\})$ coincides with the first answer. Whereas the conclusion, as per which $(\{X_1\})$ and $(\{X_2\})$ are not pairwise decidable and, as a result, a propositionally noncontextual quantum theory is not possible, corresponds to the second answer.

Evidently, the lack of pairwise decidability in the truth-value assignment even when the $\{0, 1\}$-valued observables encoding propositions about the state of the system are commensurable, puts in doubt whether the elements of a bounded lattice can be always regarded as having a truth-valued interpretation. What is more, this may suggest that quantum logic (as a modified version of propositional logic) cannot be stretched to adequately account for (at least) some measurement processes corresponding to answering yes-no questions about the state of the quantum system. Otherwise stated, not only the distributive laws of propositional logic have no universal validity in quantum theory but perhaps the truth-value assignment also.

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$^7$One can easily notice that the description of a double-slit interference experiment presented above bears a great deal of similarity to Einstein’s example of a particle confined to a two-chambered box. See the detailed analysis of this example in [10].
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