COMMUNICATION-OPTIMAL TILINGS FOR PROJECTIVE NESTED LOOPS WITH ARBITRARY BOUNDS

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ABSTRACT. Reducing communication - either between levels of a memory hierarchy or between processors over a network - is a key component of performance optimization (in both time and energy) for many problems, including dense linear algebra [BCD+14], particle interactions [DGK+13], and machine learning [DD18, GAB+18]. For these problems, which can be represented as nested-loop computations, previous tiling based approaches [CDK+13, DR16] have been used to find both lower bounds on the communication required to execute them and optimal rearrangements, or blockings, to attain such lower bounds. However, such general approaches have typically assumed the problem sizes are large, an assumption that is often not met in practice. For instance, the classical (# arithmetic operations)/(cache size)\(^{1/2}\) lower bound for matrix multiplication [HK81, BCD+14] is not tight for matrix-vector multiplications, which must read in at least \(O(#\text{ arithmetic operations})\) words of memory; similar issues occur for almost all convolutions in machine learning applications, which use extremely small filter sizes (and therefore, loop bounds).

In this paper, we provide an efficient way to both find and obtain, via an appropriate, efficiently constructible blocking, communication lower bounds and matching tilings which attain these lower bounds for nested loop programs with arbitrary loop bounds that operate on multidimensional arrays in the projective case, where the array indices are subsets of the loop indices. Our approach works on all such problems, regardless of dimensionality, size, memory access patterns, or number of arrays, and directly applies to (among other examples) matrix multiplication and similar dense linear algebra operations, tensor contractions, \(n\)-body pairwise interactions, pointwise convolutions, and fully connected layers.

1. INTRODUCTION

Many structured computations, including dense linear algebra, \(n\)-body problems, and many machine learning kernels, can be expressed as a collection of nested loops, where each iteration accesses elements from several multidimensional arrays, indexed by some function of the current loop iteration function:

\[
\text{for } x_1 \in [L_1], \cdots, \text{for } x_d \in [L_d] : \\
\text{perform operations on } A_1[\phi_1(x_1, \ldots, x_d)], \ldots, A_n[\phi_n(x_1, \ldots, x_d)].
\]

(1.1)

where \([L_1]\) represents the set \([1, \ldots, L_1]\). For many such problems, the time and energy costs of communication - that is, moving data between different levels of the memory hierarchy, or between different cores or processors - can significantly outweigh the cost of computation in practice [BCD+14]. For example, communication-optimized implementations of matrix multiply [HK81, BCD+14], \(n\)-body problems [DGK+13], and convolutional neural nets [GAB+18], among others, have significantly outperformed their non-communication-optimized counterparts. Therefore, rearranging the order in which we perform these operations by dividing the nested loops into subsets called tiles which are executed in sequence can lead to significantly improved results in practice.

Date: March 3, 2020.

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Most previous applied work, including that cited above, has been focused on finding communication-optimal tilings and lower bounds for specific problems. While this is useful for commonly used kernels whose optimizations can impact performance across a large number of applications (e.g. matrix multiply, convolutions), it is less practicable to develop new theory for and hand-optimize algorithms whose applications fall into smaller niches. This has stymied research into, for instance, unconventional neural net architectures such as capsule networks [HSF18], which require optimized kernels to test at scale but lack such kernels due to being unproven and not widely used [BI19].

Progress has also been made [CDK+13, DRI16] in generalizing some of these techniques by considering communication patterns via the Brascamp-Lieb inequalities, which apply to any loop nest where the array indices are affine functions of the loop indices (i.e. the $\phi_i$ above are affine). These methods provide both communication lower bounds and constructions for tilings for such problems.

Unfortunately, the above lines of work have largely ignored situations when certain loop bounds ($L_i$, above) are small. In this case, the methods can produce weak lower bounds and infeasible tilings. Take, for instance, the case of matrix multiplication:

$$\text{for } \{x_1, x_2, x_3\} \in [L_1] \times [L_2] \times [L_d]$$

$$A_1(x_1, x_3) = A_2(x_1, x_2) \times A_3(x_2, x_3)$$

Existing combinatorial and geometric [BCD+14], techniques states that a lower bound on the communication between a cache of size $M$ and main memory required to execute this set of instructions is

$$\Omega \left( \frac{L_1L_2L_3}{M^{1/2}} \right)$$

words of memory, and may be attained by rewriting the nested loops as follows:

$$\text{for } \{o_1, o_2, o_3\} \in [0..L_1/B_1 - 1] \times [0..L_2/B_2 - 1] \times [0..L_3/B_3 - 1]$$

$$\text{for } \{i_1, i_2, i_3\} \in [B_1] \times [B_2] \times [B_d]$$

$$x_1 = B_1o_1 + i_1$$

$$x_2 = B_2o_2 + i_2$$

$$x_3 = B_2o_2 + i_2$$

$$A(x_1, x_3) = A_2(x_1, x_2) \times A_3(x_2, x_3)$$

where the tile (the three inner loops) has dimensions $B_1 = B_2 = B_3 \approx \sqrt{M/3}$.

However, when $L_1 < \sqrt{M/3}$, this tiling becomes infeasible. Furthermore, the lower bound also ceases to be useful. For instance, when $L_3 = 1$, corresponding to a matrix-vector multiplication, the minimum communication needed to evaluate this multiplication is at least $L_1L_2$, since $A_2$ must be read in its entirety. However, the previous lower bound evaluates to $\Omega \left( \frac{L_1L_2}{M^{1/2}} \right)$, which is clearly unachievable.

[DD18] addresses this situation for convolutions, finding a separate lower bound (and a corresponding, feasible, tiling) for the case when the filter size is small (as they often are in most CNNs). In this paper, we apply the techniques from [DD18] to find a general communication lower bound and optimal tiling for arbitrary loop bounds in the case where the array accesses are all subsets of the loop bounds (the so-called “projective case”, which applies to most dense linear algebra applications, as well as point convolutions), and in doing so we prove that the optimal tile shape for a projective loop nest is always a rectangle. We review the proof in the large-bound case in Section 3 [BI19] present a stronger communication lower bound that encompasses bounds of arbitrary size in Section 4 [BI19] and present a linear program that gives the actual tiling required to achieve this lower bound (proving that it is tight) in Section 5. We then conclude with several examples and a discussion in Sections 6 [BI19] and 7.
2. Problem Setup, Preliminaries, Notation, and Definitions

Define \([n]\) to be the set \([1, 2, \ldots, n]\), and \([m, n]\) to be the set \([m, m + 1, \ldots, n - 1, n]\).

Formally, we will concern ourselves with the following \(d\)-level nested-loop program, which consists of operations on the elements of the \(d_1, \ldots, d_n\)-dimensional arrays \(A_1, \ldots, A_n\) indexed by affine functions \(\phi_i : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d_i}\) for \(i \in [n]\):

\[
\text{for } x_1 \in [L_1], \ldots, \text{for } x_d \in [L_d] : 
\phi_i (x_1, \ldots, x_d), \ldots, \phi_n (x_1, \ldots, x_d)
\]

for \(i \in [n]\) :

perform operations on \(A_1 [\phi_1 (x_1, \ldots, x_d)], \ldots, A_n [\phi_n (x_1, \ldots, x_d)]\)

This representation includes many commonly used matrix and tensor operations, including most linear algebra operations, tensor contractions, and convolutional neural nets.

We will assume that each \(x_i\) is present in the support of at least one of the \(\phi_i\); this assumption may be made without loss of generality as in [CDK+13].

Let us formally model the machine as follows: suppose we have a processor attached to a cache of size \(M\), which is in turn connected to a slow memory of unlimited size. The processor may only perform operations on elements of the arrays present in the cache, and we wish to find a reordering of the operations in (2.1) that minimizes the amount of communication between the cache and the slow memory.

[CDK+13] provides a tight lower bound for communication complexity in this model when \(L_1, \ldots, L_d\) are sufficiently large, as follows: First, represent each operation in (2.1), indexed by \(x_1, \ldots, x_d\), as the point indexed by the vector \((x_1, \ldots, x_d) \in \mathbb{Z}^d\). As a result, the entire set of operations represented by (2.1) can be treated as the hyper-rectangle of \(x \in [L_1] \times \ldots \times [L_d]\). Furthermore, note that the element of array \(A_i\) of memory required for the operation indexed by \((x_1, \ldots, x_d)\) is \(\phi_i (x_1, \ldots, x_d)\); in particular, given a set \(S \subset \mathbb{Z}^d\) of operations, the elements of \(A_i\) it requires are indexed by \(\phi_i (S)\). As a result, it suffices to find a lower bound on the number of subsets (or an upper bound on the size of a single subset) needed to tile the hyper-rectangle, with each one corresponding to a segment of the program that can be executed without going back to main memory. To satisfy this condition, we require that each subset ("tile") \(S\) satisfy the condition:

\[
|\phi_i (S)| \leq M
\]

as we cannot use more than \(M\) words memory in a computation without going to slow memory.

Invoking the discrete Brascamp-Lieb inequality [BCCT10, CDK+13], we get that any such tile has volume at most \(M^{\sum_{i \in [n]} s_i}\), where the \(s_i\) are the solutions to the linear program:

\[
\begin{align*}
\min \quad & \sum_{i \in [1..n]} s_i \quad \text{s.t.} \\
\sum_{j=1}^{n} s_i \text{rank}(\phi_i (H)) & \geq \text{rank}(H) \\
& \forall \text{subgroups } H \leq \mathbb{Z}^d
\end{align*}
\]

This implies that the minimum number of tiles needed to cover the entire hyper-rectangle is at least \(\prod_{i \in [1..d]} L_i / M^{\sum_{i \in [n]} s_i}\). Since each tile corresponds to an execution of a subset of operations without going back to slow memory, and we must complete all operations in the 'hypercube', the total number of words transferred between slow and fast memory must be at least

\[
\Omega \left( \frac{\prod_{i \in [d]} L_i}{M^{\sum_{i \in [n]} s_i - 1}} \right)
\]

An explicit construction of a tile shape that achieves this lower bound is described in [RD16]. We will review this later in the paper for the projective case.
In this section, we review the techniques used to construct optimal tilings and find communication lower bounds for nested-loop programs with large indices. Our approach in this section will be the building block for what we do in Sections 2 and 3. We limit our attention to cases where the index functions, $\phi_i$, are projections: that is, their output is a subset of the input. For convenience, let the indices of the output of $\phi_i$ be denoted $\supp(\phi_i)$. For instance, if $\phi(x_1, \ldots, x_5) = (x_1, x_4)$, then $\supp(\phi) = \{1, 4\}$.

In order to find the communication lower bound, it suffices to solve the LP (2.2). This linear program has one constraint for each subgroup $H \leq \mathbb{Z}^d$, since the number of such subgroups is infinite, determining a finite closed form for inequalities using a brute-force enumeration of all possible $H$ is impossible. Note, however, that since $\text{rank}(\phi_i(H)) \leq \text{rank}(H) = d$, the number of non-unique constraints is at most $(d + 1)^{n+1}$. In the general, continuous case - that is, for arbitrary affine $\phi$, with $H$ ranging over subgroups of $\mathbb{R}^d$ rather than $\mathbb{Z}^d$ - an algorithm guaranteed to terminate in finite (but unbounded) time was given by Valdimarrson [Val10]. A separation oracle for the resulting polytope was given in [GGdOW16], which immediately implies an algorithm for enumerating the relevant constraints in double-exponential time.

In the case where $\phi_i$ are projections, however, a simple, closed-form listing of the constraints is given by Theorem 6.6 of [CDK+13], which states that it suffices to check that the inequality $\sum_{i=1}^n s_i \text{rank}(\phi_i(H)) \geq \text{rank}(H)$ holds for all $H$ in the set of subgroups $\{e_1, \ldots, e_d\}$, where $e_i$ is the subgroup comprised of all vectors with zero entries at all indices except for $i$.

Therefore, this LP reduces to:

$$\text{min} \sum s_j \text{ s.t.} \quad 1 \leq \sum_{j \text{ s.t. } \supp(\phi_i) \ni j} s_j \quad \forall i \in [1..d]$$

Thinking of the $\phi_i$ as 0-1 vectors with 1s in the indices contained in its support, and letting $\bar{s}$ denote the vector $[s_1, ..., s_n]^T$, we can rewrite the linear program as (omitting nonnegativity constraints) as follows: minimize $\bar{T}^T \bar{s}$ subject to:

$$\left[ \begin{array}{c} \phi_1 \\ \vdots \\ \phi_n \end{array} \right] \bar{s} \geq \bar{T}.$$  

The solution to this linear program, which we denote $k_{HBL}$, immediately gives us the communication lower bound $\prod_{i} L_i / M^{k_{HBL}}$.

Now that we have a lower bound, we would like to find an actual tiling that attains it in order to show that it is tight. Let us ansatz (following Loomis-Whitney, etc.) that the optimal tile is a hyperrectangle of dimensions $b_1 \times \ldots \times b_d$, where the $b_i$ are constants which we wish to determine. We wish to select a tile whose volume (that is, $\prod_{i \in [1..d]} b_i$) is as large as possible, but we are subject to memory limitations: the subsets of each array that are used must fit in cache. Since the subsets of array $A$, required to complete the operations in this hyperrectangle are of size $\prod_{i \in \supp(\phi_i)} b_i$, we obtain the constraint (again, ignoring constant factors) $\prod_{i \in \supp(\phi_i)} b_i \leq M$. Taking logs base $M$ and letting $\lambda_i$ denote $\log_M b_i$, we obtain the following linear program: maximize $\bar{T}^T [\lambda_1, ..., \lambda_d]$ subject to:

$$\left[ \begin{array}{c} -\phi_1 \\ \vdots \\ -\phi_n \end{array} \right] [\lambda_1] \leq \bar{T}.$$  

Taking the dual gives us (3.2), which implies that this tiling obtains the lower bound.
Notice that we did not encode the constraint that $b_i \leq L_i$ in this linear program. Although this does not change the result when $L_i$ is assumed to be very large, this does not always hold, and the lower bound computed by 3.2 is not always tight. In the following section, we modify this approach to give tight lower bounds for arbitrarily-sized inputs.

4. The Lower Bound

4.1. One Small Index. We will start our approach to small loop bounds by considering the case when all loop indices but one are assumed to be bounded by arbitrarily large values. Our approach will be to (a) find an upper bound for a tile restricted to single “slice” of the iteration space formed by fixing the loop index with a small bound, (b) calculate an upper bound for the entire tile by summing individual slice bounds together over all possible values of the same index, and (c) divide the total number of operations by the aforementioned quantity to achieve a communication lower bound.

Let us first consider the case where a single loop bound - say, $L_1$, the upper bound on $x_1$ - is small, and the others are large. We may assume without loss of generality that $L_1 \leq M$; if the opposite is true, then $L_1$ would be large enough for the analysis of Section 3 to apply, as any tile whose memory footprint is at most $M$ would fit in the $L_1$ dimension. Furthermore, suppose without loss of generality that $\phi_1, ..., \phi_p$ (for some integer $p$) all contain $x_1$ and $\phi_{p+1}, ..., \phi_n$ do not. We will now find a communication lower bound for the subset of instructions whose slice of operations with $x_1$ fixed can be found by using LP 3.1 with the $\phi$ replaced with $\phi'$, to compute an upper bound for the max tile size...

$$\min \sum \hat{s}_j \text{ s.t. } 1 \leq \sum_{j \text{ s.t. supp}(\phi') \ni i} \hat{s}_j \forall i \in [1,d]$$

This amounts to removing the first row in the constraint matrix of the LP 3.2:

$$\begin{bmatrix} [\text{remove first row}] \\ \phi_1 & \cdots & \phi_n \\ \vdots \\ \hat{s}_1 \vdots \hat{s}_d \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \vdots \\ \hat{s}_d \end{bmatrix} \geq \begin{bmatrix} \bar{1} \\ \vdots \\ \bar{1} \end{bmatrix}$$

To find a upper bound for the size of a tile, we sum over the upper bounds for the size each of its slices, each of which corresponds to a single value of $x_1$. Let $\phi_{1|x_1=k}, ..., \phi_{n|x_1=k}$ be the functions with $x_1$ fixed to $k$. Then, the maximum tile size is found by maximizing the following quantity (with $V$ representing the tile):

$$\sum_{i \in [L_1]} |\phi_{1|x_1=i}(V)|^\hat{s}_1 \cdots |\phi_{n|x_1=i}(V)|^\hat{s}_n = M\sum_{i \in [L_1]} |\phi_{1|x_1=i}(V)|^\hat{s}_1 \cdots |\phi_{p|x_1=i}(V)|^\hat{s}_p$$

subject to:

$$\sum_{i \in [L_1]} |\phi_{j|x_1=i}(V)| \leq M \forall j \in [p].$$

We bound (4.1) subject to the constraints (4.2), and compute the maximum tile size, as follows:
Lemma 1. The maximum tile size for a tile $V$, subject to the constraints that (a) that $\phi_i(V) \leq M$ for all $i$ and (b) the set of all distinct $x_1$-coordinates of elements of $V$ is at most $L_1$ in cardinality (i.e. the tile fits inside the loop bounds), is bounded above by $M^\kappa$, where

$$\kappa = \max \left\{ \sum_{i=1}^{n} \hat{s}_i + \beta_1 \left( 1 - \sum_{i=1}^{p} \hat{s}_i \right), \sum_{i=1}^{n} \hat{s}_i \right\} .$$

Proof. There are three cases:

1. If $\sum_{i \in [p]} \hat{s}_i < 1$, the maximum of the quantity (4.1) is achieved when we distribute the weight across terms in the sum, i.e. for all $j \in [1..p]$, let $|\phi|_{x_1=i}(V) = M/L_1$ for all $i \in [1..L_1]$, which leads to a tile size of $M^\kappa$ where

$$\kappa := \sum_{i=1}^{n} \hat{s}_i + \beta_1 \left( 1 - \sum_{i=1}^{p} \hat{s}_i \right)$$

and $\beta_1 = \log_M L_1$.

2. If $\sum_{i \in [p]} \hat{s}_i > 1$, the maximum is achieved when we concentrate the entire weight into one term of the sum (i.e. for all $j \in [1..p]$, let $|\phi|^r_{x_1 = i}(V) = M$ for some $i'$ and let $|\phi|^r_{x_1 = i}(V) = 0$ for $i \neq i'$), which leads to a tile size of $M^\kappa$ where

$$\kappa := \sum_{i=1}^{n} \hat{s}_i .$$

3. If $\sum_{i \in [p]} \hat{s}_i = 1$, then both (4.3) and (4.4) are equal. Furthermore, since the only difference between $\hat{s}$ and $s$ is that the latter must satisfy the additional constraint $\sum_{i \in [1..p]} s_i \geq 1$ in the constraint (which is satisfied in this case by $\hat{s}$ as well), we get an upper bound of $M^\kappa \sum_{i \in [1..p]} \hat{s}_i = M^\kappa \sum_{i \in [1..p]} s_i$ immediately from (3.1).

For convenience, denote $|\phi|^r_{x_1 = x'_i}(V)|$, the slice of $V$ corresponding to $x'_i$, as $y_{i,x'_i}$. We want to maximize

$$\sum_{x_1=1}^{L_1} y_{i,x_1} \cdots y_{p,x_1}$$

subject to

$$\sum_{x_1=1}^{L_1} y_{i,x_1} - M \leq 0 \quad \forall i \in [p] .$$

Without loss of generality, assume all the $\hat{s}_i$ are positive; if $\hat{s}_i = 0$, then we can remove $y_{i,x_1}$ from both the statement of the maximization problem (e.g. by setting it to 1 for all $x_i$) and from the quantities (4.3) and (4.4) without affecting the rest of the proof.

Since any slack in any one of the above inequalities can be removed by increasing one of the $y_{i,x_1}$, and doing so will only increase the quantity we’re trying to maximize, we can take these inequalities to be equalities. The Lagrange multipliers for this problem are:

$$\mathcal{L} = \sum_{x_1=1}^{L_1} y_{i,x_1} \cdots y_{p,x_1}$$

subject to

$$\sum_{x_1=1}^{L_1} y_{i,x_1} - M \leq 0 \quad \forall i \in [p] .$$

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subject to

$$\sum_{x_1=1}^{L_1} y_{i,x_1} - M \leq 0 \quad \forall i \in [p] .$$
Setting the gradient (with respect to both \( y_{i,j} \) and \( \lambda_i \)) to 0, and looking at the derivative with respect to \( y_{i,j} \), we get:

\[
\frac{\lambda_i y_{i,j}}{s_i} = \frac{\lambda_j y_{j,i}}{s_j}
\]  

(4.5)

These equations are invariant in \( j \): that is, no matter which value \( j \) we fix \( x_1 \) to, the set of equations that \( y_{i,j} \) must satisfy are identical (this intuitively follows from symmetry across the \( x_i \)).

As a result, we may assume \( \lambda_i \neq 0 \); if it is in fact zero, then the quantity we’re trying to maximize would be zero, which clearly cannot be the case since we can construct a tile containing only one element (i.e. with our objective being 1) that satisfies all the constraints of the maximization problem.

In particular, \( \lambda_i y_{i,j} / s_i = y_{i,j}^{\delta_1...\delta_p} \) must remain invariant as \( i \) varies (with a fixed \( j \)), which implies that for any \( i_1, i_2, j \),

\[
\frac{\lambda_{i_1} y_{i_1,j}}{s_{i_1}} = \frac{\lambda_{i_2} y_{i_2,j}}{s_{i_2}}
\]

implying that the ratio between \( y_{i,j} \) for two different values of \( i \) is independent of the \( j \) (i.e. slice) we choose, remaining fixed at

\[
\frac{y_{i_1,j}}{y_{i_2,j}} = \frac{\lambda_{i_2} s_{i_1}}{\lambda_{i_1} s_{i_2}}
\]

Therefore, the point we’re trying to solve for satisfies this relationship:

\[
y_{i,j} = \frac{\lambda_i s_{i,j}}{\lambda_j s_{j}} y_{1,j}
\]

(4.6)

For any given \( j \), one of two cases must hold: either \( y_{i,j} = 0 \) for all \( i \) (in which case the tile does not intersect at all with the slice \( x_1 = j \)) or all \( y_{i,j} \) are nonzero, and we can substitute (4.6) into (4.5) to get:

\[
\frac{\lambda_i}{s_i} = y_{1,j}^{-1+\sum \delta_k} \prod_{k=1}^{p} \frac{s_{k,j}}{y_{k,j}} \prod_{k=1}^{p} \left( \frac{\lambda_i s_{i,j}}{\lambda_j s_{j}} \right)^{\delta_k}
\]

Canceling \( \lambda_j / s_j \) from both sides, and moving the first term in the last expression over to the left, we get

\[
y_{1,j}^{1-\sum \delta_k} = \left( \frac{\lambda_1}{s_1} \right)^{\sum \delta_k-1} \prod_{k=1}^{p} \left( \frac{s_k}{\lambda_k} \right)^{\delta_k}
\]

We may assume that \( 1 - \sum \delta_k \) is nonzero, as the case when it is zero is covered by case (3) above. Therefore, since the right hand side is independent of \( j \), it follows that all nonzero values of \( y_{1,j} \) are equal. Since \( y_{1,j} \) determines the value of \( y_{i,j} \) for all \( i \) via (4.6), it follows that each \( y_{i,j} \) must
either be (a) equal to some nonzero constant independent of \( j \) or (b) be equal to zero, if and only if all \( y_{i,j} \) for the same \( j \) must also be zero.

Let the number of \( j \) such that \( y_{i,j} \neq 0 \) be \( \theta \), which must fall between 1 and \( L_1 \) inclusive (since the number of slices is at most equal to the loop bound corresponding to the dimension we’re summing over). Therefore, in order to satisfy (a), the remaining \( y_{i,j} \) must be equal to

\[
y_{i,j} = \frac{M}{\theta}.
\]

Substituting this into (4.1), we get that the max tile size is:

\[
M^{\sum_{i \in [p+1,n]} \hat{s}_i} \prod_{i=1}^{p} \left( \frac{M}{\theta} \right)^{\hat{s}_i} = M^{\sum_{i=1}^{n} \hat{s}_i - \sum_{i=1}^{p} \hat{s}_i}
\]

so the log (base \( M \)) of tile size is:

\[
\sum_{i=1}^{n} \hat{s}_i + (\log_M \theta) \left( 1 - \sum_{i=1}^{p} \hat{s}_i \right).
\]

Therefore, either \( 1 - \sum_{i=1}^{p} \hat{s}_i \) is positive, in which case the maximum occurs when we set \( \theta \) to \( L_1 \), giving (recall that \( \beta_1 = \log_M L_1 \)):

\[
\sum_{i=1}^{n} \hat{s}_i + \beta_1 \left( 1 - \sum_{i=1}^{p} \hat{s}_i \right),
\]

or \( 1 - \sum_{i=1}^{p} \hat{s}_i \) is negative, in which case the maximum occurs at \( \theta = 1 \), in which case we get

\[
\sum_{i=1}^{n} \hat{s}_i.
\]

as desired. \( \square \)

4.2. **Multiple small bounds.** We now generalize the proof Section 4.1 to the case where multiple loop bounds are taken to be small.

Suppose that the loops indexed by \( x_i \) have bounds \( L_i \). Let \( R_j \subseteq \{1..n\} \) denote the set of indices \( i \) such that \( \supp(\phi_{i,j}) \) contains \( x_j \).

As before, our approach considers the communication lower bound for a “slice” - that is, a subset of the iteration polytope formed by restricting certain loop indices to fixed values - and summing these slice lower bounds over all possible values of the fixed indices. This time, however, each slice will be formed by simultaneously fixing multiple indices, which we assume without loss of generality are \( x_1 \) through \( x_q \) (the following argument holds for any \( q \), and is independent of the actual value of \( q \)). As was the case in the single-variable case, an upper bound on max tile size for a single slice is given by \( M^{\sum_{i \in \{1..n\}} \hat{s}_i} \), where \( \hat{s}_i \) are any nonnegative numbers that satisfy:

\[
1 \leq \sum_{j \text{ s.t. } \supp(\phi_j') \ni x_i} \hat{s}_j
\]

where \( \phi_j' \) now corresponds to removing \( x_1, ..., x_{q-1} \) from \( \phi_j \) (or, alternatively, chopping off the first \( q \) rows of the HBL LP constraint matrix (3.2))

We now develop an analog to Lemma 1 in order to maximize the sum of the slices over \( \{x_1, ..., x_q\} \in \{1..L_1\} \times ... \times \{1..L_q\} \). Our main result is as follows:

**Theorem 2.** Let \( q \in [1..d] \), and \( \hat{s}_i \) be any nonnegative numbers satisfying

\[
1 \leq \sum_{j \text{ s.t. } \supp(\phi_j') \ni x_i} \hat{s}_j
\]
where $\phi_j$ is obtained by removing $x_1, \ldots, x_q$ from $\phi$. Then $M^k$, where

$$
k = \sum_{i=1}^n \bar{s}_i + \sum_{j \in [q] \text{ s.t. } \sum_{i \in R_j} \bar{s}_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} \bar{s}_i \right) \right]
$$

represents an upper bound on the tile size.

Notice that this theorem holds for all possible $q$, as well as reorderings of the variables. As a result, this lemma in fact generates $2^d$ separate upper bounds for tile size (one for each subset $\mathcal{Q}$ of indices that we hold to be small). Therefore, the smallest upper bound on tile size (which corresponds to the largest lower bound on communication) we can achieve in this manner is $M^k$ for

$$
k = \min_{\mathcal{Q} \subseteq [d]} \sum_{i=1}^n \bar{s}_{\mathcal{Q}, i} + \sum_{j \in \mathcal{Q} \text{ s.t. } \sum_{i \in R_j} \bar{s}_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} \bar{s}_i \right) \right]
$$

where $\bar{s}_{\mathcal{Q}, i}$ is the solution to the HBL LP (3.2) with the rows indexed by elements of $\mathcal{Q}$ removed.

**Proof.** By induction on $q$. The base case, for $q = 1$, is simply Lemma 1.

Let $\bar{s}'_{i}$ be defined as $\bar{s}_{[q-1], i}$. Suppose for induction that $M^k$, for

$$
k = \sum_{i=1}^n \bar{s}'_{i} + \sum_{j \in [q-1] \text{ s.t. } \sum_{i \in R_j} \bar{s}'_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} \bar{s}'_i \right) \right]
$$

represents an upper bound on the tile size.

We start by finding an upper bound on the tile size, as before, by summing over several “slices”, each being defined as the subset of the elements where $x_1$ through $x_q$ are set to fixed values.

We begin by generalizing the notion of slices to the case where multiple indices may be small. As before, let $\phi_i[x_1, \ldots, x_q]$ denote $\phi_i$ with $x_j$ fixed to $\hat{x}_j$ for all $j \in [q]$. By definition, as $\phi_i$ only depends on indices in its support, $\phi_i[x_1, \ldots, x_q]$ must be identical to $\phi_{i \cap \text{supp}(\phi_i)}$.

We wish to maximize the size of the entire tile - that is, the sum of all the sizes of the slices:

$$
\sum_{x_1 \in [1..L_1], \ldots, x_q \in [1..L_q]} |\phi_1[x_1, \ldots, x_q](V)|^{\bar{s}_1} \cdots |\phi_n[x_1, \ldots, x_q](V)|^{\bar{s}_n}
$$

subject to the memory constraints

$$
\sum_{x_k \in [1..L_q] \text{ for } k \in [1..q] \setminus \text{supp}(\phi_i)} |\phi_i[x_1, \ldots, x_q](V)| \leq M \quad \forall i \in \bigcup_{j \in [1..q]} R_j.
$$

As before, we will simplify our notation by defining $y_i[x_1, \ldots, x_q] := |\phi_i[x_1, \ldots, x_q](V)|$. Our optimization problem therefore can be rewritten as maximizing:

$$
\sum_{x_1 \in [1..L_1], \ldots, x_q \in [1..L_q]} y_1^{\bar{s}_1} \cdots y_n^{\bar{s}_n}
$$

subject to the constraints:

$$
1 \leq \sum_{x_k \in [1..L_q] \text{ for } k \in [q] \setminus \text{supp}(\phi_i)} y_i[x_1, \ldots, x_q] \leq M \quad \forall i \in \bigcup_{j \in [1..q]} R_j.
$$

The definition of $\phi_i[x_1, \ldots, x_q]$ (and therefore of $y_i[x_1, \ldots, x_q]$) requires us to further impose an additional constraint on the solution: for all $i$, the value of $y_i[x_1, \ldots, x_q]$ must remain independent of
If we never end up with an component of \( x \) value of \( C \) (i.e. the leftmost inequality in (4.11)) without affecting correctness. Furthermore, in order to make it easier to reason about the constraints (4.9), we will multiply them all by the appropriate values in order to ensure that the sum is over the same set of variables: \( x_1 \) through \( x_q \):

\[
\prod_{j \notin \supp(\phi)} L_j \leq \sum_{x_1 \in \{1..L_1\}, \ldots, x_q \in \{1..L_q\}} y_i,_{x_1,\ldots,x_q} \leq M \prod_{j \notin \supp(\phi)} L_j \quad \forall i \in \bigcup R_j.
\]

Since our goal is to find an upper bound on the tile size, which is the result of this constrained maximization problem, we can remove the lower bound constraints on \( \sum_{x_1 \in [1..L_1], \ldots, x_q \in [1..L_q]} y_i,_{x_1,\ldots,x_q} \) (i.e. the leftmost inequality in (4.11)) without affecting correctness.

The resulting problem is almost identical to that of Lemma II, except with different limits (one may think of this ‘flattening’ the \( q \)-dimensional tensor \( x_1, \ldots, x_q \) into a single vector in order to get a single sum as we did in the previous section). Recall that none of the steps we used to compute the maximum in our proof of Lemma II actually used the value of the right sides of the constraints, since all those constants were all differentiated away as a constant factor when taking gradients; as a result, the same result applies here. Specifically, the maximum is obtained at a point specified as follows: select some subset \( \mathcal{S} \subseteq \{1..L_1\} \times \cdots \times \{1..L_q\} \) of integer tuples, which represent \( x_i \)-indices for which \( y_i,_{x_1,\ldots,x_q} \) will be nonzero. For each \( \{x_1, \ldots, x_q\} \) in \( \mathcal{S} \), \( y_i,_{x_1,\ldots,x_q} \) must be equal to a constant value independent of \( \{x_1, \ldots, x_q\} \). In order to maximize (4.10), we set constraint (4.9) to obtain:

\[
y_i,_{x_1,\ldots,x_q} = \frac{M}{|\mathcal{S}|} \quad \forall i
\]

where \( \mathcal{S} \) is \( \phi_i \) (restricted to \( x_1 \ldots x_q \)) applied to \( \mathcal{S} \).

For indices not in \( \mathcal{S} \), set \( y_i,_{x_1,\ldots,x_q} \) to zero for all \( i \). The resulting upper bound for tile size is therefore:

\[
\sum_{x_1,\ldots,x_q \in \mathcal{S}} \prod_i \left( \frac{M}{|\mathcal{S}|} \right)^{\delta_i} = |\mathcal{S}| \prod_i \left( \frac{M}{|\mathcal{S}|} \right)^{\delta_i} = \frac{|\mathcal{S}|^{\sum \delta_i}}{\prod_i |\mathcal{S}|^{\delta_i}} M^{\sum \delta_i}
\]

where the first equality is a result of the independence of the summand with \( x \), with the number of nonzero terms in the sum being \( |\mathcal{S}| \).

Claim: without loss of generality, we can assume that \( \mathcal{S} \) is a rectangle; that is, it can be written as set \( C_1 \times \cdots \times C_q \) for some sets \( C_i \subseteq [L_i] \).

Proof of claim: Suppose not. Then there exist points \( x', x'' \in \mathcal{S} \) such there exists some point \( x^* \notin \mathcal{S} \), where each \( x^*_j \) is equal to \( x'_j \) for all \( j \) except a single value \( j^* \), at which it takes on the value of \( x^*_{j^*} \). To see why this is true, take any two distinct \( x', x'' \in \mathcal{S} \), and repeatedly change one component of \( x' \) to match the corresponding component of \( x'' \), stopping when either \( x' = x'' \), or \( x' \notin \mathcal{S} \). In the latter case, set \( x^* = x' \), and let \( \hat{x}' \) denote its immediate predecessor in this process. If we never end up with an \( x^* \) for any distinct pairs of \( x' \) and \( x'' \) in \( \mathcal{S} \), then \( \mathcal{S} \) must be a rectangle.
Our goal will be to show that this configuration is suboptimal. Consider the set of functions $\phi_i$ for $i \in R_\ell$, that is, the set of functions containing $x_j$. 

Let us consider the following categories, distinguished by how $\phi_i$ maps $x'$, $x''$, and $x^*$. 

1. $\phi_i(x') = \phi_i(x'') = \phi_i(x^*)$. Notice that replacing $x''$ with $x^*$ in $\mathcal{S}$ either reduces $|\mathcal{S}_i|$ by one (if there is no other $x^*$ such that $\phi_i(x^*) = \phi_i(x'')$) or keeps it the same (if such an $x^*$ exists; notice that in the latter case, adding $x^*$ to $\mathcal{S}$ keeps $|\mathcal{S}_i|$ constant. We denote these cases (1a) and (1b) respectively.

2. $\phi_i(x') = \phi_i(x'') = \phi_i(x^*)$. Analogously to case (1), replacing $x'$ with $x^*$ either reduces $\mathcal{S}_i$ (case (2a)) or keeps it the same (case (2b)).

3. $\phi_i$ maps $x'$, $x''$, and $x^*$ to three distinct points.

4. $\phi_i(x') = \phi_i(x'') = \phi_i(x^*)$. Notice that this category must be empty: If $x'' = x^*$, then by definition this quantity is also $x^*$; therefore, going to $*$ from $''$ can only make the number of agreements better under any projection.

In the above categories, $i$ satisfying (1) and (4) implies that $i \notin R_\ell$, while $i$ satisfying (2) and (3) implies that $i \in R_\ell$. We will show that $\mathcal{S}$ is suboptimal by providing strict improvements on it.

1. If there are any $i$ in category (1a), we replace $x''$ with $x^*$ in $\mathcal{S}$, reducing $|\mathcal{S}_i|$. In order to see how this change affects the values of $\mathcal{S}_i$ for other $i$, we first note that for other $i \notin R_\ell$, $\phi_i(x') = \phi_i(x^*)$, so this change can only keep constant or decrease $|\mathcal{S}_i|$ for such $i$. For all $i$ in any of the other categories - (1b), (2ab), (3), or (4) - $|\mathcal{S}_i|$ remains the same. Therefore, as the value of $|\mathcal{S}_i|$ either remains the same or decreases (with at least one strict decrease), and $|\mathcal{S}|$ remains constant, we obtain a strict increase in the value of $X_{i,x}$.

2. If some $i$ falling into category (3): Denote the set of $i$ such that $\phi_i$ maps $x'$, $x''$, and $x^*$ onto different values as $Q$. We will split into two cases, based on the values of $\sum_{i \in R_\ell} s_i$:

   a. Suppose $\sum_{i \in R_\ell} s_i \geq 1$. Consider the assignment to the $y_{i,x}$ given by $\mathcal{S}$; its objective $(4.13)$ is:

   $$\sum_{x_1 \in [L_1], \ldots, x_{\ell-1} \in [L_{\ell-1}], x_{\ell+1} \in [L_{\ell+1}], \ldots, x_q \in [L_q]} \left( \sum_{x_{\ell} \in [L_{\ell}]} y_{1,\{x_1,\ldots,x_q\}}^{s_1} \cdots y_{R_i,\{x_1,\ldots,x_q\}}^{s_{R_i}} \right)$$

   Factoring the innermost term into terms that are constant w.r.t. $x_{\ell}$ and those that are not, we can rewrite this as:

   $$\sum_{x_1 \in [L_1], \ldots, x_{\ell-1} \in [L_{\ell-1}], x_{\ell+1} \in [L_{\ell+1}], \ldots, x_q \in [L_q]} \left( \prod_{i \in \{1,\ldots,q\}} y_{i,\{x_1,\ldots,x_q\}}^{s_i} \sum_{x_{\ell} \in [L_{\ell}]} \prod_{i \in R_\ell} y_{i,\{x_1,\ldots,x_q\}}^{s_i} \right).$$

   Let us restrict our attention a single “slice”: that is, an instance of the term

   $$\sum_{x_{\ell} \in [L_{\ell}]} \prod_{i \in R_\ell} y_{i,\{x_1,\ldots,x_q\}}^{s_i}$$

   with fixed values for $x_1$ through $x_q$, excluding $x_{\ell}$. By equality constraints, we get that all the nonzero values of $y_{i,\{x_1,\ldots,x_q\}}^{s_i}$ must be equal to a constant independent of $x_1, \ldots, x_q$ (but dependent on $i$). Let $m_i = \sum_{x_{\ell} \in [L_{\ell}]} y_{i,\{x_1,\ldots,x_q\}}$, and $d$ denote the number of $x_{\ell}$ such that $(x_1, \ldots, x_q)$, with all coordinates except $x_{\ell}$ set to our fixed values, are in $\mathcal{S}$ (and therefore, nonzero terms in the above sum $(4.14)$); this restricts the nonzero values of
For all \( i \), we require that the value of \( \sigma \) strictly increases the objective. Without loss of generality, we will assume that the latter is not required to be \( \sum_{i \in R} s_i \geq 1 \), which holds regardless in this case. Therefore, we can replace \( s_i \) with \( \hat{s}_i \) in order to completing the induction for the entire proof of Lemma 2 in this particular case.

(b) Now suppose \( \sum_{i \in Q} \hat{s}_i < 1 \). As \( Q \subseteq R \), it immediately follows that \( \sum_{i \in R} \hat{s}_i \leq \sum_{i \in R} s_i \), both \( \hat{s}_i \) and \( s_i \) must map to the same point, or \( \hat{s}_i = s_i \). As \( \alpha, \beta, \gamma \in S \), both \( y_{i', x'} \) and \( y_{i', x''} \) must be nonzero, so \( y_{i', x'} \) must be nonzero as well. As nonzero values of \( y_{i, x} \) are independent of \( x \) for all \( i \), we must have

\[ y_{i', x'} = y_{i', x''} = y_{i', x'} \]

For all \( i \), let the value of \( y_{i, x'} + y_{i, x''} + y_{i, x'} \) be denoted \( \mu_i \), and define \( k', k'', k^* \) such that \( y_{i, x'} = k' \mu_i \) (and likewise for \( k'', k^* \)); our starting configuration, with \( S \) containing \( x', x'' \) but not \( x \), is \( k' = k'' = 1/2, k^* = 0 \). So as not to break any constraints, we will require that the value of \( y_{i, x'} + y_{i, x''} + y_{i, x'} \) stay constant, so we will enforce \( k' + k'' + k^* = 1 \). The contribution of these three tiles to the objective is:

\[
\prod_{i \in Q} y_{i, x'}^\hat{s}_i \prod_{i \in Q} y_{i, x'}^\hat{s}_i \prod_{i \in Q} y_{i, x''}^\hat{s}_i \prod_{i \in Q} y_{i, x''}^\hat{s}_i \prod_{i \in Q} y_{i, x'}^\hat{s}_i \prod_{i \in Q} y_{i, x'}^\hat{s}_i
\]
with equality following from (4.15). We substitute $y_{i,x'} = k'\mu_i$ and the corresponding definitions for $y_{i,x''}, y_{i,x^*}$ to rewrite the above expression as:

$$
\left( \prod_{i \notin Q} (k'\mu_i)^{\delta_i} + \prod_{i \in Q} (k''\mu_i)^{\delta_i} + \prod_{i \in Q} (k^*\mu_i)^{\delta_i} \right) \prod_{i \in Q} y_{i,x'}^{\delta_i}
$$

$$
= \left( k'\sum_{i \in Q} \mu_i^{\delta_i} + k''\sum_{i \in Q} \mu_i^{\delta_i} + k^*\sum_{i \in Q} \mu_i^{\delta_i} \right) \prod_{i \in Q} y_{i,x'}^{\delta_i}
$$

$$
= \left( k'\sum_{i \in Q} \delta_i + k''\sum_{i \in Q} \delta_i + k^*\sum_{i \in Q} \delta_i \right) \prod_{i \in Q} y_{i,x'}^{\delta_i}
$$

We will leave $y_{i,x'}$ constant for all $i \notin Q$, and we will not vary $\mu_i$, the sum of $y_{i,x'}, y_{i,x''},$ and $y_{i,x^*}$, so it suffices to maximize

$$k'\sum_{i \in Q} \delta_i + k''\sum_{i \in Q} \delta_i + k^*\sum_{i \in Q} \delta_i$$

subject to

$$k' + k'' + k^* = 1.$$  

As $\sum_{i \in Q} \delta_i < 1$, the solution to this maximization problem is obtained by setting $k' = k'' = k^* = 1/3$; all other assignments (including the current one) are suboptimal. As we do not vary any $y_{i,x}$ for $i \notin Q$ and any $x$, this change does not affect the constraints corresponding to any other $\phi_i$ than those in $Q$, which all must be still satisfied as we do not vary $\mu_i$; therefore, both these assignments satisfy the constraints (4.9). Therefore the current assignment under $\mathcal{S}$, with $k^*$ set to 0 and $k', k''$ set to 1/2, must be suboptimal, providing us with our contradiction in this case.

(3) If there exists $i$ in category (2a), but none in (1a) and (3), we will replace $x'$ with $x^*$ in $\mathcal{S}$, decreasing $|\mathcal{S}|$ by one. The values of $|\mathcal{S}_i|$ for other $i$, in this case, either also decrease (for other $i$ falling in case (2a)), remain the same (for $i$ falling in cases (1b), (2b), (4)), therefore corresponding to a strict improvement in the value of (4.13).

(4) If we have $i$ in categories (1b), (2b), or (4) (but none in categories (1a), (2a), or (3), all of which were dealt with in earlier cases) add $x^*$ to $\mathcal{S}$; this does not change any value of $\mathcal{S}$, but increases $\mathcal{S}$ by 1, leading to a strictly improved solution.

Each of these cases (except case (3a), which uses the inductive hypothesis), presents a strict improvement to the value of (4.13). Therefore, $\mathcal{S}$ must not be optimum, providing a contradiction. We can therefore conclude that optimum value of $\mathcal{S}$ must have no triple $x', x'' \in \mathcal{S}$, $x^* \notin \mathcal{S}$ such that $x^*$ agrees with $x'$ everywhere except one coordinate where it agrees with $x''$, and therefore $\mathcal{S}$ must be a rectangle, as desired. }

Now that we’ve shown that $\mathcal{S}$ is a rectangle, let us assume that its dimensions are $\rho_1, \ldots, \rho_j$. Then $\mathcal{S}$ has cardinality $\prod_{i \in [n]} \rho_i$ and $\mathcal{S}$ has cardinality $\prod_{i \in [n] \cap \text{supp}(\phi_i)} \rho_j$. Substituting into (4.13), we get:

$$\frac{|\mathcal{S}|}{\prod_i |\mathcal{S}_i|^\delta_i} M_{\Sigma_i \delta_i} = \frac{\prod_{i \in [n]} \rho_i}{\prod_{i \in [d]} \left( \prod_{j \in [n] \cap \text{supp}(\phi_i)} \rho_j \right) \delta_i} M_{\Sigma_i \delta_i}$$

$$= \frac{\prod_{i \in [n]} \rho_i}{\prod_{i \in [n]} \left( \prod_{j \in R \rho_j} \rho_j^\delta_i \right)} M_{\Sigma_i \delta_i}$$

$$= \prod_{j \in [n]} \rho_j^{1-\Sigma_i \delta_i} M_{\Sigma_i \delta_i}$$
Since we have full control over the value of $\rho$, we can maximize the value of this expression by setting the $\rho$ to their maximum possible value, $L_i$ if $1 - \sum_{i \in R_j} \hat{s}_i \geq 0$, and to their minimum possible value, 1, if $1 - \sum_{i \in R_j} \hat{s}_i \leq 0$.

Therefore, the maximum value of our objective (4.8) is obtained at:

$$M \sum_i \hat{s}_i \prod_{j \in [q]} \prod_{\sum_{i \in R_j} \hat{s}_i \leq 1} L_j^{1 - \sum_{i \in R_j} \hat{s}_i}$$

or equivalently, $M^k$ where

$$k = \sum_{i=1}^n \hat{s}_i + \sum_{j \in [q]} \prod_{\sum_{i \in R_j} \hat{s}_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} \hat{s}_i \right) \right].$$

as desired.

Finally, we need to modify our solution to satisfy (4.10) with no change to the objective value.

Let $y_{l,\{x_1, \ldots, x_q\}}'$ be $\max_{x_j \text{ s.t. } j \in \text{supp}(\phi_{a\{x_1, \ldots, x_q\}})} y_{l,\{x_1, \ldots, x_q\}}$, which takes on the value $M/|\mathcal{S}|$ if there is some nonzero element of $\mathcal{S}$ that matches $(x_1, \ldots, x_q)$ at the indices in the support of $\phi_{a}$, and is zero otherwise. In order to show that this modification does not change the value of objective (4.8), it suffices to show that

$$y_{l,\{x_1, \ldots, x_q\}}^{\hat{s}_1} \cdots y_{n,\{x_1, \ldots, x_q\}}^{\hat{s}_n} = \left( y_{l,\{x_1, \ldots, x_q\}}' \right)^{\hat{s}_1} \cdots \left( y_{n,\{x_1, \ldots, x_q\}}' \right)^{\hat{s}_n}$$

Suppose $(x_1, \ldots, x_q) \in \mathcal{S}$. Both sides are nonzero, and by equality constraint it is obvious that they must be the same.

Suppose $(x_1, \ldots, x_q) \notin \mathcal{S}$. Clearly the left is zero. Recall that $\mathcal{S}$ is a rectangle; that is, it can be written as set $\{(x_1, \ldots, x_q) : x_i \in C_i \forall i\}$ for some sets $C_i \subseteq [L_i]$. By definition, there must exist some $k$ such that $x_k \notin C_k$. There must be some some $j'$ such that $\phi_j'$ contains $x_k$; by definition, $y_{j',\{x_1, \ldots, x_k, \ldots, x_q\}}$ - and therefore, the entire right-hand-side of (4.17) - must be zero as well.

Furthermore, in order to show that this solution does not violate any of the constraints, consider

$$\sum_{x_k \in [1..L_k] \text{ for } k \in \text{supp}(\phi_l)} y_{l,\{x_1, \ldots, x_q\}}'$$

By definition, at most $\mathcal{S}_l$ of these terms may be nonzero, and each since must have value $M/|\mathcal{S}_l|$, this term must be at most $M$, as desired.

Notice that this proof works if we fix any subset of $1..q$ rather than the entire set. In other words, we can freely replace the sum from 1 to $q$ with a sum over any subset of 1 to $q$ and still get a valid upper bound (by changing the sum from $j \in [q]$ to summing over a subset of $[q]$ in equation (4.16)).

5. TILING CONSTRUCTION

In this section, we describe an explicit construction of a tiling that achieves the upper bound on tile size (and therefore achieves the lower bound on computation) from section 4.

Consider the LP that gives us the tiling in this case. We start with the linear program (3.3) and add constraints requiring that the blocks be no larger than the loop bounds (in log-space, $\lambda_i \leq \beta_i$):
(5.1) \[
\max \sum_{i \in d} \lambda_i \text{ s.t.} \sum_{i \text{ s.t. } x_i \in \text{supp}(\phi)} \lambda_i \leq 1 \quad \forall j \in [n]
\]
\[
\lambda_i \leq \beta_i \quad \forall i \in [q]
\]
\[
\lambda_i \geq 0 \quad \forall i \in [d]
\]

**Theorem 3.** The rectangular tile with dimensions given by the solution to (5.1) has cardinality equal to one of the upper bounds for tile size from Section 4 for a loop program defined by the \(\phi_i\); in other words, the solution to (5.1) equals

(5.2) \[
\sum_{i=1}^{n} \xi_{\mathcal{D},i} + \sum_{j \in \mathcal{D} \text{ s.t. } \sum_{i \in \mathcal{R}_j} s_i \leq 1} \beta_j \left(1 - \sum_{i \in \mathcal{R}_j} \xi_{\mathcal{D},i} \right)
\]

for some \(\mathcal{D} \subseteq [q]\), where \(\xi_{\mathcal{D},i}\) satisfies the constraints of the HBL LP (3.2) with the rows indexed by elements of \(\mathcal{D}\) removed:

(5.3) \[
\begin{pmatrix}
\phi_1 & \cdots & \phi_n
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_n
\end{pmatrix}
\geq
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\]

Let us write the constraints of (5.1) in the following fashion:

(5.4) \[
\begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_d
\end{pmatrix}
\leq
\begin{pmatrix}
1 \\
\vdots \\
\beta_q
\end{pmatrix}
\]

The dual of this linear program, with variables \(\xi_1, \ldots, \xi_q, s_1, \ldots, s_n\) is to minimize

(5.5) \[
\sum_{i \in [q]} \beta_i \xi_i + \sum_{j=1}^{n} s_j
\]

subject to

(5.6) \[
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & \phi_n \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_q \\
\xi_1 \\
\vdots \\
\xi_n
\end{pmatrix}
\leq
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\]

(as well as nonnegativity constraints \(\xi_i \geq 0\) for all \(i \in [q]\), \(s_i \geq 0\) for all \(i \in [n]\), which we omit from the matrix for brevity)

We now show that the optimal value of (5.5) is equivalent to (5.2) for some \(s_i\) satisfying (5.3).
Proof. By induction on $q$.

For the base case, suppose $q = 0$. This is just the case in Section 3. Suppose for induction that the solution to

$$\max \sum_i \lambda_i \text{ s.t.}$$

$$\sum_{i \text{ s.t. } x_i \in \text{supp}(\phi)} \lambda_i \leq 1 \quad \forall j \in [n]$$

$$\lambda_i \leq \beta_i \quad \forall i \in [q - 1]$$

takes the form

$$\sum_{i=1}^n s_{\mathcal{D},i} + \sum_{j \in \mathcal{D} \text{ s.t. } \sum_{i \in R_j} s_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} s_{\mathcal{D},i} \right) \right]$$

for some $\mathcal{D} \subseteq [q - 1]$ and $s_i$ satisfying (5.3).

Consider the LP: minimize (5.5) subject to (5.6). Denote its solution by $\zeta'_i, s'_i$; we wish to discover the minimum value of the objective (5.5).

We will rewrite the LP (5.6) in such a way that preserves the optimal value of the objective. First, we remove one variable - say, $\zeta_q$ - from it. Since there is no benefit to setting $\zeta_q$ any larger than necessary (it increases the objective (5.5), and does not come into play in any other constraints) we can fix its value as necessary to ensure that either the $q$th constraint or the nonnegativity constraint $\zeta_q \geq 0$ is tight. We have two cases:

Case 1: $\sum_{i \in R_q} s'_i \geq 1$. In this case, the $q$th constraint is satisfied at the optimal point regardless of the value of $\zeta_q$, so we may set $\zeta_q$ to 0. Now, the objective (5.5) becomes:

$$\sum_{i=1}^{q-1} \beta_i \zeta_i + \sum_{j=1}^n s_j$$

Since the $q$th constraint is the only one containing $\zeta_q$, we can delete the $q$th column on the left block of the constraint matrix (5.6) and remove $\zeta_q$ from the LP entirely. Therefore, the resulting LP is therefore exactly the dual of (5.7), which, by inductive hypothesis, has optimal objective value of the form:

$$\sum_{i=1}^n s_{\mathcal{D},i} + \sum_{j \in \mathcal{D} \text{ s.t. } \sum_{i \in R_j} s_i \leq 1} \left[ \beta_j \left( 1 - \sum_{i \in R_j} s_{\mathcal{D},i} \right) \right]$$

for $\mathcal{D} \subseteq [q - 1] \subset [q]$, and $s_{\mathcal{D},i}$ satisfying (5.3) as desired.

Case 2: $\sum_{i \in R_q} s'_i < 1$. Without loss of generality, assume this holds for $R_1$ through $R_{q-1}$ as well (if not, find $j$ such that $\sum_{x \in R_j} s'_i \geq 1$, permute the LP to swap the positions of $\zeta_j$ and $\zeta_q$, and proceed to case 1).

Therefore, we may modify the LP by setting $\zeta_1$ to $1 - \sum_{i \in R_1} s_i$ to keep it tight, and do the same with $\zeta_2$ through $\zeta_q$; this does not change the optimal objective value. Removing those constraints (since they’ve all been encoded into the objective), we get a new objective to replace (5.5) in our linear program:

$$\min \sum_{i=1}^n s_i + \sum_{j=1}^q \left[ \beta_j \left( 1 - \sum_{i \in R_j} s_i \right) \right]$$
Furthermore, since \( \sum_{i \in R_j}s_i^t < 1 \) for all \( j \in [q] \) this objective at its optimizer, \( s_1^t, ..., s_q^t \), is precisely equal to
\[
\sum_{i=1}^n s_i^t + \sum_{j \in [q]} \sum_{i \in R_j}s_i^t \left[ \beta_j \left( 1 - \sum_{i \in R_j}s_i^t \right) \right]
\]
which is of the same form as (5.2).

Furthermore, we may remove the first \( q \) constraints from (5.6), since our choices for values of \( \zeta_1, ..., \zeta_q \) guarantee that they will be satisfied. The resulting constraint matrix is identical to (5.3).

Therefore, the tile whose dimensions are given by (5.1) attains the lower bound given by Lemma 2 with \( \mathcal{Q} = [q] \), as desired.

\[ \square \]

6. Examples

We demonstrate several applications of our theory below.

6.1. Matrix-Matrix and Matrix-Vector Multiplication. We start by re-deriving the classical lower bound [HK81] for the triply-nested-loop matrix multiplication

\[
\begin{align*}
A_1(x_1, x_3) + & = A_2(x_1, x_2) \times A_3(x_2, x_3) \\
\{x_1, x_2, x_3\} & \in [L_1] \times [L_2] \times [L_3]
\end{align*}
\]

Our memory accesses are given by the functions:
\[
\begin{align*}
\phi_1(x_1, x_2, x_3) & = (x_1, x_3) \\
\phi_2(x_1, x_2, x_3) & = (x_1, x_2) \\
\phi_3(x_1, x_2, x_3) & = (x_2, x_3)
\end{align*}
\]

Therefore, the HBL LP is to minimize \( s_1 + s_2 + s_3 \) subject to
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3
\end{bmatrix}
\geq
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(6.1)

The optimal value of this LP is obtained when all the \( s_i \) are \( 1/2 \), giving a tile size upper bound of \( M^{1/2+1/2} = M^{3/2} \), which provides the standard \( L_1L_2L_3/M^{1/2} \) lower bound.

Now let us consider the case where \( L_3 \) may be small, which corresponds to problem sizes approaching matrix-vector multiplications (which occurs \( L_3 = 1 \)). In this case, our tile, which has length \( M^{1/2} \) in the \( L_3 \) dimension, cannot fit in our iteration space.

We first find a lower bound. Removing the row corresponding to \( x_3 \) from (6.1), we get that given any \( \hat{s}_i \) satisfying
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{s}_1 \\
\hat{s}_2 \\
\hat{s}_3
\end{bmatrix}
\geq
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

(6.2)

raising \( M \) to the power
\[
\max \{ \hat{s}_1 + \hat{s}_2 + \hat{s}_3, \hat{s}_1 + \hat{s}_2 + \hat{s}_3 + (\log_M L_3)(1 - \hat{s}_1 - \hat{s}_3) \}
\]

represents a valid upper bound on the tile size.

Since (6.2) is satisfied when \( \hat{s}_2 = 1 \) and \( \hat{s}_1, \hat{s}_3 = 0 \), this term becomes
\[
\max \{ 1, 1 + \log_M L_3 \}
\]
giving an upper bound of \( \max \{ M, ML_3 \} = ML_3 \) (as \( L_3 \) is always positive); therefore the communication lower bound is given by
\[
\frac{L_1 L_2 L_3}{ML_3} M = L_1 L_2.
\]
This is as expected, since we need to read at least \( L_1 L_2 \), the size of \( A_2 \), into fast memory to perform the operation.

Now let us consider the question of finding the tile. Instantiating LP (5.1) with the relevant values of \( \phi_{1,2,3} \), we get:
\[
\begin{align*}
\max & \lambda_1 + \lambda_2 + \lambda_3 \text{ s.t.} \\
& \lambda_1 + \lambda_3 \leq 1 \\
& \lambda_1 + \lambda_2 \leq 1 \\
& \lambda_2 + \lambda_3 \leq 1 \\
& \lambda_3 \leq \beta_3 = \log_M L_3 \\
\end{align*}
\]

There are two cases here: if \( \beta_3 \geq 1 \), then the last constraint is of no relevance, so the solution becomes \( 3/2 \), as in the case above.

On the other hand, if \( \beta_3 \leq 1 \), then adding the second and fourth inequalities gives
\[
\lambda_1 + \lambda_2 + \lambda_3 \leq 1 + \lambda_3 \leq 1 + \beta_3.
\]
We again split based on whether or not \( \beta_3 \geq 1/2 \); intuitively, we may consider this a question of whether the \( L_3 \) is sufficiently large (at least \( \sqrt{M} \)) to fit the \( \sqrt{M} \times \sqrt{M} \times \sqrt{M} \) tile derived above, or whether we must modify the tile’s shape to get it to fit in the \( L_3 \) dimension.

If \( \beta_3 \geq 1/2 \), then the optimum for the LP without the fourth constraint, \( \lambda_1 = \lambda_2 = \lambda_3 = 1/2 \), satisfies the fourth constraint and is therefore optimal, leading to the same \( \sqrt{M} \times \sqrt{M} \times \sqrt{M} \) as in the “large loop bound” cases discussed above.

If \( \beta_3 \leq 1/2 \), then we can set \( \lambda_3 = \beta_3 \) to make the fourth inequality tight, and then set \( \lambda_1 = 1 - \beta_3 \) and \( \lambda_2 = \beta_3 \) to tighten in addition to the first inequality in the LP; as three irredundant inequalities are tight and we only have three variables, this solution must be optimal as well. This obtains a tile size of \( M/L_3 \times L_3 \times L_3 = ML_3 \) (with a communication cost of \( L_1 L_2 \), a quantity that is equal to the size of \( A_2 \) and therefore must be optimal) as expected.

Alternatively, we could achieve the same tile size with a tile of size \( \sqrt{M} \times \sqrt{M} \times L_3 \) (corresponding to \( \lambda = \lambda_2 = 1/2, \lambda_3 = \beta_3 \)). In fact, the LP is optimized by any point between the two solutions we found previously; specifically, for any \( \alpha \leq 1 \),
\[
\begin{align*}
\lambda_1 &= \alpha/2 + (1 - \alpha)(1 - \beta_3) \\
\lambda_2 &= \alpha/2 + (1 - \alpha)\beta_3 \\
\lambda_3 &= \beta_3
\end{align*}
\]
optimizes LP (6.3); this corresponds to a tile size of:
\[
\frac{M^{1-\alpha/2}}{L_3^{1-\alpha}} \times M^{\alpha/2} L_3^{1-\alpha} \times L_3.
\]

When attempting to optimize this matrix multiplication on a real-world system, we may select any tiling from the above \( \alpha \)-parameterized family of optimal tilings in order to find one that runs well in practice (e.g. inner loops being multiples of cache line lengths or vector units).

As the communication cost’s derivation is symmetrical (i.e. it continues to be valid when we swap the subscripts) and the tile for the small-\( L_3 \) case above remains be a legal tiling if \( L_3 \) is the smallest loop index, we obtain the following tight lower bound for matrix multiplication’s communication cost:
\[
\max (L_1 L_2 L_3 / \sqrt{M}, L_1 L_2, L_2 L_3, L_1 L_3)
\]
6.2. **Tensor Contraction.** Let \(1 \leq j < k - 1 < d\). Let us consider a tensor contraction of the form

\[
A_1(x_1, \ldots, x_j, x_k, \ldots, x_d) = A_2(i_1, \ldots, i_{k-1}) \times A_3(x_{j+1}, x_d)
\]

This nested-loop model encapsulates several machine learning applications. For instance, point-wise convolutions - convolutions with \(1 \times 1\) filters, often used along depth-separable convolutions [HZC+17] to mimic the effect of standard machine learning convolutions with less memory usage, may be represented as tensor contractions:

\[
\text{for } \{b, c, k, w, h\} = 0: \{B, C, K, W, H\} - 1
\]

\[
(6.5) \quad \text{Out}(k, h, w, b) = \text{Image}(w, h, c, b) \times \text{Filter}(k, c)
\]

The same holds for fully connected convolutional layers.

The communication lower bound for the large-loop bound case is, as derived in [CDK+13], is \(L_1 \ldots L_d / \sqrt{M}\).

We instantiate the LP 5.1 to get:

\[
\begin{align*}
\max & \quad \lambda_1 + \ldots + \lambda_d \\
\text{subject to} & \\
\lambda_1 + \ldots + \lambda_j + \lambda_k + \ldots + \lambda_d & \leq 1 \\
\lambda_1 + \ldots + \lambda_{k-1} & \leq 1 \\
\lambda_{j+1} + \ldots + \lambda_d & \leq 1 \\
\lambda_1 & \leq \beta_1 = \log_M L_1 \\
& \vdots \\
\lambda_d & \leq \beta_d = \log_M L_d
\end{align*}
\]

The structure of this linear program is much like that of matrix multiplication, and it can be transformed into one identical to that for matrix multiplication. Let \(\gamma_1 = \sum_{i \in [j]} \lambda_i\), \(\gamma_2 = \sum_{i \in [j+1, k-1]} \lambda_i\), and \(\gamma_3 = \sum_{i \in [k, d]} \lambda_i\). Then we can rewrite the linear program as maximizing \(\gamma_1 + \gamma_2 + \gamma_3\) subject to:

\[
\begin{align*}
\gamma_1 + \gamma_3 & \leq 1 \\
\gamma_1 + \gamma_2 & \leq 1 \\
\gamma_2 + \gamma_3 & \leq 1 \\
\gamma_1 & \leq \sum_{i \in [j]} \beta_i \\
\gamma_2 & \leq \sum_{i \in [j+1, k-1]} \beta_i \\
\gamma_3 & \leq \sum_{i \in [k, d]} \beta_i
\end{align*}
\]

As this linear program is identical to that for matrix multiplication, it immediately follows that its optimum is either \(3/2\) or \(1 + \min \left\{ \sum_{i \in [j]} \beta_i, \sum_{i \in [j+1, k-1]} \beta_i, \sum_{i \in [k, d]} \beta_i \right\}\), whichever is smaller for the given program.
6.3. n-body Pairwise Interactions. Suppose we have a list of $n$ objects, and each object interacts with every other object. This comes up frequently in many scientific computing applications (e.g. particle simulations), as well as database joins.

The nested loops for this problem are (for some arbitrary function $f$):

$$
\text{for } \{x_1, x_2\} \in [L_1] \times [L_2] \\
A_1[x_1] = f(A_2[x_1], A_3[x_3])
$$

Instantiating (5.1) we get:

$$
\begin{align*}
\max & \lambda_1 + \lambda_2 \quad \text{s.t.} \\
& \lambda_1 \leq 1 \\
& \lambda_2 \leq 1 \\
& \lambda_1 \leq \beta_1 = \log_M L_1 \\
& \lambda_2 \leq \beta_2 = \log_M L_2
\end{align*}
$$

which gives us a maximum tile size of $\min \{M^2, L_1 M, L_2 M, L_1 L_2\}$ and a maximum communication cost of $\min \{L_1 L_2 / M, L_2, L_1, M\}$. The last term, $M$, is a result of the assumption in our model that each tile carries $M$ words of memory into cache. Therefore, it is important to note that if total amount of memory required to execute the program without going back to main memory is less than $M$, the output of the program will still be $M$, when in the actual cost is in fact the sum of the sizes of the matrices.

7. Discussion and Future Work

In this paper, we have shown a systematic, efficiently computable way of determining optimal tilings for projective loop nests of arbitrary size, and used it to rederive several tight lower bounds that have hitherto largely been computed by a problem-specific approach.

Our approach reveals some structural properties of the tile as well: All such loop nests share an optimal tile shape (rectangles). Furthermore, as the optimal tile’s dimension for any projective loop nest is the solution to a linearly parameterized linear program, its cardinality for a given loop nest must be of the form $M^{f(L_1, \ldots, L_d)}$ for some piecewise linear function $f$. In fact, for a given loop nest, we may programmatically find a closed form of $f$ by feeding LP (5.1), which calculates the dimensions of the tile, into a multiparametric linear program solver, e.g. that of [BBM03], as in [DD18]. This piecewise-linear structure has also been previously shown to hold for convolutions [DD18], and we conjecture that this property holds even in the general, non-projective case as well.

The immediate application we see for our approach is as compiler optimization to automatically block projective nested loops. While many such common loops have already been extensively optimized in high-performance libraries (and some of these optimizations have been implemented in compilers, e.g.icc’s --opt-matmul flag), our techniques are fully general - applying to applications (e.g. pairwise interactions) that do not fit this mold - and do not require programmers to have any familiarity with specific high performance libraries, only access to a compiler with the right optimizations.

Furthermore, as the memory model we use can be generalized to multiprocessor machines (as in [Kni15], following the approach of [ITT04]), our work also provides evidence for the intuition that the best way to split projective loop-nest tasks up on a multiprocessor system is to assign each processor a rectangular subset of the iteration space.

Our work is intended as a first step towards generally optimizing non-projective nested loops, such as those found in neural nets, image processing, and other similar structured computations, many of which lack well-studied high-performance implementations [BI19]. Algorithms to find
such tilings - and the shapes thereof - are known\footnote{Such algorithms, which enumerate all the constraints of the HBL linear program, are in general hard (double exponential in $n$ and $d$, as of the time of publication of this paper). However, as the cost only needs to be incurred once (e.g. during a computation of a highly performance sensitive kernel), and as $n$ and $d$ tend to be relatively small in practice, this is less of an impediment than it might appear at first glance.} for problems with large indices \cite{DR16, CDK+13}; however, a general method for addressing the small-bound case, which occurs in many applications (including most machine learning ones, where, for instance, filter sizes tend to vary), is still unknown, and is left to future work.

\section*{Acknowledgements}

We would like to thank Tarun Kathuria for helpful discussions.

This material is based upon work supported by the US Department of Energy, Office of Science under Award Numbers 7081675 and 1772593; Cray, under Award Number 47277; and DARPA, under Award Number FA8750-17-2-0091.

\section*{References}

\footnotesize

\begin{itemize}
  \item \cite{BBM03}  F. Borrelli, A. Bemporad, and M. Morari. Geometric algorithm for multiparametric linear programming. \textit{Journal of Optimization Theory and Applications}, 118(3):515–540, Sep 2003.
  \item \cite{BCCT10}  J. Bennett, A. Carbery, M. Christ, and T. Tao. Finite bounds for Hölder-Brascamp-Lieb multilinear inequalities. \textit{Math. Res. Lett.}, 17(4):647–666, 2010.
  \item \cite{BCD+14}  G. Ballard, E. Carson, J. Demmel, M. Hoemmen, N. Knight, and O. Schwartz. Communication lower bounds and optimal algorithms for numerical linear algebra. \textit{Acta Numerica}, 23:1–155, 2014.
  \item \cite{BI19}  Paul Barham and Michael Isard. Machine learning systems are stuck in a rut. In \textit{Proceedings of the Workshop on Hot Topics in Operating Systems}, HotOS ’19, pages 177–183, New York, NY, USA, 2019. ACM.
  \item \cite{CDK+13}  M. Christ, J. Demmel, N. Knight, T. Scanlon, and K. Yelick. Communication lower bounds and optimal algorithms for programs that reference arrays - part 1. arxiv.org/abs/1308.0068, 2013.
  \item \cite{DD18}  James Demmel and Grace Dinh. Communication-optimal convolutional neural nets. \textit{CoRR}, abs/1802.06905, 2018.
  \item \cite{DGK+13}  M. Driscoll, E. Georganas, P. Koanantakool, E. Solomonik, and K. Yelick. A communication-optimal n-body algorithm for direct interactions. In 2013 IEEE 27th International Symposium on Parallel and Distributed Processing, pages 1075–1084, May 2013.
  \item \cite{DR16}  James Demmel and Alex Rusciano. Parallelepipeds obtaining HBL lower bounds. Technical Report UCB/EECS-2016-162, EECS Department, University of California, Berkeley, Nov 2016.
  \item \cite{GAB+18}  Evangelos Georganas, Sasikanth Avancha, Kunal Banerjee, Dhiraj D. Kalamkar, Greg Henry, Hans Pabst, and Alexander Heinecke. Anatomy of high-performance deep learning convolutions on SIMD architectures. \textit{CoRR}, abs/1808.05567, 2018.
  \item \cite{GGdOW16}  Ankit Garg, Leonid Gurvits, Rafael Mendes de Oliveira, and Avi Wigderson. Algorithmic aspects of brascamp-lieb inequalities. \textit{CoRR}, abs/1607.06711, 2016.
  \item \cite{HK81}  Jia-Wei Hong and H. T. Kung. I/O complexity: The red-blue pebble game. In \textit{Proceedings of the Thirteenth Annual ACM Symposium on Theory of Computing}, STOC ’81, page 326–333, New York, NY, USA, 1981. Association for Computing Machinery.
  \item \cite{HSF18}  Geoffrey E Hinton, Sara Saabour, and Nicholas Frosst. Matrix capsules with EM routing. In \textit{International Conference on Learning Representations}, 2018.
  \item \cite{HZC+17}  Andrew G. Howard, Menglong Zhu, Bo Chen, Dmitry Kalenichenko, Weijun Wang, Tobias Weyand, Marco Andreetto, and Hartwig Adam. Mobilenets: Efficient convolutional neural networks for mobile vision applications, 2017.
  \item \cite{ITT04}  Dror Irony, Sivan Toledo, and Alexander Tiskin. Communication lower bounds for distributed-memory matrix multiplication. \textit{J. Parallel Distrib. Comput.}, 64(9):1017–1026, September 2004.
  \item \cite{Kni15}  Nick Knight. Communication-Optimal Loop Nests. PhD thesis, EECS Department, University of California, Berkeley, Aug 2015.
  \item \cite{RD16}  A. Rusciano and J. Demmel. Parallelepipeds obtaining HBL lower bounds. arxiv.org/abs/1611.05944, 2016.
  \item \cite{Val10}  S. Valdimarsson. The Brascamp-Lieb polyhedron. \textit{Canadian J. Math.}, 62(4):870–888, 2010.
\end{itemize}