Semiclassical Prediction of Large Spectral Fluctuations in Interacting Kicked Spin Chains

Maram Akila*, Boris Gutkin†*, Peter Braun*, Daniel Waltner* and Thomas Guhr*

*: Faculty of Physics, University of Duisburg-Essen, Lotharstr. 1, 47048 Duisburg, Germany
†: Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany
E-mail: maram.akila@uni-due.de
E-mail: boris.gutkin@uni-due.de

Abstract. While plenty of results have been obtained for single-particle quantum systems with chaotic dynamics through a semiclassical theory, much less is known about quantum chaos in the many-body setting. We contribute to recent efforts to make a semiclassical analysis of many-body systems feasible. This is nontrivial due to both the enormous density of states and the exponential proliferation of periodic orbits with the number of particles. As a model system we study kicked interacting spin chains employing semiclassical methods supplemented by a newly developed duality approach. We show that for this model the line between integrability and chaos becomes blurred. Due to the interaction structure the system features (non-isolated) manifolds of periodic orbits possessing highly correlated, collective dynamics. As with the invariant tori in integrable systems, their presence lead to significantly enhanced spectral fluctuations, which by order of magnitude lie in-between integrable and chaotic cases.

Keywords: spin chains, many-body semi-classics
1. Introduction

For many years the field of quantum chaos has revolved mostly around questions regarding the spectral statistics of single to few body Hamiltonian systems \[1, 2\]. As has been realized, already in the 80’s, the type of spectral statistics crucially depends on the system’s underlying classical dynamics. In particular, for fully chaotic systems the spectral statistics turn out to be universal, see \[3, 4\], and are well described by Random Matrix Theory (RMT) \[5\]. The desire to understand this connection led to the development of methods based on the semiclassical Gutzwiller trace formula which allows one to treat spectral correlations in systems with fully chaotic dynamics \[6, 7, 8\].

In recent years the focus of research has shifted from single to many-body systems and there is a natural inclination to explore their spectral properties using semiclassical tools \[9, 10, 11, 12\]. These systems possess several distinct dynamical features which set them apart from the single particle case. One of them is the possibility of collective dynamics which is found in countless physical systems, ranging from Bose Einstein condensation \[13, 14\] over superparamagnetism \[15\] to nuclei \[16, 17, 18, 19\]. This phenomenon can appear only when the system possesses a significant number of particles \(N\). So, as opposed to the single particle semiclassical limit, where only the Planck’s constant tends to zero, the many-body problem should take into account the existence of another (large) parameter \(N\). This interplay has a profound impact on the resulting spectrum of the system. In this paper we continue our recent effort \[20\] to explore spectral properties of the kicked spin chains through their classical dynamics when the number of spins \(N\) becomes large.

While eigenenergies \(\{E_1, E_2, \ldots\}\) of a generic integrable Hamiltonian behave like a bunch of statistically independent numbers, spectra of chaotic systems exhibit a much more rigid structure. A simple way to validate this statement is to look at the spectral density \(d(E)\), where the smooth and the oscillating parts are separated:

\[
d(E) = \sum_{n=1}^{\infty} \delta(E - E_n) = \tilde{d}(E) + d_{osc}(E).
\]

Recall that \(d(E)\) should be understood as a distribution which requires smoothing with an appropriate test function to remove its singular nature \[2\]. A natural expectation is that due to the rigidity of the spectra the smoothed oscillating part \(d_{osc}\) for Hamiltonians with chaotic dynamics should be significantly smaller than for integrable ones.

The impact of classical dynamics on the magnitude of \(d_{osc}\) can easily be understood through a semiclassical theory. To this end we consider the Fourier transform, \(\text{Tr} \hat{U}(t) = \int_{-\infty}^{\infty} d(E)e^{-itE/\hbar} \, dE\), of the density of states which is given by traces of the time evolution propagator \(\hat{U}(t)\). In the semiclassical limit \(\hbar \to 0\) those are related to the periodic orbits (POs) of the corresponding classical system,

\[
\text{Tr} \hat{U}(t) = \sum_n e^{-itE_n/\hbar} \sim \sum_\gamma A_\gamma e^{iS_\gamma/\hbar},
\]

where the sum on the right-hand side runs over all POs \(\gamma\) possessing a period \(t\). Their associated action is denoted by \(S_\gamma\) and the prefactor \(A_\gamma\) depends on the stability of
the PO. In the case of integrable systems with $D$ degrees of freedom POs are not isolated but rather reside on $D$ dimensional tori. For these systems the prefactor scales as $A_\gamma \sim \hbar^{-(D+1)/2}$. If, on the contrary, the orbit is fully isolated one observes no scaling, $A_\gamma \sim \hbar^0$. Furthermore, if the orbit has (non-trivial) marginal directions the corresponding scaling is of an intermediate type, $A_\gamma \sim \hbar^{-\alpha}$, where the fractional exponent $\alpha$ depends on the type of bifurcation \[21, 22, 23\]. For purely chaotic systems all periodic orbits are isolated, therefore the oscillating term in the semiclassical limit scales as $d_{osc} \sim \hbar^{-1}$ independently of the system dimensions. On the other hand for fully integrable $N$-particle systems the resulting spectral oscillations have significantly larger scales of the order $d_{osc} \sim \hbar^{-(D+1)/2}$ which clearly grow with the number of particles, as $D \propto N$.

The above line of reasoning holds only for the pure semiclassical limit, where $\hbar \to 0$ while the number of particles $N$ is held fixed. Clearly, the purely thermodynamic limit, where $N \to \infty$ at fixed $\hbar$, corresponds to a very different physical picture. For instance, in the case of interacting bosons the first and the second limit give rise to two very different classical systems – they correspond to the first and second quantization treatment of the system Hamiltonian, respectively \[10, 12\]. In the present paper we address the problem of spectral fluctuations in the case when both $\hbar^{-1}$ and $N$ are large. Within a rigorous mathematical framework this corresponds to a double limit, where simultaneously $\hbar \to 0$ and $N \to \infty$ is taken while some relation between both parameters is kept. Specifically, we ask whether an analog of (2) holds in this case and if yes, what the magnitude of the prefactors $A_\gamma$ is. As we show, the answer is in general affirmative, but with a twist – for the model under consideration the sum on the right hand side of (2) turns out to be strongly dominated by a very particular class of POs with a low spatial period.

Going from few to many-particle systems is a hard challenge, especially so in the semiclassical limit \[24\]. The primary reason for this is pretty obvious: for a $N$-body system the density of states grows exponentially with $N$, \textit{i.e.}, $\bar{d} \sim \hbar^{-D+1}$. Already for moderate $N$ a full resolution of the quantum spectrum is beyond the scope of any numerical or experimental approach. As a result, the direct calculation of the left hand side in (2) seems intractable. To the best of our knowledge an explicit comparison of periodic orbits and quantum spectra was never attempted before beyond few-body systems like the helium atom ($N = 2$) \[25, 26\]. Similar constraints hold for the overall field of many-body quantum mechanics. Many of the currently employed standard methods try to circumvent this limitation with approximate approaches which reduce the complexity of the system’s Hilbert space. For instance, mean-field methods treat effective single particle systems, see \textit{e.g.} \[27\], and matrix product states restrict the possible maximal entanglement between different particles \[28, 29\]. These methods are typically well suited to treat systems in vicinity to the ground state, but fail in the bulk of the spectrum, where the eigenstates of the system are fully entangled. Since we are interested in the semiclassical limit of the Hamiltonian, a different idea is needed to reduce the complexity of the Hilbert space. To this end we employ a recently developed
approach, see [30, 31, 20], to evaluate traces of the time evolution operator for kicked chain-like systems with local interactions. This method, for discrete maps and integer times $T$, is based on the exact duality relation

$$\text{Tr} \hat{U}^T = \text{Tr} \hat{W}^N,$$

which connects traces of the unitary Floquet evolution $\hat{U}$ in time to those of a dual, non-unitary evolution $\hat{W}$ in space. Crucially, the dimension of $\hat{W}$ depends only on $T$ and is therefore small if times are short. This allows an effective numerical calculation of the left hand side of (2) provided that the considered time $T$ encompasses only several periods. On a more fundamental level, by virtue of its low dimension the operator $\hat{W}$ is much better suited for studies of large scale many-body spectral fluctuations in comparison to the original time evolution $\hat{U}$.

Originally, the duality approach has been developed for the Kicked Ising Chain model with a fixed spin quantum number of $j = 1/2$ [31]. In the present paper we focus on a natural extension of this setting to an arbitrary $j$. This model can be seen as a chain of $N$ Kicked Tops coupled through nearest neighbour interaction, as detailed in section 2. The semiclassical limit is attained by sending the effective Planck’s constant $\hbar_{\text{eff}} = j^{-1}$ to zero, i.e., $j \to \infty$. While the classical dynamics of a single Kicked Top ranges from integrable to almost fully chaotic, its $N$ body extension never exhibits full hyperbolicity. This means, in particular, that a typical PO possesses both elliptic and hyperbolic directions. As we show in the body of the paper POs of the system, in general, can be separated into two classes – isolated POs and non-isolated PO manifolds of dimensions 2 and 4. Depending on the choice of parameters these PO manifolds may have a short spatial period $N_{(p)}^{(\gamma)}$ which can be seen as a signature of highly correlated, collective dynamics. Furthermore, we explore (2) for 1 and 2 time steps by investigating the spectrum of the dual operators $\hat{W}$. Our main result is that $\text{Tr} \hat{U}^2$ is strongly dominated by a small number of these PO manifolds, while all isolated POs are suppressed in the large $N$ limit. The factors $A_\gamma \sim \hbar^{-\alpha(N)}$ associated with each PO manifold exhibit a very large scaling exponent $\alpha(N) = \alpha_0 N$, growing linearly with $N$. The maximum scaling $\alpha_0$ appears to be 1/4 which is exactly half of the corresponding value for the integrable case. Moreover, the model discriminates between different chain lengths. It shows particularly strong spectral oscillations for chains of length $N = N_{(p)}^{(\gamma)} k$ with $k \in \mathbb{N}$, where the prime length $N_{(p)}^{(\gamma)}$ is solely defined by the PO manifolds.

The paper is structured as follows. In the next section we introduce the Kicked Spin Chain model, which in comparison to [20] is extended by a non-zero torsion $V$. Its periodic orbits are then studied in section 3. The formulation and proof of the spectral duality relation (3) is given in section 4. In section 5 the semiclassical trace formula is studied numerically for the aforementioned one and two time steps $T$ of dynamical evolution in the case of $V = 0$. We demonstrate that for $T = 2$ the right hand side of (2) is indeed strongly dominated by collective PO manifolds in the large $N$ limit. In the subsequent section 6 we explain these empirical findings via a semiclassical theory of the dual operator $\hat{W}$. We extend the above results to the case of $V \neq 0$ in section
Finally, the conclusions are presented in section 8. Technical details are relegated to the appendices.

2. Model

Throughout this section we introduce the Kicked Spin Chain model \[32, 33, 34, 35\] for general spin quantum numbers \(j\). The Hamiltonian

\[
\hat{H}(t) = \hat{H}_I + \hat{H}_K \sum_{T=-\infty}^{\infty} \delta(t - T)
\]

(4)

describes the general dynamics of \(N\) spins where \(\hat{H}_I\) describes the (Ising) coupling between spins and \(\hat{H}_K\) provides local kicks acting on each spin separately. Time (in between kicks) is measured in terms of integer unit steps. The corresponding time (Floquet) evolution operator for a single time step is thus given by

\[
\hat{U} = \hat{U}_I \hat{U}_K \quad \text{with} \quad \hat{U}_{I,K} = e^{-i(j+1/2)\hat{H}_{I,K}}.
\]

(5)

Therein \((j + 1/2)\), as detailed later, takes on the role of the inverse Planck constant \(\hbar^{-1}\).

As a side remark, an exchange of \(\hat{H}_I\) and \(\hat{H}_K\) in (4) leads to the same evolution \(\hat{U}\).

2.1. Kicked Top

To begin with we recall the Kicked Top [2] as the \(N = 1\) limit of the model. The corresponding Hamiltonians are

\[
\hat{H}_{K}^{(KT)} = \frac{2 \vec{b} \cdot \hat{\vec{S}}}{j+1/2},
\]

(6)

\[
\hat{H}_{I}^{(KT)} = \frac{4J^{(KT)}}{(j + 1/2)^2} (\hat{S}_z)^2
\]

(7)

with the spin operator \(\hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)^T\) for spin quantum number \(j\), i.e., \((\hat{\vec{S}})^2 = j(j + 1)\). The Ising part contains a non-trivial quadratic term which can be thought of as a shear or torsion. This term singles out the \(z\)-direction and therefore the magnetic field \(\vec{b}\) in \(\hat{H}_{K}^{(KT)}\) can be restricted, without loss of generality, to the \(xz\)-plane, \(\vec{b} = (b^x, 0, b^z)^T = b(\sin \varphi, 0, \cos \varphi)^T\), where \(\varphi\) is the angle between the magnetic field and the \(z\)-axis, \(\tan \varphi = b^x/b^z\). To avoid a dependence of the coupling parameter strengths on \(j\) in the classical limit we rescale both Hamiltonians by their respective powers in \(\hat{\vec{S}}\). Minimal uncertainty is given in terms of \(j\) for spin coherent states, which therefore replaces \(\hbar^{-1}\) as a measure of Planck cell size.

The corresponding classical model can be found replacing \(\hat{\vec{S}}\) by a spin vector \(\sqrt{j(j + 1)} \vec{n}, \|\vec{n}\| = 1\), precessing on the Bloch sphere. The relation to the canonical coordinates \((q,p)\) is given by [2, 36]

\[
\vec{n} = \left(\sqrt{1-p^2} \cos q, \sqrt{1-p^2} \sin q, p\right)^T
\]

(8)
Figure 1. Upper hemisphere ($n^z > 0$) of the classical phase-space for the kicked top, see eq. (10), after 200 iterations for several hundred randomly chosen initial points. Parameters are chosen as $J^{(KT)} = 0.7, b = 0.9\sqrt{2} \approx 1.27$. The angle varies with $\varphi = 0, 0.2, \pi/4$ (from left to right).

with the corresponding Hamiltonian

$$H(q,p) = 4Jp^2 + 2\left(\hat{b}^z p + b^z \sqrt{1-p^2} \cos q\right) \sum_{T=-\infty}^{\infty} \delta(t-T)$$

having only a single degree of freedom.

The classical action of the kick onto $\vec{n}$ is a rotation around the $\vec{b}$-axis by the angle $2\hat{b}$, denoted by $R_{\hat{b}}(2\hat{b})$. The Ising part also acts as a rotation around the $z$-axis, however, its angle depends on the value of $n^z$ creating the torsion. Combining both rotations one finds the new position of $\vec{n}$ after a single time step as

$$\vec{n}(T+1) = R_z(4J^{(KT)}n^z) R_{\hat{b}}(2b) \vec{n}(T).$$

Figure 1 provides Poincare sections of this dynamics for different values of $\varphi$. In the case of $\varphi = 0$, left panel, the system is integrable and the phase-space is filled by tori corresponding to fixed actions. Changing the angle slightly leads to a breakup of those tori, leaving the system in a mixed state including remaining regular islands, while for $\varphi = \pi/4$ the system is (almost) fully chaotic.

In several cases it turns out to be useful to decompose the $\vec{b}$-rotation into three rotations around the coordinate axes,

$$R_{\hat{b}}(2b) = R_z(\alpha) R_x(\beta) R_z(\gamma) = R_z(\alpha - \pi/2) R_y(\beta) R_z(\gamma + \pi/2).$$

The angles $\alpha, \beta, \gamma$ are the corresponding Euler angles of this decomposition and for our choice of $\vec{b}$, namely $\hat{b}^y = 0$, they are given by

$$\alpha = \gamma, \quad b^z \tan(\pi/2 - \alpha) = b \cot \beta, \quad \cos \beta = \left(\frac{b^z}{b}\right)^2 + \left(\frac{\hat{b}^y}{b}\right)^2 \cos 2\hat{b}.$$

Due to $R_z(x) R_z(y) = R_z(x+y)$ this allows us, classically, to express the whole dynamics in terms of alternating $x, z$ or $y, z$ rotations, respectively.
2.2. Kicked Spin Chain

For the $N$-body extension of the Kicked Top into a one dimensional chain we introduce a homogeneous bilinear coupling between neighbouring spins,

$$
\hat{H}_I = \sum_{n=1}^{N} \left( \frac{4J \hat{S}_{n+1}^z \hat{S}_n^z}{(j+1/2)^2} + \frac{4V (\hat{S}_n^z)^2}{(j+1/2)^2} \right),
$$

with the inter-spin Ising coupling $J$ and an additional local non-linearity governed by $V$. Throughout the paper we mostly restrict ourselves to the special case $V=0$ and assume this condition if not stated otherwise. Discussion of the $V \neq 0$ extension is relegated to section 7. The kicked part is kept local and identical to (6),

$$
\hat{H}_K = \sum_{n=1}^{N} \frac{2 \vec{b} \cdot \hat{\vec{S}}_n}{j + 1/2}.
$$

Boundary conditions for the interaction are chosen periodic, i.e., $\hat{S}_{N+1}^z = \hat{S}_1^z$, making the system translation invariant. In consequence, the special case of $N=1$ of the Kicked Spin Chain corresponds to the Kicked Top above with $J^{(KT)} = J + V$.

The modifications on the classical side compared to (10) and (9) are straightforward. The Hamiltonian (9) is adjusted to

$$
H(\vec{q},\vec{p}) = \sum_{n=1}^{N} \left[ 4Jp_{n+1}p_n + 4Vp_n^2 + 2 \left( b^z p_n + b^x \sqrt{1 - p_n^2} \cos q_n \right) \sum_{T=-\infty}^{\infty} \delta(t-T) \right],
$$

which includes additional interaction between neighbouring momenta. The Hamiltonian equations of motions give rise to the rotation of $N$ classical spin vectors $\vec{n}_m$,

$$
\vec{n}_m(T+1) = R_z(4J (\chi_m + 2V/J n^z_m)) R_x(2b) \vec{n}_m(T).
$$

In this case the angle of rotation, $\chi_m = n^z_{m-1} + n^z_{m+1}$, encodes the bilinear interaction between the spins.

The system remains integrable for $b^x = 0$. In the special case of $j=1/2$ the Kicked Spin Chain (for $V = 0$) possesses another (non-trivial) integrable regime for $b^z = 0$ [37, 38, 31] which ceases to exist for higher spin quantum numbers.

3. Periodic Orbits

For the semiclassical analysis knowledge of the PO actions and their stabilities is essential. In the limit of large $j$ the trace of the propagator can be expressed by a Gutzwiller-type of sum over POs of period $T$:

$$
\text{Tr} \, \hat{U}^T \sim \sum_{\gamma(T)} A_{\gamma} e^{i(j+1/2)S_{\gamma}}.
$$

This relation was explicitly derived for spin systems in [39]. Here $S_{\gamma}$ is the classical action as given in Appendix A and the prefactor $A_{\gamma}$ is determined by the stability of the orbit. If the orbit is sufficiently isolated in phase-space, it is given by

$$
A_{\gamma} = \frac{T_{(\gamma)} e^{iG_{\gamma}}}{|\text{det} (M_{\gamma} - I)|}.
$$
KIC semiclassical fluctuations

where $G_\gamma$ is the Maslov phase and $M_\gamma$ is the monodromy matrix determining the stability of the orbit under small perturbations.

After establishing basic properties of the POs due to the system’s chain like structure we look, in more detail, at the case of integrable dynamics in section 3.2. The general case, with $V = 0$, is covered by section 3.3 where the primary focus is on manifolds of non-isolated POs which play a crucial role in the subsequent semiclassical analysis.

3.1. General Properties

A periodic orbit $\gamma$ of duration $T_\gamma$ for $N_\gamma$ spins is a set $\{\vec{n}_m\}_{m=1}^{N_\gamma}$ of Bloch vectors satisfying

$$\vec{n}_m(T_\gamma) = \vec{n}_m(0), \quad \text{where} \quad \vec{n}_m(0) = \vec{n}_m, \quad \vec{n}_m(T_\gamma) = \left( R_m(J, V, \vec{b}) \right)^{T_\gamma} \vec{n}_m(0)$$

with the classical propagation matrix $R_m(J, V, \vec{b}) = R_z(4J\chi_m + 8Vn_z^m) R_{\vec{b}}(2b)$, compare section 2.2. As $\gamma$ is a valid orbit for $T_\gamma$ time steps it will, by further repetition, also be a valid orbit for $kT_\gamma$ time steps ($k \in \mathbb{N}$). This is a direct consequence of the system’s translation invariance in time. The minimal number of time steps required to close the orbit (for the first time) is the primitive time period $T^{(P)}_\gamma$. Such an orbit leads to $T^{(P)}_\gamma$ different fixed point solutions to (19) corresponding to changed initial starting points along the orbit.

By construction we have a translational symmetry not only in time but also along the chain direction. Accordingly, a periodic orbit of the $N$ spin system induces, by repetition, an orbit for a $kN$ particle system with the same parameters. For instance, every periodic orbit of the Kicked Top is also a periodic orbit of the Kicked Spin Chain for $J^{(KT)} = J + V$. We introduce the primary spatial period $N^{(P)}_\gamma$ as the minimal number of spins required to accommodate the orbit $\gamma$. The cyclic permutation of the motion of individual spins along the chain does, due to the translation symmetry, not change the overall dynamics and any given orbit is thus part of a family of $N^{(P)}_\gamma$ identical orbits with identical action and stabilities.

Periodic orbits can be expressed in terms of repetitions of the prime orbits which encompasses the minimal number of particles and time steps necessary to accommodate it. These types of repetitions imply a linear scaling of the action $S_\gamma$ of an orbit,

$$S_\gamma = r^{(T)}_\gamma r^{(N)}_\gamma S^{(P)}_\gamma,$$

where $S^{(P)}_\gamma$ is the action of the prime orbit and

$$r^{(T)}_\gamma = \frac{T}{T^{(P)}_\gamma}, \quad r^{(N)}_\gamma = \frac{N}{N^{(P)}_\gamma}$$

are the repetitions in time and space, respectively. The actions $S_\gamma$ can be calculated as the sum of local spherical areas swept by the $\vec{n}_i$’s on the Bloch’s spheres. The specific calculations are relegated to Appendix A.

Numerics shows that for a generic choice of parameters most of the orbits comprise both hyperbolic and elliptic directions. In other words, for a typical $\gamma$ the set of
eigenvalues of the corresponding monodromy matrix $M_{\gamma}$ includes ones for which $|\lambda_i| > 1$, as well as ones with $|\lambda_i| = 1$. The relation (18) breaks down when one of the directions becomes marginal, i.e., one of the eigenvalues turns into 1 and changes from hyperbolic to elliptic, or vice versa, under infinitesimal change of the system parameters. If an orbit is marginal this also holds for its repetitions in time and space. We comment further on non-isolated orbits in section 5.

3.2. Integrable Case

For $b^x = 0$ all rotations are around the $z$-axis and therefore commute with the Hamiltonian making the system integrable. As a result the dynamics of the kicked system for arbitrary times is equivalent to one at fixed time, e.g. $T = 1$ with rescaled system parameters $J \rightarrow JT$ and $b^z \rightarrow b^z T$. Moreover, the flow induced by the Hamiltonian $\hat{H}_I + \hat{H}_K$ for time $T$ is identical to the evolution of the kicked system for $T$ time steps with the same parameters. For the classical trajectories $p_n = \text{const.}$ holds for each $n$ and periodic orbits form $N$ dimensional manifolds. To close a trajectory in phase-space after $T$ iterations it is sufficient that the total change in angles $\Delta q_n$ is a multiple of $2\pi$,

$$\Delta q_n = 4T(J(p_{n-1} + p_{n+1}) + 2Vp_n) + 2b^z T = 2\pi m_n .$$

(22)

The $m_n \in \mathbb{Z}$ is a local winding number for spin $n$. Since the momenta are bounded, $|p_n| \leq 1$, $\chi_n = p_{n-1} + p_{n+1}$ resides within the interval $[-2, +2]$. Therefore, this equation has no solution if, for instance, $b^z > 4(J+V)$ and $4(J+V) + b^z < \pi/T$. In such cases the system does not posses any classical periodic orbits of period $T$ or shorter. If all parameters (times $T$) are sufficiently small, the first accessible winding number is necessarily zero. With increasing time $T$ the number of possible $m_n$ grows linearly and with it the number of possible (distinct) periodic orbits grows algebraically. With respect to $N$ the number of periodic orbits is determined by all admissible combinations of the winding numbers. If there is more than one allowed $m_n$ the growth is thus exponential in $N$. This exponential growth also holds for non-integrable parameter choices.

3.3. General Case

A perturbation of the integrable model by a non-zero $b^x$ breaks up the $N$-dimensional periodic tori into isolated periodic orbits and some low dimensional manifolds of non-isolated periodic orbits. We first comment on the general properties of the isolated ones and later detail on the manifolds which, as it turns out, play a significant role in the semiclassical treatment of the corresponding quantum model.

3.3.1. Isolated Periodic Orbits. The observed exponential proliferation of periodic orbits for increasing $N$ within the integrable model carries over to the general case. For large $N$ the stabilities of orbits are of a mixed type, i.e., both hyperbolic and elliptic directions are present in the same orbit. The behavior of the prefactors $|A_{\gamma}|$,
however, substantially depends on the time \( T \). For a generic set of parameters and \( T = 1 \) a typical orbit is well isolated, so that \( |A_\chi| \) is an exponentially small quantity. On the other hand, for \( T = 2 \) we found many \( \gamma \)'s for whom a large number of eigenvalues of \( \mathbf{M}_\chi \) are close to \( 1 \). In other words, an essential number of periodic orbits is almost marginal implying quite small determinants \( \det(1 - \mathbf{M}_\chi) \). In such cases the approximation \([18]\) is no longer applicable.

### 3.3.2. Periodic Orbit Manifolds.

Besides the isolated orbits the case \( T = 2 \) \((V = 0)\) also features four dimensional manifolds of periodic orbits, \( i.e., \) regions in phase-space where every point constitutes a periodic orbit. As we explain below, this phenomenon occurs when the length of the spin chain is equal to \( N = 4k \), \( k \in \mathbb{N} \). This peculiar condition can be traced back to a special feature of the four-spin system whose periodic orbits, by repetition, also induce periodic orbits of larger systems with \( N = 4k \). According to \([19]\) for \( V = 0 \) and \( N = 4 \) the time evolution of the first and the third spin vectors \( \vec{n}_1, \vec{n}_3 \) are provided by one and the same rotation matrix \( R_z(4J\chi_1)R_y(2b) \). This immediately implies that the scalar product \((\vec{n}_1 \cdot \vec{n}_3)\) is a conserved quantity. Similarly, \((\vec{n}_2 \cdot \vec{n}_4)\) is preserved, as well. In other words, the \( N = 4 \) spin chain possess two integrals of motion. Particularly, in the case of \( b^z = 0 \) the system is over integrable having 6 integrals of motion rather than 4: In addition to the four momenta \( p_i, i = 1, \ldots, 4 \) the differences between coordinates \( q_{1,2} - q_{3,4} \) are conserved under time evolution.

**4D manifolds** — In the general case we provide an explicit construction of periodic orbit manifolds. Since the dynamics of spin \( i \) depends exclusively on the time evolution of the variable \( \chi_i = n_{i-1}^z + n_{i+1}^z \), any trajectory satisfying the condition

\[
R_z(4J\chi^{(1)}_i)R_y(2b)R_z(4J\chi^{(2)}_i)R_y(2b) = 1, \quad i = 1, \ldots, N, \tag{23}
\]

where \( \chi^{(1)}_i, \chi^{(2)}_i \) are the values at the time-steps \( t = 1, 2 \), respectively, is automatically periodic. The most simple way to satisfy this condition is to assume that \( 4J\chi^{(1)}_i \mod 2\pi = 4J\chi^{(2)}_i \mod 2\pi = \chi \) is constant for all spins. This implies that \((R_z(4J\chi^{(1)}_i)R_y(2b))^2 = 1\) such that \( R_z(4J\chi^{(1)}_i)R_y(2b) = R_y(\pi) \) is a rotation about \( \pi \) around some axis \( \zeta \). This forces the value of \( \chi^{(1)}_i \) to satisfy the following equation:

\[
b^z \tan \left( 2J\chi^{(t)}_i \right) = b \cot b, \quad i = 1, \ldots, N, \quad t = 1, 2. \tag{24}
\]

Fixing the values of \( \chi \) by eq. \([24]\) imposes restrictions onto the positions of each spin at each time-step \( t = 1, 2 \),

\[
\chi^{(1)}_i = n_{i-1}^z + n_{i+1}^z, \quad \chi^{(2)}_i = -2\sin^2b \sin \varphi \cos \varphi \left( n_{i-1}^z + n_{i+1}^z \right) + \sin \varphi \sin 2b \left( n_{i-1}^y + n_{i+1}^y \right) \tag{25}
\]

\[
+ \left( \cos 2b \sin^2 \varphi + \cos^2 \varphi \right) \left( n_{i-1}^z + n_{i+1}^z \right), \tag{26}
\]

where the constants \( \chi^{(t)}_i \) satisfy \([24]\) for all \( i \) and \( t \). The second equation results from the fact that the second time step \( \chi^{(2)}_i \) is obtained from the original spin vectors via a rotation, \( \chi^{(2)}_i = \vec{e}_z \cdot \mathbf{R}_z(2b) (\vec{n}_{i-1} + \vec{n}_{i+1}) \) (the \( z \)-component is not changed by \( \mathbf{R}_z(4J\chi^{(t)}_i) \) and it thus does not need to be considered). For any sequence of \( 2N \) solutions of \([24]\)
obeying the conditions $-2 \leq \chi_i^{(t)} \leq +2$, the equations (25), (26) fix a 4-dimensional manifold of initial conditions for periodic orbits. An example of such a periodic orbit is given in figure 2 which shows that the relative motion between the spins is frozen due to the identical $R_z(4J\chi_i^{(t)})$. All periodic orbits belonging to these manifolds have one and the same action provided by an elegant formula,

$$S_{\text{man}} = J \sum_{i=1}^{N} \sum_{t=1}^{2} \chi_i^{(t)} \chi_{i+1}^{(t)},$$

whose explanation is relegated to Appendix B.

We can distinguish three different regimes, where (24) has none, one or several solutions in the interval $-2 \leq \chi_i^{(t)} \leq +2$, each having unique consequences for the system behavior. The first case occurs, when $J$ or $b$ are sufficiently small, bringing the model close to the integrable/non-interacting regime. Most of the paper is devoted to the single manifold regime, where (24) admits one unique solution such that $\chi_i^{(t)} = \chi$ for all $i, t$. In this case the action of the manifold orbits is given by

$$S = N S_{\text{man}} \quad \text{with} \quad S_{\text{man}} = 2J\chi^2 \quad \text{and} \quad N = 4k.$$  

In this case equation (26) reduces, using $\chi_i^{(2)} = \chi_i^{(1)} = (n_{i-1}^x + n_{i+1}^x)$, to the simpler form

$$\chi = \left(n_{i-1}^x + n_{i+1}^x\right) \cot \varphi + \left(n_{i-1}^y + n_{i+1}^y\right) \frac{\cot b}{\sin \varphi}.$$  

For the case of several possible solutions $\chi_i^{(t)} = \chi + m_i^{(t)}(\pi/2J)$, $m_i^{(t)} \in \mathbb{Z}$ of eq. (24), the number of different manifolds of periodic orbits starts to grow exponentially with $N$. This can be understood if we compare the role of the $m_i^{(t)}$ to the spin winding numbers in the integrable case. There the number of orbits with respect to $N$ was determined by the exponentially growing amount of different possible combinations of winding numbers. In a similar way we can exchange the possible values of $m_i^{(t)}$ along the spin chain leading to the exponential growth of different periodic orbit manifolds.

2D manifolds — So far, we considered the cases where (23) holds for all spins. For the existence of manifolds it is sufficient to demand this condition for only half of the spins, e.g. the odd indexed ones. In this case the manifold will be only two dimensional
as we are sparing out half of the chain. For the even indexed spins this implies that they still fulfill (25). While this ensures that the trajectories of the odd spins are closed regardless of their initial conditions, we need further restrictions to ensure that also the trajectories of the even ones are periodic. An exemplary way to realize this, already present for 4 spins, is by aligning the even spins along the rotational axis, \( i.e., R_z(4J\chi_i^{(0)})R_b(2b)\vec{n}_i = \vec{n}_i \), where \( i \) is even. From the components of this equation we derive several constraints, one on the angles of each of the two even spin, 

\[
\sin (q_i + \gamma) = \frac{p_i \tan \beta/2}{\sqrt{1 - p_i^2}},
\]

which is surprisingly independent of the Ising interaction. Therein \( \beta, \gamma \) are the Euler angles of the kick rotation as defined in (12). Another constraint fixes the value of \( \chi_i \) and therefore the axis along which the spins are aligned. As before for the 4D manifolds this value has to be maintained by the other set of spins, in this case the odd indexed ones, at both time steps. The resulting equations,

\[
p_{i-1} + p_{i+1} = \frac{q_i}{2J} \mod \pi/(2J),
\]

\[
(p_{i-1} + p_{i+1}) \tan \beta/2 = \sqrt{1 - p_{i-1}^2} \sin (q_{i-1} + \gamma) + \sqrt{1 - p_{i+1}^2} \sin (q_{i+1} + \gamma) \mod \pi/(2J \sin \beta),
\]

are, up to the differing value of \( \chi_i \), similar in nature to the ones used in (25) and (26), including the possible multiplicity in the values of \( \chi \). The last constraints concern the values of the odd \( \chi_{i\pm 1} \), due to demanding the original manifold condition (23) they are given by (24) and have to be fixed via the even indexed spins. They are aligned along the same axis and we thus find

\[
p_i = \frac{\chi_{i\pm 1}}{2}.
\]

These six constraints fully fix the even spins, (30) and (33) hold for each of the spins separately, while (31) and (32) impose only two further conditions.

**3.4. Weak coupling regime.**

In the special case of the non-interacting regime \( J = 0 \), all periodic orbits are given by compositions of solutions for the single spin case. As a result, a non-interacting spin chain becomes fully chaotic if the corresponding \( N = 1 \) Kicked Top possesses chaotic dynamics, as happens for certain choices of the parameters \( \vec{b} \) and \( V \). In such a case each periodic orbit of the non-interacting spin chain is isolated and fully hyperbolic. This situation still persists after introducing a weak coupling \( J \) between the spins, at least, if \( T \) is sufficiently short. In this regime periodic orbits are fully hyperbolic and can be related to their non-interacting counterparts making their identification an easy achievable goal. Accordingly, a leading order semiclassical approximation (17) works significantly better for weakly interacting spin chains in comparison to the general case, where the dynamics is plagued by bifurcations.
4. Duality Relation

The dynamics of chain-like models with nearest neighbour interactions is governed by Hamiltonian equations which are local both in time and particle indices \( n, t \), respectively. This suggests that in certain situations it might be useful to reverse the roles of \( n \) and \( t \) looking at \( t \) as particle and \( n \) as time index, see [30]. We first illustrate this on the classical level and later extend these ideas to the quantum setting.

4.1. Classical Duality

It is a simple observation that, in general, one and the same set of Newtonian equations with nearest neighbor interactions,

\[
q_{n,t+1} = \phi(q_{n,t}, q_{n,t-1}, q_{n-1,t}, q_{n+1,t})
\]

leads to two possible dynamical systems. The first one is provided by the conventional symplectic map \( \Phi: (\vec{q}_t, \vec{p}_t) \to (\vec{q}_{t+1}, \vec{p}_{t+1}), \vec{q}_t = (q_{1,t}, \ldots q_{n,t}), \vec{p}_t = (p_{1,t}, \ldots p_{n,t}) \) describing the propagation of the system in time. On the other hand, the same set of equations can be used to connect the “future” coordinate \( q_{n+1,t} \) in space through its spacial predecessors:

\[
q_{n+1,t} = \tilde{\phi}(q_{n,t}, q_{n-1,t}, q_{n,t-1}, q_{n,t+1}).
\]

Under the condition that such an inversion is unique this defines the second map \( \tilde{\Phi}: (\vec{q}_n, \vec{p}_n) \to (\vec{q}_{n+1}, \vec{p}_{n+1}), \vec{q}_n = (q_{n,1}, \ldots q_{n,T}), \vec{p}_n = (p_{n,1}, \ldots p_{n,T}) \) which we call dual. It corresponds to the propagation in “space”, i.e., in particle index, rather than in time. For the case of the considered spin chain such a dual map can be defined if \( b \) and \( J \) are sufficiently small as \([35]\) possesses a unique solution only in this case. Both maps are, except for some special cases (see [30]), quite different. In particular \( \tilde{\Phi} \) is typically not even symplectic. Nevertheless, the two maps posses one and the same set of periodic orbits. Indeed, \( \Phi \) for a chain of length \( T \) and \( \tilde{\Phi} \) for a chain of length \( N \) have the same set of fixed points for \( N \) (respectively \( T \)) steps of dynamical evolution. In other words, both \( \tilde{\Phi} \) and \( \Phi \) can, in principle, be used to find periodic orbits of the system. In the next section we show how the above classical duality reappears in the quantum setting.

4.2. Quantum Duality

A central object of our calculations are the traces of \( \hat{U}^T \) which encode information on the quantum spectrum. Straightforward calculations of this quantity are not possible for long chains due to exponentially growing matrix dimension, \( \text{dim} \hat{U} = (2j+1)^N \times (2j+1)^N \). Even for the smallest spin quantum number \( j = 1/2 \) only spectra of chains with around 20 spins are easily accessible. In [30] it was observed that this problem can be, in fact, circumvented due to the exact relation

\[
\text{Tr} \hat{U}^T = \text{Tr} \hat{W}^N
\]
which identifies the traces of the quantum time evolution operator with those of a dual one \( \hat{W} \) of dimension \((2j + 1)^T \times (2j + 1)^T\). Informally speaking, the evolution operators \( \hat{U} \) and \( \hat{W} \) can be regarded as quantizations of \( \Phi \) and \( \tilde{\Phi} \), respectively. In contrast to \( \hat{U} \) the dual operator \( \hat{W} \) is in general non-unitary due to the non-symplectic nature of its classical counterpart \( \tilde{\Phi} \). Most significantly, \( \hat{W} \) has a rather small (\( N \)-independent) dimension, as long as the considered time \( T \) is short. This allows the calculation of Tr \( \hat{U}^T \) for small \( T \) and arbitrary \( N \), even if \( j \) is relatively large.

In [31] such a duality was shown for the \( j = 1/2 \) Kicked Ising Chain. Here, we extend it to a broad class of kicked systems with nearest neighbor interactions and arbitrary \( j \). To this end we consider the \((2j + 1)^N\) dimensional product basis,

\[
|\bar{\sigma}\rangle = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \ldots \otimes |\sigma_N\rangle, \tag{37}
\]

with discrete single particle states \( |\sigma_n\rangle \in \{ | -j \rangle, | -j + 1 \rangle, \ldots | j \rangle \} \). It is assumed that the time evolution of the system can be split into two parts, \( \hat{U} = \hat{U}_I \hat{U}_K \), where \( \hat{U}_I \) and \( \hat{U}_K \) correspond to interaction and kick, respectively. Further on, we assume that the interaction part \( \hat{U}_I \) couples only nearest-neighbours, is diagonal and in addition translation invariant with respect to the particle number in the basis (37). Its matrix elements are thus given by

\[
\langle \bar{\sigma} | \hat{U}_I | \bar{\sigma}' \rangle = \exp \left( \sum_{n=1}^{N} f_I (\sigma_n, \sigma_{n+1}) \right) \delta_{\bar{\sigma}, \bar{\sigma}'}, \tag{38}
\]

where we introduced the function \( f_I \) which represents the interaction between neighboring spins. For our choice of Hamiltonian, eq. (13), it is given by

\[
f_I (\sigma_n, \sigma_{n+1}) = \frac{-4i}{j + 1/2} \left( J_{\sigma_n} \sigma_{n+1} + V_{\sigma_n}^2 \right). \tag{39}
\]

The kicking part \( \hat{U}_K \) is subjected to only one constraint that it has to be local:

\[
\hat{U}_K = \bigotimes_{n=1}^{N} \hat{u}_K \quad \text{with} \quad \langle \sigma | \hat{u}_K | \sigma' \rangle = e^{f_K(\sigma, \sigma')}. \tag{40}
\]

Here the function \( f_K \) may be arbitrary, as long as \( \hat{u}_K \) is a unitary matrix. For the case considered in the paper, see (14), we obviously find:

\[
f_K(\sigma, \sigma') = \ln \langle \sigma | \exp \left( -2i \vec{b} \cdot \hat{S} \right) | \sigma' \rangle. \tag{41}
\]

We now introduce said dual matrix \( \hat{W} = \hat{W}_I \hat{W}_K \) via

\[
\langle \bar{\sigma} | \hat{W}_I | \bar{\sigma}' \rangle = \exp \left( \sum_{t=1}^{T} f_K (\sigma_t, \sigma_{t+1}) \right) \delta_{\bar{\sigma}, \bar{\sigma}'}, \tag{42}
\]

\[
\hat{W}_K = \bigotimes_{t=1}^{T} \hat{w}_K \quad \text{with} \quad \langle \sigma | \hat{w}_K | \sigma' \rangle = e^{f_K(\sigma, \sigma')}, \tag{43}
\]

where we exchanged the position of \( f_I \) and \( f_K \) and consider a chain of \( T \) spins. In contrast to the work of [30], this new operator is non-unitary, this also holds for \( \hat{W}_I \) and
\( \hat{W}_K \) separately. For the model at hand, it can be given in a more explicit form. The new interaction part retains a diagonal structure,

\[
\langle \vec{\sigma} | \hat{W}_I | \vec{\sigma}' \rangle = \delta_{\vec{\sigma}, \vec{\sigma}'} T \prod_{t=1}^{T} \langle \sigma_t | \exp \left( -2i \vec{b} \cdot \vec{S} \right) | \sigma_{t+1} \rangle,
\]

which is fully determined by the kick part of the original model. Contrary, the original interaction provides the form of the dual kick:

\[
\hat{W}_K = \bigotimes_{t=1}^{T} w_K, \quad \langle \sigma' | w_K | \sigma \rangle = \exp \left( -4i \left( J\sigma \sigma' + V\sigma^2 \right) / j + 1/2 \right).
\]

We discuss the spectrum of \( \hat{W}_I \hat{W}_K \) in section 6.

To recover the trace duality we rewrite the traces on the left side of eq. (36) as 2D partition function by inserting identities for the different times,

\[
\text{Tr} \hat{U}^T = \sum_{\{\vec{\sigma}(t)\}} \langle \vec{\sigma}(1) | \hat{U} | \vec{\sigma}(T) \rangle \langle \vec{\sigma}(T) | \hat{U} | \vec{\sigma}(T-1) \rangle \cdots \langle \vec{\sigma}(1) \rangle,
\]

and expressing this further as a sum over all possible combinations of \( \sigma_{n,t} \in \{-j, \ldots, j\} \) per time–step and spin index,

\[
\text{Tr} \hat{U}^T = \sum_{\sigma_{n,t} \in \{-j, \ldots, j\}} \exp \left( i \sum_{n=1}^{N} \sum_{t=1}^{T} f_I(\sigma_{n,t}, \sigma_{n+1,t}) + f_K(\sigma_{n,t}, \sigma_{n,t+1}) \right). \tag{47}
\]

Since the result is symmetric under the exchange \( n \leftrightarrow t, N \leftrightarrow T, f_I \leftrightarrow f_K \), an analogous procedure leads to the same expression for \( \text{Tr} \hat{W}^N \). In the context of 2D classical Ising models the operators \( \hat{U} \) and \( \hat{W} \) are nothing more than transport operators along the “temporary” and “spatial” directions which express the partition function in two different ways.

Finally, let us comment on a certain peculiarity of the integrable case (\( b^\times = 0 \)). Due to the identity \( \hat{U}(J, \vec{b})^T = \hat{U}(TJ, T\vec{b}) \) the evolution for \( T \) time steps can be equivalently thought of as one for a single time step with rescaled parameters. Therefore, the dual operator always takes on the form of a \((2j+1) \times (2j+1)\) matrix (rather than \((2j+1)^T \times (2j+1)^T\)) for a single spin system:

\[
\hat{W}_{nm} = \exp \left( -i \frac{4JT}{j + 1/2} nm - i \frac{4JV}{j + 1/2} m^2 - 2iTb^\times n \right), \tag{48}
\]

where the indices \( n, m \) run from \(-j\) to \(+j\).

5. Action Spectrum

For quantum Hamiltonian systems the underlying classical POs can be revealed by taking an appropriate Fourier transform of the spectral density with respect to an energy like parameter [1, 40, 41, 42]. For quantum maps, however, energy is not defined. Still, it is possible to extract classical POs out of traces of the quantum evolution taking a Fourier transform over the inverse of the effective Planck’s constant, see e.g., [2, 8, 43].
The linear scaling with the spin quantum number $j$ in the exponent of (17) makes it a good quantity for such a Fourier transformation. Employing this procedure we obtain

$$\rho(S) = \frac{1}{j_{\text{cut}}} \sum_{j=1}^{j_{\text{cut}}} e^{-i(j+1/2)S} \text{Tr} \hat{U}^T \sim \frac{1}{j_{\text{cut}}} \sum_{\gamma(T)} A_\gamma \delta_{j_{\text{cut}}}(S - S_\gamma),$$

where $\delta_{j_{\text{cut}}}$ stands for a periodized approximation of the $\delta$-distribution with width $\sim \pi / j_{\text{cut}}$ and height $j_{\text{cut}}$. The cut-off $j_{\text{cut}}$ is introduced in order to keep the dimension of $\hat{U}$, or more precisely $\hat{W}$, numerically accessible. As $j_{\text{cut}} \to \infty$ the function $\rho(S)$ resolves the classical orbit actions $S_\gamma$ for orbits of period $T$, up to a modulus of $2\pi$. It is worth noting that for technical reasons the sum in (49) is restricted to integer values of $j$. The inclusion of half-integer $j$ allows, in principle, the resolution of $S_\gamma$ up to a modulus of $4\pi$. We do not pursue this issue further.

In this section we numerically study the action spectrum $\rho(S)$ for one and two time steps. To this end we first evaluate the spectrum of the dual operator $\hat{W}$ for $T = 1, 2$ and then calculate $\text{Tr} \hat{U}^T$ using the duality relation (36). As a result, we are able to obtain $\rho(S)$ for an arbitrarily large spin chain and some finite $j_{\text{cut}}$.

5.1. The effect of bifurcations

Recall that for isolated POs the prefactors $A_\gamma$ are given by (18). Accordingly, if $\det(M_\gamma - 1) \neq 0$ for all POs of period $T$ the function $\rho(S)$ does not scale with $j_{\text{cut}}$. On the other hand, when a periodic orbit changes its stability type, one of the corresponding eigenvalues turns into $\Lambda_i = 1$. This immediately implies a divergence of (18). In this case the linearized dynamics in terms of $M_\gamma$ is insufficient to describe the weight of an orbit to the sum in (17). Instead, higher orders have to be taken into account in the form of uniform approximations [22, 23]. The adjusted $A_\gamma$ has the scaling $j^\alpha$, where the exponent $\alpha > 0$ depends on the type of bifurcation. To demonstrate how bifurcations affect the action spectrum $\rho(S)$ we show, as an example, a (isochronous) pitchfork bifurcation for single particle systems on the left hand side of figure 3, where the algebraic scaling $\alpha = 1/4$ is clearly observed. In addition, we provide a slightly detuned system which shows at first algebraic growth and for larger $j_{\text{cut}}$ tends towards saturation.

So far, studies of the bifurcation effects on the quantum spectrum have been mostly restricted to systems with a single degree of freedom [23, 44, 22, 45]. While an exact bifurcation is a singular event nearly bifurcating orbits with $\Lambda_i \approx 1$ are generic in many-body systems with mixed dynamics. In general, for $N$-body systems the number of elliptic directions increases with the number of degrees of freedom $N$. Assuming that phases of the corresponding (elliptic) eigenvalues of $M_\gamma$ are distributed uniformly, the probability to come close to one should grow with $N$. While one might argue that in the limit $j \to \infty$ equation (18) must be recovered for such nearly bifurcating orbits, this is only true for the pure semiclassical limit with fixed $N$. In practice this is never the
$KIC$ semiclassical fluctuations

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.8\textwidth]{figure3.png}
\end{center}
\caption{(color online) Left hand side: Scaling of the action spectrum peak height $|\rho(S_\gamma)|$ for $N=1, T=2$ over the cut-off parameter $j_{\text{cut}}$. Shown are three cases, for a position which features an (isochronous) pitchfork bifurcating orbit ($J=0.7, b^x \approx 0.94$ and $b^z \approx 0.90$), the same orbit with slightly detuned couplings ($J=0.68$) and a generic (i.e., isolated) orbit for the detuned parameters. Right hand side: Dependence of the height of $|\rho(S_\gamma)|$ on the cut-off parameter $j_{\text{cut}}$ for two selected orbits of the action spectrum for $N=7$ particles given in figure 4. The selected orbit shown in blue is the largest one in figure 4 at $S_\gamma \approx 5.77$, which also shows the strongest deviations from the semiclassical prediction. The other one (orange) is a small ghost orbit at $S_\gamma \approx 2.75$ for the same parameters. While the first one saturates, for large $j_{\text{cut}}$, to a limiting value, the ghost decays exponentially.}
\end{figure}

case as $j$ is necessarily finite. In other words, for a limit where both $N$ and $j$ tend to infinity the prefactors $A_\gamma$ (resp. $\rho(S)$) might still possess a non trivial scaling $j^{\alpha(N)}$ due to the presence of quasi-marginal directions.

5.2. Single Time Step

In the case of $T=1$ the spectrum of $\hat{W}$ can be easily calculated for a relatively large cut-off parameter $j_{\text{cut}} \sim 10^4$, while the number of periodic orbits grows weaker with $N$ in comparison to longer times. This allows a good resolution of the action spectrum for moderate spin chain lengths and isolated POs. Figure 4 shows the (absolute) action spectrum $|\rho(S)|$, see (49), for identical parameters but different numbers of spins. The upper row depicts numerical calculations based on the spectrum of the dual quantum operators and colored bars therein mark the positions of classical periodic orbits. For comparison the lower row contains a semiclassical approximation for which we use the right hand side of (17) instead of the actual traces in (49). In contrast to the upper row this one relies solely on classical information – actions $S_\gamma$ of the POs and their stabilities $A_\gamma$ provided by eq. (18).

5.2.1. $N = 1$. The left panels shows the single particle case of the Kicked Top, which features only two periodic orbits for such short times. The broader peak to the right is a ghost orbit (emerging for larger $J$) which is naturally not reproduced in the lower panel. Otherwise the agreement is excellent.
5.2.2. N = 7. The middle panel shows $|\rho(S)|$ for $N=7$ spins, containing significantly more orbits with very good agreement between the classical positions of their actions and the corresponding peaks of $|\rho(S)|$. As $N$ is prime these orbits necessarily possess either $N^{(P)} = 1$ or $N^{(P)} = 7$ marked by different colors in fig. 4. Naturally, POs with $N^{(P)} = 1$ are just repetitions of POs encountered in the $N = 1$ case.

Comparison to the semiclassical approximation shows good agreement for approximately half of the POs, but the others exhibit some deviations in height due to the proliferating bifurcations. This is most apparent for the highest peak at $S_\gamma \approx 5.77$ (which height is deliberately cut in the lower panel). The particular PO contains 6 elliptic, 4 mixed and 4 purely hyperbolic directions, with $|\ln \Lambda| \approx 0.86$ bringing it sufficiently close to a bifurcation. To check it in more details we take a look at the peak heights as a function of $j_{\text{cut}}$, see the right hand side of fig. 4. Indeed, the function shows strong oscillations due to the existence of accompanying orbits with close actions and saturation is achieved only for considerably high values of $j_{\text{cut}}$.

5.2.3. N = 19. For the right hand panels in figure 4 the number of spins is increased further to $N=19$. In this case we are no longer able to resolve individual orbits despite an increased $j_{\text{cut}}$. The semiclassical approximation, based on $\sim 2000$ found orbits, resembles the actual function $|\rho(S)|$ only for some of the largest peaks. Given the huge amount of underlying POs the clear structure of the action spectrum with only a few dominant peaks is quite remarkable. The positions of these peaks indeed correspond to the POs with the largest prefactors $A_\gamma$. In fig. 4 we mark only orbits $\gamma$ which surpass
Figure 5. (color online) Approximate action spectrum plotted over the action $S$ and the chosen cut-off parameter $j_{\text{cut}}$. Upper row corresponds to $T=1$ lower to $T=2$, left column features $N=1$, right $N=10$. Parameters are given by $J=0.7$ and $b^x=b^z=0.9$. $j_{\text{cut}}$ is necessarily integer, the graphic shows an interpolation. Noise in the upper right panel is caused by computational difficulties of the visualization.

a fixed threshold, $|\det (M_\gamma - 1)|^{-1/2} > 10^{-3}$. As one can see, their positions coincide with the largest spikes of $|\rho(S)|$.

5.3. Two Time Steps

For two time steps ($T = 2$) the number of POs is substantially larger in comparison to the $T = 1$ case with the same parameters. Furthermore, many of them are close to bifurcations, making a semiclassical reconstruction of the spectrum even harder. In addition, significantly smaller achievable values of $j_{\text{cut}}$ limit our resolution. We illustrate this with a direct comparison of the $T = 1$ and $T = 2$ cases in figure 5 where we depict the action spectrum using $j_{\text{cut}}$ as additional variable. For the $T = 1$ cases in the upper row we reach sufficiently high values of $j_{\text{cut}}$ to observe both the initial interference of nearby orbits and saturation of $|\rho(S_\gamma)|$ for larger $j_{\text{cut}}$. As stated previously the $N=1$ case (left) also contains a separated ghost orbit whose decay becomes apparent in this visualization. For the $T = 2$ case it is no longer feasible to resolve such scales and we, instead, only observe the initial growth associated with close to bifurcation orbits.

In contrast to the previous subsection we therefore omit the semiclassical reconstruction but provide, in figure 5, the numerically calculated action spectra for different particle numbers and identical system parameters. Remarkably, for chain lengths divisible by 4 the action spectrum $\rho(S)$ turns out to be strongly dominated by the PO manifolds. This means that for $N = 4k, k \in \mathbb{N}$ one observes only few
Figure 6. (color online) Absolute value $|\rho(S)|$ of the approximate action spectrum, eq. (49), over $S$ for $T = 2$ time steps using $j_{\text{cut}} = 114$. The system parameters are $J = 0.7$ and $b^x = b^z = 0.9$, only the number $N$ of spins is varied. Coloured lines indicate classical orbit positions, the color corresponds to the primitive period: $N(S) \gamma = 1$ (orange), $N(S) \gamma = N$ (blue) and purple otherwise. Green lines correspond to $N S_{\text{man}}$, see eq. (28), indicative of the manifolds position. In the cases of $N = 4$ and $N \geq 6$ only selected orbits are shown, see text. The arrows indicate the position of an $N(S) \gamma = 1$ orbit close to an isochronous pitchfork bifurcation, compare fig. 3. Its impact can be followed up to $N \approx 19$

strong peaks exactly at the positions of the PO manifolds actions, (28), while all other POs are essentially suppressed. Furthermore, for these length sequences $\rho(S)$ exhibits a particularly large magnitude.

As one can check, the height of the peaks at $S_{\text{man}}$ follows a scaling law,

$$|\rho(S_{\text{man}})| \sim (j_{\text{cut}})^{\alpha(N)} \quad \alpha(N) \sim \alpha_0 N,$$

with a constant $\alpha_0$ only weakly dependent on the system parameters (for further details on its value see section 6.1). This scaling is shown in figure 7 in comparison to the integrable case, where $\alpha(N) = N/2$. Clearly visible is a strong enhancement whenever the particle number is $N = 4k$ i.e., when the PO manifolds appear. However, a linear growth of scaling with $N$ is a general trend, independent of whether the particle number is a multiple of four or not. Compare e.g. the general magnitude for the $N = 7$ case in
figure [6] to the case of $N = 9$, in both cases the manifold is absent. In contrast, $T = 1$ shows no scaling of $\alpha$ with $N$. A slight, visible decay for this case in figure 7 can be attributed to strong interference between neighboring orbits, which influences the actual results.

![Figure 7](image.png)

**Figure 7.** (color online) Estimated scaling exponent $\alpha$ of the largest peak in the action spectrum for various particle numbers and fixed system parameter $J = 0.7$ and $b^x = b^z = 0.9$ for $T = 1, 2$. In the integrable case $b^x = 0$ is chosen. For numerical fitting the heights in the range $j_{\text{cut}} = 95$ to 114 (for $T = 2$) and $j_{\text{cut}} = 200$ to 400 (for $T = 1$ and integrable) are taken into account. For $T = 1$ a close inspection shows that the value of $\alpha$ is not yet fully saturated but instead slightly negative.

While the scaling (50) in the integrable system is easily understood in terms of the classical $N$ dimensional invariant tori, recall the growth of $A_j \sim j^{(N-1)/2}$, the increase of $\alpha$ with $N$ in the case of the four-dimensional PO manifolds seems to be, at first, a perplexing phenomenon, given that the number of their marginal directions does not grow with $N$. In the strict semiclassical limit $j \to \infty$ with fixed $N$ the existence of four marginal directions would imply only the constant scaling $\alpha(N) = 2$. The anomalously large scaling in the double limit case can be attributed to the increase of quasi-marginal directions for which the corresponding Lyapunov exponents are close to one. A hand-waving, qualitative explanation of (50) can be attempted in terms of counting quasi-marginal directions, for whom the Lyapunov exponents are near zero.

As numerics shows, their numbers do indeed grow with $N$, but correct accounting of such directions is already a challenge for single particle systems, see [22 44]. Taking into account contributions of all nearly bifurcating orbits for a large $N$ seems to be an extremely difficult problem and we avoid this path in what follows. Rather, we will provide an explanation for (50) through the study of spectral properties of the dual operator $\hat{W}$.

### 6. Spectrum of the Dual Operator

The question of the anomalously large spectral fluctuations associated with the PO manifolds, specifically its scaling with $N$ as observed in the last section, can be addressed in terms of the largest eigenvalues of the dual operator $\hat{W}$. Indeed, for large $N$ the traces...
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Figure 8. (color online) Comparison between the single time step action spectrum \(|\rho(S)|\) (blue curves) and an approximated variant using only the largest eigenvalue in (51) instead of the full traces. The four panels correspond to \(N=5\) (left) and \(N=20\) (right), the upper row uses a low cut-off, \(j_{\text{cut}} = 50\), the lower one features \(j_{\text{cut}} = 500\). Parameters are given by \(J=0.7\) and \(b^x=b^z=0.9\).

of \(\hat{W}^N\) are dominated by their largest eigenvalues,

\[
\text{Tr} \hat{U}^T = \text{Tr} \hat{W}^N = \sum_l \tilde{\lambda}_l^N (1 + O(e^{-\delta N})), \quad \delta > 0 ,
\]

where the sum can be restricted to several eigenvalues \(\tilde{\lambda}_l\) with the maximal absolute value. The validity of this approximation greatly depends on the magnitude of \(N\). In figure 8 we depict both the actual action spectrum (blue curve) and an approximate result (orange), for which we leave in the sum (51) only the largest eigenvalue. The agreement between the two curves greatly improves with the number of spins \(N\). Besides \(N\) also \(j\) plays a role as it governs the dimension of \(\hat{W}\) and therefore the gap \(\delta\) between the largest eigenvalues and their successors.

6.1. Numerical Findings

As has been explained above, it is of crucial importance to understand how the largest eigenvalues of \(\hat{W}\) depend on \(j\) in the semiclassical limit \(j \to \infty\). Below we provide the results of a numerical study of the dual operator spectrum and give their explanations based on a semiclassical theory in the next section.

For only a single time step \(T=1\) the spectrum \(\{\tilde{\lambda}_i | i = 1, \ldots, 2j + 1\}\) of \(\hat{W}\) is uniformly distributed in the angular direction, see figure 10 for a generic example. As the operator is non unitary, the eigenvalues are not restricted to the unit circle and, in fact, many of them reside close to the origin indicating the non-unitary nature of the
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Figure 9. (color online) Phase of the trace of the dual operator for differing spin quantum numbers $j$ and $J = 0.7$, $b^x = b^z = 0.9$ where we consider $T = 2$ time-steps for $N = 56$ particles. The (rescaled) contribution of the manifold’s action (gray line proportional to $(j + 1/2)S_{\text{man}} \bmod 2\pi)$ is clearly visible and works more accurately for larger $j$.

Figure 10. (color online) Eigenvalue spectrum $\tilde{\lambda}$ of $\hat{W}$ for $T = 1$ in the complex plane. System parameters are chosen as $J = 0.7$ and $b^x = b^z = 0.9$ with $j_{\text{cut}} = 4700$.

dual evolution. For two time steps, $T = 2$, the dual spectrum $\{\tilde{\lambda}_i | i = 1, \ldots, (2j + 1)^2\}$ has a similar rotationally invariant distribution in the regime where no PO manifolds exist, see fig. [11]. In sharp contrast, a pronounced structure emerges whenever PO manifolds are present. To illustrate this, figure [11] shows the dual spectrum in the regime where either only one or several PO manifolds exist. The spectral distribution has a remarkable cross-like shape(s) indicating an approximate four-fold rotational symmetry, which becomes more and more pronounced for the largest eigenvalues as $j \to \infty$. This symmetry singles out sequences $N = 4k$, where, according to (51), the sum of the largest eigenvalues adds up coherently. On the contrary, for $N \neq 4k$ the sum of the largest eigenvalues vanishes to the leading order in $j$, thus significantly reducing the magnitude of the spectral fluctuations. To make a quantitative prediction it is, therefore, natural to look at the largest $\tilde{\lambda}_i$ as functions of $j$. Focusing on the regime where only one PO
Figure 11. (color online) To the left the spectrum of the dual operator for $T = 2$ and $j = 100$ is presented in the complex plane. The right column shows the scaling of its largest eigenvalue in dependence of $j$ with a numerical fit of $\alpha_0$ where applicable. The middle column depicts the eigenvector corresponding to the largest eigenvalue (for $j = 80$). Therein, the blue rim corresponds to the boundary of the classically allowed region as specified in eq. (59). The parameters are chosen as $J = 0.2$ and $b^x = b^z = 0.3$ (first row) where no manifold is present. In the second row $J = 0.6$ with $b^x = b^z = 0.9$ represents the single manifold regime leading to $\alpha_0 \approx 0.21$. The purple bar shown with the eigenvector is the solution to $\sigma_1 + \sigma_2 = g$ where $g$ is given by (62) for $p = 0$. In the last row $J = 0.8$, $b^x = b^z = 0.9$, where several manifolds exist. The scaling lies in between $0.17 \leq \alpha_0 \leq 0.23$, to guide the eye the shown dashed line corresponds to $\alpha_0 \approx 0.21$. The middle figure in this case shows two different eigenvectors, colored blue and orange, that correspond to the largest eigenvectors of two different crosses shown on the left figure. The endpoints of the purple arrows represent the semiclassical predictions for the localization centers of the first (shown in blue) eigenvector. The two arrows correspond to the parameters $p_1 = 0$, $p_2 = -1$ and $p_1 = -1$, $p_2 = 0$, respectively, see Appendix C for details. The intersection points of the purple bar with the ellipse boundary indicate localization centers of the second eigenvector (shown in orange). It corresponds to $p_1 = p_2 = 0$, as in the single manifold case.
manifold exists, we find that the phases of the four largest dual eigenvalues are given by
\[ \arg \tilde{\lambda}_{max,l} = (j + 1/2)S_{man} + \frac{\pi l}{2} + O(1/j) \quad l \in \{1, 2, 3, 4\}. \] (52)
As predicted, in the cases \( N = 4k \) the \( \pi l/2 \) parts in the phase cancel under summation of the eigenvalues. Remarkably, to the leading order in \( j \), the phases are determined by the prime action \( S_{man} \) of the PO manifold, (28). Such a connection is reminiscent of the Bohr-Sommerfeld quantisation rule for the spectrum of integrable Hamiltonian systems. Furthermore, the absolute values of the largest eigenvalues scale algebraically with \( j \),
\[ |\tilde{\lambda}_{max}| \propto j^{\alpha_0}(1 + O(1/j)). \] (53)
This explains the linear dependence of \( \alpha(N) \) on \( N \), i.e., \( \alpha(N) \sim \alpha_0 N \) in (50). The same scaling carries over to the traces \( \text{Tr} \hat{W}^N \) even for \( N \neq 4k \) where a similar linear growth of \( \alpha \) with \( N \) is observed, but with a constant negative offset, see figure 7.

In the regime of a single PO manifold, the contribution of the four largest eigenvalues is sufficient to get the total phase of the trace even for large powers in \( N \), improving with increased \( j \), see figure 9. Therefore, for \( T = 2 \) the whole essential information about the spectral fluctuations in the system is stored in two parameters: \( S_{man} \) and \( \alpha_0 \). Additional PO manifolds contribute other quadruples of eigenvalues \( \tilde{\lambda}_{max,l}^{(\ell)} \) with a similar scaling of the absolute value \( |\tilde{\lambda}_{max,l}^{(\ell)}| \) but (possibly) different phases \( (j + 1/2)N S_{man}^{(\ell)} + l\pi/2, l = 1, \ldots, 4 \), where \( S_{man}^{(\ell)} \) is the action of the respective PO manifold. As a result, the total contribution in the traces of the evolution operator for \( N = 4k \) is given by:
\[ \text{Tr} \hat{U}^2 = 4j^{N\alpha_0} \sum_{\ell} C_{\ell} e^{i(j+1/2)N S_{man}^{(\ell)} (1 + O(1/j))}, \] (54)
where the sum is over the distinct PO manifolds.

A straightforward inspection of the eigenvectors corresponding to the maximal eigenvalues of \( \hat{W} \) reveals their remarkable localization properties, see fig. 11. These eigenvectors comprise two parts,
\[ \tilde{\psi} = \tilde{\psi}_q + \tilde{\psi}_p, \] (55)
of which \( \tilde{\psi}_q \) is sharply localized in the \( |\sigma_1\rangle \otimes |\sigma_2\rangle \) basis while \( \tilde{\psi}_p \) is localized in the momentum basis \( |\vec{\sigma}_1\rangle \otimes |\vec{\sigma}_2\rangle \), where
\[ |\vec{\sigma}\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{\sigma = -j}^{2j+1} e^{i2\pi \sigma \vec{\sigma} / (2j+1)} |\sigma\rangle. \] (56)

6.2. Semiclassical Theory for \( T = 2 \)
To understand the form of \( \tilde{\lambda}_{max,l} \) and the localization properties of the corresponding eigenvectors, let us first recall the product structure of the dual operator \( \hat{W}_I \hat{W}_K \). For \( V = 0 \) the form (43) of the kick part \( \hat{W}_K \) is reminiscent of the kernel of the Fourier transformation, such that the correspondence becomes exact if \( J = \pi/4 \). The action
Figure 12. (color online) Absolute square of the Wigner small $d$-matrix $d^\beta_{\sigma_1 \sigma_2}$ for an angle $\beta \approx 1.48$ ($b^x = b^z = 0.9$), and $j = 80$.

of this part on a coherent state, localized in both momenta and coordinates, can be interpreted as an exchange of position and momenta values. The interaction part $\hat{W}_I$, eq. (44), is instead given by a product of transition elements of a unitary rotation induced by a constant magnetic field

$$\langle \sigma_1 \sigma_2 | \hat{W}_I | \sigma_1 \sigma_2 \rangle = \langle \sigma_1 | e^{-2i \beta \hat{S}_x} | \sigma_2 \rangle \langle \sigma_2 | e^{-2i \beta \hat{S}_y} | \sigma_1 \rangle = e^{-i(\sigma_1 + \sigma_2)(\alpha + \gamma - \pi)} (d^\beta_{\sigma_1 \sigma_2})^2.$$ (57)

Here $\alpha, \beta, \gamma$ (with $\alpha = \gamma$) are the Euler angles given in (12) and

$$d^\beta_{\sigma \sigma'}(\beta) = \langle \sigma | e^{-i \beta \hat{S}_x} | \sigma' \rangle$$ (58)

is Wigner’s small $d$-matrix. Conveniently, the uniform semiclassical limit of $d^\beta_{\sigma \sigma'}$ is well known [46]. When $j \to \infty$, the function $d^\beta_{\sigma \sigma'}$ is supported within the elliptic region,

$$\sigma'^2 + \sigma^2 - 2\sigma' \sigma \cos \beta \leq (j + 1/2)^2 \sin^2 \beta,$$ (59)

where it scales as $d^\beta_{\sigma \sigma'} \sim j^{-1/2}$ at a finite distance from the boundary and exponentially decays outside of the region [59], see fig. 12. In the semiclassical limit any eigenvector of the dual matrix $\hat{W}$ must, therefore, reside in the classically allowed region given by (59). The largest values of $d^\beta_{\sigma \sigma'}$ are attained along the ellipse boundary. Here one generically finds $d^\beta_{\sigma \sigma'} \sim j^{-1/3}$ while in the vicinity of the four tangent points of the boundary, where in addition either $\sigma$ or $\sigma'$ take on values of $\pm j$, the scaling is $d^\beta_{\sigma \sigma'} \sim j^{-1/4}$. As we show below, such enhanced scaling at the boundary of (59) is responsible for large spectral fluctuations in the model at $T = 2$.

To analyze the spectrum of $\hat{W}$ it turns out to be rather instructive to treat its second power which can be represented as the product

$$\hat{W}^2 = \hat{W}_I \hat{W}_0$$ (60)

of the diagonal matrix $\hat{W}_I$ and an “almost permutation” $\hat{W}_0 = \hat{W}_K \hat{W}_I \hat{W}_K$. More specifically, we show in Appendix C that, for the regime where only a single PO manifold exists, the second factor can be split into the product of two matrices $\hat{W}_0 = (2j + 1)\hat{P}\hat{G}$, where $\hat{P}$ is the following truncated permutation,

$$\langle \sigma_1 \sigma_2 | \hat{P} | \sigma'_1 \sigma'_2 \rangle = \delta_{\sigma_1 + \sigma'_1, g_1} \delta_{\sigma_2 + \sigma'_2, g_2},$$ (61)

and $\hat{G}$ is a band diagonal matrix whose elements are of order 1 near the diagonal and scale as $j^{-1}$ away of it. Here, the constants $g_1, g_2$ are given by

$$g_i = [(2j + 1)(\pi(1 + 2p_i) - 2\gamma)/8J], \quad i \in \{1, 2\},$$ (62)
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with \( p_i \)'s being integers such that \(-2j \leq g_i \leq 2j\) holds. In the regime of only a single PO manifold the last condition determines \( p_1 \) (resp. \( g_i \)) uniquely i.e., \( p_1 = p_2 = p \) (resp. \( g_1 = g_2 = g \)). For the sake of simplicity of exposition we focus below on this particular case and later briefly comment on the extension of the results to the regimes where multiple PO manifolds exist.

To simplify the problem further we substitute \( \hat{G} \) with the unity matrix and consider the spectrum of \((2j + 1)\hat{W}_I\hat{P}\) instead. Recalling the diagonal structure of \( \hat{W}_I \) it is straightforward to see that the eigenvectors of \((2j + 1)\hat{W}_I\hat{P}\) take a simple form:

\[
\psi(\sigma_1, \sigma_2) = C_1|\sigma_1, \sigma_2\rangle \pm C_2|g - \sigma_1, g - \sigma_2\rangle, \quad (63)
\]

with the corresponding eigenvalues \( \Lambda_{(\sigma_1, \sigma_2)} \) given by

\[
\frac{\Lambda_{(\sigma_1, \sigma_2)}}{2j + 1} = \pm \left( \langle \sigma_1, \sigma_2 | \hat{W}_I | \sigma_1, \sigma_2 \rangle |g - \sigma_1, g - \sigma_2| \hat{W}_I |g - \sigma_1, g - \sigma_2 \rangle \right)^{\frac{1}{2}}. \quad (64)
\]

This in turn can be written down in terms of Wigner's \( d \)-functions as

\[
\Lambda_{(\sigma_1, \sigma_2)} = \pm \left( 2j + 1 \right) \left( 2j + 1 \right) d_{\sigma_2, \sigma_1}(\beta) d_{g - \sigma_2, g - \sigma_1}(\beta), \quad (65)
\]

where we have explicitly separated the complex phase from the amplitude. Having at hand the approximate spectrum \((63, 65)\) of the operator \( \hat{W}^2 \) we can straightforwardly write down the corresponding eigenvalues and eigenvectors for \( \hat{W} \),

\[
\psi \approx \psi(\sigma_1, \sigma_2) \pm \Lambda_{(\sigma_1, \sigma_2)}^{-1/2} \hat{W} \psi(\sigma_1, \sigma_2), \quad \hat{\lambda} \approx \pm \Lambda_{(\sigma_1, \sigma_2)}^{1/2}. \quad (66)
\]

The first term \( \psi(\sigma_1, \sigma_2) \) is sharply localized in the \((\sigma_1, \sigma_2)\) space. In contrast, the second term \( \hat{W} \psi(\sigma_1, \sigma_2) \) is localized in momentum space \((\bar{\sigma}_1, \bar{\sigma}_2)\) due to the presence of the \( \hat{W}_K \) factor in \( \hat{W} \). This is in agreement with the previous numerical observation \((55)\). (It is important to emphasize that the actual eigenstates of \( \hat{W}^2 \) have a finite support, while the eigenstates of the approximation \( \hat{W}_I\hat{P} \) are point-like localized.)

In order to find the largest eigenvalues of \( \hat{W}_I\hat{P} \) (resp. \( \hat{W} \)) we need to look for \((\sigma_1, \sigma_2)\) such that \(|\Lambda_{(\sigma_1, \sigma_2)}|\) reaches its maximum value. By eq. \((65)\) this happens whenever both \((\sigma_1, \sigma_2)\) and \((g - \sigma_1, g - \sigma_2)\) belong to the boundary of the ellipse \((59)\). In other words, the localization points of the corresponding eigenvectors are located at the intersection points between the line \( \sigma_1 + \sigma_2 = g \) and the ellipse boundary. Figure \(11\) shows such an eigenvector \( \tilde{\psi} \) of \( \hat{W} \) corresponding to its largest eigenvalue as well as the respective line together with the ellipse boundary. As can be observed, the localization points of \( \tilde{\psi} \) are, indeed, in a good agreement with the above prediction. By eqs. \((65, 66)\) the phases of the four largest eigenvalues of \( \hat{W} \) are given by

\[
(2\gamma - \pi)g/2 + \frac{l\pi}{2} = \left( 2J\chi^2(j + 1/2) + \frac{l\pi}{2} \right) \mod 2\pi, \quad l = 1, 2, 3, 4, \quad (67)
\]

where on the left hand side we used the identity \( 2J\chi = (\pi/2 - \gamma) \mod \pi, \) see \((12)\). After taking into account the expression \((28)\) for the actions of the PO manifolds this immediately yields the previously, empirically found eq. \((52)\). The same approach can be used to evaluate the absolute value of \( \hat{\lambda} \). By eqs. \((65, 66)\) we have \(|\Lambda_{(\sigma_1, \sigma_2)}|^{1/2} \sim j^{\alpha_0}\), with \( \alpha_0 = 1/4 \) if \( \sigma_1, \sigma_2 \) belong to the tangent points of the ellipse and \( \alpha_0 = 1/6, \)
Figure 13. (color online) From left to right: spectrum of the dual operator \((j = 100)\), corresponding largest eigenvector \((j = 80)\) and scaling of the largest eigenvalue in dependence of \(j\) (numerical slope \(a_0 = 0.252 \pm 0.004\)). Compare also figure 11 for further information. Parameters are chosen as \(T = 2, J = 0.45, b^x = 0.789802\) and \(b^z = 0.483691\) such that the eigenvector localizes at the tangent points, see text.

otherwise. In fig. 13 we check this prediction for specially tuned parameters such that the localization points are at the tangent points of the ellipse boundary. In this case the maximum possible scaling \(a_0 = 1/4\) is clearly observed, see fig. 13 (right). On the other hand, for generic parameters the scaling exponent is typically above the naive prediction 1/6. This is probably a consequence of the fact that some (small) portion of the eigenstate \(\psi\) is always localized at the tangent points of the ellipse. A detailed investigation of this question would require taking into account the precise structure of \(\hat{G}\) which is beyond the scope of the present paper.

So far, we considered the case of the single manifold regime. In the parameter regime for multiple PO manifolds several combinations of different integers \((p_1,p_2)\) exist, see Appendix C such that the corresponding \(g_1,g_2\) satisfy the conditions \(-2j \leq g_i \leq 2j, i = 1,2\). As a result, the matrix \(P\) is provided by a sum of permutations – each one corresponds to some particular solution \((p_1,p_2)\). To find the spectrum of eigenvalues of \(\hat{W}\) one follows the same procedure as in the single manifold case. Accordingly, for the largest eigenvalues of \(\hat{W}\) both points \((\sigma_1,\sigma_2)\) and \((g_1 - \sigma_1, g_2 - \sigma_2)\) should belong to the boundary of the ellipse (59). This condition defines a pair of points on the ellipse boundary for each solution \((p_1,p_2)\). All these points serve as centers of localization for the corresponding eigenvectors \(\psi\), see fig. 11.

7. Kicked Spin Chain Model for \(V \neq 0\)

So far, we considered a particular case of the Kicked Spin Chain model as we set \(V = 0\) in the interaction part of the Hamiltonian (13). In this section we allow an arbitrary strength \(V\) of the quadratic term. The kick part of the dual operator is now reminiscent of the kernel of the so called fractional Fourier transformation [47, 48]. As we show below, the core result of the previous sections – the emergence of the anomalously large spectral fluctuations for chain lengths \(4k, k \in \mathbb{N}\) – reappears here again with a peculiar twist. For spin chains of the length \(N = N_0k, k \in \mathbb{N}\) the anomalously large fluctuations,
dominated by PO manifolds, emerge when the ratio between interaction and torsion strength \( \mu = V/J \) attains the following set of values:

\[
\mu = -\cos \frac{2\pi p}{N_0} \quad p \in \{1, 2, \ldots, N_0 - 1\}. \tag{68}
\]

In other words, the model possesses large spectral fluctuations for spin chains of lengths \( N = N_0 k \) with an arbitrary \( N_0 \) when the parameter \( \mu \) is tuned according to (68).

From this perspective the previous \( V = 0 \) case is merely a special one corresponding to \( N_0 = 4 \). To illustrate this we show in figure 14 the action spectrum for \( T = 2 \) at a ratio of \( \mu = 1/2 \) where we find strong peaks for every \( N = 3k \). The spectrum of the corresponding dual operator \( \hat{W} \) shows a three-fold symmetry, see figure 15. As one can observe, the absolute value of the largest eigenvalue scales algebraically with \( j \), which explains the large spectral fluctuations in that case. Furthermore, the corresponding eigenvector looks structurally similar to what occurred in the \( V = 0 \) case. As a further example, for \( N_0 = 5 \) and \( \mu = (\sqrt{5} + 1)/4 \) one has large spectral fluctuations for all chains with \( N = 5k \), see fig. 16. The dual operator in this case, see figure 17, has a 5-fold symmetry.

To explain condition (68) we turn back to the classical dynamics of the model. Looking at periodic orbits of the system for \( V \neq 0 \) we again find

\[
(R_i (4J\chi_n) R_k)^2 = 1, \quad n = 1, \ldots, N_0 \tag{69}
\]
as conditions for the existence of PO manifolds, where we introduced \( \chi_n = p_{n-1} + 2mp_n + p_{n+1} \), compare also with (16). The above conditions fix uniquely (up to an addition of factors \((4J)^{-1}2\pi k_n, k_n \in \mathbb{Z}\)) the variables \( \chi_n \), but not necessarily the \( p_n \). The PO manifolds emerge whenever the \( N_0 \) conditions (69) do not resolve the set \( p_1, \ldots, p_{N_0} \) uniquely. For instance, in the case of 3 spins and \( \mu = 1/2 \) one finds that all \( \chi_n \) are identical to \( p_1 + p_2 + p_3 \). This linear dependence explains the emergence of classical PO manifolds. We can extend this line of reasoning to arbitrary \( N \) and \( \mu \) based on the
Figure 15. (color online) Spectrum of the dual operator containing a non-linear part for $j=100$ (left panel), corresponding largest eigenvector ($j=80$) (middle) and scaling of the largest eigenvalue in dependence of $j$ (numerical slope $a_0 \approx 0.167$). Parameters are chosen as $T=2$, $J=\sqrt{\pi}/3$, $V=\sqrt{\pi}/6$ and $b^x=b^z=0.9$.

Figure 16. (color online) Absolute value $|\rho(S)|$ of the action spectrum over $S$ for $T=2$ using $j_{\text{cut}}=100$. The particle number is indicated in the upper right corners and for $N=6,7$ only orbits with $A_\gamma>0.5$ are shown. The other system parameters are $J=0.5$, $V=(\sqrt{5}+1)/8$ and $b^x=b^z=0.9$. For the color coding see figure 6.

cyclic $N_0 \times N_0$ dimensional band matrix

$$
\mathcal{Z} = \begin{pmatrix}
2\mu & 1 & 0 & \cdots & 0 & 1 \\
1 & 2\mu & 1 & 0 & \cdots & 0 \\
0 & 1 & 2\mu & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 1 & 2\mu
\end{pmatrix}
$$

(70)

connecting the $\chi_n$ and $p_n$ variables via $\vec{\chi} = \mathcal{Z}\vec{p}$. PO manifolds appear whenever $\mathcal{Z}$ is not of full rank. This happens if one of the eigenvalues of $\mathcal{Z}$, $z_n = 2\mu + 2\cos \frac{2\pi n}{N_0}$, $n \in \{1, 2, \ldots, N_0\}$.

(71)

satisfies the condition $z_n = 0$ for some $n$. As one can easily see, this immediately implies (68). All eigenvalues in (71) are doubly degenerate, except $z_{N_0}$ and $z_{N_0/2}$ (for even $N_0$). Accordingly, for all $\mu \neq \pm 1$ from the set (68) we have a freedom to choose two (continuous) parameters $\eta_1^{(i)}, \eta_2^{(i)}$ at each time step $i = 1, 2$ such that $p_n^{(i)}(\eta_1^{(i)}, \eta_2^{(i)})$, while the $\chi_n^{(i)}$ are independent of $\eta_1^{(i)}, \eta_2^{(i)}$. This yields 4-dimensional PO manifolds parametrized by $\eta_1^{(1)}, \eta_2^{(1)}, \eta_1^{(2)}, \eta_2^{(2)}$. In the case $\mu = 1$ the corresponding
Figure 17. (color online) Spectrum of the dual operator containing a non-linear part for \( j = 100 \) (left panel), corresponding largest eigenvector \( (j = 80) \) (middle) and scaling of the largest eigenvalue in dependence of \( j \) (numerical slope \( a_0 \approx 0.167 \)). Parameters are chosen as \( T = 2, J = 1/2, V = (\sqrt{5} + 1)/8 \) and \( b^x = b^z = 0.9 \).

Figure 18. (color online) Spectrum of the dual operator for large \( \mu = 12 \) (left panel, \( j = 100 \)). Besides \( V = 12J \) all other parameters are identical to figure 15. The slope indicated in the rightmost panel is \( a_0 \approx 0.03 \). The middle panel shows the eigenvector corresponding to the largest eigenvalue at \( j = 80 \).

eigenvalue \( z_{N_0/2} \) is non-degenerate and the dimension of the PO manifold is 2 rather than 4 while the spectrum of the dual operator is distributed isotropically. Finally, for \( \mu = -1 \) the parameters \( \chi_n \) satisfy \( \sum \chi_n = 0 \) (by definition) which immediately implies \( 4J\chi_n = 0 \mod 2\pi \) for each \( n \). A simple substitution of this value back into (69) shows that for \( \mu = -1 \) this equation might hold only if \( R_S \) is a rotation by \( \pi \) itself. Therefore, for a generic value of the magnetic field and \( \mu = -1 \) PO manifolds do not exist.

Remarkably, as \( N_0 \) runs through all integer numbers, the set of \( \mu \) values defined by (68) becomes dense in the interval \([-1, 1]\). Informally speaking this implies that in the parameter space we are always “arbitrary close” to PO manifolds for \( \mu \in [-1, 1] \). This in turn suggests that relatively large spectral fluctuations should be observed for any set of parameters with \( |\mu| < 1 \). Indeed, for such parameters we observe a non-trivial scaling \( j^\alpha \) of the largest dual eigenvalues, with the values of \( \alpha \) similar to the \( V = 0 \) case. On the other hand, for \( |\mu| > 1 \) this scaling turns out to be close to zero, see fig. 18.
Figure 19. (color online) Visualization of the two possible ways of semiclassical analysis in chain-like kicked systems. On the left hand side the standard approach of quantum chaos is shown, here the semiclassical trace formula is directly applied to the unitary time evolution $U^T$ of the $N$-spin chain. The right hand side illustrates the dual approach. In this case the traces of the time evolution operator are first rewritten through traces of spatial (i.e., along the chain) evolution $W^N$. For the second step the semiclassical analysis is applied to the spectrum of the non-unitary operator $\hat{W}$.

8. Conclusion

Although many-body systems played a pronounced role in the foundation of quantum chaos its later on development has been mainly restricted to few particle systems. This limitation is seemingly related to one of the key semiclassical tools, the trace formula, which connects traces of quantum evolution operators with periodic orbits (POs) of the underlying classical system. For systems with few degrees of freedom this approach is applicable to a very wide range of time scales, including the Heisenberg times, where the phenomenon of spectral universality holds. On the other hand, an increase in the number of particles $N$ leads to an exponential proliferation of POs on the classical side of the problem and, simultaneously, to an exponential growth of the effective Hilbert space dimension (resp. density of states) on the quantum side. Thus, it becomes apparent that the conventional quantum chaos path, illustrated on the left hand side of figure 19, should fail, in general, to reproduce correctly the classical-quantum correspondence in a limit where both $N$ and $\hbar^{-1}$ grow simultaneously.

Still, as we show in the present paper, the situation is not entirely hopeless, as long as one is interested in the short time scales of many-body evolution. The key ingredient of our approach is the duality relation which connects traces of the unitary evolution $U^T$ to those of the non-unitary operator $\hat{W}$. Crucially, the dimension of $\hat{W}$ is independent of $N$ and remains small for short evolution times. This drastically reduces the complexity of the problem from the numerical point of view. What is even more important, the duality relation opens up a second path, illustrated on the right hand side of figure 19, suitable to address large $N$ systems. Instead of treating the spectrum of the original unitary evolution $\hat{U}$ we can apply semiclassical techniques to the dual operator $\hat{W}$.

We focus on the trace formula for a model of a long, interacting, kicked spin chain in the regime of very short times $T = 1$ and $T = 2$ while the spin quantum number $j$ plays the role of the inverse Planck’s constant. The most significant result of our study is the
observation of extremely large spectral fluctuations for model parameters where non-
isolated 4-dimensional manifolds of POs appear. This POs feature a short spacial period
and can be interpreted as signatures of collective dynamics. In particular, this happens
if $T = 2$ and the ratio $\mu$ between the inter-spin coupling strength and the on-side torsion
is tuned to satisfy the relation (68). Furthermore, provided that the spin chain length
$N$ is large and divisible by the spatial prime period of the PO manifolds, the trace
formula is completely dominated by them while all isolated POs are suppressed. As our
analysis shows, the contributions of PO manifolds to the trace formula comes with large
prefactors $|A_\gamma| \sim j^{\alpha(N)}$ which exponentially grow with $N$: $\alpha(N) = \alpha_0 N$. This explains
the dominance of such structures over isolated POs, where $\alpha_0 = 0$. The exponent $\alpha_0$
weakly depends on the system parameters and ranges between 0 and $1/4$. A similar
growth is observed in fully integrable system with $\alpha_0 = 1/2$. Informally speaking, this
result puts our model somewhere in between fully integrable and fully chaotic systems
(where $\alpha_0 = 0$), as far as long range spectral fluctuations are of concern.

The above statements nicely illustrate a pronounced difference between the pure
semiclassical limit $j \to \infty$ with fixed $N$ and one where $N$ tends to infinity together with $j$.
In the first case $\alpha(N)$ would be bounded by half of the marginal directions of the
PO manifold’s stability matrices, i.e., it would not grow with $N$. The linear growth of
$\alpha(N)$ in the double limit case can be traced down to a growing number of quasi-marginal
directions, whose contribution to the trace formula is hard to evaluate for finite $j$. The
duality approach accounts them in a systematic and quantitative way based on features
of the dual operator’s spectrum.

What remains an open question is the dependence of the spectral fluctuations on $T$.
So far, the analysis has been limited to the two shortest times $T = 1, 2$ and it would be of
interest to assert how (or whether) large spectral oscillations exist for larger $T$. Another
type of open questions left beyond the scope of the present paper concerns the generality
of the phenomenon. For instance, whether it appears in other kicked models, or in
Hamiltonian systems with continuous time evolution. More specifically, one would like
to understand whether the existence of PO manifolds is a necessary/sufficient condition
for large spectral oscillations in general.

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Appendix A. Classical Action

The classical action, as used in (20), of the system contains two contributions, a part
stemming from the interaction ($S_I$) and one from the local kicking ($S_K$). Along the
orbit, or similarly on any other trajectory, it may be split according to

$$S_\gamma = \sum_{t=0}^{T-1} S_K(\vec{q}(t), \vec{p}(t); \vec{q}(t+\epsilon), \vec{p}(t+\epsilon)) + S_I(\vec{p}(t+\epsilon))$$ (A.1)
where the kick is restricted to times $t$ to $t + \epsilon$ with $\epsilon \to 0$ and $\vec{q}, \vec{p}(t + \epsilon)$ are the positions of the spins directly after its application.

As long as the rotation of the spins is around the $z$-axis as in the interaction part, only the $q$ component changes while $p$ remains constant. This makes the evaluation of $S_I$ straightforward and for the contribution to a single time step we find

$$S_I(\vec{p}(t + \epsilon)) = \int_{\vec{q}(t+\epsilon)}^{\vec{q}(t+\epsilon+\epsilon)} \vec{p} \cdot d\vec{q} - \int_{t+\epsilon}^{t+\epsilon+\epsilon} H(\vec{q}, \vec{p}) \, d\tau$$

$$= \sum_{n=1}^{N_q} \Delta q_n(t + \epsilon) p_n(t + \epsilon) + \Delta t H_I(p(t + \epsilon))$$

$$= 4 \sum_{n=1}^{N_q} \left( J p_{n+1}(t + \epsilon) p_n(t + \epsilon) + V(p_n(t + \epsilon))^2 \right),$$

wherein $\Delta t \to 1$ and $\Delta q_n(t + \epsilon) = 4J(p_{n-1}(t + \epsilon) + p_{n+1}(t + \epsilon)) + 8Vp_n(t + \epsilon)$. Throughout one type of dynamics “energy” is conserved and thus $H_I$ is constant along the trajectory segment. From a conceptual point of view $dp_i dq_i$ are the area elements on the Bloch sphere and the integrals thus measure the area swept by the spin vectors $\vec{n}_i$.

The kicking part is given by the Larmor rotation of all spins about $\vec{b}$ around the same angle. Is is local for every spin and its action is thus a sum of single spin actions. But, besides the integrable case ($b^z = 0$) both $p$ and $q$ change. However, we may change our coordinate system $(q, p) \to (Q, P)$ into a basis where the rotation is around the $Z$-axis instead of $\vec{b}$. The transformations are given by

$$q(Q, P) = \arctan \frac{\sqrt{1 - P^2} \sin Q}{\sqrt{1 - P^2} \cos \varphi \cos Q - P \sin \varphi},$$

$$p(Q, P) = P \cos \varphi + \sqrt{1 - P^2} \sin \varphi \cos Q$$

with respect to the angle $\varphi$ between the magnetic field and the $z$-direction. The inverse transformation is given by $\varphi \to -\varphi$. Neglecting particle indices we may cast the integral part of $S_K$ into

$$\int_{q(t)}^{q(t+\epsilon)} p \, dq = \int_t^{t+\epsilon} p(\tau) \dot{q}(\tau) \, d\tau$$

$$= \int_t^{t+\epsilon} p(Q(\tau), P) \left( \partial_\tau q(Q(\tau), P) \right) \, d\tau$$

$$= \int_{Q(t)}^{Q(t+\epsilon)} p(Q, P) \left( \partial_Q q(Q, P) \right) \, dQ$$

$$= \Phi(Q(t + \epsilon), P) - \Phi(Q(t), P),$$

for which we use that $P$ is constant under rotation. The change in angle, $Q(t + \epsilon) = Q(t) + \Delta Q$, is given by the rotation matrix $R_\gamma$ and independent of $\epsilon$. After some calculation the antiderivative $\Phi$ may be found as

$$\Phi(Q, P) = QP + \arctan w_-(Q, P) - \arctan w_+(Q, P)$$

with $w_\pm = \frac{P \cos \varphi + \sqrt{1 - P^2} \sin \varphi \pm 1}{P \cos \varphi} \tan \frac{Q}{2}$. 

$\textit{KIC semiclassical fluctuations}$
While using this equation one has to keep track of increased winding numbers when \( Q \) passes from \(+\pi\) to \( -\pi\). The remaining part of \( S_K \) is the \((\text{time})\) integral over \( H_K \). Again, along the segment \( H_K \) is constant and may be evaluated at an arbitrary point. Although the time interval of the kick tends to zero the delta distribution is adjusted such that the integral remains of unit measure. This part will compensate the \( P \Delta Q \) contribution from the previous integral. As a side remark, for \( \varphi = 0 \) we find \( S_K = 0 \).

**Appendix B. Kick Action for 2 Time Steps**

For periodic orbits with \( T^{(p)}_{\gamma} = 2 \) the kick action \( S_K^{(n,1)} \) for the \( n \)-th spin at the first time step is identical to \( \pm S_K^{(n,2)} \) at the other time step. It can therefore either add up to double its value or cancel all together. In the case of the 4D manifolds this cancellation, occurring for all of the spins, leads to their simple action formula \([27]\). To understand this property we have to look at trajectories (not necessarily parts of periodic orbits) connecting two different values of \( p \), \( p_i \rightarrow p_f \), under the action of \( R_\gamma \). For simplicity, we restrict our argument to a single spin. Generically, there are either none or two, and only two, trajectories \( z_{1,2} \),

\[
(q_{1,2}^i, p^i) \rightarrow (q_{1,2}^i, p^f), \tag{B.1}
\]

connecting the initial and final momenta (compare with the spin rotation about the \( y \)-axis relevant for the evaluation of the Wigner \( d \)-function in \([19]\)). As we show subsequently the action along the two trajectories fulfills

\[
S_K(z_1) = -S_K(z_2). \tag{B.2}
\]

This is important as for any \( T^{(p)}_{\gamma} = 2 \) periodic orbit the spin, in the second time step has to return via \( p_f \rightarrow p^i \) along any one of the two possible trajectories \( z'_{1,2} \):

\[
(-q_{1,2}^i, p^f) \rightarrow (-q_{1,2}^i, p^i) \tag{B.3}
\]

which are time reversed reflections of \( z_{1,2} \) flipped perpendicular to the \( xz \)-plane. Due to symmetry we find the associated actions \( S_K(z_{1,2}) = S_K(z'_{1,2}) \), compare equations \([A.3]\) and \([A.6]\). A periodic orbit with the first kick segment given by, for example, \( z_1 \) may close either via \( z'_1 \) or \( z'_2 \) as its second segment. In the first case the actions of the kicks will add up, it is further easy to show that the orbit will then be highly symmetric with all its four points in the same plane orthogonal to the field. On the contrary, if the orbit is composed of \( z_1 \) and \( z'_2 \) the overall kick action is zero.

To prove \((B.2)\) we point out that the action is path independent and we may safely use the Euler decomposition, see \([11]\), of the rotation into \( z, x \) and \( z \)-rotations. The \( z \) contributions lead to vanishing actions and only the \( x \) part has to be dealt with. Denoting the corresponding segments of the two trajectories by either \((q_{1x}^i, p^f) \rightarrow (q_{1x}^f, p^f) \) or \((q_{2x}^i, p^f) \rightarrow (q_{2x}^f, p^f) \) one may from purely geometrical reasons conclude that \( q_{2x}^f = \pi - q_{1x}^i \). In other words, the second possible trajectory segment connecting two different \( p \) values under \( R_x \) is obtained by reflection at the \( yz \)-plane. Using the rotated coordinate system \( Q, P \) aligned to the field, see \([Appendix A]\) we find that
\( Q_{1x}^i = Q_{2x}^i, \ P_{2x}^i = -P_{1x}^i \) corresponds to this reflection. Looking at \( \{A.6\} \) for \( \varphi = \pi/2 \) it is straightforward to see that \( P \to -P \) leads to a sign change in \( \Phi \), which concludes the proof of \( \{B.2\} \).

A generic PO consists of both types of spins, those for which the (local) kick action cancels as well as those where it adds up leading still to a non-trivial result for the overall \( S_K \). What remains to be argued is that for the manifolds’ orbits only the cancelling type occurs. To make this plausible, let us again look at a single spin \( \vec{n}_1 \) of the manifold. It is mapped under time evolution onto

\[
\vec{n}_2 = R_I R_y \vec{n}_1 ,
\]

where by construction of the manifold we may assume \( R_I \) to be a fixed, given matrix independent of our concrete choice of \( \vec{n}_1 \). proving our statement by contradiction, let us assume that the new vector belongs to those mirror reflected trajectories that have identical action. In this case it may also be obtained as

\[
\vec{n}_2 = P_y R_y \vec{n}_1 ,
\]

where \( P_y \) denotes the reflection along the \( xz \)-plane. While these two equations can be satisfied for single vectors \( \vec{n}_1 \) for the manifold it would have to be satisfied for the set of linear independent vectors residing on it. Thus, we would require that a rotation equates a reflection, \( R_I = P_y \), which can not be satisfied. Therefore, orbits on \( T_\gamma(\manifold) = 2 \) manifolds have to feature vanishing \( S_K \) contributions wherever the dimension of the manifold (locally) does not collapse. As a closing remark, while the construction of the point \( \vec{n}_2 \) belonging to the cancelling trajectory is slightly more involved it necessarily involves a further reflection \( P_x \) and two reflections can be expressed by a rotation.

\[
\langle \sigma_1 \sigma_2 | \hat{W}_I | \sigma_1 \sigma_2 \rangle = \langle \sigma_1 | e^{-2i\vec{\theta} \cdot \vec{S}} | \sigma_2 \rangle \langle \sigma_2 | e^{-2i\vec{\theta} \cdot \vec{S}} | \sigma_1 \rangle = e^{-i(\sigma_1 + \sigma_2)(\alpha + \gamma - \pi)} (d_{\sigma_1 \sigma_2}^\gamma(\beta))^2 . \tag{B.6}
\]

Here \( \alpha, \beta, \gamma \) (with \( \alpha = \gamma \)) are the Euler angles given in \( \{12\} \) and

\[
d_{\sigma \sigma'}(\beta) = \langle \sigma | e^{-i\beta \hat{S}_y} | \sigma' \rangle , \tag{B.7}
\]

Appendix C. Dual matrix spectrum

In this appendix we provide an approximation for the spectrum of the dual evolution \( \hat{W} \). Rather than consider the dual operator itself it is instructive to analyze the spectrum of its square \( \hat{W}^2 = \hat{W}_I \hat{W}_0 \), with \( \hat{W}_0 = \hat{W}_K \hat{W}_I \hat{W}_K \). The idea is that the operator \( \hat{W}_0 \) can be thought of as an approximate permutation. To see this we notice that its matrix elements can be written down as

\[
\langle \sigma_1 \sigma_2 | \hat{W}_0 | \sigma_1' \sigma_2' \rangle = \sum_{m_1 = -j}^{j} \sum_{m_2 = -j}^{j} \exp \left[ -i4J(m_1 \sigma_1 + m_2 \sigma_2) \right] \frac{1}{j + 1/2} \langle m_1 | e^{-2i\vec{\theta} \cdot \vec{S}} | m_2 \rangle \cdot \exp \left[ -i4J(m_1 \sigma_1' + m_2 \sigma_2') \right] \frac{1}{j + 1/2} \langle m_2 | e^{-2i\vec{\theta} \cdot \vec{S}} | m_1 \rangle . \tag{C.1}
\]
By using $\hat{S}^z$ operators it can be rewritten as
\[
\langle \sigma_1 \sigma_2 | \hat{W}_0 | \sigma'_1 \sigma'_2 \rangle = \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} \langle m_1 | e^{-i \kappa_1 \hat{S}^z} | m_1 \rangle \langle m_1 | e^{-2i \vec{b} \cdot \hat{S}} | m_2 \rangle \cdot \langle m_2 | e^{-i \kappa_2 \hat{S}^z} | m_2 \rangle \langle m_2 | e^{-2i \vec{b} \cdot \hat{S}} | m_1 \rangle = \text{Tr} \left( e^{-i \Phi(\vec{n}, \hat{S})} \right),
\]
where
\[
\kappa_1 = \frac{2 \pi a (\sigma_1 + \sigma'_1)}{2j + 1}, \quad \kappa_2 = \frac{2 \pi a (\sigma_2 + \sigma'_2)}{2j + 1}, \quad a = 4J/\pi,
\]
and the operator
\[
e^{-i \Phi(\vec{n}, \hat{S})} := e^{-i \kappa_1 \hat{S}^z} e^{-2i \vec{b} \cdot \hat{S}} e^{-i \kappa_2 \hat{S}^z} e^{-2i \vec{b} \cdot \hat{S}},
\]
describes rotation around some axis $\vec{n}$ by an angle $\Phi$. From the last representation it follows
\[
\langle \sigma_1 \sigma_2 | \hat{W}_0 | \sigma'_1 \sigma'_2 \rangle = \sum_{m=-j}^{j} e^{-im\Phi} = \frac{\sin(j + 1/2)\Phi}{\sin \Phi/2}. \quad (C.2)
\]

The rotation angle $\Phi := \Phi(\kappa_1, \kappa_2)$ can be straightforwardly determined through the relationship:
\[
\text{Tr} \left( R_z(\kappa_1) R_y(2\vec{b}) R_z(\kappa_2) R_y(2\vec{b}) \right) = 1 + 2 \cos \Phi
\]
with $R_z(\kappa_1)$, $R(2\vec{b})$ being rotations along $z$ and $\vec{b}$ directions, respectively. At this point it is convenient to use the Euler decomposition $R(2\vec{b}) = R_z(\alpha) R_x(\beta) R_z(\alpha)$ leading to
\[
\text{Tr} \left( R_z(\kappa_1 + \theta) R_x(\beta) R_z(\kappa_2 + \theta) R_x(\beta) \right) = 1 + 2 \cos \Phi(\kappa_1 + \theta, \kappa_2 + \theta), \quad (C.3)
\]
with $2\alpha = \theta$. This allows to evaluate the function $\Phi(x, y)$ explicitly:
\[
2 \cos \Phi(x, y) = \cos x \cos y (1 + \cos^2 \beta) - 2 \sin x \sin y \cos \beta - (\cos x + \cos y + 1) \sin^2 \beta.
\]

Note that the matrix elements (C.2) are of the order $2j + 1$ if $\Phi \approx 0$ and of the order 1, otherwise. The solutions of the equations $\Phi(\kappa_1 + \theta, \kappa_2 + \theta) = 0$ are provided by all $\kappa_1, \kappa_2$ such that
\[
R_z(\kappa_1 + \theta) R_x(\beta) = (R_z(\kappa_2 + \theta) R_x(\beta))^{-1}. \quad (C.4)
\]

After writing down the left and the right hand side of this equation in the matrix form and comparing them element-wise (see e.g., [54]) we conclude that both rotations about the $z$-direction must be by $\pi$ modulo $2\pi$:
\[
\kappa_1 + \theta = \pi + 2\pi p_1, \quad \kappa_2 + \theta = \pi + 2\pi p_2, \quad (C.5)
\]
with $p_1, p_2 \in \mathbb{Z}$, or equivalently:
\[
\frac{\sigma_1 + \sigma'_1}{2j + 1} = \frac{1 + 2p_1 - \theta/\pi}{2a}, \quad \frac{\sigma_2 + \sigma'_2}{2j + 1} = \frac{1 + 2p_2 - \theta/\pi}{2a}. \quad (C.6)
\]
Since $-j \leq \sigma, \sigma' \leq j$, the above solutions exist only if the interval $[-2a + \theta/\pi, 2a + \theta/\pi]$ contains a point from $\{1 + 2k | k \in \mathbb{Z}\}$. This is, in fact, precisely the condition for the existence of 4-dimensional manifolds. In particular, for the case $b_z = 0$ this condition reduces to $|a| < 1/2$. 

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**Single PO manifold.** In what follows we will consider parameters $a, \theta$ such that (C.6) admits at most one solution $\sigma_1, \sigma_2 \in [-j, j]$, $p_1 = p_2 = p$ for each pair $\sigma'_1, \sigma'_2$. In that case we can write $\hat{W}_0 = (2j + 1) \hat{G}$, where
\begin{equation}
\langle \sigma_1 \sigma_2 | \hat{P} | \sigma'_1 \sigma'_2 \rangle = \delta_{\sigma_1 + \sigma'_1, a} \delta_{\sigma_2 + \sigma'_2, g}.
\end{equation}
\begin{equation}
-j \leq g = \left[ \frac{(2j + 1)(1 + 2p - \theta/\pi)}{2a} \right] \leq j
\end{equation}
is a truncated permutation while $\hat{G}$ has a band like structure. The last matrix has approximately unity elements on the diagonal $\langle \sigma_1 \sigma_2 | \hat{G} | \sigma_2 \sigma_1 \rangle \approx 1$, while its off-diagonal elements are highly fluctuating with absolute values decaying as distance from the diagonal grows:
\begin{equation}
|\langle \sigma_1 \sigma_2 | \hat{G} | \sigma'_1 \sigma'_2 \rangle| \sim \left( (\sigma'_1 - \sigma_1)^2 + (\sigma_2 - \sigma'_2)^2 \right)^{-1/2}.
\end{equation}

To facilitate the study of the spectrum of $\hat{W}$ we make a crude approximation $\hat{G} \approx 1$ (resp. $\hat{W}_0 \approx (2j + 1) \hat{P}$) in the body of the paper. The above approximation amounts to picking up the largest element from each row of the matrix $\hat{W}_0$. Since $\hat{W}_i$ is a diagonal matrix and $\hat{P}$ is a permutation, the eigenvectors of $\hat{W}_i \hat{P}$ take a simple form:
\begin{equation}
\psi_{(\sigma_1, \sigma_2)} = C_1 |\sigma_1 \sigma_2 \rangle \pm C_2 |g - \sigma_1 g - \sigma_2 \rangle,
\end{equation}
with
\begin{equation}
(C_1/C_2)^2 = \langle \sigma_1 \sigma_2 | \hat{W}_i | \sigma_1 \sigma_2 \rangle / \langle g - \sigma_1 g - \sigma_2 | \hat{W}_i | g - \sigma_1 g - \sigma_2 \rangle
\end{equation}
and the corresponding eigenvalues $\tilde{\Lambda}_{(\sigma_1, \sigma_2)}$ given by
\begin{equation}
\tilde{\Lambda}_{(\sigma_1, \sigma_2)}^2 = \langle \sigma_1 \sigma_2 | \hat{W}_i | \sigma_2 \sigma_1 \rangle \langle g - \sigma_1 g - \sigma_2 | \hat{W}_i | g - \sigma_1 g - \sigma_2 \rangle.
\end{equation}
This in turn can be written down in terms of Wigner $d$-functions as
\begin{equation}
\tilde{\Lambda}_{(\sigma_1, \sigma_2)} = e^{i(\theta - \pi)g} d_{\sigma_2, \sigma_1}^{\theta} d_{g - \sigma_2, g - \sigma_1}^{\pi}. \tag{C.12}
\end{equation}

**Multiple PO manifolds.** In this case eq. (C.6) admits multiple solutions corresponding to several different combinations of $(p_1, p_2)$. Each pair $(p_1, p_2)$ determines uniquely the pair of constants
\begin{equation}
g_1 = \left[ \frac{(2j + 1)(1 + 2p_1 - \theta/\pi)}{2a} \right], \quad g_2 = \left[ \frac{(2j + 1)(1 + 2p_2 - \theta/\pi)}{2a} \right]
\end{equation}
such that $\hat{W}_0$ can be thought as an approximate sum of permutations (if only the largest elements in each row are left), i.e., $\hat{W}_0 \approx \hat{P}$, where
\begin{equation}
\hat{P} = (2j + 1) \sum_{(p_1, p_2)} \hat{P}_{(p_1, p_2)}, \tag{C.13}
\end{equation}
\begin{equation}
\langle \sigma_1 \sigma_2 | \hat{P}_{(p_1, p_2)} | \sigma'_1 \sigma'_2 \rangle = \delta_{\sigma_1 + \sigma'_1, g_1} \delta_{\sigma_2 + \sigma'_2, g_2}.
\end{equation}

As opposed to the single manifold case, even within the above approximation it seems to be impossible to provide explicit formula for the spectrum of $\hat{W}_i \hat{P}$ for generic system parameters. However, after the crossover from the regime of single PO...
manifold to one of multiple PO manifolds there exists a certain range of parameters where permutations $\hat{P}_{(p_1, p_2)}$ are mutually orthogonal:

$$\hat{P}_{(p'_1, p'_2)} \hat{P}_{(p_1, p_2)} = 0, \quad \text{for} \quad (p'_1, p'_2) \neq (p_1, p_2).$$

In this case the total spectrum of $\hat{W}_I \hat{P}$ is composed of subspectra of the operators of $\hat{W}_I \hat{P}_{(p_1, p_2)}$ and can be easily evaluated. As in the single manifold case, the eigenvectors take a simple form

$$\psi_{(\sigma_1, \sigma_2)} = C_1 |\sigma_1 \sigma_2 \rangle \pm C_2 |g_1 - \sigma_1 g_2 - \sigma_2 \rangle,$$

with the corresponding eigenvalues given by:

$$\tilde{\Lambda}_{(\sigma_1, \sigma_2)} = e^{i(\theta - \pi)(g_1 + g_2)/2} d_{\sigma_2, \sigma_1} d_{g_2 - \sigma_2, g_1 - \sigma_1} \beta.$$  \hspace{1cm} (C.15)

Note that the eigenstates with the largest eigenvalues must be localized at the boundary of the ellipse [59]. The localization points $(\sigma_1, \sigma_2)$ are, therefore, determined by the demand that both points $(g_1 - \sigma_1, g_2 - \sigma_2)$ and $(\sigma_1, \sigma_2)$ belong to the ellipse boundary. To see that these are also eigenstates of $\hat{W}_I \hat{P}$ it is sufficient to notice that action of other permutations $\hat{P}_{(p'_1, p'_2)} \neq \hat{P}_{(p_1, p_2)}$ on the states (C.14) brings them to zero.

The numerical computation of the actual spectrum of the operator $\hat{W}_I \hat{W}_0$ shows that the localization points of its eigenvectors associated with the highest eigenvalues are indeed have the same localization points as the states (C.14), see fig. [11]. Furthermore, as can be seen on the same figure, the bulk of the spectrum is composed of a number of cross-like structures. Each such cross is associated with one of the pairs $(p_1, p_2)$ in the sum (C.14).

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