RELATIONSHIP BETWEEN THE PRIME-COUNTING FUNCTION AND A UNIQUE PRIME NUMBER SEQUENCE

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ABSTRACT. In mathematics, the prime counting function \( \pi(x) \) is defined as the function yielding the number of primes less than or equal to a given number \( x \). In this paper, we prove that the asymptotic limit of a summation operation performed on a unique subsequence of the prime numbers yields the prime number counting function \( \pi(x) \) as \( x \) approaches \( \infty \). We also show that the prime number count \( \pi(n) \) can be estimated with a notable degree of accuracy by performing the summation operation on the subsequence up to a limit \( n \).

1. GENERATING \( P' \) AND \( P'' \)

Consider the prime number subsequence of higher order \([2]\)

\[
P' = \{p'\} = \{2, 5, 7, 13, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 71, \ldots \}.
\]

It was discovered \([3]\) that \( P' \) can be generated via an alternating sum of the prime number subsequences of increasing order, i.e.,

\[
P' = \left\{ (-1)^{n-1} \left\{ p^{(n)} \right\} \right\}_{n=1}^{\infty}
\]

where the right-hand side of Eq. (1.1) is an expression of the alternating sum.

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The prime number subsequences of increasing order \([4]\) in Expression 1.2 are defined as

\[
\{ p^{(1)} \} = \{ p_n \}_{n=1}^\infty = \{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots \}
\]

\[
\{ p^{(2)} \} = \{ p_{pp} \}_{n=1}^\infty = \{ 3, 5, 11, 17, 31, 41, 59, 67, 83, 109, 127, \ldots \}
\]

\[
\{ p^{(3)} \} = \{ p_{ppp} \}_{n=1}^\infty = \{ 5, 11, 31, 59, 127, 179, 277, 331, \ldots \}
\]

\[
\{ p^{(4)} \} = \{ p_{pppp} \}_{n=1}^\infty = \{ 11, 31, 127, 277, 709, \ldots \}
\]

\[
\{ p^{(5)} \} = \{ p_{ppppp} \}_{n=1}^\infty = \{ 31, 127, 709, \ldots \}
\]

and so on and so forth. It is noted for clarification that the operation performed on the right-hand side of Eq. 1.1 denotes an alternating sum of the entire sets of prime number subsequences of increasing order.

The prime number subsequence of higher order \(P'\) can also be generated by performing an alternating sum of the individual elements across the sets. To illustrate this, we arrange the subsequences in Expression 1.2 side-by-side and sum elements laterally across the rows to create the new \(P'\) subsequence term-by-term as follows:
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| (row) | $+p^{(1)}$ | $-p^{(2)}$ | $+p^{(3)}$ | $-p^{(4)}$ | $+p^{(5)}$ | $-p^{(6)}$ | ... | $p'$ |
|-------|------------|------------|------------|------------|------------|------------|-----|-------|
| (1)   | 2          | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 2   |
| (2)   | 3          | 3          | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 0   |
| (3)   | 5          | 5          | 5          | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 5   |
| (4)   | 7          | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 7   |
| (5)   | 11         | 11         | 11         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 0   |
| (6)   | 13         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 13  |
| (7)   | 17         | 17         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 0   |
| (8)   | 19         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 19  |
| (9)   | 23         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 23  |
| (10)  | 29         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 29  |
| (11)  | 31         | 31         | 31         | 31         | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 31  |
| ...   | ...        | ...        | ...        | ...        | ...        | ...        | ... | ...   |

Table 1. Alternating Sum of $p^{(n)}$

Thus, the infinite prime number subsequence $\mathbb{P}'$ of higher order \[2\] emerges in the rightmost column of Table1:

$$\mathbb{P}' = \{p'\} = \{2, 5, 7, 13, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 71, ...\}.$$ 

The prime number subsequence of higher order $\mathbb{P}'$ can also be generated by the N-sieve \[3\]. We now demonstrate how that is accomplished. Starting with $n = 1$, choose the prime number with subscript 1 (i.e., $p_1 = 2$) as the first term of the subsequence and eliminate that prime number from the natural number line. Then, proceed forward on $\mathbb{N}$ from 1 to the next available natural number. Since 2 was eliminated from the natural number line in the previous step, one moves forward to the next available natural number that has not been eliminated, which is 3. 3 then becomes the subscript for the next $\mathbb{P}'$ term which is $p_3 = 5$, and 5 is then eliminated from the natural number line, and so on and so forth. Such a sieving operation has been carried out in the chart below for the natural numbers 1 to 100.
Thus, we may optionally designate \( P' \), which has been created via the N-sieve operation above, by the following notation \[3\] to indicate that the natural numbers \( N \) have been sieved to produce this prime number subsequence:

\[
\lfloor N \rfloor = P' = \{2, 5, 7, 13, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 71, \ldots \}.
\]

Regardless of the method used to generate \( P' \), when the prime numbers in this unique subsequence are applied as indexes to the set of all prime numbers \( P \), one obtains the next higher-order prime number subsequence

\[
P'' = \{p''\} = \{3, 11, 17, 41, 67, 83, 109, 127, 157, 191, 211, 241, \ldots \}.
\]

By definition \[3\], the sequence \( P'' \) can be generated via the expression

\[
P'' = \left\{ (1 + 1)^{\frac{1}{2}} \right\}_{n=2}^{\infty}
\]

where an expansion of the right-hand side of Eq. 1.3 is the alternating sum

\[
\left\{ p^{(2)} \right\} - \left\{ p^{(3)} \right\} + \left\{ p^{(4)} \right\} - \left\{ p^{(5)} \right\} + \left\{ p^{(6)} \right\} - \ldots.
\]

Further, it has been shown \[3\] that the subsequences \( P' \) and \( P'' \) when added together form the entire set of prime numbers \( P \):

\[
P = P' + P''.
\]

We sketch a short proof of Eq. 1.5 here:
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Proof.

It has been established [3] that

$$P' = \left\{ (-1)^{n-1} \left\{ p^{(n)} \right\} \right\}_{n=1}^{\infty} = \left\{ p^{(1)} \right\} - \left\{ p^{(2)} \right\} + \left\{ p^{(3)} \right\} - ...$$

and

$$P'' = \left\{ (-1)^{n} \left\{ p^{(n)} \right\} \right\}_{n=2}^{\infty} = \left\{ p^{(2)} \right\} - \left\{ p^{(3)} \right\} + \left\{ p^{(4)} \right\} - ... .$$

Then,

$$P' + P'' = \left\{ p^{(1)} \right\} - \left\{ p^{(2)} \right\} + \left\{ p^{(3)} \right\} - ...$$

$$+ \left\{ p^{(2)} \right\} - \left\{ p^{(3)} \right\} + \left\{ p^{(4)} \right\} - ... = \left\{ p^{(1)} \right\} = P.$$

□

An interesting property was observed in the relationship between the set of all prime numbers $P$ and the complement prime number sets $P'$ and $P''$. Since $P'' = P P'$, Eq. 1.5 can be restated as

$$P'' = P - \{2, 5, 7, 13, 19, 23, 29, ... \}$$

$$= \{ p_2, p_3, p_5, p_{13}, p_{19}, p_{23}, p_{29}, ... \} = P P'$$

where the prime numbers of the subsequence $P'$ form the indexes for the complement set of primes $P''$ such that

$$P'' = P P' = \{ p_k \mid k \in P' \}.$$  

We now calculate the average gap size for $P'$ at $\infty$.

2. AVERAGE GAP OF $P'$

We will now derive the asymptotic density for the prime number subsequence $P'$ assuming that $1/ \ln n$ is the asymptotic density of the set of all prime numbers $P$ at $\infty$. We approach this task via
alternately adding and subtracting the prime number densities (or "probabilities" as they are also called) of the prime number subsequences of increasing order to arrive at a value for the density of $P'$. We begin by recalling [3] that the prime number subsequence $P'$ is formed by the alternating series

$$P' = \left\{ (-1)^{n-1} \left\{ p^{(n)} \right\} \right\}_{n=1}^{\infty} = \left\{ p^{(1)} \right\} - \left\{ p^{(2)} \right\} + \left\{ p^{(3)} \right\} - \ldots$$

where

$$\left\{ p^{(k)} \right\} = \left\{ p_{p \cdot p n} \right\} (p "k" times).$$

Broughan and Barnett [4] have shown that for the general case of higher-order superprimes $p_{p \cdot p n}$, the asymptotic density is approximately

$$\frac{n}{p_{p \cdot p n}} \sim \frac{n}{n (\ln n)^k} \sim \frac{1}{(\ln n)^k}$$

for large $n \in \mathbb{N}$.

Now, assuming that $\ln n$ is the asymptotic limit of the gap size for the set of all prime numbers $\mathbb{P}$ at $\infty$, we derive an expression for the density $d$ for the prime number subsequence $P'$ at $\infty$. We begin with the geometric series

$$S = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots = \frac{1}{1 + x} (|x| < 1).$$

Then let

$$T = -S + 1$$

so that

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$$T = x - x^2 + x^3 - x^4 + x^5 + ...$$

$$= \frac{-1}{1 + x} + 1$$

$$= \frac{x}{1 + x}.$$

Now substitute $\frac{1}{\ln n}$ for $x$ to get

$$\frac{1}{\ln n} - \frac{1}{(\ln n)^2} + \frac{1}{(\ln n)^3} - \frac{1}{(\ln n)^4} + ...$$

so that

$$T = d_{P'} \approx \frac{1}{\ln n} - \frac{1}{(\ln n)^2} + \frac{1}{(\ln n)^3} - \frac{1}{(\ln n)^4} + ...$$

$$= \frac{1}{\ln n + 1}. \tag{2.1}$$

Based on our assumption that $1/ \ln n$ is the asymptotic limit of the density of the set of all prime numbers $\mathbb{P}$ as $n \to \infty$, Eq. (2.2) provides us with the density (or probability) of the occurrence of the primes $\mathbb{P}'$ at $\infty$. Thus, the average gap $g$ between prime numbers in the subsequence $\mathbb{P}'$ on the natural number line as $n \to \infty$ is the inverse of the density $d$ of $\mathbb{P}'$ so that

$$g_{P'} = \frac{1}{d_{P'}} \approx \frac{1}{\ln n} - \frac{1}{(\ln n)^2} + \frac{1}{(\ln n)^3} - \frac{1}{(\ln n)^4} + ...$$

$$= \ln n + 1.$$

Since it has been shown via the N-sieving operation [3] that the prime number subsequence $\mathbb{P}'$ has fewer primes than the set of all prime numbers $\mathbb{P}$ at $\infty$, it intuitively follows that the average gap size for $\mathbb{P}'$ will always be larger than the gap size for $\mathbb{P}$ at $\infty$. 7
3. ESTIMATING $\pi(x)$ VIA $\mathbb{P}'$

When we remove the prime number subsequence $\mathbb{P}''$ from the set of all prime numbers $\mathbb{P}$, we create the prime number subsequence $\mathbb{P}'$ [3]. Further, it was shown in the previous section that the limit of the average gap size between the prime numbers $\mathbb{P}'$ is $\ln n + 1$ as $n$ tends toward $\infty$. Thus, the increase made to the average gap between the prime numbers $\mathbb{P}'$ when discounting the primes $\mathbb{P}''$ on the natural number line is unity at $\infty$. This is equivalently stated by Eq. 3.1 wherein the average gap size for the set of all prime numbers $\mathbb{P}$ is subtracted from the average gap size for the set of all prime numbers $\mathbb{P}'$ to yield the average contribution that the removal of the prime numbers $\mathbb{P}''$ adds to to produce the average gap size for $\mathbb{P}'$ at $\infty$.

$$g_{\mathbb{P}'} - g_{\mathbb{P}} = 1. \quad (3.1)$$

Fig. 1 provides a visual representation of the operation of Eq. 3.1 on the natural number line by showing that in all intervals where an element of $\mathbb{P}''$ exists, the removal of $\mathbb{P}''$ and replacing with a null integer placeholder iteratively increases the average gap size between the remaining prime numbers $\mathbb{P}'$ on the natural number line by unity as that operation is carried out to $\infty$.

![Fig. 1 – Prime Gaps on the Natural Number Line](image-url)
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And since

$$\sum_{k=1}^{x} \left( p_{p'_{k}} - p_{(p'_{k}-1)} \right)$$

represents the prime gaps [5]

$$\{ p_{p'_n} - p_{(p'_n-1)} \} = \{ 1, 4, 4, 4, 6, 4, 2, 14, 6, 10, 12, 2, \ldots \}$$

which have been counted as placeholders among the set of all prime numbers $\mathbb{P}$ (thereby increasing the gap size between the remaining prime numbers $\mathbb{P}'$ from $\ln n$ to $\ln n + 1$ at $\infty$), we arrive at the asymptotic limit

$$\pi(x) \sim \sum_{k=1}^{x} \left( p_{p'_{k}} - p_{(p'_{k}-1)} \right),$$

the sum of which approximates the prime number count $\pi(x)$ for the set of all prime numbers $\mathbb{P}$ at $\infty$.

**Theorem 3.1.** The prime number counting function $\pi(x)$ is asymptotically equivalent to an operation performed on a unique subsequence of the prime numbers in that

$$\pi(x) \sim \sum_{k=1}^{x} \left( p_{p'_{k}} - p_{(p'_{k}-1)} \right)$$

which states that the magnitude of the gaps contributed by an operation performed on the unique prime number subsequence $p_{p'}$ as $x$ approaches $\infty$ is asymptotically equivalent to the total number of primes counted by $\pi(x)$ as $x$ approaches $\infty$.

**Proof.** We begin with the asymptotic limit of the prime counting function [1]

$$\pi(x) \sim \frac{x}{\ln x} \quad (3.2)$$

to show that as $x \to \infty$,

$$\lim_{x \to \infty} \frac{\pi(x)}{\sum_{k=1}^{x} \left( p_{p'_{k}} - p_{(p'_{k}-1)} \right)} = 1. \quad (3.3)$$
In order to evaluate Limit 3.3 we need to express both the numerator and denominator in terms of \( x \) and \( \ln x \). The asymptotic limit of the prime counting function in terms of \( x \) and \( \ln x \) is defined in 3.2, and a careful examination of Fig. 1 reveals that the denominator of 3.3 can be expressed as

\[
x \left[ 1 - \frac{\ln x}{\ln x + 1} \right]
\]

so that

\[
\lim_{{x \to \infty}} \frac{\pi(x)}{\sum_{k=1}^{x} \left( p_{p_k} - p_{(p_k - 1)} \right)} = \frac{\frac{x}{\ln x}}{x \left[ 1 - \frac{\ln x}{\ln x + 1} \right]} = \frac{\ln x + 1}{\ln x}.
\]

And clearly,

\[
\lim_{{x \to \infty}} \frac{\ln x + 1}{\ln x} = 1.
\]

We also show that the asymptotic limit of the ratio of \( \pi(x) \) to the complement of the sum in the denominator of Limit 3.3, or

\[
x - \sum_{k=1}^{x} \left( p_{p_k} - p_{(p_k - 1)} \right),
\]

converges to zero at \( \infty \), implying that the complement Expression 3.5 is infinitely larger than the count of prime numbers for infinitely large \( x \).

**Theorem 3.2.** The prime number counting function \( \pi(x) \) is asymptotically equivalent to zero when evaluated against the complement expression

\[
x - \sum_{k=1}^{x} \left( p_{p_k} - p_{(p_k - 1)} \right)
\]

as \( x \) approaches \( \infty \).

**Proof.** We again begin with the asymptotic limit of the prime counting function [1] in 3.2 to show that as \( x \to \infty \),
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$$\lim_{x \to \infty} \frac{\pi(x)}{x - \sum_{k=1}^{x} (p'_{k} - p'_{k-1})} = 0. \quad (3.6)$$

To transform the denominator of the Limit $3.6$ to an expression in terms of $x$ and $\ln x$, we recognize that when one subtracts $3.4$ from $x$, we have

$$x - x \left[1 - \frac{\ln x}{\ln x + 1}\right] = x \left[1 - \frac{1}{\ln x + 1}\right]$$

so that

$$\lim_{x \to \infty} \frac{\pi(x)}{x - \sum_{k=1}^{x} (p'_{k} - p'_{k-1})} = \frac{x}{x \left[1 - \frac{1}{\ln x + 1}\right]} = \frac{\ln x + 1}{(\ln x)^2}.$$ 

And clearly,

$$\lim_{x \to \infty} \frac{\ln x + 1}{(\ln x)^2} = 0.$$ 

□

4. APPROXIMATION OF $\pi(n)$ FOR $n < \infty$

It was discovered using Mathematica that the prime number count can be estimated with a notable degree of accuracy (within the bounds of a multiplicative constant) by performing the aforementioned operation on the prime number subsequence of higher-order up to a finite integer $p'_{p'_{N}}$, i.e.,

$$\pi(p'_{p'_{N}}) \approx C_3 * \sum_{n=1}^{N} \left(p'_{p'_{n}} - p'_{p'_{n-1}}\right). \quad (4.1)$$

The results of Eq. 4.1 tabulated below begin at $p'_{p'_{N}} \approx 100$ and incrementally go up to $p'_{p'_{N}} \approx 10E6$:
An observance of the data in the table above reveals that the constant $C_3$ appears to oscillate rather tightly around the value

$$\frac{\pi \sqrt{3}}{6}$$

which happens to be the densest packing density possible for identically-sized circles in a plane. This would imply (at least within the range of $p_{p_N'}$ in the table) that the ratio of the prime counting function $\pi(p_{p_N'})$ to the sum of the gaps counted by

$$\sum_{n=1}^{N} \left( p_{p_n'} - p_{(p_n'-1)} \right)$$

closely approximates the density of identical circles packed as tightly as possible in a hexagonal packing arrangement in a plane. More study is needed to determine the convergence or divergence of the constant $C_3$. 

| $p_{p_N'}$ | $\pi(p_{p_N'})$ | $\sum_{n=1}^{N} p_{p_n'} - p_{(p_n'-1)}$ | $C_3$ |
|-----------|----------------|---------------------------------|-------|
| 1E02      | 25             | 23                              | 1.08696 |
| 1E03      | 168            | 187                             | 0.89840 |
| 1E04      | 1,229          | 1,319                           | 0.93177 |
| 1E05      | 9,592          | 10,651                          | 0.90057 |
| 1E06      | 78,498         | 86,249                          | 0.91013 |
| 2E06      | 148,933        | 165,133                         | 0.90190 |
| 3E06      | 216,816        | 239,893                         | 0.90380 |
| 4E06      | 283,146        | 312,563                         | 0.90588 |
| 5E06      | 348,513        | 384,277                         | 0.90693 |
| 6E06      | 412,849        | 455,401                         | 0.90656 |
| 7E06      | 476,648        | 525,917                         | 0.90632 |
| 8E06      | 539,777        | 595,285                         | 0.90675 |
| 9E06      | 602,489        | 665,345                         | 0.90553 |
| 10E6      | 664,579        | 733,389                         | 0.90618 |
5. CONCLUSION

In this paper, we derived an expression for the asymptotic limit of the prime-counting function $\pi(x)$ as a function of the prime number subsequence of higher order $P'$. We further showed that the expression derived is a good approximation (to within a constant $C_3$) of the prime counting function $\pi(n)$ up to any positive real $N \leq 10^6$.

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