Abstract

In this work, we investigate the noncommutative Dirac oscillator (NCDO) with a permanent electric dipole moment (EDM) in the presence of an electromagnetic field. We consider a radial magnetic field generated by anti-Helmholtz coils and the uniform electric field of the Stark effect. Next, we determine the exact solutions of the system, given by the Dirac spinor and by the relativistic energy spectrum. We note that such a spinor is written in terms of the generalized Laguerre polynomials, and such a spectrum is a linear function on the potential energy $U$ and depends on the quantum numbers $n$ and $m$, spin parameter $s$, and of four angular frequencies: $\omega$, $\bar{\omega}$, $\omega_\theta$, and $\omega_\eta$, where $\omega$ is the frequency of the DO, $\bar{\omega}$ is a type of ‘cyclotron frequency’, and $\omega_\theta$ and $\omega_\eta$ are the NC frequencies of position and momentum. We graphically analyze the behavior of the spectrum as a function of these frequencies for three different values of $n$, with and without the influence of $U$. Besides, as an interesting result, another interpretation can be given for the origin of the nonminimal coupling of the DO, which would be through a neutral fermion with EDM in a radial magnetic field. Finally, we also analyze the nonrelativistic limit of our results, and comparing our problem with other works, we verified that our results generalize some particular cases in the literature.

1. Introduction

In 1989, Moshinsky and Szczepaniak developed the first relativistic version of the quantum harmonic oscillator (QHO) for spin-1/2 fermions, which became known as the Dirac oscillator (DO) [1, 2]. To build the DO, we must insert into the free Dirac equation (DE) a nonminimal coupling given by: $p \rightarrow p - im_0\omega/\beta r$, where $p = -i\hbar\nabla$ is the momentum operator, $m_0$ is the rest mass, $\omega$ is the angular frequency, $\beta$ is one of the Dirac matrices, and $r$ is the position vector (operator) [1–3]. In particular, such an interaction term is not interesting just because it generates the QHO with a strong spin–orbit coupling in the nonrelativistic limit, but because it may also have applications in quantum chromodynamics (QCD) [3–5]. For example, since the interaction or confinement potential has a term linear in the position, implies that the interaction term of the DO could serve to model quark confinement in QCD [3–6]. In addition, it has already been shown that such an interaction term arises through the DE for a nucleon (i.e. neutron or a proton) with a magnetic dipole moment (MDM) interacting with external electromagnetic fields [3–5, 7].

Since it was introduced in the literature, several papers about the DO have been made in different areas of physics, such as in thermodynamic [8–10], physics–mathematics [4, 11–13], quantum optics [14–17], graphene physics [10, 18–21], etc, and has also been studied in different contexts, such as in quantum phase transitions [22, 23], hidden supersymmetry [4], Aharonov–Bohm–Coulomb system [24], spin effects [25–27], rotating frames and cosmic strings [28, 29], minimal length scenario [23], quantum rings [20, 30], etc. In 2013, the DO was verified experimentally by J. A. Franco–Villafañe et al [31], where the experimental apparatus was based on a microwave system consisting of a chain of coupled dielectric disks. In addition to the DO, there are also other
types of relativistic oscillators in the literature (but not yet experimentally proven), i.e. the Klein–Gordon oscillator (spin-0 bosons) [32] and the Duffin-Kemmer–Petiau oscillator (spin-0 and spin-1 bosons) [33].

Recently, the DO has been studied with Lorentz symmetry violation [34], Aharonov–Bohm and Aharonov–Casher effects [35, 36], thermal properties [37], position-dependent mass [38], etc.

In 1947, H. S. Snyder proposed the idea of quantized spacetimes, thus introducing the concept of noncommutative space (NCS) or noncommutative spacetime (NCST) as we know today [39, 40]. For Snyder, although the Minkowski spacetime is a continuum, this assumption is not required by Lorentz invariance, consequently, a model of a Lorentz invariant discrete spacetime inspired by quantum mechanics is possible. In [41, 42], some interesting consequences of quantized spacetimes are discussed (i.e. grand unified theory). In [43–47], a NCS is considered as a possible scenario for the short-distance behavior of some physical theories (i.e. quantum gravity). So, in order to generalize the already well-established NCS, the concept of noncommutative phase space (NCPS) was later introduced, where now both spaces obey a rigorous mathematical formulation: the noncommutative geometry (NCG) [43, 48–50]. In essence, a NCPS is based on the assumption that the position and momentum operators are noncommutative (NC) variables and must satisfy $[\hat{x}_\mu^N, \hat{p}_\nu^N] \neq 0$ and $[\hat{p}_\mu^N, \hat{p}_\nu^N] = 0$. For more details about the phenomenology of NCG, please see [45, 46, 51–53], where supposed signatures of NC were investigated (i.e. decay of kaons, photon-neutrino interaction, etc). Also, a NCS (or a NCPS) has a wide range of applications: in QCD [34], quantum electrodynamics (QED) [55], black holes [36], quantum cosmology [37], Shannon entropy [58], graphene [59], quantum wells [60], DO and QHO [61, 62], etc. Recently, the NCPS has been used in the quantum Hall effect (QHE) [63], and in the Pauli oscillator (PO) [64].

In classical electrodynamics, an electric dipole moment (EDM) is defined from the first-order term of the multipole expansion of the electrostatic potential for a discrete or continuous charge distribution; and consists of two equal and opposite charges that are infinitesimally close [65, 66]. Mathematically, such an EDM is defined as $d \equiv \alpha r$ (discrete case), where $\alpha$ is the total charge, or $d \equiv \int \int \int \rho(r')dV'$ (continuous case), where $\rho(r')$ is the charge density [65, 66]. However, in some atoms (Xe-129, Hg-199, Ra-225, ...) [67–71] and molecules (H2O, HCl, CO, ...) [65, 66] in which the centers of positive and negative charge do not coincide, then there will be a permanent EDM, given by $d_0 = d_0\hat{n}$, where $\hat{n} = \frac{\mathbf{n}}{|\mathbf{n}|}$ is a unit vector (can be oriented in any direction in space).

In particular, such an EDM (intrinsic or non-null EDM) is already ‘born’ with the atom or molecule itself. Also, even in atoms and molecules with a null EDM, the action of an external electric field can separate the centers of positive and negative charges and thus produce an induced EDM [65, 66], which is defined as $d \equiv \alpha E$, where $\alpha$ is the polarizability and $E$ is an external electric field [65, 66].

From the point of view of particle physics, nuclei and particles (Dirac fermions) ‘can’ also have a non-null permanent EDM, given by $d_N$ and defined as $d_N \equiv d_N\hat{N}$ ($d_N = \text{const}$.), where $\hat{N} = \frac{\mathbf{J}}{|\mathbf{J}|}$, being $\mathbf{J}$ the (spin) angular momentum, i.e., $d_N$ must be parallel or antiparallel to the angular momentum (cannot be oriented in any direction in space) [72–74]. Here, an EDM cannot exist unless both parity (P) and time-reversal (T) invariance are violated (broken), where both nonrelativistic and relativistic Hamiltonians for the EDM, given by $H_{\text{EDM}}^{\text{Pauli}} = -d_N \mathbf{J} \cdot \mathbf{E}$ (Pauli Hamiltonian) and $H_{\text{EDM}}^{\text{Dirac}} = -d_N \mathbf{\gamma} \cdot \mathbf{E}$ (Dirac Hamiltonian), are odd under P and T transformations [72–74]. However, due to current experimental upper limits, EDMs must be extremely small, imply that so far no EDMs have been observed [72–76]. But as has already been said, EDMs may be non-null because P and T are in fact violated in nature [72–76]. For example, if we assume CPT invariance as an exact symmetry of nature (unconditionally valid or unbroken symmetry), then the T-violation is equivalent to CP violation (observed in meson decays), i.e. EDM is a direct signal of T-violation and also of CP violation [72–76]. Also, EDMs have already been employed for a possible search for dark matter [77], and for an explanation of the baryonic asymmetry problem, since the Standard Model cannot produce the observed asymmetry.

This paper has as its goal to investigate the bound-state solutions of the NCDO with a permanent EDM in the presence of an external electromagnetic field in the $(2 + 1)$-dimensional Minkowski spacetime. With respect to the electromagnetic field, here we consider a radial magnetic field and linear at $r$ (polar radial distance) generated by the so-called anti-Helmholtz coils, and by the uniform electric field of the Stark effect, in which is described by a uniform electric field in the z-direction. In particular, we believe that our problem could perhaps be applied to graphene (2D Dirac matter) if one day its MDM is measured. Now, with respect to bound-state solutions, we mainly focus on the exact solutions of an eigenvalue equation, given by the eigenfunctions (Dirac spinor and the spinorial wave function) and on energy eigenvalues (relativistic and nonrelativistic energy spectrum). Besides, we also consider the ‘spin’ of the planar Dirac fermion, described by a parameter $s$, the so-called spin parameter, where $s = +1$ is for the spin ‘up’, and $s = -1$ is for the spin ‘down’. In this way, we can say that we are going to investigate the bound-state solutions under the influence of spin effects, NC effects, and electromagnetic effects, respectively.
The structure of this work is organized as follows. In section 2, we start our discussion by initially introducing the total Lagrangian (density) of the system, which is the sum of two Lagrangians: the free Dirac Lagrangian, and the Lagrangian for neutral fermions with EDM. From this total Lagrangian, we obtain the equation of motion for the NCDO, both in linear and quadratic form. In section 3, we analyze the asymptotic behavior of our second-order differential equation, where we get the bound-state solutions, given by the NC Dirac spinor and the relativistic energy spectrum. In section 4, we analyze the nonrelativistic limit of our results, where we get the equation of motion for the nonrelativistic NCDO as well as the nonrelativistic bound-state solutions, given by the spinorial wave function and the nonrelativistic spectrum. Finally, in section 5 we present our conclusions and some remarks.

2. The noncommutative Dirac oscillator in the (2+1)-dimensional Minkowski spacetime

To obtain the equation of motion for the NCDO with EDM in the presence of an external electromagnetic field (linear and quadratic form), we will first derive the DE from the total Lagrangian (density) of our problem. Therefore, in quantum field theory (QFT) the Lagrangian for the free Dirac field (spin-1/2 free relativistic fermions) in (3 + 1)-dimensions is written as follows (Cartesian coordinates in SI units) [78]

$$L_0 = \Psi(i\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi, \quad (\mu = 0, 1, 2, 3),$$

where \(\Psi = \Psi_0(r, \mathbf{r}) = \Psi_0(r, \Psi_0, \Psi_0)^T \in \mathbb{C}^4\) is the four-component Dirac spinor (four-element column vector or Dirac field), and it is interpreted as a superposition of spin-up/down particles and spin-up/down antiparticles, \(\Psi \equiv \Psi^\dagger \gamma^0\) is the adjoint spinor (Dirac adjoint) of \(\Psi\), and \(\gamma^\mu (\gamma^\mu, \gamma^\nu) \equiv (\beta, \beta\alpha)\) are the gamma matrices (\(\beta, \alpha\) are the Dirac matrices), which satisfies the anticommutation relation of the Clifford Algebra: \(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I_{4\times4}\) being \(I_{4\times4}\) a 4 \times 4 unit matrix. Explicitly, \(\gamma^0\) (Hermitian), \(\gamma\) (anti-Hermitian), and \(\alpha\) (Hermitian) are defined in the standard Dirac representation as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \otimes I_{2 \times 2}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} = i\sigma_2 \otimes \sigma, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} = \sigma_1 \otimes \sigma,$$

where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) are the 2 \times 2 Pauli matrices, which satisfies: \(\sigma_0 = \sigma_1\) and \(\sigma_0 = \sigma_1 = 2\delta_{ij} + \delta_{ij}\) with \(\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3 = 0\). Explicitly, these Pauli matrices take the form

$$\sigma_0 = I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now, considering Dirac fermions (i.e. neutrons) with an EDM given by \(d_f = \pm |d_f|\), we have the following Lagrangian [72–74]

$$L_{EDM} = -\frac{d_f}{2} F_{\mu\nu} \gamma_5 F_{\mu\nu} \Psi, \quad (\mu, \nu = 0, 1, 2, 3),$$

where \(\sigma^{\mu\nu} = \frac{1}{2} (\gamma^\mu, \gamma^\nu) = i\gamma^\mu \gamma^\nu\) is an antisymmetric tensor (Hermitian tensor), \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor (electromagnetic field strength) with \(A_\mu = (A_0, \mathbf{A})\) being the electromagnetic four-potential (potential four-vector), and \(\gamma_5 = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3\) is the fifth gamma matrix (Hermitian matrix), also called of chirality or handedness matrix and satisfies: \(\{\gamma_5, \gamma^\mu\} = 0\), and \((\gamma^5)^2 = I_{4 \times 4}\). Explicitly, this matrix is defined as follows (in Dirac representation)

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \otimes I_{2 \times 2}.$$

Therefore, the total Lagrangian is written as

$$L_T = L_0 + L_{EDM} = \Psi \left( i\hbar\gamma^\mu \partial_\mu - \frac{d_f}{2} \sigma^{\mu\nu} \gamma_5 F_{\mu\nu} - m_0c^2 \right) \Psi.$$

Using now the Euler–Lagrange equation for \(\Psi\), given by [78]

$$\frac{\partial}{\partial \hat{\psi}} \frac{\partial L_T}{\partial (\partial \hat{\psi})} - \frac{\partial L_T}{\partial \hat{\psi}} = 0,$$

we obtain as a result the following tensorial DE for a neutral fermion with EDM

$$\left( \gamma^\mu p_\mu - \frac{d_f}{2c} \sigma^{\mu\nu} \gamma_5 F_{\mu\nu} - m_0c^2 \right) \Psi = 0,$$
where \( p_\mu = i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial x^\mu} = (P_\mu - p) \) is the four-momentum operator, and \( x^\mu = (ct, \mathbf{r}) \) is the four-position operator. In particular, here we have basically the DE with a type of nonminimal coupling, given by the second term of Eq. (8) (another type of nonminimal coupling is for the MDM [7, 78, 80]), where the EDM is also sometimes called of 'nonminimal coupling constant’. So, based on the fact that \( \gamma^\mu p_\mu = \gamma^\mu \partial_\mu - \gamma \cdot p \), and \( \sigma^{\mu \nu} F_{\mu \nu} = 2(\frac{2}{\hbar} \mathbf{S} \cdot \mathbf{E} - \alpha \cdot \mathbf{B}) \), Eq. (8) is rewritten in the form (differential DE)

\[
\left[ c \alpha \cdot (\mathbf{p} + idf \beta \mathbf{B}) - \frac{2df}{\hbar} \beta \mathbf{S} \cdot \mathbf{E} + \beta m_0 c^2 - i\hbar \frac{\partial}{\partial t} \right] \Psi = 0,
\]

or in terms of the total Hamiltonian (operator) of the system \( H_{\text{DE}} \), as

\[
id\frac{\partial \Psi}{\partial t} = H_{\text{DE}} \Psi = [H_0 + H_{\text{EDM}}] \Psi,
\]

where \( H_0 = c \alpha \cdot \mathbf{p} + \beta m_0 c^2 \) is the free Dirac Hamiltonian, \( H_{\text{EDM}} = -df / \beta \left( \frac{2}{\hbar} \mathbf{S} \cdot \mathbf{E} + ic \alpha \cdot \mathbf{B} \right) \) is the Dirac Hamiltonian for the EDM, or simply the Hamiltonian of the EDM [74], being \( \mathbf{S} = \frac{1}{2} \Sigma \) the spin operator (vector), where \( \Sigma = I_{2 \times 2} \otimes \sigma = (\Sigma^1, \Sigma^2, \Sigma^3) \), with \( \Sigma^j = \frac{1}{2} \epsilon^{ijk} \gamma_k \), \( \mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \) is the electric field, and \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic field.

On the other hand, here we assume that our system is a stationary quantum system; consequently, the Dirac spinor can be written as follows [15, 78]

\[
\Psi = e^{-iE_0t/\hbar} \psi,
\]

where \( E \) is the relativistic total energy, and \( -\infty < t < \infty \) is the temporal coordinate. Besides, modifying the momentum in the form \( \mathbf{p} \rightarrow \mathbf{p} - im_0 \omega \mathbf{r} \), we then have from Eq. (9) the following (time-independent) stationary DO with EDM

\[
\left[ c \alpha \cdot (\mathbf{p} - im_0 \omega \mathbf{r} + idf \beta \mathbf{B}) - \frac{2df}{\hbar} \beta \mathbf{S} \cdot \mathbf{E} + \beta m_0 c^2 - E \right] \psi = 0,
\]

or in the form of an eigenvalue equation, as

\[
\hat{H}_{\text{DO}} \psi = [H_{\text{DO}} + H_{\text{EDM}}] \psi = E \psi,
\]

where \( \hat{H}_{\text{DO}} \) is the Hamiltonian of the DO with EDM or the total Hamiltonian of the DO, being \( H_{\text{DO}} = c \alpha \cdot (\mathbf{p} - im_0 \omega \mathbf{r}) + \beta m_0 c^2 \) the usual DO Hamiltonian.

Now, let us consider a fermion confined exclusively to a surface (plane), that is, subject to purely planar dynamics. So, making use of the symmetry under \( z \) translations of the system (translational invariance in the \( z \)-direction), we can reduce a four-component DE to a two-component DE in such a way that we can exclude the \( z \) degree of freedom by imposing \( p_z = 0 \) (or \( p_z \psi = 0 \) [7, 14, 15, 25–27, 80, 81]). In that way, in \( 2 + 1 \)-dimensions where \( \alpha = \Sigma \sigma \) and \( \gamma^0 = \sigma_3 \), we have

\[
[\sigma \cdot (\mathbf{p} - im_0 \omega \sigma_3 \mathbf{r} + idf \sigma_3 \mathbf{B}) - d f \sigma_3 \sigma \cdot \mathbf{E} + \sigma \cdot m_0 c^2 - E] \psi = 0,
\]

where \( \mathbf{p} = (p_x, p_y) = (\mathbf{p}_x, \mathbf{p}_y) \) is the momentum operator, \( \mathbf{r} = (x_1, x_2) = (x, y) \) is the position vector, \( \psi \) is now a two-component Dirac spinor, and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) = (\sigma_x, \sigma_y, \sigma_z) \), where the spin parameter \( s \) takes the values \pm 1, being \( s = +1 \) for the spin 'up' (\( \uparrow \)), and \( s = -1 \) for the spin 'down' (\( \downarrow \)), respectively [25–29]. In particular, this parameter was introduced in the literature in 1990 through two articles published by C. R. Hagen: Aharonov–Bohm (AB) Scattering of Particles with Spin, and Exact Equivalence of Spin–1/2 Aharonov–Bohm and Aharonov–Casher (AC) Effects, and it was a way to include the 'spin' in planar fermions (both the AB and AC effects are intrinsically planar phenomena) [80, 81].

Now, let us choose the form of the external electromagnetic field. So, in polar coordinates (or Cartesian coordinates), we consider a radial magnetic field and linear at \( r = \sqrt{x^2 + y^2} \approx 0 \) (radial coordinate) generated by anti-Helmholtz coils, given by: \( \mathbf{B} = \Phi \mathbf{r} = \Phi \varrho \mathbf{\hat{r}} \) (also called of anti-Helmholtz magnetic-field), where \( \Phi = \frac{B_0}{R} > 0 \) is a 'linear density of magnetic field', being \( B_0 \) the averaged field strength when \( r \) is equal to the radius \( R \), and \( \mathbf{\hat{r}} \) is a unit vector \([82–87]\). In particular, such a magnetic field is produced by a pair of identical Helmholtz coils whose electric currents are equal and flow in opposite directions (anti-Helmholtz configuration) and is used in problems involving ferrofluids and magnetoreological fluids [82–87], magnetic traps for neutral atoms and molecules [88–91], synthesis of cold antihydrogen atoms [92, 93], fabrication of electromagnetic scanning micromirrors [94], etc. In addition, radial magnetic fields are also important in astrophysics and solar physics [95–98]. Now, with respect to the electric field, we consider the uniform electric field of the Stark effect, given by: \( \mathbf{E} = E_0 \mathbf{\hat{z}} \), where \( E_0 > 0 \) is the field strength, and \( \mathbf{\hat{z}} \) is a unit vector in the \( z \)-direction [79], that is, we have a uniform electric field along the \( z \)-direction that crosses perpendicularly the plane (and is analogous to what happens in the quantum Hall effect in graphene where a uniform magnetic field along the \( z \)-direction crosses perpendicularly the plane) [99, 100]. In particular, the Stark effect arises due to the
interaction of the EDM of atoms or molecules with a uniform external electric field, and there are two types of Stark effect: the linear (for a permanent EDM) and the quadratic (for an induced EDM). In this paper, we have the ’first case’.

With the well-defined electromagnetic field, Eq. (14) becomes (in Cartesian coordinates)

\[ [c\sigma_0(p_x - iAx\sigma_3) + sc\sigma_2(p_y - iAy\sigma_3) = dfE_0 + \sigma_1m_0c^2 - E] \psi = 0, \]

where \( A \equiv (m_0\omega - df\Phi) > 0 \), and we use the fact that \( B = (\Phi x, \Phi y) \).

So, to obtain the (linear) NCDO, it is necessary to write the momentum and position operators, as well as the spinor, in a NCPS. Having done that, we have the following NCDO

\[ [c\sigma_0(p_x^{NC} - iAx^{NC}\sigma_3) + sc\sigma_2(p_y^{NC} - iAy^{NC}\sigma_3) = dfE_0 + \sigma_1m_0c^2 - E] \psi^{NC} = 0, \]

or explicitly, as

\[ [c\lambda\sigma_0(p_x + is\sigma_3) - m_0c\Omega_2(x + is\sigma_1) - (E + dfE_0) + \sigma_1m_0c^2] \psi^{NC} = 0, \]

where \( \lambda \equiv (1 + \frac{\Delta\omega}{2\eta}) > 0, \Omega \equiv (\frac{\lambda}{m_0} + \frac{m}{2m_0}) > 0, \) and \( \psi^{NC} \) is our NC Dirac spinor, and we use the fact that \( p_x^{NC} \) and \( x^{NC} \) are written as follows (please see appendix)

\[ p_x^{NC} = p_x + \frac{\eta}{2\hbar}y, \quad p_y^{NC} = p_y - \frac{\eta}{2\hbar}x, \]

\[ x^{NC} = x - \frac{\theta}{2\hbar}p_y, \quad y^{NC} = y + \frac{\theta}{2\hbar}p_x, \]

in which they satisfy

\[ [\hat{x}^{NC}, \hat{p}_y^{NC}] = i\hbar \theta, \quad [\hat{x}^{NC}, \hat{x}^{NC}] = i\hbar \eta, \quad [\hat{p}_x^{NC}, \hat{p}_y^{NC}] = i\hbar \eta \]

with \( \theta_{ij} = \theta_{ji} = \eta_{ij} = \eta_{ji} \) being antisymmetric constants ‘tensors’ (real deformation parameters), \( \epsilon_{ij} \) is the Levi-Civita symbol, and \( \theta \) and \( \eta \) (\( \theta, \eta \in \mathbb{R}_+ \)) are the position and momentum NC parameters with dimensions of length-squared and momentum-squared, respectively. From a phenomenological point of view, these two NC parameters can have the following values: \( \theta \leq 4.0 \times 10^{-40} \text{ m}^2 \) and \( \eta \leq 1.76 \times 10^{-64} \text{ kg}^2 \text{ m}^2 \text{ s}^{-2} \) [60].

Now, let us get the quadratic NCDO, given by a second-order differential equation, where it is from this equation that we will determine the bound-state solutions for the fermionic system. For this, we will define an ansatz for the original Dirac spinor according to Refs. [26, 101, 102], which can be defined in terms of another spinor as follows

\[ \psi^{NC} \equiv O_\Phi \phi^{NC} = [c\lambda\sigma_0(p_x + is\sigma_3) - m_0c\Omega_2(x + is\sigma_1) - (E + dfE_0) + \sigma_1m_0c^2] \phi^{NC}, \]

where \( O_\Phi \) is a ’Dirac operator’, and has a similar form to the NCDO. In particular, the spinor (21) allows us to obtain a second-order differential equation in a direct way without the need to obtain two coupled differential equations for the radial functions (radial components of the spinor), in other words, we just took a different path (but equivalent) to arrive at the same result (here we find it more advantageous to use spinor in the form (21)). Furthermore, in [103] a spinor (or Dirac operator) similar to ours was used to solve the relativistic hydrogen atom with position-dependent mass, something that would be much more difficult to achieve if solved via coupled equations [78].

Then, substituting (21) in (17), we get the following quadratic NCDO

\[ [c^2\hat{x}^2 + \hat{p}_y^2 + m_0^2c^4\Omega^2(x^2 + y^2) - 2m_0^2c^3\Omega(\hbar\sigma_3 + sL_z) - (E + dfE_0)^2 + m_0^2c^4] \phi^{NC} = 0, \]

or in the form of an eigenvalue equation, as

\[ H_{\text{quad.}}^{\text{NCDO}} \phi^{NC} = \left[ \frac{\Omega}{\lambda}(\hbar\sigma_3 + sL_z) \right] \phi^{NC} = \left[ \frac{(E + dfE_0)^2 - m_0^2c^4}{2m_0^2c^3\Omega^2} \right] \phi^{NC}, \]

where

\[ H_{\text{quad.}}^{\text{NCQHO-like}} = \frac{\hat{p}_x^2}{2m_0} + \frac{1}{2} \left( \frac{m_0\Omega^2}{\lambda^2} \right) \hat{r}^2, \]

with \( H_{\text{quad.}}^{\text{NCQHO-like}} \) being the Hamiltonian of the quadratic NCDO, \( H_{\text{NCQHO-like}}^{\text{NCQHO-like}} \) is the noncommutative quantum harmonic oscillator (NCQHO)-like Hamiltonian in two-dimensions (since \( \lambda \) is dimensionless and \( \Omega \) has angular frequency dimension), and \( \hat{p}, \hat{r}, \) and \( L_z \) are given by \( \hat{p} = \hat{p}_x + \hat{p}_y, \hat{r} = x^2 + y^2, \) and \( L_z = \hat{x}p_y - \hat{y}p_x \) (z-component of the orbital angular momentum operator L). It is worth noting that in the absence of the EDM (\( df = 0 \), or of the electromagnetic field \( \mathbf{B} = \mathbf{E} = 0 \)), and of the NCPS (\( \theta = \eta = 0 \)), with \( \varphi = (\psi, \psi^\dagger) \in \mathbb{C}^2 \), we reduce equation (23) to the particular case of literature (DO in the position space) [12, 15, 25].
On the other hand, since we want to obtain the bound-state solutions for the NCDO in polar coordinates, we must write \( \mathbf{p}, \mathbf{r}, \) and \( L_z \) as follows

\[
\mathbf{p} = -i\hbar \left( \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\phi}{r} \frac{\partial}{\partial \phi} \right), \quad \mathbf{r} = r \hat{\mathbf{r}}, \quad L_z = -i\hbar \frac{\partial}{\partial \phi},
\]

where \( 0 \leq \phi \leq 2\pi \) is the angular coordinate and \( r = \sqrt{x^2 + y^2} \), with \( 0 \leq r < \infty \), is the polar radial coordinate, respectively. A good choice for the spinor \( \varphi^{NC} \) is given by \[ [21, 25, 26]

\[
\varphi^{NC}(r, \phi) = \begin{pmatrix} e^{im\phi}f_+(r) \\ e^{i(m+s)\phi}f_-(r) \end{pmatrix}, \quad (f_+(r) \neq f_-(r)),
\]

where \( f_\pm(r) \) are real radial functions (spinorial radial components), and \( m = 0, \pm 1, \pm 2, \ldots \) is the orbital magnetic quantum number (or simply magnetic quantum number). Here, we wrote our spinor in the form \( (26) \) to be in agreement with the particular case in the literature. Also, a spinor in the form \( \varphi^{NC} = e^{im\phi}(f_+, f_-)^T \) would no longer be consistent since that would imply \( f_+(r) = f_-(r) \) as solutions, which is not allowed.

Therefore, substituting \( (26) \) and \( (25) \) into \( (22) \), we get the following second-order differential equation for the NCDO (radial or quadratic NCDO)

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\Gamma_\kappa}{r^2} - \Lambda r^2 + E_\kappa \right] f_\kappa(r) = 0,
\]

where we define

\[
\Gamma_\kappa = \left( m + s \frac{1}{2} - \frac{\kappa}{2} \right), \quad E_\kappa = \frac{(E + d_f E_0)^2 - m_0^2 c^4}{(\hbar c)^2} + 2\epsilon_{\kappa}(\Gamma_\kappa + s\kappa), \quad \Lambda = \frac{m_0^2 \Omega}{\hbar \lambda},
\]

with \( \kappa = \pm 1 \) being a parameter that describes the two components of the spinor, where \( \kappa = +1 \) describes a ‘particle with spin up or down’ (\( s = \pm 1 \)), and \( \kappa = -1 \) describes an ‘antiparticle with spin up or down’ (\( s = \pm 1 \)), respectively. So, it is worth mentioning that in the absence of the EDM \( (d_f = 0) \), or of the electromagnetic field \( (B = E = 0) \), and also of the NCPS \( (\theta = \eta = 0) \), we reduce Eq. \( (27) \) to the particular case of literature \[ [25] \].

Already for \( d_f = 0 \), and without the ‘spin’ \( (s = \pm 1) \), with \( x = i\hbar \frac{\partial}{\partial \rho} \), \( y = i\hbar \frac{\partial}{\partial \phi} \), we reduce Eq. \( (27) \) (for \( \kappa = +1 \)) also to a particular case of literature (NCDO in the momentum space) \[ [61] \].

### 3. Bound-state solutions: two-component Dirac spinor and the relativistic energy spectrum

To solve exactly Eq. \( (27) \), let us introduce a new (dimensionless) variable in our problem, given by: \( \rho = \Lambda r^2 \) (\( \Lambda = |\Lambda| > 0 \)). Thus, through a change of variable, Eq. \( (27) \) becomes

\[
\left[ \rho \frac{d^2}{d\rho^2} + \frac{d}{d\rho} - \frac{\Gamma_\kappa}{4\rho} - \rho \frac{4}{4\rho} - E_\kappa \right] f_\kappa(\rho) = 0.
\]

Now, analyzing the asymptotic (limit) behavior of Eq. \( (29) \) for \( \rho \to 0 \) and \( \rho \to \infty \), we have a (regular) solution to this equation given by the following ansatz

\[
f_\kappa(\rho) = C_\kappa \rho^{\pm|\Gamma_\kappa|/2} e^{-E_\kappa(\rho)}, \quad (|\Gamma_\kappa| \geq 0),
\]

where \( C_\kappa > 0 \) are normalization constants, \( F_\kappa(\rho) \) are unknown functions to be determined.

Substituting \( (30) \) in \( (29) \), we have a second-order differential equation for \( F_\kappa(\rho) \) as follows

\[
\left[ \rho \frac{d^2}{d\rho^2} + (|\Gamma_\kappa| - \rho) \frac{d}{d\rho} - \tilde{E}_\kappa \right] F_\kappa(\rho) = 0,
\]

where

\[
|\Gamma_\kappa| = |\Gamma_\kappa| + 1, \quad \tilde{E}_\kappa = \frac{|\Gamma_\kappa|}{2} - \frac{E_\kappa}{4\Lambda}.
\]

According to the literature \[ [25, 27, 102] \], Eq. \( (31) \) is the well-known generalized Laguerre equation, whose solution are the so-called generalized Laguerre polynomials, written as \( F_\kappa(\rho) = L_{|\Gamma_\kappa|}^{\tilde{E}_\kappa}(\rho) \). There is actually another solution to Eq. \( (31) \), however, was rejected because it is irregular at the origin \[ [25, 27, 102] \]. However, for Eq. \( (32) \) to generate the Laguerre polynomials (finite polynomials) and not an infinite series, the quantity \( \tilde{E}_\kappa \) must be equal to a non-positive integer, i.e. \( \tilde{E}_\kappa = -n \), where \( n = n_r = 0, 1, 2, \ldots \) is the radial quantum number. Therefore, from this quantization condition \( (\tilde{E}_\kappa = -n) \), we obtain as a result the following relativistic energy spectrum (relativistic energy levels) for the NCDO with EDM in the presence of an external electromagnetic field.
in the $(2 + 1)$-dimensional Minkowski spacetime

\[
E_{n,m,s,\chi}^\chi = U + \chi m_0 c^2 \sqrt{1 + \frac{2\hbar \eta}{m_0 c^2} \left( 1 + s \frac{\omega - \sigma \bar{\omega}}{\omega_\eta} \right) + \left( \omega - \sigma \bar{\omega} + s \omega_\eta \right)},
\]

(33)

where

\[
U \equiv -d_f E_\eta = -\sigma |d_f| E_\eta, \quad (\sigma = \pm 1),
\]

and

\[
N = N_{eff} \equiv \left[ 2n + 1 - \kappa + \left( m + s \frac{1 - \kappa}{2} \right) - s \left( m + s \frac{1 - \kappa}{2} \right) \right] \geq 0,
\]

(35)

being $\chi = \pm 1$ a parameter (‘energy parameter’) in which it describes the positive energy states ($\chi = +1$), or simply a particle (neutron or DO), as well as the negative energy states ($\chi = -1$), or simply an antiparticle (antineutron or anti-DO), and $N$ is a total or effective quantum number since it depends on all the others (and it is also very similar to what appears in [102]). Then, we clearly see that the spectrum (33) is a linear function on the (electric) potential energy $U$ (an energy generated by the interaction of the EDM with the external electric field $E$) [65, 66], where $\sigma = +1$ is for $d_f > 0$ and $\sigma = -1$ is for $d_f < 0$, and also depends explicitly on the quantum numbers $n$ and $m$, spin parameter $s$, and of four angular frequencies, given by: $\omega$, $\bar{\omega} = \omega_{\text{EDM}}$, $\omega_\eta$, and $\omega_\eta$, where $\omega$ is the well-known frequency of the NCDO, $\bar{\omega} = \frac{|d_f| \Phi}{m_0} > 0$ is a type of ‘cyclotron frequency’ or ‘EDM frequency’ (in this case $\sigma = \pm 1$ describes the revolution direction of the corresponding classical motion), and

\[
\omega_\eta \equiv \frac{2\hbar}{mc^2} > 0 \quad \text{and} \quad \omega_\eta \equiv \frac{n}{2m_0} > 0 \quad \text{are the NC frequencies of position and momentum, respectively}. \nonumber
\]

Therefore, we can say that the first term of (33) is the non-quantized (continuous) part of the spectrum (unaffected by NC parameters) and the second term is the quantized (discrete) part (affected by NC parameters). In addition, as a direct consequence of energy $U$, the spectrum is asymmetric (there is a ‘break of symmetry’ in the spectrum), i.e. the energies of the particle and antiparticle are not equal ($E^+ \neq |E^+|$). Still about the energy $U$, and making an analogy with the classical case (the classical EDM), $U$ can be positive: $U > 0$ (the energy is highest), which means an EDM antiparallel to the electric field, or negative: $U < 0$ (the energy is lowest), in which means an EDM parallel to the electric field, respectively. On the other hand, we notice that even in the absence of the DO ($\omega = 0$) and of the EDM ($d_f = 0$), the spectrum still remains quantized due to the presence of $\omega_\eta$ or $\eta$ (not of $\omega_\eta$), in which such NC frequency or NC parameter acts as a kind of ‘NC field (potential)’, and whose spectrum is given by

\[
E_{n,m,s,\chi}^\chi = \chi m_0 c^2 \sqrt{1 + \frac{2\hbar \eta}{m_0 c^2} \tilde{N}},
\]

(36)

where $\tilde{N} \equiv 2n + 1 - \kappa + \left( m + s \frac{1 - \kappa}{2} \right) - s \left( m + s \frac{1 - \kappa}{2} \right)$ (once now $\Lambda = \omega_\eta$, and $\Lambda = |\Lambda|$), in which is the relativistic spectrum for a ‘free’ Dirac fermion in a NCPS with the NC only of the momentum (not yet studied in the literature). Already in the absence of the DO ($\omega = 0$) and of the NCPS ($\theta = \eta = 0$), the spectrum also remains quantized (due to the presence of $\omega$ or $d_f$), and is written as

\[
E_{n,m,s,\chi}^\chi = U + \chi m_0 c^2 \sqrt{1 + \frac{2\hbar \eta}{m_0 c^2} \tilde{N}},
\]

(37)

where $\tilde{N} \equiv 2n + 1 + \sigma \kappa + \left( m + s \frac{1 - \kappa}{2} \right) + s \left( m + s \frac{1 - \kappa}{2} \right)$ (once now $\Lambda = -\sigma \omega_\eta$, and $\Lambda = |\Lambda|$), in which is the relativistic spectrum of a ‘free’ neutral Dirac fermion with EDM (not yet studied in the literature).

Furthermore, it is also interesting to note that the conditions for $\lambda$ and $\Omega$, which in the spectrum (33) are given by the following expressions

\[
\lambda = \left( 1 + s \frac{\omega - \sigma \bar{\omega}}{\omega_\eta} \right) > 0,
\]

(38)

and

\[
\Omega = (\omega - \sigma \bar{\omega} + s \omega_\eta) > 0,
\]

(39)

results in possible values which $\theta$ (or $\omega_\eta$) and $\eta$ (or $\omega_\eta$) must satisfy, and are given as follows

\[
\theta > \frac{2s\hbar}{(\sigma \bar{\omega} - \omega)}, \quad \text{or} \quad \omega_\eta < s(\sigma \bar{\omega} - \omega), \quad (s^2 = 1),
\]

(40)

and

\[
\eta > 2s\hbar m_0 (\sigma \bar{\omega} - \omega), \quad \text{or} \quad \omega_\eta > s(\sigma \bar{\omega} - \omega), \quad (s^2 = 1).
\]

(41)

Before we analyze graphically (via 2D graphs) and in detail the behavior of the spectrum (33) as a function of the four angular frequencies for different values of $n$ and $m$, with and without the influence of $U$ (or $E_\eta$), it is interesting that we first analyze one of the most important and striking aspects of two-dimensional energy
spectra (or three-dimensional) [1, 2, 25], which is their degeneracy or the degenerate states of the system. In that way, we verify that this degeneracy depends on the values (signs) of $s$ and $m$, however, is not affected by the parameter $\kappa$ (or $\sigma$). Therefore, in table 1 we have four possible configurations for the degeneracy depending on the values of $s$ and $m$ as well as the respective values of $N$.

According to table 1, we see that the spectrum can be infinitely (for $sm \geq 0$) or finitely (for $sm < 0$) degenerate, and therefore, there can be a finite or infinite number of degenerate states (states that share the same energy eigenvalue) depending on the values of $s$ and $m$ [1, 9, 25, 27]. In particular, an infinite (accidental) degeneracy arises when the spectrum (all the energy levels) only depends on the quantum number $n$ (there are infinitely degenerate states), while a finite degeneracy arises when the spectrum (all the energy levels) depends on both the quantum numbers $n$ and $m$ (there are finitely degenerate states) [1, 9, 25, 27]. For example, fixing a given value of $\kappa$ (preferably $\kappa = +1$), we can define a new quantum number from $n$ and $m$, such as: $l = n + m$ or $l = |n + m| (m \geq 0)$. In that way, from the number $l$ we can determine the expression for the total degree of (finite) degeneracy, given by: $\Omega(l) = \sum_{|m|=l} (2m + 1) = (l + 1)^2$, where $2|m| + 1 (m < 0$ or $m \geq 0$ is the (finite) number of degenerate states of the system.

So, based on the information about the degeneracy we get table 2, where we have four possible configurations for the spectrum depending on the values of $s$ and $m$, in which we define for simplicity that $B \equiv \frac{4}{|m|}$ and $\bar{n} \equiv (n + 1 - \frac{2}{\bar{r}})$.

According to table 2, we see that for $sm \geq 0$ (configs. 1 and 4), the spectra increase as a function of $n$, while for $sm < 0$ (configs. 2 and 3), the spectra increase as a function of $n$ and $m$, respectively. With respect to the maximal and minimal spectrum, the maximal spectrum (highest allowed energies) is reached for $s = +1$ and $m < 0$ with $\sigma = -1$ and $\kappa = 1$ (config. 2), while the minimal spectrum (lowest allowed energies) is reached for $s = -1$ and $m < 0$ with $\sigma = -1$ and $\kappa = +1$ (config. 4). Furthermore when $U \rightarrow 0$ (without electric field) all spectra from table 2 become symmetrical (for each specific configuration), and therefore, we have: $E^+ = |E|$ > 0. On the other hand, comparing the spectrum (33), or the spectra from table 2, with some works of the literature, we verified that in the absence of the EDM ($d_2 = 0$), or of the electromagnetic field ($B = E = 0$), and of the NCPS ($\theta = \eta = 0$), we obtain the usual spectrum of the DO for $\kappa = +1$ with $sm \geq 0$ or $sm < 0$ [14, 15, 25, 27], and for $\kappa = -1$ with $sm \geq 0$ [12]. Already for $d_2 = 0$ with $\kappa = +1$ and $sm \geq 0$, we obtain the spectrum of the NCDO [61]. It is interesting to note that for $d_2 = \theta = \eta = 0$ with $\kappa = +1$ and $sm \geq 0$, we obtain also the spectrum of the DO in $(3 + 1)$-dimensions for $j = 1$ (1/2 = 1/2, 3/2, ... $E^{(3+1)D}_{n,j=1/2} = E^{(3+1)D}_{n,j=3/2}$), where $j$ is the total angular momentum quantum number that arises from $J^z$ (total angular momentum squared), $l$ is the orbital (azimuthal) quantum number that arises from $L^z$ (orbital angular momentum squared), and 1/2 is the spin-up [1, 2, 9, 13].

However, the spectrum for $j = 1$ (1/2 = 1/2, 3/2, ... (spin down) is larger for the $(3 + 1)$-dimensional case ($E^{(3+1)D}_{n,j=0;k=0} > E^{(3+1)D}_{n,j=1/k}$ and $E^{(3+1)D}_{n,j=1/k} = E^{(3+1)D}_{n,j=1/2}$), as it should be. Now, for $j = 1$ (1/2 and $sm \geq 0$, the DO does not distinguish a $(2 + 1)$-dimensional universe from a $(3 + 1)$-dimensional (an ‘anomaly’). From the above, we clearly see that our relativistic spectrum generalizes some particular cases of the literature.

Now, we can analyze graphically the behavior of the relativistic spectrum as a function of the four angular frequencies for different values of $n$ and $m$ (without loss of generality, here we will adopt only $m = -1$). Actually,

### Table 1. Degeneracy depends on the values of $s$ and $m$.

| Configuration | $s$ | $m$ | $N$ | Degeneracy |
|---------------|----|----|----|------------|
| 1             | +1 | $m \geq 0$ | $2n + 1 - \kappa$ | infinite |
| 2             | +1 | $m < 0$ | $2n + 1 - \kappa + 2|m + \frac{1 - \kappa}{2}|$ | finite |
| 3             | -1 | $m \geq 0$ | $2n + 1 - \kappa + 2|m - \frac{1 - \kappa}{2}|$ | finite |
| 4             | -1 | $m < 0$ | $2n + 1 - \kappa$ | infinite |

### Table 2. Relativistic energy spectra for the degenerate states of the particle and antiparticle.

| Configuration | Relativistic energy spectrum | Degeneracy |
|---------------|-----------------------------|------------|
| 1             | $E^{(+)}_{n,m,s} = U + \chi m c^2 \left(1 + \bar{n}B \right) \left(1 - \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right)$ | infinite |
| 2             | $E^{(-)}_{n,m,s} = U + \chi m c^2 \left(1 + \bar{n}B \right) \left(1 - \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right)$ | finite |
| 3             | $E^{(+)}_{n,m,s} = U + \chi m c^2 \left(1 + \bar{n}B \right) \left(1 - \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right)$ | finite |
| 4             | $E^{(-)}_{n,m,s} = U + \chi m c^2 \left(1 + \bar{n}B \right) \left(1 - \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} - \omega)}{\omega - \sigma \bar{\omega} - \omega_0} \right) \left(1 + \frac{(\omega - \sigma \bar{\omega} + \omega)}{\omega - \sigma \bar{\omega} + \omega_0} \right)$ | infinite |
such analysis will only be for three frequencies, because as we will see soon, \( \omega \) and \( \bar{\omega} \) have the same behavior (for \( \sigma = -1 \)). Besides, for the sake of practicality, we will focus our attention, for example, on the maximal spectrum (configure 2). Therefore, first considering the particle, we have figure 1, where shows the behavior of the energies as a function of \( \omega \) for the ground state \((n = 0)\) and the first two excited states \((n = 1, 2)\), with and without the presence of potential energy \( U(U > 0) \), in which we take for simplicity that \( h = c = m_0 = |d_f| = \Phi = \theta = \eta = 1 \) (a ‘toy model’).

According to figure 1, we see that the energies increase linearly as a function of \( \omega \) (the higher \( \omega \) the higher \( E_n^f(\omega) \)), and their values are larger with the increase of \( n \) (as it should be) and in the presence of \( U \). Therefore, the function of the electric field \( E_0 \) is to increase the energies of the particle (and also increases linearly as a function of \( E_0 \) if the quantized part remains constant). In addition, the graph of \( E_n^f(\bar{\omega}) \) versus \( \bar{\omega} \) must be identical to figure 1, and as \( \bar{\omega} \propto B_0 \), implies that the energies also increase linearly as a function of the magnetic field. In fact, the behavior between both graphs happens because these two frequencies are ‘indistinguishable or interchangeable variables’ \( (\omega \leftrightarrow \bar{\omega}) \), that is, we can exchange one frequency for the other and the spectra/graphs will be the same (however, this is only true for \( d_f < 0 \) or \( \sigma = -1 \), or if \( d_f > 0 \) and \( B < 0 \)). Thus, we can give another interpretation (another possible explanation) for the origin of the nonminimal coupling of the DO, which would be through a neutral fermion with EDM interacting with a radial magnetic field and linear at \( r \), where \( d_f B \rightarrow -m_0 \omega r \) (please see Eq. (12)). From a physical point of view, this interpretation would be the magnetic analog (a ‘dual effect’ or ‘duality transformation’) of the current nonminimal coupling (neutral fermion with MDM interacting with a radial electric field and linear at \( r \)), where \( \mu E \rightarrow -m_0 \omega^2 r \) \((\mu > 0 \) and \( E < 0)\) [3–5, 7]. It is also worth mentioning that as \( \omega \) increases the energy levels become more spaced out (either for the case \( U > 0 \) or \( U = 0 \)), i.e. the energy difference between two consecutive levels gets bigger \((\Delta E_n = E_{n+1} - E_n \) increases).

In figure 2, we have the behavior of the energies of the antiparticle as a function of \( \omega \) for the ground state and the first two excited states (with \( \hbar = c = m_0 = |d_f| = \Phi = \theta = \eta = 1 \)). According to this figure, we see that the energies increase linearly as a function of \( \omega \) (the higher \( \omega \) the higher \( E_n^\bar{\alpha}(\omega) \)), and their values are larger with the increase of \( n \), however, their values are smaller in the presence of \( U \). Therefore, the function of the electric field \( E_0 \) is to decrease the energies of the antiparticle. So, comparing now both the energies of the particle and antiparticle, we see that the energies of the particle are always greater than those of the antiparticle for \( U > 0 \), and equal for \( U = 0 \) (symmetrical spectra). Besides, analogous to the case of the particle here the energy levels also become more spaced out.

In figure 3, we have the behavior of the energies of the particle \((\chi = +1)\) and antiparticle \((\chi = -1)\) as a function of \( \omega_{\bar{\alpha}} \) for three different values of \( n \) \((n = 0, 1, 2)\), with \( h = c = m_0 = \Phi = \omega = \eta = U = 1 \). According to this figure, we see that the energies decrease as a function of \( \omega_{\bar{\alpha}} \), however, increase as a function of \( \theta \), since \( \omega_{\bar{\alpha}} \propto \frac{1}{\theta} \). Therefore, the function of \( \omega_{\bar{\alpha}} \) is to decrease the energies (the bigger \( \omega_{\bar{\alpha}} \) the smaller \( |E_n^{\alpha}(\omega_{\bar{\alpha}})| \)), while the function of \( \theta \) is to increase (the higher \( \theta \) the higher \( |E_n^{\alpha}(\omega_{\bar{\alpha}})| \)). By way of illustration, in the limit \( \omega_{\bar{\alpha}} \rightarrow 0 (\theta \rightarrow \infty) \) the energies increase infinitely, while on the limit \( \omega_{\bar{\alpha}} \rightarrow \infty (\theta \rightarrow 0) \) the energies tend to constant values, respectively. Furthermore, we also see that the energies are larger with the increase of \( n \) (as it should be). So, comparing both the energies of the particle and antiparticle, we see that the energies of the particle are always

![Figure 1. Graph of \( E_n^f(\omega) \) versus \( \omega \) for three different values of \( n \) with \( U \neq 0 (E_0 = 1) \) and \( U = 0 (E_0 = 0) \).](image-url)
greater than those of the antiparticle, except when $\omega \theta \rightarrow 0$ (or $\theta \rightarrow \infty$), which is when energy levels get close enough ($\Delta E \approx 0$). Now, unlike the graph in figure 1 and 2, here the energy levels have a nearly constant spacing as $\omega \theta$ increases ($\Delta E \approx \text{const}$).

Already in figure 4, we have the behavior of the energies of the particle ($\chi = +1$) and antiparticle ($\chi = -1$) as a function of $\omega \eta$ for three different values of $n (n = 0, 1, 2)$, where $\hbar = c = m_0 = \Phi = \omega = \theta = U = 1$. According to this figure, we see that the energies increase as a function of $\omega \eta$, and also of $\eta$, since $\omega \eta \propto \eta$. Therefore, the function of $\omega \eta$ (or $\eta$) is to increase the energies (the higher $\omega \eta$ or $\eta$ the higher $|E^\lambda_n(\omega \eta)|$). Furthermore, we also see that the energies are larger with the increase of $n$ (as it should be). So, comparing both the energies of the particle and antiparticle, we see that the energies of the particle are always greater than those of the antiparticle.

Analogous to the graph in figure 1 and 2, here the energy levels also become more spaced out as $\omega \eta$ increases ($\Delta E$ increases).

Now, let’s concentrate our attention on the form of the NC Dirac spinor to the relativistic bound states. To obtain such a spinor, we must first find the form of $\psi^{NC}$. So, using the fact that the variable $\rho$ is written as $\rho = \tilde{\Lambda} r^2$, implies that we can rewrite (30) as

$$f_\lambda (r) = \tilde{C}_\lambda r^{1/4} e^{-\frac{\Lambda^2 r^2}{4}} L_{\lambda n}^{(4)} (\Lambda r^2)$$

where $\tilde{C}_\lambda \equiv C_\lambda \Lambda^{1/4} > 0$ are new normalization constants.
From the radial functions \((42)\), the spinor \((26)\) takes the following form

\[
\psi^{NC}(r, \phi) = \begin{cases} 
\bar{C}_+ e^{im\phi}e^{-\frac{am^2}{2}\Gamma_n}L_n^{|m|}(\Lambda r^2) \\
\bar{C}_- e^{i(m+s)\phi}e^{-\frac{am^2}{2}\Gamma_n^{(1)}}L_n^{(1)}(\Lambda r^2) 
\end{cases}
\]

(43)

Now, let’s find the form of the spinor \(\psi^{NC}\). However, we need to write such a spinor in polar coordinates, where \(f = \arctan \left( \frac{y}{x} \right)\), and \((x, y) = r(\cos \phi, \sin \phi)\). In that way, using

\[
\begin{aligned}
 p_x + is_3p_y &= -i\hbar e^{ims_3\phi}\left( \frac{\partial}{\partial r} + is_3 \frac{\partial}{r \partial \phi} \right), \\
x + is_3y &= re^{ims_3\phi},
\end{aligned}
\]

(44)

the spinor \(\psi^{NC}\) becomes

\[
\psi^{NC} \equiv \begin{cases} 
-i\hbar\lambda\sigma_1 e^{ims_3\phi}\left( \frac{\partial}{\partial r} + is_3 \frac{\partial}{r \partial \phi} \right) - m_0c\Omega\sigma_2 e^{ims_3\phi} + (E + d_1E_0) + s_3m_0c^2 \bigg] \psi^{NC},
\end{cases}
\]

(45)

or explicitly, as

\[
\psi^{NC} \equiv \begin{cases} 
E + d_1E_0 + m_0c^2 \\
-i\hbar\lambda\sigma_1 e^{ims_3\phi}\left( \frac{\partial}{\partial r} + s_3 \frac{\partial}{r \partial \phi} \right) - im_0c\Omega \bigg] \psi^{NC},
\end{cases}
\]

(46)

where we use the fact that \(e^{ims_3\phi} = \text{diag}(e^{is_3\phi}, e^{-is_3\phi})\), and also the Pauli matrices.

Therefore, replacing \((43)\) in \((46)\), and knowing that \(\Psi_{NC}^{\text{Dirac}} = e^{-iE\tau / \hbar} \psi^{NC}\), we obtain the following two-component NC Dirac spinor

\[
\Psi_{NC}^{\text{Dirac}}(t, r, \phi) = \begin{cases} 
E + d_1E_0 + m_0c^2 \\
-i\hbar\lambda\sigma_1 e^{ims_3\phi}\left( \frac{\partial}{\partial r} + s_3 \frac{\partial}{r \partial \phi} \right) - im_0c\Omega \bigg] \psi^{NC},
\end{cases}
\]

(47)

where we define

\[
F_\kappa(r) \equiv \bar{C}_e(E + d_1E_0 + \kappa m_0c^2)\Gamma_n^{(1)}L_n^{|m|}(\Lambda r^2), \quad (\kappa = \pm 1),
\]

(48)

and

\[
G_\kappa(r) \equiv \hbar\lambda\bar{C}_e\Gamma_n^{(1)}L_n^{|m|}(\Lambda r^2)\left( -\frac{\kappa\Gamma_n^{(1)}}{r} + (1 - \kappa)\Lambda r \right) = \kappa\Lambda rL_n^{|m|}(\Lambda r^2).
\]

(49)

It should be noted that our Dirac spinor simultaneously incorporates the positive and negative values of the quantum number \(m\), which does not happen, for example, in ref. [27]. However, the temporal part is equal, given by \(e^{-iE\tau / \hbar}\). From a practical point of view, one of the advantages of we have a spinor with this characteristic is the possibility of calculating the physical observables faster and more directly than if we had two spinors.
4. Nonrelativistic limit

Here, let’s investigate the nonrelativistic limit of our results, which is basically the low-energy limit, and it occurs when the speed of light tends to infinity ($c \to \infty$). To achieve this, we can use a standard prescription widely used in the literature. Thus, in such a prescription we need to consider that most of the total energy of the system is concentrated in the rest energy of the particle [14, 25, 27, 78], consequently, it implies that we can write the relativistic energy $E$ as: $E = m_0c^2 + \varepsilon$, in which $m_0c^2$ must satisfy two conditions, given by: $m_0c^2 \gg \varepsilon$ and $m_0c^2 \gg U$, respectively. Therefore, using this prescription in Eq. (23), we get the following Pauli-like NCQHO (nonrelativistic NCDO) with EDM in the presence of an external electromagnetic field in the 2D Euclidean space

$$H^{D}_{\text{Pauli-like}} \varphi^{NC} = [\hat{H}^{2D}_{\text{NCQHO-like}} - \Lambda \Omega (\hbar s_3 + s L_z) - d_f E_0] \varphi^{NC} = \varepsilon \varphi^{NC},$$

(50)

or with the temporal dependence (general case), as

$$\hat{H}^{2D}_{\text{Pauli-like}} \Psi^{NC}_{\text{Pauli-like}} = [\hat{H}^{2D}_{\text{NCQHO-like}} - \Lambda \Omega (\hbar s_3 + s L_z) - d_f E_0] \Psi^{NC}_{\text{Pauli-like}} = i\hbar \frac{\partial \Psi^{NC}_{\text{Pauli-like}}}{\partial t},$$

(51)

where

$$\hat{H}^{2D}_{\text{NCQHO-like}} \equiv \lambda^2 \hat{H}^{2D}_{\text{QHO-like}} = \lambda^2 \left[ \frac{\hat{p}_r^2}{2m_0} + \frac{1}{2} \left( \frac{m_0 \Omega^2}{\lambda^2} \right) \hat{r}^2 \right] = \frac{\hat{p}_r^2}{2M} + \frac{1}{2} M \Omega^2 \mathbf{r}^2, \quad (\Omega^2 \equiv \lambda^2 \Omega^2),$$

(52)

with $\hat{H}^{2D}_{\text{Pauli-like}}$ being the Hamiltonian of the Pauli-like NCQHO, $\hat{H}^{2D}_{\text{NCQHO-like}}$ is also a NCQHO-like Hamiltonian, $M = M^{NC} \equiv \frac{m_0}{\lambda^2}$ is a ‘NC effective mass’, $\Psi^{NC}_{\text{Pauli-like}} = e^{-i\varepsilon/\hbar} \varphi^{NC}$ is the NC spinorial wave function (two-component NC Pauli-like spinor), where $s = +1$ describes a particle with spin ‘up’ and $s = -1$ describes a particle with spin ‘down’. Besides, the second term in (56) is a constant that shifts all energy levels (but does not affect the eigenfunctions), and the third term is the spin–orbit coupling term [1]. Summarizing, the nonrelativistic limit of the DO in (2 + 1)-dimensions (commutative or NC) results in the 2DQHO with a strong spin–orbit coupling term with all levels shifted by the factor $\Lambda \Omega$ [25]. So, comparing Eq. (56) with the literature, we verified that in the absence of the EDM ($d_f = 0$), or of the electromagnetic field ($B = E = 0$), and of the NCPS ($\theta = \eta = 0$), we obtain the Pauli-like QHO with $s = \pm 1$ [25] and $s = \pm 1$ (spinless) [15]. Already for $\omega = \theta = \eta = E_0 = 0$ and $s = +1$, and considering the lower component of the spinor ($\kappa = -1$) with $e^{im_0}$ (consistent from a nonrelativistic point of view), we obtain the Schrödinger equation (SE) for a spinless neutral particle with EDM in the presence of an external magnetic field, where now the magnetic field is written as $B = \rho_m e_r$, with $\rho_m$ being a magnetic charge density [104]. Explicitly, this SE is basically the nonrelativistic limit of Eq. (27) for $\kappa = -1$ with $e^{im_0}$.

On the other hand, for the Hamiltonian $H^{D}_{\text{Pauli-like}}$ without EDM ($d_f = 0$) to be reduced to the particular case of the literature (with a spin–orbit coupling) we must make some modification in $\lambda^2$ and $\Omega^2$ and in the product $\lambda \Omega$. In that way, using the following approximations with $\Theta \equiv \frac{\lambda}{2\hbar}, \tilde{\Theta} \equiv \frac{\lambda}{2\hbar}$ and $d_f = 0$

$$\lambda^2 = \left( 1 + \frac{\omega^2}{\omega_0^2} \right)^{\frac{1}{2}} \approx \left( 1 + \frac{\omega^2}{\omega_0^2} \right) = \left( 1 + m_0^2 \omega^2 \Theta^2 \right),$$

(53)

$$\Omega^2 \approx \left( \omega + s \omega_0 \right)^2 \approx \left( \omega^2 + s \omega_0^2 \right) = \left( \omega^2 + \frac{\tilde{\Theta}^2}{m_0^2} \right),$$

(54)

$$\lambda \Omega = \left( 1 + s \frac{\omega}{\omega_0} \right) \left( \omega + s \omega_0 \right) \approx \left( \omega + s \omega_0 + \frac{s \omega^2}{\omega_0} \right) = \left( \omega + s \frac{\tilde{\Theta}}{m_0^2} + s m_0 \omega^2 \Theta \right),$$

(55)

Eq. (50) can be rewritten as

$$H^{D}_{\text{Pauli-like}} \varphi^{NC} = [H_{\theta,\Theta,\tilde{\Theta}} - \omega (\hbar s_3 + s L_z)] \varphi^{NC} = \varepsilon \varphi^{NC},$$

(56)

where

$$H_{\theta,\Theta,\tilde{\Theta}} = \frac{\hat{p}_r^2}{2M_0} + \frac{1}{2} M_0 \Omega^2 \Theta^2 \mathbf{r}^2 - S_{\theta,\Theta,\tilde{\Theta}} L_z,$$

(57)

being

$$\frac{1}{M_0} \equiv \left( \frac{1}{m_0} + m_0 \omega^2 \Theta^2 \right), \quad \Omega_{\theta,\Theta,\tilde{\Theta}} \equiv \left( \frac{1}{m_0} + m_0 \omega^2 \Theta^2 \right) \left( m_0 \omega^2 + \frac{\tilde{\Theta}^2}{m_0^2} \right),$$

(58)
Table 3. Nonrelativistic energy spectra for the degenerate states of the particle.

| Configuration | Nonrelativistic energy spectrum | Degeneracy |
|---------------|-----------------------------------|------------|
| 1             | $\varepsilon_{n, \pm} = U + \vec{A} (1 + \frac{\omega - \sigma \hat{\omega}}{\omega_0})(\omega - \sigma \hat{\omega} + \omega_0)$ | infinite   |
| 2             | $\varepsilon_{n, \pm; \pm} = U + \vec{B} (1 + \frac{\omega - \sigma \hat{\omega}}{\omega_0})(\omega - \sigma \hat{\omega} + \omega_0)(\theta + |m + \frac{\kappa}{2}|)$ | finite     |
| 3             | $\varepsilon_{n, \pm; -} = U + \vec{B} (1 - \frac{\omega - \sigma \hat{\omega}}{\omega_0})(\omega - \sigma \hat{\omega} - \omega_0)(\theta + |m - \frac{\kappa}{2}|)$ | finite     |
| 4             | $\varepsilon_{n, -} = U + \vec{A} (1 - \frac{\omega - \sigma \hat{\omega}}{\omega_0})(\omega - \sigma \hat{\omega} - \omega_0)$ | infinite   |

and

$$S_{\Theta, \vec{B}} \equiv s\left(\frac{\Theta}{m_0} + m_0 \omega^2 \Theta\right). \quad (59)$$

Therefore, for $s = +1$ we obtain the Hamiltonian of ref. [62], while for $\Theta = 0$ and $s = +1$, we obtain the Hamiltonian of ref. [64] for the 2D case ($p_x = z = 0$).

Thus, using the standard prescription again, but now in (33), we obtain as a result the following nonrelativistic energy spectrum (nonrelativistic energy levels) for the Pauli-like NCQHO with EDM in the presence of an external electromagnetic field

$$\varepsilon_{n, m, s} = U + \hbar N\left(1 + s\frac{(\omega - \sigma \hat{\omega})}{\omega_0}\right)(\omega - \sigma \hat{\omega} + sw_0), \quad (60)$$

where

$$U \equiv -d_f E_0 = -\sigma|d_f|E_0, \quad (\sigma = \pm 1), \quad (61)$$

and

$$N = N_{\text{eff}} \equiv \left[2n + 1 - \kappa + \left|m + s\frac{1 - \kappa}{2}\right| - s\left|m + s\frac{1 - \kappa}{2}\right|\right] \geq 0. \quad (62)$$

Based on the spectrum (60), we get table 3, where we have four possible configurations for the spectrum depending on the values of $s$ and $m$ (analogous to the relativistic case for the particle), where we define for simplicity that $\tilde{n} \equiv n + \frac{1 - \kappa}{2}$ and $\tilde{B} \equiv 2\tilde{\eta}$.

We note that the spectrum (60), or the spectra from table 3, has some similarities and some differences with the relativistic case. For example, unlike the relativistic case, the spectrum (60):

- can admit negative or positive energy states (since we can have $N = 0$). However, as we have seen in the previous section, $U$ positive (negative) means an EDM antiparallel (parallel) to the electric field;
- depends linearly (is a linear function) on the quantum numbers $n$ and $m$;
- depends quadratically (is a quadratic function) on the frequencies $\omega$ and $\hat{\omega}$ (as we saw, both are interchangeable variables);
- can describe a particle with spin 'up' ($s = \kappa = +1$) or 'down' ($s = \kappa = -1$).

Now, similar to the relativistic case, the spectrum (60):

- depends linearly on $U$ (however, the concept of the asymmetric spectrum does not apply here), and is greater (for $\sigma = -1$) in the presence of such energy (we will see this soon via graphs);
- has a finite or infinite degeneracy (depending on the values of $s$ and $m$);
- remains quantized even in the absence of the DO ($\omega = 0$) and of the EDM ($d_f = 0$), or of the DO and of the NCPS ($\Theta = \eta = 1$);
- can satisfy: $\theta > \frac{2\hbar}{(\sigma \hat{\omega} - \omega)}$ and $\eta > 2s/m_0(\sigma \hat{\omega} - \omega)$;
- has the maximal spectrum for $s = +1$ and $m < 0$ with $\sigma = -1$ and $\kappa = \pm 1$ (figure 2), while the minimal spectrum is for $s = -1$ and $m < 0$ with $\sigma = \kappa = +1$ (figure 4);
- increases as a function of $n, m, \omega, \hat{\omega}$, and $\omega_{j\rho}$ and decreases as a function of $\omega_0$ (we will see this soon via graphs).
Furthermore, comparing the spectrum (60), or the spectra from table 3, with some works of the literature, we verified that in the absence of the EDM ($d_f = 0$), and of the NCPS ($\theta = \eta = 0$), we obtain the usual spectrum of the Pauli-like QHO for $\kappa = +1$ [25]. It is interesting to note that for $d_f = 0 = \eta = 0$ with $\kappa = +1$ and $sm \geq 0$, we obtain also the spectrum of the Pauli-like QHO in three dimensions for $j = l + 1/2$ ($E_{n,0,\kappa=+1}^{3D} = E_{n,j=+1/2}^{3D}$). However, the spectrum for $j = l - 1/2$ (spin down) is larger for the three-dimensional case ($E_{n,0,\kappa=+1}^{3D} < E_{n,j=-1/2}^{3D}$ and $E_{n,0,\kappa=+1}^{3D} < E_{n,j=-1/2}^{3D}$), as it should be. So, for $j = l + 1/2$ and $sm \geq 0$, the Pauli-like QHO does not distinguish a two-dimensional universe from a three-dimensional (an ‘anomaly’ in this case). In addition, taking the nonrelativistic limit of the spectrum (37) with $\kappa = -1, s = -1, m - 1 \rightarrow m$ (comes from the fact that now the angular part of the wave function is given by $e^{im\varphi}$), $U = 0$ (without electric field), and $B \rightarrow \frac{\hbar}{2}\varepsilon_\pi$, we obtain the nonrelativistic spectrum of a ‘free’ neutral particle with EDM in the presence of an external magnetic field [104]. From the above, we see clearly that our nonrelativistic spectrum generalizes some particular cases of the literature.

Now, let’s analyze graphically the behavior of the nonrelativistic spectrum as a function of the four angular frequencies (for three actually, since $\omega \leftrightarrow \bar{\omega}$ when $\sigma = -1$) for three different values of $n$ ($n = 0, 1, 2$). Analogous to the relativistic case, here we will also choose the maximal spectrum, which is achieved for $s = +1$ and $m < 0$ with $\sigma = -1$ and $\kappa = \pm 1$ (config. 2). Therefore, we have figure 5, which shows the behavior of the energies as a function of $\omega$ for the ground state ($n = 0$) and the first two excited states ($n = 1, 2$), with and without the presence of potential energy $U(U \geq 0)$, in which we take for simplicity that $\hbar = c = m_0 = |d_f| = \Phi = \theta = \eta = 1$.

According to graph 5, we see that the energies increase quadratically as a function of $\omega$ (or $\bar{\omega}$), and their values are larger with the increase of $n$ and in the presence of $U$. Then, analogous to the relativistic case, here the energy difference between two consecutive levels gets bigger ($\Delta E_\omega$ increases ‘aggressively’) as $\omega$ increases (either for the case $U \equiv 0$ or $U = 0$). In figure 6, we have the behavior of the energies as a function of $\omega_\rho$ with and without the presence of $U(U \geq 0)$, where $n = 0, 1, 2$, and $\hbar = c = m_0 = \Phi = \omega = \eta = 1$. According to this figure, we see that the energies decrease as a function of $\omega_\rho$, but increase as a function of $\bar{\omega}_\rho$, where their values are larger with the increase of $n$, and in the presence of $U$. However, unlike the relativistic case, here this increase or decrease is more ‘gentle’ (less ‘aggressively’). Now, unlike the graph in figure 5, here the energy levels have a nearly constant spacing as $\omega_\rho$ increases ($\Delta E \approx \text{const}$).

Already in figure 7, we have the behavior of the energies as a function of $\bar{\omega}_\rho$ with and without the presence of $U(U \geq 0)$, where $n = 0, 1, 2$, and $\hbar = c = m_0 = \Phi = \omega = \theta = 1$. According to this figure, we see that the energies increase (linearly) as a function of $\omega_\rho$ (or $\eta$), where their values are larger with the increase of $n$ (as it should be) and in the presence of $U$. However, unlike the relativistic case, here this increase is more ‘aggressive’. Besides, analogous to the graph in figure 5, here the energy levels also become more spaced out as $\omega_\rho$ increases ($\Delta E$ increases ‘aggressively’).

To end this section, let’s now obtain the NC spinorial wave function (NC Pauli-like spinor) for the nonrelativistic bound states of the system. In particular, this function can be obtained in two different ways (but equivalent), namely: solving directly Eq. (50), or starting directly from the function (43). Here, we chose this second option because it is the least labor-intensive way. Therefore, we have the following NC spinorial wave...
function

\[ \Psi^{\text{NC}}_{\text{Pauli-like}} = \left( \begin{array}{c} C' e^{i(m+\sigma)\phi - \frac{\omega t}{\hbar}} L_n^{m+\sigma}(\Lambda r^2) \\ C' e^{i(m+\sigma)\phi - \frac{\omega t}{\hbar}} L_n^{m+\sigma}(\Lambda r^2) \end{array} \right), \]

where \( C' > 0 \) are new (nonrelativistic) normalization constants.

5. Conclusion

In this paper, we investigate the bound-state solutions of the NCDO with EDM in the presence of an external electromagnetic field in \((2 + 1)\)-dimensions. With respect to the electromagnetic field, we consider a radial magnetic field and linear at \( r \) generated by anti-Helmholtz coils, and the uniform electric field of the Stark effect. Defining the original Dirac spinor in terms of another spinor via a 'Dirac operator', we get at a second-order differential equation. Next, we introduce a new variable in this equation, and then we analyze the asymptotic behavior of the resultant equation for \( \rho \to 0 \) and \( \rho \to \infty \), we obtain as a result the generalized Laguerre equation. From this equation, we get the bound-state solutions, given by the two-component NC Dirac spinor and the relativistic energy spectrum.
So, we verify that the spinor is written in terms of the generalized Laguerre polynomials, and the spectrum is a linear function on the potential energy $U$ (generated by the interaction of the EDM with the electric field), and also depends explicitly on the quantum numbers $n$ and $m$, the spin parameter $s$, and of four angular frequencies, given by: $\omega$, $\bar{\omega}$, $\omega_0$, and $\bar{\omega}_0$, where $\omega$ is the well-known frequency of the DO, $\bar{\omega}$ is a type of cyclotron frequency”, and $\omega_0$ and $\bar{\omega}_0$ are the NC frequencies of position and momentum. In addition, some characteristics of the spectrum are: is asymmetric due to $U$, still remains quantized even in the absence of the DO ($\omega = 0$) and of the EDM ($d_f = 0$), or of the DO and of the NCPS ($\theta = \eta = 1$), has a finite or infinite degeneracy, and the maximal values are reached for $s = +1$ and $m < 0$ with $d_f < 0$, while the minimal values are for $s = -1$ and $m < 0$ with $d_f > 0$.

Furthermore, we also analyze graphically the behavior of the spectrum as a function of the four angular frequencies for three different values of $n$. For example, in the graph of $|E^g(\omega)|$ versus $\omega$, we verify that the energies increase linearly as a function of $\omega$. On the other hand, the graph of $|E^g(\bar{\omega})|$ versus $\bar{\omega}$ (we omit this graph for simplicity) is identical to the of $|E^g(\omega)|$ versus $\omega$, and as $\bar{\omega} \propto B_0$, implies that the energies increase linearly as a function of the magnetic field. In fact, the behavior between both graphs happens because these two frequencies are ‘interchangeable variables’ ($\omega \leftrightarrow \bar{\omega}$), i.e. we can exchange one frequency for the other and the graphs will be the same (for $d_f < 0$). Consequently, from this, we can give another interpretation for the origin of the nonminimal coupling of the DO, which would be through a neutral fermion with EDM interacting with a radial magnetic field.

Now, in the graph of $|E^g(\omega_0)|$ versus $\omega_0$ we verify that the energies decrease as a function of $\omega_0$, however, increase as a function of $\theta$. Already in the graph of $|E^g(\bar{\omega}_0)|$ versus $\bar{\omega}_0$ we verify that the energies increase as a function of $\omega_0$ (or $\eta$). Besides, in all these graphs we also verify that the energies of the particle are always greater than those of the antiparticle (but both increase as $n$ increases), where the energies of the particle (antiparticle) are larger (smaller) in the presence of $U$. Thus, comparing our relativistic spectrum with some other works, we verify that our results generalize some particular cases of the literature.

Finally, we also study the nonrelativistic limit of our results, where such a limit is obtained when considering that most of the total energy is concentrated in the rest energy of the particle. Therefore, as a direct consequence of this limit, we get the Pauli-like NCQHO in two dimensions, which is basically the NCQHO with a strong spin–orbit coupling term and a constant that shifts all energy levels, and whose solution is the NC spinorial wave function.

Furthermore, in the presence of $U$, still remains quantized even in the absence of the DO ($\omega = 0$) and of the EDM ($d_f = 0$), or of the DO and of the NCPS ($\theta = \eta = 1$), has a finite or infinite degeneracy, and the maximal energies are reached for $s = +1$ and $m < 0$ with $d_f < 0$, while the minimal energies are for $s = -1$ and $m < 0$ with $d_f > 0$.

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Data availability statement

This paper has no associated data or the data will not be deposited.

Appendix. The two-dimensional noncommutative phase space

In usual two-dimensional (2D) quantum mechanics, a quantum phase space (commutative phase space) is defined by substituting the classical canonical variables of position and momentum $(x_i, p_i)$ by their respective Hermitian (quantum) operators, now written as $\hat{x}_i$ and $\hat{p}_i$, where obey the following Heisenberg (canonical) commutation relations (usual or ordinary Heisenberg algebra) [43]

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad (i, j = 1, 2 = x, y),$$

(A1)
and whose Heisenberg uncertainty relations (Heisenberg uncertainty principle for position and momentum) are written as follows

$$\Delta \hat{x}_i \Delta \hat{p}_j \geq \frac{\hbar}{2} \delta_{ij}, \quad \Delta \hat{p}_i \Delta \hat{x}_j = 0, \quad \Delta \hat{p}_i \Delta \hat{p}_j = 0,$$

(A2)

where $\delta_{ij} = \delta_{ij}$ is the Kronecker delta (Euclidean metric). In essence, the uncertainty principle states that we cannot measure simultaneously and with high precision two operators that do not commute with each other (incompatible operators), and therefore, the more we know about one operator (or observable), the less we know about the other (and vice versa).

Now, in order to define a noncommutative quantum phase space (NCQPS), or simply a NCPS [58–60], the relations given in (A1) must then obey the following deformed Heisenberg (noncanonical) commutation relations (NC or deformed Heisenberg algebra)

$$[\hat{x}^\text{NC}_i, \hat{p}^\text{NC}_j] = i\hbar \delta_{ij} \left(1 + \frac{\theta_{ij}}{4\hbar^2}\right), \quad [\hat{x}^\text{NC}_i, \hat{x}^\text{NC}_j] = i\hbar \theta_{ij}, \quad [\hat{p}^\text{NC}_i, \hat{p}^\text{NC}_j] = i\eta_{ij},$$

(A3)

where the NC operators $\hat{x}^\text{NC}_i$ and $\hat{p}^\text{NC}_i$ are defined as

$$\hat{x}^\text{NC}_i \equiv \hat{x}_i - \frac{1}{2\hbar} \theta_{ij} \hat{p}_j, \quad \hat{p}^\text{NC}_i \equiv \hat{p}_i + \frac{1}{2\hbar} \eta_{ij} \hat{x}_j, \quad (\hat{x}_i = \delta_{ij} \hat{x}^1; \hat{p}_i = \delta_{ij} \hat{p}^1 = -i\partial_j),$$

(A4)

with $\theta_{ij} \equiv \eta_{ij}$ and $\eta_{ij} \equiv \eta_{ij}$ being antisymmetric constants ‘tensors’ (real deformation parameters), $\epsilon_{ij} = - \epsilon_{ij}$ is the Levi-Civita symbol (a pseudotensor), and $\theta > 0$ and $\eta > 0$ ($\theta, \eta \in \mathbb{R}^*$) are the position and momentum NC parameters (NC parameter for the coordinate and momentum space) with dimensions of length-squared and momentum-squared, respectively. From a phenomenological point of view, these two NC parameters can have the following values: $\theta \approx 4 \times 10^{-40}$ m$^2$ and $\eta \approx 1.76 \times 10^{-63}$ kg m$^{-2}$ s$^{-2}$ [60].

In particular, the relationship between the set of NC variables $[\hat{x}^\text{NC}_i, \hat{p}^\text{NC}_j]$ (here is not the anticommutator) and the set of usual variables $[\hat{x}_i, \hat{p}_j]$ is a consequence of a noncanonical linear transformation, known as Darboux transformation or Seiberg-Witten map [58–60]. However, all physical observables are entirely independent of the chosen map (how it should be). Furthermore, the NCS (NC only in position) causes a change in the usual product of two arbitrary functions $F(r)$ and $G(r)$, where now such a product is called star product or Moyal product, and whose definition is given as follows [60]

$$F(r) \ast G(r) = F(r) e^{i\theta/2\hbar (\hat{\partial}_r \hat{\partial}_r - \hat{\partial}_r \hat{\partial}_r)} G(r) = F(r) e^{i\theta/2\hbar (\hat{\partial}_r \hat{\partial}_r - \hat{\partial}_r \hat{\partial}_r)} G(r).$$

(A5)

In fact, in the absence of the NCS ($\theta = 0$), the star product is simply the usual product $F(r)G(r)$. Also, the uncertainty relations for the NC case are now written as

$$\Delta \hat{x}^\text{NC}_i \Delta \hat{p}^\text{NC}_j \geq \frac{\hbar}{2} \delta_{ij}, \quad \Delta \hat{x}^\text{NC}_i \Delta \hat{x}^\text{NC}_j \geq \frac{1}{2} |\theta_{ij}|, \quad \Delta \hat{p}^\text{NC}_i \Delta \hat{p}^\text{NC}_j \geq \frac{1}{2} |\eta_{ij}|,$$

(A6)

where

$$\hbar_{\text{eff}} = \hbar^\text{NC} = \hbar (1 + \xi),$$

(A7)

with $\hbar_{\text{eff}}$ being the effective, deformed or NC Planck constant, and the parameter $\xi$ is given by: $\xi \equiv \frac{\hbar^\text{NC}}{\hbar}$. For $\xi \ll 1$, or in the (commutative) limit $\xi \rightarrow 0$, we recover the usual uncertainty relations. For a more detailed discussion of the possible hypothetical values of $\xi$, please see [60], where the NC gravitational quantum well was studied.

Last but not least, the NCPS can also be expanded to include the $(2 + 1)$-dimensional Minkowski spacetime of the relativistic quantum mechanics [60], which is given by following relations

$$[\hat{\xi}^\text{NC}_\mu, \hat{\xi}^\text{NC}_\nu] = i\hbar \left(\delta_{\mu\nu} + \frac{1}{4\hbar^2} \theta^\nu_{\mu} \eta_{\rho\sigma} \right), \quad [\hat{\xi}^\text{NC}_\mu, \hat{\xi}^\text{NC}_\sigma] = i\hbar \theta^\rho_{\mu} \eta_{\rho\sigma}, \quad [\hat{p}^\text{NC}_\mu, \hat{p}^\text{NC}_\nu] = i\hbar \eta_{\mu\nu},$$

(A8)

where

$$\hat{\xi}^\text{NC}_\mu = \hat{\xi}_\mu - \frac{1}{2\hbar} \theta^\rho_{\mu} \hat{p}_\rho, \quad \hat{p}^\text{NC}_\mu = \hat{p}_\mu + \frac{1}{2\hbar} \eta_{\rho\sigma} \hat{\xi}_\rho, \quad (\hat{\xi}_\mu = \delta_{\mu} \hat{\xi}^1; \hat{p}_\mu = \delta_{\mu} \hat{p}^1),$$

(A9)

and

$$F(r, \theta) \ast G(r, \theta) \equiv F(r) e^{i\theta/2\hbar (\hat{\partial}_r \hat{\partial}_r - \hat{\partial}_r \hat{\partial}_r)} G(r),$$

(A10)

being $\hat{g}_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1)$ the flat metric tensor (Minkowski metric) and $\mu, \nu = 0, 1, 2 = t, x, y$. In addition, in this work, we consider the NC only in the spatial components of $\hat{\xi}^\text{NC}_\mu$ and $\hat{p}^\text{NC}_\mu$ (space-like NC or spatial NC), where it implies: $\theta_{0\mu} = \eta_{0\mu} = 0$; otherwise, the unitary, causality or locality of the quantum mechanics would not be preserved [43, 49, 60].
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