COMMUTATIVITY EQUATIONS AND THEIR TRIGONOMETRIC SOLUTIONS

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Abstract. We consider commutativity equations $F_i F_j = F_j F_i$ for a function $F(x^1, \ldots, x^N)$, where $F_i$ is a matrix of the third order derivatives $F_{ikl}$. We show that under certain non-degeneracy conditions a solution $F$ satisfies the WDVV equations. Equivalently, the corresponding family of Frobenius algebras has the identity field $e$.

We study trigonometric solutions $F$ determined by a finite collection of vectors with multiplicities, and we give an explicit formula for $e$ for all the known such solutions. The corresponding collections of vectors are given by non-simply laced root systems or are related to their projections to the intersection of mirrors.

1. Introduction

A celebrated system of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations for a prepotential function $F(x) = F(x^1, \ldots, x^N)$ has the form

$$F_{ij} g^{kl} F_{lmn} = F_{mik} g^{kl} F_{ljn}, \tag{1.1}$$

where

$$F_{ij} = \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k},$$

and $G = (g^{kl})$ is a constant symmetric $N \times N$ matrix. These equations appeared in topological field theories \cite{27} and they are in the core of Frobenius manifolds theory \cite{9}. In these considerations one normally has the property that the components of the flat metric $G^{-1}$ can be represented as

$$(G^{-1})_{ij} = \sum_{k=1}^{N} e^k F_{kij} \tag{1.2}$$

for some vector field $e = \sum_{k=1}^{N} e^k(x) \partial_{x^k}$ which is the identity field for the corresponding family of Frobenius algebras. For example, in the Frobenius manifolds theory one normally has $e = \partial_{x^1}$, which is flat with respect to the metric $G^{-1}$. In the case of almost dual Frobenius manifold on the space of orbits of a finite Coxeter group the field $e$ is not constant, it is proportional to the Euler vector field \cite{11}.

It is also of interest to consider equations (1.1) without the additional assumption (1.2) which expresses the metric $G^{-1}$ as a linear combination of the third order derivatives of the prepotential. Indeed, in the case of $G$ being the identity matrix the corresponding equations (1.1) have the form of the commutativity equations

$$F_i F_j = F_j F_i, \tag{1.3}$$

where $F_i$ is the $N \times N$ matrix with matrix entries $(F_i)_{kl} = F_{ikl} = \frac{\partial^3 F}{\partial x^i \partial x^k \partial x^l}$.

Equations (1.3) appeared in the study of $N = 4$ supersymmetric mechanical system (see \cite{28}). For a suitable ansatz for the supercharges the supersymmetry algebra relations are
satisfied provided that equations (1.3) hold. Existence of the identity field or rather, more specifically, additional relations of the form $\sum_i x^i F_{ijk} = -\delta_{jk}$ lead to further superconformal symmetry (see [28] and also [5], where the relation with the WDVV equations is emphasized).

In this paper we are interested in commutativity equations (1.3) and the additional condition of the existence of a vector field $e = \sum_{i=1}^N e^k \partial_x^k$ such that $e(F_{ij}) = \delta_{ij}$. This vector field is the identity vector field for a family of algebras depending on $x$. One of our main results provides a sufficient condition on $F$ which ensures that $e$ exists. The components of the field can then be expressed via determinants of the matrices whose entries are the third order derivatives of the prepotential $F$. Similarly, for a general constant matrix $G$ we establish a representation of $G^{-1}$ as a linear combination of the matrices $F_i$ as a consequence of equations (1.1) (see Sections 6 and 7).

There is an interesting class of solutions of the equations (1.1), (1.2) determined by finite collections $A$ of vectors. The corresponding prepotential has the form

$$F = \sum_{\alpha \in A} (\alpha, x)^2 \log(\alpha, x), \quad x \in V. \tag{1.4}$$

In the case when $A$ is a root system such solutions of the WDVV equations appeared in [18]. They are almost dual prepotentials for the finite group orbit spaces Frobenius manifolds [11]. Such solutions also appear in four-dimensional Seiberg–Witten theory as perturbative parts of the corresponding prepotentials [17]. More generally, solutions of the form (1.4) exist for special configurations of vectors known as $\vee$-systems introduced by Veselov in [26]. This class of solutions was studied further in [7,13,14,23]. Thus it was shown that the class is closed under the operations of taking subsystems and projections of $A$, and such solutions have to do with Dubrovin’s almost duality on the discriminant strata. Connection of these solutions to the supersymmetric mechanics was explored in [16]. More generally, one may also consider solutions of the form (1.4) for the commutativity equations (1.1) (without extra condition (1.2)). The corresponding (irreducible) configurations of vectors $A$ can be shown to be the complex Euclidean version of $\vee$-systems introduced in [14].

There are also interesting trigonometric solutions of the equations (1.1), (1.2) of the form

$$F = \sum_{\alpha \in A} c_\alpha f((\alpha, x)) + Q(x, y), \tag{1.5}$$

where function $f = f(z)$ satisfies $f'''(z) = \cot z$, $c_\alpha \in \mathbb{C}$ and $Q$ is a cubic polynomial depending on the additional variable $y$. Solutions of this form for reduced root systems and Weyl-invariant multiplicities were obtained by Hoevenaars and Martini in [19] (see also [24] and [4] for more details). They appear as almost dual prepotentials for the extended affine Weyl groups orbit spaces [10,12], see [22] for type $A_N$. Such solutions also appeared in five-dimensional Seiberg–Witten theory as perturbative parts of prepotentials [17]. In the case of simply laced root systems these solutions describe quantum cohomology of resolutions of simple $A, D, E$ singularities [6]. Solutions of the form (1.5) for general configurations $A$ were initially studied in [15] where a closely related notion of the trigonometric $\vee$-system was introduced. Similarly to the rational case, we showed in [2] that this class of solutions is closed under restrictions and that a subsystem of a trigonometric $\vee$-system is also a trigonometric $\vee$-system. The restriction procedure for the classical root systems recovers solutions obtained by Pavlov from reductions of Egorov hydrodynamic chains [20].

There are also elliptic versions of some of these solutions considered by Riley and Strachan in [21,25].
It appears that solutions of the form (1.5) with $Q = 0$ of the WDVV equations (1.1), (1.3) may also exist. Such a solution for the root system $B_N$ appeared in [19] and it was generalized to $BC_N$ in [3]. The corresponding metric $G$ is the identity so the commutativity equations (1.3) hold as well.

Solutions of the form (1.5) with $Q = 0$ for the commutativity equations (1.3) for the root systems $A = F_4, G_2$ were obtained in [2]. The corresponding multiplicities are Weyl invariant but they have to satisfy a linear relation. A multi-parameter deformation of the root systems $A$ (1.3) may also exist. Such a solution for the root system $BC_N$ was also obtained in [3]. It is unclear whether there are more Frobenius manifold structures associated with such solutions.

In this paper we study solutions of the commutativity equations (1.3) of the form (1.5) with $Q = 0$. Thus we give a $\vee$-system version of conditions which the corresponding configuration of vectors has to satisfy, which we call a Euclidean trigonometric $\vee$-system (see Section 2). We also show that restrictions of solutions of the commutativity equations give new solutions and that a subsystem of a Euclidean trigonometric $\vee$-system is also a Euclidean trigonometric $\vee$-system (see Sections 3, 5). In Section 4 we clarify relations of Euclidean trigonometric $\vee$-systems to other versions of rational and trigonometric $\vee$-systems. All the known irreducible solutions of the commutativity equations (1.3) of the form (1.5) with $Q = 0$ are the non-simply laced root systems $BC_N, F_4, G_2$ with a relation between invariant multiplicities as well as restrictions of such solutions to the intersection of mirrors (in the case of $BC_N$ one can also extend analytically integer parameters defining the restriction). In all these cases we give an explicit uniform formula for the corresponding identity field $e$ in Section 8. Existence of the identity field implies that we also get new solutions of WDVV equations (1.1), (1.3) in the case of root system $F_4$ and its projections.

2. Commutativity equations and Euclidean trigonometric $\vee$-systems

Let $A$ be a finite set of non-zero vectors in a Euclidean space $V \cong \mathbb{C}^N, N \in \mathbb{N}$, with the bilinear inner product $(\cdot, \cdot)$. Let $c: A \to \mathbb{C}$ be the (multiplicity) function. We denote $c_\alpha := c(\alpha)$ for $\alpha \in A$. We assume that $A$ belongs to a lattice of rank $N$. For each vector $\alpha \in A$ let us introduce the set of its collinear vectors from $A$:

$$\delta_\alpha := \{ \gamma \in A: \gamma \sim \alpha \}.$$ 

Let $\delta \subseteq \delta_\alpha$ and $\alpha_0 \in \delta_\alpha$. Then for any $\gamma \in \delta$ we have $\gamma = k_\gamma \alpha_0$ for some $k_\gamma \in \mathbb{R}$. Note that $k_\gamma$ depends on the choice of $\alpha_0$ and different choices of $\alpha_0$ give rescaled collections of these parameters. Define $C_\delta^{\alpha_0} := \sum_{\gamma \in \delta} c_\gamma k_\gamma^2$. Note that $C_\delta^{\alpha_0} \neq 0$ if and only if $C_\delta^{\alpha_0} \neq 0$ for any $\tilde{\alpha}_0 \in \delta$.

We define strings (or series) of vectors as follows (cf. [15]).

For any $\alpha \in A$ let us distribute all the vectors in $A \setminus \delta_\alpha$ into a disjoint union of $\alpha$-strings

$$A \setminus \delta_\alpha = \bigcup_{s=1}^k \Gamma_\alpha^s,$$

where $k \in \mathbb{N}$ depends on $\alpha$. These strings $\Gamma_\alpha^s$ are determined by the property that for any $s = 1, \ldots, k$ and for any two covectors $\gamma_1, \gamma_2 \in \Gamma_\alpha^s$ one has either $\gamma_1 + \gamma_2 = m\alpha$ or $\gamma_1 - \gamma_2 = m\alpha$ for some $m \in \mathbb{Z}$. We assume that the strings are maximal, that is if $\gamma \in \Gamma_\alpha^s$ for some $s \in \mathbb{N}$, then $\Gamma_\alpha^s$ must contain all the covectors of the form $\pm \gamma + m\alpha \in A$ with $m \in \mathbb{Z}$. Note that if for some $\beta \in A$ there is no $\gamma \in A$ such that $\beta + \gamma = m\alpha$ for $m \in \mathbb{Z}$, then $\beta$ itself forms a single $\alpha$-string.
By replacing some vectors from $A$ with their opposite ones and keeping the multiplicity unchanged one can get a new configuration whose vectors belong to a half-space. We will denote such a system by $A_+$. If this system contains repeated vectors $\alpha$ with multiplicities $c_\alpha^i$ then we replace them with the single vector $\alpha$ with multiplicity $c_\alpha := \sum_i c_\alpha^i$.

Let us now define a Euclidean trigonometric $\vee$-system in analogy with a trigonometric $\vee$-system \cite{15}.

**Definition 2.1.** The pair $(A, c)$ is called a Euclidean trigonometric $\vee$-system if for all $\alpha \in A$ and for all $\alpha$-strings $\Gamma_\alpha$, one has the relation

$$
\sum_{\beta \in \Gamma_\alpha} c_\beta(\alpha, \beta)\alpha \land \beta = 0. \tag{2.1}
$$

Consider a function $F$ given by the formula

$$
F = \sum_{\alpha \in A} c_\alpha f((\alpha, x)), \tag{2.2}
$$

where the function $f$ is given by

$$
f(z) = \frac{1}{6}iz^3 + \frac{1}{4}Li_3(e^{-2iz})
$$

so that $f'''(z) = \cot z$. We are interested in configurations $(A, c)$ such that the commutativity equations

$$
F_iF_j = F_jF_i, \quad i, j = 1, \ldots, N, \tag{2.3}
$$

hold, where $F_i$ is the $N \times N$ matrix with entries

$$
(F_i)_{pq} = F_{ipq} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_q}.
$$

The following statement establishes invariance of the commutativity equations under the action of the group of orthogonal transformations $O(N, \mathbb{C})$. Summation from 1 to $N$ over repeated indices will be assumed throughout unless indicated otherwise.

**Proposition 2.2.** Suppose a function $F = F(x^1, \ldots, x^N)$ satisfies commutativity equations \cite{13}. Let $C = (C^k_i) \in O(N, \mathbb{C})$, and let

$$
\bar{x}^k = C^k_i x^i, \tag{2.4}
$$

where $\bar{x}^1, \ldots, \bar{x}^N$ is a new coordinates system. Then $\bar{F}(\bar{x}) = F(x)$ satisfies commutativity equations

$$
\bar{F}_i\bar{F}_j = \bar{F}_j\bar{F}_i, \quad i, j = 1, \ldots, N, \tag{2.5}
$$

where

$$
(\bar{F}_i)_{pq} = \frac{\partial^3 \bar{F}}{\partial x^i \partial x^j \partial x^q}.
$$

**Proof.** Since $\partial_{x^i} = C^k_i \partial_{\bar{x}^k}$, we have $F_{ijk} = C^k_i C^j_l C^m_k \bar{F}_{ijk}$. Then commutativity equations $F_{ijk}F_{klm} = F_{mjk}F_{kli}$ in the new coordinates take the form

$$
C^k_i C^j_l C^m_k \bar{F}_{ijk} \bar{F}_{klm} = C^m_i C^j_l C^k_m \bar{F}_{mjk} \bar{F}_{ijkl} \tag{2.6}
$$

Now we multiply both sides of equality (2.6) by $\hat{C}_\alpha^k \hat{C}_\beta^j \hat{C}_\gamma^l \hat{C}_\epsilon^m$, where $\hat{C} = C^{-1}$ so that $\hat{C}_\alpha^k C^\beta_k = \delta^\beta_\alpha$. We get

$$
C^k_i C^\beta_k \hat{F}_{\alpha\beta\gamma\epsilon} = C^k_i C^\beta_k \hat{F}_{\alpha\beta\gamma\epsilon}. \tag{2.7}
$$
For an orthogonal transformation $C$ we have $C_k^\dagger C_{\tilde k}^a = \delta_k^a$. Hence equality (2.7) reduces to (2.5).

We are going to establish a relation between solutions (2.2) of the commutativity equations (2.3) and Euclidean trigonometric $\vee$-systems. The following two lemmas hold.

**Lemma 2.3.** The commutativity equations (2.3) for the function (2.2) are equivalent to the identity
\[
\sum_{\alpha,\beta \in A} c_\alpha c_\beta (\alpha, \beta) \cot(\alpha, x) \cot(\beta, x) B_{\alpha,\beta}(a,b) \alpha \wedge \beta = 0, \tag{2.8}
\]
for all $a, b \in V$, where $B_{\alpha,\beta}(a,b) = \alpha \wedge \beta = (\alpha, a)(\beta, b) - (\alpha, b)(\beta, a)$.

**Lemma 2.4.** Suppose that identity (2.8) holds for any $a, b \in V$. Suppose also that $C_{\delta_0}^{\alpha_0} \neq 0$ for any $\alpha \in A, \delta \subseteq \delta_\alpha, \alpha_0 \in \delta_\alpha$. Then $A$ is a Euclidean trigonometric $\vee$-system.

Proofs of Lemmas 2.3 and 2.4 are similar to the proofs of analogous statements in [2] for the case of the trigonometric $\vee$-system (see also [1]).

Note that if $A$ is a Euclidean trigonometric $\vee$-system then the left-hand side of identity (2.8) is non-singular. Since all vectors from $A$ belong to an $N$-dimensional lattice then the left-hand side of identity (2.8) is a rational function in suitable exponential variables which has degree zero and therefore is a constant. In order to find this constant, by changing some of the vectors from $A$ to their opposite ones we can assume that all vectors from $A$ belong to a half-space, hence form a positive system $A_+$. Then in an appropriate limit in a cone $\cot(\alpha, x) \rightarrow i$ for all $\alpha \in A_+$ and the identity (2.8) reduces to
\[
\sum_{\alpha,\beta \in A_+} c_\alpha c_\beta (\alpha, \beta) B_{\alpha,\beta}(a,b) \alpha \wedge \beta = 0.
\]

From these considerations we get the following result.

**Theorem 2.5.** Suppose that a configuration $(A,c)$ satisfies the condition $C_{\delta_0}^{\alpha_0} \neq 0$ for any $\alpha \in A, \delta \subseteq \delta_\alpha, \alpha_0 \in \delta_\alpha$. Then the commutativity equations (2.3) for the prepotential (2.2) imply the following two conditions:

1. $A$ is a Euclidean trigonometric $\vee$-system,
2. $\sum_{\alpha,\beta \in A_+} c_\alpha c_\beta (\alpha, \beta) B_{\alpha,\beta}(a,b) \alpha \wedge \beta = 0$ for all $a, b \in V$.

Conversely, if a configuration $(A,c)$ satisfies conditions (1), (2) then commutativity equations (2.3) hold.

Root systems of Weyl groups provide examples of Euclidean trigonometric $\vee$-systems.

**Proposition 2.6.** A root system $A = \mathcal{R}$ with Weyl-invariant multiplicity function $c$ is a Euclidean trigonometric $\vee$-system.

Proof. Fix $\alpha \in \mathcal{R}$. Take any $\beta \in \mathcal{R}$, and let $\gamma = s_\alpha \beta = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \alpha$. Since $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ we get that $\beta, \gamma \in \Gamma_\alpha^s$ for some $s$. We have
\[
c_\beta = c_\gamma, \quad (\alpha, \beta) = -(\alpha, \gamma), \quad \alpha \wedge \beta = \alpha \wedge \gamma.
\]

Hence the contribution of $\beta$ and $\gamma$ to the sum in (2.1) cancel each other. \qed
In general root systems $A = R$ with invariant multiplicities do not satisfy condition (2) in Theorem 2.5. It has been shown in [4] that this condition is satisfied for root systems $R = BC_N, F_4, G_2$ with special invariant multiplicities.

Solutions of commutativity equations can be applied to construct $N = 4$ supersymmetric mechanical systems. Hamiltonians corresponding to root systems $R = BC_N, F_4, G_2$ were given explicitly in [4].

3. Subsystems of a Euclidean trigonometric $\vee$-system

Now we consider subsystems of a Euclidean trigonometric $\vee$-system.

**Definition 3.1.** Let $A \subset V$ be a finite collection of vectors. A subset $B \subseteq A$ is called a subsystem if

$$B = A \cap W$$

for a linear subspace $W \subseteq V$. The subsystem $B$ is called reducible if $B$ is a disjoint union of two non-empty subsystems $B = B_1 \cup B_2$. The subsystem $B$ is called irreducible if it is not reducible.

If $c$ is a multiplicity function for $A$ then we will equip a subsystem $B \subseteq A$ with the multiplicity function which is the restriction of the multiplicity function $c$ on $B$.

Assume that the linear span $\langle B \rangle = W$. We say that the subsystem $B$ is non-isotropic if the restriction of the inner product $(\cdot, \cdot)$ onto $W$ is non-degenerate.

**Theorem 3.2.** Any non-isotropic subsystem of a Euclidean trigonometric $\vee$-system is also a Euclidean trigonometric $\vee$-system.

The proof of Theorem 3.2 is similar to the proof in [2] of the analogous statement for the trigonometric $\vee$-system, see also [1].

4. Relation with other types of $\vee$-systems

4.1. Relation with trigonometric $\vee$-systems. For a finite subset $A \subset V$ with a multiplicity function $c: A \to \mathbb{C}$, consider a bilinear form $G_{A,c}$ on $V$ given by

$$G_{A,c}(x, y) = \sum_{\alpha \in A} c_\alpha(\alpha, x)(\alpha, y), \quad x, y \in V,$$

where $c_\alpha := c(\alpha)$. Following an analogy with the rational case (see [14] and subsection 4.2 below), we say that the pair $(A, c)$ is well-distributed in $V$ if the bilinear form (4.1) is proportional to the form $(\cdot, \cdot)$. The pair $(A, c)$ is called a trigonometric $\vee$-system if it satisfies the relations

$$\sum_{\beta \in \Gamma^*_\alpha} c_\beta G_{A,c}(\alpha, \beta) \alpha \wedge \beta = 0$$

for all $\alpha \in A$ and all $\alpha$-strings $\Gamma^*_\alpha$.

Now let $(A, c)$ be a Euclidean trigonometric $\vee$-system. Define a linear operator $M: V \to V$ as

$$M = \sum_{\beta \in A} c_\beta \beta \otimes \beta,$$

that is, $M(u) = \sum_{\beta \in B} c_\beta(\beta, u)\beta$ for any $u \in V$. The following statement takes place.

**Lemma 4.1.** Let $(A, c)$ be a Euclidean trigonometric $\vee$-system. Assume that the linear span $\langle A \rangle = V$. Then
(1) Any \( \alpha \in \mathcal{A} \) is an eigenvector of \( M \).
(2) The vector space \( V \) can be decomposed as
\[
V = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \quad k \in \mathbb{N},
\]
where \( M|_{V_i} = \lambda_i I, \lambda_i \in \mathbb{C}, \) and \( I \) is the identity operator, and \( \lambda_i \neq \lambda_j \) for \( i \neq j \).

The proof of Lemma 4.1 is similar to the proof in [2] for the trigonometric \( \lor \)-system case (see also [1]).

Since \( \mathcal{A} \subset V = V_1 \oplus \cdots \oplus V_k \), then \( \mathcal{A} \) can be represented as a disjoint union
\[
\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_k,
\]
where \( \mathcal{A}_i := \mathcal{A} \cap V_i \subset V_i \). The following two lemmas relate the strings of vectors in \( \mathcal{A} \) and its components \( \mathcal{A}_i \).

**Lemma 4.2.** Let \( \mathcal{A} \) be a Euclidean trigonometric \( \lor \)-system. Let \( \alpha \in \mathcal{A} \) be such that \( \alpha \in V_i \) for some \( i \). Consider an \( \alpha \)-string \( \Gamma^s_\alpha \) in \( \mathcal{A}_i \) and let \( \beta \in \Gamma^s_\alpha \). Then \( \Gamma^s_\alpha \subset V_i \) or \( \Gamma^s_\alpha \subseteq \{ \pm \beta \} \).

**Proof.** For \( \beta \in \Gamma^s_\alpha \) we have two possible cases.

Case (i) \( \beta \in V_i \). Then for any \( \gamma \in \Gamma^s_\alpha \) we have that \( \gamma = m\alpha + \varepsilon \beta \in V_i \) for some \( m \in \mathbb{Z} \) and \( \varepsilon = \pm 1 \). Hence \( \Gamma^s_\alpha \subset V_i \).

Case (ii) \( \beta \notin V_i \). Hence \( \beta \in V_j \) for some \( j \neq i \). Then for any \( \gamma \in \Gamma^s_\alpha \) we have that \( \gamma \in V_i \) or \( \gamma \in V_j \) since decomposition (4.3) is the direct sum. Note that \( \gamma \notin V_i \) as otherwise we will have \( \beta = m\alpha + \varepsilon \gamma \in V_i \), for some \( m \in \mathbb{Z} \) and \( \varepsilon = \pm 1 \), which is a contradiction. Note also that \( \gamma \notin V_j \) unless \( \gamma = \pm \beta \) as otherwise we have \( m\alpha = \beta + \varepsilon \gamma \in V_j \) for some \( m \in \mathbb{Z} \) and \( \varepsilon = \pm 1 \), which is a contradiction. Hence \( \Gamma^s_\alpha \subseteq \{ \pm \beta \} \). \( \square \)

**Lemma 4.3.** Let \( \alpha, \beta \in \mathcal{A}_i \). Let \( ^A\Gamma^s_\alpha, ^A\Gamma^t_\alpha \) be the \( \alpha \)-strings in \( \mathcal{A} \) and \( \mathcal{A}_i \) respectively containing \( \beta \). Then the set \( ^A\Gamma^s_\alpha \) is equal to the set \( ^A\Gamma^t_\alpha \).

**Proof.** Let \( \gamma \in ^A\Gamma^s_\alpha \). Then \( \gamma = m\alpha + \varepsilon \beta \in \mathcal{A}_i \), for some \( m \in \mathbb{Z} \) and \( \varepsilon = \pm 1 \). Thus \( \gamma \in ^A\Gamma^t_\alpha \) by the maximality of \( ^A\Gamma^t_\alpha \). Hence \( ^A\Gamma^s_\alpha \subseteq ^A\Gamma^t_\alpha \). The opposite inclusion is obvious. Therefore \( ^A\Gamma^s_\alpha = ^A\Gamma^t_\alpha \). \( \square \)

Note that the operator \( M \) is symmetric: \( (M(u), v) = (u, M(v)) \) for any \( u, v \in V \). Hence its eigenspaces are orthogonal.

**Proposition 4.4.** We have \( (u, v) = 0 \) for any \( u \in V_i \) and \( v \in V_j \) such that \( i \neq j \).

The following statement takes place.

**Lemma 4.5.** Restriction \( (\cdot, \cdot)_i \) of the bilinear form \( (\cdot, \cdot) \) onto the subspace \( V_i \) is non-degenerate.

**Proof.** Suppose that \( v \in V_i \) satisfies \( (v, u)_i = 0 \) for all \( u \in V_i \). By Proposition 4.3 we have \( (v, u) = 0 \) for all \( u \in V \). Hence \( v = 0 \) since \( (\cdot, \cdot) \) is non-degenerate. \( \square \)

The following statement relates the Euclidean trigonometric \( \lor \)-systems and the trigonometric \( \lor \)-systems.

**Theorem 4.6.** If \( \mathcal{A} \) is a Euclidean trigonometric \( \lor \)-system then the subsystem \( \mathcal{A}_i = \mathcal{A} \cap V_i \) is well-distributed in the subspace \( V_i \) with the bilinear form \( (\cdot, \cdot)_i \) for all \( i \). Furthermore, if the bilinear form
\[
G_{\mathcal{A}_i}(u, v) = \sum_{\alpha \in \mathcal{A}_i} c_\alpha(\alpha, u)(\alpha, v), \quad u, v \in V_i
\]
is non-degenerate on $V_i$ (equivalently, $G_{A,c}$ is non-zero), then $A_i$ is a trigonometric $\vee$-system. Moreover, $A$ is a trigonometric $\vee$-system if the form $G_{A,c}$ is non-degenerate.

**Proof.** By Lemma 4.1 we have $M|_{V_i} = \lambda_i I$. Hence for any $u \in V_i$ and $v \in V$ we have

$$G_{A,c}(u, v) = (M(u), v) = \lambda_i(u, v).$$  

(4.4)

Note also that by Proposition 4.4 we have that $G_{A,c}$ is non-degenerate on $V_i$. Thus the subsystem $A_i$ is reducible. Finally, for any $u \in V_i$ and $v \in V$ we have

$$G_{A,c}(u, v) = (M(u), v) = \lambda_i(u, v).$$  

(4.5)

Thus the subsystem $A_i$ is well-distributed in the subspace $V_i$.

Let us now assume that $G_{A,c}$ is non-degenerate on $V_i$, that is $\lambda_i \neq 0$. Let $\alpha \in A_i$. Consider an $\alpha$-string $\Gamma_\alpha$ in $A_i$. Then by Lemmas 4.2, 4.3 and formulas (4.4), (4.5) we have

$$\sum_{\beta \in \Gamma_\alpha} c_\beta G_{A,c}(\alpha, \beta) \alpha \wedge \beta = \lambda_i \sum_{\beta \in \Gamma_\alpha} c_\beta (\alpha, \beta) \alpha \wedge \beta = 0$$  

(4.6)

since $A$ is a Euclidean trigonometric $\vee$-system. This proves that $A_i$ is a trigonometric $\vee$-system. Finally, for $\alpha \in A_i$ let us consider its $\alpha$-string $A_{\Gamma_\alpha}$ in $A$. If $A_{\Gamma_\alpha} \subset V_i$ then $\sum_{\beta \in A_{\Gamma_\alpha}} c_\beta G_{A,c}(\alpha, \beta) \alpha \wedge \beta = 0$ by Lemma 4.3 and (4.4). If $A_{\Gamma_\alpha} \not\subset V_i$ then $A_{\Gamma_\alpha} \subseteq \{\pm \beta\}$ for some $\beta \in V_j$, $j \neq i$, by Lemma 4.2. Then $G_{A,c}(\alpha, \beta) = \lambda_\alpha(\alpha, \beta) = \lambda_j(\alpha, \beta)$ by (4.4). Hence $G_{A,c}(\alpha, \beta) = 0$ and the trigonometric $\vee$-system condition holds. □

Let $U \subseteq V$ be a linear subspace such that $\langle A \cap U \rangle = U$. The following statement takes place.

**Proposition 4.7.** Let $(A, c)$ be a Euclidean trigonometric $\vee$-system. Then the set of vectors $A \cap U$ with the multiplicity function $c|_{A \cap U}$ is well-distributed in $U$ or the system $A \cap U$ is reducible.

**Proof.** Define a linear operator $M_U : U \to U$ by

$$M_U := \sum_{\beta \in A \cap U} c_\beta \beta \otimes \beta.$$  

Let $\alpha \in A \cap U$. Let us sum up the Euclidean trigonometric $\vee$-condition (2.1) over $\alpha$-strings which belong to the subspace $U$. Then

$$\sum_{\beta \in A \cap U} c_\beta (\beta, \alpha) \beta = M_U(\alpha) = \lambda \alpha$$

for some $\lambda = \lambda(\alpha)$. Suppose that $A \cap U$ is irreducible. Then $\lambda$ does not depend on $\alpha$ and $M_U = \lambda I$. Therefore

$$\sum_{\beta \in A \cap U} c_\beta (\beta, u)(\beta, v) = (M_U(u), v) = \lambda(u, v),$$

and the pair $(A \cap U, c|_{A \cap U})$ is well-distributed. □

Suppose that $G_{A,c}$ is non-degenerate and define the vector $\alpha^\vee \in V$ by the relation

$$G_{A,c}(\alpha^\vee, x) = (\alpha, x)$$  

(4.7)

for all $x \in V$. Now assume that $\alpha \in V_i$ in which case we also have $\alpha^\vee \in V_i$ for some $i$. Then by Lemma 4.1 we have

$$G_{A,c}(\alpha^\vee, x) = (M_U(\alpha^\vee), x) = \lambda_i(\alpha^\vee, x).$$  

(4.8)
Hence from relations (4.7), (4.8) we have that \( \alpha \vee = \lambda^{-1}_i \alpha \). Therefore if the pair \((A, c)\) satisfies conditions (4.2) then for all \( \alpha \)-strings \( \Gamma^*_\alpha \) we have

\[
\sum_{\beta \in \Gamma^*_\alpha} c_\beta G_{A,c}(\alpha^\vee, \beta^\vee) \alpha \land \beta = \lambda^{-2}_i \sum_{\beta \in \Gamma^*_\alpha} c_\beta G_{A,c}(\alpha, \beta) \alpha \land \beta = 0.
\]

These conditions coincide with the definition of the trigonometric \( \vee \)-system given in [15] so the two definitions are equivalent. Let us now introduce an inner product \( \langle \cdot, \cdot \rangle \) on \( V \) as

\[
\langle u, v \rangle := G_{A,c}(u^\vee, v^\vee), \quad u, v \in V.
\]

The following statement is immediate.

**Proposition 4.8.** Let \((A, c)\) be a trigonometric \( \vee \)-system. Then \((A, c)\) is a Euclidean trigonometric \( \vee \)-system with respect to the bilinear form (4.9).

### 4.2. Relation with complex Euclidean \( \vee \)-systems.

Following [14], let us recall the notion of the (rational) complex Euclidean \( \vee \)-system. Let \( V \) be a complex vector space with a non-degenerate bilinear form \( (\cdot, \cdot) \). Let \( A \subset V \) be a finite set of vectors. Consider the canonical form

\[
G_A^*(x, y) = \sum_{\alpha \in A} (\alpha, x)(\alpha, y), \quad x, y \in V.
\]

Suppose that \( G_A^* \) is proportional to the form \( (\cdot, \cdot) \). Let \( \pi \subseteq V \) be a two-dimensional subspace such that \( \langle A \cap \pi \rangle = \pi \). \( A \) is said to be a (rational) complex Euclidean \( \vee \)-system if for any such \( \pi \) the subsystem \( B = A \cap \pi \) is reducible or the corresponding form \( G_B^*|_\pi \) is proportional to \( (\cdot, \cdot)|_\pi \).

**Proposition 4.9.** Let \( F = \sum_{\alpha \in A} (\alpha, x)^2 \log(\alpha, x) \). Suppose that \( F \) satisfies the commutativity equations and \( A \) is irreducible. Then \( A \) is a (rational) complex Euclidean \( \vee \)-system. Moreover, if for a two-dimensional plane \( \pi \subset V \) the subsystem \( A \cap \pi \) is reducible then the corresponding directions are orthogonal with respect to the form \( (\cdot, \cdot) \).

Proof is similar to [14]. The substitution of \( F \) into the commutativity equations gives the condition

\[
\sum_{\alpha, \beta \in A} \frac{(\alpha, \beta)}{(\alpha, x)(\beta, x)} (\alpha \land \beta)(\alpha \land \beta) = 0.
\]

It implies that for any two-dimensional plane \( \pi \)

\[
\sum_{\beta \in \pi \cap A} (\alpha, \beta) \alpha \land \beta = 0.
\]

The following statement relates the Euclidean trigonometric \( \vee \)-system and the (rational) complex Euclidean \( \vee \)-system.

**Proposition 4.10.** Let \((A, c)\) be an irreducible Euclidean trigonometric \( \vee \)-system. Then the set of vectors \( \sqrt{c} \alpha, \alpha \in A \), is a (rational) complex Euclidean \( \vee \)-system.

Proof. Firstly, since \( A \) is irreducible then by Lemma [14] we have \( A = A_1 \subset V = V_1 \) and \( M|_{V_1} = \lambda_1 I \). Then by Theorem [14] we have that \( A_1 = A \) is well-distributed.

Secondly, by Proposition [14] we have that any subsystem of \( A \) is well-distributed or reducible, which implies the statement. \( \square \)
5. Restricted solutions of commutativity equations

In this Section we apply the restriction procedure to a given solution to the commutativity equations. This gives new solutions of the commutativity equations.

Let $B = A \cap W$ be a subsystem of $A$ for some $n$-dimensional linear subspace $W = \langle B \rangle \subset V$. Define

$$W_B := \{ x \in V : (\beta, x) = 0 \quad \forall \beta \in B \}.$$ 

Let $(\cdot, \cdot)_B$ be the restriction of $(\cdot, \cdot)$ on $W_B$, and assume that it is non-degenerate. Let $S \subset B$ be a basis of $W$. Let $f_1, \ldots, f_n$ be an orthonormal basis of the space $W_B$, and let $\xi = (\xi^1, \ldots, \xi^n)$ be the corresponding orthonormal coordinates in $W_B$. Define $M_A = V \setminus \bigcup_{\alpha \in A} \Pi_\alpha$, and $M_B = W_B \setminus \bigcup_{\alpha \in A \setminus B} \Pi_\alpha$, where $\Pi_\alpha = \{ x \in V : (\alpha, x) = 0 \}$. The following statement shows that the class of solutions of commutativity equations corresponding to Euclidean $\vee$-systems is closed under the restrictions.

**Theorem 5.1.** Assume that prepotential (2.2) satisfies commutativity equations (2.3). Suppose that $C^\alpha_\delta \neq 0$ for any $\alpha \in S, \alpha_0 \in \delta, \delta \subseteq \delta_\alpha$. Then the prepotential

$$F_B = \sum_{\alpha \in A \setminus B} c_\alpha f((\alpha, \xi)), \quad \xi \in M_B,$$

(5.1)

satisfies the commutativity equations

$$(F_B)_i (F_B)_j = (F_B)_j (F_B)_i, \quad i, j = 1, \ldots, n,$$

where $(F_B)_i$ is the $n \times n$ matrix with entries

$$((F_B)_i)_{pq} = (F_B)_{ipq} = \frac{\partial^3 F_B}{\partial \xi^i \partial \xi^p \partial \xi^q}.$$ 

**Proof.** First for any $u = (u^1, \ldots, u^N), v = (v^1, \ldots, v^N) \in V$ let us consider the vector fields

$$\partial_u = u^i \partial_{x^i}, \partial_v = v^i \partial_{x^i} \in T_x M_A.$$

We define the following multiplication on the tangent space $T_x M_A$:

$$\partial_u \ast \partial_v = u^i v^j F_{ijk} \partial_{x^k}. \quad \text{(5.2)}$$

It is easy to check that the associativity of the multiplication $\ast$ is equivalent to the commutativity equations (2.3). From the formula (2.2) we have

$$F_{ijk} = \sum_{\alpha \in A} c_\alpha (\alpha, f_i)(\alpha, f_j)(\alpha, f_k) \cot(\alpha, x).$$

Hence multiplication (5.2) takes the form

$$\partial_u \ast \partial_v = \sum_{\alpha \in A} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \partial_\alpha.$$ 

(5.3)

By identifying $V \cong T_x V$, we have

$$u \ast v = \sum_{\alpha \in A} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \alpha.$$ 

Consider now a point $x_0 \in M_B$ and two tangent vectors $u_0, v_0 \in T_{x_0} M_B$. We extend vectors $u_0$ and $v_0$ to two local analytic vector fields $u(x), v(x)$ in the neighbourhood $U$ of $x_0$ that are tangent to the subspace $W_B$ at any point $x \in M_B \cap U$ such that $u_0 = u(x_0)$ and $v_0 = v(x_0)$. The proof of the next lemma is similar to the proof of [2 Lemma 4.1] (see also [13 Lemma 1] for the rational case, and [3] for the trigonometric case).
Lemma 5.2. The limit of the product \( u(x) \ast v(x) \) exists when vector \( x \) tends to \( x_0 \in M_B \) and it satisfies

\[
u_0 \ast v_0 = \sum_{\alpha \in A \setminus B} c_\alpha(\alpha, u_0)(\alpha, v_0) \cot(\alpha, x_0) \alpha.
\]

In particular, the product \( u_0 \ast v_0 \) is determined by vectors \( u_0 \) and \( v_0 \) only.

The following lemma holds and it shows that multiplication (5.3) is closed on the tangent space \( T_{x_0}M_B \).

Lemma 5.3. Let \( u, v \in T_{x_0}M_B \) where \( x_0 \in M_B \). Then \( u \ast v \in T_{x_0}M_B \).

The proof of Lemma 5.3 is similar to the proof of [2, Lemma 4.2]. It uses an argument analogous to [2] which claims that the following identity holds for any \( a, b \in V \) if \( \tan(\alpha, x) = 0 \):

\[
\sum_{\beta \in A \setminus \alpha} c_\beta(\alpha, \beta) \cot(\beta, x) B_{\alpha, \beta}(a \otimes b) \alpha \land \beta = 0.
\]

Then for \( u, v \in T_{x_0}M_B, x_0 \in M_B \), the product (5.3) takes the form

\[
\partial_u \ast \partial_v = \sum_{\alpha \in A \setminus B} c_\alpha(\alpha, u)(\alpha, v) \cot(\alpha, x_0) \partial_\alpha.
\]

By using the orthonormal basis \( f_1, \ldots, f_n \) of \( W_B \) we rearrange \( \partial_\alpha \) as

\[
\partial_\alpha = \sum_{k=1}^n (\alpha, f_k) \partial f_k.
\]

Hence for \( x_0 = \xi = \sum_{i=1}^n \xi^i f_i \) we have

\[
\partial_{f_i} \ast \partial_{f_j} = \sum_{\alpha \in A \setminus B} \sum_{k=1}^n c_\alpha(\alpha, f_i)(\alpha, f_j)(\alpha, f_k) \cot(\alpha, \xi) \partial f_k = \sum_{k=1}^n \tilde{F}_{ijk} \partial f_k, \quad i, j = 1, \ldots, n,
\]

where \( \tilde{F}(\xi) = \sum_{\alpha \in A \setminus B} c_\alpha(f, \xi) = F_B \). Now multiplication (5.4) is associative and it is easy to check that its associativity is equivalent to the commutativity equations

\[
\tilde{F}_i \tilde{F}_j = \tilde{F}_j \tilde{F}_i, \quad i, j = 1, \ldots, n.
\]

As an application of Theorem 5.1 we get the following solutions of the commutativity equations starting from a solution for the root system \( BC_N \). Let \( q, r, s \in \mathbb{C}, m = (m_1, \ldots, m_n) \in (\mathbb{C}^\times)^n \). Suppose

\[
r = -8s - 2q(N - 2)
\]
with $N = \sum_{i=1}^{n} m_i$. Define the configuration $\text{BC}_n(q, r, s; m) \subset \mathbb{C}^n$ consisting of the following vectors $\alpha$ with the multiplicities $c_\alpha$:

\[
\begin{align*}
& m_i^{-1/2}e_i, \quad \text{with multiplicity } rm_i, \quad 1 \leq i \leq n, \\
& 2m_i^{-1/2}e_i, \quad \text{with multiplicity } sm_i + \frac{1}{2}qm_i(m_i - 1), \quad 1 \leq i \leq n, \\
& m_i^{-1/2}e_i \pm m_j^{-1/2}e_j, \quad \text{with multiplicity } qm_im_j, \quad 1 \leq i < j \leq n.
\end{align*}
\] (5.5)

Note that in the case $m_i = 1$ for all $i = 1, \ldots, n$, configuration (5.5) reduces to the positive half $\text{BC}_n^+$ of the root system $\text{BC}_n$. Consider the function

\[
\widetilde{F} = \sum_{\alpha \in \text{BC}_n(q, r, s; m)} c_\alpha f((\alpha, x)), \quad x \in \mathbb{C}^n.
\] (5.6)

It follows from Theorem 5.1 and it was shown earlier in [3] that function (5.6) satisfies the commutativity equations

\[
\widetilde{F}_i\widetilde{F}_j = \widetilde{F}_j\widetilde{F}_i \quad i, j = 1, \ldots, n.
\]

Consider now the positive half $F_4^+$ of the root system $F_4$ given by

\[
F_4^+ = \{e_i (1 \leq i \leq 4), \ e_i \pm e_j (1 \leq i < j \leq 4), \ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.
\] (5.7)

Let $r$ be the multiplicity of short roots and let $q$ be the multiplicity of long roots. A basis of simple roots consists of $\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

Up to an orthogonal transformation there are two projected systems in dimension three and four projected systems in dimension two. Firstly, we give details of the three-dimensional projections of $F_4^+$. There are two different projections of $F_4^+$ along the root system $A_1$. The first one $(F_4, A_1)_1$ is obtained by projecting to the hyperplane $\alpha_3 = 0$. The second one $(F_4, A_1)_2$ is obtained by projecting to the hyperplane $\alpha_2 = 0$. Hence we have the following three-dimensional projected systems of the positive root system $F_4^+$.

- The projected system $(F_4, A_1)_1$ consists of the following vectors:

\[
\begin{align*}
& e_i, \quad \text{with multiplicity } r + 2q, \quad 1 \leq i \leq 3, \\
& e_i \pm e_j, \quad \text{with multiplicity } q, \quad 1 \leq i < j \leq 3, \\
& \frac{1}{2}(e_1 \pm e_2 \pm e_3), \quad \text{with multiplicity } 2r.
\end{align*}
\]
• The projected system \((F_4, A_1)_2\) consists of the following vectors (after doing a change of variables and renaming vectors):

\[
e_1, e_2, \quad \text{with multiplicity } r,
\]
\[
\sqrt{2}e_3, \quad \text{with multiplicity } q,
\]
\[
\frac{\sqrt{2}}{2}e_3, \quad \text{with multiplicity } 2r,
\]
\[
e_1 \pm e_2, \quad \text{with multiplicity } q,
\]
\[
\frac{1}{2}(e_1 \pm e_2), \quad \text{with multiplicity } 2r,
\]
\[
e_1 \pm \frac{\sqrt{2}}{2}e_3, e_2 \pm \frac{\sqrt{2}}{2}e_3, \quad \text{with multiplicity } 2q,
\]
\[
\frac{1}{2}(e_1 \pm e_2 \pm \sqrt{2}e_3), \quad \text{with multiplicity } r.
\]

Secondly, we give details of the two-dimensional projections of \(F_4^+\). There are two different projections of \(F_4^+\) along the root system \(A_2\). The first one \((F_4, A_2)_1\) is obtained by projecting to the plane \(\alpha_1 = \alpha_2 = 0\). The second one \((F_4, A_2)_2\) is obtained by projecting to the plane \(\alpha_3 = \alpha_4 = 0\). There is also a projected configuration \((F_4, A_1^2)\) along the subsystem \(A_1 \times A_1\) to the plane \(\alpha_1 = \alpha_3 = 0\), and there is a projected configuration \((F_4, B_2)\) along the subsystem \(B_2\) to the plane \(\alpha_2 = \alpha_3 = 0\). These configurations have the following explicit form

• The configuration \((F_4, A_2)_1\) consists of vectors \(\alpha\) with the corresponding multiplicities \(c_\alpha\) given as follows:

\[
e_1, \quad \text{with multiplicity } r,
\]
\[
\frac{1}{\sqrt{3}}e_2, \quad \text{with multiplicity } 3r,
\]
\[
\frac{2}{\sqrt{3}}e_2, \quad \text{with multiplicity } 3q,
\]
\[
e_1 \pm \frac{1}{\sqrt{3}}e_2, \quad \text{with multiplicity } 3q,
\]
\[
\frac{1}{2}(e_1 \pm \frac{1}{\sqrt{3}}e_2), \quad \text{with multiplicity } 3r,
\]
\[
\frac{1}{2}(e_1 \pm \frac{3}{\sqrt{3}}e_2), \quad \text{with multiplicity } r.
\]
- The configuration \((F_4, A_1^2)\) consists of vectors \(\alpha\) with the corresponding multiplicities \(c_\alpha\) given as follows:

\[
\begin{align*}
e_1, & \quad \text{with multiplicity } r + 2q, \\
\sqrt{2}e_2, & \quad \text{with multiplicity } q, \\
\frac{1}{2}e_1, & \quad \text{with multiplicity } 4r, \\
\frac{\sqrt{2}}{2}e_2, & \quad \text{with multiplicity } 2(r + 2q), \\
e_1 \pm \frac{1}{\sqrt{2}}e_2, & \quad \text{with multiplicity } 2q, \\
\frac{1}{2}(e_1 \pm 2\sqrt{2}e_2), & \quad \text{with multiplicity } 2r.
\end{align*}
\]

Configurations \((F_4, A_2)\) and \((F_4, B_2)\) are equivalent to the root systems \(G_2\) and \(BC_2\), respectively, the corresponding solutions of the commutativity equations were found in [4]. Theorem 5.1 gives new solutions of the commutativity equations which are listed in the next statement.

**Theorem 5.4.** Let \((A, c)\) be one of the configurations \((F_4, A_1)_1, (F_4, A_1)_2, (F_4, A_2)_1, (F_4, A_1^2)\) described above. Then the function \(F = \sum_{\alpha \in A} c_\alpha f((\alpha, x))\) satisfies the commutativity equations, where \(x \in \mathbb{C}^3\) for the first two configurations and \(x \in \mathbb{C}^2\) for the last two configurations and parameters \(r, q\) satisfy the condition \(r = -2q\) or \(r = -4q\).

6. **Commutativity equations and WDVV equations**

In this Section we investigate the relation between the commutativity equations and the WDVV equations. Let \(V \cong \mathbb{C}^N, N \geq 2\). Let \(F = F(x^1, \ldots, x^N)\) be a function on \(V\). We recall that it was proven in [18] (see also [17]) that the generalized WDVV equations

\[
F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad i, j, k = 1, \ldots, N
\]

can be written equivalently in the form

\[
F_i B^{-1} F_j = F_j B^{-1} F_i, \quad i, j = 1, \ldots, N, \tag{6.1}
\]

where \(B\) is any non-degenerate matrix of the form \(B = \sum_{i=1}^{N} A^k F_k\) for some functions \(A^k\). If the matrix \(B\) happens to be a multiple of the identity matrix then WDVV equations (6.1) reduce to the commutativity equations

\[
F_i F_j = F_j F_i, \quad i, j = 1, \ldots, N. \tag{6.2}
\]

A natural question to investigate is when there exists such a linear combination \(B\). We give an answer in this Section.

Let us assume that a function \(F = F(x^1, \ldots, x^N)\) satisfies the commutativity equations (6.2). Let us denote by \([F_i, F_j]_{(a,b)}\) the \((a, b)\)-entry of the commutator \([F_i, F_j]\) or, more explicitly,

\[
[F_i, F_j]_{(a,b)} = \sum_{m=1}^{N} (F_{iam} F_{jbm} - F_{ibm} F_{jam}). \tag{6.3}
\]
The equality \([F_i, F_k]_{(i,j)} = 0\) implies that
\[
F_{ijk}F_{iii} = \sum_{m=1}^{N} F_{ijm}F_{ikm} - \sum_{m \neq i} F_{imn}F_{jkm}. \tag{6.4}
\]

Observe the equality of matrix entries \([F_a, F_b]_{(i,j)} = [F_i, F_j]_{(a,b)}\). Introduce the notation
\[
[F_i, F_j]_{(a,b)} = F_{iam}F_{jbm} - F_{ibm}F_{jam},
\]
where there is no summation over \(m\) in the right-hand side. Define a matrix \(B^N = (B_{ij})_{i,j=1}^N\) with the entries given as a linear combination of the third order derivatives of \(F\):
\[
B_{ij} = \sum_{k=1}^{N} A^k F_{kij}, \tag{6.5}
\]
for some functions \(A^k = A^k(x^1, \ldots, x^N)\). Now we will investigate when there exists such a combination \(B\) so that equations \((6.2)\) imply equations \((6.1)\). For that it is sufficient to deduce that the matrix \(B\) is proportional to the identity.

Fix \(i_0 \in \mathbb{N}, 1 \leq i_0 \leq N\). Let \(P\) be the \((N-1) \times N\) matrix \(P = (F_{i_0ij}), 1 \leq i, j \leq N, i \neq i_0\). Define
\[
A^k = (-1)^{k+1} \det P_k, \tag{6.6}
\]
where the matrix \(P_k\) is obtained from the matrix \(P\) by removing its \(k\)-th column.

The following statement takes place.

**Lemma 6.1.** For any function \(F = F(x^1, \ldots, x^N)\) which satisfies the commutativity equations \((6.2)\) the following relation holds
\[
\det \begin{pmatrix}
F_{aiar} & F_{biar} & F_{ciar} \\
F_{aiot} & F_{biot} & F_{ciot} \\
F_{art} & F_{brt} & F_{crt}
\end{pmatrix} = - \sum_{m \neq t} \det \begin{pmatrix}
F_{aiam} & F_{biom} & F_{ciom} \\
F_{arm} & F_{brm} & F_{crm}
\end{pmatrix},
\]
where \(1 \leq a, b, c \leq N, 1 \leq r < t \leq N, \) and \(r, t \neq i_0\).

**Proof.** By applying the first row expansion and the commutativity equations we get
\[
\det \begin{pmatrix}
F_{aiar} & F_{biar} & F_{ciar} \\
F_{aiot} & F_{biot} & F_{ciot} \\
F_{art} & F_{brt} & F_{crt}
\end{pmatrix} = F_{aior} \det \begin{pmatrix}
F_{biot} & F_{ciot} \\
F_{brt} & F_{crt}
\end{pmatrix} - F_{biar} \det \begin{pmatrix}
F_{aiot} & F_{ciot} \\
F_{art} & F_{crt}
\end{pmatrix}
\]
\[
+ F_{ciar} \det \begin{pmatrix}
F_{aiot} & F_{biot} \\
F_{art} & F_{brt}
\end{pmatrix} = F_{aior} \sum_{m \neq t} [F_{ij}, F_{rj}]_{(b,c)}^{(m)} - F_{biar} \sum_{m \neq t} [F_{ij}, F_{rj}]_{(a,c)}^{(m)} + F_{ciar} \sum_{m \neq t} [F_{ij}, F_{rj}]_{(a,b)}^{(m)}
\]
\[
= - \sum_{m \neq t} \det \begin{pmatrix}
F_{aiar} & F_{biar} & F_{ciar} \\
F_{aiot} & F_{biot} & F_{ciot} \\
F_{art} & F_{brt} & F_{crt}
\end{pmatrix}.
\]
\[\square\]

The following statement takes place.
Lemma 6.2. Suppose that a function $F = F(x^1, \ldots, x^N)$ satisfies the commutativity equations (6.2). Let $1 \leq r < t \leq N$, $r, t \neq i_0$. Let $N \times N$ matrix $Q$ be obtained from the matrix $P$ by inserting the $i_0$-th row $R_{i_0} = (F_{1i_0}, \ldots, F_{N i_0})$. Then $\det Q = 0$.

Proof. Let $D = \det Q$. Let $R_i$ denote the $i$-th row in the matrix $Q$. Let us perform the Laplace expansion of $D$ along the rows

$$R_r = (F_{i_0 1r}, F_{i_0 2r}, \ldots, F_{i_0 Nr}),$$

$$R_t = (F_{i_0 1t}, F_{i_0 2t}, \ldots, F_{i_0 Nt}),$$

and the row $R_{i_0}$. For subsets $S, T \subset [N] := \{1, \ldots, N\}$ we define $Q_{ST}$ to be the submatrix of $Q$ defined by deleting rows $S$ and columns $T$. Let $I = \{r, t, i_0\}$, $J = \{a, b, c\}$ for some $1 \leq a < b < c \leq N$. By applying the Laplace expansion to $Q$ we get

$$D = \varepsilon \sum_J \sigma_J \det Q_{IJ} \det \begin{pmatrix} F_{aio} & F_{bi0} & F_{ci0} \\ F_{ai0} & F_{bi0} & F_{ci0} \\ F_{ai0} & F_{bi0} & F_{ci0} \end{pmatrix},$$

where $\varepsilon = \pm 1$ is determined by the relative order of $r, t$ and $i_0$, and $\sigma_J = (-1)^s$ with $s = \sum_{i \in I} i + \sum_{j \in J} j$. Now by Lemma 6.1 the determinant (6.7) can be rewritten as

$$D = -\varepsilon \sum_J \sigma_J \det Q_{IJ} \left( \sum_{m \neq t} \det \begin{pmatrix} F_{aio} & F_{bi0} & F_{ci0} \\ F_{ai0} & F_{bi0} & F_{ci0} \\ F_{ai0} & F_{bi0} & F_{ci0} \end{pmatrix} \right),$$

$$= -\sum_{m \neq t} \det Q^{(m)},$$

where the matrix $Q^{(m)}$ is obtained from the matrix $Q$ by replacing rows $R_{i_0}$ and $R_t$ as follows:

$$R_{i_0} \rightarrow \tilde{R}_{i_0} = (F_{1m}, \ldots, F_{Nm}),$$

$$R_t \rightarrow \tilde{R}_t = (F_{i_0 1m}, \ldots, F_{i_0 Nm}).$$

Note that the row $\tilde{R}_t$ is equal to the $m$-th row of the matrix $Q^{(m)}$. Hence $\det D = 0$. \qed

The following statement takes place.

Proposition 6.3. Assume that the function $F = F(x^1, \ldots, x^N)$ satisfies the commutativity equations (6.2). Assume also that the rank of the matrix $P$ is $N - 1$. Then matrix $B$ with the entries given by formulae (6.5), (6.6) is diagonal.

Proof. Let us assume that $i_0 = 1$, the general case can be dealt with similarly. Consider the system of linear equations

$$B_{1m} = \sum_{k=1}^N A^k F_{1km} = 0,$$

for some functions $A^k = A^k(x^1, \ldots, x^N)$. The system (6.8) represents a homogeneous system of $N - 1$ linear equations in variables $A^k$. The assumption that the rank of the matrix $P = (F_{ij})$, where $2 \leq i \leq N$, $1 \leq j \leq N$, is $N - 1$ implies that the system (6.8) has a non-trivial solution. Moreover, this unique solution is up to proportionality.

Now, fix $2 \leq s \leq N$. The direct substitution of the functions $A^k$ given by formula (6.6) into the right-hand side of relation (6.8) gives a row expansion of the determinant of a matrix.
with the repeated rows, hence the equation $B_{1s} = 0$ is satisfied. Note also that $A^k \neq 0$ for some $k$ since the rank of the matrix $P$ is $N - 1$.

Now we will show that off-diagonal entries $B_{rt} = 0$, where $2 \leq r < t \leq N$. In order to do so, we add a row corresponding to the non-diagonal entry $B_{rt}$ to the coefficient matrix $P$ of the linear system (6.8) and we will show that the resulting matrix is singular. This will imply the existence of a non-trivial solution to the resulting system of $N$ equations. Indeed, as the first $N - 1$ equations have a unique solution given by (6.6) up to proportionality, it also has to solve the last equation. Thus we consider equations (6.8) together with

$$B_{rt} = \sum_{k=1}^{N} A^k F_{krt} = 0$$

(6.9)

as a system of linear equations for functions $A^k$. Its coefficient matrix is singular by Lemma 6.2. This proves the statement.

□

The following statement gives a further property of the matrix $B$.

**Proposition 6.4.** Under the assumptions of Proposition 6.3 we have

$$B_{11} = B_{pp}$$

for all $1 \leq p \leq N$.

**Proof.** Let us assume that $i_0 = 1$, the general case can be dealt with similarly. Let us first consider the case $p = 2$. Since the matrix $P$ has rank $N - 1$, this implies that there exists some $q$ ($1 \leq q \leq N$) such that $F_{12q} \neq 0$. Following the idea of the proof of Proposition 6.3 let us consider the following set of homogeneous equations:

$$B_{1m} = 0, \quad 2 \leq m \leq N,$$

$$F_{12q}(B_{11} - B_{22}) = 0.$$  

(6.10)

It is sufficient to show that the coefficient matrix $M$ corresponding to equations (6.10) considered as linear equations for $A^k$ is singular. We have

$$M = \begin{pmatrix}
F_{112} & F_{212} & \cdots & F_{N12} \\
F_{113} & F_{213} & \cdots & F_{N13} \\
\vdots & \vdots & \ddots & \vdots \\
F_{11N} & F_{21N} & \cdots & F_{N1N} \\
F_{12q}(F_{111} - F_{122}) & F_{12q}(F_{211} - F_{222}) & \cdots & F_{12q}(F_{N11} - F_{N22})
\end{pmatrix}.$$

Let $D = \det M$. From the identity (6.4) we have

$$F_{12q}F_{111} = \sum_{m=1}^{N} F_{12m}F_{1qm} - \sum_{m=2}^{N} F_{11m}F_{2qm}.$$  

(6.11)

Similarly, we have

$$F_{12q}F_{222} = \sum_{m=1}^{N} F_{12m}F_{2qm} - \sum_{m \neq 2} F_{1qm}F_{22m}.$$  

(6.12)
Let \( R_i \) denote the \( i^{th} \) row in the matrix \( M \). We have
\[
R_i = (F_{11(i+1)}, F_{21(i+1)}, \ldots, F_{N1(i+1)}), \quad 1 \leq i \leq N - 1, \\
R_N = (r_{N1}, r_{N2}, \ldots, r_{NN}),
\]
where
\[
\begin{align*}
    r_{N1} &= \sum_{m \neq 2} F_{1qm} F_{12m} - \sum_{m \neq 1} F_{11m} F_{2qm}, \\
    r_{N2} &= \sum_{m \neq 2} F_{1qm} F_{22m} - \sum_{m \neq 1} F_{12m} F_{2qm}, \\
    r_{Nk} &= F_{12q} (F_{k11} - F_{k22}), \quad 3 \leq k \leq N
\end{align*}
\]
by applying formulas (6.11), (6.12). Now let us perform the following row operation on the matrix \( M \) and let \( \tilde{M} \) be the resulting matrix:
\[
R_N \rightarrow \tilde{R}_N = R_N - F_{11q} R_1 + \sum_{i=2}^{N} F_{2qi} R_{i-1}.
\]
Let \( \tilde{r}_{Nk} \) be the \( k^{th} \) element in the row \( \tilde{R}_N \) of the matrix \( \tilde{M} \). We have
\[
\begin{align*}
    \tilde{r}_{N1} &= \sum_{m \neq 1,2} F_{1qm} F_{12m}, \\
    \tilde{r}_{N2} &= \sum_{m \neq 1,2} F_{1qm} F_{22m}, \\
    \tilde{r}_{Nk} &= \sum_{m=1}^{N} F_{2qm} F_{1km} - F_{12q} F_{22k} - F_{11q} F_{12k}, \quad 3 \leq k \leq N
\end{align*}
\]
Let \( S_{2m} = (F_{12m}, F_{22m}, \ldots, F_{N2m}) \), where \( 3 \leq m \leq N \). By Lemma 6.2 row \( S_{2m} \) is a linear combination of the rows of the matrix \( P \). Therefore one can add the row \( S_{2m} \) to the last row of the matrix \( \tilde{M} \) without changing its determinant \( D \). Let us add the rows \( -F_{1qm} S_{2m}, m = 3, \ldots, N \), consecutively to the last row of \( \tilde{M} \). The last row \( (\tilde{r}_{N1}, \tilde{r}_{N2}, \ldots, \tilde{r}_{NN}) \) of the resulting matrix has the form
\[
\begin{align*}
    \hat{r}_{N1} &= 0, \quad \hat{r}_{N2} = 0, \\
    \hat{r}_{Nk} &= \sum_{m=1}^{N} F_{2qm} F_{1km} - \sum_{m=1}^{N} F_{1qm} F_{2km} = -[F_1, F_2]_{(q,k)}, \quad 3 \leq k \leq N
\end{align*}
\]
by formula (6.3). It follows from the commutativity equations that \( D = 0 \). This proves that \( B_{11} = B_{22} \). Similarly, one can prove that \( B_{11} = B_{pp} \) for all \( p \). \( \square \)

As a corollary of Propositions 6.3 and 6.4 the following statement takes place.

**Theorem 6.5.** Under the assumptions of Proposition 6.3 the matrix \( B \) given by formulas (6.5), (6.6) is proportional to the identity matrix.

We also have the following result.

**Proposition 6.6.** Under the assumptions of Proposition 6.3 suppose also that there exists a non-degenerate linear combination \( G = \eta^k F_k \) for some functions \( \eta^k, (1 \leq k \leq N) \). Then matrix \( B \) given by formulas (6.5), (6.6) is a non-zero multiple of the identity matrix.
Proof. From Theorem 6.5 we know that the matrix $B$ is proportional to the identity matrix. It remains to show that $B$ is not the zero matrix. Let $B_{ij} = A^kF_{ijk} = h\delta_{ij}$ for some function $h = h(x)$. Assume that $h = 0$. Then $A^kF_{ijk} = 0$. Hence $\eta^lA^kF_{ijk} = 0$, which means that the non-zero vector $(A^1, \ldots, A^N)$ belongs to the kernel of the form $G$ (cf. a similar argument in \[14\]). Therefore $G$ is degenerate, which contradicts the assumption. Hence $h \neq 0$ and the statement follows. □

The following theorem is a corollary of Theorem 6.5 and Proposition 6.6, and it explains that a function $F$ satisfying the commutativity equations also solves the WDVV equations.

Theorem 6.7. Under the assumptions of Proposition 6.3 suppose that there exists a non-degenerate linear combination $G = \eta^kF_k$ for some functions $\eta^k$. Then $F$ is a solution of WDVV equations (6.1) where the matrix $B$ is given by formulas (6.5), (6.6).

Remark 6.8. Note that under the assumptions of Theorem 6.7, function $F$ also satisfies the generalized WDVV equations

\[ F_iF_j^{-1}F_k = F_kF_j^{-1}F_i, \quad i, j, k = 1, \ldots, N \]

provided that matrices $F_j$ are non-degenerate. Indeed these equations follow from equations (6.1) by the result from \[18\] (see also \[17\]). It also follows that $F$ satisfies the WDVV equations

\[ F_iG^{-1}F_j = F_jG^{-1}F_i, \quad i, j = 1, \ldots, N \]

for any non-degenerate linear combination $G = a^iF_i$.

7. Existence of the identity field

In this Section we define a natural multiplication on the tangent plane $T_xV$ associated with a solution $F$ of the commutativity equations. We find the identity vector field of this multiplication and establish that it is proportional to the vector field $A^k\partial_{x^k}$, where functions $A^k$ are given by formula (6.6). Thus we will express the identity vector field in terms of $F$.

For any functions $u = (u^1, \ldots, u^N), v = (v^1, \ldots, v^N): V \to V$, consider vector fields $\partial_u = u^i\partial_{x^i}, \partial_v = v^i\partial_{x^i} \in \Gamma(TV)$. Let us define the following multiplication on the tangent space $T_xV$ for generic $x \in V$:

\[ \partial_u \ast \partial_v = u^i v^j F_{ijk}\partial_{x^k}. \] (7.1)

Note that multiplication (7.1) defines a commutative associative algebra on $T_xV$ if $F$ satisfies commutativity equations (6.2).

Consider a vector field

\[ e = e^k\partial_{x^k}, \] (7.2)

where $e^k = e^k(x^1, \ldots, x^N)$ are some functions. Consider an $N \times N$ matrix $B = (B_{ij})_{i,j=1}^N$ given by

\[ B_{ij} = e(F_{ij}) = e^kF_{ijk}, \quad i, j = 1, \ldots, N. \] (7.3)

Proposition 7.1. The following statements are equivalent:

1. The matrix $B$ with entries given by (7.3) is equal to the identity matrix,
2. The vector field $e$ given by formula (7.2) is the identity vector field of the multiplication (7.1).
Proof. From relations (7.1), (7.2) and (7.3) we have
\[ e \ast \partial_v = e^i v^j \partial_{x^j} = e^i v^j F_{ijk} \delta^{kl} \partial_{x^l} = B_{jkl} v^j \delta^{kl} \partial_{x^l}. \] 
(7.4)
Let us firstly assume that \( B_{jkl} = \delta_{jk} \). Then relation (7.4) reduces to
\[ e \ast \partial_v = v^j \partial_{x^j} = \partial_v. \]
That is statement (2) follows from (1).
Secondly, assume that \( e \) is the identity vector field of the multiplication (7.1). Then from relation (7.3) we have
\[ e \ast \partial_v = B_{jkl} v^j \partial_{x^k} = \partial_v = v^j \partial_{x^j}. \]
This implies that \( B_{jkl} = \delta_{jk} \), that is statement (1) holds. \( \square \)

Proposition 7.1 allows us to reformulate Theorem 6.5 as follows.

**Theorem 7.2.** Under the assumptions of Proposition 6.3 there exists the identity vector field \( e = e^k \partial_{x^k} \) for the multiplication (7.1). It has the form \( e^k = h^{-1} A^k \), where \( A^k \) is given by formula (6.6) and \( h = A^k F_{kii} \) (for any \( i = 1, \ldots, N \)).

Now we are going to generalize Theorem 6.7 to the case of an arbitrary constant metric \( g \) in place of the standard metric \( \delta_{ij} \). Thus we start with equations of the form
\[ F_{ij \alpha \beta} g^{\alpha \beta} F_{\beta \delta \epsilon} = F_{ij \alpha \beta} g^{\alpha \beta} F_{\beta \delta \epsilon}, \quad \text{quad} 1 \leq i, j, k, l \leq N. \] 
(7.5)
We will show that matrix \((g_{\alpha \beta})_{\alpha, \beta=1}^N\) can be represented as a linear combination of the matrices \( F_i \) under some non-degeneracy assumptions.

**Theorem 7.3.** Let \( F = F(x^1, \ldots, x^N) \) be a function on \( V \) which satisfies equations (7.5) for some constant symmetric non-degenerate matrix \((g_{\alpha \beta})\). Define new coordinates
\[ y^i = \tilde{C}^i_j x^j, \] 
(7.6)
where the matrix \( \tilde{C} = (\tilde{C}^i_j) \) satisfies the relations \( \tilde{C}^\alpha_i \tilde{C}^\beta_j g^{ij} = \delta^{\alpha \beta} \), where \( 1 \leq \alpha, \beta \leq N \). Let \( \tilde{F}(y) = F(x) \). Suppose that there exists \( i_0, 1 \leq i_0 \leq N, \) such that the matrix \((\tilde{F}_{aij}(y))\) has rank \( N-1 \), where \( \tilde{F}_{aij} = \frac{\partial \tilde{F}}{\partial y^i y^j} g^{ij} \) and \( 1 \leq i, j \leq N, i \neq i_0 \). Then there exists a unique vector field \( e = e^k(x) \partial_{x^k} \) such that
\[ e(F_{lm}) = e^k F_{klm} = g_{lm}, \]
where \((g_{lm})\) is the inverse matrix for \((g^{\alpha \beta})\).

**Proof.** Let \( C = \tilde{C}^{-1} = (C^i_k) \). Then \( x^i = C^i_j y^j \). We also have
\[ \partial_{x^j} = \tilde{C}^i_j \partial_{y^i}, \quad \partial_{y^i} = C^i_j \partial_{x^j}. \] 
(7.7)
From (7.7) we have the following relations:
\[ F_j(x) = \tilde{C}^i_j \partial_{y^i} \tilde{F}(y) = \tilde{C}^i_j \tilde{F}_j(y). \]
Hence,
\[ F_{pjk}(x) = \tilde{C}^m_p \tilde{C}^i_j \tilde{C}^l_k \tilde{F}_{mjl}(y). \] 
(7.8)
By multiplying relation (7.8) by \( C^i_a C^j_b C^k_c \) we get
\[ \tilde{F}_{abc}(y) = C^p_a C^q_b C^r_c F_{pjk}(x). \]
By applying relation (7.8) we rewrite equation (7.5) as
\[ \hat{C}_p^i \hat{C}_q^j \hat{C}_r^a \hat{C}^b \tilde{F}_{pqr} g^{\alpha \beta} \hat{C}_a^\beta \hat{C}_b^\alpha \hat{C}_r \hat{C}_q \hat{C}_p \tilde{F}_{abd} = \hat{C}_s^m \hat{C}_k^a \hat{C}_q \hat{C}_r^\alpha \hat{C}^b \tilde{F}_{sqr} g^{\alpha \beta} \hat{C}_a^\beta \hat{C}_b^\alpha \hat{C}_r \hat{C}_q \hat{C}_p \tilde{F}_{abd}. \] (7.9)

It follows from the relation \( \hat{C}_i^\alpha \hat{C}_j^\beta g^{ij} = \delta^{\alpha \beta} \) that\( \hat{C}_l^a \hat{C}_m^b \delta_{lm} = g_{ab}. \) (7.10)

Hence we reduce equation (7.9) to
\[ \hat{C}_p^i \hat{C}_b^k \tilde{F}_{pqr} \tilde{F}_{rbd} = \hat{C}_s^m \hat{C}_q \hat{C}_r^\alpha \hat{C}^b \tilde{F}_{sqr} \tilde{F}_{rbd}. \] (7.11)

By multiplying equation (7.11) by \( C_n^i C_m^k \) we get
\[ \tilde{F}_{nqr} \tilde{F}_{rmd} = \tilde{F}_{mqr} \tilde{F}_{rmd}, \]
that is \( \tilde{F}_m \) and \( \tilde{F}_n \) commute. Now since the rank of the matrix \( (\tilde{F}_{ioj}) \) is \( N - 1 \), it follows by Theorem 6.5 that there exists a unique vector field \( e = e^i(y) \partial_y \) such that
\[ e(\tilde{F}_{\alpha \beta}(y)) = e^j(y) \tilde{F}_{j\alpha \beta}(y) = \delta_{\alpha \beta}. \]

From relation (7.8) we have
\[ C_j^i \partial_x F_{ab}(x) = \hat{C}_a^i \hat{C}_b^m \tilde{F}_{jlm}(y). \]

This equation implies that
\[ e(F_{ab}(x)) = e^j C_j^i \partial_x F_{ab}(x) = e^j \tilde{F}_{jlm}(y) \hat{C}_a^i \hat{C}_b^m = \delta_{lm} \hat{C}_a^i \hat{C}_b^m = g_{ab} \]
by relation (7.10). This proves the theorem. \( \square \)

**Remark 7.4.** We note that the maximality of the rank of the matrix \( P \) is sufficient but not necessary for the existence of the identity field. Indeed, in the case of two-dimensional Frobenius manifold consider the function \( F \) given by
\[ F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + f(t^2) \] (7.12)
with some function \( f(t^2) \). We have equation \( F_1 G F_2 = F_2 G F_1 \), where the matrix entries \( (F_i)_{kl} = \frac{\partial^3 F}{\partial t^i \partial t^k \partial t^l} \), and
\[ G = G^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = F_1. \] (7.13)

Now let
\[ C = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{i} & 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} i & -\frac{i}{2} \\ 1 & \frac{1}{2} \end{pmatrix} \]
be the matrices of the change of variables such that
\[ t^i = C_i^i x^j, \quad x^i = \hat{C}_i^j t^j, \] (7.14)
where \((x^1, x^2)\) is a new coordinate system and the matrix \(G = (g^{ij})\) satisfies the relation 
\[ \bar{C}_i^\alpha \bar{C}_j^\beta g^{ij} = \delta^{\alpha\beta}. \]
Let \(\bar{F}(x) = F(t)\). Then we have
\[
\bar{F}_{112} = \frac{\partial^3 \bar{F}}{\partial x^1 \partial x^1 \partial x^2} = \frac{1}{4} - f'''(ix^1 + x^2),
\]
\[
\bar{F}_{122} = \frac{\partial^3 \bar{F}}{\partial x^1 \partial x^2 \partial x^2} = -\frac{i}{4} + if'''(ix^1 + x^2).
\]
Note that the matrix \((\bar{F}_{112}, \bar{F}_{122})\) has rank zero if \(f(t^2) = \frac{1}{24}(t^2)^3 = \frac{1}{24}(ix^1 + x^2)^3\). Nonetheless \(e = \partial_{t^1} = i\partial_{x^1} + \partial_{x^2}\) is the identity field.

Remark 7.5. The maximality of the rank of the matrix \(P\) condition may be satisfied in the case of a family of non-semisimple algebras. An example is given by prepotential \((7.12)\) with \(f = 0\), as it follows from considerations in Remark \(7.4\)

8. Identity field for non-simply laced root systems and their projections

We are going to relate the identity field for a solution of commutativity equations and a restriction of such a solution. Firstly, we have the following statement. 

Lemma 8.1. Let \(F(x)\) be a function and \(e = e^k \partial_{x^k}\) be a vector field such that \(e(F_{ij}) = \delta_{ij}\), where \(1 \leq i, j \leq N\). Let \(\tilde{x}^k = C_k^i x^i\) for a matrix \(C = (C_k^i) \in O(N, \mathbb{C})\). Let \(\bar{F}(\tilde{x}) = F(x)\). Then \(e(\tilde{F}_{\mu\nu}) = \delta_{\mu\nu}\), where \(1 \leq \mu, \nu \leq N\).

Proof. We have
\[ e^k F_{ijk} = \delta_{ij}. \quad (8.1) \]
By relation \((2.4)\) we have \(\partial_{x^i} = C_i^k \partial_{x^k}\). Hence we have
\[ F_{ijk} = C_k^i C_j^i \tilde{F}_{\mu\nu}. \quad (8.2) \]
Then by formula \((8.2)\) relation \((8.1)\) can be written as
\[ e^k C_k^i C_j^i \tilde{F}_{\mu\nu} = \delta_{ij}. \quad (8.3) \]
Let \(\tilde{C} = C^{-1} = (\tilde{C}_j^i)\). Multiply equality \((8.3)\) by \(\tilde{C}_\mu^i \tilde{C}_\nu^j\). We get
\[ e^k C_k^i \tilde{F}_{\mu\nu} = \delta_{\mu\nu}. \quad (8.4) \]
since \(\tilde{C} \in O(N, \mathbb{C})\). Hence equality \((8.4)\) becomes
\[ e^k C_k^i \tilde{F}_{\mu\nu} = \delta_{\mu\nu}. \quad (8.5) \]
Note that \(e = e^k C_k^i \partial_{x^i}\). We have by relation \((8.5)\) that \(e(\tilde{F}_{\mu\nu}) = \delta_{\mu\nu}\) as required. 

Let \(B = A \cap W\) be a subsystem of \(A\) for some \(n\)-dimensional linear subspace \(W = \langle B \rangle \subseteq V\). Let 
\[ W_B := \{ x \in V : \langle \beta, x \rangle = 0 \quad \forall \beta \in B \}. \]
Let \(f_1, \ldots, f_n\) be an orthonormal basis of the space \(W_B\), and let \(\xi^1, \ldots, \xi^n\) be the corresponding orthonormal coordinates in \(W_B\). Let us extend the orthonormal basis in \(W_B\) to an orthonormal basis \(f_1, \ldots, f_n, f_{n+1}, \ldots, f_N\) in \(V\) and let \(\xi^1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_N\) be the corresponding orthonormal coordinates in \(V\). The following statement takes place.
Proposition 8.2. Let a function $F$ be given by formula (2.2). Let $e = e(z), z \in V$, be a vector field such that $e(F_{ij}) = \delta_{ij}$ for all $i, j = 1, \ldots, N$, where $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$. Let $F(x^1, \ldots, x^N)$ be such that $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$. Let $\xi = e|_{W_B} \in \Gamma(T_B W_B)$. Then $\xi = e|_{W_B} \in \Gamma(T_B W_B)$. Let $F_{ij}$ be such that $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$ and function $F_B$ is given by formula (5.1).

Proof. Let $C = (C^k_i) \in O(N, \mathbb{C})$ be such that $\xi^k = C^k_i x^i$. By Proposition 8.1 we have $e(\hat{F}_{ij}) = \delta_{ij}$, where $\hat{F}_{ij} = \frac{\partial^2 F}{\partial \xi^i \partial \xi^j}$, and $i, j = 1, \ldots, N$. Hence $\hat{e}(\hat{F}_{ij}|_{W_B}) = \delta_{ij}, 1 \leq i, j \leq n$, which implies the statement since $\hat{F}|_{W_B} = F_B$ and $\hat{F}_{ij}|_{W_B} = (F_B)_{ij}$. □

In the next proposition we give a formula for the identity field for the multiplication (7.1) corresponding to the root system $F_4$, see [1] for a proof.

Proposition 8.3. The matrix $B = h^{-1} \sum_{k=1}^{4} B^k F_k$ is the identity matrix in dimension four in the following cases:

- $F$ has the form (2.2) corresponding to $A = F_4^+$ with the condition $r = -2q, q \neq 0$, where
  \[
  B^k = \sin x^k \left( \cos x^k (-1 + \sum_{i \neq k} \cos 2x^i) - 2 \prod_{i \neq k} \cos x^i \right), \quad k = 1, 2, 3, 4,
  \]
  \[
  h(x) = 6q + \frac{1}{2} \sum_{\alpha \in F_4^+} c_{\alpha} \cos (2\alpha, x).
  \]

- $F$ has the form (2.2) corresponding to $A = F_4^+$ with the condition $r = -4q, q \neq 0$, where
  \[
  B^k = \sin x^k \left( \cos x^k + 2 \prod_{i \neq k} \cos x^i \right), \quad k = 1, 2, 3, 4,
  \]
  \[
  h(x) = -q \left( 6 + \sum_{i=1}^{4} \cos 2x^i + 8 \prod_{i=1}^{4} \cos x^i \right).
  \]

Solutions of the WDVV equations corresponding to the root system $BC_n$ and its deformation $BC_n(q, r, s; m)$ were found in [3]. In the case of the root system $F_4$ and its projections we get new solutions of the WDVV equations.

Theorem 8.4. Function (2.2) corresponding to $A = F_4^+$ or any of its 3-dimensional projections $(F_4, A_1)_1$, $(F_4, A_1)_2$ satisfies WDVV equations (6.1) if $r = -2q$ or $r = -4q, q \neq 0$.

Proof. It was proven in [4] that function (2.2) for the collection $A = F_4^+$ satisfies commutativity equations (6.2) if $r = -2q$ or $r = -4q$. For $A = F_4^+$ the statement follows by Proposition 8.3. It is easy to see that for the three-dimensional restrictions $A = (F_4, A_1)_{1,2}$ the assumptions of Proposition 8.2 hold. The statement follows. □

Now we give the identity vector field for all the non-simply laced root systems as well as their projections. In the case of root system $F_4$ it can be checked that the components of the identity field given by the next theorem are equal to $h^{-1} B^k$ given by Proposition 8.3 (see [1]).
Theorem 8.5. Let function $F$ be given by (2.2). Consider a vector field $e$ given by

$$e = c_0 H^{-1} \sum_{\alpha \in \mathcal{A}} \bar{c}_\alpha \sin(2(\alpha, x)) \partial_\alpha$$

for some $c_0, \bar{c}_\alpha \in \mathbb{C}$ and

$$H = H_0 + \sum_{\alpha \in \mathcal{A}} \bar{c}_\alpha \sin^2(\alpha, x)$$

for some $H_0 \in \mathbb{C}$. Then $e(F_{ij}) = \delta_{ij}$ if

1. $\mathcal{A} = F_4^+$ given by formula (5.7) or $\mathcal{A}$ is one of the projections $(F_4, A_1)_1$, $(F_4, A_1)_2$, $(F_4, A_2)_1$, $(F_4, A_2)_2$, $(F_4, B_2)_1$, $(F_4, A_1^2)$, and

$$r = -2q \neq 0, \quad c_0 = -\frac{1}{4q}, \quad H_0 = 0, \quad \bar{c}_\alpha = c_\alpha \forall \alpha \in \mathcal{A},$$

2. $\mathcal{A}$ is the same as in (1) and

$$r = -4q \neq 0, \quad c_0 = \frac{1}{4q}, \quad H_0 = 36q, \quad \bar{c} = c_\alpha|_{q=0} \forall \alpha \in \mathcal{A},$$

3. $\mathcal{A} = G_2^+$ and

$$p = -3q \neq 0, \quad c_0 = -\frac{1}{9q}, \quad H_0 = 0, \quad \bar{c}_\alpha = c_\alpha \forall \alpha \in G_2^+, \quad \text{where } q \text{ is the multiplicity of the long roots } \sqrt{3}e_1, \frac{1}{2}(\sqrt{3}e_1 \pm 3e_2) \text{ and } p \text{ is the multiplicity of the short roots } e_2, \frac{1}{2}(\sqrt{3}e_1 \pm e_2),$$

4. $\mathcal{A}$ is the same as in (3), and

$$p = -9q \neq 0, \quad c_0 = \frac{1}{9q}, \quad H_0 = 27q, \quad \bar{c} = c_\alpha|_{q=0} \forall \alpha \in G_2^+,$$

5. $\mathcal{A} = BC_n(q, r, s; m), q \neq 0, n \geq 2,$ and

$$r = -8s - 2q \left( \sum_{i=1}^{n} m_i - 2 \right), \quad c_0 = -\frac{1}{4q}, \quad H_0 = \frac{r(2s - q)}{q}, \quad \bar{c}_\alpha = c_\alpha|_{q=s=0} \forall \alpha \in BC_n(q, r, s; m).$$

6. $\mathcal{A} = BC_1^+$ with $c_{\pm e_1} = r, c_{\pm 2e_1} = s$ and

$$c_0 = -\frac{1}{2(r + 8s)}, \quad H_0 = -\frac{r(r + 4s)}{r + 8s}, \quad \bar{c}_{e_1} = r, \bar{c}_{2e_1} = 0.$$

Theorem 8.5 follows from the identity

$$\sum_{\alpha, \beta \in \mathcal{A}} \bar{c}_\alpha c_\beta (\alpha, \beta)(\beta, u)(\beta, v) \sin(2\alpha, x) \cot(\beta, x) = c_0^{-1}H(u, v)$$

for any $u, v \in V$ for each case specified in Theorem 8.5. By Proposition 8.2 it is sufficient to establish identity (8.7) for the case when $\mathcal{A}$ is a (non-simply laced) root system. Indeed it is easy to see that the vector field $e$ given by (8.6) for $\mathcal{A} = F_4$, $BC_N$ satisfies the condition $e|_W \in \Gamma(T_s W)$ for any intersection of mirrors $W$. It is also clear that the restricted vector field $e|_W$ has the form (8.6) for the corresponding projections of the root system $\mathcal{A}$.

Case by case proof of the identity (8.7) for $\mathcal{A} = F_4$ and $\mathcal{A} = G_2$ is contained in [1]; see [3] for $\mathcal{A} = BC_N$. We are not aware of a uniform proof of Theorem 8.5 or the identity (8.7).
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