

**Study of** $B_c^* \to \psi(1S, 2S)P, \eta_c(1S, 2S)P$ **weak decays**

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Abstract

Motivated by the potential prospects of the $B_c^*$ meson samples at hadron colliders, the bottom-changing $B_c^* \to \psi(1S, 2S)P, \eta_c(1S, 2S)P$ weak decays are first studied with the perturbative QCD approach, where $P = \pi$ and $K$. It is found that branching ratio of the CKM-favored $B_c^* \to J/\psi\pi$ decay is about $\mathcal{O}(10^{-8})$, which might be measurable at the future LHC experiments.
I. INTRODUCTION

The $B_c^*$ meson consists of two heavy quarks with different flavor numbers, i.e., $\bar{b}c$ ($b\bar{c}$) for the $B_{c}^{*+}$ ($B_{c}^{*-}$) meson. The $B_{c}^*$ meson is a spin-triplet ground state ($n^{2s+1}L_J = 1^3S_1$). The $B_{c}^*$ meson lies below the $BD$ meson pair threshold. What is more, the mass difference $m_{B_{c}^*} - m_{B_c} \simeq 50 \text{ MeV}$ [1] is far less than the mass of pion. So, the $B_{c}^*$ meson decays via the strong interaction are completely forbidden. However, the $B_{c}^*$ meson decays through the electromagnetic and weak interactions are allowable within the standard model of elementary particles. The dominant magnetic dipole (M1) transition, $B_{c}^* \rightarrow B_c \gamma$, is strongly suppressed by the compact phase spaces, which results in a lifetime $\tau_{B_{c}^*} \sim O(10^{-18} \text{ s})$ [2]. Besides, the $B_{c}^*$ meson carries explicitly nonzero bottom and charm quantum numbers ($B = C = \pm 1$). Hence, the $B_{c}^*$ meson can decay by means of the flavor-changing weak transitions.

The $B_{c}^*$ meson weak decays, similar to the $B_c$ meson weak decays [3–9], can be divided into three classes: (1) the $b$ quark decay with the $c$ quark as a spectator, (2) the $c$ quark decay with the $b$ quark as a spectator, and (3) the $b$ and $c$ quarks annihilation into a virtual $W^\pm$ boson. The $B_{c}^*$ meson has a large mass. In addition, both constituent quarks of the $B_{c}^*$ meson can decay individually. Therefore, the $B_{c}^*$ meson has abundant weak decay channels. However, the $B_{c}^*$ meson weak decays have received much less attention in the past. There is no experimental measurement report [10] and few theoretical investigations concerned with the $B_{c}^*$ weak decay. Fortunately, with the high luminosity and large production cross section of the $B_{c}^*$ meson [11–14] at the running LHC, a huge amount of the $B_{c}^*$ meson data samples would be accumulated. Some of the $B_{c}^*$ meson weak decays might be explored in the future. The $B_{c}^*$ meson provides another laboratory to study the heavy flavor weak decay.

In this paper, we will study the nonleptonic $B_{c}^* \rightarrow \psi(1S, 2S)P, \eta_c(1S, 2S)P$ decays with the perturbative QCD (pQCD) approach [15–17], where $P = \pi$ and $K$. Our motivations are as follows. Firstly, due to the development of experimental instruments and technology, final states of the charmonium and the charged pion and/or kaon are easy to identify experimentally. With the advancement of high energy hadron collider experiments, the $B_{c} \rightarrow \psi(1S, 2S)P$ decays have been observed [10, 18–23]. Due to the production cross section $\sigma(B_{c}^*) \gtrsim \sigma(B_c)$ in hadron collisions [11–14], hopefully, it is anticipated that the $B_{c}^* \rightarrow \psi(1S, 2S)P, \eta_c(1S, 2S)P$ decays might be observed experimentally in the future. A theoretical study on the $B_{c}^* \rightarrow \psi(1S, 2S)P, \eta_c(1S, 2S)P$ decays is necessary to provide the future
experimental investigation with an immediate reference. Secondly, due to the relations of
$\sigma(B_c^*) \gtrsim \sigma(B_c)$ and $m_{B_c^*} \simeq m_{B_c}$,[1], one possible background for the $B_c$ meson decays might come from the $B_c^*$ meson decays into the same final states. Hence, the study of the $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays will provide some useful information for the experimental analysis on the $B_c \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays. Thirdly, as it is well known, the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{cb}|$ could be determined from the semileptonic decays of the $B$ meson to the $D^{(*)}$ meson. However, there exists a more than 3.0$\sigma$ discrepancy between the values from exclusive and inclusive determinations$^a$[10].

The $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays are induced actually by the $b \to c$ (or $\bar{b} \to \bar{c}$) transition at the quark level. The weak interaction coupling and the decay amplitudes are proportional to the CKM matrix element $|V_{cb}|$ [see Eq.(11) or Eqs.(37)-(39)]. Hence, the $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays, together with nonleptonic $B \to D^{(*)}X$ decays and semileptonic $B \to D^{(*)}\ell\nu$ decays, are expected to give more stringent constraints on the CKM matrix element $|V_{cb}|$, other parameters extracted from the $B$ meson decays, and contributions from possible new physics. Fourthly, owing to the same dynamical mechanism of the bottom quark decay, many phenomenological models used for the $B$ meson decays could, in principle, be generalized and applied to the $B_c^*$ meson weak decays. The practical applicability and reliability of the pQCD approach can be reevaluated with the $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays. Further, the $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ decays provide an opportunity to study polarization effects involved in the vector meson decays.

This paper is organized as follows. The theoretical framework and decay amplitudes with the pQCD approach are presented in Section II. Section III is devoted to the numerical results and discussion. The last section is a summary.

II. THEORETICAL FRAMEWORK

A. The effective Hamiltonian

By means of the operator product expansion and the renormalization group (RG) method, the effective Hamiltonian responsible for the $B_c^* \to \psi(1S,2S)P$, $\eta_c(1S,2S)P$ weak decays

$^a$ The values of the CKM element $|V_{cb}|$ obtained from inclusive and exclusive determinations are $|V_{cb}| = (42.2 \pm 0.8) \times 10^{-3}$ and $(39.2 \pm 0.7) \times 10^{-3}$, respectively [10].
is expressed in terms of four-quark operators with the process-independent couplings of the Wilson coefficients \( C_i \) \[24\],

\[
\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} V_{cq}^* V_{ub} \left\{ C_1 (\bar{b}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta q_\beta)_{V-A} + C_2 (\bar{b}_\alpha c_\beta)_{V-A} (\bar{u}_\beta q_\alpha)_{V-A} \right\} + \text{h.c.},
\]

(1)

where the Fermi coupling constant \( G_F \approx 1.166 \times 10^{-5} \text{ GeV}^{-2} \) \[10\]; current operator \((\bar{q}_1 q_2)_{V-A} = \bar{q}_1 \gamma_\mu (1-\gamma_5) q_2\); \( \alpha \) and \( \beta \) are color indices. The CKM factor \( V_{cb}^* V_{ub} \) can be written in terms of the Wolfenstein parameters \[10\], i.e.,

\[
V_{cb}^* V_{ud} = A \lambda^2 - \frac{1}{2} A \lambda^4 - \frac{1}{8} A \lambda^6 + \mathcal{O} (\lambda^8), \quad \text{for } P = \pi; \tag{2}
\]

\[
V_{cb}^* V_{us} = A \lambda^3 + \mathcal{O} (\lambda^8), \quad \text{for } P = K. \tag{3}
\]

The auxiliary parameter \( \mu \) separates the physical contributions into two parts. The hard contributions above the scale \( \mu \) are summarized into the Wilson coefficients \( C_i (\mu) \). Due to the properties of asymptotic freedom of QCD forces, the Wilson coefficients are, in principle, computable order by order with the RG equation improved perturbation theory as long as the scale \( \mu \) is not too small \[24\]. The physical contributions below the scale \( \mu \) are included in the hadronic matrix elements (HME) where the local four-quark operators are sandwiched between initial and final states. The participating hadrons are bounds states of partons. With the participation of the strong interaction in the transition from quarks to hadrons, especially, the presence of long-distance QCD effects and the entanglement of nonperturbative and perturbative contributions, how to properly evaluate HME is one of major tasks for a serious phenomenology of weak decays of heavy flavor hadrons.

**B. Hadronic matrix elements**

Phenomenologically, one has to turn to some approximation or assumption for the HME calculation. The Lepage-Brodsky approach \[25\] is usually applied to a hard scattering process, where a hadron transition matrix element is generally written as a convolution integral of hadron wave functions reflecting the nonperturbative contributions and hard scattering amplitudes containing perturbative contributions. In order to wipe out the endpoint singularities appearing in the collinear approximation \[26, 28\] and suppress the soft contributions, as it is argued with the pQCD approach \[13, 17\], the transverse momentum \( k_T \) of valence quarks should be retained and Sudakov factor \( e^{-S} \) should be introduced for each of the wave
functions. Finally, the $B^*_c$ weak decay amplitude can be expressed as a multidimensional integral of many parts \cite{16, 17}, including the hard effects enclosed by the Wilson coefficients $C_i$, the heavy quark decay amplitudes $H_i$, and the universal wave functions $\Phi$,

\[
\mathcal{A} \sim \sum_i \int \prod_j dk_j C_i(t) H_i(t, k_j) \Phi_j(k_j) e^{-S_j},
\]

where $t$ is a typical scale, $k_j$ is the momentum of a valence quark.

C. Kinematic variables

In the rest frame of the $B^*_c$ meson, the light-cone kinematic variables are defined as follows.

\[
p_1 = \frac{m_1}{\sqrt{2}} (1, 1, 0), \quad (5)
\]

\[
p_2 = (p_2^+, p_2^-, 0), \quad (6)
\]

\[
p_3 = (p_3^-, p_3^+, 0), \quad (7)
\]

\[
k_i = x_i p_i + (0, 0, \vec{k}_iT), \quad (8)
\]

\[
p_i^\pm = (E_i \pm p)/\sqrt{2}, \quad (9)
\]

\[\epsilon_i^\parallel = \frac{1}{\sqrt{2}} (-1, 1, 0), \quad (10)
\]

\[\epsilon_i^\perp = \frac{1}{m_2} (p_2^+ - p_2^-, 0), \quad (11)
\]

\[\epsilon_{i,2} = (0, 0, \vec{1}), \quad (12)
\]

\[s = 2 p_2 \cdot p_3, \quad (13)
\]

\[t = 2 p_1 \cdot p_2 = 2 m_1 E_2, \quad (14)
\]

\[u = 2 p_1 \cdot p_3 = 2 m_1 E_3, \quad (15)
\]

\[s t + s u - t u = 4 m_1^2 p^2, \quad (16)
\]

where for variables including the momentum $p_i$, mass $m_i$, energy $E_i$, and longitudinal (transverse) polarization vector $\epsilon_i^\parallel (\epsilon_i^\perp)$, the subscript $i = 1, 2, 3$ stands for the $B^*_c$ meson, charmonium $\psi (\eta_c)$, and pseudoscalar meson $P$, respectively; $k_i$ is the momentum of a valence quark; $x_i$ and $k_i T$ are the longitudinal momentum fraction and transverse momentum, respectively; $p$ is the center-of-mass momentum of final states; $s$, $t$ and $u$ are the Lorentz invariant parameters. The kinematic variables are displayed in Fig[2](a).
D. Wave functions

It is seen from Eq. (4) that wave functions are essential input parameters with the pQCD approach. And in general, wave functions are universal, i.e., process independent. Following the notations in Refs. [29, 30], wave functions of participating mesons are defined as

\[
\langle 0 | \bar{b}_i(0) c_j(z) | B_c^p (p, e^\parallel) \rangle = \frac{f_{B_c^p}}{4} \int d^4 k e^{-i k \cdot z} \left\{ \phi^\parallel \left[ m_{B_c^p} \Phi_{B_c^p}^V(k) - \Phi_{B_c^p}^T(k) \right] \right\}_{ji},
\]

\[
\langle 0 | \bar{b}_i(0) c_j(z) | B_c^p (p, e^\perp) \rangle = \frac{f_{B_c^p}}{4} \int d^4 k e^{-i k \cdot z} \left\{ \phi^\perp \left[ m_{B_c^p} \Phi_{B_c^p}^V(k) - \Phi_{B_c^p}^T(k) \right] \right\}_{ji},
\]

\[
\langle \psi(p, e^\parallel) | c_i(0) \bar{c}_j(z) | 0 \rangle = \frac{f_\psi}{4} \int d^4 k e^{+i k \cdot z} \left\{ \phi^\parallel \left[ m_\psi \Phi_\psi^V(k) + \Phi_\psi^T(k) \right] \right\}_{ji},
\]

\[
\langle \psi(p, e^\perp) | c_i(0) \bar{c}_j(z) | 0 \rangle = \frac{f_\psi}{4} \int d^4 k e^{+i k \cdot z} \left\{ \phi^\perp \left[ m_\psi \Phi_\psi^V(k) + \Phi_\psi^T(k) \right] \right\}_{ji},
\]

\[
\langle \eta_c(p) | c_i(0) \bar{c}_j(z) | 0 \rangle = \frac{i f_{\eta_c}}{4} \int d^4 k e^{+i k \cdot z} \left\{ \gamma_5 \left[ \hat{\phi} \Phi_{\eta_c}^a(k) + m_{\eta_c} \Phi_{\eta_c}^a(k) \right] \right\}_{ji},
\]

\[
\langle P(p) | \bar{u}_i(0) q_j(z) | 0 \rangle = \frac{i f_P}{4} \int d^4 k e^{+i k \cdot z} \left\{ \gamma_5 \left[ \hat{\phi} \Phi_P^a(k) + \mu_P \Phi_P^a(k) + \mu_P (\not{t} + \not{n} - 1) \Phi_P^t(k) \right] \right\}_{ji},
\]

where \( f_{B_c^p}, f_\psi, f_{\eta_c}, \) and \( f_P \) are decay constants; wave functions of \( \Phi_\psi^{V,T} \) and \( \Phi^a \) are twist-2; \( \Phi^{V,t} \) and \( \Phi^{P,t} \) are twist-3; \( \mu_P = m_2^2/(m_u + m_d) \) is the chiral parameter; \( n_+ \) and \( n_- \) are the positive and negative null vectors, respectively.

For the light pseudoscalar meson \( P \), only the leading twist (twist-2) distribution amplitude (DA) \( \phi^a_P(x) \) is involved in our calculation (see the Appendix). And the normalized DA \( \phi^a_P(x) \) has the following general structure \([30]\):

\[
\phi^a_P(x) = 6 x \bar{x} \left\{ 1 + \sum_{n=1} a^P_{n} C_n^{3/2}(t) \right\},
\]

where \( \bar{x} = 1 - x \) and \( t = x - \bar{x} \); the Gegenbauer moment \( a^P_{n} \) is a nonperturbative parameter. The Gegenbauer polynomials are expressed as:

\[
C_1^{3/2}(t) = 3 t, \quad C_2^{3/2}(t) = \frac{3}{2} (5 t^2 - 1), \quad \ldots
\]

The \( B_c^* \) meson and charmonium \( \psi \) and \( \eta_c \) consist of two heavy flavors. The motion of the valence quarks in these mesons should be nearly nonrelativistic. Taking a similar treatment of the nonrelativistic heavy quarkonium system \([31, 34]\), DAs for the \( B_c^* \) meson and charmonium can be written as

\[
\phi^{V,T}_{B_c^*}(x) = A x \bar{x} \exp \left\{ - \frac{\bar{x} m_b^2 + x m_c^2}{8 \omega_t^2 x \bar{x}} \right\},
\]
\[ \phi_{B^*_c}(x) = B (\bar{x} - x)^2 \exp \left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \omega_l^2 x \bar{x}} \right\}, \] (26)

\[ \phi_{B^*_c}^V(x) = C \left\{ 1 + (\bar{x} - x)^2 \right\} \exp \left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \omega_l^2 x \bar{x}} \right\}, \] (27)

\[ \phi_{\eta_c(1S)}^a(x) = \phi_{\psi(1S)}^a(x) = D x \bar{x} \exp \left\{ - \frac{m_c^2}{8 \omega_s^2 x \bar{x}} \right\}, \] (28)

\[ \phi_{\psi(1S)}^t(x) = E (\bar{x} - x)^2 \exp \left\{ - \frac{m_c^2}{8 \omega_s^2 x \bar{x}} \right\}, \] (29)

\[ \phi_{\psi(1S)}^V(x) = F \left\{ 1 + (\bar{x} - x)^2 \right\} \exp \left\{ - \frac{m_c^2}{8 \omega_s^2 x \bar{x}} \right\}, \] (30)

\[ \phi_{\eta_c(1S)}^p(x) = G \exp \left\{ - \frac{m_c^2}{8 \omega_s^2 x \bar{x}} \right\}, \] (31)

\[ \phi_{\psi(2S)}^{v,t,V,T}(x) = H \phi_{\psi(1S)}^{v,t,V,T}(x) \left\{ 1 + \frac{m_c^2}{2 \omega_s^2 x \bar{x}} \right\}, \] (32)

\[ \phi_{\eta_c(2S)}^{a,p}(x) = I \phi_{\eta_c(1S)}^{a,p}(x) \left\{ 1 + \frac{m_c^2}{2 \omega_s^2 x \bar{x}} \right\}, \] (33)

where parameter \( \omega_i \simeq m_i \alpha_s(m_i) \) determines the average transverse momentum of valence quarks according to the power counting rules of nonrelativistic QCD effective theory [35–37]; parameters \( A, B, C, D, E, F, G, H, I \) in Eqs. (25–33) could be explicitly determined with the following normalization conditions,

\[ \int_0^1 dx \phi_{B^*_c}^{v,t,V,T}(x) = 1, \] (34)

\[ \int_0^1 dx \phi_{\psi}^{v,t,V,T}(x) = 1, \] (35)

\[ \int_0^1 dx \phi_{\eta_c}^{a,p}(x) = 1. \] (36)

The shape lines of the normalized DAs of participating mesons are illustrated in Fig. 1. It is clearly seen from Fig. 1 that (1) a broad peak appears at \( x < 0.5 \) region for DAs of the \( B^*_c \) meson. (2) The shape lines of DAs for the \( \psi(1S,2S) \) and \( \eta_c(1S,2S) \) mesons are symmetric versus \( x \), which agree basically with the postulated scenario that patrons share momentum fractions according to their masses. (3) The differences between DAs of \( \phi_{\pi}^a \) and \( \phi_{\pi}^p \) arise from the flavor symmetry breaking effects representing by the Gegenbauer moment \( a_1^K \neq 0 \). (4) Owing to the exponential functions, the shape lines of DAs in Eqs. (25–33) fall quickly down to zero at endpoints \( x, \bar{x} \to 0 \). So the DAs of Eqs. (25–33) will give an effective cut for the soft contributions from the endpoints.
π and the emitted nonfactorizable topologies (c,d) where one gluon is exchanged between the spectator quark and the B meson. The helicity amplitudes for the B decay are displayed in Fig.2, including factorizable emission topologies, (a,b) where one gluon couples the B meson with the recoiled ηc meson, nonfactorizable topologies (c,d) where one gluon is exchanged between the spectator quark and the emitted π meson.

E. Decay amplitudes

The Feynman diagrams for the $B_c^* \to \eta_c \pi^+$ decay are displayed in Fig.2 including factorizable topologies (a,b) where one gluon couples the $B_c^*$ meson with the recoiled $\eta_c$ meson, nonfactorizable topologies (c,d) where one gluon is exchanged between the spectator quark and the emitted $\pi$ meson.

The Lorentz invariant amplitudes for the $B_c^* \to \psi P$, $\eta_c P$ decays are written as

$$A(B_c^* \to \psi P) = i \mathcal{F} f_{\psi} \sum_i \left\{ A_{iL}(\epsilon_{B_c^*}^\parallel, \epsilon_{\psi}^\parallel) + A_{iN}(\epsilon_{B_c^*}^\perp, \epsilon_{\psi}^\perp) + i A_{iT} \varepsilon_{\alpha\beta\mu\nu} P_{B_c^*}^\alpha P_{\psi}^\beta \epsilon_{B_c^*}^\mu \epsilon_{\psi}^\nu \right\},$$

$$A(B_c^* \to \eta_c P) = \mathcal{F} f_{\eta_c} \sum_i A_{i,P},$$

$$\mathcal{F} = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud} \pi C_F N_c f_{B_c^*} f_P,$$

where the subscript $i$ on $A_{i,j}$ corresponds to one of the indices in Fig.2 the subscript $j$ refers to different helicity amplitudes; and the expressions of building blocks $A_{i,j}$ are collected in the Appendix. The helicity amplitudes for $B_c^* \to \psi P$ decays are defined as

$$\mathcal{M}_0 = -\mathcal{F} \sum_i A_{iL}(\epsilon_{B_c^*}^\parallel, \epsilon_{\psi}^\parallel),$$
\[ \mathcal{M}_\parallel = \sqrt{2} F \sum_i A_{i,N}, \]  \hspace{1cm} (41) \]
\[ \mathcal{M}_\perp = \sqrt{2} F m_{B_c^*} p \sum_i A_{i,T}. \]  \hspace{1cm} (42) \]

III. NUMERICAL RESULTS AND DISCUSSION

In the rest frame of the $B_c^*$ meson, branching ratios for the $B_c^* \to \psi P, \eta_c P$ decays are defined as
\[
\mathcal{B}(B_c^* \to \psi P) = \frac{1}{24 \pi} \frac{p}{m_{B_c^*}^2 \Gamma_{B_c^*}} \{|\mathcal{M}_0|^2 + |\mathcal{M}_\parallel|^2 + |\mathcal{M}_\perp|^2\}, \hspace{1cm} (43)
\]
\[
\mathcal{B}(B_c^* \to \eta_c P) = \frac{1}{24 \pi} \frac{p}{m_{B_c^*}^2 \Gamma_{B_c^*}} |\mathcal{A}(B_c^* \to \eta_c P)|^2, \hspace{1cm} (44)
\]
where $\Gamma_{B_c^*}$ is the full width of the $B_c^*$ meson.

| TABLE I: The numerical values of input parameters. |
|--------------------------------------------------|
| CKM parameter[^10] | $A = 0.811 \pm 0.026$, | $\lambda = 0.22506 \pm 0.00050$, |
| $m_{B_c^*} = 6332 \pm 9$ MeV[^1], | $m_{\psi(1S)} = 3096.900 \pm 0.006$ MeV[^10], | $\Lambda_{QCD}^{(5)} = 210 \pm 14$ MeV[^10], |
| $m_b = 4.78 \pm 0.06$ GeV[^10], | $m_{\psi(2S)} = 3686.097 \pm 0.025$ MeV[^10], | $\Lambda_{QCD}^{(4)} = 292 \pm 16$ MeV[^10], |
| $m_c = 1.67 \pm 0.07$ GeV[^10], | $m_{\eta_c(1S)} = 2983.4 \pm 0.5$ MeV[^10], | $a_1^\pi = 0$[^30], |
| $f_{B_c^*} = 422 \pm 13$ MeV[^38], | $m_{\eta_c(2S)} = 3639.2 \pm 1.2$ MeV[^10], | $a_2^\pi (1$ GeV $) = 0.25 \pm 0.15$[^30], |
| $f_{\pi} = 130.2 \pm 1.7$ MeV[^10], | $m_{\pi} = 139.57$ MeV[^10], | $a_1^K (1$ GeV $) = 0.06 \pm 0.03$[^30], |
| $f_K = 155.6 \pm 0.4$ MeV[^10], | $m_K = 493.677 \pm 0.016$ MeV[^10], | $a_2^K (1$ GeV $) = 0.25 \pm 0.15$[^30], |

[^1]More predictions of the $B_c^*$ meson mass with different models can be found in Table II of Ref.[39].

The numerical values of some input parameters are listed in Table I. If it is not specified explicitly, their central values will be used in the calculation.

The decay constant $f_\psi$ is related with the branching ratio for the leptonic decay $\psi \to e^+e^-$ through the formula[^40]
\[
\mathcal{B}(\psi \to e^+e^-) = \frac{4 \pi Q_e^2 \alpha_{em}^2 f_\psi^2}{3 m_\psi \Gamma_\psi}, \hspace{1cm} (45)
\]
where $Q_e$ is the charge of the charm quark in unit of $|e|$; $\alpha_{em}$ is the fine-structure constant of the electromagnetic interaction; $\Gamma_\psi$ is full decay width of the $\psi$ meson. From the available experimental data of both $\mathcal{B}(\psi(1S) \to e^+e^-) = (5.971 \pm 0.032)\%$ and $\mathcal{B}(\psi(2S) \to e^+e^-) = \ldots$
one can obtain $f_{\psi(1S)} = (416.2 \pm 7.4) \text{ MeV}$ and $f_{\psi(2S)} = (294.6 \pm 7.2) \text{ MeV}$, respectively. The decay constant $f_{\eta_c}$ can be extracted from the branching ratio for the $\eta_c$ meson decay into two photons using the formula [40]

$$\mathcal{B}(\eta_c \to \gamma \gamma) = \frac{4 \pi Q_c^2 \alpha_{em}^2 f_{\eta_c}^2}{m_{\eta_c} \Gamma_{\eta_c}}.$$  \hfill (46)$$

With the up-to-date data of $\mathcal{B}(\eta_c(1S) \to \gamma \gamma) = (1.59 \pm 0.13) \times 10^{-4}$ and $\mathcal{B}(\eta_c(2S) \to \gamma \gamma) = (1.9 \pm 1.3) \times 10^{-4}$ [10], one can obtain $f_{\eta_c(1S)} = (337.7 \pm 18.2) \text{ MeV}$ and $f_{\eta_c(2S)} = (243.1 \pm 127.4) \text{ MeV}$, respectively.

Besides, the full width of the $B_c^+$ meson, $\Gamma_{B_c^+}$, is also an essential input parameter. Because the electromagnetic radiation process $B_c^+ \to B_c \gamma$ dominates the $B_c^+$ meson decay, an approximation $\Gamma_{B_c^+} \simeq \Gamma(B_c^+ \to B_c \gamma)$ will be used here. However, unfortunately, the photon from the $B_c^+ \to B_c \gamma$ process is not hard enough, so, it is fairly challenging to identify experimentally. The information on $\Gamma(B_c^+ \to B_c \gamma)$ comes mainly from theoretical estimations. Theoretically, the partial decay width of the spin-flip M1 transition process has the expression [2],

$$\Gamma(B_c^+ \to B_c \gamma) = \frac{4}{3} \alpha_{em} k_{\gamma}^3 \mu_h^2,$$  \hfill (47)$$

where $k_{\gamma}$ is the photon momentum in the rest frame of the $B_c^+$ meson; $\mu_h$ is the M1 moment of the $B_c^+$ meson. There are plenty of theoretical predictions on $\Gamma(B_c^+ \to B_c \gamma)$ with different approaches, such as various potential models [41–49]. However, because of the incomprehension about $\mu_h$, these estimations suffer from large uncertainties, $\Gamma(B_c^+ \to B_c \gamma) \simeq 20 \sim 80 \text{ eV}$ [41–49] (see the numbers in Tables 3 and 6 in Ref. 2). To give a quantitative estimation, a ballpark guess $\Gamma_{B_c^+} \simeq (50 \pm 30) \text{ eV}$ will be employed here for the moment, where an assumed uncertainty is given to be marginally consistent with previous results [41–49].

**TABLE II:** Branching ratios for the $B_c^+ \to \psi P$, $\eta_c P$ decays, where the theoretical uncertainties come from scale $(1 \pm 0.1)t_i$, mass $m_c$ and $m_b$, the CKM parameters, and $\Gamma_{B_c^+}$, respectively.

| final states | branching ratio | final states | branching ratio |
|--------------|----------------|--------------|----------------|
| $\psi(1S)\pi$ | $(9.16^{+0.98+1.13+0.68+13.74}_{-0.45-0.25-0.65-3.43}) \times 10^{-8}$ | $\psi(2S)\pi$ | $(3.21^{+0.39+0.26+0.24+4.81}_{-0.17-0.32-0.23-1.20}) \times 10^{-8}$ |
| $\eta_c(1S)\pi$ | $(2.22^{+0.28+0.06+0.16+3.32}_{-0.12-0.08-0.16-0.83}) \times 10^{-8}$ | $\eta_c(2S)\pi$ | $(4.59^{+0.57+0.63+0.34+6.88}_{-0.25-0.86-0.33-1.72}) \times 10^{-9}$ |
| $\psi(1S)K$ | $(7.28^{+0.80+0.57+0.58+10.93}_{-0.36-0.46-0.55-2.73}) \times 10^{-9}$ | $\psi(2S)K$ | $(2.37^{+0.28+0.08+0.19+3.55}_{-0.13-0.18-0.18-0.89}) \times 10^{-9}$ |
| $\eta_c(1S)K$ | $(1.67^{+0.21+0.04+0.13+2.51}_{-0.09-0.07-0.13-0.63}) \times 10^{-9}$ | $\eta_c(2S)K$ | $(3.42^{+0.43+0.45+0.27+5.12}_{-0.19-0.66-0.26-1.28}) \times 10^{-10}$ |
Our numerical results are presented in Table II, where the uncertainties come from the typical scale $(1 \pm 0.1)t_i$, mass $m_c$ and $m_b$, the CKM parameters, and the decay width $\Gamma_{B^*_c}$, respectively. The followings are some comments.

(1) Because of the hierarchical relations between the CKM matrix elements $|V_{ud}| > |V_{us}|$, branching ratios for the $B^*_c \to \psi K, \eta_c K$ decays are generally an order of magnitude less than those for the $B^*_c \to \psi \pi, \eta_c \pi$ decays with the same charmonium in the final states, i.e.,

$$\mathcal{B}r(B^*_c \to \psi \pi) > \mathcal{B}r(B^*_c \to \psi K),$$  \hspace{1cm} (48)

$$\mathcal{B}r(B^*_c \to \eta_c \pi) > \mathcal{B}r(B^*_c \to \eta_c K).$$  \hspace{1cm} (49)

Due to the hierarchical relations between decay constants $f_{\psi(1S)} > f_{\psi(2S)}$ and $f_{\eta_c(1S)} > f_{\eta_c(2S)}$, along with relatively compact phase spaces for final $\psi(2S)P, \eta_c(2S)P$ states with respect to those for the $\psi(1S)P, \eta_c(1S)P$ states, there are some hierarchical relations, i.e.,

$$\mathcal{B}r(B^*_c \to \psi(1S)P) > \mathcal{B}r(B^*_c \to \psi(2S)P),$$  \hspace{1cm} (50)

$$\mathcal{B}r(B^*_c \to \eta_c(1S)P) > \mathcal{B}r(B^*_c \to \eta_c(2S)P),$$  \hspace{1cm} (51)

for the same final pseudoscalar meson $P$.

In addition, due to the conservation of angular momentum, there are more wave amplitudes contributing to the $B^*_c \to \psi P$ decays than the only $p$-wave amplitudes contributing to the $B^*_c \to \eta_c P$ decays. So there are some hierarchical relations, i.e.,

$$\mathcal{B}r(B^*_c \to \psi(1S)P) > \mathcal{B}r(B^*_c \to \eta_c(1S)P),$$  \hspace{1cm} (52)

$$\mathcal{B}r(B^*_c \to \psi(2S)P) > \mathcal{B}r(B^*_c \to \eta_c(2S)P),$$  \hspace{1cm} (53)

for the same final pseudoscalar meson $P$.

(2) The branching ratios of the $B^*_c \to \psi P, \eta_c P$ decays are several orders of magnitude less than the branching ratios of the $B_c \to \psi P, \eta_c P$ decays \[7, 8\]. So, the possible influence from the $B^*_c$ meson decays could be safely neglected when the $B_c \to \psi P, \eta_c P$ decays are studied experimentally. On the other hand, with the improvement of detection ability and analytical techniques, rare $B$ decay modes with branching ratio $\sim O(10^{-8})$, such as the $B^0 \to K^+K^-$ decay \[50\], can be accessible at the LHCb experiments now. The branching ratios of the $B^*_c \to \psi \pi$ decays can reach up to $O(10^{-8})$. In addition, according to the estimation of Ref. \[14\], the production cross section of the $B^*_c$ meson is about 30 nb at LHC. It is
promisingly expected to have more than $10^{10} \ B^*_c$ meson samples, corresponding to hundreds of the $B^*_c \rightarrow \psi(1S)\pi, \psi(2S)\pi$ decays, per $\text{ab}^{-1}$ data accumulated at LHC. The possible background from the $B_c \rightarrow \psi\pi$ decays might, in principle, be excluded from the invariant mass of final states. So, even given the detection efficiency, the $B^*_c \rightarrow J/\psi\pi$ decay is also measurable, although very challenging, at the future LHC experiments.

(3) The spectator quarks in the $B^*_c \rightarrow \psi$ and $B^*_c \rightarrow \eta_c$ transitions are the heavy charm quark. It is usually assumed that the charm quark in the $B^*_c$ meson and charmonium might be close to on-shell, and the gluons emitted or absorbed by the spectator quarks might be soft. It is natural to question the validity of perturbative calculation and the practicability of pQCD approach. In order to eliminate the doubts, it is necessary to check how many shares come from the perturbative domain. The contributions to branching ratio $\mathcal{B}r(B^*_c \rightarrow J/\psi\pi)$ from different $\alpha_s/\pi$ region are displayed in Fig. 3. It is clearly seen that more than 85% (95%) contributions to branching ratio $\mathcal{B}r(B^*_c \rightarrow J/\psi\pi)$ come from the $\alpha_s/\pi \leq 0.2 \ (0.3)$ regions, which implies that the perturbative calculation with the pQCD approach is feasible and credible. The small Wilson coefficient $C_1$ and the small coupling $\alpha_s$ at a higher scale $\mu$ will account for the small percentage from the $\alpha_s/\pi \leq 0.1$ region. The tiny share from the $\alpha_s/\pi \geq 0.5$ region is caused by the serious suppression on soft contributions from many factors, such as Sudakov factor, DAs for the $B^*_c$ meson and charmonium. In addition, a preferable convention to choose the scale as the largest one of all virtualities of internal particles [see Eq.(A27) and Eq.(A28)] is employed to ensure the perturbative calculation
with the pQCD approach.

(4) The theoretical predictions have large uncertainties. With the present predictions on branching ratios of the $B_c^* \to \psi P, \eta_c P$ decays, strict constraints on parameters (such as the CKM matrix element $|V_{cb}|$ and decay width $\Gamma_{B^*_c}$) cannot be obtained. A global fit with more observables seems to be necessary. The first uncertainty from the typical scale $\mu$ might, in principle, be reduced by the inclusion of higher order corrections to HMEs. The decay amplitudes are closely related with wave functions [see Eq.(4)], and parameters of $m_b$ and $m_c$ have much influence on wave functions used here. The third uncertainty arises mainly from the Wolfenstein parameter $A$, i.e., $\frac{2 |A|}{A} \sim 6.4\%$. And a large uncertainty comes from the indefinite decay width $\Gamma_{B^*_c}$. The uncertainties from $m_b$ and $m_c$, the CKM parameters, and $\Gamma_{B^*_c}$ are expected to reduce greatly through either the relative ratio of branching ratios or other observables, such as the polarization fractions $f_{0,\parallel,\perp} = \frac{|M_{0,\parallel,\perp}|^2}{|M_0|^2 + |M_\parallel|^2 + |M_\perp|^2}$. Our studies show that the dominant contributions come from the factorizable topologies. There are large cancellations between the nonfactorizable contributions. Thus the effects from possible new physics should be imperceptible. The more dedicated studies are deserved in the future.

IV. SUMMARY

It is expected that there would be a huge amount of the $B_c^*$ meson data samples at the LHC, and there would be a realistic possibility to search for the $B_c^*$ meson weak decays in the future. In this paper, the nonleptonic $B_c^* \to \psi(1S,2S)P, \eta_c(1S,2S)P$ decays are studied first with a phenomenological pQCD approach, in order to offer a ready reference for the future experimental analysis. It is found that branching ratio for the $B_c^* \to J/\psi \pi$ decay is about $\sim \mathcal{O}(10^{-8})$, which could be accessible at the future experiments.

Appendix A: Amplitude building blocks for the $B_c^* \to \psi P, \eta_c P$ decays

$$\mathcal{A}_{a,P} = 2 m_1 (\epsilon_1 \cdot p_2) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f (\alpha, \beta_1, b_1, b_2) \alpha_s (t_a) \times E_f (t_a) a_1 (t_a) \phi_{B^*_c}^a (x_1) \{ \phi_{\eta_c}^a (x_2) (m_1^2 x_2 + m_2^2 x_2) + \phi_{\eta_c}^p (x_2) m_2 m_b \},$$  \hspace{1cm} (A1)
\[ \mathcal{A}_{a,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_a, b_1, b_2) \alpha_s(t_a) E_f(t_a) \times a_1(t_a) \phi_{B_c}^v(x_1) \left\{ \phi_v^v(x_2) \left( m_1^2 s \bar{x}_2 + m_2^2 t x_2 \right) + \phi_v^\nu(x_2) m_2 m_b u \right\}, \] (A2)

\[ \mathcal{A}_{a,N} = m_1 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_a, b_1, b_2) \alpha_s(t_a) \times E_f(t_a) a_1(t_a) \phi_{B_c}^V(x_1) \left\{ \phi_v^V(x_2) m_2 (u - s x_2) + \phi_v^T(x_2) m_b s \right\}, \] (A3)

\[ \mathcal{A}_{a,T} = 2 m_1 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_a, b_1, b_2) \alpha_s(t_a) \times E_f(t_a) a_1(t_a) \phi_{B_c}^V(x_1) \left\{ \phi_v^V(x_2) m_2 \bar{x}_2 + \phi_v^T(x_2) m_b \right\}, \] (A4)

\[ \mathcal{A}_{b,P} = 2 m_1 (\epsilon_1 \cdot p_2) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_b, b_2, b_1) \alpha_s(t_b) E_f(t_b) \times \alpha_s(t_b) E_f(t_b) \left\{ \phi_{B_c}^v(x_1) \left[ \phi_{B_n}^v(x_2) 2 m_2 m_c - \phi_{B_n}^a(x_2) m_1 m_c \right] + \phi_{B_c}^a(x_1) \left[ \phi_{B_n}^a(x_2) 2 m_2 m_c - \phi_{B_n}^a(x_2) (m_2^2 \bar{x}_1 + m_3^2 x_1) \right]\right\} a_1(t_b), \] (A5)

\[ \mathcal{A}_{b,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_b, b_2, b_1) \alpha_s(t_b) E_f(t_b) \times a_1(t_b) \phi_v^v(x_2) \left\{ \phi_v^v(x_1) (m_2^2 u \bar{x}_1 - m_2^2 t x_1) + \phi_{B_c}^v(x_1) m_1 m_c s \right\}, \] (A6)

\[ \mathcal{A}_{b,N} = m_2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_b, b_2, b_1) E_f(t_b) \times \alpha_s(t_b) a_1(t_b) \phi_v^V(x_2) \left\{ \phi_v^V(x_1) m_1 (s - u x_1) + \phi_{B_c}^T(x_1) m_c u \right\}, \] (A7)

\[ \mathcal{A}_{b,T} = 2 m_2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_f(\alpha, \beta_b, b_2, b_1) \alpha_s(t_b) \times E_f(t_b) a_1(t_b) \phi_v^v(x_2) \left\{ \phi_v^v(x_1) m_1 \bar{x}_1 + \phi_{B_c}^T(x_1) m_c \right\}, \] (A8)

\[ \mathcal{A}_{c,P} = \frac{2 m_1 (\epsilon_1 \cdot p_2)}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \alpha_s(t_c) \left\{ \phi_{B_c}^v(x_1) \phi_{B_n}^a(x_2) s (x_2 - \bar{x}_3) + \phi_{B_c}^T(x_1) \phi_{B_n}^a(x_2) m_2 (x_1 - x_2) \right\} \phi_{B_n}^v(x_3) E_n(t_c) C_2(t_c), \] (A9)
\begin{equation}
A_{c,L} = \frac{1}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_c) E_n(t_c) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) 4 m_1^2 p^2 (x_2 - x_3) \right. \\
+ \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) (u x_1 - s x_2 - 2 m_3^2 x_3) \bigg\} C_2(t_c), 
\end{equation}

\begin{equation}
A_{c,N} = \frac{1}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_c) E_n(t_c) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ m_1^2 s (x_1 - x_3) + m_2^2 u (x_3 - x_2) \right\}, 
\end{equation}

\begin{equation}
A_{c,T} = \frac{2}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_c) E_n(t_c) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ m_1^2 (x_1 - x_3) + m_2^2 (x_3 - x_2) \right\}, 
\end{equation}

\begin{equation}
A_{d,P} = \frac{2 m_1 (\epsilon^1 p_2)}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) m_1 m_2 (x_1 - x_2) \\
+ \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) (2 m_2 x_2 + s x_3 - t x_1) \right\} \alpha_s(t_d) C_2(t_d) \phi_p^x(x_3), 
\end{equation}

\begin{equation}
A_{d,L} = \frac{1}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_d) E_n(t_d) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) 4 m_1^2 p^2 (x_3 - x_1) \\
- \phi_{B_c^x}^\alpha(x_1) \phi_{\psi}^\alpha(x_2) m_1 m_2 (u x_1 - s x_2 - 2 m_3^2 x_3) \right\} C_2(t_d), 
\end{equation}

\begin{equation}
A_{d,N} = \frac{1}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_d) E_n(t_d) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ m_1^2 s (x_3 - x_1) + m_2^2 u (x_2 - x_3) \right\}, 
\end{equation}

\begin{equation}
A_{d,T} = \frac{2}{N_c} \int_0^1 \! dx_1 \int_0^1 \! dx_2 \int_0^1 \! dx_3 \int_0^\infty \! db_1 \int_0^\infty \! b_2 db_2 \int_0^\infty \! b_3 db_3 \alpha_s(t_d) E_n(t_d) \times H_n(\alpha, \beta_c, b_1, b_2, b_3) \phi_p^x(x_3) \left\{ m_1^2 (x_3 - x_1) + m_2^2 (x_2 - x_3) \right\}, 
\end{equation}

where the subscript $i$ of $A_{i,j}$ corresponds to the indices of Fig. 2, the subscript $j$ refers to possible helicity amplitudes.
The function $H_{f,n}$ and Sudakov factor $E_{f,n}$ are defined as

$$H_f(\alpha, \beta, b_i, b_j) = K_0(b_i \sqrt{-\alpha}) \left\{ \theta(b_i - b_j) K_0(b_j \sqrt{-\beta}) I_0(b_j \sqrt{-\beta}) + (b_i \leftrightarrow b_j) \right\}, \quad (A17)$$

$$H_n(\alpha, \beta, b_1, b_2, b_3) = \left\{ \theta(-\beta) K_0(b_3 \sqrt{-\beta}) + \frac{\pi}{2} \theta(\beta) \left[ i J_0(b_3 \sqrt{\beta}) - Y_0(b_3 \sqrt{\beta}) \right] \right\} \times \left\{ \theta(b_2 - b_3) K_0(b_2 \sqrt{-\alpha}) I_0(b_2 \sqrt{-\alpha}) + (b_2 \leftrightarrow b_3) \right\} \delta(b_1 - b_2), \quad (A18)$$

$$E_f(t) = \exp\{-S_{B_f^c}(t) - S_{\psi,\eta_c}(t)\}, \quad (A19)$$

$$E_n(t) = \exp\{-S_{B_n^c}(t) - S_{\psi,\eta_c}(t) - S_P(t)\}, \quad (A20)$$

$$S_i(t) = s(x_i, b_i, p_i^+) + s(\bar{x}_i, b_i, p_i^+) + 2 \int_{1/b_i}^t \frac{d\mu}{\mu} \gamma_q, \quad (A21)$$

where $I_0, J_0, K_0$ and $Y_0$ are Bessel functions; $\gamma_q = -\alpha_s/\pi$ is the quark anomalous dimension; the expression of $s(x, b, Q)$ can be found in of Ref. [15]; $\alpha$ and $\beta_i$ are virtualities of gluon and quarks. The definitions of the particle virtuality and typical scale $t_i$ are given as follows.

$$\alpha = x_1^2 m_1^2 + x_2^2 m_2^2 - x_1 x_2 t, \quad (A22)$$

$$\beta_a = m_1^2 + x_2^2 m_2^2 - x_2 t - m_b^2, \quad (A23)$$

$$\beta_b = m_2^2 + x_1^2 m_1^2 - x_1 t - m_c^2, \quad (A24)$$

$$\beta_c = \alpha + x_3^2 m_3^2 - x_1 x_3 u + x_2 x_3 s, \quad (A25)$$

$$\beta_d = \alpha + x_3^2 m_3^2 - x_1 x_3 u + x_2 x_3 s, \quad (A26)$$

$$t_{a,b} = \max\{\sqrt{-\alpha}, \sqrt{\beta_{a,b}}, 1/b_1, 1/b_2\}, \quad (A27)$$

$$t_{c,d} = \max\{\sqrt{-\alpha}, \sqrt{\beta_{c,d}}, 1/b_2, 1/b_3\}. \quad (A28)$$

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