Deep ReLU neural network approximation in Bochner spaces and applications to parametric PDEs

Dinh Dũng \textsuperscript{a}, Van Kien Nguyen\textsuperscript{b}, and Duong Thanh Pham\textsuperscript{c}

\textsuperscript{a}Information Technology Institute, Vietnam National University, Hanoi
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
Email: dinhzung@gmail.com

\textsuperscript{b}Faculty of Basic Sciences, University of Transport and Communications
No.3 Cau Giay Street, Lang Thuong Ward, Dong Da District, Hanoi, Vietnam
Email: kiennv@utc.edu.vn

\textsuperscript{c}Vietnamese German University,
Ring road 4, Quarter 4, Thoi Hoa Ward, Ben Cat Town, Binh Duong Province, Vietnam
Email: duong.pt@vgu.edu.vn

December 15, 2022

Abstract

We investigate non-adaptive methods of deep ReLU neural network approximation in Bochner spaces $L_2(U^\infty, X, \mu)$ of functions on $U^\infty$ taking values in a separable Hilbert space $X$, where $U^\infty$ is either $\mathbb{R}^\infty$ equipped with the standard Gaussian probability measure, or $I^\infty := [-1,1]^\infty$ equipped with the Jacobi probability measure. Functions to be approximated are assumed to satisfy a certain weighted $\ell_2$-summability of the generalized chaos polynomial expansion coefficients with respect to the measure $\mu$. We prove the convergence rate of this approximation in terms of the size of approximating deep ReLU neural networks. These results then are applied to approximation of the solution to parametric elliptic PDEs with random inputs for the lognormal and affine cases.

Keywords and Phrases: High-dimensional approximation in Bochner spaces; Deep ReLU neural networks; Parametric elliptic PDEs with random inputs.

Mathematics Subject Classifications (2020): 65C30, 41A25, 68T07, 65C20, 41A63.

1 Introduction

The aim of the present paper is to construct deep ReLU neural networks for approximation of functions in Bochner spaces and to apply the obtained results to approximation of solutions to parametric and stochastic elliptic PDEs with lognormal or affine inputs. We investigate the convergence rate of this approximation in terms of the size of approximating deep ReLU neural networks.

*Corresponding author
The universal approximation capacity of neural networks has been known since the 1980’s ([10, 34, 22, 6]). In recent years, deep neural networks have been rapidly developed and successfully applied to a wide range of fields. The main advantage of deep neural networks over shallow ones is that they can output compositions of functions cheaply. Since their application range is getting wider, theoretical analysis revealing reasons of these significant practical improvements attracts substantial attention [2, 17, 41, 49, 50]. In the last several years, there has been a number of interesting papers that addressed the role of depth and architecture of deep neural networks in approximating functions that possess special regularity properties such as analytic functions [19, 39], differentiable functions [44, 52], oscillatory functions [29], functions in Sobolev or Besov spaces [1, 26, 30, 53]. High-dimensional approximations by deep neural networks have been studied in [40, 48, 14], and their applications to high-dimensional PDEs in [47, 20, 42, 24, 25, 27]. Most of these papers used deep ReLU (Rectified Linear Unit) neural networks since the rectified linear unit is a simple and preferable activation function in many applications. The output of such neural networks is continuous piece-wise linear function which are easily and cheaply computed. We refer the reader to the recent surveys [18, 43] for various problems and aspects of neural network approximation and bibliography.

In computational uncertainty quantification, the problem of efficient numerical approximation for parametric and stochastic partial differential equations (PDEs) has been of great interest and achieved significant progress in recent years. There is a vast number of works on this topic to not mention all of them. We point out just some works [3, 5, 4, 7, 8, 9, 12, 13, 21, 33, 54] which are directly related to our paper.

Recently, a number of works have been devoted to various problems and methods of deep neural network approximation for parametric and stochastic PDEs with affine inputs on the compact set \( \mathbb{I}^\infty := [-1, 1]^{\infty} \) and with the condition on uniform ellipticity, such as dimensionality reduction [51], deep neural network expression rates for the Taylor generalized polynomial chaos expansion (gpc) of solutions to parametric elliptic PDEs [45], reduced basis methods [37] the problem of learning the discretized parameter-to-solution map in practice [23], Bayesian PDE inversion [31], etc.

Let \( D \subset \mathbb{R}^d \) be a bounded Lipschitz domain. Consider the diffusion elliptic equation

\[
- \text{div}(a \nabla u) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0, \tag{1.1}
\]

for a given fixed right-hand side \( f \) and a spatially variable scalar diffusion coefficient \( a \). Denote by \( V := H_0^1(D) \) the energy space and \( H^{-1}(D) \) the dual space of \( V \). Assume that \( f \in H^{-1}(D) \) (in what follows this preliminary assumption always holds without mention). If \( a \in L_\infty(D) \) satisfies the ellipticity assumption

\[
0 < a_{\min} \leq a \leq a_{\max} < \infty,
\]

by the well-known Lax-Milgram lemma, there exists a unique solution \( u \in V \) to the equation (1.1) in the weak form

\[
\int_D a \nabla u \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in V.
\]

Partial differential equations (PDEs) with parametric and stochastic inputs are a common model used in science and engineering. Depending on the nature of the modeled object, the parameters involved in them may be either deterministic or random. Random nature reflects the uncertainty in various parameters presented in the physical phenomenon modelled by the equation. For the equation (1.1), we consider the diffusion coefficients having a parametric form \( a = a(y) \), where
\( y = (y_j)_{j \in \mathbb{N}} \) is a sequence of real-valued parameters ranging in the set \( U^\infty \) which is either \( \mathbb{R}^\infty \) or \( I^\infty \). Denote by \( u(y) \) the solution to the parametric diffusion elliptic equation

\[
- \text{div}(a(y) \nabla u(y)) = f \quad \text{in} \quad D, \quad u(y)|_{\partial D} = 0.
\] (1.2)

The resulting solution operator maps \( y \in U^\infty \) to \( u(y) \in V \). The objective is to achieve a numerical approximation of this complex map by a small number of parameters with a guaranteed error in a given norm.

In the present paper, we consider the lognormal case when \( U^\infty = \mathbb{R}^\infty \) and the diffusion coefficient \( a \) is of the form

\[
a(y) = \exp(b(y)), \quad \text{with} \quad b(y) = \sum_{j=1}^{\infty} y_j \psi_j,
\] (1.3)

and \( y_j \) are i.i.d. standard Gaussian random variables, and the affine case when \( U^\infty = I^\infty \) and the diffusion coefficient \( a \) is of the form

\[
a(y) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j.
\] (1.4)

Here \( \bar{a} \in L_\infty(D) \) and \( \psi_j \in L_\infty(D) \) for both the cases.

Let us briefly describe the problem setting and main contribution of the present paper.

We are interested in the problem of deep ReLU neural network approximation of the solution \( u(y) \) to parametric and stochastic elliptic PDEs (1.2) with lognormal inputs (1.3) or affine inputs (1.4).

Recently, for parametric elliptic PDEs (1.2) with random inputs, we have investigated related problems of sparse-grid polynomial interpolation approximation [13], and of deep ReLU neural network approximation based on polynomial interpolations [11]. A unified approach suggested in [13, 11] to those problems, is to study non-adaptive methods of sparse-grid polynomial interpolation and deep ReLU neural network approximations in general Bochner spaces which are able to be applied to parametric and stochastic elliptic PDEs. Following this approach, in the present paper, we construct non-adaptive methods of deep ReLU neural network approximations in Bochner spaces \( L_2(U^\infty, X, \mu) \) of functions on \( U^\infty \) taking values in a separable Hilbert space \( X \), where \( U^\infty \) is either \( \mathbb{R}^\infty \) equipped with the standard Gaussian probability measure \( \mu = \gamma \), or \( I^\infty := [-1, 1]^\infty \) equipped with the Jacobi probability measure \( \mu = \nu_{a,b} \). Differing from the deep ReLU neural network approximation in [11] which relies on polynomial interpolations, the approximation of functions \( v \in L_2(U^\infty, X, \mu) \) in the present paper is based on an \( m \)-term truncation

\[
S_m v(y) := \sum_{j=1}^{m} v_{s_j} P_{s_j}(y)
\]

of the orthonormal generalized polynomial chaos (gpc) expansion with respect to the measure \( \mu \):

\[
v(y) = \sum_{s \in F} v_s P_s(y), \quad v_s \in X,
\] (1.5)

where \( F \) is the set of all families of non-negative integers \( s = (s_j)_{j \in \mathbb{N}} \) with finite support. Functions \( v \) to be approximated are assumed to satisfy a certain weighted \( \ell_2 \)-summability of the
coefficients \((v_s)_{s \in S}\) of the expansion (1.5). For every integer \(n \geq 4\), we construct a non-adaptive deep ReLU neural network \(\Phi_n = (\phi_s)_{j=1}^m\) on \(\mathbb{R}^m\) with \(m = O(n/\log n)\), having size not greater than \(n\) and \(m\) outputs so that the approximation of \(v\) by the function

\[
\Phi_n(y) := \sum_{j=1}^m v_s \phi_s(y)
\]

gives the twofold error bounds:

\[
\|v - \Phi_n\|_{L^2(\mathbb{U}_\infty, X, \mu)} = O\left(m^{-1/q}\right) = O\left((n/\log n)^{-1/q}\right).
\]

(1.6)

Here, the error bounds are much sharper than those established in [11] for deep ReLU neural network approximation based on polynomial interpolations.

These results are then applied to approximation in the space \(L^2(\mathbb{U}_\infty, V, \mu)\) of the solution \(u(y)\) to parametric and stochastic elliptic PDEs (1.2) with the lognormal inputs (1.3) and the affine inputs (1.4). They are also applied to the approximation of holomorphic functions on \(\mathbb{R}^\infty\) in the present paper, but we believe that these results are applicable to holomorphic functions on \(\mathbb{R}^\infty\).

We give some comments on the difference of our contribution from the directly related paper [45]. In [45], the authors proved an error bound \(O\left(n^{-(1/q - 1/2)}/(\log n \log \log n)\right)\) of the uniform approximation (or equivalently, of the approximation in the norm of Bochner space \(L_\infty(\mathbb{R}^\infty, V, \lambda)\) with the uniform Lebesgue \(\lambda\)) by deep ReLU neural networks of size \(n\) of solution to parametric elliptic PDE (1.2) with affine inputs (1.4), based on the non-orthogonal Taylor gpc expansion. In the present paper, we improved this result as \(O\left(n^{-(1/q - 1/2)}/(\log n)\right)\) and extended this approximation to the Bochner space \(L_2(\mathbb{U}_\infty, V, \nu_{a,b})\) with the Jacobi probability measure, based on the orthonormal Jacobi gpc expansion. The error bound of this extended approximation is as in the right-hand side of (1.6). The most principal difference of the present paper from [45] is the results on the approximation in the norm of Bochner space \(L_2(\mathbb{R}^\infty, V, \gamma)\) by deep ReLU neural networks of size \(n\) of solution to parametric elliptic PDE (1.2) on non-compact set \(\mathbb{R}^\infty\) with lognormal inputs (1.3), based on the orthonormal Hermite gpc expansion. The construction of approximating deep ReLU neural networks and proof of the bound for the approximation error are much more technically complicated. Our main idea of constructing the approximating deep ReLU neural networks \(\phi_n\) as well as the proof of the error bound (1.6) is that we first approximate the solution \(u \in L_2(\mathbb{R}^\infty, V, \gamma)\) by the truncation \(S_m u\) with error bound \(O(m^{-1/q})\). Then we construct a deep ReLU neural network \(\phi_n\) on finite-dimensional space \(\mathbb{R}^m\) that approximate \(S_m u\) with error \(O(m^{-1/q})\). Finally we estimate the size of the deep neural networks \(\phi_n\) as \(n = O(m \log m)\). The construction and proofs of the bounds are based on the weighted \(\ell_2\)-summability of the gpc Hermite expansion coefficients of \(u\), the known realization of approximate multiplication by deep ReLU neural networks, and some known results on Gaussian-weighted polynomial approximation on \(\mathbb{R}^m\). The last but not least difference is, as noticed all of the results of the present paper are deduced from a general theory of deep ReLU neural network in Bochner spaces.

Notice that the error bound of this deep ReLU neural network approximation is comparable to the error bound of the best approximation of \(v\) by (linear non-adaptive) \(n\)-term truncations of the gpc expansion as well as of the approximation by the particular truncation \(S_n u\) which is \(O(n^{-1/q})\) [4, 13]. However, a deep ReLU neural network represents a continuous piece-wise linear function defined on a number of polyhedral subdomains and therefore, is easily generated.
and computed in numerical implementation. For some PDEs, it has been shown that deep neural networks are capable of representing solutions without incurring the curse of dimensionality, see, for instance, [36, 28, 35]. Therefore deep ReLU neural networks are one of the preferable tools in numerical solving (parametric) PDEs. We refer the reader to [23, 37, 45] for further discussion on the role of neural networks in the approximation of solutions to parametric PDEs.

In the present paper, we are concerned about the parametric approximability for parametric and stochastic elliptic PDEs. Therefore, the results themselves do not yield a practically realizable approximation since they do not cover the approximation of the gpc expansion coefficients which are functions of the spatial variable. Moreover, the approximant $\Phi_n$, as we can see, is not a real deep ReLU network, but just a combination of the gpc coefficients and the components of a deep ReLU network. Naturally, it would be desirable to study the problem of full neural network approximation of the solution to parametric and stochastic elliptic PDEs by combining the spatial and parametric domains based on fully discrete approximation in [3, 13]. We will discuss this problem in a forthcoming paper.

The paper is organized as follows. In Section 2, we present a necessary knowledge about deep ReLU neural networks. Section 3 is devoted to the investigation of non-adaptive methods of deep ReLU neural network approximation in Bochner space with Gaussian measure and applications to approximations of the solution to the parameterized diffusion elliptic equation (1.2) with lognormal inputs (1.3), and of holomorphic functions on $\mathbb{R}^\infty$. In Section 4, we study non-adaptive methods of deep ReLU neural network approximation in Bochner space with Jacobi measure and applications to approximations of the solution to the parameterized diffusion elliptic equation (1.2) with affine inputs (1.4).

### Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers and $\mathbb{N}_0 := \{s \in \mathbb{Z}: s \geq 0\}$. We denote $\mathbb{R}^\infty$ the set of all sequences $y = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathbb{R}$. Denote by $\mathbb{F}$ the set of all sequences of non-negative integers $s = (s_j)_{j\in\mathbb{N}}$ such that their support $\nu_s := \text{supp}(s) := \{j \in \mathbb{N}: s_j > 0\}$ is a finite set. We use $(e^j)_{j\in\mathbb{N}}$ for the standard basis of $\ell^2(\mathbb{N})$. For a set $G$, we denote by $|G|$ the cardinality of $G$. We use letters $C$ and $K$ to denote general positive constants which may take different values, and $C_{\alpha,\beta,...}$ and $K_{\alpha,\beta,...}$ when we want to emphasize the dependence of these constants on $\alpha, \beta, ...$, or when this dependence is important in a particular situation.

### 2 Deep ReLU neural networks

In this section, we present some necessary definitions and elementary facts on deep ReLU neural networks. There is a wide variety of neural network architectures and each of them is adapted to specific tasks. As in [52], we will use deep feed-forward neural networks which allow connections between neurons in a layer with neurons in any preceding layers (but not in the same layer). This improves the bound of the weight of parallelization network via the weights of the component networks (see Lemma 2.2 below) in comparing with the standard deep feed-forward neural networks which allows the connection between neurons only in neighboring layers (comp. [45]).

In deep neural network approximation, we will employ the ReLU activation function that is defined by $\sigma(t) := \max\{t, 0\}$, $t \in \mathbb{R}$. We will use the notation $\sigma(x) := (\sigma(x_1), \ldots, \sigma(x_d))$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

**Definition 2.1** Let $d, L \in \mathbb{N}$, $L \geq 2$, $N_0 = d$, and $N_1, \ldots, N_L \in \mathbb{N}$. Let $W^\ell = \left(\omega_i^\ell\right)_{i,j} \in \mathbb{R}^{(N_\ell)^2}$.
\[ N^\ell \times \left( \sum_{i=0}^{\ell-1} N_i \right), \quad \ell = 1, \ldots, L, \quad N^\ell \times \left( \sum_{i=0}^{\ell-1} N_i \right) \] matrices, and \( b^\ell \in \mathbb{R}^{N_\ell} \). A ReLU neural network \( \Phi \) with input dimension \( d \), output dimension \( N_L \) and \( L \) layers is a sequence of matrix-vector tuples

\[ \Phi = ((W^1, b^1), \ldots, (W^L, b^L)) \],

in which the following computation scheme is implemented

\[
\begin{align*}
    z^0 &:= x \in \mathbb{R}^d; \\
    z^\ell &:= \sigma \left( W^\ell \left( z^0, \ldots, z^{\ell-1} \right)^T + b^\ell \right), \quad \ell = 1, \ldots, L-1; \\
    z^L &:= W^L \left( z^0, \ldots, z^{L-1} \right)^T + b^L.
\end{align*}
\]

We call \( z^0 \) the input and with an ambiguity denote \( \Phi(x) := z^L \) the output of \( \Phi \) and in some places we identify a ReLU neural network with its output. We will use the following terminology.

- The number of layers \( L(\Phi) = L \) is the depth of \( \Phi \);
- The number of nonzero \( w^\ell_{i,j} \) and \( b^\ell_j \) is the size of \( \Phi \) and denoted by \( W(\Phi) \);
- When \( L(\Phi) \geq 3 \), \( \Phi \) is called a deep neural network, and otherwise, a shallow neural network.

The following two results are easy to verify from the definition above. We also refer the reader to [30, Remark 2.9 and Lemma 2.11] for further remarks and comments.

**Lemma 2.2 (Parallelization)** Let \( N \in \mathbb{N}, \lambda_j \in \mathbb{R}, j = 1, \ldots, N \). Let \( \Phi_j, j = 1, \ldots, N \) be deep ReLU neural networks with input dimension \( d \). Then we can explicitly construct a deep ReLU neural network denoted by \( \Phi \) so that

\[
\Phi(x) = \begin{pmatrix} \Phi_1(x) \\ \vdots \\ \Phi_N(x) \end{pmatrix}, \quad x \in \mathbb{R}^d.
\]

Moreover, we have

\[
W(\Phi) \leq \sum_{j=1}^{N} W(\Phi_j) \quad \text{and} \quad L(\Phi) = \max_{j=1,\ldots,N} L(\Phi_j).
\]

The network \( \Phi \) is called the parallelization network of \( \Phi_j, j = 1, \ldots, N \) and denoted by \( \Phi = (\Phi_j)_{j=1}^{N} \).

**Lemma 2.3 (Concatenation)** Let \( \Phi_1 \) and \( \Phi_2 \) be two ReLU neural networks such that output layer of \( \Phi_1 \) has the same dimension as input layer of \( \Phi_2 \). Then, we can explicitly construct a ReLU neural network \( \Phi \) such that \( \Phi(x) = \Phi_2(\Phi_1(x)) \) for \( x \in \mathbb{R}^d \). Moreover we have

\[
W(\Phi) \leq W(\Phi_1) + W(\Phi_2) \quad \text{and} \quad L(\Phi) = L(\Phi_1) + L(\Phi_2).
\]

The network \( \Phi \) is called the concatenation network of \( \Phi_1 \) and \( \Phi_2 \).

The following lemma is easily deduced from [45, Propositions 3.1 and 3.3] and [40, Proposition 2].
Lemma 2.4 Let $\ell \in \mathbb{N}^d$. For every $\delta \in (0, 1)$, $d \in \mathbb{N}$, $d \geq 2$, we can explicitly construct a deep ReLU neural network $\Phi_P$ so that

$$
\sup_{x \in [-1,1]^d} \left| \prod_{j=1}^{d} x_j^{\ell_j} - \Phi_P(x) \right| \leq \delta.
$$

Furthermore, if $x_j = 0$ for some $j \in \{1, \ldots, d\}$ then $\Phi_P(x) = 0$ and there exists a constant $C > 0$ independent of $\delta$ and $d$ such that

$$
W(\Phi_P) \leq C(1 + |\ell|_1 \log(|\ell|_1 \delta^{-1})) \quad \text{and} \quad L(\Phi_P) \leq C(1 + \log |\ell|_1 \log(|\ell|_1 \delta^{-1})).
$$

For $j = 0, 1$, let $\varphi_j$ be the continuous piece-wise linear functions with break points $\{-2, -1, 1, 2\}$ such that $\text{supp}(\varphi_j) = [-2, 2]$, $\varphi_0(x) = 1$ and $\varphi_1(x) = x$ if $x \in [-1, 1]$. Notice that $\varphi_j$ can be realized exactly by a shallow ReLU neural network (still denoted by $\Phi_j$), i.e.,

$$
\varphi_0(x) = \sigma(x - 2) - 3\sigma(x - 1) + 4\sigma(x) - 3\sigma(x + 1) + \sigma(x + 2),
$$

and

$$
\varphi_1(x) = \sigma(x - 2) - 2\sigma(x - 1) + 2\sigma(x + 1) - \sigma(x + 2).
$$

Lemma 2.5 Let $\ell \in \mathbb{N}^d$. Let $\varphi$ be either $\varphi_0$ or $\varphi_1$. For every $\delta \in (0, 1)$, $d \in \mathbb{N}$, we can explicitly construct a deep ReLU neural network $\Phi$ so that

$$
\sup_{x \in [-2,2]^d} \left| \prod_{j=1}^{d} \varphi_j^{x_j} - \Phi(x) \right| \leq \delta.
$$

Furthermore, $\text{supp}(\Phi) \subset [-2,2]^d$ and there exists a constant $C > 0$ independent of $\delta$ and $d$ such that

$$
W(\Phi) \leq C\left(1 + |\ell|_1 \log(|\ell|_1 \delta^{-1})\right) \quad \text{and} \quad L(\Phi) \leq C\left(1 + \log |\ell|_1 \log(|\ell|_1 \delta^{-1})\right).
$$

(2.1)

Proof. This simple lemma is proven in [11]. For completeness, let us recall the proof. The network $\Phi$ is constructed as a concatenation of $\{\varphi(x_j)\}_{j=1}^{d}$ and $\Phi_P$. Notice that

$$
\{\varphi(x_j)\}_{j=1}^{d} \subset [-1,1]^d, \quad \forall x \in [-2,2]^d.
$$

Hence, the estimate (2.1) follows directly from Lemmata 2.3 and 2.4 and the estimates $W(\varphi_0) \leq 10$, $W(\varphi_1) \leq 8$. \qed

3 Approximation in Bochner spaces with Gaussian measure

In this section, we investigate non-adaptive methods of deep ReLU neural network approximation in the Bochner space $L_2(\mathbb{R}^\infty, X, \gamma)$ with Gaussian measure $\gamma$ of functions on $\mathbb{R}^\infty$ taking values in a separable Hilbert space $X$ and satisfying a weighted $\ell_2$-summability of the Hermite gpc expansion coefficients. We construct such methods and prove convergence rates of the approximation by them. These methods are constructed via the truncations on finite-dimensional supercubes of the truncated Hermite gpc expansion of functions in $L_2(\mathbb{R}^\infty, X, \gamma)$ to be approximated. The results are then applied to the approximation of the solution to the parametric elliptic PDEs (1.2) with lognormal inputs (1.3) on $\mathbb{R}^\infty$ and of holomorphic maps on $\mathbb{R}^\infty$. 


3.1 Approximation by truncations of the Hermite gpc expansion

We first recall a concept of infinite tensor product of probability measures. Let $\mu(y)$ be a probability measure on $U$. We introduce the probability measure $\mu(y)$ on $U^\infty$ as the infinite tensor product of the probability measures $\mu(y_j)$:

$$
\mu(y) := \bigotimes_{j \in \mathbb{N}} \mu(y_j), \quad y = (y_j)_{j \in \mathbb{N}} \in U^\infty.
$$

If $\rho(y)$ is the density of $\mu(y)$, i.e., $d\mu(y) = \rho(y)dy$, then we write

$$
d\mu(y) := \bigotimes_{j \in \mathbb{N}} \rho(y_j)dy_j, \quad y = (y_j)_{j \in \mathbb{N}} \in U^\infty.
$$

(For details on infinite tensor product of probability measures, see, e.g., [32, pp. 429–435].)

Let $X$ be a separable Hilbert space. The probability measure $\mu(y)$ induces the Bochner space $L_2(U^\infty, X, \mu)$ of $\mu$-measurable mappings $v$ from $U^\infty$ to $X$ for which the norm

$$
\|v\|_{L_2(U^\infty, X, \mu)} := \left( \int_{U^\infty} \|v(\cdot, y)\|_X^2 \, d\mu(y) \right)^{1/2} < \infty.
$$

We consider approximations in the space $L_2(R^\infty, X, \gamma)$ with the infinite tensor product standard Gaussian probability measure $\gamma$ on $R^\infty$. More precisely, let $\gamma(y)$ be the probability measure on $R$ with the standard Gaussian density:

$$
\gamma(y) := g(y)dy, \quad g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.
$$

Then the infinite tensor product standard Gaussian probability measure $\gamma(y)$ on $R^\infty$ can be defined by

$$
d\gamma(y) := \bigotimes_{j \in \mathbb{N}} g(y_j)dy_j.
$$

We make use of the abbreviation $L_2(X) := L_2(R^\infty, X, \gamma)$.

A powerful strategy for approximation of functions $v \in L_2(X)$ is based on truncations of the Hermite gpc expansion

$$
v(y) = \sum_{s \in F} v_s H_s(y), \quad v_s \in X,
$$

where

$$
H_s(y) = \bigotimes_{j \in \mathbb{N}} H_{s_j}(y_j), \quad v_s := \int_{R^\infty} v(y) H_s(y) \, d\gamma(y), \quad s \in F,
$$

with $(H_k)_{k \in \mathbb{N}_0}$ being the Hermite polynomials normalized according to $\int_R |H_k(y)|^2 g(y)dy = 1$. Here recall that $F$ denotes the set of all sequences of non-negative integers $s = (s_j)_{j \in \mathbb{N}}$ such that their support $v_s := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. Notice that $(H_s)_{s \in F}$ is an orthonormal basis of $L_2(R^\infty, \gamma) := L_2(R^\infty, R, \gamma)$. Moreover, for every $v \in L_2(X)$ represented by the series (3.1), the Parseval’s identity holds

$$
\|v\|_{L_2(X)}^2 = \sum_{s \in F} \|v_s\|_X^2.
$$
For \( s, s' \in \mathbb{F} \), the inequality \( s' \leq s \) means that \( s'_j \leq s_j \) for \( j \in \mathbb{N} \). A set \( \Lambda \subset \mathbb{F} \) is called downward closed if the inclusion \( s \in \mathbb{F} \) yields the inclusion \( s' \in \mathbb{F} \) for every \( s' \in \mathbb{F} \) such that \( s' \leq s \). A sequence \((\sigma_s)_{s \in \mathbb{F}}\) is called increasing if \( \sigma_{s'} \leq \sigma_s \) when \( s' \leq s \). Denote by \((e^j)_{j \in \mathbb{N}}\) the standard basis of \( \ell_2(\mathbb{N}) \).

In this subsection, as the first preliminary step in deep ReLU neural network approximation, we study the approximation of \( v \in \mathcal{L}_2(X) \) by truncations \( S_\Lambda \) on a finite set \( \Lambda \subset \mathbb{F} \) of the Hermite gpc expansion (3.1). In the second preliminary step, we study the approximation of \( v \) by the truncations of \( S_\Lambda \) on finite-dimensional supercube. In these approximations, functions \( v \) are assumed to satisfy a certain weighted \( \ell_2 \)-summability condition as described in Assumption \( A_{q, \theta} \) below.

For \( \theta \geq 0 \), we define the sequence
\[
p_s(\theta) := \prod_{j \in \mathbb{N}} (1 + s_j)^\theta, \quad s \in \mathbb{F}.
\] (3.2)

For \( 0 < q < \infty \) and \( \theta \geq 0 \), we say that \( v \in \mathcal{L}_2(X) \) represented by the series (3.1), satisfies Assumption \( A_{q, \theta} \) \((A_q = A_{q, 0} \) for short\) if

**Assumption \( A_{q, \theta} \)** There exist a constant \( M \) and an increasing sequence \( \sigma = (\sigma_s)_{s \in \mathbb{F}} \) of positive numbers such that \( \sigma_{e^{i'}} \leq \sigma_{e^i} \) if \( i' < i \), and that
\[
\left( \sum_{s \in \mathbb{F}} (\sigma_s \| v_s \|_X)^2 \right)^{1/2} \leq M < \infty, \quad \text{with} \quad (p_s(\theta) \sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F}).
\] (3.3)

We would like to emphasize that the weighted \( \ell_2 \)-summability condition (3.3) involving the function \( p_s(\theta) \) with \( \theta > 0 \) play a vital role in the proofs of the main results of the present paper. For example, the proof of Theorem 3.6 requires Assumption \( A_{q, \theta} \) with \( \theta \geq 4/q \). Notice that if \( v \) satisfies Assumption \( A_{q, \theta} \), then \( v \) also satisfies Assumption \( A_{q, \theta'} \) for every \( \theta' < \theta \). In what follows, we will use this property without mention.

Assume that \( 0 < q < \infty \) and \( \sigma = (\sigma_s)_{s \in \mathbb{F}} \) is an increasing sequence of positive numbers. For \( \xi > 0 \), we introduce the set
\[
\Lambda(\xi) := \{ s \in \mathbb{F} : \sigma_s^q \leq \xi \}.
\] (3.4)

For a function \( v \in \mathcal{L}_2(X) \) represented by the series (3.1), we define the truncation
\[
S_{\Lambda(\xi)} v := \sum_{s \in \Lambda(\xi)} v_s H_s.
\] (3.5)

**Lemma 3.1** For every \( v \in \mathcal{L}_2(X) \) satisfying Assumption \( A_q \) and for every \( \xi > 1 \), there holds
\[
\| v - S_{\Lambda(\xi)} v \|_{\mathcal{L}_2(X)} \leq M \xi^{-1/q}.
\]

**Proof.** Applying the Parseval’s identity, noting (3.5), (3.4) and Assumption \( A_q \), we obtain
\[
\| v - S_{\Lambda(\xi)} v \|_{\mathcal{L}_2(X)}^2 = \sum_{s : \sigma_s^q > \xi^{1/q}} \| v_s \|_X^2 = \sum_{s : \sigma_s^q > \xi^{1/q}} (\sigma_s \| v_s \|_X)^2 \sigma_s^{-2} \leq \xi^{-2/q} \sum_{s \in \mathbb{F}} (\sigma_s \| v_s \|_X)^2 = M^2 \xi^{-2/q}.
\]

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The set $\Lambda(\xi)$ determining the truncation $S_{\Lambda(\xi)}$ plays an important role in deep ReLU neural network approximation in the Bochner space $L^2(X)$. Let us present some properties of this set which will be used in the following. Observe that under Assumption A$q$, the set $\Lambda(\xi)$ is finite and downward closed. We define the following numbers:

$$m_1(\xi) := \max_{s \in \Lambda(\xi)} |s|_1,$$

and

$$m(\xi) := \max \{ j \in \mathbb{N} : \exists s \in \Lambda(\xi) \text{ such that } s_j > 0 \}.$$

By this definition we have

$$\bigcup_{s \in \Lambda(\xi)} \nu_s \subset \{1, 2, \ldots, m(\xi)\},$$

where $\nu_s := \text{supp}(s) := \{ j \in \mathbb{N} : s_j > 0 \}$ denotes the support of $s$.

We put for $(\sigma^{-1}_s)_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$

$$K_q := \max (1, K'_q) \quad \text{with} \quad K'_q := \sum_{s \in \mathbb{F}} \sigma^{-q}_s,$$

and for $(p_s(\theta)\sigma^{-1}_s)_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$

$$K_{q,\theta} := \max (1, K'_{q,\theta}) \quad \text{with} \quad K'_{q,\theta} := \left( \sum_{s \in \mathbb{F}} p_s(\theta)^q \sigma^{-q}_s \right)^{\frac{1}{q}}.$$

Lemma 3.2 Let $\xi \geq 1$ and $(\sigma_s)_{s \in \mathbb{F}}$ be a sequence of positive numbers. Then we have the following.

(i) Assume that $(\sigma^{-1}_s)_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$. The set $\Lambda(\xi)$ is finite and it holds

$$|\Lambda(\xi)| \leq K_q\xi.$$

(ii) Assume that $(p_s(\theta)\sigma^{-1}_s)_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ for some $\theta > 0$. There holds

$$m_1(\xi) \leq K_{q,\theta}\xi^\frac{1}{q}.$$

Proof. Notice that $1 \leq \sigma^{-q}_s \xi$ for every $s \in \Lambda(\xi)$. This implies (i):

$$|\Lambda(\xi)| = \sum_{s \in \Lambda(\xi)} 1 \leq \sum_{s \in \Lambda(\xi)} \xi \sigma^{-q}_s \leq K_q\xi.$$

Moreover, we have that $1 \leq s_j$ for every $j \in \nu_s$. Hence, we derive the inequalities

$$\max_{s \in \Lambda(\xi)} |s|_1^{\theta q} \leq \sum_{s \in \Lambda(\xi)} \left( \prod_{j \in \nu_s} (1 + s_j) \right)^{\theta q} \leq \sum_{s \in \Lambda(\xi)} p_s(\theta)^q \xi \sigma^{-q}_s \leq K'_{q,\theta}\xi,$$

which prove (ii).
Lemma 3.3 Let $0 < q < \infty$ and $(\sigma_s)_{s \in \mathbb{F}}$ be an increasing sequence of positive numbers. Assume that $(\sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ and $\sigma_{e^i} \leq \sigma_{e^{i'}}$ if $i' < i$. Then there holds for $\xi \geq 1$

$$m(\xi) \leq K_q \xi. \quad (3.11)$$

Proof. The statement is obvious if $\Lambda(\xi) = \emptyset$. Therefore we assume that $\Lambda(\xi) \neq \emptyset$. Noting (3.7), there is a $s \in \Lambda(\xi)$ such that $s_{m(\xi)} \geq 1$. Then we have $e_{m(\xi)} \leq s$. Here recall that $e_{m(\xi)}$ is the $m(\xi)$th vector in the standard basis of $\ell_2(\mathbb{N})$. Since $(\sigma_s)_{s \in \mathbb{F}}$ be an increasing sequence, we have $\Lambda(\xi)$ is downward closed and therefore $e_{m(\xi)} \in \Lambda(\xi)$. From the definition (3.4) of $\Lambda(\xi)$ and the assumption in the lemma, we obtain

$$\sigma_{e_1}^q \leq \sigma_{e_2}^q \leq \ldots \leq \sigma_{e_{m(\xi)}}^q \leq \xi.$$ 

Thus, $e_1, \ldots, e_{m(\xi)}$ belong to $\Lambda(\xi)$. This yields the inequality $|\Lambda(\xi)| \geq m(\xi)$ which together with the inequality $|\Lambda(\xi)| \leq K_q \xi$ in Lemma 3.2(i) proves (3.11). \hfill \Box

Notice that if Assumption $A_q$ holds, then $m(\xi)$ is finite and, therefore, the truncation $S_{\Lambda(\xi)}v$ of the series (3.1) can be seen as a function on $\mathbb{R}^m$. In this section, for $\xi > 1$, we use the letters $\omega$ and $m$ only for the notation

$$\omega := [K_q, \theta \xi], \quad m := m(\xi). \quad (3.12)$$

Let us introduce the truncation $S_{\Lambda(\xi)}^\omega v$ of $S_{\Lambda(\xi)}v$ on the supercube

$$B_\omega^m := [-2\sqrt{\omega}, 2\sqrt{\omega}]^m \subset \mathbb{R}^m.$$ 

For a function $\varphi$ defined on $\mathbb{R}$, we denote by $\varphi^\omega$ the truncation of $\varphi$ on $B_\omega^1$, i.e.,

$$\varphi^\omega(y) := \begin{cases} \varphi(y) & \text{if } y \in B_\omega^1 \\ 0 & \text{otherwise}. \end{cases}$$

If $\nu_s \subset \{1, \ldots, m\}$, we put

$$H^\omega_s(y) := \prod_{j=1}^m H^\omega_{s_j}(y_j), \quad y \in \mathbb{R}^m.$$ 

We have $H^\omega_s(y) = \prod_{j=1}^m H_{s_j}(y_j)$ if $y \in B^m_\omega$ and $H^\omega_s(y) = 0$ otherwise.

For a function $v \in L_2(X)$ represented by the series (3.1), noting the truncation $S_{\Lambda(\xi)}v$ given by (3.5) and (3.4), we define

$$S_{\Lambda(\xi)}^\omega v := \sum_{s \in \Lambda(\xi)} v_s H^\omega_s. \quad (3.13)$$

Below in this subsection, we use letters $C$ and $K$ to denote various constants which may depend on the parameters $M, q, \sigma, \theta$, as mentioned in Theorem 3.6.

In what follows, for convenience we consider $\mathbb{R}^m$ as the subset of all $y \in \mathbb{R}^\infty$ such that $y_j = 0$ for $j = m + 1, \ldots$. If $f$ is a function on $\mathbb{R}^m$ taking values in a Hilbert space $X$, then $f$ has an extension to $\mathbb{R}^{m'}$ for $m' > m$ or to $\mathbb{R}^\infty$ which is denoted again by $f$, by the formula $f(y) = f\left((y_j)_{j=0}^m\right)$ for $y = (y_j)_{j=1}^m$ or $y = (y_j)_{j=\in N}$, respectively.
The tensor product of standard Gaussian probability measures $\gamma(y)$ on $\mathbb{R}^m$ is defined by

$$d\gamma(y) := \bigotimes_{j=1}^{m} g(y_j) d(y_j).$$

For a $\gamma$-measurable subset $\Omega$ in $\mathbb{R}^m$, the spaces $L_2(\Omega, X, \gamma)$ and $L_2(\Omega, \gamma)$ are defined in the usual way.

Under the assumptions of Lemma 3.3, $S_{\Lambda(\xi)}^\omega$ can be seen as a function on $\mathbb{R}^m$, where we recall that $m := m(\xi)$. The function $S_{\Lambda(\xi)}^\omega$ can be treated as the truncation of $S_{\Lambda(\xi)}^\omega$ on the supercube $B_\omega^m$. For $g \in L_2(\mathbb{R}^m, X, \gamma)$, we have $\|g\|_{L_2(\mathbb{R}^m, X, \gamma)} = \|g\|_{L_2(\mathbb{R}^\infty, X, \gamma)}$ in the sense of extension of $g$. We will make use of these facts without mention.

We are interested in the estimating in terms of parameter $\xi$ the error of the approximation of $v \in L_2(X)$ by $S_{\Lambda(\xi)}^\omega$. To this end, we need some bound for the $L_2(\mathbb{R}^m \setminus B_\omega^m, \gamma)$ norm of polynomials on $\mathbb{R}^m$.

**Lemma 3.4** Let $\varphi(y) = \prod_{j=1}^{m} \varphi_j(y_j)$ for $y \in \mathbb{R}^m$, where $\varphi_j$ is a polynomial in the variable $y_j$ of degree not greater than $\omega$ for $j = 1, \ldots, m$. Then there holds

$$\|\varphi\|_{L_2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} \leq C m \exp(-K\omega) \|\varphi\|_{L_2(\mathbb{R}^m, \gamma)},$$

where the positive constants $C$ and $K$ are independent of $\omega, m$ and $\varphi$.

**Proof.** The proof of the lemma relies on the following inequality which is an immediate consequence of [38, Theorem 6.3]. Let $\psi$ be a polynomial of degree at most $\ell$. Applying [38, Theorem 6.3] for polynomial $\psi(\sqrt{\omega}t)$ with weight $\exp(-t^2)$ (in this case $a_\ell = \sqrt{\ell}$, see [38, Page 41]) and $\kappa = \sqrt{2} - 1$ we get

$$\|\psi\|_{L_2(\mathbb{R} \setminus [-\sqrt{\ell}, \sqrt{\ell}], \gamma)} \leq C \exp(-K\ell) \|\psi\|_{L_2([-\sqrt{\ell}, \sqrt{\ell}], \gamma)}$$

for some positive numbers $C$ and $K$ independent of $\ell$ and $\psi$. Hence, for a polynomial $\psi$ of degree not greater than $\omega$ we obtain

$$\|\psi\|_{L_2(\mathbb{R} \setminus B_\omega, \gamma)} \leq C \exp(-K\omega) \|\psi\|_{L_2([-\sqrt{\omega}, \sqrt{\omega}], \gamma)} \leq C \exp(-K\omega) \|\psi\|_{L_2(\mathbb{R}, \gamma)} . \quad (3.14)$$

We denote

$$I_j := \{y = (y_i)_{i=1}^{m} \in \mathbb{R}^m : |y_j| > 2\sqrt{\omega}\}, \quad j = 1, \ldots, m.$$ 

Since

$$\mathbb{R}^m \setminus B_\omega^m = \bigcup_{j=1}^{m} I_j,$$

we have

$$\|\varphi\|_{L_2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} \leq \sum_{j=1}^{m} \|\varphi\|_{L_2(I_j, \gamma)} = \sum_{j=1}^{m} \left( \|\varphi_j\|_{L_2(\mathbb{R} \setminus B_\omega, \gamma)} \prod_{i \neq j} \|\varphi_i\|_{L_2(\mathbb{R}, \gamma)} \right). \quad (3.15)$$

Applying (3.14) for the polynomials $\varphi_j$, for $j = 1, \ldots, m$, whose degree is not greater than $\omega$ we obtain

$$\|\varphi_j\|_{L_2(\mathbb{R} \setminus B_\omega, \gamma)} \leq C \exp(-K\omega) \|\varphi_j\|_{L_2(\mathbb{R}, \gamma)}$$
with some positive constants $C$ and $K$ independent of $\omega$, $m$ and $\varphi$. This together with (3.15) yields

$$\|\varphi\|_{L^2(\mathbb{R}^m \setminus B_m, \gamma)} \leq C \exp (-K\omega) m \prod_{i=1}^{m} \|\varphi_i\|_{L^2(\mathbb{R}, \gamma)} = Cm \exp (-K\omega) \|\varphi\|_{L^2(\mathbb{R}^m, \gamma)}.$$ 

Lemma 3.5 \textit{Let} $0 < q < \infty$ \textit{and} $\theta \geq 1/q$. \textit{For every} $v \in \mathcal{L}_2(X)$ \textit{satisfying Assumption $A_{q,\theta}$ and for every} $\xi > 1$, \textit{there holds}

$$\|v - S_{\Lambda(\xi)}^\omega v\|_{\mathcal{L}^2_2} \leq C \xi^{-1/q},$$

\textit{where the positive constant} $C$ \textit{is independent of} $v$ \textit{and} $\xi$.

\textbf{Proof.} Since $S_{\Lambda(\xi)}v$ and $S_{\Lambda(\xi)}^\omega v$ can be considered as functions on $\mathbb{R}^m$, we have

$$\|S_{\Lambda(\xi)}v - S_{\Lambda(\xi)}^\omega v\|_{\mathcal{L}^2_2} = \|S_{\Lambda(\xi)}v - S_{\Lambda(\xi)}^\omega v\|_{L^2_2(\mathbb{R}^m \setminus B_m, X, \gamma)},$$

and consequently,

$$\|v - S_{\Lambda(\xi)}^\omega v\|_{\mathcal{L}^2_2} \leq \|v - S_{\Lambda(\xi)}v\|_{\mathcal{L}^2_2} + \|S_{\Lambda(\xi)}v - S_{\Lambda(\xi)}^\omega v\|_{L^2_2(\mathbb{R}^m \setminus B_m, X, \gamma)}. \tag{3.17}$$

From Lemma 3.1, it is sufficient to show that the second term in the right-hand side is bounded by $C \xi^{-1/q}$. By the equality

$$\|H_s - H_s^\omega\|_{L^2_2(\mathbb{R}^m, \gamma)} = \|H_s\|_{L^2_2(\mathbb{R}^m \setminus B_m, \gamma)}, \quad s \in \Lambda(\xi),$$

and the triangle inequality we obtain

$$\|S_{\Lambda(\xi)}v - S_{\Lambda(\xi)}^\omega v\|_{L^2_2(\mathbb{R}^m \setminus B_m, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \|v_s\|_X \|H_s - H_s^\omega\|_{L^2_2(\mathbb{R}^m, \gamma)}$$

$$= \sum_{s \in \Lambda(\xi)} \|v_s\|_X \|H_s\|_{L^2_2(\mathbb{R}^m \setminus B_m, \gamma)}. \tag{3.18}$$

Notice that for every $s \in \Lambda(\xi)$,

$$H_s(y) = \prod_{j=1}^{m} H_{s_j}(y_j), \quad y \in \mathbb{R}^m,$$

where $H_{s_j}$ is a polynomial in variable $y_j$, of degree not greater than $m_1(\xi)$. By Lemma 3.2(ii), (3.12) and the assumption $\theta \geq 1/q$ we have that

$$m_1(\xi) \leq |K_{q,\theta} \xi^\omega| \leq |K_{q,\theta}| = \omega.$$

Applying Lemma 3.4 gives

$$\|H_s\|_{L^2_2(\mathbb{R}^m \setminus B_m, \gamma)} \leq Cm \exp (-K\omega) \|H_s\|_{L^2_2(\mathbb{R}^m, \gamma)} = Cm \exp (-K\omega),$$

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where the positive constants $C$ and $K$ are independent of $\omega$, $m$ and $s$. This together with (3.18) and the Cauchy–Schwarz inequality yields that

$$
\|S_{\Lambda(\xi)}^P - S_{\Lambda(\xi)}^\omega v\|_{L_2(\mathbb{R}^m \setminus B_m^\omega, X, \gamma)} \leq C m \exp(-K\omega) \sum_{s \in \Lambda(\xi)} \|v_s\|_X
\leq C m \exp(-K\omega) \sup_{s \in F} \sigma_s^{-1} \sum_{s \in \Lambda(\xi)} \sigma_s \|v_s\|_X
\leq C m \exp(-K\omega) \sum_{s \in \Lambda(\xi)} (\sigma_s \|v_s\|_X)^2 \left(\sum_{s \in \Lambda(\xi)} (\sigma_s \|v_s\|_X)^2\right)^{1/2}.
$$

(3.19)

Using (3.12), Assumption $A_{q,\theta}$ and Lemmata 3.2 and 3.3 we finally obtain

$$
\|S_{\Lambda(\xi)}^P - S_{\Lambda(\xi)}^\omega v\|_{L_2(\mathbb{R}^m \setminus B_m^\omega, X, \gamma)} \leq C \xi^{3/2} \exp(-K\xi) \leq C \xi^{-1/q}.
$$

(3.20)

This is the bound of the second term and therefore the stated result is proved.

### 3.2 Approximation by deep ReLU neural networks

In this subsection, we construct deep ReLU neural networks which can be used to approximate functions in $L^2(X)$, and prove bounds of the error of the approximation by them.

The main result in this subsection is read as follows.

**Theorem 3.6** Let $v \in L^2(X)$ satisfy Assumption $A_{q,\theta}$ with $\theta \geq 4/q$. Then for every $\xi > 2$, we can construct a deep ReLU neural network $\Phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{R}^m$, $m \leq \lfloor K_{q,\xi} \rfloor$, having the following properties.

(i) The deep ReLU neural network $\Phi_{\Lambda(\xi)}$ is independent of $v$;

(ii) The input and output dimensions of $\Phi_{\Lambda(\xi)}$ are at most $m$;

(iii) The components $\phi_s$, $s \in \Lambda(\xi)$, of $\Phi_{\Lambda(\xi)}$ are deep ReLU neural networks on $\mathbb{R}^{|\nu_s|}$ with $|\nu_s| \leq K_{q,\theta} \xi^{1/\theta q}$, having support contained in the super-cube $[-T, T]^{|\nu_s|}$, where $T := 4 \sqrt{\lfloor K_{q,\theta} \rfloor}$;

(iv) $W(\Phi_{\Lambda(\xi)}) \leq C \xi \log \xi$;

(v) $L(\Phi_{\Lambda(\xi)}) \leq C \xi^{1/\theta q}$;

(vi) If $\phi_s$ is extended to the whole $\mathbb{R}^\infty$ by $\phi_s(y) = \phi_s\left((y_j)_{j \in \nu_s}\right)$ for $y = (y_j)_{j \in \mathbb{N}} \in \mathbb{R}^\infty$, the approximation of $v$ by the function

$$
\Phi_{\Lambda(\xi)}(y) := \sum_{s \in \Lambda(\xi)} v_s \phi_s(y)
$$

(3.21)

gives the error estimate

$$
\|v - \Phi_{\Lambda(\xi)}v\|_{L_2(X)} \leq C \xi^{-1/q}.
$$

(3.22)

Here the positive constants $C = C_{m,q,\sigma,\theta}$ are independent of $v$ and $\xi$. 


Proof. For convenience we brief the proof into several steps. Recall that throughout the present paper, we use letters \( C \) and \( K \) to denote general positive constants which may take different values.

**Step 1.** In this step, we construct a deep ReLU neural network \( \phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)} \) on \( \mathbb{R}^m \) and prove the claims (i)–(iii) in Theorem 3.6.

We preliminarily notice the following. Suppose that \( \phi_{\Lambda(\xi)} \) and therefore, the function \( \Phi_{\Lambda(\xi)} \) are already constructed. Due to the inequality

\[
\|v - \Phi_{\Lambda(\xi)} v\|_{L^2(X)} \leq \|v - S^\omega_{\Lambda(\xi)} v\|_{L^2(X)} + \|S^\omega_{\Lambda(\xi)} v - \Phi_{\Lambda(\xi)} v\|_{L^2(B^m, \nu)} + \|\Phi_{\Lambda(\xi)} v\|_{L^2(\mathbb{R}^m \setminus B^m, \nu)},
\]

the claim (vi) will be proven if we show the bound \( C\xi^{-1/q} \) for the three terms in the right-hand side. This bound has been shown for first term as in Lemma 3.5. This tell us that we should construct \( \phi_{\Lambda(\xi)} \) on \( \mathbb{R}^m \) for approximating \( S^\omega_{\Lambda(\xi)} v \) by \( \Phi_{\Lambda(\xi)} v \), and prove that the second and third terms in (3.23) are bounded by \( C\xi^{-1/q} \).

Let us begin to construct \( \phi_{\Lambda(\xi)} \). Since the function \( S^\omega_{\Lambda(\xi)} v \) is a linear combination of the truncated Hermite polynomials \( H^\omega_s, s \in \Lambda(\xi) \), on the supercube \( B^m \subset \mathbb{R}^m \), we will use the construction in Lemma 2.5 to design a deep ReLU neural network \( \phi_s \) for approximating each \( H^\omega_s \) with an appropriate accuracy. Then the deep ReLU neural network \( \phi_{\Lambda(\xi)} \) formed by parallelization as described in Lemma 2.2 will be desired.

It is well-known that for each \( s \in \mathbb{N}_0 \), the univariate Hermite polynomial \( H_s \) can be written as

\[
H_s(x) = \sqrt{s! \sum_{\ell=0}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^{\ell}}{\ell!(s-2\ell)!} x^{s-2\ell}} := \sum_{\ell=0}^{s} a_{s,\ell} x^{\ell}.
\]

Then for each \( s \in \Lambda(\xi) \) we have

\[
H_s(y) = \prod_{j=1}^{m} H_{s_j}(y_j) = \sum_{s=0}^{s} \left( \prod_{j=1}^{m} a_{s_j,\ell_j} \right) y^\ell = \sum_{\ell=0}^{s} a_{\ell} y^\ell,
\]

where we put \( a_{\ell} := \prod_{j=1}^{m} a_{s_j,\ell_j} \) and \( y^\ell := \prod_{j=1}^{m} y_j^{\ell_j} \).

From (3.13) and (3.25) we get for every \( y \in B^m \omega \),

\[
S^\omega_{\Lambda(\xi)} v(y) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^{s} a_{\ell} \left( y^\ell \right)^\omega = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^{s} a_{\ell} \left( 2\sqrt{\omega} \right)^{\ell_1} \prod_{j \in \nu_\ell} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j}.
\]

We will use this form of representation of \( S^\omega_{\Lambda(\xi)} v \) for constructing \( \phi_{\Lambda(\xi)} \) and \( \Phi_{\Lambda(\xi)} \).

Let \( \ell \in \mathbb{F} \) be such that \( 0 \leq \ell \leq s \). By definition we have \( \nu_\ell \subset \nu_s \). By changing variables

\[
x = \frac{y}{2\sqrt{\omega}}, \quad x \in \nu_s,
\]

we have

\[
\prod_{j \in \nu_\ell} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j} = \prod_{j \in \nu_\ell} \varphi_j \left( \frac{y_j}{2\sqrt{\omega}} \right) \prod_{j \in \nu_\ell \setminus \nu_0} \varphi_0 \left( \frac{y_j}{2\sqrt{\omega}} \right) = h_{s,\ell}(x), \quad y \in B^{|\nu_\ell|}_{\omega}.
\]
where
\[ h_{s,\ell}(x) := \prod_{j \in \nu_\ell} \varphi_j^\ell(x_j) \prod_{j \in \nu_\ell \setminus \nu_\ell} \varphi_0(x_j), \tag{3.28} \]
\( \varphi_0 \) and \( \varphi_1 \) are the piece-wise linear functions defined before Lemma 2.5. Hence by Lemma 2.5, for every \( 0 \leq \ell \leq s \), with
\[ \delta_s^{-1} := \xi^{1/q+1/2} p_s(1) (2\sqrt{\omega})|s| \max_{0 \leq \ell \leq s} \{|a_\ell|\}, \tag{3.29} \]
there exists a deep ReLU neural network \( \phi_{s,\ell} \) on \( \mathbb{R}^{|\nu_s|} \) with \( \text{supp}(\phi_{s,\ell}) \subseteq [-2,2]|\nu_s| \) such that
\[ \sup_{y \in B_{|\nu_s|}} \left| h_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) - \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq \delta_s, \tag{3.30} \]
and therefore,
\[ \sup_{y \in B_{|\nu_s|}} \left| \prod_{j \in \nu_s} \left( \frac{y_j}{2\sqrt{\omega}} \right) \right| - \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq \delta_s, \tag{3.31} \]
and
\[ \text{supp} \left( \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right) \subseteq B_{|\nu_s|}. \tag{3.32} \]

Considering the right-hand side of (3.29), we have \( \omega \geq \xi > 1 \) and therefore,
\[ (2\sqrt{\omega})|s| \max_{0 \leq \ell \leq s} \{|a_\ell|\} \geq 2|a_s| |a_s| \geq 1. \tag{3.33} \]
The last inequality can be proven by using (3.24), (3.25) and Stirling’s approximation. With the definition (3.29), this yields that
\[ |\ell|_1 + |\nu_s \setminus \nu_\ell| \leq |s|_1 \leq p_s(1) \leq \delta_s^{-1}. \]
Hence, the size and the depth of \( \phi_{s,\ell} \) are bounded as
\[ W(\phi_{s,\ell}) \leq C \left( 1 + |s|_1 (\log |s|_1 + \log \delta_s^{-1}) \right) \leq C \left( 1 + |s|_1 \log \delta_s^{-1} \right) \tag{3.34} \]
and
\[ L(\phi_{s,\ell}) \leq C \left( 1 + \log |s|_1 (\log |s|_1 + \log \delta_s^{-1}) \right) \leq C \left( 1 + \log |s|_1 \log \delta_s^{-1} \right). \tag{3.35} \]

For approximating \( H^s \), we define the deep ReLU neural network \( \phi_s \) on \( \mathbb{R}^{|\nu_s|} \) by
\[ \phi_s(y) := \sum_{0 \leq \ell \leq s} a_\ell (2\sqrt{\omega})^{|\ell|_1} \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right), \quad y \in \mathbb{R}^{|\nu_s|}, \tag{3.36} \]
which is a parallelization of the component deep ReLU neural networks \( \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \).

We define \( \phi_{\Lambda(\ell)} := (\phi_s)_{s \in \Lambda(\ell)} \) as the deep ReLU neural network on \( \mathbb{R}^m \) realized by parallelization \( \phi_s, s \in \Lambda(\ell) \).
In what follows, according to the above convention, for \( s \in \Lambda(\xi) \), in some places we identify the functions \( \phi_{s,\ell} \) and \( \phi_s \) on \( \mathbb{R}^{|\nu_s|} \) with their extensions on \( \mathbb{R}^m \) or on \( \mathbb{R}^\infty \) due to the inclusions \( \nu_s \subset \{1, \ldots, m\} \subset \mathbb{N} \).

The input dimension of \( \phi_{\Lambda(\xi)} \) is not greater than \( m \) which is at most \( \lfloor Kq\xi \rfloor \) by Lemma 3.3. The output dimension of \( \phi_{\Lambda(\xi)} \) is the number \( |\Lambda(\xi)| \) which is at most \( \lfloor Kq\xi \rfloor \) by Lemma 3.2(i).

From (3.32) it follows the inclusion
\[
supp(\phi_{s}) \subset B_{4\omega}^{\nu_s}, \quad s \in \Lambda(\xi),
\]
and from Lemma 3.2(ii) the inequality \( |\nu_s| \leq \lfloor Kq,\theta\xi \rfloor^\frac{1}{q} \). The claims (i)–(iii) are proven.

**Step 2.** In this step, we prove the claim (vi) in Theorem 3.6.

Due to the inequality (3.23) and Lemma 3.5 to prove the claim (vi) it is sufficient to show the bounds
\[
\|S_{\Lambda(\xi)}^\omega y - \Phi_{\Lambda(\xi)} y\|_{L^2(B_{2\omega}^m, X, \gamma)} \leq C\xi^{-1/q},
\]
and
\[
\|\Phi_{\Lambda(\xi)} y\|_{L^2(\mathbb{R}^m \setminus B_{2\omega}^m, X, \gamma)} \leq C\xi^{-1/q},
\]
where the positive constants \( C \) are independent of \( v \) and \( \xi \).

From the equality
\[
\Phi_{\Lambda(\xi)}(y) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^s a_{s,\ell} \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right)
\]
and (3.26), (3.30), (3.36), similarly to (3.19), we prove (3.38):
\[
\|S_{\Lambda(\xi)}^\omega y - \Phi_{\Lambda(\xi)} y\|_{L^2(B_{2\omega}^m, X, \gamma)} = \left\| \sum_{s \in \Lambda(\xi)} v_s H_s^\omega - \sum_{s \in \Lambda(\xi)} v_s \phi_s(y) \right\|_{L^2(B_{2\omega}^m, X, \gamma)}
\leq \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s |a_{s,\ell}| \left( 2\sqrt{\omega} \right)^{|s|1} \delta_s
\leq \xi^{-1/q-1/2} \sum_{s \in \Lambda(\xi)} \|v_s\|_X \leq C\xi^{-1/q}.
\]

We now verify (3.39). We first prove the following auxiliary inequality
\[
\left| \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq 2, \quad \forall y \in \mathbb{R}^m.
\]

Due to (3.32), it is sufficient to prove this inequality for \( y \in B_{4\omega}^{\nu_s} \). Observe that the inequalities (3.33) and \( p_s(1) \geq 1 \) yield \( \delta_s \leq 1 \). On the other hand, from the definition of \( h_{s,\ell} \) it follows
\[
\sup_{y \in B_{4\omega}^{\nu_s}} \left| h_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq 1
\]

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By using the last two inequalities, from (3.30) we prove (3.40) for $y \in B_{4\omega}^{1/2}$.

By (3.36) and (3.40) we have that

$$\|\Phi(\xi)v\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, X, \gamma)} \leq C \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s |a_{s,\ell}(2\sqrt{\omega})|^{|s|/2} \|\phi_{s,\ell}(2\sqrt{\omega})\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, \gamma)} \leq 2 \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s |a_{s,\ell}(2\sqrt{\omega})|^{|s|/2} \|1\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, \gamma)}.$$

Applying Lemma 3.4 to the polynomial $\varphi(y) = 1$, we get

$$\|\Phi(\xi)v\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, X, \gamma)} \leq C m \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s |4\omega|^{|s|/2} \exp(-K\omega) |a_{s,\ell}| = C m \sum_{s \in \Lambda(\xi)} \|v_s\|_X |4\omega|^{|s|/2} \exp(-K\omega) \sum_{\ell=0}^s |a_{s,\ell}|.$$

In order to estimate the sum $\sum_{\ell=0}^s |a_{s,\ell}|$, we need an inequality for the coefficients of Hermite polynomials. By the representation (3.24) of $H_s$, $s \in \mathbb{N}_0$, there holds

$$\sum_{\ell=0}^s |a_{s,\ell}| \leq \sqrt{s!} \sum_{\ell=0}^{\lfloor s/2 \rfloor} \frac{2^{-\ell}}{\ell!(s-2\ell)!} \leq \sqrt{s!} \frac{\sqrt{s}}{2} \sum_{\ell=0}^{\lfloor s/2 \rfloor} 2^{-\ell} \leq \sqrt{s!}.$$

Indeed, this inequality is obvious with $s = 0, 1, 2, 3$. When $s \geq 4$ we have $\frac{1}{\ell!(s-2\ell)!} \leq \frac{1}{2}$ for all $\ell = 0, \ldots, \lfloor s/2 \rfloor$. Therefore,

$$\sum_{\ell=0}^s |a_{s,\ell}| \leq \sqrt{s!} \sum_{\ell=0}^{\lfloor s/2 \rfloor} \frac{2^{-\ell}}{\ell!(s-2\ell)!} \leq \sqrt{s!} \frac{\sqrt{s}}{2} \sum_{\ell=0}^{\lfloor s/2 \rfloor} 2^{-\ell} \leq \sqrt{s!}.$$

It follows from (3.41) that

$$\sum_{\ell=0}^s |a_{s,\ell}| = \sum_{\ell=0}^s \prod_{j=1}^m |a_{s_j,\ell_j}| \leq \prod_{j=1}^m \sum_{\ell=0}^{s_j} |a_{s_j,\ell_j}| \leq \prod_{j=1}^m \sqrt{s_j!}, \quad (3.42)$$

and hence,

$$\sum_{\ell=0}^s |a_{s,\ell}| \leq \prod_{j=1}^m \sqrt{s_j!} \leq \prod_{j=1}^m s_j^{s_j/2} = |s_1|^{s_1/2}. \quad (3.43)$$

By using this estimate and Lemma 3.2, we can continue the estimation of $\|\Phi(\xi)v\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, X, \gamma)}$ as

$$\|\Phi(\xi)v\|_{L_2(\mathbb{R}^m \setminus B_{w_0}^m, X, \gamma)} \leq C m \sum_{s \in \Lambda(\xi)} \|v_s\|_X |4\omega|^{m_1} \frac{m_1}{2} \exp(-K\omega) m_1^{m_1/2}$$

$$\leq C m |\Lambda(\xi)|^{1/2} \left( \sum_{s \in \Lambda(\xi)} \|v_s\|_X^2 \right)^{1/2} |4\omega|^{m_1} \frac{m_1}{2} \exp(-K\omega) m_1^{m_1/2}$$

$$\leq C m \xi^{1/2} |4\omega|^{m_1} \frac{m_1}{2} \exp(-K\omega) m_1^{m_1/2},$$

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where \( m_1 = m_1(\xi) \). We have from the inequality \( \frac{1}{m_1} \leq \frac{1}{\xi} \) and Lemma 3.2 that \( m_1 \leq K_\theta \xi^{1/4} \), and from Lemma 3.3 that \( m \leq K_\theta \xi \). Taking account of the choice of \( \omega \), we derive the estimate
\[
\| \Phi_{\Lambda(\xi)} \|_{L_2(\mathbb{R}^n \setminus B_m, \gamma)} \leq C \xi^{3/2} (4K_\theta \xi^{1/4})^{K_\theta \xi^{1/4}/2} (K_\theta \xi^{1/4})^{K_\theta \xi^{1/4}/2} \exp(-KK_\theta \xi),
\]
which implies (3.39). This completes the proof of the claim (vi).

**Step 3.** In this step, we prove the claims (iv) and (v) in Theorem 3.6. Namely, we prove that for the size of \( \Phi_{\Lambda(\xi)} \),
\[
W(\Phi_{\Lambda(\xi)}) \leq C \xi \log \xi,
\]
and for the depth of \( \Phi_{\Lambda(\xi)} \),
\[
L(\Phi_{\Lambda(\xi)}) \leq C \xi^{1/\theta q} \log \xi,
\]
where the positive constants \( C \) are independent of \( \nu \) and \( \xi \).

By Lemma 2.2 and (3.34) the size of \( \Phi_{\Lambda(\xi)} \) is estimated as
\[
W(\Phi_{\Lambda(\xi)}) = \sum_{(s, \ell) \in \Lambda^\ast(\xi)} L(\phi_{s, \ell}) \leq C \sum_{(s, \ell) \in \Lambda^\ast(\xi)} (1 + |s|_1 \log \delta_s^{-1})
\]
where
\[
\Lambda^\ast(\xi) := \{(s, \ell) \in \mathcal{F} \times \mathcal{F} : s \in \Lambda(\xi) \text{ and } 0 \leq \ell \leq s \}.
\]
From (3.29) and the inequality \( \log \xi \geq 1 \) we derive that
\[
\log(\delta_s^{-1}) \leq C \left( \log \xi + \log p_s(1) + |s|_1 \log \omega + \log \left( \max_{0 \leq \ell \leq s} |a_{\ell s}| \right) \right).
\]
Noting that \( \log \omega \geq \log \xi \) and \( |s|_1^2 \geq |s|_1 \) for all \( s \in \mathcal{F} \), we obtain
\[
\sum_{(s, \ell) \in \Lambda^\ast(\xi)} (1 + |s|_1 \log \delta_s^{-1}) \leq C \left( \sum_{(s, \ell) \in \Lambda^\ast(\xi)} |s|_1 \log p_s(1)
\right.
\]
\[
+ \log \omega \sum_{(s, \ell) \in \Lambda^\ast(\xi)} |s|_1^2 + \sum_{(s, \ell) \in \Lambda^\ast(\xi)} |s|_1 \log \left( \max_{0 \leq \ell \leq s} |a_{\ell s}| \right) \right).
\]
To estimate the terms in the right-hand side we need the following auxiliary assertion. Let \( \tau \geq 0 \), \( 0 < q < \infty \), and \( \Lambda^\ast(\xi) \) be defined in (3.47). Assume \( \left( p_s \left( \frac{\tau + 1}{q} \right) \sigma_s^{-1} \right)_{s \in \mathcal{F}} \in \ell_q(\mathcal{F}) \). Then there holds
\[
\sum_{(s, \ell) \in \Lambda^\ast(\xi)} p_s(\tau) \leq C \xi.
\]
Indeed, from the definition of the set \( \Lambda(\xi) \) and the functions \( p_s(\tau) \) we derive
\[
\sum_{(s, \ell) \in \Lambda^\ast(\xi)} p_s(\tau) = \sum_{s \in \Lambda(\xi)} \sum_{\ell=0}^{s} p_s(\tau) \leq \xi \sum_{s \in \mathcal{F}} \sum_{\ell=0}^{s} p_s(\tau) \sigma_s^{-q}
\]
\[
= \xi \sum_{s \in \mathcal{F}} \left( \prod_{j=1}^{m} (1 + s_j) \right) p_s(\tau) \sigma_s^{-q} \leq \xi \sum_{s \in \mathcal{F}} p_s(\tau + 1) \sigma_s^{-q} \leq C \xi.
\]

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Observe that by the definitions (3.2) we have $\log p_s(1) \leq |s|_1$ and $|s|^k \leq p_s(k)$ for $k \in \mathbb{N}$. Hence, for the first and second terms on the right-hand side of (3.49), since $(p_s(\frac{4}{q})\sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ from (3.50) we derive that
\[
\sum_{(s,\ell) \in \Lambda^*(\xi)} |s|_1 \log p_s(1) \leq \sum_{(s,\ell) \in \Lambda^*(\xi)} |s|^2 \leq \sum_{(s,\ell) \in \Lambda^*(\xi)} p_s(2) \leq C\xi
\]
and
\[
\log \omega \sum_{(s,\ell) \in \Lambda^*(\xi)} |s|^2 \leq \log \omega \sum_{(s,\ell) \in \Lambda^*(\xi)} p_s(2) \leq C\xi \log \omega \leq C\xi \log \xi,
\]
where in the last inequality we note that $\omega = [K_{q,\theta} \xi]$, see (3.12). Now we turn to the third term in (3.49). The inequalities (3.42) imply
\[
\log \left(\max_{0 \leq \ell \leq s} |a_\ell| \right) \leq \log \left(\prod_{j=1}^m s_j!\right) \leq \sum_{j=1}^m \log(s_j!) \leq \sum_{j=1}^m s_j^2 \leq p_s(2).
\]
Using (3.50) again we obtain
\[
\sum_{(s,\ell) \in \Lambda^*(\xi)} |s|_1 p_s(2) \leq \sum_{(s,\ell) \in \Lambda^*(\xi)} p_s(3) \leq C\xi,
\]
since $(p_s(\frac{4}{q})\sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$. This together with (3.51) and (3.52) yields
\[
\sum_{(s,\ell) \in \Lambda^*(\xi)} (1 + |\ell|_1 \log \delta_s^{-1}) \leq C \xi \log \xi,
\]
which combined with (3.46) gives (3.44).

We now prove (3.45). By Lemma 2.2 and (3.35) the depth of $\phi_{\Lambda(\xi)}$ is bounded as
\[
L(\phi_{\Lambda(\xi)}) = \max_{(s,\ell) \in \Lambda^*(\xi)} L(\phi_{s,\ell}) \leq C \max_{(s,\ell) \in \Lambda^*(\xi)} (1 + \log |s|_1 \log \delta_s^{-1}).
\]
Due to (3.48), this inequality can be modified as
\[
L(\phi_{\Lambda(\xi)}) \leq C \max_{s \in \Lambda(\xi)} (\log |s|_1) \max_{(s,\ell) \in \Lambda^*(\xi)} (\log \delta_s^{-1}).
\]
From Lemma 3.2 we obtain
\[
\max_{s \in \Lambda(\xi)} (\log |s|_1) \leq C \log \xi.
\]
We have by (3.48) that
\[
\max_{(s,\ell) \in \Lambda^*(\xi)} (\delta_s^{-1}) \leq C \left(\log \xi + \max_{s \in \Lambda(\xi)} \log p_s(1) + \log(2\omega) \max_{s \in \Lambda(\xi)} |s|_1 + \max_{0 \leq \ell \leq s} |a_\ell| \right).
\]
For the second and third terms on the right-hand side, we have by the well-known inequality $\log p_s(1) \leq |s|_1$ and Lemma 3.2,
\[
\max_{s \in \Lambda(\xi)} \log p_s(1) \leq \max_{s \in \Lambda(\xi)} |s|_1 \leq C_{\xi^{1/\theta q}}
\]
and
\[ \log(2\omega) \max_{s \in \Lambda(\xi)} |s|_1 \leq C \xi^{1/\theta_q} \log \xi. \]

Now we turn to the fourth term in (3.55). From (3.43) it follows that
\[ \log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \leq \log \left( |s|_1 \log |s|_1 \right). \]

Hence,
\[ \max_{(s, \ell) \in \Lambda^*(\xi)} \log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \leq \max_{s \in \Lambda(\xi)} (|s|_1 \log |s|_1) \leq C \xi^{1/\theta_q} \log \xi. \]

This together with (3.54)–(3.2) yields (3.45).

The proof of Theorem 3.6 is complete.

**Theorem 3.7** Let \( v \in L_2(X) \) satisfy Assumption A_{q, \theta} with \( \theta \geq 4/q \). Then for every integer \( n > 2 \), we can construct a deep ReLU neural network \( \phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)} \) on \( \mathbb{R}^m \) with \( m := \left\lfloor K_n \frac{n}{\log n} \right\rfloor \) for some positive constant \( K_q \), having the following properties.

(i) The deep ReLU neural network \( \phi_{\Lambda(\xi_n)} \) is independent of \( v \);

(ii) The input and output dimensions of \( \phi_{\Lambda(\xi_n)} \) are at most \( m \);

(iii) The components \( \phi_s, s \in \Lambda(\xi_n) \), of \( \phi_{\Lambda(\xi_n)} \) are deep ReLU neural networks on \( \mathbb{R}^{\nu_s} \) with \( |\nu_s| \leq C_\delta n^\delta \), having support contained in the supercube \([-T, T]^{\nu_s}\), where \( T := C_\delta \sqrt{\frac{n}{\log n}} \) and \( \delta := \frac{1}{q \theta} \);

(iv) \( W(\phi_{\Lambda(\xi_n)}) \leq n \);

(v) \( L(\phi_{\Lambda(\xi_n)}) \leq C \delta n^\delta \);

(vi) The approximation of \( v \) by \( \Phi_{\Lambda(\xi_n)}v \) defined as in Theorem 3.6(iv), gives the error estimates
\[ \|v - \Phi_{\Lambda(\xi_n)}v\|_{L_2(X)} \leq C n^{-1/q} \leq C \left( \frac{n}{\log n} \right)^{-1/q}. \]

Here the positive constants \( C = C_{M, q, \sigma, \theta} \) are independent of \( v \) and \( n \).

**Proof.** For a given integer \( n > 1 \), we choose \( \xi_n \geq 2 \) as the maximal number satisfying the inequality \( C \xi_n \log \xi_n \leq n \), where \( C \) is the positive constant in the claim (iii) of Theorem 3.6. It is easy to verify that there exists a positive constant \( C_1 \) independent of \( n \) such that
\[ C \xi_n \log \xi_n \leq n \leq C_1 \xi_n \log \xi_n. \]  

Hence, \( C_2 \log \xi_n \leq \log n \leq C_3 \log \xi_n \) for some positive constants \( C_2, C_3 \). From the last inequalities and (3.56) we derive that there exist positive constants \( C_4 \) and \( C_5 \) independent of \( n \) such that
\[ C_4 \frac{n}{\log n} \leq \xi_n \leq C_5 \frac{n}{\log n}. \]

From Theorem 3.6 with \( \xi = \xi_n \) we deduce the desired results. \( \square \)
3.3 Application to parameterized elliptic PDEs with lognormal inputs

In this section, we apply the results in the previous section to deep ReLU neural network approximation of the solution $u(y)$ to the parametric elliptic PDEs (1.1) with lognormal inputs (1.3). This is based on a weighted $\ell_2$-summability of the series $\|u_s\|_V$ in following lemma which combines [4, Theorems 3.3 and 4.2] and [13, Lemma 5.3].

**Lemma 3.8** Let $\theta$ be an arbitrary nonnegative number and $(p_s(\theta))_{s \in F}$ the sequence given as in (3.2). Let $0 < q < \infty$ and $(\rho_j)_{j \in \mathbb{N}}$ be an increasing sequence of positive numbers such that $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$. Assume further that

$$\sum_{j \in \mathbb{N}} \rho_j |\psi_j| < \infty.$$  

Then we have that for any $\eta \in \mathbb{N}$,

$$\sum_{s \in F} (\sigma_s^2 \|u_s\|_V)^2 < \infty \text{ with } (p_s(\theta)\sigma_s^{-1})_{s \in F} \in \ell_q(\mathbb{F}),$$

where

$$\sigma_s^2 := \sum_{\|s'\|_{\mathbb{N}}} \left( \frac{s}{s'} \right) \prod_{j \in \mathbb{N}} \rho_j^{2s_j}. \quad (3.57)$$

This weighted $\ell_2$-summability result leads to significant improvements of the convergence rate in the case when the component functions $\psi_j$ have limited overlaps such as splines, finite elements or wavelet bases (for details, see [4]). Our result for the solution $u$ to the parametric elliptic PDEs (1.1) with lognormal inputs (1.3) is read as follows.

**Theorem 3.9** Under the assumptions of Lemma 3.8, let $0 < q < \infty$ and $\delta$ be arbitrary positive number. Then for every integer $n > 2$, we can construct a deep ReLU neural network $\Phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \left\lfloor K\frac{n}{\log n} \right\rfloor$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\Phi_{\Lambda(\xi_n)}$ is independent of $u$;

(ii) The input and output dimensions of $\Phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) The components $\phi_s, s \in \Lambda(\xi_n)$, of $\Phi_{\Lambda(\xi_n)}$ are deep ReLU neural networks on $\mathbb{R}^{|\nu_s|}$ with $|\nu_s| \leq C_\delta n^\delta$, having support contained in the super-cube $[-T, T]^{\nu_s}$, where $T := C_\delta \sqrt{\frac{n}{\log n}}$;

(iv) $W(\Phi_{\Lambda(\xi_n)}) \leq n$;

(v) $L(\Phi_{\Lambda(\xi_n)}) \leq C_\delta n^\delta$

(vi) The approximation of $u$ by $\Phi_{\Lambda(\xi_n)}u$ defined as in Theorem 3.6(iv), gives the error estimates

$$\|u - \Phi_{\Lambda(\xi_n)}u\|_{L_2(V)} \leq Cm^{-1/q} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.$$
Here the positive constants $C$ and $C_\delta$ are independent of $u$ and $n$.

**Proof.** Since $(\rho_j)_{j \in \mathbb{N}}$ is an increasing sequence of positive numbers, it is easily seen that $\sigma = (\sigma_s)_{s \in \mathbb{F}}$ is an increasing sequence and that $\sigma_{e'} \leq \sigma_{e}$ if $i' < i$. Therefore Assumption A$q,\theta$ is satisfied. To prove the theorem we apply Theorem 3.7 to the solution $u$. Without loss of generality we can assume that $\delta \leq 1/4$. We take first the number $\theta := 1/\delta q$ satisfying the inequality $\theta \geq 4/q$, and then choose a number $\eta \in \mathbb{N}$ satisfying the inequality $\eta > 2(\theta + 1)/q$. By using Lemma 3.8 one can check that for $X = V$ and the sequence $(\sigma_s)_{s \in \mathbb{F}}$ defined as in (3.57), $u \in L^2(V)$ satisfies the assumptions of Theorem 3.7 which proves the theorem.

### 3.4 Application to approximation of holomorphic functions

In this section we show that some holomorphic functions satisfy the assumption and therefore we can explicitly construct deep ReLU neutral network to approximate them. Let us first introduce the concept of “$(b, \xi, \varepsilon, X)$-holomorphic functions” which has been introduced in [16].

We recall the concept of “$(b, \xi, \varepsilon, X)$-holomorphic functions” which has been introduced in [16]. For $N \in \mathbb{N}$ and a positive sequence $\rho = (\rho_j)_{j=1}^N$ is $(b, \xi)$-admissible if

$$\sum_{j=1}^N b_j \rho_j \leq \xi. \quad (3.59)$$

A function $v \in L^2(X)$ is called $(b, \xi, \varepsilon, X)$-holomorphic if

(i) for every $N \in \mathbb{N}$ there exists $v_N : \mathbb{R}^N \to X$, which, for every $(b, \xi)$-admissible $\rho$, admits a holomorphic extension (denoted again by $v_N$) from $\mathcal{S}(\rho) \to X$; furthermore, for all $N < M$

$$v_N(y_1, \ldots, y_N) = v_M(y_1, \ldots, y_N, 0, \ldots, 0) \quad \forall (y_j)_{j=1}^N \in \mathbb{R}^N, \quad (3.60)$$

(ii) for every $N \in \mathbb{N}$ there exists $\varphi_N : \mathbb{R}^N \to \mathbb{R}_+$ such that $\|\varphi_N\|_{L^2(\mathbb{R}^N; \gamma)} \leq \varepsilon$ and

$$\sup_{\rho \text{ is } (b, \xi)\text{-adm.}} \sup_{z \in \mathcal{B}(\rho)} \|v_N(y + z)\|_X \leq \varphi_N(y) \quad \forall y \in \mathbb{R}^N,$$

(iii) with $\tilde{v}_N : \mathbb{R}^\infty \to X$ defined by $\tilde{v}_N(y) := v_N(y_1, \ldots, y_N)$ for $y \in \mathbb{R}^\infty$ it holds

$$\lim_{N \to \infty} \|v - \tilde{v}_N\|_{L^2(X)} = 0.$$

The following key result on weighted $\ell_2$-summability of $(b, \xi, \varepsilon, X)$-holomorphic functions has been proven in [16, Corollary 4.9].

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Theorem 3.10  Let $v$ be $(b, \xi, \varepsilon, X)$-holomorphic for some $b \in \ell_p(\mathbb{N})$ with $0 < p < 1$. Let $\eta \in \mathbb{N}$ and let the sequence $\rho = (\rho_j)_{j \in \mathbb{N}}$ be defined by

$$\rho_j := b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|b\|_{\ell_p(\mathbb{N})}.$$ 

Assume that $b$ is a decreasing sequence and that $b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|b\|_{\ell_p(\mathbb{N})} > 1$ for all $j \in \mathbb{N}$. Then $v$ satisfies Assumption $A_{q, \theta}$ for $q := \frac{p}{1-p}$, $\sigma = \sigma(\rho, \eta) := (\sigma_s)_{s \in \mathbb{F}}$ given by (3.57), and $M := \varepsilon C_{b, \xi}$ with some positive constant $C_{b, \xi}$.

The condition $b$ is a decreasing sequence implies $\rho$ is an increasing sequence and therefore $\sigma_{i'} \leq \sigma_i$ if $i' < i$. Moreover $\rho_j = b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|b\|_{\ell_p(\mathbb{N})} > 1$ for all $j \in \mathbb{N}$ and $\sigma = \sigma(\rho, \eta) := (\sigma_s)_{s \in \mathbb{F}}$ given by (3.57) imply that $(p_s(\theta) \sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ for any $\theta > 0$. For a proof we refer the reader again to [15, Lemma 5.3].

In the same way as the proof of Theorem 3.9, from Theorem 3.10 we derive

Theorem 3.11  Let $v$ be $(b, \xi, \varepsilon, X)$-holomorphic for some $b \in \ell_p(\mathbb{N})$ with $0 < p < 1$, and let $\delta$ be an arbitrary positive number. Then, with the notations of Theorem 3.10, for every integer $n > 2$, we can construct a deep ReLU neural network $\Phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \left\lfloor \frac{K n}{\log n} \right\rfloor$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\Phi_{\Lambda(\xi_n)}$ is independent of $v$;

(ii) The input and output dimensions of $\Phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) The components $\phi_s$, $s \in \Lambda(\xi_n)$, of $\Phi_{\Lambda(\xi_n)}$ are deep ReLU neural networks on $\mathbb{R}^{|\nu_s|}$ with $|\nu_s| \leq C \delta n^\delta$, having support contained in the super-cube $[-T, T]^{|\nu_s|}$, where $T := C \delta \sqrt{\frac{n}{\log n}}$;

(iv) $W(\Phi_{\Lambda(\xi_n)}) \leq n$;

(v) $L(\Phi_{\Lambda(\xi_n)}) \leq C \delta n^\delta$;

(vi) The approximation of $v$ by $\Phi_{\Lambda(\xi_n)} v$ defined as in Theorem 3.6(iv), gives the error estimates

$$\|v - \Phi_{\Lambda(\xi_n)} v\|_{L_2(\mathbb{X})} \leq C m^{-(1/p-1)} \leq C \left( \frac{n}{\log n} \right)^{(1/p)-1}.$$ 

Here the positive constants $C$, $C_\delta$ and $C_\delta'$ are independent of $v$ and $n$.

We notice some important examples of $(b, \xi, \varepsilon, X)$-holomorphic functions which are solutions to parametric PDEs equations and which were studied in [16].

Formally, replacing $y = (y_j)_{j \in \mathbb{N}} \in \mathbb{R}^\infty$ in the coefficient $a(y)$ in (1.3) by $z = (z_j)_{j \in \mathbb{N}} = (y_j + i\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\infty$, the real part of $a(z)$ is

$$\Re[a(z)] = \exp \left( \sum_{j \in \mathbb{N}} y_j \psi_j(x) \right) \cos \left( \sum_{j \in \mathbb{N}} \xi_j \psi_j(x) \right).$$

(3.61)
We find that $\Re \{ a(z) \} > 0$ if
\[ \left\| \sum_{j \in \mathbb{N}} \xi_j \psi_j \right\|_{L_\infty(D)} < \frac{\pi}{2}. \]

This observation motivate the study of the analytic continuation of the solution map $y \mapsto u(y)$ to $z \mapsto u(z)$ for complex parameters $z = (z_j)_{j \in \mathbb{N}}$ where each $z_j$ lies in the strip
\[ S_j (\rho) := \{ z_j \in \mathbb{C} : |\Im z_j| < \rho_j \} \quad (3.62) \]
and where $\rho_j > 0$ and $\rho = (\rho_j)_{j \in \mathbb{N}} \in (0, \infty)\mathbb{N}$ is any sequence of positive numbers such that
\[ \left\| \sum_{j \in \mathbb{N}} \rho_j \psi_j \right\|_{L_\infty(D)} < \frac{\pi}{2}. \]

For further detail of this continuation we refer to [16, Proposition 3.8].

In general, let $b(y)$ be defined as in (1.3) and $\mathcal{V}$ a holomorphic map from an open set in $L_\infty(D)$ to $X$. Then function compositions of the type
\[ v(y) = \mathcal{V}(\exp(b(y))) \]
are $(b, \xi, \varepsilon, X)$-holomorphic under certain conditions [16, Proposition 4.11]. This allows us to apply Theorem 3.11 for deep ReLU neural network approximation of solutions $v(y) = \mathcal{V}(\exp(b(y)))$ as $(b, \xi, \varepsilon, X)$-holomorphic functions on various function spaces $X$, to a wide range of parametric and stochastic PDEs with lognormal inputs. Such function spaces $X$ are high-order regularity spaces $H^s(D)$ and corner-weighted Sobolev (Kondrat’ev) spaces $K^s(D)$ ($s \geq 1$) for the parametric elliptic PDEs (1.1) with lognormal inputs (1.3); spaces of solutions to linear parabolic PDEs with lognormal inputs (1.3); spaces of solutions to linear elastostics equations with lognormal modulus of elasticity; spaces of solutions to Maxwell equations with lognormal permittivity.

After the first version of the present paper appeared in ArXiv website, we have been informed about the paper [46] on deep ReLU neural network approximation of holomorphic functions, in a private communication with its authors. Subsection 3.4 has been added in the second ArXiv version after the paper [46] appeared. The results on convergence rate of this subsection improved the results of [46] by the help of using deep ReLU neural networks connecting neurons in a layer with neurons in preceding layers (see Section 2).

4 Approximation in Bochner spaces with Jacobi measure

The theory of non-adaptive deep ReLU neural network approximation in Bochner spaces with Gaussian measure, which has been discussed in Section 3 can be generalized and extended to other situations. In this section, we investigate non-adaptive methods of deep ReLU neural network approximation in Bochner spaces with Jacobi probability measure. Functions to be approximated satisfy a weighted $\ell_2$-summability of their Jacobi gpc expansion coefficients. We construct such methods and prove the convergence rate of the approximation by them. These methods are constructed via the truncated Jacobi gpc expansion of functions. The results are then applied to the approximation of the solutions to parametric elliptic PDEs (1.2) with affine inputs (1.4).
4.1 Approximation by truncations of the Jacobi gpc expansion

For given $a, b > -1$, we consider the infinite tensor product of the Jacobi probability measure on $\mathbb{I}^\infty$

$$d\nu_{a,b}(y) := \bigotimes_{j \in \mathbb{N}} \delta_{a,b}(y_j) dy_j,$$

where

$$\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)}.$$

If $v \in L_2(X) := L_2(\mathbb{I}^\infty, X, \nu_{a,b})$ for a separable Hilbert space $X$, we consider the orthonormal Jacobi gpc expansion of $v$ of the form

$$v = \sum_{s \in F} v_s J_s(y), \quad (4.1)$$

where

$$J_s(y) = \bigotimes_{j \in \mathbb{N}} J_{s_j}(y_j), \quad v_s := \int_{\mathbb{I}^\infty} v(y) J_s(y) d\nu_{a,b}(y),$$

and $(J_k)_{k \geq 0}$ is the sequence of Jacobi polynomials on $\mathbb{I} := [-1, 1]$ normalized with respect to the Jacobi probability measure, i.e., $\int_{\mathbb{I}} |J_k(y)|^2 \delta_{a,b}(y) dy = 1$. One has the Rodrigues’ formula

$$J_k(y) = c_{0,0}^a \frac{c_{k}^a}{k!2^k(1-y)^{-a}(1+y)^{-b}} \frac{d^k}{dy^k} ((y^2 - 1)^k(1-y)^a(1+y)^b),$$

where $c_{0,0}^a := 1$ and

$$c_{k}^a := \sqrt{(2k + a + b + 1)k!\Gamma(k + a + b + 1)\Gamma(a + 1)\Gamma(b + 1)} \Gamma(k + a + 1)\Gamma(k + b + 1)\Gamma(a + b + 2), \quad k \in \mathbb{N}. \quad (4.2)$$

Examples corresponding to the values $a = b = 0$ are the family of the Legendre polynomials, and to the values $a = b = -1/2$ the family of the Chebyshev polynomials.

Throughout this section, if $f$ is a function on $\mathbb{I}^m$ taking values in a Hilbert space $X$, then $f$ has an extension to $\mathbb{I}^{m'}$ for $m' > m$ or to $\mathbb{I}^\infty$ which is denoted again by $f$, by the formula

$$f(y) = f\left((y_j)_{j=0}^m\right) \text{ for } y = (y_j)_{j=1}^{m'} \text{ or } y = (y_j)_{j \in \mathbb{N}},$$

Assumption B Let $0 < q < \infty$, $c_{k}^{a,b}$ be defined as in (4.2) and let $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 such that $(\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$. For $v \in L_2(X)$ represented by the series (4.1), there exists a sequence of positive numbers $(\rho_j)_{j \in \mathbb{N}}$ such that $c_{k}^{a,b} \rho_j^{-k} \leq \delta_j^{-k}$ for $k, j \in \mathbb{N}$ and

$$\left(\sum_{s \in F} (\sigma_s \|v_s\|_X)^2\right)^{1/2} \leq M < \infty,$$

where

$$\sigma_s := c_{s}^{-1} \prod_{j \in \mathbb{N}} \rho_j^{s_j}, \quad c_s := \prod_{j \in \mathbb{N}} c_{s_j}. \quad (4.3)$$
In this subsection, for the function \( v \in L_2(X) \) represented by the series (4.1) and the sequence \((\sigma_s)_{s \in F}\) given as in (4.3), we consider the approximation of \( v \) by the truncation

\[
S_{\Lambda(\xi)} v := \sum_{s \in \Lambda(\xi)} v_s J_s,
\]

where \( \Lambda(\xi) \) is defined by the formula (3.4) for the sequence \((\sigma_s)_{s \in F}\) given as in (4.3).

In the same way as the proof of Lemma 3.1, we prove

**Lemma 4.1** For every \( v \in L_2(X) \) satisfying Assumption B and for every \( \xi > 1 \), there holds

\[
\|v - S_{\Lambda(\xi)} v\|_{L_2(X)} \leq M \xi^{-1/q}.
\]

We will need some auxiliary results for further use. The following lemma is a direct consequence of [13, Lemma 6.2].

**Lemma 4.2** Let \( 0 < q < \infty \) and \( \theta \) and \( \lambda \) be arbitrary nonnegative real numbers. Assume that \( \rho = (\rho_j)_{j \in \mathbb{N}} \) be a sequence of numbers strictly larger than 1 such that \((\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})\). Then for the sequences \((\sigma_s)_{s \in F}\) and \((p_s(\theta, \lambda))_{s \in F}\) given as in (4.3) and (3.2), respectively, we have

\[
\sum_{s \in F} p_s(\theta, \lambda)\sigma_s^{-q} < \infty.
\]

**Lemma 4.3** Let \( 0 < q < \infty \), \( c_k^{a, b} \) be defined as in (4.2) and let \((\delta_j)_{j \in \mathbb{N}} \) be a sequence of numbers strictly larger than 1 such that \((\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})\). Assume that there exists a sequence of positive number \((\rho_j)_{j \in \mathbb{N}} \) such that \( c_k^{a, b} \rho_j^{-k} < \delta_j^{-k} \), \( k, j \in \mathbb{N} \). For the sequence \((\sigma_s)_{s \in F}\) given as in (4.3), let \( m_1(\xi) \) be the number defined by (3.6). Then we have for every \( \xi \geq 2 \)

\[
m_1(\xi) \leq C \log \xi,
\]

with the positive constant \( C \) independent of \( \xi \).

**Proof.** The proof relies on Lemmata 3.2 and 4.2 and a technique from the proof of [45, Lemma 2.8(ii)]. Fix a number \( p \) satisfying \( 0 < p < q \) and let the sequence \((\beta_s)_{s \in F}\) be given by

\[
\beta_s^{-1} := \begin{cases} \max(\sigma_s^{-1}, j^{-1/p}) & \text{if } s = e^j, \\ \sigma_s^{-1} & \text{otherwise}. \end{cases}
\]

Notice that the sequence \((\alpha_s^{-1})_{s \in F}\) defined by

\[
\alpha_s^{-1} := \begin{cases} j^{-1/p} & \text{if } s = e^j, \\ 0 & \text{otherwise}, \end{cases}
\]

belongs to \( \ell_q(F) \). On the other hand, from Lemma 4.2 one can see that the sequence \((\sigma_s^{-1})_{s \in F}\) belongs to \( \ell_q(F) \). This implies that the sequence \((\beta_s^{-1})_{s \in F}\) belongs to \( \ell_q(F) \). Hence, by Lemma 3.2 the set \( \Lambda_{\beta}(\xi) := \{ s \in F : \beta_s^q \leq \xi \} \) is finite. Notice also that \((\beta_s)_{s \in F}\) is increasing and \( \Lambda_{\beta}(\xi) \) is downward closed. Put \( n := |\Lambda_{\beta}(\xi)| \). Then the set \( \Lambda_{\beta}(\xi) \) contains \( n \) largest elements of \((\beta_s)_{s \in F}\). Therefore by the construction of \((\beta_s)_{s \in F}\) we have

\[
\min_{s \in \Lambda_{\beta}(\xi)} \beta_s^{-1} = \beta_{s_n}^{-1} \geq n^{-1/p}.
\]
Since \( c_{k}^{a,b} \rho_j^{-k} \leq \delta_j^{-k} \), \( k, j \in \mathbb{N} \) and \((\delta_j)_{j \in \mathbb{N}} \) be a sequence of numbers strictly larger than 1 and \((\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_{q}(\mathbb{N})\), there exists \( \delta < 1 \) such that \( c_{k}^{a,b} \rho_j^{-k} \leq \delta \) for \( k, j \in \mathbb{N} \). Therefore have for \( r > 1 \),

\[
\sup_{|s| = r} \beta_s^{-1} = \sup_{|s| = r} \sigma_s^{-1} \leq \delta^r.
\]

Let \( \bar{r} > 1 \) be an integer such that \( n^{-1/p} > \delta^\bar{r} \). Then one can see that

\[
\max_{s \in \Lambda_{\beta}(\xi)} |s| < \bar{r}.
\]

For the function \( g(t) := \delta^t \), its inverse is defined as \( g^{-1}(x) = \frac{\log x}{\log \delta} \). Hence we get \( \bar{r} < g^{-1}(n^{-1/p}) \), and consequently,

\[
\max_{s \in \Lambda_{\beta}(\xi)} |s| < g^{-1}(n^{-1/p}) \leq C \log n = C \log |\Lambda_{\beta}(\xi)|.
\]

By Lemma 3.2 we obtain the inequality \( |\Lambda_{\beta}(\xi)| \leq C \xi \) which together with the inclusion \( \Lambda(\xi) \subset \Lambda_{\beta}(\xi) \) proves (4.4).

**Lemma 4.4** Let the Jacobi polynomial \( J_s \) be written in the form

\[
J_s(y) = \sum_{\ell=0}^{s} a_{s,\ell}y^\ell,
\]

then

\[
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq K_{a+b}\delta^s.
\]

**Proof.** It is well-known that for each \( s \in \mathbb{N} \), the univariate Jacobi polynomial \( J_s \) can be written as

\[
J_s(y) = \frac{\Gamma(a + s + 1)}{s!\Gamma(a + b + s + 1)} \sum_{m=0}^{s} \left( \begin{array}{c} s \\ m \end{array} \right) \frac{\Gamma(a + b + s + m + 1)}{\Gamma(a + m + 1)} \left( \frac{y - 1}{2} \right)^m,
\]

where \( \Gamma \) is the gamma function. From \( \frac{n-1}{2} = \frac{n-2+1}{2} \) we see that \( |a_{s,\ell}| \) is equal to the coefficient of \( (y-2)\ell \). Therefore, choosing \( y = 3 \) we get

\[
\sum_{\ell=0}^{s} |a_{s,\ell}| = \frac{\Gamma(a + s + 1)}{s!\Gamma(a + b + s + 1)} \sum_{m=0}^{s} \left( \begin{array}{c} s \\ m \end{array} \right) \frac{\Gamma(a + b + s + m + 1)}{\Gamma(a + m + 1)}.
\]

Let

\[
B(x, y) := \int_{0}^{1} t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

be the beta function. It is decreasing in \( x \) and in \( y \). Hence for \( m \leq s \),

\[
\frac{\Gamma(a + b + s + m + 1)}{\Gamma(a + m + 1)} = \frac{\Gamma(b + s)}{B(a + m + 1, b + s)} \leq \frac{\Gamma(b + s)}{B(a + s + 1, b + s)} = \frac{\Gamma(a + b + 2s + 1)}{\Gamma(a + s + 1)}.
\]

This gives

\[
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq \frac{\Gamma(a + b + 2s + 1)}{s!\Gamma(a + s + 1)} \sum_{m=0}^{s} \left( \begin{array}{c} s \\ m \end{array} \right) = 2^s \frac{\Gamma(a + b + 2s + 1)}{s!\Gamma(a + b + s + 1)}.
\]
By using Stirling’s formula for the gamma function $\Gamma(x + \alpha) \sim \Gamma(x)x^\alpha$, we get

$$
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq C \left( \frac{(2e)^s(a+b+s+1)^s}{s^s} \right) \leq C(2e)^s \left( \frac{a+b+1}{s} + 1 \right)^s \leq K_{a+b}6^s.
$$

### 4.2 Approximation by deep ReLU neural networks

**Theorem 4.5** Let $v \in L_2(X)$ satisfy Assumption B. Then for every integer $n \geq 4$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \lceil \frac{K}{\log n} \rceil$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ is independent of $v$;

(ii) The input and output dimensions of $\phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) $W(\phi_{\Lambda(\xi_n)}) \leq n$;

(iv) $L(\phi_{\Lambda(\xi_n)}) \leq C \log n \log \log n$;

(v) The components $\phi_s, s \in \Lambda(\xi_n)$, of $\phi_{\Lambda(\xi_n)}$ are deep ReLU neural networks on $\mathbb{R}^{\nu_s}$. If $\Phi_{\Lambda(\xi_n)}v$ is defined as in Theorem 3.6(iv) with replacing $\mathbb{R}^\infty$ by $\mathbb{I}^\infty$, then the approximation of $v$ by $\Phi_{\Lambda(\xi_n)}v$ gives the error estimates

$$
\|v - \Phi_{\Lambda(\xi_n)}v\|_{L_2(X)} \leq Cm^{-1/q} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.
$$

Here the positive constants $C$ are independent of $v$ and $n$.

**Proof.** Similar to the proof of Theorem 3.7, this theorem is deduced from a counterpart of Theorem 3.6 for the case $\mathbb{I}^\infty$. It states that for every $\xi \geq 4$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{I}^m$ with $m \leq \lfloor K_q \xi \rfloor$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi)}$ is independent of $v$;

(ii) The input and output dimensions of $\phi_{\Lambda(\xi)}$ are at most $m$;

(iii) $W(\phi_{\Lambda(\xi)}) \leq C \xi \log \xi$;

(iv) $L(\phi_{\Lambda(\xi)}) \leq C \xi \log \log \xi$;

(v) The components $\phi_s, s \in \Lambda(\xi)$, of $\phi_{\Lambda(\xi)}$ are deep ReLU neural networks on $\mathbb{R}^{\nu_s}$. If $\Phi_{\Lambda(\xi)}v$ is defined as in Theorem 3.6(iv) with replacing $\mathbb{R}^\infty$ by $\mathbb{I}^\infty$, then the approximation of $v$ by $\Phi_{\Lambda(\xi)}v$ gives the error estimate

$$
\|v - \Phi_{\Lambda(\xi)}v\|_{L_2(X)} \leq C\xi^{-1/q}.
$$
Here the positive constants $C$ are independent of $v$ and $\xi$.

The proofs of these claims are similar to the proof of Theorem 3.6, but simpler due to the compact property of $I^\infty$. To prove the claims (i)–(v) let us follow the steps in that proof.

**Step 1.** In this step, we construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{R}^m$ and prove the claims (i) and (ii).

Suppose that $\phi_{\Lambda(\xi)}$ and therefore, the function $\Phi_{\Lambda(\xi)}$ are already constructed. Due to the inequality

$$
\| v - \Phi_{\Lambda(\xi)}v \|_{L_2(X)} \leq \| v - S_{\Lambda(\xi)}v \|_{L_2(X)} + \| S_{\Lambda(\xi)}v - \Phi_{\Lambda(\xi)}v \|_{L_2(X)},
$$

the claim (iv) will be proven if we show the bound $C\xi^{-1/4}$ for the two terms in the right-hand side. This bound has been shown for first term as in Lemma 4.1. By Lemma 3.3 $S_{\Lambda(\xi)}v$ can be considered as a function on $\mathbb{R}^m$. Therefore, to prove the claim (iv) we will construct $\phi_{\Lambda(\xi)}$ on $\mathbb{R}^m$ which is able to approximate $S_{\Lambda(\xi)}v$ with an error bounded by $C\xi^{-1/4}$.

From (4.5) for each $s \in \mathbb{F}$ we have

$$
J_s(y) = \sum_{\ell=0}^s a_{s, \ell} y^\ell,
$$

where $a_{s, \ell} := \prod_{j=1}^m a_{s_j, \ell_j}$ and $y^\ell := \prod_{j=1}^m y_j^{\ell_j}$. Hence, we get for every $y \in \mathbb{R}^m$,

$$
S_{\Lambda(\xi)}v(y) := \sum_{s \in \Lambda(\xi)} v_s J_s(y) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^s a_{s, \ell} \prod_{j \in \nu_\ell} y_j^{\ell_j}.
$$

(4.10)

We have

$$
\prod_{j \in \nu_\ell} y_j^{\ell_j} = \prod_{j \in \nu_\ell} \varphi_1^{\ell_j} (y_j) \prod_{j \in \nu_s \setminus \nu_\ell} \varphi_0 (y_j) = h_{s, \ell}(y), \quad y \in [\nu_s],
$$

(4.11)

where

$$
h_{s, \ell}(y) := \prod_{j \in \nu_\ell} \varphi_1^{\ell_j} (y_j) \prod_{j \in \nu_s \setminus \nu_\ell} \varphi_0 (y_j),
$$

$\varphi_0$ and $\varphi_1$ are the piece-wise linear functions defined before Lemma 2.5. By applying Lemma 2.5 to the product in the right-hand side, for every $\ell$ with $0 \leq \ell \leq s$, with

$$
\delta_s^{-1} := \xi^{1/4 + 1/2} p_s(1) \sum_{\ell=0}^s |a_{s, \ell}|,
$$

(4.12)

there exists a deep ReLU neural network $\phi_{s, \ell}$ on $[\nu_s]$ such that

$$
\sup_{y \in [\nu_s]} | y^\ell - \phi_{s, \ell}(y) | \leq \delta_s,
$$

and similarly to (3.34) and (3.35), the size and depth of $\phi_{s, \ell}$ are bounded as

$$
W(\phi_{s, \ell}) \leq C \left( 1 + |s|_1 \log \delta_s^{-1} \right)
$$

(4.13)

and

$$
L(\phi_{s, \ell}) \leq C \left( 1 + \log |s|_1 \log \delta_s^{-1} \right).
$$

(4.14)
Let the deep ReLU neural network \( \phi_s \) on \( \| v_s \| \) be defined by

\[
\phi_s := \sum_{\ell=0}^{s} a_{\ell} \varphi_{s, \ell},
\]

which is a parallelization of component networks \( \varphi_{s, \ell} \). We define \( \phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)} \) as the deep ReLU neural network realized by parallelization \( \phi_s, s \in \Lambda(\xi) \).

The claim (i) follows from the construction. The claim (ii) can be proven in the same way as the proof of the claim (ii) in Theorem 3.6.

**Step 2.** In this step, we prove the claim (v).

From the equality

\[
\Phi_{\Lambda(\xi)}(y) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^{s} a_{\ell} \phi_{s, \ell}(y),
\]

(4.10) and (4.12), similarly to (3.19), we have that

\[
\| S_{\Lambda(\xi)} v - \Phi_{\Lambda(\xi)} v \|_{L_2(X)} \leq \sum_{s \in \Lambda(\xi)} \| v_s \|_X \sum_{\ell=0}^{s} |a_{\ell}| \delta_s \leq \xi^{-1/q-1/2} \sum_{s \in \Lambda(\xi)} \| v_s \|_X \leq C \xi^{-1/q}
\]

which together with Lemma 4.1 and (4.9) proves the claim (iv).

**Step 3.** In this step, we prove the claims (iii) and (iv).

By Lemma 2.2 and (4.13) the size of \( \phi_{\Lambda(\xi)} \) is estimated as

\[
W(\phi_{\Lambda(\xi)}) = \sum_{s \in \Lambda(\xi)} \sum_{\ell=0}^{s} L(\phi_{s, \ell}) \leq C \sum_{(s, \ell) \in \Lambda^*(\xi)} (1 + |s|_1 \log \delta_s^{-1}) \quad (4.15)
\]

where \( \Lambda^*(\xi) \) is given as in (3.47).

From (4.12), the inequality \( \log \xi \geq 1 \) and Lemma 4.4 we derive that

\[
\log(\delta_s^{-1}) \leq C \left( (1/q + 1/2) \log \xi + \log p_s(1) + \log \left( \sum_{\ell=0}^{s} |a_{\ell}| \right) \right)
\]

\[
\leq C \left( \log \xi + \log p_s(1) + |s|_1 \right).
\]

Noting \( |s|_1^2 \geq |s|_1 \) for all \( s \in \mathbb{F} \), we obtain

\[
\sum_{(s, \ell) \in \Lambda^*(\xi)} (1 + |s|_1 \log \delta_s^{-1}) \leq C \left( \xi \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 + \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log p_s(1) \right)
\]

\[
+ \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1^2 \right)
\]

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Hence, in a way similar to the proof of (3.53) we obtain
\[ \sum_{(\ell, s) \in \Lambda^*(\xi)} (1 + |s_1| \log \delta_s^{-1}) \leq C \xi \log \xi, \]
which combined with (4.15) gives the claim (iii).

We now prove the claim (iv). In same way as the proof of (3.54) we derive that the depth of \( \phi_{\Lambda(\xi)} \) is bounded as
\[ L(\phi_{\Lambda(\xi)}) \leq C \max_{s \in \Lambda(\xi)} (\log |s_1|) \max_{(s, \ell) \in \Lambda^*(\xi)} (\log \delta_s^{-1}). \] (4.16)

By Lemma 4.3 we obtain
\[ \max_{s \in \Lambda(\xi)} (\log |s_1|) \leq C \log \log \xi. \] (4.17)

We have by (4.2) that
\[ \max_{(s, \ell) \in \Lambda^*(\xi)} (\delta_s^{-1}) \leq C \left( \log \xi + \max_{s \in \Lambda(\xi)} \log p_s(1) + \max_{s \in \Lambda(\xi)} \left( \sum_{\ell=0}^{s} |a_\ell| \right) \right). \] (4.18)

For the second on the right-hand side, we have by the well-known inequality \( \log p_s(1) \leq |s_1| \) and Lemma 4.3,
\[ \max_{s \in \Lambda(\xi)} \log p_s(1) \leq \max_{s \in \Lambda(\xi)} |s_1| \leq C \log \xi. \] (4.19)

Now we turn to the third term in (4.18). From Lemma 4.4 it follows that
\[ \log \left( \sum_{\ell=0}^{s} |a_\ell| \right) \leq \log \left( (6K_{a,b})^{|s_1|} \right) \leq C |s_1|. \]
Hence, again by Lemma 4.4
\[ \max_{s \in \Lambda(\xi)} \log \left( \sum_{\ell=0}^{s} |a_\ell| \right) \leq C \max_{s \in \Lambda(\xi)} |s_1| \leq C \log \xi. \]
This together with (4.16)–(4.19) yields the claim (iv).

The proof of Theorem 4.5 is complete.

### 4.3 Application to parameterized elliptic PDEs with affine inputs

We now apply Theorem 4.5 to approximation of the solution \( u(y) \) to the parameterized elliptic PDEs (1.1) with affine inputs (1.4).

**Theorem 4.6** Let \( 0 < q < \infty, c^{a,b}_k \) be defined as in (4.2) and let \( (\delta_j)_{j \in \mathbb{N}} \) be a sequence of numbers strictly larger than 1 such that \( (\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N}) \). Let \( \bar{a} \in L_{\infty}(D) \) and \( \text{ess inf} \bar{a} > 0 \). Assume that there exists a sequence of positive numbers \( (\rho_j)_{j \in \mathbb{N}} \) such that \( c^{a,b}_k \rho_j^{-k} < \delta_j^{-k}, k, j \in \mathbb{N}, \) and
\[ \frac{\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_{\infty}(D)}}{\bar{a}} < 1. \] (4.20)

Then for every integer \( n > 1 \), we can construct a deep ReLU neural network \( \phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)} \) on \( \mathbb{R}^m \) with \( m := \left\lfloor \frac{K_n}{\log \xi_n} \right\rfloor \) for some positive constant \( K \), having the following properties.
(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ is independent of $u$;

(ii) The input and output dimensions of $\phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) $W(\phi_{\Lambda(\xi_n)}) \leq n$;

(iv) $L(\phi_{\Lambda(\xi_n)}) \leq C \log n \log \log n$;

(v) The components $\phi_s$, $s \in \Lambda(\xi_n)$, of $\phi_{\Lambda(\xi_n)}$ are deep ReLU neural networks on $\mathbb{R}^{|\nu_s|}$. If $\Phi_{\Lambda(\xi_n)}u$ is defined as in Theorem 3.6(iv) with replacing $\mathbb{R}^\infty$ by $\mathbb{I}^\infty$, then the approximation of $u$ by $\Phi_{\Lambda(\xi_n)}u$ gives the error estimates

$$\|u - \Phi_{\Lambda(\xi_n)}u\|_{L_2(V)} \leq C m^{-1/q} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.$$

Here the positive constants $C$ are independent of $u$ and $n$.

Proof. It has been proven in [5] that under the assumptions of the theorem, for the sequence $(\sigma_s)_{s \in \mathbb{F}}$ given as in (4.3),

$$\sum_{s \in \mathbb{F}} (\sigma_s \|u_s\|_V)^2 < \infty.$$  

This means that Assumption B holds for $v = u$ with $X = V$. Hence, applying Theorem 4.5 to $u$, we prove the theorem.

We next discuss the approximation by deep ReLU neural networks for parametric elliptic PDEs with affine inputs and error measured in the uniform norm of $L_\infty(\mathbb{I}^\infty, V)$ by using $m$-term truncations of the Taylor gpc expansion of $u$. This problem has been studied in [45] for the particular case of the uniform probability measure $\nu_{0,0}$.

If for the sequence $(\rho_j)_{j \in \mathbb{N}}$ of numbers strictly larger than 1 we have the condition 4.20 and if $(\rho_j^{1/j})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$ for some $0 < q < 2$, then the solution $u$ to the parameterized elliptic PDEs (1.1) with affine inputs (1.4) can be decomposed in the Taylor gpc expansion

$$u = \sum_{s \in \mathbb{F}} t_s y_s, \quad t_s = \frac{1}{s!} \partial^s u(0)$$

with

$$\left( \sum_{s \in \mathbb{F}} (\sigma_s \|t_s\|_V)^2 \right)^{1/2} \leq C < \infty,$$

where

$$\sigma_s := \prod_{j \in \mathbb{N}} \rho_j^{s_j},$$

see [5, Theorem 2.1]. Moreover, the sequence $(\|t_s\|_V)_{s \in \mathbb{F}}$ is $\ell_p$-summable with $p = \frac{2q}{2+q} < 1$. We define

$$S_{\Lambda(\xi)}v := \sum_{s \in \Lambda(\xi)} t_s y_s,$$

where $\Lambda(\xi)$ is given by the formula (3.4). The following theorem is an improvement of [45, Theorem 3.9].
Theorem 4.7 Let \( \bar{a} \in L_\infty(D) \) and \( \operatorname{ess} \inf \bar{a} > 0 \). Assume that there exists an increasing sequence \((\rho_j)_{j \in \mathbb{N}}\) of numbers strictly larger than 1 such that the sequence \((\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})\) for some \( q \) with \( 0 < q < 2 \), and there holds the condition (4.20). Then for every integer \( n \geq 4 \), we can construct a deep ReLU neural network \( \Phi_m(x) := (\phi_s)_{s \in \Lambda(x)} \) on \( \mathbb{R}^m \) with \( m := \left\lfloor K \frac{n}{\log n} \right\rfloor \) for some positive constant \( K \), having the following properties.

(i) The deep ReLU neural network \( \Phi_m(x) \) is independent of \( u \);
(ii) The input and output dimensions of \( \Phi_m(x) \) are at most \( m \);
(iii) \( W(\Phi_m(x)) \leq n \);
(iv) \( L(\Phi_m(x)) \leq C \log n \log \log n \);
(v) The components \( \phi_s, s \in \Lambda(x) \), of \( \Phi_m(x) \) are deep ReLU neural networks on \( \mathbb{R}^{\lceil q \rceil} \). If \( \Phi_m(x)u \) is defined as in Theorem 3.6(iv) with replacing \( \mathbb{R}^\infty \) by \( \mathbb{R}^\infty \) and \( u_s \) by \( t_s \), then the approximation of \( u \) by \( \Phi_m(x)u \) gives the error estimates

\[
\| u - \Phi_m(x)u \|_{L_\infty(\mathbb{R}^\infty, V)} \leq C m^{1-1/q-1/(2)} \leq C \left( \frac{n}{\log n} \right)^{-1/q-1/2}.
\]

Here the positive constants \( C \) are independent of \( u \) and \( n \).

Proof. This theorem can be proven in a way similar to the proof of Theorem 4.6. Let us give a brief proof. Given \( \xi \geq 3 \), we have the Cauchy-Schwarz inequality and Lemma 4.2 that

\[
\| u - S_m(x)u \|_{L_\infty(\mathbb{R}^\infty, V)} \leq \sum_{s \notin \Lambda(x)} \| t_s \|_V \leq \left( \sum_{s > \xi^{1/q}} (\sigma_s \| t_s \|_V^2)^{1/2} \right) \left( \sum_{s > \xi^{1/q}} \sigma_s^{-q} \sigma_s^{-(2-q)} \right)^{1/2} \leq C \xi^{-1/q-1/2} \left( \sum_{s \in \mathbb{F}} \sigma_s^{-q} \right)^{1/2} \leq C \xi^{-1/q-1/2}.
\]

Put \( \delta := \xi^{-1/q-1/2} \). For every \( s \in \Lambda(x) \), by Lemma 2.5 there exists a deep ReLU neural network \( \phi_s \) on \( \mathbb{R}^{\lceil q \rceil} \), such that

\[
\sup_{y \in \mathbb{R}^{\lceil q \rceil}} | y^s - \phi_s(y) | \leq \delta,
\]

and the size and depth of \( \phi_s \) are bounded as

\[
W(\phi_s) \leq C (1 + |s|_1 \log \delta^{-1}) \leq C (1 + |s|_1 \log \xi)
\]

and

\[
L(\phi_s) \leq C (1 + \log |s|_1 \log \delta^{-1}) \leq C (1 + \log |s|_1 \log \xi).
\]

We define \( \Phi_m(x) := (\phi_s)_{s \in \Lambda(x)} \) as the deep ReLU neural network realized on \( \mathbb{R}^m \) by parallelization of \( \phi_s, s \in \Lambda(x) \). Consider the approximation of \( u \) by

\[
\Phi_m(x)u := \sum_{s \in \Lambda(x)} t_s \phi_s.
\]
Then by the inclusion \( \| t_s \|_{V} \in \ell_p(\mathbb{F}), \ p \in (0, 1) \) and (4.21), we have

\[
\| u - \Phi_{\Lambda(\xi)} u \|_{L_\infty(\mathbb{I}, V)} \leq \| u - S_{\Lambda(\xi)} u \|_{L_\infty(\mathbb{I}, V)} + \| S_{\Lambda(\xi)} u - \Phi_{\Lambda(\xi)} u \|_{L_\infty(\mathbb{I}, V)} \\
\leq C\xi^{-(1/q-1/2)} + \sum_{s \in \Lambda(\xi)} \| t_s \|_{V} \| y_s - \phi_s \|_{L_\infty(\mathbb{I}, V)} \tag{4.22}
\]

where the positive constants \( C \) may be different and are independent of \( u \) and \( \xi \). By the construction of \( \phi_{\Lambda(\xi)} \) we have

\[
W(\phi_{\Lambda(\xi)}) \leq \sum_{s \in \Lambda(\xi)} C\xi^{-(1/q-1/2)} + C\xi^{-(1/q-1/2)} \sum_{s \in \Lambda(\xi)} \| t_s \|_{V} \leq C\xi^{-(1/q-1/2)},
\]

where in the last estimate we used Lemmata 3.2(i) and 4.2. Similarly, we have

\[
L(\phi_{\Lambda(\xi)}) \leq \max_{s \in \Lambda(\xi)} L(\phi_s) \leq C \max_{s \in \Lambda(\xi)} \left( 1 + \log |s|_1 \log \xi \right) \leq C \log \xi \log \log \xi, \tag{4.23}
\]

where in the last estimate we used Lemma 4.3. Now following argument at the end of the proof of Theorem 3.9, by using (4.22)–(4.24) we obtain the existence of a number \( \xi_n \) for a given \( n \geq 4 \) for which there hold the claims (i)–(v) in the theorem.

\[ \square \]

**Acknowledgments.** A part of this work was done when the authors were working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). They would like to thank the VIASM for providing a fruitful research environment and working condition.

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