INVERSE MEAN CURVATURE FLOW INSIDE A CONE IN WARPED PRODUCTS

LI CHEN, JING MAO*, NI XIANG AND CHI XU

Abstract. Given a convex cone in the prescribed warped product, we consider hypersurfaces with boundary which are star-shaped with respect to the center of the cone and which meet the cone perpendicularly. If those hypersurfaces inside the cone evolve along the inverse mean curvature flow, then, by using the convexity of the cone, we can prove that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of round sphere as time tends to infinity.

Keywords: Inverse mean curvature flow, cone, warped products.

MSC: Primary 53C44, Secondary 53C42, 35B45, 35K93.

1. Introduction

Unlike the mean curvature flow, which is a shrinking flow, the inverse mean curvature flow (IMCF for short), which says a submanifold evolves along its outward normal direction with a speed equal to the reciprocal of the mean curvature, in general is an expanding flow.

A classical result for IMCF is due to Gerhardt [5], who proved that if a closed, smooth, star-shaped hypersurface with strictly positive mean curvature evolves along the IMCF, then the flow exists for all the time and, after rescaling, the evolving hypersurfaces converge to a round sphere as time tends to infinity (see also [19]). The star-shaped assumption is important in the study of IMCF, since it allows us to equivalently transform the evolution equation of IMCF, which, in local coordinates of the initial submanifold, corresponds to a system of second-order parabolic partial differential equations (PDEs for short) with some specified initial conditions, into a scalar second-order parabolic PDE which seems to be relatively easy to deal with.

In general, singularities may occur in finite time for non-starshaped submanifolds evolving under the IMCF. By defining a notion of weak solutions to IMCF, Huisken and Ilmanen [8, 9] proved the Riemannian Penrose inequality by using the IMCF approach. One of the reasons why people pay attention to the study of IMCF is that one can use IMCF to derive some interesting geometric inequalities like what Huisken and Ilmanen have done. In fact, there already exist some results like this. For instance, we know that for a closed convex surface \( \mathcal{M} \) in the 3-dimensional Euclidean space \( \mathbb{R}^3 \), the classical Minkowski inequality is

\[
\int_{\mathcal{M}} H d\mu \geq \sqrt{16\pi} |\mathcal{M}|,
\]

where \( H \) is the mean curvature, \( |\mathcal{M}| \) is the area of \( \mathcal{M} \), and \( d\mu \) is the volume density of \( \mathcal{M} \). This result can be generalized to the high dimensional case. In fact, for a convex hypersurface \( \mathcal{M} \) in \( \mathbb{R}^n \), one has

\[
\int_{\mathcal{M}} H d\mu \geq (n-1)|S^{n-1}| \frac{1}{n-1} |\mathcal{M}|^{\frac{n-2}{n-1}},
\]

* Corresponding author.
where $|S^{n-1}|$ denotes the area of the unit sphere in $S^{n-1}$ in $\mathbb{R}^n$. By using the method of IMCF, the above Minkowski inequality has been proven to be valid for mean convex and star-shaped hypersurfaces in $\mathbb{R}^n$ also (cf. [3] [7]). Also using the method of IMCF, Brendle, Hung and Wang [11] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurfaces in the $n$-dimensional ($n \geq 3$) anti-de Sitter-Schwarzschild manifold, which generalized the related conclusions in the Euclidean space mentioned above.

The first and second authors have been working on IMCF for several years and have also obtained some interesting results. For instance, Chen and Mao [2] considered the evolution of a smooth, star-shaped and $F$-admissible ($F$ is a 1-homogeneous function of principle curvatures satisfying some suitable conditions) embedded closed hypersurface in the $n$-dimensional anti-de Sitter-Schwarzschild manifold along its outward normal direction has a speed equal to $1/F$ (clearly, this evolution process is a natural generalization of IMCF, and we call it inverse curvature flow. We write as ICF for short), and they proved that this ICF exists for all the time and, after rescaling, the evolving hypersurfaces converge to a sphere as time tends to infinity. This interesting conclusion has been improved by Chen, Mao and Zhou [3] to the situation that the ambient space is a warped product $I \times_{\lambda(r)} N^n$ with $I$ an unbounded interval of $\mathbb{R}$ (i.e., the set of real numbers) and $N^n$ a Riemannian manifold of nonnegative Ricci curvature.

Suppose $N$ and $B$ are semi-Riemannian manifolds with metrics $g_N$ and $g_B$, and let $f > 0$ be a smooth function on $N$. The warped product $N \times_f B$ is the product manifold furnished with the metric tensor $g = \pi^*(g_N) + (f \circ \pi)^2 \sigma^*(g_B)$, where $\pi$ and $\sigma$ are the projections of $N \times B$ onto $N$ and $B$, respectively. $N$ and $B$ are called the base and the fiber of $N \times_f B$, respectively. Clearly, $I \times_{\lambda(r)} N^n$ mentioned above is a special warped product with base $N^n$ and fiber $I \subset \mathbb{R}$. Comparing with general Riemannian manifolds, warped products have some interesting and useful properties (see, for instance, [13] Appendix A) or [16] Appendix A). We call special warped products $[0, \ell) \times_{f(x)} S^{n-1}$, $f(0) = 0$, $f'(0) = 1$, $f|_{(0, \ell)} > 0$, $0 < \ell < \infty$ are $n$-dimensional ($n \geq 2$) spherically symmetric manifolds with the base point $p := \{0\} \times_{f(0)} S^{n-1}$ (also known as generalized space forms). Especially, if $\ell = \infty$, $f(x) = x$, then $[0, \ell) \times_{f(x)} S^{n-1} \equiv \mathbb{H}^n(-k)$, i.e., the $n$-dimensional hyperbolic space with constant sectional curvature $-k < 0$; if $\ell = \frac{\pi}{\sqrt{k}}$, $f(x) = \frac{\sin(\sqrt{k}x)}{\sqrt{k}}$, then after endowing a one-point compactification topology, the closure of $[0, \ell) \times_{f(x)} S^{n-1}$ equals $S^n(k)$, i.e., the $n$-dimensional sphere with constant sectional curvature $k > 0$. Spherically symmetric manifolds are very nice mode spaces which can be used to successfully improve some classical results in Riemannian geometry (for example, Cheng’s eigenvalue comparison theorem, Bishop’s volume comparison theorem, and so on). For more details on this topic, we refer readers to, for instance, [4] [13] [14] [15] [16].

Marquardt [17] successfully proved that if an $n$-dimensional ($n \geq 2$) compact $C^{2,\alpha}$-hypersurface with boundary, which meets a given cone in $\mathbb{R}^{n+1}$ perpendicularly and is star-shaped with respect to the center of the cone, evolves along the IMCF, then the flow exists for all the time and, after rescaling, the evolving hypersurfaces converge to a piece of the round sphere as time tends to infinity. Based on our experience in [2] [3], we would like to know “if we replace the ambient space $\mathbb{R}^{n+1}$ in [17] by a warped product $I \times_{\lambda(r)} N^n$ with $I \subset \mathbb{R}$ an unbounded interval, whether the IMCF exists for all the time or not? What about the convergence if we have the long-time existence?” The purpose of this paper is trying to answer this question.

Assume that, as before, $I$ is an unbounded interval of $\mathbb{R}$ and $N^n$ is a Riemannian manifold with metric $g_N$. Naturally, $I \times_{\lambda(r)} N^n$ is a warped product with the warping function $\lambda(r)$ defined on $I$ and the metric given as follows

$$g = dr \otimes dr + \lambda^2(r)g_N.$$
Let $M^n \subset N^n$ be a portion of $N^n$ such that $\Sigma^n := \{(y(x), x) \in I \times_{\lambda(r)} N^n | y(x) > 0, x \in \partial M^n\}$ be the boundary of a smooth convex cone. We can prove the following conclusion.

**Theorem 1.1.** Let $I \times_{\lambda(r)} N^n$ be an $(n+1)$-dimensional $(n \geq 2)$ warped product with the warping function $\lambda(r)$ satisfying $\lambda(r) > 0$, $\lambda'(r) \leq C$ and $0 \leq \lambda^{1+\alpha}(r)\lambda''(r) \leq C$ for some positive constants $\alpha$, $C$ on $I^\circ$ (if $I$ has an endpoint $a$, then $I^\circ = I \setminus \{a\}$; if $I$ does not have endpoint, then $I^\circ = I$), where $I$ denotes an unbounded interval of $\mathbb{R}$ and $N^n$ is an $n$-dimensional Riemannian manifold with nonnegative Ricci curvature. Let $\Sigma^n \subset I \times_{\lambda(r)} N^n$ be the boundary of a smooth, convex cone that is centered at some interior point of $M^n$ and has the outward unit normal vector $\mu$. Let $F_0 : M^n \to I \times_{\lambda(r)} N^n$ such that $M^n_0 := F_0(M^n)$ is a compact $C^{2,\alpha}$-hypersurface which is star-shaped with respect to the center of the cone and has a strictly positive principal curvature. Assume furthermore that $M^n_0$ meets $\Sigma^n$ orthogonally. That is

$$F_0(\partial M^n) \subset \Sigma^n,$$

where $\overline{v}_0$ is the outward unit normal to $M^n_0$. Then there exists a unique embedding $F \in C^{2+\alpha,1+\alpha}(M^n \times (0, \infty), I \times_{\lambda(r)} N^n) \cap C^{\infty}(M^n \times (0, \infty), I \times_{\lambda(r)} N^n)$ with $F(\partial M^n,t) \subset \Sigma^n$ for $t \geq 0$, satisfying the following system

\[
\begin{cases}
\frac{\partial F}{\partial t} = \nu \circ F & \text{in } M^n \times (0, \infty) \\
\langle \mu \circ F, \overline{v} \circ F \rangle = 0 & \text{on } \partial M^n \times (0, \infty) \\
F(\cdot, 0) = F_0 & \text{on } M^n
\end{cases}
\]

where $\overline{v}$ is the unit normal vector to $M^n_0 := F(M^n, t)$ pointing away from the center of the cone and $H$ is the scalar mean curvature of $M^n_0$. Moreover, after area-preserving rescaling, the rescaled solution $F(\cdot, t)$ converges smoothly to an embedding $F_\infty$, mapping $M^n$ into a piece of a geodesic sphere.

**Remark 1.1.** Clearly, if $\lambda(r) = r$, $N^n = \mathbb{S}^n$ (i.e., the $n$-dimensional Euclidean unit sphere), $I = [0, \infty)$, then $I^\circ = (0, \infty)$ and $I \times_{\lambda(r)} N^n = \mathbb{R}^{n+1}$, Theorem [11] here degenerates into [18 Theorem 1]. That is, our Theorem [11] covers [18 Theorem 1] as a special case.

This paper is organized as follows. The geometry of star-shaped hypersurfaces in the warped product $I \times_{\lambda(r)} N^n$ will be discussed in Section 2, and we will use the fact that star-shaped hypersurfaces $M^n_0$ can be written as graphs over $M^n \subset N^n$ to transform the first evolution equation of the system $(\sharp)$ into a scalar second-order parabolic PDE, which leads to the short-time existence of the IMCF. $C^0$ and gradient estimates will be derived in Sections 3 and 4, respectively. Higher regularity and convergence of the solution of $(\sharp)$ will be shown in the last section.

### 2. Preliminary Facts

In this section, we would like to give some basic facts first such that our conclusions can be explained clearly and understood well.

We want to describe the hypersurface $M^n_t$ at time $t$ as a graph over $M^n \subset N^n \subset I \times_{\lambda(r)} N^n$, and then we can make ansatz

$$\overline{F} : M^n \times [0, T) \to I \times_{\lambda(r)} N^n : (x,t) \to (u(x,t), x)$$

for some function $u : M^n \times [0, T) \to I \subset \mathbb{R}$. Since the initial $C^{2,\alpha}$-hypersurface is star-shaped, there exists a scalar function $u_0 \in C^{2,\alpha}(M^n_0)$ such that the $F_0 : M^n \to I \times_{\lambda(r)} N^n$ has the form $x \mapsto (u_0(x), x)$. Set $\tilde{M}^n_t := \overline{F}(M^n, t)$. Define $p := \overline{F}(x,t)$ and assume that a point on $M^n$ is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. Let $\partial_i$ be the corresponding
coordinate vector fields on $M^n \subset N^n$ and $\sigma_{ij} = g_N(\partial_i, \partial_j)$ be the metric on $M^n \subset N^n$. Let $u_i = D_i u$, $u_{ij} = D_j D_i u$, and $u_{ijk} = D_k D_j D_i u$ denote the covariant derivatives of $u$ with respect to the metric $g_N$ and let $\nabla$ be the Levi-Civita connection of $\tilde{M}^n_t$ with respect to the metric $\tilde{g}$ induced from the metric $g$ of the warped product $I \times_{\lambda(t)} N^n$. The tangent vector on $\tilde{M}^n_t$ is
\[ \tilde{e}_i = \partial_i + D_i u \partial_r \]
and the corresponding outward unit normal vector is given by
\[ \tilde{v} = \frac{1}{v} \left( \partial_r - \frac{1}{\lambda^2} \nabla^j u \partial_j \right), \]
where $\nabla^j u = \sigma^{ij} \nabla_i u$, and $v := \sqrt{1 + \lambda^{-2} |\nabla u|^2}$ with $\nabla u$ the gradient of $u$. Clearly, we know that the induced metric $\tilde{g}$ on $\tilde{M}^n_t$ has the form
\[ g_{ij} = \lambda^2 \sigma_{ij} + \nabla_i u \cdot \nabla_j u \]
and its inverse is given by
\[ g^{ij} = \frac{1}{\lambda^2} \left( \sigma^{ij} - \frac{\nabla^i u \nabla^j u}{1 + |\nabla u|^2} \right) = \frac{1}{\lambda^2} \left( \sigma^{ij} - \frac{\nabla^i u \nabla^j u}{v^2} \right). \]
Let $h_{ij} dx_i \otimes dx_j$ be the second fundamental form of $\tilde{M}^n_t$, and then we have
\[ h_{ij} = \langle \nabla \tilde{e}_i \tilde{e}_j, \tilde{v} \rangle = -\frac{1}{v} \left( u_{i,j} - \lambda \sigma_{ij} - 2 \lambda' \frac{\lambda}{\lambda u} u_{ij} \right), \]
where $u_{i,j}$ is the covariant derivative of $u$. Define a new function $\varphi(x,t) = \int_c^{u(x,t)} \frac{1}{\lambda(s)} ds$, where $x = x(\xi^1, \ldots, \xi^n)$, and then the second fundamental form can be rewritten as
\[ h_{ij} = \frac{\lambda}{v} (\lambda' \sigma_{ij} + \varphi_i \varphi_j) - \varphi_{i,j} \]
and
\[ h^i_j = g^{ik} h_{kj} = \frac{\lambda'}{\lambda v} \delta^i_j - \frac{1}{\lambda v} \overline{\sigma}_{ik} \varphi_{k,j} \quad \text{with} \quad \overline{\sigma}_{ij} = \sigma^{ij} - \frac{\varphi_i \varphi_j}{v^2}. \]
Naturally, the scalar mean curvature is given by
\[ H = \sum_{i=1}^{n} h^i_i = \frac{n \lambda'}{\lambda v} - \frac{1}{\lambda v} \sum_{i=1}^{n} \left( \sum_{k=1}^{n} \overline{\sigma}_{ik} \varphi_{k,i} \right). \]
Based on the above facts and [17], we can get the following existence and uniqueness for the IMCF $(\tilde{z})$.

**Lemma 2.1.** Let $F_0, I, \lambda$ be as in Theorem 1.1. Then there exist some $T > 0$, a unique solution $u \in C^{2+\alpha,1+\alpha}(M^n \times [0,\infty), I) \cap C^\infty(M^n \times (0,\infty), I)$, where $\varphi(x,t) = \int_c^{u(x,t)} \frac{1}{\lambda(s)} ds$, of the following system
\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{v}{\lambda^2} - \frac{v^2}{n \lambda^2 - \sigma^V u_{ij}} & \text{in } M^n \times (0,T) \\
\nabla u \varphi = 0 & \text{on } \partial M^n \times (0,T) \\
\varphi(\cdot,0) = \varphi_0 := \int_c^{u(x,0)} \frac{1}{\lambda(s)} ds & \text{on } M^n,
\end{cases}
\]
and a unique map $\psi : M^n \times [0,T] \to M^n$, which has to be bijective for fixed $t$ and has to satisfy $\psi(\partial M^n,t) = \partial M^n$, such that the map $F$ defined by
\[ F : M^n \times [0,T] \to I \times_{\lambda(t)} N^n : (x,t) \mapsto F(\psi(x,t),t) \]
has the same regularity as stated in Theorem 1.1 and is the unique solution to $(\tilde{z})$. 
Remark 2.1. As pointed out in [17], for immersed hypersurfaces in a Riemannian manifold and for arbitrary smooth supporting hypersurfaces $\Sigma^n$, one can also get the short time existence of $(\tilde{\sharp})$. Naturally, for immersed hypersurfaces in a warped product $I \times_{\lambda(r)} N^n$, we definitely have the short time existence result.

Let $T^*$ be the maximal time such that there exists some $u \in C^{2,1} (M^n, [0, T^*)) \cap C^\infty (M^n, (0, T^*))$ which solves $(\tilde{\sharp})$. In the sequel, we will prove a priori estimates for those admissible solutions on $[0, T^*)$ where $T < T^*$.

3. $C^0$ estimate

In this section, we will use the evolution equation of $\varphi$ to get some estimates for $\lambda$ and $\dot{\varphi}$.

Lemma 3.1. If $\varphi$ satisfies $(\tilde{\sharp})$, and the warping function $\lambda(r)$ satisfies $\lambda(r) > 0$, $\lambda'(r) > 0$, $\lambda''(r) \geq 0$ on $I^0$, then we have

$$\lambda(\inf u(\cdot, 0)) \leq \lambda(u(x, t)) e^{-\frac{t}{n}} \leq \lambda(\sup u(\cdot, 0)), \quad \forall x = (\xi^1, \ldots, \xi^n) \in M^n \subset N^n, t \in [0, T].$$

Proof. If $\varphi$ could reach its maximum at the boundary, by Hopf’s Lemma, it follows that the derivative of $\varphi$ along the outward unit normal vector must be strictly greater than 0, which is contradict with the boundary condition $\nabla_\mu \varphi = 0$. Therefore, $\varphi$ must attain its maximum at interior points. The same situation happens when $\varphi$ reaches its minimum. By the chain rule, it is easy to get $\dot{\lambda} = \lambda' \dot{u} = \lambda' \lambda \dot{\varphi}$. Hence, the evolution equation of $\lambda$ is the following

$$(3.1) \quad \frac{1}{\lambda' \lambda} \dot{\lambda} = \frac{v^2}{n \lambda' - \tilde{\sigma}^{ij} \varphi_{i,j}}.$$

By the facts

$$\lambda_i = \lambda' u_i = \lambda' \lambda \varphi_i$$

and

$$\lambda_{i,j} = \lambda'' \lambda \varphi_i + (\lambda')^2 \varphi_i \varphi_j + \lambda' \lambda \varphi_{i,j},$$

we know that when $\varphi$ gets its maximum or minimum, the same situation happens to $\lambda$. When $\lambda$ gets its maximum, the Hessian of $\varphi$ is negative definite, which means $\tilde{\sigma}^{ij} \varphi_{i,j} \leq 0$. Therefore, when $\lambda$ gets its maximum, we have

$$\frac{1}{\lambda'(\sup u(\cdot, 0)) \lambda} \dot{\lambda} \leq \frac{1}{n \lambda'(\sup u(\cdot, 0))},$$

which implies

$$\frac{1}{\lambda} \dot{\lambda} \leq \frac{1}{n}.$$

Integrating both sides of the above inequality, we can get

$$(3.2) \quad \lambda \leq \lambda(\sup u(\cdot, 0)) e^{\frac{t}{n}}.$$

When $\lambda$ gets its minimum, $\tilde{\sigma}^{ij} \varphi_{i,j}$ is positive definite. By a similar way, we can obtain

$$(3.3) \quad \lambda(\inf u(\cdot, 0)) e^{\frac{t}{n}} \leq \lambda.$$

Combining $(3.2)$ and $(3.3)$ yields the conclusion of Lemma 3.1 directly. \qed

Remark 3.1. Clearly, Lemma 3.1 tells us that in the evolving process of the IMCF $(\tilde{\sharp})$, the rescaled warping function $\lambda(u(x, t)) e^{-\frac{t}{n}}$ can be controlled from both below and above, which implies that one might expect some good convergence for evolving hypersurfaces after rescaling.
Lemma 3.2. If \( \varphi \) satisfies (\( \tilde{z} \)) and \( \lambda \) satisfies \( \lambda(r) > 0, \lambda'(r) > 0, 0 \leq \lambda^{1+\alpha}(r)\lambda''(r) \leq C \) for some positive constants \( \alpha, C \) on \( I^\circ \), then there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \leq \dot{\varphi} \leq C_2.
\]

Proof. Set \( Q(\nabla \varphi, \nabla^2 \varphi, \lambda') := \frac{v^2}{n\lambda - \sigma^2(\varphi_{ij})} \). Differentiating both sides of the first evolution equation of (\( \tilde{z} \)), it is easy to get that \( \dot{\varphi} \) satisfies
\[
\begin{align*}
\frac{\partial \dot{\varphi}}{\partial t} &= Q^{ij} \nabla_{ij} \dot{\varphi} + Q^k \nabla_k \dot{\varphi} - \frac{n\lambda'' \lambda \dot{\varphi}}{(n\lambda - \sigma^2(\varphi_{ij}))^2} \\
\nabla \mu \dot{\varphi} &= 0 \\
\dot{\varphi}(\cdot, 0) &= \dot{\varphi}_0
\end{align*}
\]
(3.4)
in \( M^n \times (0, T) \),
on \( \partial M^n \times (0, T) \),
on \( M^n \).

Similar to the argument in the proof of Lemma 3.1, it follows that \( \dot{\varphi} \) must reach its maximum and the minimum at interior points by applying Hopf’s Lemma to (3.4). Therefore, at the point where \( \dot{\varphi} \) gets its maximum, we have
\[
\dot{\varphi}_i = 0, \quad \dot{\varphi}_{ij} \leq 0.
\]
(3.5)

Conversely, at the point where \( \dot{\varphi} \) gets its minimum, we have
\[
\dot{\varphi}_i = 0, \quad \dot{\varphi}_{ij} \geq 0.
\]
(3.6)

Set \( \dot{\varphi}_{\text{max}} := \sup_{M^n} \dot{\varphi}(\cdot, t), \quad \dot{\varphi}_{\text{min}} := \inf_{M^n} \dot{\varphi}(\cdot, t) \).

Combining (3.4), (3.5) and the assumptions for the warping function \( \lambda \), we can obtain the evolution equation of \( \dot{\varphi}_{\text{max}} \) as follows
\[
\frac{\partial \dot{\varphi}_{\text{max}}}{\partial t} = Q^{ij} \nabla_{ij} \dot{\varphi}_{\text{max}} - \frac{n\lambda'' \dot{\varphi}_{\text{max}}}{\lambda T^2} \leq -\frac{n\lambda'' \dot{\varphi}^3}{v^2} \leq 0,
\]
which implies
\[
\dot{\varphi} \leq \dot{\varphi}_{\text{max}} \leq \sup_{M^n} \dot{\varphi}(\cdot, 0) := C_2.
\]
(3.7)

On the other hand, since \( 0 < \lambda^{1+\alpha}(r)\lambda''(r) \leq C \), we have
\[
0 < \lambda''(r)\lambda(r) \leq Ce^{-\frac{\alpha t}{n}}
\]
by applying Lemma 3.1. In general, the constant \( C \) in the above inequality should be different from the one in the assumption of Lemma 3.2. However, for convenience, we use the same symbol. Then, combining (3.4), (3.6) and the assumptions for \( \lambda \), we have
\[
\frac{\partial \dot{\varphi}_{\text{min}}}{\partial t} = Q^{ij} \nabla_{ij} \dot{\varphi}_{\text{min}} - \frac{n\lambda'' \dot{\varphi}_{\text{min}}^3}{v^2} \geq -\frac{n\lambda'' \dot{\varphi}_{\text{max}}^3}{v^2} \geq -Ce^{-\frac{\alpha t}{n}} \dot{\varphi}
\]
by applying Lemma 3.1. The maximum principle tells us that \( \dot{\varphi} \) is bounded from below by the solution of ordinary differential equation (we write as ODE for short)
\[
\frac{\partial f}{\partial t} = -Ce^{-\frac{\alpha t}{n}} f,
\]
with \( f(0) := \inf_{M^n \subset N^n} \dot{\varphi}(\cdot, 0) \). By a straight calculation, we get the solution of the ODE (3.8) as follows
\[
f = f(0)e^{\frac{C_0}{n}(e^{-\frac{\alpha t}{n}} - 1)} \geq C_1 := f(0)e^{\frac{C_0}{n}} > 0.
\]
Therefore, we have
\[
\dot{\varphi} \geq \dot{\varphi}_{\text{min}}(t) \geq f(t) \geq C_1.
\]
Together with (3.7), the conclusion of Lemma 3.2 follows.

Remark 3.2. By the $C^0$-estimate, we know that IMCF preserves the convexity during the evolving process, which implies $H > 0$ for all $t \in [0, T]$. This fact has been shown in [20, Theorem 3.5].

4. Gradient Estimate

In this section, the gradient estimate will be shown.

Lemma 4.1. If $\varphi$ satisfies (3.9), $\lambda$ satisfies $\lambda(r) > 0$, $\lambda'(r) > 0$, $0 \leq \lambda^{1+\alpha}(r) \lambda''(r) \leq C$ for some positive constants $\alpha$, $C$ on $I^0$, and the Ricci curvature of $N^m$ is nonnegative, then we have

$$|\nabla \varphi| \leq C_3,$$

where $C_3$ is a nonnegative constant depending on $\sup_M \varphi(\cdot, 0)$, i.e., the supremum of $\varphi(x, t)$ at the initial time $t = 0$.

Proof. Set $\psi = \frac{|\nabla \varphi|^2}{2}$. Then differentiating $\psi$ with respect to $t$ and together with (3.9), we have

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} \nabla_m \varphi \nabla^m \varphi = \nabla_m \dot{\varphi} \nabla^m \varphi = \nabla_m Q \nabla^m \varphi,$$

which is equivalent with

$$\frac{\partial \psi}{\partial t} = \left( Q_{ij} \nabla_{ij} \varphi + Q^k \nabla_{km} \varphi - \frac{v^2 \lambda'' \lambda \nabla_m \varphi}{(n \lambda' - \bar{\sigma} \varphi_{i,j})^2} \right) \nabla^m \varphi.$$

By straightforward calculation, we have

$$\frac{\partial \psi}{\partial t} = \left( Q_{ij} \nabla_{ij} \varphi + Q^k \nabla_{km} \varphi - \frac{2n \lambda'' \psi}{\lambda H^2} \right) \nabla^m \varphi$$

and

$$\nabla_{ij} \psi = \nabla_j (\nabla_{mj} \varphi \nabla^m \varphi) = \nabla_{mj} \varphi \nabla^m \varphi + \nabla_{mi} \varphi \nabla_j^m \varphi = (\nabla_{ij} \varphi + R_{limj}^l \nabla^m \varphi + \nabla_{mi} \nabla_j^m \varphi).$$

On the other hand, applying the Ricci identity, we can obtain

$$\nabla_{ij} \varphi \nabla^m \varphi = \nabla_{ij} \psi - R_{limj}^l \nabla_l^m \varphi = \nabla_{mi} \nabla_j^m \varphi.$$

Combining (4.1), (4.2) and (4.3) yields

$$\frac{\partial \psi}{\partial t} = Q_{ij} \nabla_{ij} \varphi + \left( Q^k - \frac{\nabla_k \psi}{\lambda^2 \varphi H^2} \right) \nabla_k \psi - \frac{1}{\lambda^2 H^2} \sigma_{ij} R_{limj} \nabla^l \varphi \nabla^m \varphi + \frac{R_{limj} \nabla^l \varphi \nabla^m \varphi \nabla^m \varphi}{\lambda^2 H^2 \varphi^2} - \frac{|\nabla^2 \varphi|^2}{\lambda^2 H^2} - \frac{2n \lambda'' \psi}{\lambda H^2}.$$

Since the curvature tensor is antisymmetric with respect to the indices $i$ and $j$, we have $R_{limj} \nabla^l \varphi \nabla^m \varphi \nabla^m \varphi \nabla^m \varphi = 0$. Besides, the nonnegativity of the Ricci curvature yields

$$\sigma_{ij} R_{limj} \nabla^l \varphi \nabla^m \varphi = Ric(\nabla \varphi, \nabla \varphi) \geq 0.$$

By Lemmas 3.1, 3.2 and the fact $\dot{\varphi} = \frac{v}{\lambda \dot{\varphi}}$, we have

$$\lambda H^2 = \lambda \left( \frac{v}{\lambda \dot{\varphi}} \right)^2 = \frac{\lambda^2 + |\nabla r|^2}{\lambda \dot{\varphi}^2} \leq \frac{\lambda^2 + 1}{\lambda \dot{\varphi}^2}.$$
which implies

\[-2n\lambda'' \quad \geq \quad -2nC_1^2 \lambda'' \quad \geq \quad -2nC_1^2 \lambda \quad \geq \quad -2nC_1^2 C \lambda^{-(2+\alpha)} \]

\[\geq \quad -2nC_1^2 C \left( \lambda \left( \inf_{M^n} u(\cdot, 0) \right) \right)^{-(2+\alpha)} e^{-\frac{(2+\alpha)t}{n}} \]

\[= \quad -C_4 e^{-\frac{(2+\alpha)t}{n}}, \]

where \(C_4 := 2nC_1^2 C \left( \lambda \left( \inf_{M^n} u(\cdot, 0) \right) \right)^{-(2+\alpha)} \). Putting the above three facts into (4.3) results in

\[
\frac{\partial \psi}{\partial t} \leq Q^{ij} \nabla_{ij} \psi + \left( Q^k - \frac{\nabla^k \psi}{\lambda^2 v^2 H^2} \right) \nabla_k \psi - C_4 e^{-\frac{(2+\alpha)t}{n}} \psi.
\]

Choose an orthonormal frame \(\{e_1, \cdots, e_n\} \) at \(x \in \partial M^n\) such that \(e_1, \cdots, e_{n-1} \in T_x \partial M^n\) and \(e_n = \mu\), and then we can obtain

\[
\nabla_{\mu} \psi = \nabla_{e_n} \psi = \sum_{i=1}^n \nabla_{e_i, e_n} \varphi \nabla_{e_i} \varphi = \sum_{i=1}^{n-1} (\nabla_{e_i, e_n} \varphi - (\nabla_{e_i} e_n) \varphi) \nabla_{e_i} \varphi
\]

\[= \quad - \sum_{i=1}^{n-1} (\nabla_{e_i} e_n, e_j) \nabla_{e_j} \varphi \nabla_{e_i} \varphi
\]

\[= \quad - \sum_{i=1}^{n-1} h_{\partial M^n} \nabla_{e_i} \varphi \nabla_{e_j} \varphi \leq 0,
\]

where \(h_{\partial M^n}\) is the second fundamental form of \(\partial M^n\). The last inequality holds because of the convexity of the cone \(\Sigma^a\). Therefore, together with (4.5), we know that \(\psi\) satisfies

\[
\begin{cases}
\frac{\partial \psi}{\partial t} \leq Q^{ij} \nabla_{ij} \psi + \left( Q^k - \frac{\nabla^k \psi}{\lambda^2 v^2 H^2} \right) \nabla_k \psi - C_4 e^{-\frac{(2+\alpha)t}{n}} \psi & \text{in } M^n \times (0, T) \\
\nabla_{\mu} \varphi \leq 0 & \text{on } \partial M^n \times (0, T) \\
\psi(\cdot, 0) = \frac{|\nabla \varphi(\cdot, 0)|^2}{2} = \frac{|\nabla \varphi_0|^2}{2} & \text{on } M^n.
\end{cases}
\]

Then using the maximum principle, we have \(\psi \leq \sup_{M^n} \frac{|\nabla \varphi_0|^2}{2}\), which, together with Lemma 3.1, yields the desired gradient estimate.

\[\square\]

5. Higher regularity and Convergence

In this section, the convergence and the higher regularity of the IMCF (\(\hat{F}\)) will be discussed after the area-preserving rescaling. We consider the rescaling \(\hat{F} = \eta(t) F(x, t)\), where \(F\) is the parameterization of the graph \(M_t^n\), and \(\eta(t)\) is the smooth function with respect to \(t\) satisfying

\[\int_{\tilde{M}_t^n} d\tilde{\mu} = |M_0|,
\]

where \(\tilde{M}_t^n\) is the rescaled hypersurface, \(d\tilde{\mu}\) is the volume element of \(\tilde{M}_t^n\), \(|M_0|\) denotes the area of the initial hypersurface \(M_0\). Recall that the induced metric and the second fundamental form
of $M^n_t$ are given by
\[ g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle, \quad h_{ij} = \left\langle \nabla u, \frac{\partial^2 F}{\partial x^i \partial x^j} \right\rangle, \]
so the corresponding induced metric and the second fundamental form of $\tilde{M}^n_t$ should be
\[ \hat{g}_{ij} = \eta^2 g_{ij}, \quad \hat{h}_{ij} = \eta^2 h_{ij}, \quad \hat{g}^{ij} = \eta^{-2} g^{ij}. \]
Differentiating both sides of (5.1) and then we can get
\[ \frac{d}{dt} \int_{M^n_t} \hat{d}\mu = \int_{M^n_t} \left( \frac{\partial}{} \frac{\partial g_{ij}}{\partial t} \right) d\mu = \int_{M^n_t} \left( \eta^{1/2} g_{ij} + \frac{1}{H} h_{ij} \eta^{1/2} \right) g^{ij} \eta^{-2} d\mu = 0, \]
which implies that
\[ \int_{M^n_t} (n\eta^{-1}\eta' + 1) d\mu = 0. \]
Therefore, we have
\[ n\eta^{-1}\eta' + 1 = 0, \]
and then solving the above ODE, together with $\eta(0) = 1$, yields $\eta(t) = e^{-\frac{1}{n}t}$.

In order to get the long time existence for the IMCF $(\tilde{\varphi})$, we do the rescaling $\tilde{u} = ue^{-\frac{1}{n}t}$ for the graphic function $u(x, t)$ of the evolving hypersurface $M^n_t$ in the base part of the warped product $I \times \lambda(r) N^n$. Through this process, we can obtain the following result.

**Lemma 5.1.** Let $u$ be an admissible solution of $(\tilde{\varphi})$ and let $\Sigma^n$ be a smooth, convex cone. If $\lambda$ satisfies $\lambda(r) > 0$, $0 < \lambda'(r) \leq C$, $0 \leq \lambda^{1+\alpha}(r) \lambda(\varphi) \leq C$ for some positive constants $\alpha$, $C$ on $I^0$, and the Ricci curvature of $N^n$ is nonnegative, then there exist some $\beta > 0$ and some $D > 0$ such that
\[ [\nabla \tilde{u}]_\beta + [\partial \tilde{u}/\partial t]_\beta + [\tilde{H}]_\beta \leq D \left( \| u_0 \|_{C^{2+\alpha}(M^n)}, n, \beta, M^n \right), \]
where $[f]_\beta := [f]_{x, \beta} + [f]_{t, \beta/2}$ is the sum of the Hölder coefficients of $f$ with respect to $x$ and $t$ in the domain $M^n \times [0, T]$.

**Proof.** First, we try to have the priori estimates for $|\nabla \varphi|$ and $|\partial \varphi/\partial t|$. That is because the priori estimates for $|\nabla \varphi|$ and $|\partial \varphi/\partial t|$ imply a bound for $[\tilde{u}]_{x, \beta}$ and $[\tilde{u}]_{t, \beta/2}$, which, together with [10 Chapter 2, Lemma 3.1], can give the bound for $[\nabla \tilde{u}]_{x, \beta/2}$ provided a bound for $[\nabla \tilde{u}]_{x, \beta}$ obtained. Since, after rescaling, $[\nabla \tilde{u}]$ and $\partial \tilde{u}/\partial t$ can be written as
\[ \nabla \tilde{u} = \nabla u \cdot e^{-t/n}, \quad \tilde{u} = e^{-\frac{t}{n}} - \frac{1}{n} \tilde{u}, \]
which implies $\nabla \tilde{u} = \lambda \nabla \varphi \cdot e^{-\frac{t}{n}}$. Then it is sufficient to bound $[\nabla \varphi]_{x, \beta}$ if one wants to bound $[\nabla \tilde{u}]_{x, \beta}$. In order to get this bound, we fix $x$ and rewrite the first evolution equation of $(\tilde{\varphi})$ as follows
\[ (5.2) \quad \text{div}_\nu \left( \frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) = \frac{n\lambda'}{\sqrt{1 + |\nabla \varphi|^2}} - \frac{\sqrt{1 + |\nabla \varphi|^2}}{\varphi}, \]
which is an elliptic PDE endowed with the Neumann boundary condition $\nabla_{\nu} \varphi = 0$. Clearly, the RHS of (5.2) is bounded since $\varphi, \nabla \varphi$ are bounded (see Lemmas 3.2 and 4.1 respectively) and $0 < \lambda' \leq C$. Besides, the RHS of (5.2) is also a measurable function in $x$. Therefore, by similar calculations to those in [11], Chapter 4, §6 (interior estimate) and Chapter 10, §2 (boundary estimate), we can get a Morrey estimate, which yields the estimate for $[\nabla \varphi]_{x, \beta}$. 
Note that \( \hat{u} = \lambda \varphi e^{-\alpha t/n} \). So, we need to bound \( \varphi \) if we want to get a bound for \( |\partial \hat{u}/\partial t|_\beta \). By
the first equation of (3.3), it is not difficult to get the evolution equation for \( \varphi \) with respect to the induced rescaled metric as follows

\[
\frac{\partial \hat{\varphi}}{\partial t} = \text{div} \left( \frac{\nabla \hat{\varphi}}{H^2} \right) - 2 \hat{\varphi}^{-1} \frac{|\nabla \hat{\varphi}|^2}{H^2} - n \frac{\lambda \hat{\varphi}^3}{v},
\]

Note that the Neumann condition \( \nabla \nu \hat{\varphi} = 0 \) implies that the interior and boundary estimates are basically the same. Then, together with the strict positivity of \( \hat{\varphi} \), we can define the test function \( \chi = \xi^2 \hat{\varphi} \) and the support of \( \xi \) has to be chosen away from the boundary for the interior estimate. Integration by parts and Young’s inequality results in

\[
\frac{1}{2} \left\| \varphi \xi \right\|_{2, M^n}^2 \bigg|_{t_0}^{t_1} + \frac{1}{\max \|H^2\|} \int_{t_0}^{t_1} \int_{M^n} \xi^2 |\nabla \hat{\varphi}|^2 d\mu dt \leq \int_{t_0}^{t_1} \int_{M^n} \left[ \varphi^2 \xi \frac{\partial \xi}{\partial t} + \frac{\xi^2 |\nabla \hat{\varphi}|^2}{2 \max \|H^2\|} + \frac{2 \max \|H^2\|\varphi^2 |\nabla \xi|^2}{\min \|H^4\|} \right],
\]

which implies

\[
\frac{1}{2} \left\| \varphi \xi \right\|_{2, M^n}^2 \bigg|_{t_0}^{t_1} + \frac{1}{\max \|H^2\|} \int_{t_0}^{t_1} \int_{M^n} \xi^2 |\nabla \hat{\varphi}|^2 d\mu dt \leq \left( 1 + \frac{2 \max \|H^2\|}{\min \|H^4\|} \right) \int_{t_0}^{t_1} \int_{M^n} \hat{\varphi}^2 \left[ \xi \frac{\partial \xi}{\partial t} + |\nabla \xi|^2 \right],
\]

where, as before, \( \max(\cdot) \) and \( \min(\cdot) \) denote the supremum and the infimum of a prescribed quantity over \( M^n \subset N^n \) respectively. Then, similar to [10], Chapter 5, §1 (interior estimate) and §7 (boundary estimate), the boundedness for \( [\hat{\varphi}]_\beta \) can be obtained, and moreover, all local interior and boundary estimates are independent of \( T \).

The estimate of \( \hat{H} \) follows from the estimates for \( \lambda, \nabla \varphi, \hat{\varphi} \) and the identity \( \hat{\varphi} \lambda e^{-t/n} \hat{H} = v = \sqrt{1 + |\nabla \varphi|^2} \).

Applying Lemma 5.1 we can get the following higher-order estimates.

**Lemma 5.2.** Let \( u \) be an admissible solution of (1.1) and let \( \Sigma^n \) be a smooth, convex cone. If \( \lambda \) satisfies \( \lambda(r) > 0, 0 < \lambda'(r) \leq C, 0 \leq \lambda^{1+\alpha}(r) \lambda'(r) \leq C \) for some positive constants \( \alpha, C \) on \( I^0 \), and the Ricci curvature of \( N^n \) is nonnegative, then for every \( t_0 \in (0, T) \), there exists some \( \beta > 0 \) such that

\[
\left\| \hat{u} \right\|_{C^{2+\beta,1+\beta/2}(M^n \times [0,T])} \leq D \left( \left\| u \right\|_{C^{2+\alpha}(M^n)}, n, \beta, M^n \right)
\]

and

\[
\left\| \hat{u} \right\|_{C^{2k+\beta,k+\beta/2}(M^n \times [t_0,T])} \leq D \left( \left\| u(\cdot,t_0) \right\|_{C^{2+\alpha}(M^n)}, n, \beta, M^n \right).
\]

**Proof.** Since \( \varphi(x,t) = \int_{c}^{u(x,t)} \frac{ds}{\lambda(s)} \), we know that the bound of \( \varphi \) leads to the bound of \( u \). So, we try to estimate \( \varphi \). Rewrite (5.1) as follows

\[
\frac{\partial \varphi}{\partial t} = \frac{1}{H^2} \Delta \varphi + \left( \frac{2 \sqrt{1 + |\nabla \varphi|^2}}{\lambda H} - \frac{n \lambda'}{\lambda^2 H^2} \right).
\]

By Lemma 5.1 we know that (5.3) is a uniformly parabolic PDE with Hölder continuous coefficients. Therefore, by [10] Chapter IV, Theorem 5.3], which shows \( \varphi \) is \( C^{2+\beta,1+\beta/2} \), and the
linear theory in [12] Chapter 4, which implies the second-order bound, we have

\begin{equation}
\| \tilde{u} \|_{C^{2+\beta,1+\beta/2}} \leq D (\| u_0 \|_{C^{2+\alpha}(M^n)}, n, \beta, M^n).
\end{equation}

Differentiating both sides of (5.3) with respect to $t$ and $\xi_i$, $1 \leq i \leq n$, respectively, one can easily get evolution equations of $\dot{\varphi}$ and $\varphi_i$ respectively, which, using the estimate (5.4), can be treated as uniformly parabolic PDEs on the time interval $[t_0, T]$. At the initial time $t_0$, all compatibility conditions are satisfied and the initial function $\varphi(\cdot, t_0)$ is smooth, which implies a $C^{3+\beta,(3+\beta)/2}$ estimate for $\varphi$, and a $C^{1+\beta,1+\beta/2}$ estimate for $\dot{\varphi}$. So, we have the $C^{4+\beta,2+\beta/2}$ estimate for $\tilde{u}$. From [12] Chapter 4, Theorem 4.3, Exercise 4.5 and the above argument, it is not difficult to know that the constant are independent of $T$. Higher regularity can be proven by induction over $k$. \qed

Then the long-time existence and the convergence can be discussed.

**Lemma 5.3.** let $\Sigma^n$ be a smooth, convex cone. Let $u$ be an admissible solution of $\tilde{u}$ and let $T^*$ be the maximal existence time. Then $T^* = \infty$, and the rescaled solution $\tilde{F}(\cdot, t) = F(\cdot, t)e^{-\frac{1}{t}}$ ($F(\cdot, t)$ is the embedding map mentioned in Theorem [12]) converges smoothly to an embedding $\tilde{F}_\infty$ mapping $M^n$ into a piece of a geodesic sphere.

**Proof.** By Lemma 5.2 we know that the Hölder norms of $u = \tilde{u}e^{t/n}$ cannot blow up as $T$ tends to the maximal time $T^* < \infty$, which implies that $u$ can be extended to a solution to $\tilde{u}$ in $[0,T^*)$. The short time existence result (see Lemma 2.1 and the higher-order estimates (see Lemma 5.2) imply the existence of a solution beyond $[0,T^*)$ which is smooth away from $t = 0$. This is a contradiction. Therefore, we have $T^* = \infty$.

By Lemmas 3.1 and 4.1 we can get the following estimate

$$|Du| \leq Ke^{-\gamma t},$$

where $K$ is a positive constant depending only on $C_3 \lambda (\inf u(\cdot,0))$, $\gamma$ is a constant depending only on $\alpha$ and the dimension $n$. By the Arzelà-Ascoli theorem, we know that every subsequence of $\tilde{u}$ converges to a constant function $\omega_\infty$ in $C^1(M^n)$. Assume that $\tilde{u}$ convergent to the constant $\omega_\infty$ in $C^{\beta}(M^n)$. Since $\tilde{u}$ is uniformly bounded in $C^{\beta+1}$ and $\tilde{u}$ is Hölder continuous, by the Arzelà-Ascoli theorem we know that there exists a subsequence convergent to $\omega_\infty$ in $C^{\beta+1}(M^n)$. Then we can get the conclusion that every subsequence must converge, and the limit has to be $\omega_\infty$. Therefore, $\tilde{u}$ converges to $\omega_\infty$ in $C^{\beta+1}(M^n)$ and $C^\infty$ convergence follows by the induction.

Finally, we accurately describe the asymptotic behavior of rescaled hypersurfaces as time tends to infinity. Recall that $h_{ij}^t = \frac{1}{\lambda e}(\lambda' \delta_{ij} - \tilde{\sigma}^{ik} \varphi_{kj})$. So, by the assumption $0 < \lambda' \leq C$, Lemmas 3.1 and 5.2 it follows that

$$|\tilde{h}_{ij}^t - \delta_{ij}| \leq e^{-\frac{\lambda' t}{\lambda e}} (\lambda' - 1) \delta_{ij} + \frac{1}{\lambda e} e^{-\frac{\lambda' t}{\lambda e}} \tilde{\sigma}^{ik} \varphi_{kj}$$

$$\leq e^{-(\frac{\lambda' + 1}{\lambda e}) t} (\frac{\lambda'}{\lambda e} \delta_{ij} + \frac{1}{\lambda e} e^{-\frac{\lambda' t}{\lambda e}} \tilde{\sigma}^{ik} \varphi_{kj})$$

$$\leq C_5 e^{-\delta t}$$

for some positive constant $C_5$ depending on $C$, $D$, $C_3$, $\lambda (\inf u(\cdot,0))$, and some constant $\delta$ depending on $\alpha$ and $n$. So, from the above argument, we know that after rescaling, the evolving hypersurfaces converge smoothly to a piece of a geodesic sphere as time tends to infinity. \qed

Theorem [12] follows naturally from Lemmas 2.1 and 5.3.
Acknowledgments. This research was supported in part by the National Natural Science Foundation of China (Grant Nos. 11201131, 11401131 and 11101132) and Hubei Key Laboratory of Applied Mathematics (Hubei University).

References

[1] S. Brendle, P.-K. Hung and M.-T. Wang, A Minkowski inequality for hypersurfaces in the anti-de Sitter-Schwarzschild manifold, Commun. Pure Appl. Math. 69 (2016) 124–144.
[2] L. Chen and J. Mao, Non-parametric inverse curvature flows in the AdS-Schwarzschild manifold, The Journal of Geometric Analysis, DOI:10.1007/s12220-017-9848-6.
[3] L. Chen, J. Mao and H.-Y. Zhou, Inverse curvature flows in warped product manifolds, preprint.
[4] P. Freitas, J. Mao and I. Salavessa, Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds, Calc. Var. Partial Differential Equations 51 (2014) 701–724.
[5] C. Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differ. Geom. 32 (1990) 299–314.
[6] P.-F. Guan and J.-F. Li, The quermassintegral inequalities for k-convex starshaped domains, Adv. Math. 221 (2009) 1725–1732.
[7] P.-F. Guan, X.-N. Ma, N. Trudinger and X.-H. Zhu, A form of Alexandrov-Fenchel inequality, Pure Appl. Math. Q. 6 (2010) 999–1012.
[8] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differ. Geom. 59 (2001) 353–437.
[9] G. Huisken and T. Ilmanen, Higher regularity of the inverse mean curvature flow, J. Differ. Geom. 80 (2008) 433–451.
[10] O.-A. Ladyzenskaja, V.-A. Solonnikov and N.-N. Ural’ceva, Linear and Quasilinear Equations of Parabolic Type, AMS, New York, 1968.
[11] O.-A. Ladyzenskaja and N.-N. Ural’ceva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.
[12] G.-M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore, 1996.
[13] J. Mao, Eigenvalue estimation and some results on finite topological type, Ph.D. thesis, IST-UTL, 2013.
[14] J. Mao, Eigenvalue inequalities for the p-Laplacian on a Riemannian manifold and estimates for the heat kernel, J. Math. Pures Appl. 101(3) (2014) 372–393.
[15] J. Mao, Volume comparison theorems for manifolds with radial curvature bounded, Czech. Math. J. 66(1) (2016) 71–86.
[16] J. Mao, F. Du and C.-X. Wu, Eigenvalue Problems on Manifolds, Scientific Press, Beijing, 2017.
[17] T. Marquardt, Inverse mean curvature flow for hypersurfaces with boundary, Ph.D. thesis, FU-Berlin, 2012.
[18] T. Marquardt, Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone, J. Geom. Anal. 23 (2013) 1303–1313.
[19] J. Urbas, On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures, Math. Z. 205 (1990) 355–372.
[20] H.-Y. Zhou, Inverse mean curvature flow in warped product manifolds, available online at arXiv:1609.09665v3.