A brief comment on the similarities of the IR solutions for the ghost propagator DSE in Landau and Coulomb gauges

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This brief note is devoted to reconcile the conclusions from a recent analysis of the IR solutions for the ghost propagator Dyson-Schwinger equations in Coulomb gauge with previous studies in Landau gauge.

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I. INTRODUCTION

In a series of papers [1–3], we studied the IR behaviour of the solutions for the ghost propagator Dyson-Schwinger equation (GPDSE) in Landau gauge. A similar analysis has been very recently carried out in Coulomb gauge [4]. This brief note is devoted to reconcile the apparently discrepant pictures resulting from the studies in Landau and Coulomb gauges mentioned above.

II. THE GPDSE IN COULOMB AND LANDAU GAUGES

As was explained in detail in refs. [1–3], the low-momentum behavior for the Landau gauge ghost dressing function can be inferred from the analysis of the Dyson-Schwinger equation for the ghost propagator (GPDSE) which can be written as follows:

\[
\frac{1}{F(k^2, \mu^2)} = \frac{1}{F(p^2, \mu^2)} + N_C \, g^2(\mu^2) \, H_1(\mu^2),
\]

where \( F(k^2, \mu^2) \) is the ghost propagator dressing function renormalized at the subtraction point, \( \mu^2 \), \( H_1 \) is the ghost-colinear non-perturbative ghost-gluon form factor (usually assumed to be 1) and

\[
I(k^2) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{F(q^2, \mu^2)}{q^2} \left( \frac{(k \cdot q)^2}{k^2} - q^2 \right) \left[ \Delta \left( \frac{(q - k)^2}{(q - k)^2} \right) - \Delta \left( \frac{(q - p)^2}{(q - p)^2} \right) \right] \right)
\]

with a renormalized gluon propagator,

\[
\Delta(q^2, \mu^2) = \frac{B(\mu^2)}{q^2 + M^2} \simeq \frac{B(\mu^2)}{M^2} \left( 1 - \frac{q^2}{M^2} + \cdots \right).
\]

where the Schwinger mechanism [7] is invoked to generate, via the fully dressed non-perturbative three-gluon vertex, a dynamical gluon mass, \( M(q^2) \), that we will approximate by its zero-momentum value in the IR domain [1].

On the other hand, the authors of ref. [4] recently performed a similar study of the GPDSE in Coulomb gauge. In particular, they applied the same strategy followed to investigate the low-momentum Landau-gauge ghost dressing function in ref. [1] and took the Gribov’s equal-time spatial gluon propagator dressing function [2],

\[
G_T(\vec{k}^2) = \frac{1}{2} \sqrt{\vec{k}^2 + m^4},
\]

1 This is shown to be a good low-momentum approximation for the running mass in ref. [5].

2 In very good agreement with the Euclidean SU(2) lattice results obtained for small lattice couplings in ref. [6].
as the input required to build a kernel and solve the GPDS, again with the approximation of replacing the fully
dressed spatial ghost-gluon vertex by the bare one (this is, also in Coulomb gauge, an exact result in the limit of a
vanishing incoming ghost up to all perturbative orders [5]). Thus, the GPDS can be rewritten as follows:

\[
\frac{1}{F(\vec{k}^2, \mu^2)} = \frac{1}{F(p^2, \mu^2)} - N_C \frac{g^2(\mu)}{(4\pi)^2} \int_0^\infty \frac{dq^2}{q^2} F(q^2, \mu^2) \left( I(\vec{k}^2, q^2; m) - I(p^2, q^2; m) \right),
\]

where \( I \) represents the angular integration,

\[
I(\vec{k}^2, q^2; m) = \int_{-1}^1 dz (1 - z^2) \left( 1 + \frac{\vec{k}^2}{p^2} - 2z \sqrt{\frac{\vec{k}^2}{p^2}} \right)^{-1/2} \left[ \left( 1 + \frac{\vec{k}^2}{p^2} - 2z \sqrt{\frac{\vec{k}^2}{p^2}} \right) + \frac{m^4}{p^4} \right]^{-1/2}.
\]

It should be emphasized that the ghost propagator dressing function in Coulomb gauge is strictly independent of the
energy, \( k_1^2 \), as a non-perturbative result of the Slavnov-Taylor identities [5].

After assuming a pure powerlaw behaviour, \( F(q^2) \sim (q^2)^{\alpha_F} \), for the dressing function and analyzing
asymptotically both Eqs. (1,5), one is left in both gauges with the two following well-known cases: (i) \( \alpha_F = 0 \) ("decoupling"),
that means zero-momentum finite ghost dressing function; and (ii) \( \alpha_F \neq 0 \) ("scaling"), where the low-momentum
behavior of the gluon propagator, \( \Delta(q^2) \sim (q^2)^{\alpha_G-1} \), forces the ghost dressing function to diverge at low-momentum
through the scaling condition: \( 2\alpha_F + \alpha_G = 0 \). As well in Landau as in Coulomb gauge, a massive gluon propagator
via the Schwinger mechanism or the Gribov’s formula for the equal-time spatial dressing lead to \( \alpha_G = 1 \) and thus
\( \alpha_F = -1/2 \). In particular, an elaborated asymptotical analysis of eq. (1) leaves us with [2] :

\[
F(q^2, \mu^2) \simeq \begin{cases} 
 \left( \frac{10\pi^2}{N_C H_1 g_R(\mu^2) B(\mu^2)} \right)^{1/2} \left( \frac{M^2}{q^2} \right)^{1/2}, & \text{if } \alpha_F \neq 0, \\
F(0, \mu^2) \left( 1 + \frac{N_C H_1}{16\pi} \bar{\sigma}_T(0) \right)^{1/2} \left[ \ln \frac{q^2}{M^2} - \frac{11}{6} \right] + O\left( \frac{q^4}{M^4} \right), & \text{if } \alpha_F = 0.
\end{cases}
\]

If \( \alpha_F \neq 0 \), the perturbative strong coupling defined in the Taylor scheme [10], \( \alpha_T = g_T^2/(4\pi) \), reaches a constant at zero-momentum,

\[
\lim_{q^2 \to 0} \alpha_T(q^2) = \lim_{q^2 \to 0} \left( \frac{g^2(\mu^2)}{4\pi} q^2 \Delta(q^2, \mu^2) F^2(q^2, \mu^2) \right) = \frac{5\pi}{2N_C H_1}.
\]

as can be obtained from Eqs.(2,7). In the case \( \alpha_F = 0 \), the subleading correction to the non-zero finite value for the
zero-momentum ghost dressing function, given by eq. (7), is controlled by the well-defined zero-momentum limit of
\( \bar{\sigma}_T(q^2) = (M^2/q^2) \alpha_T(q^2) \), which is an extension of the non-perturbative effective charge definition from the gluon
propagator [11] to the Taylor ghost-gluon coupling [5].

The same two cases result from the analysis of eq. (5) for the Coulomb gauge in ref. [3], where a ghost propagator
dressing function behaving asymptotically as either a constant or \( F(\vec{k}^2) \sim (\vec{k}^2)^{-1/2} \) is analytically found and confirmed
by a numerical study.

III. THE FAMILY OF SOLUTIONS AND THE DIALING PARAMETER

The GPDS in eq. (1) with the input of a gluon propagator borrowed from lattice QCD calculations is numerically
solved in ref. [1] and both kinds of solutions in eq. (7) were shown to happen controlled by the size of the coupling
at the renormalization point, \( g(\mu) \). In QCD, one needs to provide a physical scale and a standard manner to proceed
is by fixing the size of the coupling at a given momentum scale. This can be seen as a boundary condition to solve the
DSEs. Thus, for any coupling, \( g(\mu) \), below some critical value, \( g_{\text{crit}} \), an infinite number of regular or decoupling
solutions for the ghost dressing, behaving as eq. (7) indicates, were found; for \( g(\mu) = g_{\text{crit}} \), a unique critical or scaling
solution behaving as eq. (7) was found. It appeared not to be other solutions otherwise. In ref. [1], for a subtraction
point \( \mu = 1.5 \text{ GeV} \), a critical coupling \( g_{\text{crit}} \approx 3.33 \) and a very good description of ghost propagator lattice data with
a regular solution of eq. (1) for \( g(\mu) \approx 3.11 \) were obtained. These results were also recently confirmed [3] by studying
the coupled system of ghost and gluon propagator DSE in the PT-BFM scheme [13]. This last work payed attention
to the critical solution limit by studying how \( F(0, \mu^2) \) diverges as \( g(\mu) \to g_{\text{crit}} \approx 1.31 \), with a subtraction point \( \mu = 10 \)
GeV. One can now apply the perturbative definition of the Taylor strong coupling in eq. (8) and compute this coupling with the gluon and ghost solutions of [3] in order to see how the critical limit is approached. This is shown in Fig. 1, where it can be also seen that all the curves for \(\alpha_T\) obtained for different values of \(g(\mu)\) tend to join each other as \(q^2/\mu^2\) increases (right). As explained in [3], the scaling solution cannot be obtained in the PT-BFM scheme with “massive” gluons but only appears as an end-point for the family of regular or decoupling solutions. In Fig. 1, the curve for the critical limit is obtained by rescaling, up to giving \(\alpha_T(0)\) from eq. (8) with \(H_1 = 1\) at zero-momentum, the results at the critical limit for \(q^2\Delta(q^2)F^2(q^2)\) numerically obtained in ref. [1], not by solving the coupled DSE system but by applying the lattice gluon propagator as an input to solve the GPDE. Indeed, the critical value for the coupling at \(\mu = 10\) GeV can be read from the critical curve in Fig. 1 and one gets \(g(\mu) \simeq 1.56\), in fairly good agreement with the value of ref. [3].

On the other hand, the numerical analysis of eq. (5) for the Coulomb gauge in ref. [4] also shows both regular and critical solution to happen but controlled by \(F(0, \mu)\) (or \(\Gamma(0, \mu) = 1/F(0, \mu)\)) as a boundary condition with the size of the coupling fixed to be \(g^2(\mu) = \tilde{g}^2 = 4\pi \times 0.1187\) for \(N_C = 3\). In that case, regular or decoupling solutions correspond to finite values of \(F(\mu^2)\) and the divergent limit, \(F(0, \mu^2) \to \infty\), provides us with the unique critical or scaling solution (a similar pattern is claimed to be also found in Landau gauge for the authors of ref. [12]). Again, the lattice results for the ghost propagator seem to agree with a regular solution with \(F(0, \mu^2) \simeq 10\), although much larger values for the boundary condition cannot be ruled out. Furthermore, the authors of ref. [4] demonstrate that the ghost dressing obtained from eq. (6) for any \(F(0, \mu)\) behave as \(e \cdot (\vec{k}^2)\gamma_g\) at asymptotically large \(\vec{k}^2/m^2\); where \(m\) is the Gribov mass scale in eq. (4). \(\gamma_g\) is the leading-order ghost-anomalous dimension and \(e\) is the same coefficient for all the solutions. In other words, in the perturbative domain, the perturbative behaviour is recovered and the differences between values of \(F(\vec{k}^2, \mu^2)\) for different arbitrary inputs of \(F(0, \mu^2)\) vanish (This can be clearly seen in Fig. 2 of [4]). Then, they conclude that the boundary condition is not connected to the renormalization (at least in the perturbative regime).

We are thus left with either a family of Landau-gauge DSE solutions dialed by the size of the coupling at the renormalization point or a family of Coulomb-gauge ones dialed by the zero-momentum ghost dressing value as a boundary condition not connected to the renormalization. How both pictures can be reconciled? The key point stems from the different renormalization prescriptions applied to the ghost propagator in both analyses. In the Landau-gauge analysis of refs. [1][3], the standard MOM prescription, where the Green functions are required to take their tree-level expression at the renormalization point and for some particular kinematical choice (this implies \(F(\mu^2, \mu^2) = 1\)), is the one applied. The prescription applied to the ghost propagator by the authors of ref. [4] is defined by their eq. (3.20) for the renormalization constant \(Z_c(\Lambda, [g, \Gamma(0)])\), where \(\Gamma(0) = 1/F(0, \mu^2)\). In particular, this renormalization constant depends on the boundary condition, \(\Gamma(0)\), in such a manner that the value for this boundary condition is rescaling the ghost dressing function (and, as can be clearly seen in Fig. 2 of [4], it does not take the tree-level value, 1, as happens in MOM prescription for the subtraction point). In the following, we will show how, depending on the renormalization prescription, both patterns can be found for the family of solutions for eq. (4) in Landau gauge.
Let's consider a MOM solution of eq. (1) for arbitrary coupling, $g(\mu)$; let's then apply the following transformation:

$$g(\mu) \rightarrow s \ g(\mu), \quad F(q^2, \mu^2) \rightarrow \frac{1}{s} F(q^2, \mu^2).$$

(9)

The properties of eq. (1) (the same happens of course for eq. (5)) guarantee that, for any $s$ being a c-number, the transformed dressing function verifies the DSE equation with the transformed coupling (of course, MOM prescription implies $s = 1$). Then, if one chooses $s = \frac{1}{g}/g(\mu)$ and apply the transformation to every solution of the MOM family, we will be left with a one-to-one correspondence between these solutions and the new ones

$$F(q^2, \mu^2) \equiv \frac{g(\mu)}{g} F(q^2, \mu^2),$$

(10)

for the fixed coupling $g$, which can be identified by the zero-momentum value, $F(0, \mu^2)$. This new family of transformed solutions obeys the same pattern of the Coulomb-gauge eq. (5). It is interesting to notice that the strong coupling defined in the Taylor scheme can be also obtained from the transformed solutions as

$$\alpha_T(q^2) \equiv \frac{g^2(\mu)}{4\pi} q^2 \Delta(q^2, \mu^2) F(q^2, \mu^2) \equiv \frac{g^2(\mu)}{4\pi} q^2 \Delta(q^2, \mu^2) F(q^2, \mu^2),$$

(11)

although it is obvious that neither $F$ nor the coupling are in MOM scheme. More interesting enough is to realize from eq. (11) that: (i) $F$ does not depend on $\mu$, as far as one applies the same renormalized gluon propagator to obtain any solution for arbitrary coupling, $g(\mu)$, as done in [4]; (ii) the transformed ghost dressing function for the critical MOM solution $(g(\mu) \rightarrow g_{crit})$ corresponds to the scaling solution for the critical boundary condition, $F(0, \mu^2) \rightarrow \infty$ and diverges as $F(q^2, \mu^2) \sim (\mu^2)^{-1/2}$ for $q^2 \rightarrow 0$, as it clearly results from eq. (8), (iii) finally, as the difference between the Taylor strong couplings computed for any two arbitrary values of $g(\mu)$ vanishes at asymptotically large momentum (see Fig. [1]), the same should happen to $F$ for different values of $F(0, \mu^2)$. All this is claimed by the authors of [4] to identify the family of solutions for the Coulomb-gauge eq. (3).

IV. CONCLUSIONS

We thus conclude that the behaviour of the family of Coulomb-gauge GPDSE solutions in ref. [4] is analogous to the one described in refs. [1,3] for Landau gauge, although not renormalized in MOM scheme but after applying the transformation of eq. (9). The input parameter for the solutions in ref. [4] is the zero-momentum ghost dressing, which can be interpreted as a boundary condition and put in connection with the Gribov problem. On the other hand, for Landau gauge and MOM scheme, $g(\mu)$ is related to the strong coupling in Taylor scheme while the size of the fixed coupling, $g$, after applying eq. (9) is physically meaningless. Thus, as done in ref. [3], the critical value for $g(\mu)$ can be used to derive a critical value for $\Lambda_{\overline{MS}}$ which can be compared with lattice evaluations or experimental determinations to investigate whether the critical solution can be ruled out.

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