ON THE CANONICAL RING OF COVERS
OF SURFACES OF MINIMAL DEGREE

FRANCISCO JAVIER GALLEGOL
AND
B. P. PURNAPRAJNA

INTRODUCTION

Let \( \varphi : X \to Y \) be a generically finite morphism. The purpose of this paper is to show how the \( \mathcal{O}_Y \)-algebra structure on \( \varphi_* \mathcal{O}_X \) controls algebro-geometric aspects of \( X \) like the ring generation of graded rings associated to \( X \) and the very ampleness of line bundles on \( X \). As the main application of this we prove some new results for certain regular surfaces \( X \) of general type. Precisely, we find the degrees of the generators of the canonical ring of \( X \) when the canonical morphism of \( X \) is a finite cover of a surface of minimal degree. These results complement results of Ciliberto [Ci] and Green [G]. The techniques of this paper also yield different proofs of some earlier results, such as Noether’s theorem for certain kind of curves and some results on Calabi-Yau threefolds that had appeared in [GP2].

The canonical ring of surfaces of general type have attracted the attention of several geometers. Kodaira (see [Kod]) first proved that \( |K_X \otimes m| \) embeds a minimal surface of general type \( X \) as a projectively normal variety for all \( m \geq 8 \). This was later improved by Bombieri (see [Bo]) who proved the same result if \( m \geq 6 \) and by Ciliberto (see [Ci]), who lowered the bound to \( m \geq 5 \). Recently the authors proved (see [GP1]) more general results on projective normality and higher syzygies for adjunction bundles for an algebraic surface. As a corollary of these results they recovered and improved the results of Bombieri and Ciliberto on projective normality and extended them to higher syzygies.

An important class of minimal surfaces of general type are those whose canonical divisor is base-point-free. If one goes by the results proved so far surfaces with base-point-free canonical bundle fall into one of these two categories: those
whose canonical morphism maps onto a surface of minimal degree and those
whose canonical morphism doesn’t map onto a surface of minimal degree. The
latter have been studied by Ciliberto (see [Ci]) and (see Green [G]). The former
are studied in this article. Green and Ciliberto proved this nice result regarding
the generators of the canonical ring of \( X \):

Let \( X \) be a regular surface of general type with a base point free canonical
divisor. Assume that the canonical morphism \( \varphi \) satisfies the following conditions:

1. \( \varphi \) does not map \( X \) generically 2 : 1 onto \( \mathbb{P}^2 \).
2. \( \varphi(S) \) is not a surface of minimal degree (other than \( \mathbb{P}^2 \)).

Then the the canonical ring of \( X \) is generated in degree less than or equal to
2.

In the present article we deal with surfaces of general type \( X \) whose canonical
morphism \( \varphi \) maps \( X \) onto a surface of minimal degree \( Y \). These surfaces have
been studied in the works of Horikawa (see [H1], [H2], [H3] and [H4]), Catanese
(see [Ca]) and Konno (see [Kon]) among others, where they play a central role
in the classification of surfaces of general type with small \( c_2 \), in questions about
degenerations and the moduli of surfaces of general part. The study of these
surfaces have a direct bearing on the study of linear series on threefolds such as
Calabi-Yau threefolds as the results in [OP] and authors results in [GP2] show.

The study the canonical rings of these surfaces is carried out in Section 2.
We determine the precise degrees of the generators of its canonical ring (see
Theorem 2.1). The answer depends on the degree of \( \varphi \) and the degree of \( Y \).
As a corollary of our result and the result of Ciliberto and Green, we find that
conditions (1) and (2) above characterize the regular surfaces of general type
with base point free canonical bundle whose canonical ring is generated in degree
less than or equal to 2.

As we said at the beginning, in order to study the canonical ring of \( X \) we
will use the existence of a generically finite morphism \( \varphi \) from \( X \) to a variety \( Y \).
The morphism \( \varphi \) is the canonical morphism of \( X \) and \( Y \) is a surface of minimal
degree, that is, a nondegenerate surface in projective space whose degree is
equal to its codimension plus 1. The classification of these surfaces is classically
known: they are (linear) \( \mathbb{P}^2 \), the Veronese surface in \( \mathbb{P}^5 \), smooth rational scrolls
or cones over one of them (see [EH]). Thus the surface \( Y \) is simpler than \( X \). A
measurement of its simplicity is that its general hyperplane section is a smooth,
rational normal curve. Therefore in Section 2 we see how the algebra structure
of \( \varphi_* \mathcal{O}_X \) governs the multiplicative structure of the canonical ring of \( X \) and we
use this to study its ring generators.
Section 3 is devoted to constructing examples. We recall some known examples of surfaces of general type mapping to a surface of minimal degree and construct some new ones. We also show that certain kinds of examples of finite canonical morphisms are not possible. For example, we show that odd degree covers of smooth rational scrolls or cyclic covers of degree bigger than 3 of surfaces of minimal degree do not exist.

In Section 4, we show another example that illustrates the relation between the algebra structure of $\pi_*O_X$ given by a finite morphism $\pi$ and the canonical ring of $X$. As an application of this, we give a different proof of Noether’s theorem for curves general among those possessing an effective theta-characteristic. In this case the finite morphism $\pi$ is induced by the complete linear series of a theta-characteristic of a curve mapping to a rational curve. In Section 5, we give yet another illustration of this philosophy and give a different proof of results on Calabi-Yau threefolds proved in [GP2].

Finally we expand on these ideas in two forthcoming articles, [GP3] and [GP4]. In the first we study the canonical ring of higher dimensional varieties of general type whose canonical morphism maps onto a variety of minimal degree. One of results in [GP3] shows that the converse of the theorem of Ciliberto and Green for surfaces stated above is false for higher dimensional varieties of general type. In the second we carry out a detailed study of homogeneous rings associated to line bundles on trigonal curves.

**Convention.** We will work over an algebraic closed field of characteristic 0.

1. **Preliminaries**

In this section we will recall some known facts about the push forward of the structure sheaf of a variety by a flat, finite morphism. We summarize these facts below and refer for the proof to [HM], Section 2.

Let $X$ and $Y$ be algebraic varieties over a field $k$ and let $n$ a natural number which does not divide $\text{car}(k)$. Let $\pi : X \rightarrow Y$ be a finite, flat morphism of degree $n$. We have the folowing facts:

1.1. The sheaf $\pi_*O_X$ is a rank $n$ locally free sheaf on $Y$ of algebras over $O_Y$.

1.2. There exists a map

$$\frac{1}{n} \text{tr} : \pi_*O_X \rightarrow O_Y$$
of sheaves of $\mathcal{O}_Y$ modules defined locally as follows: Given $\alpha \in \pi_*\mathcal{O}_X$ we consider the homomorphism of $\mathcal{O}_Y$-modules

$$\pi_*\mathcal{O}_X \xrightarrow{\alpha} \pi_*\mathcal{O}_X$$

induced by multiplication by $\alpha$. Then we define $\frac{1}{n}\text{tr}(\alpha)$ as the trace of such homomorphism divided by $n$.

1.3. $\frac{1}{n}\text{tr}$ is surjective, in fact, the map $\mathcal{O}_Y \hookrightarrow \pi_*\mathcal{O}_X$ induced by $\pi$ is a section of $\frac{1}{n}\text{tr}$. Therefore the sequence

$$0 \rightarrow E \rightarrow \pi_*\mathcal{O}_X \xrightarrow{\frac{1}{n}\text{tr}} \mathcal{O}_Y$$

splits. $E$ is the kernel of $\frac{1}{n}\text{tr}$ and locally consists of the trace 0 elements of $\pi_*\mathcal{O}_X$. We will call $E$ the trace-zero module of $\pi$.

1.4. $\pi_*(\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_Y$-algebras, therefore it has a multiplicative structure. Its multiplication map is an $\mathcal{O}_Y$-bilinear map

$$[\mathcal{O}_Y \oplus E] \otimes [\mathcal{O}_Y \oplus E] \rightarrow \mathcal{O}_Y \oplus E$$

made of four components. The first component

$$\mathcal{O}_Y \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y \oplus E$$

is given by the multiplication in $\mathcal{O}_Y$ and therefore goes to $\mathcal{O}_Y$. The components

$$\mathcal{O}_Y \otimes E \rightarrow \mathcal{O}_Y \oplus E$$

$$E \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y \oplus E$$

are given by the left and right module structure of $E$ over $\mathcal{O}_Y$ and therefore go to $E$. Finally there is a fourth component

$$E \otimes E \rightarrow \mathcal{O}_Y \oplus E$$

which factorizes through

$$S^2E \rightarrow \mathcal{O}_Y \oplus E$$

for multiplication in $\pi_*\mathcal{O}_Y$ is commutative.
2. Covers of surfaces of minimal degree.

Our purpose is to study the generators of the canonical ring of certain surfaces of general type. Precisely we are interested in studying those regular surfaces of general type whose canonical divisor is base point free and such that the image of the canonical morphism is a variety of minimal degree. We obtain the following

**Theorem 2.1.** Let \( S \) be a regular surface of general type with at worst canonical singularities and such that its canonical bundle \( K_S \) is base-point-free. Let \( \varphi \) be the canonical morphism of \( S \). Let \( n \) be the degree of \( \varphi \) and assume that the image of \( \varphi \) is a surface of minimal degree \( r \).

1) if \( n = 2 \) and \( r = 1 \) (i.e., if \( \varphi \) is generically \( 2:1 \) onto \( \mathbb{P}^2 \)), the canonical ring of \( S \) is generated by its part of degree 1 and one generator in degree 4;
2) if \( n \neq 2 \) or \( r \neq 1 \), the canonical ring of \( S \) is generated by its part of degree 1, \( r(n-2) \) generators in degree 2 and \( r-1 \) generators in degree 3.

The knowledge of how many linearly independent generators are needed in each degree is obtained from the knowledge of the image of the multiplication maps of global sections of powers of the canonical bundle. We study those multiplication maps by studying similar maps of a curve \( C \) in \( |K_S| \). Thus we will first prove the following

**Proposition 2.2.** Let \( C \) be a smooth curve. Let \( \theta \) be a base-point-free line bundle on \( C \) such that \( \theta^\otimes 2 = K_C \). Let \( \pi \) be the morphism induced by \( |\theta| \), let \( n \) be the degree of \( \pi \) and assume that \( \pi(C) \) is a rational normal curve of degree \( r \). Let \( \beta(s,t) \) be the multiplication map

\[
H^0(\theta^\otimes s) \otimes H^0(\theta^\otimes t) \rightarrow H^0(\theta^\otimes s+t), \text{ for all } s,t > 0 .
\]

The codimension of the image of \( \beta(s,t) \) in \( H^0(\theta^\otimes s+t) \) is as follows:

a) If \( r = 1 \), the codimension is:
   a.1) \( n - 2 \), for \( s = t = 1 \),
   a.2) 0, for \( s = 2, t = 1 \), i.e., \( \beta(2,1) \) surjects.
   a.3) 1, for \( s = 3, t = 1 \).
   a.4) 1, for \( s = t = 2, n = 2 \) and 0 if \( n > 2 \).
   a.5) 0, for \( s \geq 4, t = 1 \), i.e., \( \beta(s,1) \) surjects for all \( s \geq 4 \).

b) If \( r > 1 \), the codimension is:
   b.1) \( r(n-2) \), for \( s = t = 1 \).
   b.2) \( r - 1 \), for \( s = 2, t = 1 \).
b.3) \(0, \text{ for } s \geq 3, t = 1, \text{i.e., } \beta(s, 1) \text{ surjects for all } s \geq 3.\)

Moreover, if \(r = 1\) and \(n = 2\), then the image of \(\beta(2, 2)\) and the image of \(\beta(3, 1)\) are equal.

In order to prove Proposition 2.2 we will use the following

**Lemma 2.3.** Let \(C, \theta\) and \(\pi\) as in the statement of Proposition 2.2. Then

\[
\pi_* O_C = O_{\mathbb{P}^1} \oplus (n-2)O_{\mathbb{P}^1} (-r-1) \oplus O_{\mathbb{P}^1} (-2r-2).
\]

**Proof.** Since the image of \(\pi\) is smooth and of dimension 1, \(\pi\) is flat. Then \(\pi_* O_C = O_{\mathbb{P}^1} \oplus E\) as \(O_{\mathbb{P}^1}\)-modules, with \(E\) vector bundle over \(\mathbb{P}^1\). We now show that \(E = (n-2)O_{\mathbb{P}^1} (-r-1) \oplus O_{\mathbb{P}^1} (-2r-2)\).

We have \(\pi_* \theta = \pi_* O_C \otimes O_{\mathbb{P}^1} (r)\) and \(\pi_* K_C = \pi_* O_C \otimes O_{\mathbb{P}^1} (2r)\), by projection formula. Any vector bundle over \(\mathbb{P}^1\) splits, hence

\[
\pi_* (O_C) = O_{\mathbb{P}^1} \oplus E = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} (a_1) \oplus \ldots \oplus O_{\mathbb{P}^1} (a_{n-1}),
\]

for some negative integers \(a_1, \ldots, a_{n-1}\) (\(C\) is connected). Then \(h^1(K_C) = 1\) implies that exactly one of the \(a_i\)'s, let us say \(a_{n-1}\), satisfies \(a_{n-1} + 2r = -2\). On the other hand, since \(\pi\) is induced by the complete linear series \(|\theta|\), \(h^0(\theta) = r + 1 = h^0(O_{\mathbb{P}^1}(r))\), so \(a_i + r \leq -1\) for all \(1 \leq i \leq n-2\). Finally, since degree of \(\theta\) is \(g(C) - 1\), \(h^1(\theta) = h^0(\theta) = r + 1\). Since \(h^1(O_{\mathbb{P}^1} (-r-2)) = r + 1, a_i + r \geq -1\) for all \(1 \leq i \leq n-2\), so \(a_i + r = -1\) for all \(1 \leq i \leq n-2\). \(\square\)

(2.4) **Proof of Proposition 2.2.** In Lemma 2.3 we have completely determined the structure of \(\pi_* O_C\) as \(O_{\mathbb{P}^1}\)-module. Now we look at the structure of \(\pi_* O_C\) as \(O_{\mathbb{P}^1}\)-algebra. If \(n = 2\), it is completely determined by the branch divisor of \(\pi\) on \(\mathbb{P}^1\); since in this case \(\pi\) is cyclic. If \(n > 2\), we observe the following:

(2.4.1) For some \(1 \leq i, j \leq n-2\), the projection of the map

\[
O_{\mathbb{P}^1} (a_i) \otimes O_{\mathbb{P}^1} (a_j) \longrightarrow \pi_* O_C \text{ to } O_{\mathbb{P}^1} (-2r-2)
\]

is surjective, in fact, it is an isomorphism.

This is so because otherwise \(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} (a_1) \oplus \ldots \oplus O_{\mathbb{P}^1} (a_{n-2})\) would be an integral subalgebra of \(\pi_* O_C\), free over \(O_{\mathbb{P}^1}\) of rank \(n-1\). Then \(n-1\) should divide \(n\), which is not possible if \(n > 2\).
Now we will use our knowledge of \( \pi_*O_X \) to study the maps \( \beta(s,r) \) which appear in the statement of the proposition. We will write \( \beta_s \) in place of \( \beta(s,1) \). Let \( R_l = H^0(\theta^{\otimes l}) \). Then, since \( \theta = \pi^*O_{P^1}(r) \), by projection formula

\[
R_1 = H^0(O_{P^1}(r)),
\]

\[
R_l = H^0(O_{P^1}(lr) \oplus (n-2)H^0(O_{P^1}((l-1)r-1)) \oplus H^0(O_{P^1}((l-2)r-2)) \text{ and } R_{l+1} = H^0(O_{P^1}((l+1)r)) \oplus (n-2)H^0(O_{P^1}(lr-1)) \oplus H^0(O_{P^1}((l-1)r-2)).
\]

Therefore an element of \( R_l \), i.e., a global section of \( H^0(\theta^{\otimes l}) \) is a sum of \( n \) components, one in each piece of the above decomposition of \( R_l \). On the other hand, the product of an element of \( R_l \) belonging to one of the blocks with an element of \( R_1 \) is determined by the ring structure of \( O_{P^1} \) and by the module structure of \( E \). More precisely, the restriction of \( \beta_l \) to \( H^0(O_{P^1}(lr)) \oplus H^0(O_{P^1}(r)) \) maps, in fact isomorphically, onto \( H^0(O_{P^1}((l+1)r)) \). The restriction of \( \beta_l \) to each of the blocks \( H^0(O_{P^1}((l-1)r-1)) \otimes H^0(O_{P^1}(r)) \) maps to the corresponding \( H^0(O_{P^1}((l-1)r-2)) \). This restriction is 0 if \( (l-1)r-1 \) is negative and an isomorphism otherwise. Likewise, the restriction of \( \beta_l \) to \( H^0(O_{P^1}((l-2)r-2)) \otimes H^0(O_{P^1}(r)) \) goes to \( H^0(O_{P^1}((l-1)r-2)) \), being 0 if \( (l-2)r-2 \) is negative and an isomorphism otherwise. Therefore it is crucial to tell which blocks of a given \( R_l \) are 0. We have

\[
R_1 = H^0(O_{P^1}(r));
\]

\[
R_2 = H^0(O_{P^1}(2r)) \oplus (n-2)H^0(O_{P^1}(r-1)); \text{ and if } l \geq 3,
\]

\[
R_l = H^0(O_{P^1}(lr)) \oplus (n-2)H^0(O_{P^1}((l-1)r-1)) \oplus H^0(O_{P^1}((l-2)r-2)).
\]

All the direct summands appearing in the above formulae are nonzero, except \( H^0(O_{P^1}((l-2)r-2)) \) when \( l = 3 \) and \( r = 1 \) and \( (n-2)H^0(O_{P^1}((l-1)r-1)) \) for all \( l \) and all \( r \) when \( n = 2 \). We now determine the image of \( \beta_l \). If \( l = 1 \), the image of \( \beta_1 \) is \( H^0(O_{P^1}(2r)) \), which has codimension \( (n-2)r \) in \( R_2 \). If \( l = 2 \), the image of \( \beta_2 \) is \( H^0(O_{P^1}(3r)) \oplus H^0((n-2)O_{P^1}(2r-1)) \) which has codimension \( r-1 \) in \( R_3 \). If \( l = 3 \) and \( r \geq 2 \) or if \( l \geq 4 \), the image of \( \beta_l \) is all \( R_l \), i.e., \( \beta_l \) surjects. All this proves a.1), a.2), a.5) and b). If \( r = 1 \) the image of \( \beta(3,1) \) is \( H^0(O_{P^1}(4r)) \oplus (n-2)H^0(O_{P^1}(3r-1)) \), which has codimension 1 in \( R_4 \). This proves a.3). If \( r = 1 \) and \( n = 2 \), the image of \( \beta(2,2) \) is \( H^0(O_{P^1}(4r)) \), which has codimension 1 in \( R_4 \). This proves the first claim in a.4) and the last sentence of Proposition 2.2. Finally, if \( n > 2 \), recall (see 2.4.1) that for some \( 1 \leq i,j \leq n-2 \), the projection of the map

\[
O_{P^1}(a_i) \otimes O_{P^1}(a_j) \longrightarrow \pi_*O_C
\]
to $\mathcal{O}_{\mathbb{P}^1}(-4)$ is surjective, in fact, it is an isomorphism. Then if $n > 2$ the image of $\beta(2, 2)$ is all $R_4$. This proves the second part of a.4). □

**Remark 2.5.** Note that $\theta^{\otimes 2} = K_C$. Then a proof of a.4), alternate to the one given above, can be obtained from Noether’s Theorem and from the base-point-free pencil trick. The way how Noether’s theorem is related to the algebra structure of $\pi_*\mathcal{O}_C$ will be clear in Section 4, where we will give a different, simple proof of this classical result in certain particular cases.

>From Proposition 2.2 we obtain the following

**Corollary 2.6.** Let $C$ be a smooth curve. Let $\theta$ be a base-point-free line bundle on $C$ such that $\theta^{\otimes 2} = K_C$. Let $\pi$ be the morphism induced by $|\theta|$, let $n$ be the degree of $\pi$ and assume that $\pi(C)$ is a rational normal curve. Let $R$ be $\bigoplus_{l=0}^{\infty} H^0(\theta^{\otimes l})$. Then

1) if $r = 1$ and $n = 2$, the ring $R$ is generated by its part of degree 1 and one generator in degree 4;
2) if $r = 1$ and $n > 2$, the ring $R$ is generated by its part of degree 1 and $n - 2$ generators in degree 2;
3) if $r > 1$, the ring $R$ is generated by its part of degree 1, $r(n - 2)$ generators in degree 2 and $r - 1$ generators in degree 3.

**Proof:** To know in what degrees we need generators we look at the maps $\beta(s, t)$ of multiplication of sections. Precisely the number of generators needed in degree $l+1$ is the codimension in $R_{l+1}$ of the sum of the images of $\beta(l, 1), \beta(l-1, 2), \ldots, \beta([\frac{l+1}{2}], [\frac{l+1}{2}])$. In particular $R$ is generated in degree less than or equal to $l$ if $\beta_k$ surjects for all $k \geq l$. Thus 1) follows from part a) of Proposition 2.2 and from the fact that the images of $\beta(3, 1)$ and $\beta(2, 2)$ are equal. 2) follows likewise from part a) of Proposition 2.2 (note that in this case $\beta(2, 2)$ surjects). Finally 3) follows from part b) of Proposition. □

(2.7) **Proof of Theorem 2.1:** The proof rests on Proposition 2.2. The idea is “to lift” the generators of $R$ to the canonical ring of $S$. Let us define

$$H^0(K_S^{\otimes s}) \otimes H^0(K_S^{\otimes r}) \xrightarrow{\alpha(s,t)} H^0(K_S^{\otimes s+t})$$

and let also denote $\alpha(s, 1)$ as $\alpha_s$. As in the case of $R$, the images of $\alpha(s, t)$ will tell us the generators of each graded piece of the canonical ring of $S$. In fact it will suffice to prove the following:
(a) If \( r = 1 \) and \( n = 2 \), \( \alpha_l \) surjects for all \( l \geq 1 \), except if \( l = 3 \). The images of \( \alpha_3 = \alpha(3, 1) \) and \( \alpha(2, 2) \) are equal and have codimension 1 in \( H^0(K_S^\otimes 4) \).

(b) If \( r = 1 \) and \( n > 2 \), \( \alpha_l \) surjects for all \( l \geq 1 \), except if \( l = 1, 3 \). The image of \( \alpha_1 \) has codimension \( n - 2 \) in \( H^0(K_S^\otimes 2) \). The map \( \alpha(2, 2) \) is surjective.

(c) If \( r \geq 2 \), \( \alpha_l \) is surjective if \( l \geq 3 \). The image of \( \alpha_1 \) has codimension \( r(n - 2) \) in \( H^0(K_S^\otimes 2) \). The image of \( \alpha_2 \) has codimension \( r - 1 \) in \( H^0(K_S^\otimes 3) \).

Thus we proceed to prove (a), (b), (c). Recall that \( Y \) is an irreducible variety of minimal degree, and in particular, normal. On the other hand the locus formed by the points of \( Y \) with non finite fibers has codimension 2. Thus using Bertini’s Theorem we can choose a smooth curve \( C \) of \( |K_S| \) such that the restriction of the canonical morphism of \( S \) to \( C \) is finite (and flat) onto a smooth rational normal curve of degree \( r \). Let us denote by \( \theta \) the restriction of \( K_S \) to \( C \). By adjunction \( K_C = \theta^\otimes 2 \). Since \( K_S \) is base-point-free so is \( \theta \). Finally, since \( H^1(O_X) = 0 \), \( \pi \) is induced by the complete linear series \( |\theta| \) and therefore \( C, \theta \) and \( \pi \) satisfies the hypothesis of Proposition 2.2.

We prove first the statements in (a), (b) and (c) regarding the maps \( \alpha_l \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^0((K_S^\otimes l) \otimes H^0(O_S)) & \rightarrow & H^0((K_S^\otimes l) \otimes H^0(K_S)) \\
\downarrow & & \downarrow \alpha_l \\
H^0(K_S^\otimes l) & \rightarrow & H^0(K_S^\otimes (l+1))
\end{array}
\]

The right most horizontal arrows are surjective because \( H^1(O_S) = 0 \), by Serre duality and by Kawamata-Viehweg vanishing. The left hand side vertical arrow trivially surjects. The right hand side vertical arrow is the composition of the map \( H^0(K_S^\otimes l) \otimes H^0(\theta) \rightarrow H^0(\theta^\otimes l) \otimes H^0(\theta) \), which is surjective for all \( l \geq 1 \) again because \( H^1(O_S) = 0 \), by Serre duality and by Kawamata-Viehweg vanishing, and the map \( \beta_l \) of multiplication of global sections on \( C \), studied in Proposition 2.2. Then it follows from chasing the diagram that the map \( H^0(K_S^\otimes (l+1)) \rightarrow H^0(\theta^\otimes (l+1)) \) maps the image of \( \alpha_l \) onto the image of \( \beta_l \) and that the codimension of the image of \( \beta_l \) in \( H^0(\theta^\otimes (l+1)) \) is equal to the codimension of \( \alpha_l \) in \( H^0(K_S^\otimes (l+1)) \). This, together with Proposition 2.2, a.1, a.2, a.3, a.5 and b, proves the claims in (a), (b) and (c) concerning the codimensions of the images of the maps \( \alpha_l \).

Thus the only things left to prove are the claims about \( \alpha(2, 2) \) when \( r = 1 \). We consider now this commutative diagram.
The right most horizontal arrows are surjective because $H^1(\mathcal{O}_S) = 0$ and by Serre duality, and by Kawamata-Viehweg vanishing. The left hand side vertical arrow surjects, as we have already proven. The right hand side vertical arrow is the composition of the map $H^0(K_S^\otimes 2) \otimes H^0(\theta^\otimes 2) \rightarrow H^0(\theta^\otimes 4)$, which is surjective because $S$ is regular and by Serre duality, and the map $\beta(2,2)$ of multiplication of global sections on $C$. Then it follows from chasing the diagram that the map $H^0(K_S^\otimes 4) \rightarrow H^0(\theta^\otimes 4)$ maps the image of $\alpha(2,2)$ onto the image of $\beta(2,2)$ and that the codimension of the image of $\beta(2,2)$ in $H^0(\theta^\otimes 1)$ is equal to the codimension of the image of $\alpha(2,2)$ in $H^0(K_S^\otimes 4)$. On the other hand, we know that the image of $\beta(2,2)$ and of $\beta_3 = \beta(3,1)$ are equal of codimension 1 in $H^0(\theta^\otimes 4)$, if $r = 1$ and $n = 2$. Thus we conclude that the images of $\alpha(3,1)$ and $\alpha(2,2)$ in $H^0(K_S^\otimes 4)$ are also equal and of codimension 1. Finally, if $r = 1$ and $n > 2$, $\beta(2,2)$ surjects by Proposition 2.2.a.4. Thus we conclude that if $r = 1$ and $n > 2$, then $\alpha(2,2)$ surjects. □

Theorem 2.1 complements known results on generation of the canonical ring of smooth, regular surfaces of general type. Ciliberto and Green (cf. [G], Theorem 3.9.3, and [Ci]) proved that, given a smooth surface of general type with $h^1(\mathcal{O}_S) = 0$ and $K_S$ globally generated and being $\varphi$ the canonical morphism, a sufficient condition for the canonical ring of $S$ to be generated in degree less than or equal to 2 is that none of the following happen:

1. $\varphi$ maps $S$ generically $2 : 1$ onto $\mathbb{P}^2$.
2. $\varphi(S)$ is a surface of minimal degree (other than $\mathbb{P}^2$).

As a corollary of Ciliberto and Green result and of Theorem 2.1 we obtain the following

**Corollary 2.8.** Let $S$ be a smooth regular surface of general type and such that $K_S$ is globally generated. Let $\varphi$ be the canonical morphism of $S$. The canonical ring of $S$ is generated in degree less than or equal to 2 if and only if none of the following happens:

1. $\varphi$ maps $S$ generically $2 : 1$ onto $\mathbb{P}^2$.
2. $\varphi(S)$ is a surface of minimal degree (other than $\mathbb{P}^2$).
3. Examples of surfaces of general type

In this section we construct some new examples of surfaces of general type which satisfy the hypothesis of Theorem 2.1. The easiest way one could think of producing examples would be to build suitable cyclic covers of surfaces of minimal degree. However, as next remark shows, only low degree cyclic covers can be induced by the canonical morphism of a regular surface, so we have to employ other means to construct some new examples.

**Proposition 3.1.** Let $X$ be a surface of general type with at worst canonical singularities and with base-point-free canonical bundle. Assume that the complete canonical series of $X$ restricts to a complete linear series on a general hyperplane section (e.g., if $X$ is regular). Let $\varphi : X \rightarrow Y$ be the canonical morphism to a surface of minimal degree. Let $n$ be the degree of $\varphi$. Assume that, on the complement $U$ of a codimension 2 closed subset of $Y$,

$$f_* \mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1} \oplus \cdots \oplus L^{1-n}.$$

Then $n = 2$ or 3.

**Proof.** Let $H$ be a general hyperplane section of $Y$ contained in $U$ and let $C$ be the inverse image of $H$ by $\varphi$. Then $C$ is a smooth irreducible member of $|K_X|$. By assumption the morphism $\varphi|_C : C \rightarrow H$ is induced by the complete linear series of a line bundle $\theta$. By adjunction $\theta \otimes 2 = K_C$. Thus $C$, $\theta$ and $\varphi|_C$ satisfy the hypothesis of Lemma 2.3 and

$$(\varphi|_C)_* \mathcal{O}_C = \mathcal{O}_H \oplus (n-2)N^{-1} \oplus N^{-2}.$$ 

On the other hand $(\varphi|_C)_* \mathcal{O}_C$ is equal to the restriction of $\varphi_* \mathcal{O}_X$ to $H$, i.e, to

$$\mathcal{O}_H \oplus (L \otimes \mathcal{O}_H)^{-1} \oplus \cdots \oplus (L \otimes \mathcal{O}_H)^{1-n}.$$ 

This is only possible if $n = 2$ or 3. □

**Corollary 3.2.** Let $X$ be a regular surface of general type with at worst canonical singularities and with base-point-free canonical bundle. Let $Y$ be the image of $X$ by its canonical morphism $X \xrightarrow{\varphi} Y$. If $Y$ is a surface of minimal degree and $\varphi$ is a cyclic cover, then the degree of $\varphi$ is 2 or 3.

The next proposition also rules out many possible examples of covers of odd degree:
Proposition 3.3. Let $X$ be a surface of general type with at worst canonical singularities whose canonical divisor is base-point-free. Let $\varphi$ be a morphism induced by a subseries of $|K_X|$. If $\varphi$ is generically finite onto a smooth scroll $Y \subset \mathbb{P}^N$, then the degree of $\varphi$ is even. In particular, there are not generically finite covers of odd degree of smooth rational normal scrolls, induced by subseries of $K_X$.

Proof. Let $f$ be a fiber of $Y$ and let $C$ be a section of $Y$. Let $-d = C^2$. Since $Y$ is a scroll, its hyperplane section is linearly equivalent to $C+mf$, for some integer $m$. Then $K_X = \varphi^*(C+mf)$. Then \(\deg \varphi = (\varphi^* f) \cdot (\varphi^* C) = (\varphi^* f) \cdot (K_X + \varphi^* f)\), which is an even number. \(\square\)

Now we mention some examples of regular minimal surfaces $X$ whose canonical morphism $\varphi$ maps onto a variety of minimal degree and produce some new ones.

The cases when $\varphi$ is generically finite and has degree 2 and 3 have been completely studied by Horikawa and Konno (see [H1], Theorem 1.6, [H2], Theorem 2.3.I, [H3], Theorem 4.1 and [Kon], Lemma 2.2 and Theorem 2.3). As it turns out there exist generically double covers of linear $\mathbb{P}^2$, the Veronese surface, smooth rational normal scrolls $S(a,b)$ with $b \leq 4$ and cones over rational normal curves of degree 2, 3 and 4 and generically triple covers of $\mathbb{P}^2$ (in particular cyclic triple covers of $\mathbb{P}^2$ ramified along a sextic with suitable singularities) and of the cones over rational normal curves of degree 2 and 3. Horikawa (see [H4], Theorem 2.1) also describes all generically finite quadruple covers $X \xrightarrow{\varphi} Y$, where $X$ is smooth, minimal regular surface, $\varphi$ is the canonical morphism of $X$ and $Y$ is linear $\mathbb{P}^2$.

The examples of Horikawa and Konno just reviewed are examples of covers of degree less than or equal to 3 of surfaces of minimal degree and quadruple covers of $\mathbb{P}^2$. We now construct three new sets of examples of regular surfaces of general type which are quadruple covers of surfaces of minimal degree under the canonical morphism. These examples are 4:1 covers of smooth rational normal scrolls isomorphic to the Hirzebruch surfaces $F_0$ and $F_1$ and of quadric cones in $\mathbb{P}^3$.

Example 3.4. We construct finite quadruple covers $X \xrightarrow{\varphi} Y$, where $X$ is a smooth minimal regular surface of general type, $\varphi$ is the canonical morphism of $X$ and $Y$ is a smooth rational scroll $S(m,m)$, $m \geq 1$.

Let $f$ be a fiber of one of the fibrations of $\mathbb{P}^1$ and let $f'$ be a fiber of the other fibration. Then $Y$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and it is embedded in $\mathbb{P}^{2m+1}$ by $|f + mf'|$.
or by $|f' + mf|$. If $Y$ is embedded by $|f + mf'|$, let $a_1, a_2, b_1$ and $b_2$ satisfy the following: either $a_1 = 1, a_2 = 2, b_1 = m + 1$ and $b_2 = 1$ or $a_1 = 2, a_2 = 1, b_1 = 1$ and $b_2 = m + 1$. If $Y$ is embedded by $|f' + mf|$, let $a_1, a_2, b_1$ and $b_2$ satisfy the following: either $b_1 = 1, b_2 = 2, a_1 = m + 1$ and $a_2 = 1$ or $b_1 = 2, b_2 = 1, a_1 = 1$ and $a_2 = m + 1$. Let $D_i$s be smooth divisors linearly equivalent to $2(a_i f + b_i f')$ intersecting at $D_1 \cdot D_2$ distinct points. Those divisors exist because by the choices of $a_1, a_2, b_1$ and $b_2$, both $2(a_1 f + b_1 f')$ and $2(a_2 f + b_2 f')$ are base-point-free. Let $X' \xrightarrow{\varphi_1} Y$ be the double cover of $Y$ ramified along $D_1$. Since $D_1$ is smooth, so is $X'$. Let $D_2'$ be the inverse image in $X'$ of $D_2$ by $\varphi_1$. Since $D_2$ is smooth and meets $D_1$ at distinct points, $D_2'$ is also smooth. Let $X \xrightarrow{\varphi_2} X'$ be the double cover of $X'$ ramified along $D_2'$. Since $X'$ and $D_2'$ are both smooth, so is $X$. Let us call $\varphi = \varphi_1 \circ \varphi_2$. Now we will show that $X$ is a regular surface of general type, that $K_X = \varphi^* O_Y(1)$ and that $\varphi$ is induced by the complete canonical series of $X$. First we find out the structure of $\varphi_* O_X$ as module over $O_Y$. Recall that $\varphi_2_* O_X = O_Y \oplus \varphi_1^* O_Y(-a_2 f - b_2 f')$. Then

$$\varphi_* O_X = \varphi_1_* O_X \oplus \varphi_1^* O_Y(-a_2 f - b_2 f').$$

Since $\varphi_1_* O_X = O_Y \oplus O_Y(-a_1 f - b_1 f')$, then by projection formula we have

$$\varphi_* O_X = O_Y \oplus O_Y(-a_1 f - b_1 f') \oplus O_Y(-a_2 f - b_2 f') \oplus O_Y(-(a_1 + a_2)f - (b_1 + b_2)f').$$

We see now that $X$ is regular. Recall that $H^1(O_X) = H^1(\varphi_* O_X)$. Our choice of $a_1, a_2, b_1$ and $b_2$ implies that $a_1 f + b_1 f'$ and $a_2 f + b_2 f'$ are both base-point-free and big, thus by Kawamata-Viehweg vanishing,

$$H^1(O_Y(-a_1 f - b_1 f')) = H^1(O_Y(-a_2 f - b_2 f')) = H^1(O_Y(-(a_1 + a_2)f - (b_1 + b_2)f')) = 0.$$

Then, since $H^1(O_Y)$ also vanishes so does $H^1(\varphi_* O_X)$ and $H^1(O_X)$. We now compute $K_X$. Since $\varphi_2$ is a double cover ramified along $D_2$, $K_X = \varphi_2^*(K_X' \otimes \varphi_1^*(O_Y(a_2 f + b_2 f')))$. By a similar reason, $K_X' = \varphi_1^*(K_Y \otimes O_Y(a_1 f + b_1 f'))$. Then $K_X = \varphi^*(K_Y \otimes O_Y((a_1 + a_2)f + (b_1 + b_2)f'))$. Since $K_Y = O_Y(-2f - 2f')$, it follows again from the choices of $a_1, a_2, b_1$ and $b_2$ that $K_X = \varphi^* O_Y(1)$. Finally, to see that $\varphi$ is induced by the complete canonical linear series of $X$ we compute $H^0(K_X)$. We do the computation in the case $O_Y(1) = O_Y(f + mf')$. The case $O_Y(1) = O_Y(m f + f')$ is analogous. Since $K_X = \varphi^* O_Y(1)$,

$$H^0(K_X) = H^0(O_Y(1)) \oplus H^0(O_Y((1 - a_1)f + (m - b_1)f')) \oplus H^0(O_Y((1 - a_2)f + (m - b_2)f')) \oplus H^0(O_Y((1 - a_1 - a_2)f + (m - b_1 - b_2)f')).$$

Example 3.5. We construct finite quadruple covers $X \xrightarrow{\varphi} Y$, where $X$ is a smooth regular surface of general type with base-point-free canonical bundle, $\varphi$ is the canonical morphism of $X$ and $Y$ is a smooth rational scroll $S(m-1,m)$, $m \geq 2$.

Let $C_0$ be a minimal section of $F_1$ and let $f$ be one of the fibers. Then $Y$ is $F_1$ and it is embedded in $\mathbb{P}^{2m}$ by $|C_0 + mf|$. Let $a_1, a_2, b_1$ and $b_2$ satisfy the following: either $a_1 = 1, a_2 = 2, b_1 = m+1$ and $b_2 = 2$ or $a_1 = 2, a_2 = 1, b_1 = 2$ and $b_2 = m+1$.

Let $D_1$s be smooth divisors linearly equivalent to $2(a_iC_0 + b_i f)$ intersecting at $D_1 \cdot D_2$ distinct points. The fact that such divisors exist follows from our choice of $a_1, a_2, b_1$ and $b_2$, which implies that the linear systems of $D_1$ and $D_2$ are base-point-free. Let $X' \xrightarrow{\varphi_1} Y$ be the double cover of $Y$ ramified along $D_1$. Since $D_1$ is smooth, so is $X'$. Let $D_2'$ be the inverse image in $X'$ of $D_2$ by $\varphi_1$. Since $D_2'$ is smooth and meets $D_1$ transversally, $D_2'$ is also smooth. Let $X \xrightarrow{\varphi_2} X'$ the double cover of $X'$ ramified along $D_2'$. Since $X'$ and $D_2'$ are both smooth, so is $X$. Let us call $\varphi = \varphi_1 \circ \varphi_2$. Now we will show that $X$ is a regular surface of general type, that $K_X = \varphi^*O_Y(1)$ and that $\varphi$ is induced by the complete canonical series of $X$. First we find out the structure of $\varphi_*O_X$ as module over $O_Y$. Recall that $\varphi_*O_X = O_{X'} \oplus \varphi_1^*O_Y(-a_2C_0 - b_2 f)$. Then

$$\varphi_*O_X = \varphi_1_*O_{X'} \oplus \varphi_1^*(\varphi_1^*O_Y(-a_2C_0 - b_2 f)) .$$

Since $\varphi_1_*O_{X'} = O_Y \oplus O_Y(-a_1C_0 - b_1 f)$, then by projection formula we have

$$\varphi_*O_X = O_Y \oplus O_Y(-a_1C_0 - b_1 f) \oplus O_Y(-a_2C_0 - b_2 f) \oplus O_Y(-(a_1 + a_2)C_0 - (b_1 + b_2) f) .$$

We see now that $X$ is regular. Recall that $H^1(O_X) = H^1(\varphi_*O_X)$. Our choices of $a_1, a_2, b_1$ and $b_2$ imply that $a_1C_0 + b_1 f$ and $a_2C_0 + b_2 f$ are both base-point-free and big divisors, thus by Kawamata-Viehweg vanishing,

$$H^1(O_Y(-a_1C_0 - b_1 f)) = H^1(O_Y(-a_2C_0 - b_2 f)) = H^1(O_Y(-(a_1 + a_2)C_0 - (b_1 + b_2) f)) = 0 .$$
Then, since $H^1(\mathcal{O}_Y)$ also vanishes so does $H^1(\varphi_*\mathcal{O}_X)$ and therefore $H^1(\mathcal{O}_X)$. We now compute $K_X$. Since $\varphi_2$ is a double cover ramified along $D'_2$, $K_X = \varphi_2^*(K_{X'} \otimes \varphi_1^*(\mathcal{O}_Y(a_2C_0 + b_2f)))$. By similar reason, $K_{X'} = \varphi_1^*(K_Y \otimes \mathcal{O}_Y(a_1C_0 + b_1f))$. Then $K_X = \varphi_1^*(K_Y \otimes \mathcal{O}_Y((a_1 + a_2)C_0 + (b_1 + b_2)f))$. Since $K_Y = \mathcal{O}_Y(-2C_0 - 3f)$, it follows from our choice of $a_1, a_2, b_1$ and $b_2$ that $K_X = \varphi^*\mathcal{O}_Y(1)$. Finally, to see that $\varphi$ is induced by the complete canonical linear series of $X$ we compute $H^0(K_X)$. Since $K_X = \varphi^*\mathcal{O}_Y(1)$,

$$
H^0(K_X) = H^0(\mathcal{O}_Y(1)) \oplus H^0(\mathcal{O}_Y((1-a_1)C_0 + (m-b_1)f)) \oplus H^0(\mathcal{O}_Y((1-a_2)C_0 + (m-b_2)f)) \oplus H^0(\mathcal{O}_Y((1-a_1-a_2)C_0 + (m-b_1-b_2)f)).
$$

Again, by the choices of $a_1, a_2, b_1$ and $b_2$, the last three direct sums of the above expression are 0, so $\varphi$ is indeed induced by the complete canonical series of $K_X$.

□

**Remark 3.6.** With the same arguments, if we allow certain mild singularities in $D_1$ and $D_2$, one can construct examples of covers of $\mathbb{F}_0$ and $\mathbb{F}_1$ with at worst canonical singularities.

Finally we construct an example of a quadruple cover of a singular surface of minimal degree.

**Example 3.7.** We construct an example of a smooth, generically finite, quadruple cover $X \xrightarrow{\varphi} Z$ of the quadric cone $Z$ in $\mathbb{P}^3$, where $X$ is a regular surface of general type whose canonical divisor is base-point-free, and $\varphi$ is its canonical morphism.

Let $Y = \mathbb{F}_2$. Let $C_0$ be the minimal section of $Y$ and let $f$ be a fiber of $Y$. Let $D_1$ be a smooth divisor on $Y$, linearly equivalent to $2C_0 + 6f$ and meeting $C_0$ transversally. Let $D_2$ be a smooth divisor on $Y$ linearly equivalent to $3C_0 + 6f$ and meeting $D_1$ transversally. Such divisors $D_1$ and $D_2$ exist because $2C_0 + 6f$ is very ample and $3C_0 + 6f$ is base-point-free. Note also that, since $(3C_0 + 6f) \cdot C_0 = 0$, $C_0$ and $D_2$ do not meet. Let $X' \xrightarrow{\varphi_1} Y$ be the double cover of $Y$ along $D_1$. Since $D_1$ is smooth, so is $X'$. Since $D_1$ meets $C_0$ at two distinct points, the pullback $C'_0$ of $C_0$ by $\varphi_1$ is a smooth line with self-intersection $-4$. Let $D'_2$ be the pullback of $D_2$ by $\varphi_1$. Since $D_1$ and $D_2$ meet transversally, $D'_2$ is smooth and since $D_2$ and $C_0$ do not meet, neither do $D'_2$ and $C'_0$. Let $L'_2$ be the pullback of $2C_0 + 3f$ by $\varphi_1$. Let $X \xrightarrow{\varphi_2} X'$ be the double cover of $X'$ along $D'_2 \cup C'_0$. Since $D'_2 \cup C'_0$ is smooth, so is $X'$. Let us denote $\varphi = \varphi_1 \circ \varphi_2$. Then

$$
\varphi_*\mathcal{O}_X = \varphi_{1*}\varphi_{2*}\mathcal{O}_X = \varphi_{1*}(\mathcal{O}_{X'} \oplus L'_2) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-C_0 - 3f) \oplus \mathcal{O}_Y(-2C_0 - 3f) \oplus \mathcal{O}_Y(-3C_0 - 6f). \quad (3.7.1)
$$
Since $-C_0 - 3f$ and $-3C_0 - 6f$ are big and base-point-free, by Kawamata-Viehweg vanishing and Serre duality, $H^1(\mathcal{O}_Y(-C_0 - 3f)) = H^1(\mathcal{O}_Y(-3C_0 - 6f)) = 0$. By Serre duality $H^1(\mathcal{O}_Y(-2C_0 - 3f)) = H^1(\mathcal{O}_Y(-f))^* = 0$. Then, since $H^1(\mathcal{O}_Y) = 0$, $X$ is regular. Arguing as in Example 3.4 and Example 3.5 we see that

$$K_X = \varphi^*(K_Y \otimes \mathcal{O}_Y(3C_0 + 6f)) = \varphi^*\mathcal{O}_Y(C_0 + 2f)(3.7.2).$$

Now we compute $H^0(K_X)$. Using projection formula and 3.7.1 and 3.7.2 we obtain that

$$H^0(K_X) = H^0(\mathcal{O}_Y(C_0 + 2f)) \oplus H^0(\mathcal{O}_Y(-f)) \oplus H^0(\mathcal{O}_Y(-C_0 - f)) \oplus H^0(\mathcal{O}_Y(-2C_0 - 4f)) = H^0(\mathcal{O}_Y(C_0 + 2f)).$$

Thus the canonical morphism of $X$ is the composition of $\varphi$ and the morphism $Y \overset{\phi}{\rightarrow} Z \subset \mathbb{P}^3$, induced by the complete linear series of $C_0+2f$. Since $\phi$ contracts $C_0$, the canonical morphism of $X$ is not finite, but it is generically finite of degree 4 onto $Z$, which is a surface of minimal degree as we wanted.

On the other hand, if $C''_0$ is the pullback of $C_0$ by $\varphi$, then $C''_0$ is a smooth line with self-intersection $-2$. Thus the morphism $\phi \circ \varphi$ also factorizes as $\varphi' \circ \psi$, where

$$X \overset{\psi}{\rightarrow} \overline{X}$$

is the morphism from $X$ to its canonical model $\overline{X}$ and

$$\overline{X} \overset{\varphi'}{\rightarrow} Z$$

is the canonical morphism of $\overline{X}$. Thus $\varphi'$ is an example of a finite, $4:1$ canonical morphism from regular surface of general type with canonical singularities onto a singular surface of minimal degree. □

4. THE CANONICAL RING OF A CURVE

In this section we study a very well known ring, the canonical ring of a curve. It was proved by Noether that the canonical ring of smooth curve $C$ is generated in degree 1 if and only if the curve is non hyperelliptic. Our purpose is to show again the link between the structure of this canonical ring and the structure of $\mathcal{O}_C$ as an algebra over $\mathcal{O}_{\mathbb{P}^1}$ via a suitable morphism from $C$ to $\mathbb{P}^1$. Precisely, we will look at curves endowed with certain finite morphisms to a rational curve.
First, we will consider hyperelliptic curves and its $2 : 1$ morphism to $\mathbb{P}^1$. Second, we will consider curves having a base-point-free theta-characteristic inducing a morphism from the curve to a rational normal curve. The latter class of curves include hyperelliptic curves but also other curves far more general: those curves with a base-point-free theta characteristic with two linearly independent sections. We will give a new proof of Noether’s theorem in these two cases.

**Theorem 4.1.** Let $C$ be a smooth hyperelliptic curve of genus $g$.

1) If $g = 2$, then its canonical ring is generated by its elements of degree 1, and by 1 element of degree 3.

2) If $g \geq 3$, then its canonical ring is generated by its elements of degree 1, and by $g - 2$ elements of degree 2.

**Proof.** Let $L$ be the base-point-free $g^1_2$ on $C$ and let $\pi : C \rightarrow \mathbb{P}^1$ be the morphism induced by $|L|$. The cover $\pi$ is double, therefore $\pi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g - 1)$ and the structure of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g - 1)$ as algebra over $\mathcal{O}_{\mathbb{P}^1}$ is well known: the elements of $\mathcal{O}_{\mathbb{P}^1}$ multiply among themselves by the product in $\mathcal{O}_{\mathbb{P}^1}$, the elements of $\mathcal{O}_{\mathbb{P}^1}(-g - 1)$ via the module structure of $\mathcal{O}_{\mathbb{P}^1}(-g - 1)$ over $\mathcal{O}_{\mathbb{P}^1}$, and finally the multiplication of elements of $\mathcal{O}_{\mathbb{P}^1}(-g - 1)$ is dictated by the branch divisor on $\mathbb{P}^1$ and produces elements of $\mathcal{O}_{\mathbb{P}^1}$. Recall now that $K_C = L \otimes^{g-1}$. Then $\pi_* K_C^{\otimes n} = \mathcal{O}_{\mathbb{P}^1}(n(g - 1)) \oplus \mathcal{O}_{\mathbb{P}^1}((n - 1)(g - 1) - 2)$. This implies for instance that $H^0(K_C) = H^0(\mathcal{O}_{\mathbb{P}^1}(g - 1))$ and $H^0(K_C^{\otimes 2}) = H^0(\mathcal{O}_{\mathbb{P}^1}(2(g - 1))) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g - 3))$. Thus the image of

$$H^0(K_C) \otimes H^0(K_C) \overset{\alpha_1}{\longrightarrow} H^0(K_C^{\otimes 2})$$

is $H^0(\mathcal{O}_{\mathbb{P}^1}(2(g - 1)))$, which has codimension $g - 2$ in $H^0(K_C^{\otimes 2})$. On the other hand, since $H^0(\mathcal{O}_{\mathbb{P}^1}((n - 1)(g - 1) - 2)) \neq 0$ if $n \geq 2$, except for $n = g = 2$, we see that the map

$$H^0(K_C^{\otimes n}) \otimes H^0(K_C) \overset{\alpha_n}{\longrightarrow} H^0(K_C^{\otimes n+1})$$

surjects for all $n \geq 2$, except if $n = g = 2$. If $g = 2$, $\alpha_2$ does not surject, since its image is $H^0(\mathcal{O}_{\mathbb{P}^1}(3))$, which has codimension 1 in $H^0(K_C^{\otimes 3})$. □

**Theorem 4.2.** Let $C$ be a smooth curve of genus $g \geq 3$ possessing a base-point-free line bundle $\theta$ such that $\theta^{\otimes 2} = K_C$ and such that $|\theta|$ induces a morphism of degree $n$ onto a rational normal curve of degree $r$. Then the canonical ring
of $C$ is generated in degree 1 unless $r = \frac{n-1}{2}$, equivalently, unless $n = 2$. In particular, if $r \neq \frac{n-1}{2}$, then $C$ is non-hyperelliptic.

Proof. Let $\pi$ be the morphism induced by $|\theta|$. The pair $(C, \theta)$ satisfies the hypothesis of Lemma 2.3. Thus we know that

$$E = (n-2)\mathcal{O}_{\mathbf{P}^1}(-r-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2r-2).$$

Recall also (cf. 2.4.1) that

For some $1 \leq i, j \leq n-2$, the projection of the map

$$\mathcal{O}_{\mathbf{P}^1}(a_i) \otimes \mathcal{O}_{\mathbf{P}^1}(a_j) \rightarrow \pi_*\mathcal{O}_C$$ to $\mathcal{O}_{\mathbf{P}^1}(-2r-2)$ is surjective,

in fact, it is an isomorphism.

We want to discuss the surjectivity of the maps

$$H^0(K_C^{\otimes m}) \otimes H^0(K_C) \xrightarrow{\gamma_m} H^0(K_C^{\otimes m+1})$$

for all $m \geq 1$. The map $\gamma_m$ is in fact the map $\beta(2m, 2)$ defined in Proposition 2.2. Recall that

$$H^0(K_C) = H^0(\mathcal{O}_{\mathbf{P}^1}(2r)) \oplus (n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(r-1))$$

and

$$H^0(K_C^{\otimes 2}) = H^0(\mathcal{O}_{\mathbf{P}^1}(4r)) \oplus (n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(3r-1)) \oplus H^0(\mathcal{O}_{\mathbf{P}^1}(2r-2)).$$

Then the restriction of $\beta(2, 2)$ to $H^0(\mathcal{O}_{\mathbf{P}^1}(2r)) \otimes H^0(\mathcal{O}_{\mathbf{P}^1}(2r))$ surjects onto $H^0(\mathcal{O}_{\mathbf{P}^1}(4r))$. Thus if $n = 2$, the image of $\beta(2, 2)$ is $H^0(\mathcal{O}_{\mathbf{P}^1}(4r))$, which has codimension $2r-1$. Note that, since in this case $C$ is hyperelliptic, $r = \frac{n-1}{2}$ (and $g$ is therefore odd), so the codimension of the image of $\beta(2, 2)$ in $H^0(K_C^{\otimes 2})$ is $g-1$, as seen in Theorem 4.1. If $n \geq 3$, then restriction of $\beta(2, 2)$ to $H^0(\mathcal{O}_{\mathbf{P}^1}(2r)) \otimes (n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(r-1))$ surjects onto $(n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(3r-1))$. Finally, it follows from 2.4.1 that the image of the restriction of $\beta(2, 2)$ to $(n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(r-1)) \otimes (n-2)H^0(\mathcal{O}_{\mathbf{P}^1}(r-1))$ projects onto $H^0(\mathcal{O}_{\mathbf{P}^1}((2r-2))$. Therefore $\beta(2, 2)$ surjects if $n \geq 3$. On the other hand

$$H^0(K_C^{\otimes m}) = H^0(\mathcal{O}_{\mathbf{P}^1}(2mr)) \oplus (n-2)H^0(\mathcal{O}_{\mathbf{P}^1}((2m-1)r-1)) \oplus H^0(\mathcal{O}_{\mathbf{P}^1}((2m-2)r-2)),$$

therefore $\beta(2m, 2)$ surjects for all $m \geq 2$. □

Not every curve $C$ of genus $g$ has a theta-characteristic satisfying the hypothesis of Theorem 4.2. The curves with theta-characteristics with a positive even number of sections form a divisor $\mathcal{M}_g^1$ in $\mathcal{M}_g$ (see [Be] and [F]). Moreover, the theta-characteristic of a general curve of $\mathcal{M}_g^1$ is base-point-free (see [T]). Thus we can deduce from this and the above proposition Noether’s theorem for the case in which $C$ is a general curve of $\mathcal{M}_g^1$. 
Corollary 4.3. Let $C$ a smooth curve of genus $g \geq 3$, general in $\mathcal{M}_g^1$. Then the canonical ring of $C$ is generated in degree 1. In particular, $C$ is non-hyperelliptic.

5. Homogeneous rings of Calabi-Yau threefolds

In this section we give a new, different proof of the following result contained in [GP2] as part of Theorems 1.4 and 1.7 and Corollary 1.8 of that article. The arguments we use will give yet another illustration of how the algebra structure induced on $\mathcal{O}_X$ is linked to the very ampleness and normal generation of a line bundle.

Theorem 5.1. Let $X$ be a Calabi-Yau threefold and let $B$ an ample and base-point-free line bundle such that $h^0(B) = 4$. Let $S$ be a divisor in $|B|$ and let $C$ be any smooth curve $C \in |B \otimes \mathcal{O}_S|$. The following are equivalent:

1. $B \otimes ^2$ satisfies property $N_0$.
2. $B \otimes ^3$ satisfies property $N_0$.
3. The sectional genus of $B$ is bigger than 3.
4. The curve $C$ is non-hyperelliptic.

Proof. We will denote by $\varphi$ the morphism induced by $|B|$ onto $\mathbb{P}^3$. First of all we observe that, by [M], Theorem 2,

$$H^0(B \otimes ^n) \otimes H^0(B) \longrightarrow H^0(B \otimes ^{n+1})$$

surjects if $n \geq 4$. This implies that

$$H^0(B \otimes ^n) \otimes H^0(B \otimes ^n) \longrightarrow H^0(B \otimes ^{2n})$$

surjects if $n \geq 4$. Therefore, $B \otimes ^2$ satisfies property $N_0$ if and only if

$$H^0(B \otimes ^2) \otimes H^0(B \otimes ^2) \longrightarrow H^0(B \otimes ^4)$$

surjects. Analogously, $B \otimes ^3$ satisfies property $N_0$ if and only if

$$H^0(B \otimes ^3) \otimes H^0(B \otimes ^3) \longrightarrow H^0(B \otimes ^6)$$

surjects. The proof goes on through three steps.

(5.1.1) The vector bundle $\varphi_* \mathcal{O}_X$. The morphism $\pi$ is finite of degree $n \geq 2$. The push down of $\mathcal{O}_X$ by $\varphi$, $\varphi_* \mathcal{O}_X$, is isomorphic to $\mathcal{O}_{\mathbb{P}^3} \oplus E$, where $E$ is a vector bundle of rank $n - 1$ on $\mathbb{P}^3$. Since $h^1(B \otimes ^n) = h^2(B \otimes ^n) = 0$ for all $n \in \mathbb{Z}$, using
projection formula we obtain from Horrocks’ criterion that $E$ splits as a direct sum of line bundles. On the other hand $B \otimes O_S = K_S$, and the restriction of $|B|$ is the complete canonical series of $S$, for $h^1(\mathcal{O}_X) = 0$. In addition, $h^1(\mathcal{O}_S) = 0$. Let us denote by $\pi$ the restriction of $\varphi$ to $C$. We are under the hypothesis of Lemma 2.3, therefore

$$\pi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus (n-2)\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-4).$$

This implies that

$$\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3} \oplus (n-2)\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-4).$$

Let us call $E_1 = (n-2)\mathcal{O}_{\mathbb{P}^3}(-2)$ and $E_2 = \mathcal{O}_{\mathbb{P}^3}(-4)$.

**5.1.2** Relationship between the algebra structure of $\varphi_* \mathcal{O}_X$ and the normal generation of $B \otimes 2$ and $B \otimes 3$. We study now the $\mathcal{O}_{\mathbb{P}^3}$-algebra structure of $\varphi_* \mathcal{O}_X$ in relation with the surjectivity of the maps

$$H^0(B \otimes 2) \otimes H^0(B \otimes 2) \xrightarrow{\alpha} H^0(B \otimes 4)$$

and

$$H^0(B \otimes 3) \otimes H^0(B \otimes 3) \xrightarrow{\beta} H^0(B \otimes 6).$$

Recall that $B \otimes 2$ (resp. $B \otimes 3$) is normally generated if and only if $\alpha$ (resp. $\beta$) surjects. Recall that $H^0(B \otimes 2) = H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \oplus H^0(E(2)) = H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \oplus H^0(E_1(2))$ and $H^0(B \otimes 4) = H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \oplus H^0(E(4)) = H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \oplus H^0(E_1(4)) \oplus H^0(E_2(4))$. Then we can see the map $\alpha$ as direct sum of:

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \xrightarrow{\gamma} H^0(\mathcal{O}_{\mathbb{P}^3}(4))$$

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \otimes H^0(E_1(2)) \xrightarrow{\delta} H^0(E_1(4))$$

$$H^0(E_1(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \xrightarrow{\zeta} H^0(E_1(4))$$

$$H^0(E_1(2)) \otimes H^0(E_1(2)) \xrightarrow{\eta} H^0(\mathcal{O}_{\mathbb{P}^3}(4) \oplus E(4)).$$

The map $\gamma$ is induced by ring multiplication on $\mathcal{O}_{\mathbb{P}^3}$ and it is therefore surjective. The maps $\delta$ and $\epsilon$ are induced by module multiplication and are also surjective. Therefore $\alpha$ surjects if and only if the composition of $\eta$ with the projection to $H^0(E_2(4))$ is surjective. Now the map $\eta$ depends on the way in which
elements of \( E \) multiply among themselves. Let us denote by \( \mu \) the morphism \( E \otimes E \to \mathcal{O}_{\mathbb{P}^3} \oplus E \) induced by the ring structure of \( \varphi_* \mathcal{O}_X \). Now, the composition of \( \eta \) with the projection to \( H^0(E_2(4)) \) is surjective if and only if

\[ (*) \quad \mu \text{ induces an isomorphism from at least one of the components of } E_1 \otimes E_1 \text{ isomorphic to } \mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \text{ onto the component } E_2. \]

The same argument proves that \( \beta \) is surjective if and only if \((*)\) holds.

**References**

[Be] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.

[Bo] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes. Et. Sci. Publ. Math. **42** (1973), 171–219.

[Ca] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Differential Geom. **19** (1984), 483–515.

[Ci] C. Ciliberto, *Sul grado dei generatori dell’anello di una superficie di tipo generale*, Rend. Sem. Mat. Univ. Politec. Torino **41** (1983)

[EH] D. Eisenbud and J. Harris, *On varieties of minimal degree (A centennial account)*, Algebraic Geometry, Bowdoin 1985, Amer. Math. Soc. Symp. in Pure and App. Math. **46** (1987), 1–14.

[F] H.M. Farkas, *Special divisors and analytic subloci of Teichmüller space*, Amer. J. Math. **88** 1966 881–901.

[GP1] F.J. Gallego and B.P. Purnaprajna, *Projective normality and syzygies of algebraic surfaces*, J. Reine Angew. Math. **506** (1999), 145–180.
[GP2] F.J. Gallego and B.P. Purnaprajna, *Very ampleness and higher syzygies for Calabi-Yau threefolds*, Math. Ann. 312 (1998), no. 1, 133–149.

[GP3] F.J. Gallego and B.P. Purnaprajna, *Canonical Covers of varieties of minimal degree*, in preparation.

[GP4] F.J. Gallego and B.P. Purnaprajna, *On the rings of trigonal curves*, in preparation.

[G] M.L. Green, *The canonical ring of a variety of general type*, Duke Math. J. 49 (1982), 1087–1113.

[HM] D. Hahn and R. Miranda, *Quadruple covers of algebraic varieties*, J. Algebraic Geom. 8 (1999), 1–30.

[H1] E. Horikawa, *Algebraic surfaces of general type with small $c_1^2$. I*, Ann. of Math. (2) 104 (1976), 357–387.

[H2] E. Horikawa, *Algebraic surfaces of general type with small $c_1^2$. II*, Invent. Math. 37 (1976), 121–155.

[H3] E. Horikawa, *Algebraic surfaces of general type with small $c_1^2$. III*, Invent. Math. 47 (1978), 209–248.

[H4] E. Horikawa, *E. Algebraic surfaces of general type with small $c_1^2$. IV*, Invent. Math. 50 (1978/79), 103–128.

[Kod] K. Kodaira, *Pluricanonical systems on algebraic surfaces of general type*, J. Math. Soc. Japan 20 (1968), 170–192.

[Kon] K. Konno, *Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$*, Math. Ann. 290 (1991), 77–107.

[M] D. Mumford, *Varieties defined by quadratic equations*, Corso CIME in Questions on Algebraic Varieties, Rome (1970), 30–100.

[OP] K. Oguiso and T. Peternell, *On polarized canonical Calabi-Yau threefolds*, Math. Ann. 301 (1995), 237–248.

[T] M. Teixidor i Bigas, *Half-canonical series on algebraic curves*, Trans. Amer. Math. Soc. 302 (1987), 99–115.