Expansions of tree amplitudes for Einstein–Maxwell and other theories

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The expansions of tree-level scattering amplitudes for one theory into amplitudes for another theory, which have been studied in recent work, exhibit hidden connections between different theories that are invisible in the traditional Lagrangian formalism of quantum field theory. In this paper, the general expansion of tree Einstein–Maxwell amplitudes into the Kleiss–Kuijf basis of tree Yang–Mills amplitudes has been derived by applying a method based on differential operators. The obtained coefficients are shared by the expansion of tree \( \phi^4 \) amplitudes into tree BS (bi-adjoint scalar) amplitudes and the expansion of tree special Yang–Mills scalar amplitudes into tree BS amplitudes, as well the expansion of tree Dirac–Born–Infeld amplitudes into tree non-linear sigma model amplitudes.

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1. Introduction

Modern research into S-matrices has exhibited remarkable properties of scattering amplitudes that are not evident upon inspecting traditional quantum field theory. The expansion of tree-level scattering amplitudes of one theory into those of another theory, which hints at hidden connections between different theories, is a significant example. Such unexpected expansions were first implied by the well known Kawai–Lewellen–Tye (KLT) relation [1], which represents the tree-level amplitudes of gravity as the double copy of color-ordered tree amplitudes of Yang–Mills (YM) theory,

\[
A_G = \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} A_{YM}^L(n-1, n \sigma, 1) S[\sigma | \tilde{\sigma}] A_{YM}^R(1, \tilde{\sigma}, n-1, n),
\]  

(1)

where \( S_{n-3} \) denotes permutations on \( n - 3 \) external gluons and \( S[\sigma | \tilde{\sigma}] \) stands for the kinematic kernel. One can arrive at the expansion of gravitational amplitudes into YM ones by summing over all \( S_{n-3} \) permutations \( \sigma \) (or \( \tilde{\sigma} \)) in the relation (1). The Cachazo–He–Yuan (CHY) formula proposed in 2013 indicates a much richer web of expansions among a wide range of theories [2–6]. In the CHY framework, the \( n \)-point tree-level amplitudes arise as a multi-dimensional contour integral over \( n \) auxiliary complex variables, formally written as

\[
A_n = \int d\mu_n I^{CHY},
\]

(2)
where the auxiliary variables are fully localized by the scattering equations. The integrand $I^{\text{CHY}}$ depends on the theory under consideration, which can always be factorized as two ingredients:

$$I^{\text{CHY}} = I^L \times I^R. \quad (3)$$

The integrands for gravity and YM take the formulae $I^L_G = \text{Pf}'\Psi(k_i, \epsilon_i, z_i)$, $I^R_G = \text{Pf}'\tilde{\Psi}(k_i, \tilde{\epsilon}_i, z_i)$, and $I^L_{\text{YM}} = \text{Pf}'\Psi(k_i, \epsilon_i, z_i)$, $I^R_{\text{YM}} = \text{Pf}'\Psi(k_i, \epsilon_i, z_i)$, respectively. Here $\Psi(k_i, \epsilon_i, z_i)$ is a $2n \times 2n$ anti-symmetric matrix depends on external momenta $k_i$, polarization vectors $\epsilon_i$, as well as auxiliary variables $z_i$. $\text{Pf}'\Psi(k_i, \epsilon_i, z_i)$ denotes the reduced Pfaffian of the matrix $\Psi(k_i, \epsilon_i, z_i)$. $\text{PT}(\alpha)$ is the so-called Parke–Taylor factor:

$$\text{PT}(\alpha) = \frac{1}{(z_{a_1} - z_{a_2})(z_{a_2} - z_{a_3}) \cdots (z_{a_n} - z_{a_1})}. \quad (4)$$

Notice that the polarization tensors of gravitons are expressed as the product of polarization vectors $\epsilon_i^\mu \tilde{\epsilon}_i^\nu$. In the framework, the expansion can be understood as expanding the reduced Pfaffian $\text{Pf}'\Psi(k_i, \epsilon_i, z_i)$ into the Parke–Taylor factors $\text{PT}(\alpha)$. One can go further since the CHY integrands reflect the double copy structure for a variety of other theories beyond gravity. For instance, the integrands for Einstein–Yang–Mills theory (EYM), Einstein–Maxwell theory (EM), Born–Infeld theory (BI) also carry $I^L_{\text{EM}} = \text{Pf}'\Psi(k_i, \epsilon_i, z_i)$ [6]; thus the amplitudes of these theories can also be expanded into YM ones by expanding $I^R_{\text{YM}}$ into the Parke–Taylor factors.

Although conceptually simple and straightforward, the evaluation of the explicit coefficients of expansion is rather complicated. In order to overcome the inadequacy, various methods have been developed from different angles [7–17]. Among these investigations, the recently proposed approach based on differential operators is efficient and systematic [17]. The differential operators introduced by Cheung et al. [18] link on-shell tree amplitudes for a series of theories together, and organize them into an elegant unified web\(^1\). In this unified web, tree amplitudes of other theories can be generated by acting proper differential operators on tree amplitudes of gravity; thus the connections between amplitudes of different theories can be represented by differential equations. With these relations, expansions of tree amplitudes can be arrived at via two paths. One is solving the corresponding differential equations, together with considering some physical constraints such as the gauge invariance condition. Through the manipulation, the expansion of tree amplitudes for EYM and gravity into tree pure YM amplitudes\(^2\) has been derived efficiently [17]. Another way is getting the expansion of other amplitudes by applying corresponding differential operators on the given expansion of gravitational amplitudes since amplitudes of other theories arise from amplitudes of gravity via differential operators. Along this path, the expansion of BI amplitudes into YM ones has been obtained directly, and the expansion of YM amplitudes into bi-adjoint scalar (BS) ones has also been discussed [17].

It is worth emphasizing that the connections between amplitudes of different theories, which were previously reflected by differential operators, are now established by expansions.

The unified web based on differential operators includes many more theories than gravity, EYM, BI, and YM; thus it is natural to expect that the same method can be applied to other theories in the web, and expansions for these theories could also be found. The current short paper is devoted to

\(^1\) A similar web for CHY integrands of various theories was provided earlier in Ref. [6], and the relation between the two pictures was established in Refs. [19,20].

\(^2\) More precisely, the expansion of amplitudes of EYM and gravity into the Kleiss–Kuijf (KK) basis. The KK basis will be explained in the next section.
applying the method based on differential operators to tree amplitudes of Einstein–Maxwell theory (EM), $\phi^4$ theory, and special Yang–Mills scalar theory (sYMS), as well as Dirac–Born–Infeld theory (DBI). By applying appropriate operators on the expansion of tree gravitational amplitudes, it is simple to obtain the expansion of tree EM amplitudes into the Kleiss–Kuijf (KK) basis. Then, the expansion of tree $\phi^4$ amplitudes into tree BS amplitudes, the expansion of tree sYMS amplitudes into tree BS amplitudes, and the expansion of tree DBI amplitudes into tree non-linear sigma model (NLSM) amplitudes can be read out straightforwardly by acting operators on the expansion of EM amplitudes. Thus the web of the expansions, which reflects the deep connections between different theories, has been generalized to include EM, YM, $\phi^4$, sYMS, BS, and DBI as well as NLSM.

This paper is organized as follows. In Sect. 2, we give a brief review of differential operators and the formula of the expansion of gravitational amplitudes into the KK basis. These ingredients serve as background for the work in this paper. With these preparations, we derive the general expression of the expansion of tree EM amplitudes into the KK basis in Sect. 3. As by-products, the expansion of $\phi^4$ amplitudes into BS ones, the expansion of sYMS amplitudes into BS ones, and the expansion of DBI amplitudes into NLSM ones are also provided in this section. Some explicit examples are given in Sect. 4. Finally, we end with a summary and discussion in Sect. 5.

2. Backgrounds

In this section, we review some already known results that are crucial for discussions in later sections. Firstly, we introduce differential operators, which are the main tools in this paper. Secondly, we review the KK basis of pure YM amplitudes, and the explicit formula of the expansion of tree gravitational amplitudes into the KK basis.

2.1. Differential operators

The differential operators defined by Cheung et al. act on variables constructed by Lorentz contractions of external momenta and polarization vectors, establishing unifying relations for a variety of theories [18–20]. There are three kinds of basic operators:

1. Trace operator:

$$ T_{ij}^\epsilon \equiv \frac{\partial}{\partial (\epsilon_i \cdot \epsilon_j)}, \quad (5) $$

where $\epsilon_i$ is the polarization vector of the $i$th external leg.

2. Insertion operator:

$$ T_{ikj}^\epsilon \equiv \frac{\partial}{\partial (\epsilon_k \cdot k_i)} - \frac{\partial}{\partial (\epsilon_k \cdot k_j)}, \quad (6) $$

where $k_i$ denotes the momentum of the $i$th external leg.

3. Longitudinal operator:

$$ L_i^\epsilon \equiv \sum_{j \neq i} k_i \cdot k_j \frac{\partial}{\partial (\epsilon_i \cdot k_j)}, \quad L_{ij}^\epsilon \equiv -k_i \cdot k_j \frac{\partial}{\partial (\epsilon_i \cdot \epsilon_j)}. \quad (7) $$

Here the upper index $\epsilon$ means that the operators are defined through polarization vectors $\epsilon_i$. For gravitons with two copies of the polarization vectors $\epsilon_i$, for another independent copy $\tilde{\epsilon}_i$. 

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Four kinds of combinatorial operators can be defined as products of these basic operators:

1. For a length-$m$ ordered set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of external particles, the operator $T^\epsilon[\alpha]$ is given as

$$T^\epsilon[\alpha] \equiv T^{\epsilon}_{\alpha_1\alpha_m} \cdot \prod_{i=2}^{m-1} T^{\epsilon}_{\alpha_{i-1}\alpha_i\alpha_m}, \quad (8)$$

which generates the color ordering $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

2. For $n$ external particles, the operator $L^\epsilon$ is defined as

$$L^\epsilon \equiv \prod_i L^\epsilon_i, \quad \tilde{L}^\epsilon \equiv \sum_{\rho \in \text{pair}} \prod_{ij \in \rho} L^\epsilon_{ij}. \quad (9)$$

The two definitions $L^\epsilon$ and $\tilde{L}^\epsilon$ are not equivalent to each other at the algebraic level. However, when acting on appropriate on-shell physical amplitudes, two combinations $L^\epsilon \cdot T^\epsilon_{ab}$ and $\tilde{L}^\epsilon \cdot T^\epsilon_{ab}$, with subscripts of $L^\epsilon_i$ and $L^\epsilon_{ij}$ running through all nodes in $\{1, 2, \ldots, n\}/\{a, b\}$, give the same effect and have a meaningful physical interpretation.

3. For a length-$2m$ set $I$, the operator $T^\epsilon_{X_{2m}}$ is defined by

$$T^\epsilon_{X_{2m}} \equiv \sum_{\rho \in \text{pair}} \prod_{k,j \in \rho} T^\epsilon_{ijk} \cdot \quad (10)$$

4. The operator $T^\epsilon_{X_{2m}}$ is defined as

$$T^\epsilon_{X_{2m}} \equiv \sum_{\rho \in \text{pair}} \prod_{k,j \in \rho} \delta_{l_k l_j} T^\epsilon_{ikjk} \cdot \quad (11)$$

where $\delta_{l_k l_j}$ forbids the interaction between particles with different flavors.

An explanation for the notation $\sum_{\rho \in \text{pair}} \prod_{k,j \in \rho} l_k l_j$ is necessary. One can divide the length-$2m$ set $I$ into $m$ pairs, then the set $\rho$ of pairs for this partition can be written as

$$\rho = \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\}, \quad (12)$$

with conditions $i_1 < i_2 < \cdots < i_m$ and $i_t < j_t, \forall t$. Under such partitions, $\prod_{k,j \in \rho}$ stands for the product of $T^\epsilon_{ikjk}$ for all pairs $(i_k, j_k)$ in $\rho$, and $\sum_{\rho \in \text{pair}}$ denotes the summation over all possible partitions.

The combinatorial operators above translate the tree amplitudes of gravity into tree amplitudes of other theories. Here we only focus on the generated theories that will be considered later, and list them in Table 1. In this table, EMf stands for EM amplitudes in which photons carry flavors. Sets $\{h\}_a$, $\{p\}_a$, $\{g\}_a$, and $\{s\}_a$ denote gravitons, photons, gluons, and scalars respectively, where $a$ is the length of the set. The notation $(\cdot \cdot \cdot | i'_1, \ldots, i'_n)$ stands for the additional color ordering $(i'_1, \ldots, i'_n)$ among all external particles. In the notation $(\{\alpha\}; \{\beta\})$, $\{\alpha\}$ are particles with lower spin, while $\{\beta\}$ are particles with higher spin. The upper index of $A$ denotes the polarization vectors of external particles by the following rule: $\tilde{\epsilon}$ are only carried by particles in $\{\beta\}$, while $\epsilon$ are carried by all particles.
The KK relation (13) indicates that an arbitrary color-ordered YM amplitude can be expanded into color-ordered YM amplitudes with two external legs fixed at both ends. In this sense, \((n-2)\)! amplitudes with two legs fixed at both ends can be chosen as the complete basis (the so-called KK basis). Actually, the KK bases are not independent of each other due to the Bern–Carrasco–Johansson (BCJ) relations among them [22]. The BCJ relations suggest truly independent bases that include \((n-3)\)! amplitudes with three legs are fixed. However, as discussed in Ref. [17], the coefficients contain poles when expanding to the BCJ basis. If we hope that the coefficients are rational functions of Lorentz invariant kinematic variables and all physical poles are included in the basis, the only candidate is the KK basis. In other words, for rational coefficients, the KK basis is independent.

With the KK basis introduced above, now we discuss the expansion of tree gravitational amplitudes into such a basis. The expansion of gravitational amplitudes into color-ordered YM ones in the

### Table 1. Unifying relations.

| Amplitude | Operator acts on \(A^\epsilon_{\text{YM}}(\{h\}_a)\) |
|-----------|-------------------------------------|
| \(A^\epsilon_{\text{YM}}(\{p\}_{2n}; \{h\}_{n-2m})\) | \(T^\epsilon_{\text{YM}}\) |
| \(A^\epsilon_{\text{YM}}(\{p\}_{2n}; \{h\}_{n-2m})\) | \(T^\epsilon_{\text{YM}}\) |
| \(A^\epsilon_{\text{YM}}(\{i\}_1, \ldots, \{i\}_n)\) | \(T^\epsilon[i_1, \ldots, i_n] \cdot T^\epsilon_{\text{YM}}\) |
| \(A^\epsilon_{\text{YM}}(\{i\}_1, \ldots, \{i\}_n)\) | \(T^\epsilon[i_1, \ldots, i_n] \cdot T^\epsilon_{\text{YM}}\) |
| \(A^\epsilon_{\text{YM}}(\{i\}_1, \ldots, \{i\}_n)\) | \(T^\epsilon[i_1, \ldots, i_n] \cdot T^\epsilon_{\text{YM}}\) |
| \(A^\epsilon_{\text{YM}}(\{i\}_1, \ldots, \{i\}_n)\) | \(T^\epsilon[i_1, \ldots, i_n] \cdot T^\epsilon_{\text{YM}}\) |

\[A^\epsilon_{\text{YM}}(1, \{a\}, \{b\}) = \sum_{\{\}} (-)^{|b|} A^\epsilon_{\text{YM}}(1, \{a\}\cup\{b\}^T, n),\]  

\[\sum_{\{\}} A^\epsilon_{\text{YM}}(1, \{2, 3\}\cup\{4, 5\}, 6) = A^\epsilon_{\text{YM}}(1, 2, 3, 4, 5, 6) + A^\epsilon_{\text{YM}}(1, 2, 4, 3, 5, 6) + A^\epsilon_{\text{YM}}(1, 2, 4, 5, 3, 6) + A^\epsilon_{\text{YM}}(1, 2, 5, 3, 6) + A^\epsilon_{\text{YM}}(1, 4, 2, 3, 5, 6) + A^\epsilon_{\text{YM}}(1, 4, 2, 5, 3, 6) + A^\epsilon_{\text{YM}}(1, 4, 5, 2, 3, 6).\]
framework of ordered splitting is given as \[^{3}\] [17]

\[
A^\epsilon,\vec{\epsilon}_{G}(\{h\}_n) = \sum_{\{Or_l\}} \sum_{l} (-)^{\text{Or}_l} \left( F_0 \prod_{l=1}^{t} L_i \right) A_{\text{YM}}^\epsilon(1, \text{Or}_0 \sqcup \text{Or}_1 \sqcup \cdots \sqcup \text{Or}_t, n),
\]

where elements in \{h\}_n are labeled as \{h\}_n = \{1, 2, \ldots, n\}, and \text{Or}_0 denotes the length of the set \text{Or}_0. Some explanations are in order. The ordered splitting for \(n\) elements used here is a little different from that used for EYM amplitudes in Refs. [13,14,16], which can be defined as follows. First, a reference ordering for elements should be given, for instance from that used for EYM amplitudes in Refs. [13,14,16], which can be defined as follows. First, a leg \(i\) is the sum of the momenta of external legs satisfying two conditions: (1) legs on the LHS of the framework of ordered splitting is given as [17]

\[
\text{L}_l = \gamma_{l}, \gamma_{l-1}, \ldots, \gamma_1 \cdot Z_{\gamma_1},
\]

while \(F_0\) for \text{Or}_0' = \{\gamma_1, \gamma_2, \ldots, \gamma_t\} is given by

\[
F_0 = \vec{\epsilon}_{h_1} \cdot f_{\gamma_1} \cdot f_{\gamma_2} \cdot \ldots \cdot f_{\gamma_n} \cdot \vec{\epsilon}_{h_n}.
\]

Here \(f_{\gamma_i}^{\mu\nu}\) stands for the strength tensor \(f_{\gamma_i}^{\mu\nu} \equiv k_{\gamma_i}^{\mu} \vec{\epsilon}_{\gamma_i}^{\nu} - \vec{\epsilon}_{\gamma_i}^{\mu} k_{\gamma_i}^{\nu}\). The combinatory momentum \(Z_i^{\mu}\) is the sum of the momenta of external legs satisfying two conditions: (1) legs on the LHS of the leg \(i\) in the color-ordered YM amplitude, (2) legs on the LHS of \(i\) in the labeled chain defined by the ordered splitting. The labeled chain used here for a given ordered splitting is the ordered

\[^{3}\] Explicit formulae for the recursive expansion of gravity amplitudes have not been provided in Ref. [17]. Instead, the authors gave

\[
A^\epsilon,\vec{\epsilon}_{G}(\{h\}_n) = \sum_{\{h\}_n} \vec{\epsilon}_{h_1} \cdot f_{h_1} \cdot B_f + (\vec{\epsilon}_{h_1} \cdot \vec{\epsilon}_{h_2}) A_{\text{YM}}^\epsilon(\{h\}_n/\{h_1, h_2\}),
\]

and

\[
B_f^{\mu} = \sum_{\alpha/\beta} (\vec{\epsilon}_{h_2} \cdot f_{\alpha} \cdots f_{\alpha_2})^{\mu} A_{\text{YM}}^\epsilon(h_1, h_2, \alpha_2, \ldots, \alpha_n, h_2; \{h\}_n/\{h_1 \cup h_2 \cup \alpha\}).
\]

To achieve the result (15), one need to be careful about the sign. Since tensors \(f_{\gamma_i}^{\mu\nu}\) are anti-symmetric, we have

\[
\vec{\epsilon}_{h_1} \cdot f_{\gamma_f} \cdot B_f = -\vec{\epsilon}_{h_1} \cdot f_{\gamma_f} \cdot (-f_{\alpha_2}) \cdots (-f_{\alpha_n}) \cdot \vec{\epsilon}_{h_2}.
\]

Tensors \((-f_{\gamma_i}^{\mu\nu})\) cause the factor \((-)^{\text{Or}_1}\) in Eq. (15).
set \{1, \text{Or}_0, \text{Or}_1, \ldots, \text{Or}_t, n\}. The summation over the shuffles \(\sum_{\text{shuffles}}\) was defined previously when introducing the KK basis. The summation \(\sum_{\{\text{Or}_j\}}\) means summing over all possible ordered splittings.

In formula (15), the gravitational amplitude is expanded into color-ordered YM amplitudes with two legs fixed at both ends; thus the coefficients of expansion into the KK basis can be read out conveniently. Formula (15) serves as the starting point for computations in the next section.

Before ending this subsection, we emphasize that, in the expansion (15), the basis only carries polarization vectors \(\epsilon\), and all polarization vectors \(\tilde{\epsilon}\) are included in the coefficients.

3. Derivation of expansions

In this section, we derive the general formulae for expansions of tree amplitudes. We first apply operators \(T_{X_{2m}}^Z\) and \(T_{X_{2m}}^\tilde{Z}\) on both sides of Eq. (15) simultaneously to obtain the ordered splitting formula for the expansion of tree EM amplitudes into YM ones. Then, we propose the rule of getting the coefficients of the KK basis. Finally, we explain that coefficients of the expansion of gravitational amplitudes into the KK basis are shared by the expansion of \(\phi^4\) amplitudes into BS ones and the expansion of sYMS amplitudes into BS ones, as well as the expansion of DBI amplitudes into NLSM ones.

3.1. Expansion of EM amplitudes into YM ones: ordered splitting formula

We start with tree amplitudes of general Einstein–Maxwell theory in which photons do not carry any flavor. To generate such amplitudes, one can act the operator \(T_{X_{2m}}^Z\) on gravitational amplitudes, which can be seen in Table 1. Now we are going to apply this operator on both sides of Eq. (15). On the LHS, the operator \(T_{X_{2m}}^Z\) transmutes the gravitational amplitude into the EM one. On the RHS, since basis depends on polarization vectors \(\epsilon_i\) and all \(\tilde{\epsilon}_i\) are included in coefficients, the operator \(T_{X_{2m}}^Z\) only modifies the coefficients. Thus, the action of the operator provides the expansion of EM amplitudes into pure YM amplitudes, with the coefficients determined by acting the operator on coefficients in the expansion of gravitational amplitudes.

As a warm-up, we first restrict ourselves to the special case in which all external particles of the EM amplitude are photons, i.e., \(2m = n\). Under this condition, subscripts \(i\) and \(j\) of \(T_{ij}^Z\) in \(T_{X_{2m}}^Z\) run over all external particles. To study the effect of \(T_{X_{2m}}^Z\), let us consider the operator \(\prod_{i_k j_k \in \rho} T_{i_k j_k}^Z\) under a given partition \(\rho\). If a term on the RHS of Eq. (15) contains all \((\tilde{\epsilon}_{i_k} \cdot \tilde{\epsilon}_{i_j})\) with \(i_k j_k \in \rho\), the operator \(\prod_{i_k j_k \in \rho} T_{i_k j_k}^Z\) turns all these \((\tilde{\epsilon}_{i_k} \cdot \tilde{\epsilon}_{i_j})\) into 1. Otherwise, the term will be annihilated by \(\prod_{i_k j_k \in \rho} T_{i_k j_k}^Z\). Thus, in each term that can survive under the action of \(T_{X_{2m}}^Z\), every polarization vector must contract with another one. This requirement indicates that not all ordered splittings are allowed; each subset in the ordered splitting must take an even length according to the observation that each \((\tilde{\epsilon}_{i_k} \cdot \tilde{\epsilon}_{i_j})\) only occurs in \(L_l\) or \(F_0\). Acting \(T_{X_{2m}}^Z\) on terms corresponding to these selected splittings, one can get the non-vanishing contributions. For proper ordered splittings with even lengths, \(L_l\) and \(F_0\) contain

\[
(-)^{|\text{Or}_l|} \frac{1}{2} (\tilde{\epsilon}_{\gamma_1} \cdot \tilde{\epsilon}_{\gamma_{r-1}})(k_{\gamma_{r-1}} \cdot k_{\gamma_{r-2}}) \cdots (\tilde{\epsilon}_{\gamma_4} \cdot \tilde{\epsilon}_{\gamma_3})(k_{\gamma_3} \cdot k_{\gamma_2})(\tilde{\epsilon}_{\gamma_2} \cdot \tilde{\epsilon}_{\gamma_1})(k_{\gamma_1} \cdot Z_{\gamma_1}),
\]

and

\[
(-)^{|\text{Or}_l|} \frac{1}{2} (\tilde{\epsilon}_{1} \cdot \tilde{\epsilon}_{\gamma_1})(k_{\gamma_1} \cdot k_{\gamma_2})(\tilde{\epsilon}_{\gamma_2} \cdot \tilde{\epsilon}_{\gamma_3})(k_{\gamma_3} \cdot k_{\gamma_4}) \cdots (\tilde{\epsilon}_{\gamma_{r-2}} \cdot \tilde{\epsilon}_{\gamma_{r-1}})(k_{\gamma_{r-1}} \cdot k_{\gamma_r})(\tilde{\epsilon}_{\gamma_r} \cdot \tilde{\epsilon}_{n}),
\]
respectively. After turning all \((\tilde{e}_{i_k} \cdot \tilde{e}_{j_k})\) into 1, we get the expansion of the tree EM amplitude as

\[
A_{\text{EM}}^{\ell\varnothing}(p,n; \varnothing) = \sum_{|\mathcal{O}|_{\text{even}}} \sum_{\mathcal{U}} (-)^{|\mathcal{O}|_1} \left( E_0 \prod_{l=1}^t N_l \right) A_{\text{YM}}(1, \mathcal{O}_0^l \cup \mathcal{O}_1^l \cup \cdots \cup \mathcal{O}_t^l, n),
\]

(20)

where

\[
E_0 = (-)^{|\mathcal{O}|_0} (k_{p_1} \cdot k_{p_2})(k_{p_3} \cdot k_{p_4}) \cdots (k_{p_{n-1}} \cdot k_{p_n}),
\]

(21)

and

\[
N_l = (-)^{|\mathcal{O}|_l} (k_{p_{l+1}} \cdot k_{p_{l-1}}) \cdots (k_{p_3} \cdot k_{p_2})(k_{p_1} \cdot Z_{p_1}).
\]

(22)

If the set \(\{\mathcal{O}_0^l\}\) is empty, we have \(E_0 = 1\).

Then we turn to the general case that the \(n\)-point tree EM amplitude contains \(2m\) photons and \((n-2m)\) gravitons as external particles. Since the two special gravitons 1 and \(n\) in Eq. (15) can be chosen arbitrarily, we assume that both 1 and \(n\) are turned into photons. Let us consider a given partition \(\rho = \{(i_1, f_1), (i_2, f_2), \ldots, (i_m, f_m)\}\) with \(i_1 < i_2 < \cdots < i_m\) and \(i_k < j_k, \forall k\) for external photons. Under the action of corresponding \(\prod_{(i_k, f_k) \in \rho} T_{i_k j_k}^{\rho}\) in the operator \(T_{Y_{2m}}^{\rho}\), all non-vanishing terms on the RHS of Eq. (15) must contain all \((\tilde{e}_{i_k} \cdot \tilde{e}_{j_k})\) with \((i_k, j_k) \in \rho\). This indicates that each pair in \(\rho\) should appear in one subset of the ordered splitting at nearby positions, according to the definition of \(L_t\) and \(F_0\). More explicitly, if \(\mathcal{O}_i\) includes \(i_k\), it must take the form \(\{i_1, i_k, i_{k+1}, \ldots\} \) or \(\{i_1, i_{k-1}, i_k, \ldots\}\). If \(\mathcal{O}_i\) does not contain \(j_k\), or contains \(j_k\) but \(i_k\) and \(j_k\) are not nearby, the corresponding term on the RHS of Eq. (15) is annihilated by \(\prod_{(i_k, j_k) \in \rho} T_{i_k j_k}^{\rho}\). We use the notation \(\mathcal{O}_i^{\rho}\) to denote subsets under these proper ordered splittings. Having determined the ordered splittings, now we consider the effect of acting the operator on the corresponding coefficients. For \(\mathcal{O}_0^p\), since \(F_0\) includes \(\tilde{e}_{i_1}^\mu\) and \(\tilde{e}_{n}^\mu\) in the first and last positions, the action of \(\prod_{(i_k, f_k) \in \rho} T_{i_k j_k}^{\rho}\) turns the vectors \((\tilde{e}_{i_1} \cdot f_{a})^\mu\) and \((f_b \cdot \tilde{e}_{n})^\mu\) into \(-k_a^\mu\) and \(k_b^\mu\), respectively. For other pairs in \(\mathcal{O}_0^p\), the tensors \((f_i \cdot f_j)^{\mu\nu}\) are turned into \(-k_i^{\mu}k_j^{\nu}\). All \((-\cdot)^{n_p_{0} - 1}\) signs give rise to \((-)^{n_p_{0}}|H_{0}|\), where \(|H_{0}|\) is the number of photon pairs in \(\mathcal{O}_0^p\). Thus for the subset \(\mathcal{O}_0^p\) we get \((-)^{n_p_{0} - 1}G_0\), where \(G_0\) can be obtained from \(F_0\) by the replacement

\[
(\tilde{e}_{i_1} \cdot f_{a})^\mu \rightarrow k_a^\mu, \quad (f_b \cdot \tilde{e}_{n})^\mu \rightarrow k_b^\mu, \quad (f_i \cdot f_j)^{\mu\nu} \rightarrow k_i^{\mu}k_j^{\nu}.
\]

(23)

If \(\mathcal{O}_0^p = \varnothing\), we have \((\tilde{e}_{i_1} \cdot \tilde{e}_{n}) \rightarrow 1\). For other \(\mathcal{O}_i^p\), if \(i_k\) in one pair \((i_k, j_k)\) is at the last position of the subset, i.e., appears as \(\{i_{k-1}, i_k\}\), then the vector \((\tilde{e}_{i_k} \cdot f_j)^\mu\) will be turned into \(-k_j^\mu\). For other cases, the tensors \((f_i \cdot f_j)^{\mu\nu}\) are turned into \(-k_i^{\mu}k_j^{\nu}\). Thus, one can obtain \((-)^{|H_{1}|}H_{1}\), where \(H_{1}\) can be obtained from \(L_t\) via the replacement

\[
(\tilde{e}_{i_k} \cdot f_j) \rightarrow k_j^\mu, \quad (f_i \cdot f_j)^{\mu\nu} \rightarrow k_i^{\mu}k_j^{\nu}.
\]

(24)

Collecting these results together, we find that the contribution of an individual splitting can be expressed as

\[
\sum_{\mathcal{U}} (-)^{|\mathcal{O}_0| + m - 1} \left( G_0 \prod_{l=1}^t H_l \right) A_{\text{YM}}(1, \mathcal{O}_0^p \cup \mathcal{O}_1^p \cup \cdots \cup \mathcal{O}_t^p, n); \quad (25)
\]

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thus the full expansion is given by

\[ A_{EM}^{\epsilon, \bar{\epsilon}}(\{p\}_{2m}; \{h\}_{n-2m}) = \sum_{\rho} \sum_{|Or_\rho|} \sum_{\bigcup l} (-)^{|Or_\rho|+m-1} \left( G_0 \prod_{l=1}^t H_l \right) A_{YM}^{\epsilon, \bar{\epsilon}}(1, Or_0^\rho \cup Or_1^\rho \cup \cdots \cup Or_t^\rho, n), \tag{26} \]

where the summation \( \sum_{|Or_\rho|} \) is over all ordered splittings corresponding to a special partition \( \rho \) of photons, and \( \sum_{\rho} \) is over all partitions due to the definition of the operator \( T_{A_{2m}}^\epsilon \). When all external particles are photons, it can be verified straightforwardly that the general formula (26) is reduced to the special one (20).

At the end of this subsection, we discuss the expansion of EM amplitudes in which photons carry flavors. For this case, the operator \( T_{A_{2m}}^\epsilon \) is replaced by \( T_{A_{2m}}^\epsilon \), and a contraction (\( \epsilon_{ik} \cdot \bar{\epsilon}_{jk} \)) is permitted only when two photons carry the same flavor. The constraints from \( \delta_{ik} \delta_{jk} \) lead to the conclusion that only partitions satisfying \( \delta_{ik} \delta_{jk} = 1 \) for all \( i_k, j_k \in \rho \) provide non-vanishing contributions. Thus, the expansion of these amplitudes in the ordered splitting formula can be obtained from formula (26) by restricting the summation \( \sum_{\rho} \) on proper partitions.

### 3.2. Expansion of EM amplitudes into the KK basis

Formula (26) provides the expansion of tree EM amplitudes into pure YM amplitudes in the framework of ordered splitting. To obtain the expansion into the KK basis, one can extract coefficients of the KK basis with the desired color ordering from the obtained expansion in the ordered splitting formula. An alternative way is to reconstruct the corresponding ordered splittings from the desired color ordering in the KK basis directly, without requiring the given expansion in the ordered splitting formula, as will be discussed in this subsection. The procedure for the expansion of EYM amplitudes is provided in Refs. [13,14,16]. The manipulation is similar but a little different for EM amplitudes.

Now we propose the algorithm.

Assuming that the color ordering in the KK basis is \((1, 2, 3, \ldots, n - 1, n)\), and the reference ordering is chosen to be \( n < i_1 < i_2 < \cdots < i_{n-1} \). Subsequently, for a given partition \( \rho = \{(i_1, j_1), \ldots, (i_m, j_m)\} \), one can determine the corresponding ordered splittings as follows.

- **First step:** List all possible ordered subsets \( Or_0^\rho = \{1, \gamma_1, \gamma_2, \ldots, \gamma_r, n\} \), respecting the color ordering in the KK basis, i.e., \( \gamma_1 < \gamma_2 < \cdots < \gamma_r \). In addition, if a photon is included in \( Or_0^\rho \), its partner in \( \rho \) should also be included in \( Or_0^\rho \), and the positions of the two photons in \( Or_0^\rho \) are nearby.

- **Second step:** For each \( Or_0^\rho \), remove its elements in \( \{1, 2, \ldots, n\} \). Then, for the remaining elements in \( \{1, 2, \ldots, n\}/Or_0^\rho \), we select the lowest element \( h_1 \) in the reference ordering and construct all possible ordered subsets \( Or_i^\rho = \{\gamma_1', \gamma_2', \ldots, \gamma_r', h_1\} \), satisfying \( h_1 < \gamma_1' < \gamma_2' < \cdots < \gamma_r' < h_1 \). The subset \( Or_i^\rho \) also satisfies the condition that each photon contained in \( Or_i^\rho \) occurs near its partner in \( \rho \).

- **Repeat the second step** until complete ordered splitting is achieved.

All proper ordered splittings for all partitions can be found via the manipulation mentioned above. After generating ordered splittings, the coefficient for the particular YM amplitude \( A_{YM}^{1, 2, \ldots, n} \)

\[ A_{EM}^{\epsilon, \bar{\epsilon}}(\{p\}_{2m}; \{h\}_{n-2m}) = \sum_{\rho} \sum_{|Or_\rho|} \sum_{\bigcup l} (-)^{|Or_\rho|+m-1} \left( G_0 \prod_{l=1}^t H_l \right) A_{YM}^{\epsilon, \bar{\epsilon}}(1, Or_0^\rho \cup Or_1^\rho \cup \cdots \cup Or_t^\rho, n), \tag{26} \]
can be obtained by summing factors \((-)^{|Or_0|+m-1} C_0 \prod_{\lambda=1}^l H_\lambda\) over all correct ordered splittings and all partitions. Some simple examples will be presented in the next section to illustrate the algorithm more explicitly.

So far, the expansion of EM amplitudes into the KK basis can be formally represented as

\[
A_{\text{EM}}^\varepsilon (p_{2m}; h_{n-2m}) = \sum_{\sigma \in S_{n-2}} C^\varepsilon (\sigma, m, \rho) A_{\text{YM}}^\varepsilon (1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n), \tag{27}
\]

where \(\sigma\) stands for permutations among \((n - 2)\) elements. The coefficients \(C^\varepsilon (\sigma, m, \rho)\) depend on polarization vectors \(\varepsilon_i\), permutation \(\sigma\), and number of photon pairs \(m\), as well as allowed partitions \(\rho\). Of course, it also depends on external momenta although the dependence is implicit in formula (27). Because of the dependence on possible partitions, formula (27) is correct for EM amplitudes whether or not the photons carry flavor.

### 3.3. Expansions of the \(\phi^4\), sYMS, and DBI amplitudes

In this subsection, we will identify that the coefficients \(C^\varepsilon (\sigma, m, \rho)\) in the expression (27) are also the coefficients in the expansion of \(\phi^4\) amplitudes into BS amplitudes and the expansion of sYMS amplitudes into BS amplitudes, as well as the expansion of DBI amplitudes into NLSM amplitudes.

Let us come to \(\phi^4\) theory whose amplitudes can be generated by acting the operator \(T^\varepsilon [i_1' \cdots i_n']\) on EM amplitudes \(A_{\text{EM}}^\varepsilon (p_{n}; \emptyset)\) that all external particles are photons without flavor, due to relations (27) - (29) in Table 1. If one sets the LHS of Eq. (27) to be \(A_{\text{EM}}^\varepsilon (p_{n}; \emptyset)\), and acts the operator \(T^\varepsilon [i_1' \cdots i_n']\) on both sides of Eq. (27) simultaneously, the LHS gives the \(\phi^4\) amplitude \(A_{\phi^4}(i_1', \ldots, i_n')\). For the RHS, since the operator \(T^\varepsilon [i_1' \cdots i_n']\) is defined via polarization vectors \(\varepsilon_i\), it only affects the KK bases \(A_{\text{YM}}^\varepsilon (1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n)\) and transmute them into BS amplitudes \(A_{\text{BS}}(1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n; i_1', \ldots, i_n')\), as shown in Table 1. Then, we get the expansion of \(\phi^4\) amplitudes into BS ones as

\[
A_{\phi^4}(i_1', \ldots, i_n') = \sum_{\sigma \in S_{n-2}} C(\sigma, m, \rho) A_{\text{BS}}(1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n; i_1', \ldots, i_n'). \tag{29}
\]

Coefficients \(C(\sigma, m, \rho)\) are products of Lorentz contractions of external momenta, which can be understood as sewing 4-point vertexes of \(\phi^4\) theory into 3-point vertexes of BS theory by eliminating propagators.

In the above discussion, if we act the operator \(T^\varepsilon [i_1' \cdots i_n']\) on EMf amplitudes \(A_{\text{EMf}}^\varepsilon (p_{2m}; h_{n-2m})\) in which photons carry flavors, the generated amplitudes are amplitudes of special YM theory, which describe the low-energy effective action of coincident D-branes. Thus we also have

\[
A_{\text{SYM}}^\varepsilon (s_{2m}; g_{n-2m}; i_1', \ldots, i_n') = \sum_{\sigma \in S_{n-2}} C^\varepsilon (\sigma, m, \rho) A_{\text{BS}}(1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n; i_1', \ldots, i_n'). \tag{30}
\]

Notice that the allowed partitions \(\rho\) for Eqs. (29) and (30) are different because of the constraints from \(\delta_{i_1 k} i_k\) for the second one.
Similarly, by acting the operator $L^e \cdot T^e_{ab}$ on both sides of Eq. (27), one gets the expansion of DBI amplitudes into NLSM ones as

$$A^e_{DBI}([s]_{2m}; [p]_{n-2m}) = \sum_{\sigma \in S_{n-2}} C^e(\sigma, m, \rho)A^\text{NLSM}(1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, n).$$  \hspace{1cm} (31)

If formulae (29), (30), and (31) are correct expansions, the basis used in them must be complete and independent. Now we explain that there are KK-like relations among color-ordered BS and NLSM amplitudes; thus this condition is satisfied. As pointed out in Ref. [17], the KK relation can be derived by using differential operators. Indeed, one can regard the KK relation as the inference of the algebraic property of the differential operator $T^e[\alpha]$. To see this, we rewrite the operator $T^e[\alpha]$ for a length-$n$ set $[\alpha]$ as

$$T^e[\alpha] \equiv T^e_{\alpha_1\alpha_n} \cdot \prod_{i=2}^{n-1} T^e_{\alpha_{i-1}\alpha_i\alpha_n}$$

$$= T^e_{\alpha_1\alpha_n} \cdot \left( \prod_{i=2}^{k} T^e_{\alpha_{i-1}\alpha_i\alpha_n} \right) \cdot \left( \prod_{j=k+1}^{n-1} T^e_{\alpha_{j-1}\alpha_j\alpha_n} \right)$$

$$= T^e_{\alpha_1\alpha_n} \cdot \left( \prod_{i=2}^{k} T^e_{\alpha_{i-1}\alpha_i\alpha_n} \right) \cdot \left( \prod_{j=k+1}^{n-1} T^e_{\alpha_{n} \alpha_{n-1}} \right).$$  \hspace{1cm} (32)

The operator $T^e_{\alpha_1\alpha_n} \cdot \left( \prod_{i=2}^{k} T^e_{\alpha_{i-1}\alpha_i\alpha_n} \right)$ generates the color ordering $(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_n)$, which is equivalent to $(\alpha_n, \alpha_1, \alpha_2, \ldots, \alpha_k)$ due to the cyclic symmetry, and the operator $\left( (-)^{n-k-1} \prod_{j=k+1}^{n-1} T^e_{\alpha_{n} \alpha_{n-1}} \right)$ can be interpreted as inserting $\{\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_{k+1}\}$ between $\alpha_n$ and $\alpha_k$ in $(\alpha_n, \alpha_1, \alpha_2, \ldots, \alpha_k)$ [18–20]. Setting $\alpha_n = 1, \alpha_k = n, \{a\} = \{\alpha_1, \ldots, \alpha_{k-1}\}, \{b\} = \{\alpha_{k+1}, \ldots, \alpha_{n-1}\}$, and applying this operator on the $n$-point gravitational amplitude $A^e_G([h]_n)$, one gets the KK relation (13) immediately. Since the algebraic relation (32) is general, it is not surprising that similar relations exist among color-ordered amplitudes of other theories beyond YM. Replacing $A^e_G([h]_n)$ in the above derivation by $T^e[i_1' \cdots i_n']A^e_G([h]_n)$ and $L^e \cdot T^e A^e_G([h]_n)$, the KK-like relations for BS amplitudes and NLSM amplitudes can be obtained as

$$A_{\text{BS}}(1, \{a\}, n, \{b\}|i'_1, \ldots, i'_n) = \sum_{\{i\}} (-)^{|b|} A_{\text{BS}}(1, \{a\} \cup \{b\}, T, n|i_1', \ldots, i'_n),$$  \hspace{1cm} (33)

and

$$A_{\text{NLSM}}(1, \{a\}, n, \{b\}) = \sum_{\{i\}} (-)^{|b|} A_{\text{NLSM}}(1, \{a\} \cup \{b\}, T, n).$$  \hspace{1cm} (34)

Thus, BS amplitudes and NLSM amplitudes with the color ordering $(1, \{a\} \cup \{b\}, T, n)$ share the completeness and independence of the KK basis; therefore they can be chosen as the proper basis for rational coefficients. Consequently, one can conclude that formulae (29), (30), and (31) are correct expansions for $\phi^4$, sYMS, and DBI amplitudes, respectively.

4. Examples

Next, we provide some examples to illustrate the expansions obtained in the previous section. Since the expansion of $\phi^4$ amplitudes into BS amplitudes, the expansion of sYMS amplitudes into BS
amplitudes, and the expansion of DBI amplitudes into NLSM amplitudes share the same coefficients with the expansion of EM amplitudes into the KK basis, we only consider EM amplitudes in this section.

4.1. 4-point EM amplitude $A^\epsilon_{EM}(\{1, 2, 3, 4\}; \emptyset)$

The first example is the 4-point EM amplitude $A^\epsilon_{EM}(\{1, 2, 3, 4\}; \emptyset)$ in which all external particles are photons in the set \{1, 2, 3, 4\}. The reference ordering is chosen as $4 < 2 < 3 < 1$. We first consider the case in which photons do not carry flavor. Our result (20) requires the length of each subset in the ordered splitting to be even, and $Or_0 = \{1, \ldots, 4\}$; thus only three splittings \{\{1, 4\}, \{3, 2\}, \{1, 3, 2, 4\}\}, and \{\{1, 2, 3, 4\}\} are proper. After evaluating $E_0$ and $N_i$, we get the expansion in the ordered splitting formula:

$$A^\epsilon_{EM}(\{1, 2, 3, 4\}; \emptyset) = -(k_3 \cdot Z_3)A^\epsilon_{YM}(1, 3, 2, 4) - (k_2 \cdot k_3)A^\epsilon_{YM}(1, 3, 2, 4) - (k_2 \cdot k_3)A^\epsilon_{YM}(1, 2, 3, 4).$$

The formula of expansion depends on the choice of KK basis, i.e., the choice of two special legs, which are fixed at both ends in the color ordering, and the choice of reference ordering. For instance, if we chose the basis $A^\epsilon_{YM}(1, \sigma_3, \sigma_4, 2)$, and the reference ordering $2 < 3 < 4 < 1$, a similar manipulation yields

$$A^\epsilon_{EM}(\{1, 2, 3, 4\}; \emptyset) = -(k_4 \cdot Z_4)A^\epsilon_{YM}(1, 4, 3, 2) - (k_3 \cdot k_4)A^\epsilon_{YM}(1, 4, 3, 2) - (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 4, 2).$$

As a verification of self-consistency, we need to prove that the two expressions (35) and (36) are equivalent. We first use the observation that $Z_3 = Z_4 = k_1$, together with the momentum conservation law and the on-shell condition $k_i^2 = 0$, to turn Eq. (35) into

$$E_1 = (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 2, 4) - (k_2 \cdot k_3)A^\epsilon_{YM}(1, 2, 3, 4),$$

and Eq. (36) into

$$E_2 = (k_4 \cdot k_2)A^\epsilon_{YM}(1, 4, 3, 2) - (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 4, 2).$$

Using the ordered reversed identity

$$A^\epsilon_{YM}(i_1, i_2, \ldots, i_n) = (-)^n A^\epsilon_{YM}(i_n, i_{n-1}, \ldots, i_1),$$

together with the cyclic symmetry of color ordering, we have

$$A^\epsilon_{YM}(1, 4, 3, 2) = A^\epsilon_{YM}(2, 3, 4, 1) = A^\epsilon_{YM}(1, 2, 3, 4);$$

therefore

$$E_1 - E_2 = (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 2, 4) + (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 4, 2) + (k_1 \cdot k_2)A^\epsilon_{YM}(1, 2, 3, 4),$$

where we employ $k_2 \cdot (k_3 + k_4) = -(k_2 \cdot k_1)$. Then, we apply the cyclic symmetry and ordered reversed identity again to get

$$(k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 2, 4) + (k_3 \cdot k_4)A^\epsilon_{YM}(1, 3, 4, 2)$$

$$= (k_3 \cdot k_4)A^\epsilon_{YM}(4, 1, 3, 2) + (k_3 \cdot k_4)A^\epsilon_{YM}(4, 3, 1, 2).$$
To continue, we use the well known fundamental BCJ relation

\[(k_3 \cdot (k_4 + k_1))A_Y^e(4, 1, 3, 2) + (k_3 \cdot k_4)A_Y^e(4, 3, 1, 2) = 0, \tag{43}\]

to arrive at

\[(k_3 \cdot k_4)A_Y^e(1, 3, 2, 4) + (k_3 \cdot k_4)A_Y^e(1, 3, 4, 2) = -(k_3 \cdot k_1)A_Y^e(4, 1, 3, 2) = -(k_3 \cdot k_4)A_Y^e(1, 3, 2, 4). \tag{44}\]

Then we use the fundamental BCJ relation

\[(k_3 \cdot k_1)A_Y^e(1, 3, 2, 4) + (k_3 \cdot (k_1 + k_2))A_Y^e(1, 2, 3, 4) = 0 \tag{45}\]

to obtain

\[(k_3 \cdot k_4)A_Y^e(1, 3, 2, 4) + (k_3 \cdot k_4)A_Y^e(1, 3, 4, 2) = -(k_3 \cdot (k_1 + k_2))A_Y^e(1, 2, 3, 4) = -(k_3 \cdot k_4)A_Y^e(1, 3, 2, 4). \tag{46}\]

Putting this back into Eq. (41) we get

\[E_1 - E_2 = ((k_1 \cdot k_2) - (k_3 \cdot k_4))A_Y^e(1, 2, 3, 4). \tag{47}\]

Since

\[2k_1 \cdot k_2 = (k_1 + k_2)^2 = (k_3 + k_4)^2 = 2k_3 \cdot k_4, \tag{48}\]

we finally get

\[E_1 - E_2 = 0; \tag{49}\]

thus, although they seem different, the two expressions are indeed equivalent to each other.

Now we turn to the case in which photons carry flavors. Suppose there are two flavors labeled by 1 and 2 of external photons, 1 is carried by 1 and 3, another one 2 is carried by 2 and 4. Then, the corresponding partition is \((\{1, 3\}, \{2, 4\})\), thus only the splitting \((\{1, 3\}, \{2, 4\})\) is allowed, which yields the expansion

\[A_Y^{e, \EE}(\{1, 2, 3, 4\}; \emptyset) = -(k_2 \cdot k_3)A_Y^e(1, 3, 2, 4). \tag{50}\]

Until now expansions in this subsection have been given in the ordered splitting formula. To get expansions into the KK basis, one can identify coefficients of the KK basis via the procedure proposed in Sect. 3.2. Since 1 and 4 are fixed at two ends in the color ordering of the KK basis, there are two color orderings \((1, 2, 3, 4)\) and \((1, 3, 2, 4)\) that need to be considered. For the partitions \((\{1, 2\}, \{3, 4\})\), the ordered splitting for color ordering \((1, 2, 3, 4)\) can be constructed as \((\{1, 2, 3, 4\})\), while the ordered splitting for ordering \((1, 3, 2, 4)\) does not exist. For the partition \((\{1, 3\}, \{2, 4\})\), the proper ordered splitting for color ordering \((1, 2, 3, 4)\) does not exist, while the splitting for color ordering \((1, 3, 2, 4)\) can be found as \((\{1, 3, 2, 4\})\). For the partition \((\{1, 4\}, \{2, 3\})\), the ordered splitting for color ordering \((1, 2, 3, 4)\) does not exist, while the splitting for color ordering \((1, 3, 2, 4)\) can be constructed as \((\{1, 4\}, \{3, 2\})\). Then coefficients for the KK basis under each given partition are

\[
\begin{align*}
\{1, 2, (3, 4)\} & : \quad C(2, 3) = -(k_2 \cdot k_3), \quad C(3, 2) = 0, \\
\{1, 3, (2, 4)\} & : \quad C(2, 3) = 0, \quad C(3, 2) = -(k_2 \cdot k_3), \\
\{1, 4, (2, 3)\} & : \quad C(2, 3) = 0, \quad C(3, 2) = -(k_3 \cdot Z_3). \quad \tag{51}
\end{align*}
\]
where \(C(2, 3)\) and \(C(3, 2)\) denote the coefficients of \(A_{\text{YM}}^\epsilon(1, 2, 3, 4)\) and \(A_{\text{YM}}^\epsilon(1, 3, 2, 4)\). The above results can be verified directly in formulae (35) and (50).

### 4.2. 5-point EM amplitude \(A_{\text{EM}}^\epsilon(\{1, 5\}; \{2, 3, 4\})\)

The second example is the 5-point amplitude \(A_{\text{EM}}^\epsilon(\{1, 5\}; \{2, 3, 4\})\), which includes two photons 1, 5 and three gravitons 2, 3, 4. For this case, there is no need to distinguish whether the photons carry flavor or not. We chose the reference ordering as \(5 < 1 < 3 < 4 < 2\). Then we have following proper ordered splittings: \((\{1, 5\}, [4, 2, 3])\), \((\{1, 5\}, [2, 4, 3])\), \((\{1, 5\}, [4, 3, 2])\), \((\{1, 5\}, [2, 3], [4])\), \((\{1, 5\}, [3], [2, 4])\), as well as \((\{1, 5\}, [3], [4], [2])\). For these splittings, we have

\[
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot F_2 \cdot F_4 \cdot Z_4, \\
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot F_4 \cdot F_2 \cdot Z_2, \\
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot F_4 \cdot Z_4, \quad H_2 = \bar{c}_2 \cdot Z_2, \\
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot F_2 \cdot Z_2, \quad H_2 = \bar{c}_4 \cdot Z_4, \\
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot Z_3, \quad H_2 = \bar{c}_4 \cdot F_2 \cdot Z_2, \\
G_0 = 1, \quad H_1 = \bar{c}_3 \cdot Z_3, \quad H_2 = \bar{c}_4 \cdot Z_4, \quad H_3 = \bar{c}_2 \cdot Z_2, 
\]

(52)

and the expansion in the framework of ordered splitting is expressed as

\[
A_{\text{EM}}^\epsilon(\{1, 5\}; \{2, 3, 4\}) = (\bar{c}_3 \cdot f_2 \cdot f_4 \cdot Z_4)A_{\text{YM}}^\epsilon(1, 4, 2, 3, 5) + (\bar{c}_3 \cdot f_4 \cdot f_2 \cdot Z_2)A_{\text{YM}}^\epsilon(1, 2, 4, 3, 5) \\
+ \sum_{\mu} (\bar{c}_3 \cdot f_4 \cdot Z_4)(\bar{c}_2 \cdot Z_2)A_{\text{YM}}^\epsilon(1, \{4, 3\}|\mu|2, 5) \\
+ \sum_{\mu} (\bar{c}_3 \cdot f_2 \cdot Z_2)(\bar{c}_4 \cdot Z_4)A_{\text{YM}}^\epsilon(1, \{2, 3\}|\nu|4, 5) \\
+ \sum_{\mu} (\bar{c}_4 \cdot f_2 \cdot Z_2)(\bar{c}_3 \cdot Z_3)A_{\text{YM}}^\epsilon(1, 3|\nu|2, 4, 5) \\
+ \sum_{\mu} (\bar{c}_3 \cdot Z_3)(\bar{c}_4 \cdot Z_4)(\bar{c}_2 \cdot Z_2)A_{\text{YM}}^\epsilon(1, 3|\nu|4|\mu|2, 5), 
\]

(53)

by using the general formula (26).

Then we consider the expansion into the KK basis. To illustrate the algorithm provided in Sect. 3.2, we construct ordered splittings for the color ordering \((1, 2, 4, 3, 5)\) in the KK basis. For the current case, there is only one partition \((\{1, 5\})\). At the first step, the ordered subset \(\text{Or}_0^p\), which contains two photons 1 and 5 at two ends and two nearby photons, has only one candidate \(\text{Or}_0^p = \{1, 5\}\). Then, the lowest element in the set of remaining gravitons \((1, 2, 3, 4, 5)\)/\(\text{Or}_0^p = \{2, 3, 4\}\) is 3. At the second step, we construct all possible subsets \(\text{Or}_1^p\) containing 3 as the last element: \(\text{Or}_1^p = \{3\}, \text{Or}_1^p = \{4, 3\}, \text{Or}_1^p = \{2, 3\}, \text{Or}_1^p = \{2, 4, 3\}\). Notice that \((4, 2, 3)\) is not the correct choice of \(\text{Or}_1^p\) since it does not satisfy the color ordering \((1, 2, 4, 3, 5)\) in the KK basis. At the third step, we construct \(\text{Or}_2^p\) for all \((1, 2, 3, 4, 5)\)/\(\text{Or}_0^p/\text{Or}_1^p \neq \emptyset\). For \(\text{Or}_1^p = \{3\}\), the set of remaining elements is \((1, 2, 3, 4, 5)\)/\(\text{Or}_0^p/\text{Or}_1^p = \{2, 4\}\), then \(\text{Or}_2^p = \{4\}\) and \(\text{Or}_2^p = \{2, 4\}\) can be constructed. For \(\text{Or}_1^p = \{4, 3\}\), we have \(\text{Or}_2^p = \{2\}\). For \(\text{Or}_2^p = \{2, 3\}\), we have \(\text{Or}_2^p = \{4\}\). At the final step, for \(\text{Or}_1^p = \{3\}, \text{Or}_2^p = \{4\}, \text{we have Or}_3^p = \{2\}..
The above recursive construction can be summarized as

\[
\begin{align*}
{1, 5}, {3} & \rightarrow \begin{cases} 
{1, 5}, {3}, {4} \rightarrow \{1, 5, {3}, {4}, {2} 
\end{cases} \\
{1, 5} & \rightarrow \begin{cases} 
{1, 5}, {4}, {3} \rightarrow \{1, 5, {4}, {3}, {2} 
{1, 5}, {2}, {3} \rightarrow \{1, 5, {2}, {3}, {4} 
{1, 5}, {2}, {4}, {3} 
\end{cases}
\end{align*}
\]

One can construct ordered splittings for other color orderings in a similar way. After depicting \(G_0\) and \(H_I\) for each splitting, we get coefficients of the KK basis as follows:

\[
C(2, 3, 4) = (\bar{\epsilon}_3 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_4 \cdot Z_4) \mid_{\{1,5\},\{2,3\},\{4\}} + (\bar{\epsilon}_4 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_3 \cdot Z_3) \mid_{\{1,5\},\{3\},\{2,4\}}
\]

\[
+ (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

\[
C(2, 4, 3) = (\bar{\epsilon}_3 \cdot f_4 \cdot Z_2) (\bar{\epsilon}_4 \cdot Z_4) \mid_{\{1,5\},\{2,4,3\}} + (\bar{\epsilon}_3 \cdot f_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{4,3\},\{2\}}
\]

\[
+ (\bar{\epsilon}_3 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_4 \cdot Z_4) \mid_{\{1,5\},\{2,3\},\{4\}} + (\bar{\epsilon}_4 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_3 \cdot Z_3) \mid_{\{1,5\},\{3\},\{2,4\}}
\]

\[
+ (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

\[
C(3, 2, 4) = (\bar{\epsilon}_3 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_3 \cdot Z_3) \mid_{\{1,5\},\{3\},\{2,4\}} + (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

\[
C(3, 4, 2) = (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

\[
C(4, 2, 3) = \bar{\epsilon}_3 \cdot f_2 \cdot f_4 \cdot Z_4 \mid_{\{1,5\},\{4,2,3\}} + (\bar{\epsilon}_3 \cdot f_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{4,3\},\{2\}}
\]

\[
+ (\bar{\epsilon}_3 \cdot f_2 \cdot Z_2) (\bar{\epsilon}_4 \cdot Z_4) \mid_{\{1,5\},\{2,3\},\{4\}} + (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

\[
C(4, 3, 2) = (\bar{\epsilon}_3 \cdot f_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{4,3\},\{2\}} + (\bar{\epsilon}_3 \cdot Z_3) (\bar{\epsilon}_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) \mid_{\{1,5\},\{3\},\{4\},\{2\}}
\]

where \(C(i, j, k)\) denotes the coefficient of the color-ordered YM amplitude \(A_{YM}^\epsilon(1, i, j, k, 5)\). Since the combinatorial momentum \(Z_i\) depends not only on the color ordering but also on the ordered splitting, we have explicitly written down the ordered splitting for each term.

4.3. 5-point EM amplitude \(A_{EM}^{\epsilon \bar{\epsilon}}(\{1, 3, 4, 5\}; \{2\})\)

The final example is the 5-point amplitude \(A_{EM}^{\epsilon \bar{\epsilon}}(\{1, 3, 4, 5\}; \{2\})\), which contains four photons 1, 3, 4, 5 and one graviton 2. For this amplitude, we still chose the reference ordering \(5 \prec 1 \prec 3 \prec 4 \prec 2\). We start with the case in which photons do not carry flavor. Then the allowed ordered splittings for each partition can be found as follows: For the partition \(\{1, 5\}, (3, 4)\), the proper splittings are \(\{1, 5\}, \{2, 4, 3\}\) and \(\{1, 5\}, \{4, 3\}, \{2\}\); for the partition \(\{1, 4\}, (3, 5)\), the proper splittings are \(\{1, 4, 2, 3, 5\}\) and \(\{1, 4, 3, 5, \{2\}\}; for the partition \(\{1, 3\}, (4, 5)\), the proper splittings are
{\{1, 3, 2, 4, 5\}} and {\{1, 3, 4, 5\}, \{2\}}. For these ordered splittings, we have

\[
G_0 = 1, \quad H_1 = k_4 \cdot f_2 \cdot Z_2, \\
G_0 = 1, \quad H_1 = k_4 \cdot Z_4, \quad H_2 = \bar{\epsilon}_2 \cdot Z_2, \\
G_0 = k_4 \cdot f_2 \cdot k_3, \\
G_0 = k_4 \cdot k_3, \quad H_1 = \bar{\epsilon}_2 \cdot Z_2, \\
G_0 = k_3 \cdot f_2 \cdot k_4, \\
G_0 = k_3 \cdot k_4, \quad H_1 = \bar{\epsilon}_2 \cdot Z_2.
\]

(56)

Thus, applying the general formula (26), we get the expansion in the ordered splitting formula:

\[
A_{\text{EM}}^\epsilon (\{1, 3, 4, 5\}, \{2\}) = -(k_4 \cdot f_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, 2, 4, 3, 5) - \sum_{\sqcup} (k_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, \{4, 3\}\cup\{2, 5\})
\]

\[
+ (k_4 \cdot f_2 \cdot k_3) A_{\text{YM}}^\epsilon (1, 4, 2, 3, 5) - \sum_{\sqcup} (k_4 \cdot k_3) (\bar{\epsilon}_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, \{4, 3\}\cup\{2, 5\})
\]

\[
+ (k_3 \cdot f_2 \cdot k_4) A_{\text{YM}}^\epsilon (1, 3, 2, 4, 5) - \sum_{\sqcup} (k_3 \cdot k_4) (\bar{\epsilon}_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, \{3, 4\}\cup\{2, 5\}).
\]

(57)

Then we turn to the case in which photons carry flavors. Assuming that there are two flavors labeled 1 and 2, 1 is carried by photons 1 and 5, while 2 is carried by 3 and 4. One can find that the proper ordered splittings for the current case has only two candidates {\{11, 51\}, \{2, 42, 32\}} and {{11, 51}, \{42, 32\}, \{2\}}. Then the expansion in the ordered splitting formula is given as

\[
A_{\text{EMF}}^\epsilon (\{11, 32, 42, 51\}; \{2\}) = -(k_4 \cdot F_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, 2, 4, 3, 5)
\]

\[
- \sum_{\sqcup} (k_4 \cdot Z_4) (\bar{\epsilon}_2 \cdot Z_2) A_{\text{YM}}^\epsilon (1, \{4, 3\}\cup\{2, 5\}).
\]

(58)

Finally, we consider the coefficients of the KK basis. We choose the color ordering (1, 2, 4, 3, 5) as the example to illustrate the algorithm proposed in Sect. 3.2. The recursive construction of ordered splittings can be summarized as follows:

- Partition \{(1, 5), (3, 4)\}:

\[
\{1, 5\} \rightarrow \begin{cases} 
\{1, 5\}, \{4, 3\} \rightarrow \{1, 5\}, \{4, 3\}, \{2\} \\
\{1, 5\}, \{2, 4, 3\} \end{cases}.
\]

(59)

At the first step, \{1, 5\} is the only choice in which Or_{10} contain photons 1 and 5 at both ends with two nearby photons. At the second step, the choices \{1, 5\}, \{2, 3\} and \{1, 5\}, \{4, 2, 3\} are excluded since the photons 3 and 4 should appear at nearby positions in the same subset, because of the pair (3, 4) in the partition. The second one \{1, 5\}, \{4, 2, 3\} also violates the desired color ordering (1, 2, 4, 3, 5).

- Partition \{(1, 4), (3, 5)\}:

\[
\{1, 4, 3, 5\} \rightarrow \{1, 4, 3, 5\}, \{2\}.
\]

(60)
At the first step, we drop the candidate Or$^p_0 = \{1, 4, 2, 3, 5\}$ that violates the color ordering $(1, 2, 4, 3, 5)$.

- Partition $\{(1, 3), (4, 5)\}$: The correct ordered splitting does not exist since both Or$^p_0 = \{1, 3, 4, 5\}$ and Or$^p_0 = \{1, 3, 2, 4, 5\}$ violate the color ordering $(1, 2, 4, 3, 5)$.

Before ending this subsection, we list the coefficients of the KK basis for each partition:

- Partition $\{(1, 5), (3, 4)\}$:
  \[
  C(2, 3, 4) = 0, \quad C(3, 2, 4) = 0, \quad C(3, 4, 2) = 0, \\
  C(4, 2, 3) = -(k_4 \cdot Z_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,3,4,5\}}, \quad C(4, 3, 2) = -(k_4 \cdot Z_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,5,4,3\}}, \\
  C(2, 4, 3) = -k_4 \cdot f_2 \cdot Z_2\big|_{\{1,3,4,5\}} - (k_4 \cdot Z_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,5,4,3\}}. \tag{61}
  \]

- Partition $\{(1, 4), (3, 5)\}$:
  \[
  C(2, 3, 4) = 0, \quad C(3, 2, 4) = 0, \quad C(3, 4, 2) = 0, \\
  C(4, 2, 3) = -(k_4 \cdot k_3)(\vec{e}_2 \cdot Z_2)\big|_{\{1,4,3,5\}}, \quad C(4, 3, 2) = -(k_4 \cdot k_3)(\vec{e}_2 \cdot Z_2)\big|_{\{1,4,3,5\}}, \\
  C(4, 2, 3) = k_4 \cdot f_2 \cdot k_3\big|_{\{1,4,2,3,5\}} - (k_4 \cdot k_3)(\vec{e}_2 \cdot Z_2)\big|_{\{1,4,3,5\}}. \tag{62}
  \]

- Partition $\{(1, 3), (4, 5)\}$:
  \[
  C(2, 3, 4) = 0, \quad C(4, 2, 3) = 0, \quad C(4, 3, 2) = 0, \\
  C(2, 3, 4) = -(k_3 \cdot k_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,3,4,5\}}, \quad C(3, 4, 2) = -(k_3 \cdot k_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,3,4,5\}}, \\
  C(3, 2, 4) = k_3 \cdot f_2 \cdot k_4\big|_{\{1,3,2,4,5\}} - (k_3 \cdot k_4)(\vec{e}_2 \cdot Z_2)\big|_{\{1,3,4,5\}}. \tag{63}
  \]

## 5. Summary

We demonstrate how to obtain the expansion of tree EM amplitudes into the KK basis of tree YM amplitudes efficiently by applying proper differential operators in this paper. The coefficients for the KK basis in the expansion are shared by the expansion of tree $\phi^4$ amplitudes into tree BS amplitudes, the expansion of tree sYMS amplitudes into tree BS amplitudes, and the expansion of tree DBI amplitudes into tree NLSM amplitudes, as has been explained in detail. These expansions exhibit connections among amplitudes of different theories that are invisible from the point of view of Feynman rules, and serve as dual representations of unifying relations described by differential operators.

The method used in Ref. [17] and this paper can also be applied to other theories linked by differential operators. One of our future directions is to derive expansions of other theories via this method and construct a complete web for expansions.

Interestingly, for the expansions of EM amplitudes obtained in the current paper, the manifest gauge invariance is missing for all gravitons$^5$. The loss of manifest gauge invariance is a general feature for expansions of amplitudes into the KK basis. As discussed in Ref. [17], for the expansion of single-trace EYM amplitudes, the manifest gauge invariance for all gravitons can be ensured when expanding into the BCJ basis rather than the KK basis, with the cost that coefficients contain poles.

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$^5$ When saying “gauge invariance” for a particular particle $i$, we mean the Ward’s identity that the amplitude vanishes under the replacement $\vec{e}_i \rightarrow k_i$. 

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For EM amplitudes, reproduction of the manifest gauge invariance for gravitons is a significant problem.

The expansions of amplitudes not only provide the theoretical understanding of connections between different theories, but are also of benefit for practical calculations. For example, since the evaluation of YM amplitudes is much easier than that of EM amplitudes, one can calculate YM amplitudes at the first step, and get EM amplitudes through the expansions. The obtained EM amplitudes may be used to study the quantum corrections of the behavior of photons in a gravitational field, such as gravitational light bending and Hawking radiation.

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