Trace Anomaly Inflation in Brane Induced Gravity

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ABSTRACT: In the present letter we find that Starobinsky’s inflationary solution is also valid in the Dvali-Gabadadze-Porrati (DGP) model where a 3-brane is embedded in 5-dimensional Minkowski bulk. We show that such a solution is typically not supported by the Self-Accelerated branch of the model, giving therefore a natural selection of the conventional branch of solutions. In the absence of brane induced Einstein-Hilbert term the SA branch is always selected out. We then study the linearized modes around all such de Sitter brane solutions finding perturbative stability for a range of parameters of the brane QFT.

KEYWORDS: gravity, physics of the early universe, cosmology with extra dimensions.
1. Introduction

The DGP model of brane induced gravity [1], a model for modification of gravity at large distances (cosmology at late times) provides an appealing alternative explanation for the current accelerated expansion of the Universe [2, 3].

In this article we study the very early Universe in brane induced gravity. Inspired by the seminal work of Starobinsky [4] we consider the localized matter as being the high energy limit of an asymptotically free QFT (such as QCD) and search for inflationary de Sitter solutions (see [5] for a more recent discussion about Starobinsky’s solution). A thorough study of quantum effects of brane conformal field theories in the physics of the early (brane) Universe, has already been done for several other models of brane world: it includes the RS model [6] as well as other variants that involve the inclusion of higher curvature bulk terms, a richer bulk matter content, as well as supersymmetric setups and generalizations to dS bulks and FRW brane cosmology [7].

The main motivation of the present study of inflation in brane-induced gravity models, relies on the observation that if the DGP model successfully incorporated inflation then, interestingly enough, it would mimic the whole cosmological paradigm: an initial meta-stable inflationary solution that would decay into “regular” FRW cosmology. In the high energy limit, when the brane curvature satisfies $R > M_{GUT}^2$, the brane stress tensor is well approximated by the trace anomaly-generated stress tensor quadratic in brane curvatures acting as a source for the junction condition associated to the 4d modified Einstein equations of motion. Such a junction condition will, as usual, lead to the brane Friedman equation.

We analyze the cosmology resulting from the model, with and without the inclusion of an induced non-conformal EH term on the brane. We then proceed to study the perturbative spectrum and find the conditions on the parameters of the brane quantum field theory such that there is a period of inflation that is long enough.
In the original DGP model there is a branch of "self-accelerated" solutions [2] in which late time acceleration could be driven solely by the effect of the modified gravitation without the need for any energy density. This branch of solutions, whose consistency with data has been extensively studied (see, e.g., [9]), appears to be unstable both perturbatively [10, 11] (see, however, the possible caveats to this conclusion [12]) and non-perturbatively (with hints in the exact solutions [13, 14] and a recent discussion in [15] - see also, [16]). A welcome feature that results from our approach is that once the quantum effects that drive inflation are included the conventional (as opposed to the self-accelerated) branch of solutions of the model is naturally selected, if the induced Planck mass is small. On the other hand if the induced Planck mass is large the self-accelerated branch results perturbatively unstable.

2. The Setup

We consider a five-dimensional action which includes the Einstein-Hilbert term $S_{EH}$ and a bulk matter action $S_B$ which we will assume to give rise to a perfect fluid type of stress tensor $S_M^{MN} = -2g^{MR}\delta S_B/\delta g^{RN} = \text{diag}(-\rho_B, p_B, p_B, p_B, p_y)$. We will be interested in studying brane-cosmological bulk solutions of the form

$$ds^2 = -N^2(\tau, y)d\tau^2 + A^2(\tau, y)d\Omega_k^2 + B^2(\tau, y)dy^2 \quad (2.1)$$

where $y$ is the coordinate orthogonal to the hypersurface $\{y = 0\}$ that we will refer to as "the brane" and $d\Omega_k^2 = \Omega_{mn}(x^m)dx^mdx^n$ is a unit radius maximally-symmetric space, characterizing the brane spatial section. We will assume that some matter is confined on the brane. In particular, we will consider the backreaction caused by the renormalized stress tensor of a set of massless conformal quantum fields on an isotropic homogeneous (brane) universe, along the lines of what originally done by Starobinsky [4]. Such massless quantum fields are to be understood as the high energy limit of the (massive) fields of an asymptotically free QFT. The result will be twofold: on the one hand we generalize Starobinsky’s idea to the DGP brane world model. On the other hand we study brane-induced gravity (DGP model [1]) in case when the matter stress tensor on the brane is dominated by the trace-anomaly-induced one.

It was shown [17] that, assuming there is no bulk-brane energy exchange, the $\tau \tau$ and $yy$ bulk equations of motion take the simple form

$$F' = -\frac{2\kappa^2}{3} A'A^3\rho_B \quad (2.2)$$

$$\dot{F} = \frac{2\kappa^2}{3} AA^3p_y \quad (2.3)$$

where

$$F(t, y) = \left(\frac{A'A}{B}\right)^2 - \left(\frac{AA}{N}\right)^2 - kA^2, \quad (2.4)$$

---

$^1$We denote $'= \partial_\tau$ and $'= \partial_y$. 
whereas the $\tau y$ equation of motion reads

$$\frac{N'}{N} \frac{\dot{A}}{A} - \frac{\dot{N}}{N} \left( \frac{\dot{A}}{N} \right)' = 0 .$$

Equation (2.2) can be integrated and yields

$$F = -\frac{\kappa^2}{6} \rho_B A^4 - C ,$$

where $C$ is an integration constant that was found \[18\] to be related to the bulk Weyl tensor. Hence,

$$\left( \frac{A'}{AB} \right)^2 = \left( \frac{\dot{A}}{AN} \right)^2 + \frac{k}{A^2} - \frac{C}{A^4} - \frac{\kappa^2}{6} \rho_B .$$

We will be interested in the case of an empty, Minkowski bulk for which $\rho_B = C = 0$ so that $F = 0$ and the leftover Einstein’s equation is also automatically satisfied and (2.7) reduces to

$$\left( \frac{A'}{AB} \right)^2 = \left( \frac{\dot{A}}{AN} \right)^2 + \frac{k}{A^2} .$$

Hence,

$$\frac{A'}{AB} = -\epsilon \sqrt{\left( \frac{\dot{A}}{AN} \right)^2 + \frac{k}{A^2}} ,$$

which projected upon the brane becomes

$$\frac{a'}{ab} = -\epsilon \sqrt{\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2}} ,$$

where

$$a(\tau) = A(\tau, 0) ,$$

$$b(\tau) = B(\tau, 0) ,$$

$$n(\tau) = N(\tau, 0) \equiv 1 .$$

The l.h.s. of (2.10) is related to the extrinsic curvature on the brane that is defined by

$$K_{\mu\nu} = e^M_{\mu} e^N_{\nu} \nabla_M n_N |_{\Sigma} ,$$

where

$$e^M_{\mu} = \frac{\partial X^M}{\partial x^\mu} = \delta^M_{\mu} ,$$

$$e^M_{\mu} n_M = 0 ,$$

$$e^M_{\mu} e^N_{\nu} = \delta^M_{\nu} .$$
are respectively the tangent vectors characterizing the brane embedding in the bulk (we choose a static gauge) and the (unit) normal vector to the brane

\[ n_M = (0, 0, B) , \]  \hspace{1cm} (2.17)

so that

\[ K^0_0 = \frac{n'}{b} , \]  \hspace{1cm} (2.18)

\[ K^m_n = \frac{a'}{ab} \delta^m_n = -\epsilon \sqrt{\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \delta^m_n} , \quad \epsilon = \pm 1 , \]  \hspace{1cm} (2.19)

is the expression for the extrinsic curvature tensor, where we have made use of (2.10). The Israel junction condition links the latter to the stress tensor of the matter localized on the brane

\[ K_{\mu\nu} - \gamma_{\mu\nu} K = -\frac{1}{2M^3} T_{\mu\nu} , \]  \hspace{1cm} (2.20)

where the lhs is evaluated at \( y = 0^+ \) and \( \gamma_{\mu\nu} \) is the induced metric on the brane, namely,

\[ ds^2 = -d\tau^2 + a(\tau)^2 d\Omega^2_k \equiv \gamma_{\mu\nu}(x)dx^\mu dx^\nu . \]  \hspace{1cm} (2.21)

Assuming the brane matter to be a perfect fluid, from the spatial components of the extrinsic curvature we get

\[ \epsilon \sqrt{\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2}} = \frac{1}{6M^2} \rho . \]  \hspace{1cm} (2.22)

If the localized matter is conformal the (v.e.v. of the) energy momentum tensor reduces to the one generated by the trace anomaly \[4, 13, 20\]

\[ T^A_{\mu\nu} = \tilde{\alpha} \left( -R_{\mu}^{\quad \sigma} R_{\nu\sigma} + \frac{2}{3} RR_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R^2_{\alpha\beta} - \frac{1}{4} g_{\mu\nu} R^2 \right) \\
+ \frac{\tilde{\beta}}{6} \left( -2 \nabla_\mu \nabla_\nu R + 2g_{\mu\nu} \Box R + 2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 \right) \equiv \tilde{\alpha} \Theta^1_{\mu\nu} + \tilde{\beta} \Theta^2_{\mu\nu} , \]  \hspace{1cm} (2.23)

where \( \tilde{\alpha} = k_2/6!(2\pi)^2 \) and \( \tilde{\beta} = k_3/6!(2\pi)^2 \) and all tensors are calculated according to the induced metric, and yield

\[ \epsilon \sqrt{\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2}} = \frac{1}{2M^3} \left\{ \tilde{\alpha} \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right)^2 \right\} \\
+ \tilde{\beta} \left[ -\left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a}}{a} \left( \frac{\ddot{a}}{a} \right) - 2 \frac{\dot{a}}{a} \left( \left( \frac{\ddot{a}}{a} \right)^2 + \frac{k}{a^2} \right) \right] + \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right)^2 \} , \]  \hspace{1cm} (2.24)

that is the brane Friedman equation which can be written as

\[ \epsilon \nu^{1/2} a^2 = \frac{1}{2M^3} \left[ \tilde{\alpha} \nu^2 + \tilde{\beta} \left( \frac{nu_{aa}}{a^2} - 2nu_{a}^2 - \frac{nu_{aa}}{4a^2} - kv_{aa} + \frac{3kv_{a}}{a} \right) \right] , \]  \hspace{1cm} (2.25)
by means of the transformation $v(a) = (a\dot{a})^2 + ka^2$. One possible set of solutions of the latter equation can be achieved by setting $v = H^2 a^4$ (with $H^2$ constant), upon which the term in the round parenthesis vanishes (lower “indices” $a$ denote derivatives w.r.t. $a$); the leftover terms uniquely fix the constant,

$$
(a\dot{a})^2 + ka^2 = H^2 a^4, \\
\dot{a}H^3 = 2M^3\epsilon,
$$

(2.26)

and select the allowed branch, that is $\epsilon = +1$ for $k_2 > 0$ and $\epsilon = -1$ in the opposite case. In other words de Sitter type of solution

$$
a(\tau) = \begin{cases} 
  a_0 \exp(Ht), & k = 0 \\
  H^{-1}\cosh(Ht), & k = +1 \\
  H^{-1}\sinh(Ht), & k = -1
\end{cases}
$$

(2.27)

holds in our brane world setup. However, since

$$
k_2 = N_S + 11N_F + 62N_V > 0,
$$

(2.28)

only the conventional branch ($\epsilon = +1$) is allowed for such a solution. In fact, a negative contribution to $k_2$ can be obtained from quantum fields that have a wrong sign in the kinetic term, in other words ghosts: since the trace anomaly in four dimensions comes from a triangle diagram describing the correlation function of three stress tensors, changing the sign of the action for the quantum fields changes that of the stress tensor and in turn that of the anomaly. The previous argument clearly raises an issue concerning the (quantum) stability of the branch $\epsilon = -1$, in the present setup. Such a branch corresponds to the so-called self-accelerated branch of the original DGP model \cite{2}. Hence, the argument we gave above would a priori seem to confirm the more pessimistic points of view. Note however that such an argument does not imply that the self-accelerated branch is altogether excluded by (2.24); in fact the second term on r.h.s. of (2.24) is not positive definite - it is related to the trivial anomaly - and thus it might allow for solutions even for $\epsilon = -1$.

Before concluding this section let us also point out that in the above solution the inflation rate $H$ is set in by the fundamental scale $M$ and depends upon 4d physics only through the numerical combination $k_2$. In the (DGP) brane world scenario the fundamental scale is taken to be much smaller that the 4d Planck mass: therefore the curvature scale $R \sim H^2$ is also much smaller than $M_P^2$ and thus lower than $M_{GUT}^2$, so that the validity of the approximation (2.23) appears not to be guaranteed. However, we will see in the next
section that the inclusion of an induced non-conformal Einstein-Hilbert term on the brane enhances the value of the de Sitter scale. A brane-induced E-H term is also a necessary ingredient in order to recover 4d gravity between brane massive sources.

2.1 Adding a non-conformal Einstein-Hilbert term on the brane

In this subsection we consider the inclusion of a non-conformal induced EH term on the brane. Such an induced term is part of the effective action of matter fields coupled to gravity \[1\]. However it was shown that a brane EH term can be produced also at the classical level on a tensionful brane if the bulk action includes higher curvature terms \[22\].

The presence of an induced E-H term on the brane does not necessarily mean that the inflationary phase is no longer there. In fact as we now show we can have a steeper inflationary phase whose scale is set in by a combination of both \(M\) and induced Planck mass \(M_P\). Note in fact that the anomaly generated stress tensor, “ameliorated” with the localized Einstein tensor coming from the induced EH term, is a good approximation for “large” brane curvature, \(M_P^2 > R > M_{GUT}^2\). The first inequivalence guarantees that brane loop corrections are under control. Note also that, although bulk loops are controlled by the fundamental scale \(M\), the expansion is under control there as well since the bulk is (a patch of) flat Minkowski space. The aforementioned inflationary phase could be unstable as pointed out by Starobinsky and Vilenkin \[4, 8\] and the universe would thus finally decay to “regular” FRW cosmology. We discuss about stability in the next sections.

In this section we thus consider the inclusion of an Einstein-Hilbert term on the brane

\[
S_4 = \frac{M_P^2}{2} \int d^4x \sqrt{-g} R_4
\]

so that (2.26) becomes

\[
\epsilon m_c + H = \frac{\dot{\alpha} H^3}{M_P^2}
\]

(2.30)

with \(m_c = 2M^3/M_P^2\) being the crossover scale of the DGP model. Denoting \(z = H/M\lambda\), with \(\lambda = (2\tilde{\alpha})^{-1/3}\), equation (2.31) can be rewritten as a cubic equation

\[
z^3 - 2\lambda \left(\frac{M_P}{M}\right)^2 z - 4\epsilon = 0
\]

(2.31)

that is of the form that allows for direct Cardano’s solution

\[
\frac{H}{M\lambda} = \left[2\epsilon + \sqrt{4 - \left(\frac{2\lambda}{3} \left(\frac{M_P}{M}\right)^2\right)^3}\right]^{1/3} + \left[2\epsilon - \sqrt{4 - \left(\frac{2\lambda}{3} \left(\frac{M_P}{M}\right)^2\right)^3}\right]^{1/3}
\]

(2.32)

For \(\epsilon = +1\) there is one real positive solution regardless of the value of the quantity \(\lambda (M_P/M)^2\).

For \(\epsilon = -1\) and \(0 < \lambda (M_P/M)^2 < 3 \cdot 2^{-1/3}\), there is no positive solution whereas for \(\lambda (M_P/M)^2 > 3 \cdot 2^{-1/3}\) there are two positive solutions. Therefore, if the induced
Planck mass is large enough, both branches would seem to allow expanding de Sitter solutions \((2.27)\). In fact, in the limit where \(M_P \gg M\) one recovers Starobinsky’s expansion rate \(H_S \sim M_P/\sqrt{\alpha}\). Note also that the expansion rate for the conventional (self-accelerated) branch corresponds to \(\frac{\delta H^2}{M_P^2} > 1\) \(\frac{\delta H^2}{M_P^2} < 1\). In other words a phenomenological selection “criterion” that, a priori, seems to pick the conventional branch is the highest dS expansion rate. However, in such a case higher order corrections (in \(H/\Lambda\)) to the quantum stress tensor are not guaranteed to be under control, since the induced Plack mass is related to the UV cutoff of the localized theory as \(M_P^2 \sim \Lambda^2 N\), with \(N\) being (roughly) the total number of fields in the theory \((N \lesssim k_2)\) and thus \(H_S \gtrsim \Lambda\). An hypothetical way out to such a problem might be the inclusion of bulk higher curvature terms that could allow to lower the dS scale. Let us also remind the reader that, in order for quantum gravitational corrections to be under control, \(H_S < M_P\), the total number of fields must be extremely large such that \(k_2 > 6!\) \(^4\)

3. Perturbations of the de Sitter brane solution

In this section we consider small perturbations around the de Sitter brane solution introduced previously. Let us first point out that the stress tensor \(\Theta^1_{\mu\nu}\), as defined in \((2.23)\), is not conserved in a generic background and should be supplemented by a term \(R^{\alpha\beta}C_{\mu\alpha\nu\beta}\), with \(C_{\mu\alpha\nu\beta}\) being the Weyl tensor. However, linear perturbations of such a term around a maximally-symmetric background, vanish: we can thus safely omit the latter in what follows.

The bulk equations and junction conditions are given by:

\[
R_{MN} = 0 , \\
G_{\mu\nu} - m_c (K_{\mu\nu} - g_{\mu\nu} K) = \frac{T^A_{\mu\nu}}{M_P^2} + \frac{T_{\mu\nu}}{M_P^2} ,
\]

where we added to the r.h.s. of \((3.2)\) an extra subleading \(T_{\mu\nu}\) component.

We want to solve for the perturbations around the \(k = 0\) de Sitter background solution (see the appendix) in a convenient gauge, namely,

\[
g_{\mu\nu} = N^2 (y) \gamma_{\mu\nu} + \tilde{h}_{\mu\nu} \, , \quad g_{yy} = 1 \quad g_{\mu y} = 0 \, , \quad \nabla^{\mu} \tilde{h}_{\mu\nu} = \nabla_\nu \tilde{h} .
\]

where \(N(y) = 1 - \epsilon H |y|\), \(\gamma_{\mu\nu}\) has curvatures \(R_{\mu\nu} = 3H^2 \gamma_{\mu\nu}\) and \(R = 12H^2\) and \(\epsilon = \pm 1\) \((\epsilon = +1\) is the conventional branch: \(|y| \leq H^{-1}\)).

Let us first gather some results:

\[
\delta R_{\mu\nu} = \frac{1}{2} \nabla_\mu \nabla_\nu \tilde{h} - \frac{1}{2} \Box \tilde{h}_{\mu\nu} + 4H^2 \tilde{h}_{\mu\nu} - H^2 \gamma_{\mu\nu} \tilde{h} ,
\]

\[
\delta R = -3H^2 \tilde{h} ,
\]

\(^4\)A severe parametrical constraint comes from the observational limit on the amplitude of long-wavelength gravitational waves, requiring \(k_3 \gg k_2\), that appears quite unnatural. However notice that \(k_3\) is the coefficient of the trivial part of the anomaly and is thus arbitrary.
Thus,
\[
\delta G_{\mu\nu} = -\frac{1}{2} \left( \Box \tilde{h}_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \tilde{h} \right) - 2H^2 \tilde{h}_{\mu\nu} + \frac{1}{2} H^2 \gamma_{\mu\nu} \tilde{h},
\]
(3.6)

From the $yy$ equation we get
\[
\delta \gamma_{\mu\nu} \delta K_{\mu\nu} = -\epsilon H \tilde{h}_{\mu\nu} / 2, \]
(3.7)

where $\alpha = \tilde{\alpha} H \gamma_{\mu\nu}$ and, for future reference, $\beta = \tilde{\beta} H^2 / M_P^2$. Therefore, the junction conditions (3.2) and its trace yield:
\[
\frac{1}{2} \left( m_c \partial_y + (1 - 2\alpha - 4\beta) \Box - 4H^2(1 - \frac{3}{2} \alpha - 2\beta) \right) \tilde{h}_{\mu\nu} = \frac{1}{M_P^2} \left( T_{\mu\nu} - \frac{1}{3} \gamma_{\mu\nu} T \right) - \frac{1}{2} \left( (1 - 2\alpha - 6\beta) \gamma_{\mu\nu} - (1 - 2\alpha - 2\beta) H^2 \gamma_{\mu\nu} \right) \tilde{h},
\]
(3.8)

Note that for non vanishing $\beta$ the trace $\tilde{h}$ becomes a propagating degree of freedom.

A way towards the solution of this system is through the shift to transverse traceless metric perturbations $h_{\mu\nu}$ by introducing the brane bending mode $\varphi(x)$:
\[
\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{2N}{H} \Pi^+_{\mu\nu} \varphi,
\]
(3.10)

with $\Pi^\pm_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \pm \gamma_{\mu\nu} H^2$.

The system (3.8), (3.9) and the bulk $\mu\nu$ equations becomes:
\[
\frac{1}{2} \left( m_c \partial_y + (1 - 2\alpha - 4\beta) \Box - 4H^2(1 - \frac{3}{2} \alpha - 2\beta) \right) h_{\mu\nu} = \frac{1}{M_P^2} \left( T_{\mu\nu} - \frac{1}{3} \gamma_{\mu\nu} T \right) + (-\epsilon m_c - 2H(\alpha + 4\beta)) \Pi^+_{\mu\nu} \varphi - 2\frac{\beta}{H} \Pi^+_{\mu\nu} Q \varphi,
\]
(3.11)

\[
(\epsilon H m_c + 2\alpha H^2 - 2\beta \Box) Q \varphi = \frac{HT}{3M_P^2},
\]
(3.12)

\[
(N^2 \partial_y^2 - \mathcal{O}) h_{\mu\nu} = 0,
\]
(3.13)

where we use
\[
\mathcal{O} = 4H^2 - \Box,
\]
(3.14)

\[
Q = -4H^2 - \Box,
\]
(3.15)

and the identity
\[
\mathcal{O} \Pi^+_{\mu\nu} \varphi = \Pi^-_{\mu\nu} Q \varphi.
\]
(3.16)
valid for any scalar $\phi$. Combining (3.11) and (3.13) into a single equation gives:

$$m \epsilon \partial_y^2 - \left(\frac{m \epsilon}{N^2} + 2 \delta(y)\right) \mathcal{O} - 2 \delta(y) \left((2 \alpha + 4 \beta) \Box - 2 H^2 (3 \alpha + 4 \beta)\right) h_{\mu \nu} =$$

$$-4 \delta(y) \left[\frac{1}{M_P^2} \left(T_{\mu \nu} - \frac{1}{3} \gamma_{\mu \nu} T\right) - (\epsilon m \epsilon + 2 H (\alpha + 4 \beta)) \Pi^+_{\mu \nu} \phi - 2 \frac{\beta}{H} \Pi^+_{\mu \nu} Q \phi\right],$$

which by making use of the condition (2.30) derived from the background equations, i.e., $\epsilon m \epsilon = (\alpha - 1) H$, can be solved to yield

$$\frac{1}{2} h_{\mu \nu} = S(O) \left(\frac{1}{M_P^2} \left(T_{\mu \nu} - \frac{1}{3} \gamma_{\mu \nu} T\right) - (\epsilon m \epsilon + 2 H (\alpha + 4 \beta)) \Pi^+_{\mu \nu} \phi - 2 \frac{\beta}{H} \Pi^+_{\mu \nu} Q \phi\right).$$

We recover $\tilde{h}_{\mu \nu}$ via (3.10) with $\phi$ given by the solution of (3.12).

Notice that we have correlated the two possible solutions (3.20) with the two branches of the model via the $\epsilon$ parameter. This implies a choice of boundary conditions at $y = \infty$: had these been uncorrelated, non-normalizable modes would be contributing to the amplitudes implying the presence of sources far away into the bulk.

Having the solution for the perturbations (3.18) we now concentrate on the one graviton exchange amplitude between brane localized sources $T$ and $T'$ (we follow the approach presented in [12]; see [24] for a pedagogical introduction). To this end, it is useful to decompose the corresponding conserved energy momentum tensors $T_{\mu \nu}$ and $T'_{\mu \nu}$ into transverse, longitudinal and trace parts:

$$T_{\mu \nu} = T^{TT}_{\mu \nu} + \frac{1}{4} \gamma_{\mu \nu} T + \frac{1}{3} P_{\mu \nu} \frac{1}{Q} T.$$

where $P_{\mu \nu} = \nabla_\mu \nabla_\nu - \frac{1}{4} \gamma_{\mu \nu} \Box$ (for which the identity $P_{\mu \nu} f(Q) \phi = f(O) P_{\mu \nu} \phi$ holds for any smooth $f$ and scalar $\phi$).

We obtain the following expressions:

$$\frac{1}{2} \tilde{h}_{\mu \nu} = \frac{1}{M_P^2} S(O) T^{TT}_{\mu \nu} + \frac{N}{H} \Pi^+_{\mu \nu} \phi,$$

$$\varphi = -\frac{1}{3 \xi H M_P^2} \left(\frac{1}{Q} - \frac{2 \beta}{2 \beta Q - \xi H^2}\right) T,$$

where $\xi = 1 - 3 \alpha - 8 \beta$.

Next, let us introduce the Lichnerowicz operator $\Delta$ with the following properties

$$(\Delta - 4 H^2) T^{TT}_{\mu \nu} = O T^{TT}_{\mu \nu},$$

$$\varphi = \gamma_{\mu \nu} \phi = \gamma_{\mu \nu} Q \phi,$$

$$(\Delta - 4 H^2) P_{\mu \nu} \phi = P_{\mu \nu} Q \phi.$$
This gives,
\[
S(O)T_{\mu\nu}^{TT} = S(\Delta - 4H^2)T_{\mu\nu} - \frac{1}{4} S(Q)\gamma_{\mu\nu}T - \frac{1}{3} \nabla_\mu \nabla_\nu \frac{S(Q)}{Q}T + \frac{1}{12} \gamma_{\mu\nu} \Box S(Q)T,
\]
\[
T'^{\mu\nu}S(O)T_{\mu\nu}^{TT} = T'^{\mu\nu}S(\Delta - 4H^2)T_{\mu\nu} + \frac{1}{12} T' \left( \frac{\Box}{Q} - 3 \right) S(Q)T + \nabla_\mu (\cdots). \quad (3.27)
\]

We compute the amplitude,
\[
\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-\gamma} T'^{\mu\nu} \tilde{h}_{\mu\nu},
\]
obtaining:
\[
\mathcal{A} = \frac{1}{M_P^2} \int d^4x \sqrt{-\gamma} \left\{ T'^{\mu\nu}S(\Delta - 4H^2)T_{\mu\nu}
- \frac{1}{3} T' \left[ \left( 1 + \frac{H^2}{Q} \right) S(Q) + \frac{N}{\xi Q} - \frac{2\beta N}{\xi (2\beta Q - \xi H^2)} \right] T \right\}. \quad (3.29)
\]

Let us focus on the conventional branch (\(\epsilon = +1\)). In this case, there could be poles at \(Q = 2H^2\) (vanishing denominator of \(S(Q)\)), at \(Q = 0\) and at \(2\beta Q = \xi H^2\). We will also discuss below the interpretation of the other zero of the denominator of \(S(Q)\).

The \(Q = 2H^2\) (\(\Box = -6H^2 \equiv -2\Lambda\)) pole, together with the one at \(\Delta = 6H^2\) corresponds to a massless spin-2 state on the de Sitter background (with the correct tensorial structure).

The would be \(Q = 0\) (\(\Box = -4H^2\)) pole has vanishing residue (at that point \(H^2S(0) + N/\xi = 0\)).

To go further into the analysis, let us find another expression for the amplitude. We define a complex variable \(w \equiv -z - H^2/4\) such that \(\tilde{S}(w) = S(-w - H^2/4)\) has a branch cut in the positive real \(w\) axis. We get the alternative form
\[
\tilde{S}(w) = \frac{1}{2\pi i} \int \frac{ds}{s - w} \tilde{S}(s) - \frac{1}{w_0 - w} \lim_{s \to w_0} (s - w_0) \tilde{S}(s) \quad (3.30)
\]
via a contour integral \(\Gamma\) which goes around the cut just below and above the positive \(w\) axis and closes at infinity through a counterclockwise circle. The residue of the pole in the first Riemann sheet, \(w_0\), plus the jump in the imaginary part across the cut give
\[
\tilde{S}(w) = \frac{c}{w_0 - w} + \int_0^\infty \frac{ds}{s - w} \rho(s), \quad \rho(s) = \frac{1}{\pi} \frac{(\alpha - 1)H\sqrt{s}}{(1 - \alpha)^2H^2s + ((1 - 2\alpha - 4\beta)s + 3H^2(\frac{1}{4} - \alpha - 3\beta))^2}. \quad (3.31)
\]
\[\]
or equivalently,
\[
S(Q \equiv -\Box - 4H^2) = \frac{c}{-\Box - 2\Lambda} + \int_{\frac{1}{2}H}^\infty \frac{dm}{m^2 - \Box - 2\Lambda} \rho \left( m^2 - 9H^2/4 \right). \quad (3.32)
\]
\[
\]
The simple pole at \( w = w_0 \equiv -9H^2/4 \) in \((3.31)\) corresponds to the massless state \((\Box + 2\Lambda = 0 \text{ in } (3.34))\) while the integral has contributions from the continuum of massive KK modes \(0 \leq w < \infty\), i.e., \(9H^2/4 \leq \Box + 2\Lambda < \infty\) with positive definite spectral density \((3.33)\) as long as \(\alpha > 1\) (a resonance representing the other zero in the denominator of \(S\) - see \((3.19)\)). With these conventions, the massless state has positive norm as long as \(c\) is positive, which for the regime of interest for us, namely, \(\alpha \gtrsim 1\), implies \(\beta \lesssim -1/4\).

The \(2\beta Q = \xi H^2\) pole has positive residue (corresponding to an extra scalar state with positive norm) as long as \(\xi\) is positive, which for \(\alpha \gtrsim 1\) coincides with the positivity of norm of the massless state. The squared mass (eigenvalue of \(\Box + 2H^2\)) of this extra state is

\[
m_h^2 = -2H^2 - \frac{\xi}{2\beta}H^2.
\]

By recalling the relation to the parameters \(k_2\) \((2.28)\) and \(k_3\) in \([4]\)

\[
\xi = 1 - \frac{1}{2880\pi^2} (3k_2 + 8k_3) \left( \frac{H}{M_P} \right)^2,
\]

\[
\beta = \frac{k_3}{2880\pi^2} \left( \frac{H}{M_P} \right)^2,
\]

we see that a modest negative \(k_3\) partially compensating \(k_2\) in \((3.36)\) leads to a theory free of ghost-like instabilities around a de Sitter background. Furthermore, in the original Starobinsky setup a future instability is found \([4, 8]\) due to a light tachyon. This slow instability which could be responsible for the exit from a long inflationary phase persists in our setup. Keeping \(\xi > 0\) and focusing in the regime \(\alpha \gtrsim 1\), \(\beta \lesssim -1/4\) gives a tachyonic mass \(m_h^2 \sim -2H^2\) to the extra scalar state.

On the self-accelerated branch \((\epsilon = -1)\) the pole \(w = w_0\) sits in the first Riemann sheet provided one chooses the opposite prescription for \(\sqrt{-w}\) \([12]\). With this choice we obtain that the spectral density has the same form as in \((3.33)\) though now is negative definite, since \(\alpha < 1\): it can be interpreted as due to a tower of negative-norm KK states.

4. Conclusions

We considered the high energy limit of asymptotically-free brane matter in Minkowski bulk and studied Starobinsky’s trace-anomaly driven inflationary solutions. We studied perturbations about such de Sitter solutions and found that the system could display no instabilities. In particular, in the conventional branch there is a range of parameters (dictated by the number of scalar, vector and fermionic components) of the brane QFT such that there are no ghost-like excitations but a slow instability to end the inflationary phase.

In absence of any induced brane EH term, we found that only the conventional branch allows the aforementioned solution, becoming naturally selected. In the presence of an
induced EH term the phenomenon is more complex: for small induced Planck mass, $M_P < M\tilde{\alpha}^{1/6}$, only the conventional branch is allowed, whereas for big induced Planck mass $M_P > M\tilde{\alpha}^{1/6}$ both branches are allowed. In particular in the limit $M_P \gg M$ both branches converge to the Starobinsky’s solution. For the self-accelerated branch we have $H \lesssim H_S$ that corresponds to $\alpha \lesssim 1$ ($\alpha = 1$ being Starobinsky’s solution), whereas for the conventional branch we have $H \gtrsim H_S$ and $\alpha \gtrsim 1$. However, the self-accelerated branch turns out to be unstable regardless of the value of $M_P$.

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**A. Bulk solutions**

We briefly sketch the derivation of the bulk solutions corresponding to the dS cosmologies studied above. As shown by [17, 2] the assumption of zero bulk-brane energy exchange (cfr. eq. (2.3)) allows to write

$$\frac{\dot{A}}{N} = \alpha(\tau).$$

Thus setting the brane “clock” such that $N(\tau, 0) = 1$, and using the bulk equation of motion (2.4) we get

$$A(\tau, y) = a(\tau) - \epsilon|y|\sqrt{k + \dot{a}^2}$$

and therefore, using the explicit solutions described in the main part of the paper we obtain

$$A(\tau, y) = a(\tau) \left(1 - \epsilon H|y|\right)$$

and thus

$$ds^2 = N(y)^2d\tilde{s}^2 + dy^2 = N(y)^2\left(-dT^2 + a^2(\tau) d\Omega_k^2\right) + dy^2,$$

with $|y| < H^{-1}$ for the conventional branch $\epsilon = +1$ and $|y| < \infty$ for the self-accelerated branch $\epsilon = -1$. In conformal coordinated the latter reads

$$ds^2 = e^{-\epsilon H|z|}\left(-dT^2 + a^2(\tau) d\Omega_k^2 + dz^2\right)$$

with $z \in \mathbb{R}$.
References

[1] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485, 208 (2000) [arXiv:hep-th/0005016]; G. R. Dvali and G. Gabadadze, Phys. Rev. D 63, 065007 (2001) [arXiv:hep-th/0008054].

[2] C. Deffayet, Phys. Lett. B 502, 199 (2001) [arXiv:hep-th/0010186].

[3] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D 65, 044023 (2002) [astro-ph/0105068].

[4] A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).

[5] J. C. Fabris, A. M. Pelinson and I. L. Shapiro, Nucl. Phys. B 597, 539 (2001) [Erratum-ibid. B 602, 644 (2001)] [arXiv:hep-th/0009197].

[6] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D 62, 043501 (2000) [arXiv:hep-th/0003052]; S. Nojiri and S. D. Odintsov, Phys. Lett. B 484, 119 (2000) [arXiv:hep-th/0004097].

[7] S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D 62, 064006 (2000) [arXiv:hep-th/0001192]; S. Nojiri, O. Obregon and S. D. Odintsov, Phys. Rev. D 62, 104003 (2000) [arXiv:hep-th/0005127]; S. Nojiri and S. D. Odintsov, JHEP 0007, 049 (2000) [arXiv:hep-th/0006232]; S. Nojiri and S. D. Odintsov, Phys. Rev. D 64, 023502 (2001) [arXiv:hep-th/0102032]; S. Nojiri and S. D. Odintsov, JHEP 0112, 033 (2001) [arXiv:hep-th/0107134]; K. Koyama and J. Soda, JHEP 0105, 027 (2001) [arXiv:hep-th/0110164].

[8] A. Vilenkin, Phys. Rev. D 32, 2511 (1985).

[9] C. Deffayet, S. J. Landau, J. Raux, M. Zaldarriaga and P. Astier, Phys. Rev. D 66, 024019 (2002) [arXiv:astro-ph/0201164]; A. Lue, R. Scoccimarro and G. Starkman, Phys. Rev. D 69, 044005 (2004) [arXiv:astro-ph/0307034]; Phys. Rev. D 69, 124015 (2004) [arXiv:astro-ph/0401515]; J. S. Alcaniz, D. Jain and A. Dev, Phys. Rev. D 66, 067301 (2002) [arXiv:astro-ph/0206448]; E. V. Linder, Phys. Rev. D 72, 043529 (2005) [arXiv:astro-ph/0507263]; L. Knox, Y. S. Song and J. A. Tyson, arXiv:astro-ph/0503644.

G. Dvali, A. Gruzinov and M. Zaldarriaga, Phys. Rev. D 68, 024012 (2003) [arXiv:hep-ph/0212069]; M. Fairbairn and A. Goobar, Phys. Lett. B 642, 432 (2006) [arXiv:astro-ph/0511009]; K. Koyama and R. Maartens, JCAP 0601, 016 (2006) [arXiv:astro-ph/0511634]; I. Sawicki and S. M. Carroll, arXiv:astro-ph/0510364; R. Maartens and E. Majerotto, Phys. Rev. D 74, 023004 (2006) [arXiv:astro-ph/0603353].

[10] M. A. Luty, M. Porrati and R. Rattazzi, JHEP 0309, 029 (2003) [arXiv:hep-th/0303116]; A. Nicolis and R. Rattazzi, JHEP 0406, 059 (2004) [arXiv:hep-th/0404159]; K. Koyama, Phys. Rev. D 72, 123511 (2005) [arXiv:hep-th/0503191]; D. Gorbunov, K. Koyama and S. Sibiryakov, Phys. Rev. D 73, 044016 (2006) [arXiv:hep-th/0512097]; K. Koyama and F. P. Silva, Phys. Rev. D 75, 084040 (2007) [arXiv:hep-th/0702169].

[11] C. Charmousis, R. Gregory, N. Kaloper and A. Padilla, JHEP 0610, 066 (2006) [arXiv:hep-th/0604086].

[12] C. Deffayet, G. Gabadadze and A. Iglesias, JCAP 0608, 012 (2006) [arXiv:hep-th/0607099].

[13] N. Kaloper, Phys. Rev. Lett. 94, 181601 (2005) [Erratum-ibid. 95, 059901 (2005)] [arXiv:hep-th/0501028]; Phys. Rev. D 71, 086003 (2005) [Erratum-ibid. D 71, 129905 (2005)] [arXiv:hep-th/0502035].

– 13 –
[14] G. Dvali, G. Gabadadze, O. Pujolas and R. Rahman, Phys. Rev. D 75, 124013 (2007) [arXiv:hep-th/0612016].
[15] R. Gregory, N. Kaloper, R. C. Myers and A. Padilla, JHEP 0710, 069 (2007) [arXiv:0707.2666 [hep-th]].
[16] K. Izumi, K. Koyama, O. Pujolas and T. Tanaka, Phys. Rev. D 76, 104041 (2007) [arXiv:0706.1980 [hep-th]].
[17] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477, 285 (2000) [arXiv:hep-th/9910219].
[18] T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D 62, 024012 (2000) [arXiv:gr-qc/9910076]; S. Mukohyama, T. Shiromizu and K. i. Maeda, Phys. Rev. D 62, 024028 (2000) [Erratum-ibid. D 63, 029901 (2001)] [arXiv:hep-th/9912287].
[19] P. C. W. Davies, S. A. Fulling, S. M. Christensen and T. S. Bunch, Annals Phys. 109, 108 (1977); P. C. W. Davies, Phys. Lett. B 68, 402 (1977).
[20] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D 63, 083504 (2001) [arXiv:hep-th/0010232]; S. W. Hawking and T. Hertog, Phys. Rev. D 65, 103515 (2002) [arXiv:hep-th/0107088].
[21] M. J. Duff, Nucl. Phys. B 125, 334 (1977); F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. B 389, 53 (1993) [arXiv:hep-th/9208059].
[22] O. Corradini, A. Iglesias, Z. Kakushadze and P. Langfelder, Phys. Lett. B 521, 96 (2001) [arXiv:hep-th/0108055].
[23] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84, 2778 (2000) [arXiv:hep-th/9911055].
[24] G. Gabadadze, arXiv:hep-th/0408118.
[25] M. Porrati, Phys. Lett. B 498, 92 (2001) [arXiv:hep-th/0011152].