ON PUSHED WAVEFRONTS OF MONOSTABLE EQUATION WITH UNIMODAL DELAYED REACTION

Karel Hasík, Jana Kopfová, Petra Nábělková
Mathematical Institute, Silesian University, 746 01 Opava, Czech Republic
Sergei Trofimchuk∗
Instituto de Matemática, Universidad de Talca, Casilla 747, Talca, Chile

Abstract. We study the Mackey-Glass type monostable delayed reaction-diffusion equation with a unimodal birth function \( g(u) \). This model, designed to describe evolution of single species populations, is considered here in the presence of the weak Allee effect (\( g(u_0) > g'(0)u_0 \) for some \( u_0 > 0 \)). We focus our attention on the existence of slow monotonic traveling fronts to the equation: under given assumptions, this problem seems to be rather difficult since the usual positivity and monotonicity arguments are not effective. First, we solve the front existence problem for small delays, \( h \in [0, h_p] \), where \( h_p \) (given by an explicit formula) is optimal in a certain sense. Then we take a representative piece-wise linear unimodal birth function making possible explicit computation of traveling fronts. In this case, we find out that a) increase of delay can destroy asymptotically stable pushed fronts; b) the set of all admissible wavefront speeds has usual structure of a semi-infinite interval \([c^*, +\infty)\); c) for each \( h \geq 0 \), the pushed wavefront is unique (if it exists); d) pushed wave can oscillate slowly around the positive equilibrium for sufficiently large delays.

1. Introduction. In this work, we consider the Mackey-Glass type delayed reaction-diffusion equation

\[ u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x)), \quad u \geq 0, \quad (t, x) \in \mathbb{R}^2, \quad h \geq 0, \quad (1) \]

widely used to model evolution of single species populations (see [4] for details and further references). In such an ecological interpretation of (1), \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a birth function, and if we further assume that \( g'(0) \geq 1 \) and that the equation \( g(x) = x \) has just two solutions, 0 and \( \kappa > 0 \), this equation belongs to the class of the so-called monostable population models. Clearly, monostable equation (1) has exactly two equilibria, \( u = 0 \) and \( u = \kappa \). It can also have positive wave solutions \( u(t, x) = \phi(x + ct), \phi(-\infty) = 0, \liminf_{t \to +\infty} \phi(t) > 0 \), corresponding to the transition regimes between these two equilibria. These solutions called semi-wavefronts (or wavefronts if, in addition, \( \phi(+\infty) = \kappa \)) characterize spatial propagation of the species and thus are quite significant from the ecological point of view. The description of the set \( \mathcal{C}(h) \) of all possible velocities of semi-wavefronts (for each fixed \( h > 0 \)) is of evident practical importance, this problem is also one of fundamental

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interest in the traveling wave theory. It follows from (1) that $C(h)$ depends solely on the properties of $g$ (for each fixed $h \geq 0$).

In particular, it is known that $C(h)$ is a closed unbounded interval, $C(h) = [c_*(h), +\infty)$ whenever $C^1$-smooth $g$ is either monotone on $[0, \kappa]$ or it satisfies the following two (sub-tangency and unimodality) conditions:

$$g(x) \leq g'(0)x, \quad x \geq 0,$$

(2)

(U) there exists $0 < \theta < \kappa$ such that $g$ increases on the interval $[0, \theta]$ and is decreasing otherwise; we are assuming also that $g'(0) > 1$.

Then real number $c_*(h)$ belonging to the boundary of $C(h) = [c_*(h), +\infty)$, is called the minimal (or critical) speed of propagation, it has a special status in the theory and applications. In particular, occupation of a new environment by invasive species is realized precisely with the minimal speed.

It should be noted that the assumption (U) of unimodal shape of $g$ is quite natural from the ecological point of view [4] and can barely be considered as an essential limitation. The situation when $g$ is increasing on $[0, \kappa]$ is considerably simpler for mathematical analysis due to the availability of comparison techniques. Owing to the previous studies (cf. [11, 17, 22]) properties of waves in the monotone model (1) are well understood so we will assume in the sequel that $g$ is non-monotone and has unimodal shape. In such a case, the topological structure of $C(h)$ can be potentially more complex when, in addition, $g$ does not satisfy inequality (2) (recall that (2) is an important ecological restriction excluding the weak Allee effect\(^1\)). In [3, question (iii), p. 107], S. Ai posed a question about the existence of the minimal speed in its usual meaning of a unique boundary point of $C(h)$, for monotone fronts to monostable non-monotone delayed equations. Theorem 1.3 in [3] answers partially this question for some special models with distributed delays which admit transformation of associated delayed profile equations into the systems of ordinary differential equations. Fenichel’s geometric singular perturbation theory was instrumental in proving this result in [3] (which, however, does not apply to equations with discrete delays).

Consequently, the delayed equation (1) with the unimodal birth function $g$ which does not satisfy the sub-tangency assumption (2) is both an interesting and a challenging object to study. Even such starting point for the research as the question about connectedness of the set $C(h)$ for $h > 0$ should still be answered. In view of our previous discussion, we will say that the propagation speed $c$ (and the associated traveling wave) is critical if it belongs to the boundary of the set $C(h)$. It is natural to expect that the critical wavefronts correspond to the key transition regimes in the model, and if the set $C(h)$ is not connected, equation (1) can have multiple special modes of propagation. The problem concerning the uniqueness of the critical semi-wavefronts for equation (1) seems to be very difficult in both monostable and bistable cases, cf. [2, 3], however, one can also expect that it can be solved at least in the case of small delays (in the spirit of the proverb “small delays are harmless”). In this paper, we are presenting the first result in this direction. In fact, it is the optimal one whenever we are concerned with the monotone wavefronts:

\(^1\)On the other hand, (2) is essential for the analysis of (1), allowing to invoke positivity arguments based on non-negativity of the function $g'(0)x - g(x)$. 
Theorem 1.1. Suppose that $g$ satisfies (U), \( \min_{u \in [0, \kappa]} g'(u) = g'(\kappa) < 0 \), and
\[
|g'(u) - g'(0)| \leq Au^\gamma, \quad u \in [0, \delta],
\] (3) for some \( A > 0, \delta \in (0, \theta), \gamma \in (0, 1] \). Define \( h_* > 0 \) as the unique real solution of the equation \( 1 = |g'(\kappa)| h e^{h+1} \). Then for each fixed \( h \in [0, h_*] \) there is a positive number \( c_* = c_*(h) \) (called the minimal speed of propagation) such that equation (1):

(a) possesses a unique monotone wavefront \( u(t, x) = \phi_c(x + ct) \) for every \( c \geq c_* \);

(b) does not have any semi-wavefront propagating at the velocity \( c < c_* \).

The above result concerning the existence of \( c_*(h) \) satisfying both requirements (a) and (b) fails to hold if \( h > h_* \).

Theorem 1.1 shows that the inclusion of ‘small’ delays into model (1) does not change the usual structure of an unbounded interval \([c_*, +\infty)\) of the set \( C \) of all admissible speeds for semi-wavefronts. Importantly, the above result presents simple formula for the exact upper bound \( h_* \) for the size of the ‘small’ delay (observe that \( h_* \to +\infty \) if \( g'(\kappa) \to 0^- \)). The existing literature on the subject presents various perturbation techniques to treat the case of small delays. Specifically, here we would like to mention the Wu and Zou method from [21] and the Ou and Wu approach in [14]. The aforementioned works show that the existence of the wavefront for non-delayed equation (1) propagating at speed \( c > c_*(0) \) implies, under rather weak shape conditions on \( g \), the existence of some positive \( h_0(c) \) such that this wavefront persists for all \( h \in [0, h_0(c)] \). Nevertheless, these results do not allow to establish the connectedness of the set \( C(h) \) even for small \( h \). To have a better idea of what \( C(h) \) may look like, we study in this paper an explicitly solvable ‘toy’ model with piece-wise linear (but discontinuous) unimodal birth function \( g \) shown on Figure 1.

Note that condition (2) in this case reads as \( k \geq 3 \).

Figure 1. Toy model: piece-wise linear birth function \( g \).

As several previous works show (e.g. see [7, 13, 20]), such a kind of nonlinear birth functions \( g \) allows to detect all essential geometric features of traveling waves that appear in the unimodal models. In Section 5, we show that for each \( k \in (1, 3) \) all traveling fronts to equation (1) considered with \( g \) given on Figure 1 can be determined in an explicit way. This leads to the following conclusion confirming all results of Theorem 1.1 (as well as of Theorem 1.3 below) and suggesting that the simple topological structure \( C(h) = [c_*(h), +\infty) \) of the set of all admissible semi-wavefront speeds could also hold for unimodal equation (1).
Let \( c = c_\#(h) \) be the unique positive number for which the characteristic equation
\[
\chi(z, c) := z^2 - cz - 1 + g'(0)e^{-zch} = 0.
\] has a double positive root (so that \( c > c_\# \) implies that (4) has exactly two real solutions \( 0 < \lambda_2 < \lambda_1, \lambda_j = \lambda_j(c) \).

**Theorem 1.2.** Suppose that \( k \in (1, 3) \) and take \( g \) as on Figure 1. Then there exists a continuous decreasing function \( c_* : [0, +\infty) \rightarrow (0, +\infty) \) such that \( c_*(h) \) is the minimal speed of propagation in the sense that equation (1) has a wavefront solution propagating with the speed \( c \) if and only if \( c \geq c_*(h) \). Furthermore, if \( k \in (1, 5/3) \), then there is some maximal \( h_p = h_p(k) \in (0, +\infty) \) such that \( c_*(h) > c_\#(h) \) for all \( h \in [0, h_p) \). Finally, each wavefront is unique (up to translation) and for each fixed \( h \geq 0 \) equation (1) has at most one non-linearly determined wavefront (i.e. wavefront with asymptotic representation (6) given in Theorem 1.3 below).

Some numerical and geometrical evidences suggest that for \( k \) close to \( 5/3 \), \( h_p(k) \) is finite, but if \( k \) goes closer to 1 then \( h_p(k) = +\infty \) (compare the cases \( k = 1.5 \) and \( k = 1.2 \) in Section 5). In other words, if ‘non-subtangency’ of the birth function \( g(u) \) at 0 is relatively strong then, independently on the size of the delay, all minimal wavefronts are non-linearly determined. However, when ‘non-subtangency’ of \( g(u) \) at 0 is relatively weak (in our toy model this surely happens if \( k \geq 1.5 \)), then all minimal wavefronts become linearly determined once the delay surpasses the critical value \( h_p \). This change is important for the dynamics of (1) because the non-linearly determined wavefronts have better stability properties. See also [6] for other arguments.

Theorems 1.1 and 1.2 raise the question of whether the minimal speed \( c_*(h) \) can be calculated explicitly from (1) for each fixed \( g \) and given \( h \geq 0 \). It is well known that if we assume (2) then \( c_*(h) = c_\#(h) \). Without (2), the computation of \( c_*(h) \) can be regarded as a very difficult task even for non-delayed models [5, 9, 23]. It is known that \( c_*(h) \in [c_\#(h), c^*(h)] \), where \( c = c^*(h) \) is the unique positive number for which the equation
\[
z^2 - cz - 1 + g'(0)e^{-zch} = 0, \quad g'_+ := \sup_{x \geq 0} g(x)/x,
\] has a double positive root. As we can see, in general, \( c_*(h) \) depends on the whole nonlinearity \( g \) and not only on the value of its derivative at 0. The critical wavefront \( u(t, x) = \phi_+(x + c_*(h)t) \) is called pushed if \( c_*(h) > c_\#(h) \). Previous studies of monotone model (1) showed that pushed wavefronts have better stability properties in comparison with non-critical waves, cf. [22]. Particularly, this is due to the fast exponential decay at \(-\infty\) of the profile \( \phi_+(t) \). Our next result shows that the latter characteristic property of pushed wavefronts is also valid if the delay is relatively small:

**Theorem 1.3.** Assume all the condition of Theorem 1.1 and take some \( c > c_*(h) \) for \( h \in [0, h_\#] \). Then the following asymptotic representation is valid (for an appropriate \( s_0 \) and some \( \sigma > 0 \)):
\[
(\phi_c, \phi'_c)(t + s_0, c) = e^{\lambda_2t}(1, \lambda_2) + O(e^{(\lambda_2+\sigma)t}), \quad t \rightarrow -\infty.
\] If, in addition, \( g \in C^{1, \gamma}[0, \kappa], c = c_*(h) > c_\#(h) \) and \( h \in [0, h_\#] \), then
\[
(\phi_c, \phi'_c)(t + s_0, c) = e^{\lambda_1t}(1, \lambda_1) + O(e^{(\lambda_1+\sigma)t}), \quad t \rightarrow -\infty.
\]
Theorem 1.3 refines the following well-known statement concerning the asymptotic representations of wavefront’s profile at $-\infty$:

**Proposition 1.** Suppose that $g$ satisfies (U) and (3) and let $u = \phi(x + ct)$ be a semi-wavefront for (1). If, in addition, $c > c_\#(h)$, then the following asymptotic representation is valid (for an appropriate $s_0$, $j \in \{1, 2\}$ and some $\sigma > 0$):

$$(\phi, \phi')(t + s_0, c) = e^{\lambda_j t}(1, \lambda_j) + O(e^{(\lambda_j + \sigma)t}), \quad t \to -\infty. \quad (7)$$

If $c = c_\#(h)$, then there are some nonnegative $A, B$ such that $A + B > 0$ and

$$(\phi, \phi')(t + s_0, c) = (-At + B)e^{\lambda_j t}(1, \lambda_j) + O(e^{\lambda_j t}), \quad t \to -\infty. \quad (8)$$

The structure of the remainder of this paper is as follows. Section 2 introduces suitable definitions of fundamental solutions for two linear integro-differential operators and then analyzes their properties and some relations existing between them. The analysis in this section offers an additional insight into the properties of fundamental solutions earlier established in [16] by applying alternative, more technical approach. The use of fundamental solutions as well as ‘base functions’ from [17] are the key elements in the proof of Theorem 1.1 given in Section 3. Next, Section 4 contains short proofs of Proposition 1 and Theorem 1.3: we note that the proof of the latter theorem was substantially shortened due to studies realized in [10] (compare with the proof of Theorem 1.4 in [17]). Such a simplification, however, required additional $C^{1,2}$-smoothness property of $g(u)$. Finally, in Section 5 we present detailed analytical and numerical studies of a ‘toy’ model and prove Theorem 1.2.

In Appendix, where the characteristic function of the variational equation at the positive steady state is analyzed, we further improve some results established in [10, Lemma 1.1]. The obtained improvement is used in the next section.

2. **A convolution factorization of the fundamental solution.** Suppose that $u(x, t) = \phi(x + ct)$ is a wavefront solution of equation (1). Then its profile $\phi$ satisfies the boundary value problem

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = \kappa. \quad (9)$$

By linearizing the above differential equation around the positive equilibrium $\kappa$, we obtain

$$y''(t) - cy'(t) - y(t) + g'(\kappa)y(t - ch) = 0.$$

Considering exponential solutions $y(t) = e^{zt}$ of the latter equation, we find that $z$ should satisfy $\chi_\kappa(z) = 0$, where the characteristic function $\chi_\kappa$ is given by

$$\chi_\kappa(z) = z^2 - cz - 1 + g'(\kappa)e^{-zch}.$$

We will analyze the situation when $\chi_\kappa$ has exactly three real zeros, one positive and two negative (counting multiplicity), $\mu_3 \leq \mu_2 < 0 < \mu_1$. In such a case, every complex zero $\mu_j$ of $\chi_\kappa$ is simple [20, Lemma A.2] and has its real part $\Re \mu_j < \mu_2$ [10, Lemma 1.1]. Importantly, the latter estimate can be improved: in Appendix, we show that actually $\Re \mu_j < \mu_3$ for each complex zero $\mu_j$ of $\chi_\kappa$.

The set $D_\kappa$ of all points $(h, c) \in \mathbb{R}^2_+$ for which $\chi_\kappa$ has three real zeros was described in [10, Lemma 1.1 and Theorem 2.3]:

$$D_\kappa = [0, h_*] \times \mathbb{R}_+ \cup \{(h, c) \in \mathbb{R}^2_+ : c \leq c_\kappa(h), \ h > h_*\},$$
where \( h_* > 0 \) was defined in Theorem 1.1 and \( c_{\kappa} : (h_*, +\infty) \to (0, +\infty), c_{\kappa}(h_*) = +\infty, c_{\kappa}(+\infty) = 0 \), is a decreasing smooth function implicitly defined by

\[
\frac{2 + \sqrt{c^4 h^2 + 4 c^2 h^2 + 4}}{c c^2 h^2|y'(\kappa)|} = \exp \left( \frac{\sqrt{c^4 h^2 + 4 c^2 h^2 + 4 - c^2 h}}{2} \right), \quad h > h_*.
\] (10)

For each \((h, c) \in D_{\kappa}\) we introduce the following integro-differential operators:

\[
(D_1 y)(t) = y'(t) - cy'(t) - y(t) + g'(\kappa) y(t - ch),
\]

\[
(D_2 y)(t) = y'(t) - (c - \mu_2) y(t) - g'(\kappa) e^{-\theta_{ch}^\mu_2} \int_{-ch}^0 e^{-\mu_2 y}(t + s) ds.
\]

**Lemma 2.1.** The operators \( D_1 \) and \( D_2 \) commute and \( D y = D_1 D_2 y = D_2 D_1 y \) for every \( y \in C^2(\mathbb{R}, \mathbb{R}) \).

**Proof.** By a straightforward computation and integration by parts we obtain

\[
D_1 D_2 y = y''(t) - (c - \mu_2) y'(t) - g'(\kappa) e^{-\theta_{ch}^\mu_2} \int_{-ch}^0 e^{-\mu_2 y}(t + s) ds
\]

\[
- \mu_2 \left( y'(t) - (c - \mu_2) y(t) - g'(\kappa) e^{-\theta_{ch}^\mu_2} \int_{-ch}^0 e^{-\mu_2 y}(t + s) ds \right) =
\]

\[
y'' - cy' + (\mu_2 c - \mu_2^2 - g'(\kappa) e^{-\theta_{ch}^\mu_2}) y(t) + g'(\kappa) y(t - ch) = D_y.
\]

Similarly,

\[
D_2 D_1 y = D_2 (y' - \mu_2 y) = (y'' - \mu_2 y'') - (c - \mu_2) (y' - \mu_2 y)
\]

\[
- g'(\kappa) e^{-\theta_{ch}^\mu_2} \int_{-ch}^0 e^{-\mu_2 y}(t + s) ds = D_y.
\]

Again, to prove the latter equality, we have to integrate by parts. \( \square \)

**Definition 2.2.** Consider \((h, c) \in D_{\kappa}\). We define the fundamental solution \( \psi(t) \) of equation \( D_2 y = \delta(t) \) where \( \delta(t) \) is the Dirac \( \delta \)-function in the following way:

\[
\psi(t) = \frac{-\mu_1 - \mu_2}{\chi_{\kappa}'(\mu_1)} e^{\mu_{1 t}}, \quad t < 0,
\]

and if \( t > 0 \) then \( \psi(t) \) coincides with solution of the functional differential equation

\[(D_2 y)(t) = 0 \text{ subject to the initial conditions}
\]

\[
y(0) = \psi(0^-) + 1 = 1 - \frac{\mu_1 - \mu_2}{\chi_{\kappa}'(\mu_1)}, \quad y(s) = \psi(s), \quad s \in [-ch, 0).
\] (11)

In this way, \( D_2 \psi)(t) = 0 \) for all \( t \neq 0 \) and \( \psi(0) - \psi(0^-) = 1. \)

**Lemma 2.3.** The fundamental solution \( \psi(t) \) is negative: \( \psi(t) < 0 \) for all \( t \in \mathbb{R} \) and exponentially decaying at \( \pm \infty. \)

**Proof.** Clearly, \( \chi_{\kappa}'(\mu_1) > 0 \) and therefore \( \psi(t) < 0 \) for all \( t < 0. \) Next, since

\[
0 = \chi_{\kappa}(\mu_1) - \chi_{\kappa}(\mu_2) = (\mu_1 + \mu_2 - c)(\mu_1 - \mu_2) - \theta e^{-\theta_{ch}} (\mu_1 - \mu_2), \quad \theta \in (\mu_2, \mu_1),
\]

we find that

\[
\mu_1 + \mu_2 - c = \theta e^{-\theta_{ch}} < \theta e^{-\mu_{1ch}},
\]

and therefore \( \psi(0) < 0 \) because of

\[
\chi_{\kappa}'(\mu_1) \psi(0) = \chi_{\kappa}'(\mu_1) - (\mu_1 - \mu_2) = \mu_1 + \mu_2 - c - \theta e^{-\theta_{ch}} < 0.
\]
We claim that \( \psi(t) < 0 \) for all \( t > 0 \).

**Step 1.** First, assuming that \( c < c_\kappa(h) \), we find an appropriate asymptotic representation of \( \psi \) at \( +\infty \). As a solution of linear functional differential equation, \( \psi(t) \) has at most exponential growth at \( +\infty \) (see [8, Section 1.3]) and therefore we can apply the Laplace transform method to

\[
\psi'(t) - (c - \mu_2)\psi(t) - g'(\kappa)e^{-ch\mu_2} \int_{-c}^{0} e^{-\mu_2 s} \psi(t + s) ds = 0,
\]

taking into account conditions (11). Let \( \Psi(z) \) denote the Laplace transform of \( \psi(t) \). After some easy computations we get

\[
0 = z\Psi(z) - \psi(0) - (c - \mu_2)\Psi(z) - \frac{g'(\kappa)(e^{-zch} - e^{-\mu_2 ch})}{\mu_2 - z} \Psi(z) + A(z)
\]

where entire function \( A(z) \) is given by

\[
A(z) = \frac{g'(\kappa)e^{-ch\mu_2}(\mu_1 - \mu_2)}{\chi'_{\kappa}(\mu_1)} \int_{-c}^{0} e^{-(\mu_2 - z)v} dv \int_{0}^{v} e^{(\mu_1 - z)s} ds.
\]

The following properties of \( A(z) \) can be easily checked:

\[
A(\mu_1) = \psi(0), \quad A(\mu_2) = A(\mu_3) = \psi(0) - 1.
\]

Consequently, for each pair \((h, c) \in \mathbb{R}^2_+\) such that \( c < c_\kappa(h) \), the function

\[
\Psi(z) = \frac{z - \mu_1}{\chi_{\kappa}(z)} \left( \psi(0) - A(z) \right),
\]

is meromorphic on \( \mathbb{C} \) and analytic on the half-plane \( \Re z > \mu_3 \). Thus the function

\[
\Psi_1(z) = \Psi(z) - \frac{\mu_3 - \mu_2}{\chi'_{\mu_3}(z)}
\]

is analytic on the half-plane \( \Re z > \mu_3 - \delta \) for sufficiently small \( \delta > 0 \). Observe that the fraction in the above representation corresponds to the Laplace transform of the negative eigenfunction

\[
e_3(t) = \frac{\mu_3 - \mu_2}{\chi'_{\mu_3}} e^{\mu_3 t}
\]

for the operators \( D \) and \( D_2 \).

Next, \( \psi(t) - e_3(t) \) is \( C^2 \)-smooth for \( t > 2ch \) and we find that \( (D(\psi - e_3))(t) = (D_1D_2(\psi - e_3))(t) = 0 \) for all \( t > 2ch \). Taking into account the analyticity of \( \Psi_1(z) \) on the half-plane \( \Re z > \mu_3 - \delta \) for sufficiently small \( \delta > 0 \), in view of [12, Proposition 7.2], we find that

\[
\psi(t) - e_3(t) = O(e^{(\mu_3 - \delta)t}), \quad t \to +\infty.
\]

This means that

\[
\psi(t) = \frac{\mu_3 - \mu_2}{\chi'_{\mu_3}} e^{\mu_3 t} + O(e^{(\mu_3 - \delta)t}) < 0, \quad t \to +\infty,
\]

so that \( \psi(t) < 0 \) for all sufficiently large \( t > 0 \).

Suppose now that \( \psi(t_0) \geq 0 \) for some \( t_0 > 0 \). Consider the family of functions

\[
\psi(t, p) = \psi(t) + pe_3(t), \quad p \geq 0.
\]
Since \( c_3(t) < 0 \), \( t \in \mathbb{R} \), there exists the smallest \( p_0 \geq 0 \) such that \( \psi(t, p_0) \leq 0 \) for all \( t \geq 0 \). Then \( \psi(t_1, p_0) = \psi'(t_1, p_0) = 0 \) for some leftmost \( t_1 > 0 \). Clearly, \( D_2 \psi(t, p_0) = 0 \) for all \( t > 0 \). In particular,

\[
0 = D_2 \psi(t_1, p_0) = \psi'(t_1, p_0) - (c - \mu_2) \psi(t_1, p_0) - g'(\kappa) e^{-\kappa \mu_2} \int_{-\infty}^{0} e^{-\mu_2 s} \psi(t_1 + s, p_0) ds
\]

\[
= -g'(\kappa) e^{-\kappa \mu_2} \int_{-\infty}^{0} e^{-\mu_2 s} \psi(t_1 + s, p_0) ds < 0,
\]

a contradiction proving that actually \( \psi(t) < 0 \) for all \( t \geq 0 \).

**Step 2.** Now, consider the case when \( c = c_\kappa(\bar{h}) \) and take an increasing sequence of positive \( c_j \) converging to \( c \). We will use the notation \( \psi(t, h, c) \) to show dependence of the fundamental solution of equation \( D_2 y = \delta(t) \) on parameters \( h, c \). In view of continuous dependence of solutions of the functional differential equation \( D_2 y = 0 \) on parameters and initial data, we obtain that \( \psi(t, h, c_j) \) converges to \( \psi(t, h, c) \) uniformly on compact subsets of \([0, +\infty]\). Consequently, by Step 1, we conclude that \( \psi(t, h, c) \leq 0 \) for all \( t \geq 0 \). Suppose that \( \psi(t_1, \bar{h}, c) = 0 \) for some leftmost \( t_1 \). Then \( t_1 > 0 \) and \( \psi'(t_1, \bar{h}, c) = 0 \). However, arguing as at the end of Step 1, we get immediately the contradiction \( 0 = D_2 \psi(t_1, \bar{h}, c) < 0 \). Hence, \( \psi(t, h, c) < 0 \) for all \( t \geq 0 \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded continuous function, then the convolution

\[
y(t) = \int_{\mathbb{R}} \psi(t - s) f(s) ds = \frac{\mu_1 - \mu_2}{\lambda_k(\mu_1)} \int_{t}^{+\infty} e^{\mu_1 (t-s)} f(s) ds + \int_{-\infty}^{t} \psi(t - s) f(s) ds,
\]

is bounded and continuously differentiable function satisfying the functional differential equation \( D_2 y = f \). We can use this fact to solve the second-order equation \( D_2 y = f \). Indeed, the equation \( D_1 (D_2 y) = f \) has a unique bounded solution

\[
(D_2 y)(t) = \int_{-\infty}^{t} e^{\mu_2 (t-s)} f(s) ds = (\theta * f)(t),
\]

where

\[
\theta(t) = e^{\mu_2 t}, \quad t \geq 0, \quad \theta(t) = 0, \quad t < 0,
\]

is the fundamental solution of \( D_1 y = \delta(t) \). Consequently, the equation \( D_1 (D_2 y) = f \) has a unique bounded solution

\[
y = \psi * (\theta * f) = (\psi * \theta) * f.
\]

The function \( N = \psi * \theta \) is called the fundamental solution of the equation \( D_2 y = \delta(t) \), clearly, \( N(t) < 0 \) for all \( t \in \mathbb{R} \), this property was earlier established in [16] by using an alternative (and more technical) approach.

3. **Proof of Theorem 1.1.** By [16, Theorem 8], for \( h \in [0, h_*] \) each traveling wave \( u(t, x) = \phi(x + ct) \) has strictly increasing profile \( \phi(t) \), moreover, \( \phi'(t) > 0 \) for all \( t \in \mathbb{R} \) (see also [19, Lemma 6]). The same theorem in [16] assures that \( \phi(t) \) is unique up to translation. Next, if \( h > h_* \) then there exists \( c > c^*(h) \) such that \( (h, c) \notin D_0 \). Due to Theorem 1.7 in [10], traveling wave propagating with the speed \( \bar{c} \) is not monotone. This establishes the optimal nature of the bound \( h_* \). In this way, we have only to prove that for each fixed \( h \in [0, h_*] \), the set of all possible wave speeds is a connected interval of the form \([c_\kappa(h), +\infty)\). The next assertion provides one of the key arguments for the proof of Theorem 1.1:
Lemma 3.1. Suppose that $g$ satisfies (3) with (U) and $(h, c) \in \mathcal{D}_\kappa$ (so that $\phi'(t) > 0$) be such that

\[ 1 + hg'(\kappa)e^{-\mu_2(c)ch} > 0. \] (12)

Then

\[ \phi'(t) + hg'(\kappa)\phi'(t - \bar{c}h) > 0, \quad t \in \mathbb{R} \]

for every wavefront $\phi(x + ct)$ of equation (1) which propagates with the speed $c$ and each $\bar{c}$ for which $1 + hg'(\kappa)e^{-\mu_2(c)ch} > 0$.

Proof. Set

\[ \mathcal{D}_* = \{(h, c) \in \mathbb{R}_+^2 : 1 + hg'(\kappa)e^{-\mu_2(c)ch} > 0\}. \]

Then the boundary of $\mathcal{D}_*$ consists of the union of the half-line $(0, c), c \geq 0$, with the interval $(0, h), 0 \leq h < -1/g'(\kappa)$, and with the graph $\Gamma_1$ defined by the system of equations

\[ 1 + hg'(\kappa)e^{-\mu_2ch} = 0, \quad \mu^2 - c\mu - 1 + g'(\kappa)e^{-\mu_2ch} = 0. \] (13)

On the other hand, the graph $\Gamma_2$ of $c = c_\kappa(h)$ is defined by the system

\[ 2\mu - c - chg'(\kappa)e^{-\mu_2ch} = 0, \quad \mu^2 - c\mu - 1 + g'(\kappa)e^{-\mu_2ch} = 0. \]

If $\Gamma_1$ and $\Gamma_2$ intersect at some point $(\bar{h}, \bar{c})$ then necessarily $\mu = 0$, a contradiction. Thus we conclude that $\Gamma_1$ belongs to the interior of the set $\mathcal{D}_\kappa$. In fact, solving (13), we find that

\[ \mu = \frac{c - \sqrt{c^2 + 4 + 4/h}}{2} = \frac{1}{ch} \ln(h|g'(\kappa)|), \]

from which we obtain the equation $c = c(h)$ for the curved part of the boundary of $\mathcal{D}_*$:

\[ c(h) = \frac{-\ln(h|g'(\kappa)|)}{\sqrt{h(1 + h + \ln(h|g'(\kappa)|))}}, \quad h < h_* := 1/|g'(\kappa)|. \] (14)

Clearly, $c = c(h)$ is a strictly decreasing function with $c(h^+_*) = +\infty, \ c(h^*) = 0$. Therefore condition (12) is automatically satisfied for all $h \in [0, h_*]$.

Now, if $u(t, x) = \phi(x + ct)$ is a monotone wavefront to (1), then $\phi$ satisfies

\[ \phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0. \]

Set $z(t) = \phi'(t) > 0$. By differentiating the latter equation, we find that

\[ z''(t) - c z'(t) - z(t) + g'(\phi(t - ch))z(t - ch) = 0, \]

or, equivalently,

\[ (D_2D_1)z(t) = z''(t) - c z'(t) - z(t) + g'(\phi(t - ch))z(t - ch) = b(t), \]

where

\[ b(t) = a(t)z(t - ch) \leq 0, \quad a(t) = g'(\kappa) - g'(\phi(t - ch)) \leq 0, \quad t \in \mathbb{R}. \]

Consequently,

\[ (D_1z)(t) = z'(t) - \mu_2 z(t) = (\psi * b)(t) \geq 0, \]

so that $(z(t)e^{-\mu_2t})' \geq 0$ and

\[ z(t - ch)e^{-\mu_2(t - \bar{c}h)} = \phi'(t - \bar{c}h)e^{-\mu_2(t - \bar{c}h)} \leq \phi'(t)e^{-\mu_2t} = z(t)e^{-\mu_2t}, \quad t \in \mathbb{R}. \]

Hence, $\phi'(t - \bar{c}h)e^{\mu_2\bar{c}h} \leq \phi'(t)$ and

\[ \phi'(t) + hg'(\kappa)\phi'(t - \bar{c}h) \geq \phi'(t - \bar{c}h)e^{\mu_2\bar{c}h} + hg'(\kappa)\phi'(t - \bar{c}h) = \phi'(t - \bar{c}h)(e^{\mu_2\bar{c}h} + hg'(\kappa)) > 0, \quad t \in \mathbb{R}. \]

This completes the proof of Lemma 3.1. \qed
Corollary 1. Let wavefront $u = \phi(x + c_0 t)$ be such that $c_0, h$ satisfy (12). Then

$$\phi'(t) + h\phi'(\kappa)\phi'(t - \dot{c} h) > 0, \quad t \in \mathbb{R},$$

whenever $\dot{c} \in [c_0, c_0 + \nu]$ and $\nu > 0$ is sufficiently small number.

Now, fix some $h \in (0, h_*)$ and consider

$$\mathcal{C}(h) := \{c \geq 0 : \text{equation (1) has a wavefront propagating at the velocity } c\}.$$

It is known from [18], that $\mathcal{C}(h)$ contains the subinterval $[c^*(h), +\infty)$ while

$$c_* := \inf \mathcal{C}(h) > c_0 > 0.$$  

It is easy to see that $\mathcal{C}(h)$ is closed so that $c_* \in \mathcal{C}(h)$. Assume that $c_0 \in \mathcal{C}(h) \cap [c^*(h), c_*)$ and let $u(t, x) = \phi(x + c_0 t)$ be a wavefront solution,

$$\phi''(t) - c_0 \phi'(t) - \phi(t) + g(\phi(t - c_0 h)) = 0.$$  

Then take $\nu$ as in Corollary 1 and let $c' \in [c^*(h), c_*)$, $c' - c_0 \in (0, \nu)$ be small enough to satisfy $(1 + \gamma)\lambda_2(c') > \lambda_2(c_0)$. Note that $\lambda_2(c)$ is a decreasing function of $c$. To simplify the notation, we will write $\lambda_2' := \lambda_2(c')$, $\lambda_2 := \lambda_2(c_0)$. For the reader’s convenience, the proof of of Theorem 1.1 is divided into several steps.

Step I (Properties of an auxiliary function $\phi_\sigma$). Set $\phi_\sigma(t) := \sigma \phi(t)$, where $\sigma > 1$ is close to 1. We have

$$E(t, \sigma) := \phi''_\sigma(t) - c'_\sigma \phi'_\sigma(t) - \phi_\sigma(t) + g(\phi_\sigma(t - c_\sigma h)) =$$

$$\phi''_\sigma(t) - c_0 \phi'_\sigma(t) - \phi_\sigma(t) + g(\phi(t - c_0 h)) + [(c_0 - c') \phi'_\sigma(t) + g(\phi_\sigma(t - c'_\sigma h)) - g(\phi(t - c_0 h))] =$$

$$(c_0 - c') \phi'_\sigma(t) + g(\phi_\sigma(t - c'_\sigma h)) - g(\phi(t - c_0 h)).$$

By our assumptions, $|g(x) - g'(0) x| \leq A x^{1+\gamma}$, $x \in [0, \delta]$. Take some $t_*$ such that $\phi(t_* - c_0 h) < \delta/2$. Then for all $1 < \sigma \leq 2$ and $t \leq t_*$,

$$g(\phi_\sigma(t - c'_\sigma h)) - g(\phi(t - c_0 h)) \leq 6A(\phi(t - c_0 h))^{1+\gamma}.$$  

On the other hand, from Proposition 1 we know that

$$(c_0 - c') \phi'_\sigma(t) = (c_0 - c') \zeta \phi(t)(1 + o(1)) = (c_0 - c') \zeta \phi(1 + o(1))$$

for $\zeta \in \{\lambda_1(c_0), \lambda_2(c_0)\}$ and $t \to -\infty$. As a consequence, there exists $T_1 \leq t_*$ (which does not depend on $\sigma$) such that, for all $\sigma \in [1, 2]$,

$$E(t, \sigma) < 0, \quad t \leq T_1.$$  

Due to assumption (U), the function $G(u) := g(u)/u$ has negative derivative on some interval $\mathcal{O} = (\theta', +\infty) \supset [\theta, \kappa]$. Thus

$$G(u) - G(v) = G'(w)(u - v) < 0, \quad u > w > v \geq \theta'.$$

Observe that $\theta'$ does not depend on $\sigma$. Since from the very beginning we can fix $c'$ sufficiently close to $c_0$ to have $\phi(T_2 - c'_\sigma h), \phi(T_2 - c_\sigma h) \in (\theta', \theta)$, for some $T_2$ (which depends only on $\phi, c_0$), we obtain that for $t \geq T_2$ and every $\sigma > 1$, it holds

$$E(t, \sigma) = (c_0 - c') \phi'_\sigma(t) + g(\phi_\sigma(t - c'_\sigma h)) - g(\phi(t - c_\sigma h)) =$$

$$(c_0 - c') \phi'_\sigma(t) + (\phi(t - c'_\sigma h)) + g(\phi(t - c_\sigma h)) + g(\phi(t - c'_\sigma h)) - g(\phi(t - c_\sigma h)) \leq$$

$$(c_0 - c') \phi'_\sigma(t) + h \phi'(\kappa) \phi'(t - c'_\sigma h) =$$

$$(c_0 - c') \phi'(t) + h \phi'(\kappa) \phi'(t - c'_\sigma h) < 0, \quad \text{for some } \dot{c}' \in (c_0, c').$$
Finally, since \( \phi_{\sigma}(t-c'h), \phi_{\sigma}(t-c_0h) \in (0, \theta) \), for all small \( \sigma \) and \( t \leq T_2 \), we conclude that

\[
E(t, \sigma) = (c_0 - c')\phi_{\sigma}'(t) + g(\phi_{\sigma}(t-c'h)) - \sigma g(\phi(t-c_0h)) < \\
(c_0 - c')\phi_{\sigma}'(t) + g(\phi_{\sigma}(t-c_0h)) - \sigma g(\phi(t-c_0h)) < 0
\]

uniformly on \([T_1, T_2]\) for all small \( \sigma - 1 > 0 \).

All the above shows that \( E(t, \sigma) < 0 \) for all \( t \in \mathbb{R} \) and each \( \sigma > 1 \) sufficiently close to 1.

**Step II (Construction of an upper solution).** By Step I, we can choose \( c', \sigma > 1 \) in such a way that \( E(t, \sigma) < 0, t \in \mathbb{R} \). For \( a := b^2, b \in (0, 1], \), set \( \phi_b(t) := \phi_{\sigma}(t) + ae^{\lambda_2 t} + be^{\lambda_2 t} \), where \( \lambda_2 = \lambda_2(c') \), \( \lambda_2 = \lambda_2(c_0) \). Let \( T_3 = T_3(b) \) be that unique point where \( \phi_b(T_3(b)) = \kappa \), then \( T_3(b) \leq T_3(0) \) for all \( b \geq 0 \). It is clear that \( \phi_b'(T_3) > 0 \) and that \( T_3(b) \to T_3(0) \) as \( b \to 0 \). Next, we find that

\[
E_+(t, b) := \phi_b'(t) - c'\phi_b'(t) - \phi_b(t) + g(\phi_b(t-c'h)) = E(t, \sigma) + b\chi(\lambda_2, c')e^{\lambda_2 t} + \\
g(\phi_{\sigma}(t-c'h) + ae^{\lambda_2(t-c'h)} + be^{\lambda_2(t-c'h)}) - g(\phi_{\sigma}(t-c'h)) - g'(0)(ae^{\lambda_2(t-c'h)} + be^{\lambda_2(t-c'h)})
\]

\[
\leq E(t, \sigma) + b\chi(\lambda_2, c')e^{\lambda_2 t} + \\
A(\lambda_2, c') + 3ae^{\lambda_2(t-c'h)}(1 + 1) \phi_{\sigma}'(t-c'h) + 2b^2 e^{\lambda_2 t} |t-c'h|
\]

\[
\leq E(t, \sigma) + b\chi(\lambda_2, c')e^{\lambda_2 t} + \\
A(\lambda_2, c') + 3ae^{\lambda_2(t-c'h)}(1 + 1) \phi_{\sigma}'(t-c'h) + 2b^2 e^{\lambda_2 t} |t-c'h|
\]

for some positive \( \nu_1, C_j \) and negative \( T_4 \) (which does not depend on \( b \)). Since \( \chi(\lambda_2, c') < 0 \), we may choose \( T_4 \) is such a way that \( E_+(t, b) < 0 \) for all \( t \leq T_4, b \in (0, 1] \). On the other hand, we know that, uniformly on each compact interval, \( E_+(t, b) \to E(t, \sigma), b \to 0^+ \). Therefore \( E_+(t, b) < 0 \) for all \( t \leq T_3(0^+) + 1 \) for all sufficiently small \( b \).

Consider now \( C^\infty \)-smooth non-increasing function \( \psi(t) \) such that \( \psi(t) = 1 \) for all \( t \leq T_3(0) + 1 \) and \( \psi(t) = 0 \) for all \( t \geq T_3(0) + 1 + c'h \). We define an upper solution \( \phi_+ \) by

\[
\phi_+(t) := \phi_{\sigma}(t) + (ae^{\lambda_2 t} + be^{\lambda_2 t})\psi(t).
\]

Observe that, for all small \( b \), the function \( \phi_+(t) \) is increasing and

\[
\phi_+(t) - c'\phi_+(t) + g(\phi_+(t-c'h)) = \begin{cases} 
E(t, \sigma) < 0, & \text{for all } t \geq T_3(0) + 1 + c'h; \\
E_+(t, b) < 0, & \text{for } t \leq T_3(0) + 1.
\end{cases}
\]

Since uniformly on \([T_3(0) + 1, T_3(0) + 1 + c'h]\),

\[
\lim_{b \to 0^+} (\phi_+(t) - c'\phi_+(t) + g(\phi_+(t-c'h))) = E(t, \sigma) < 0,
\]

we conclude that, for all small \( b > 0 \),

\[
\phi_+(t) - c'\phi_+(t) + g(\phi_+(t-c'h)) < 0, \quad t \in \mathbb{R}.
\]

**Step III (Construction of a lower solution).** Consider the following concave monotone linear rational function

\[
p(x) := \frac{g(x)}{1 + Bx} \leq \frac{g(0)}{x}, \quad x \geq 0, \quad B := \frac{g(0)}{\theta} - 1, \quad p(0) = 0, \quad p(\theta) = \theta,
\]

where \( \theta \) is a large positive number.
and set \( g_- (x) := \min \{ g(x), p(x) \} \). It is clear that \( g_- \) is continuous and increasing on \([0, \theta]\) and that
\[
g_-'(0) = g'(0), \quad g_-(0) = 0, \quad g_-(\theta) = \theta, \quad g_-(x) \leq g'(0)x, \quad x \geq 0.
\]
Moreover, in some right neighborhood of 0,
\[
|g_-(u)/u - g'(0)| \leq A'u^\gamma, \quad u \in (0, \delta'],
\]
for some \( A' > 0, \ \delta' > 0 \). As we have mentioned in the introduction, this implies the existence of a monotone positive function \( \phi_- \), \( \phi_-(-\infty) = 0, \ \phi_-(+\infty) = \theta \), satisfying the equation
\[
\phi''(t) - c'\phi'_-(t) - \phi_-(t) + g_-(\phi_-(t - c'h)) = 0.
\]
Due to the property \( g_-(x) \leq g'(0)x, \ \ x \geq 0 \), we also know that
\[
(\phi_-, \phi'_-)(t + t_0, c) = e^{\lambda^2 t}(1, \lambda^2) + O(e^{(\lambda^2 + \sigma)t}), \quad t \to -\infty.
\]
Finally, since \( g_-(x) \leq g(x) \) we obtain that
\[
\phi''(t) - c'\phi'_-(t) - \phi_-(t) + g(\phi_-(t - c'h)) \geq 0.
\]

**Step IV (Iterations).** Comparing asymptotic representations of monotone functions \( \phi_-(t) \) and \( \phi_+(t) \) at \( +\infty \) and \( -\infty \), we find easily that
\[
\phi_-(t + s_1) \leq \phi_+(t), \quad t \in \mathbb{R},
\]
for some appropriate \( s_1 \). Simplifying, we will suppose that \( s_1 = 0 \).

Using the fundamental solution \( N(t), \int_{\mathbb{R}} N(s)ds = 1/(g'(\kappa) - 1) \), defined in Section 2, we can rewrite the profile equation (9) with \( c = c' \) in the following equivalent forms
\[
(D'\phi)(t) = g'(\kappa)\phi(t - c'h) - g(\phi(t - c'h)), \quad \phi(-\infty) = 0, \ \phi(+\infty) = \kappa,
\]
(here we use the notation \( (D' y)(t) := y''(t) - c' y'(t) - y(t) + g'(\kappa)y(t - c'h) \)), and \( \phi(t) = (N \phi)(t), \ \phi(-\infty) = 0, \ \phi(+\infty) = \kappa, \) where
\[
(N \phi)(t) := \int_{\mathbb{R}} N(t - s) (g'(\kappa)\phi(s - c'h) - g(\phi(s - c'h))) ds.
\]
Since \( N(t) < 0, \ t \in \mathbb{R}, \) and, by our assumption, \( \min_{u \in [0, \kappa]} g'(u) = g'(\kappa) < 0 \), the integral operator \( N \) is increasing on \( C(\mathbb{R}, [0, \kappa]) \), i.e.
\[
0 \leq (N \phi)(t) \leq (N \psi)(t) \leq \kappa, \quad \text{whenever } \phi(t) \leq \psi(t), \ t \in \mathbb{R}, \ \phi, \psi \in C(\mathbb{R}, [0, \kappa]).
\]
In addition, the properties of functions \( \phi_-(t) \) and \( \phi_+(t) \) guarantee that
\[
\phi_-(t) \leq (N \phi_-(t)) \leq (N^2 \phi_-(t)) \leq \cdots \leq (N^2 \phi_+(t)) \leq (N \phi_+(t)) \leq \phi_+(t), \quad t \in \mathbb{R}.
\]
By standard arguments (e.g. see [16] for details), the latter implies the existence of a monotone continuous function \( \phi(t) = \lim_{k \to +\infty} (N^k \phi_+)(t) \) such that
\[
(N \phi)(t) = \phi(t), \quad \phi_-(t) \leq \phi(t) \leq \phi_+(t), \quad t \in \mathbb{R}.
\]
This amounts to the existence of a wavefront propagating at velocity \( c' \). Moreover, the latter estimations show that, for some \( s_0 \) and positive \( \delta \),
\[
\phi(t + s_0) = e^{\lambda^2 t} + O(e^{(\lambda^2 + \delta)t}), \quad t \to -\infty.
\]
Finally, to prove that \( \mathcal{C}(\kappa) \) coincides with the interval \( [c_-, \infty) \), let us consider the open set \( \mathcal{O} = [c_-, \infty) \setminus \mathcal{C}(\kappa) \). If \( \mathcal{O} \neq \emptyset \), we take one connected component of \( \mathcal{O} \), say \( (c_0, c_1) \). Since \( c_0 \in \mathcal{C}(\kappa) \), there is some \( c' \in (c_0, c_1) \) such that \( c' \in \mathcal{C}(\kappa) \), in contradiction to the definition of \( \mathcal{O} \). Therefore \( \mathcal{C}(\kappa) = [c_-, \infty) \).
Remark 1. (I) It is easy to see that the result established in this section is slightly stronger that the assertion of Theorem 1.1. Indeed, we have proved that even for \( h > h^* \) sufficiently close to \( h^* \) there is a positive number \( c_* = c_*(h) \) such that equation (1) (a) possesses a unique monotone wavefront \( u(t, x) = \phi_c(x + ct) \) for every \( c \in [c_*, c(h)] \), where \( c(h) \) is given by (14); (b) does not have any semi-wavefront propagating at the velocity \( c < c_* \). It is instructive to note that the main conclusion of Theorem 1.3 in [3] is of the same kind.

(II) Take some monotone wavefront \( \phi(x + ct) \) and consider the expression

\[
F(t, c) := \phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)).
\]

Then \( F(t, c_0) \equiv 0 \),

\[
F_t(t, c_0) = -(\phi'(t) + hg'(\phi(t - c_0h)))\phi'(t - c_0h)),
\]

so that if we want the inequality \( E(t, \sigma) < 0 \) to be satisfied for all small \( c' - c_0 > 0 \), \( \sigma - 1 > 0 \), we need to assure the following property of a wavefront:

\[
\phi'(t) + hg'(\phi(t - c_0h))\phi'(t - c_0h) > 0, \quad t \in \mathbb{R}.
\]

Next, it is well known (e.g. see [10, Lemma 4.3]) that the monotonicity of \( \phi(t) \) and the assumption \( g'(x) \geq g'(\kappa), x \in [0, \kappa] \), imply that

\[
\lim_{t \to +\infty} [\phi'(t - c_0h)/\phi'(t)] = e^{-\mu_2 c_0 h}.
\]

All this shows that condition (12) is nearly optimal for the construction of an upper solution for the perturbed profile equation (with \( c' > c_0 \) close to \( c_0 \)) from \( \phi(t) \).

4. Proofs of Proposition 1 and Theorem 1.3.

Proof of Proposition 1. Due to [19, Lemma 6], we have that \( \phi'(s) > 0 \) for all \( s \) from some infinite interval \( (-\infty, \sigma) \). The exponential decay of \( \phi(t) \) at \( -\infty \) is assured by [1, Lemma 3 (ii)]. Therefore there is \( \delta > 0 \) such that

\[
g(\phi(t - ch)) = [g'(0) + r(t)]\phi(t - ch), \quad \text{where} \quad r(t) := \frac{g(\phi(t - ch))}{\phi(t - ch)} - g'(0) = o(e^{\delta t}).
\]

On the other hand, it is easy to see that the convergence \( \phi(t) \to 0, t \to -\infty \), is not super-exponential, cf. [18, Theorem 5.4 and Remark 5.5]. Now we can proceed as in [18, Remark 5.5] (where [12, Proposition 7.2] should be used) to obtain asymptotic formulas (7), (8).

Proof of Theorem 1.3. It follows from Theorem 1.1 and Remark 1 (I) that we can associate a unique monotone traveling front \( u = \phi_c(x + ct) \) satisfying the normalization condition \( \phi_c(0) = \theta \) to each pair \( (h, c) \in D_* \) such that \( c \geq c_*(h) \). The uniqueness of solutions \( \phi_c(t) \) implies that \( \phi_c(t) \) is a continuous function of \( c, t \). Therefore for every \( c' > c_*(h) \), \( (h, c') \in D_* \), there exists some interval \([c_0, c']\) with \( c_0 > c_*(h) \) such that

\[
1 + hg'(\kappa)e^{-\mu_2 (c_0 - h)} > 0, \quad \text{for all} \quad \bar{c} \in [c_0, c'],
\]

as well as \( \phi_{c_0}(T_2 - c' h), \phi_{c_0}(T_2 - c_0 h) \in (\theta', \theta), \) for some \( T_2 \). Arguing now as in Section 3, we find that \( \phi_c'(t) \geq \phi_-(t + t_*), \quad t \in \mathbb{R}, \) for some \( t_* \). This yields the representation (15) and, consequently, the required formula (5).

Next, suppose that \( (h, c_0) \in D_* \) and \( c_* = c_*(h) > c_d(h) \). Proposition 1 shows that if (6) does not hold for these parameters then the critical profile \( \phi_{c_0}(t) \) must satisfy the asymptotic relation (5). Assuming this relation to hold, we find from Theorem 1.7 in [10] that equation (1) has a monotone wavefront \( \phi_c(x + ct) \) for
all $c$ from some open interval containing $c_*$ (note that the first part of our proof together with the latter assumption imply that the set $D_{01}$ defined in [10] contains the interval $\{h\} \times [c_*(h), c(h))$). However, this contradicts the minimality property of $c_*(h)$.

5. A toy model: delay turns pushed waves into pulled waves. In this section, aiming at understanding the structure of the set of all admissible speeds $C(h)$ defined in the introduction and, partially, the dynamics of model (1), we consider the following piece-wise linear equation

$$u_t(t, x) - u_{xx}(t, x) + u(t, x) = \begin{cases} ku(t - h, x), & u(t - h, x) \in [0, 1), \\ 4 - u(t - h, x), & u(t - h, x) \geq 1, \end{cases}$$

(16)

where the slope $k$ of the birth function at zero satisfies $k \in (1, 3)$ and $\kappa = 2$ is the positive equilibrium (see Fig. 1, note that condition (2) holds if we choose $k \geq 3$). This kind of equations, sometimes nicknamed ‘toy models’, is frequently used in the theory of traveling waves, cf. [7, 13, 20]. The advantage of (16) is that such a remarkable and difficult to detect solution of the delayed equation such as its positive traveling wave, can be found explicitly by using the Laplace transform.

Now, since (16) has discontinuous right hand-side, we define the positive profile $\phi(t)$, $\phi(-\infty) = 0$, $\phi(+\infty) = 2$, of its wavefront $u(t, x) = \phi(x + ct)$ as $C^1$—smooth and piece-wise analytical solution of the delayed differential equation

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0,$$  

where $g(u) = \begin{cases} ku, & u \in [0, 1), \\ 4 - u, & u \geq 1. \end{cases}$

Lemma 5.1. If $\phi(t)$ is a wavefront profile for (16), then $\phi(t) < 3$ for all $t \in \mathbb{R}$.

Proof. By standard arguments, we obtain the following integral representation

$$\phi(t) = \int_{\mathbb{R}} K_1(t - s)g(\phi(s - ch))ds,$$

where $K_1(s) > 0, s \in \mathbb{R}, \int_{\mathbb{R}} K_1(s)ds = 1$.

Thus the lemma follows from the fact that $g(\phi(s - ch)) \leq 3$ and $g(\phi(-\infty)) = 0$. 

We first consider $h = 0$. It is straightforward to see that the roots of the characteristic equation at the positive equilibrium, $\chi_*(z) = z^2 - cz - 1 - e^{-zh} = 0$, $h = 0$, are given by the formula

$$\mu_{1,2}(c) := \frac{1}{2} \left( c \pm \sqrt{c^2 + 8} \right)$$

and satisfy $\mu_2 < 0 < \mu_1$, while the roots of the characteristic equation at the zero steady state, $\chi(z, c) = z^2 - cz - 1 + ke^{-zh} = 0$, $h = 0$, $c \geq c_# = 2\sqrt{k - 1}$, are

$$\lambda_{1,2}(c) := \frac{1}{2} \left( c \pm \sqrt{c^2 - 4(k - 1)} \right),$$

where $0 < \lambda_2 \leq \lambda_1$. Therefore, if $c \geq c_#(k) = 2\sqrt{k - 1}$, then the point $(0, 0)$ on the phase plane $(\phi, \phi')$ is an unstable node and the point $(2, 0)$ is a saddle point. Clearly, each wave profile $\phi(t)$ corresponds to a unique heteroclinic connection $(\phi, \phi')$ between these equilibria of (17) on the phase plane diagram. Since the stable manifold of the saddle point is given by the equation $\phi' = \mu_2(\phi - 2)$, we obtain easily its value $-\mu_2$ at $\phi = 1$, i.e.

$$\phi'|_{\phi=1} = \frac{1}{2} \left( \sqrt{c^2 + 8} - c \right).$$
Next, in the non-delayed case, the graph \((\phi, \phi')\) of the pushed wave is given by 

\[ \phi' = \lambda_1 \phi \quad \text{for} \quad \phi \leq 1, \] 

see \cite[Section 2.6]{15}. Since, in addition, the profile \(\phi\) is \(C^1\)-continuous, we obtain the following compatibility condition at \(\phi = 1\):

\[
\lambda_1 = \frac{1}{2} \left( c + \sqrt{c^2 - 4(k - 1)} \right) = \frac{1}{2} \left( \sqrt{c^2 + 8} - c \right).
\]

It is easy to see that this equation can be solved only when \(k \leq 5/3\) yielding the following relation between the speed of the pushed wave \(c_*(k)\) and the slope \(k\):

\[
c_*(k) = \frac{1 - k}{\sqrt{2(3 - k)}}, \quad k \in \left( 1, \frac{5}{3} \right),
\]

which is applicable since \(c_*(k) > c_#(k)\) for \(k \in (1, 5/3)\).

Hence, in general, the minimal speed \(c_*\) of propagation in the non-delayed model (16) is given by

\[
c_* = \begin{cases} 
\frac{1 + k}{\sqrt{2(3 - k)}}, & k \in (1, 5/3], \quad \text{(pushed critical wave),} \\
\frac{2 \sqrt{k - 1}}{5}, & k > 5/3, \quad \text{(pulled critical wave).}
\end{cases}
\] (18)

Now, if \(k \in (1, 5/3)\), then the continuity argument suggests that equation (16) has pushed minimal wavefront for all small \(h > 0\). Characteristic property (6) of this wave suggests how its explicit determination can be obtained. Indeed, let \(\phi(t)\) be the profile of the minimal front propagating with the speed \(c = c_* > c_#\).

Obviously, there exists the rightmost \(t_0\) such that \(\phi(t) \in (0, 1)\) for all \(t \leq t_0 - ch\), and \(\phi(t_0 - ch) = 1\). Set for simplicity \(t_0 = 0\). Then for all \(t \leq 0\), the wave profile \(\phi\) is a positive solution of the linear equation

\[
\phi''(t) - c\phi'(t) - \phi(t) + k\phi(t - ch) = 0.
\]

For \(c > c_#\), the characteristic equation \(\chi(z, c) = 0\) has two positive real roots \(0 < \lambda_2 < \lambda_1\) which dominate each other (complex) root in the sense that \(\Re\lambda_j < \lambda_2\), e.g. see \cite[Lemma 2.3]{18}. Then the positivity of \(\phi\) and its pushed character imply that

\[
\phi(t) = e^{\lambda_1(t + ch)}, \quad t \leq 0.
\]

Let us suppose now that \(\phi(t) \geq 1\) for all \(t \geq -ch\). This assumption is automatically satisfied if \(\phi(t)\) is monotone on \(\mathbb{R}\) and we will also prove later in this section that each wavefront (not necessarily pushed) to (16) normalized by \(\phi(-ch) = 1\) has to satisfy \(\phi(t) > 1\) for all \(t > -ch\). Then for \(t > 0\) the profile \(\phi\) verifies

\[
\psi''(t) - c\psi'(t) - \psi(t) - \psi(t - ch) = 0.
\]

The change of variables \(\psi = \phi - 2\) transforms the latter equation into

\[
\psi''(t) - c\psi'(t) - \psi(t) - \psi(t - ch) = 0.
\] (19)

The \(C^1\)-continuity of \(\phi\) also implies that

\[
\psi(t) = e^{\lambda_1(t + ch)} - 2, \quad t \in [-ch, 0],
\]

\[
\psi(0) = e^{\lambda_1ch} - 2, \quad \psi'(0) = \lambda_1 e^{\lambda_1ch}.
\]

Applying the Laplace transform \((L\psi)(z) = \int e^{-zt}\psi(t)dt\) to (20), we get

\[
\chi_{\kappa}(z)(L\psi)(z) = \psi'(0) + z\psi(0) - cz\psi(0) + e^{-zch}\int_{-ch}^{0} e^{-zt}\psi(t)dt.
\]

(21)
By using the Rouché theorem, it is easy to find that \( \chi_\kappa(z) \) has a unique positive zero \( \mu_1 \) while other characteristic values have negative real parts. Therefore \( \lim \psi(t) = 0, \ t \to +\infty, \) if and only if \( (L\psi)(\mu_1) = 0. \) The last equation has the form

\[
\lambda_1 e^{\lambda_1 c t} + (\mu_1 - c)(e^{\lambda_1 c t} - 2) + e^{-\mu_1 c t} \int_{-c}^{0} e^{-\mu_1 t} \psi(t) dt = 0,
\]

which can be transformed (by using the relations \( \chi(\lambda_1, c) = \chi_\kappa(\mu_1) = 0 \)) into

\[
\frac{\lambda_1(c)}{\mu_1(c)} = \frac{3 - k}{4}.
\]  

(22)

A simple calculation shows that formula (22) agrees with (18) with \( h = 0 \) while our previous discussion suggests that (22) must have at least one solution \( c_\ast(h) \) close to \( c_\ast(0) \) for all small \( h > 0. \) For further analysis of (22) note that, being simple zeros of the characteristic quasi-polynomials, \( \lambda_1(c) \) and \( \mu_1(c) \) are analytical functions of \( c. \) In addition, \( T(c) := \lambda_1(c)/\mu_1(c) \) is an increasing function of \( c. \) Indeed,

\[
T(c) := \frac{\lambda_1(c)}{\mu_1(c)} = \frac{\lambda_1(c) - \epsilon_1(c)}{\lambda_1(c) + \epsilon_2(c)} = 1 - \frac{2\epsilon_1(c)}{\sqrt{c^2 + 4 + \epsilon}}.
\]

where \( \tilde{\lambda}_1(c) = (\sqrt{c^2 + 4} + \epsilon)/2 \) satisfies \( z^2 - cz - 1 = 0 \) and \( \epsilon_1(c), \epsilon_2(c) \) are associated complementary functions. We claim that each \( \epsilon_j(c) \) is decreasing, \( \epsilon_j(+\infty) = 0 \) and this implies the monotonic character of \( T(c). \) To prove the monotonicity of \( \epsilon_1(c), \)

consider the interval \( [\lambda_1(c_1), \tilde{\lambda}_1(c_1)] \) for positive \( c_1 < c_2, \) the parabola \( y_1(z) = z^2 - c_1 z - 1 \) as well as the shifted parabola \( \tilde{y}_2(z) = y_2(z + \alpha), \) \( y_2(z) = z^2 - c_2 z - 1, \) where \( \alpha > 0 \) is chosen to comply with \( y_1(\lambda_1(c_1)) = \tilde{y}_2(\lambda_1(c_1)). \) It is clear that the graph of \( \tilde{y}_2(z) \) is a shifted (horizontally and downward) copy of the graph of \( y_1(z). \)

Hence, \( \tilde{y}_1(z) < \tilde{y}_2(z) \) for \( z \in [\lambda_1(c_1), \lambda_1(c_1)] \) and we can conclude that parabola \( \tilde{y}_2 \) intersects the abscissa axis at some point from the interval \( (\lambda_1(c_1), \tilde{\lambda}_1(c_1)) \) so that its intersection with the graph of \( y = -ke^{-c_2 h(z + \alpha)} = 0 \) belongs to the same interval.

This means that \( \epsilon_1(c_1) = \tilde{\lambda}_1(c_1) - \lambda_1(c_1) > \tilde{\lambda}_1(c_2) - \lambda_1(c_2) = \epsilon_2(c_2). \) The fact that \( \epsilon_2(c) \) decreases can be proved in a similar way. The property \( \epsilon_j(+\infty) = 0, \ j = 1, 2, \) is geometrically obvious.

For a fixed \( h \geq 0, \) the function \( T(c) \) is defined for all \( c \geq c_\#(h) \) (note that \( \lambda_1(c) \) is not defined for \( c < c_\#(h) \)). As we have seen, \( T(+\infty) = 1 > (3 - k)/4 \) so that the unique pushed wave to (16) exists if and only if \( T(c_\#(h)) \leq (3 - k)/4. \)

Therefore, for each \( k \in (1, 5/3), \) the critical speed \( c_\ast(h) \) is well defined by (22) and satisfies \( c_\ast(h) > c_\#(h) \) on some maximal interval \( h \in [0, h_p], \) with \( h_p \in (0, +\infty) \) depending on \( k. \)

It is known that \( c_\#(h) \) is a decreasing function of \( h \) (see e.g. [10, Lemma 1.2]). The minimal speed \( c_\ast(h) \) has the same property when the birth function \( g \) increases between 0 and \( \kappa, \) cf. [11, Lemma 3.5]. The next result shows that the monotonic nature of \( c_\ast(h) \) is also preserved in our unimodal toy model:

**Lemma 5.2.** The above defined function \( c_\ast : [0, h_p] \to (0, +\infty) \) is continuous and strictly decreasing.

**Proof.** Since the quotient \( \lambda_1/\mu_1 \) is a smooth function of \( c, h \) and equation (22) has a unique solution for each \( h \in [0, h_p], \) continuity of \( c_\ast \) follows. Next, consider the roots \( \lambda_1 = \lambda_1(c, h) \) and \( \mu_1 = \mu_1(c, h), \) this time also indicating their dependence on
PUSHED WAVEFRONTS FOR A MONOSTABLE NON-MONOTONE DELAYED MODEL

It is easy to see that for fixed speed $c$ and delays $h_1 < h_2$, $h_j \in [0, h_p]$, 
\[ \frac{\lambda_1(c, h_1)}{\mu_1(c, h_1)} < \frac{\lambda_1(c, h_2)}{\mu_1(c, h_2)}, \]
so $\lambda_1/\mu_1$ is an increasing function of delay as well as velocity. Therefore, since 
\[ \frac{\lambda_1(c_*(h_2), h_2)}{\mu_1(c_*(h_2), h_2)} = \frac{3 - k}{4} = \frac{\lambda_1(c_*(h_1), h_1)}{\mu_1(c_*(h_1), h_1)} < \frac{\lambda_1(c_*(h_1), h_2)}{\mu_1(c_*(h_1), h_2)}, \]
we conclude that $c_*(h_2) < c_*(h_1)$.

An important question concerns the possibility of the intersection of the graphs of $c_*(h)$ and $c_#(h)$. If this happens and $c_*(0) > c_#(0)$ but $c_*(h') = c_#(h')$ for some finite $h' > 0$, the delay is transforming minimal pushed fronts into pulled fronts

In a similar fashion, we can investigate the intersection of the curves $c = c_*(h)$ and $c = c_#(h)$. If this happens and $c_*(0) < c_#(0)$ but $c_*(h') > c_#(h')$ for some finite $h' > 0$, the delay is changing the monotonicity of pushed fronts. Particularly, the profile of the wavefront propagating with the minimal speed $c_*(h')$ is slowly oscillating around the positive equilibrium, cf. [19]. The same situation was observed for akin bistable wavefronts [2, 20] to (1). Arguing as in the previous paragraphs, we find that it suffices to indicate $k \in (1, 5/3)$ and $h > 0$ such that 
\[ T_1(h) = \frac{\lambda_1(c_#(h), h)}{\mu_1(c_#(h), h)} > \frac{3 - k}{4}, \]
clearly, for such a particular $h$, equation (22) does not have a solution. Here we can use the limit value $T_1(+\infty) = \lambda_\infty/\mu_\infty$, (cf. [10, Lemma 1.2]), where
\[ \lambda_\infty = \sqrt{1 + \frac{1}{\rho^2} - \frac{1}{\rho}}, \quad \rho = \sqrt{w_+(2 + w_+),} \]
with $w_+$ being the unique positive root of the equation 
\[ e^{-w}(2 + w) = 2/k, \]
and $\mu_\infty$ is the unique positive root for
\[ \mu^2 - 1 = e^{-\mu\rho}. \]

In a similar fashion, we can investigate the intersection of the curves $c = c_*(h)$ and $c = c_#(h)$. If this happens and $c_*(0) < c_#(0)$ but $c_*(h') > c_#(h')$ for some finite $h' > 0$, the delay is changing the monotonicity of pushed fronts. Particularly, the profile of the wavefront propagating with the minimal speed $c_*(h')$ is slowly oscillating around the positive equilibrium, cf. [19]. The same situation was observed for akin bistable wavefronts [2, 20] to (1). Arguing as in the previous paragraphs, we find that it suffices to indicate $k \in (1, 5/3)$ and $h' > 0$ such that 
\[ T_2(h') = \frac{\lambda_1(c_*(h'), h')}{\mu_1(c_*(h'), h')} < \frac{3 - k}{4}. \]

The latter implies $c_*(h') < c_#(h')$. If the curves $c = c_*(h)$ and $c = c_#(h)$ do not intersect, we can use the limit value $T_2(+\infty) = \hat{\lambda}_\infty/\hat{\mu}_\infty$, (cf. [10, Lemma 1.1]), where $\hat{\lambda}_\infty > 0$ and $\hat{\mu}_\infty < 0$ can be found from the equations 
\[ \lambda^2 - 1 + ke^{-\hat{\lambda}} = 0, \quad \mu^2 - 1 = e^{-\mu\hat{\rho}} \]
resp., where $\hat{\rho} = \sqrt{w_-(2 + w_-)}$ and $w_-$ is the unique negative root of the equation 
\[ e^{-w}(2 + w) = -2. \]

In the two examples given below, some numerical and geometrical evidences suggest that $T_1(h)$ is an increasing function, so that there can be at most one point of intersection of the graphs of $c_#$ and $c_*$. Take first $k = 1.5$. Then $w_+ = 0.7088 \ldots$, 

\[ h. \]
\[ \rho = 1.3856 \ldots, \lambda_\infty = 0.5115 \ldots, \mu_\infty = 1.1031 \ldots \quad \text{and} \quad \lambda_\infty/\mu_\infty = 0.4637 \ldots > (3 - 1.5)/4 = 0.375 \]. In this particular case, the minimal wavefront is pulled for all \( h > 0.3379 \ldots \) and is pushed for all \( h < 0.3379 \ldots \), see Figure 2 (left). Next, consider \( k = 1.2 \). Then \( w_+ = 0.3388 \ldots, \rho = 0.8901 \ldots, \lambda_\infty = 0.3806 \ldots, \mu_\infty = 1.1639 \ldots \), so that \( \lambda_\infty/\mu_\infty = 0.3269 \ldots < (3 - 1.2)/4 = 0.45 \). In this particular case, the minimal wavefront is pushed for all \( h \geq 0 \), see Figure 2 (right) and compare \( T_1(4) = 0.3141 \ldots \) with \( T_1(+\infty) = 0.3269 \ldots \). If additionally \( h > h_{osc} = 3.25 \ldots \), after crossing the level \( u = 1 \), this pushed wavefront slowly oscillates \([19]\) around it. Here \( h_{osc} \) is determined from the equation \( T_2(h_{osc}) = (3 - k)/4 = 0.45 \), the graphs of \( c_\# \) and \( c_\# \) have an intersection. Hence, contrarily to the first case (when \( k = 1.5 \)), delay is not changing the pushed nature of minimal wavefronts when we have relatively ‘strong’ non-subtangency of \( g \) at 0 (\( g \) is taken with \( k = 1.2 \)).

The following result completes the above discussion by showing that for every \( h \geq 0 \) the set \( C(h) \) of all possible velocities of semi-wavefronts for model (16) has the usual structure of semi-infinite interval \([c', +\infty)\).

**Theorem 5.3.** Suppose that \( k \in (1, 3) \) is such that (22) has solution \( c_\# \geq c_\# \). Then \( c_\# \) is the minimal speed of propagation in the sense that equation (16) has a wavefront solution propagating with the speed \( c \) if and only if \( c \geq c_\# \). Furthermore, \( c_\# \) is the minimal speed of propagation if (22) does not have a positive root \( c \).

**Proof.** Clearly, each non-critical (i.e. \( c \neq c_\# \)) wavefront profile normalized by the condition \( \phi(-ch) = 1 \) should be of the form
\[ \phi(t) = e^{\lambda_2(t+ch)p + (1 - p)e^{\lambda_1(t+ch)}}, \quad t \leq 0, \]
with some appropriate \( p \geq 0 \). Since \( \lambda_2 < \lambda_1 \), we have that \( \phi(t) > 0, \ t \leq -ch \).

Similarly to the case of pushed wavefronts (\( p = 0 \)), relation (21) yields the following determining equation for the admissible speeds \( c > c_\# \):
\[ \lambda_2 e^{\lambda_2 ch p} + (1 - p) \lambda_1 e^{\lambda_1 ch} + (\mu_1 - c)(e^{\lambda_2 ch p} + (1 - p)e^{\lambda_1 ch} - 2) + \\
e^{-\mu_1 ch} \int_{-ch}^0 e^{-\mu_1 t} \left( e^{\lambda_2(t+ch)p} + (1 - p)e^{\lambda_1(t+ch)} - 2 \right) dt = 0. \]
After a straightforward computation, we find the following relation between \( p, c, k \):
\[
\frac{1 - p}{1 - \lambda_1/\mu_1} + \frac{p}{1 - \lambda_2/\mu_1} = \frac{4}{1 + k},
\]
from which
\[
p = \frac{4\lambda_1/\mu_1 - (3 - k)(\mu_1 - \lambda_1 - \lambda_2)}{(1 + k)\lambda_1 - \lambda_2} \leq \frac{\mu_1 - \lambda_2}{\lambda_1 - \lambda_2}.
\]
(24)
where \( \lambda_2 < \lambda_1 < \mu_1 \) for \( c > c_\# \). Formula (24) shows the uniqueness of the profile \( \phi \) normalized by \( \phi(-ch) = 1 \) (equivalently, the uniqueness of \( p \)) for each fixed admissible \( c \). We will write \( \phi(t, c) \) to indicate the dependence of \( \phi \) on \( c \). Since \( p \) is a continuous function of \( c \), we conclude that \( \phi(t, c) \) depends continuously on \( t, c \).

Using (24), in accordance with [19], we obtain that \( \phi'(t, c) > 0 \) for all \( t \leq -ch \).

To prove the latter fact, it suffices to establish the positivity of \( \phi'(-ch, c) \):
\[
(1 + k)\phi'(-ch, c) = (3 - k)(\mu_1 - \lambda_1 - \lambda_2) + \frac{4\lambda_1\lambda_2}{\mu_1} > 0.
\]
Hence, for each \( c \geq c_\# \) satisfying the latter inequality, the initial part \( \phi(t, c) \) given by (23) can be continued for \( t > ch \) as a solution of (19), with \( \phi(+\infty, c) = 2 \). Here we are assuming that \( \phi(t, c) > 1 \) for all \( t > -ch \), this assumption is automatically satisfied when \( \phi(t) \) is monotone increasing on \( \mathbb{R} \) (i.e. \( \phi(h, c) \in D_\ast \)). In the general case, connect two points
\[
(0, c'), (h, c) \in D_\ast = \{(h, c) : c \geq c_\#(h) \text{ and } \lambda_1(c)/\mu_1(c) \geq (3 - k)/4 \}
\]
with some continuous path \( (h(s), c(s)), s \in [0, 1], \) lying in \( D_\ast \). Let \( \phi(t, s) := \phi(t, c(s)) \) be the corresponding family of profiles, it depends continuously on \( t, s \). Suppose now that \( \phi(t', 1) \leq 1 \) for some \( t' > -ch \). Since \( \phi'(t, 0) > 0 \), \( t \in \mathbb{R}, \) and \( \phi(+\infty, s) = 2 \), there exist the smallest value of \( s_1 \) and some \( t_1 > -ch \) such that \( \phi(t_1, 1) = 1 \) and \( \phi'(t_1, s_1) = 0 \), \( \phi''(t_1, s_1) \geq 0 \). This actually implies that \( t_1 > 0 \) since otherwise \( \phi(t, s_1) \) has two critical points on \( (-\infty, 0) \) which is not possible in view of (23). But then (19) and Lemma 5.1 lead to the following contradiction:
\[
0 = \phi''(t_1, s_1) - c(s_1)\phi'(t_1, s_1) - \phi(t_1, s_1) + 4 - \phi(t_1 - c(s_1)h(s_1), s_1) > 1 + 4 - 3 = 0.
\]
Hence, \( \phi(t, s) > 1 \) for all \( t > -ch \), \( s \in [0, 1] \) that legitimizes the construction of a wavefront to (17) by \( C^1 \)-continuously gluing its initial part (23) with the unique piece of the corresponding solution for equation (19). This finalizes the proof of Theorem 5.3.

On Figure 3, we present solution \( u(t, x) \) of the Cauchy problem
\[
u_0(t, x) = \begin{cases} 0, & \text{as } x < 0, \ t \in [-h, 0], \\ 2, & \text{as } x \geq 0, \ t \in [-h, 0], \end{cases}
\]
to equation (16) at the indicated sequence \( t = t_j \) of moments. The numerical simulations are based on the Crank-Nicholson method which is second-order accurate in both spatial and temporal directions. The spatial step size is chosen as \( \Delta x = 0.05 \) in the computational interval \( x \in [-25, 25] \) together with the Dirichlet boundary conditions \( u(t, -25) = 0 \) and \( u(t, 25) = 2 \). The temporal step size is \( \Delta t = 0.01 \).
It is known from [22] that in the monotone case (i.e. when the birth function $g$ is increasing on $[0, \kappa]$), solution $u(t, x)$ of (25) exponentially rapidly converges to the pushed wavefront. In sight of the above developed theory of equation (16), it is natural to expect that its solution $u(t, x)$ also will converge to the pushed wavefront. Comparison of our theoretical and numerical results corroborates this fact. For example, on Figure 3 (left) we present snapshots of solution $u(t, x)$ already stabilized around the pushed wavefront for model (16) considered with $h = 0.5$ and $k = 1.2$. By our theory, in this case the profile of pushed wave is monotone. The right part of Figure 3 shows a magnified fragment of the leading edge of pushed wave for parameters $h = 6$ and $k = 1.2$. Our theory predicts that the profile of wavefront must be non-monotone in this case. In good accordance with the theory, in this case, the numerics presents a profile of pushed wave oscillating around the positive equilibrium (the amplitude of oscillations is relatively small).

![Figure 3. Snapshots of solution $u(t, x)$ to the Cauchy problem (25), $k = 1.2$, converging to the pushed wavefront, at the indicated sequence of times $t = t_j$. The cases $h = 0.5$ (left), $h = 6$ (right).](image)

Furthermore, we have compared the theoretical values $c_\#(h)$, $c_*(h)$ and the speeds $c_{ns}(h)$ obtained from the numerical simulations for different values of delay $h$:

| $h$ | $c_\#(h)$ | $c_*(h)$ | $c_{ns}(h)$ | $h$ | $c_\#(h)$ | $c_*(h)$ | $c_{ns}(h)$ |
|-----|------------|----------|-------------|-----|------------|----------|-------------|
| 0.5 | 0.5720     | 0.6562   | 0.6377      | 3.5 | 0.1922     | 0.2091   | 0.2112      |
| 1   | 0.4270     | 0.4770   | 0.4662      | 4   | 0.1733     | 0.1883   | 0.1892      |
| 1.5 | 0.3420     | 0.3779   | 0.3746      | 4.5 | 0.1579     | 0.1713   | 0.1727      |
| 2   | 0.2860     | 0.3138   | 0.3165      | 5   | 0.1450     | 0.1571   | 0.1572      |
| 2.5 | 0.2458     | 0.2687   | 0.2688      | 5.5 | 0.1340     | 0.1452   | 0.1461      |
| 3   | 0.2157     | 0.2351   | 0.2353      | 6   | 0.1246     | 0.1348   | 0.1346      |

In the table, we can observe a good agreement between theoretical and numerical values of the minimal speed for monostable wavefronts. Clearly, the minimal wave has pushed character. In any event, as seen from a point of view of rigorous analytical proofs, such a convergence of solution (25) to a pushed wavefront remains a difficult open problem.
6. Appendix. As it was established in [20, Lemma A.2] and [10, Lemma 1.1], for each pair \((h, c) \in D_\kappa\), the characteristic function \(\chi_\kappa\) has exactly three real zeros, one positive and two negative (counting multiplicity), \(\mu_3 \leq \mu_2 < 0 < \mu_1\). In addition, every complex zero \(\mu_j\) of \(\chi_\kappa\) is simple [20, Lemma A.2] and has its real part \(\Re \mu_j < \mu_2\) [10, Lemma 1.1]. We claim that actually \(\Re \mu_j < \mu_3\) for each complex zero \(\mu_j\) of \(\chi_\kappa\).

Indeed, fix \(c = \bar{c}\) and consider the unique value \(h^* > 0\) such that \((h^*, \bar{c})\) belongs to the boundary of \(D_\kappa\) (i.e. \(c_\kappa(h^*) = \bar{c}\), cf. Section 2). Our claim is trivially valid for the parameters \((h, c) = (h^*, \bar{c})\) because of \(\mu_2 = \mu_3\). Moreover, since each half-plane \(\Re z \geq \alpha\) contains at most finite number of zeros of \(\chi_\kappa\) and \(\mu_j(h) \not\in \mathbb{R}\), being a simple complex zero, depends continuously on \(h > 0\), the above claim is also valid for all \(h\) from some maximal interval \([h_\delta, h^*]\). If \(h_\delta > 0\) then \(\Re \mu_k(h_\delta) = \mu_3\) for some index \(k\). In this way, there are at least three zeros of \(\chi_\kappa\) having the same real part. However, by [20, Lemma A.2], this situation cannot occur.

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E-mail address: Karel.Hasik@math.slu.cz
E-mail address: Jana.Kopfova@math.slu.cz
E-mail address: petra.nabelkova@math.slu.cz
E-mail address: trofimch@inst-mat.utalca.cl