NOTE ON THE BIJECTIVITY OF THE PAK-STANLEY LABELLING

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1. Introduction

This article has the sole purpose of presenting a simple, self-contained and direct proof of the fact that the Pak-Stanley labeling is a bijection. The construction behind the proof is subsumed in a forthcoming paper [1], but an actual self-contained proof is not explicitly included in that paper.

Let \( n \) be a natural number and consider the Shi arrangement of order \( n \), the union \( \mathcal{S}_n \) of the hyperplanes of \( \mathbb{R}^n \) defined, for every \( 1 \leq i < j \leq n \), either by equation \( x_i - x_j = 0 \) or by equation \( x_i - x_j = 1 \). The regions of the arrangement are the connected components of the complement of \( \mathcal{S}_n \) in \( \mathbb{R}^n \). Jian Yi Shi [5] introduced in literature this arrangement of hyperplanes and showed that the number of regions is \( (n+1)^{n-1} \).

On the other hand, \( (n+1)^{n-1} \) is also the number of parking functions of size \( n \), which were defined (and counted) by Alan Konheim and Benjamin Weiss [3]. These are the functions \( f : [n] \to [n] \) such that

\[ \forall i \in [n], \ |f^{-1}(\{i\})| \geq i \]

or, equivalently, such that, for some \( \pi \in \mathfrak{S}_n \), \( f(i) \leq \pi(i) \) for every \( i \in [n] \) (as usual, \( [n] := \{1, \ldots, n\} \) and \( \mathfrak{S}_n \) is the set of permutations of \( [n] \)).

The Pak-Stanley labeling [7] consists of a function \( \lambda \) from the set of regions of \( \mathcal{S}_n \) to the set of parking functions of size \( n \).

We define [0] := \( \emptyset \) and, for \( i, j \in \mathbb{N}, [i, j] := [j] \setminus [i-1] \), so that \([i, j] = \{i, i+1, \ldots, j\}\) if \( i \leq j \) and \([i, j] = \emptyset\) otherwise. Finally, \([i] = [1, i]\) for every integer \( i \geq 0 \) as stated before.

Let \( A \subseteq [n] \), say \( A =: \{a_1, \ldots, a_m\} \) with \( a_1 < \cdots < a_m \) and let \( W_A \) be the set of words of form \( w = a_{\alpha_1} \cdots a_{\alpha_m} \) for some permutation \( \alpha \in \mathfrak{S}_m \). If \( 1 \leq i < j \leq m \), we distinguish the subword \( w(i : j) := a_{\alpha_i} \cdots a_{\alpha_j} \) from the set \( w([i, j]) := \{a_{\alpha_i}, \ldots, a_{\alpha_j}\} \).

Similarly, we define \( w^{-1} : A \to [m] \) through \( w^{-1}(w) = i \) for every \( i \in [m] \).

**Definition 1.1.** Given a word \( w = w_1 \cdots w_k \in W_A \) and a set \( I = \{[o_1, c_1], \ldots, [o_k, c_k]\} \) with \( 1 \leq o_i < c_i \leq m \) for every \( i \in [k] \) and \( o_1 < o_2 < \cdots < o_k \), we say that the pair \( P = (w, I) \) is a valid pair if

- \( w_{o_i} > w_{c_i} \) for every \( i \in [k] \);
- \( c_1 < c_2 < \cdots < c_k \).

An \( A \)-parking function is a function \( f : A \to [m] \) for which

\[
\forall j \in [m], \ |f^{-1}([j])| \geq j.
\]
We denote by $\text{PF}_A$ the set of $A$-parking functions. Of course, for $f: A \to [m]$, $f \in \text{PF}_A$ if and only if $f \circ \iota_A$ is a parking function, where $\iota_A: [m] \to A$ is such that $\iota_A(i) = a_i$. A particular case occurs when

$$\forall j \in [m], \ f(a_j) \leq j.$$ 

In this case, we say that $f$ is $A$-central. We denote by $\text{CF}_A$ the set of $A$-central parking functions. We call contraction of intervals $w$, the valid pair $(I, w)$ in this way to a (unique) region of $S_n$.

Consider, for a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \mathcal{S}_n$, the (unique) permutation $w \in \mathcal{S}_n$ such that $x_{w_1} < \cdots < x_{w_n}$, and consider the set $\mathcal{I} = \{[o_1, c_1], \ldots, [o_m, c_m]\}$ of all maximal intervals $I_i = [o_i, c_i]$ with $o_i < c_i$ for $i = 1, \ldots, k$, such that

- $w_{o_i} > w_{c_i}$;
- for every $\ell, m \in I_i$ with $\ell < m$ and $w_{\ell} > w_m$, $0 < x_{w_m} - x_{w_{\ell}} < 1$.

Then, clearly $(w, \mathcal{I})$ is a valid pair that does not depend on the particular point $x$ that we have chosen. More precisely, if a similar construction is based on a different point $y \in \mathbb{R}^n \setminus \mathcal{S}_n$ then at the end we obtain the same valid pair if and only if $x$ and $y$ are in the same region of $\mathcal{S}_n$. Finally, it is not difficult to see that every valid pair corresponds in this way to a (unique) region of $\mathcal{S}_n$.

**Example 2.1** (example p. 484, ad.). Let $w = 843967125$ and $\mathcal{I} = \{[1, 6], [3, 8], [6, 9]\}$. The valid pair $(w, \mathcal{I})$ corresponds to the region

$$\left\{ (x_1, \ldots, x_9) \in \mathbb{R}^9 \mid x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5, \right.$$ 

$$x_8 + 1 > x_7, x_3 + 1 > x_2, x_7 + 1 > x_5, \right.$$ 

$$x_4 + 1 < x_1, x_6 + 1 < x_5 \right\}$$

where also $x_8 + 1 > x_6$ (since $x_7 > x_6$) and $x_8 + 1 < x_1$ (since $x_8 < x_4$), for example.

Let $R_0$ be the region corresponding to the valid pair $(w, \mathcal{I})$ where $w = n(n - 1) \cdots 2 1$ and $\mathcal{I} = \{[1, n]\}$, so that $(x_1, \ldots, x_n) \in R_0$ if and only if $0 < x_i - x_j < 1$ for every $0 \leq i < j < n$.

In the Pak-Stanley labeling $\lambda$, the label of $R_0$ is, using the one-line notation, $\lambda(R_0) = 11 \cdots 1$. Furthermore,

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(1) Note that the order is reversed relatively to Stanley’s paper [7].

(2) The fact that $0 < x_{w_m} - x_{w_\ell}$ already follows from the fact that $w_{\ell} > w_m$. 

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2. The Pak-Stanley labeling

Igor Pak and Richard Stanley [7] created a (bijective) labeling of the regions of the Shi arrangement with parking functions that may be defined as follows.

In the Pak-Stanley labeling $\lambda$, we say that $\lambda$ is the region corresponding to the valid pair $(\mathcal{I}, \lambda)$ and

$$\hat{w}(a) := w^{-1}(a) - \left\{ b \in A \mid b > a, w^{-1}(b) < w^{-1}(a) \right\}.$$ 

Note that indeed $\hat{w} \in \text{CF}_A$, since $\hat{w}(a) = |w([w^{-1}(a)]) \cap [a]|$.

For example, $843967 = 346789$. In fact, $843967(3) = 1$ since $w^{-1}(3) = 3$ and $w([3]) \cap [3] = \{8, 4, 3\} \cap [3] = \{3\}$, but, for instance, $843967(6) = 3$ since $w^{-1}(6) = 5$ and $w([5]) \cap [6] = \{3, 4, 6\}$.

When $A = [n]$, the $A$-central parking functions are simply central parking functions.
• if the only hyperplane that separates two regions, $R$ and $R'$, has equation $x_i = x_j$ ($i < j$) and $R_0$ and $R$ lie in the same side of this plane, then $\lambda(R') = \lambda(R) + \varepsilon_j$ (as usual, the $i$-th coordinate of $\varepsilon_j$ is either 1, if $i = j$, or 0, otherwise);

• if the only hyperplane that separates two regions, $R$ and $R'$, has equation $x_i = x_j + 1$ ($i < j$) and $R_0$ and $R$ lie in the same side of this plane, then $\lambda(R') = \lambda(R) + \varepsilon_i$.

Thus, given a region $R$ of $S_n$ with associated valid pair $P = (w, \{[o_1, c_1], \ldots, [o_m, c_m]\})$, if $f = \lambda(R)$ and $i = w_j$, then, counting the planes of equation $x_{w_k} - x_i = 0$ or $x_i - x_{w_k} = 1$ that separate $R$ and $R_0$, respectively, we obtain (cf. [7])

$$f_i = 1 + \left| \{ k < j \mid w_k < i \} \right| + \left| \{ k < j \mid w_k > i, \ no \ \ell \in [m] \ satisfies \ j, k \in [o_\ell, c_\ell] \} \right| .$$

(2.3)

Hence, if $j \notin [o_1, c_1], \ldots, [o_m, c_m]$, (2.4) $f_i = j$;

in this case, let $o_p(i) = o_p(w_j) := j$. Otherwise, if $k \leq m$ is the least integer for which $j \in [o_k, c_k]$, (2.5) $f_i = o_k - 1 + w(o_k : c_k)(i) .$

and we define $o_p(i) := o_k$.

In Figure 1, we represent $S_3$ with each region $R$ labeled with $\lambda(R)$.

By requiring the validity of equations (2.4) and (2.5) under the same conditions, we extend $\lambda$ to every valid pair $P = (w, J)$, where $w \in W_4$ for some $A \subseteq [n]$. Note that in this way we still obtain an $A$-parking function $f = \lambda(w, J)$.

Moreover, if $1 \leq k < \ell \leq |A|$ then $o_p(w_k) \leq o_p(w_\ell)$. If, in addition, $w_k > w_\ell$, then (2.6) $f(w_k) \leq f(w_\ell)$.

In fact, $f(w_\ell) = \ell - \left| \{ o_p(w_\ell) \leq \ell \mid w_\ell > w_\ell \} \right| \geq k - \left| \{ o_p(w_k) \leq j \leq k \mid w_\ell > w_\ell \} \right| = f(w_k)$, since the size of the set $\{ o_p(w_\ell) \leq j \leq \ell \mid w_\ell > w_\ell \} \setminus \{ o_p(w_k) \leq j \leq k \mid w_\ell > w_\ell \}$, which is equal to $\{ k \leq j \leq \ell \mid w_k > w_\ell \}$, is clearly less than or equal to $\ell - k$.

Example 2.7 (continued). Let again $R$ be the region of $S_3$ associated with the valid pair $(843967125, \{[1, 6], [3, 8], [6, 9]\})$. Writing with a variant of Cauchy’s two-line notation, we have, corresponding to the intervals $[1, 6]$, $[3, 8]$ and $[6, 9]$, respectively, $w(1: 6) = 843967$ and $f_1 = 843967 = 113414$, $f_2 = 396712 = 121232$, $f_3 = 7125 = 1131$ and, finally, $f = \lambda(R) = 341183414$, which we also write $843967125$ (3) (cf. Figure 1).

Similarly, for $A = [9] \setminus \{8, 4\}$, we may consider $f = \lambda(3967125, \{[1, 6], [4, 7]\})$, the $A$-parking function $\underline{3967125} = 1216232$.

3. Injectivity of $\lambda$

The proof of the injectivity of $\lambda$ is based on the following lemma, where a particular case is considered. Beforehand, we introduce a new concept.

(3) Note that, for example, the central parking function $1132 = \underline{2413}$ corresponds to $2413$. 


Definition 3.1. Let \( w \in W_A \) for a subset \( A \) of \([n]\), consider the poset of inversions of \( w \), \( \text{inv}(w) := \{(i,j) \mid i < j, w_i > w_j\} \), ordered so that \((i,j) \leq (k,\ell)\) if and only if \([i,j] \subseteq [k,\ell]\). Then, define \( \text{maxinv}(w) \) as the set of maximal elements of \( \text{inv}(w) \).

Lemma 3.2. Let \( A \subseteq [n], v,w \in W_A \), and suppose that \( P = (v,\mathcal{I}) \) is a valid pair. If \( \lambda(v,\mathcal{I}) = \hat{w} \), then \( v = w \) and \( \mathcal{I} = \text{maxinv}(v) \).

Proof. We first prove that \( v = w \). Let \( A = \{a_1,\ldots,a_m\} \) with \( a_1 < \cdots < a_m \), and suppose that, for \( \pi,\rho \in \mathfrak{S}_m \), \( v = a_{\pi_1}a_{\pi_2}\cdots a_{\pi_m} \) and \( w = a_{\rho_1}a_{\rho_2}\cdots a_{\rho_m} \), and that, for some \( 1 \leq \ell \leq n \), \( \pi_i = \rho_i \) whenever \( 1 \leq i < \ell \) but, contrary to our assumption, \( \pi_\ell \neq \rho_\ell \).

Finally, define \( j, k > \ell \) such that \( \rho_\ell = \pi_j \) and \( \pi_\ell = \rho_k \) and \( x := a_{\pi_\ell}, y := a_{\rho_\ell} \). Graphically, we have

\[
\begin{align*}
v &= w_1 \cdots w_{\ell-1} x = v_\ell v_{\ell+1} \cdots y = v_j \cdots v_m \\
w &= w_1 \cdots w_{\ell-1} y = w_\ell w_{\ell+1} \cdots x = w_k \cdots w_m
\end{align*}
\]

Then, for \( a = o_P(y) < j \),

\[
\hat{w}(y) = \ell - \left| \{1 \leq i < \ell \mid w_i > y\} \right| \\
= j - \left| \{a \leq i < j \mid v_i > y\} \right|
\]

and hence

\[
j - \ell = \left| \{\ell \leq i < j \mid v_i > y\} \right| - \left| \{1 \leq a < \ell \mid w_i > y\} \right|.
\]
This means that, for every \( i \) with \( \ell \leq i < j \), \( w_i > y \) (and, in particular, \( x > y \)) and that, for every \( i \) with \( 1 \leq i < a \), \( w_i \leq y \). On the other hand, for \( b = o_P(x) \leq \ell \),
\[
\hat{w}(x) = k - \left| \left\{ 1 \leq i < k \mid w_i > x \right\} \right| = \ell - \left| \left\{ b = i \leq \ell \mid w_i > x \right\} \right|
\]
and
\[
k - \ell = \left| \left\{ \ell \leq i < k \mid w_i > x \right\} \right| + \left| \left\{ 1 \leq i < b \mid w_i > x \right\} \right|
\]
Note that \( b \leq a \) since \( \ell < j \) and \( P \) is a valid pair. Then, \( \left\{ 1 \leq i < b \mid w_i > x \right\} = \emptyset \) and \( w_i > x \) for every \( i \) with \( \ell \leq i < j \). In particular, \( y > x \), which is absurd. We now leave it to the reader to prove that \( \mathcal{I} = \maxinv(v) \).

**Corollary 3.3.** Let \( A \subseteq [n] \). The function \( \mathcal{C}_A : \mathcal{W}_A \rightarrow \mathcal{CF}_A \) : \( w \mapsto \hat{w} \) is a bijection.

*Proof.* Since \( |\mathcal{W}_A| = |\mathcal{CF}_A| = |A|! \), the result follows from the last lemma, since \( \mathcal{C}_A \) is injective.

**Definition 3.4.**

- We denote the inverse of \( \mathcal{C}_A \) by \( \varphi_A : \mathcal{CF}_A \rightarrow \mathcal{W}_A \).
- Given an \( A \)-parking function \( f : A \rightarrow [n] \), the center of \( f \), \( Z(f) \), is the (unique\(^{(4)}\)) maximal subset \( Z \) of \( A \) such that the restriction of \( f \) to \( Z \) is \( Z \)-central. Let \( \zeta := |Z| \) and note that \( \zeta \neq 0 \) since \( f^{-1}(1) \subseteq Z \) and \( |f^{-1}(1)| \geq 1 \). Finally, let \( f_Z : Z \rightarrow [n] \) be the restriction of \( f \) to its center.

**Lemma 3.5.** Let \( f = \lambda(w, \mathcal{I}) \) for a valid pair \( P = (w, \mathcal{I}) \), where \( w \in \mathcal{W}_A \) for \( A \subseteq [n] \) with \( m = |A| \).

**3.5.1.** Let, for some \( p \geq 0 \), \( \mathcal{I} = \{ [o_1, c_1], \ldots, [o_p, c_p] \} \) with \( o_1 < \cdots < o_p \). Then,
\[
f_Z = w(1; \zeta)
\]
and, in particular, \( w([\zeta]) = Z \). Moreover, \( \maxinv(w(1; \zeta)) = \{ [o_1, c_1], \ldots, [o_j, c_j] \} \) for some \( 0 \leq j \leq p \).

**3.5.2.** For every \( j \in [m] \), \( w_j \in Z(f) \) if and only if
\[
f(w_j) = 1 + \left| \left\{ k < j \mid w_k < w_j \right\} \right|.
\]

*Proof.*

**3.5.1** We start by proving the second statement, namely that \( w([\zeta]) = Z \). Note that \( w_1 \in f^{-1}\{1\} \subseteq Z \) and suppose, contrary to our claim, that, for some \( k < \zeta \) which we consider as small as possible, \( w_k \notin Z \). Again, let \( \ell > k \) be as small as possible with \( w_\ell \in Z \) and define \( v = w(1; k) \).

We now consider the “restriction” \( w^* \) of \( w \) to \( Z \), that is, the subword of \( w \) obtained by deleting all the elements of \( [n] \setminus Z \), and let
\[
w' := \varphi_Z(f_Z) \in \mathcal{W}_Z.
\]

\(^{(4)}\)Note that if the restriction of \( f \) to \( X \) is \( X \)-central and the restriction of \( f \) to \( Y \) is \( Y \)-central for two subsets \( X \) and \( Y \) of \( A \), then the restriction of \( f \) to \( (X \cup Y) \) is also \((X \cup Y)\)-central.
By Lemma 3.2, \( w^* = w' \) and \( k - f(w_i) \) is the number of integers greater than \( w_i \) that precede it in \( w^* \). This means that \( w_k, \ldots, w_{\ell-1} > w_\ell \) and that \( o(w_\ell) \leq k \). Hence, \( k - f(w_k) \) is also the number of integers greater than \( w_k \) that precede it in \( w \), and so \( \hat{v} \) is the restriction of \( f \) to \( w([k]) \), and \( a \in Z \), a contradiction. Now, the result follows also from Lemma 3.2.

We have proven that the “initial parts” of both \( w \) and \( \mathcal{J} \) are characterized by \( f \). Let \( m = |A| \), consider \( c \in \mathbb{N} \) such that \( 1 < c \leq \zeta \), and define \( \hat{w} := w(c:m) \); define also \( \hat{\mathcal{J}} := \emptyset \) if \( j = p \), for \( j, p \) defined as in the statement of Lemma 3.5 and \( \hat{\mathcal{J}} := \{ \hat{I}_1, \ldots, \hat{I}_{p-j} \} \), where

\[
\hat{I}_1 := [1, c_{j+1} - c + 1], \ldots, \hat{I}_{p-j} := [a_p - c + 1, c_p - c + 1],
\]

if \( p > j \). Suppose that, for some such \( c \), \( f \) also determines \( \hat{f} := \lambda(\hat{w}, \hat{\mathcal{J}}) \). This proves our promised result (by induction on \( |A| \)) and shows how to proceed for actually finding \( w \in \mathcal{S}_n \) and \( \mathcal{J} \), given \( f = \lambda(w, \mathcal{J}) \): we find the center \( Z \) of \( f \), build \( \varphi_Z(f_Z) \in W_Z \) and \( \hat{f} \), find the center \( \hat{Z} \) of \( \hat{f} \), build \( \varphi_{\hat{Z}}(f_{\hat{Z}}) \in W_{\hat{Z}} \) and \( \hat{\mathcal{J}} \), etc.

**Definition 3.6.** Given a parking function \( f \in PF_A \), \( f = \lambda(w, \mathcal{J}) \), \( m := |A| \), \( Z := Z(f) \), and \( \zeta := |Z| < m \),

- let \( b := \min f(A \setminus Z) \) and \( a := \max(f^{-1}(\{b\}) \setminus Z) \);
- if \( b > \zeta \), let \( c := b \);
  - if \( b \leq \zeta \), let \( c \) be the greatest integer \( i \in [\zeta] \) for which

\[
(3.7) \quad i + |w([i, \zeta]) \cap [a - 1]| = b.
\]

- let \( X := w([c - 1]) \) (\( X \subseteq Z \) by Lemma 3.5);
- let \( \tilde{f} : A \setminus X \rightarrow [m - c + 1] \)

\[
x \mapsto \begin{cases} f(x) - |X \cap [x - 1]|, & \text{if } x \in Z; \\ f(x) - c + 1, & \text{otherwise}. \end{cases}
\]

**Lemma 3.7.** With the definitions above,

- **3.7.1.** \( a = w_{\zeta+1} \) and \( a \in Z(\tilde{f}) \);
- **3.7.2.** \( Z \setminus X \subseteq Z(\tilde{f}) \);
- **3.7.3.** \( c = o_{(\hat{w}, \hat{\mathcal{J}})}(a) \) and
- **3.7.4.** \( \tilde{f} = \lambda(\hat{w}, \hat{\mathcal{J}}) \).

**Proof.** If \( b > \zeta \), then \( X = Z \) and all the statements follow directly from the definitions. Hence, we consider that \( b \leq \zeta \). We start by seeing that \( c \) is well defined. Define \( h : [\zeta] \rightarrow \mathbb{N} \) by

\[
h(i) = i + |w([i, \zeta]) \cap [a - 1]|.
\]

Then, for every \( i < \zeta \), since \( w([i, \zeta]) = \{w_i\} \cup w([i+1, \zeta]) \), \( h(i+1) \) either equals \( h_i \) or \( h_i + 1 \), depending on whether \( w_i \) is either less than \( a \) or greater than \( a \). Since \( h(\zeta) \geq \zeta \geq b \), by definition, all we have to prove is that \( h(1) < b \), or, equivalently, that \( 1 + |Z \cap [a - 1]| < f_a \). But \( f_a \leq 1 + |Z \cap [a - 1]| \) implies that the restriction of \( f \) to \( Z' := Z \cup \{a\} \) is \( Z' \)-central, by Lemma 3.5.2, which, since \( a \not\in Z \), contradicts the maximality of \( Z \). Note that the set of values of \( i \) for which (3.7) holds true is an interval, and that its maximum, \( c \), is the only one that is greater than \( a \). By definition of \( a \) and by Lemma 3.5.1, \( a = w_{\zeta+1} \), for if \( x = w_k \) and \( a = w_\ell \) with \( \ell > k \) and \( x > a \), then \( f(x) \leq b \), by (2.9), and \( x \in Z(\tilde{f}) \).
Now, let \( g = \lambda(\tilde{w}, \tilde{\beta}) \) for \( \tilde{w} \) and \( \tilde{\beta} \) as defined before. If \( x \in A \setminus Z \), by definition of \( \lambda \), viz. \( \text{Proposition } 3.8 \), \( g(x) = f(x) - c + 1 = \tilde{f}(x) \). In particular, \( g(a) = 1 + \lfloor \tilde{w}((\zeta - c + 1) \cap [a - 1]) \rfloor \). Hence, by Lemma \( 3.5.2 \), \( a \in Z(g) \). Now, Lemma \( 3.5.1 \) implies that \( Z \setminus X \), the set of elements on the left side of \( a \) in \( \tilde{w} \), is a subset of \( Z(g) \), and that \( c = a(w, \tilde{\beta})(a) \). Now, the last result, viz. \( g = \tilde{f} \), follows immediately, since for \( x = w, j \) with \( c \leq j \leq \zeta \), \( f(x) = 1 + \lfloor w([j]) \cap [x - 1] \rfloor \) and \( g(x) = 1 + \lfloor \tilde{w}((j - c + 1) \cap [x - 1]) \rfloor \).

This concludes the proof of our main result.

**Proposition 3.8.** The Pak-Stanley labeling is injective. \( \square \)

### 4. Inverse

It is easy to directly prove Corollary \( 3.3 \) and even to explicitly define \( \varphi_A \), the inverse of \( C_A \). Nevertheless, we consider here a method that we find very convenient, and particularly well-suited to our purpose, the s-parking. Note that a similar method is given by the depth-first search version of Dhar’s burning algorithm defined by Perkinson, Yang and Yu \( [4] \). In fact, it may be proved that \( Z(f) \) is the set of \( \zeta \) visited vertices before the first back-tracking, and that \( w(1; \zeta) \) is given by the order in which the vertices are visited.

**Definition 4.1.** Let again \( A =: \{a_1, \ldots, a_m\} \) with \( a_1 < \cdots < a_m \) and \( f : A \to \{m\} \). For every \( i \in [m] \), define the set \( A_i := \{a_1, \ldots, a_i\} \), and define recursively the bijection \( w^i : A_i \to [i] \) as follows.

- \( w^1 : a_1 \mapsto 1 \) (necessarily);
- for \( 1 < j \leq i \leq m \),
  - \( j < i \), \( w^i(a_j) = \begin{cases} w^{i-1}(a_j), & \text{if } w^{i-1}(a_j) < f(a_i) \\ 1 + w^{i-1}(a_j), & \text{if } w^{i-1}(a_j) \geq f(a_i) \end{cases} \)
  - \( w^i(a_i) = f(a_i) \);

Finally, let \( \psi : [m] \to A \) be the inverse of \( w^m : A \to [m] \). We call \( S(f) := \psi \) (viewed as the word \( \psi(1) \cdots \psi(m) \)) the s-parking of \( f \).

This operation resembles placing books on a bookshelf, where in step \( i \) we want to put book \( a_i \) at position \( f(a_i) \) — and so we must shift right every book already placed in a position greater than or equal to \( f(a_i) \). For example, if \( A = \{3, 4, 6, 7, 8, 9\} \subseteq [9] \) and \( f = 346789 \), then \( S(f) = 843967 \). On the other hand, if \( B = \{1, 2, 3, 6, 7, 9\} \) and \( g = 1231231 \), then \( S(g) = 396712 \). Finally, let \( C = \{1, 2, 5, 7\} \) and \( h = 1231 \), so that \( S(h) = 7125 \). The three constructions are used in the next example. See Figure \( 2 \) where a parking function \( f \) is represented on the top rows by orderly stacking in column \( i \) the elements of \( f^{-1}(i) \) (cf. \( 2 \)), and row \( j \) below the horizontal line is the inverse of \( w^j \). Note that \( (1.1) \) implies that \( w^1 \) is indeed a bijection for \( i = 1, \ldots, m \).

**Lemma 4.2.** Given \( A \) and \( f \) as in the previous definition, \( f = \overline{S(f)} \). Conversely, given \( A \) and \( w \in W_A \), \( w = S(\overline{w}) \).

**Proof.** Let \( w = S(f) \) and \( \psi = w^{-1} \) and note that, when we s-park \( f \), each element \( a_i \) of \( A \) is put first at position \( f(a_i) \), and it is shifted one position to the right by an element \( a_j \) if and only if \( j > i \) and \( \pi_j < \pi_i \); it ends at position \( \psi_i \). Hence, \( f = \overline{S(f)} = \overline{\tilde{w}}. \) Then \( S \) is the inverse of \( C_A \), that is, \( S = \varphi_A \). \( \square \)
Example 2.1 (Conclusion). Let us recover the valid pair $P = \lambda^{-1}(f)$ out of $f = 341183414$. In the first column, on the right, the elements of the center of $f$ are written in italic and $a$ is written in boldface. The last column may be obtained by s-parking, as represented in Figure 2.

|   |   |   |   |
|---|---|---|---|
| 8 | 4 | 9 | 9 |
| 3 | 6 | 7 | 7 |
| 3 |   |   | 1 |
| 4 | 3 | 6 | 1 |
| 4 | 3 | 6 | 7 |
| 8 | 4 | 3 | 6 |
| 8 | 4 | 3 | 9 |

$P = (843967125, \{[1, 6], [3, 8], [6, 9]\})$.

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