EQUIDISTRIBUTION OF EXPANDING DEGENERATE MANIFOLDS
IN THE SPACE OF LATTICES

NIMISH A. SHAH AND PENGYU YANG

Abstract. For the space of unimodular lattices in a Euclidean space, we give necessary
and sufficient conditions for equidistribution of expanding translates of any real-analytic
submanifold under a diagonal flow. This work extends the earlier result of Shah in the
case of non-degenerate submanifolds.

We apply the above dynamical result to show that if the affine span of a real-analytic
submanifold in a Euclidean space satisfies certain Diophantine and arithmetic conditions,
then almost every point on the manifold is not Dirichlet-improvable.

1. Introduction

1.1. Background. Let $G$ be a Lie group, $\Gamma$ a discrete subgroup of $G$, and $Y$ a smooth
submanifold of $G/\Gamma$. In 2002, Margulis [Mar02] asked the following question in a chapter
of the book A panorama of number theory or the view from Baker’s garden:

(Q) What is the distribution of $gY$ in $G/\Gamma$ when $g$ tends to infinity in $G$?

He further divided this question into two subquestions:

(Q1) What is the behavior of $gY$ ‘near infinity’ in $G/\Gamma$?

(Q2) What is the distribution of $gY$ in the ‘bounded part’ of $G/\Gamma$?

These questions are connected to various problems in Diophantine approximation and
number theory, for instance, the lattice counting problems [DRS93, EM93, EMS96],
the Sprindžuk conjecture on very well approximable (VWA) vectors [Spr80, KM98]
and Dirichlet-improvable (DI) matrices [DS70a, DS70b, KW08, Sha09a].

In this paper, we are particularly interested in the distribution of translates $g_t Y$, where
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If $Y$ is an open subset of the full expanding horosphere, using the thickening method
originating in Margulis’ thesis and the mixing property of the diagonal flows in homoge-

When $Y$ is a real-analytic curve, Kleinbock and Margulis studied (Q1) for $G = \text{SL}_n(\mathbb{R})$
in the seminal work [KM98], and later the first named author studied (Q2) for $G = \text{SL}_n(\mathbb{R})$
in [Sha09a, Sha10] and $G = \text{SO}(n, 1)$ in [Sha09b]. Later Aka, Breuillard,

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Rosenzweig, and de Saxcé [ABRS18] generalized the results in [KM98] to Grassmannians; in particular, they discovered that the obstruction to non-divergence are certain Schubert subvarieties called constraining pencils. Based on these works, in [Yan20], the second named author studied the case where $G$ is a semisimple algebraic group and defined unstable Schubert varieties as a natural generalization of the constraining pencils, which are potential obstructions to the non-escape of mass. We say that $Y$ is non-degenerate if $Y$ is not contained in any unstable Schubert variety. In [Yan20], it was shown that translates of a non-degenerate curve $Y$ has no escape of mass and a necessary and sufficient condition for equidistribution of the translates $g_tY$ is given (again assuming non-degeneracy).

Now, the case where $Y$ is degenerate remains to be studied. The first nontrivial case is $G = \text{SL}_3(\mathbb{R})$. In a preprint [CY19], S. Chow and L. Yang obtained an effective equidistribution result where $Y$ is a Diophantine line, and the translating elements are in a specific cone of the full diagonal subgroup. Later, in a joint work [KdSSY23] of authors with D. Kleinbock and N. de Saxcé, the flow $\text{diag}(e^{2t}, e^{-t}, e^{-t})$ was studied, and necessary and sufficient conditions for both the non-escape-of-mass and equidistribution were provided. For $G = \text{SL}_4(\mathbb{R})$, R. Shi and B. Weiss [SW17] gave an example of a line defined over a real quadratic number field, such that the translates are stuck in a fixed compact set; in particular, they do not get equidistributed.

In this paper, we consider the case where $G = \text{SL}_n(\mathbb{R})$ and $g_t = \text{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t})$, and we study translates of a degenerate real-analytic submanifold of the expanding horosphere; here nondegeneracy means $Y$ is contained in a proper affine subspace of the horosphere, which is isomorphic to $\mathbb{R}^{n-1}$ in this case. We provide necessary and sufficient Diophantine or arithmetic conditions on the affine span of $Y$ for both non-escape-of-mass and equidistribution.

1.2. Statement of the main results. Let $G = \text{SL}_n(\mathbb{R})$, $\Gamma = \text{SL}_n(\mathbb{Z})$ and $X = G/\Gamma$. Let $\mu_X$ denote the unique $G$-invariant probability measure on $X$. Let 

$$g_t = \text{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t}),$$

so that the expanding horospherical subgroup of $G$ associated to $g_t$ is

$$U^+ = \{ g \in G \mid g_{-t}g_t \to e, t \to +\infty \} = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right\}. \quad (1.1)$$

Let $x_0 = \varepsilon \Gamma \in X$. Let $B$ be an open ball in a finite-dimensional Euclidean space. Let $\phi : B \to U^+ \cong \mathbb{R}^{n-1}$ be a nonconstant real-analytic map. Let $\lambda$ be a probability measure that is absolutely continuous with respect to the Lebesgue measure on the Euclidean space with support contained in $B$. Let $\lambda_\phi$ denote the push-forward of $\lambda$ under the map $s \mapsto \phi(s)x_0$ from $B$ to $X$. Let $g_t\lambda_\phi$ denote the push-forward of $\lambda_\phi$ by $g_t$, i.e. $g_t\lambda_\phi(E) = \lambda_\phi(g_t^{-1}E)$ for any measurable $E \subset X$. We are interested in the limiting distribution of the family $\{g_t\lambda_\phi\}_{t \geq 0}$ of translated measures.

We say that a family $\{\lambda_i\}_{i \in \mathcal{F}}$ of probability measures on $X$ have no escape of mass if for every $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $\lambda_i(K) > 1 - \varepsilon$ for all $i \in \mathcal{F}$. 
Furthermore, for $\mathcal{F} = \mathbb{N}$ or $\mathbb{R}_{\geq 0}$, we say that a family $\{\lambda_i\}_{i \in \mathcal{F}}$ of probability measures on $X$ get equidistributed in $X$ if

$$\int f \, d\lambda_i \xrightarrow{i \to \infty} \int f \, d\mu_X, \forall f \in C_c(X);$$

that is, $\lambda_i$ converges to $\mu_X$ with respect to the weak-* topology as $i \to \infty$.

Our main results give criteria for the non-escape of mass and equidistribution of the translates $\{g_t \lambda_\phi\}_{t \geq 0}$. It turns out that these phenomena depend only on the affine span of $\phi(B)$.

Let us first parametrize the affine span $L_\phi$ of $\phi(B)$. Since $\phi$ is nonconstant, the dimension of $L_\phi$ is $d - 1$ for some $2 \leq d \leq n$.

Suppose $d < n$, then by permuting the coordinates, which commutes with the $g_t$-action, we may assume that $L_\phi$ is of the form

$$L_\phi = \{(x, \tilde{x} A_\phi) \mid x \in \mathbb{R}^{d-1}\},$$

where $A_\phi \in M_{d,n-d}(\mathbb{R})$. Here $\tilde{x} = (1, x)$ for any row vector $x \in \mathbb{R}^{d-1}$. We wish to phrase our criteria in terms of certain Diophantine and arithmetic properties of $A_\phi$ or $L_\phi$. Therefore, we introduce the following definitions.

Let $m, l$ be positive integers, and let $\|\cdot\|$ denote the sup-norm. Given a positive real number $r$, let $W_r(m, l)$ denote the set of matrices $A \in M_{m,l}(\mathbb{R})$ for which there exists $C > 0$ such that the inequality

$$\|A q + p\| \leq C \|q\|^{-r}. \quad (1.2)$$

has infinitely many solutions $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^l$.

Similarly, let $W'_r(m, l)$ denote the set of matrices $A \in M_{m,l}(\mathbb{R})$ for which (1.2) has a nonzero solution $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^l$ for every $C > 0$.

We will prove the following criterion for the non-escape of mass.

**Theorem 1.1.** The translates $\{g_t \lambda_\phi\}_{t \geq 0}$ have no escape of mass if and only if one of the following holds:

1. $d = n$.
2. $d < n$ and $A_\phi \notin W'_{n-1}(d, n-d)$.

It follows from the main theorem of [BD86, page 353] that the Hausdorff dimension of $W'_{n-1}(d, n-d)$ is $d(n-d) - d + 1$, which is strictly smaller than $d(n-d)$ since $d \geq 2$.

Before stating our equidistribution criteria, we will introduce some notations. Let $N = \binom{n-2}{2}$. Given any $A = \left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_{n-2} \\ b_1 & b_2 & \cdots & b_{n-2} \end{array}\right) \in M_{2,n-2}(\mathbb{R})$, we define an associated matrix $A^{\text{ext}} \in M_{2n-3,N}(\mathbb{R})$ in the following way. In the block matrix form, write

$$A^{\text{ext}} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad (1.3)$$
where the matrices $X = (x_{k,ij})_{1 \leq k \leq n-2}$, $Y = (y_{k,ij})_{1 \leq k \leq n-2}$, $Z = (z_{i1}, \ldots, z_{ij}, \ldots)_{1 \leq i < j \leq n-2}$ are given by

$$
x_{k,ij} = \begin{cases} 
a_j & k = i \\
-a_i & k = j \\
0 & k \neq i, j
\end{cases}, \quad y_{k,ij} = \begin{cases} 
b_j & k = i \\
-b_i & k = j \\
0 & k \neq i, j
\end{cases}, \quad z_{ij} = a_j b_i - a_i b_j.
$$

We will prove the following criterion for equidistribution.

**Theorem 1.2.** The translates $\{g_t \lambda_\phi\}_{t \geq 0}$ get equidistributed in $X$ if and only if none of the following occurs:

1. $d < n$ and $A_\phi \in \mathcal{W}_{n-1}(d, n - d)$.
2. There exist integers $r \geq d$, $m \geq 2$ with $rm = n$, and a number field $\mathbb{K} \subset \mathbb{R}$ with $[\mathbb{K} : \mathbb{Q}] = m$, such that $\mathcal{L}_\phi$ is contained in some $(r - 1)$-dimensional affine subspace of $\mathbb{R}^{n-1}$ which is defined over $\mathbb{K}$.
3. $n \geq 4$ is even, $d = 2$, and $A_{\phi}^{\text{ext}} \in \mathcal{W}_{n-2}(2n - 3, N)$.

When $n$ is a prime number, the (2) and (3) in the above theorem do not occur. Hence, we have the following:

**Corollary 1.3.** Suppose that $n$ is a prime number. Then exactly one of the following holds:

1. The translates $\{g_t \lambda_\phi\}_{t \geq 0}$ get equidistributed in $X$.
2. $d < n$ and $A_\phi \in \mathcal{W}_{n-1}(d, n - d)$.

Compared to previous works [Sha09a, Yan20, KdSSY23], the main difficulty and novelty of this article is the classification of maximal intermediate subgroups, which arise when we apply Ratner’s theorem and linearization technique. We use tools in geometric invariant theory to deal with non-closed group orbits in linear representations. In the remaining cases, where we have closed orbits, we first classify the maximal intermediate subgroup over an algebraically closed field and then use Galois cohomology to classify the possible $\mathbb{Q}$-groups that could arise.

If $\phi$ is only assumed to be a smooth map, then non-divergence and equidistribution no longer depend only on the affine span of the submanifold, and one may need to impose some local conditions on the derivatives of $\phi$. The main difficulty in working with smooth curve case is that the $(C, \alpha)$-good properties needed for the linearization technique do not work for translates of a fixed piece of a smooth curve. To address this issue, one studies expanding translates of shrinking curves as done in [Sha09c, SY23, SY24] for non-degenerate smooth curves. However, it is unclear how to adapt this article’s geometric invariant theory technique to study the translates of shrinking curves.

### 1.3. An application to Dirichlet-improvable vectors on degenerate manifolds.

The motivation for our study comes from the Diophantine approximation. Denote by $\| \cdot \|$ the supremum norm on $\mathbb{R}^{n-1}$, where $n \geq 2$ (unless specified otherwise, all the norms will be taken to be the supremum norm). Let $\mathcal{T} = \{T_i\} \subset \mathbb{R}$ such that $T_i \to \infty$ as $i \to \infty$, and let $0 < \delta \leq 1$. Following Davenport and Schmidt [DS70a], let $\text{DI}(\mathcal{T}, \delta)$ denote the
set of vectors \( \mathbf{x} \in \mathbb{R}^{n-1} \) such that for all large \( T \in \mathcal{T} \), the system of inequalities

\[
\begin{align*}
|\mathbf{x} \cdot \mathbf{q} + p| &\leq \delta T^{-(n-1)} \\
|\mathbf{q}| &\leq T
\end{align*}
\]

has a solution \((p, \mathbf{q})\), where \( p \in \mathbb{Z} \) and \( \mathbf{q} \in \mathbb{Z}^{n-1} \setminus \{0\} \). Similarly, let \( \text{DI}'(\mathcal{T}, \delta) \) denote the set of vectors \( \mathbf{x} \in \mathbb{R}^{n-1} \) such that for all large \( T \in \mathcal{T} \), the system of inequalities

\[
\begin{align*}
\|\mathbf{q}\mathbf{x} + \mathbf{p}\| &\leq \delta T^{-1} \\
|\mathbf{q}| &\leq T^{n-1}
\end{align*}
\]

has a solution \((\mathbf{p}, \mathbf{q})\), where \( \mathbf{p} \in \mathbb{Z}^{n-1} \) and \( \mathbf{q} \in \mathbb{Z} \setminus \{0\} \). By Dirichlet’s theorem \( \text{DI}(\mathcal{T}, 1) = \mathbb{R}^{n-1} \) and \( \text{DI}'(\mathcal{T}, 1) = \mathbb{R}^{n-1} \). By a theorem of Davenport and Schmidt, for any \( 0 < \delta < 1 \), the sets \( \text{DI}(\mathcal{T}, \delta) \) and \( \text{DI}'(\mathcal{T}, \delta) \) are Lebesgue null, see [DS70b, KW08]. After Davenport and Schmidt [DS70a], let \( B \) be an open ball in a finite-dimensional Euclidean space and \( \phi : B \rightarrow \mathbb{R}^{n-1} \) be a \( C^\infty \) map, and we want to know under what condition on \( \phi \) one can say that for Lebesgue a.e. \( s \in B \), we have \( \phi(s) \notin \text{DI}(\mathcal{T}, \delta) \cup \text{DI}'(\mathcal{T}, \delta) \) for any \( \delta < 1 \); in other words, the Dirichlet’s theorem cannot be improved for \( \phi(s) \) for a.e. \( s \in B \). In a series of articles [Sha09a, Sha10, SY23], it was shown that the above non-improvability statement holds when the affine span of \( \phi(B) \) equals \( \mathbb{R}^{n-1} \). In [KdSSY23] it was proved that if \( n = 3 \) and \( \phi(B) \) is not contained in a rational line in \( \mathbb{R}^2 \), then for Lebesgue almost all \( s \in B \), \( \phi(s) \notin \text{DI}([N], \delta) \cup \text{DI}'([N], \delta) \) for any \( \delta < 1 \). In this article, we generalize the result for all \( n \geq 3 \) in terms of Diophantine and algebraic properties of the affine span of \( \phi(B) \).

As an application of our dynamical result Theorem 1.2, we will obtain the following using Dani’s correspondence [Dan85, KW08].

**Theorem 1.4.** Let \( \phi : B \rightarrow \mathbb{R}^{n-1} \) be a real-analytic map. Suppose that none of (1)(2)(3) in Theorem 1.2 occurs. Let \( \mathcal{T} = \{T_i\} \subset \mathbb{R} \) such that \( T_i \rightarrow \infty \) as \( i \rightarrow \infty \), then for Lebesgue-a.e. \( s \in B \), \( \phi(s) \notin \text{DI}(\mathcal{T}, \delta) \cup \text{DI}'(\mathcal{T}, \delta) \) for any \( \delta < 1 \). In fact, for a.e. \( s \in B \), for every \( \delta > 0 \), there exists a sequence \( i_j \rightarrow \infty \) such that (1.4) and (1.5) are not solvable for \( \mathbf{x} = \phi(s) \) and \( T = T_i \) for all \( j \).

**Proof.** Let \( \{e_i : 1 \leq i \leq n\} \) denote the standard basis of \( \mathbb{R}^n \) and \( \mathbf{w} \) be the matrix such that \( \mathbf{w}e_i = e_{n-i+1} \) for all \( i \). Let \( L = G \times G \) and \( \Lambda = \Gamma \times \Gamma \). Let \( \rho(g) = (g, \mathbf{w}(g^{-1}) \mathbf{w}^{-1}) \) for all \( g \in G \). Then \( \rho : G \rightarrow L \) is an injective homomorphism. Then \( \rho(\Gamma) \subset \Lambda \) and \( \rho(\Gamma) \) is a lattice in \( \rho(G) \). Let \( \bar{\rho} : G/\Gamma \rightarrow L/\Lambda \) be the map \( \bar{\rho}(g) = \rho(g)\Lambda \) for all \( g \in G \). Then \( \bar{\rho} \) is a continuous injective proper map. Now for any \( f \in C_c(L/\Lambda) \), we have \( f \circ \bar{\rho} \in C_c(G/\Gamma) \).

So, by Theorem 1.2, we get

\[
\lim_{t \rightarrow \infty} \frac{1}{|B|} \int_B f(\rho(a_t)\rho(\phi(s))\Lambda) \, ds = \lim_{t \rightarrow \infty} \frac{1}{|B|} \int_B f \circ \rho(a_t\phi(s))\Gamma \, ds = \int_{G/\Gamma} f \circ \rho \, d\mu_X = \int_{L/\Lambda} f \, d\mu_{\bar{\rho}(X)},
\]

where \( \mu_{\bar{\rho}(X)} \) is the unique \( \rho(G) \) invariant probability measure on the closed set \( \bar{\rho}(G/\Gamma) \cong \rho(G)/\rho(\Gamma) \) of \( L/\Lambda \). Having shown this, we argue exactly as in Section 2 of [Sha09a] replacing [Sha09a, Theorem 1.3] by the above deduction. \( \square \)
The readers are referred to [KdSSY23, Theorem 1.5] or [SY20, Section 1.2] for a discussion on the deduction of Theorem 1.4 from Theorem 1.2 using Dani’s correspondence [Dan85, KW08]. In fact, for the map φ as in the statement of Theorem 1.4, one also obtains the conclusion of [Sha09a, Theorem 1.4] which is a generalization in the sense of simultaneous non-Dirichlet improvability along a sequence of natural numbers.

Remark 1.5. As we have noted above, if n is a prime number, then (2) or (3) in Theorem 1.2 do not occur. This is the situation in [KdSSY23], where n = 3.

1.4. Notation. In this paper, all algebraic groups are assumed to be affine.

We use boldface capital letters $G, H, F$, etc. to denote algebraic groups (over $\mathbb{Q}$ if not specified) and use Roman capital letters $G, H, F$, etc. to denote the groups of real points, and $G^0, H^0, F^0$, etc. to denote their connected components of identity, respectively.

Let $K$ be a field contained in $\mathbb{R}$ and $G$ an algebraic group over $K$. In this paper, a representation of $G$ always means a finite-dimensional algebraic representation, i.e., a pair $(\rho, V)$ where $V$ is a vector space over $K$ and $\rho : G \rightarrow \text{GL}(V)$ is a morphism of algebraic groups over $K$. We also say that $V$ is a $G$-module. Sometimes, we call $\rho$ or $V$ a representation of $G$ for simplicity. We say that $\rho$ is faithful if the kernel of $\rho$ is trivial and that $\rho$ is irreducible if there is no non-trivial $G$-invariant proper subspace of $V$.

Given a positive integer $r$, let $I_r$ denote the identity $r \times r$ matrix.

Let $X$ be a variety with $G$ acting morphically over $K$, i.e. the $G$-action is given by a morphism $G \times X \rightarrow X$. Following [Bor91, §1.1.7], we define the transporter between subsets $M$ and $N$ of $X(\mathbb{R})$:

$$\text{Tran}_G(M, N) = \{ g \in G \mid gM \subset N \}.$$  

(1.6)

If $M$ and $N$ are defined over $K$, then $\text{Tran}_G(M, N)$ is also defined over $K$. Let $\text{Tran}_G(M, N)$ denote the $K$-points of $\text{Tran}_G(M, N)$.

We write $A = O(B)$ or $A \ll B$ or $B \gg A$ if $A \leq CB$ for some constant $C > 0$, and $A \sim B$ if $C^{-1}B \leq A \leq CB$ for some constant $C \geq 1$. For sequences $\{A_i\}_{i \in \mathbb{N}}$ and $\{B_i\}_{i \in \mathbb{N}}$ we write $A_i = o(B_i)$ if there exists a sequence $c_i \rightarrow 0$ such that $A_i = c_i B_i$.

For a finite field extension $K/\mathbb{F}$ and an algebraic group $G$ over $K$, we write $\text{Res}_{K/\mathbb{F}} G$ for the restriction of scalar (or Weil restriction) of $G$ from $K$ to $\mathbb{F}$.

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2. Instability in invariant theory

In this section, we define and study the unstable vectors in representations. Our main tool is Kempf’s numerical criterion from geometric invariant theory.

Let $G$ be a connected reductive algebraic group defined over a field $K$ of characteristic 0. There exists a $G(K)$-invariant norm $\| \cdot \|$ on the set $X_*(G) := \text{Hom}(\mathbb{G}_m, G)$ of cocharacters of $G$ defined over $K$.

Let $\rho : G \rightarrow \text{GL}(V)$ be a representation of $G$ defined over $K$. Let us first recall some notations from [Kem78]. For any $\lambda \in X_*(G)$ and $v \in V$, we have the decomposition
\[ v = \sum_{i \in \mathbb{Z}} v_i, \text{ where } \lambda(t)v_i = t^i v_i \text{ for all } i. \] Let 
\[ m(v, \lambda) = \min \{ i \in \mathbb{Z} : v_i \neq 0 \}. \] (2.1)

We say that a nonzero vector \( v \) is \textit{unstable} if the Zariski closure of \( Gv \) contains the origin. Given any \( \lambda \in X_*(G) \), we associate a \textit{parabolic subgroup} \( P(\lambda) \) defined over \( K \), and its \textit{unipotent radical} \( U^+(\lambda) \) defined over \( K \) such that for any field \( \mathbb{F} \supseteq K \) we have
\[ P(\lambda)(\mathbb{F}) = \{ g \in G(\mathbb{F}) : \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \} \] (2.2)
\[ U^+(\lambda)(\mathbb{F}) = \{ g \in G(\mathbb{F}) : \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} = 1 \}. \] (2.3)

Let us recall the following important theorem of Kempf [Kem78, Theorem 4.2]. The form we state here is a special case of Kempf’s original theorem.

**Theorem 2.1** (Kempf, 1978). Let \( 0 \neq v \in V(K) \) be an unstable vector. Then

1. The function \( \lambda \mapsto m(v, \lambda)\|\lambda\| \) has a positive maximum value \( B_v \).
2. Let \( \Lambda_v \) be the set of indivisible \( K \)-cocharacters \( \lambda \) such that \( m(v, \lambda) = B_v\|\lambda\| \). Then
   (a) \( \Lambda_v \) is non-empty.
   (b) There is a parabolic \( K \)-subgroup \( P_v \) of \( G \) such that \( P_v = P(\lambda) \) for any \( \lambda \in \Lambda_v \).
   (c) \( \Lambda_v \) is a principal homogeneous space under conjugation by the \( K \)-points of the unipotent radical of \( P_v \).
   (d) Any \( K \)-maximal torus of \( P_v \) contains the image of a unique member of \( \Lambda_v \).
   (e) For any \( g \in G(K) \), \( \Lambda_{gv} = g\Lambda_v g^{-1} \) and \( P_{gv} = gP_v g^{-1} \). [Kem78, Corollary 3.5].
   (f) For any \( l \in P_v(K) \), \( \Lambda_{vl} = \Lambda_v \) and \( P_{vl} = P_v \). (Consequence of (2c) and (2e).)

If necessary, we will write \( \Lambda_v^G \) instead of \( \Lambda_v \) to allow the ambient group to vary.

In the remaining part of the section, we will further assume that \( G \) is a connected \( K \)-split semisimple group\(^1\). We pick a maximal \( K \)-split torus \( S \) of \( G \). Then \( S \) is also a maximal torus in \( G \) defined over \( K \).

**Proposition 2.2.** Let \( V \) be a representation of \( G \) defined over \( K \). Let \( v \) be an unstable vector in \( V(K) \). Then there exists an irreducible representation \( W \) of \( G \) defined over \( K \), a highest weight vector \( w \in W(K) \), an element \( g_0 \in G(K) \), and constants \( C > 0, \beta > 0 \) such that for any \( g \in G \) one has
\[ \|gg_0w\| \leq C\|gv\|^\beta. \] (2.4)

First we will recall some definitions, facts, and notation.

2.1. **Root system and highest weights.** Let \( X_*(S) = \text{Hom}(G_m, S) \cong \mathbb{Z}^r \) denote the group of cocharacters of \( S \) defined over \( K \). Let \( X^*(S) = \text{Hom}(S, G_m) \cong \mathbb{Z}^r \) denote the group of characters of \( S \) defined over \( K \). We have a pairing \( \langle \cdot, \cdot \rangle : X^*(S) \times X_*(S) \to \text{Hom}(G_m, G_m) \cong \mathbb{Z} \) given by \( \langle \chi, \lambda \rangle = \chi \circ \lambda \); that is, \( \chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle} \) for any \( t \in \mathbb{C}^* \). This pairing is a \textit{perfect} (\textit{dual}) \textit{pairing}, meaning the canonical inclusions \( X^*(S) \hookrightarrow \text{Hom}(X_*(S), \mathbb{Z}) \) and \( X^*(S) \hookrightarrow \text{Hom}(X^*(S), \mathbb{Z}) \), induced by the pairing, are isomorphisms.

For the following material, the reader may refer to [C24, Appendix G. Root datum].

\(^1\)For the purpose of this article, it is sufficient to focus on the case of \( G = SL_n \) and \( K = \mathbb{Q} \).
The group $\mathcal{W} = N_{G}(S)/Z_{G}(S) = N_{G}(S)/S$ is called the $\mathbb{K}$-Weyl group of $G$ relative to $S$.

Let $\Phi = \Phi(G, S) \subset X^{\ast}(S)$ denote the set of (non-trivial) $\mathbb{K}$-roots on $S$ for the Adjoint action of $S$ on the Lie algebra of $G$. For each $\alpha \in \Phi$, $G_{\alpha} := [Z_{G}((\ker \alpha)^{\circ}), Z_{G}((\ker \alpha)^{\circ})]$ is isomorphic to $SL_{2}$ or $PGL_{2}$ over $\mathbb{K}$, where $^{\circ}$ denote the irreducible component of the identity. We pick a unique co-root $\hat{\alpha} \in X_{\ast}(S)$ such that $\hat{\alpha}(G_{\alpha})$ is in $S \cap G_{\alpha}$ and $\langle \alpha, \hat{\alpha} \rangle = 2$. And we pick $n_{\alpha} \in N_{G}(S)(\mathbb{K}) \cap G_{\alpha}(\mathbb{K})$ that conjugates $\hat{\alpha}$ to its inverse. We denote its image in $\mathcal{W}$ by $w_{\alpha}$. We denote the actions of $w_{\alpha}$ on $X^{\ast}(S)$ and $X_{\ast}(S)$ by $s_{\alpha}$ and $s_{\hat{\alpha}}$, respectively. Then $s_{\alpha}(\chi) = \chi - \langle \chi, \hat{\alpha} \rangle \alpha$ and $s_{\hat{\alpha}}(\delta) = \delta - \langle \alpha, \delta \rangle \hat{\alpha}$. Also, $\langle s_{\alpha}(\chi), \lambda \rangle = \langle \chi, s_{\hat{\alpha}}(\lambda) \rangle$. Moreover, the Weyl group $\mathcal{W}$ is generated by $\{w_{\alpha} : \alpha \in \Phi\}$ (see [C24, Corollary G.2.11]).

There exists a positive definite bilinear form $\langle \cdot, \cdot \rangle$ on $X_{\ast}(S)$ taking values in $\mathbb{Z}$, and it is invariant under the Weyl group $\mathcal{W}$. For example, let

$$\langle \lambda, \lambda' \rangle := \sum_{\alpha \in \Phi(G, S)} \langle \alpha, \lambda \rangle \langle \alpha, \lambda' \rangle.$$ 

Let $\| \lambda \| := \sqrt{\langle \lambda, \lambda \rangle}$. Being $\mathcal{W}$-invariant, the norm $\| \cdot \|$ on $X_{\ast}(S)$ extends uniquely to a $G(\mathbb{K})$-invariant norm on $X_{\ast}(G)$. This is so because the canonical injection

$$\mathcal{W} \backslash X_{\ast}(S) \to G(\mathbb{K}) \backslash X_{\ast}(G)$$

is a surjection [Bor69, 20.19] and [Kem78, Lemma 2.1].

Using this bilinear form and the perfect pairing $\langle \cdot, \cdot \rangle$, we can identify $X_{\ast}(S)$ with $X^{\ast}(S)$, via the map $\delta \mapsto \hat{\delta}$, defined by

$$\langle \hat{\delta}, \lambda \rangle = \langle \delta, \lambda \rangle, \forall \lambda \in X_{\ast}(S).$$

This identification gives a positive definite $\mathcal{W}$-invariant integral bilinear form on $X^{\ast}(S)$, also denoted by $\langle \cdot, \cdot \rangle$. Then for all $\alpha \in \Phi$, $(\hat{\alpha})^\wedge = 2\alpha/(\alpha, \alpha)$ (see [C24, Prop.G.2.5]).

There exists $\lambda_{0} \in X_{\ast}(S)$ such that $\langle \alpha, \lambda_{0} \rangle \neq 0$ for all $\alpha \in \Phi$. Let $\Phi^{+} = \{ \alpha \in \Phi : \langle \alpha, \lambda_{0} \rangle > 0 \}$. There exists a unique $\Delta = \{ \alpha_{1}, \alpha_{2}, \ldots, \alpha_{r} \} \subset \Phi^{+}$, called the set of simple roots relative to $\lambda_{0}$, such that every element of $\Phi^{+}$ can be expressed as a non-negative integral combination of elements of $\Delta$, and $\Delta$ is a basis of $X^{\ast}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$. Then $\{ \hat{\alpha}_{i} : i = 1, \ldots, r \}$ is a $\mathbb{Q}$-basis of $X_{\ast}(S) \otimes \mathbb{Q}$. The elements of its dual basis $\{ \mu_{1}, \ldots, \mu_{r} \} \subset X^{\ast}(S)$ are called the $\mathbb{K}$-fundamental weights; that is, $\langle \mu_{i}, \hat{\alpha}_{j} \rangle$ is 0 if $i \neq j$ and is 1 if $i = j$.

Given $\delta \in X_{\ast}(S)$, we have $\delta = \sum_{i=1}^{r} k_{i}\mu_{i}$, where

$$k_{i} = \langle \hat{\delta}, \hat{\alpha}_{i} \rangle \in \mathbb{Z}$$

$$= \langle \delta, \alpha_{i} \rangle = \langle \hat{\alpha}_{i}, \delta \rangle = \langle (\hat{\alpha}_{i})^\wedge, \delta \rangle = \frac{2\langle \alpha_{i}, \alpha_{i} \rangle}{\langle \alpha_{i}, \alpha_{i} \rangle} \langle \alpha_{i}, \delta \rangle = \frac{2}{\langle \alpha_{i}, \alpha_{i} \rangle} \langle \alpha_{i}, \delta \rangle.$$ (7.7)

We will call the elements of the set

$$X_{\ast}(S)^{+} = \{ \lambda \in X_{\ast}(S) : \langle \alpha, \lambda \rangle \geq 0, \forall \alpha \in \Delta \}$$ (7.8)

**dominant cocharacters**, and

$$X_{\ast}(S) = \mathcal{W} \cdot X_{\ast}(S)^{+}.$$ (7.9)

The elements of the following set are called **dominant integral weights**: $$X^{\ast}(S)^{+} = \{ \chi \in X^{\ast}(S) : \langle \chi, \hat{\alpha} \rangle \geq 0, \forall \alpha \in \Delta \}.$$
In other words, a character on \( S \) is a dominant integral weight if and only if it is a non-negative integral linear combination of the fundamental weights. By (2.7),

\[
\delta \in X_*(S)^+ \iff \hat{\delta} \in X^*(S)^+.
\]

Let \( N^+ = U^+(\lambda_0) \), see (2.3). It is a maximal unipotent subgroup of \( G \). Also, \( U^+(\lambda) = N^+ \) for all \( \lambda \in X_*(S) \) such that \( \langle \alpha, \lambda \rangle > 0 \) for all \( \alpha \in \Phi^+ \). So, \( N^+ \) depends only on \( \Phi^+ \).

A nonzero vector, say \( w \), in a finite-dimensional representation of \( G \) is called a highest weight vector if it is fixed by \( N^+ \), and \( S \) acts on the line containing \( w \) via a character, called a highest weight. Any highest weight is a dominant integral weight. Conversely, we have the following:

**Highest weight theorem.** Every dominant integral weight is a highest weight of a unique (up to isomorphism) irreducible, finite-dimensional representation of \( G \) defined over \( K \).

Moreover, any irreducible \( K \)-representation of \( G \) admits a unique highest weight, and the corresponding weight space is one dimensional and defined over \( K \).

We fix a maximal compact subgroup \( K \) of \( G \) such that \( S = S(\mathbb{R}) \) is invariant under the Cartan involution of \( G \) associated with \( K \). By Iwasawa decomposition, \( G = K S^0 N^+ \). Without loss of generality, we may assume that any norm on a finite-dimensional representation of \( G \) over \( \mathbb{R} \) considered in this section is \( K \)-invariant.

2.1.1. **Example.** Let \( G = SL_n \) over \( K = \mathbb{Q} \) and \( S \) be the full diagonal subgroup of \( SL_n \). Then \( X_*(S) \cong \mathbb{Z}^{n-1} \), where any \( \delta \in X_*(S) \) is given by \( \delta(t) = \text{diag}(t^{m_1}, \ldots, t^{m_n}) \), where \( (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1} \), and \( m_n = -(m_1 + \cdots + m_{n-1}) \). Then \( \Phi = \{ \alpha_{i,j} : i \neq j, 1 \leq i, j \leq n \} \), where \( \alpha_{i,j}(\text{diag}(t_1, \ldots, t_n)) = t_i/t_j \). Then \( G_{\alpha_{i,j}} \) is the copy of \( SL_2 \) corresponding to the coordinates \((i',j')\), where \( i',j' \in \{i,j\} \). So,

\[
\alpha_{i,j}(t) = \text{diag}(1, \ldots, t, \ldots, t^{-1}, \ldots, 1)
\]

where \( i \)-th entry is \( t \), \( j \)-the entry is \( t^{-1} \). One chooses \( \Delta = \{ \alpha_i := \alpha_{i,i+1} : 1 \leq i \leq n-1 \} \). Then the fundamental weight \( \mu_i(\text{diag}(t_1, \ldots, t_n)) = t_1 \cdots t_i \).

We consider the bilinear form \( \langle , \rangle \) on \( X_*(S) \cong \mathbb{Z}^{n-1} \) given by

\[
((m_1, \ldots, m_{n-1}), (m'_1, \ldots, m'_{n-1})) = \sum_{i=1}^{n} m_i m'_i,
\]

where \( m_n, m'_n \) are as defined as above. It is invariant under the Weyl group of \( S \); the Weyl group is represented by permutations of the standard basis.

We note that given \( \delta \) as above corresponding to \((m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1} \), we have

\[
\hat{\delta} = \sum_{i=1}^{n-1} (m_i - m_{i+1}) \mu_i.
\]

Thus, \( \delta \) is a dominant integral weight if and only if \( m_1 \geq m_2 \geq \cdots \geq m_n \).

Also, \( N^+ \) is the group of all \( n \times n \) upper triangular matrices with 1 in all the diagonal entries. We choose \( K = SO(n) \). Then \( G = K S^0 N^+ \); here, \( S^0 \) consists of all diagonal matrices in \( G \) with positive entries.

Let \( V \) denote the standard representation of \( SL_n \) with the standard basis \( e_1, \ldots, e_n \). Then for each \( 1 \leq i \leq n-1 \), \( W_i = \wedge^i V \) is an irreducible representation of \( SL_n \) defined
over $\mathbb{Q}$ with the highest weight $\mu_i$ and $w_i := e_1 \wedge \ldots \wedge e_i \in W_i(\mathbb{Q})$ is a highest weight vector.

**Proof of Proposition 2.2.** Given a nonzero unstable $v \in V(\mathbb{K})$, by Theorem 2.1, we pick $\lambda \in \Lambda_v$. By (2.5) and (2.9), we can pick $g_0 \in G(\mathbb{K})$ such that $\delta := g_0^{-1} \lambda g_0 \in X_\epsilon(S)^{\prime}$. Let $v' = g_0^{-1}v$. Then $\delta \in X_\epsilon(S)^+ \cap \Lambda_{v'}$.

Let $\delta \in X^*(S)$ be as in (2.6). Then by (2.7), $\hat{\delta}$ is a dominant integral weight. Therefore, by the highest weight theorem, we pick an irreducible representation $W$ of $G$ defined over $\mathbb{K}$ and a vector $w \in W(\mathbb{K}) \neq 0$ such that $w$ is fixed by $N^+$ and $S$ acts on the line containing $w$ via the character $\hat{\delta}$.

Let $\beta = (\delta, \delta)/m(v', \delta) > 0,$

where the function $m(\cdot, \cdot)$ is defined in (2.1) and $m(v', \delta) > 0$ by Theorem 2.1(1). Let $g_0 = (n_1 g_1)^{-1} \in G(\mathbb{K})$. Then $g_0 v' = v$. So, to prove (2.4), it suffices to show that there exists $C > 0$ such that for any $g \in G$ we have

$$\|gw\| \leq C\|gv'\|^\beta.$$ 

We argue by contradiction. Suppose there exists a sequence $g_i$ in $G$ such that

$$\lim_{i \to \infty} \|g_i v'\|^\beta/\|g_i w\| = 0.$$ (2.10)

Let $S_1 = \ker(\hat{\delta})^\circ$, which is a $\mathbb{K}$-split subtorus of $S$ of rank $r - 1$. Then $S^0 = S(\mathbb{R})^0 = \delta(\mathbb{R}_{>0})S_1(\mathbb{R})^0$. The centralizer of $\delta$ in $G$, denoted by $Z_G(\delta)$, is a connected reductive $\mathbb{K}$-subgroup of $G$ (see [C24, Prop.1.4.3]). Let $U_1 = N^+ \cap Z_G(\delta)$. Then $N^+ = U_1 \ltimes U^+(\delta)$ and $N^+ = U_1 U^+(\delta)$. Let $H = S_1[Z_G(\delta), Z_G(\delta)]$. Then $H$ is a connected reductive $\mathbb{K}$-subgroup, and $S_1$ is a maximal torus of $H$. We note that $U_1 \subset [Z_G(\delta), Z_G(\delta)]$. Therefore, by the Iwasawa decomposition,

$$G = KS^0N^+ = K\delta(\mathbb{R}_{>0})S_1U_1U^+(\delta) = K\delta(\mathbb{R}_{>0})HU^+(\delta).$$

We express each $g_i = k_i \delta(\tau_i) h_i u_i$, where $k_i \in K$, $\tau_i > 0$, $h_i \in S_1 U_1 \subset H$, and $u_i \in U^+(\delta)$.

Then, since the norm is $K$-invariant and $w$ is fixed by $S_1$ and $N^+$, by (2.6),

$$\|g_i w\| = \|\delta(\tau_i) w\| = |\cdot\|w\| = \tau_i^{(\delta, \delta)} \|w\| = \tau_i^{(\delta, \delta)} \|w\|. $$ (2.11)

Now we express $V = \oplus_{i \in \mathbb{Z}} V_i$, where for each $i$,

$$V_i = \{ x \in V : \delta(t)x = t^i x, \forall t \in \mathbb{C}^* \}$$

is defined over $\mathbb{K}$. Let $\pi : V \to V_{m(v', \delta)}$ be the corresponding projection, which is defined over $\mathbb{K}$. Since $H$ centralizes $\delta(G_m)$, each $V_i$ is $H$-invariant, and hence $\pi$ is $H$-equivariant.

We claim that

$$\pi(auv') = \pi(v'), \forall u \in U^+(\delta).$$ (2.12)

To prove this claim, let $u \in U^+(\delta)$. Then $\delta(t)u \delta(t)^{-1} \to e$ as $t \to 0$. By the definition of $m(v', \delta)$, see (2.1), $v' = \sum_{i \geq m(v', \delta)} v'_i$, where $v'_i \in V_i$ for each $i$. Now, for each $i$,

$$t^{-i} \delta(t)uv'_i = t^{-i}(\delta(t)u \delta(t)^{-1})(\delta(t)v'_i) = (\delta(t)u \delta(t)^{-1})v'_i \to v'_i, \text{ as } t \to 0,$$

and hence $uv'_i \in v'_i + \oplus_{j > i} V_j$. Therefore,

$$uv' \in v'_{m(v', \delta)} + \oplus_{j > m(v', \delta)} V_j.$$
Hence, $\pi(uv') = v'_{m(v', \delta)} = \pi(v')$, which proves the claim.

Since $g_i = k_i(\tau_i)h_iu_i$, we get

$$
\|g_iu'\| = \|\delta(\tau_i)h_iu_iu'\| \\
\geq \|\pi(\delta(\tau_i)h_iu_iu')\| \\
= \|\delta(\tau_i)\pi(h_iu_iu')\| \\
= \tau_i^{m(v', \delta)}\|h_i\pi(u_iu')\| \\
= \tau_i^{m(v', \delta)}\|h_i\pi(v')\|, \text{ by } (2.12).
$$

Combining (2.11) and (2.13),

$$
\|g_iu'\|^\beta/\|g_iw\| = \frac{\tau_i^{m(v', \delta)}\|h_i\pi(v')\|^\beta}{\tau_i^{\delta(\delta)}\|w\|} = \frac{\|h_i\pi(v')\|^\beta}{\|w\|},
$$

as $m(v', \delta) = (\delta, \delta)$. So by (2.10), $\|h_i\pi(v')\| \to 0$ as $i \to \infty$. Therefore $\pi(v') \in V(\mathbb{K})$ is H-unstable in $V_{m(v', \delta)}$.

We apply Theorem 2.1 to the representation $V_{m(v', \delta)}$ of $H$. Since $S_1$ is a maximal $\mathbb{K}$-split torus of $H$, there exists $l \in H(\mathbb{K})$ such that $\Lambda^H_{m(\pi(v'))}$ contains a unique element of $X^*_+(S_1)$, say $\delta_l$. Since $S_1 \subset \ker(\hat{\delta})$,

$$
(\delta, \delta_l) = (\hat{\delta}, \delta_l) = \hat{\delta} \circ \delta_l = 0 \in \text{Hom}(G_m, G_m).
$$

Since $l \in H \subset Z_G(\delta) \subset P(\delta) = P_{\nu}$, by Theorem 2.1 we have $\Lambda_{l_{\nu}} = \Lambda_{\nu}$ and thus $\delta \in \Lambda_{l_{\nu}}$. By the definition of $\Lambda_{\nu}$ in Theorem 2.1, we have $B_{lv'} = \frac{m(lv', \delta)}{\|\delta_l\|}$. Also, $m(lv', \delta) = m(v', \delta)$.

For any positive integer $N$, let $\delta_N = N\delta + \delta_l \in X^*_+(S)$. For $N$ large enough, we claim that

$$
\frac{m(lv', \delta_N)}{\|\delta_N\|} > \frac{m(lv', \delta)}{\|\delta\|} = B_{lv'},
$$

which will contradict the maximality of $B_{lv'}$.

To prove the claim, consider the weight space decomposition $V = \bigoplus V_\chi$, where $S$ acts on $V_\chi$ by multiplication via the character $\chi$ of $S$. For $v \in V$, let $v_\chi$ denote its $V_\chi$ component in the above decomposition. Let

$$
\Xi = \{\chi \in X^*(S) \ | \ (lv'_\chi) \neq 0 \text{ and } \langle \chi, \delta \rangle = m(lv', \delta) = m(v', \delta)\} = \{\chi \in X^*(S) \ | \ (\pi(lv'))_\chi \neq 0\}.
$$

Since $\delta \in \Lambda_{l_{lv'}}$, there exists $R \in \mathbb{Z}$ such that for any $\chi \in X^*(S) \setminus \Xi$ such that $(lv'_\chi) \neq 0$, we have $\langle \chi, \delta \rangle \geq m(lv'_\chi, \delta) + 1$, and $\langle \chi, \delta_l \rangle \geq R$. Therefore, we can pick $N_0 \in \mathbb{N}$ such that for all $\chi \in X^*(S) \setminus \Xi$ such that $(lv'_\chi) \neq 0$, we have

$$
\frac{\langle \chi, \delta_N \rangle}{\|\delta_N\|} \geq \frac{m(lv', \delta) + 1 + R/N}{\|\delta\| + \|\delta_l\|/N} \geq \frac{m(lv', \delta) + 1/2}{\|\delta\|}, \forall N \geq N_0.
$$

So, to prove (2.15), it suffices to show that for any $\chi \in \Xi$, for all sufficiently large $N$,

$$
\frac{\langle \chi, \delta_N \rangle}{\|\delta_N\|} > \frac{\langle \chi, \delta \rangle}{\|\delta\|}.
$$

(2.16)
To prove (2.16), we define an auxiliary function:

$$f(s) = \frac{\langle \chi, \delta + s \cdot \delta_l \rangle^2}{\|\delta + s \cdot \delta_l\|^2} = \frac{\langle \chi, \delta \rangle^2 + 2s \langle \chi, \delta \rangle \langle \chi, \delta_l \rangle + s^2 \langle \chi, \delta_l \rangle^2}{(\delta, \delta) + 2s(\delta, \delta_l) + s^2(\delta_l, \delta_l)}.$$  

Compute its derivative at 0:

$$f'(0) = \frac{2\langle \chi, \delta \rangle \langle \chi, \delta_l \rangle (\delta, \delta) - 2(\delta, \delta_l) \langle \chi, \delta \rangle^2}{(\delta, \delta)^2}.$$  

Since $\delta_l \in \Lambda_{\text{tr}(v')}^H = \Lambda_{\pi(v')}^H$ and $(\pi(v'))_1 \neq 0$, we know that $\langle \chi, \delta_l \rangle > 0$. Also, we know that $\langle \chi, \delta \rangle = m(v', \delta) > 0$. And $(\delta, \delta_l) = 0$ by (2.14). Therefore $f'(0) > 0$. Hence, for $N$ large, we have

$$f(1/N) > f(0),$$  

and (2.16) follows because each side of (2.17) is the square of each side of (2.16) respectively. \qed 

**Remark 2.3.** Our proof of Proposition 2.2 does not use the irreducibility of $W$. We proved the following: Let $V$ be a $\mathbb{K}$-representation of $G$ and suppose that $v \in V(\mathbb{K})$ is a nonzero $G$-unstable vector. By Theorem 2.1, pick a $g_0 \in G(\mathbb{K})$ such that $g_0^{-1} \Lambda_v g_0 \cap X_*(S)^+ = \{\delta\}$. Let $\beta = (\delta, \delta)/m(g_0^{-1}v, \delta) > 0$. Let $\delta \in X^*(S)^+$ be such that $\langle \delta, \lambda \rangle = (\delta, \lambda)$ for all $\lambda \in X_*(S)$. Let $W$ be any $\mathbb{K}$-representation of $G$ with a highest weight $w \in W(\mathbb{K})$ corresponding to weight $\delta$. Then there exists a $C > 0$ such that $\|gg_0w\| \leq C\|gv\|^{\beta}$ for all $g \in G$.

We recall a lemma from [Kem78].

**Lemma 2.4 ([Kem78], Lem.1.1).** Let $X$ be an affine $G$-scheme over $\mathbb{K}$ and let $Y$ be a closed $G$-subscheme of $X$ over $\mathbb{K}$. Then there exists a $G$-equivariant morphism $f : X \to W$ over $\mathbb{K}$, where $W$ is a finite dimensional representation of $G$ over $\mathbb{K}$, such that $Y = f^{-1}(0)$.

**Corollary 2.5.** Let $V$ be a representation of $G$ over $\mathbb{K}$, and $v \in V(\mathbb{K})$ such that $Gv$ is not Zariski closed. Then there exists an irreducible representation $W$ of $G$ defined over $\mathbb{K}$, a highest weight vector $w \in W(\mathbb{K})$ with nonzero weight $\mu \in X^*(S)^+$, an element $g_0 \in G(\mathbb{K})$, and a constant $\beta > 0$ with the following property: for any $R > 0$, there exists a constant $C > 0$ such that for any $g \in G$,

$$\text{if } \|gv\| \leq R, \text{ then } \|gg_0w\| \leq C\|gv\|^\beta.$$  

**Proof.** By closed orbit lemma [Bor91, I.1.8], $Gv$ is Zariski open in its Zariski closure $\text{Zcl}(Gv)$ in $V$. Since $v \in V(\mathbb{K})$, $G(\mathbb{K})$ is Zariski dense in $G$ (see [Bor91, V.18.3]), and $G$ acts on $V$ over $\mathbb{K}$, we have $G(\mathbb{K})v$ is Zariski dense in $Gv$. Therefore $Y = \text{Zcl}(Gv) \setminus Gv$ is a non-empty Zariski closed subset of $V$ defined over $\mathbb{K}$.

Therefore, by Lemma 2.4, there exists a finite dimensional representation $V_1$ of $G$ defined over $\mathbb{K}$ and a $G$-equivariant morphism $f : V \to V_1$ defined over $\mathbb{K}$ such that $f(v) \neq 0$, and $f(Y') = \{0\}$. Since $Y \subset \text{Zcl}(Gv)$, we have $0 \in \text{Zcl}(Gf(v))$. Therefore $f(v)$ is unstable in $V_1$. Also $f(v) \in V_1(\mathbb{K})$. Therefore, by Proposition 2.2, there exist an irreducible representation $W$ of $G$ defined over $\mathbb{K}$, and a highest weight vector $w \in W(\mathbb{K})$, an element $g_0 \in G(\mathbb{K})$, and constants $\beta > 0$ and $C_1 > 0$ such that for any $g \in G$ we have

$$\|gg_0w\| \leq C_1\|gf(v)\|^\beta.$$
Lemma 2.6. Let $W$ be an irreducible representation of $G$ defined over $\mathbb{K}$ with the highest weight $n_1\mu_1 + \cdots + n_r\mu_r$, where each $n_i \in \mathbb{Z}_{\geq 0}$, and let $w \in W(\mathbb{K})$ be a highest weight vector. Let $N = n_1 + \cdots + n_r$. Let $\Omega$ be a non-empty compact subset of $G$ whose Zariski closure is irreducible. Then there exists a constant $C > 0$ such that for any $h_1, h_2 \in G$,

$$
\sup_{\omega \in \Omega} \|h_1\omega h_2 w\| \geq C \cdot \left( \min_{1 \leq i \leq r} \sup_{\omega \in \Omega} \|h_1\omega \omega h_2 w_i\| \right)^N.
$$

Proof. By Iwasawa decomposition, for $g \in G$, we can write $g = ktu$ for $k \in K, t \in S^0$ and $u \in N^+$. Being highest weights, $w_i$'s and $w$ are fixed by $N^+$. Without loss of generality, we may assume that the norms on $W_i$'s and $W$ are induced by $K$-invariant inner products. Therefore

$$
\|gw_i\| = \mu_i(t)\|w_i\|, \forall i, \text{ and } \|gw\| = \left( \prod_{i=1}^{r} \mu_i(t)^{n_i} \right) \|w\|.
$$

We re-scale the inner product on $W$ such that $\|w\| = \prod_{1 \leq i \leq r} \|w_i\|^{n_i}$. Then

$$
\|gw\| = \prod_{1 \leq i \leq r} \|gw_i\|^{n_i}, \forall g \in G.
$$

Now let $F(g) = \|h_1gh_2w\|^2$ and $F_i(g) = \|h_1gh_2w_i\|^2, \forall i$. Then $F$ and $F_i$'s are regular functions on $G$, and $F(g) = \prod_{1 \leq i \leq r} F_i(g)^{n_i}$. Let $Z$ be the Zariski closure of $\Omega$ in $G$; by our assumption, $Z$ is an irreducible algebraic set. We use the norm $\|F\| = \sup_{\omega \in \Omega} |F(\omega)|$ on the space of regular functions on $Z$.

We claim that for any positive integers $d_1$ and $d_2$, there exists a constant $c = c(d_1, d_2) > 0$ such that for any polynomials $E_1$ and $E_2$ of degrees $d_1$ and $d_2$ respectively on $Z$, we have $\|E_1E_2\| \geq c \|E_1\| \|E_2\|$. Indeed, by homogeneity, we only need to check this for $\|E_1\| = \|E_2\| = 1$, and then the possible values of $\|E_1E_2\|$ form a compact subset of $\mathbb{R}_{>0}$.

Therefore, there exists a constant $C > 0$ such that

$$
\|F\| \geq C \prod_{1 \leq i \leq r} \|F_i\|^{n_i} \geq C \cdot \left( \min_{1 \leq i \leq r} \|F_i\| \right)^N.
$$

□

3. Expansion in linear representations

In this section, we first describe a certain transporter and then prove several results on expansion in linear representations of special linear groups.
3.1. Description of a transporter. Let $G$ be a connected reductive algebraic group over $\mathbb{R}$. Let $\delta : G_m \to G$ be a cocharacter of $G$ defined over $\mathbb{R}$. Let $P(\delta)$ denote the parabolic subgroup of $G$ associated to $\delta$ as in (2.2). Let

$$U^-(\delta) = U^+(\delta^{-1}) = \{g \in G : \lim_{t \to 0} \delta(t)^{-1}g\delta(t) = e\},$$

which is called the expanding horospherical subgroup of $G$ associated to $\delta(t)$ for $|t| < 1$.

Lemma 3.1. Let $H$ be a reductive $\mathbb{R}$-subgroup of $G$ containing $\delta(G_m)$. Then

$$H \cap P(\delta)U^-(\delta) = (P(\delta) \cap H)(U^-(\delta) \cap H).$$

Proof. Let $T$ denote a maximal $\mathbb{R}$-split torus of $H$ containing $\delta(G_m)$. Then $P(\delta) \cap H$ is a parabolic subgroup of $H$ containing $T$ and $U^-(\delta) \cap H$ is the unipotent radical of $P(\delta) \cap H$. Let $N$ denote the normalizer of $T$ in $H$. Then, by Bruhat decomposition on $H$ (see [Bor91, 21.15]),

$$H = (P(\delta) \cap H)N(U^-(\delta) \cap H).$$

Let $h \in H$. Then $h \in (P(\delta) \cap H)nu$ for some $n \in N$ and $u \in U^-(\delta) \cap H$. Since $nTn^{-1} = T \subset P(\delta) \cap H$, and $\delta(\mathbb{R}^*) \subset T$, we have

$$(P(\delta) \cap H)n\delta(\mathbb{R}^*) = (P(\delta) \cap H)n.$$

Therefore, in the quotient space $(P(\delta) \cap H)H$, as $t \to 0$, we have

$$(P(\delta) \cap H)h\delta(t) = (P(\delta) \cap H)(n\delta(t))(\delta(t)^{-1}u\delta(t)) \to (P(\delta) \cap H)n.$$

Further, suppose that $h \in P(\delta)U^-(\delta)$. Then, as $t \to 0$, we have

$$P(\delta)h\delta(t) \to P(\delta)$$

in $P(\delta) \setminus G$. Since $(P(\delta) \cap H)H$ is compact, the natural injection $(P(\delta) \cap H)H \hookrightarrow P(\delta) \setminus G$ is a proper continuous map. Therefore, by (3.3), as $t \to 0$, we have

$$(P(\delta) \cap H)h\delta(t) \to (P(\delta) \cap H)$$

in $(P(\delta) \cap H)H$.

So, by (3.2), we conclude that $(P(\delta) \cap H)n = (P(\delta) \cap H)$. Hence

$$h \in (P(\delta) \cap H)nu \subset (P(\delta) \cap H)(U^-(\delta) \cap H).$$

\[\square\]

Let $W$ be a Zariski closed subset of $G$ containing the identity $e$ of $G$. We write $W^\circ$ for the union of irreducible components of $W$ containing $e$. Let $V$ be a finite-dimensional representation of $G$ defined over $\mathbb{R}$. Let $V_{\geq 0}(\delta)$ denote the direct sum of non-negative weight spaces of $V$ with respect to $\delta$; that is, the weight spaces where $\delta(t)$ acts as non-negative powers of $t$. In other words,

$$V_{\geq 0}(\delta) = \{v \in V : \lim_{t \to 0} \delta(t)v \text{ exists in } V\} = \{v \in V : \lim_{t \to 0} \|\delta(t)v\| < \infty\}.$$

As in (1.6),

$$\text{Tran}_G(w, V_{\geq 0}(\delta)) := \{g \in G : gw \in V_{\geq 0}(\delta)\}.$$

Lemma 3.2. Suppose that $w \in V(\mathbb{R}) \setminus \{0\}$ is such that the isotropy group $G_w$ of $w$ is reductive and contains $\delta(G_m)$. Then we have the following sets of equalities:

$$\text{Tran}_G(w, V_{\geq 0}(\delta)) \cap P(\delta)U^-(\delta) = P(\delta)G_w \cap P(\delta)U^-(\delta) = P(\delta)(G_w \cap U^-(\delta)),$$

$$(\text{Tran}_G(w, V_{\geq 0}(\delta))^\circ = (P(\delta)G_w)^\circ = P(\delta)G_w^\circ).$$

(3.4)

(3.5)
Proof. We write \( Z = \text{Tran}_G(w, V_{>0}(\delta)) \), which is Zariski closed by [Bor91, Sect.I.1.7]. Since \( w \) is fixed by \( \delta(G_m) \), \( Z \) contains the identity \( e \). Using the definition of \( P(\delta) \), it is straightforward to verify that \( P(\delta)V_{>0}(\delta) = V_{>0}(\delta) \). Therefore \( P(\delta)Z = Z = ZG_w \). In particular, \( P(\delta)G_w \subset Z \). Hence 
\[
P(\delta)G \cap P(\delta)U^{-}(\delta) \subset Z \cap P(\delta)U^{-}(\delta).
\]

Since \( \delta(G_m) \) is contained in the reductive subgroup \( G_w \) of \( G \), by Lemma 3.1, 
\[
G_w \cap P(\delta)U^{-}(\delta) = (G_w \cap P(\delta))(G_w \cap U^{-}(\delta)).
\]

Hence 
\[
P(\delta)G_w \cap P(\delta)U^{-}(\delta) = P(\delta)(G_w \cap U^{-}(\delta)).
\]

Since \( P(\delta)Z = Z \), we have \( Z \cap P(\delta)U^{-}(\delta) = P(\delta)(Z \cap U^{-}(\delta)) \). Therefore, to justify (3.4), it remains to show that 
\[
Z \cap U^{-}(\delta) \subset G_w.
\]

To prove this, since \( U^{-}(\delta) \) is a unipotent algebraic subgroup of \( G \), the orbit \( U^{-}(\delta)w \) is Zariski closed in \( V \), see [Bir71, Theorem 12.1]. Hence, the map \( f : U^{-}(\delta)/(G_w \cap U^{-}(\delta)) \to V \), given by \( f(u(G_w \cap U^{-}(\delta))) = uw \) for all \( u \in U^{-}(\delta) \), is a well-defined, injective, proper, and continuous.

Let \( u \in Z \cap U^{-}(\delta) \). Then \( uw \in V_{>0}(\delta) \). So, \( \lim_{t \to 0} \delta(t)uw \) exists in \( V \). Since \( \delta(G_m) \) fixes \( w \), 
\[
\delta(t)uw = (\delta(t)u\delta(t)^{-1})w = f((\delta(t)u\delta(t)^{-1})(G_w \cap U^{-}(\delta)))
\]
converges in \( V \) as \( t \to 0 \). Since \( f \) is a proper map, there exists a compact set \( \Omega \subset U^{-}(\delta) \) and \( r > 0 \) such that \( \delta(t)u\delta(t)^{-1} \subset \Omega(G_w \cap U^{-}(\delta)) \) for all \( |t| < r \). Hence \( u \in (\delta(t)^{-1}\omega\delta(t))(G_w \cap U^{-}(\delta)) \) for all \( |t| < r \). From (3.1), since \( U^{-}(\delta) \) is finite dimensional, we deduce that \( \delta(t)^{-1}\Omega\delta(t) \to \{ e \} \) as \( t \to 0 \). Therefore, \( u \in G_w \). Thus, \( Z \cap U^{-}(\delta) \subset G_w \). This completes the proof of (3.4).

Since \( PU^{-}(\delta) \) is Zariski open dense in \( G \) containing \( e \), (3.5) follows from (3.4). \( \square \)

3.2. Expansion in linear representations of special linear groups. Let \( d \geq 2 \) be an integer. Let \( G = \text{SL}_d \) and \( G = \text{SL}_d(\mathbb{R}) \). Let \( \rho : G \to GL(V) \) be a representation of \( G \). Let \( P_1 \) be the parabolic subgroup of \( G \) which is the stabilizer of the \( (d-1) \)-space spanned by \( e_2, \ldots, e_d \) in the standard representation. In this subsection, we prove several results on the expansion phenomenon in representations of \( G \).

We first recall a variant of Shah’s basic lemma (cf. [Sha09a, Corollary 4.6]):

**Lemma 3.3.** Let \( G = \text{SL}_d \) over \( \mathbb{R} \) for \( d \geq 2 \), and let \( \lambda : G_m \to G \) be the multiplicative one-parameter subgroup defined by \( \lambda(t) = \text{diag}(t^{-(d-1)}, t, \ldots, t) \) for all \( t \in G_m \). Consider a representation of \( G \) on a finite-dimensional vector space \( V \) defined over \( \mathbb{R} \). Then for any \( v \in V(\mathbb{R}) \) such that \( v \) is not fixed by \( G \), the image of the transporter \( \text{Tran}_G(v, V_{>0}(\lambda)) \) under the natural projection \( \pi : G \to P_1 \backslash G \) is contained in a union of at most two proper linear subspaces of \( P_1 \backslash G \cong \mathbb{P}^{d-1}(\mathbb{R}) \).

**Proof.** We note that \( P_1 = P(\lambda) \). Let \( S \) be the full diagonal subgroup of \( G \). We choose a set of simple roots such that \( \lambda \in X_+(S)^+ \), see (2.9). For example, take \( t = \text{diag}(t_1, \ldots, t_d) \mapsto t_i t_i^{-1} \), for \( 1 \leq i \leq d-1 \), as the set of simple roots. Let \( W = N_G(S)/S \) and \( W_P = N_{P_1}(S)/S \). Let \( W_P = W/W_P \). Let \( \{ \sigma_1, \ldots, \sigma_d \} \subset N_G(S)(\mathbb{Q}) \) denote the set of representatives of \( W_P \), where \( \sigma_i \cdot \lambda(t) \) has entry \( t^{-(d-1)} \) at the \( i \)-th diagonal position, and
the rest of the diagonal entries are $t$. Let $B$ be the standard minimal parabolic subgroup of $G$ such that $B \subset P_1$; that is, $B$ is the group of all lower triangular matrices in $G$.

We first consider the case where $Gv$ is not Zariski closed. By Corollary 2.5, there exists a finite-dimensional irreducible representation $W$ of $G$ defined over $\mathbb{R}$, a highest weight vector $w \in W(\mathbb{R})$ with nonzero weight $\mu \in X^*(S)^+$, an element $g_0 \in G(\mathbb{R})$, and $\beta > 0$ such that for any $R > 0$ there exists a constant $C = C_R > 0$ such that

$$\forall g \in G, \text{ if } \|gw\| \leq R, \text{ then } \|g_0gw\| \leq C\|gw\|^\beta.$$  \hspace{1cm} (3.6)

From this, we will deduce the following:

$$\text{Tran}_G(v, V_{\geq 0}(\lambda)) \subset \text{Tran}_G(g_0w, W_{\geq 0}(\lambda)) = \text{Tran}_G(w, W_{\geq 0}(\lambda))g_0^{-1}. \hspace{1cm} (3.7)$$

To verify this, let $g \in G$ be such that $gv \in V_{\geq 0}(\lambda)$. Then $\|\lambda(t)gv\| \leq R$ for all $0 < |t| \leq t_0$, for some $t_0 > 0$ and $R > 0$. Then by (3.6), $\|\lambda(t)g_0w\| \leq CR^\beta$ for all $0 < |t| \leq t_0$. Therefore $g_0gw \in W_{\geq 0}(\lambda)$. So $g_0 \in \text{Tran}_G(w, W_{\geq 0}(\lambda))$. This verifies (3.7).

Suppose $g \in \text{Tran}_G(w, W_{\geq 0}(\lambda))$. Then $gw \in W_{\geq 0}(\lambda)$. Consider the Bruhat decomposition (see [Bor91, 21.15]) $G = \sqcup_{\sigma \in \mathcal{W}} P_1 \sigma^{-1} B$, and suppose $g \in P_1 \sigma^{-1} B$. Since $w$ is fixed by $B$, and $W_{\geq 0}(\lambda)$ is $P_1$-invariant, it follows that $\sigma^{-1} w \in W_{\geq 0}(\lambda)$. Since

$$\lambda(t)\sigma^{-1} w = \sigma^{-1}(\sigma\lambda(t)\sigma^{-1})w = \sigma^{-1}\mu(\sigma\lambda(t)\sigma^{-1})w = t^{(\mu, \sigma \cdot \lambda)} \sigma^{-1} w, \hspace{1cm} \forall t \in \mathbb{R},$$

we conclude that $\langle \mu, \sigma \cdot \lambda \rangle \geq 0$. Note that $\mu$ is of the form $t \mapsto t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$, where $a_1 \leq \cdots \leq a_d$ are integers, not all 0, satisfying $a_1 + \cdots + a_d = 0$. So $a_d > 0$. Let $j$ be the largest integer such that $a_j \leq 0$. So $j < d$. For any $1 \leq k \leq d$,

$$\langle \mu, \sigma_k \cdot \lambda \rangle = (a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_d) \cdot (1, \ldots, 1, -(d-1), 1, \ldots, 1) = -a_k d.$$  

Therefore, since $\langle \mu, \sigma \cdot \lambda \rangle \geq 0$, we get $\sigma \in \{\sigma_1, \ldots, \sigma_j\}$. Hence,

$$\text{Tran}_G(w, W_{\geq 0}(\lambda)) \subset \bigcup_{i=1}^j P_i \sigma_i^{-1} B = P_1 H_j,$$

where $H_j$ denotes the stabilizer of $e_1 \wedge \cdots \wedge e_j$ in $\bigwedge^j \mathbb{R}^n$. Note that the image of $P_1 H_j$ under $\pi$ is a proper linear subspace of $P_1 \backslash G \cong \mathbb{P}^{d-1}(\mathbb{R})$ of dimension $j - 1 < d - 1$. Since the $G$-action sends linear subspaces to linear subspaces of the same dimensions, by (3.7), the image of $\text{Tran}_G(w, V_{\geq 0}(\lambda))$ under $\pi$ is also contained in a proper linear subspace.

Now assume that $Gv$ is Zariski closed. Take any $g \in \text{Tran}_G(v, V_{\geq 0}(\lambda))$. Then $\lambda(t)gv$ converges to some $w \in Gv$ as $t \to 0$. In particular, $w$ is fixed by $\lambda(G_m)$. Also, since $Gw = Gv$ is Zariski closed, $G/G_w \cong Gw$ is an affine variety. And since $G$ is reductive, it follows from Matsushima’s criterion (see [Bor69, 7.10]) that $Gw$ is reductive. So, by Lemma 3.2, $\text{Tran}_G(w, V_{\geq 0}(\lambda))$ is contained in the union of $P(\lambda)(G_w \cap U^-)$ and the complement of $P(\lambda)U^-(\lambda)$. Since $G$ is a simple Lie group, the smallest reductive subgroup of $G$ containing $\lambda(t)$, for some $0 < |t| < 1$, and the expanding horospherical subgroup $U^-(\lambda)$ of $\lambda(t)$ (see (3.1)) in $G$ equals $G$. Since $w$ is not $G$-fixed, and $\lambda(\mathbb{R}^+ \subset G_w$ we get that $G_w \cap U^-(\lambda)$ is a proper subgroup of $U^-(\lambda)$. Now the images of $P(\lambda)(G_w \cap U^-(\lambda))$ and the complement of $P(\lambda)U^-(\lambda)$ under the projection $\pi$ are proper linear subspaces of $P_1 \backslash G \cong \mathbb{P}^{d-1}(\mathbb{R})$. \hfill $\Box$

The following technical lemma will be useful.
Lemma 3.4. Let $G = \text{SL}_d(\mathbb{R})$ for $d \geq 2$, and $g_t = \text{diag}(e^{(d-1)t}, e^{-t}, \ldots, e^{-t})$. Let $\Omega$ be a compact subset of $G$, such that the image $\pi(\Omega)$ of $\Omega$ under the natural projection $\pi : G \to P_1 \setminus G$ is not contained in a union of any two proper linear subspaces of $P_1 \setminus G \cong \mathbb{P}^{d-1}(\mathbb{R})$. Let $V$ be a finite-dimensional real representation of $G$ with a norm $\| \cdot \|$. There exists $D > 0$ such that for any $v \in V$,

$$\sup_{\omega \in \Omega} \| g_t \omega v \| \geq D \| v \|. \quad (3.8)$$

In fact, for any $v \in V$ not fixed by $G$, there exists $C(v) > 0$ such that for all $t \geq 0$,

$$\sup_{\omega \in \Omega} \| g_t \omega v \| \geq C(v) e^t \| v \|. \quad (3.9)$$

Moreover, if there is no nonzero $G$-fixed vector in $V$, then there exists a constant $C > 0$ such that for all $v \in V$, and $t \geq 0$,

$$\sup_{\omega \in \Omega} \| g_t \omega v \| \geq C e^t \| v \|. \quad (3.10)$$

Proof. Let $\lambda$ be as in Lemma 3.3. Since $\pi(\Omega)$ is not contained in a union of any two proper linear subspaces of $P^{d-1}(\mathbb{R})$, it follows from Lemma 3.3 that the set $\Omega v$ is not contained in $V_{\geq 0}(\lambda)$. Hence

$$C(v) := \sup_{\omega \in \Omega} \| \pi_{< 0}(\omega v) \| > 0, \quad (3.11)$$

where $\pi_{< 0} : V \to V_{< 0}(\lambda)$ is the $\lambda$-equivariant projection to the direct sum of the eigenspaces where $\lambda(s)$ acts as a negative power of $s \in \mathbb{R}^*$. Since $g_t = \lambda(e^{-t})$ for all $t \in \mathbb{R}$,

$$V_{< 0}(\lambda) = \{ v \in V : g_{-t}v = \lambda(e^t)v \to 0 \text{ as } t \to \infty \}$$

For any $w \in V_{< 0}(\lambda)$ and any $t \geq 0$, putting $s = e^{-t} \leq 1$,

$$\| g_t w \| = \| \lambda(s)w \| \geq s^{-1} \| w \| = e^t \| w \|. \quad (3.12)$$

Now we have

$$\sup_{\omega \in \Omega} \| g_t \omega v \| \geq \sup_{\omega \in \Omega} \| \pi_{< 0}(g_t \omega v) \| \geq \sup_{\omega \in \Omega} \| g_t \pi_{< 0}(\omega v) \| \geq e^t \sup_{\omega \in \Omega} \| \pi_{< 0}(\omega v) \| \geq C(v) e^t \| v \|. \quad (3.13)$$

where the first equality follows from the $\lambda$-equivariance of $\pi_{< 0}$. This proves (3.9).

Note that we have a well-defined map $f : \mathbb{P}(V) \to \mathbb{R}_{\geq 0}$ given by $[v] \mapsto C(v)$. Suppose there is no nonzero $G$-fixed vector in $V$, then the image of $f$ is contained in $\mathbb{R}_{\geq 0}$. Since $f$ is continuous, the image of $f$ is compact. Hence (3.10) holds for $C := \min_{v \neq 0} C(v) > 0$.

One obtains (3.8) by expressing $V$ as a direct sum of a $G$-fixed subspace and a complementary $G$-invariant subspace and then applying (3.10) to the complementary subspace. \qed
4. Consequences of boundedness in linear representations

Let $G = \text{SL}_n$, and let $\alpha_i$ be the root $\text{diag}(t_j) \mapsto t_it_{i+1}^{-1}$ for $1 \leq i \leq n - 1$. Then $\Delta = \{-\alpha_i\}_{1 \leq i \leq n-1}$ form a set of simple roots. Let $P_i$ be the maximal parabolic subgroup of $G$ associated to $\Delta \setminus \{-\alpha_i\}$ for $1 \leq i \leq n - 1$, and let $P_i = P_i(\mathbb{R})$. In block matrix form,

$$P_i = \left\{ \begin{pmatrix} A_{i \times i} & 0_{i \times (n-i)} \\ C_{(n-i) \times i} & D_{(n-i) \times (n-i)} \end{pmatrix} \in \text{SL}_n(\mathbb{R}) \right\}. \quad (4.1)$$

Let $U_i$ be the unipotent radical of $P_i$. Let $L_i$ be the marked Levi subgroup of $P_i$, which is of the form

$$L_i = \left\{ \begin{pmatrix} A_{i \times i} & 0_{i \times (n-i)} \\ 0_{(n-i) \times i} & D_{(n-i) \times (n-i)} \end{pmatrix} \in \text{SL}_n(\mathbb{R}) \right\}. \quad (4.2)$$

Let

$$H_i = \left\{ \begin{pmatrix} A \\ I_{n-i} \end{pmatrix} : A \in \text{SL}_i(\mathbb{R}) \right\} \quad (4.3)$$

We recall some notations from the introduction. Let $g_t = \text{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t})$. Let $\mathcal{M}$ be a compact connected analytic subset of $P_1 \setminus G \cong \mathbb{P}^{n-1}(\mathbb{R})$, and let $\mathcal{L}_M$ denote the linear span of $\mathcal{M}$ in $P_1 \setminus G$, and suppose the dimension of $\mathcal{L}_M$ is $d - 1$. Since the $(d - 1)$-dimensional linear subspaces of $P_1 \setminus G$ are parameterized by $P_1 \setminus G$, $\mathcal{L}_M$ can be written as $P_1 P_d g_M$ for some $g_M \in G$, where the class $[g_M] = P_d g_M$ parametrizes $\mathcal{L}_M$. Then, we can write

$$\mathcal{M} = \Omega g_M, \quad (4.4)$$

where $\Omega$ is a compact subset of $P_1 P_d = P_1 L_d = P_1 H_d$. Without loss of generality, we may assume that $\Omega$ is a compact subset of $H_d$. We also know that the image of $\Omega$ in $P_1 \setminus G \cong \mathbb{P}^{n-1}(\mathbb{R})$ has linear span exactly equal to $P_1 \setminus P_1 L_d$.

Suppose that there exists a representation $V$ of $G$, a sequence $t_i \to \infty$, a sequence $\{\gamma_i\}$ in $\Gamma$, a nonzero vector $v_0 \in V(\mathbb{Q})$, and a constant $C > 0$ such that

$$\sup_{\omega \in \Omega} \|g_t \omega g_M \gamma_i v_0\| \leq C, \forall i. \quad (\star)$$

The main goal of this section is to derive consequences of $(\star)$. More precisely, we will find those $g_M$ for which $(\star)$ could possibly hold.

**Proposition 4.1.** Suppose there exists a representation $V$ of $G$, a sequence $t_i \to \infty$, a sequence $\{\gamma_i\}$ in $\Gamma$, a nonzero vector $v_0 \in V(\mathbb{Q})$ not fixed by $G$, and a constant $C > 0$ such that $(\star)$ holds. Then $d < n$, and at least one of the following three statements holds:

1. There exists $t_i' \to \infty$, $C' > 0$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\sup_{\omega \in \Omega} \|g_t' \omega g_M v_i\| \leq C', \forall i.$$

2. There exist integers $r \geq d$, $m \geq 2$, and a number field $K \subset \mathbb{R}$ such that $[K : \mathbb{Q}] = m$, $n = mr$, and $g_M \in P_d P_r G(K)$.

3. $n \geq 4$ is even, $d = 2$, and there exists $C' > 0$ and $w_0 \in \wedge^2 \mathbb{Z}^n$ such that

$$\sup_{\omega \in \Omega} \|g_t \omega g_M \gamma_i w_0\| \leq C', \forall i.$$
We remark that \( w_0 \) is usually not decomposable (see Remark 4.3 for a definition). The rest of the section is devoted to proving Proposition 4.1.

**Decomposition of the translating flow.** Let

\[
b_t = \begin{pmatrix}
e^{-\frac{nd}{d}n}I_d \\
e^{-t}I_{n-d}
\end{pmatrix}, \quad c_t = \begin{pmatrix}
e^{-\frac{nd}{d}n}I_{d-1} \\
e^{-\frac{nt}{d}}I_{n-d}
\end{pmatrix}
\]  

One has \( g_t = c_tb_t \). Since \( b_t \) centralizes \( L_d \), we can rewrite (\( \spadesuit \)) as

\[
\sup_{i \in \mathbb{N}} \sup_{\omega \in \Omega} \| c_t \omega b_t g_M \gamma_i v_0 \| < \infty.
\]  

Let

\[
M_d = \left\{ \begin{pmatrix} A & n \\ t & I_{n-d} \end{pmatrix} : A \in \text{GL}_d \text{ and } \det A = t^{(n-d)} \right\} \subset G.
\]  

The next lemma will be important to analyze the cases when \( Gv_0 \) is Zariski closed in \( V \).

**Lemma 4.2.** Suppose that (\( \spadesuit \)) holds and that \( Gv_0 \) is Zariski closed in \( V \). Then the set \( \{b_t g_M \gamma_i v_0 \}_{i \in \mathbb{N}} \) has an accumulation point, say \( v_\infty \in Gv_0 \subset V \setminus \{0\} \). Moreover, \( v_\infty \) is fixed by \( H_d \). In particular, a conjugate of the stabilizer of \( v_0 \) contains \( H_d \), and hence \( d < n \).

Furthermore, if \( \{\gamma_i v_0 : i \in \mathbb{N}\} \) is bounded, then \( v_\infty \) is fixed by \( M_d \), and hence the stabilizer of \( v_0 \) contains a conjugate of \( M_d \).

**Proof.** Applying (3.8) of Lemma 3.4 for \( H_d \) and \( \Omega \), from (\( \spadesuit \)) we conclude that

\[
\sup_{i \in \mathbb{N}} \| b_t g_M \gamma_i v_0 \| < \infty.
\]

Hence, after passing to a subsequence, \( b_t g_M \gamma_i v_0 \rightarrow v_\infty \) for some \( v_\infty \in V \).

Since \( Gv_0 \) is Zariski closed, we have that \( Gv_0 \) is closed in the Hausdorff topology (see e.g. [PR94, Section 3.2]). Therefore, there exists \( g \in G \) such that \( v_\infty = gv_0 \). We write \( F = Gv_0 \) and \( H = Gv_\infty \), then we know \( F \) is a reductive \( \mathbb{Q} \)-subgroup of \( G \) by Matsushima’s criterion (see [Bor69, 7.10]), and \( F = g^{-1}Hg \).

Furthermore, we claim that \( v_\infty \) is fixed by \( H_d \). Indeed, when considering the restricted representation to \( H_d \), we have a decomposition \( V = V_0 \oplus V_1 \), where \( V_0 \) is the subspace of all \( H_d \)-fixed vectors and \( V_1 \) is the \( H_d \)-invariant complement of \( V_0 \) in \( V \). Let \( \pi_0 \) and \( \pi_1 \) be the \( H_d \)-equivariant projections from \( V \) to \( V_0 \) and \( V_1 \), respectively. Suppose that \( \pi_1(v_\infty) \neq 0 \), then there exists \( \delta > 0 \) such that \( \| \pi_1(b_t g_M \gamma_i v_0) \| > \delta \) for all sufficiently large \( i \). By (3.10) of Lemma 3.4, there exists a constant \( C > 0 \) such that for all \( v \in V_1 \setminus \{0\} \) and all \( t \geq 0 \),

\[
\sup_{\omega \in \Omega} \| c_t \omega v \| \geq Ce^t\| v \|.
\]

Hence \( \sup_{\omega \in \Omega} \| c_t \omega b_t g_M \gamma_i v_0 \| \rightarrow \infty \) as \( t \rightarrow \infty \), but this contradicts (\( \spadesuit \)). Hence we must have \( \pi_2(v_\infty) = 0 \), and thus \( v_\infty \) is fixed by \( H_d \).

We note that if \( d = n \), then \( G = H_d \) fixes \( v_0 \), which contradicts the assumption of Proposition 4.1. Therefore, \( d < n \).

Further, suppose that \( \{\gamma_i v_0 : i \in \mathbb{N}\} \) is bounded, then after passing to a subsequence, we may assume that \( \gamma_i v_0 = \gamma_i v_0 \) for all \( i \geq i_0 \). It follows that \( b_t g_M \gamma_i v_0 \rightarrow v_\infty \) as \( i \rightarrow \infty \). Hence \( v_\infty \) is fixed by \( \{b_t\} \). \( \square \)
We will prove Proposition 4.1 by analyzing the following three cases separately in the next three subsections:

1. \( G v_0 \) is not Zariski closed in \( V \).
2. \( G v_0 \) is Zariski closed, and \( \gamma_i v_0 \not\to \infty \). Hence we may assume \( \gamma_i v_0 = v_0 \in V(\mathbb{Q}) \) for all \( i \), by passing to a subsequence and replacing \( v_0 \) with \( \gamma_i v_0 \).
3. \( G v_0 \) is Zariski closed, and \( \gamma_i v_0 \to \infty \) as \( i \to \infty \).

4.1. Case (1). Under the assumption that \( G v_0 \) is not Zariski closed, we analyze (♠).

The next lemma allows us to pass from fundamental representations of \( SL_n \) to the standard representation. Such a result appears in an unpublished draft by David Simmons in response to a question asked by Kleinbock in [Kle08, Section 6.1]; the draft was communicated to us by Dmitry Kleinbock. Another proof of the result due to Emmanuel Breuillard was communicated to us by Nicolas de Saxcé. Both the proofs follow a similar line of geometric arguments. We will provide an algebraic proof using Plücker relations.

**Plücker relations.** Let \( k \in \{1, \ldots, n-1\} \). For \( v \in \wedge^k \mathbb{R}^n \), we can write

\[
v = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} C_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k},
\]

where \( \{e_1, \ldots, e_n\} \) denotes the standard basis of \( \mathbb{R}^n \), and define \( \|v\| = \sup_{1 \leq i_1 < \cdots < i_k} |C_{i_1 \cdots i_k}| \). We have that \([v]\) is in the image of the Plücker embedding \( Gr(k,n)(\mathbb{R}) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{R}^n) \) if and only if the coordinates \( C_{i_1 \cdots i_k} \) satisfy the following Plücker relations: For any two ordered sequences

\[
\mathcal{I} = (i_1, \ldots, i_{k-1}), \text{ where } i_1 < \cdots < i_{k-1} \text{ and } \tag{4.8}
\]

\[
\mathcal{J} = (j_1, \ldots, j_{k+1}), \text{ where } j_1 < \cdots < j_{k+1} \text{, we have } \tag{4.9}
\]

\[
\sum_{l=1}^{k+1} (-1)^l C_{i_1 \cdots i_{k-1} j_l} C_{j_1 \cdots j_{l-1} j_{l+1}} = 0, \tag{4.10}
\]

where for a permutation \( \sigma \) of \( (i_1 < \cdots < i_k) \), \( C_{\sigma(i_1) \cdots \sigma(i_k)} := \text{sgn}(\sigma) C_{i_1 \cdots i_k} \). Here, we set \( C_{i_1 \cdots i_k} = 0 \) if \( i_1, \ldots, i_k \) have repetitions.

**Remark 4.3.** We say that a nonzero \( v \in \wedge^k \mathbb{R}^n \) is decomposable if \( v = v_1 \wedge \cdots \wedge v_k \) for some linearly independent \( v_1, \ldots, v_k \in \mathbb{R}^n \). In this case, let \( \Delta_v \) denote the \( \mathbb{Z} \)-span of \( \{v_1, \ldots, v_k\} \) and \( \text{Covol}(\Delta_v) \) denote the co-volume of the lattice \( \Delta_v \) in its \( k \)-dimensional \( \mathbb{R} \)-span. Then \( \text{Covol}(\Delta_v) \) equals the Euclidean norm of \( v \) with respect to its coordinates as in (4.7), see [Dan78, Lemma 1.4]. Hence

\[
\text{Covol}(\Delta_v) \leq \|v\| \leq \sqrt{\binom{n}{k}} \text{Covol}(\Delta_v). \tag{4.11}
\]

**Lemma 4.4.** There exists \( C = C(n, d, \Omega) > 0 \) such that the following holds: for any \( c > 0 \), for any \( k \in \{1, \ldots, n-1\} \), any nonzero decomposable \( w = w_1 \wedge \cdots \wedge w_k \in \wedge^k \mathbb{R}^n \), where \( w_1, \ldots, w_k \in \mathbb{R}^n \), and any \( t > (\log C + \log c - \log \|w\|)/(n-k) \), if

\[
\sup_{\omega \in \Omega} \|g_\omega w\| \leq c, \tag{4.12}
\]
then there exists a nonzero $v$ in the $\mathbb{Z}$-span of $\{w_1, \ldots, w_k\}$ such that
\[
\sup_{\omega \in \Omega} \|g_t \omega v\| \leq C e^{1/k}, \text{ where } t' = ((n - k)t - \log c + \log \|w\|)/n.
\]

Proof. In this proof, all the implicit constants in \( \ll \) depend only on $n$, $d$, $\Omega$, and $k$.

Let $V_1$ denote the span of \( \{e_1, \ldots, e_d\} \) and $V_2$ denote the span of \( \{e_{d+1}, \ldots, e_n\} \). We have the decomposition
\[
\wedge^k \mathbb{R}^n = \oplus_i \wedge^i V_1 \otimes \wedge^{k-i} V_2.
\]

Let $\pi_i$ denote the projection from $\wedge^k \mathbb{R}^n$ to $W_i := \wedge^i V_1 \otimes \wedge^{k-i} V_2$. Note that $b_t$ acts on $W_i$ by scalar multiplication by $e^{(ni/d-k)t}$. Notice that $g_t \omega w = c_i_\wedge b_t w$.

For $i \geq 1$ such that $W_i$ is non-trivial, we apply Lemma 3.4 for $L_d$ and $c_t$. We note that the only non-negative eigenvalue of $c_t$ on $W_i$ is $e^{(n-ni/d)t}$. Then there exists $C_1 > 0$ such that from (4.12) we conclude that $\|\pi_t(b_t w)\| \leq C_1 e^{-(n-ni/d)t}$. Since $\pi_i$ is $b_t$-equivariant, we have $\|\pi_i(w)\| \leq C_1 e^{-(n-k)t}$. For any choice of $C \geq C_1$, by our assumption,
\[
(n - k)t > \log C + \log c - \log \|w\|.
\]

Therefore $\|\pi_i(w)\| < \|w\|$ for all $i \geq 1$. Since we consider the sup-norm,
\[
\|w\| = \|\pi_0(w)\|.
\]

Now $H_d$ acts trivially on $W_0$, so $\|\pi_0(w)\| \leq C e^{kt}$. Write $\|\pi_0(w)\| = C \delta t$, where $\delta \leq k$. We claim that for all $i \geq 1$ we have
\[
\|\pi_i(w)\| \ll C e^{-(i(n-k)-(i-1)\delta)t}. \tag{4.13}
\]

We verify this claim by induction. The base case $i = 1$ is already established. For the inductive step, suppose that (4.13) holds for $1 \leq i \leq m - 1$. We write
\[
w = \sum_{i_1 < i_2 < \cdots < i_k} C_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

Since we are taking the sup-norm, there exist $d + 1 \leq p_1 < p_2 < \cdots < p_k \leq n$ and $1 \leq q_1 < q_2 < \cdots < q_m \leq d + 1 \leq q_{m+1} < \cdots < q_k \leq n$ such that $\|\pi_0(w)\| = |C_{p_1 \ldots p_k}|$ and $\|\pi_m(w)\| = |C_{q_1 \ldots q_k}|$. By (4.8), for the two ordered sequences $\mathcal{I} = (q_1 < \cdots < q_{k-1})$ of size $k - 1$ and $\mathcal{J} = (q_k < p_1 < \cdots < p_k)$ of size $k + 1$, and the triangle inequality, we have
\[
\|\pi_m(w)\| \|\pi_0(w)\| = |C_{q_1 \ldots q_k}| \cdot |C_{p_1 \ldots p_k}|
\]
\[
\leq \sum_{l=1}^k |C_{q_1 \ldots q_{k-1} p_l} \cdot C_{q_k p_1 \ldots p_l}|
\]
\[
\leq k \|\pi_{m-1}(w)\| \|\pi_1(w)\|
\]

Hence by the induction hypothesis, we have
\[
\|\pi_m(w)\| \ll \|\pi_{m-1}(w)\| \|\pi_1(w)\| \|\pi_0(w)\|^{-1}
\]
\[
\ll C e^{-(m-1)(n-k)-(m-2)\delta t} C e^{-(n-k)t} (C e^{\delta t})^{-1}
\]
\[
= C e^{-(m(n-k)-(m-1)\delta)t}.
\]

Therefore (4.13) holds for $i = m$, and the claim is verified.
It follows from (4.13) that
\[
\|b_{n-k+\delta}w\| \ll c e^{(\delta - \frac{k(d(n-k))}{n}) t}. \tag{4.14}
\]

Since \(w = w_1 \land \ldots \land w_k \neq 0\), the \(\mathbb{Z}\)-span of \(\{w_1, \ldots, w_k\}\), say \(\Delta_w\), is a lattice in its \(k\)-dimensional \(\mathbb{R}\)-span. In view of (4.11) and (4.14), the covolume of the lattice \(b_{n-k+\delta}w\) in its \(k\)-dimensional \(\mathbb{R}\)-span is \(\ll c e^{(\delta - \frac{k(d(n-k))}{n}) t}\). Therefore by Minkowski’s convex body theorem, there exists a nonzero \(v \in \Delta_w\) such that
\[
\|b_{n-k+\delta}v\| \ll c^{1/k} e^{(\delta - \frac{k(d(n-k))}{n}) t}.
\]

Since \(\delta \leq k\), we have
\[
\frac{\delta}{k} \frac{d(n-k+\delta)}{n} \leq -(d-1) \frac{n-k+\delta}{n}.
\]

Let \(t' = \frac{n-k+\delta}{n} t = ((n-k)t + \delta t)/n\), and we recall that \(\delta t = \log \|\pi_0(v)\| - \log c = \log \|v\| - \log c\). It follows that
\[
\|b_{dt'}v\| \ll c^{1/k} e^{-(d-1)t'}.
\]

We write \(v = v_1 + v_2\), where \(v_1\) belongs to the \(\mathbb{R}\)-spans of \(\{e_1, \ldots, e_d\}\) and \(v_2\) belongs to the \(\mathbb{R}\)-spans of \(\{e_{d+1}, \ldots, e_n\}\). By (4.5), \(b_{dt'}v_1 = e^{(n-d)t'}v_1\) and \(b_{dt'}v_2 = e^{dt'}v_2\). Therefore
\[
\|v_1\| \ll c^{1/k} e^{-(n-1)t'} \text{ and } \|v_2\| \ll c^{1/k} e^{t'}.
\]

Since \(\Omega\) is a compact subset of \(H_d\),
\[
\sup_{\omega \in \Omega} \|\omega v_1\| \ll c^{1/k} e^{-(n-1)t'} \text{ and } \Omega v_2 = v_2.
\]

Since \(\|g_{t'}\| \leq e^{(n-1)t'}\) and \(g_{t'}v_2 = e^{-t'}v_2\), we get
\[
\sup_{\omega \in \Omega} \|g_{t'}\omega v\| \ll c^{1/k},
\]
and the conclusion of the lemma holds. \(\square\)

**Corollary 4.5.** Let \(k \in \{1, \ldots, n-1\}\). Let \(W_k = \bigwedge^k \mathbb{R}^k\) be the \(k\)-th exterior power of the standard representation of \(G\) and \(w_k = e_1 \land \cdots \land e_k\). Suppose that there exists \(t_i \to \infty\) and \(\gamma_i \in \Gamma\) such that
\[
\sup_{\omega \in \Omega} \|g_{t_i} \omega g_{\mathcal{M}} \gamma_i w_k\| \to 0, \text{ as } i \to \infty. \tag{4.15}
\]
Then there exists \(t_i' \to \infty\) and \(v_i \in \mathbb{Z}^n \setminus \{0\}\) such that
\[
\sup_{\omega \in \Omega} \|g_{t_i'} \omega g_{\mathcal{M}} v_i\| \to 0, \text{ as } i \to \infty. \tag{4.16}
\]

**Proof.** Let \(C > 0\) be as in Lemma 4.4. Let \(c_i = \sup_{\omega \in \Omega} \|g_{t_i} \omega g_{\mathcal{M}} \gamma_i w_k\|\). Note that \(g_{\mathcal{M}} \Gamma w_k\) is a discrete subset of \(W_k \setminus \{0\}\); so \(\|g_{\mathcal{M}} \gamma_i w_k\|\) has a uniform lower bound for all \(i\). Hence, for every sufficiently large \(t_i\), the assumption of Lemma 4.4 is satisfied. We take \(t_i' = ((n-k)t_i - \log c_i + \log \|g_{\mathcal{M}} \gamma_i w_k\|)/n\). By Lemma 4.4, we know that (4.16) holds. Finally, since \(t_i \to \infty\), \(c_i \to 0\), and \(\inf_i \|g_{\mathcal{M}} \gamma_i w_k\| > 0\), we have \(t_i' \to \infty\). \(\square\)

Recall the definitions of \(L_d\) and \(H_d\) given by (4.2) and (4.3).
Proposition 4.6. Let $V$ be a representation of $G = \text{SL}_n$, and let $v_0 \in V(\mathbb{Q})$ such that $Gv_0$ is not Zariski closed. Let $\Omega$ be a non-empty compact subset of $H_d$ whose Zariski closure is irreducible. Let $g_M \in G$. Suppose that there exists $C > 0$, $t_i \to \infty$ and $\gamma_i \in \Gamma$ such that

$$\sup_{\omega \in \Omega} \| g_{t_i \omega} g_M \gamma_i v_0 \| \leq C, \forall i.$$ (4.17)

Then there exist $R > 0$, $v_i \in \mathbb{Z}^n \setminus \{0\}$, and $t_i' \to \infty$ such that

$$\sup_{\omega \in \Omega} \| g_{t'_i \omega} g_M v_i \| \leq R, \forall i.$$ (4.18)

Moreover, if $P_1 \backslash P_1 \Omega$ is not contained in a union of two proper linear subspaces of $P_1 \backslash P_1 H_d$, then $d < n$.

Proof. By Corollary 2.5, there exists an irreducible representation $W$ of $G$ defined over $\mathbb{Q}$, a highest weight vector $w \in W(\mathbb{Q})$, an element $g_0 \in G(\mathbb{Q})$ such that, due to (4.17), we can pick a constant $D > 0$ such that

$$\sup_{\omega \in \Omega} \| g_{t_i \omega} g_M \gamma_i g_0 w \| \leq D, \forall i.$$ (4.19)

Combined with Lemma 2.6 and Section 2.1.1, this implies that there exists $D' > 0$ depending only on $\Omega$, $w$ and $D$, such that for each $i$, there exists $k \in \{1, \ldots, n-1\}$ such that

$$\sup_{\omega \in \Omega} \| g_{t_i \omega} g_M \gamma_i g_0 w_k \| \leq D',$$ (4.20)

where $w_k = e_1 \wedge \cdots \wedge e_k \in W_k(\mathbb{Z}) = \wedge^k \mathbb{Z}^n$. By passing to a subsequence, we may assume that $k$ is a constant for all $i$.

Since $g_0 \in G(\mathbb{Q})$, pick $N \in \mathbb{Z}_{>0}$ such that $Ng_0$ is a $n \times n$ matrix with integer entries. Let $w_i = N \cdot \gamma_i g_0 w_k \in \wedge^k \mathbb{Z}^n \setminus \{0\}$. Then (4.19) implies that

$$\sup_{\omega \in \Omega} \| g_{t_i \omega} g_M w_i \| \leq ND'.$$ (4.20)

we note that $w_i = (N\gamma_i g_0 e_1) \wedge \cdots \wedge (N\gamma_i g_0 e_k)$, where $N\gamma_i g_0 e_j \in \mathbb{Z}^n$ for each $j$, and $\log \| w_i \| \geq 0$. Now (4.18) follows from (4.20) and Lemma 4.4.

Furthermore, suppose that the image of $\Omega$ in $P_1 \backslash G$ is not contained in a union of any two proper linear subspaces of $P_1 \backslash P_1 H_d$, we claim that $d < n$. Indeed, suppose $d = n$, then $H_d = G$. Since $G$ has no nonzero fixed vector in $\wedge^k \mathbb{R}^n$ for any $1 \leq k \leq n - 1$, the constant in (3.9) of Lemma 3.4 applied to the $G$ action on $\wedge^k \mathbb{R}^n$ is uniform in $v$. This contradicts (4.19). \hfill \Box

We apply Proposition 4.6 to conclude from (♠) that $d < n$ and there exists $t'_i \to \infty$, $C' > 0$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\sup_{\omega \in \Omega} \| g_{t'_i \omega} g_M v_i \| \leq C', \forall i.$$ i.e. (1) of Proposition 4.1 holds.
4.2. Case (2). $Gv_0$ is Zariski closed, and $\gamma_i v_0 = v_0$ for all $i$. So (•) takes the form

$$\sup_{i \in \mathbb{N}} \sup_{\omega \in \mathbb{Z}} \|c_i \omega b_i g_M v_0\| < \infty. \quad (4.21)$$

By Lemma 4.2, $d < n$ and as $i \to \infty$,

$$b_i v_0 \to v_\infty \in Gv_0, \quad (4.22)$$

and $v_\infty$ is fixed by $M_d$, which denotes the group generated by $\{b_i\}$ and $H_d$.

Let $H$ denote the stabilizer of $v_\infty$ in $G$, and $F$ the real points of the stabilizer $F$ of $v_0$ in $G$. Pick $g \in G$ such that $v_\infty = gv_0$. Then

$$M_d \subset H = gFg^{-1}. \quad (4.23)$$

Let $\lambda(\tau) = \text{diag}(\tau^{-(n-d)} I_d, \tau^d I_{n-d})$ be a multiplicative 1-parameter subgroup corresponding to $b_t$. Here $\lambda(\tau) = b_t$ for any $\tau > 0$ and $t = -d \log \tau$; so $\tau \to 0$ correspond to $t \to \infty$. So, in view of (4.22), $g_M v_0 = g_M g^{-1} v_\infty \in V_{\geq 0}(\lambda)$, and hence $g_M g^{-1} \in \text{Tran}_G(v_\infty, V_{\geq 0}(\lambda))$. Note that $P(\lambda) = P_d$, see (4.1).

We have $b_t g_M g^{-1} v_\infty \to v_\infty$ as $i \to \infty$. Since $Gv_\infty$ is closed, the orbit map $G/H \to Gv_\infty$ is open. Hence $b_t g_M g^{-1} \in g_i H$ for some $g_i \to e$ in $G$. Since $H$ has only finitely many connected components, by passing to a subsequence and replacing $g$ by an element of $Hg$, without loss of generality, we may assume that $b_t g_M g^{-1} \in g_i H^0$ and $g_i \to e$ in $G$. We apply Lemma 3.2 to conclude that

$$b_t g_M g^{-1} \in \text{Tran}_G(w, V_{\geq 0}(\delta))^\circ \subset P_d H,$$

and note that $b_t \in P_d$. Hence we have

$$g_M \in P_d H g = P_d gF. \quad (4.24)$$

As noted in the proof of Lemma 3.2, $F$ is a reductive $\mathbb{Q}$-subgroup of $G$. By our assumption, $v_0$ is not $G$ fixed, so $F$ is a proper subgroup of $G$.

The rest of the subsection is devoted to classifying the pair $(g, F)$.

**Lemma 4.7.** The reductive group $F$ is contained in a proper $\overline{\mathbb{Q}}$-parabolic subgroup of $G$.

**Proof.** Suppose not, then $F$-action on $\overline{\mathbb{Q}}^n$ is irreducible. Let $\rho : F \to \text{GL}(\mathbb{C}^n)$ denote the restriction of the standard representation of $\text{SL}_n$. Since $H_d \subset F$, we have that $\rho(F)$ contains $\text{SL}(W_1) \times 1_{W_2}$, where $W_1$ is the subspace $\mathbb{C}^n$ whose last $(n-d)$-coordinates are zero and $W_2$ is the subspace of $\mathbb{C}^n$ whose first $d$-coordinates are zero. Since $F \neq \text{SL}_n$ over $\overline{\mathbb{Q}}$, by Theorem A.7 we have that $n$ is even, $d = 2$, and $F \cong \text{Sp}_n$. Hence $F$ does not contain any conjugate of $\{g_t\}_{t \in \mathbb{R}}$; indeed, any diagonalizable element of $\text{Sp}_n$ has eigenvalues $\tau_{\pm}^1, \ldots, \tau_{\pm}^{1/2}$, and thus cannot be conjugated to $g_t$ if $t \neq 0$. This contradicts the fact that $\{g_t\}_{t \in \mathbb{R}} \subset M_d \subset H = gFg^{-1}$. 

As a consequence of Lemma 4.7, the reductive $\mathbb{Q}$-group $Z_G(F)$, the centralizer of $F$ in $G$, has a non-trivial $\overline{\mathbb{Q}}$-cocharacter. By [Bor91, Theorem 18.2], $Z_G(F)$ has a nontrivial maximal torus defined over $\overline{\mathbb{Q}}$, say $T$. Then $Z_G(T)$ is a proper reductive $\mathbb{Q}$-subgroup of $G$. By a theorem of Chevalley (see e.g. [PR94, Theorem 2.15]), there exists a $\mathbb{Q}$-representation $W$ of $G$ and a vector $w$ in $\overline{W}(\mathbb{Q})$ such that $G_w = Z_G(T)$ and $G_w$ is Zariski closed in $W$. 
Since \(Gv_0\) is closed in \(V\), the map \(hF \mapsto hv_0, \forall h \in G\), from \(G/F \to V\), is a proper map. Hence, the condition (4.21) is equivalent to the statement that \(c_t \Omega b_t g_M F/F\) stays in a fixed compact subset of \(G/F\) for all \(t\). Since \(F \subset Z_G(T)\), we deduce that \(c_t \Omega b_t g_M Z_G(T)/Z_G(T)\) stays in a fixed compact subset of \(G/Z_G(T)\) for all \(t\). Therefore

\[
sup_{i \in \mathbb{N}} \sup_{\omega \in \Omega} \|c_t \omega b_t g_M w\| < \infty. \tag{4.25}
\]

We can run the same arguments in this subsection with (4.25) in place of (4.21), and \(Z_G(T)\) in place of \(F\). We know that \(Z_G(T)\) is a Levi subgroup of a proper \(\mathbb{Q}\)-parabolic subgroup of \(G\). Hence by replacing \(F\) with \(Z_G(T)\), we may assume that \(F\) is a Levi subgroup of a \(\overline{\mathbb{Q}}\)-parabolic subgroup of \(G\). More explicitly, there exist positive integers \(n_1, n_2, \ldots, n_m\) and an element \(l \in G(\overline{\mathbb{Q}})\) such that \(n = n_1 + \cdots + n_m\), and

\[
F = l \begin{pmatrix} GL_{n_1} & & \\ & \ddots & \\ & & GL_{n_m} \end{pmatrix} l^{-1} \cap G. \tag{4.26}
\]

It follows that

\[
Z_G(F) = \left\{ l \begin{pmatrix} t_1 I_{n_1} & & \\ & \ddots & \\ & & t_m I_{n_m} \end{pmatrix} l^{-1} : \prod_{i=1}^m t_i^{n_i} = 1 \right\} \subset F. \tag{4.27}
\]

In particular, \(Z_G(F)\) equals \(Z(F)\), the center of \(F\).

**Lemma 4.8.** Suppose the \(\mathbb{Q}\)-torus \(Z_G(F) = Z(F)\) is \(\mathbb{Q}\)-isotropic. Then there exists a representation \((\rho', W)\) of \(G\) and a rational nonzero vector \(w_0 \in W\) such that \(w_0\) is unstable in \(W\), and

\[
sup_{\omega \in \Omega} \|g_t \omega g_M w_0\| \to 0 \text{ as } t \to \infty. \tag{4.28}
\]

**Proof.** Our assumption is equivalent to saying that \(F\) is contained in a proper \(\mathbb{Q}\)-parabolic subgroup of \(G\), say \(P\). Without loss of generality, we may assume that \(P\) is \(\mathbb{Q}\)-maximal. Then there exists an integer \(1 \leq k \leq n - 1\) and a nonzero \(\mathbb{Q}\)-vector \(u\) in the \(k\)-th exterior power \(W\) of the standard representation of \(G\), such that \(P\) acts on the line \([u]\) via a \(\mathbb{Q}\)-character \(\chi\) of \(P\).

Let \(g\) be as in (4.24). Then \(Hg = gF\) and \(g_M \in P_d Hg\). By its definition, \(\Omega\) is a compact subset of \(L_d \subset P_d\), see (4.1). Therefore \(\Omega g_M \subset P_d Hg\). We note that \(H_d \subset M_d \subset G_{v_{\infty}} = H\). Since \(P_d \subset P_1 H_d\), we conclude that \(P_d H \subset P_1 H\). Therefore \(\Omega g_M \subset P_1 Hg = P_1 gF\). Therefore there exists a compact set \(\Omega_1 \subset P_1\) and \(\Omega_2 \subset F\) such that \(\Omega g_M \subset \Omega_1 g_0 \Omega_2\). Then \(\Omega_1' := \bigcup_{t \geq 0} g_t \Omega_1 g_t^{-1}\) is a compact subset of \(P_1\), and

\[
g_t \Omega g_M \subset \Omega_1' g_t g_0 \Omega_2, \forall t \geq 0. \tag{4.29}
\]

Since \(g_t \in H = gFg^{-1}\), we have \(g^{-1} g_t g \in F \subset P\) for all \(t \in \mathbb{R}\), and thus \(g^{-1} g_t g\) acts on \([u]\) via multiplication by \(\chi(g^{-1} g_t g)\). Notice that in the \(k\)-th exterior power \(W\) of the standard representation of \(G\), the only two weights of \(g_t\) are \(t \mapsto e^{(n-k)t}\) and \(t \mapsto e^{-kt}\). Hence \(g_t\) cannot fix \(gu\) for \(t \neq 0\), and it follows that \(\chi(g^{-1} g_t g) \neq 1\) for all \(t \neq 0\). If \(\chi(g^{-1} g_t g) < 1\), we take \(w_0 = u\); if \(\chi(g^{-1} g_t g) > 1\), we replace \(P\) by its opposite parabolic subgroup containing \(F\). In either case, \(g^{-1} g_t g w_0 \to 0\) as \(t \to \infty\).
Therefore by (4.29), since element of $\Omega_2 \subset F \subset P$, $(g^{-1} g_0) \subset P$, and elements of $P$ act on $w_0$ by scalars, we get
\[
g_t \Omega g_{\omega, w_0} = \Omega' g_t g \Omega_2 w_0 = \Omega' g \Omega_2 (g^{-1} g_0) w_0, \forall t \geq 0.
\] (4.30)
Since $\Omega'_1$ and $\Omega_2$ are compact, (4.28) holds.

**Remark 4.9.** Suppose $Z_G(F)$ is not $\mathbb{Q}$-anisotropic. Then by (4.28) of Lemma 4.8, the orbit $Gw_0$ is not Zariski closed in $W$, and hence by our discussion in subsection 4.1, (1) of Proposition 4.1 would hold.

Therefore now we assume that $Z_G(F) = Z(F)$ is a $\mathbb{Q}$-anisotropic torus of $G$.

Let $\mathbb{L}$ denote the splitting field of the torus $Z(F)$. Its Galois group is denoted by $\mathcal{G} = \text{Gal}(\mathbb{L}/\mathbb{Q})$. Recall that $n_1, n_2, \ldots, n_m$ are the sizes of the blocks in $F$ as in (4.26).

**Lemma 4.10.** Suppose that $Z_G(F) = Z(F)$ is a $\mathbb{Q}$-anisotropic torus of $G$. Then there exists a positive integer $r$ such that $n_1 = \cdots = n_m = r$.

**Proof.** The standard representation $E$ of $\mathcal{G}$ has a weight space decomposition with respect to $Z(F)$
\[
E = \bigoplus_{i=1}^m E_{\chi_i},
\] (4.31)
where $\chi_i \in X^*(Z(F))$ are characters of $Z(F)$ defined over $\mathbb{L}$, and $E_{\chi_i}$ consists of vectors on which $Z(F)$ acts as $\chi_i$. After relabelling, we have $\dim E_{\chi_i} = n_i$. In view of (4.27), the only relation between the $\chi_i$'s is $\prod_{i=1}^m \chi_i^{n_i} = 1$.

The $\mathcal{G}$-action on $X^*(Z(F))$ is given by
\[
(\sigma \chi)(t) = \sigma \circ \chi \circ \sigma^{-1}(t), \quad \forall \sigma \in \mathcal{G}.
\]
Then, for all $\sigma \in \mathcal{G}$ and $i \in \{1, \ldots, n\}$, $\sigma E_{\chi_i} = E_{\sigma \chi_i}$, and hence $\dim E_{\chi_i} = \dim E_{\sigma \chi_i}$. Therefore to show that $n_1 = \cdots = n_m$, it is sufficient to verify that $\mathcal{G}$-action on the set $\{\chi_1, \ldots, \chi_m\}$ is transitive. Indeed, suppose $\mathcal{X}$ is a $\mathcal{G}$-invariant proper subset, then $\prod_{\chi_i \in \mathcal{X}} \chi_i^{n_i} = 1$, because it is a $\mathbb{Q}$-character and $Z(F)$ is $\mathbb{Q}$-anisotropic. But this gives us an additional relation between $\chi_1, \ldots, \chi_m$, a contradiction.

By Lemma 4.10 there exists $l \in \text{GL}_n(\mathbb{L})$ such that for any $z \in Z(F)$ we have
\[
z = l \begin{pmatrix} \chi_1(z) I_r \\ \vdots \\ \chi_m(z) I_r \end{pmatrix} l^{-1}.
\] (4.32)
Hence $F = l H_{r,m} l^{-1}$, where
\[
H_{r,m} = \begin{pmatrix} \text{GL}_r \\ \vdots \\ \text{GL}_r \end{pmatrix} \cap \mathcal{G}.
\] (4.33)
It follows that $Z(F) = l T_{r,m} l^{-1}$, where
\[
T_{r,m} = Z(H_{r,m}) = \begin{pmatrix} t_1 I_r \\ \vdots \\ t_m I_r \end{pmatrix} \cap \mathcal{G}.
\]
4.2.1. Description of $l$. From the proof of Lemma 4.10 we know that $\mathcal{G}$ acts transitively on $\{\chi_1, \ldots, \chi_m\}$. Let $\mathcal{H}$ be the stabilizer of $\chi_1$, so $[\mathcal{G} : \mathcal{H}] = m$. Let $K = \mathbb{L}^H$ be the fixed field of $\mathcal{H}$. Then $[K : \mathbb{Q}] = [\mathcal{G} : \mathcal{H}] = m$. Let $\{\sigma_1, \ldots, \sigma_m\}$ be the set of embeddings $K \hookrightarrow \mathbb{C}$. We choose and fix a basis $\{x_1, \ldots, x_m\}$ of $K$ as a $\mathbb{Q}$-vector space, and let $l_0 \in \text{GL}_n(\mathbb{L})$ be such that

$$l_0^{-1} = \begin{pmatrix} \sigma_1(x_1)I_r & \cdots & \sigma_1(x_m)I_r \\ \vdots & \ddots & \vdots \\ \sigma_m(x_1)I_r & \cdots & \sigma_m(x_m)I_r \end{pmatrix} \in \text{GL}_n(\mathbb{L}).$$  

(4.34)

Remark 4.11. We claim that $l_0H_{r,m}l_0^{-1}$ is a $\mathbb{Q}$-subgroup of $G$. Indeed, for any $\sigma \in \mathcal{G}$, we have $\sigma(l_0^{-1}) = w_\sigma \cdot l_0^{-1}$, where the permutation matrix $w_\sigma$ is an element of $N_{\text{GL}_n}(H_{r,m})$, see (4.33). Then $\sigma(l_0H_{r,m}l_0^{-1}) = l_0w_\sigma^{-1}H_{r,m}w_\sigma l_0^{-1} = l_0H_{r,m}l_0^{-1}$. Hence $l_0H_{r,m}l_0^{-1}$ is defined over $\mathbb{Q}$. Moreover, $l_0H_{r,m}l_0^{-1}$ is isomorphic to $\text{Res}_{K/\mathbb{Q}}^{(1)} \text{GL}_r$ as $\mathbb{Q}$-groups.

Lemma 4.12. There exists $g_\mathbb{Q} \in \text{GL}_n(\mathbb{Q})$ such that $F = g_\mathbb{Q}l_0H_{r,m}l_0^{-1}g_\mathbb{Q}^{-1}$. In particular, later, we will assume that $l = g_\mathbb{Q}l_0$.

Proof. Equivalently, we shall prove that $Z(F) = g_\mathbb{Q}l_0T_{r,m}l_0^{-1}g_\mathbb{Q}^{-1}$. We use the Galois cohomology approach, and the readers are referred to [PR94, Section 1.3] and [Ser94] for more details.

Let $G' = \text{GL}_n$. Write $N'_{r,m} = N_{G'}(T_{r,m})$ and $H'_{r,m} = Z_{G'}(T_{r,m})$. Consider the homogeneous variety $X = G'/N'_{r,m}$. Since $F = lH_{r,m}l_0^{-1}$, we have $Z(F) = lT_{r,m}l_0^{-1}$ which is also defined over $\mathbb{Q}$. For any $\sigma \in \mathcal{G}$, we have $\sigma(Z(F)) = Z(F)$, and it follows that $l^{-1}\sigma(l) \in N'_{r,m}(\mathbb{L})$. Hence $[l] \in X$ is fixed by $G'$, and therefore we have $[l] \in X(\mathbb{Q})$. Hence it suffices to prove that $[l]$ and $[l_0]$ are on the same $G'(\mathbb{Q})$ orbit in $X(\mathbb{Q})$.

By [Ser94, Prop.36, Cor.1], we have an exact sequence of $G$-sets

$$G'(\mathbb{Q}) \to X(\mathbb{Q}) \xrightarrow{\partial} H^1(G, N'_{r,m}(\mathbb{L})) \to H^1(G, G'(\mathbb{L})).$$

Since the first Galois cohomology of $G' = \text{GL}_n$ vanishes by Hilbert’s theorem 90, the above exact sequence tells us that the $G'(\mathbb{Q})$-orbits on $X(\mathbb{Q})$ are parameterized by $H^1(G, N'_{r,m}(\mathbb{L}))$. Now $\partial[l]$ and $\partial[l_0]$ are represented by the 1-cocycles $b_l: \sigma \to l^{-1}\sigma(l)$ and $b_{l_0}: \sigma \to l_0^{-1}\sigma(l_0)$. So, it suffices to show that $b_l = b_{l_0}$.

On the other hand, we consider the Weyl group

$$W_{r,m} = N_{G'}(T_{r,m})/Z_{G'}(T_{r,m}) = N'_{r,m}/H'_{r,m} \cong \mathfrak{S}_m,$$

where $\mathfrak{S}_m$ denotes the symmetric group of permutations on $m$ elements. We have another exact sequence of $G$-sets

$$H^1(G, H'_{r,m}(\mathbb{L})) \to H^1(G, N'_{r,m}(\mathbb{L})) \xrightarrow{\partial} H^1(G, W_{r,m}).$$

By Shapiro’s Lemma and Hilbert’s theorem 90, we have

$$H^1(Q, b_{l_0}H'_{r,m}(\mathbb{Q})) = H^1(Q, \text{Res}_{\mathbb{Q}/\mathbb{Q}}(\text{GL}_r)(\mathbb{Q})) = H^1(K, \text{GL}_r(\mathbb{Q})) = 1,$$

where $b_{l_0}H'_{r,m}$ is the twist of $H'_{r,m}$ by the 1-cocycle $b_{l_0}$ as defined in [Ser94, Ch. 1, §5] or [PR94, Page 23]. Therefore as $\mathbb{Q}$-groups,

$$b_{l_0}H'_{r,m} \cong l_0H'_{r,m}l_0^{-1} \cong \text{Res}_{\mathbb{Q}/\mathbb{Q}}(\text{GL}_r),$$

where $\text{Res}_{\mathbb{Q}/\mathbb{Q}}(\text{GL}_r)$ is the base change of $\text{GL}_r$ to $\mathbb{Q}$. Hence $l_0H_{r,m}l_0^{-1} \cong \text{Res}_{\mathbb{Q}/\mathbb{Q}}(\text{GL}_r)$.
where the first isomorphism follows from the definition of the twist, and the second isomorphism follows from the arguments as in Remark 4.11.

Since the inflation-restriction sequence

\[ 1 \rightarrow H^1(\mathcal{G}, b_0 H^r_{t,m}(\mathbb{L})) \rightarrow H^1(\mathbb{Q}, b_0 H^r_{t,m}(\mathbb{Q})) \rightarrow H^1(\mathbb{L}, b_0 H^r_{t,m}(\mathbb{Q})) \]

is exact (see [PR94, Page 25]), we have \( H^1(\mathcal{G}, b_0 H^r_{t,m}(\mathbb{L})) = 1 \). Hence, by [Ser94, Cor. 2 of Prop. 39], the fibre \( \iota^{-1}(\iota(b_0)) \) contains only one element \( b_0 \). Hence it suffices to show that \( \iota(b_l) = \iota(b_0) \).

It is clear that the \( \mathcal{G} \)-action on \( \mathcal{W}_{r,m} \) is trivial, and thus \( H^1(\mathcal{G}, \mathcal{W}_{r,m}) \) is identified with the conjugacy classes of the group homomorphisms from \( \mathcal{G} \) to \( \mathcal{W}_{r,m} \). Note that \( \mathcal{W}_{r,m} \) is naturally identified with the symmetric group of the set \( \{ \chi_1, \ldots, \chi_m \} \), as well as the set \( \{ \sigma_1, \ldots, \sigma_m \} \). Hence \( \iota(b_l) \) and \( \iota(b_0) \) induce \( \mathcal{G} \)-actions on these two sets respectively. Once we show that these induced actions are conjugate to the natural Galois actions, then we are done.

On one hand, by (4.32), for any \( z \in Z(\mathcal{F})(\mathbb{Q}) \) and any \( \sigma \in \mathcal{G} \), we have

\[
\begin{bmatrix}
\chi_1(z)I_r \\
\vdots \\
\chi_m(z)I_r
\end{bmatrix}
\begin{bmatrix}
l \\
\vdots \\
l
\end{bmatrix}^{-1} = z = \sigma(z)
\]

so

\[
\begin{bmatrix}
\chi_1(z)I_r \\
\vdots \\
\chi_m(z)I_r
\end{bmatrix}
\begin{bmatrix}
l \\
\vdots \\
l
\end{bmatrix} = \sigma(l)
\begin{bmatrix}
\chi_1(z)I_r \\
\vdots \\
\chi_m(z)I_r
\end{bmatrix}
\]

and thus \( \iota(b_l)(\sigma) \cdot \chi_i = \sigma \chi_i \) for \( 1 \leq i \leq m \).

On the other hand,

\[
\sigma^{-1}(l_0) = 
\begin{bmatrix}
\sigma_1(x_1)I_r & \cdots & \sigma_1(x_m)I_r \\
\vdots & \ddots & \vdots \\
\sigma_m(x_1)I_r & \cdots & \sigma_m(x_m)I_r
\end{bmatrix}
\]

and thus \( \iota(b_0)(\sigma) \cdot \sigma_i = \sigma \sigma_i \) for \( 1 \leq i \leq m \).

Hence, up to conjugation, both actions indeed coincide with the natural Galois actions respectively. \( \square \)

We defined \( M_d \) in (4.6). Let

\[
M_{-d} = \left\{ \begin{bmatrix} tI_d & A \end{bmatrix} : A \in \text{GL}_{n-d}, \det A = t^{-d} \right\}.
\]

We note that \( Z(M_d) = M_{-d} \) and \( Z(M_{-d}) = M_d \). By (4.23), we have \( H = g F g^{-1} \subset M_d \). Taking centralizers on both sides, we get \( g Z(F) g^{-1} \subset M_{-d} \). Consider the right action of \( G \) on the space \( \mathbb{C}^n \) of \( n \)-dimensional row vectors, and let \( \{ e_1, \ldots, e_n \} \) be its standard basis. Then for each \( z \in Z(F) \), \( g z g^{-1} \) acts on the span of \( \{ e_1, \ldots, e_d \} \) by a scalar multiple. Hence \( (\mathbb{C} e_1 + \cdots + \mathbb{C} e_d)g \) is contained in \( E_{\chi}(\mathbb{C}) \) for some \( \chi \in \{ \chi_1, \ldots, \chi_m \} \), where \( E_{\chi} \) is
Then there exist integers by (4.24), that is, \(\leq\) of Proposition 4.1, it will be used in the proof of the converse of Proposition 4.1 later. By relabelling \(\chi_1, \ldots, \chi_m\) we may assume \(\chi = \chi_1\) and \(\chi\) is defined over \(\mathbb{K}\). Hence, \(\mathbb{K}\) is a subfield of \(\mathbb{R}\).

Now pick \(f_{d+1}, \ldots, f_r \in \mathbb{R}^n\) such that \(\{e_1, \ldots, e_d, f_{d+1}, \ldots, f_r\}\) form a basis of \(E_{\chi}(\mathbb{R})g^{-1}\). Take any \(g' \in G\) such that \(e_i g' = e_i\) and \(e_j g' = f_j\) for \(1 \leq i \leq d\) and \(d + 1 \leq j \leq r\). It follows that \(g' \in P_d\) and

\[
g'gZ(F)g^{-1}g'^{-1} \supset \left( tI_r \begin{smallmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{smallmatrix} \right) \cap G.
\]

Taking centralizers in \(G\), we get

\[
g'gFg^{-1}g'^{-1} \subset \left( \begin{smallmatrix} \text{GL}_r \\ \text{GL}_{n-r} \end{smallmatrix} \right) \cap G \subset P_r.
\]

The linear span of the first \(r\) rows of \(g'g\) is \(E_{\chi}\), and thus is defined over \(\mathbb{K} \subset \mathbb{R}\). In other words, we have \([g'g] \in (P_r \setminus G)(\mathbb{K}) = P_r(\mathbb{K}) \setminus G(\mathbb{K})\) (see e.g. [Bor91, Prop. 20.5]). Hence by (4.24),

\[
g_M \in P_dF = P_dg'gF = P_dg'gFg^{-1}g'^{-1}g'g \subset P_dP_rG(\mathbb{K}).
\]

In other words, (2) of Proposition 4.1 holds.

We now summarize the above discussion in the following lemma.

**Lemma 4.13.** Under the assumptions of Proposition 4.1, suppose that \(Gv_0\) is Zariski closed and \(\gamma_i v_0 = v_0\) for all \(i\). Further suppose that (1) of Proposition 4.1 does not hold; that is,

\[
\inf_{\gamma \in \Gamma} \sup_{\omega \in \Omega} \|g_\omega g_M \gamma e_1\| \rightarrow \infty \text{ as } t \rightarrow \infty.
\]

Then there exist integers \(r \geq d\), \(m \geq 2\), and a number field \(\mathbb{K} \subset \mathbb{R}\) such that \([\mathbb{K} : \mathbb{Q}] = m\), \(n = mr\), and \(g_M \in P_dP_rG(\mathbb{K})\); that is, (2) of Proposition 4.1 holds.

We also prove a lemma in the converse direction. Although it is not used in the proof of Proposition 4.1, it will be used in the proof of the converse of Proposition 4.1 later. For \(1 \leq i \leq n - 1\), let

\[
Q_i = \left( \begin{smallmatrix} I_i \\ \text{Mat}_{(n-i)}i \text{ } \text{ } \text{SL}_{n-i} \end{smallmatrix} \right).
\]

Let \(L_i, g_M, L_M\) be as at the beginning of the section, and recall that \(d = \dim L_M + 1\).

**Lemma 4.14.** Suppose that there exist integers \(r \geq d\), \(m \geq 2\), and a number field \(\mathbb{K} \subset \mathbb{R}\) such that \([\mathbb{K} : \mathbb{Q}] = m\), \(n = mr\), and \(g_M \in P_dP_rG(\mathbb{K})\). Then, at least one of the following holds:

1. There exist a \(\mathbb{Q}\)-subgroup \(F \cong \mathbb{Q} \text{ Res}_{\mathbb{K}/\mathbb{Q}}^{(1)} \text{ GL}_r\) of \(G\) and an element \(g_0 \in G\) such that \(M_r \subset g_0Fg_0^{-1} \subset L_r\) and \(g_{\mathbb{K}} \in Q_r g_0\).
2. \(L_M\) is contained in a proper linear subspace of \(\mathbb{P}^{n-1}(\mathbb{R})\) which is defined over \(\mathbb{Q}\).
Proof. We identify $P_r \backslash G$ with $\text{Gr}(r, n)$ by mapping $P_r g$ to the span of the first $r$ rows of $g$. As before we choose and fix a basis $\{x_1, \ldots, x_m\}$ of $K$ as a $\mathbb{Q}$-vector space. Consider the $r \times n$ matrix

$$B = (x_1 I_r \quad \cdots \quad x_m I_r).$$

Its columns give a $\mathbb{Q}$-basis of $K^n$. Let $[g_K]$ be the top $r \times n$ block of $g_K$. We discuss the following two possibilities.

Suppose first that the columns of $[g_K]$ are $\mathbb{Q}$-linearly dependent, then $\mathcal{L}$ is contained in a proper linear subspace of $\mathbb{P}^{n-1}(\mathbb{R})$ defined over $\mathbb{Q}$, where $\mathcal{L}$ denotes the $(r-1)$-dimensional linear subspace parametrized by $P_r g_K$. Recall that $\mathcal{L}_M$ is the $(d-1)$-dimensional linear subspace of $\mathbb{P}^{n-1}(\mathbb{R})$ parametrized by $P_d g_M$. Since $g_M \in P_d P_r g_K$, we have $\mathcal{L}_M \subset \mathcal{L}$. Hence (2) of the lemma holds.

Now suppose that the columns of $[g_K]$ are linearly independent over $\mathbb{Q}$. Then there exists $g_0 \in \text{GL}_n(\mathbb{Q})$ such that $[g_K] = B g_0$. Recall that after possibly relabelling the $\chi_i$’s, we have assumed that $\chi_1$ is defined over $\mathbb{K} \subset \mathbb{R}$, and $\sigma_1$ is the inclusion of $\mathbb{K}$ into $\mathbb{R}$. Therefore $B$ consists of the first $r$ row vectors of $l_0^{-1}$, where $l_0$ is as in (4.34). Hence we have $P_r g_K = P_r l_0^{-1} g_0$. Now take $F = g_0^{-1} l_0 H_{r,m}^{-1} q_{g_0}$, and by Remark 4.11 we know that $l_0 H_{r,m}^{-1}$ is defined over $\mathbb{Q}$. Hence $F$ is defined over $\mathbb{Q}$. We again consider the eigenspaces of $Z(F)$ in the standard representation. Since $E_{\chi_1}$ is defined over $\mathbb{K} \subset \mathbb{R}$, $W := \bigoplus_{i=2}^n E_{\chi_i}$ is also defined over $\mathbb{R}$. Let $g_0 \in G$ be the matrix such that the block of the first $r$ rows and the remaining $(n-r)$ rows form a basis of $W$. Then $g_K \in Q_r g_0$. One can also verify that $Z(L_r) \subset g_0 Z(F) g_0^{-1} \subset M_r$. By taking centralizers, we see that (1) of the lemma holds.

Corollary 4.15. Under the assumptions of Lemma 4.14, suppose (1) of Lemma 4.14 holds. Then, there exists a compact subset $\Sigma$ of $G$ such that $g_r \Omega g_m$ is contained in $\Sigma F$ for all $t \geq 0$, where $\Omega$ is as in (4.4).

Proof. By assumption, we write $g_M = p_d p_r g_K$ for some $p_d \in P_d$ and $p_r \in P_r$. Observe that $P_1 P_d P_r = Q_1 M_r$. Since $\Omega$ is a compact subset of $P_1 P_d$, there exist compact sets $\Omega_1 \subset Q_1$ and $\Omega_2 \subset M_r$ such that $\Omega p_d p_r \subset \Omega_1 \Omega_2$. Hence $\Omega g_m = \Omega p_d p_r g_K \subset \Omega_1 \Omega_2 g_K$. Since $\Omega_1 \subset Q_1 \subset P_1$ is compact, $g_0 \Omega_1 g_0^{-1}$ is contained in a fixed compact set for all $t \geq 0$. Hence, it suffices to show that $g_t \Omega_2 g_K F/F$ is contained in a fixed compact set of $G/F$ for all $t \geq 0$. By our assumption, we can write $g_K = q_r g_0$ for some $q_r \in Q_r$. Hence, there exists a compact set $\Omega_3 \subset M_r$ and $u_r \in U_r$ such that $g_t \Omega_2 g_K = g_t \Omega_2 g_0 = g_t u_r \Omega_3 g_0$. Since $u_r \in U_r \subset P_1$, we have $g_t u_r g_t^{-1} \rightarrow e$ as $t \rightarrow \infty$. Hence, it suffices to show that $g_t \Omega_3 g_0 F/F$ is contained in a fixed compact set of $G/F$ for all $t \geq 0$. Since $g_t \in M_r$, we have $g_t \Omega_3 \subset M_r \subset g_0 F g_0^{-1}$. Hence, $g_0 \Omega_3 g_0 F/F = g_0 F/F$ is a single point in $G/F$, and the proof is finished.

4.3. Case (3). $G_{v_0}$ is Zariski closed, and $\gamma_i v_0 \rightarrow \infty$ as $i \rightarrow \infty$.

As before, let $F = G_{v_0}$. Since $\gamma_i v_0 \rightarrow \infty$, $v_0$ is not fixed by $G$, and $F$ is a proper subgroup of $G$. By Lemma 4.2, $d < n$ and we know that a conjugate of $F(C)$ contains $H_d$, where $F(C)$ denotes the $C$ points of $F$. Hence, this enables us to use Theorem A.7 to classify $F_C$, which is obtained from $F$ by extension of scalars to $C$.

We first show that if $F_C$ is contained in a proper parabolic subgroup of $G_C$, then one can reduce this case to Case (1) as in Section 4.1 or Case (2) as in Section 4.2.
Lemma 4.16. Let $\rho : G \to \text{GL}(V)$ be a representation of $G$, and $v_0 \in V$ such that $Gv_0$ is Zariski closed. We assume that (\textbullet) holds for $(\rho, V, v_0)$, i.e.
\[
\sup_{\omega \in \Omega} \|c_t \omega b_t g_M \gamma_i v_0 \| \leq C, \ \forall i. \tag{4.37}
\]
Let $F = Gv_0$. Suppose that $F_C$ is contained in a proper parabolic subgroup of $G_C$, then there exists a nonzero $w_0 \in \mathfrak{g}_\mathbb{Q}$ such that (\textbullet) holds for $(\text{Ad}, \mathfrak{g}, w_0)$, i.e.
\[
\sup_{\omega \in \Omega} \|c_t \omega b_t g_M \gamma_i w_0 \| \leq C, \ \forall i. \tag{4.38}
\]

Proof. Suppose that $F_C$ is contained in a proper parabolic subgroup of $G_C$. Since $F$ is reductive, it is contained in a Levi subgroup of that parabolic, and thus $Z_G(F)_C$ contains a non-trivial multiplicative one-parameter subgroup. Since $v_0 \in V(\mathbb{Q})$, $F$ is defined over $\mathbb{Q}$, and hence $Z_G(F)$ is a nontrivial reductive subgroup of $G$ defined over $\mathbb{Q}$. It follows that $\text{Lie}(Z_G(F))$ is a non-trivial Lie algebra over $\mathbb{Q}$. Hence we may pick a nonzero element $w_0 \in \text{Lie}(Z_G(F))(\mathbb{Q})$. Since $F$ stabilizes $w_0$, we have a continuous $G$-equivariant map $G/F \to \mathfrak{g}$ which sends $gF$ to $gw_0$. In particular, it sends compact sets to compact sets. Hence (4.37) implies (4.38). \hfill \Box

Lemma 4.17. Consider the adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$, and suppose that there exists a nonzero $w_0 \in \mathfrak{g}$ such that (4.38) holds. Then the set $\{\gamma_i w_0\}$ is bounded.

Proof. Recall from the beginning of the section that $P_1 \Omega$ is not contained in any proper linear subspace of $P_1 \setminus P_1 L_d \cong (P_1 \cap H_d) \setminus H_d$. Since $\Omega$ is analytic and connected, we deduce that $P_1 \Omega$ is not contained in a union of finitely many proper linear subspaces of $(P_1 \cap H_d) \setminus H_d$. Applying Lemma 3.3 to $H_d$, we know that for any $0 \neq v \in \mathfrak{g}$, the set $\Omega v$ cannot be contained in $\mathfrak{g}_{<0}(c_t) := \{X \in \mathfrak{g} : \text{Ad}(c_t)(X) \to 0 \text{ as } t \to \infty \}$. By (4.5),
\[
\mathfrak{g}_{<0}(g_t) = \bigoplus_{i=2}^n \mathfrak{g}_{ij} \subset \bigoplus_{i=2}^d \mathfrak{g}_{ij} = \mathfrak{g}_{<0}(c_t),
\]
where $\mathfrak{g}_{ij}$ is the 1-dimensional linear subspace spanned by the elementary matrix whose only nonzero entry lies in the $i$-th row and $j$-th column. Therefore $\Omega v \not\subset \mathfrak{g}_{<0}(g_t)$ for any nonzero $v$. Let $\pi_{\geq0}$ denote the $g_t$-equivariant projection from $\mathfrak{g}$ to $\mathfrak{g}_{\geq0}(g_t)$, where $\mathfrak{g}_{\geq0}(g_t) := \{X \in \mathfrak{g} : \text{Ad}(g_t)(X) \text{ is convergent as } t \to 0 \}$. Then by compactness of $\mathbb{P}(\mathfrak{g})$, the map $\mathbb{R}v \mapsto \sup_{\omega \in \Omega} \|\pi_{\geq0}(\omega v)/\|v\|\$ has a positive minimum. Therefore, we can pick $D > 0$ such that for any $v \in \mathfrak{g}$ and any $t \geq 0$, we have
\[
\sup_{\omega \in \Omega} \|g_t \omega v\| \geq D\|v\|.
\]
Hence
\[
\sup_{\omega \in \Omega} \|c_t \omega b_t g_M \gamma_i w_0\| = \sup_{\omega \in \Omega} \|g_t \omega g_M \gamma_i w_0\| \geq D\|g_M \gamma_i w_0\|.
\]
So, by (4.38), $\{g_M \gamma_i w_0\}$ is bounded. Hence $\{\gamma_i w_0\}$ is bounded. \hfill \Box

If $F_C$ is contained in any proper parabolic subgroup of $G_C$, then by Lemma 4.16 and Lemma 4.17, we are reduced to the Case (1) or Case (2). Therefore, we assume that $F_C$ is not contained in any parabolic subgroup of $G_C$.

Lemma 4.18. Suppose that $F_C$ is not contained in any proper parabolic subgroup of $G_C$. Then $n$ is even, $d = 2$, and there exists $L \in \text{GL}_n(\mathbb{C})$ such that $F_C = L \text{Sp}_n L^{-1}$. 
Proof. It follows from our assumption that the inclusion $F_C \subset G_C = \text{SL}_n$ gives a faithful irreducible representation of $F_C$ on $\mathbb{C}^n$. Furthermore, Lemma 4.2 implies that a conjugate of $F_C$ contains $(H_d)_C$. Hence by Theorem A.7 we know that $n$ is even, $d = 2$, and $F_C = \text{Sp}(\mathbb{C}^n, \omega)$ for some sympletic form $\omega$ on $\mathbb{C}^n$. Let $l_C$ be the transformation matrix from the standard basis of $\mathbb{C}^n$ to a symplectic basis of $(\mathbb{C}^n, \omega)$. Then we have $F_C = l_C \text{Sp}_n l_C^{-1}$.

Lemma 4.19. Suppose that $F_C = l_C \text{Sp}_n l_C^{-1}$ for some $l_C \in \text{GL}_n(\mathbb{C})$. Then there exists $l_Q \in \text{GL}_n(\mathbb{Q})$ such that $F = l_Q \text{Sp}_n l_Q^{-1}$.

Proof. Note that the normalizer of $\text{Sp}_n$ in $\text{GL}_n$ is $\text{GSp}_n$. Consider the homogeneous variety $X = \text{GL}_n / \text{GSp}_n$ of conjugates of $\text{Sp}_n$ in $\text{GL}_n$. Then $F_C = [l_C] \in X$. Since $F$ is defined over $\mathbb{Q}$, we have that $F_C \in X(\mathbb{Q})$. To prove the lemma, we only need to show that $\text{GL}_n(\mathbb{Q})$ acts transitively on $X(\mathbb{Q})$. By [Ser94, Corollary 1 of Proposition 36], it suffices to show that the kernel of $H^1(\mathbb{Q}, \text{GSp}_n) \to H^1(\mathbb{Q}, \text{GL}_n)$ is trivial.

We consider following the exact sequence of algebraic groups:

$$1 \to \text{Sp}_n \to \text{GSp}_n \to \text{Gm} \to 1.$$ 

Since $H^1(\mathbb{Q}, \text{Sp}_n) = H^1(\mathbb{Q}, \text{Gm}) = 1$, we also have $H^1(\mathbb{Q}, \text{GSp}_n) = 1$. Hence, we are done.

By combining the discussions on all three cases, we will complete:

Proof of Proposition 4.1. Firstly, suppose that $Gv_0$ is not Zariski closed. We apply Proposition 4.6 to conclude that there exists $t_i' \to \infty$, $C' > 0$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\sup_{\omega \in \Omega} \|g_{t'_i} \omega g_{\lambda} v_i\| \leq C', \forall i;$$

that is, (1) of Proposition 4.1 holds.

Now suppose that $Gv_0$ is Zariski closed, and $\gamma_i v_0 = v_0$ for all $i$. By Lemma 4.13, we know that either (1) or (2) of Proposition 4.1 holds.

Finally, suppose that $Gv_0$ is Zariski closed, and $\gamma_i v_0 \to \infty$ as $i \to \infty$. We first assume that $F_C$ is contained in a proper parabolic subgroup of $G_C$. By Lemma 4.16 and Lemma 4.17, we can reduce to Case (2) in Section 4.2 and again by Lemma 4.13 we have either (1) or (2) of Proposition 4.1 occurs.

Now we assume that $F_C$ is not contained in any proper parabolic subgroup of $G_C$. By Lemma 4.18 and Lemma 4.19, there exists $l_Q \in \text{GL}_n(\mathbb{Q})$ such that $F = l_Q \text{Sp}_n l_Q^{-1}$; we note that $n$ is even. Let $w_0 = l_Q(e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{n-1} \wedge e_n)$, so that $G_{w_0} = F$. Multiplying $w_0$ by a positive integer, we may assume that $w_0 \in \wedge^2 \mathbb{Z}^n$. Hence (3) of Proposition 4.1 holds.

5. Interpretation of Diophantine and arithmetic conditions

In this section, we reformulate our Diophantine and arithmetic properties of $A \in M_{d,n-d}(\mathbb{R})$ in a group theoretic manner.

Let $d$ be an integer such that $2 \leq d \leq n - 1$. Recall that for any $A \in M_{d,n-d}(\mathbb{R})$, we define the following affine subspace of $\mathbb{R}^{n-1}$:

$$\mathcal{L}_A = \{(x, \tilde{x} A) \mid x \in \mathbb{R}^{d-1}\},$$

where $\tilde{x} := (1, x) \in \mathbb{R}^d$ for any $x \in \mathbb{R}^{d-1}$. 

We write
\[ u_A := \begin{pmatrix} I_d & A \\ 0 & I_{n-d} \end{pmatrix}, \tag{5.2} \]
and we note that \( L_A = L_0 u_A \).

As in (4.4), let \( \Omega \) be a compact subset of \( P_1 L_d \) such that the linear span of \( P_1 \Omega \) in \( P_1 \setminus G \) is \( P_1 L_d \).

Recall that we have defined the following two flows in \( SL_n \):
\[ b_t = \left( e^{\frac{-a-d}{d} t} I_d \quad e^{-t} I_{n-d} \right), \quad c_t = \left( e^{\frac{n-d}{d} t} I_d \quad e^{-\frac{n}{d} t} I_{d-1} \quad I_{n-d} \right), \tag{5.3} \]
so that \( g_t = b_t c_t \). We identify the expanding horospherical group \( U^+ \) of \( g_1 \) with \( \mathbb{R}^{n-1} \) in view of (1.1).

We need the following elementary fact, whose proof is left to the reader:

**Lemma 5.1.** Let \( \Sigma \) be a compact subset of \( \mathbb{R}^k \) whose linear span is the full \( \mathbb{R}^k \). Then there exists \( C > 1 \) such that for all \( v \in \mathbb{R}^k \),
\[ C^{-1} \|v\| \leq \sup_{X \in \Sigma} |X \cdot v| \leq C \|v\|, \]
where \( \cdot \) denotes the inner product.

The following two lemmas relate the Diophantine properties of \( A \) with certain bounds in the standard representation of \( SL_n \). We recall the definitions of \( \mathcal{W} \) and \( \mathcal{W}' \) from Section 1.2.

**Lemma 5.2.** Given \( 0 < r < d \). The following are equivalent:

1. \( A \in \mathcal{W}_{n-d+r}(d, n - d) \).
2. There exist \( R > 0 \), \( t_i \to \infty \) and \( v_i \in \mathbb{Z}^n \setminus \{0\} \) such that
\[ \|b_{di} u_A v_i\| \leq Re^{-rt_i}. \tag{5.4} \]
3. There exist \( C > 0 \), \( t_i \to \infty \) and \( v_i \in \mathbb{Z}^n \setminus \{0\} \) such that
\[ \sup_{\omega \in \Omega} \|g_{ti} u_A v_i\| \leq Ce^{(d-r-1)t_i}. \tag{5.5} \]

**Proof.** Write \( v_i = \left( \begin{pmatrix} p_i \\ q_i \end{pmatrix} \right) \) where \( p_i \in \mathbb{Z}^d \) and \( q_i \in \mathbb{Z}^{n-d} \setminus 0 \). Then \( u_A v_i = \left( \begin{pmatrix} Aq_i + p_i \\ q_i \end{pmatrix} \right) \). One can verify directly that (1) and (2) are equivalent to the following: there exists \( t'_i \to \infty \) such that
\[ \|Aq_i + p_i\| = O(e^{-(n-d+r)t'_i}) \text{ and } \|q_i\| = e^{(d-r)t'_i} \tag{5.6} \]
with the standard big-O notation. One can also verify that (5.6) implies (3). We now show that (3) implies (5.6).

Since \( P_1 L_d = P_1 (U^+ \cap L_d) \), without loss of generality we may assume that \( \Omega \) is a compact subset of \( U^+ \cap L_d \cong \mathbb{R}^{d-1} \) which is not contained in any proper affine subspace.
of $\mathbb{R}^{d-1}$. Let $\psi$ denote the identification of $U^+ \cap L_d$ with $\mathbb{R}^{d-1}$, and $\widetilde{\psi} : U^+ \cap L_d \to \mathbb{R}^d$ is given by $\omega \mapsto (1, \psi(\omega))$. By computation,

$$\omega u_A v_i = \begin{pmatrix} \widetilde{\psi}(\omega) \cdot (Aq_i + p_i) \\ (Aq_i + p_i)_2 \\ \vdots \\ (Aq_i + p_i)_d \\ q_i \end{pmatrix}$$

By our assumption on $\Omega$, $\widetilde{\psi}(\Omega)$ is compact and spans $\mathbb{R}^d$. Applying Lemma 5.1 to $\Sigma = \widetilde{\psi}(\Omega)$ we have

$$\sup_{\omega \in \Omega} \| g_t \omega u_A v_i \| \asymp e^{(n-1)t} \| Aq_i + p_i \| + e^{-t_i} || q_i || .$$

It follows that (3) is equivalent to (5.6).

Analogous to the above lemma, we also have the following:

**Lemma 5.3.** Given $0 < r < d$. The following are equivalent:

1. $A \in W_{n \cdot d \cdot r}(d, n - d)$.
2. There exists $R_i \to 0$, $t_i \to \infty$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that
   $$\| b_{di} u_A v_i \| \leq R_i e^{-t_i} .$$
3. There exists $C_i \to 0$, $t_i \to \infty$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that
   $$\sup_{\omega \in \Omega} \| g_t \omega u_A v_i \| \leq C_i e^{(d-r-1)t_i} .$$

**Proof.** Write $v_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix}$. Arguing in the same way as in Lemma 5.2, one can show that (1),(2), and (3) are all equivalent to the following: there exists $t'_i \to \infty$ such that

$$\| Aq_i + p_i \| = o(e^{-(n-d+r)t'_i})$$

and $\| q_i \| = e^{(d-r)t'_i}$

with the standard small-notation.

For the next lemma, we assume that $d = 2$, and prove an elementary result concerning the exterior square of the standard representation of $\text{SL}_n$.

Now that $A \in M_{2,n-2}(\mathbb{R})$, and let $N = \begin{pmatrix} n-2 \\ 2 \end{pmatrix}$. Let $A^{\text{ext}} \in M_{2n-3,N}(\mathbb{R})$ be as defined in (1.3).

**Lemma 5.4.** Let $W$ be the exterior square of the standard representation of $\text{SL}_n$. The following are equivalent:

1. $A^{\text{ext}} \in W_{d,n \cdot d \cdot 2}(2n-3, N)$.
2. There exists $C > 0$, $t_i \to \infty$ and $w_i \in W(\mathbb{Z}) \setminus \{0\}$ such that
   $$\sup_{\omega \in \Omega} \| g_t \omega u_A w_i \| \leq C .$$

(5.10)
Proof. We note that the \( w_i \) is not necessarily decomposable, and thus we cannot apply Lemma 4.4.

Let \( V \) be the standard representation of \( SL_n \); let \( V_1 \) and \( V_2 \) be the linear spans of \( e_1, e_2 \) and \( e_3, \ldots, e_n \) respectively. Let \( W_1 = V_1 \cap V_1 + V_1 \otimes V_2 \), \( W_2 = V_2 \cap V_2 \). We have \( W = W_1 \oplus W_2 \), and let \( \pi_1, \pi_2 \) be the projections from \( W \) to \( W_1, W_2 \) respectively. Let \( \psi \) be the natural identification of \( (U^+ \cap L_2) \) with \( \mathbb{R} \) and let \( \tilde{\psi} : U^+ \cap L_2 \to \mathbb{R}^2 \) be given by \( \omega \mapsto (1, \psi(\omega)) \). Let \( 3 \leq i \leq n \). Then \( V_1 \otimes \mathbb{R} e_i \cong \mathbb{R}^2 \) is \( (U^+ \cap L_2) \)-stable. And we have

\[
\omega(a \cdot e_1 \cap e_i + b \cdot e_2 \cap e_i) = (\hat{\psi}(\omega) \cdot (a, b)) e_1 \cap e_i + b \cdot e_2 \cap e_i, \forall (a, b) \in \mathbb{R}^2.
\]

Since \( \psi \) is nonconstant, \( \hat{\psi}(\Omega) \) spans \( \mathbb{R}^2 \). Applying Lemma 5.1 to \( \Sigma = \hat{\psi}(\Omega) \) and \( k = 2 \), one can show that

\[
\sup_{\omega \in \Omega} \| \omega w \| \asymp \| w \|, \forall w \in V_1 \otimes \mathbb{R} e_i.
\]

And, since \( U^+ \cap L_2 \) acts trivially on \( V_1 \cap V_1 \) and on \( W_2 \), for any \( w \in W \) we have

\[
\sup_{\omega \in \Omega} \| \pi_j(\omega w) \| = \sup_{\omega \in \Omega} \| \omega \pi_j(w) \| \asymp \| \pi_j(w) \|, \text{ where } j = 1, 2.
\]

Hence

\[
\sup_{\omega \in \Omega} \| g_i \omega u_A w_i \| \asymp e^{(n-2)t} \sup_{\omega \in \Omega} \| \pi_1(\omega u_A w_i) \| + e^{-2t} \sup_{\omega \in \Omega} \| \pi_2(\omega u_A w_i) \|
\]

\[
\asymp e^{(n-2)t} \| \pi_1(u_A w_i) \| + e^{-2t} \| \pi_2(u_A w_i) \|.
\]

Hence (5.10) is equivalent to

\[
\begin{cases}
\| \pi_1(u_A w_i) \| = O(e^{-(n-2)t}) \\
\| \pi_2(u_A w_i) \| = O(e^{2t})
\end{cases}
\]  

(5.11)

For simplicity, we fix \( i \) and write \( w, t \) for \( w_i, t_i \). By taking a smaller \( t \) if necessary, we rewrite (5.11) as

\[
\begin{cases}
\| \pi_1(u_A w) \| = O(e^{-(n-2)t}) \\
\| \pi_2(u_A w) \| \asymp e^{2t}
\end{cases}
\]  

(5.12)

Write \( w = \begin{pmatrix} p \\ q \end{pmatrix} \), where \( p \in \mathbb{Z}^{2n-3} \) and \( q \in \mathbb{Z}^N \). We now check that (5.12) is equivalent to

\[
\| A^{ext} q + p \| = O(\| q \|^{-\frac{2n-2}{2}}).
\]  

(5.13)

For convenience we rewrite

\[
A = \begin{pmatrix} a_3 & a_4 & \cdots & a_n \\ b_3 & b_4 & \cdots & b_n \end{pmatrix}.
\]

For all \( 3 \leq i \leq n \) and \( i < j \leq n \), we have

\[
\begin{align*}
    u_A(e_1 \cap e_2) &= e_1 \cap e_2 \\
    u_A(e_1 \cap e_i) &= e_1 \cap e_i + b_i e_1 \cap e_2 \\
    u_A(e_2 \cap e_i) &= e_2 \cap e_i - a_i e_1 \cap e_2 \\
    u_A(e_i \cap e_j) &= e_i \cap e_j + (a_i b_j - a_j b_i) e_1 \cap e_2 + a_i e_1 \cap e_j + b_i e_2 \cap e_j - a_j e_1 \cap e_i - b_j e_2 \cap e_i.
\end{align*}
\]
We write \( w = \sum_{1 \leq k < l \leq n} C_{kl} e_k \wedge e_l \), and compute
\[
u_A w = \sum_{3 \leq i < j \leq n} C_{ij} e_i \wedge e_j + \sum_i (C_{1i} + \sum_{j<i} C_{ji} a_j - \sum_{j>i} C_{ij} a_j) e_i \wedge e_i
+ \sum_i (C_{2i} + \sum_{j<i} C_{ji} b_j - \sum_{j>i} C_{ij} b_j) e_2 \wedge e_i\]
\[+ (\sum_{i<j} (a_i b_j - a_j b_i) C_{ij} + \sum_i b_i C_{1i} - \sum_j a_j C_{2j} + C_{12}) e_1 \wedge e_2.\]

Now (5.12) is equivalent to
\[
\begin{aligned}
C_{1i} + \sum_{j<i} C_{ji} a_j - \sum_{j>i} C_{ij} a_j &= O(e^{-(n-2)t}) \\
C_{2i} + \sum_{j<i} C_{ji} b_j - \sum_{j>i} C_{ij} b_j &= O(e^{-(n-2)t}) \\
\sum_{i<j} (a_i b_j - a_j b_i) C_{ij} + \sum_i b_i C_{1i} - \sum_j a_j C_{2j} + C_{12} &= O(e^{-(n-2)t}) \\
C_{ij} &= O(e^{2t})
\end{aligned}
\tag{5.14}
\]

Therefore from the first two equations, we obtain
\[
\sum_i b_i C_{1i} - \sum_j a_j C_{2j} = -2 \sum_{i<j} (a_i b_j - a_j b_i) C_{ij} + O(e^{-(n-2)t}).
\]

Inserting this data in the third equation, we conclude that (5.14) is equivalent to
\[
\begin{aligned}
C_{1i} + \sum_{j<i} C_{ji} a_j - \sum_{j>i} C_{ij} a_j &= O(e^{-(n-2)t}) \\
C_{2i} + \sum_{j<i} C_{ji} b_j - \sum_{j>i} C_{ij} b_j &= O(e^{-(n-2)t}) \\
- \sum_{i<j} (a_i b_j - a_j b_i) C_{ij} + C_{12} &= O(e^{-(n-2)t}) \\
C_{ij} &= O(e^{2t})
\end{aligned}
\tag{5.15}
\]

By our definition of \( A^\text{ext} \), (5.15) is equivalent to (5.13); and by our definition of \( W \), (5.13) is equivalent to \( A^\text{ext} \in W_{n-2}(2n-3, N). \)

\[\square\]

**Lemma 5.5.** Let \( r \geq d \) be an integer. Let \( K \subset \mathbb{R} \) be a real field. The following are equivalent:

1. \( L_A \) is contained in some \((r - 1)\)-dimensional affine subspace of \( \mathbb{R}^{n-1} \) which is defined over \( K \).
2. \( L_A \) is contained in some \((r - 1)\)-dimensional linear subspace of \( \mathbb{P}^{n-1}(\mathbb{R}) \) which is defined over \( K \).
3. \( u_A \in P_d P_r G(K) \).

**Proof.** (1) \( \iff \) (2). This is obvious.

(2) \( \iff \) (3). Let \([x_1 : \cdots : x_n]\) be the homogeneous coordinates of \( \mathbb{P}^{n-1} \). Let \( P_0 \) be the \((r - 1)\)-dimensional linear subspace of \( \mathbb{P}^{n-1} \) defined by \( x_{r+1} = \cdots = x_n = 0 \). Now (2) is equivalent to \( L_A = L_0 u_A \subset P_0 g_K \) for some \( g_K \in G(K) \). Hence it suffices to show that for any \( g \in G \), \( g \in P_d P_r \) if and only if \( L_0 g \subset P_0 \). This can be checked directly. \( \square \)
6. Sharp conditions for non-escape of mass and equidistribution

In this section, we will obtain sufficient representation theoretic conditions for the non-escape of mass and equidistribution. These representation theoretic conditions were interpreted in terms of Diophantine or arithmetic conditions in Section 5. We will then prove that these conditions are also necessary for the non-escape of mass or equidistribution.

6.1. Non-escape of mass and proof of Theorem 1.1. Let \( \phi \) and \( \lambda_\phi \) be as in Section 1.2. We can write \( \phi(B) = \Omega u_A \), where \( A = A_\phi \in M_{d,n-d}(\mathbb{R}) \), and \( \Omega \) is a compact subset of \( P_1L_d \) such that the linear span of \( P_1\Omega \) in \( P_1 \setminus G \) is \( P_1L_d \). Hence \( \Omega \) satisfies the condition in the beginning of Section 4.

The following theorem is due to Dani-Margulis [DM93] and Kleinbock-Margulis [KM98]. It provides a representation-theoretic criterion for no escape of mass to infinity. What we state here is a consequence of their original theorem.

Let \( W_k = \wedge^k \mathbb{R}^n \) and \( w_k = e_1 \wedge \cdots \wedge e_k \).

**Theorem 6.1** (c.f. [KM98, Theorem 5.2]). Fix a norm \( \| \cdot \| \) on \( W_k \). For any \( \epsilon > 0 \) and \( R > 0 \), there exists a compact set \( K \subset G/\Gamma \) such that for any \( t > 0 \) and any ball \( J \subset B \), one of the following holds:

\( \begin{align*}
(1) & \text{ There exist } k \in \{1, \ldots, n-1\} \text{ and } \gamma \in \Gamma \text{ such that } \\
& \sup_{s \in J} \| g_t \phi(s) \gamma w_k \| < R; \\
(2) & |\{s \in J : g_t \phi(s)x_0 \in K\}| \geq (1-\epsilon)|J|.
\end{align*} \)

A key ingredient in the proof is a growth property called the \((C, \alpha)\)-good property of the family of functions \( \{B \ni s \mapsto \|g_t \phi(s)w\| : t \geq 1, w \in W_k\} \), see [KM98, Proposition 3.4], [Sha09a, Section 3.2] and [Sha09b, Section 2.1].

**Proof of Theorem 1.1.** Suppose that the sequence of \( g_t \)-translates of \( \lambda_\phi \) has the escape of mass. Then there exists \( \epsilon > 0 \) and \( t_i \to \infty \) such for any compact set \( K \subset X \), we have \( g_{t_i} \lambda_\phi(K) \leq 1-\epsilon \). By Theorem 6.1, after passing to a subsequence, there exists \( k \in \{1, \ldots, n-1\} \) and \( \gamma_i \in \Gamma \) such that \( \sup_{s \in B} \| g_{t_i} \phi(s) \gamma_i w_k \| < 1/i \). By Proposition 4.6, \( d < n \). Now by Corollary 4.5, there exists \( t'_i \to \infty \) and \( v_i \in \mathbb{Z}^n \setminus \{0\} \) such that \( \sup_{s \in B} \| g_{t'_i} \phi(s)v_i \| \to 0 \) as \( i \to \infty \). We apply Lemma 5.3 (taking \( r = d-1 \)) to conclude that \( A_\phi \in W_{n-1}'(d, n-d) \).

Conversely, suppose that \( d < n \) and \( A_\phi \in W_{n-1}'(d, n-d) \). Again by Lemma 5.3, there exists \( t_i \to \infty \) and \( v_i \in \mathbb{Z}^n \setminus \{0\} \) such that \( \sup_{s \in B} \| g_{t_i} \phi(s)v_i \| \to 0 \) as \( i \to \infty \). By Mahler’s compactness criterion, for any compact subset \( K \), we have \( g_{t_i} \lambda_\phi(K) = 0 \) for all large \( i \). Hence the sequence of \( g_t \)-translates of \( \lambda_\phi \) has escape of mass. \( \square \)

6.2. Consequence of the failure of equidistribution. We suppose that there is no escape of mass for \( \{g_t \lambda_\phi : t > 0\} \), and thus any limit measure is a probability measure on \( X \). Now suppose that as \( t \to \infty \), \( g_t \lambda_\phi \) does not get equidistributed on \( X = G/\Gamma \) with respect to the \( G \)-invariant probability measure \( \mu_X \). Let \( x_0 = \mathbb{Z}^n \subset G/\Gamma \). Then there exists a sequence \( t_i \to \infty \), a function \( f \in C_c(G/\Gamma) \) and an \( \epsilon > 0 \) such that

\[ \left| \int_B f(g_{t_i} \phi(s)x_0) \, d\lambda(s) - \int f \, d\mu_X \right| > \epsilon. \] (6.1)
Since there is no escape of mass, by further passing to a subsequence, we assume that there exists a probability measure $\mu_\phi$ on $X$ such that $g_t\lambda_\phi \to \mu_\phi$ as $i \to \infty$.

We will analytically modify $\phi$ on the left by elements of the centralizer of the flow $\{g_t\}$, to get a new map $\psi : B \to G$ and its corresponding measure $\lambda_\psi$, concentrated on $\psi(B)x_0$, which is also the twist of $\lambda_\phi$ by the same elements in the centralizer. Then after passing to a subsequence, $g_t\lambda_\psi \to \mu_\psi$ for some probability measure $\mu_\psi$ on $X$. We intend to modify $\phi$ in such a way that $\mu_\psi$ is invariant under a nontrivial unipotent subgroup of $U$. Hence by Ratner’s measure classification theorem [Ra91], each ergodic component of the limit measure is homogeneous, and hence, if $\mu_\psi$ is not $G$-invariant, then it will be strictly positive on the image in $G/T$ of a $U$-invariant proper algebraic subvariety of $G$. Then, we can apply the linearization technique developed by Dani and Margulis [DM93] to obtain certain algebraic conditions as a consequence of (6.1).

6.2.1. Twisting $\phi$ by centralizer of $\{g_t\}$ and invariance under a unipotent subgroup. In view of (1.1) in Section 1.2, given any row vector $v \in \mathbb{R}^{n-1}$, let

$$u(v) = \begin{pmatrix} 1 \\ v \\ I_{n-1} \end{pmatrix} \in U^+.$$ 

We may assume that $B$ is an open ball in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, and let $\tilde{\phi} : B \to \mathbb{R}^{n-1}$ be the analytic map such that $\phi(s) = u(\tilde{\phi}(s))$ for all $s \in B$. Consider the derivative map $D\tilde{\phi} : B \to \text{Hom}(\mathbb{R}^k, \mathbb{R}^{n-1})$. Since $\phi$ is nonconstant and analytic, there exists $1 \leq r \leq n-1$ such that the rank of $D\tilde{\phi}(s)$ is $r$ for all $s \in B$ outside a closed subset of zero Lebesgue measure. Let $U_r$ denote subspace spanned by the first $r$ standard basis vectors of $\mathbb{R}^{n-1}$. By expressing the absolutely continuous measure $\lambda$ on $\mathbb{R}^k$ as a countable sum of absolutely continuous measures with small supports, without loss of generality, we may assume that the ball $B$ is small enough so that the rank of $D\tilde{\phi}(s)$ is $r$ for all $s \in B$, and there exists an analytic function $\tilde{\zeta} : B \to \text{SL}(n-1, \mathbb{R})$ such that

$$D\tilde{\phi}(s)(\mathbb{R}^k) = U_r \cdot \tilde{\zeta}(s), \ \forall s \in B, \ (6.2)$$

where $\text{SL}(n-1, \mathbb{R})$ acts on $\mathbb{R}^{n-1}$ on the right. For all $s \in B$, define

$$\zeta(s) = \begin{pmatrix} 1 \\ \tilde{\zeta}(s) \end{pmatrix} \in Z_G(\{g_t\}).$$

Then

$$u(v)\zeta(s) = \zeta(s)u(v\tilde{\zeta}(s)), \ \forall v \in \mathbb{R}^{n-1}, \ \forall s \in B, \ (6.3)$$

Define $\psi : B \to G$ by

$$\psi(s) = \zeta(s)\phi(s) = \zeta(s)u(\tilde{\phi}(s)), \ \forall s \in B. \ (6.4)$$

Proposition 6.2. Let $v \in U_r \subset \mathbb{R}^{n-1}$, $f \in C_c(G/\Gamma)$ and $x \in G/\Gamma$. Then for all $t > 0$,

$$\int_B f(u(v)g_t\psi(s)x) \, d\lambda(s) = \int_B f(g_t\psi(s)x) \, d\lambda(s) + o_t(1),$$

where $o_t(1) \to 0$ as $t \to \infty$. 
Proof. Let \( t > 0 \) and \( s \in B \). Then

\[
\begin{align*}
    u(v)g_t \psi(s) &= g_t u(e^{-nt}v)\psi(s) \\
    &= g_t u(e^{-nt}v)\zeta(s) u(\tilde{\phi}(s)), \text{ by (6.4)} \\
    &= g_t \zeta(s) u(e^{-nt}v\zeta(s) + \tilde{\phi}(s)), \text{ by (6.3)} \\
    &= g_t \zeta(s) u(e^{-nt}D\tilde{\phi}(s)(\tilde{v}) + \tilde{\phi}(s)), \text{ for some } \tilde{v} \in \mathbb{R}^k \text{ by (6.2)} \\
    &= g_t \zeta(s) u(\tilde{\phi}(s + e^{-nt}\tilde{v}) + o(e^{-nt})), \text{ due to Taylor expansion} \\
    &= g_t \zeta(s) u(o(e^{-nt}u(\tilde{\phi}(s + e^{-nt}\tilde{v})) \\
    &= u(o_t(1))g_t \zeta(s) u(\tilde{\phi}(s + e^{-nt}\tilde{v})), \text{ as } g_t u(w) = u(e^{nt}w)g_t \\
 &= u(o_t(1))g_t \zeta(s)(s + e^{-nt}\tilde{v})^{-1}\psi(s + e^{-nt}\tilde{v}), \text{ by (6.4)} \\
 &= (I + o_t(1))g_t \psi(s + e^{-nt}\tilde{v}), \text{ as } \zeta(B) \subset Z_G(g_t).
\end{align*}
\]

Since \( f \) is uniformly continuous on \( G/\Gamma \), we have

\[
    f(u(v)g_t \psi(s)x) = f(g_t \psi(s + e^{-nt}\tilde{v})x) + o_t(1).
\]

Since \( \lambda \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^k \) with support contained in \( B \), there exists \( h \in L^1(\mathbb{R}^k) \) such that \( d\lambda(s) = h(s)ds \) for Lebesgue a.e. \( s \in \mathbb{R}^k \). For any \( w \in \mathbb{R}^k \), let \( h_w(s) = h(s - w) \) for all \( s \in \mathbb{R}^{k-1} \). Then \( \|h_w - h\|_1 \to 0 \) as \( w \to 0 \) in \( \mathbb{R}^k \). Therefore

\[
\begin{align*}
    \int_B f(u(v)g_t \psi(s)x) d\lambda(s) \\
    &= \int_B f(g_t \psi(s + e^{-nt}\tilde{v})x) d\lambda(s) + o_t(1) \\
    &= \int_{\mathbb{R}^k} f(g_t \psi(s + e^{-nt}\tilde{v})x) h(s) ds + o_t(1) \\
    &= \int_{\mathbb{R}^k} f(g_t \psi(s)x) h(s - e^{-nt}\tilde{v}) ds + o_t(1) \\
    &= \int_{\mathbb{R}^k} f(g_t \psi(s)x) h(s) ds + O(||f||_\infty ||h_{e^{-nt}\tilde{v}} - h||_1) + o_t(1) \\
    &= \int_B f(g_t \psi(s)x) d\lambda(s) + o_t(1).
\end{align*}
\]

Ratner’s theorem and linearization technique for \((C, \alpha)\)-good maps. Now let \( \lambda_\psi \) be the probability measure, which is the pushforward of \( \lambda \) under the map \( s \mapsto \psi(s)x_0 \) from \( B \) to \( X = G/\Gamma \). We suppose that we are given a sequence \( t_i \to \infty \) such that \( g_{t_i} \lambda_\psi \) converges to a probability measure, say \( \mu \), on \( G/\Gamma \) as \( i \to \infty \) with respect to the weak-* topology; that is, for any \( f \in C_c(G/\Gamma) \), we have

\[
\lim_{i \to \infty} \int_B f(g_{t_i} \psi(s)x_0) d\lambda(s) \to \int_X f d\mu.
\]
Then by Proposition 6.2 we conclude that \( \mu \) in invariant under \( U_r = \{ u(v) : v \in U_r \subset \mathbb{R}^n \} \). Therefore, by Ratner’s classification of ergodic invariant measures [Ra91], almost every \( U_r \)-ergodic component of \( \mu \) is a periodic (homogeneous) measure on \( G/\Gamma \).

Therefore, we can argue as in [KdSSY23, Propositions 6.3-6.4] using the linearization technique from Dani-Margulis [DM93, Proposition 4.2], which uses polynomial growth properties of one-dimensional unipotent orbits in a vector space. Since we are working with translates of analytic manifolds, here we use similar \((C, \alpha)\)-good growth properties introduced by Kleinbock and Margulis [KM98] for the family of analytic functions

\[
\{ s \mapsto \xi(g_v \psi(s)v) : t \in \mathbb{R}, v \in V, \xi \in V^* \}
\]
on \( B \), where \( V \) is any finite dimensional representation of \( G \). To take care of higher dimensional manifolds (when \( k > 1 \)), we argue as in [Sha94, Theorem 5.2, Proposition 5.4] and obtain the following linear dynamical boundedness condition:

Let \( g \) denote the Lie algebra of \( G = \text{SL}_n(\mathbb{R}) \) with its natural \( \mathbb{Q} \)-structure. For each \( 1 \leq d < \dim g \), let \( V_d = \wedge^d g \).

**Proposition 6.3.** Suppose that the \( U_r \)-invariant limiting measure \( \mu \) is not \( G \)-invariant. Then after passing to a sequence of \( \{ t_i \} \), there exists \( 1 \leq d < \dim(G) \), a nonzero vector \( v_0 \in V_d(\mathbb{Q}) \) which is not fixed by \( G \), a sequence \( \{ \gamma_i \} \subset \Gamma = \text{SL}_n(\mathbb{Z}) \), and a constant \( C_1 > 0 \) such that the following holds:

\[
\sup_{s \in B} \| g_{t_i} \psi(s) \gamma_i v_0 \| \leq C_1, \forall i.
\]  

(6.5)

Since \( \psi(s) = \zeta(s) \phi(s) \) and \( \zeta(s) \) commutes with \( \{ g_t \} \), by shrinking \( B \) a little, if needed, we can assume that \( \{ \zeta(s) : s \in B \} \) is contained in a compact set, so there exists a constant \( C > 0 \) such that

\[
\sup_{s \in B} \| g_{t_i} \phi(s) \gamma_i v_0 \| \leq C, \forall i.
\]

(\( \heartsuit \))

### 6.3. Getting failure of equidistribution.
In this subsection, we show that if any one of the conditions (1),(2), or (3) in Theorem 1.2 holds, then equidistribution would fail.

We need the following consequence of the reduction theory for arithmetic subgroups.

**Lemma 6.4.** Let \( F \) be a connected reductive \( \mathbb{Q} \)-subgroup of a connected reductive \( \mathbb{Q} \)-group \( G \). Suppose that \( \text{rank}_\mathbb{Q} F < \text{rank}_\mathbb{Q} G \). Let \( \Gamma \subset G(\mathbb{Q}) \) be a discrete subgroup of \( G \) commensurable with \( G(\mathbb{Z}) \). Then, for any compact set \( C \subset G \), \( CF_T \) is a closed proper subset of \( G \).

**Proof.** By [Bor69, 7.7], there exists a finite dimensional representation \( V \) of \( G \) defined over \( \mathbb{Q} \) and a vector \( v \in V(\mathbb{Q}) \) such that the stabilizer of \( v \) is \( F \). Since \( \Gamma \) is commensurable with \( G(\mathbb{Z}) \), the coordinates of elements of \( \Gamma v \) have bounded denominators with respect to a \( \mathbb{Q} \)-basis of \( V \). Therefore, \( \Gamma v \) is discrete in \( V \). Therefore, \( \Gamma F \) is closed in \( G \), because \( \Gamma F \) is the inverse of \( \Gamma v \) under the map \( g \mapsto g v \). Hence, \( FT \) is closed in \( G \).

Let \( T \) be a maximal \( \mathbb{Q} \)-split torus of \( F \). By our assumption, there exists a maximal \( \mathbb{Q} \)-split torus \( S \) in \( G \) containing \( T \) as a proper subtorus [Bor91, V.15.4]. So we pick a non-trivial \( \mathbb{Q} \)-character \( \beta \in X^*(S) \) such that \( T \subset \ker \beta \) [Bor69, III.8.2(c)].

Let \( P \) be a minimal \( \mathbb{Q} \)-parabolic subgroup of \( G \) containing \( S \). Let \( \Delta \subset X^*(S) \) denote the corresponding set of simple roots of \( G \) with respect to \( S \), for the ordering that is associated to \( P \). Let \( A = S(\mathbb{R})^0 \). Let \( A_1 = \{ a \in A : \alpha(a) \leq 1, \forall \alpha \in \Delta \} \). Let \( W \subset N_G(S)(\mathbb{Q}) \) be a
finite set of representatives the Weyl group \( N_G(S)(\mathbb{Q})/Z_G(S)(\mathbb{Q}) \) of \( G \) with respect to \( S \). Then \( A = \bigcup_{w \in W} wA_1w^{-1} \).

Now \( \beta : A \to \mathbb{R}^*_0 \) is a nontrivial continuous homomorphism. So, \( \ker \beta \) is a strictly lower dimensional closed subgroup of \( A \). Since \( \text{Int}(A_1) \), the interior of \( A_1 \), is a nonempty open subset of \( A \), and we pick

\[
a \in \text{Int}(A_1) \setminus \bigcup_{w \in W} wA_1w^{-1}(\ker \beta)w.
\]

Then, for all \( \alpha \in \Delta \), \( \alpha(a) < 1 \), and for all \( w \in W \), \( 0 < \beta(waw^{-1}) \neq 1 \). Let \( a_n' = a^n \) for all \( n \in \mathbb{N} \). Then, as \( n \to \infty \),

\[
\alpha(a_n') \to 0, \forall \alpha \in \Delta, \quad \text{and} \quad \beta(wa_n'w^{-1}) \to 0 \text{ or } \beta(wa_n'w^{-1}) \to \infty, \forall w \in W.
\]

Let \( P_F \) be a minimal \( \mathbb{Q} \)-parabolic subgroup of \( F \) containing \( T \). As a consequence of the reduction theory due to Borel and Harish-Chandra [Bor69, 13.1], there exists a finite set \( \Sigma_F \subset F(\mathbb{Q}) \) such that \( F = \mathfrak{S}_F \Sigma_F(F \cap \Gamma) \), where \( \mathfrak{S}_F \) is a Siegel subset of \( F \) with respect to \( T \) and a choice of a minimal \( \mathbb{Q} \)-parabolic subgroup of \( F \) containing \( T \). And by [Bor69, 12.2, 12.3], \( \mathfrak{S}_F \subset C_FT^0 \), where \( C_F \) is a compact subset of \( F \). Thus, we have

\[
F = C_FT^0\Sigma_F(F \cap \Gamma).
\]

Now, suppose that \( G = CFT \) for some compact set \( C \subset G \). Then, after passing to a subsequence, we write

\[
a'_n = c'_n t_n \sigma' \gamma'_n,
\]

where \( c'_n \subset CC_F \), \( t_n \in T^0 \) and \( \sigma' \in \Sigma_F \), and \( \gamma'_n \in F \cap \Gamma \).

Since \( T^0 \subset A = \bigcup_{w \in W} wA_1w^{-1} \), after passing to a subsequence, there exists \( w' \in W \) such that

\[
s_n := w'^{-1} t_n w' \in A_1.
\]

So we get

\[
a'_n = (c'_n w'^{-1}) s_n (w' \sigma') \gamma'_n.
\]

Let \( \mathfrak{N} \) be the unipotent radical of \( P \), defined over \( \mathbb{Q} \). Then \( P = Z_G(S) \ltimes \mathfrak{N} \). Let \( M \) be a maximal \( \mathbb{Q} \)-anisotropic subgroup of \( Z_G(S) \). Then \( Z_G(S) = MS \), and in fact, \( Z_G(S)(\mathbb{R})^0 = M^0A \) and \( P^0 = M^0AN \). Let \( K \) be a maximal compact subgroup of \( G \) such that the corresponding Cartan involution preserves \( A \). Then, \( G = KP^0 = KM^0AN \), and the map \( (KM^0) \times A \times N \to G \) given by the group multiplication is a homeomorphism, see [BJ07, (2.3)]. Now we express

\[
w'(c'_n)^{-1} = k_n m_nb_nv_n,
\]

where \( k_n \in K \), \( m_n \in M^0 \), \( b_n \in A \), and \( v_n \in N \). Since \( W(CC_F)^{-1} \) is compact, the sequence \( \{b_n\} \) is a relatively compact subset of \( A \).

Combining, (6.9) and (6.10), since \( m_n \in Z_G(A) \), we get

\[
s_n(w' \sigma') \gamma'_n = (w'c'_n)^{-1} a'_n = k_n m_n b_nv_n a'_n = (k_n b_n a'_n)[a'_n^{-1} m_n v_n a'_n].
\]

Since \( M \) is \( \mathbb{Q} \)-anisotropic, \( M^0N/(M^0N \cap \Gamma) \) is compact. Let \( \Omega \) be a compact neighborhood of \( e \) in \( M^0N \) such that \( M^0N = \Omega(M^0N \cap \Gamma) \). Therefore, \( a'_n^{-1} m_n v_n a'_n = \omega_n \gamma''_n \) for sequences \( \{\omega_n\} \subset \Omega \) and \( \{\gamma''_n\} \subset M^0N \cap \Gamma \). Inserting this expression in (6.11) we get

\[
s_n \sigma \gamma_n = k_n a_n \omega_n,
\]
where $\sigma = w'\sigma' \in G(\mathbb{Q})$, $\gamma_n = \gamma'_n \gamma''_n \in \Gamma$, and $a_n = b_n a'_n$. Since $\{b_n\}$ is relatively compact in $A$, by (6.6), $a_n \in A_1$ for all large $n$. Let $\mathcal{G}_0 = KA_1 \Omega$. By (6.8) and (6.12),

$$\mathcal{G}_0 \sigma \gamma_n \cap \mathcal{G}_0 \neq \emptyset, \forall n \gg 1.$$ 

Now, $\mathcal{G}_0$ is a normal Siegel set in $G$ with respect to the $\mathbb{Q}$-parabolic $P$ and our choice of the maximal compact subgroup $K$, see [Bor69, 12.3]. Therefore, by Siegel property [Bor69, 15.5], which is generalization of a theorem of Siegel [Bor69, 4.6] for $G = SL_n(\mathbb{R})$, we get $\{\gamma_n\}$ is a finite set. Hence, after passing to a subsequence, we may assume that $\gamma_n = \gamma$ for all $n$.

By Bruhat decomposition [Bor69, 11.4], $G = NWZ_G(A)N$. So we can express

$$\sigma \gamma = vwzu,$$

where $v, u \in N$, $w \in W$, and $z \in AM = Z_G(A)$. Therefore, by (6.12)

$$k_n a_n (\omega_n u^{-1}z^{-1}) = (s_n v s_n^{-1}) s_n w.$$ 

Now, the sequences $\{a_n(\omega_n u^{-1}z^{-1})a_n^{-1}\} \subset MN$ and $\{s_n v s_n^{-1}\} \subset N$ are relatively compact, as $a_n, s_n \in A_1$. Thus,

$$c_n a_n = w^{-1}s_n w \in A,$$

where $c_n = w^{-1}(s_n v s_n^{-1})^{-1} k_n (a_n u^{-1}z^{-1} a_n^{-1})$. Moreover, $\{c_n\}$ is a relatively compact subset of $A$. Since $T \subset \ker \beta$ and $\{t_n\} \subset T^0$, by (6.8),

$$1 = \beta(t_n) = \beta(w' s_n w'^{-1}) = \beta(w' w^{-1} w^{-1} w^{-1}) = \beta(w_1 c_n w_1^{-1}) \beta(w_1 a_n w_1^{-1}),$$

where $w_1 = w' w \in W$. This contradicts (6.7), because $\{\beta(w_1 c_n w_1^{-1})\}$ is a relatively compact subset of $\mathbb{R}_{>0}^*$. Therefore, $CF T \neq G$ for any compact set $C$. \hfill $\square$

Let the notations be the same as in Theorem 1.2. We take $\Omega = \phi(B) v_A^{-1}$.

**Lemma 6.5.** Suppose $d < n$ and $A_\phi \in W_{n-1}(d, n - d)$, then the measures $g_t \lambda_\phi$ do not get equidistributed in $X$ as $t \to \infty$.

**Proof.** Suppose $A_\phi \in W_{n-1}(d, n - d)$, then by Lemma 5.2 where we take $r = d - 1$, there exists $C > 0$, $t_i \to \infty$ and $0 \neq v_i \in \mathbb{Z}^n$ such that $\sup_{s \in B} ||g_t \phi(s)v_i|| \leq C$. Without loss of generality, we may assume that all the $v_i$ are primitive; hence $v_i = \gamma_i e_1$ for $\gamma_i \in \Gamma$. We discuss the following two cases.

Suppose first there exists $c > 0$ such that $\inf_{s \in B} ||g_t \phi(s)v_i|| \geq c$ for all $i$. Let $\mu$ be a weak-* limit of a subsequence of $\{g_t \lambda_\phi\}$, and $E$ be the support of $\mu$. We claim that $E$ is not equal to $X$. Indeed, take $M > 0$ large enough such that $M^{-1} < c$ and $M > C$. Then the unimodular lattice $\mathbb{Z}M^{-1}e_1 + \mathbb{Z}Me_2 + \cdots + \mathbb{Z}Me_n$ is not in $E$, as every primitive vector in this lattice has length either $M^{-1}$ or at least $M$. Hence $\{g_t \lambda_\phi\}_{t \geq 0}$ do not get equidistributed in $X$.

Now suppose that after passing to a subsequence $\lim_{i \to \infty} \inf_{s \in B} ||g_t \phi(s)v_i|| = 0$. Let $V_1 = \mathbb{R}e_1$ and $V_2 = \mathbb{R}e_2 + \cdots + \mathbb{R}e_n$ be eigenspaces of $g_t$ with eigenvalues $e^{(n-1)t}$ and $e^{-t}$ respectively. Let $\pi_1$ (resp. $\pi_2$) be the projection from $\mathbb{R}^n$ to $V_1$ (resp. $V_2$). By our assumption we have $\pi_2(\phi(s_i)v_i) = o(e^{t_i})$ for every $i$ and some $s_i \in B$. Since $\phi(s) \in U^+$, we have

$$\pi_2(\phi(s)v_i) = \pi_2(\phi(s_i)v_i) = o(e^{t_i}), \forall s \in B.$$
On the other hand, since \( \sup_{s \in B} \|g_t(\phi(s)v_i)\| \leq C \), we have \( \pi_1(\phi(s)v_i) = O(e^{-(n-1)t}) \) for all \( s \in B \). Hence there exists a sequence \( t_i' \to \infty \) such that \( \sup_{s \in B} \|g_{t'}(\phi(s)v_i)\| \to 0 \) as \( i \to \infty \).

By Mahler’s compactness criterion, we know that \( g_{t'} \phi(B) \) leave any fixed compact set, and \( g_{t'} \lambda_0 \) weak-* converge to 0. In particular, in this case, we also have that \( \{g_t \lambda_0\}_{t \geq 0} \) do not get equidistributed in \( X \).

\[ \square \]

**Lemma 6.6.** Suppose that there exist integers \( r \geq d, m \geq 2 \) with \( rm = n \), and a number field \( K \subset \mathbb{R} \) with \( [K : \mathbb{Q}] = m \), such that \( \mathcal{L}_\phi \) is contained in some \((r-1)\)-dimensional affine subspace of \( \mathbb{R}^{n-1} \) which is defined over \( K \). Then \( \{g_t \lambda_0\}_{t \geq 0} \) do not get equidistributed in \( X \).

**Proof.** By Lemma 5.5, \( u_A \in P_tP_G(K) \). If \( \mathcal{L}_\phi \) is contained in some proper affine subspace of \( \mathbb{R}^{n-1} \) defined over \( \mathbb{Q} \), then \( g_t \phi(B) \) leave any fixed compact set as \( t \to \infty \), and equidistribution fails in this case. Otherwise, by Lemma 4.14 and Corollary 4.15 there exists a compact subset \( \Sigma \) of \( G \) such that \( g_t \phi(B) \subset \Sigma \Gamma_t \) for all \( t \geq 0 \), where \( F = F(\mathbb{R}) \) and \( F \cong R_{e_K/Q}^{(1)} \). Since \( \text{rank}_Q(F) = r - 1 < n - 1 = \text{rank}_Q(G) \), by Lemma 6.4 we know that \( \Sigma \Gamma_t \) is a proper closed subset of \( X = G/\Gamma \), and thus \( \{g_t \lambda_0\}_{t \geq 0} \) do not get equidistributed in \( X \).

\[ \square \]

**Lemma 6.7.** Suppose that \( n \geq 4 \) is even, \( d = 2 \) and \( A_{\phi}^{ext} \in W_{2,2}(2n-3, N) \). Then \( \{g_t \lambda_0\}_{t \geq 0} \) do not get equidistributed in \( X \).

**Proof.** Let \( V \) be the standard representation of \( \text{SL}_n \), and \( W = \wedge^2 V \). By Lemma 5.4, there exists \( C > 0 \), \( t_i \to \infty \) and \( w_i \in W(\mathbb{Z}) \setminus \{0\} \) such that
\[
\sup_{\omega \in \Omega} \sup_{s \in B} \|g_t(\omega u_Aw_i)\| = \sup_{s \in B} \|g_t(\phi(s)w_i)\| \leq C, \quad \forall i.
\]

Without loss of generality we may assume that \( w_i \) is primitive for every \( i \). By the natural isomorphism \( \wedge^2 V \cong \wedge^2 (V^*)^* \), we identify \( W \) with the space of alternating bilinear forms on \( V^* \). Hence we can talk about the rank of an element in \( W \), which is an even number. After passing to a subsequence, we may assume that all the \( w_i \) have the same rank \( 2k \). We discuss the following two cases.

Suppose first that \( 2k = n \). Equivalently, \( w_i \) has full rank for every \( i \). Hence the Pfaffian \( \text{Pff}(w_i) \) is a nonzero integer for every \( i \). Since Pfaffian is a \( \text{SL}_n \)-invariant, by (6.13) we know that \( \text{Pff}(w_i) \) are bounded. Therefore, after passing to a subsequence we may assume that \( \text{Pff}(w_i) \) is a constant, and thus all the \( w_i \) are on the same \( \text{SL}_n(\mathbb{R}) \)-orbit \( Y \). Since \( Y \) is a \( \text{SL}_n \)-homogeneous variety defined over \( \mathbb{Q} \), by a theorem of Borel and Harish-Chandra, there are only finitely many \( \text{SL}_n(\mathbb{Z}) \)-orbits on \( Y(\mathbb{Z}) \).  \(^2\) After further passing to a subsequence, we may assume that all the \( w_i \) are on the same \( \text{SL}_n(\mathbb{Z}) \)-orbit. We write \( w_i = \gamma_i w_0 \), where \( \gamma_i \in \Gamma \) and \( w_0 \in W(\mathbb{Z}) \) is of full rank. Let \( F = F_{w_0} \) be the isometry group of \( w_0 \), which is isomorphic to \( \text{Sp}_n \) as a \( \mathbb{Q} \)-group. We have an equivariant homeomorphism from \( Gw_0 \) to \( G/F \). Now by (6.13), there exists a compact subset \( \Sigma \) of \( G \) such that \( g_t \phi(B) \subset \Sigma \Gamma_t \) for all \( t \geq 0 \). Since \( \text{rank}_Q(F) = \frac{n}{2} < n - 1 = \text{rank}_Q(G) \), by Lemma 6.4 we know that \( \Sigma \Gamma_t \) is a proper closed subset of \( X = G/\Gamma \), and thus \( \{g_t \lambda_0\}_{t \geq 0} \) do not get equidistributed in \( X \).

\(^2\)As a referee has pointed out, the number of \( \text{SL}_n(\mathbb{Z}) \)-orbits in \( Y(\mathbb{Z}) \) is equal to the number of ways of writing the Pfaffian as an \( k \)-fold product of positive integers.
Now suppose that \(2k < n\). Consider the \(\text{SL}_n\)-module morphism

\[
f : \text{Sym}^k W \to \wedge^{2k} V
\]

which sends \((v_1 \wedge v_2) \cdot (v_3 \wedge v_4) \cdots (v_{2k-1} \wedge v_{2k})\) to \(v_1 \wedge v_2 \wedge \cdots \wedge v_{2k}\). Since each \(w_i\) is of rank \(2k\), we know that \(w_i := f(w_i^k)\) is a pure tensor in \(\wedge^{2k} V\). Equation (6.13) now implies that there exists \(C' > 0\) such that for all \(i\),

\[
\sup_{s \in B} \|g_t \phi(s) w_i\| \leq C'.
\]

By Lemma 4.4 there exist nonzero vectors \(v_i \in \mathbb{Z}^n\), a constant \(C'' > 0\), and \(t'_i \to \infty\) such that

\[
\sup_{s \in B} \|g_{t'_i} \phi(s) v_i\| \leq C''.
\]

By the proof of Lemma 6.5, we know that \(\{g_t \lambda_\phi\}_{t \geq 0}\) do not get equidistributed in \(X\). □

6.4. **Proof of Theorem 1.2.** Finally, we prove our main theorem on equidistribution, namely Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that \(\{g_t \lambda_\phi\}_{t \geq 0}\) do not get equidistributed in \(X\). If \(\{g_t \lambda_\phi\}_{t \geq 0}\) has escape of mass, then by Theorem 1.1 we have \(n < d\) and \(A_\phi \in \mathcal{W}'_{d,n-1}(d, n - d) \subset \mathcal{W}_{d,n-1}(d, n - d)\); that is, (1) of the theorem holds. Now assume that there is no escape of mass, then there exists a sequence \(t_i \to \infty\) such that \(g_{t_i} \lambda_\phi\) weak-* converge to a probability measure on \(X\) which is not the Haar measure. Hence by Proposition 6.3 we know that there exists a representation \(V\) of \(G\), a sequence \(t_i \to \infty\), a sequence \(\{\gamma_i\}\) in \(\Gamma\), a nonzero vector \(v_0 \in V(Q)\) which is not fixed by \(G\), and a constant \(C > 0\) such that (♠) holds. Therefore \(d < n\), and (1) or (2) or (3) of Proposition 4.1 holds. Applying Lemma 5.2, Lemma 5.5 and Lemma 5.4, we know that (1), (2), and (3) of Proposition 4.1 are equivalent to (1), (2), and (3) of Theorem 1.2, respectively. Therefore, (1) or (2) or (3) of Theorem 1.2 holds.

Conversely, suppose that (1) or (2) or (3) of Theorem 1.2 holds. Then by Lemma 6.5, Lemma 6.6, and Lemma 6.7, \(\{g_t \lambda_\phi\}_{t \geq 0}\) do not get equidistributed in \(X\). □

**Appendix A. A classification theorem**

In this section, we briefly recall the definition and basic properties of root systems and weights and then prove a classification theorem that is used for describing intermediate subgroups. Some statements are provided without proof, and readers are referred to [Hum78, Chapter III, VI] for a detailed discussion.

**A.1. Classification of root systems with a certain property.** Let \(E\) be a Euclidean space with inner product \((\cdot, \cdot)\). A reflection with respect to a vector \(\alpha \in E\) is the isometry of \(E\) given by \(\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha\). Write \(\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\).

**Definition A.1.** A subset \(\Phi\) of the Euclidean space \(E\) is called a (reduced) root system in \(E\) if the following axioms are satisfied:

1. \(\Phi\) is finite, spans \(E\), and does not contain 0.
2. If \(\alpha \in \Phi\), the only multiples of \(\alpha\) in \(\Phi\) are \(\pm \alpha\).
3. If \(\alpha \in \Phi\), the reflection \(\sigma_\alpha\) leaves \(\Phi\) invariant.
4. If \(\alpha, \beta \in \Phi\), then \(\langle \beta, \alpha \rangle \in \mathbb{Z}\).
Let $\Phi$ be a root system in $E$. The Weyl group of $\Phi$ is the subgroup of $\text{GL}(E)$ generated by the reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$, and is denoted by $W$. The lattice generated by $\Phi$ in $E$ is called the root lattice, and is denoted by $\Lambda_r$.

A subset $\Delta$ of $\Phi$ is called a set of simple roots if it is a basis of $E$ and each root $\beta \in \Phi$ can be written as a nonnegative or nonpositive integral combination of elements in $\Delta$. We call $\beta$ positive or negative respectively.

We call $\Phi$ irreducible if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other. It is known that the Weyl group acts irreducibly on an irreducible root system. In an irreducible root system, at most two root lengths occur. All roots of the same length are in the same orbit of the Weyl group $W$. In case $\Phi$ is irreducible, with two distinct root lengths, we speak of long roots and short roots. If all roots are of equal length, it is conventional to call all of them long.

Let $\Phi$ be a root system in a Euclidean space $E$, with Weyl group $W$. Let $\Lambda$ be the set of all $\lambda \in E$ for which $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha) (\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$, and call its elements weights. $\Lambda$ is a lattice in $E$, and is called the weight lattice. Fix a system of simple roots $\Delta \subset \Phi$, and define $\lambda \in \Lambda$ to be dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$. Let $\Lambda^+$ denote the set of dominant weights. Suppose $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. The fundamental weights are $\omega_1, \ldots, \omega_n$ such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol.

We call a subset $\Pi$ of $\Lambda$ saturated if for all $\lambda \in \Pi$, $\alpha \in \Phi$ and $i$ between 0 and $\langle \lambda, \alpha \rangle$, the weight $\lambda - i \alpha$ also lies in $\Pi$. Any saturated set is stable under $W$.

Given a root system $\Phi$, we have a decomposition

$$\Phi = \Phi_1 \amalg \cdots \amalg \Phi_k,$$

where $\Phi_i$’s are irreducible and orthogonal to each other.

**Theorem A.2.** Let $\Phi = \Phi_1 \amalg \cdots \amalg \Phi_k$ be a root system with Weyl group $W$, and let $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$ be the corresponding weight lattice. Let $\Pi = \{\lambda_1, \ldots, \lambda_n\}$ be a saturated subset of $\Lambda$ with a highest weight. Suppose there exists a root $\alpha \in \Phi_1$ such that

$$\langle \lambda_i, \alpha \rangle = \begin{cases} 1 & i = 1 \\ -1 & i = 2 \\ 0 & 3 \leq i \leq n \end{cases}.$$ (A.1)

Then there is a choice of simple roots such that $\lambda_1$ is the highest weight of $\Pi$; $\Pi$ is contained in the span of $\Phi_1$; and one of the following holds:

1. $\Phi_1 = A_{n-1}$, and $\lambda_1 \in \{\omega_1, \omega_{n-1}\}$.
2. $n$ is even, $\Phi_1 = C_2^\perp$, and $\lambda_1 = \omega_1$.

**Remark A.3.** The dominant weights correspond to the highest weights of irreducible representations of semisimple Lie algebras. The two cases in the theorem correspond to (1) standard and its contragradient representation of $\mathfrak{sl}_n$, and (2) standard representation of $\mathfrak{sp}_n$.

We make some preparations before proving Theorem A.2.

**Lemma A.4.** Under the assumption of Theorem A.2:

1. there is a choice of simple roots such that $\lambda_1$ is the highest weight of $\Pi$;
(2) $\Pi$ is contained in the span of $\Phi_1$.

Proof. We may choose simple roots $\Delta = \Delta_1 \cdots \Delta_r$ such that $\alpha$ is a dominant weight, i.e. $(\beta, \alpha) \geq 0$ for all $\beta \in \Delta$. Under this choice, we have that $\lambda_1$ is the highest weight of $\Pi$. Indeed, by assumption $\Pi$ has a highest weight $\lambda$ minimal if and only if the $W_i$, i.e. $(\beta, \alpha)$, $\lambda$ is contained in the convex hull of $\Phi_1$, see [Hum78, Section 13.4 Lemma B]. For any $\beta \in \Phi \setminus \Phi_1$, we have $(\beta, \alpha) = 0$, and it follows that $\langle \sigma_\beta(\lambda_1), \alpha \rangle = \langle \lambda_1, \alpha \rangle = 1$. Hence $\sigma_\beta(\lambda_1) = \lambda_1$ for all $\beta \in \Phi \setminus \Phi_1$. Note that $W$ is generated by the simple reflections in $\Delta$, and that simple reflections in $\Delta_i$ and $\Delta_j$ commute if $i \neq j$. Hence we have $W \cdot \lambda_1 = W_1 \cdot \lambda_1$, where $W_1$ is the Weyl group of $\Phi_1$. Finally, $W_1 \cdot \lambda_1$ is contained in $E_1$, and $\Pi$ is contained in the convex hull of $W_1 \cdot \lambda_1$, thus also in $E_1$. \hfill $\square$

In the next lemma, we will need the following notion (see [Hum78, P72, Section 13 Exercise 13]): we call $\lambda \in \Lambda^+$ minimal (or minuscule) if $\mu \in \Lambda^+, \mu \prec \lambda$ implies that $\mu = \lambda$. Each coset of $\Lambda_r$ in $\Lambda$ contains precisely one minimal $\lambda$. One can show that $\lambda$ is minimal if and only if the $W$-orbit of $\lambda$ is saturated with highest weight $\lambda$, if and only if $\lambda \in \Lambda^+$ and $(\lambda, \beta) = 0, 1, -1$ for all roots $\beta$.

Lemma A.5. Under the assumption of Theorem A.2, one can choose a system of simple roots such that $\lambda_1$ is minimal in $\Lambda^+_1$.

Proof. By Lemma A.4, we may choose a set of simple roots such that $\lambda_1 \in \Lambda^+_1$. By the discussion above, it suffices to show that

$$\langle \lambda_1, \beta \rangle = 0, \pm 1, \quad \forall \beta \in \Phi.$$

Suppose that $|\langle \lambda_1, \beta \rangle| \geq 2$, consider the string

$$\{\lambda_1, \ldots, \lambda_1 - \langle \lambda_1, \beta \rangle \beta\}.$$

This string has length at least 3, and pairing it with $\alpha$, we get a finite arithmetic progression of length at least 3. By Equation (A.1), the only possibility is that $\langle \lambda_1, \beta \rangle = 2$ and $\lambda_1 - 2\beta = \lambda_2$. Since $\Pi$ is saturated, there exists $i$ such that $\lambda_i = \sigma_i(\lambda_1)$. Again by Equation (A.1) we have $i = 2$ and $\lambda_2 = \sigma_\alpha(\lambda_1) = \lambda_1 - \alpha$. But this implies $\alpha = 2\beta$, contradicting that the root system $\Phi$ is reduced. \hfill $\square$

Now we are in a position to prove Theorem A.2

Proof of Theorem A.2. The first assertion is proved in Lemma A.4. By Lemma A.5, we have that $\lambda_1$ is minimal.

There is a complete (finite) list of minimal weights in each irreducible root system (c.f. [Hum78, P72, Section 13 Exercise 13]). We consider each case separately to verify whether Equation (A.1) holds. This case-by-case verification requires some efforts involving long and elementary calculations. Finally, one can show that Equation (A.1) only holds for $(\Phi, \lambda_1)$ being $(A_{n-1}, \omega_1)$, $(A_{n-1}, \omega_{n-1})$, or $(C_{2n}, \omega_1)$. \hfill $\square$
A.2. Classification of intermediate Lie subalgebras. Let \( \mathfrak{g} \) be a semi-simple Lie algebra over an algebraically closed field of characteristic 0, and let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). Let \( V \) be a finite-dimensional \( \mathfrak{g} \)-module. Then the \( \mathfrak{h} \)-action on \( V \) is diagonalizable, and we have the decomposition \( V = \bigoplus V_{\lambda} \), where \( \lambda \) runs over \( \mathfrak{h}^* \) and \( V_{\lambda} = \{ v \in V \mid h.v = \lambda(h)v, \forall h \in H \} \). If \( V_{\lambda} \) is nonzero, we call \( \lambda \) a weight of \( V \), and \( V_{\lambda} \) a weight space. The set of weights of \( V \) is saturated with a highest weight. The highest weight yields a one-to-one correspondence between dominant integral weights and irreducible representations of \( \mathfrak{g} \).

Let \( \mathfrak{f}_{12} \) be the Lie subalgebra of \( \mathfrak{sl}_n \) consisting of traceless matrices in the upper-left \( 2 \times 2 \) block.

**Theorem A.6.** Let \( \varrho: \mathfrak{g} \to \text{End}(V) \) be a finite-dimensional faithful irreducible representation of a semisimple Lie algebra \( \mathfrak{g} \). Suppose that under an identification of \( \text{End}(V) \) with \( \mathfrak{sl}_n \), \( \varrho(\mathfrak{g}) \) contains \( \mathfrak{f}_{12} \). Then \( \mathfrak{g} \) is simple, and moreover one of the following holds:

1. \( \mathfrak{g} = \mathfrak{sl}_n \), and \( V \) is the standard representation or the contragradient representation of the standard representation of \( \mathfrak{g} \).
2. \( n \) is even, \( \mathfrak{g} = \mathfrak{sp}_n \), and \( V \) is the standard representation of \( \mathfrak{g} \).

**Proof.** Since \( \mathfrak{g} \) is semisimple, \( \varrho(\mathfrak{g}) \) is contained in \( \mathfrak{sl}_n \). Let \( \mathfrak{f} = \varrho(\mathfrak{g}) \). Take a chain of Cartan subalgebras \( \mathfrak{h}_{12} \subset \mathfrak{h}_i \subset \mathfrak{h} \) in the chain \( \mathfrak{f}_{12} \subset \mathfrak{f} \subset \mathfrak{sl}_n \). By taking a suitable basis \( \{e_1, e_2, \ldots, e_n\} \) of \( V \) we may assume \( \mathfrak{h} \) consists of diagonal matrices, and then \( \mathfrak{h}_{12} \) consists of elements of the form \( \text{diag}(a, -a, 0, \ldots, 0) \). Now let \( \alpha \) be the character \( \text{diag}(a_1, a_2, \ldots, a_n) \mapsto a_1 - a_2 \). Since \( \mathfrak{f} \) contains \( E_{12} \) and \( \mathfrak{h}_f \) acts on \( E_{12} \) via \( \alpha \), we have that \( \alpha \) is a root of \( \mathfrak{f} \) with respect to \( \mathfrak{h}_f \). Let \( \Pi_f = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) be the weights of \( V \) with respect to \( \mathfrak{h}_f \) counted with multiplicity, such that \( e_1, e_2, \ldots, e_n \) are their weight vectors respectively. Since \( \mathfrak{h}_f \) is a Euclidean subspace of \( \mathfrak{h}^* \), we have

\[
\langle \lambda_i, \alpha \rangle = \begin{cases} 1 & i = 1 \\ -1 & i = 2 \\ 0 & 3 \leq i \leq n \end{cases}
\]

Hence we can apply Theorem A.2 to conclude that \( \varrho \) factors through a simple factor \( \mathfrak{g}_1 \) of \( \mathfrak{g} \). But \( (\varrho, V) \) is faithful, hence \( \mathfrak{g}_1 = \mathfrak{g} \) and \( \mathfrak{g} \) is simple. We apply Theorem A.2 again, and the conclusion of the theorem follows. Indeed, note that in each highest weight module associate to each dominant weight appearing in (1)(2) of Theorem A.2, all weight spaces are one-dimensional. Hence \( \{\lambda_1, \ldots, \lambda_n\} \) are distinct, and the cardinality of \( \Pi_f \) is \( n \). \( \square \)

A.3. Classification of intermediate subgroups. Using the above classification theorem on intermediate Lie subalgebras, we are able to obtain the following classification of intermediate subgroups, which we are interested in.

**Theorem A.7.** Let \( G \) be a reductive group over an algebraically closed field \( \mathbb{K} \) of characteristic 0. Let \( \rho: G \to \text{GL}(V) \) be a faithful irreducible representation of \( G \), such that \( \rho(G) \) is contained in \( \text{SL}(V) \). Suppose that there are linear subspaces \( W_1 \) and \( W_2 \) of \( V \) with \( V = W_1 \bigoplus W_2 \) and \( \dim W_1 \geq 2 \), such that \( \text{SL}(W_1) \times 1_{W_2} \) is contained in \( \rho(G) \). Then one of the following holds:

1. \( \rho(G) = \text{SL}(V) \).
(2) $n$ is even and $\dim W_1 = 2$; there exists a symplectic form $\omega$ on $V$ such that $\rho(G) = \text{Sp}(V, \omega)$.

Proof. We first make a few reductions. Let $G^0$ be the identity component of $G$. Suppose the theorem holds for $G^0$, then one easily sees that $G = G^0$. Therefore, without loss of generality, we may assume that $G$ is connected. Also, it suffices to prove the theorem for $\dim W_1 = 2$. We first prove the theorem for $G$ semisimple.

Let $n = \dim V$. We take the differential of $\rho$ and get a Lie algebra representation $d\rho: \mathfrak{g} \to \text{End}(V)$. By Theorem A.6, either $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{g} = \mathfrak{sp}_n$ ($n$ even).

If $\mathfrak{g} = \mathfrak{sl}_n$ and $(d\rho, V)$ is the standard or the contragradient representation of $\mathfrak{g}$, then $d\rho$ lifts to the standard or the contragradient representation of the simply-connected group $\text{SL}_n$, which is faithful. Hence $G = \text{SL}_n$ and $\rho(G) = \text{SL}(V)$.

If $\mathfrak{g} = \mathfrak{sp}_n$ and $(d\rho, V)$ is the standard representation of $\mathfrak{g}$, then $d\rho$ lifts to the standard representation of the simply-connected group $\text{Sp}_n$, which is faithful. Hence $G = \text{Sp}_n$ and $\rho(G) = \text{Sp}(V, \omega)$ for some symplectic form $\omega$ on $V$.

For general $G$, consider $G^\text{der} = [G, G]$, and the conditions of the theorem still hold for $G^\text{der}$. Then $G^\text{der}$ satisfies either (1) or (2), and $G = G^\text{der}$ as $\rho$ is faithful. □

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The Ohio State University, Columbus, OH 43210
Email address: shah@math.osu.edu

Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing, China 100871
Email address: yangpengyu@amss.ac.cn