Involutions of $\text{sl}(2,k)$ and non-split, three-dimensional simple Lie algebras

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Abstract

We give a process to construct non-split, three-dimensional simple Lie algebras from involutions of $\text{sl}(2,k)$, where $k$ is a field of characteristic not two. Up to equivalence, non-split three-dimensional simple Lie algebras obtained in this way are parametrised by a subgroup of the Brauer group of $k$ and are characterised by the fact that their Killing form represents $-2$. Over local and global fields we re-express this condition in terms of Hilbert and Legendre Symbols and give examples of three-dimensional simple Lie algebras which can and cannot be obtained by this construction over the field of rationals.

1 Introduction

It is well-known that the non-split three-dimensional simple real Lie algebra $\mathfrak{su}(2)$ can be constructed from $\mathfrak{sl}(2,\mathbb{R})$ equipped with a Cartan involution. In this article, we generalise this process to fields $k$ of characteristic not two and obtain non-split three-dimensional simple Lie algebras from $\mathfrak{sl}(2,k)$ equipped with a certain type of involution.

To this end, in Section 3, we start from an involution $\sigma$ of $\mathfrak{sl}(2,k)$ such that the linear map $\text{ad}(x)$ is not diagonalisable for all fixed points $x$ of $\sigma$. We show in Theorem 3.1 that the Lie algebra obtained by our construction is non-split if and only if $K(x,x)$ is not a sum of two squares in $k$ for all fixed points $x$ of $\sigma$, where $K$ is the Killing form of $\mathfrak{sl}(2,k)$. Three-dimensional simple Lie algebras which can be obtained by this construction are characterised by the fact that their Killing form is anisotropic and represents $-2$ (Proposition 3.7).

In Section 2 we recall how to associate a quaternion algebra to a three-dimensional simple Lie algebra. Quaternion algebras do not define a subgroup of the Brauer group $\text{Br}(k)$, however, the set of quaternion algebras associated to the non-split three-dimensional simple Lie algebras constructed from involutions of $\mathfrak{sl}(2,k)$ does (Proposition 3.9 and Theorem 3.10). Finally in Section 4 we characterise which non-split three-dimensional simple Lie algebras are obtainable over local and global fields in terms of Hilbert and Legendre Symbols (Propositions 4.5 and 4.7) and we give explicit rational three-dimensional simple Lie algebras which can and cannot be obtained by this construction in Example 4.12.

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Throughout this paper, the field $k$ is always of characteristic not two.

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2 Generalities about three-dimensional simple Lie algebras

In this section, we recall some results about three-dimensional simple Lie algebras (see [Jac58] or [Mal92] for details) and describe their correspondence with quaternion algebras.

**Definition 2.1.** Let \( \{x, y, z\} \) be the canonical basis of \( k^3 \) and let \( \alpha, \beta \in k^* \). Define an antisymmetric bilinear bracket \([ \ , \ ] : k^3 \times k^3 \to k^3 \) by
\[
[x, y] := z, \quad [y, z] := \alpha x, \quad [z, x] := \beta y
\]
and denote the algebra \((k^3, [ \ , \ ])\) by \( L(\alpha, \beta) \).

The algebra \( L(\alpha, \beta) \) is a three-dimensional simple Lie algebra, its Killing form is \( < -2\beta, -2\alpha, -2\alpha\beta > \) and we have the following result:

**Proposition 2.2.** If \( s \) is a three-dimensional simple Lie algebra, then there exist \( \alpha, \beta \in k^* \) such that \( s \) is isomorphic to \( L(\alpha, \beta) \).

**Remark 2.3.** A ternary quadratic form \( q \) is isometric to the Killing form of a three-dimensional simple Lie algebra if and only if \( \text{disc}(q) = [-2] \in k^*/k^{\times}2 \).

The Lie algebra \( sl(2, k) \) is isomorphic to \( L(-1, 1) \). We say that a three-dimensional simple Lie algebra \( s \) is split if it is isomorphic to \( sl(2, k) \). If there exists a non-zero \( h \in s \) such that \( \text{ad}(h) \) is diagonalisable then \( s \) is split.

**Proposition 2.4.** Let \( s \) be a three-dimensional simple Lie algebra, \( K \) be its Killing form and \( h \in s \). The characteristic polynomial of \( \text{ad}(h) \) is \( -X(X^2 - \frac{K(h,h)}{2}) \). In particular, \( \text{ad}(h) \) is diagonalisable if and only if \( \frac{K(h,h)}{2} \) is a non-zero square in \( k \).

**Proof.** Straightforward calculation.

\( \square \)

It is well-known that the imaginary part of a quaternion algebra \( H \) is a three-dimensional simple Lie algebra for the bracket defined by the commutator. Let \( \alpha, \beta \in k^* \). Recall ([Vig80],[Lam05]) that the quaternion algebra \((\frac{a+bi}{\alpha}, \frac{cj+d}{\beta})\) is the \( k \)-algebra on two generators \( i, j \) with the defining relations:
\[
i^2 = \alpha, \quad j^2 = \beta, \quad ij = -ji.
\]

We note that the element \( ij \) verifies \( (ij)^2 = -\alpha\beta \). Furthermore, \( \{1, i, j, ij\} \) form a \( k \)-basis for \((\frac{a+bi}{\alpha}, \frac{cj+d}{\beta})\) and \((\frac{a+bi}{\alpha}, \frac{cj+d}{\beta})\) is a central simple unital four-dimensional associative, non-commutative composition algebra for the norm form
\[
N(a + bi + cj + dij) = (a + bi + cj + dij)(a + bi + cj + dij) = a^2 - \alpha\beta^2 - \beta c^2 + \alpha\beta d^2,
\]
where \((a + bi + cj + dij) := a - (bi + cj + dij)\) for all \( a + bi + cj + dij \in (\frac{a+bi}{\alpha}, \frac{cj+d}{\beta})\).

The imaginary part of the quaternion algebra \((\frac{a+bi}{\alpha}, \frac{cj+d}{\beta})\) together with the commutator is isomorphic to the Lie algebra \( L(-\beta, -\alpha) \) since \( \text{Im}(\frac{a+bi}{\alpha}, \frac{cj+d}{\beta}) = \text{Span} < i, j, ij > \) and
\[
\begin{align*}
\frac{i}{2} \cdot \frac{j}{2} &= \frac{ij}{2}, \\
\frac{j}{2} \cdot \frac{ij}{2} &= -\beta \cdot \frac{i}{2}, \\
\frac{ij}{2} \cdot \frac{i}{2} &= -\alpha \cdot \frac{j}{2}.
\end{align*}
\]
Conversely, from a three-dimensional simple Lie algebra \( s \) we can reconstruct a quaternion algebra as follows.

**Definition 2.5.** Let \( s \) be a three-dimensional simple Lie algebra, \( K \) be its Killing form and \( H(s) \) be the vector space defined by \( H(s) := k \oplus s \). Define the product \( \cdot : H(s) \times H(s) \to H(s) \) by:

\( a) \) for \( a, b \in k \), \( a \cdot b := ab \) (the field product on \( k \));

\( b) \) for \( a \in k \) and \( v \in s \), \( a \cdot v := v \cdot a := av \) (the scalar multiplication of \( k \) on \( s \)).
c) for $v, w \in \mathfrak{s}$,
\[
v \cdot w := \frac{K(v, w)}{8} \cdot 1 + \frac{[v, w]}{2}.
\]

We define a norm form on $\mathcal{H}(\mathfrak{s})$ by
\[
N(x) := x \cdot \tau = (a^2 - \frac{K(v, v)}{8}) \cdot 1 \quad \forall x = a \cdot 1 + v \in \mathcal{H}(\mathfrak{s}),
\]
where $\tau := a \cdot 1 - v$.

**Proposition 2.6.** Let $\mathfrak{s}$ be a three-dimensional simple Lie algebra. The vector space $\mathcal{H}(\mathfrak{s}) = k \oplus \mathfrak{s}$ with the product $\cdot$ above is a quaternion algebra. Furthermore, if $\mathfrak{s} \cong L(\alpha, \beta)$ then $\mathcal{H}(\mathfrak{s}) \cong \left( \frac{-\beta - \alpha}{k} \right)$.

**Proof.** Straightforward calculation using the identity
\[
K([v, w], [v, w]) = \frac{1}{2} (K(v, w)^2 - K(v, v)K(w, w)) \quad \forall v, w \in \mathfrak{s}.
\]

**Example 2.7.**

a) Over $\mathbb{R}$, the Lie algebra $\mathfrak{su}(2)$ is the imaginary part of the classical quaternion algebra $\mathbb{H}$.

b) The Lie algebra $\mathfrak{sl}(2, k)$ is the imaginary part of the split quaternion algebra $M_2(k) \cong \left( \frac{-1}{k} \right)$.

The following is a résumé of the correspondence between three-dimensional simple Lie algebras, their Killing forms and their associated quaternion algebras.

**Proposition 2.8.** For $\alpha, \beta, \alpha', \beta' \in k^*$ the following are equivalent:

a) the Lie algebras $L(\alpha, \beta)$ and $L(\alpha', \beta')$ are isomorphic;

b) the quaternion algebras $\left( \frac{-\alpha - \beta}{k} \right)$ and $\left( \frac{-\alpha' - \beta'}{k} \right)$ are isomorphic;

c) the quadratic forms $< \beta, \alpha, \alpha \beta >$ and $< \beta', \alpha', \alpha' \beta >$ are isometric.

**Corollary 2.9.** Two three-dimensional simple Lie algebras are isomorphic if and only if their Killing forms are isometric. In particular, a three-dimensional simple Lie algebra is split if and only if its Killing form is isotropic.

### 3 Construction of non-split three-dimensional simple Lie algebras from involutions of $\mathfrak{sl}(2,k)$

In this section we give a construction of non-split three-dimensional simple Lie algebras from involutions of $\mathfrak{sl}(2, k)$ and characterise those which one can be obtained in this way. We show that non-split three-dimensional simple Lie algebras constructed from these involutions define a particular subgroup of the Brauer group $B(k)$ of $k$. We first introduce some notation. Let
\[
k^*_{-1} := \{ x^2 + y^2 \mid x, y \in k \} \setminus \{ 0 \}.
\]

This is a subgroup of $k^*$ and if $-1 \in k^{*2}$, then $k^*/k^*_{-1} \cong \{ 1 \}$ since
\[
\left( \frac{1 + \Delta}{2} \right)^2 + \left( \sqrt{1 - \frac{1 - \Delta}{2}} \right)^2 = \Delta \quad \forall \Delta \in k^*.
\]

Let $\mathfrak{s}$ be a split three-dimensional simple Lie algebra, $K$ be its Killing form and $\sigma$ be a non-trivial involutive automorphism of $\mathfrak{s}$ such that
\[
\frac{K(x, x)}{2} \neq 1 \in \frac{k^*}{k^{*2}} \quad \forall x \in \mathfrak{s}^*.
\]
To this data, we are going to associate another three-dimensional simple Lie algebra \( s' \). Let \( \Lambda \in k^* \) be such that

\[
[\Lambda] = \left[ \frac{K(x, x)}{2} \right] \in \frac{k^*}{k^{*2}} \quad \forall x \in s^*,
\]

and \( \lambda \) be a square root of \( \Lambda \) in a non-trivial quadratic extension of \( k \). Since \( \sigma \) is involutive we have

\[
s \cong I \oplus p,
\]

where \( I \) is the one-dimensional eigenspace for the eigenvalue \( 1 \) and \( p \) is the two-dimensional eigenspace for the eigenvalue \( -1 \). Let \( s' \) be the three-dimensional simple \( k \)-Lie algebra

\[
s' := I \oplus \lambda p
\]

with the Lie bracket extended from \( s \):

\[
[a + \lambda b, c + \lambda d] = ([a, c] + \Lambda[b, d]) + \lambda([b, c] + [a, d]) \quad \forall a + \lambda b, c + \lambda d \in s'.
\]

**Theorem 3.1.** Let \( s \) be a split three-dimensional simple Lie algebra, \( K \) be its Killing form and \( \sigma \) be a non-trivial involutive automorphism of \( s \) such that

\[
\left[ \frac{K(x, x)}{2} \right] \neq 1 \in \frac{k^*}{k^{*2}} \quad \forall x \in s^*.
\]

The three-dimensional simple Lie algebra \( s' \) associated to \( (s, \sigma) \) by the construction above is non-split if and only if

\[
\left[ \frac{K(x, x)}{2} \right] \neq 1 \in \frac{k^*}{k^{*1}} \quad \forall x \in s^*.
\]

**Proof.** We first prove the following lemma.

**Lemma 3.2.** Let \( x \) be a non-zero element of a split three-dimensional simple Lie algebra \( s \). Then, there exists \( h \) in \( s \) such that \( \text{ad}(h) \) is diagonalisable and which is orthogonal to \( x \).

**Proof.** Let \( \{h, e, f\} \) be a standard \( sl(2, k) \)-triple and \( K \) be the Killing form of \( s \). Since \( \text{Span} < e, f > \) is a hyperbolic plane, there exists \( x' \in \text{Span} < e, f > \) such that \( K(x, x) = K(x', x') \). This implies that there exists \( g \in \text{SO}(s) \) such that \( g(x') = x \). Since \( h \) is orthogonal to \( x' \), \( g(h) \) is orthogonal to \( x \), we have \( K(g(h), g(h)) = K(h, h) \) and so the linear map \( \text{ad}(g(h)) \) is diagonalisable. \( \square \)

By Lemma 3.2, there exist \( h, e, f \in s \) such that \( \{h, e, f\} \) is a standard basis of \( s \cong sl(2, k) \) and \( x \in \text{Span} < e, f > \) where \( x \) is a non-zero fixed point of \( \sigma \). Since \( \sigma \) is involutive and \( \sigma(h) = -h \) we obtain that \( \sigma \) is a reflection on the hyperbolic plane \( \text{Span} < e, f > \) and so there exists \( a \in k^* \) such that \( \sigma(e) = af \) and \( \sigma(f) = \frac{1}{a}e \). The eigenspaces \( I \) and \( p \) are

\[
I = \text{Span} < e + af >, \quad p = \text{Span} < h, e - af >,
\]

and so

\[
s' = I \oplus \lambda p = \text{Span} < e + af > \oplus \text{Span} < \lambda h, \lambda(e - af) >.
\]

We now calculate the structure constants of \( s' \):

\[
\frac{\lambda h}{2}, \frac{e + af}{2} = \frac{\lambda(e - af)}{2},
\]

\[
\frac{e + af}{2}, \frac{\lambda(e - af)}{2} = -\frac{\lambda h}{2},
\]

\[
\frac{\lambda(e - af)}{2}, \frac{\lambda h}{2} = -\frac{\lambda(e + af)}{2}.
\]

Since

\[
[\Lambda] = \left[ \frac{K(s, x)}{2} \right] = \left[ \frac{K(s(e + af), e + af)}{2} \right] = [a] \in \frac{k^*}{k^{*2}}
\]

it follows that \( s' \) is isomorphic to \( L(-\Lambda, -\Lambda) \). The quadratic form \( < 2\Lambda, 2\Lambda, -2\Lambda^2 > \) is isometric to \( < -2, 2\Lambda, 2\Lambda > \) and so, by Proposition 2.8, the Lie algebra \( L(-\Lambda, -\Lambda) \) is isomorphic to \( L(-\Lambda, 1) \).
Lemma 3.3. Let $\Delta, \Delta' \in k^*$. The Lie algebras $L(-\Delta, 1)$ and $L(-\Delta', 1)$ are isomorphic if and only if $[\Delta] = [\Delta']$ in $k^*/k^*_{-1}$. In particular, $L(-\Delta, 1)$ is split if and only if $\Delta$ is a sum of two squares.

Proof. By Proposition 2.8, $L(-\Delta, 1)$ is isomorphic to $L(-\Delta', 1)$ if and only if $< 1, -\Delta, -\Delta >$ is isometric to $< 1, -\Delta', -\Delta' >$. By Witt’s cancellation Theorem, $< 1, -\Delta, -\Delta >$ is isometric to $< 1, -\Delta', -\Delta' >$ if and only if $< \Delta, \Delta >$ is isometric to $< \Delta', \Delta' >$. Since they have the same discriminant, they are isometric if and only if they represent a common element (Proposition 5.1 p.15 in [Lam05]), in other words if and only if $[\Delta] = [\Delta'] \in k^*/k^*_{-1}$.

This theorem motivates the following definition.

Definition 3.4. Let $\mathfrak{s}$ be a split three-dimensional simple Lie algebra, let $K$ be its Killing form and let $\sigma$ be an automorphism of $\mathfrak{s}$. We say that $\sigma$ is of Cartan type if and only if:

$$\sigma \neq \text{Id}, \quad \sigma^2 = \text{Id}, \quad [K(x, x)] \neq 1 \in k^*/k^*_{-1} \quad \forall x \in \mathfrak{s}^\sigma.$$

Two automorphisms of Cartan type $\sigma$ and $\sigma'$ are said to be equivalent if

$$[K(x, x)] = [K(x', x')] \in k^*/k^*_{-1} \quad \forall x \in \mathfrak{s}^\sigma, \forall x' \in \mathfrak{s}^\sigma'.$$

Remark 3.5. a) The Killing form $K$ of a split three-dimensional simple Lie algebra $\mathfrak{s}$ represents all the elements of $k$. Hence for all $\alpha \in k^*$ such that $[\alpha] \neq 1 \in k^*/k^*_{-1}$, there exists an automorphism of Cartan type $\sigma$ of $\mathfrak{s}$ such that

$$K(x, x) = \alpha \quad \forall x \in \mathfrak{s}^\sigma.$$

b) If $k = \mathbb{R}$, an automorphism $\sigma$ of $\mathfrak{sl}(2, \mathbb{R})$ is of Cartan type if and only if $\sigma$ is a Cartan involution.

c) If $x \in \mathfrak{s}$ satisfies to $[K(x, x)] \neq 1 \in k^*/k^*_{-1}$ then $\frac{K(x, x)}{2}$ is not a square.

We now study the non-split three-dimensional simple Lie algebras which can be obtained by the construction above.

Definition 3.6. A non-split three-dimensional simple Lie algebra $\mathfrak{s}'$ is said to be obtainable if there exists an automorphism of Cartan type $\sigma$ of a split three-dimensional simple Lie algebra $\mathfrak{s}$ such that $\mathfrak{s}'$ is isomorphic to the Lie algebra associated to $(\mathfrak{s}, \sigma)$ by the construction above.

We now summarise various conditions for a non-split three-dimensional simple Lie algebra to be obtainable in the following proposition.

Proposition 3.7. Let $\mathfrak{s}'$ be a non-split three-dimensional simple Lie algebra and $K$ be its Killing form. The following are equivalent:

a) $\mathfrak{s}'$ is obtainable,

b) there exist $x, h \in \mathfrak{s}'$ such that

$$h \bot x, \quad [K(x, x)] \neq 1 \in k^*/k^*_{-1} \quad \text{and} \quad [K(h, h)] = [K(x, x)] \in k^*/k^*_{2},$$

c) $\mathfrak{s}'$ is isomorphic to $L(-\Delta, -\Delta)$ for some $\Delta \in k^*$, 

d) the Killing form of $\mathfrak{s}'$ represents $-2$. 

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Proof. Conditions a) and b) are equivalent by construction. As we saw in the proof of Theorem 3.1, conditions a), b) imply condition c). Conversely, if \( s' \cong L(-\Delta, -\Delta) \) for some \( \Delta \in k^* \), we have \( [\Delta] \neq 1 \in k^*/k_{-1}^* \) since \( s' \) is non-split. By Remark 3.5, \( L(-\Delta, -\Delta) \) is obtainable and so Conditions a), b) and c) are equivalent. We now show that Conditions a), b) and c) are equivalent to Condition d). If the Killing form \( K \) of \( s \) represents \(-2\), there exists \( \delta \) and \( \gamma \) in \( k^* \) such that \( K \) is isometric to \( < -2, \delta, \gamma > \). Since \( disc(K) = [-2] \in k^*/k_{-2} \) and \( disc(< -2, \delta, \gamma >) = [-2\delta\gamma] \in k^*/k_{-2} \) we have \( [\gamma] = [\delta] \in k^*/k_{-2} \) and then \( K \) is isometric to the quadratic form \( < -2, \delta, \delta > \) which is isometric to the Killing form of \( L(\frac{-\delta}{2}, \frac{-\delta}{2}) \). Hence \( s \) is isomorphic to \( L(\frac{-\delta}{2}, \frac{-\delta}{2}) \). Conversely, the Killing form of \( L(\delta, \delta) \) is isometric to \( < -2\delta, -2\delta, -2 > \) and hence represents \(-2 \). \( \square \)

Consider the Brauer group \( B(k) \) of \( k \). The elements of \( B(k) \) are in 1 : 1 correspondence with the isomorphism classes of central division algebras over \( k \) (for details see Chap.IV of [Lam05]). Non-isomorphic quaternion algebras represent different elements of \( B(k) \) and, we now consider the elements of \( B(k) \) represented by the quaternion algebras constructed from obtainable non-split three-dimensional simple Lie algebras (see Section 2):

**Definition 3.8.** Let \( B(k) \) be the Brauer group of \( k \). Define

\[
H(k) := [\mathcal{H}(sl(2, k))] \cup \{ [\mathcal{H}(s)] \in B(k) \mid s \text{ is an obtainable non-split three-dimensional simple Lie algebra} \}.
\]

A quaternion algebra is of order 2 in the Brauer group but, in general, the set of classes of quaternion algebras in \( B(k) \) is not a subgroup. However, the set \( H(k) \) of classes of quaternion algebras associated to obtainable non-split three-dimensional simple Lie algebra \( s \) is a subgroup of \( B(k) \).

**Proposition 3.9.** The set \( H(k) \) is a subgroup of the Brauer group \( B(k) \) isomorphic to \( k^*/k_{-1}^* \).

Proof. Let \( s \) and \( s' \) be obtainable non-split three-dimensional simple Lie algebras. By Proposition 3.7, there exist \( \Delta \) and \( \Delta' \) in \( k^* \) such that \( s \cong L(-\Delta, 1) \) and \( s' \cong L(-\Delta', 1) \) and by Proposition 2.6 the quaternion algebras \( \mathcal{H}(s) \) and \( \mathcal{H}(s') \) are isomorphic respectively to \( (\frac{-1}{k}\Delta)^\perp \) and \( (\frac{-1}{k}\Delta')^\perp \). By Linearity (see Theorem 2.11 p.60 in [Lam05]) we have

\[
(\frac{-1}{k}\Delta)^\perp \otimes (\frac{-1}{k}\Delta')^\perp \cong (\frac{-1}{k}\Delta \Delta') \otimes M_2(k).
\]

Hence, in the Brauer group we have

\[
[(\frac{-1}{k}\Delta)^\perp \otimes (\frac{-1}{k}\Delta')^\perp] = [(\frac{-1}{k}\Delta \Delta')]
\]

and so \( H(k) \) is a subgroup of \( B(k) \). Furthermore the map \( [(\frac{-1}{k}\Delta)^\perp] \mapsto \Delta \) defines a group isomorphism between \( H(k) \) and \( k^*/k_{-1}^* \). \( \square \)

We can summarise the correspondences between automorphisms of Cartan type of \( sl(2, k) \), obtainable non-split three-dimensional simple Lie algebras, \( k^*/k_{-1}^* \) and \( H(k) \) as follows.

**Theorem 3.10.** We have the following correspondences

\[
\begin{align*}
\left\{ \text{equivalence classes of automorphisms of Cartan type of } sl(2, k) \right\} & \leftrightarrow \left\{ \text{isomorphism classes of obtainable non-split three-dimensional simple Lie algebras} \right\} \\
& \leftrightarrow \left\{ \text{elements in } k^*/k_{-1}^* \right\} \\
& \leftrightarrow \left\{ \text{elements in the subgroup } H(k) \text{ of the Brauer group } B(k) \text{ of } k \right\}.
\end{align*}
\]

4 **Criteria to be obtainable over local and global fields**

In this section, after recalling definitions, we characterise which non-split three-dimensional simple Lie algebras are obtainable over local and global fields in terms of the Hilbert symbol and the Legendre symbol (see [Vig80] for details about quaternion algebras over local and global fields). We also give examples of obtainable and unobtainable non-split three-dimensional simple Lie algebras over the field of rationals.
Definition 4.1. Let $\alpha, \beta \in k^*$. We define the Hilbert symbol $(\alpha, \beta) \in \{\pm 1\}$ as follows:

$$(\alpha, \beta) := \begin{cases} 
1 & \text{if the binary form } \alpha \alpha > \text{ represents } 1, \\
-1 & \text{otherwise}.
\end{cases}$$

Proposition 4.2. Let $\alpha, \beta \in k^*$. The Lie algebra $L(\alpha, \beta)$ is split if and only if $(-\alpha, -\beta) = 1$.

Proof. By Corollary 2.9 the Lie algebra $L(\alpha, \beta)$ is split if and only if $<-2\beta, -2\alpha, -2\alpha \beta>$ is isotropic. Since the norm form of the imaginary part of the quaternion algebra $(\frac{-\beta, \alpha}{k})$ is isometric to $<-(-\beta), -(\alpha), -(\alpha)(-\beta)>$, then by Theorem 2.7 p.58 of [Lam05] we have that $L(\alpha, \beta)$ is split if and only if $(-\alpha, -\beta) = 1$. \hfill \Box

We now introduce the Legendre symbol.

Definition 4.3. For an odd prime $p$ and $a \in \mathbb{Z}$, the Legendre symbol is defined by:

$$\left(\frac{a}{p}\right) := \begin{cases} 
0 & \text{if } p \text{ divides } a, \\
1 & \text{if } a \text{ is a square modulo } p, \\
-1 & \text{otherwise}.
\end{cases}$$

Remark 4.4. There is a formula for the Legendre symbol:

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod{p}.$$  

If $K_p$ is a non-dyadic local field, we denote by $\overline{K_p}$ its residue class field and denote by $\nu_p$ its valuation. Recall that for any prime $p$, the fields $\mathbb{Q}_p$ and $\mathbb{F}_p(\langle t \rangle)$ are examples of local fields whose residue class field is isomorphic to $\mathbb{F}_p$. The Hilbert symbol over a local field can be re-written in terms of the Legendre symbol as follows. Let $\alpha, \beta \in K_p^*$. We note $a = \nu_p(\alpha)$ and $b = \nu_p(\beta)$. By Corollary p.211 of [Ser68] we have

$$(\alpha, \beta) = \left((-1)^{ab} \frac{\alpha^b}{\beta^a}\right)^{\nu_p^{-1}}.$$

In particular if $|K_p|$ is prime, we have

$$(\alpha, \beta) = \left((-1)^{ab} \frac{\alpha^b}{\beta^a}\right)^{3}.$$  

By Proposition 2.8 and by Theorem 2.2 p.152 of [Lam05], there is up to isomorphism only one non-split three-dimensional simple Lie algebra over $K_p$. The standard model is $L(-u, -u)$ where $u, \pi \in K_p$, $\nu_p(u) = 0$, $\bar{u} \notin K_p^{-2}$ and $\nu_p(\pi) = 1$.

Proposition 4.5. Let $K_p$ be a non-dyadic local field and $\overline{K_p}$ be its residue class field. A non-split three-dimensional simple Lie algebra over $K_p$ is obtainable if and only if $|K_p| \equiv 3 \pmod{4}$.

Proof. We first need the following Lemma

Lemma 4.6. We have $-1 \in K_p^2$ if and only if $|\overline{K_p}| \equiv 1 \pmod{4}$ if and only if $K_p^* / k_p^* \equiv 1$. 

Proof. We have $-1 \in K_p^2$ if and only if $|\overline{K_p}| \equiv 1 \pmod{4}$ by Corollary 2.6 p.154 of [Lam05]. We have $K_p^* / k_p^* \equiv 1$ if and only if the quadratic form $<1, 1, -\Delta>$ is isotropic for all $\Delta \in K_p^*$. The quadratic form $<1, 1, -\Delta>$ is isotropic for all $\Delta \in K_p^*$ if and only if $(-1, \Delta) = 1$ for all $\Delta \in K_p^*$. However, since $K_p$ is a local field, $(-1, \Delta) = 1$ for all $\Delta \in K_p^*$ if and only if $-1$ is a square by Proposition 7 p.208 of [Ser68]. \hfill \Box
Let $\mathfrak{s}$ be a non-split three-dimensional simple Lie algebra. If $|k_p| \equiv 1 \pmod{4}$, then by the previous lemma and the proposition 3.7 there is no automorphism of Cartan type of $\mathfrak{s}(2,k)$. If $|k_p| \equiv 3 \pmod{4}$, then by Lemma 4.6, there exists an automorphism of Cartan type $\sigma$ of $\mathfrak{s}(2,k)$. Let $\mathfrak{s}'$ be the non-split three-dimensional simple Lie algebra associated to $(\mathfrak{s}(2,k),\sigma)$ by the construction of Section 3. Since there is up to isomorphism one non-split three-dimensional simple Lie algebra over $k_p$, the Lie algebras $\mathfrak{s}$ and $\mathfrak{s}'$ are isomorphic and so $\mathfrak{s}$ is obtainable.

Recall that the global fields are the number fields and the finite extensions of the function fields $\mathbb{F}_q(t)$. Using the Hasse-Minkowski theorem ([Lam05] p.170) and the previous proposition we obtain the following characterisation of obtainable non-split three-dimensional simple Lie algebras over global fields.

**Proposition 4.7.** Let $k$ be a global field. A non-split three-dimensional simple Lie algebra $\mathfrak{s}$ is obtainable if and only if it satisfies the following conditions:

a) over every non-archimedean completion $k_p$ of $k$ such that $|k_p| \equiv 1 \pmod{4}$, the Killing form of $\mathfrak{s}$ is isotropic,

b) the Killing form of $\mathfrak{s}$ represents $-2$ over all dyadic completions.

**Remark 4.8.** Condition b) is automatically satisfied if $k$ is of characteristic not two and a finite extension of a function field.

**Proof.** Using Proposition 3.7, the Lie algebra $\mathfrak{s}$ is obtainable if and only if its Killing form $K$ represents $-2$. Moreover, $K$ represents $-2$ if and only if the quadratic form $K \perp <2>$ is isotropic. By the Hasse-Minkowski theorem ([Lam05] p.170) we know that $K \perp <2>$ is isotropic over $k$ if and only if $K \perp <2>$ is isotropic over every completion $k_p$ of $k$ (including the dyadic completions). We now show that this condition is automatically satisfied for archimedean completions and non-archimedean completions $k_p$ such that $|k_p| \equiv 3 \pmod{4}$.

If $| \cdot |_P$ is an archimedean absolute value on $k$ then $k_P$ is either $\mathbb{R}$ or $\mathbb{C}$. If $k_P = \mathbb{C}$, the quadratic form $K \perp <2>$ is isotropic and if $k_P = \mathbb{R}$, the signature of $K \perp <2>$ is indefinite and then isotropic.

Using Propositions 3.7 and 4.5 we have that over every non-archimedean completion $k_P$ of $k$ such that $|k_p| \equiv 3 \pmod{4}$ the quadratic form $K \perp <2>$ is isotropic. Finally, using again Proposition 4.5 we have that over every non-archimedean completion $k_P$ of $k$ such that $|k_p| \equiv 1 \pmod{4}$ the quadratic form $K \perp <2>$ is isotropic if and only if $K$ is isotropic. This complete the proof of the proposition.

This result can be re-expressed as follows.

**Corollary 4.9.** Let $k$ be a global field. Let $L(\alpha,\beta)$ be a non-split three-dimensional simple Lie algebra, where $\alpha, \beta \in k^*$. The Lie algebra $L(\alpha,\beta)$ is obtainable if and only if it satisfies the following conditions:

a) over every non-archimedean non-dyadic completion $k_P$ of $k$ such that $|k_p| \equiv 1 \pmod{4}$ and such that $v_P(\alpha)$ or $v_P(\beta)$ is non-zero we have

\[(\alpha,\beta)_{k_P} = 1\]

(2)

where $v_P$ is the valuation associated to $k_P$.

b) the quadratic form $<\alpha,\beta,\alpha\beta,-1>$ is isotropic over all dyadic completions.

**Proof.** Let $K$ be the Killing form of $L(\alpha,\beta)$. By Remark 4.2, the quadratic form $K$ is isotropic over a non-archimedean completion $k_P$ of $k$ such that $|k_p| \equiv 1 \pmod{4}$ if and only if $(-\alpha,-\beta)_{k_P} = 1$. But since $-1$ is a square in $k_p$ by Lemma 4.6 this is equivalent to have $\alpha,\beta)_{k_P} = 1$.

Over $\mathbb{Q}$ this implies the following result:

**Proposition 4.10.** Let $L(\alpha,\beta)$ be a non-split three-dimensional simple Lie algebra over $\mathbb{Q}$, where $\alpha, \beta \in \mathbb{Q}^*$. The Lie algebra $L(\alpha,\beta)$ is obtainable if and only if for every prime $p \equiv 1 \pmod{4}$ such that $v_p(\alpha)$ or $v_p(\beta)$ is non-zero we have

\[\left(\frac{\alpha^{v_p(\beta)}\beta^{v_p(\alpha)}}{p}\right) = 1 \pmod{p}\]

where $v_p$ is the $p$-adic valuation associated to the prime $p$. 

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Remark 4.11. For fixed $\alpha, \beta \in \mathbb{Q}^*$, the number of primes $p$ such that $v_p(\alpha)$ or $v_p(\beta)$ is non-zero is finite.

Proof. The dyadic completion of $\mathbb{Q}$ is $\mathbb{Q}_2$ and 

$$\text{disc}(\langle \alpha, \beta, \alpha \beta, -1 \rangle_{\mathbb{Q}_2}) = -1 \notin \mathbb{Q}_2^*$$

by Corollary p.40 in [Cas78]. Hence, $\langle \alpha, \beta, \alpha \beta, -1 \rangle_{\mathbb{Q}_2}$ is isotropic by Lemma 2.6 p.59 of [Cas78] and so the Killing form of $L(\alpha, \beta)$ represents $-2$ over $\mathbb{Q}_2$. We know that the non-archimedean completions of $\mathbb{Q}$ are the $p$-adic fields $\mathbb{Q}_p$ and the residue class field of $\mathbb{Q}_p$ is isomorphic to $\mathbb{F}_p$. Then, $[\sqrt[p]{q}] \equiv 1 \pmod{4}$ if and only if $p \equiv 1 \pmod{4}$. If $p \equiv 1 \pmod{4}$, from Equation (1), we have

$$(\alpha, \beta)_{k_p} = \left( \frac{3^{v_p(\beta)} \sqrt[p]{q}}{p} \right).$$

In particular, if $v_p(\alpha) = 0$ and $v_p(\beta) = 0$, then the condition (2) is automatically satisfied. \qed

Here are some examples of obtainable and unobtainable non-split three-dimensional simple Lie algebras over the field of rationals using Proposition 4.10.

Example 4.12. Suppose that $k = \mathbb{Q}$.

a) If $\alpha, \beta > 0$, then $(-\alpha, -\beta) = -1$ and so $L(\alpha, \beta)$ is non-split by Proposition 4.2. In particular, the Lie algebras $L(2,3)$, $L(2,5)$ and $L(3,25)$ are non-split. The Lie algebra $L(2,3)$ is obtainable since there is no prime $p \equiv 1 \pmod{4}$ such that $v_p(2)$ or $v_p(3)$ is non-zero. The Lie algebra $L(2,5)$ is unobtainable since for the prime $p = 5$, we have $v_5(2) = 0$, $v_5(5) = 1$ and

$$\left( \frac{3^{v_5(5)} \sqrt[5]{q}}{5} \right) = 2^2 = 1 \pmod{5}.$$  

The Lie algebra $L(3,25)$ is obtainable since for the prime $p = 5$, we have $v_5(3) = 0$, $v_5(25) = 2$ and

$$\left( \frac{3^{v_5(25)} \sqrt[5]{q}}{5} \right) = 9^2 = 1 \pmod{5}.$$  

b) Using Proposition 2.8 and Example 2.17 p.63 of [Lam05] we have that the Lie algebra $L(3,-5)$ is non-split. Since for the prime $p = 5$, we have $v_5(3) = 0$, $v_5(-5) = 1$ and

$$\left( \frac{3^{v_5(-5)} \sqrt[5]{q}}{5} \right) = 3^2 = 1 \pmod{5},$$

then $L(3, -5)$ is unobtainable.

c) Let $p$ be an odd prime. We know from Example 2.14 p.62 of [Lam05] and Proposition 2.8 that $L(1,-p)$ is non-split if and only if $p \equiv 3 \pmod{4}$. If $p \equiv 3 \pmod{4}$, then the non-split Lie algebra $L(1,-p)$ is obtainable since there is no prime $p' \equiv 1 \pmod{4}$ such that $v_{p'}(1)$ or $v_{p'}(-p)$ is non-zero.

d) Let $p$ be an odd prime. We know from Example 2.15 p.62 of [Lam05] and Proposition 2.8 that $L(2,-p)$ is non-split if and only if $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$. If $p \equiv 7 \pmod{8}$, then the non-split Lie algebra $L(2, -p)$ is always obtainable since there is no prime $p' \equiv 1 \pmod{4}$ such that $v_{p'}(2)$ or $v_{p'}(-p)$ is non-zero. If $p \equiv 5 \pmod{8}$, the non-split Lie algebra $L(2, -p)$ is always unobtainable since for the prime $p$, we have $v_p(2) = 0$, $v_p(-p) = 1$ and

$$\left( \frac{2}{p} \right) = -1 \pmod{p}$$

by the Quadratic Reciprocity Law (see p.181 in [Lam05]).
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