Compressively certifying quantum measurements

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We introduce a reliable compressive procedure to uniquely characterize any given low-rank quantum measurement using a minimal set of probe states that is based solely on data collected from the unknown measurement itself. The procedure is most compressive when the measurement constitutes pure detection outcomes, requiring only an informationally complete number of probe states that scales linearly with the system dimension. We argue and provide numerical evidence showing that the minimal number of probe states needed is even generally below the numbers known in the closely-related classical phase-retrieval problem because of the quantum constraint. We also present affirmative results with polarization experiments that illustrate significant compressive behaviors for both two- and four-qubit detectors just by using random product probe states.

I. INTRODUCTION

Along with states and processes, measurements play a fundamental role in the formalism of quantum mechanism. As a consequence, physical effects observed from quantum protocols are, logically, sensitive to the actual mechanisms of the detectors [1], especially precision-sensitive protocols [2, 3] and measurement-based quantum computation [4–6]. Unambiguous characterization of these elements is hence crucial to ensure the correct functioning of protocols in which they are employed [7–12].

More precisely, a quantum measurement is modeled by a set of positive outcome operators that sum to unity, which is also known as a positive operator-valued measure (POVM). Characterizing such a POVM entails the identification of all outcome operators by initializing input probe states and inferring these operators from the corresponding measurement data. For d-dimensional quantum systems, d2 probe states are necessary for this task with arbitrary measurements. However, as practical measurements of high tomographic power correspond to (nearly-)pure outcomes [13–17], exploiting this extreme rank deficiency can significantly reduce the number of probe states. Previously, there have been proposals based on the idea of compressed sensing [18, 19] to reduce the measurement settings required to reconstruct low-rank quantum states [20–23] and processes [24–26]. These proposals, nonetheless, require the correct knowledge about the maximal rank of the unknown state or process in order to use a highly specific compressed-sensing measurement, which is difficult to justify in realistic scenarios. To the best of our knowledge, there is no compressed-sensing proposal developed for detector tomography.

Recently, a novel paradigm for compressive quantum state and process tomography [27–30] that does not depend on any spurious a priori information about the state or process of interest, and provides a built-in verification method that certifies if the characterization is truly unique from the collected data. We shall use the underlying theoretical framework to formulate compressive quantum detector tomography (CQDT) as the representative approach to generally efficient measurement characterization. Interestingly, CQDT generalizes a rather extensive literature on phase-retrieval studies [31–34] where independent low-rank (positive) matrices are reconstructed from classical intensity measurements, which offers interesting mathematical results for us to benchmark our compressive scheme.

In what follows, we shall present the theory of CQDT and demonstrate its performance with several examples of low-rank quantum measurements. Furthermore, we show that the probe states needed to carry out CQDT can be very general, and the minimal number of them can even be lower than the minimal number required in phase-retrieval problems, a close cousin to the problem of CQDT, but without collective operator constraints (such as the unit-sum and positivity constraints for POVMs). In particular, we highlight that this minimal number scales linearly with d for all rank-1 measurements instead of the usual quadratic behavior. To showcase CQDT in realistic physical settings, we present experimental data for both two-qubit and four-qubit measurements performed using polarization encoding and confirm that the resulting reconstructions are still highly compressive with real data.

II. COMPRESSION QUANTUM DETECTOR TOMOGRAPHY

For d-dimensional systems, any quantum measurement, or POVM, is defined as a set of M d-dimensional positive operators that resolve the identity \( \sum_{j=0}^{M-1} \Pi_j = \mathbb{1} \). Data collected with such a measurement on a given quantum state \( \rho \) are statistically distributed according to the probabilities \( p_j \) dictated by Born’s rule—\( p_j = \text{tr}(\rho \Pi_j) \). The Hermiticity of \( \Pi_j \) implies that one needs minimally \( d^2 \) probe states to uniquely re-

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construct the unknown POVM if no other additional steps are carried out.

On the other hand, common POVMs designed for quantum-information protocols are either pure or at most highly rank-deficient. To put things into perspective, for rank-$r$ operators, specified by $O(rd) \ll d^2$ parameters for $r \ll d$, it should in principle be possible to utilize $O(rd)$ probe states to uniquely characterize every single POVM element $\Pi_j$. The reconstruction is also said to be informationally complete (IC). The purpose of CQDT is to carry out this task without additional information about the unknown POVM (which includes its rank). It includes the important uniqueness certification method that directly inspects all data to check if a reconstruction derived from said data is unique or not.

To find the IC set of distinct probe states for characterizing an unknown rank-$r$ POVM in CQDT from ground up, we formulate an iterative procedure that first feeds the POVM with a randomly chosen probe state $\rho_1$. Next, the collected data $\nu_1 = (\nu_1^{(1)}, \nu_2^{(1)}, \ldots, \nu_M^{(1)})^T$, independently distributed among the $M$ POVM outcome elements, are used to obtain the optimal physical probabilities $\hat{\rho}_1$ that are “nearest” to $\nu_1$, where the caret denotes an estimator. This automatically defines a convex set $C_1$ of POVMs that are consistent with $\hat{\rho}_1$. The logical followup is then to verify if $C_1$ has zero volume, namely whether it contains just a single POVM. Since only one probe state is used, $C_1$ clearly has finite volume, so the next probe state distinct from the first is chosen and CQDT repeats, where this time the convex set $C_2$ that is consistent with the probabilities $\{\hat{\rho}_1, \hat{\rho}_2\}$ is certified for uniqueness, and so forth (see the schematic in Fig. 1).

During the $L$th step of the iteration, for the sake of demonstration, we may take the optimal column of probabilities $\hat{\nu}_l$ as the constrained least-squares (LS) solution to the distance metric

$$D = \sum_{l'=1}^{L} ||\hat{\nu}_l - \nu_l||^2 \quad \text{s. t.} \quad \tilde{\Pi}_j \geq 0, \quad \sum_{j=0}^{M-1} \tilde{\Pi}_j = I,$$

although other statistical options like the maximum-likelihood principle [35–37] may also be applied. After which the uniqueness certification is carried out by computing an indicator function $s_{\text{CVX}}$ over the convex set $C_L$ of POVMs that are consistent with $\{\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_L\}$. A straightforward way to do this is to define $s_{\text{CVX}} = f_{\text{max}} - f_{\text{min}}$, where $f = \sum_{j=0}^{M-1} \text{tr} \{\tilde{\Pi}_j Z_j\}$ and $Z_j$ are fixed but randomly-chosen full-rank positive operators. Both function optimization are carried out according to the POVM constraints and LS constraints

$$\{ \text{tr} \{\rho_L \tilde{\Pi}_j\} = \hat{\rho}_j L, \sum_{j=0}^{M-1} \hat{\rho}_j L = 1 \}.$$

Following Ref. [27], it can be shown that if $s_{\text{CVX}} = 0$, then $C_L$ contains only a single unique POVM that satisfies the LS probabilities, and this is when we shall denote the IC number of probe states $L_{\text{IC}} = L$. We note that all the physical constraints in both the LS optimization and $s_{\text{CVX}}$ computation can be conveniently integrated into semidefinite programs, which are generally polynomially efficient optimization algorithms [38].

![FIG. 1. CQDT as an easy iterative procedure. A probe state is sent to the unknown POVM and its corresponding and all previous measurement data are collectively analyzed to see if they lead to a unique POVM characterization. If this is not the case, another probe state distinct from all the already chosen ones is next sent to the POVM and the procedure is repeated until a unique reconstruction is obtained.](image)

![FIG. 2. Plots of the IC number of probe states ($L_{\text{IC}}$) against dimension (d) for varying values of POVM rank $r$ and $M$. (a) CQDT on rank-1 POVMs requires only a $L_{\text{IC}} = 4d - 4$ that scales linearly in $d$, whereas $L_{\text{IC}} = 4d^2 - 4$ when $d < 2r$ and linearly in $d$ when $d > 2r$. (b,c,d) More specifically, in comparison with the results reported in [31, 34] (dashed curve) for phase retrieval, the typical number of probe states required to compressively reconstruct rank-$r$ POVMs (dotted-dashed curve) is lower as the actual POVM space is much smaller than the product of Hermitian-operator spaces due to both the positivity and unit-sum constraints. The numerical estimates of $L_{\text{IC}}$ pertaining to the linear regime when $d < 2r$ are quoted in the legends. All graphs stabilize at the fitted functions and is verified with $M = 5d^2$ (not shown in the figure panels). All error regions are constructed from 10 randomly generated square-root POVMs, which are entangled measurements, and their noiseless probabilities.](image)
FIG. 3. Experimental setup. Photons at 810 nm are generated by SPDC from a 3 mm Type I β-barium borate crystal pumped with a 405 nm CW laser at 50 mW, on two modes selected by interference filters with FWHM = 7.3 nm and single mode fibers. Separable probe states are prepared by means of a quarter-wave plate (QWP) at angle $\varphi_1$ followed by a half-wave plate (HWP) at angle $\varphi_2$ polarization rotations on one qubit in this order and a QWP at angle $\varphi_2$ followed by a HWP at angle $\varphi_2$ on the other qubit in the same order. After which, the two photons are then sent through a partially polarizing beam splitter (PPBS) with transmittivities $T_{90} = 1$ and $T_{45} = 1/3$, acting as a controlled Z (CZ) gate. Two further PPBSs with the same transmittivities, rotated by 90°, are employed to compensate for the unbalance in the amplitudes of the two polarization components [39]. A projective measurement is then performed on each photon by means of a HWP ($\vartheta_{m_1}$) for one output and HWP ($\vartheta_{m_2}$) for the other, and polarizing beam splitters (PBSs). The photons are then collected with single-mode fibres and sent to two avalanche photodiodes (APDs) for detection.

III. BENCHMARKING AGAINST LOW-RANK PHASE-RETRIEVAL PROBLEMS

There is another field of study that is closely related to the problem of CQDT—the phase-retrieval problem that finds the IC set of complex signals $\{\phi_1, \phi_2, \ldots\}$ to uniquely identify an unknown Hermitian matrix $\mathbf{H}$ in some fixed computational basis through the respective intensity measurements $\phi_1^\dagger \mathbf{H} \phi_1 = y_1$ [31–34]. It was conjectured in [31] and later proven in [34] that the IC number of signals needed to uniquely characterize a rank-$r$ $\mathbf{H}$ of known $r$ is $L_{IC}^r = (4dr - 4d^2)\eta(r - d/2) + d^2\eta(d/2 - r)$ in terms of the usual Heaviside step function $\eta(\cdot)$.

This expression remains the same even when one attempts to recover a set of low-rank Hermitian matrices $\sum_j \mathbf{H}_j = \mathbf{1}$ that sum to the identity matrix, since this constraint merely reflects the linear dependence in the intensities $\phi_1^\dagger \mathbf{H}_j \phi_1 = y_{j,l}$ with respect to the index $l$ and does not reduce the number of independent parameters that specify the individual matrices $\mathbf{H}_j$ except for one of them. The situation becomes starkly different when $\mathbf{H}_j \geq 0$, which is that of CQDT. The positivity constraint imposed on all matrices now heavily restricts the ranges of parameters these matrices are collectively allowed to possess in order for the unit-sum constraint to remain true. Therefore, just like quantum states and processes, compressive methods are highly effective on quantum measurements because of the positivity constraint.

To gain a physical understanding of CQDT in the absence of statistical noise ($\bar{p}_i = p_i$), Fig. 2 charts the characteristic behaviors of $L_{IC}$ with respect to the Hilbert-space dimension $d$ for low-rank POVMs. The compressive effect arising with low-rank POVMs can be observed from Fig. 2, with $L_{IC} = 4d - 4 = O(d)$ for rank-$1$ POVMs in the limit of large number ($M$) of measurement outcomes where all projectors behave approximately as independent rank-$1$ operators despite the unit-sum constraint. Additionally, this number is believed to be near optimal [34]. In this case, $L_{IC} \rightarrow L_{IC}^r$ asymptotically since any rank-$1$ Hermitian operator $\mathbf{P}_j = |\phi_j\rangle\langle\phi_j|$ can be written as a real-scalar multiple $(\alpha_j)$ of a projector $|\phi_j\rangle\langle\phi_j|$, and the only difference between rank-$1$ phase-retrieval and CQDT is the constraint $\alpha_j > 0$ for all $j$ such that enforcing this constraint does not reduce the number of parameters needed to be specified. On closer inspection of the asymptotic values of $L_{IC}$ with increasing rank $r < d$, as shown in Figs. 2(b,c,d), it turns out that $L_{IC} < L_{IC}^r$ even in the large-$M$ limit. This time, unlike the $r = 1$ case, imposing positivity on all $r$ eigenvalues of every rank-$r$ operator significantly reduces the volume of all individual linear-operator spaces. We emphasize that Fig. 2 illustrates results based on randomly chosen square-root POVMs, which are “pretty good” measurements when employed in quantum-state discrimination problems [40–42] and is interestingly equivalent to *Haar-random POVMs* introduced recently in [43] (see also Appendix A for a brief recipe to generate them). The enhancement in the compressibility of CQDT as a consequence of operator constraints is a rather general quantum phenomenon [21] that manifests itself in any sort of physical measurements.
values of the waveplates angles vary in the interval $-\pi/2 \leq \vartheta_1, \vartheta_2 \leq \pi/2$ and $-\pi/4 \leq \varphi_1, \varphi_2 \leq \pi/4$ (see Appendix B for further details).

The measurement relies on a controlled Z (CZ) gate, which is implemented by means of a partially polarizing beam splitter (PPBS) [44-47], acting as $U_{\text{CZ}} = |1\rangle \langle 0| \otimes \sigma_z + |0\rangle \langle 1| \otimes \mathbb{I}$ in terms of the Pauli operator $\sigma_z$. After the gate, a projective measurement is eventually performed for each qubit by means of a HWP at an angle $\vartheta_{m_1}$ for the first qubit, another HWP at an angle $\vartheta_{m_2}$ for the second, and polarizing beam splitters (PBSs). We consider four different POVMs, $\mathcal{M}_i^{(r=1)} = \{\Pi_i^j = \langle \psi_j^{i} | \psi_j^{i} \rangle\}_{j=0}^{2}$ for $1 \leq i \leq 4$, where

$$\langle \psi_j^{i} | = U_{\text{CZ}}[U_{\text{HWP}}(\vartheta_{m_1}) \otimes U_{\text{HWP}}(\vartheta_{m_2})]|l\rangle_1|l'\rangle_2$$

with $\{l,l'\} \in \{00,01,10,11\}$, obtained by fixing the projection on the first qubit at $\vartheta_{m_1} = 22.5^\circ$ (quoted in degrees), and adopting for the second qubit the four settings $\vartheta_{m_2} = 0^\circ, 7.5^\circ, 14^\circ, 22.5^\circ$. This amounts to vary from a separable measurement when $\vartheta_{m_2} = 0^\circ$, to an entangling one when $\vartheta_{m_2} = 22.5^\circ$. We also perform CQDT on rank-2 POVMs that are defined by linear combinations of the basis outcomes inasmuch as $\mathcal{M}_i^{(r=2)} = \{\Pi_i^j = \langle \psi_j^{i} | \psi_j^{i} \rangle + \langle \psi_j^{i} \rangle_{\otimes 1}\}_{j=0}^{5}$, where $\otimes$ is addition modulo 4.

The performance of CQDT in terms of the uniqueness measure $\Delta_{\text{ent}}$ and target POVM fidelity is demonstrated in Fig. 4. The IC number of probe states $L_{\text{IC}}$, which is obtained at the value of $L$ for which $\Delta_{\text{ent}}$ first drops below some small pre-chosen threshold, for both ranks $r = 1$ and 2 match well with the simulation values in Fig. 2. To compute the POVM fidelity, we choose to compare the POVM Choi-Jamiołkowski operator [48] since the corresponding fidelity would then be invariant under arbitrary permutations of the POVM element label. For instance, the POVMs $\mathcal{M} = \{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ and $\mathcal{M}' = \{\Pi_4, \Pi_3, \Pi_1, \Pi_2\}$ are treated as the one and the same measurement and should therefore give a unit mutual fidelity (see Appendix C for the technical details of the POVM fidelity computation).

To unveil how significantly compressive CQDT can get for high-dimensional systems, we also look at the performance on four-qubit POVMs. These are derived by considering product measurements of the previous two-qubit POVMs. The rank-1 and 2 two-qubit POVMs are respectively mapped to...

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**IV. EXPERIMENTAL CONFIRMATION**

We formally demonstrate CQDT using an experimental setup as shown in Fig. 3. Two qubits are encoded in the polarization degree of freedom for photon pairs generated via SPDC, with $|H\rangle \equiv |1\rangle$ and $|V\rangle \equiv |0\rangle$. By means of half wave plates (HWPs) and quarter wave plates (QWPs) we prepare twenty random two-qubit probe states as $U_{\text{HWP}}(\vartheta_1)U_{\text{QWP}}(\varphi_1) \otimes U_{\text{HWP}}(\vartheta_2)U_{\text{QWP}}(\varphi_2)|1\rangle_1|1\rangle_2$, where the values of the waveplates angles vary in the interval $-\pi/2 \leq \vartheta_1, \vartheta_2 \leq \pi/2$ and $-\pi/4 \leq \varphi_1, \varphi_2 \leq \pi/4$.

The measurement relies on a controlled Z (CZ) gate, which is implemented by means of a partially polarizing beam splitter (PPBS) [44-47], acting as $U_{\text{CZ}} = |1\rangle \langle 0| \otimes \sigma_z + |0\rangle \langle 1| \otimes \mathbb{I}$ in terms of the Pauli operator $\sigma_z$. After the gate, a projective measurement is eventually performed for each qubit by means of a HWP at an angle $\vartheta_{m_1}$ for the first qubit, another HWP at an angle $\vartheta_{m_2}$ for the second, and polarizing beam splitters (PBSs). We consider four different POVMs, $\mathcal{M}_i^{(r=1)} = \{\Pi_i^j = \langle \psi_j^{i} | \psi_j^{i} \rangle\}_{j=0}^{2}$ for $1 \leq i \leq 4$, where

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their four-qubit counterparts $\mathcal{M}_i^{(r=1)} = \{\Pi_{jk} = |\psi_{ij}\rangle\langle\psi_{ij}| \otimes |\psi_{ik}\rangle\langle\psi_{ik}|\}_{j,k=0}^3$ and $\mathcal{M}_i^{(r=2)} = \{\Pi_{jk} = (|\psi_{ij}\rangle\langle\psi_{ij}| \otimes |\psi_{ik}\rangle\langle\psi_{ik}| + |\psi_{ij}\rangle\langle\psi_{ij+1}| \otimes |\psi_{ik}\rangle\langle\psi_{ik+1}|)/2\}_{j,k=0}^3$. The CQDT performance for these four-qubit product measurement bases are shown in Fig. 5. Owing to a stronger statistical noise and product structures of the POVMs, we find that $L_{IC}$ is less than the corresponding estimated values in Fig. 2.

In both aforementioned figures, the fidelity is always less than one at $L = L_{IC}$ because of statistical fluctuation in the data. On this note, it is instructive to recall that previous studies of overcomplete quantum tomography [17, 49–54] has led to an understanding that measuring probe states of numbers beyond $L_{IC}$ should generally lead to an improvement in reconstruction fidelity. This is evidently observed in both Figs. 4 and 5.

V. CONCLUSION

We have successfully formulated and demonstrated a highly compressive quantum detector tomography scheme that allows us to characterize any set of low-rank measurements with an extremely small set of probe states relative to the square of the Hilbert-space dimension. We have shown that our compressive scheme can even outperform known phase-retrieval algebraically both kinds of measurements have identical distributions. These measurements can be generated as follows:

There exists a simple routine to generate a POVM $\{\Pi_j\}$ whose elements $\sum_j \Pi_j = 1$ sum to the identity. For a rank-$r$ POVM of $M$ elements:

Appendix A: Square-root measurements

Square-root measurement

1. Generate a set of $M$ operators $A_j$ represented by $d \times r$ complex matrices whose entries are independently and identically distributed according to the standard Gaussian distribution.

2. Define $S = \sum_{j=0}^{M-1} A_j A_j^\dagger$.

3. Define $\Pi_j = S^{-1/2} A_j A_j^\dagger S^{-1/2}$.

The above set of operators then form a POVM and is commonly coined the square-root measurement. Recently it has been shown that such measurements are in fact equivalent to Haar-random POVMs considered in [43], in the sense that algebraically both kinds of measurements have identical distributions. These measurements can be generated as follows:

Haar-random measurement

1. Begin with the standard basis $\{|0\rangle, |1\rangle, \ldots, |M-1\rangle\}$ that spans the vector space $\mathbb{C}^M$.

2. Randomly sample an $rM \times d$ isometry operator $V$ ($V^\dagger V = 1$) from the Haar distribution under the condition $d \leq rM$. This can be done by first generating an $rM \times rM$ complex matrix $A$, then computing the QR decomposition $A = QR$ and defining the random Haar-distributed $rM \times rM$ unitary matrix $U_{Haar} = QL$, where $L = R_{\text{diag}} \otimes |R_{\text{diag}}|$, $R_{\text{diag}} = \text{diag}\{R\}$ and $\otimes$ denotes the Hadamard division. Finally, we represent $V$ as the $rM \times d$ block of $U_{Haar}$.

3. Define $\Pi_j = V^\dagger |j\rangle\langle j| \otimes 1_r$, for $0 \leq j \leq M-1$, where $1_r$ is the $r$-dimensional identity operator.

Appendix B: State preparation

In the following table, we report the wave plate settings for the preparation of the 20 random two-qubit probe states. The
same list of configurations is used to generate the 400 random four-qubit probe states through the tensor product of all possible two-qubit state pairs out of these 20 probe states.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{State} & \text{QWP}(\varphi_1) & \text{HWP}(\vartheta_1) & \text{QWP}(\varphi_2) & \text{HWP}(\vartheta_2) \\
\hline
1 & -25.95^\circ & 27.46^\circ & -42.30^\circ & 76.53^\circ \\
2 & 38.98^\circ & -9.14^\circ & 17.29^\circ & -36.51^\circ \\
3 & -19.24^\circ & 20.93^\circ & -1.52^\circ & -60.21^\circ \\
4 & -2.80^\circ & -35.81^\circ & 17.10^\circ & 4.65^\circ \\
5 & -14.86^\circ & 24.84^\circ & 1.90^\circ & -13.63^\circ \\
6 & -13.00^\circ & 68.55^\circ & -5.75^\circ & -42.08^\circ \\
7 & 15.05^\circ & 10.52^\circ & -34.70^\circ & 57.50^\circ \\
8 & 27.17^\circ & 30.09^\circ & -27.33^\circ & 50.72^\circ \\
9 & 41.99^\circ & 7.78^\circ & 0.73^\circ & -80.72^\circ \\
10 & -18.63^\circ & 28.91^\circ & 0.49^\circ & 20.20^\circ \\
11 & -42.46^\circ & -8.43^\circ & 35.33^\circ & -58.36^\circ \\
12 & -0.42^\circ & -80.64^\circ & 6.60^\circ & -79.93^\circ \\
13 & 36.78^\circ & 83.32^\circ & -20.74^\circ & 22.32^\circ \\
14 & 24.02^\circ & -55.00^\circ & 9.93^\circ & -80.20^\circ \\
15 & 21.65^\circ & 7.80^\circ & -10.16^\circ & 6.07^\circ \\
16 & -27.10^\circ & 76.29^\circ & -11.84^\circ & 75.04^\circ \\
17 & 32.08^\circ & -39.84^\circ & -41.19^\circ & -86.63^\circ \\
18 & 3.96^\circ & 86.10^\circ & 11.22^\circ & -1.26^\circ \\
19 & -21.22^\circ & 75.01^\circ & 35.88^\circ & 68.69^\circ \\
20 & 14.54^\circ & 42.89^\circ & 41.69^\circ & 68.41^\circ \\
\hline
\end{array}
\]

**TABLE I.** Experimental angular configurations (in degrees) for all optical wave plates responsible for generating the two-qubit probe states.

We start by defining the unique square-root operators \( K_j = \sqrt{\Pi_j} \) out of the POVM elements. In the language of quantum dynamics, these form a set of Kraus operators that collectively describe the state-reduction map for the probe state \( \rho : \rho \mapsto K_j \rho K_j^\dagger / \rho_j \). We may then describe the POVM as a whole with a \( d^2 \)-dimensional Choi-Jamiołkowski operator \( E \) by defining

\[
E = \frac{1}{d} \sum_{i=0}^{d-1} \sum_{l'=0}^{d-1} \sum_{j=0}^{M-1} K_j |l \rangle |l' \rangle K_j^\dagger \otimes |l \rangle \langle l' |. \tag{C1}
\]

Since \( \text{tr} \{ E \} = 1 \), we may now define the POVM fidelity \( F \) of two different Choi-Jamiołkowski operator \( E \) and \( E' \) in exactly the same way as we usually do for quantum states—by means of the function \( F = \text{tr}\{ (E^{1/2} E'^{1/2})^{1/2} \}^2 \) that is symmetric in \( E \) and \( E' \).

It is obvious that by construction, \( F \) is invariant under the ordering of measurement outcomes. This benefit is, however, accompanied by an important disclaimer. Namely, \( E \) is not a one-to-one representation of any POVM. This is because Eq. \((C1)\) is a result of a unidirectional mapping \( \{ \Pi_j \} \mapsto E \) and in the course of this procedure, information about the individual \( \Pi_j \)s are lost; while \( \{ \Pi_j \} \) guarantees a unique \( E \), a given \( E \) can be obtained from an infinitely many sets of Kraus operators \([48]\). Unlike quantum processes where the Kraus operators are just mathematical representations of the unique operator \( E \), quantum measurements correspond to physically singled-out Kraus operators by construction. So, although the Choi-Jamiołkowski operator is ideal for computing the fidelity between two POVMs, (C)QDT cannot be performed with this operator.
