A closed formula for illiquid corporate bonds
and an application to the European market

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Abstract

We deduce a simple closed formula for illiquid corporate coupon bond prices when liquid bonds with similar characteristics (e.g. maturity) are present in the market for the same issuer. The key model parameter is the time-to-liquidate a position, i.e. the time that an experienced bond trader takes to liquidate a given position on a corporate coupon bond.

The option approach we propose for pricing bonds illiquidity is reminiscent of the celebrated work of Longstaff (1995) on the non-marketability of some non-dividend-paying shares in IPOs. This approach describes a quite common situation in the fixed income market: it is rather usual to find issuers that, besides liquid benchmark bonds, issue some other bonds that either are placed to a small number of investors in private placements or have a limited issue size.

The model considers interest rate and credit spread term structures and their dynamics. We show that illiquid bonds present an additional liquidity spread that depends on the time-to-liquidate aside from credit and interest rate parameters.

We provide a detailed application for two issuers in the European market.

Keywords: Corporate coupon bonds, liquidity, TRACE, time-to-liquidate.

JEL Classification: C51, G12, H63.

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1 Introduction

The natural question that arises when dealing with liquidity is: “How long does it take to liquidate a given position?” Despite the relevance of this question, not only has there not yet appeared, in the financial industry, a unique modeling framework, but not even a standard language for addressing liquidity. Unfortunately liquidity problems are, in general, really complicated. There are several aspects of asset liquidity, including tightness (i.e. bid–ask spread, the transaction cost incurred in case of a small liquidation), market impact (i.e. the average response of prices to a trade, see, e.g. Bouchaud et al. (2008)), market elasticity (i.e. how rapidly a market regenerates the liquidity removed by a trade) and the time-to-liquidate a position. In this paper we focus on corporate bonds. In the literature, few are the models available for corporate bonds: the main modeling approach has been introduced by Jarrow (2001) that view illiquidity as an exogenously determined component in bond yield in addition to the default risk component; significant evidence has been found for this component in the corporate bond spread due to illiquidity (see, e.g. Longstaff et al. 2005, Dick-Nielsen et al. 2012).

Traditional liquidity measures have been developed for the equity market within Market Impact Models (see, e.g. Lillo et al. 2003, Bouchaud et al. 2008, Gatheral 2010, and references therein) with a particular focus on stocks with larger capitalization: execution typically takes place in a timeframe from minutes to hours, only in some cases within a horizon of a few days. However these liquidity measures are not applicable to securities, such as many corporate bonds, that do not trade on a regular basis. In this case a complete representation of asset liquidity could be not feasible, for several reasons, such as i) the market is still largely OTC and bid–ask quotes are not available for many corporate bonds, ii) trading costs often decrease with trade size (see, e.g. Edwards et al. 2007) and iii) the time-to-liquidate a position can be some weeks, or even months, in some cases. Moreover the bond market can be very differentiated even for the same issuing institution: some bonds can be very illiquid while some others, even with similar characteristics (e.g. the same time to maturity), trade every day, with trading activity far from being uniform over time but mostly concentrated on recently issued bonds (‘on-the-run’ issues).

Focus on the bond market liquidity was boosted following the regulatory effort to introduce more transparency in the bond market. In the U.S.A., starting from the 1st of July, 2002, information on the prices and the volumes of completed transactions have been publicly disclosed for a significant set of corporate bonds. The National Association of Security Dealers (NASD, and after July 2007 the Financial Industry Regulatory Authority, FINRA) mandated transparency in the corporate bond

2 Practitioners well know that publicly disclosed quotes are often not true commitments to trade at that price but rather just indications (i.e. ‘indicative’ quotes).

3 In the corporate bond market a difference of some orders of magnitude with respect to large cap stocks is observed: “a typical US large cap stock, say Apple as of November 2007, had a daily turnover of around 8bn USD” with an “average of 6 transactions per second and on the order of 100 events per second affecting the order book” (see Bouchaud et al. 2008, p. 76).
market through the Trade Reporting and Compliance Engine (TRACE) program; under TRACE, all trades for corporate bonds in USD must be reported within 15 minutes of execution (see, e.g. Bessembinder et al. 2006, Dick-Nielsen et al. 2012, and references therein). Thanks to the transactional data provided by TRACE, Schestag et al. (2016) were able to apply to the bond market eight competing liquidity measures and to benchmark the effectiveness of thirteen liquidity proxies that only need daily information. On the basis of TRACE data for a period of eight years (2002-2010), Helwege et al. (2014) draw some relevant conclusions on liquidity premium: once they control for the credit risk component of the bond price thanks to the measure of the difference in the spreads between pairs of bonds with the same characteristics they can measure the sheer liquidity premium, and they find that it is time-varying and related to systematic risk.

The impact following the introduction of the TRACE program on bond market liquidity was mixed, as discussed by Asquith et al. (2013), with a decrease in daily price standard deviation, but also a parallel marked decrease in trading activity. Such evolution drove the slump of fixed-income revenues and the decline in profits of large dealers (see, e.g. Bessembinder et al. 2006).

In Europe, the observed evolution in the U.S.A. bond market after the introduction of TRACE has caused a lively debate within European institutions (Glover 2014), with consequent delay in the enforcement of mandatory transparency rules in the European Union. After several years of haggling between policy makers, the ruling of bond market transparency has been included within the update of the Markets in Financial Instruments Directive (also known as MiFID II) approved in April 2014 and binding since January 2018. The European Securities and Markets Authority (ESMA) is in charge of collecting transaction data from dealers and disclose information on bond liquidity. Since compulsory data collection started in January 2018, it is too early to draw significant conclusions from the analysis of time series of ESMA transaction data. However, recently a team of ESMA economists (De Renzis et al. 2018) has analyzed liquidity in the most liquid segment of the EU corporate bond market on the basis of a transactional dataset. They analyze the overall liquidity of the European bond market and their conclusion confirms the mixed results observed in the U.S.A. market by Asquith et al. (2013). In particular they show, in line with the results of Galliani et al. (2014), that the liquidity is positively related to the outstanding amount and negatively related with measures of market volatility with statistical significance.

The econometric analyses both in the U.S.A and in Europe show evidence of a split market, with large differences in the liquidity of debt securities traded in the same marketplace and this points to a relevant research question, i.e. the estimation of the sheer liquidity premium represented by the difference in price between comparable bonds which differ only in term of market liquidity. Moreover, in a framework where the effectiveness of econometric techniques is hampered by the sparsity of trades and by non-stationary data (see, e.g. Helwege et al. 2014), we deem useful to resort to a theoretical model that distinguishes between the credit and liquidity component of the spread and clarifies the impact of volatility on the liquidity component. Clearly, it is crucial in this context to have a model price for illiquid coupon bonds that takes into account, with a parsimonious modeling approach, a precise measure of illiquidity. In this paper we simplify significantly the problem for corporate coupon bonds by addressing just one single aspect of market liquidity: the time-to-liquidate a given

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4Six transaction cost measures, one price impact measure, and one price dispersion measure.

5Same issuer, same coupon type and both coupon amount and maturity within a narrow range.

6Markit EUR iBoxx Corporate database representing the segment of the 40 most liquid European corporate bonds. This database includes only the most liquid segment in the Investment Grade market for a time-window of 2 years.
position (hereinafter ttl), and we propose a closed formula for the liquidity component of corporate bond spreads defined as the difference in bond yields between a bond with limited liquidity and a very liquid bond of the same issuer. This approach presents several advantages: liquidity is considered an intrinsic characteristic of each single issue, it can vary over time and it depends on the size. Liquidity is expressed in terms of a price discount (or equivalently in terms of a liquidity spread) as a simple closed formula of a single one-dimensional parameter, the ttl. This is the time lag that, at a given value date and for a given size, an experienced trader needs to liquidate the position.

This problem is reminiscent of the celebrated work of Longstaff (1995) on the non-marketability of some non-dividend-paying shares in IPOs. More recently, Koziol and Sauerbier (2007) tackle a similar problem in the case of a risk-free zero-coupon (ZC) bond with maturity $T$. In this paper we propose an alternative modeling approach that allows an extension to default risk and coupon payments, the most common situation in the fixed income market.

We consider an application to the European market, where the problem of pricing illiquidity is even more significant than in the U.S. market, due to the differences in mandatory transparency requirements explained above. European companies had the equivalent of 8.4 trillion of bonds outstanding in various currencies in May 2014, up from 6.3 trillion at the beginning of 2008 (Glover 2014), so that the European bond market is almost as large as that of the U.S. Moreover, in Europe, it is relatively frequent to observe private placements to institutional investors, where a single issue is detained by a very limited pool of bondholders, and, especially in the financial sector, there are several bonds with small issue sizes aimed either at retail investors or at private-banking clients of a banking institution. Often no market price is available in these cases. For these reasons, the calibration of a liquidity model, especially in Europe, is often a challenging task. The proposed formula, besides bond characteristics (maturity, coupon, sinking features, etc...), depends on standard market quantities, such as i) the observed risk-free interest curve ii) issuers credit spread term-structure and iii) bond volatility. We show model calibration at a given value date $t_0$ for two European issuers in the financial sector and the liquidity spread curves that are obtained for different values of the time-to-liquidate.

The contributions of this paper to the existing literature on illiquid coupon bonds are threefold. First, it provides a simple closed formula for illiquid corporate coupon bonds that relates the time-to-liquidate a position and bond volatility to the price difference with respect of the corresponding liquid bond. Second, it clarifies, via a detailed calibration on some examples in the European market, the relevance of a parsimonious modeling approach and the relative importance of model parameters in liquidity spread. Third, it allows quantifying the liquidity impact in terms of prices for corporate debt of an insufficiently transparent market and it suggests some policy implications: this study highlights the importance of implementing a post-trade transparency in which the dissemination of information (trade time, volume, and price) should be extended to all corporate bonds. Having transparent market information on both liquid and less liquid bonds with similar characteristics would allow a complete quantification of the liquidity impact on corporate prices.

The remainder of this paper is organized as follows. In Section 2 we describe the model set-up and the liquidity problem formulation. In Section 3 we deduce the closed formula and in Section 4 we show how to calibrate the model parameters on real market data for two European bond issuers. In Section 5 we make some concluding remarks.
2 The model

The modeling framework includes two main sets of financial ingredients: we should i) outline the model set-up for corporate bonds, introducing the interest rate and the credit components, and ii) describe how illiquidity affects corporate bond prices. Then, we need to specify the interest rate and the credit dynamics in line with the parsimony requirement, as discussed in the Introduction: our aim is to model as parsimoniously as possible, due to lack of an abundance of accurate data sources. This section is divided into three parts. The first subsection is devoted to describing the model set-up for corporate bonds, while in the following we focus on the modeling of illiquidity. In the last subsection we specify the dynamics for interest rates and the credit spreads. In each subsection we present and discuss one of the main modeling assumptions.

2.1 The model set-up

We model interest rates and credits according to a zero-recovery model: default for an obligor \( C \) (the issuer of interest) is modeled via a Cox process \( N_t \) with intensity \( \lambda_t \), while the risk-free interest rate \( r_t \) follows a continuous dynamics. In this subsection we briefly describe the notation, we introduce the Defaultable HJM model via Assumption 1, we present the corporate coupon bond within this model and we define the simplest derivative contract: the defaultable forward ZC bond.

The notation is close to the one used in standard textbooks (see, e.g. Schönbucher 2003, Ch.6). We choose the value date \( t_0 = 0 \). We consider the background filtration \( (\mathcal{G}_t)_{t \geq 0} \) generated by a \( d \)-dimensional Brownian motion \( W_t \), with \( dW_t^{(j)} dW_t^{(l)} = \rho_{jl} dt \) for \( j, l = 1, \ldots, d \) and \( \rho \in \mathbb{R}^{d \times d} \) the instantaneous correlation matrix. The risk-free interest rate \( r_t \) and the intensity \( \lambda_t \) are processes adapted to \( (\mathcal{G}_t)_{t \geq 0} \). Default for an obligor \( C \) is modeled via a Cox process \( N_t \) with intensity \( \lambda_t \), i.e. conditional on the background filtration \( (\mathcal{G}_t)_{t \geq 0} \), \( N_t \) is an inhomogeneous Poisson process with intensity \( \lambda_t \). The quantity \( dN_t \) indicates the number of jumps between \( t^- \) and \( t \) it is equal to 1 if a jump occurs and zero otherwise. We define \( (\mathcal{F}^N_t)_{t \geq 0} \) to be the filtration generated by \( N_t \). The full filtration is obtained by combining this one and the background filtration (see, e.g. Schönbucher 2003):

\[
(\mathcal{F}_t)_{t \geq 0} = (\mathcal{G}_t)_{t \geq 0} \vee (\mathcal{F}^N_t)_{t \geq 0}.
\] (1)

Market practitioners model corporate bond spreads via Zeta-spreads: this is equivalent, from a modeling perspective, to considering zero recovery and to stating that the default probability models the whole credit risk for the obligor \( C \). In particular we consider a zero-recovery model as a limit case of a fractional recovery model. A defaultable ZC with Fractional Recovery (FR) between time \( t \) and maturity \( T \), \( B_q(t, T) \), is the price of a defaultable ZC where, if a default occurs at \( t_d \), the value of the defaultable asset is \( 1 - q \) times its pre-default value, with \( 0 < q < 1 \), i.e.

\[
B_q(t_d, T) = (1 - q)B_q(t_d^-, T).
\]

A defaultable ZC with zero recovery can be seen as a particular case of a ZC with FR when \( q \) tends to 1 from below (see, e.g. Schönbucher 2003). Often, it is simpler to use this modeling perspective for a generic \( q \) and then consider the limit for \( q \) close to 1. This is the approach we follow in this
Risk-free ZC $B(t_0, T)$ and defaultable ZC (with zero recovery) $\overline{B}(t_0, T) := B_{q=1^-}(t_d, T)$ are related to the default time $t_d$ and the stochastic discount $D(t_0, T) := \exp - \left( \int_{t_0}^{T} r_s ds \right)$ via

$$
\begin{aligned}
B(t_0, T) &:= \mathbb{E}[D(t_0, T)|\mathcal{F}_0] \\
\overline{B}(t_0, T) &:= \mathbb{E}[D(t_0, T)1_{t_d>T}|\mathcal{F}_0]. 
\end{aligned}
$$

The model we consider in this paper is a generalization of the model of [Heath et al. 1992] to the defaultable case (Schönbucher 1998, Duffie and Singleton 1999) and it is called the Defaultable HJM model (hereinafter, DHJM).

**Assumption 1**: (DHJM)

The following dynamics under the risk-neutral measure are assumed for the risk-free ZC and the defaultable ZC for every $t \in (0, T]$

$$
\begin{aligned}
\frac{dB(t, T)}{B(t, T)} &= r_t dt + \sigma(t, T) \cdot dW_t \\
\frac{d\overline{B}_q(t, T)}{\overline{B}_q(t^-, T)} &= (r_t + q\lambda_t) dt + \overline{\sigma}(t, T) \cdot dW_t - q dN_t
\end{aligned}
$$

with $B(t_0, T)$ and $\overline{B}_q(t_0, T)$ their initial conditions at value date $t_0$. The instantaneous rate $r_t$ and the intensity $\lambda_t$ satisfy

$$
\begin{aligned}
r_t &= -\frac{\partial \ln B(t_0, t)}{\partial t} + \frac{1}{2} \int_{t_0}^{t} \frac{\partial}{\partial t} \sigma(s, t)^2 ds - \int_{t_0}^{t} \frac{\partial}{\partial t} \sigma(s, t) \cdot dW_s \\
(r_t + q\lambda_t) &= -\frac{\partial \ln \overline{B}_q(t_0, t)}{\partial t} + \frac{1}{2} \int_{t_0}^{t} \frac{\partial}{\partial t} \overline{\sigma}(s, t)^2 ds - \int_{t_0}^{t} \frac{\partial}{\partial t} \overline{\sigma}(s, t) \cdot dW_s.
\end{aligned}
$$

where the volatilities $\sigma(t, T)$ and $\overline{\sigma}(t, T)$ are $d$-dimensional vectors of deterministic functions of time with $\sigma(T, T) = \overline{\sigma}(T, T) = 0 \in \mathbb{R}^d$. We indicate with $x \cdot y$ the scalar product between two vectors $x, y \in \mathbb{R}^d$ and with $x^2$ the scalar product $x \cdot px; x \in \mathbb{R}^d$.

DHJM focuses on the Zero recovery case $\overline{B}(t, T)$ with $q = 1^-$

In the DHJM, the main relation between the defaultable ZC and stochastic discount is

$$
\overline{B}(t_0, T) = \mathbb{E}[\overline{D}(t_0, T)|\mathcal{F}_0]
$$

where $\overline{D}(t_0, T) := \exp - \left( - \int_{t_0}^{T} (r_s + \lambda_s) ds \right)$ is called the defaultable stochastic discount.

With **Assumption 1** it is possible to model exactly the features of a default with Fractional Recovery. Considering a realization of the processes between $t_0$ and $t$ and using the Generalized Itô lemma (see, e.g. Appendix A), we get that the value of the defaultable ZC at a generic time $t$ starting from the initial condition is

$$
\overline{B}(t, T) = \overline{B}(t_0, T) (1 - q)^{\lambda_t} \exp \left\{ \int_{t_0}^{t} \left[ r_s + q\lambda_s - \frac{1}{2} \overline{\sigma}^2(s, T) \right] ds + \int_{t_0}^{t} \overline{\sigma}(s, T) \cdot dW_s \right\}.
$$
Thus, the default of \( B(t, T) \) occurs when the Poisson process jumps and the jump size is
\[
\Delta B(t, T) = B(t, T) - B(t^{-}, T) = -B(t^{-}, T) q dN_t
\]
i.e. the ZC loses a fraction \( q \) of its pre-default value; in particular the case with \( q = 1^- \) describes the Zero-recovery model we are interested in.

In this study we focus on fixed rate bonds that are not callable, puttable, or convertible. A corporate coupon bond of the obligor \( C \) at value date \( t_0 = 0 \) is
\[
\overline{P}(t_0, T; c, t) := \sum_{i=1}^{N} c_i \overline{B}(t_0, t_i) .
\]
(5)

In the definition of a corporate coupon bond (5), the price depends on the set of flows \( c := \{c_i\}_{i=1, \ldots, N} \) and the set of payment dates \( t := \{t_i\}_{i=1, \ldots, N} \). The \( i^{th} \) payment \( c_i \) at time \( t_i \) for \( i < N \) is the coupon payment with the corresponding daycount, while the last payment at \( t_N = T \) has the bond face value added to the coupon payment. A corporate coupon bond \( \overline{P} \) always indicates invoice (or dirty) prices, as in standard fixed income modeling.

We conclude this subsection introducing the simplest derivative contract. Let us define the forward defaultable ZC bond and its main properties in the DHJM.

**Definition 2.1.** The forward defaultable ZC bond at time \( t \) is a derivative contract with a reference obligor \( C \) and with three times \( t, \tau \) and \( T \) s.t. \( t \leq \tau \leq T \). This forward contract is characterized by the payment in \( \tau \) of
\[
\begin{cases}
\overline{B}(t; \tau, T) & \text{if the obligor } C \text{ has not defaulted up to time } \tau \\
0 & \text{otherwise}
\end{cases}
\]
in order to receive 1 in \( T \) if the obligor \( C \) has not defaulted up to time \( T \) (and zero otherwise). The amount \( \overline{B}(t; \tau, T) \), also called forward defaultable ZC bond price, is established in \( t \).

In Figure we show the flows that characterize a forward defaultable ZC bond.

**Lemma 2.2.** The forward defaultable ZC bond price is related to a defaultable ZC via
\[
\overline{B}(t; \tau, T) = \frac{\overline{B}(t, T)}{\overline{B}(t, \tau)} .
\]
(6)

**Proof.** See Appendix A.

From **Definition 2.1** of a forward defaultable ZC, we observe that \( \overline{B}(\tau; \tau, T) = \overline{B}(\tau, T) \), i.e. the forward defaultable bond price tends to the default bond price as time \( t \) tends to \( \tau \). We indicate with \( \overline{P}(t; \tau, T; c, t) \) the forward defaultable coupon bond corresponding to (5); clearly in the forward \( \overline{P}(t; \tau, T; c, t) \), only coupons with payment dates \( t_i > \tau \) appear.
Figure 1: We show the flows that characterize a forward defaultable ZC bond paid at $\tau$ if no default event occurs up to $\tau$. A forward defaultable ZC price is established at time $t$. The contract gives the right to receive 1 if no default event occurs up to $T$.

**Lemma 2.3.** In the DHJM, the dynamics for the forward defaultable ZC is

$$
\frac{d\bar{B}(t; \tau, T)}{\bar{B}(t^-; \tau, T)} = \frac{d\bar{B}(t; \tau, T)}{\bar{B}(t; \tau, T)} = \left[\sigma(t, T) - \sigma(t, \tau)\right] \cdot \left[dW_t + \rho \sigma(t, \tau) \, dt\right] \quad t \in [0, \tau]
$$

or equivalently

$$
\bar{B}(t; \tau, T) = \bar{B}(t_0; \tau, T) \exp\left\{ -\frac{1}{2} \int_{t_0}^t \left[\sigma^2(s, T) - \sigma^2(s, \tau)\right] \, ds + \int_{t_0}^t \left[\sigma(s, T) - \sigma(s, \tau)\right] \cdot dW_s\right\}.
$$

**Proof.** See Appendix A.

The above lemma states that the dynamics of the forward defaultable ZC bond price is continuous and it is, *mutatis mutandis*, the same as the corresponding dynamics for a risk-free ZC (see, e.g. Musiela and Rutkowski 2006).

**Lemma 2.4.** In the DHJM, the defaultable stochastic discount between $t$ and $\tau \geq t$ $\bar{D}(t, \tau)$ is related to the corresponding defaultable ZC via the relation

$$
\bar{D}(t, \tau) = \bar{B}(t, \tau) \exp\left\{ -\frac{1}{2} \int_t^\tau \sigma^2(s, \tau) \, ds + \int_t^\tau \sigma(s, \tau) \cdot dW_s\right\}.
$$

**Proof.** See Appendix A.

**Lemma 2.3** and **Lemma 2.4** correspond to the two standard properties of HJM models (see, e.g. Musiela and Rutkowski 2006); these properties also hold in the case of defaultable bonds described by *Assumption 1*. Hereinafter, we consider **Lemmas 2.3** and **2.4** only in the limit $q \uparrow 1$ that corresponds to the Zero-recovery model we are interested in.

A consequence of these two lemmas is that it is possible to introduce a $\tau$-defaultable-forward measure (hereinafter also $\tau$-forward measure), s.t. the process

$$
W_t^{(\tau)} := W_t + \int_{t_0}^t \rho \sigma(s, \tau) \, ds
$$
is a $d$-dimensional Brownian motion under the new forward measure. We indicate with $E^{\tau}[\cdot]$ the expectation under the $\tau$-forward measure. In the $\tau$-forward measure, $B(t; \tau, T)$ is a martingale, and the dynamics for the forward defaultable ZC has a particularly simple form:

$$d\overline{B}(t; \tau, T) = \overline{B}(t; \tau, T) v(t; \tau, T) \cdot dW^\tau_t$$

with $v(t; \tau, T) := \sigma(t, T) - \sigma(t, \tau)$.

In the next subsection we describe the modeling framework on how illiquidity affects corporate coupon bonds.

2.2 Problem formulation

Let us consider a hypothetical investor who holds, at value date $t_0 = 0$, an illiquid corporate bond, i.e. he needs some time in order to liquidate a position with a given size of that bond. We assume that this investor is an experienced trader with a better information than other market players on that particular corporate market segment at value date $t_0$: he is able to sell the position in the illiquid bond after a time-to-liquidate $\tau$ at the same price as a liquid bond with the same characteristics (issuer, coupons, payment dates); we assume that this experienced trader knows all the features of the bonds of that issuer and all the potential clients that could be interested in buying the bond he holds.

After the seminal paper of Kyle [1985] the assumption that some market players are better informed than the other agents is rather common when analyzing from a theoretical perspective specific trading mechanisms and the price formation process of some assets. In particular, this liquidity problem is reminiscent of the celebrated work of Longstaff [1995], which compares with an option approach the value at $t_0$ of an illiquid security with that of a liquid one with equal future cash flows after $\tau$; as an example, Longstaff [1995] focuses attention on a non-dividend paying stock in IPOs. An additional feature characterizes the hypothetical investor in Longstaff [1995]: he is able to sell the liquid security “with perfect timing” during $[t_0, \tau]$. Also in this paper the experienced trader holds a perfect market timing ability: he is able to sell the liquid asset “when the price of the security reaches its maximum” (cf. Longstaff [1995], p.1768) if the issuer does not default before the time-to-liquidate $\tau$ and to sell it at value date otherwise. The additional value of the liquid security over the illiquid one is calculated by regarding the optimal strategy of this hypothetical investor.

More recently, Longstaff’s results have been extended by Koziol and Sauerbier [2007], who tackle a similar problem in the case of a risk free zero coupon bond with maturity $T$. The authors consider the illiquidity premium in the case of a risk-free short rate which follows a Vasicek [1977] model; the ZC is modeled as a geometric Brownian motion (GBM) process, similarly to the stocks in Longstaff [1995]. The ZC can be traded at a given set of dates, established at value date $t_0$; the illiquidity price in their study is obtained via a (numerically intense) Monte Carlo technique. Unfortunately, as already mentioned in the Introduction, the approach of Koziol and Sauerbier [2007] can not be extended easily to the case of interest. In this paper we consider a coupon bearing bond in the presence of credit risk: we obtain illiquid bond prices via a simple closed formula.

Hence, when dealing with fixed income securities, we have to consider that bonds pay coupons and that they can default. Two are the cases of interest: either the corporate issuer defaults after the time-to-liquidate $\tau$ or it defaults before $\tau$. 
In the former case, in order to compare two assets with the same future cash flows at $\tau$ it is better to consider the corresponding forward security introduced in previous subsection. The selling price for the experienced trader, able to sell the forward coupon bond with optimal timing, is

$$M_\tau := \max_{t_0 \leq t \leq \tau} P(t; \tau, T; c, t) ;$$

this price is paid at time $\tau$ as in the forward defaultable bond case.

In the latter case, the illiquid position, due to the zero recovery, has a value equal to zero in $\tau$ (and then also in the value date); the liquid bond is sold immediately at its price at value date $P(t_0; c, t)$. As we show in the numerical example in section 4, the contribution of this term is smaller than the others for some orders of magnitude. Let us recall that a default event in such a narrow time interval is very rare, since i) the time-to-liquidate is few months in the most illiquid cases and some weeks more generally; and ii) the typical corporate issuer we are considering in this study, with both liquid and illiquid issues, is investment grade.

Summing up the two possibilities for the liquid and illiquid bonds are

|         | liquid | illiquid |
|---------|--------|----------|
| $t_d \leq \tau$ | $P(t_0; c, t)$ | 0 | paid in $t_0$ |
| $t_d > \tau$     | $M_\tau$   | $P(\tau; c, t)$ | paid in $\tau$ |

Longstaff’s idea is very intuitive: the main limitation of holding an illiquid bond, compared with a comparable issue of the same corporate entity, is related to the impossibility for a while to sell the bond and convert its value into cash. The time-to-liquidate is the main exogenous model parameter: it models the liquidity restriction as an opportunity cost for this hypothetical investor.

**Assumption 2:** (Illiquidity price).

The illiquidity price $\Delta_\tau$ is defined as the value at $t_0$ of the difference between liquid and illiquid prices. Its present value equals

$$\Delta_\tau := \mathbb{E} \left[ D(t_0, \tau) \mathbb{I}_{t_d > \tau} (M_\tau - P(\tau; T; c, t)) + \mathbb{I}_{t_d \leq \tau} P(t_0; c, t) | \mathcal{F}_0 \right] \diamond \quad (10)$$

Thus, according to the Longstaff criteria, the illiquidity price is defined as the difference, for the hypothetical investor, of the selling prices of the liquid and illiquid forward asset prices plus a correction in the (rare) event of issuer default before the time-to-liquidate.

**Remark 2.5.** The above definition does not consider the case when coupon payments take place between the value date $t_0$ and $\tau$. We have already underlined that the time-to-liquidate is, even in the most illiquid cases, a few months, and then at most one coupon payment could be present in the time interval $(t_0, \tau)$. The first coupon, when paid before $\tau$, can be separated by the other flows in the coupon bond; a technique known in the market place as coupon stripping. In practice, corporate bond traders consider that payment, i.e. within a short lag in the future, very liquid. We assume that this coupon makes the same contribution to both the liquid and illiquid coupon bonds, i.e. maintains only its interest rate and credit risk components; thus, this coupon does not appear in the illiquidity price $\Delta_\tau$ in (10). Hereinafter, we consider in the corporate coupon bond only the coupons after the time-to-liquidate, i.e. in definition (5) the first coupon in the sum is the first one paid after $\tau$. 
Lemma 2.6. Within the DHJM model of Assumption 1, the price of illiquidity is equal to

\[
\Delta_\tau = \mathbb{E} \left[ \overline{D}(t_0, \tau) M_{\mathcal{G}_0} \right] - \mathbb{E} \left[ \overline{D}(t_0, \tau) \overline{P}(\tau; T; c, t) | \mathcal{G}_0 \right] + (1 - \mathcal{P}(t_0, \tau)) \overline{P}(t_0; c, t) = \\
= \overline{B}(t_0, \tau) \left\{ \mathbb{E}^{(\tau)} [M_{\mathcal{G}_0}] - \mathbb{E}^{(\tau)} [\overline{P}(\tau; c, t) | \mathcal{G}_0] \right\} + (1 - \mathcal{P}(t_0, \tau)) \overline{P}(t_0; c, t) \quad .
\]

(11)

where \( \mathcal{P}(t_0, \tau) \) is the issuer survival probability up to the time-to-liquidate.

Proof. See Appendix A.

Hence, we have shown that illiquidity price \( \Delta_\tau \) depends on a sum of relatively simple terms apart from \( \mathbb{E}^{(\tau)} [M_{\mathcal{G}_0}] \). Even this term can be computed with a closed formula, once we select a model within the class of DHJM, as done in the Assumption 3 discussed in the next subsection.

2.3 A parsimonious model selection

Let us observe that the quantity \( \mathbb{E}^{(\tau)} [M_{\mathcal{G}_0}] \) in the price of illiquidity \( \Delta_\tau \) (11) does not depend separately on \( r_t \) and \( \lambda_t \), but depends only on their combination

\[
\bar{\tau}_t := r_t + \lambda_t .
\]

This property holds for whichever DHJM model is selected (i.e. whatever \( \sigma(t, T) \) and \( \overline{\sigma}(t, T) \) are chosen) for the dynamics (3) of the risk-free ZC curve \( B(t, T) \) and the defaultable ZC curve \( \overline{B}(t, T) \).

As discussed in the Introduction, the main driver for model selection is parsimony when dealing with illiquid corporate bonds, due to the poorness of the data set and model calibration issues. One of the simplest models within this set was proposed by Schönbucher (2000), where both \( r_t \) and \( \lambda_t \) follow two correlated 1-dimensional Hull and White (1990) models

\[
\begin{aligned}
\begin{cases}
    r_t &= \varphi_t + x^{(1)}_t \\
    \lambda_t &= \psi_t + x^{(2)}_t
\end{cases}
\end{aligned}
\]

where \( x^{(1)}_t \) and \( x^{(2)}_t \) are two correlated Ornstein–Uhlenbeck (OU) processes with zero mean and zero initial value; \( \varphi_t \) and \( \psi_t \) are two deterministic functions of time. This model has the main advantage of allowing an elementary separate calibration of the zero-rates (via \( \varphi_t \)) and the Zeta-spread (via \( \psi_t \)).

A consequence of the observation that only the dynamics for \( \tau_t \) matters for liquidity, makes us consider an even simpler model with only one OU driver, as stated in the following assumption.

Assumption 3: (Parsimonious dynamics).

We model the risk-free interest rate \( r_t \) and the intensity \( \lambda_t \) as

\[
\begin{aligned}
\begin{cases}
    r_t &= \varphi_t + (1 - \hat{\gamma}) x_t \\
    \lambda_t &= \psi_t + \hat{\gamma} x_t
\end{cases}
\end{aligned}
\]

(12)

with \( \varphi_t, \psi_t \) two deterministic functions of time and \( x_t \) an Ornstein–Uhlenbeck (OU) process with zero mean and initial value

\[
\begin{aligned}
\begin{cases}
    dx_t &= -\hat{\alpha} x_t dt + \hat{\sigma} dW_t \\
    x_0 &= 0
\end{cases}
\end{aligned}
\]

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where $\hat{\gamma} \in [0, 1]$, while $\hat{a}, \hat{\sigma}$ are two positive constant parameters.

This assumption is in line with day-to-day practice. In the marketplace, often one cannot observe derivative instruments that allow calibrating separately the volatility of the risk-free curve and the volatility of the credit spread, and there is not enough information to discriminate between the two dynamics. One can associate a fraction $\hat{\gamma}$ of the total dynamics to the credit component and the remaining fraction to the interest rate component. Conversely, the two initial curves (risk-free zero-rate and Zeta-spread) can be easily calibrated separately on market data and the integrals of $\varphi_t, \psi_t$ between $t_0$ and a given maturity $T$ are related to these two curves up to $T$. We do not report these relations here because we provide final formulas in terms of $B(t_0, T)$ and $\overline{B}(t_0, T)$, and both curves can be calibrated directly from market data.

A consequence of Assumption 3 is that the rate $\bar{r}_t$

$$\bar{r}_t = \varphi_t + \psi_t + x_t$$

is modeled according to a Hull–White (HW) model with volatility (see, e.g., Brigo and Mercurio [2007])

$$\sigma(t, T) = \frac{\hat{\sigma}}{\hat{a}} \left(1 - e^{-\hat{a}(T-t)}\right) \in \mathbb{R} \quad t \leq T$$

and then the volatility $v(t; \tau, t_i) = \bar{\sigma}(t, t_i) - \bar{\sigma}(t, \tau)$, defined in equation (9), is a separable function in the times $t$ and $t_i$, i.e.,

$$v(t; \tau, t_i) = \zeta_i \nu(t)$$

with $\zeta_i := (\hat{\sigma}/\hat{a}) \left[1 - e^{-\hat{a}(t_i - \tau)}\right]$ and $\nu(t) := e^{-\hat{a}(\tau - t)}$ with $t_0 \leq t \leq \tau \leq t_i$.

In the next section we show that, under Assumptions 1, 2 and 3, it is possible to compute the price of illiquidity $\Delta_\tau$ via a closed formula and it is possible to associate a liquidity spread as a component of the corporate bond spread in addition to the credit spread.

3 A closed formula for illiquid corporate coupon bonds

In this section we present the main result of this paper: the illiquidity price $\Delta_\tau$ (10) can be evaluated directly via a simple closed formula obtained using valuation techniques from option-pricing theory. This result is far from being obvious. A defaultable forward coupon bond $\overline{P}(t; \tau, T; c, t)$ is the sum of forward defaultable ZCs $\{\overline{B}(t; \tau, t_i)\}_{i=1,\ldots,N}$, each one following the dynamics (7) and then described as a GBM (8). No known closed formula exists for the running maximum of a sum of GBMs. In order to get the closed formula, we take the following steps. First we prove that a lower and an upper bound of (10) can be computed via closed formulas. Then we show, calibrating the model parameters for two European issuers, that the difference between the upper and lower bounds is negligible for all practical purposes. We can then use one of the two bounds as the closed-form solution we are looking for; in this section we prove the existence of these bounds and in the next section we show the tightness of their difference.
Lemma 3.1. The following inequalities hold:

\[ \sum_{i} c_i B(t^*; \tau, t_i) \leq \max_{t \in [t_0, \tau]} \left\{ \sum_{i} c_i B(t; \tau, t_i) \right\} \leq \sum_{i} c_i \max_{t \in [t_0, \tau]} B(t; \tau, t_i) \quad \forall t^* \in [t_0, \tau], \ t_i \geq \tau \]

where the sum over \( i \) is limited to all coupons with payment date \( t_i > \tau \) larger than \( \tau \).

Proof. The left inequality is obvious since the maximum value of a function on the time interval \([t_0, \tau]\) is greater than the same function’s values at any other time \( t^* \) in the interval. The right inequality is due to the fact that the maximum of a sum is less than or equal to the sum of the maxima ♣

In particular we can choose \( t^* \) equal to the time-location

\[ t^* = \min \left\{ t' \mid B(t'; \tau, t_N) = \max_{t \in [t_0, \tau]} B(t; \tau, t_N) \right\} , \quad (15) \]

i.e. equal to the (first) time when the last forward ZC, \( B(t; \tau, t_N) \), reaches its maximum in the interval \([t_0, \tau]\).

In the next theorem we prove that the expected values of these lower and upper bounds have a simple form; in Section 4 we show that they can be considered equal for all practical purposes.

Theorem 3.2. Lower and upper bounds for the illiquidity price \( \Delta_\tau \) \( (10) \) are:

\[ \sum_{i=1}^{N} c_i B(t_0, t_i) (\pi^u_i(\tau) - \mathcal{P}(t_0, \tau)) \leq \Delta_\tau \leq \sum_{i=1}^{N} c_i B(t_0, t_i) (\pi^l_i(\tau) - \mathcal{P}(t_0, \tau)) \]

where the sum is limited to the payment dates \( t_i \) and

\[ \pi^u_i(\tau) := \frac{4 + \Sigma^2_i(\tau)}{2} \Phi \left( \frac{\Sigma_i(\tau)}{\sqrt{2}} \right) + \frac{\Sigma_i(\tau)}{\sqrt{2}} \exp \left( -\frac{\Sigma^2_i(\tau)}{8} \right) \]

\[ \pi^l_i(\tau) := \int_{0}^{1} d\eta \frac{e^{-\frac{1}{8} \Sigma^2_i(\tau)}}{\pi \sqrt{1-\eta} \sqrt{\eta}} e^{-\frac{\eta}{2} \Sigma_i(\tau)} \Phi \left[ \frac{\sqrt{1-\eta}}{2} \Sigma_i(\tau) \right] \]

\[ \left\{ 1 + \sqrt{\frac{\pi (1-\eta)}{2}} \Sigma_i(\tau) e^{\frac{1-\eta}{8} \Sigma^2_i(\tau)} \right\} \]

\[ \left\{ 1 + \sqrt{\frac{\pi \eta}{2}} (2 \Sigma_i(\tau) - \Sigma_i(\tau)) e^{\frac{\eta}{8} (2 \Sigma_i(\tau) - \Sigma_i(\tau))^2} \Phi \left[ \frac{\sqrt{\eta}}{2} (2 \Sigma_i(\tau) - \Sigma_i(\tau)) \right] \right\} \quad (16) \]

The cumulated volatility is

\[ \Sigma^2_i(\tau) := \int_{t_0}^{\tau} \sigma^2(s; \tau, t_i) \, ds = \zeta^2 \frac{1 - e^{-2a(\tau-t_0)}}{2a} \]

where \( \zeta \) is defined in equation \( \{14\} \), \( \Phi[\bullet] \) is the standard normal CDF and the issuer survival probability up to the time-to-liquidate is

\[ \mathcal{P}(t_0, \tau) = \exp \left\{ -\int_{t_0}^{\tau} \psi_s \, ds - \frac{1}{2} \zeta^2 \int_{t_0}^{\tau} \sigma^2(s, \tau) \, ds \right\} , \quad (17) \]

with \( \psi_s \) the deterministic part of the intensity introduced in \( \{12\} \) and \( \sigma(s, \tau) \) is defined in \( \{13\} \).
This theorem is the key result of this paper: it indicates a lower and an upper bound for the price of illiquidity $\Delta_r$. In Section 4 we show for two issuers that these bounds appear to be very tight, being on the order of $10^{-6}$ the face value in the worst case: their difference can be considered negligible for all practical purposes.

Let us observe that, in the above expression, the stochastic part is exactly the same $\forall i$ and differs only for the $\zeta_i$ term in the deterministic drift part, where $\zeta_i$ is, once one considers parameters calibrated with market data, smaller than all other terms and does not change significantly for the majority of the coupon bond payment dates. Due to these two arguments, the time-location of the maximum is exactly the same for all forward ZCs and equal to $t^*$ in (15) in most scenarios.

In practice, either of the two closed form solutions can be used indifferently, and in particular, the simplest expression of the two bounds, i.e. the upper bound. This fact allows defining, in an elementary way, a liquidity basis as done in the next subsection.

### 3.1 The liquidity basis

A consequence of the above theorem and of the tightness of the difference between the two bounds is that the illiquid corporate coupon price is

$$ P_{\tau}(t_0, T; c, t) := P(t_0, T; c, t) - \Delta_r = \sum_{i=1}^{N} c_i \bar{B}(t_0, t_i) \left( 1 + P(t_0, \tau) - \pi_i^U(\tau) \right) $$ (18)

where $\pi_i^U(\tau)$ is defined in (16) and $P(t_0, \tau)$ can be found in (17). We can also define an illiquid ZC as

$$ \bar{B}_{\tau}(t_0, t_i) := B(t_0, t_i) \left( 1 + P(t_0, \tau) - \pi_i^U(\tau) \right) $$

and the liquidity basis

$$ L_{\tau}(t_i) := -\frac{1}{t_i - t_0} \ln \left( \frac{\bar{B}_{\tau}(t_0, t_i)}{\bar{B}(t_0, t_i)} \right) = -\frac{1}{t_i - t_0} \ln \left( 1 + \frac{\pi_i^U(\tau)}{P(t_0, \tau)} \right) . $$ (19)

Thus, we can decompose the illiquid ZC bond into three components: risk-free discount, credit, and liquidity:

$$ \bar{B}_{\tau}(t_0, T) = e^{-R(T-t_0)} e^{-Z(T)(T-t_0)} e^{-L_{\tau}(T)(T-t_0)} $$
where $R(T)$ is the Zero rate and $Z(T)$ is the Zeta spread. This corresponds to what is done by practitioners in their day-to-day activities: they add a basis related to liquidity to the bond credit spread. The main advantage of the model presented in this study is that, given the ttl for the illiquid corporate bond position of interest, it allows associating a liquidity basis to some bond characteristics (e.g. coupon payment dates) and to the volatility of the corresponding liquid bond, via the two parameters $\hat{a}$ and $\hat{\sigma}$.

It is useful to underline that the liquidity basis depends mainly on the volatility parameters, and it is only slightly affected by the credit component (but not by the rates component). The liquidity component in the ZC price $\left(1 + \mathcal{P}(t_0, \tau) - \pi_i^U(\tau)\right)$ is mainly a function of the cumulated volatility $\Sigma_i(\tau)$.

4 An application to the financial sector in the European bond market

In this section we illustrate the impact of illiquidity applying formula (18) to obligations with different maturities issued by two of the main financial institutions in Europe. We also show that the difference between the upper and lower bounds of the illiquidity price is negligible for all practical purposes.

4.1 Calibration of model parameters

The two European financial institutions in Europe that we consider in this study are BNP Paribas S.A. (hereinafter BNPP) and Banco Santander S.A. (Santander) on 10 September 2015 (value date). The settlement date is 14 September 2015. At value date, BNPP was rated A and Santander A- according to S&P.

| maturity    | coupon (%) | clean price |
|-------------|------------|-------------|
| 27-Nov-2017 | 2.875      | 105.575     |
| 12-Mar-2018 | 1.500      | 102.768     |
| 21-Nov-2018 | 1.375      | 102.555     |
| 28-Jan-2019 | 2.000      | 104.536     |
| 23-Aug-2019 | 2.500      | 106.927     |
| 13-Jan-2021 | 2.250      | 106.083     |
| 24-Oct-2022 | 2.875      | 110.281     |
| 20-May-2024 | 2.375      | 106.007     |

Table 1: Clean prices for BNPP liquid bonds. Senior unsecured benchmark issues with maturity less than or equal to 10 years. Coupons are annual with day-count convention Act/Act. Prices are end-of-day mid-prices on 10 September 2015.

7The settlement date is equal to two business days after the value date for both the interest rate and credit products in the Euro-zone.
As discussed in Section 3, the closed formula for illiquid bond prices, besides the bond characteristics (maturity, payment dates, coupons, sinking features, time-of-liquidate, etc...), includes the observed i) zero-rate curve, ii) credit spread term-structure for the issuer of interest, and iii) bond volatility. These “ingredients” can be calibrated with the market data following standard techniques.

First, the risk-free curve we consider is the OIS curve as the market standard; it has been bootstrapped from OIS quoted rates. Quotes at value date are provided by Bloomberg. The discount curve \( B(t_0, T) \) is bootstrapped following the standard procedure; OIS rates and discount factors are reported in [Baviera (2017)]

Second, in order to construct the Zeta-spread curve, i.e. the (liquid) credit component in the spread, we consider all senior unsecured benchmark issues (i.e. with issue size larger than €500 million) with maturity less than or equal to 10 years. Coupons are paid annually with the Act/Act day-count convention for all bonds in both sets. The closing day mid-prices are reported in Tables 1 and 2.

For each one of the two issuers, its time-dependent Zeta-spread curve

\[
Z(T) := -\frac{1}{T-t_0} \ln \frac{B(t_0, T)}{B(t_0, T)}
\]

can be bootstrapped from liquid bond invoice prices (see, e.g. [Schönbucher 2003]). Invoice prices are obtained adding the accrual to the clean price. We assume a constant Zeta-spread curve up to the maturity of the bond with the lowest maturity and we use a linear interpolation rule on Zeta-spread afterwards; the day-count convention for Zeta-spreads is Act/365, as the market standard.

Finally, the volatility parameters (\( \hat{a} \), \( \hat{\sigma} \) and \( \hat{\gamma} \)) should be calibrated on options on corporate bonds. Unfortunately, prices on liquid options on BNPP and Santander bonds are not available in the market at value date. We consider a proxy in order to calibrate the volatility parameters; we notice that at value date both banks are Systemically Important Financial Institutions (SIFI) and belong to the panel of banks contributing to the Euribor rate. The dynamics of the spread between the Euribor and the OIS curve can be considered a good proxy of the dynamics of the average credit spread for

| maturity       | coupon (%) | clean price |
|----------------|------------|-------------|
| 27-Mar-2017    | 4.000      | 105.372     |
| 04-Oct-2017    | 4.125      | 107.358     |
| 15-Jan-2018    | 1.750      | 102.766     |
| 20-Apr-2018    | 0.625      | 99.885      |
| 14-Jan-2019    | 2.000      | 103.984     |
| 13-Jan-2020    | 0.875      | 99.500      |
| 24-Jan-2020    | 4.000      | 112.836     |
| 14-Jan-2022    | 1.125      | 98.166      |
| 10-Mar-2025    | 1.125      | 93.261      |

Table 2: Clean prices for Santander senior unsecured benchmark issues with maturity less than or equal to 10 years. Coupons are annual with day-count convention Act/Act. Prices are end-of-day mid-prices at value date.
financial institutions with the above characteristics. As mentioned in Grbac and Runggaldier (2015), this spread models the risk related to the Euro interbank market, and default risk is one important component of this interbank risk. Let us underline that we use this proxy to calibrate only volatility parameters, while credit spreads are calibrated on issuer liquid bond market.

ATM swaptions on Euribor swap rates are very liquid in Europe: we can use these OTC option contracts at $t_0$ as a proxy, in order to calibrate the volatility parameters. Swaption ATM normal volatilities are provided by Bloomberg; their values in $t_0$ and the calibration procedure are reported in Baviera (2017). Calibrated values are $\hat{a} = 12.94\%$, $\hat{\sigma} = 1.26\%$ and $\hat{\gamma} = 0.04\%$.

In the two cases analyzed, as already mentioned in Section 2, the correction to include the default risk up to ttl is small. All survival probabilities $\mathcal{P}(t_0, \tau)$ are close to 1: we report in Table 3 the default probabilities $1 - \mathcal{P}(t_0, \tau)$ in the time interval of interest. All values are of order $10^{-4}$.

|       | BNPP       | Santander  |
|-------|------------|------------|
| 2w    | $1.27 \times 10^4$ | $1.80 \times 10^4$ |
| 2m    | $5.42 \times 10^4$  | $7.73 \times 10^4$  |

Table 3: Default probabilities $1 - \mathcal{P}(t_0, \tau)$ for BNPP and Santander for the two ttl of 2 weeks and 2 months.

The correction due to the default risk up to ttl is negligible in the liquidity spread: this fact justifies the decomposition of the bond spread in the three components risk-free, credit and liquidity proposed in Section 3.1.

### 4.2 Illiquid bond prices

In this section we show that, considering two sets of illiquid bonds with the same characteristics as the liquid bonds (e.g. coupons and payments dates) and ttl equal to either two weeks or two months, the difference between the lower and upper bounds for the illiquidity price $\Delta_\tau$ is on the order of $10^{-8}$ times the face value. Figure 2 presents this difference for BNPP, and Figure 3 for Santander. Moreover, we consider the bond with longest maturity within the Santander set (i.e. the one with the largest difference) and in Figure 4 we plot the difference between the two bounds with ttl equal to two months for a wide range of volatility parameters around the estimated values: $\hat{a} \in (0, 30\%)$ and $\hat{\sigma} \in (0, 4\%)$. We observe that this difference is, in the worst case, of less than 1 Euro for every million of face value. This difference is the maximum error we make if we evaluate the illiquidity price $\Delta_\tau$ with one of these bounds. It is negligible for all practical purposes. This fact allows us to consider indifferently either the lower or the upper bound as a closed-form solution for $\Delta_\tau$.

In Section 3 we have shown that a liquidity basis (19) could be added to each ZC in order to take into account liquidity. Practitioners often consider a liquidity yield spread as the term that should be added to the yield in order to obtain the illiquid bond price (18)

$$\overline{P}_\tau(t_0, T; c, t) =: \sum_{i=1}^{N} c_i e^{-[\mathcal{Y}(T)+\mathcal{L}_\tau(T)](t_i-t_0)}$$

where $\mathcal{Y}(T)$ is the yield of the corresponding liquid bond $\overline{P}(t_0, T; c, t)$ and $\mathcal{L}_\tau(T)$ the liquidity yield spread for ttl equal to $\tau$. 

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Figure 2: Difference between the upper and lower bounds for the illiquidity price $\Delta_\tau$ for BNPP bonds. We consider illiquid bonds with the same characteristics (e.g. coupons, payment dates) as the bonds in Table 1 with ttl equal to two weeks (continuous blue line and squares) and two months (dashed red line and triangles). This difference is on the order of $10^{-8}$ times the face value in the worst-case, and so it is negligible for all practical purposes.

Figure 3: Difference between the upper and lower bounds for the illiquidity price $\Delta_\tau$ for Santander bonds. We consider illiquid bonds with the same characteristics (e.g. coupons, payment dates) as the bonds in Table 2 with ttl equal to two weeks (continuous blue line and squares) and two months (dashed red line and triangles). This difference is on the order of $10^{-9}$ times the face value.
Figure 4: Difference between the upper and lower bounds for the illiquidity price $\Delta r$ for Santander bond with the longest maturity in Table 2 varying $\hat{a} \in (0\%, 30\%)$ and $\hat{\sigma} \in (0\%, 4\%)$. We have considered a large range of possible values of the two key volatility parameters around the estimated values of $\hat{a} = 12.94\%$ and $\hat{\sigma} = 1.26\%$. We consider illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to two months.

In Figures 5 and 6 we show the liquidity yield spread for BNPP and Santander for different bond maturities and ttl equal to two weeks and two months.

Lastly, it is useful to observe that the liquidity spreads obtained with the technique described in this paper are of the same order of magnitude of those observed in econometric studies in the U.S. market for bonds of similar maturity (2y up to 10y) and similar ratings. For example, Dick-Nielsen et al. report post-subprime crisis liquidity spreads in the U.S. market between 24.7 basis points (bps) and 105.4 bps for A rated issuers and between 55.0 bps and 175.1 bps for BBB (see, e.g. Dick-Nielsen et al. [2012], table B2, panel B).

5 Conclusions

In this paper we have proposed a closed formula (18) for illiquid corporate coupon bonds when the corresponding liquid credit curve can be observed in the market for the same issuer. This formula is obtained by bounding from above and below the illiquidity price (10). Calibrating the model parameters with market data, we have shown that these two bounds coincide for all practical purposes.

This formula clarifies that illiquidity is an intrinsic component of the bond spread, and hence of the bond price. In the presence of a liquid credit curve, it is possible to disentangle the two components, credit and liquidity, in the observed spread over the risk free rate. In particular, we have shown
Figure 5: BNPP bond yields. We consider all benchmark issues with maturity less than 10y described in Table 1 and their yields (continuous blue line and squares). We show also the yield obtained for illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to two weeks (dashed red line and triangles) and two months (dotted green line and circles).

Figure 6: Santander bond yields. We consider all benchmark issues with maturity lower than 10y in Table 2 and their yields (continuous blue line and squares). We show also the yield obtained for illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to two weeks (dashed red line and triangles) and ttl equal to two months (dotted green line and circles).
that the liquidity spread depends mainly on the bond volatility and on the time-to-liquidate a given position (via a cumulated volatility).

This closed formula \( (18) \) is very simple. Besides a set of parameters that can be easily calibrated from liquid market data, the model includes just one additional parameter: the “time-to-liquidate”. It can be used by practitioners for different possible applications; let us mention some of them. This model can support traders in their day-to-day activities. On the one hand, the \( \text{ttl} \) parameter can be evaluated \textit{ex ante} by an experienced trader with a deep knowledge of the characteristics of that particular illiquid market (concentration, frequency for similar trades with similar characteristics observed in the recent past) who desires to liquidate a given position; the formula gives a theoretical background for the market practice of adding a liquidity spread to the bond yields either when pricing illiquid issues or when receiving them as collateral. On the other hand, the formula can also be used to get an “implied time-to-liquidate” from market quotes if both liquid and illiquid prices are available, translating observable spreads into a time lag for liquidating a position and hence providing an interesting piece of information for market participants.

Moreover, the model can also be useful to risk managers and regulators. The \( \text{ttl} \) can be easily backtested \textit{ex post} by risk managers, who, thanks also to the transaction data made recently available, can measure the average time needed for liquidating a position in an illiquid corporate bond of a given size. By offering an explicit relationship between the market volatility and the liquidity premium, it also gives a theoretical background for setting haircuts for illiquid bonds accepted as collateral in dependence from the market volatility and offers to risk managers a way to set Vega limits on illiquid bonds positions: the proposed approach clarifies that the cumulated volatility is the key driver of the liquidity basis.

Lastly, this study also allows drawing some policy implications for the European bond market, in which public reporting of trade data just started in 2018 and is currently limited to summary information at single issue (ISIN) level. TRACE reports, for a selected set of bonds, information about the trades executed (trade time, volume, and price) in the US market but it does not reveal quotes. We show how the time-to-liquidate is linked by a simple, but not simplistic, model to the price of liquidity and, in the bond market, the time-to-liquidate is the natural quantity to be estimated from the traded volumes per unit time. We highlight therefore the relevance of public reporting of trade volumes per unit time, since it could reduce significantly the opaqueness of the market and hence the total cost of debt for corporate issuers.

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## 6 Notation and Shorthands

| Symbol          | Description                                                                 |
|-----------------|----------------------------------------------------------------------------|
| $\bar{a}, \bar{\sigma}, \bar{\gamma}$ | parameters in short rate $r_t$ and intensity $\lambda_t$ dynamics (12) |
| $B(t,T)$        | risk-free zero-coupon (ZC) bond at $t$ with maturity $T$                    |
| $\bar{B}(t,T)$  | defaulatable ZC bond at $t$ with maturity $T$ and zero recovery             |
| $\bar{B}(t;\tau,T)$ | illiquid defaulatable ZC bond at $t_0$ with zero recovery and ttl equal to $\tau$ |
| $\bar{C}(t;\tau,T)$ | forward ZC bond                                                               |
| $c = \{c_i\}_{i=1,...,N}$ | defaulatable coupon bond flows (coupons and face value)                     |
| $\bar{P}(t_0,T;c,t)$ | defaulatable coupon bond at $t_0$ with maturity $T$                        |
| $\bar{P}(t_0;0,T;c,t)$ | illiquid defaulatable coupon bond at $t_0 = 0$ with maturity $T$ and ttl equal to $\tau$ |
| $\bar{P}(t;\tau,T;c,t)$ | forward defaulatable coupon bond at $t$, paid at $\tau$, with maturity $T$ |
| $\bar{P}(t_0,\tau)$ | issuer survival probability up to the time-to-liquidate                     |
| $\Delta_{\tau}$ | price of illiquidity with a ttl equal to $\tau$                            |
| $D(t,T)$        | stochastic discount factor, equal to $\exp\left(-\int_t^T r_s \, ds\right)$ |
| $\bar{D}(t,T)$  | defaulatable stochastic discount factor, equal to $\exp\left(-\int_t^T \tau_s \, ds\right)$ |
| $\mathbb{E}[\bullet]$ & $\mathbb{E}^{\tau}[\bullet]$ | expectation under the risk neutral & under the $\tau$-defaulatable forward measure |
| $\mathcal{F}_t$ | full filtration                                                               |
| $\mathcal{F}_N$ | filtration generated by $\mathcal{N}_t$                                     |
| $\mathcal{G}_t$ | background filtration                                                         |
| $q$             | Loss-Given-Default in a FR model; $q \to 1^-$ reproduces the zero-recovery model |
| $L_{\tau}(t_i)$ | Liquidity basis for a ZC with maturity $t_i$                                |
| $\mathcal{L}_{\tau}(T)$ | Liquidity yield spread for a coupon bond with maturity $T$ and ttl $\tau$ |
| $N$             | number of defaulatable bond coupons                                           |
| $\mathcal{N}_t$ | jump process that models default of the corporate issuer                     |
| $\lambda_t$     | stochastic intensity at time $t$                                             |
| $r_t$           | stochastic short rate at time $t$                                            |
| $\bar{r}_t$     | defaulatable short rate at time $t$, defined as $r_t + \lambda_t$           |
| $\rho$          | instantaneous correlation matrix in $\mathbb{R}^{d \times d}$ s.t. $dW_t^{(i)} dW_t^{(j)} = \rho_{ij} \, dt$ |
| $\sigma(t,T)$   | HJM risk-free ZC volatility between $t$ and $T$ in $\mathbb{R}^d$           |
| $\bar{\sigma}(t,T)$ | HJM defaulatable ZC volatility between $t$ and $T$ in $\mathbb{R}^d$       |
| $\Sigma_i(\tau)$ | cumulated volatility s.t. $\Sigma_i(\tau) := \int_{t_0}^\tau v^2(s;\tau,t_i) \, ds$ |
| $t_0$           | value date                                                                   |
| $t_d$           | default time                                                                  |
| $\tau$          | time-to-liquidate (ttl)                                                      |
| $t = \{t_i\}_{i=1,...,N}$ | payment dates of the defaulatable coupon bond with maturity $t_N \equiv T$ |
| $v(t;\tau,T)$   | equal to $\sigma(t,T) - \sigma(t,\tau)$                                    |
| $W_t$           | vector of correlated Brownian motion in $\mathbb{R}^d$ s.t. $dW_t^{(i)} dW_t^{(j)} = \rho_{ij} \, dt$ |
| $x \cdot y$     | scalar product between $x, y \in \mathbb{R}^d$                             |
| $x^2$           | an abbreviation for scalar prod. $x \cdot \rho x$ with $x \in \mathbb{R}^d$ and $\rho \in \mathbb{R}^{d \times d}$ correlation |
| $\mathcal{Y}(T)$ | yield of the liquid bond $\bar{P}(t_0,T;c,t)$ with maturity $T$             |

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Shorthands

CDF : Cumulative Distribution Function
FR : fractional-recovery
DHJM : Defaultable HJM framework
GBM : Geometric Brownian motion
MiFID : Markets in Financial Instruments Directive
pdf : probability density function
s.t. : such that
ttl : time-to-liquidate
w.r.t. : with respect to
TRACE : Trade Reporting and Compliance Engine
ZC : zero-coupon
Appendix A

Proof of Lemma 2.2. It is enough to use the definition of a forward defaultable ZC and to impose that the NPV is zero at time \( t \) (see also Figure 1).

\[
\mathbb{E} \left[ D(t, \tau) \mathbb{1}_{t_d > \tau} \mathcal{B}(t, \tau, T) | \mathcal{F}_t \right] = \mathbb{E} \left[ D(t, T) \mathbb{1}_{t_d > T} | \mathcal{F}_t \right] .
\]

The proposition is proven after observing that \( \mathcal{B}(t, \tau, T) \) is a known quantity at \( t \), since the forward price is established at time \( t \), and using definition (2).

Proof of Lemma 2.3. A direct application of the Generalized Itô Lemma, using the dynamics (3) and equation (6).

Proof of Lemma 2.4. Given the definition of \( \mathcal{D}(t_0, T) \), this is a straightforward computation after integrating \( r_s + q \lambda_s \) in (4) for \( s \) between \( t_0 \) and \( T \) and taking the limit \( q \nearrow 1 \).

Proof of Lemma 2.6. This is an application of Girsanov’s theorem on the risk neutral measure and the \( \tau \)-forward measure. Using Lemma 2.4, we get that

\[
\mathbb{E} \left[ D(t, \tau) \mathbb{1} \right] = \mathcal{B}(t, \tau) \mathbb{E}^{(\tau)} \left[ \mathbb{1} \right]
\]

proving the lemma.

The following technical lemma is needed in order to prove Theorem 3.2.

Lemma. The joint probability of i) the maximum \( y := \max[x(t); t \in (0, T)] \) and ii) its time location \( \theta \in (0, T) \), where \( x(t) := ct + W(t) \) is a 1-dimensional Wiener process with drift \( c t \) where \( c \in \mathbb{R} \), is

\[
p(\theta, y; c) = \frac{1}{\pi} \frac{y}{\sqrt{T - \theta} \theta^{3/2}} e^{-\frac{y^2 + cy}{2\theta}} \left\{ 1 - \sqrt{2\pi (T - \theta)} c e^{\frac{c^2 (T - \theta)}{2}} \Phi \left[ -c \sqrt{T - \theta} \right] \right\}
\]

with \( y = x(\theta) > 0 \) and \( \theta \in (0, T) \).

Proof. Consider the density \( p(\theta, y; x, c, \sigma^2) \) in equation (1.5) in Shepp (1979), where \( x \) is the endpoint \( x(T) \). The marginal distribution is obtained by setting \( \sigma = 1 \) and by integrating over \( x \in (-\infty, y) \).
Proof of Theorem 3.2. The upper bound is obvious given Lemma 3.1 and after observing that each ZC \( \mathcal{B}(t; \tau, t_i) \) in Equation (9) is a martingale under the \( \tau \)-forward measure and follows a GBM with volatility \( v(t; \tau, t_i) \). Thus, the expected value of the running maximum of the \( i \)th martingale \( \mathcal{B}(t; \tau, t_i) \) for \( t \in [t_0, \tau] \) takes the form \( \mathcal{B}(t_0; \tau, t_i) \pi_i \) (see, e.g. Longstaff 1995, and references therein).

The lower bound is the sum over \( i \) of the expected values of the \( \mathcal{B}(t^*; \tau, t_i) \) computed at time \( t^* \) s.t. \( \mathcal{B}(t^*; \tau, t_N) \) reaches its (first) maximum for a given realization of the process. In this case, using the separability property of the volatility (14), we get

\[
E(\tau) \left\{ \mathcal{B}(t^*; \tau, t_i) \right\} = \mathcal{B}(t_0; \tau, t_i) E(\tau) \left\{ \exp \left[ \zeta_i \left( -\frac{1}{2} \zeta_i \int_{t_0}^{t^*} \nu^2(s) \, ds + \int_{t_0}^{t^*} \nu(s) \, dW(\tau)(s) \right) \right] \right\}.
\]

By means of the change of time

\[
\tilde{t} := \tilde{t}(t) := \int_{t_0}^t \nu^2(s) \, ds \in (0, \tilde{\tau})
\]

where \( \tilde{\tau} \) stands for \( \tilde{t}(\tau) \), we get \( dW(\tau)(\tilde{t}) = \nu(t) \, dW(\tau)(t) \) and

\[
E(\tau) \left\{ \mathcal{B}(t^*; \tau, t_i) \right\} = \mathcal{B}(t_0; \tau, t_i) E(\tau) \left\{ \exp \left[ \zeta_i \left( -\frac{1}{2} \zeta_i \theta + W(\tau)(\theta) \right) \right] \right\}
\]

\[
= \mathcal{B}(t_0; \tau, t_i) E(\tau) \left\{ \exp \left[ \zeta_i \left( -\frac{1}{2} (\zeta_i - \zeta_N) \theta + x(\theta) \right) \right] \right\}
\]

where we have defined the drifted Brownian motion \( x(\tilde{t}) := -\zeta_N \tilde{t}/2 + W(\tau)(\tilde{t}) \) and \( x(\theta) \) its maximum value for \( \tilde{t} \in (0, \tilde{\tau}) \) with

\[
\theta := \tilde{t}(t^*)
\]

Let us observe that

\[
E(\tau) \left\{ \exp \left[ \zeta_i \left( -\frac{1}{2} (\zeta_i - \zeta_N) \theta + x(\theta) \right) \right] \right\} = \int_0^{\tilde{\tau}} d\theta \int_0^{+\infty} dy \, p \left( \theta; y; \frac{-\zeta_N}{2} \right) e^{\zeta_i (-\frac{1}{2} (\zeta_i - \zeta_N) \theta + y)}
\]

where \( p(\theta; y; -\zeta_N/2) \) was deduced in the previous technical lemma for a generic drift \( c \). After computing the integral w.r.t. \( y \), and some algebra, we get the result ♣.