Abstract

We consider two different constructions of higher brackets. First, based on a Grassmann-odd, nilpotent $\Delta$ operator, we define a non-commutative generalization of the higher Koszul brackets, which are used in a generalized Batalin-Vilkovisky algebra, and we show that they form a homotopy Lie algebra. Secondly, we investigate higher, so-called derived brackets built from symmetrized, nested Lie brackets with a fixed nilpotent Lie algebra element $Q$. We find the most general Jacobi-like identity that such a hierarchy satisfies. The numerical coefficients in front of each term in these generalized Jacobi identities are related to the Bernoulli numbers. We suggest that the definition of a homotopy Lie algebra should be enlarged to accommodate this important case. Finally, we consider the Courant bracket as an example of a derived bracket. We extend it to the “big bracket” of exterior forms and multi-vectors, and give closed formulas for the higher Courant brackets.

Keywords: Batalin-Vilkovisky Algebra; Homotopy Lie Algebra; Koszul Bracket; Derived Bracket; Courant Bracket.
1 Introduction

It is well-known [34] that in general the symmetrized, multiple nested Lie brackets

\[ [[[Q, a_1], a_2], \ldots, a_n], \]

where \( Q \) is a fixed nilpotent Lie algebra element \([Q, Q] = 0\), do not obey the original homotopy Lie algebra definition of Lada and Stasheff [22]. Several papers have been devoted to tackle this in special situations. For instance, Voronov considers the projection of above nested, so-called derived brackets
(1.1) into an Abelian subalgebra [34, 17, 35, 2]. In this paper we stay in the non-Abelian setting and observe that although a multiple nested bracket (1.1) does not obey the generalized Jacobi identities of Lada and Stasheff [22], it is – after all – very close. It turns out that one may organize the nested Lie brackets (1.1) in such a way that all the terms in the generalized Jacobi identities of Lada and Stasheff appear, but as a caveat, with different numerical prefactors related to the Bernoulli numbers.

The paper is organized as follows. In Section 2 we widen the definition of a homotopy Lie algebra by basing it on a generalized bracket product to allow for more general prefactors. In Section 3 we consider the Koszul bracket hierarchy [21, 1, 4, 11], and solve a long-standing problem of providing an ordering prescription for a construction of higher Koszul brackets for a non-commutative algebra \( A \), in such a way that the higher brackets form a homotopy Lie algebra. It turns out that in the non-commutative Koszul construction, the emerging homotopy Lie algebra is of the original type considered by Lada and Stasheff [22], cf. Theorem 3.5. On the other hand, the new types of (generalized) homotopy Lie algebras with non-trivial prefactors will be essential for the derived bracket hierarchies (1.1) studied in Section 4. Section 5 is devoted to the Courant bracket [13], which is a two-bracket defined on a direct sum of the tangent and the cotangent bundle \( T^\ast M \) over a manifold \( M \),

\[
\left[ X \oplus \xi, Y \oplus \eta \right]_H = \left[ X, Y \right] \oplus \left( \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} d(i_Y \xi - i_X \eta) + i_X i_Y H \right), \tag{1.2}
\]

where \( H \) is a closed “twisting” three-form. This bracket has many interesting applications, for instance Hitchin’s generalized complex geometry [15, 16]. In hindsight, the importance of the Courant bracket can be traced to the fact that it belongs to a derived homotopy Lie algebra [28, 29, 30] related to the exterior de Rham complex. Section 6 contains further theoretical aspects of homotopy Lie algebras.

2 Homotopy Lie Algebras

Let \( \text{Sym}_\epsilon^\bullet A := T^\bullet A / I \) denote a graded* symmetric tensor algebra over a graded vector space \( A \), where \( I \subseteq T^\bullet A \equiv \bigoplus_{n \geq 0} T^n A \) is the two-sided ideal generated by the set

\[
\left\{ b \otimes a - (-1)^{(\epsilon_a + \epsilon_b)(\epsilon_b + \epsilon_c)} a \otimes b \big| a, b \in A \right\} \subseteq T^2 A \equiv A \otimes A. \tag{2.1}
\]

Here \( \epsilon \in \{0, 1\} \) modulo 2 is a fixed “suspension parity”. We let the symbols “\( \otimes \)” and “\( \circ \)” denote the un-symmetrized and the symmetrized tensor product in the tensor algebras \( T^\bullet A \) and \( \text{Sym}_\epsilon^\bullet A \), respectively. In practice we shall focus on the symmetric tensor product “\( \circ \)”, and the only important thing is, that two arbitrary vectors \( a, b \in A \), with Grassmann parities \( \epsilon_a \) and \( \epsilon_b \), commute in \( \text{Sym}_\epsilon^\bullet A \) up to the following sign convention:

\[
b \circ a = (-1)^{(\epsilon_a + \epsilon_b)(\epsilon_b + \epsilon_c)} a \circ b \in \text{Sym}_\epsilon^\bullet A. \tag{2.2}
\]

A \( \bullet \)-bracket \( \Phi : \text{Sym}_\epsilon^\bullet A \to A \) is a collection of multi-linear \( n \)-brackets \( \Phi^n : \text{Sym}_\epsilon^n A \to A \), where \( n \in \{0, 1, 2, \ldots\} \) runs over the non-negative integers. In addition a \( \bullet \)-bracket \( \Phi \) carries an intrinsic Grassmann parity \( \epsilon_\Phi \in \{0, 1\} \). Detailed explanations of sign conventions are relegated to Subsection 2.1. We now introduce a \( \bullet \)-bracket product denoted with a “\( \circ \)”.

*Adjectives from supermathematics such as “graded”, “super”, etc., are from now on implicitly implied. We will also follow commonly accepted superconventions, such as, Grassmann parities are only defined modulo 2, and “nilpotent” means “nilpotent of order 2”.
Definition 2.1 Let there be given a set of complex numbers $c^n_k$ where $n \geq k \geq 0$. The “$\circ$” product $\Phi \circ \Phi' : \text{Sym}^* \mathcal{A} \to \mathcal{A}$ of two $\bullet$-brackets $\Phi, \Phi' : \text{Sym}^* \mathcal{A} \to \mathcal{A}$ is then defined as

$$(\Phi \circ \Phi')(a_1, \ldots, a_n) := \sum_{k=0}^{n} \frac{c^n_k}{k!(n-k)!} \sum_{\pi \in S_n} (-1)^{\epsilon_{\pi,a}} \Phi^{n-k+1} \left( \Phi^k(a_{\pi(1)}, \ldots, a_{\pi(k)}), a_{\pi(k+1)}, \ldots, a_{\pi(n)} \right)$$

for $n \in \{0, 1, 2, \ldots\}$.

Definition 2.2 The “$\circ$” product is non-degenerate if the complex coefficients $c^n_k$, $n \geq k \geq 0$, satisfy

$$\forall n \in \{0, 1, 2, \ldots\} \exists k \in \{0, 1, \ldots, n\} : c^n_k \neq 0 .$$

A priori we shall not assume any other properties of this product, like for instance associativity or a pre-Lie property. See Subsection 6.1 for further discussions of potential product properties. The aim of this paper is to determine values of the $c^n_k$ coefficients that lead to useful products, guided by important examples. We first generalize an important definition of Lada and Stasheff [22].

Definition 2.3 A vector space $\mathcal{A}$ with a Grassmann-odd $\bullet$-bracket $\Phi : \text{Sym}^* \mathcal{A} \to \mathcal{A}$ is a (generalized) homotopy Lie algebra if the $\bullet$-bracket $\Phi$ is nilpotent with respect to a non-degenerate “$\circ$” product, $\Phi \circ \Phi = 0$, $\epsilon_{\Phi} = 1$.

The infinite hierarchy of nilpotency relations behind (2.5) are also known as “generalized Jacobi identities” [23] or “main identities” [36, 11]. The first few relations will be displayed in detail in Subsection 2.2. In the original homotopy Lie algebra definition of Lada and Stasheff [22] the product coefficients are fixed to be

$$c^n_k = 1 ,$$

(2.6)

cf. Subsection 2.2. We shall not assume (2.6) because important examples are incompatible with this restriction, cf. Section 4. Instead we adapt the non-degeneracy condition (2.4). The $A_\infty$ definition [31] can similarly be generalized. We note that we shall in general lose an auxiliary description of a homotopy Lie algebra in terms of a nilpotent co-derivation, cf. Subsection 6.3.

Our goal is to determine universal values of the $c^n_k$ coefficients that generate important classes of homotopy Lie algebras. By the word “universal” we mean that a particular set of $c^n_k$ coefficients works within an entire class of $\bullet$-brackets. For instance, the Bernoulli numbers will play an important rôle for the so-called derived brackets, cf. Section 4.

The above algebraic homotopy Lie algebra construction has a geometric generalization to vector bundles $E = \coprod_{p \in M} E_p$ over a manifold $M$, where each fiber space $E_p$ is a homotopy Lie algebra. However, for most of this paper, it is enough to work at the level of a single fiber.

2.1 Sign Conventions

The sign factor $(-1)^{\epsilon_{\pi,a}}$ in the product definition (2.3) arises from introducing a sign (2.2) each time two neighboring elements of the symmetric tensor $a_1 \odot \ldots \odot a_n$ are exchanged to form a permuted tensor $a_{\pi(1)} \odot \ldots \odot a_{\pi(n)}$, i.e. working in $\text{Sym}^n \mathcal{A}$, we have

$$a_{\pi(1)} \odot \ldots \odot a_{\pi(n)} = (-1)^{\epsilon_{\pi,a}} a_1 \odot \ldots \odot a_n \in \text{Sym}^n \mathcal{A} .$$

(2.7)
In detail, the sign conventions are

\[ \epsilon_{\Phi \Phi'} = \epsilon_{\Phi} + \epsilon_{\Phi'} , \]  
\[ \epsilon(\Phi^n(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \epsilon a_i + (n-1)\epsilon + \epsilon_{\Phi} , \]  
\[ \Phi^n(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (-1)^{(\epsilon_{\epsilon} + \epsilon_{\epsilon} + \epsilon_{\epsilon})} \Phi^n(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) , \]  
\[ \Phi^n(a_1, \ldots, a_i \lambda, a_{i+1}, \ldots, a_n) = (-1)^{\epsilon \lambda \lambda} \Phi^n(a_1, \ldots, a_i, \lambda a_{i+1} + \ldots, a_n) , \]  
\[ \Phi^n(\lambda a_1, a_2, \ldots, a_n) = (-1)^{\epsilon \lambda \lambda} \Phi^n(a_1, a_2, \ldots, a_n) , \]  
\[ \Phi^n(a_1, \ldots, a_n \lambda) = \Phi^n(a_1, \ldots, a_n) \lambda , \]  
\[ a \lambda = (-1)^{\epsilon \lambda \lambda} \lambda a , \]  

where \( \lambda \) is a supernumber of Grassmann parity \( \epsilon_\lambda \). It is useful to memorize these sign conventions by saying that a symbol \( \Phi \) carries Grassmann parity \( \epsilon_\Phi \), while a “comma” and tensor-symbols \( \otimes \) and \( \otimes' \) carry Grassmann parity \( \epsilon \). Note however that the zero-bracket \( \Phi^0 \) has Grassmann parity \( \epsilon_{\Phi} + \epsilon \). The bracket product \( \otimes \) is Grassmann-even, \( \epsilon(\otimes) = 0 \), cf. eq. (2.8). While the sign implementation may vary with authors and applications, we stress that the Grassmann-odd nature of a \( \bullet \)-bracket \( \Phi \) is an unavoidable, characteristic feature of a homotopy Lie algebra, cf. eq. (2.5).

We remark that one could in principle bring different kinds of Grassmann parities \( \epsilon^{(i)} \) into play, where an upper index \( i \in I \) labels the different species. In that case the eq. (2.2) should be replace by

\[ b \otimes a = a \otimes b \prod_{i \in I} (-1)^{(\epsilon^{(i)}_{a} + \epsilon^{(i)}_{b})} (\epsilon^{(i)}_{a} + \epsilon^{(i)}_{b}) . \]  

As an example the exterior form degree could be assigned to a different type of parity. This could provide more flexible conventions for certain systems. Nevertheless, we shall only consider one type of parity in this paper for the sake of simplicity.

### 2.2 Connection to Lie Algebras

The importance of the homotopy Lie algebra construction is underscored by the fact that the two-bracket \( \Phi^2(a, b) \) of a Grassmann-odd \( \bullet \)-bracket \( \Phi \) gives rise to a Lie-like bracket \([,] \) of opposite parity \( \epsilon' := 1 - \epsilon \),

\[ [a, b] := (-1)^{(\epsilon_a + \epsilon_c)} \Phi^2(a, b) , \quad a, b \in \mathcal{A} , \quad \epsilon_{\Phi} = 1 . \]  

(This particular choice of sign is natural for a derived bracket, cf. Section 4. Note that in the context of the Koszul bracket hierarchy and Batalin-Vilkovisky algebras the opposite sign convention is usually adapted, i.e. \( [a, b] := (-1)^{(\epsilon_a + \epsilon_c)} \Phi^2(a, b) \), cf. Section 3.) The bracket (2.16) satisfies bi-linearity and skewsymmetry

\[ \epsilon([a, b]) = \epsilon_{a} + \epsilon_{b} + \epsilon' , \]  
\[ [\lambda a, b] = \lambda [a, b] \mu , \]  
\[ [a \lambda, b] = (-1)^{\epsilon \lambda} [a, \lambda b] , \]  
\[ [b, a] = (-1)^{(\epsilon_{a} + \epsilon_{a})} (\epsilon_{a} + \epsilon_{c}) [a, b] , \]  

where \( \lambda, \mu \) are supernumbers. The failure (if any) of the Jacobi identity

\[ \sum_{a, b, c \text{ cycl.}} (-1)^{(\epsilon_{a} + \epsilon_{c}) (\epsilon_{c} + \epsilon_{c})} [a, b, c] = (-1)^{\epsilon_{a} + \epsilon' (\epsilon_{a} + \epsilon') (\epsilon_{c} + \epsilon')} \text{Jac}(a, b, c) \]  

is
is measured by the Jacobiator $\text{Jac} : \text{Sym}^3\mathcal{A} \to \mathcal{A}$, defined as

$$\text{Jac}(a_1, a_2, a_3) := \frac{1}{2} \sum_{\pi \in S_3} (-1)^{\epsilon_{\pi,0}} \Phi^2(\Phi(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)})).$$  \hspace{1cm} (2.22)

The first few nilpotency relations (2.5) are

$$c_0^n \Phi^1(\Phi^0) = 0, \hspace{1cm} (2.23)$$

$$c_0^1 \Phi^2(\Phi^0, a) + c_1^1 \Phi^1(\Phi^1(a)) = 0, \hspace{1cm} (2.24)$$

$$c_0^2 \Phi^3(\Phi^0, a, b) + c_1^2 \left[ \Phi^2(\Phi^1(a), b) + (-1)^{\epsilon_{a,b}} \Phi^2(a, \Phi^1(b)) \right] + c_2^1 \Phi^1(\Phi^2(a, b)) = 0, \hspace{1cm} (2.25)$$

$$c_0^3 \Phi^4(\Phi^0, a, b, c) + c_1^3 \text{Jac}(a, b, c) + c_2^3 \Phi^1(\Phi^3(a, b, c)) + c_3^1 = 0, \hspace{1cm} (2.26)$$

and so forth. If all the $c_k^n$ coefficients are equal to 1 this becomes the homotopy Lie algebra of Lada and Stasheff [22]. (We shall ignore the fact that Lada and Stasheff [22] do not include a zero-bracket $\Phi^0$ in the definition and they use another sign convention.) If $c_0^1 = 0$ the one-bracket $\Phi^1$ becomes nilpotent, cf. eq. (2.24), so in this case (ignoring the fact that we have not defined an integer grading) the one-bracket $\Phi^1$ essentially gives rise to a complex $(\mathcal{A}, \Phi^1)$. Note that the Jacobi identity (2.26) is modified by the presence of higher brackets. In this paper we work under the hypothesis that the characteristic features of a homotopy Lie algebra is formed by the Grassmann-odd and nilpotent nature of the $\bullet$-bracket $\Phi$, i.e. the mere existence of the $c_k^n$ coefficients, rather than what particular values those $c_k^n$ coefficients might have.

### 2.3 Rescaling

A couple of general remarks about the prefactors in the “c” product (2.3) is in order. First of all, the denominator $k!(n-k)!$ has been included to conform with standard practices. (Traditionally homotopy Lie algebras are explained via un-shuffles [22], a notion we shall not use in this paper. The combinatorial factor $k!(n-k)!$ disappears when recast in the language of un-shuffles.) It is convenient to introduce an equivalent scaled set of coefficients $b_k^n$ that are always assumed to be equal to the $c_k^n$ coefficients multiplied with the binomial coefficients,

$$b_k^n \equiv \binom{n}{k} c_k^n, \hspace{1cm} 0 \leq k \leq n. \hspace{1cm} (2.27)$$

We shall often switch back and forth between the “b” and the “c” picture using eq. (2.27).

Secondly, we remark that the nilpotency relations (2.31), and hence the coefficients $c_k^n$, may always be trivially scaled

$$c_k^n \to \lambda_n c_k^n, \hspace{1cm} (2.28)$$

where $\lambda_n, n \in \{0, 1, 2, \ldots\}$, are non-zero complex numbers. Also, if one allows for a re-normalization of the bracket definition $\Phi^n \to \Phi^n/\lambda_n$, and one scales the coefficients $c_k^n \to \lambda_k \lambda_{n-k+1} c_k^n$ accordingly, the nilpotency relations are not changed. In practice, one works with a fixed convention for the normalization of the brackets, so the latter type of scaling is usually not an issue, while the former type (2.28) is a trivial ambiguity inherent in the definition (2.5). When we in the following make uniqueness claims about the $c_k^n$ coefficients in various situations, it should always be understood modulo the trivial scaling (2.28).
2.4 Polarization

The product definition (2.3) may equivalently be written in a diagonal form

\[(\Phi \circ \Phi)^n(a^{\otimes n}) = \sum_{k=0}^{n} b_k^n \Phi^{n-k+1} (\Phi^{k}(a^{\otimes k}) \circ a^{\otimes (n-k)}) , \quad \epsilon(a) = \epsilon . \quad (2.29)\]

That eq. (2.29) follows from the product definition (2.3) is trivial. The other way follows by collecting the string \(a_1 \otimes \ldots \otimes a_n\) of arguments into a linear combination

\[a = \sum_{k=0}^{n} \lambda_k a_k , \quad \epsilon(a) = \epsilon , \quad (2.30)\]

where the supernumbers \(\lambda_k, k \in \{0, 1, \ldots, n\}\), have Grassmann parity \(\epsilon(\lambda_k) = \epsilon(a_k) + \epsilon\). The product definition (2.3) then follows by inserting the linear combination (2.30) into eq. (2.29), considering terms proportional to the \(\lambda_1 \lambda_2 \ldots \lambda_n\) monomial, and using the appropriate sign permutation rules. The general lesson to be learned is that the diagonal carries all information. This polarization trick is not new, as one can imagine; see for instance Ref. [3] and Ref. [35]. The crucial point is that the nilpotency relations (2.5) may be considered on the diagonal only,

\[(\Phi \circ \Phi)^n(a^{\otimes n}) = 0 , \quad \epsilon(a) = \epsilon , \quad n \in \{0, 1, 2, \ldots\} . \quad (2.31)\]

Eq. (2.31) will be our starting point for subsequent investigations.

We mention in passing that a construction involving a pair of \(\bullet\)-brackets \(\Phi^a, a \in \{1, 2\}\), sometimes referred to as an “\(Sp(2)\)-formulation” [11], can always be deduced from polarization of \(\Phi = \sum_{a=1}^{2} \lambda_a \Phi^a\), \(\epsilon(\lambda_a) = 0\).

3 The Koszul Bracket Hierarchy

The heart of the following construction goes back to Koszul [21, 1, 4] and was later proven to be a homotopy Lie algebra in Ref. [11].

3.1 Basic Settings

Consider a graded algebra \((\mathcal{A}, \cdot)\) of suspension parity \(\epsilon \in \{0, 1\}\), satisfying bi-linearity and associativity,

\[
\begin{align*}
\epsilon(a \cdot b) &= \epsilon_a + \epsilon_b + \epsilon , \\
(\lambda a) \cdot (b \mu) &= \lambda (a \cdot b) \mu , \\
(a \lambda) \cdot b &= (-1)^{\epsilon \lambda} a \cdot (\lambda b) , \\
(a \cdot b) \cdot c &= a \cdot (b \cdot c) ,
\end{align*}
\]

where \(\lambda, \mu\) are supernumbers and \(a, b, c \in \mathcal{A}\) are algebra elements. Let there also be given a fixed algebra element \(e\) of Grassmann parity \(\epsilon(e) = \epsilon\) and a Grassmann-odd, linear operator \(\Delta : \mathcal{A} \to \mathcal{A}\), also known as a Grassmann-odd endomorphism \(\Delta \in \text{End}(\mathcal{A})\). Note that the algebra product “\(\cdot\)” carries Grassmann parity, cf. eq. (3.1). This implies for instance that a power \(a^n := a \cdot \ldots \cdot a\) of an element \(a \in \mathcal{A}\) has Grassmann parity \(\epsilon(a^n) = n \epsilon(a) + (n-1) \epsilon, n \in \{1, 2, 3, \ldots\}\). Let \(L_a, R_a : \mathcal{A} \to \mathcal{A}\) denote the left and the right multiplication map \(L_a(b) := a \cdot b\) and \(R_a(b) := b \cdot a\) with an algebra element \(a \in \mathcal{A}\), respectively. The Grassmann parity of the multiplication maps \(L_a, R_a \in \text{End}(\mathcal{A})\) is in both cases \(\epsilon(L_a) = \epsilon(a) + \epsilon = \epsilon(R_a)\).
3.2 Review of the Commutative Case

In this Subsection 3.2 we assume that the algebra \( \mathcal{A} \) is commutative.

**Definition 3.1** If the algebra \( \mathcal{A} \) is commutative, the Koszul \( n \)-bracket \( \Phi^n_{\Delta} \) is defined [11] as multiple, nested commutators acting on the algebra element \( e \),

\[
\Phi^n_{\Delta}(a_1, \ldots, a_n) := \underbrace{[[[\ldots [\Delta, L_{a_1}], \ldots], L_{a_n}]_n \text{ commutators}}_{n \text{ commutators}} e , \quad \Phi^0_{\Delta} := \Delta(e) .
\] (3.5)

Here \([S, T] := ST - (-1)^{e_S e_T} TS\) denotes the commutator of two endomorphisms \( S, T \in \text{End}(\mathcal{A}) \) under composition. One easily verifies that this definition is symmetric in the arguments \((a_1, \ldots, a_n)\) by using the Jacobi identity for the commutator-bracket in \( \text{End}(\mathcal{A}) \).

**Proposition 3.2** In the commutative case the Koszul \( \bullet \)-bracket \( \Phi_{\Delta} \) satisfies a recursion relation with only three terms [1]

\[
\Phi^{n+1}_{\Delta}(a_1, \ldots, a_n, a_{n+1}) = \Phi^n_{\Delta}(a_1, \ldots, a_n \cdot a_{n+1}) - \Phi^n_{\Delta}(a_1, \ldots, a_n) \cdot a_{n+1} - (-1)^{e_n+e} \Phi^n_{\Delta}(a_1, \ldots, a_{n+1}) \cdot a_n
\] (3.6)

for \( n \in \{1, 2, 3, \ldots\} \).

Note that the one-bracket \( \Phi^1_{\Delta} \) can not be expressed recursively in terms of the zero-bracket \( \Phi^0_{\Delta} \) alone.

**Proof of Proposition 3.2:** Observe that

\[
\Phi^n_{[\Delta, L_{a_n}]}(a_1, \ldots, a_n) - \Phi^n_{[\Delta, L_{a_n}]}(a_1, \ldots, a_n) = (-1)^{e_n+e} \Phi^n_{L_{a_n}[\Delta, L_{a_n}]}(a_1, \ldots, a_n) + (-1)^{e_n+e}+(a \leftrightarrow b)
\] (3.7)

for \( n \in \{0, 1, 2, \ldots\} \). The wanted eq. (3.6) emerges after relabelling of eq. (3.7) and use of the definition (3.5). We mention for later that eq. (3.7) also makes sense in a non-commutative setting.

\[\square\]

The main example of the Koszul construction is with a bosonic suspension parity \( \epsilon = 0 \), cf. eq. (2.2), with \( e \) being an algebra unit, and where \( \Delta \in \text{End}(\mathcal{A}) \) is a nilpotent, Grassmann-odd, linear operator, \( \Delta^2 = 0, \epsilon(\Delta) = 1 \). This is called a generalized Batalin-Vilkovisky algebra by Akman [1, 6, 10]. If furthermore the higher brackets vanish, i.e. \( \Phi^n_{\Delta} = 0, n \geq 3 \), then the \( \Delta \) operator is by definition an operator of at most second order, and \((\mathcal{A}, \Delta)\) becomes a Batalin-Vilkovisky algebra [14, 26]. We give an explicit example in eq. (5.46). The zero-bracket \( \Phi^0_{\Delta} = \Delta(e) \) typically vanishes in practice. For a Batalin-Vilkovisky algebra with a non-vanishing zero-bracket \( \Phi^0_{\Delta} \), see Ref. [12].

3.3 The Intermediate Case: \( \text{Im}(\Delta) \subseteq Z(\mathcal{A}) \)

We would like to address the following two questions:

1. Which choices of the bracket product coefficients \( c^n_k \) turn the Koszul construction (3.5) into a (generalized) homotopy Lie Algebra?

2. Does there exist a non-commutative version of the Koszul construction?
As we shall see in Subsection 3.6 the answer to the second question is yes. For practical purposes, it is of interest to seek out intermediate cases that are no longer purely commutative, but where the non-commutative obstacles are manageable. In this and the following Subsections 3.3-3.5 we make the simplifying assumption that the image of the $\Delta$ operator lies in the center of the algebra, i.e.

$$\text{Im}(\Delta) \subseteq Z(A),$$

where the center $Z(A)$ is, as usual,

$$Z(A) := \{ a \in A \mid \forall b \in A : b \cdot a = (-1)^{(\ell_a+\ell)(\ell_b+\ell)}a \cdot b \}.$$  

(3.8)

The full non-commutative case is postponed until Subsection 3.6. The case (3.8) clearly includes the commutative case, and it turns out that the treatment of the first question from this intermediate perspective is completely parallel to the purely commutative case.

**Definition 3.3** If the assumption (3.8) is fulfilled, the Koszul $n$-bracket is defined as symmetrized, nested commutators acting on the algebra element $e$,

$$\Phi^n_\Delta(a_1, \ldots, a_n) := \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\ell_{\pi,a}} \left[ \ldots [\Delta, L_{a_{\pi(1)}}, \ldots], L_{a_{\pi(n)}} \right] e, \quad \Phi^0_\Delta := \Delta(e).$$

(3.10)

In general, all the information about the higher brackets is carried by the diagonal,

$$\Phi^n_\Delta(a^{\otimes n}) = \left[ \ldots [\Delta, L_a], \ldots, L_a \right] e = \sum_{k=0}^{n} \binom{n}{k} (-L_a)^k \Delta L_a^{n-k}(e), \quad \epsilon(a) = e.$$  

(3.11)

**Proposition 3.4** If the assumption (3.8) is fulfilled, the Koszul brackets $\Phi_\Delta$ satisfy the recursion relation

$$\Phi^n_\Delta(a_1, \ldots, a_n) = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[ \prod_{k=i+1}^{n} (-1)^{(\ell_i+\ell)(\ell_k+\ell)} \right] \left[ \prod_{\ell=j}^{n} (-1)^{(\ell_j+\ell)(\ell_{\ell+\ell})} \right] \Phi^{n-1}_\Delta(a_1, \ldots, \hat{a}_i, \ldots, a_n, a_j \cdot a_i + (-1)^{(\ell_i+\ell)(\ell_j+\ell)}a_i \cdot a_j) - \frac{2}{n} \sum_{1 \leq i \leq n} \left[ \prod_{k=i+1}^{n} (-1)^{(\ell_i+\ell)(\ell_k+\ell)} \right] \Phi^{n-1}_\Delta(a_1, \ldots, \hat{a}_i, \ldots, a_n) \cdot a_i$$  

(3.12)

for $n \in \{2, 3, 4, \ldots\}$.

For instance the two-bracket $\Phi^2_\Delta$ can be defined via the one-bracket $\Phi^1_\Delta$ as

$$\Phi^2_\Delta(a, b) = \frac{1}{2} \Phi^1_\Delta(a \cdot b) - \Phi^1_\Delta(a) \cdot b + (-1)^{(\ell_a+\ell)(\ell_b+\ell)}(a \leftrightarrow b).$$

(3.13)

The recursion relations are more complicated than in the commutative case. Whereas the commutative recursion relations (3.6) involve only three terms, the number of terms now grows quadratically with the number $n$ of arguments. Loosely speaking, one may say that the recursion relations dissolve as one moves towards full-fledged non-commutativity, cf. Subsection 3.6. This is fine since recursion relations are anyway not an essential ingredient of a homotopy Lie algebra, although at a practical level they can be quite useful.
Proof of Proposition 3.4: Note that eq. (3.7) still holds in this case:
\[ \Phi^n_{[\Delta, L_a]}(a^\odot n) - \Phi^n_{[[\Delta, L_a], L_a]}(a^\odot n) = \Phi^n_{2L_a[\Delta, L_a]}(a^\odot n), \quad \epsilon(a) = \epsilon, \] (3.14)
for \( n \in \{0, 1, 2, \ldots \} \). This leads to
\[ \Phi^{n+1}_\Delta((a^\odot(n+1)) = \Phi^n_\Delta((a \odot a^\odot(n-1)) - 2a \cdot \Phi^n_\Delta(a^\odot n) \]
\[ = \Phi^n_\Delta(a^\odot(n-1) \odot (a \cdot a)) - 2\Phi^n_\Delta(a^\odot n) \cdot a, \quad \epsilon(a) = \epsilon, \] (3.15)
for \( n \in \{1, 2, 3, \ldots \} \). The recursion relation (3.12) now follows from polarization of eq. (3.15), cf. Subsection 2.4.

\[ \Box \]

3.4 Nilpotency Relations

We now return to the first question in Subsection 3.3. More precisely, we ask which coefficients \( c^n_k \) could guarantee the nilpotency relations (2.31), if one is only allowed to additionally assume that the \( \Delta \) operator is nilpotent in the sense that
\[ \Delta R_c \Delta = 0? \] (3.16)

Note that the criterion (3.16) reduces to the usual nilpotency condition \( \Delta^2 = 0 \) if \( e \) is a right unit for the algebra \( A \). The generic answer to the above question is given by the following Theorem 3.5.

Theorem 3.5 Let there be given a set of \( c^n_k \) product coefficients, \( n \geq k \geq 0 \). The nilpotency relations (2.31) are satisfied for all Koszul brackets \( \Phi_\Delta \) that have a nilpotent \( \Delta \) operator (in the sense of eq. (3.16)), if and only if
\[ \forall n \in \{0, 1, \ldots \} : \quad c^n_k = c^n \] (3.17)
is independent of \( k \in \{0, \ldots, n\} \).

Bearing in mind the trivial rescaling (2.28), this solution (3.17) is essentially \( c^n_k = 1 \) in perfect alignment with the requirement (2.6) in the original definition of Lada and Stasheff [22]. So there is no call for a new bracket product “\( \odot \)” to study the Koszul bracket hierarchy. This no-go statement obviously remains valid when considering the general non-commutative case, cf. Subsection 3.6, since the general \( \Phi_\Delta \) bracket (3.30) should in particular reproduce all the severely limiting situations where the condition (3.8) holds.

Proof of Theorem 3.5 when assuming eq. (3.8): We start with the “only if” part. To see eq. (3.17), first note that for two mutually commuting elements \( a, b \in A \) with \( \epsilon(a) = \epsilon \),
\[ \Phi^{n+1}_\Delta(b \odot a^\odot n) = \Phi^n_{[\Delta, L_b]}(a^\odot n) \]
\[ = \sum_{i=0}^{n} \binom{n}{i} (-L_a)^i \Delta L_a^{n-i}(b \cdot e) - (-1)^{\epsilon_b + \epsilon} b \cdot \Phi^n_\Delta(a^\odot n). \] (3.18)

Putting \( b = \Phi^n_k(a^\odot k) \) with Grassmann parity \( \epsilon_b = 1 - \epsilon \), the element \( b \) commutes with \( a \) because of eq. (3.8), and the \( n' \)th square bracket becomes
\[ (\Phi_\Delta \circ \Phi_\Delta)^n(a^\odot n) = \sum_{k=0}^{n} b^n_k \sum_{i=0}^{n-k} \binom{n-k}{i} (-L_a)^i \Delta L_a^{n-k-i}(\Phi^n_k(a^\odot k) \cdot e) \]
\[
+ \sum_{k=0}^{n} b_k^n \Phi_k^\Delta (a^\odot k) \cdot \Phi_{n-k}^\Delta (a^\odot (n-k)) .
\] (3.19)

The two sums on the right-hand side of eq. (3.19) are of different algebraic natures, because the two \(\Delta\)'s are nested in the first sum, while in the second sum they are not. In general, to ensure the nilpotency relations (2.31), one should therefore impose that the two sums vanish separately. (To make this argument sound one uses that nilpotency relations (2.31) hold for all possible choices of \(A\), \(\Delta\) and \(e\) satisfying eq. (3.16).) The vanishing of the second sum just imposes a symmetry

\[
b_k^n = b_{n-k}^n
\] (3.20)

among the \(b_k^n\) coefficients, because the family of brackets \(\Phi_k^\Delta (a, \ldots, a), k \in \{0, 1, \ldots, n\}\), mutually commute in a graded sense, which in plain English means: anti-commute. We shall see shortly that the symmetry (3.20) is superseded by stronger requirements coming from the first sum. After some elementary manipulations the first sum reads

\[
\sum_{i=0}^{n} (-L_a)^i \sum_{j=0}^{n-i} \Delta L_a^{n-i-j} \left( \sum_{k=0}^{n-i-j} \frac{n-i-j}{k} \right) \left( \sum_{k=0}^{n-j} c_{k+j}^n \right) (-1)^k R_e \Delta L_a^j (e) .
\] (3.21)

Terms where non-zero powers of \(L_a\) are sandwiched between the two \(\Delta\)'s are bad, as the nilpotency condition (3.16) does not apply to them. Accordingly, the expression inside the square brackets in eq. (3.21) must vanish for such terms. In detail, there should exist complex numbers \(c_{(j)}^n\), \(0 \leq j \leq n\), such that

\[
\forall i, j, n : 0 \leq i, j \leq n \Rightarrow \sum_{k=0}^{n-i-j} c_{k+j}^n \left( \frac{n-i-j}{k} \right) (-1)^k = c_{(j)}^n \delta_{n,i+j} .
\] (3.22)

(The complex numbers \(c_k^n\) and \(c_{(j)}^n\) should not be confused.) Putting \(j = 0\) and \(m = n - i \in \{0, \ldots, n\}\), the eq. (3.22) reduces to

\[
\forall m, n : 0 \leq m \leq n \Rightarrow \sum_{k=0}^{m} c_k^n \left( \frac{m}{k} \right) (-1)^k = c_{(0)}^n \delta_{m,0} .
\] (3.23)

This in turn implies that \(c_k^n\) can only depend on \(n\),

\[
\forall k \in \{0, \ldots, n\} : c_k^n = c_{(0)}^n ,
\] (3.24)

which establishes the claim (3.24). For completeness let us mention that if one inserts the solution (3.24) back into eq. (3.22) one gets similarly,

\[
\forall j \in \{0, \ldots, n\} : c_{(j)}^n = c_{(0)}^n .
\] (3.25)

The “if” part of the proof follows easily by going through above reasoning in reversed order, but it is also a consequence of Theorem 3.6 below.

\[\square\]

### 3.5 Off-Shell with respect to the Nilpotency Condition

One may summarize the discussions of the last Subsection 3.4 in the following Theorem 3.6.

**Theorem 3.6** A Koszul •-bracket \(\Phi_\Delta\) satisfies a square identity [11]

\[
\Phi_\Delta \circ \Phi_\Delta = \Phi_{\Delta R_e \Delta} ,
\] (3.26)

where “\(\circ\)” here refers to the ordinary bracket product (2.3) with \(c_k^n = 1\).
We stress that this identity holds without assuming the nilpotency condition (3.16). It is instructive to see a direct proof of this square identity (3.26) that uses a generating function and polarization to minimize the combinatorics. In the case of the Koszul $\bullet$-bracket $\Phi_\Delta$ the generating function is just the ordinary exponential function “exp”. This is implemented as a formal series of “exponentiated brackets”,

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_\Delta^n (a \otimes_n) = \left( e^{-[L_a, \cdot]} \right) e = e^{-L_a} \Delta e^{L_a} (e) , \quad \epsilon (a) = \epsilon .
\] (3.27)

Conversely, one may always extract back the $n'$th bracket $\Phi_\Delta^n$ by identifying terms in eq. (3.27) that has homogeneous scaling degree $n$ under scaling $a \to \lambda a$ of the argument $a$.

**Proof of Theorem 3.6 when assuming eq. (3.8):** First note that for two mutually commuting elements $a, b \in A$ with $\epsilon (a) = \epsilon$,

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_\Delta^{n+1} (b \otimes a \otimes_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi_\Delta [\Delta, L_b] (a \otimes_n) \\
= e^{-L_a} \Delta e^{L_a} (b \cdot e) - (1)^{\epsilon_b} b \cdot e^{-L_a} \Delta e^{L_a} (e) .
\] (3.28)

Putting $b = e^{-L_a} \Delta e^{L_a} (e)$ with Grassmann parity $\epsilon_b = 1 - \epsilon$, the element $b$ commutes with $a$ because of eq. (3.8), the element $b$ is nilpotent $b \cdot b = 0$, and the exponentiated left-hand side of eq. (3.26) becomes

\[
\sum_{n=0}^{\infty} \frac{1}{n!} (\Phi_\Delta \circ \Phi_\Delta)^n (a \otimes_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k! (n-k)!} \Phi_\Delta^{n-k+1} \left( \Phi_\Delta^k (a \otimes_k) \circ a \otimes (n-k) \right) \\
= e^{-L_a} \Delta e^{L_a} \left( e^{-L_a} \Delta e^{L_a} (e) \cdot e \right) + e^{-L_a} \Delta e^{L_a} (e) \cdot e^{-L_a} \Delta e^{L_a} (e) \\
= e^{-L_a} \Delta R e \Delta e^{L_a} (e) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi_\Delta^n (a \otimes_n) ,
\] (3.29)

which is just the exponentiated right-hand side of eq. (3.26).

\[\square\]

### 3.6 The General Non-Commutative Case

We now consider the general case without the assumption (3.8).

**Definition 3.7** In the general non-commutative case the Koszul $n$-brackets is defined as

\[
\Phi_\Delta^n (a_1, \ldots, a_n) := \sum_{i,j,k \geq 0 \atop i+j+k = n} \frac{B_{i,j,k}}{i! j! k!} \sum_{\pi \in S_n} (-1)^{\epsilon_{\pi, a} + \epsilon_{\pi (1)} + \ldots + \epsilon_{\pi (i)} + \ldots + \epsilon_{\pi (n)}} a_{\pi (1)} \cdot \ldots \cdot a_{\pi (i)} \\
\cdot \Delta \left( a_{\pi (i+1)} \cdot \ldots \cdot a_{\pi (i+j)} \cdot e \right) \cdot a_{\pi (i+j+1)} \cdot \ldots \cdot a_{\pi (n)} ,
\] (3.30)

where the $B_{i,j,k}$ coefficients are given through the generating function

\[
B (x, y) = \sum_{i,j=0}^{\infty} B_{i,j} \frac{x^i y^j}{i! j!} = \frac{x - y}{e^x - e^y} = B (y, x) \\
= 1 - \frac{x}{2} + y - \frac{1}{2} \left( \frac{x^2}{6} + \frac{2xy}{3} + \frac{y^2}{6} \right) - \frac{1}{3!} \left( \frac{x^3 y}{6} + \frac{x y^2}{2} \right) \\
+ \frac{1}{4!} \left( -\frac{x^4}{30} + \frac{2x^3 y}{15} + \frac{4x^2 y^2}{5} + \frac{2xy^3}{15} - \frac{y^4}{30} \right) + \ldots .
\] (3.31)
The $B_{k,\ell}$ coefficients are related to the Bernoulli numbers $B_k$ via
\[ B(x, y) = e^{-x}B(y-x) = e^{-y}B(x-y) , \] (3.32)
cf. eq. (4.14), or in detail,
\[ B_{k,\ell} = (-1)^k \sum_{i=0}^{k} \binom{k}{i} B_{i+\ell} = B_{\ell,k} , \quad k, \ell \in \{0, 1, 2, \ldots \} . \] (3.33)

The first few brackets read
\[ \Phi^0 = \Delta(e) , \] (3.34)
\[ \Phi^1(a) = \Delta(a \cdot e) - \frac{1}{2} \Delta(e) \cdot a - \frac{1}{2} (-1)^{\epsilon_a+\epsilon} a \cdot \Delta(e) , \] (3.35)
\[ \Phi^2(a, b) = \frac{1}{2} \Delta(a \cdot b \cdot e) - \frac{1}{2} \Delta(a \cdot e) \cdot b - \frac{1}{2} (-1)^{\epsilon_a+\epsilon} a \cdot \Delta(b \cdot e) + \frac{1}{12} \Delta(e) \cdot a \cdot b \\
+ \frac{1}{3} (-1)^{\epsilon_a+\epsilon} a \cdot \Delta(e) \cdot b + \frac{1}{12} (-1)^{\epsilon_a+\epsilon} a \cdot b \cdot \Delta(e) + (-1)^{\epsilon_a+\epsilon} \epsilon(a \leftrightarrow b) . \] (3.36)

In general, all the information about the higher brackets is carried by the diagonal,
\[ \Phi^n(a^{\circ n}) = \sum_{i, j, k \geq 0 \atop i+j+k = n} \binom{n}{i, j, k} B_{i,k} a^i \cdot \Delta(a^j \cdot e) \cdot a^k , \quad \epsilon(a) = \epsilon . \] (3.37)

The definition (3.37) is consistent with the previous definition (3.11) for the intermediate case (3.8). This is because $B(x, x) = e^{-x}$, or equivalently,
\[ \sum_{k=0}^{n} \binom{n}{k} B_{k,n-k} = (-1)^n . \] (3.38)

The formal series of exponentiated brackets may be compactly written
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n(a^{\circ n}) = B(L_a, R_a)\Delta e^{L_a}(e) = B(R_a - L_a) e^{-L_a} \Delta e^{L_a}(e) , \quad \epsilon(a) = \epsilon . \] (3.39)

The latter expression shows that all the “.” products in the bracket definition (3.30) can be organized as commutators from either $\mathcal{A}$ or $\text{End}(\mathcal{A})$, except for the dot “.” in front of the fixed element $e$.

In the general non-commutative case the $\Phi^{n+1}_\Delta$ bracket can not be expressed recursively in terms of the $\Phi^n_\Delta$ bracket alone, although there are exceptions. Most notably, the three-bracket $\Phi^3_\Delta$ can be expressed purely in terms of the two-bracket $\Phi^2_\Delta$,
\[ \Phi^3(a_1, a_2, a_3) = \frac{1}{6} \sum_{\pi \in S_3} (-1)^{\epsilon_{\pi,a}} \left[ \Phi^2\left(a_{\pi(1)} \cdot a_{\pi(2)}, a_{\pi(3)}\right) - \frac{1}{2} \Phi^2\left(a_{\pi(1)}, a_{\pi(2)}\right) \cdot a_{\pi(3)} \\
- \frac{1}{2} (-1)^{\epsilon_{\pi(a_1)}} a_{\pi(1)} \cdot \Phi^2\left(a_{\pi(2)}, a_{\pi(3)}\right) \right] . \] (3.40)

(Of course, one may always replace appearances of $\Delta$ in definition (3.30) with zero and one-brackets, i.e. $\Delta(e) = \Phi^0_\Delta$, and $\Delta(a \cdot e) = \Phi^1_\Delta(a) + \frac{1}{2} \Phi^0_\Delta(a) a + \frac{1}{2} (-1)^{\epsilon_a+\epsilon} a \cdot \Phi^0_\Delta$, and in this way express the $n$-bracket $\Phi^n_\Delta$ in terms of lower brackets, in this case $\Phi^0_\Delta$ and $\Phi^1_\Delta$.)

Our main assertion is that the square identity (3.26) in Theorem 3.6 holds for the fully non-commutative $\Phi_\Delta$ bracket definition (3.30), i.e. without assuming eq. (3.8). The Theorem 3.5 is also valid in the general situation.
Proof of Theorem 3.6 in the general case: First note that for two elements \( a, b \in \mathcal{A} \) with \( \epsilon(a) = \epsilon \),

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_{\Delta}^{n+1}(b \circ a^{\otimes n}) = B(L_a, R_a) \Delta(E(L_a, R_a)b \cdot e) + \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(R)} a^i \cdot \Delta e^{L_a}(e) \cdot a^j \cdot b \cdot a^k
\]

\[+ (-1)^{i+j+k} \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(L)} a^i \cdot b \cdot a^j \cdot \Delta e^{L_a}(e) \cdot a^k ,
\]

(3.41)

where

\[
E(x, y) := \frac{e^x - e^y}{x - y} = \sum_{k, \ell=0}^{\infty} \frac{x^ky^\ell}{(k+\ell+1)!} ,
\]

(3.42)

\[
B_{i,j,k}^{(R)} := \frac{B_{i+j+k+1}}{i!(j+k+1)!} ,
\]

(3.43)

\[
B^{(R)}(x, y, z) := \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(R)} x^iy^jz^k = \frac{B(x, y) - B(x, z)}{y - z} ,
\]

(3.44)

\[
B_{i,j,k}^{(L)} := \frac{B_{i+j+k+1}}{(i+j+k+1)!k!} ,
\]

(3.45)

\[
B^{(L)}(x, y, z) := \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(L)} x^iy^jz^k = \frac{B(x, z) - B(y, z)}{x - y} .
\]

(3.46)

Putting \( b = B(L_a, R_a) \Delta e^{L_a}(e) \) with Grassmann parity \( \epsilon_b = 1 - \epsilon \), the exponentiated left-hand side of eq. (3.26) becomes

\[
\sum_{n=0}^{\infty} \frac{1}{n!} (\Phi_{\Delta} \circ \Phi_{\Delta})^n(a^{\otimes n}) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \Phi_{\Delta}^{n-k+1} \left( \Phi_{\Delta}^{k}(a^{\otimes k}) \circ a^{\otimes (n-k)} \right)
\]

\[= B(L_a, R_a) \Delta \left( E(L_a, R_a)B(L_a, R_a)\Delta e^{L_a}(e) \cdot e \right) + \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(R)} a^i \cdot \Delta e^{L_a}(e) \cdot a^j \cdot B(L_a, R_a)\Delta e^{L_a}(e) \cdot a^k
\]

\[- \sum_{i,j,k=0}^{\infty} B_{i,j,k}^{(L)} a^i \cdot B(L_a, R_a)\Delta e^{L_a}(e) \cdot a^j \cdot \Delta e^{L_a}(e) \cdot a^k .
\]

(3.47)

This should be compared with the exponentiated right-hand side of eq. (3.26),

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_{\Delta}^{n}(a^{\otimes n}) = B(L_a, R_a) \Delta R_\epsilon \Delta e^{L_a}(e) .
\]

(3.48)

The two sides (3.47) and (3.48) are equal provided that the following two conditions are met

\[
E(x, y)B(x, y) = 1 ,
\]

(3.49)

\[
B^{(R)}(x, y, z)B(y, z) = B(x, y)B^{(L)}(x, y, z) .
\]

(3.50)

The first equation (3.49) has a unique solution for \( B(x, y) \) given by \( 1/E(x, y) \), leaving no alternative to eq. (3.31). It is remarkable that this unique solution (3.31) satisfies the non-trivial second criterion (3.50) as well, as one may easily check by inspection, thereby ensuring the existence of the non-commutative Koszul construction.
The Derived Bracket Hierarchy

In this Section we consider an important class of $\bullet$-brackets that naturally requires a non-trivial bracket product (2.3) in order to satisfy the nilpotency relations (2.5), namely the so-called derived $\bullet$-brackets. As we shall soon see in eq. (4.6) below, the derived brackets are composed of nested Lie brackets in a simple manner. This should be contrasted with the non-trivial definition (3.30) of the non-commutative Koszul hierarchy that – among other things – involved the Bernoulli numbers. Nevertheless, in a strange twist, while the $c^n_k$ coefficients in the nilpotency relations (2.5) are all simply 1 for the non-commutative Koszul hierarchy, the $c^n_k$ coefficients will be considerably more complicated for the derived hierarchy and involve – of all things – the Bernoulli numbers!

4.1 Definitions

We abandon the associative “$\cdot$” structure considered in Subsection 3.1, and consider instead a Lie algebra $(\mathcal{A},[\ ,\ ])$ of parity $\epsilon \in \{0,1\}$, satisfying bi-linearity, skewsymmetry and the Jacobi identity,

$$\epsilon([a, b]) = \epsilon_a + \epsilon_b + \epsilon \ , \quad (4.1)$$

$$[\lambda a, b\mu] = \lambda[a, b]\mu \ , \quad (4.2)$$

$$[a\lambda, b] = (-1)^{\epsilon_a\lambda}[a, b] \ , \quad (4.3)$$

$$[b, a] = -(-1)^{(\epsilon_a + \epsilon)(\epsilon_b + \epsilon)}[a, b] \ , \quad (4.4)$$

$$0 = \sum_{a,b,c \text{ cycl.}} (-1)^{(\epsilon_a + \epsilon)(\epsilon_b + \epsilon)}[[[Q, a]], b, c] \ , \quad (4.5)$$

where $\lambda, \mu$ are supernumbers. Let there be given a fixed Lie algebra element $Q \in \mathcal{A}$.

**Definition 4.1** The derived $n$-bracket $\Phi^n_Q$, $n \in \{0,1,2,\ldots\}$, is defined as [7]

$$\Phi^n_Q(a_1, \ldots, a_n) := \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\epsilon \pi,a} [[[Q, a_{\pi(1)}], \ldots, a_{\pi(n)}], a] \ , \quad \Phi^0_Q := Q \ . \quad (4.6)$$

Note that the zero-bracket $\Phi^0_Q = Q$ is just the fixed Lie algebra element $Q$ itself. Since we are ultimately interested in $\bullet$-brackets $\Phi_Q$ that carry an odd intrinsic Grassmann parity $\epsilon(\Phi_Q) = 1$, cf. eq. (2.5), we shall demand from now on that the Grassmann parity $\epsilon_Q$ of the fixed Lie algebra element $Q$ is opposite of the suspension parity $\epsilon$,

$$\epsilon_Q = 1 - \epsilon \ . \quad (4.7)$$

All the information is again carried by the diagonal,

$$\Phi^n_Q(a^{\otimes n}) = [\Phi^{n-1}_Q(a^{\otimes (n-1)}), a] = [[[Q, a], \ldots, a], a] = (-\text{ad}_a)^n Q \ , \quad \epsilon(a) = \epsilon \ , \quad (4.8)$$

where we have defined the adjoint action $\text{ad} : \mathcal{A} \to \text{End}(\mathcal{A})$ by $(\text{ad}_a)(b) := [a, b]$.

**Proposition 4.2** A derived $\bullet$-bracket $\Phi_Q$ satisfies the recursion relation

$$\Phi^n_Q(a_1, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} \left[ \prod_{k=i+1}^{n} (-1)^{(\epsilon_i + \epsilon)(\epsilon_k + \epsilon)} \right] \Phi^{n-1}_Q(a_{i}, \ldots, \hat{a}_i, \ldots, a_n), a_i \quad (4.9)$$

for $n \in \{1,2,3,\ldots\}$. 


Proof of Proposition 4.2: The recursion relation (4.9) follows from polarization of eq. (4.8), cf. Subsection 2.4.

As we saw in Subsection 2.2 there is a Lie-like bracket of opposite parity $\epsilon_Q = 1 - \epsilon$ given by
\[
[a, b]_Q := (-1)^{\epsilon_a + \epsilon_Q} \Phi_Q^n(a, b) = \frac{1}{2}[[a, Q], b] + \frac{1}{2}[a, [Q, b]] = -(-1)^{(\epsilon_a + \epsilon_Q)(\epsilon_b + \epsilon_Q)}[b, a]_Q .
\] (4.10)
Thus the derived $\bullet$-bracket $\Phi_Q$ gives rise to an interesting duality $[\ , \ ] \rightarrow [\ , \ ]_Q$ between brackets of even and odd parity [8]. The suspension parity $\epsilon$ was introduced in the first place in eq. (2.2) to bring the even and odd brackets on equal footing, and we see that the formalism embraces this symmetry. The bracket (4.10) is known as a (skew-symmetric, inner) derived bracket [18, 19, 20]. The outer, derived $\bullet$-bracket hierarchies are modeled after the properties of the inner hierarchies, and will be discussed elsewhere.

4.2 Nilpotency versus Square Relations

We would like to analyze which coefficients $\epsilon_k^n$ could guarantee the nilpotency relations (2.31), if we are only allowed to additionally assume that $Q$ is nilpotent in the Lie bracket sense,
\[
[Q, Q] = 0 .
\] (4.11)
The $n\text{'}th$ square bracket $(\Phi_Q \circ \Phi_Q)^n(a^{\odot n})$ is nothing but a linear combination of terms built out of $n+1$ nested Lie brackets $[\ , \ ]$, whose $n+2$ arguments consist of two $Q$'s and $n$ $a$'s. The only such term that the nilpotency condition (4.11) annihilates, is the term
\[
\Phi_{[Q, Q]}^n(a^{\odot n}) = \left[\ldots[[Q, Q], a], \ldots, a\right] = (-\text{ad}a)^n [Q, Q]
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left[\Phi_Q^{n-k}(a^{\odot (n-k)}), \Phi_Q^k(a^{\odot k})\right] ,
\] (4.12)
for $n \in \{0, 1, 2, \ldots\}$. Therefore, instead of imposing the nilpotency condition (4.11), it is equivalent to let the $n\text{'}th$ square bracket $(\Phi_Q \circ \Phi_Q)^n(a^{\odot n})$ be proportional to the term (4.12), i.e.
\[
(\Phi_Q \circ \Phi_Q)^n(a^{\odot n}) = \alpha_n \Phi_{[Q, Q]}^n(a^{\odot n}) ,
\] (4.13)
where $\alpha_n$ is a proportionality factor that depends on $n \in \{0, 1, 2, \ldots\}$. This off-shell strategy with respect to the nilpotency condition (4.11) has also been promoted in Ref. [9] in a similar context. Since one may trivially scale the nilpotency relations (2.31) with a non-zero complex number, cf. Subsection 2.3, it is enough to study the square relation (4.13) with a proportionality factor equal to either $\alpha_n = 1$ or $\alpha_n = 0$. The case $\alpha_n = 1$ is a set of coupled, non-homogeneous linear (also known as affine) equations in the $c_k^n$ product coefficients. The analogous homogeneous problem corresponds to letting the proportionality factor be $\alpha_n = 0$, while continuing not to require nilpotency (4.11) of $Q$.

4.3 Solution

In this Subsection we present the complete solution to the square relation (4.13). To this end, let $B_k$ be the Bernoulli numbers, $k \in \{0, 1, 2, \ldots\}$, generated by
\[
B(x) = \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = 1 - \frac{x}{2} + \frac{x^2}{6} \frac{1}{2!} - \frac{x^4}{30} \frac{4!}{4!} + \mathcal{O}(x^6) ,
\] (4.14)
and let us for later convenience define the negative Bernoulli numbers as zero,

\[ 0 = B_{-1} = B_{-2} = B_{-3} = \ldots . \]  

**Theorem 4.3** Let there be given a set of \( c^n_k \) coefficients with \( n \geq k \geq 0 \). The square relations

\[ \Phi_Q \circ \Phi_Q = \Phi_{[Q,Q]} \]

are satisfied for all derived \( \bullet \)-brackets \( \Phi_Q \), if and only if

\[ c^n_k = B_k + \delta_{k,1} + c^{n(H)}_k, \quad n \geq k \geq 0, \]

where the homogeneous part \( c^{n(H)}_k \) solves the corresponding homogeneous equation

\[ \Phi_Q \circ \Phi_Q = 0. \]

For a given \( n \in \{0, 1, 2, \ldots \} \), the solution space for the homogeneous problem (4.18) is \([n + 1]/2\) dimensional. A basis \( c^{n(H)}_{k(m)} \) of solutions, labelled by an integer \( m \in \{0, 1, \ldots , [n - 1]/2\} \), is

\[ c^{n(H)}_{k(m)} = \binom{k}{m} B_{k-m} - \binom{k}{n-m} B_{k-n+m} = - (m \leftrightarrow n-m). \]

In practice, it is easier to let the label \( m \in \{0, 1, \ldots , n\} \) run all the way to \( n \), and work with an over-complete set of solutions. Also introduce a generating polynomial

\[ c^{n(H)}_k(t) := \sum_{m=0}^{n} c^{n(H)}_{k(m)} t^m. \]

One may reformulate the solution eqs. (4.17) and (4.19) with the help of a generating function

\[ c(x, y) := \sum_{k, \ell=0}^{\infty} c^{k+\ell} x^k y^\ell = \sum_{k, \ell=0}^{\infty} \frac{t^{k+\ell}}{(k+\ell)!} x^k y^\ell. \]

The particular solution \( c^{n(H)}_k = B_k + \delta_{k,1} \) is then

\[
\begin{align*}
    c(x, y) &= B(x)e^{x+y} - B(-x)e^y = (B(x)+x)e^y \\
    &= 1 + \left( \frac{x}{2} + y \right) + \frac{1}{2} \left( \frac{x^2}{6} + xy + y^2 \right) + \frac{1}{3!} \left( \frac{x^2 y}{2} + \frac{3xy^2}{2} + y^3 \right) \\
    &\quad + \frac{1}{4!} \left( \frac{x^4}{30} + x^2 y^2 + 2xy^3 + y^4 \right) + \ldots ,
\end{align*}
\]

and the homogeneous solution is

\[ c^{(H)}(x, y, t) = B(x)e^{xt+y} - B(xt)e^{x+y}, \]

where \( B \) is the Bernoulli generating function, cf. eq. (4.14). Eqs. (4.22) and (4.23) are our main result of Section 4. We emphasize that the square relation (4.16) and its homogeneous counterpart (4.18) are satisfied with these \( c^n_k \) product solutions without assuming the nilpotency condition (4.11). Also note that none of these solutions are consistent with the ordinary product (2.6). The solution shows that attempts to fit the derived bracket hierarchy (4.6) into the original homotopy Lie algebra definition (2.6) are bound to be unnatural and will only work in special situations.
Proof of Theorem 4.3: To derive the solution eqs. (4.17) and (4.19), first note that for two elements \( a, b \in A \) with \( \epsilon(a) = \epsilon \),

\[
\Phi_Q^{n+1}(b \odot a^{\odot n}) = \frac{1}{n+1} \sum_{\ell=0}^{n} \left[ \ldots \left[ [\Phi_Q^{\ell}(a^{\odot \ell}), b], \ldots \right], a \right]_{n-\ell+1 \text{ Lie brackets}}
\]

\[
= \frac{1}{n+1} \sum_{\ell=0}^{n} (-1)^{n-\ell} [\Phi_Q^{\ell}(a^{\odot \ell}), b]
\]

(4.24)

for \( n \in \{0, 1, 2, \ldots \} \). Letting \( b = \Phi_Q^{\ell}(a^{\odot k}) \) this becomes

\[
\Phi_Q^{n+1} \left( \Phi_Q^{\ell}(a^{\odot k}) \odot a^{\odot n} \right) = \sum_{i=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n \\ i \end{array} \right) \left[ \Phi_Q^{n-i}(a^{\odot (n-i)}), \Phi_Q^{\ell+i}(a^{\odot (k+i)}) \right].
\]

(4.25)

After some elementary manipulations the \( n'\text{th} \) bracket product reads

\[
(\Phi_Q \diamond \Phi_{Q'})^n(a^{\odot n}) = \sum_{k=0}^{n} b_k^n \left( \Phi_Q^{k-n+1}(a^{\odot k}) \odot a^{\odot (n-k)} \right)
\]

\[
= \sum_{k=0}^{n} \sum_{i=0}^{k} \frac{b_k^n}{i+1} \left( \begin{array}{c} n-k+i \\ i \end{array} \right) \left[ \Phi_Q^{n-k}(a^{\odot (n-k)}), \Phi_{Q'}^{k}(a^{\odot k}) \right]
\]

\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{i=0}^{k} \frac{c_{k-i}^{k}}{i+1} \left( \begin{array}{c} k \\ i \end{array} \right) \left[ \Phi_Q^{n-k}(a^{\odot (n-k)}), \Phi_{Q'}^{k}(a^{\odot k}) \right].
\]

(4.26)

Combining eqs. (4.12), (4.16) and (4.26) with \( Q' = Q \), one derives

\[
\forall k, n : \quad 0 \leq k \leq n \quad \Rightarrow \quad \sum_{i=0}^{k} \frac{c_{k-i}^{k}}{i+1} \left( \begin{array}{c} k \\ i \end{array} \right) - 1 = - (k \leftrightarrow n-k).
\]

(4.27)

The right-hand side of eq. (4.27) comes from the symmetry

\[
\left[ \Phi_Q^{k}(a^{\odot k}), \Phi_Q^{\ell}(a^{\odot \ell}) \right] = (k \leftrightarrow \ell), \quad k, \ell \in \{0, 1, 2, \ldots \}.
\]

(4.28)

This symmetry is the origin of the non-trivial homogeneous solutions. Let us first assume that the left-hand side of (4.27) vanishes. The unique solution for this case reads

\[
c_k^n = B_k + \delta_{k,1},
\]

(4.29)

which establishes eq. (4.17). We now focus on the homogeneous part of eq. (4.27). First note that for a fixed \( m \in \{0, \ldots, n\} \), the equation

\[
\sum_{i=0}^{k} \frac{c_{k-i}^{k}}{i+1} \left( \begin{array}{c} k \\ i \end{array} \right) = \delta_{k,m},
\]

(4.30)

with a Kronecker delta function source on the right-hand side, has the unique solution

\[
c_k^n = \left( \begin{array}{c} k \\ m \end{array} \right) B_{k-m}.
\]

(4.31)

Hence a \((k \leftrightarrow n-k)\) skewsymmetric version of the Kronecker delta function source, i.e.

\[
\sum_{i=0}^{k} \frac{c_{k-i}^{k}}{i+1} \left( \begin{array}{c} k \\ i \end{array} \right) = \delta_{k,m} - \delta_{k,n-m}
\]

(4.32)

will have the unique solution eq. (4.19).
4.4 Ward-like and Jacobi-like Identities

We now discuss particular useful solutions, i.e. non-trivial identities with as few terms as possible. The \(n\)th square bracket \((\Phi_Q \circ \Phi_Q)^n(a, \ldots, a)\) typically has \(n + 1\) terms on the diagonal. Here we shall use the freedom in the homogeneous part to kill most of these terms.

Consider first the Ward solution

\[
c^n_k = B_k + \delta_{k,1} - c_{k(0)}^n = \delta_{k,1} + \delta_{k,n} , \quad n \geq k \geq 0 ,
\]

or equivalently using eq. (4.21),

\[
c(x, y) = xe^y + e^x .
\]

It corresponds to a hierarchy of Ward identities

\[
n \Phi^n_Q \left( \Phi^n_Q(a) \circ a^{(n-1)} \right) + \Phi^n_Q \left( \Phi^n_Q(a^n) \right) = \Phi^n_{[Q,Q]}(a^n) , \quad \epsilon(a) = \epsilon ,
\]

where \(n \in \{0, 1, 2, \ldots\} .\) (The name "Ward identity" refers to a similar identity encountered in String Theory.) After polarization the Ward identities are

\[
\sum_{k=1}^{n} (-1)^{\epsilon_1 + \cdots + \epsilon_{k-1} + (k-1)\epsilon} \Phi^n_Q \left( a_1, \ldots, a_{k-1}, \Phi^n(a_k), a_{k+1}, \ldots, a_n \right) + \Phi^n_Q \left( \Phi^n_Q(a_1, \ldots, a_n) \right) = \Phi^n_{[Q,Q]}(a_1, \ldots, a_n) .
\]

The first few Ward identities read,

\[
\Phi^n_Q(\Phi^n_Q) = \Phi^n_{[Q,Q]} , \quad 2\Phi^n_Q(\Phi^n_Q(a)) = \Phi^n_{[Q,Q]}(a) ,
\]

\[
\Phi^n_Q \left( \Phi^n_Q(a), b \right) + (-1)^{\epsilon_1 + \epsilon_2} \Phi^n_Q \left( a, \Phi^n_Q(b) \right) + \Phi^n_Q \left( \Phi^n_Q(a, b) \right) = \Phi^n_{[Q,Q]}(a, b) .
\]

Consider next the solution

\[
c^n_k = B_k + \delta_{k,1} - \frac{2}{n} c^n_{k(1)} + \sum_{m=0}^{n} \frac{m}{n} c^n_{k(m)}
\]

\[
= \frac{4}{n} \delta_{k,2} + \frac{2}{n} \delta_{k,n-1} - \delta_{k,n} , \quad 0 \neq n \geq k \geq 0 ,
\]

or equivalently using eq. (4.21),

\[
c(x, y) - 1 = 2xe^y + (2y - x)E(x)
\]

\[
= 2 \left( \frac{x}{y} \right)^2 \left( 1 + (y - 1)e^y \right) + \left( \frac{2y}{x} - 1 \right) (e^x - 1) .
\]

Here we have defined

\[
E(x) := \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{1}{B(x)} .
\]

The solution (4.40) corresponds to generalized Jacobi identities

\[
2n \Phi^n_Q \left( \Phi^n_Q(a \circ a) \circ a^{(n-1)} \right) + 2\Phi^n_Q \left( \Phi^n_Q(a^n) \circ a \right) - \Phi^n_Q \left( \Phi^n_Q(a^{(n+1)}) \right) = \Phi^n_{[Q,Q]}(a^{(n+1)}),
\]

\[
\sum_{k=1}^{n} (-1)^{\epsilon_1 + \cdots + \epsilon_{k-1} + (k-1)\epsilon} \Phi^n_Q \left( a_1, \ldots, a_{k-1}, \Phi^n(a_k), a_{k+1}, \ldots, a_n \right) + \Phi^n_Q \left( \Phi^n_Q(a_1, \ldots, a_n) \right) = \Phi^n_{[Q,Q]}(a_1, \ldots, a_n) .
\]

\[
\Phi^n_Q(\Phi^n_Q(a)) = \Phi^n_{[Q,Q]}(a) ,
\]

\[
\Phi^n_Q \left( \Phi^n_Q(a), b \right) + (-1)^{\epsilon_1 + \epsilon_2} \Phi^n_Q \left( a, \Phi^n_Q(b) \right) + \Phi^n_Q \left( \Phi^n_Q(a, b) \right) = \Phi^n_{[Q,Q]}(a, b) .
\]
where \( n \in \{0, 1, 2, \ldots \} \). The first few read,

\[
2\Phi_Q^2 (\Phi_Q^0, a) - \Phi_Q^1 (\Phi_Q^1 (a)) = \Phi_{[Q,Q]}^1 (a) , \quad (4.44)
\]

\[
\Phi_Q^2 (\Phi_Q^1 (a), b) + (-1)^{a+b} \Phi_Q^2 (a, \Phi_Q^1 (b)) + \Phi_Q^1 (\Phi_Q^2 (a, b)) = \Phi_{[Q,Q]}^2 (a, b) , \quad (4.45)
\]

\[
2 \text{Jac}(a, b, c) - \Phi_Q^1 (\Phi_Q^3 (a, b, c)) = \Phi_{[Q,Q]}^3 (a, b, c) , \quad (4.46)
\]

where \( \text{Jac}(a, b, c) \) is the Jacobiator, cf. eq. (2.22). It is worth mentioning that the zero-bracket \( \Phi_Q^0 \), which normally complicates a homotopy Lie algebra, cf. eqs. (4.38), (4.39) and (4.46), therefore, in the nilpotent case \([Q,Q] = 0\), the Grassmann-odd one-bracket \( \Phi_Q^1 \) is nilpotent (4.38), and it obeys a Leibniz rule (4.36) with respect to the \( n \)-bracket \( \Phi_Q^n \). And perhaps most importantly, the two-bracket satisfies a generalized Jacobi identity (4.46) that only differs from the original Jacobi identity by a \( \Phi_Q^2 \)-exact term.

By transcribing the work of Courant [13] to this situation, one may define the notion of a Dirac subalgebra.

**Definition 4.4** A Dirac subalgebra is a subspace \( \mathcal{L} \subseteq \mathcal{A} \) that is:

1. Maximal Abelian with respect to the original Lie bracket \([\ , \ ]\),

2. Closed under the two-bracket \( \Phi_Q^2 \), i.e. \( \Phi_Q^2 (\mathcal{L}, \mathcal{L}) \subseteq \mathcal{L} \), or equivalently, \([\mathcal{L}, \mathcal{L}]_Q \subseteq \mathcal{L} \), cf. eq. (4.10).

It follows immediately from the bracket definition (4.6) that all the higher brackets \( \Phi_Q^n \), \( n \in \{3, 4, 5, \ldots \} \) vanish on a Dirac subalgebra \( \mathcal{L} \). In particular, the Jacobi identity for the two-bracket \( \Phi_Q^2 \) is satisfied in a Dirac subalgebra \( \mathcal{L} \), cf. eq. (4.46). We also point out a connection to Courant algebroids, where the generalized Jacobi identity (4.46) translates into the first (out of five) defining properties for the Courant algebroid, cf. Ref. [24].

## 5 The Courant Bracket

The Courant bracket [13] has received much interest in recent years primarily due to Hitchin generalized complex geometry [16]. A broad introduction to the subject can be found in the PhD theses of Roytenberg [29] and Gualtieri [15]. In this Section we give three different constructions of the (skewsymmetric) Courant bracket as a derived bracket, and we elaborate on its connection to homotopy Lie algebras [28, 29]. One well-known construction [18, 19, 20, 29] relies on an operator representation, see Subsection 5.1, and two partially new constructions rely on an even and an odd Poisson bracket [25, 29, 30], respectively, see Subsections 5.2-5.4.

The Courant bracket is defined on vectors and exterior forms as

\[
[X,Y]_H = [X,Y] + (-1)^{ev} i_X i_Y H = -(-1)^{ev} \epsilon \epsilon [Y,X]_H , \quad X, Y \in \Gamma (TM) , \quad (5.1)
\]

\[
[X,\eta]_H = \frac{1}{2} (\mathcal{L}_X + i_X d) \eta = (i_X d + \frac{(-1)^{ex}}{2} d i_X) \eta = (\mathcal{L}_X - \frac{(-1)^{ex}}{2} d i_X) \eta = \text{Jac}(\theta, X)_H , \quad (5.2)
\]

\[
[\xi,\eta]_H = 0 , \quad \xi, \eta \in \Gamma (\bigwedge (T^* M)) , \quad (5.3)
\]

with a closed twisting form \( H \in \Gamma (\bigwedge (T^* M)) \). (We shall ignore the fact that the twisting form \( H \) is zero in the original Courant bracket [13].) The Courant bracket (5.1)-(5.3) does not satisfy the Leibniz
rule nor the Jacobi identity. It is therefore natural to ask what is the significance of these formulas, in particular eq. (5.2)? In hindsight the answer is, that there exists a homotopy Lie algebra structure behind the Courant bracket, that makes the Jacobi identity valid modulo Q-exact terms, cf. eq. (4.46).

And underneath the homotopy Lie algebra structure, there is a Grassmann-odd nilpotent Hamiltonian vector field that generalizes the de Rham exterior derivative. As we shall see in Subsection 5.6 below, the higher Courant brackets are naturally defined on multi-vectors and exterior forms via the derived bracket hierarchy (4.6). When restricted to only vector fields, functions and one-forms, the $L_\infty$ hierarchy truncates and becomes an $L_3$ algebra [23]. Roytenberg and Weinstein [28, 29] define a related set of higher brackets through a homological resolution [5].

Finally, let us mention for completeness that people often consider a non-skewsymmetric version of the Courant bracket (which is also called a Dorfman bracket and is related to Loday/Leibniz algebras), partly to avoid handling the Jacobi identity directly, and partly to simplify the axioms for a Courant algebroid, cf. Ref. [29]. Nevertheless, at the end of the day, it is often the skewsymmetrized bracket that is relevant for applications. More importantly, our underlying derived bracket definition (4.6) is manifestly skewsymmetric, and hence we shall here only treat skewsymmetric brackets. We shall take advantage of polarization to shortcut the lengthy calculations that is normally associated with the skewsymmetric bracket, cf. Subsection 2.4.

5.1 Review of the Operator Representation

Perhaps the simplest realization of the Courant bracket as a (skewsymmetric, inner) derived bracket is the following operator construction due to Kosmann-Schwarzbach [18, 19, 20, 29]. Consider a $d$-dimensional bosonic manifold $M$, and let $x^i, i \in \{1, 2, \ldots, d\}$, denote local bosonic coordinates in some coordinate patch $U \subseteq M$. To avoid cumbersome notation, we shall often take the liberty to write local objects, which formally only live on $U$, as if they are living on the whole manifold $M$. Furthermore, we shall regard exterior forms

$$\eta = \eta(x^i, c^i) \in \Gamma(\bigwedge^\bullet(T^*M)) \cong C^\infty(E), \quad E \equiv \Pi TM,$$

as functions in the variable $(x^i, c^i)$ on the parity-inverted tangent bundle $E \equiv \Pi TM$ by identifying the basis of one-forms $dx^i \equiv c^i$ with the fermionic variables $c^i$. Notice that an $n$-form $\eta = \frac{1}{n!} \eta_{i_1 \ldots i_n} c^{i_1} \ldots c^{i_n}$ has Grassmann parity $\epsilon_\eta = n$ modulo 2, if the coordinate functions $\eta_{i_1 \ldots i_n} = \eta_{i_1 \ldots i_n}(x)$ are bosonic. Now let the Lie algebra $\mathcal{A}$ of Section 4 be the Lie algebra $\mathcal{A} = \text{End}(C^\infty(E))$ of operators acting on the above functions (5.4), and let the Lie bracket for $\mathcal{A}$ be given by the commutator bracket $[\cdot, \cdot]$. We shall often identify an exterior form $\eta$ with the left multiplication operator $L_\eta \in \mathcal{A}$, i.e. the operator that multiplies from the left with $\eta$. For the fixed Lie algebra element of Section 4, one now chooses a twisted version

$$D := d + H_{\text{odd}} \in \mathcal{A}$$

of the de Rham exterior derivative $d$ on the manifold $M$,

$$d = c^i \frac{\partial}{\partial x^i}, \quad \epsilon(d) = 1.$$  

Here $H_{\text{odd}} \in \Gamma(\bigwedge^{\text{odd}}(T^*M))$ is an odd closed exterior form, i.e. linear combinations of forms of odd form-degree, that are closed. The oddness condition ensures that $D$ carries definite Grassmann-parity. The $D$ operator is nilpotent,

$$[D, D] = 2dH_{\text{odd}} = 0,$$

because the exterior form $H_{\text{odd}}$ is closed. Parity-inverted vector fields $\tilde{X}$ (where the tilde “$\sim$” denotes parity-inversion, cf. eq. (5.27) below) are now represented as first-order differential operators,

$$\tilde{i}_X = X^i \frac{\partial}{\partial c^i}, \quad \epsilon(\tilde{i}_X) = 1 - \epsilon_X.$$

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One may now construct the derived $\cdot$-bracket hierarchy of $D$. The zero-bracket $\Phi^0_D = D$ is the generator $D$ itself. The one-bracket $\Phi^1_D = [D, \cdot]$ is a nilpotent, Grassmann-odd operator, cf. eq. (4.38). It is just the de Rham exterior derivative on exterior forms,
\[
\Phi^1_D(\eta) = [D, \eta] = d\eta ,
\]
individually of $H_{\text{odd}}$. On vectors, the one-bracket $\Phi^1_D$ becomes a sum of a Lie derivative and a contracted term,
\[
(-1)^{\epsilon_X} \Phi^1_D(i_X) = \mathcal{L}_X + i_X H_{\text{odd}} .
\]
Here and below, we repeatedly make use of the Cartan relations
\[
2d^2 = [d, d] = 0 , \quad [i_X, i_Y] = 0 , \quad \mathcal{L}_X = [i_X, d] , \quad i_{[X,Y]} = [\mathcal{L}_X, i_Y] .
\]
The two-bracket $\Phi^2_D$ gives rise to an odd version $[\cdot, \cdot]_D$ of the Courant bracket [18, 19, 20], cf. eq. (4.10).

**Definition 5.1** The odd Courant bracket $[\cdot, \cdot]_D$ is defined as the derived bracket,
\[
[a, b]_D := (-1)^{\epsilon_a+1} \Phi^2_D(a, b) = \frac{1}{2}[[a, D], b] + \frac{1}{2}[a, [D, b]] = -(-1)^{(\epsilon_a+1)(\epsilon_b+1)}[b, a]_D , \quad a, b \in \mathcal{A} = \text{End}(C^\infty(E)) .
\]

In this formulation, the Courant bracket (5.1)-(5.3) reads
\[
[i_X, i_Y]_D = i_{[X,Y]} + (-1)^{\epsilon_X} i_X i_Y H_{\text{odd}} , \quad X, Y \in \Gamma(TM) ,
\]
\[
[i_X, \eta]_D = \frac{1}{2}(\mathcal{L}_X + i_X d)\eta ,
\]
\[
[\xi, \eta]_D = 0 , \quad \xi, \eta \in \Gamma(\bigwedge^\bullet(T^*M)) .
\]

### 5.2 Symplectic Structure

It is also possible to build the Courant bracket (5.1)-(5.3) as a derived bracket with the help of a Poisson bracket $\{\cdot, \cdot\}$. Consider a $d$-dimensional bosonic base manifold $M$, let $x^i, i \in \{1, 2, \ldots, d\}$, denote local bosonic coordinates in coordinate patches, and let $p_i = \partial_i \in \Gamma(TM)$ for symmetric multi-vectors, $i \in \{1, 2, \ldots, d\}$. We now define a Poisson bracket on the cotangent bundle $T^*M$ as
\[
\{p_i, x^j\} = \delta^j_i = -\{x^j, p_i\} ,
\]
\[
\{x^i, x^j\} = 0 ,
\]
where we are going to fix the fundamental Poisson bracket $\{p_i, p_j\}$ a little bit later. Within the $(x^i, p_j)$ sector, it is consistent to put $\{p_i, p_j\}$ to zero, but an extension below to other sectors of the Whitney sum $E \equiv T^*M \oplus E \oplus E^* \supseteq T^*M$ will complicate matters, cf. eq. (5.20) and Table 1. (The sign of the Poisson bracket (5.16), which is opposite of the standard physics conventions, has been chosen to minimize appearances of minus signs.)

We next introduce the notation $b_i \equiv \tilde{\partial}_i \in \Gamma(E)$ for the local fermionic basis of skewsymmetric multi-vectors, where as before the tilde “$\sim$” represents a parity-inversion, cf. eq. (5.27). The skewsymmetric multi-vectors
\[
\pi = \pi(x^i, b_i) \in \Gamma(\bigwedge^\bullet(TM)) \cong C^\infty(E^*) , \quad E^* \cong \Pi T^*M ,
\]
can be identified with functions on the parity-inverted cotangent bundle $\Pi T^*M \cong E^\ast$. Similarly, as explained in Subsection 5.1, we write $c^i \equiv dx^i \in \Gamma(E^\ast)$ for the local fermionic one-forms that constitute a basis for the exterior forms. Recall also that exterior forms (5.4) can be regarded as functions on the parity-inverted tangent bundle $E \equiv \Pi TM$. Note that the natural symmetric pairing

$$\langle \bar{\partial}_i, dx^j \rangle = \delta^j_i = \langle dx^j, \bar{\partial}_i \rangle$$

between exterior forms and multi-vectors can equivalently be viewed as a canonical Poisson bracket of fermions,

$$\{ b_i, c^j \} = \delta^j_i = \{ c^j, b_i \}, \quad \{ c^i, c^j \} = 0 = \{ b_i, b_j \}.$$ (5.18)

The idea is now to regard the two Poisson brackets (5.16) and (5.18) as part of the same symplectic structure on a 4d dimensional manifold, which we take to be the total space $E \equiv T^*M \oplus E \oplus E^\ast$ of the 3d dimensional vector bundle $E \to M$ with local coordinates $(x^i; p_i, c^i, b_i)$, and let the combined symplectic structure play the rôle of the Lie algebra structure $[\cdot, \cdot]$ of Section 4. Obviously this idea implies that one should fix the Poisson bracket in the cross-sectors between the even and the odd coordinates. In the end, it turns out that the cross-sector assignments do not matter, as they do not enter the Courant bracket in pertinent sectors. However to be specific, we shall model the cross-sectors over an arbitrary connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. In detail, the Poisson brackets between bosonic and fermionic variables are

$$\{ x^i, b_j \} = 0 = \{ b_j, x^i \}, \quad \{ x^i, c^j \} = 0 = \{ c^j, x^i \},$$

$$\{ p_i, b_j \} = \Gamma^k_{ij} b_k = -\{ b_j, p_i \}, \quad -\{ p_i, c^j \} = \Gamma^k_{ik} c^k = \{ c^j, p_i \},$$ (5.19)

cf. Table 1. To ensure the Jacobi identity for the $\{ \cdot, \cdot \}$ bracket, one finally defines the $\{ p_i, p_j \}$ sector to be

$$-\{ p_i, p_j \} = R^k_{\ell ij} c^\ell b_k =: \tilde{R}_{ij},$$ (5.20)

where

$$R^k_{\ell ij} = \frac{\partial \Gamma^k_{\ell ij}}{\partial x^\ell} + \Gamma^k_{im} \Gamma^m_{\ell ij} - (i \leftrightarrow j)$$ (5.21)

is the Riemann curvature tensor. For instance, the second Bianchi identity

$$0 = \sum_{i,j,k \text{ cycl.}} \left( \frac{\partial R^m_{nij}}{\partial x^k} + \Gamma^m_{kl} R^l_{nij} - \Gamma^k_{kn} R^m_{\ell ij} \right)$$ (5.22)

guarantees that the Jacobi identity holds in the $(p_i, p_j, p_k)$ sector.

A simpler picture emerges if one introduces the momentum variables [32, 33]

$$P_i := p_i + \Gamma^k_{ij} c^\ell b_k.$$ (5.23)

Table 1: The Poisson bracket $\{ \cdot, \cdot \}$.

|       | $x^3$ | $p_j$ | $c^j$ | $b_j$ |
|-------|-------|-------|-------|-------|
| $x^4$| 0     | $-\delta^j_i$ | 0     | 0     |
| $p_i$| $\delta^j_i$ | $-R_{ij}$ | $-\Gamma^j_{ik} c^k$ | $\Gamma^k_{ij} b_k$ |
| $c^i$| 0     | $\Gamma^i_{jk} c^k$ | 0     | $\delta^j_i$ |
| $b_i$| 0     | $-\Gamma^k_{ji} b_k$ | $\delta^j_i$ | 0     |
Table 2: The Poisson bracket $\{\cdot, \cdot\}$ in Darboux coordinates.

|   | $x^i$ | $P_j$ | $c^j$ | $b_j$ |
|---|-------|-------|-------|-------|
| $x^i$ | 0     | $-\delta^i_j$ | 0     | 0     |
| $P_j$ | $\delta^i_j$ | 0     | 0     | 0     |
| $c^j$ | 0     | 0     | 0     | $\delta^j_i$ |
| $b_j$ | 0     | 0     | $\delta^j_i$ | 0     |

Remarkably, the quadruple $(x^i, P_j, c^j, b_j)$ are local Darboux coordinates for the Poisson bracket $\{\cdot, \cdot\}$, cf. Table 2. The corresponding symplectic two-form is just the canonical two-form

$$\omega = dx^i \wedge dP_j + dc^j \wedge db_j, \quad d\omega = 0,$$

(5.24) cf. Table 3. Here $d$ denotes the de Rham exterior derivative on the Whitney sum $E$, $d = \partial_{x^i} + \partial_{p^i} + \partial_{c^j} + \partial_{b_j}$, $\epsilon = 0$, (5.25) which should not be confused with the de Rham exterior derivative $d$ on the base manifold $M$, cf. eq. (5.6). One may choose the symplectic potential $\vartheta$ to be

$$-\vartheta = P_idx^i - b_idc^i = dx^i P_i + dc^i b_i, \quad d\vartheta = \omega,$$

(5.26) Let us address the issue of coordinate transformations in the base manifold $M$. Because of the presence of the $\Gamma_{ij}^k$ symbol in the definition (5.23), the momentum variables $P_i$ do not have the simple co-vector transformation law that for instance the variables $p_i$ and $b_i$ enjoy. Nevertheless, the local expression (5.26) for the one-form $\vartheta$ is invariant. Hence $\vartheta$ is a globally defined symplectic potential for the $(2d|2d)$ symplectic manifold $E$ with an exact symplectic two-form $\omega = d\vartheta$. Moreover, the Whitney sum $E \cong T^*E$ may be identified with the cotangent bundle $T^*E$, where $E \equiv \Pi TM$ is the parity-inverted tangent bundle with local coordinates $(x^i, c^j)$. More precisely, the momentum variables $(P_i, b_i)$ can be identified with the fiber coordinates of $T^*E$, and $(P_i, b_i)$ are co-vectors in that sense [32, 33]. On the other hand, if one instead had started with the cotangent bundle $T^*E$ rather than the Whitney sum $E$, one would have gotten the symplectic structure (5.24) for free, without the use of a connection $\nabla$. Depending on the application, it is useful to do just that. The catch, is, that the $P_i$ variables follow a more complicated set of transformation rules than the $p_i$ variables.

There is a natural tilde isomorphism $\sim: \Gamma(TM) \to \Gamma(\Pi TM)$, which maps vectors to parity-inverted vectors,

$$\Gamma(TM) \ni X = X^i p_i \quad \sim \quad \epsilon_X \quad X^i b_i =: \tilde{X} \in \Gamma(\Pi TM), \quad \epsilon_{\tilde{X}} = 1 - \epsilon_X.$$

(5.27)

Table 3: The symplectic two-form $\omega$ in eq. (5.24) consists of the inverse matrix of Table 1.
In particular, vectors $X$ and parity-inverted vectors $\tilde{X}$ are bosons and fermions, respectively, if the coordinate functions $X^i = X^i(x)$ are bosonic, as is normally the case. More generally, we define the tilde operation $\sim: C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ to be the right derivation

$$\tilde{a} := (a \frac{\partial}{\partial p_i})b_i + (a \frac{\partial}{\partial b_i})p_i , \quad a = a(x^i, p_i, c^i, b_i) \in C^\infty(\mathcal{E}) , \quad \epsilon(\sim) = 1 . \quad (5.28)$$

The Poisson bracket on vectors and exterior forms mimics the Lie bracket, the covariant derivative and the interior product (cocontraction),

$$\{X, Y\} = \left( X^i \frac{\partial Y^j}{\partial x^i} p_j - (-1)^{\epsilon_X \epsilon_Y} (X \leftrightarrow Y) \right) - X^i Y^j \tilde{R}_{ij} =: [X, Y] - \{X, Y\}^\sim , \quad (5.29)$$

$$\{X, \tilde{Y}\} = X^i (\frac{\partial Y^j}{\partial x^i} b_j + \Gamma^k_{ij} Y^j b_k) =: (\nabla_X Y)^k b_k =: (\nabla_X \tilde{Y})^\sim , \quad (5.30)$$

$$\{\tilde{X}, \tilde{Y}\} = 0 , \quad \{X, \eta\} = X^i (\frac{\partial \eta}{\partial x^i} - \frac{\partial}{\partial x^i} - \Gamma^i_{jk} c^j \Gamma^k_{ij} p_k) =: \nabla_X \eta , \quad (5.31)$$

$$\{\tilde{X}, \eta\} = X^i \frac{\partial \eta}{\partial c^i} =: i_X \eta , \quad \{\xi, \eta\} = 0 , \quad \xi, \eta \in \Gamma(\Lambda^*(TM)) . \quad (5.32)$$

The sign conventions are,

$$\epsilon(i_X) = 1 - \epsilon_X = \epsilon_{\tilde{X}} , \quad i_{\lambda X} \eta = \lambda i_X \eta , \quad (\lambda X)^\sim = \lambda \tilde{X} , \quad X\lambda = (-1)^{\epsilon_X \epsilon_Y} \lambda X , \quad (5.33)$$

where $\lambda$ is a supernumber.

### 5.3 Anti-Symplectic Structure

There is a dual formulation in terms of an odd Poisson bracket, also known as an anti-bracket and traditionally denoted as $(\cdot, \cdot)$ in the physics literature. The symbol $(\cdot, \cdot)$ will be reserved for the even Poisson bracket introduced in the last Subsection 5.2. The anti-bracket $(\cdot, \cdot)$ on $E^* \subseteq \mathcal{E}$ is given as

$$\begin{align*}
(b_i, x^j) &= \delta^j_i = - (x^j, b_i) , \\
(x^i, x^j) &= 0 , \\
-(b_i, b_j) &= R^k_{ij} c^i p_k =: R_{ij} .
\end{align*} \quad (5.34)$$

|       | $x^j$ | $b_j$ | $c^j$ | $p_j$ |
|-------|-------|-------|-------|-------|
| $x^i$ | 0     | $-\delta^i_j$ | 0     | 0     |
| $b_i$ | $\delta^i_j$ | $-R_{ij}$ | $-\Gamma^k_{ij} c^k$ | $\Gamma^k_{ij} p_k$ |
| $c^i$ | 0     | $\Gamma^k_{ik} c^k$ | 0     | $-\delta^i_j$ |
| $p_i$ | 0     | $-\Gamma^k_{ik} p_k$ | $\delta^i_j$ | 0     |

Table 4: The anti-bracket $(\cdot, \cdot)$. 

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Table 5: The anti-bracket $(\cdot, \cdot)$ in Darboux coordinates.

|   | $x^j$ | $B_i$ | $c^j$ | $p_i$ |
|---|-------|-------|-------|-------|
| $x^i$ | 0 | $-\delta^j_i$ | 0 | 0 |
| $B_i$ | $\delta_i^j$ | 0 | 0 | 0 |
| $c^j$ | 0 | 0 | 0 | $-\delta^j_i$ |
| $p_i$ | 0 | 0 | $\delta_i^j$ | 0 |

It is well-known that in the flat case $R^k_{ij}=0$, this anti-bracket (5.36) is just the Schouten-Nijenhuis bracket

$$(\pi, \rho)_{SN} := \pi \left( \frac{\partial r}{\partial b_i} \frac{\partial}{\partial x^i} - \frac{\partial r}{\partial x^i} \frac{\partial}{\partial b_i} \right) \rho ,$$

when restricting to skew-symmetric multi-vectors $\pi, \rho \in \Gamma(\wedge^\bullet(TM)) \cong C^\infty(E^*)$. Similar to eq. (5.18), the natural skew-symmetric pairing $\langle \partial_i, dx^j \rangle = -\langle dx^j, \partial_i \rangle$ can be modelled over an anti-bracket,

$$(p_i, c^j) = \delta^j_i = -(c^j, p_i) ,$$

$$(c^i, c^j) = 0 = (p_i, p_j) .$$

The antibracket $(\cdot, \cdot)$ is extended to the Whitney sum $\mathcal{E}$ by fixing the cross-sectors as

$$(x^i, p_j) = 0 = (p_j, x^i) ,$$

$$(x^i, c^j) = 0 = (c^j, x^i) ,$$

$$(b_i, p_j) = \Gamma^k_{ij} p_k = -(p_j, b_i) ,$$

$${-}(b_i, c^j) = \Gamma^k_{ik} c^k = (c^j, b_i) ,$$

cf. Table 4.

When one defines the anti-field variables

$$B_i := b_i + \Gamma^k_{ij} c^j p_k ,$$

the quadruple $(x^i, B_i, c^j, p_i)$ become local Darboux coordinates for the anti-bracket $(\cdot, \cdot)$, cf. Table 5. The corresponding anti-symplectic two-form reads

$$\tilde{\omega} = dx^i \wedge dB_i + dc^j \wedge dp_i ,$$

$$d\tilde{\omega} = 0 ,$$

cf. Table 6. One may choose the following globally defined anti-symplectic potential

$$-\tilde{\vartheta} = B_i dx^i + p_i dc^i = dx^i B_i + dc^i p_i ,$$

$$d\tilde{\vartheta} = \tilde{\omega} .$$

Table 6: The anti-symplectic two-form $\tilde{\omega}$ in eq. (5.42) consists of the inverse matrix of Table 4.
Table 7: A list of manifolds in diagram (5.43).

| Manifold                          | Local coordinates | Structure            |
|-----------------------------------|-------------------|----------------------|
| Base manifold $M$                 | $(x^i)$           | Sympl. pot. $-p_idx^i$|
| Cotangent bundle $T^*M$           | $(x^i, p_i)$      | Anti-sympl. pot. $-b_idx^i$|
| Parity-inv. tang. bdl. $E \equiv \Pi TM$ | $(x^i, \dot c^i)$ | Symplectic potential $-P_idx^i + c^idb_i$|
| Parity-inv. cot. bdl. $\Pi T^*M \cong E^*$ | $(x^i, b_i)$     | Anti-symplectic potential $\tilde{\vartheta} = -B_idx^i - p_idc^i$ |
| Parity-inverted tangent bundle $\Pi T(E^*)$ | $(x^i, \dot{b_i}^i; \dot{c^i}; -P_i)$ | Symplectic potential $\vartheta = \dot{P_i}^i - \dot{b_i}^i$ |
| Cotangent bundle $T^*E$           | $(x^i, b_i; P_i, \dot c^i)$ | Anti-symplectic potential $\vartheta = -B_idx^i + b_idc^i$ |

Note that the anti-symplectic potential $\tilde{\vartheta}$ is – as the name suggests – equal to the symplectic potential (5.26) acted upon with the tilde operator “∼”, cf. eq. (5.28). The $(2d|2d)$ anti-symplectic manifold $\mathcal{E} \cong \Pi T^*E$ may be identified with the parity-inverted cotangent bundle $\Pi T^*E$. More precisely, the anti-field variables $(B_i, p_i)$ can be identified with the fiber coordinates of $\Pi T^*E$.

We here display a commutative diagram of various bundle isomorphisms, so-called Legendre transformations and canonical projection maps possible [29, 30, 32, 33]

$$
\begin{array}{c|c|c|c|c}
\text{Legendre} & T^*(E^*) & \cong & T^*E & \cong & \mathcal{E} & \cong & \Pi T^*E & \cong & \Pi T(T^*M) \\
\text{(if } \nabla \text{)} & \downarrow & \downarrow & \checkmark & \downarrow & \downarrow & \checkmark & \downarrow & \checkmark & \downarrow \\
\Pi T(E^*) & \rightarrow & E \oplus E^* & \rightarrow & M & \leftarrow & T^*M \oplus E & \leftarrow & \Pi T(T^*M) \\
\end{array}
$$

(5.43)

cf. Table 7. We shall actually only use the bundle isomorphism $T^*E \cong (\mathcal{E}; \nabla) \cong \Pi T^*E$, corresponding to the upper middle part of the diagram (5.43). These bundle identifications will from now on often be used without explicitly mentioning it, as it will be clear from the context whether they have been applied or not. The rest of the diagram (5.43) is shown for the sake of completeness. The Grassmann-odd tilde transformation (5.28) (which exchanges the fiber coordinates $p_i \leftrightarrow b_i$ and $P_i \leftrightarrow B_i$), is responsible for the apparent reflection symmetry along a vertical symmetry axis in the diagram (5.43).
We mention in passing that the anti-bracket may be encoded in a commutative Koszul hierarchy

\[ (-1)^{i^a}(a, b) = \Phi_\Delta^2(a, b) = [[\Delta, L_a], L_b]1 \, , \quad a, b \in C^\infty(\Pi T^*E) \, , \]

of an odd, nilpotent, second-order \( \Delta \) operator

\[
\Delta = \frac{\partial^i}{\partial x^i} \frac{\partial}{\partial t_i} + \frac{\partial}{\partial e^i} \frac{\partial}{\partial p_i}
\]

\[ = (\frac{\partial}{\partial x^i} - \Gamma^j_{ij} - \Gamma^k_{ij} p_k + \Gamma^j_{ik} c^k \frac{\partial}{\partial c^j} + \frac{1}{2} R_{ij} \frac{\partial}{\partial b_j} + \frac{\partial}{\partial e^j} \frac{\partial}{\partial p_i}) \, . \]

\[ \text{cf. Subsection 3.2. Both } \Delta \text{ formulas (5.45) and (5.46) are invariant under coordinate transformations in the base manifold } M \text{ without the use of a volume form } [12]. \text{ This is due to a balance of bosonic and fermionic degrees of freedom. The latter } \Delta \text{ formula (5.46), which uses the bundle isomorphism } \Pi T^*E \cong \mathcal E, \text{ cf. diagram (5.43), is presumably new. Note that the higher Koszul brackets and the Koszul zero-bracket vanish, i.e. } \Phi^\text{n}_\Delta = 0 \text{ for } n \geq 3 \text{ and for } n = 0, \text{ so this is an example of a commutative Batalin-Vilkovisky algebra.} \]

The anti-bracket on vectors and exterior forms mimics the covariant derivative, the Lie bracket and the interior product, as was the case for the even Poisson bracket \{\cdot, \cdot\}. However in the anti-bracket case the rôles of \( X \) and \( \bar X \) are exchanged,

\[ (X, Y) = 0 \, , \quad X, Y \in \Gamma(TM) \, , \]

\[ (\bar X, Y) = X^i (\frac{\partial Y^j}{\partial x^i} p_j + \Gamma^k_{ij} Y^j p_k) =: (\nabla_X Y)^k p_k =: \nabla_X Y \, , \]

\[ (\bar X, \bar Y) = \left( X^i \frac{\partial Y^j}{\partial x^i} b_j - (1)^{\epsilon_X \epsilon_Y} (X \leftrightarrow Y) \right) - X^i Y^j R_{ij} \]

\[ = [X, Y]^\sim - R(X, Y) \, , \]

\[ (X, \eta) = X^i \frac{\partial \eta}{\partial c^i} =: i_X \eta \, , \]

\[ (\bar X, \eta) = X^i \frac{\partial \eta}{\partial x^i} - \Gamma^j_{ik} c^k \frac{\partial \eta}{\partial c^j} \eta =: \nabla_X \eta \, , \]

\[ (\xi, \eta) = 0 \, , \quad \xi, \eta \in \Gamma(\bigwedge^*(T^*M)) \, . \]

\[ \text{5.4 Derived Brackets} \]

We now use the even and odd symplectic structure \{\cdot, \cdot\} and \( \langle \cdot, \cdot \rangle \) from Subsections 5.2-5.3 to define a homotopy Lie algebra structure on \( C^\infty(T^*E) \) and \( C^\infty(\Pi T^*E) \), respectively. Using the bundle isomorphisms of diagram (5.43), we then obtain two homotopy Lie algebra structures on the same algebra \( C^\infty(\mathcal E) \). The even and odd symplectic structure \{\cdot, \cdot\} and \( \langle \cdot, \cdot \rangle \) will here both play the rôles of the Lie algebra structure \{\cdot, \cdot\} of Section 4. It comes in handy that we have left the parity \( \epsilon \in \{0, 1\} \) of the \{\cdot, \cdot\} bracket open, so that we readily can model both even and odd brackets. Similarly, we will have two generators for the derived \( \bullet \)-brackets, a Grassmann-odd generator \( Q \) and a Grassmann-even generator \( S \), which (despite the notation) both will play the rôle of the fixed Lie algebra element \( Q \) of Section 4. The new \( Q \) and \( S \) are defined as [25]

\[ Q := c^i p_i + H_{\text{odd}} = c^i p_i + \frac{1}{2} c^j c^k T^j_{ij} b_k + H_{\text{odd}} \, , \quad \epsilon_Q = 1 \, , \]

\[ S := c^i B_i + H_{\text{even}} = c^i b_i + \frac{1}{2} c^j c^k T^j_{ij} p_k + H_{\text{even}} \, , \quad \epsilon_S = 0 \, , \]

(5.53)
where the exterior forms \( H_{\text{odd}} \in \Gamma(\Lambda^{\text{odd}}(T^*M)) \) (resp. \( H_{\text{even}} \in \Gamma(\Lambda^{\text{even}}(T^*M)) \)) are closed Grassmann-odd (resp. Grassmann-even) forms. The local expressions for \( Q \in C^\infty(T^*E) \) and \( S \in C^\infty(\Pi T^*E) \) are invariant under coordinate transformations in the base manifold \( M \), and hence define global scalars. Here \( T^k_{ij} \) in eq. (5.53) is the torsion tensor,

\[
T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.
\]  

(5.54)

The generators \( Q \) and \( S \) are nilpotent in the Poisson and anti-bracket sense,

\[
\{Q, Q\} = 2c^i \partial H_{\text{odd}} \partial x^i = 2dH_{\text{odd}} = 0,  
\]

(5.55)

\[
(S, S) = 2c^i \partial H_{\text{even}} \partial x^i = 2dH_{\text{even}} = 0,  
\]

(5.56)

respectively, because \( H_{\text{odd}} \) and \( H_{\text{even}} \) are closed. (The nilpotency (5.55) and (5.56) are also due to the first Bianchi identity,

\[
\sum_{i,j,k \text{ cycl.}} R^\ell_{kij} = \sum_{i,j,k \text{ cycl.}} \left( \frac{\partial T^\ell_{ij}}{\partial x^k} + \Gamma^\ell_{km} T^m_{ij} \right),  
\]

(5.57)

if one starts from the latter expressions in eq. (5.53) that depends on \( p_i \) and \( b_i \) explicitly.) We stress that the nilpotent generators \( Q \) and \( S \), and similarly, the even and odd Poisson structures \( \{\cdot, \cdot\} \) and \( \langle \cdot, \cdot \rangle \), are on completely equal footing, regardless of what the notation might suggest. (The notation is inspired by the physics literature on constrained dynamics; for instance the eq. (5.56) resembles the Classical Master Equation of Batalin and Vilkovisky.)

The generators \( Q \) and \( S \) turn \( C^\infty(T^*E) \) and \( C^\infty(\Pi T^*E) \) into homotopy Lie algebras with derived \( \bullet \)-brackets \( \Phi_Q \) and \( \Phi_S \), respectively. The zero-bracket \( \Phi_Q^0 = Q \) is the generator \( Q \) itself. The one-bracket \( \Phi_Q = \{Q, \cdot\} \) is a nilpotent, Grassmann-odd, Hamiltonian vector field on \( T^*E \), cf. eq. (4.38). Similarly for the \( \Phi_S \) bracket hierarchy. For exterior forms \( \eta \in C^\infty(E) \) (which can be viewed as functions on any of the three bundles \( E, T^*E \) and \( \Pi T^*E \) mentioned in Table 7), the one-brackets are just the de Rham exterior derivative \( d \),

\[
\Phi_Q(\eta) = \{Q, \eta\} = c^i \frac{\partial \eta}{\partial x^i} = d\eta,  
\]

\[
\Phi_S(\eta) = (S, \eta) = \langle S, \eta \rangle  
\]

(5.58)

independently of \( H_{\text{odd}}, H_{\text{even}} \) and \( \nabla \). On vectors \( X \in \Gamma(TM) \), the one-bracket \( \Phi_Q \) reads

\[
\Phi_Q(X) = \{Q, X\} = dX^i + c^j(\Gamma^i)^k_j X^j P_k - c^j c^j \partial_i (\Gamma^i)^k_j b_k X^\ell - (-1)^{\epsilon \xi} \nabla_X H_{\text{odd}},  
\]

\[
\Phi^1_S(X) = (S, X) = dX^i + c^j(\Gamma^i)^k_j X^j B_k - (-1)^{\epsilon \xi} c^j c^j \partial_i (\Gamma^i)^k_j p_k X^\ell - (-1)^{\epsilon \xi} \nabla_X H_{\text{even}},  
\]

\[
\Phi_Q(\bar{X}) = \{Q, \bar{X}\} = (-1)^{\epsilon \xi} X^i P_i + dX^i + (-1)^{\epsilon \xi} \imath_X H_{\text{odd}} \in C^\infty(T^*E)  
\]

\[
(-1)^{\epsilon \xi} X^i + \nabla^i X^i + (-1)^{\epsilon \xi} \imath_X H_{\text{odd}},  
\]

\[
\Phi^1_S(\bar{X}) = (S, \bar{X}) = B_i X^i + dX + (-1)^{\epsilon \xi} \imath_X H_{\text{even}} \in C^\infty(\Pi T^*E)  
\]

\[
(-1)^{\epsilon \xi} X^i + \nabla^i X^i + (-1)^{\epsilon \xi} \imath_X H_{\text{even}},  
\]

(5.59) \hspace{1cm} (5.60) \hspace{1cm} (5.61) \hspace{1cm} (5.62)

where \( \nabla = c^i \nabla_i \) is the connection one-form, \( R^k_{\ell ij} = \frac{1}{2} c^c c^j R^k_{cij} \) is the curvature two-form, and \( \nabla^i \) denotes the transposed connection, which is defined as \( (\Gamma^i)^k_j := \Gamma^k_{ij} \). The geometric importance of these one-brackets is underscored by the fact that their adjoint action (in the Poisson or anti-bracket sense) reproduce the Lie derivative on exterior forms

\[
\{\Phi_Q^1(\bar{X}), \eta\} = \{Q, \bar{X}, \eta\} + (-1)^{\epsilon \xi} \{\bar{X}, Q, \eta\} \]  

(5.63)
independently of $H_{\text{odd}}$, $H_{\text{even}}$ and $\nabla$. The two-brackets $\Phi^2_Q$ and $\Phi^2_S$ give rise to an odd and an even Courant bracket, $(\cdot, \cdot)_Q$ and $\{\cdot, \cdot\}_S$, respectively, cf. eq. (4.10).

**Definition 5.2** The odd and even Courant brackets $(\cdot, \cdot)_Q$ and $\{\cdot, \cdot\}_S$ are defined as the derived brackets,

$$(a, b)_Q := (-1)^{e_a+1} \Phi^2_Q(a, b) = \frac{1}{2}\{a, Q\}, b\} + \frac{1}{2}\{a, Q, b\} \quad a, b \in C^\infty(T^*E),$$

$$(a, b)_S := (-1)^{e_a+1} \Phi^2_S(a, b) = \frac{1}{2}\{(a, S), b\} + \frac{1}{2}\{(a, S), b\} \quad a, b \in C^\infty(\Pi T^*E),$$

respectively.

Notice the complete democracy among the even and odd brackets in the Definition 5.2. By restricting to vector fields and exterior forms one finds the celebrated formulas (5.1)-(5.3) for the Courant bracket,

$$\{X, Y\}_S = \{X, Y\} = [X, Y] + (-1)^{\nu} i_X i_Y H_{\text{even}}, \quad X, Y \in \Gamma(TM),$$

$$(\tilde{X}, \tilde{Y})_Q = [X, Y] + (-1)^{\nu} i_X i_Y H_{\text{odd}},$$

$$\{X, \eta\}_S = (\tilde{X}, \eta)_Q = \frac{1}{2}(\mathcal{L}_X + i_X d)\eta,$$

$$\{\xi, \eta\}_S = 0,$$

$$\{\xi, \eta\}_Q = 0, \quad \xi, \eta \in \Gamma(\wedge(T^*M)),$$

which do not use the bundle isomorphisms of diagram (5.43), and hence are independent of $\nabla$; plus one finds the following formulas

$$(X, Y)_Q = \frac{1}{2} \nabla_X [\nabla^t Y] - \frac{1}{2} \nabla^t X (\nabla^t Y)^i - \frac{(-1)^{\nu}}{2} \nabla_X \nabla_Y H_{\text{odd}} + \mathcal{O}(c^2 b) - (-1)^{e_X+1} (e_Y+1) (X \leftrightarrow Y),$$

$$(\tilde{X}, \tilde{Y})_S = \frac{1}{2} \nabla_X [\nabla^t \tilde{Y}] + \frac{(-1)^{\nu}}{2} \left( \nabla^t \tilde{X} (\nabla^t Y)^i - \nabla_X \nabla_Y H_{\text{even}} \right) + \mathcal{O}(c^2 p) - (-1)^{e_X+1} (e_Y+1) (X \leftrightarrow Y),$$

$$(X, \eta)_Q = (\tilde{X}, \eta)_S = \frac{1}{2} ([\nabla_X, d] + \nabla_X d)\eta = \nabla_X d\eta - \frac{(-1)^{\nu}}{2} d\nabla_X \eta,$$

which do use the bundle isomorphisms of diagram (5.43), and whose right-hand sides are second-order differential operators and not particularly illuminating. The lesson to be learned, is, that one should stick to vector fields without parity-inversion $X \in \Gamma(TM) \subseteq C^\infty(\Pi T^*E)$ for the $\Phi_S$ brackets on the algebra $C^\infty(\Pi T^*E)$, and to vector fields with parity-inversion $\tilde{X} \in \Gamma(E) \subseteq C^\infty(T^*E)$ for the $\Phi_Q$ brackets on the algebra $C^\infty(T^*E)$.

### 5.5 Discussion

Obviously, we have only scratched the surface. One can for instance also calculate what the Courant brackets should be on skewsymmetric multi-vectors. For instance, the one-bracket $\Phi^1_Q$ on a multi-vector $\pi \in \Gamma(\bigwedge(T^*E)) \cong C^\infty(E^*) \subseteq C^\infty(T^*E)$ reads

$$\Phi^1_Q(\pi) = d\pi + (P_i + H_{\text{odd}} \frac{\partial}{\partial c^i}) \frac{\partial}{\partial b^i} \pi \in C^\infty(T^*E)$$

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The Courant two-bracket \((\cdot, \cdot)_Q\) becomes a twisted version of the Schouten-Nijenhuis bracket (5.37),
\[
(\pi, \rho)_Q = (\pi, \rho)_{SN} + (\pi \frac{\partial}{\partial b_i} H_{odd} \frac{\partial}{\partial c_j})(\rho \frac{\partial}{\partial b_j} H_{odd} \frac{\partial}{\partial c_i}) \in C^\infty(T^*E) , \quad \pi, \rho \in \Gamma(\bigwedge^\bullet(TM)) ,
\]
which does not depend on the \(P_i\) variables. However, the Courant bracket \((\pi, \eta)_Q\) between a higher-order skew-symmetric multi-vector, \(\pi\), and a higher-order form, \(\eta\), does not close on such objects, but will in general also depend on the \(P_i\) variables. A complete treatment of the Courant bracket would therefore include an investigation of form-valued multi-vectors, where the word “multi-vector” here should be understood in a generalized sense that depends on both the bosonic and fermionic generators, \(p_i \equiv \partial_i \in \Gamma(TM)\) and \(b_i \equiv \tilde{\partial}_i \in \Gamma(IITM)\), or the analogues obtained via the bundle identifications of diagram (5.43). It is then natural, in turn, to allow the twisting \(H_{odd}\) (resp. \(H_{even}\)) to be an odd (resp. even) form-valued multi-vector as well.

The even or odd Poisson construction may be generalized further to an arbitrary Poisson or anti-Poisson manifold with a nilpotent function \(Q\) or \(S\) satisfying \(\{Q, Q\} = 0\) or \(\{S, S\} = 0\), respectively.

Finally, let us mention that the odd bracket \((\cdot, \cdot)_Q\) may be viewed as a classical counterpart of the odd operator bracket \([\cdot, \cdot]_D\) in the spirit of deformation quantization. In detail, one defines a hat quantization map “\(^{\wedge}\)” that takes functions \(a\) (also known as symbols) to normal-ordered differential operators \(\hat{a}\),
\[
C^\infty(T^*E) \ni a = a(x^i, c^j, P_i, b_i) \xrightarrow{^{\wedge}} \hat{a} = : a(x^i, c^j, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial c^j}) : = a \exp\left(\frac{\partial}{\partial P_i} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial b_i} \frac{\partial}{\partial c^i}\right) \Bigg|_{P_i = 0, b_i = 0} \in \text{End}(C^\infty(E)) .
\]

The colons in eq. (5.76) indicate normal-ordering, which means that the derivatives are ordered to the rightmost position. Examples of the “\(^{\wedge}\)” quantization map are
\[
\hat{\eta} = \eta , \quad \hat{X} = i_X , \quad \hat{Q} = D , \quad \hat{\partial}_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij} c^j \frac{\partial}{\partial c^k} = \nabla_i ,
\]
\begin{equation}
\text{cf. eqs. (5.4), (5.8), (5.53) and (5.5). The algebra isomorphism } \hat{a} \circ \hat{b} = \hat{a \star b} \text{ between corresponding associative algebras is given by the } \star \text{ product},
\end{equation}
\[
a \star b = (a \exp(\frac{\partial}{\partial P_i} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial b_i} \frac{\partial}{\partial c^i}) b) \in C^\infty(T^*E) , \quad a, b \in C^\infty(T^*E) .
\]

The outer parenthesis on the right-hand side of eq. (5.78) indicates that the expression should be interpreted as a function, i.e. a zeroth-order differential operator. The “\(^{\wedge}\)” quantization map (5.76) is clearly related to the symplectic structure (5.24). In fact, the \(\star\) commutator \([a, b]_\star \equiv a \star b - (-1)^{\epsilon_a \epsilon_b} b \star a\) is a quantum deformation of the Poisson bracket \(\{a, b\}\), where the Planck constant here is set equal to the value \(-2\pi i\), that is, \(i\hbar = 1\). In the same way we get the new result, that the Courant operator bracket
\[
[\hat{a}, \hat{b}]_D = \frac{1}{2} (\{a, Q\} + [a, [Q, b]]_\star)^\wedge , \quad a, b \in C^\infty(T^*E) ,
\]
\begin{equation}
\text{where the odd bracket } [\cdot, \cdot]_\star \text{ on odd objects is defined as in eq. (5.51) and (5.52).}
\end{equation}
is a quantum deformation of the classical Courant bracket \((a, b)_Q = \frac{1}{2}\{\{a, Q\}, b\} + \frac{1}{2}\{a, \{Q, b\}\} \). 

Recall that the bracket \((\cdot, \cdot)_Q\) and the operator counterpart \([\cdot, \cdot]_D\) are both odd brackets. They both realize the Courant bracket (5.1)-(5.3) with the help of parity-inverted vectors, either directly via \(\tilde{X} \in C^\infty(T^*E)\), or via the contraction \(i_X \in \text{End}(C^\infty(E))\). They are also both capable of reproducing the same twisting with an odd closed form \(H_{\text{odd}}\). On the other hand, the possibility of twisting the Courant bracket with an even closed form \(H_{\text{even}}\) seems to have gone unnoticed so far in the literature. The even bracket \(\{\cdot, \cdot\}_S\) realizes just this possibility.

### 5.6 Higher Brackets

It is a major point that the derived bracket construction (4.6) of the Courant bracket automatically provides us with an infinite tower of higher Courant brackets and a host of nilpotency relations (4.13) corresponding to all the allowed values of the \(e^n_k\) product coefficients found in Section 4, like for instance the Ward identity (4.36) and the generalized Jacobi identity (4.43). In this Subsection we calculate the higher Courant brackets in pertinent sectors. The connection \(\nabla\) and the bundle isomorphisms of diagram (5.43) are not used in the rest of the paper.

**Proposition 5.3** The higher Courant brackets among vectors read

\[
\Phi^n_S(X_1, \ldots, X_n) = i_{X_1} \cdots i_{X_n} H_{\text{even}}, \quad X_1, \ldots, X_n \in \Gamma(TM),
\]

\[
\Phi^n_Q(\tilde{X}_1, \ldots, \tilde{X}_n) = \Phi^n_D(i_{X_1}, \ldots, i_{X_n}) \quad \epsilon(X_1) = 0, \ldots, \epsilon(X_n) = 0, \quad \epsilon(\tilde{X}_1) = 0, \ldots, \epsilon(\tilde{X}_n) = 0 \quad \text{(5.80)}
\]

for \(n \in \{3, 4, 5, \ldots\}\). The Courant three-bracket between two vectors \(X, Y \in \Gamma(TM)\) with Grassmann-parity \(\epsilon_X = 0 = \epsilon_Y\) and one exterior form \(\eta \in \Gamma(\Lambda^s(T^*M))\), is

\[
\Phi^3_S(X, Y, \eta) = \Phi^3_Q(\tilde{X}, \tilde{Y}, \eta) = \Phi^3_D(i_X, i_Y, \eta)
\]

\[
= \frac{1}{3} \left( i_{[X,Y]} + \frac{1}{2} (i_X \mathcal{L}_Y - i_Y \mathcal{L}_X) + i_X i_Y d \right) \eta
\]

\[
= \left( \frac{1}{2} (i_X \mathcal{L}_Y - i_Y \mathcal{L}_X) + \frac{1}{3} di_X i_Y \right) \eta
\]

\[
= \frac{1}{2} \left( i_X i_Y d - \frac{1}{3} di_X i_Y + i_{[X,Y]} \right) \eta \quad \text{(5.81)}
\]

The higher Courant brackets of vectors and one exterior form are given recursively as

\[
\Phi^{n+1}_S(X_1, \ldots, X_n, \eta) = \Phi^{n+1}_Q(\tilde{X}_1, \ldots, \tilde{X}_n, \eta) = \Phi^{n+1}_D(i_{X_1}, \ldots, i_{X_n}, \eta)
\]

\[
= \frac{1}{n+1} \sum_{1 \leq i \leq n} (-1)^{i-1} i_{X_i} \Phi^i_Q(\tilde{X}_1, \ldots, \tilde{X}_n, \eta) \quad \text{(5.82)}
\]

for \(n \in \{3, 4, 5, \ldots\}\) and \(\epsilon(X_1) = 0, \ldots, \epsilon(X_n) = 0\) bosonic, or directly as

\[
\Phi^{n+1}_Q(\tilde{X}_1, \ldots, \tilde{X}_n, \eta) = \frac{6}{(n+1)n(n-1)} \sum_{1 \leq i < j \leq n} (-1)^{(n-1-i)+(n-j)}
\]

\[
\times i_{X_1} \cdots \hat{i}_{X_1} \cdots \hat{i}_{X_2} \cdots \hat{i}_{X_{j-1}} \cdots i_{X_n} \Phi^3_Q(\tilde{X}_i, \tilde{X}_j, \eta) \quad \text{(5.83)}
\]

for \(n \in \{2, 3, 4, \ldots\}\). Brackets between vectors and exterior forms with more than one exterior form as argument vanish.
vanishing of higher brackets with two or more exterior forms follows from eqs. (5.34) and (5.52).

Proof of Proposition 5.3: The calculations are most efficiently done along the diagonal $X^\otimes n$ with the vector $X$ taken to be a fermion, $\epsilon_X = 1$. Hence the parity-inverted vector $\tilde{X}$ and $i_X$ are bosons. One finds

$$
\begin{align*}
\Phi^2_S(X \otimes X) &= -[X, X] + i_X^2 H_{\text{even}}, \\
\Phi^2_Q(\tilde{X} \otimes \tilde{X}) &= -[X, X] + i_X^2 H_{\text{odd}}, \\
\Phi^n_D(i_X \otimes i_X) &= -i_{[X, X]} + i_X^2 H_{\text{even}}, \\
\Phi^n_S(X^\otimes n) &= (-i_X)^n H_{\text{even}}, \\
\Phi^n_S(\tilde{X}^\otimes n) &= (-i_X)^n H_{\text{odd}},
\end{align*}
$$

for $n \in \{3, 4, 5, \ldots \}$. These facts imply that the expansion eq. (4.24) truncates after only three terms,

$$
\Phi^n_Q(\tilde{X}^\otimes (n-1) \otimes \eta) = \frac{1}{n} \sum_{k=0}^{n-1} (-\text{ad} \tilde{X})^{n-1-k} \{ \Phi^k_Q(\tilde{X}^\otimes k), \eta \}
$$

with $\epsilon_X = 1$ and $n \in \{3, 4, 5, \ldots \}$. Similarly for the $S$ and $D$ hierarchies. In the $n=3$ case this reads

$$
\begin{align*}
\Phi^3_S(X \otimes X \otimes \eta) &= \Phi^3_Q(\tilde{X} \otimes \tilde{X} \otimes \eta) = \Phi^n_D(i_X \otimes i_X \otimes \eta) = \frac{1}{3} \left( i_X^2 d + i_X L_X - i_{[X, X]} \right) \eta \\
&= \left( i_X L_X + \frac{1}{3} d i_X^2 \right) \eta = \frac{1}{2} \left( i_X^2 d - \frac{1}{3} d^2 i_X - i_{[X, X]} \right) \eta, \quad \epsilon_X = 1.
\end{align*}
$$

Now apply polarization $X = \sum_{i=1}^n \lambda^{(i)} X^{(i)}$ with $\epsilon(\lambda^{(i)}) = 1$ and $\epsilon(X^{(i)}) = 0$, cf. Subsection 2.4. The vanishing of higher brackets with two or more exterior forms follows from eqs. (5.34) and (5.52).

\[\square\]

5.7 B-transforms

In this Subsection the $B$-transforms [15, 16] is generalized to the higher brackets. The $B$-transforms in this context are canonical or anti-canonical transformations generated by even or odd exterior forms $B_{\text{even}}$ or $B_{\text{odd}}$, respectively. It follows immediately from the derived bracket definition (4.6) that

$$
\begin{align*}
e^{-[B_{\text{even}}, \cdot]} \Phi^D_{D'}(a_1, \ldots, a_n) &= \Phi^D_{D'} \left( e^{-[B_{\text{even}}, \cdot]} a_1, \ldots, e^{-[B_{\text{even}}, \cdot]} a_n \right), \\
e^{-[B_{\text{even}}, \cdot]} \Phi^Q_{Q'}(a_1, \ldots, a_n) &= \Phi^Q_{Q'} \left( e^{-[B_{\text{even}}, \cdot]} a_1, \ldots, e^{-[B_{\text{even}}, \cdot]} a_n \right), \\
e^{-[B_{\text{odd}}, \cdot]} \Phi^S_{S'}(a_1, \ldots, a_n) &= \Phi^S_{S'} \left( e^{-[B_{\text{odd}}, \cdot]} a_1, \ldots, e^{-[B_{\text{odd}}, \cdot]} a_n \right),
\end{align*}
$$

where we have defined

$$
D' := e^{-[B_{\text{even}}, \cdot]} D, \quad Q' := e^{-[B_{\text{even}}, \cdot]} Q, \quad S' := e^{-[B_{\text{odd}}, \cdot]} S.
$$

The $B$-transforms are a symmetry of the derived brackets, i.e.

$$
D' = D, \quad Q' = Q, \quad S' = S,
$$

33
if the $B$-forms are closed,

$$[D, B_{\text{even}}] = dB_{\text{even}} = 0, \quad \{Q, B_{\text{even}}\} = dB_{\text{even}} = 0, \quad (S, B_{\text{odd}}) = dB_{\text{odd}} = 0.$$ (5.94)

In this way it becomes obvious, that closed $B$-transforms are algebra automorphisms for the full Courant $\bullet$-bracket hierarchy [15, 16].

6 Supplementary Formalism

Section 6 is an open-ended investigation, where we make contact to some notions, that could be useful for future studies. More specifically, we go back to the original setup of Section 2 and touch on some of the theoretical aspects of a homotopy Lie algebra $\mathcal{A}$, such as properties of the bracket product $\circ$ and co-algebraic structures on $\text{Sym}^\bullet \mathcal{A}$.

6.1 Pre-Lie Products

What properties should one demand of the $\circ$ product? Associativity is too strong: This is not even fulfilled for the ordinary product with coefficients $c^n_k = 1$. The next idea is to let the product $\circ$ be pre-Lie. To measure the non-associativity one usually defines the associator

$$\text{ass}(\Phi, \Phi', \Phi'') := (\Phi \circ \Phi') \circ \Phi'' - \Phi \circ (\Phi' \circ \Phi'').$$ (6.1)

**Definition 6.1** The bracket product $\circ$ is pre-Lie if for all $\bullet$-brackets $\Phi, \Phi', \Phi'' : \text{Sym}^\bullet \mathcal{A} \rightarrow \mathcal{A}$ the associator is symmetric in the last two entries,

$$\text{ass}(\Phi, \Phi', \Phi'') = (-1)^{\epsilon_{\Phi'} \epsilon_{\Phi''}} \text{ass}(\Phi, \Phi'', \Phi').$$ (6.2)

For a pre-Lie product $\circ$ the commutator

$$[\Phi, \Phi'] := \Phi \circ \Phi' - (-1)^{\epsilon_{\Phi'} \epsilon_{\Phi''}} \Phi' \circ \Phi$$

becomes a Lie bracket that satisfies the Jacobi identity, hence the name “pre-Lie”. One may simplify the pre-Lie condition (6.2) by polarization into

$$\text{ass}(\Phi, \Phi', \Phi') = 0, \quad \epsilon_{\Phi'} = 1,$$ (6.4)

cf. Subsection 2.4. We now give a necessary and sufficient condition in terms of the $c^n_k$ coefficients for the $\circ$ product to be pre-Lie.

**Proposition 6.2** A $\circ$ product is pre-Lie if and only if the $c^n_k$ product coefficients satisfy the following two conditions

$$\forall k, \ell, m \geq 0 : \begin{cases} c_k^{k+\ell+m} c_{\ell+1}^{\ell+1} = c_k^{k+\ell+m} c_{k+\ell}^k, \\ c_k^{k+\ell+m} c_{\ell+1}^{\ell+1} = (k \leftrightarrow \ell). \end{cases}$$ (6.5)

**Proof of Proposition 6.2:** To see eq. (6.5), first note that for two vectors $a, b \in \mathcal{A}$ with $\epsilon(a) = \epsilon$,

$$(\Phi \circ \Phi')^{n+1} (b \odot a^{\odot n}) = \sum_{k=0}^{n} \binom{n+1}{k+1} \Phi^{n-k+1} (\Phi^{k+1} (b \odot a^{\odot k}) \odot a^{\odot (n-k)})$$
Theorem 6.3

The \( \text{“non-zero complex numbers} \ \lambda \ \text{We would like to find the possible} \)

Next insert the two expressions (6.7) and (6.8) into the pre-Lie condition (6.4) with \( \Phi' = \Phi'' \text{ odd. By comparing coefficients one derives} \)

On the other hand,

Next insert the two expressions (6.7) and (6.8) into the pre-Lie condition (6.4) with \( \Phi' = \Phi'' \text{ odd. By comparing coefficients one derives} \)

The two conditions (6.5) follow by translating (6.9) into the \text{“c” picture with the help of eq. (2.27).} \)

We would like to find the possible \( c^n_k \text{ coefficients that solves the two necessary and sufficient pre-Lie} \)

\textbf{Theorem 6.3} \text{ The \text{“c” product is pre-Lie and satisfies the condition (6.10), if and only if there exist} non-zero complex numbers} \( \lambda_n, n \in \{0,1,2,\ldots\}, \text{ such that} \)

In other words, a generic pre-Lie product is essentially just an ordinary product \( c^n_k = 1 \text{ with rescaled} \)

\textit{Proof of Theorem 6.3:} It is simple to check that the solution (6.11) satisfies the two pre-Lie conditions (6.5) and the technical condition (6.10). Now let us prove the other direction. Putting \( k = 0 \text{ in the first of the two conditions (6.5), one gets after relabelling} \)

\begin{align*}
\forall k, n : \quad 0 \leq k \leq n & \quad \Rightarrow \quad c_0^n c_{k+1}^n = c_0^k c_k^n .
\end{align*}
One may apply eq. (6.12) twice to produce
\[ \forall n \in \{0, 1, 2, \ldots\} : \quad c_0^n c_0^{n+1} c_2^{n+2} = c_0^n c_0^0 c_0^1 . \]  

Equation (6.13) and the assumption (6.10) lead to
\[ \forall n \in \{0, 1, 2, \ldots\} : \quad c_0^n \neq 0 \]  

by an inductive argument. Therefore one may define non-zero numbers
\[ \lambda_n := \frac{c_0^{n-1}}{\prod_{k=0}^{n-1} c_0^k} = c_0^{n-1} \lambda_{n-1}, \quad \lambda_0 := 1. \]  

Repeated use of (6.12) leads to the solution (6.11). Interestingly in the generic case (6.10), one does not need the second of the two pre-Lie conditions (6.5) to derive the solution (6.11).

\[ \square \]

### 6.2 A Co-Product

Similar to the “◦” bracket product construction (2.3) one may define a co-product \( \triangle \) on \( \text{Sym}^\bullet A \).

**Definition 6.4** Let there be given a set of complex numbers \( \gamma_k^n \) with \( n \geq k \geq 0 \). The co-product \( \triangle : \text{Sym}^\bullet A \to \text{Sym}^\bullet A \otimes \text{Sym}^\bullet A \) is then defined as
\[
\triangle(a_1 \odot \ldots \odot a_n) := \sum_{k=0}^{n} \frac{\gamma_k^n}{k!(n-k)!} \sum_{\pi \in S_n} (-1)^{\epsilon(\pi)} (a_{\pi(1)} \odot \ldots \odot a_{\pi(k)}) \odot (a_{\pi(k+1)} \odot \ldots \odot a_{\pi(n)}) \]  

for \( n \in \{0, 1, 2, \ldots\} \).

We denote the coefficients \( \gamma_k^n, n \geq k \geq 0 \), with a Greek \( \gamma \) to stress that they in general differ from the “◦” bracket product coefficients \( c_k^n \). Recall that “\( \odot \)” and “\( \odot \)” denote the un-symmetrized and symmetrized tensor product in the tensor algebras \( T^\bullet A \) and \( \text{Sym}^\bullet A \), respectively, cf. eq. (2.1). The Grassmann parity of the co-product \( \triangle \), and the bracket product “◦” are assumed to be bosonic,
\[ \epsilon(\triangle) = \epsilon(\odot) = 0, \]  

while the parity of the symmetrized and the un-symmetrized tensor products “\( \odot \)” and “\( \odot \)” follows the suspension parity,
\[ \epsilon(\odot) = \epsilon(\odot) = \epsilon. \]  

The co-product definition (6.16) is by polarization equivalent to
\[ \triangle(a^{\odot n}) = \sum_{k=0}^{n} \beta_k^n a^{\odot k} \odot a^{\odot (n-k)}, \quad \epsilon(a) = \epsilon, \]  

where we assume that an analogue of eq. (2.27) holds for the Greek co-product coefficients \( \beta_k^n \) and \( \gamma_k^n \),
\[ \beta_k^n \equiv \binom{n}{k} \gamma_k^n, \quad 0 \leq k \leq n. \]  

The standard co-product \( \triangle \) on \( \text{Sym}^\bullet A \) corresponds to coefficients \( \gamma_k^n = 1 \), cf. Ref. [27].
Definition 6.5 A co-product $\Delta$ is co-associative if
\[
(1 \otimes \Delta) \Delta = (\Delta \otimes 1) \Delta .
\] (6.21)

Co-associativity is equivalent to
\[
\forall k, \ell, m \geq 0 : \beta^k_{\ell+m} \beta^\ell_m = \beta^k_{\ell+m} \beta^k_{\ell},
\] (6.22)
which generically looks like $\beta^k_{\ell+m} = \frac{\lambda_k \lambda^m_{\ell}}{\lambda_{k+m}}$. Similarly, a co-product $\Delta$ is co-commutative if $\beta^k = \beta^{n-k}$.

Definition 6.6 A linear operator $\delta : \text{Sym}^* A \rightarrow \text{Sym}^* A$ is a co-derivation if
\[
\Delta \delta = (\delta \otimes 1 + 1 \otimes \delta) \Delta .
\] (6.23)

6.3 A Lifting

We now describe a lifting map “$\sim$”: $\Phi \mapsto \tilde{\Phi}$ for $\bullet$-brackets.

\[
\text{Sym}^* A \quad \sim \quad \text{Sym}^* A
\]

One co-product plays a special rôle in this lifting. This is the co-product that corresponds to the “$\circ$” bracket product itself, i.e. when the co-product coefficients $\gamma^k_l$ are equal to the “$\circ$” product coefficients $c^k_l$. Let this particular co-product be denoted by a triangle $\circ \triangle$ with a “$\circ$” on top.

Definition 6.7 Let there be given a bracket product “$\circ$” and a $\bullet$-bracket $\Phi : \text{Sym}^* A \rightarrow A$. The lifted bracket $\tilde{\Phi} : \text{Sym}^* A \rightarrow \text{Sym}^* A$ is defined as
\[
\tilde{\Phi} := \text{Sym}^* (\Phi \otimes 1) \circ \triangle ,
\] (6.25)

or written out,
\[
\tilde{\Phi}(a_1 \circ \ldots \circ a_n) := \sum_{k=0}^{n} \frac{c^n_k}{k!(n-k)!} \sum_{\pi \in S_n} (-1)^{\varepsilon_{\pi}} \Phi^k(a_{\pi(1)} \circ \ldots \circ a_{\pi(k)}) \circ a_{\pi(k+1)} \circ \ldots \circ a_{\pi(n)}
\] (6.26)

for $n \in \{0, 1, 2, \ldots\}$.

The definition (6.26) is by polarization equivalent to
\[
\tilde{\Phi}(\epsilon \circ a) = \sum_{k, \ell \geq 0} \frac{c^{k+\ell}}{k! \ell!} \Phi^k(a^{\circ k}) \circ a^{\circ \ell} , \quad \epsilon(a) = \epsilon ,
\] (6.27)

where we have defined a formal exponentiated algebra element as
\[
e^{\circ a} := \sum_{n \geq 0} \frac{1}{n!} a^{\circ n} , \quad \epsilon(a) = \epsilon .
\] (6.28)

In case of a standard bracket product “$\circ$” with coefficients $c^n_k = 1$, the lifting (6.26) becomes the standard lifting [27] of a $\bullet$-bracket $\Phi$ to a co-derivations $\delta = \tilde{\Phi}$.
Proposition 6.8 Let there be given a bracket product \( \circ \) and a co-product \( \triangle \). All lifted brackets \( \hat{\Phi} \) are co-derivations, if and only if

\[
\forall k, \ell, m \geq 0 : \begin{cases}
\gamma_{k+\ell+m} c_k^{k+\ell+m} = \gamma_{k+\ell} c_k^{k+\ell}, \\
\gamma_{k+\ell+m} c_k^{k+\ell+m+1} = \gamma_{k+\ell} c_k^{k+\ell+m}.
\end{cases}
\tag{6.29}
\]

The two co-derivation conditions (6.29) become identical if the co-product \( \triangle \) is co-commutative. The Proposition 6.8 suggests that one should adjust the co-product \( \triangle \) according to which \( \circ \) product one is studying. For instance, if the bracket product coefficients \( c_k^n = \lambda_k \lambda_{n-k+1}/\lambda_n \) are of the generic pre-Lie form (6.11), it is possible to satisfy the two co-derivation conditions (6.29) by choosing co-product coefficients of the form \( \gamma_k^n = \lambda_k \lambda_{n-k}/\lambda_n \). This choice of co-product is at the same time both co-associative and co-commutative.

Proof of Proposition 6.8: First note that for two vectors \( a, b \in A \) with \( \epsilon(a) = \epsilon \),

\[
\triangle(b \circ a^{\circ n}) = \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sum_{m=0}^{n} \beta_{k+\ell+m}^{n+1} (b \circ a^{\circ k}) \circ a^{\circ (n-k)}
\tag{6.30}
\]

for \( n \in \{0, 1, 2, \ldots \} \). Therefore

\[
\hat{\Phi}(a^{\circ n}) = \sum_{k, \ell, m \geq 0} b_k^n \left[ \beta_{k+\ell+m+1}^{\ell+1} \frac{\ell+1}{\ell+m+1} (\hat{\Phi}(a^{\circ k}) \circ a^{\circ \ell}) \circ a^{\circ m} + \beta_{\ell+1}^{\ell+1+m+1} \frac{\ell+1}{\ell+m+1} a^{\circ \ell} \circ (\hat{\Phi}(a^{\circ k}) \circ a^{\circ m}) \right].
\tag{6.31}
\]

On the other hand,

\[
(\hat{\Phi} \circ 1)(a^{\circ n}) = \sum_{k, \ell, m \geq 0} \beta_{k+\ell+m}^n b_k^{k+\ell} (\hat{\Phi}(a^{\circ k}) \circ a^{\circ \ell}) \circ a^{\circ m},
\tag{6.32}
\]

and

\[
(1 \circ \hat{\Phi})(a^{\circ n}) = \sum_{k, \ell, m \geq 0} \beta_{\ell+1}^n b_k^{k+\ell} a^{\circ \ell} \circ (\hat{\Phi}(a^{\circ k}) \circ a^{\circ m}).
\tag{6.33}
\]

Next insert the three expressions (6.31), (6.32) and (6.33) into the co-derivative definition (6.23) with \( \delta = \hat{\Phi} \). By comparing coefficients one derives

\[
\forall k, \ell, m \geq 0 : \begin{cases}
b_{k+\ell+m} \beta_{k+\ell+m+1}^{k+\ell+m} \frac{\ell+1}{\ell+m+1} = \beta_{k+\ell} b_{k+\ell+m}^{k+\ell+m}, \\
b_{k+\ell+m} \beta_{\ell+1}^{k+\ell+m+1} \frac{m+1}{\ell+m+1} = \beta_{k+\ell} b_{k+\ell+m}^{k+\ell+m}.
\end{cases}
\tag{6.34}
\]

The two conditions (6.29) follow by translating (6.34) into the \( \sim \) picture with the help of eq. (2.27). 

\[ \square \]

Proposition 6.9 A bracket product \( \circ \) is pre-Lie, if and only if the lifting map \( \sim \) is an algebra homomorphism, i.e. for all brackets \( \hat{\Phi}, \Phi' : \text{Sym}^* A \to A \),

\[
(\Phi \circ \Phi') \sim = \hat{\Phi} \Phi'.
\tag{6.35}
\]
Proof of Proposition 6.9: First note that for two vectors \( a, b \in A \) with \( \epsilon(a) = \epsilon \),

\[
\tilde{\Phi}(b \odot e^{\odot a}) = \sum_{k, \ell \geq 0} \frac{k^{\ell+1}}{\ell!} \Phi^{k+1}(b \odot a^{\odot k}) \odot a^{\odot \ell} + \sum_{k, \ell \geq 0} \frac{\ell^{k+1}}{\ell!} \Phi^k(a^{\odot k}) \odot b \odot a^{\odot \ell} .
\] (6.36)

Therefore the composition of two lifted brackets \( \tilde{\Phi} \) and \( \tilde{\Phi}' \) is

\[
\tilde{\Phi}\tilde{\Phi}'(e \odot a) = \sum_{k, \ell, m \geq 0} \frac{\ell^{k+1}}{\ell! \ell'! m!} \Phi^{\ell+1}(\Phi^{k+1}(a^{\odot k}) \odot a^{\odot \ell}) \odot a^{\odot m} + \sum_{k, \ell, m \geq 0} \frac{m^{k+1}}{\ell'! \ell! m!} \Phi^{m}(a^{\odot m}) \odot a^{\odot k} .
\] (6.37)

On the other hand,

\[
(\Phi \circ \Phi')(e^{\odot a}) = \sum_{k, \ell, m \geq 0} \frac{\ell^{k+1}}{\ell! \ell'! m!} \Phi^{\ell+1}(\Phi^{k+1}(a^{\odot k}) \odot a^{\odot \ell}) \odot a^{\odot m} .
\] (6.38)

By comparing coefficients in eqs. (6.37) and (6.38) one sees that the condition (6.35) is equivalent to the two pre-Lie conditions (6.5).

\[\square\]

6.4 Normalization

One may get back the \( \bullet \)-bracket \( \Phi : \text{Sym}^\bullet \mathcal{A} \to \mathcal{A} \) from its lifted bracket \( \tilde{\Phi} : \text{Sym}^\bullet \mathcal{A} \to \text{Sym}^\bullet \mathcal{A} \) with the help of the projection maps \( \pi_n : \text{Sym}^n \mathcal{A} \to \text{Sym}^n \mathcal{A} \). In particular,

\[
\pi_1 \circ \tilde{\Phi}(a_1 \odot \ldots \odot a_n) = c_n^n \Phi(a_1 \odot \ldots \odot a_n) .
\] (6.39)

**Definition 6.10** A “\( \circ \)” bracket product is normalized if

\[
\forall n \in \{0, 1, 2, \ldots\} : c_n^n = 1 .
\] (6.40)

Obviously, a normalized “\( \circ \)” product is non-degenerate, cf. eq. (2.4). Moreover, in the normalized case one may sharpen eq. (6.39) into

\[
\Phi = \pi_1 \circ \tilde{\Phi} ,
\] (6.41)

so that the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Sym}^\bullet \mathcal{A} & \xrightarrow{\tilde{\Phi}} & \text{Sym}^\bullet \mathcal{A} \\
\Phi \downarrow & \simeq & \pi_1 \\
\mathcal{A} & \xrightarrow{\pi_1} & \\
\end{array}
\] (6.42)

In the normalized case the product of brackets (2.3) may be related to composition of the lifted brackets as

\[
\Phi \circ \Phi' = \tilde{\Phi} \tilde{\Phi}' = \pi_1 \tilde{\Phi} \tilde{\Phi}' .
\] (6.43)
This carries the advantage that composition, unlike the bracket product, is born associative. It is natural to ask, what $c^n_k$ coefficients would satisfy a nilpotency condition for a lifted bracket
\[ \tilde{\Phi} \tilde{\Phi} = 0 , \quad \epsilon \Phi = 1 ? \] (6.44)

Contrary to the nilpotency relations (2.5) for the “◦” product, which are first-order equations in the $c^n_k$ coefficients, the nilpotency conditions (6.44) are quadratic in the $c^n_k$ coefficients. We end this discussion with a corollary.

**Corollary 6.11** For a normalized pre-Lie product “◦”, the bracket $\Phi$ is nilpotent with respect to the bracket product “◦”, if and only if the lifted bracket $\tilde{\Phi}$ is nilpotent with respect to composition, i.e.
\[ \Phi \circ \Phi = 0 \iff \tilde{\Phi} \tilde{\Phi} = 0 . \] (6.45)

### 6.5 The Ward Solution Revisited

As an example let us consider the Ward solution (4.33) of the derived bracket hierarchy, but this time normalized according to the eq. (6.40),
\[ c^n_k = \begin{cases} 1 & \text{for } k = 1 \lor k = n , \\ 0 & \text{otherwise} , \end{cases} \] (6.46)
or equivalently using eq. (4.21),
\[ c(x, y) = x(e^y - 1) + e^x . \] (6.47)

This solution is identical to the original solution (4.33) and (4.34), except for the fact that we have divided the first Ward identity (4.38) with 2, which is always permissible, cf. Subsection 2.3.

**Proposition 6.12** The Ward solution (6.46) is pre-Lie, normalized and satisfies the nilpotency relations for the derived bracket hierarchy, i.e. $[Q, Q] = 0 \Rightarrow \Phi_Q \circ \Phi_Q = 0$.

We conclude that there is a non-empty overlap between the derived solutions found in Section 4, the pre-Lie property (6.2) and the normalization condition (6.40).

**Proof of Proposition 6.12:** The Ward solution (6.46) is obviously normalized, cf. eq. (6.40). We saw in Section 4 that it satisfies the nilpotency relations for the derived bracket hierarchy. The pre-Lie property (6.5) may either be checked directly, or perhaps more enlightening, one may consider product coefficients $c^n_k = \lambda_k \lambda_{n-k+1} / \lambda_n$ of the generic pre-Lie form (6.11) with
\[ \lambda_k = \begin{cases} 1 & \text{for } k \in \{0,1\} , \\ \epsilon^k & \text{for } k \in \{2,3,4,\ldots\} , \end{cases} \] (6.48)

where $\epsilon$ is a non-zero complex number. One may easily see that the solution (6.11) becomes the Ward solution (6.46) in the limit $\epsilon \to 0$. Hence the Ward solution (6.46) is also pre-Lie by continuity.
7 Conclusions

The paper contains the following main new results:

- We found a non-commutative generalization (3.30) of the higher Koszul brackets, such that they form a homotopy Lie algebra.
- We found the most general nilpotency relations for the derived bracket hierarchy, cf. Theorem 4.3.
- We defined and calculated the higher Courant brackets, cf. Proposition 5.3.

A common platform for all of these topics is provided by a (generalized) homotopy Lie algebra that allows for arbitrary non-degenerate prefactors \(c^k_n\) in the nilpotency relations. These prefactors can equivalently be viewed as a non-standard bracket product “\(\circ\)”, cf. eq. (2.3). The generalization is desirable because the original definition (2.6) of Lada and Stasheff [22] excludes important systems, for instance the derived bracket hierarchy, which in all other respects has the hallmarks of an \(L_\infty\) algebra, cf. Section 4. In Section 6 we have analyzed the (generalized) homotopy Lie algebras further, and we have displayed their co-algebra structures. The question remains whether the (generalized) homotopy Lie algebra definition (2.4) considered in this paper is the final say. For instance, should one demand the pre-Lie property (6.2) of a homotopy Lie algebra definition? As we saw in Subsection 6.5 the pre-Lie property carries a small, but non-empty, overlap with the solutions (4.22) and (4.23) to the derived bracket hierarchy. A complete answer will require further studies and definitely more examples.

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