The high-Re stratified wake of a slender body and its comparison with a bluff body wake

Jose L. Ortiz-Tarin¹, Sheel Nidhan¹, Sutanu Sarkar¹†

¹Department of Mechanical and Aerospace Engineering, University of California San Diego, CA 92093, USA

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The high-Reynolds number stratified wake of a slender body is studied using a high-resolution hybrid simulation. The wake generator is a 6:1 prolate spheroid with a tripped boundary layer, the diameter-based body Reynolds number is \( Re = U_\infty D/\nu = 10^5 \), and the body Froude numbers are \( Fr = U_\infty/ND = \{2, 10, \infty\} \). The wake defect velocity \( (U_d) \) decays following three stages with different wake decay rates (Spedding 1997) as for a bluff body. However, the transition points among stages do not follow the expected \( Nt = Nx/U_\infty \) values. Comparison with the wake of a circular disk in similar conditions (Chongsiripinyo & Sarkar 2020) quantifies the influence of the wake generator - bluff versus slender - in stratified flow. The strongly stratified \( Fr = 2 \) wake is in a resonant state. The steady lee waves strongly modulate the mean flow and, relative to the disk, the \( Fr = 2 \) spheroid wake shows an earlier transition from the non-equilibrium (NEQ) stage to the quasi two-dimensional (Q2D) stage. The NEQ-Q2D transition is followed by a sharp increase in the turbulent kinetic energy and horizontal wake meanders. At \( Fr = 10 \), the start of the NEQ stage is delayed. Transfers between kinetic energy and potential energy reservoirs (both mean and turbulence) are analyzed and the flows are compared in phase space (local Froude and Reynolds number as coordinates). Overall, the results of this study point to the difficulty of finding a universal framework for stratified wake evolution, independent of the features of the body, and provide insights into how buoyancy effects depend on the wake generator.

1. Introduction

Due to their low drag coefficients, slender bodies are extensively used in aerospace and naval applications. Multiple studies have described the flow around these bodies focusing on the drag force, the boundary layer, and the flow separation (Wang 1970; Costis et al. 1989; Wang et al. 1990; Chesnakas & Simpson 1994; Fu et al. 1994; Constantinescu et al. 2002; Wikström et al. 2004). However, despite their presence in many underwater applications, only a few works have looked into the wake of a slender body (Chevray 1968; Jiménez et al. 2010; Kumar & Mahesh 2018) and, only recently, the far wake of a slender body has been studied (Ortiz-Tarin et al. 2021).

The near wake of a slender body with a turbulent boundary layer (TBL) is characterized by having a small recirculation region. The recirculation region is surrounded by a ring of small scale turbulence that emerges from the boundary layer and does not show strong vortex

† Email address for correspondence: sarkar@ucsd.edu
shedding (Jiménez et al. 2010; Posa & Balaras 2016; Kumar & Mahesh 2018; Ortiz-Tarin et al. 2021). As a result, the wake is thin and develops slowly compared to the wake of bluff bodies. These particular features of the slender body high-Re near wake lead to interesting effects further downstream: (i) despite having a smaller drag coefficient than bluff bodies, the defect velocity \( U_d = U_\infty - U \) of the slender body wake can be larger than that of a bluff body for a long downstream distance, (ii) the turbulent kinetic energy of the wake shows an off-center radial peak at the location where the turbulent boundary layer separates – instead of a Gaussian profile with a central peak, and (iii) helical instabilities come into play only in the intermediate and far field of the wake. These particularities affect the scaling laws of the wake. In a domain spanning 80D the defect velocity, the kinetic energy, and the dissipation do not follow the classic high-Re scaling and they decay differently than bluff body wakes (Ortiz-Tarin et al. 2021) exhibiting a non-equilibrium scaling of dissipation (Vassilicos 2015; Dairay et al. 2015).

The few studies that look into slender body wakes assume that the body moves in an unstratified environment, where the density of the surrounding fluid is constant. However, in a realistic underwater marine environment the effect of density stratification due to salinity and temperature can become relevant. Density stratification suppresses vertical motions, triggers the formation and sustenance of coherent structures, and leads to the radiation of internal gravity waves. More importantly, in a stratified environment, the wake of a submersible lives longer than in an unstratified environment, i.e., it takes more time for the flow disturbance to die out (Spedding 2014). The study of stratified wakes has been nearly exclusively focused on the flow past bluff bodies (Lin & Pao 1979; Hanazaki 1988; Lin et al. 1992; Chomaz et al. 1992; Orr et al. 2015; Pal et al. 2017) and underwater topography (Drazin 1961; Castro et al. 1983; Baines 1998). Here, we study the influence of stratification on the high-Re wake of a prolate spheroid with a turbulent boundary layer.

The strength of ambient stratification is measured by the body-based Froude number \( Fr = U_\infty / ND \). This is the ratio between the convective frequency of the flow, \( U_\infty / D \) – where \( U_\infty \) is the freestream velocity and \( D \) the diameter of the body – and the buoyancy frequency \( N \). In the wake of ocean submersibles, \( Fr \sim O(1 - 10^2) \). However, since the velocity deficit in the wake \( U_d(x) \) decays with the streamwise distance and the wake width \( L(x) \) increases, the Froude number defined with local variables \( Fr_l = U_d / NL \) decreases as the flow evolves. Thus, even in a weakly stratified environment, eventually all wakes are affected by stratification.

Since the relative strength of stratification increases locally as the flow develops, the evolution of the stratified wake is multistage. Based on the measurements of \( U_d \) and \( L \) in initially weakly stratified bluff body wakes, Spedding (1997) identified three different regimes in the stratified wake evolution: (i) a 3D regime where the effects of stratification are negligible and the wake decays similar to its unstratified counterpart, (ii) a non-equilibrium regime (NEQ) spanning approximately \( Nt \approx 2 - 50 \) (here \( Nt \) is the non-dimensional buoyancy time) where buoyancy starts suppressing vertical motions and the wake decays as \( U_d \sim x^{-1/4} \), (iii) and finally a quasi-two-dimensional regime (Q2D) where buoyancy dominates the flow, the wake is organized into vortices with primarily horizontal motion and the defect velocity decays decays sharper than in the NEQ stage, e.g., \( U_d \sim x^{-3/4} \) in several studies.

Notice that the arrival of the wake into each of the three stages in its evolution depends on the value of \( Nt \), which is equivalent to a downstream distance of \( x / Fr \) from the wake generator. At high Froude number, the downstream distance required to reach the NEQ and Q2D regions can become very large. Consequently, the size of the computational domain required to access these regimes rapidly becomes computationally unfeasible. To circumvent these limitation temporal simulations were used in the study of Gourlay et al. (2001). Temporal simulations use a reference frame moving with the wake where time correlates with streamwise distance.

in a fixed reference frame. By assuming that the streamwise development of the flow is slow, periodic boundary conditions can be used and the equations are advanced in time without the need of introducing the wake generator. This reduces the computational cost significantly. Most of the studies that have contributed to our current understanding of stratified wakes use temporal simulations (Gourlay et al. 2001; Dommermuth et al. 2002; Brucker & Sarkar 2010; Diamessis et al. 2011; De Stadler & Sarkar 2012; Abdilghanie & Diamessis 2013; Redford et al. 2015; Rowe et al. 2020).

The main drawback of the temporal model is the influence of its initialization. Since the flow at the wake generator is not solved, the starting profiles of the mean and turbulence have to be assumed. These simulations lack some specific features that are generated due to the body, e.g., steady lee waves, near-wake buoyancy effects, and the vortical structures shed from the boundary layer. Even when it is tempting to assume that body specific features are lost far from the body, the universality of the wake decay has remained elusive to experiments (Bevilaqua & Lykoudis 1978; Wygnanski et al. 1986; Redford et al. 2012), even in unstratified wakes. An alternative to temporal simulations are body inclusive simulations that retain the wake generator dependent features at the expense of a higher computational cost and a limited domain size (Orr et al. 2015; Chongsiripinyo et al. 2017; Pal et al. 2017).

To the best of authors’ knowledge, Ortiz-Tarin et al. (2019) performed the first study of a stratified flow past a slender body that investigates the near and intermediate wake dynamics. Their analyses reveal that at $Fr \sim O(1)$ the type of separation and the subsequent wake establishment is strongly dependent on the characteristic frequency of the lee waves and the aspect ratio of the body. When half the wavelength of the steady lee waves ($\lambda = 2\pi Fr$) matches the length of the slender body ($L$), the separation of the boundary layer is inhibited by buoyancy effects. Based on this condition, a critical Froude number can be defined $Fr_c = L/D\pi$. When $Fr > Fr_c$ stratification suppresses the generation of turbulence in the near wake, when $Fr \approx Fr_c$ buoyancy strongly limits the flow separation and can lead to a relaminarization of the wake at low Reynolds numbers. Finally, when $Fr < Fr_c$, the lee waves enlarge the separation region and there might be an increase in the turbulence intensities in the wake. When $Fr \approx Fr_c$, the wake is in a resonant state with both the separation and the wake dimensions being strongly modulated by the steady lee waves (Hunt & Snyder 1980; Chomaz et al. 1993; Ortiz-Tarin et al. 2019).

As mentioned before, the use of body inclusive simulations has one major limitation, i.e., the high computational cost. Due to the high resolution required to solve the boundary layer of the wake generator, the downstream domain is limited and thus the possibility of looking into the far wake gets significantly restricted, particularly at high $Re$. VanDine et al. (2018) presented a hybrid spatially-evolving model, which builds on the hybrid temporally-evolving model of Pasquetti (2011), and addresses most of the aforementioned problems. The hybrid method uses inflow conditions generated from a well-resolved body-inclusive simulation to perform a separate temporal simulation in the case of Pasquetti (2011) or spatially-evolving simulation in the work of VanDine et al. (2018) without including the body. By doing so, the amount of required points is substantially reduced since the flow near the body does not have to be resolved. This important reduction of the computational cost allows one to extend the domain farther downstream to gain insight in the far wake.

Here, we use a hybrid method that combines a body-inclusive simulation and a spatially evolving body-exclusive simulation to study the stratified high-$Re$ far wake of a slender body for the first time. The Reynolds number is set to $Re = U_\infty D/\nu = 10^5$ and two levels of stratification are used, $Fr = U_\infty/ND = 2$ and 10. The simulation at $Fr = 10$ allows us to study the evolution of a weakly stratified wake in a domain that spans $80D$. Additionally, $Fr = 2$ is chosen because it is close to the critical Froude number for a 6:1 prolate spheroid $Fr_c = (L/D)/\pi = 6/\pi$. At the critical Froude number, the size of the separation region is
strongly reduced by the lee waves (Ortiz-Tarin et al. 2019). These choices also allow us to compare our results with the findings of Chongsiripinyo & Sarkar (2020) (hereafter referred as CS20) regarding the stratified wake of a disk.

In CS20, the stratified wake of a disk at $Re = 5 \times 10^4$ is studied at $Fr = 2, 10, 50, \infty$. Apart from a detailed analysis of the decay rates of the mean and turbulent quantities, CS20 links the general evolution of stratified homogeneous turbulence (Brethouwer et al. 2007; de Bruyn Kops & Riley 2019) with the evolution of the wake turbulence. As the disk wake evolves, the influence of buoyancy is ‘felt’ by the turbulent motions at progressively smaller scales. First the mean flow and the large scales and later the r.m.s. velocities are affected by stratification. Simultaneously, the horizontal eddies start gaining energy. Based on the strength of these effects, three distinct stages can be identified: weakly, intermediate and strongly stratified turbulence. In CS20, the transition between these regimes is examined and parameterized using local Froude and Reynolds numbers. Zhou & Diamessis (2019) also examined these transitions and their link with the evolution of stratified homogeneous turbulence using temporal simulations.

The present work is the continuation of Ortiz-Tarin et al. (2021) – referred to as ONS21 – where the unstratified wake of a 6:1 prolate spheroid with a turbulent boundary layer was studied and compared with a large number of simulations and experiments. In our previous study we found that the particularities of the slender body wake, e.g., small recirculation region, low entrainment, large defect velocity, bimodal distribution of the turbulent kinetic energy, among others affect the wake decay significantly. In this study we are analyzing how these features affect the evolution of the stratified wake. We also analyze the simulations of CS20 to closely compare our results with the stratified bluff body wake.

Some of the questions we want to answer are: do the stratified decay laws and their transition points depend on the shape of wake generator? how does the turbulence evolve in stratified slender body wakes and are there difference with bluff body wakes? how does the phase-space evolution of the stratified turbulence compares between bluff and slender body wakes? In broader terms, we attempt to find whether a turbulent stratified wake retains some imprint of the wake generator in the mean and turbulence evolution.

A description of the solver and the methodology is given in §2. The wakes are visualized in §3. The decay of the mean wake properties is analyzed in §4. Finally, the evolution of the turbulence and the phase-space analysis of the wake are presented in sections §5 and §6, respectively. The study is concluded in §7.

2. Methodology

To study the far wake of a slender body at a high Reynolds number we use a hybrid simulation. The hybrid model combines two simulations: body-inclusive (BI) that solves the flow past the wake generator and body-exclusive (BE) that resolves the intermediate and far wakes. Here, we use a spatially-evolving simulation following the procedure validated by VanDine et al. (2018). In the implementation, data from a selected cross-plane in the BI simulation is interpolated on to a new grid and used as an inlet boundary condition for the BE stage. This procedure allows us to alleviate the natural stiffness of the wake problem. Whereas the BI simulation is designed to capture the turbulent boundary layer and the flow separation, the BE simulation resolves the turbulence in the wake. Both the grid size and the time step required to solve the turbulent boundary layer are much smaller than those needed in the intermediate and far wakes. This method leads to significant savings in computational cost without compromising accuracy.

The setup and the solver here are the ones used in ONS2021 with the addition of stratification. Both simulations solve the three-dimensional Navier-Stokes with the Boussinesq
approximation in cylindrical coordinates. The solver uses a third-order Runge-Kutta method combined with second-order Crank-Nicolson to advance the equations in time. Second-order-accurate central differences are used for the spatial derivatives in a staggered grid. A wall-adapting local eddy viscosity (WALE) is used to properly capture the turbulent boundary layer dynamics (Nicoud & Ducros 1999). Both the BI and BE simulations use Dirichlet boundary conditions at the inflow, convective outflow and Neumann at the radial boundary. Similarly to Ortiz-Tarin et al. (2019) a sponge layer is added to the boundaries to avoid the spurious reflection of gravity waves.

An immersed boundary method (Balaras 2004; Yang & Balaras 2006) is used to resolve the flow past a 6:1 prolate spheroid at zero angle of attack. A numerical bump is introduced on the surface of the body to accelerate the transition of the boundary layer to turbulence. The annular bump is located where the surface favorable pressure gradient is nearly zero. This location is found at approximately $0.5\cdot D$ from the nose. The radial extent of the bump is $0.002\cdot D$ ($\sim 15$ wall units) and the streamwise extent is $0.1\cdot D$.

The stratification is set by a linear background density profile characterized by the Froude number, $Fr = U_\infty/ND$, where $N$ is the buoyancy frequency. Three levels of stratification are simulated: $Fr = 2$, $10$ and $\infty$. $Fr = 2$ is close to the critical Froude number $Fr_c = 6/\pi$ for the 6:1 spheroid at which the suppression of turbulence in the wake by stratification is optimal (Ortiz-Tarin et al. 2019). $Fr = 10$ is a moderate level of stratification closer to oceanic values. Finally $Fr = \infty$ is the unstratified case which will be used as a reference (ONS21).

The cylindrical coordinate system is $(x, r, \theta)$ with the origin at the body center. For convenience, the Cartesian coordinate system $(x, y, z)$ will also be used, where $z$ is the vertical direction aligned with gravity, $y$ is the spanwise direction, and $x$ is the streamwise direction.

The BI grid is designed to resolve the turbulent boundary layer and the small-scale wake turbulence. The turbulent boundary layer is resolved with $\Delta x^+ = 40$, $\Delta r^+ = 1$, and $r \Delta \theta^+ = 32$. There are 10 points in the viscous sublayer and 130 across the buffer and log layers. In the wake, the peak ratio between the grid size and the Kolmogorov length $\eta = (\nu^3/\epsilon)^{1/4}$, in both BI and BE domains is $\max(\Delta x/\eta) = 7.5$, $\max(\Delta r/\eta) = 6$, and $\max(r \Delta \theta/\eta) = 5$. The mean velocities and turbulence intensities within the turbulent boundary layer were validated against existing studies (Posa & Balaras 2016; Kumar & Mahesh 2018). Additionally, the unstratified wake decay coincides with existing numerical and experimental works on slender body wakes (see figure 1 of ONS21).

The domain size in the stratified cases is large so that internal gravity waves are weak before reaching the sponge region near the walls. The total number of grid points across BI and BE domains is approximately 1.5 billion in the unstratified case and 2 billion in the stratified simulations. Tables 1 and 2 include the most relevant parameters of BI and BE simulations, respectively. Further details on the grid design including comparison with experiments and the ratio between the grid sizes and $\eta$ can be found in section 2 of ONS21.

Once the flow has reached statistically steady state, the statistics are obtained by temporal

| Case | $Re$ | $Fr$ | $L_r$ | $L_\theta$ | $L_x^-$ | $L_x^+$ | $N_r$ | $N_\theta$ | $N_x$ |
|------|------|------|-------|------------|---------|---------|------|-----------|------|
| 1    | $10^5$ | $\infty$ | 5      | $2\pi$     | 8       | 15      | 746  | 512       | 2560 |
| 2    | $10^5$ | 10    | 60    | $2\pi$     | 20      | 30      | 848  | 512       | 3072 |
| 3    | $10^5$ | 2     | 60    | $2\pi$     | 20      | 30      | 848  | 512       | 3072 |

Table 1: Parameters of the body-inclusive simulation of prolate 6:1 spheroid. $L_x^-$ and $L_x^+$ are the upstream and downstream distances from the wake generator.
averaging, denoted by $\langle \cdot \rangle$. Instantaneous quantities are written with lower case, mean quantities with upper case, and fluctuations with prime. In the stratified cases the average is performed over $270D/U_\infty$, approximately three flow-throughs. In the unstratified simulation flow statistics are obtained through the temporal (over $100D/U_\infty$) as well as azimuthal averaging. Apart from temporal averaging, some statistics are obtained from cross-wake area integration denoted by $\{ \cdot \}$. Unless otherwise indicated, the integral is performed over a cross-section of radius $4D$. All the flow statistics presented here die out well before they reach the limit of the integrated region.

Reported velocities and lengths are normalized with the free-stream velocity $U_\infty$ and the body minor axis $D$, respectively. The normalized streamwise distance from the center of the body $x$ is also measured as a function of the buoyancy frequency and the time. The time in the $Nt$ axis refers to time measured by an observer attached to the mean flow that sees the body move at a speed $U_\infty$. A Galilean transformation yields $x/Fr = Nt$.

To compare the stratified wake of the 6:1 spheroid with that of a bluff body we use the body-inclusive disk wake simulations of CS20. The solver used in CS20 is the same as the one used here although, instead of using the WALE closure model, CS20 uses a variant of dynamic Smagorinsky. The eddy-viscosity model was changed in the spheroid simulations since WALE was demonstrated to capture the behavior of the turbulent boundary layer with the resolution used in the present wall-resolved LES. Both sets of simulations are very well resolved and have a small subgrid contribution – see ONS21 and CS20 – hence the validity of the comparison. Further details of the simulations can be found in ONS21 and CS20. The main parameters of disk simulations are shown in table 3.

3. Visualizations

Figure 1 shows instantaneous snapshots of the near wake of a spheroid and a disk at $Fr = 2$ and $Fr = 10$. At both $Fr$, the near wake structure of the two bodies is very different. Compared to the spheroid with turbulent boundary layer (TBL), the disk wake has a large recirculation region ($\approx 2D$), as shown by the red isolines in figure 1. This large recirculation region oscillates (Rigas et al. 2014) and generates a vortex shedding structure that is advected downstream (Nidhan et al. 2020). In a spheroid with TBL, the recirculation region is very

| Case | $Re$ | $Fr$ | $L_r$ | $L_\theta$ | $x_e$ | $L_x$ | $N_r$ | $N_\theta$ | $N_x$ |
|------|------|------|-------|-----------|------|-------|-------|-----------|-------|
| 1    | $10^5$ | $\infty$ | 10 | $2\pi$ | 6 | 80 | 479 | 256 | 4608 |
| 2    | $10^5$ | 10 | 57 | $2\pi$ | 9 | 90 | 619 | 256 | 4608 |
| 3    | $10^5$ | 2 | 57 | $2\pi$ | 9 | 90 | 619 | 256 | 4608 |

Table 2: Parameters of the body-exclusive simulations. $x_e$ is the extraction location of the BI simulations that is fed as inlet to the BE simulations.

| Case | $Re$ | $Fr$ | $L_r$ | $L_\theta$ | $L^-_x$ | $L^+_x$ | $N_r$ | $N_\theta$ | $N_x$ |
|------|------|------|-------|-----------|---------|---------|-------|-----------|-------|
| 1    | $5 \times 10^4$ | 15.14 | $2\pi$ | 30.19 | 125.51 | 364 | 256 | 4608 |
| 2    | $5 \times 10^4$ | 10 | 80 | $2\pi$ | 30.19 | 125.51 | 529 | 256 | 4608 |
| 3    | $5 \times 10^4$ | 2 | 80 | $2\pi$ | 30.19 | 125.51 | 529 | 256 | 4608 |

Table 3: Parameters of the disk simulations (CS20).
small (~ 0.1D) and is surrounded by the small scale turbulence of the boundary layer. As a result, the near wake is highly organized and large-scale oscillations are not observed in the near wake (Jiménez et al. 2010; Kumar & Mahesh 2018; Ortiz-Tarin et al. 2021). Only further downstream, does the wake begin to show a helical structure. This change in the structure of the slender body wake has been found to lead to a change in the decay rate and dissipation scaling in the unstratified wake (ONS21). In the following sections, we will analyze how the differences between the near wake of a disk and that of a spheroid lead to distinct trends of mean and turbulence evolution in a stratified environment. But first, let us describe different snapshots of the spheroid intermediate and far wakes. Snapshots of the disk intermediate and far wakes can be found in CS20.

Figure 2 shows an instantaneous visualization of the spheroid Fr = 2 wake in the center-vertical and center-horizontal planes. One of the distinctive features of the spheroid wake is that at Fr \( \sim O(1) \) the separation of the boundary layer can be strongly modulated by the steady lee waves (Ortiz-Tarin et al. 2019). This interaction between the lee waves and the wake is particularly strong when the Froude number is close to a critical Froude number Fr_c = AR/\pi, where AR is the body aspect ratio. When Fr \( \approx Fr_c \), half the wavelength of the lee wave (\( \lambda / D = 2\pi Fr \)) coincides with the length of the body and the size of the separation region is reduced. The flow is then in what is called resonant or saturated lee wave regime, (Hanazaki 1988; Chomaz et al. 1992). At low Reynolds numbers, this effect can lead to the relaminarization of the turbulent wake (Pal et al. 2016; Ortiz-Tarin et al. 2019). In the present case, figure 2(a) reveals that, even at Re = 10^5, the wake height is strongly modulated by the waves, although the wake is not relaminarized due to the high Re of the flow. For example, the wake height exhibits oscillations with a wavelength of \( \lambda / D = 2\pi Fr = 4\pi \). The modulation of the wake by the waves leads to an unusual configuration in the intermediate wake (x = 20 – 40) where the wake width L_H is smaller than the wake height L_V, figure 2(a) and (c). In these figures, sinuous oscillations are observed only in the horizontal plane.
Figure 2: Instantaneous contours of streamwise velocity of the spheroid Fr = 2 wake in center-vertical (a,b) and center-horizontal planes (c,d).

(figure 2d) due to strong stratification. These horizontal sinuous instabilities contrast with the lee wave induced varicose modulation in the vertical plane. As the wake evolves, the $L_H < L_V$ configuration transitions to the expected $L_H > L_V$. In this late region, the small-scale turbulence of the boundary layer has been dissipated and a layered-layer structure is observed in the vertical plane, figure 2(b). The qualitative trends of $L_H$ and $L_V$ discussed here are quantified in §4.

The main features of the $Fr = 10$ wake can be observed in the instantaneous snapshots of figure 3. The near wake, figure 3(a,c), is thin and carries the small scale turbulence generated in the boundary layer. Similar to the unstratified wake of ONS21, in the $x < 20$ region, it has a quasi-cylindrical structure. Only after $x \approx 20$, a helical structure develops. In the unstratified wake, the oscillation found at $x \approx 20$ is present until the end of the domain. Here, the $Fr = 10$ wake does not show major oscillations after $x \approx 30$. Stratification restrains the vertical motions in the wake and enhances the horizontal spread as can be seen in the visualization of the late wake, figure 3(b,d). Unlike the $Fr = 2$ wake, the horizontal and vertical $Fr = 10$ wake extent grow monotonically with increasing downstream distance.
4. Evolution of the mean flow in spheroid and disk wakes

4.1. Evolution of the mean defect velocity $U_d$

The decay rate of the mean defect velocity $U_d = U_\infty - U$ shows the different stages in the evolution of a wake. In a stratified environment, wakes transverse the 3D, NEQ and Q2D regimes (Spedding 1997). Figure 4 compares the decay of $U_d$ among the unstratified, $Fr = 10$, and $Fr = 2$ spheroid and disk wakes. To facilitate a one-to-one comparison, we present the disk data in the domain $3 \leq x \leq 80$, coinciding with the domain of the spheroid wake. Since $x$ is measured from the center of the body, the location of $x = 3$ is in the near wake for the disk and is at the terminus of the body for the spheroid. The unstratified spheroid wake (figure 4a) shows a transition between the classical high-$Re$ decay $U_d \sim x^{-2/3}$ to $U_d \sim x^{-6/5}$ at $x \approx 20$ coinciding with the development of a helical structure (ONS21). The $Fr = \infty$ disk wake decays as $U_d \sim x^{-0.9}$ from $10 \leq x \leq 65$ and transitions to the classical high-$Re$ decay of $x^{-2/3}$ afterward (CS20), as shown in figure 4(b). Compared to the disk, the $Fr = 10$ and $Fr = \infty$ spheroid wakes have a higher value of $U_d$, owing to weaker near-wake entrainment and the slower development of slender body wakes.
In the weakly stratified $Fr = 10$ regime, the defect velocity of the spheroid wake (figure 4a) evolves similarly to the unstratified wake until $Nt \approx 3.5$, when the decay rate slows down. However, at the same value of $Fr = 10$ but for the disk wake (figure 4b), $U_d$ deviates from the unstratified case at $Nt \approx 1$. Based on $U_d$, the end of the 3D region and the beginning of the NEQ region of the spheroid wake occurs at $x \approx 30$ whereas in the disk it occurs at $x \approx 10$. We discuss the reason behind this delayed deviation of the spheroid-wake $U_d$ from its unstratified counterpart in §5.

At $Fr = 2$, there are significant differences in $U_d$ evolution between the disk and spheroid wakes. In the $Fr = 2$ spheroid wake, $U_d$ shows an increased decay rate from the beginning. Although not shown here, the wake establishment is affected similarly to the $Fr = 1$ wake of the 4:1 spheroid of Ortiz-Tarin et al. (2019), where there was no 3D regime. The boundary layer evolution on the body and the separation are affected by stratification. At $Nt \approx \pi$, there is a sudden change in the decay rate due to the lee-wave-induced oscillatory modulation (Pal et al. 2017) observed in the $3 \leq x \leq 10$ region. This oscillatory modulation gets weaker downstream as the lee-wave amplitude decreases with the distance from the source.

At $Nt \approx \pi$, the wake transitions to the NEQ stage wherein $U_d$ exhibits a slower decay compared to both the preceding stage and the following stage which commences at $Nt \approx 15$. Fitting a power law to the NEQ stage results in wake decay with $x^{-0.266}$. This decay is close to the -1/4 decay in the NEQ regime found in the experiments of Spedding (1997) and later in numerical simulations (Diamessis et al. 2011; Brucker & Sarkar 2010; Redford et al. 2015; Pal et al. 2017). At $Nt \approx 15$, the spheroid wake transitions to the Q2D regime with a sharper decay and a power-law fit results in $U_d \sim x^{-0.72}$, which is close to the $x^{-0.75}$ behavior established by Spedding (1997) for the Q2D regime. The $Fr = 2$ disk wake shows a very different behavior. Until at least $x = 125$ ($Nt = 62.5$) – the full extent of the computational domain – the disk wake exhibits no transition to the Q2D regime. Instead, after transitioning to the NEQ regime with a power law of $U_d \sim x^{-0.18}$, the disk wake stays in that regime.

Thus, the NEQ regime in the spheroid wake at $Fr = 2$ is significantly shortened compared to the disk wake, with it starting at $Nt \approx \pi$ and ending early at $Nt \approx 15$ when Q2D commences. In the experiments of Spedding (1997), the NEQ regime is reported to last until $Nt \approx 40$. Other temporal studies have found that the span of the NEQ regime depends on the Reynolds number. For example, in temporal simulations, Diamessis et al. (2011) found an increase of the NEQ duration to $Nt \approx 50$ when the Reynolds number increased to $Re = 10^5$.  

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Figure 4: Decay of the peak defect velocity in (a) spheroid and (b) disk. The red dashed line in (a) indicates the decay of the $Fr = 2$ centerline defect velocity. For all other cases, centerline and maximum $U_d$ coincide. Note that the origin of the $Nt$ scale is 1.5 for $Fr = 2$ and is 0.3 for $Fr = 10$.  

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Figure 5: Wake dimensions measured using the mean defect velocity $U_d$ for the spheroid (a,c) and disk (b,d) wakes in center-vertical (a,b) and center-horizontal (c,d) planes. The legends are same as in figure 4.

Figure 6: Instantaneous radial velocity contours of the $Fr = 2$ spheroid wake in the (a) center-vertical and (b) center-horizontal planes.

Brucker & Sarkar (2010) found a transition to a Q2D-type power law at $Nt \approx 100$. Only Redford et al. (2015) observed an earlier transition around $Nt \approx 25$. The reasons behind the early arrival of the Q2D regime in the spheroid $Fr = 2$ wake will be discussed in §5.

4.2. Evolution of the mean lengthscales $L_H$ and $L_V$

The evolution of the mean wake dimensions in the spheroid and disk wakes is shown in figure 5(a,c) and (b,d), respectively. $L$ is defined such that $U_\infty - U(L) = \frac{1}{2} U_d$. The subscripts $\{V, H\}$ indicate that these measures have been taken in the vertical and horizontal planes so that they represent the half height and the half width.

The wake of a slender body is generally thinner than its bluff body counterpart. Compared to the disk wake of CS20, the present unstratified wake is smaller by a factor of about 3 – contrast black lines in figure 5(a) to (b). The difference in wake size stems from the different near-wake features. Here, the initial non-dimensional wake width is around 0.2 whereas its value for the disk is around 0.7. This observation agrees well with the scaling
proposed by Tennekes & Lumley (1972) and used in stratified wake experiments by Meunier & Spedding (2004), where the wake dimensions behind a body with diameter $D$ scale with the drag coefficient $\sqrt{C_D}$. We find that $C_D^{\text{disk}} \approx 1.11$ and $C_D^{\text{spheroid}} \approx 0.13$ resulting in $(C_D^{\text{disk}}/C_D^{\text{spheroid}})^{0.5} \approx 3.2$. Besides the initial dimensions, the near-wake growth rates of the spheroid and disk are also very different. Whereas in the $x = 3 - 20$ region the spheroid unstratified wake grows as $L \sim x^{0.2}$, the disk wake grows as $L \sim x^{0.45}$. Later, the growth rate of both wakes becomes comparable but the difference in size is already established and dictated by the near wake.

The evolution of the $Fr = 10$ spheroid wake height ($L_V$) is similar to its unstratified counterpart until $Nt \approx 3.5$ where the growth of $L_V$ slows down. While $L_V$ remains almost constant beyond $Nt \approx 4$, $L_H$ keeps increasing with a growth rate of $\sim x^{1/3}$. In the $Fr = 10$ disk wake, the deviation from the $Fr = \infty$ case happens at $x \approx 20$ ($Nt \approx 2$). Interestingly, after $Nt \approx 2$, the disk wake shows a continuous decrease in wake height. $L_H$ of both spheroid and disk wakes at $Fr = 10$ closely follow the trend of the corresponding unstratified wake. See figures 5(c,d).

The wake dimensions at $Fr = 2$ for both disk and spheroid show oscillations with a wavelength of $\lambda/D = 2\pi Fr$. This reveals the influence of the steady lee waves especially on the wake height, see figure 5(a,b). From $Nt = 1 - 15$ the oscillations of $L_V$ and $L_H$ are consistent with the conservation of momentum deficit, i.e., to counteract the contraction of $L_V$ caused by buoyancy, $L_H$ is enhanced. Note that these initial oscillations are of similar amplitude in both disk and spheroid. However, the relative change over the initial wake dimensions is much more pronounced in the spheroid wake ($\sim 10$ times) owing to its initial thinness. The influence of the lee waves on the spheroid $Fr = 2$ wake dimensions is illustrated by radial velocity contours in figure 6. The wake width contracts significantly in the region where the vertical velocity of the lee wave induces a rapid increase of wake height. Starting at $Nt = 20$, the width of the spheroid wake shows a rapid growth $L_H \sim x^{1.3}$ corresponding with: (i) the development of the horizontal wavy motions observed in figure 2(d) and (ii) the arrival of the Q2D stage with $U_d \sim x^{-3/4}$. The growth of $L_V$ remains constant at a rate of $x^{0.25}$. Both, the very rapid growth of $L_H$ and the sustained increase in $L_V$ of the spheroid wake, are very different from the trends in the $Fr = 2$ disk wake. In the disk wake, we find that, instead of an increase, the vertical height exhibits a decrease ($L_V \sim x^{-0.18}$) at $x \gtrsim 20$. Furthermore, $L_H$ grows at $x^{1/3}$, a moderate rate relative to its rapid growth rate in the disk wake.

4.3. Comparison of flow topology between stratified spheroid and disk wakes

The difference between the spheroid and disk with regards to the evolution of mean length scales ($L_V$ and $L_H$), particularly at $Fr = 2$, points toward qualitative differences in the flow topology. To further characterize these differences in the $Fr = 2$ wake, figure 7 shows contours of mean streamwise velocity ($U$) at different streamwise locations for the spheroid (top two rows) and the disk (bottom row). In each panel of figure 7, the right half shows $U_x$ and the left half shows turbulent kinetic energy (TKE), $E_T^K = (\langle u_x^2 \rangle + \langle u_y^2 \rangle + \langle u_z^2 \rangle)/2$.

For the sake of brevity, we have not included contours of the $Fr = 10$ disk and spheroid wakes since their topology is similar – the mean can be well approximated by a vertically-squeezed two-dimensional Gaussian while the TKE evolves from a bimodal (off-center peaks) distribution in the radial direction to a Gaussian at intermediate to late streamwise distances. In the case of the disk, the TKE evolves as a two-dimensional Gaussian right beyond the recirculation region.

We first discuss the disk wake (bottom row). The mean shows a monotonic spread in the horizontal direction and resembles the shape of an ellipse or a two-dimensional
Figure 7: $Fr = 2$ wake of spheroid (a-f) and disk (g-i) at different streamwise locations $x$. Contours of mean streamwise velocity in the right half and TKE in the left half of each contour. Contour limits are between the minimum (red) and maximum (white) values of the respective quantity at a given $x$ with ten levels in between. Radial extent span till $r = 1$ and $r = 4$ for the spheroid and disk contours, respectively. The disk wake is larger than the spheroid wake as can be inferred from the $r = 1$ circle in the bottom row.

Gaussian squeezed in the vertical direction. This shape does not change until the end of the computational domain at $x \approx 125$. The TKE for the disk wake also has a similar vertically squeezed appearance.

Turning to the spheroid wake, we find that its turbulence topology is different from that of the mean. In the region $5 \leq x \leq 15$, TKE shows two off-center peaks reminiscent of the TBL shedding from a slender body (Jiménez et al. 2010; Posa & Balaras 2016; Kumar & Mahesh 2018; Ortiz-Tarin et al. 2021) while mean $U_x$ shows a single central peak. At $x = 20$, we see the start of a horizontal contraction of the mean velocity in the central region of the wake and the emergence of a ‘butterfly’ shape reminiscent of the separation and wake patterns observed in Ortiz-Tarin et al. (2019), where the $Fr_c = 4/\pi \approx 1$ wake of a 4:1 spheroid was studied. Note that in this stage, the wake is thinner than taller, i.e., $L_H < L_V$. At $x \approx 20$, TKE starts transitioning from a bimodal distribution to a single peak near the center-horizontal plane. In the next section, we will analyze how the horizontal contraction of the mean wake between $x = 20$ and $x = 40$ results in an increased horizontal mean shear, resulting in the maximum TKE being produced close to the center-horizontal plane. This leads to a transition in the TKE topology from bimodal distribution to a squeezed-Gaussian distribution at $x \geq 40$. By $x \approx 60$, $U$ has been organized into two distinct layers while TKE
is sustained between these two vertically off-center layers. Note that the multi-layered mean flow structure at late $x$ in the $Fr = 2$ spheroid wake is reminiscent of the layered structure of the Q2D regime Spedding (1997).

Previously, temporal simulations (Gourlay et al. 2001; Brucker & Sarkar 2010; Redford et al. 2015) have shown that the mean and the turbulence can evolve differently. Indeed, the effect of buoyancy is ‘felt’ very differently by the large and the small scales in the flow. The general trend is that, in the late wake, the turbulence occupies a smaller and smaller vertical fraction of the mean effect as time passes (Redford et al. 2014). Instead, the finding here for the spheroid wake is the combined effect of having a wake in saturated-lee-wave state and initial off-center peaks of TKE, established by the TBL separation. These characteristics of the flow have not been not captured in temporal simulations since they have not accounted for the wake generator and the steady lee waves.

5. Evolution of the turbulent flow in spheroid and disk wakes

The energy of the flow is contrasted between spheroid and disk wake in this section. The turbulent kinetic energy (TKE also denoted as $E^K_T$), turbulent potential energy (TPE, $E^P_T$), mean kinetic energy (MKE, $E^K_M$) and mean potential energy (MPE, $E^P_M$) are defined as:

$$E^K_T = \frac{1}{2} \left( \langle u_x^2 \rangle + \langle u_y^2 \rangle + \langle u_z^2 \rangle \right), \quad E^P_T = \frac{\gamma \rho}{\rho_0} \frac{\rho'}{\rho_0} \langle \rho' \rangle/2,$$

$$E^K_M = \frac{1}{2} \left( U_d^2 + \langle u_y^2 \rangle + \langle u_z^2 \rangle \right), \quad E^P_M = \gamma \rho_d^2/2,$$

where $\gamma = g^2/\rho_0^2 N^2$. In what follows, trends of area-integrated values, denoted by $\{\cdot\}$, of these energy measures are reported. Area-integrated quantities are preferred because the peaks of mean and turbulence in stratified slender wakes are often times off-centered as can be seen in figure 7. The integration allows for a uniform comparison across cases.

5.1. Evolution of TKE, spectra, and PE-to-KE ratios

The evolution of the area-integrated TKE of the disk and spheroid wakes is shown in figure 8. The most noticeable aspect is that, for all $Fr$ numbers, $\{E^K_T\}$ in spheroid wakes is an order of magnitude smaller than in corresponding disk wakes. Although it is not shown here, we found a similar result for the area-averaged TKE (over an area of $r = 4D$ cross-section), instead of area-integrated TKE.

The decay of the unstratified wake is studied in more detail in ONS21 for the spheroid and CS20 for the disk. In unstratified flow, $\{E^K_T\}$ decays following $E^K_T \sim x^{-2/5}$ for the spheroid and the disk wake follows $\{E^K_T\} \sim x^{-2/3}$. Both decay rates can be readily obtained from the decay of the peak TKE and the growth of $L$. Specifically, in the case of the spheroid $L \sim x^{3/5}$ and $E^K_T \sim x^{-8/5}$ and, in the case of the disk, $L \sim x^{1/3}$, and peak TKE $E^K_T \sim x^{-3/3}$.

At $Fr = 10$ and for both the spheroid and the disk, $\{E^K_T\}$ deviates from the unstratified case at $Nt \approx 1$, see figure 8(a,b). Interestingly, while the TKE is affected by stratification at $Nt \approx 1$ for both wake generators, the mean flow showed a different behavior. In the disk $Fr = 10$ wake, $U_d$ deviated from its unstratified counterpart at $Nt \approx 1$ while, in the spheroid, this change occurred later at $Nt \approx 3$. The onset of deviations from the unstratified case will be explained in more detail in the following subsections.

At $Fr = 2$, there is a striking influence of the wake generator on the evolution of TKE. Whereas in the disk wake, the TKE decays monotonically, the far wake of the spheroid displays a rapid increase.

The disk wake shows a monotonic decay in $\{E^K_T\}$ and its individual components throughout
Compared to the horizontal components, \( \{E_{Kz}^T\} \) shows a sharper decay after \( Nt \approx 10 \) and turbulence anisotropy progressively increases. In the spheroid \( Fr = 2 \) wake, \( \{E_{Kx}^T\} \) decays rapidly until \( Nt \approx 10 \). However, after \( Nt \approx 10 \), the decay slows down and is followed by a period of sustained growth starting at \( Nt \approx 20 \) and lasting until the end of the domain. The region of TKE growth coincides with the development of the large-scale horizontal motions observed in figure 2(d) and the rapid growth of \( L_H \) shown in figure 5(c). It also coincides with the accelerated decay rate of \( U_d \) starting at \( Nt \approx 20 \).

Notice that right before the start of the rapid increase of TKE, the \( Fr = 2 \) wake has a configuration where \( L_H < L_V \) (figure 7d). The horizontal response of the flow to the strong lee waves is what sustains this configuration. It is only after their strength subsides that the control on the wake is released to allow the horizontal wavy motion to develop. The rapid development of this motion coincides with the rapid increase in horizontal TKE, namely \( \{E_{Ky}^T\} \) and \( \{E_{Kx}^T\} \) as seen in figure 8(c,e). To the best of the authors’ knowledge this is the first wake study, resolving the flow at the body, that observes an increase of fluctuation energy with downstream distance instead of its usual decrease.

To further quantify the horizontal wavy motions observed in figure 2(d), the energy spectra of the spanwise velocity are computed at the centerline. These spectra are compared between locations before (figure 9a) and after (figure 9b) the start of the the TKE increase associated with the Q2D regime. The spectra before \( x < 40 \) do not show preferential energization of the low frequencies. This finding is consistent with the visualizations in figure 2 where the intermediate wake does not show any sign of large-scale motions. Beyond \( x = 40 \), however, spectra show a strong peak at Strouhal number \( St \approx 0.35 \) (figure 9b). This value of Strouhal number agrees with the approximate wavelength of structures in figure 2(d), where the
To summarize, the arrival of the Q2D regime in the $Fr = 2$ wake is accompanied by a strong increase in TKE (figure 8a) and the appearance of large scale motions in the center-horizontal plane (figure 2d and 9b).

In the $Fr = 2$ disk wake, the vortex shedding mode at $St = fD/U_\infty \approx 0.13 - 0.14$ is dominant throughout the whole domain. The horizontal meanders, which are prevalent in the spheroid wake, are absent in the disk wake at least until the end of the domain at $x/D = 125$. Since the vortex shedding mode, its long-lasting effect on the wake, and its internal wave field are described in detail by Nidhan et al. (2022), we do not discuss these aspects further.

The spheroid wake has significantly lower TKE content relative to the disk wake and also a different distribution of the mean momentum. The content of potential energy relative to that of kinetic energy is also of interest. Figure 10 shows the ratio of area-integrated potential energy (PE) to kinetic energy (KE). Both fluctuating (10a) and mean (figure 10b) components are shown at $Fr = 2$ and 10 for both wake generators.
At $Fr = 10$, the turbulent PE-to-KE ratio increases steadily in both disk and spheroid cases indicating an increasing influence of buoyancy on turbulence (figure 10a). In the $Fr = 2$ wakes, the turbulent PE-to-KE ratio peak around $x \approx 30$ in both cases and decay afterward. By the end of the measurement region at $x = 80$, the turbulent PE-to-KE ratios are similar across $Fr = 2$ and 10 wakes. Thus, stratification and body shape do not qualitatively affect the ability of turbulence to stir the density field in the intermediate and far wake. Quantitatively, the TPE-to-TKE ratios are somewhat higher for the disk relative to the spheroid for both wakes.

The mean PE-to-KE ratios in the $Fr = 10$ wakes (figure 10b) are minuscule compared to their turbulent counterparts. At $Fr = 2$, the mean-based ratio is much larger, particularly close to the wake generators, pointing towards a strong influence of the steady lee waves. Both disk and spheroid mean-based ratios oscillate with a characteristic lengthscale corresponding to the wavelength of steady lee waves at $Fr = 2$. It is particularly revealing as to how much larger is the magnitude of $\frac{\langle E_P^M \rangle}{\langle E_K^M \rangle}$ in the spheroid wake compared to the disk, as it explains why the spheroid flow is so strongly modulated by the lee waves. It is worth noting that comparison of the absolute value of MPE between the disk and the spheroid reveals that it is the disk that has the larger MPE, an order of magnitude larger. The amplitude of the lee wave generated by the disk is larger than that of the spheroid by about a factor of 2.

5.2. Analyses of TKE budget terms

To quantitatively understand the origin of the TKE increase in the $Fr = 2$ spheroid wake, we look into the different terms of the TKE transport equation:

$$U_i \frac{\partial E_T^K}{\partial x_i} + \frac{\partial T_i}{\partial x_i} = P - \varepsilon + B,$$

where $P$ is the turbulent production, $\varepsilon$ is the turbulent dissipation and $B$ denotes the turbulent buoyancy flux. These quantities are defined by:

$$P = -\langle u'_i u'_j \rangle \frac{\partial U_i}{\partial x_j}, \quad \varepsilon = 2\nu \langle s'_{ij} s'_{ij} \rangle - \langle \tau^{rs}_{ij} s'_{ij} \rangle, \quad B = -\frac{g}{\rho_0} \langle \rho' u'_z \rangle,$$

where $s_{ij} = (\partial_j u_i + \partial_i u_j)/2$ is the strain-rate tensor and $\tau^{rs}_{ij} = -2\nu_s s_{ij}$ is the subgrid stress tensor. The contribution of the subgrid term to the TKE transport equation is found to be small.

The turbulent production is a source of TKE and a sink in the MKE equation. The turbulent buoyancy flux transfers energy between TKE and TPE and the turbulent dissipation is a sink of TKE. Along with the sinks and sources of energy, there is a term responsible for the spatial redistribution of TKE, the turbulent transport

$$T_i = \frac{1}{2} \langle u'_i u'_j u'_j \rangle + \langle u'_i p' \rangle - 2\nu \langle u'_i s'_{ij} \rangle - \langle u'_j \tau^{rs}_{ij} \rangle.$$

The turbulent transport redistributes energy, primarily in the $y$-$z$ plane, and its contribution to the area-integrated budget is negligible.

Figure 11 shows the evolution of the area-integrated terms in the TKE budget of the spheroid wake. We do not present the TKE budget terms in the disk wakes here as they are presented and discussed in detail by CS20.

The production of the $Fr = 10$ spheroid wake decays at a rate comparable to its unstratified counterpart until $Nt \approx 3$ when it starts to deviate (figure 11a). $U_d$ in the $Fr = 10$ spheroid wake also starts deviating around $Nt \approx 3$ (figure 4a). At $Nt \approx 3$, the contribution of $P_{xz} = \langle u'_x u'_z \rangle \partial_z U_x$ by vertical shear starts declining due to the reduction of $\langle u'_x u'_z \rangle$. At $Nt \approx 4.5$, 

Stratified slender body wakes
the contribution of the production \( P_{xy} = \langle u'_x u'_y \rangle \partial_z U_x \) by horizontal shear overcomes \( P_{xz} \). Also, the decay rate of \( P_{xy} \) reduces compared to the unstratified case.

The reduction of \( P_{xz} \) in the \( Fr = 10 \) wake coincides with the transition of \( U_d \) to a slower decay rate. Here the NEQ regime starts when \( P_{xz} \) rapidly reduces – before \( P_{xy} \) becomes larger than \( P_{xz} \). At the same location that \( P_{xz} \) starts decaying rapidly, there is a maximum in the buoyancy flux \( B \) and \( L_V \) stops increasing. Buoyancy is starting to affect the wake decay. The decay of \( \epsilon \) is similar to that of the unstratified wake until \( Nt \approx 3 \) where the decay rate starts increasing.

The initial value of \( P \) in the \( Fr = 2 \) wake is higher than in the unstratified case. The distinct separation pattern and the vertical contraction of the mean flow (figure 5a) lead to an increase in the vertical shear and hence \( P_{xz} \). This initially high value of \( P_{xz} \) rapidly decays as \( L_V \) increases and the vertical shear is reduced. Note that the beginning of the \( U_d \)-based NEQ regime at \( Nt \approx \pi \) also coincides with this reduction of \( P_{xz} \). However, the mechanism at \( Fr = 2 \) is different from that of the \( Fr = 10 \) wake. Whereas at \( Fr = 10 \), the reduction of \( \langle u'_x u'_z \rangle \) causes the decay of \( P_{xz} \), at \( Fr = 2 \) the decay of \( \langle u'_x u'_z \rangle \) is accompanied by a sudden reduction of \( \partial_z U_x \). The reduction in vertical shear is driven by the expansion of \( L_V \) induced by lee waves (figure 5a).
As the modulation of the wake by the lee waves continues, the horizontal production is enhanced by the strong reduction of $L_H$ in the NEQ region during $7 < x < 20$. The value of $\{P_{xy}\}$ overtakes $\{P_{xz}\}$ at $Nt \approx 5$, which is located close to the overtaking location ($Nt \approx 4.5$) for the $Fr = 10$ wake. The magnitude of $\{P_{xy}\}$ remains nearly constant until the end of the domain. Figure 11(f) shows the maximum horizontal mean shear in the central streamwise-horizontal plane for both wakes. The mean shear in the $Fr = 2$ wake increases in the region $10 < x < 30$, exactly where the TKE decay starts plateauing, see figure 8(a). The enhanced mean horizontal shear at $Fr = 2$ prevents the horizontal production from decaying monotonically unlike $Fr = 10$.

Since the turbulent dissipation keeps decreasing (figure 11c), the value of $\{P\}/\{\varepsilon\}$ becomes $> 1$ explaining the rapid increase of TKE in the $Fr = 2$ wake. Note that the $\{P\}/\{\varepsilon\}$ ratio oscillates with the characteristic wavelength of the lee waves revealing the strong influence of wave-related buoyancy effects on the energetics of the $Fr = 2$ wake.

One of the features of the arrival of the NEQ regime reported in previous studies is the radiation of internal gravity waves (Abdilghanie & Diamessis 2013; De Stadler & Sarkar 2012; Rowe et al. 2020). In these spheroid wakes, we find that the integrated wave flux remains negligible compared to the other terms in the TKE budget and hence is not shown here. The small magnitude of the turbulent wave flux is consistent with the findings of Meunier et al. (2018) who found that the magnitude of the wake-generated waves depends on the body drag coefficient – and the 6:1 spheroid has a very low drag compared to bluff bodies.

5.3. Early arrival of the Q2D regime in the $Fr = 2$ spheroid wake

A key difference between the $Fr = 2$ disk and spheroid wakes is the early arrival of the Q2D regime in the spheroid wake. Whereas in the disk wake, CS20 did not observe a transition to the Q2D in a domain that extended up to $x = 125$ ($Nt = 62.5$), here we observe a transition at $x \approx 40$ ($Nt \approx 20$). The early transition in the spheroid wake is a consequence of the strong modulation of the intermediate wake by buoyancy, an effect that occurs for bodies with large aspect ratio ($L/D$) and, specifically, when $Fr$ is near its critical value $Fr_c = L/D\pi$.

Figure 5(c) shows that $L_H$ in the $Fr = 2$ spheroid wake contracts in the region $5 < x < 30$ as a response to the expansion of $L_V$ (figure 5a) by steady lee waves. This phenomenon leads to the ‘butterfly’ shaped structure of mean $U_x$ (figure 7(d,e)) which was also observed at a lower Reynolds number, $Re = 10^4$ in Ortiz-Tarin et al. (2019). In that study, which was performed at $Re = 10^4$, the spheroid wake at critical $Fr_c$ relaminarized. Here, at a higher $Re = 10^5$, the flow response at the resonant state is quite different. The constriction of $L_H$ leads to an enhancement in the mean horizontal shear shown in figure 11(f). This enhancement significantly slows down the decay of horizontal production (figure 11b). While $\varepsilon$ continues to decay, $\{P\}/\{\varepsilon\}$ becomes $> 1$ leading to an increase of TKE for $x > 40$ (figure 8a). In figure 7(d-f), we also see the maximum TKE location moving to the wake axis from its off-center location in the near wake. It is worth noting that the enhanced $\{P\}$ that acts as a source of TKE is a sink for the mean energy. The sharp increase in TKE leads to a faster decay of $U_d$, and the $Fr = 2$ wake transitions to the Q2D regime early on, at $x \approx 40$.

In the disk wake, $L_H$ is initially 3–4 times larger than in the spheroid wake. While the lee wave modulation of $L_H$ is present in the disk wake as well (figure 5d), its amplitude relative the original value of $L_H$ is quite small. Hence the horizontal mean shear (not shown here) and the horizontal production in the $Fr = 2$ disk wake continue to decay unlike in the spheroid wake.

The arrival of the Q2D regime in the spheroid wake is also accompanied by distinctive features of the Q2D regime reported in previous literature. Figure 2(d) shows lateral meanders in the late $Fr = 2$ wake similar to the lateral meanders in temporal simulations in the literature.
Ortiz, Nidhan and Sarkar (Brucker & Sarkar 2010; Diamessis et al. 2011), albeit the temporal-simulation meanders occur much later in \(Nt\) units. As noted during the discussion of spectra, the waviness in the late wake has a characteristic frequency \(St \approx 0.35\). The mean wake in the Q2D regime has a layered topology (figure 7f) as reported by Spedding (2002) and Chongsiripinyo et al. (2017). The turbulence state in the Q2D regime is characterized by weak vertical fluctuations \(u'_z \ll u'_h\) (Spedding 1997) with \(\{E_{K_h}^T\}/\{E_{K_h}\} < 0.1\) at \(x > 60\) – where subscript \(h\) denotes the horizontal component of the fluctuations. Since the Q2D regime of the spheroid wake is in a relatively early phase, pancake vortices do not appear until the end of the simulation domain, \(x = 80\).

5.4. Late transition to NEQ regime in the \(Fr = 10\) spheroid wake

At \(Fr = 10\), the main difference between the disk and spheroid wakes is the location at which \(U_d\) deviates from the unstratified counterpart, i.e., the transition point to the NEQ regime. In the spheroid wake this transition occurs around \(Nt \approx 3\), whereas in the disk it occurs at \(Nt \approx 1\), see figure 4 (a).

To understand better how this transition occurs we can look into the TKE and MKE budget terms. In broad terms, in stratified wakes the mean and the turbulence deviate from their unstratified counterparts when buoyancy starts affecting these quantities. In the MKE budget, buoyancy can directly affect the turbulence production and MKE-to-MPE transfer. Likewise in the TKE budget, buoyancy controls the transfer to TPE through buoyancy flux \(B\). In the \(Fr = 10\) spheroid wake, we find the MKE to MPE transfer to be weak initially from \(3 < x < 20\), as confirmed by the ratios in figure 10(b). Hence, the only other way stratification come into play is through buoyancy effects on production. In figure 11(a) we see that the \(Fr = 10\) spheroid production starts deviating from the unstratified counterpart at \(x \approx 30\) \((Nt \approx 3)\). This is also the location where \(U_d\) deviates from the unstratified scaling in figure 4(a), thus explaining the later onset of NEQ transition in the spheroid wake.

Turning to the disk wake at \(Fr = 10\), we find a key difference with the spheroid wake is that the mean buoyancy flux is non-negligible, positive, and monotonically increasing from \(x \approx 10\) onward, indicating a net transfer of energy from MPE to MKE. As a result, \(U_d\) as well as the MKE (figure 17a of CS20) start deviating from the unstratified counterpart at \(x \approx 10\) or \(Nt \approx 1\).

6. Evolution of the local flow state and its trajectory in phase space

In previous sections, we showed how the spheroid and the disk wake do not transition between the 3D, NEQ and Q2D regimes at the same \(Nt\). In this section, we examine the evolution of key local non-dimensional numbers describing the mean and fluctuating state to explore their roles. These non-dimensional numbers are local, streamwise-varying measures of stratification (Froude number) and the dynamical range of inertially dominated scales (Reynolds number). We also plot the trajectory of each wake in the Froude-Reynolds phase space.

The mean vertical Froude number \((Fr_v)\) and the turbulent vertical and horizontal Froude numbers \((Fr_v, Fr_h)\) are defined as

\[
Fr_v = \frac{U_d}{2NL_v}, \quad Fr_v = \frac{u'_h}{Nl_v}, \quad Fr_h = \frac{u'_h}{Nl_{Hk}}. \tag{6.1}
\]

where \(u'_h = (\langle u'^2_x \rangle + \langle u'^2_y \rangle)^{1/2}\) is the r.m.s. of the horizontal fluctuations and \(l^2_v = \langle \partial_z u'_y \rangle / \langle (\partial_z u'_x)^2 + (\partial_z u'_y)^2 \rangle \) is a vertical turbulent length scale (Riley & DeBruynKops 2003).

The mean Froude number is defined consistently with the global Froude number \((Fr = \ldots\)

Stratified slender body wakes

\[ \frac{U}{ND}, \text{this is, with the wake full height } 2L_V. \] The turbulent Froude numbers are defined with turbulence length scales in the energy-containing range. Since the horizontal turbulent integral length scale \( l_h \) is not easy to compute in a spatially evolving flow we use \( L_{Hk} \) as a surrogate following CS20.

The mean vertical Froude number \( Fr_V \) (figure 12a) becomes \( O(1) \) in both disk and spheroid wakes at the location at which the decay of \( U_d \) slows down signaling the beginning of the NEQ regime, marked by \( Nt \approx 1 \) for the disk and \( Nt \approx 3 \) for the spheroid. For the \( Fr = 2 \)

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\[ Fr = 10 \]

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cases, $Fr_v$ in the spheroid wake starts dropping faster beyond $x \approx 40$ – the streamwise location where the wake transitions from NEQ to Q2D regime.

The turbulent Froude numbers play an analogous role to those based on the mean. When their values become $O(1)$, turbulence starts being affected by buoyancy. $Fr_v$ is defined using $l_v$, which has a significance to shear instability. Defined with $l_v$, $Fr_v$ is proportional to $Ri^{-1/2}$, where $Ri$ is the Richardson number of the fluctuating shear (CS20, Riley & deBruynKops (2003)). When $Fr_v$ becomes $O(1)$ (figure 12b) is also when the spanwise and vertical components of the TKE start showing anisotropy of the turbulence stress tensor and $E^T_{Kx} > E^T_{Kz}$, as shown in figure 8 for the $Fr = 2$ wakes. See also figure 8 of CS20.

In figure 12(c), the $Fr = 10$ wakes of both disk and spheroid reach $Fr_h \sim O(1)$ at $x \approx 10 - 20$, the location at which the $\{E^T_K\}$ starts deviating from its unstratified counterpart (figure 8(a,b)). In the $Fr = 2$ spheroid wake, $Fr_h < 1$ throughout the domain and $\{E^T_K\}$ deviates from the unstratified decay from the very beginning in both disk and spheroid wakes. Overall, we find that $Fr_h \sim O(1)$ is a good indicator of the deviation of $\{E^T_K\}$ from the unstratified counterpart. In contrast, $Fr_v \sim O(1)$ marks the location at which turbulence anisotropy between the vertical and the spanwise components starts growing. Figure 12(d) shows the evolution of turbulence Reynolds number, $Re_h = u_h' L_{Hk} / \nu$ with $u_h'$ being the intensity of horizontal turbulent fluctuations. The value of $Re_h$ (figure 12d) does not change much as the wake evolves, remaining at $O(10^4)$ for the disk wake and at $O(10^3)$ for the spheroid wake.

A consolidated view of the evolution of the state of fluctuations is provided in phase space (Brethouwer et al. 2007; Zhou & Diamessis 2019; de Bruyn Kops & Riley 2019; Chongssiripinyo & Sarkar 2020). In the phase-space portrait, one axis measures the buoyancy effect on the large scales through the turbulent Froude number ($Fr_h$) and the other axis is a measure of Reynolds number that is not $Re_h$ but one which accounts for buoyancy in addition to inertia. Ozmidov-scale eddies are the largest eddies unrestrained by buoyancy. The Ozmidov scales $l_o = (\epsilon / N^3)^{1/2}$ and $u_o = (\epsilon / N)^{1/2}$ lead to the definition of $Re_b = u_o l_o / \nu = \epsilon / \nu N^2$ as the so-called buoyancy Reynolds number. Another convenient measure of Reynolds number, which does not require explicit computation of the turbulent dissipation rate, is the buoyancy-weighted Reynolds number, $Re_h Fr_h^2$ (Billant & Chomaz 2001; Riley & deBruynKops 2003). The Reynolds numbers based on buoyancy tend to decrease with downstream distance as buoyancy progressively increases in importance and limits the range of scales which are susceptible to 3D turbulent motions. As long as $Re_h Fr_h^2 > O(1)$, viscous effects do not dominate.

Following CS20, figure 13 shows the evolution all four wakes in the $Re_h Fr_h^2 - Fr_h$ phase space, where $Re_h = u_h' L_{Hk} / \nu$, $u_h'$ being the intensity of horizontal turbulent fluctuations. The flow evolves in the direction of the arrow from a state where buoyancy effects are weak (almost negligible) to a region characterized by the presence of stratified turbulence. Within the state of stratified turbulence, three different regimes can be demarcated: weakly, intermittently, and strongly stratified turbulence (WST, IST, and SST). Unlike the case of freely decaying turbulence, the mean velocity is also important here. Therefore, CS20 elected to distinguish between WST (where buoyancy begins to affect the mean velocity) and IST (where the turbulence anisotropy begins to be affected) as we do here. The SST regime is one where buoyancy effects on the large scales is very strong ($Fr_h \ll 1$) but nevertheless the value of $Re_h Fr_h^2$ is sufficiently large so that viscous effects are not dominant.

The slope in the $Re_h Fr_h^2 - Fr_h$ plane is similar for both disk and spheroid wakes in the weak buoyancy and WST stages. However, there are significant differences in the way the spheroid wakes transverse the phase space. The spheroid wake starts out thinner than the disk wake. Therefore, it takes a longer streamwise development for the spheroid wake to have
the same value of $Fr_h$ as the disk wake and, therefore, the local Reynolds number becomes lower than for the disk at the same value of $Fr_h$.

The spheroid has a body-based Reynolds number ($Re = 10^5$) which is larger than the corresponding disk value by a factor of 2. Nevertheless, turbulence in the spheroid wake is unable to access either the the IST regime or the SST regime while the disk wake is able to access these regimes of stratified turbulence. Furthermore, the increase in TKE, which is not observed in the $Fr = 2$ disk wake, reverses the trajectory of the $Fr = 2$ spheroid wake. To the authors’ best knowledge, a reversing trend of phase-space trajectory has not been seen before in the stratified turbulence literature. These differences reveal that the phase space evolution, at least for $Fr \leq 10$, depends on the features of the wake generator, e.g., its aspect ratio or the type of BL separation.

Based on the phase-space portrait alone, one may hastily conclude that stratified slender body wakes always experience a weaker buoyancy effect relative to bluff bodies. This is true for the $Fr = 10$ case (at least for the limited $Nt$ simulation time) in terms of the relative amount of TPE (figure 10) and also since the deviation of the mean-based vertical scale ($L_V$) from its unstratified benchmark is smaller for the spheroid wake. But, we have also shown that for the $Fr = 2$ spheroid wake, compared to the disk wake, there is a much earlier onset of the Q2D regime. The mean and turbulence quantities are highly intertwined and the strong modulation of the mean by steady lee waves of the high aspect-ratio spheroid in the $Fr = 2$ case ultimately leads to an early entry into the Q2D regime. Hence, it is not entirely accurate to say that the slender body wakes are weakly affected by stratification. The present work calls for a need to generalize the parameter space of \{\textit{Re}, \textit{Fr}\} in turbulent shear flows to account for the mean flow field (also possibly its instabilities) to build a more comprehensive understanding of buoyancy effects in shear flows. In the case of wakes, the shape of the body generator is brought into play through the mean flow as shown here.

7. Discussion and final remarks

The high-Reynolds number stratified wake of a slender body has been studied using a high-resolution hybrid simulation. The wake generator is a 6:1 prolate spheroid with a tripped turbulent boundary layer, the diameter-based Reynolds number is $Re = 10^5$ and the Froude numbers, namely $Fr = U/ND = \{2, 10, \infty\}$, take moderate to large values. By comparing the spheroid wake with the disk wake of Chongsiripinyo & Sarkar (2020) (referred to as CS20), we are able to study the influence of the wake generator - slender versus bluff - in the establishment and evolution of stratified wakes.

The near wake of the 6:1 prolate spheroid with a turbulent boundary layer is characterized by a small recirculation region ($\sim 0.1D$). The recirculation region is surrounded by small-scale turbulence that emerges from the boundary layer and does not show strong vortex shedding (Jiménez \textit{et al.} 2010; Posa & Balaras 2016; Kumar & Mahesh 2018; Ortiz-Tarin \textit{et al.} 2021). As a result, the wake is much thinner and develops slower than the wake of a bluff body like the disk, which has a large recirculation region ($\sim 2D$) and vortex shedding from the body (Nidhan \textit{et al.} 2020). These body-dependent features of the near wake were recently shown to affect the decay of the far wake in environments without density stratification (Ortiz-Tarin \textit{et al.} 2021). In the present stratified simulations we also find substantial differences in the decay of the disk and spheroid wake. Particularly, we find that the starting locations of the non-equilibrium (NEQ) and the following quasi-2D (Q2D) regions of wake deficit velocity depend on the wake generator.

At $Fr = 2 \approx (L/D)/\pi$, the wake of a 6:1 prolate spheroid is in a resonant state. The half wavelength of the lee waves is equal to the body length and, as a result, the flow separation and the wake are strongly modulated by the waves. Whereas previous works had described
this regime in laminar-separation configurations of a sphere (Hanazaki 1988; Chomaz et al. 1992) and a 4:1 spheroid (Ortiz-Tarin et al. 2019), the present results show that the influence of the waves persists even at high Reynolds numbers and with the separation of a turbulent boundary layer. At $Fr = 2$, the flow and the turbulence in the spheroid wake evolve very differently from the disk wake. Both the lack of strong shedding in the near wake (Ortiz-Tarin et al. 2021) and the strong modulation of the mean flow by the lee waves, lead to a wake with vertical and horizontal profiles of mean velocity that depart strongly from Gaussian. These features are not observed in the disk wake at $Fr = 2$, which shows a vertically-squeezed Gaussian topology and a weak imprint of lee waves on the wake dimensions.

At $Fr = 2$, both disk and spheroid wakes transition to the NEQ regime at $Nt \approx \pi$. However the transition to the Q2D regime - with enhanced wake decay - is very different; whereas the spheroid wake transitions at $Nt \approx 15$, the disk wake does not access the Q2D regime in a domain that spans $Nt \approx 60$. Other bluff bodies, e.g., the towed sphere (Spedding et al. 1996; Spedding 1997) show transition to the Q2D regime at $Nt \approx 50$, also delayed with respect to the spheroid wake. The early transition to the Q2D regime of the spheroid wake is driven by its strong modulation – horizontal contraction and expansion of the wake width– in response to the vertical contraction and expansion by the lee waves. This modulation has a particularly strong effect on the slender wake of a spheroid where the horizontal contraction is a large fraction of the wake width. The early start of the Q2D regime in the spheroid wake is accompanied by a sustained increase of turbulent kinetic energy (TKE), driven by an increase of the horizontal mean shear which acts on the turbulence of the separated boundary layer. The TKE increase is limited to the horizontal velocity with the spanwise component being strongest, having almost an order of magnitude larger energy than the vertical. Although coherent vortical structures and spanwise flapping are seen in the horizontal motion, pancake eddies are not seen until the end of the domain at $x/D = 80$.

At $Fr = 10$ also, there are differences between the disk and the spheroid wakes. Particularly in the spheroid wake, the beginning of the NEQ stage occurs later, at $Nt \approx 3$ instead of $Nt \approx 1$ ($x \approx 30$ instead of $x \approx 10$). The difference in the start of the NEQ can be attributed to the value of the local mean Froude number $Fr_V = U_d/2NL_V$. As noted previously, the spheroid wake is thinner than the disk wake, the mixing in the near wake is weaker, and as a result the defect velocity in the intermediate wake is larger. These features increase the value of the spheroid wake local Froude number and delay the onset of the buoyancy effect that gives rise to the NEQ regime. Additionally, the analysis of the energy transfers in the mean kinetic energy (MKE) balance shows that the onset of buoyancy effect in the disk wake is associated with the transfer from mean potential energy (MPE) to MKE while that for the spheroid wake is linked to the reduction in the transfer from MKE to TKE. For a slender body such as a spheroid, the separation is streamlined so that the change in mean vertical location of a fluid particle as it moves from the fore to the lee of the body is small, keeping the MPE small. Therefore, unlike a bluff body such as a disk, the MPE reservoir is not influential insofar as buoyancy effects on the wake kinetic energy.

Taking the unstratified case as a base line, the effect of buoyancy in the spheroid $Fr = 10$ wake is observed earlier (at $Nt \approx 1$) on the decay of the TKE than its effect (at $Nt \approx 3$) on the decay of the mean defect velocity. In the spheroid wake at $Fr = 10$, the transfer from TKE to TPE is responsible for the enhanced decay of TKE at $Nt \approx 1$ while the transfer from MKE to TKE is responsible for the enhanced decay of the mean defect velocity at $Nt \approx 3$. The introduction of the potential energy reservoir introduces additional routes for energy transfer and, therefore, buoyancy might not affect the mean and turbulent components of kinetic energy at the same time. Intuitively, one would expect that buoyancy first affects the largest scales (e.g., the mean velocity) and progressively the smaller scales of velocity fluctuations (TKE) start feeling the influence of buoyancy. This intuition, borrowed from the
study of homogeneous stratified turbulence, was supported for the disk wake but does not translate well to the evolution of the slender body wake.

Meunier & Spedding (2004) compared the evolution far into the stratified wake, up to $x \approx 8000$, among several body shapes. The body Reynolds number was $Re = 5000$. The defect velocity showed difference among the various bodies until $Nt \approx 30$ but these differences reduced with increasing $Nt$ so that by $Nt \approx 50$ the laboratory data suggested a universal decay with the Q2D power-law exponent. In contrast, the present result for the $Fr = 2$ wake shows an earlier onset of the Q2D regime at $Nt \approx 15$. The value of $Re = 10^5$ is larger here and it is possible that features linked to the early onset, i.e., the instability that leads to horizontal meanders and also the enhanced TKE production, are inhibited by viscous damping at the lower $Re$ of the experiments. The present simulations do not extend into the very far wake reached in the experiments. Future numerical or experimental work at higher $Re$ that probes the very far wake would clearly be useful. To do so in simulations, it will be efficient to utilize the temporal model and initialize it with data from a hybrid simulation of the type conducted here.

The simulations show that the buoyancy timescale $Nt$ alone is not sufficient to determine the state of the wake decay for both generators. However, we find that the value of the local turbulent and mean Froude numbers can be a good proxy to describe some aspects of the wake state. For both disk and spheroid wakes, $FrV = U_d/2NL_v$ becomes $O(1)$ at the location at which the decay of $U_d$ slows down; $Fr_h = u'_h/NLHk \sim O(1)$ marks the location at which the area-integrated TKE of the stratified wake starts deviating from the unstratified case; and $Fr_V = u'_h/Nl_v \sim O(1)$ signals the location at which anisotropy between the different TKE components starts growing.

The buoyancy-weighted Reynolds number ($Re_h Fr_h^2$) has been used widely in stratified flow as a convenient surrogate for the buoyancy Reynolds number ($Re_b = \varepsilon/\nu N^2$) since it displays similar trends during the flow’s evolution and the two quantities can be shown to be proportional using classical inviscid scaling of the turbulent dissipation rate. The surrogacy is true for the stratified wakes considered here except for the $Fr = 2$ wake after its entry into the stage of Q2D wake decay. The horizontal fluctuation energy, therefore $Fr_h$, increases owing to horizontal meanders and flapping of the flow. However, $\varepsilon$ continues to decrease, albeit at a reduced rate relative to the NEQ regime. The value of $Re_b = O(10)$ is not high in the Q2D regime realized here at $Fr = 2$. It remains to be seen if, in the Q2D regime at even higher body-based Reynolds number, the equivalence between $Re_h Fr_h^2$ and $Re_b$ is recovered and whether the unusual upward trajectory seen here in $\{Fr_h, Re_h Fr_h^2\}$ phase space is also seen in $\{Fr_h, Re_b\}$ space. The duration of the upward trajectory in phase space until the eventual downward shift toward the viscous regime is also of interest.

The differences between bluff body (disk) and slender body (6:1 spheroid) wakes illustrate the difficulty of finding a universal scaling for the high-$Re$ stratified wake. The initial magnitude of $U_d$ for different wake generators and levels of stratification can be roughly scaled with the global $Fr$ and the body drag coefficient (Meunier & Spedding 2004). However, the start and the duration of the NEQ regime cannot be assumed to be independent of the wake generator. We find that rather than a particular $Nt$, the local mean Froude number is a good proxy for the onset of the NEQ regime in the mean defect velocity and the values of local turbulent Froude number provide guidance for the behavior of TKE, e.g., the onset of buoyancy effect as well as the location at which the ratio of vertical to horizontal TKE starts decreasing. We are unable to connect Froude number to the Q2D regime transition of the wake. More numerical and experimental work spanning different wake generators, different sources of turbulence including freestream turbulence, and longer downstream distances
will be instrumental in building a comprehensive picture of the effect of initial/boundary conditions on subsequent wake evolution.

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**Author Contributions.** J.L.O.-T. and S.N. contributed equally to this paper.

**REFERENCES**

Abdilghanie, A. M. & Diamessis, P. J. 2013 The internal gravity wave field emitted by a stably stratified turbulent wake. *J. Fluid Mech.* 720, 104–139.

Baines, P. G. 1998 *Topographic Effects in Stratified Flows.* Cambridge University Press.

Balaras, E. 2004 Modeling complex boundaries using an external force field on fixed Cartesian grids in large-eddy simulations. *Comput. Fluids* 33, 375–404.

Bevilaqua, P. M. & Lykoudis, P. S. 1978 Turbulence memory in self-preserving wakes. *J. Fluid Mech.* 89 (3), 589–606.

Billant, P. & Chomaz, J. M. 2001 Self-similarity of strongly stratified inviscid flows. *Phys. Fluids* 13 (6), 1645–1651.

Brethouwer, G., Billant, P., Lindborg, E. & Chomaz, J.-M. 2007 Scaling analysis and simulation of strongly stratified turbulent flows. *J. Fluid Mech.* 585, 343–368.

Brucker, K. A. & Sarkar, S. 2010 A comparative study of self-propelled and towed wakes in a stratified fluid. *J. Fluid Mech.* 652, 373–404.

de Bruyn Kops, S. M. & Riley, J. J. 2019 The effects of stable stratification on the decay of initially isotropic homogeneous turbulence. *J. Fluid Mech.* 860, 787–821.

Castro, I. P., Snyder, W. H. & Marsh, G. L. 1983 Stratified flow over three-dimensional ridges. *J. Fluid Mech.* 135, 261–282.

Chesnakas, C. J. & Simpson, R. L. 1994 Full three-dimensional measurements of the cross-flow separation region of a 6:1 prolate spheroid. *Exp. Fluids* 17, 68–74.

Chevray, R. 1968 The turbulent wake of a body of revolution. *J. Basic Eng.* 90, 275–284.

Chomaz, J. M., Bonneton, P., Butet, A., Perrier, M. & Hofffinger, E. J. 1992 Froude number dependence of the flow separation line on a sphere towed in a stratified fluid. *Phys. Fluids* 4 (2), 254–258.

Chomaz, J. M., Bonneton, P. & Hofffinger, E. J. 1993 The structure of the near wake of a sphere moving horizontally in a stratified fluid. *J. Fluid Mech.* 254 (II), 1–21.

Chongsiripinyo, K., Pal, A. & Sarkar, S. 2017 On the vortex dynamics of flow past a sphere at Re = 3700 in a uniformly stratified fluid. *Phys. Fluids* 29 (2), 020704.

Chongsiripinyo, K. & Sarkar, S. 2020 Decay of turbulent wakes behind a disk in homogeneous and stratified fluids. *J. Fluid Mech.* 885.

Constantinescu, G. S., Pasinato, H., Wang, Y. Q., Forsythe, J. R. & Squires, K. D. 2002 Numerical investigation of flow past a prolate spheroid. *J. Fluids Eng.* 124 (4), 904–910.

Costis, C. E., Telionis, D. P. & Hoang, N. T. 1989 Laminar separating flow over a prolate spheroid. *J. Aircr.* 26 (9), 810–816.

Dairay, T., Oelligado, M. & Vassilicos, J. C. 2015 Non-equilibrium scaling laws in axisymmetric turbulent wakes. *J. Fluid Mech.* 781, 166–195.

De Stadler, M. B. & Sarkar, S. 2012 Simulation of a propelled wake with moderate excess momentum in a stratified fluid. *J. Fluid Mech.* 692, 28–52.

Diamessis, P. J., Speeding, G. R. & Domaradzki, J. A. 2011 Similarity scaling and vorticity structure in high-Reynolds-number stably stratified turbulent wakes. *J. Fluid Mech.* 671, 52–95.

Dommernuth, D. G., Rottman, J. W., Innis, G. E. & Novikov, E. A. 2002 Numerical simulation of the wake of a towed sphere in a weakly stratified fluid. *J. Fluid Mech.* 473 (473), 83–101.

Drazin, P. G. 1961 On the steady flow of a fluid of variable density past an obstacle. *Tellus* 13 (2), 239–251.

Fu, T. C., Shekarriz, A., Katz, J. & Huang, T. T. 1994 The flow structure in the lee of an inclined 6:1 prolate spheroid. *J. Fluid Mech.* 269, 79–106.

Gourlay, M. J., Arendt, S. C., Fritts, D. C. & Werne, J. 2001 Numerical modeling of initially turbulent wakes with net momentum. *Phys. Fluids* 13 (12), 3783–3802.
Turbulent Wake. J. Fluid Mech. 179–187.

Geometry in the wake of a towed sphere in a stably stratified fluid. J. Fluid Mech. 454, 5–31.

Sphere at a subcritical Reynolds number of 3700 and moderate Froude number. J. Fluid Mech. 853, 537–563.

A loss of memory in stratified momentum wakes. Phys. Fluids 16 (2), 298–305.

Subgrid-scale stress modelling based on the square of the velocity gradient tensor. Flow Turbul. Combust. 62, 183–200.

Spectral POD analysis of the turbulent wake of a disk at Re=50000. Phys. Rev. Fluids .

Analysis of coherence in turbulent stratified wakes using spectral proper orthogonal decomposition. J. Fluid Mech. 934.

Numerical simulations of the near wake of a sphere moving in a steady, horizontal motion through a linearly stratified fluid at Re = 1000. Phys. Fluids 27 (3), 035113.

Stratified flow past a prolate spheroid. Phys. Rev. Fluids 094803, 1–28.

High-Reynolds number wake of a slender body. J. Fluid Mech. 261, 333–374.

Regeneration of turbulent fluctuations in low-Froude number flow over a sphere at Reynolds number of 3700. J. Fluid Mech. 804 R2, 1–11.

Direct numerical simulation of stratified flow past a sphere at a subcritical Reynolds number of 3700 and moderate Froude number. J. Fluid Mech. 826, 5–31.

Temporal/spatial simulation of the stratified far wake of a sphere. Comput. Fluids 40 (1), 179–187.

A numerical investigation of the wake of an axisymmetric body with appendages. J. Fluid Mech. 792, 470–498.

On the universality of turbulent axisymmetric wakes. J. Fluid Mech. 710, 419–452.

Direct Numerical Simulation of Weakly Stratified Turbulent Wake. Nasa Technical Memorandum 218523 (September), 568–609.

A numerical study of a weakly stratified turbulent wake. J. Fluid Mech. 776, 568–609.

Low-dimensional dynamics of a turbulent axisymmetric wake. J. Fluid Mech. 755, R51–R511. 2003 Dynamics of turbulence strongly influenced by buoyancy. Phys. Fluids 15 (7), 2047–2059.

Internal gravity wave radiation from a stratified turbulent body. J. Fluid Mech. 888, 1–26.

The evolution of initially turbulent bluff-body wakes at high internal Froude number. J. Fluid Mech. 337, 283–301.

Vertical structure in stratified wakes with high initial froude number. J. Fluid Mech. 454, 71–112.

Wake Signature Detection. Annu. Rev. Fluid Mech. 46 (1), 273–302.

Turbulence, similarity scaling and vortex geometry in the wake of a towed sphere in a stably stratified fluid. J. Fluid Mech. 314, 53–103.
VanDine, A., Chongsiripinyo, K. & Sarkar, S. 2018 Hybrid spatially-evolving DNS model of flow past a sphere. *Comput. Fluids* **171**, 41–52.

Vassilicos, J. C. 2015 Dissipation in Turbulent Flows. *Annu. Rev. Fluid Mech.* **47** (1), 95–114.

Wang, K. C. 1970 Three-dimensional boundary layer near the plane of symmetry of a spheroid at incidence. *J. Fluid Mech.* **43**, 187–209.

Wang, K. C., Zhou, H. C., Hu, C. H. & Harrington, S. 1990 Three-dimensional separated flow structure over prolate spheroids. *Proc. R. Soc. London, Ser. A: Mathematical and Physical Sciences* **429** (1876), 73–90.

Wikström, N., Svennberg, U., Alin, N. & Fureby, C. 2004 Large eddy simulation of the flow around an inclined prolate spheroid. *J. Turbul.* **5** (29).

Wygnanski, W., Champagne, F. & Marasli, B. 1986 *On the large-scale structures in two-dimensional, small-deficit, turbulent wakes*, , vol. 168.

Yang, J. & Balaras, E. 2006 An embedded-boundary formulation for large-eddy simulation of turbulent flows interacting with moving boundaries. *J. Comput. Phys.* **215** (1), 12–40.

Zhou, Q. & Diameissis, P. J. 2019 Large-scale characteristics of stratified wake turbulence at varying Reynolds number. *Physical Review Fluids* **4** (8), 1–30.