POINTWISE WAVE BEHAVIOR OF THE NAVIER-STOKES EQUATIONS IN HALF SPACE

LINGLONG DU
Department of Applied Mathematics, Donghua University
Shanghai 201620, China

HAITAO WANG*
Institute of Natural Sciences and School of Mathematical Sciences
Shanghai Jiao Tong University
Shanghai 200240, China

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Abstract. In this paper, we investigate the pointwise behavior of the solution for the compressible Navier-Stokes equations with mixed boundary condition in half space. Our results show that the leading order of Green’s function for the linear system in half space are heat kernels propagating with sound speed in two opposite directions and reflected heat kernel (due to the boundary effect) propagating with positive sound speed. With the strong wave interactions, the nonlinear analysis exhibits the rich wave structure: the diffusion waves interact with each other and consequently, the solution decays with algebraic rate.

1. Introduction. The Navier-Stokes equations are the fundamental system in the fluid dynamics. Qualitative and quantitative studies on the fluid dynamic theory will help us to understand the physical phenomena much better. Most of the interesting phenomena in the fluid dynamics are related to the presence of a physical boundary, such as slip boundary layer, thermal creep flow, and curvature effects. They can be understood only with the knowledge of the interaction between the fluid waves and the boundary layer. In this paper, we begin to study the one dimensional isentropic Navier-Stokes equations. We focus on the structure of long time solution in the sense of pointwise wave propagation. There have been some essential progress in developing the pointwise estimate of long time solution for the initial-boundary value problem of different models such as [1, 2, 4, 6, 7, 13] and the references therein.

The isentropic Navier-Stokes equations in Eulerian coordinate with mixed boundary condition in half line are

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* Corresponding author: Haitao Wang, haitaowang.math@gmail.com.
\[
\begin{aligned}
\rho_t + m_x = 0, & \quad x \in \mathbb{R}_+, \quad t > 0, \\
m_t + \left( \frac{m^2}{\rho} + p(\rho) \right) = \nu \left( \frac{m}{\rho} \right)_{xx}, \\
(a_1 \partial_x m + a_2 m)|_{x=0} = 0,
\end{aligned}
\]

where \(\rho(x, t) > 0\) is the density, \(m(x, t)\) represents the momentum, and \(p(\rho)\) represents the pressure. \(\nu\) is the viscosity, \(a_1\) and \(a_2\) are constants. Here we only treat the case \(a_1a_2 < 0\). The boundary condition of Dirichlet type (\(a_1 = 0\)) and Neumann type (\(a_2 = 0\)) are much simpler. For the case of \(a_1a_2 > 0\), the linearized system is unstable. These are explained in Section 2. To overcome the complexity of the mixed type condition, we first construct Green’s function for the linearized initial-boundary value problem.

We linearize the above system around \(\rho = 1\) and \(m = 0\), and denote the perturbation still by \(\rho\) and \(m\):

\[
\begin{aligned}
\rho_t + m_x = 0, \\
m_t + c^2 \rho_x = \nu m_{xx} + Q, \\
(a_1 \partial_x m + a_2 m)|_{x=0} = 0,
\end{aligned}
\]

where \(c = \sqrt{\rho'(1)}\) is the sound speed, and the nonlinear term

\[Q \equiv \tilde{Q}_x = - \left[ \frac{m^2}{1 + \rho} + p(1 + \rho) - p(1) - p'(1) \rho + \nu \left( \frac{\rho m}{1 + \rho} \right)_{xx} \right].\]

In [10], the authors proved the global existence for the initial-boundary value problem of compressible Navier-Stokes equations in three-dimensional half space and showed the convergence of the solution to the equilibrium state by using the energy estimates. Later, [11, 12] obtained the decay rate of the solution for the half space \(\mathbb{R}^n_+\) for \(n \geq 2\) with Dirichlet boundary condition. They proved that if the initial perturbation of constant state is in \(H^{s+l} \cap L^1\), where \(s = \left[ \frac{n}{2} \right] + 1\) and \(l\) is a nonnegative integer, then the \(L^p\) norm solution of linearized problem has the optimal \(O(t^{-\frac{n}{2}(1 - \frac{l}{p})})\) decay rate with \(p \in [2, \infty]\). Here we refine the \(L^p\) estimates into more accurate pointwise estimates and give a better understanding of local nonlinear wave interactions in one dimensional case. For the high dimensional case, we leave it to the future, however.

The Green’s function for the linearized initial-boundary value problem satisfies the following equation system

\[
\begin{aligned}
& \begin{pmatrix}
\partial_t + \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix} & \partial_x - \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \partial_{xx}
\end{pmatrix} \mathcal{G}(x, t; y) = 0, x > 0, y > 0, t > 0, \\
& \mathcal{G}(x, 0; y) = \delta(x - y)I, \\
& (-a_1 \partial_t - a_2) \mathcal{G}(0, t; y) = 0.
\end{aligned}
\]

Here \(I\) is \(2 \times 2\) identity matrix and \(\delta(x - y)\) is the Dirac function. The boundary condition has been rewritten in view of \(\partial_x m = -\partial_t \rho\).

Our first result is about the pointwise structure of the Green’s function \(\mathcal{G}(x, t; y)\):

**Theorem 1.1.** When \(a_1a_2 < 0\), or \(a_1 = 0\), or \(a_2 = 0\), there exists a constant \(C\) such that the Green’s function \(\mathcal{G}(x, t; y)\) of linearized system (3) has the following
estimates for all $0 \leq x, y < \infty$, $t \geq 0$:

$$
|D^a_x \left( G(x, t; y) - e^{-\frac{x^2}{2t}} (\delta(x - y) - \delta(x + y)) \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right) \right) |
\leq O(1)t^{-\alpha/2} \left( e^{\frac{(x-y-ct)^2}{2\nu t}} + e^{\frac{(x-y+ct)^2}{2\nu t}} + e^{\frac{(x+y+ct)^2}{(2a+2r)t}} \right)
$$

$$
+ O(1) e^{-(x^2+y^2)/C} + O(1) e^{-(x^2+y^2)/C}.
$$  \hspace{1cm} (4)

Here $\alpha$ is a non-negative integer.

The pointwise estimate for the Green’s function allows us to give the time-asymptotic behavior of the solution to the nonlinear initial-boundary value problem:

**Theorem 1.2.** Assume that the initial data $(\rho_0, m_0)$ satisfying $\| (\rho_0 - 1, m_0) \|_{H^s(\mathbb{R}^+)} \leq \varepsilon_0$ and

$$
|D^a_x (\rho_0 - 1, m_0)(x)| = O(1)\varepsilon_0(1 + x^2)^{-r/2}, \quad r > 1/2,
$$

for $\alpha \leq 1$. Then for $\varepsilon_0$ sufficiently small, there exists a unique global classical solution to the problem (1) with the mixed boundary condition where $a_1 a_2 < 0$ or $a_1 = 0$, or $a_2 = 0$. Moreover, the solution has the following pointwise estimates for $\alpha \leq 1$,

$$
|D^a_x (\rho - 1, m)(x, t)| \leq O(1)\varepsilon_0(1 + t)^{-\alpha/4} \left[ (x - c(t + 1))^2 + (t + 1) \right]^{-1/2},
$$

here $c = \sqrt{p'}(1)$ is the sound speed.

**Remark 1.3.** Comparing to [12], we get the optimal pointwise estimate for $\rho$ and $m$. For $\alpha = 1$, we only get extra $(1 + t)^{-1/4}$ time decay rate for the solution, this is due to the effects of boundary and closure of the nonlinear interactions. There still leaves room for improvement. We will investigate it further in the future.

**Corollary 1.4.** Under the assumptions in Theorem 1.2, we have the following optimal $L^p(\mathbb{R}^+)$ estimates of the solution

$$
\| (\rho - 1, m)(\cdot, t) \|_{L^p(\mathbb{R}^+)} \leq O(1)\varepsilon_0(1 + t)^{-\frac{1}{4}(1 - \frac{1}{p})}, \quad p \in (1, \infty].
$$

The proof of the nonlinear estimates is based on the pointwise description of Green’s function for the linearized initial-boundary value problem and the Duhamel’s principle. The nonlinearity in one dimensional problem is much stronger than that in higher dimensional cases, so we need more accurate Green’s function to identify the exact leading order. The leading term of the fundamental solution for the linearized Cauchy problem is the convective heat kernel. With the help of the connection between fundamental solution and Green’s function in the transformed space, extra mirror term is obtained due to the boundary effect for Green’s function. Thus the leading waves of Green’s function propagate in two directions: one are heat kernel and reflected heat kernel propagating with positive sound speed; the other is the heat kernel propagating with negative sound speed. Due to the strong nonlinear interaction of these waves with source terms and nonlinear terms, the solution decays with algebraic rate.

Throughout this paper we use $O(1)$, $C$, $D_0$, $D$, $\cdots$ to denote universal positive constants. Denote by $L^p$ and $W^{m,p}$ the usual Lebesgue and Sobolev spaces on $\mathbb{R}^+$ and $H^m = W^{m,2}$, with norms $\| \cdot \|_{L^p}, \| \cdot \|_{W^{m,p}}, \| \cdot \|_{H^m}$, respectively.
The rest of paper is arranged as follows: In Section 2, we construct Green’s function for the initial-boundary value problem. The pointwise structures of the nonlinear solution are studied in Section 3. Section 4 contains some useful lemmas for the nonlinear coupling of different diffusion waves.

2. The Green’s function for the initial-boundary value problem.

2.1. Fundamental solution for the Cauchy problem. The necessary preliminary for the construction of Green’s function is the fundamental solution of Cauchy problem. The fundamental solution for linearized isentropic Navier-Stokes equations solves the system

\[
\begin{cases}
\left( \partial_t + \frac{c^2}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \partial_x - \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \partial_x^2 \right) G(x,t) = 0, & x \in \mathbb{R}, \ t > 0, \\
G(x,0) = \delta(x) I.
\end{cases}
\]  

(5)

The pointwise estimate of \( G(x,t) \) is studied by [8, 14]. It can be estimated by studying the inverse Fourier transform of \( \mathcal{F}[G](\xi,t) \)

\[
\mathcal{F}[G](\xi,t) = \begin{pmatrix}
\frac{\sigma_+ e^{-s+t} - e^{-s-t}}{\sigma_+ - \sigma_-} & -i\xi e^{s+t} - e^{s-t} \\
-i\xi e^{-s+t} - e^{-s-t} & \frac{\sigma_+ e^{s+t} - e^{s-t}}{\sigma_+ - \sigma_-}
\end{pmatrix},
\]

where

\[
\sigma_\pm = -\frac{1}{2} \xi (\nu \xi \pm \sqrt{\nu^2 \xi^2 - 4c^2}).
\]

Therefore, we have

\[
G(x,t) = e^{-\frac{2c^2}{\nu} t} \delta(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + e^{-\frac{(x+c) e^{\xi} t}{\nu t}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2c} \\ \frac{c}{2} & \frac{1}{2} \end{pmatrix} + e^{-\frac{(x-c) e^{-\xi} t}{\nu t}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2c} \\ -\frac{c}{2} & \frac{1}{2} \end{pmatrix}
+ O(1) t + 1 \right]^{-\frac{1}{2}} t^{-\frac{1}{2}} \left[ e^{-\frac{(x+c) e^{\xi} t}{\nu t}} + e^{-\frac{(x-c) e^{-\xi} t}{\nu t}} \right] + O(1) e^{-c(|x|+t)/C},
\]

where \( D > 0 \) is a constant.

2.2. The connection between Green’s function and fundamental solution. Introduce the Fourier transform and Laplace transform for \( f(x,t) \):

\[
\mathcal{F}[f](\xi,t) = \int_{\mathbb{R}} e^{-i\xi x} f(x,t) dx, \\
\mathcal{L}[f](x,s) = \int_0^\infty e^{-st} f(x,t) dt.
\]

It turns out that the fundamental solution and the Green’s function are closely related in the transformed variables. Taking Laplace transform in \( t \) and Fourier transform in \( x \) to the first equation in (5), denoting the transformed variables by \( s \) and \( \xi \) respectively, and representing the transformed one by \( \mathcal{F}[\mathcal{L}[G]](\xi,s) \), we obtain

\[
\begin{pmatrix} s & \frac{i\xi}{c^2} \\ ic^2 \xi & s + \nu \xi^2 \end{pmatrix} \mathcal{F}[\mathcal{L}[G]](\xi,s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Invert the matrix to yield

\[
\mathcal{F}[\mathcal{L}[G]](\xi,s) = \frac{1}{s^2 + (c^2 + \nu s) \xi^2} \begin{pmatrix} s + \nu \xi^2 & -i\xi \\ -ic^2 \xi & s \end{pmatrix}.
\]
Taking inverse Fourier transform to $\xi$ and using
\[ \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} e^{-\lambda|x|} d\xi = \frac{e^{-\lambda|x|}}{2\lambda}, \]
where $\lambda = \lambda(s) = s/\sqrt{\nu s + c^2}$, we have
\[ \mathcal{L}[G](x,s) = \frac{1}{\nu s + c^2} \left( \nu \delta(x) + \frac{c^2 s}{\nu s + c^2} \frac{e^{-\lambda|x|}}{2\lambda} \frac{\text{sgn}(x)e^{-\lambda|x|}}{2\lambda s} \right). \]
In particular, when $\bar{x} > 0$,
\[ \mathcal{L}[G](-\bar{x},s) = \frac{1}{\nu s + c^2} \left( \frac{c^2 s}{\nu s + c^2} \frac{1}{2\lambda s} \right) e^{-\lambda \bar{x}}. \]

Now we consider the initial-boundary value problem, which is the linearization of (2),
\[ \begin{cases}
\rho_t + m_x = 0, & x > 0, \ t > 0, \\
m_t + c^2 \rho_x = \nu m_{xx}, \\
(\rho(x,0) \ m(x,0)) = (\rho_0(x) \ m_0(x)), \\
(-a_1 \rho_t + a_2 m)|_{x=0} = 0.
\end{cases} \tag{6} \]
Making use of the fundamental solution to take care the initial data,
\[ (\bar{\rho} \ \bar{m})(x,t) \equiv \int_0^\infty G(x-y,t) \begin{pmatrix} \rho_0(y) \\ m_0(y) \end{pmatrix} dy, \]
then the functions $(\rho - \bar{\rho}, m - \bar{m})$, which will be denoted by $(\hat{\rho}, \hat{m})$, satisfy the following homogeneous initial data problem
\[ \begin{cases}
\hat{\rho}_t + \hat{m}_x = 0, & x > 0, \ t > 0, \\
\hat{m}_t + c^2 \hat{\rho}_x = \nu \hat{m}_{xx}, \\
(\hat{\rho}(x,0) \ \hat{m}(x,0)) = 0, \\
(-a_1 \hat{\rho}_t + a_2 \hat{m})|_{x=0} = -(-a_1 \partial_t \ a_2) \int_0^\infty G(-y,t) \begin{pmatrix} \rho_0(y) \\ m_0(y) \end{pmatrix} dy \equiv b(t).
\end{cases} \]
Differentiating the first equation with respect to $t$, the second equation with respect to $x$, and by suitable combinations we find that
\[ \hat{\rho}_t - c^2 \hat{\rho}_{xx} = \nu \hat{\rho}_{txx}, \quad \hat{m}_t - c^2 \hat{m}_{xx} = \nu \hat{m}_{txx}. \]

Taking Laplace transform in $t$ and using homogenous initial data to get
\[ \mathcal{L}[\hat{\rho}]_{xx} = \frac{s^2}{\nu s + c^2} \mathcal{L}[\hat{\rho}], \quad \mathcal{L}[\hat{m}]_{xx} = \frac{s^2}{\nu s + c^2} \mathcal{L}[\hat{m}]. \]
Solving the ODE and dropping out the divergent mode as $x \to +\infty$, we have
\[ \mathcal{L}[\hat{\rho}](x,s) = \mathcal{L}[\hat{\rho}_0](s) e^{-\lambda x}, \quad \mathcal{L}[\hat{m}](x,s) = \mathcal{L}[\hat{m}_b](s) e^{-\lambda x}, \]
where $\lambda(s) = s/\sqrt{\nu s + c^2}$ is defined as before, $\hat{\rho}_0, \hat{m}_b$ represent the Dirichlet data. On the other hand, from the first equation of (6) and boundary relationship we have
\[ s\mathcal{L}[\hat{\rho}_0] - \lambda \mathcal{L}[\hat{m}_b] = 0, \quad -a_1 s\mathcal{L}[\hat{\rho}_0] + a_2 \mathcal{L}[\hat{m}_b] = \mathcal{L}[b](s), \]
which imply that
\[ \mathcal{L}[\hat{\rho}_0] = \frac{\lambda\mathcal{L}[b](s)}{s(a_2 - a_1\lambda)}, \quad \mathcal{L}[\tilde{m}_0] = \frac{\mathcal{L}[b](s)}{a_2 - a_1\lambda}, \]

hence
\[ \left( \frac{\mathcal{L}[\hat{\rho}]}{\mathcal{L}[\tilde{m}]} \right)(x, s) = \frac{1}{a_2 - a_1\lambda} \left( \frac{\lambda}{1} \right) \mathcal{L}[b](s) e^{-\lambda x}. \]

Thus the solution
\[ \left( \frac{\mathcal{L}[\rho]}{\mathcal{L}[m]} \right)(x, s) = \left( \frac{\mathcal{L}[\hat{\rho}]}{\mathcal{L}[\tilde{m}]} \right)(x, s) + \left( \frac{\mathcal{L}[\hat{\rho}]}{\mathcal{L}[\tilde{m}]} \right)(x, s) \]
\[ = \int_0^\infty \mathcal{L}[G](x - y, s) + \frac{1}{a_2 - a_1\lambda} \left( \frac{\lambda}{1} \right) (a_1 s - a_2) \mathcal{L}[G](-y, s) e^{-\lambda x} \left( \rho_0(y) \right) m_0(y) dy. \]

Therefore the transformed Green’s function \( \mathcal{L}[G](x, s; y) \) is
\[ \mathcal{L}[G](x, s; y) = \mathcal{L}[G](x - y, s) + \frac{1}{a_2 - a_1\lambda} \left( \frac{\lambda}{1} \right) (a_1 s - a_2) \mathcal{L}[G](-y, s) e^{-\lambda x}. \]

Direct calculation shows that the second term in \( \mathcal{L}[G](x, s; y) \) is
\[ \frac{1}{a_2 - a_1\lambda} \left( \frac{\lambda}{1} \right) (a_1 s - a_2) \mathcal{L}[G](-y, s) e^{-\lambda x}. \]

On the other hand we notice that for \( x > 0 \) and \( y > 0 \)
\[ \mathcal{L}[G](x + y, s) = \frac{1}{\nu s + c^2} \left( \frac{c^2 e^{\frac{c^2}{\nu s + c^2}}}{\nu s + c^2} \right) \mathcal{L}[G](x, s) e^{-\lambda (x+y)} \]
\[ = \frac{1}{2} \left( \frac{c^2}{\nu s + c^2} \right) \mathcal{L}[G](x, s) e^{-\lambda (x+y)} \].

Therefore the Green’s function in transformed variable is
\[ \mathcal{L}[G](x, s; y) = \mathcal{L}[G](x - y, s) + \frac{a_2 + a_1\lambda}{a_2 - a_1\lambda} \mathcal{L}[G](x + y, s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

In summary, the solution of the system (6) is
\[ \left( \begin{pmatrix} \rho \\ m \end{pmatrix} \right)(x, t) \]
\[ = \int_0^\infty \left[ G(x - y, t) + \mathcal{L}^{-1} \left[ \frac{a_2 + a_1\lambda}{a_2 - a_1\lambda} \right] * G(x + y, t) \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \rho_0(y) \\ m_0(y) \end{pmatrix} dy. \]

For \( a_1a_2 > 0 \), it is easy to see that \( \frac{a_2 + a_1\lambda(s)}{a_2 - a_1\lambda(s)} \) has a pole in right half complex plane, which leads to an exponential growth in time. In particular, for Dirichlet boundary condition \( a_1 = 0 \), we have
\[ G(x, t; y) = G(x - y, t) + G(x + y, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \]
for Neumann boundary condition $a_2 = 0$, we have
\[
\mathbb{G} (x; t; y) = G (x - y, t) - G (x + y, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let us focus on the nontrivial case that $a_1 a_2 < 0$. Taking inverse Laplace transform with respect to $t$, we have
\[
\mathbb{G} (x; t; y) = G (x - y, t) + G_{\text{mir}} (x + y, t).
\]
Here the subscript “mir” stands for “mirror” since this part is analogous to mirror image of original fundamental solution,
\[
G_{\text{mir}} (x + y, t) \equiv L^{-1} \left[ \frac{a_2 + a_1 \lambda}{a_2 - a_1 \lambda} \mathcal{L} \{G\} (x + y, s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].
\]

Now let us compute the inverse Laplace transform for the mirror part of Green’s function. Note that
\[
\frac{a_2 + a_1 \lambda}{a_2 - a_1 \lambda} \mathcal{L} \{G\} (x + y, s) = -\mathcal{L} \{G\} (x + y, s) + \frac{2a_2}{a_2 - a_1 \lambda} \mathcal{L} \{G\} (x + y, s).
\]
Let
\[
g (x, t) \equiv L^{-1} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \mathcal{L} \{G\}\right] (x, t),
\]
then function $g (x, t)$ satisfies
\[
(a_2 + a_1 \partial_x) g = 2a_2 G (x, t).
\]
Solving this ODE gives
\[
g (x, t) = 2\gamma \int_{x}^{\infty} e^{-\gamma (z - x)} G (z, t) \, dz = 2\gamma \int_{0}^{\infty} e^{-\gamma z} G (x + z, t) \, dz,
\]
where $\gamma \equiv -\frac{2a_1}{a_2} > 0$. Therefore
\[
G_{\text{mir}} (x + y, t) = \left( -G (x + y, t) + 2\gamma \int_{0}^{\infty} e^{-\gamma z} G (x + y + z, t) \, dz \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
By the expression of the fundamental solution and noticing that Dirac-delta function parts vanish for $x > 0$ and $y > 0$, we have
\[
G_{\text{mir}} (x + y, t) = - \left( e^{- \frac{(x + y + c t)^2}{4 \nu t}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{- \frac{(x + y - c t)^2}{4 \nu t}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \right)
+ 2\gamma \left\{ \frac{E (x + y, t; -c, 2\nu)}{\sqrt{\nu t}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{E (x + y, t; c, 2\nu)}{\sqrt{\nu t}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}
+ O \left( (t + 1)^{-1/2} t^{-1/2} \left( e^{- \frac{(x + y - c t)^2}{4 \nu t}} + e^{- \frac{(x + y + c t)^2}{4 \nu t}} \right) \right).
\]
Here the function $E$ is defined as follows:

**Lemma 2.1.** Define function
\[
E (x, t; \lambda, D_0) \equiv \int_{0}^{\infty} e^{-\gamma z} e^{- \frac{(x + z - \lambda t)^2}{4 \nu t}} \, dz.
\]
Let $\gamma > 0$, $\lambda > 0$. Then there exists a constant $C$ such that for any given $\varepsilon > 0$
\[
\frac{\partial^k}{\partial x^k} E (x, t; \lambda, D_0) \leq O \left( t^{-k/2} e^{- \frac{(x - \lambda t)^2}{4 (D_0 + \varepsilon) t}} + e^{- \frac{|x + t| \varepsilon}{t}} \right).
\]
Proof. Straightforward computation shows that

\[ E(x, t; \lambda, D_0) = \frac{\sqrt{\pi} D_0 t}{2} e^{\gamma(x - \lambda t) + \frac{\gamma^2 D_0 t}{4}} \text{Erfc} \left( \frac{x - \lambda t + \frac{\gamma D_0 t}{2}}{\sqrt{D_0 t}} \right), \]

where \( \text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt. \)

We consider the following cases:

Case 1. \( \frac{x - \lambda t + \frac{\gamma D_0 t}{2}}{\sqrt{D_0 t}} \leq 0; \)  
Case 2. \( 0 < \frac{x - \lambda t + \frac{\gamma D_0 t}{2}}{\sqrt{D_0 t}} < K; \)  
Case 3. \( \frac{x - \lambda t + \frac{\gamma D_0 t}{2}}{\sqrt{D_0 t}} \geq K. \)

Here the positive constant \( K \) is chosen to guarantee the validity of asymptotic expansion of \( \text{Erfc} \) function. In fact, any \( K \geq 2 \) will do.

For case 1, we have

\[ x - \lambda t = \frac{x - \lambda t}{3} + \frac{2(x - \lambda t)}{3} \leq \frac{x - \lambda t}{3} - \frac{\gamma D_0 t}{3}, \]

which implies

\[ E(x, t; \lambda, D_0) = O(1) \sqrt{t} e^{-\frac{2(x - \lambda t)}{3} - \frac{\gamma^2 D_0 t}{12}} = O(1) e^{-\frac{|x + t|}{6}}. \]

For case 2,

\[ -\frac{\gamma D_0 t}{4} < x - \lambda t + \frac{\gamma D_0 t}{4} < -\frac{\gamma D_0 t}{4} + K \sqrt{D_0 t}, \]

so

\[ E(x, t; \lambda, D_0) \leq O(1) \sqrt{t} e^{-\frac{\gamma^2 D_0 t}{12} + K \gamma \sqrt{D_0 t}} \leq O(1) e^{-\frac{t}{6}} \leq O(1) e^{-\frac{|x + t|}{6}}. \]

The second inequality comes from the fact that the function exponentially decays in \( t \) when \( t \) large and bounded when \( t \) small. The last inequality is due to \( x \) is bounded by \( t \) for case 2, so we can sacrifice part of the time decay to gain space decay.

For case 3, using expansion of \( \text{Erfc} \) function

\[ \text{Erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi} x} + O(1) \frac{e^{-x^2}}{x^3}, \]

we find that

\[ E(x, t; \lambda, D_0) = O(1) e^{-\frac{(x - \lambda)^2}{\gamma D_0 t}} \frac{t}{x - \lambda t + \frac{\gamma D_0 t}{2}}. \]

There are two subcases:

Case (3.a). \( x - \lambda t \geq K \sqrt{D_0 t} - \frac{\gamma D_0 t}{2} \) and \( |x - \lambda t| \leq \frac{\gamma D_0 t}{4}; \)
Case (3.b). \( x - \lambda t \geq K \sqrt{D_0 t} - \frac{\gamma D_0 t}{2} \) and \( |x - \lambda t| > \frac{\gamma D_0 t}{4}. \)

For the first one, the numerator and denominator are comparable, hence

\[ E(x, t; \lambda, D_0) = O(1) e^{-\frac{(x - \lambda)^2}{\gamma D_0 t}}. \]

For the second one, absorbing factor \( t \) by exponential decay yields that

\[ E(x, t; \lambda, D_0) = O(1) e^{-\frac{(x - \lambda)^2}{\gamma D_1 t}}, \]

where \( D_1 \) can be chosen as any positive number slightly larger than \( D_0. \)

For derivatives we just need to absorb the extra terms into exponential function. Thus the lemma is proved. \( \square \)
Remark 2.2. The function \( E(x,t;\lambda D_0) \) is the solution of the following equation
\[
 f_t + \lambda f_x = D_0 f_{xx},
\]
\[
 f(x,0) = \begin{cases} 
 e^{-\gamma|x|} & x \leq 0, \\
 0 & x > 0.
\end{cases}
\]
The above lemma just states that the function behaves like the heat kernel.

Summarizing, the Green’s function of the linearized initial-boundary value problem (3) can be represented as
\[
 G(x; t; y) = G(x - y, t) - G(x + y, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + g(x + y, t), 0 \leq x, y < \infty,
\]
and satisfies the estimates (4) given in Theorem 1.1.

Remark 2.3. The third term in the right hand side of (4) represents the reflections wave traveling with velocity \(-c\), which is resulted from the presence of boundary.

3. Nonlinear stability. A priori energy estimate for the half space problem for three-dimensional case was done by [10], here the problem with mixed boundary condition in half line is similar, we just omit the proof of the existence and focus on deriving the pointwise asymptotic behavior of the nonlinear problem directly.

From Theorem 1.1 and Duhamel’s principle applied to (2), one has the integral representation of the solution
\[
 U(x,t) = \int_0^\infty G(x,t;y)U_0(y)\,dy + \int_0^t \int_0^\infty G(x,t-s;y) \begin{pmatrix} 0 \\ Q(y,s) \end{pmatrix} \,dy\,ds
\]
\[
 \equiv I(x,t) + N(x,t).
\]
where \( U(x,t) = (\rho(x,t) \quad m(x,t))^T \). The first term is from initial data, the second term represents the nonlinear coupling. From the assumptions on the initial data satisfies for \( \alpha = 0, 1 \),
\[
 |\partial_x^\alpha U_0| \leq C\varepsilon_0 (1 + x^2)^{-r}, \quad r > 1/2.
\]
The initial data gives the structure of \( \partial_x^\alpha I \) for \( \alpha \leq 1 \),
\[
 |\partial_x^\alpha I(x,t)| \leq |\partial_x^\alpha \int_0^\infty G_S(x,t;y)U_0(y)\,dy| + \left| \int_0^\infty \partial_x^\alpha G_L(x,t;y)U_0(y)\,dy \right|
\]
\[
 \equiv I_1^\alpha + I_2^\alpha.
\]
Here \( G_S \) is the short wave parts, corresponds to singular part in Green’s function \( G \); while \( G_L \) is the long wave part which dominates the large time behavior. They are given respectively by
\[
 G_S(x,t;0) = e^{-\frac{x^2}{\nu t}}(\delta(x-y) - \delta(x+y)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
 |\partial_x^\alpha G_L(x,t;y)| = O(1)t^{-\alpha/2} \left( e^{-\frac{(x-y+ct)^2}{2\nu t}} + e^{-\frac{(x-y-ct)^2}{2\nu t}} + e^{-\frac{(x+y-ct)^2}{2\nu t}} \right)
\]
\[
 + O(1)e^{-(|x-y|+t)/C} + O(1)e^{-(|x+y|+t)/C}.
\]
In the rest of our paper, following the method in [3, 5], we change all \( t \) in the expression of \( G_L \) to \( t + 1 \) to avoid the singular point if necessary.
By representation of singular part $G_S$ and initial condition,
\[
\mathcal{I}_1^0 (x, t) \leq Ce^{-\frac{2\alpha}{\nu(t+1)}} |\partial_x^2 U_0(x)| \leq C\varepsilon_0 e^{-\frac{\alpha}{\nu(t+1)}(x^2 + 1)^{-r}}.
\]

To estimate $\mathcal{I}_2^0$, we make use of the estimate for regular part of Green’s function $\partial_x^a G_L$. For $\alpha = 0, 1$, by Lemma 4.1 with replacing $x$ by $x - ct$ or $x + ct$, we have
\[
\mathcal{I}_2^0 (x, t) = |\partial_x^a \int_0^\infty G_L (x, t; y) U_0 (y) dy| 
\leq C\varepsilon_0 \int_0^\infty (t + 1)^{-\frac{3}{2}} \left[ \frac{e^{-\frac{(x+y-s)^2}{4\nu(t+1)}}}{\sqrt{\nu(t+1)}} + \frac{e^{-\frac{(x-y-s)^2}{4\nu(t+1)}}}{\sqrt{\nu(t+1)}} \right] (1 + y^2)^{-r} dy 
\leq C\varepsilon_0 (1 + t)^{-\frac{3}{2}} \left\{ (t + 1 + (|x| - c(t+1))^2)^{-\frac{r}{2}} + e^{-\frac{(x-c)^2}{4t(t+1)}} \right\}.
\]
It is easy to see that $\mathcal{I}_1^0$ has the same estimates as that of $\mathcal{I}_2^0$.

Now we introduce the following notations:
\[
\theta^a (x, t; \lambda, D) \equiv (t + 1)^{-\alpha/2} e^{-\frac{|x-y-(x+1)|^2}{4\nu(t+1)}},
\]
\[
\psi^a (x, t; \mu) \equiv \left( \sqrt{t+1} + |x - \mu(t+1)| \right)^{-\alpha}.
\]
Set
\[
A_0 (x, t) = \psi^1 (x, t; c) + \psi^1 (x, t; -c),
\]
then there exists some positive constant $\varepsilon_0$ such that
\[
|\mathcal{I} (x, t)| \leq \varepsilon_0 \varepsilon_0 A_0 (x, t), \quad |\partial_x \mathcal{I} (x, t)| \leq \varepsilon_0 \varepsilon_0 (t+1)^{-1/2} A_0 (x, t).
\]
Introduce
\[
M(T) = \sup_{0 < \xi < T} \{ \|U A_0^{-1}\|_{L^\infty} + \| (t+1)^{1/4} U_2 A_0^{-1}\|_{L^\infty} \},
\]
then we have
\[
|U (x, t)| \leq M(T) A_0 (x, t), \quad |U_x (x, t)| \leq M(t) (t+1)^{-1/4} A_0 (x, t).
\] (7)
Our goal is to show the uniform boundedness of $M(t)$ and thereby the pointwise decay of the solution as expressed in (7).

**Remark 3.1.** Note that there is a discrepancy between the time decay rates of the terms calculated from initial data and that of our ansatz. Actually we make this ansatz mainly for the closure of nonlinear estimate.

Now we consider the nonlinear coupling and its derivatives $\partial_x^a N$ for $\alpha \leq 1$. To this end, we need the estimate of nonlinear term $Q (x, t) = \tilde{Q}_x$. Straightforward computation shows that
\[
\left| \tilde{Q} (x, t) \right| = O (1) M^2(t) \left[ \psi^2 (x, t; c) + \psi^2 (x, t; -c) \right].
\]
Now we begin to estimate the nonlinear coupling for $\alpha = 0, 1$.
\[
\partial_x^a N (x, t) = \int_0^t \int_0^\infty \partial_x^a G (x, t-s; y) \left( \begin{array}{c} 0 \\ Q (y, s) \end{array} \right) dy ds 
= \int_0^t \int_0^\infty \partial_x^a G_S (x, t-s; y) \left( \begin{array}{c} 0 \\ Q (y, s) \end{array} \right) dy ds + \int_0^t \int_0^\infty \partial_x^a G_L (x, t-s; y) \left( \begin{array}{c} 0 \\ \partial_y Q (y, s) \end{array} \right) dy ds 
\equiv \partial_x^a N_1 + \partial_x^a N_2.
\]
From the matrix multiplication of $G_S$ and $(0, Q)^T$, we have
\[ |\partial_y^p N_1| = \int_0^t \int_0^\infty \partial_y^p G_S(x, t - s; y) \begin{pmatrix} 0 \\ Q(y, s) \end{pmatrix} ds dy = 0. \]

For $N_2$, we integrate by parts
\[ |N_2| = \left| \int_0^t \int_0^\infty G_L(x, t - s; y) \partial_y \tilde{Q}(y, s) ds dy \right| \leq \left| \int_0^t \int_0^\infty \partial_y G_L(x, t - s; y) \tilde{Q}(y, s) ds dy \right| + \left| \int_0^t G_L(x, t - s; y) \tilde{Q}(y, s) \bigg|_{y=0}^\infty ds \right| \leq O(1) M^2(t) \int_0^t \int_0^\infty (t - s)^{-1} \left[ e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} + e^{-\frac{(x-y+c(t-s))^2}{2(t-s)}} + e^{-\frac{(x+y-c(t-s))^2}{2(t-s)}} \right] \left[ e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} + e^{-\frac{(x+y+c(t-s))^2}{2(t-s)}} + e^{-\frac{(x-y+c(t-s))^2}{2(t-s)}} \right] (1 + s)^{-2} ds + O(1) M^2(t) \int_0^t \int_0^\infty (t - s)^{-1/2} \left[ e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} + e^{-\frac{(x+y+c(t-s))^2}{2(t-s)}} + e^{-\frac{(x-y+c(t-s))^2}{2(t-s)}} \right] \left[ e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} + e^{-\frac{(x+y+c(t-s))^2}{2(t-s)}} + e^{-\frac{(x-y+c(t-s))^2}{2(t-s)}} \right] (1 + s)^{-2} ds = N_{2,1} + N_{2,2}. \]

For $N_{2,1}$, we first consider
\[ \int_0^t \int_0^\infty (t - s)^{-1} e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} \psi^2(y, s; c) ds dy. \]
Noting $\psi^2(y, s; c) \leq C(t + 1)^{-1/4} \psi^{3/2}(y, s; c)$, thus we apply Lemma 4.2 with $\alpha = 2$, $\alpha' = 0$ and $\beta = \frac{1}{2}$ to obtain it is bounded by
\[ O(1) \left[ \theta^1(x, t; c, 2\nu + \varepsilon) + (t + 1)^{1/4} \psi^{3/2}(x, t; c) \right] \leq O(1) \psi^1(x, t; c). \]

As for the interaction between waves with different propagation speeds, say
\[ \int_0^t \int_0^\infty (t - s)^{-1} e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} \psi^2(y, s; -c) ds dy, \]
we apply Lemma 4.3 with $\alpha = 2$, $\alpha' = 0$ and $\beta = \frac{1}{2}$ to dominate it by
\[ O(1) \left[ \theta^{3/2}(x, t; c, 2\nu + \varepsilon) + (t + 1)^{1/4} \left[ (x-c(t+1))^2 + (t+1)^{3/2} \right]^{-3/4} \right] + (t + 1)^{1/4} \psi^1(x, t; -c) \left[ (x-c(t+1))^2 + (t+1)^2 \right]^{-1/4} + \psi^{1/2}(x, t; c) \psi^{1/2}(x, t; -c) \]
\[ \leq O(1) \left[ \psi^1(x, t; c) + \psi^1(x, t; -c) \right]. \]

The other terms in $N_{2,1}$ can be estimated similarly, hence we have
\[ N_{2,1} = O(1) M^2(t) \left[ \psi^1(x, t; c) + \psi^1(x, t; -c) \right] = O(1) M^2(t) A_0(x, t). \]

For $N_{2,2}$, here we only compute one of them since the others can be treated identically.
\[ (I) = \int_0^t \int_0^\infty e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} Q(y, s) ds dy + \int_{t/2}^t \int_0^\infty e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} \tilde{Q}(y, s) ds dy. \]


Therefore

\[ H \]

Hence we have

\[ K \]

Together with (8) to yield

\[ L \]

where

\[ M \]

For

\[ N \]

is suitably large, which follows that

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]

\[ \]
For $\partial_2 N_2$, we can use integration by parts to transfer the derivative in $y$ to derivative in $x$ to get

$$|\partial_2 N_2| = \left| \int_0^t \int_0^\infty \partial_2 G_L (x, t-s; y) \partial_y \tilde{Q} (y, s) \, dy \, ds \right| \leq \left| \int_0^t \int_0^\infty \partial_y^2 G_L (x, t-s; y) \tilde{Q} (y, s) \, dy \, ds \right| + \left| \int_0^t \partial_2 G_L (x, t-s; y) \tilde{Q} (y, s) \big|_{y=0} \, ds \right| = N_{2,a} + N_{2,b}.$$  

Again applying Lemma 4.2 and Lemma 4.3 with $\alpha = 3$, $\alpha' = 0$ and $\beta = 1/2$ we find that

$$N_{2,a} = O(1) M^2 (t) \int_0^t \int_0^\infty (t-s)^{-3/2} \left[ e^{-\frac{(x-y+c(t-s))^2}{2(t-s)}} + e^{-\frac{(x-y-c(t-s))^2}{2(t-s)}} + e^{\frac{(x+y-c(t-s))^2}{2(t-s)}} \right] \times [\psi^2 (y, s; c) + \psi^2 (y, s; -c)] \, dy \, ds \leq O(1) M^2 (t) \sum_{\lambda = \pm c} \left\{ \psi^2 (x, t; \lambda, D_0) + (t+1)^{-1/4} \log (t+1) \psi^2 (x, t; \lambda) \right\}$$

$$+ O(1) M^2 (t) \sum_{\lambda = \pm c} \left\{ \psi^5/3 (x, t; \lambda, D_0) + \psi^3/2 (x, t; \lambda) \right\} \leq O(1) M^2 (t) (1+t)^{-1/4} A_0 (x, t).$$

$N_{2,b}$ can be treated similarly as the term $N_{2,2}$ hence we omit the details here,

$$N_{2,b} \leq O(1) M^2 (t) (1+t)^{-1/2} A_0 (x, t).$$

Therefore we conclude that

$$\partial_2 N_2 \leq O(1) M^2 (t) (1+t)^{-1/2} A_0 (x, t).$$

Combine above estimates to obtain

$$M(t) \leq C \varepsilon_0 + CM^2 (t),$$

this together with the smallness of $\varepsilon_0$ and the continuity of $M(t)$ lead to $M(t) \leq C$ for $t \geq 0$, that is

$$|U| \leq CA_0, \quad |U_x| \leq C(t+1)^{-1/4} A_0.$$

This completes the proof of Theorem 1.2.

4. Appendix. In the Appendix, we collect some computational lemmas for wave coupling, which are used in previous proofs.

**Lemma 4.1.** Let $D_0 > 0$, $r > 1/2$. Then for any given $E > D_0$, we have

$$I = \int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{2(t+y^2)}} (1+y^2)^{-r} \, dy = O(1) \left[ \frac{e^{-\frac{x^2}{2(t+1)}}}{\sqrt{t+1}} + (t+1+x^2)^{-r} \right].$$

**Proof.** We consider the following two cases:

**Case 1.** $|x| \leq \sqrt{t+1};$

**Case 2.** $|x| > \sqrt{t+1}.$
For case 1, one has
\[ I \leq \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} \left(1 + y^2\right)^{-r} dy \leq \frac{O(1)}{\sqrt{t+1}}. \]

For case 2, we decompose the integration region into two parts,
\[
I = \int_{|y|<\frac{|x|}{\sqrt{t+1}}} e^{-\frac{(x-y)^2}{2(t+s+1)}} \left(1 + y^2\right)^{-r} dy + \int_{|y|>\frac{|x|}{\sqrt{t+1}}} e^{-\frac{(x-y)^2}{2(t+s+1)}} \left(1 + y^2\right)^{-r} dy
\]
\[
\leq O(1) \int_{|y|<\frac{|x|}{\sqrt{t+1}}} e^{-\frac{y^2}{2(t+s+1)}} \left(1 + y^2\right)^{-r} dy + O(1) \int_{|y|>\frac{|x|}{\sqrt{t+1}}} e^{-\frac{(x-y)^2}{2(t+s+1)}} \left(1 + x^2\right)^{-r} dy
\]
\[
\leq O(1) \left[ e^{-\frac{x^2}{2(t+1)}} + (t+1+x^2)^{-r} \right].
\]

This completes the proof. \(\square\)

**Lemma 4.2** ([9]). Let \(\alpha \geq \alpha' \geq 0, \alpha - \alpha' < 3, \beta \geq 0, \nu > 0\) and \(\lambda\) be a constant. Then for any given \(\varepsilon > 0\), and all \(-\infty < x < \infty, t \geq 0\), we have
\[
\int_{0}^{t} \int_{-\infty}^{\infty} (t-s)^{-\left((\alpha-\alpha')/2\right)} (t-s+1)^{-\alpha'/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2(t-s+1)}} \cdot (s+1)^{-\beta/2} \psi^{3/2}(y,s;\lambda) dy ds
\]
\[
= O(1) \left[ \theta^\gamma (x,t;\lambda,\nu+\varepsilon) + (t+1)^{-\sigma/2} \psi^{3/2} (x,t;\lambda) \right]
\]
\[
+ \begin{cases} O(1) \theta^\gamma (x,t;\lambda,\nu+\varepsilon) \log (t+1) & \text{for } \beta = 3/2, \\
0, \quad & \text{otherwise,}
\end{cases}
\]
\[
+ \begin{cases} O(1) (t+1)^{-\sigma/2} \psi^{3/2} (x,t;\lambda) \log (t+1) & \text{for } \alpha = 3 \text{ or } \beta = 2, \\
0, \quad & \text{otherwise,}
\end{cases}
\]
where \(\gamma = \alpha + \min (\beta, \frac{3}{2}) - \frac{3}{2}\), and \(\sigma = \min (\alpha, 3) + \min (\beta, 2) - 3\).

**Lemma 4.3** ([9]). Let the constants \(\alpha \geq 1, \alpha' \geq 0, 0 \leq \alpha - \alpha' < 3, \beta \geq 0, \nu > 0\) and \(\lambda \neq \lambda'\). Then for any given \(\varepsilon > 0\), \(K > 2|\lambda - \lambda'|\), and all \(-\infty < x < \infty, t \geq 0\), we have
\[
\int_{0}^{t} \int_{-\infty}^{\infty} (t-s)^{-\left((\alpha-\alpha')/2\right)} (t-s+1)^{-\alpha'/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2(t-s+1)}} \cdot (s+1)^{-\beta/2} \psi^{3/2}(y,s;\lambda') dy ds
\]
\[
= O(1) \theta^\gamma (x,t;\lambda,\nu+\varepsilon) + O(1)(t+1)^{-\sigma/2}
\]
\[
\cdot \left[ (x-\lambda(t+1))^2 + (t+1)^3(3/4 - (1/3) \min(\beta, 2)) \right]^{-3/4}
\]
\[
+ O(1) (t+1)^{-\sigma/2} \left[ \psi^{3/2} (x,t;\lambda') \right]^{(1/3) \min(\alpha, 3)}
\]
\[
\cdot \left[ (x-\lambda(t+1))^2 + (t+1)^2 \right]^{-3(1/4)(1-\min(\alpha, 3))} \cdot \begin{cases} 1 & \text{if } \alpha \neq 3 \\
1 + \log (t+1) & \text{if } \alpha = 3
\end{cases}
\]
\[
+ O(1) |x-\lambda(t+1)|^{-1/2} \min(\beta, 5/2) - 1/4 \cdot |x-\lambda'(t+1)|^{-(1/2)\alpha-1} \cdot char \left\{ \min (\lambda, \lambda') (t+1) + K\sqrt{t+1} \leq x \leq \max (\lambda, \lambda') (t+1) - K\sqrt{t+1} \right\}
\]
NAVIER-STOKES EQUATIONS IN HALF SPACE

\[ \begin{align*}
&\theta(\alpha, (x, t; \lambda, \nu + \varepsilon)) \log (t + 1) \\
&+ O(1) \begin{cases}
\theta^\alpha (x, t; \lambda, \nu + \varepsilon) \log (t + 1) & \text{if } \beta = 3/2, \\
(t + 1)^{- (1/2)(\alpha - 1)} \psi_{3/2}^\beta (x, t; \lambda) \log (t + 1) & \text{if } \beta = 2, \\
0, & \text{otherwise},
\end{cases}
\end{align*} \]

where \( \gamma = \alpha + 1/2 \min \left( \beta, 3 \right) - 3/4 \), \( \sigma = \alpha + \min(\beta, 2) - 3 \), \( \sigma' = \min(\alpha, 3) + \beta - 3 \).

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E-mail address: matdl@dhu.edu.cn
E-mail address: haitaowang.math@gmail.com