BLOCKS WITH A QUATERNION DEFECT GROUP OVER A 2-ADIC RING: THE CASE $\tilde{A}_4$

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Abstract. Except for blocks with a cyclic or Klein four defect group, it is not known in general whether the Morita equivalence class of a block algebra over a field of prime characteristic determines that of the corresponding block algebra over a $p$-adic ring. We prove this to be the case when the defect group is quaternion of order 8 and the block algebra over an algebraically closed field $k$ of characteristic 2 is Morita equivalent to $k\tilde{A}_4$. The main ingredients are Erdmann’s classification of tame blocks [6] and work of Cabanes and Picaronny [4, 5] on perfect isometries between tame blocks.

Introduction

Throughout these notes, $\mathcal{O}$ is a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic 2 and with quotient field $K$ of characteristic 0. According to Erdmann’s classification in [6], if $G$ is a finite group and if $b$ is a block of $\mathcal{O}G$ having the quaternion group $Q_8$ of order 8 as defect group, then the block algebra $kGb$ is Morita equivalent to either $kQ_8$ or $k\tilde{A}_4$ or the principal block algebra of $k\tilde{A}_5$, where here $b$ is the canonical image of $b$ in $kG$. In the first case the block is it nilpotent (cf. [3]), and it follows from Puig’s structure theorem of nilpotent blocks in [8] that $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}Q_8$. In the remaining two cases one should expect that $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}\tilde{A}_4$ or the principal block algebra of $\mathcal{O}\tilde{A}_5$, respectively. We show this to be true in one of these two cases under the assumption that $K$ is large enough:

Theorem A. Let $G$ be a finite group, and let $b$ be a block of $\mathcal{O}G$ having a quaternion defect group of order 8. Denote by $\tilde{b}$ the image of $b$ in $kG$. Assume that $KG\tilde{b}$ is split. If $kGb$ is Morita equivalent to $k\tilde{A}_4$ then $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}\tilde{A}_4$.

By Cabanes-Picaronny [4, 5], in the situation of Theorem A there is a perfect isometry between the character groups of $\mathcal{O}Gb$ and of $\mathcal{O}\tilde{A}_4$. Thus Theorem A is a consequence of the following slightly more general Theorem which characterises $\mathcal{O}Gb$ in terms of its center, its character group and $k\tilde{A}_4$; see the end of this section for more details regarding the notation.
Theorem B. Let $A$ be an $O$-free $O$-algebra such that $K \otimes A$ is split semi-simple and such that $k \otimes A$ is Morita equivalent to $k \tilde{A}_4$. Assume that there is an isometry $\Phi : \text{ZIrr}_K(A) \cong \text{ZIrr}_K(O \tilde{A}_4)$ which maps $\text{Proj}(A)$ to $\text{Proj}(O \tilde{A}_4)$ such that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \text{Irr}_K(A)$ induces an $O$-algebra isomorphism of the centers $Z(A) \cong Z(O \tilde{A}_4)$. Then $A$ is Morita equivalent to $O \tilde{A}_4$.

Theorem B is in turn a consequence of the more precise Theorem C, describing $A$ in terms of generators and relations:

Theorem C. Let $A$ be a basic $O$-free $O$-algebra such that $K \otimes A$ is split semi-simple and such that $k \otimes A$ is isomorphic to $k \tilde{A}_4$. Assume that there is an isometry $\Phi : \text{ZIrr}_K(A) \cong \text{ZIrr}_K(O \tilde{A}_4)$ which maps $\text{Proj}(A)$ to $\text{Proj}(O \tilde{A}_4)$ such that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \text{Irr}_K(A)$ induces an $O$-algebra isomorphism of the centers $Z(A) \cong Z(O \tilde{A}_4)$. Then $A$ is isomorphic to the unitary $O$-algebra with set of generators $\{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$ of $A$, such that $e_0, e_1, e_2$ are pairwise orthogonal idempotents whose sum is 1 and satisfying the following relations:

\[
\begin{align*}
\beta &= e_0 \beta = \beta e_1, \quad \gamma = e_1 \gamma = \gamma e_0; \\
\delta &= e_1 \delta = \delta e_2, \quad \eta = e_2 \eta = \eta e_1; \\
\lambda &= e_2 \lambda = \lambda e_0, \quad \kappa = e_0 \kappa = \kappa e_2; \\
\beta \delta &= -2\kappa + \kappa \lambda \kappa; \\
\eta \gamma &= -2 \lambda + \lambda \kappa \lambda; \\
\delta \lambda &= -2 \gamma + \gamma \beta \gamma; \\
\kappa \eta &= -2 \beta + \beta \gamma \beta; \\
\lambda \beta &= -2 \eta + \eta \delta \eta; \\
\gamma \kappa &= -2 \delta + \delta \eta \delta; \\
\gamma \beta \delta &= -4 \delta + 2 \delta \eta \delta; \\
\delta \eta \gamma &= -4 \gamma + 2 \gamma \beta \gamma; \\
\lambda \kappa \eta &= -4 \eta + 2 \eta \delta \eta; \\
\beta \gamma \kappa &= -4 \kappa + 2 \kappa \lambda \kappa; \\
\eta \delta \lambda &= -4 \lambda + 2 \lambda \kappa \lambda; \\
\kappa \lambda \beta &= -4 \beta + 2 \beta \gamma \beta; \\
\lambda \gamma \beta &= -4 \eta + 2 \eta \delta \eta; \\
\beta \delta \lambda &= -8 \beta + 4 \beta \gamma \beta; \\
\delta \lambda \beta &= -8 \delta + 4 \delta \eta \delta; \\
\lambda \beta \delta &= -8 \lambda + 4 \lambda \kappa \lambda; \\
\gamma \lambda \beta &= -4 \gamma + 2 \gamma \beta \gamma; \\
\gamma \delta \beta &= -8 \delta + 4 \delta \eta \delta; \\
\lambda \beta \delta &= -8 \lambda + 4 \lambda \kappa \lambda; \\
\delta \lambda \beta &= -8 \delta + 4 \delta \eta \delta; \\
\lambda \beta \delta &= -8 \lambda + 4 \lambda \kappa \lambda.
\end{align*}
\]

When reduced modulo 2, these relations seem to be more than those occurring in Erdmann’s work [6] over $k$ (we recall these more precisely in §2); but they are not, since the extra relations over $k$ can be deduced from those given by Erdmann. We need to add in extra relations over $O$ in order to ensure that the algebra we construct is $O$-free of the right rank.

Since $O \tilde{A}_4$ fulfills the hypotheses of Theorem C it follows that $A \cong O \tilde{A}_4$, hence Theorem C indeed implies Theorem B. The proof of Theorem C is given at the end of Section 2.
**Notation.** If \( A \) is an \( \mathcal{O} \)-algebra such that \( K \otimes \mathcal{O} \) is split semi-simple, we denote by \( \text{Irr}_K(A) \) the set of characters of the simple \( K \otimes \mathcal{O} \)-modules, viewed as central functions from \( A \) to \( \mathcal{O} \) and we denote by \( \text{Irr}_k(k \otimes \mathcal{O}) \) the set of isomorphism classes of simple \( k \otimes \mathcal{O} \)-modules. We denote by \( \text{ZIrr}_K(A) \) the group of characters of \( A \), and we denote by \( \text{Proj}(A) \) the subgroup of \( \text{ZIrr}_K(A) \) generated by the characters of the projective indecomposable \( A \)-modules. We denote by \( L_0(A) \) the subgroup of \( \text{ZIrr}_K(A) \) of all elements which are orthogonal to \( \text{Proj}(A) \) with respect to the usual scalar product in \( \text{ZIrr}_K(A) \). For any \( \chi \in \text{Irr}_K(A) \), we denote by \( e(\chi) \) the corresponding primitive idempotent in \( Z(K \otimes \mathcal{O}) \). If \( A = OG \) for some finite group \( G \) we have the well-known formula

\[
e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x.
\]

We refer to [1, 2] for the concept and basic properties of perfect isometries, and to [9] for general block theoretic background material.

1 **Characters and perfect isometries of \( \mathcal{O}\tilde{A}_4 \)**

We identify \( \tilde{A}_4 = Q_8 \rtimes C_3 \). Let \( t \) be a generator of \( C_3 \) and let \( y \) be an element of order 4 in \( Q_8 \). Set \( z = y^2 \); that is, \( z \) is the unique central involution of \( \tilde{A}_4 \). Then the seven elements \( 1, z, y, t, t^2, tz, t^2z \) are a complete set of representatives of the conjugacy classes in \( \tilde{A}_4 \).

Let \( \omega \) be a primitive third root of unity in \( \mathcal{O} \). The character table of \( \tilde{A}_4 \) is as follows:

|   | 1 | z | y | t | t^2 | tz | t^2z |
|---|---|---|---|---|-----|-----|-----|
| \( \eta_0 \) | 1 | 1 | 1 | 1 | 1   | 1   | 1   |
| \( \eta_1 \) | 1 | 1 | 1 | \( \omega \) | \( \omega^2 \) | \( \omega \) | \( \omega^2 \) |
| \( \eta_2 \) | 1 | 1 | 1 | \( \omega^2 \) | \( \omega \) | \( \omega^2 \) | \( \omega \) |
| \( \eta_3 \) | 3 | 3 | -1 | 0  | 0   | 0   | 0   |
| \( \eta_4 \) | 2 | -2 | 0  | -\( \omega^2 \) | -\( \omega \) | -\( \omega^2 \) | \( \omega \) |
| \( \eta_5 \) | 2 | -2 | 0  | -\( \omega \) | -\( \omega^2 \) | \( \omega \) | \( \omega^2 \) |
| \( \eta_6 \) | 2 | -2 | 0  | -1 | -1  | 1   | 1   |

The algebra \( \mathcal{O}\tilde{A}_4 \) has three simple modules \( T_0, T_1, T_2 \), up to isomorphism. Choosing for \( T_0 \) the trivial module and after possibly exchanging the notation for \( T_1, T_2 \), the
ordinary decomposition matrix of $O\tilde{A}_4$ is as follows:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
$$

The Cartan matrix of $O\tilde{A}_4$ is the product of the decomposition matrix with its transpose, hence equal to

$$
\begin{pmatrix}
4 & 2 & 2 \\
2 & 4 & 2 \\
2 & 2 & 4
\end{pmatrix}
$$

Let $e_0, e_1, e_2$ be primitive idempotents in $O\tilde{A}_4$ such that $O\tilde{A}_4 e_i$ is a projective cover of $T_i, 0 \leq i \leq 2$. By the above decomposition matrix, the characters of the projective indecomposable $O\tilde{A}_4$-modules $O\tilde{A}_4 e_i$ are

$$
\begin{align*}
\eta_0 + \eta_3 + \eta_4 + \eta_5, \\
\eta_1 + \eta_3 + \eta_4 + \eta_6, \\
\eta_2 + \eta_3 + \eta_5 + \eta_6,
\end{align*}
$$

respectively. Their norm is 4, and the differences of any two different characters of projective indecomposable $O\tilde{A}_4$-modules yields the following further elements in $\text{Proj}(O\tilde{A}_4)$ having also norm 4:

$$
\begin{align*}
\eta_0 - \eta_1 + \eta_5 - \eta_6, \\
\eta_0 - \eta_2 + \eta_4 - \eta_6, \\
\eta_1 - \eta_2 + \eta_4 - \eta_5.
\end{align*}
$$

It is easy to check, that up to signs, these are all elements in $\text{Proj}(O\tilde{A}_4)$ having norm 4.

A self-isometry $\Phi$ of $\text{ZIrr}_K(O\tilde{A}_4)$ maps every $\eta_i$ to $\epsilon_i \eta_{\pi(i)}$ for some signs $\epsilon_i \in \{1, -1\}$ and a permutation $\pi$ of $\{0, 1, \ldots, 6\}$. In other words, $\Phi$ is determined by the permutation $\tau$ of the set $\{1, -1\} \times \{0, 1, \ldots, 6\}$ satisfying $\tau(1, i) = (\epsilon_i, \pi(i))$ and $\tau(-1, i) = (-\epsilon_i, \pi(i))$ for all $i, 0 \leq i \leq 6$. If we write $i, -i$ instead of $(1, i), (-1, i)$, respectively, this becomes $\tau(i) = \epsilon_i \pi(i)$ and $\tau(-i) = -\epsilon_i \pi(i)$, with the usual cancellation rules for signs. In this way, every self-isometry $\Phi$ of $\text{ZIrr}_K(O\tilde{A}_4)$ gets identified to a permutation of the set of symbols $\{i, -i|0 \leq i \leq 6\}$.

A perfect self-isometry of $\text{ZIrr}_K(O\tilde{A}_4)$ preserves necessarily $\text{Proj}(O\tilde{A}_4)$. The next Proposition implies that the converse is true, too:
Proposition 1.1. The group of all perfect self-isometries of \(Z\text{Irr}_K(O\tilde{A}_4)\) is equal to the group of all self-isometries of \(Z\text{Irr}_K(O\tilde{A}_4)\) which preserve \(\text{Proj}(O\tilde{A}_4)\). This group is generated by \(-\text{Id}\) together with the set of permutations

\[(0,1,2)(4,6,5),\]
\[(1,2)(4,5),\]
\[(2,-3)(5,-6).\]

Every algebra automorphism of \(O\tilde{A}_4\) induces a permutation on \(\text{Irr}_K(O\tilde{A}_4)\) which is in fact a perfect isometry on \(Z\text{Irr}_K(O\tilde{A}_4)\). Since \(\eta_1\) has degree 1, it is an algebra homomorphism from \(O\tilde{A}_4\) to \(O\), and hence the map sending \(x \in O\tilde{A}_4\) to \(\eta_1(x)x\) is an algebra automorphism of \(O\tilde{A}_4\) whose inverse sends \(x \in O\tilde{A}_4\) to \(\eta_2(x)x\). The following statement is an immediate consequence from the character table of \(O\tilde{A}_4\):

Lemma 1.2. Let \(\gamma\) be the algebra automorphism of \(O\tilde{A}_4\) defined by \(\gamma(x) = \eta_1(x)x\) for all \(x \in O\tilde{A}_4\). The permutation \(\pi\) of \(\{0,1,\ldots,6\}\) defined by \(\eta_i \circ \gamma = \eta_{\pi(i)}\) is equal to \(\pi = (0,1,2)(4,6,5)\).

The anti-automorphism of \(O\tilde{A}_4\) sending \(x \in \tilde{A}_4\) to \(x^{-1}\) induces also a permutation of the set \(\text{Irr}_K(O\tilde{A}_4)\), and this is also a perfect isometry (this holds for any finite group). This permutation can also be read off the character table:

Lemma 1.3. Let \(\iota\) be the algebra anti-automorphism of \(O\tilde{A}_4\) mapping \(x \in \tilde{A}_4\) to \(x^{-1}\). The permutation \(\pi\) of \(\{0,1,\ldots,6\}\) defined by \(\eta_i \circ \iota = \eta_{\pi(i)}\) is equal to \(\pi = (1,2)(4,5)\).

Proof of 1.1. The first two permutations are perfect isometries by 2.2 and 2.3, respectively. An easy but painfully long verification shows that the bicharacter sending \((g,h) \in \tilde{A}_4 \times \tilde{A}_4\) to

\[\eta_0(g)\eta_0(h) + \eta_1(g)\eta_1(h) - \eta_2(g)\eta_3(h) - \eta_3(g)\eta_2(h) + \eta_4(g)\eta_4(h) - \eta_5(g)\eta_6(h) - \eta_6(g)\eta_5(h)\]

is perfect; that is, its value at any \((g,h)\) is divisible in \(O\) by the orders of \(C_{\tilde{A}_4}(g)\) and \(C_{\tilde{A}_4}(h)\) and it vanishes if exactly one of \(g, h\) has odd order. Thus the isometry given by the permutation \((2,-3)(5,-6)\) is perfect. It remains to show that these permutations, together with \(-\text{Id}\), generate the group of all self-isometries which preserve \(\text{Proj}(O\tilde{A}_4)\).

We described above a complete list of all elements in \(\text{Proj}(O\tilde{A}_4)\) having norm 4. Since the characters of the projective indecomposable modules are in that list, a self-isometry \(\Phi\) of \(Z\text{Irr}_K(O\tilde{A}_4)\) preserves \(\text{Proj}(O\tilde{A}_4)\) if and only if it permutes this set of norm 4 elements.
Let $\Phi$ be a self-isometry of $Z\text{Irr}_K(\mathcal{O}\bar{A}_4)$ which preserves $\text{Proj}(\mathcal{O}\bar{A}_4)$. Then $\Phi$ preserves also the group $L^0(\mathcal{O}\bar{A}_4)$ of generalised characters which are orthogonal to all characters in $\text{Proj}(\mathcal{O}\bar{A}_4)$. Up to signs, the complete list of elements in $L^0(\mathcal{O}\bar{A}_4)$ having norm 3 is

$$
\eta_0 + \eta_1 - \eta_4 , \eta_0 + \eta_2 - \eta_5 , \eta_0 - \eta_3 + \eta_6 ,
$$

$$
\eta_1 + \eta_2 - \eta_6 , \eta_1 - \eta_3 + \eta_5 , \eta_2 - \eta_3 + \eta_4 .
$$

Up to signs again, the complete list of elements in $L^0(\mathcal{O}\bar{A}_4)$ having norm 4 is

$$
\eta_0 + \eta_1 + \eta_2 - \eta_3 ,
$$

$$
\eta_0 - \eta_1 - \eta_5 + \eta_6 , \eta_0 - \eta_2 - \eta_4 + \eta_6 , \eta_0 + \eta_3 - \eta_4 - \eta_5 ,
$$

$$
\eta_1 - \eta_2 - \eta_4 + \eta_5 , \eta_1 + \eta_3 - \eta_4 - \eta_6 , \eta_2 + \eta_3 - \eta_5 - \eta_6 .
$$

The first norm 4 element in this list, $\eta_0 + \eta_1 + \eta_2 - \eta_3$, is the only norm 4 element which is orthogonal to all other norm 4 elements in $L^0(\mathcal{O}\bar{A}_4)$. Thus $\Phi$ has to permute the characters $\eta_0, \eta_1, \eta_2, \eta_3$ amongst each other.

Suppose first that $\Phi$ fixes $\eta_3$. Then, by composing $\Phi$ with a suitable product of powers of the first two permutations in the statement, we may assume that $\Phi$ fixes $\eta_0, \eta_1, \eta_2$ up to signs. By considering the first of the above norm 4 elements in $L^0(\mathcal{O}\bar{A}_4)$ we get that $\Phi$ fixes $\eta_0, \eta_1, \eta_2$ all with positive signs. By considering the norm 3 elements in $L^0(\mathcal{O}\bar{A}_4)$, it follows that $\Phi$ fixes also $\eta_4, \eta_5$ and $\eta_6$ with positive signs. Thus a self-isometry of $Z\text{Irr}_K(\mathcal{O}\bar{A}_4)$ which preserves $\text{Proj}(\mathcal{O}\bar{A}_4)$ and which fixes $\eta_3$ is in the group generated by the set of two permutations $(0, 1, 2)(4, 6, 5)$ and $(1, 2)(4, 5)$.

Suppose next that $\Phi$ does not fix $\eta_3$. By precomposing $\Phi$ with a suitable power of $(0, 1, 2)(4, 6, 5)$ we may assume that $\Phi$ sends $\eta_2$ to $-\eta_3$. By composing $\Phi$ with a suitable power of $(0, 1, 2)(4, 5, 6)$ we may assume that $\Phi$ fixes $\eta_0$, up to a sign. Since $\Phi$ preserves the norm 4 element $\eta_0 + \eta_1 + \eta_2 - \eta_3$, we necessarily have $\Phi(\eta_0) = \eta_0$. Then $\Phi$ maps $\eta_1$ either to $\eta_1$ or $\eta_2$ (with positive signs, again because of that same norm 4 element). In the first case, $\Phi$ fixes both $\eta_0, \eta_1$, and by checking the norm 3 elements in $L^0(\mathcal{O}\bar{A}_4)$ one gets $\Phi = (2, -3)(5, -6)$. In the second case, again checking on norm 3 elements, one gets $\Phi = (1, 2, -3)(4, 5, -6)$, but this is already the product of $(1, 2)(4, 5)$ and $(2, -3)(5, -6)$. □

2 The algebra $A$

Let $A$ be a basic $\mathcal{O}$-algebra fulfilling the hypotheses of Theorem B; that is, $K \otimes A$ is split semi-simple, $k \otimes A$ is isomorphic to $k\bar{A}_4$, and there is an isometry $Z\text{Irr}_K(A) \cong Z\text{Irr}_K(\mathcal{O}\bar{A}_4)$ mapping $\text{Proj}(A)$ to $\text{Proj}(\mathcal{O}\bar{A}_4)$ and inducing an isomorphism $Z(A) \cong Z(\mathcal{O}\bar{A}_4)$. There is a “compatible choice” for these isomorphisms:
Proposition 2.1. There is an algebra isomorphism $\alpha : k \otimes A \cong k\tilde{A}_4$ and an isometry $\Phi : \text{ZIrr}_K(A) \cong \text{ZIrr}_K(O\tilde{A}_4)$ mapping $\text{Proj}(A)$ to $\text{Proj}(O\tilde{A}_4)$ with the following properties:

(i) $\Phi$ maps $\text{Irr}_K(A)$ onto $\text{Irr}_K(O\tilde{A}_4)$; that is, all signs are $+1$.

(ii) The map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \text{Irr}_K(A)$ induces an isomorphism $Z(A) \cong Z(O\tilde{A}_4)$.

(iii) For any primitive idempotents $e \in A$ and $f \in O\tilde{A}_4$ and every $\chi \in \text{Irr}_K(A)$ such that $\alpha(e) = f$ we have $\chi(e) = \Phi(\chi)(f)$; that is, $A$ and $O\tilde{A}_4$ have the same decomposition matrices through $\alpha$ and $\Phi$.

Proof. The $O$-rank of $A$ is 24 and also the sum of the squares of the seven irreducible $K$-linear characters of $A$; thus every irreducible character of $A$ has degree smaller than 5. Also, there is no character of degree 4 because $24 - 4^2 = 8$ cannot be written as a sum of six squares of the six remaining characters. But there must be a character of degree 3; if not, 24 would be the sum of seven squares all either 1 or 4, which is not possible. Thus the squares of the six remaining characters add up to $24 - 3^2 = 15$, and the only way to do this is with three characters of degree 1 and three characters of degree 2.

This proves that the character degrees of the irreducible characters of $A$ and of $O\tilde{A}_4$ coincide for some bijection $\text{Irr}_K(A) \cong \text{Irr}_K(O\tilde{A}_4)$. Since the decomposition matrix of $A$ multiplied with its transpose yields the Cartan matrix of $A$ - which is equal to that of $k\tilde{A}_4$ - the algebra $A$ has in fact the same decomposition matrix as $O\tilde{A}_4$ for a suitable bijection $\Phi : \text{Irr}_K(A) \cong \text{Irr}_K(O\tilde{A}_4)$ and the bijection $\text{Irr}_k(k \otimes A) \cong \text{Irr}_k(k\tilde{A}_4)$ induced by $\alpha$. Extend $\Phi$ to a $Z$-linear isomorphism $\text{ZIrr}_K(A) \cong \text{ZIrr}_K(O\tilde{A}_4)$, still denoted by $\Phi$. By construction, $\Phi$ sends the characters of the projective indecomposable $A$-modules to the characters of the projective indecomposable $O\tilde{A}_4$-modules; in particular, $\Phi$ maps $\text{Proj}(A)$ to $\text{Proj}(O\tilde{A}_4)$. It remains to see that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \text{Irr}_K(A)$ induces an isomorphism $Z(A) \cong Z(O\tilde{A}_4)$. For any $i$, $0 \leq i \leq 6$, denote by $\chi_i$ the irreducible character of $A$ such that $\Phi(\chi_i) = \eta_i$. As in the proof of 1.1, we have a distinguished norm 4 element in $L^0(A)$ which is orthogonal to all other norm 4 elements in $L^0(A)$, namely $\chi_0 + \chi_1 + \chi_2 - \chi_3$. Thus, if $\Psi : \text{ZIrr}_K(A) \cong \text{ZIrr}_K(O\tilde{A}_4)$ is some isometry mapping $\text{Proj}(A)$ to $\text{Proj}(O\tilde{A}_4)$ and inducing an isomorphism $Z(A) \cong Z(O\tilde{A}_4)$, then $\Psi(\chi_0 + \chi_1 + \chi_2 - \chi_3) = \pm(\eta_0 + \eta_1 + \eta_2 - \eta_3)$. By Proposition 1.1, there is a perfect self-isometry $\mu$ of $\text{ZIrr}_K(O\tilde{A}_4)$ such that $\Phi = \mu \circ \Psi$. □

Remark 2.2. If we assume that $A$ is Morita equivalent to some block algebra with $Q_8$ as defect group, then Proposition 2.1 follows also from the work of Cabanes and Picaronny in [4, 5].

Since $k \otimes A \cong k\tilde{A}_4$, the quiver of $A$ is the same as that of $k\tilde{A}_4$, thus of the following form:
Write $\bar{a}$ for the image of $a \in A$ in $\bar{A} = k \otimes O \bar{A}_4$. The generators $\beta, \gamma, \delta, \kappa, \lambda, \eta$ can be chosen such that their images in $\bar{A}$ fulfill the following relations:

\[
\bar{\beta} \bar{\delta} = \bar{\kappa} \bar{\lambda} , \\
\bar{\eta} \bar{\gamma} = \bar{\lambda} \bar{\kappa} , \\
\bar{\delta} \bar{\lambda} = \bar{\gamma} \bar{\beta} , \\
\bar{\kappa} \bar{\eta} = \bar{\beta} \bar{\gamma} , \\
\bar{\lambda} \bar{\beta} = \bar{\eta} \bar{\delta} , \\
\bar{\gamma} \bar{\kappa} = \bar{\delta} \bar{\eta} \bar{\delta}
\]

and

\[
\bar{\gamma} \bar{\delta} \bar{\eta} = \bar{\delta} \bar{\eta} \bar{\gamma} = \bar{\lambda} \bar{\kappa} \bar{\eta} = 0 .
\]

In order to determine the algebra structure of $A$, we have to "lift" these relations over $O$.

We fix an algebra isomorphism $\alpha : k \otimes A \cong k \bar{A}_4$ and an isometry $\Phi : Z \text{Irr}_K(A) \cong Z \text{Irr}_K(O \bar{A}_4)$ satisfying the conclusions of Proposition 2.1. We denote by $\chi_i$ the unique irreducible $K$-linear character of $A$ such that $\Phi(\chi_i) = \eta_i$ for all $i$, $0 \leq i \leq 6$.

The characters $\eta_0, \eta_1, \eta_2, \eta_3$ of $O \bar{A}_4$ have height zero, the characters $\eta_4, \eta_5, \eta_6$ have height one. Thus, via the isomorphism of the centers induced by $\Phi$, it follows that for $0 \leq i \leq 3$ we have $8e(\chi_i) \in A$, and for $4 \leq j \leq 6$ we have $4e(\chi_j) \in A$. We can in fact describe an $O$-basis of $Z(A)$ in terms of the centrally primitive idempotents $e(\chi_i)$. The strategy is now to play off the descriptions of $Z(k \otimes A)$ in terms of the generators in the quiver and of $Z(A)$ in terms of the centrally primitive idempotents $e(\chi_i)$.
Lemma 2.3. The following elements of $Z(K \otimes A)$ are all contained in the radical $J(Z(A))$:

\begin{align*}
    s &= 2e(\chi_4) + 2e(\chi_5) + 2e(\chi_6), \\
    z_0 &= 4e(\chi_2) + 4e(\chi_3) + 2e(\chi_4), \\
    z_1 &= 4e(\chi_1) + 4e(\chi_3) + 2e(\chi_5), \\
    z_2 &= 4e(\chi_0) + 4e(\chi_3) + 2e(\chi_6), \\
    y_0 &= 4e(\chi_1) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_5), \\
    y_1 &= 4e(\chi_0) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_6), \\
    y_2 &= 4e(\chi_0) + 4e(\chi_1) + 2e(\chi_5) + 2e(\chi_6).
\end{align*}

Moreover, for any two different $i, j$ in $\{0, 1, 2\}$ the set

\[ \{1, z_i, z_j, s, 8e(\chi_3), 4e(\chi_{i+4}), 4e(\chi_{j+4})\} \]

is an $O$-basis of $Z(A)$.

Proof. In view of Proposition 2.1 we may assume that $A = O\tilde{A}_4$. This is just an explicit verification, using the character table of $\tilde{A}_4$. One verifies first that $z_0 \in A$. By symmetry, this implies that $z_1, z_2$ are also in $A$. Then $y_0 = z_0 + z_1 - 8e(\chi_3)$ is in $A$, similarly for the $y_1, y_2$. An equally easy computation shows that $s \in A$. Thus all the given elements belong to $Z(A)$. None of these elements is invertible, so they all belong to $J(Z(A))$ because $Z(A)$ is local.

In order to see the last statement on the basis of $Z(A)$, we may assume that $i = 0$ and $j = 1$. For any $x \in \tilde{A}_4$ denote by $\overline{x}$ the conjugacy class sum of $x$ in $O\tilde{A}_4$. The orthogonality relations imply the well-known formula

\[ \overline{x} = \sum_{0 \leq m \leq 6} \frac{\chi_m(\overline{x}^{-1})}{\chi_m(1)} e(\chi_m). \]

Thus, for the seven conjugacy classes in $\tilde{A}_4$, we have

\begin{align*}
    1 &= e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) + e(\chi_4) + e(\chi_5) + e(\chi_6); \\
    \overline{z} &= e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) - e(\chi_4) - e(\chi_5) - e(\chi_6); \\
    \overline{y} &= 6e(\chi_0) + 6e(\chi_1) + 6e(\chi_2) - 2e(\chi_3); \\
    \overline{t} &= 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) - 2\omega e(\chi_4) - 2\omega^2 e(\chi_5) - 2e(\chi_6); \\
    \overline{t^2} &= 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) - 2\omega^2 e(\chi_4) - 2\omega e(\chi_5) - 2e(\chi_6); \\
    \overline{t\overline{z}} &= 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) + 2\omega e(\chi_4) + 2\omega^2 e(\chi_5) + 2e(\chi_6); \\
\end{align*}
\[ t^2 z = 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) + 2\omega^2 e(\chi_4) + 2\omega e(\chi_5) + 2e(\chi_6). \]

We show that they are all in the \( O \)-linear span of the elements in the set
\[
\{1, z_0, z_1, s, 8e(\chi_3), 4e(\chi_4), 4e(\chi_5)\}.
\]

Note first that
\[
z_2 = 4 \cdot 1 - z_0 - z_1 - s + 8e(\chi_3),
\]
\[4e(\chi_6) = 2s - 4e(\chi_4) - 4e(\chi_5)\]
are in the \( O \)-linear span of this set. One easily verifies now that
\[
\bar{z} = 1 - s, \\
y = 6 \cdot 1 - 3s - 8e(\chi_3), \\
\bar{t} = \omega z_0 + \omega^2 z_1 + z_2 - 4\omega e(\chi_4) - 4\omega^2 e(\chi_5) - 4e(\chi_6), \\
\bar{t}^2 = \omega^2 z_0 + \omega z_1 + z_2 - 4\omega^2 e(\chi_4) - 4\omega e(\chi_5) - 4e(\chi_6), \\
\bar{t}z = \omega z_0 + \omega^2 z_1 + z_2, \\
\bar{t}^2 \bar{z} = \omega^2 z_0 + \omega z_1 + z_2.
\]

This concludes the proof of 2.3 \( \Box \)

The center of \( \bar{A} = k \otimes_A \bar{O} \) can easily be described in terms of the generators in the quiver of \( A \):

**Lemma 2.4.** The following set is a \( k \)-basis of \( Z(\bar{A}) \).
\[
\{1, \bar{\beta} \bar{\gamma} + \bar{\gamma} \bar{\beta}, \bar{\kappa} \lambda + \bar{\lambda} \bar{\kappa}, \bar{\eta} \bar{\delta} + \bar{\delta} \bar{\eta}, \bar{\beta} \bar{\delta} \lambda, \bar{\delta} \bar{\lambda} \bar{\beta}, \bar{\lambda} \bar{\beta} \bar{\delta}\}.
\]

**Proof.** Straightforward verification, using \((\bar{\beta} \bar{\gamma})^2 = \bar{\beta} \bar{\delta} \bar{\lambda} \) and the similar relations for the other elements in the given set. \( \Box \)

**Proposition 2.5.** For any primitive idempotent \( e \) in \( A \) we have \( Z(A)e = eAe \). Moreover,
(i) the set \( \{e_0, z_0e_0, z_1e_0, 4e(\chi_4)e_0\} \) is an \( O \)-basis of \( e_0Ae_0 \).
(ii) the set \( \{e_1, z_0e_1, z_2e_1, 4e(\chi_4)e_1\} \) is an \( O \)-basis of \( e_1Ae_1 \);
(iii) the set \( \{e_2, z_1e_2, z_2e_2, 4e(\chi_5)e_2\} \) is an \( O \)-basis of \( e_2Ae_2 \).

**Proof.** Since \( Z(A) \cong Z(O \bar{A}_4) \) and \( Z(\bar{A}) \cong Z(k\bar{A}_4) \), the canonical map \( A \to \bar{A} \) maps \( Z(A) \) onto \( Z(\bar{A}) \) and hence \( Z(A)e \) onto \( Z(\bar{A})\bar{e} \). By Nakayama’s Lemma, it suffices to show that \( Z(\bar{A})\bar{e} = \bar{e}A\bar{e} \). Now \( \dim_k(\bar{e}A\bar{e}) = 4 \) by the Cartan matrix, and so we have only to show that \( \dim_k(Z(\bar{A})\bar{e}) = 4 \). By the symmetry of the quiver of \( A \), we may assume that \( e \) corresponds to the vertex labelled 0. Then the set \( \{\bar{e}, \bar{\beta} \bar{\gamma}, \bar{\kappa} \lambda, \bar{\beta} \bar{\delta} \lambda\} \) is a
One constructs \( U_R \) the unique submodule of \( k \). Thus the set given in (i) generates \( e_0Ae_0 \) as \( \mathcal{O} \)-module, by the first statement and by the \( \mathcal{O} \)-basis of \( Z(A) \) described in 2.3. Now we have

\[
\begin{align*}
8e(\chi_3)e_0 &= 2z_0e_0 - 4e(\chi_4)e_0 , \\
4e(\chi_5)e_0 &= 2z_0e_0 - 2z_1e_0 + 4e(\chi_4)e_0 , \\
s_0 &= (z_1 - z_0 + 4e(\chi_4))e_0 .
\end{align*}
\]

Thus the set given in (i) generates \( e_0Ae_0 \) as \( \mathcal{O} \)-module, and hence is a basis since the \( \mathcal{O} \)-rank of \( e_0Ae_0 \) is 4. The same arguments show (ii), (iii). \( \square \)

**Proposition 2.6.** We can choose the generators \( \beta, \gamma, \delta, \eta, \lambda, \kappa \) in such a way that

(i) \( A_A \gamma \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_0 \) with character \( \chi_3 + \chi_4 \);

(ii) \( A_A \lambda \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_0 \) with character \( \chi_3 + \chi_5 \);

(iii) \( A_A \eta \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_1 \) with character \( \chi_3 + \chi_6 \);

(iv) \( A_A \beta \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_1 \) with character \( \chi_3 + \chi_4 \);

(v) \( A_A \kappa \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_2 \) with character \( \chi_3 + \chi_5 \);

(vi) \( A_A \delta \) is the unique \( \mathcal{O} \)-pure submodule of \( Ae_2 \) with character \( \chi_3 + \chi_6 \).

**Proof.** We are going to prove (i); by the symmetry of the quiver of \( A \) one gets all other statements. Observe first that \( A_A \gamma \) is the unique 5-dimensional submodule of \( Ae_0 \) with composition factors \( 2[S_0], 2[S_1], [S_2] \). Indeed, the set \( \{ \bar{\gamma}, \bar{\beta}\bar{\gamma}, \bar{\eta}\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma}, \bar{\beta}\gamma\bar{\gamma} \} \) is a \( k \)-basis of \( A_A \gamma \), and we have \( \bar{\gamma}, \bar{\beta}\bar{\gamma} \in \bar{e}_0Ae_0 \), yielding the two composition factors isomorphic to \( S_0 \), we have \( \bar{\beta}\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma} \in \bar{e}_1Ae_0 \), yielding the two composition factors isomorphic to \( S_1 \), and finally \( \bar{\eta}\bar{\gamma} \in \bar{e}_2Ae_0 \), yielding the remaining composition factor isomorphic to \( S_2 \). One checks that there is no other submodule with exactly these composition factors. Now there is exactly one \( \mathcal{O} \)-pure submodule \( U \) of \( Ae_0 \) whose reduction modulo \( J(\mathcal{O}) \) has composition series \( 2[S_0] + 2[S_1] + [S_2] \), namely the unique \( \mathcal{O} \)-pure submodule of \( Ae_0 \) with character \( \chi_3 + \chi_4 \); this is a direct consequence of the decomposition matrix. One constructs \( U \) as follows: write \( K \otimes Ae_0 = X_0 \oplus X_3 \oplus X_4 \oplus X_5 \), where \( X_j \) is the unique submodule of \( K \otimes Ae_0 \) with character \( \chi_j \) for \( j \in \{0, 3, 4, 5\} \), and then \( U = Ae_0 \cap (X_3 \oplus X_4) \). Take now for \( \gamma \) any inverse image in \( U \) of \( \bar{\gamma} \). Then \( A_A \gamma \subseteq U \) and \( U \subseteq A_A \gamma + J(\mathcal{O})U \). Thus \( A_A \gamma = U \) by Nakayama’s Lemma. \( \square \)
Corollary 2.7. If the generators $\beta, \gamma, \delta, \eta, \lambda, \kappa$ are chosen such that they fulfill the conclusions of 2.6 then, with the notation of 2.3, the following hold.

(i) $y_0 \delta = y_0 \eta = 0$.
(ii) $y_1 \lambda = y_1 \kappa = 0$.
(iii) $y_2 \gamma = y_2 \beta = 0$.

Proposition 2.8. We can choose the generators $\beta, \gamma, \delta, \eta, \lambda, \kappa$ such that the following holds:

\[
\begin{align*}
\beta \gamma &= z_0 e_0 = 4e(\chi_3)e_0 + 2e(\chi_4)e_1; \\
\gamma \beta &= z_0 e_1 = 4e(\chi_3)e_1 + 2e(\chi_4)e_1; \\
\delta \eta &= z_2 e_1 = 4e(\chi_3)e_1 + 2e(\chi_6)e_1; \\
\eta \delta &= z_2 e_2 = 4e(\chi_3)e_2 + 2e(\chi_6)e_2; \\
\lambda \kappa &= z_1 e_2 = 4e(\chi_3)e_2 + 2e(\chi_5)e_2; \\
\kappa \lambda &= z_1 e_0 = 4e(\chi_3)e_0 + 2e(\chi_5)e_0; \\
\beta \delta \lambda &= \kappa \eta \gamma = 8e(\chi_3)e_0; \\
\delta \lambda \beta &= \gamma \kappa \eta = 8e(\chi_3)e_1; \\
\lambda \beta \delta &= \eta \gamma \kappa = 8e(\chi_3)e_2.
\end{align*}
\]

Proof. In view of the decomposition matrix of $A$ we have $e_0 = e(\chi_0)e_0 + e(\chi_3)e_0 + e(\chi_4)e_0 + e(\chi_5)e_0$. Moreover, the elements $e(\chi_0)e_0, e(\chi_3)e_0, e(\chi_4)e_0, e(\chi_5)e_0$ are $K$-linearly independent because they are pairwise orthogonal idempotents in $K \otimes A$. Similar statements hold for $e_1, e_2$.

We assume a choice of generators fulfilling 2.6. We have $A\beta \gamma \subseteq A\gamma$, and the submodule $A\gamma$ of $Ae_0$ has character $\chi_3 + \chi_4$ by 2.6. Thus $\beta \gamma$ is a $K$-linear combination of $e(\chi_3)e_1$ and $e(\chi_4)e_1$. But also $\beta \gamma$ is an $O$-linear combination of the basis elements $e_1, z_0 e_1 z_1 e_1, 4e(\chi_4)e_1$ given in 2.5 in which none of $\chi_1, \chi_5$ shows up. Therefore $\beta \gamma$ is in fact an $O$-linear combination of the elements $z_0 e_0, 4e(\chi_4)e_0$; say

\[
\beta \gamma = (\mu_0 z_0 e_0 + \nu_0 e(\chi_4))e_0 = (4\mu_0 e(\chi_3) + 2(\mu_0 + 2\nu_0)e(\chi_4))e_0
\]

for some coefficients $\mu_0, \nu_0 \in O$. Hence

\[
(\beta \gamma)^2 = (16\mu_0^2 e(\chi_3) + 4(\mu_0 + 2\nu_0)^2 e(\chi_4))e_0.
\]

Now $(\beta \gamma)^2 \neq 0$, and therefore $\mu_0 \in O^\times$. Set now

\[
a_0 = 1 + \nu_0 \mu_0^{-1} y_0.
\]
Since \( y_0 \in J(Z(A)) \) by 2.3 we have \( a_0 \in Z(A)^\times \). A trivial verification, comparing coefficients, shows that we have

\[
\beta \gamma = \mu_0 z_0 a_0 e_0 .
\]

Since \( \gamma = e_1 \gamma = \gamma e_0 \), multiplying this with \( \gamma \) on the left yields

\[
\gamma \beta \gamma = \mu_0 z_0 a_0 e_1 \gamma .
\]

Now both \( \gamma \beta \) and \( \mu_0 z_0 a_0 e_1 \) are contained in the pure submodule \( A\beta \) of \( Ae_1 \) with character \( \chi_3 + \chi_4 \), by 2.6 and the nature of the element \( z_0 \). Right multiplication by \( \gamma \) on this submodule is therefore injective (the annihilator of \( \gamma \) in \( Ae_1 \) is the pure submodule with character \( \chi_1 + \chi_6 \)). Hence the previous equality implies also the equality

\[
\gamma \beta = \mu_0 z_0 a_0 e_1 .
\]

In an entirely analogous way one finds scalars \( \mu_1, \mu_2 \in \mathcal{O}^\times \) such that, setting \( a_1 = 1 + \nu_1 \mu_1^{-1} y_1 \) and \( a_2 = 1 + \nu_2 \mu_2^{-1} y_2 \), one gets the equalities

\[
\delta \eta = \mu_2 z_2 a_2 e_1 , \quad \eta \delta = \mu_2 z_2 a_2 e_2 ,
\]

\[
\lambda \kappa = \mu_1 z_1 a_1 e_2 , \quad \kappa \lambda = \mu_1 z_1 a_1 e_0 .
\]

Moreover, the equalities in 2.7 imply the following equalities:

\[
a_0 \delta = \delta , \quad a_0 \eta = \eta ,
\]

\[
a_1 \lambda = \lambda , \quad a_1 \kappa = \kappa ,
\]

\[
a_2 \gamma = \gamma , \quad a_2 \beta = \beta .
\]

If we replace now \( \beta \) by \( a_0 \beta \), this is not going to change the properties stated in 2.6 and also this is not changing the relations over \( k \) of the quiver. Similarly, we can replace \( \delta \) by \( a_2 \delta \) and \( \lambda \) by \( a_1 \lambda \). Then the generators \( \beta, \gamma, \delta, \eta, \lambda, \kappa \) still fulfill 2.6, and in addition, we have now the following equalities:

\[
\beta \gamma = \mu_0 z_0 e_0 , \quad \gamma \beta = \mu_0 z_0 e_1 ,
\]

\[
\delta \eta = \mu_2 z_2 e_1 , \quad \eta \delta = \mu_2 z_2 e_2 ,
\]

\[
\lambda \kappa = \mu_1 z_1 e_2 , \quad \kappa \lambda = \mu_1 z_1 e_0 .
\]

We have to get rid of the scalars \( \mu_0, \mu_1, \mu_2 \). Since \( \chi_3 \) is the only character appearing in the characters of all projective indecomposable \( A \)-modules we have

\[
\beta \delta \lambda = 8 \mu e(\chi_3)e_0
\]
for some $\mu \in \mathcal{O}$. Then actually $\mu \in \mathcal{O}^\times$ because $\bar{\beta}\bar{\delta}\bar{\lambda} \neq 0$. Moreover, $\beta\delta\lambda\beta = 8\mu e(\chi_3)\beta$, and hence also

$$\delta\lambda\beta = 8\mu e(\chi_3)e_1.$$ 

The same argument applied again yields

$$\lambda\beta\delta = 8\mu e(\chi_3)e_2.$$ 

Applying this argument to the arrows in the quiver in the opposite direction implies that there is $\mu' \in \mathcal{O}^\times$ such that

$$\kappa\eta\gamma = 8\mu' e(\chi_3)e_0,$$

$$\eta\gamma\kappa = 8\mu' e(\chi_3)e_2,$$

$$\gamma\kappa\eta = 8\mu' e(\chi_3)e_1.$$ 

Now $\bar{\beta}\bar{\delta}\bar{\lambda} = \bar{\kappa}\bar{\lambda}\bar{\kappa} = \bar{\kappa}\bar{\eta}\bar{\gamma}$, and hence $\mu' = \mu(1 + \nu)$ for some $\nu \in J(\mathcal{O})$. Note that we can always multiply any of the generators by any scalar in $1 + J(\mathcal{O})$ without modifying the relations over $k$. Thus, if we replace $\kappa$ by $(1 + \nu)\kappa$, we may assume that $\mu' = \mu$.

Since the set $\{\kappa, \kappa\lambda\kappa\}$ is an $\mathcal{O}$-basis of $e_0Ae_2$, we can write

$$\beta\delta = a\kappa + b\kappa\lambda\kappa$$

for some unique scalars $a, b \in \mathcal{O}$. Multiplying this by $\lambda$ yields

$$8\mu e(\chi_3)e_0 = \beta\delta\lambda = a\kappa\lambda + b(\kappa\lambda)^2 = (a\mu_1z_1 + b\mu_2^2z_1^2)e_0.$$ 

By comparing the coefficients at $e(\chi_3)e_0$ and $e(\chi_5)e_0$ of the left and right expression in this equality, we get the equations

$$8\mu = 4a\mu_1 + 16b\mu_1^2,$$

$$0 = 2a\mu_1 + 4b\mu_1^2.$$ 

An easy computation shows that $b = \frac{\mu_2}{\mu_1^2}$. Moreover, since $\bar{\beta}\bar{\delta}\bar{\lambda} = (\bar{\kappa}\bar{\lambda})^2$ we have $\bar{a} = 0$ and $\bar{b} = 1_k$, hence $b = \frac{\mu_2}{\mu_1^2} \in 1 + J(\mathcal{O})$. By repeating the same argument we find also that the coefficients $\frac{\mu_2}{\mu_1^2}, \frac{\mu_2}{\mu_1^2}$ are in $1 + J(\mathcal{O})$.

Next, we compute $\beta\delta\lambda\kappa\eta\gamma$ in two different ways: on one hand we have

$$(\beta\delta\lambda)(\kappa\eta\gamma) = 64\mu^2 e(\chi_3)e_0,$$

and on the other hand we have

$$\beta(\delta(\lambda\kappa)\eta)\gamma = \mu_0\mu_1\mu_2z_0z_1z_2e(\chi_3)e_0 = 64\mu_0\mu_1\mu_2e(\chi_3)e_0.$$
Together we get
\[ \mu^2 = \mu_0 \mu_1 \mu_2 . \]
Thus \( \frac{\mu_2}{\mu_0 \mu_1} \in 1 + J(\mathcal{O}) \). Similarly, \( \frac{\mu_1 \mu_2}{\mu_0 \mu_1} = \frac{\mu_1}{\mu_0} \in 1 + J(\mathcal{O}) \). But then also \( \frac{\mu_1 \mu_2}{\mu_0 \mu_1} \in 1 + J(\mathcal{O}) \). Since \( 2 \in J(\mathcal{O}) \) this implies that \( \frac{\mu_1}{\mu_0} \in 1 + J(\mathcal{O}) \). Thus\( \mu_1 \mu_2 \in 1 + J(\mathcal{O}) \). Similarly, \( \mu_1, \mu_0, \mu_2 \in 1 + J(\mathcal{O}) \). So we can replace \( \beta \) by \( \mu_0^{-1} \beta \), or equivalently, we can assume that \( \mu_0 = 1 \). Similarly, we can assume that \( \mu_1 = \mu_2 = 1 \). Then \( \mu^2 = 1 \). If \( \mu = -1 \) we multiply all generators by \( -1 \); since \( 2 \in J(\mathcal{O}) \), this does not change the relations over \( k \), but it does change the sign of any of the above expressions \( \beta \delta \lambda \) etc. involving three generators. Therefore, we can also assume that \( \mu = 1 \).

□

We can now prove Theorem C from the introduction.

**Proof of Theorem C.** We assume a choice of generators of \( A \) fulfilling Proposition 2.8. We show that \( A \) satisfies the relations given in Theorem C. Those in the first three lines are obvious. Since the set \( \{ \kappa, \kappa \lambda \kappa \} \) is an \( \mathcal{O} \)-basis of \( e_0 A e_2 \), we can write
\[ \beta \delta = a \kappa + b \kappa \lambda \kappa \]
for some unique scalars \( a, b \in \mathcal{O} \). Multiplying this by \( \lambda \) yields
\[ 8 e(\chi_3) e_0 = \beta \delta \lambda = a \kappa \lambda + b (\kappa \lambda)^2 = (4a + 16b) e(\chi_3) e_0 + (2a + 4b) e(\chi_5) e_0 . \]
By comparing the coefficients at \( e(\chi_3) e_0 \) and \( e(\chi_5) e_0 \) of the left and right expression in this equality, we get the equations
\[ 8 = 4a + 16b , \]
\[ 0 = 2a + 4b . \]
Thus the coefficients \( a, b \) have values
\[ a = -2 , b = 1 , \]
and from this we get the following relation in the statement of Theorem C:
\[ \beta \delta = -2 \kappa + \kappa \lambda \kappa . \]
In exactly the same way we get the following five relations in the Theorem:
\[ \eta \gamma = -2 \lambda + \lambda \kappa \lambda , \]
\[ \delta \lambda = -2 \gamma + \gamma \beta \gamma , \]
\[ \kappa \eta = -2 \beta + \beta \gamma \beta , \]
\[ \lambda \beta = 2 \eta + \eta \delta \eta, \]
\[ \gamma \kappa = -2 \eta + \delta \eta \delta. \]

A similar technique is going to yield the remaining relations: write \( \gamma \beta \delta = c \delta + d \delta \eta \delta \) for some unique \( c, d \in \mathcal{O} \); as before, this is possible since \( \{ \delta, \delta \eta \delta \} \) is an \( \mathcal{O} \)-basis of \( e_1 A e_2 \). Multiplying by \( \eta \) yields
\[ \gamma \beta \delta \eta = c \delta \eta + d (\delta \eta)^2 = cz_2 e_1 + dz_2^2 e_1. \]

The left side is equal to \( (\gamma \beta)(\delta \eta) = z_0 z_2 e_1 \), so comparing coefficients yields now
\[ 16 = 4c + 16d, \]
\[ 0 = 2c + 4d, \]
and this implies \( c = -4 \) and \( d = 2 \). Thus we get indeed
\[ \gamma \beta \delta = -4 \delta + 2 \delta \eta \delta \]
as claimed. The remaining relations of this type follow in exactly the same way.

Now consider the last three relations. Write \( \beta \delta \lambda \beta = r \beta + s \gamma \beta \), for \( r, s \in \mathcal{O} \). Then
\[ \beta \delta \lambda \beta \gamma = r \beta \gamma + s \beta \gamma \beta \gamma. \]
So
\[ 32 \epsilon (\chi_3) e_0 = (4r + 16s) \epsilon (\chi_3) e_0 + (2r + 4s) \epsilon (\chi_4) e_0 \]
which yields \( s = 4 \) and \( r = -8 \). The remaining two relations follow in exactly the same way. Thus \( A \) satisfies all relations given in Theorem C.

Let \( \tilde{A} \) be the \( \mathcal{O} \)-algebra described by the generators and relations given in Theorem C. There is a surjective algebra morphism from \( \tilde{A} \) to \( A \). In order to show that \( \tilde{A} \) and \( A \) are isomorphic it suffices therefore to show that the cardinality of a minimal generating set for \( A \) as an \( \mathcal{O} \)-module is at most 24. Thus it suffices to check that the set
\[ S := \{ e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa, \]
\[ \beta \gamma, \gamma \beta, \delta \eta, \lambda \kappa, \kappa \lambda, \]
\[ \beta \gamma \beta, \gamma \beta \gamma, \delta \eta \delta, \lambda \kappa \lambda, \kappa \lambda \kappa, \]
\[ \beta \delta \lambda, \delta \lambda \beta, \lambda \beta \delta \} \]
spans \( \tilde{A} \) as \( \mathcal{O} \)-module. This is an easy consequence of the given relations; we give some details for the convenience of the reader: Let
\[ \mathcal{G} = \{ e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa \} \]
From the given relations it is immediate that for any two elements \( x, y \) of \( \mathcal{G} \), \( xy \) is in the \( \mathcal{O} \)-span of \( S \). Thus it suffices to show that for any two elements \( x, y \) of \( \mathcal{G} - \{ e_0, e_1, e_2 \} \) and any element \( u \) of \( S - \{ e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa \} \), \( xu \) and \( uy \) are
in the $O$-span of $S$. From the given relations we may also assume that $u$ is one of \(\beta\gamma\beta, \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa\) or one of \(\beta\delta\lambda, \delta\lambda\beta, \lambda\beta\delta\).

First, note that the relations \(\kappa\eta = -2\beta + \beta\gamma\beta\) and \(\delta\lambda = -2\gamma + \gamma\beta\gamma\) give that \(\kappa\eta\gamma = \beta\delta\lambda\). Similarly, we get \(\eta\gamma\kappa = \lambda\delta\beta\) and \(\gamma\kappa\eta = \delta\beta\delta\).

Now suppose \(u = \beta\gamma\beta\). Then we may assume that \(x\) is one of \(\gamma\) or \(\lambda\) and that \(y\) is one of \(\gamma\) or \(\delta\). The relation \(\kappa\eta = -2\beta + \beta\gamma\beta\) gives \(\gamma\kappa\eta = -2\gamma + \gamma\beta\gamma\), hence \(\gamma\beta\gamma\) is in the $O$-span of $S$. The relation \(\kappa\eta = -2\beta + \beta\gamma\beta\) also gives \(\lambda\kappa\eta = -2\lambda\beta + \lambda\beta\gamma\beta\).

It follows from the relation \(\lambda\kappa\eta = -4\eta + 2\eta\delta\eta\) that \(\lambda\beta\gamma\beta\) is in the $O$-span of $S$. We show similarly that \(\beta\gamma\beta\gamma\) and \(\beta\gamma\beta\delta\) are in the $O$-span of $S$.

The cases \(u = \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa\) are handled analogously.

Now suppose \(u = \beta\delta\lambda\). Then we may assume that \(x\) is one of \(\lambda\) or \(\gamma\) and \(y\) is one of \(\beta\) or \(\kappa\). The relation \(\gamma\beta\delta\lambda = -8\lambda + 4\lambda\kappa\lambda\) shows that \(\lambda\beta\delta\lambda\) is in the $O$-span of $S$. From the relation \(\gamma\beta\delta\lambda = -4\delta + 2\delta\eta\delta\), we get \(\gamma\kappa\eta = -2\delta + \delta\eta\delta\), we get \(\delta\eta\delta\lambda = \gamma\kappa\lambda + 2\delta\lambda\). Hence \(\delta\eta\delta\lambda\) is in the $O$-span of $S$, and so is \(\gamma\beta\delta\lambda\). We argue similarly to show that \(\beta\delta\lambda\beta\) and \(\beta\delta\lambda\kappa\) are in the $O$-span of $S$.

The cases \(u = \delta\lambda\beta\) and \(u = \lambda\beta\delta\) are handled in the same fashion. \(\square\)

**Remark 2.9.** An interesting consequence of 2.5 is the structure of $eAe$ for any primitive idempotent $e$ in $A$. We have an $O$-algebra isomorphism

$$eAe \cong O[X, Y]/\langle X^2 - Y^2 - 2(X - Y), XY - 2X^2 + 4X \rangle;$$

indeed, we may assume that $e = e_0$, and then the assignment $X \mapsto z_0 e_0$, $Y \mapsto z_1 e_0$ induces the required isomorphism. In particular, we have an isomorphism of $k$-algebras

$$\bar{e}\bar{A}\bar{e} \cong k[X, Y]/\langle X^2 - Y^2, XY \rangle.$$  

This is, by Erdmann [6, III.1, III.3], up to isomorphism the unique 4-dimensional symmetric $k$-algebra which is not isomorphic to the group algebra of the Klein four group. One might be tempted to ask whether any symmetric $O$-algebra is the endomorphism algebra of some projective module of some block algebra.

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