AN APPLICATION OF INTERPOLATION INEQUALITIES BETWEEN THE DEVIATION OF CURVATURE AND THE ISOPERIMETRIC RATIO TO THE LENGTH-PRESERVING FLOW

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Abstract. In recent work of Nagasawa and the author, new interpolation inequalities between the deviation of curvature and the isoperimetric ratio were proved. In this paper, we apply such estimates to investigate the large-time behavior of the length-preserving flow of closed plane curves without a convexity assumption.

1. Introduction. A number of papers have been devoted to the study of curvature flows with non-local term. These are called non-local curvature flows. Gage [3] and Jiang-Pan [4] studied such flows

\[ \partial_t f = \kappa - \frac{1}{L} \left( \int_0^L \kappa \cdot \nu \, ds \right) \nu, \]

\[ \partial_t f = \kappa - \frac{L}{2A} \nu, \]

respectively. Here \( f = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2 \) is a function such that \( \text{Im} f \) is a closed plane curve with rotation number 1 and \( s \) is the arc-length parameter, \( \nu = (-f_2, f_1) \) is the inward unit normal vector, \( \kappa = f'' \) is the curvature vector. The (signed) area \( A \) is given by

\[ A = -\frac{1}{2} \int_0^L f \cdot \nu \, ds. \]

Along the flow (1), \( A \) is preserved and the length of the evolving curves is non-increasing. Hence we call it the area-preserving flow. Along the flow (2), \( A \) is non-decreasing if initial (signed) area is positive and the length of the evolving curves is non-increasing if the initial curve is convex. In [3] and [4], it was proved that a simple closed strictly convex initial curve remains so along the flow, and the evolving curve converges to a circle in each non-local curvature flow. However there are very few results for non-local curvature flows when initial curve is not convex. Hence we would like to know the behavior of evolving curves not assuming convexity. To do this, we consider as follows.

2020 Mathematics Subject Classification. Primary: 53A04, 53E10, 35B40, 35K55.

Key words and phrases. Length-preserving flow, isoperimetric ratio, curvature, interpolation inequalities.
The curvature $\kappa = \kappa \cdot \nu$ is positive when $\text{Im} f$ is convex. Since the curve has rotation number 1, the deviation of curvature is

$$\tilde{\kappa} = \kappa - \frac{1}{L} \int_0^L \kappa ds = \kappa - \frac{2\pi}{L}.$$ 

For a non-negative integer $\ell$, we set

$$I_\ell = L^{2\ell+1} \int_0^L |\tilde{\kappa}(\ell)|^2 ds,$$

which is a scale invariant quantity (cf. [1]). It is important to estimate $I_\ell$ for the global analysis of evolving curves. We have the Gagliardo-Nirenberg inequalities

$$I_\ell \leq C I_m^{1-\frac{\ell}{m}},$$

where $0 \leq \ell \leq m$ and $C$ is constant and independent of $L$. Such inequalities are very useful but only these are not sufficient to estimate $I_0$ because these inequalities use $I_0$. Hence we need a different type of inequality to estimate $I_\ell$ for $\ell \geq 0$.

The curve along the flow (1) or (2) is expected to converge to a circle when the initial curve is close to a circle (in some sense) even if it is not convex. If it is true, the isoperimetric ratio $\frac{4\pi A}{L^2}$ converges to 1 as $t \to \infty$. Taking this into consideration, we introduce the quantity

$$I_{-1} = 1 - \frac{4\pi A}{L^2},$$

which is also scale invariant, and is non-negative by the isoperimetric inequality. $I_{-1}$ can be expressed using the integral of $\kappa^{-1}$ when $\kappa \neq 0$, for details of the proofs, see [7]. Hence we use the notation $I_{-1}$.

Several inequalities for $I_0, I_{-1}$ were derived by Nagasawa and the author in [7]. Using these inequalities, $I_\ell$ can be interpolated by $I_{-1}$ and $I_m$ for $\ell \in \{0, 1, \ldots, m\}$. The authors applied the inequalities to the flow (1) and (2), and they showed that, assuming global existence, solutions of each flow become convex in finite time and converge exponentially to a circle even if the initial curve is not strictly convex.

The purpose of this paper is to consider the large-time behavior of the length-preserving flow

$$\partial_t f = \kappa - \left( \frac{1}{2\pi} \int_0^L \|\kappa\|^2 ds \right) \nu, \quad (3)$$

where $\| \cdot \|$ is the Euclid norm. It is an interesting question whether the same result can be obtained even if the behavior of the area and the length is different from the flow (1) and (2). We have

$$\frac{dL}{dt} = -\int_0^L \partial_t f \cdot \kappa ds = -\int_0^L \|\kappa\|^2 ds + \frac{1}{2\pi} \int_0^L \|\kappa\|^2 ds \int_0^L \kappa ds = 0,$$

$$\frac{dA}{dt} = -\int_0^L \partial_t f \cdot \nu ds = \frac{L}{2\pi} \int_0^L \tilde{\kappa}^2 ds \geq 0.$$ 

Hence $L$ is preserved and $A$ is non-decreasing along the flow. Furthermore, above equality never holds except the trivial case $\tilde{\kappa} \equiv 0$. Therefore the circle is the stationary solution. This flow was firstly studied by Ma-Zhu [5], who proved that a simple closed strictly convex initial curve remains so along the flow, and the evolving curve converges to a circle. Their method is not applicable without the convexity.
Hence we investigate the large-time behavior of evolving curves not assuming the convexity.

In section 2, we introduce the several inequalities which was proved in [7]. Furthermore, we present the results for the flow (1) and (2) given in [7].

In section 3, we investigate the length-preserving flow (3).

2. Known results. In this section we introduce some results which were established in [7], for details of the proofs, see [7].

2.1. Several inequalities. Clearly, we have that $\tilde{\kappa} \equiv 0$ implies $\text{Im}f$ is a round circle, which attains the minimum $I_{-1} = 0$. This suggests that $I_{-1}$ can be dominated by certain quantities involving $\tilde{\kappa}$. Indeed, we have

$$I_{-1} = \frac{L^2 - 4\pi A}{L^2} = \frac{1}{L^2} \int_0^L (-Lf \cdot \kappa + 2\pi f \cdot \nu) \, ds$$

$$= -\frac{1}{L} \int_0^L \tilde{\kappa}(f \cdot \nu) ds$$

$$= -\frac{1}{L} \int_0^L \tilde{\kappa} \left( f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds \right) ds$$

and

$$\left| f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds \right| \leq L.$$

Thus it holds that

$$0 \leq I_{-1} \leq I_0^2.$$

However, since $f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds = 0$ when $\tilde{\kappa} \equiv 0$, it seems that the above inequality can be improved.

**Theorem 2.1.** We have

$$8\pi^2 I_{-1} \leq I_0 \leq I_{-1}^{\frac{3}{2}} \left[ L^3 \int_0^L \left\{ \kappa^3 \tilde{\kappa} + (\kappa')^2 \right\} ds \right]^{\frac{1}{2}}.$$

Two equalities only hold in the trivial case $\tilde{\kappa} \equiv 0$.

Considering a complex-valued function

$$f = f_1 + if_2,$$

and the Fourier series expansion of $f$, we can prove this Theorem. From this inequality, we have the new interpolation inequalities.

**Theorem 2.2.** Let $0 \leq \ell \leq m$. There exists a positive constant $C = C(\ell, m)$ independent of $L$ such that

$$I_{\ell} \leq C \left( I_{-1}^{\frac{m-\ell}{m}} I_m + I_{-1}^{\frac{m-\ell}{m}} I_{m+1}^{\frac{\ell+1}{m+1}} \right)$$

holds.
2.2. Applications to some geometric flows. In this subsection we give applications of our inequalities to the asymptotic analysis of geometric flows of closed plane curves. One of the flows is a curvature flow (2) with a non-local term first studied by Jiang-Pan [4], and another is the area-preserving curvature flow (1) considered by Gage [3]. If the initial curve is convex, then the flows exist for all time, preserving the convexity, and the curve approaches a round circle; this was shown in [4, 3]. The local existence of flows without a convexity assumption was shown by Ševčovič-Yazaki [8]. However, the large-time behavior for this case is still open. It seems evolving curves may develop singularities in finite time for some non-convex initial curves [6], but, on the other hand, the global existence for a certain initial non-convex curve was shown in [8]. Escher-Simonett [2] showed the global existence and investigated the large-time behavior of the area-preserving curvature flow for initial data close to a circle and without a convexity assumption. In this subsection, we present the results for the large-time behavior of the flows without a convexity assumption assuming the global existence.

We consider the flows (1) and (2). Observe that the equations which \( f \) satisfies are

\[
\partial_t f = \partial_s^2 f - \frac{2\pi}{L} R \partial_s f,
\]

\[
\partial_t f = \partial_s^2 f - \frac{L}{2A} R \partial_s f,
\]

where

\[
R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Since these are parabolic equations with a non-local term, \( f \) is smooth for \( t > 0 \) as long as the solution exists. Hence by shifting the initial time, the initial data is smooth. Then we have the following theorem.

**Theorem 2.3.** Assume that \( f \) is a global solution of (1) or (2) such that the initial rotation number is 1 and the initial (signed) area is positive. Then for each \( \ell \in \mathbb{N} \cup \{-1, 0\} \), there exist \( C_\ell > 0 \) and \( \lambda_\ell > 0 \) such that

\[
I_\ell(t) \leq C_\ell e^{-\lambda_\ell t}.
\]

Furthermore there exist \( A_\infty \) and \( L_\infty \) such that \( A(t) \) converges to \( A_\infty \) and \( L(t) \) converges to \( L_\infty \) as \( t \to \infty \).

The key to show this Theorem is how to estimate \( I_0 \). The derivative of \( I_0 \) can be estimated using \( I_{-1} \) from Theorem 2.2. Therefore, if we find the estimate of \( I_{-1} \), we also find the estimate of \( I_0 \). The estimate of \( I_{-1} \) is easier than \( I_0 \). Hence Theorem 2.2 is important. Furthermore, from this Theorem, we have the following Theorem.

**Theorem 2.4.** Let \( f \) be as in Theorem 2.3, and we define a complex-valued function by

\[
f = f_1 + if_2.
\]

Let \( f(s,t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)(t) \varphi_k(s) \) be the Fourier expansion for any fixed \( t > 0 \), where

\[
\varphi_k(s) = \frac{1}{\sqrt{L}} \exp \left( \frac{2\pi i k s}{L} \right), \quad \hat{f}(k) = \int_0^L f(t) \overline{\varphi_k} ds.
\]
Set
\[ c(t) = \frac{1}{\sqrt{L(t)}}(\Re \hat{f}(0)(t), \Im \hat{f}(0)(t)), \]
and define \( r(t) \geq 0 \) and \( \sigma(t) \in \mathbb{R}/2\pi \mathbb{Z} \) by
\[ \hat{f}(1)(t) = \sqrt{L(t)} r(t) \exp \left( i \frac{2\pi \sigma(t)}{L(t)} \right), \]
where \( \Re \hat{f}(0)(t) \) is the real part of \( \hat{f}(0)(t) \) and \( \Im \hat{f}(0)(t) \) is the imaginary part of \( \hat{f}(0)(t) \). Furthermore we set
\[ \hat{f}(\theta, t) = f(L(t) \theta - \sigma(t), t), \quad (\theta, t) \in \mathbb{R}/\mathbb{Z} \times [0, \infty). \]

Then the following claims hold.

1. There exists \( c_\infty \in \mathbb{R}^2 \) such that
\[ \| c(t) - c_\infty \| \leq C e^{-\gamma t}. \]

2. The function \( r(t) \) converges exponentially to the constant \( L_\infty \) as \( t \to \infty \):
\[ \left| r(t) - \frac{L_\infty}{2\pi} \right| \leq C e^{-\gamma t}. \]

3. There exists \( \sigma_\infty \in \mathbb{R}/2\pi \mathbb{Z} \) such that
\[ |\sigma(t) - \sigma_\infty| \leq C e^{-\gamma t}. \]

4. For any \( k \in \mathbb{N} \cup \{0\} \) there exist \( C_k > 0 \) and \( \gamma_k > 0 \) such that
\[ \| \hat{f}(\cdot, t) - \hat{f}_\infty \|_{C^k(\mathbb{R}/\mathbb{Z})} \leq C_k e^{-\gamma_k t}, \]

where
\[ \hat{f}_\infty(\theta) = c_\infty + \frac{L_\infty}{2\pi} (\cos 2\pi \theta, \sin 2\pi \theta). \]

5. For sufficiently large \( t \), \( \text{Im} \hat{f}(\cdot, t) \) is the boundary of a bounded domain \( \Omega(t) \). Furthermore, there exists \( T_* \geq 0 \) such that \( \Omega(t) \) is strictly convex for \( t \geq T_* \).

6. Let \( D_{r_\infty}(c) \) be the closed disk with center \( c_\infty \) and radius \( r_\infty \). Then we have
\[ d_H(\Omega(t), D_{r_\infty}(c_\infty)) \leq C e^{-\gamma t}, \]
where \( d_H \) is the Hausdorff distance.

7. Let \( b(t) = \frac{1}{A(t)} \int_{\Omega(t)} x \, dx \) be the barycenter of \( \Omega(t) \). Then we have
\[ \| A(t)(b(t) - c(t)) \| \leq C e^{-\gamma t}. \]

3. The length-preserving flow. We consider the flow (3). If the initial curve is convex, then the flow exists for all time keeping the convexity, and the curve approaches a round circle; this was shown in [5]. We have the same results for the flows (1) and (2) without convexity assumption assuming the global existence. In this subsection, we give a proof of this fact.

First we prove the exponential decay of \( I_{-1} \).

**Theorem 3.1.** Assume that \( f \) is a global solution of (3) such that the initial rotation number is 1 and the initial (signed) area is positive. Then there exist \( C > 0 \) and \( \lambda > 0 \) such that
\[ I_{-1}(t) \leq C e^{-\lambda t}. \]
Proof. Since
\[
\frac{dA}{dt} = - \int_0^L \partial_t f \cdot \nu ds = \frac{L}{2\pi} \int_0^L \kappa^2 ds = \frac{I_0}{2\pi}.
\] (4)
from Theorem 2.1, we have
\[
\frac{d}{dt} I_{-1} = -4\pi \frac{dA}{L^2} = -\frac{2}{L^2} I_0 \leq -\frac{16\pi^2}{L^2} I_{-1}.
\]
Therefore, we have the desired conclusion. □

Next we show the exponential decay of \( I_\ell \) for \( \ell \in \mathbb{N} \).

**Theorem 3.2.** Let \( f \) be as in Theorem 3.1. For each \( \ell \in \mathbb{N} \cup \{0\} \), there exist \( C_\ell > 0 \) and \( \lambda_\ell > 0 \) such that
\[
I_\ell(t) \leq C_\ell e^{-\lambda_\ell t}.
\]

Proof. We initially consider the behavior of \( I_0 \). Since
\[
I_0 = L \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds = L \int_0^L \kappa^2 ds - 4\pi^2,
\]
we have
\[
\frac{d}{dt} I_0 = L \frac{d}{dt} \int_0^L \kappa^2 ds
\]
\[
= L \int_0^L 2\kappa \partial_t \kappa ds + L \int_0^L \kappa^2 \partial_t (ds)
\]
\[
= L \int_0^L 2\kappa \left( \partial_t^2 \kappa + \kappa^2 - \frac{\kappa^2}{2\pi} \int_0^L \kappa^2 ds \right) ds
\]
\[
= L \int_0^L \left( 2\partial_t^2 \kappa + \kappa^3 \right) \left( \kappa - \frac{1}{2\pi} \int_0^L \kappa^2 ds \right) ds
\]
\[
= L \int_0^L \left( 2\partial_t^2 \kappa + \kappa^3 \right) \left( \kappa - \frac{1}{2\pi} \int_0^L \kappa^2 ds \right) ds
\]
\[
= -2L \int_0^L (\partial_t \kappa)^2 ds + L \int_0^L \kappa^3 \kappa ds - \frac{L}{2\pi} \int_0^L \kappa^3 ds \int_0^L \kappa^2 ds
\]
\[
= -\frac{2}{L^2} I_1 + L \int_0^L \left( \kappa^4 + \frac{6\pi}{L} \kappa^3 + \frac{12\pi^2}{L^2} \kappa^2 \right) ds
\]
\[
- \frac{L}{2\pi} \int_0^L \left( \kappa^3 + \frac{6\pi}{L} \kappa^2 + \frac{8\pi^3}{L^3} \right) ds \int_0^L \kappa^2 ds.
\]
Hence we obtain
\[
\frac{d}{dt} I_0 + \frac{3}{L^2} I_0^2 + \frac{4\pi^2}{L^2} I_0 + \frac{2}{L^2} I_1 \leq \frac{1}{L^2} \int_0^L \left( L^3 \kappa^4 + 6\pi L^2 \kappa^3 + 12\pi^2 L \kappa^2 \right) ds
\]
\[
= \frac{1}{L^2} \int_0^L \left( L^3 \kappa^4 + 6\pi L^2 \kappa^3 + 12\pi^2 L \kappa^2 \right) ds
\]
\[
= \frac{1}{L^2} \int_0^L \left( L^3 \kappa^4 + 6\pi L^2 \kappa^3 + 12\pi^2 L \kappa^2 \right) ds
\]
By the Gagliardo-Nirenberg inequalities, we have
\[
\frac{1}{L^2} \int_0^L \left( L^3 \kappa^4 + 6\pi L^2 \kappa^3 + 12\pi^2 L \kappa^2 \right) ds \leq \frac{C}{L^2} \left( I_1^3 I_0^2 + I_1^2 I_0^3 + I_0 \right).
\]
Applying Young’s inequalities, Theorem 2.2 and Theorem 3.1, we have
\[ I_1^2 I_3^2 \leq \epsilon I_1 + C_I I_0^3, \]
\[ I_4^4 I_5^4 \leq \epsilon I_1 + C_I I_0^3 \leq \epsilon (I_1 + I_0) + C_I I_0^3 \leq C\epsilon I_1 + C_I I_0^3, \]
\[ I_0 \leq C I_1^2 \left( I_1 + I_0 \right) \leq C \left( I_1^2 + \epsilon \right) I_1 + C_I I_1 \]
\[ \leq C \left( I_1^2 + \epsilon \right) I_1 + C_I e^{-\lambda t}, \]
for any \( \epsilon > 0 \). Hence the first term on the right-hand side of (5) is estimated above by
\[ C \left( I_1^2 + \epsilon \right) I_1 + \frac{C_I}{L^2} (I_0^3 + e^{-\lambda t}). \]
Moreover we have, by Young’s inequality,
\[ -\frac{1}{2\pi L^2} \left( L^2 \int_0^L \tilde{\kappa}^3 ds \right) \left( L \int_0^L \tilde{\kappa}^2 ds \right) \leq \frac{C}{L^2} I_1^2 I_0^3 I_0 = \frac{C}{L^2} I_1^2 I_0^3 \]
\[ \leq \frac{\epsilon}{L^2} I_1 + \frac{C_I}{L^2} I_0^3. \]
Taking \( \epsilon \) sufficiently small and \( t \) sufficiently large, by Theorem 3.1, we have
\[ \frac{d}{dt} I_0 + \frac{3}{L^2} I_0^3 + \frac{4\pi^2}{L^2} I_0 + \frac{C_1}{L^2} I_1 \leq \frac{C_2}{L^2} I_0^3 + \frac{C_3}{L^2} e^{-\lambda t}. \]
(6)
It is obvious that there exists \( T_1 > 0 \) satisfying
\[ \int_{T_1}^{\infty} \frac{C_3}{L^2} e^{-\lambda t} dt < \frac{3}{2C_2}. \]
Furthermore, we have
\[ \int_0^{\infty} I_0 dt \leq C \]
by integrating (4), because \( A \) is non-decreasing. Hence there exists \( T_2 \geq T_1 \) such that \( I_0(T_2) < \frac{3}{2C_2} \). We would like to show \( I_0(t) < \frac{3}{C_2} \) for \( t \geq T_2 \). To do this, we argue by contradiction. Then there exists \( T_3 > T_2 \) such that
\[ I_0(t) < \frac{3}{C_2} \text{ for } t \in [T_2, T_3], \text{ and } I_0(T_3) = \frac{3}{C_2}. \]
It follows from (6) that
\[ \frac{d}{dt} I_0 \leq \frac{C_3}{L^2} e^{-\lambda t} \]
for \( t \in [T_2, T_3] \). Hence
\[ I_0(T_3) = I_0(T_2) + \int_{T_2}^{T_3} \frac{d}{dt} I_0 dt < \frac{3}{2C_2} + \int_{T_1}^{\infty} \frac{C_3}{L^2} e^{-\lambda t} dt < \frac{3}{C_2}. \]
This contradicts \( I_0(T_3) = \frac{3}{C_2} \). Hence we show
\[ I_0(t) < \frac{3}{C_2} \]
for \( t \geq T_2 \). Therefore, from (6), we obtain
\[ \frac{d}{dt} I_0 + \frac{C_1}{L^2} I_1 \leq \frac{C_3}{L^2} e^{-\lambda t}. \]
By Wirtinger’s inequalities, we have
\[ \frac{d}{dt} I_0 + \frac{C_4}{L^2} I_0 \leq \frac{C_3}{L^2} e^{-\lambda t}. \]
Thus the assertion for \( \ell = 0 \) with some positive \( \lambda_0 \) has been proved.
Next we consider the behavior of \( I_\ell \) for \( \ell \in \mathbb{N} \). Set
\[ J_{k,p} = \left\{ \left. L^{(1+k)p-1} \int_0^L |\partial_t^{k}\tilde{\eta}|^p \, ds \right\}^{1/p} \right\}.
By the Gagliardo-Nirenberg inequalities we have
\[ J_{k,p} \leq C J_{m,p} I_{0,2}^{-\theta} = C J_{m} I_{0,2}^{\frac{1-\theta}{2}} \]
for \( k \in \{0, 1, \ldots, m\} \), \( p \geq 2 \). Here \( C \) is independent of \( L \), and \( \theta = \frac{1}{m} \left( k - \frac{1}{p} + \frac{1}{2} \right) \in [0, 1] \). For \( k \in \mathbb{N} \cup \{0\} \) and \( m \in \mathbb{N} \), let \( P^k_m(\tilde{\eta}) \) be any linear combination of the type
\[ P^k_m(\tilde{\eta}) = \sum_{i_1, \ldots, i_m = k} c_{i_1, \ldots, i_m} \partial_t^{i_1}\tilde{\eta} \cdots \partial_t^{i_m}\tilde{\eta} \]
with universal, constant coefficients \( c_{i_1, \ldots, i_m} \). Similarly we define \( P^k_0 \) as a universal constant. We can show
\[ \partial_t \partial_t^{k}\tilde{\eta} = \partial_t^{k+2}\tilde{\eta} + \sum_{m=0}^{2} L^{-(2-m)} P^k_{m+1}(\tilde{\eta}) + \sum_{m=0}^{1} L^{-(1-m)} P^k_{m+1}(\tilde{\eta}) \int_0^L \tilde{\eta}^2 \, ds \]
by induction on \( k \). Hence we have
\[ \frac{d}{dt} I_\ell = 2L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \partial_t^{\ell}\tilde{\eta} \, ds + L^{2\ell+1} \int_0^L (\partial_t^{\ell}\tilde{\eta})^2 \partial_t^{\ell}(ds) \]
\[ = -2L^{2\ell+1} \int_0^L (\partial_t^{\ell+1}\tilde{\eta})^2 \partial_t^{\ell}(ds) + 2L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \left( \sum_{m=0}^{1} L^{-(1-m)} P^\ell_{m+1}(\tilde{\eta}) \right) \int_0^L \tilde{\eta}^2 \, ds \]
\[ + 2L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \left( \sum_{m=0}^{1} L^{-(1-m)} P^\ell_{m+1}(\tilde{\eta}) \right) \int_0^L \tilde{\eta}^2 \, ds \]
\[ - L^{2\ell+1} \int_0^L (\partial_t^{\ell}\tilde{\eta})^2 \partial_t^{\ell}(ds) + \frac{L^{2\ell+1}}{2\pi} \int_0^L (\partial_t^{\ell}\tilde{\eta})^2 \partial_t^{\ell}(ds) \int_0^L \tilde{\eta}^2 \, ds \]
\[ = -2L^{2\ell+1} \int_0^L (\partial_t^{\ell+1}\tilde{\eta})^2 \, ds + 2 \sum_{m=0}^{2} L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \left( \sum_{m=0}^{1} L^{-(1-m)} P^\ell_{m+1}(\tilde{\eta}) \right) \int_0^L \tilde{\eta}^2 \, ds \]
\[ + 2 \sum_{m=0}^{1} L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \left( \sum_{m=0}^{1} L^{-(1-m)} P^\ell_{m+1}(\tilde{\eta}) \right) \int_0^L \tilde{\eta}^2 \, ds. \]
We define \( \Phi_m \) to be
\[ \Phi_m = L^{2\ell+1} \int_0^L \partial_t^{\ell}\tilde{\eta} \left( \sum_{m=0}^{1} L^{-(1-m)} P^\ell_{m+1}(\tilde{\eta}) \right) \int_0^L \tilde{\eta}^2 \, ds. \]
When \( m = 0 \), we have
\[ \Phi_0 = \frac{C}{L^2} I_\ell \]
since $P_1^\ell(\tilde{\kappa}) = c\partial_s^\ell \tilde{\kappa}$. When $m = 1$, we have
\[
\Phi_1 = L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} L^{-1} P_2^\ell(\tilde{\kappa}) ds.
\]
Also $P_2^\ell(\tilde{\kappa})$ is a linear combination of $(\partial_s^k \tilde{\kappa})(\partial_s^{\ell-k} \tilde{\kappa})$ with $k = 0, \cdots, \ell$. By Hölder’s inequality, we have
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-1} P_2^\ell(\tilde{\kappa}) \right| \leq \sum_{k=0}^\ell \frac{C}{L^2} J_{\ell,3} J_{k,3} J_{\ell-k,3},
\]
and (7) yields
\[
J_{j,3} \leq CI_{\ell+1}^{1+\theta(j,3)} \frac{L^{j}}{\ell^{j+1}}, \quad \theta(j,3) = \frac{j}{\ell} + \frac{1}{\ell+1}.
\]
Hence applying Young’s inequality, we obtain
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-1} P_2^\ell(\tilde{\kappa}) ds \right| \leq \frac{C}{L^2} J_{\ell,4} J_{j,4} J_{\ell-j,4}
\leq \frac{C}{L^2} I_{\ell+1}^{2\ell+1} I_0^{2\ell+3} \leq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C\epsilon}{L^2} I_{\ell+1}^{2\ell+3}
\]
for any $\epsilon > 0$. When $m = 2$, we obtain
\[
\Phi_2 = L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} P_3^\ell(\tilde{\kappa}) ds.
\]
We can estimate this term similarly. Indeed, since $P_3^\ell(\tilde{\kappa})$ is a linear combination of
\[
\partial_s^\ell \tilde{\kappa}, \partial_s^k \tilde{\kappa}, \text{and} \partial_s^{\ell-j-k} \tilde{\kappa},
\]
we have
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) P_3^\ell(\tilde{\kappa}) \right| \leq \sum_{m=0}^{\ell} \sum_{j+k+m}^{\ell \geq 0} \frac{C}{L^2} J_{\ell,j} J_{j,k} J_{\ell-j-k,4}
\leq \frac{C}{L^2} I_{\ell+1}^{2\ell+1} I_0^{2\ell+3} \leq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C\epsilon}{L^2} I_{\ell+1}^{2\ell+3}
\]
for any $\epsilon > 0$. Hence we obtain
\[
\frac{d}{dt} I_\ell + \frac{C_1}{L^2} I_{\ell+1} \leq \frac{C_2}{L^2} I_\ell + \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_3}{L^2} I_0^{2\ell+3} + \frac{C_4}{L^2} I_0
\]
From Theorem 2.2 and Young’s inequality, we have
\[
\frac{C}{L^2} I_\ell \leq \frac{C}{L^2} \left( I_{\ell+1}^{\frac{1}{\ell+1}} + I_{\ell+1}^{\frac{1}{\ell+1}} \right) \leq \frac{C}{L^2} \left( (I_{\ell+1}^{\frac{1}{\ell+1}} + C I_{\ell+1}^{\frac{1}{\ell-1}} + C I_{\ell+1}^{\frac{1}{\ell-1}}) \right).
\]
Taking $\epsilon$ sufficiently small and $t$ sufficiently large, we have
\[
\frac{d}{dt} I_\ell + \frac{C_3}{L^2} I_{\ell+1} \leq \frac{C}{L^2} \left( I_0^{\frac{2\ell+2}{\ell+1}} + I_0^{2\ell+3} + I_0^{(2\ell+4)} + I_{\ell-1} \right).
\]
By Wirtinger’s inequalities, we have
\[
\frac{d}{dt} I_\ell + \frac{C_2}{L^2} I_\ell \leq \frac{C}{L^2} \left( I_0^{\frac{2\ell+5}{\ell+1}} + I_0^{2\ell+3} + I_0^{(2\ell+4)} + I_{\ell-1} \right).
\]
Since we have already shown that $I_{\ell-1}$ and $I_0$ decay exponentially as $t \to \infty$, we show
\[
\frac{d}{dt} I_\ell + \frac{C_2}{L^2} I_\ell \leq \frac{C_3}{L^2} e^{-\lambda t}.
\]
Hence we obtain the desired conclusion for $\ell \in \mathbb{N}$. □
We can prove the following theorem in a similar way to the proof of Theorem 2.4, where we use Theorem 3.1 instead of Theorem 2.3.

**Theorem 3.3.** The claims (1)–(7) in Theorem 2.4 also hold for global solutions of the length-preserving flow.

**Acknowledgments.** The author expresses their appreciation to Professor Shigetoshi Yazaki and Professor Tetsuya Ishiwata for sharing information of related articles, and for discussions. The author also would like to express their gratitude to Professor Neal Bez for English language editing.

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Received January 2019; 1st revision February 2019; final revision February 2020.

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