Strong gravity Lense–Thirring precession in Kerr and Kerr–Taub–NUT spacetimes

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Received 6 November 2013, revised 29 January 2014
Accepted for publication 7 February 2014
Published 5 March 2014

Abstract
An exact expression derived in the literature for the rate of dragging of inertial frames (Lense–Thirring (LT) precession) in a general stationary spacetime, is reviewed. The exact LT precession frequencies for Kerr, Kerr–Taub–NUT (Newman–Unti–Tamburino) and Taub–NUT spacetimes are explicitly derived. Remarkably, in the case of the zero angular momentum Taub–NUT spacetime, the frame-dragging effect is shown not to vanish, when considered for spinning test gyroscopes. The result becomes sharper for the case of vanishing ADM mass of that spacetime. We clarify how our results are consistent with claims in the recent literature of null orbital plane precession for NUT spacetimes.

Keywords: strong, precession, Lense–Thirring
PACS numbers: 04.20.$-q$, 04.20.Cv, 04.20.Jb

(Some figures may appear in colour only in the online journal)

1. Introduction
Stationary spacetimes with angular momentum (rotation) are known to exhibit an effect called Lense–Thirring (LT) precession whereby locally inertial frames are dragged along the rotating spacetime, making any test gyroscope in such spacetimes precess with a certain frequency called the LT precession frequency [1]. This frequency has been shown to decay as the inverse cube of the distance of the test gyroscope from the source [2] for large enough distances where curvature effects are small, and known to be proportional to the angular momentum of the source. Most earlier analyses of the LT effect [3] assume large distances ($r \gg M$, $M$ is the ADM mass of the rotating spacetime due to a compact object) for the test gyroscope.
Such weak field analyses lead to the standard result for LT precession frequency in the weak field approximation, given by [3, 4]

$$\Omega_{\text{LT}} = \frac{1}{r^3} [3(\hat{J} \cdot \hat{r})\hat{r} - \hat{J}]$$

(1)

where, $\hat{r}$ is the unit vector along $r$ direction. In a recent work reported in [5], an alternative approach based on solving the geodesic equations of the test gyroscope numerically, once again within the weak gravitational field approximation, is used to compute the frame-dragging effect for galactic-centre black holes. In another very recent related work [6], Hackman and Lämmerzahl have given an expression of LT precession valid up to first order in the Kerr parameter $a$ for a general axisymmetric Plebański–Demiański spacetime. The LT precession rate has also been derived [7, 8] through solving the geodesic equations for both Kerr and Kerr–de-Sitter spacetimes at the polar orbit. These results are not applicable for orbits which lie in orbital planes other than the polar plane. We understand that observations of precession due to locally inertial frame-dragging have so far been possible only for spacetimes whose curvatures are small enough; e.g., the LT precession in the Earth’s gravitational field which was probed recently by the LAGEOS experiment [9] and also by Gravity Probe B [10]. Though there has been so far no attempt to measure LT precession effects due to frame-dragging in strong gravity regimes [11], observational prospects of LT precession in strong gravity situations have been discussed in [12]. The problem of accretion onto compact objects also stands to be influenced by strong gravity physics, especially by an understanding of LT precession under such conditions. A recent work by Stone and Loeb [13] has estimated the effect of weak-field LT precession on accreting matter close to compact accreting objects. Modifications due to strong gravity LT precession to such situations are not without interest.

In this paper, we present a detailed analysis of the frame-dragging phenomenon in Kerr–Taub–NUT (Newman–Unti–Tamburino) spacetime, where the NUT charge is an additional feature with interesting consequences. The Kerr and Taub–NUT spacetimes emerge as special cases of this analysis. Also, the oft-quoted weak-field result (1) (in a ‘Copernican’ frame) for the LT precession rate is readily obtained from this general result, inserting the metric for the desired spacetime.

The paper is accordingly organized as follows: in section 2, we present a brief derivation of LT precession frequency in any stationary spacetime, following [14, 15]. In section 3, we derive the exact LT precession rates in Kerr–Taub–NUT and Taub–NUT spacetimes. This is followed by a detailed discussion of the Taub–NUT spacetime and how one observes a nonvanishing LT precession despite a vanishing Kerr parameter, provided one looks at spinning test gyroscopes. A possible analytic extension of the Taub–NUT spacetime is also considered, to delineate the properties of the ‘horizon’. The rate of the LT precession in Kerr spacetime is next derived in section 4, without invoking either the weak gravity approximation or an approximation involving the Kerr parameter. The weak-field approximation is then shown to emerge straightforwardly from our general formulation. We end in section 5 with a summary and a discussion on future outlook.

### 2. Derivation of Lense–Thirring precession frequency

Let us consider an observer at rest in a stationary spacetime with a timelike Killing field $K$. The observer moves along an integral curve $\gamma(\tau)$ of $K$. So, her four velocity can be written as

$$u = (-K^2)^{-\frac{1}{2}} K.$$  

(2)

We can now choose an orthonormal tetrad $e_a$ along $\gamma$ which is Lie-transported:

$$L_K e_a = 0$$

(3)
where \( \alpha = 0, 1, 2, 3 \). As \( e_0 \) is just \( u = \dot{\gamma} \) (where, ‘dot’ denotes the differentiation with respect to \( \tau \) ), \( u \) is perpendicular to \( e_1, e_2, e_3 \) axes. We also have

\[
\langle e_\alpha, e_\beta \rangle = \eta_{\alpha\beta},
\]

where, \( \langle, \rangle \) this symbol implies the scalar product and \( \eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \). We can interpret \( e_\alpha \) as axes at rest. This choice is what is sometimes known as the ‘Copernican’ frame.

We know that the spin of the gyroscope precesses with respect to that axes of rest and we are interested in the change of the spin relative to this system. We know that torsion

\[
T(K, e_i) = \nabla_K e_i - \nabla_{e_i} K = [K, e_i] = 0.
\]

We know

\[
\omega_{ij} = \langle \nabla_u e_i, e_j \rangle
\]

in where the \( \omega_{ij} \) related with the angular velocity \( \Omega^i \) as

\[
\omega_{ij} = \epsilon_{ijl} \Omega^l.
\]

Now, using equations (6) and (2) we get,

\[
\omega_{ij} = (-K^2)^{-\frac{1}{2}} \langle e_j, \nabla_K e_i \rangle.
\]

The gyroscopic precesses with the angular velocity \( \Omega \) relative to the tetrad frame \( e_a \), \( \Omega \) is considered as the angular velocity or the precession rate of the LT precession. As \( [K, e_i] = L_K e_i = 0 \), we get from the equation (5) is

\[
\nabla_K e_i = \nabla_{e_i} K.
\]

Substituting this result in equation (8) we get,

\[
\omega_{ij} = (-K^2)^{-\frac{1}{2}} \frac{1}{2} [\nabla \tilde{K}(e_i, e_j) - \nabla \tilde{K}(e_j, e_i)]
\]

so,

\[
\omega_{ij} = \frac{1}{4} (-K^2)^{-\frac{1}{2}} d \tilde{K}(e_i, e_j).
\]

So, the exact LT frequency of precession of test gyroscopes in strongly curved stationary spacetimes, analysed within a Copernican frame, is expressed as a co-vector given in terms of the timelike Killing vector fields \( K \) of the stationary spacetime, as (in the notation of \([14, 15]\))

\[
\tilde{\Omega} = \frac{1}{2K^2} \ast (\tilde{K} \wedge d\tilde{K})
\]

or,

\[
\Omega^\mu = \frac{1}{2K^2} \eta^{\nu\rho\sigma} K_\nu \partial_\rho K_\sigma,
\]

where, \( \eta^{\nu\rho\sigma} \) represent the components of the volume-form in spacetime and \( \tilde{K} \) and \( \tilde{\Omega} \) denote the one-form of \( K \) and \( \Omega \), respectively. \( \tilde{\Omega} \) will vanish for stationary spacetimes if and only if \( (\tilde{K} \wedge d\tilde{K}) \) does.

For the general stationary spacetime, we can use the coordinate basis form of \( K = \partial_0 \) and the co-vector components are easily seen to be \( K_\mu = g_{\mu 0} \). This co-vector could also be written in the following form

\[
\tilde{K} = g_{00} \, dx^0 + g_{0i} \, dx^i.
\]

The spatial components of the precession rate (in the chosen frame) are

\[
\Omega^i = \frac{1}{2} \frac{\epsilon_{i\beta}}{g_{00} \sqrt{-g} [g_{00} g_{0j} - g_{04} g_{00}] - g_{04} g_{00} g_{00,j}]}
\]
The vector field corresponding to the LT precession co-vector in (16) can be expressed as
\[
\Omega = \frac{1}{2} \epsilon_{ij} \left[ g_{0i,j} \left( \partial_j - \frac{g_{0i}}{g_{00}} \partial_0 \right) - \frac{g_{0i,j}}{g_{00}} \right].
\] (17)

The remarkable feature of the above equation (17) is that it is applicable to any arbitrary stationary spacetime which is non-static; it gives us the exact rate of LT precession in such a spacetime. For instance, a Taub–NUT [16]-[17] spacetime with vanishing ADM mass is known to be non-rotating, but still has an angular momentum (dual or ‘magnetic’ mass [18]); we use equation (17) to compute the LT precession frequency in this case as well.

3. Lense–Thirring precession in Kerr–Taub–NUT spacetime

3.1. The Kerr–Taub–NUT spacetime $a \neq 0, n \neq 0$

The Kerr–Taub–NUT spacetime is geometrically a stationary, axisymmetric vacuum solution of Einstein equation with Kerr parameter ($a$) and NUT charge ($n$). If the NUT charge vanishes, the solution reduces to the Kerr geometry. The metric of the Kerr–Taub–NUT spacetime is[19]
\[
ds^2 = -\frac{\Delta}{\rho^2} (dt - A \, d\phi)^2 + \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \, d\theta^2 + \frac{1}{\rho^2} \sin^2 \theta (a \, dt - B \, d\phi)^2.
\] (18)

With
\[
\Delta = r^2 - 2Mr + a^2 - n^2, \quad p^2 = r^2 + (n + a \cos \theta)^2,
\]
\[
A = a \sin^2 \theta - 2n \cos \theta, \quad B = r^2 + a^2 + n^2.
\] (19)

As the spacetime has an intrinsic angular momentum (due to Kerr parameter $a$), we can expect a non-zero frame-dragging effect. We get from equation (17), the LT precession rate in Kerr–Taub–NUT spacetime is
\[
\vec{\Omega}_{\text{LT}}^{\text{KTN}} = \sqrt{\frac{\Delta}{\rho^2}} \left[ \frac{\rho^2}{(\rho^2 - 2Mr - n^2)} \hat{r} + \frac{\rho^2}{\rho^2} \hat{r} \right],
\] (20)

where $\rho^2 = r^2 + a^2 \cos^2 \theta$. In contrast to the Kerr spacetime, where the source of the LT precession is the Kerr parameter (specific angular momentum) $a$, the Kerr–Taub–NUT spacetime has an extra somewhat surprising feature: the LT precession does not vanish even for vanishing Kerr parameter $a = 0$, so long as the NUT charge $n \neq 0$. This means that though the orbital angular momentum ($J$) of this spacetime vanishes, the spacetime does indeed exhibit an intrinsic spinlike angular momentum (at the classical level itself) which we discuss below in more detail. One can show that inertial frames are dragged along this orbitally non-rotating spacetime with the precession rate
\[
\vec{\Omega}_{\text{LT}}^{\text{KTN}} = \frac{n \sqrt{\Delta}|_{a=0}}{\rho^3} \hat{r}.
\] (21)

where, $p^2 = r^2 + n^2$. Notice that the precession rate is independent of $\theta$ and also that it vanishes when the NUT charge vanishes, as already alluded to above. In fact, for $a = n = 0$, the Kerr–Taub–NUT spacetime reduces to the static Schwarzschild spacetime which of course does not cause any inertial frame dragging. We consider this curious phenomenon in somewhat more detail in the next subsection.
3.2. The Taub–NUT spacetime

The Taub–NUT spacetime is geometrically a stationary, non-rotating vacuum solution of Einstein equation with NUT charge \( n \). The Einstein–Hilbert action requires no modification to accommodate this NUT charge or ‘dual mass’ which is perhaps an intrinsic feature of general relativity, being a gravitational analogue of a magnetic monopole in electrodynamics [20].

Consider the line element (of NUT spacetime), which is presented by Newman et al [21]

\[
\begin{align*}
\text{d}s^2 = & -f(r) \left( \text{d}t + 4n \sin^2 \frac{\theta}{2} \text{d}\phi \right)^2 + \frac{1}{f(r)} \text{d}r^2 + (r^2 + n^2)(\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) \\
\text{where,} \quad f(r) = & \frac{r^2 - 2Mr - n^2}{r^2 + n^2}.
\end{align*}
\]

Here, \( M \) represents the ‘gravitoelectric mass’ or ‘mass’ and \( n \) represents the ‘gravitomagnetic mass’ or ‘dual’ (or ‘magnetic’) mass of this spacetime. It is obvious that the spacetime (22) is not invariant under time reversal \( t \to -t \), signifying that it must have a sort of ‘rotational sense’, once again analogous to a magnetic monopole in electrodynamics. One is thus led to the conclusion that the source of the nonvanishing LT precession is this ‘rotational sense’ arising from a nonvanishing NUT charge. Without the NUT charge, the spacetime is clearly hypersurface orthogonal and frame-dragging effects vanish, as already mentioned above. This ‘dual’ mass has been investigated in detail in [22], who also refer to it as an ‘angular momentum monopole’ [18] in Taub–NUT spacetime.

In the Schwarzschild coordinate system, \( f(r) = 0 \) at

\[
r = r_\pm = M \pm \sqrt{M^2 + n^2}
\]

\( r_\pm \) are similar to horizons in this geometry in the sense that \( f(r) \) changes sign from positive to negative across the horizon and the radial coordinate \( r \) changes from spacelike to timelike. But is \( r = r_\pm \) an event horizon in the sense of the event horizon of Schwarzschild spacetime? We shall focus on this issue momentarily. For the present, we note that the LT precession rate (which can be easily obtained from equation (21) also) is given by

\[
\Omega_{\text{LT}}^{\text{MTN}} = \frac{n(r^2 - 2Mr - n^2)^{\frac{1}{2}}}{(r^2 + n^2)^{\frac{3}{2}}} \hat{r}.
\]

It is clear that \( \Omega_{\text{LT}}^{\text{MTN}} = 0 \) on \( r = r_\pm \), in contrast to the LT precession frequency in the standard Kerr spacetime which is maximum closest to the event horizon! Further, if we plot the magnitude of the precession rate as a function of the radial coordinate for \( r > r_+ \), as obtained from (25), one obtains the profile like figure 1.

Thus, the precession rate is maximum around \( r = 5 \), but it sharply drops for \( r \to r_+ \) and vanishes on the ‘horizon’.

3.3. Analytic extension of Taub–NUT spacetime

As the metric (22) blows up at \( r = r_\pm \), we should perhaps try a different co-ordinate system where it is smooth on the ‘horizon’. Following [23], wherein an analytic extension of the metric (22) has been attempted, one obtains the transformed metric

\[
\begin{align*}
\text{d}s^2 = & (r^2 + n^2)(\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) + F^2[du_\pm^2 - dv_\pm^2 - (2n/r_\pm)(u_\pm du_\pm - v_\pm dv_\pm) \cos \theta \text{d}\phi] \\
& - (n/r_\pm)^2(u_\pm^2 - v_\pm^2) \cos^2 \theta \text{d}\phi^2.
\end{align*}
\]
where,

\[ F^2 = 4r_\pm^4 (r^2 + n^2)^{-1} \left( \frac{r - r_\pm}{r_\pm} \right) \frac{1 - \sqrt{\frac{r}{r_\pm}}}{1 + \sqrt{\frac{r}{r_\pm}}} \exp \left( - \frac{r}{r_\pm} \right) \]  \hspace{1cm} (27)

\[ u_\pm = \left( \frac{r - r_\pm}{r_\pm} \right)^{1/2} \left( \frac{r - r_\pm}{r_\pm} \right)^{i\pi / 4} \exp \left( \frac{r}{2r_\pm} \right) \cosh \left( \frac{t}{2r_\pm} \right) \]  \hspace{1cm} (28)

\[ v_\pm = \left( \frac{r - r_\pm}{r_\pm} \right)^{1/2} \left( \frac{r - r_\pm}{r_\pm} \right)^{i\pi / 4} \exp \left( \frac{r}{2r_\pm} \right) \sinh \left( \frac{t}{2r_\pm} \right). \]  \hspace{1cm} (29)

In \( u, v \) co-ordinate system \( r \) could be redefined as

\[ u_\pm^2 - v_\pm^2 = \left( \frac{r - r_\pm}{r_\pm} \right)^{1/2} \left( \frac{r - r_\pm}{r_\pm} \right)^{i\pi / 4} \exp \left( \frac{r}{2r_\pm} \right). \]  \hspace{1cm} (30)

Recall that locally every spherically symmetric four dimensional spacetime has the structure \( \mathbb{R}^2 \otimes S^2 \) where \( \mathbb{R}^2 \) is a two dimensional Lorentzian spacetime. In this Taub–NUT case, the attempted analytic extension discussed immediately above leads to a vanishing of the two dimensional Lorentzian metric on the ‘horizon’ \( r = r_\pm \), in contrast to the Schwarzschild metric. This might be taken to imply that perhaps the null surface \( r = r_\pm \) is not quite an event horizon; rather it is a null surface where ingoing future-directed null geodesics appear to terminate, as already noticed in [24]. So, physical effects on this null hypersurface might not be easy to compute, as a result of which the apparent vanishing of the LT precession on this hypersurface is to be taken with a pinch of salt.

The NUT spacetime, for the mass \( M = 0 \) is also well-defined (see, for example, appendix of [18]). We can also write down the precession rate only for massless dual mass (NUT charge \( n \) can be regarded as dual mass) solutions of NUT spacetime. This turns out to be

\[ \hat{\Omega}_{LT}^{\text{TN}} = \frac{n(r^2 - n^2)^{1/2} r_\pm}{(r^2 + n^2)^2}. \]  \hspace{1cm} (31)
At the points $r = \pm n$, the LT precession vanishes akin to the previous case, but the same caveats apply here as well. In figure 1, we observe that for $n = 3$, the LT precession starts for $r > 3$ and continues to infinity. Setting $\frac{d\Omega_{LT}}{dt} = 0$, we get that $\Omega_{LT}$ is maximum at $r = \sqrt{2}n$. In our figure 1, this value is $r = 3\sqrt{2} = 4.24$ m. Now, we are not interested for $r < 3$. Our formulas are not comfortable in that regions and $r < r_{+}$ is also not well-defined for Taub–NUT spacetimes. From our precession rate formulas (21, 25, 31) at dual mass spacetimes we can see that the precession rate ($\Omega_{LT}$) is the same, starting from the polar region to the equatorial plane for a fixed distance. $\Omega_{LT}$ depends only on distance ($r$) of the test gyroscope from the ‘dual mass’.

At this point we refer to a recent paper by Kagramanova et al [24] where it is claimed that the LT effect in fact vanishes everywhere/everywhen in the Taub–NUT spacetime. In that paper, timelike geodesic equations in this spacetime are investigated. The orbital plane precession frequency ($\Omega_{\phi} - \Omega_0$) is computed, following the earlier work of [25, 26], and a vanishing result ensues. This result is then interpreted in [24] as a signature for a null LT precession in the Taub–NUT spacetime.

We would like to submit that what we have focused on in this paper is quite different from the ‘orbital plane precession’ considered in [24]. Using a ‘Copernican’ frame, we calculate the precession of a gyroscope which is moving in an arbitrary integral curve (not necessarily geodesic). Within this frame, an untorqued gyro in a stationary but not static spacetime held fixed by a support force applied to its centre of mass, undergoes LT precession. Since the Copernican frame does not rotate (by construction) relative to the inertial frames at asymptotic infinity (‘fixed stars’), the observed precession rate in the Copernican frame also gives the precession rate of the gyro relative to the fixed stars. It is thus, more an intrinsic property of the classical spin of the spacetime (as an untorqued gyro must necessarily possess), in the sense of a dual mass, rather than an orbital plane precession effect for timelike geodesics in a Taub–NUT spacetime. The dual mass is like the Saha spin of a magnetic monopole in electrodynamics [20], which may have a vanishing orbital angular momentum, but to which a spinning electron must respond in that its wavefunction acquires a geometric phase.

More specifically, in our case, we consider the gyroscope equation [14] in an arbitrary integral curve

$$\nabla_u S = \langle S, a \rangle u$$

(32)

where, $a = \nabla_u u$ is the acceleration, $u$ is the four velocity and $S$ indicates the spacelike classical spin four vector $S' = (0, \vec{S})$ of the gyroscope. For geodesics $a = 0 \Rightarrow \nabla_u S = 0$.

In contrast, Kagramanova et al [24] consider the behaviour of massive test particles with vanishing spin $S = 0$ [27], and compute the orbital plane precession rate for such particles, obtaining a vanishing result. We are thus led to conclude that because two different situations are being considered, there is no inconsistency between our results and theirs.

In summary, we have noted in this subsection several subtleties of computing the LT precession rate on and near the ‘horizon’ of a Taub–NUT spacetime, and our results are consistent with earlier literature where geodesic incompleteness on this null hypersurface has been noted.

4. Lense–Thirring precession in Kerr spacetime

One can now use equation (17) to calculate the angular momentum of a test gyroscope in a Kerr spacetime to get the LT precession in a strong gravitational field. In Boyer–Lindquist
coordinates, the Kerr metric is written as,
\[
\begin{align*}
\text{d}s^2 &= -(1 - \frac{2Mr}{\rho^2}) \text{d}t^2 - \frac{4Mr \sin^2 \theta}{\rho^2} \text{d}\phi \text{d}t + \frac{\rho^2}{\Delta} \text{d}r^2 + \rho^2 \text{d}\theta^2 \\
&\quad + \left( r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta \text{d}\phi^2
\end{align*}
\]  
(33)

where, \(\alpha\) is Kerr parameter, defined as \(\alpha = \frac{J}{M}\), the angular momentum per unit mass and
\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2. 
\]  
(34)

For the Kerr spacetime, the only nonvanishing \(g_{0i} = g_{0\phi}, i = \phi\) and \(j, l = r, \theta\); substituting these in equation (17), the precession frequency vector is given by
\[
\Omega_{\text{LT}} = \frac{1}{2\sqrt{-g}} \left[ \left( g_{\phi r,r} - \frac{g_{\phi r}}{g_{00}} g_{00,r} \right) \partial_\theta - \left( g_{\phi \theta,\theta} - \frac{g_{\phi \theta}}{g_{00}} g_{00,\theta} \right) \partial_r \right] 
\]  
(35)

where, the various metric components can be read off from equation (33). Likewise,
\[
\sqrt{-g} = \rho^2 \sin \theta. 
\]  
(36)

In order to make numerical predictions for the LT precession frequency in a strong gravity domain, we need to transform the precession frequency formula from the coordinate basis to the orthonormal ‘Copernican’ basis: first note that
\[
\Omega_{\text{LT}} = \Omega^\theta \partial_\theta + \Omega^r \partial_r 
\]  
(37)
\[
\Omega_{\text{LT}}^2 = g_{rr}(\Omega^r)^2 + g_{\theta \theta}(\Omega^\theta)^2. 
\]  
(38)

Next, in the orthonormal ‘Copernican’ basis at rest in the rotating spacetime, the tetrad vector \(e_0 = u\), the tangent vector along the integral curve of the timelike Killing vector \(K\). In this basis, with our choice of polar coordinates, \(\Omega_{\text{LT}}\) can be written as
\[
\tilde{\Omega}_{\text{LT}} = \sqrt{g_{rr}} \hat{r} + \sqrt{g_{\theta \theta}} \hat{\theta} 
\]  
(39)

where, \(\hat{r}\) is the unit vector along the direction \(r\). For the Kerr metric,
\[
\Omega^\theta = -aM \sin \theta \frac{\rho^2 - 2r^2}{\rho^4(\rho^2 - 2Mr)} 
\]  
(40)
\[
\Omega^r = 2aM \cos \theta \frac{r\Delta}{\rho^4(\rho^2 - 2Mr)}. 
\]  
(41)

Substituting the values of \(\Omega^r\) and \(\Omega^\theta\) in equation (39), we get the following expression of LT precession rate in Kerr spacetime
\[
\tilde{\Omega}_{\text{LT}}^2 = 2aM \cos \theta \frac{\rho^4(\rho^2 - 2Mr)}{r^\Delta \rho^3(\rho^2 - 2Mr)} \hat{r} - aM \sin \theta \rho^2 - 2r^2 \hat{\theta}. 
\]  
(42)

The magnitude of this vector is
\[
\Omega_{\text{LT}}(r, \theta) = \frac{aM}{\rho^3(\rho^2 - 2Mr)} [4\Delta r^2 \cos^2 \theta + (\rho^2 - 2r^2)^2 \sin^2 \theta]^\frac{1}{2}. 
\]  
(43)

This is the LT precession rate where no weak gravity approximation has been made. It should therefore be applicable to any rotating spacetime like rotating black hole etc.

In the weak-field limit \((r \gg M)\), equation (42) reduces to
\[
\tilde{\Omega}_{\text{LT}}(r, \theta) = \frac{J}{r^3} [2 \cos \theta \hat{r} + \sin \theta \hat{\theta}] 
\]  
(44)

where, \(\theta\) is the colatitude. The resemblance of this equation with equation (1) is unmistakeable.
We can visualize the difference between strong and weak gravity LT precession through a graphical representation. In figure 2, we draw two graphs, the red one is for $\Omega_{\text{weakLT}} = \frac{2aM}{r^3}$ and the green is for $\Omega_{\text{strongLT}} = \frac{2aMr}{(r^2+a^2)^{3/2} \sqrt{r^2-2Mr+a^2}}$ at $\theta = 0$. We see that $\Omega_{\text{strongLT}}$ is much greater than $\Omega_{\text{weakLT}}$ for small $r$, i.e., near the compact body. As $r$ increases, the red and the green lines overlap with each other, i.e., the weak gravity approximation emerges as a reasonable approximation.

In a numerical comparison between calculated values of $\Omega_{\text{strongLT}}$ and $\Omega_{\text{weakLT}}$ for typical compact objects, with those of the Sun and the Earth, the effect of strong gravity is seen to exceed by 30% the weak-gravity precession rate in general for all compact objects, while being roughly of the same order for the weak-gravity sources. This strongly motivates deeper observational probes of strong gravity LT precession of compact objects.

5. Summary and discussion

The analyses presented above has two important features: (a) the Lense–Thirring (LT) precision frequency of a gyroscope in a ‘Copernican’ frame within a Kerr spacetime is computed without any assumption on the angular momentum parameter or indeed the curvature of spacetime. The only comparable attempt in the literature is that in [28], which however is not the same computation as ours, and the result is not the same in terms of metric coefficients. (b) The result derived in equation (17) is in fact valid, not just for axisymmetric spacetimes, but also for general non-static stationary spacetimes, once again without any assumptions about the curvatures involved. This result, we believe is applicable to a very large class of strong gravity systems.
While most textbook calculations of the LT precession in the weak field approximation, the book of Misner et al [28] must also be mentioned. Here, the orbital angular velocity for locally non-rotating observers in a Kerr–Newman spacetime is given in equation (33.24) as an exercise. This formula does not appear to be restricted to the weak-field approximation. However, from an astrophysical standpoint, it is not clear that the computed angular frequency corresponds to what might be measured as the LT precession in a strong gravity situation, because it has been derived in a locally non-rotating frame which the authors amply clarify is not a Copernican frame. A naïve limiting procedure does not appear to reduce this frequency to the standard weak-field result (1) in ‘Copernican’ frames quoted in most other textbooks for the LT precession rate in a weak gravitational field.

The substantial difference between the LT precession frequency, arising in strong gravity regime and the standard, weak field precession rate for inertial frame dragging ought to provide a strong motivation for their measurement in space probes planned for the near future. The fascinating world of gravitational effects associated with strongly gravitating compact objects may provide the best yet dynamical observational signatures of general relativity. In this paper, the focus has not been on understanding the effect of strong gravity LT precession on emission mechanism of pulsars and x-ray emission from black holes and neutron stars. We expect nontrivial modifications to arise from incorporation of frame-dragging effects in the theoretical analyses of these phenomena. We hope to report on this in the near future. There are other additional avenues of further work currently being explored: the most general axisymmetric solution of Einstein’s equation given by the Plebański–Demiański metric has been investigated for an understanding of the LT precession in this case [29].

Acknowledgments

We thank S Bhattacharjee, R Nandi, A Majhi, P Pradhan and especially D Bandyopadhyay for illuminating and helpful discussions, and also for guiding us through the literature on pulsar data. One of us (PM) thanks S Bhattacharya for discussions regarding observability of the LT precession in the strong gravity regime. We also thank L Iorio, C Lämmerzahl and N Stone for useful correspondence regarding this paper and for bringing their work relevant to the issues discussed here, to our attention. Last but not the least, one of us (CC) thanks V Kagramanova and J. Kunz for gracious hospitality during an academic visit and for invaluable discussions regarding the subject of this paper. CC is also grateful to the Department of Atomic Energy (DAE, Govt of India) for the financial assistance.

Note added. As suggested by an anonymous referee, incorporating multipole moment corrections (as example, for multipole corrections to the Schwarzschild metric see [30]) to the Kerr metric to determine the strong gravity LT precession rate near neutron star surfaces is a very interesting project on which we hope to report elsewhere.

References

[1] Schiff L I 1960 Am. J. Phys. 28 340
[2] Lense J and Thirring H 1918 Phys. Z. 19 156–63
[3] Hartle J B 2009 Gravity: An Introduction to Einstein’s General Relativity (Delhi: Pearson)
[4] Iorio L 2011 Astrophys. Space Sci. 331 351–95
[5] Kannan R and Saha P 2009 Astrophys. J. 690 1553
[6] Hackmann E and Lämmerzahl C 2012 Phys. Rev. D 85 044049
[7] Kranotis G V 2004 Class. Quantum Grav. 21 4743–69
[8] Kranotis G V 2007 Class. Quantum Grav. 24 1775–808
[9] Ciufolini I and Pavlis E C 2004 Nature 431 958
[10] F Everitt C W et al 2011 Phys. Rev. Lett. 106 221101
[11] Bhattacharya S 2012 Lecture at Conf. on Advances in Astroparticle Physics and Cosmology (March, Darjeeling)
[12] Stella L and Possenti A 2009 Space Sci. Rev. 148 105
[13] Stone N and Loeb A 2012 Phys. Rev. Lett. 108 061302
[14] Straumann N 2009 General Relativity with Applications to Astrophysics (Berlin: Springer)
[15] Lightman A P, Press W H, Price R H and Teukolsky S A 1979 Problem book in Relativity and Gravitation (Princeton, NJ: Princeton university press)
[16] Taub A H 1951 Ann. Math. 53 3
[17] Newman E, Tamburino L and Unti T 1963 J. Math. Phys. 4 7
[18] Ramaswamy S and Sen A 1981 J. Math. Phys. 22 2612
[19] Miller J G 1973 J. Math. Phys. 14 486
[20] Lynden-Bell D and Nouri-Zonoz M 1996 arXiv:9612049v1 [gr-qc]
[21] Misner C W 1963 J. Math. Phys. 4 924
[22] Ramaswamy S and Sen A 1986 Phys. Rev. Lett. 57 8
[23] Miller J G, Kruskal M D and Godfrey B B 1971 Phys. Rev. D 4 2945
[24] Kagramanova V, Kunz J, Hackmann E and Lämmerzahl C 2010 Phys. Rev. D 81 124044
[25] Drasco S and Hughes S A 2004 Phys. Rev. D 69 044015
[26] Fujita R and Hikida W 2009 Class. Quantum Grav. 26 135002
[27] Kagramanova V 2013 private communication
[28] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco: Freeman)
[29] Chakraborty C and Pradhan P 2013 Eur. Phys. J. C 73 2536
[30] Glendenning N K and Weber F 1994 Phys. Rev. D 50 3836