LETTER

Reconciling edge states with compressible stripes in a ballistic mesoscopic conductor

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Abstract

The well-known Landauer–Buttiker (LB) picture used to explain the quantum Hall effect uses the concept of (chiral) edge states that carry the current. In their seminal 1992 article, Chklovskii, Shklovskii and Glazman (CSG) showed that the LB picture does not account for some very basic properties of the gas, such as its density profile, as it lacks a proper treatment of the electrostatic energy. They showed that, instead, one should consider alternated stripes of compressible and incompressible phases. In this letter, we revisit this issue using a full solution of the quantum-electrostatic problem of a narrow ballistic conductor, beyond the CSG approach. We recover the LB channels at low field and the CSG compressible/incompressible stripes at high field. Our results have important implications for the propagation of edge magneto-plasmons.

Electrostatic energy is very often the largest energy scale in a physical situation. Yet, the electrostatic landscape is equally often taken for granted as an external potential, which may result in a wrong physical picture. A well known example is the quantum Hall effect [1] (QHE) that has been largely discussed using the concept of edge states in a non-interacting Landauer–Buttiker (LB) picture [2, 3]. Despite being very successful for the understanding of e.g. the quantification of the Hall resistance and the vanishing longitudinal resistance, the LB picture also fails spectacularly to describe basic physics such as the density profile of the electron gas. In a series of articles [4–6] that culminated with the work of Chkhlovskii, Shkhlovskii and Glazman (CSG) [7–9], it was recognized that the LB picture should be revisited. It was shown that the interplay between quantum mechanics and electrostatics leads to the emergence of compressible and incompressible stripes, a concept related, yet somehow different, to the original edge states. An important effort has been devoted to the experimental observation of these stripes [10–20]. CSG work, as well as a large fraction of the subsequent literature [21–30] was based on the Thomas–Fermi approximation which is suitable at high magnetic field but inadequate at low field where the LB approach is expected to work well. The self-consistent problem can also be tackled directly in presence of a finite temperature [31]. More recent works improved on Thomas–Fermi by incorporating a Gaussian broadening of the Landau levels [32–34]. Solving the full self-consistent electrostatic-quantum problem, beyond the above approximations and at low temperature, is a difficult task however, as the presence of the Landau levels (and the associated Dirac comb for the density of states) makes the set of equations highly nonlinear. In this letter, we use a newly developed numerical technique capable of handling this problem [35] and explore how the LB channels present at low field evolve into CSG compressible stripes at high magnetic field. Using the solution of the full self-consistent problem, we find that in a large region of the parameter space the system is in an ‘hybrid’ phase that borrows features from both the LB and CSG pictures.
The LB picture of the QHE regime (and its failure)

We consider the infinite wire geometry of figure 1(a): a two-dimensional electron gas at the interface between GaAs and GaAlAs is placed under a perpendicular magnetic field $B$ and confined to quasi-1D through two gates situated a few tens of nm above the gas. In the effective mass approximation, the electronic wave function $\psi(x, y)$ is described by a simple Schrödinger equation

$$\frac{1}{2m^*}(\hbar \nabla^2 - e\vec{A})\psi(x, y) - eU(x)\psi(x, y) = E\psi(x, y),$$

where the vector potential takes the form $\vec{A} = Bx\hat{y}$ (Landau gauge) and $U(x)$ is the electrostatic potential ($m^*$: effective mass, $e$ electron charge). In the LB picture, the function $U(x)$ is an external input of the problem. The general solution of equation (1) takes the form of plane waves $\psi_n(x, y) = \psi_n(x) e^{ik_y}$ along the $y$-direction with momentum $k_y$. In the absence of $U(x)$, these states are simply the Landau levels [36]: equally spaced highly degenerate dispersively levels $E_n(k_y) = \hbar \omega_c (n + 1/2)$ with states $\psi_n(x)$ that are exponentially localized along the $x$-direction around $x_n = n \hbar \omega_c / B$ (where $\omega_c = eB/m^*$: cyclotron frequency). Applying an external confining potential $U(x)$ around the edges of the sample (such as the one shown in the inset of figure 1(c)) provides the usual LB picture for the edge states of the QHE. For a slowly varying $U(x)$ (Thomas–Fermi approximation), the spectrum becomes dispersive and follows the potential

$$E_n(k_y) \approx \hbar \omega_c (n + 1/2) - eU(x_n)$$

$E_n(k_y)$ crosses the Fermi energy ($E_F = 0$) and provides conducting LB channels. The localized channels are the edge states of the system. The corresponding band structure is shown in figure 1(b). These edge states are localized around $x_n$ and chiral (the velocity $v_k = (1/\hbar) dE/dk$ is positive for the edge states on the right of the sample and negative on the other edge). The associated velocity is directly linked to the confining potential $v_k \approx (\hbar/2m^*) dU(x)/dx$.

The main problem with the LB picture can be seen in figure 1(c) where the associated electronic density profile $n(x)$ is shown. In the bulk of the system there are exactly two filled Landau levels, hence the electronic density is given by their corresponding degeneracy $n(x) = 2/(2\pi\hbar^2) = 2eB/h$. As one moves towards the edge of the sample, one reaches the point where the second Landau level is not filled any more and the density drops to $n(x) = 1/(2\pi\hbar^2)$ and eventually to $n(x) = 0$. In this picture, the density is essentially set by the magnetic field $B$. However, the typical energy associated with the field $\hbar \omega_c$ is of the order of 10 meV which is several order of magnitude smaller than the electrostatic energy that would be needed to deform whatever electronic density $n(x, B = 0)$ was there at $B = 0$ into the one of figure 1(c). One concludes that the potential $U(x)$ should not be considered as an external input but rather as the solution of the Poisson equation.

![Figure 1](image_url)

**Figure 1.** Non-interacting picture of QHE. (a) side view of the system (infinite along the $y$ direction). (b) Dispersion relation $E_n(x = k_B^2)$ for the three lowest Landau levels. (c) Density profile $n(x)$ (thin line) with Fermi level $E_F = 0$ and $B = 1$ T. The results of (b) and (c) have been calculated using a direct numerical solution of equation (1) with the external potential $U(x)$ shown in the inset of (c).
\[ \Delta U(\vec{r}) = -\frac{e}{\epsilon} n^3(\vec{r}), \]
\[ \frac{dU}{dz}(z = 0^+) = \frac{dU}{dz}(z = 0^-) = -\frac{e}{\epsilon} n(x), \]

where \( n^3(\vec{r}) \) is the (3D) dopant density and \( n(x) \) the (2D) density of the electronic gas. In our wire geometry, the electronic density is given by
\[ n(x) = \sum_p \int \frac{dk}{2\pi} |\psi_{nk}(x)|^2 f(E(k)) \]

which closes our system of equation (\( f[E] = 1/(e^{E/k_BT} + 1) \) is the Fermi function). In the QHE regime, equations (1), (3), (4) and (5) form a highly nonlinear set of equations that we solve numerically. We refer to [35] for details of the calculations. A more precise model would replace the 2D electronic density \( n(x) \) by a 3D density \( n(x, z) \propto n(x) e^{-z^2/(\lambda^2)} \) (with \( \lambda \approx 10 \text{ nm} \)) to account for the finite transverse width of the 2DEG. We have found that the results are almost insensitive to the precise value of \( \lambda \) or the transversal shape. Note that the self-consistent electrostatic quantum problem neglects the correlation energy which is of the order of 10 \( \mu \text{eV} \) [37] and is responsible for e.g. the fractional QHE effect [38] at very low temperature.

**The CSG picture of compressible and incompressible stripes**

The strength of CSG argument is that the entire physical picture can be constructed from simple considerations, essentially from the dispersion relation equation (2) that relates the energy of the Landau level \( E_n(k) \) to its spatial position \( x_n \). Let us suppose that we know the density profile \( n(x, B = 0) \) in the absence of magnetic field. We further assume that this profile (which results from the interplay between the electrostatic and kinetic energy) is only weakly affected by the presence of the magnetic field (as argued above the cyclotron energy \( h\omega_c \) is small compared to the electrostatic energy needed to strongly modify the density profile). For a generic value of the field, it follows that the filling fraction \( \nu \equiv n_0 2\pi l_B^2 \) of the Landau levels in the bulk of the sample \( (n_0 = n(x, B = 0) \) has generically a non-integer value e.g. \( \nu = 2.4 \). From that statement, it follows that a Landau level (in this example the third one) lies exactly at the Fermi energy since it is only partially occupied. This is a very different situation from the non-interacting picture discussed above. As one gets away from the center of the sample towards the edge, the filling factor \( \nu(x) \equiv n(x, B = 0)/2\pi l_B^2 \) decreases until the third Landau level is totally depleted and one starts to deplete the second Landau level. Depleting the second Landau level implies that it sits at the Fermi energy, hence a sharp rise of the electrostatic energy (of amplitude \( h\omega_c \)). In the small region where this sharp rise occurs, the density is constant (no available level at the Fermi energy). This region is an incompressible stripe. Continuing towards the edge, we hence obtain a set of compressible stripes separated by incompressible stripes. The blue lines of figure 2 show the resulting dispersion relation (lower panels) and density profile (upper panels) for three different value of the magnetic field. The blue lines of figure 2 resemble very strongly the cartoon shown in figure 1 of the CSG paper. However, they correspond to a full self-consistent calculation in the Thomas–Fermi approximation.

**Hybrid phase at intermediate fields**

We now turn to the full self-consistent solution of the problem equations (1), (3), (4) and (5) without using the Thomas–Fermi approximation. Note that in what follows, the Landau levels are supposed to be fully spin polarized, we do not discuss the magnetic instabilities. The results are shown with orange lines in figure 2. At high field (middle and right panels), the full solution bears strong similarities with the Thomas–Fermi result and one gets alternated compressible and incompressible stripes with the middle of the sample being compressible (middle panels) or incompressible (right panel, corresponds to a quantized plateau of conductance) [9]. One qualitative difference is the absence of well defined plateaus of the density in the incompressible region. This is due to the fact that the Landau levels spread over a width \( l_B \) which is not infinitely small compared to the width of the incompressible stripes (e.g. \( l_B \approx 13 \text{ nm} \) at \( B = 3.73 \text{ T} \)). At very low magnetic fields (not shown), one fully recover the LB picture with well defined propagating channels that cross the Fermi level. We find that the transition between the LB picture and the CSG one at high field happens in two stages: first, the formation of Landau levels that get pinned at the Fermi level; secondly the evolution of the edge states into compressible stripes. In the corresponding intermediate field range, the system is in an intermediate ‘hybrid’ phase with well defined edge states (similar to those shown in figure 1) yet with a central compressible stripe that remains pinned at the Fermi level. This situation is illustrated in the left panel of figure 2 (\( B = 2.2 \text{ T} \), corresponding to \( \nu \approx 4.5 \)). We define the ‘hybrid’ phase as the regime where the LB picture for the edge states is fully applicable yet the bulk of the system is described by the CSG scenario.
Magneto-conductance of ballistic wires

To gain further insight, we now turn to a discussion of the current that flows upon applying a small bias voltage across the wire. We focus on the out-of-equilibrium current: the additional current that is induced by the small bias voltage. There is also an equilibrium current density that flows inside the incompressible regions and that results in a zero net total current. Note that the question of where does the out-of-equilibrium current flow is ill-defined in the Thomas–Fermi approximation, at least at very small temperature. Indeed the current must flow in the compressible regions since transport requires available states at the Fermi level. Yet in the compressible regions, the velocity \( v_\text{f} = \frac{1}{\hbar} \frac{dE}{dk} \) vanishes, hence one would expect the current to do the same. This small paradox (which points to the current being concentrated to the edge of the compressible stripes) is resolved by using the full self-consistent solution. In a perfectly ballistic system where all conducting channels are perfectly transmitted, the two-terminals conductance \( g \) is given by a simple form of the Landauer formula

\[
g \sim e^2/h \nu (B)
\]

where the heaviside function \( \theta(x) \) selects the channels with positive velocities. At zero temperature, this formula provides the well-known quantization of conductance: using \( \nu_k = (1/h) dE/dk \) and the fact that \( d\nu / dE \to -\nu(E) \) one finds that \( g \) simply counts the number of bands that cross the Fermi level (in unit of \( e^2/h \)). The CSG situation where there is a degenerate band exactly at the Fermi level is a new situation as it can lead to non-quantized conductance even in a ballistic conductor. Indeed, assuming that \( E_n(k) \) is a (very slowly) increasing function of \( k \) from \( k = 0 \) to \( k = \infty \), one gets \( g = (e^2/h) \sum_n \int f [E(k = 0)] - f [E(k = \infty)] \) which translates into \( g = (e^2/h) \nu \). Since the central stripe is in general not fully filled, this results in a non quantized conductance that scales as \( \sim 1/B \) except for the plateaus occuring when the central region is incompressible. This situation corresponds to a regime where the temperature is very small compared to the cyclotron frequency, yet large compared to the small variations of the electric potential in the central Landau level. It is in sharp contrast with the zero temperature limit where quantization is always expected.

This is illustrated in the inset of figure 3(a) where we plot the conductance versus temperature in the hybrid phase: we identify three regimes: a regime of large temperature (with respect to \( \hbar \omega_c \)) and accordingly large conductance; a regime of low temperature where one recovers the LB quantization; and an intermediate regime without quantization \( g = (e^2/h)\nu(B) \). The crossover between the later two is strongly system dependent: it depends on the small curvature of the electric field in the bulk of the sample, hence on the strength of the gate.
One of the key findings is that the QHE appears in two stages, leading to the hybrid phase described above. A rough estimate of the parametric regime where this hybrid phase is expected goes as follows. The typical width \( w \) of a compressible stripe is set by the density profile \( n(x, B = 0) \). If \( x_0 \) defines the position of an integer filling \( \nu = n(x_0, B = 0) = 2n_0 \), then \( n(x_0 + w, B = 0) = 2n_0 = \nu - 1 \). Noting \( d \) the typical distance with which \( n(x_0, B = 0) \) falls from its bulk value to zero, we get \( \nu \sim (d/\nu) \). \( d \) depends strongly on the electrostatics of the problem; it is of the order of the distance between the gate and the gas, \( d \sim 140 \text{ nm} \) in our example. The crossover from the compressible stripe behavior to the LB like edge state is expected when quantum fluctuations are large enough, i.e. \( w \sim l_\text{B} \). This translates into \( \nu_0 \sim (n_0d^2)^{1/3} \). Hence, for filling fraction larger than \( \nu_0 \sim (n_0d^2)^{1/3} \) one expects LB like edge states while for higher magnetic field, one recovers the CSG stripes. In our (rather typical) example, the hybrid scenario corresponds to most Hall plateaus \( \nu > (n_0d^2)^{1/3} \approx 3 \).

As the LB edge states and CSG stripes are both associated with one quantum of conductance, it remains to discuss the difference between the two situations in actual observables. We expect an important difference in the propagation of voltage pulses \[ \text{[citation]} \] (edge magneto-plasmons). Indeed, the plasmon velocity is proportional to the Fermi velocity (up to a renormalization factor due to Coulomb interaction [40, 41]) of the corresponding mode. For a LB edge state, this velocity \( v_\text{pl} = (1/h)E/dk \) is a well defined quantity that depends on the confining potential i.e. \( v_\text{pl} \sim h/(dm^2) \). We find typical values \( v_\text{pl} \sim 10^7 \text{ m/s} \) consistent with the values quoted in the literature. The situation is drastically different for a CSG compressible stripe where the velocity vanishes in the middle of the stripe and sharply rises on its boundaries. If follows that the average velocity in the stripe drops down upon entering the CSG regime. Perhaps more importantly, the velocity now strongly depends on \( k \) which results in an important spreading of the excitation between the slow part in the middle of the stripe and the faster part toward its edges. Hence, we expect that at high field, voltage pulses will get highly distorted in sharp contrast with the behavior in the hybrid phase at lower field. We anticipate that the hybrid phase is more favorable for the propagation of pulses than the CSG phase.

**Figure 3.** (a) Conductance \( g(B) \) as a function of the magnetic field for a confinement \( V_g = -1.6 \text{ V} \) at \( T = 1 \text{ K} \) (thin blue line). The dotted line indicates the \( g = n_0 e/B \) law, horizontal lines indicate quantized values of the conductance while the vertical lines the expected position of the Hall plateaus from the bulk density \( n_0 = n_0 h/(en) \). Inset: conductance \( g(T) \) as a function of the temperature (log scale) for a confinement \( V_g = -0.75 \text{ V} \). The vertical line indicates \( k_B T = \hbar \nu_0 \). (b) Same as (a) for \( V_g = -0.75 \text{ V} \). The round, diamond and triangle symbols mark, respectively, the conductance at \( B = 2.2, 3.73 \) and 4.8 T of figure 2. Inset: zoom of the main panel at small field. \( T = 0 \text{ K} \).
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