A VERSION OF HÖRMANDER’S THEOREM FOR
MARKOVIAN ROUGH PATHS

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Abstract

We consider a rough differential equation of the form
\[ dY_t = \sum_{i=1}^{d} V_i(Y_t) dX^i_t + V_0(Y_t) dt, \]
where \( X_t \) is a Markovian rough path. We demonstrate that if the vector fields \((V_i)_{0 \leq i \leq d}\) satisfy the parabolic Hörmander’s condition, then \( Y_t \) admits a smooth density with a Gaussian type upper bound, given that the generator of \( X_t \) satisfy certain non-degenerate conditions. The main new ingredient of this paper is the study of a non-degenerate property of the Jacobian process of \( X_t \).

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1 Introduction

We consider rough differential equations in \( \mathbb{R}^d \) of the form
\[ dY_t = \sum_{i=1}^{d} V_i(Y_t) dX^i_t + V_0(Y_t) dt, \]
\[ Y_0 = y_0 \in \mathbb{R}^d, \ t \in [0,1]. \] (1)

Over the past decade, rough differential equations driven by Gaussian processes (i.e., \( X_t \) is a Gaussian rough path) have been extensively studied. The existence and smoothness of the density of \( Y_t \) is among the most important questions and have attracted a lot attentions.
The case where $X_t$ is given by the Stratonovich Brownian rough path is equivalent to the classical probabilistic Hörmander’s theorem studied by P. Malliavin [16]. In this case, it is possible to prove that $Y_t$ admits a smooth density is equivalent to the hypoellipticity of the differential operator given by

$$L = \frac{1}{2} \sum_{i=1}^{d} V_i^2 + V_0.$$ 

L. Hörmander [13] was the first to formulate a sufficient condition on $\{V_i\}_{0 \leq i \leq d}$ to ensure the hypoellipticity of $L$, known today as the parabolic Hörmander’s condition. The parabolic Hörmander’s condition, which we introduce now, has become a fundamental setting in many areas including probability, geometry and PDE.

**Definition 1.1.** Let $\{V_i\}_{0 \leq i \leq d}$ be a collection of smooth vector fields on $\mathbb{R}^d$. Define

$$W^0 = \{V_1, \ldots, V_d\},$$

$$W^{k+1} = \bigcup_{0 \leq i \leq d} \{[V, V_i] : V \in W^k\}, \quad k \in \mathbb{Z}^+.$$ 

We say $\{V_i\}_{0 \leq i \leq d}$ satisfy the parabolic Hörmander’s condition if we can find an integer $k_0 \geq 0$ such that for every $x \in \mathbb{R}^d$

$$\text{Span} \left\{ \bigcup_{0 \leq k \leq k_0} W^k(x) \right\} = \mathbb{R}^d.$$ 

With the previous definition, the probabilistic Hörmander’s theorem by P. Malliavin [16] can be stated as follows.

**Theorem 1.2.** Let $W_t$ be a standard Brownian motion on $\mathbb{R}^d$ and consider

$$dY_t = \sum_{i=1}^{d} V_i(Y_t) \circ dW_t^i + V_0(Y_t)dt, \quad Y_0 = y_0 \in \mathbb{R}^d, \quad t \in [0, 1].$$

Assume that $\{V_i\}_{0 \leq i \leq d}$ are smooth vector fields with bounded derivatives of all orders. If $\{V_i\}_{0 \leq i \leq d}$ satisfy the parabolic Hörmander’s condition, then for all $t \in (0, 1]$, $Y_t$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$.

**Remark 1.3.** Theorem 1.2 is still true when the Stratonovich integral is replaced by Itô’s integral in the equation.

Under the same assumptions as theorem 1.2 with $W_t$ replaced by a fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$, F. Baudoin and M. Hairer [2] proved that $Y_t$ admits a smooth density. With recent developments on Gaussian rough paths, we can consider [1] with $X_t$ given by a general non-degenerate Gaussian rough path. It was proved in [3], that the $Y_t$ admits a density if $\{V_i\}_{0 \leq i \leq d}$ satisfy the parabolic Hörmander’s condition. Smoothness of the density was proved later in [4] after the tail estimate for the associated Jacobian process and a deterministic Norris’s lemma were established (see [5], [12]).

The major tool that all above-mentioned results heavily rely on is the Malliavin calculus, which is a successful application of differential measure theory to Radon Gaussian measures. Despite being sufficiently flexible to obtain related results for a number of extensions of the original problem, Malliavin calculus cannot be applied to problems without an underlying
Gaussian structure. As a result, contrary to the rapid developments of its Gaussian counterpart, the study of rough differential equations driven by Markovian rough paths progresses rather slowly. In [6], I. Chevyrev and M. Ogrodnik used analysis on manifolds to prove that $Y_t$ admits a density with respect to any smooth measure with assumptions strictly stronger than parabolic Hörmander’s condition on the vector fields. The question of the smoothness of the density is still open.

In this paper, we consider (1) with $X_t$ given by a Markovian rough path. Our goal is to prove that, with the parabolic Hörmander’s assumption on the vector fields, $Y_t$ admits a smooth density with a Gaussian type upper bound. Unlike Gaussian processes, a general Markov process $X_t$ may not have a smooth density. Thus, certain regularity assumption is necessary for the coefficients of its generator. We will see very soon that this extra regularity assumption provides an underlying Gaussian structure, which, in turn, makes Malliavin calculus applicable.

Now we can introduce our basic settings. We fix two constants $0 < \lambda < \Lambda$. Let $a(x)$ be a non-constant measurable function from $\mathbb{R}^d$ to the space of symmetric matrices which are uniformly elliptic with respect to $\lambda$ and bounded by $\Lambda$, i.e.,

$$\lambda|\xi|^2 \leq \langle \xi, a(x)\xi \rangle \leq \Lambda|\xi|^2$$

for any $\xi \in \mathbb{R}^d$ and almost every $x \in \mathbb{R}^d$, where $|\xi|$ is the Euclidean norm. We use $\Xi^{\lambda,\Lambda}$ to denote all the functions that satisfy (2). Define the associated differential operator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \left( a_{i,j} \partial_j \right)$$

with domain $\text{Dom}(L) = \{ f \in H^2(\mathbb{R}^d) \mid Lf \in L^2 \}$. Let $\{X_t\}_{t \geq 0}$ be the Markov process generated by $L$ and define $Y_t$ as the solution to

$$dY_t = \sum_{i=1}^{d} V_i(Y_t) dX^i_t + V_0(Y_t) dt, \quad Y_0 = y_0 \in \mathbb{R}^d, \quad t \in [0,1].$$

We can now state our assumptions. First two are standard.

**Assumption 1.4.** The vector fields $\{V_i\}_{0 \leq i \leq d}$ are smooth and bounded together with all their derivatives.

**Assumption 1.5.** The vector fields $\{V_i\}_{0 \leq i \leq d}$ satisfy the parabolic Hörmander’s condition.

Our next assumption is about the regularity of $a(x)$ that we mentioned earlier.

**Assumption 1.6.** The function $a(x)$ is smooth and bounded together with all their derivatives.

The smoothness assumption on $a(x)$, though seems a bit restrictive at the first look, is actually necessary. Indeed, we can let $\{V_i\}_{0 \leq i \leq d}$ be the columns of the identity matrix on $\mathbb{R}^d$ and let $V_0 = 0$. Obviously $\{V_i\}_{0 \leq i \leq d}$ satisfy Assumptions [1.4] and [1.5]. If we aim for a Hörmander’s type theorem we must have that $Y_t = X_t$ has a smooth density. On the other hand, we know from classical analysis that, to obtain $k$th-order derivative on the density of $X_t$, one needs $a(x)$ to be at least $(k-1)$th-order continuously differentiable. Consequently, the smoothness of $a(x)$ is a necessary.
It is well-known that if $a(x)$ is smooth, $X_t$ is a diffusion process, whose stochastic differential equation can be written as follows:

$$dX_t = \sum_{i=1}^{d} A_i(X_t) dW_t^i + B(X_t) dt,$$

where $A = \sqrt{a}$ and $B = \nabla \cdot a$.

Central to all previous cases is the non-degeneracy of the driving signal $X_t$. It is not surprising that $A(x)$, which itself is an elliptic system, guarantees that $X_t$ is non-degenerate in certain sense. However, we will see later that, exclusive to the non-Gaussian case, the non-degeneracy of the Malliavin derivative of $X_t$ is also needed (see remark 3.5). This motivates our final assumption. We want to emphasize that the next assumption is not necessary if $X_t$ generated by (3) is a Gaussian process (e.g., when $a(x)$ is constant). Hence, it constitutes the biggest difference between non-Gaussian case and Gaussian case.

**Assumption 1.7.** Let $A(x)$ be the unique square root of $a(x)$. For a fixed positive constant $C_J$, the differential of $A(x)$, which is a smooth map $dA(x) : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, does not vanish and satisfy the following uniform non-degenerate condition

$$\left| v^T \cdot dA(x) \right|^2 \geq C_J |v|^2, \forall x, v \in \mathbb{R}^d.$$

The main result of this paper is the following:

**Theorem 1.8.** Assume $a \in \Xi^{\lambda, \Lambda}$, let $X_t$ be the Markov process whose generator is given by (3) with canonical rough lift $X_t$. Consider the rough differential equation

$$Y_t = y_0 + \sum_{i=1}^{d} \int_0^t V_i(Y_s) dX_s^i + \int_0^t V_0(Y_s) ds, \ y_0 \in \mathbb{R}^d, \ t \in [0, 1].$$

Suppose that $a(x)$ is not constant and **Assumptions 1.4, 1.5, 1.6 and 1.7** are satisfied. Then for any $t \in (0, 1]$, $Y_t$ has a smooth density $p_{Y_t}(y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, $p_{Y_t}(y)$ has the following Gaussian type upper bound,

$$p_{Y_t}(y) \leq C_1(t) \exp \left( -\frac{C_2(y - y_0)^2}{t} \right).$$

The fundamental argument for our proof is again the classical Malliavin calculus adapted to the rough paths theory, which involves the study of the Malliavin derivative of $Y_t$ and a small ball estimate for $X_t$. The main difficulty in the our case is the extra integral structure in the Malliavin derivative of $Y_t$, which never appeared in any previous cases as explained in remark 3.4. We will tackle this by developing a non-degenerate property of the Jacobian process of $X_t$.

There is another important point that we would like to emphasis. We shall see later that the majority of our proofs can be completely carried out in the language of classical stochastic calculus. This is due to the fact that $X_t$ and its Malliavin derivative are both diffusion and rough integrals against diffusion coincide with the corresponding Stratonovich integrals. We will frequently take advantage of this fact in our proofs. From this perspective, our results can be viewed as an application of stochastic calculus to a rough path problem. However, we will continue to present our results in the rough paths setting. The
reason is twofold. On one hand, formula \([\Box]\), which is crucial for future studies in more
general settings, can only be generalized as rough integrals. On the other hand, results
like proposition \([\Box, \Box]\) are, in fact, deterministic and point-wise, which are stronger than their
stochastic counterparts. We prefer to keep them in this stronger form.

2 Preliminary material

2.1 Rough paths

For \(\alpha \in (\frac{1}{3}, \frac{1}{2}]\), we define the space of \(\alpha\)-Hölder rough path on \(\mathbb{R}^d\), in symbols \(\mathcal{C}^\alpha([0,1], \mathbb{R}^d)\),
as those pairs \((X, X) =: X \in C([0,1], \mathbb{R}^d \otimes (\mathbb{R}^d)^{\otimes 2})\) such that
\[
\|X\|_\alpha = \sup_{s \neq t \in [0,1]} \frac{\|X_{s,t}\|}{|t - s|^\alpha} < +\infty, \quad \|X\|_{2\alpha} = \sup_{s \neq t \in [0,1]} \frac{\|X_{s,t}\|}{|t - s|^{2\alpha}} < +\infty,
\]
where \(X_{s,t} = X_t - X_s\) and similarly for \(X_{s,t}\). Moreover, for \(0 \leq s < u \leq t \leq 1\),
\[
X_{s,t} = X_{s,u} + X_{u,t} + X_{s,u} \otimes X_{u,t}.
\]
We equip \(\mathcal{C}^\alpha([0,1], \mathbb{R}^d)\) with the rough metric
\[
\rho_\alpha(X, Y) = \sup_{s \neq t \in [0,1]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^\alpha} + \sup_{s \neq t \in [0,1]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^{2\alpha}},
\]
and define \(\rho_\alpha(X) := \rho_\alpha(X, 0)\). Note that if \(X \in BV([0,1]; \mathbb{R}^d)\), \(X\) can be canonically
defined as
\[
X_{s,t} = \int_s^t X_{s,r} dX_r,
\]
where the integral is understood as Riemann–Stieltjes integral and \((X, X) \in \mathcal{C}^\alpha([0,1], \mathbb{R}^d)\)
for any \(\alpha \in (0,1]\). A rough path \(X \in \mathcal{C}^\alpha([0,1], \mathbb{R}^d)\) is said to be a geometric \(\alpha\)-Hölder
rough path if we can find a sequence \(\{X^k\}_{k \geq 1} \in BV([0,1]; \mathbb{R}^d)\) such that
\[
\lim_{k \to \infty} \rho_\alpha(X, X^k) \to 0. \tag{4}
\]

Another important notion is the so called controlled rough path. Let \(\alpha \in (\frac{1}{3}, \frac{1}{2}]\) and
\(X_t \in C^\alpha([0,1], \mathbb{R}^d)\) be an \(\alpha\)-Hölder continuous path. For a Banach space \(W\), we say \(Y_t \in C^\alpha([0,1], W)\), is controlled by \(X_t\) if we can find \(Y'_t \in C^\alpha([0,1], L(\mathbb{R}^d, W))\) such that the
remainder term \(R^Y\) given by
\[
Y_{s,t} = Y'_t X_{s,t} + R^Y_{s,t},
\]
satisfies \(\|R^Y\|_{2\alpha} < +\infty\), where \(L(\mathbb{R}^d, W)\) is the space of all linear operators from \(\mathbb{R}^d\) to
\(W\). We denote the space of all \(W\)-valued controlled paths \(\mathcal{D}^{2\alpha}_X(W)\), and endow it with the semi-norm
\[
\|Y, Y'\|_{X,2\alpha} = \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.
\]
\(\mathcal{D}^{2\alpha}_X\) becomes a Banach space with the norm \((Y, Y') \mapsto |Y_0| + |Y'_0| + \|Y, Y'\|_{X,2\alpha}\). Controlled
paths are stable under composition with with regular functions. In fact, it is easy to check
that if \(f \in C^2_b(W)\), then \(f(Y_t) \in \mathcal{D}^{2\alpha}_X\) with
\[
f(Y_t)' = f'(Y_t)Y'_t.
\]
If \( Y_t \) is controlled by \( X_t \) with \( W = L(\mathbb{R}^d, V) \) for a Banach space \( V \). Then the rough integral of \( Y_t \) against \( X_t \) is defined as
\[
\int_0^t Y_s dX_s := \lim_{\Delta_n \to 0} \sum_i Y_{t_{n_i}} \cdot X_{t_{n_i}, t_{n_{i+1}}} + Y'_{t_{n_i}} \cdot X_{t_{n_i}, t_{n_{i+1}}},
\]

where \( \{t_{n_i}\}_{i \geq 1} \) is a sequence of partitions of \([0, t]\) with mesh \( \Delta_n \). Note that we use the canonical injection \( L(\mathbb{R}^d, L(\mathbb{R}^d, V)) \to L(\mathbb{R}^d \otimes \mathbb{R}^d, V) \) in writing \( Y'_{t_{n_i}} \cdot X_{t_{n_i}, t_{n_{i+1}}} \). The rough integral can be seen as a map from \( \mathcal{D}^2(V) \) to \( \mathcal{D}^2(V) \). Since
\[
\left( \int_0^t Y_s dX_s, Y_t \right) \in \mathcal{D}^2(V).
\]

This map is in fact continuous and we have the estimate
\[
\left\| \left( \int_0^t Y_s dX_s, Y_t \right) \right\|_{X,2\alpha} \leq \|Y\|_{\alpha} + \|Y'\|_{L_{\infty}} \|X\|_{2\alpha} + C \left( \|X\|_{\alpha} \|R^Y\|_{2\alpha} + \|Y'\|_{\alpha} \|X\|_{2\alpha} \right),
\]
where \( C \) is a positive constant depends on \( \alpha \).

Finally, let \( f \in C^2_b(V, L(\mathbb{R}^d, V)) \). We may consider the rough differential equation driven by \( X \) given by
\[
Z_t = z_0 + \int_0^t f(Z_s) dX_s.
\]
A process \( Z_t \in V \) is said to be a solution \([3]\) if \( Z_t \in \mathcal{D}^2(V) \) and the integral in \([3]\) holds as a rough integral. We may also consider more general equations like
\[
dY_t = \sum_{i=1}^d V_i(Y_t) dX^i_t + V_0(Y_t) dt, \quad Y_0 = y_0 \in \mathbb{R}^d, \quad t \in [0, 1].
\]
For simplicity, We will follow the usual convention and write
\[
Y_t = \pi_V(0, y_0; X)(t), \quad t \in [0, 1].
\]

The map \( \pi \) is called the Itô-Lyons map. Moreover, if \( X \) is a geometric \( \alpha \)-Hölder rough path, then we have
\[
\lim_{k \to \infty} \left\| \pi_V(0, y_0; X) - \pi_V(0, y_0; X^k) \right\|_{\alpha} = 0,
\]
where \( \{X^k\}_{k \geq 1} \) is any sequence such that \([1]\) is satisfied.

### 2.2 Markovian rough paths

Central to our purpose is the Markovian rough paths. Let \( X_t \) be the Markov process generated by \([3]\). For Let \( t \in (0, 1] \) and \( \{D_n\}_{n \geq 1} \) be a sequence of increasing partitions of interval \([0, t]\) with \( \Delta_n \to 0 \). Define
\[
K_{i,j}^n(X)_t = \sum_{t_k \in D_n} \frac{X^i_{t_{k+1}} + X^i_{t_k}}{2} (X^j_{t_{k+1}} - X^j_{t_k}),
\]
then \( K_{i,j}(X)_t := \lim_{n \to \infty} K_{i,j}^n(X)_t \) exists in probability, and the couple \( X_t = (X_t, K_t) \) is a geometric rough path in \( \mathcal{C}^\alpha([0, 1], \mathbb{R}^d) \) for any \( \alpha \in (0, 1/2) \) (see \([15]\)). \( X_t \) is called the canonical rough lift of \( X_t \).

**Remark 2.1.** It is worth mentioning that there are other equivalent ways to construct the rough lift of \( X_t \), see for example \([7]\).
2.3 Malliavin calculus

We collect some basic materials in Malliavin calculus and refer to [18] for a complete exploration.

Let $\mathcal{F}_t$ be the filtration generated by a Brownian motion $W_t$ and $\mathcal{H} = W^{1,2}_0([0, 1])$ be the Cameron-Martin space of $W_t$. A $\mathcal{F}_1$-measurable random variable $F$ is said to be cylindrical if it has the form

$$F = f(W_{t_1}, W_{t_2}, \cdots, W_{t_n}),$$

where $f \in C^\infty_b(\mathbb{R}^n, \mathbb{R})$ and $\{t_i\}_{1 \leq i \leq n} \in [0, 1]$. We denote the collection of all cylindrical random variables $\mathcal{S}$.

The Malliavin derivative of $F$ is defined as

$$DF = \sum_{i=1}^n \partial_i f(W_{t_1}, W_{t_2}, \cdots, W_{t_n}) 1_{[0, t_i]},$$

where $1_{[0, t_i]}$ is the indicator function of interval $[0, t_i]$. For $h \in \mathcal{H}$, we have the following relation between directional derivatives of $F$ and $DF$:

$$D_h F := \langle DF, \dot{h} \rangle_{L^2([0, 1]; \mathbb{R}^d)} = \lim_{\epsilon \to 0} \frac{f(W_{t_1} + \epsilon h_{t_1}, \cdots, W_{t_n} + \epsilon h_{t_n}) - f(W_{t_1}, \cdots, W_{t_n})}{\epsilon},$$

where $\dot{h}$ is the derivative of $h$. It is often useful to use the canonical isometry between $\mathcal{H}$ and $L^2([0, 1])$ and take $DF$ as an element in $\mathcal{H}$. (In fact, on abstract Wiener spaces, Malliavin derivatives are defined to be random variables in the corresponding Cameron-Martin space; the usual setting for Brownian motion is a very special case.) One could then iterate the previous definition the define the $n$-th Malliavin derivative of $D^n F$ which takes value in $\mathcal{H}^\otimes n$.

For any $p \geq 1$, it is possible to prove that $D^n$ is closable from $\mathcal{S}$ to $L^p(\Omega; \mathcal{H}^\otimes n)$. We denote by $\mathbb{D}^{n,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,p} = \left( \mathbb{E}|F|^p + \sum_{i=1}^n \mathbb{E}\left\|D^i F\right\|_{\mathcal{H}^\otimes i}^p \right)^{\frac{1}{p}},$$

and $\mathbb{D}^\infty = \cap_{n \geq 1} \cap_{p \geq 1} \mathbb{D}^{n,p}$.

If the vector fields $\{A_i\}_{1 \leq i \leq d}, B \in C^\infty_b$, then for any $t \in [0, 1]$, $X_t$ defined by

$$dX_t = \sum_{i=1}^d A_i(X_t) dW^i_t + B(X_t) dt, X_0 = x_0 \in \mathbb{R}^d$$

belongs to $\mathbb{D}^\infty$. Moreover, let $A = (A_1, A_2, \cdots, A_d)$ then $DX_t$ satisfies

$$D_r X_t = A(X_r) + \int_0^t DA_i(X_s) \cdot D_r X_s dW^i_s + \int_0^t DB(X_s) \cdot D_r X_s ds,$$

for $r \leq t$ and $D_r X_t = 0$ for $s > t$ a.s.

Recall the Jacobian process associated with $X_t$ is given by

$$J^X_{t \leftarrow 0} = I_{d \times d} + \sum_{i=1}^d \int_0^t DA_i(X_s) \cdot J^X_{s \leftarrow 0} dW^i_s + \int_0^t DB(X_s) \cdot J^X_{s \leftarrow 0} ds, \quad t \in [0, 1].$$
The inverse of $J_{t-0}^X$, written as $J_{0-t}^X$, is well defined, and we have the following composition property:

$$J_{t-u}^X \cdot J_{u-s}^X = J_{t-s}^X,$$

for $s, u, t \in [0, 1]$. The following integrability result is well-known and we refer to theorem 7.2 and remark 7.3 of [4] for a proof.

**Proposition 2.2.** Suppose that $\{A_i\}_{1 \leq i \leq d}, B \in C_b^\infty(\mathbb{R}^d)$. Define $\Phi_t = (X_t, J_{t-0}^X, J_{0-t}^X)$. Then for any $\gamma \in (0, \frac{1}{2})$, we have $\|\Phi\|_{\gamma} \in L^p(\Omega)$ for any $p \geq 1$. In particular, $\|\Phi\|_{\infty} = \sup_{t \in [0,1]}|\Phi(t)| \in L^p(\Omega)$ for any $p \geq 1$.

By uniqueness, it is straightforward to see that, for any $r \leq t$ we have

$$D_tX_t = J_{t-r}^X A(X_r). \tag{6}$$

Another fundamental object in Malliavin calculus is given by the next

**Definition 2.3.** Let $F = (F^1, F^2, \ldots, F^d)$ be a random vector with whose components belong to $\mathbb{D}^{1,1}$. The Malliavin matrix $\Gamma(F)$ is defined as

$$\Gamma(F) = (\langle DF^i, DF^j \rangle_{\mathcal{H}})_{1 \leq i, j \leq d}.$$

An estimate on the Malliavin matrix allows one to prove the existence and smoothness of density.

**Proposition 2.4.** Let $F = (F^1, F^2, \ldots, F^d)$ be a random vector whose components belong to $\mathbb{D}^{\infty}$. If $(\det \Gamma(F))^{-1} \in L^p(\Omega)$ for all $p \geq 1$, then $F$ has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$.

### 3 Malliavin derivative of $Y_t$

This section is devoted to the computation of the Malliavin derivative of $Y_t$ defined in theorem 1.8. Let $\{D_n\}_{n \geq 1}$ be a sequence of increasing partitions of $[0, t]$, with the mesh $\Delta_n \to 0$. For any process $Z$ defined on $[0, t]$, we use $Z^n$ to represent the piece-wise linear approximation of $Z$ along $D_n$.

Similar to $X_t$, the associated Jacobian process $J_{t-0}^Y$ is a $d \times d$ matrix-valued process given by

$$dJ_{t-0}^Y = \sum_{i=1}^d DV_i(Y_t)J_{t-0}^Y dX_t^i + DV_0(Y_t)J_{t-0}^Y dt, \quad J_{0-0}^Y = I_{d \times d}, \quad t \in [0, 1].$$

Now let us write

$$Y_t(\omega) = y_0 + \sum_{i=1}^d \int_0^t V_i(Y_s) dX_s^i(\omega) + \int_0^t V_0(Y_s) ds, \quad y_0 \in \mathbb{R}^d, \quad t \in [0, 1],$$

where $\omega$ represents the underlying Brownian paths. For every $h \in \mathcal{H}$, we have by definition

$$D_hY_t^j = (DY_t^j, \dot{h})_{L^2([0,1])} = \lim_{\epsilon \to 0} \frac{Y_{t}^j(\omega + \epsilon h) - Y_{t}^j(\omega)}{\epsilon}, \quad 1 \leq j \leq d. \tag{7}$$
Here

\[ Y_t(\omega + \epsilon h) = y_0 + \int_0^t V_0(Y_s(\omega + \epsilon h))ds + \sum_{i=1}^d \int_0^t V_i(Y_s(\omega + \epsilon h))dX(\omega + \epsilon h)_s. \]

Note that \( X(\omega + \epsilon h) \) is the diffusion given by

\[ X_t(\omega + \epsilon h) = x_0 + \sum_{i=1}^d \int_0^t A_i(X_s(\omega + \epsilon h))d(W + \epsilon h)_s + \int_0^t B(X_s(\omega + \epsilon h))ds. \]

Hence, the rough lift \( X(\omega + \epsilon h)_t \) is well defined.

**Proposition 3.1.** Let \( J^Y_{t \leq 0} \) be the Jacobian process of \( Y_t \). Under assumptions of theorem \[L.8\] and for all \( t \in [0,1], h \in \mathcal{H}, \) we have

\[ D_h Y_t = \sum_{i=1}^d J^Y_{t \leq 0} \int_0^t J^Y_{s \leq 0} V_i(Y_s)dD_h X^i_s. \]

almost surely.

**Remark 3.2.** Although this result is straightforward, to the author’s best knowledge it is not written in any standard reference. We include a proof for completeness.

**Proof.** We first note that, \( X^n \) has bounded variation almost surely. Thus, \( \pi_V(0, y_0; X^n) \) is just the classical Itô map and we will simply write it as \( \pi_V(0, y_0; X^n) \). We know that \( \pi_V(0, y_0; X) = \lim_{n \to \infty} \pi_V(0, y_0; X^n) \) uniformly. As a result

\[ \lim_{\epsilon \to 0} \frac{Y_t(\omega + \epsilon h) - Y_t(\omega)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\pi_V(0, y_0; (X + \epsilon h)^n)(t) - \pi_V(0, y_0; X^n)(t)}{\epsilon}. \]

On the other hand, we have

\[ \lim_{\epsilon \to 0} \frac{(X + \epsilon h)^n - X^n}{\epsilon} = D_h X^n. \]

By Duhamel’s principle of ODE (see for example \[8\] section 4.1 and 4.2), we have

\[ \lim_{\epsilon \to 0} \frac{\pi_V(0, y_0; (X + \epsilon h)^n)(t) - \pi_V(0, y_0; X^n)(t)}{\epsilon} = \sum_{i=1}^d \int_0^t J^Y_{s \leq 0} V_i(Y_s) d(D_h X^n)_s, \]

where \( Y_n(t) = \pi_V(0, y_0; X^n)(t) \). Finally, sending \( n \) to infinity gives

\[ \lim_{\epsilon \to 0} \frac{Y_t(\omega + \epsilon h) - Y_t(\omega)}{\epsilon} = \sum_{i=1}^d \int_0^t J^Y_{s \leq 0} V_i(Y_s) dD_h X^i_s. \]

\[ \square \]

**Corollary 3.3.** Let \( J^Y_{t \leq 0} \) be the Jacobian process of \( Y_t \). Under assumptions of theorem \[L.8\] and for all \( t \in [0,1] \) and \( 1 \leq j \leq d, \) we have

\[ D^j_r Y_t = \sum_{i=1}^d J^Y_{t \leq 0} \int_0^t J^Y_{s \leq 0} V_i(Y_s)dD^j_r X^i_s. \]
Proof. By definition we have
\[ D_h X_t^j = \sum_{j=1}^{d} \langle D_h^j X_t^i, \hat{h}_t^j \rangle_{L^2([0,1])}. \]

With same notations as previous proposition, we have
\[ \sum_{i=1}^{d} \int_0^t J^Y_{t-s} V_i(Y_n(s))d(D_h X^i)_s = \sum_{i=1}^{d} \langle \sum_{i=1}^{d} \int_0^t J^Y_{t-s} V_i(Y_n(s))d(D^j_h X^i)_s, \hat{h}_t^j \rangle_{L^2([0,1])}. \]

Taking the limit and we immediately have
\[ D_r Y_t = \sum_{i=1}^{d} J^Y_{t-0} \int_0^t J^Y_{0-s} V_i(Y_s)dD_r X^i_s. \]

Remark 3.4. We can make a quick comparison between (8) and (9). Note that if \( a(x) \equiv I_{d \times d}, \) then \( X_t \) is a standard Brownian motion and \( D_r X_t = I_{d \times d} \cdot 1_{[0,t]} \cdot r \). If we fix \( r \), \( D_r X_t \) is diagonal and a pure jump process with a jump at \( r \). By (8) we have
\[ D_r^1 Y_t = \sum_{i=1}^{d} J^Y_{t-0} \int_0^t J^Y_{0-s} V_i(Y_s)dD^1_r X^i_s = J^Y_{t-0} V_j(Y_t), \]
which recovers (9). We see that in (9), the rough integral degenerates into evaluation at a single point and this happens when \( X_t \) is Gaussian; because Malliavin derivatives of Gaussian processes are always pure jump processes when \( r \) is fixed. Our goal is to study the more general rough integral formula (8); that is why we assume that \( a(x) \) is not constant.

Remark 3.5. Now is a good time to further discuss the motivation of Assumptions 1.7. One crucial step in the proof of a Hörmander’s type theorem is the implication that
\[ \left\{ \inf_{\|v\|_v=1} v^T \cdot \langle DY_t, DY_t \rangle_{H} \cdot v \leq \epsilon \right\} \Rightarrow \left\{ \|J^Y_{0-s} V_i(Y_s)\|_\infty \leq \epsilon^\alpha \right\}, \quad (10) \]
for some \( \alpha > 0 \) and \( 1 \leq i \leq d \). When \( X_t \) is Brownian motion, this step was done using a non-property of the \( L^2 \) norm (see lemma A.3 of [4]). For the case where \( X_t \) is a non-degenerate Gaussian process, this is done by an interpolation inequality (see theorem 6.9 of [3]). Since we have \( D_r Y_t = J^Y_{t-0} V_j(Y_t) \) when \( X_t \) is Gaussian, in all previous cases (10) can be roughly understood as
\[ \left\{ \inf_{\|v\|_v=1} v^T \cdot \langle DY_t, DY_t \rangle_{H} \cdot v \leq \epsilon \right\} \Rightarrow \left\{ \|DY_t\|_\infty \leq \epsilon^\alpha \right\}. \]

However, in our case, due to the integral representation (8), same type of argument only gives
\[ \left\{ \inf_{\|v\|_v=1} v^T \cdot \langle DY_t, DY_t \rangle_{H} \cdot v \leq \epsilon \right\} \Rightarrow \left\{ \left\| \int_0^t J^Y_{0-s} V_i(Y_s)dD_r X^i_s \right\|_\infty \leq \epsilon^\beta \right\}. \]

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for some $\beta > 0$, $1 \leq i \leq d$. We still need to further justify that
\[
\begin{aligned}
\left\{ \left\| \int_0^t J_0^{Y_i} V_i(Y_s) dD_r X_i^s \right\| \leq \epsilon^\beta \right\} \Rightarrow \left\{ \left\| J_0^{Y_i} V_i(Y_s) \right\|_\infty \leq \epsilon^\alpha \right\}.
\end{aligned}
\] (11)

To prove (11), it requires $D_r X_s = J_{X_t}^X A(X_t)$ to be non-degenerate. Since $A$ is an elliptic system, the behavior of $D_r X_s$ essentially depends on the Jacobian process of $X_t$. Recall
\[
dJ_{t+0}^X = \sum_{i=1}^d DA_i(X_t) J_{t+0}^X W_i + DB(X_t) J_{t+0}^X dt.
\]
Thus, a control on $(DA_1, DA_2, \ldots, DA_d) = dA$ will be sufficient to give the desired non-degeneracy. From this perspective, Assumptions [17] is very natural.

4 Small ball estimate

This section is devoted to developing technical tools necessary for the proof of our main result.

4.1 Hölder roughness of diffusion

Norris’ type lemmas are crucial in the proofs of Hörmander’s type theorems. In classical stochastic analysis theory, it is nothing but a quantitative version of the Doob-Meyer decomposition of semi-martingales. In the context of rough differential equations, we are going to use a deterministic version of Norris’ lemma, which first appeared in [12] and was improved in [14]. We start with a definition.

**Definition 4.1.** Let $\theta \in (0, 1)$. A path $X : [0, 1] \to \mathbb{R}^d$ is called $\theta$-Hölder rough if there exists a constant $c > 0$ such that for every $s$ in $[0, 1]$, every $\epsilon$ in $(0, \frac{1}{2}]$, and every $\phi \in \mathbb{R}^d$ with $|\phi| = 1$, there exists $t$ in $[0, 1]$ such that $\epsilon/2 < |t - s| < \epsilon$ and
\[
|\langle \phi, X_{s,t} \rangle| > c\epsilon^\theta.
\]
The largest such constant is called the modulus of $\theta$-roughness, and is denoted by $L_\theta(X)$.

Now we can state the Norris type result that we are going to use and refer to [14] for a proof.

**Proposition 4.2.** Let $X$ be a geometric $\alpha$-Hölder rough path and assume that $X$ is a $\theta$-Hölder rough with $2\alpha > \theta$. Let $Y$ be a $\mathbb{R}^d$-valued path controlled by $X$, and set
\[
Z_t = \sum_{i=1}^d \int_0^t Y_i^s dX_i^s + \int_0^t b_s ds,
\]
where $b$ is an $\alpha$-Hölder continuous function. Then there exists constants $l > 0$ and $q > 0$ such that, setting
\[
A := 1 + L_\theta(X)^{-1} + \rho_\alpha(X) + \|Y\|_\alpha + \|Y'\|_\alpha + \|b\|_\alpha,
\]
one has the bound
\[
\|Y\|_\infty + \|b\|_\infty \leq M A^q \|Z\|_l
\]
where $M$ depends on the dimension $d$ and $Y$. 11
We aim to prove Hölder roughness for $X_t$ in this subsection. Let $v \in \mathbb{R}^d$ with $|v| = 1$, then $v^T X_{s,t}$ is given by

$$v^T X_{s,t} = \int_s^t v^T A_i(X_l) dW_i^l + \int_s^t v^T B(X_l) dl, \forall t \in [s,1].$$

We are able to prove the following small ball estimate.

**Lemma 4.3.** If $\{A_i\}_{1 \leq i \leq d}$ form an elliptic system, then for any $s \in [0,1)$ and $k \in (0,1)$, we can find $\epsilon_0 > 0$ and constants $C, C' > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $\delta \in (0,1)$

$$\mathbb{P}\left( \inf_{\|v\|=1} \sup_{t \in [s,s+\delta]} |v^T X_{s,t}| \leq \epsilon \right) \leq C \exp \left\{ -C' \frac{\delta}{\epsilon^{2-2k}} \right\}.$$

*Proof.* For a fixed $v \in \mathbb{R}^d$ and $s \in [0,1)$, define $M_{s,r}(v) = v^T X_{s,r}$ for $s \leq r$. Then

$$\sup_{t \in [s,s+\delta]} |v^T X_{s,t}| = \sup_{r \in [s,s+\delta]} |M_{s,r}|.$$

Let $M_{s,r}^m$ be the local martingale part of $M_{s,r}$. By uniform ellipticity, we can find a constant $C_1 > 0$ such that

$$\langle M_{s,r}^m, M_{s,r}^m \rangle_{s,s+\delta} \geq \delta \inf_{r \in [s,s+\delta]} \sum_{i=1}^d \left| v^T A_i(X_r) \right|^2 \geq \delta C_1,$$

where the bracket means the quadratic variation. We can therefore deduce that

$$\mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}| \leq \epsilon \right) = \mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}| \leq \epsilon, \langle M_{s,r}^m, M_{s,r}^m \rangle_{s,s+\delta} \geq C_1 \delta \right).$$

By Itô’s formula, we have

$$M_{s,r}^2 = 2 \sum_{i=1}^d \int_s^r M_{s,l} \cdot v^T A_i(X_l) dW_i^l + 2 \int_s^r M_{s,l} \cdot v^T B(X_l) dl + \sum_{i=1}^d \int_s^r \left| v^T A_i(X_l) \right|^2 dl. \quad (13)$$

Since $\{A_i\}_{1 \leq i \leq d}, B \in C^\infty_b$, we can find $C_B > 0$ such that

$$\left\{ \sup_{r \in [s,s+\delta]} |M_{s,r}| \leq \epsilon \right\} \Rightarrow \left\{ \sup_{r \in [s,s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T B(X_l) dl \right| \leq \delta C_B \epsilon \right\} \& \left\{ \int_s^{s+\delta} M_{s,l}^2 \left| v^T A_i(X_l) \right|^2 dl \leq \delta C_B^2 \epsilon^2 \right\}. \quad (14)$$

By the second term of the last implication we have the following decomposition for $k \in (0,1)$ and $1 \leq i \leq d$

$$\mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}|^2 \leq \epsilon^2 \right) = \mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2 \left| v^T A_i(X_l) \right|^2 dl \leq \delta C_B^2 \epsilon^2 \right)$$

$$= \mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2 \left| v^T A_i(X_l) \right|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s,s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_l) dW_i^l \right| > \delta \epsilon^k \right)$$

$$+ \mathbb{P}\left( \sup_{r \in [s,s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2 \left| v^T A_i(X_l) \right|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s,s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_l) dW_i^l \right| \leq \delta \epsilon^k \right). \quad (15)$$

(16)
For (15), we have by the exponential inequality for martingales (see, [19], p. 153) that

\[ P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2|v^T A_i(X_i)|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s, s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_i) dW_l^i \right| > \delta \epsilon^k \right) \leq P \left( \int_s^{s+\delta} M_{s,l}^2|v^T A_i(X_i)|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s, s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_i) dW_l^i \right| > \delta \epsilon^k \right) \leq 2 \exp \left\{ -\frac{\delta}{C_B^2 \epsilon^{2-2k}} \right\}.

Thus, we have

\[ P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2|v^T A_i(X_i)|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s, s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_i) dW_l^i \right| \leq \delta \epsilon^k \right) \geq P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2 \right) - 2 \exp \left\{ -\frac{\delta}{C_B^2 \epsilon^{2-2k}} \right\}.

On the other hand, by using (14) and (13) we see that

\[ \left\{ \begin{array}{l} \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \int_s^{s+\delta} M_{s,l}^2|v^T A_i(X_i)|^2 dl \leq \delta C_B^2 \epsilon^2, \sup_{r \in [s, s+\delta]} \left| \int_s^r M_{s,l} \cdot v^T A_i(X_i) dW_l^i \right| \leq \delta \epsilon^k \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \sup_{r \in [s, s+\delta]} \sum_{i=1}^d \int_s^r \left| v^T A_i(X_i) \right|^2 dl \leq \epsilon^2 + 2C_B \delta \epsilon + d \delta \epsilon^k \end{array} \right\}. \tag{18} \]

From (17) and (18) we can deduce that

\[ P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \sup_{r \in [s, s+\delta]} \sum_{i=1}^d \int_s^r \left| v^T A_i(X_i) \right|^2 dl \leq \epsilon^2 + 2C_B \delta \epsilon + d \delta \epsilon^k \right) \geq P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2 \right) - 2 \exp \left\{ -\frac{\delta}{C_B^2 \epsilon^{2-2k}} \right\}. \tag{19} \]

By (12), there exists \( \epsilon_1 > 0 \) such that when \( \epsilon < \epsilon_1 \) we have

\[ \sup_{r \in [s, s+\delta]} \sum_{i=1}^d \int_s^r \left| v^T A_i(X_i) \right|^2 dl = \langle M_{s,r}^m, M_{s,r}^m \rangle_{s,s+\delta} \geq \delta C_1 > \epsilon^2 + 2C_B \delta \epsilon + d \delta \epsilon^k. \tag{20} \]

Combining (19) and (20) gives

\[ P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}| \leq \epsilon, \langle M_{s,r}^m, M_{s,r}^m \rangle_{s,s+\delta} \geq \delta C_1 \right) \leq P \left( \sup_{r \in [s, s+\delta]} |M_{s,r}|^2 \leq \epsilon^2, \sup_{r \in [s, s+\delta]} \sum_{i=1}^d \int_s^r \left| v^T A_i(X_i) \right|^2 dl > \epsilon^2 + 2C_B \delta \epsilon + d \delta \epsilon^k \right) \leq 2 \exp \left\{ -\frac{\delta}{C_B^2 \epsilon^{2-2k}} \right\}. \]
Up to this point, all of our computations are done with $v$ fixed. We shall conclude with a compactness argument. The idea is similar to that of [17] (page 127). Observe that if \( \sup_{t-s \leq \delta} |X_{s,t}| \) is uniformly bounded, then \( \sup_{t-s \leq \delta} |v^T X_{s,t}| \) is Lipschitz as a function of $v$. Moreover, since the unit ball of $\mathbb{R}^d$ is compact, we can cover it with balls of radius $a < 1$ and the number of these ball can be chosen to be less than $C/a^d$ for some constant $C$. So, we have for any $\tau \in (0, \frac{1}{2})$

\[
\mathbb{P}\left( \inf_{\|v\|=1} \sup_{t-s \leq \delta} |v^T X_{s,t}| \leq \epsilon \right) \leq \frac{C_2 \theta^d}{\epsilon^d} \sup_{\|v\|=1} \mathbb{P}\left( \sup_{t-s \leq \delta} |v^T X_{s,t}| \leq 2\epsilon \right) + \mathbb{P}\left( \sup_{t-s \leq \delta} |X_{s,t}| > \theta \right)
\leq \frac{C_2 \theta^d}{\epsilon^d} \exp\left\{ -\frac{\delta}{C^2_B\epsilon^{2-2k}} \right\} + C_4 \exp\{-C_5 \theta^d\}.
\]

We used the exponential integrability of $\sup_{s \in [0,1]} |X_s|$ in the last inequality (see for example proposition 2.9 of [17]). Finally, set
\[
\theta = \frac{\sigma_{\frac{1}{2}}}{\epsilon^{2-2k}},
\]
then we can find $\epsilon_0 \leq \epsilon_1$ such that for $\epsilon < \epsilon_0$

\[
\mathbb{P}\left( \inf_{\|v\|=1} \sup_{t-s \leq \delta} |v^T X_{s,t}| \leq \epsilon \right) \leq \frac{C_2 \delta^{\frac{\theta}{2}}}{\epsilon^{2-2k}} \exp\left\{ -\frac{\delta}{C^2_B\epsilon^{2-2k}} \right\} + C_4 \exp\{-C_5 \delta \}
\leq C_6 \exp\{-C_7 \delta \}
\]

Proposition 4.4. Under the assumptions of the previous lemma, for any $\theta \geq \frac{1}{2}$, and $k \in (0,1)$ such that $\theta > \frac{1}{2-2k}$, we have

\[
\mathbb{P}(L_\theta(X) < \epsilon) \leq C_1 \exp\{-C_2 \epsilon^{2k-2}\}.
\]

In particular, $L_\theta^{-1}(X) \in L^p(\Omega)$, for any $p \geq 1$.

Proof. Let us define
\[
D_\theta(X) := \inf_{\|v\|=1} \inf_{n \geq 1} \inf_{1 \leq 2^n s \leq 2^n t \in I_{l,n}} \sup_{s,t \in I_{l,n}} \frac{|v^T X_{s,t}|}{2^{-n\theta}};
\]
and
\[
\hat{D}_\theta(X) := \inf_{\|v\|=1} \inf_{n \geq 1} \inf_{s = \frac{l}{2^n} \leq \frac{l + 1}{2^n} t \in I_{l,n}} \sup_{s,t \in I_{l,n}} \frac{|v^T X_{s,t}|}{2^{-n\theta}},
\]
where
\[
I_{l,n} = \left[ \frac{l}{2^n}, \frac{l + 1}{2^n} \right].
\]
Obviously we have $D_\theta(X) \geq \hat{D}_\theta(X)$. The exact same argument of lemma 3 of [12] can be applied here, from which we can deduce

\[
L_\theta(X) > \frac{1}{2 \cdot 8^\theta} D_\theta(X) \geq \frac{1}{2 \cdot 8^\theta} \hat{D}_\theta(X).
\]
Thus, it suffices to give estimate for $\hat{D}_\theta(X)$. By definition, we have

$$\mathbb{P}(\hat{D}_\theta(X) < \epsilon) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \mathbb{P}(\inf_{\|v\|=1} \sup_{s=\frac{1}{2^m}, t \in I_{l,n}} |v^T X_{s,t}| < \epsilon).$$

When $\epsilon$ is sufficiently small, we can apply the previous lemma to get

$$\mathbb{P}(\hat{D}_\theta(X) < \epsilon) \leq C_1 \sum_{n=1}^{\infty} 2^n \exp \left\{ -C_2 2^{n(2-2k-1)} \right\}.$$

Since $\theta > \frac{1}{2} - 2k$, we can find $C_3, C_4 > 0$ uniformly over $\epsilon \leq 1$, $n \geq 1$, such that

$$2^n \exp \left\{ -C_2 2^{n(2-2k-1)} \right\} \leq \exp \left\{ C_3 - C_4 n \epsilon^{2k-2} \right\}.$$

Thus,

$$\mathbb{P}(\hat{D}_\theta(X) < \epsilon) \leq C_1 \sum_{n=1}^{\infty} \exp \left\{ C_3 - C_4 n \epsilon^{2k-2} \right\} \leq C_5 \exp \left\{ -C_6 \epsilon^{2k-2} \right\},$$

and the proof is finished.

4.2 Non-degenerate property of Jacobian processes

Now, we move on to the study of non-degenerate property of the Jacobian process of $X_t$.

For the sake of conciseness, we adopt the notions from [11] and introduce the following

**Definition 4.5.** A family of sets $\{E_\epsilon\}_{\epsilon \in [0,1]} \in \mathcal{F}$ is said to be “almost true” if for any $p \geq 1$, we can find $C_p > 0$ such that

$$\mathbb{P}(E_\epsilon) \geq 1 - C_p \epsilon^p.$$

Similarly for “almost false”. Given two such families of events $A$ and $B$, we say that “$A$ almost implies $B$” and we write $A \Rightarrow_\epsilon B$ if $A \setminus B$ is almost false.

It is straightforward to check that these “almost” implications are transitive and invariant under any reparametrisation of the form $\epsilon \mapsto \epsilon^\alpha$ for $\alpha > 0$.

**Remark 4.6.** A typical situation where this definition naturally appears is as follows. Suppose that $Z$ is a random variable in some probability space such that $\mathbb{E}|Z|^p < \infty$ for any $p \geq 1$, then

$$\mathbb{P}\left(|Z| > \frac{1}{\epsilon}\right) \leq \mathbb{E}|Z|^p \epsilon^p.$$

In other words, $\left\{\frac{1}{\epsilon} \leq \frac{1}{\epsilon} \right\}$ is almost true. We will simply write it as $|Z| \leq C \epsilon$.

Our next result establishes a non-degenerate property of Jacobian process $J^X$.

**Proposition 4.7.** Let $f(s) \in \mathbb{R}^d$ be a diffusion controlled by $J^X$, such that for any $\gamma \in (0, \frac{1}{2})$ we have $\|f\|_\gamma \in L^p(\Omega)$ for any $p \geq 1$. Then

$$\left\{ \sup_{r \in [0,t]} \left| \int_0^r f(s)^T dJ^X_{s+0} \right| \leq \epsilon \right\} \Rightarrow_\epsilon \left\{ \sup_{r \in [0,t]} |f(r)| \leq C \epsilon^\alpha \right\},$$

for some constant $C > 0$ and $\alpha \in (0,1)$. 
Proof. Since $J^X$ is a semi-martingale, we can consider

$$M_t := \int_0^t f(s)^T \circ dJ^X_{s\downarrow t-0}. $$

One has the following representation (recall $f^m$ means the local martingale part of $f$).

$$M_r = \sum_{i=1}^d \int_0^r f(s)^T \cdot DA_i(X_s) J^X_{s\downarrow t-0} dW^i_s + \int_0^r f(s)^T \cdot DB(X_s) J^X_{s\downarrow t-0} ds + \frac{1}{2} \langle f^m, M^m \rangle_{[0,r]}.$$

By our assumptions and proposition 4.8, it is easy to see $M_r$ verifies the assumption of lemma 4.11 of [10], from which we have

$$\left\{ \sup_{r \in [0,t]} |M_r| \leq \epsilon \right\} \Rightarrow \epsilon \left\{ \max_{1 \leq i \leq d} \sup_{r \in [0,t]} \left| f(s)^T \cdot DA_i(X_s) J^X_{s\downarrow t-0} \right| < \epsilon \alpha' \right\}. \quad (21)$$

for some $\alpha' \in (0,1)$. Moreover, since

$$\left| J^X_{s\downarrow t-0} \right| = \left\{ \inf_{|v| = 1} \left| J^X_{s\downarrow t-0} \cdot v \right| \right\}^{-1},$$

we have, by proposition 2.2, for any $\eta > 0$

$$\mathbb{P} \left( \inf_{s \in [0,\tilde{t}]} \left| v^T \cdot J^X_{s\downarrow t-0} \right| < \epsilon \eta |v| \right) = \mathbb{P} \left( \|\Phi\|_\infty > \frac{1}{\epsilon \eta} \right) \leq C_\theta \epsilon^{\eta p}.$$ 

Thus, for $s \in [0,\tilde{t}]$ we have

$$\sum_{i=1}^d \left| f(s)^T \cdot DA_i(X_s) J^X_{s\downarrow t-0} \right|^2 \geq \epsilon^2 \eta \sum_{i=1}^d \left| f(s)^T \cdot DA_i(X_s) \right|^2. \quad (22)$$

Finally, Assumption 1.7 gives

$$\sum_{i=1}^d \left| f(s)^T \cdot DA_i(X_s) \right|^2 \geq C_f |f(s)|^2. \quad (23)$$

Combining (21) (22) (23) and use the fact that $\eta$ is arbitrary, we deduce

$$\left\{ \sup_{r \in [0,t]} \left| \int_0^r f(s)^T dJ^X_{s\downarrow t-0} \right| \leq \epsilon \right\} \Rightarrow \epsilon \left\{ \sup_{s \in [0,t]} |f(s)| \leq C_\epsilon \alpha \right\}$$

for some $\alpha \in (0,1)$.

With the previous proposition, we can show the desired non-degenerate property of $D_r X_s$ as a family of processes index by $r$.

Proposition 4.8. Let $f(s) \in \mathbb{R}^d$ be a diffusion controlled by $D_r X$ for any $r \in [0,1]$ and that for any $\gamma \in (0,\frac{1}{2})$, we have $\|f\|_\gamma \in L^p(\Omega)$ for all $p \geq 1$. Then for any $t \in [0,1]$ we have

$$\left\{ \sup_{r \in [0,t]} \left| \int_0^t f(s)^T dD_r X_s \right| \leq \epsilon \right\} \Rightarrow \epsilon \left\{ \sup_{s \in [0,t]} |f(s)| \leq \epsilon \alpha \right\}.$$
Proof. As before, we can write
\[
\int_0^t f(s)^T D_r X_s = \int_r^t f(s)^T \circ dJ_{s+t-0}^X \cdot J_{0-\epsilon r}^X A(X_r).
\]
Since \(A\) is an elliptic system, we see immediately
\[
\left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \cdot J_{0-\epsilon r}^X A(X_r) \right| \geq C_1 \left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \right|.
\tag{24}
\]
Moreover, we have
\[
|J_{0-\epsilon r}^X| = \left\{ \inf_{|r| = 1} |J_{r-\epsilon}^X \cdot v| \right\}^{-1},
\]
which implies that for any \(\beta > 0\)
\[
\mathbb{P} \left( \inf_{|r| = 1} |J_{r-\epsilon}^X \cdot v| \leq \epsilon^\beta \right) \leq \mathbb{P}(\|\Phi\|_\infty \geq \frac{1}{\epsilon^\beta}) \leq C_\epsilon \epsilon^p.
\]
By taking transpose, we have that
\[
\left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \right| \geq \epsilon \epsilon^\beta \left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \right|.
\tag{25}
\]
Combining (24) and (25) gives
\[
\left| \int_0^t f(s)^T D_r X_s \right| \geq \epsilon \epsilon^\beta \left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \right|.
\]
As a result, we have for any \(p \geq 1\) that
\[
\left\{ \sup_{r \in [0,t]} \left| \int_0^t f(s)^T D_r X_s \right| \leq \epsilon \right\} \Rightarrow \epsilon \left\{ \sup_{r \in [0,t]} \left| \int_r^t f(s)^T \circ dJ_{s+t-0}^X \right| \leq C_1 \epsilon^{1-\beta} \right\} \Rightarrow \left\{ \sup_{r \in [0,t]} \left| \int_0^t f(s)^T \circ dJ_{s+t-0}^X \right| \leq C_2 \epsilon^{1-\beta} \right\}
\]
From proposition 4.7, we deduce
\[
\left\{ \sup_{r \in [0,t]} \left| \int_0^t f(s)^T \circ dJ_{s+t-0}^X \right| \leq C_2 \epsilon^{1-\beta} \right\} \Rightarrow \epsilon \left\{ \sup_{r \in [0,t]} \left| f(r) \right| \leq C_3 \epsilon^{(1-\beta)\alpha} \right\}
\]
for some \(\alpha \in (0,1)\). Since \(\beta\) is arbitrary and almost true implications are transitive, our result follows.

We prepare another lemma for next section.

**Lemma 4.9.** If \(f(s) \in \mathbb{R}^{d \times d}\) is a diffusion process controlled by \(D_r X_s\) such that
\[
\|f\|_\infty, \left\| f^M \right\|_\infty \in L^p(\Omega), \ \forall p \geq 1.
\]
Then
\[
F(r) = \int_0^t f(s) dD_r X_s
\]
is \(\gamma\)-Hölder continuous for any \(\gamma < \frac{1}{2}\) and \(\|F\|_\gamma \in L^p(\Omega)\) for all \(p \geq 1\).
Proof. We have
\[ F(r) = \int_0^t f(s) dD_r X_s = \int_0^t f(s) \circ dD_r X_s = \int_r^t f(s) \circ dD_r X_s \text{ a.s.} \]
where we used the property that \( D_r X_s = 0 \text{ a.s. for } s < r \). As a result, for \( 0 \leq u \leq v \leq t \) we have
\[ F(u) - F(v) = \int_u^t f(s) \circ d(D_u X_s - D_v X_s) + \int_u^t f(s) \circ dD_u X_s. \]
The first term on the right hand side can be written as
\[ \Gamma_1 = \int_u^t f(s) \circ dJ_{s-t} X_s \cdot (J_{0+u} X_s - J_{0+v} X_s). \]
Similarly, we can write the second term as
\[ \Gamma_2 = \int_u^v f(s) \circ dJ_{s-t} X_s \cdot J_{0+u} X_u. \]
For \( \Gamma_1 \) we have for any \( \alpha \in (0, \frac{1}{2}) \) and \( p \geq 1 \)
\[
\mathbb{E}\left| \Gamma_1 \right|^p \leq 2^p \cdot \mathbb{E}\left\{ \sup_{t \in [0,1]} \int_0^t f(s) \circ dJ_{s-t} X_s \left( C_B \left\| \Phi \right\|_{\alpha} (u-v)^\alpha + \left\| \Phi \right\|_\infty C_B (v-u)^\alpha \right)^p \right\}
\leq 2^{2p-1} \cdot \mathbb{E}\left\{ \sup_{t \in [0,1]} \int_0^t f(s) \circ dJ_{s-t} X_s \left( C_B + \left\| \Phi \right\|_\infty + \left\| \Phi \right\|_{\alpha} \right)^p \right\} |v-u|^{\alpha p}.
\]
For \( \Gamma_2 \) we have for any \( p \geq 1 \)
\[
\mathbb{E}\left| \Gamma_2 \right|^p \leq \mathbb{E}\left\{ C_B^p \left\| \Phi \right\|_\infty^p \int_u^v f(s) \circ dJ_{s-t} X_s \right|^p \leq C_B^p \mathbb{E}\left\{ \left\| \Phi \right\|_\infty^{2p} \right\} \mathbb{E}\left\{ \int_u^v f(s) \circ dJ_{s-t} X_s \right|^p \right\}^{\frac{2p}{p}}.
\]
By Burkholder-Davis-Gundy inequality, standard computation gives that for any \( q \geq 2 \)
\[
\mathbb{E}\left\{ \int_u^v f(s) \circ dJ_{s-t} X_s \right|^q \right\} \leq \mathbb{E}\left\{ \int_u^v f(s) d(J_{s-t} X_s)^m + \int_u^v f(s) d(J_{s-t} X_s)^b + \frac{1}{2} (f^m, (J_{s-t} X_s)^m)_{u,v} \right\}^q \right\}
\leq C_q \mathbb{E}\left\{ \left\| f \right\|_\infty^q \left\| \Phi \right\|_\infty^q C_B^q |v-u|^{\frac{q}{2}} + \left\| f \right\|_\infty^q \left\| \Phi \right\|_\infty^q C_B^q |v-u|^q \right\} + C_q' (\mathbb{E}\left\{ f^m \right\}^q)^{\frac{1}{2}} (\mathbb{E}\left\{ \left\| \Phi \right\|_\infty^q C_B^{2q} \right\}^q |v-u|^{\frac{q}{2}} + \left\| f \right\|_\infty^q \left\| \Phi \right\|_\infty^q C_B^q |v-u|^{q^2 q}) \right\}
\leq P_q (\mathbb{E}\left\{ f^m \right\}^q)^{\frac{1}{2}} (\mathbb{E}\left\{ \left\| \Phi \right\|_\infty^q C_B^{2q} \right\}^q |v-u|^{\frac{q}{2}} + \left\| f \right\|_\infty^q \left\| \Phi \right\|_\infty^q C_B^q |v-u|^{q^2 q}),
\]
where \( P_q \) is some polynomial depends on \( q \). It is also relative easy to check that, under our assumptions
\[
\sup_{t \in [0,1]} \int_0^t f(s) \circ dJ_{s-t} X_s \in L^p(\Omega), \quad \forall p \geq 1.
\]
Hence, combining our estimate for \( \Gamma_1 \) and \( \Gamma_2 \) gives
\[
\mathbb{E}\left| F(v) - F(u) \right|^p \leq 2^{2p-1} (\mathbb{E}\left| \Gamma_1 \right|^p + \mathbb{E}\left| \Gamma_2 \right|^p) \leq C_p |v-u|^{\alpha p}.
\]
We conclude with Kolmogorov continuity theorem and Besov–Hölder embedding (see theorem A.10 of [8]). \( \Box \)
5 Existence of smooth density and Gaussian type upper bound

5.1 Malliavin smoothness and integrability

Let $X_t, Y_t$ be the processes defined in theorem [18]. We need to show $Y_t \in D^\infty$ before we can apply proposition [2.4]. In general, it is not an easy task to show the Malliavin smoothness of solution to a rough differential equation (see [4], proposition 7.5 and the note after). This result has been obtained for Gaussian rough paths in [14], but the technique used there is very difficult to generalize to non-Gaussian rough paths. It would be interesting to investigate further in this direction.

For our purpose, however, we can again use the fact that rough integrals against $X_t$ coincide with Stratonovich integrals against $X_t$ to our advantage. Indeed, the couple process $(X_t, Y_t)$ is solution to a stochastic differential equation driven by Brownian motion. Since the vector fields $\left\{ A_i \right\}_{1 \leq i \leq d}, B, \left\{ V_i \right\}_{0 \leq i \leq d} \in C_0^\infty(\mathbb{R}^d)$, we immediately have $(X_t, Y_t) \in D^\infty$.

The Jacobian process of $(X_t, Y_t)$, is given by

$$J^{X,Y}_{t \leftarrow 0} = \begin{bmatrix} \frac{\partial X_t}{\partial X_0} & \frac{\partial X_t}{\partial Y_0} \\ \frac{\partial Y_t}{\partial X_0} & \frac{\partial Y_t}{\partial Y_0} \end{bmatrix} = \begin{bmatrix} J^X_{t \leftarrow 0} & 0 \\ J^Y_{t \leftarrow 0} & J^Y_{t \leftarrow 0} \end{bmatrix}$$

with inverse

$$J^{X,Y}_{0 \leftarrow t} = \begin{bmatrix} J^X_{0 \leftarrow t} & 0 \\ -J^Y_{0 \leftarrow t} \frac{\partial Y_t}{\partial X_0} & J^Y_{0 \leftarrow t} \end{bmatrix}.$$  

Define

$$U_t = (X_t, Y_t, J^{X,Y}_{t \leftarrow 0}, J^{X,Y}_{0 \leftarrow t}),$$

then by proposition [2.2], we know for any $0 < \gamma < \frac{1}{2}$ the $\gamma$-Hölder constant $\|U_t\|_\gamma \in L^p(\Omega)$ for all $p \geq 1$. Since $Y_t$ is solution to a rough differential equation driven by $X_t$ with $C_0^\infty$ vector fields, $Y_t$ is automatically a rough path controlled by $X_t$. Along the same lines of proposition 8.1 and corollary 8.2 of [4], we know $\|Y, Y'\|_{X,2\gamma}$ is in $L^p(\Omega)$ for any $p \geq 1$. So we have

**Proposition 5.1.** Under the assumptions of theorem [18]. For any $\gamma \in (0, \frac{1}{2})$, define

$$\mathcal{L}_\theta = 1 + L_\theta(X)^{-1} + \|M_t\|_\gamma + \|Y, Y'\|_{X,2\gamma} + \rho_\gamma(X),$$

then $\mathcal{L}_\theta \in L^p(\Omega)$ for all $p \geq 1$.

5.2 Proof of main results

In order to prove theorem [18] by lemma 2.31 of [18] and proposition [2.4] it boils down to get an estimate on

$$\mathbb{P} \left( \inf_{\|v\|=1} v^T \Gamma(Y_t)v < \epsilon \right)$$

for every $t \in (0,1]$. Note that, we can write

$$\Gamma(Y_t) = J^Y_{t \leftarrow 0} \cdot C(Y_t) \cdot (J^Y_{t \leftarrow 0})^T,$$

where $C(Y_t)$ is the so called reduced Malliavin matrix of $Y_t$. Since $J^Y_{t \leftarrow 0}$ already verifies proposition [2.4] it suffices to prove our estimate for $C(Y_t)$. 

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For a fixed unit vector \( v \in \mathbb{R}^d \), we define
\[
f^i_v(s) = v^T \cdot J^Y_{0 \leq s} V_i(Y_s).
\]
With this and our previous computation (8), we have
\[
v^T C_t(Y)v = \sum_{j=1}^{d} \int_{0}^{1} \left| \sum_{i=1}^{d} \int_{0}^{t} f^i_v(s) dD^i_s X^i_s \right|^2 dt.
\]

**Proof of theorem 1.8.** It is easily checked that \( f^i_v(s) \) satisfy the assumptions of lemma 4.9. So for any \( \gamma \in (0, \frac{1}{2}) \), let
\[
F^{i,j}(r) = \sum_{i=1}^{d} \int_{0}^{r} f^i_v(s) dD^i_s X^i_s,
\]
we have \( R = \|F\|_{\gamma} \in L^p(\Omega) \) for all \( p \geq 1 \). When \( v^T C_t(Y)v < 1 \), by lemma A.3 of [9] we have
\[
\sup_{r \in [0,1]} \left| \sum_{i=1}^{d} \int_{0}^{t} v^T J^Y_{0 \leq s} V_i(Y_s)dD^i_s X^i_s \right| \leq 2R^{\frac{2\gamma}{2\gamma+1}} \left( v^T C_t(Y)v \right)^{\frac{1}{2\gamma+1}} + 2(\gamma^T C_t(Y)v)^{\frac{1}{2\gamma+1}}.
\]
(26)

We can deduce from (26) that there exists some constant \( C > 0 \) such that
\[
\sup_{r \in [0,1]} \int_{0}^{t} \Xi_s dD_r X_s \leq C(2R^{\frac{2\gamma}{2\gamma+1}} + 2(\gamma^T C_t(Y)v)^{\frac{1}{2\gamma+1}}),
\]
where \( \Xi_s \in \mathbb{R}^{d \times d} \) with \( (\Xi_s)_{ji} = f^i_v(s) \). Now by proposition 4.8 we are able to find \( \alpha > 0 \) such that
\[
\left\{ v^T C_t(Y)v < \epsilon \right\} \Rightarrow \left\{ \sup_{s \in [0,t]} |f^i_v(s)| \leq (2R^{\frac{2\gamma}{2\gamma+1}} + 2)^\alpha \epsilon^{\frac{\alpha}{2\gamma+1}} \right\}.
\]
(27)

The key observation is that
\[
f^i_v(t) = v^T J^Y_{0 \leq t} V_i(Y_t) = v^T V_i(y_0) + \int_{0}^{t} [V_i, V_0](Y_s) ds + \sum_{j=1}^{d} \int_{0}^{t} [V_j, V_i](Y_s) dX^i_s.
\]
By proposition 4.4 and 4.2, there exist some \( q, l > 0 \)
\[
\| [V_i, V_0] \|_{\infty} & \| [V_j, V_i] \|_{\infty} \leq M L^q_\theta \| f_v \|^l.
\]
By induction, we can see
\[
\left\| v^T J^Y_{0 \leq t} W(Y_t) \right\|_{\infty} \leq C L^m_\theta \| f_v \|^{n(k)}
\]
for all \( W \in \mathcal{W}_k \), where \( m(k), n(k) \) are constants only depend on \( k \). Since \( \{V_i\}_{0 \leq i \leq d} \) satisfy the parabolic Hömander's condition, we can find \( a_0 > 0 \) such that
\[
a_0 = \inf_{\|v\|=1} \sum_{W \in W_{0 \leq k \leq k_0}} \left\| v^T W(y_0) \right\| \leq \inf_{\|v\|=1} \sum_{W \in \mathcal{W}_k} \left\| v^T J^Y_{0 \leq t} W(Y_s) \right\|_{\infty}
\]
\[
\leq \inf_{\|v\|=1} \left\| v^T J^Y_{0 \leq t} W(Y_t) \right\|_{\infty}
\]
\[
\leq \inf_{\|v\|=1} C L^{m(k)}_\theta \| f_v \|^{n(k)}. \tag{28}
\]
From (27) and (28) we have
\[
\left\{ \inf_{\|v\|=1} v^T C_t(Y) v < \epsilon \right\} \Rightarrow \left\{ a_0 \leq C \mathcal{L}_\theta^m(k) \left( 2 \mathcal{R}^{\frac{2\gamma}{2\gamma+1}} + 2 \right) \alpha^n(k) \epsilon^{\frac{\alpha(n(k))}{2\gamma+1}} \right\}. \tag{29}
\]

On the other hand, since \( \mathcal{L}_\theta, \mathcal{R} \in L^p(\Omega) \) for any \( p \geq 1 \), we have
\[
\mathbb{P}\left\{ a_0 \leq C \mathcal{L}_\theta^m(k) \left( 2 \mathcal{R}^{\frac{2\gamma}{2\gamma+1}} + 2 \right) \alpha^n(k) \epsilon^{\frac{\alpha(n(k))}{2\gamma+1}} \right\} \leq C_p \epsilon^p. \tag{30}
\]

Hence, combining (29) (30), we see that when \( \epsilon \) is sufficiently small
\[
\mathbb{P}\left( \inf_{\|v\|=1} v^T C_t(Y) v \leq \epsilon \right) \leq C_p \epsilon^p. \tag{31}
\]

This finishes the proof of existence of a smooth density.

Let \( p_{Y_t}(y) \) be the density of \( Y_t \), we have the following upper bound (see proposition 2.1.4 and 2.1.5 in [18])
\[
p_{Y_t}(y) \leq C \cdot \mathbb{P}\left( \sup_{s \in [0,t]} |Y_s - y_0| > |y - y_0| \right)^\frac{1}{2} \left\| \det \left( \Gamma(Y_t) \right)^{-1} \right\|_{L^k(\Omega)}^m \left\| D Y_t \right\|_{h,\rho}^l,
\]
for some constants \( m, l, h, \rho, k \). Consider \((X_t, Y_t)\) as solution to a stochastic differential equation driven by Brownian motion, then by proposition 2.10 from [1] we have
\[
\mathbb{P}\left( \sup_{s \in [0,t]} |Y_s - y_0| > |y - y_0| \right)^\frac{1}{2} \leq \exp \left( -\frac{t}{C(y - y_0)^2} \right).
\]

We have just proved that \( \det(\Gamma(Y_t))^{-1} \in L^p(\Omega) \) for all \( p \geq 1 \). Finally, for \( \|D Y_t\|_{h,\rho}^l \) the exact argument of lemma 4.1 in [1] applies to \((X_t, Y_t)\). Therefore \( \|D Y_t\|_{h,\rho}^l < C(t) \), for some positive constant \( C(t) \), and the proof is finished. \( \square \)

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**References**

[1] F. Baudoin, E. Nualart, C. Ouyang, and S. Tindel, *On probability laws of solutions to differential systems driven by a fractional Brownian motion*, Ann. Probab. 44 (2016), no. 4, 2554–2590. MR3531675

[2] Fabrice Baudoin and Martin Hairer, *A version of Hörmander’s theorem for the fractional Brownian motion*, Probab. Theory Related Fields 139 (2007), no. 3-4, 373–395. MR2322701

[3] Thomas Cass and Peter Friz, *Densities for rough differential equations under Hörmander’s condition*, Ann. of Math. (2) 171 (2010), no. 3, 2115–2141. MR2680405

[4] Thomas Cass, Martin Hairer, Christian Litterer, and Samy Tindel, *Smoothness of the density for solutions to Gaussian rough differential equations*, Ann. Probab. 43 (2015), no. 1, 188–239. MR3298472

[5] Thomas Cass, Christian Litterer, and Terry Lyons, *Integrability and tail estimates for Gaussian rough differential equations*, Ann. Probab. 41 (2013), no. 4, 3026–3050. MR3112937
[6] Ilya Chevyrev and Marcel Ogrodnik, A support and density theorem for Markovian rough paths, Electron. J. Probab. 23 (2018), Paper No. 56, 16. MR3814250

[7] Peter Friz and Nicolas Victoir, On uniformly subelliptic operators and stochastic area, Probab. Theory Related Fields 142 (2008), no. 3-4, 475–523. MR2438699

[8] Peter K Friz and Nicolas B Victoir, Multidimensional stochastic processes as rough paths: theory and applications, Vol. 120, Cambridge University Press, 2010.

[9] M. Hairer and N. S. Pillai, Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion, Ann. Inst. Henri Poincaré Probab. Stat. 47 (2011), no. 2, 601–628. MR2814425

[10] Martin Hairer, On Malliavin’s proof of Hörmander’s theorem, Bull. Sci. Math. 135 (2011), no. 6-7, 650–666. MR2838095

[11] Martin Hairer and Jonathan C. Mattingly, A theory of hypoellipticity and unique ergodicity for semi-linear stochastic PDEs, Electron. J. Probab. 16 (2011), no. 23, 658–738. MR2786645

[12] Martin Hairer and Natesh S. Pillai, Regularity of laws and ergodicity of hypoelliptic SDEs driven by rough paths, Ann. Probab. 41 (2013), no. 4, 2544–2598. MR3112925

[13] Lars Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171. MR222474

[14] Yuzuru Inahama, Malliavin differentiability of solutions of rough differential equations, J. Funct. Anal. 267 (2014), no. 5, 1566–1584. MR3229800

[15] Antoine Lejay, Stochastic differential equations driven by processes generated by divergence form operators. I. A Wong-Zakai theorem, ESAIM Probab. Stat. 10 (2006), 356–379. MR2247926

[16] Paul Malliavin, Stochastic calculus of variation and hypoelliptic operators, Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976), 1978, pp. 195–263. MR536013

[17] James Norris, Simplified Malliavin calculus, Séminaire de Probabilités, XX, 1984/85, 1986, pp. 101–130. MR942019

[18] David Nualart, The Malliavin calculus and related topics, Second, Probability and its Applications (New York), Springer-Verlag, Berlin, 2006. MR2200233

[19] Daniel Revuz and Marc Yor, Continuous martingales and brownian motion, Vol. 293, Springer Science & Business Media, 2013.

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