Thermodynamical multihair and phase transitions of 4-dimensional charged Taub-NUT-AdS spacetimes

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Abstract

We study the behavior of phase transitions for the four-dimensional charged Taub-NUT-AdS spacetime with the Newman-Unti-Tamburino (NUT) parameter interpreted as the thermodynamic multihair in the extended thermodynamic phase space, and mainly focus on the effects of the NUT parameter on the phase transitions. We find that there is an upper bound on the value of the NUT parameter beyond which the corresponding physical inflection point or critical point will not exist, and the thermodynamic trihair interpretation of the NUT parameter would admit a little larger upper bound than the thermodynamic bihair interpretation. Moreover, as long as the NUT parameter is vanishingly small, the analogy to the van der Waals liquid/gas phase transition is valid irrespective of the multihair characteristics of the NUT parameter. However, as the NUT parameter increases to be comparable to the electric charge, such analogy to the van der Waals system will be broken, and the corresponding inflection point is not a thermodynamic critical point any more. For a large NUT parameter, there are frequent occurrences of the zeroth order phase transition in the case of the thermodynamic bihair interpretation, while only the first order phase transition happens in the case of the thermodynamic trihair interpretation.

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I. INTRODUCTION

It has been more than four decades since the concept of black hole entropy was well established in the pioneering work of Bekenstein and Hawking [1, 2]. Since then, a wide variety of phenomena concerning the thermodynamic aspects of black holes have been studied, which provide deep insights into the quantum nature of gravity. It was found that black holes can exhibit some incredibly rich thermal phase structures and phase transitions between different spacetime geometries. Particularly, for black holes in Anti-de Sitter (AdS) space, which are thermodynamically stable due to fact that AdS space acts as a confined cavity, their thermodynamical properties are quite different from those in asymptotically flat or de Sitter space. The first study about the thermodynamics of AdS black holes was reported by Hawking and Page [3], who found that there could be a phase transition between the stable large black hole and the thermal gas for Schwarzschild black holes in AdS space. It is worth pointing out that the thermodynamic behavior of black holes in AdS space is often utilized in strongly coupled gauge theories by means of AdS/CFT correspondence [4–8].

More recently, motivated by the basic thermodynamic scaling argument, it was realized that the negative cosmological constant should be interpreted as the pressure in the extended black hole thermodynamics and its conjugate quantity is just the thermodynamic volume [9]. With this proposal, some interesting thermodynamic phenomena and rich phase structures quite analogous to that of the van der Waals fluid are discovered for charged AdS black holes [9–11], e.g., the liquid/gas transition for the van der Waals fluid is analogous to the small/large black hole phase transition. A lot of more interesting phase transition phenomena for black holes in AdS space have been demonstrated as well, including triple points and reentrant phase transitions [12, 13], multiple re-entrant phase transitions [14], isolated critical points [15, 16] and a super fluid phase transition [17]. These thermodynamic phenomena exhibit novel chemical-type phase behaviors, leading to the naming of black hole chemistry [18, 19].

On the other hand, asymptotically locally flat spacetimes endowed with a nonzero NUT charge posed a great challenge to the study of thermodynamics. The Taub-Newman-Unti-Tamburino (Taub-NUT) solution as a simple non-radiating exact solution to Einstein’s field equations was firstly discovered by Taub in 1951 [20], and subsequently rediscovered as a candidate for a black hole in the generalization of the Schwarzschild space-time [21].
The Lorentzian Taub-NUT metric is one of the most intriguing solutions since it features a string like singularity (usually called Misner string singularity) on the polar axis with closed timelike curves in its vicinity, and carries a peculiar type of the NUT charge (the Misner gravitational charge) which is somewhat analogous to the magnetic monopole. In order to avoid these singularities, Misner once imposed a periodical condition for the time coordinate in the Lorentzian Taub-NUT metric [22]. Such a periodical identification, however, may create closed timelike curves everywhere, rendering the maximal extension of the spacetime problematic [23, 24].

Recently, it has been argued that the Taub-NUT-type spacetime with the presence of Misner strings is less pathological than previously thought. The Misner strings are transparent for geodesics, which could render the spacetime geodesically complete and be free of causal pathologies for freely falling observers [25–28]. Based on these arguments, a lot of efforts have been made to formulate a consistent and reasonable thermodynamical first law for the Lorentzian Taub-NUT-type spacetimes with the presence of Misner strings [28–32]. In such developments, a pair of new conjugate quantities related to the NUT parameter are introduced as a crucial step to obtain the first law. However, it has been argued in Ref. [33] that the thermodynamical quantity associated the NUT parameter in the aforementioned studies of the thermodynamics does not possess the conventional characters of global charges measured at infinity.

Therefore, it was suggested that the NUT parameter should be considered as a thermodynamic multihair rather than a single feature of the physical source, and then the appropriate thermodynamic first law for Taub-NUT-type spacetimes could be naturally formulated by first deriving the Christodoulou-Ruffini-type squared-mass formula without imposing any constraint condition.

In this paper, based on the thermodynamical first law derived in Ref. [33], we perform a study on the behavior of possible phase transitions for the 4-dimensional Lorentzian Reissner-Nordström-Taub-NUT-AdS (RN-Taub-NUT-AdS) spacetime in the extended thermodynamical phase space, and examine the analogy to the van der Waals-like liquid/gas transition. We also attempt to seek the influence of the NUT parameter as a poly-facet on the corresponding phase structure and the phase transition and make a cross-comparison of the role of thermodynamic bihair and trihair played by the NUT parameter in phase transitions. The organization of the paper is as follows. In next section, we will review the
thermodynamics for the RN-Taub-NUT-AdS spacetime with the NUT parameter interpreted as a thermodynamic bihair (angular momentum and NUT charge) or a thermodynamic trihair (angular momentum, NUT charge and magnetic mass). In Sec.II, we investigate the influence of the NUT parameter on the phase transitions with the NUT parameter interpreted as a thermodynamic bihair and thermodynamic trihair, respectively. This allows a comparison between the behaviors of phase transitions for the $N$-bihair solution and the $N$-trihair solution. Meanwhile, the presence of the critical point or inflection point is also discussed in detail. Finally, we end up in Sec.IV by summarizing the results.

II. THE FIRST LAW WITH THE NUT PARAMETER INTERPRETED AS A THERMODYNAMIC MULTIHAIR

For the 4-dimensional RN-Taub-NUT-AdS spacetime with a nonzero cosmological constant and a pure electric charge, the metric can be written in following form with the Misner strings symmetrically distributed along the polar axis [21, 33–37]

$$ds^2 = -\frac{f(r)}{r^2 + N^2}(dt + 2N \cos \theta d\phi)^2 + \frac{r^2 + N^2}{f(r)}dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$f(r) = r^2 - 2Mr - N^2 + Q^2 + \frac{(r^4 + 6N^2r^2 - 3N^4)}{\ell^2}$$

and the one-form electromagnetic potential

$$A = \frac{Qr}{r^2 + N^2}(dt + 2N \cos \theta d\phi).$$

Here, the parameters $M, Q, N$ and $\ell$ represent the electric mass, the electric charge, the NUT charge and the AdS radius, respectively. In the discussions which follow, we only consider the case in which the charges, i.e., $Q$ and $N$, are all positive for simplicity. Introducing $J_N = NM$ as the “angular momentum”, the second physical feature of the NUT parameter, and after some algebraic manipulations, the Christodoulou-Ruffini-type squared-mass formula [38] can be derived straightforwardly

$$M^2 = \frac{\pi}{45} \left[ \left(1 + \frac{32\pi}{3}PN^2\right) \left(\frac{S}{\pi} - 2N^2\right) + Q^2 + \frac{8S^2P}{3\pi} \right]^2 + \frac{J_N^2\pi}{S},$$

where the thermodynamic pressure $P = 3/(8\pi\ell^2)$ and the entropy $S = \pi(r_+^2 + N^2)$ with $r_+$ representing the radius of the outer horizon.
The differentiation of Eq. (4) leads to the first thermodynamic law with the NUT parameter interpreted as a thermodynamic bihair (N-bihair solution) \[33\]

\[dM = TdS + \omega_h dJ_N + \Psi_h dN + \Phi dQ + V dP,\] 

where the corresponding conjugate thermodynamic variables satisfy

\[T = \frac{f'(r_+)}{4\pi(r_+^2 + N^2)} = \frac{r_+ - M + 2r_+(r_+^2 + 3N^2)/\ell^2}{2\pi(r_+^2 + N^2)}, \quad \Phi = \frac{Qr_+}{r_+^2 + N^2}, \]

\[\Psi_h = \frac{4N_R(r_+^2 - 3N^2) - 2N_R\ell^2}{(r_+^2 + N^2)\ell}, \quad \omega_h = \frac{N}{r_+^2 + N^2},\]

\[V = \frac{4\pi r_+(r_+^4 + 6N^2r_+^2 - 3N^4)}{3(r_+^2 + N^2)}.\] 

Here, the value of the NUT parameter \(N\) cannot be overwhelmingly large in comparison with \(Q\) or \(r_+\) so as to hold the thermodynamic volume positive. In other words, the horizon radius \(r_+\) should be larger than \(\sqrt{2\sqrt{3} - 3N}\) to meet the requirement of a positive volume. In the following discussions, we will assume that this condition is satisfied, in accordance with the usual analysis of the van der Waals liquid-gas system. If a positive thermodynamic volume is assumed, one can choose to interpret the NUT parameter \(N\) as the thermodynamic trihair, i.e., the angular momentum, NUT charge and magnetic mass. Thus, in the metric described by Eq. (1), another consistent first law of thermodynamics can also be obtained by assuming angular momentum \(J_N = NM\) and magnetic mass \(\tilde{M} = N(1 + 4N^2/\ell^2)\). Note here that the positivity of \(N\) ensures the positivity of \(\tilde{M}\). Through some algebraical manipulations, the thermodynamic volume can be derived as \(\tilde{V} = 4\pi r_+(r_+^2 + 3N^2)/3\) via appropriately adjusting the identity \(f(r_+) = 0\) and the expression of the square-mass formula. Concretely, the corresponding first law for the \(N\)-trihair solution can be written as \[33\]

\[dM = TdS + \omega_h dJ_N + \Phi dQ + \tilde{\Psi}_h dN + \xi d\tilde{M} + \tilde{V} dP,\] 

where the additional or new conjugate thermodynamic quantities satisfy

\[\tilde{\Psi}_h = -\frac{2N_R}{r_+^2 + N^2} + \frac{4N^2 - \ell^2}{\ell^2}\xi, \quad \xi = \frac{r_+(r_+^2 - 3N^2)}{4N(r_+^2 + N^2)}.\] 

It is worth pointing out here that the expression of the thermodynamic volume \(\tilde{V} = 4\pi r_+(r_+^2 + 3N^2)/3\) is completely identical to the result given in Refs. [28–30] where the first law is formulated by introducing a new thermodynamic NUT charge together with its
conjugate quantity that is not a global conventional charge but a combination of parameters \( N, Q, \ell \) and \( r_+ \). It is easy to find that no matter how large the NUT parameter will be, the thermodynamic volume \( \tilde{V} \) is always nonnegative. In comparison with the case of the \( N \)-bihair solution, the thermodynamic volume of the \( N \)-trihair solution approaches zero only in the condition \( r_+ = 0 \).

To gain a better understanding of the influence of the NUT parameter on the thermodynamic properties of the RN-Taub-NUT-AdS spacetime, we will next discuss the possible phase structure and the phase transition in the cases of the \( N \)-bihair solution and \( N \)-trihair solution respectively.

III. THE PHASE TRANSITIONS WITH THE NUT PARAMETER BEING A THERMODYNAMIC MULTIHAIR

In above discussions, the equation of state and the corresponding thermodynamic quantities have been derived explicitly. We are now going to explore the influence of the NUT parameter on the phase structure of the 4-dimensional charged Taub-NUT-AdS spacetimes, and examine the analogy to the van der Waals liquid/gas transition.

A. \( N \)-bihair solution

It is easy to find that when the NUT parameter vanishes the metric (1) reduces to the 4-dimensional charged AdS black holes. For a nonzero NUT parameter, Eq. (6) can be translated into the equation of state for the charged Taub-NUT-AdS spacetime with fixed \( Q \) and \( N \), that is

\[
P = \frac{r_+ T}{2(N^2 + r_+^2)} - \frac{1}{8\pi (N^2 + r_+^2)} + \frac{Q^2}{8\pi (N^2 + r_+^2)^2}, V = \frac{4\pi r_+ (r_+^4 + 6N^2 r_+^2 - 3N^4)}{3(r_+^2 + N^2)}. \tag{9}
\]

Recalling that the corresponding critical point is determined by the inflection point in the pressure-volume diagram, we have

\[
\frac{\partial P}{\partial V} = 0, \quad \frac{\partial^2 P}{\partial V^2} = 0. \tag{10}
\]

To facilitate the study on the possible behavior of phase transitions in the pressure-volume diagram, the nonnegative \( V^{1/3} \) can be approximately treated as the specific volume in the following discussions due to \( V^{1/3} \propto (r_+^2 + 5N^2/3)/r_+ \) for a not too large value of \( N \).
Since the parameters $Q$ and $N$ interwind in the equation of state, the exact expression of the inflection point solution for Eq. (10) is quite complicated. However, for the special case of $Q \gg N$, the critical point can be approximated as

$$P_c \approx \frac{1}{96\pi Q^2} + \frac{N^2}{216\pi Q^4}, \quad V_c \approx 8\sqrt{6}\pi Q^3 + \sqrt{\frac{8}{3}\pi QN^2}, \quad T_c \approx \frac{\sqrt{6}}{18\pi Q} + \frac{\sqrt{6}N^2}{72\pi Q^3},$$

(11)

where the subscript “c” represents the corresponding quantity satisfying the condition for a critical point. As $N$ increases to be large enough, the thermodynamic volume at the inflection point would turn into negative. As result, the corresponding inflection point will become unphysical. Therefore, the additional constraints $V > 0$, $P > 0$ and $T > 0$ should be considered in Eq. (10). It is easy to verify that there is an upper bound on the NUT parameter $N$ for a fixed eclectic charge $Q$, that is $N_b = \sqrt{3\sqrt{3} - 1}Q/2 \approx 0.8264Q$. If $N > N_b$, the corresponding inflection point will be located at where the volume is negative (i.e., $V_c < 0$), which means that there is no longer a physically meaningful critical point or inflection point.

According to Eq. (9), it is easy to find when the NUT parameter $N$ is very small ($N \ll N_b$) the possible phase structure is analogous to that of the 4-dimensional RN-AdS black holes in the extended phase space, where the analogous liquid/gas phase transition of the van der Waals system occurs [9]. In order to show these properties in detail, the corresponding pressure-volume diagram and the behavior of the Gibbs free energy that is defined by $G = M - TS$ in the extended thermodynamics for a fixed $Q$ are illustrated in Fig.(1). As shown in Fig.(1), there obviously exists a first order phase transition between the small black hole and large black hole for $T < T_c$ or $P < P_c$, and the inflection point obeying condition (10) represents the critical point of the phase transition whose approximate expression is given by Eq. (11).

As the value of $N$ increases gradually but still remains smaller than $N_b$, the corresponding phase structure deviates somewhat from the van der Waals phase structure. Since the vanishing thermodynamic volume ($V = 0$) itself is more likely to be located at the phase transition point or inflection point and plays a crucial role in the behavior of the phase transition, it is worth exploring firstly the properties of the thermodynamic temperature in the limit of a vanishing volume. According to the equation of state (9), it is easy to find that for $N > \sqrt{1 + \sqrt{3}Q/2} \approx 0.8264Q$, the temperature $T$ at the point of $V = 0$ holds positive for any nonnegative pressure, while for $N \leq \sqrt{1 + \sqrt{3}Q/2}$, the temperature at $V = 0$ would
FIG. 1: The plots for the $N$-bihair solution with $Q = 1.0, N = 0.01Q$. (a) The pressure-volume diagram (the isotherm) is displayed with $T_c \approx 0.0433$ and $T_0 \approx 0.0306$. Note that if $T < T_0$, a part of the isotherm corresponds to a negative pressure. (b) The behavior of the Gibbs free energy is depicted in the $G-T$ plane with $P_c \approx 0.0033$. Its behavior in general is analogous to the situation of RN-AdS black holes. Here the black arrows indicate the increasing horizon $r_+$.

be either positive or negative, strongly depending on the magnitude of pressure $P$. After some simple algebraic manipulations, such pressure can be straightforwardly obtained

$$P_0 = \frac{Q^2 - 2(\sqrt{3} - 1)N^2}{64\pi(2 - \sqrt{3})N^4}. \quad (12)$$

If $P > P_0$, the temperature at $V = 0$ is positive for $N \leq \sqrt{1 + \sqrt{3}Q/2}$; however, for $P < P_0$, the temperature at $V = 0$ will be negative for $N \leq \sqrt{1 + \sqrt{3}Q/2}$. The latter case implies that $V = 0$ will make the temperature be of no physical sense. These properties can be straightforwardly captured in Figs. (2) and (4) that follow. Indeed, if we want to explore the interesting role played by $N$ and the vanishingly small $V$ on the thermodynamical phase transition behavior, the non-negativeness of the thermodynamical temperature is a necessary constraint. Therefore, we can examine, without loss generality, the corresponding behavior of phase transitions for the $N$-bihair solution via appropriately choosing certain values of $N$ based on the above discussions.

In Fig. (2), with $N = 0.7Q < \sqrt{1 + \sqrt{3}Q/2}$, we find as we expect that only if the pressure is large enough, can the point $V = 0$ occur as an obstruction of the Gibbs free energy in the $G-T$ diagram. Interestingly, there, aside from the first order phase transition
for $P < 0.6289 P_c$, exists a zeroth order phase transition at a certain large constant pressure (e.g., $P = 100P_c > P_0$). The zeroth order phase transition is not a real phase transition, but it is discovered that the global minimum of the Gibbs free energy is discontinuous in the $G$-$T$ diagram [11]. The details of the zeroth order phase transition are depicted in Fig. (3). Therefore, one may conclude that the point $V = 0$ located on the minimal branch of the Gibbs free energy for the $N$-bihair solution may have important effects on the possible phase transitions. Besides these, it should be pointed out the inflection point here is no longer the usual thermodynamic terminal critical point of the first order phase transition.

As shown in Fig. (4), with $N = Q$ satisfying $\sqrt{1 + \sqrt{3}Q}/2 < N < N_b$, the phase transitions

\begin{figure}[h]
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\begin{subfigure}{0.45\textwidth}
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\includegraphics[width=\textwidth]{fig2a.pdf}
\caption{(a)}
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\includegraphics[width=\textwidth]{fig2b.pdf}
\caption{(b)}
\end{subfigure}
\caption{The plots for the case of the $N$-bihair solution with $Q = 1.0, N = 0.7Q$. (a) The pressure-volume diagram is displayed with $T_c \approx 0.0505$ and $T_0 \approx 0.0337$. (b) The behavior of the Gibbs free energy is displayed with $P_c \approx 0.0044$, where the additional inserted small plot has $P = 100P_c$ with the solid circle point denoting $V = 0$. It should be pointed out that the inflection point here is no longer the usual terminal critical point of the first order phase transition due to the presence of a small cusp at $P = P_c$ for the Gibbs free energy. Actually, for small constant pressure ($P < 0.6289P_c$ in numerical evaluation), there exists a first order phase transition between the small and large black hole, and there seems to be a zeroth order phase transition for $0.6289P_c < P < P_c$ and $P > 44.50P_c$. Aside from these regions of $P$, no phase transition occurs. Note the black arrows indicate the increasing horizon radius $r_+$, and a zoomed-in view about the zeroth order phase transition is displayed in Fig.(3).}
\end{figure}
FIG. 3: A zoomed view of the zeroth order phase transition in the case of $N$-bihair solution with $Q = 1.0, N = 0.7Q, P_c \approx 0.0044$ for (a) $P = 0.8P_c$ and for (b) $P = 100P_c$. The red solid lines indicate the global minimal branch of the Gibbs free energy, while the black solid lines denote the non-minimal branch. The black dashed line and the circle point in right plot stand for $V < 0$ and $V = 0$, respectively. Obviously, there is a jump for the global minimum of Gibbs free energy at temperature $T_z$, which means the occurrence of zeroth order phase transition between the small and large black hole.

structure becomes quite rich in comparison with the case of $N < \sqrt{1 + \sqrt{3Q/2}}$. There still exist possible zeroth and first order phase transitions in the $G$-$T$ diagram, and it is highly possible that the points of $V = 0$ just correspond to the zeroth order phase transition points as long as the constant pressure $P$ is not too small. Moreover, as shown in Fig. (5), where the possible phase transition points are marked by vertical red dashed lines, besides the large/small black hole phase transition, there exist some complicated phase transitions, e.g., the large/small/large/medium black hole phase transition, the large/small/medium black hole phase transition and the large/medium/small black hole phase transition. Here, the inflection point in this case is again no longer the usual terminal critical point of the first order phase transition, and the behavior of the phase transition is quite different from that of the van der Waals phase transition.

As for a large $N$, we can set $N = 2Q > N_b$ without loss of generality, then there is no longer physical inflection point for the thermodynamic system (see Fig. (6)). And if the pressure is large enough, all the points of $V = 0$ in the $G$-$T$ diagram correspond to the
FIG. 4: The plots for the case of the $N$-bihair solution with $Q = 1.0, N = Q$. The isotherms are displayed with $T_c = 1/(4\pi)$ and $T_0 = 1/(8\pi)$ in (a). The behavior of the Gibbs free energy is depicted with $P_c = 1/(32\pi)$ in (b), where the dashed colored lines and circle points denote $V < 0$ and $V = 0$, respectively. The phase transitions are quite complicated; concretely, for $0 < P \leq 0.2618 P_c$, there exists a first order phase transition, for $P \in (0.2618 P_c, 0.3030 P_c)$, two first order phase transitions and one zeroth order phase transition, for $P \in [0.3030 P_c, 0.3183 P_c]$, a first order phase transition and a zeroth order phase transition, for $P \in (0.3183 P_c, \sqrt{3} P_c/2) \cup (\sqrt{3} P_c/2, 0.9306 P_c)$, two zeroth order phase transitions, and for $P \geq 0.9306 P_c$, only one zeroth order phase transition. The corresponding details of phase transitions are analyzed in Fig. (5) as a supplement. Here the black arrows indicate the increasing horizon $r_+$,

points of the zeroth order phase transition between the small and large black hole.

B. $N$-trihair solution

Let us now turn to the $N$-trihair solution. Similarly, the equation of state can be written as

$$P = \frac{r_+ T}{2(N^2 + r_+^2)} - \frac{1}{8\pi(N^2 + r_+^2)} + \frac{Q^2}{8\pi(N^2 + r_+^2)^2}, \quad \tilde{V} = \frac{4\pi r_+(r_+^2 + 3N^2)}{3}. \tag{13}$$

The critical point (inflection point) can still be obtained by solving the equation

$$\frac{\partial P}{\partial V} = 0, \quad \frac{\partial^2 P}{\partial V^2} = 0. \tag{14}$$

To make the inflection have physical sense, constraints $\tilde{V} > 0$, $P > 0$ and $T > 0$ should also be imposed on Eq. (14).
FIG. 5: Some zoomed-in views of the possible phase transitions for the $N$-bihair solution with $Q = 1.0, N = Q$ and $P_c = 1/(32\pi)$. In all plots, the red solid lines highlight the global minimal branches of the Gibbs free energy, while the black solid lines denote the non-minimal branches with the black dashed lines and the circle points indicating $V < 0$ and $V = 0$, respectively. Here, $T_{z1}, T_{z2}$ indicate the corresponding temperatures of the zeroth order phase transition and $T_{f1}, T_{f2}$ correspond the temperatures of the first order phase transition. More interestingly, for the case of $P = 0.28P_c$, there exists a reentrant-like phase transition behavior: the large/small/large/medium black hole phase transition as the temperature continuously decreases.

After some straightforward algebraical manipulations, we find that there also exists an upper bound on $N$ (denoted by $N_t$ ), that is $N_t = 3Q/2\sqrt{2} \approx 1.0607Q$ which is slightly larger than the previous one for the $N$-bihair situation. It should be emphasized that for $N > N_t$ the inflection point would become unphysical, since a large value of $N$ would render the pressure at the inflection point negative rather than the volume. For a special case of $N \ll Q$, the inflection point can be simplified to

$$P_c \approx \frac{1}{96\pi Q^2} + \frac{N^2}{216\pi Q^4}, \tilde{V}_c \approx 8\sqrt{6}\pi Q^3 - 2\sqrt{6}\pi QN^2, T_c \approx \frac{\sqrt{6}}{18\pi Q} + \frac{\sqrt{6}N^2}{72\pi Q^3}. \quad (15)$$

According to the equation of state (13) and Eq. (15), it is easy to deduce that for small
FIG. 6: The plots for the case of $N$-bihair solution with $Q = 1.0, N = 2Q$. The isotherms and Gibbs free energy are displayed in (a) and (b), respectively. Note that the dashed lines correspond to negative pressure in (a) and negative volume in (b), respectively. For $P \leq 0.0041$, there is a first order phase transition, while for $P > 0.0041$, the points of $V = 0$, denoted by circle points, indicate the zeroth order phase transitions. The black arrows indicate the increasing horizon $r_+$. 

$N$ the phase structure of the $N$-trihair solution is quite analogous to that of the $N$-bihair solution. If $N$ is not too small, according to the foregoing discussion, the influence of $N$ on the phase transitions would become important. Without loss generality, we could choose $N = \{Q, 2Q\}$ for convenience in comparison with the $N$-bihair solution. In Fig. (7), the isotherms for the $N$-trihair solution with $N = Q$ and $N = 2Q$ are plotted, respectively. We can see that the pressure $P$ is an increasing function of the volume $\tilde{V}$ in the regime of small volume, especially in comparison with the $N$-bihair solution at $N = Q$ (Fig. (4(a))), and such characters of increasing function are quite apparent. One may analyze this property from a pure mathematical point of view. Differentiate the state equations (9)(13), we have

$$
\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial P}{\partial r_+}\right)_{r_+} \frac{\partial r_+}{\partial V} = -\frac{3}{16\pi^2} \frac{(N^2 - 2Q^2)r_+ + r_+^3 + 2N^4T\pi - 2\pi Tr_+^4}{(N^2 + r_+^2)(3N^6 - 21N^4r_+^2 - 11N^2r_+^4 - 3r_+^6)} \tag{16}
$$

and

$$
\left(\frac{\partial P}{\partial V}\right)_T = \frac{r_+(N^2 - 2Q^2 + r_+^3)}{16\pi^2(N^2 + r_+^2)^4} + \frac{(N^2 - r_+^2)T}{8\pi(N^2 + r_+^2)^3}. \tag{17}
$$

For the $N$-bihair solution, whether $(\partial P/\partial V)_T$ is negative or positive near the point $V = 0$ (i.e., $r_+ = \sqrt{2\sqrt{3} - 3N}$) should be dependent on both the NUT parameter and the temperature, while for the $N$-trihair solution, for vanishingly small $\tilde{V}$ ($r_+ \to 0$), Eq. (17)
can be approximated as
\[
\left( \frac{\partial P}{\partial \tilde{V}} \right)_T \approx \frac{T}{8N^4 \pi} + \frac{r_+(N^2 - 2Q^2)}{16N^8 \pi^2}.
\] (18)

Thus, the thermodynamic pressure \( P \) at a constant temperature is an increasing function of \( \tilde{V} \) in the regime of small volume as long as \( N \) is not vanishingly small.

![Graphs](a) and (b)

**FIG. 7:** The pressure-volume diagram for the \( N \)-trihair solution at various temperature with fixed \( Q = 1.0, N = Q \) in (a) and \( Q = 1.0, N = 2Q \) in (b). Here, \( T_c = 1/(4\pi) \), \( T_0 = 1/(8\pi) \) and the dashed lines represent the negative pressure.

For a cross-comparison of the pressure-volume diagram between the \( N \)-bihair and \( N \)-trihair solutions, the isotherms at temperature \( T = T_c \) for both the \( N \)-bihair and \( N \)-trihair situation are plotted in Fig. (8) as a supplement to analyze above property.

In Fig. (9), the behavior of the Gibbs free energy for the \( N \)-trihair solution with \( N = \{Q, 2Q\} \) is depicted in detail. It is not difficult to find that all the curves are identical to those for the \( N \)-bihair solution except for that the dashed curves of the Gibbs free energy correspond to the constraint of a positive volume. Here, it should be emphasized that as long as \( r_+ > 0, \tilde{V} > 0 \) will hold. Therefore, the constraint of \( \tilde{V} > 0 \) is trivial and independent of \( N \), which leads to that the first order phase transition occurs for the \( N \)-trihair solution rather than the zeroth phase transition. The detailed phase transition points are indicated by vertical dashed red lines. However, the analogy to the van der Waals liquid-gas phase transition has been broken by the presence of a not too small \( N \). The inflection points, if exist, do not indicate the usual critical points of thermodynamic phase transitions any more.
FIG. 8: The isotherms at the temperature $T = T_c$ for both $N$-bihair and $N$-trihair solution with fixed $Q = 1.0$.

FIG. 9: The Gibbs free energy is plotted as a function of temperature in the $G$-$T$ plane for the $N$-trihair solution. Here, we set $Q = 1.0, N = Q$ in (a) and $Q = 1.0, N = 2Q$ in (b). The vertical dashed colored lines or the black circle dots indicate the corresponding temperatures where the first order phase transition would occur at a certain constant pressure. The black arrows indicate the increasing horizon radius $r_+$. 
IV. CONCLUSION

In the extended thermodynamic phase space, we have studied the thermodynamical phase structure of the 4-dimensional Lorentzian RN-Taub-NUT-AdS spacetime, including the possible phase transitions, critical points and analogy to the van der Waals fluid. Armed with the cognition that the NUT parameter may possess multiple physical characters and the consistent thermodynamical first law formulated in Ref. [33], we analyze the influence of the NUT parameter on the possible phase transitions of such NUT-type spacetime in detail.

When the NUT parameter $N$ is interpreted as a thermodynamic bihair, we find that there is an upper bound on the value of $N$ for fixed electric charge $Q$, that is $N_b = \sqrt{3\sqrt{3} - 1}Q/2$, beyond which the corresponding physical inflection point or critical point will not exist. If the NUT parameter $N$ is very small ($N \ll N_b$), there seems to be a first order phase transition between the small and large black hole, which is quite analogous to the van der Waals phase transition between liquid and gas. As $N$ increases gradually, the analogy to the van der Waals system is broken somewhat. The zeroth order phase transition between the small and large black hole may occur, and the inflection point obtained by solving the condition for a critical point is not the usual thermodynamic critical point. Especially for $N = Q$, the constraint of a positive volume may give rise to a zeroth order phase transition. As a result, there exists rich phase structures and more complicated reentrant-like phase transitions. With $N$ increasing beyond the upper bound $N_b$, it is found that the first order phase transition and the inflection point will disappear completely for a large enough positive pressure. Instead, only the zeroth order phase transition is likely to happen.

Regarding the $N$-trihair solution, one may also find a upper bound, $N_t = 3Q/2\sqrt{2}$, which is a little larger than the upper bound $N_b$ for the $N$-bihair solution. For small $N(N < Q)$, the small/large black hole phase transition, similar to the $N$-bihair solution, will appear like the van der Waals liquid/gas phase transition. As $N$ increases to be comparable to $Q$, in contrast to the situation of the $N$-bihair solution, the phase structure then will become relatively simple, and it seems that only the first order phase transition between the small and large black hole occurs since the mere constraint of a positive volume cannot give rise to the zeroth order phase transition for the $N$-trihair solution. Particularly, when $N$ goes beyond its upper bound $N_t$, there is neither the zeroth order phase transition nor the physical inflection point or critical point. Besides these, for not too small $N$, it is obvious
that the pressure along the isotherm for the $N$-trihair solution behaves as an increasing function of $\tilde{V}$ in the regime of small volume. Therefore, one may conclude that not only the magnitude but also the thermodynamic multihair characters of the NUT parameter can influence the phase structure and the phase transition of the 4-dimensional RN-Taub-NUT-AdS spacetime. The thermodynamical analogy to the van der Waals system is valid only in the case of a vanishingly small NUT parameter.

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