Division of an Angle into Equal Parts and Construction of Regular Polygons by Multi-Fold Origami

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Abstract. This article analyses geometric constructions by origami when up to \( n \) simultaneous folds may be done at each step. It shows that any arbitrary angle can be \( m \)-sected if the largest prime factor of \( m \) is \( p \leq n + 2 \). Also, the regular \( m \)-gon can be constructed if the largest prime factor of \( \phi(m) \) is \( q \leq n + 2 \), where \( \phi \) is Euler’s totient function.

1. Introduction

Two classic construction problems of plane geometry are the division of an arbitrary angle into equal parts and the construction of regular polygons [14]. It is well known that the use of straight edge and compass allows for the bisection of angles and the constructions of regular \( m \)-gons if \( m = 2^a p_1 p_2 \cdots p_k \), where \( a, k \geq 0 \) and each \( p_i \) is a distinct odd prime of the form \( p_i = 2^{h_i} + 1 \). It is also known that origami extends the constructions by allowing for the trisection of angles and the constructions of regular \( m \)-gons if \( m = 2^{a_1} 3^{a_2} p_1 p_2 \cdots p_k \), where \( a_1, a_2, k \geq 0 \) and each \( p_i \) is a distinct prime of the form \( p_i = 2^{h_i} 3^{h_i} + 1 > 3 \) [1].

Standard origami constructions are performed by a sequence of elementary single-fold operations, one at a time. Each elementary operation solves a set of specific incidences constraints between given points and lines and their folded images [1, 2, 8]. A total of eight elementary operations may be defined and stated as in Table 1 [12]. The operations can solve arbitrary cubic equations [3, 7], and therefore they can be applied to related construction problems such as the duplication of the cube [15] and those mentioned above [3, 4, 5].

The range of origami constructions may be extended further by using multi-fold operations, in which up to \( n \) simultaneous folds may be performed at each step [2], instead of single folds. In the case of \( n = 2 \), the set of possible elementary operations increases to 209 or more (the exact number has still not been determined). It has been shown that 2-fold origami allows for the geometric solution of arbitrary septic equations [9], quintisection of an angle [10] and construction of the regular hendecagon [13].

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Table 1. Single-fold operations [12]. $\mathcal{O}$ denotes the medium in which folds are performed; e.g., a sheet of paper, fabric, plastic, metal or any other foldable material.

| #   | Operation                                                                 |
|-----|---------------------------------------------------------------------------|
| 1   | Given two distinct points $P$ and $Q$, fold $\mathcal{O}$ to place $P$ onto $Q$. |
| 2   | Given two distinct lines $r$ and $s$, fold $\mathcal{O}$ to align $r$ and $s$. |
| 3   | Fold along a given a line $r$.                                            |
| 4   | Given two distinct points $P$ and $Q$, fold $\mathcal{O}$ along a line passing through $P$ and $Q$. |
| 5   | Given a line $r$ and a point $P$, fold $\mathcal{O}$ along a line passing through $P$ to reflect $r$ onto itself. |
| 6   | Given a line $r$, a point $P$ not on $r$ and a point $Q$, fold $\mathcal{O}$ along a line passing through $Q$ to place $P$ onto $r$. |
| 7   | Given two lines $r$ and $s$, a point $P$ not on $r$ and a point $Q$ not on $s$, where $r$ and $s$ are distinct or $P$ and $Q$ are distinct, fold $\mathcal{O}$ to place $P$ onto $r$, and $Q$ onto $s$. |
| 8   | Given two lines $r$ and $s$, and a point $P$ not on $r$, fold $\mathcal{O}$ to place $P$ onto $r$, and to reflect $s$ onto itself. |

Thus, the purpose of this article is to analyze the general case of $n$-fold origami with arbitrary $n \geq 1$ and determine what angle divisions and regular polygons can be obtained.

2. Single- and multi-fold origami

An $n$-fold elementary operation is the resolution of a minimal set of incidence constraints between given points, lines, and their folded images, that defines a finite number of sets of $n$ fold lines [2]. For the case of $n = 1$, all possible elementary operations are those listed in Table 1. An example of operation for $n = 2$ is illustrated in Fig. 1.

Any number of $n_i$-fold operations, $i = 1, 2, \ldots, k$, may be gather together and considered as a unique $n$-fold operation, with $n = \sum_{i=1}^{k} n_i$. Thus, we define $n$-fold origami as the construction tool consisting of all the $k$-fold elementary operations, with $1 \leq k \leq n$.

The medium on which all folds are performed is assumed to be an infinite Euclidean plane. Points are referred by their Cartesian $xy$-coordinates or by identifying them as complex numbers, as convenient. A point or complex number is said to be $n$-fold constructible iff it can be constructed starting from numbers 0 and 1 and applying a sequence of $n$-fold operations. It has been shown that the set of constructible numbers in $\mathbb{C}$ by single-fold origami is the smallest subfield of $\mathbb{C}$ that is closed under square roots, cube roots and complex conjugation [1]. An immediate corollary is that the field $\mathbb{Q}$ of rational numbers is $n$-fold constructible, for any $n \geq 1$.

The present analysis is based on the following version of a previous theorem by Alperin and Lang [2].
Figure 1. A two-fold operation [13]. Given two points $P$ and $Q$ and three lines $\ell$, $r$, $s$, simultaneously fold along a line $\gamma$ to place $P$ onto $r$, and along a line $\delta$ to place $Q$ onto $s$ and to align $\ell$ and $\gamma$.

Theorem 1. The real roots of any $m$th-degree polynomial with $n$-fold constructible coefficients are $n$-fold constructible if $m \leq n + 2$.

Proof. The real roots of any $m$th-degree polynomial may be obtained by Lill’s method [11, 7, 17]. It consists of defining first a right-angle path from and origin $O$ to a terminus $T$, where the lengths and directions of the path’s segments are given by the non-zero coefficients of the polynomial. Next, a second right-angle path with $m$ segments between $O$ and $T$ is constructed by folding, and this construction demands the execution of $m - 2$ simultaneous folds, if $m \geq 3$, or a single fold, if $m \leq 3$. The first intersection (from $O$) between both paths is the sought solution.

Details of the method may be found in the cited references. An example for solving $x^5 - a = 0$ is shown in Fig. 2. □

It must be noted that the roots of 5th- and 7th-degree polynomials may be obtained by 2-fold origami, instead of the 3- and 5-fold origami, respectively, predicted by the above theorem [16, 9]. Therefore, Theorem 1 only posses a sufficient condition on the number of simultaneous folds required.

3. Angle section

Let us consider first the case of division into any prime number of parts.

Lemma 2. Any angle may be divided into $p$ equal parts by $n$-fold origami if $p$ is a prime and $p \leq n + 2$.

Proof. Let $\ell$ be a line forming an angle $\theta$ with the $x$-axis on the plane. Then, point $P(\cos \theta, 0)$ may be constructed as shown in Fig. 3.

Consider next the multiple angle identity

$$\cos(p\alpha) = T_p(\cos \alpha)$$ (1)
Figure 2. Geometrical solution of $x^3 - a = 0$ by 3-fold origami. Set perpendicular segments $OQ$ and $QT$ with respective lengths 1 and $a$, line $p$ parallel to $QT$ at a distance of 1, and line $q$ parallel to $OQ$ at a distance of $a$. Next, construct Lill’s path $OA$, $AB$, $BC$, $CD$, $DT$ by performing three simultaneous folds: fold $\chi_1$ places point $O$ onto line $p$, fold $\chi_2$ is perpendicular to $\chi_1$ and passes through the intersection of $\chi_1$ with the direction line of $OQ$ (point $B$), and fold $\chi_3$ is perpendicular to $\chi_2$, passes through the intersection of $\chi_2$ with the direction line of $QT$ (point $C$), and places point $T$ onto line $q$. Point $A$ is at the intersection of $\chi_1$ with the direction line of $QT$, and the length of $QA$ is $\sqrt[3]{a}$.

where $T_p$ is the $p$th Chebyshev polynomial of the first kind, defined by

$$T_0(x) = 1,$$  \hspace{1cm} (2)

$$T_1(x) = x,$$  \hspace{1cm} (3)

$$T_{p+1}(x) = 2xT_p(x) - T_{p-1}(x).$$  \hspace{1cm} (4)

Letting $\theta = pa$, then Eq. (1) is a $p$th-degree polynomial equation on $x = \cos(\theta/p)$ with integer (constructible) coefficients. According to Theorem 1, the equation may be solved by $p - 2$-fold origami, if $p \geq 3$, or single-fold origami, if $p \leq 3$. Then, a line $\ell'$ forming an angle $\theta/p$ may be constructed from $\cos(\theta/p)$ by reversing the procedure in Fig. 3.

The lemma is easily extended to the general case of division into an arbitrary number of parts.

**Theorem 3.** Any angle may be divided into $m \geq 2$ equal parts by $n$-fold origami if the largest prime factor $p$ of $m$ satisfies $p \leq n + 2$.

**Proof.** Let $m = p_1p_2 \cdots p_k$, where each $p_i$ is a prime and $p_i \leq n + 2$. Then, the theorem is proved by induction over $k$ and applying Lemma 2.
Division of an angle and construction of regular polygons by multi-fold origami

Figure 3. Construction for Lemma 2. Given points $O(0,0)$, $Q(1,0)$, and line $\ell$ forming an angle $\theta$ with $OQ$: (1) fold along a line ($\chi_1$) to place $\ell$ onto $OQ$, and next (2) fold along a perpendicular ($\chi_2$) to $OQ$ passing through $Q'$. The intersection of $OQ$ and $\chi_2$ is $P = (\cos \theta, 0)$.

Again, we remark that the above theorem only possesses a sufficient condition on the number of multiple folds required. For $m = 5$, it predicts $n = 3$; however, a solution using only 2-fold origami has been published [10].

**Example 1.** Any angle may be divided into 11 equal parts by 9-fold origami.

4. Regular polygons

The analysis follows similar steps to previous treatments on geometric constructions by single-fold origami and other tools [6, 18, 19].

Consider an $m$-gon ($m \geq 3$) circumscribed in a circle with radius 1 and centered at the origin in the complex plane. Its vertices are given by the $m$th-roots of unity, which are the solutions of $z^m - 1 = 0$.

Let us recall that an $m$th root of unity is primitive if it is not a $k$th root of unity for $k < m$. The primitive $m$th roots are solutions of the $m$th cyclotomic polynomial

$$\Phi_m(z) = \prod_{1 \leq k \leq m \atop \gcd(k,m) = 1} \left(z - e^{2\pi ik/m}\right).$$

This polynomial has degree $\phi(m)$, where $\phi$ is Euler’s totient function; i.e., $\phi(m)$ is the number of positive integers $k \leq m$ that are coprime to $m$. A property of any $m$th primitive root $\xi_m$ is that all the $m$ distinct roots may be obtained as $\xi_m^k$, for $k = 0, 1, \ldots, m-1$. This property provides a convenient way to construct the regular $m$-gon.

**Lemma 4.** The regular $m$-gon is $n$-fold constructible if a primitive $m$th root of unity is $n$-fold constructible.
Proof. Let $\xi_m = e^{i\theta}$ be a primitive $m$th root of unity. Then, $\xi_m^k = e^{ik\theta}$ and therefore all roots may be constructed from $\xi_m$ by applying rotations of an angle $\theta$ around the origin. The rotations may be performed by single-fold origami, as shown in Fig. 4. Once all the roots have been constructed, segments connecting consecutive roots may be created by single folds. □

Next, we state a sufficient condition for the $n$-fold constructability of a number $\alpha \in \mathbb{C}$.

Lemma 5. A number $\alpha \in \mathbb{C}$ is $n$-fold constructible if there is a field tower $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k \subset \mathbb{C}$, such that $\alpha \in F_k$ and $[F_j : F_{j-1}] \in \{2, 3, \ldots, n+2\}$ for each $j = 1, 2, \ldots, k$.

Proof. The theorem is proved by induction over $k$. If $k = 0$, then $\alpha \in F_0 = \mathbb{Q}$ is constructible by single-fold origami [1], and therefore is $n$-fold constructible for any $n \geq 1$.

Next, assume that $F_{k-1}$ is $n$-fold constructible. Let $\alpha \in F_k$, then $\alpha$ is a root of a minimal polynomial $p$ with coefficients in $F_{k-1}$, and its degree divides $[F_k : F_{k-1}]$. If $\alpha$ is real, then it may be constructed by $n$-fold origami (Theorem 1). If not, then its complex conjugate $\bar{\alpha}$ is also a root of $p$. The real and imaginary parts of $\alpha$, $\Re(\alpha) = (\alpha + \bar{\alpha})/2$ and $\Im(\alpha) = (\alpha - \bar{\alpha})/2$, respectively, are in $F_k$ and therefore they are real roots of minimal polynomials $p_{\Re}$ and $p_{\Im}$ with coefficients in $F_{k-1}$. Again, the degrees of both $p_{\Re}$ and $p_{\Im}$ divide $[F_k : F_{k-1}]$ and hence $\Re(\alpha)$ and $\Im(\alpha)$ are $n$-fold origami constructible.

Using the above lemmas, we finally obtain a sufficient condition for the constructability of the regular $m$-gon.

Theorem 6. The regular $m$-gon is $n$-fold constructible if the largest prime factor $p$ of $\phi(m)$ satisfies $p \leq n + 2$. 

Figure 4. Given $O = (0, 0)$, $Q = (1, 0)$ and $P = (\cos \theta, \sin \theta)$, a fold along line $\chi$ passing through $O$ and $Q$ places $Q$ on $Q' = (\cos 2\theta, \sin 2\theta)$. 

Proof. Let $\xi_m = e^{i\theta}$ be a primitive $m$th root of unity. Then, $\xi_m^k = e^{ik\theta}$ and therefore all roots may be constructed from $\xi_m$ by applying rotations of an angle $\theta$ around the origin. The rotations may be performed by single-fold origami, as shown in Fig. 4. Once all the roots have been constructed, segments connecting consecutive roots may be created by single folds. □
Proof. Let $\phi(m) = p_1 p_2 \cdots p_k$, where each $p_i$ is a prime and $p_i \leq n + 2$, and $\xi_m$ be a primitive $m$th root of unity. The Galois group $\Gamma$ of the extension $\mathbb{Q}(\xi_m) : \mathbb{Q}$ is abelian and has order $\phi(m)$ [18]. Therefore, it has a series of normal subgroups $1 = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_r = \Gamma$ where each factor $\Gamma_{j+1}/\Gamma_j$ is abelian and has order $p_i$ for some $1 \leq i \leq k$. By the Galois correspondence, there is a field tower $\mathbb{Q}(\xi_m) = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_r = \mathbb{Q}$ such that $[K_j : K_{j+1}] = p_i$. Thus, by Lemma 5, $\xi_m$ is $n$-fold constructible, and by Lemma 4, the $m$-gon is $n$-fold constructible. \qed

Example 2. The totient of 199 is $\phi(199) = 2 \cdot 3^2 \cdot 11$. Therefore, the regular 199-gon may be constructed by 9-fold origami.

5. Final comments

Gleason [6] noted that any regular $m$-gon may be constructed if, in addition to straight edge and compass, a tool to $p$-sect any angle is available for every prime factor $p$ of $\phi(m)$. The above results match his conclusion: if $n$-fold origami can $p$-sect any angle for every prime factor $p$ of $\phi(m)$, then, by Lemma 2, the largest prime factor is $p_{\text{max}} \leq n + 2$. By Theorem 6, the $m$-gon can be constructed.

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