GENERALIZED DRINFELD POLYNOMIALS FOR HIGHEST WEIGHT VECTORS OF THE BOREL SUBALGEBRA OF
THE $SL_2$ LOOP ALGEBRA

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In a Borel subalgebra $U(B)$ of the $sl_2$ loop algebra, we introduce a highest weight vector $\Psi$. We call such a representation of $U(B)$ that is generated by $\Psi$ highest weight. We define a generalization of the Drinfeld polynomial for a finite-dimensional highest weight representation of $U(B)$. We show that every finite-dimensional highest weight representation of the Borel subalgebra is irreducible if the evaluation parameters are distinct. We also discuss the necessary and sufficient conditions for a finite-dimensional highest weight representation of $U(B)$ to be irreducible.

1. Introduction

In the classical analogue of the Drinfeld realization of the quantum $sl_2$ loop algebra, $U_q(L(sl_2))$, the Drinfeld generators, $\bar{x}^\pm_k$ and $\bar{h}_k$ for $k \in \mathbb{Z}$, satisfy the following defining relations\textsuperscript{1,2,9}:

\begin{align*}
\,[\bar{h}_j, \bar{x}^\pm_k] &= \pm 2 \bar{x}^\pm_{j+k}, & \,[\bar{x}^+_j, \bar{x}^-_k] &= \bar{h}_{j+k}, \\
\,[\bar{h}_j, \bar{h}_k] &= 0, & \,[\bar{x}^\pm_j, \bar{x}^\mp_k] &= 0, \quad \text{for } j, k \in \mathbb{Z}.
\end{align*}

(1.1)

In a representation of $U(L(sl_2))$, a vector $\Omega$ is called a highest weight vector if $\Omega$ is annihilated by generators $\bar{x}^k_+$ for all integers $k$ and such that $\Omega$ is a simultaneous eigenvector of every generator of the Cartan subalgebra, $\bar{h}_k$ ($k \in \mathbb{Z}$)\textsuperscript{1,2}. We call a representation of $U(L(sl_2))$ highest weight if it is generated by a highest weight vector. For a finite-dimensional irreducible representation we associate a unique polynomial through the highest weight $\bar{x}^\pm_k$. It is shown that any given irreducible highest weight representation is finite-dimensional if and only if it has the Drinfeld polynomial\textsuperscript{1}.

Recently it was shown that the XXZ spin chain at roots of unity has the $sl_2$ loop algebra symmetry\textsuperscript{5,7,10,11,12}. Fabricius and McCoy has conjectured...
that every Bethe ansatz eigenstate should be highest weight of the \( sl_2 \) loop algebra, and also that the Drinfeld polynomial can be derived from the Bethe state. It is explicitly shown that regular XXZ Bethe states in some sectors are indeed highest weight. However, it is still nontrivial how to connect the highest weight vector with the Drinfeld polynomial. In fact, the Drinfeld polynomial is defined for an irreducible representation not for a highest weight vector. Furthermore, there exist finite-dimensional highest weight representations that are reducible and indecomposable. It has been shown that a given highest weight representation is irreducible if the evaluation parameters are distinct. Here, we shall define evaluation parameters in §3. Thanks to the theorem, we solve the connection problem at least for the case of distinct evaluation parameters.

In this paper, we discuss a generalization of the theorem to the case of a highest weight representation of a Borel subalgebra of \( U(L(sl_2)) \). The generalization should play a key role in the study of the spectral degeneracy of the XXZ spin chain under twisted boundary conditions. Let us consider the subalgebra generated by generators \( h_0, x_0^+ \) and \( x_1^- \) satisfying the relations (1.1). We call it a Borel subalgebra of \( U(L(sl_2)) \), and denote it by \( U(B) \). It has the following generators:

\[
h_k, x_k^+ \text{ for } k \in \mathbb{Z}_{\geq 0}, \quad x_k^- \text{ for } k \in \mathbb{Z}_{>0}.
\]

We define a highest weight vector of the Borel subalgebra \( U(B) \) by such a vector \( \Psi \) that satisfies the following relations:

\[
x_k^+ \Psi = 0, \quad h_k \Psi = d_k \Psi, \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]

We call the representation of \( U(B) \) generated by \( \Psi \) highest weight and the set \( \{d_k\} \) the highest weight. Here we note that \( d_0 \) is not necessarily an integer, since \( x_{-1}^- \) does not exist in \( U(B) \). In §2 of the present paper, we derive a useful recursive relation of \( x_k^- \Psi \) for \( k \in \mathbb{Z}_{>0} \). In §3 we introduce a generalization of the Drinfeld polynomial for a finite-dimensional highest weight representation of the Borel subalgebra \( U(B) \). In §4 we show that every highest weight representation of the Borel subalgebra with distinct and nonzero evaluation parameters is irreducible.

Throughout the paper, we denote by \( \Psi \) a highest weight vector of the Borel subalgebra \( U(B) \) with highest weight \( d_k \) and by \( V_B \) the representation generated by it, i.e. \( V_B = U(B)\Psi \). We also assume that \( V_B \) is finite-dimensional.
2. Sectors of $V_B$ and nilpotency

Lemma 2.1. Let us define the sector of $h_0 = d_0 - 2n$ in $V_B$ for an integer $n \geq 0$ by the subspace consisting of vectors $v_n \in V_B$ such that $h_0 v_n = (d_0 - 2n) v_n$. Here we recall $h_0 \Psi = d_0 \Psi$. Then, $V_B$ is given by the direct sum of such sectors. Any vector $v_n$ in the sector of $h_0 = d_0 - 2n$ is expressed as a linear combination of monomial vectors $x_j^- \cdots x_n^- \Psi$.

Proof. It is clear from the PBW theorem.

We note that generator $x_1^-$ is nilpotent in any $V_B$.

Definition 2.1. We say that generator $x_1^-$ is nilpotent of degree $r$ in $V_B$, if $(x_1^-)^{r+1} \Psi = 0$, while $(x_1^-)^j \Psi \neq 0$ for $0 < j \leq r$.

The degree $r$ of nilpotency for generator $x_1^-$ gives the largest $n$ for non-vanishing sectors of $h_0 = d_0 - 2n$, as shown in the next proposition.

Proposition 2.1. If generator $x_1^-$ is nilpotent of degree $r$, then the sector of $h = d_0 - 2r$ is one-dimensional: every monomial vector in the sector is proportional to $(x_1^-)^r \Psi$ with some constant $C_{k_1, \ldots, k_r}$:

$$x_{k_1}^- \cdots x_{k_r}^- \Psi = C_{k_1, \ldots, k_r} (x_1^-)^r \Psi, \text{ for } k_1, \ldots, k_r \in \mathbb{Z}_{>0}. \quad (2.1)$$

Furthermore, sectors of $h = d_0 - 2n$ for $n > r$ are of zero-dimensional. For instance, we have $x_{k_1}^- \cdots x_{k_r+1}^- \Psi = 0$ for $k_1, \ldots, k_r+1 \in \mathbb{Z}_{>0}$.

Proof. Setting $m = r$ in lemma 2.3, we have eq. (2.1). For the case of $n > r$ we show it from lemma 2.3 where we set $m = n$.

Let $B_+$ be such a subalgebra of $U(B)$ that is generated by $x_k^+$ for $k \in \mathbb{Z}_{>0}$. We define $(X)^{(n)}$ by $X^n = X^n/n!$.

Lemma 2.2. Let $m$ and $t$ be integers satisfying $0 \leq t \leq m + 1$. In the Borel subalgebra $U(B)$, for $k_1, \ldots, k_t, n \in \mathbb{Z}_{>0}$, and $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$x_\ell^+ (x_n^-)^{(m+1-t)} x_{k_1}^- \cdots x_{k_t}^-$$

$$= -x_{\ell+2n}^- (x_n^-)^{(m-t-1)} x_{k_1}^- \cdots x_{k_t}^- + (x_n^-)^{(m-t)} x_{k_1}^- \cdots x_{k_t}^- h_{\ell+n}$$

$$+ \sum_{j=1}^{t} (x_n^-)^{(m+1-t)} \prod_{i=1, i \neq j}^{t} x_{k_i}^- \cdot h_{\ell+k_j} + (-2) \sum_{j=1}^{t} (x_n^-)^{(m-t)} x_{\ell+n+k_j}^- \prod_{i=1, i \neq j}^{t} x_{k_i}^- \mod U(B)B_+ \quad (2.2)$$
Lemma 2.3. Suppose that $x_1^-$ is nilpotent of degree $r$ in $V_B$, and $m$ be an integer with $m \geq r$. Let us take a positive integer $p$ satisfying $p \leq m$. We have

$$ (x_1^-)^{m-p}x_{k_1}^- \cdots x_{k_p}^- \Psi = A_{k_1, \ldots, k_p}^{(r)} (x_1^-)^m \Psi, \quad (2.3) $$

for any set of positive integers $k_1, \ldots, k_p$.

Proof. We prove (2.3) by induction on $p$ by making use of eq. (2.2).

Lemma 2.4. The following recursive formulas on $n$ hold for $n > 0$:

(A$_n$): $$(x_1^+)^{(n-1)} (x_1^-)^{(n)} = \sum_{j=1}^{n} (-1)^{j-1} x_j^- (x_0^+)^{(n-j)} (x_1^-)^{(n-j)} \mod U(B)B_+.$$

(B$_n$): $$(n(x_0^+)^{(n)}(x_1^-)^{(n)}) = \sum_{j=1}^{n} (-1)^{j-1} h_j (x_0^+)^{(n-j)} (x_1^-)^{(n-j)} \mod U(B)B_+.$$

(C$_n$): $$[h_1, (x_0^+)^{(m)} (x_1^-)^{(m)}] = 0 \mod U(B)B_+ \text{ for } m \leq n.$$

Making use of (B$_n$) of lemma 2.4 inductively, we show that $\Psi$ is a simultaneous eigenvector of operators $(x_0^+)^{(m)} (x_1^-)^{(n)}$ for $n > 0$. For a given positive integer $k$, we denote by $\lambda_k$ the eigenvalue: $(x_0^+)^{(k)} (x_1^-)^{(k)} \Psi = \lambda_k \Psi$.

Lemma 2.5. If $x_1^-$ is nilpotent of degree $r$ in $V_B$, we have

$$ x_{r+1}^- \Psi = \sum_{j=1}^{r} (-1)^{r-j} \lambda_{r+1-j} x_j^- \Psi. \quad (2.4) $$

Moreover, it leads to the following:

$$ x_{r+1+p}^- \Psi = \sum_{j=1}^{r} (-1)^{r-j} \lambda_{r+1+p-j} x_j^- \Psi, \quad \text{for } p \in \mathbb{Z}_{\geq 0}. \quad (2.5) $$

Proof. Relation (2.4) is derived from (A$_{r+1}$) of lemma 2.4. Making use of $x_{r+1+n}^- = (-2)^{-1} [h_n, x_{r+1}^-]$ and (2.4), we derive (2.5).

Proposition 2.2. Suppose that $x_1^-$ is nilpotent of degree $r$ in $V_B$. In the sector of $h_0 = d_0 - 2n$ with $0 \leq n \leq r$, every vector is expressed as a sum of monomial vectors $x_{k_1}^- \cdots x_{k_n}^- \Psi$ for integers $k_1, k_2, \ldots, k_n$ satisfying $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq r$.

Proof. It is clear from (2.5).
3. Generalized Drinfeld Polynomials $P_\Psi(u)$ for $V_B$

**Definition 3.1.** Suppose that $x_1^-$ is nilpotent of degree $r$ in $V_B$. We define a polynomial $P_\Psi(u)$ by

$$P_\Psi(u) = \sum_{k=0}^{r} \lambda_k (-u)^k.$$  (3.1)

**Definition 3.2.** If polynomial $P_\Psi(u)$ of $V_B$ is factorized as

$$P_\Psi(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k},$$  (3.2)

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively, then we call $a_j$ the evaluation parameters of highest weight vector $\Psi$. We denote by $a$ the set of $s$ parameters, $a_1, a_2, \ldots, a_s$.

We note that $r$ is given by the sum: $r = m_1 + \cdots + m_s$. Let us define parameters $\hat{a}_i$ for $i = 1, 2, \ldots, r$, as follows:

$$\hat{a}_i = a_k \quad \text{if} \quad m_1 + m_2 + \cdots + m_{k-1} < i \leq m_1 + m_2 + \cdots + m_{k-1} + m_k.$$  (3.3)

Then, the set $\hat{a} = \{\hat{a}_j | j = 1, 2, \ldots, r\}$ corresponds to the set of evaluation parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, r$.

4. Generators with parameters

4.1. Loop algebra generators with parameters

Let $A$ be a set of parameters such as $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. We define generators with $m$ parameters $x_m^\pm(A)$ and $h_m(A)$ as follows $^6$:

$$x_m^\pm(A) = \sum_{k=0}^{m} (-1)^k x_{m-k}^\pm \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k},$$

$$h_m(A) = \sum_{k=0}^{m} (-1)^k h_{m-k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}.$$  (3.4)

In terms of generators with parameters we generalize the defining relations of the $sl_2$ loop algebra. Let $A$ and $B$ are arbitrary sets of $m$ and $n$ parameters, respectively. The operators with parameters satisfy the following:

$$[x_m^+(A), x_n^-(B)] = h_{m+n}(A \cup B), \quad [h_m(A), x_n^+(B)] = \pm 2x_m^\pm(A \cup B).$$  (3.5)
By using the relations (3.5), it is straightforward to show the following:

\[
[x^+_m(A), (x^-_m(B))^n] = (x^-_m(B))^{(n-1)}h_{\ell+m}(A \cup B)
- x^-_{\ell+2m}(A \cup B)(x^-_m(B))^{(n-2)},
\]

\[
[h_\ell(A), (x^{\pm}_m(B))^n] = \pm 2(x^{\pm}_m(B))^{(n-1)}x^{\pm}_m(A \cup B).
\]  

(3.6)

Here the symbol \((X)^{(n)}\) denotes the \(n\)th power of operator \(X\) divided by the \(n\) factorial, i.e. \((X)^{(n)} = X/n!\).

Let the symbol \(\alpha\) denote a set of \(m\) parameters, \(\alpha_j\) for \(j = 1, 2, \ldots, m\). We denote by \(A_j\) the set of all the parameters except for \(\alpha_j\), i.e. \(A_j = \alpha \setminus \{\alpha_j\} = \{\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m\}\). We introduce the following symbol:

\[
\rho^{\pm}_j(\alpha) = x^{\pm}_{m-1}(A_j) \quad \text{for} \quad j = 1, 2, \ldots, m.
\]  

(3.7)

Here we note the following:

**Lemma 4.1.** If \(x^-_n(A)\Omega = 0\) for some set of \(n\) parameters, \(A\), then we have \(x^-_{n+m}(A \cup B)\Omega = 0\) for any set of \(m\) parameters, \(B\).

Hereafter, we denote by \(a^{\otimes m}_j\) the set of parameter \(a_j\) with multiplicity \(m\), i.e. \(a^{\otimes m}_j = \{a_j, a_j, \ldots, a_j\}\). Moreover, in the case of \(m = 1\), we write \(x^\pm_1(a^{\otimes 1}_j)\) simply as \(x^\pm_1(a_j)\).

**4.2. Borel subalgebra generators with parameters**

In the case of the Borel subalgebra \(U(B)\), we do not have generator \(x^-_0\) in \(U(B)\). In order to introduce generators with parameters for \(U(B)\), we thus need some trick.

For a given set of \(m\) parameters, \(\alpha_j\) for \(j = 1, 2, \ldots, m\), we introduce the extended set of parameters as follows:

\[
\alpha^{(n)} = \alpha \cup \{0^{\otimes n}\}.
\]  

(3.8)

Here we recall that \(a^{\otimes n}\) denotes the set of \(a\) with multiplicity \(n\). We also introduce the following symbols:

\[
\rho^{\pm}_j(\alpha^{(1)}) = x^{\pm}_m(A_j^{(1)}) \quad \text{for} \quad j = 1, 2, \ldots, m.
\]  

(3.9)

It is easy to show

\[
\frac{\sum_{j=1}^n \rho^{\pm}_j(\alpha^{(1)})}{\prod_{k=1;k\neq j}^{m} a_{kj}} = x^{\pm}_{m+1-n}(\{\alpha_{n+1}, \ldots, \alpha_m\} \cup \{0\}) \quad (1 \leq n \leq m).
\]  

(3.10)
It follows inductively on \( n \) that \( x_k^- \) for \( 1 \leq k \leq m \) are expressed in terms of linear combinations of \( \rho_j^- (\alpha^{(1)}) \) with \( 1 \leq j \leq m \).

The reduction relation (2.4) is expressed as
\[
x_{r+1}^- (\hat{a}^{(1)}) \Psi = 0.
\]

However, if we have
\[
x_s^- (\hat{a}^{(1)}) \Psi = 0,
\]
making use of (3.11), we can express monomial vector \( x_{j_1}^- x_{j_2}^- \cdots x_{j_n}^- \Psi \) of any set of positive integers, \( j_1, \ldots, j_n \), as a linear combination of \( \rho_{k_1}^- (\alpha^{(1)}) \rho_{k_2}^- (\alpha^{(1)}) \cdots \rho_{k_n}^- (\alpha^{(1)}) \Psi \) over some sets of integers with \( 1 \leq k_1, \ldots, k_n \leq r \).

5. Highest weight representations

5.1. The case of distinct evaluation parameters

Let us discuss the case where all the evaluation parameters \( \hat{a}_j \) have multiplicity 1, i.e. \( m_j = 1 \) for \( j = 1, \ldots, s \). We call it the case of distinct evaluation parameters. Here we note that \( s = r \). We therefore have
\[
x_{s+1}^- (\hat{a}^{(1)}) \Psi = 0.
\] (3.12)

**Lemma 5.1.** If all evaluation parameters \( \hat{a}_j \) are distinct \((m_j = 1 \text{ for all } j)\), we have
\[
\left( \rho_j^- (\alpha^{(1)}) \right)^2 \Psi = 0.
\] (3.13)

**Proof.** First, we show
\[
x_0^+ (\rho_j^- (\alpha^{(1)}))^2 \Psi = 0.
\] (3.14)

From eq. (3.6) we have
\[
x_0^+ (\rho_j^- (\alpha^{(1)}))^2 \Psi = x_s^- (A_j^{(1)}) h_s (A_j^{(1)}) \Psi - x_{2s}^- (A_j^{(1)} \cup A_j^{(1)}) \Psi.
\]

We set \( a_0 = 0 \). In terms of \( a_{kj} = a_k - a_j \), we have
\[
h_s (A_j^{(1)}) \Psi = \prod_{k=0; k \neq j}^s a_{jk} \Psi,
\]
and using eq. (3.12) and lemma 4.1 we have
\[
x_{2s}^- (A_j^{(1)} \cup A_j^{(1)}) \Psi = a_{j0} \prod_{k=1; k \neq j}^s a_{jk} x_s^- (A_j^{(1)}) \Psi.
\]
We thus obtain eq. (3.14). Secondly, we apply \((x_0^+)^{(r-1)}(x^-_1(a_j))^{(r-1)}\) to 
\((\rho^-_j(a^{(1)}))^2\Psi\). The product is given by zero since it is out of the sectors of \(V_\Psi\) due to the fact that \((r-1) + 2 > r\) and proposition 2.1:
\[
(x_0^+)^{(r-1)}(x^-_1(a_j))^{(r-1)}\left(\rho^-_j(a^{(1)})\right)^2 \Psi = 0.
\]
We then show that the left-hand-side is given by
\[
\rho^-_j(a^{(1)})^2(x_0^+)^{(r-1)}(x^-_1(a_j))^{(r-1)}\Psi = \prod_{k=1;k\neq j}^r a_{kj} \times \left(\rho^-_j(a^{(1)})\right)^2 \Psi.
\]
Here, through induction on \(n\) and using \(B_n\) of lemma (2.4), we show
\[
[[x_0^+]^n(x^-_1(a_j))^n] \left(\rho^-_j(a^{(1)})\right)^2 \Psi = 0 \quad (n \leq r - 1).
\]
Since \(a_{kj} \neq 0\) for \(k \neq j\), we obtain eq. (3.13).

**Lemma 5.2.** Let \(x^-_1\) be nilpotent of degree \(r\) in \(V_B\). In the sector of \(h_0 = d_0 - 2n\) for an integer \(n\) with \(0 \leq n \leq r\), every vector \(v_n\) is written as
\[
v_n = \sum_{1 \leq j_1 < \cdots < j_n \leq s} C_{j_1,\ldots,j_n} \prod_{t=1}^n \rho^-_{j_t}(a^{(1)}) \Psi.
\]  
(3.15)
Suppose that \(\lambda_r \neq 0\). Then, if \(v_n\) is zero, all the coefficients \(C_{j_1,\ldots,j_n}\) in (3.15) are given by zero.

**Proof.** In terms of \(\rho^-_j(a^{(1)})\), any vector in the sector is expressed as a linear combination of \(\rho^-_{j_1}(a^{(1)}) \cdots \rho^-_{j_n}(a^{(1)})\Psi\). From lemma 5.1 we may assume \(1 \leq j_1 < \cdots < j_n \leq s\). For a set of integers with \(1 \leq i_1,\ldots,i_n \leq s\), multiplying both sides of eq. (3.15) with \(\rho^+_{i_1}(a^{(1)}) \cdots \rho^+_{i_n}(a^{(1)})\), we have
\[
\rho^+_{i_1}(a^{(1)}) \cdots \rho^+_{i_n}(a^{(1)})v_n = C_{i_1,\ldots,i_n} \prod_{t=1}^n \prod_{k=0;k \neq t}^s a_{i_k}^2 \times \Psi
\]
Therefore, if \(v_n = 0\), all the coefficients \(C_{j_1,\ldots,j_n}\) are given by zero.

From lemmas 5.1, 5.2 and proposition 2.1 we have the following:

**Prop 5.1.** If evaluation parameters \(\hat{a}_j\) of \(\Psi\) are distinct, the set of vectors \(\prod_{t=1}^n \rho_{j_t}(a^{(1)})\Psi\) for \(1 \leq j_1 < \cdots < j_n \leq s\) gives a basis of the sector of \(h_0 = d_0 - 2n\) in \(V_B\).

**Theorem 5.1.** Let \(V_B\) denotes the finite-dimensional representation of \(U(B)\) generated by a highest weight vector \(\Psi\). If \(x^-_1\) is nilpotent of degree \(r\) in \(V_B\) and \(\Psi\) has distinct and nonzero evaluation parameters \(a_1,\ldots,a_r\), then \(V_B\) is irreducible.
Proof. We show that every nonzero vector of $V_B$ has such an element of the loop algebra that maps it to $\Psi$. Suppose that there is a nonzero vector $v_n$ in the sector of $h_0 = d_0 - 2n$ that has no such element. Then, we have
\[
x_{k_1}^+ \cdots x_{k_n}^+ v_n = 0
\] (3.16)
for all monomial elements $x_{k_1}^+ \cdots x_{k_n}^+$. Here $v_n$ is expressed in terms of the basis vectors $\rho_{j_1}(a^{(1)}) \cdots \rho_{j_n}(a^{(1)}) \Psi$ with coefficients $C_{j_1, \ldots, j_n}$ and $1 \leq j_1 < \cdots < j_n \leq s$, as in (3.15). Then, by the same argument as in lemma 5.2 we show that all the coefficients $C_{j_1, \ldots, j_n}$ vanish. However, this contradicts with the assumption that $v_n$ is nonzero. It therefore follows that $v_n$ has such an element that maps it to $\Psi$. We thus obtain the theorem.

5.2. The case of degenerate evaluation parameters

Let us discuss a general criteria for a finite-dimensional highest weight representation to be irreducible.

Theorem 5.2. Recall that $V_B$ is a finite-dimensional representation of the Borel subalgebra $U(B)$ generated by a highest weight vector $\Psi$ that has evaluation parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$. Suppose that $x_{-1}^-$ is nilpotent of degree $r$ and the evaluation parameters are nonzero, i.e. $a_1 a_2 \cdots a_s \neq 0$. We also recall that $\mathcal{a}$ denotes the set of evaluation parameters: $\mathcal{a} = \{a_1, a_2, \ldots, a_s\}$. Then, $V_B$ is irreducible if and only if
\[
x_{s+1}^- (a^{(1)}) \mid R; \Phi \rangle = 0,
\]
where $\mathcal{a}$ denotes the set of evaluation parameters $a_1, a_2, \ldots, a_s$. Then, it follows from theorem 5.2 that $V_B$ is irreducible, and the degenerate multiplicity of $\mid R; \Phi \rangle$ is given by $(m_1 + 1)(m_2 + 1) \cdots (m_s + 1)$. We prove it by generalizing the proof of theorem 5.1 (cf. Ref. 6).

Theorem 5.2 plays an important role when we discuss the spectral degeneracy of the twisted XXZ spin chain at roots of unity associated with the Borel subalgebra $U(B)$ of the $sl_2$ loop algebra. Here the spin chain satisfies the twisted boundary conditions. We show in some sectors that a regular Bethe ansatz eigenvector $\mid R; \Phi \rangle$ is a highest weight vector of the Borel subalgebra $U(B)$ for some twist angle $\Phi$. It is nontrivial whether the highest weight representation $V_B$ generated by $\mid R; \Phi \rangle$ is irreducible or not. Suppose that $x_{-1}^-$ is nilpotent of degree $r$ in $V_B$, $\mid R; \Phi \rangle$ has nonzero evaluation parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$, where $m_1 + \cdots + m_s = r$, and we have the following relation:
\[
x_{s+1}^- (a^{(1)}) \mid R; \Phi \rangle = 0,
\]
where $\mathcal{a}$ denotes the set of evaluation parameters $a_1, a_2, \ldots, a_s$. Then, it follows from theorem 5.2 that $V_B$ is irreducible, and the degenerate multiplicity of $\mid R; \Phi \rangle$ is given by $(m_1 + 1)(m_2 + 1) \cdots (m_s + 1)$.
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References

1. V. Chari and A. Pressley, Quantum Affine Algebras, Commun. Math. Phys. 142 (1991) 261–283.
2. V. Chari and A. Pressley, Quantum Affine Algebras at Roots of Unity, Representation Theory 1 (1997) 280–328.
3. V. Chari and A. Pressley, Weyl modules for classical and quantum affine algebras, Representation Theory 5 (2001) 191–223.
4. T. Deguchi, The $sl_2$ loop algebra symmetry of the twisted transfer matrix of the six-vertex model at roots of unity, J. Phys. A 37 (2004) 347–358.
5. T. Deguchi, Regular XXZ Bethe states as highest weight vectors of the $sl_2$ loop algebra, cond-mat/0503564.
6. T. Deguchi, The Six-Vertex Model at Roots of Unity and some Highest Weight Representations of the $sl_2$ Loop Algebra, to appear in Ann. Henri Poincaré (2006 Birkhäuser Verlag Basel/Switzerland) (cond-mat/0603112).
7. T. Deguchi, K. Fabricius and B. M. McCoy, The $sl_2$ Loop Algebra Symmetry of the Six-Vertex Model at Roots of Unity, J. Stat. Phys. 102 (2001) 701–736.
8. T. Deguchi and K. Kudo, Spectral degeneracy of the twisted XXZ spin chain, in preparation.
9. V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Doklady 36 (1988) 212–216.
10. K. Fabricius and B. M. McCoy, Bethe's Equation Is Incomplete for the XXZ Model at Roots of Unity, J. Stat. Phys. 103(2001) 647–678.
11. K. Fabricius and B. M. McCoy, Completing Bethe’s Equations at Roots of Unity, J. Stat. Phys. 104(2001) 573–587.
12. K. Fabricius and B. M. McCoy, Evaluation Parameters and Bethe Roots for the Six-Vertex Model at Roots of Unity, in Progress in Mathematical Physics Vol. 23 (MathPhys Odyssey 2001), edited by M. Kashiwara and T. Miwa, (Birkhäuser, Boston, 2002) 119–144.
13. N. Jacobson, Lie algebras (Wiley, New York, 1962)
14. C. Korff, The twisted XXZ chain at roots of unity revisited, J. Phys. A 37 (2004) 1681–1689.