POSITIVE EXPONENTIAL SUMS AND ODD POLYNOMIALS

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Abstract. Given an odd integer polynomial \( f(x) \) of a degree \( k \geq 3 \), we construct a non-negative valued, normed trigonometric polynomial with the spectrum in the set of integer values of \( f(x) \) not greater than \( n \), and a small free coefficient \( a_0 = O((\log n)^{-1/k}) \). This gives an alternative proof for the maximal possible cardinality of a set \( A \), so that \( A - A \) does not contain an element of \( f(x) \). We also discuss other interpretations and an ergodic characterization of that bound.

1. Introduction

We consider polynomials \( f(x) = \alpha_k x^k + \ldots + \alpha_1 x \) with integer coefficients, satisfying:

\[
\text{For all } j \text{ even, } \alpha_j = 0; \text{ and the leading coefficient is } \alpha_k > 0.
\]

The main result of the paper is the following:

**Theorem 1.** Given an integer polynomial \( f \) of a degree \( k \geq 3 \) satisfying (1.1), there exist cosine polynomials

\[
T(x) = b_0 + \sum_{0 < f(j) \leq N} b_{f(j)} \cos(2\pi f(j)x),
\]

such that for all \( x \), \( T(x) \geq 0 \), and such that all coefficients \( b_j \) are non-negative, normed (i.e. \( \sum b_j = 1 \)), and that \( b_0 = O((\log N)^{-1/k}) \).

We now discuss the background and implications of that result. Denote by \( T(D) \) the set of all cosine polynomials with the spectrum in a set of positive integers \( D \cup \{0\} \), such that \( T(x) \geq 0, T(0) = 1 \); and let \( T(D) \) be the subset of \( T(D) \) with non-negative coefficients. Kamae and Mendès France in \([5]\) introduced the notion of van der Corput sets (VdC sets; or correlative sets), if \( \inf_{T \in T(D)} b_0 = 0 \) (\( b_0 \) is the free coefficient). One can also define VdC\(^+\) sets as those for which \( \inf_{T \in T(D)} b_0 = 0 \). Let \( \gamma(n), \gamma^+(n) \) be the arithmetic functions which measure how quickly a set is becoming a van der Corput set:

\[
\gamma(n) = \inf_{T \in T(D \cap [1,n])} b_0
\]
and \( \gamma^+(n) \) analogously for \( T^+(D) \) (and then \( \gamma(n) \leq \gamma^+(n) \)). Theorem 1 can now be rephrased as follows: for the sets of values of an odd integer polynomial \( f \),

\[
\gamma^+(n) = O((\log n)^{-1/k}).
\]
Kamae and Mendès France, Ruzsa and Montgomery (8, 12, 13) described various characterizations of van der Corput sets and the function $\gamma(n)$, mainly related to uniform distribution properties of the set $D$. In particular, in [5] it was shown that van der Corput sets are intersective sets, and Ruzsa showed in [13] that an upper bound for the function $\gamma$ is also an upper bound for the intersective property. Following that, a Corollary of Theorem 1 is the following:

Corollary 1. Let $f$ be an integer polynomial satisfying (1.2). Suppose that $N$ is an integer and that $A \subset \{1, \ldots, N\}$ is such that the difference between any two elements of $A$ is never an integer value of $f$. Then $|A| = O(N(\log N)^{-1/k})$.

This gives another proof of the upper bound for the difference property of odd polynomials, where the best current result (by Lucier [6], valid for all polynomials) is $N(\log N)^{-1/(k-1+o(1))}$. Montgomery set a problem in [8] for finding any upper bounds for the van der Corput property for any ”interesting” sets, such as the set of squares and more generally the set of values of an integer polynomial. Ruzsa in [12] announced the $\gamma(n) = O((\log n)^{-1/2})$ bound for the set of squares, but the proof was never published. One of the authors in [15, 16] proved bounds $\gamma(n) \leq \gamma^+(n) = O((\log n)^{-1/3})$ for the set of perfect squares, and $\gamma(n) \leq \gamma^+(n) = O((\log n)^{-1+o(1)})$ for the set of shifted primes. We also note that Theorem 1 can be extended to all integer polynomials of degree $k \geq 3$, but for now with the bound only $\gamma^+(n) = O((\log \log n)^{-1/k^2})$ (10).

Ruzsa showed that, by using only non-negative coefficients in the case of squares, one can not do better than $O((\log n)^{-1})$. We extend the same argument to show that the Theorem 1 is close to optimal if only non-negative coefficients are used:

Theorem 2. Let $f(x) = \beta x^k$, $\beta > 0$ an integer, and $k \geq 3$ an odd integer. Then

$$\gamma^+(n) \geq (1/\varphi(k) + o(1))(\log n)^{-1}.$$ (1.4)

It is hoped that one can improve the van der Corput and intersective sets bounds by constructing cosine polynomials also with negative coefficients. Ruzsa and Matolcsi have recently announced progress in this direction in the case of perfect squares; and also discussed this in a more general setting of commutative groups (7).

The Theorems 1, 2 have an ergodic-theoretical interpretation, as was noted in [11]:

Corollary 2. Let $f$ be an integer polynomial of a degree $k \geq 3$ satisfying (1.7), $H$ an arbitrary real Hilbert space, $U$ an unitary operator on $H$, and $P$ the projection to the kernel of $U - I$. If $x \in H$ is such that $Px \neq 0$, then there exists a positive integer $f(j)$ such that $(U^{f(j)}x, x) > 0$.

Furthermore, if $(Px, x) > \gamma^+(n)(x, x)$, then there exists such $f(j) \leq n$, where $\gamma^+(n)$ is the best such bound valid universally for all $H, U$, with bounds (1.3), and (1.4) in the case $f(x) = \alpha_k x^k$.

The structure of the paper. We first introduce some notation related to the polynomial $f$. The degree of $f$ will be always denoted by $k$, and let $l \geq 1$ be the smallest index so that $\alpha_l > 0$. Let $c(f) = (\alpha_k, \ldots, \alpha_l)$ be the content of the polynomial. Without loss of generality we always assume that for $x \geq 1$, $f(x) \geq 1$, and that $f(j)$, $j \geq 1$ is an increasing sequence (if not, we find the smallest $j_0$ such that it holds for $j \geq j_0$, and modify all the estimates by skipping the first $j_0$ values of $f$, this impacting only the implicit constants in the estimates).
Let $F_n(x)$ be the normed Fejér’s kernel
\[
F_n(x) = \frac{1}{n} + 2 \sum_{j=1}^{n} \left( \frac{1}{n} - \frac{j}{n^2} \right) \cos(2\pi jx),
\]
and then $F_n(x) \geq 0$, and $F_n(0) = 1$. The key tool in our construction will be, following the idea of I. Ruzsa, construction of a polynomial of the type (1.2) which approximates $F_n(x)$. We may further restrict "allowable" indices $j$ to those with an integer $d$ as a factor, and define
\[
G_{n,d}(x) = \frac{2}{K} \sum_{\alpha_k(dj)^k \leq n} \alpha_k kd^k j^{k-1} \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} \right) \cos(2\pi f(dj)x),
\]
where $K$ is chosen so that $G_{n,d}(0) = 1$ ($K$ will be close to 1 for $n$ large enough; and will be estimated in Section 2).

The structure of the proof is as follows: let $S(f, q)$ be the complete trigonometric sum
\[
S(f, q) = \frac{q-1}{q} \sum_{s=0}^{q-1} e(f(s)/q),
\]
where $e(x) = \exp(2\pi ix)$. We will need the reduced complete trigonometric sum over the multipliers of an integer $d$:
\[
S_d(f, q) = \sum_{s=0}^{q-1} e(f(ds)/q).
\]
For $q$ small (the major arc estimates), we show that
\[
G_{n,d}(a/q + \kappa) = \frac{1}{q} S_d(a f, q) F_n(\kappa) + \text{error term},
\]
where the error term is small for small $\kappa$ and large $n$ as compared to $q, d$. For large $q$, we show by partial summation and by using the well-known Vinogradov’s trigonometric sum estimates that $G_{n,d}(x)$ is small. The key step is the averaging step: we choose the constants $d_0, \ldots, d_s$ and normalized weights $w_0, \ldots, w_s$ such that for any $q$, $\sum w_j S_d(a f, q) \geq -\delta$, $\delta = O((\log N)^{-1/k})$. Here $N$ is the size of the largest non-zero coefficient in the family of polynomials $G_{n,d_j}$. The estimate $b_0 = O((\log N)^{-1/k})$ follows from this and the size of the error term.

Unfortunately, for polynomials which are not odd, this approach seems to fail as (1.7) does not hold. Namely, there is another factor difficult to control if one can not a-priori claim that the imaginary part of $S(f, q)$ is 0, as is the case for odd polynomials.

We prove Theorem 1 in Sections 2-5, and Theorem 2 in Section 6.

2. The major arcs

We will use the notation $O_{k}(.)$, $\ll_{k}$, $O_f(.)$, $\ll_f$ when the implicit constant depends implicitly on the degree $k$ or the coefficients of the polynomial $f$ (including the degree) respectively. We will often use the following relations: If $x, y \geq 1$ are
integers such that \( f(x), f(y) \ll f(n) \), then

\[
\begin{align*}
(2.1) \quad f(x) &= O_f(x^k) = \\
(2.2) \quad &\alpha_k x^k + O_f(n^{1-1/k}), \\
(2.3) \quad |f(x) - f(y)| &= |x - y|O_f(n^{1-1/k}) = \\
(2.4) \quad &\alpha_k |x - y|x^{k-1} + |x - y|^2O_f(n^{1-2/k}).
\end{align*}
\]

The relations above can be computed easily by using \( \beta = 2(|\alpha_{k-1}| + \cdots + |\alpha_1|)/\alpha_k \) and the relation \( \alpha_k x^k \leq 2f(x) \) if \( x \geq 1 \) and \( x \geq \beta \).

The following result by Chen [3] and Nechaev [9] gives a bound for the complete trigonometric sums.

**Lemma 1.** Let \( f \) be an integer polynomial of a degree \( k \geq 2 \). Then

\[
S(f, q) = O_k((c(f), q)^{1/k} q^{1-1/k}).
\]

We will need the next bound for the content of the polynomial \( f \) when it goes over the multipliers of an integer \( d \):

**Lemma 2.** Let \( f \) be an integer polynomial of a degree \( k \geq 1 \), \( d \) an integer and \( g(x) = f(dx) \), for all \( x \). Then

\[
c(g) \leq d^k |\alpha_1|^k.
\]

**Proof.** We first assume that \( f \) is a primitive polynomial (i.e. \( c(f) = 1 \)). One can than easily show that \( c(g) \leq d^k (d, \alpha_1)^{k-1} \). If \( f \) is not primitive, we apply the previous result to the polynomial \( f/c(f) \) and get

\[
c(f/c(f)) \leq d^k (d, \alpha_1/c(f))^{k-1} \leq d^k |\alpha_1|^{k-1}.
\]

The claim now follows from \( c(g) = c(f/c(f))c(f) \) and \( c(f) \leq |\alpha_1| \). \( \square \)

We now state the major arcs estimate.

**Proposition 1.** Let \( G_{n,d}(x) \) be a trigonometric polynomial as in (1.5) for some integer polynomial \( f \) of a degree \( k \geq 3 \) satisfying (1.1), and \( n, d \) positive integers. Let \( x = a/q + \kappa \). Then

\[
G_{n,d}(x) = \frac{1}{q} S_d(a, q) F_n(\kappa) + O_f(dqn^{-1/k}(1 + |\kappa|n)).
\]

**Proof.** Without loss of generality we assume that \( dqn^{-1/k} \leq \alpha_k^{-1/k} \) (otherwise the error term is of the order 1 and the claim is trivial). By writing \( \text{Re}(e(y)) \) instead of \( \cos(2\pi y) \) and appropriate grouping, we get

\[
G_{n,d}(x) = \text{Re} \sum_{s=0}^{q-1} \frac{1}{q} e(f(ds)a/q) \frac{2}{K} \sum_{\alpha_k(dj)^k \leq n \atop j \equiv s(\text{mod } q)} \alpha_k qkd^{k-1} \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} \right) e(f(dj)\kappa).
\]
We fix all the parameters and constants. Let $B/2$ be the inner sum in the expression above, $A = B/K$, and let

$$C = 2 \sum_{\alpha_k(dj)^k \leq n, \beta \equiv s \pmod{q}} e(f(dj)\kappa) \sum_{t = f(dj)}^{f(d(j+q)q) - 1} \left( \frac{1}{n} - \frac{t}{n^2} \right),$$

$$D = 2 \sum_{f(dj) \leq n, \beta \equiv s \pmod{q}} e(f(dj)\kappa) \sum_{t = f(dj)}^{f(d(j+q)) - 1} \left( \frac{1}{n} - \frac{t}{n^2} \right),$$

$$F^*_n(\kappa) = \frac{1}{n} + 2 \sum_{t=1}^{n} \left( \frac{1}{n} - \frac{t}{n^2} \right) e(tk).$$

Note that $F_n(x) = \text{Re} F^*_n(x)$, but $F^*_n$ has in general a non-zero imaginary part.

If $m$ is chosen so that $\alpha_k(dm)^k \leq n < \alpha_k(d(m+1))^k$, then one easily gets

$$|\alpha_k d^k m^k - n| \leq \alpha_k d^k ((m+1)^k - m^k) \leq \alpha_k k^2 (m^k - (m+1)^k).$$

We now have

(2.5) $\alpha_k d^k m^k = n + O_f(d n^{1-1/k}).$

Using that and $\sum_{1 \leq j \leq m} j^{k-1} = (1/k)m^k + O_k(m^{k-1})$, we estimate $K$:

$$K = 2 \sum_{\alpha_k(dj)^k \leq n} \alpha_k d^k (m^k + O_k(m^{k-1})) - \frac{1}{n^2} \alpha_k^2 d^2 (m^2 + O_k(m^{2k-1})) =$$

(2.6) $= 1 + O_f(d n^{1/k}).$

Similarly, by using the elementary fact that

(2.7) $\sum_{1 \leq j \leq m, j \equiv s \pmod{q}} j^{k-1} = O_k \left( \frac{1}{q} m^k + m^{k-1} \right),$

one gets that $|B| \leq 1 + O_f(d q n^{-1/k})$. The assumption $d q n^{-1/k} \ll f \ll 1$ implies $B = O_f(1)$, thus

(2.8) $A - B = B(1/K - 1) = O_f(d n^{-1/k}).$

If $1 \leq j \leq m$ and $f(dj) \leq t \leq f(d(j + q)) - 1$, then (2.2) and (2.3) imply that

$$t = \alpha_k(dj)^k + O_f(d q n^{1-1/k}).$$
Using that and (2.4), we get
\[
\sum_{t=f(dj)}^{f(d(j+q))} \left( \frac{1}{n} - \frac{t}{n^2} \right) = (f(d(j+q)) - f(dj)) \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} + O_f(dqn^{-1/k}) \right) = (\alpha_kqkd^k_jqk^{-1} + O_f(d^2q^2n^{-2/k})) \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} + O_f(dqn^{-1/k}) \right) = \alpha_kqkd^k_jqk^{-1} \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} \right) + \alpha_kqkd^k_jqk^{-1}O_f(dqn^{-1/k}) + O_f(d^2q^2n^{-2/k}).
\]
By summing the previous over all the summands "j" in the definition of C, we get
\[
|B - C| = O_f(dqn^{-1/k}) \sum_{\alpha_k(dj)^k \leq n, j \equiv s(\text{mod } q)} \alpha_kqkd^k_jqk^{-1} + O_f(d^2q^2n^{-2/k}) \sum_{\alpha_k(dj)^k \leq n, j \equiv s(\text{mod } q)} 1.
\]
Now (2.7) implies
\[
(2.9) \quad B - C = O_f(dqn^{-1/k}).
\]
From (2.5), as \(\alpha_kd^k(m+q)^k - \alpha_kd^km^k \leq \alpha_kd^kqk(m+q)^{k-1} + d(m+q) \ll f^{-1}(n^{1/k})\), we get \(\alpha_kd^k(m+q)^k = n + O_f(dqn^{-1/k})\). Therefore (2.2) implies
\[
(2.10) \quad f(d(m+q)) = n + O_f(dqn^{-1/k}).
\]
Choose \(m_s\) so that \(f(dm_s) \leq n < f(dm_s + 1)\). Assume that \(m_s \leq m\) (the second case is proved analogously). If \(f(dm_s + 1) \leq t \leq f(dm(m+q)) - 1\), then (2.10) implies
\[
(2.11) \quad t = n + O_f(dqn^{-1/k}).
\]
Similarly as before, one shows that
\[
(2.12) \quad f(dm(m+1)) = n + O_f(dn^{-1/k})
\]
and
\[
(2.13) \quad f(dm_s + q) = n + O_f(dqn^{-1/k}).
\]
Now, \(C\) and \(D\) only differ in the number of summands \((1/n - t/n)^2\), thus by (2.11), one gets
\[
|C - D| \leq 2 \sum_{t=f(d(m_s+1))}^{f(d(m_s+q))} \left| \frac{1}{n} - \frac{t}{n^2} \right| = O_f(dqn^{-1/k})(f(d(m+q)) - f(dm_s + 1))).
\]
Using (2.10) and (2.12), we deduce that
\[
(2.14) \quad C - D = O_f(dqn^{-1/k}).
\]
We now estimate \(D - F_n^s(x)\). If \(1 \leq j \leq m_s\) and \(f(dj) \leq t \leq f(d(j+q)) - 1\), the relations \(|e(x) - e(y)| \leq 2\pi|x - y|\) and (2.3) imply
\[
|e(f(dj)\kappa) - e(t\kappa)| \leq 2\pi|\kappa|(f(d(j+q)) - f(dj)) = O_f(|\kappa|dqn^{-1/k}).
\]
Comparing $D$ and $F_n^\ast(\kappa)$, we see that
\[
|D - F_n^\ast(\kappa)| \leq \frac{1}{n} + 2 \sum_{t=1}^{f(ds)-1} \left| \frac{1}{n} - \frac{t}{n^2} \right| |e(t\kappa)|
\]
\[+ 2 \sup_{1 \leq j \leq m_\ast, j \equiv 0 \pmod{q}} \left| e(f(dj)\kappa) - e(t\kappa) \right| \sum_{t=f(ds)}^{n} \left| \frac{1}{n} - \frac{t}{n^2} \right|
\]
\[+ 2 \sum_{t=n+1}^{f(d(m_\ast+q))-1} \left| \frac{1}{n} - \frac{t}{n^2} \right| |e(f(dj)\kappa)|.
\]
Using the previous two relations, (2.11) and (2.13), we conclude that
\[D - F_n^\ast(\kappa) = O_f(dq n^{-1/k}) + O_f(dq |\kappa| n^{1-1/k}).
\]

Now note that $S_d(a, f, q) = \sum_{s=0}^{q-1} e(f(ds)a/q)$ is real. The claim now follows by combining (2.8), (2.9), (2.14) and (2.15). \hfill \Box

Let $\mathcal{M}(Q, R)$ denote the major arcs, namely the set of all $x \in \mathbb{R}$ which can be approximated by a rational $a/q$, $(a, q) = 1$, where $q \leq Q$, so that $|x - a/q| \leq 1/qR$, and let $\mathfrak{m}(Q, R) = \mathbb{R} \setminus \mathcal{M}(Q, R)$ be the minor arcs. We also define a function $\tau(d, q)$ which will describe the behavior of the principal part of the major arcs estimate:

\[
\tau^*(d, q) = \begin{cases} 1, & \text{if } q \mid d^k \\ -c_0|\alpha_1|^{1-1/k} & \text{otherwise,} \end{cases}
\]

\[
\tau(d, q) = \max\{\tau^*(d, q), -1\},
\]
where $r = q/(q, d^k)$ and $c_0$ is the implicit constant in Lemma\[\text{[1]}\] Lemma\[\text{[1]}\] and Lemma\[\text{[2]}\] imply that $S_d(a, f, q)/q \geq \tau(d, q)$. As $F_n(\kappa)$ is non-negative, we can now put all the results of this section together:

**Corollary 3. The major arcs estimate.** Let $G_{n,d}(x)$ be a trigonometric polynomial as in (1.7) for some integer polynomial $f$ of a degree $k \geq 3$ satisfying (1.7). Assume $1 \leq Q < R$ are given. Let $x \in \mathcal{M}(Q, R)$, $x = a/q + \kappa$, $(a, q) = 1$, $q \leq Q$. Then
\[
G_{n,d}(x) \geq \tau(d, q)F_n(\kappa) + O_f(dn^{-1/k}(Q + n/R)).
\]

3. The minor arcs

We derive the following Lemma from the well-known estimates of Vinogradov.

**Lemma 3.** Let $f(x) = \alpha_k x^k + \ldots + \alpha_1 x$ be an integer polynomial. If $m, d, 1 \leq Q < R$ are constants so that $Q \geq \alpha_k d^k m^{1/k}$ and $x \in \mathfrak{m}(Q, R)$, then
\[
\sup_{1 \leq m_* \leq m} \left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \ll_f \left( d^{-k} QR \right)^{1/(k-1/k)} + m^{1-\rho},
\]
where $\rho = 1/(8k^2 (\log k + 1.5 \log \log k + 4.2))$.

**Proof.** We write $g(j)$ instead of $f(dj)x$, or more precisely: let $g(y) = \beta_k y^k + \ldots + \beta_1 y$ where $\beta_j = \alpha_j d^j x$. Let $1 \leq m_* \leq m$. Applying Vinogradov exponential sum bounds (see [17], Section 11), it is easy to see that if $Q \geq \alpha_k d^k m_*^{1/k}$, $m_* \geq \ldots$
(\alpha_k^{-1}d^{-k}QR)^{1/(k-1/k)} and \( x \in \mathfrak{m}(Q, R) \), then \( \beta = (\beta_k, \ldots, \beta_1) \) is of the second class, thus

\begin{equation}
(3.1) \quad \left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \ll_f m_*^{1-\rho} \leq m^{1-\rho}.
\end{equation}

Trivially for any \( m_* \),

\begin{equation}
(3.2) \quad \left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \leq m_*.
\end{equation}

The claim now follows by summing (3.2) for the cases \( m_* \geq (\alpha_k^{-1}d^{-k}QR)^{1/(k-1/k)} \) and (3.2) for the cases \( m_* \leq (\alpha_k^{-1}d^{-k}QR)^{1/(k-1/k)} \).

**Proposition 2. The minor arcs estimate.** Let \( G_{n,d}(x) \) be the trigonometric polynomial as in (1.5) for some odd integer polynomial \( f \) and \( 1 \leq Q < R \) constants. Let \( x \in \mathfrak{m}(Q, R) \). Then assume that \( d \leq n^{1/k} \) and \( \alpha_k d^k n^{1/k^2} \leq Q \).

\[ G_{n,d}(x) \ll_f n^{-1/k} (QR)^{1/(k-1/k)} + dn^{-\rho/k}. \]

**Proof.** Choose \( m \) so that \( \alpha_k d^k m^k \leq n < \alpha_k d^k (m + 1)^k \). Then

\begin{equation}
(3.3) \quad \frac{d^k m^k - 1}{n} + \frac{d^2 k m^{2k-1}}{n^2} \ll_f dn^{-1/k}.
\end{equation}

We introduce the notation

\[ g(j) = \alpha_k kd^k j^{k-1} \left( \frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2} \right), \]

\[ h(j) = \cos(2\pi f(dj)x) = \text{Re} e(f(dj)x). \]

By partial integration, using the notation \( \Delta g(j) = g(j+1) - g(j), H(j) = \sum_{i=1}^j h(i) \), we get

\[ G_{n,d}(x) = \frac{2}{K} \sum_{j=1}^m g(j) h(j) = \frac{2}{K} \left( g(m) H(m) - \sum_{j=1}^{m-1} \Delta g(j) H(j) \right) \ll_f \frac{1}{K} \left( \frac{d^k m^{k-1}}{n} + \frac{d^2 k m^{2k-1}}{n^2} \right) \sup_{1 \leq m_* \leq m} |H(m_*)| \]

By (2.0) and as \( d \leq n^{1/k}, 1/K = O_f(1) \). As \( Q \geq \alpha_k d^k n^{1/k^2} \) and as \( m_*, n^{1/k} \leq m \), we get \( Q \geq \alpha_k d^k m^{1/k} \). We now combine Lemma 3 and (3.3):

\[ G_{n,d}(x) \ll_f dn^{-1/k} \left( (d^{-k}QR)^{1/(k-1/k)} + n^{1/k-\rho/k} \right) \leq n^{-1/k} (QR)^{1/(k-1/k)} + dn^{-\rho/k}. \]

\[ \square \]
4. Cancelling out the leading term

Recall the definition of the functions $\tau^*(d, q)$, $\tau(d, q)$ in Section 2, estimating the principal part of the major arcs estimate. For clarity of presentation, denote by $\alpha = \alpha_0|\alpha|$, $\beta = 1/k$, and then

$$
\tau(d, q) = \begin{cases} 
1 & \text{if } q|d^l \\
\max\{-\alpha r^\beta, -1\} & \text{otherwise},
\end{cases}
$$

where $r = q/(d^l)$. We use in this section only the facts that $\alpha > 0$, $0 < \beta < 1$.

**Theorem 3. Averaging.** Assume $\delta > 0$ is given. Then there exist integer constants $s > 0$, $d_0 < d_1 < \ldots < d_s$, $d_s = O(\exp(c_1\delta^{-1/\beta}))$, $c_1$ depending only on $\alpha, \beta$, and a real constant $\lambda > 0$ such that for any integer $q$,

$$
\frac{1}{\Lambda} \sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq -\delta,
$$

where $\Lambda = 1 + \lambda + \ldots + \lambda^s$.

We first discuss the case when $q$ is a prime power $q = p^k$, which encodes the key idea of this section. If $p$ is a prime, then

$$
\tau^*(p^j, p^k) = \begin{cases} 
1, & k \leq jl \\
-\alpha p^{-\beta(k-jl)} & \text{otherwise}.
\end{cases}
$$

**Lemma 4.** Say $p$ is a prime, and $\mu$ any real constant satisfying

$$
1 > \mu \geq \frac{\alpha + 1}{\alpha + p^\beta}.
$$

Then for any positive integer constants $s, k$,

$$
\sum_{j=0}^{s} \mu^j \tau^*(p^j, p^k) \geq -\mu^{s+1}/(1-\mu).
$$

**Proof.** It can be easily deduced from (4.2) that

$$
\frac{\mu p^{\beta l}}{\mu p^{\beta l} - 1} \geq \frac{1}{1-\mu}.
$$

Assume $m$ is the largest integer so that $m \leq s$, $ml < k$. Then for all $j \leq m$,

$$
\tau^*(p^j, p^k) = -\alpha p^{\beta(j-k)}.
$$

Denote the left side of (4.3) by $A_s$. We first apply $k \geq ml + 1$, then (4.4) and finally (4.5):

$$
A_m = -\alpha \sum_{j=0}^{m} \mu^j p^{\beta(j-k)} \geq -\alpha \sum_{j=0}^{m} \mu^j p^{\beta(j-m-1)} = -\alpha \mu^m p^{-\beta} \sum_{j=0}^{m} (\mu p^{\beta})^{-j} \geq -\alpha \mu^m p^{-\beta} \sum_{j=0}^{\infty} (\mu p^{\beta})^{-j} = -\frac{\alpha p^{\beta(l-1)}}{\mu p^{\beta l} - 1}\mu^{m+1} \geq -\frac{\mu^{m+1}}{1-\mu}.
$$

The case $m = s$ is now proved. If $m < s$, then for $m < j \leq s$, $\tau^*(p^j, p^k) = 1$, thus

$$
A_s = A_m + \sum_{j=m+1}^{s} \mu^j \geq -\frac{\mu^{m+1}}{1-\mu} + \sum_{j=m+1}^{s} \mu^j = -\mu^{s+1}/(1-\mu).
$$

□
We now improve Lemma 4 and (4.2), so that also for small $p$, $\mu$ can be close to $1/2$.

**Lemma 5.** Say $p$ is a prime, $1 > \mu > 1/2$ and $a \geq 1$ an integer satisfying

\begin{equation}
\label{eq:4.6}
p^{\beta al} \geq \frac{\alpha p^{-\beta}(1 - \mu) + 2\mu - 1}{\mu(2\mu - 1)}.
\end{equation}

Then for any positive integers $s, k$,

\begin{equation}
\label{eq:4.7}
\sum_{j=0}^{s} \mu^j \tau(p^{a_j}, p^k) \geq -\mu^{s+1}/(1 - \mu).
\end{equation}

**Proof.** We follow the steps of the proof of Lemma 4, and first note that (4.6) implies

\begin{equation}
\label{eq:4.8}
-\alpha p^{-\beta} + \mu p^{\beta al} - 1 \geq -\frac{\mu}{1 - \mu}.
\end{equation}

Denote the left side of (4.7) by $B_m$. Let $m$ be the largest integer so that $m \leq s$, $am \leq k$. Then for all $j \leq m$, $\tau(p^{a_j}, p^k) \geq -\alpha p^{\beta(a_j-k)}$. In the calculation below we apply that and the following facts respectively: $\tau \geq -1$ for $j = m$; $k \geq am + 1$; then (4.8) and finally (4.9). We thus have

\begin{align*}
B_m & \geq -\alpha \sum_{j=0}^{m-1} \mu^j p^{\beta(a_j-k)} - \mu^m \\
& \geq -\alpha \sum_{j=0}^{m-1} \mu^j p^{-\beta(al+1) - j} - \mu^m \\
& \geq -\mu^m \frac{\alpha p^{-\beta} + \mu p^{\beta al} - 1}{\mu p^{\beta al} - 1} \geq -\frac{\mu^{m+1}}{1 - \mu}.
\end{align*}

The rest of the proof is analogous to the proof of Lemma 4. \qed

We now set $\lambda = 1/2^\beta$, and combine Lemmas 4 and 5 to find the prime power components of $d_j$ in Theorem 4.1.

**Lemma 6.** There exist a constant $c_2$ depending only on $\alpha, \beta$, so that the following holds: for any positive integer $s$ and prime number $p \leq 2^s$, there exist integers $0 = a_0 \leq a_1 \leq \ldots \leq a_s$ such that for any positive $k$,

\begin{equation}
\label{eq:4.10}
\sum_{j=0}^{s} \lambda^j \tau(p^{a_j}, p^k) \geq -\frac{1 + \alpha}{1 - \lambda} \lambda^{s+1},
\end{equation}

\begin{equation}
\label{eq:4.11}
p^{a_s} < 2^{c_2 s}.
\end{equation}

**Proof.** We will distinguish small and large primes, and will apply below Lemma 4 for large, and Lemma 5 for small primes. Let $p_* = p_*(\alpha, \beta)$ be the smallest prime such that (1.2) holds for $p = p_*$ and $\mu = \lambda = 1/2^\beta$ (and then it holds for all $p \geq p_*$). Let $a_* = a_*(\alpha, \beta)$ be the smallest integer so that (4.6) holds for $p = 2$, $a = a_*$ and $\mu = \lambda = 1/2^\beta$ (and then it holds for all primes $p$ and the same $a = a_*$). We distinguish two cases:
(i) Assume $p$ is small, i.e. $p < p_*$. Then we set $a_j = a_* j$. Because of definition of $a_*$, we can apply Lemma 5 and get

\begin{equation}
\sum_{j=0}^{s} \lambda^j \tau(p^a, p^b) \geq -\frac{1}{1-\lambda} \lambda^{s+1}.
\end{equation}

We also see that

\begin{equation}
p^{a_*} < p^{a_* s}.
\end{equation}

(ii) Let $p$ be large, that means $p^* \leq p \leq 2^s$. We find an integer $q$ so that $p^q \leq p < p^{q+1}$, and let $b, r$ be the quotient and the reminder of dividing $s$ by $q$, thus $s = bq + r$. Let $a_j = \lfloor j/q \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than $x$. First note that the function $f(x) = (\alpha + x^q)/(\alpha + x)^q$ is increasing for $x \geq 1$ (e.g. by differentiating). Now applying this, the definition of $p_*$ and $p \geq p^q$, we get

\begin{equation}
\lambda^q \geq \left(\frac{\alpha + 1}{\alpha + p^q}\right)^q \geq \frac{\alpha + 1}{\alpha + p^q} \geq \frac{\alpha + 1}{\alpha + p^q}.
\end{equation}

Denote the right side of (4.11) with $C_s(k)$ and let $C_s^*(k)$ be the same sum with $\tau^*$ instead of $\tau$. We can now apply Lemma 6 with $\mu = \lambda^q = 1/2^q$, and get

\begin{equation}
C_{bq-1}(k) = \sum_{j=0}^{b-1} (1 + \lambda + \ldots + \lambda^{q-1}) \mu^j \tau^*(p^j, p^k) \geq -\frac{1 + \lambda + \ldots + \lambda^{q-1}}{1-\mu} \mu^b = -\sum_{j=bq}^{\infty} \lambda^j.
\end{equation}

We analyze two cases. Suppose $k \leq bl$. Then $\tau^*(p^j, p^k) = 1$. We use (4.15) and get

\begin{align*}
C_s^*(k) &= C_{bq-1}(k) + \sum_{j=bq}^{s} \lambda^j \geq -\sum_{j=s+1}^{\infty} \lambda^j = -\frac{1}{1-\lambda} \lambda^{s+1}.
\end{align*}

Now assume $k > bl$. Then $\tau^*(p^j, p^k) \leq 0$ and also for all $j \leq b-1$, $\tau^*(p^j, p^k) = p^{-\beta} \tau^*(p^j, p^{k-1})$. We now get from (4.15) that

\begin{equation}
C_{bq-1}(k) = p^{-\beta} C_{bq-1}(k-1) \geq -p^{-\beta} \sum_{j=bq}^{\infty} \lambda^j.
\end{equation}

It is easy to deduce from (4.14) that

\begin{equation}
-p^{-\beta} \geq -\lambda^q.
\end{equation}

As $\tau^*(p^b, p^k) \geq -\alpha p^{-\beta}$, because of (4.16), (4.17) and finally $bq + q \geq s + 1$, we get

\begin{align*}
C_s^*(k) &= C_{bq-1}(k) + \sum_{j=bq}^{s} \lambda^j \tau^*(p^b, p^k) \geq -p^{-\beta} \sum_{j=bq}^{\infty} \lambda^j - \sum_{j=bq}^{s} \lambda^j \alpha p^{-\beta} \geq -\sum_{j=bq+q}^{\infty} \lambda^j \alpha p^{-\beta} \geq -\sum_{j=bq+q}^{\infty} \lambda^j \alpha p^{-\beta} \\
&\geq -(1 + \alpha) \sum_{j=bq+q}^{\infty} \lambda^j \geq -\frac{1 + \alpha}{1-\lambda} \lambda^{s+1}.
\end{align*}

As $C_s(k) \geq C_s^*(k)$, we see that (4.10) holds in both cases. Finally,

\begin{equation}
p^{a_*} = p^b < p^{(q+1)} \leq p^{2s}.
\end{equation}

We get (4.11) from (4.13) and (4.18), with $c_2 = \max{2, a_*} \log_2 p_*$. □
We now show why the left side of \((4.1)\) can be reduced to analysis of a prime factor.

**Lemma 7.** Say \(d_0, d_1, \ldots, d_s\) is a sequence of integers such that \(d_j|d_{j+1}\). Then for each integer \(q\), there exists a prime \(p\) such that for all \(j\),

\[
\tau(d_j, q) \geq \tau(p^{a_i}, p^k),
\]

where \(p^{a_i}, p^k\) are the factors in the prime decomposition of \(d_j, q\) respectively.

**Proof.** If \(q = 1\), then \(k = 0\), so both sides of \((4.19)\) are equal to 1. Assume now that \(q > 1\). Let \(m + 1\) be the smallest index such that \(q|d_{m+1}\) (if there is no such \(m\), we set \(m = s\)). If \(m = 0\), then \(q|d_j\) for all \(j\), so both sides of \((4.19)\) are equal to 1. In that case, we choose any prime \(p\) in the decomposition of \(q\).

Now say 1 \(\leq m \leq s\), and let \(r = q/(q, d_m)\) and let \(p\) be any prime in the prime decomposition of \(r\). For \(j \geq m + 1\), both sides of \((4.19)\) are equal to 1. For \(j \leq m\), it is straightforward to check \((4.19)\). \(\square\)

We now complete the proof of Theorem 3. Recall that \(\lambda = 1/2^\beta\). Let \(c_3\) be the largest of the constants \((1 + \alpha)/(1 - \lambda)\) and \(\alpha/\lambda\), and choose \(s\) so that

\[
(4.20) \quad \lambda\delta/c_3 \leq \lambda^{s+1} \leq \delta/c_3.
\]

Let \(\Lambda = 1 + \lambda + \cdots + \lambda^s, m = 2^s\) and 2 = \(p_1 < p_2 < \ldots < p_t\) be all the prime numbers between 1 and \(m\), and let \(a_i^j\) be the exponents constructed in Lemma 6 associated to the prime \(p_i, i = 1, \ldots, t, j = 0, \ldots, s\). We set

\[
d_j = \prod_{i=1}^{t} p_i^{a_i^j}.
\]

Let \(p\) be the smallest prime number constructed in Lemma 7. If \(p \leq m\), then \(p = p_i\), for some \(i = 1, \ldots, t\). Now applying Lemma 7, Lemma 6, \((4.20)\) and \(\Lambda \geq 1\), we deduce that for any positive integer \(q\),

\[
\frac{1}{\Lambda} \sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq \frac{1}{\Lambda} \sum_{j=0}^{s} \lambda^j \tau(p_i^{a_i^j}, p^k) \geq -\frac{1 + \alpha}{(1 - \lambda)\Lambda} \lambda^{s+1} \geq -\delta.
\]

Now assume that \(p > m\). Then Lemma 7 and \((4.20)\) imply that

\[
\frac{1}{\Lambda} \sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq -\alpha p^{-\beta} \geq -\delta.
\]

We deduce that \((4.1)\) holds. Now we estimate \(d_s\). By \((4.20)\) and the definition of \(m\), we get \(m \leq (c_3/\delta)^{1/\beta}\) and thus

\[
(4.21) \quad s \leq \frac{1}{\log 2} \log(c_3/\delta)^{1/\beta}.
\]

The prime number theorem implies that \(t \leq c_4(m/\log m)\), for some constant \(c_4\), so

\[
(4.22) \quad t \leq c_4 \frac{(c_3/\delta)^{1/\beta}}{\log(c_3/\delta)^{1/\beta}}.
\]

Finally, by applying \((4.11), (4.21)\) and \((4.22)\), we get that \(d_s \leq \exp(c_2 st) \leq \exp(c_1 \delta^{-1/\beta})\), where \(c_1 = c_2 c_3^{1/\beta} c_4/\log 2\).
5. Proof of Theorem \[\text{[1]}\]

We choose \(\delta > 0\), and find first, by applying the Theorem \[\text{[8]}\] the constants \(d_0, \ldots, d_s\) and \(\lambda > 0\) such that for all integers \(q\),

\[
\frac{1}{\Lambda} \sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq -\delta/2,
\]

(5.1)

\[
d_s \leq \exp(c_5\delta^{-k}),
\]

(5.2)

where \(\Lambda = 1 + \lambda + \cdots \lambda^s\) and \(c_5\) depends on the degree and the coefficients of the polynomial \(f\). Let \(c_6\) and \(c_7\) be the implicit constants from Corollary \[\text{[3]}\] and Proposition \[\text{[2]}\] respectively. To streamline the calculations below, we define \(\Lambda = \lambda^{1 + 1} + 1\) max\{\(c_5, c_6, c_7\)/c_5\} and \(d_s = c_8 \exp(c_5\delta^{-k})\). Then it is easy to check that

\[
\max\{c_6, c_7\}(\alpha_k + 1)d_s^{-1} \leq \frac{\delta}{2}
\]

(5.3)

and that \(d_j \leq d_s\) for all \(j = 1, \ldots, s\). Compiling the constraints and the error terms from Corollary \[\text{[3]}\] and Proposition \[\text{[2]}\] we see that it is now enough to choose the constants \(n, Q, R\) so that:

\[
c_6_d_s n^{-1/k}(Q + n/R) \leq \frac{\delta}{2},
\]

\[
d_s \leq n^{1/k},
\]

\[
\alpha_k d_s^k n^{1/k^2} \leq Q,
\]

\[
c_7 \left(n^{-1/k} (Q R)^{1/(k-1/k)} + d_s n^{-\rho/k}\right) \leq \frac{\delta}{2},
\]

where \(\rho = 1/(8k^2(\log k + 1.5 \log \log k + 4.2))\). One can check using \[\text{[5, 7]}\] that the choice \(n = d_s^k, Q = \alpha_k d_s^{1.5k^2}\) and \(R = d_s^{k - k^2 + k^2 - 2.5k^4}\) satisfies all these relations.

We now define the cosine polynomial

\[
T(x) = \delta + \frac{1 - \delta}{\Lambda} \sum_{j=0}^{s} \lambda^j G_{n,d_j}(x).
\]

Clearly \(T(0) = 1\). Now for \(x \in \mathfrak{M}(Q, R)\), Corollary \[\text{[3, 5.1]}\] with the choice of constants above imply \(T(x) \geq \delta + (1 - \delta)(-\delta/2 - \delta/2) \geq 0\). Similarly for \(x \in \mathfrak{m}(Q, R)\), \(T(x) \geq 0\). Choose \(m_j\) such that \(\alpha_k d_s^k m_j \leq n < \alpha_k d_s^k (m_j + 1)\), for \(j = 0, \ldots, s\), and let \(m = \max\{m_0, \ldots, m_s\}\). For given \(\delta > 0\), the largest non-zero coefficient of the polynomial \(T\) is of the order at most \(N = P(d_s m)\). From \[\text{[2.11]}\], we get \(N = O_f(\exp(c_5(k + k^8)\delta^{-k}))\), thus \(\delta = O_f((\log N)^{-1/k})\).

6. Proof of the lower bound

The proof of the lower bound mimics the construction of I. Ruzsa in the case of \(f(x) = x^2\) \[\text{[14]}\].

**Lemma 8.** Let \(k \geq 3\) be an odd integer, \(\beta > 0\) an integer, and \(p \equiv 1(\text{mod } k)\) a prime, \(p > \beta\). Then there exists a collection of integers \(a_1, a_2, \ldots, a_s, s = (p - 1)/k\) such that for any integer \(j, (j, p) = 1\),

\[
\sum_{i=1}^{s} \cos(2\pi \beta j^k a_i/p) \leq -\sqrt{s/(k - 2)}.
\]
Proof. Assume without loss of generality that $\beta = 1$ (we can do it as $(\beta, p) = 1$). As the congruence $x^k \equiv y^k (\mod p)$ has $k$ solutions for any $y$ relatively prime with $p$, we can divide the set of $p - 1$ reduced residue classes mod $p$ into $k$ equivalence classes $Q_1, ..., Q_k$ of size $s = (p - 1)/k$, defined as: $a_1 \sim a_2$ if for some $j$, $(j, p) = 1$, $a_1a_2^{-1} \equiv j^k (\mod p)$ (the $a_2^{-1}$ is the multiplicative inverse of $a_2$ mod $p$). As $k$ is odd, $a \sim -a$. We conclude that the sum $A_m$, defined below is real, on the left-hand side independent of $j$, $(j, p) = 1$ and on the right-hand side independent of $a \in Q_m$:

\begin{equation}
A_m = \sum_{a \in Q_m} e(j^k a/p) = \frac{1}{k} \sum_{j=1}^{p-1} e(j^k a/p).
\end{equation}

By definition,

\begin{equation}
\sum_{m=1}^{k} A_m = \sum_{a,(a,p)=1} e(j^k a/p) = -1.
\end{equation}

We now evaluate $\sum_{m=1}^{k} A_m^2$ by using the right-hand side of (6.1), and get

\begin{equation}
\sum_{m=1}^{k} s k^2 A_m^2 = \sum_{m=1}^{k} \left( p-1 \right) \sum_{a=1}^{p-1} e(x^k a/p) \sum_{x=1}^{p-1} e((x^k - y^k)a/p) = \sum_{x,y=1}^{p-1} \sum_{a=1}^{p-1} e((x^k - y^k)a/p) = (kp - p + 1)(p - 1),
\end{equation}

where we used that $x^k \equiv y^k$ has $k$ solutions mod $p$ for $x, y$ relatively prime with $p$. Now suppose all $A_m \geq -c$ for some $c \geq 0$. If there are $k^-$ numbers $A_m < 0, 1 \leq k^- \leq k$, then $\sum A_m < 0$, $A_m < 0$, and by using (6.2), $\sum_{m,A_m\leq0} A_m^2 \leq (k^- c - 1)^2$. Combining that with (6.3), we easily get $c \geq \sqrt{s/(k - 2)}$. Now we can find $A_m \leq -\sqrt{s/(k - 2)}$, and choose $a_1, ..., a_s$ to be the elements of $Q_m$.

We now complete the proof of Theorem 2. Choose any cosine polynomial $f(x)$, $T(x) \geq 0$ and $T(0) = 1$, with $f(x) = \beta x^k$. By calculating $\sum_{a=1}^{k} T(a/p), p \equiv 1(\mod k), p > \beta$, applying Lemma 8 and noting that $1 + \sqrt{s(k - 2)} \leq \sqrt{p}$, we get

\[ \sum_{p \mid j} a_j \geq \frac{1}{\sqrt{p}}. \]

We multiply this by $\log p$ and sum for $p \leq m$. We get

\[ \sum_{p \mid j, p \equiv 1(\mod k), p \leq m} \log p \geq \sum_{p \mid j, p \equiv 1(\mod k), p \leq m} \log p. \]

By the theorem on primes in arithmetic sequences, the sum on the right side is $\sqrt{n}/\phi(k) + o(1)$. On the left-hand side, the coefficient of $a_j$, $j > 0$ is $\leq \log j \leq \log n$ (where $n$ is the largest non-zero coefficient of $T(x)$), and the coefficient of $a_0$ is
m(1/ϕ(k) + o(1)), where ϕ(k) is the Euler’s totient function. By writing ε = o(1), we get

\[ ma_0 \left( \frac{1}{\phi(k)} + \varepsilon \right) + \log n \geq \left( \frac{2}{\phi(k)} - \varepsilon \right) \sqrt{n}. \]

We express \(a_0\), minimize over \(\sqrt{m}\) and obtain \(a_0 \geq \frac{(1/\phi(k) - \varepsilon)}{\log n}\).

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