Bell’s Theorem - Why Inequalities, Correlations?

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Abstract

It is shown that Bell’s counterfactuals admit joint quasiprobability distributions (i.e. joint distributions exist, but may not be non-negative). A necessary and sufficient condition for the existence among them of a true probability distribution (i.e. non-negative) is Bell’s inequalities. This, in turn, is a necessary condition for the existence of local hidden variables. The treatment is amenable to generalization to examples of 'nonlocality without inequalities'.

1 Introduction

Bell’s derivation of his famous inequalities [2] of forty years ago hardly leaves room for improvement in terms of conciseness or elegance (see, however [9] for a particularly clear derivation and discussion). It is the purpose of this paper to give a straightforward ('brute force') derivation of them starting from very simple assumptions. The inequalities automatically follow as a necessary and sufficient condition for the impossibility of Bell type local realism, for the situation he envisaged[1].

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To demonstrate the approach, let us first consider a toy problem. Let us assume we have two spin-1/2 particles in an EPR-Bohm\cite{4,5} state, i.e. their spins are in a singlet state, and their position state corresponds to one of of them being localized in the vicinity of an observer named Alice, and the other in the vicinity of a remote observer named Bob. Let us now assume that Alice has chosen a particular measurement to perform that can return either of two possible outcomes. Similarly Bob has chosen another such binary measurement. For convenience, let’s label the two outcomes of each measurement as $\pm 1$. Suppose further that Alice’s outcomes have probabilities $P[A = +1] = p_{A}^{+}$, $P[A = -1] = p_{A}^{-}$ where $p_{A}^{\pm}$ are determined by some theory (in particular we would be interested by outcomes predicted by Quantum Mechanics, but that is immaterial at this point). Bob’s outcomes have probabilities $p_{B}^{\pm}$. We now pose the question: can a joint probability, $\{P_{ab}^{AB} | a, b \in \{+1, -1\}\}$ be defined such that the marginal probabilities it generates for $A, B$ coincide with $P^{A}$ and $P^{B}$. In other words, $P^{AB}$ has to satisfy:

\begin{align*}
P^{AB}_{++} + P^{AB}_{+-} &= P^{A}_{+} \\
(P^{AB}_{-+} + P^{AB}_{--} &= P^{A}_{-}) \\
(P^{AB}_{++} + P^{AB}_{-+} &= P^{B}_{+} \\
(P^{AB}_{--} + P^{AB}_{-+} &= P^{B}_{-}) \\
P^{AB}_{++} + P^{AB}_{-+} + P^{AB}_{-+} + P^{AB}_{--} &= 1 \quad (1)
\end{align*}

and

\begin{align*}
P_{ab}^{AB} \geq 0, \ a, b \in \{+1, -1\} \quad (2)
\end{align*}

The equations in parentheses are easily seen to be redundant - they are implied by the rest, and by the fact that the marginals, being distributions, sum to 1.

We can answer in the affirmative immediately, since the ‘product probability’ $P_{ab}^{AB} \equiv P^{A}_{a}P^{B}_{b}$ is indeed always a well defined probability, and has the desired marginals. This also implies that $P^{AB}$ is not, in general, uniquely defined, since we could start by choosing a manifestly non-product joint distribution and it would be different than the product distribution defined by
its marginals. However, the point is, that Eqs. (11) can be solved directly. These solutions can be called quasiprobabilities, since they are not necessarily non-negative. Inequalities (2) are just the statement that the distribution correspond to a probability (i.e. be non-negative). Since the constraints on $P^{AB}$ can always be satisfied, there is no dependence on any particular assumptions about the marginal distributions. No matter what Quantum Mechanics predicts, it can be mimicked by a local realistic model (see next section).

We shall see below that in Bell’s scenario - the same as the toy problem, but now Alice can choose to measure either $A_1$ or $B_1$, and Bob either $B_2$ or $C_2$. The variables $B_1$ and $B_2$ will be chosen in a special way, and now the predictions of Quantum Mechanics will be important. While Quantum Mechanics does not define a joint probability $P^{A_1B_1B_2C_2}$ ($a, b, c, d \in \{\pm 1\}$), it does predict the joint distributions $P^{A_1B_2}, P^{A_1C_2}, P^{B_1C_2}$ and $P^{B_1B_2}$ ($B_1, 2$ will be defined such that $P_{ab}^{B_1B_2}$ will be nonzero only for $a = -b$). It will be seen that while the equations for $P^{A_1B_1B_2C_2}$ generalizing (1) can still be satisfied, the inequalities generalizing (2) can only be satisfied for some marginal distributions. In particular, for some choices of variables $A, B, C$, those constraints will clash with the predictions of Quantum Mechanics. For binary valued pairs of variables, the probability distributions can be stated in terms of the expectation values of the products: $\langle AB \rangle \equiv E(A_1 B_2)$, etc. (see below).

When expressed in this way, the constraints become identical to Bell’s famous inequalities. For $n$-valued measurables ($n > 2$) or more than two particles, linear correlations would no longer suffice. The fact that any marginal single observable distributions are compatible with a joint probability distribution remains true when we increase the number of such observables, and the number of values they can take, as long as both remain finite. The simple construction of a joint product distribution carries over to this case. Note that the single observables can be composite (e.g. vectors) but the different observables should be defined independently of each other (unlike $(A, B_2)$ and $(B_1, C)$, $B_1 = -B_2$ used by Bell).

When we go to the continuous case, this no longer holds. As shown in [11], one way to define the Wigner quasiprobability distribution for the state of a particle with a one dimensional continuous degree of freedom, is simply to require that it generate the correct marginal distributions for all variables of the form $x_\theta \equiv \cos \theta x + \sin \theta p$ (for a very good exposition see [12]). These conditions define Wigner’s distribution $W(q, p)$ uniquely. In general $W$ takes both positive and negative values. Coherent states, which are arguably the
closest to being “classical”, are a notable exception. “Scrödinger cat” states display strong oscillations with notable negative dips. The analogy with what follows seems to be more than coincidental.

Following Bell’s seminal paper, it was shown that with three particles in a particular entangled state (GHZ state \[6\] and for a particular choice of observables, one can get a contradiction with the assumption of existence a local realist joint probability, not involving inequalities. Instead, the existence of a local realist theory implies the existence of single events that violate the predictions of QM. More recently, Hardy has shown\[8\] that one could get similar results even with two spin-1/2 particles. In his example, local realism implies that either some events must exist that violate the predictions of quantum theory, or other events (outcomes of measurements) should never occur. Thus these tests involve no inequalities, besides perhaps whether some probability be larger than 0. Surprisingly, Hardy’s construction works for almost all entangled states, the only exceptions being the maximally entangled ones! Those are precisely the states that display maximal violation of Bell’s inequalities.

The direct derivation of Bell’s inequalities provided here, shows that for maximally entangled states, no scheme involving (essentially) 3 observables (as in Bell’s original derivation) can do better than give precisely Bell’s statistical inequalities as the local realist predictions. It might be hoped that a general analysis of the 4 observable case along these lines could shed some light on this intriguing complementarity between Bell’s and Hardy’s examples.

2 The Condition for Existence of Quasi-Probabilities

Let us assume a hidden variable \(\lambda\) exists, that determines the outcome of the measurement of \(A = \vec{\alpha} \cdot \vec{\sigma}_1^A\): \(A = f(\vec{\alpha}, \lambda)\), and similarly for \(B = \vec{\beta} \cdot \vec{\sigma}_i^B\) \((i \in \{1, 2\})\) and \(C = \vec{\gamma} \cdot \vec{\sigma}_2^C\). In other words, \(A_i, B_i\) and \(C_i\) are random variables in the same space. The statistics of \(\lambda\) determine a well defined probability distribution \(P(A_1 = a, B_2 = b, C_2 = c)\) \((a, b, c \in \{-1, 1\})\). We have suppressed \(B_1\), because it will be assumed that, with probability 1, \(B_2 = -B_1\). In what follows, we will use the notation \(P_{+++}^{ABC} = P(A_1 = +1, B_2 = +1, C_2 = +1)\), etc. While it is assumed that this common distribution exists, it is also assumed that only two of the variables are simultaneously experimentally accessible. We note that \(P^{ABC}\) determines the marginal distributions: \(P^{AB}, P^{AC}\) and
$$P^{BC} = P^{B_{1}C} = P^{(-B_{2})C} \quad (P_{ab}^{AB} = \sum_{c=+,-} P_{abc}^{ABC}, \ldots).$$

The marginal probabilities are given by quantum theory (and verified experimentally). The problem is to find \{P_{abc}^{ABC}\}_{a,b,c=+,-} satisfying:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
P_{++}^{ABC} \\
P_{+-}^{ABC} \\
P_{-+}^{ABC} \\
P_{+++}^{ABC} \\
P_{-++}^{ABC} \\
P_{-++}^{ABC} \\
P_{---}^{ABC} \\
P_{-+-}^{ABC} \\
P_{--+}^{ABC} \\
P_{---}^{ABC}
\end{pmatrix}
= 
\begin{pmatrix}
P_{BC}^{++} \\
P_{BC}^{+-} \\
P_{BC}^{-+} \\
P_{BC}^{+++} \\
P_{BC}^{++-} \\
P_{BC}^{+-+} \\
P_{BC}^{-+-} \\
P_{BC}^{---} \\
P_{BC}^{--+} \\
P_{BC}^{---}
\end{pmatrix}
$$

(3)

Which would make it a quasiprobability distribution. Note that \(P_{--}, P_{-+}, P_{-+}\) do not appear on the right hand side. That is because the equations for those lines are automatically satisfied when the others are, by the normalization of the marginal probabilities (i.e. those equations were removed because they depend linearly on the rest). Referring this matrix equation as \(Mx = p\), we note that \(M's\) rank is only 7, so that if a solution exists, it is not unique. In other words, the homogeneous equation \(Mx = 0\) has a unique solution (up to a multiplicative constant), \(x_{h} = (-1, 1, 1, -1, 1, -1, 1, 1, 1, 1)\). The condition for the existence of solutions is that the vector \(p\) be in the column-space of \(M\), or equivalently, orthogonal to its orthogonal complement. This orthogonal complement is spanned by the vectors

\n\{(-1, -1, 0, 0, 0, 0, 1, 0, 0, 0), (0, 0, 0, -1, -1, 0, 1, 0, 0, 0), (-1, 0, -1, 1, 0, 1, 0, 0, 0, 0)\}\n
The condition is therefore that the marginal probabilities satisfy the equations:

$$
\begin{align*}
P_{++}^{BC} + P_{--}^{BC} &= P_{++}^{AB} + P_{--}^{AB} \\
P_{++}^{AC} + P_{--}^{AC} &= P_{++}^{AB} + P_{--}^{AB} \\
P_{++}^{BC} + P_{--}^{BC} &= P_{++}^{AC} + P_{--}^{AC}
\end{align*}
$$

(4)

These are simple consistency requirements. For example, the first equation follows from the requirement that both sides be equal to \(P_{++}^{ABC} + P_{--}^{ABC} + \ldots\)
$P_{+++} + P_{ABC}$. It will be seen in the next section that, remarkably, these conditions are satisfied by Bell’s counterfactuals for a singlet state.

When these equations are satisfied, the family of quasiprobabilities, $x$, is given by $x = M^+ p + cx_h$, where $M^+$ is the pseudoinverse of $M$, and $x_h$ is the same as above, and $c$ any real number.

The expression for $M^+$ is:

$$
\begin{pmatrix}
\frac{1}{4} & -\left(\frac{1}{2}\right) & -\left(\frac{1}{8}\right) & \frac{1}{4} & -\left(\frac{1}{2}\right) & -\left(\frac{1}{8}\right) & \frac{1}{4} & -\left(\frac{1}{2}\right) & -\left(\frac{1}{8}\right) & \frac{1}{8}
\\
-\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & -\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & -\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & -\left(\frac{1}{20}\right)
\\
-\left(\frac{1}{20}\right) & \frac{1}{8} & \frac{13}{40} & -\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & -\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & -\left(\frac{1}{20}\right)
\\
\frac{1}{20} & -\left(\frac{9}{40}\right) & -\left(\frac{9}{40}\right) & \frac{9}{20} & \frac{3}{8} & -\left(\frac{1}{20}\right) & \frac{1}{8} & \frac{13}{40} & \frac{1}{8} & \frac{13}{40}
\\
\frac{7}{20} & -\left(\frac{3}{20}\right) & -\left(\frac{3}{20}\right) & \frac{1}{20} & 1 & -\left(\frac{1}{20}\right) & \frac{13}{40} & \frac{1}{8} & \frac{13}{40} & \frac{1}{8}
\\
-\left(\frac{1}{20}\right) & \frac{7}{8} & -\left(\frac{1}{20}\right) & \frac{1}{20} & -\left(\frac{9}{40}\right) & -\left(\frac{9}{40}\right) & \frac{9}{20} & \frac{3}{8} & \frac{1}{20} & -\left(\frac{1}{20}\right)
\\
-\left(\frac{3}{20}\right) & -\left(\frac{1}{20}\right) & \frac{3}{8} & -\left(\frac{1}{20}\right) & \frac{1}{4} & -\left(\frac{3}{20}\right) & \frac{3}{8} & \frac{1}{4} & -\left(\frac{3}{20}\right) & \frac{3}{8}
\\
-\left(\frac{1}{4}\right) & -\left(\frac{3}{8}\right) & -\left(\frac{3}{8}\right) & -\left(\frac{1}{4}\right) & -\left(\frac{3}{8}\right) & -\left(\frac{3}{8}\right) & -\left(\frac{1}{4}\right) & -\left(\frac{3}{8}\right) & -\left(\frac{3}{8}\right) & \frac{1}{8}
\\
\end{pmatrix}
$$

Due to the symmetries in Bell’s problem, the equation will look much simpler for that case.

Finally, a probability distribution also has to satisfy the 8 inequalities: $P_{abc}^{ABC} = (M^+ p + cx_h)_{abc} \geq 0 \ (a, b, c = +, -)$. In the next section it will be shown that, for Bell’s problem, these are equivalent to Bell’s inequalities.

## 3 Bell’s Counterfactuals: The Singlet State

Let our two-spin-$\frac{1}{2}$ system be in the singlet state $|\psi_-\rangle$, and the observables $A, B, C$ be defined as above. Then the two-observable common distribution functions are equal to:

$$P_{ab}^{A_1B_2} = \langle \pi_a^{A_1} \pi_b^{B_2} \rangle_{\psi_-} (a, b = +, -) \tag{6}$$

where $\pi_a^{A_1}$ is the projection operator $\frac{1 + \vec{a} \cdot \vec{\sigma}}{2}$, etc. So,

$$P_{ab}^{A_1B_2} = \frac{1}{4} \langle 1 + a \vec{a} \cdot \vec{\sigma}_1 + b \vec{b} \cdot \vec{\sigma}_2 + ab(\vec{a} \cdot \vec{\sigma}_1)(\vec{b} \cdot \vec{\sigma}_2) \rangle_{\psi_-}$$

$$= \frac{1}{4} \langle 1 + ab((\vec{a} \cdot \vec{\sigma}_1)(\vec{b} \cdot \vec{\sigma}_2))_{\psi_-} \rangle = \frac{1}{4}(1 + ab(AB)) \tag{7}$$
((\vec{a} \cdot \vec{\sigma}) (\vec{\beta} \cdot \vec{\sigma}_2))_{\psi_-} = -\vec{a} \cdot \vec{\beta}.

To summarize,

\[
\begin{align*}
P^{AB}_{++} &= P^{AB}_{--} = \frac{1}{4}(1 + (A_1B_2)) \\
P^{AB}_{+-} &= P^{AB}_{-+} = \frac{1}{4}(1 - (A_1B_2)); \\
P^{AC}_{++} &= P^{AC}_{--} = \frac{1}{4}(1 + (A_1C_2)) \\
P^{AC}_{+-} &= P^{AC}_{-+} = \frac{1}{4}(1 - (A_1C_2)); \\
P^{BC}_{++} &= P^{BC}_{--} = \frac{1}{4}(1 + (B_1C_2)) \\
P^{BC}_{+-} &= P^{BC}_{-+} = \frac{1}{4}(1 + (B_1C_2));
\end{align*}
\]

(8)

Note the opposite signs in the last two lines. That is due to the fact that we had defined \(P^{BC} \equiv P^{B_2C_2} = P^{(-B_1)C_2} \).

It is now straightforward to see that equations (4) are satisfied. We are thus assured of the existence of our quasiprobabilities.

Finally, the inequalities \(8P^{ABC}_{abc} = 8(M^++ cx_h)_{abc} \geq 0 \) \((a, b, c = +, -)\), become:

\[
\begin{align*}
1 + (AB) + (AC) - (BC) - c &\geq 0 \\
1 + (AB) - (AC) + (BC) + c &\geq 0 \\
1 - (AB) + (AC) + (BC) + c &\geq 0 \\
1 - (AB) - (AC) - (BC) - c &\geq 0 \\
1 - (AB) - (AC) - (BC) + c &\geq 0 \\
1 - (AB) + (AC) + (BC) - c &\geq 0 \\
1 + (AB) - (AC) + (BC) - c &\geq 0 \\
1 + (AB) + (AC) - (BC) + c &\geq 0 \\
1 + (AB) + (AC) - (BC) + c &\geq 0
\end{align*}
\]

(9)

The question: is there any value of \(c\) such that all these inequalities are simultaneously satisfied.

The first and last equations imply:

\[
1 + (AB) \geq -((AC) - (BC))
\]

(10)
Similarly, the second and second to last inequalities imply:

\[ 1 + \langle AB \rangle \geq + (\langle AC \rangle - \langle BC \rangle). \]  

(11)

Hence, together they are nothing other than Bell’s famous inequality:

\[ 1 + \langle AB \rangle \geq |\langle AC \rangle - \langle BC \rangle|. \]  

(12)

The remaining four inequalities imply:

\[ 1 - \langle AB \rangle \geq |\langle AC \rangle + \langle BC \rangle| \]  

(13)

which is inequality 12 with the substitution \( B \mapsto -B \).

Conversely, the last two inequalities imply that inequalities 9 are satisfied for \( c = 0 \).

## 4 Conclusion

Because of the symmetries of the singlet state of two spin-1/2 particles, and the existence of just two measurement results for each single variable - the joint distributions of two variables \( \{A, B\} \) can be expressed in terms of a single parameter, the linear correlation \( \langle AB \rangle \). Therefore, the most general inequalities on the distributions of the pairs \( \{A, B\}, \{A, C\}, \{B, C\} \) for them to be generated as marginals of a (hypothetical) joint distribution of \( \{A, B, C\} \) can be stated in terms of \( \langle AB \rangle, \langle AC \rangle \) and \( \langle BC \rangle \). These are Bell’s inequalities. However, the underlying assumptions behind the derivation are much more obvious when stated in terms of the distributions rather than the correlations. Furthermore, there is a subtle psychological danger of confusing the general question of correlations (i.e. any statistical dependence) with the much more restricted sense of ‘linear correlations’. Once observables with more than two eigenvalues are considered, these two concepts become quite distinct.

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[1] The main point of this paper is that Bell’s inequalities are equivalent to the conditions imposed by the assumptions that the joint probabilities for the three pairs of observables considered ($\{A, B\}, \{A, C\}, \{B, C\}$) can be considered marginals of a (hypothetical) joint probability of the three observables $\{A, B, C\}$. It turns out that this was shown by Wigner already in 1970 [13]. Pitowsky [14] has shown that George Boole’s ‘conditions of possible experience’ [15] enunciated over a century before Bell’s seminal paper, when applied to the problem at hand, lead to Bell’s inequalities in this way. Unfortunately, this derivation of the inequalities, which appears to me to be the most direct, seems to be largely unknown even to people in the field, or at least routinely overlooked in textbooks and in reviews on the subject.

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