Updated $1/N_c$ expansion analysis of $[56, 2^+]$ and $[70, \ell^+]$ baryon multiplets

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Abstract

The mass spectra of the $[56, 2^+]$ and $[70, \ell^+]$ multiplets, both belonging to the $N = 2$ band, is reviewed in the $1/N_c$ expansion method. Previous studies, separately made for each multiplet, are presently updated to the 2014 Particle Data Group. The mass formula including corrections up to $O(1/N_c)$ and first order in SU(3) flavor symmetry breaking, has the same independent operator basis in both cases. A special emphasis is made on the role of the SU(3) symmetry breaking operators $B_i$ ($i = 1, 2, 3$). This can allow for multiplet assignment of $\Lambda$ and $\Sigma$ hyperons, which generally is quite difficult to make. Tentative assignments of hyperons with two- and one-star resonances are made to the $[70, \ell^+]$ multiplet. Another important aim is to find out whether or not a common value of the coefficient $c_1$ of the dominant operator in the mass formula, can well fit the present data in both multiplets. A negative answer, which is here the case, implies distinct Regge trajectories for symmetric and mixed symmetric states.

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I. INTRODUCTION

The $1/N_c$ expansion method proposed by 't Hooft [1] and applied to baryons by Witten [2], where $N_c$ is the number of colors, is based on the discovery that, for $N_f$ flavors, the ground state baryons display an exact contracted SU($2N_f$) spin-flavor symmetry in the large $N_c$ limit of QCD [3, 4]. The Skyrme model, the strong coupling theory [5] and the static quark model share a common underlying symmetry with QCD baryons in the large $N_c$ limit [6].

Presently the $1/N_c$ expansion method is considered to be a model independent, powerful and systematic tool for baryon spectroscopy. It has been applied with great success to the ground state baryons ($N = 0$ band), described by the symmetric representation $56$ of SU(6) [4, 7–12]. At $N_c \to \infty$ the ground state baryons are degenerate. At large, but finite $N_c$, the mass splitting starts at order $1/N_c$ as first observed in Ref. [6].

The extension of the $1/N_c$ expansion method to excited states requires the symmetry group SU($2N_f$) $\times$ O(3) [13], in order to introduce orbital excitations. There is no a priori justification for this symmetry. However, in practice, the experimentally observed resonances can approximately be classified as SU($2N_f$) $\times$ O(3) multiplets, grouped into excitation bands, $N = 1, 2, 3, ...$, each band containing a number of SU(6) $\times$ O(3) multiplets, as is done in quark models. In addition, lattice QCD studies have shown that the number of each spin and flavor states in the lowest energy bands is in agreement with the expectations based on a weakly broken SU(6) $\times$ O(3) symmetry [14], used in quark models and in the treatment of excited states in large $N_c$ QCD, as is here the case.

The extension of the $1/N_c$ expansion to excited states has been theoretically supported by Pirjol and Yan [15] who derived consistency conditions for excited states, similar to those for the ground state [3, 4]. Later on, the extension of the method to excited states was legitimized by Cohen and Lebed [16] who proved the compatibility between the meson-nucleon scattering picture and the quark model type picture, both leading to identical degeneracy patterns, giving rise to towers of states.

Some symmetric multiplets of SU(6) $\times$ O(3) classification, in particular $[56, 2^+]$ and $[56, 4^+]$, containing two and four units of orbital excitations, were analyzed by analogy to the ground state in Refs. [17] and [18] respectively. In this case the splitting starts at order $1/N_c$ as well.
The situation is technically more complicated for mixed symmetric states. Two approaches have been proposed so far. The first one is based on the Hartree approximation and describes the $N_c$ quark system as a ground state symmetric core of $N_c - 1$ quarks and an excited quark [19]. This implies the split of SU$(2N_f)$ generators into two parts, one acting on the core and the other on the excited quark. Naturally, the number of generators entering the mass formula becomes larger, hence the applicability of the method beyond the $N = 1$ band becomes more problematic [20].

The second procedure, where the Pauli principle is implemented to all $N_c$ identical quarks has been proposed in Refs. [21, 22]. There is no physical reason to separate the excited quark from the rest of the system. The method can straightforwardly be applied to all excitation bands $N$. It requires the knowledge of the matrix elements of all the SU$(2N_f)$ generators acting on mixed symmetric states described by the partition $(N_c - 1, 1)$. In both cases the mass splitting starts at order $N_c^0$. A discussion on the comparison between the two methods and various applications can be found in Ref. [23]. In the following we apply the procedure of Refs. [21, 22] to analyze baryons thought to belong to the mixed symmetric $[70, \ell^+]$ multiplet and reanalyze the symmetric multiplet $[56, 2^+]$, which means that the same basis operators is used in the mass formula in both cases, which is a novel aspect of this study.

In the next section we recall results previously obtained for multiplets in the $N = 2$ band. In Sec. II we introduce the basis operators used in the mass formula and in Sec. IV we derive or recall the analytic forms of the matrix elements of the basis operators. Sec. V is devoted to a numerical fit which gives the values of the dynamical coefficients entering the mass formula followed by a discussion of the obtained resonance masses. Conclusions are drawn in the last section. In Appendix A we recall the analytic formula of the single particle spin-orbit operator used for mixed symmetric $[70, \ell^+]$ multiplets. In Appendix B we present analytic details of the tensor operator $O_6$. In Appendix C we discuss the mixing between spin quartets and doublets in a simplified model. In Appendix D we show details of the analytic calculations of the SU(3) flavor breaking operators $B_2$ and $B_3$.

II. PREVIOUS STUDIES OF THE $N = 2$ BAND IN THE $1/N_c$ EXPANSION

The $N = 2$ band has the following multiplets $[56', 0^+], [56, 2^+], [70, 0^+], [70, 2^+]$ and $[20, 1^+]$. The observed resonances are usually assigned to the symmetric $[56]$ or the mixed
symmetric $[70]$ SU(6) multiplets. The antisymmetric SU(6) multiplet $[20, 1^+]$ has been ignored so far, on the basis that it does not have a real counterpart. However, two new resonances $N(2100)1/2^+$ and $N(2040)3/2^+$, presently included in the 2014 Particle Data Group [24], were recently assigned to the $[20, 1^+]$ multiplet, by an educated guess, see Ref. [25] Table 23. Before drawing any conclusion, stronger experimental evidence is required to confirm the existence of $N(2100)1/2^+$ and $N(2040)3/2^+$.

The multiplet $[56', 0^+]$ describes states with a radial excitation, in particular the Roper resonance. It was the first to be studied in the large $N_c$ limit [26], by using a simplified mass formula of the Gürsey-Radicati type. The analysis was free of any assumption regarding the wave function except its symmetry in SU(6). Strong decay widths were calculated as well.

The analysis of the $[56, 2^+]$ baryon masses has first been performed in Ref. [17]. It has been reconsidered in Ref. [18] with nearly identical results and the analysis has been extended to the higher multiplet $[56, 4^+]$ of the $N = 4$ band in the same paper.

The $[70, 0^+]$ and $[70, 2^+]$ baryon masses were first analyzed in Ref. [27] for $N_f = 2$ and extended in Ref. [20] to $N_f = 3$, both studies being performed within the symmetric core + excited quark procedure [19]. The $[70, \ell^+]$ ($\ell = 0, 2$) multiplets were revisited [28] within the approach of Ref. [21] where the Pauli principle was fully taken into account.

In Refs. [27] and [28] Regge-type trajectories have been drawn for the most dominant coefficient in the mass formula, denoted in the following by $c_1$ and somewhat conflicting results have been obtained. The trajectories were drawn as a function of the band number $N = 0, 1, 2, 3$ and 4. While in Ref. [27] a single trajectory has been obtained (note that large $N_c$ results for the $N = 3$ band were not available yet), in Ref. [28] two distinct, nearly parallel, Regge trajectories have been obtained, the lower one for symmetric $[56]$-plets and the higher one for mixed symmetric $[70]$-plets.

In this work we wish to clarify the issue, whether or not one or two distinct trajectories stem from the $1/N_c$ expansion, one for symmetric the other for mixed symmetric states. For this purpose we combine together the analysis of the $[56, 2^+]$ and $[70, \ell^+]$ multiplets of the $N = 2$ band. An important aspect is that presently we use the same set of linearly independent operators in the mass formula, which was not the case before. Details are given in the following sections. We do not include the $[56', 0^+]$ multiplet, associated with states having a radial excitation, because they can deteriorate the numerical fit, due to their location in the spectrum, too low from the other states.
Another incentive to perform this analysis was that the band number $N$ appeared to be a good quantum number for the spin-independent part of semirelativistic quark models \cite{29,31}. Therefore plotting $c_1^2$ as a function of $N$ seems meaningful, inasmuch as $c_1^2$ simulates the effect of the kinetic and the confinement parts of quark model Hamiltonians.

Presently we use the data of the 2014 Particle Data Group \cite{24}, which sometimes give more precise values for the resonance masses with smaller error bars than before. For example $N(1720)^{3+}_2$ has a mass of $1725 \pm 25$ MeV as compared to $1700 \pm 50$ MeV in the 2002 Particle Data Group \cite{32}, used in Ref. \cite{17}. The changes are due to a more complex analysis of all major photo-production of mesons in a coupled-channel partial wave analysis as done, for example, in Ref. \cite{33}.

III. THE MASS OPERATOR

The general form of the mass operator, where the SU(3) symmetry is broken, has first been proposed in Ref. \cite{11} as

$$ M = \sum_i c_i O_i + \sum_i d_i B_i. \quad (1) $$

The operators $O_i$ are defined as the scalar products

$$ O_i = \frac{1}{N_{c-1}} O^{(k)}_i \cdot O^{(k)}_{SF}, \quad (2) $$

where $O^{(k)}_i$ is a $k$-rank tensor in SO(3) and $O^{(k)}_{SF}$ a $k$-rank tensor in SU(2)-spin, but invariant in SU($N_f$). Thus $O_i$ is rotational invariant. For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms. The $k = 1$ tensor has three components, which are the generators $L^i$ of SO(3). The components of the $k = 2$ tensor operator of SO(3) read

$$ L^{(2)ij} = \frac{1}{2} \{L^i, L^j\} - \frac{1}{3} \delta_{i,j} \bar{L} \cdot \bar{L}. \quad (3) $$

The operators $B_i$ break the SU(3) flavor symmetry and are defined to have zero expectation values for nonstrange baryons.

The angular momentum-independent operators up to $O(N_c^{-1})$ are

$$ O_1 = N_c \mathbb{1}, \quad O_3 = \frac{1}{N_c} S \cdot S, \quad O_4 = \frac{1}{N_c} (T \cdot T - \frac{N_c(N_c+6)}{12}), \quad (4) $$

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where $S^i, T^a$ and $G^{ia}$ are the SU(6) spin, flavor and spin-flavor operators. The definition of $O_4$ where the term $N_c(N_c + 6)/12$ has been subtracted is necessary for including SU(3) singlets \[22\]. This definition gives the same matrix elements as the isospin operator $\frac{1}{N_c} T \cdot T$ restricted to nonstrange baryons within SU(4) symmetry \[21\]. The form of $O_4$ is consistent with Eq. (5.12) of Ref. \[9\].

The operators containing $O^{(k)}_\ell$ tensors are the spin-orbit operator $O_2$ and the tensor operator $O_6$, the latter being defined as

$$O_6 = \frac{1}{N_c} L^{(2)ij} G^{ia} G^{ja}. \tag{5}$$

For the $[56]$-plets the spin-orbit operator $O_2$ is defined in terms of angular momentum $L^i$ components acting on the whole system as in Ref. \[17\] and is order $O(1/N_c)$

$$O_2 = \frac{1}{N_c} L \cdot S, \tag{6}$$

while for the $[70]$-plets it is defined as a single-particle operator $\ell \cdot s$ of order $O(N_c^0)$, as used previously \[19, 20, 28\]. Note that $O_6$ is normalized differently as compared to Ref. \[34\]. Actually, what matters in the mass formula is the product $c_i O_i$. Note also that $O_6$ is order $O(N_c^0)$ in the $[70, 1^-]$ multiplet \[34\] which is an important issue in the study of the compatibility between the quark-shell picture and meson-nucleon scattering picture \[35\]. The existing compatibility legitimates the extension of the $1/N_c$ expansion to excited baryons \[36\].

In the context of our approach, where the baryon is treated as a system of $N_c$ quarks irrespective of its spin-flavor symmetry, the SU(3) breaking operators are defined as

$$B_1 = n_s, \tag{7}$$

where $n_s$ is the number of strange quarks and

$$B_2 = \frac{1}{N_c} (L^i G^{is} - \frac{1}{2\sqrt{3}} L \cdot S), \tag{8}$$

$$B_3 = \frac{1}{N_c} (S^i G^{is} - \frac{1}{2\sqrt{3}} S \cdot S), \tag{9}$$

where the angular momentum operator $L^i$, the spin operator $S^i$ and the component 8 of the spin-flavor operator $G^{is}$ act on the entire system of $N_c$ quarks.
TABLE I. Matrix elements of $O_i$ for SU(3) octets and decuplets belonging to the [56, 2$^+$] multiplet.

|                | $O_1$ | $O_2$ | $O_3$ | $O_6$ |
|----------------|-------|-------|-------|-------|
| $^28[56, 2^+]$ | $N_c$ | $-\frac{3}{2N_c}$ | $\frac{3}{4N_c}$ | 0     |
| $^28[56, 2^+]$ | $N_c$ | $\frac{1}{N_c}$   | $\frac{3}{4N_c}$ | 0     |
| $^410[56, 2^+]$| $N_c$ | $-\frac{9}{2N_c}$ | $\frac{15}{4N_c}$ | $\frac{7}{2N_c}$ |
| $^410[56, 2^+]$| $N_c$ | $-\frac{3}{N_c}$  | $\frac{15}{4N_c}$ | 0     |
| $^410[56, 2^+]$| $N_c$ | $-\frac{1}{2N_c}$ | $\frac{15}{4N_c}$ | $-\frac{5}{2N_c}$ |
| $^410[56, 2^+]$| $N_c$ | $\frac{3}{N_c}$   | $\frac{15}{4N_c}$ | $\frac{1}{N_c}$  |

IV. MATRIX ELEMENTS

A. The multiplet [56, 2$^+$]

In Table I we reproduce the analytic forms obtained for the operators $O_i$ in terms of $N_c$ for the multiplet [56, 2$^+$]. The first is the trivial spin-flavor singlet operator $O_1$ defined by Eq. (1) with matrix elements equal to $N_c$ in all cases. For symmetric states the spin-orbit operator $O_2$ and the spin operator $O_3$ are of order $O(N_c^{-1})$. From Eq. (6) it follows that the matrix elements of $O_2$ are given by the usual formula

$$\langle O_2 \rangle = \frac{1}{2N_c}[J(J+1)-L(L+1)-S(S+1)], \quad (10)$$

where $L$ is the angular momentum of the whole system. The matrix elements of the spin operator $O_3$ are trivially equal to $\frac{1}{N_c}S(S+1)$. We are reminded that for symmetric states the matrix elements of the isospin operator $O_4$ are equal to those of the spin operator, thus are not included in Table I.

We also include the operator $O_6$, defined by Eq. (5). In the [56, 2$^+$] multiplet it contributes only to the decuplet resonances. The general analytic form of the matrix elements of $O_6$ were derived in Ref. [34] and for convenience the formula is reproduced in Appendix B. The calculations require knowledge of the matrix elements of the SU(6) generators for
TABLE II. Expectation values of SU(3) breaking operators for strange octets of the \([56, 2^+]\) multiplet. Here, \(a_J = 1, -2/3\) for \(J = 3/2, 5/2\) respectively, from Ref. \[17\].

| \(B_1\) | \(B_2\) | \(B_3\) |
|--------|--------|--------|
| \(2\Lambda_J\) | \(1\) | \(3\sqrt{3}a_J\) | \(-3\sqrt{3}\) |
| | | \(\frac{4N_c}{8N_c}\) | |
| \(2\Sigma_J\) | \(1\) | \(-\sqrt{3}a_J\) | \(\sqrt{3}\) |
| | | \(\frac{4N_c}{8N_c}\) | |
| \(2\Xi_J\) | \(2\) | \(\sqrt{3}a_J\) | \(-\sqrt{3}\) |
| | | \(\frac{N_c}{2N_c}\) | |

spin-flavor symmetric states, namely the isoscalar factors \([N_c] \times [21^4] \rightarrow [N_c]\), derived in Table I of Ref. \[37\].

Although \(G^a\) acts sometimes as a coherent operator, introducing an extra power of \(N_c\) in mixed symmetric states, this is not the case for spin-symmetric states, so that the matrix elements of \(O_6\) are of order \(1/N_c\), as one can see from Table II

In the mass formula \(\Pi\) we have included three first-order SU(3) symmetry breaking operators \(B_1, B_2\) and \(B_3\) (denoted by \(\bar{B}_1, \bar{B}_2\) and \(\bar{B}_3\) in Ref. \[17\]). Their nonvanishing matrix elements, were calculated in Ref. \[17\] where the matrix element of the first term of \(B_3\) of Eq. \(\Psi\) was obtained from the formula

\[
\langle S'G^{is} \rangle = \frac{1}{4\sqrt{3}}[3I(I + 1) - S(S + 1) - \frac{3}{4}n_s(n_s + 2)],
\]

later proven for symmetric states in Ref. \[37\] where \(n_s\) is the number of strange quarks.

For octets the nonvanishing expectation values of \(B_i\) are reproduced in Table II which shows that the effect of \(B_2\) depends on \(J\). At \(J = 3/2\) it increases the mass of \(\Lambda\) and lowers the mass of \(\Sigma\) while for \(J = 5/2\) it does the other way round. \(B_3\) has the role of lowering the mass of \(\Lambda\) while increasing the mass of \(\Sigma\), irrespective of \(J\). In all \(B_2\) and \(B_3\) remove the degeneracy due to \(B_1 = 1\).

For the \(4^{10}\) strange members of the \([56, 2^+]\) multiplet the expectation values found in Ref. \[17\] for \(B_2\) and \(B_3\) can be written in a compact form, as one can see in Appendix D. These are

\[
B_2 = -\frac{n_s}{2\sqrt{3}N_c}\langle L \cdot S\rangle,
\]

and

\[
B_3 = -\frac{n_s}{2\sqrt{3}N_c}\langle S \cdot S\rangle.
\]
Eqs. (12) and (13) give equal space splitting between the decuplet members at fixed $J$. Note that the operator $B_2$ can raise or lower the mass depending on the sign of $\langle L \cdot S \rangle$. From Eqs. (12) and (13) and the definitions of $O_2$ and $O_3$ it follows that the expectation values of $O_2$, $O_3$, $B_2$ and $B_3$ satisfy the relation

\[
\frac{B_2}{B_3} = \frac{O_2}{O_3},
\]

which holds in fact for both the octet and the decuplet of the $[56, 2^+]$ multiplet [18].

B. The multiplet $[70, \ell^+]$

In Table III we reproduce the analytic forms of the matrix elements of the operators $O_i$ included in the mass formula for the multiplet $[70, \ell^+]$. They are successively listed for all possible octets, decuplets and SU(3)-flavor singlets of this multiplet. Details of the derivation of these analytic forms as a function of $N_c$ can be found in Ref. [28]. Accordingly, the spin-orbit operator $O_2$ is the single-particle operator

\[
O_2 = \ell \cdot s = \sum_{i=1}^{N_c} \ell(i) \cdot s(i),
\]

the matrix elements of which are of order $N_c^0$ and are calculated in Appendix A.

Note that in the case of mixed symmetric states the matrix elements of the operator $O_6$ are $O(N_c^0)$, in contrast to the symmetric case where they are $O(N_c^{-1})$, and nonvanishing only for octets, while for the symmetric case they are nonvanishing for decuplets. Thus, at large $N_c$ the splitting starts at order $O(N_c^0)$ for mixed symmetric states due both to $O_2$ and $O_6$, while in the symmetric core + excited quark procedure [19] several operators are order $O(N_c^0)$ [38].

The SU(3) flavor breaking operators $B_i$ are the same as for the symmetric multiplet. The operator $B_1$ is given by Eq. (7). The matrix elements of $B_2$ and $B_3$ were calculated as explained in Appendix D. Results for octets and decuplets at some fixed $J$ are exhibited in Table IV.

For practical purposes we have summarized these results by two simple analytic formulas valid at $N_c = 3$. The diagonal matrix elements of $B_2$ take the following form

\[
B_2 = -n_s \frac{\langle L \cdot S \rangle}{6\sqrt{3}},
\]

which holds in fact for both the octet and the decuplet of the $[56, 2^+]$ multiplet [18].
TABLE III. Matrix elements of $O_i$ operators for SU(3)-flavor octet, decuplet and singlet states in $[70, \ell^+]$ multiplet.

|       | $O_1$  | $O_2$  | $O_3$  | $O_4$  | $O_6$  |
|-------|--------|--------|--------|--------|--------|
| $^4\!8[70, 2^+]^{7^+}/2$ | $N_c$  | 1      | $15/4N_c$ | $3/4N_c$ | $-N_c-1/4N_c$ |
| $^2\!8[70, 2^+]^{5^+}/2$ | $N_c$  | $(2N_c-3)/3N_c$ | 3      | $3/4N_c$ | 0 |
| $^4\!8[70, 2^+]^{5^+}/2$ | $N_c$  | $1/6$  | 15      | $3/4N_c$ | $5(N_c-1)/8N_c$ |
| $^4\!8[70, 0^+]^{3^+}/2$ | $N_c$  | 0      | 15      | 3      | 0 |
| $^2\!8[70, 2^+]^{3^+}/2$ | $N_c$  | $-2N_c-3/2N_c$ | 3      | $3/4N_c$ | 0 |
| $^4\!8[70, 2^+]^{3^+}/2$ | $N_c$  | -1     | 15      | $3/4N_c$ | 0 |
| $^2\!8[70, 0^+]^{1^+}/2$ | $N_c$  | 0      | 3       | 3      | 0 |
| $^4\!8[70, 2^+]^{1^+}/2$ | $N_c$  | $3/2$  | 15      | 3      | $7(N_c-1)/8N_c$ |
| $^2\!10[70, 2^+]^{5^+}/2$ | $N_c$  | $-1/3$ | 3       | $15/4N_c$ | 0 |
| $^2\!10[70, 2^+]^{3^+}/2$ | $N_c$  | 1/2    | 3       | $15/4N_c$ | 0 |
| $^2\!10[70, 0^+]^{1^+}/2$ | $N_c$  | 0      | 3       | $15/4N_c$ | 0 |
| $^2\!1[70, 2^+]^{5^+}/2$ | $N_c$  | 1      | 3       | $-2N_c+3/4N_c$ | 0 |
| $^2\!1[70, 2^+]^{3^+}/2$ | $N_c$  | $-3/2$ | 3       | $2N_c+3/4N_c$ | 0 |
| $^2\!1[70, 0^+]^{1^+}/2$ | $N_c$  | 0      | 3       | $2N_c+3/4N_c$ | 0 |

where $\langle L \cdot S \rangle$ is the expectation value of the spin-orbit operator with the angular momentum operator acting on the whole system. Similarly the diagonal matrix elements of $B_3$ take the simple analytic form

$$B_3 = -n_s S(S + 1)/6\sqrt{3}, \quad (17)$$

where $S$ is the total spin. The contribution of $B_3$ is always negative, otherwise vanishing for nonstrange baryons. These formulas can be applied to $^2\!8\ell^+$, $^4\!8\ell^+$, $^2\!10\ell^+$ and $^2\!11/2\ell^+$ baryons of the $[70, \ell^+]$ multiplet. From Eqs. (16) and (17) it follows that Eq. (14) is satisfied for the $[70, 2^+]$ multiplet as well.
TABLE IV. Matrix elements of the SU(3) breaking operators $B_i$ for strange baryons for the $^48_7/2$, $^48_5/2$, $^28_3/2$, $^210_5/2$, $^210_1/2$ sectors and the singlet $^2\Lambda_{1/2}$ of the $[70, \ell^+)$ multiplet.

| Sector | $B_1$ | $B_2$ | $B_3$ |
|--------|-------|-------|-------|
| $^4\Lambda_{7/2}$ | 1 | $-\frac{\sqrt{3}}{2N_c}$ | $\frac{5\sqrt{3}}{8N_c}$ |
| $^4\Sigma_{7/2}$ | 1 | $\frac{\sqrt{3}}{N_c} - 9$ | $\frac{5\sqrt{3}}{N_c} - 9$ |
| $^4\Xi_{7/2}$ | 2 | $-\frac{2\sqrt{3}}{3} \frac{1}{N_c - 1}$ | $-\frac{5\sqrt{3}}{6} \frac{1}{N_c - 1}$ |
| $^4\Lambda_{5/2}$ | 1 | $\frac{\sqrt{3}}{12N_c}$ | $\frac{5\sqrt{3}}{8N_c}$ |
| $^4\Sigma_{5/2}$ | 1 | $-\frac{\sqrt{3}}{36N_c} \frac{N_c - 9}{N_c - 1}$ | $\frac{5\sqrt{3}}{24N_c} \frac{N_c - 9}{N_c - 1}$ |
| $^4\Xi_{5/2}$ | 2 | $\frac{\sqrt{3}}{9} \frac{1}{N_c - 1}$ | $-\frac{5\sqrt{3}}{6} \frac{1}{N_c - 1}$ |
| $^2\Lambda_{3/2}$ | 1 | $-\frac{\sqrt{3}}{4N_c} \frac{N_c - 9}{N_c + 3}$ | $\frac{\sqrt{3}}{8N_c} \frac{N_c - 9}{N_c + 3}$ |
| $^2\Sigma_{3/2}$ | 1 | $\frac{\sqrt{3}}{12N_c} \frac{N_c + 3}{N_c - 1}$ | $-\frac{\sqrt{3}}{24N_c} \frac{N_c + 3}{N_c - 1}$ |
| $^2\Xi_{3/2}$ | 2 | $-\frac{\sqrt{3}}{3N_c} \frac{N_c^2 - 12N_c + 9}{(N_c - 1)(N_c + 3)}$ | $\frac{\sqrt{3}}{6N_c} \frac{N_c^2 - 12N_c + 9}{(N_c - 1)(N_c + 3)}$ |
| $^2\Sigma^*_{5/2}$ | 1 | $-\frac{\sqrt{3}}{18N_c} \frac{5N_c + 9}{N_c + 5}$ | $-\frac{\sqrt{3}}{24N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Xi^*_{5/2}$ | 2 | $\frac{\sqrt{3}}{9N_c} \frac{5N_c + 9}{N_c + 5}$ | $-\frac{\sqrt{3}}{12N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Omega_{5/2}$ | 3 | $-\frac{\sqrt{3}}{6N_c} \frac{5N_c + 9}{N_c + 5}$ | $\frac{\sqrt{3}}{8N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Sigma^*_{1/2}$ | 1 | 0 | $-\frac{\sqrt{3}}{24N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Xi^*_{1/2}$ | 2 | 0 | $-\frac{\sqrt{3}}{12N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Omega_{1/2}$ | 3 | 0 | $-\frac{\sqrt{3}}{8N_c} \frac{5N_c + 9}{N_c + 5}$ |
| $^2\Lambda'_{1/2}$ | 1 | 0 | $-\frac{3\sqrt{3}}{8N_c} \frac{N_c - 1}{N_c + 3}$ |
V. FIT AND DISCUSSION

Presently we perform a consistent analysis of the experimentally known resonances supposed to belong either to the symmetric [56, 2+] multiplet or to the mixed symmetric multiplet [70, ℓ+] with ℓ = 0 or 2, by using the same operator basis. Results of the fitted coefficients $c_i$ and $d_i$ are exhibited in Table V together with the values of $\chi^2_{\text{dof}}$ for each multiplet.

For the [56, 2+] multiplet the values of the coefficients $c_i$ and $d_i$ are quite close to those of Ref. [17]. The spin-orbit coefficient $c_2$ is about twice as small in the present case but one should take into account the contribution of the operator $O_6$, also depending on the angular momentum, which is absent in Ref. [17]. The PDG2014 data give a somewhat larger $\chi^2_{\text{dof}}$ as compared to the PDG2002 data used in Ref. [17] where $\chi^2_{\text{dof}}$ was reported to be 0.7. But this does not much affect the coefficients $c_i$ or $d_i$. In the fit for the [56, 2+] multiplet we have ignored the poorly known resonance $\Sigma(1840)3/2^+$ by analogy with Ref. [17].

The results for the [70, ℓ+] multiplet can only roughly be compared to those Table I, Fit 2 of Ref. [28], because $B_2$ and $B_3$ were missing there. Note that the factor 15 of $O_6$ has been removed here, which explains the larger value of $c_6$ now. In fact the product $c_6O_6$ matters in the mass. The values of $c_2$ are similar to Ref. [28]. The $1/N_c$ corrections are dominated by $O_3$ and $O_4$. The sum of $c_3$ and $c_4$ is comparable to that of $c_3$ in [56, 2+] where $O_3$ and $O_4$ contribute equally.

An important remark is that the values of the most dominant coefficient $c_1$ are different for the two multiplets. The $c_1$ of [70, ℓ+] is higher by about 85 MeV. This implies that two distinct Regge trajectories are expected for the symmetric and mixed symmetric multiplets as already hinted in Ref. [28].

As mentioned in the Introduction, in quark models $c_1^2$ would correspond to the kinetic plus the confinement parts of the spin-independent Hamiltonian [23, 29]. In a semirelativistic model it happens that the multiplet [70, ℓ+] lies above [56, ℓ+] when the hyperfine interaction is switched off. For example, in Ref. [39] it was explicitly shown that the [70, 4+] multiplet is about 50 MeV higher than the [56, 4+] multiplet. Therefore the present results for $c_1$ hint at a qualitative agreement with the quark model.
TABLE V. List of dominant operators and their coefficients (MeV) from the mass formula (1) obtained in numerical fits for the \([56, 2^+]\) in column 2 and \([70, \ell^+]\) multiplets in columns 3 and 4 respectively. The spin-orbit operator \(O_2\) is defined by Eq. (6) for \([56, 2^+]\) and by Eq.(15) for \([70, \ell^+]\).

| Operator \(O_i\) | \([56, 2^+]\) | \([70, \ell^+]\) |
|------------------|--------------|-----------------|
| \(O_1 = N_c \mathbb{1}\) | 542 ± 2 | 627 ± 10 |
| \(O_2\) spin-orbit \(\text{Eq.}(10)\) | 7 ± 10 | \(\text{Eq.}(15)\) 69 ± 26 |
| \(O_3 = \frac{1}{N_c} S^i S^i\) | 233 ± 11 | 88 ± 31 |
| \(O_4 = \frac{1}{N_c} \left[ T^a T^a - \frac{1}{12} N_c (N_c + 6) \right]\) | | 127 ± 21 |
| \(O_6 = \frac{1}{N_c} L^{(2)ij} G^{i\alpha} G^{j\alpha}\) | 6 ± 19 | 72 ± 71 |
| \(B_1 = n_s\) | 205 ± 14 | 76 ± 31 |
| \(B_2 = \frac{1}{N_c} (L^i G^{i\alpha} - \frac{1}{2\sqrt{3}} L_i^i S^\alpha)\) | 97 ± 40 | -172 ± 106 |
| \(B_3 = \frac{1}{N_c} (S^i G^{i\alpha} - \frac{1}{2\sqrt{3}} S^i S^\alpha)\) | 197 ± 69 | 279 ± 144 |
| \(\chi^2_{\text{dof}}\) | 1.63 | 1.48 |

A. The multiplet \([56, 2^+]\)

The partial contribution and the calculated total mass obtained from the fit are presented in Table VII. The experimental masses are taken from the 2014 version of the Review of Particle Properties (PDG) [24], except for \(\Delta(1905)5/2^+\) where we used the mass of Ref. [17] which gives a smaller \(\chi^2_{\text{dof}}\), but does not much change the fitted values of \(c_i\) and \(d_i\). As expected, the most important subleading contribution comes from the spin operator \(O_3\). The contributions of the angular momentum-dependent operators \(O_2\) and \(O_6\) are comparable, but small. Among the SU(3) breaking terms, \(B_1\) is dominant. An important remark is that in the \([56, 2^+]\) multiplet \(B_2\) and \(B_3\) lift the degeneracy of \(\Lambda\) and \(\Sigma\) baryons in the octets, which is not the case for the \([70, \ell^+]\) multiplet.
TABLE VI. Partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion with operators of Tables I and II. The last two columns give the empirically known masses and the 2014 status in the Review of Particles Properties [24].

| Partial contrib. (MeV) | Total (MeV) | Experiment (MeV) | Name, Status |
|-----------------------|-------------|------------------|--------------|
| $^2\Lambda[56, 2^+]/2$ | 1626 - 4 | 58 0 0 0 0 | 1680 ± 9 | $N(1720)\frac{3^+}{2}$ **** |
| $^2\Lambda[56, 2^+]/2$ | 1626 - 2 | 58 0 0 0 0 | 1685 ± 5 | $N(1680)\frac{5^+}{2}$ **** |
| $^2\Sigma[56, 2^+]/2$ | 205 - 28 | -42 0 0 0 0 | 1821 ± 5 | $\Lambda(1820)\frac{5^+}{2}$ **** |
| $^2\Xi[56, 2^+]/2$ | 205 9 | 14 0 0 0 0 | 1915 ± 17 | $\Sigma(1915)\frac{3^+}{2}$ **** |
| $^2\Omega[56, 2^+]/2$ | 410 -37 | -57 0 0 0 0 | 2002 ± 13 | |
| $^4\Sigma[56, 2^+]$ | 1626 - 7 | 291 0 0 0 0 | 1914 ± 27 | $\Delta(1910)\frac{1^+}{2}$ **** |
| $^4\Sigma[56, 2^+]$ | 291 0 | 0 0 0 0 0 | 1935 ± 35 | $\Delta(1920)\frac{1^+}{2}$ **** |
| $^4\Xi[56, 2^+]$ | 205 -28 | 14 0 0 0 0 | 2032 ± 8 | $\Sigma(2030)\frac{1^+}{2}$ **** |
| $^4\Omega[56, 2^+]$ | 615 14 | -213 0 0 0 0 | 2238 ± 58 | |
| $^4\Omega[56, 2^+]$ | 205 -56 | -142 0 0 0 0 | 2188 ± 44 | |
| $^4\Omega[56, 2^+]$ | 615 -84 | -213 0 0 0 0 | 2326 ± 63 | |
| $^4\Xi[56, 2^+]$ | 1626 - 1 | 291 -5 | 0 0 0 0 | 1926 ± 12 | $\Delta(1950)\frac{1^+}{2}$ **** |
| $^4\Omega[56, 2^+]$ | 205 -28 | -71 0 0 0 0 | 2032 ± 8 | $\Sigma(2030)\frac{1^+}{2}$ **** |
| $^4\Omega[56, 2^+]$ | 410 -56 | -142 0 0 0 0 | 2138 ± 15 | |
| $^4\Omega[56, 2^+]$ | 615 -84 | -213 0 0 0 0 | 2244 ± 26 | |

B. The multiplet $[70, \ell^+]$

The 2014 version of the Review of Particle Properties (PDG) [24] incorporates the new multichannel partial wave analysis of the Bonn-Gatchina group [33]. According to the Bonn-Gatchina group the resonance $P_{13}(1900)$ has been upgraded from two to three stars with a
Breit-Wigner mass of $1905 \pm 30$ MeV. The resonance $N(2000)5/2^+$ has been split into two two-star resonances, namely $N(1860)5/2^+$ and $N(2000)5/2^+$, with masses indicated in Table VII. There is a new one-star resonance $N(2040)3/2^+$ observed in the decay $J/\psi \to p\bar{p}\pi^0$ [40]. There is also a new two-star resonance $N(1880)1/2^+$ observed by the Bonn-Gatchina group with a mass of $1870 \pm 35$ MeV [33].

As compared to Ref. [28] where only 11 resonances have been included in the numerical fit, here we consider 16 resonances, having a status of three, two or one star. This means that we have tentatively added the resonances $\Xi(2120)?^*$, $\Sigma(2070)5/2^{++}$, $\Sigma(1940)?^*$, $\Xi(1950)?^{+++}$, and $\Sigma(2080)3/2^{++}$. The masses and the error bars considered in the fit correspond to averages over those data taken into account in the particle listings, except for a few which favor specific experimental values cited in the headings of Table VII. For example the value of the mass of the $N(1880)1/2^{++}$ resonance is taken identical to that of Ref. [33], the other data mentioned in the listings being ignored. For $\Delta(2000)5/2^{+++}$ and $\Sigma(1880)1/2^{++}$ we averaged over two and eight experimental values, respectively, indicated in the 2014 version of PDG.

We have ignored the $N(1710)1/2^{+++}$ and the $\Sigma(1770)1/2^{++}$ resonances, the theoretical argument being that their masses are too low, leading to unnatural sizes for the coefficients $c_i$ or $d_i$ [41]. On the experimental side one can justify the removal of the $N(1710)1/2^{+++}$ resonance due to the latest GWU analysis of Arndt et al. [42] where it has not been seen. This is a controversial resonance. We have also ignored the $\Delta(1750)1/2^{++}$ resonance, considered previously [20], inasmuch as, neither Arndt et al. [42] nor Anisovich et al. [33] find evidence for it.

The partial contributions and the calculated total masses obtained from the fit are presented in Table VII. One can see that the fit is good except for $\Sigma(1880)1/2^{++}$ where the calculated mass is too high, perhaps suggesting that the average over the eight resonances indicated in the particle listings is not quite adequate.

Regarding the contribution of various operators we note that the good fit for $N(1880)1/2^{+++}$ was due to contribution of the spin-orbit operator $O_2$ of $-103$ MeV and of the operator $O_6$ which contributed with $-42$ MeV. The good fit also suggests that $\Sigma(1940)?^*$ and $\Xi(1950)?^{+++}$ assigned by us to the $^2[70, 2^+]3/2^+$ multiplet is well justified and that these resonances may have $J^P = 3/2^+$, hopefully to be confirmed experimentally in future analyses.
TABLE VII. Partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion with operators of Tables III and IV. The last two columns give the empirically known masses and status from the 2014 Review of Particles Properties [24] unless specified by (A) from [33], (L) from [43], (Z) from [44], (G1) from [45], (B) from [46], (AB) from [40], (G2) from [47].

| Part. contrib. (MeV) | Total (MeV) | Experiment (MeV) | Name, status |
|----------------------|-------------|------------------|--------------|
| $4N[70, 2^+1/2^+]$  | 1882        | 2105 ± 19        | $N(1990)7/2^+$ *** |
| $4\Lambda[70, 2^+1/2^+]$ | 76          | 2100 ± 30(L)     | $\Lambda(2020)7/2^+$ * |
| $4\Xi[70, 2^+1/2^+]$ | 152         | 2130 ± 8         | $\Xi(2120)7^+$ * |
| $4N[70, 2^+5/2^+]$  | 1882 - 12  | 2042 ± 41        | $N(2000)5/2^+$ *** |
| $4\Lambda[70, 2^+5/2^+]$ | 76         | 2009 ± 40        | $\Lambda(2110)5/2^+$ *** |
| $4\Sigma[70, 2^+1/2^+]$ | 1882 - 103 | 1955 ± 32        | $\Sigma(1940)?$ |
| $2N[70, 2^+5/2^+]$  | 1882        | 1902 ± 22        | $N(1900)3/2^+$ *** |
| $2\Sigma[70, 2^+3/2^+]$ | 0           | 1933 ± 11        | $\Sigma(1940)7^+$ * |
| $2\Xi[70, 2^+3/2^+]$ | 0           | 1964 ± 70        | $\Xi(1950)7^+$ *** |
| $4N[70, 0^+3/2^+]$  | 1882        | 2024 ± 20        | $N(2040)3/2^+$ *** |
| $4\Sigma[70, 0^+3/2^+]$ | 76          | 2100 ± 23        | $\Sigma(2080)3/2^+$ *** |
| $2\Delta[70, 2^+5/2^+]$ | 1882         | 2086 ± 37        | $\Delta(2000)5/2^+$ *** |
| $2\Sigma^*[70, 0^+3/2^+]$ | 1882         | 2119 ± 25        | $\Sigma(1880)1/2^+$ *** |
| $2\Lambda^*[70, 0^+1/2^+]$ | 1882         | 1865 ± 19        | $\Lambda(1810)1/2^+$ *** |
The $1/N_c$ expansion is based on the SU(6) symmetry which naturally allows a classification of excited baryons into octets, decuplets and singlets. In Table VII the experimentally known resonances are presented. In addition some predictions are made for unknown resonances. Many of the partners in a given supermultiplet are not known. Note that $\Lambda$ and $\Sigma$ are degenerate in our approach.

The present findings can be compared to the suggestions for assignments in the $[70, \ell^+]$ multiplet made in Ref. [25] as educated guesses. The assignment of $\Sigma(1880) 1/2^{++*}$ as a $[70, 0^+] 1/2^+$ decuplet resonance is confirmed as well as the assignment of $\Lambda(1810) 1/2^{++*}$ as a flavor singlet.

However, we are at variance with Ref. [25] regarding $\Lambda(2110) 5/2^{++*}$ as a partner of $N(2000) 5/2^{++*}$ in a spin quartet. Our suggestion is that $\Lambda(2110) 5/2^{++*}$ is a member of a spin doublet, together with $N(1860) 5/2^{++}$ and $\Sigma(2070) 5/2^*$. We also disagree that $N(1900) 3/2^{++*}$ is a member of a spin quartet. We propose it as a partner of $\Sigma(1940) ?^{*}$ and $\Xi(1950) ?^{*}$ in a spin doublet.

However, one has to keep in mind that at the same $J$ spin doublets and quartets can mix, for example for $N[70, 2^+]$ at $J = 3/2$ or $5/2$. The mixing would be due to the off-diagonal matrix elements of the spin-orbit operator $O_2$ and the tensor operator $O_6$. A qualitative simplified discussion is given in Appendix C. We plan further studies on this subject in the future.

The problem of assignment is not trivial. Within the $1/N_c$ expansion method Ref. [17] suggests that $\Sigma(2080) 3/2^{++*}$ and $\Sigma(2070) 5/2^{++*}$ could be members of two distinct decuplets in the $[56, 2^+]$ multiplet. It would be interesting to further investigate more hyperons hopefully based on more extended and reliable data.

Note that the resonance $N(2040) 3/2^{+}$ is here identified as a member of a spin quartet in the $[70, 0^+]$ multiplet while Crede and Roberts [25] interpret it as a member of the SU(6) antisymmetric $[20, 1^+]$ multiplet, ignored so far, as not being physically possible.

Finally, we mention that although the operators $B_2$ and $B_3$ have different analytic forms at arbitrary $N_c$, as seen from Table IV they acquire identical values at $N_c = 3$ for $\Lambda$ and $\Sigma$ in octets, which means that they cannot lift the degeneracy between these hyperons, as happens for the $[56, 2^+]$ multiplet. One can lift this degeneracy by introducing a new
operator

\[ B_4 = \frac{1}{N_c} \sum_i T^i T^i - O_4, \]  

(18)
as proposed in Ref. [34]. Presently this is not necessary inasmuch as the experimental data are too scarce and not accurate enough. In addition, in some multiplets the hierarchy of masses as a function of the strangeness is contrary to expectations, for example for the multiplet \( ^4[70,2^+]^5/2^+ \). This requires further investigation.

VI. SUMMARY AND CONCLUSIONS

The value obtained for the coefficient \( c_1 \) is about 85 MeV larger in the \( [70, \ell^+] \) multiplet than in the \( [56, 2^+] \) multiplet. This implies that two distinct Regge-type trajectories are expected for the symmetric and mixed symmetric multiplets, consistent with previous literature [28, 48]. The spin-orbit coefficient for the \( [70, \ell^+] \) multiplet, is similar to our previous work [23]. The spin and flavor operators are two-body and bring important contributions to the masses. As one can see from Table III the expectation values of \( O_4 \) are positive for octets and decuplets and of order \( N_c^{-1} \), as in SU(4), and negative and of order \( N_c^0 \) for flavor singlets, which makes its role rather subtle in the numerical fit, improving the singlet masses. The contribution of the operator \( O_6 \) containing an SO(3) tensor is important especially for \( [70, \ell^+] \) multiplet. Together with the spin-orbit it may lead to the mixing of doublets and quartets to be considered in further studies when the accuracy of data will increase. The incorporation of \( B_2 \) and \( B_3 \) in the mass formula of the \( [70, \ell^+] \) multiplet brings more insight into the SU(6) multiplet classification of excited baryons in the \( N = 2 \) band.

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Appendix A: Matrix elements of \( O_2 \) for the \( [70, \ell^+] \) multiplets

The expression of the matrix elements of the single-particle spin-orbit operator of Eq. (15) was first given in Ref. [19]. In Ref. [36] it was reproduced in a more concise form, written
below in a slightly different form. Like for the operator $O_6$ given in the next appendix, every matrix element can be factorized into a part dependent on $\ell, S, S', J$ and another one, independent of $\ell$ and $J$, but dependent on $N_c$. One has

$$\langle (\lambda' \mu') Y'I'I'_3; \ell' S' J J_3 | \ell \cdot s | (\lambda \mu) Y I I_3; \ell S J J_3 \rangle =$$

$$\delta_{\ell' \ell} \delta_{\lambda' \lambda} \delta_{\mu' \mu} \delta_{Y' Y} \delta_{I'I} \delta_{J J_3} C_{so}(\ell, S, S', J) F_{so}(N_c, S, S'), \tag{A1}$$

where

$$C_{so}(\ell, S, S', J) = (-1)^{J+\ell-I} \frac{3}{2} (2S + 1)(2S' + 1) \ell(\ell + 1)(2\ell + 1) \left\{ \begin{array}{c} \ell \\ S \\ S' \\ J \end{array} \right\}. \tag{A2}$$

and

$$F_{so}(N_c, S, S') = \sum_{\eta=\pm 1} (-1)^{(1/2-\eta/2)} \left\{ \begin{array}{ccc} 1/2 & 1/2 \\ S_c & S & S' \end{array} \right\} C_{\rho \eta} C_{\rho \eta}. \tag{A3}$$

where $\rho = S - I$, $\eta = 1$ for $I_c = I + 1/2$, $\eta = -1$ for $I_c = I - 1/2$ and

$$C_{0+} = \sqrt{\frac{S[N_c + 2(S + 1)]}{N_c(2S + 1)}}, \tag{A4}$$

$$C_{0-} = -\sqrt{\frac{(S + 1)(N_c - 2S)}{N_c(2S + 1)}},$$

for the $S = I$ nonstrange states.

Note that the matrix elements obtained from Eq. (A1) have an extra factor $3/2$ as compared to those derived in Ref. [20] for mixed symmetric multiplets. The reason is that in Ref. [20] we have used an approximate orbital wave function for simplicity. The second term of Eq. (5) of that reference was neglected as being of order $N_c^{-1/2}$. Equation (A1) contains the contribution of that part of the wave function. Also note that Eq. (A1) has been independently derived in Ref. [49] where the coefficients $c_{\rho \eta}$ were identified with isoscalar factors of the permutation group $S_{N_c}$.

**Appendix B: Expectation values of $O_6$**

The operator $O_6$, defined by Eq. (5), is proportional to $L^{(2)} \cdot G \cdot G$, where the SO(3) rank-two tensor $L^{(2)}$ is defined by Eq. (3). For a given $\ell$, its matrix elements can be rewritten in
the following factorized form

\[ \langle (\lambda \mu') Y' I' I_3; \ell S' J J_3 | (-1)^{i+j+a} L^{(2)ij} G^{-i-a} | (\lambda \mu) Y I I_3; \ell S J J_3 \rangle = \delta_{\ell \ell'} \delta_{\lambda \lambda'} \delta_{\mu \mu'} \delta_{Y' Y} \delta_{I' I} C(\ell, S, S', J) F(N_c, S, S'), \]  

(B1)

which multiplied by the factor \( \frac{1}{N_c} \) gives the matrix elements of \( O_6 \). The formula (B1) contains a factor independent of \( N_c \) which we have denoted by \( C(\ell, S, S', J) \). This is

\[ C(\ell, S, S', J) = \delta_{SS'} (-1)^{J+\ell-S} \times \frac{1}{2} \sqrt{\frac{5\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)}{6}} \sqrt{(2S+1)(2S'+1)} \left\{ \begin{array}{ccc} \ell & \ell & 2 \\ S & S' & J \end{array} \right\}. \]  

(B2)

This factor can be used to calculate matrix elements of other symmetric multiplets than \([56, 2^+]\) by using the property

\[ \frac{\langle O_6 | [56, \ell^+] \rangle}{\langle O_6 | [56, 2^+] \rangle} = \frac{C(\ell, S, S', J)}{C(2, S, S', J)}. \]  

(B3)

The other factor of Eq. (B1), let us denote by it \( F(N_c, S, S') \), is independent of \( \ell \) but dependent on \( N_c \) and of the representation \([f]\) of SU(6), containing the Casimir operator \( C_{SU(6)}^{N_c} \). This factor is

\[ F(N_c, S, S') = C_{[f]}^{SU(6)} \sum_{S''} (-1)^{(S-S'')} \left\{ \begin{array}{ccc} 1 & 1 & 2 \\ S & S' & S'' \end{array} \right\} \times \sum_{\rho, \lambda'', \mu''} \left( [f] \begin{array}{cc} [21^4] \\ (\lambda'' \mu'') S'' \end{array} \right) \left( [f] \begin{array}{cc} [21^4] \\ (\lambda \mu) S \end{array} \right) \left( [f] \begin{array}{cc} [f] \\ (\lambda'' \mu'') S'' \end{array} \right) \left( [f] \begin{array}{cc} [f] \\ (\lambda \mu) S' \end{array} \right). \]  

(B4)

In the present case, we deal with symmetric states of \( N_c \) quarks which means that we have to take \([f] = [N_c]\). The Casimir operator is \( C_{SU(6)}^{N_c} = \frac{5}{12} N_c (N_c + 6) \). The sum over \( \rho, \lambda'' \) and \( \mu'' \) contains isoscalar factors derived in Ref. [37] and presented in Table I of that reference.

For symmetric states \([f] = [N_c]\) it happens that the sum over \( S'' \) is such that \( F(N_c, S, S') \) becomes \( O(N_c^0) \), and the order of \( O_6 \) is given by the factor \( \frac{1}{N_c} \). In other words it means that the three-flavor case is more subtle than the two-flavor case, as already pointed out in Ref. [50], because \( G^{ia} \) does not have the same \( N_c \) dependence in the flavor weight diagram.
Appendix C: Mixing of quartets and doublets

Mixing between doublets and quartets with the same $J$ is possible due to off-diagonal matrix elements of $O_2$ and $O_6$. Introducing a mixing angle $\theta_J$ one can define

$$
|N_J(\text{upper})\rangle = \cos \theta_J |^4N_J\rangle + \sin \theta_J |^2N_J\rangle,
|N_J(\text{lower})\rangle = -\sin \theta_J |^4N_J\rangle + \cos \theta_J |^2N_J\rangle,
$$

where $|^4N_J\rangle$ and $|^2N_J\rangle$ are the theoretical states used in the fit and upper and lower are the physical states with upper and lower energies. One can give an estimate of the mixing angle $\theta_J$ to order $O(N_c^0)$ which can be obtained from a simplified mass formula including $O_1$, $O_2$ and $O_6$, the first being order $O(N_c)$, the two others order $O(N_c^0)$. In Table VIII we exhibit the off-diagonal matrix elements of $O_2$ and $O_6$ as a function of $N_c$. Like the diagonal matrix elements, they are also order $O(N_c^0)$.

TABLE VIII. Off-diagonal matrix elements of $O_2$ and $O_6$ for all states belonging to the $[70, 2^+]$ multiplet.

|       | $O_2$ | $O_6$ |
|-------|-------|-------|
| $^2\!^8S_7 ^2\!^8P_7$ | $-\sqrt{\frac{N_c+3}{4N_c}}$ | $-\frac{7}{8}\sqrt{\frac{N_c+3}{N_c}}$ |
| $^2\!^8S_7 ^2\!^8P_5$ | $\frac{1}{3}\sqrt{\frac{7(N_c+3)}{2N_c}}$ | $\frac{7}{32}\sqrt{\frac{N_c+3}{N_c}}$ |

Then we have to take $N_c \to \infty$ in all matrix elements of $O_2$ and $O_6$. As an example we show here the matrix of the $N_{5/2}$ states

$$
M_{N_{5/2}}^\ell = \begin{pmatrix}
    c_1N_c + \frac{2}{3}c_2 & \frac{\sqrt{7}}{2} \left( -\frac{1}{3}c_2 + \frac{1}{4}c_6 \right) \\
    \frac{\sqrt{7}}{2} \left( -\frac{1}{3}c_2 + \frac{1}{4}c_6 \right) & c_1N_c - \frac{1}{6}c_2 + \frac{5}{8}c_6
\end{pmatrix},
$$

(C2)

This suggests that the general form of a $2 \times 2$ matrix to be diagonalized is

$$
M_{N_J}^\ell = \begin{pmatrix}
    A & B \\
    B & C
\end{pmatrix},
$$

(C3)
so the mixing angle turns out to be

$$\tan 2\theta = -\frac{2B}{C - A}. \quad (C4)$$

Replacing $A, B$ and $C$ by their values from Eq. (C2), one obtains

$$\tan 2\theta_{5/2} = -\frac{\sqrt{7/2}(-\frac{2}{3}c_2 + \frac{1}{2}c_6)}{-\frac{5}{6}c_2 + \frac{5}{8}c_6} \quad (C5)$$

Using the coefficients $c_2$ and $c_6$ from Table IV one obtains $\tan 2\theta_{5/2} \approx -1.49$, which gives $\theta_{5/2} \approx -28$ degrees. This is quite a large mixing angle. In the real case, one has to introduce corrections of order $1/N_c$. However the mixing angles are completely unknown experimentally for the $[70, 2+]$, contrary to the $[70, 1^-]$ multiplet (for the most recent analysis see Ref. [51]). So, a comparison between theory and experiment is not yet possible.

**Appendix D: Breaking operators**

Here we present some details of the calculation of the diagonal matrix elements of the SU(3) breaking operators $B_2$ and $B_3$ presented in Table IV. In the context of our approach, where the baryon is treated as a system of $N_c$ quarks irrespective of its spin-flavor symmetry, they are defined as

$$B_2 = \frac{1}{N_c}(L^i G^{is} - \frac{1}{2\sqrt{3}}L \cdot S), \quad (D1)$$

and

$$B_3 = \frac{1}{N_c}(S^i G^{is} - \frac{1}{2\sqrt{3}}S \cdot S), \quad (D2)$$

where the angular momentum operator $L^i$ acts on the entire system of $N_c$ quarks. The matrix elements of the operators $L^i G^{is}$ and $S^i G^{is}$ have been obtained in Ref. [34]. Their analytic expressions are

$$\langle (\lambda'\mu')Y'I'I_3\ell' S'JJ_3|(-1)^iL^i G^{-is}|(\lambda\mu)Y'I'I_3; \ell SJJ_3\rangle =$$

$$\delta_{\ell'\ell}(-1)^{\ell+S'+J}\sqrt{\frac{C_{SU(6)}^{[f]}}{2}}\ell(\ell + 1)(2\ell + 1)(2S' + 1) \begin{pmatrix} S' & \ell & J \\ \ell & S & 1 \end{pmatrix} \sum_{\rho} \begin{pmatrix} (\lambda\mu) \begin{pmatrix} 11 \end{pmatrix} \\ YI \end{pmatrix} \begin{pmatrix} (\lambda'\mu') \begin{pmatrix} 11 \end{pmatrix} \\ YI \end{pmatrix}_{\rho} \begin{pmatrix} [f] & [21^4] \\ (\lambda\mu)S \begin{pmatrix} (11)1 \end{pmatrix} & (\lambda'\mu')S' \end{pmatrix}_{\rho}, \quad (D3)$$
and

\[
\langle (\lambda'\mu')Y'I'I_3'; \ell' S' JJ_3' \vert (-1)^J S' G^{-i\hbar} \vert (\lambda\mu)Y II_3; \ell S JJ_3 \rangle = \delta_{\ell\ell'} \delta_{S'S}
\]

\[
\times \sqrt{\frac{C_{SU(6)}^{[\lambda\mu]} }{2}} \sqrt{S(S+1)} \sum_{\rho} \left( \begin{array}{c}
\frac{\lambda\mu}{11} \\
YI \\
00
\end{array} \frac{\lambda'\mu'}{YI} \right) \rho \left( \begin{array}{c}
[f] \\
[21^4] \\
[f]
\end{array} \frac{(\lambda\mu)S}{(11)1} \frac{(\lambda'\mu')S}{\rho} \right),
\]

(D4)

respectively. Note that the factor \(\sqrt{\ell(\ell+1)}\) appearing in Eq. (D3) is missing in Eq. (D4) of Ref. [34].

The matrix elements of \(S^i G^{i8}\) for mixed symmetric \([70, 2^+]\) states have been straightforwardly obtained from the analytic forms of the matrix elements of \(S^i G^{i8}\) exhibited in Table X of Ref. [34], where they were derived in the context of the multiplet \([70, 1^-]\), but they can be applied to any angular momentum \(\ell\) and parity. These analytic forms are simple ratios of polynomials in the variables \(N_c\), the isospin \(I\) and the strangeness \(S\).

From the definitions (D3) and (D4) one can see that the expectation values of \(L^i G^{i8}\) and \(S^i G^{i8}\) are related by

\[
\langle L^i G^{i8} \rangle = \delta_{S'S}(-1)^{\ell+S'+J} \sqrt{\frac{\ell(\ell+1)(2\ell+1)(2S'+1)}{S(S+1)}} \left\{ \begin{array}{c}
S' \\
\ell \\
J
\end{array} \frac{S}{\ell} \frac{\ell}{S} \frac{1}{1} \right\} \langle S^i G^{i8} \rangle,
\]

(D5)

which can help to easily find the entries for \(L^i G^{i8}\) using \(S^i G^{i8}\) from Table X of Ref. [34] and the corresponding 6j coefficients. Note also that the expectation values of \(S^i G^{i8}\) are independent of \(J\), as seen from Table X of Ref. [34]. The expectation values of \(L^i G^{i8}\) for \(J'\) and \(J\) at fixed \(S\) can be obtained from the ratio of the corresponding 6j coefficients.

\[
\frac{\langle L^i G^{i8} \rangle_{J'}}{\langle L^i G^{i8} \rangle_{J}} = (-1)^{J'-J} \left\{ \begin{array}{c}
S \\
\ell \\
J'
\end{array} \frac{S}{\ell} \frac{\ell}{S} \frac{1}{1} \right\}.
\]

(D6)

The use of (D6) can simplify the calculation of \(B_2\).

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