DISCRETE CONLEY INDEX THEORY FOR
ZERO DIMENSIONAL BASIC SETS

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Abstract. In this article the discrete Conley index theory is used to study
diffeomorphisms on closed differentiable n-manifolds with zero dimensional hy-
perbolic chain recurrent set. A theorem is established for the computation of
the discrete Conley index of these basic sets in terms of the dynamical infor-
mation contained in their associated structure matrices. Also, a classification
of the reduced homology Conley index of these basic sets is presented using
its Jordan real form. This, in turn, is essential to obtain a characterization of
a pair of connection matrices for a Morse decomposition of zero-dimensional
basic sets of a diffeomorphism.

1. Introduction. The algebraic and topological techniques in dynamical systems
have played a significant role as can be verified, for example, in Morse Theory and
more recently in Conley’s Index Theory. This theory was introduced by C. Conley in
the late seventies and has been further developed in a variety of contexts. Initially
defined for continuous dynamical systems on compact sets, see [5], [17] and [18] and
later for discrete dynamical systems.

The Conley index was originally defined for isolated invariant sets $S$ of continuous
flows and defined to be the homotopy type of the pointed space obtained from
a filtration pair associated to $S$. It was then proven that the Conley index is
independent of the choice of the filtration pair and is a homotopy invariant. This
is precisely the difficulty in the discrete case since there are examples where the
homotopy type of the filtration pair of $S$ may depend on the choice of the pair.

Motivated by this difficulty several definitions were presented for the discrete case.
See [7], [11], [16] and [21]. The version we adopt herein is due to Richeson
[15] and called the reduced homological Conley index. The reason behind this

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choice is twofold: firstly it is computationally easier to work with and secondly, it contains the information that is needed in the definition of connection matrix pairs. This index will be defined more precisely in Section 2. It is defined for an isolated invariant set $S$ of a continuous map $f$ as

$$\text{Con}^*(S) = (CH_*(S), \chi_*(S)),$$

where $CH_*(S) = \bigcap_{n>0} ((f_P)_*)^n(H_*(N_L, [L]))$ and $\chi_*(S) : CH_*(S) \to CH_*(S)$ is an isomorphism induced by $(f_P)_*$. The map $(f_P)_*$ is the homology induced map of the pointed space map $f_p : N_L \to N_L$ associated to the filtration pair $P = (N, L)$.

The Conley index has the crucial property of continuation which guarantees the invariance of the index under small $C^0$ perturbations of the system. See [7] and [15]. The continuation property together with the fact that fitted diffeomorphisms are $C^0$-dense on the set of diffeomorphisms on closed manifolds, see [20], motivated our investigation of the discrete Conley index for zero dimensional basic sets.

Another crucial tool in Conley index theory is the connection matrix. Franzosa in [8],[9] and [10] developed the connection matrix theory for Morse decompositions of flows. This theory has proven to be useful in the study of the connection orbits between Morse sets as well as the detection of bifurcation behaviour, see [13]. Independently and around the same time, Bartiomejczyk and Dzedzej in [1] and Richeson in [15], considered a generalization of this theory for discrete dynamical systems. In [1], the connection matrix $\Delta$ uses solely the information contained in $CH^*$ of the discrete Conley index $\text{Con}^* = (CH^*, \chi^*)$. On the other hand, the generalization presented in [15] is actually a pair of connection matrices $(\Delta, a)$ where $\Delta$ uses the information given by $CH^*$ and $a$ uses the information given by the isomorphism $\chi^*$. A disadvantage of the connection matrix presented in is that it only identifies degree one connections whereas the benefit in considering the pair of connection matrices in [14] and [15] is that the existence of both degree zero and degree one connections can be detected and thus becomes a richer dynamical invariant.

Herein, we work with the pair of connection matrices $(\Delta, a)$. As with the Conley index, the main property of the connection matrix pair is its invariance under continuation. This confers to it a robust feature, more specifically invariance under $C^0$ perturbations of the system. The advantage being that, in order to study the pair of connection matrices of diffeomorphisms on a closed manifold $M$, Diff($M$) it suffices to consider fitted diffeomorphisms which are $C^0$-dense in Diff($M$), see [20]. In fact, we will consider an even larger class that contains the fitted diffeomorphisms, i.e. diffeomorphisms which admit a finite zero-dimensional basic set decomposition. For more details see [6].

In this article, Section 2 provides the background information needed throughout this work. In Section 3, one of the main results, Theorem 3.2, establishes a relation between a homological invariant, namely the Conley index, and a dynamical invariant obtained from the structure matrix of a zero dimensional basic set. In Section 4 a characterization of the isomorphism of the Conley index of a zero dimensional basic set by using the real Jordan form of its structure matrix is obtained in Theorem 4.1. Furthermore, in Section 5, by using Theorem 3.2, an alternative formulation for the results in [6] on the homology zeta function and on Morse inequalities of diffeomorphisms with hyperbolic chain recurrent sets on closed manifolds is obtained.
Theorem 3.2 will also be essential in obtaining important characterization results for connection matrices of diffeomorphisms with zero dimensional basic sets. In Section 6, we prove in Theorem 6.2 that the connection matrix pair \((\Delta, a)\) has, in some sense, complementary information, that is, in some cases \(a\) captures dynamical information and \(\Delta\) does not and vice-versa. Furthermore, in Section 7, Theorem 7.1 establishes that the matrix \(a\) gives no information for time-one maps, i.e., any detectable dynamical behaviour captured by the connection matrix pair \((\Delta, a)\) is registered solely in \(\Delta\). In Section 8, another main result is obtained in Theorem 8.2 which is a global characterization for diffeomorphisms whose hyperbolic chain recurrent set is a union of zero-dimensional basic sets.

2. Background.

2.1. Diffeomorphism with hyperbolic chain recurrent set and zero dimensional basic sets. In this section we will present some basic definitions and theorems on the dynamical behaviour of diffeomorphisms \(f : M \rightarrow M\) where \(M\) is a closed manifold with metric \(d(\cdot, \cdot)\).

Definition 2.1. Let \(f : M \rightarrow M\) be a diffeomorphism. A point \(x \in M\) is called chain recurrent if given \(\varepsilon > 0\) there exists \(x_1 = x, x_2, \ldots, x_{n-1}, x_n = x\) (with \(n = n(\varepsilon)\)) such that 
\[
    d(f(x_i), x_{i+1}) < \varepsilon, \quad \forall 1 \leq i < n.
\]

Define the chain recurrent set, \(\mathcal{R}(f)\), as the set of all chain recurrent points of \(f\).

By Smale’s spectral decomposition, Theorem 2.3 one has that if \(\mathcal{R}(f)\) is hyperbolic it admits a basic set decomposition.

Definition 2.2. Let \(f : M \rightarrow M\) be a diffeomorphism. A compact \(f\)-invariant set \(\Lambda\) is said to have a hyperbolic structure provided that the tangent bundle of \(M\) restricted to \(\Lambda\) can be written as a Whitney sum \(T\Lambda M = E^u \oplus E^s\) of subbundles \(Df\)-invariant, and there are constants \(C > 0, \lambda \in (0, 1)\), such that 
\[
    \|Df^n(v)\| \leq C\lambda^n \|v\|, \quad \forall v \in E^s, n \geq 0, \text{ and}
\]
\[
    \|Df^n(v)\| \geq C^{-1}\lambda^{-n} \|v\|, \quad \forall v \in E^u, n \geq 0
\]

Theorem 2.3 (Smale’s Spectral Decomposition, [19]). Let \(f : M \rightarrow M\) be a diffeomorphism such that its chain recurrent set \(\mathcal{R}(f)\) has a hyperbolic structure. Then \(\mathcal{R}(f)\) is a finite disjoint union of compact invariant sets \(\Omega_1, \Omega_2, \ldots, \Omega_n\) and each \(\Omega_i\) contains an orbit of the system which is dense in \(\Omega_i\).

The \(\Omega_i\)’s of Theorem 2.3 are called basic sets of \(f\), and the fiber dimension of the bundle \(E^u_{\Omega_i}\) is called the index of \(\Omega_i\).

Given a zero dimensional basic set \(\Omega\) of a diffeomorphism \(f : M \rightarrow M\) with hyperbolic chain recurrent set, one can associate to the dynamics of \(f\) in \(\Omega\), a matrix with 0, 1 and \(-1\) entries, called the structure matrix.

In order to define the structure matrix one needs the relation between a zero dimensional basic set and its subshift of finite type. We refer the reader to [3].

Definition 2.4. The subshift of finite type determined by a finite set \(S\) and the relation \(\rightarrow\) is a homeomorphism \(\sigma : \Sigma \rightarrow \Sigma\), where \(\Sigma \subset \prod_{i=-\infty}^{\infty} S\) is defined as

...
\[\Sigma = \{s = (\ldots, s_{-1}, s_0, s_1, \ldots) \mid s_i \to s_{i+1} \text{ for all } i\} \text{ and } \sigma(s) = s' \text{ where } s'_i = s_{i-1}, \text{ so } \sigma \text{ shifts to the right.}\]

The subshift of finite type \(\sigma\) determined by \(\mathcal{S}\) and \(\to\) is called a **vertex shift** associated to the matrix \(A\) if one numbers the elements of \(\mathcal{S}\) from 1 to \(n\) (\(n\) is the cardinality of \(\mathcal{S}\) and the matrix \(A_{n\times n}\) is given as follows:

\[
A_{ij} = \begin{cases} 
1, & \text{if } s_i \to s_j \\
0, & \text{otherwise}
\end{cases}
\]

Bowen proves in [3] that if \(\Omega\) is a zero dimensional basic set of a diffeomorphism \(f\) with hyperbolic \(\mathcal{R}(f)\), then \(f\) restricted to \(\Omega\) is topologically conjugate to a vertex shift \(\sigma(G) : \Sigma_G \to \Sigma_G\) by \(h : \Sigma_G \to \Omega\) that has the property that the locally constant function

\[
\Delta(x) = \begin{cases} 
1 & \text{if } Df_x : E^n_x \to E^n_{f(x)} \text{ preserves orientation}, \\
-1 & \text{if } Df_x : E^n_x \to E^n_{f(x)} \text{ reverses orientation}
\end{cases}
\]

is constant in \(h(C(k))\) for each \(k\), where \(C(k) = \{c \in \Sigma_G \mid c_0 = k\}\). Hence, one defines \(\Delta_k\) as the value of \(\Delta(x)\) in \(h(C(k))\).

**Definition 2.5.** Define the **structure matrix** \(A\) for a basic set \(\Omega\) by \(A_{jk} = \Delta_k G_{jk}\).

Next we present a result of Bowen and Franks [4] which associates a structure matrix to a homological invariant.

**Definition 2.6.** Let \(f : M \to M\) be a diffeomorphism with hyperbolic chain recurrent set with basic sets \(\{\Omega_i\}_{i=0}^{\infty}\), then a **filtration associated to** \(f\) is a collection of submanifolds \(M_0 \subset M_1 \subset \cdots \subset M_n = M\) such that

1. \(f(M_i) \subset \text{int}(M_i)\);
2. \(\Omega_i = \bigcap_{n=-\infty}^{\infty} f^n(M_i \setminus M_{i-1})\).

The existence of a filtration follows easily from the existence of a smooth Lyapunov function \(\phi : M \to \mathbb{R}\) for the diffeomorphism \(f\), [6].

**Definition 2.7.** Let \(h : V \to V\) and \(h' : V' \to V'\) be two endomorphisms where \(V\) and \(V'\) are vector spaces.

1. Define the **nonnilpotent part** \(h^+\) of \(h\) as the map induced by \(h\) in the quotient space \(V / V_0\) where \(V_0 = \{v \in V \mid h^k(v) = 0 \text{ for some } k > 0\}\).
2. One says that \(h\) and \(h'\) are **conjugate** if there exists an isomorphism \(\Psi : V \to V'\) such that \(h' \circ \Psi = \Psi \circ h\).

**Theorem 2.8** ([4]). Suppose that the diffeomorphism \(f : M \to M\) has hyperbolic chain recurrent set, \(\{M_i\}\) is a filtration associated to \(f\) and, \(\Omega_i\) is a zero dimensional basic set of index \(u\). If \(A\) is a structure matrix \(n \times n\) for \(\Omega_i\) and \(F\) is a field, then

1. The nonnilpotent part \(A^+\) of \(A : F^n \to F^n\) is conjugate to the nonnilpotent part \(f^+_u\) of \(f_u : H_u(M_i, M_{i-1}; F) \to H_u(M_i, M_{i-1}; F)\), and
2. the map \(f_{uk} : H_k(M_i, M_{i-1}; F) \to H_k(M_i, M_{i-1}; F)\) is nilpotent for all \(k \neq u\).
2.2. Reduced homological Conley index. Let $U$ be an open subset of a locally compact metric space $X$ and $f : U \to X$ a continuous map.

Define the maximal invariant subset of $N$, $\text{Inv}(N)$ as the set of points $x \in N$ such that there exists a complete orbit $\{x_n\}^\infty_{n=0} \subset N$ with $x_0 = x$ and $f(x_n) = x_{n+1}$ for all $n$. One says that a set $S$ is an isolated invariant set if there exists an isolating neighborhood $N$ (i.e., a compact $N$ with $\text{Inv}(N) \subset \text{int}(N)$) such that $S = \text{Inv}(N)$.

Given an isolated invariant set $S$ there exists a pair of compact spaces $(N, L)$ with $L \subset N$ contained in the interior of the domain of $f$ with the following properties:

1. $\overline{N \setminus L}$ is an isolating neighborhood of $S$,
2. $L$ is a neighborhood of $N^− = \{x \in N \mid f(x) \notin \text{int}(N)\}$ in $N$,
3. $f(L) \cap \overline{N \setminus L} = \emptyset$

See [7]. This compact pair is called filtration pair.

Let $P = (N, L)$ be a filtration pair of an isolated invariant set $S$ of a continuous map $f$. The map $f$ induces a continuous base-point preserving map $f_P : N_L \to N_L$ with the property $[L] \subset \text{int} f^{-1}([L])$, where $N_L$ is the quotient space $N/L$ and $[L]$ is the collapsed set that is taken as base point (see [7]). This map is the pointed space map associated to $P$.

**Definition 2.9.** Define reduced homological Conley index of $S$ as

$$\text{Con}_*(S) = (CH_*(S), \chi_*(S)),$$

where $CH_*(S) = \bigcap_{n>0} ((f_P)_*)^n(H_*(N_L, [L]))$ and $\chi_*(S) : CH_*(S) \to CH_*(S)$ is the automorphism induced by $(f_P)_* : H_*(N_L, [L]) \to H_*(N_L, [L])$ which is the homology induced map of the pointed space map $f_P : N_L \to N_L$.

Henceforth, whenever we mention the Conley index we mean the reduced homological Conley index defined above.

2.3. Connection matrix pair. In preparation for the connection matrix pair definition we need to present some basic concepts. See [13] and [14] for more details.

We say that a relation $\prec$ is a partial order on a finite set $\mathcal{P}$ provided for all $\pi, \pi' \in \mathcal{P}$

1. $\pi < \pi'$ never holds, and
2. $\pi < \pi'$ and $\pi'' < \pi''$ implies $\pi < \pi''$.

On the other hand, $\prec$ is a total ordering, if, in addition, for any $\pi \neq \pi'$ we have that $\pi < \pi'$ or $\pi' < \pi$.

Suppose $\mathcal{P}$ is a finite set with partial order $\prec$.

1. A subset $I \subset \mathcal{P}$ is called an interval if $\pi < \pi'' < \pi'$ and $\pi, \pi' \in I$ then $\pi'' \in I$.
   Denote by $\mathcal{I} = \mathcal{I}(\mathcal{P}, \prec)$ the set of all intervals in $\mathcal{P}$.
2. An interval $I$ is called attracting if $\pi'' < \pi'$ and $\pi' \in I$ implies $\pi'' \in I$.
   Denote by $\mathcal{A} = \mathcal{A}(\mathcal{P}, \prec)$ the set of all attracting intervals in $\mathcal{P}$.
3. An ordered collection of disjoint intervals $(I_1, \ldots, I_n)$ is called an adjacent $n$-tuple if $I_1 \cup \cdots \cup I_n \in \mathcal{I}(\mathcal{P})$ and for any $\pi \in I_i$ and $\pi' \in I_j$ with $i < j$ we have that $\pi' \not\prec \pi$.
   Denote by $\mathcal{I}_n = \mathcal{I}_n(\mathcal{P}, \prec)$ the set of all adjacent $n$-tuples.

**Definition 2.10.** A collection of disjoint isolated invariant sets $\mathcal{M}(\mathcal{P}, \prec) = \{M_\pi \subset S \mid \pi \in \mathcal{P}\}$ is a Morse decomposition of an isolated invariant set $S$ if for every $x \in S$ and every solution $\sigma : \mathbb{Z} \to S$ for $x$ we have either
1. \( \sigma(Z) \subset M_{z} \) for some \( \pi \in P \), or
2. \( \omega(\sigma) \subset M_{x} \) and \( \alpha(\sigma) \subset M_{x'} \) for some \( \pi < \pi' \).

For any interval \( I \subset P \) define the set

\[
M_{I} = \bigcup_{\pi \in I} M_{\pi} \cup \bigcup_{\pi', \pi'' \in I} C(M_{\pi'}, M_{\pi''}; S),
\]

where \( C(M_{\pi'}, M_{\pi''}; S) \) is the set of connecting orbits from \( M_{\pi'} \) to \( M_{\pi''} \), and we define it to be

\[
\{ x \in S \setminus (M_{\pi'} \cup M_{\pi''}) \mid \exists \text{ solution } \sigma : Z \to S \text{ with } \alpha(\sigma) \subset M_{\pi'}, \omega(\sigma(x)) \subset M_{\pi''} \}.
\]

A well known property of Morse decompositions is that if \( (I, J) \in I_{2}(P, <) \) then \( (M_{I}, M_{J}) \) is an attractor-repeller decomposition of \( M_{I,J} \).

Given a Morse decomposition \( M(P, <) = \{ M_{\pi} \subset S \mid \pi \in P \} \) there exists a collection of compact sets in \( X \) such that it is possible to extract a filtration pair for each \( M_{I} \). \(^{[15]} \) Such a collection, which we define subsequently, is called a Morse set filtration.

**Definition 2.11.** A collection of compact sets \( N(P, <) = \{ N(I) \subset X \mid I \in A \} \) is called a Morse set filtration for \( M(P, <) \) provided that for any attracting intervals \( I, J \subset P \) the following conditions hold:

1. \( (N(I), N(\emptyset)) \) is a filtration pair for \( M_{I} \),
2. \( N(I) \cap N(J) = N(I \cap J) \), and
3. \( N(I) \cup N(J) = N(I \cup J) \).

We now proceed to define a connection matrix pair \( (\Delta, a) \) which makes use of the notion of braid isomorphisms. The basic idea is that for a pair of graded matrices \( (\Delta, a) \) to be a connection matrix pair, the graded module braids with endomorphism obtained for both \( (\Delta, a) \) and for the Conley indices of the Morse sets be isomorphic.

**Definition 2.12.** A graded module braid with endomorphism is a collection \( H = \{ (H(I), a(I)) \mid I \in I \} \) of graded modules and linear endomorphisms such that the following conditions are satisfied.

1. For each \( I \in I \), there exists a graded module \( H(I) \) and a linear endomorphism \( a(I) : H(I) \to H(I) \) of zero degree.
2. For all \( (I, J) \in I_{2} \) there exists a long exact sequence of modules and endomorphisms

\[
\cdots \to (H^{n}(J), a^{n}(J)) \to (H^{n}(IJ), a^{n}(IJ)) \to (H^{n}(I), a^{n}(I)) \delta^{n} \cdots
\]

3. For all \( (I, J, K) \in I_{3} \) the braid diagram in Figure [1] commutes.

**Definition 2.13.** Suppose \( H(P) = \{ (H(I), a(I)) \mid I \in I \} \) and \( H'(P) = \{ (H'(I), a'(I)) \mid I \in I \} \) are graded module braids with endomorphisms. Define a graded module braid isomorphism \( \Psi : H \to H' \) to be a collection of isomorphisms \( \psi^{*}(I) : H(I) \to H'(I) \) defined for all \( I \in I \) such that for \( (I, J) \in I_{2} \) the following diagram commutes.

\[
\begin{array}{cccccc}
\cdots & \delta & \to & (H(J), a(J)) & \to & (H(IJ), a(IJ)) & \to & (H(I), a(I)) & \delta & \cdots \\
\downarrow \psi^{*}(J) & & & & & & & & \downarrow \psi^{*}(IJ) & \\
\cdots & \delta & \to & (H'(J), a(J)) & \to & (H'(IJ), a'(IJ)) & \to & (H'(I), a'(I)) & \delta & \cdots 
\end{array}
\]

Next we define the connection matrix pair for a Morse decomposition \( M(P, <) \) of an isolated invariant set \( S \).
Definition 2.14. The pair of graded matrices $(\Delta, a)$ is called a connection matrix pair for the Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ if the graded module braid with endomorphism

$$\mathcal{H}(\mathcal{M}) = \{\text{Con}^k(M_I) = (CH^k(M_I), \chi^k(M_I)) \mid I \in \mathcal{I}\}.$$

is isomorphic to the graded module braid with endomorphism

$$\mathcal{H}\Delta(\mathcal{M}) = \{(H^k(C\Delta(I)), (a^k(I))^*) \mid I \in \mathcal{I}\},$$

where $C\Delta(I) = \bigoplus_{\pi \in I} CH^*(M_\pi), \Delta(I)$ is a cochain complex and $a(I)$ is the associated cochain map.

In other words, $(\Delta, a)$ is a connection matrix pair for $\mathcal{M}(\mathcal{P}, <)$ of $S$ if for all $(I, J, K) \in \mathcal{I}_3$ we have that the following graded module braids with endomorphisms 1 and 2 are isomorphic, see Figure 2 and Figure 3.
We denote the set of connection matrix pairs for $\mathcal{M}(\mathcal{P}, <)$ by $\mathcal{CM}(\mathcal{M}(\mathcal{P}, <))$.

The proposition that follows is due to Richeson, see [13], which we state in terms of Morse sets.

**Proposition 1.** Let $f : M \to M$ be a diffeomorphism and $(\Delta, a)$ a connection matrix pair for a Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ of an isolated invariant set $S$ of $f$. Let $\pi, \pi' \in \mathcal{P}$ with $\pi < \pi'$ be adjacent elements. If one of the following occurs:

1. $\Delta^*(\pi', \pi) \neq 0$ or
2. $CH^*(M_{\pi\pi'}) \not\cong CH^*(M_\pi) \oplus CH^*(M_{\pi'})$ or
3. $\chi^*(M_{\pi\pi'})$ not conjugated to $\chi^*(M_\pi) \oplus \chi^*(M_{\pi'})$

then $\mathcal{C}(M_{\pi'}, M_\pi; M_{\pi'}) \neq \emptyset$.

### 3. The Conley index and the structure matrix

In this section we prove the relation between the Conley index isomorphism of a zero dimensional basic set with the nonnilpotent part of its structure matrix.

**Lemma 3.1.** Let $M$ be a closed manifold. Suppose $f : M \to M$ is a diffeomorphism with hyperbolic chain recurrent set, $\{\Omega_i\}_{i=0}^n$ are basic sets of $f$ and $\{M_i\}_{i=0}^n$ is an associated filtration. Then

1. $P_i = (M_i, M_{i-1})$ is a filtration pair for $\Omega_i$, $\forall i = 0, \ldots, n$.
2. $f_{*k} : H_k(M_i, M_{i-1}) \to H_k(M_i, M_{i-1})$ and $(f_{P_i})_{*k} : H_k(M_i/M_{i-1}, [M_{i-1}]) \to H_k(M_i/M_{i-1}, [M_{i-1}])$ are conjugate for all $k$, where $f_{P_i}$ is the pointed space map associated to $P_i = (M_i, M_{i-1})$.

**Proof.** 1. Let $\phi : M \to \mathbb{R}$ be a smooth Lyapunov function such that $M_i = \phi^{-1}((-\infty, c_i])$ for all $i = 0, \ldots, n$. Then $M_i$ is compact and

$$\text{Inv}(M_i \setminus M_{i-1}) = \bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus M_{i-1}) = \Omega_i,$$

$M_i^- = \{x \in M_i | f(x) \notin \text{int} M_i \} = \emptyset \subset M_{i-1}$, and $f(M_{i-1}) \cap M_i \setminus M_{i-1} = \emptyset$, i.e., $(M_i, M_{i-1})$ is a filtration pair for $\Omega_i$, $\forall i = 0, \ldots, n$. 

---

**Figure 3. Graded module braids with endomorphisms 2**
2. Since the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{f} & N \\
\downarrow q & & \downarrow q \\
\downarrow L & & \downarrow L \\
L & \xrightarrow{f_P} & L
\end{array}
\]

is commutative (\(q : N \rightarrow \frac{N}{L}\) is a quotient map), it follows that \(q \circ f = f_P \circ q\).

Therefore, \(q_\ast \circ f_\ast = (f_P)_\ast \circ q_\ast\), i.e., \(f_\ast\) is conjugate to \((f_P)_\ast\).

\(\square\)

**Remark 1.** For an endomorphism \(f\) we denote the generalized kernel by \(gKer(f)\) and the generalized image by \(gIm(f)\).

**Theorem 3.2.** Let \(M\) be a closed manifold and \(f : M \rightarrow M\) be a diffeomorphism. Suppose \(f\) has a hyperbolic chain recurrent set and \(\Omega_i\) is a zero dimensional basic set of index \(u\). If \(A\) is an \(n \times n\) structure matrix for \(\Omega_i\), and \(\mathbb{F}\) is a field, then

1. \(\chi_u(\Omega_i)\) is conjugate to the nonnilpotent part \(A^+\) of \(A : \mathbb{F}^n \rightarrow \mathbb{F}^n\);
2. \(\text{Con}_k(\Omega_i) = (0, 0)\) for all \(k \neq u\).

where \(\text{Con}_u(\Omega_i) = (CH_u(\Omega_i), \chi_u(\Omega_i))\) is the Conley index of \(\Omega_i\).

**Proof.** Let \(\{M_j\}\) be a filtration associated to \(f\). It is well known that \(A^+\) is conjugate to the nonnilpotent part \(f^+\) of \(f\) : \(H_u(M_i, M_{i-1}) \rightarrow H_u(M_i, M_{i-1})\) (see Theorem 2.8). By Lemma 3.1 we have that

\[
f_{k} : H_k(M_i, M_{i-1}) \rightarrow H_k(M_i, M_{i-1})
\]

and

\[
(f_P)_k : H_k(M_i/M_{i-1}, [M_{i-1}]) \rightarrow H_k(M_i/M_{i-1}, [M_{i-1}])
\]

are conjugate for all \(k\) and consequently \(f^+_u\) and \((f_P)^+_u\) are also conjugate.

We will show that \((f_P)^+_u = \chi_u(\Omega_i)\). Since, \((H_u(M_i/M_{i-1}, [M_{i-1}]))_0\) is the set \(\{v \in H_u(M_i/M_{i-1}, [M_{i-1}]) \mid (f_P)^+u(v) = 0 \text{ for some } k > 0\}\) then \((H_u(M_i/M_{i-1}, [M_{i-1}]))_0 = gKer((f_P)_u)\). Hence,

\[
\frac{H_u(M_i/M_{i-1}, [M_{i-1}])}{(H_u(M_i/M_{i-1}, [M_{i-1}]))_0} = \frac{H_u(M_i/M_{i-1}, [M_{i-1}])}{gKer((f_P)_u)} = (gIm(f_P)_u) = CH^+(\Omega_i).
\]

Therefore, \(\chi_u(\Omega_i) = (f_P)_u\) and this proves item 1.

It is well known that the map \(f_{k}\) is nilpotent for all \(k \neq u\) (see Theorem 2.8). Hence, \((f_P)_k\) is also nilpotent for all \(k \neq u\). Thus,

\[
CH_k(\Omega_i) = \frac{H_k(M_i/M_{i-1}, [M_{i-1}])}{gKer((f_P)_k)} \approx 0, \forall k \neq u,
\]

which implies that \(\chi_k(\Omega_i) = 0\) for all \(k \neq u\).

\(\square\)

**Remark 2.** An immediate consequence of our Theorem 3.2 and the Lemma 6.2 of \[7\] is that if \(\dim W^s(\Omega_i) < k\) or \(\dim W^s(\Omega_i) < (\dim M) - k\), then \(\text{Con}_k(\Omega_i) = (0, 0)\).

**Remark 3.** Note that any Smale diffeomorphism on a closed manifold satisfies Theorem 3.2. Moreover, if this diffeomorphism is Morse-Smale we have that \(\chi_u(\Omega_i)\) is conjugate to

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\pm 1 & 0 & 0 & \cdots & 0
\end{pmatrix}_{m \times m}
\]
Since, in general, computing the nonnilpotent part of a matrix can involve an increasing number of computations as the matrix gets larger, a simple example is provided in order to illustrate the use of Theorem 3.2. We will contrast this theorem with Theorem 4.1 which makes the calculations more straightforward.

Example 3.1. We will compute the Conley index of a Smale Horseshoe, Figure 4, embedded in $S^2$ using Theorem 3.2.

![Figure 4. Smale’s Horseshoe](image)

A structure matrix of the basic set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(Q)$ of index 1 is given by

$$
\begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix}
$$

with respect to the set of handles $H = H_0 \cup H_1$.

Observe that $A^+ = 0$, since $A^2 = 0$. Hence, the Conley index of Smale’s Horseshoe is

$$Con_q(\Lambda) = (0, 0), \forall q.$$

Observe that the advantage of this approach is that we do not make use of the filtration pair nor the pointed space map in order to compute the Conley index. We make use solely of the dynamical information contained in the structure matrix of Smale’s Horseshoe.

4. The Conley index and the real Jordan form of a structure matrix.

In this section, we show that the isomorphism $\chi^*$, of the Conley index of a zero dimensional basic set of a diffeomorphism with hyperbolic chain recurrent set, is conjugate to the nonnilpotent part of an associated structure matrix. From a practical point of view, working with the structure matrix in its real Jordan form makes the computation of the nilpotent and nonnilpotent part of the matrix effortless. The information of the nonnilpotent part of this matrix is easily retrieved by a quick analysis of the non zero eigenvalues and the nilpotent part by the zero eigenvalues.

We prove the following theorem:

**Theorem 4.1.** Let $M$ be a closed manifold and $f : M \rightarrow M$ a diffeomorphism with a hyperbolic chain recurrent set and $\Omega$ a zero dimensional basic set of index $u$. Let $A$ be an $n \times n$ structure matrix for $\Omega$ and $\lambda_1, \ldots, \lambda_r$ their eigenvalues pairwise disjoint, then $\chi_u(\Omega)$ is conjugate to the real Jordan form of matrix $A$ after removing the
Jordan blocks corresponding to zero eigenvalue. In other words, $\chi_u(\Omega)$ is conjugate to

\begin{align*}
1. & \quad J = \begin{pmatrix}
\lambda_1 & 0 \\
\lambda_2 & \lambda_3 \\
\vdots & \ddots \\
0 & \cdots & \lambda_n
\end{pmatrix} \quad \text{if } r = n \text{ and } \lambda_i \neq 0, \forall i = 1, \ldots, n. \\
2. & \quad J = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_r \\
\lambda_{r+1} & \ddots \\
0 & \cdots & \lambda_n
\end{pmatrix} \quad \text{if } r = n \text{ and } \lambda_r = 0. \\
3. & \quad J = \begin{pmatrix}
J_{\lambda_1} & 0 \\
J_{\lambda_2} & \ddots \\
0 & \cdots & J_{\lambda_r}
\end{pmatrix} \quad \text{if } r < n \text{ and } \lambda_i \neq 0, \forall i = 1, \ldots, r. \\
4. & \quad J = \begin{pmatrix}
J_{\lambda_1} \\
\vdots \\
J_{\lambda_{r-1}} \\
J_{\lambda_r} & \ddots \\
0 & \cdots & \lambda_n
\end{pmatrix} \quad \text{if } r < n \text{ and } \lambda_r = 0.
\end{align*}

where for each $i = 1, \ldots, r$ we have that $s_i$ is the geometric multiplicity of the eigenvalue $\lambda_i$, $n_i$ is the algebraic multiplicity of the eigenvalue $\lambda_i$ and $J_{\lambda_i}$ has the form

\begin{align*}
J_{\lambda_i} &= \begin{pmatrix}
J_{\lambda_i,1} & J_{\lambda_i,2} \\
& \ddots \\
& & J_{\lambda_i,s_i}
\end{pmatrix}_{n_i \times n_i}
\end{align*}

where each block $J_{\lambda_i,j}$ is a real Jordan block associated to the eigenvalue $\lambda_i$ which has one of the following forms:

\begin{align*}
\begin{pmatrix}
\lambda_i & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda_i
\end{pmatrix}
\end{align*}
if $\lambda_i$ is a real eigenvalue or

$$
\begin{pmatrix}
D & I & 0 & \cdots & 0 & 0 \\
0 & D & I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & D & I \\
0 & 0 & 0 & \cdots & 0 & D
\end{pmatrix}
$$

if $\lambda_i = a + bi$ is a complex eigenvalue, where $D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Let

$$
J = \begin{pmatrix}
J_{\lambda_1,1} & \ldots & 0 \\
& \ddots & \vdots \\
& \vdots & J_{\lambda_1,s_1} \\
0 & \ldots & J_{\lambda_r,1}
\end{pmatrix}
$$

be the real Jordan form of $A$, where each $s_i$ is the geometric multiplicity of the eigenvalue $\lambda_i$ and each $J_{\lambda_i,j}$ is the real Jordan block associated to the eigenvalue $\lambda_i$.

Denote by $n_{i,j}$ the size of the real Jordan block $J_{\lambda_i,j}$ with $j = 1, \ldots, s_i$ and $i = 1, \ldots, r$. By the construction of the real Jordan form, we have that $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,s_i}$. Since, $n = \sum_{i=1}^r \sum_{j=1}^{s_i} n_{i,j}$, then

$$
F^n = [V_{\lambda_1,1} \oplus \cdots \oplus V_{\lambda_1,s_1}] \oplus [V_{\lambda_2,1} \oplus \cdots \oplus V_{\lambda_2,s_2}] \oplus \cdots \oplus [V_{\lambda_r,1} \oplus \cdots \oplus V_{\lambda_r,s_r}]
$$

where

$$
V_{\lambda_1,1} = \bigoplus_{i=1}^{n_{1,1}} F, \quad V_{\lambda_1,2} = \bigoplus_{i=1}^{n_{1,2}} F, \quad \cdots \quad V_{\lambda_1,s_1} = \bigoplus_{i=1}^{n_{1,s_1}} F \\
V_{\lambda_2,1} = \bigoplus_{i=1}^{n_{2,1}} F, \quad V_{\lambda_2,2} = \bigoplus_{i=1}^{n_{2,2}} F, \quad \cdots \quad V_{\lambda_2,s_2} = \bigoplus_{i=1}^{n_{2,s_2}} F \\
\vdots \\
V_{\lambda_r,1} = \bigoplus_{i=1}^{n_{r,1}} F, \quad V_{\lambda_r,2} = \bigoplus_{i=1}^{n_{r,2}} F, \quad \cdots \quad V_{\lambda_r,s_r} = \bigoplus_{i=1}^{n_{r,s_r}} F
$$

Thus,

$$
\ker J^k = [\ker J_{\lambda_1,1}^k \oplus \cdots \oplus \ker J_{\lambda_1,s_1}^k] \oplus \cdots \oplus [\ker J_{\lambda_r,1}^k \oplus \cdots \oplus \ker J_{\lambda_r,s_r}^k], \forall k > 0,
$$

$$
g\ker J = [g\ker J_{\lambda_1,1} \oplus \cdots \oplus g\ker J_{\lambda_1,s_1}] \oplus \cdots \oplus [g\ker J_{\lambda_r,1} \oplus \cdots \oplus g\ker J_{\lambda_r,s_r}],
$$

$$
g\im J = [g\im J_{\lambda_1,1} \oplus \cdots \oplus g\im J_{\lambda_1,s_1}] \oplus \cdots \oplus [g\im J_{\lambda_r,1} \oplus \cdots \oplus g\im J_{\lambda_r,s_r}].
$$
Therefore,

\[
J^+ = \begin{pmatrix}
J_{\lambda_1,1}^+ & & & 0 \\
& \ddots & & \\
& & J_{\lambda_s,s}^+ & \\
0 & & & J_{\lambda_r,r}^+
\end{pmatrix}
\]

Hence, in order to calculate the nonnilpotent part, \(J^+\), of the real Jordan form \(J\) of \(A\), it suffices to compute the nonnilpotent part of the real Jordan blocks that make up \(J\). In other words, it suffices to compute the nonnilpotent part of the following real Jordan blocks associated to an eigenvalue \(\lambda\):

- \(J(1) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0
\end{pmatrix}_{p \times p}
\) if \(\lambda \neq 0\)

- \(J(2) = \begin{pmatrix}
D & I & 0 & \cdots & 0 & 0 \\
0 & D & I & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & D & I \\
0 & 0 & 0 & \cdots & 0 & D
\end{pmatrix}_{p \times p}
\) if \(\lambda = 0\)

- \(J(3) = \begin{pmatrix}
\lambda^k & C_1\lambda^{k-1} & \cdots & C_p-2\lambda^{k-(p-2)} & C_p-1\lambda^{k-(p-1)} & C_p-2\lambda^{k-(p-2)} \\
0 & \lambda^k & \cdots & C_p-3\lambda^{k-(p-3)} & C_p-2\lambda^{k-(p-2)} & C_p-3\lambda^{k-(p-3)} \\
0 & 0 & \cdots & \lambda^k & C_1\lambda^{k-1} & \cdots \\
0 & 0 & \cdots & 0 & \lambda^k & C_2\lambda^{k-2}
\end{pmatrix}_{p \times p}
\)

\((b \neq 0)\), where \(D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\) and \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\).

Firstly, we show that \(J(1)^+ = J(1)\). Since \(\lambda \neq 0\), we have that

\[
J(1)^k = \begin{pmatrix}
\lambda^k & C_1\lambda^{k-1} & \cdots & C_p-2\lambda^{k-(p-2)} & C_p-1\lambda^{k-(p-1)} & C_p-2\lambda^{k-(p-2)} \\
0 & \lambda^k & \cdots & C_p-3\lambda^{k-(p-3)} & C_p-2\lambda^{k-(p-2)} & C_p-3\lambda^{k-(p-3)} \\
0 & 0 & \cdots & \lambda^k & C_1\lambda^{k-1} & \cdots \\
0 & 0 & \cdots & 0 & \lambda^k & C_2\lambda^{k-2}
\end{pmatrix}_{p \times p}
\]

where \(C_i = \binom{k}{i} = \frac{k(k-1)\cdots(k-(i-1))}{i!}, \forall i = 1, \ldots, p-1\). If \(k \geq p\), then \(C_i \neq 0\), for all \(i = 1, \ldots, p-1\) and if \(k < p\), then there exists \(k \leq i_0 < p\) such that \(C_i \neq 0\) if \(i = 1, \ldots, i_0\) and \(C_i = 0\) for \(i \geq i_0\). Hence, \(gKer J(1) = \{(0, \ldots, 0)\}\) and thus we conclude that \(J(1)^+ = J(1)\).

Note that the matrix \(J(2)\) is nilpotent, consequently \(gKer J(2) = \mathbb{F}^p\), which implies that \(J(2)^+ = 0\).
Finally, we will prove that $J(3)^+ = J(3)$. Indeed, since

$$J(3)^k = \begin{pmatrix}
D^k & C_1 D^{k-1} & \cdots & C_{p'-2} D^k \cdot (p'-2) & C_{p'-1} D^k \cdot (p'-1) \\
0 & D^k & \cdots & C_{p'-3} D^k \cdot (p'-3) & C_{p'-2} D^k \cdot (p'-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & D^k & C_1 D^{k-1} \\
0 & 0 & \cdots & 0 & D^k
\end{pmatrix}$$

where $p' = \frac{p}{2}$ and $C_i = \binom{k}{i} = \frac{k(k-1) \cdots (k-(i-1))}{i!}$, $\forall i = 1, \ldots, p'-1$, we have that $\det(J(3)^k) = \det(D^k) \cdot \cdots \cdot \det(D^k) \neq 0$. This follows because $\det(D^k) \neq 0$ which is easily proved by induction on $k$. In this way, the columns of $J(3)^k$ form a linearly independent set. Therefore, $gKerJ(3) = \{(0, \ldots, 0)\}$, which implies that $J(3)^+ = J(3)$.

In what follows we present an example which confirms the practicability of the computation of the Conley index using Theorem 4.1, specially in the cases of larger structure matrices.

**Example 4.1.** Consider four rectangles $H_1$, $H_2$, $H_3$ and $H_4$ in $\mathbb{R}^2$ and the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ shown in Figure 5, where $f(H_1) = V_1$, $f(H_2) = V_2$, $f(H_3) = V_3$ and $f(H_4) = V_4$.

![Figure 5. fitted diffeomorphism](image)
A structure matrix of the basic set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U_1 \cup U_2 \cup U_3 \cup U_4)$ of index 1 is

$$A = \begin{pmatrix}
1 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}$$

when we take as a set of handles $H = H_1 \cup H_2 \cup H_3 \cup H_4$.

Since the real Jordan form of $A$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

we have by Theorem 4.1 that $\chi_1(\Lambda)$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and therefore,

$$\text{Con}_q(\Lambda) = \begin{cases}
(\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}), & q = 1 \\
(0, 0), & q \neq 1
\end{cases}$$

Moreover, since $A = \bigcap_{n \in \mathbb{Z}} f^n(H_1)$ and $R = \bigcap_{n \in \mathbb{Z}} f^n(H_2 \cup H_3 \cup H_4)$ form a Morse decomposition of $\Lambda$, $\chi_1(A) = \chi_1(R) = \text{Id}$ and $\chi_1(\Lambda)$ is not conjugated to the sum $\chi_1(A) \oplus \chi_1(R)$ we conclude by using Proposition 1 that there is a connecting orbit between $R$ and $A$.

Observe that if we were to compute the nonnilpotent part of the structure matrix in this example it would have been necessary many more calculations than the ones needed by using Theorem 4.1.

5. Conley index, homology zeta function and morse inequalities. In this section we present two basic tools using the Conley index, namely the homology zeta function and the Morse inequalities which are generally used as an obstruction to the existence of diffeomorphisms realizing an a priori collection of dynamical data (see [2]). Our motivation in this section is different since our aim is to show how to calculate the homology zeta function and Morse Inequalities using the Conley index presented in Subection 2.2. The example of a diffeomorphism on a torus at the end of this section illustrates this calculation.

**Proposition 2.** Let $M$ be a closed manifold. Suppose $f : M \to M$ is a diffeomorphism with hyperbolic chain recurrent set and $\{\Omega_i\}_{i=0}^n$ all the basic sets of $f$, then the homology zeta function of $f|_{\Omega_i}$ is

$$Z_i(f) = \prod_{k=0}^{\dim M} \det(I - \chi_k(\Omega_i)t)^{(-1)^{k+1}}$$

where $\text{Con}_k(\Omega_i) = (CH_k(\Omega_i), \chi_k(\Omega_i))$ is the Conley index with coefficients in $\mathbb{R}$ of the basic sets $\Omega_i$, $\forall i = 0, \ldots, l$.

**Proof.** Let $\{M_i\}$ be a filtration associated to $f$. We proved in Lemma 3.1 that

$$(f|_M^*)_{k} : H_k(M_i/M_{i-1}, [M_{i-1}]; \mathbb{R}) \to H_k(M_i/M_{i-1}, [M_{i-1}]; \mathbb{R})$$
and
\[ f_{sk} : H_k(M_i, M_{i-1}; \mathbb{R}) \to H_k(M_i, M_{i-1}; \mathbb{R}) \]
are conjugated by the isomorphism \( q_{sk} : H_k(M_i, M_{i-1}; \mathbb{R}) \to H_k(M_i/M_{i-1}, [M_{i-1}]; \mathbb{R}) \) induced by the quotient map \( q : M_i \to M_i/M_{i-1} \). Thus,
\[
\det (I - (f_P)_{sk}t) = \det (I - (q_{sk} \circ (f_{sk})^{-1})) t \\
= \det (I - (q_{sk} \circ (f_{sk} t) \circ q_{sk}^{-1})) t \\
= \det (q_{sk} \circ (I - f_{sk} t) \circ q_{sk}^{-1}) t \\
= \det (q_{sk}) \det (I - f_{sk} t) \det (q_{sk})^{-1} t \\
= \det (I - f_{sk} t)
\]

On the other hand, there exists a base \( H_k(M_i/M_{i-1}, [M_{i-1}]; \mathbb{R}) \) such that
\[
[(f_P)_{sk}]_\beta = \begin{pmatrix} (f_P)_{sk}^+ & 0 \\ 0 & (f_P)_{sk}^- \end{pmatrix}
\]
where \( (f_P)_{sk}^- \) is the nilpotent part of \( (f_P)_{sk} \). Hence,
\[
\det (I - (f_P)_{sk}) = \det (I - (f_P)_{sk}^+) \cdot \det (I - (f_P)_{sk}^-) = \det (I - (f_P)_{sk}^+)
\]
follows since \( (f_P)_{sk}^- \) being nilpotent implies that \( \det (I - (f_P)_{sk}^-) = 1 \).
Therefore,
\[
\det (I - \chi_k(\Omega_i)t) = \det (I - (f_P)_{sk}^+) = \det (I - (f_P)_{sk}) = \det (I - f_{sk} t).
\]

By using our Proposition 2 and Theorem 6.5 of [6], we obtain the following proposition.

**Proposition 3.** If \( f : M \to M \) has hyperbolic chain recurrent set and their basic sets are homologically split at \( q \) over \( \mathbb{R} \), then there exists an integer polynomial \( P(t) \) such that
\[
P(t)^{(-1)^q} \prod_{u(t) \subseteq q} Z_i(f) = \prod_{k=0}^{q} \det (I - \chi_k(M)t)^{(-1)^{k+1}}
\]
where
\[
Z_i(f) = \prod_{k=0}^{\dim M} \det (I - \chi_k(\Omega_i)t)^{(-1)^{k+1}}
\]
and \( \text{Con}_k(\Omega_i) = (CH_k(\Omega_i), \chi_k(\Omega_i)) \) is the Conley index with coefficients in \( \mathbb{R} \) of the basic set \( \Omega_i, \forall i = 0, \ldots, l \).

**Proof.** Since \( P = (M, \emptyset) \) is a filtration pair of \( M \), we have that \( f_{sk} : H_k(M; \mathbb{R}) \to H_k(M; \mathbb{R}) \) which is induced by \( f \) and \( (f_P)_{sk} : H_k(M, \emptyset; \mathbb{R}) \to H_k(M, \emptyset; \mathbb{R}) \) which is induced by \( f_P \) are isomorphic and therefore \( \det (I - f_{sk} t) = \det (I - \chi_k(M)t) \).

The construction of the following diffeomorphism of Example 5.1 can be found in [6].

**Example 5.1.** Consider the diffeomorphism \( f : T^2 \to T^2 \) illustrated in Figure 6.

The diffeomorphism \( f \) will have an attracting fixed point \( p \), a repelling fixed point \( \infty \) in the deleted disc \( D_\infty \) and is constructed in such a way that the two strips \( H = h_1 \cup h_2 \) form a set of hyperbolic handles of \( f \). Thus, \( \mathcal{R}(f) = \{p\} \cup \Lambda \cup \{\infty\} \), where \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H) \).
The structure matrices of $p$, $\Lambda$ and $\infty$ are respectively $A_0 = (1)$, $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and $A_2 = (1)$. Hence, by Theorem 3.2, the Conley indices of the basic sets of $f$ are

$Con_q(p) = \begin{cases} (\mathbb{R}, Id), & q = 0 \\ (0, 0), & q \neq 0 \end{cases}$, $Con_q(\infty) = \begin{cases} (\mathbb{R}, Id), & q = 2 \\ (0, 0), & q \neq 2 \end{cases}$
Con_q(Λ) = \begin{cases} 
\left( \mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right), & q = 1 \\
(0,0), & q \neq 1 
\end{cases}

Moreover, by Proposition 2, the homology zeta functions associated to the basic sets are

\[ Z_p(f) = \det (I - \chi_0(p)t)^{-1} \det (I - \chi_1(p)t) \det (I - \chi_2(p)t)^{-1} = (1 - t)^{-1} (1) \]

\[ Z_{\Lambda}(f) = \det (I - \chi_0(\Lambda)t)^{-1} \det (I - \chi_1(\Lambda)t) \det (I - \chi_2(\Lambda)t)^{-1} = (1) \det \left( \begin{array}{cc} 1 & -t \\ t & (1 - t) \end{array} \right) (1) = (1 - t + t^2) \]

\[ Z_{\infty}(f) = \det (I - \chi_0(\infty)t)^{-1} \det (I - \chi_1(\infty)t) \det (I - \chi_2(\infty)t)^{-1} = (1) (1 - t)^{-1} = (1 - t)^{-1} \]

Now, since

Con_q(T^2) = \begin{cases} 
(\mathbb{R}, Id), & q = 0 \\
\left( \mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right), & q = 1 \\
(\mathbb{R}, Id), & q = 2 \\
(0,0), & \text{otherwise} 
\end{cases}

and the basic sets of f are homologically split at 1 over \( \mathbb{R} \) we have that, by Proposition 3, there exists an integer polynomial \( P(t) \) such that

\[ P(t)^{-1} \prod_{u(i) \leq 1} Z_i(f) = \prod_{k=0}^{1} \det (I - \chi_k(M)t)^{(-1)^{k+1}} \]

Furthermore, we have that \( P(t) = 1 \), i.e.,

\[ \prod_{u(i) \leq 1} Z_i(f) = \prod_{k=0}^{1} \det (I - \chi_k(M)t)^{(-1)^{k+1}} \]

since

\[ \prod_{u(i) \leq 1} Z_i(f) = Z_0(f)Z_1(f) = (1 - t)^{-1} (t^2 - t + 1) \]

\[ \prod_{k=0}^{1} \det (I - \chi_k(M)t)^{(-1)^{k+1}} = \det (I - \chi_0(T^2)t)^{-1} \det (I - \chi_1(T^2)t) = (1 - t)^{-1} (t^2 - t + 1) \]

Of course the existence of this polynomial was expected since we started off with a diffeomorphism on a torus satisfying the hypothesis of Proposition 3. We also observe, in this example, that \( \chi_1(\Lambda) \) is not conjugate to \( \chi_1(p) \oplus \chi_1(\infty) \) and hence by Proposition 3 \( C(\infty, p) \neq \emptyset \).
6. Properties of connection matrix pairs. In this section we prove properties satisfied by connection matrix pairs that allows the detection of connecting orbits between Morse sets.

**Definition 6.1.** Let $A$ and $B$ be two fixed endomorphism. We say that two endomorphism $g$ and $f$ are related by $\sim_{AB}$, $g \sim_{AB} f$, if and only if,

$$
\begin{pmatrix}
A & 0 \\
g & B
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
A & 0 \\
f & B
\end{pmatrix}
$$

are conjugate.

The above is an equivalence relation. We denote the equivalence class $\sim_{AB}$ by $[g]$ where $g$ is an element of the class.

**Theorem 6.2.** Let $f : M \to M$ be a diffeomorphism and $(\Delta, a)$ a connection matrix pair of a Morse decomposition $M(\mathcal{P}, <)$ of an isolated invariant set $S$ of $f$ with partial order $<$. 

(a) If $<$ is the $f$-defined order and $\pi, \pi' \in \mathcal{P}$ are two elements that are not related by $<$, then $\Delta(\pi, \pi') = \Delta(\pi', \pi) = 0$ and $[a(\pi, \pi')] = [a(\pi', \pi)] = 0$.

(b) If $\pi < \pi'$ in $\mathcal{P}$ are adjacent elements such that the submatrix $\Delta(\pi')$ of $\Delta$ is null and the entry $[a(\pi', \pi)]$ of $a$ is non-zero, then $C(M_{\pi'}, M_{\pi}) \neq \emptyset$.

(c) Let $\pi < \pi'$ be two adjacent elements in $\mathcal{P}$ such that

$$
CH^k(M_\pi) \neq 0, \quad q = k \quad \text{and} \quad CH^q(M_{\pi'}) = \begin{cases}
0, & q = k \\
CH^q(M_{\pi'}) \neq 0, & q \neq k
\end{cases}
$$

If $\Delta(\pi', \pi) \neq 0$, then $a(\pi', \pi) = 0$.

**Proof.** (a) Since $\pi, \pi'$ are not related by the $f$-defined order, we have that $\Delta(\pi', \pi) = 0$, $CH^q(M_{\pi'}) = CH^q(M_\pi) \oplus CH^q(M_{\pi'})$ and $\chi^q(M_{\pi'})$ is conjugated to $\chi^q(M_\pi) \oplus \chi^q(M_{\pi'})$. Otherwise there would be connections between $M_\pi$ and $M_{\pi'}$ (see Proposition [1]).

Hence, since $\Delta(\pi', \pi) = 0$, we have that

$$
\begin{pmatrix}
\chi(M_\pi) & 0 \\
a(\pi', \pi) & \chi(M_{\pi'})
\end{pmatrix} = a(\pi, \pi') \equiv (a(\pi, \pi'))^* \sim \chi(M_{\pi'})
$$

and therefore $[a(\pi', \pi)] = 0$. Similarly we prove that $[a(\pi, \pi')] = 0$.

(b) We have, by hypothesis, that if $\Delta(\pi, \pi') = 0$, then $a(\pi, \pi') = (a(\pi, \pi'))^*$. Moreover, since $I = \{\pi, \pi'\}$ is an interval, then $(a(\pi, \pi'))^*$ is conjugate to $\chi(M_{\pi'})$ and consequently $a(\pi, \pi')$ is conjugate to $\chi(M_{\pi'})$.

On the other hand, since $(M_{\pi}, M_{\pi'})$ is an attractor-repeller decomposition for $M_{\pi, \pi'}$, then $a(\pi, \pi') = \begin{pmatrix}
\chi(M_\pi) & 0 \\
a(\pi', \pi) & \chi(M_{\pi'})
\end{pmatrix}$. Hence, by hypothesis, $[a(\pi', \pi)] \neq 0$, subsequently,

$$
\begin{pmatrix}
\chi(M_\pi) & 0 \\
a(\pi', \pi) & \chi(M_{\pi'})
\end{pmatrix}
$$

is not conjugate to

$$
\begin{pmatrix}
\chi(M_\pi) & 0 \\
0 & \chi(M_{\pi'})
\end{pmatrix}
$$

1 We say that $<$ is an $f$-defined order if: $\pi < \pi' \Leftrightarrow \exists \pi_0 = \pi, \pi_1, \ldots, \pi_n = \pi' \in \mathcal{P}$ such that $C(M_{\pi_{j-1}}, M_{\pi_j}) \neq \emptyset$ for all $j = 1, \ldots, n$.
which implies that \( \chi(M_{\pi'}) \) is not conjugate to \( \chi(M_\pi) \oplus \chi(M_{\pi'}) \), i.e., \( C(M_\pi', M_\pi) \neq \emptyset \) (Proposition 1).

(c) By hypothesis, since \( \Delta(\pi', \pi) \neq 0 \), then there is a \( q \) such that

\[
\Delta^q(\pi', \pi) : CH^q(M_\pi) \to CH^{q+1}(M_{\pi'})
\]

is non-zero. Thus, \( s = k + 1 \neq k \).

Hence,

\[
a^q(\pi', \pi) : CH^q(M_\pi) \to CH^q(M_{\pi'})
\]

and \( s = k + 1 \), then

\[
\begin{array}{ll}
CH^q(M_\pi) = 0, & \text{if } q \neq k \\
CH^q(M_{\pi'}) = 0, & \text{if } q = k
\end{array}
\]

which implies \( a(\pi', \pi) = 0 \).

Under the hypothesis of item (b) of Theorem 6.2 and Proposition 1 it is required that \( f \) be a diffeomorphism, however it is easy to see that continuity suffices.

It is interesting to see that in the case of two-manifolds, depending on the choice of the Morse decomposition, we obtain information only for \( \Delta \), as is shown in the following proposition.

**Proposition 4.** Let \( M \) be a compact two-manifold and \( f : M \to M \) be a diffeomorphism. Let \( \mathcal{M}(\mathcal{P}, <) = \{ M_i \mid i = 1, 2, 3 \} \) be a Morse decomposition of \( M \) such that \( M_i \) is the set of all basic sets of index \( i - 1 \) and \( < \) is an \( f \)-defined order. Suppose that

\[
Con^q(M_1) = \begin{cases} (CH^0(M_1), \chi^0(M_1)), & q = 0 \\ (0, 0), & q \neq 0 \end{cases}
\]

\[
Con^q(M_2) = \begin{cases} (CH^1(M_2), \chi^1(M_2)), & q = 1 \\ (0, 0), & q \neq 1 \end{cases}
\]

Con^q(M_3) = \begin{cases} (CH^2(M_3), \chi^2(M_3)), & q = 2 \\ (0, 0), & q \neq 2 \end{cases}

then the pair \( (\Delta, a) \) which follows is a connection matrix pair for the Morse decomposition \( \mathcal{M}(\mathcal{P}, <) \)

\[
\Delta^q = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ \delta^0(2, 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta^1(3, 2) & 0 \end{pmatrix}, & q = 0 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \chi^1(M_3) & 0 \end{pmatrix}, & q = 1 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & q = 2 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & q = 2 \end{cases},
\]

\[
a^q = \begin{cases} \begin{pmatrix} \chi^0(M_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & q = 0 \\ \begin{pmatrix} \chi^1(M_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, & q = 1 \\ \begin{pmatrix} \chi^2(M_3) \\ 0 \end{pmatrix}, & q = 2 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & q = 2 \end{cases}.
\]

Proof. A connection matrix pair for a Morse decomposition \( \mathcal{M}(\mathcal{P}, <) \) has the following form

\[
\Delta = \begin{pmatrix} 0 & 0 & 0 \\ \Delta(2, 1) & 0 & 0 \\ \Delta(3, 1) & \Delta(3, 2) & 0 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a(1, 1) & 0 & 0 \\ a(2, 1) & a(2, 2) & 0 \\ a(3, 1) & a(3, 2) & a(3, 3) \end{pmatrix}
\]

It is easy to see that \( \Delta(3, 1) = 0, a(2, 1) = a(3, 1) = a(3, 2) = 0 \).

We have that \( \Delta^0(2, 1) \) and \( \Delta^1(3, 2) \) are possible non-zero entries of \( \Delta^0 \) and \( \Delta^1 \).

In fact,

\[
\Delta^0(2, 1) : CH^0(M_1) \to CH^1(M_2)
\]
and if \( q \neq 0 \), then \( CH^0(M_1) = 0 \Rightarrow \Delta^0(2, 1) \equiv 0 \). If \( q = 0 \), then \( CH^0(M_1) \) and \( CH^1(M_2) \) can be non-zero. Thus, \( \Delta^0(2, 1) \equiv \delta^0(2, 1) \) and
\[
\Delta^0(3, 2) : CH^0(M_2) \to CH^{q+1}(M_3)
\]
and if \( q \neq 1 \), then \( CH^q(M_2) = 0 \Rightarrow \Delta^0(3, 2) \equiv 0 \). If \( q = 1 \), then \( CH^1(M_2) \) and \( CH^2(M_3) \) can be non-zero. Hence, \( \Delta^1(3, 2) \equiv \delta^1(3, 2) \).

Furthermore, for each \( k = 0, 1, 2 \), we have that \( a^q(k+1, k+1) = 0 \) if \( q \neq k \) and \( a^k(k+1, k+1) \) is conjugate to \( \chi^k(M_{k+1}) \). Indeed, we have that \( a^q(k+1, k+1) : CH^q(M_{k+1}) \to CH^q(M_{k+1}) \), and if \( q \neq k \), then \( CH^q(M_{k+1}) = 0 \Rightarrow a^q(k+1, k+1) \equiv 0 \). Since \( a^k(k+1, k+1) : CH^k(M_{k+1}) \to CH^k(M_{k+1}) \) and \( \Delta(k+1) \equiv 0 \), then \( a^k(k+1, k+1) \equiv (a^k(k+1, k+1))^* \sim \chi^k(M_{k+1}) \).

Note that in order to detect connections between Morse sets of the same index, i.e., connections of degree zero, we should consider a coarser Morse decomposition.

In [12], a uniqueness theorem for connection matrices associated to a Morse-Smale flow without periodic orbits is presented. In what follows we prove a similar result for the connection matrix pair of a Morse-Smale diffeomorphisms without periodic orbits.

**Theorem 6.3.** Let \( M \) be a smooth manifold and \( f : M \to M \) be a gradient-like Morse-Smale diffeomorphism\(^2\) without periodic orbits. Let \( S \) be an isolated invariant set of \( f \) which has a Morse decomposition \( (P, <) \) with \(< \) an \( f \)-defined order, where each Morse set \( M_\pi \) is a basic set of \( f \). Then the connection matrix pair \( (\Delta, a) \), over a field \( \mathbb{F} \), for \( (P, <) \) is unique in the equivalence class of each entry \( a(\pi, \pi') \) of the matrix \( a \) given by the relation
\[
\sim_{\chi(M_\pi), \chi(M_{\pi'})}
\]
presented in Definition 6.1.

**Proof.** Denote by \( W^u(\pi) \) and \( W^s(\pi) \) the unstable and stable manifolds of \( M_\pi \), respectively.

Let \( \pi' < \pi \) in \( P \) and \(< \) an \( f \)-defined order. Note that if \( M_\pi \) is a basic set of \( f \), i.e., is a hyperbolic fixed point, then
\[
CH^q(M_\pi) = \begin{cases} \mathbb{F}, & q = k \\ 0, & q \neq k \end{cases}
\]
for some \( k \).

**Case 1:** Suppose that \( \pi, \pi' \) are adjacent elements.

The entry \( \Delta(\pi, \pi') \) of \( \Delta \) is determined uniquely by the reduced cohomological Conley index sequence associated to the attractor-repeller pair \((M_{\pi'}, M_\pi)\)
\[
\cdots \delta^* \to Con^k(M_{\pi'}) \to Con^k(M_{\pi''}) \to Con^k(M_{\pi'}) \delta^* \to Con^{k+1}(M_\pi) \to \cdots
\]
If \( \Delta^q(\pi, \pi') \neq 0 \), by item (c) of Theorem 6.2 then \( a(\pi, \pi') = 0 \). Now, if \( \Delta(\pi, \pi') = 0 \), then \( \Delta(\pi'\pi) \equiv 0 \) and so \( (a(\pi'\pi))^* \equiv a(\pi'\pi) \). Hence,
\[
a(\pi'\pi) = \begin{pmatrix} \chi(M_{\pi'}) \\ a(\pi, \pi') \\ a(\pi'\pi) \end{pmatrix} \sim \chi(M_{\pi'\pi})
\]
and \( \chi(M_{\pi'\pi}) \) is determined uniquely by the reduced cohomological Conley index sequence associated to the attractor-repeller pair \((M_{\pi'}, M_\pi)\)
\[
\cdots \delta^* \to Con^k(M_{\pi'}) \to Con^k(M_{\pi''}) \to Con^k(M_{\pi'}) \delta^* \to Con^{k+1}(M_\pi) \to \cdots
\]
Therefore, \([a(\pi, \pi')]\) is determined uniquely by \( \chi(M_{\pi'\pi}) \).

---

\(^2\)A Morse-Smale diffeomorphism \( f \) is said to be **gradient-like** if for all periodic points \( x \neq y \) we have that: \( W^u(x) \cap W^s(y) \neq \emptyset \Rightarrow \dim(W^u(x)) > \dim(W^u(y)) \).
Case 2: Suppose that $\pi, \pi'$ are not adjacent elements.

Reineck, in [12], showed that if $\pi, \pi'$ are not adjacent elements, then $\Delta(\pi, \pi') = 0$. We will show that, in this case, $a(\pi, \pi') = 0$.

Since $\pi, \pi'$ are not adjacent elements, then there is an element $\pi'' \in P$ such that $\pi' < \pi'' < \pi$. Let $i, j$ and $k$ be such that $CH^i(M_{\pi''}) = F$, $CH^j(M_{\pi''}) = F$ and $CH^k(M_{\pi}) = F$.

So $\pi' < \pi''$, $W^s(\pi')$ and $W^u(\pi'')$ intersect transversally, and we have that $i < j$. Likewise, we have that $j < k$.

Hence, $i \leq j - 1 \leq (k - 1) - 1 = k - 2$ and consequently $i \neq k$. Therefore,

$$o^q(\pi', \pi) : CH^q(M_{\pi'}) \to CH^q(M_{\pi})$$

is null for all $q$.

\[ \square \]

7. Connection matrix pairs of Time-one maps of flows. The aim of this section is to present results on connection matrix pairs in the special case of diffeomorphisms which are time-one maps of flows. We prove in Theorem 7.1 that the connection matrix $\Delta$ of the connection matrix pair $(\Delta, a)$ will be the same as the connection matrix of the flow. Furthermore, under certain conditions we have that the off diagonal entries of $a$ are zero. In the case of time-one maps of Morse-Smale flows, these off diagonal entries of $a$ are always zero as is shown in Proposition 6.

**Theorem 7.1.** Let $\varphi$ be a flow on a locally compact metric space $M^n$, we have that $\varphi_1 : M^n \to M^n$ is a diffeomorphism. Let $S$ be an isolated invariant set and $\mathcal{M}(P, <) = \{M_\pi | \pi \in P\}$ a Morse decomposition of $S$ and $(\Delta, a)$ a connection matrix pair for $\mathcal{M}(P, <)$.

(a) If $<\pi$ is the flow order, we have that $\Delta$ is a connection matrix, in the continuous setting, for $\mathcal{M}(P, <)$. 

(b) If $\pi < \pi'$ are two adjacent elements in $P$ such that $\Delta(\pi', \pi) = 0$, then $a(\pi', \pi) = 0$.

**Proof.** (a) Let $I \in \mathcal{I}$. Since $\varphi_1$ is a time-one map of flow $\varphi$, then $\text{Con}^k(M_I) = (CH^k(M_I), \chi^k(M_I)) = (H^k(M_I), Id)$ (see [11]).

As $(\Delta, a)$ is a connection matrix pair for $\mathcal{M}(P, <)$, thus the

$$\mathcal{H} (\mathcal{M}) = \{ \text{Con}^k(M_I) = (H^k(M_I), Id) | I \in \mathcal{I} \}$$

is isomorphic to graded module braid with endomorphism

$$\mathcal{H} \Delta (\mathcal{M}) = \{(H^k(C\Delta(I)), (a^k(I))^*) | I \in \mathcal{I} \}.$$ 

Hence, we have that the graded module braids with endomorphism as in the continuous case,

$$\mathcal{H} (\mathcal{M}_{\text{continuous}}) = \{ \text{Con}^k(M_I) = H^k(M_I) | I \in \mathcal{I} \}$$

and

$$\mathcal{H} \Delta (\mathcal{M}_{\text{continuous}}) = \{ H^k(C\Delta(I)) | I \in \mathcal{I} \}.$$ 

are also isomorphic. Therefore, $\Delta$ is a connection matrix, in continuous sense, for the Morse decomposition $\mathcal{M}(P, <)$ of $S$.

(b) Let $\pi < \pi'$ be two adjacent elements in $P$ such that $\Delta(\pi', \pi) = 0$, thus

$$a(\pi'\pi') \equiv (a(\pi'\pi'))^* \sim \chi(M_{\pi'}) = 0 \text{ or } Id$$

(see [11]). Therefore, $a(\pi', \pi) = 0$.  \[ \square \]
We would like to know what happens when the hypothesis of Theorem 7.1 changes from $\Delta(\pi', \pi) = 0$ to $\Delta(\pi', \pi) \neq 0$. We do not know what happens to the entry $a(\pi', \pi)$ of the matrix $a$, since in this case we do not have the identification $a(\pi' \pi') \equiv (a(\pi' \pi'))^*$. In the case of Morse-Smale flows, the connection matrix $a$ will not give any information on connecting orbits since in this case there do not exist zero degree connections.

**Proposition 5.** Let $f : M \to M$ be a time-one map of a Morse-Smale flow with $k$ basic sets. Consider the Morse decomposition of $M$, $\mathcal{M}(\mathcal{P}, <)$, where each Morse set is a basic set of $f$. Let $(\Delta, a)$ be a connection matrix pair for $\mathcal{M}(\mathcal{P}, <)$. Then the entries $a(i, j)$ of $a$ with $i \neq j$ are zeros, i.e., the graded matrix $a$ has the following form:

$$a = \begin{pmatrix} a^1(1, 1) & 0 & \cdots & 0 \\ 0 & a^2(2, 2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k(k, k) \end{pmatrix}$$

Moreover, each entry $a^q(i, i)$ is conjugate to the identity if $q$ equals the index of the basic set $M_i$ and conjugate to zero otherwise.

8. **Connection matrix pair for zero-dimensional basic sets decomposition.** In this section we prove Theorem 8.2 which characterizes the connection matrix pair for diffeomorphisms with zero-dimensional hyperbolic chain recurrent under an index duality hypothesis which requires that $\text{Con}_q(\Lambda) = \text{Con}^q(\Lambda)$. We prove in Corollary 1 that this condition is satisfied for basic sets of Morse-Smale diffeomorphisms and for $\Lambda(k) = \bigcap_{n \in \mathbb{Z}} f^n(H(k))$ of fitted diffeomorphisms with respect to a handle set $\bigcup_k H(k)$.

**Definition 8.1.** Let $M$ be a compact manifold and $f : M \to M$ be a diffeomorphism. For each $0 \leq k \leq n$ let

$$\Lambda_{k,1}, \ldots, \Lambda_{k,s_k}$$

all basic set of index $k$ of $f$.

Let $M_{i,j} = \Lambda_{i,j}$ with $0 \leq i \leq n$ and $1 \leq j \leq s_i$, $\mathcal{P} = \{(i, j) \mid 0 \leq i \leq n \ e \ 1 \leq j \leq s_i\}$ and $<$ an $f$-defined order, we have that $\mathcal{M}(\mathcal{P}, <)$ is a Morse decomposition for $M$ which we refer to a Morse decomposition of basic sets ordered by index.

Note that $(i, j) < (i', j')$ whenever $i < i'$, where $<'$ is a total order $0 <' 1 <' \cdots <' n$.

In the following Theorem the characterize a pair of connection matrices for a Morse decomposition of zero-dimensional basic sets.

**Theorem 8.2.** Let $M$ be closed manifold and $f : M \to M$ be a diffeomorphism with hyperbolic chain recurrent set. Suppose that for each basic set $\Lambda$ of $f$ we have that $\Lambda$ is zero-dimensional and $\text{Con}_q(\Lambda) = \text{Con}^q(\Lambda), \forall q$. Let $\mathcal{M}(\mathcal{P}, <)$ be the Morse decomposition of basic sets ordered by their index. Then a connection matrix pair $(\Delta, a)$ for $\mathcal{M}(\mathcal{P}, <)$ has the following form as in Figure 7, where for each $i = 1, \ldots, s_q$ we have that $A_i^+$ is the nonnilpotent part of the structure matrix $A_i$ associated to the basic set $\Lambda_{q,i}$. 
The entries of the highlighted square are maps of $\text{CH}^q(\Omega_{q,i})$ to $\text{CH}^{q+1}(\Omega_{(q+1),j})$ for some $i \in \{1, \ldots, s_q\}$ and some $j \in \{1, \ldots, s_{(q+1)}\}$.

Figure 7. Connection matrix pair for zero-dimensional basic sets decomposition
Proof. By Theorem 3.2 we have

\[ \text{Con}^q(M_\pi) = 0, \forall q \neq u(\pi) \]

where \( u(\pi) \) is the index of the basic set \( M_\pi \).

We will compute the entries of \( \Delta \),

\[ \Delta^q(\pi', \pi) : CH^q(M_\pi) \to CH^q(M_{\pi'}) \]

Let \( \pi, \pi' \in \mathcal{P} \) with \( \pi < \pi' \), then we have one of the following cases:

Case 1: If \( M_\pi \) has index \( u \) and \( M_{\pi'} \) has index \( u + r \) with \( r \neq 1 \), then \( \Delta^q(\pi', \pi) = 0 \).

Indeed, if \( q \neq u \), then \( CH^q(M_\pi) = 0 \) which implies that \( \Delta^q(\pi', \pi) = 0 \). On the other hand, if \( q = u \), then \( CH^{u+1}(M_{\pi'}) = 0 \), since \( u + 1 \neq u + r \), consequently \( \Delta^u(\pi', \pi) = 0 \).

Case 2: If \( M_\pi \) has index \( u \) and \( M_{\pi'} \) has index \( u + 1 \), then

\[ \Delta(\pi, \pi') = \begin{pmatrix} 0 & 0 \\ \Delta(\pi', \pi) & 0 \end{pmatrix} \]

and in this case \( \Delta(\pi', \pi) \neq 0 \) or \( \Delta(\pi', \pi) = 0 \).

We will compute the entries of \( a^q(\pi', \pi) : CH^q(M_\pi) \to CH^q(M_{\pi'}) \)

Let \( \pi, \pi' \in \mathcal{P} \) with \( \pi < \pi' \), we have one of the following cases:

Case 1: If \( M_\pi \) has index \( u \) and \( M_{\pi'} \) has index \( s \) with \( u \neq s \), then \( a(\pi', \pi) = 0 \).

In fact, if \( q \neq u \), then \( CH^q(M_\pi) = 0 \) which implies that \( a^q(\pi', \pi) = 0 \). Conversely, if \( q = u \), then \( CH^u(M_\pi) = 0 \). Since \( u \neq s \), then \( a^u(\pi', \pi) = 0 \).

Case 2: If \( M_\pi \) and \( M_{\pi'} \) have the same index \( u \), then \( a^q(\pi', \pi) = 0, \forall q \neq u \) and

\[ a^u(\pi \pi') \sim \begin{pmatrix} A(M_{\pi})^+ & 0 \\ a^u(\pi', \pi) & A(M_{\pi'})^+ \end{pmatrix} \]

where \( A(M_{\pi})^+ \) is the nonnilpotent part of the structure matrix \( A(M_\pi) \) associated to the basic set \( M_\pi \) and \( A(M_{\pi'})^+ \) is the nonnilpotent part of the structure matrix \( A(M_{\pi'}) \) associated to the basic set \( M_{\pi'} \).

Indeed, if \( q \neq u \), then \( CH^q(M_{\pi'}) = CH^q(M_{\pi'}) = 0 \) which implies that \( a^q(\pi', \pi) = 0 \). Conversely, if \( q = u \), then \( a^u(\pi', \pi) = 0 \) or \( a^u(\pi', \pi) \neq 0 \). By Theorem 3.2 \( \chi^u(M_{\pi'}) \sim A(M_{\pi'})^+ \) and \( \chi^u(M_{\pi'}) \sim A(M_{\pi'})^+ \). On the other hand, since \( \Delta(\pi) = \Delta(\pi') = 0 \), and so \( a^u(\pi) \equiv (a^u(\pi))^* \sim \chi^u(M_{\pi'}) \sim A(M_{\pi'})^+ \) and \( a^u(\pi') \equiv (a^u(\pi'))^* \sim \chi^u(M_{\pi'}) \sim A(M_{\pi'})^+ \).

Thus,

\[ a^u(\pi \pi') \sim \begin{pmatrix} A(M_{\pi})^+ & 0 \\ a^u(\pi', \pi) & A(M_{\pi'})^+ \end{pmatrix} \]

\[ \square \]

Lemma 8.3. Let \( S \) be an isolated invariant set of a diffeomorphism \( f \). If there is a filtration pair \( P = (N, L) \) of \( S \) such that the pointed space \( N_L \) has the same homotopy type of \( S^{k_1} \vee \cdots \vee S^{k_v} \), then

\[ \text{Con}_q(S) = \text{Con}^q(S), \forall q. \]

Proof. Since \( N_L \cong S^{k_1} \vee \cdots \vee S^{k_v} \), we have \( H_* (N_L) = H^*(N_L) = (f)_* \sim (f)^* \).

So, \( gKer((f)^*) = gKer((f)_*) \) which implies that

\[ CH^*(N_L) = \frac{H^*(N_L)}{gKer((f)^*)} = \frac{H_*(N_L)}{gKer((f)_*)} = CH_*(N_L) \]

and therefore \( \text{Con}_q(S) = \text{Con}^q(S), \forall q. \)

\[ \square \]
Corollary 1. Let $M$ be an $n$-dimensional closed manifold and $f : M \to M$ be a diffeomorphism.

(a) If $f$ is Morse-Smale, then for all basic sets $\Omega$ of $f$ we have that $\text{Con}_q(\Omega) = \text{Con}^q(\Omega)$, $\forall q$.

(b) If $f$ is fitted with respect to $\bigcup_{k} H(k)$, then, for all $k$, $\text{Con}_q(\Lambda(k)) = \text{Con}^q(\Lambda(k))$, $\forall q$, where $\Lambda(k) = \bigcap_{n \in \mathbb{Z}} f^n(H(k))$.

Proof. Let $\Omega$ be a periodic orbit $\{p_1, \ldots, p_s\}$ of period $s$ or be a set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H(k))$ with $H(k) = \bigcup_{i=1}^{s} h_i$. If $k = \dim W^u(p)$, we have that

$$N = (D^k_1 \times D^{n-k}_1) \cup \cdots \cup (D^k_s \times D^{n-k}_s)$$

and

$$L = (D^k_1 \times \partial D^{n-k}_1) \cup \cdots \cup (D^k_s \times \partial D^{n-k}_s)$$

is a filtration pair for $\Omega$. Thus, $N_L \simeq \bigvee_s S^k$. Therefore, by Lemma 8.3 the desired result follows. 

Corollary 2. Under the hypothesis of Theorem 8.2, the connection matrix pair $(\Delta, a)$ for $\mathcal{M}(\mathcal{P}, \prec)$ can be written in the following form as shown in Figure 8, where for each $0 \leq j \leq n$ and each $i = 1, \ldots, s_j$ we have that $A^{i+}_j$ is the nonnilpotent part of the structure matrix $A^j_i$ associated to the basic set $\Lambda_{j,i}$.

![Figure 8](image_url)

**Figure 8.** Representation of a connection matrix pair for a zero-dimensional basic set decomposition

The connection matrix pair presented in Corollary 2 where the collection of matrices of $\Delta$ form a block auxiliary diagonal below the main block diagonal containing the blocks of the collection of matrices of $a$ is very convenient to work with.
We foresee that this will be useful for algebraic calculations, for example to explore continuation properties.

**Corollary 3.** Let $f : M \to M$ be a diffeomorphism on a closed manifold with Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ of an isolated invariant set $S$, then there is a fitted diffeomorphism $\tilde{f}$ on $M$, $C^0$-close to $f$, and a Morse decomposition $\tilde{\mathcal{M}}(\mathcal{P}, <)$ of an isolated invariant set $\tilde{S}$ of $\tilde{f}$ such that

$$\mathcal{C}M(\mathcal{M}(\mathcal{P}, <)) = \mathcal{C}M(\tilde{\mathcal{M}}(\mathcal{P}, <)).$$

**Example 8.1.** Consider five rectangles $H_1, H_2, H_3, H_4$ and $H_5$ in $\mathbb{R}^2$ and the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ shown in Figure 9 where $f(H_1) = V_1$, $f(H_2) = V_2$, $f(H_3) = V_3$, $f(H_4) = V_4$ and $f(H_5) = V_5$.

Thus, if $\mathcal{P} = \{1, 2, 3\}$, $1 < 3$ and $2 < 3$ then

$$\mathcal{M}(\mathcal{P}, <) = \left\{ M_1 = \bigcap_{n \in \mathbb{Z}} Q_1, M_2 = \bigcap_{n \in \mathbb{Z}} Q_2, M_3 = \bigcap_{n \in \mathbb{Z}} Q_3 \right\}$$
is a Morse decomposition for the invariant isolated set $S = \bigcap_{n \in \mathbb{Z}} (Q_1 \cup Q_2 \cup Q_3)$.

The Conley index of the sets $M_1$, $M_2$, $M_3$, $M_{13}$, $M_{12}$ and $M_{23}$ are

$$\text{Con}^q(M_1) = \text{Con}^q(M_2) = \text{Con}^q(M_3) = \begin{cases} (\mathbb{Q}, \text{Id}), & q = 1 \\ (0, 0), & q \neq 1 \end{cases}$$

$$\text{Con}^q(M_{13}) = \text{Con}^q(M_{23}) = \begin{cases} (\mathbb{Q} \oplus \mathbb{Q}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}), & q = 1 \\ (0, 0), & q \neq 1 \end{cases}$$

$$\text{Con}^q(M_{12}) = \begin{cases} (\mathbb{Q} \oplus \mathbb{Q}, \text{Id}_{2 \times 2}), & q = 1 \\ (0, 0), & q \neq 1 \end{cases}$$

Therefore, by using Theorem 8.2, the pair of graded matrices $(\Delta, a)$

$$\Delta^q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \forall q$$

$$a^q = \begin{cases} \begin{pmatrix} \text{Id} & 0 & 0 \\ 0 & \text{Id} & 0 \\ \text{Id} & \text{Id} & \text{Id} \end{pmatrix}, & q = 1 \\ 0, & q \neq 1 \end{cases}$$

is a connection matrix pair for the Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ of $S$.

Observe that if we only have the information on the pair of connection matrices of this system it is possible, by using Theorem 8.2, to conclude the existence of a connecting orbit between $M_3$ and $M_1$ once it is known that $\Delta(1, 3) \equiv 0$ and $[a(3, 1)] \neq 0$. Analogously, we conclude the existence of a connecting orbit between $M_3$ and $M_2$. Note that these connections are degree zero connections detected by the matrix $a$ in the matrix connection pair while the matrix $\Delta$ being null provides no information on the existence of connections. This example illustrates the sharp difference between the connection matrix theory in the discrete dynamical setting and in the continuous dynamical setting. In the latter case, the connection matrix would not have detected connections. A consequence of Corollary 8.3 is that this connection is still detected by the pair of connection matrices after the system undergoes a $C^0$ perturbation.

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