Random walk on $p$–adics in glassy systems

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We show that $p$–adic analysis provides a quite natural basis for the description of relaxation in hierarchical glassy systems. For our purposes, we specify the Markov stochastic process considered by S. Albeverio and W. Karwowski. As a result we have obtained a random walk on $p$–adic integer numbers, which provide the generalization of Cayley tree proposed by Ogielski and Stein. The temperature-dependent power-law decay and the Kohlrausch law are derived.

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I. INTRODUCTION

The growing interest in spin glassy dynamics has stimulated intense research in experimental [1–4] as well as theoretical domain [5–17]. The theoretical studies have been successfully developed in the following two directions:

(i) scaling theory for growing domains and droplets [5–7]
(ii) the hierarchical structures in ultrametric spaces for relaxation dynamics [8–17].

The goal of this paper is to investigate the second item (ii), using $p$–adic analysis. The temperature cycle experiments [1–4] are usually interpreted by the existence of a continuous hierarchy in spin glassy dynamics. The hierarchical organization of metastable states depends on the temperature. It is assumed that at given temperature $T$ the system occupies restricted part of the phase space inside one of the free energy valleys. After cooling down to temperature $T - \Delta T$, where $\Delta T$ is rather small in the range $1^\circ K - 2^\circ K$, such a part of phase space is divided into several smaller descendent valleys separated by energy barriers. If the temperature is lowered the system has its new landscape of valleys but the information about the previous states is not lost (memory effect). So, after heating the system to temperature $T$ all descend states merge together into much smaller numbers of their ancestor states. The system in temperature $T$ continues the relaxation process. After heating the system to critical temperature $T_c$ all occupied states merge into unique common ancestor state and phase space looks as in paramagnetic phase.

From temperature cycling experiments it can also be seen, that the relaxation process at temperature $T$ during time $t$ is practically the same as the process at temperature $T' = T - \Delta T$ during some shorter time $t' < t$. The free energy barriers separating valleys at temperature $T'$ are higher than those at the temperature $T$ and corresponding amount of time $t$ needed to occupy the respective part of the phase space at temperature $T$ is smaller than in the case of temperature $T'$. Hierarchical structure of metastable states corresponds to such structure expressed in terms of pure states in the sense of Parisi solution [17]. If we lower the temperature the energy barriers separating states are becoming higher, even infinite.

The dominant process in the relaxation dynamics has been described as the hopping between the physical states [4]. Such mechanism leads to the studies of the models with a hierarchy-constrained dynamics with ultrametric topology. These models did appear very useful for the description of complex systems, called glassy systems, with highly degenerate metastable states.

In this paper we consider dynamics for glassy systems using a random walk on $p$–adic numbers. We specify, to our purposes, the Markov stochastic process, considered by S. Albeverio and W. Karwowski [18, 19], for the description of dynamics in ultrametric spaces. We use the space of $p$–adic integers, which for our purposes is an appropriate example of ultrametric space. We should notice that the $p$–adic integers may be represented by the bottom of the regular, infinite Cayley tree. Let us recall that in Cayley tree of order $p$ every branch at each level splits into $p$ other branches. For the illustration we present on Fig. 1 the Cayley tree with $p = 2$ from which we obtain, after infinity
number of splitting, the 2-adic integers as describing the bottom of the tree.
We can introduce a random walk in p-adic integers $\mathbb{Z}_p$ analogously as a random walk in a ultrametric space represented by leaves of regular, finite Cayley tree. The pure physical states of the system are represented by these leaves of infinite Cayley tree. In order to construct the Markov process on p-adic integers we firstly consider Markov process on p-adic balls with the same finite radius and finally we contract the ball radius to zero. In that way we obtain transition probabilities in the space of p-adic integers which correspond to the ones considered by Ogielski and Stein [9], in a case of finite ultrametric space.

The plan of the paper is the following. In Section 2 we introduce basic mathematical notions and the properties of p-adic integers. We describe as well a Markov process on p-adic integers. In Section 3 we pass to the physical application. We consider the thermally activated random walk on ultrametric space described as a Markov process on the space of p-adic balls. The temperature-dependent power-law decay and the Kohlrausch law are derived. Let us notice finally that recently one observes an increasing interest in the application of p-adic numbers in mathematical physics [20]. The p-adic analysis was used to study stochastic processes [20, 21], especially to ultrametric jump diffusion [22–24].

II. CONSTRUCTION OF MARKOV PROCESS ON THE p-ADIC INTEGERS.

Before we consider a Markov process with p-adic integers as a state space let us first introduce some notation and basic properties of p-adic numbers. More details may be found, for example in [25] or [20].

A. p-adic integers

Let $p$ be an arbitrary prime number and let $\mathbb{Z}_p$ denote the set of p-adic integers. A p-adic integer is a formal series $\sum_{i \geq 0} a_i p^i$ with coefficients $a_i$ satisfying $0 \leq a_i \leq p - 1$. With this definition, a p-adic integer $a = \sum_{i \geq 0} a_i p^i$ can be identified with the sequence $(a_i)_{i \geq 0}$ of its coefficients.

In order to introduce a distance between p-adic integers $a$ and $b$ let us first consider an order of a p-adic integer. The order of a p-adic integers $a = (a_i)_{i \geq 0}$ is the smallest $m$ for which $a_m \neq 0$

$$ord_p(a) = \min\{i : a_i \neq 0\},$$

(1)
with the convention that maximum of the empty set is equal infinity. Notice that if \(x, y\) are \(p\)-adic integers then

\[
\text{ord}_p(xy) = \text{ord}_p(x) + \text{ord}_p(y),
\]

\[
\text{ord}_p(x + y) \geq \min(\text{ord}_p(x), \text{ord}_p(y)).
\]

Now, in term of the function \(\text{ord}_p(\cdot)\), we may introduce in the space of \(p\)-adic integers the norm \(||\cdot||_p\)

\[
||x||_p = p^{-\text{ord}_p(x)},
\]

and the \(p\)-adic metrics

\[
d_p(x, y) = ||x - y||_p = p^{-\text{ord}_p(x - y)}.
\]

It is easy to check that the function \(d_p\) is a metric, and moreover due to inequality (2b), fulfills the following ultametric condition

\[
d_p(x, y) \leq \max(d_p(x, z), d_p(z, y)).
\]

For \(M \geq 0\) and \(a \in \mathbb{Z}_p\) we may define a closed \(p\)-adic ball \(B(a, M)\) with center \(a\) and radius \(p^{-M}\)

\[
B(a, M) = \{ x \in \mathbb{Z}_p : d_p(x, a) \leq p^{-M} \}.
\]

If centre of the ball \(a\) has \(p\)-adic representation \(a = \sum_{j=0}^{\infty} a_j p^{-j}\), then the ball \(B(a, M)\) is completely determined by

\[
\{a\}_M = a_M, a_{M-1}, \ldots, a_0.
\]

Let us recall some basic properties of the \(p\)-adic balls:

(i) for any \(p\)-adic balls \(B(a, M)\) and \(B(b, M)\) we have

\[
B(a, M) \cap B(b, M) = \emptyset \text{ or } B(a, M) = B(b, M),
\]

which means that any \(p\)-adic balls are disjoint, or one is enclosed in another.

(ii) every point of the ball is the centre of this ball

\[
\text{if } x \in B(a, M) \text{ then } B(x, M) = B(a, M).
\]

(iii) each ball \(B(a, M)\) of radius \(p^{-M}\) may be represented as a finite union of disjoint balls \(B(a_i, M+1)\) of radius \(p^{-(M+1)}\)

\[
B(a, M) = \bigcup_{i=0}^{p^{-1}} B(a_i, M + 1),
\]

(iv) (iii) implies that for disjoint balls \(B(a_i, M)\) of radius \(p^{-M}\)

\[
\mathbb{Z}_p = \bigcup_{i=0}^{p^{M-1}} B(a_i, M).
\]

Let us finally define the distance between two arbitrary balls \(B_1\) and \(B_2\) as

\[
d_p(B_1, B_2) = \inf \{d_p(x, y) : x \in B_1, y \in B_2 \}.
\]

With such definition two balls with representations \(\{a\}_M \equiv a_M, a_{M-1}, \ldots, a_k, a_{k-1}, \ldots, a_0\) and \(\{b\}_M \equiv b_M, b_{M-1}, \ldots, b_k, a_{k-1}, \ldots, a_0\), differ for the first time at \(k\)-th position, we have

\[
d_p(\{a\}_M, \{b\}_M) = p^{-k}.
\]
B. Random walk on $p$–adic balls

Let $\{X(t), t \geq 0\}$ be a Markov process with the state space $Z_p$ and transition rates between states depending only on $p$–adic distance between these states. To define such a process we follow general construction given in [19]. In order to do this we represent $Z_p$ as a finite union of disconnected balls $B_i^M$ with the radius $p^{-M}$

$$Z_p = \bigcup_{i=0}^{p^{M-1}} B_i^M$$ (14)

For a fixed $M$ we may consider a finite state space Markov process $\{X_M(t), t \geq 0\}$ with the state space $E_M = \{B_0^M, B_1^M, \ldots, B_{p^M-1}^M\}$, set of $p$–adic balls with the radius $p^{-M}$, and transition probabilities from ball $B_i^M$ at time 0 to ball $B_j^M$ at time $t$, $P_{i,j}^{(M)}(t)$, defined as

$$P_{i,j}^{(M)}(t) = P \left( X_M(t) = B_j^M | X_M(0) = B_i^M \right)$$

$$= P \left( X(t) \in B_j^M | X(0) \in B_i^M \right).$$ (15)

Transition probabilities $P_{i,j}^{(M)}(t)$, $(i, j = 0, 1, \ldots, p^M - 1)$) are solutions of the system of Kolmogorov equations

$$\frac{d}{dt} P_{i,j}^{(M)}(t) = -q_{ij}^{(M)} P_{i,j}^{(M)}(t) + \sum_{0 \leq k \leq p^M-1, k \neq j} q_{ij,k}^{(M)} P_{k,j}^{(M)}(t),$$ (16)

with the initial condition $P_{i,j}(0) = \delta_{ij}$, where $q_{ij}^{(M)}$ the infinitesimal transition probability and $q_{ij,k}^{(M)}$ the intensity of stay in state $i$ of the process are defined for any pair $i, j$ of different states represented by balls $B_i^M$, $B_j^M$ with radius $p^{-M}$, by

$$q_{ij}^{(M)} = \lim_{h \to 0} \frac{P_{i,j}^{(M)}(h)}{h},$$ (17)

and for any state $i$, by

$$q_i^{(M)} = \lim_{h \to 0} \frac{1 - P_{i,j}^{(M)}(h)}{h}.$$ (18)

Observe, that each ball $B_i^M$ is a union of disjoint balls $B_{ik}^{M+1}$, $(k = 1, \ldots, p)$ of radius $p^{-(M+1)}$. We stress here, that $q_{ij,k}^{(M)}$ depend only on $p$–adic distance between balls $B_i^M$ and $B_j^M$. For this reason we may represent transition probabilities of the process $X_M(t)$ by appropriate transition probabilities of the process $X_{M+1}(t)$

$$P_{i,j}^{(M)}(t) = P \left( X_M(t) = B_j^M | X_M(0) = B_i^M \right)$$

$$= P \left( X(t) \in \bigcup_{k=1}^{p} B_{jk}^{M+1} | X(0) \in \bigcup_{k=1}^{p} B_{ik}^{M+1} \right)$$

$$= p P \left( X_{M+1}(t) = B_j^{M+1} | X_{M+1}(0) = B_i^{M+1} \right)$$

$$= p P_{i,j}^{(M+1)}(t))$$ (19)

This equality leads to the recurrence relation for the local characteristics $q_{i,j}^{(M)}$ and $q_{i,j}^{(M+1)}$ of the processes $\{X_M(t)\}$ and $\{X_{M+1}(t)\}$. Let $\text{dist}_p(B_i^M, B_j^M) = p^{-n}$, then we define

$$q_{i,j}^{(M)} \equiv p^{-M} u(-M, M - n).$$ (20)
we may notice that definition of \( q_{i,j}^{(M)} \) and (19) implies

\[
u(-M + 1, m - 1) = u(-M, m).
\] (21)

Finally taking into account that \( p^{-M} u(M, m) \) represents probability intensity transition of the Markov process and (21) we may write

\[
u(-M, m) = a(-M + m - 1) - a(-M + m),
\] (22)

where \( \{a(-n)\}, n = 0, 1, 2, \ldots \) is a sequence of positive numbers such that

\[a(-n) \geq a(-n + 1)
\] (23)

Proceeding in the similar way as in [19] we obtain the solution of the Kolmogorov equations for the Markov process with local characteristics \( q_{i,j}^{(M)} \) given by (20) and (21) in the form

\[
P^{(M)}_{i,j}(t) = p^{-M} + \frac{p - 1}{p} \sum_{i=0}^{M-1} p^{-i} \exp(tW_{-M,i+1}),
\] (24)

if \( \text{dist}_p(B_i^M, B_j^M) = p^{-M+m} \) then

\[
P^{(M)}_{i,j}(t) = p^{-M} + \frac{p - 1}{p} \sum_{i=0}^{M-1} p^{-(m+i)} \exp(tW_{-M,m+i+1}) - p^{-m} \exp(tW_{-M,m})
\] (25)

where

\[
W_{-M,j} = - \sum_{k=j}^{M-1} (u(-M,k) - u(-M,k+1))p^{-M+k}.
\] (26)

Now we define the transition probabilities of the Markov process on \( \mathbb{Z}_p \) in the following way.

For any \( x \in \mathbb{Z}_p \) let

\[
P_t(x; B(a,M)) \equiv P \{ X_M(t) \in B(x, M) \mid X_M(0) \in B(a, M) \},
\] (27)

By arguments similar to those in [19] we may prove that there exists continuous time Markov stochastic process \( \{X(t), t \geq 0\} \) with state space \( \mathbb{Z}_p \) and transition probabilities \( P_t(x; B) \) given by (27).

Observe that (27) gives us direct connection between Markov processes \( \{X_M(t), t \geq 0\} \) on \( p \)-adic balls with radius \( p^{-M} \), and Markov process \( \{X(t), t \geq 0\} \) on \( \mathbb{Z}_p \).

III. DYNAMICS AS A THERMAL HOPPING IN \( p \)-ADIC SPACE.

In this section we shall apply the mathematical considerations from Section 2 to the physical system with the hierarchy of states which can be linked with Cayley tree structure. By studying the dynamics of such the systems temperature-dependent power law decay and the Kohlrausch law are derived. Transitions between states are thermally activated. The height of the energy barriers, \( \Delta_k \) \( k = 1, 2, \ldots \), which the system overcomes can be ordered in the increasing sequence \( \Delta_1 < \Delta_2 < \cdots < \Delta_k \ldots \). The time evolution of the system is described by a random walk on the space of states. The simplest model of such dynamics is the model proposed by Ogielski and Stein [9]. They consider a regular Cayley tree with \( M \) levels and fixed branching ratio \( p \). The total number of leaves, points on the bottom of the tree, is \( n = p^M \). The natural ultrametric distance \( d(k, l) \) between leaves \( k \) and \( l \), is defined as equal to the height \( m, m = 0, 1, \ldots, M \) of their closest common ancestor. Now, identifying the states \( x \) and \( y \) separated by the energy barrier \( \Delta_m \) with leaves \( k \) and \( l \), the probability of moving from state \( x \) to state \( y \) may be defined as equal to transition probability from leaf \( k \) to leaf \( l \) separated by the ultrametric distance \( m \). Thus dynamics in the space of states separated by the energy barrier may be studied in terms of appropriate Markov process involving the end points of Cayley tree, as a space of states. It is a nontrivial observation that the probability transition intensities between states are depending on their ultrametric distance. Due to hierarchical structure of the state space a probability
intensities matrix of the process has Parisi matrix structure. Parisi matrix has regular form, and for illustration we present the case $p = 2$.

$$
\begin{bmatrix}
\epsilon_0 & \epsilon_1 & E_1 \\
\epsilon_1 & \epsilon_0 & E_2 \\
E_1 & E_2 & \epsilon_0 & \epsilon_1 & \epsilon_0
\end{bmatrix},
$$

(28)

where $E_i$ is the matrix with all elements equal to $\epsilon_i$.

One can observe that end points of a regular Cayley tree with $M$ levels and fixed branching ratio $M$ may be represented as a set of disconnected balls $\{B_0^M, \ldots, B_{p^{M-1}}^M\}$ with radius $p^{-M}$ covering $Z_p$.

Let us consider now a special case of Markov process on $p-$adic integers. We assume that transition probability intensities of the process $\{X(t), t \geq 0\}$ depend on $p-$adic distance only. For this process we have a corresponding Markov chain $\{X_M(t), t \geq 0\}$ with the set of disconnected balls $\{B_0^M, \ldots, B_{p^{M-1}}^M\}$ with the radius $p^{-M}$ covering $Z_p$, as a state space. If we enumerate these balls in such a way that $\text{dist}_p(B_0^M, B_j^M)$ increase with $i$ then $Q = \{q_{ij}\}$, $(0 \leq j \leq p^M - 1, 0 \leq i \leq p^{M-1})$, the matrix of transition probability intensities of the process $\{X_M(t), t \geq 0\}$ has Parisi matrix form. By appropriate choice of $u(-M, k)$, we obtain process studied in [9].

Let for $k = 1, 2, \ldots, M - 1$

$$\epsilon_k = p^{-M}u(-M, k) \equiv q_{ij}^{(M)},$$

(29)

where $q_{ij}^{(M)}$ is probability transition intensity of a jump from the ball $B_i^M$ to the ball $B_j^M$, separated by $p-$adic distance $p^{-(M-k)}$.

From (24) we have that process which at time $0$ starts from a ball $B_0^M$ will be found at this ball at time $t$ with probability

$$P_t(B_0^M, B_0^M) = p^{-M} + \frac{p-1}{p} \sum_{i=1}^{M-1} p^{-i} \exp(-tW_{-M,i+1})$$

(30)

which together with (26) and (29) gives

$$P_t(B_0^M, B_0^M) = p^{-M} + \frac{p-1}{p} \sum_{i=0}^{M-1} p^{-i} \exp(-t \sum_{k=i+1}^{M} (\epsilon_k - \epsilon_{k+1})p^k)$$

(31)

For a special case $p = 2$ equation (31) has the same form as corresponding equation (6) from [9]

$$P_t(B_0^M, B_0^M) = 2^{-M} + \frac{1}{2} \sum_{i=0}^{M-1} 2^{-i} (\exp(-t[2a_{i+1} + \sum_{k=i+2}^{M} a_k]))$$

(32)

where $a_k = 2^{k-1} \epsilon_k$ represents the probability intensity of a jump an ultrametric distance $k$ from a starting sit at Cayley tree, while in our case $a_k$ represents the probability transition intensity of a jump to any ball of radius $2^{-M}$ at $2-$adic distance $2^{-(M-k)}$.

Finally we contract the ball radiuses to zero and performing procedure analgocial to [9] and specifying the form of energy barriers and probability intensities of crossing a barriers we are able to compare different scenario. The most simple case is a sequence of barriers linearly growing with $p-$adic distance $\Delta_k = \Delta k$ for some positive constant $\Delta$ and $a_k = e^{-\Delta k/T}$ and for fixed temperature $T$. In this case (31) gives us for large time $t$ a temperature-dependent power law.

$$\lim_{M \to \infty} P_t(B_0^M, B_0^M) \sim t^{-\Delta \ln 2/\Delta}.$$  

(33)

For a sequence of energy barriers which grows in slower way, i.e $\Delta_k = \Delta \ln k$, for some positive constant $\Delta$ and $a_k = e^{-\Delta \ln k/T}$ and for fixed temperature $T$, the probability $P_t(B_0^M, B_0^M)$ fulfills, for large $t$, the Kohlrausch law

$$\lim_{M \to \infty} P_t(B_0^M, B_0^M) \sim \exp(-t^{T/\Delta}).$$

(34)
IV. CONCLUDING REMARKS

In our paper use random walk framework and we show that \( p \)-adic analysis is very natural tool to describe the relaxation process in a glassy systems. \( p \)-adic space has a well defined ultrametric topology which we employ in this paper. We do not use the random walk along the Cayley tree [9] because one can measure the ultrametric distance between the physical states directly on the bottom of the tree.

For the description of dynamics of the hierarchical systems besides of the random walk on \( p \)-adic numbers one can alternatively use the process of jump diffusion on \( p \)-adic numbers which description employs the pseudodifferential operators [26] in \( p \)-adic space. In these both approaches one obtains the equivalent physical results [22–24].

It seems that the \( p \)-adic analysis is a good tool not only for the description of the dynamics of the glassy systems, but also for another hierarchical processes like the evolution of fractals [27], the avalanches [28], protein folding [29] ect.. It is interesting to notice that the \( p \)-adic space inherently includes the natural hierarchy: \( p \)-adic balls can be covered by the smaller ones disjoint balls. The hierarchy of these nested balls correspond to the hierarchy of the scales of the configuration rearrangements, as it is seen from the scaling theory for the growing domains and droplets [5–7]. The droplets may be broken into smaller ones when temperature decrease.

It is also interesting to notice that by the mapping of any Cayley tree onto its bottom states one can obtain the integer \( p \)-adic space and further changing the fractional part of the main ancestors we obtain another integer \( p \)-adic space. \( p \)-adic space consists of infinite number of these integer \( p \)-adic spaces. In conclusion one can comment that the memory effects in spin glassy systems it seems to be are well described by \( p \)-adic topology.

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