ON THE QUASI-CONFORMAL CURVATURE TENSOR OF AN ALMOST KENMOTSU MANIFOLD WITH NULLITY DISTRIBUTIONS

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Abstract. The objective of the present paper is to characterize quasi-conformally flat and $\xi$-quasi-conformally flat almost Kenmotsu manifolds with $(k, \mu)$-nullity and $(k, \mu)'$-nullity distributions, respectively. Also we characterize almost Kenmotsu manifolds with vanishing extended quasi-conformal curvature tensor and extended $\xi$-quasi-conformally flat almost Kenmotsu manifolds such that the characteristic vector field $\xi$ belongs to the $(k, \mu)$-nullity distribution.

Keywords: Almost Kenmotsu manifold, Einstein manifold, Weyl conformal curvature tensor, Quasi-conformal curvature tensor, Extended quasi-conformal curvature tensor.

1. Introduction

Let $M$ be a $(2n + 1)$-dimensional Riemannian manifold with metric $g$ and let $T(M)$ be the Lie algebra of differentiable vector fields in $M$. The Ricci operator $Q$ of $(M, g)$ is defined by

\begin{equation}
\label{1.1}
g(QX, Y) = S(X, Y),
\end{equation}

where $S$ denotes the Ricci tensor of type $(0, 2)$ on $M$ and $X, Y \in T(M)$. The Weyl conformal curvature tensor $C$ is defined by

\begin{equation}
\label{1.2}
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2n(2n - 1)}[g(Y, Z)X - g(X, Z)Y],
\end{equation}

for $X, Y, Z \in T(M)$, where $R$ and $r$ denote the Riemannian curvature tensor and scalar curvature of $M$, respectively.

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255
For a \((2n + 1)\)-dimensional Riemannian manifold, the quasi-conformal curvature tensor \(\tilde{C}\) is given by
\[
\tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
- \frac{r}{2n + 1}a + 2b[g(Y, Z)X - g(X, Z)Y],
\]
where \(a\) and \(b\) are two scalars. The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [21]. If \(a = 1\) and \(b = -\frac{1}{2n-1}\), then the quasi-conformal curvature tensor reduces to conformal curvature tensor.

A \((2n + 1)\)-dimensional Riemannian manifold will be called a manifold of the quasi-constant curvature if the Riemannian curvature tensor \(\tilde{R}\) of type \((0, 4)\) satisfies the condition
\[
\tilde{R}(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
+ q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)]
+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z),
\]
where \(\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)\), \(p, q\) are scalars and there exists a unit vector field \(\rho\) satisfying \(g(X, \rho) = T(X)\). The notion of the quasi-constant curvature for Riemannian manifolds was introduced by Chen and Yano [4].

At present, the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of \(k\)-nullity distribution was introduced by Gray [10] and Tanno [15] in the study of Riemannian manifolds \((M, g)\), which is defined for any \(p \in M\) and \(k \in \mathbb{R}\) as follows:
\[
N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},
\]
for any \(X, Y \in T_pM\), where \(T_pM\) denotes the tangent vector space of \(M\) at any point \(p \in M\) and \(R\) denotes the Riemannian curvature tensor of type \((1, 3)\). Blair, Koufogiorgos and Papantonio [1] introduced the generalized notion of \(k\)-nullity distribution, named \((k, \mu)\)-nullity distribution on a contact metric manifold \((M^{2n+1}, \phi, \xi, \eta, g)\), which is defined for any \(p \in M\) and \(k, \mu \in \mathbb{R}\) as follows:
\[
N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]
+ \mu[g(Y, Z)hX - g(X, Z)hY]\},
\]
where \(h = \frac{1}{2}L\xi\phi\) and \(L\) denotes the Lie differentiation.

In [7] Dileo and Pastore introduce the notion of \((k, \mu)\)'-nullity distribution, another generalized notion of \(k\)-nullity distribution, on an almost Kenmotsu manifold \((M^{2n+1}, \phi, \xi, \eta, g)\), which is defined for any \(p \in M^{2n+1}\) and \(k, \mu \in \mathbb{R}\) as follows:
\[
N_p(k, \mu)' = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]
+ \mu[g(Y, Z)h'X - g(X, Z)h'Y]\},
\]
where \( h' = h \circ \phi \).

A differentiable \((2n + 1)\)-dimensional manifold \( M \) is said to have a \((\phi, \xi, \eta)\)-structure or an almost contact structure, if it admits a \((1, 1)\) tensor field \( \phi \), a characteristic vector field \( \xi \), and a 1-form \( \eta \) satisfying ([2],[3]),

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]

(1.8)

where \( I \) denotes the identity endomorphism. Here also \( \phi \xi = 0 \) and \( \eta \circ \phi = 0 \) hold; both can be derived from (1.8) easily.

If a manifold \( M \) with a \((\phi, \xi, \eta)\)-structure admits a Riemannian metric \( g \) such that

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for any vector fields \( X, Y \) of \( T_pM^{2n+1} \), then \( M \) is said to be an almost contact metric manifold. The fundamental 2-form \( \Phi \) on an almost contact metric manifold is defined by \( \Phi(X, Y) = g(X, \Phi Y) \) for any \( X, Y \) of \( T_pM^{2n+1} \). The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the \((1, 2)\)-type torsion tensor \( N_\phi \), defined by \( N_\phi = [\phi, \phi] + 2d\eta \otimes \xi \), where \([\phi, \phi] \) is the Nijenhuis torsion of \( \phi \) [2]. Recently in ([7],[8],[9],[13],[14]), an almost contact metric manifold such that \( \eta \) is closed and \( d\Phi = 2\eta \wedge \Phi \) are studied and called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also, Kenmotsu manifolds can be characterized by \((\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \), for any vector fields \( X, Y \). It is well known [11] that a Kenmotsu manifold \( M^{2n+1} \) is locally a warped product \( I \times f N^{2n} \) where \( N^{2n} \) is a Kähler manifold, \( I \) is an open interval with coordinate \( t \) and the warping function \( f \), defined by \( f = ce^t \) for some positive constant \( c \). Let us denote the distribution orthogonal to \( \xi \) by \( \mathcal{D} \) and defined by \( \mathcal{D} = \ker(\eta) = \text{Im}(\phi) \). In an almost Kenmotsu manifold, since \( \eta \) is closed, \( \mathcal{D} \) is an integrable distribution.

At each point \( p \in M \), we have

\[
T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}
\]

where \( \{\xi_p\} \) is 1-dimensional linear subspace of \( T_p(M) \) generated by \( \xi_p \). Then the Weyl conformal curvature tensor \( C \) is a map:

\[
C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi\}.
\]

Three particular cases can be considered as follows:

1. \( C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\} \), that is, the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero.
2. \( C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \), that is, the projection of the image of \( C \) in \( \{\xi\} \) is zero.
3. \( C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\} \), that is, when \( C \) is restricted to \( \phi(T_p(M)) \times \phi(T_p(M)) \), the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero, which is equivalent to \( \phi^2 C(\phi X, \phi Y) \phi Z = 0 \).
Definition 1.1. [22] A contact metric manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) is said to be \(\xi\)-conformally flat if the linear operator \(C(X,Y)\) is an endomorphism of \(\phi(T(M))\), that is, if
\[
C(X,Y)\phi(T(M)) \subset \phi(T(M)).
\]
Then it follows immediately that

Proposition 1.1. [22] On a contact metric manifold \((M^{2n+1}, \phi, \xi, \eta, g)\), the following conditions are equivalent.

(a) \(M^{2n+1}\) is \(\xi\)-conformally flat,
(b) \(\eta(C(X,Y)Z) = 0\),
(c) \(\phi^2 C(X,Y)Z = -C(X,Y)Z\),
(d) \(C(X,Y)\xi = 0\),

where \(X, Y, Z \in T(M)\).

Almost Kenmotsu manifolds have been studied by several authors such as Dileo and Pastore ([7]-[9]), Wang and Liu ([16]-[20]), De and Mandal ([5], [6], [12]) and many others. In the present paper we like to study quasi-conformal curvature tensor of almost Kenmotsu manifolds with \((k, \mu)\) and \((k, \mu)'\)-nullity distributions, respectively. Also, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended \(\xi\)-quasi-conformally flat almost Kenmotsu manifolds with \((k, \mu)\)-nullity distribution.

The paper is organized as follows:

In Section 2, we give a brief account on almost Kenmotsu manifolds with \(\xi\) belonging to the \((k, \mu)\)-nullity distribution and \(\xi\) belonging to the \((k, \mu)'\)-nullity distribution. Section 3 deals with quasi-conformally flat and \(\xi\)-quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field \(\xi\) belonging to the \((k, \mu)\)-nullity distribution. As a consequence of the main result, we obtain several corollaries. Section 4 is devoted to the study of quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field \(\xi\) belonging to the \((k, \mu)'\)-nullity distribution. In the final section, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended \(\xi\)-quasi-conformally flat almost Kenmotsu manifolds with \((k, \mu)\)-nullity distribution.

2. Almost Kenmotsu manifolds

Let \(M^{2n+1}\) be an almost Kenmotsu manifold. We denote by \(h = \frac{1}{2} \mathcal{L}_\xi \phi\) and \(l = R(\cdot, \xi)\xi\) on \(M^{2n+1}\). The tensor fields \(l\) and \(h\) are symmetric operators and satisfy the following relations [13]:

\[
(2.1) \quad h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0,
\]

\[
(2.2) \quad \nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi = 0),
\]
Almost Kenmotsu Manifolds with Nullity Distributions

\( \phi l - l = 2(h^2 - \phi^2), \) (2.3)

\( R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla Y \phi h) X - (\nabla X \phi h) Y, \) (2.4)

for any vector fields \( X, Y. \) The \((1,1)\)-type symmetric tensor field \( h' = h \circ \phi \) is anti-commuting with \( \phi \) and \( h'\xi = 0. \) Also it is clear that ([7], [18])

\( h = 0 \iff h' = 0, \quad h'^2 = (k + 1)\phi^2 (\iff h^2 = (k + 1)\phi^2). \) (2.5)

3. Quasi-conformally flat almost Kenmotsu manifolds with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution

In this section we study quasi-conformally flat and \( \xi \)-quasi-conformally flat almost Kenmotsu manifolds with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution. From (1.6) we obtain

\( R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \) (3.1)

where \( k, \mu \in \mathbb{R}. \) Before proving our main results in this section we first state the following:

**Lemma 3.1.** ([7]) Let \( M^{2n+1} \) be an almost Kenmotsu manifold of dimension \((2n + 1).\) Suppose that the characteristic vector field \( \xi \) belonging to the \((k, \mu)\)-nullity distribution. Then \( k = -1, h = 0 \) and \( M^{2n+1} \) is locally a wrapped product of an open interval and an almost Kähler manifold.

In view of Lemma 3.1 it follows from the equation (3.1),

\( R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \) (3.2)

\( R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \) (3.3)

\( S(X, \xi) = -2n\eta(X), \) (3.4)

\( Q\xi = -2n\xi, \) (3.5)

for any vector fields \( X, Y \) on \( M^{2n+1}. \)

**Theorem 3.1.** An almost Kenmotsu manifold \( M^{2n+1} \) with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution is quasi-conformally flat if and only if the manifold is locally isometric to the hyperbolic space \( \mathbb{H}^{2n+1}(-1). \)

**Proof:** Let us first consider the manifold \( M^{2n+1} \) which is quasi-conformally flat, that is,

\( \tilde{C}(X, Y)Z = 0, \) (3.6)
for any vector fields $X, Y, Z$ on $M^{2n+1}$.

From (1.3) we have

$$\tilde{R}(X, Y, Z, W) = \frac{b}{a} S(X, Z)g(Y, W) - S(Y, Z)g(X, W)$$

$$+ S(Y, W)g(X, Z) - S(X, W)g(Y, Z) + \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

(3.7)

Putting $Z = \xi$ in the above equation and using (3.2) and (3.4) we get

$$\eta(X)g(Y, W) - \eta(Y)g(X, W) = \frac{b}{a} \left[ -2n\eta(X)g(Y, W) + 2n\eta(Y)g(X, W) 
+ S(Y, W)\eta(X) - S(X, W)\eta(Y) \right]$$

$$+ \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] [g(Y, W)] \eta(X)$$

$$- g(Y, W)\eta(X)].$$

(3.8)

Putting $Y = \xi$ in the above equation we obtain after simplification

$$S(X, W) = \alpha g(X, W) + \beta \eta(X)\eta(W),$$

where $\alpha = \frac{a}{b} \left[ \frac{2bn}{a} + \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] + 1 \right]$ and $\beta = \frac{a}{b} \left[ - \frac{4bn}{a} - \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] - 1 \right]$. Therefore, we have $\alpha + \beta = -2n$.

Now using the above relation, (3.9) implies

$$r = 2n(\alpha - 1).$$

(3.10)

In [7], Dileo and Pastore proved that in an almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution the sectional curvature $K(X, \xi) = -1$. From this we get in an almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution the scalar curvature $r = -2n(2n + 1)$. Using this value of $r$ we obtain from (3.10), $\alpha = -2n$. This implies $\beta = 0$.

Hence (3.9) reduces to

$$S(X, W) = -2ng(X, W).$$

(3.11)

From (3.7) we obtain

$$aR(X, Y)Z = -b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

$$+ \frac{r}{2n+1} \left[ \frac{a}{2n} + 2b \right] [g(Y, Z)X - g(X, Z)Y].$$

(3.12)

Using the value of $r$ and (3.11) in (3.12) yields

$$R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y],$$

which implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Conversely, suppose that the manifold is locally isometric to the hyperbolic space.
\( \mathbb{H}^{2n+1}(-1) \). That is, (3.13) holds. Contracting \( X \) in (3.13) yields
\[
S(Y, Z) = -2ng(Y, Z).
\]
Hence (3.13) and (3.14) together implies \( \tilde{C}(X, Y)Z = 0 \). That is, the manifold is quasi-conformally flat. Hence the theorem is proved.

Now, if \( a = 1 \) and \( b = -\frac{1}{2n} \), then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Hence we can state the following:

**Corollary 3.1.** An almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution is conformally flat if and only if the manifold is locally isometric to the hyperbolic space \( \mathbb{H}^{2n+1}(-1) \).

The above corollary has been proved by De and Mandal [5].

**Theorem 3.2.** An almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution is \( \xi \)-quasi-conformally flat if and only if the manifold is an Einstein manifold.

**Proof:** Let us consider a manifold that is \( \xi \)-quasi-conformally flat. That is,
\[
\tilde{C}(X, Y)\xi = 0,
\]
which implies
\[
aR(X, Y)\xi = -b[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] + \frac{r}{2n+1}\left[\frac{a}{2n} + 2b[2g(Y, \xi)X - g(X, \xi)Y]\right].
\]
Using (3.2) and (3.4) and \( r = -2n(2n + 1) \) we get from the above equation
\[
\eta(Y)QX - \eta(X)QY = -2n[\eta(Y)X - \eta(X)Y],
\]
Putting \( Y = \xi \) in the above equation we obtain
\[
QX = -2nX,
\]
which implies \( S(X, Y) = -2ng(X, Y) \). That is, the manifold is Einstein. Conversely, assume that the manifold is Einstein. Then there exists a scalar \( \lambda \) such that
\[
S(X, Y) = \lambda g(X, Y).
\]
In an almost Kenmotsu manifold with \((k, \mu)\)-nullity distribution, the scalar curvature \( r = -2n(2n + 1) \). This implies \( \lambda = -2n \). Now
\[
\tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b[2g(Y, Z)X - g(X, Z)Y]\right].
\]
Using (3.18) we get
\( \tilde{C}(X, Y) = a[R(X, Y)Z + (g(Y, Z)X - g(X, Z)Y)] \).
Putting \( Z = \xi \) in the above equation and using (3.2) we obtain
\( \tilde{C}(X, Y)\xi = 0 \),
which implies that the manifold is \( \xi \)-quasi-conformally flat.

If \( a = 1 \) and \( b = -\frac{1}{2n-1} \), then the quasi-conformal curvature tensor reduces to
conformal curvature tensor.

Thus we are in a position to state the following:

**Corollary 3.2.** An almost Kenmotsu manifold with \( (k, \mu) \)-nullity distribution is
\( \xi \)-conformally flat if and only if it is Einstein.

### 4. Quasi-conformally flat almost Kenmotsu manifolds with \( \xi \) belonging to the \( (k, \mu) \)′-nullity distribution

In this section we study \( \xi \)-quasi-conformally flat almost Kenmotsu manifolds with
\( \xi \) belonging to the \( (k, \mu) \)′-nullity distribution. Let \( X \in \mathcal{D} \) be the eigen vector of
\( h' \) corresponding to the eigen value \( \lambda \). Then from (2.5) it is clear that
\( \lambda^2 = -(k + 1) \), a constant. Therefore \( k \leq -1 \) and \( \lambda = \pm \sqrt{-k - 1} \). We denote by \( [\lambda]' \) and \( [-\lambda]' \)
the corresponding eigenspaces related to the non-zero eigen value \( \lambda \) and \( -\lambda \) of \( h' \), respectively. Before presenting our main theorem we recall some results:

**Lemma 4.1.** (Prop. 4.1 and Prop. 4.3 of [7]) Let \( (M^{2n+1}, \phi, \xi, \eta, g) \) be an almost Kenmotsu manifold such that \( \xi \) belongs to the \( (k, \mu) \)′-nullity distribution and \( h' \neq 0 \). Then \( k < -1, \mu = -2 \) and \( \text{Spec} (h') = \{0, \lambda, -\lambda\} \), with 0 as a simple eigen value and \( \lambda = \sqrt{-k - 1} \). The distributions \( [\xi] \oplus [\lambda]' \) and \( [\xi] \oplus [-\lambda]' \) are integrable with totally geodesic leaves. The distributions \( [\lambda]' \) and \( [-\lambda]' \) are integrable with totally umbilical leaves. Furthermore, the sectional curvatures are given by the following:

(a) \( K(X, \xi) = k - 2\lambda \) if \( X \in [\lambda]' \) and
\( K(X, \xi) = k + 2\lambda \) if \( X \in [-\lambda]' \),
(b) \( K(X, Y) = k - 2\lambda \) if \( X, Y \in [\lambda]' \);
\( K(X, Y) = k + 2\lambda \) if \( X, Y \in [-\lambda]' \) and
\( K(X, Y) = -(k + 2) \) if \( X \in [\lambda]', Y \in [-\lambda]' \),
(c) \( M^{2n+1} \) has a constant negative scalar curvature \( r = 2n(k - 2n) \).

**Lemma 4.2.** (Lemma 3 of [16]) Let \( (M^{2n+1}, \phi, \xi, \eta, g) \) be an almost Kenmotsu manifold with \( \xi \) belonging to the \( (k, \mu) \)′-nullity distribution. If \( h' \neq 0 \), then the Ricci operator \( Q \) of \( M^{2n+1} \) is given by
\( Q = -2n\eta \otimes \xi - 2nh' \).

Moreover, the scalar curvature of \( M^{2n+1} \) is \( 2n(k - 2n) \).
From (1.7) we have,

\[ R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \]

where \( k, \mu \in \mathbb{R} \). Also we get from (4.2)

\[ R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \]

Contracting \( X \) in (4.2), we have

\[ S(Y, \xi) = 2nk\eta(Y). \]

Moreover, in an almost Kenmotsu manifold with \((k, \mu)\)'-nullity distribution

\[ \nabla_X \xi = X - \eta(X)\xi + h'X \]

and

\[ (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) + g(h'X, Y) \]

holds.

**Theorem 4.1.** A \((2n+1)\)-dimensional \((n > 1)\) quasi-conformally flat almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution is either conformally flat or of a quasi-constant curvature.

**Proof:** Let us assume that the manifold \( M^{2n+1} \) is quasi-conformally flat, that is,

\[ \tilde{C}(X, Y)Z = 0, \]

for any vector fields \( X, Y, Z \) on \( M^{2n+1} \).

From (1.3) we have

\[
\begin{align*}
    a\tilde{R}(X, Y, Z, W) &= b[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\
                        &+ \frac{r}{(2n + 1)} \left[ \frac{\alpha}{2n} + 2b[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \right].
\end{align*}
\]

Putting \( Z = \xi \) in the above equation and using (4.2) and (4.4) we have

\[
\begin{align*}
    ak[\eta(Y)g(X, W) - \eta(X)g(Y, W)] + a\mu[\eta(Y)g(h'X, W) - \eta(X)g(h'Y, W)] \\
    = b[2nk\eta(X)g(Y, W) - 2nk\eta(Y)g(X, W) - \eta(Y)S(X, W) + \eta(X)S(Y, W)] \\
    + \frac{r}{(2n + 1)} \left[ \frac{\alpha}{2n} + 2b[\eta(Y)g(X, W) - \eta(X)g(Y, W)] \right].
\end{align*}
\]
Putting \( Y = \xi \) in the above equation and using (4.4) we get after simplifying
\[
S(X, W) = \left[-2nk + \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - \frac{ak}{b}\right]g(X, W)
\]
\[(4.10)\]
\[\quad + \left[4nk - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) + \frac{ak}{b}\right]\eta(X)\eta(W) - \frac{am}{b}g(h'X, W).
\]
Let us denote
\[(4.11)\]
\[A = -2nk + \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - \frac{ak}{b},\]
and
\[(4.12)\]
\[B = 4nk - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) + \frac{ak}{b}.
\]
Then, we see that
\[(4.13)\]
\[A + B = 2nk.
\]
Putting \( X = W = e_i \) in (4.10), where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i, i = 1, 2, 3, ..., (2n+1) \), we get
\[(4.14)\]
\[r = A(2n+1) + B.
\]
From (4.13) and (4.14) we get
\[(4.15)\]
\[A = \frac{r}{2n} - k.
\]
From (4.11) and (4.15), it follows that
\[-2nk + \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - \frac{ak}{b} = \frac{r}{2n} - k.
\]
The above relation gives
\[(4.16)\]
\[(a + 2nb - b)(r - 2nk(2n+1)) = 0.
\]
Hence, either \( a + 2nb - b = 0 \) or \( r = 2nk(2n+1) \).

Let us suppose that \( a + 2nb - b = 0 \). Then we see that \( b = -\frac{a}{2n+1} \). Hence, from (1.3), it follows that \( \tilde{C}(X, Y)Z = aC(X, Y)Z \), where \( C(X, Y)Z \) is the Weyl conformal curvature tensor. So, in this case, the quasi-conformally flat manifold is conformally flat.
From (4.1) and (4.17), it follows that

$$S(X, W) = 2nk^2g(X, W) - \frac{a\mu}{b}g(h'X, W).$$

Using (4.17) in (4.8) yields

$$\tilde{R}(X, Y, Z, W) = k[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
- \mu [g(h'X, Z)g(Y, W) - g(h'Y, Z)g(X, W)]
+ g(h'Y, W)g(X, Z) - g(h'X, W)g(Y, Z).$$

(4.18)

From (4.1) and (4.17), it follows that

$$g(h'X, W) = l[g(X, W) - \eta(X)\eta(W)],$$

(4.19)

where $$l = \frac{2nk^2(k+1)}{n^2 - 2nb} = -\frac{nk^2(k+1)}{n+nb}$$, by Lemma 4.1.

Using (4.19) in (4.18) we get

$$\tilde{R}(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
+ g(g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W)]
+ g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z),$$

(4.20)

where $$p = k - 4l$$ and $$q = 2l$$.

This completes the proof.

5. Extended quasi-conformal curvature tensor of an almost Kenmotsu manifold with \((k, \mu)\)-nullity distribution

In this section we study vanishing extended quasi-conformal curvature tensor and extended \(\xi\)-quasi-conformally flat almost Kenmotsu manifolds with \(\xi\) belonging to \((k, \mu)\)-nullity distribution.

The extended form of quasi-conformal curvature tensor can be written as

$$\tilde{C}_c(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
- \frac{r}{2n + 1}[g(Y, Z)X - g(X, Z)Y]
- \eta(Y)\tilde{C}(X, Z) - \eta(Z)\tilde{C}(X, Y)\xi.$$  

(5.1)

Theorem 5.1. In an almost Kenmotsu manifold with \(\xi\) belonging to \((k, \mu)\)-nullity distribution, the extended quasi-conformal curvature tensor vanishes if and only if the manifold is locally isometric to the hyperbolic space \(\mathbb{H}^{2n+1}(-1)\).

Proof: Putting \(Y = Z = \xi\) and supposing that the extended quasi-conformal tensor vanishes, we get from (5.1)

$$aR(X, \xi)\xi + b[S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi] + (a + 4nb)(X - \eta(X)\xi)
- \eta(X)\tilde{C}(\xi, \xi)\xi - \tilde{C}(X, \xi)\xi - \tilde{C}(X, \xi)\xi = 0.$$  

(5.2)
Now, using (3.4) and (3.5) the above equation reduces to
\[ bQX = -2nbX + 2\tilde{C}(X, \xi)\xi. \]  
(5.3)

Now, Using (3.2), (3.4) and (3.5) we obtain
\[ \tilde{C}(X, \xi) = 2nbX + bQX. \]  
(5.4)

Putting the value of \( \tilde{C}(X, \xi) \xi \) in (5.3) we get
\[ QX = -2nX, \]  
(5.5)

which implies
\[ S(X, Y) = -2ng(X, Y). \]  
(5.6)

This shows that the manifold is Einstein. Since, the extended quasi-conformal curvature tensor vanishes, we have from (5.1)
\[ aR(X, Y)Z = -b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \]
\[ -(a + 4nb)[g(Y, Z)X - g(X, Z)Y] \]
\[ + \eta(X)\tilde{C}(\xi, Y)Z + \eta(Y)\tilde{C}(X, \xi)Z + \eta(Z)\tilde{C}(X, Y)\xi. \]  
(5.7)

Now, making use of (3.3), (3.4), (3.5) and (5.5) we obtain
\[ \tilde{C}(\xi, Y)Z = 0, \tilde{C}(X, \xi)Z = 0. \]

Again since the manifold is Einstein, we have from Theorem 3.2
\[ \tilde{C}(X, Y)\xi = 0. \]

Putting these values in (5.7) and using (5.6) we get
\[ R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y]. \]  
(5.8)

This implies that the manifold is locally isometric to the hyperbolic space \( \mathbb{H}^{2n+1}(-1) \).

Conversely, suppose that the manifold is locally isometric to the hyperbolic space \( \mathbb{H}^{2n+1}(-1) \). That is, (5.8) holds.

Contracting X in (5.8) yields
\[ S(Y, Z) = -2ng(Y, Z). \]  
(5.9)

Now, as shown earlier in this theorem
\[ \tilde{C}(\xi, Y)Z = \tilde{C}(X, \xi)Z = \tilde{C}(X, Y)\xi = 0. \]

Then, making use of (5.8), (5.9) and the above values, we obtain from (5.1) that
\[ \tilde{C}_e(X, Y)Z = 0. \]

Hence the theorem is proved.
Theorem 5.2. An almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution is extended $\xi$-quasi-conformally flat if and only if the manifold is Einstein.

Proof: Suppose $\tilde{C}(X, Y)\xi = 0$ and putting $Y = \xi$, we get from (5.1)

\[ aR(X, \xi)\xi + b[S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi] + (a + 4nb)(X - \eta(X)\xi) - \eta(X)\tilde{C}(\xi, \xi)\xi - \tilde{C}(X, \xi)\xi = 0. \]

(5.10)

Now, using (3.4) and (3.5) the above equation reduces to

\[ bQX = -2nbX + 2\tilde{C}(X, \xi)\xi. \]

(5.11)

Now, Using (3.2), (3.4) and (3.5) we obtain

\[ \tilde{C}(X, \xi)\xi = 2nbX + bQX. \]

(5.12)

Putting the value of $\tilde{C}(X, \xi)\xi$ in (5.11) we get

\[ QX = -2nX, \]

(5.13)

which implies that the manifold is Einstein.

Conversely, if the manifold is Einstein then obviously $\tilde{C}_e(X, Y)\xi = 0$.

Hence the theorem is established.

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