Possibility of an adiabatic transport of an edge Majorana through an extended gapless region

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In the context of slow quenching dynamics of a $p$-wave superconducting chain, it has been shown that a Majorana edge state can not be adiabatically transported from one topological phase to the other separated by a quantum critical line. On the other hand, the inclusion of a phase factor in the hopping term, that breaks the extended time reversal invariance, results in an extended gapless region between two topological phases. We show that for a finite chain with an open boundary condition there exists a non-zero probability that an edge Majorana can be adiabatically transported from one topological phase to the other across this gapless region following a slow quench of the superconducting term; this happens for an optimum transit time, that is proportional to the system size and diverges for a thermodynamically large chain. We attribute this phenomenon to the mixing of the Majorana only with low-lying inverted bulk states.

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The $p$-wave superconducting chain, introduced by Kitaev, has become a topic of immense interest in recent years for its fascinating topological properties. These studies lie at the interface of condensed matter physics, quantum information processing, decoherence and quantum computation. The remarkable property of the model is that the topological phase hosts zero energy Majorana modes at the ends of an open chain as the midgap excitations between positive and negative energy bulk states; a topological phase is separated by a quantum critical line from the other topological and non-topological phases as also happens in a topological insulators. It has been proposed that Majorana states can possibly be achieved by the proximity effect between the surface state of a strong topological insulator and a $s$-wave superconductor. The experimental realization of the zero-energy Majorana modes have been found recently in nanowires coupled to superconductors. The hybridization of Majorana fermions has also been observed experimentally through the zero-bias anomalies in the differential conductance of an InAs nanowire coupled to superconductor.

On the other hand, given the recent interest in the non-equilibrium quenching dynamics of quantum many body systems across quantum critical points (QCPs) (for review articles see), the studies involving quenching dynamics of a topological system across a QCP have emerged as a rapidly growing field of research. Especially, the quenching dynamics of a topological insulator and the $p$-wave superconducting chain have been explored in this connection. We note that the dynamical generation, formation and manipulation of edge Majorana states for a driven system have also been studied extensively.

Bermudez et al., addressed the question whether an adiabatic transport of an initial edge Majorana state from one topological phase to the other is possible when the hopping amplitude of the $p$-wave superconducting Hamiltonian is slowly varied in a linear fashion with time. It has been found that such an adiabatic transportation of edge Majorana is forbidden as it gets completely delocalized throughout the chain when the system reaches the QCP separating the two topological phases.

We here consider a modified $p$-wave superconducting chain with a complex hopping term which breaks the extended time reversal symmetry (ETRS) of the model as well as generates an extended gapless phase separating two topological phases as introduced in Ref. In this communication, we probe the question of transporting an edge Majorana adiabatically from one topological phase to the other for a finite chain which is driven across this extended quantum critical region. Our most significant observation is that indeed there exists a finite probability for Majorana edge state to tunnel adiabatically through the intermediate gapless region when the superconducting gap parameter is tuned in a linear fashion with a finite quenching rate. We emphasize at the outset that this adiabatic transport is only possible for an optimal transit time that the system requires to traverse the gapless region. To the best of our knowledge, our work is the first one which points to the possibility of the adiabatic passage of an edge Majorana from one phase to the other under a slow quenching of a finite Majorana chain.

The model we consider here is defined by the Hamiltonian of a 1D $p$-wave superconductor with a complex hopping term,

$$H = \sum_{n=1}^{N-1} \left[ -w_0 e^{i \phi} f_n^\dagger f_{n+1} - w_0 e^{-i \phi} f_{n+1}^\dagger f_n \right] + \Delta (f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger) - \sum_{n=1}^{N} \mu (f_n^\dagger f_n - 1/2),$$

where $w_0$, $\phi$, $\Delta$ and $\mu$ are nearest-neighbor hopping amplitude, phase of the hopping term, superconducting gap and chemical potential, respectively with $N$ being the number of lattice sites. The annihilation and creation operators $f_n$ ($f_n^\dagger$), defined at the lattice site $n$, satisfy
the fermionic anti-commutation relations \( \{ f_m, f_n \} = 0 \) and \( \{ f_m, f_n^\dagger \} = \delta_{mn} \).

In order to explore the topological properties of the model, we use a real space representation of an open chain Hamiltonian (1) in terms of the two Majorana operators \( a_n \) and \( b_n \) at each site are defined as

\[
\begin{align*}
    f_n &= \frac{1}{2} (a_n + ib_n), \\
    f_n^\dagger &= \frac{1}{2} (a_n - ib_n),
\end{align*}
\]

where \( a_n \) and \( b_n \) are real and Hermitian and satisfy the relations \( \{ a_m, a_n \} = \{ b_m, b_n \} = 2 \delta_{mn} \) and \( \{ a_m, b_n \} = 0 \). This allows us to write the Hamiltonian with an open boundary condition in the following form

\[
H = -\frac{i}{2} \sum_{n=1}^{N} \left[ w_0 \cos \phi (a_{n}b_{n+1} + a_{n-1}b_{n}) - \Delta(a_{n}b_{n+1} - a_{n-1}b_{n}) + w_0 \sin \phi (a_{n}a_{n+1} + b_{n}b_{n+1}) \right] + \sum_{n=1}^{N} \mu a_{n}b_{n}.
\]

The Hamiltonian (5) exhibits a conspicuous band inversion phenomena near zero energy levels within the region bounded by \( \Delta/w_0 = -\sin \phi \) and \( \Delta/w_0 = \sin \phi \); for the detailed analysis of the spectrum see the supplementary material. Furthermore, for any non-zero value of \( \phi \) the Majorana modes \( a_n \) and \( b_n \) cannot be decoupled. We note that for \( \phi = 0 \), the topological phase I (phase II) is characterized by the presence of an isolated zero-energy Majorana mode \( a_1(b_1) \) at the left edge and \( b_N(a_N) \) at right edge when \( \Delta = w_0 \) and \( \mu = 0 \). We set \( w_0 = 1 \) for the rest of the paper.

We consider the dynamics of the above Majorana chain under the quenching scheme \( \Delta(t) = -1 + 2t/\tau \) along the path \( \mu = 0 \). The non-linear time variation can also be considered for the model (6). The quenching dynamics of the model with a PBC (as given in Eq. 2) under the above quenching protocol, results in a defect density which is given by Kibble-Zurek scaling law with a modified quenching rate (7).

Let us now focus on the more interesting situation when we start from an initial zero energy Majorana edge state at \( \Delta(0) = 0 \) (i.e., in phase II) and investigate the possibility of its adiabatic transport to the other topological phase. By numerical integration of the time dependent Schrödinger equation we shall estimate following probabilities at the final instant \( t = \tau \): probability of Majorana getting excited to the positive energy band \( P_{\text{def}} \) (negative energy band \( P_{\text{neg}} \))

\[
P_{\text{def(neg)}} = \sum_{\epsilon^+>0(\epsilon^-<0)} |\langle \epsilon^+(-) | \Psi(\tau) \rangle |^2,
\]

where \( |\epsilon^+(-) \rangle \) corresponds to the positive and negative energy eigenstates of the final Hamiltonian within phase I with \( \Delta = 1 \) and \( |\Psi(\tau) \rangle \) is the time-evolved Majorana state at \( t = \tau \). On the other hand, a non-zero value of the probability \( P_m \) defined as \( P_m = |\langle \epsilon_0 | \Psi(\tau) \rangle |^2 \), where \( |\epsilon_0 \rangle \) denotes the zero energy edge Majorana state with \( \Delta = 1 \), indicates a finite probability of the adiabatic transport of the edge Majorana from phase II to phase I.

![Figure 1](image-url)
Interestingly, we find that although $P_{m}$ exhibits dips. Here, $N = 100$.

The variation of $P_{\text{def}}$ and $P_{m}$ as a function of $\tau$ for different values of $\phi$ with a system size $N = 100$ is shown in Fig. 2(b) and Fig. 2(a); clearly, there is no obvious scaling relation with $\tau$ as compared to the PBC scenario. Interestingly, we find that although $P_{\text{def}}$ remains fixed at 0.5 for small values of $\tau$, there exists a characteristic $\tau_{c}$ (denoted by $\tau_{c}$) for which the first significant dip in $P_{\text{def}}$ occurs followed by a few drops. At $\tau = \tau_{c}$, on the other hand, we find the first prominent peak (see Fig. 2(b)) in $P_{m}$ which implies a finite probability of an adiabatic tunneling of the initial edge Majorana following a quench from phase II to phase I. This can be contrasted to the case $\phi = 0$: $P_{m}$ stays zero for all values of $\tau$ which means the adiabatic transport of the edge Majorana is completely forbidden and $P_{\text{def}}$ is always equal to 1/2 implying that the edge Majorana gets completely delocalized within the bulk modes as reported in Ref. 23. Moreover, $P_{\text{def}}$ and $P_{\text{neg}}$ fall on top of each other signifying that evolved edge Majorana modes get delocalized within the same number of positive and negative energy interior bulk states with an equal probability for all values of $\tau$ (see Fig. 3).}

Furthermore, analyzing the results presented in Figs. 2(a, b), one can establish a relation between $\phi$ and $\tau_{c}$, which dictates the positions of the first significant dip (peak) in $P_{\text{def}}$ ($P_{m}$). We observe that the value of $\tau_{c}$ increases with decreasing $\phi$. Defining two instants of time when the system enters and leaves the gapless phase as $t_{e}$ and $t_{o}$, respectively, the passage time through the gapless phase is found to be $\Delta t = t_{o} - t_{e} = \pi \sin \phi$. We observe that there exists an optimal value of $\Delta t$ which is independent of $\phi$ (for a given $N$) when $P_{m}$ becomes non-zero; this is then related to the characteristic value of $\tau$ as

$$\Delta t_{\text{opt}} = \tau_{c} \sin \phi. \quad (7)$$

It is noteworthy, that $\tau_{c}$ itself depends on $\phi$ and diverges when $\phi \to 0$, so that $P_{m} = 0$, for all values of $\tau$ suggesting the impossibility of an adiabatic transport of the edge Majorana in that case. We plot $\ln \tau_{c}$ against $\ln \sin \phi$ in Fig. 3(b) which establishes the relation between $\tau_{c}$ and $\sin \phi$ as given in Eq. (7) when $\Delta t_{\text{opt}}$ is fixed.

A close observation of Fig. 2(a, b) suggests that there exists a set of relations of the same form like Eq. (7) for the passage times $\Delta t_{\text{op}1}, \Delta t_{\text{op}2}, \ldots, \Delta t_{\text{opn}}$ ($\Delta t_{\text{op}(n-1)} < \Delta t_{\text{opn}}$) associated with the peaks in $P_{m}$; where $n$ is the number of peaks in $P_{m}$. The vanishing adiabatic transition probability $P_{m}$ in the limit $\tau < \tau_{c}$, suggests the existence of a threshold value of transit time $\Delta t_{\text{th}} = \Delta t_{\text{opt}}$, which is a function of $N$, below which the adiabatic tunneling of edge Majorana is forbidden. The variation of $P_{m}$, on the other hand, is shown as a function of $\tau$ for different values of system size with $\phi = \pi/5$ in Fig. 4(a). Fig. 4(b) shows that $\tau_{c}$ increases linearly with $N$ for a fixed $\phi$.

![FIG. 3: (Color online) (a)Plots of $P_{\text{def}}, P_{\text{neg}}$ and $P_{m}$ with $\tau$ for $\phi = \pi/10$ show that all of them add up to unity. (b) The plot shows a linear variation of $\ln(\tau_{c})$ as a function of $\ln(\sin \phi)$ with slope $(\approx -0.9)$ nearly equal to -1.](image)

![FIG. 4: (Color online) (a)Plots of $P_{m}$ as a function of $\tau$ for different values of $N$ with a fixed $\phi = \pi/5$. $P_{m}$ shows peak at different values of $\tau \geq \tau_{c}$. (b) The figure shows a log-log plot between $\tau_{c}$ and $N$ for the above $\phi$ with slope $(\approx 0.94)$ nearly equal to 1 confirming $\tau_{c} \sim N$.](image)

We shall make a conjecture for the adiabatic transport of edge Majorana based on the following observations: the optimum transit time $\Delta t_{\text{opt}}$ governing the adiabatic passage, essentially depends only on the system size $N$ whereas it marginally depends on the length of the quenching for the path $\mu = 0$, and the phase $\phi$ of the complex hopping term. We propose that for passage times of the order of (or greater than) $\Delta t_{\text{th}}$, the time evolved Majorana starts delocalizing only within the
inverted energy levels (near zero energy) present inside the gapless region. For $\Delta t < \Delta t_{\text{th}}$, in contrary, the initial Majorana state interacts with all the energy levels including the inverted levels (see Fig. 5) prohibiting the possibility of an adiabatic transfer. The probability of getting an edge Majorana at the other phase after the quenching is maximum when the time evolved Majorana state interacts with a minimal number of inverted bands so that the associated wave-function is closely related to the equilibrium wave-function of the edge Majorana at the other phase. This signature of efficiency in adiabatic tunneling is reflected in the Fig. (2b, 4a) where the first significant peak height signature of efficiency in adiabatic tunneling is reflected. The plot signifies that in the limit of $\tau_c \ll \Delta t$, the Majorana gets delocalized only with the inverted bands.

In short, the conjecture is the following: in the limit of small $\tau$, the Majorana gets delocalized over all the levels and hence there is no possibility of recombination after passage through the gapless phase. On the other hand, for finite $\tau$ (when $\tau$ is around $\tau_c$ and above that), the Majorana, in fact, gets delocalized only with the inverted region. Remarkably, there exists some optimal values of the passage time within the gapless phase for which the Majorana recombines partially from the inverted bands and one gets an adiabatic transfer to the other phase with a finite probability. Moreover, the adiabatic transport for an optimum transit time is more probable when the time evolved Majorana mixes with a minimal number of inverted bands.

Finally, the question remains why do $\tau_c$ and hence $\Delta t_{\text{top}}$ increase with the system size as shown in Fig. 1 (4). The energy difference between two consecutive energy levels within the “inverted” region, decreases as $N$ increases, leading to an increase in the characteristic relaxation time of the system and therefore a higher value of $\tau_c$ would be necessary for an adiabatic passage.

In summary, we discuss the quench dynamics of a modified version of a 1D $p$-wave superconducting chain by varying the superconducting gap term in a linear fashion in time. We observe that there exist a finite probability of tunneling of an edge Majorana from one topological phase to the other at certain characteristic $\tau$ and $\phi$, for an optimum transit time through the intermediate extended gapless region. We attribute the phenomenon of tunneling to the mixing of the time evolved Majorana states only with the inverted energy levels which is only possible above a threshold transit time that the system requires to cross the gapless region. Furthermore, there is a possible recombination of the Majorana state for some optimal values of the passage time. Interestingly, for an infinite system (in the thermodynamic limit), the threshold passage time $\Delta t_{\text{th}}$ diverges and hence the adiabatic transport is impossible.

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30 For more details see supplementary material.
Supplementary Material on “Possibility of an adiabatic transport of an edge Majorana through an extended gapless region”

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Here, we give some details that are not presented in the main text. In Sec. II we first analyze the energy levels of the Hamiltonian (1) with periodic as well as open boundary conditions. We also discuss about the irreducible coupling between a and b Majoranas for any non-zero \( \phi \). We then present quench dynamics of the model with the periodic boundary condition (in Sec. III) and find that the density of defect satisfies the Kibble-Zurek scaling law with a renormalized rate rescaled by the phase of the complex hopping term. In Sec. IV we describe the quenching of edge Majorana considering a non-linear time variation of superconducting interaction term. In Sec. V we propose an argument in favour of localization of edge Majorana in real space by considering the model in momentum space.

I. THE SPECTRUM WITH PERIODIC, OPEN BOUNDARY CONDITIONS AND AN IRREDUCIBLE COUPLING BETWEEN a AND b MAJORANAS

Here, we shall analyze the energy spectrum of the Hamiltonian (5) with periodic and open boundary conditions as shown in Fig. (S1a) and Fig. (S1b), respectively. The presence of two zero-energy lines in the spectrum of Hamiltonian with open boundary condition signifies that the phase I and II host two zero-energy Majorana modes at each end of the chain. In contrary, the system with periodic boundary condition does not have any edge Majorana mode. The diagram also shows that two inverted cones for both the positive and negative levels are present near zero energy. There is an inversion where the outer most bulk energy level, that becomes nearest to zero energy line at \( \xi = -\sin \phi \), bends towards the interior bulk up to \( \xi = 0 \) without crossing the next energy level; after that it again bends in the opposite direction and becomes closest to the zero energy line at \( \xi = \sin \phi \) above which it again becomes the outer most bulk energy level. The range of \( \xi \) within which bending of energy levels occurs is decreasing as one moves away from the zero energy level towards the interior bands. As a result an inverted cone with vertex at \( \xi = \Delta = 0 \) appears at both side of the zero energy line. The inverted cones persist upto \( \xi = \Delta = \pm \sin \phi \) (other two vortices of the cone). Cone like structures are also appearing deep inside (far away from zero energy) the positive and negative energy levels. The interior cone like structures are missing for \( \phi = \pi/2 \) as the inverted cones (both side of the zero energy) eat up that interior region. One can also see that the number of inverted levels increases with increasing \( N \) and \( \phi \) (see Fig. S2).

![FIG. S1: (Color online) Plot shows the variation of energy levels as a function of parameter \( \xi = \frac{\Delta}{w_0} \) (with \( w_0 = 1 \)) for periodic (a) and open (b) boundary conditions with \( \phi = \pi/10 \) and \( N = 100 \).](image)

One can see that a and b Majorana particles are coupled to each other in an irreducible manner which is an outcome of the non-zero phase in the complex hopping term. The Heisenberg equations of motion for the zero-energy Majorana
modes \((a_n \text{ and } b_n)\) using \([S1]\) are then given by

\[
i \dot{a}_n = -[H, a_n] = 0, \quad i \dot{b}_n = -[H, b_n] = 0,
\]

\[
w_0 \cos \phi (b_{n+1} + b_{n-1}) - \Delta (b_{n+1} - b_{n-1}) + \mu b_n + w_0 \sin \phi (a_{n+1} - a_{n-1}) = 0,
\]

\[
w_0 \cos \phi (a_{n+1} + a_{n-1}) - \Delta (a_{n+1} - a_{n-1}) + \mu a_n + w_0 \sin \phi (b_{n+1} - b_{n-1}) = 0. \tag{S1}
\]

By numerically diagonalizing the Hamiltonian \([S1]\), one can show that \(a_1\) and \(b_1\) are indeed coupled though the probability of \(a_1\) is much higher than having a \(b_1\) at the left edge of the chain if one chooses the same set of parameter values as given in the main text. An identical situation occurs at the right edge of the chain where both \(a_N\) and \(b_N\) exist with the probability of \(b_N\) being much higher than that of \(a_N\). A topological invariant number which is different in two topological phases for this ETRS broken Hamiltonian has been introduced in the Ref. \([S6]\).

### II. QUENCHING DYNAMICS OF THE PERIODIC CHAIN

In this section, we will study the quenching dynamics of the Hamiltonian \([\text{I}]\) with a periodic boundary condition (when the Majorana edge states do not exist) choosing a linear time variation of the superconducting term of the form \(\Delta(t) = -1 + 2 t / \tau\), where \(\tau (\gg 1)\) is the inverse of quenching rate and time \(t\) runs from 0 to \(\tau\); without any loss of generality, we shall choose the path \(\mu / w_0 = 0\). As a result, the system is quenched from phase II to phase I in Fig. \([\text{I}]\) through the gapless region. The vertical span of the gapless region is maximum for the chosen path.

The quenching dynamics through an extended gapless phase leads to some interesting observations (e.g., exponentially decaying defect density with quenching rate) which have studied extensively\([S2]\). Here, we shall estimate the defect density as a function of \(\tau\) and \(\phi\) following the above mentioned adiabatic quench through the extended gapless region.

In momentum space, the Hamiltonian \([\text{I}]\) gets decoupled to different \(k\)-modes: \(H = \sum_{k=0}^{N} h_k\) with \(h_k\) being a \(2 \times 2\) matrix \([2]\); the reduced \(2 \times 2\) space spanned by the basis vectors \(|0\rangle\) (with no quasi-particle) and \(|k, -k\rangle\) (with quasi-particles having opposite momenta \(k\) and \(-k\)). Along the chosen quenching path \(\mu = 0\), there exist a finite number of degenerate critical momentum modes for which energy gap vanishes within the gapless region with \(k_c = \sin^{-1}(\pm \cos \phi/\sqrt{1 - \Delta^2})\). For the positive interval lying between \(0 < k_c < \pi\), these modes are ranging symmetrically around \(k_c = \pi/2\) (critical mode at the boundary of the gapless region \(\Delta = \pm \sin \phi\) ) starting from a degenerate critical mode \(k_c = \pi/2 + \phi\) (critical mode at the center of the gapless region \(\Delta = 0\) ) to the other degenerate critical mode \(k_c = \pi/2 - \phi\) (critical mode at the center of the gapless region \(\Delta = 0\) ) same as the negative side of the interval lying between \(-\pi < k_c < 0\) where \(k_c = -\pi/2\) is the central critical mode. To derive the scaling of the
defect density, we shall make resort to the Landau-Zener (LZ) transition formula following an appropriate unitary transformation we can re-write the $2 \times 2$ matrix $h_k$ as
\[ h_k(t) = (2 \sin \phi \sin k) I + (2 \Delta(t) \sin k) \sigma^x + (2 \cos \phi \cos k) \sigma^y, \] (S2)
where the time dependent parameter $\Delta(t)$ is shifted to the diagonal.

The time evolution of the system under the above mentioned quenching scheme is governed by the time dependent Schrödinger equation
\[ i \frac{\partial |\psi_k(t)\rangle}{\partial t} = h_k |\psi_k(t)\rangle, \] (S3)
where at any instant $t$, the state $|\psi_k(t)\rangle$ can be written as $|\psi_k(t)\rangle = u_k(t)|1_k\rangle + v_k(t)|2_k\rangle$, where $u_k(t)$ and $v_k(t)$ are time-dependent amplitudes and we have chosen the initial condition: $u_k(0) = 1$ and $v_k(0) = 0$. The point to note here that $|1_k\rangle$ and $|2_k\rangle$ can be written as a linear combination of $|0\rangle$ and $|k, -k\rangle$.

The above Schrödinger equation can be solved analytically for each momentum mode and an exact form of excitation probability ($p_k$) at final time ($t \to \infty$) can be obtained using the LZ non-adiabatic transition probability. The Hamiltonian $S2$ consists of two parts: the term with the identity operator that does not play any role in time evolution through essential to achieve the extended gapless region while the dynamics is dictated by the $2 \times 2$ LZ term. One can then readily obtain the probability of defect at the final state for each mode
\[ p_k = e^{-2\pi\gamma_k}, \] (S4)
where $\gamma_k = \delta_k^2 |\frac{d}{dt}(E_1 - E_2)|$, $\delta_k = 2 \cos \phi \cos k$ and $E_{1,2} = \pm 2\Delta(t) \sin k$. In the thermodynamic limit, one can calculate the defect density by integrating the $p_k$ over the $k$ modes lying within the 1st Brillouin zone
\[ n = \frac{2\pi}{N} \int_{-\pi}^{\pi} p_k dk \sim \frac{1}{\pi \cos \phi \sqrt{\tau}} \sim \frac{1}{\pi \sqrt{\tau}} \frac{1}{\sqrt{1 - W_d^2/4}}. \] (S5)

![FIG. S3](image-url)

**FIG. S3**: (Color online) (a) The logarithm of defect density $\ln n$ with the logarithm of quench time $\ln \tau$ for $\phi = \pi/4$ and $\pi/10$ are plotted. (b) The plot shows the variation of $\ln n$ with $\ln \cos \phi$ for a quench time $\tau = 200$ which confirms the $\phi$ dependence of $n$ given in S6.

In deriving the above relation, one has to use the fact that the maximum contribution to the defect comes from the modes close to the “critical” mode, for which the gap vanishes for the LZ part, $\sim \sqrt{\Delta^2 \sin^2 k + \cos^2 \phi \cos^2 k}$ which vanishes for $k = \pi/2$ when $\Delta = 0$. In other words, the dynamics is completely insensitive to the gapless region generated by the identity term of $S2$.

The scaling of the defect density as given in Eq. S6 clearly satisfies the KZ scaling $n \sim \tau^{-\nu d/\nu z + 1}$, where $d$ is the spatial dimensionality, with $d = z = \nu = 1$. However, we would like to highlight a subtle point here: when comparing with the conventional KZ scaling obtained for $\phi = 0$, we observe that in the scaling S6, effectively the parameter $\tau$ gets renormalized to $\tau_{\text{eff}} = \tau \cos^2 \phi$ as a consequence of the phase term in the hopping amplitude. The scaling of $n$ as obtained via direct numerical integration of Eq. S3 is presented in Fig. S3 which indeed corroborates the analytical prediction.

The above study can be generalized to the non-linear quenching5 of the form $\Delta = -1 + 2(t/\tau)\alpha$, where $t$ goes from 0 to $\tau$, and $\alpha > 0$; the defect density has been found to satisfy a scaling relation $n \sim 1/(\cos \phi \tau^{\alpha/(\alpha + 1)})$, where the result for the linear case is retrieved for $\alpha = 1$. 
III. ADIABATIC TRANSPORT OF EDGE MAJORANA IN A POWER LAW QUENCH

![Graph](image)

FIG. S4: (Color online) (a) Plots of $P_m$ for a non-linear time variation with $\alpha = 1.5$ as a function of $\tau$ with different values of $\phi$. (b) Plots of $P_m$ as a function of $\tau$ for different values of $\phi$ with $\alpha = 2$. The right most panel shows the plot of a linear variation of $\ln \tau_c$ as a function of $\ln \sin \phi$ with slope ($=-0.77$) nearly equal to -0.75 for $\alpha = 2$. The plot justifies that $\tau_c$ is proportional to $\sin^\alpha (\alpha + 1)/2\alpha$ for a fixed $\Delta t_{op}$. Here $N = 100$.

We generalize the quenching scheme to a non-linear quenching of the superconducting gap parameter using the protocol $\Delta = -1 + 2(t/\tau)^\alpha$ and then examine the adiabatic transport probability of edge Majorana with quench rate. Here as well, we find a finite tunneling probability of the edge Majorana from phase II to phase I for a characteristic quench time $\tau_c$ which is a function of $\phi$ and the non-linearity $\alpha$ of the quenching scheme (see Fig. (S4)). In fact, one can propose a scaling form $\Delta t = \sin \phi \, \tau^{2\alpha/\alpha+1} f(\tau/\tau_c)$ which has been constructed from a dimensionless combination of the hopping amplitude ($w_0 = 1$) and $\tau$. Here we can see that as $\tau < \tau_c$, the scaling function $f(\tau/\tau_c) \sim \text{constant}$, implying $\Delta t < \Delta t_{th}$ which is in accordance with our results that the adiabatic transportation is forbidden below a threshold transit time. On the other limit, $\tau \geq \tau_c$, $f(\tau/\tau_c) \sim (\tau_c/\tau)^{2\alpha/\alpha+1}$. Optimal transit times look like $\Delta t_{op} = \sin \phi \, \tau_c^{2\alpha/\alpha+1} \geq \Delta t_{th}$. As a result, $P_m$ shows first significant peak at $\tau_c$ followed by a few peaks for $\tau > \tau_c$. The variation of $\tau_c$ with $\phi$ is shown in Fig. (S4) which closely matches with this predicted scaling form. Therefore, we can say that the possibility of adiabatic transport of edge Majoranas for this model shows up even for a non-linear quenching.

IV. SIGNATURE OF LOCALIZATION OF MAJORANA

In this section, we analyze the Hamiltonian with PBC (given in Eq. (S2)) further. We consider the LZ part of the Hamiltonian which can be written as

$$h_k^{LZ} = \xi_k \, s^z + \delta_k \, s^y,$$

(S6)

with $\xi_k = 2\Delta \sin k$ and $\delta_k = 2 \cos \phi \cos k$. The above Hamiltonian can be expressed in terms of the Bogoliubov operators which reduces the Hamiltonian to a diagonal one. The transformation relations are given by

$$b_k = u_k \, f_k - iv_k \, f_k^\dagger,$$

$$b_{-k} = u_k \, f_{-k} + iv_k \, f_{-k}^\dagger,$$

(S7)

where $b_k$ and $b_{-k}$ are Bogoliubov fermionic operators which satisfy the anti-commutation relations. The parameters ($u_k$ and $v_k$) satisfy the relation $|u_k|^2 + |v_k|^2 = 1$ for each $k$-mode and are defined as

$$u_k = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi_k}{\varepsilon_k}}, \quad v_k = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi_k}{\varepsilon_k}},$$

(S8)

where $\varepsilon_k = \sqrt{\xi_k^2 + \delta_k^2}$. Finally, the Hamiltonian in Eq. (2) can be written in term of the Bogoliubov fermions as

$$H = \sum_{k>0} \varepsilon_k \, b_k^\dagger b_k$$

(S9)
For $\phi = 0$, the values of $u_k$ and $v_k$ are the same ($= 1/\sqrt{2}$) at the critical point ($\Delta = 0$). The equilibrium edge Majorana states at the critical point can be cast only by an equal superposition of positive and negative energy bulk excitations associated with the $k_c = \pm \pi/2$ over the Bogoliubov vacuum. As a consequence of that the edge Majorana gets equally delocalized throughout the chain in real space as soon as system crosses the gapless critical line separating two topological phases.

For $\phi \neq 0$, on the other hand, there is an extended gapless region bounded by two horizontal lines $\Delta = \pm \sin\phi$ (see the phase diagram). Within the gapless phase, $u_k \neq v_k$, for any $\Delta \neq 0$ while they are only equal when $\Delta = 0$. Therefore, we have a large number of Bogoliubov operators (given in Eq. (S7)) associated with the critical modes $(\pi/2 - \phi$ to $\pi/2 + \phi$ and $-\pi/2 - \phi$ to $-\pi/2 + \phi$) within the gapless region. Consequently, in the momentum space the system is not highly localized near only two bulk gapless critical modes as happens for the case of $\phi = 0$. The equilibrium edge Majorana states can not be written by an unequal superposition of positive and negative energy bulk excitation associated with the critical momentum modes present inside the gapless region for any non-zero $\Delta$. This implies that the edge Majoranas do not delocalize uniformly throughout the chain when the system crosses the extended gapless quantum critical region separating two topological phases.

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