EXISTENCE OF TWO VIEW CHIRAL RECONSTRUCTIONS

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ABSTRACT. A fundamental question in computer vision is whether a set of point pairs is the image of a scene that lies in front of two cameras. Such a scene and the cameras together are known as a chiral reconstruction of the point pairs. In this paper we provide a complete classification of $k$ point pairs for which a chiral reconstruction exists. The existence of chiral reconstructions is equivalent to the non-emptiness of certain semialgebraic sets. We describe these sets and develop tools to certify their non-emptiness. For up to three point pairs, we prove that a chiral reconstruction always exists while the set of five or more point pairs that do not have a chiral reconstruction is Zariski-dense. We show that for five generic point pairs, the chiral region is bounded by line segments in a Schl"afli double six on a cubic surface with 27 real lines. Four point pairs have a chiral reconstruction unless they belong to two non-generic combinatorial types, in which case they may or may not.

1. INTRODUCTION

A fundamental question in computer vision is whether a set of point pairs $P = \{(u_i, v_i) : i = 1, \ldots, k\}$ is the image of a set of world points $q_i$ that are visible in two cameras. If we ignore the constraint (as is commonly done) that the points $q_i$ need to lie in front of the cameras, we get a projective reconstruction [12]. In reality though, cameras can only see points in front of them. A reconstruction that obeys this additional constraint is known as a chiral reconstruction [1, 11]. The aim of this paper is to give a complete answer to the question: Given a set of point pairs $P = \{(u_i, v_i), i = 1, \ldots, k\}$, when does $P$ admit a chiral reconstruction?

Under the assumption that the points in each image are distinct, we prove the following facts.

(1) A set of at most three point pairs always has a chiral reconstruction.
(2) A set of four point pairs has a chiral reconstruction unless the configurations are of two specific non-generic types, in which case a chiral reconstruction may not exist.
(3) Five or more point pairs can fail to have a chiral reconstruction with positive probability (in particular, even if they are in general position).
(4) For five sufficiently generic point pairs, the problem translates to finding points in semialgebraic regions on a real cubic surface whose boundaries are segments of lines in a Schl"afli double six of real lines, creating an unexpected bridge to classical results in algebraic geometry.

The study of chirality was initiated in [11] with follow up work by Werner, Pajdla and others [21, 22, 23, 24]. There is no agreement on the name for reconstructions that are chiral. Hartley in [11] and Werner & Pajdla in [24] call them strong realizations. Werner in [21, 22, 24] calls them oriented projective reconstructions. Here and in our previous work [1], we prefer the term chiral reconstruction. In [11], Hartley shows that chiral reconstructions are a special class of quasi-affine reconstructions. See [1 Section 5] for a detailed account of quasi-affine reconstructions in the context of our approach to chirality. Hartley’s work was done using projective geometry. In later work, such as [21, 22] and [23], the authors use oriented projective geometry [10], which also explains their use of the term oriented projected reconstructions. Following Hartley, our work uses projective geometry. Justifications of our choice of framework can be found in [1 Section 1]. We now comment briefly on how our results relate to the existing literature in computer vision and give more detailed citations in later sections.

A specific example of five point pairs in general linear position that do not admit a chiral reconstruction was given in [22] and appears again in [21]. Our result (3) shows that, in fact, the set of five point pairs without a chiral reconstruction is Zariski dense. The case of four point pairs covered in result (2) is the most involved since it does not assume genericity, and covers all possible configurations of four point pairs. We show that chiral reconstructions can fail in this case only in degenerate situations. To the best of our knowledge, both results (2) and (3) are novel. Result (1) says that three point pairs admit a chiral reconstruction unconditionally. While this is not too hard to prove, it also does not appear in the literature and completes the story.
Result (4) establishes a new connection between chiral reconstructions of five generic point pairs and the classical theory of cubic surfaces from algebraic geometry. The cubic surface in \( \mathbb{P}^3 \) arises naturally in our set up and the two camera planes, modeled as \( \mathbb{P}^2 \times \mathbb{P}^2 \), can be obtained by blowing down this surface. The papers \cite{22} and \cite{21} study chirality in the setting of \( \mathbb{P}^2 \times \mathbb{P}^2 \) using tools from oriented matroids. It was shown in \cite{22} that the chiral regions are bounded by certain conics. Our results show that these conics are obtained by blowing down a Schläfli double six of real lines on the cubic surface, allowing the chiral regions to be seen as semialgebraic subsets of the cubic surface bounded by lines. Both frameworks produce a sixth pair of points \( \{u_i, v_i\} \) whose existence was derived in \cite{22} using the conics in \( \mathbb{P}^2 \times \mathbb{P}^2 \). Our results show that the cubic surface is the blow up of the six points \( \{u_i, v_i\} \) in each \( \mathbb{P}^2 \).

The blow up/blow down maps create a new interpretation of the results in \cite{22} while also offering a unified picture of chirality that transfers seamlessly between the space of camera epipoles and the space of fundamental matrices of \( \mathcal{P} \). We draw from, and build on, methods from the above mentioned papers from computer vision, and our own paper \cite{1}. Our main tools are centered in complex and real algebraic geometry as well as semialgebraic geometry.

This paper is organized as follows. Formal definitions of projective and chiral reconstructions can be found in Section 3. In Section 4 we introduce the inequalities imposed by chirality and develop tools to certify them. Along the way we prove that a set of at most three point pairs always has a chiral reconstruction.

In Section 5 we prove that a set of four point pairs has a chiral reconstruction when the point configurations in each view have sufficiently similar geometry, and in particular, when they are in general position. The bad cases fall into two non-generic combinatorial types. In particular, the probability of choosing four point pairs that fail to have a chiral reconstruction is zero.

In Section 6 we show that when \( k > 4 \), point pairs that are in general linear position may not have a chiral reconstruction. Specific examples of this type when \( k = 5 \) were known to Werner \cite{22} and as mentioned before, there are close connections between our work and that of Werner’s \cite{21,22,23,24}. We make two new contributions for the case \( k = 5 \). In Section 7 we show that one can decide the existence of a chiral reconstruction when \( k = 5 \) and \( \mathcal{P} \) is generic, by checking 20 discrete points. We use this test to show that the set of five point pairs that do not admit a chiral reconstruction is Zariski-dense. In other words, five point pairs do not have a chiral reconstruction with positive probability. A set of six or more point pairs can have a chiral reconstruction only if any subset of five point pairs among them have one. Hence, for any value of \( k > 5 \), there will be point configurations without a chiral reconstruction. Our second contribution in Section 7 is to show that the case of \( k = 5 \) is intimately related to the theory of cubic surfaces from classical algebraic geometry. Indeed, \( \mathcal{P} \) creates a Schläfli double six of 12 lines on a cubic surface, all of whose 27 lines are real. These lines determine the boundary of the semialgebraic regions corresponding to chiral reconstructions.

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2. Background and Notation

We now introduce some background and notation that will be needed in the paper. Let \( \mathbb{P}^n \) and \( \mathbb{P}^n_{\mathbb{R}} \) denote \( n \)-dimensional projective space over complex and real numbers respectively. More generally, we write \( \mathbb{P}(V) \) for the projective space over a vector space \( V \), which is the set of lines in \( V \). We write \( \mathbf{a} \sim \mathbf{b} \) if \( \mathbf{a} \) and \( \mathbf{b} \) are the same points in projective space, and reserve \( \mathbf{a} = \mathbf{b} \) to mean coordinate-wise equality. A projective camera is a matrix in \( \mathbb{P}(\mathbb{R}^{3 \times 4}) \) of rank three, i.e., a \( 3 \times 4 \) real matrix defined only up to scaling, hence naturally a point in the projective space over the vector space \( \mathbb{R}^{3 \times 4} \). Usually, an affine representative of a projective camera is fixed and we block-partition such a matrix as \( A = \begin{bmatrix} G & \mathbf{t} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \) where \( G \in \mathbb{R}^{3 \times 3} \) and \( \mathbf{t} \in \mathbb{R}^3 \). The center of \( A \) is the unique point \( \mathbf{c}_A \in \mathbb{P}^3_{\mathbb{R}} \) such that \( A \mathbf{c}_A = 0 \). The camera \( A \) is a rational map from the "world" \( \mathbb{P}^3 \) to the "camera plane" \( \mathbb{P}^2 \) sending a "world point" \( \mathbf{q} \) to its "image" \( A \mathbf{q} \). It is defined everywhere except at \( \mathbf{c}_A \). Consider the hyperplane at infinity in \( \mathbb{P}^3 \), \( L_{\infty} = \{ \mathbf{q} \in \mathbb{P}^3 : \mathbf{n}_x^\top \mathbf{q} = 0 \} \), as an oriented hyperplane in \( \mathbb{R}^4 \) with fixed normal \( \mathbf{n}_x = (0, 0, 0, 1)^\top \). The camera \( A \) is said to be finite if \( \mathbf{c}_A \) is a finite point, i.e., \( \mathbf{c}_A \notin L_{\infty} \). A special representative of a camera center can be obtained by Cramer’s rule where the \( i \)th coordinate of \( \mathbf{c}_A \) is the determinant of the submatrix of \( A \) obtained by dropping the \( i \)th column. In particular, for a finite camera \( A = \begin{bmatrix} G & \mathbf{t} \end{bmatrix} \), the Cramer’s rule center is \( \mathbf{c}_A = \det(G)(-G^{-1}\mathbf{t}, 1)^\top \).

Throughout this paper we use the Cramer’s rule representation of \( \mathbf{c}_A \). The camera \( A = \begin{bmatrix} G & \mathbf{t} \end{bmatrix} \) is finite if and only if \( \det(G) \neq 0 \), and all cameras in this paper will be finite.
The principal plane of a finite camera \( A = [G \ t] \) is the hyperplane \( L_A := \{q \in \mathbb{P}^3 : A_3 \cdot q = 0\} \), where \( A_3 \cdot q \) is the third row of \( A \), i.e. it is the set of points in \( \mathbb{P}^3 \) that image to infinite points in \( \mathbb{P}^2 \). Note that the camera center \( c_A \) lies on \( L_A \). We regard \( L_A \) as an oriented hyperplane in \( \mathbb{R}^4 \) with normal vector \( n_A := \det(G)A^+_3 \), which we call the principal ray of \( A \). The \( \det(G) \) factor ensures that the normal vector of the principal plane does not flip sign under a scaling of \( A \). The depth of a finite point \( q \) in a finite camera \( A \) is defined as (see [12])

\[
\text{depth}(q; A) := \left( \frac{1}{\| \det(G) \| \| G_A \|} \right) \frac{(n_A^\top q)}{(n_A^\top n_A)}.
\]

Note that the sign of \( \text{depth}(q; A) \) is unaffected by scaling \( q \) and \( A \). The depth of a finite point \( q \in \mathbb{P}^3 \) in a finite camera \( A \) defined in Equation (1) is zero if and only if \( n_A^\top q = 0 \), which happens if and only if \( q \) lies on the principal plane \( L_A \). Otherwise, \( n_A^\top q \neq 0 \) and \( \text{sgn}(\text{depth}(q; A)) = \text{sgn}(\langle n_A^\top q \rangle) \) is either positive or negative. It is then natural to say that a finite point \( q \) is in front of the camera \( A \) if \( \text{depth}(q; A) > 0 \), see [11]. Since only the sign of \( \text{depth}(q; A) \) matters, we refer to this sign as the chirality of \( q \) in \( A \), denoted as \( \chi(q; A) \), which is either 1 or -1.

The above notion of chirality was introduced by Hartley in the seminal paper [11], where he was concerned with a pair of cameras, see also [12, Chapter 21]. In [11], the definition of chirality was extended to all points in \( \mathbb{P}^3 \), finite and infinite, and defined for an arrangement of cameras. Here is the two camera version we need.

**Definition 2.1.** Let \((A_1, A_2)\) be a pair of finite projective cameras. Then the chiral domain of \((A_1, A_2)\), is the Euclidean closure in \( \mathbb{P}^3 \) of the set

\[ \{q \in \mathbb{P}^3 \mid q \text{ finite}, \ \text{depth}(q, A_1) > 0, \ \text{depth}(q, A_2) > 0\}. \]

A point \( q \in \mathbb{P}^3 \) is said to have chirality 1 with respect to \((A_1, A_2)\), denoted as \( \chi(q; (A_1, A_2)) = 1 \), if and only if \( q \) lies in the chiral domain of \((A_1, A_2)\).

In this paper we will be concerned with a pair of finite non-coincident cameras \((A_1, A_2)\) by which we mean that their centers are distinct. We will see that one can always take \( A_1 = [I \ 0] \), and then the conditions of finite and non-coincident imply that \( A_2 = [G \ t] \) where \( G \in GL_3 \) and \( t \neq 0 \). The pair \((A_1, A_2)\) gives rise to the unique (up to scale) real, rank two fundamental matrix \( X = [t] \times G \) where

\[
[t]_\times = \begin{bmatrix}
0 & -t_3 & t_2 \\
t_3 & 0 & -t_1 \\
-t_2 & t_1 & 0
\end{bmatrix}.
\]

The skew-symmetric matrix \([t]_\times\) represents the cross product with \( t \) as a linear map; that means that for \( t, r \in \mathbb{R}^3 \) we have \( t \times r = [t]_\times r = [r]^\top t \). Also, \( \text{rank}([t]_\times) = 2 \) if and only if \( t \neq 0 \). We are only ever interested in properties of fundamental matrices (mostly rank and kernels) that remain unaffected by scaling and will therefore mostly consider them up to scaling which is to say as a point of \( \mathbb{P}(\mathbb{R}^3) \). We chose not to introduce a different notation to distinguish the matrix itself from the line it spans.

The epipole pair of the cameras \((A_1, A_2)\) is \((e_1, e_2) \in \mathbb{P}^2 \times \mathbb{P}^2 \) where \( e_1 \) is the image of the center \( c_2 \) in \( A_1 \), and \( e_2 \) is the image of the center \( c_1 \) in \( A_2 \). The line joining the centers \( c_1 \) and \( c_2 \) is called the baseline of the camera pair \((A_1, A_2)\). Note that all world points on the baseline (with the exception of the respective camera centers) image to the epipole in each camera.

### 3. Projective and Chiral Reconstructions

Throughout this paper our input is a collection of point pairs \( \mathcal{P} = \{\{u_i, v_i\} : i = 1, \ldots, k\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \). Each \( u_i \) (and \( v_i \)) is the homogenization of a point in \( \mathbb{R}^2 \) by adding a last coordinate one. Hence \((u_i, v_i)\) is a pair of finite points in \( \mathbb{P}^2 \times \mathbb{P}^2 \) with a fixed representation. We will also assume that all \( u_i \) (and all \( v_i \)) are distinct.

In this section we formally define projective and chiral reconstructions of \( \mathcal{P} \), and characterize their existence. We then set up the geometric framework within which these reconstructions will be studied in this paper.

#### 3.1. Projective reconstructions.

**Definition 3.1.** A projective reconstruction of \( \mathcal{P} \) consists of a pair of projective cameras \( A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3 \times 4}) \), world points \( Q = \{q_1, \ldots, q_k\} \subset \mathbb{P}^3 \) and non-zero scalars \( w_{1i}, w_{2i} \) such that \( A_1 q_i = w_{1i} u_i \) and \( A_2 q_i = w_{2i} v_i \) for \( i = 1, \ldots, k \). If the cameras and world points are all finite, then \((A_1, A_2, Q)\) is called a finite projective reconstruction of \( \mathcal{P} \).
The basics of projective reconstructions can be found in [12, Chapters 9 & 10]. Theorem 3.1 in [17] proves that if \( P \) has a projective reconstruction then it also has a finite projective reconstruction with \( A_1 = \begin{bmatrix} I & 0 \end{bmatrix} \).

We now recall the necessary and sufficient conditions for the existence of a finite projective reconstruction of \( P \). For a point \( e \in \mathbb{P}^2 \), let \( e(u_1, \ldots, u_k) \) denote the set of lines joining \( e \) to each \( u_i \). The following geometric characterization is well-known [12, 18, 22, 23].

**Theorem 3.2.** [12 Section 9.4], [18 Section 2.4] The set of point pairs \( P \) has a projective reconstruction \((A_1, A_2, Q)\) if and only if there exist points \( e \) so that the following conditions hold:

1. \( e(u_1, \ldots, u_k) \) is a finite projective reconstruction with \( e \).
2. \( e(u_1, \ldots, u_k) \) is \( e \)-regular.
3. \( e(u_1, \ldots, u_k) \) is \( e \)-regular in the sense of [17].
4. \( e(u_1, \ldots, u_k) \) is \( e \)-regular in our sense if and only if it is \( e \)-regular in the sense of [17].

The points \( e_1, e_2 \) in the above theorem are the epipoles of the camera pair \((A_1, A_2)\) in the reconstruction. We now give a second characterization of the existence of a projective reconstruction in terms of *fundamental matrices*: Below, we will abuse notation and not distinguish between the matrix and the line it spans in the space of matrices (or in other words the corresponding point in projective space). All relevant properties of the matrices are invariant under scaling.

**Definition 3.3.** (1) A fundamental matrix of \( P \) is a rank two matrix \( X \in \mathbb{P}(\mathbb{R}^{3 \times 3}) \) such that

\[
X_v^T X u_i = 0, \quad \text{for} \quad i = 1, \ldots, k.
\]

The linear equations \( X_v^T X u_i = 0 \) in \( X \) are called the epipolar equations of \( P \).

(2) Given a rank two matrix \( X \in \mathbb{P}(\mathbb{C}^{3 \times 3}) \) and a pair \((u, v)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \), such that \( X_v^T X u = 0 \), we say that \( X \) is \((u, v)\)-regular if \( X_v^T X u = 0 \), and if only if \( X_v^T X u = 0 \). i.e., \( u \) and \( v \) simultaneously generate the right and left kernels of \( X \), or neither generate a kernel.

(3) A \( P \)-regular fundamental matrix is a fundamental matrix of \( P \) that is \((u_i, v_i)\)-regular for each point pair in \( P \).

It is commonly believed that \( P \) has a projective reconstruction if and only if it has a fundamental matrix. However, a bit more care is needed as in the following theorem.

**Theorem 3.4.** [17 Theorem 4.6] There exists a finite projective reconstruction of \( P \) with two non-coincident cameras, and \( A_1 = \begin{bmatrix} I & 0 \end{bmatrix} \), if and only if there exists a \( P \)-regular fundamental matrix.

**Remarks 3.5.** (1) Let \( A_1 = \begin{bmatrix} I & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} G & t \end{bmatrix} \) be the two finite non-coincident cameras in the projective reconstruction. Then recall that \( G \in \text{GL}_3 \), \( t \neq 0 \), and the cameras correspond to the unique fundamental matrix \( X = [t]_X G \) up to scale. The epipoles of the cameras are \( e_1 \sim G^{-1} t \) and \( e_2 \sim t \) which generate the right and left kernels of \( X \). Further, \( G \) defines a homography of \( \mathbb{P}^2 \) that sends \( e_1 \) to \( e_2 \), and the line \( e_1 u_i \) to the line \( e_2 v_i \). Hence \( G \) encodes the epipolar line homography of Theorem 3.2.

(2) Conversely, any (rank two) fundamental matrix \( X \in \mathbb{P}(\mathbb{C}^{3 \times 3}) \) can be factored as \( X = [t]_X G \) for some \( t \in \mathbb{R}^3 \) and \( G \in \text{GL}_3 \) and yield a pair of cameras \((A_1 = \begin{bmatrix} I & 0 \end{bmatrix}, A_2 = \begin{bmatrix} G & t \end{bmatrix})\) whose epipoles generate the left and right kernels of \( X \). See [12 Section 9.5] for more details on the correspondence between fundamental matrices and camera pairs. These cameras can then be used to reconstruct a set of world points \( Q \) if \( X \) is \( P \)-regular. The resulting projective reconstruction is said to be associated to \( X \).

(3) A fundamental matrix is \( P \)-regular if and only if for each \( i \), either \( u_i \) and \( v_i \) are both epipoles of the cameras or neither are. Indeed, this is a necessary condition for a projective reconstruction since if \( u_i \sim e_1 \), then its reconstruction \( q_i \in \mathbb{P}^3 \) lies on the baseline of the cameras and hence images to \( e_2 \) in camera \( A_2 \) requiring \( v_i \sim e_2 \). This subtlety is often overlooked, and it is common to equate the existence of a projective reconstruction of \( P \) to the existence of a fundamental matrix of \( P \).

(4) Lastly, we remark that [17 Theorem 4.6] is stated using a different notion of regularity. However, a fundamental matrix \( X \) is \( P \)-regular in our sense if and only if it is \([t]_X X, t\)-regular in the sense of [17] and hence the above theorem is exactly [17 Theorem 4.6].

We now discuss the geometry encoded in Theorem 3.4 which will set the foundation for the work in this paper. Even though fundamental matrices are real, we will work over \( \mathbb{C} \) to allow for methods from complex algebraic geometry, and will specialize to \( \mathbb{R} \) as needed. A matrix \( X \in \mathbb{P}(\mathbb{C}^{3 \times 3}) \) can be identified with a point in \( \mathbb{P}^8 \) by concatenating its rows. Under this identification we let \( R_2 \subset \mathbb{P}^8 \) be the determinantal hypersurface of matrices in \( \mathbb{P}(\mathbb{C}^{3 \times 3}) \) of rank at most two, and \( R_3 \) be its subvariety of rank one matrices. As projective subvarieties of \( \mathbb{P}^8 \), \( \dim R_2 = 7 \), \( \dim R_3 = 6 \), \( \dim R_1 = 4 \). For a point pair \((u, v)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \), let \( L((u,v)) \) denote the hyperplane in \( \mathbb{P}^8 \):

\[
L((u,v)) = \{ X \in \mathbb{P}^8 : v^T X u = \langle X, vu^T \rangle := \text{Tr}(X^T vu^T) = 0 \}
\]
where $\langle \cdot , \cdot \rangle$ denotes the Frobenius inner product on matrices. Let $\mathcal{L}_P = \bigcap_{i=1}^k L_{(u_i, v_i)}$. Generically, $\mathcal{L}_P$ is a linear space in $\mathbb{P}^8$ of codimension $k$.

**Definition 3.6.** The variety $R_2 \cap \mathcal{L}_P$ in $\mathbb{P}^8$ is the epipolar variety of $P$.

Under sufficient genericity of $P$, $\dim (R_2 \cap \mathcal{L}_P) = 7 - k$ and $\deg (R_2 \cap \mathcal{L}_P) = 3$. Hence, the epipolar variety is empty when $k \geq 8$, consists of three points when $k = 7$, and infinitely many points when $k < 7$.

For $(u_i, v_i) \in P$, consider the following five-dimensional linear spaces of $\mathbb{P}^8$ that are in fact in $R_2$:

$$\tilde{W}_{u_i} := \{ X \in \mathbb{P}^8 : X u_i = 0 \} \quad \text{and} \quad \tilde{W}^{v_i} := \{ X \in \mathbb{P}^8 : v_i^T X = 0 \}.$$ Their intersections with the epipolar variety are the linear spaces: $W_{u_i} := \mathcal{L}_P \cap \tilde{W}_{u_i}$ and $W^{v_i} := \mathcal{L}_P \cap \tilde{W}^{v_i}$, each of which generically has dimension $6 - k$ since $\tilde{W}_{u_i}, \tilde{W}^{v_i} \subset L_{(u_i, v_i)}$.

**Definition 3.7.** The linear space $W_{u_i}$ (resp. $W^{v_i}$) will be called the $u_i$ (resp. $v_i$) **wall** and the intersection $W_{u_i} \cap W^{v_i}$ will be called the **$(u_i, v_j)$ corner**.

When $i \neq j$, a $(u_i, v_j)$ corner has dimension $5 - k$ generically since there are at most 5 independent equations among $v_j^T X = 0 = X u_i$. Thus a wall has codimension one and a corner has codimension two in the epipolar variety, generically. For non-generic data $P$, all the dimensions computed above may be larger.

The second condition in Theorem 2.4 can now be rephrased as the existence of a rank $2$ matrix $X$ in the epipolar variety $\mathcal{L}_P$ such that for each $i$, $X$ is either in the $(u_i, v_i)$ corner or in the complement of $W_{u_i} \cup W^{v_i}$. Note that a rank two $X$ can lie in at most one $(u_i, v_j)$ corner because the points $u_i$ (and $v_j$) are pairwise distinct.

Going forward, we will work both in $\mathbb{P}^8$, the space of fundamental matrices, and in $\mathbb{P}^2 \times \mathbb{P}^2$, the space of epipoles. These spaces are related by the **adjoint map**,

$$\text{adj} : \mathbb{P}^8 \dashrightarrow \mathbb{P}^8, \quad X \mapsto \text{adj}(X)$$

where $\text{adj}(X) = \text{cof}(X)^T$ and $\text{cof}(X)$ is the cofactor matrix of $X$. If $X \in R_2$ then $X \cdot \text{adj}(X) = \text{adj}(X) \cdot X = 0$ and thus, if $\text{rank}(X) = 2$, then all non-zero rows (resp. columns) of $\text{adj}(X)$ are multiples of each other and generate the left (resp. right) kernel of $X$. Since generators of the right and left kernels of a fundamental matrix represent epipoles, the adjoint map provides a convenient connection between epipole space and fundamental matrix space.

**3.2. Chiral reconstructions.** A physical constraint on a true reconstruction $(A_1, A_2, Q)$ is that the reconstructed world points in $Q$ must lie in front of the cameras $A_1$ and $A_2$. Recall from the Introduction that this means we require $Q$ to lie in the chiral domain of $(A_1, A_2)$ or equivalently, $\chi(q_i; (A_1, A_2)) = 1$ for all $q_i \in Q$. A full development of multiview chirality can be found in [1]. For this paper, we use the following inequality description of the chiral domain for two views from [1, Theorem 1] as a definition. The cited result shows that these inequalities cut out the Euclidean closure of the set in Definition 2.1 (under the mild assumption that it has non-empty interior).

**Definition 3.8.** A chiral reconstruction of $P$ is a projective reconstruction $(A_1, A_2, Q)$ of $P$ with finite non-coincident cameras such that for all $i$,

$$(n_x^T q_i)(n_1^T q_i) \geq 0, \quad (n_x^T q_i)(n_2^T q_i) \geq 0, \quad \text{and} \quad (n_1^T q_i)(n_2^T q_i) \geq 0$$

where $n_x = (0, 0, 0, 1)^T$ and $n_i$ is the principal ray of $A_i$.

Recall that two projective reconstructions $(A_1, A_2, Q)$ and $(A_1', A_2', Q')$ are **projectively equivalent** if they are related by a homography of $\mathbb{P}^3$, i.e., there is a $H \in \text{GL}_4$ such that $A_1' = A_1 H^{-1}$ and $Q' = H Q := \{ H q_i, i = 1, \ldots, k \}$. A projective reconstruction which is not chiral can sometimes be transformed into a chiral reconstruction by a homography [1, 11, 23]. We recall the conditions under which this is possible.

**Theorem 3.9.** Consider a finite projective reconstruction of $P$ with non-coincident cameras $A_1 = [I \ 0], A_2 = [G \ t]$, and world points $Q = \{ Q_1, \ldots, Q_k \} \subset \mathbb{P}^3$. Then the following are equivalent.

1. There exists a projectively equivalent chiral reconstruction $(A_1 H^{-1}, A_2 H^{-1}, HQ)$ of $P$.
2. $(n_1^T q_i)(n_2^T q_i)$ has the same sign for all $i$.
3. $w_{1;2} = w_{1;3}$ has the same sign for all $i$.

Furthermore, if no $q_i$ lies on the baseline of $(A_1, A_2)$, then (1), (2), (3) are equivalent to

4. $(t \times v_i)^T(t \times Gu_i)$ has the same sign for all $i$. 

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Proof. The equivalence of (1) and (2) is Theorem 8 in [1]. The equivalence of (1) and (3) is Theorem 17 in [1]. The epipoles of the given cameras are \( e_1 \sim G^{-1}t \) and \( e_2 \sim t \), and hence if no world point lies on the baseline, \( G^{-1}t \neq u_i \) (equivalently, \( t \neq Gu_i \)) and \( t \neq v_i \) for any \( i \). Also, since \( A_1,A_2 \) are non-coincident and \( t \neq 0 \), Theorem 3.2 and Item 1 imply that \( t,v_i,Gu_i \) are collinear. Therefore, \( (t \times v_i), (t \times v_i)^{\top}G \neq 0 \) for all \( i \). The equivalence of (3) and (4) can then be derived from the same arguments as in Lemma 7 in [1]. □

If a world point lies on the baseline, then its images \((u_i,v_i)\) in \( A_1,A_2 \) are the epipoles of the cameras, and the expression in (4) becomes zero. However, since any point on the baseline can serve as the world point \( q_i \), we can control the sign of \((n_1^i q_i, n_2^i q_i)\) as shown next.

**Lemma 3.10.** For a pair of non-coincident cameras \((A_1, A_2)\) whose baseline is not contained in either principal plane \( L_{A_1} \) and \( L_{A_2} \), there exist \( q_+ \) and \( q_- \) on the baseline such that \((n_1^i q_+)(n_2^i q_+) > 0 \) and \((n_1^i q_-)(n_2^i q_-) < 0 \).

Proof. Since \( c_1 \in L_{A_1}, n_1^i c_1 = 0 \). On the other hand, since the baseline is not contained in \( L_{A_2} \), \( c_1 \neq c_2, c_1 \notin L_{A_2}, \) and \( (n_2^i c_1) \neq 0 \). By continuity, there exist perturbations \( q_+ \) and \( q_- \) of \( c_1 \) on the baseline such that \((n_1^i q_+)(n_2^i q_+) > 0 \) and \((n_1^i q_-)(n_2^i q_-) < 0 \).

**Remarks 3.11.** For reconstructions where both epipoles are finite, the hypothesis of Lemma 3.10 is satisfied. Indeed, if for instance the baseline was contained in the principal plane \( L_{A_1} \), then \( c_2 \in L_{A_1} \) and so \( e_1 \sim A_1 c_2 \) would be an infinite point.

We now have a necessary and sufficient condition for the existence of a chiral reconstruction.

**Lemma 3.12.** There exists a chiral reconstruction of \( \mathcal{P} \) if and only if there exist \( t \in \mathbb{R}^3 \setminus \{0\} \) and \( G \in GL_3 \) such that \([t], G \) is a \( \mathcal{P} \)-regular fundamental matrix and

\[
(t \times v_i)\top(t \times Gu_i))(t \times v_j)\top(t \times Gu_j) > 0 \quad \text{for all} \quad 1 \leq i < j \leq k.
\]

Proof. Suppose \((A_1, A_2, Q)\) is a chiral reconstruction of \( \mathcal{P} \) with non-coincident finite cameras where \( A_1 = [I \ 0] \). Then \( A_2 = [G \ t] \) for some \( t \in \mathbb{R}^3 \setminus \{0\} \) and \( G \in GL_3 \), and by Theorem 3.4 \([t], G \) is a \( \mathcal{P} \)-regular fundamental matrix associated to \( A_1 \) and \( A_2 \). For all \( i \) such that \( q_i \) is not on the baseline, \((t \times v_i)\top(t \times Gu_i)\) has the same sign by Theorem 3.9. If some world point \( q_i \) is on the baseline, then its image \((u_i, v_i)\) is the pair of epipoles \((G^{-1}t, t)\), and hence \((t \times v_i)\top(t \times Gu_i) = 0 \). Therefore, the inequalities in (3) hold.

Conversely, suppose there exist \( t \in \mathbb{R}^3 \setminus \{0\} \) and \( G \in GL_3 \) such that \([t], G \) is a \( \mathcal{P} \)-regular fundamental matrix and the inequalities (3) hold. By Theorem 3.4, there exist world points \( Q \) such that \((A_1 = [I \ 0], A_2 = [G \ t], Q)\) is a projective reconstruction of \( \mathcal{P} \). Let \( Q \subseteq Q \) be the set of world points not on the baseline of \( A_1 \) and \( A_2 \). Since the inequalities (3) hold, the quadruple products \((t \times v_i)\top(t \times Gu_i)\) have the same sign for all \( q_i \in Q \). By Theorem 3.9 there exists a constant \( \sigma \in \{-1, 1\} \) such that \( \sigma = \text{sgn}(n_1^i q_i, n_2^i q_i) \) for all \( q_i \in Q \).

If some point \( q_i \in Q \) lies on the baseline, then \( q_i \)'s images to the pair of epipoles \( G^{-1}t \) and \( t \) in the two cameras and hence \((t \times v_i)\top(t \times Gu_i) = 0 \). By Lemma 3.10 we may replace \( q_i \) by some world point \( q_i' \) on the baseline such that \( \text{sgn}(n_1^i q_i', n_2^i q_i') = \sigma \). Let \( Q' \) be the modification of \( Q \) obtained by replacing all world points on the baseline as above, but keeping all other world points intact. By construction, \( \text{sgn}(n_1^i q_i', n_2^i q_i') = \sigma \) for all \( q_i' \in Q' \). The transformed reconstruction \((A_1, A_2, Q')\) is projectively equivalent to a chiral reconstruction by Theorem 3.9 □

Lemma 3.12 implies that for a chiral reconstruction to exist, there must be \( \mathcal{P} \)-regular fundamental matrices that satisfy the inequalities (3). In the next section, we examine these inequalities to understand the regions of the epipolar variety in which fundamental matrices that lead to chiral reconstructions live.

### 4. Chiral tools

In this section we develop tools to prove the existence of chiral reconstructions. In Section 4.1 we describe the semialgebraic chiral epipolar region of fundamental matrices associated to chiral reconstructions of \( \mathcal{P} \). In Section 4.2 we show how inequalities defining the chiral epipolar region can be checked in epipole space. Even if not stated explicitly, we are working over \( \mathbb{R}_p \) when dealing with inequalities. We combine these tools in Section 4.3 to prove that a set of three point pairs has a chiral reconstruction. In Section 4.4 we show how the walls and corners of the epipolar variety can be used to decide if a set of more than three point pairs has a chiral reconstruction.
4.1. The chiral epipolar region. By Lemma 3.12, a fundamental matrix must satisfy \( g_i(X) \) to yield a chiral reconstruction. In this section, we describe the strict subset of the epipolar variety satisfying these constraints.

**Definition 4.1.** Let \( X \in \mathbb{P}(\mathbb{R}^{3 \times 3}) \) denote a real \( 3 \times 3 \)-matrix up to scaling. For each point pair \((u_i, v_i)\), the \( i \)-th chiral polynomial is

\[
g_i(X) := v_i^T [-t]_X X u_i = (t \times v_i)^T X u_i
\]

where \( t^T X = 0 \). The set of all \( g_i g_j(X) = g_i(X) g_j(X) \geq 0 \) are called the chiral epipolar inequalities of \( \mathcal{P} \). Here, the same representative \( t \) for the left-kernel of \( X \) must be used in \( g_i \) and \( g_j \).

The \( i \)-th chiral polynomial is, strictly speaking, not a polynomial because there is no way to write a generator of the left kernel of a matrix \( X \) as a polynomial expression that works for every \( 3 \times 3 \) matrix of rank 2. To be technically precise, it is a section of a line bundle on the quasi-projective variety of \( 3 \times 3 \) matrices of rank exactly two. We avoid these technicalities and argue that the chiral epipolar inequalities are well-defined in an elementary way using the adjoint: Writing \( t \) with \( t^T X = 0 \) in terms of \( X \) is the composition of the adjoint map \( \text{adj}: \mathcal{R}_2 \backslash \mathcal{R}_1 \rightarrow \mathbb{P}^8 \), whose image is \( \mathbb{P}^2 \times \mathbb{P}^2 \) in its Segre embedding, with the projection \( \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \). So locally, \( t \) is given as a row of the adjoint matrix \( \text{adj}(X) \) (but only on the open set where that row is non-zero). The entries of the matrix \([-t]_X \) are polynomials of degree three in the entries of \( X \). This shows that the inequalities \( g_i g_j(X) \geq 0 \) are locally of degree six in the entries of \( X \). Also, if two rows \( t \) and \( t' \) of \( \text{adj}(X) \) are non-zero and differ by a negative multiple \( \lambda \in \mathbb{R}_{<0} \), i.e. \( t = \lambda t' \), the sign of \( g_i g_j(X) \) does not change because it essentially differs by \( \lambda^2 \). Therefore the sign of \( g_i g_j \) is well-defined for every real \( 3 \times 3 \) matrix of rank two up to scaling, i.e. for every fundamental matrix. The set of real matrices \( X \) in \( \mathcal{R}_2 \) for which \( g_i g_j(X) \geq 0 \) is a semi-algebraic subset of \( \mathbb{P}(\mathbb{R}^{3 \times 3}) \) in the following sense: There is an open affine cover of \( \mathcal{R}_2 \backslash \mathcal{R}_1 \) (by sets on which we can write \( t \) as a polynomial function of \( X \)), such that the inequalities \( g_i g_j(X) \geq 0 \) become polynomial and hence define a semi-algebraic set in each open subset of (the real points in) this cover. On the intersection of any two open sets in the cover, the regions cut out by these inequalities agree.

We now show that the chiral polynomial \( g_i(X) \) records the quadruple product \( (t \times v_i)^T (t \times Gu_i) \) from Lemma 3.12.

**Lemma 4.2.** If \( X = [t]_G \), then \( g_i(X) = (t \times v_i)^T (t \times Gu_i) \) for each \( i \).

**Proof.** \( g_i(X) = v_i^T [-t]_X X u_i = ([t]^T X v_i)^T [t]_X X u_i = (t^T X v_i)^T [t]_X X u_i = (t \times v_i)^T (t \times Gu_i) \). \( \Box \)

The next theorem, which is analogous to Theorem 3 in [23], now follows from Lemma 3.12 and Lemma 4.2.

**Theorem 4.3.** There exists a chiral reconstruction of \( \mathcal{P} \) if and only if there exists a \( \mathcal{P} \)-regular fundamental matrix \( X \) such that \( g_i g_j(X) \geq 0 \) for all \( 1 \leq i < j \leq k \).

**Definition 4.4.** The chiral epipolar region of \( \mathcal{P} \) is the set of \( \mathcal{P} \)-regular fundamental matrices \( X \) such that \( g_i g_j(X) \geq 0 \) for all \( 1 \leq i < j \leq k \).

The chiral epipolar region of \( \mathcal{P} \) is contained in the semialgebraic subset of the real part of the epipolar variety \( \mathcal{R}_2 \cap \mathcal{L}_\mathcal{P} \) that is cut out by the chiral epipolar inequalities. It is not necessarily equal to this set because the chiral epipolar region additionally requires the fundamental matrices to be \( \mathcal{P} \)-regular. However, since \( \mathcal{P} \)-regularity only fails on a proper algebraic subset, if the chiral epipolar region has non-empty interior, the boundary of the interior is determined by the points where the chiral epipolar inequalities change sign, which we study next.

**Lemma 4.5.** Let \( X \) be a fundamental matrix of \( \mathcal{P} \). Then \( g_i(X) = 0 \) if and only if \( X \in W_{u_i} \) or \( X \in W^{v_i} \).

**Proof.** Clearly, \( g_i(X) = v_i^T [-t]_X X u_i = 0 \) if \( X u_i = 0 \). If \( v_i^T X = 0 \), then \( v_i \) and \( t \) are collinear and therefore \( v_i [-t]_X X u_i = 0 \), which implies \( g_i(X) = 0 \). For the other implication, we know that \( v_i^T X u_i = 0 \) and \( v_i^T [-t]_X X u_i = 0 \), where the three vectors \( v_i, u_i, t \) are real and non-zero. We assume that \( X u_i \neq 0 \) and show that \( v_i^T X = 0 \). We know \( X u_i \) is orthogonal to \( v_i \). Therefore, it must be collinear with \( v_i \times (v_i \times t) \), which is the same as \( (v_i^T t) v_i - (v_i^T v_i) t \). We also know that \( t^T X = 0 \), which implies that \( t \) is also orthogonal to \( X u_i \), hence also to \( v_i \times (v_i \times t) \). The dot product \( t^T ((v_i^T t) v_i - (v_i^T v_i) t) = 0 \), i.e. \( (v_i^T t)^2 = (v_i^T v_i)(t^T t) \). The Cauchy-Schwarz inequality implies that \( t \) and \( v_i \) are collinear, which implies the claim \( v_i^T X = 0 \). \( \Box \)

The goal of the paper is to understand when the chiral epipolar region of \( \mathcal{P} \) is non-empty, or equivalently, when \( \mathcal{P} \) has a chiral reconstruction. When \( k = 7 \), generically \( \mathcal{R}_2 \cap \mathcal{L}_\mathcal{P} \) consists of three points and it is easy to check if the real points lie in the chiral epipolar region of \( \mathcal{P} \). Therefore, our focus will be on values of \( k \leq 7 \).
4.2. Translating to epipole space. In this section, we show how we can check the validity of chiral epipolar inequalities in $\mathbb{P}_2^2 \times \mathbb{P}_2^2$, the space of epipoles. Consider the $(1,1)$-homogeneous quadratic polynomial

$$D_{ij}(u, v) := \det \begin{bmatrix} u_i & u_j & u \\ v_i & v_j & v \end{bmatrix}$$

where $(u, v) \in \mathbb{P}_2^2 \times \mathbb{P}_2^2$. Note that $D_{ij}(u, v) = 0$ if and only if either factor is zero which is if and only if $u_i, u_j, u$ are collinear or $v_i, v_j, v$ are collinear. Werner uses the quantities $D_{ij}(u, v)$ to impose an orientation constraint on the epipolar line homography described in our Theorem 5.2 see [22] Section 5 and [21] Section 6.5. We will show that $D_{ij}(u, v)$ is closely related to the products of chiral polynomials $g_ig_j(X)$ where $u$ and $v$ generate the right and left kernels of a fundamental matrix $X$. We rely on the following well known identity [5][21].

**Lemma 4.6.** Suppose $q_1, q_2, q_3 \in \mathbb{R}^4$. Let $A = \begin{bmatrix} G & t \end{bmatrix}$ be a finite camera with Cramer’s rule center $c_A = \det(G)(G^{-1}t, 1)$. Then $\det \begin{bmatrix} Aq_1 & Aq_2 & Aq_3 \end{bmatrix} = \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} c_A$.

**Lemma 4.7.** Consider a projective reconstruction $(A_1, A_2, Q)$ of $P$, i.e., $A_1q_i = w_1i, u_i$ and $A_2q_i = w_2q_i$ where $w_{ij} \neq 0$. Suppose $D_{ij}(-A_1e_2, A_2e_1) \neq 0$ where $e_i$ is the Cramer’s rule center of $A_i$. Then

$$\det \begin{bmatrix} Aq_{12} & Aq_{13} \end{bmatrix} = \det \begin{bmatrix} q_1 & q_2 \end{bmatrix} c_A.$$

**Proof.** Expand $D_{ij}(-A_1e_2, A_2e_1)$ as follows.

1. $D_{ij}(-A_1e_2, A_2e_1) = -\det \begin{bmatrix} u_i & u_j & A_1e_2 \\ v_i & v_j & A_2e_1 \end{bmatrix} = -\det \begin{bmatrix} \frac{1}{w_1}A_1q_i & \frac{1}{w_2}A_2q_i & \frac{1}{w_2}A_2q_j & \frac{1}{w_2}A_2e_2 \\ \frac{1}{w_2}q_i & \frac{1}{w_2}q_j & c_1 & c_2 \end{bmatrix}$

2. $= -\det \begin{bmatrix} \frac{1}{w_1}A_1q_i & \frac{1}{w_2}A_2q_i & \frac{1}{w_2}A_2q_j & \frac{1}{w_2}A_2e_2 \\ \frac{1}{w_2}q_i & \frac{1}{w_2}q_j & c_1 & c_2 \end{bmatrix}$

3. $= \frac{1}{w_1w_2}(\det \begin{bmatrix} q_i & q_j & c_1 & c_2 \end{bmatrix})^2$.

Equation 3.3 follows from Equation 3.5 by applying Lemma 4.6 to both determinants in the product. Since $D_{ij}(-A_1e_2, A_2e_1) \neq 0$ by assumption, we conclude that $\det D_{ij}(-A_1e_2, A_2e_1) = \det(sgn(w_1w_2))(w_1w_2)$.

**Lemma 4.8.** Let $X$ be a fundamental matrix of $P$. Suppose $D_{ij}(\text{adj}(X)t, t) \neq 0$ where $t^TX = 0$. Then $\det D_{ij}(\text{adj}(X)t, t) = \det(g_ig_j(X))$.

**Proof.** Write $X = \begin{bmatrix} t \end{bmatrix}_xG$ for $t \in \mathbb{R}^3 \setminus \{0\}$ and some $G \in \text{GL}_3$. We know that $t$ and $\text{adj}(X)t$ generate the one-dimensional left and right kernels of $X$, respectively. Since $D_{ij}(\text{adj}(X)t, t) \neq 0$, we know that $t \neq 0, \text{adj}(X)t \neq 0$, $\text{adj}(X)t$ is not collinear with $u_i$ and $u_j, t$ and $v_i \neq v_i$. In particular, this means that neither $u_i$ nor $u_j$ is a right kernel of $X$ and neither $v_i$ nor $v_j$ is the left kernel of $X$. It follows that $X$ is $(u_i, v_i)$ regular and $(u_j, v_j)$ regular and neither is the epipole pair. By Theorem 3.4 there exists a finite projective reconstruction $(A_1 = [I \ 0], A_2 = [G \ t], \{q_i, q_j\})$ of $(\{u_i, v_i\}, \{u_j, v_j\})$ such that the world points $q_i, q_j$ are not on the baseline.

Lemma 4.7 implies that $\det D_{ij}(-A_1e_2, A_2e_1) \neq 0$ and $\det D_{ij}(-A_1e_2, A_2e_1) = \det(g_ig_j(X))$. Finally note that $A_2e_1 = t$ and $-A_1e_2 = \det(G)(G^{-1}t)$ which is a positive multiple of $\text{adj}(X)t$. Indeed, $\text{adj}(X)t = \text{adj}[[t]_xG]t = \det(G)\text{adj}[[t]_x]t = \det(G)G^{-1}(tt^t)t = \|t\|^2\det(G)(G^{-1}t)$.

Substituting $\text{adj}(X)t$ for $-A_1e_2$ and $t$ for $A_2e_1$, the result follows.

The computation in the previous proof, in particular 3.9, shows that if $t$ is a non-zero generator of the left kernel of a fundamental matrix $X$ then $\text{adj}(X)t$ is a non-zero generator of the right kernel of $X$.

Note that $\det \begin{bmatrix} u_i & u_j & A_1e_2 \end{bmatrix}$ can be zero without $u_i$ or $u_j$ being the epipole $A_1e_2$. Indeed, by Lemma 4.6 this happens whenever $q_i, q_j, c_1, c_2$ are coplanar. On the other hand, Lemma 4.8 implies that $g_i$ vanishes at $X$ if and only if $u_i$ or $v_i$ is an epipoles of $X$. Therefore, $D_{ij}(\text{adj}(X)t, t)$ may vanish even when $g_ig_j(X) \neq 0$.

Lemma 4.9 shows that knowing the specific generators of the kernels of $X$, i.e., $t$ and $\text{adj}(X)t$, respectively, is enough to compute the sign of the chiral epipolar inequalities. Note that a choice of generator $t$ for the left kernel
of $X$ determines a signed generator $\text{adj}(X)t$ for the right kernel. When $D_{ij}(\text{adj}(X)t, t)$ does not vanish, we can use it to infer the validity of chiral epipolar inequalities via Lemma 4.3 and hence argue for the existence of a chiral reconstruction of $\mathcal{P}$. We now identify a situation where we can use any generators of the kernels of $X$ in $D_{ij}$.

**Definition 4.9.** Suppose $X$ is a fundamental matrix of $\mathcal{P}$. Define $I(X)$ to be the set of indices $i$ such that $g_i(X) \neq 0$, i.e., the index set of inactive chiral polynomials at $X$. Let $\mathcal{P}_{I(X)}$ be the subset of point pairs in $\mathcal{P}$ indexed by $I(X)$.

**Theorem 4.10.** Let $X$ be a fundamental matrix of $\mathcal{P}$ where $e_1$ and $e_2$ generate the right and left kernels of $X$, respectively. Suppose $|I(X)| \geq 3$, and $D_{ij}(e_1, e_2) \neq 0$ for all $i, j \in I(X)$. Then there exists a chiral reconstruction of $\mathcal{P}_{I(X)}$ associated to $X$ if and only if $D_{ij}(e_1, e_2)$ has the same sign for all $i, j \in I(X)$.

**Proof.** Suppose there exists a chiral reconstruction of $\mathcal{P}_{I(X)}$ associated to $X$. Then by Theorem 4.3, $g_i g_j(X) \geq 0$ for all $i, j \in I(X)$. In fact, $g_i g_j(X) > 0$ for all $i, j \in I(X)$ since if $g_i g_j(X) = 0$ for some $i, j$ while $D_{ij}(\text{adj}(X)t, t) \neq 0$, we would contradict Lemma 4.3. Indeed, if $D_{ij}(e_1, e_2) \neq 0$ for some kernel generators $e_1, e_2$, it remains non-zero for any other pair of kernel generators. By Theorem 4.3, $D_{ij}(\text{adj}(X)t, t)$ has the same sign for all $i, j$, and since $\text{adj}(X)t$ and $t$ are (non-zero) generators of the right and left kernels of $X$, the result follows.

Conversely, suppose $D_{ij}(e_1, e_2)$ has the same non-zero sign for all $i, j \in I(X)$ where $e_1$ and $e_2$ generate the right and left kernels of $X$, respectively. Then $e_1 = \lambda \text{adj}(X)e_2$ for some non-zero $\lambda$ by (10). By Lemma 4.3, we know

$$\text{sgn} \, D_{ij}(e_1, e_2) = \lambda \text{sgn} \, D_{ij}(\text{adj}(X)e_2, e_2) = \lambda \text{sgn} \, g_i g_j(X)$$

for all $i, j$. This shows that $g_i g_j(X)$ has the same sign for all $i, j \in I(X)$. Since $|I(X)| \geq 3$, this common sign cannot be negative and hence $g_i g_j(X) > 0$ for all $i, j \in I(X)$. These strict inequalities also imply that $X$ is $\mathcal{P}_{I(X)}$-regular. Then by Theorem 4.3, there is a chiral reconstruction of $\mathcal{P}_{I(X)}$ associated to $X$. □

We remark that $D_{ij}(e_1, e_2)$ does not have a well-defined sign on $\mathbb{P}^2_\mathbb{R} \times \mathbb{P}^2_\mathbb{R}$ because it is linear in $e_1$ and $e_2$. To get an inequality description of chirality in epipole space, we can take pairwise products $D_{ij}(e_1, e_2)D_{ik}(e_1, e_2)$ which are quadratic in each $\mathbb{P}^2_\mathbb{R}$ factor. If $D_{ij}(e_1, e_2)D_{ik}(e_1, e_2) > 0$, then $g_i g_k(X) > 0$ for any fundamental matrix $X$ with epipoles $e_1$ and $e_2$. However, since $D_{ij}(e_1, e_2)$ may vanish even when $g_i g_j(X)$ does not, we observe that $D_{ij}(e_1, e_2)D_{ik}(e_1, e_2) > 0$ for all triples $i, j, k$ is not equivalent to $g_i g_j(X) > 0, g_i g_k(X) > 0$ and $g_j g_k(X) > 0$. Due to this subtlety, we primarily study chirality using $g_i g_j(X) > 0$ in $\mathbb{P}^2_\mathbb{R}$ as opposed to $D_{ij}D_{ik} > 0$ in $\mathbb{P}^2_\mathbb{R} \times \mathbb{P}^2_\mathbb{R}$.

### 4.3 Three point pairs always have a chiral reconstruction

In this section, we apply the tools developed so far to show that there is always a chiral reconstruction when $|\mathcal{P}| = 3$, and hence also when $|\mathcal{P}| \leq 3$ since $\mathcal{P}$ can have a chiral reconstruction only if all its subsets have one. We begin with two technical lemmas.

**Lemma 4.11.** Suppose $a_1, a_2, a_3$ are three non-collinear points in $\mathbb{R}^3$. Then for each of the eight elements in $\{+, -\}^3$, there is an $e \in \mathbb{R}^3$ such that $a_1, a_2, a_3, e$ are in general position (no three in a line) and

$$\sigma = (\text{sgn} |\det[a_1, a_2, e]|, \text{sgn} |\det[a_1, a_3, e]|, \text{sgn} |\det[a_2, a_3, e]|).$$

Further, for each $\sigma \in \{+, -\}^3$, the corresponding choices of $e$ come from an open polyhedral cone in $\mathbb{R}^3$.

**Proof.** The expression $\det[a_1, a_2, e] = l_{ij}(e)$ is the linear form whose kernel is the span of $a_i$ and $a_j$. Since $a_1, a_2, a_3$ are non-collinear, the hyperplanes cut out by $l_{12}(e), l_{13}(e), l_{23}(e)$ divide $\mathbb{R}^3$ into eight regions, each of which is a polyhedral cone. The interiors of these cones correspond to the eight sign patterns $\sigma$. □

For $v_1, v_2 \in \mathbb{R}^3$, let $\text{cone}(v_1, v_2) := \{\lambda v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \geq 0\}$ be the convex cone spanned by $v_1$ and $v_2$.

**Lemma 4.12.** Suppose $v_1, v_r, t$ are points in $\mathbb{R}^3$ on an affine line $L$. If $t \neq \text{cone}(v_1, v_r)$, then for all $w_1, w_2 \in \text{cone}(v_1, v_r)$, $(t \times w_1)^T(t \times w_2) > 0$.

**Proof.** If $t \neq \text{cone}(v_1, v_r)$, then either $v_1 \in \text{cone}(t, v_r)$ or $v_r \in \text{cone}(v_1, t)$. Suppose $v_1 \in \text{cone}(t, v_r)$. Since $w_1, w_2 \in \text{cone}(v_1, v_r)$ and $\text{cone}(v_1, v_r) \subseteq \text{cone}(t, v_r)$, we know $w_1, w_2 \in \text{cone}(t, v_r)$. Write $w_1 = \lambda_1 t + \lambda_2 v_r$ and $w_2 = \lambda_1 t + \mu_2 v_r$ where $\lambda_1, \lambda_2 \geq 0$. Since $w_1 \neq t, \lambda_2, \mu_2 > 0$. The result follows from direct computation using that $t \neq v_r$:

$$(11) \quad (t \times w_1)^T(t \times w_2) = (t \times (\lambda_1 t + \lambda_2 v_r))^T(t \times (\mu_1 t + \mu_2 v_r)) = \lambda_2 \mu_2(t \times v_r)^T(t \times v_r) > 0.$$  

Similar reasoning applies if $v_r \in \text{cone}(v_1, t)$. □

**Theorem 4.13.** If $|\mathcal{P}| = 3$ then $\mathcal{P}$ has a chiral reconstruction.
Proof. We break the proof into two parts:

1. Suppose $\mathcal{U} = \{u_1, u_2, u_3\}$ or $V = \{v_1, v_2, v_3\}$ is in general position, say $\mathcal{U}$ is non-collinear. Choose $e_2$ not on the line spanned by $v_i$ and $v_j$ for any $i, j$, so that $\det \begin{bmatrix} v_i & v_j & e_2 \end{bmatrix} \neq 0$ for all $i, j$. By Lemma 4.11 there exists an $e_1$ such that $D_{ij}(e_1, e_2)$ has the same non-zero sign for all $i, j$. Since $k = 3$ and $e_1$ and $e_2$ are chosen from open regions, $W_{e_1} \cap W_{e_2}$ contains at least one rank two matrix $X$. By construction, this $X$ is a $P$-regular fundamental matrix with epipoles $e_1, e_2$ and $I(X) = \{1, 2, 3\}$. By Theorem 4.10 $X$ yields a chiral reconstruction of $P$.

2. Suppose both $\mathcal{U}$ and $V$ are collinear and consider the affine lines $L_{\mathcal{U}}$ and $L_V$ in $\mathbb{R}^3$ spanned by these points, which all have last coordinate 1. Let $u_i, u_r$ be the furthest left and right points on the $L_{\mathcal{U}}$ line, so that the third point lies strictly between $u_i$ and $u_r$. Similarly let $v_i$, $v_r$ be the furthest left and right points on the $L_V$ line. Let $t \in L_V \setminus \text{cone}(v_i, v_r)$ and choose $G \in GL_3$ such that $Gu_i = v_i$ and $Gu_r = v_r$. Define $X = \{t\} G$. Since $t$, $v_i$, $Gu_i$ are collinear for all $i$, the $i$th epipolar equation is satisfied. Since the chosen epipoles for $X$ do not coincide with any data points, $X$ is a $P$-regular fundamental matrix. By construction $Gu_i \in \text{cone}(v_i, v_r)$ for each $i$. Combining Lemma 4.2 and Lemma 4.12, it follows that $g_i(X) > 0$ for each $i$, and there is a chiral reconstruction of $P$ associated to $X$ by Theorem 4.3. \hfill \qed

4.4. Walls and Corners. To understand the existence of chiral reconstructions when $|P| \geq 4$, we need one more tool that we now develop. Recall that the chiral epipolar region of $P$ is the set of $P$-regular fundamental matrices that live in the semialgebraic subset of the real epipolar variety cut out by the chiral epipolar inequalities. Lemma 4.5 implies that the chiral epipolar region is bounded by the $W_{e_1}, W_{e_2}$ walls. The fundamental matrices on walls are generally $P$-irregular and do not correspond to a reconstruction. However, we show that $P$-irregular fundamental matrices that are smooth points of the epipolar variety and yield partial chiral reconstructions, can be perturbed to $P$-regular fundamental matrices that yield chiral reconstructions of $P$.

Lemma 4.14. Suppose $R_2 \cap L_P$ is irreducible. If $X$ is a smooth fundamental matrix that is $(u_i, v_i)$-irregular, then there is a tangent direction $d \in T_X(R_2 \cap L_P)$ such that the directional derivative $D_d g_i(X) \neq 0$.

Proof. Suppose $X$ is a smooth fundamental matrix of $P$. Smoothness implies that the tangent space at $X$ to the epipolar variety has the same dimension as the variety. If $X$ is $(u_i, v_i)$-irregular for some $i$, then $X$ is in exactly one of $W_{u_i}$ or $W_{v_i}$. Since $R_2 \cap L_P$ is irreducible, each wall must be an embedded component of strictly smaller dimension. This means that the wall’s tangent space is strictly contained in the tangent space of the epipolar variety at $X$. Therefore, we can choose a direction $d$ tangent to the epipolar variety at $X$ which is not tangent to the wall which contains $X$. Lemma 4.5 implies that $g_i$ vanishes on the real part of the epipolar variety only on the walls. By construction $D_d g_i(X) \neq 0$. \hfill \qed

The following lemma is needed for Theorem 4.16 below, but its proof might be best understood after Section 7.

Lemma 4.15. Suppose $|P| \leq 5$ and $R_2 \cap L_P$ is irreducible. If a wall $W_{u_i}$ (or $W_{v_i}$) contains a matrix of rank two, then a generic point $Y$ on the wall is a smooth point of $R_2 \cap L_P$.

Proof. We reduce to the case of $\dim(L_P) = 3$ as follows. If the wall contains a smooth point, then so will its intersection with generic data planes $L_{(u_i, v_j)}$. Therefore, cutting with sufficiently many of these, using Bertini’s Theorem, we can assume that $L_P$ has dimension three, $R_2 \cap L_P$ is an irreducible cubic surface in $\mathbb{P}^3$, and $W_{u_i}$ is a line on it. Suppose for contradiction that $R_2 \cap L_P$ is singular at every point in $W_{u_i}$.

The cubic surfaces which are singular along a line have been classified, see e.g. [3] in particular Case E]. We show that $R_2 \cap L_P$ cannot be any of these types, essentially because it contains too many intersecting lines. Indeed, $W_{u_i}$ intersects $W_{v_{m}}$ as long as $l \neq m$ because the equations $Xu_l = 0$ and $v_{m}^\top X = 0$ impose at most three additional conditions on the three dimensional $L_P$. Additionally, the assumption that the wall $W_{u_i}$ contains a matrix of rank two implies that this wall does not coincide with $W_{u_j}$ for $j \neq i$.

The first examples of cubic surfaces singular along a line are the cones over a singular plane cubic curve. Our epipolar variety cannot be such a surface because it contains intersecting lines with distinct intersection points, which these cones do not. There are only two other types of cubic surfaces (up to change of coordinates) that are singular along a line.

The next type that is singular along a line, contains a one-dimensional family of lines, and one more line. A representative is given by the equation $w^2y + x^2z$, which contains the lines $\mathcal{V}(w, x), \mathcal{V}(y, z)$ and a family of lines that
form a twisted cubic in the Plücker quadric $\text{Gr}(2, 4)$ of lines in $\mathbb{P}^3$. The lines in the family are pairwise skew. This type of singular cubic surface only contains two lines that intersect lines in the family, which is inconsistent with the intersection pattern of lines on $\mathcal{R}_2 \cap \mathcal{L}_P$. The last relevant type is represented by the equation $w^2y + w^2x + x^3$. It is singular along the line $\mathcal{V}(x, w)$ and it contains a one-dimensional family of lines. This family of lines is a twisted cubic curve in $\text{Gr}(2, 4)$ and they are mutually skew, which also does not fit the intersection pattern on our epipolar variety.

This discussion of cases shows that the epipolar variety cannot be singular along an entire line if it is an irreducible cubic surface in $\mathbb{P}^3$.

Recall $I(X)$ and $\mathcal{P}_{I(X)}$ from Definition 4.9.

**Theorem 4.16.** Suppose $|\mathcal{P}| \leq 5$ and $\mathcal{R}_2 \cap \mathcal{L}_P$ is irreducible. Then there exists a chiral reconstruction of $\mathcal{P}$ if and only if $\mathcal{P}_{I(X)}$ has a chiral reconstruction associated to some smooth $X \in \mathcal{R}_2 \cap \mathcal{L}_P$.

**Proof.** Suppose there exists a chiral reconstruction of $\mathcal{P}$. By Theorem 4.3, there exists a fundamental matrix $X$, which is $(u_i, v_i)$-regular for all $i$ and $g_ig_j(X) \geq 0$ for all $1 \leq i < j \leq k$. If $g_ig_j(X) > 0$ for all $i, j$, then the semialgebraic subset of the epipolar variety described by the chiral epipolar inequalities has non-empty interior. Since every non-empty, open semialgebraic set in an algebraic variety intersects its smooth locus, there is a smooth $X' \in \mathcal{R}_2 \cap \mathcal{L}_P$ associated to a smooth fundamental matrix, finishing the proof of the first implication.

Conversely, suppose there is a chiral reconstruction of $\mathcal{P}_{I(X)}$ associated to a smooth $X \in \mathcal{R}_2 \cap \mathcal{L}_P$. If $\mathcal{P} = \mathcal{P}_{I(X)}$, there is nothing to show. Otherwise, we have $g_i(X) = 0$ for some $i$, which means that $X$ lies on some walls by Lemma 4.5. Smoothness of $X$ implies that $X$ must have rank two, so its left and right kernels are one-dimensional. Since the $u_i$'s and $v_j$'s are distinct, this means $X$ may lie on at most one wall $W_{u_i}$ and at most one wall $W_{v_j}$. In other words, $I(X)$ must be one of $[k]\{i\}$ or $[k]\{i,j\}$.

Suppose $I(X) = [k]\{i\}$. We argue locally: Choose an affine chart around $X$ such that $g_j(X) > 0$ for all $j \in [k]\{i\}$, which is possible because we know that $g_jg_l(X) > 0$ for all pairs $j, \ell \in [k]\{i\} = I(X)$. If $X$ is $W_{u_i} \cap W_{v_j}$, it is $\mathcal{P}$-regular and is associated to a chiral reconstruction. Otherwise, Lemma 4.14 implies that there is a tangent direction $d \in T_X(\mathcal{R}_2 \cap \mathcal{L}_P)$ such that $D_{d}\gamma(X) \neq 0$. Then we have $D_d(g_ig_j)(X) = (D_d\gamma_i)(X)g_j(X)$ because $g_i(X) = 0$. Since $g_j(X) > 0$ in our affine chart, the sign of the directional derivative of $g_ig_j$ is determined by $(D_d\gamma_i)(X)$, which is independent of $j$. By flipping the sign of $d$, if necessary, we can assume that $(D_d\gamma_i)(X) > 0$. By Taylor approximation, there is a nearby smooth point $X' \in \mathcal{R}_2 \cap \mathcal{L}_P$ such that $g_jg_l(X') > 0$ for all $j, \ell \in [k]$. By Theorem 4.3, there is a chiral reconstruction of $\mathcal{P}$ corresponding to $X'$.

Suppose $I(X) = [k]\{i,j\}$. Choose an affine chart around $X$ such that $g_l(X) > 0$ for all $l \in [k]\{i,j\}$, which is possible because we know that $g_ig_m(X) > 0$ for all pairs $l, m \in [k]\{i,j\} = I(X)$. Choose linearly independent tangent vectors $d_i, d_j \in T_X(\mathcal{R}_2 \cap \mathcal{L}_P)$ such that the directional derivatives $D_{d_i}g_l(X) \neq 0$ and $D_{d_j}g_l(X) \neq 0$. Choose a linear combination $d \in \text{Span}(d_i, d_j)$ such that $(D_d\gamma_i)(X) > 0$ and $(D_d\gamma_j)(X) > 0$. Since $g_i(X) > 0$ in our affine chart, the sign of the directional derivatives $(D_d\gamma_i)(X)$ and $(D_d\gamma_j)(X)$ at $X$ are determined by $(D_d\gamma_i)(X)$ and $(D_d\gamma_j)(X)$, respectively. By Taylor approximation, there is a nearby smooth point $X' \in \mathcal{R}_2 \cap \mathcal{L}_P$ such that $g_ig_m(X') > 0$ for all $l, m \in [k]$. By Theorem 4.3, there is a chiral reconstruction of $\mathcal{P}$ associated to $X'$.

To apply Theorem 4.16, it is useful to understand the smooth locus of the epipolar variety which we describe next. For a more general discussion of tangent spaces to rank varieties we refer to [10, Example 14.16].

**Lemma 4.17.** Suppose $X \in \mathcal{R}_2 \cap \mathcal{L}_P$ is a rank two matrix such that $u$ spans the right kernel of $X$ and $v$ spans the left kernel of $X$. Then $X$ is a smooth point of $\mathcal{R}_2 \cap \mathcal{L}_P$ if and only if $uv^T$ does not lie in the span of $\{v_iu_i^T : i = 1, \ldots, k\} \subset \mathbb{P}^3$.

**Proof.** The gradient of the determinant of $X$ is the cofactor matrix of $X$, which is $\text{adj}(X)^T$. Since $uv^T$ and $\text{adj}(X)$ are collinear, the tangent hyperplane to $\mathcal{R}_2$ at $X$ is

$$T_XR_2 = \{M \in \mathbb{P}^3 : \langle M, uv^T \rangle = 0 \}.$$
5. Four Point Pairs

In the previous section we saw that any set of three point pairs will always have a chiral reconstruction. In this section we completely analyze the case of four point pairs. In Theorem 5.13 we prove that if the point configurations in the two views have the same rank, then they admit a chiral reconstruction. Otherwise, a chiral reconstruction can fail to exist (Theorem 5.14). Our main tool will be Theorem 4.16 which requires understanding when the epipolar variety is reducible. We do this in Section 5.1 which leads to our chirality results in Section 5.2.

5.1. Irreducibility of the epipolar variety. Suppose the epipolar variety $\mathcal{R}_2 \cap \mathcal{L}_P$ is reducible. Then $\mathcal{L}_P$ is not contained in $\mathcal{R}_2$ since otherwise, $\mathcal{R}_2 \cap \mathcal{L}_P = \mathcal{L}_P$ which is irreducible. Therefore, $\mathcal{R}_2 \cap \mathcal{L}_P$ is a proper subvariety of $\mathcal{L}_P$. Further, any irreducible component of $\mathcal{R}_2 \cap \mathcal{L}_P$ has codimension one in $\mathcal{L}_P$ since $\det(X) = 0$ is the only additional condition to those defining $\mathcal{L}_P$. Conversely, if $C$ is a proper subvariety of $\mathcal{R}_2 \cap \mathcal{L}_P$ of codimension one in $\mathcal{L}_P$, then $C$ is an irreducible component of $\mathcal{R}_2 \cap \mathcal{L}_P$. Indeed, if $C$ lies in some irreducible component of $\mathcal{R}_2 \cap \mathcal{L}_P$ of dimension larger than $\dim(C)$, then $C$ has codimension at least two in $\mathcal{L}_P$, which is a contradiction. Lastly, since $\mathcal{R}_2 \cap \mathcal{L}_P$ has degree three, it must have a linear component if it is reducible and this linear component is a subspace of $\mathcal{R}_2$. Thus to understand whether $\mathcal{R}_2 \cap \mathcal{L}_P$ is reducible, we need to understand the linear subspaces of $\mathcal{R}_2$ that have codimension one in $\mathcal{L}_P$.

A subspace $C \subset \mathbb{C}^{3 \times 3}$ is said to have rank at most $k$ if $k$ is the maximum rank of matrices in $C$. The following definition is from [8] for $3 \times 3$ matrices.

Definition 5.1. Let $C \subset \mathbb{C}^{3 \times 3}$ be a subspace of rank at most $k$. Then $C$ is called a compression space if there exists a subspace $V \subseteq \mathbb{C}^3$ of codimension $k_1$ and a subspace $W \subseteq \mathbb{C}^3$ of dimension $k_2$ such that $k_1 + k_2 = k$, and every $X \in C$ maps $V$ into $W$ (compresses $V$ into $W$). We refer to $C$ as a $(k_1, k_2)$-compression space.

Note that every subspace of a $(k_1, k_2)$-compression space is again a $(k_1, k_2)$-compression space. The following theorem from [8] attributed to Atkinson [2] tells us what subspaces of $\mathcal{R}_2$ look like.

Theorem 5.2. [8] Theorem 1.1] A vector space of matrices in $\mathbb{C}^{3 \times 3}$ of rank at most two is either a compression space or a subspace of the linear space of $3 \times 3$ skew-symmetric matrices.

Theorem 5.3. Suppose $|P| = 4$. Then the epipolar variety $\mathcal{R}_2 \cap \mathcal{L}_P$ is reducible if and only if $\mathcal{L}_P$ contains a compression space of rank at most two and codimension one.

Proof. By the above discussion, if $\mathcal{R}_2 \cap \mathcal{L}_P$ is reducible, then it has a linear component $C$ that is a subspace of $\mathcal{R}_2$ and has codimension one in $\mathcal{L}_P$. Since $\dim(\mathcal{L}_P) \geq 4$ when $|P| = 4$, and the space of $3 \times 3$ skew-symmetric matrices has dimension two, by Theorem 5.2 $C$ must be a compression space of rank at most two. Conversely, if $C$ is a compression space in $\mathcal{L}_P$ of rank at most two and codimension one, then $C \subseteq \mathcal{R}_2 \cap \mathcal{L}_P$, and since $C$ has codimension one in $\mathcal{L}_P$, $C$ must be an irreducible component of $\mathcal{R}_2 \cap \mathcal{L}_P$. \qed

Example 5.4. Consider the three point pairs given by the columns of the following matrices:

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using Macaulay2 one can compute that $\mathcal{R}_2 \cap \mathcal{L}_P = Q \cup C$ where $Q$ has degree two and $C$ has degree one. The ideal of $C$ is $\langle x_{21}, x_{23}, x_{31}, x_{33} \rangle$, so every $X \in C$ has the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & x_{32} & 0 \end{pmatrix}.$$

Since $\dim(C) = 4$ and $\dim(\mathcal{L}_P) = 5$, $C$ has codimension one in $\mathcal{L}_P$. The linear space $C$ is a $(1,1)$-compression space of rank at most two; each $X \in C$ compresses the line spanned by the columns of $U$ into the orthogonal complement of the line spanned by the columns of $V$.\]
Suppose we now enlarge the set of point pairs to

\[
U = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

Now \( R_2 \cap L_P \) has a linear component \( C' \) consisting of matrices of the form

\[
\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & x_{32} & 0 \end{pmatrix}
\]

such that \( 5x_{11} + x_{12} + x_{13} + 7x_{22} + x_{32} = 0. \)

This subspace \( C' \) is clearly a subspace of the compression space \( C \) and has codimension one in the new \( L_P \). The linear condition arises from the epipolar equation \( \langle X, v_4u_4^T \rangle = 0 \) imposed by the new point pair.

The main result of this subsection is the following which will allow us to use Theorem 4.10 to establish chirality results for four point pairs in the next subsection. A set of points in \( \mathbb{R}^2 \) is said to be in general (linear) position if no three of them are collinear.

**Theorem 5.5.** Suppose \( |P| = 4 \). If for all triples of distinct indices \( i, j, k \), \( u_i, u_j, u_k \) or \( v_i, v_j, v_k \) are in general position, then \( R_2 \cap L_P \) is irreducible.

Theorem 5.5 will follow from Theorem 5.3 if under the assumptions of the theorem, there are no compression spaces of rank at most two and codimension one in \( L_P \). We will now prove that this is indeed the case by understanding the orthogonal complements of compression spaces of rank at most two.

Two vector spaces \( C_1 \) and \( C_2 \) in \( \mathbb{C}^{3\times 3} \) are equivalent if they have the same linear transformations up to a change of bases in the domain \( \mathbb{C}^3 \) and codomain \( \mathbb{C}^3 \), i.e., if there exists \( A, B \in \text{GL}_3 \) such that \( C_2 = \{ BXA : X \in C_1 \} \). The following classification is straightforward.

**Lemma 5.6.** Let \( C \) be a \((k_1, k_2)\)-compression space of rank at most two in \( \mathbb{C}^{3\times 3} \).

1. If \((k_1, k_2) = (2, 0)\) then \( C \) is equivalent to a vector space of matrices with a zero last column.
2. If \((k_1, k_2) = (0, 2)\) then \( C \) is equivalent to a vector space of matrices with a zero last row.
3. If \((k_1, k_2) = (1, 1)\) then \( C \) is equivalent to a vector space of matrices of the form

\[
\begin{pmatrix}
0 & 0 & * \\
0 & 0 & * \\
* & * & *
\end{pmatrix}
\]

Call a \((k_1, k_2)\)-compression space maximal if it is not a proper subspace of another \((k_1, k_2)\)-compression space. In particular, a \((k_1, k_2)\)-compression space is maximal if the equations cutting it out are precisely the conditions that impose the zeros guaranteed in their standard forms. Non-maximal compression spaces satisfy further conditions on their potentially non-zero coordinates. The following lemmas help us understand maximal compression spaces.

**Lemma 5.7.** Suppose \( u, v \in \mathbb{C}^3 \setminus \{0\} \). Then \( vu^T \in S := \text{Span}\{b_1a_1^T, b_2a_2^T, \ldots, b_na_n^T\} \) if and only if \( u \in \text{Span}\{a_i\}_{i=1}^m \) and \( v \in \text{Span}\{b_i\}_{i=1}^n \).

**Proof.** Suppose \( vu^T \in S \). Then multiplying \( vu^T \) on the right by \( u \), we get that \( v \in \text{Span}\{b_1, \ldots, b_n\} \). Similarly, multiplying \( vu^T \) on the left by \( v^T \), we get that \( u \in \text{Span}\{a_1, \ldots, a_m\} \). Conversely, if \( u = \lambda_1a_1 + \ldots, \lambda_na_n \) and \( v = \mu_1v_1 + \ldots, \mu_nv_n \), then it is immediate that \( vu^T \in S \). Note that for this last statement to hold, it was important that all possible outerproducts of the form \( b_ia_i^T \) are present in \( S \).

**Lemma 5.8.** Let \( C \subseteq \mathbb{C}^{3\times 3} \) be a maximal \((k_1, k_2)\)-compression space of rank at most two and let \( C^\perp \subseteq \mathbb{C}^{3\times 3} \) be its orthogonal complement.

1. If \((k_1, k_2) = (2, 0)\), then \( C^\perp = \text{Span}\{b_1a_1^T, b_2a_2^T, b_3a_3^T\} \) where \( a_1, a_2, a_3 \) are linearly independent vectors in \( \mathbb{C}^3 \). Similarly if \((k_1, k_2) = (0, 2)\), then \( C^\perp = \text{Span}\{ba_1^T, ba_2^T, ba_3^T\} \) where \( b \in \mathbb{C}^{3\setminus\{0\}} \) and \( a_1, a_2, a_3 \) are linearly independent vectors in \( \mathbb{C}^3 \). In both cases, as a projective space, \( \text{dim}(C^\perp) = 2 \) and all matrices in \( C^\perp \) have rank one.
2. If \((k_1, k_2) = (1, 1)\), then \( C^\perp = \text{Span}\{b_1a_1^T, b_2a_2^T, b_3a_3^T\} \) where \( b_1, b_2, b_3 \) are linearly independent and \( a_1, a_2, a_3 \) are linearly independent vectors in \( \mathbb{C}^3 \). As a projective space, \( \text{dim}(C^\perp) = 3 \) and the rank one matrices in it cut out a variety isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).
Proof. (1) Let \( \overline{C} \) be the maximal \((2,0)\)-compression space in standard coordinates, consisting of all matrices with a zero last column. Then \((\overline{C})^+ \) is spanned by \( e_1 e_1^T, e_2 e_2^T, e_3 e_3^T \). By Lemma 5.6, a general \((2,0)\)-compression space \( C = \{X \in \mathbb{C}^{3 \times 3} : AX = C \} \) for a pair of fixed invertible matrices \( A, B \). Therefore, 
\[
0 = \langle BXA, e_1 e_3^T \rangle = \langle BXA, e_2 e_3^T \rangle = \langle BXA, e_3 e_3^T \rangle \text{ for all } X \in C.
\]
Taking the first dot product:
\[
0 = \langle BXA, e_1 e_3^T \rangle = Tr(A^T X^T B^T e_1 e_3^T) = Tr(X^T (B^T e_1) (Ae_3)^T) = \langle X, (B^T e_1) (Ae_3)^T \rangle.
\]
Setting \( Ae_1 = a_i \) and \( B^T e_j = b_j \), we get that \( C^+ = \text{Span}\{a_1 a_1^T, b_2 a_3^T, b_3 a_3^T\} \). Since \( B \) is invertible, \( b_1, b_2, b_3 \) are independent, and no row of \( A \) is all zero. Hence, \( C^+ \) is spanned by 3 independent rank one matrices and as a projective space, \( \dim(C^+) = 2 \). Further, any linear combination of these three rank one matrices looks like \( \lambda b_1 a_1^T + \mu b_2 a_3^T + \nu b_3 a_3^T = (\lambda b_1 + \mu b_2 + \nu b_3) a_1^T \) which is a rank one matrix. Therefore, all matrices in \( C^+ \) have rank one. The proof for \((0,2)\)-compression spaces is analogous.

(2) By the same reasoning as in the previous case, if \( C \) is a \((1,1)\)-compression space, then \( C^+ = \text{Span}\{b_1 a_1^T, b_2 a_2^T, b_2 a_3^T, b_3 a_2^T\} \). Since \( C \) is maximal, \( \dim(C^+) = 3 \) as a projective space, and hence \( C^+ \cong \mathbb{P}^3 \). Since the linear combinations
\[
\lambda_1 b_1 a_1^T + \lambda_2 b_2 a_2^T + \mu_1 b_2 a_3^T + \mu_2 b_2 a_3^T = (\lambda_1 b_1 + \mu_1 b_2) (\gamma_1 a_1 + \nu_1 a_2)^T
\]
are rank one matrices, there is a \( \mathbb{P}^1 \times \mathbb{P}^1 \) worth of rank 1 matrices in \( C^+ \) which forms a subvariety of dimension two. By Lemma 5.7, there are no more rank one matrices in \( C^+ \).

From now on we will denote a maximal compression space of rank at most two by \( C \) and a subspace of it by \( C' \). If a non-maximal compression space \( C' \) appears as an irreducible component of \( \mathbb{R}^2 \cap \mathcal{L}_P \), then \((\overline{C})^+ \) will be spanned by the generators of \( C^+ \) along with some data matrices \( v, u \) as in Example 5.4. In particular a basis of \((\overline{C})^+ \) is the union of the basis for \( C^+ \) from Lemma 5.8 and some data matrices.

Lemma 5.9. Suppose for all distinct indices \( i, j, k, u_i, u_j, u_k \) or \( v_i, v_j, v_k \) are in general position. Let \( C' \) be a \((k_1, k_2)\)-compression space contained in a maximal \((k_1, k_2)\)-compression space \( C \).

(1) If \( C' \) is of type \((2,0)\) or \((0,2)\), then \( C^+ \) contains at most one data matrix.

(2) If \( C' \) is of type \((1,1)\), then \( C^+ \) contains at most two data matrices.

Proof. (1) If \( C' \) is a \((2,0)\) compression space, then by Lemma 5.8, \( C^+ = \text{Span}\{b_1 a_1^T, b_2 a_2^T, b_3 a_3^T\} \) for some \( a \neq 0 \) and \( b_1, b_2, b_3 \) which are linearly independent. By Lemma 5.7, \( v_i u_i^T \in C^+ \) if and only if \( u_i \in \text{Span}\{a\} \). Since point pairs are distinct and have last coordinate equal to one, at most one \( u_i \in \text{Span}\{a\} \), hence at most one \( v_i u_i^T \in C^+ \). The case \((0,2)\) is argued similarly.

(2) If \( C' \) is a \((1,1)\) compression space, then by Lemma 5.8, \( C^+ = \text{Span}\{b_1 a_1^T, b_2 a_1^T, b_2 a_2^T, b_2 a_3^T\} \) for some linearly independent \( a_1, a_2 \) and \( b_1, b_2 \). By Lemma 5.7, \( v_i u_i^T \in C^+ \) if and only if \( u_i \in \text{Span}\{a_1, a_2\} \) and \( v_j \in \text{Span}\{b_1, b_2\} \). If there are three data matrices in \( C^+ \), say corresponding to indices \( i, j, k \), then \( u_i, u_j, u_k \in \text{Span}\{a_1, a_2\} \) and \( v_i, v_j, v_k \in \text{Span}\{b_1, b_2\} \). However this contradicts our general position assumption, so we conclude that \( C^+ \) may contain at most two data matrices.

We need one final lemma to show that the general position assumption in Theorem 5.5 prevents the existence of codimension one compression spaces of rank at most two in \( \mathcal{L}_P \) when \( |P| = 4 \). The following simple fact about our input point pairs will be useful.

Lemma 5.10. If for \( i \neq j \), \( w u_i^T = v_j u_j^T \) then \( w = v_j \) and \( u_i = u_j \). Similarly, if for \( i \neq j \), \( v_i u_i^T = v_j w^T \) then \( w = u_i \) and \( v_i = v_j \).

Proof. Since the \( u_i \)'s have last coordinate 1, if \( w u_i^T = v_j u_j^T \), the last columns of the two matrices are \( w \) and \( v_j \). Therefore, \( w = v_j \) and hence \( u_i = u_j \). The second statement is proved similarly.

Lemma 5.11. Suppose \( |P| = 4 \) and for all distinct indices \( i, j, k, u_i, u_j, u_k \) or \( v_i, v_j, v_k \) are in general position. Let \( C' \) be a \((k_1, k_2)\)-compression space of rank at most two and codimension one in \( \mathcal{L}_P \), and let \( C \) be a maximal \((k_1, k_2)\)-compression space containing \( C' \). 

(1) If \( C' \) is of type \((2,0)\) or \((0,2)\), then \( C^+ \) contains at least two data matrices.

(2) If \( C' \) is of type \((1,1)\), then \( C^+ \) contains at least three data matrices.
Proof. By the general position assumption, at least three of the data matrices are linearly independent, and hence \( \dim(L_P) = 4 \) or 5.

1. Suppose \( C' \) is a maximal \((2, 0)\)-compression space of rank at most two and codimension one in \( L_P \). Then by Lemma 5.18 \((C')^\perp = \text{Span}\{b_1a^T, b_2a^T, b_3a^T\}\) and \( \dim(C') = 5 \). This implies that \( \dim(L_P) = 6 \) which is a contradiction.

So suppose \( C' \) is a non-maximal \((2, 0)\)-compression space of rank at most two and codimension one in \( L_P \), and \( C \) is a maximal \((2, 0)\)-compression space containing \( C' \). Define

\[
M_C = \begin{pmatrix} b_1a^T \\ b_2a^T \\ b_3a^T \end{pmatrix}, \quad M_D = \begin{pmatrix} v_1u_1^T \\ v_2u_2 \\ v_3u_3^T \\ v_4u_4^T \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} M_C \\ M_D \end{pmatrix}.
\]

(a) \( \dim(L_P) = 5 \): Then \( \dim(C') = 4 \) and \((C')^\perp\) has a basis of cardinality four. We may assume without loss of generality that \((C')^\perp = \text{Span}\{b_1a^T, b_2a^T, b_3a^T, v_1u_1^T\}\). By Lemma 5.7, \( u_1 \) is a multiple of \( a \). Since \( \dim(L_P) = 5 \), \( \text{rank}(M_D) = 3 \), and \( C' \) has codimension one in \( L_P \), \( \text{rank}(M) = 4 \) and so \( v_2u_2, v_3u_3, v_4u_4 \in (C')^\perp \). We need to argue that for \( i > 1 \), \( v_i u_i^T = \text{Span}\{b_1a^T, b_2a^T, b_3a^T\} \).

Suppose for \( i > 1 \) \( v_i u_i^T \) equals the linear combination

\[
\alpha_1 b_1 a^T + \alpha_2 b_2 a^T + \alpha_3 b_3 a^T + \beta v_i u_i^T = (\sum \alpha_j b_j)^T (u_i^T).
\]

If \( \beta = 0 \) then we are done, so suppose \( \beta \neq 0 \). If all the \( \alpha_j \)'s are zero, then \( v_i u_i^T \) is a multiple of \( v_1 u_1^T \) which is impossible by Lemma 5.10. If at least one \( \alpha_j \neq 0 \), then \( \alpha_j = \beta_1 \). In this case, \( v_i u_i^T \) looks like \( v_i u_i^T + \delta v_i u_i^T \). Again, by Lemma 5.10, this cannot equal \( v_i u_i^T \) for any \( i = 2, 3, 4 \). Therefore, it must be that for \( i > 1 \), \( v_i u_i^T \in C^\perp \).

(b) \( \dim(L_P) = 4 \): In this case, \( \dim(C') = 3 \), \( \text{rank}(M_D) = 4 \) and \( \text{rank}(M) = 5 \). Without loss of generality, \( v_3 u_3^T, v_4 u_4^T \in (C')^\perp = \text{Span}\{b_1a^T, b_2a^T, b_3a^T, v_1 u_1^T, v_2 u_2^T\} \). We need to argue that \( v_3 u_3^T, v_4 u_4^T \in \text{Span}\{b_1a^T, b_2a^T, b_3a^T\} \).

Suppose for \( i \in \{3, 4\} \), \( v_i u_i^T \) is a linear combination of the form

\[
\alpha_1 b_1 a^T + \alpha_2 b_2 a^T + \alpha_3 b_3 a^T + \beta v_i u_i^T = (\sum \alpha_j b_j)^T (u_i^T).
\]

If all the \( \alpha_j \)'s are 0 then \( v_i u_i^T \) lies in the span of the other two data matrices which contradicts that \( \text{rank}(M_D) = 4 \). If all \( \beta_j \)'s are 0 then we are done. So assume that at least one \( \alpha_j \) and one \( \beta_j \) are non-zero. If only one \( \beta_j \) is non-zero, the claim follows from the same argument as in the previous case. So suppose \( \beta_1, \beta_2 \neq 0 \). By Lemma 5.7, neither \( u_1 \) nor \( u_2 \) are multiples of \( a \) which means that the second matrix in the product has rank at least two. The first matrix also has rank at least two since \( v_1 \) and \( v_2 \) are linearly independent and hence \( (C')^\perp \) has rank at least two and cannot equal \( v_i u_i^T \) for \( i = 3, 4 \).

The \((0, 2)\) case is argued similarly.

2. Suppose \( R_2 \cap L_P \) has an irreducible component \( C' \) which is a \((1, 1)\)-compression space.

(a) \( \dim(L_P) = 5 \): In this case, \( \dim(C') = 4 \) and \( C' = C \) is a maximal compression space. Therefore, \( (C')^\perp = C^\perp = \text{Span}\{b_1a_i^T, b_2a_i^T, b_3a_i^T, b_4a_i^T\} \) and \( \text{rank}(M) = 4 \). The matrix \( M_C \) now has rows \( b_1a_i^T, b_2a_i^T, b_3a_i^T, b_4a_i^T \). Since it already has rank four, it must be that all the data matrices are in \( C^\perp \).

They are in fact in the \( \mathbb{F}^1 \times \mathbb{F}^1 \) subvariety of \( C^\perp \) containing all the rank one matrices.

(b) \( \dim(L_P) = 4 \): In this case \( \dim(C') = 3 \) which means that without loss of generality, \( (C')^\perp = \text{Span}\{b_1a_i^T, b_1a_2^T, b_2a_2^T, b_3a_3^T, b_4a_4^T\} \). Also, \( \text{rank}(M) = 5 \) and hence \( v_2 u_2^T, v_3 u_3^T, v_4 u_4^T \in (C')^\perp \).

We need to argue that these matrices in fact lie in \( C^\perp = \text{Span}\{b_1a_1^T, b_2a_2^T, b_3a_3^T, b_4a_4^T\} \). Suppose for \( i \in \{2, 3, 4\} \), \( v_i u_i^T \) is a linear combination of the form

\[
\alpha_{11} b_1 a_1^T + \alpha_{12} b_2 a_2^T + \alpha_{21} b_2 a_1^T + \alpha_{22} b_3 a_2^T + \beta v_1 u_1^T = (b_1 b_2 v_1) \begin{pmatrix} \alpha_{11} a_1^T + \alpha_{12} a_2^T \\ \alpha_{21} a_1^T + \alpha_{22} a_2^T \end{pmatrix} \beta u_1^T
\]

with \( \beta \neq 0 \). As before, some \( \alpha_{ij} \) must also be non-zero. By Lemma 5.7, \( v_1 \neq \text{Span}\{b_1, b_2\} \) or \( u_1 \neq \text{Span}\{a_1, a_2\} \). Suppose \( v_1 \neq \text{Span}\{b_1, b_2\} \). Then the first matrix in \( (14) \) has rank two and hence
Equivalently, we break the proof into three parts:

Proof. If configurations of the types in Figure 1e and Figure 1f for which a chiral reconstruction can fail to exist.

Rank captures the geometry of $\mathcal{U}$ and $\mathcal{V}$. Indeed, if $\text{rank}(\mathcal{U}) = 1$, then all $u_i$ are coincident as points in $\mathbb{P}^2$. If $\text{rank}(\mathcal{U}) = 2$, then all $u_i$ are collinear in $\mathbb{P}^2$, and if $\text{rank}(\mathcal{U}) = 3$, then some three $u_i$ are non-collinear in $\mathbb{P}^2$. The assumption that points in $\mathcal{U}$ and $\mathcal{V}$ are distinct implies that $\text{rank}(\mathcal{U}), \text{rank}(\mathcal{V}) \geq 2$. The spirit of Theorem 5.13, the main result of this section, is that when $\mathcal{U}$ and $\mathcal{V}$ have similar geometry, then $\mathcal{P}$ has a chiral reconstruction. In particular, a chiral reconstruction exists for the combinatorial types in Figure 1a, Figure 1b, and Figure 1c where $\text{rank}(\mathcal{U}) = \text{rank}(\mathcal{V}) = 3$ and for the type in Figure 1d where $\text{rank}(\mathcal{U}) = \text{rank}(\mathcal{V}) = 2$. We will present examples of configurations of the types in Figure 1a and Figure 1f for which a chiral reconstruction can fail to exist.

Theorem 5.13. If $|\mathcal{P}| = 4$ and $\text{rank}(\mathcal{U}) = \text{rank}(\mathcal{V})$, then $\mathcal{P}$ has a chiral reconstruction.

Proof. We break the proof into three parts:

(1) Suppose $\text{rank}(\mathcal{U}) = \text{rank}(\mathcal{V}) = 2$. Then the points in both $\mathcal{U}$ and $\mathcal{V}$ are collinear as in Figure 1d even as points in $\mathbb{R}^3$. Assume without loss of generality that the $v_i$ points appear in the order $v_1, v_2, v_3, v_4$ along the affine line $L$ they span in $\mathbb{R}^3$. The ordering of $v_i$ induces an ordering of the $u_i$. Let $l, r \in \{1, 2, 3, 4\}$ be such that for all $i \in \{1, 2, 3, 4\}$, $u_i \in \text{cone}(u_l, u_r)$, and define $G \in GL_3$ such that $Gu_i = v_1$ and $Gu_r = v_4$. This forces $v_1, Gu_1 \in \text{cone}(v_1, v_4)$ for all $i$. For $t \in L \setminus \text{cone}(v_1, v_4)$, $(t \times v_i) \perp (t \times Gu_i)$, proving the result.
(2) Suppose \( \text{rank}(U) = \text{rank}(V) = 3 \) and for all \( i, j, k, u_i, u_j, u_k \) or \( v_i, v_j, v_k \) are in general position. Then by Theorem 5.14, \( R_2 \cap L_P \) is irreducible, and hence by Theorem 5.16, it suffices to show that there is a smooth rank two matrix \( X \in R_2 \cap L_P \) for which \( P_{I(X)} \) has a chiral reconstruction.

We first claim that there is an ordering of point pairs such that \( u_1, u_2, u_3 \) are in general position and \( v_4 \) does not lie on any of the lines \( L_{12}, L_{13}, \) and \( L_{23} \) where \( L_{ij} \) is the line spanned by \( v_i \) and \( v_j \). Indeed, since rank \( (V) = 3 \), there is an ordering of the point pairs so that \( v_4 \) does not lie on any of the lines \( L_{12}, L_{13}, \) or \( L_{23} \). If \( u_1, u_2, u_3 \) are in general position for this ordering, we are done. Otherwise, \( u_1, u_2, u_3 \) collinear and so \( v_1, v_2, v_3 \) must be in general position by assumption, and hence \( v_1, v_2, v_3, v_4 \) are in general position. Also, since rank \( (U) = 3 \), \( u_1, u_2, u_4 \) must be non-collinear. Since \( v_3 \) cannot be on the lines \( L_{12}, L_{14}, L_{24}, \) swapping point pairs with indices 3 and 4 gives the desired result.

Reorder point pairs so that \( u_1, u_2, u_3 \) are in general position and \( v_4 \) does not lie on any of \( L_{12}, L_{13}, \) and \( L_{23} \). Then choosing \( e_2 = \mathbf{v}_4 \) forces \( \det[\mathbf{v}_1 \mathbf{v}_2 e_2], \det[\mathbf{v}_1 \mathbf{v}_3 e_2], \det[\mathbf{v}_2 \mathbf{v}_3 e_2] \) to all be non-zero. Removing \( u_4 \) from \( U \) leaves three points in general position, so we may pick \( e_1 \) by Lemma 4.11 so that \( \det[\mathbf{u}_i u_j e_1] \det[\mathbf{v}_i \mathbf{v}_j e_2] > 0 \) for all \( 1 \leq i < j \leq 3 \). Since the positivity of these determinant products is an open condition, there is a open polyhedral cone from which such an \( e_1 \) can be chosen. Consider the system

\[
X e_1 = 0, \quad v_4^T X u_i = 0, \quad i = 1, 2, 3, 4
\]

which consists of at most eight linearly independent equations. Therefore, this system has a solution \( X \in \mathbb{P}^8 \). From Lemma 4.17, we know the tangent space to \( R_2 \) at \( X \) has normal \( v_4 e_2^T \) and \( X \) is a smooth point of the epipolar variety if \( v_4 e_2^T \) does not lie in \( \text{Span}(v_i u_j^T, i = 1, 2, 3, 4) \). If \( v_4 e_2^T \in \text{Span}(v_i u_j^T, i = 1, 2, 3, 4) \), there is a minimal subset of the five matrices, including \( v_4 e_2^T \) that are dependent. Placing the vectorizations of these matrices in the rows of a matrix, we get that all its maximal minors are zero. Each of these minors, set to zero, is a linear equation in \( e_1 \) and together they cut out a subspace in \( \mathbb{P}^2 \). Since we were able to choose \( e_1 \) from an open polyhedral cone, there is a choice of \( e_1 \) that avoids this subspace and the rank one variety. Hence, \( X \) can be chosen (by choosing \( e_1 \) appropriately) to be a smooth rank two fundamental matrix and we are done by Theorem 4.16.

(3) Suppose \( \text{rank}(U) = \text{rank}(V) = 3 \) and there exist \( i, j, k \) for which both \( u_{ij}, u_k \) and \( v_i, v_j, v_k \) are collinear. Without loss of generality we may assume that \( \{i, j, k\} = \{1, 2, 3\} \), and that \( v_1, v_2, v_3 \) appear in this order on the affine line \( L \) they span in \( \mathbb{R}^3 \). Let \( l, r \in \{1, 2, 3\} \) be such that \( u_l, u_r, u_3 \in \text{cone}(u_1, u_2) \), and define \( G \) by \( G(u_l) = v_1, G(u_r) = v_3, \) and \( G(u_3) = v_4 \). Then for a \( t \in L \setminus \text{cone}(v_1, v_3) \), set \( X = [t]_4 G \). Since, \( v_i, t, \) and \( G u_i \) are collinear for each \( i, X \) satisfies all the epipolar equations \( v_i^T X u_i = 0 \). Also, since \( v_l, G u_i \in \text{cone}(v_1, v_3) \) for \( i = 1, 2, 3, g_1(X), g_2(X), g_3(X) > 0 \) by Lemma 4.2 and Lemma 4.12. Finally, \( g_4(X) = (t \times v_4)^T (t \times G u_4) = (t \times v_4)^T (t \times v_4) > 0 \), and so \( X \) strictly satisfies all chiral epipolar inequalities and \( P \) has a chiral reconstruction by Theorem 4.3.

We conclude by showing that for four point pairs, Theorem 5.13 is best possible in the following sense.

**Theorem 5.14.** For the two combinatorial types where \( \text{rank}(U) \neq \text{rank}(V) \), there are instances of \( P \) without a chiral reconstruction.

We give examples to prove Theorem 5.14. Recall that the epipolar line homography of Theorem 3.2 cannot send coincident lines to distinct lines or vice versa. Therefore if \( X \) is a \( P \)-regular fundamental matrix with right and left kernels generated by \( e_1 \) and \( e_2 \), respectively, then \( e_1, u_i, u_j \) are collinear if and only if \( e_2, v_i, v_j \) are collinear.

**Example 5.15.** Consider the arrangement in Figure 2, where \( \text{rank}(U) = 3 \neq \text{rank}(V) = 2. \)

![Figure 2](image-url)
Let $L$ be the line in $\mathbb{P}^2$ spanned by $(0, 0, 1)^\top$ and $(1, 0, 1)^\top$. Suppose there is a fundamental matrix $X$ for $\mathcal{P}$ with $e_2^\top X = 0 = X e_1$, and suppose $e_2 = (e_{21}, e_{22}, e_{23})^\top$ and $e_1 = (e_{11}, e_{12}, e_{13})^\top$. Then $e_2 \notin L$ (equivalently, $e_{22} \neq 0$) because then the set of lines $e_2(v_1, v_2, v_3, v_4)$ consists of a repeated line, while the set of lines $e_1(u_1, u_2, u_3, u_4)$ contains at least two distinct lines for any choice of $e_1$. If $e_2 \in \mathbb{P}_L$, then $e_2(v_1, v_2, v_3, v_4)$ consists of four distinct lines, and hence $e_1(u_1, u_2, u_3, u_4)$ must also have four distinct lines. In particular, $e_1 \notin L$, or equivalently, $e_{12} \neq 0$. These restrictions imply that $D_{12}(e_1, e_2) = 2e_{12}e_{22}$, $D_{13}(e_1, e_2) = 2e_{12}e_{22}$, $D_{23}(e_1, e_2) = -e_{12}e_{22}$, are all non-zero. It is then clear that they cannot all have the same sign. Therefore, by Theorem 4.10, there is no chiral reconstruction of $\mathcal{P}$.

The epipolar variety in Example 5.15 is reducible; $W^{v_4}$ is a linear component containing the wall $W_{u_2}$. Let $t = v_4$ and define $G \in \text{GL}_3$ such that $Gu_1 = v_1$ and $Gu_2 = v_4$. Then $X = [t]_G$ is a fundamental matrix of $\mathcal{P}$. Indeed, $v_4^\top X = 0$ and $X u_2 = [v_4]_G u_2 = [v_4]_G v_4 = 0$ and therefore $X$ satisfies the second and fourth epipolar equations. It is straightforward to check that $v_1, t, Gu_1$ are collinear and $v_3, t, Gu_3$ are collinear, so $X$ also satisfies the first and third epipolar equations. By construction, $g_2(X) = q_4(X) = 0$, and by Lemma 4.2 and Lemma 4.12 $g_1 g_3(X) > 0$. Thus, $X$ is a fundamental matrix of $\mathcal{P}$ that satisfies all the chiral epipolar inequalities $g_i g_j(X) \geq 0$ for all $1 \leq i < j \leq 4$. However, $X$ lies in the corner $W_{u_2} \cap W^{v_4}$ and is not $\mathcal{P}$-regular and all neighboring matrices to $X$ on $W^{v_4}$ are also $\mathcal{P}$-irregular. Therefore, we cannot use Theorem 4.10 to perturb $X$ to a regular fundamental matrix.

Example 5.15 illustrates why the irreducibility of the epipolar variety is needed in Theorem 4.16. The next example shows that a chiral reconstruction may not exist even when the epipolar variety is irreducible.

**Example 5.16.** Consider the arrangement in Figure 3 where $\text{rank}(U) = 3 \neq \text{rank}(V) = 2$.

$$U = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

**Figure 3**

By Theorem 5.5 this epipolar variety is irreducible. Suppose $\mathcal{P}$ has a fundamental matrix $X$ with left epipole $e_2 = (e_{21}, e_{22}, e_{23})^\top$ and right epipole $e_1 = (e_{11}, e_{12}, e_{13})^\top$. By Theorem 3.2 $e_2$ cannot be on the line spanned by the $v$ points since there is no $e_1$ for which the set of lines $e_1(u_1, \ldots, u_4)$ consists of coincident lines. For any $e_2 = (e_{21}, e_{22}, e_{23})^\top$ not on the line spanned by the $v$ points, the vector

$$(\det[v_i, v_j, e_2] : 1 \leq i, j \leq 4) = (2e_{22}, e_{22}, 3e_{22}, -e_{22}, e_{22}, 2e_{22})$$

which has sign pattern $++-++$ (up to sign) since $e_{22} \neq 0$, and for any $e_1 = (e_{11}, e_{12}, e_{13})^\top$,

$$(\det[u_i, u_j, e_1] : 1 \leq i, j \leq 4) = (e_{12}, -e_{11} + e_{12}, -e_{11}, -e_{11} + e_{13}, -e_{11} - e_{12} + e_{13}, -e_{11} + e_{13}).$$

We can guarantee a chiral reconstruction if

$e_{12} > 0, \quad -e_{11} + e_{12} > 0, \quad -e_{11} > 0, \quad -e_{11} + e_{13} < 0, \quad -e_{11} - e_{12} + e_{13} > 0, \quad -e_{12} + e_{13} > 0.$

However, this implies that $e_{12} > e_{11} > e_{13} > e_{12}$ which is a contradiction. Similarly, we cannot achieve the sign pattern $---+++--$ either. What might still be possible is to choose $e_1$ so that some of the determinants $\det[u_i, u_j, e_1]$ are zero, or equivalently, $e_1$ lies on the line joining two of the $u$ points. This will not work since then two coincident rays will have to map to two non-coincident rays under the homography in Theorem 3.2. Therefore, $\mathcal{P}$ has no chiral reconstruction.

Unlike in Example 5.15, there is no fundamental matrix for Example 5.16 that satisfies all chiral epipolar inequalities. Indeed, Theorem 4.16 implies that there cannot be a smooth $X$ such that $\mathcal{P}_I(X)$ has a chiral reconstruction associated to $X$. This rules out the possibility that there is an irregular $X$ with respect to some index that satisfies the chiral epipolar inequalities.
6. Five Point Pairs

We have seen so far that three point pairs unconditionally have a chiral reconstruction as do four point pairs unless their geometry is special and dissimilar. In this section we will see that five point pairs can fail to have a chiral reconstruction even if both sets of points are in general position. Specific examples of such point pairs were known to Werner [22]. We will show that, in fact, five point pairs do not have a chiral reconstruction with positive probability. We will use Theorem 4.16 to devise a simple test for chirality in this section, and connect to the classical theory of cubic surfaces in the next section.

Throughout this section, let $\mathcal{P} = \{(u_i, v_i) : i = 1, \ldots, 5\}$ be a set of five generic point pairs in the sense that $\mathcal{L}_\mathcal{P}$ is a generic subspace in $\mathbb{P}^8$ of codimension five. In this case, $\mathcal{L}_\mathcal{P}$ misses the four-dimensional variety $\mathcal{R}_1$, and by Bertini’s Theorem, the epipolar variety $\mathcal{R}_2 \cap \mathcal{L}_\mathcal{P}$ is a smooth, irreducible cubic surface. Each wall $W_{u_i}$ or $W_{v_j}$ is a line on $\mathcal{R}_2 \cap \mathcal{L}_\mathcal{P}$. We first see how these lines intersect.

**Lemma 6.1.** Let $\mathcal{P}$ be a set of five generic point pairs. Then

1. $W_{u_i}$ and $W_{u_j}$, and similarly $W_{v_i}$ and $W_{v_j}$, do not intersect for all $i \neq j$.
2. $W_{u_i}$ and $W_{v_j}$ do not intersect for all $i$.
3. The corner $W_{u_i} \cap W_{v_j}$ consists of a unique real rank two matrix for $i \neq j$.

**Proof.**

1. Recall that for $i \neq j$, $u_i \neq u_j$, and so there is no rank two $X$ such that $Xu_i = X u_j$.
2. Since $\widehat{W}_{u_i} \cap \widehat{W}_{v_i} \subset L_{(u_i, v_i)}$, $W_{u_i} \cap W_{v_i}$ is the intersection of the three-dimensional linear space $\widehat{W}_{u_i} \cap \widehat{W}_{v_i}$ with the generic codimension four linear space $\bigcap_{\lambda \neq i} L_{(u_\lambda, v_\lambda)}$ in $\mathbb{P}^8$, and hence empty.
3. When $i \neq j$, $W_{u_i} \cap W_{v_j}$ is the intersection of the three-dimensional linear space $\widehat{W}_{u_i} \cap \widehat{W}_{v_j}$ with the codimension three linear space $\bigcap_{\lambda \neq i,j} L_{(u_\lambda, v_\lambda)}$ in $\mathbb{P}^8$. This intersection is zero-dimensional and since the defining equations are linear with real coefficients, it consists of a single real matrix $X$. Generically, the intersection will miss $\mathcal{R}_1$, so we conclude that $X$ is a real rank two matrix.

**Remarks 6.2.** Concretely, when $i \neq j$, the rank two matrix $W_{u_i} \cap W_{v_j}$ is $X = [v_j] \times H$ where $H$ is the unique homography sending $u_i$ to $v_j$ and $u_i$ to $v_i$ for all $l \neq i, j$. It is easy to verify that this $X$ satisfies the epipolar equations and is the unique point in $W_{u_i} \cap W_{v_j}$.

**Definition 6.3.** Let $C$ be the set of all wall intersections on the epipolar variety:

$$C = \bigcup_{1 \leq i,j \leq k} W_{u_i} \cap W_{v_j}.$$ 

We call the points in $C$ the corners of the epipolar variety.

Lemma 6.1 shows that $C$ consists of 20 distinct fundamental matrices of $\mathcal{P}$. However, they do not correspond to full projective reconstructions of $\mathcal{P}$ because necessarily the $(u_i, v_j)$ corner is neither $(u_i, v_i)$ regular nor $(u_j, v_j)$ regular. Despite this fact, we argue that checking the signs of the chiral epipolar inequalities at these corners suffices to determine whether $\mathcal{P}$ has a chiral reconstruction.

**Theorem 6.4.** Let $\mathcal{P}$ be a generic set of five point pairs. Then $\mathcal{P}$ has a chiral reconstruction if and only if $\mathcal{P}_{I(x)}$ has a chiral reconstruction for some 20 corners $X \in C$.

**Proof.** Suppose $\mathcal{P}$ has a chiral reconstruction. Then Theorem 4.13 implies that there is a $\mathcal{P}$-regular fundamental matrix $X$ such that $g_i g_j(X) \geq 0$ for all $i, j$. By Lemma 6.1, the epipolar variety does not contain the $(u_i, v_i)$ corner for any $i$. Therefore regularity with respect to all image pairs implies $g_i(X) \neq 0$ for all $i$, meaning $g_i g_j(X) > 0$ for all $i, j$. Let $U$ be the connected component of the semialgebraic subset of the epipolar variety described by $g_i g_j \geq 0$ that contains $X$. The chiral epipolar inequalities only change sign along walls. Either $U$ is the entire epipolar variety or its boundary is contained in the walls. Every wall contains four corners and every interval on a wall that is contained in $U$ is bounded by corners (or unbounded, in which case it contains all four corners on the wall). So $U$ contains corners and every corner $X \in U$ lies in $C$ and corresponds to a chiral reconstruction of the subset $\mathcal{P}_{I(x)}$ of point pairs.

Conversely, suppose $\mathcal{P}_{I(x)}$ has a chiral reconstruction associated to some $X \in C$. The epipolar variety $\mathcal{R}_2 \cap \mathcal{L}_\mathcal{P}$ is smooth, irreducible, and consists only of rank two matrices. Hence the points in $C$ are smooth fundamental matrices and Theorem 4.16 implies there is a chiral reconstruction of $\mathcal{P}$. □
Theorem 4.10 allows us to infer the sign of all non-zero chiral epipolar inequalities at a corner \( X \in C \) directly using the input point pairs. The next corollary makes this precise.

**Corollary 6.5.** Let \( \mathcal{P} \) be a set of five generic point pairs. Then \( \mathcal{P} \) has a chiral reconstruction if and only if there is some \((u_i, v_j)\) corner such that \( D_{lm}(u_i, v_j)\), \( D_{ln}(u_i, v_j)\), and \( D_{mn}(u_i, v_j)\) have the same sign for \( l, m, n \in [5]\setminus\{i, j\}\).

**Proof.** Since \( \mathcal{P} \) is generic, no three \( u_i \) are collinear and no three \( v_i \) are collinear. Hence, the expressions \( D_{lm}(u_i, v_j)\), \( D_{ln}(u_i, v_j)\), and \( D_{mn}(u_i, v_j)\) are all non-zero for pairwise distinct \( l, m, n \in [k]\setminus\{i, j\}\). There is some \((u_i, v_j)\) corner \( X \in C \) such that \( D_{lm}(u_i, v_j)\), \( D_{ln}(u_i, v_j)\), and \( D_{mn}(u_i, v_j)\) have the same sign if and only if \( \mathcal{P}_{l,m,n} = \mathcal{P}_{l,m,n} \) has a chiral reconstruction associated to \( X \) (Theorem 4.10) if and only if \( \mathcal{P} \) has a chiral reconstruction (Theorem 6.4). \( \square \)

Corollary 6.5 gives an algorithm to check whether five generic point pairs have a chiral reconstruction. We evaluate the sign of three polynomials on the input point pairs at each of the 20 corners. The signs are the same at a corner if and only if it lies in the boundary of the chiral epipolar region of \( \mathcal{P} \). We illustrate the procedure on an example.

**Example 6.6.** Let

\[
U = \begin{pmatrix} 0 & 0 & 4 & 2 & 2 \\ 0 & 4 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2 & 2 & 4 & 0 & 1 \\ 1 & 3 & 0 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\]

and \( \mathcal{P} = \{ (u_i, v_i) \} \) where \( u_i \) is the \( i \)th column of \( U \) and \( v_i \) is the \( i \)th column of \( V \). By Corollary 6.5 and the following table, \( \mathcal{P} \) has no chiral reconstruction.

| \( i \) | \( j \) | \( D_{lm}(u_i, v_j) \) | \( D_{ln}(u_i, v_j) \) | \( D_{mn}(u_i, v_j) \) | \( i \) | \( j \) | \( D_{lm}(u_i, v_j) \) | \( D_{ln}(u_i, v_j) \) | \( D_{mn}(u_i, v_j) \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | -16 | -64 | 20 | 3 | 4 | -64 | 36 | 20 |
| 1 | 3 | -32 | -56 | 32 | 3 | 5 | -32 | -12 | 20 |
| 1 | 4 | 64 | 40 | -96 | 4 | 1 | 16 | -8 | -4 |
| 1 | 5 | 112 | -40 | -32 | 4 | 2 | 16 | 8 | -28 |
| 2 | 1 | -16 | -4 | 12 | 4 | 3 | 32 | -4 | -28 |
| 2 | 3 | -32 | 8 | -32 | 4 | 5 | -16 | -4 | 28 |
| 2 | 4 | 64 | -24 | -32 | 5 | 1 | -16 | 16 | -16 |
| 2 | 5 | -16 | 24 | -32 | 5 | 2 | 48 | -16 | 16 |
| 3 | 1 | 16 | -8 | -12 | 5 | 3 | 32 | -16 | 16 |
| 3 | 2 | 16 | 24 | -20 | 5 | 4 | -32 | 48 | -16 |

Now consider the following modification of the above example obtained by perturbing \( v_5 \):

\[
U = \begin{pmatrix} 0 & 0 & 4 & 2 & 2 \\ 0 & 4 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2 & 2 & 4 & 0 & 4 \\ 1 & 3 & 0 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\]

We compute the products \( D_{lm}, D_{ln}, D_{mn} \) at all 20 corners \( (u_i, v_j) \) and find that

\[
D_{14}(u_2, v_3) = -32, \quad D_{15}(u_2, v_3) = -64, \quad D_{45}(u_2, v_3) = -64,
\]

and hence the \( (u_2, v_3) \) corner lies in the boundary of the chiral epipolar region, so these point pairs have a chiral reconstruction. We point out that the same-signness property is not symmetric in the sense that if it holds for \((u_i, v_j)\), it need not hold for \((u_j, v_i)\). Indeed,

\[
D_{14}(u_3, v_2) = 16, \quad D_{15}(u_3, v_2) = -48, \quad D_{45}(u_3, v_2) = 16.
\]

The point pairs in (15) are a slight modification of [22, Figure 1] for which the epipolar variety is singular. Our modification makes the variety smooth, a property of interest in the next section. Furthermore, since sufficiently small perturbations of point pairs do not change the signs of \( D_{lm}, D_{ln}, \) and \( D_{mn} \) at any \((u_i, v_j)\) corner, our methods show that having no chiral reconstruction is an open condition. This leads to the following conclusion.

**Theorem 6.7.** The set of five point pairs without a chiral reconstruction is Zariski dense in the space of all five point pairs.

Finally, embedding the point pairs in (15) into instances of six or more point pairs one can construct larger configurations with no chiral reconstruction.

20
7. Connections to Classical Algebraic Geometry

In this section, we discuss the connection between five point pairs and the classical theory of real cubic surfaces in projective 3-space. This point of view is yet another way to study the space of epipoles, which goes back to Klein and Segre [20]. General references for cubic surfaces are still mostly classical books like [14], [15], or [20]. More modern accounts of classical facts about cubic surfaces can be found in [7] or [19]. Some of the history going back to Cayley and Salmon is discussed in [6].

As in Section 6, we consider sets of five point pairs \( P \) that are generic though some of our discussion below might generalize to more singular situations. In the first three subsections, we use the following as a running example.

**Example 7.1.** Consider the point correspondences \( P = \{(u_i, v_i), \ i = 1, \ldots, 5\} \) where \( u_i \) is the \( i \)th column of \( U \) and \( v_i \) is the \( i \)th column of \( V \) shown below:

\[
U = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 3 & 5 & -1 & -3 & 1 \\ 0 & 0 & -2 & -2 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\]

**7.1. From two images to a cubic surface.** Given five generic point pairs \( P = \{(u_i, v_i) \in \mathbb{P}^2 \times \mathbb{P}^2 | i = 1, 2, 3, 4, 5\} \), the epipolar variety \( \mathcal{R}_2 \cap \mathcal{L}_P \subset \mathcal{L}_P \cong \mathbb{P}^3_{\mathbb{R}} \) is a smooth cubic surface and its defining equation comes with a determinantal representation. Indeed, \( \mathcal{R}_2 \subset \mathbb{P}^5_{\mathbb{R}} \) is defined by \( \det(X) = 0 \); to get the equation of the intersection with the linear space \( \mathcal{L}_P \subset \mathbb{P}^5_{\mathbb{R}} \), we compute a basis \( (M_0, M_1, M_2, M_3) \) of \( \mathcal{L}_P \) and plug a general linear combination \( z_0 M_0 + z_1 M_1 + z_2 M_2 + z_3 M_3 \) of this basis into the equation \( \det(X) = 0 \) to obtain

\[ \mathcal{R}_2 \cap \mathcal{L}_P = \{ (z_0 : z_1 : z_2 : z_3) \in \mathbb{P}^3_{\mathbb{R}} | \det(M(z_0, z_1, z_2, z_3)) = 0 \} \]

where the entries of \( M \) are linear forms in \( z_0, z_1, z_2, z_3 \).

**Example 7.2.** The variety \( \mathcal{R}_2 \cap \mathcal{L}_P \) is cut out by \( \det(M(z)) = 0 \) where

\[
M(z) = \begin{pmatrix} -z_0 - z_1 + z_2 + 4z_3 & -z_2 - z_3 & -z_2 + z_3 \\ z_0 - z_1 & z_0 + z_1 + 3z_2 - z_0 + 2z_1 + 2z_2 - z_3 \\ z_0 + 3z_1 + z_2 + 2z_3 & z_2 + 5z_3 & 5z_2 - 5z_3 \end{pmatrix}.
\]

The coefficient matrices \( M_0, M_1, M_2, M_3 \) in the linear matrix polynomial \( M(z) = z_0 M_0 + z_1 M_1 + z_2 M_2 + z_3 M_3 \) form a basis of \( \mathcal{L}_P \).

**7.2. The 27 lines on a cubic surface.** The cubic surface \( S = \mathcal{R}_2 \cap \mathcal{L}_P \subset \mathbb{P}^3_{\mathbb{R}} \) contains special lines coming from the input point pairs, namely the walls \( W_{u_i} \) and \( W_{v_i} \). From Lemma 6.1, we know the lines \( W_{u_i} \) do not intersect each other and the lines \( W_{v_i} \) do not intersect each other. Additionally the lines \( W_{u_i} \) intersect the lines \( W_{v_i} \) in corners, but they do not all intersect pairwise. We have 10 lines in a very special configuration: It turns out that this is only possible if these 10 lines are contained in a real Schl"afli double six on \( S \).

**Definition 7.3.** A Schl"afli double six on a smooth cubic surface \( S \subset \mathbb{P}^3_{\mathbb{R}} \) is a pair of six-tuples of lines

\[ \{(\ell_1, \ell_2, \ldots, \ell_6), (\ell'_1, \ell'_2, \ldots, \ell'_6)\} \]

on \( S \) such that the six lines in each tuple are pairwise disjoint and \( \ell_i \cap \ell'_j \) is non-empty if and only if \( i \neq j \).

Every smooth cubic surface contains 27 complex lines in total and they can be organized into 36 different Schl"afli double six configurations. The combinatorics behind this has been studied extensively. A discussion of the Schl"afli double sixes is summarized in [4, Section 2]. For a development via the blow-up construction, see [13, Chapter V, Section 4]. With this approach, it is straightforward to check that a pair of five tuples of lines \( (\ell_1, \ldots, \ell_5), (\ell'_1, \ldots, \ell'_5) \) on a smooth cubic surface with \( \ell_i \cap \ell_j = \emptyset, \ell'_i \cap \ell'_j = \emptyset \) (for all \( i \neq j \)), and \( \ell_i \cap \ell'_j = \emptyset \) if and only if \( i = j \), uniquely determines a Schl"afli double six.

The fact that our cubic surfaces, that appear as epipolar varieties, always contain a Schl"afli double six consisting of real lines, means that they are all of the same real topological type. Schl"afli, Klein, and Zeuthen classified the real topologies of cubic surfaces. There are five possible types. The real classification of cubic surfaces is summarized in [4, Theorem 5.4] and discussed in detail in [9, pp. 40–55]. Epipolar varieties are always of type \( F_1 \) containing 27 real lines. Indeed, cubic surfaces of type \( F_1, F_3 \), or \( F_5 \) do not contain enough real lines to have a real Schl"afli double six (namely only seven, three, and three). Type \( F_2 \) surfaces contain 15 real lines, which would be enough for a Schl"afli double six. These surfaces are the blow-up of \( \mathbb{P}^2_{\mathbb{R}} \) in four real points and one conjugate pair of complex points. The 15
lines are the four real exceptional divisors over the four real points, the strict transforms of the six real lines joining any pair of the four real points, the strict transform of the one real line joining the conjugate pair of complex points and the strict transforms of the four conics passing through all sets of five points that is complementary to one of the real points. It is straightforward to check that there are no six mutually skew lines among these 15 real lines.

Every smooth cubic surface that arises as the epipolar variety of five point pairs is the blow-up of the real projective plane $\mathbb{P}^2_R$ in six real points in general position. (The other types are obtained by blowing up $\mathbb{P}^2_R$ in six complex points that are invariant under complex conjugation with different numbers of fixed points, namely 0, 2, and 4. The last remaining type is not isomorphic to a blow-up of $\mathbb{P}^2_R$ over the reals.) This is curious because we start with five point pairs that specify 10 lines on the epipolar variety, but these 10 lines determine two other lines via the unique Schlafli double six containing the 10. This prescribes one more point in every image, as we will see below. In [22], Werner shows one way to construct this sixth point in each image using the five point pairs. We discuss two other ways to explain the sixth point (see Example 7.4 and Example 7.5).

Example 7.4. The four-dimensional linear space $\text{Span}\{v_1 u_1^\top, v_2 u_2^\top, \ldots, v_5 u_5^\top\}$ intersects the four-dimensional, degree six variety $R_1$ in six points. The sixth point is $v_0 u_0^\top$ where $v_0 = (-3, -12, 5)$ and $u_0 = (18, 11, 17)^\top$.

The Schlafli double six specified by the lines $W_i$ and $W_j$ for $0 \leq i, j \leq 5$ accounts for twelve of the 27 lines on the epipolar variety. For $i \neq j$, $W_i$ and $W_j$ span a tritangent plane, $\pi_i^j$. The tritangent plane intersects the epipolar variety in $W_i u_i$, $W_j u_j$, and one additional

$$W_i := \{X \in R_2 \cap L_P : X u = 0 \text{ and } v^\top X = 0 \text{ for some } u \in \text{Span}\{u_i, u_j\}, v \in \text{Span}\{v_i, v_j\}\}.$$ 

The plane $\pi_i^j$ coincides with $\pi_i^j$, hence the line $W_i^j$ coincides with $W_i^j$ for all $i, j$. The distinct lines $W_i^j$ for $0 \leq i < j \leq 5$ account for the remaining 15 real lines on the epipolar variety.

7.3. The determinantal representations of a cubic surface. From the classical point of view, we can start with six real points in the real projective plane $\mathbb{P}^2_R$ and obtain a cubic surface of the correct topological type. Each of these surfaces is the epipolar variety for five generic points pairs but every cubic surface is compatible with many second images. To determine the second image, we go via a determinantal representation; different choices of a determinantal representation result in different second images (even modulo projective transformations).

Given six real points $u_0, \ldots, u_5$ in $\mathbb{P}^2_R$, we get a cubic surface $S \subset \mathbb{P}^3_R$ that is the blow-up of $\mathbb{P}^2_R$ in these six points by considering the rational map $\mathbb{P}^2_R \dashrightarrow \mathbb{P}^3_R$ given by the four-dimensional space of cubics that vanish at the six points. Fixing a basis $c_0, c_1, c_2, c_3$ of this space, the map is given by $x \mapsto (c_0(x) : c_1(x) : c_2(x) : c_3(x))$ and is defined on $\mathbb{P}^2_R \setminus \{u_0, \ldots, u_5\}$. The closure of the image of this map is the cubic surface $S$ in $\mathbb{P}^3_R$ (which is determined up to change of coordinates on $\mathbb{P}^2_R$ by the six points in $\mathbb{P}^2_R$).

We cannot determine the second image from this surface alone. However, this information is specified by a determinantal representation of the surface. Determinantal representations are closely related to the Hilbert-Burch matrix as is explained in [4] Section 3 (see also [9]). The vanishing ideal $I$ of six points in $\mathbb{P}^2_R$ in general linear position is generated by the four-dimensional linear space of cubics vanishing on it. Its minimal free resolution looks like

$$0 \to F \to G \to I \to 0$$

where $F$ is a free, graded $\mathbb{R}[x_0, x_1, x_2]$-module of rank three and the map from $F$ to $G$ is given by a $4 \times 3$ Hilbert-Burch matrix $L(x_0, x_1, x_2)^\top$ with entries linear in $x_0, x_1, x_2$. Every such matrix gives a determinantal representation of $S$ via a simple computation: Determine the $3 \times 3$ matrix $M$ with entries in $\mathbb{R}[z_0, z_1, z_2, z_3]$ such that

$$L(x_0, x_1, x_2)^\top \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = M(z_0, z_1, z_2, z_3)^\top \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

and then $S = \{(z_0, z_1, z_2, z_3) \in \mathbb{P}^3_R \mid \det(M(z)) = 0\}$. Since $M(z)^\top$ is also a determinantal representation of $S$, we get another Hilbert-Burch matrix $L'(x_0, x_1, x_2)^\top$ with linear entries such that

$$L'(x_0, x_1, x_2)^\top \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = [M(z_0, z_1, z_2, z_3)]^\top \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. $$
Therefore a determinantal representation determines six real points \( u_i \) in the first image cut out by the \( 3 \times 3 \) minors of \( L \) and six real points \( v_i \) in the second image cut out by the \( 3 \times 3 \) minors of \( L' \).

**Example 7.5.** The Hilbert Burch matrix of \( M(z) \) is

\[
L = \begin{pmatrix}
-x_0 & -x_0 & x_0 - x_1 - x_2 & 4x_0 - x_1 + x_2 \\
-x_0 + x_1 + 2x_2 & x_0 - x_1 + 2x_2 & 3x_1 + x_2 & -x_2 \\
x_0 & 3x_0 & x_0 + x_1 + 5x_2 & 2x_0 + 5x_1 - 5x_2
\end{pmatrix}
\]

while the Hilbert-Burch matrix of \( M(z)^\top \) is

\[
L' = \begin{pmatrix}
-x_0 + x_1 + 2x_2 & -x_0 - x_1 + 3x_2 & x_0 + x_2 & 4x_0 + 2x_2 \\
x_1 & -x_0 - x_1 + 3x_2 & x_0 + x_2 & 4x_0 + 2x_2 \\
-x_1 & 2x_1 & x_0 - x_1 + 5x_2 & 2x_0 + 5x_1 - 5x_2
\end{pmatrix}
\]

We can compute the six real points in each image from the Hilbert-Burch matrices, because the \( 3 \times 3 \) minors vanish exactly at these points. The zeros of these ideals of minors are \( u_0, \ldots, u_5 \) where \( u_0 = (18, 11, 17)^\top \) and \( v_0, \ldots, v_5 \) where \( v_0 = (-3, -12, 5)^\top \), respectively. This gives yet another way to compute \( u_0 \) and \( v_0 \).

7.4. **From a determinantal representation to epipoles.** Given a determinantal representation \( M(z) \) of a cubic surface \( S \subset \mathbb{P}^3_\mathbb{R} \), consider the map \( S \rightarrow \mathbb{P}^2_\mathbb{R} \times \mathbb{P}^2_\mathbb{R} \) \( z \mapsto \text{adj}(M(z)) \). Since the matrix \( M(z) \) has rank two for every point \( z \in S \), the image lies inside \( \mathbb{P}^2_\mathbb{R} \times \mathbb{P}^2_\mathbb{R} \) embedded via the standard Segre embedding in \( \mathbb{P}^5_\mathbb{R} \). With its given determinantal representation, the image of the cubic surface \( S \) under the adjoint map is the space of epipoles in two images consistent with \( P \).

From our study of the chiral epipolar region in Section 6, we are interested in the lines \( W_u \) and \( W^\vee \) on the cubic surface corresponding to \( P \). We compute the images of these lines under the adjoint map, connecting our work with Werner’s results in epipole space [22]. The image of \( W_u \) under the adjoint map is \( \{ u_i \} \times C^1 \), where \( C^1 \subset \mathbb{P}^2_\mathbb{R} \) is a curve of degree two, namely the conic passing through \( v_j \) for \( i \neq j \). Indeed, for every matrix \( X \in W_u \), the vector \( u_i \) is in the right kernel, by definition, whereas the left kernel varies. This left kernel varies along a conic in \( \mathbb{P}^2_\mathbb{R} \) because every entry of the adjoint matrix of a \( 3 \times 3 \) matrix is a quadric. So intersecting the image of \( W_u \) under the adjoint map with a line in \( \mathbb{P}^2_\mathbb{R} \) translates, by pulling back via the adjoint map, to the intersection of a quadric hypersurface in \( \mathbb{P}^4_\mathbb{R} \) with the line \( W_u \), which therefore consists of two intersection points. Symmetrically, the image of \( W^\vee \) is \( C_j \times \{ v_j \} \), where \( C_j \subset \mathbb{P}^2_\mathbb{R} \) is the conic passing through \( u_j \) for \( i \neq j \). This behaviour is special and occurs only for the walls in the cubic surface. The image of the other lines \( W_i^j \) under the adjoint is contained in \( \text{Span}\{u_i, u_j\} \times \text{Span}\{v_i, v_j\} \).

The conics \( C_i \) in the first image and \( C^j \) in the second image are described in Section 4 of [22]. These conics divide each image into cells of possible epipoles with piecewise conic boundaries. Each cell is uniquely characterized by the subset of conics that participate to form its boundary or equivalently by the subset of points that belong to its boundary. Werner’s test for the existence of a chiral reconstruction involves looking for “allowed segments” of the conics \( C_i \) and \( C^j \). In our work, we translate this question to studying the intersections of lines, i.e., the \( \{ u_i, v_j \} \) corners in the cubic surface in the preimage of the adjoint. In the following example, we show how our chirality test on corners relates to Werner’s allowed segments in the space of epipoles.

**Example 6.6 continued.** Consider the five point pairs from [16]:

\[
U = \begin{pmatrix}
0 & 0 & 4 & 2 & 2 \\
0 & 4 & 0 & 1 & 3 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

and

\[
V = \begin{pmatrix}
2 & 2 & 4 & 0 & 4 \\
1 & 3 & 0 & 4 & 4 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

We find that the same sign condition of Corollary 6.5 holds at the following corners: \( (u_2, v_3) \), \( (u_2, v_4) \), \( (u_3, v_1) \), \( (u_4, v_1) \), and \( (u_4, v_3) \). These corners are in the boundary of the chiral epipolar region of \( P \) which lives in the cubic surface \( \mathcal{R}_2 \cap \mathcal{L}_P \). Blowing down with the adjoint map, we get the non-empty region of epipoles in \( \mathbb{P}^2_\mathbb{R} \times \mathbb{P}^2_\mathbb{R} \) that correspond to a chiral reconstruction. The boundary of the region in the first image is defined by \( C_3 \) and \( C_4 \) at \( u_2 \), \( C_1 \) and \( C_2 \) at \( v_3 \), and \( C_1 \) and \( C_3 \) at \( u_4 \). The boundary of the region in the second image is defined by \( C^3 \) and \( C^4 \) at \( v_1 \), \( C^2 \) and \( C^4 \) at \( v_3 \), and \( C^2 \) and \( C^3 \) at \( v_4 \). See Figure 4.

We remark that our test for chirality requires more than just the cubic surface \( S \) determined by six real points \( u_i \) in one image \( \mathbb{P}^2_\mathbb{R} \). For different determinantal representations of \( S \), the six points in the other image may differ. In order to check the chiral epipolar inequalities, it is necessary to fix the second image, for example, by fixing a determinantal representation of \( S \).
It is also necessary to specify five out of the six point pairs to test for chirality. As the next example shows, while some sets of five point pairs may have a chiral reconstruction, another set associated with the same determinantal representation may not. These observations point out the intricate dependence of chirality on arithmetic information.

Example (6.6 continued). Consider the five point pairs from (15):

\[ U = \begin{pmatrix} 0 & 0 & 4 & 2 & 2 \\ 0 & 4 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 2 & 2 & 4 & 0 & 1 \\ 1 & 3 & 0 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \]

We compute the sixth associated point pair \( u_0 = (504/281, 300/281, 1)^\top \) and \( v_0 = (68/97, 300/97, 1)^\top \). Note that the need for all point pairs to have last coordinate one fixes the representatives of \( u_0 \) and \( v_0 \). By computing the relevant products \( D_{lm}, D_{ln}, D_{mn} \) at the 20 corners, we find that \( \{ (u_i, v_i) \}_{i=1}^5 \) does not have a chiral reconstruction. However, suppose we consider a different subset of five point pairs

\[ \hat{P}_5 = \{ (u_i, v_i) : i \in \{0, 1, 2, 3, 4\} \}. \]

Some corners now satisfy the same sign condition of Corollary 6.5 hence \( \hat{P}_5 \) has a chiral reconstruction. In fact, the subset of five point pairs \( \hat{P}_i \) which omits index \( i \) has a chiral reconstruction when \( i = 1, 2, 3, 5 \) and has no chiral reconstruction when \( i = 0, 4 \). The cubic surface \( R_2 \cap L_{\hat{P}_i} \) is the same for all \( i \) and the determinantal representations are all equivalent, but whether a chiral reconstruction exists or not depends on which five point pairs we choose.

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