Scalar field fluctuations in Pre–Big–Bang Cosmologies

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Abstract

Jordan–Brans–Dicke theories with a linearized potential for the scalar field are investigated in the framework of the stochastic approach. The fluctuations of this field are examined and their backreaction on the classical background is described. We compute the mode functions and analyze the time evolution of the variance of the stochastic ensemble corresponding to the full quantum scalar field in the pre–big–bang regime. We compute fluctuations of the term discriminating between the two branches of solutions present in the theory. We find, both analytically and upon direct integration of the stochastic equations of motion, that the dispersion of these fluctuations grows to achieve the magnitude of the term separating the two classical solutions. This means that the ensembles representing classical solutions which belong to different branches do overlap; this may provide a quantum mechanical realization at the level of field theory to change among solutions belonging to different branches.

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1 Introduction

In general scalar tensor theories of gravity, such as Jordan–Brans–Dicke (JBD) theories, the gravitational coupling constant is replaced by a dynamical field \[ \omega \]. The theories are parameterized by a dimensionless kinetic coupling parameter \( \omega \). In the limit \( \omega \to \infty \) these theories go over to Einstein gravity. From time–delay experiments it is known that \( \omega \) must be larger than \( \sim 500 \) at the present epoch \([2]\). It is conceivable that in the early universe the actual value of \( \omega \) differed significantly from that bound in being itself a function of the Planck mass, i. e. of the JBD field \( \Phi = M_p^2 = G^{-1} \)(see e. g. \([3]\)). Being part of the dynamical evolution in the early stages \( \Phi \) is assumed to finally acquire a finite value which is consistent with observations of \( G \) and \( \omega \) today.

Theories of gravity arising from Kaluza–Klein theories or supergravity may appear as a JBD theory after compactification to four dimensions with \( \omega \) of the order of unity \([4]\) with the square root of the metric of the compactified dimensions acting as a JBD field on the uncompactified space–time manifold. On the other hand, the universal part of the low energy effective string action \([5]\) can be viewed as a JBD theory with the specific choice \( \omega = -1 \) after a field redefinition \( \Phi = e^{-\phi} \) and \( \phi \) being the dilaton of string theory. In this context a (nonperturbatively) generated potential for the corresponding scalar field is taken into account which can effectively act as a cosmological constant and drive a period of inflation. This has been extensively studied in the literature \([5]\). Recently the possibility of having a kinetic type of inflation (in contrast to that driven by a potential term of the JBD field or an inflaton) has been an active field of investigation \([3, 7]\).

Especially much work has been devoted to the special case of the string low energy effective action (\( \omega = -1 \)). It has been observed that there are two different cosmological solutions (one defined for \( t > t_0 \) and the other for \( t < t_0 \)), related by scale factor duality (SFD) and separated from each other by a curvature singularity, which offer a quite attractive pre–big–bang scenario \([8]\). Due to the negative kinetic coupling one of the pre–big–bang solutions undergoes a superinflationary period totally driven by the kinetic energy term. This stage is supposed to be followed after the big bang by an ordinary radiation dominated expanding Robertson–Walker universe.

On tree–level, the curvature singularity prevents a smooth transition from the inflationary phase to the radiation dominated universe (graceful exit problem) \([9]\). Later it was proven \([10]\) that a dilaton potential cannot trigger such an exit. However, the singularity is thought to be a property of the low energy effective action which would be absent if higher order corrections were taken into account thus intertwining these two phases \([11]\).

As an alternative approach quantum cosmological techniques have been applied to the scenario in the regime where the tree–level effective action is still considered to be a good approximation \([12]\).

SFD is part of a larger \( O(d, d) \)–symmetry \([13]\) present in low energy action and reduces to SFD in the case of a homogeneous and isotropic universe. A potential for the dilaton as well as a curvature term break SFD. In many situations one can have an approximate symmetry since a spatial curvature term can safely be ignored at very early times and the influence of the dilaton potential is suppressed by the kinetic term as in the case of the
superinflationary branch close to the curvature singularity. General scalar tensor theories such as the JBD theory don’t possess this kind of $O(d,d)$-symmetry which is clearly of stringy origin. Nevertheless solutions can be found which form two disconnected branches separated by a field dependent discriminant; they are (apart from singularities) defined for all times such that one always finds similar pre–big–bang scenarios as those described above for the stringy case, but one also faces similar problems. In JBD theories, i. e. those with arbitrary but constant $\omega$, such periods of kinetic inflation are present only if the parameter $\omega$ is negative [3]. These models will be the most interesting ones for our purpose [14].

Quantum fluctuations in JBD theory have been studied on inflationary backgrounds, i. e. those fulfilling the slow–roll conditions, in [15], where also density perturbations have been computed; similar analyses have been performed on more general Robertson–Walker backgrounds, but without potential, or alternatively for an exponential potential in the context of extended inflation in the Einstein frame [16]. For the string cosmological scenario ($\omega = -1$) scalar and tensor perturbations have been computed [17] and the exit problem via quantum backreaction has been investigated in [18]. We point out that a full analysis of the backreaction problem in JBD theory on general cosmological backgrounds including a generic potential (and also a matter contribution to the energy density) is still lacking. One step into that direction shall be the subject of this paper. For the problem to be tractable analytically one could e. g. expand the solutions for small values of the potential or for small times where the influence of the potential can be linearized.

The intention of this paper is to study the dynamics of quantum fluctuations of the JBD field and its backreaction on the background fields by means of a stochastic analysis. For this purpose we split the field and its first time derivative (which will be treated as an independent stochastic variable) into sub– and super–horizon sized modes. This technique has been applied to similar problems before [19]. In this framework it is possible to trace the time evolution of the long wavelength (coarse grained) fields which we interpret as the classical background under the influence of the stochastic noise operators. It is possible to write down a system of stochastic differential equations (Langevin equations) and the corresponding Fokker–Planck equation for the probability distribution of the stochastic variables, i. e. the classical background field and its first time derivative. The Langevin equations can be integrated numerically by a well known method [20, 21]. For the Fokker–Planck equation one can in principle search for an analytic solution.

The stochastic method is suitable to investigate the effects of quantum effects on the background quantities as long as the size of the fluctuations are under control. It is a well defined technique and can serve as an alternative to the minisuperspace approach of examining the exit problem of string cosmologies. Since the stochastic differential equations can be integrated for any form of the potential one can hope to make statements about the influence of the scalar field potential in this context. However, we argue that a linearized potential (which shows up as a cosmological term in the action) will be sufficient as a first approximation. For the linear potential we will also investigate the mode functions
on that background and their dependence on the potential and the parameter \( \omega \) since they will finally enter the stochastic differential equations. By comparing the relative size of the mode functions and the zero mode it is also possible to check the validity of the procedure.

With the above tools we are able to study the backreaction of the fluctuations on the dynamical level. We check up on the rôle of a large potential and examine how it decouples from the dynamics in the limit when it becomes small. We compute the variance of the scalar field fluctuations and its backreaction on the Hubble parameter which we will finally compare with its discriminant separating the two different background solutions. If both are approximately equal, it is no longer possible to unambiguously talk about two distinct branches.

In section 2 we give the basic equations of JBD cosmology and fix our conventions. The setup of the stochastic treatment of the scalar field fluctuations are presented in section 3. We discuss the general background solutions for a linear potential in JBD cosmology and give expressions for the mode functions on that background in section 4 together with numerical solutions for variable \( \omega \) and strengths of the potential; we analyze the exit problem of the superinflationary branch in the presence of quantum fluctuations of a scalar field. The results are summarized in section 5.

2 Basics of Jordan–Brans–Dicke cosmology

We start with the following action of the JBD theory,

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \Phi \mathcal{R} - \omega \frac{\partial_\mu \Phi \partial^\mu \Phi}{\Phi} - V(\Phi) + 16\pi \mathcal{L}_m \right],
\]

including matter and a finite potential energy term for the JBD scalar field \( \Phi \). \( \mathcal{R} \) is the Ricci scalar of the metric. In the following we keep \( \omega \) as an arbitrary parameter of the theory assuming that it is independent of the field \( \Phi \). Since the physics is more obvious in the JBD frame, we will not change into the conformally related Einstein frame.

Varying the action (1) with respect to \( g^{\mu\nu} \) and \( \Phi \) yields the following two equations of motion

\[
R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \frac{8\pi}{\Phi} T_{\mu\nu} + \frac{\omega}{\Phi^2} (\partial_\mu \Phi)(\partial_\nu \Phi) + \frac{1}{\Phi} \nabla_\mu \nabla_\nu \Phi
\]

\[
- g_{\mu\nu} \left[ \frac{\Box \Phi}{\Phi} + \frac{\omega}{2\Phi^2} (\partial_\alpha \Phi)(\partial^\alpha \Phi) + \frac{V(\Phi)}{\Phi^2} \right],
\]

\[
\mathcal{R} = \omega \frac{(\partial_\mu \Phi)(\partial^\mu \Phi)}{\Phi^2} - 2\omega \frac{\Box \Phi}{\Phi} + \frac{\partial V(\Phi)}{\partial \Phi},
\]

where \( T_{\mu\nu} \) is the energy–momentum tensor of the matter sector. Combining the trace of equation (2) with equation (3) one arrives at a Klein–Gordon equation for the JBD
scalar \( \Phi \),
\[
\Box \Phi = \frac{8\pi}{2\omega + 3} T_m - \frac{2}{2\omega + 3} V(\Phi) + \frac{1}{2\omega + 3} \frac{\partial V(\Phi)}{\partial \Phi} \Phi , \tag{4}
\]
with \( T_m \equiv T_m^{\mu \mu} \) being the trace of the energy–momentum tensor. For the homogeneous and isotropic metric we use an ansatz of the Robertson–Walker type defined by
\[
g_{\mu \nu} dx^\mu dx^\nu = dt^2 - a^2(t) d\Omega^2(k) . \tag{5}
\]
Here \( a(t) \) is the scale factor of the universe and \( d\Omega^2(k) \) stands for the volume element of the 3–dimensional spacelike hypersurface of constant \( t \) and \( k = \pm 1,0 \). In this metric an ideal fluid is described in its rest frame by \( T_m = \rho - 3p \) (with \( \rho \) and \( p \) denoting the matter energy density and pressure respectively). The equations (2) and (3) in the metric (5) consist of only two linearly independent equations, the third one serving as a constraint. We choose the 0–0 component of (2) in the metric (5) which reads
\[
H^2 = \frac{8\pi \rho}{3\Phi} + \frac{\omega \dot{\Phi}^2}{6 \Phi^2} - H \frac{\dot{\Phi}}{\Phi} - \frac{V(\Phi)}{6 \Phi} + \frac{\omega (\nabla \Phi)^2}{6 a^2 \Phi^2} + \frac{1}{3 a^2 \Phi} - \frac{k}{a^2} , \tag{6}
\]
where we have used the definition of the Hubble parameter, \( H \equiv \dot{a}/a \) and an overdot denoting a time derivative. For later purpose it will be useful to write (6) as an explicit equation for the Hubble parameter,
\[
H = \frac{1}{2} \frac{\dot{\Phi}}{\Phi} \pm \sqrt{\frac{8\pi \rho}{3\Phi} + \frac{2\omega + 3}{12} \frac{\dot{\Phi}^2}{\Phi^2} - \frac{V(\Phi)}{6 \Phi} + \frac{\omega (\nabla \Phi)^2}{6 a^2 \Phi^2} + \frac{1}{3 a^2 \Phi} - \frac{k}{a^2} , \tag{7}
\]
which shows that there are two different background solutions separated by a discriminant given by the square root in (7). By setting \( \omega = -1 \) for a homogeneous and free field in a universe described by \( \rho = k = 0 \) it becomes apparent that the + sign (– sign) in (7) corresponds to the (–) branch ((+) branch) of reference [9] defined only for \( t > 0 \) (\( t < 0 \)). Therefore, the discriminant distinguishes between a superinflationary and a collapsing phase of the (+) branch for negative times and between an ordinary radiation dominated expanding universe and a collapsing one of the (–) branch for positive times. At each instant of time the separation originating from the square root of (7) is, however, equal to the separation between the two different branches if they were defined for all times. In subsection 4.3 we shall compare the discriminant with the fluctuations in the JBD field.

Equivalently, the Klein–Gordon equation (4) in the Robertson–Walker metric (5) is given by
\[
\ddot{\Phi} + 3H \dot{\Phi} - \frac{\Delta \Phi}{a^2} - \frac{8\pi}{2\omega + 3} T_m + \frac{2}{2\omega + 3} V(\Phi) - \frac{1}{2\omega + 3} \frac{\partial V(\Phi)}{\partial \Phi} \Phi = 0 . \tag{8}
\]
The system of equations (4)–(8) contains also spatial inhomogeneities of the scalar field \( \Phi(x,t) \) which we assume to be small but non negligible and living in a homogeneous universe described by the scale factor \( a(t) \). Since the purpose of this paper is to study the behaviour of inhomogeneous modes of \( \Phi \) we separate them from the homogeneous
background, i.e. the zero mode of $\Phi$. We decompose the scalar field $\Phi$ into a homogeneous background and an inhomogeneous part according to

$$\Phi(x, t) = \Phi_0(t) + \delta(x, t), \quad \text{(9)}$$

and expand the latter within a box of volume $v = l^3$ into its modes as follows:

$$\delta(x, t) = l^{-3/2} \sum_k \left[ a_k \varphi_k(t) e^{i k \cdot x} + a_k^* \varphi_k^*(t) e^{-i k \cdot x} \right]. \quad \text{(10)}$$

The sum is performed over all wave vectors $k = \frac{2\pi n}{l} \neq 0$ with $n \in \mathbb{Z}^3$. The equation of motion for the homogeneous zero mode $\Phi_0(t)$ is clearly given by (8) with the spatial derivative dropped. On the other hand, substituting the ansatz (9) into the Klein–Gordon equation (8), expanding to linear order in $\delta(x, t)$ and using the equations of motion for the background $\Phi_0(t)$ we immediately arrive at the equation of motion for the mode functions $\varphi_k(t)$,

$$\ddot{\varphi}_k + 3H \dot{\varphi}_k + \left[ \frac{|k|^2}{a^2} + \frac{1}{2\omega + 3} \left( V'\Phi_0 - V''\Phi_0 \Phi_0 \right) \right] \varphi_k = 0, \quad \text{(11)}$$

with $V'(\Phi_0)$ denoting $\frac{\partial V(\Phi)}{\partial \Phi}|_{\Phi = \Phi_0}$, etc. This equation is independent of the curvature term $k/a^2$ which means that the topology of the spatial section enters only via the background quantities.

For the rest of the paper we will neglect the spatial curvature term ($k = 0$) as well as the contribution of ordinary matter to the energy density ($\rho = p = 0$). For completeness we give the background solutions in the case where in addition the potential is neglected. The equations of motion (6) and (8) allow the following power–law solutions which are defined for all times except $t = 0$:

$$\Phi_0(t) \sim |t|^{r(\omega)} \quad \text{and} \quad a(t) \sim |t|^{q(\omega)}, \quad \text{(12)}$$

with the exponents given by

$$r(\omega) = \frac{1 \pm \sqrt{9 + 6\omega}}{4 + 3\omega} \quad \text{and} \quad q(\omega) = \frac{1 - r(\omega)}{3}. \quad \text{(13)}$$

After the field redefinition $\Phi = e^{-\phi}$ we recover the known result for the low energy string action ($\omega = -1$) \cite{9}

$$\phi(t) \sim (\mp \sqrt{3} - 1) \ln |t| \quad \text{and} \quad a(t) \sim |t|^{\mp \frac{1}{\sqrt{3}}}, \quad \text{(14)}$$

in which case it is invariant under SFD. Comparing (13) with (14) it is clear that the + sign in front of the square root in (13) corresponds to the accelerated expansion in pre–big–bang cosmology, i.e. the superinflationary part of the (+) branch. For $t > 0$ the part of the (−) branch which corresponds to the − sign in (13) can be smoothly connected.

\footnote{The singularity can certainly also be at $t = t_0$. For simplicity we set $t_0 = 0$.}
to a FRW universe with decelerated expansion and can therefore account for our present universe. We note that there is also a constant background solution for the vanishing potential, i.e. \( \Phi_0 = H = 0 \). We will not be interested in that possibility, however, since it implies a static universe.

For the zero potential background the mode functions can be directly read off from equation (11) and given in terms of Hankel functions,

\[
\varphi_k(t) \propto |t|^{-\alpha(\omega)} C_\nu(\omega) \left( \beta(\omega) |t|^{\gamma(\omega)} \right),
\]

where for shorthand we introduced the quantities

\[
\alpha(\omega) = -\frac{r(\omega)}{2}, \quad \beta(\omega) = \frac{1}{1-q(\omega)} |k|, \quad \gamma(\omega) = 1-q(\omega), \quad \nu(\omega) = \pm \frac{1-3q(\omega)}{2}. \quad (16)
\]

By \( a_0 \) we denote any initial value for the scale factor which we can set to one for simplicity. \( C_\nu \) stands for any linear combination of the Hankel functions \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) with the two integration constants \( c_1(k) \) and \( c_2(k) \) multiplying the respective Hankel functions. Choosing a certain linear combination corresponds to making a specific choice for the vacuum state; it is subject to the Klein–Gordon normalization condition \( |c_2(k)|^2 - |c_1(k)|^2 = 1 \).

The correct Minkowski limit (positive frequencies) can be obtained by the simplest choice \( c_1 = 0 \) and \( c_2 = 1 \) for all wavenumbers which is the vacuum state we shall adopt in the following. Choosing the negative sign in the above expression for \( \nu(\omega) \) gives the non-trivial solution \( \Phi_0 \sim |t|^{r(\omega)} \) in the limit \( k \to 0 \), whereas the positive sign yields the constant solution.

3 Stochastic equations and correlation functions

In this section we will derive the modified equations of motion in the stochastic approach and the correlation functions between the various stochastic field operators. For this purpose we split the field and its first time derivative in sub- and super-horizon sized parts \cite{19, 21},

\[
\begin{align*}
\Phi(x, t) &= \Phi_<(x, t) + \Phi_>(x, t) \\
\dot{\Phi}(x, t) &= v_<(x, t) + v_>(x, t),
\end{align*}
\]

and decompose each of them into Fourier modes according to

\[
\begin{align*}
\Phi_>(x, t) &= l^{-3/2} \sum_k \theta[\pm (k - \varepsilon aH)] \left( a_k \varphi_k(t) e^{ik \cdot x} + a_k^\dagger \varphi_k^*(t) e^{-ik \cdot x} \right) \quad (18) \\
v_>(x, t) &= l^{-3/2} \sum_k \theta[\pm (k - \varepsilon aH)] \left( a_k \dot{\varphi}_k(t) e^{ik \cdot x} + a_k^\dagger \dot{\varphi}_k^*(t) e^{-ik \cdot x} \right). \quad (19)
\end{align*}
\]

The quantities \( \Phi_< \) and \( v_< \) are the long-wavelength (coarse-grained) fields and correspond to the lower sign in the \( \theta \) functions of equation (18) and (14), whereas \( \Phi_> \) and \( v_> \) are the short-wavelength fluctuations (referring to the upper sign). The parameter \( \varepsilon \) denotes the
splitting point and should be of the order of unity if we choose the horizon length as our effective physical cutoff. We define the stochastic noise operators

\[
\chi(x,t) = \varepsilon a(H^2 + \dot{H}) l^{-3/2} \sum_k \delta(k - \varepsilon aH) \left( a_k \Phi_k(t) e^{ikx} + a_k^\dagger \dot{\Phi}_k(t) e^{-ikx} \right)
\]

\[
\xi(x,t) = \varepsilon a(H^2 + \dot{H}) l^{-3/2} \sum_k \delta(k - \varepsilon aH) \left( a_k \dot{\Phi}_k(t) e^{ikx} + a_k^\dagger \Phi_k(t) e^{-ikx} \right).
\]

In the following we will aim at deriving a set of stochastic differential equations (Langevin equations) describing the system given by the action (I). We regard the short–wavelength quantities \( \Phi_\geq(x,t) \) and \( v_\geq(x,t) \) as small perturbations around the coarse–grained fields \( \Phi_\leq(x,t) \) and \( v_\leq(x,t) \) which we will finally interpret as semiclassical random variables whose evolution we follow under the influence of the stochastic noise terms (20). It is easy to see that

\[
\dot{\Phi}_\leq(x,t) = v_\leq(x,t) + \chi(x,t), \tag{21}
\]

where we have used the relation

\[
\frac{\partial}{\partial t} \theta[\mp(k - \varepsilon aH)] = \pm \varepsilon a(H^2 + \dot{H}) \delta(k - \varepsilon aH). \tag{22}
\]

In the same way we get \( \dot{\Phi}_\geq(x,t) = v_\geq(x,t) - \chi(x,t) \) and hence \( \dot{\Phi}(x,t) = v_\geq(x,t) + v_\leq(x,t) \). Equation (24) is the first stochastic differential equation since it contains only coarse–grained fields (our random variables) and the stochastic noise.

It is more difficult to obtain a Langevin equation for \( v_\leq(x,t) \). We write \( \dot{v}_\leq(x,t) = \dot{\Phi}(x,t) - \dot{\Phi}_\geq(x,t) \) and get

\[
\dot{v}_\leq(x,t) = -3H\dot{\Phi}(x,t) + \frac{\Delta\Phi(x,t)}{a^2} - \frac{1}{2\omega + 3} \left( 2V(\Phi) - V'(\Phi) \Phi(x,t) \right) + \xi(x,t) - l^{-3/2} \sum_k \theta(k - \varepsilon aH) \left( a_k \dot{\Phi}_k(t) e^{ikx} + a_k^\dagger \Phi_k(t) e^{-ikx} \right), \tag{23}
\]

where we used the equation of motion for the full (inhomogeneous) quantum field (8) and the definition of \( v_\geq(x,t) \), equation (19). The first line of equation (23) can be split into the coarse–grained and short wavelength fields according to (17). We will expand the potential and its derivative around the coarse–grained fields,

\[
2V(\Phi) - V'(\Phi) \Phi = 2V_0 - V_0' \Phi_\geq + \left[ V_0' - V_0'' \Phi_\leq \right] \Phi_\leq + \mathcal{O}(\Phi_\geq^2), \tag{24}
\]

where we have used the notation \( V_0 \equiv V(\Phi_0), \) \( V_0' \equiv \partial V(\Phi)/\partial \Phi|_{\Phi=\Phi_0}, \) etc. To simplify the expressions we will approximate \( \Phi_0 \approx \Phi_\leq \) and \( v_0 \approx v_\leq \) which means that effectively we neglect all spatial gradients of the coarse–grained field \( \Phi_\leq \). Later we will see that this is a very good approximation; it gets exact in the limit \( \varepsilon \to 0 \). This identification will cancel the terms linear in the small quantities \( \Phi_\geq \) and \( v_\geq \) in equation (23). In the second line of (23) the second time derivatives of the mode functions are eliminated via their equation.
of motion (11).
This together with the approximation mentioned above gives the following differential equation for the coarse–grained field \( v_\ll (x, t) \),

\[
\dot{v}_\ll(x, t) = -3Hv_\ll(x, t) - \frac{1}{2\omega + 3} \left( 2V_0 - V_0' \Phi_\ll(x, t) \right) + \xi(x, t) \tag{25}
\]
up to second order in the small quantities.

The equations (21) and (25) constitute a system of stochastic differential equations for the fields \( \Phi_\ll(x, t) \) and \( v_\ll(x, t) \) which we regard as independent random variables subject to the two independent stochastic forces \( \chi(x, t) \) and \( \xi(x, t) \). Effects of spatial variability of the fields enter only through the stochastic noise operators. For a full description of the system an equation similar to (7) should be added to the two Langevin equations; the total energy density we assume to be stored mainly in the background fields. Therefore, we neglect energy contributions arising from the small quantities \( \Phi_\gg(x, t) \) and \( v_\gg(x, t) \).

\[
H = -\frac{1}{2} \frac{v_\ll}{\Phi_\ll} \pm \sqrt{\frac{2\omega + 3}{12} \frac{v_\ll^2}{\Phi_\ll^2} - \frac{V_0}{6 \Phi_\ll}}. \tag{26}
\]

The set of equations (21), (25) and (26) can be integrated by modeling the stochastic noise terms by a random number generator. Statistical independence of the numbers requires a Gaussian distribution. Integrating Langevin equations of that type is a well known problem and one can use standard techniques [20].

The correlation functions between the stochastic noise operators can be calculated in a straightforward way,

\[
\langle 0 | \chi(x, t) \chi(x', t') | 0 \rangle = \frac{1}{2\pi^2} \epsilon^3 a^3 H^4 \left( 1 + \frac{\dot{H}}{H^2} \right) \frac{\sin(\epsilon aH|x - x'|)}{\epsilon aH|x - x'|} \times \left| \varphi_k(t) \right|^2 \bigg|_{k = \epsilon a|H|} \delta(t - t'), \tag{27}
\]
and similar expressions for the correlation functions \( \langle 0 | \xi(x, t) \xi(x', t') | 0 \rangle \) and \( \langle 0 | \chi(x, t) \chi(x', t') | 0 \rangle \) and \( \langle 0 | \xi(x, t) \chi(x', t') | 0 \rangle \), for which \( |\varphi_k(t)|^2 \) is replaced by \( |\dot{\varphi}_k(t)|^2 \), \( \varphi_k(t) \dot{\varphi}_k(t) \) and \( \dot{\varphi}_k(t) \varphi_k^*(t) \) respectively. In the same way one derives the only non trivial commutator for the noise operators,

\[
[\chi(x, t), \xi(x', t')] = \frac{1}{2\pi^2} \epsilon^3 a^3 H^4 \left( 1 + \frac{\dot{H}}{H^2} \right) \frac{\sin(\epsilon aH|x - x'|)}{\epsilon aH|x - x'|} \times
\]

\[
(\varphi_k(t) \dot{\varphi}_k^*(t) - \dot{\varphi}_k(t) \varphi_k^*(t)) \bigg|_{k = \epsilon a|H|} \delta(t - t'). \tag{28}
\]
4 Mode functions with a cosmological term

The aim of the following section is to analyze the mode functions of the JBD field on a background evolving under a linearized version of the potential $V(\Phi)$ in the action (1). This linear term can be interpreted as a nonvanishing cosmological constant, or as a mass term for the canonical mass dimension 1 field $\chi$ defined through $\chi^2 = \Phi$. We choose a linear potential first of all because it is tractable analytically. Moreover, since our main point of interest is the superinflationary branch before the singularity, it is not the minima of the potential, which are modeled by nonlinear terms, which control inflation. Actually, the influence of the potential on the dynamics is naively expected to be subdominant, because, for small values of $\omega$, the grander part of the energy is stored in the kinetic term of the scalar field. However, the strength of the potential can affect the rate of evolution which in turn can change the magnitude of quantum (represented here as stochastic) fluctuations. To study this effect in the system not too far away from the singularity the linearized version of the potential term is expected to serve sufficiently well. The mode functions enter e.g. in the correlation functions (27) and the equations for the variance of the fluctuations. Background solutions for a finite cosmological constant have been given for the first time in [22] and applied to pre–big–bang cosmologies in [14]. For completeness we give a short review of the derivation in the following subsection.

4.1 Background solutions for non zero cosmological constant

Using the ansatz $V(\Phi) = -2\lambda \Phi$ for the linearized potential the 0–0 component and the i–i component of the Einstein equation (2) and the Klein–Gordon equation (4) read respectively (spatial curvature and matter contributions are omitted)

\begin{align*}
3H^2 - \lambda &= \frac{\omega}{2} \Psi^2 - 3H \Psi \\
-2 \frac{\ddot{a}}{a} - H^2 + \lambda &= \frac{\omega}{2} \Psi^2 + \frac{\dot{\Phi}_0}{\Phi_0} + 2H \Psi \\
\frac{\dot{\Phi}_0}{\Phi_0} + 3H \Psi &= \frac{2\lambda}{2\omega + 3},
\end{align*}

where we have used the definitions $H = \dot{a}/a$ and $\Psi = \dot{\Phi}_0/\Phi_0$. Defining the following functions

\begin{align*}
f &= \Phi_0 a^3 \\
h &= \dot{\Phi}_0 a^3 \\
g &= \ddot{\Phi}_0 a^3
\end{align*}

and noting that $\ddot{a}/a = \dot{H} + H^2$ and $\ddot{\Phi}_0/\Phi_0 = \dot{\Psi} + \Psi^2$, we find that the linear combination of equations (29) – (30) – (31) reduces to the familiar form

\begin{align*}
\ddot{f} - b^2 f &= 0 \\
\text{where} \quad b^2 &= 2\lambda \frac{3\omega + 4}{2\omega + 3}.
\end{align*}
The general solution of this equation is \( f = A_+ e^{bt} + A_- e^{-bt} \) with constants of integration \( A_+ \) and \( A_- \) to be determined later. The l.h.s of equation (31) is equal to \( \dot{h}/f \) so that it can be written as

\[
\dot{h} - d^2 f = 0 \quad \text{where} \quad d^2 = \frac{2\lambda}{2\omega + 3},
\]

with the first integral \( h = C + d^2 \int f \, dt \) (\( C \) being an arbitrary constant). From (32) we see that \( \dot{f} = h + 3Hf \) which can be solved for \( H \) and since the functions \( f \) and \( h \) are already known it can be integrated to yield the scale factor \( a \),

\[
H = \frac{\dot{f} - h}{3f} \quad \text{and} \quad a = \tilde{a}_0 |f|^\frac{1}{\omega} \exp \left[ -\frac{1}{3} \int \frac{h}{f} \, dt \right].
\]

From (32) we can read off that \( \Psi = \dot{\Phi}/\Phi = h/f \) and by integration we get

\[
\Phi_0 = \tilde{\Phi}_0^{(0)} \exp \left[ \int \frac{h}{f} \, dt \right].
\]

In general the Hubble parameter \( H \) will have a singularity. Without loss of generality we can fix it to be at \( t = 0 \); this requires to choose \( A_+ = -A_- \equiv \pm A \) with \( A > 0 \). Then \( f = \pm 2A \sinh(bt) \) and the sign ambiguity determines the two branches of solutions. It turns out that this fixes the ratio \( C/A \) as well meaning that the solutions for \( H, a \) and \( \Phi_0 \) that we get by successive integration of the above equations are only fulfilled if \( C/A \) takes the value

\[
\frac{C}{A} = 2 \sqrt{\frac{6\lambda}{3\omega + 4}}.
\]

We fix the constants of integration \( \tilde{a}_0 \) and \( \tilde{\Phi}_0^{(0)} \) by making contact with the well known results for \( \omega = -1 \) in the limit \( \lambda \to 0 \), \( a = a_0 |t|^{1/\sqrt{3}} \) and \( \phi_0 = \phi_0^{(0)} - (1 + \sqrt{3}) \ln |t| \), \( \phi_0 \) being again the stringy dilaton, \( \phi_0 = -\ln \Phi_0 \).

The relevant background quantities parameterized by \( \omega \) and \( \lambda \) are

\[
H(t) = \frac{\sqrt{2\lambda} (\omega + 1)}{\sqrt{2\omega + 3} \sqrt{3\omega + 4}} \coth (bt) \mp \frac{\sqrt{2\lambda}}{\sqrt{3} \sqrt{3\omega + 4}} \sinh^{-1} (bt),
\]

\[
a(t) = a_0 2^{\mp \sqrt{6\omega + 9} / 3\omega + 12} |b|^{-\omega} \sinh(bt) \left| \tanh \left( \frac{bt}{2} \right) \right|^{\frac{1}{2} - \sqrt{6\omega + 9} / 3\omega + 12},
\]

\[
\Phi_0(t) = \Phi_0^{(0)} 2^{\mp \sqrt{6\omega + 9} / 3\omega + 12} |b|^{-\omega} \sinh(bt) \left| \tanh \left( \frac{bt}{2} \right) \right|^{\frac{1}{2} - \sqrt{6\omega + 9} / 3\omega + 12}.
\]
where we have used $q(\omega)$ and $r(\omega)$ as defined in (13). Choosing $\omega = -1$ these equations reduce to

$$H(t) = \mp \sqrt{\frac{2\lambda}{3}} \sinh^{-1}(\sqrt{2}\lambda t) \longrightarrow \mp \frac{1}{\sqrt{3}} \frac{1}{t} \quad (41)$$

$$a(t) = a_0 2^{\mp \frac{1}{\sqrt{3}}} |\sqrt{2}\lambda|^{\frac{1}{\sqrt{3}}} \left| \tanh \left( \frac{1}{2} \sqrt{2}\lambda t \right) \right|^{\mp \frac{1}{\sqrt{3}}} \longrightarrow a_0 |t|^{\mp \frac{1}{\sqrt{3}}} \quad (42)$$

$$\Phi_0(t) = \Phi_0^{(0)} 2^{\pm \sqrt{3}} |\sqrt{2}\lambda|^{-1\pm \sqrt{3}} \left| \sinh(\sqrt{2}\lambda t) \right| \left| \tanh \left( \frac{1}{2} \sqrt{2}\lambda t \right) \right|^{\pm \sqrt{3}} \quad (43)$$

$$\longrightarrow \Phi_0^{(0)} \left| t \right|^{1\pm \sqrt{3}},$$

where the expressions after the arrow are valid in the limit $\sqrt{\lambda} t \ll 1$. The prefactors in the equations (39) and (40) are essentially the constants $\tilde{a}_0$ and $\tilde{\Phi}_0^{(0)}$ and have been chosen such that in the special case $\omega = -1, \lambda = 0$ we get the simplest possible initial values. All the upper signs correspond collectively to the $(+)$ branch showing accelerated expansion for $t < 0$ and the lower signs to the $(-)$ branch with decelerated expansion for $t > 0$.

One comment is in order concerning the sign of the potential. Up to now it was assumed that $\lambda$ is positive; this is indeed not necessary: If $\lambda$ is negative the correct expressions are obtained by substituting $b \rightarrow |b|$ or $\lambda \rightarrow |\lambda|$ respectively and simultaneously replacing all hyperbolic functions by their trigonometric counterparts.

### 4.2 Mode functions for non zero cosmological constant

Being equipped with the full expressions for the background with linearized potential we are now in a position to investigate the influence of this potential on the mode functions determined by equation (11). It possible to examine the dependence on the JBD parameter $\omega$ equally well. It is clear that the equation of motion for the mode functions in the background (38) – (40) cannot be integrated analytically in full generality. However, it is possible to examine the system in the limit $\sqrt{\lambda} t \ll 1$. This can either be the small potential limit for unconstrained time parameter or an expansion around $t = 0$ valid for general $\lambda$. In the small $\lambda$ limit it can be studied how the potential effectively decouples from the solutions. Finally we will perform a numerical integration of the most general equation (11) and compare the results with the approximate formulae. We will later investigate the region of validity of the stochastic approach.

First we want to give the asymptotic expressions of the solutions to (11) in the limit $bt \gg 1$ for positive $\lambda$ (this is equivalent to $\sqrt{\lambda} t \gg 1$ for all values of $\omega$ we are interested in); in this limit the functions $H(t)$ and $a(t)$ in (38) and (39) can be simplified significantly. We find one exponentially growing and one decaying mode, $\varphi_k^{(1)} \sim e^{\Omega_1 t}$ and $\varphi_k^{(2)} \sim e^{\Omega_2 t}$.
with the constants $\Omega_1$ and $\Omega_2$ given by

$$\Omega_{1,2} = -\frac{1}{2} \left( \frac{3 \sqrt{2}\lambda (\omega + 1)}{\sqrt{2\omega + 3\sqrt{3}}(3\omega + 4)} \pm \sqrt{8\lambda + \frac{18\lambda (\omega + 1)^2}{(2\omega + 3)(3\omega + 4)}} \right).$$  \hfill (44)

The two constants $\Omega_1$ and $\Omega_2$ are the same for both branches. For $\omega = -1$ this reduces to $\Omega_{1,2} = \pm \sqrt{2\lambda}$; for that case we also find a correction,

$$\Omega_{1,2} = \pm \sqrt{2\lambda \left( 1 - \frac{|k|^2}{a_0^2} \frac{2^{-1+\frac{1}{\sqrt{3}}} \lambda^{-1+\frac{1}{\sqrt{3}}}}{2\omega + 3} \right)},$$  \hfill (45)

where the $\pm$ under the root distinguishes the (+) branch (upper signs) from the (−) branch (lower signs) and the $\pm$ in front of the root refers to the two different constants $\Omega_1$ and $\Omega_2$. We note that similar asymptotic solutions can not be given for negative $\lambda$ because the trigonometric functions are undetermined at infinity.

Next we can try to solve equation (11) for small arguments $bt \ll 1$ where the hyperbolic and trigonometric functions show the same behaviour. The approximate equation of motion for the mode functions now read

$$\ddot{\varphi}_k + \frac{3q(\omega)}{|t|} \dot{\varphi}_k + \left[ \frac{|k|^2}{a_0^2} |t|^{-2q(\omega)} - \frac{2\lambda}{2\omega + 3} \right] \varphi_k = 0,$$  \hfill (46)

and is valid for all $\omega$ not too close to the singular point $-3/2$; an overdot denotes a derivative with respect to $|t|$. The dependence on $\lambda$ and its sign enters only through the last term in the brackets. It is clear that the solutions only depend on the modulus of $t$. For $k = 0$ we obtain an expression which passes over to the zero mode in that limit and for $\lambda = 0$ we recover the free solution (15).

Unfortunately this equation is still too difficult to solve in general the trouble coming from the irrational powers of the first term in the bracket; it cannot even be solved as a power law series. However, for the most interesting case of $\omega = -1$ on the (+) branch the exponent can well be approximated by $-2q(+) (\omega = -1) = 2/\sqrt{3} \approx 1.15$. So one can look for a solution with the exponents 1 and 2 and compare the two cases. Later we will comment on the validity of this simplification. We want to solve for $\omega = -1$ the equation

$$\ddot{\varphi}_k - \frac{\sqrt{3}}{|t|} \dot{\varphi}_k + \left[ \frac{|k|^2}{a_0^2} |t|^m - 2\lambda \right] \varphi_k = 0,$$  \hfill (47)

with $m$ replacing 1 or 2. This equation can be solved via the power law series

$$\varphi_k(t) = |t|^r \sum_{n=0}^{\infty} c_n |t|^n.$$  \hfill (48)
The exponent $r$ and the coefficients $c_n$ can be computed in the standard way. For $m = 1$ we get the two linearly independent solutions

$$\varphi^{(1)}_k(t) = c_0 \left[ 1 - \frac{1 + \sqrt{3}}{2} \lambda |t|^2 - \frac{2 + \sqrt{3} |k|^2}{3 a_0^2} |t|^3 \right. - \frac{3 + 2 \sqrt{3}}{12} \lambda^2 |t|^4 - \frac{23 + 9 \sqrt{3} |k|^2 a_0^2}{390} \lambda |t|^5 - \left. \ldots \right] \quad (49)$$

$$\varphi^{(2)}_k(t) = c_0 |t|^{1 + \sqrt{3}} \left[ 1 + \frac{3 - \sqrt{3}}{6} \lambda |t|^2 - \frac{4 - \sqrt{3} |k|^2}{39} a_0^2 |t|^3 \right. + \frac{9 - 4 \sqrt{3}}{132} \lambda^2 |t|^4 - \frac{381 - 157 \sqrt{3} |k|^2 a_0^2}{12870} \lambda |t|^5 + \left. \ldots \right], \quad (50)$$

and similarly for $m = 2$,

$$\varphi^{(1)}_k(t) = \tilde{c}_0 \left[ 1 - \frac{1 + \sqrt{3}}{2} \lambda |t|^2 - \frac{3 + 2 \sqrt{3}}{24} \left( 2 \lambda^2 - (1 - \sqrt{3}) \frac{|k|^2}{a_0^2} \right) |t|^4 \right. - \frac{21 + 13 \sqrt{3}}{1584} \left( 2 \lambda^2 - (7 - 3 \sqrt{3}) \frac{|k|^2}{a_0^2} \right) \lambda |t|^6 - \left. \ldots \right] \quad (51)$$

$$\varphi^{(2)}_k(t) = \tilde{c}_0 |t|^{1 + \sqrt{3}} \left[ 1 + \frac{3 - \sqrt{3}}{6} \lambda |t|^2 + \frac{9 - 4 \sqrt{3}}{264} \left( 2 \lambda^2 - (3 + \sqrt{3}) \frac{|k|^2}{a_0^2} \right) |t|^4 \right. + \frac{75 - 37 \sqrt{3}}{36432} \left( 2 \lambda^2 - (13 + 3 \sqrt{3}) \frac{|k|^2}{a_0^2} \right) \lambda |t|^6 + \left. \ldots \right]. \quad (52)$$

These are two linearly independent solutions for each case with two different arbitrary constants $c_0$ and $\tilde{c}_0$ parameterising them. Differences between the solutions for $m = 1$ and $m = 2$ are of third order in $|t|$. It can be read off that the zero mode in the free case ($\lambda = 0$) is a constant or goes to zero as $\sim |t|^{1 + \sqrt{3}}$ in accordance with what we found in the previous discussion. Comparing with equation (43) the two yet undetermined constants are fixed to be $c_0 = \tilde{c}_0 = \Phi_0^{(0)}$. We are not interested in the constant solution because we want to avoid a static universe.

Recursively, the higher order coefficients can be computed according to the following relations:

$$c_n = \frac{2 \lambda c_{n-2} - \frac{|k|^2}{a_0^2} c_{n-3}}{r^2 + r (2n - 1 - \sqrt{3}) + n(n - 1 - \sqrt{3})} \quad \text{for} \quad m = 1 \quad (53)$$

$$c_n = \frac{2 \lambda c_{n-2} - \frac{|k|^2}{a_0^2} c_{n-4}}{r^2 + r (2n - 1 - \sqrt{3}) + n(n - 1 - \sqrt{3})} \quad \text{for} \quad m = 2.$$ 

In figure 1 the results of a numerical integration of equation (11) are shown for $\omega = -1$ and the linear potential discussed in this section. The qualitative picture is the same,
Figure 1: Mode functions for $\omega = -1$. In (a) they are shown for various wavenumbers $|k|$ as indicated in the figure for negative times. The potential was set to $\lambda = 1$. The zero mode corresponds to the bold line. In (b) the exact solution is compared with the approximate ones given in equation (50) and (52) for $\lambda = 1$ and $|k| = 10$. Accordance is achieved for small $t$.

however, for general $\omega$: Starting from a power law growth for the zero mode as $|t|$ is getting larger, this behaviour is converted into oscillations with increasing amplitude and frequency after a characteristic time for the mode functions with larger $k$. This initial growth is well approximated by the power law series solutions as can be seen in (b) where the full untruncated solutions (50) and (52) are shown. The mode functions are similar for all $k$ around the origin and $\varphi_k/\Phi_0 \rightarrow 1$ as one approaches zero. We point out that for the linear potential it is not necessary that the higher mode functions are much smaller than the zero mode, because in this case equation (11) is exact and there are no corrections $\propto |\varphi_k|^2$ etc.

Away from $\omega = -1$ we shall be contented with the qualitative picture given by the numerical integration shown in figure 4. Although $\omega$ is a free parameter of the theory we are especially interested in $\omega$ lying around $-1$. Higher dimensional theories in $D$ dimensions compactified to a maximally symmetric internal manifold yielding a $d + 1$ dimensional Kaluza–Klein cosmology can be viewed as a JBD theory described by a parameter $\omega = -1 + \frac{1}{D}$; the rôle of the JBD field is played by $\sqrt{g^{(D)}_{mn}}$ [3]. Going from $\omega = -1$ towards somewhat larger values accessible to the above formula it is apparent that the amplitude of the mode functions are significantly suppressed. They become
enormously large if $\omega$ drops sightly below $-1$.

### 4.3 Scalar field fluctuations

The presence of two different solutions, a generic feature of all JBD theories, have their origin in the sign ambiguity in front of the square root of equation (7). As long as only the classical theory is considered the root is a well defined discriminant separating the two branches. Fluctuations in the JBD field can, via backreactions, influence the homogeneous degrees of freedom such that their behaviour has to be described in terms of stochastic quantities as well (e.g. mean value, variance). As soon as the induced fluctuations in $H$ are comparable in size with the root distinguishing the two classical paths it is conceivable that the system undergoes a transition from one trajectory to the other and a strict distinction between the two trajectories present in the classical picture is no longer justified.

The subject of this section is to make this idea more precise. Let us begin with a vanishing potential. According to our choice of the vacuum the mode functions are given by $\varphi_k = |t|^{-\alpha} H_\nu^\beta(\beta |t|^{\nu})$ together with equation (16) in which we shall always choose the negative sign for $\nu$ in order to deal with the cosmologically interesting solutions only; in this case
The following relations will turn out to be useful:

\[
\frac{|\dot{\varphi}_k|^2}{|\varphi_k|^2} = \frac{r^2}{t^2} + \frac{|k|^2}{a_0^2} t^{2\gamma - 2} \left| \frac{H^{(2)}_{\nu-1}(\beta | t | \gamma)}{H^{(2)}_\nu(\beta | t | \gamma)} \right|^2 + 2r \frac{|k|}{a_0} |t|^{\gamma - 2} \text{Re} \left( \frac{H^{(2)}_{\nu-1}(\beta | t | \gamma)}{H^{(2)}_\nu(\beta | t | \gamma)} \right)
\]

\[
\frac{\varphi_k \dot{\varphi}_k^*}{|\varphi_k|^2} = \frac{r}{t} + \frac{|k|}{a_0} |t|^{\gamma} \frac{H^{(1)}_{\nu-1}(\beta | t | \gamma)}{H^{(1)}_\nu(\beta | t | \gamma)}
\]

\[
\frac{\varphi_k \dot{\varphi}_k}{|\varphi_k|^2} = \frac{r}{t} + \frac{|k|}{a_0} |t|^{\gamma} \frac{H^{(2)}_{\nu-1}(\beta | t | \gamma)}{H^{(2)}_\nu(\beta | t | \gamma)}.
\]  

As was demonstrated e. g. in [21] the variance in the fields can be given, due to stochastic differential equations, by the correlation functions of the corresponding noise operators. They are given by equation (27) where we shall adopt the specific choice \( \varepsilon = 1 \). For simplicity we neglect the spatial dependence of the correlation functions (i. e. we consider local fluctuations only) and define the variance of the stochastic field operators to be the following positive definite expressions

\[
\langle \delta \Phi^2 \rangle = \lim_{|x-x'| \to 0} \langle 0 | \chi(x, t) \chi(x', t') | 0 \rangle \bigg|_{t=t', \varepsilon=1}
\]

\[
\langle \delta v^2 \rangle = \lim_{|x-x'| \to 0} \langle 0 | \xi(x, t) \xi(x', t') | 0 \rangle \bigg|_{t=t', \varepsilon=1}
\]

\[
\langle \delta \Phi \delta v \rangle = \lim_{|x-x'| \to 0} \langle 0 | \frac{1}{2} [\chi(x, t) \xi(x', t') + \xi(x, t') \chi(x, t)] | 0 \rangle \bigg|_{t=t', \varepsilon=1}
\]

\[
\langle \delta v \delta \Phi \rangle = \lim_{|x-x'| \to 0} \langle 0 | \frac{1}{2} [\xi(x, t) \chi(x', t') + \chi(x, t') \xi(x, t)] | 0 \rangle \bigg|_{t=t', \varepsilon=1}
\]

where the cross terms have been defined in this way in order to avoid complex quantities.

Using equation (27) together with the given mode functions we finally get with the help of (54)

\[
\langle \delta \Phi^2 \rangle = \frac{1}{2\pi^2} \left| \frac{1-q}{q} \right| a^3 H^{4} \left( \frac{\Phi_0}{\Phi_0^{(0)}} \right) \left| H^{(2)}_\nu \left( \frac{q}{1-q} \right) \right|^2
\]

\[
\langle \delta v^2 \rangle = \langle \delta \Phi^2 \rangle H^2 \left| \frac{r^2}{q^2} + \frac{H^{(2)}_{\nu-1} \left( \frac{q}{1-q} \right)}{H^{(2)}_\nu \left( \frac{q}{1-q} \right)} \right|^2 + 2r \left| \frac{2r}{|q|} \text{Re} \left( \frac{H^{(2)}_{\nu-1} \left( \frac{q}{1-q} \right)}{H^{(2)}_\nu \left( \frac{q}{1-q} \right)} \right) \right|
\]

\[\] \[\]  

\[\text{In an earlier analysis [21] the semiclassical limit } \varepsilon \to 0 \text{ had to be chosen because it is only in this limit that the mode functions can be approximated by those of a massless free scalar field. Since we don’t use such an approximation here we need not perform the limit } \varepsilon \to 0.\]
\[ \langle \delta \Phi \delta v \rangle = \langle \delta \Phi^2 \rangle |H| \left| \frac{r}{|q|} + \text{Re} \left( \frac{H_{\nu-1}^{(2)} \left( \frac{q}{1-q} \right)}{H_{\nu}^{(2)} \left( \frac{q}{1-q} \right)} \right) \right| = \langle \delta v \delta \Phi \rangle . \]

For the last equation the commutator (28) was used. The explicit time dependence is reexpressed in terms of the background variables, equation (12). The above equations depend on an overall parameter \( a_0 \) which we shall take to be of the order of unity.

An especially interesting example is the pre–big–bang scenario of [9] which is modeled by \( \omega = -1 \). In the following we will consider the superinflationary (+) branch realized by \( q = -1/\sqrt{3} \) and negative \( t \). As can be seen from the first line of the above system together with the time dependence of the background quantities, the variance of \( \Phi \) grows for the (+) branch like \((-t)^{3/2}\) as one approaches the curvature singularity at \( t = 0 \). The additional factors of \( H \) and \( H^2 \) in the expressions for \( \langle \delta \Phi \delta v \rangle \) and \( \langle \delta v^2 \rangle \) respectively make those variances grow even faster, i.e. \( \langle \delta \Phi \delta v \rangle \propto (-t)^{-4} \) and \( \langle \delta v^2 \rangle \propto (-t)^{-5} \) because the Hubble parameter scales like \(-q/(-t)\). Hence, the growth of the fluctuations is not bounded from above and one can easily imagine them dominating over the classical evolution. In particular, if \( \delta H_{\text{rms}} \equiv \langle \delta H^2 \rangle^{1/2} \), the fluctuation of the Hubble parameter induced via backreactions, is of the same order of magnitude as the square root of equation (57) there is no need to assume that the field will stay on its classical path. At this point the field can be imagined to be part of either branch.

Next we will derive a quantitative estimate for the occurrence of this effect. The quantity which distinguishes between the two branches is the expression under the square root of equation (57) which we will denote as \( A \),

\[ A = \frac{2\omega + 3}{12} \frac{\Phi_0^2}{\Phi} - \frac{V(\Phi_0)}{6 \Phi_0} . \]  

(57)

Note that \( A \) is completely given by the homogeneous field; we are looking for its variation induced by substituting \( \Phi_0 \rightarrow \Phi_0 + \delta \Phi \) and \( v_0 \rightarrow v_0 + \delta v \). The variation of \( A \) to second order gives

\[
\delta A = \frac{1}{6 \Phi_0} \left( \frac{V(\Phi_0)}{\Phi_0} - V'(\Phi_0) - (2\omega + 3) \frac{v_0^2}{\Phi_0^2} \right) \delta \Phi + \frac{2\omega + 3}{6 \Phi_0} \frac{v_0}{\Phi_0} \delta v
\]
\[ + \frac{1}{6 \Phi_0} \left( - \frac{V'(\Phi_0)}{\Phi_0^2} + \frac{V''(\Phi_0)}{\Phi_0} - \frac{1}{2} \frac{V''(\Phi_0) + 3(2\omega + 3) \frac{v_0^2}{\Phi_0^2}}{2 \Phi_0^2} \right) \delta \Phi^2
\]
\[ + \frac{(2\omega + 3)}{6 \Phi_0} \frac{1}{2 \Phi_0} \delta v^2 - \frac{(2\omega + 3)}{6 \Phi_0} \frac{2v_0}{\Phi_0^2} \delta \Phi \delta v . \]

(58)

Since we neglected the spatial dependence of the correlation functions, the above variation is due to local fluctuations of \( \Phi \) and \( v \) and it is consistent to consider the expression (57) with the spatial derivatives dropped. The variance of the quantity \( A \) is

\[
\langle \delta A^2 \rangle = \frac{1}{36 \Phi_0^2} \left( \frac{V(\Phi_0)}{\Phi_0} - V'(\Phi_0) - (2\omega + 3) \frac{v_0^2}{\Phi_0^2} \right)^2 \langle \delta \Phi^2 \rangle + \frac{(2\omega + 3)^2}{36 \Phi_0^2} \frac{v_0^2}{\Phi_0^2} \langle \delta v^2 \rangle
\]

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\[ + \frac{(2\omega + 3)}{36\Phi_0^4} \frac{2v_0}{\Phi_0} \left( \frac{V(\Phi_0)}{\Phi_0} - V'(\Phi_0) - (2\omega + 3) \frac{\dot{v}_0^2}{\Phi_0^2} \right) \langle \delta \Phi \dot{\delta} \rangle. \] (59)

We shall identify \( \langle \delta \Phi^2 \rangle \), \( \langle \delta v^2 \rangle \) and \( \langle \delta \Phi \dot{\delta} \rangle \) with the expressions given in equation (56). As a condition for an overlap of the two distinct classical trajectories induced by stochastic effects we require \( \delta A_{\text{rms}} / A \sim 1 \). For general \( \omega \) and a vanishing potential we get

\[
\frac{\delta A_{\text{rms}}}{A} = 2 \sqrt{\frac{\langle \delta \Phi^2 \rangle}{\Phi_0^2} + \frac{\langle \delta v^2 \rangle}{v_0^2} - 2 \frac{\langle \delta \Phi \dot{\delta} \rangle}{\Phi_0 v_0}}. \] (60)

In the model considered so far \( \delta A_{\text{rms}} / A \sim \delta H_{\text{rms}} / H \) (apart from a trivial factor 2 coming from the square root) which means that right at the point when an overlap becomes significant a large fraction of the total energy is stored in fluctuations and spatial gradients may no longer be neglected. Although no assumption is necessary concerning the ratio \( \varphi_k / \Phi_0 \) for a vanishing or a linearized version of the potential the energy density due to the fluctuations should not be dominating over the contribution coming from the background. From this point of view the condition \( \delta A_{\text{rms}} / A \sim 1 \) will be met at the border of validity of this method.

Setting \( \omega = -1 \) we find a quantitative estimate by inserting the expressions derived in equation (56) into (60),

\[
\frac{\delta A_{\text{rms}}}{A} = \frac{1}{\pi} \frac{\sqrt{2}}{\sqrt{3 + \sqrt{3}}} \left( \Phi_0^{(0)} \right)^{-\frac{1}{2}} \left( H^2 a^2 \Phi_0^{-\frac{1}{2}} \right) \left| H_{\nu-1}^{(2)} \left( \frac{1}{1 + \sqrt{3}} \right) \right|\]

\[
= \frac{1}{3\pi} \frac{\sqrt{2}}{\sqrt{3 + \sqrt{3}}} \frac{a_0^2}{\Phi_0^{(0)}} \left| H_{\nu-1}^{(2)} \left( \frac{1}{1 + \sqrt{3}} \right) \right| (-t)^{-\left(\frac{5}{2} + \sqrt{3}\right)}, \] (61)

with the index of the Hankel function given by \( \nu = -\frac{1}{2} + \frac{\sqrt{3}}{2} \) and \( \left| H_{\nu-1}^{(2)} \left( \frac{1}{1 + \sqrt{3}} \right) \right| \approx 7.49 \). In the second line we have used the time dependence of the background fields for the pre–big–bang branch. The result depends on the initial values \( a_0 \) and \( \Phi_0^{(0)} \) which we assume to be of the order of one. If we had not made the choice \( \varepsilon = 1 \) at the beginning the result would also depend on \( \varepsilon^{3/2} \) which can always be absorbed by the initial values. No matter what the prefactors are, the strong time dependence with the power of \( -\frac{5}{2} - \sqrt{3} \approx -4.23 \) ensures that the condition \( \delta A_{\text{rms}} / A \sim 1 \) is unavoidably met at a time which can be read off from equation (61); this time is \( \sim 1 \) if the initial condition \( a_0^3 / \Phi_0^{(0)} \) is of the order of one.

The same as was demonstrated above for the stringy example \( \omega = -1 \) can very easily be generalized for arbitrary \( \omega \) (again restricting ourselves to a vanishing potential). Equation (61) is valid for general \( \omega \). The structure of \( \delta A_{\text{rms}} / A \) is very similar to that of (61),

\[
\frac{\delta A_{\text{rms}}}{A} = \frac{\sqrt{2}}{\pi} \left| \frac{1 - q(\omega)}{q(\omega)} \right| \left| H_{\nu}^{(2)} \left( \frac{q(\omega)}{1 - q(\omega)} \right) \right| F(\omega) \left( \Phi_0^{(0)} \right)^{-\frac{1}{2}} \left( H^2 a^2 \Phi_0^{-\frac{1}{2}} \right), \] (62)
with \( F(\omega) \) denoting a function of \( \omega \) which is approximately one in the region of interest. Using the general background solutions for zero potential (12) and (13) it is obvious that 
\[
\delta A_{\text{rms}}/A \sim (-t)^{2+\frac{1}{2n}} F(\omega) + \frac{1}{n} \rho(\omega)
\]
for the (+) branch. The exponent, being a function of \( \omega \), is always negative (as it should be) and a monotonically growing function increasing from \(-\frac{5}{2} - \sqrt{3} \) at \( \omega = -1 \) to \(-\frac{5}{2} \) at \( \omega = 0 \); this is again a sufficiently fast growing function for the value of \( \delta A_{\text{rms}}/A \) becoming one at a time which is of the order of unity. The \( \omega \)-dependent prefactor of equation (62) shows the remarkable feature that it has a singular point at \( \omega = 0 \) where it becomes plus infinity. This means that approaching \( \omega = 0 \) from either direction drastically enhances the ratio \( \delta A_{\text{rms}}/A \) such that the overlap condition is easily fulfilled.

In trying to explore the influence of a linearized potential on the ratio \( \delta A_{\text{rms}}/A \) one would have to refer to the approximate mode functions we found in (50) and (52) for \( \omega = -1 \) and insert them into the corresponding equation which is similar to (60). From the resulting formulae it is, however, hard to see how a small but non zero potential influences the effect described above. Therefore, we shall rather directly integrate the coupled stochastic equations of motion (21), (25) and (26) using the approximate mode functions valid for \( \sqrt{\lambda t} < 1 \). The discretized version of the equations can be written in the form

\[
\Phi_{n+1} = \Phi_n + v_n \delta t + \frac{1}{3\pi} \sqrt{1 + \sqrt{3}} \frac{a_0^{3}}{2} |\Phi_0^{(0)}| a_0^{3} \left( 1 + \frac{3 - \sqrt{3}}{6} \lambda |t_n|^2 \right) \times |t_n|^{-1+\frac{\sqrt{3}}{2}} \sqrt{\delta t} w_n^{(1)}
\]

\[
v_{n+1} = v_n - 3 H_n v_n \delta t + 2 \lambda \Phi_n \delta t + \frac{1}{3\pi} \frac{1 + \sqrt{3}}{2} |\Phi_0^{(0)}| a_0^{3} \times \left( 1 + \left( \frac{3 - \sqrt{3}}{6} + \frac{3 - \sqrt{3}}{3} \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \right) \lambda |t_n|^2 \right) |t_n|^{-2+\frac{\sqrt{3}}{2}} \sqrt{\delta t} w_n^{(2)}
\]

\[
H_n = -\frac{1}{2} v_n \pm \sqrt{\left| \frac{1}{12} \Phi_n^2 + \frac{\lambda}{3} \right|},
\]

with \( n \) being the step number and \( \delta t \) the step size. To simulate a discrete Wiener process we use the two independent random Gaussian deviates \( w_n^{(1)} \) and \( w_n^{(2)} \) of unit variance and zero mean,

\[
\langle w_n^{(i)} \rangle = 0 \quad \text{and} \quad \langle w_m^{(i)} w_n^{(j)} \rangle = \delta_{ij} \delta_{mn}.
\]

The above discretized equations are in agreement with Itô’s definition [23] of the stochastic noise. It is assumed that other interpretations of the noise term have only small effects on the qualitative behaviour of the system [21]. The Gaussian deviates are produced by a standard method [24] from a very well tested random number generator.
As an example, the results of the numerical integration of the equations (63) – (65) are shown in figure 3. The integration was done for the (+) branch (referring to the lower sign in equation (65)) and negative $t$. Both plots show two classical trajectories (obtained by switching off the stochastic noise terms) corresponding to accelerated expansion ($H > 0$) and accelerated contraction ($H < 0$). On the classical level there is no transition possible between the regions of positive and negative $H$. In both examples the initial conditions of all stochastic trajectories are chosen to emerge from the same starting point which is also the starting point of the classical path for $H > 0$. In (a) the free parameter $|\Phi^0(0)| a_0^{3/2}$ is varied which effectively sets the strength of the fluctuations. While smoothly turning on the fluctuations by increasing the value of $|\Phi^0(0)| a_0^{3/2}$ the stochastic trajectory bends towards the second classical path characterized by negative $H$. The situation is of course not attractive; on the other hand this provides an example that via backreactions two a priori disconnected regions in the space of solutions to the classical equations of motion are no longer separated when sizable fluctuations enter the analysis. Certainly not all stochastic trajectories behave as the one depicted in (a). In plot (b) the path labelled by ”5.0” in (a) is redrawn together with other realizations, i.e. all the parameters are held fixed but the initialization of the random number generator is different. The dispersion of the whole ensemble of stochastic trajectories grows much faster than the difference between the two distinct classical solutions; this is in accordance with what we found analytically in this subsection.

The condition $\sqrt{\lambda} t \ll 1$ is fulfilled at any instance of the integration; moreover, the relative size of the fluctuations are always kept under control during the integration meaning that $\delta \Phi/\Phi \ll 1$ and $\delta v/v \ll 1$ are satisfied for almost all points; thus violently fluctuating intervals where the whole procedure becomes doubtful are excluded. As was already anticipated in [9] on the classical level, we again find from the numerical integration that the influence of a linear potential on the dynamics during the phase of kinetic inflation is negligible also when fluctuations are included. The drastic deviations of the stochastic trajectories from the classical ones are present also for zero potential and are qualitatively similar. From equations (63) and (64) it follows that the fluctuations in $v$ dominate over those in $\Phi$ by one power in $t$ as one approaches $t = 0$. Indeed we find from the numerics that the stochastic noise term in equation (63) is subdominant.

On the basis of the intuitions associated with the standard inflationary paradigm one would expect the size of the fluctuations to be set by the Hubble parameter alone. However, in our case we can see that the results depend somewhat on the parameter $|\Phi^0(0)| a_0^{3/2}$. This is not too surprising, as there is another mass scale in the problem, namely the Planck scale which is actually the dynamical variable equivalent to $\Phi$, hence the dependence on initial conditions can be seen as the dependence on the initial value of the effective Planck scale. Since there is no canonical way of determining a unique, internal set of initial conditions for the pre–big–bang epoch, the size of the fluctuations can only be given in terms of these parameters. The qualitative picture remains however unchanged no matter what their specific values might be, as long as they are not too exotic. From figure 3 one can draw the conclusion that the choice of the initial conditions merely fixes the instant of
Figure 3: The numerical integration of the stochastic equations of motion (63) – (65) for the (+) branch of the \( \omega = -1 \) theory are shown (\( \lambda = 1 \) has been chosen). In both plots the result of the classical equations (bold lines) are compared with the stochastic trajectories. The Hubble parameter is given as a function of (negative) time. (a) Various stochastic paths parameterized by the effective fluctuation strength \( |\Phi^{(0)}_0| a_0^{3/2} \) (as indicated) are compared. (b) The specific path "5.0" of (a) for different randomly chosen initializations of the random number generator. All stochastic trajectories emerge from the same starting point being also the starting point of the upper \( (H > 0) \) classical solution.

time, when the stochastic trajectory bends away from the original classical solution. Finally, we should comment on the assumed value of the splitting parameter \( \varepsilon = 1 \). In investigations of potential driven inflationary scenarios with a period very close to the actual de Sitter epoch one tends to go with the value of \( \varepsilon \) to zero. This is usually very helpful in simplifying calculations, but in our opinion not so well justified on physical grounds, as strictly speaking in this limit all the Fourier modes except the zero mode are considered “fluctuations”. Hence here we decided to keep \( \varepsilon \) finite and of the order one which singles out the causal horizon as the splitting scale at any instant of the evolution. It is clear that \( \varepsilon \) could have been chosen smaller - this would mean that we shift more and more degrees of freedom into the fluctuating part of the field. It turns out that over a wide range of finite \( \varepsilon \) the results of the stochastic analysis are indeed physically equivalent.
5 Summary and Conclusion

Jordan–Brans–Dicke (JBD) theories with a linearized potential for the JBD field are investigated in the framework of a stochastic analysis which is capable of taking fluctuations of this field into account; their backreactions on the classical background are examined. We split the JBD scalar in sub- and super-horizon parts treating the long wavelength modes as the background quantity whose time evolution is subject to the short wavelength fluctuations. We derive the stochastic equations of motion for the system together with the two point correlation functions of the stochastic noise operators; they finally set the strength of the random force term in the Langevin equations which, in principle, can be integrated for any potential.

For a non vanishing potential we give the scalar field mode functions for several limiting cases; they finally enter the expressions for the correlation functions. We compute the mode functions in the limit of $\sqrt{\lambda} t \gg 1$ for general $\omega$ and give the approximate solutions for $\sqrt{\lambda} t \ll 1$ for the $\omega = -1$ model in terms of power law series; this allows one to study how the mode functions are modified if a small but finite potential is switched on. The mode functions can not be given analytically in the most general case; Therefore, we integrate the system numerically and analyze the influence of general $\lambda$ and $\omega$. We find that all mode functions are regular at $t = 0$ and turn from a power law growth to an oscillatory regime after a time $t$ given by the ratio $|k|/\lambda$. Moreover, we demonstrate that they strongly depend on the value of $\omega$ chosen with a significant increase when $\omega$ drops even slightly below $-1$.

After defining the variances of the random variables in terms of the correlation functions we explicitly give their form in the zero potential limit and for the stringy model $\omega = -1$. Arguing that two distinct classical solutions can no longer be distinguished as soon as the dispersion of the fluctuations is of the same order of magnitude as the separation between the two classical trajectories, we show for general $\omega$ that this condition is met for all cases as $t = 0$ is approached. We point out that this fact is not due to the mode functions (since they remain finite for $t \to 0$), but due to a combination of the background fields which gets singular close to $t = 0$. We find a strong enhancement of this effect for $\omega \to 0$. This is where many theories of gravity arising from Kaluza–Klein theories after compactification to four dimensions are located [4].

Finally, for $\omega = -1$, we perform an integration of the stochastic equations of motion for a finite potential demonstrating how the stochastic ensemble corresponding to the full quantum scalar field evolves under the influence of the scalar field fluctuations. The dynamics is such that the ensembles representing classical solutions which belong to disconnected solutions start overlapping. We again find that the dispersion of the fluctuations grows to achieve the magnitude of the term separating the two classical solutions. This phenomenon can be interpreted as the quantum mechanical realization of connecting classically disconnected solutions at the level of field theoretical considerations.
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