Some asymptotic properties of random walks on homogeneous spaces

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Abstract

Let $G$ be a connected semisimple real Lie group with finite center, and $\mu$ a probability measure on $G$ whose support generates a Zariski-dense subgroup of $G$. We consider the right $\mu$-random walk on $G$ and show that each random trajectory spends most of its time at bounded distance of a well-chosen Weyl chamber. We infer that if $G$ has rank one, and $\mu$ has a finite first moment, then for any discrete subgroup $\Lambda \subseteq G$, the $\mu$-walk and the geodesic flow on $\Lambda \backslash G$ are either both transient, or both recurrent and ergodic, thus extending a well known theorem due to Hopf-Tsuji-Sullivan-Kaimanovich dealing with the Brownian motion.

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1 Introduction

This paper studies the asymptotic properties of random walks on semisimple Lie groups or their quotients. This topic has been developed for 60 years. The heart of the subject is the theory of random walks on linear groups worked out by Furstenberg, Kesten, Guivarc’h, and many others, to transpose classical limit theorems for Markov chains on $\mathbb{Z}^d$ to the context of linear random walks. It recently led to spectacular applications to walks on finite volume homogeneous spaces, such as Eskin-Margulis Theorem establishing the non escape of mass for any starting point, or later Benoist-Quint’s classification of stationary probability measures (see also [14]).

Our paper adds to this network of ideas by enriching the general theory of walks on linear groups and inferring a recurrence criterion for walks in infinite volume.

A seminal result due to Furstenberg is that a trajectory of a Zariski-dense random walk on a semisimple Lie group goes to infinity in a specific direction, given by a point on the flag variety. In the very concrete setting of a walk on $PSL_2(\mathbb{R})$, seen as the unitary bundle of the Poincaré disk $D$, this means that every walk trajectory converges toward a limit point on the boundary $\partial D$. Furstenberg’s result has since been transposed to the context of Gromov hyperbolic spaces. More precisely, Ancona considers the Brownian motion on a Gromov hyperbolic manifold and shows that the distance between a Brownian trajectory and its limit geodesic ray grows at most logarithmically. Analogous results for walks on hyperbolic groups are proven in [2, 26, 29, 30].

The first theorem of our paper completes this panel of results by claiming that the distance between a random trajectory and its asymptotic geodesic ray (or more generally asymptotic Weyl chamber) is most of the time bounded.

Let $G$ be a connected semisimple real Lie group with finite center, $\mu$ a probability measure on $G$. The (right) $\mu$-random walk on $G$ is defined by the transition probabilities

$$p(g, h) = \mu(g^{-1}h)$$

A trajectory starting from a point $x_0 \in G$ is thus obtained as a sequence $(x_0b_1\ldots b_n)_{n\geq 1}$ where the $b_i \in G$ are independent $\mu$-distributed increments. We make the assumption that the subgroup $\Gamma_\mu = \langle \text{supp } \mu \rangle \leq G$ generated by the support of $\mu$ is Zariski-dense in $G$. Denoting by $\mathfrak{g}$ the Lie algebra of $G$, this means that every polynomial function on $\text{End}(\mathfrak{g})$ which vanishes on the adjoint representation $\text{Ad}\Gamma_\mu$ is also null on $\text{Ad}G$. 
**Theorem A** (Bounded deviations). Let $G$ be a connected semisimple real Lie group with finite center, and $\mu$ a probability measure on $G$ such that $\Gamma_\mu$ is Zariski-dense in $G$. Set $B = G^{\mathfrak{R}^2}$, $\beta = \mu^{\mathfrak{R}^2}$, fix a maximal compact subgroup $K \subseteq G$, and a left invariant metric $d$ on $G$.

To every $b \in B$, one can associate a Weyl Chamber $C(b) \subseteq G/K$ such that the following holds. For all $\varepsilon > 0$, there exists a constant $R > 0$, such that for $\beta$-almost every $b \in B$,

$$
\liminf_{n \to +\infty} \frac{1}{n} \mathbb{E}[t \{ i \in [1,n], \, d(b_1 \ldots b_i, C(b)) \leq R \}] > 1 - \varepsilon
$$

Here $d(b_1 \ldots b_i, C(b))$ refers to the distance between $b_1 \ldots b_i$ and $C(b)$ seen as a right $K$-invariant subset of $G$.

Recall that when $G$ has rank 1 (e.g. $G = SO_\varepsilon(d,1)$ or $G = SU(d,1)$), then a Weyl chamber of $G/K$ is just a geodesic ray for the symmetric space structure of $G/K$. In higher rank, it corresponds to a convex cone in a maximal flat of $G/K$. The result we prove is actually slightly more precise than Theorem A: the map $b \mapsto C(b)$ is explicit in terms of the Cartan decompositions of $(b_1 \ldots b_n)_{n \geq 1}$, and we bound the distance between $b_1 \ldots b_n$ and a particular point in $C(b)$ (see Section 2.5).

In concrete linear algebra terms, for $G = SL_d(\mathbb{R})$, $K = SO_d(\mathbb{R})$, we can set $\mathfrak{a}^+ = \{ t = \text{diag}(t_1, \ldots, t_d), \, t_1 \geq \cdots \geq t_d, \, \sum t_i = 0 \}$ and choose $C(b)$ of the form $C(b) = k_x(b) \exp(\mathfrak{a}^+)K$ where $k_x(b) \in K$. Writing $b_1 \ldots b_n K = k_n(b) \exp(t_n(b))K$ where $t_n(b) \in \mathfrak{a}^+$ is the so-called Cartan projection, our deviation result bounds the difference of angle $k_n^{-1}(b)k_x(b)$ in terms of $t_n(b)$: for all $\varepsilon > 0$, for $\beta$-typical $b \in B$, there is a subset $S_{\varepsilon,b} \subseteq \mathbb{N}^*$ of asymptotic density at least $(1-\varepsilon)$ such that

$$(k_n^{-1}(b)k_x(b))_{i,j} = O_{\varepsilon}(\exp(t_n(b)_i - t_n(b)_j)) \quad \text{for } n \in S_{\varepsilon,b}, \, i > j$$

In particular $k_n^{-1}(b)k_x(b)$ converges to $\text{Id}$ (up to sign of coefficients) at speed $O_{\varepsilon}(\exp(-\min_{i \neq j} |t_n(b)_i - t_n(b)_j|))$ along $n \in S_{\varepsilon,b}$. More information on the asymptotic behavior of $t_{n}(b)$ can be found in [3] (in particular [3, Th. 10.9]).

Finally, we emphasize that no moment assumption is made on $\mu$ in Theorem A. In contrast, the logarithmic bounds of [2] [26] [9] [30] all rely strongly on the hypothesis of a finite exponential moment (or at least the Hölder regularity of the harmonic measure). Note also that Theorem A is already new in the case where the $\mu$-walk is a (discrete) Brownian motion.

Our second theorem considers the $\mu$-random walk induced on a quotient $X = \Lambda \backslash G$, where $\Lambda$ is a discrete subgroup of $G$, and characterizes the situations of recurrence or transience in terms of the geodesic flow. A first result of this kind was obtained by Tsuji [33], who built on Hopf’s alternative [20] [21].
to prove that the Brownian motion and the geodesic flow on a hyperbolic surface are either both recurrent ergodic, or both transient. Sullivan [31] extended Hopf-Tsuji Theorem to hyperbolic manifolds of arbitrary dimension, and Kaimanovich [22] pushed it even further, dealing with walks on rank one symmetric spaces. However, all these theorems concern a Brownian motion or at least a spread-out random walk, i.e. a $\mu$-walk such that $\mu$ (or a convolution power) is absolutely continuous with respect to the Haar measure on $G$. In this paper, we extend them to walks determined by an arbitrary Zariski-dense probability measure $\mu$ with a finite first moment, meaning that

$$\int_G \log ||\text{Ad}|| \, d\mu(g) < \infty$$

Let us now prepare our statement by recalling some basic definitions. More details can be found in [4, 1.2-1.3].

Assume $G$ has rank one and let $\mathfrak{a} \subseteq \mathfrak{g}$ be a Cartan subspace of $G$ which is orthogonal to the Lie algebra of $K$ for the Killing form. For $t \in \mathfrak{a}$, set $a_t = \exp(t) \in G$. The one-parameter subgroup $(a_t)_{t \in \mathfrak{a}}$ acts by right multiplication on $X$, inducing a flow that we call the geodesic flow. This terminology is natural as any geodesic path on the locally symmetric space $X/K$ is of the form $t \mapsto xa_tK$ for a suitable $x \in X$ [28, Proposition 4.3].

Let $F \subseteq X$ be a compact set of positive Haar measure. The Green function of the $\mu$-walk associated to $F$, denoted by $G_\mu(\cdot, F) : X \to [0, +\infty]$, estimates the average time spent in $F$ by the $\mu$-trajectories starting at a given point. We may define in a similar way a Green function for the geodesic flow $G(\cdot, F) : X \to [0, +\infty]$. The precise formulas are

$$G_\mu(x, F) = \int_B \sum_{n \geq 0} 1_F(x b_1 \ldots b_n) \, d\beta(b) \quad G(x, F) = \int_{\mathfrak{a}^+} 1_F(x a_t) \, dt$$

where $\mathfrak{a}^+ \subseteq \mathfrak{a}$ is a fixed Weyl chamber of $\mathfrak{a}$.

The $\mu$-walk on $X$ is recurrent (resp. transient) if for almost-every $x \in X$, $\beta$-almost every $b \in B$, the trajectory $(xb_1 \ldots b_n)_{n \geq 0}$ sub-converges to $x$ (resp. leaves every compact). It is equivalent to say that for every $F$, almost-every $x \in X$, one has $G_\mu(x, F) \in \{0, +\infty\}$ (resp. $G_\mu(x, F) < +\infty$).

The $\mu$-walk on $X$ is ergodic if the subgroup $\Gamma_\mu$ of $G$ generated by the support of $\mu$ acts ergodically on $X$ for the Haar measure. In the context of a recurrent random walk, this amounts to say that $G_\mu(x, F) = \infty$ for every $F$ and almost-every $x \in X$.

Analogous definitions of recurrence, transience, ergodicity hold for the geodesic flow on $X$. 

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Theorem B (Recurrence criterion). Let $G$ be a connected simple real Lie group of rank one, $\Lambda \subseteq G$ a discrete subgroup, $X = \Lambda \backslash G$. Let $\mu$ be a probability measure on $G$ with a finite first moment and with $\Gamma_\mu$ Zariski-dense in $G$.

Then the $\mu$-walk and the geodesic flow on $X$ are either both recurrent ergodic, or both transient with locally integrable Green functions.

A striking consequence is that the recurrence properties of a walk on a rank-one homogeneous space do not depend on the generating measure $\mu$ but only on the geometry of the space. Also note that in the particular case where $G = Sp(1,m)$ for some $m \geq 2$, or $G = F_4^{-20}$, the geodesic flow (or equivalently the $\mu$-walk) is always transient as long as $\Lambda$ has infinite covolume in $G$ (see Theorem 4.1 and the subsequent remark).

Organization of the paper.

Section 2 is dedicated to Theorem A. It is the occasion to set the notations, recall the dynamics of Zariski-dense random walks on the flag variety of $G$, and introduce a parametrization of the set of maximal flats of $G/K$ which will also be useful in the rest of the paper.

Section 3 is dedicated to Theorem B. The proof that the transience of the geodesic flow on $X$ implies the transience of the walks is entirely contained in the section. The converse relies on the framework that emerged in Section 2 more specifically Corollary 2.6. Both statements use renewal results for the Cartan projection on rank one simple real Lie groups.

Section 4 is an appendix that recalls the Hopf alternative for the geodesic flow, usually stated in terms of Poincaré series, and explains how it can be formulated via the Green functions of the flow.

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2 Random trajectories and asymptotic Weyl chambers

This section is dedicated to the proof Theorem A

2.1 Notations and strategy

Throughout Section 2 we denote by $G$ a connected non-compact semisimple real Lie group with finite center, $K \subseteq G$ a maximal compact subgroup, $\mathfrak{g}, \mathfrak{k}$ their respective Lie algebras, $\mathfrak{a} \subseteq \mathfrak{g}$ a Cartan subspace orthogonal to $\mathfrak{k}$ for the Killing form, $\mathfrak{a}^+ \subseteq \mathfrak{a}$ a closed Weyl chamber of $\mathfrak{a}$, and for $t \in \mathfrak{a}$ we set $a_t = \exp(t) \in G$. Recall that in this context, every element $g \in G$ can be written as $g = k_g a_t l_g$ where $k_g, l_g \in K$, $t_g \in \mathfrak{a}^+$. This is called the Cartan
decomposition, and the element $t_g$ is unique and called the Cartan projection of $g$.

Let $\mu$ be a probability measure on $G$ and $\Gamma_\mu \subseteq G$ the subgroup generated by its support. We suppose that $\Gamma_\mu$ is Zariski-dense in $G$, but make no moment assumption on $\mu$. We denote by $(B, \beta) = (G^N, \mu^\otimes N)$ the space of instructions guiding the $\mu$-walk on $G$.

The proof of Theorem A can be summarized as follows. We fix for every $b \in B$, $n \geq 0$ a Cartan decomposition

$$b_1 \ldots b_n = k_n(b) a_{t_n(b)} l_n(b)$$

Section 2.2 guarantees that we can choose those decompositions such that for $\beta$-almost every $b \in B$, the sequence $(k_n(b)) \in K^N_\mu$ has a limit $k_\infty(b) \in K$. We aim to control the distance $d(b_1 \ldots b_n, k_\infty(b) a_{t_n(b)})$ given a left $G$-invariant riemannian metric on $G$. To this end, we define the flag varieties $G/P^-$, $G/P$, denote by $\xi^- = P^-/P^-$, $\xi_0 = P/P$ their basepoints, and set $\mathcal{F}^+ = G.(\xi^- \cdot \xi_0)$ the $G$-orbit of $(\xi^- \cdot \xi_0)$ in $G/P^- \times G/P$. Section 2.3 introduces a map $(\xi^- \cdot \xi) \mapsto F(\xi^- \cdot \xi)$ from $\mathcal{F}^+$ to the space of maximal flats in $G/K$. It is a $G$-equivariant cover such that $F(\xi^- \cdot \xi_0) = \exp(a)K$. We see in Section 2.4 that for some constant $C_0 > 0$, every $\xi^- \in G/P^-$, $\beta$-almost every $b \in B$, large enough $n \geq 0$,

$$d(b_1 \ldots b_n, k_\infty(b) a_{t_n(b)}) \leq C_0 d(b_1 \ldots b_n, F(\xi^- \cdot \xi_b)) + C_0$$

where $\xi_b = k_\infty(b) \xi_0$ is the limit point of $(b_1 \ldots b_n)$ on the flag variety $G/P$. Finally, in Section 2.5, we use Birkhoff Ergodic Theorem to control the right-hand side of the previous inequality and conclude the proof of Theorem A.

Remark. The method of proof is inspired by Ledrappier’s paper [26] establishing logarithmic deviation on free groups between sample paths and their asymptotic geodesic rays, under a finite exponential moment condition. As pointed out by the referee, it is also related to [32] in which Tiozzo proves sublinear deviation of sample paths under very general assumptions.

2.2 Dynamics of walks on the flag variety

We recall here basic facts about the dynamics of the $\mu$-walk on the flag variety of $G$. The existence of the limit direction $k_\infty$ introduced in 2.1 is also justified.

Let us begin with the definition of the flag variety. For $\alpha \in a^*$, set

$$\mathfrak{g}_\alpha := \{ s \in \mathfrak{g}, \forall t \in a, (\text{ad } t)(s) = \alpha(t)s \}$$

As the action of $(\text{ad } t)_{t \in a}$ on $\mathfrak{g}$ is simultaneously diagonalizable, we can write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$
where $\Phi := \{\alpha \in \mathfrak{a}^* \setminus \{0\}, \mathfrak{g}_\alpha \neq \{0\}\}$ is the root system of $\mathfrak{a}$. Denote by $\Phi^+ \subseteq \Phi$ the subset of positive roots given by $\mathfrak{a}^+$, namely $\Phi^+ = \{\alpha \in \Phi, \alpha(\mathfrak{a}^+) \subseteq \mathbb{R}^+\}$. Set $u = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and let $P = N_G(\mathfrak{g}_0 \oplus u)$ be the subgroup of elements $g \in G$ whose adjoint action preserves the subspace $\mathfrak{g}_0 \oplus u$. The flag variety of $G$ is defined as the quotient $\mathcal{P} = G/P$. We set $\xi_0 = P/P$ the standard base point.

It will often be convenient to work with a concrete realisation of $\mathcal{P}$ as some $G$-orbit in a product of projective spaces. To this end, we recall the following fact \cite[Lemma 6.32]{8}. Let $\Pi$ be the basis of $\Phi$ prescribed by $\Phi^+$.

**Fact 1.** There exists a family $(V_\alpha, \rho_\alpha)_{\alpha \in \Pi}$ of proximal irreducible algebraic representations of $G$ such that

- denoting by $\xi_\alpha \in \mathbb{P}(V_\alpha)$ the line of highest weight of $(V_\alpha, \rho_\alpha)$, we have a $G$-equivariant embedding
  \[ \mathcal{P} \hookrightarrow \prod_{\alpha \in \Pi} \mathbb{P}(V_\alpha), \ g\xi_0 \mapsto (\rho_\alpha(g)\xi_\alpha)_{\alpha \in \Pi} \]

- the highest weights $(\chi_\alpha)_{\alpha \in \Pi}$ form a basis of $\mathfrak{a}^*$. Moreover $\chi_\alpha - \alpha$ is also a weight of $(V_\alpha, \rho_\alpha)$.

We equip each $V_\alpha$ with a scalar product that is $\rho_\alpha(K)$-invariant and such that every element of $T\rho_\alpha(\mathfrak{a})$ is self-adjoint \cite[Lemma 6.33]{8}.

We know by \cite[Proposition 10.1]{8} that $\mathcal{P}$ admits a unique stationary probability measure, we call it $\nu_{\mathcal{P}}$. This measure gives no mass to proper subvarieties, and is proximal: for $\beta$-almost every $b \in B$, the limit measure $\nu_{\mathcal{P},b} = \lim(b_1 \ldots b_n)_{*} \nu_{\mathcal{P}}$ is a Dirac mass $\nu_{\mathcal{P},b} = \delta_{\xi_b}$. We then have $\nu_{\mathcal{P}} = \int_B \delta_{\xi_b} d\beta(b)$. The $\alpha$-coordinate map $\psi_\alpha : \mathcal{P} \to \mathbb{P}(V_\alpha), g\xi_0 \mapsto \rho_\alpha(g)\xi_\alpha$ sends $\nu_{\mathcal{P}}$ to the unique $\mu$-stationary probability measure $\nu_{\mathbb{P}(V_\alpha)}$ on $\mathbb{P}(V_\alpha)$. It is atom free and proximal, with limit measures $\nu_{\mathbb{P}(V_\alpha),b} = \delta_{\xi_{b,\alpha}}$ where $\xi_{b,\alpha} = \psi_\alpha(\xi_b)$.

Recall that we have fixed for every $b \in B$, $n \geq 0$, a Cartan decomposition $b_1 \ldots b_n = k_n(b)\mathfrak{a}_+l_n(b)$. We show in Lemma 2.1 that the sequence $(k_n(b)\xi_0)_{n \geq 0}$ converges in $\mathcal{P}$. As $\mathcal{P} = K/M$ where $M = Z_K(\mathfrak{a})$, this result ensures we can always choose our decompositions so that $(k_n(b))_{n \geq 0}$ converges in $K$, justifying the definition of $k_\infty$ given in Section 2.1.

**Lemma 2.1.** For $\beta$-almost every $b \in B$, one has the convergence in $\mathcal{P}$

\[ k_n(b)\xi_0 \xrightarrow[n \to +\infty]{} \xi_b \]

**Proof.** It is enough to argue for each coordinate, i.e. show that for each $\alpha \in \Pi$,

\[ \rho_\alpha(k_n(b))\xi_\alpha \xrightarrow[n \to +\infty]{} \xi_{b,\alpha} \]
To lighten the proof, we just write $g$ for $\rho_\alpha(g)$.

Notice that

$$\frac{b_1 \ldots b_n}{\|b_1 \ldots b_n\|} = k_n(b) \frac{a_{l_n(b)}}{\|a_{l_n(b)}\|} l_n(b)$$

[8] Proposition 4.7] states that any accumulation point in $\text{End}(V_\alpha)$ of the sequence $(\frac{b_{1 \ldots b_n}}{\|b_{1 \ldots b_n}\|})_{n \geq 1}$ must be of rank one with image $\xi_{b,\alpha}$. This yields that the sequence $\frac{a_{l_n(b)}}{\|a_{l_n(b)}\|}$ converges to the orthogonal projection on the line of heighest weight $\xi_\alpha$. Let $k'_\alpha(b)$ be some limit value of the sequence $k_n(b)$, and $\sigma : \mathbb{N} \to \mathbb{N}$ an extraction such that $k_{\sigma(n)}(b) \to k'_\alpha(b)$ and $l_{\sigma(n)}(b)$ converges in $K$. Then $\frac{b_{1 \ldots b_{\sigma(n)}}}{\|b_{1 \ldots b_{\sigma(n)}}\|}$ converges as well and its limit has image $k'_\alpha(b)\xi_\alpha$. Hence $\xi_{b,\alpha} = k'_\alpha(b)\xi_\alpha$, which proves the lemma.

2.3 Parametrization of maximal flats

Let $P^-$ be the opposite flag variety of $G$ defined by setting $u^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$, $P^- = N_G(\mathfrak{g}_0 + u^-)$ and $\mathcal{P}^- = G/P^-$. Let $\xi_0^- = P^-, \xi_0^+ = P$ be the base points of $\mathcal{P}^-$, $\mathcal{P}^+$ and denote by $\mathcal{F}^+ = G.(\xi_0^-, \xi_0)$ the (open) $G$-orbit of $(\xi_0^-, \xi_0)$ in $\mathcal{P}^- \times \mathcal{P}$. The following lemma associates to every pair of flags $(\xi^-, \xi) \in \mathcal{F}^+$ a maximal flat of $G/K$.

Lemma 2.2. There exists a unique $G$-equivariant map

$$\mathcal{F}^+ \to \{\text{maximal flats of } G/K\}$$

$$(\xi^-, \xi) \mapsto F(\xi^-, \xi)$$

such that $(\xi_0^-, \xi_0) \mapsto \exp(\mathfrak{a})K$. Moreover this map is a Galois cover whose group of deck transformations is the Weyl group of $G$.

Remark. The map $F$ is also used in [23] Section 10] to describe the Poisson boundary of walks on discrete subgroups of semi-simple Lie groups, via the strip approximation method.

Proof. We can identify $\mathcal{F}^+$ to $G/(P^- \cap P) = G/Z_G(\mathfrak{a})$ and the set of maximal flats of $G/K$ to $G/N_G(\mathfrak{a})$. The map we are to define is just the quotient projection map $G/Z_G(\mathfrak{a}) \to G/N_G(\mathfrak{a})$. Moreover the Weyl group $W = Z_G(\mathfrak{a}) \backslash N_G(\mathfrak{a})$ is finite and acts freely on $G/Z_G(\mathfrak{a})$ by right multiplication, so the above projection map is a Galois cover whose group of deck transformations is $W$.

We now check that for every $\xi^- \in \mathcal{P}^-$, and $\beta$-almost every $b \in B$, the maximal flat $F(\xi^-, \xi_b)$ is well defined. To this end, we begin with the following criterion

\[ \text{ } \]
Lemma 2.3. Let \(g_1, g_2 \in G\).

\[
(g_1 \xi_0^-, g_2 \xi_0) \in \mathcal{F}^+ \iff g_1^{-1} g_2 \in P^- P
\]

Proof.

\[
(g_1 \xi_0^-, g_2 \xi_0) \in \mathcal{F}^+ \iff (\xi_0^-, g_1^{-1} g_2 \xi_0) \in G.(\xi_0^-, \xi_0) \\
\iff \exists h \in G, h \xi_0^- = \xi_0^- \text{ and } h \xi_0 = g_1^{-1} g_2 \xi_0 \\
\iff \exists h \in P^-, h^{-1} g_1^{-1} g_2 \in P
\]

\[\blacksquare\]

Lemma 2.4. For every \(\xi^- \in \mathcal{P}^-, \) and \(\beta\)-almost every \(b \in B, \) one has \((\xi^-, \xi_b) \in \mathcal{F}^+\).

Proof. Write \(\xi^- = g_1 \xi_0^-\) for some \(g_1 \in G.\) According to Lemma 2.3, we need to check that for \(\beta\)-almost every \(b \in B, \)

\[
\xi_b \in g_1 P^- \xi_0
\]

Bruhat decomposition \([10, \text{ Theorem 5.15}]\) guarantees that \(\mathcal{P} = \sqcup_{w \in W} g_1 P^- w \xi_0\) where \(W\) stands for the Weyl group. It is thus enough to show that for \(w \in W \setminus \{0\}, \) we have \(\nu_{\mathcal{P}}(g_1 P^- w \xi_0) = 0.\)

As \(w \neq 0, \) we have \(w \xi_0 \neq \xi_0, \) so by Fact 1, there exists \(\alpha \in \Pi\) such that \(\rho_\alpha(w) \xi_\alpha \neq \xi_\alpha.\) As \(\rho_\alpha(w)\) permutes the weights of \(V_\alpha,\) it has to send \(\xi_\alpha\) to \(V_\alpha^-\) the unique \(a\)-invariant complementary subspace of \(\xi_\alpha.\) But \(V_\alpha^-\) is stable under \(\rho_\alpha(P^-).\) To sum up, we have

\[
\psi_\alpha(g_1 P^- w \xi_0) \subseteq \mathbb{P}(\rho_\alpha(g_1) V_\alpha^-)
\]

where \(\psi_\alpha\) is the \(\alpha\)-coordinate projection \(\mathcal{P} \to \mathbb{P}(V_\alpha), g\xi_0 \mapsto \rho_\alpha(g) \xi_\alpha.\) As the action of \(G\) on \(V_\alpha\) is irreducible, the stationary measure \(\nu_{\mathcal{P}(V_\alpha)} = \psi_\alpha \cdot \nu_{\mathcal{P}}\) gives no mass to proper projective spaces \([8, \text{ Lemma 4.6}]\). In particular, the above yields \(\nu_{\mathcal{P}}(g_1 P^- w \xi_0) = 0.\) Finally \(\nu_{\mathcal{P}}(g_1 P^- \xi_0) = 1.\)

\[\blacksquare\]

2.4 Distance formula

The goal of Section 2.4 is to prove Corollary 2.6 which bounds the distance from \(b_1 \ldots b_n\) to \(k_x \cdot (b) a t \) by the distance from \(b_1 \ldots b_n\) to a well-chosen maximal flat of \(G/K.\) Corollary 2.6 will follow from geometric (non-random) considerations encapsulated in Proposition 2.5. We endow \(G\) with a left \(G\)-invariant Riemannian metric, and denote by \(d\) the distance induced on \(G.\) For \(\xi \in \mathcal{P},\) we write \(\xi = k_\xi \xi_0\) where \(k_\xi \in K.\) Recall that for \(x \in G,\) we denote by \(x = k_x a t \) where \(k_x, t \in K, t \in a^+\) a Cartan decomposition of \(x.\) The Cartan projection \(t\) is uniquely defined, and \(k_x, k_\xi\) are uniquely defined in \(K/M (\text{where } M = Z_K(a))\) as long as \(t\) is in the interior of \(a^+.\)
**Proposition 2.5.** There exists $C_0 > 0$ such that for all $(\xi^-, \xi) \in \mathcal{F}^+$, all sequence $(x_n) \in G^\mathbb{N}$ such that $\inf_{a \in \Phi^+} \alpha(t_{x_n}) \to +\infty$ and $k_{x_n} \to k_\xi$ in $K/M$, we have for $n$ large enough,

$$d(x_n, k_\xi a_{t_{x_n}}) \leq C_0 d(x_n, F(\xi^-, \xi)) + C_0$$

**Corollary 2.6.** There exists $C_0 > 0$ such that for all $\xi^- \in \mathcal{P}^-$, for $\beta$-almost every $b \in B$, for large enough $n \geq 0$,

$$d(b_1 \ldots b_n, k_x(b) a_{t_{x_n}(b)}) \leq C_0 d(b_1 \ldots b_n, F(\xi^-, \xi_b)) + C_0$$

We first prove Proposition 2.5. It relies on the following technical lemma.

**Lemma 2.7.** There exists a constant $C_1 > 0$ such that for every $u \in \exp(u)$, there exists a neighborhood $V_K \subseteq K$ of the neutral element $e$ in $K$ such that for all $s, t \in a, k \in V_K$,

$$d(a_s, a_t) \leq C_1 d(ua_s, ka_t) + C_1$$

Let us see first how to deduce Proposition 2.5 from Lemma 2.7.

**Proof of Proposition 2.5.** Let $(\xi^-, \xi) \in \mathcal{F}^+$ and $(x_n)$ as in Proposition 2.5. Using Section 2.3,

$$F(\xi^-, \xi) = k_\xi F(k_{x_n}^{-1} \xi^-, \xi_0) = k_\xi u_{\xi^-} \exp(a) K$$

for some element $u_{\xi^-} \in \exp(u)$. The assumption $\inf_{a \in \Phi^+} \alpha(t_{x_n}) \to +\infty$ implies that $a_{t_{x_n} u_{\xi^-} a_{t_{x_n}}} \to 0$ as $n$ goes to infinity. Hence we can write for $n \geq 0, s \in a$,

$$d(x_n, k_\xi a_{t_{x_n}}) = d(x_n, k_\xi u_{\xi^-} a_{t_{x_n}}) + o(1)$$

$$\leq d(x_n, k_\xi u_{\xi^-} a_s) + d(a_s, a_{t_{x_n}}) + o(1)$$

Using the assumption that $k_{x_n} \to k_\xi$ in $K/M$ and Lemma 2.7, we have for large $n \geq 0$, every $s \in a$,

$$d(a_s, a_{t_{x_n}}) \leq C_1 d(k_{x_n}^{-1} k_{x_n} a_{t_{x_n}}, u_{\xi^-} a_s) + C_1$$

$$= C_1 d(k_{x_n} a_{t_{x_n}}, k_\xi u_{\xi^-} a_s) + C_1$$

$$\leq C_1 d(x_n, k_\xi u_{\xi^-} a_s) + C_2$$

where $C_2 = C_1 \text{diam} K + C_1$, which leads to

$$d(x_n, k_\xi a_{t_{x_n}}) \leq (1 + C_1) d(x_n, k_\xi u_{\xi^-} a_s) + C_2 + o(1)$$

Choosing $s$ to realise the infimum, we obtain for large enough $n \geq 0$,

$$d(x_n, k_\xi a_{t_{x_n}}) \leq (1 + C_1) d(x_n, F(\xi^-, \xi)) + C_2 + 1$$

which concludes the proof. \qed
We now need to show Lemma 2.7.

Proof of Lemma 2.7. Notice first that if \( d_1, d_2 \) are the distances induced on \( G \) by two left \( G \)-invariant Riemannian metrics, then there exists a constant \( R > 0 \) such that
\[
\frac{1}{R} d_2 \leq d_1 \leq R d_2
\]
Hence, in order to prove Lemma 2.7, we can specify \( d \) as follows. Let \( s = \mathfrak{t}^\perp \) be the orthogonal of \( \mathfrak{t} \) for the Killing form \( \mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \). We know by [8] that \( \mathcal{K} \) is negative definite on \( \mathfrak{k} \) and positive definite on \( \mathfrak{s} \). In particular we have a decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \) and we may define a scalar product on \( \mathfrak{g} \) by setting \( \langle \cdot, \cdot \rangle = -\mathcal{K}(\cdot, \cdot) \) where \( \theta = \text{Id}_k \oplus -\text{Id}_s \) is the opposition involution map. We endow \( G \) with left \( G \)-invariant metric that coincides with \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) and write \( d \) the corresponding distance map.

It is then a standard exercise to check that for \( s, t \in \mathfrak{a} \),
\[
d(e, a_s) = ||s||
\]
where \( ||\cdot|| \) is the euclidean norm associated to \( \langle \cdot, \cdot \rangle \). In particular, for any \( g, h \in G \),
\[
||\kappa(g^{-1}h)|| - 2\text{diam}K \leq d(g, h) \leq ||\kappa(g^{-1}h)|| + 2\text{diam}K
\]
where \( \kappa : G \to \mathfrak{a}^+ \) denotes the Cartan projection map.

This inequality means that we may reformulate Lemma 2.8 as follows.

Lemma 2.7 bis. There exists a constant \( C_2 > 0 \) such that for every \( u \in \exp(u) \), there exists a neighborhood \( V_K \subseteq K \) of the neutral element \( e \) in \( K \) such that for all \( s, t \in \mathfrak{a} \), \( k \in V_K \)
\[
||s - t|| \leq C_2||\kappa(a^{-s}uka_t)|| + C_2
\]

Proof of Lemma 2.7 bis. We use the representations \( (V_\alpha, \rho_\alpha)_{\alpha \in \Pi} \) introduced in Fact 1. For \( \alpha \in \Pi \), let \( v_\alpha \in \xi_\alpha \) be a vector in the line of heighest weight of \( V_\alpha \) such that \( ||v_\alpha|| = 1 \). Recall from [8 Section 6.8] that \( \rho_\alpha(u)(v_\alpha) = v_\alpha \). In particular, there exists a neighborhood \( V_{K,\alpha} \subseteq K \) of \( e \) in \( K \) such that for all \( k \in V_{K,\alpha} \), we have \( \rho_\alpha(uk)v_\alpha = v'_\alpha + v''_\alpha \) with \( v'_\alpha \in \xi_\alpha \), \( v''_\alpha \in \xi_\alpha^\perp \) and \( ||v''_\alpha|| \geq \frac{1}{2} \).
Let \( s, t \in \mathfrak{a} \) and \( k \in V_{K,\alpha} \).
\[
||\rho_\alpha(a^{-s}uka_tv_\alpha)|| \geq \frac{1}{2} e^{\chi_\alpha(t-s)}
\]
which leads to
\[
\chi_\alpha(t - s) \leq \log ||\rho_\alpha(a^{-s}uka_t)|| + \log 2
\]
However, \( \log ||\rho_\alpha(a^{-s}uka_t)|| = \chi_\alpha(\kappa(a^{-s}uka_t)) \leq ||\chi_\alpha|| ||\kappa(a^{-s}uka_t)|| \), so
\[
\chi_\alpha(t - s) \leq ||\chi_\alpha|| ||\kappa(a^{-s}uka_t)|| + \log 2
\]
As $\kappa(g) = \kappa(g^{-1})$, we can apply the previous argument to $(a_{-s}uka_t)^{-1}$ to strengthen the previous inequality and get for $s, t \in a, k \in V_{K,\alpha}$ neighborhood of $e$ in $K$,

$$|\chi_\alpha(t - s)| \leq ||\chi_\alpha|| ||\kappa(a_{-s}uka_t)|| + \log 2$$

Now assuming $k \in V_K = \cap_{\alpha \in \Pi} V_{K,\alpha}$ and summing over $\alpha \in \Pi$,

$$\sum_{\alpha \in \Pi} |\chi_\alpha(t - s)| \leq (\sum_{\alpha \in \Pi} ||\chi_\alpha||) ||\kappa(a_{-s}uka_t)|| + \#\Pi \log 2 \quad (1)$$

As the weights $(\chi_\alpha)_{\alpha \in \Pi}$ form a basis of $a^*$, there exists a constant $C > 0$, depending only on $(\chi_\alpha)_{\alpha \in \Pi}$ and $||.||$, such that

$$||t - s|| \leq C \sum_{\alpha \in \Pi} |\chi_\alpha(t - s)| \quad (2)$$

Inequalities (1) and (2) together prove Lemma 2.7 bis. \hfill \Box

We now turn to the proof of Corollary 2.6. In order to apply Proposition 2.5, we prove

**Lemma 2.8.** For $\beta$-almost every $b \in B$, for every $\alpha \in \Phi^+$,

$$\alpha(t_n(b)) \xrightarrow{n \to +\infty} +\infty$$

**Proof of Lemma 2.8.** We only need to show that for every $\alpha \in \Pi$, $\beta$-almost every $b \in B$,

$$\alpha(t_n(b)) \xrightarrow{n \to +\infty} +\infty$$

Consider again the representations $(V_\alpha, \rho_\alpha)$ introduced in Fact 1. Arguing as in Lemma 2.1, we see that $\rho_\alpha(a_{tn}(b)) ||\rho_\alpha(a_{tn}(b))||$ converges to the orthogonal projection on the line of highest weight $\xi_\alpha$ in $V_\alpha$. In particular, given a vector $w_\alpha$ in the weight space of $\chi_\alpha - \alpha$, we get $\rho_\alpha(a_{tn}(b))||w_\alpha|| \to 0$. Noticing that $||\rho_\alpha(a_{tn}(b))|| = e^{\chi_\alpha(t_n(b))}$, the latter can be rewritten as

$$e^{-\alpha(t_n(b))} \xrightarrow{n \to +\infty} 0$$

which concludes the proof. \hfill \Box

**Proof of Corollary 2.6.** It follows from the combination of Proposition 2.5 and Lemmas 2.4, 2.8, 2.11.
2.5 Proof of Theorem [A]

We conclude Section 2 with the proof of Theorem [A]. We actually show the following more detailed version.

Theorem [A] bis. Keep the notations of Section 2.1. In particular, $G$ is a connected semisimple real Lie group with finite center, $\mu$ a probability measure on $G$ with $\Gamma_\mu$ Zariski-dense in $G$, set $B = G^N$, $\beta = \mu^\otimes N$ and for $\beta$-almost every $b \in B$, choose a Cartan decomposition $b_1 \ldots b_n = k_n(b)a_t(b)l_n(b)$ with $k_n(b)$ converging in $K$, and let $k_\infty(b) = \lim k_n(b)$.

Then for all $\varepsilon > 0$, there exists a constant $R > 0$ such that for $\beta$-almost every $b \in B$,

$$\liminf_{n \to +\infty} \frac{1}{n} \varepsilon \{ i \in [1,n], \ d(b_1 \ldots b_i, k_\infty(b)a_t(b)) \leq R \} > 1 - \varepsilon$$

Proof. In view of Corollary 2.6 it is enough to prove that for all $\varepsilon > 0$, there exists a constant $R > 0$ and an element $\xi^- \in P^-$, such that for $\beta$-almost every $b \in B$,

$$\liminf_{n \to +\infty} \frac{1}{n} \varepsilon \{ i \in [1,n], \ d(b_1 \ldots b_i, F(\xi^-, \xi_b)) \leq R \} > 1 - \varepsilon$$

The observation that

$$d(b_1 \ldots b_i, F(\xi^-, \xi_b)) = d(e, F(b_1^{-1} \ldots b_i^{-1}\xi^-, \xi_{Tb})) \quad (3)$$

where $T : B \to B, b = (b_i)_{i \geq 1} \mapsto (b_{i+1})_{i \geq 1}$ is the one-sided shift, motivates the following.

Let $\tilde{\mu}$ be the image of $\mu$ under the inversion map $g \mapsto g^{-1}$, and $\nu_{\beta^-}$-the $\tilde{\mu}$-stationary probability measure on $P^-$. Define $T^+ : B \times \mathcal{P}^- \to B \times \mathcal{P}^-, (b, \xi^-) \mapsto (Tb, b_1^{-1}\xi^-)$. The ergodicity of the $\tilde{\mu}$-walk on $\mathcal{P}^-$ is a consequence of [8] Proposition 4.7] and means that the dynamical system $(B \times \mathcal{P}^-, \beta \otimes \nu_{\beta^-}, T^+)$ is measure-preserving and ergodic [8] Proposition 2.14]. Define (almost everywhere) a function $f : B \times \mathcal{P}^- \to [0, +\infty]$ setting

$$f(b, \xi^-) = \begin{cases} 1 & \text{if } d(e, F(\xi^-, \xi_b)) \leq R \\ 0 & \text{otherwise} \end{cases}$$

Notice that $f$ is measurable\(^1\) and that we may choose $R > 0$ large enough so that $\beta \otimes \nu_{\beta^-}(f) > 1 - \varepsilon$. In this case, Birkhoff Ergodic Theorem implies that

$$\frac{1}{n} \sum_{i=1}^{n} f \circ (T^+)^i(b, \xi^-) > 1 - \varepsilon$$

\(^1\)To check this, observe that the map \(\phi : \mathcal{P}^+ \to \mathbb{R}^+, (g_{\xi_0}^{-}, g_{\xi_0}) \mapsto d(e, F(g_{\xi_0}^{-}, g_{\xi_0}) = d(g^{-1}, F(\xi_{\xi_0}^{-}, \xi_0))\) is continuous, hence its extension to $\mathcal{P}^+ \times \mathcal{P}$ by setting \(\phi = +\infty\) on the (closed) complement $\mathcal{P}^- \times \mathcal{P} \times \mathcal{P}^+$ is measurable. Now the measurability of $f$ follows from the measurability of $B \to \mathcal{P}, b \mapsto \xi_b$. 

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which can be rewritten as
\[ \frac{1}{n} \sum_{i \in [1, n]} d(b_i \ldots b_1, F(\xi^-, \xi_b)) \leq R > 1 - \varepsilon \]
This concludes the proof. \( \square \)

3 Recurrence criterion

The goal of this section is to prove our second theorem announced in the introduction.

**Theorem 3.** Let \( G \) be a connected simple real Lie group of rank one, \( \Lambda \subseteq G \) a discrete subgroup, \( X = \Lambda \backslash G \). Let \( \mu \) be a probability measure on \( G \) with a finite first moment and \( \Gamma_\mu \) Zariski-dense in \( G \).

Then the \( \mu \)-walk and the geodesic flow on \( X \) are either both recurrent ergodic, or both transient with locally integrable Green functions.

We will use freely the notations of \( \mathbb{2.1} \) and always be in the setting of Theorem 3. In particular, \( \mathfrak{a} \) denotes a Cartan subspace of dimension 1, that we will identify with \( \mathbb{R} \) via the linear isomorphism sending \( 1 \in \mathbb{R} \) to the element \( v_0 \in \mathfrak{a}^+ \) of norm 1 for the Killing form. In this regard, for \( t \in \mathbb{R} \), we have by definition \( a_t = \exp(tv_0) \in G \).

As we shall explain in Section 4, the dichotomy presented in Theorem 3 is already known for the geodesic flow:

**Fact 2.** The geodesic flow \( (a_t)_{t \in \mathbb{R}} \) on \( X \) is either recurrent ergodic, or transient with locally integrable Green functions.

Hence Theorem 3 is equivalent to the following propositions that we will prove independently in the next sections.

**Proposition 3.1.** If the geodesic flow on \( X \) is recurrent and ergodic, then it is also the case of the \( \mu \)-walk on \( X \).

Denote \( \mathcal{P}(X) \) the collection of subsets \( F \subseteq X \) such that \( F \) is relatively compact and has positive Haar measure.

**Proposition 3.2.** If the Green functions of the geodesic flow \( G(\cdot, F)_{F \in \mathcal{P}(X)} \) are locally integrable, then it is also the case of the Green functions \( G_\mu(\cdot, F)_{F \in \mathcal{P}(X)} \) of the \( \mu \)-walk on \( X \).

For the proofs to come, it will be useful to embed \( G \) in a linear group \( SL(V_0) \) via a faithful irreducible proximal algebraic representation. \( V_0 \) will be endowed with a \( K \)-invariant scalar product \( \langle \cdot, \cdot \rangle_0 \) such that \( \mathfrak{a} \) is self-adjoint \( \mathbb{\cite{Lemma 6.33}} \).
3.1 Renewal theory

It happens that both Propositions 3.1 and 3.2 rely on renewal results for the Cartan projection of the right random walk on $G$. The role of this section is to state and prove these results.

We first give some context. A renewal theorem considers a transient random walk and estimates the average time spent in a given bounded subset when the latter degenerates. The standard case of a non-arithmetic walk on $\mathbb{R}$ can be found in [25]. It was generalized by Kesten in [24] to the Iwasawa cocycle for linear random walks.

**Theorem** (Renewal Theorem for the Iwasawa cocycle, [24]). Let $d \geq 2$ and $m$ be a probability measure on $SL_d(\mathbb{R})$ with a finite first moment and such that $\Gamma_m := \langle \text{supp} m \rangle$ is strongly irreducible and unbounded. Denote by $\lambda_m > 0$ the first Lyapunov exponent of $m$ and by $(S_n)_{n \geq 0}$ the left $\mu$-random walk on $SL_d(\mathbb{R})$ starting at $Id$. Then, for any interval $I \subseteq \mathbb{R}$ and vector $v \in \mathbb{R}^d \setminus \{0\}$,

$$E(\sharp \{n \geq 0, \log ||S_n v|| \in I + t\}) \to \frac{\text{leb}(I)}{\lambda_m}$$

**Remark.** A more precise statement is proven in [17], and the speed of convergence is estimated in [27] under the assumption that $m$ has an exponential moment.

Now consider the $\mu$-walk on our rank one simple Lie group $G$. The assumption that $\mu$ has a finite first moment implies that for $\beta$-almost every $b \in B$,

$$t_n(b) \to_{n \to +\infty} n \lambda_{\mu}$$

where $\lambda_{\mu} > 0$ is the first Lyapunov exponent of $\mu$ (see [3]). In view of the above renewal theorem, it is natural to conjecture the following renewal statement for the Cartan projection:

$$E_{\beta}(\sharp \{n \geq 0, t_n \in I + t\}) \to_{n \to +\infty} \frac{\text{leb}(I)}{\lambda_{\mu}}$$

(4)

This is known to be true if $\mu$ has a finite exponential moment [27] but the case where $\mu$ has only a finite first moment is still open. We prove two propositions (3.3,3.5) that can be seen as first steps to show the convergence (4).

The first proposition guarantees that for any point $x \in X$ which has a recurrent orbit $(x_{\alpha t})_{t \geq 0}$ under the geodesic flow, the sequence $(x_{\alpha t_n(b)})_{n \geq 0}$ is also recurrent for $\beta$-almost every $b \in B$ (see Lemma 3.9).

**Proposition 3.3.** Let $I \subseteq \mathbb{R}$ be a large enough bounded interval. For any subset $S \subseteq \mathbb{R}_+$ containing arbitrary large real numbers, for $\beta$-almost every $b \in B$,

$$\sharp \{n \geq 0, t_n(b) \in I + S\} = +\infty$$

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The proof Proposition 3.3 relies on Lemma 3.4 according to which the probability that the Cartan projection of a \( \mu \)-trajectory \((gb_1 \ldots b_n)\) meets the translate \( I + s \) is close to 1 as long as \( s \) is large enough. Recall from 2.1 that \( t_g \geq 0 \) denotes the Cartan projection of an element \( g \in G \).

**Lemma 3.4.** Let \( \varepsilon > 0 \) and \( I \subseteq \mathbb{R} \) a large enough bounded interval. Then for every \( g \in G \),

\[
\liminf_{s \to +\infty} \beta\{b \in B, \exists n \geq 0, t_{gb_1 \ldots b_n} \in I + s\} > 1 - \varepsilon
\]

**Proof of Lemma 3.4.** We use the representation \( G \subseteq SL(V_0) \) introduced earlier in Section 3. Our assumptions on the scalar product \( \langle ., . \rangle_0 \) of \( V_0 \) imply that the product \( gb_1 \ldots b_n \) and its adjoint

\[
S_n(b, g) = t' b_n \ldots t' b_1 t' g
\]

have the same Cartan projection \( t_{gb_1 \ldots b_n} \). Hence the norm of \( S_n(b, g) \) seen as an operator on \( V_0 \) is of the form \( | | S_n(b, g) || = e^{c_0 t_{gb_1 \ldots b_n}} \) where \( c_0 = \log | | a_1 | | > 0 \). Lemma 3.4 can then be restated as : for every \( \varepsilon > 0 \), there exists \( I \subseteq \mathbb{R} \) bounded interval such that for all \( g \in G \),

\[
\liminf_{s \to +\infty} \beta\{b \in B, \exists n \geq 0, \log | | S_n(b, g) || \in I + s\} > 1 - \varepsilon
\]

We know by [24 (1.17)] this statement is true for the Iwasawa cocycle : there exists a bounded interval \( J \) such that for all \( g \in G, v \in V \setminus \{0\} \), for all \( s > s_{g,v}, \)

\[
\beta\{b \in B, \exists n \geq 0, \log | | S_n(b, g) v \rangle \in J + s\} > 1 - \varepsilon/2
\]

Moreover, arguing by contradiction, we can infer from [8, Corollary 4.8] that the difference between the Iwasawa cocycle and the Cartan projection is ultimately bounded : there exists constants \( R > 0 \) such that for all unit vector \( v \in V_0 \)

\[
\beta\{b \in B, \forall n \geq 0, \log | | t' b_n \ldots t' b_1 || - \log | | t' b_n \ldots t' b_1 v \rangle | | < R \} > 1 - \varepsilon/2
\]

In particular, choosing for each \( g \in G \) a unit vector \( v_g \in V_0 \) such that \( | | t' gv_g \rangle | | \geq \frac{1}{2} | | g \rangle | | \), and setting \( R' = R + \log 2, \)

\[
\beta\{b \in B, \forall n \geq 0, \log | | S_n(b, g) \rangle \rangle - \log | | S_n(b, g)v_g \rangle \rangle < R' \} > 1 - \varepsilon/2
\]

Consider now an interval \( I \) that contains the \( R' \)-neighborhood of \( J \). Then using (5) and (7), for all \( g \in G, s > s_{g,v_g}, \)

\[
\beta\{b \in B, \exists n \geq 0, | | S_n(b, g) \rangle \rangle \in I + s\}
\]

\[
\geq \beta\{b \in B, \exists n \geq 0, \log | | S_n(b, g)v_g \rangle \rangle \in J + s
\]

and \( \log | | S_n(b, g) \rangle \rangle - \log | | S_n(b, g)v_g \rangle \rangle < R' \}

\[
\geq 1 - \varepsilon
\]
We have finally obtained (5), hence the lemma.

**Proof of Proposition 3.3.** Let $\varepsilon \in ]0, 1[$ and $I \subseteq \mathbb{R}$ as in Lemma 3.4. Let $S \subseteq \mathbb{R}_+$ be a subset containing arbitrarily large real numbers. Define by induction a family of stopping times $(n_k : B \to \mathbb{N})_{k \in \mathbb{N}}$ as follows:

- $n_0 = 0$
- Applying Lemma 3.4 with $g = b_1 \ldots b_{n_k(b)}$, choose $n_{k+1}(b) > n_k(b)$ for which
  $$\beta\{a \in B, \exists n \in \mathbb{N} \ni n_k(b), n_{k+1}(b) \}, t_{b_1 \ldots b_{n_k(b)}a_{n_k(b)+1} \ldots a_n} \in I + S\} > 1 - \varepsilon$$
  and such that $n_{k+1}$ is a measurable function of the product $b_1 \ldots b_{n_k(b)}$.

Now observe that if $1 \leq k_1 < \cdots < k_N$ are distinct integers, then by the Markov property

$$\beta\{b \in B, \forall i \in ]1, N], \forall n \in \mathbb{N} \ni n_{k_i-1}, n_{k_i} \}, t_n(b) \notin I + S\} \leq (1 - \varepsilon)^N$$

Hence, given any infinite sequence of integers $1 \leq k_1 < \cdots < k_i < \ldots,$

$$\beta\{b \in B, \forall i \geq 1, \forall n \in \mathbb{N} \ni n_{k_i-1}, n_{k_i} \}, t_n(b) \notin I + S\} = 0$$

This equality implies the statement of the lemma.

The second proposition will be used to compare the Green functions of the geodesic flow and of the $\mu$-walk on $X$. It states that the average time spent by the Cartan projection $(t_n(b))$ of a trajectory $(b_1 \ldots b_n)$ in a given interval $I$ of $\mathbb{R}$ is bounded by a constant that only depends on $\text{leb}(I)$.

**Proposition 3.5.** Let $I \subseteq \mathbb{R}$ be a bounded interval. Then

$$\sup_{t \in \mathbb{R}} \mathbb{E}_\beta(\#\{n \geq 0, t_n \in I + t\}) < \infty$$

The idea is that the Cartan projection of a trajectory $(b_1 \ldots b_n)_{n \geq 0}$ has a low probability to come back to an interval once it has gone past it. We formalize this in Lemma 3.6.

**Lemma 3.6.** Let $R, \varepsilon > 0$. There exists $n_0 \geq 0$ such that for every $g \in G$,

$$\beta\{b \in B, \forall n \geq n_0, t_{gb_1 \ldots b_n} \geq t_g + R\} \geq 1 - \varepsilon$$

**Proof of Lemma 3.6.** We use again the representation $G \subseteq SL(V_0)$ introduced earlier in the section. Our assumptions on the scalar product $\langle ., . \rangle_0$ of $V_0$
imply that the product $gb_1 \ldots b_n$ and its adjoint $S_n(b, g) = \iota b_n \ldots \iota b_1 \iota g$ have the same Cartan projection $t_{gb_1 \ldots b_n}$. Hence

$$||S_n(b, g)|| = e^{c_0 t_{gb_1 \ldots b_n}}$$

where $c_0 = \log ||a_1|| > 0$. Lemma 3.6 can then be rephrased as: for any $C, \varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $g \in G$,

$$\beta(b \in B, \forall n \geq n_0, ||S_n(b, g)|| \geq C ||\iota g||) \geq 1 - \varepsilon \quad (8)$$

To prove (8), assume by contradiction there exist $C, \varepsilon > 0$, a sequence of integers $p_n \to \infty$ and elements $p_g \in G$ such that for all $k \geq 0$,

$$\beta(b \in B, \exists n \geq N_k, ||S_n(b, g_k)|| < C ||\iota g_k||) \geq \varepsilon \quad (9)$$

Up to extraction, one may also suppose that the normalized sequence $(f_k ||g_k||)$ converges to an endomorphism $f_\infty \in \text{End}(V_0)$. By (9), there exists a set $B' \subseteq B$ of measure at least $\varepsilon$ such that for every $b \in B'$, there are sequences of integers $(k_i), (n_i)$ going to infinity and satisfying

$$||S_{n_i}(b, g_k)|| < C ||\iota g_k||$$

leading to

$$||\iota b_{n_i} \ldots \iota b_1 f_\infty|| < ||\iota b_{n_i} \ldots \iota b_1 (f_\infty - \frac{\iota g_k}{||g_k||})|| + C = o(||\iota b_{n_i} \ldots \iota b_1||)$$

where the last equality is true for almost every $b$. But this yields a contradiction with [8, Corollary 4.8]. Hence we have (8), and the lemma follows.

We can now conclude the section with the proof of Proposition 3.5.

**Proof of Proposition 3.5.** Denote by $N_I : B \to \mathbb{N} \cup \{\infty\}, b \mapsto \sharp\{n \geq 0, t_n(b) \in I\}$ the function that counts the time spent in $I$ for the Cartan projection of a $\mu$-trajectory on $G$. We want to bound above the expectation of $N_I$. Let $R > \text{leb}(I)$, $\varepsilon \in ]0, 1[$, and $n_0 \geq 0$ as in Lemma 3.6. We are going to show that for all $k \geq 0$,

$$\beta(b \in B, N_I(b) \geq kn_0 + 1) \leq \varepsilon^k \quad (10)$$

Once (10) is established, it is easy to conclude:

$$\int_B N_I d\beta = \sum_{n \geq 1} \beta(N_I \geq n) \leq \sum_{k \geq 0} n_0 \beta(N_I \geq kn_0 + 1) \leq n_0 \frac{1}{1 - \varepsilon}$$

and the constant $n_0 \frac{1}{1 - \varepsilon}$ depends on $I$ only via the choice of $R$ which is solely bound to satisfy $R > \text{leb}(I)$.
Let us now prove (10). We introduce a sequence of stopping times \((\tau_i : B \to \mathbb{N} \cup \{\infty\})_{i \geq 1}\) indicating the first hitting time of the interval \(I\) by the sequence \((t_n(b))_{n \geq 0}\), then its successive return times separated by at least \(n_0\) steps.

\[
\tau_1 = \inf\{n \geq 0, t_n \in I\}, \quad \tau_{i+1} = \inf\{n \geq \tau_i + n_0, t_n \in I\}
\]

Observe that

\[
\beta\{b \in B, N_I(b) \geq kn_0 + 1\} \leq \beta\{b \in B, \tau_{k+1}(b) < \infty\} = \sum_{j \in \mathbb{N}^k} \beta\{b \in B, (\tau_i(b))_{i \in k} = j, \tau_{k+1}(b) < \infty\}
\]

The inequality \(R > \text{leb}(I)\) yields for every \(j = (j_1, \ldots, j_k) \in \mathbb{N}^k\) the inclusion

\[
\{b \in B, (\tau_i(b))_{i \in k} = j, \tau_{k+1}(b) < \infty\} \subseteq \{b \in B, (\tau_i(b))_{i \in k} = j, \exists m \geq j_k + n_0, t_m(b) < t_{j_k}(b) + R\}
\]

Using the Markov property and Lemma 3.6, we infer that

\[
\beta\{b \in B, (\tau_i(b))_{i \in k} = j, \tau_{k+1}(b) < \infty\} \leq \beta\{b \in B, (\tau_i(b))_{i \in k} = j\} \epsilon
\]

Summing over every \(j \in \mathbb{N}^k\), and iterating the process, we obtain

\[
\beta\{b \in B, \tau_{k+1}(b) < \infty\} \leq \beta\{b \in B, \tau_k(b) < \infty\} \epsilon \leq \cdots \leq \epsilon^k
\]

Hence, as announced,

\[
\beta\{b \in B, N_I(b) \geq kn_0 + 1\} \leq \epsilon^k
\]

\[\square\]

### 3.2 Recurrence and ergodicity

In this section we prove Proposition 3.1: we assume the geodesic flow \((a_t)_{t \in \mathbb{R}}\) on \(X\) to be recurrent ergodic and show that the \(\mu\)-random walk on \(X\) is recurrent ergodic as well.

#### 3.2.1 Recurrence

We begin with the recurrence of the walk.

**Lemma 3.7.** *The \(\mu\)-walk on \(X\) is recurrent*

Consider a large compact set \(\tilde{L} \subseteq X\). We aim to show that for almost every \(x \in X\), and \(\beta\)-almost every \(b \in B\), there exist infinitely many times \(n \geq 0\) such that

\[
xb_1 \ldots b_n \in \tilde{L}
\]
Endow $G$ with a left invariant Riemannian metric, $X$ with the quotient metric, denote by $\xi^0 = P^-/P^-$ the base point of the flag variety $\mathcal{P}^-$. According to Corollary 2.6, there exists a constant $C_0 > 0$ such that for $\beta$-almost every $b \in B$, large enough $n \geq 0$,

$$d(b_1 \ldots b_n, k_\mathcal{P}^- (b)a_{t_n(b)}) \leq C_0 d(b_1 \ldots b_n, F(\xi^0_-, \xi_b)) + C_0$$

Fix a compact subset $L \subseteq X$ and a constant $R > 0$ (to be specified below), and assume $\tilde{L}$ contains the $C_0(R + 1)$-neighborhood of $L$. In this case, we just need to show that for almost every $x \in X$, $\beta$-almost every $b \in B$, there exists infinitely many times $n \geq 0$ such that

$$xk_\mathcal{P}^- (b)a_{t_n(b)} \in L \quad \text{and} \quad d(b_1 \ldots b_n, F(\xi^0_-, \xi_b)) \leq R \quad (11)$$

The difficulty is that the set of return times in $L$ given by

$$\{n \geq 0, xk_\mathcal{P}^- (b)a_{t_n(b)} \in L\}$$

has null density in $\mathbb{N}$, hence we can not say directly that it intersects

$$\{n \geq 0, d(b_1 \ldots b_n, F(\xi^0_-, \xi_b)) \leq R\}$$

even if the latter has a density close to one (by the proof of Theorem A).

A first important observation is that we can ignore the term $k_\mathcal{P}^- (b)$. More precisely, using Fubini’s Theorem and equation 3, the statement (11), hence Lemma 3.7, reduces to the following.

**Lemma 3.8.** We can choose the parameters $(L, R)$ such that for almost every $x \in X$, $\beta$-almost every $b \in B$, infinitely many times $n \geq 0$,

$$xa_{t_n(b)} \in L \quad \text{and} \quad d(e, F(b^{-1}_n \ldots b^{-1}_1 \xi^0_-, \xi_{T^nb})) \leq R$$

This reduction is crucial because it separates the effects of the $n$ first instructions $(b_1, \ldots, b_n)$ and of the tail $T^nb$, thus allowing to argue conditionally to the situation at time $n$. To show Lemma 3.8 we use the following strategy: Prove that for almost every $(x, b)$, the sequence $(xa_{t_n(b)})_{n \geq 0}$ meets $L$ infinitely often (Lemma 3.9). Show that among those $n$, infinitely many satisfy

$$d(e, F(b^{-1}_n \ldots b^{-1}_1 \xi^0_-, \xi_{T^nb})) \leq R$$

To obtain the latter, we justify in Lemma 3.10 that we may replace $\xi_{T^nb}$ by the term $b_{n+1} \ldots b_{n+k_n} \xi_n$ where $k_n \geq 0$ is a large integer, and $\xi_n$ is a random point on the flag variety $\mathcal{P}$, then we use the Markov property to conclude (together with Lemma 3.11).

Let us begin with the statements and proofs of the three lemmas advertised at the moment.
Lemma 3.9. We can choose the compact set \( L \subseteq X \) such that for almost every \( x \in X \), \( \beta \)-almost every \( b \in B \), there exists infinitely many times \( n \geq 0 \) for which

\[
xa_{t_n(b)} \in L
\]

Proof. Let us first specify the compact set \( L \). According to Proposition 3.3, there exists a constant \( c > 0 \) such that if \( I = (0, c) \) and \( (s_k) \to +\infty \) then for \( \beta \)-almost every \( b \in B \),

\[
\mathbb{P}\{n \geq 0, t_n(b) \in \bigcup_{k \geq 0} I + s_k\} = +\infty \quad (12)
\]

Set \( c' = \max_{|t| \leq c} d(e, a_t) \) where \( d \) refers to the metric on \( G \). Fix some point \( y_0 \in X \) and set \( L = \{y \in X, d(y_0, y) \leq c' + 1\} \) the set of vectors \( y \in X \) whose distance (in \( X \)) to \( y_0 \) is less than \( c' + 1 \).

Let \( E \subseteq X \) be the set of elements \( x \in X \) such that there exists a sequence of real numbers \( (s_k) \to +\infty \) for which \( xa_{s_k} \to y_0 \). The assumption that the geodesic flow on \( X \) is recurrent ergodic implies that \( E \) has full measure in \( X \).

Let \( x \in E \). Then the set

\[
\{t > 0, xa_t \in L\}
\]

contains a subset of the form \( \bigcup_{k \geq 0} I + s_k' \) where \( (s_k') \to +\infty \). Hence, by (12), for \( \beta \)-almost every \( b \in B \), the sequence \( (xa_{t_n(b)}) \) meets \( L \) infinitely often.

Lemma 3.10. Let \( \delta > 0 \). There exists a sequence of integers \( (k_n) \in \mathbb{N}^{\mathbb{N}} \) such that for \( \beta \)-almost-every \( b \in B \) and \( \nu^{\mathbb{N}}_{\mathbb{P}} \)-almost every \( (\xi_n) \in \mathcal{P}^{\mathbb{N}} \), for large enough \( n \geq 0 \),

\[
d(b_{n+1} \ldots b_n \xi_n, \xi_{T^b}) \leq \delta
\]

Proof. We have for \( \beta \)-almost every \( b \in B \),

\[
(b_1 \ldots b_n) \nu_{\mathbb{P}} \xrightarrow{n \to +\infty} \nu_{\mathbb{P}, b} = \delta_{\xi_b}
\]

hence

\[
\nu_{\mathbb{P}}\{\xi \in \mathcal{P}, d(b_1 \ldots b_n \xi, \xi_b) > \delta\} \xrightarrow{n \to +\infty} 0
\]

Integrating in \( b \in B \), we obtain

\[
\beta \otimes \nu_{\mathbb{P}}\{(b, \xi) \in \mathcal{P}, d(b_1 \ldots b_n \xi, \xi_b) > \delta\} \xrightarrow{n \to +\infty} 0
\]

Extracting a subsequence whose sum is finite, we obtain \( (k_n) \in \mathbb{N}^{\mathbb{N}} \) such that

\[
\sum_{n \geq 0} \beta \otimes \nu_{\mathbb{P}}^{\mathbb{N}}\{(b, \xi) \in B \times \mathcal{P}^{\mathbb{N}}, d(b_1 \ldots b_{k_n} \xi_n, \xi_b) > \delta\} < \infty
\]

The observation that \( d(b_1 \ldots b_{k_n} \xi_n, \xi_b) \) and \( d(b_{n+1} \ldots b_{n+k_n} \xi_n, \xi_{T^b}) \) have the same law, combined with Borel-Cantelli Lemma, lead to the statement in Lemma 3.10.
Lemma 3.11. There exists a constant $R' > 0$ such that for every $\xi^- \in \mathcal{P}^-$,
\[ \nu\{\xi \in \mathcal{P}, d(e, F(\xi^-, \xi)) \leq R'\} > 2/3 \]

Proof. We know from Lemma 2.4 that for every $\xi^- \in \mathcal{P}^-$, there exists a constant $R_{\xi^-} > 0$ such that
\[ \nu\{\xi \in \mathcal{P}, d(e, F(\xi^-, \xi)) \leq R_{\xi^-}\} > 2/3 \quad (13) \]
We need to show that $R_{\xi^-}$ may be chosen independently of $\xi^-$. To see this, notice that the function $F_{\xi^-} \to [0, +\infty]$, $(\xi^-, \xi) \mapsto d(e, F(\xi^-, \xi))$ is continuous and proper, as it can be identified with the quotient map $G/Z_G(a) \to [0, +\infty], gZ_G(a) \mapsto d(g^{-1}, \exp(a)K)$. In particular for all $C > 0$, Heine Theorem gives a constant $\delta > 0$ such that for all $(\xi^-, \xi), (\eta^-, \eta) \in \mathcal{P}^+$ with $d(\xi^-, \eta^-) \leq \delta$, $d(\xi, \eta) \leq \delta$,
\[ d(e, F(\xi^-, \xi)) \leq C \implies d(e, F(\eta^-, \eta)) \leq C + 1 \quad (14) \]
(13) and (14) together imply that the constant $R_{\xi^-}$ can be chosen uniformly on a neighborhood of $\xi^-$. The compactness of $\mathcal{P}^+$ then leads to a uniform constant $R' > 0$ as in the statement of the lemma.

We now prove Lemma 3.8.

Proof of Lemma 3.8. We first make preparations to replace later the term $\xi_{T^n b}$ by $b_{n+1} \ldots b_{n+k_n} \xi_n$ where $k_n \geq 0$ is a large integer and $\xi_n$ is a random point on the flag variety $\mathcal{P}$. Let $R' > 0$ as in Lemma 3.11 set $R = R' + 1$.
As we saw in the proof of Lemma 3.11 there exists a constant $\delta > 0$ such that for all $(\xi^-, \xi') \in \mathcal{P}^+$, all $\xi \in \mathcal{P}$ with $d(\xi', \xi) \leq \delta$, one has
\[ d(e, F(\xi^-, \xi')) \leq R' \implies d(e, F(\xi^-, \xi)) \leq R \]
Choose a sequence $(k_n) \in \mathbb{N}^{\mathbb{N}^*}$ as in Lemma 3.10.

We now proceed to the proof. Let $L \subseteq X$ be as in Lemma 3.9 and fix a vector $x \in X$ such that for almost every $b \in B$, the set $\mathcal{N}_{x,b} := \{ n \geq 0, x_{t_n(b)} \in L \}$ has infinite cardinal. Define by induction a sequence of stopping times $\tau_i : B \to \mathbb{N} \cup \{\infty\}$ setting
\[ \begin{align*}
\tau_1(b) &:= \inf\{ n \geq 0, n \in \mathcal{N}_{x,b} \} \\
\tau_{i+1}(b) &:= \inf\{ n \geq \tau_i(b) + k_{\tau_i(b)} + 1, n \in \mathcal{N}_{x,b} \}
\end{align*} \]
Given some integers \( i_1 > i_0 \geq 0 \), one has by the Markov property and Lemma 3.11

\[
\left( \frac{1}{3} \right)^{i_1-i_0+1} \geq \beta \otimes \nu^{\otimes \mathbb{N}} \{(b, (\xi_i)) \in B \times \mathcal{P}, \ \forall i \in [i_0, i_1] \},
\]

\[d(e, F(b_{r_1(b)}^{-1} \ldots b_{r_1(b)+1}^{-1}\xi_0, b_{r_1(b)+1} \ldots b_{r_1(b)+k_{r_1(b)}}(\xi_i)) > R')\]

Letting \( i_1 \) go to \( +\infty \), we deduce that for \( \beta \)-almost every \( b \in B \), there exists \( i \geq i_0 \) such that

\[d(e, F(b_{r_1(b)}^{-1} \ldots b_{r_1(b)+1}^{-1}\xi_0, b_{r_1(b)+1} \ldots b_{r_1(b)+k_{r_1(b)}}(\xi_i)) \leq R'\]

As \( i_0 \) can be chosen arbitrarily large, we obtain that for almost-every \( b \in B \), almost every \( (\xi_i) \in \mathcal{P}^{\mathbb{N}} \), there exists infinitely many integers \( i \geq 0 \) such that

\[d(e, F(b_{r_1(b)}^{-1} \ldots b_{r_1(b)+1}^{-1}\xi_0, b_{r_1(b)+1} \ldots b_{r_1(b)+k_{r_1(b)}}(\xi_i)) \leq R' \]  \( (15) \)

But our choice of \( (k_{r_1}) \) guarantees that for large enough \( i \geq 0 \),

\[d(b_{r_1(b)+1} \ldots b_{r_1(b)+k_{r_1(b)}}(\xi_i, \xi_{T^{r_1(b)}})) \leq \delta \]  \( (16) \)

By \( (15), (16) \) and the definition of \( \delta \), we can conclude: for almost-every \( b \in B \), there exists infinitely many integers \( i \geq 0 \) such that

\[d(e, F(b_{r_1(b)}^{-1} \ldots b_{r_1(b)+1}^{-1}\xi_0, \xi_{T^{r_1(b)}}) \leq R\]

This finishes the proof of Lemma 3.8, yielding Lemma 3.7.

\[\Box\]

3.2.2 Ergodicity

We now prove the ergodicity of the \( \mu \)-walk on \( X \).

**Lemma 3.12.** The \( \mu \)-walk on \( X \) is ergodic.

The key idea is that the subgroup \( \Gamma_\mu \) generated by the support of \( \mu \) must contain loxodromic elements, whose action on \( X \) is (almost) conjugate to the geodesic flow, hence ergodic. Recall that an element \( g_0 \in G \) is loxodromic if it can be written, up to conjugation, as \( g_0 = ma_c \) where \( m \in K \), \( c > 0 \), and \( ma_c = a_c m \) (see also [8, Section 6.10]).

**Lemma 3.13.** The action of a loxodromic element \( g_0 \) on \( X \) is conservative ergodic for the Haar measure.

In this statement the conservativity of \( g_0 \) means that for almost-every point \( x \in X \), the sequence \( (xg_0^n)_{n \geq 0} \) subconverges to \( x \) (see [1] Section 1.1 for more details).
Proof of Lemma 3.13. One can assume that \( g_0 = ma_c \) where \( m \in K, \ c > 0, \) and \( ma_c = a_c m. \) In particular, the recurrence of the geodesic flow on \( X \) implies the conservativity of \( g_0. \) Its ergodicity follows by standard arguments (given for the geodesic flow in [1, Theorem 7.4.3]). We explain them briefly. Denote by \( \lambda \) a Haar measure on \( X, \) let \( f, p \in L^1(X, \lambda) \) with \( p > 0, \) \( \lambda(p) = 1. \) Hopf Ergodic Theorem [1, 2.2.5] and the conservativity of \( g_0 \) imply the almost-sure convergence:

\[
\frac{\sum_{k=0}^{n-1} f(g_0^k)}{\sum_{k=0}^{n-1} p(g_0^k)} \xrightarrow{n \to \infty} \mathbb{E}_{p\lambda}\left(\frac{f}{p}\right)_{\mathcal{I}}
\]

where \( \mathbb{E}_{p\lambda}(\frac{f}{p}|\mathcal{I}) \) is the conditional expectation of \( f/p \) for the probability measure \( p\lambda \) and with respect to the \( \sigma \)-algebra \( \mathcal{I} \) of the \( \lambda \)-a.e. \( g_0 \)-invariant subsets of \( X. \) We need to show this \( \sigma \)-algebra is \( \lambda \)-trivial, which amounts to say that for every choice of \( f, p, \) the limit \( \Phi_{f,p} \) is \( \lambda \)-a.e. constant. Endow \( G \) with a Riemannian metric that is \( G \)-left invariant and \( K \)-right invariant, and equip \( X \) with the quotient metric. Arguing as in [1, 7.4.3] we can assume that \( p, \) then \( f, \) are regular enough so that \( \Phi_{f,p} \) is constant along the stable or unstable manifolds of \( g_0. \) More precisely, denote by \( U \subseteq G \) (resp. \( U^- \)) unipotent connected subgroup of \( G \) whose Lie algebra is \( u \) (resp. \( u^- \)). Then for \( x \in X, \ u \in U, \)

\[
d(xu g_0^k, x' g_0^k) = d(xua_c^k, x'a_c^k) \xrightarrow{k \to \infty} 0
\]

and the same goes for \( U^- \) and \( k \to -\infty. \) By our choice of \( p \) and \( f, \) this yields for every \( u \in U \cup U^- \) the almost-sure equality

\[
\Phi_{f,p}(u) = \Phi_{f,p} \quad (\lambda \text{-a.e.})
\]

As \( U \) and \( U^- \) together generate \( G, \) the \( \lambda \)-a.e. invariance of \( \Phi_{f,p} \) by \( U \) and \( U^- \) implies its \( \lambda \)-a.e. invariance by a countable dense subset of \( G, \) hence by \( G. \) The map \( \Phi_{f,p} \) is then \( \lambda \)-a.e. constant. \( \square \)

Proof of Lemma 3.12. The subgroup \( \Gamma_\mu \) generated by the support of \( \mu \) is Zariski-dense in \( G, \) so it must contain some loxodromic element \( g_0 \) (see [8, Prop. 6.11]). By Lemma 3.13, the action of \( g_0 \) on \( X \) is ergodic, hence so is the action of \( \Gamma_\mu. \) This proves the ergodicity of the \( \mu \)-walk on \( X. \) \( \square \)
3.3 Transience

In this section we prove Proposition 3.2:

**Proposition 3.2.** If the Green functions of the geodesic flow $G(., F)_{F \in \mathcal{P}(X)}$ are all locally integrable, then it is also the case of the Green functions $G_\mu(., F)_{F \in \mathcal{P}(X)}$ of the $\mu$-walk on $X$.

**Proof.** Let $E, F \subseteq X$ be compact $K$-invariant subsets of $X$ and $\lambda$ a Haar measure on $X$. We can write

$$
\int_E G_\mu(x, F) \, d\lambda(x) = \int_E \int_{B_{n \geq 0}} 1_F(xb_1 \ldots b_n) \, d\beta(b) \, d\lambda(x)
$$

$$
= \int_E \int_{B_{n \geq 0}} 1_F(xa_{t_n(b)}) \, d\beta(b) \, d\lambda(x) \quad (17)
$$

where the last inequality comes from the $K$-invariance of $E, F$ and $\lambda$.

Let $F' \subseteq X$ be a compact set such that $\bigcup_{t \in [0, 1]} F a_t \subseteq F'$. Then

$$
1_F(xa_{t_n(b)}) \leq \int_{\mathbb{R}^+} 1_{F'}(xa_t) 1_{[t_n(b), t_n(b)+1]}(t) \, dt \quad (18)
$$

Combining (17) and (18), we obtain

$$
\int_E G_\mu(x, F) \, d\lambda(x) \leq \int_E \int_{\mathbb{R}^+} \int_{B_{n \geq 0}} 1_{F'}(xa_t) 1_{[t_n(b), t_n(b)+1]}(t) \, dt \, d\beta(b) \, d\lambda(x)
$$

$$
= \int_E \int_{\mathbb{R}^+} 1_{F'}(xa_t) \left[ \int_{B_{n \geq 0}} 1_{[t_n(b), t_n(b)+1]}(t) \, d\beta(b) \right] \, d\lambda(x)
$$

The term between brackets estimates the average time spent by the Cartan projection of a $\mu$-trajectory on $G$ in the interval $[t-1, t]$. By Proposition 3.5 it is less than a constant $R \in [0, +\infty[$ that does not depend on $t$ but only on the initial data $(G, K, a^+, \mu)$. Finally, we get

$$
\int_E G_\mu(x, F) \, d\lambda(x) \leq R \int_{\mathbb{R}^+} 1_{F'}(xa_t) \, dt \, d\lambda(x)
$$

$$
= R \int_E G(x, F') \, d\lambda(x)
$$

$$
< +\infty
$$

$\square$

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4 Appendix: Hopf dichotomy for the geodesic flow

In this appendix, we justify the following fact used in Section 3. The notations are those of Section 3. In particular $G$ is a connected simple real Lie group of rank one, $\Lambda \subseteq G$ is a discrete subgroup, and $X = \Lambda \backslash G$.

**Fact 2** The geodesic flow $(a_t)_{t \in \mathbb{R}}$ on $X$ is either recurrent ergodic, or transient with locally integrable Green functions.

This result is already known but usually stated differently using the notion of Poincaré series, as in Theorem 4.1 below. We explain here why Fact 2 is a reformulation of Theorem 4.1. The point is that the Poincaré series of $\Lambda$ at the maximal exponent expresses, up to a multiplicative constant, the integral of the Green function of the geodesic flow on a $K$-orbit in $X$ (Lemma 4.3).

Recall first the notion of Poincaré series. Endow $G/K$ with its standard structure of symmetric space, i.e. with its unique left $G$-invariant Riemannian metric that coincides with the Killing form on $T_{K/K}G/K \cong \mathfrak{t}$. Write $d$ the corresponding distance map on $G/K$. Given points $z_1, z_2 \in G/K$, and a positive real number $s > 0$, the Poincaré series of $\Lambda$ at $(z_1, z_2, s)$ is defined as

$$p(z_1, z_2, s) = \sum_{g \in \Lambda} e^{-sd(z_1, gz_2)}$$

Observe that the convergence or divergence of the series $p(z_1, z_2, s)$ does not depend on the points $z_1, z_2$ but only on the parameter $s$. It is then natural to introduce the number

$$\delta_\Lambda = \inf \{s > 0, p(z_1, z_2, s) < \infty\}$$

known as the critical exponent of $\Lambda$. As $\Lambda$ is discrete, $\delta_\Lambda$ is less or equal to the exponential growth rate of the volume of balls in $G/K$, given by

$$\delta_G = \lim_{R \to +\infty} \frac{1}{R} \log(V_R)$$

where $V_R > 0$ is the Riemannian volume of a ball of radius $R$ in $G/K$. The case of equality $\delta_\Lambda = \delta_G$ expresses that the orbits of $\Lambda$ are not too sparse in $G/K$. As we see below, it is a necessary condition for the geodesic flow on $X$ to be recurrent, but it is not sufficient in general. For instance, if $G = PSL_2(\mathbb{R})$ and $X/K$ is a $\mathbb{Z}^d$-cover of a compact hyperbolic surface with $d \geq 3$, then the geodesic flow on $X$ is transient [29] but $\delta_\Lambda = \delta_{PSL_2(\mathbb{R})} = 1$ [13]. The following result claims that we can strengthen slightly the condition that $\delta_\Lambda$ is maximal to characterize the situations of recurrence/transience. It is usually called Hopf-Tsuji-Sullivan Theorem, but it is actually due to Kaimanovich in the context of rank-one symmetric spaces.
Theorem 4.1. [22, Theorem 3.3] The geodesic flow \((a_t)\) on \(X\) is recurrent ergodic if and only if \(p(z_1, z_2, \delta_G) = +\infty\), and is transient otherwise.

Remark. If \(G = Sp(1, m)\) for some \(m \geq 2\), or \(G = F_4^{-20}\), then [12, Theorem 4.4] claims that a discrete subgroup \(\Lambda \subseteq G\) of infinite covolume satisfies \(\delta_\Lambda < \delta_G\). In particular, by Theorem 4.4 the geodesic flow on \(X = \Lambda \backslash G\) is transient. According to Theorem 4.1, the same holds true for walks on \(X\) given by a probability measure \(\mu\) on \(G\) with finite first moment and \(\Gamma_\mu\) Zariski-dense in \(G\).

We now explain why Fact 2 is a reformulation of Theorem 4.1. We freely identify any subset of \(G/K\) to a right \(K\)-invariant subset of \(G\). Given \(z \in G/K\), \(\varepsilon > 0\), we denote by \(B(z, \varepsilon)\) the ball of center \(z\) and radius \(\varepsilon\) in the symmetric space \(G/K\). We also set \(\lambda_K\) the Haar probability measure on \(K\). Finally, given \(s, t \geq 0\), \(c > 1\) we write \(s = c^{\pm 1}t\) if \(s \in [c^{-1}t, ct]\).

Lemma 4.2. For all \(\varepsilon > 0\), there exists \(c > 1\) such that for \(z_1, z_2 \in G/K\), \(g_1 \in z_1\),

\[
\int_K G(g_1k, B(z_2, \varepsilon))d\lambda_K(k) = c^{\pm 1}e^{-\delta_G d(z_1, z_2)}
\]  \hspace{1cm} (19)

Proof. Let \(m\) be the Liouville measure on the symmetric space \(G/K\). According to Helgason’s book [19, Theorem 5.8], there exists a constant \(r > 0\) such that for all non-negative function measurable functions \(f : G/K \to \mathbb{R}_+\),

\[
\int_{G/K} f \, dm = r \int_{K \times \mathbb{R}^+} f(ka_tK)\sigma(t) \, d\lambda_K(k) \, dt
\]  \hspace{1cm} (20)

where \(\sigma(t) = \Pi_{a \in \Phi^+} \sinh(a(t\nu_0))^{\dim \Phi^+}\), with \(\nu_0 \in a^+\) unique vector of norm 1 (see beginning of [4]). We must have the equivalence \(\sigma(t) \sim r'e^\delta_G t\) for some \(r' \in \{1/2, 1/4\}\) as \(t\) goes to \(+\infty\), and in particular, there exists a constant \(R > 0\) such that for all \(t > 1\),

\[
\sigma(t) = R^{\pm 1} e^{\delta_G t}
\]

Let us now check (19). We can assume that \(g_1 = e\) and \(d(z_1, z_2) > 1 + \varepsilon\). Specifying \(f\) in (20) to be the characteristic function of the ball \(B(z_2, \varepsilon)\) in \(G/K\), we obtain

\[
V_\varepsilon = r \int_{K \times \mathbb{R}^+} 1_{B(z_2, \varepsilon)}(ka_tK)\sigma(t) \, d\lambda_K(k) \, dt
\]

\[
= rR^{\pm 1} \int_{K \times \mathbb{R}^+} 1_{B(z_2, \varepsilon)}(ka_tK)e^{\delta_G t} \, d\lambda_K(k) \, dt
\]

\[
= r(Re^\varepsilon)^{\pm 1} e^{\delta_G d(z_2, 0)} \int_K G(k, B(z_2, \varepsilon)) \, d\lambda_K(k)
\]

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and finally,
\[
\int_K G(k, B(z_2, \varepsilon))d\lambda_K(k) = V_\varepsilon r^{-1}(Re^\varepsilon)^{1+1}e^{-\delta_G d(z_2, 0)}
\]

We now use the previous lemma to show that the Poincaré series of \( \Lambda \) with parameter \( s = \delta_G \) expresses the average of the Green function of the geodesic flow on a \( K \)-orbit in \( X \). Given a point \( p \in X/K \) and \( r > 0 \), denote by \( B(p, r) \) the ball of center \( p \) and radius \( r \) in \( X/K \) for the quotient metric, and define \( r_X(p) \) as the supremum of the real numbers \( r > 0 \) such that the preimage of \( B(p, r) \) in \( G/K \) is a collection of disjoint open balls (on which \( \Lambda \) acts transitively). In the case where \( \Lambda \) has no torsion, the action of \( \Lambda \) on such a collection of balls is simply transitive, and \( r_X(p) \) is called the injectivity radius of \( X \) at \( p \). In general, we only know that the action of \( \Lambda \) has finite stabilizer (by discreteness), and write \( N_X(p) \in \mathbb{N}/\{0\} \) its cardinal.

**Lemma 4.3.** Let \( \varepsilon > 0 \) and \( N \in \mathbb{N} \setminus \{0\} \). There exists a constant \( C > 1 \) such that for any \( p_1, p_2 \in X/K \) with \( r_X(p_2) > \varepsilon \) and \( N_X(p_2) \leq N \), and any \( z_1, z_2 \in G/K \), \( x_1 \in X \) such that \( p_1 = \Lambda z_1 = x_1 K \), \( p_2 = \Lambda z_2 \), one has
\[
\int_K G(x_1 k, B(p_2, \varepsilon))d\lambda_K(k) = C^{\pm 1}p(z_1, z_2, \delta_G)
\]

**Proof.** The assumption that \( r_X(p_2) > \varepsilon \) means that the preimage in \( G/K \) of \( B(p_2, \varepsilon) \) is the disjoint union \( \bigcup_{g \in \Lambda} B(g z_2, \varepsilon) \) where each ball appears with multiplicity \( N_X(p_2) \). Hence, given \( g_1 \in G \) such that \( x_1 = \Lambda g_1 \), we can write
\[
\int_K G(x_1 k, B(p_2, \varepsilon))d\lambda_K(k) = \frac{1}{N_X(p_2)} \sum_{g \in \Lambda} \int_K G(g_1 k, B(g z_2, \varepsilon))d\lambda_K(k)
\]
\[
= c^{\pm 1} \frac{1}{N_X(p_2)} \sum_{g \in \Lambda} e^{\delta_G d(z_1, g z_2)}
\]
\[
= c^{\pm 1} \frac{1}{N_X(p_2)} p(z_1, z_2, \delta_G)
\]
if \( c > 1 \) is chosen as in Lemma 4.2.

It is now easy to conclude that Fact 2 is a reformulation of Theorem 4.1:

**Fact 2 \( \iff \) Theorem 4.1** We check that the Poincaré series \( p(z_1, z_2, \delta_G) \) is finite if and only if the Green functions of the geodesic flow are locally integrable. Let \( z_2 \in G/K \), \( p_2 = \Lambda z_2 \in X/K \) its projection on \( X/K \), \( (\varepsilon, N) \in \mathbb{R}_{>0} \times \mathbb{N} \) such that \( r_X(p_2) > \varepsilon \) and \( N_X(p_2) \leq N \), and write \( C > 1 \) the associated constant of Lemma 4.3. Let \( E \subseteq X \) be a right \( K \)-invariant compact
subset. Lemma 4.3, together with the $K$-invariance of $E$ and of the Haar measure $\lambda$ on $X$, implies that
\[
\int_E G(x, B(p, \varepsilon)) d\lambda(x) = \int_K \int_E G(xk, B(p, \varepsilon)) d\lambda_K(k) d\lambda(x) \\
= \int_E C^{\pm 1} p(z, z_2, \delta_G) d\lambda(x) \\
= (C e^{\delta_G R})^{\pm 1} \lambda(E) p(z_1, z_2, \delta_G)
\] (21)
where $z \in G/K$ is any lift of $x$ (i.e. satisfies $\Lambda z = xK$), $z_1$ is the lift of some fixed arbitrary point in $E$, and $R$ is the diameter of the projection of $E$ in $X/K$.

The equation (21) implies that $G(., B(p, \varepsilon))$ is locally integrable if and only if the poincaré series $p(z_1, z_2, \delta_G)$ is finite. Notice this is also true if one replaces $B(p, \varepsilon)$ by any relatively compact subset $F \subseteq X$ with positive measure. More precisely, if $p(z_1, z_2, \delta_G) < \infty$ then, covering $F$ by a finite number of balls $B(p, \varepsilon)$ with $\varepsilon < r_X(p_2)$, we infer from above that $G(., F)$ is locally integrable. Conversely, if $p(z_1, z_2, \delta_G) = +\infty$, then by Theorem 4.1 the geodesic flow is recurrent ergodic, and as $F$ has positive measure, we necessarily have $G(., F) = +\infty$ almost everywhere.

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