Doubly Threshold Graphs for Social Network Modeling

Vida Ravanmehr, Gregory J. Puleo, Sadegh Bolouki, Olgica Milenkovic
Coordinated Science Laboratory
University of Illinois, Urbana-Champaign
Urbana, IL, 61801.
Email: {vidarm,puleo,bolouki,milenkov}@illinois.edu

Abstract—Threshold graphs are recursive deterministic network models that capture properties of social and economic interactions. One drawback of these graph families is that they have limited constrained generative attachment rules. To mitigate this problem, we introduce a new class of graphs termed Doubly Threshold (DT) graphs which may be succinctly described through vertex weights that govern the existence of edges via two inequalities. One inequality imposes the constraint that the sum of weights of adjacent vertices has to exceed a specified threshold. The second inequality ensures that adjacent vertices have a bounded difference of their weights. We provide a conceptually simple characterization and decomposition of DT graphs and analyze their forbidden induced subgraphs which we compare to those of prototypical social networks. We also present a method for performing vertex weight assignments on DT graphs that satisfy the defining constraints and conclude our exposition with an analysis of the intersection number, diameter and clustering coefficient of DT graphs.

I. INTRODUCTION

The problem of analyzing complex behaviors of large social, economic and biological networks based on generative recursive and probabilistic models has been the subject of intense study in graph theory, machine learning and statistics. In these settings, one often assumes the existence of attachment and preference rules for network formation, or imposes constraints on subgraph structures as well as vertex and edge features that govern the creation of network communities [1], [2], [3], [4], [5], [6]. Models of this type have been used to successfully predict network dynamics and topology fluctuations, infer network community properties and preferences, determine the bottlenecks and rates of spread of information and commodities and elucidate functional and structural properties of individual network modules [7], [8], [9].

Several measures for assessing the quality of graph family models for social or biological networks include the vertex degree distribution, the graph diameter and clustering coefficient (i.e., density of triangles). The vertex degree distribution describes the number of vertices of each degree in the graph, and is usually assumed to follow a power-law, with the probability of an arbitrary vertex having degree \( d \) scaling as \( d^{-\gamma} \), for some parameter \( \gamma > 0 \). The diameter of a graph is the length of the longest shortest path between any two vertices of a graph, and is known to be a small constant for most known social and biological networks [2]. The clustering index constraint ensures that the model correctly contains a large number of network motifs (e.g., triangles) known to exist in both biological and social networks (Here, we use the word “motif” to refer to a small induced subgraph which appears in networks with a probability significantly higher than that predicted by some random generative model). In his comprehensive study of social network motifs, Ugander [10] determined the frequency of induced subgraphs with three and four vertices in a large cohort of interaction and friendship networks. In addition to showing that triangles (cliques with three vertices, \( K_3 \)) and cliques with four vertices, \( K_4 \), are the most prominent network motifs, Ugander also established the existence of strong anti-motifs, highly infrequent induced subgraphs. For example, cycles of length four \( (C_4) \) represent the least likely induced subgraphs in social networks.

Here, we take a new approach to deterministic network modeling by proposing a new family of graph structures that have small diameter, avoid anti-motifs of real social and biological networks and include graphs with good clustering coefficients. The graphs in question, termed Doubly Threshold (DT) graphs, may be succinctly defined as follows: Each vertex is assigned a nonnegative weight. An edge between two vertices exists if the sum of the vertex weights exceeds a certain threshold, while at the same time, the difference between the weights remains bounded by another prescribed threshold.

DT graphs may be seen as a generalization of two classes of graphs: Threshold and Unit Interval graphs. Threshold graphs were introduced by Chvatal and Hammer [11] in order to solve a set-packing problem; they are defined by the first generative property of DT graphs, stating that an edge between two vertices exists if and only if the sum of their weights exceeds a predetermined threshold. Threshold graphs are often referred to as “rich people networks,” as the weights may be associated with wealth, in which case the generative rule is interpreted as “rich people always know each other” or “everyone knows the rich people” [4]. As discussed in more details in the next section, the vertex weight assignment of threshold graphs can be easily obtained from the degrees of the vertices; furthermore, graphs in this family may be constructed recursively, by adding what are called isolated or dominating vertices based on simple attachment rules. Most importantly, the graphs may be completely quantified by three forbidden induced subgraphs, \( 2K_2 \), \( C_4 \) and \( P_4 \) (see Fig. [1]).
As a result, threshold graphs have been used as social network models and have also found other applications in aggregation of inequalities, synchronization and cyclic scheduling [12].

On the other hand, unit interval graphs were introduced in [13]. These graphs are also defined in terms of constrained vertex weights, so that the difference of the weights of every two adjacent vertices lies below a predefined threshold. [1]

DT graphs, as already pointed out, combine the threshold and unit interval graph constraints in a manner that ensures that two vertices are connected by an edge if and only if their joint wealth (i.e., sum of weights) is above some threshold \( \alpha \), and their weights (weights) do not disagree by more than some other threshold \( \beta \). As such, DT graphs may be interpreted as networks of “sufficiently wealthy people within the same economic class”, where the same economic class constraint arises from the requirement that the weights are of comparable size. Many other interpretations are possible, for example by assuming that the weights represent the number of papers published in a given topic area and links describing coauthorship (two researchers are likely to be coauthors of a paper on a given topic if they jointly published a certain number of papers in the field and are of matching research experience), two people are likely to coinvest in a project if they total wealth exceeds the value of the investment capital needed and their investment contributions are comparable etc.

As will be shown in subsequent sections, DT graphs also have special hierarchical community structures that capture a number of important properties of real social networks. Furthermore, they may be modeled in a probabilistic setting, in which one assumes that the vertices satisfying the constraints are connected by an edge with probability \( p \gg 1/2 \), while vertices not satisfying the constraints are not connected with probability \( q \gg 1/2 \). Furthermore, as will be shown in subsequent sections, DT graphs avoid induced subgraphs that are also avoided in real social networks, such as \( C_4 \). Although the focus of this work is on combinatorial DT graphs with scalar weights and their analysis, the underlying model may be easily generalized to include vector weights, and weight assignments that follow some distribution (e.g., uniform or Gaussian).

The paper is organized as follows. In Section II we briefly review relevant definitions and concepts from graph theory and introduce DT graphs. In Section III we characterize the topological properties of DT graphs and some of their forbidden induced subgraphs, and describe a simple decomposition of the graphs. This decomposition allows one to find a vertex weight assignment that satisfies the DT graph constraints. In Section IV we provide a polynomial-time algorithm for identifying whether a graph is DT or not. In Section V using the previously devised DT graphs decomposition, we first describe a number of forbidden induced subgraphs of DT graphs and then provide closed formulas for the intersection number and the clustering coefficient of DT graphs as well as a bound on the diameter of DT graphs.

## II. PRELIMINARIES AND BACKGROUND

We start by introducing relevant definitions and by providing an overview of basic properties of threshold graphs. Throughout the paper, \( \mathbb{R} \) is used to denote the set of real numbers, while \( \mathbb{R}^+ \) is used to denote the set of positive real numbers.

Let \( G(V, E) \) be an undirected graph, with vertex set \( V = \{1, \ldots, n\} \) and edge set \( E \). Two vertices \( i, j \in V \), \( i \neq j \), are said to be adjacent if there exists an edge in \( E \), herein denoted by \( e_{ij} \), connecting them. For every \( i \in V \), we denote by \( \mathcal{N}(i) \) the set of the vertices adjacent to \( i \), i.e.,

\[
\mathcal{N}(i) \triangleq \{ j \in V \mid e_{ij} \in E \}.
\]

The cardinality of \( \mathcal{N}(i) \), denoted by \( d(i) \), is referred to as the degree of vertex \( i \).

**Definition 1.** A graph \( G(V, E) \) is called a threshold graph if there exists a fixed \( T \in \mathbb{R}^+ \), and a weight function \( w : V \to \mathbb{R}^+ \), such that for all distinct \( i, j \in V \):

\[
e_{ij} \in E \iff w(i) + w(j) \geq T.
\]

We will refer to such a threshold graph as a \((T, w)\) graph [12].

Threshold graphs may be equivalently defined as those graphs that avoid \( C_4, P_4 \) and \( 2K_2 \) as induced subgraphs (see Fig. 1). Furthermore, threshold graphs may be generated using a recursive procedure, by sequentially adding an isolated vertex (a vertex not connected to any previously added vertices) or a dominating vertex (a vertex connected to all previously added vertices).

Threshold graphs can also be alternatively characterized via what is called the *vicinal preorder* \( \mathbf{R} \) [12], defined on the set \( V \) of vertices of \( G \) as:

\[
i \mathbf{R} j \iff \mathcal{N}(i) \setminus \{j\} \subset \mathcal{N}(j).
\]

The preorder \( \mathbf{R} \) on \( V \), defined in (3), is total if it is a binary relation which is transitive and for any pair of vertices \( i, j \), one has \( i \mathbf{R} j \) or \( j \mathbf{R} i \). Given a threshold graph with threshold \( T \) and vertex weights \( w \), it is straightforward to show that

\[
i \mathbf{R} j \iff w(i) \leq w(j).
\]

Therefore, since the preorder \( \leq \) on the set \( \mathbb{R}^+ \) is total, the preorder \( \mathbf{R} \) on \( V \) is total as well. It turns out that the converse
is also true \cite{12}, i.e., if the preorder \( R \) on \( V \) is total, then \( G \) is a threshold graph. To see why this is true, let \( \delta_1 < \ldots < \delta_m \) represent all the distinct, positive degrees of the vertices of \( G \), and set \( \delta_0 = 0 \). For all \( i, 0 \leq i \leq m \), define
\[
D_i \triangleq \{ i \in V \mid d(i) = \delta_i \}.
\] (5)

Notice that \( (D_0, \ldots, D_m) \) forms a partition of \( V \), known as the degree partition of \( V \). Define the vertex weight function \( w \) according to \( w(i) = j, \forall i \in D_j, 0 \leq j \leq m \), and set the threshold to \( T = m + 1 \). One can then show that the threshold \( T \) and the aforedescribed weight function \( w \) satisfy \cite{2}, implying that \( G \) is a threshold graph \cite{12}.

**Proposition 1.** A graph \( G(V, E) \) is a threshold graph if and only if the preorder \( R \) on \( V \), defined in \( (2) \), is total.

Unit interval graphs can similarly be defined in terms of constrained vertex weights as follows:

**Definition 2.** A graph \( G(V, E) \) is called a unit interval graph if there exist a fixed \( T \in \mathbb{R}^+ \), and a weight function \( w : V \to \mathbb{R}^+ \) such that for all distinct \( i, j \in V \):
\[
e_{ij} \in E \iff |w(i) - w(j)| \leq T.
\] (6)

**Definition 3.** Given a connected graph \( G(V, E) \), a distance decomposition of \( V \) is a partition \( (C_0, C_1, \ldots, C_m) \), \( m \geq 0 \), of \( V \) in which
\[
C_l \triangleq \left\{ i \in V \mid \min_{j \in C_0} \text{dist}(i, j) = l \right\}, \quad \forall l, 1 \leq l \leq m,
\] (7)

where \( \text{dist}(i, j) \) is the length of the shortest path between \( i \) and \( j \) in the graph \( G \).

Equivalently, a distance decomposition may be generated starting from \( C_0 \), and then recursively creating \( C_l \), \( 1 \leq l \leq m \), according to
\[
C_l \triangleq \left\{ i \in V \setminus \bigcup_{l'=0}^{l-1} C_{l'} \mid \exists j \in C_{l'-1} : e_{ij} \in E \right\}.
\] (8)

Simply put \( C_1 \) as the set of vertices adjacent to \( C_0 \) in \( G \), excluding \( C_0 \); \( C_2 \) as the set of vertices adjacent to \( C_1 \) in \( G \), excluding \( C_0 \) and \( C_1 \), and so on. Clearly, there is no edge between \( C_l \) and \( C_l' \), \( 0 \leq l, l' \leq m \), if \( |l - l'| \geq 2 \).

We introduce next a new family of graphs, termed doubly threshold graphs, which combine the properties of threshold and unit interval graphs.

**Definition 4.** A graph \( G(V, E) \) is termed a doubly threshold (DT) graph if there exist two fixed parameters \( \alpha \geq \beta \in \mathbb{R}^+ \) and a weight function \( w : V \to \mathbb{R}^+ \), such that for all distinct \( i, j \in V \),
\[
e_{ij} \in E \iff \begin{cases} w(i) + w(j) \geq \alpha, \\
|w(i) - w(j)| \leq \beta. \end{cases}
\] (9)

We will refer to graphs with the above defining properties as

\(^2\)With a slight abuse of terminology, we use the term “partition” although \( D_0 \) may be empty.

\[\text{Fig. 2: An example of a DT graph, along with a weight assignment for the parameters } \alpha = 10 \text{ and } \beta = 2.\]

\((\alpha, \beta, w)\)-DT graphs.

Figure 2 illustrates a DT graph with \( \alpha = 10 \) and \( \beta = 2 \), along with a possible weight assignment. Note that there exists an edge between the two vertices labeled by 5 and 7, as \( 5 + 7 = 12 > \alpha = 10 \) and \( |5 - 7| = 2 \leq \beta = 2 \), and there is no edge between the vertices labeled by 4 and 7 as \( |4 - 7| = 3 > \beta = 2 \).

**III. Characterization of DT Graphs**

We characterize next the structure of a general connected DT graph \( G(V, E) \) with parameters \( (\alpha, \beta, w) \). We first note that for every \( e_{ij} \in E \), from the two inequalities in \((9)\), one must have \( \min\{w(i), w(j)\} \geq \frac{\alpha - \beta}{2} \). Thus, noticing that every vertex has at least one neighbor as the graph is connected, we have the following proposition.

**Proposition 2.** For every \( i \in V \), \( w(i) \geq \frac{\alpha - \beta}{2} \).

We now proceed to demonstrate that if \( G \) is not a unit interval graph, its set of vertices \( V \) has a distance decomposition \( (C_0, \ldots, C_m) \) with a special structure. For this purpose, define:
\[
C_0 \triangleq \left\{ i \in V \mid w(i) \in \left[ \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right] \right\}.
\] (10)

**Proposition 3.** The subgraph induced by \( V \setminus C_0 \) is a unit interval graph with parameters \( (\beta, w) \). Consequently, if \( C_0 \) is the empty set, then \( G \) is a unit interval graph.

**Proof:** According to \((6)\) and \((9)\), it suffices to show that \( w(i) + w(j) \geq \alpha, \forall i, j \in V \setminus C_0, i \neq j \). To prove this, one needs to simply note that \( w(i) \) and \( w(j) \) are both greater than or equal to \( \frac{\alpha + \beta}{2} \), according to Proposition \((2)\) and the definition of \( C_0 \) in \((10)\).

Assume next that \( C_0 \) is non-empty. Without loss of generality, we can assume that for every \( i, j \in C_0 \),
\[
\mathcal{N}(i) \setminus \{j\} = \mathcal{N}(j) \setminus \{i\} \Rightarrow w(i) = w(j).
\] (11)

In fact, if for some \( i, j \in C_0 \), \((11)\) does not hold, i.e., if \( \mathcal{N}(i) \setminus \{j\} = \mathcal{N}(j) \setminus \{i\} \) but \( w(i) \neq w(j) \), one may modify the weights assigned to \( i \) and \( j \) so that both equal \( \max\{w(i), w(j)\} \) if \( e_{ij} \in E \), or both equal \( \min\{w(i), w(j)\} \) if \( e_{ij} \notin E \). The modified weights \( w \) still satisfy condition \((9)\) for all distinct \( i, j \in V \). Furthermore, the set \( C_0 \) remains unchanged. By repeating this process, a weight function \( w \) emerges for which \((11)\) is satisfied for every \( i, j \in C_0 \). We also point out that \((11)\) can also be assumed to hold for
every \(i, j \in V\) for which \(e_{ij} \in E\). In fact, if for some \(i, j \in V, e_{ij} \in E\), (11) is violated, one may modify the weights assigned to \(i\) and \(j\) to \(\max\{w(i), w(j)\}\) and repeat the reassignment procedure until (11) is satisfied for all \(i, j \in V\) such that \(e_{ij} \in E\).

Having defined \(C_0\) in (10), let \((C_0, \ldots, C_m)\) be the distance decomposition of \(V\) starting with \(C_0\) as previously defined. Then, the following result holds.

**Proposition 4.** The vicinal preorder \(R_0\) defined on the set \(C_0\) as

\[ i \sim R_0 j \iff \mathcal{N}(i) \setminus \{j\} \subset \mathcal{N}(j), \]

(12)
is total.

**Proof:** It suffices to show that for the preorder \(R_0\) to be total on \(C_0\), for every distinct \(i, j \in C_0\), one has to have

\[ i \sim R_0 j \iff w(i) \leq w(j). \]

(13)

Recall that the preorder \(\preceq\) is total on \(\mathbb{R}^+\). From (12) and (13), it therefore suffices to prove that for every distinct \(i, j \in C_0\) one has

\[ w(i) \leq w(j) \iff \mathcal{N}(i) \setminus \{j\} \subset \mathcal{N}(j). \]

(14)

(\(\Rightarrow\)) Assume that \(w(i) \leq w(j)\). We prove for every \(k \in V\setminus \{j\}\) the following fact: If \(e_{ik} \in E\), then \(e_{jk} \in E\). We consider two different cases:

1. If \(k \in C_0\), from the definition of \(C_0\) in (10), \(|w(j) - w(k)| \leq \beta\). Moreover, since \(e_{ik} \in E\), \(w(i) + w(k) \geq \alpha\).

Thus, \(w(j) + w(k) \geq \alpha\). Therefore, \(e_{jk} \in E\).

2. If \(k \in V\setminus C_0\), \(w(i) \leq w(j) < w(k)\). Thus, \(e_{ik} \in E\), we have:

\[ \begin{align*}
  w(j) + w(k) &\geq w(i) + w(k) \geq \alpha, \\
  |w(j) - w(k)| &\leq |w(i) - w(k)| \leq \beta,
\end{align*} \]

which implies \(e_{jk} \in E\).

(\(\Leftarrow\)) Assume that \(\mathcal{N}(i) \setminus \{j\} \subset \mathcal{N}(j)\). We prove that \(w(i) \leq w(j)\). If \(\mathcal{N}(i) \setminus \{j\} = \mathcal{N}(j) \setminus \{i\}\), from (11), we have \(w(i) = w(j)\). Thus, \(\mathcal{N}(i) \setminus \{i\} \in C\). Then, there exists \(k \in V\setminus \{i, j\}\) such that \(e_{ik} \notin E\) and \(e_{jk} \in E\). We show that \(w(i) < w(j)\) by considering the following two cases:

1. If \(k \in C_0\), from the definition of \(C_0\) in (10), both \(|w(i) - w(k)| \leq \beta\) and \(|w(j) - w(k)| \leq \beta\) are satisfied. Thus, since \(e_{ik} \notin E\) and \(e_{jk} \in E\), according to (9), we must have \(w(i) + w(k) < \alpha\) and \(w(j) + w(k) \geq \alpha\), which immediately results in \(w(i) < w(j)\).

2. If \(k \in V\setminus C_0\), from Proposition 2 and the definition of \(C_0\) in (10), \(w(k) \geq \frac{\alpha + \beta}{2}\). On the other hand, since \(i, j \in C_0\), both \(w(i)\) and \(w(j)\) are greater than or equal to \(\frac{\alpha + \beta}{2}\). Thus, \(w(i) + w(k) \geq \alpha\) and \(w(j) + w(k) \geq \alpha\).

Therefore, since \(e_{ik} \notin E\) and \(e_{jk} \in E\), according to (9), we must have \(|w(i) - w(k)| \leq \beta\) and \(|w(j) - w(k)| \leq \beta\). Recall that \(w(k) \geq \frac{\alpha + \beta}{2}\), which implies that \(w(k) \geq \max\{w(i), w(j)\}\). Hence, \(w(k) - w(i) > \beta\) and \(w(k) - w(j) \leq \beta\), which together imply \(w(i) < w(j)\).

Next, we give a characterization of the subgraphs induced by \(C_l\) and define a preorder on the vertices in \(C_l\) for all \(1 \leq l \leq m\) in Propositions 5 and 6.

**Proposition 5.** For every \(l, 1 \leq l \leq m\), the subgraph of \(G\) induced by \(C_l\) is a clique.

**Proof:** From the recursive relation (8), for every \(l, 1 \leq l \leq m\), we have:

\[ w(i) \leq \max_{j \in C_{l-1}} w(j) + \beta, \forall i \in C_l. \]

(16)

Using the definition of \(C_0\) in (10) and the inequality (16), one may prove by induction on \(l\) that \(\forall l, 1 \leq l \leq m\),

\[ w(i) \in \left[ \frac{\alpha + (2l-1)\beta}{2}, \frac{\alpha + (2l+1)\beta}{2} \right], \forall i \in C_l. \]

(17)

Thus, for every distinct \(i, j \in C_l\), we have \(|w(i) - w(j)| < \beta\) and

\[ w(i) + w(j) \geq \frac{\alpha + (2l-1)\beta}{2} + \frac{\alpha + (2l+1)\beta}{2} \geq \alpha. \]

(18)

Therefore, according to (9), we must have \(e_{ij} \in E\). Hence, the subgraph induced by \(C_l\) for \(1 \leq l \leq m\), is a clique.

**Proposition 6.** The preorder \(R_l\), defined on the set \(C_l, 1 \leq l \leq m\), according to

\[ i \sim R_l j \iff \begin{cases} 
\mathcal{N}(i) \cap C_{l-1} \subset \mathcal{N}(i) \cap C_{l-1}, \\
\mathcal{N}(i) \cap C_{l+1} \subset \mathcal{N}(j) \cap C_{l+1}, \\
\mathcal{N}(i) \cap C_l \subset \mathcal{N}(j) \cap C_l,
\end{cases} \]

(19)
is total.

**Proof:** Since the preorder \(\preceq\) on \(\mathbb{R}^+\) is total, it suffices to show that:

\[ w(i) \leq w(j) \iff \begin{cases} 
\mathcal{N}(i) \cap C_{l-1} \subset \mathcal{N}(i) \cap C_{l-1}, \\
\mathcal{N}(i) \cap C_{l+1} \subset \mathcal{N}(j) \cap C_{l+1}, \\
\mathcal{N}(i) \cap C_l \subset \mathcal{N}(j) \cap C_l,
\end{cases} \]

(20)

(\(\Rightarrow\)) Assume that \(w(i) \leq w(j)\). We first show that \(\mathcal{N}(i) \cap C_{l-1} \subset \mathcal{N}(j) \cap C_{l-1}\). Let \(k \in \mathcal{N}(i) \cap C_{l-1}\) be arbitrary. Since \(k \in C_{l-1}\), we must have

\[ w(k) < w(i) \leq w(j). \]

On the other hand, since \(k \in \mathcal{N}(j)\), we also have \(w(j) - w(k) \leq \beta\). Thus, \(w(i) - w(k) \leq \beta\). Moreover, since

\[ w(i) \geq \frac{\alpha + (2l-1)\beta}{2} \quad \text{and} \quad w(k) \geq \frac{\alpha + (2l-3)\beta}{2}, \]

we have \(w(i) + w(k) \geq \alpha + (2l-2)\beta \geq \alpha\). Hence, according to (9), \(e_{ik} \in E\). We now show that \(\mathcal{N}(i) \cap C_{l+1} \subset \mathcal{N}(j) \cap C_{l+1}\).

For an arbitrary \(k \in \mathcal{N}(i) \cap C_{l+1}\), similar to the previous argument, we have \(w(i) \leq w(k) < w(k)\) and \(w(k) - w(i) \leq \beta\). Thus, \(w(k) - w(j) \leq \beta\). Moreover, \(w(k) + w(j) \geq \alpha\), and according to (9), \(e_{jk} \in E\).

(\(\Leftarrow\)) Assume that both inclusion relations of (20) hold. Moreover, assume to the contrary of the claimed assumption...
that \( w(j) < w(i) \). From part \((\Rightarrow)\) of the proof, we conclude

\[
\begin{aligned}
N(j) \cap C_{l-1} &\subset N(j) \cap C_{l-1}, \\
N(j) \cap C_{l+1} &\subset N(i) \cap C_{l+1}.
\end{aligned}
\]

From (21) and the two inclusion relations of (20), we obtain

\[
N(i) \cap (C_{l-1} \cup C_{l+1}) = N(j) \cap (C_{l-1} \cup C_{l+1}).
\]

Recall next that since \( i, j \in C_l \), their neighbors can only be in \( C_{l-1}, C_l \) and \( C_{l+1} \), while \( C_i \) is a clique. Thus, from (22), we conclude that \( N(i) \setminus \{j\} = N(j) \setminus \{i\} \). Therefore, according to (11), we must have \( w(i) = w(j) \). Hence, the proof follows by contradiction.

A distance decomposition of a DT graph is shown in Fig. 3. We show next that the properties of connected DT graphs established in Propositions 3-6 are also sufficient for a graph to be DT.

**Theorem 1.** A connected graph \( G(V, E) \) is a DT graph if and only if it is a unit interval graph or if there is a distance decomposition \( (C_0, C_1, \ldots, C_m) \), for some \( m \geq 0 \), for which all the following statements hold true:

(i) The vicinal preorder \( R_0 \) defined on the set \( C_0 \) as

\[
i R_0 j \iff N(i) \setminus \{j\} \subset N(j),
\]

is total.

(ii) For every \( l, 1 \leq l \leq m \), the subgraph of \( G \) induced by \( C_l \) is a clique.

(iii) The preorder \( R_l \) defined on the set \( C_l \), \( 1 \leq l \leq m \), as

\[
i R_l j \iff \begin{cases} N(j) \cap C_{l-1} \subset N(i) \cap C_{l-1}, \\ N(i) \cap C_{l+1} \subset N(j) \cap C_{l+1}, \end{cases}
\]

is total, where we enforce \( C_{m+1} = \emptyset \).

Note that the “only if” part is an immediate result of the properties established in Propositions 3-6. Thus, it remains to show the “if” part.

Let \( \alpha \geq \beta > 0 \) be arbitrary. If \( G \) is a unit interval graph, there is a weight function \( w : V \rightarrow \mathbb{R}^+ \) such that \( G \) is a unit interval graph with parameters \((\beta, w')\). By defining \( w' = w + \frac{\alpha}{2} \), it is straightforward to conclude that \( G \) is a \((\alpha, \beta, w')\)-DT graph. Therefore, assume that a distance decomposition \( (C_0, C_1, \ldots, C_m) \), where \( m \geq 0 \), exists and satisfies (ii)-(iii). We construct a weight function \( w : V \rightarrow \mathbb{R}^+ \) that establishes that \( G(V, E) \) is a \((\alpha, \beta, w')\)-DT graph. We first assign weights to the vertices in \( C_0 \) and then to those in \( C_l \)’s, \( 1 \leq l \leq m \).

**Step 1:** For the weight assignments of \( C_0 \), we first show that the subgraph of \( G \) induced by \( C_0 \) is a threshold graph.

Defining a preorder \( R_0' \) over \( C_0 \) according to

\[
i R_0' j \iff (N(i) \cap C_0) \setminus \{j\} \subset N(j) \cap C_0,
\]

we have

\[
i R_0 j \Rightarrow i R_0' j.
\]

Thus, since \( R_0 \) is total on \( C_0 \) according to (i), \( R_0' \) is also a total order on \( C_0 \). Therefore, according to Proposition 1, the subgraph of \( G \) induced by \( C_0 \) is a threshold graph.

**Lemma 1.** There exists a weight assignment \( w(i) \) for each \( i \in C_0 \) with the following properties:

1. The subgraph of \( G \) induced by \( C_0 \) is a threshold graph with parameters \((\alpha, w)\).
2. \( w(i) \neq w(j) \), \( \forall i, j \in C_0, i \neq j \).
3. \( w(i) \in \left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) \), \( \forall i \in C_0 \).

**Proof:** Recall the notion of the degree partition of the vertices of a graph from the argument leading to Proposition 1. Let \( (D_0, \ldots, D_m) \) be the degree partition of \( C_0 \) in the subgraph of \( G \) induced by \( C_0 \). We start with defining the weight function \( w : C_0 \rightarrow \mathbb{R}^+ \) as \( w(i) = j \) for every \( i \in D_j \), \( 0 \leq j \leq m' \). The subgraph of \( G \) induced by \( C_0 \) is a threshold graph with parameters \((m' + 1, w)\). We now modify the weight function \( w \) in such a way that it meets the criteria 1-3 of Lemma 1. First, for every \( i \in C_0 \), we modify \( w(i) \) to \( w(i) + \epsilon_i \), where \( 0 < \epsilon_i < 1/2 \), in such a way that the modified weights of every two distinct vertices in \( C_0 \) are different. The subgraph of \( G \) induced by \( C_0 \) remains a threshold graph with parameters \((m' + 1, w)\), a fact which may be verified by observing that \( m' + 1 \) is an integer; the starting weights of the assignment were all integer-valued; and the modified weights are obtained from the previous weights by adding to them a value smaller than 1/2. In the next step, we divide all the weights obtained in the previous step by \( |C_0| \) to obtain a threshold graph with parameters \((1, w')\), where \( w(i) = w(j) \) for every distinct \( i, j \in C_0 \), and where all the weights are in \((0, 1)\). Multiplying the weights by \( \beta \) and then adding \( \frac{\alpha - \beta}{2} \) to them, the subgraph of \( G \) induced by \( C_0 \) becomes a threshold graph with parameters \((\alpha, w)\) where \( w \) satisfies all the three criteria of Lemma 1.

In conclusion, the weight assignments of \( C_0 \) meet all three criteria of Lemma 1. We also point out that \( \forall i, j \in C_0 \),

\[
w(i) \leq w(j) \Rightarrow i R_0 j.
\]

**Step 2:** Let a constant \( \epsilon > 0 \) be such that it satisfies the following two inequalities:

\[
\epsilon < \min_{i, j \in C_0} \left\{ |w(i) - w(j)| : w(i) \neq w(j) \right\},
\]

\[
\epsilon < \frac{n}{n + 1} \min_{i \in C_0} \left\{ w(i) - \frac{\alpha - \beta}{2} \right\}.
\]

Notice that since \( w \) satisfies Criteria 2 and 3 of Lemma 1, an \( \epsilon > 0 \) such as described above exists. Then, for every \( l, 1 \leq l \leq m \), we define the vertex weights for \( C_l \) recursively as

Fig. 3: Decompositional structure of a DT graph.
follows: $\forall i \in C_l$, 
\begin{align*}
w(i) & \triangleq \beta + \left( \min \left\{ w(k) \mid k \in N(i) \cap C_{l-1} \right\} \right) \\
& - \frac{\epsilon}{n(n+1)^{l-1}} \left( 1 - \frac{|N(i)\cap C_{l-1}|}{n+1} \right),
\end{align*} 
(30)
where we note that $C_{m+1}$ was assumed to be the empty set. Observing that (27) holds for every $i, j \in C_0$, by induction on $l$, it is clear from (28) that for every $i, j \in C_l$, $1 \leq l \leq m$, 
\begin{equation*}
w(i) \leq w(j) \iff i \mathbf{R}_e j.
\end{equation*} 
(31)

Having defined the vertex weights, we are now ready to prove that $G(V, E)$ is an $(\alpha, \beta, w)$-DT graph. i.e., that the condition (9) is satisfied for every distinct $i, j \in V$. We consider the following cases:

Case 1: Let $i, j \in C_0$. We know that the subgraph of $G$ induced by $C_0$ is a threshold graph with parameters $(\alpha, w)$. Therefore, 
\begin{equation*}
e_{ij} \in E \iff w(i) + w(j) \geq \alpha.
\end{equation*} 
(32)
By noticing from the third criterion of Lemma 1 that both $w(i)$ and $w(j)$ lie in the interval $\left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right)$, we have $|w(i) - w(j)| \leq \beta$. This fact, together with (32), implies (9).

Case 2: Let $i \in V \setminus C_0$. We first state and prove the following lemmas.

Lemma 2. For every $C_l$, $0 \leq l \leq m$, and every $k' \in C_l$, we have 
\begin{equation*}
\alpha + \frac{(2l-1)\beta}{2} + \frac{\epsilon}{n(n+1)^{l-1}} < w(k') < \alpha + \frac{(2l+1)\beta}{2}.
\end{equation*} 
(33)

Proof: We prove the inequalities in (33) by induction on $l$. For $l = 0$, the first inequality of (33) is an immediate result of (29), while the second inequality follows from the third criterion of Lemma 1. We now assume that (33) holds for $l - 1$, $1 \leq l \leq m$, and prove that it also holds for $l$. To prove the first inequality of (33), we observe that 
\begin{equation*}
\min \left\{ w(k) \mid k \in N(k') \cap C_{l-1} \right\} \\
\geq \min_{k \in C_{l-1}} w(k) \\
> \frac{\alpha + (2l-3)\beta}{2} + \frac{\epsilon}{n(n+1)^{l-2}},
\end{equation*} 
(34)
where in the second inequality of (34), we used the induction hypothesis. Furthermore, 
\begin{equation*}
|N(k') \cap C_{l+1}| \geq 0.
\end{equation*} 
(35)
Using inequalities (34) and (35) in the recursive relation (30) results in the first inequality of (33). For the second inequality, by noticing that $|N(i) \cup C_{l+1}| \leq n$, one may use (30) to obtain 
\begin{equation*}
w(k') \leq \beta + \min_{k \in C_{l-1}} w(k) \\
< \beta + \frac{\alpha + (2l-1)\beta}{2} = \frac{\alpha + (2l+1)\beta}{2}.
\end{equation*} 
(36)
In the second inequality, we used the induction hypothesis for $l - 1$.

Lemma 3. For every $C_l$, $0 \leq l \leq m$, we have 
\begin{equation*}
\frac{\epsilon}{(n+1)^l} \leq \min_{w(k'), w'' \in C_0} \left\{ |w(k') - w(k'')| \mid w(k') \neq w(k'') \right\}.
\end{equation*} 
(37)

Proof: The proof follows by induction on $l$. For $l = 0$, (37) reduces to (28). We now assume that (37) holds for some $l - 1$, $1 \leq l \leq m$ and prove it for $l$. For $w(k')$ to be different from $w(k'')$, according to the recursive relation (30), at least one of the following relations must hold:
\begin{equation*}
N(k') \cap C_{l-1} \neq N(k'') \cap C_{l-1},
\end{equation*} 
(38)
\begin{equation*}
|N(k') \cap C_{l+1}| \neq |N(k'') \cap C_{l+1}|.
\end{equation*} 
(39)
Recalling (11), $R_e$ as defined in (24) is total on $C_l$. With no loss of generality, assume that $k' R_e k''$, which results in 
\begin{equation*}
N(k') \cap C_{l+1} \subseteq N(k') \cap C_{l-1},
\end{equation*} 
(40)
\begin{equation*}
|N(k') \cap C_{l+1}| \leq |N(k'') \cap C_{l+1}|.
\end{equation*} 
(41)
Case 1: If (38) holds, from (40) one has 
\begin{equation*}
N(k'') \cap C_{l-1} \subseteq N(k') \cap C_{l-1},
\end{equation*} 
(42)
which implies that 
\begin{equation*}
\min \left\{ w(k) \mid k \in N(k') \cap C_{l-1} \right\} \\
< \min \left\{ w(k) \mid k \in N(k'') \cap C_{l-1} \right\}.
\end{equation*} 
(43)

Notice that the difference between the two expressions on the opposite side of inequality (43) is at least $\epsilon/(n+1)^{l-1}$ by the induction hypothesis. Using this observation and (41) in the recursive relation (30) results in 
\begin{equation*}
w(k'') - w(k') \geq \epsilon/(n+1)^{l-1} > \epsilon/(n+1)^l.
\end{equation*} 
(44)
Case 2: If (39) holds, from (41), we have 
\begin{equation*}
|N(k') \cap C_{l+1}| < |N(k'') \cap C_{l+1}|,
\end{equation*} 
(45)
where the difference between the two expressions on the opposite side of inequality (45) is at least 1. We also know from (40) that 
\begin{equation*}
\min \left\{ w(k) \mid k \in N(k') \cap C_{l-1} \right\} \\
\leq \min \left\{ w(k) \mid k \in N(k'') \cap C_{l-1} \right\}.
\end{equation*} 
(46)
Using (45) and (46) in (30), we have 
\begin{equation*}
w(k'') - w(k') \geq \frac{\epsilon}{(n+1)^{l-1}} \left( \frac{1}{n+1} \right) = \frac{\epsilon}{(n+1)^l},
\end{equation*} 
which completes the proof. 

Recall that we wish to show that condition (9) is satisfied for every $i \in V \setminus C_0$, and $j \in V$. Without loss of generality, assume next that $i \in C_l$, $1 \leq l \leq m$, and $j \in C_{l'}$, where $0 \leq l' \leq l$. We analyze the cases $l' \leq l - 2$, $l' = l - 1$, and $l' = l$ as follows.

1. If $l' \leq l - 2$, we know from the defining property of the distance decomposition $(C_0, C_1, \ldots, C_m)$ that $e_{ij} \notin E$. On the other hand, according to Lemma 2, $w(i) - w(j) > \beta$. 

2. If $l' = l - 1$, we have 
\begin{equation*}
w(i) - w(j) \geq \beta.
\end{equation*} 
(47)

3. If $l' = l$, we have 
\begin{equation*}
w(i) - w(j) \geq \beta.
\end{equation*} 
(48)
Thus, the condition (9) holds.

2. If \( l' = l - 1 \), we consider two possibilities: \( e_{ij} \in E \) and \( e_{ij} \notin E \). If \( e_{ij} \in E \), from (30) we have
\[
    w(i) \leq \beta + \left( \min \{ w(k) \mid k \in \mathcal{N}(i) \cap C_{l-1} \} \right) = \beta + w(j).
\]
On the other hand, according to Lemma 2 we conclude that \( w(i) + w(j) \geq \alpha \). Thus, (7) holds. If \( e_{ij} \notin E \), then \( j \neq j' \) where
\[
    j' = \arg\min \{ w(k) \mid k \in \mathcal{N}(i) \}.
\]
If \( w(j') > w(j) \), then from Lemma 3
\[
    w(j') - w(j) > \frac{\epsilon}{(n + 1)^{l-1}}.
\]
Thus, from the recursive relation (30), it is straightforward to show that \( w(i) > w(j) + \beta \). As a result, condition (3) is satisfied. The inequality \( w(j') \leq w(j) \) is impossible, since otherwise from (31) and \( e_{ij'} \in E \), one would have \( e_{ij} \in E \).

3. If \( l' = l \), then \( e_{ij} \in E \) according to (2). On the other hand, from Lemma 4 we obtain that both \( w(i) + w(j) \geq \alpha \) and \( |w(i) - w(j)| \leq \beta \) are satisfied. Hence, (9) holds.

This completes the proof of Theorem 1.

IV. A POLYNOMIAL-TIME ALGORITHM FOR IDENTIFYING DT GRAPHS

Having characterized DT graphs and assigned weights to a DT graph given \( \alpha \) and \( \beta \), we are now ready to describe a polynomial-time algorithm for checking if a given graph \( G(V,E) \) is DT or not. In fact, the algorithm finds a distance decomposition satisfying conditions (i)-(iii) of Theorem 1 for a DT graph which is not a unit interval graph. If \( G \) is a DT graph, the algorithm finds a forbidden induced subgraph in \( G \) (see Fig. 5 of Section V for a more in depth discussion of forbidden subgraphs in DT graphs) or shows that there does not exist a distance decomposition satisfying conditions (i)-(iii) of Theorem 1 in \( G \). Throughout the remainder of the section, we first provide necessary definitions and concepts we need for the algorithm and then proceed to describe the polynomial-time algorithm itself.

We start by recalling forbidden subgraphs of unit interval graphs [16] and the definition of semi unit interval graphs.

**Lemma 4.** A graph is unit interval if and only if it is chordal (i.e., if it avoids cycles of length longer than three) and contains none of the graphs of Fig. 4 as induced subgraphs (the graphs are known as \( K_{1,3} \) (Fig. 4(a)), a sun graph (Fig. 4(b)) and a net graph (Fig. 4(c))).

**Definition 5.** A graph \( G(V,E) \) is semi unit interval if it is chordal and has no induced subgraph isomorphic to a net or a sun.

The following lemma immediately follows from Definition 5.

**Lemma 5.** If \( G(V,E) \) is a DT graph, then \( G \) is semi unit interval.

**Proof:** It is easy to see that there is no distance decomposition satisfying conditions (i)-(iii) of Theorem 1 for \( C_4 \), a sun and a net. (Connections between DT, unit interval and chordal graphs are discussed in more detail in Section V-A.)

**Definition 6.** For some \( p \in V \), a \( p \)-max partition is a partition \( (V_T, V_U) \) of \( V \) arising from some weight function \( w \) satisfying \( w(p) = \max \{ w(i) : i \in V_T \} \). Henceforth, we denote the subgraph of \( G \) induced by the set \( V_T \) by \( G[V_T] \) and the set induced by the set \( V_U \) by \( G[V_U] \).

**Definition 7.** Let \( G(V,E) \) be a semi unit interval graph, let \( p \in V \), and let \((V_T, V_U)\) be a partition of \( V \). We say that \((V_T, V_U)\) is \( p \)-admissible if all the following conditions hold:

1) No two vertices of \( V_T \) are incomparable in the vicinal preorder,
2) Among the vertices of \( V_T \), \( p \) is maximal in the vicinal preorder,
3) For every \( i \in V_T \), the set \( \mathcal{N}(i) \cap V_U \) is a clique, and
4) There are no induced subgraphs isomorphic to a bull (Fig. 5(a)) or \( K_{1,3} \) with any of the partitions of \( V \) into \( V_T \) and \( V_U \) as shown in Fig. 5 where a vertex in \( V_T \) is denoted by \( \circ \) and a vertex in \( V_U \) is denoted by \( \bullet \).
Theorem 2. Let $G(V, E)$ be a semi unit interval graph. A partition $(V_T, V_U)$ of $V$ is p-max if and only if it is p-admissible.

Proof: First we show that if $(V_T, V_U)$ is p-max, then it is p-admissible. Properties (1)-(3) follow immediately from Theorem 1. It remains to show that none of the forbidden partitions appear in the graph.

- Suppose that $X$ is the vertex set of a forbidden bull, and let $i$ and $j$ be the vertices of degree 1. If $i$ or $j$ is adjacent to some vertex of $V_T$, then the set of neighbors of $V_T$ does not form a clique, which is a contradiction to Theorem 1. If neither $i$ nor $j$ is adjacent to a vertex of $V_T$, then both $i$ and $j$ have distance exactly 2 from $V_T$; by Theorem 1 this means that $i$ and $j$ should be adjacent, which is not the case.

- Since $G[V_U]$ is a unit interval graph and $K_{1,3}$ is a forbidden induced subgraph for unit interval graphs, there cannot be any $K_{1,3}$ for which all vertices lie in $V_U$.  

- Suppose that $X$ is the vertex set of a $K_{1,3}$ with all vertices in $V_U$ except for a single leaf vertex $i \in V_T$. Let $j$ be the center vertex and let $k_1, k_2$ be the leaves that lie in $V_U$. Since for $p \in V_U, q \in V_T$ we have $e_{pq} \in E$ if and only if $|w(p) − w(q)| ≤ β$, we see that $w(j) ≤ w(k_1), w(j) ≤ w(k_2)$, since $e_{ik_1} \notin E$ and $e_{ik_2} \notin E$. Without loss of generality, assume that $w(j) ≤ w(k_1) ≤ w(k_2)$. Now $e_{j,k_1}, e_{j,k_2} \in E$ implies $e_{k_1,k_2} \notin E$, which is not the case.

Now, let $(V_T, V_U)$ be a p-admissible partition of $V$. The partitioning conditions immediately imply that $G[V_T]$ is a threshold graph and $G[V_U]$ is a unit interval graph. Let $C_0 = V_T$ and, for $l ≥ 1$, define $C_l$ as in [7], i.e.,

$$C_l = \left\{ i \in V \mid \min_{j \in C_l} \text{dist}(i, j) = l \right\}.$$ 

We establish next that each $C_l$ is a clique. Condition (3) immediately implies that $C_1$ is a clique. Assuming that $C_l$ is a clique, we show that $C_{l+1}$ is also a clique. Let $i, j \in C_{l+1}$ and suppose that $i$ and $j$ are nonadjacent. Each of the vertices $i$ and $j$ have at least one neighbor in $C_l$.

**Case 1**: The vertices $i$ and $j$ have a common neighbor $k \in C_l$. The vertex $k$ has a neighbor $q \in C_{l−1}$; now, $i, j, k, q$ is an induced $K_{1,3}$ subgraph with the center $k \in V_U$ and at least two leaves $i, j$ in $V_U$; but this configuration is forbidden.

**Case 2**: The vertices $i$ and $j$ have no common neighbor in $C_l$. Let $i' \in N(i) \cap C_l$ and let $j' \in N(j) \cup C_l$. Since $C_l$ is a clique, $e_{i', j'} \in E$. If $l = 1$, then we have that $p \in N(i') \cap N(j')$, since $p$ is maximal in the vicinal preorder on $V_T$. Now $p, i, j, i', j'$ induces a forbidden partitioning of vertices of the bull. If $l > 1$, let $k \in N(i') \cap C_{l−1}$. If $e_{k,j'} \notin E$, then $i', j', i, k$ is an induced $K_{1,3}$ with $i'$ as its center and all its vertices in $V_U$, which is forbidden. If $e_{k,j'} \in E$, then since $l − 1 ≥ 1$, we see that $k$ has some neighbor $q \in C_{l−2}$. Since $q$ cannot be adjacent to any of $i', j', i, j$, we see that $i, j, i', j', k, q$ induces a net. This contradicts the assumption that $G$ is semi unit interval.

Recall the definition of the preorder $R_\beta$ on $C_l$ given in [24]. We must show that $R_\beta$ is a total preorder on each $\ell$. Let $i, j \in R_\beta$ and suppose to the contrary that $i, j$ are incomparable in $R_\beta$. There are four possibilities (in fact, only two possibilities, up to symmetry), each of which we may eliminate.

**Case 1**: One has $N(i) \cap C_{l−1} \not\subset N(j)$ and $N(j) \cap C_{l−1} \not\subset N(i)$. In this case, there exist $i', j' \in C_{l−1}$ with $i' \in N(i) \setminus N(j)$ and $j' \in N(j) \setminus N(i)$. If $l > 1$, then since $C_{l−1}$ and $C_l$ are cliques, this implies that $ii'j'$ induces a $C_4$ in $G$, contradicting the assumption that $G$ is chordal. If $l = 1$, then this implies $ii'j'$ are vertices of $V_T$ that are incomparable in the vicinal preorder, contradicting the assumption that $(V_T, V_U)$ is p-admissible.

**Case 2**: One has $N(i) \cap C_{l−1} \not\subset N(j)$ and $N(j) \cap C_{l−1} \not\subset N(i)$. By symmetry, this is covered by Case 1.

**Case 3**: One has $N(i) \cap C_{l−1} \not\subset N(j)$ and $N(j) \cap C_{l−1} \not\subset N(i)$. Take $i_1 \in (N(i) \cap C_{l−1}) \setminus N(j)$ and $i_2 \in (N(i) \cap C_{l−1}) \setminus N(i)$. Observe that $e_{i_1,i_2} \notin E$, since $i_1$ is at distance $l−1$ from every vertex of $V_T$. Hence $\{i, i_2, i_1\}$ is an induced $K_{1,3}$ in $G$, with only the vertex $i_2$ possibly in $V_T$; this is a forbidden induced partitioning.

**Case 4**: One has $N(j) \cap C_{l−1} \not\subset N(i)$ and $N(i) \cap C_{l−1} \not\subset N(i)$. By symmetry, this is covered by Case 3.

Having defined all the necessary concepts, we now describe a polynomial-time algorithm that, given a graph $G$, either finds a partition of the vertex set into $(V_T, V_U)$ satisfying the conditions of Theorem 1 or proves that no such partition exists. The algorithm is:

1. If $G$ is not chordal, or if $G$ has an induced net or sun, output no. If $G$ has any isolates, let $G'$ be the graph with the isolates removed, and output the result for $G'$.
2. Check whether $G$ itself is unit interval, and if so, output $(V_T, V_U) = (\emptyset, V)$ and stop.
3. If $G$ is not a unit-interval graph, then for each vertex $p$, check whether there is a p-max decomposition $(V_T, V_U)$.
   If so, output any such partition; otherwise, output “no”.

To carry out the third step, we seek properties of p-max partitions.

**Lemma 6.** If $(V_T, V_U)$ is a p-max partition, then for every $i \in V_T$, we have $N(i) \subset N(p) \cup \{p\}$.

**Proof:** Let $w$ be a weight function witnessing that $(V_T, V_U)$ is p-max. For any $k \in N(i) \setminus \{p\}$, we have $w(p) + w(k) ≥ w(i) + w(k) ≥ w(i)$; since all vertices in $V_T$ are distance $\beta$ of each other, this implies that if $k \in V_T$, then $k \in N(p)$. If $k \in V_U$, then we also have $|w(k) − w(p)| = w(k) − w(p) ≤ w(k) − w(i) = |w(k) − w(i)|$, hence $k \in N(p)$.

**Corollary 1.** If $(V_T, V_U)$ is a p-max partition and $i$ is a vertex with $N(i) \not\subset N(p) \cup \{p\}$, then $i \in V_U$.

Let $W = \{i \in V(G) : N(i) \subset N(p) \cup \{p\}\}$. By Corollary 1, we have $V_T \subset W$ for any p-max partition $(V_T, V_U)$.
Lemma 7. Let $p \in V$ and let $(V_T, V_U)$ be a partition of $V$ such that:

1) $p \in V_T$,
2) all vertices of $V \setminus W$ are in $V_U$, and
3) all vertices in $V_U$ that are adjacent to $p$ form a clique.

If $(V_T, V_U)$ is not $p$-admissible, then either $G$ has a forbidden bull in which $p \in V_T$, or $G$ has a forbidden induced $K_{1,3}$ in which some vertex of $V \setminus W$ is the center vertex.

Proof: First, let $S$ be the vertex set of an induced forbidden bull, with vertices labeled as shown in Fig. 6. Since $i \in V_T$, we have $i \in W$. Therefore, $\{j_1, j_2\} \subset N(p)$. Since $e_{j_1, k_1}, e_{j_2, k_1} \notin E$ and since the vertices in $V_U$ that are incident to $p$ form a clique, we see that $k_1, k_2 \notin N(p)$. Therefore, $(S \setminus \{i\}) \cup \{p\}$ also induces a forbidden bull.

Now let $S$ be the vertex set of a forbidden induced subgraph $K_{1,3}$. Since the vertices in $V_U$ that are in $N(p)$ form a clique, at most one leaf vertex of $S$ lies in $N(p) \cap V_U$. In particular, the center vertex of $S$ has a neighbor outside $N(p) \cup \{p\}$, which implies that the center vertex does not lie in $W$, by Corollary 1.

We now define a 2SAT instance that models the vertex partition problem.

Definition 8. Given a semi unit interval graph $G$ and a vertex $p \in V$, we define a 2SAT instance as follows:

(i) For each $i \in V$, define a variable $x_i$, with the intended interpretation that $x_i$ should be true if and only if $i \in V_T$ in the partition;
(ii) We add a clause $(x_p \lor x_p)$, and for each $i \in V \setminus W$, we add a clause $(\neg x_i \lor \neg x_i)$;
(iii) For each nonadjacent pair of vertices $i, j \in N(p)$, we add a clause $(x_i \lor x_j)$;
(iv) For each pair of vertices $i, j$ that are incomparable in the vicinal preorder, we add a clause $(\neg x_i \lor \neg x_j)$;
(v) For every pair of vertices $i, j$ that are the leaves of some induced bull with $p$ as the degree-2 vertex, we add a clause $(x_i \lor x_j)$;
(vi) For every copy of $K_{1,3}$ with the center vertex $k \in V \setminus W$ with leaves $i, j, q$, we add three clauses $(x_i \lor x_j), (x_i \lor x_q), (x_j \lor x_q)$.

Theorem 3. For any semi unit interval graph $G$ and any $p \in V$, $G$ has a $p$-max partition if and only if the associated 2SAT instance is satisfiable.

Proof: First, suppose that $G$ has a $p$-max partition $(V_T, V_U)$. Consider the 2SAT assignment obtained by letting $x_i$ be true if and only if $i \in V_T$. We verify that all clauses of the 2SAT instance are satisfied:

- By Corollary 1 all clauses added in step (ii) are satisfied.
- Since in a $p$-max partition, the vertices in $V_U$ that are adjacent to $p$ form a clique, and all clauses added in step (iii) are satisfied.
- Since in a $p$-max partition the vicinal preorder is total on $V_T$, all clauses added in step (iv) are satisfied.
- Since a $p$-max partition omits the forbidden induced subgraphs of Definition 7 all clauses added in steps (v) and (vi) are satisfied.

On the other hand, suppose that the 2SAT instance is satisfiable. Let $(V_T, V_U)$ be the partition obtained by putting $i \in V_T$ if and only if $x_i$ is true. We verify that $(V_T, V_U)$ is $p$-admissible. Properties (1)-(3) of Definition 7 are easy to verify:

1) No two vertices of $V_T$ are incomparable in the vicinal preorder, since this would violate a clause added in step (iv).
2) Step (ii) guarantees that only vertices of $W$ can be in $V_T$, so $p$ is maximal among the vertices of $T$ in the vicinal preorder.
3) If for some $i \in V_T$, the set $N(i) \cap V_U$ is not a clique, then by the maximality of $N(p)$, we also have that $N(p) \cap V_U$ is not a clique, which would violate a clause added in step (iii).

To verify Property (4) of Definition 7 we first observe that satisfying the clauses added in steps (ii) and (iii) implies that $(V_T, V_U)$ satisfies the hypothesis of Lemma 7. Hence, if $G$ has a forbidden induced bull given in part (4) of Definition 7 then by Lemma 7 we can find such a forbidden subgraph with $p$ as the vertex of degree 2, which violates a clause added in step (v). Likewise, if $G$ has a forbidden induced $K_{1,3}$ given in part (4) of Definition 7 then by Lemma 7, we can find some forbidden $K_{1,3}$ whose center lies in $V \setminus W$, violating some clause added in step (vi).

V. FORBIDDEN SUBGRAPHS, INTERSECTION NUMBER, DIAMETER AND CLUSTERING COEFFICIENTS OF DT GRAPHS

We now turn our attention to analyzing properties of DT graphs that are relevant in the context of social network modeling. In particular, we study (a) forbidden induced subgraph structures for DT graphs, which appear to be uncommon structures in social and economic networks; (b) the intersection number of DT graphs, which is of relevance for latent feature modeling and inference in social networks [2], [17], [18]; (c) the diameter of DT graphs, capturing relevant connectivity properties of networks; and (d) the clustering coefficient, providing a normalized count of the number of triangles in the graphs.
A. Forbidden Induced Subgraphs of DT Graphs

Using the decomposition of DT graphs, it is straightforward to determine the structure of some of their forbidden induced subgraphs. We start with the following two simple results which we implicitly used in the previous section.

**Lemma 8.** DT graphs are $C_4$-free.

**Proof:** It suffices to show that $C_4$ is not a DT graph. First of all, since interval graphs are chordal, and hence avoid cycles of length greater than three, $C_4$ is not a unit interval graph. Moreover, it is easy to check that in any distance decomposition $(C_0, C_1, \ldots, C_m)$ of the vertices of $C_4$, at least one of the conditions (i), (ii) or (iii) in Theorem 1 is not met. More precisely, condition (i) asserts that either $|C_0| = 1$ or that $|C_0| = 2$ and that the two vertices in $C_0$ are not adjacent. If $|C_0| = 1$, then $C_1$ cannot be a clique, i.e., condition (ii) is violated. If $|C_0| = 2$ and the two vertices of $C_0$ are non-adjacent, then $C_1$ contains the other two non-adjacent vertices, which again violates condition (ii). Hence, from Theorem 1 $C_4$ is not a DT graph.

Using the same approach as the one used to prove Lemma 8 it can be seen that DT graphs are free of any cycle of length greater than four. As a result, we have the following corollary.

**Corollary 2.** DT graphs are chordal.

In addition to avoiding $C_4$ induced subgraphs, DT graphs may also be easily shown to avoid the subgraphs depicted in Fig. 1. Subgraph avoidance is, in general, is most easily established using techniques similar to those described in the proof of Lemma 8. For instance, given that DT graphs are chordal, the forbidden subgraphs of chordal graphs are automatically inherited by DT graphs. As another example, with regards to the subgraph $[a)$, it can be easily seen that for any choice of vertices satisfying condition (i) of Theorem 1 condition (ii) of Theorem 1 is not met and therefore, there is no a distance decomposition $(C_0, C_1, \ldots, C_m)$ satisfying the conditions of Theorem 1. Unfortunately, it appears difficult to characterize all forbidden subgraphs of DT graph. Since DT graphs are chordal, they are free of all cycles of length greater than three, but it is not clear if there exist finitely many forbidden subgraphs that are not cycles.

A final remark is in place. The distance decomposition of DT graphs depicted in Fig. 1 reveals an important hierarchical community structure in DT graphs, where subsets of vertices in the basic community – the threshold graph – belong to highly connected (clique) communities, which in turn have vertices that belong to highly connected communities, etc.

B. Intersection Number of a DT Graph

We start by providing relevant definitions regarding intersection graphs and intersection representations [16].

**Definition 9.** Let $F = \{S_1, \ldots, S_n\}$ be a family of arbitrary sets (possibly with repetition). The intersection graph associated with $F$ is an undirected graph with vertex set $F$ and the property that $S_\alpha$ is adjacent to $S_\beta$ if and only if $\alpha \neq \beta$ and $S_\alpha \cap S_\beta \neq \emptyset$.

We note that every graph can be represented as an intersection graph [19].

**Definition 10.** The intersection number of a graph $G(V, E)$ is the cardinality of a minimal set $S$ for which $G$ is the intersection graph of the family of subsets of $S$. The intersection number of $G$ is denoted by $i(G)$.

Equivalently, the intersection number equals the smallest number of cliques needed to cover all of the edges of $G$ [20], [21]. A set of cliques with this property is known as an edge clique cover. In fact, an edge clique cover of $G$ is any family $Q = \{Q_1, \ldots, Q_k\}$ of complete subgraphs of $G$ such that every edge of $G$ is in at least one of $E(Q_1), \ldots, E(Q_k)$, i.e. $e_{ij} \in E(G)$ implies that $e_{ij} \in \bigcup_{k=1}^{K} E(Q_k)$ [16].

Scheinerman and Trens [22] gave an algorithm to compute the intersection number of chordal graphs in polynomial time. Since DT graphs are chordal, it is possible to apply the Scheinerman–Trens algorithm to compute the intersection number of DT graphs. In this section, however, we present an explicit formula for the intersection number of DT graphs.

**Theorem 4.** Let $G$ be a $(\alpha, \beta, \omega)$-DT graph, and let $1, \ldots, n$ be the vertices of $G$, ordered so that $\omega(1) \leq \omega(2) \leq \cdots \leq \omega(n)$. For each $i \in \{1, \ldots, n\}$, let $\mathcal{N}_i^\omega = \{j > i : e_{ij} \in E\}$.
If 

\[ S = \{ i \in V(G) : N_i^+ \text{ is nonempty and } \{ i \} \cup N_i^+ \not\subset N_{i-1}^+ \}, \]

then \( i(G) = |S| \).

**Proof:** For each \( i \in S \), let \( C_i = \{ i \} \cup N_i^+ \). We claim that \( \{ C_i \}_{i \in S} \) is an edge clique cover of \( G \). Let \( e_{jk} \) be any edge of \( G \), with \( j < k \), and let \( i \) be the largest element of \( S \) satisfying \( i \leq j \). (Such an element must exist, since if \( \min S < j \), then \( d(i) = 0 \) for all \( i \leq j \), contradicting the existence of the edge \( e_{jk} \).) Since \( k \in N_j^+ \), the definition of \( S \) implies that \( \{ j \} \cup N_j^+ \not\subset N_{j-1}^+ \). Hence \( e_{jk} \in E(C_i) \). This implies that \( i(G) \leq |S| \).

To show that \( i(G) \geq |S| \), we give a set \( X \) of \( |S| \) edges such that any clique in \( G \) contains at least one edge in \( X \). For each \( i \in S \), the definition of \( S \) implies that we may fix a vertex \( u_i \in N_i^+ \) such that \( \{ i, u_i \} \not\subset N_{i-1}^+ \). By the definition of a \((\alpha, \beta, \gamma)\)-DT graph, this yields \( \{ i, u_i \} \not\subset N_i^+ \) for all \( r < i \). Let \( X = \{ u_i : i \in S \} \). Clearly \( |X| = |S| \).

Now suppose that \( i, j \) are distinct members of \( S \), with \( i < j \), and let \( C \) be a clique of \( G \) containing \( e_{jj} \). The choice of \( j \) implies that \( \{ j, j^* \} \not\subset N_i^+ \). Since \( i < j < j^* \), we have \( \{ j, j^* \} \not\subset N_i \), so \( i \not\in C \), and in particular \( e_{ii} \not\in E(C) \). Thus, every clique of \( G \) contains at most one edge of \( X \), so that \( i(G) \geq |S| \). □

**C. Diameter of a DT graph**

We start with a formal definition of the diameter of a graph [23].

**Definition 11.** The diameter of a graph \( G(V, E) \), denoted by \( D(G) \), is the longest shortest path between any two vertices of the graph.

Using the decomposition theorem for DT graphs, one can prove the following claim.

**Theorem 5.** Let \( G(V, E) \) be a connected DT graph that is not unit interval and let \( (C_0, C_1, \cdots, C_m) \) be a distance decomposition of \( G \). If \( m \geq 1 \), then \( D(G) = m + \lambda \), where \( \lambda \in \{0, 1\} \).

**Proof:** Clearly \( D(G) \geq m \), since a vertex in \( C_m \) and a vertex in \( C_0 \) have distance at least \( m \). Thus, it suffices to show that \( D(G) \leq m + 1 \).

First we claim that for all \( i, j \in C_0 \), we have \( \min \text{dist}(i, j) \leq 2 \). This follows from the fact that connected threshold graphs have diameter at most two [12].

Next we claim that for all \( i \in C_0 \) and \( j \in C_l \), where \( 1 \leq l \leq m \), we have \( \min \text{dist}(i, j) \leq l + 1 \). Let \( P \) be a path with \( l - 1 \) edges from \( j \) to some vertex \( k \in C_1 \). (Such a path necessarily exists, since for each \( r \), any vertex in \( C_r \) has a neighbor in \( C_{r-1} \).) If \( i \) has some neighbor \( q \in C_1 \), then \( Pqi \) (or \( Pi \) if \( q = i \)) is a \( j, i \)-path of length at most \( l + 1 \). Otherwise, let \( q \) be a \( R_0 \)-maximal vertex of \( C_0 \); since \( G \) is connected, we have \( i \in N(q) \), so that \( Pqi \) is again a path of length at most \( l + 1 \).

Finally, we claim that if \( i \in C_l \) and \( j \in C_l \), where \( r \geq l \), then \( \min \text{dist}(i, j) \leq (r - l) + 1 \). Let \( P \) be a path with \( r - l \) edges from \( j \) to a vertex \( k \in C_l \). If \( k \neq i \), then \( Pi \) is a \( j, i \)-path of length \( r - l + 1 \).

In all cases, we have \( \min \text{dist}(i, j) \leq m + 1 \). □

The diameter of most known social networks is known to be a small constant: For example, the *Small world phenomena* [24] suggests that the diameter of the underlying graphs is close to 6. Since the diameter of a DT graph with distance decomposition \( (C_0, C_1, \cdots, C_m) \) is at most \( m + 1 \), the question arises whether or not a given DT graph has a decomposition with \( m \leq 5 \).

To answer this question, we use the decomposition algorithm described in the previous section. We know that in a \( p \)-max partition, the vertices that possibly lie in \( V_T \) are the vertices at distance at most \( 2 \) from \( p \). In particular, if \( m \geq 2 \) and there is a vertex at distance greater than \( m \) from \( p \), then that vertex has to be in \( V_U \) in any \( p \)-max partition, and will therefore be in a clique at distance \( m + 1 \) from the threshold graph. Conversely, any vertex at distance at least \( m + 1 \) from the threshold graph is also at distance \( m + 1 \) from \( p \).

So, for \( m \geq 2 \), there is a partition with at most \( m \) layers in the unit-interval graph if and only if there is some vertex \( p \) such that (1) every vertex is within distance \( m \) of \( p \), and (2) the graph has a \( p \)-max partition.

For the special case \( m = 1 \), it is no longer necessary that every vertex is within distance \( 1 \) of \( p \), but the only way this is possible is if every vertex at distance \( 2 \) from \( p \) is in \( V_T \). These vertices are isolated vertices in the threshold graph. Therefore, for each such vertex, we can add \((x_v \lor x_u)\) as an additional constraint to the 2SAT problem and search for a vertex \( p \) such that the modified 2SAT problem has a solution. This would produce the desired decomposition.

For the special case \( m = 0 \), one only needs to check whether the graph is a threshold graph without isolates, which is straightforward to do, and as already mentioned, such graphs have diameter at most 2.

**D. The Clustering Coefficient of DT Graphs**

The global clustering coefficient of a graph is defined based on counts of triplets of vertices [25], [26] (A triplet consists of three connected vertices). A triangle in the graph includes three closed triplets, one centered on each of the vertices.

More formally, the global clustering coefficient is defined as:

\[ C = \frac{3 \times \text{number of triangles}}{\text{number of triplets}} \] (47)

\[ = \frac{\text{number of closed triplets}}{\text{number of triplets}}. \] (48)

To calculate the clustering coefficient of a DT graph, we again use a distance decomposition of the graph, say \( (C_0, C_1, \cdots, C_m) \). Assume that \( G(V, E) \) is a connected DT graph with \( |V| = n \) vertices, and assume that using [23] and [24], the order of the vertices of \( G \) has been established as \( 1, 2, \cdots, n \). In case that two vertices are assigned the same ranking within the order, we randomly break the tie. Let \( \{d_1, \cdots, d_n\} \) be a set in which \( d_i \) is the degree of the \( i \)-th vertex.
vertex in $G$, for $i = 1, \ldots, n$; i.e. $d_i = |N_i|$. Now, we provide closed formulas for the numerator and the denominator of (48).

- Clearly, the number of triplets in each graph equals $\sum_{i=1}^{n} \binom{d_i}{2}$. So, the denominator of (48) may be obtained using the sequence of the vertices of the graph.
- The numerator in the expression for the clustering coefficient may be found using the properties of DT graphs. Recall the definition of $N_i^+$ in Theorem[8] and let $d_i^+ = |N_i^+|$. Then, the number of the triangles equals $\sum_{i=1}^{n} \binom{d_i^+}{2}$. To see this, first consider a vertex $i$ and the set $N_i^+$. Let $j_1, j_2 \in N_i^+$, where $j_1 < j_2$. Let $i \in C_i$. If $j_1, j_2 \in C_i$, or $j_1, j_2 \in C_{i+1}$, then a triangle is formed by $i, j_1$ and $j_2$, as $C_i$ and $C_{i+1}$ form cliques. If $j_1 \in C_i$ and $j_2 \in C_{i+1}$, then $e_{j_1, j_2} \in E$. From $w_i + w_j \geq \alpha$ and $j_2 > j_1$, we have $w_{j_1} + w_{j_2} \geq \alpha$. From $|w_i - w_j| \leq \beta$ and $|w_i - w_{j_2}| \leq \beta$ we arrive at $|w_{j_1} - w_{j_2}| \leq \beta$. Therefore, $e_{j_1, j_2} \in E$. This proves the following lemma.

**Lemma 9.** Let $G(V, E)$ be a connected DT graph with $n$ vertices. Assume that the order of the vertices of $G$ has been established as $1, 2, \ldots, n$ using (23) and (24). Let $d_i$ and $d_i^+$ be $|N_i|$ and $|N_i^+|$, respectively. Then,

$$C = \frac{3 \sum_{i=1}^{n} \binom{d_i}{2}}{\sum_{i=1}^{n} \binom{d_i^+}{2}}.$$  

(49)

**Acknowledgment:** The authors gratefully acknowledge funding from the NIH BD2K Targeted Software Program, under the contract number U01 CA198943-02, and the NSF grants IOS1339388 and CCF 1117980. The authors would also like to thank Hoang Dau, Pan Li and Hussein Tabatabei Yazdi at the University of Illinois for helpful discussions.

**REFERENCES**

[1] A.-L. Barabási and R. Albert, “Emergence of scaling in random networks,” Science, vol. 286, no. 5439, pp. 509–512, 1999.
[2] M. O. e. Jackson, “Social and economic networks,” Princeton university press, vol. 3, 2008.
[3] W. Richards and O. Macindoe, “Decomposing social networks,” in IEEE Second International Conference on Social Computing (SocialCom), Minneapolis, MN, Aug. 2010, pp. 114–119.
[4] P. Diaconis, S. Holmes, and S. Janson, “Threshold graph limits and random threshold graphs,” Internet Mathematics, vol. 5, no. 3, pp. 267–320, 2008.
[5] M. Bradonjic, A. Hagberg, and A. G. Percus, “The structure of geographical threshold graphs,” Internet Mathematics, vol. 5, no. 1-2, pp. 113–119, 2008.
[6] E. P. Reilly and S. E. R., “Random threshold graphs,” The electronic journal of combinatorics, vol. 16, pp. 1–32, Oct. 2009.
[7] C. E. Tsourakakis, “Provably fast inference of latent features from networks,” in International World Wide Web Conference Committee, Florence, Italy, May 2015, pp. 111–112.
[8] K. Miller, M. I. Jordan, and T. L. Griffiths, “Nonparametric latent feature models for link prediction,” in Advances in Neural Information Processing Systems 22, 2009, pp. 1276–1284.
[9] K. Palla, Z. Ghahramani, and D. A. Knowles, “An infinite latent attribute model for network data,” in Proceedings of the 29th International Conference on Machine Learning (ICML-12), New York, NY, USA, 2012, pp. 1607–1614.
[10] J. Ugander, L. Backstrom, and J. Kleinberg, “Subgraph frequencies: Mapping the empirical and extremal geography of large graph collections,” in Proceedings of 22nd International World Wide Web Conference, 2013.