GALOIS ACTIONS FOR SEMIFIELD EXTENSIONS
AND GALOIS COVERINGS ON TROPICAL CURVES

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Abstract. For a semifield extension $T/S$, an action of a finite group $G$ on $T$ is Galois if (1) the $G$-invariant subsemifield of $T$ is $S$ and (2) subgroups of $G$ whose invariant semifields coincide are equal. We show that for a finite harmonic morphism between tropical curves $\varphi: \Gamma \to \Gamma'$ and an isometric action of a finite group $G$ on $\Gamma$, $\varphi$ is $G$-Galois if and only if the natural action of $G$ on the rational function semifield $\text{Rat}(\Gamma)$ of $\Gamma$ induced by the action of $G$ on $\Gamma$ is Galois for the semifield extension $\text{Rat}(\Gamma)/\varphi^*(\text{Rat}(\Gamma'))$, where $\varphi^*(\text{Rat}(\Gamma'))$ stands for the pull-back of $\text{Rat}(\Gamma')$ by $\varphi$.

1. Introduction

We call an injective semiring homomorphism between semifields $S \hookrightarrow T$ a semifield extension, and write it as $T/S$. For a semifield extension $T/S$, an action of a finite group $G$ on $T$ is Galois if (1) the $G$-invariant subsemifield $T^G$ of $T$ is $S$ and (2) subgroups of $G$ whose invariant semifields coincide are equal. For an intermediate semifield $M$ of $T/S$, we write as $G_M$ the subgroup of $G$ such that the restriction of its every element on $M$ is the identity map of $M$. For semifield extensions, an analogue of Galois correspondence holds:

**Theorem 1.1** (Galois correspondence for semifield extensions). Let $T/S$ be a semifield extension. Fix an action of a finite group $G$ on $T$. Let $A$ be the set of all intermediate semifields $M$ of $T/S$ such that $M = T^G_M$. Let $B$ be the set of all subgroups of $G$. Then, if the action of $G$ on $T$ is Galois for $T/S$, then the maps $\Phi : A \to B; M \mapsto G_M$ and $\Psi : B \to A; H \mapsto T^H$ satisfy $\Psi \circ \Phi = \text{id}_A$, $\Phi \circ \Psi = \text{id}_B$ and reverse the inclusion relations, where $\text{id}_A$ (resp. $\text{id}_B$) denotes the identity map of $A$ (resp. $B$). Moreover, for any $M \in A$, the natural action of $G_M$ on $T$ is Galois for $T/M$.

One of the biggest differences from field extensions is that for the natural action of the automorphism group $\text{Aut}(T/S)$ of a semifield extension $T/S$ on $T$, even the invariant subsemifield of $T$ by $\text{Aut}(T/S)$ is $S$, Theorem 1.1 may not hold. Here, $\text{Aut}(T/S)$ is the group of all elements of $T/S$. 

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automorphisms of $T$ whose restrictions on $S$ are the identity map of $S$. It occurs due to that the automorphism group of a semifield extension is more complicated than that of a field extension (see the following example). Hence we cannot drop the second condition of the definition of Galois actions for semifield extensions.

**Example 1.2.** The $T$-algebra automorphism group $\text{Aut}_T(\text{Rat}(\Gamma'))$ of the rational function semifield $\text{Rat}(\Gamma')$ of a tropical curve $\Gamma'$ is isomorphic to the automorphism group $\text{Aut}(\Gamma')$ of $\Gamma'$ by \cite[Corollary 1.3]{4}. Here, $T$ is the tropical semifield $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and a tropical curve is a metric graph that may have edges of length $\infty$. $\text{Aut}(\Gamma')$ coincides with the isometry group of $\Gamma'$ (except points at infinity). Hence, Artin’s theorem, which states that for a finite group $G$ of automorphisms of a field $L$ and the invariant subfield $K$ of $L$ by $G$, the extension $L/K$ is a finite Galois extension with Galois group $G$, clearly does not hold for semifield extensions.

The following theorem gives a relation between Galois coverings on tropical curves and Galois actions for semifield extensions:

**Theorem 1.3.** Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves and $G$ a finite group isometrically acting on $\Gamma$. Then, $\varphi$ is $G$-Galois if and only if the action of $G$ on the rational function semifield $\text{Rat}(\Gamma')$ of $\Gamma'$ naturally induced by the action of $G$ on $\Gamma$ is Galois for the semifield extension $\text{Rat}(\Gamma)/\varphi^*(\text{Rat}(\Gamma'))$.

Here, finite harmonic morphisms are morphisms of our category of tropical curves (see Section 2 for more details), $\varphi^*(\text{Rat}(\Gamma'))$ stands for the pull-back of $\text{Rat}(\Gamma')$ by $\varphi$, and “$\varphi$ is $G$-Galois” means that (1) $\varphi$ is a finite harmonic morphism of degree $|G|$ (the order of $G$) and (2) the action of $G$ on $\Gamma$ induces a transitive action on every fiber and (3) every stabilizer subgroup of $G$ with respect to all but a finite number of points is trivial.

This paper is organized as follows. In Section 2, we prepare basic definitions related to semirings and tropical curves which we need later. Section 3 gives proofs of Theorems 1.1 and 1.3. In that section, we also consider sufficient conditions such that finite group actions become Galois under some assumptions.

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2. Preliminaries

2.1. Semirings and congruences. In this paper, a semiring is a commutative semiring with the absorbing neutral element 0 for addition and the identity 1 for multiplication such that 0 ≠ 1. If every nonzero element of a semiring \( S \) is multiplicatively invertible, then \( S \) is called a semifield. A semiring \( S \) is additively idempotent if \( x + x = x \) for any \( x \in S \). An additively idempotent semiring \( S \) has a natural partial order, i.e., for \( x, y \in S \), \( x \geq y \) if and only if \( x + y = x \).

A map \( \phi : S_1 \to S_2 \) between semirings is a semiring homomorphism if for any \( x, y \in S_1 \),
\[
\phi(x + y) = \phi(x) + \phi(y), \quad \phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad \phi(0) = 0, \quad \text{and} \quad \phi(1) = 1.
\]
A semiring homomorphism \( \phi : S_1 \to S_2 \) is a semiring isomorphism if \( \phi \) is bijective. A semiring automorphism of \( S \) is a semiring isomorphism \( S \to S \).

Given a semiring homomorphism \( \phi : S_1 \to S_2 \), we call the pair \((S_2, \phi)\) (for short, \( S_2 \)) a \( S_1 \)-algebra. For a semiring \( S_1 \), a map \( \psi : (S_2, \phi) \to (S_2', \phi') \) between \( S_1 \)-algebras is a \( S_1 \)-algebra homomorphism if \( \psi \) is a semiring homomorphism and \( \phi' = \psi \circ \phi \). When there is no confusion, we write \( \psi : S_2 \to S_2' \) simply.

The set \( T := R \cup \{-\infty\} \) with two tropical operations:
\[
a \oplus b := \max\{a, b\} \quad \text{and} \quad a \odot b := a + b,
\]
where both \( a \) and \( b \) are in \( T \), becomes a semifield. Here, for any \( a \in T \), we handle \(-\infty\) as follows:
\[
a \oplus (-\infty) = (-\infty) \oplus a = a \quad \text{and} \quad a \odot (-\infty) = (-\infty) \odot a = -\infty.
\]
\( T \) is called the tropical semifield.

Let \( S \) be a semiring. A subset \( E \subset S \times S \) is a congruence on \( S \) if it is a subsemiring of \( S \times S \) that defines an equivalence relation on \( S \). The kernel of a semiring homomorphism \( \phi : S_1 \to S_2 \) is the congruence \( \ker(\phi) = \{(x, y) \in S_1 \times S_1 \mid \phi(x) = \phi(y)\} \).

2.2. Tropical curves. In this paper, a graph is an unweighted, undirected, finite, connected nonempty multigraph that may have loops. For a graph \( G \), the set of vertices is denoted by \( V(G) \) and the set of edges by \( E(G) \). The degree of a vertex is the number of edges incident to it. Here, a loop is counted twice. A leaf end is a vertex of degree one. A leaf edge is an edge incident to a leaf end.

A tropical curve is the underlying topological space of the pair \((G, l)\) of a graph \( G \) and a length function \( l : E(G) \to R_{\geq0} \cup \{\infty\} \), where \( l \) can take the value \( \infty \) on only leaf edges, together with an identification of each edge \( e \) of \( G \) with the closed interval \([0, l(e)]\). When \( l(e) = \infty \), the interval \([0, \infty) \) is the one point compactification of the interval \([0, \infty) \) and the leaf end of \( e \) must be identified with \( \infty \). We regard this not just as a topological space but as almost a metric space. The
distance between $\infty$ and any other point is infinite. If $E(G) = \{e\}$ and $l(e) = \infty$, then we can identify either leaf ends of $e$ with $\infty$. When a tropical curve $\Gamma$ is obtained from $(G, l)$, the pair $(G, l)$ is called a model for $\Gamma$. There are many possible models for $\Gamma$. A model $(G, l)$ is loopless if $G$ is loopless. Let $\Gamma_\infty$ denote the set of all points of $\Gamma$ identified with $\infty$. An element of $\Gamma_\infty$ is called a point at infinity. The valence of a point $x$ of $\Gamma$ is the minimum number of the connected components of $U \setminus \{x\}$ with all neighborhoods $U$ of $x$. Remark that this “valence” is defined for a point of a tropical curve and the “valence” in the first paragraph of this subsection is defined for a vertex of a graph, and these are compatible with each other. We frequently identify a vertex (resp. an edge) of $G$ with the corresponding point (resp. the corresponding closed subset) of $\Gamma$. The relative interior $e^\circ$ of an edge $e$ is $e \setminus \{v, w\}$ with the endpoint(s) $v, w$ of $e$.

2.3. **Rational functions and chip-firing moves.** Let $\Gamma$ be a tropical curve. A continuous map $f : \Gamma \to R \cup \{\pm \infty\}$ is a rational function on $\Gamma$ if $f$ is a piecewise affine function with integer slopes, with a finite number of pieces and that can take the value $\pm \infty$ at only points at infinity, or a constant function of $-\infty$. $\text{Rat}(\Gamma)$ denotes the set of all rational functions on $\Gamma$. For rational functions $f, g \in \text{Rat}(\Gamma)$ and a point $x \in \Gamma \setminus \Gamma_\infty$, we define

$$(f \oplus g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (f \odot g)(x) := f(x) + g(x).$$

We extend $f \oplus g$ and $f \odot g$ to points at infinity to be continuous on whole $\Gamma$. Then both are rational functions on $\Gamma$. Note that for any $f \in \text{Rat}(\Gamma)$, we have

$$f \oplus (-\infty) = (-\infty) \oplus f = f$$

and

$$f \odot (-\infty) = (-\infty) \odot f = -\infty.$$ 

Then $\text{Rat}(\Gamma)$ becomes a semifield with these two operations. Also, $\text{Rat}(\Gamma)$ becomes a $T$-algebra with the natural inclusion $T \hookrightarrow \text{Rat}(\Gamma)$. Let $\text{Aut}_T(\text{Rat}(\Gamma))$ denote the set of all $T$-algebra automorphisms of $\text{Rat}(\Gamma)$. Then $\text{Aut}_T(\text{Rat}(\Gamma))$ has a group structure. Note that for $f, g \in \text{Rat}(\Gamma)$, $f = g$ means that $f(x) = g(x)$ for any $x \in \Gamma$.

A subgraph of a tropical curve is a compact subset of the tropical curve with a finite number of connected components. Let $\Gamma_1$ be a subgraph of a tropical curve $\Gamma$ which has no connected components consisting of only a point at infinity, and $l$ a positive number or infinity. The chip-firing move by $\Gamma_1$ and $l$ is defined as the rational function $\text{CF}(\Gamma_1, l)(x) := -\min(l, \text{dist}(x, \Gamma_1))$ with $x \in \Gamma$, where $\text{dist}(x, \Gamma_1)$ stands for the distance between $x$ and $\Gamma_1$. 

2.4. Finite harmonic morphisms and Galois coverings. Let $\varphi : \Gamma \to \Gamma'$ be a continuous map between tropical curves. $\varphi$ is a finite harmonic morphism if there exist loopless models $(G, l)$ and $(G', l')$ for $\Gamma$ and $\Gamma'$, respectively, such that (1) $\varphi(V(G)) \subset V(G')$ holds, (2) $\varphi(E(G)) \subset E(G')$ holds, (3) for any edge $e \in G$, there exists a positive integer $\deg_e(\varphi)$ such that for any points $x, y$ of $e$, $\text{dist}(\varphi(x), \varphi(y)) = \deg_e(\varphi) \cdot \text{dist}(x, y)$ holds, and (4) for every vertex $v$ of $G$, the sum $\sum_{e \in E(G); e \to v', v \in e} \deg_e(\varphi)$ is independent of the choice of $e' \in E(G')$ incident to $\varphi(v)$. This sum is denoted by $\deg_v(\varphi)$. Then, the sum $\sum_{v \in V(G); v \to v'} \deg_v(\varphi)$ is independent of the choice of vertex $v'$ of $G'$. It is said the degree of $\varphi$. If both $\Gamma$ and $\Gamma'$ are singletons, we regard $\varphi$ as a finite harmonic morphism that can have any number as its degree. Note that for any $x \in \Gamma$, we can choose loopless models $(G, l), (G', l')$ above so that $x \in V(G)$ and $\varphi(x) \in V(G')$.

Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. For $f \in \text{Rat}(\Gamma')$, the push-forward of $f$ is the function $\varphi_*(f) : \Gamma' \to \mathbb{R} \cup \{\pm \infty\}$ defined as follows: for $x' \in \Gamma' \setminus \Gamma'_{\infty}$,

$$\varphi_*(f)(x') := \sum_{x \in \Gamma : \varphi(x) = x'} \deg_x(\varphi) \cdot f(x).$$

We continuously extend $\varphi_*(f)$ on $\Gamma'_{\infty}$. Then, $\varphi_*(f)$ is a rational function on $\Gamma'$. The pull-back $\varphi^*(f')$ of $f' \in \text{Rat}(\Gamma')$ is the rational function $f' \circ \varphi$ on $\Gamma$.

Remark 2.1. Let $\varphi : \Gamma \to \Gamma'$ be a map between tropical curves. Then $\varphi$ is a continuous map whose restriction on $\Gamma \setminus \Gamma'_{\infty}$ is an isometry if and only if it is a finite harmonic morphism of degree one. In this paper, we will use the word "a finite group $G$ isometrically acts on a tropical curve $\Gamma$" as the meaning that $G$ continuously acts on $\Gamma$ and it is isometric on $\Gamma \setminus \Gamma'_{\infty}$.

Remark 2.2. Let $\Gamma$ be a tropical curve and $G$ a finite group isometrically acting on $\Gamma$. Let $\Gamma/G$ be the quotient space (as topological space) and $\pi_G : \Gamma \to \Gamma/G$ be the natural surjection. Fix a loopless model $(V, E, l)$ ($V$ is a set of vertices and $E$ is a set of edges) for $\Gamma$ compatible with the action of $G$ on $\Gamma$, i.e., for any $g \in G$, $g(V) = V$ holds. Let $V' := \pi_G(V), E' := \pi_G(E)$ and for any $e \in E$, $l'(\pi_G(e)) := |G_e| \cdot l(e)$, where $G_e$ denotes the stabilizer subgroup of $G$ with respect to $e$. Then, $(V', E', l')$ gives $\Gamma/G$ a tropical curve structure and is a loopless model for the quotient tropical curve $\Gamma/G$. By loopless models $(V, E, l)$ and $(V', E', l')$ for $\Gamma$ and $\Gamma/G$, respectively, $\pi_G$ is a finite harmonic morphism of degree $|G|$.

Definition 2.3 ([3, Definition 4.1]). Let $\Gamma$ be a tropical curve and $G$ a finite group. An isometric action of $G$ on $\Gamma$ is Galois if there exists a finite subset $U'$ of $\Gamma/G$ such that for any $x' \in (\Gamma/G) \setminus U'$, $|\pi^{-1}_G(x')| = |G|$ holds.
Definition 2.4 ([3, Definition 4.2]). Let $\varphi : \Gamma \to \Gamma'$ be a map between tropical curves. $\varphi$ is Galois if there exists a Galois action of a finite group $G$ on $\Gamma$ such that there exists a finite harmonic morphism of degree one $\theta : \Gamma/G \to \Gamma'$ satisfying $\varphi \circ g = \theta \circ \pi_G$ for any $g \in G$. Then, we say that $\varphi$ is a $G$-Galois covering on $\Gamma'$ or just $G$-Galois.

3. Main results

Throughout this paper, we assume that an action of a finite group $G$ on a semifield $T$ induces a group homomorphism from $G$ to the automorphism group of $T$.

Definition 3.1 (Semifield extensions). Let $T, S$ be semifields. We call an injective semiring homomorphism $S \hookrightarrow T$ a semifield extension, and write it as $T/S$. We frequently identify a semifield extension $T/S$ with the inclusion $S \subset T$ via the injection $S \hookrightarrow T$.

Let $T/S$ be a semifield extension. Let $M$ be a semifield. A pair of injective semiring homomorphisms $S \hookrightarrow M$, $M \hookrightarrow T$ compatible with $T/S$ is called an intermediate semifield of $T/S$. We frequently identify the intermediate semifield with the inclusion $S \subset M \subset T$ via the injections above. We also call $M$ an intermediate semifield of $T/S$.

We call an automorphism of $T$ whose restriction on $S$ is the identity map of $S$ an automorphism of $T/S$. Let $\text{Aut}(T/S)$ denote the set of all automorphisms of $T/S$. Then, $\text{Aut}(T/S)$ becomes a group. We call it the automorphism group of $T/S$.

Let $G$ be a finite group. For an action of $G$ on $T$, we call the subset of $T$ whose each element is fixed by all elements of $G$ the $G$-invariant semifield, and write it as $T^G$. Then, $T^G$ naturally becomes an intermediate semifield of $T/S$. The action of $G$ on $T$ is Galois for $T/S$ if (1) $T^G = S$ and (2) subgroups of $G$ whose invariant semifields coincide are equal. For an intermediate semifield $M$ of $T/S$, we write as $G_M$ the subset of $G$ such that the restriction of its every element on $M$ is the identity map of $M$. Then, $G_M$ becomes a group.

Remark 3.2. Note that there exists a non-injective semiring homomorphism between semifields. In fact, the map $T \to B; t \neq -\infty \mapsto 0; -\infty \mapsto -\infty$ is a non-injective semiring homomorphism. Here, $B$ is the boolean algebra ($\{\neg \infty, 0\}, \text{max}, +$), which is a subsemifield of $T$.

Proposition 3.3. Let $T/S$ be a semifield extension. Fix an action of a finite group $G$ on $T$. Let $A$ be the set of all intermediate semifields $M$ of $T/S$ such that $M = T^{G_M}$. Let $B$ be the set of $G^H_T$ for any subgroup $H$ of $G$. Then, the maps $\Phi : A \to B; M \mapsto G_M$ and $\Psi : B \to A; G^H_T \mapsto T^{G^H_T}$ satisfy $\Psi \circ \Phi = \text{id}_A$, $\Phi \circ \Psi = \text{id}_B$ and reverse the inclusion relations.

Proof. It is straightforward. □
Remark 3.4. Let $T/S$ be a semifield extension. Let $G$ be a finite group acting on $T$. When $T^G = S$, the following are equivalent:

1. the action of $G$ on $T$ is Galois for $T/S$,
2. for any subgroups $H_1, H_2$ of $G$, if $T^{H_1} = T^{H_2}$, then $H_1 = H_2$,
3. for any subgroups $H_1, H_2$ of $G$, if $G_{T^{H_1}} = G_{T^{H_2}}$, then $H_1 = H_2$,
4. for any subgroups $H_1, H_2$ of $G$, if $\text{Aut}(T^{H_1}) = \text{Aut}(T^{H_2})$, then $H_1 = H_2$, and
5. for $T/S$, the Galois correspondence holds.

Proof. (1) $\iff$ (2) : clear.

(2) $\Rightarrow$ (3) : we shall show the contraposition. Assume that there exist distinct subgroups $H_1, H_2$ of $G$ such that $G_{T^{H_1}} = G_{T^{H_2}}$. For any $a \in T^{H_1}$ and $f \in G_{T^{H_2}}$, $f(a) = a$ holds since $G_{T^{H_1}} = G_{T^{H_2}}$. Since $H_2$ is a subgroup of $G_{T^{H_2}}$, we have $a \in T^{H_2}$. The converse inclusion is shown in a similar way, we have the conclusion.

(3) $\Rightarrow$ (4) : since the intersection of $G$ and $\text{Aut}(T^{H_1})$ is $G_{T^{H_1}}$, by (3), we have $H_1 = H_2$.

(4) $\Rightarrow$ (2) : for any subgroups $H_1, H_2$ of $G$, if $T^{H_1} = T^{H_2}$, then since $\text{Aut}(T^{H_1}) = \text{Aut}(T^{H_2})$, we have $H_1 = H_2$ by (4).

(1) $\Rightarrow$ (5) : it is given by Theorem 3.3.

(5) $\Rightarrow$ (2) : for any subgroups $H_1, H_2$ of $G$, assume that $T^{H_1} = T^{H_2}$ holds. Since $G_{T^{H_1}} = G_{T^{H_2}}$, by Theorem 3.3 we have $H_1 = H_2$. □

Theorem 3.5. Let $T/S$ be a semifield extension. Let $G$ be a finite group acting on $T$. Let $M$ be an intermediate semifield of $T/S$ such that $M = T^{G_M}$. Then, the following are equivalent:

1. $G_M$ is a normal subgroup of $G$,
2. for any $g \in G$, $g(M) \subset M$,
3. for any $g \in G$, $g(M) = M$,
4. there exists a subgroup $H$ of $\text{Aut}(M/S)$ such that for any $g \in G$, there exists $h \in H$ that coincides with the restriction $g|_M$, and
5. there exists a subgroup $H'$ of $\text{Aut}(M/S)$ such that for any $m \in M$, the orbits $Gm$ and $H'm$ coincide.

Moreover, if the action of $G$ on $T$ is Galois for $T/S$, then $M'' = S$ holds and $H$ can be choosen to be the natural action of $H$ on $T$ is Galois for $T/M$, and if the natural action of $H$ on $T$ is Galois for $T/M$, then $H$ is isomorphic to the quotient group $G/G_M$.

Proof. (1) $\Rightarrow$ (2) : for any $g \in G$, $g' \in G_M$, $m \in M$, as $g^{-1}g'g \in G_M$, we have $g^{-1}g'g(m) = m$. Thus, $g'(g(m)) = g(m) \in T^{G_M} = M$ holds.
(2) \implies (3): since \( g \) is arbitrary element of \( G \), we have also \( g^{-1}(M) \subset M \). Hence, \( M = g(g^{-1}(M)) \subset g(M) \) holds.

(3) \implies (4): we can define a group homomorphism \( \phi : G \to \text{Aut}(M/S) \) as \( g \mapsto g|_M \). The image \( \text{Im}(\phi) \) is the desired group.

(4) \implies (5): it is enough to choose the subgroup \( \{ g|_M \mid g \in G \} \) of \( H \) as \( H' \).

(5) \implies (2): it is clear.

(2) \implies (1): for any \( g \in G \), \( g' \in G_M \), \( m \in M \), since \( g(m) \in M \), we have \( g^{-1}g'g(m) = g^{-1}(g'(g(m))) = g^{-1}(g(m)) = m \). Therefore, we have \( g^{-1}g' \in G_M \).

Assume that the action of \( G \) on \( T \) is Galois for \( T/S \). It is clear that \( M^{H'} \supseteq S \) holds. For any \( a \in M^{H'} \), since \( \{ a \} = H'a = Ga \), by the orbit-stabilizer theorem (cf. [5, Chapter 6]), we have \( G_a = G \). Thus, \( a \in T^G = S \) holds. This means that \( M^{H'} = S \).

Assume that the action of \( G \) on \( T \) is Galois for \( T/S \). Let \( H = \{ g|_M \mid g \in G \} \). Note that it is \( \text{Im}(\phi) \). Let \( H_1, H_2 \) be subgroups of \( H \) satisfying \( M^{H_1} = M^{H_2} \). Let \( G_i := \phi^{-1}(H_i) \). Since

\[
M^{H_i} = \{ m \in M \mid \text{for any } h_i \in H_i, h_i(m) = m \} = \{ m \in M \mid \text{for any } g_i \in G_i, g_i(m) = m \} = \{ t \in T \mid \text{for any } g_i \in G_i, g_i(t) = t \} \cap M = T^{G_i} \cap M,
\]

we have \( T^{G_1} \cap M = T^{G_2} \cap M \). Since the kernel \( \ker(\phi) \) of \( \phi \) and \( G_M \) coincide, \( G_M \) is a subgroup of \( G_i \). Hence, \( M = T^{G_M} \supseteq T^{G_i} \) hold. Therefore, we have \( T^{G_1} \cap M = T^{G_i} \), and thus, \( T^{G_1} = T^{G_2} \).

Since the action of \( G \) on \( T \) is Galois for \( T/S \), \( G_1 \) must be \( G_2 \). Thus, \( H_1 = \phi(G_1) = \phi(G_2) = H_2 \) hold, and hence, the action of \( H \) on \( M \) is Galois for \( M/S \).

Assume that the action of \( H \) on \( M \) is Galois for \( M/S \). By assumption, \( H' := \{ g|_M \mid g \in G \} \) is a subgroup of \( H \). By the same argument above, since \( T^G = S = M^{H'} \supseteq M^{H} \supseteq S \), we have \( M^{H'} = M^{H} \).

Since the action of \( H \) on \( M \) is Galois for \( M/S \), we have \( H' = H \). Since \( H' = \text{Im}(\phi) \) and \( \ker(\phi) = G_M \), \( H' \) is isomorphic to \( G/G_M \). In conclusion, \( H \) is isomorphic to \( G/G_M \).

We give a sufficient condition that for a semifield extension \( T/S \), a given finite group action on \( T \) is Galois:

**Proposition 3.6.** Let \( T/S \) be a semifield extension. Fix an action of a finite group \( G \) on \( T \). If \( T^G = S \) holds and if for any subgroup \( H \) of \( G \), there exists an element \( a \in T \) whose stabilizer subgroup \( G_a \) is \( H \), then the action of \( G \) on \( T \) is Galois for \( T/S \).

**Proof.** Let \( H_1, H_2 \) be subgroups of \( G \). Assume that \( T^{H_1} = T^{H_2} \) holds. There exist \( a, b \in T \) whose stabilizer subgroups are \( H_1, H_2 \), respectively. Hence, we have \( H_1 = \bigcap_{c \in T^{H_1}} G_c = \bigcap_{d \in T^{H_2}} G_d = H_2 \). \( \square \)
Proposition 3.7. Let \( \varphi_1 : \Gamma \to \Gamma'_1 \) be a finite harmonic morphism between tropical curves. Then, the pull-backs \( \varphi_1^*(\text{Rat}(\Gamma'_1)) \) and \( \varphi_2^*(\text{Rat}(\Gamma'_2)) \) coincide if and only if there exists a finite harmonic morphism of degree one \( \varphi_{12} \) satisfying \( \varphi_1 = \varphi_{12} \circ \varphi_2 \).

Proof. We show the if part. Let \( f \in \varphi_1^*(\text{Rat}(\Gamma'_1)) \). There exists \( f' \in \text{Rat}(\Gamma'_1) \) such that \( f = \varphi_1^*(f') \). Hence we have \( f = \varphi_1^*(f') = f' \circ \varphi_1 = f' \circ \varphi_{12} \circ \varphi_2 \). Since \( \varphi_{12} \) is a finite harmonic morphism, we have \( f' \circ \varphi_{12} \in \text{Rat}(\Gamma'_2) \), and thus \( f \in \varphi_{12}^*(\text{Rat}(\Gamma'_2)) \). Since \( \varphi_{12} \) is a finite harmonic morphism of degree one, it is bijective and the inverse map \( \varphi_{12}^{-1} \) is also a finite harmonic morphism of degree one. Therefore, we have the inverse inclusion by the same argument.

We show the only if part. Let \( (V, E, l), (V'_1, E'_1, l'_1) \) be loopless models for \( \Gamma, \Gamma'_1 \), respectively, such that \( \varphi_1(V) = V'_1 \). Let \( (\tilde{V}, \tilde{E}, \tilde{l}), (V'_2, E'_2, l'_2) \) be loopless models for \( \Gamma, \Gamma'_2 \), respectively, such that \( \varphi_2(V) = V'_2 \). For any \( x \in \Gamma \setminus (V \cup \tilde{V}) \), there exist \( e \in E \) and \( \tilde{e} \in \tilde{E} \) containing \( x \). Then, we have \( \text{deg}_e(\varphi_1) = \text{deg}_e(\varphi_1) = \text{deg}_e(\varphi_2) = \text{deg}_e(\varphi_2) \) by the definition of pull-back of rational functions. In fact, there exists a positive number \( \varepsilon \) such that \( \varphi_1^*(\text{CF}(\{\varphi_1(x)\}, \varepsilon)) \) (resp. \( \varphi_2^*(\text{CF}(\{\varphi_2(x)\}, \varepsilon)) \)) has slope \( \text{deg}_e(\varphi_1) \) (resp. \( \text{deg}_e(\varphi_2) \)) on \( \varphi_1^{-1}(U'_1) \cap e \) (resp. \( \varphi_2^{-1}(U'_2 \cap \tilde{e}) \)), where \( U'_i \) is the \( \varepsilon \)-neighborhood of \( \varphi_i(x) \). Since these slopes \( \text{deg}_e(\varphi_1) \) and \( \text{deg}_e(\varphi_2) \) are the minimum (absolute values of) slopes other than zero on \( e \) and \( \tilde{e} \), respectively, and \( \varphi_1^*(\text{Rat}(\Gamma'_1)) = \varphi_2^*(\text{Rat}(\Gamma'_2)) \), \( \text{deg}_e(\varphi_1) \) must be \( \text{deg}_e(\varphi_2) \) (cf. \cite[Remark 3.3.24]{Remark}). Also, by the definition of the push-forward of rational functions, with a sufficiently small positive number \( \delta \), we have \( (\varphi_1)_*(\text{CF}(\{x\}, \delta) \circ \delta) = \text{CF}(\{\varphi_1(x)\}, \delta) \circ (\text{deg}_e(\varphi_1) \cdot \delta) \) and \( (\varphi_2)_*(\text{CF}(\{x\}, \delta) \circ \delta) = \text{CF}(\{\varphi_2(x)\}, \delta) \circ (\text{deg}_e(\varphi_2) \cdot \delta) \). Since \( \varphi_i \) is continuous, the map \( \varphi_{12} : \Gamma'_2 \to \Gamma'_1; \varphi_2(y) \to \varphi_1(y) \) is a finite harmonic morphism of degree one, where \( y \in \Gamma \).

Corollary 3.8. Let \( \varphi : \Gamma \to \Gamma' \) be a finite harmonic morphism between tropical curves. Let \( G \) be a finite group isometrically acting on \( \Gamma \). Then, \( \text{Rat}(\Gamma)^G = \varphi^*(\text{Rat}(\Gamma')) \) if and only if \( \varphi \) is \( G \)-Galois.

Proof. Let \( (V, E, l), (V', E', l') \) be loopless models for \( \Gamma, \Gamma' \), respectively, such that \( \varphi(V) = V' \). The if part follows from \cite[Remark 3.3.24]{Remark} since for any \( e \in E \), \( \text{deg}_e(\varphi) = 1 \). We shall show the only if part. If \( \text{Rat}(\Gamma)^G = \varphi^*(\text{Rat}(\Gamma')) \), then for any \( e \in E \), \( \text{deg}_e(\varphi) = 1 \). For any \( x \in \Gamma \setminus \Gamma_\infty \) and any \( \varepsilon > 0 \), since \( \varphi^*(\text{CF}(\{\varphi(x)\}, \varepsilon)) \) takes zero at and only at each element of \( \varphi^{-1}(\varphi(x)) \) and is \( G \)-invariant, we have \( Gx = \varphi^{-1}(\varphi(x)) \). Thus, \( \varphi \) is \( G \)-Galois.

Proof of Theorem 7.3 The if part follows from Corollary 3.8. We shall show the only if part. By Corollary 3.8 \( \text{Rat}(\Gamma)^G = \varphi^*(\text{Rat}(\Gamma')) \). Let \( G_1, G_2 \) be subgroups of \( G \). Assume that \( \text{Rat}(\Gamma)^{G_1} = \text{Rat}(\Gamma)^{G_2} \) holds. Let \( \Gamma'_i \) be the quotient tropical curve of \( \Gamma \) by \( G_i \). By \cite[Theorem 1.1]{Theorem}, the natural surjection \( \pi_{G_i} : \Gamma \to \Gamma'_i \) is \( G_i \)-Galois. Hence, there exists a
finite harmonic morphism of degree one $\pi_{12}$ satisfying $\pi_{G_1} = \pi_{12} \circ \pi_{G_2}$. Thus, by [3, Theorem 1.1] again, we have $G_1 = G_2$, which completes the proof. \qed

Note that by [4, Corollary 1.3], the automorphism group of a tropical curve $\Gamma$ is isomorphic to $\text{Aut}_T(\text{Rat}(\Gamma))$. Here, an automorphism of $\Gamma$ is a finite harmonic morphism of degree one $\Gamma \to \Gamma$. Hence, the following corollary holds:

**Corollary 3.9.** Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves and $G$ a finite group isometrically acting on $\text{Rat}(\Gamma)$. Then, $\varphi$ is $G$-Galois for the natural action of $G$ on $\Gamma$ if and only if the action of $G$ on $\text{Rat}(\Gamma)$ is Galois for $\text{Rat}(\Gamma)/\varphi^*(\text{Rat}(\Gamma'))$.

**Proposition 3.10.** Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. Let $G$ be a finite group isometrically acting on $\Gamma$. If $\varphi$ is $G$-Galois, then for any subgroup $H$ of $G$, there exists a rational function $f$ on $\Gamma$ whose stabilizer subgroup is $H$.

**Proof.** If $\Gamma$ (and hence $\Gamma'$) is a singleton, then the assertion is clear. Assume that $\Gamma$ (and thus $\Gamma'$) is not a singleton. Let $\Gamma''$ be the quotient tropical curve of $\Gamma$ by $H$. Let $x'' \in \Gamma''$ be a two valent point. There exists a positive real number $\varepsilon$ such that the $\varepsilon$-neighborhood $U''$ of $x''$ consists of only two valent points. Since $\varphi$ is $G$-Galois, the stabilizer subgroup of $G$ with respect to the pull-back $\pi_H^*(\text{CF}(\{x''\}, \varepsilon))$ is $H$. \qed

By this proposition and Proposition 3.6, we have the following corollary:

**Corollary 3.11.** Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. Let $G$ be a finite group isometrically acting on $\Gamma$. Then, $\text{Rat}(\Gamma)^G = \varphi^*(\text{Rat}(\Gamma'))$ and for any subgroup $H$ of $G$, there exists $g \in \text{Rat}(\Gamma)$ whose stabilizer subgroup is $H$ if and only if the natural action of $G$ on $\text{Rat}(\Gamma)$ is Galois for $\text{Rat}(\Gamma)/\varphi^*(\text{Rat}(\Gamma'))$.

![Figure 1](image.png)

**Figure 1.** On each figure, black dots (resp. lines) stand for vertices (resp. edges).

**Remark 3.12.** Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves. Let $G$ be a finite group isometrically acting on $\Gamma$. Then, even if the natural action of $G$ on $\text{Rat}(\Gamma)$ is Galois for $\text{Rat}(\Gamma)/\text{Rat}(\Gamma)^G$, it may not be Galois for $\text{Rat}(\Gamma)/\varphi^*(\text{Rat}(\Gamma'))$. See Example 3.13.
Example 3.13. Let $G$ be the graph consisting of three vertices $v_1, v_2, v_3$, two multiple edges $e_1, e_2$ between $v_1$ and $v_2$, and one edge $e_3$ between $v_2$ and $v_3$ (the left figure of Figure 1). Let $\Gamma$ be the tropical curve obtained from $(G, l)$, where $l(E(G)) = \{1\}$. Let $\sigma$ be the permutation $(e_1 e_2)$. The group $\langle \sigma \rangle$ generated by $\sigma$ naturally acts on $\Gamma$. Let $\Gamma'$ be the quotient tropical curve of $\Gamma$ by $\langle \sigma \rangle$. Note that the pair of the quotient graph $G' := G/\langle \sigma \rangle$ and the length function $l' : E(G') \to R \cup \{\infty\}; [e_1] \mapsto 1; [e_3] \mapsto 2$ is a model for $\Gamma'$, where $[e_i]$ denotes the equivalence class of $e_i$. The natural surjection $\pi_{\langle \sigma \rangle} : \Gamma \to \Gamma'$ is not $\langle \sigma \rangle$-Galois. Also, the natural action of $\langle \sigma \rangle$ on $\text{Rat}(\Gamma)$ is not Galois for $\text{Rat}(\Gamma)/\pi_{\langle \sigma \rangle}^*(\text{Rat}(\Gamma'))$ but $\text{Rat}(\Gamma)/\text{Rat}(\Gamma)^{\langle \sigma \rangle}$.

Remark 3.14. Let $\Gamma$ be a tropical curve. Then, even an isometric action of a finite group $G$ on $\Gamma$ is faithful, the natural action of $G$ on $\text{Rat}(\Gamma)$ may not be Galois for $\text{Rat}(\Gamma)/\text{Rat}(\Gamma)^G$. See Example 3.15.

Example 3.15. Let $G$ be the graph consisting of two vertices and three multiple edges $e_1, e_2, e_3$ between them (the right figure of Figure 1). Let $l$ be the length function such that $l(E(G)) = \{1\}$ and $\Gamma$ the tropical curve obtained from $(G, l)$. The symmetric group of degree three $\Sigma_3$ isometrically and faithfully acts on $\Gamma$ in a natural way. The invariant subsemifields by the permutation $(e_1 e_2 e_3)$ and by $\Sigma_3$ coincide. On the other hand, clearly $\langle (e_1 e_2 e_3) \rangle \neq \Sigma_3$. Hence the natural action of $\Sigma_3$ on $\text{Rat}(\Gamma)$ is not Galois for $\text{Rat}(\Gamma)/\text{Rat}(\Gamma)^{\Sigma_3}$.

Proposition 3.16. Let $S$ be an additively idempotent semiring and $\phi : S \to S$ an automorphism of $S$. If $S$ is totally ordered with respect to the natural partial order, then $\phi$ is the identity of $S$ or the order of $\phi$ is infinite.

Proof. Assume that $\phi$ is not the identity of $S$. Then, there exists an element $s \in S \setminus \{0, 1\}$ such that $\phi(s) \neq s$. Since $S$ is totally ordered, $\phi(s) + s = s$ or $\phi(s) = \phi(s)$. If $\phi(s) + s = s$, then $\phi^2(s) = \phi(s)$. By repeating the same argument, we have $s \geq \phi(s) \geq \phi^2(s) \geq \cdots$. Since $\phi(s) \neq s$, these are distinct. Hence, in this case, the cardinality of $\langle \phi \rangle s$ is infinite. When $\phi(s) + s = \phi(s)$, by the same argument, the cardinality of $\langle \phi \rangle s$ is also infinite. □

By Proposition 3.16, for a semifield extension $T/S$, if $T$ is an additively idempotent semifield totally ordered with respect to the natural partial order and an action of a finite group $G$ on $T$ is Galois for $T/S$, then $G$ is trivial and $T = S$.

Finally, we consider another sufficient condition that for a semifield extension $T/S$, a given finite group action on $T$ is Galois under some assumptions. Let $U$ be an additively idempotent semifield. Assume that $U$ is totally ordered with respect to the natural partial order. Let $T$ be a finitely generated semifield over $U$. Let $t_1, \ldots, t_n$ be generators of $T$. Then, the $U$-algebra homomorphism $\psi : U(X_1, \ldots, X_n) \to T$ defined
by $X_i \mapsto t_i$ is surjective, where $U(X_1, \ldots, X_n)$ denotes the semifield of all fractions of all polynomials with coefficients in $U$ and each $X_i$ is an indeterminate. By \cite[Proposition 2.4.4]{[1]}, $\psi$ induces a semiring isomorphism $\phi : U(X_1, \ldots, X_n)/\ker(\psi) \to T$. Let $V$ be the set \{ $u \in (U \setminus \{0\})^n$ | for any $(f, f') \in \ker(\psi), f(u) = f'(u)$ \}. Assume that for any two elements $[f], [f'] \in U(X_1, \ldots, X_n)/\ker(\psi)$, $[f] = [f']$ if and only if for any $v \in V$, $f(v) = f'(v)$ holds. Let $G$ be a finite group. Assume that an action of $G$ on $V$ induces the action of $G$ on $U(X_1, \ldots, X_n)/\ker(\psi)$ such that for any $g \in G$, $[f] \in U(X_1, \ldots, X_n)/\ker(\psi)$, $v \in V$, $g([f])(v) = f(g^{-1}(v))$. Since $U(X_1, \ldots, X_n)/\ker(\psi)$ is a $U$-algebra, this induced action induces a group homomorphism from $G$ to the $U$-algebra automorphism group of $U(X_1, \ldots, X_n)/\ker(\psi)$.

**Proposition 3.17.** In the above setting, if there exists an element $v$ of $V$ whose stabilizer subgroup of $G$ is trivial, then the action of $G$ on $T$ is Galois for $T/T^G$.

**Proof.** Let $v = (v_1, \ldots, v_n)$ and

$$f(X_1, \ldots, X_n) := \left[ \sum_{i=1}^{n} \left\{ v_i^{-1} \cdot X_i + (v_i^{-1} \cdot X_i)^{-1} \right\} \right]^{-1}.$$ 

For $u = (u_1, \ldots, u_n) \in V$, since $U$ is totally ordered, if $u \neq v$, then there exists $i$ such that $v_i^{-1} \cdot u_i \neq 0$. Hence, $v_i^{-1} \cdot u_i$ or $(v_i^{-1} \cdot u_i)^{-1}$ is bigger than 0. Thus, we have

$$f(u) = \begin{cases} 0 & \text{if } u = v, \\ < 0 & \text{if } u \neq v. \end{cases}$$

For any subgroup $H$ of $G$, let $f_H := \sum_{h \in H} h(f)$. By definition, $f_H$ takes zero at and only at elements of $Hv$, and values less than zero at any other elements of $V$. Therefore, the stabilizer subgroup of $G$ with respect to $f_H$ is $H$, which completes the proof by Proposition 3.6. \hfill \Box

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