Novel and Unique Expression for the
Radiation Reaction Force,
Relevance of Newton’s Third Law and Tunneling

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March 2014

Abstract: We derive the radiation reaction by taking into account that the acceleration of
the charge is caused by the interaction with some heavy source particle. In the non relativistic
case this leads, in contrast to the usual approach, immediately to a result which is Galilei
invariant. Simple examples show that there can be small regions of extremely low velocity
where the energy requirements cannot be fulfilled, and which the charged particle can only
cross by quantum mechanical tunneling. We also give the relativistic generalization which
appears unique. The force is a four-vector, but only if the presence of the source is taken
into account as well. It contains no third derivatives of the position as the Lorentz-Abraham-
Dirac equation, and consequently no run away solutions. All examples considered so far give
reasonable results.

1 Introduction: What has gone wrong?

“The problem of radiation reaction and the self force is the oldest unsolved mystery in physics”
as stated in the review article of Hammond [1]. We refer to this article for details about history,
suggested solutions, and literature.

Let us very briefly recall the standard derivation of the reaction force, the short comings in
this derivation, as well as the problems arising with the result obtained in this way.

The starting point is the well known Larmor formula for the energy loss per time of a
charged particle, accelerated by some external force:

\[ \frac{dE}{dt} = - \frac{2}{3} \frac{e^2}{c^3} \dot{v}^2. \]  

(1)

One argues that this energy loss has to be compensated by the power due to a radiation reaction
force \( f_{\text{rad}} \) which acts on the particle:

\[ f_{\text{rad}} v = - \frac{2}{3} \frac{e^2}{c^3} \dot{v} \cdot \dot{v}. \]  

(2)

Subsequently one integrates this relation over some time interval and performs a partial inte-
gration:

\[ \int_{t_1}^{t_2} f_{\text{rad}} \dot{v} dt = - \frac{2}{3} \frac{e^2}{c^3} \int_{t_1}^{t_2} \dot{v}^2 dt = \frac{2}{3} \frac{e^2}{c^3} \left( \int_{t_1}^{t_2} \ddot{v} dt - \dot{v} \dot{v} |_{t_1}^{t_2} \right). \]  

(3)

One now assumes that \( \ddot{v} \dot{v} \) vanishes at the end points and boldly concludes

\[ f_{\text{rad}} = \frac{2}{3} \frac{e^2}{c^3} \ddot{v}. \]  

(4)
Finally one can write down the relativistic generalization $F_{\text{rad}}^\mu$ of the radiation reaction force:

$$F_{\text{rad}}^\mu = \frac{2 e^2}{3 c^3} \left( \frac{d^3 u^\mu}{d\tau^2} + \frac{du^\mu}{d\tau} \frac{du^\nu}{d\tau} u^\mu c^2 \right),$$  

with $u^\mu$ the velocity four vector and $\tau$ the proper time. The second term has been added in order to ensure $u^\mu F_{\text{rad}}^\mu = 0$. The corresponding equation of motion is often called the Lorentz-Abraham-Dirac (LAD-) equation [2][3][4].

Maliciously speaking one may say that the Larmor formula (1) is the very last formula in this chain which is correct. Already (2), though presented in an innumerable number of text books, must obviously be wrong - from the simple reason that it is not Galilei invariant!

Under a non relativistic boost, i.e. a Galilei transformation of the form $x = x' + wt$, forces and accelerations are invariant, while velocities change according to $v = v' + w$. Clearly eq. (2) is not Galilei invariant.

The assumptions made after the partial integration are as well dubious and rarely ever fulfilled. The integrated terms $\dot{v}v_t|_{t_1}$ are again not Galilei invariant and only vanish for rather special cases. Nevertheless one appears astonished when the resulting equation, applied to situations where the assumption is definitely not fulfilled, gives unphysical runaway solutions. It is also not mandatory to conclude from the equality of the integrals to the equality of the integrands. And, finally, one only gets an information on the component of the force parallel to the velocity. The resulting equation of motion is also strange because it contains the third time derivative of the position, a rather unfamiliar feature. For the initial value problem one has to prescribe not only position and velocity, but also the acceleration.

In the present work we will approach and solve the problem in a direct way. An essential point is to give up the unphysical concept of an external field which is implicitly present in the usual procedure. Instead we assume that the “external” force arises through the interaction with a second (very heavy) source particle. We work directly with the equations for momentum and energy, without performing partial integrations or other manipulations, or needing any assumptions about the internal structure of the charge.

In the non relativistic case treated in sect. 2 our procedure corrects immediately the wrong equation (2), introduces the difference between the velocities of particle and source, and leads to a result which is Galilei invariant. No third derivatives $\dot{x}$ appear, one has the standard initial value problem of prescribing position and momentum, no run away solutions show up. For applications we assume, of course, that the mass of the source particle is very large, so it becomes essentially a static source. We apply our equation to three simple problems: Constant external force, harmonic oscillator, and circular motion. Some of the solutions show a remarkable property, at first sight a serious defect, but indeed an inevitable phenomenon which can be well understood and will be explained in detail later. In some (very tiny) regions there is no solution because the energy requirements cannot be fulfilled. The mathematics reacts by producing a complex solution in these regions. This indicates that we have to leave the field of classical physics here and must enter the world of quantum mechanics. The charged particle has to tunnel through a classically forbidden region.

In sect. 3 we generalize to relativistic motion. We construct a four-force which we consider as unique. It has all the properties which are required. The manifestly covariant form, as was to be expected from the non relativistic case, can only be obtained if one considers the source particle as well. We apply our equation to the linear relativistic motion in a constant electric field. This leads to a reasonable result.

Sect. 4 summarizes the properties of the radiation reaction force obtained here.
2 Non relativistic equation

We abandon the unphysical concept of an external force as reason for the acceleration and replace it by the force created by a large mass $M$ situated at $X$, which interacts with the considered particle of mass $m$ and charge $e$ via a potential $\varphi(x - X)$. The velocities of the particle and the source are denoted by $v$ and $V$. The source particle $M$ has no charge for simplicity.

For non relativistic motion one can ignore the radiation of momentum, there is only radiation of energy. Momentum conservation then gives $mv + M\dot{V} = 0$, while the energy loss per time becomes

$$\frac{d}{dt} \left( \frac{m}{2} v^2 + \frac{M}{2} V^2 + \varphi \right) = m\dot{v}v + M\dot{V}V + (v - V)\nabla\varphi$$

$$= (v - V)(m\dot{v} + \nabla\varphi)$$

$$= -\frac{2e^2}{3c^3} v^2. \quad (6)$$

Now the total force $f_{tot}$ is the sum of the force $f_{M\rightarrow m}$ which the source particle $M$ exerts on particle $m$, and the radiation reaction force $f_{rad}$, i.e. $f_{tot} = f_{M\rightarrow m} + f_{rad}$. Using the equation of motion $mv = f_{tot}$ and the relation $\nabla\varphi = -f_{M\rightarrow m}$, one gets $m\dot{v} + \nabla\varphi = f_{tot} - f_{M\rightarrow m} = f_{rad}$, and we are left with

$$(v - V)f_{rad} = -\frac{2e^2}{3c^3} v^2. \quad (7)$$

The auxiliary potential $\varphi$ has completely disappeared. Thus the equation for the radiation force becomes

$$f_{rad} = -\frac{2e^2}{3c^3} v \frac{v - V}{(v - V)^2}. \quad (8)$$

One could have added a term $f_\perp$ which is orthogonal to $v - V$. But from the vectors which are available one can only construct $(v - V) \times (\dot{v} - \dot{V})$ which is an axial vectors and therefore forbidden by parity.

Clearly invariance under non relativistic boosts is now restored, and the reason for the failure in the usual derivation is also clear. Performing the same steps as before, but eliminating the charged particle $m$ instead of the source $M$, one finds that there is not only a force on the particle under consideration but also a force on the source particle. Using the unphysical concept of an external field ignores one of the fundamental laws of physics: Isaac Newton’s third law $actio = reactio$.

The message learned from the above considerations is: **It is not possible to determine the radiation reaction force without knowing the motion of the source which causes the acceleration.**

But now a curiosity arises, in our opinion not a paradox, but nevertheless a rather strange consequence. Consider, for instance, a linear potential, i.e. a constant force. This can be approximately created by a far away source $M$. Imagine we could measure the radiation reaction force in a lab. Then, by comparison with (8), we could determine the relative velocity between particle and source, even without seeing the source! The situation becomes even more confusing if the force is due to several sources which are in motion with respect to each other. Plenty of conceptual problems to think about.
We will first give two simple applications for one dimensional problems and compare our results with those from the non relativistic LAD equation (4). For applications we take, of course, the limit \( M/m \to \infty \) and can accordingly choose a convenient system in which \( X = \text{const.} \) and \( V = 0 \), such that \( \varphi \) becomes a function of \( x \) only.

The equation of motion for one dimensional problems then reads

\[
m\ddot{x} = -\varphi'(x) - m\tau_c \frac{\dot{x}^2}{x}, \tag{9}\]

where \( \tau_c \) denotes the characteristic time

\[
\tau_c = \frac{2}{3} \frac{e^2}{c^3 m}, \tag{10}\]

not to be confused with the proper time \( \tau \) in the following section. Solving the quadratic equation (9) for \( \ddot{x} \) gives

\[
\ddot{x} = -\frac{\dot{x}}{2\tau_c} \left( 1 \pm \sqrt{1 - \frac{4\tau_c \varphi'}{m \dot{x}}} \right) \to \left\{ -\frac{\dot{x}}{\tau_c}, -\frac{\varphi'}{m} \right\} \text{ for } \tau_c \to 0. \tag{11}\]

Only the lower sign for the square root gives the correct limit for \( \tau_c \to 0 \), therefore we will always choose this sign in the following. One also realizes the possibility of a complex square root. Note that the differential equation is non linear, we don’t have the possibility of constructing real solutions by superposition. We will clarify the origin of this phenomenon when treating the example of a constant external force.

**Constant external force**

We consider a charge in a field with constant acceleration \(-g\). The equation of motion is (this equation was also used by Hammond [5], without, however, mentioning the classically forbidden regions)

\[
m\dot{v} = -mg - m\tau_c \frac{v^2}{v}, \tag{12}\]

or

\[
\dot{v} = -\frac{v}{2\tau_c} \left( 1 - \sqrt{1 - \frac{4g\tau_c}{v}} \right). \tag{13}\]

This is a first order separable differential equation for \( v(t) \). It can be explicitly integrated to obtain \( t \) as function of \( v \), but the result is not very enlightening. A qualitative discussion is more informative.

For negative \( v \) the differential equation is perfectly well behaved. This corresponds to the case that we simply drop the charge in the (say) gravitational field. For positive \( v \), however, i.e. if we throw the charge upwards, there is no longer a real solution if \( 0 < v < v_c \), where we introduced the critical velocity

\[
v_c = 4g\tau_c. \tag{14}\]

This behavior, though very strange at first sight, can be well understood. If the charge moves upwards it gains potential energy, furthermore it has to provide the energy for the radiation. These two energies have to be compensated by a corresponding loss of kinetic energy. But near the turning point the velocity is so small that it is no longer possible to decrease the kinetic energy sufficiently. There is no longer a solution for the energy requirement, one would need a negative kinetic energy. The mathematics consequently answers with a complex solution! For
negative velocities there is no problem. When the particle falls downwards it losses potential energy which can be used to provide the energy for the radiation.

Clearly classical physics breaks down here. We are confronted with a typical tunneling problem, where the particle has to tunnel through a classically forbidden region. We also expect a certain probability for reflection instead of transmission. For all practical situations one may safely ignore this (extremely small!) forbidden region. We did not attempt a quantum mechanical calculation of the tunneling. The problem is awkward, due to the lack of a Lagrangian or Hamiltonian formulation.

The problematic of the classically forbidden region disappears in a perturbative treatment of first order in $\tau_c$. More precisely, because (12) can be written in the parameter free form $\frac{d\tilde{v}}{d\tilde{t}} = -\frac{1}{\tilde{v}} - \frac{(d\tilde{v}/d\tilde{t})^2}{\tilde{v}}$, with $\tilde{v} = v/g\tau_c$, this means that $t \gg \tau_c$ with $t$ chosen such that the turning occurs near $t = 0$. One finds the perturbative solution

$$\dot{\tilde{v}}(t) = -g(1 - \tau_c/t).$$

This implies that $x(t)$ is finite everywhere, $v(t)$ has a logarithmic singularity, and $\dot{v}(t)$ a pole. If one keeps away from the classically forbidden region one has an excellent approximation to the exact solution.

Of course one can also check the energy balance $\Delta E_{pot} + \Delta E_{kin} + \Delta E_{rad} = 0$ in first order of $\tau_c$ explicitly. The frequently asked question “Where does the energy come from if the particle permanently radiates?” has a simple answer. The radiation has the effect that, at a given position, the particle has a smaller velocity than it would have without radiation, or, formulated the other way round, it reaches a certain velocity only at a lower altitude.

We compare with the LAD equation (4), $m\ddot{x} = -mg + m\tau_c\dot{x}$, or $a = -g + \tau_c\dot{a}$. It has the solution $a = -g + a_i\exp(t/\tau_c)$, i.e. either a non radiating solution for $a_i = 0$, or a run away solution for $a_i \neq 0$.

Harmonic oscillator

The equation of motion is

$$m\ddot{x} = -m\omega^2 x - m\tau_c\dot{x}^2/\dot{x}. \tag{16}$$

An ansatz of the form $x(t) = x_i\exp(\lambda t)$ leads to two complex solutions but these are useless for constructing a real solution because the equation is non linear and we are not allowed to build a superposition.

Remembering the previous discussion we expect that there will be no real solution close to the extrema where the velocity is small. Again the problem is less drastic in lowest order perturbation theory. One finds

$$x(t) = x_0 \cos \omega t \exp[-\tau_c(\omega^2 t/2 - \omega \tan \omega t \ln |\sin \omega t|)] \tag{17}.$$

We compare with the LAD equation $m\ddot{x} = -m\omega^2 x + m\tau_c \dot{x}$. This is easily solved by the ansatz $x(t) = x_0 \exp(\lambda t)$ with $\lambda$ a solution of the cubic equation $\lambda^2 + \omega^2 - \tau_c\lambda^3 = 0$. There is always one positive solution for $\lambda$ which represents the run away solution. Furthermore there is a pair of complex conjugate solutions with negative real part which lead to damped oscillations. If $\tau_c$ is treated in lowest order one finds $\lambda = \pm i\omega - \tau_c\omega^2/2$. This term is also present in our perturbative solution (17), but the latter contains further terms which take into account the varying radiation during the oscillation. The LAD equation, on the other hand, averages the whole process.
If we ignore the subtleties near the classically forbidden regions around the extrema we can derive a nice general property. Consider two consecutive zeros, located at \( t_1 \) and \( t_2 = t_1 + T/2 \). While \( x_1 = x_2 = 0 \), the velocities are different, \( v_2 = -\exp(-\alpha/2)v_1 \), say. But since (16) is a second order differential equation the solution is fixed by the initial values \( x_i, v_i \). Furthermore, if \( x(t) \) is a solution of (16), then this is also the case for \( \text{const} \cdot x(t) \). Therefore one can immediately conclude \( x(t + T/2) = (-1)^n \exp(-n\alpha/2) x(t) \) for any integer \( n \).

Knowledge within one half period is sufficient to know the whole solution.

Circular motion

Consider a charged particle which is forced to move in a circle of fixed radius \( r \), and no other external forces present. For the radial and azimuthal components of velocity and acceleration one has, using the simplification \( \dot{r} = 0 \),

\[
v_r = 0, \quad a_r = -v_\phi^2/r, \quad a_\phi = \dot{v}_\phi. \tag{19}
\]

Therefore \( a^2 = v_\phi^4/r^2 + \dot{v}_\phi^2 \). This leads to the following differential equation for \( v_\phi \equiv v \):

\[
m\dot{v} = -m\tau_c(v_\phi^3 + \frac{\dot{v}^2}{v}), \tag{20}
\]

or

\[
\dot{v} = -\frac{v}{2\tau_c}(1 - \sqrt{1 - 4\tau_c^2v^2/r^2}). \tag{21}
\]

The radicand is positive for any non relativistic motion and macroscopic radius. We don’t look for exact solutions but are content with an expansion in \( \tau_c \) to first order where we have to solve \( \dot{v} = -\tau_c v^3/r^2 \), resulting in

\[
v = v_i/\sqrt{1 + 2\tau_c v_i^2(t - t_i)/r^2}. \tag{22}
\]

In lowest order of \( \tau_c \) and for \( v \ll c \) this agrees with the result of Hammond \[1\] as well as with the solution from the LAD equation. This is not surprising because in this example approximately \( \dot{v} \perp v \), therefore the neglection of the integrated terms \( \dot{v}v|_{t_1}^{t_2} \) is justified.

3 Relativistic equation

The relativistic generalization of the Larmor formula (1) reads

\[
\frac{dP^\mu_{\text{rad}}}{d\tau} = -K u^\mu, \tag{23}
\]

with

\[
K = -\frac{2}{3} \frac{e^2}{c^3} \frac{du_\mu}{d\tau} \frac{du^\nu}{d\tau} = m\tau_c \left[ \frac{(dv/d\tau)^2}{1 - v^2} + \frac{(v\dot{v}/d\tau)^2}{(1 - v^2)^2} \right], \tag{24}
\]

where \( d\tau = \sqrt{1 - v^2} \, dt \) denotes the proper time of the particle, and \( u^\mu = (1/\sqrt{1 - v^2})(1, v) \) the four-velocity. With the exception of the coefficient in \( K \) we put \( c = 1 \) everywhere.
We closely follow the non relativistic treatment. For deriving the formula we use a simple model with an instantaneous potential \( \varphi(x - X) \). This model does not claim any physical relevance. It is translation invariant but not Lorentz invariant. Nevertheless it will lead us to a covariant formula for the reaction force. The potential \( \varphi(x - X) \) is only an auxiliary construct. As in the non relativistic case it will completely disappear in the final formula.

The momentum and energy of our system are

\[
\begin{align*}
P &= \frac{m}{\sqrt{1-v^2}} v + \frac{M}{\sqrt{1-V^2}} V, \\
P^0 &= \frac{m}{\sqrt{1-v^2}} v + \frac{M}{\sqrt{1-V^2}} + \varphi(x - X).
\end{align*}
\]

(25)

(26)

Now apply the time derivative \( d/dt = \sqrt{1-v^2} \, d/d\tau \) to these equations. For the derivation we can specialize to a one dimensional motion where the formulae simplify. The differentiation can be easily performed, the result has to be identical to \(-K\sqrt{1-v^2} \, u^\mu = -K(1, v)\). This leads to the two equations

\[
\begin{align*}
\frac{m}{(1-v^2)^{3/2}} \frac{dv}{dt} + \frac{M}{(1-V^2)^{3/2}} \frac{dV}{dt} &= -Kv, \\
\frac{m}{(1-v^2)^{3/2}} \frac{dv}{dt} + \frac{M}{(1-V^2)^{3/2}} V \frac{dV}{dt} + (v - V) \varphi' &= -K.
\end{align*}
\]

(27)

(28)

Eliminating the term with \( M \) from the momentum equation (27) and introducing into the energy equation (28) one obtains

\[
(v - V) \left( \frac{m}{(1-v^2)^{3/2}} \frac{dv}{dt} + \varphi' \right) = -K(1 - V v).
\]

(29)

We now work with the four-forces. As in the non relativistic case we have \( F_{tot} = F_{M \rightarrow m} + F_{\text{rad}} \), and we can use the equation of motion for \( F_{tot} \) and the connection between \( \varphi' \) and \( F_{M \rightarrow m} \),

\[
F_{tot} = \frac{m}{(1-v^2)^2} \frac{dv}{dt}, \quad \text{and} \quad \varphi' = -\sqrt{1-v^2} F_{M \rightarrow m},
\]

(30)

therefore

\[
\left( \frac{m}{(1-v^2)^{3/2}} \frac{dv}{dt} + \varphi' \right) = \sqrt{1-v^2}(F_{tot} - F_{M \rightarrow m}) = \sqrt{1-v^2} F_{\text{rad}}.
\]

(31)

One is left with

\[
F_{\text{rad}} = -K \frac{1 - V v}{\sqrt{1-v^2}(v - V)}.
\]

(32)

This can be written in a manifestly covariant form. In the non relativistic limit the force is proportional to the velocity, therefore \( F_{\text{rad}}^\mu \) has to contain a term proportional to the four-velocity \( u^\mu \). In order to fulfill the condition \( u_\mu F_{\text{rad}}^\mu = 0 \) it must appear in the combination \( u^\mu - U^\mu / U u \). One has

\[
u^m - U^m / U u = \frac{v^m - V^m + v^2 V^m - V v^m}{\sqrt{1-v^2}(1 - V v)}
\]

(33)
Therefore (32), which refers to the special case $V \parallel v$, can be written as the spatial part of
\[
F_{\mu}^{\text{rad}} = \frac{K}{(u^{\nu} - U^{\nu}/uU)^2} (u^{\mu} - U^{\mu}/Uu),
\]
with $K$ given in (24). It is important to emphasize that the formulation of $F_{\mu}^{\text{rad}}$ as a four-force is only possible if one has the four-vector $U^{\mu}$ available. We are confident that this solution is unique.

For applications we will of course again consider the limit $M/m \to \infty$, so we can put $U^{\mu} \approx (1,0)$. The spatial component of the reaction force (35) then simply becomes
\[
F_{\mu}^{\text{rad}} = K \frac{v^{\mu}}{(u^{\nu} - U^{\nu}/uU)^2} = -\frac{K}{\sqrt{1 - v^2}} \frac{v^{\mu}}{v^2}.
\]
We give a simple application.

**Constant external electric field**

We consider a linear motion in a constant electric field $-E$ with $E > 0$. The minus sign was chosen for convenient comparison with the non relativistic case discussed previously. For $v \parallel \dot{v}$ the formula (24) for $K$ simplifies to $K = m\tau_c (dv/d\tau)^2/(1 - v^2)^2$, and the equation of motion, conveniently written in terms of $t$, not of $\tau$, reads
\[
m \frac{dv}{dt} = -(1 - v^2)^{3/2} eE - m\tau_c \frac{(dv/dt)^2}{(1 - v^2)^{3/2} v}.
\]
This corresponds to the previous non relativistic equation (12) with the replacements $g \to (1 - v^2)^{3/2} eE/m$, $\tau_c \to \tau_c (1 - v^2)^{-3/2}$. We can again solve for $dv/dt$ and, performing these replacements in (14), obtain the critical velocity $v_c = 4eE\tau_c/m$, and the classically forbidden region $0 < v < v_c$.

The solution of (37) without the radiation term is well known. In first order perturbation theory in $\tau_c$ the solution of the full equation is
\[
v(t) = -\left[\frac{eEt}{q} - \tau_c meE \left(\frac{1}{q^2} + \frac{m}{2q^3} \ln \frac{q - m}{q + m}\right)\right],
\]
with
\[
q = \sqrt{m^2 + (eE)^2 t^2}.
\]
The limit for large times is particularly simple:
\[
v(t) \to -\left[1 - \frac{m^2}{2(eE)^2} (1 + \frac{2\tau_c eE}{m}) \frac{1}{t^2}\right].
\]
It shows how the approach of $v$ to $-1$ is slowed down by the radiation.
Summary and conclusions

We summarize our essential results and the properties of the equations derived here.

• It is mandatory to take into account the presence of the source which causes the acceleration of the charged particle. Only then one ends up with a force with the correct transformation property.

• There can be (extremely small) regions where the velocity is so small that the energy requirements cannot be fulfilled. The charge has to pass through such regions by quantum mechanical tunneling. It is impressive how the equation responds to the appearance of classically forbidden regions.

• Because we chose a direct approach without manipulations like partial integrations our equation does not contain third derivatives $\dddot{x}$, in clear contrast to the LAD equation. Consequently there are no problems with run away solutions.

• The non relativistic equation for the radiation reaction force can be considered as unique, because it was derived in a direct straightforward way. We believe that this is also the case for the relativistic generalization.

• In all examples considered so far we obtained reasonable results.

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