Concept of Fully Dually Symmetric Electrodynamics

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It has been shown, that electromagnetic field (EM) has in general case quaternion structure, consisting of four independent fields, which differ each other by the parities under space inversion and time reversal. It follows immediately from Rainich dual symmetry of Maxwell equations and additional hyperbolic dual symmetry, established in given work. It has also been shown, that for any complex relativistic field the gauge invariant conserving quantity is two-component scalar or pseudoscalar value - complex charge. It means in applicability to EM-field, that its gauge symmetry group is determined by two-parametric group \( \Gamma(\alpha, \beta) = U_1(\alpha) \otimes R(\beta) \), where \( R(\beta) \) is abelian multiplicative group of real numbers (excluding zero). Generalized Maxwell equations for four-component EM-field are obtained on the basis of its both dual and hyperbolic dual symmetries. Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry are found. It is shown, that the only one physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the functional space. In fact it is the proof for four component structure of EM-field to be a single whole, that is confirmation along with the possibility of the representation of EM-field in four component quaternion form the necessity of given representation. It extends the overview on the nature of EM-field itself.

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I. INTRODUCTION AND BACKGROUND

A. Matrix Algebra of Complex Numbers and its some Consequences for Quantum Theory

Let us summarize some useful results from algebra of the complex numbers. The numbers 1 and \( i \) are usually used to be basis of the linear space of complex numbers over the field of real numbers. At the same time to any complex number \( a + ib \) can be set up in conformity the \([2 \times 2]\)-matrix according to biective mapping

\[
f : a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.
\]

(1)

The matrices

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

(2)

produce basis for complex numbers \( \{a + ib\}, a, b \in R \) in the linear space of \([2 \times 2]\)-matrices, defined over the field of real numbers. It is convenient often to define the space of complex numbers over the group of real positive numbers, then the dimensionality of the matrices and basis has to be duplicated, since to two unities - positive 1 and negative \(-1\) can be set up in conformity the \([2 \times 2]\)-matrices according to biective mapping

\[
\varepsilon : 1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

(3)

which allows to recreate the operations with negative numbers without recourse of negative numbers themselves. Consequently, in accordance with mapping \( \zeta \) the following \([4 \times 4]\)-matrices, so called \([0,1]\)-matrices, can be basis of complex numbers

\[
\zeta : 1 \rightarrow [e_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
i \rightarrow [e_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
-1 \rightarrow [e_3] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
-i \rightarrow [e_4] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

(4)

The choise of basis is ambiguous. Any four \([4 \times 4]\) \([0,1]\)-matrices, which satisfy the rules of cyclic recurrence

\[
i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1
\]

(5)
can be basis of complex numbers. In particular, the following $[4 \times 4]$ $[0,1]$-matrices

$$
\begin{align*}
[e_1] &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\quad
[e_2] = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},
\end{align*}
$$

$$
\begin{align*}
[e_3] &= \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\quad
[e_4] = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},
\end{align*}
$$

can also be basis of complex numbers. Naturally, the set of $[0,1]$-matrices, given by (3) is isomorphous to the set, which is given by (1). It is evident, that the system of complex numbers can be constructed by infinite number of the ways, at that cyclic basis can consist of $m$ units, $m \in \mathbb{N}$, starting from three. It is remarkable, that the conformity between complex numbers and matrices is realized by bijective mappings. It means, that there is also to existing the inverse mapping, by means of which to any square matrix, belonging to the linear space with a basis given by (1), or (3), or any other, satisfying the rules of cyclic recurrence like to (5), can be set up in conformity the complex number. In particular to any Hermitian matrix $H$ can be set up in conformity the complex number in correspondence with mapping $\xi$

$$
\xi : H \to S + iA = \begin{bmatrix}
S & -A \\
A & S
\end{bmatrix},
$$

where $S$ and $A$ are symmetric and antisymmetric parts of Hermitian matrix. Given short consideration allows to formulate the following statements.

1. Quantized free EM-field is complex field in general case.

Proof is evident and it is based on (7), if to take into account, that quantized free EM-field can be determined by Hermitian operators $\hat{E}(\vec{r},t)$ and $\hat{H}(\vec{r},t)$, representing themselves the full set of quantized free EM-field operator vector-functions, that is, they can serve for basis in corresponding operator vector-functional space (see Sec.II). Given statement can be generalized.

2. Any quantumphysical quantity is complex quantity in general case.

Proof is evident and it is based on the same relationship, since any quantumphysical quantity is determined by Hermitian operator. Therefore, two sets of observables, which are determined by real functions, correspond to any quantumphysical operator quantity in general case.

B. Additional gauge invariance of complex relativistic fields

We will argue in the next Section, that EM-field in the matter can be considered in general case to be complex field, each component in which is also complex field, that is, it has quaternion nature. In given Section we will prove the idea, that for any complex field the conserved quantity, corresponding to its gauge symmetry, that is, charge, can be in general case also complex.

Let $u(x) = \{ u_i(x) \}$, $i = \overline{1,n}$, the set of the functions of some complex relativistic field, that is, scalar, vector or spinor field, given in some space of Lorentz group representations. It is well known, that Lagrange equations for any complex relativistic field can be represented in the form of one matrix relativistic differential equation of the first order in partial derivatives, that is in the form of so called generalized relativistic equation, and analogous equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions $u^+(x) = \{ u_i^+(x) \}$. The equation for the set $u(x)$ of field functions is

$$
(\alpha_\mu \partial_\mu + \kappa u_0)u(x) = 0.
$$

Similar equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions, that is for the functions $u^+(x) = \{ u_i^+(x) \}$, $i = \overline{1,n}$, is

$$
\partial_\mu u^+(x)\alpha_\mu + \kappa u^+(x)\alpha_0 = 0.
$$

In equations (8, 9) $\alpha_\mu, \alpha_0$ are matrices with constant numerical elements. They have an order, which coincides with dimension of corresponding space of Lorentz group representation, realized by $\{ u_i(x) \}$, $i = \overline{1,n}$. In particular, they are $[n \times n]$- matrices, if $\{ u_i(x) \}$, $i = \overline{1,n}$ are scalar functions. It is evident, that the transformation

$$
u'(x) = \beta exp(\alpha)u(x),
$$

where $\alpha, \beta \in \mathbb{R}$, and analogous transformation for Hermitian conjugate functions (or complex conjugate functions in the case of scalar fields)

$$
u'^+(x) = \beta exp(-\alpha)\nu^+(x)
$$

keep Lagrange equations (8, 9) to be invariant. It is understandable that transformation of field functions by relationships (10, 11) is equivalent to multiplication of field functions by arbitrary complex number. It is well known, that given linear transformation is the simplest example of isomorphism of corresponding linear space, which is given over the field of complex numbers, onto itself, that is, in the case considered the relationships (10, 11) give isomorphism of the space of field functions. Automorphism of any linear space leads to some useful properties of the objects, which belong to given space. For instance, if to set up in a correspondence to the space of field function the affine space, then conservation laws of collinearity of the points and of simple relation of the triple of collinear points will be fulfilled by automorphism in given affine space. Consequently, we have to expect the physical consequences of given algebraic property in the case of physical spaces. Conformally to
the case considered we have in fact gauge transformation of field functions, which is more general in comparison with usually used. The set $(\beta \exp(-i\alpha))$ for all possible $\alpha$, $\beta \in R$ produces the group $\Gamma$, which is direct product of known symmetry group $U_1$, and multiplicative group $\mathbb{R}$ of all real numbers (without zero). Therefore, in the case considered the symmetry group of given complex field acquires additional parameter. So, we will have

$$\Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathbb{R}(\beta) \quad (12)$$

Let us find the irreducible representations of the group $\mathbb{R}(\beta)$. It has to be taken into account, that the group $\mathbb{R}(\beta)$ is abelian group and its irreducible representations $T(\mathbb{R})$ are onedimensional. So, the mapping

$$T : \mathbb{R} \rightarrow T(\mathbb{R}) \quad (13)$$

is isomorphism, where $T(1) = 1$. Therefore, for $\forall (\beta, \gamma)$ of pair of elements of group $\mathbb{R}(\beta)$ the following relationship takes place

$$T(\beta, \gamma) = T(\beta)T(\gamma). \quad (14)$$

Then, it is easy to show, that

$$T(\beta) = \beta^{2k+1}. \quad (15)$$

The value $\beta^{2k+1}(1)$ can be obtained from the condition

$$T(-\beta) = -T(\beta). \quad (16)$$

Consequently, we have

$$T(\beta) = \beta^{2k+1} = \exp[(2k + 1)\ln|\beta|], \quad (17)$$

where $k \in N$. Then irreducible representations of the group $\Gamma(\alpha, \beta)$ represent direct product of irreducible representations of the groups $U_1(\alpha)$ and $\mathbb{R}(\beta)$

$$T(U_1(\alpha)) \otimes T(\mathbb{R}(\beta)) = \exp(-i\alpha)\exp[(2k + 1)\ln|\beta|], \quad (18)$$

where $m, k = 0, \pm 1, \pm 2, \ldots$

It is clear, that some conserved quantity has to correspond to gauge symmetry of the field, which is determined by the group $\mathbb{R}(\beta)$. Thus we arrive at a formulation of the following statement.

3. **Conserving quantity - complex charge, which is invariant under total gauge transformations, corresponds to any complex relativistic field (scalar, vector, spinor).**

**Proof.**

Really, since generalized relativistic equations are invariant under transformations (10) and variation of action integral with starting Lagrangian is equal to zero, then variation of action integral with transformed Lagrangian in accordance with (11) will also be zero. Consequently, all the conditions of applicability of Nöther theorem, by proof of which the only invariance under Lagrange equations is sufficient, [1], are held true. We wish to pay attention to typical inaccuracy, which is abundant in the literature, consisting in that, that for applicability of Nöther theorem the Lagrangian under corresponding symmetry transformations is required. At the same time the only invariance of Lagrange equations under corresponding symmetry transformations, which certainly takes place in given case, is necessary (see proof of Nöther theorem).

According to Nöther theorem, the conserved quantity, corresponding to $\nu - th$ parameter ($\nu = \Gamma, R$) by invariance of field under some $k$-parametric symmetry group, is (see, for instance, [2]).

$$Q_\nu(\sigma) = \int_{(\sigma)} \theta_{\mu\nu} d\sigma_\mu = \text{const}, \quad (19)$$

where $\sigma$ is any spacelike hypersurface, $\sigma \subset 1R_4$ and 4-tensor $\theta_{\mu\nu}$ is determined by relation

$$\theta_{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu u_\nu)} [\partial_\mu u_\nu X_{\nu\mu} - Y_{\nu\mu}] - LX_{\mu\nu}. \quad (20)$$

in which $L$ is field Lagrangian and the matrices $X_{\mu\nu}, Y_{\mu\nu}$ are determined by matrix representations $||(I_\nu)_\mu||$ and $||(J_\nu)_k||$ of infinitesimal operators of symmetry group in coordinate space and in the space of field functions respectively in accordance with the following relationships

$$X_{\mu\nu} = (I_\nu)_{\mu\alpha} x_\alpha, Y_{\mu\nu} = (J_\nu)_{ik} u_k. \quad (21)$$

Since the value of integral in (19) does not depend on the choose of spacelike hypersurface, then usually the hypersurface, which is orthogonal immediately to time axis, is used. By given choose 4-vector $d\sigma_\mu$, representing itself infinitesimal element of spacelike hypersurface, is $\{d\sigma_\mu\} = \{0, 0, 0, d\sigma_4\}$, where $d\sigma_4 = -i\partial^4x$. Then the expression (19) gets the form

$$Q_\nu(\sigma_4) = -i \int_{(\sigma_4)} \theta_{4\nu} d^3x = \text{const}, \quad (22)$$

where the conservation of the quantity $Q_\nu(\sigma_4)$ in time is represented in explicit form, since the time can be unambiguously set in the correspondence to hypersurface $\sigma_4$ (see [2]).

In the case of the invariance of the action variation under gauge symmetry group $\mathbb{R}(\beta)$ the values $X_{\mu\nu} = 0$ (gauge transformations do not touch upon the coordinates), and, since the group $\mathbb{R}(\beta)$ is oneparametric, 4-tensor $\theta_{\mu\nu} = \theta_{\mu4} \equiv \theta_{\mu}$, that is, it represents 4-vector. Then taking into account, that in given case matrix $||(I_\nu)_\mu||$ of infinitesimal operator $I_\nu \equiv I$ represents itself real number $I = 1$, we obtain for 4-vector $\theta_\mu$ the following expression

$$\theta_\mu = \frac{\partial L}{\partial (\partial_\mu u_4)} u_4 - \frac{\partial L}{\partial (\partial_\mu u_4^*)} u_4^*, \quad (23)$$

Components of 4-vector $\theta_\mu$, which can be identified with additional 4-vector of charge-current density $\theta_\mu \equiv J_\mu^{[2]} =$
where $j^{[2]}_\mu$ is

$$j^{[2]}_\mu = i\left[\frac{\partial L}{\partial (\partial_\mu u_i)} u_i + \frac{\partial L}{\partial (\partial_\mu u_\gamma^*)} u_\gamma^*\right],$$

satisfy to continuity equation

$$\partial_\mu j^{[2]}_\mu = 0,$$

which represents itself the conservation law for 4-vector $j^{[2]}_\mu$ in differential form. It distincts from known 4-vector of charge-current density $j_\mu$ (see [2]), which is reasonable to redesignate to be $j^{[1]}_\mu$, and which is

$$j^{[1]}_\mu = -i\left[\frac{\partial L}{\partial (\partial_\mu u_i)} u_i - \frac{\partial L}{\partial (\partial_\mu u_\gamma^*)} u_\gamma^*\right]$$

by the factor $i$ and by sign of the first item. It means, that any complex field is characterized by total 4-vector $j_\mu$, which is complex and can be represented in the form

$$j_\mu = j^{[1]}_\mu + ij^{[2]}_\mu.\tag{27}$$

We see, that both real 4-vector-functions of the complex 4-current vector $j_\mu$ are differ each other the only by sign of the first item.

Conserving quantity, corresponding to [23], that is imaginary component of the charge, is equal to

$$Q^{[2]} = iQ^{[2]} = -i \int \theta_4 d^3x.\tag{28}$$

Consequently $Q^{[2]}$ is determined by relationship

$$Q^{[2]} = i \int \left[\frac{\partial L}{\partial (\partial_\mu u_i)} u_i + \frac{\partial L}{\partial (\partial_\mu u_\gamma^*)} u_\gamma^*\right] d^3x.\tag{29}$$

It is seen from relationship [29], that obtained additional charge is really purely imaginary quantity. It follows from comparison with relationship for known conserved quantity for any complex field, for instance, for Dirac field. Let us remember, that real quantity - charge $Q^{[1]}$, is the consequence of gauge symmetry, consisting in the invariance of Lagrange equations under the transformations

$$u'(x) = exp(i\alpha)u(x)\tag{30}$$

and

$$u'^+\!(x) = exp(-i\alpha)u^+\!(x).\tag{31}$$

In general case $Q^{[1]}$, (2), is

$$Q_1 = -\int \left[\frac{\partial L}{\partial (\partial_\mu u_i)} u_i - \frac{\partial L}{\partial (\partial_\mu u_\gamma^*)} u_\gamma^*\right] d^3x.\tag{32}$$

Therefore any relativistic complex field can be characterized by complex conserving quantity $Q$, that is by complex charge, which can be represented in the form

$$Q = Q^{[1]} + iQ^{[2]}\tag{33}$$

with two real components $Q^{[1]}$ and $Q^{[2]}$. The statement is proven.

From the statement 3 we obtain the consequence, which seems to be essential and it is formulated in the form of the statement 4.

4. Conserving quantity - purely imaginary charge, which is invariant under total gauge transformations, corresponds to any real relativistic field (scalar, vector, spinor).

The proof is evident, if to take into account, that any real quantity, including relativistic field, is particular case of complex quantity.

In suggestion, that analogous statements are held true for quantized fields, we can conclude, that free EM-field quantum, that is photon, possesses along with the spin by the charge, which is purely imaginary in the case of real free EM-field. It becomes now to be physically understandable rather effective realization of EM-field interaction with the matter by means of given relativistic particles.

It becomes also to be understandable qualitatively the mechanism of appearance of real part of a charge when free real EM-field enter the matter. The velocity $v$ of EM-field propagation in the matter is less in comparison with the velocity $c$ in vacuum. Consequently, hyperbolic rotation of coordinate system in, for example $(x_3, x_4)$-plane of Minkowsky space and isomorphic to it rotation in $(Q_1, Q_2)$-plane of complex charge space take place. It corresponds to appearance of real component of the charge, and it is consequence of additional hyperbolic symmetry of Maxwellian EM-field (see the next Section). The same mechanism leads to appearance of imaginary part of EM-field vector-functions and currents. Naturally it is suggested, that life time of the photons, which are entered in the matter is rather long, that is rather strong electron-photon interaction takes place.

It seems to be clear, that Maxwell equations with all complex-valued vector and scalar variables give concrete realization of the connection between dual and gauge symmetries of EM-field.

It is remarkable, that, like to mechanics, a number of conservation laws, which can have EM-field, are optional in their simultaneous fulfilment. In particular, it is evident, that by automorphic transformation of the space of EM-field functions by relationship [10] the conservation law for charge will always take place. At the same time the energy conservation law and the conservation of Poynting vector will be fulfilled, if given transformation is applied to EM-field potentials. The force characteristics, that is $\vec{E}$-, $\vec{H}$-vector functions can be used to be basis for free EM-field description, since they will represent the full set in free EM-field case. However the energy conservation law and the conservation of Poynting vector, that is mathematical construction, to which enter $\vec{E}$-, $\vec{H}$-vector functions, will not be fulfilled by transformation [10] at arbitrary $\beta$. Given situation is realized by the propagation of the EM-field in the matter with the velocity $v \neq c$, that is with the velocity, which is not
equal to light velocity in vacuum. The charge remains to be Lorentz invariant quantity (see Sec. 2), at the same time both the field characteristics, the energy and impulse (determined by Poynting vector) are not Lorentz invariant quantities. It is remarkable, that the conclusion on charge Lorentz invariance was formulated in \[3\] to be self-evident. Thus, we see, that the charge conservation law for EM-field is fulfilled even through the energy and impulse conservation laws do not take place. Therefore, the charge conservation law can be considered in given meaning to be more fundamental.

II. COMPOUND QUATERNION NATURE OF EM-FIELD WITH FOUR REAL COMPONENTS, HAVING DIFFERENT SPACE AND TIME PARITY

A. Generalized Maxwell Equations

Symmetry studies of electromagnetic (EM) field have a long history, which was starting already in 19-th century from the work of Heaviside \[4\], where the existence of the symmetry of Maxwell equations under electrical and magnetic quantities was remarked for the first time. Mathematical formulation of given symmetry gave Larmor \[5\]. It is consisting in invariance of Maxwell equations for free EM-field under the transformations

\[
\begin{align*}
\vec{E} & \rightarrow \pm \vec{H}, \quad \vec{H} \rightarrow \mp \vec{E}, \\
\end{align*}
\]

(34)

The transformations \[34\] are called duality transformations, or Larmor transformations. Larmor transformations \[34\] are particular case of the more general dual transformations, established by Rainich \[6\]. Dual transformations produce oneparametric abelian group \( U_1 \), which is subgroup of the group of chiral transformations of massless fields. Dual transformations correspond to irreducible representation of the group of chiral transformations of massless fields in particular case of quantum number \( j = 1 \) \[3\] and they are

\[
\begin{align*}
\vec{E}' & \rightarrow \vec{E} \cos \theta + \vec{H} \sin \theta \\
\vec{H}' & \rightarrow \vec{H} \cos \theta - \vec{E} \sin \theta,
\end{align*}
\]

(35)

where parameter \( \theta \) is arbitrary continuous variable, \( \theta \in [0, 2\pi] \). In fact the expression \[35\] is indication in implicit form on compound character of EM-field. Really at fixed \( \theta \) the expression \[35\] will be mathematically correct, if vector-functions \( \vec{E}, \vec{H} \) will have the same symmetry under improper rotations, that is concerning the parity \( P \) under space inversion, both be polar or axial ones, or be both consisting of polar and axial components simultaneously. Analogous conclusion takes place regarding the parity \( t \) under time reversal. The possibility to have the same symmetry, that is, the situation, when both the vector-functions \( \vec{E}, \vec{H} \) are pure polar (axial) vector-functions, or both ones \( t \)-even (\( t \)-uneven) simultaneously contradicts to experiment. Consequently it remains the variant, that vector-functions \( \vec{E}', \vec{H}' \) in the expression \[35\] are compound and consists of the components with even and uneven parities under improper rotations. It is in agreement with overview on compound symmetry structure of EM-field vector-functions \( \vec{E}(\vec{r}, t), \vec{H}(\vec{r}, t) \), \( \vec{D}(\vec{r}, t), \vec{B}(\vec{r}, t) \), consisting of both the gradient and solenoidal parts, that is uneven and even parts under space inversion in \[3\], where compound symmetry structure of EM-field vector-functions is represented to be self-evident. It corresponds also to theoretical assumption in \[7\], where along with usual choice, that is, that electric field \( \vec{E} \) is polar vector, magnetic field \( \vec{H} \) is axial vector, the alternative choice is provided. The conclusion can be easily proved, if to represent relation \[35\] in matrix form

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix}.
\]

(36)

We see, that given matrix has the form, which allows set up in conformity to it the complex number according to biective mapping like to \[1\]. Consequently, we have

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} = e^{-i\theta} \begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix},
\]

(37)

that is

\[
\begin{align*}
\vec{E}' &= \vec{E} \cos \theta - i \vec{E} \sin \theta \\
\vec{H}' &= \vec{H} \cos \theta - i \vec{H} \sin \theta.
\end{align*}
\]

(38)

It means, that to real plane, which is determined by the vectors \( \vec{E} \) and \( \vec{H} \) can be set in conformity the complex plane for the vectors \( \vec{E}' \) and \( \vec{H}' \).

It is evident now, that really both the vectors \( \vec{E}' \) and \( \vec{H}' \) are consisting of both, \( P \)-even and \( P \)-uneven components. So the first component of, for instance, \( \vec{E}' \) will be \( P \)-even under reflection in the plane situated transversely to abscess-axis, the second component will be \( P \)-uneven.

Therefore, dual transformation symmetry of Maxwell equations, established by Rainich \[6\], indicates simultaneously on both complex nature of EM-field in general case, and that both electric and magnetic fields are consisting in general case of the components with various parity under improper rotations.

Let us find the invariants of dually transformed EM-field. It is easily to show, that the following relationship is taking place

\[
\left( \vec{E}^2 - \vec{H}^2 + 2i(\vec{E}\vec{H}) \right) e^{-2i\theta} = \text{inv},
\]

(39)

that is, we have at fixed parameter \( \theta \neq 0 \) two real EM-field invariants

\[
\begin{align*}
(\vec{E}^2 - \vec{H}^2) \cos 2\theta + 2(\vec{E}\vec{H}) \sin 2\theta &= \text{inv} \\
2(\vec{E}\vec{H}) \cos 2\theta - (\vec{E}^2 - \vec{H}^2) \sin 2\theta &= \text{inv}.
\end{align*}
\]

(40)
It follows from relation (40) that, in particular, at \( \theta = 0 \) we have well known EM-field invariants
\[
(\vec{E}^2 - \vec{H}^2) = \text{inv} \\
(\vec{E} \vec{H}) = \text{inv}.
\]
(41)

It is interesting, that at \( \theta = 45^\circ \) and at \( \theta = 90^\circ \) the invariants of dually transformed EM-field are determined by the same relation (41) and by arbitrary invariants of dually transformed EM-field. It is to remark, that setting into EM-field theory of two type of charges and two type of intrinsic moments of the particles or the absorbing (dispersive) centers in condensed matter. They can be considered to be the components of complex charge or dual charge and the particles or the absorbing (dispersive) centers in condensed matter. They can be considered to be the components of complex charge or dual charge, or dual charge (in another equivalent terminology) of so called dually charged quasiparticles \([9]\) and the particles with pure imaginary electric intrinsic moments \([10]\) in condensed matter, respectively, spin-Peierls \(\pi\)-solitons and Su-Schrieffer-Heeger \(\sigma\)-solitons in carbynoids, is clear experimental proof, that EM-field theory with complex charges and complex intrinsic moments has physical content. EM-field theory with complex charges and complex intrinsic moments ceases consequently to be the only formal model, which although is very suitable for many technical calculations, but was considered up to now to be mathematical abstraction, in which magnetic charges and magnetic currents are fictitious quantities. Similar conclusion concerns the conception of complex characteristics of EM-field in the matter.

Given conception agrees well with all practice of electric circuits' calculations. Very fruitful mathematical method for electric circuits' calculations, which uses all complex electric characteristics, see for example \([11]\), was also considered earlier the only to be formal, but convenient mathematical technique. Informality of given technique gets now natural explanation. It has to be taken into account however, that in the case \( \theta = 0 \) we have well known electromodynamics with odd P-parity of electric field and even P-parity of magnetic field. In given case all the EM-field characteristics are real and by using in calculations of the complex quantities, we have always add the corresponding complex conjugate quantities. At the same time in the case \( \theta \neq 0 \) it will be incorrect, since both the real observable quantities, which corresponds to any complex EM-field characteristics have to be retained. Independent conclusion follows also from gauge invariance above considered (Sec.1). Really, the presence of complex charge means that 4-vector of current \( j_\mu \) for any complex field is complex vector. In its turn, it means, that independently on starting origin of the charges and currents in the matter [they can be result of presence of Dirac field or another complex field] all the characteristics of EM-field in the matter have also to be complex-valued. Given conclusion follows immediately from Maxwell equations, since complex current \( j_\mu \) enters explicitly Maxwell equations. It is also substantial, that Maxwell equations are invariant under the transformation of EM-field functions by relationship (10).

Let us designate the terms in (43)
\[
\vec{E} \cos \theta = \vec{E}^1, \quad \vec{E} \sin \theta = \vec{E}^2 \\
\vec{H} \cos \theta = \vec{H}^1, \quad \vec{H} \sin \theta = \vec{H}^2.
\]
(42)

The Maxwell equations for the EM-field \((\vec{E}', \vec{H}')\) in the matter in general case of both type of charged particles (that is electrically and magnetically charged), including dually charged particles are
\[
\nabla \times \vec{E}'(\vec{r}, t) = -\mu_0 \partial \vec{H}'(\vec{r}, t) / \partial t - \vec{j}'_e(\vec{r}, t),
\]
(43)
\[
\nabla \times \vec{H}'(\vec{r}, t) = \epsilon_0 \partial \vec{E}'(\vec{r}, t) / \partial t + \vec{j}'_e(\vec{r}, t),
\]
(44)
\[
(\nabla \cdot \vec{E}'(\vec{r}, t)) = \rho'_e(\vec{r}, t),
\]
(45)
\[
(\nabla \cdot \vec{H}'(\vec{r}, t)) = \rho'_g(\vec{r}, t),
\]
(46)

where \( \vec{j}'_e(\vec{r}, t), \vec{j}'_g(\vec{r}, t) \) are respectively electric and magnetic current densities, \( \rho'_e(\vec{r}, t), \rho'_g(\vec{r}, t) \) are respectively electric and magnetic charge densities. Taking into account the relation (43) and (42) the system (44), (45), (46) can be rewritten
\[
\nabla \times \left( \vec{E}'^1(\vec{r}, t) - i\vec{E}'^2(\vec{r}, t) \right) = \\
-\mu_0 \left[ \partial \vec{H}'^1(\vec{r}, t) / \partial t - i \partial \vec{H}'^2(\vec{r}, t) / \partial t \right] - \vec{j}'_g^1(\vec{r}, t) + i\vec{j}'_g^2(\vec{r}, t),
\]
(47)
\[
\nabla \times \left( \vec{H}'^1(\vec{r}, t) - i\vec{H}'^2(\vec{r}, t) \right) = \\
\epsilon_0 \left[ \partial \vec{E}'^1(\vec{r}, t) / \partial t - i \partial \vec{E}'^2(\vec{r}, t) / \partial t \right] + \vec{j}'_e^1(\vec{r}, t) - i\vec{j}'_e^2(\vec{r}, t),
\]
(48)
\[
(\nabla \cdot (\vec{E}'^1(\vec{r}, t) - i\vec{E}'^2(\vec{r}, t))) = \rho'_e^1(\vec{r}, t) - i\rho'_e^2(\vec{r}, t),
\]
(49)
\[
(\nabla \cdot (\vec{H}'^1(\vec{r}, t) - i\vec{H}'^2(\vec{r}, t))) = \rho'_g^1(\vec{r}, t) - i\rho'_g^2(\vec{r}, t),
\]
(50)

where \( j^1_e(\vec{r}, t), j^2_e(\vec{r}, t), j^1_g(\vec{r}, t), j^2_g(\vec{r}, t) \) are correspondingly electric and magnetic current densities,
which by dual transformations are obeying to relation like to \( (51) \), they are designated like to \( (12) \), \( \rho^1_e(\vec{r}, t) \), \( \rho^2_e(\vec{r}, t) \), \( \rho^1_g(\vec{r}, t) \), \( \rho^2_g(\vec{r}, t) \) are correspondingly electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact in the system of equations \( (17) \), \( (48) \), \( (49) \), \( (50) \) are integrated Maxwell equations for two kinds of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering in equations, under space inversion. So, the components \( \vec{E}^1(\vec{r}, t), \vec{H}^2(\vec{r}, t), J^1_e(\vec{r}, t) \) have uneven parity, \( \vec{E}^2(\vec{r}, t), \vec{H}^1(\vec{r}, t), J^2_e(\vec{r}, t) \) have even parity, \( \rho^1_e(\vec{r}, t), \rho^2_g(\vec{r}, t) \) are scalars, \( \rho^2_e(\vec{r}, t), \rho^1_g(\vec{r}, t) \) are pseudoscalars.

In the case, when \( J^1_e(\vec{r}, t) = 0 \), \( \rho^2_g(\vec{r}, t) = 0 \) we obtain the equations of usual singly charge electrodynamics for compound EM-field in mathematically correct form, which allows to separate the components of EM-field with various parities \( P \) under space inversion. It is remarkable, that the idea, that vector quantities, which characterize EM-field, are compound quantities and include both gradient and solenoidal parts, that is uneven and even parts under space inversion was put forward earlier in \( [3] \). At the same time in the equations of dual electrodynamics given idea was presented the only in implicit form. The representation in explicit form by equations \( (44) \), \( (45) \), \( (46) \) seems to be actual, since field vector and scalar quantities with various \( t \)- and \( P \)-parities are mathematically heterogeneous and, for instance, their simple linear combination, for instance, for \( P \)-uneven and \( P \)-even electric vector-function \( E^1(\vec{r}, t), E^2(\vec{r}, t) \)

\[
\alpha_1 E^1(\vec{r}, t) + \alpha_2 E^2(\vec{r}, t)
\]  

(51)

with coefficients \( \alpha_1, \alpha_2 \) from the field of real numbers, which is taking place in some theoretical and experimental works, is collage. Similar situation was discussed in \( [12] \) by analysis of Bloch vector symmetry under improper rotations. Mathematically the objects, which are like to \( (51) \) can exist. Actually to the set of \( \{ E^1(\vec{r}, t) \} \) and to the set of \( \{ E^2(\vec{r}, t) \} \) can be put in correspondence the affine space. However given affine space corresponds to direct sum of two usual vector spaces, consisting of different physically objects, that is, it represents in fact also collage. Really, given direct sum can be represented by direct sum of linear capsule

\[
\{ \alpha_k^1 E^1(\vec{r}, t) | \alpha_k^1 \in R, k \in N \}
\]  

(52)

representing itself three-dimensional vector space of the set \( \{ E^1(\vec{r}, t) \} \) and linear capsule

\[
\{ \alpha_k^2 E^2(\vec{r}, t) | \alpha_k^2 \in R, l \in N \}
\]  

(53)

representing itself three-dimensional vector space of the set \( \{ E^2(\vec{r}, t) \} \). It is substantial, that both the vector spaces cannot be considered to be subspaces of any three-dimensional or six-dimensional vector spaces, consisting of uniform objects. Moreover, it is evident, that the affine space, defined in that way cannot be metrizable, when considering it to be a single whole. It means in turn, that the set of objects, given by \( (51) \) are not vectors in usual algebraic meaning. Even Pythagorean theorem, for instance, cannot be used.

It can be shown, that the system, analogous to \( (17) \), \( (48) \), \( (49) \), \( (50) \) can be obtained for the second pair of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering in equations, under time reversal. Really it is easily to see, that Maxwell equations along with dual transformation symmetry, established by Rainich, given by relations \( (35) \) - \( (38) \), are symmetric relatively the dual transformations of another kind of all the vector and scalar quantities, characterizing EM-field, which, for instance, for electric and magnetic field strength vector-functions can be presented in the following matrix form

\[
\begin{pmatrix}
\vec{E}' \\
\vec{H}'
\end{pmatrix} =
\begin{pmatrix}
\cosh \vartheta & i \sinh \vartheta \\
-i \sinh \vartheta & \cosh \vartheta
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
\vec{H}
\end{pmatrix},
\]

(54)

where \( \vartheta \) is arbitrary continuous parameter, \( \vartheta \in [0, 2\pi] \). The relation \( (51) \) can be rewritten in the form

\[
\begin{pmatrix}
\vec{E}' \\
\vec{H}'
\end{pmatrix} =
\begin{pmatrix}
\cos i\vartheta & \sin i\vartheta \\
-\sin i\vartheta & \cos i\vartheta
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
\vec{H}
\end{pmatrix}.
\]

(55)

In particular, if \( \vartheta \) is polar angle of coordinate system in the plane, determined by \( \vec{E} \) and \( \vec{H} \), the transformations \( (54) \) represent themselves hyperbolic rotations in \( (\vec{E}, \vec{H}) \)-plane. Let us call the transformations \( (51) \) by hyperbolic dual transformations. It represents the interest to consider the following particular case of hyperbolic dual transformations. We can define parameter \( \vartheta \) according to relation

\[
\tanh \vartheta = \frac{V}{c} = \beta,
\]

(56)

where \( V \) is velocity of the frame of reference, moving along \( x \)-axis in 3D subspace of \( H_4 \) Minkowski space. We can also to set up in conformity to the plane \( (x_2, x_3) \) in Minkowski space, the plane \( (\vec{E}, \vec{H}) \), in which \( \vec{E}, \vec{H} \) are orthogonal and are directed along absciss and ordinate axes correspondingly (or vice versa). Then we obtain

\[
|\vec{E}'| = \frac{|\vec{E}| + \beta |\vec{H}|}{\sqrt{1 - \beta^2}}
\]

(57)

\[
|\vec{H}'| = \frac{|\vec{H}| - \beta |\vec{E}|}{\sqrt{1 - \beta^2}}
\]

(57)

(or similar relations, in which \( \vec{E}, \vec{H} \) are interchanged by places). In vector form given transformations are

\[
\vec{E}' = \frac{\vec{E} + \beta \vec{H} \times \vec{V}}{\sqrt{1 - \beta^2}},
\]

\[
\vec{H}' = \frac{\vec{H} - \beta \vec{E} \times \vec{V}}{\sqrt{1 - \beta^2}},
\]

(58)
Therefore it is seen, that \( \vec{E}, \vec{H} \) are transformed like to \( x_0 \) and \( x_1 \) coordinates (or vice versa) of the space \( 1R_4 \).

It follows from here, that both the vectors \( \vec{E}'', \vec{H}'' \) have \( t \)-even and \( t \)-uneven components in general case. We see also that Lorentz-invariance of Maxwell equations is particular case of hyperbolic dual symmetry. It means, that restriction to only Lorentz-invariance in consideration of Maxwell equations' symmetry, which is usually used, constricts the concept on the EM-field itself and it is thereby constricting the possibilities for completeness of its practical usage. Taking into account (11) we obtain the relations, which are similar to (37), which can be rewritten in the similar to (38) form, that is, we have

\[
\begin{align*}
\vec{E}'' &= \vec{E} \cos i\theta - i\vec{E} \sin i\theta \\
\vec{H}'' &= \vec{H} \cos i\theta - i\vec{H} \sin i\theta.
\end{align*}
\]

(59)

It is proof in general case, that each of two independent Maxwellian field components with even and uneven parities under space inversion is also compound and it consists of two independent components with even and uneven parities under time reversal. Then imposing designations

\[
\begin{align*}
\vec{E} \cos i\theta &= \vec{E}^{[3]}, \quad \vec{E} \sin i\theta = \vec{E}^{[4]} \\
\vec{H} \cos i\theta &= \vec{H}^{[3]}, \quad \vec{H} \sin i\theta = \vec{H}^{[4]}
\end{align*}
\]

(60)

and considering the vector-functions \( \vec{E}\[1\](\vec{r}, t), \vec{E}\[2\](\vec{r}, t) \) and \( \vec{H}[1](\vec{r}, t), \vec{H}[2](\vec{r}, t) \) to be definitional domain of the vector-functions \( \vec{E}''(\vec{r}, t), \vec{H}''(\vec{r}, t) \) correspondingly, the Maxwell equations for the components of the field \( \vec{E}\[1\], \vec{H}\[1\] \) and \( \vec{E}\[2\], \vec{H}\[2\] \) have the same form and they are

\[
\begin{align*}
\nabla \times \left( \vec{E}^{[3]}(\vec{r}, t) - i\vec{E}^{[4]}(\vec{r}, t) \right) = & -\rho_0 \frac{\partial \vec{H}^{[3]}(\vec{r}, t)}{\partial t} - i\frac{\partial \vec{H}^{[4]}(\vec{r}, t)}{\partial t} \\
& - j_{g}^{[3]}(\vec{r}, t) + i j_{g}^{[4]}(\vec{r}, t), \\
\nabla \times \left( \vec{H}^{[3]}(\vec{r}, t) - i\vec{H}^{[4]}(\vec{r}, t) \right) = & \epsilon_0 \left[ \frac{\partial \vec{E}^{[3]}(\vec{r}, t)}{\partial t} - i\frac{\partial \vec{E}^{[4]}(\vec{r}, t)}{\partial t} \right] \\
& + j_{e}^{[3]}(\vec{r}, t) - ij_{e}^{[4]}(\vec{r}, t), \\
(\nabla \cdot (\vec{E}^{[3]}(\vec{r}, t) - i\vec{E}^{[4]}(\vec{r}, t))) = & \rho_{e}^{[3]}(\vec{r}, t) - i\rho_{e}^{[4]}(\vec{r}, t), \quad (63)
\end{align*}
\]

(61)

(62)

(63)

where \( j_{e}^{[3]}(\vec{r}, t), j_{e}^{[4]}(\vec{r}, t), j_{g}^{[3]}(\vec{r}, t), j_{g}^{[4]}(\vec{r}, t) \) are, correspondingly, electric and magnetic current densities, \( \rho_{e}^{[3]}(\vec{r}, t), \rho_{e}^{[4]}(\vec{r}, t), \rho_{g}^{[3]}(\vec{r}, t), \rho_{g}^{[4]}(\vec{r}, t) \) are, correspondingly, electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact the system of equations \( (31), (22), (63) \) represent itself correctly integrated Maxwell equations for two kinds of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering equations, under time reversal. So, the components \( \vec{E}^{[3]}(\vec{r}, t), \vec{H}^{[4]}(\vec{r}, t), j_{e}^{[3]}(\vec{r}, t) \) have uneven parity, \( \vec{E}^{[4]}(\vec{r}, t), \vec{H}^{[3]}(\vec{r}, t), j_{e}^{[4]}(\vec{r}, t) \) have even parity, \( \rho_{e}^{[3]}(\vec{r}, t), \rho_{g}^{[4]}(\vec{r}, t) \) are scalars, \( \rho_{e}^{[4]}(\vec{r}, t), \rho_{g}^{[3]}(\vec{r}, t) \) are pseudoscalars. In the case, when \( j_{g}^{[3]}(\vec{r}, t) = 0, \rho_{g}^{[4]}(\vec{r}, t) = 0 \) we obtain the equations of usual singly charge electrodynamics for two components of EM-field with various parities under space inversion, at that either of the two consist also of two components of EM-field with various parity under time reversal.

It is easily to see, that invariants for EM-field, consisting of two hyperbolic dually symmetric parts, that is at \( \vartheta \neq 0 \) have the form, analogous to (39) and they can be obtained, if parameter \( \vartheta \) to replace by \( i\theta \). They are

\[
\begin{align*}
\vec{E}^{2} - \vec{H}^{2} + 2i(\vec{E} \vec{H}) e^{2\vartheta} &= \text{inv}.
\end{align*}
\]

(65)

Consequently, two real invariants at \( \vartheta \neq 0 \) have the form

\[
\begin{align*}
(\vec{E}^{2} - \vec{H}^{2}) e^{2\vartheta} &= I_{1}'' = \text{inv}, \\
2(\vec{E} \vec{H}) e^{2\vartheta} &= I_{2}'' = \text{inv}.
\end{align*}
\]

(66)

It follows from relation (66), that in both the cases, that is at \( \vartheta = 0 \) and at fixed \( \vartheta \neq 0 \), we obtain in fact well known EM-field invariants, since factor \( e^{2\vartheta} \) at fixed \( \vartheta \) seems to be insufficient. At the same time at arbitrary \( \vartheta \) the relation

\[
\frac{I_{1}''}{I_{2}''} = \frac{I_{1}}{I_{2}} = W = \text{inv}
\]

(67)

is taking place. It is seen, that the value of \( W \) is independent on \( \vartheta \). It means physically, that the absolute values of both the vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \) are changed synchronously by hyperbolic dual transformations.

So, the usage of complex number theory allows to represent correctly the electrodynamics for two photon fields, which differs by parities under space inversion or time reversal by the same single system of generalized Maxwell equations. At the same time we have two related sets, that is pairs of complex vector and scalar functions, which are ordered in their \( P \)- and \( t \)-parities. It corresponds to definition of quaternions. Really any quaternion number \( x \) can be determined according to relation

\[
x = (a_{1} + ia_{2})e + (a_{3} + ia_{4})j,
\]

(68)

where \( \{a_{m}\} \in R, m = 1,4 \) and \( e, i, j, k \) produce basis,
elements of which are satisfying the conditions

\[
(ij) = k, (ji) = -k, (ki) = j, (ik) = -j, \quad \text{and} \quad (ei) = i, (ie) = j, (ej) = k, (ke) = k.
\]  

(69)

Let us designate the quantities

\[
(\vec{E}^1[\vec{r}, t] - i\vec{E}^2[\vec{r}, t]) + (\vec{E}^3[\vec{r}, t] - i\vec{E}^4[\vec{r}, t])j = \vec{E}(\vec{r}, t)
\]

\[
(\vec{H}^1[\vec{r}, t] - i\vec{H}^2[\vec{r}, t]) + (\vec{H}^3[\vec{r}, t] - i\vec{H}^4[\vec{r}, t])j = \vec{H}(\vec{r}, t)
\]

\[
(j^e_1[\vec{r}, t] - i\vec{e}(\vec{r}, t)) + (j^e_2[\vec{r}, t] - i\vec{e}(\vec{r}, t))j = \vec{j}(\vec{r}, t)
\]

\[
(-j^e_1[\vec{r}, t] + i\vec{e}(\vec{r}, t)) + (-j^e_2[\vec{r}, t] + i\vec{e}(\vec{r}, t))j = \vec{j}(\vec{r}, t)
\]

\[
(\rho^e_1[\vec{r}, t] - i\rho^e_2[\vec{r}, t]) + (\rho^e_3[\vec{r}, t] - i\rho^e_4[\vec{r}, t])j = \rho_e(\vec{r}, t)
\]

\[
(\rho^g_1[\vec{r}, t] - i\rho^g_2[\vec{r}, t]) + (\rho^g_3[\vec{r}, t] - i\rho^g_4[\vec{r}, t])j = \rho_g(\vec{r}, t)
\]

where

\[
\vec{E}^1[\vec{r}, t], \vec{H}^2[\vec{r}, t], j^e_1[\vec{r}, t], j^e_2[\vec{r}, t]
\]

are P-uneven, t-even,

\[
\vec{E}^2[\vec{r}, t], \vec{H}^1[\vec{r}, t], j^e_2[\vec{r}, t], j^e_1[\vec{r}, t]
\]

are P-uneven, t-uneven,

\[
\vec{E}^3[\vec{r}, t], \vec{H}^4[\vec{r}, t], j^e_3[\vec{r}, t], j^e_4[\vec{r}, t]
\]

are P-even, t-even,

\[
\vec{E}^4[\vec{r}, t], \vec{H}^3[\vec{r}, t], j^e_4[\vec{r}, t], j^e_3[\vec{r}, t]
\]

are P-even, t-uneven. According to definition of quaternions \(\vec{E}(\vec{r}, t), \vec{j}(\vec{r}, t), \vec{j}(\vec{r}, t), \vec{j}(\vec{r}, t)\) are quaternions. It means, that EM-field has quaternion structure and dual and hyperbolic dual symmetry of Maxwell equations will take proper account, if all the vector and scalar quantities to represent in quaternion form. Consequently, we have

\[
[\nabla \times (\vec{E}(\vec{r}, t))] = -\mu_0 \left[ \frac{\partial \vec{j}(\vec{r}, t)}{\partial t} \right] - j^e_0(\vec{r}, t),
\]

(75)

\[
(\nabla \cdot (\vec{j}(\vec{r}, t))) = \rho_e(\vec{r}, t),
\]

(77)

\[
(\nabla \cdot (\vec{j}(\vec{r}, t))) = \rho_g(\vec{r}, t)
\]

(78)

Therefore, symmetry of Maxwell equations under dual transformations of both the kinds allows along with generalization of Maxwell equations themselves to extend the field of application of Maxwell equations. It means also, that dual electrodynamics, developed by Tomilovich and co-authors, see for instance [3], obtains additional ground. Basic field equations in dual electrodynamics [3, 7], being to be written separately for two type of independent photon fields with various parities under space inversion or time reversal, will be isomorphic to Maxwell equations in complex form. It was in fact shown partly earlier in [13, 12], where complex charge was taken into consideration. At the same time all aspect of dual symmetry, leading to four-component quaternion form of Maxwell equations seem to be representing for the first time.

### B. Cavity Dual Electrodyamics

Let us find the conserving quantities, which correspond to dual and hyperbolic dual symmetries of Maxwell equations. It seems to be interesting to realize given task on concrete practically essential example of cavity EM-field. At the same time to built the Lagrangian, which is adequate to given task it seems to be reasonable to solve the following concomitant task - to find dually symmetric solutions of Maxwell equations. It seems to be understandable, that the general solutions of differential equations can also possess by the same symmetry, which have starting differential equations, nevertheless dual symmetry of the solutions of Maxwell equations was earlier not found.

#### 1. Classical Cavity EM-Field

Suppose EM-field in volume rectangular cavity without any matter inside it and made up of perfectly electrically conducting walls. Suppose also, that the field is linearly polarized and without restriction of commonness let us choose the one of two possible polarization of EM-field electrical component \(\vec{E}(\vec{r}, t)\) along x-direction. Then the vector-function \(E_x(\vec{r}, t)\) can be represented in well known form of Fourier sine series

\[
E^{[1]}(\vec{r}, t) = E_x(\vec{r}, t)e_x = \sum_{\alpha = 1}^{\infty} A^{E}_{\alpha} q_{\alpha}(t) \sin(k_\alpha z) \bar{e}_x,
\]

(79)

where \(q_{\alpha}(t)\) is amplitude of \(\alpha\)-th normal mode of the cavity, \(\alpha \in N, k_\alpha = \alpha \pi / L, A^E_{\alpha} = \sqrt{2\omega^2_{\alpha} m_{\alpha} / V\epsilon_0}, \)
\( \omega_\alpha = \alpha \pi c / L \), \( L \) is cavity length along z-axis, \( V \) is cavity volume, \( m_\alpha \) is parameter, which is introduced to obtain the analogy with mechanical harmonic oscillator. Let us remember, that the expansion in Fourier series instead of Fourier integral expansion is determined by known diskretness of \( \vec{k} \)-space, which is the result of finiteness of cavity volume. Particular sine case of Fourier series is consequence of boundary conditions

\[
\left[ \vec{n} \times \vec{E} \right]|_S = 0, (\vec{n} \vec{H})|_S = 0,
\]

which are held true for the perfect cavity considered. Here \( \vec{n} \) is the normal to the surface \( S \) of the cavity. It is easily to show, that \( E_z(z, t) \) represents itself a standing wave along z-direction.

Let us analyse the solutions of Maxwell equations for EM-field in a cavity in comparison with known solutions from the literature to pay the attention to some mathematical details, which have however substantial physical conclusions, allowing to extend our insight to EM-field nature. For given reasons, despite on analysis simplicity, we will produce the consideration in detail.

Using the equation

\[
\epsilon_0 \frac{\partial \vec{E}(z, t)}{\partial t} = \left[ \nabla \times \vec{H}(z, t) \right],
\]

we obtain the expression for magnetic field

\[
\vec{H}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} A_\alpha E_\alpha \frac{\partial q_\alpha(t)}{\partial t} \cos(k_\alpha z) + f_\alpha(t) \vec{e}_y,
\]

where \( \{f_\alpha(t)\}, \alpha \in N, \) is the set of arbitrary functions of the time. It is evident, that the expression for \( \vec{H}(\vec{r}, t) \) is satisfying to boundary conditions. The partial solution, in which the functions \( \{f_\alpha(t)\} \) are identically zero, is always used in all the EM-field literature. However even in given case it is evident, that the Maxwellian field is complex field. Really using the equation

\[
\left[ \nabla \times \vec{E} \right] = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}
\]

it is easily to find the class of field functions \( \{q_\alpha(t)\} \). They will satisfy to differential equations

\[
\frac{d^2 q_\alpha(t)}{dt^2} + \frac{k_\alpha^2}{\mu_0 \epsilon_0} q_\alpha(t) = 0, \alpha \in N.
\]

Consequently, we have

\[
q_\alpha(t) = C_{1\alpha} e^{i \omega_\alpha t} + C_{2\alpha} e^{-i \omega_\alpha t}, \alpha \in N,
\]

where \( C_{1\alpha}, C_{2\alpha}, \alpha \in N \) are arbitrary constants. Thus, real-valued free Maxwell field equations result in well known in the theory of differential equations situation - the solutions are complex-valued functions. It means, that generally the field functions for free Maxwellian field in the cavity produce complex space. So we obtain additional independent argument, that the known conception, on the only real-quantity definiteness of EM-field, has to be corrected. On the other hand, the equation \( (81) \) has also the only real-valued general solution, which can be represented in the form

\[
q_\alpha(t) = B_\alpha \cos(\omega_\alpha t + \phi_\alpha),
\]

where \( B_\alpha, \phi_\alpha, \alpha \in N \) are arbitrary constants. It is substantial, that the functions in real-valued general solution have a definite t-parity.

Thus, we come independently on the previous consideration in Sec.I and Sec.II, Subsec.A to the conclusion, that classical Maxwellian EM-field can be both real-quantity defined and complex-quantity defined.

It is interesting, that there is the second physically substantial solution of Maxwell equations. Really, from general expression \( (82) \) for the field \( \vec{H}(\vec{r}, t) \) it is easily to obtain differential equations for \( \{f_\alpha(t)\}, \alpha \in N, \)

\[
\frac{df_\alpha(t)}{dt} + A_\alpha E_\alpha \frac{\partial^2 q_\alpha(t)}{\partial t^2} \cos(k_\alpha z) - \frac{1}{\mu_0} A_\alpha k_\alpha q_\alpha(t) \cos(k_\alpha z) = 0.
\]

The formal solution of given equations in general case is

\[
f_\alpha(t) = A_\alpha E_\alpha \sum_{\tau=1}^{\infty} A_\alpha q_\alpha(\tau) \left[ \frac{k_\alpha}{\mu_0} \int_0^t q_\alpha(\tau) d\tau - \frac{d q_\alpha(t)}{dt} \right] \cos(k_\alpha z) - \frac{1}{\mu_0} A_\alpha k_\alpha q_\alpha(t) \cos(k_\alpha z) = 0.
\]

Therefore, we have the second solution of Maxwell equations for \( \vec{H}(\vec{r}, t) \) in the form

\[
\vec{E}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} A_\alpha E_\alpha \sum_{\tau=1}^{\infty} A_\alpha q_\alpha(\tau) \cos(k_\alpha z) \vec{e}_x.
\]

The functions \( q_\alpha(t) \) and \( q_\alpha''(t) \) in relationships \( (89) \) and \( (90) \) are

\[
q_\alpha'(t) = \omega_\alpha \int_0^t q_\alpha(\tau) d\tau,
\]

\[
q_\alpha''(t) = \omega_\alpha \int_0^t q_\alpha'(\tau') d\tau'.
\]

correspondingly. Owing to the fact, that the solutions have simple form of harmonic trigonometrical functions, the second solution for electric field differs from the first solution the only by sign, that is substantial, and by inessential integration constants. Integration constants can be taken into account by means of redefinition of
factor \( m_\alpha \) in field amplitudes. It is also evident, that if vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \) are the solutions of Maxwell equations, then vector-functions \( \hat{T}\vec{E}(\vec{r}, t) \) and \( \hat{T}\vec{H}(\vec{r}, t) \), where \( \hat{T} \) is time inversion operator, are also the solutions of Maxwell equations. Moreover, if starting vector-function, to which operator \( \hat{T} \) is applied in \( t \)-even, then there is \( t \)-uneven solution, for instance for magnetic component in the form

\[
\frac{\hat{T}[t\vec{H}(\vec{r}, t)]}{t}, \quad (92)
\]

where \( t \) is time. It can be shown in a similar way, that dually symmetric solutions, which are \( P \)-even and \( P \)-uneven are also existing.

Therefore, there are the solutions with various combinations of the signs for vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \), which are realized simultaneously, that is, their linear combination with coefficients from the field \( C \) of complex numbers will represent the solution of Cauchy problem for Maxwell equations in correspondence with known theorem, that the solution of Cauchy problem for any systems of homogeneous linear equations in partial derivatives exists and it is unique in the vicinity of any point of the initial surface (in the case, when the point selected is not characteristic point and the function, which determines given hypersurface is continuously differentiable). In other words, we obtain again the agreement with Maxwell equation symmetry consideration. Given property of EM-field seems to be essential, since it permits passing for the processes, which seemingly are forbidden by CPT-theorem. For example, let us consider the resonance system EM-field plus matter in the cavity, in particular, the so called dressed state of some quasi-particles’ system. Suppose, that wave function can be factorized, matter part is \( P \)- and \( t \)-even under space and time inversion transformations, while EM-field part is \( P \)-uneven. CPT-invariance will be preserved, since EM-field has simultaneously with \( t \)-even the \( t \)-uneven component, determined by expression \( (92) \). Therefore \( t \)-parity of the function \( q_\alpha(t) \) can be various, and in the case, if we choose \( t \)-parity to be identical to the parity of the function \( q_\alpha(t) \), the solution will be different in the meaning, that the field vectors will have opposite \( t \)-parity in comparison with the first solution. It is evident, that boundary conditions are fulfilled for all the cases considered.

To built the Lagrangian we can choose the following sets of EM-field functions \( \{u^{s, \pm}_\alpha(x)\} \), \( s = 1, 2, \alpha \in N \),

\[
\begin{align*}
(u^{1, \pm}_\alpha(x)) &= \{\sqrt{\epsilon_0}A^E_\alpha \sin k_\alpha(x_3)[q_\alpha(x_4) \pm iq''_\alpha(x_4)]\} \\
(u^{2, \pm}_\alpha(x)) &= \{\sqrt{\mu_0}A^H_\alpha \cos k_\alpha(x_3)[-q'_\alpha(x_4) \pm \frac{i}{\omega_\alpha} \frac{dq_\alpha(x_4)}{dx_4}]\}
\end{align*}
\]

(93)

The functions \( \{u^{s, \pm}_\alpha(x)\} \), \( s = 1, 2, \alpha \in N \) are built from the components of the expansion in Fourier series of the fields \( \vec{E}^{(1)}(\vec{r}, t) \), \( \vec{E}^{(2)}(\vec{r}, t) \) and \( \vec{H}^{(1)}(\vec{r}, t) \), \( \vec{H}^{(2)}(\vec{r}, t) \) correspondingly. At the same time the sets \( \{u^{s, \pm}_\alpha(x)\} \), \( s = 1, 2, \alpha \in N \) produce at fixed \( x \) two orthogonal countable bases, corresponding to \( s = 1, 2 \) in two Hilbert spaces, which are formed by vectors \( \{\vec{u}^{s, \pm}_1(x), \vec{u}^{s, \pm}_2(x), \ldots\} \) for variable \( x \in R^4 \). Really scalar product of two arbitrary vectors \( \vec{u}^{s, \pm}_i(x_i), \vec{u}^{s, \pm}_j(x_j) \) and \( \vec{u}^{s, \pm}_j(x_j), \vec{u}^{s, \pm}_i(x_i) \), that is

\[
\langle \vec{u}^{s, \pm}_i(x_i) | \vec{u}^{s, \pm}_j(x_j) \rangle
\]

(94)

is equal to

\[
\sum_{\alpha=1}^{\infty} \int_0^L u^{s, \pm}_\alpha(x_{4,i}, z)u^{s, \pm}_\alpha(x_{4,j}, z)dz, s = 1, 2,
\]

(95)

that means, that it is restricted, since the sum over \( s \) represents the energy of the field in restricted volume. Consequently, the norm of vectors can be defined by the relationship

\[
||\vec{u}^{s, \pm}_i(x)|| = \sqrt{\langle \vec{u}^{s, \pm}_i(x) | \vec{u}^{s, \pm}_i(x) \rangle} = \sqrt{\sum_{\alpha=1}^{\infty} \int_0^L u^{s, \pm}_\alpha(x_{4,i}, z)u^{s, \pm}_\alpha(x_{4,j}, z)dz, s = 1, 2.}
\]

(96)

Then vector distance is

\[
d(\vec{u}^{s, \pm}_i(x_i), \vec{u}^{s, \pm}_j(x_j)) = ||\vec{u}^{s, \pm}_i(x_i) - \vec{u}^{s, \pm}_j(x_j)||. \quad (97)
\]

So we obtain, that the vectors \( \{\vec{u}^{s, \pm}_i(x)\} \), \( x \in R^4 \) produce the space \( L^2 \) and taking into account the Riss-Fisher theorem it means, that given vector space is complete, that in its turn means, that the spaces of vectors \( \{\vec{u}^{s, \pm}_i(x)\} \), \( x \in R^4, s = 1, 2 \), are Hilbert spaces. Consequently Lagrangian \( L(x) \) can be represented in the following form

\[
L(x) = \sum_{s=1}^{2} \sum_{\mu=1}^{4} \sum_{\alpha=1}^{\infty} \frac{\partial u^{\alpha, \pm}_s(x) \partial u^{\alpha, \pm}_s(x)}{\partial x_\mu} - \sum_{s=1}^{2} \sum_{\mu=1}^{4} K(x)u^{\alpha, \pm}_s(x)u^{\alpha, \pm}_s(x), \quad (98)
\]

where \( K(x) \) is factor, depending on the set of variables \( x = \{x_\mu\}, \mu = 1,4 \).

Let us find the conserving quantity, corresponding to dual symmetry of Maxwell equations. Dual transformation, determined by relation \( (30) \) is the transformation the only in the space of field three-dimensional vector-functions \( \vec{E}, \vec{H} \), (let us designate it by \( (\vec{E}, \vec{H}) \)-space) and it does not touch upon the coordinates. It seems to be convenient to define in given space the reference frame, then the transformation, given by \( (30) \) is the rotation of two component matrix vector-function

\[
||F|| = \left[ \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right]. \quad (99)
\]
Instead of two Hilbert space for two sets of vectors \{U^{[s,\pm]}(x)\}, x \in 1 R^4, s = 1, 2 we can also define one Hilbert space for row matrix vector function set

\[
\{\|U(x)\|\} = \{U^{[1,\pm]}(x)U^{[2,\pm]}(x)\}
\]  

(100)

with the set of components

\[
\{\|U_\alpha(x)\|\} = \{u_\alpha^{1,\pm}(x)u_\alpha^{2,\pm}(x)\}
\]  

(101)

where \(\alpha \in N\). In general case instead parameter \(\theta\) we can define rotation angles \(\theta_{ik}, i, k = 1, 3\) in 2D-planes of \((\vec{E}, \vec{H})\) functional space. It is evident, that \(\theta_{ik}\) are antisymmetric under the indices \(i, k = 1, 3\). According to Nöther theorem, the conserving quantity, corresponding to parameters \(\theta_{ik}\) in dual transformations (100), that is at \(\theta_{ik} = \theta_{12}\) is determined by relations like to (19) and (20). So, we obtain

\[
S_{12}^\mu = -\left[\sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)}\right] \|Y_\alpha\| + c.c.,
\]  

(102)

where \(\mu = 1, 4\) and it was taken into account, that \(\|X_\alpha\|\) in matrix relation (102), which is like to \(\|U_\alpha\|\) is equal to zero. The factor \(\frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)}\) in (102) is row matrix

\[
\frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)} = \left[\begin{array}{c}
\frac{\partial L}{\partial (\partial_\mu u_\alpha^{1,\pm})} \\
\frac{\partial L}{\partial (\partial_\mu u_\alpha^{2,\pm})}
\end{array}\right],
\]  

(103)

matrix \(\|Y_\alpha\|\) is product of matrices \(\|I_\alpha\|\) and \(\|U_\alpha(x)\|\), that is

\[
\|Y_\alpha\| = \|I_\alpha\| \left[\begin{array}{c}
u_\alpha^{1,\pm} \\
u_\alpha^{2,\pm}
\end{array}\right],
\]  

(104)

where \(\|I_\alpha\|\) is the matrix, which corresponds to infinitesimal operator of dual or hyperbolic dual transformations of \(\alpha\)-th mode of cavity EM-field. It represents in general case the product of three matrices, corresponding to rotation along three mutually perpendicular axes in 3D functional space above defined. So \(\|I_\alpha\| = \|I_{1\alpha}\|\|I_{2\alpha}\|\|I_{3\alpha}\|\).

The transformations in the form, which is given by (36) correspond to \(\theta_{23} = \theta, \theta_{12} = 0, \theta_{31} = 0\), that is \(\|I_{2\alpha}\| = \|I_{3\alpha}\| = E\), where \(E\) is unit \([2 \times 2]\)-matrix. In the absence of dispersive medium in the cavity \(\|I_\alpha\|\) will be independent on \(\alpha\). Moreover, it is easy to see that infinitesimal operator with matrix \(\|I_\alpha\|\) is the same for dual transformations, determined by (36) and hyperbolic dual transformations, determined by (34). Really \(\|I_\alpha\|\) in both the cases is

\[
\|I_\alpha\| = \left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right],
\]  

(105)

for any \(\alpha \in N\).

Conserving quantity is

\[
S_{12}^\mu = -\frac{i}{c} \int \left\{\sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)}\right\} \|Y_\alpha\| + c.c.\} d^3x
\]  

(106)

The structure of (106) unambiguously indicates, that it is the component of spin tensor [2], [14], to which dual vector component can be set in the correspondence according to relation

\[
S_1^\mu = \varepsilon_{ijk}S_2^\mu = -\varepsilon_{ijk} \frac{i}{c} \int \left\{\sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)}\right\} \|Y_\alpha\| + c.c.\} d^3x,
\]  

(107)

where \(\varepsilon_{ijk}\) is completely antisymmetric Levi-Civita 3-tensor.

Therefore we obtain, that the same physical conserving quantity corresponds to dual and hyperbolic dual symmetry of Maxwell equations. Taking into account the expressions for Lagrangian (98) and for infinitesimal operator (105), in the geometry choosed, when vector \(\vec{E}\) is directed along absciss axis, vector \(\vec{H}\) is directed along ordinate axis in \((\vec{E}, \vec{H})\) functional space, we have

\[
S_{12}^\mu = \sum_{\alpha=1}^{\infty} \left[\frac{\partial u_\alpha^{1,\pm}}{\partial x_\mu} - \frac{\partial u_\alpha^{2,\pm}}{\partial x_\mu}\right] u_\alpha^{1,\pm} + c.c.,
\]  

(108)

and

\[
S_3^\mu = \varepsilon_{312}S_2^\mu = -\frac{i}{c} \int \left\{\sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu\|U_\alpha\|)}\right\} \|Y_\alpha\| + c.c.\} d^3x,
\]  

(109)

It is projection of spin on the propagation direction. Therefore we have in given case right away physically significant quantity - spirality.

The relations (102), (107), (108), (109) show, that spin of classical relativistic EM-field in the cavity and, correspondingly, spirality are additive quantities and they represent the sum of cavity spin and spirality modes. On the connection of the conserving quantity, which is invariant of dual symmetry with spin was indicated in \([3]\), where free EM-field was considered with traditional Lagrangian, which uses vector potentials to be field functions. The result obtained together with aforementioned result in \([3]\) lift dilemma on the necessity of using of given quantity by consideration of classical EM-field. Really, the situation was to some extent paradoxical, and it can be displayed by the following conversation between two disputant physicists. "Spin exists" - has insisted the first, referring on the appearance of additional tensor component in total tensor of moment - intrinsic moment - to be consequence of Minkowsky space symmetry under Lorentz transformations, "Spin does not exists" - has insisted the second, referring on the metrized tensor of the moment, in which spin part is equal to zero \([2]\) in distinction from canonical tensor. In other words, both disputants were in one's own way right. Dual symmetry leads to unambiguous conclusion "Spin exists" and has to be taken into consideration by the solution of tasks, concerning both classical and quantum electrodynamics. Moreover spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality
in the simplest case above considered) combine two subsystems of photon fields, that is the subsystem of two fields, which have definite \(P\)-parity (even and uneven) with the subsystem of two fields, which have definite \(t\)-parity (also even and uneven) into one system. In fact we obtain the proof for four component structure of EM-field to be a single whole, that is confirmation along with the possibility of the representation of EM-field in four component quaternion form, given by \(17\), \(18\), \(19\), the necessity of given representation. It extends the overview on the nature of EM-field itself. It seems to be remarkable, that given result on the special leading significance of spin is in agreement with result in \([12]\), where was shown, that spin is quaternion vector of the state in Hilbert space, defined under ring of quaternions, corresponding to the subsystem of two fields, which have definite \(P\)-parity (even and uneven) interacting with EM-field.

It is interesting, that the charge und currents, being to be the components of 4-vector, which are transformed by corresponding representation of Lorentz group, are invariants of hyperbolic dual transformations, that is, they are also Lorentz invariants, if to take into account, that Lorentz transformations are particular case of hyperbolic dual transformations. It is seen immediately from the expressions for 4-current and it means, that observers in various inertial frames will register the same value of the charge in correspondence with conclusion in \([3]\). It is connected with invariance of Lagrange equations and equations charge by multiplication of field functions on arbitrary complex number, established in Sec.1, since by hyperbolic dual transformations the multiplication of field functions on some complex number takes place.

### 2. Quantized Cavity EM-Field

The quantization of EM-field was proposed for the first time still at the earliest stage of quantum physics \([13]\). At the same time correspondence to each mode of radiation field the quantized harmonic oscillator, was proposed for the first time by Dirac \([16]\) and it is widely used in QED including quantum optics \([17]\), it is canonical quantization. EM-field potentials are used to be field functions by canonical quantization. At the same time to describe free EM-field it is sufficient to choose immediately the observable quantities - vector-functions \(\hat{E}(\vec{r}, t)\) and \(\hat{H}(\vec{r}, t)\) - to be field functions. We use further given idea by EM-field quantization. We can start like to canonical quantization, from classical Hamiltonian, which for the first partial classical solution of Maxwell equations is

\[
\mathcal{H}^{\text{cl}}[t] = \frac{1}{2} \int \int [\epsilon_0 E_y^2(z, t) + \mu_0 H_y^2(z, t)] \, dx \, dy \, dz
\]

\[
= \frac{1}{2} \sum_{\alpha=1}^{\infty} \left[ m_\alpha \nu_\alpha^2 \rho_\alpha^2(t) + \frac{p_\alpha^2(t)}{m_\alpha} \right], \quad p_\alpha = \frac{d\rho_\alpha(t)}{dt}.
\]

(110)

So, taking into consideration the relationship for Hamiltonian \(\mathcal{H}^{\text{cl}}[t]\) we set in correspondence to canonical variables \(q_\alpha(t), p_\alpha(t)\), determined by the first partial solution of Maxwell equations, the operators by usual way

\[
[\hat{p}_\alpha(t), \hat{q}_\beta(t)] = i\hbar \delta_{\alpha\beta}
\]

\[
[\hat{q}_\alpha(t), \hat{q}_\beta(t)] = [\hat{p}_\alpha(t), \hat{p}_\beta(t)] = 0,
\]

(111)

where \(\alpha, \beta \in \mathbb{N}\). Introducing the operator functions of time \(\hat{a}_\alpha(t)\) and \(\hat{a}_\alpha^+(t)\)

\[
\hat{a}_\alpha(t) = \frac{1}{\sqrt{2m_\alpha \omega_\alpha}} [m_\alpha \omega_\alpha \hat{q}_\alpha(t) + i\hbar \hat{p}_\alpha(t)]
\]

\[
\hat{a}_\alpha^+(t) = \frac{1}{\sqrt{2m_\alpha \omega_\alpha}} [m_\alpha \omega_\alpha \hat{q}_\alpha(t) - i\hbar \hat{p}_\alpha(t)],
\]

(112)

we obtain the operator functions of canonical variables in the form

\[
\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right]
\]

\[
\hat{p}_\alpha(t) = i \sqrt{\frac{\hbar m_\alpha \omega_\alpha}{2}} \left[ \hat{a}_\alpha^+(t) - \hat{a}_\alpha(t) \right].
\]

(113)

Then EM-field operator functions are obtained right away and they are

\[
\hat{E}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_\alpha}{V \epsilon_0}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right] \sin(k_\alpha z) \right\} \vec{e}_x,
\]

(114)

\[
\hat{H}(\vec{r}, t) = i \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_\alpha}{V \mu_0}} \left[ \hat{a}_\alpha^+(t) - \hat{a}_\alpha(t) \right] \cos(k_\alpha z) \right\} \vec{e}_y,
\]

(115)

Taking into account the relationships \(114, 115\) and Maxwell equations, it is easily to find an explicit form for the dependencies of operator functions \(\hat{a}_\alpha(t)\) and \(\hat{a}_\alpha^+(t)\) on the time. They are

\[
\hat{a}_\alpha^+(t) = \hat{a}_\alpha^+(t=0) e^{i\omega_\alpha t},
\]

\[
\hat{a}_\alpha(t) = \hat{a}_\alpha(t=0) e^{-i\omega_\alpha t},
\]

(116)

where \(\hat{a}_\alpha^+(t=0), \hat{a}_\alpha(t=0)\) are constant, complex-valued in general case, operators.

It seems to be essential, that complex exponential dependencies in \(114\) cannot be replaced by the real-valued harmonic trigonometrical functions. Really, if to suggest, that

\[
\hat{a}_\alpha^+(t) = \hat{a}_\alpha^+(t=0) \cos \omega_\alpha t,
\]

(117)

then we obtain, that the following relation has to be taking place

\[
[\hat{a}_\alpha^+(t=0) - \hat{a}_\alpha(t=0)]^{-1} [\hat{a}_\alpha^+(t=0) + \hat{a}_\alpha(t=0)] = \tan \omega_\alpha t.
\]

(118)
We see, that left-hand side in relation (118) does not depend on time, right-hand side is depending. The contradiction obtained establishes an assertion. Therefore, the quantized Maxwellian EM-field is complex-valued field in full correspondence with pure algebraic conclusion in Sec.I.

Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both complex-valued and real-valued functions. The fields $\vec{E}^c(\vec{r}, t)$, $\vec{H}^c(\vec{r}, t)$ can be quantized in much the same way. The operators $\hat{a}''_\alpha(t)$, $\hat{a}''_\alpha(t)$ for quantized EM-field, corresponding to general solution of Maxwell equations, are introduced analogously to (112).

\[
\hat{a}''_\alpha(t) = \frac{1}{\sqrt{2\hbar m_\omega_\alpha}} [m_\omega_\alpha q''_\alpha(t) + i p''_\alpha(t)]
\]

\[
\hat{a}''_\alpha(t) = \frac{1}{\sqrt{2\hbar m_\omega_\alpha}} [m_\omega_\alpha q''_\alpha(t) - i p''_\alpha(t)]
\]

For the field function operators we obtain

\[
\hat{E}^c_\alpha(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar\omega_\alpha}{V\epsilon_0}} [\hat{a}''_\alpha(t) + \hat{a}''_\alpha(t)] \sin(k_\alpha z) \} \vec{e}_1,
\]

\[
\hat{H}^c_\alpha(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar\omega_\alpha}{V\mu_0}} (-i) [\hat{a}''_\alpha(t) - \hat{a}''_\alpha(t)] \cos(k_\alpha z) \} \vec{e}_2.
\]

In accordance with definition of complex quantities we can built the following combination of solutions, satisfying Maxwell equations

\[
(\hat{E}^c_1(\vec{r}, t), \hat{E}^c_2(\vec{r}, t)) \rightarrow \hat{E}^c(\vec{r}, t) + i \hat{E}^c_2(\vec{r}, t) = \hat{E}(\vec{r}, t),
\]

\[
(\hat{H}^c_1(\vec{r}, t), \hat{H}^c_2(\vec{r}, t)) \rightarrow \hat{H}^c(\vec{r}, t) + i \hat{H}^c_1(\vec{r}, t) = \hat{H}(\vec{r}, t).
\]

Consequently, the electric and magnetic field operators for quantized EM-field, corresponding to general solution of Maxwell equations, are

\[
\hat{\vec{E}}(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar\omega_\alpha}{V\epsilon_0}} [\hat{a}''_\alpha(t) + \hat{a}_\alpha(t)] \\
+ \{ \hat{a}''_\alpha(t) - \hat{a}''_\alpha(t) \} \sin(k_\alpha z) \} \vec{e}_x,
\]

and

\[
\hat{\vec{H}}(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar\omega_\alpha}{V\mu_0}} \{ [\hat{a}''_\alpha(t) - \hat{a}_\alpha(t)] \\
- \{ \hat{a}''_\alpha(t) + \hat{a}''_\alpha(t) \} \cos(k_\alpha z) \} \vec{e}_y,
\]

It is substantial, that both field operators $\hat{\vec{E}}(\vec{r}, t)$ and $\hat{\vec{H}}(\vec{r}, t)$ are Hermitian operators.

### 3. Cavity 4-currents

It represents the interest to calculate the 4-currents for given task. Let us place all the vector-functions in pairs in accordance with their parity. Then we have the following pairs

\[
(\vec{E}^c[1](\vec{r}, t), \vec{E}^c[2](\vec{r}, t)), (\vec{H}^c[1](\vec{r}, t), \vec{H}^c[2](\vec{r}, t))
\]

(126) in which both the $\vec{E}$-vectors and $\vec{H}$-vectors have the same space parity (polar and axial correspondingly) and differ each other by t-parity, t-even and t-uneven in accordance with their numbers in pairs. It means, that they trasform like to $x_1$ and $x_1$ coordinates in $1R_4$. In a similar manner can be set the vector-functions with opposite to the vector-functions in space parity

\[
(\vec{E}^c[3](\vec{r}, t), \vec{E}^c[4](\vec{r}, t)), (\vec{H}^c[4](\vec{r}, t), \vec{H}^c[3](\vec{r}, t)).
\]

(127) Then taking into account the definition of complex quantities to be pair of real defined quantities, taken in fixed order, we come in a natural way once again to concept of complex vector-functions, which describe Maxwellian EM-field equations. In other words, we have in fact the quantities

\[
\vec{E}^c[1](\vec{r}, t) + i \vec{E}^c[2](\vec{r}, t) = \vec{E}_{c,p}(\vec{r}, t),
\]

\[
\vec{H}^c[1](\vec{r}, t) + i \vec{H}^c[2](\vec{r}, t) = \vec{H}_{c,a}(\vec{r}, t),
\]

(128) and

\[
\vec{E}^c[3](\vec{r}, t) + i \vec{E}^c[4](\vec{r}, t) = \vec{E}_{c,a}(\vec{r}, t),
\]

\[
\vec{H}^c[4](\vec{r}, t) + i \vec{H}^c[3](\vec{r}, t) = \vec{H}_{c,p}(\vec{r}, t),
\]

(129) where complex plane put in correspondence to $(y, z)$ real plane, subscripts a and p mean axial and polar respectively. It seems to be convenient to determine the space of EM-field vector-functions under the ring of quaternions with another basis in comparison with basis, given by (69). We will use now the quaternion basis $\{ e_i \}, i = 0, 3$ with algebraic operations between elements, satisfying to relationships

\[
e_i e_j = \varepsilon_{ijk} e_k + \delta_{ij} e_0, e_0 e_i = e_i, e_i^2 = e_0, i, j, k = 1, 3,
\]

(130) where $\varepsilon_{ijk}$ is completely antisymmetric Levi-Chivita 3-tensor.

Let us define the vector biquaternion

\[
\vec{\Phi} = (\vec{E}^c[1] + \vec{H}^c[2]) + i(\vec{E}^c[2] - \vec{H}^c[1]),
\]

(131) which can be represented to be the sum of the biquaternion

\[
\vec{\Phi} = \vec{F} + \vec{F},
\]

(132) where $\vec{F} = \vec{E}^c[1] + i(\vec{H}^c[1]), \vec{F} = \vec{H}^c[2] + i\vec{E}^c[2]$. Then Maxwell equations for instance for two free photon fields with different t-parity are

\[
\nabla \vec{\Phi} = 0.
\]

(133)
The generalized Maxwell equations in quaternion form with quaternion basis, given by [69], can also be rewritten in fully quaternion form, if to use both the bases. It seems to be consequence of independence of basis definition for both the quaternion forms.

It is evident, that

\[ j_{\mu, \pm}(x) = j_{\mu, \pm}^{(1)}(x) + i j_{\mu, \pm}^{(2)}(x), \quad (134) \]

where subscript ± corresponds to two possibilities for definition of complex vector-functions. Along with relationships \[ (128), \quad (129) \] they can be defined by the change of addition sign in \[ (128), \quad (129) \] to opposite. The quantity \( j_{\mu, \pm}^{(1)}(x) \) is well known quantity, and it is determined by

\[
j_{\mu, \pm}^{(1)}(x) = -\frac{ie}{\hbar c} \sum_{\sigma = 1}^{2} \sum_{a = 1}^{2} \left[ \frac{\partial L(x)}{\partial (\partial_{\mu} u_{\alpha}^{s, \pm}(x))} \right] u_{\alpha}^{s, \pm}(x),
\]

where

\[ L(x) \text{ is Lagrange function and } u_{\alpha}^{s, \pm}(x), \quad s = 1, 2 \]

are

\[ u_{\alpha}^{1, \pm}(x) = \sqrt{\epsilon_0 A_0^E} \sin k_\alpha(x_3) [q_\alpha(x_4) \pm iq_\alpha''(x_4)], \]

\[ u_{\alpha}^{2, \pm}(x) = \sqrt{\mu_0 A_0^H} \cos k_\alpha(x_3) - q_\alpha'(x_4) \pm i \frac{1}{\omega_\alpha} \frac{dq_\alpha(x_4)}{dx_4}. \quad (135) \]

The functions \( u_{\alpha}^{s, \pm}(x), \quad s = 1, 2, \alpha \in N \) are built from the components of the expansion in Fourier series of the fields \( \vec{E}^{[1]}(\vec{r}, t) \), \( \vec{E}^{[2]}(\vec{r}, t) \) and \( \vec{H}^{[2]}(\vec{r}, t) \), \( \vec{H}^{[1]}(\vec{r}, t) \) correspondingly.

To determine the current density \( j_{\mu, \pm}^{(2)}(x) \) we have to take into consideration, that gauge symmetry group of EM-field is two-parametric group \( \Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathcal{R}(\beta) \), where \( \mathcal{R}(\beta) \) is abelian multiplicative group of real numbers (excluding zero). It leads also to existence for EM-field of complex 4-current densities including complex charge density component. Since the current density \( j_{\mu, \pm}^{(2)}(x) \) is

\[
j_{\mu, \pm}^{(2)}(x) = -\frac{ie}{\hbar c} \sum_{\sigma = 1}^{2} \sum_{a = 1}^{2} \left[ \frac{\partial L(x)}{\partial (\partial_{\mu} u_{\alpha}^{s, \pm}(x))} \right] u_{\alpha}^{s, \pm}(x), \quad (136) \]

where

\[ q_\alpha(t), \alpha \in N, \quad \text{for } j_{4, \pm}^{1, \pm} \text{ is equal to zero. For complex-valued functions, determined by } (\text{85}), \text{ we will have} \]

\[ j_{4, \pm}^{1, \pm}(\vec{r}, t) = \frac{8ie}{\hbar c^2 V} \sum_{\alpha = 1}^{\infty} m_\alpha \omega_\alpha^3 \left( |C_{1\alpha}|^2 - |C_{2\alpha}|^2 \right). \quad (142) \]
It is seen from (142), that \( j_{\perp}^{\pm}(\vec{r}, t) \) in the case of Maxwellian EM-field is constant, which is equal to zero at \(|C_{1\alpha}| = |C_{2\alpha}|\), that is for all real-valued functions and for complex-valued functions \( \{q_\alpha(t)\}, \alpha \in \mathbb{N} \), which differ each other by arguments of constants \( C_{1\alpha} \) and \( C_{2\alpha} \).

\[
j_{\perp}^{\pm}(\vec{r}, t) = -\frac{2e}{\hbar c^2 V} \sum_{\alpha=1}^{\infty} m_\alpha \omega_\alpha^2 \sin^2 k_\alpha z \frac{d}{dt}(|q_\alpha(t)|^2) + \omega_\alpha^4 \frac{d}{dt}(\int_0^t \int_0^{t''} q_\alpha(t') dt' dt''\big) \]

\[
\times \frac{i}{\omega_\alpha^2} \frac{d}{dt}\left[ \frac{dg_\alpha(t)}{dt}\right] \left( \frac{dg_\alpha(t)}{dt}\right)^* \int_0^t q_\alpha(t') dt'
\]

\[
+ \omega_\alpha^2 \frac{d}{dt}(\int_0^t q_\alpha(t') dt') = 0
\]

(143)

For complex-valued functions, determined by (85), we obtain

\[
j_{\perp}^{2\pm}(\vec{r}, t) = \frac{8ie}{\hbar c^2 V} \sum_{\alpha=1}^{\infty} m_\alpha \omega_\alpha^2 \cos 2k_\alpha z \times[C_{1\alpha}C_{2\alpha}^* e^{2i\omega_\alpha t} - C_{1\alpha}^* C_{2\alpha} e^{-2i\omega_\alpha t}].
\]

(144)

It can be shown, that continuity equation

\[
\frac{\partial j_{\perp}^{\pm}(x)}{\partial x_\mu} = 0
\]

(145)

is fulfilled for both general case and for Maxwellian EM-field functions considered.

### III. CONCLUSIONS

It is shown on the basis of complex number theory, that any quantumphysical quantity is complex quantity. Additional gauge invariance of complex relativistic fields was found. It is based on invariance of generalized relativistic equations under the operations of additional gauge symmetry group - multiplicative group \( \mathfrak{R} \) of all real numbers (without zero) and leads to appearance of purely imaginary component of charge. So, it was shown, that complex fields are characterized by complex charges. It gives key for correct generalization of field equations, in particular for electrodynamics. In application to EM-field it means that two-parametric group \( \Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathfrak{R}(\beta) \) determines the gauge symmetry of EM-field and that free real EM-field is characterized by purely imaginary charge.

Additional hyperbolic dual symmetry of Maxwell equations is established, which includes Lorentz-invariance to be its particular case. The essence of additional hyperbolic dual symmetry of Maxwell equations is that, that Maxwell equations along with dual transformation symmetry, established by Rainich, given by (85) - (88), are symmetric relatively the dual transformations of another kind. Hyperbolic dual transformations for electric and magnetic field strength vector functions are

\[
\begin{bmatrix}
\mathcal{E}^n \\
\mathcal{H}^n
\end{bmatrix}
= 
\begin{bmatrix}
\cosh \vartheta & i\sinh \vartheta \\
-i\sinh \vartheta & \cosh \vartheta
\end{bmatrix}
\begin{bmatrix}
\mathcal{E} \\
\mathcal{H}
\end{bmatrix},
\]

(146)

where \( \vartheta \) is arbitrary continuous parameter, \( \vartheta \in [0, 2\pi] \).

Generalized Maxwell equations are obtained on the basis of both dual and hyperbolic dual symmetries of EM-field. It is shown, that in general case both scalar and vector quantities, entering equations, are quaternion quantities, four components of which have different parities under improper rotations.

Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry and hyperbolic dual symmetry are found. It is concluded, that Maxwell equations with all quaternion vector and scalar variables give concrete connection between dual and gauge symmetries of EM-field.

The example of free classical and quantized cavity EM-field is considered. It is shown, that the same physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the geometry choosed, when vector \( \vec{E} \) is directed along absciss axis, \( \vec{H} \) is directed along ordinate axis in \((\mathcal{E}, \mathcal{H})\) functional space. Spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality in the geometry considered) combine two subsystems of photon fields, which have definite \( P \)-parity (even and uneven) with the subsystem of two fields, which have definite \( t \)-parity (also even and uneven) into one system. It is considered to be the proof for four component structure of EM-field to be a single whole, that is, it is the confirmation along with the possibility of the representation of EM-field in four component quaternion form, given by (12), (13), (14), (15), the necessity of given representation. It extends the overview on the nature of EM-field itself. It seems to be remarkable, that given result on the special leading significance of spin is in agreement with result in (12), where was shown, that spin is quaternion vector of the state in Hilbert space, defined under ring of quaternions, of any quantum system (in the frame of the chain model considered) interacting with EM-field.

New principle of EM-field quantization, which is based on choosing of immediately observable quantities - vector-functions \( \mathcal{E}(\vec{r}, t) \) and \( \mathcal{H}(\vec{r}, t) \) - to be field functions, is proposed. It is found, that quantized Maxwellian EM-field is the only complex-valued field. Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both
complex-valued and real-valued functions.

[1] Nöther E, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math-phys.Klasse, (1918) 235-257
[2] Bogush A A, Moroz L G, Introduction to the Theory of Classical Fields, M., Editorial URSS, 2004, 384
[3] Strazhev V.I, Tomilchick L.M, Electrodynamics with Magnetic Charge, Minsk, Nauka i Tekhnika, 1975, 336 pp
[4] Heaviside O, Phil.Trans.Roy.Soc.A, 183 (1893) 423-430
[5] Larmor J, Collected papers, London, 1928
[6] Rainich G Y, Trans.Am.Math.Soc., 27 (1925) 106
[7] Berezin A.V, Kurochkin Yu.A, Tolkachev Ye.A, Quaternions in Relativistic Physics, Minsk, Nauka i Tekhnika, 1989, 199 pp
[8] Landau L.D, Lifshitz E.M, Field Theory, M., Nauka, 504
[9] Yearchuck D, Yerchak Y, Kirilenko A, Popechits V, Doklady NANB, 52, N 1 (2008) 48-53
[10] Yearchuck D, Yerchak Y, Alexandrov A, Phys.Lett.A, 373 (2009) 489 - 495
[11] Andre Angot, Complements de Mathematiques, Paris, 1957, 778
[12] D.Yearchuck, Y.Yerchak, A.Dovlatova, Optics Communications, 283 (2010) 3448-3458
[13] Tolkachev E.A, Tomilchick L.M, Covariant Methods in Theoretical Physics, Minsk, 1981, pp 44-48
[14] Bogoliubov N N, Shirkov D V, Introduction in the theory of quantized fields, M., Nauka, 1973, 414 pp
[15] Born M, Jordan P, Zeitschrift fuer Physik, 34 (1925) 858-888
[16] Dirac P.AM, Proc.Roy.Soc., A, 114 (1927) 243-265
[17] Scully M O, Zubairy M S, Quantum Optics, Cambridge University Press, 1997, 650
It has been shown, that electromagnetic field (EM) has in general case quaternion structure, consisting of four independent fields, which differ each other by the parities under space inversion and time reversal. It follows immediately from Rainich dual symmetry of Maxwell equations and additional hyperbolic dual symmetry, established in given work. It has also been shown, that for any complex relativistic field the gauge invariant conserving quantity is two-component scalar or pseudoscalar value - complex charge. It means in applicability to EM-field, that its gauge symmetry group is determined by two-parametric group $\Gamma(\alpha, \beta) = U_1(\alpha) \otimes R(\beta)$, where $R(\beta)$ is abelian multiplicative group of real numbers (excluding zero). Generalized Maxwell equations for quaternion four-component EM-field are obtained on the basis of its both dual and hyperbolic dual symmetries. Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry are found. It is shown, that the only one physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the geometry, when vector $\vec{E}$ is directed along absciss axis, $\vec{H}$ is directed along ordinate axis in $(\vec{E}, \vec{H})$ functional space. In fact it is the proof for quaternion four component structure of EM-field to be a single whole, that is confirmation along with the possibility of the representation of EM-field in four component quaternion form the necessity of given representation. It extends the overview on the nature of EM-field itself. Canonical Dirac quantization method is developed in two aspects. The first aspect is the application of Dirac quantization method the only to observable quantities. The second aspect is the realization along with well known time-local quantization of space-local quantization and space-time-local quantization, which allow to establish correspondingly the time of photon creation (annihilation), the space coordinate of photon creation (annihilation) and the space and time coordinates simultaneously of photon creation (annihilation). It is shown, that Coulomb field can be quantized in 1D and 2D systems, that is it is radiation field in given low-dimensional systems.

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I. INTRODUCTION AND BACKGROUND

A. Matrix Algebra of Complex Numbers and its some Consequences for Quantum Theory

Let us summarize some useful results from algebra of the complex numbers. The numbers 1 and $i$ are usually used to be basis of the linear space of complex numbers over the field of real numbers. At the same time to any complex number $a + ib$ can be set up in conformity the $[2 \times 2]$-matrix according to biective mapping $f$

$$f : a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$  \hspace{1cm} (1)

The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ \hspace{1cm} (2)

produce basis for complex numbers $\{a + ib\}, a, b \in R$ in the linear space of $[2 \times 2]$-matrices, defined over the field of real numbers. It is convenient often to define the space of complex numbers over the group of real positive numbers, then the dimensionality of the matrices and basis has to be duplicated, since to two unities - positive 1 and negative $-1$ can be set up in conformity the $[2 \times 2]$-matrices according to biective mapping $\varepsilon$

$$\varepsilon : 1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$ \hspace{1cm} (3)

which allows to recreate the operations with negative numbers without recourse of negative numbers themselves. Consequently, in accordance with mapping $\zeta$ the following $[4 \times 4]$-matrices, so called $[0,1]$-matrices, can be basis of complex numbers

$$\zeta : 1 \rightarrow [e_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ \hspace{1cm} (4)

$$i \rightarrow [e_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$
The choice of basis is ambiguous. Any four $4 \times 4$ matrices, which satisfy the rules of cyclic recurrence

$$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$$

(6)

can be basis of complex numbers. In particular, the following $4 \times 4$ matrices

$$\begin{aligned}
[e_1'] &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \\
[e_2'] &= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \\
[e_3'] &= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \\
[e_4'] &= \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\end{aligned}$$

(7)

can also be basis of complex numbers. Naturally, the set of $[0,1]$-matrices, given by $\{1\}$ is isomorphic to the set, which is given by $\{4\}$. It is evident, that the system of complex numbers can be constructed by infinite number of the ways, at that cyclic basis can consist of $m$ units, $m \in N$, starting from three. It is remarkable, that the conformity between complex numbers and matrices is realized by biective mappings. It means, that there is also to be existing the inverse mapping, by means of which to any square matrix, belonging to the linear space with a basis given by $\{4\}$, or $\{7\}$, or any other, satisfying the rules of cyclic recurrence like to (6), can be set up in conformity the complex number. In particular to any Hermitian matrix $H$ can be set up in conformity the complex number in correspondence with mapping $\xi$.

$$\xi : H \rightarrow S + iA = \begin{bmatrix} S & -A \\ A & S \end{bmatrix}.$$ 

(8)

where $S$ and $A$ are symmetric and antisymmetric parts of Hermitian matrix. Given short consideration allows to formulate the following statements.

1. Quantized free EM-field is complex field in general case.

Proof is evident and it is based on (3), if to take into account, that quantized free EM-field can be determined by Hermitian operators $\hat{E}(r,t)$ and $\hat{H}(r,t)$, representing themselves the full set of quantized free EM-field operator vector-functions, that is, they can serve for basis in corresponding operator vector-functional space (see Sec.II). Given statement can be generalized.

2. Any quantumphysical quantity is complex quantity in general case.

Proof is evident and it is based on the same relationship, since any quantumphysical quantity is determined by Hermitian operator. Therefore, two sets of observables, which are determined by real functions, correspond to any quantumphysical operator quantity in general case.

B. Additional gauge invariance of complex relativistic fields

We will argue in the next Section, that EM-field in the matter can be considered in general case to be complex field, each component in which is also complex field, that is, it has quaternion nature. In given Section we will prove the idea, that for any complex field the conserved quantity, corresponding to its gauge symmetry, that is charge, can be in general case also complex.

Let $u(x) = \{ u_i(x) \}, \ i = \frac{1}{\sqrt{u}},$ the set of the functions of some complex relativistic field, that is, scalar, vector or spinor field, given in some space of Lorentz group representations. It is well known, that Lagrange equations for any complex relativistic field can be represented in the form of one matrix relativistic differential equation of the first order in partial derivatives, that is in the form of so called generalized relativistic equation, and analogous equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions $u^\dagger(x) = \{ u_i^\dagger(x) \}$. The equation for the set $u(x)$ of field functions is

$$(\alpha_\mu \partial_\mu + \kappa \alpha_0)u(x) = 0.$$ 

(9)

Similar equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions, that is for the functions $u^\dagger(x) = \{ u_i^\dagger(x) \}, \ i = \frac{1}{\sqrt{u}},$ is

$$\partial_\mu u^\dagger(x)\alpha_\mu + \kappa u^\dagger(x)\alpha_0 = 0.$$ 

(10)

In equations (9) $\alpha_\mu, \alpha_0$ are matrices with constant numerical elements. They have an order, which coincides with dimension of corresponding space of Lorentz group representation, realized by $\{ u_i(x) \}, \ i = \frac{1}{\sqrt{u}}$. In particular, they are $[n \times n]$-matrices, if $\{ u_i(x) \}, \ i = \frac{1}{\sqrt{u}}$ are scalar functions. It is evident, that the transformation

$$u'(x) = \beta \exp(i\alpha)u(x),$$

(11)

where $\alpha, \beta \in R$, and analogous transformation for Hermitian conjugate functions (or complex conjugate functions in the case of scalar fields)

$$u'^\dagger(x) = \beta \exp(-i\alpha)u^\dagger(x)$$

(12)

keep Lagrange equations (9) (10) to be invariant. It is understandable that transformation of field functions by relationships (11), (12) is equivalent to multiplication of field functions by arbitrary complex number. It is well known, that given linear transformation is the simplest
example of isomorphism of corresponding linear space, which is given over the field of complex numbers, onto itself, that is, in the case considered the relationships \[ (11) \] give automorphism of the space of field functions. Automorphism of any linear space leads to some useful properties of the objects, which belong to given space. For instance, if to set up in a correspondence to the space of field function the affine space, then conservation laws of collinearity of the points and of simple relation of the triple of collinear points will be fulfilled by automorphism in given affine space. Consequently, we have to expect the physical consequences of given algebraic property in the case of physical spaces. Conformably to the case considered we have in fact gauge transformation of field functions, which is more general in comparison with usually used. The set \( \{ \beta \exp(-i\alpha) \} \) for all possible \( \alpha, \beta \in R \) produces the group \( \Gamma \), which is direct product of known symmetry group \( U_1 \), and multiplicative group \( \Re \) of all real numbers (without zero). Therefore, in the case considered the symmetry group of given complex field acquires additional parameter. So, we will have

\[
\Gamma(\alpha, \beta) = U_1(\alpha) \otimes \Re(\beta)
\]  

(13)

Let us find the irreducible representations of the group \( \Re(\beta) \). It has to be taken into account, that the group \( \Re(\beta) \) is abelian group and its irreducible representations \( T(\Re) \) are onedimensional. So, the mapping

\[
T : \Re \rightarrow T(\Re)
\]  

(14)

is isomorphism, where \( T(1) = 1 \). Therefore, for \( \forall(\beta, \gamma) \) of pair of elements of group \( \Re(\beta) \) the following relationship takes place

\[
T(\beta, \gamma) = T(\beta)T(\gamma).
\]  

(15)

Then, it is easy to show, that

\[
T(\beta) = \beta^{\frac{2\pi i}{T}(1)}.
\]  

(16)

The value \( \frac{2\pi i}{T}(1) \) can be obtained from the condition

\[
T(-\beta) = -T(\beta).
\]  

(17)

Consequently, we have

\[
T(\beta) = \beta^{2k+1} = \exp[(2k+1)ln\beta],
\]  

(18)

where \( k \in N \). Then irreducible representations of the group \( \Gamma(\alpha, \beta) \) represent direct product of irreducible representations of the groups \( U_1(\alpha) \) and \( \Re(\beta) \)

\[
T(U_1(\alpha)) \otimes T(\Re(\beta)) = \exp(-ima)\exp[(2k+1)ln\beta],
\]  

(19)

where \( m, k = 0, \pm 1, \pm 2, \ldots \).

It is clear, that some conserved quantity has to correspond to gauge symmetry of the field, which is determined by the group \( \Re(\beta) \). Thus we arrive at a formulation of the following statement.

3. Conserving quantity - complex charge, which is invariant under total gauge transformations, corresponds to any complex relativistic field (scalar, vector, spinor).

Proof.

Really, since generalized relativistic equations are invariant under transformations \( (11) \) and variation of action integral with starting Lagrangian is equal to zero, then variation of action integral with transformed Lagrangian in accordance with \( (11) \) will also be zero. Consequently, all the conditions of applicability of Nöther theorem, by proof of which the only invariance under Lagrange equations is sufficient, \( [1] \), are held true. We wish to pay attention to typical inaccuracy, which is abundant in the literature, consisting in that, that for applicability of Nöther theorem the Lagrangian invariance under corresponding symmetry transformations is required. At the same time the only invariance of Lagrange equations under corresponding symmetry transformations, which certainly takes place in given case, is necessary (see proof of Nöther theorem). According to Nöther theorem, the conserved quantity, corresponding to \( \nu - th \) parameter \( (\nu = T, k) \) by invariance of field under some \( k \)-parametric symmetry group, is (see, for instance, \( [2] \)).

\[
Q_\nu(\sigma) = \int_{(\sigma)} \theta_{\mu\nu} d\sigma_\mu = \text{const},
\]  

(20)

where \( \sigma \) is any spacelike hypersurface, \( \sigma \subset 1R_4 \) and 4-tensor \( \theta_{\mu\nu} \) is determined by relation

\[
\theta_{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu u_\nu X_{\mu\nu} - Y_{\mu\nu})} - LX_{\mu\nu},
\]  

(21)

in which \( L \) is field Lagrangian and the matrices \( X_{\mu\nu}, Y_{\mu\nu} \) are determined by matrix representations \( \{ (I_\nu)_\mu \} \) and \( \{ (J_\nu)_k \} \) of infinitesimal operators of symmetry group in coordinate space and in the space of field functions respectively in accordance with the following relationships

\[
X_{\mu\nu} = (I_\nu)_{\mu\alpha} x_\alpha, Y_{\mu\nu} = (J_\nu)_k u_k.
\]  

(22)

Since the value of integral in \( (20) \) does not depend on the choose of spacelike hypersurface, then usually the hypersurface, which is orthogonal immediately to time axis is used. By given choose 4-vector \( d\sigma_\mu \), representing itself infinitesimal element of spacelike hypersurface, is \( \{ d\sigma_\mu \} = \{ 0, 0, 0, d\sigma_4 \} \), where \( d\sigma_4 = -id^3x \). Then the expression \( (20) \) gets the form

\[
Q_\nu(\sigma_4) = -i \int_{(\sigma_4)} \theta_{4\nu} d^3x = \text{const},
\]  

(23)

where the conservation of the quantity \( Q_\nu(\sigma_4) \) in time is represented in explicit form, since the time can be unambiguously set in the correspondence to hypersurface \( \sigma_4 \) (see \( [2] \)).

In the case of the invariance of the action variation under gauge symmetry group \( \Re(\beta) \) the values \( X_{\mu\nu} = 0 \)
(gauge transformations do not touch upon the coordinates), and, since the group \( \mathcal{H}(\beta) \) is oneparametric, 4-tensor \( \theta_{\mu \nu} = \theta_{\mu i} \equiv \theta_{\mu} \), that is, it represents 4-vector. Then taking into account, that in given case matrix \( \| (I_{\nu})_{\mu \nu} \| \) of infinitesimal operator \( I_{\nu} \equiv I \) represents itself real number \( I = 1 \), we obtain for 4-vector \( \theta_{\mu} \) the following expression

\[
\theta_{\mu} = - \frac{\partial L}{\partial (\partial_{\mu} u_i)} u_i - \frac{\partial L}{\partial (\partial_{\mu} u^*_1)} u^*_1, \tag{24}
\]

Components of 4-vector \( \theta_{\mu} \), which can be identified with additional 4-vector of charge-current density \( \theta_{\mu} \equiv \theta_{\mu}^{[2]} = ij_{\mu}^{[2]} \), where \( j_{\mu}^{[2]} \) is

\[
j_{\mu}^{[2]} = i \left[ \frac{\partial L}{\partial (\partial_{\mu} u_i)} u_i + \frac{\partial L}{\partial (\partial_{\mu} u^*_1)} u^*_1 \right], \tag{25}
\]

satisfy to continuity equation

\[
\partial_{\mu} j_{\mu}^{[2]} = 0, \tag{26}
\]

which represents itself the conservation law for 4-vector \( j_{\mu}^{[2]} \) in differential form. It distincts from known 4-vector of charge-current density \( j_{\mu} \) (see [2]), which is reasonable to redesignate to be \( j_{\mu}^{[1]} \), and which is

\[
j_{\mu}^{[1]} = -i \left[ \frac{\partial L}{\partial (\partial_{\mu} u_i)} u_i - \frac{\partial L}{\partial (\partial_{\mu} u^*_1)} u^*_1 \right] \tag{27}
\]

by the factor \( i \) and by sign of the first item. It means, that any complex field is characterized by total 4-vector \( j_{\mu} \), which is complex and can be represented in the form

\[
j_{\mu} = j_{\mu}^{[1]} + ij_{\mu}^{[2]} \tag{28}
\]

We see, that both real 4-vector-functions of the complex 4-current vector \( j_{\mu} \) are differ each other the only by sign of the first item.

Conserving quantity, corresponding to \( j_{\mu}^{[2]} \), that is imaginary component of the charge, is equal to

\[
Q^{[2]} = iQ^{[2]} = -i \int \theta_{\mu} d^4x. \tag{29}
\]

Consequently \( Q^{[2]} \) is determined by relationship

\[
Q^{[2]} = i \int \left[ \frac{\partial L}{\partial (\partial_{\mu} u_i)} u_i + \frac{\partial L}{\partial (\partial_{\mu} u^*_1)} u^*_1 \right] d^4x. \tag{30}
\]

It is seen from relationship \( Q^{[2]} \), that obtained additional charge is really purely imaginary quantity. It follows from comparison with relationship for known conserved quantity for any complex field, for instance, for Dirac field. Let us remember, that real quantity - charge \( Q^{[1]} \), is the consequence of gauge symmetry, consisting in the invariance of Lagrange equations under the transformations

\[
u'(x) = \exp(i\alpha) \nu(x) \tag{31}
\]

and

\[
u'(x) = \exp(-i\alpha) \nu(x). \tag{32}
\]

In general case \( Q^{[1]} \), \( Q^{[2]} \), is

\[
Q^{[1]} = - \int \left[ \frac{\partial L}{\partial (\partial_{\mu} u_i)} u_i - \frac{\partial L}{\partial (\partial_{\mu} u^*_1)} u^*_1 \right] d^4x. \tag{33}
\]

Therefore any relativistic complex field can be characterized by complex conserving quantity \( Q \), that is complex charge, which can be represented in the form

\[
Q = Q^{[1]} + iQ^{[2]} \tag{34}
\]

with two real components \( Q^{[1]} \) and \( Q^{[2]} \). The statement is proven.

From the statement 3 we obtain the consequence, which seems to be essential and it is formulated in the form of the statement 4.

4. Conserving quantity - purely imaginary charge, which is invariant under total gauge transformations, corresponds to any real relativistic field (scalar, vector, spinor).

The proof is evident, if to take into account, that any real quantity, including relativistic field, is particular case of complex quantity.

In suggestion, that analogous statements are held true for quantized fields, we can conclude, that free EM-field quantum, that is photon, possesses along with the spin by the charge, which is purely imaginary in the case of real free EM-field. It becomes now to be physically understandable rather effective realization of EM-field interaction with the matter by means of given relativistic particles.

It becomes also to be understandable qualitatively the mechanism of appearance of real part of a charge when free real EM-field enter the matter. The velocity \( v \) of EM-field propagation in the matter is less in comparison with the velocity \( c \) in vacuum. Consequently, hyperbolic rotation of coordinate system in, for example \( (x_3, x_4) \)-plane of Minkowsky space and isomorphic to it rotation in \( (Q_1, Q_2) \)-plane of complex charge space take place. It corresponds to appearance of real component of the charge, and it is consequence of additional hyperbolic symmetry of Maxwellian EM-field (see the next Section). The same mechanism leads to appearance of imaginary part of EM-field vector-functions and currents. Naturally it is suggested, that life time of the photons, which are entered in the matter is rather long, that is rather strong electron-photon interaction takes place.

It seems to be clear, that Maxwell equations with all complex-valued vector and scalar variables give concrete realization of the connection between dual and gauge symmetries of EM-field.

It is remarkable, that, like to mechanics, a number of conservation laws, which can have EM-field, are optional in their simultaneous fulfillment. In particular, it is evident, that by automorphic transformation of the space
of EM-field functions by relationship (11) the conservation law for charge will always take place. At the same time the energy conservation law and the conservation of Poynting vector will be fulfilled, if given transformation is applied to EM-field potentials. The force characteristics, that is \( \mathbf{E}' \), \( \mathbf{H}' \)-vector functions can be used to be basis for free EM-field description, since they will represent the full set in free EM-field case. However the energy conservation law and the conservation of Poynting vector, that is mathematical construction, to which enter \( \mathbf{E}' \), \( \mathbf{H}' \)-vector functions, will not be fulfilled by transformation (11) at arbitrary \( \beta \). Given situation is realized by the propagation of the EM-field in the matter with the velocity \( v \neq c \), that is, with the velocity, which is not equal to light velocity in vacuum. The charge remains to be Lorentz invariant quantity (see Sec.2), at the same time both the field characteristics, the energy and impulse (determined by Poynting vector) are not Lorentz invariant quantities. It is remarkable, that the conclusion on charge Lorentz invariance was formulated in [3] to be self-evident. Thus, we see, that the charge conservation law for EM-field is fulfilled even through the energy and impulse conservation laws do not take place. Therefore, the charge conservation law can be considered in given meaning to be more fundamental.

II. COMPOUND QUATERNION NATURE OF EM-FIELD WITH FOUR REAL COMPONENTS, HAVING DIFFERENT SPACE AND TIME PARITY

A. Generalized Maxwell Equations

Symmetry studies of electromagnetic (EM) field have a long history, which was starting already in 19-th century from the work of Heaviside [4], where the existence of the symmetry of Maxwell equations under electrical and magnetic quantities was remarked for the first time. Mathematical formulation of given symmetry gave Larmor [5]. It is consisting in invariance of Maxwell equations for free EM-field under the transformations

\[
\mathbf{E} \rightarrow \pm \mathbf{H}, \mathbf{H} \rightarrow \mp \mathbf{E},
\]

(35)
The transformations (35) are called duality transformations, or Larmor transformations. Larmor transformations (35) are particular case of the more general dual transformations, established by Rainich [6]. Dual transformations produce oneparametric abelian group \( U_1 \), which is subgroup of the group of chiral transformations of massless fields. Dual transformations correspond to irreducible representation of the group of chiral transformations of massless fields in particular case of quantum number \( j = 1 \) [3] and they are

\[
\mathbf{E}' \rightarrow \mathbf{E} \cos \theta + \mathbf{H} \sin \theta, \quad \mathbf{H}' \rightarrow \mathbf{H} \cos \theta - \mathbf{E} \sin \theta,
\]

(36)
where parameter \( \theta \) is arbitrary continuous variable, \( \theta \in [0, 2\pi] \). In fact the expression (36) is indication in implicit form on compound character of EM-field. Really at fixed \( \theta \) the expression (36) will be mathematically correct, if vector-functions \( \mathbf{E}', \mathbf{H}' \) will have the same symmetry under improper rotations, that is concerning the parity \( P \) under space inversion, both be polar or axial ones, or be both consisting of polar and axial components simultaneously. Analogous conclusion takes place regarding the parity \( t \) under time reversal. The possibility to have the same symmetry, that is, the situation, when both the vector-functions \( \mathbf{E}', \mathbf{H}' \) are pure polar (axial) vector-functions, or both ones t-even (t-uneven) simultaneously contradicts to experiment. Consequently it remains the variant, that vector-functions \( \mathbf{E}', \mathbf{H}' \) in the expression (36) are compound and consists of the components with even and uneven parities under improper rotations. It is in agreement with overview on compound symmetry structure of EM-field vector-functions (36) are particular case of the more general transformations (35) are particular case of the more general transformation symmetry of Maxwell equations, established by Rainich [6]. It is evident now, that really both the vectors \( \mathbf{E}' \) is polar vector, magnetic field \( \mathbf{H}' \) is axial vector, the alternative choice is provided. The conclusion can be easily proved, if to represent relation (35) in matrix form

\[
\begin{bmatrix}
\mathbf{E}' \\
\mathbf{H}'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\mathbf{E} \\
\mathbf{H}
\end{bmatrix}.
\]

(37)
We see, that given matrix has the form, which allows set up in conformity to it the complex number according to bijective mapping like to (1). Consequently, we have

\[
\begin{bmatrix}
\mathbf{E}' \\
\mathbf{H}'
\end{bmatrix} = e^{-i\theta} \begin{bmatrix}
\mathbf{E} \\
\mathbf{H}
\end{bmatrix},
\]

(38)
that is

\[
\mathbf{E}' = \mathbf{E} \cos \theta - i \mathbf{E} \sin \theta \\
\mathbf{H}' = \mathbf{H} \cos \theta - i \mathbf{H} \sin \theta.
\]

(39)
It means, that to real plane, which is determined by the vectors \( \mathbf{E}' \) and \( \mathbf{H}' \) can be set in conformity the complex plane for the vectors \( \mathbf{E}' \) and \( \mathbf{H}' \).

It is evident now, that really both the vectors \( \mathbf{E}' \) and \( \mathbf{H}' \) are consisting of both, \( P \)-even and \( P \)-uneven components. So the first component of, for instance, \( \mathbf{E}' \) will be \( P \)-even under reflection in the plane situated transversely to absciss-axis, the second component will be \( P \)-uneven.

Therefore, dual transformation symmetry of Maxwell equations, established by Rainich [6], indicates simultaneously on both complex nature of EM-field in general case, and that both electric and magnetic fields are consisting in general case of the components with various parity under improper rotations.
Let us find the invariants of dually transformed EM-field. It is easily to show, that the following relationship is taking place

\[ \left( \vec{E}^2 - \vec{H}^2 + 2i(\vec{E} \cdot \vec{H}) \right) e^{-2i\theta} = \text{inv}, \quad (40) \]

that is, we have at fixed parameter \( \theta \neq 0 \) two real EM-field invariants

\[ (\vec{E}^2 - \vec{H}^2) \cos 2\theta + 2(\vec{E} \cdot \vec{H}) \sin 2\theta = \text{inv} \]
\[ 2(\vec{E} \cdot \vec{H}) \cos 2\theta - (\vec{E}^2 - \vec{H}^2) \sin 2\theta = \text{inv}. \quad (41) \]

It follows from relation (41) that, in particular, at \( \theta = 0 \) we have well known \( 3 \) EM-field invariants

\[ (\vec{E}^2 - \vec{H}^2) = \text{inv} \]
\[ (\vec{E} \cdot \vec{H}) = \text{inv}. \quad (42) \]

It is interesting, that at \( \theta = 45^\circ \) and at \( \theta = 90^\circ \) the invariants of dually transformed EM-field are determined by the same relation (42) and by arbitrary \( \theta \) we have two linearly independent combinations of given known invariants.

Subsequent extension of dual symmetry for the EM-field with sources leads also to known requirement of the existence of two type of other physical quantities - two type of charges and two type of intrinsic moments of the particles or the absorbing (dispersive) centers in condensed matter. They can be considered to be the components of complex charge, or dual charge (in another equivalent terminology) of so called dually charged particles, which were described theoretically (see [3]). We have equivalent terminology (of so called dually charged particles) of complex charge, or dual charge (in another equivalent terminology) of so called dually charged particles, which were described theoretically (see [3]). We have to remark, that setting into EM-field theory of two type of charged particles or the absorbing (dispersive) centers in condensed matter, it is very suitable for many technical calculations, but was considered up to now to be mathematical abstraction, in which magnetic charges and magnetic currents are fictitious quantities. Similar conclusion concerns the conception of complex characteristics of EM-field in the matter. Given conception agrees well with all practice of electric circuits’ calculations. Very fruitfull mathematical method for electric circuits’ calculations, which uses all complex electric characteristics, see for example [11], was also considered earlier the only to be formal, but convenient mathematical technique. Formality of given technique gets now natural explanation. It has to be taken into account however, that in the case \( \theta = 0 \) we have well known electrodynamics with odd P-parity of electric field and even P-parity of magnetic field. In given case all the EM-field characteristics are real and by using in calculations of the complex quantities, we have always add the corresponding complex conjugate quantities. At the same time in the case \( \theta \neq 0 \) it will be incorrect, since both the real observable quantities, which corresponds to any complex EM-field characteristics have to be retained. Independent conclusion follows also from gauge invariance above considered (Sec.1). Really, the presence of complex charge means that 4-vector of current \( \vec{j}_\mu \) for any complex field is complex vector. In its turn, it means, that independently on starting origin of the charges and currents in the matter [they can be result of presence of Dirac field or another complex field] all the characteristics of EM-field in the matter have also to be complex-valued. Given conclusion follows immediately from Maxwell equations, since complex current \( \vec{j}_\mu \) enters explicitly Maxwell equations. It is also substantial, that Maxwell equations are invariant under the transformation of EM-field functions by relationship (11).

Let us designate the terms in (39)

\[ \vec{E} \cos \theta = \vec{E}^{[1]}, \vec{E} \sin \theta = \vec{E}^{[2]} \]
\[ \vec{H} \cos \theta = \vec{H}^{[1]}, \vec{H} \sin \theta = \vec{H}^{[2]}. \quad (43) \]

The Maxwell equations for the EM-field (\( \vec{E}^*, \vec{H}^* \)) in the matter in general case of both type of charged particles (that is electrically and magnetically charged), including dually charged particles are

\[ \nabla \times \vec{E}^{[\prime]}(\vec{r}, t) = -\mu_0 \frac{\partial \vec{H}^{[\prime]}(\vec{r}, t)}{\partial t} - i \vec{j}^{[\prime]}(\vec{r}, t), \quad (44) \]
\[ \nabla \times \vec{H}^{[\prime]}(\vec{r}, t) = \epsilon_0 \frac{\partial \vec{E}^{[\prime]}(\vec{r}, t)}{\partial t} + \vec{j}^{[\prime]}(\vec{r}, t), \quad (45) \]
\[ (\nabla \cdot \vec{E}^{[\prime]}(\vec{r}, t)) = \rho^{[\prime]}_e(\vec{r}, t), \quad (46) \]
\[ (\nabla \cdot \vec{H}^{[\prime]}(\vec{r}, t)) = \rho^{[\prime]}_g(\vec{r}, t), \quad (47) \]

where \( \vec{j}^{[\prime]}_e(\vec{r}, t), \vec{j}^{[\prime]}_g(\vec{r}, t) \) are respectively electric and magnetic current densities, \( \rho^{[\prime]}_e(\vec{r}, t), \rho^{[\prime]}_g(\vec{r}, t) \) are respectively electric and magnetic charge densities. Taking into account the relation (39) and (13) the system (44), (45), (46), (47) can be rewritten

\[ \nabla \times (\vec{E}^{[1]}(\vec{r}, t) - i \vec{E}^{[2]}(\vec{r}, t)) = \]
\[ -\mu_0 \left[ \frac{\partial \vec{H}^{[1]}(\vec{r}, t)}{\partial t} - i \partial \vec{H}^{[2]}(\vec{r}, t) \right] - \vec{j}^{[1]}_g(\vec{r}, t) + i \vec{j}^{[2]}_g(\vec{r}, t), \quad (48) \]
\[ \nabla \times (\vec{H}^{[1]}(\vec{r}, t) - i \vec{H}^{[2]}(\vec{r}, t)) = \]
\[ \epsilon_0 \left[ \frac{\partial \vec{E}^{[1]}(\vec{r}, t)}{\partial t} - i \partial \vec{E}^{[2]}(\vec{r}, t) \right] + \vec{j}^{[1]}_e(\vec{r}, t) - i \vec{j}^{[2]}_e(\vec{r}, t), \quad (49) \]
(\nabla \cdot (\vec{E}^1(\vec{r}, t) - i\vec{E}^2(\vec{r}, t))) = \rho^1_e(\vec{r}, t) - i\rho^2_g(\vec{r}, t), \quad (50)

(\nabla \cdot (\vec{H}^1(\vec{r}, t) - i\vec{H}^2(\vec{r}, t))) = \rho^1_g(\vec{r}, t) - i\rho^2_e(\vec{r}, t), \quad (51)

where \( j^1_e(\vec{r}, t) \), \( j^2_e(\vec{r}, t) \), \( j^1_g(\vec{r}, t) \), \( j^2_g(\vec{r}, t) \) are correspondingly electric and magnetic current densities, which by dual transformations are obeying to relation like to (52). They are designated like to (48), \( \rho^1_e(\vec{r}, t) \), \( \rho^2_e(\vec{r}, t) \), \( \rho^1_g(\vec{r}, t) \), \( \rho^2_g(\vec{r}, t) \) are correspondingly electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact in the system of equations (48), (49), (50), (51) can be obtained for the second pair of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering in equations, under space inversion. So, the components \( \vec{E}^1(\vec{r}, t) \), \( \vec{H}^2(\vec{r}, t) \), \( j^1_e(\vec{r}, t) \) have uneven parity, \( \vec{E}^2(\vec{r}, t) \), \( \vec{H}^1(\vec{r}, t), j^2_e(\vec{r}, t) \) have even parity, \( \rho^1_e(\vec{r}, t) \), \( \rho^2_e(\vec{r}, t) \) are scalars, \( \rho^1_g(\vec{r}, t) \), \( \rho^2_g(\vec{r}, t) \) are pseudoscalars. In the case, when \( j^2_e(\vec{r}, t) = 0 \), \( \rho^2_g(\vec{r}, t) = 0 \) we obtain the equations of usual singly charge electrodynamics for compound EM-field in mathematically correct form, which allows to separate the components of EM-field with various parities \( P \) under space inversion. It is remarkable, that the idea, that vector quantities, which characterize EM-field, are compound quantities and include both gradient and solenoidal parts, that is uneven and even parts under space inversion was put forward earlier in [3]. At the same time in the equations of dual electrodynamics given idea was presented the only in implicit form. The representation in explicit form by equations (48), (49), (50), (51) seems to be actual, since field vector and scalar functions with various t- and P-parities are mathematically heterogeneous and, for instance, their simple linear combination, for instance, for P-uneven and P-even electric and magnetic field strength vector-functions can be presented in the following matrix form

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} = \begin{bmatrix}
\cosh \theta & i \sinh \theta \\
-i \sinh \theta & \cosh \theta
\end{bmatrix} \begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix},
\]

where \( \theta \) is arbitrary continuous parameter, \( \theta \in [0, 2\pi] \). The relation (50) can be rewritten in the form

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} = \begin{bmatrix}
\cos i \theta & \sin i \theta \\
-\sin i \theta & \cos i \theta
\end{bmatrix} \begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix}.
\]

In particular, if \( \theta \) is polar angle of coordinate system in the plane, determined by \( \vec{E} \) and \( \vec{H} \), the transformations (55) represent themselves hyperbolic rotations in \((\vec{E}, \vec{H})\)-plane. Let us call the transformations (55) by hyperbolic dual transformations. It represents the interest to consider the following particular case of hyperbolic dual transformations. We can define parameter \( \theta \) according to relation

\[
tanh \theta = \frac{V}{c} = \beta,
\]

where \( V \) is velocity of the frame of reference, moving along x-axis in 3D subspace of \( 1 R_4 \) Minkowski space. We can also to set up in conformity to the plane \((x_2, x_3)\) in Minkowski space, the plane \((\vec{E}, \vec{H})\), in which \( \vec{E}, \vec{H} \) are orthogonal and are directed along abscess and ordinate axes correspondingly (or vice versa). Then we obtain

\[
|\vec{E}'| = \frac{|\vec{E}| + \beta|\vec{H}|}{\sqrt{1 - \beta^2}},
\]

\[
|\vec{H}'| = \frac{|\vec{H}| - \beta|\vec{E}|}{\sqrt{1 - \beta^2}},
\]

(or similar relations, in which \( \vec{E}, \vec{H} \) are interchanged by
Therefore it is seen, that \( \vec{E}, \vec{H} \) are transformed like to \( x_0, x_1 \) coordinates (or vice versa) of the space \( ^1 R_4 \). It follows from here, that both the vectors \( \vec{E}'', \vec{H}'' \) have \( t \)-even and \( t \)-uneven components in general case. We see also that Lorentz-invariance of Maxwell equations is particular case of hyperbolic dual symmetry. It means, that restriction to only Lorentz-invariance in consideration of Maxwell equations’ symmetry, which is usually used, constrains the concept on the EM-field itself and it is thereby constricting the possibilities for completeness of its practical usage. Taking into account (1) we obtain the relations, which are similar to (38), which can be rewritten in the similar to (39) form, that is, we have

\[
\begin{align*}
\vec{E}' &= \vec{E} \cos i\vartheta - i\vec{E} \sin i\vartheta \\
\vec{H}' &= \vec{H} \cos i\vartheta - i\vec{H} \sin i\vartheta.
\end{align*}
\]  
(60)

It is proof in general case, that each of two independent Maxwellian field components with even and uneven parities under space inversion is also compound and it consists of two independent components with even and uneven parities under time reversal. Then imposing designations

\[
\begin{align*}
\vec{E} \cos i\vartheta &= \vec{E}_{[3]}(\vec{r}, t), \quad \vec{E} \sin i\vartheta = \vec{E}_{[4]}(\vec{r}, t), \\
\vec{H} \cos i\vartheta &= \vec{H}_{[3]}(\vec{r}, t), \quad \vec{H} \sin i\vartheta = \vec{H}_{[4]}(\vec{r}, t),
\end{align*}
\]  
(61)

and considering the vector-functions \( \vec{E}_{[1]}(\vec{r}, t), \vec{E}_{[2]}(\vec{r}, t) \) and \( \vec{H}_{[1]}(\vec{r}, t), \vec{H}_{[2]}(\vec{r}, t) \) to be definitional domain for the vector-functions \( \vec{E}''(\vec{r}, t), \vec{H}''(\vec{r}, t) \) correspondingly, the Maxwell equations for the components of the field \( \vec{E}''(\vec{r}, t) \) and \( \vec{H}''(\vec{r}, t) \) have the same form and they are

\[
\begin{align*}
\nabla \times (\vec{E}_{[3]}(\vec{r}, t) - i\vec{E}_{[4]}(\vec{r}, t)) &= -\mu_0 \left[ \frac{\partial \vec{H}_{[3]}(\vec{r}, t)}{\partial t} - i \frac{\partial \vec{H}_{[4]}(\vec{r}, t)}{\partial t} \right] - j_{e, [3]}(\vec{r}, t) + ij_{g, [4]}(\vec{r}, t), \\
\nabla \times (\vec{H}_{[3]}(\vec{r}, t) - i\vec{H}_{[4]}(\vec{r}, t)) &= \epsilon_0 \left[ \frac{\partial \vec{E}_{[3]}(\vec{r}, t)}{\partial t} - i \frac{\partial \vec{E}_{[4]}(\vec{r}, t)}{\partial t} \right] + j_{e, [3]}(\vec{r}, t) - ij_{g, [4]}(\vec{r}, t),
\end{align*}
\]  
(62)

\[
(\nabla \cdot (\vec{E}_{[3]}(\vec{r}, t) - i\vec{E}_{[4]}(\vec{r}, t))) = \rho_{e, [3]}(\vec{r}, t) - i\rho_{g, [4]}(\vec{r}, t),
\]  
(64)

\[
(\nabla \cdot (\vec{H}_{[3]}(\vec{r}, t) - i\vec{H}_{[4]}(\vec{r}, t))) = \rho_{g, [3]}(\vec{r}, t) - i\rho_{e, [4]}(\vec{r}, t),
\]  
(65)

where \( j_{e, [3]}(\vec{r}, t), j_{g, [4]}(\vec{r}, t), j_{g, [3]}(\vec{r}, t), j_{g, [4]}(\vec{r}, t) \) are, correspondingly, electric and magnetic current densities, \( \rho_{e, [3]}(\vec{r}, t), \rho_{e, [4]}(\vec{r}, t), \rho_{g, [3]}(\vec{r}, t), \rho_{g, [4]}(\vec{r}, t) \) are, correspondingly, electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact the system of equations (62), (63), (64), (65) represent itself correctly integrated Maxwell equations for two kinds of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering equations, under time reversal. So, the components \( \vec{E}_{[3]}(\vec{r}, t), \vec{H}_{[3]}(\vec{r}, t), j_{e, [3]}(\vec{r}, t) \) have even parity, \( \vec{E}_{[4]}(\vec{r}, t), \vec{H}_{[4]}(\vec{r}, t), j_{e, [4]}(\vec{r}, t) \) have even parity, \( \rho_{e, [3]}(\vec{r}, t), \rho_{g, [4]}(\vec{r}, t) \) are scalars, \( \rho_{e, [4]}(\vec{r}, t), \rho_{g, [3]}(\vec{r}, t) \) are pseudoscalars. In the case, when \( j_{g, g}(\vec{r}, t) = 0, \rho_{g, g}(\vec{r}, t) = 0 \) we obtain the equations of usual singly charge electrodynamics for two components of EM-field with various parities under space inversion, at that either of the two consist also of two components of EM-field with various parity under time reversal.

It is easily to see, that invariants for EM-field, consisting of two hyperbolic dually symmetric parts, that is at \( \vartheta \neq 0 \) have the form, analogous to (40) and they can be obtained, if parameter \( \theta \) to replace by \( i\vartheta \). They are

\[
\left[ \vec{E}'^2 - \vec{H}'^2 + 2i(\vec{E}' \vec{H}') \right] e^{2\vartheta} = \text{inv}.
\]  
(66)

Consequently, two real invariants at \( \vartheta \neq 0 \) have the form

\[
\begin{align*}
(\vec{E}'^2 - \vec{H}'^2)e^{2\vartheta} &= I_1'' = \text{inv}, \\
2(\vec{E}' \vec{H}')e^{2\vartheta} &= I_2'' = \text{inv}.
\end{align*}
\]  
(67)

It follows from relation (67), that in both the cases, that is at \( \vartheta = 0 \) and at fixed \( \vartheta \neq 0 \), we obtain in fact well known EM-field invariants, since factor \( e^{2\vartheta} \) at fixed \( \vartheta \) seems to be insufficient. At the same time at arbitrary \( \vartheta \) the relation

\[
\frac{I_1''}{I_2''} = \frac{I_1}{I_2} = W = \text{inv}
\]  
(68)

is taking place. It is seen, that the value of \( W \) is independent on \( \vartheta \). It means physically, that the absolute values of both the vector-functions \( \vec{E}''(\vec{r}, t) \) and \( \vec{H}''(\vec{r}, t) \) are changed synchronously by hyperbolic dual transformations.

So, the usage of complex number theory allows to represent correctly the electrodynamics for two photon fields, which differs by parities under space inversion or time reversal by the same single system of generalized Maxwell equations. At the same time we have two related sets, that is pairs of complex vector and scalar functions, which are ordered in their \( P \)- and \( t \)-parities. It corresponds to definition of quaternions. Really any quaternion number \( x \) can be determined according to relation

\[
x = (a_1 + ia_2)e + (a_3 + ia_4),
\]  
(69)
where \( \{ a_m \} \in \mathbb{R}, m = 1, 4 \) and \( e, i, j, k \) produce basis, elements of which are satisfying the conditions

\[
(ij) = k, (ji) = -k, (ki) = j, (ik) = -j, \quad (ei) = i, (je) = j, (ek) = (ke) = k. \tag{70}
\]

Let us designate the quantities

\[
(2 E^{[1]}(r, t) - i E^{[2]}(r, t)) + (E^{[3]}(r, t) - i E^{[4]}(r, t)))j = \vec{E}(r, t)
\]

\[
(H^{[1]}(r, t) - i H^{[2]}(r, t)) + (H^{[3]}(r, t) - i H^{[4]}(r, t)))j = \vec{H}(r, t)
\]

\[
(j_e^{[-1]}(r, t) - i j_e^{[-2]}(r, t)) + (j_e^{[3]}(r, t) - i j_e^{[4]}(r, t)))j = \vec{j}_e(r, t)
\]

\[
(-j_g^{[-1]}(r, t) + i j_g^{[-2]}(r, t)) + (-j_g^{[3]}(r, t) + i j_g^{[4]}(r, t)))j = \vec{j}_g(r, t)
\]

\[
(p_{e}^{[-1]}(r, t) - i p_{e}^{[-2]}(r, t)) + (p_{e}^{[3]}(r, t) - i p_{e}^{[4]}(r, t)))j = \rho_{e}(r, t)
\]

\[
(p_{g}^{[-1]}(r, t) - i p_{g}^{[-2]}(r, t)) + (p_{g}^{[3]}(r, t) - i p_{g}^{[4]}(r, t)))j = \rho_{g}(r, t)
\]

\[
\text{where}\]

\[
E^{[1]}(r, t), H^{[2]}(r, t), j_e^{[1]}(r, t), \quad j_g^{[2]}(r, t), \rho_{e}^{[1]}(r, t), \rho_{g}^{[2]}(r, t),\]

\[
\text{are } P\text{-uneven, } t\text{-even,}\]

\[
E^{[2]}(r, t), H^{[1]}(r, t), j_e^{[2]}(r, t), \quad j_g^{[1]}(r, t), \rho_{e}^{[2]}(r, t), \rho_{g}^{[1]}(r, t),\]

\[
\text{are } P\text{-uneven, } t\text{-uneven,}\]

\[
E^{[3]}(r, t), H^{[4]}(r, t), j_e^{[3]}(r, t), \quad j_g^{[4]}(r, t), \rho_{e}^{[3]}(r, t), \rho_{g}^{[4]}(r, t),\]

\[
\text{are } P\text{-even, } t\text{-even,}\]

\[
E^{[4]}(r, t), H^{[3]}(r, t), j_e^{[4]}(r, t), \quad j_g^{[3]}(r, t), \rho_{e}^{[4]}(r, t), \rho_{g}^{[3]}(r, t),\]

\[
\text{are } P\text{-even, } t\text{-uneven. According to definition of quaternions } \vec{E}(r, t), \vec{H}(r, t), \vec{j}_e(r, t), \vec{j}_g(r, t), \rho_{e}(r, t), \rho_{g}(r, t) \text{ are quaternions. It means, that EM-field has quaternion structure and dual and hyperbolic dual symmetry of Maxwell equations will take proper account, if all the vector and scalar quantities to represent in quaternion form. Consequently, we have }\]

\[
\nabla \times (\vec{E}(r, t)) = -\mu_0 \left[ \frac{\partial \vec{H}(r, t)}{\partial t} \right] - \vec{j}_g(r, t), \tag{76}
\]

\[
\nabla \cdot (\vec{E}(r, t)) = \rho_e(r, t), \tag{78}
\]

\[
\nabla \cdot (\vec{H}(r, t)) = \rho_g(r, t)\tag{79}
\]

Therefore, symmetry of Maxwell equations under dual transformations of both the kinds allows along with generalization of Maxwell equations themselves to extend the field of application of Maxwell equations. It means also, that dual electrodynamics, developed by Tomilchick and co-authors, see for instance \[3\], obtains additional ground. Basic field equations in dual electrodynamics \[3\], \[4\], being to be written separately for two type of independent photon fields with various parities under space inversion or time reversal, will be isomorphic to Maxwell equations in complex form. It was in fact shown partly earlier in \[13\], \[14\], where complex charge was taken into consideration. At the same time all aspect of dual symmetry, leading to four-component quaternion form of Maxwell equations seem to be representing for the first time.

### III. CAVITY DUAL ELECTRODYNAMICS

Let us find the conserving quantities, which correspond to dual and hyperbolic dual symmetries of Maxwell equations. It seems to be interesting to realize given task on concrete practically essential example of cavity EM-field. At the same time to built the Lagrangian, which is adequate to given task it seems to be reasonable to solve the following concomitant task - to find dually symmetric solutions of Maxwell equations. It seems to be understandable, that the general solutions of differential equations can also possess by the same symmetry, which have starting differential equations, nevertheless dual symmetry of the solutions of Maxwell equations was earlier not found.

#### A. Classical Cavity EM-Field

Suppose EM-field in volume rectangular cavity without any matter inside it and made up of perfectly electrically conducting walls. Suppose also, that the field is linearly polarized and without restriction of commonness let us choose the one of two possible polarization of EM-field electrical component \( \vec{E}(r, t) \) along x-direction. Then the vector-function \( E_x(z, t) \vec{e}_x \) can be represented in well known form of Fourier sine series

\[
E^{[1]}(r, t) = E_x(z, t) \vec{e}_x = \sum_{\alpha=1}^{\infty} A_{\alpha}^E q_\alpha(t) \sin(k_\alpha z) \vec{e}_x, \tag{80}
\]

where \( q_\alpha(t) \) is amplitude of \( \alpha \)-th normal mode of the cavity, \( \alpha \in \mathbb{N}, k_\alpha = \alpha \pi/L, A_{\alpha}^E = \sqrt{2\pi^2 m_\alpha / V \varepsilon_0}, \)
\[ \omega_\alpha = \alpha \pi c / L, \] 
\( L \) is cavity length along z-axis, \( V \) is cavity volume, \( m_\alpha \) is parameter, which is introduced to obtain the analogy with mechanical harmonic oscillator. Let us remember, that the expansion in Fourier series instead of Fourier integral expansion is determined by known diskretness of \( \hat{k} \)-space, which is the result of finiteness of cavity volume. Particular sine case of Fourier series is consequence of boundary conditions

\[ [\vec{n} \times \vec{E}] |_{S} = 0, \ (\vec{n} \vec{H}) |_{S} = 0, \]  
which are held true for the perfect cavity considered. Here \( \vec{n} \) is the normal to the surface \( S \) of the cavity. It is easily to show, that \( E_x(z, t) \) represents itself a standing wave along z-direction.

Let us analyse the solutions of Maxwell equations for EM-field in a cavity in comparison with known solutions from the literature to pay the attention to some mathematical details, which have however substantial physical nature. For given reasons, despite on analysis simplicity, we will produce the consideration in detail.

Using the equation

\[ \varepsilon_0 \frac{\partial \vec{E}(z, t)}{\partial t} = [\nabla \times \vec{H}(z, t)] \],
we obtain the expression for magnetic field

\[ \vec{H}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} A^E_{\alpha} \varepsilon_0 \frac{d q_{\alpha}(t)}{dt} \cos(k_\alpha z) + f_{\alpha}(t) \vec{e}_y, \]  
where \( \{f_{\alpha}(t)\}, \alpha \in N, \) is the set of arbitrary functions of the time. It is evident, that the expression for \( \vec{H}(\vec{r}, t) \) is satisfying to boundary conditions. The partial solution, in which the functions \( \{f_{\alpha}(t)\} \) are identically zero, that is, \( \vec{H}(\vec{r}, t) \)

\[ \vec{H}[1](\vec{r}, t) = \sum_{\alpha=1}^{\infty} A^E_{\alpha} \varepsilon_0 \frac{d q_{\alpha}(t)}{dt} \cos(k_\alpha z) \vec{e}_y, \]

is always used in all the EM-field literature. However even in given case it is evident, that the Maxwellian field is complex field. Really using the equation

\[ [\nabla \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}, \]  
it is easily to find the class of field functions \( \{q_{\alpha}(t)\}. \)

They will satisfy to differential equations

\[ \frac{d^2 q_{\alpha}(t)}{dt^2} + \frac{k_\alpha^2}{\varepsilon_0 \mu_0} q_{\alpha}(t) = 0, \alpha \in N. \]  
Consequently, we have

\[ q_{\alpha}(t) = C_{1\alpha} e^{i \omega_\alpha t} + C_{2\alpha} e^{-i \omega_\alpha t}, \alpha \in N, \]  
where \( C_{1\alpha}, C_{2\alpha} \) are arbitrary constants. Thus, real-valued free Maxwellian field equations result in well known in the theory of differential equations situation - the solutions are complex-valued functions. It means, that generally the field functions for free Maxwellian field in the cavity produce complex space. So we obtain additional independent argument, that the known conception, on the only real-quantity definiteness of EM-field, has to be corrected. On the other hand, the equation has also the only real-valued general solution, which can be represented in the form

\[ q_{\alpha}(t) = B_\alpha \cos(\omega_\alpha t + \phi_\alpha), \]  
where \( B_\alpha, \phi, \alpha \in N \) are arbitrary constants. It is substantial, that the functions in real-valued general solution have a definite t-parity.

Thus, we come independently on the previous consideration in Sec.I and Sec.II, Subsec.A to the conclusion, that classical Maxwellian EM-field can be both real-quantity defined and complex-quantity defined.

It is interesting, that there is the second physically substantial solution of Maxwell equations. Really, from general expression for the field \( \vec{H}(\vec{r}, t) \) it is easily to obtain differential equations for \( \{f_{\alpha}(t)\}, \alpha \in N, \)

\[ \frac{df_{\alpha}(t)}{dt} + \frac{\mu_0}{\varepsilon_0} \frac{\partial^2 q_{\alpha}(t)}{\partial t^2} \cos(k_\alpha z) \]

\[ - \frac{1}{\mu_0} A^E_{\alpha} k_\alpha q_{\alpha}(t) \cos(k_\alpha z) = 0. \]

The formal solution of given equations in general case is

\[ f_{\alpha}(t) = A^E_{\alpha} \cos(k_\alpha z) \left[ \frac{k_\alpha}{\mu_0} \int_{0}^{t} q_{\alpha}(\tau) d\tau - \frac{d q_{\alpha}(t)}{dt} \right] \frac{\varepsilon_0}{k_\alpha}. \]

Therefore, we have the second solution of Maxwell equations for \( \vec{H}(\vec{r}, t) \) in the form

\[ \vec{H}[2](\vec{r}, t) = -\left\{ \sum_{\alpha=1}^{\infty} A^H_{\alpha} q'_{\alpha}(t) \cos(k_\alpha z) \right\} \vec{e}_y, \]

where \( A^H_{\alpha} = \sqrt{2 \omega^2_\alpha m_\alpha / V \mu_0} \). Similar consideration gives the second solution for \( \vec{E}(\vec{r}, t) \)

\[ \vec{E}[2](\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} A^{E'}_{\alpha} q''_{\alpha}(t) \sin(k_\alpha z) \right\} \vec{e}_x. \]

The functions \( q'_{\alpha}(t) \) and \( q''_{\alpha}(t) \) in relationships and are

\[ q'_{\alpha}(t) = \omega_\alpha \int_{0}^{t} q_{\alpha}(\tau) d\tau \]
\[ q''_{\alpha}(t) = \omega_\alpha \int_{0}^{t} q'_{\alpha}(\tau') d\tau' \]
correspondingly. Owing to the fact, that the solutions have simple form of harmonic trigonometrical functions,
the second solution for electric field differs from the first solution the only by sign, that is substantial, and by inessential integration constants. Integration constants can be taken into account by means of redefinition of factor \( m \alpha \) in field amplitudes. It is also evident, that if vector-functions \( \tilde{E}(\vec{r}, t) \) and \( \tilde{H}(\vec{r}, t) \) are the solutions of Maxwell equations, then vector-functions \( \hat{T} \tilde{E}(\vec{r}, t) \) and \( \hat{T} \tilde{H}(\vec{r}, t) \), where \( \hat{T} \) is time inversion operator, are also the solutions of Maxwell equations. Moreover, if starting vector-function, to which operator \( \hat{T} \) is applied is \( t \)-uneven solution, for instance for magnetic component in the form

\[
\hat{T}[t \tilde{H}(\vec{r}, t)], \quad (94)
\]

where \( t \) is time. It can be shown in a similar way, that dually symmetric solutions, which are \( P \)-even and \( P \)-uneven are also existing.

Therefore, there are the solutions with various combinations of the signs for vector-functions \( \tilde{E}(\vec{r}, t) \) and \( \tilde{H}(\vec{r}, t) \), which are realized simultaneously, that is, their linear combination with coefficients from the field \( C \) of complex numbers will represent the solution of Cauchy problem for Maxwell equations in correspondence with known theorem, that the solution of Cauchy problem for any systems of homogeneous linear equations in partial derivatives exists and it is unique in the vicinity of any point of the initial surface (in the case, when the point selected is not characteristic point and the function, which determines given hypersurface is continuously differentiable). In other words, we obtain again the agreement with Maxwell equation symmetry consideration. Given property of EM-field seems to be essential, since it permits passing for the processes, which seemingly are forbidden by CPT-theorem. For example, let us consider the resonance system EM-field plus matter in the cavity, in particular, the so called dressed state of some quasi-particles’ system. Suppose, that wave function can be factorized, matter part is \( P \)- and \( t \)-even under space and time inversion transformations, while EM-field part is \( P \)-uneven. CPT-invariance will be preserved, since EM-field has simultaneously with \( t \)-even the \( t \)-uneven component, determined by expression (91). Therefore \( t \)-parity of the function \( q_\alpha(t) \) can be various, and in the case, if we choose \( t \)-parity to be identical to the parity of the function \( q_\alpha(t) \), the solution will be different in the meaning, that the field vectors will have opposite \( t \)-parity in comparison with the first solution. It is evident, that boundary conditions are fulfilled for all the cases considered.

To build the Lagrangian we can choose the following sets of EM-field functions \( \{u_\alpha^{s, \pm}(x)\} \), \( s = 1, 2, \alpha \in \mathbb{N} \),

\[
\{u_\alpha^{s, \pm}(x)\} = \{\sqrt{\epsilon_0 \lambda_\alpha^s} \sin k_\alpha(x_3)[q_\alpha(x_4) \pm iq_\alpha''(x_4)]
\]

\[
\{u_\alpha^{s, \pm}(x)\} = \{\sqrt{\mu_0 \lambda_\alpha^s} \cos k_\alpha(x_3)[-q_\alpha'(x_4) \pm \frac{i}{\omega_\alpha} \frac{dq_\alpha(x_4)}{dx_4}]\}
\]

The functions \( \{u_\alpha^{s, \pm}(x)\}, s = 1, 2, \alpha \in \mathbb{N} \) are built from the components of the expansion in Fourier series of the fields \( \tilde{E}^1[\vec{r}, t], \tilde{E}^2[\vec{r}, t] \) and \( \tilde{H}^1[\vec{r}, t], \tilde{H}^2[\vec{r}, t] \) correspondingly. At the same time the sets \( \{u_\alpha^{s, \pm}(x)\}, s = 1, 2, \alpha \in \mathbb{N} \) produce at fixed \( x \) two orthogonal countable bases, corresponding to \( s = 1, 2 \) in two Hilbert spaces, which are formed by vectors \( \{\tilde{u}_\alpha^{s, \pm}(x_1), \tilde{u}_\alpha^{s, \pm}(x_2), ...\} \) for variable \( x \in 1 \mathbb{R}_4 \). Really scalar product of two arbitrary vectors \( \tilde{u}_\alpha^{s, \pm}(x_1), \tilde{u}_\alpha^{s, \pm}(x_2), ... \) is equal to

\[
\langle \tilde{u}_\alpha^{s, \pm}(x_1) | \tilde{u}_\alpha^{s, \pm}(x_2) \rangle \quad (96)
\]

that means, that it is restricted, since the sum over \( s \) represents the energy of the field in restricted volume. Consequently, the norm of vectors can be defined by the relationship

\[
\|\tilde{u}_\alpha^{s, \pm}(x)\|^2 = \sqrt{\langle \tilde{u}_\alpha^{s, \pm}(x) | \tilde{u}_\alpha^{s, \pm}(x) \rangle} = \sqrt{\sum_{\alpha=1}^{\infty} \int_0^L u_{\alpha}^{s, \pm}(x_4,i,z)u_{\alpha}^{s, \pm}(x_4,j,z)dz, s = 1, 2.} \quad (97)
\]

Then vector distance is

\[
\delta(\tilde{u}_\alpha^{s, \pm}(x_1), \tilde{u}_\alpha^{s, \pm}(x_2)) = \|\tilde{u}_\alpha^{s, \pm}(x_1) - \tilde{u}_\alpha^{s, \pm}(x_2)\|. \quad (99)
\]

So we obtain, that the vectors \( \{\tilde{u}_\alpha^{s, \pm}(x)\}, x \in 1 \mathbb{R}_4 \) produce the space \( L_2 \) and taking into account the Riss-Fisher theorem it means, that given vector space is complete, that in its turn means, that the spaces of vectors \( \{\tilde{u}_\alpha^{s, \pm}(x)\}, x \in 1 \mathbb{R}_4, s = 1, 2, \) are Hilbert spaces. Consequently Lagrangian \( L(x) \) can be represented in the following form

\[
L(x) = \sum_{s=1}^{2} \sum_{\mu=1}^{4} \sum_{\alpha=1}^{\infty} \frac{\partial u_{\alpha}^{s, \pm}(x)}{\partial x_\mu} \frac{\partial u_{\alpha}^{s, \pm}(x)}{\partial x_\mu} - \sum_{s=1}^{2} \sum_{\mu=1}^{4} \sum_{\alpha=1}^{\infty} K(x)u_{\alpha}^{s, \pm}(x)u_{\alpha}^{s, \pm}(x), \quad (100)
\]

where \( K(x) \) is factor, depending on the set of variables \( x = \{x_\mu\}, \mu = 1, 4 \).

Let us find the conserving quantity, corresponding to dual symmetry of Maxwell equations. Dual transformation, determined by relation (47) is the transformation the only in the space of field three-dimensional vector-functions \( \tilde{E}, \tilde{H} \), (let us designate it by \( (\tilde{E}, \tilde{H}) \)-space) and it does not touch upon the coordinates. It seems to be convenient to define in given space the reference frame,
then the transformation, given by (37) is the rotation of two component matrix vector-function

\[ ||F|| = \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix}. \]  \hfill (101)

Instead of two Hilbert space for two sets of vectors \( \{ \vec{U}^{i,s,\pm}(x) \} \), \( x \in 1 \mathcal{R}_4, s = 1, 2 \) we can also define one Hilbert space for row matrix vector function set

\[ \{ ||\vec{U}(x)|| \} = \{ \vec{U}^{1,\pm}(x)\vec{U}^{2,\pm}(x) \} \]  \hfill (102)

with the set of components

\[ \{ ||\vec{U}_\alpha(x)|| \} = \{ u^{1,\pm}_\alpha(x)u^{2,\pm}_\alpha(x) \} \]  \hfill (103)

where \( \alpha \in \mathbb{N} \). In general case instead parameter \( \theta \) we can define rotation angles \( \theta_{ik} \), \( i, k = \frac{1}{3} \), in 2D-planes of \( (\vec{E}, \vec{H}) \) functional space. It is evident, that \( \theta_{ik} \) are antisymmetric under the indices \( i, k = \frac{1}{3} \). According to Nöther theorem, the conserving quantity, corresponding to parameters \( \theta_{ik} \) in dual transformations (37), that is at \( \theta_{ik} = \theta_{12} \) is determined by relations like to (20) and (21). So, we obtain

\[ S^\mu_{12} = -\frac{i}{c} \int \{ \sum_{\alpha=1}^\infty \frac{\partial L}{\partial (\partial_\mu ||\vec{U}_\alpha||)} ||\vec{Y}_\alpha|| + c.c. \} d^3x \]  \hfill (104)

where \( \mu = \frac{1}{3} \) and it was taken into account, that \( ||X_\alpha|| \) in matrix relation (104), which is like to (20) is equal to zero. The factor \( \frac{\partial L}{\partial (\partial_\alpha ||\vec{U}_\alpha||)} \) in (104) is row matrix

\[ \frac{\partial L}{\partial (\partial_\mu ||\vec{U}_\alpha||)} = \begin{bmatrix} \frac{\partial L}{\partial (\partial_{\mu} u^{1,\pm}_\alpha)} \\ \frac{\partial L}{\partial (\partial_{\mu} u^{2,\pm}_\alpha)} \end{bmatrix}, \]  \hfill (105)

matrix \( ||\vec{Y}_\alpha|| \) is product of matrices \( ||\vec{I}_\alpha|| \) and \( ||\vec{U}_\alpha(x)|| \), that is

\[ ||\vec{Y}_\alpha|| = ||\vec{I}_\alpha|| \begin{bmatrix} u^{1,\pm}_\alpha \\ u^{2,\pm}_\alpha \end{bmatrix}, \]  \hfill (106)

where \( ||\vec{I}_\alpha|| \) is the matrix, which corresponds to infinitesimal operator of dual or hyperbolic dual transformations of \( \alpha \)-th mode of cavity EM-field. It represents in general case the product of three matrices, corresponding to rotation along three mutually perpendicular axes in 3D functional space above defined. So \( ||\vec{I}_\alpha|| = ||\vec{I}^{1,\pm}_{\alpha}|| ||\vec{I}^{2,\pm}_{\alpha}|| \). The transformations in the form, which is given by (37) correspond to \( \theta_{23} = \theta, \theta_{12} = 0, \theta_{31} = 0 \), that is \( ||\vec{I}^{2,\pm}_{\alpha}|| = ||\vec{I}^{3,\pm}_{\alpha}|| = E \), where \( E \) is unit \( [2 \times 2] \)-matrix. In the absence of dispersive medium in the cavity \( ||\vec{I}_\alpha|| \) will be independent on \( \alpha \). Moreover, it is easily to see, that infinitesimal operator with matrix \( ||\vec{I}_\alpha|| \) is the same for dual transformations, determined by (37) and hyperbolic dual transformations, determined by (53). Really \( ||\vec{I}_\alpha|| \) in both the cases is

\[ ||\vec{I}_\alpha|| = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \]  \hfill (107)

for any \( \alpha \in \mathbb{N} \).

Conserving quantity is

\[ S^\mu_{12} = -\frac{i}{c} \int \{ \sum_{\alpha=1}^\infty \frac{\partial L}{\partial (\partial_\mu ||\vec{U}^*_{\alpha}||)} ||\vec{Y}_\alpha|| + c.c. \} d^3x \]  \hfill (108)

The structure of (108) unambiguously indicates, that it is the component of spin tensor \( [2], [14] \), to which dual vector component can be set in the correspondence according to relation

\[ S^\mu = \varepsilon_{ijk} S^4_{jk} = \]  \hfill (109)

\[ -\frac{i}{c} \int \{ \sum_{\alpha=1}^\infty \frac{\partial L}{\partial (\partial_\mu ||\vec{U}^*_{\alpha}||)} ||\vec{Y}_\alpha||_{jk} + c.c. \} d^3x, \]  \hfill (110)

where \( \varepsilon_{ijk} \) is completely antisymmetric Levi-Civita 3-tensor.

Therefore we obtain, that the same physical conserving quantity corresponds to dual and hyperbolic dual symmetry of Maxwell equations. Taking into account the expressions for Lagrangian (100) and for infinitesimal operator (107), in the geometry chosen, when vector \( \vec{E} \) is directed along absciss axis, vector \( \vec{H} \) is directed along ordinate axis in \( (\vec{E}, \vec{H}) \) functional space, we have

\[ S^\mu_{12} = \sum_{\alpha=1}^\infty \frac{\partial u^{1,\pm}_\alpha}{\partial x_\mu} u^{2,\pm}_\alpha - \frac{\partial u^{2,\pm}_\alpha}{\partial x_\mu} u^{1,\pm}_\alpha + c.c. \]  \hfill (111)

and

\[ S^4 = \varepsilon_{312} S^4_{12} = -\frac{i}{c} \int \{ \sum_{\alpha=1}^\infty \frac{\partial L}{\partial (\partial_\mu ||\vec{U}^*_{\alpha}||)} ||\vec{Y}_\alpha|| + c.c. \} d^3x, \]  \hfill (111)

It is projection of spin on the propagation direction. Therefore we have in given case right away physically significant quantity - spirality.

The relations (104), (109), (110) show, that spin of classical relativistic EM-field in the cavity and, correspondingly, spirality are additive quantities and they represent the sum of cavity spin and spirality modes. On the connection of the conserving quantity, which is invariant of dual symmetry with spin was indicated in (3), where free EM-field was considered with traditional Lagrangian, which uses vector potentials to be field functions. The result obtained together with aforesaid result in (3) lift dilemma on the necessity of using of given quantity by consideration of classical EM-field. Really, the situation was to some extent paradoxical, and it can be displayed by the following conversation between two disputant physicists. "Spin exists" - has insisted the first, referring on the appearance of additional tensor component in total tensor of moment - intrinsic moment - to be consequence of Minkowsky space symmetry under Lorentz transformations, "Spin does not exists" - has insisted the second, referring on the metrized tensor of the moment, in which spin part is equal to zero [2] in distinction from canonical tensor. In other words, both disputants were
in one’s own way right. Dual symmetry leads to unambiguous conclusion “Spin exists” and has to be taken into consideration by the solution of tasks, concerning both classical and quantum electrodynamics. Moreover spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality in the simplest case above considered) combine two subsystems of photon fields, that is the subsystem of two fields, which have definite $P$-parity (even and uneven) with the subsystem of two fields, which have definite $T$-parity (also even and uneven) into one system. In fact we obtain the proof for four component structure of EM-field to be a single whole, that is confirmation along with the representation of Maxwell equations in quantum optics [18] corresponds to given idea. EM-field potentials are used to be field functions by canonical quantization. At the same time to describe free EM-field it is sufficient to choose immediately the observable quantities - vector-functions $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ - to be field functions. We use further given idea by EM-field quantization.

1. Time-Local Quantization of Cavity EM-Field

We can start like to canonical quantization, from classical Hamiltonian, which for the first partial classical solution of Maxwell equations is

$$\mathcal{H}^{[1]}(t) = \frac{1}{2} \iiint \left[ c_0 E^2_z(z, t) + \mu_0 H^2_y(z, t) \right] dxdydz$$

$$= \frac{1}{2} \sum_{a=1}^{\infty} \left[ m_\alpha \nu_\alpha^2 q_\alpha^2(t) + \frac{\mu_\alpha^2(t)}{m_\alpha} \right] ,$$

(112)

where

$$p_\alpha = m_\alpha \frac{dq_\alpha(t)}{dt} .$$

(113)

So, taking into consideration the relationship for Hamiltonian $\mathcal{H}^{[1]}(t)$ we set in correspondence to canonical variables $q_\alpha(t), p_\alpha(t)$, determined by the first partial solution of Maxwell equations, the operators by usual way

$$[\hat{p}_\alpha(t), \hat{q}_\beta(t)] = i\hbar \delta_{\alpha\beta}$$

$$[\hat{q}_\alpha(t), \hat{q}_\beta(t)] = [\hat{p}_\alpha(t), \hat{p}_\beta(t)] = 0,$$

(114)

where $\alpha, \beta \in N$. Introducing the operator functions of time $\hat{a}_\alpha(t)$ and $\hat{a}_\alpha^+(t)$

$$\hat{a}_\alpha(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q_\alpha(t) + i\hat{p}_\alpha(t) \right]$$

$$\hat{a}_\alpha^+(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q_\alpha(t) - i\hat{p}_\alpha(t) \right] ,$$

(115)

we obtain the operator functions of canonical variables in the form

$$\hat{q}_\alpha(t) = \sqrt{\frac{h}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right]$$

$$\hat{p}_\alpha(t) = i \sqrt{\frac{\hbar m_\alpha \omega_\alpha}{2}} \left[ \hat{a}_\alpha^+(t) - \hat{a}_\alpha(t) \right] .$$

(116)

Then EM-field operator functions are obtained right away and they are

$$\hat{E}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_\alpha}{V\epsilon_0}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right] \sin(k_\alpha z) \right\} \vec{e}_x ,$$

(117)
\[
\hat{H}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V \epsilon_0}} \left[ \hat{a}_{\alpha}^+(t) - \hat{a}_{\alpha}(t) \right] \cos(k_\alpha z) \vec{e}_y, \tag{118}
\]

Taking into account the relationships (117), (118) and Maxwell equations, it is easily to find an explicit form for the dependencies of operator functions \(\hat{a}_{\alpha}(t)\) and \(\hat{a}_{\alpha}^+(t)\) on the time. They are

\[
\hat{a}_{\alpha}^+(t) = \hat{a}_{\alpha}^+(t = 0)e^{i\omega_{\alpha} t}, \quad \hat{a}_{\alpha}(t) = \hat{a}_{\alpha}(t = 0)e^{-i\omega_{\alpha} t}, \tag{119}
\]

where \(\hat{a}_{\alpha}^+(t = 0), \hat{a}_{\alpha}(t = 0)\) are constant, complex-valued in general case, operators.

Physical sense of operator time dependent functions \(\hat{a}_{\alpha}^+(t)\) and \(\hat{a}_{\alpha}(t)\) is well known. They are creation and annihilation operator of the \(\alpha\)-mode photon. They are continuously differentiable operator functions of the \(\alpha\)-mode photon. It means, that the time of photon creation (annihilation) can be determined strictly, at the same time operator functions \(\hat{a}_{\alpha}^+(t)\) and \(\hat{a}_{\alpha}(t)\) do not carry any information on the place, that is on space coordinates of given event.

It seems to be essential, that complex exponential dependencies in (119) cannot be replaced by the real-valued harmonic trigonometrical functions. Really, if to suggest, that

\[
\hat{a}_{\alpha}^+(t) = \hat{a}_{\alpha}^+(t = 0) \cos \omega_{\alpha} t, \tag{120}
\]

then we obtain, that the following relation has to be taking place

\[
[\hat{a}_{\alpha}^+(t = 0) - \hat{a}_{\alpha}(t = 0)]^{-1}[\hat{a}_{\alpha}^+(t = 0) + \hat{a}_{\alpha}(t = 0)] = \tan \omega_{\alpha} t. \tag{121}
\]

We see, that left-hand side in relation (121) does not depend on time, right-hand side is depending. The contradiction obtained establishes an assertion. Therefore, the quantized Maxwellian EM-field is complex-valued field in full correspondence with pure algebraic conclusion in Sec.I.

Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both complex-valued and real-valued functions. The fields \(\hat{E}^{[2]}(\vec{r}, t), \hat{H}^{[2]}(\vec{r}, t)\) can be quantized in much the same way. The operators \(\hat{a}''_{\alpha}(t), \hat{a}''_{\alpha}^+(t)\) are introduced analogously to (117).

\[
\hat{a}''_{\alpha}(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q''_{\alpha}(t) + i p''_{\alpha}(t) \right] \tag{122}
\]

\[
\hat{a}''_{\alpha}^+(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q''_{\alpha}(t) - i p''_{\alpha}(t) \right]
\]

For the operators of field function we obtain

\[
\hat{E}^{[2]}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V \epsilon_0}} \left[ \hat{a}''_{\alpha}(t) + \hat{a}''_{\alpha}^+(t) \right] \sin(k_{\alpha} z) \right\} \vec{e}_1, \tag{123}
\]

\[
\hat{H}^{[2]}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V \mu_0}} \left[ \hat{a}''_{\alpha}(t) - \hat{a}''_{\alpha}^+(t) \right] \cos(k_{\alpha} z) \right\} \vec{e}_2. \tag{124}
\]

In accordance with definition of complex quantities we can built the following combination of solutions, satisfying Maxwell equations

\[
(\hat{E}^{[1]}(\vec{r}, t), \hat{E}^{[2]}(\vec{r}, t)) \rightarrow \hat{E}^{[1]}(\vec{r}, t) + i \hat{E}^{[2]}(\vec{r}, t) = \hat{\vec{E}}(\vec{r}, t), \tag{125}
\]

\[
(\hat{H}^{[2]}(\vec{r}, t), \hat{H}^{[1]}(\vec{r}, t)) \rightarrow \hat{H}^{[2]}(\vec{r}, t) + i \hat{H}^{[1]}(\vec{r}, t) = \hat{\vec{H}}(\vec{r}, t). \tag{126}
\]

Consequently, the electric and magnetic field operators for quantized EM-field, corresponding to general solution of Maxwell equations, are

\[
\hat{\vec{E}}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V \epsilon_0}} \left[ \hat{a}_{\alpha}^+(t) + \hat{a}_{\alpha}(t) \right] \right\} \sin(k_{\alpha} z) \vec{e}_x, \tag{127}
\]

and

\[
\hat{\vec{H}}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V \mu_0}} \left[ \hat{a}_{\alpha}(t) - \hat{a}_{\alpha}^+(t) \right] \right\} \cos(k_{\alpha} z) \vec{e}_y. \tag{128}
\]

It is substantial, that both field operators \(\hat{\vec{E}}(\vec{r}, t)\) and \(\hat{\vec{H}}(\vec{r}, t)\) are Hermitian operators.

The method of EM-field quantization above considered is in fact development of canonical quantization, proposed by Dirac. Further development can be made, if to take into account the independence and equal rights of all the coordinates \(x_\mu, \mu = 1, 4\) in Minkowsky space \(1 R_4\). Really all physical events are taking place on finite segment of time. It leads in application to electrodynamics to diskretness of \(\omega_-\) space of possible light frequencies like to diskretness of \(\vec{k}\)-space, which is the result of finiteness of cavity volume. It is interesting that, Dirac himself has in [17] written, that the theory proposed is not strictly relativistic, since the time everywhere is considered to be c-number instead of to consider it symmetrically with the space coordinates. From here follows unambiguously, that quantum electrodynamics, based on Dirac canonical EM-field quantization method is not fully relativistic and correspondingly it is not fully quantum theory. We will show in the next subsections the way to obtain fully quantum theory of electrodynamics.

2. Space-Local Quantization of Cavity EM-Field

We will consider for the simplicity the dependence of EM-field vector-functions the only on \(z\)-space coordinate, which is chosen in propagation direction in \(R_3 \in \mathbb{C}\).
introducing. So we will have

\[ E_y^{[1]}(z, t) = \sum_{\alpha=1}^{\infty} A^{*}_\alpha q_\alpha(z) \sin(\omega_\alpha t) \hat{e}_x, \]  

\[ H_y^{[1]}(z, t) \hat{e}_y = \{-\epsilon_0 \sum_{\alpha=1}^{\infty} [A^{*}_\alpha \omega_\alpha \cos(\omega_\alpha t) \int_0^z q_\alpha(z')dz' + H_{a0}(t)] \} \hat{e}_y. \]  

where \( q_\alpha(z), \alpha \in N, \) is \( \alpha \)-th normal mode of the 4-dimensional cavity, which include time coordinate along with space coordinates,

\[ k_\alpha = \frac{\alpha \pi}{cT}, A^{*}_\alpha = \sqrt{\frac{2\alpha^2 \omega_\alpha}{T \epsilon_0}}, \omega_\alpha = \frac{\alpha \pi}{T}. \]

\( \{H_{a0}(t), \alpha \in N, \) is the set of arbitrary functions of the time. Then the Hamiltonian can be obtained taking into account the expressions for \( E_y^{[1]}(z, t), H_y^{[1]}(z, t) \) and integrating. So we will have

\[ G^{[1]}(z) = \frac{1}{2} \sum_{\alpha=1}^{\infty} \left[ m_\alpha \omega_\alpha^2 \frac{dq_\alpha(z)}{dz}^2 + \frac{1}{c^2} \omega_\alpha^4 m_\alpha [q_\alpha(z)]^2 \right], \]  

where the case with \( \{H_{a0}(t) \equiv 0, \alpha \in N \) is chosen and

\[ q_\alpha'(z) = \int_0^z q_\alpha(z')dz'. \]  

By redefinition of the variables in accordance with relations

\[ q^{\prime \prime}_\alpha(z) = \frac{1}{c} \omega_\alpha q^{\prime \prime}_\alpha(z), \]  

\[ p^{\prime \prime}_\alpha(z) = m_\alpha \omega_\alpha \frac{dq^{\prime \prime}_\alpha(z)}{dz}, \]

the Hamiltonian \( G^{[1]}(z) \) will have the canonical form

\[ G^{[1]}(z) = \frac{1}{2} \sum_{\alpha=1}^{\infty} \left[ \frac{p^{\prime \prime}_\alpha(z)^2}{m_\alpha} + m_\alpha \omega_\alpha^2 [q^{\prime \prime}_\alpha(z)]^2 \right]. \]

It means that space coordinates’ dependent quantization of cavity EM-field can be realized in a similar manner with above described time dependent quantization. So, we can define quite analogously the quantization rules by the relationships

\[ [\hat{p}^{\prime \prime}_\alpha(z), \hat{q}^{\prime \prime}_\beta(z)] = i\lambda_0 \delta_{\alpha\beta} \]  

\[ [\hat{q}^{\prime \prime}_\alpha(z), \hat{q}^{\prime \prime}_\beta(z)] = [\hat{p}^{\prime \prime}_\alpha(z), \hat{p}^{\prime \prime}_\beta(z)] = 0, \]  

where \( \alpha, \beta \in N, \lambda_0 \) is analogue of Planck constant. It is evident from \( \lambda_0 \)-definition by \( \{130\}, \) that \( \lambda_0 \) and Planck constant have the same dimension, however their numerical coincidence seems to be unobvious, since Planck constant characterizes the "seizure" of the time by propagating of EM-field, while \( \lambda_0 \) characterises the "seizure" of the space.

The operators \( \hat{a}^{\prime \prime\prime}_\alpha(z), \hat{a}^{\prime \prime\prime\dagger}_\alpha(z) \) are defined also analogously to operators \( \hat{a}_\alpha(t), \hat{a}^{\dagger}_\alpha(t) \) and they are

\[ \hat{a}^{\prime \prime\prime}_\alpha(z) = \frac{1}{\sqrt{2m_\alpha \lambda_0 \omega_\alpha}} [m_\alpha \omega_\alpha \hat{q}^{\prime \prime\prime}_\alpha(z) + ip^{\prime \prime\prime}_\alpha(z)] \]  

\[ \hat{a}^{\prime \prime\prime\dagger}_\alpha(z) = \frac{1}{\sqrt{2m_\alpha \lambda_0 \omega_\alpha}} [m_\alpha \omega_\alpha \hat{q}^{\prime \prime\prime\dagger}_\alpha(z) - ip^{\prime \prime\prime\dagger}_\alpha(z)]. \]  

The dependencies of given scalar operator functions on coordinate \( z \) in an explicit form for Maxwellian EM-field can be easily obtained by means of solutions of Maxwell equations and they are

\[ \hat{a}^{\dagger}_\alpha(z) = \hat{a}^{\dagger}_\alpha(0) e^{ik_\alpha z}, \]  

\[ \hat{a}_\alpha(z) = \hat{a}_\alpha(0) e^{-ik_\alpha z}, \]  

where \( \hat{a}^{\dagger}_\alpha(0), \hat{a}_\alpha(0) \) are constant, complex-valued in general case, operators. Let us remark in passing, that the dependencies \( (138) \) on \( z \)-coordinate are similar to dependencies \( \hat{a}_\alpha^{\dagger}(t), \hat{a}_\alpha(t) \) on time, which are given by \( (119) \).

From relationships \( (137) \) we obtain the expressions for operators of canonical variables \( \hat{q}^{\prime \prime\prime}_\alpha(z) \) and \( \hat{p}^{\prime \prime\prime}_\alpha(z) \) in the form

\[ \hat{q}^{\prime \prime\prime}_\alpha(z) = \sqrt{\frac{\lambda_0}{2m_\alpha \omega_\alpha}} [\hat{a}^{\dagger \prime \prime\prime\dagger}_\alpha(z) + \hat{a}^{\prime \prime\prime}_\alpha(z)]; \]  

\[ \hat{p}^{\prime \prime\prime}_\alpha(z) = i\sqrt{\frac{m_\alpha \lambda_0 \omega_\alpha}{2}} [\hat{a}^{\dagger \prime \prime\prime\dagger}_\alpha(z) - \hat{a}^{\prime \prime\prime}_\alpha(z)]. \]

Then it is easy to show, that Hamilton operator \( \hat{G}^{[1]}(z) \) can be represented in the simple form

\[ \hat{G}^{[1]}(z) = \sum_{\alpha=1}^{\infty} \lambda_0 \omega_\alpha [\hat{a}^{\dagger \prime \prime\prime\dagger}_\alpha(z) \hat{a}^{\prime \prime\prime}_\alpha(z) + \frac{1}{2}], \]  

which determines physical meaning of the operators \( \hat{a}^{\dagger \prime \prime\prime\dagger}_\alpha(z) \) and \( \hat{a}^{\prime \prime\prime}_\alpha(z). \) It is evident, that they are operators of creation and annihilation of the photon at space coordinate \( z. \) So, we see, that it is possible by space coordinates’ dependent quantization to determine the place of photon creation (annihilation), however it is impossible to determine the time of photon creation (annihilation). Therefore we have reverse picture to the case of the time dependent quantization, where (see previous Subsection) it is possible to determine the time of photon creation (annihilation) and it is impossible to determine the place of photon creation (annihilation). The view of \( (140) \), which is coinciding with view of known expressions for canonical quantization, if \( \lambda_0 \) to replace by \( \hbar, \) confirms the conclusion, that dimension of constant of space coordinates’ dependent quantization and dimension of Planck constant are identical, that is \( |\lambda_0| = |\hbar|. \)
From relationships (137) and (136) we can obtain the expressions for commutation relations of the creation and annihilation operators \( \hat{a}^\alpha + (z) \) and \( \hat{a}^\alpha - (z) \). They are

\[
[\hat{a}^\alpha + (z), \hat{a}^\beta + (z)] = \delta \beta \delta, \tag{141}
\]

where \( \delta \) is unit operator, \( \alpha, \beta \in \mathbb{N} \).

It seems to be evident, that the second case of EM-field quantization, that is space coordinates’ dependent quantization is acceptable for the quantization of any Coulomb field, which has nonzeroth curl, that takes place in in 1D and 2D systems. It was passed earlier for impossible to quantize any Coulomb field, see for example [21]. The quantization of Coulomb field in lowdimensional aforesaid systems corresponds to the presence of own life of radiation Coulomb field in given systems, that is Coulomb field in lowdimensional systems has the character of radiation field and it can exist without the sources, which have created given field. Given conclusion seems to be substantial to gain a better understanding, for instance, of the properties of organic conductors, perfect nanowires and nanotubes, graphene and the systems like them, including 1D and 2D biological subsystems.

The expressions for the operators of vector-functions of EM-field are similar in their structure to expressions, given by [127], [128] and they are

\[
\tilde{\mathcal{E}}^{[1]}(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sqrt{\frac{\lambda_0}{\mu_0}} \sin \omega_\alpha t \left( \hat{a}_{\alpha}^+(z) - \hat{a}_{\alpha}^-(z) \right) \} \vec{e}_x, \tag{142}
\]

and

\[
\tilde{\mathcal{H}}^{[1]}(\vec{r}, t) = \{- \sum_{\alpha=1}^{\infty} \sqrt{\frac{\lambda_0}{\mu_0}} \cos \omega_\alpha t \left( \hat{a}_{\alpha}^+(z) + \hat{a}_{\alpha}^-(z) \right) \} \vec{e}_y, \tag{143}
\]

We see, that the field operators \( \tilde{\mathcal{E}}^{[1]}(\vec{r}, t) \) and \( \tilde{\mathcal{H}}^{[1]}(\vec{r}, t) \) are local operators in the space \( R_3 \), that allows to enter the photon wave function in coordinate representation, that is, to solve the problem, which was accepted to be unsolvable in the principle [18], [19], [20].

3. Space-Time Local Quantization of Cavity EM-Field

Let us consider general case, corresponding to discrete both \( \omega \)-space of possible light frequencies and \( \vec{k} \)-space of light wave vectors, which are result of finiteness of cavity space volume and time segment. Let us find the relations for EM-field vector-functions. In the case of cavity electrodynamics considered we have two 1D ranges of variables \( t \) and \( z \), which belong to segments \( t \in [0, T], z \in [0, L] \), that is, there is in fact to be given 2D-range \( D(t,z) \), which can be considered to be definition domain of vector-functions \( \tilde{\mathcal{E}}(\vec{r}, t) \) and \( \tilde{\mathcal{H}}(\vec{r}, t) \) of two variables \( t \) and \( z \). In the case, when given functions are absolutely integrable over both the variables \( t \) and \( z \), they can be represented in the form of multiple series, given by the relations

\[
\tilde{\mathcal{E}}(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} A_{\alpha\beta}^E q_\alpha(t) q_\beta(z) \} \vec{e}_x, \tag{144}
\]

and

\[
\tilde{\mathcal{H}}(\vec{r}, t) = \{- \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} A_{\alpha\beta}^H \frac{dq_\alpha(t)}{dt} \} \int_0^z q_\beta(z')dz' \} \vec{e}_y, \tag{145}
\]

where \( \{ q_\alpha(t) \}, \{ q_\beta(z) \}, \alpha, \beta \in \mathbb{N} \) are two systems of orthogonal functions, \( A_{\alpha\beta}^E, A_{\alpha\beta}^H \) are coefficients in given expansions, which depend only on both the indices \( \alpha \) and \( \beta \). Both two-fold series will be two-fold Fourier series, if the sets \( \{ q_\alpha(t) \}, \{ q_\beta(z) \} \) are two orthogonal systems of harmonical trigonometric functions. It is evident, that the sets \( \{ q_\alpha(t) \}, \{ q_\beta(z) \} \) are independent each other and produce bases with \( \mathbb{N}_0 \) dimension in the metrizable complete spaces \( L_2 \), which are therefore Hilbert spaces. It follows from physical meaning in the case of definite direction of propagation, that between the bases \( \{ q_\alpha(t) \}, \{ q_\beta(z) \} \) and correspondingly between both Hilbert spaces the mapping

\[
\Gamma : \{ q_\alpha(t) \} \rightarrow \{ q_\beta(z) \} \tag{146}
\]

is isomorphism, at that if there is preferential (propagation) direction in \( R_4 \)-space, both the sets have to be ordered in correspondence with running numbers. It means, that

\[
\tilde{\mathcal{E}}(\vec{r}, t) = \{ \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} A_{\alpha\beta}^E q_\alpha(t) q_\beta(z) \} \vec{e}_x = \tag{147}
\]

\[
\{ \sum_{\alpha=1}^{\infty} A_{\alpha}^E q_\alpha(t) q_\beta(z) \} \vec{e}_x
\]

and

\[
\tilde{\mathcal{H}}(\vec{r}, t) = \{- \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} A_{\alpha\beta}^H \frac{dq_\alpha(t)}{dt} \} \int_0^z q_\beta(z')dz' \} \vec{e}_y = \tag{148}
\]

\[
\{ \sum_{\alpha=1}^{\infty} A_{\alpha}^H \frac{dq_\alpha(t)}{dt} \} \int_0^z q_\beta(z')dz' \} \vec{e}_y,
\]

where \( A_{\alpha}^E, A_{\alpha}^H \) are coefficients in given expansions, which depend now the only on index \( \alpha \). We have considered the mathematical aspect. Physically the insert of Kronecker symbol \( \delta_{\alpha\beta} \) in double sum in (144), (145) corresponds to renumbering of the massive \( \{ \beta \} \) in that way, in order to \( \alpha \) and \( \beta \) have run the sets \( \{ \alpha \} \) and \( \{ \beta \} \) synchronously one after another of number growth. It is additional requirement, since although both the sets \( \{ \alpha \} \) and \( \{ \beta \} \) have the same cardinal number \( \mathbb{N}_0 \) and although the mapping (146) in view of its biectivity gives
one-to-one relation between both the sets, it can be realized along with synchronous running above indicated by infinite number of asynchronous running. The choice of synchronous running is determined by causality principle - the photons by their propagation synchronously "lock on" the space and the time. It means in its turn, that full local quantization of EM-field becomes to be possible.

It can be shown, that along with expression for \( \hat{H}(\vec{r}, t) \), given by (148), \( z \)-coordinate part can be represented in more symmetrical form like to \( t \)-coordinate part in (51), that is

\[
\hat{H}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} A_\alpha^u \left( \frac{1}{\omega_\alpha} \frac{dq_{\alpha}(t)}{dt} \left( \frac{1}{k_\alpha} \frac{dq_{\alpha}(z)}{dz} \right) \right) \hat{c}_\alpha.
\]  

(149)

where \( t \in [0, T] \), \( z \in [0, L] \) and \( A_\alpha^u \) is

\[
A_\alpha^u = \sqrt{\frac{2\omega_\alpha^2 m_\alpha}{\mu_0 V T}}.
\]

(150)

For \( \hat{E}(\vec{r}, t) \) we retain the relation, given by (147), in which \( t \in [0, T] \), \( z \in [0, L] \) and \( A_\alpha^v \) is

\[
A_\alpha^v = \sqrt{\frac{2\omega_\alpha^2 m_\alpha}{\epsilon_0 V T}}.
\]

(151)

It seems to be essential, that the segment \([0, T]\) is not arbitrary, \( T \) has to be equal to \( \frac{2}{\omega} \), that ensures the synchronization above discussed of the EM-field propagation in the space and in the time. Then discreteness of \( \omega \)-space will correspond to discreteness of \( k \)-space.

Let us designate

\[
q_\alpha(z)q_\alpha(t) = g_\alpha(z, t),
\]

\[
m_\alpha \frac{dq_{\alpha}(t)}{dt} = p_{\alpha}(t),
\]

\[
\frac{1}{\omega_\alpha} \frac{dq_{\alpha}(z)}{dz} = \frac{p_{\alpha}(z)}{\omega_\alpha},
\]

\[
p_{\alpha}(z)p_{\alpha}(t) = p_{\alpha}(z, t).
\]

(152)

Then classical Hamiltonian density is

\[
\mathcal{W}(z, t) = \frac{1}{2} \left\{ \sum_{\alpha=1}^{\infty} \left( \frac{2\omega_\alpha^2 m_\alpha}{\epsilon_0 V T} q_{\alpha}(z, t) \right)^2 \right\} + \mu_0 \left\{ \sum_{\alpha=1}^{\infty} \frac{2\omega_\alpha^2 m_\alpha}{\mu_0 V T} \frac{1}{\omega_\alpha m_\alpha} p_{\alpha}(z, t) \right\}^2 + \frac{1}{2} \left\{ \frac{2\omega_\alpha^2 m_\alpha}{V T} q_{\alpha}^2(z, t) + \sum_{\alpha \neq \beta}^{\infty} \frac{2\omega_\alpha \omega_\beta}{V T} \sqrt{m_\alpha m_\beta q_{\alpha}(z, t) q_{\beta}(z, t)} + \sum_{\alpha=1}^{\infty} \frac{1}{V T m_\alpha} p_{\alpha}^2(z, t) + \sum_{\alpha \neq \beta}^{\infty} \frac{2}{V T \sqrt{m_\alpha m_\beta}} p_{\alpha}(z, t) p_{\beta}(z, t) \right\}
\]

(153)

It seems to be evident, that by integration over 4-volume both the items with double sum will give contribution, which is equal to zero. It is consequence of orthogonality of the functions \( \{q_\alpha(t)\}, \{q_\beta(z)\} \).

It means, that the Hamiltonian density can be chosen in the canonical form

\[
\mathcal{W}^{[1]}(z, t) = \frac{1}{VT} \sum_{\alpha=1}^{\infty} \left[ m_\alpha \omega_\alpha^2 q_{\alpha}^2(z, t) + \frac{p_{\alpha}^2(z, t)}{m_\alpha} \right]
\]

(154)

Then following Dirac canonical quantization method, we have

\[
[\hat{p}_{\alpha}(z, t), \hat{q}_{\beta}(z, t)] = i\delta_{\alpha \beta},
\]

\[
[\hat{q}_{\alpha}(z, t), \hat{q}_{\beta}(z, t)] = \frac{1}{i\hbar} \delta_{\alpha \beta},
\]

\[
[\hat{p}_{\alpha}(z, t), \hat{p}_{\beta}(z, t)] = 0,
\]

(155)

It is substantial, that instead scalar value we have \( \hat{g}^{(1)}(z, t) \), that is operator function of the variables \( z \) and \( t \). Really, taking into account (152), we obtain

\[
[\hat{p}_{\alpha}(z, t), \hat{q}_{\beta}(z, t)] = \frac{1}{i\hbar} \delta_{\alpha \beta} \hat{p}_{\alpha}(z, t, \hat{q}_{\beta}(z, t)) = i\hbar \delta_{\alpha \beta} \hat{p}_{\alpha}(z, t) \hat{q}_{\beta}(z, t) + i\lambda_0 \delta_{\alpha \beta} \hat{p}_{\alpha}(z, t) \hat{q}_{\beta}(t)
\]

Therefore, \( \hat{g}^{(1)}(z, t) \) is

\[
\hat{g}^{(1)}(z, t) = i\hbar \delta_{\alpha \beta} [\hat{h}\hat{p}_{\alpha}(z) \hat{q}_{\beta}(z) + \lambda_0 \hat{p}_{\alpha}(t) \hat{q}_{\beta}(t)]
\]

(157)

It is seen, that \( \hat{g}^{(1)}(z, t) \) is dependent on both the sequence of indices \( \alpha, \beta \) (in distinction from usual case) and on the sequence of operator functions in (157). In other words there are else three operator functions of analogous structure. They are

\[
\hat{g}^{(2)}(z, t) = -i\delta_{\alpha \beta} [\hat{h}\hat{p}_{\alpha}(z) \hat{q}_{\beta}(z) + \lambda_0 \hat{p}_{\alpha}(t) \hat{q}_{\beta}(t)]
\]

(158)

\[
\hat{g}^{(3)}(z, t) = i\delta_{\alpha \beta} [\hat{h}\hat{p}_{\beta}(z) \hat{q}_{\alpha}(z) + \lambda_0 \hat{p}_{\beta}(t) \hat{q}_{\alpha}(t)]
\]

(159)

\[
\hat{g}^{(4)}(z, t) = -i\delta_{\alpha \beta} [\hat{h}\hat{p}_{\alpha}(z) \hat{q}_{\alpha}(z) + \lambda_0 \hat{p}_{\beta}(t) \hat{q}_{\beta}(t)]
\]

(160)

It seems to be convenient to define symmetrized operator \( \hat{g} \) by all the four \( \hat{g}^{(j)}(z, t) \), \( j = 1, 4 \), functions, that is

\[
\hat{g} = -i\sum_{j=1}^{4} \hat{g}^{(j)}(z, t) = -\hbar \lambda_0 \hat{e},
\]

(161)

which is scalar, multiplied on unit operator. So

\[
g = -\hbar \lambda_0
\]

(162)

The operator functions \( \hat{a}_{\alpha}(z, t) \) and \( \hat{a}_{\alpha}^+(z, t) \)

\[
\hat{a}_{\alpha}(z, t) = \frac{1}{\sqrt{2\hbar \lambda_0 m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q_{\alpha}(z, t) + i\hat{p}_{\alpha}(z, t) \right]
\]

(163)

\[
\hat{a}_{\alpha}^+(z, t) = \frac{1}{\sqrt{2\hbar \lambda_0 m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha q_{\alpha}(z, t) - i\hat{p}_{\alpha}(z, t) \right].
\]

(164)
Then the operator functions of canonical variables have the form
\[ \hat{q}_\alpha(z, t) = \sqrt{\frac{\hbar \lambda_0}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(z, t) + \hat{a}_\alpha(z, t) \right] \quad (165) \]
\[ \hat{p}_\alpha(z, t) = \frac{i}{\hbar} \sqrt{\frac{m_\alpha \omega_\alpha}{2}} \left[ \hat{a}_\alpha^+(z, t) - \hat{a}_\alpha(z, t) \right]. \quad (166) \]

It is easily to show, that operator functions \( \hat{a}_\alpha(z, t) \) and \( \hat{a}_\alpha^+(z, t) \) satisfy the following relation
\[ [\hat{a}_\alpha(z, t), \hat{a}_\alpha^+(z, t)] = -i\delta_{\alpha, 0}\hat{e} \quad (167) \]

Taking into account the expressions for \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \), given by \((167), (169)\), we have for operators of EM-field vector functions
\[ \hat{\vec{E}}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^\infty A^{\alpha E}_{\alpha} \hat{q}_\alpha(t) \hat{q}_\alpha(z) \right\} \hat{e}_x = \]
\[ \left\{ \sum_{\alpha=1}^\infty A^{\alpha E}_{\alpha} \hat{q}_\alpha(z, t) \right\} \hat{e}_x = \quad (168) \]
\[ \left\{ \sum_{\alpha=1}^\infty A^{\alpha E}_{\alpha} \sqrt{\frac{\hbar \lambda_0}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(z, t) + \hat{a}_\alpha(z, t) \right] \right\} \hat{e}_x \]
and
\[ \hat{\vec{H}}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^\infty A^{\alpha H}_{\alpha} \frac{1}{\omega_\alpha} \frac{d\hat{q}_\alpha(t)}{dt} \left( \frac{1}{k_\alpha} \frac{d\hat{a}_\alpha(z)}{dz} \right) \right\} \hat{e}_y = \]
\[ \left\{ \sum_{\alpha=1}^\infty A^{\alpha H}_{\alpha} \frac{1}{m_\alpha \omega_\alpha} \hat{p}_\alpha(t) \hat{p}_\alpha(z) \right\} \hat{e}_y, \quad (169) \]

which using the relations \((152), (166)\) can be rewritten
\[ \hat{\vec{H}}(\vec{r}, t) = \left\{ \sum_{\alpha=1}^\infty A^{\alpha H}_{\alpha} \frac{1}{\omega_\alpha} \frac{d\hat{q}_\alpha(t)}{dt} \left( \frac{1}{k_\alpha} \frac{d\hat{a}_\alpha(z)}{dz} \right) \right\} \hat{e}_y = \]
\[ i \left\{ \sum_{\alpha=1}^\infty A^{\alpha H}_{\alpha} \sqrt{\frac{\hbar \lambda_0}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(z, t) - \hat{a}_\alpha(z, t) \right] \right\} \hat{e}_y \quad (170) \]

Therefore by means of operator functions \( \hat{a}_\alpha^+(z, t), \hat{a}_\alpha(z, t) \) the local quantization of EM-field is realized, which allows to determine simultaneously along with time of creation (annihilation) of photons the space coordinate of given process.

C. Cavity 4-Currents

It represents the interest to calculate the 4-currents for given task. Let us place all the vector-functions in pairs in accordance with their parity. Then we have the following pairs
\[ (\vec{E}^{[1]}(\vec{r}, t), \vec{E}^{[2]}(\vec{r}, t)), (\vec{H}^{[2]}(\vec{r}, t), \vec{H}^{[1]}(\vec{r}, t)) \quad (171) \]
in which both the \( \vec{E} \)-vectors and \( \vec{H} \)-vectors have the same space parity (polar and axial correspondingly) and differ each other by t-parity, t-even and t-uneven in accordance with their numbers in pairs. It means, that they trasform like to \( x_1 \) and \( x_2 \) coordinates in \( 1R_4 \). In a similar manner can be set the vector-functions with opposite to the vector-functions in \((171)\) space parity
\[ (\vec{E}^{[3]}(\vec{r}, t), \vec{E}^{[4]}(\vec{r}, t)), (\vec{H}^{[4]}(\vec{r}, t), \vec{H}^{[3]}(\vec{r}, t)) \quad (172) \]

Then taking into account the definition of complex quantities to be pair of real defined quantities, taken in fixed order, we come in a natural way once again to concept of complex vector-functions, which describe Maxwellian EM-field equations. In other words, we have in fact the quantities
\[ \vec{E}^{[1]}(\vec{r}, t) + i\vec{E}^{[2]}(\vec{r}, t) = \vec{E}_{c,p}(\vec{r}, t), \]
\[ \vec{H}^{[2]}(\vec{r}, t) + i\vec{H}^{[1]}(\vec{r}, t) = \vec{H}_{c,a}(\vec{r}, t), \quad (173) \]
and
\[ \vec{E}^{[3]}(\vec{r}, t) + i\vec{E}^{[4]}(\vec{r}, t) = \vec{E}_{c,a}(\vec{r}, t), \]
\[ \vec{H}^{[4]}(\vec{r}, t) + i\vec{H}^{[3]}(\vec{r}, t) = \vec{H}_{c,p}(\vec{r}, t), \quad (174) \]

where complex plane put in correspondence to \((y, z)\) real plane, subscripts \( a \) and \( p \) mean axial and polar respectively. It seems to be convenient to determine the space of EM-field vector-functions under the ring of quaternions with another basis in comparison with basis, given by \((70)\). We will use now the quaternion basis \( \{e_i\}, i = 0, 1, 2, 3 \) with algebraic operations between elements, satisfying to relationships
\[ e_i e_j = e_{ijk} e_k + \delta_{ij} e_0, e_0 e_i = e_i, e_0^2 = e_0, i,j,k = \frac{1}{3}, 1, 2, 3 \quad (175) \]
where \( e_{ijk} \) is completely antisymmetric Levi-Chivita 3-tensor.

Let us define the vector biquaternion
\[ \vec{\phi} = (\vec{E}^{[1]} + \vec{H}^{[2]}) + i(\vec{H}^{[1]} + \vec{E}^{[2]}), \quad (176) \]
which can be represented to be the sum of the biquaternions
\[ \vec{\phi} = \vec{F} + \vec{\bar{F}}, \quad (177) \]
where \( \vec{F} = \vec{E}^{[1]} + i\vec{H}^{[1]}, \vec{\bar{F}} = \vec{H}^{[2]} + i\vec{E}^{[2]} \). Then Maxwell equations for instance for two free photon fields with different t-parity are
\[ \nabla \vec{\phi} = 0. \quad (178) \]

The generalized Maxwell equations in quaternion form with quaternion basis, given by \((70)\), can also be rewritten in fully quaternion form, if to use both the bases. It seems to be consequence of independence of basis definition for both the quaternion forms.
1. Classical Cavity 4-Currents

It is evident, that

\[ j_{\mu, \pm}(x) = j_{\mu, \pm}^{(1)}(x) + i j_{\mu, \pm}^{(2)}(x), \]  

(179)

where subscript \( \pm \) corresponds to two possibilities for definition of complex vector-functions. Along with relationships (173), (174) they can be defined by the change of addition sign in (173), (174) to opposite. The quantity \( j_{\mu, \pm}^{(1)}(x) \) is well known quantity, and it is determined by

\[ j_{\mu, \pm}^{(1)}(x) = -\frac{ie}{\hbar c} \sum_{\alpha=1}^{\infty} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (\partial_{\mu} u_{s \pm})(x)} u_{\alpha}^{s \pm}(x) \right], \]  

(180)

where \( L(x) \) is Lagrange function and \( u_{s \pm}(x), s = 1, 2 \)

\[ u_{1 \pm}(x) = \sqrt{\frac{\hbar c}{u}} \sin k_{s}(x) \left[ q_{\alpha}(x) \pm i q_{\alpha}''(x) \right] \]

\[ u_{2 \pm}(x) = \sqrt{\frac{\hbar c}{\mu}} \cos k_{s}(x) \left[ q_{\alpha}(x) - i q_{\alpha}''(x) \right] \]  

(181)

The functions \( u_{s \pm}(x), s = 1, 2, \alpha \in N \) are built from the components of the expansion in Fourier series of the fields \( \vec{E}^{[1]}(\vec{r}, t), \vec{E}^{[2]}(\vec{r}, t) \) and \( \vec{H}^{[1]}(\vec{r}, t), \vec{H}^{[2]}(\vec{r}, t) \) correspondingly.

To determine the current density \( j_{\mu, \pm}^{(2)}(x) \) we have to take into consideration, that gauge symmetry group of EM-field is two-parametric group \( \Gamma(\alpha, \beta) = U_{1}(\alpha) \otimes \mathbb{R}(\beta), \) where \( \mathbb{R}(\beta) \) is abelian multiplicative group of real numbers (excluding zero). It leads also to existence for EM-field of complex 4-currents including complex charge density component. Since the current density \( j_{\mu, \pm}^{(2)}(x) \) is

\[ j_{\mu, \pm}^{(2)}(x) = \frac{ie}{\hbar c} \sum_{\alpha=1}^{\infty} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (\partial_{\mu} u_{s \pm})(x)} u_{\alpha}^{s \pm}(x) \right], \]  

(182)

\[ -i \frac{e}{\hbar c} \sum_{\alpha=1}^{\infty} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (\partial_{\mu} u_{s \mp})(x)} u_{\alpha}^{s \pm}(x) \right]. \]

It can be easily shown, that \( j_{3 \pm}^{(1)}(\vec{r}, t) \) is always equal to zero for any set of twice continuously differentiable functions \( \{ q_{\alpha}(t) \}, \alpha \in N \). The expression for arbitrary set of twice continuously differentiable functions \( \{ q_{\alpha}(t) \}, \alpha \in N \), for \( j_{3 \pm}^{(2)}(\vec{r}, t) \) is

\[ j_{3 \pm}^{(2)}(\vec{r}, t) = -\frac{2ie}{\hbar c V} \sum_{\alpha=1}^{\infty} m_{\alpha} \omega_{\alpha}^{3} \sin 2k_{z}z \times \]

\[ \left\{ i |q_{\alpha}(t)| \pm i \omega_{\alpha}^{2} \left[ \int_{0}^{t} \int \int_{0}^{t''} q_{\alpha}(t') dt' dt'' |^{2} \right] \right\}, \]  

(183)

The relationship (183) is true for both the variants in superposition

\[ \vec{H}^{[i]}(\vec{r}, t) + i \vec{H}^{[j]}(\vec{r}, t) = \vec{H}^{[ij]}(\vec{r}, t), i \neq j, i, j = 1, 2. \]  

(184)

Taking into account relationship (87), that is the set \( \{ q_{\alpha}(t) \}, \alpha \in N \), which satisfy the Maxwell equations we will have

\[ j_{3 \pm}^{(1)}(\vec{r}, t) = \frac{8ie}{\hbar c V} \sum_{\alpha=1}^{\infty} m_{\alpha} \omega_{\alpha}^{3} \sin 2k_{z}z \times \]

\[ \left[ C_{1\alpha} C_{2\alpha}^{*} e^{2i\omega_{\alpha}t} + C_{1\alpha}^{*} C_{2\alpha} e^{-2i\omega_{\alpha}t} \right], \]  

(185)

the expression for arbitrary set of twice continuously differentiable functions \( \{ q_{\alpha}(t) \}, \alpha \in N, \) for \( j_{4 \pm}^{(1)}(\vec{r}, t) \) is

\[ j_{4 \pm}^{(1)}(\vec{r}, t) = \frac{2ie}{\hbar c V} \sum_{\alpha=1}^{\infty} m_{\alpha} \omega_{\alpha}^{3} \sin 2k_{z}z \times \]

\[ \left\{ \frac{dq_{\alpha}(t)}{dt} \mp i \frac{dq_{\alpha}''(t)}{dt} \right\} \left[ q_{\alpha}(t) \pm q_{\alpha}''(t) \right] \]

\[ + \left\{ \frac{dq_{\alpha}(t)}{dt} \pm i \frac{dq_{\alpha}''(t)}{dt} \right\} \left[ q_{\alpha}(t) \mp q_{\alpha}''(t) \right] \]

\[ + \cos^{2} k_{z}z \left[ \frac{1}{\omega_{\alpha}} |q_{\alpha}''(t) dt''^{2} \pm i\omega_{\alpha} q_{\alpha}(t) | \times \]  

\[ \left\{ \frac{1}{\omega_{\alpha}} \frac{dq_{\alpha}(t)}{dt} \mp i \omega_{\alpha} \int_{0}^{t} q_{\alpha}(t') dt' \right\} \times \]

\[ \left\{ \frac{1}{\omega_{\alpha}} \frac{dq_{\alpha}''(t)}{dt} \mp i \omega_{\alpha} \int_{0}^{t} q_{\alpha}(t') dt' \right\}, \]  

(186)

where \( q_{\alpha}''(t) = \omega_{\alpha}^{2} \int_{0}^{t} \int_{0}^{t''} q_{\alpha}(t') dt' dt'' \). It is evident from relationship (180), that in the case of real-valued sets of twice continuously differentiable functions \( \{ q_{\alpha}(t) \}, \alpha \in N \), \( j_{4 \pm}^{(1)}(\vec{r}, t) \) is equal to zero. For complex-valued functions, determined by (87), we will have

\[ j_{4 \pm}^{(1)}(\vec{r}, t) = \frac{8ie}{\hbar c V} \sum_{\alpha=1}^{\infty} m_{\alpha} \omega_{\alpha}^{3} (|C_{1\alpha}|^{2} - |C_{2\alpha}|^{2}). \]  

(187)

It is seen from (187), that \( j_{4 \pm}^{(1)}(\vec{r}, t) \) in the case of Maxwellian EM-field is constant, which is equal to zero at \( |C_{1\alpha}| = |C_{2\alpha}| \), that is for all real-valued functions and for complex-valued functions \( \{ q_{\alpha}(t) \}, \alpha \in N \), which differ...
each other by arguments of constants $C_{1\alpha}$ and $C_{2\alpha}$.

$$j_4^{2,\pm}(\vec{r}, t) = -\frac{2e}{\hbar c^2 V} \sum_{\alpha=1}^{\infty} \{m_\alpha \omega_\alpha^2 \sin^2 k_\alpha \frac{d}{dt}(q_\alpha(t))^2\} + \omega_\alpha^2 \frac{d}{dt} \left( \int_0^t \int_0^{t''} |q_\alpha(t') dt' dt''|^2 \right) + \frac{i \omega_\alpha^2 \pm \frac{d}{dt}(q_\alpha)(t) \int_0^t \int_0^{t''} |q_\alpha(t') dt' dt''|^2 \omega_\alpha^2 + m_\alpha \omega_\alpha^2 \cos^2 k_\alpha z \right) \times \left( \frac{1}{\omega_\alpha^2} \frac{d}{dt}(d\alpha(t)^2) \right) - i \frac{d}{dt}(d\alpha(t)) \int_0^t \int_0^{t'} |q_\alpha(t') dt'| \right] \right).$$

For complex-valued functions, determined by (87), we obtain

$$j_4^{2,\pm}(\vec{r}, t) = 8ie \frac{\hbar c^2 V}{\omega^2} \sum_{\alpha=1}^{\infty} m_\alpha \omega_\alpha^3 \cos 2k_\alpha z \times [C_{1\alpha} C_{2\alpha} e^{2i\omega_\alpha t} - C_{1\alpha} C_{2\alpha} e^{-2i\omega_\alpha t}].$$

It can be shown, that continuity equation

$$\frac{\partial j_4^{\pm}(x)}{\partial x} = 0$$

is fulfilled for both general case and for Maxwellian EM-field functions considered.

**D. Quantized Cavity 4-Currents**

Let us calculate the 4-currents, which correspond to quantized dual symmetric EM-field, that is to the field, which consist of two components with even and uneven parities under time reversal or space inversion of both the EM-field vector functions $\hat{E}(\vec{r}, t)$ and $\hat{H}(\vec{r}, t)$. Let us consider for distinctness the case of two-component EM-field, in which $\hat{E}(\vec{r}, t)$- components and $\hat{H}(\vec{r}, t)$- components have the same $P$-parity (even and even correspondingly) and differ each other by $t$-parity. Given choose corresponds to classical consideration inprevious subsection, that allows to compare the results for classical and quantized dual symmetric EM-field. Consequently, we can use the set of EM-field vector functions, analogous to (151), in which the operator functions are set up in conformity to canonical variables.

$$\hat{\alpha}_\alpha^{\pm}(x), s = 1, 2$$ are

$$\hat{\alpha}_\alpha^{1,\pm}(x) = \sqrt{\epsilon_0 \epsilon_\alpha} \sin k_\alpha(x_3) [\hat{q}_\alpha(x_4) \pm i \hat{q}_\alpha''(x_4)] \hat{e}_1$$

$$\hat{\alpha}_\alpha^{2,\pm}(x) = \sqrt{\mu_0 A_\alpha} \cos k_\alpha(x_3) \times$$

$$[-\hat{q}_\alpha'(x_4) \pm i \frac{1}{\omega_\alpha} \frac{d}{dx_4} \hat{q}_\alpha(x_4)] \hat{e}_2,$$

where $\hat{e}_1 \equiv \hat{e}_x$, $\hat{e}_2 \equiv \hat{e}_y$, $\hat{q}_\alpha'(x_4)$, $\hat{q}_\alpha''(x_4)$ are operator functions, which are setting up in the conformity to classical variables $q_\alpha'(x_4)$, $q_\alpha''(x_4)$, defined by (93). They are

$$\hat{q}_\alpha(t) = \omega_\alpha \int \hat{q}_\alpha(\tau) d\tau$$

$$\hat{q}_\alpha''(t) = \omega_\alpha \int \hat{q}_\alpha''(\tau') d\tau'$$

correspondingly. The functions $\hat{\alpha}_\alpha^{\pm}(x), s = 1, 2, \alpha \in N$ can be built from the components of the expansion in Fourier series of quantized dual symmetric EM-field, which consist of two components with even and uneven parities under time reversal $\hat{E}^{[1]}(\vec{r}, t)$, $\hat{E}^{[2]}(\vec{r}, t)$ and $\hat{H}^{[2]}(\vec{r}, t)$, $\hat{H}^{[1]}(\vec{r}, t)$, given by (128). Therefore we have

$$\hat{a}_\alpha^{1,\pm}(\vec{r}, t) = \sqrt{\frac{\hbar c^2 V}{\omega_\alpha}} \left( [\hat{a}_\alpha(t) + \hat{a}_\alpha^+(t)] \right) + i \left[ \hat{a}_\alpha''(t) - \hat{a}_\alpha^+''(t) \right] \cos(k_\alpha z) \hat{e}_x,$$

$$\hat{a}_\alpha^{2,\pm}(\vec{r}, t) = \sqrt{\frac{\hbar c^2 V}{\omega_\alpha}} \left( [\hat{a}_\alpha(t) - \hat{a}_\alpha^+(t)] \right) + i \left[ \hat{a}_\alpha''(t) - \hat{a}_\alpha^+''(t) \right] \cos(k_\alpha z) \hat{e}_y,$$

where superscript $\pm$ means, that in (126) and (128) by definition of complex EM-field vector function operators $\hat{E}(\vec{r}, t)$ and $\hat{H}(\vec{r}, t)$ along with the sign plus, the sign minus can be used. We also consider the case of $\hat{H}(\vec{r}, t)$ formation along with given by (128) (with both the signs in the sum) the following case

$$(\hat{H}^{[1]}(\vec{r}, t), \hat{H}^{[2]}(\vec{r}, t)) \rightarrow \hat{H}^{[1]}(\vec{r}, t) \pm i \hat{H}^{[2]}(\vec{r}, t) = \hat{H}^{\pm}(\vec{r}, t).$$

It seems to be evident, that 4-current operator can be determined by the expressions, coinciding with classical relations (179), (180), (182) in which all the physical quantities are operators. For the operator $\hat{j}_3^{1,\pm}(\vec{r}, t)$ we have

$$\hat{j}_3^{1,\pm}(\vec{r}, t) = \frac{\hat{e}}{2eV} \sum_{\alpha=1}^{\infty} k_\alpha \omega_\alpha \sin 2k_\alpha z \times$$

$$\left( [i \left[ \hat{a}_\alpha''(t) - \hat{a}_\alpha''^+(t) \right] \right) \left( [\hat{a}_\alpha(t) - \hat{a}_\alpha^+(t)] \right)^2 +$$

$$i^2 \left[ \hat{a}_\alpha''(t) - \hat{a}_\alpha''^+(t) \right] \left( [\hat{a}_\alpha(t) - \hat{a}_\alpha^+(t)] \right)^2,$$

which is equaled to zero. The same result is obtained in the case of magnetic field operator, determined by (195).
For the operator $\hat{j}^{2\pm}_3(r, t)$ we obtain the relation

$$\hat{j}^{2\pm}_3(r, t) = i m \hat{j}^{2\pm}_3(r, t) = -\frac{2ie}{cV} \sum_{\alpha=1}^{\infty} k_\alpha \omega_\alpha \sin 2k_\alpha x \times$$

$$\left\{ [\hat{a}_\alpha(t) + \hat{a}^\dagger_\alpha(t)]^2 + [\hat{a}''_\alpha(t)]^2 + [\hat{a}''''_\alpha(t)]^2 \right\},$$

(197)

which is the same for magnetic field operator, determined by (196). Therefore the operator of current density $\hat{j}^{2\pm}_3(r, t)$ is independent on sign expressions for the field operators, based on (128) and it is independent on the sequence of $\hat{H}^{[i]}(r, t)$, $i = 1, 2$, in

$$\hat{H}^{ij\pm}(r, t) = \hat{H}^{[i]}(r, t) \pm i \hat{H}^{[j]}(r, t),$$

(198)

where $i, j = 1, 2, i \neq j$.

It seems to be essential, that EM-field quantization is not bound to Maxwell equations in general case. It means, that the relations (196), (197) are true for more general fields. In the case of Maxwellian EM-field, using explicit expressions for operator scalar functions given by (119) for $\hat{a}_\alpha(t), \hat{a}_\alpha^\dagger(t)$ and similar relations for $\hat{a}''_\alpha(t)$, $\hat{a}''''_\alpha(t)$ the expression (197) has the form

$$\hat{j}^{2\pm}_3(r, t) = i m \hat{j}^{2\pm}_3(r, t) = -\frac{2ie}{cV} \sum_{\alpha=1}^{\infty} k_\alpha \omega_\alpha \sin 2k_\alpha x \times$$

$$\left\{ [\hat{a}_\alpha(t) + \hat{a}^\dagger_\alpha(t)]^2 e^{i\omega_\alpha t} + [\hat{a}''_\alpha(t) + \hat{a}''''_\alpha(t)]^2 e^{i\omega_\alpha t} \right\}.$$ 

(199)

Let us find now the fourth component of 4-vector operator of current density $\hat{j}_4^\pm(r, t)$, which determines the charge density. For real part $\hat{j}_4^{1\pm}(r, t)$ we have

$$\hat{j}_4^{1\pm}(r, t) = Re\hat{j}_4^{1\pm}(r, t) =$$

$$\pm \frac{2e}{cV} \sum_{\alpha=1}^{\infty} k_\alpha \omega_\alpha^2 \{[\hat{a}''_\alpha(t) + \hat{a}''''_\alpha(t)] - [\hat{a}_\alpha(t) + \hat{a}''_\alpha(t)]\},$$

(200)

where the expressions in braces are anticommutators. In the case of there is the connection between $\hat{a}_\alpha(t), \hat{a}^\dagger_\alpha(t)$ and $\hat{a}''_\alpha(t), \hat{a}''''_\alpha(t)$, since although they correspond to different particular solutions of Maxwell equations, the solution are related and the connection between them can be found. It leads to connection between corresponding creation and annihilation operators for two related EM-fields with different $t$-parity. It can be shown, that the following relations take place

$$\hat{a}''_\alpha(t) = \omega_\alpha^2 \int_0^t \int_0^{t'} \hat{a}_\alpha(t') dt' dt''$$

$$\hat{a}''''_\alpha(t) = \omega_\alpha^2 \int_0^t \int_0^{t'} \hat{a}''_\alpha(t') dt' dt''$$

(201)

and the relations

$$\frac{d\hat{a}_\alpha(t)}{dt} = \frac{1}{i\hbar} [\hat{a}_\alpha(t), \mathcal{H}_\alpha(t)], \in N,$$ 

(205)

where $\mathcal{H}_\alpha(t)$ is the Hamiltonian, corresponding to cavity $\alpha$-mode. It is given by relation

$$\mathcal{H}_\alpha(t) = \hbar \omega_\alpha \left[ \hat{a}^\dagger_\alpha(t) \hat{a}_\alpha(t) + \frac{1}{2} \right]$$

(207)

For calculation of derivatives of operators $\hat{a}_\alpha(t), \hat{a}''_\alpha(t), \hat{a}''''_\alpha(t)$ the relations analogous to (204) were used. Taking into account (204), (205), (206), (207) in the case of $Re\hat{j}_4^{1\pm}(r, t)$ we have

$$\frac{\partial [Re\hat{j}_4^{1\pm}(x)]}{\partial x_\mu} = \frac{\partial \hat{j}_4^{1\pm}(r, t)}{\partial x_\mu} =$$

$$\pm \frac{2e}{cV} \sum_{\alpha=1}^{\infty} \omega_\alpha^2 \{[\hat{a}_\alpha(t), \hat{a}''_\alpha(t)] + [\hat{a}''_\alpha(t), \hat{a}''''_\alpha(t)] -$$

$$[\hat{a}''''_\alpha(t), \hat{a}_\alpha(t)] - [\hat{a}_\alpha(t), \hat{a}''_\alpha(t)]\} = 0.$$
Therefore differential conservation law, given by (203), is fulfilled both for Maxwellian EM-field and in general case.

IV. CONCLUSIONS

It is shown on the basis of complex number theory, that any quantumphysical quantity is complex quantity.

Additional gauge invariance of complex relativistic fields was found. It is based on invariance of generalized relativistic equations under the operations of additional gauge symmetry group - multiplicative group $\mathbb{R}$ of all real numbers (without zero) and leads to appearance of purely imaginary component of charge. So, it was shown, that complex fields are characterized by complex charges. It gives key for correct generalization of field equations, in particular for electrodynamics. In application to EM-field it means that two-parametric group $\Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathbb{R}(\beta)$ determines the gauge symmetry of EM-field and that free real EM-field is characterized by purely imaginary charge.

Additional hyperbolic dual symmetry of Maxwell equations is established, which includes Lorentz-invariance to be its particular case. The essence of additional hyperbolic dual symmetry of Maxwell equations is that, that Maxwell equations along with dual transformation symmetry, established by Rainich, given by (36) - (39), are symmetric relatively the dual transformations of another kind. Hyperbolic dual transformations for electric and magnetic field strength vector functions are

\[
\begin{bmatrix}
\vec{E}^n \\
\vec{H}^n
\end{bmatrix} = \begin{bmatrix}
cosh \vartheta & i \sinh \vartheta \\
-i \sinh \vartheta & \cosh \vartheta
\end{bmatrix}
\begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix},
\]

where $\vartheta$ is arbitrary continuous parameter, $\vartheta \in [0, 2\pi]$.

Generalized Maxwell equations are obtained on the basis of both dual and hyperbolic dual symmetries of EM-field. It is shown, that in general case both scalar and vector quantities, entering equations, are quaternion quantities, four components of which have different parities under improper rotations.

Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry and hyperbolic dual symmetry are found. It is concluded, that Maxwell equations with all quaternion vector and scalar variables give concrete connection between dual and gauge symmetries of EM-field.

The example of free classical and quantized cavity EM-field is considered. It is shown, that the same physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the geometry choosed, when vector $\vec{E}$ is directed along absciss axis, $\vec{H}$ is directed along ordinate axis in $(\vec{E}, \vec{H})$ functional space. Spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality in the geometry considered) combine two subsystems of photon fields, which have definite $P$-parity (even and uneven) with the subsystem of two fields, which have definite t-parity (also even and uneven) into one system. It is considered to be the proof for four component structure of EM-field to be a single whole, that is, it is the confirmation along with the possibility of the representation of EM-field in four component quaternion form, given by (26), (27), (28), (29), the necessity of given representation. It extends the overview on the nature of EM-field itself. It seems to be remarkable, that given result on the special leading significance of spin is in agreement with result in [12], where was shown, that spin is quaternion vector of the state in Hilbert space, defined under ring of quaternions, of any quantum system (in the frame of the chain model considered) interacting with EM-field.

New principle of EM-field quantization is proposed. It is development of canonical Dirac quantization method, which is realized in two aspects. The first aspect consist in choosing of immediately observable quantities - vector-functions $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ - to be field functions. The second aspect is the realization along with well known time-local quantization of space-local quantization and space-time-local quantization, which allow to establish correspondingly the time of photon creation (annihilation), the space coordinate of photon creation (annihilation) and the space and time coordinates simultaneously of photon creation (annihilation). It is shown, that Coulomb field can be quantized in 1D and 2D systems, that is, it is radiation field in given low-dimensional systems.

It is found, that quantized Maxwellian EM-field is the only complex-valued field. Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both complex-valued and real-valued functions.

\[
[1] \text{Nöther E, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math-phys.Klasse, (1918) 235–257}
\]
\[
[2] \text{Bogush A A, Moroz L G, Introduction to the Theory of Classical Fields, M., Editorial URSS, 2004, 384}
\]
\[
[3] \text{Strazhev V.I, Tomilchick L.M, Electrodynamics with Magnetic Charge, Minsk, Nauka i Tekhnika, 1975, 336 pp}
\]
\[
[4] \text{Heaviside O, Phil.Trans.Roy.Soc.A,183 (1893) 423–430}
\]
\[
[5] \text{Larmor J, Collected papers, London, 1928}
\]
\[
[6] \text{Rainich G Y, Trans.Am.Math.Soc.,27 (1925) 106}
\]
\[
[7] \text{Berezin A.V, Kurochkin Yu.A, Tolkachev Ye.A, Quaternions in Relativistic Physics, Minsk, Nauka i Tekhnika, 1989, 199 pp}
\]
[8] Landau L.D, Lifshitz E.M, Field Theory, M., Nauka, 504
[9] Yearchuck D, Yerchak Y, Kirilenko A, Pepochits V, Doklady NANB, 52, N 1 (2008) 48-53
[10] Yearchuck D, Yerchak Y, Alexandrov A, Phys.Lett.A, 373 (2009) 489 - 495
[11] Andre Angot, Complements de Mathematiques, Paris, 1957, 778 pp
[12] D.Yearchuck, Y.Yerchak, A.Dovlatova, Optics Communications, 283 (2010) 3448-3458
[13] Tolkachev E.A, Tomilchik L.M, Covariant Methods in Theoretical Physics, Minsk, 1981, pp 44-48
[14] Bogoliubov N N, Shirkov D V, Introduction in the theory of quantized fields, M., Nauka, 1973, 414 pp
[15] Born M, Jordan P, Zeitschrift fuer Physik, 34(1925) 858-889
[16] Born M, Heisenberg W, Jordan P, Zeitschrift fuer Physik, 35 (1926) 557-606
[17] Dirac P.A.M, Proc.Roy.Soc.,A, 114 (1927) 243-265
[18] Scully M O, Zubairy M S, Quantum Optics, Cambridge University Press, 1997, 650
[19] Akhiezer A.I, Berestetskii V.B, Quantum Electrodynamics, M., Nauka, 1969, 623 pp
[20] Dutra S M, Cavity Quantum Electrodynamics, John Wiley and Sons, Inc., Hoboken, New Jersey, 2005, 389 pp
Concept of Fully Dually Symmetric Electrodynamics

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It has been shown, that electromagnetic field (EM) has in general case quaternion structure, consisting of four independent fields, which differ each other by the parities under space inversion and time reversal. It follows immediately from Rainich dual symmetry of Maxwell equations and additional hyperbolic dual symmetry, established in given work. It has also been shown, that for any complex relativistic field the gauge invariant conserving quantity is two-component scalar or pseudoscalar value - complex charge. It means in applicability to EM-field, that its gauge symmetry group is determined by two-parametric group \( \Gamma(\alpha, \beta) = U_1(\alpha) \otimes R(\beta) \), where \( R(\beta) \) is abelian multiplicative group of real numbers (excluding zero). Generalized Maxwell equations for four-component EM-field are obtained on the basis of its both dual and hyperbolic dual symmetries. Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry are found. It is shown, that the only one physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the geometry, when vector \( \vec{E} \) is directed along absciss axis, \( \vec{H} \) is directed along ordinate axis in \((\vec{E}, \vec{H})\) functional space. In fact it is the proof for four component structure of EM-field to be a single whole, that is confirmation along with the possibility of the representation of EM-field in four component quaternion form the necessity of given representation. It extends the overview on the nature of EM-field itself.

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I. INTRODUCTION AND BACKGROUND

A. Matrix Algebra of Complex Numbers and its some Consequences for Quantum Theory

Let us summarize some useful results from algebra of the complex numbers. The numbers 1 and \( i \) are usually used to be basis of the linear space of complex numbers over the field of real numbers. At the same time to any complex number \( a + ib \) can be set up in conformity the \([2 \times 2]\)-matrix according to biective mapping \( f \)

\[
f : a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.
\]  

(1)

The matrices

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]  

(2)

produce basis for complex numbers \( \{a + ib\} \), \( a, b \in R \) in the linear space of \([2 \times 2]\)-matrices, defined over the field of real numbers. It is convenient often to define the space of complex numbers over the group of real positive numbers, then the dimensionality of the matrices and basis has to be duplicated, since to two unities - positive 1 and negative \(-1\) can be set up in conformity the \([2 \times 2]\)-matrices according to biective mapping \( \varepsilon \)

\[
\varepsilon : 1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]  

(3)

which allows to recreate the operations with negative numbers without recourse of negative numbers themselves. Consequently, in accordance with mapping \( \zeta \) the following \([4 \times 4]\)-matrices, so called \([0,1]\)-matrices, can be basis of complex numbers

\[
\zeta : 1 \rightarrow [e_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
i \rightarrow [e_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[-1 \rightarrow [e_3] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[-i \rightarrow [e_4] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]  

(4)

The chosue of basis is ambiguous. Any four \([4 \times 4]\) \([0,1]\)-matrices, which satisfy the rules of cyclic recurrence

\[
i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1
\]  

(5)
can be basis of complex numbers. In particular, the following $[4 \times 4]$ [0,1]-matrices

$$[e_1^3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, [e_2^3] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$[e_3^3] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, [e_4^3] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

can also be basis of complex numbers. Naturally, the set of [0,1]-matrices, given by (3) is isomorphous to the set, which is given by (1). It is evident, that the system of complex numbers can be constructed by infinite number of the ways, at that cyclic basis can consist of $m$ units, $m \in N$, starting from three. It is remarkable, that the conformity between complex numbers and matrices is realized by biective mappings. It means, that there is also to be existsting the inverse mapping, by means of which to any square matrix, belonging to the linear space with a basis given by (1), or (6), or any other, satisfying the rules of cyclic recurrence like to (5), can be set up in conformity the complex number. In particular to any Hermitian matrix $H$ can be set up in conformity the complex number in correspondence with mapping $\xi$

$$\xi : H \rightarrow S + iA = \begin{bmatrix} S & -A \\ A & S \end{bmatrix},$$

where $S$ and $A$ are symmetric and antisymmetric parts of Hermitian matrix. Given short consideration allows to formulate the following statements.

1. Quantized free EM-field is complex field in general case.

   Proof is evident and it is based on (7), if to take into account, that quantized free EM-field can be determined by Hermitian operators $\hat{E}(\vec{r}, t)$ and $\hat{H}(\vec{r}, t)$, representing themselves the full set of quantized free EM-field operator vector-functions, that is, they can serve for basis in corresponding operator vector-functional space (see Sec.II). Given statement can be generalized.

2. Any quantumphysical quantity is complex quantity in general case.

   Proof is evident and it is based on the same relationship, since any quantumphysical quantity is determined by Hermitian operator. Therefore, two sets of observables, which are determined by real functions, correspond to any quantumphysical operator quantity in general case.

B. Additional gauge invariance of complex relativistic fields

We will argue in the next Section, that EM-field in the matter can be considered in general case to be complex field, each component in which is also complex field, that is, it has quaternion nature. In given Section we will prove the idea, that for any complex field the conserved quantity, corresponding to its gauge symmetry, that is charge, can be in general case also complex.

Let $u(x) = \{u_i(x)\}, i = \overline{1,n}$, the set of the functions of some complex relativistic field, that is, scalar, vector or spinor field, given in some space of Lorentz group representations. It is well known, that Lagrange equations for any complex relativistic field can be represented in the form of one matrix relativistic differential equation of the first order in partial derivatives, that is in the form of so called generalized relativistic equation, and analogous equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions $u^+(x) = \{u_i^+(x)\}$. The equation for the set $u(x)$ of field functions is

$$(\alpha_{\mu}\partial_{\mu} + \kappa u_0)a_0 = 0.$$  \hspace{1cm} (8)

Similar equation for the field with Hermitian conjugated (complex conjugated in the case of scalar fields) functions, that is for the functions $u^+(x) = \{u_i^+(x)\}$, $i = \overline{1,n}$, is

$$\partial_{\mu}u^+(x)\alpha_{\mu} + \kappa u^+(x)a_0 = 0.$$  \hspace{1cm} (9)

In equations (8, 9) $\alpha_{\mu}$, $a_0$ are matrices with constant numerical elements. They have an order, which coincides with dimension of corresponding space of Lorentz group representation, realized by $\{u_i(x)\}, i = \overline{1,n}$. In particular, they are $[n \times n]$- matrices, if $\{u_i(x)\}$, $i = \overline{1,n}$ are scalar functions. It is evident, that the transformation

$$u'(x) = \beta \exp(i\alpha)u(x),$$  \hspace{1cm} (10)

where $\alpha, \beta \in \mathbb{R}$, and analogous transformation for Hermitian conjugate functions (or complex conjugate functions in the case of scalar fields)

$$u'^+(x) = \beta \exp(-i\alpha)u^+(x)$$  \hspace{1cm} (11)

keep Lagrange equations (8, 9) to be invariant. It is understandable that transformation of field functions by relationships (10, 11) is equivalent to multiplication of field functions by arbitrary complex number. It is well known, that given linear transformation is the simplest example of isomorphism of corresponding linear space, which is given over the field of complex numbers, onto itself, that is, in the case considered the relationships (10, 11) give automorphism of the space of field functions. Automorphism of any linear space leads to some useful properties of the objects, which belong to given space. For instance, if to set up in correspondence to the space of field function the affine space, then conservation laws of collinearity of the points and of simple relation of the triple of collinear points will be fulfilled by automorphism in given affine space. Consequently, we have to expect the physical consequences of given algebraic property in the case of physical spaces. Conformally to
the case considered we have in fact gauge transformation of field functions, which is more general in comparison with usually used. The set \((β \exp(-\alpha \gamma))\) for all possible \(α, β \in R\) produces the group \(Γ\), which is direct product of known symmetry group \(U_1\), and multiplicative group \(R\) of all real numbers (without zero). Therefore, in the case considered the symmetry group of given complex field acquires additional parameter. So, we will have

\[ Γ(α, β) = U_1(α) \otimes R(β) \]  \hspace{1cm} (12)

Let us find the irreducible representations of the group \(R(β)\). It has to be taken into account, that the group \(R(β)\) is abelian group and its irreducible representations \(T(R)\) are onedimensional. So, the mapping

\[ T : R \rightarrow T(R) \]  \hspace{1cm} (13)

is isomorphism, where \(T(1) = 1\). Therefore, for \(β(β, γ)\) of pair of elements of group \(R(β)\) the following relationship takes place

\[ T(β, γ) = T(β)T(γ). \]  \hspace{1cm} (14)

Then, it is easy to show, that

\[ T(β) = β^{\frac{2k+1}{n}}. \]  \hspace{1cm} (15)

The value \(β^{\frac{2k+1}{n}}\) can be obtained from the condition

\[ T(-β) = -T(β). \]  \hspace{1cm} (16)

Consequently, we have

\[ T(β) = β^{2k+1} = \exp[(2k+1)lnβ], \]  \hspace{1cm} (17)

where \(k \in N\). Then irreducible representations of the group \(Γ(α, β)\) represent direct product of irreducible representations of the groups \(U_1(α)\) and \(R(β)\)

\[ T(U_1(α)) \otimes T(R(β)) = \exp(-ima)\exp[(2k+1)lnβ], \]  \hspace{1cm} (18)

where \(m, k = 0, ±1, ±2, ...\).

It is clear, that some conserved quantity has to correspond to gauge symmetry of the field, which is determined by the group \(R(β)\). Thus we arrive at a formulation of the following statement.

3. Conserving quantity - complex charge, which is invariant under total gauge transformations, corresponds to any complex relativistic field (scalar, vector, spinor).

Proof.

Really, since generalized relativistic equations are invariant under transformations \([10, 11]\) and variation of action integral with starting Lagrangian is equal to zero, then variation of action integral with transformed Lagrangian in accordance with \([10, 11]\) will also be zero. Consequently, all the conditions of applicability of Nöther theorem, by proof of which the only invariance under Lagrange equations is sufficient, \([11]\), are held true. We wish to pay attention to typical inaccuracy, which is abundant in the literature, consisting in that, for applicability of Nöther theorem the Lagrangian invariance under corresponding symmetry transformations is required. At the same time the only invariance of Lagrange equations under corresponding symmetry transformations, which certainly takes place in given case, is necessary (see proof of Nöther theorem). According to Nöther theorem, the conserved quantity, corresponding to \(ν - th\) parameter \((ν = \Gamma, R)\) by invariance of field under some \(k\)-parametric symmetry group, is (see, for instance, \([2]\)).

\[ Q_ν(σ) = \int_σ \theta_μνdσ_μ = \text{const}, \]  \hspace{1cm} (19)

where \(σ\) is any spacelike hypersurface, \(σ \subset 1R_4\) and 4-tensor \(θ_μν\) is determined by relation

\[ \theta_μν = \frac{∂L}{∂(∂_μu_ν)}[∂_μu_νX_μν - Y_μν] - LX_μν, \]  \hspace{1cm} (20)

in which \(L\) is field Lagrangian and the matrices \(X_μν, Y_μν\) are determined by matrix representations \(|(I_ν)μ||\) and \(|(J_ν)k||\) of infinitesimal operators of symmetry group in coordinate space and in the space of field functions respectively in accordance with the following relationships

\[ X_μν = (I_ν)μαx_α, Y_μν = (J_ν)k_iku_k. \]  \hspace{1cm} (21)

Since the value of integral in \((19)\) does not depend on the choose of spacelike hypersurface, then usually the hypersurface, which is orthogonal immediately to time axis is used. By given choose 4-vector \(dσ_μ\), representing itself infinitesimal element of spacelike hypersurface, is \(|dσ_μ|| = 0, 0, 0, dσ_4\), where \(dσ_4 = -id^4x\). Then the expression \((19)\) gets the form

\[ Q_ν(σ_4) = -i \int_σ θ_μνd^4x = \text{const}, \]  \hspace{1cm} (22)

where the conservation of the quantity \(Q_ν(σ_4)\) in time is represented in explicit form, since the time can be unambiguously set in the correspondence to hypersurface \(σ_4\) (see \([2]\)).

In the case of the invariance of the action variation under gauge symmetry group \(R(β)\) the values \(X_μν = 0\) (gauge transformations do not touch upon the coordinates), and, since the group \(R(β)\) is oneparametric, 4-tensor \(θ_μν = θ_μi ≡ θ_μ, \) that is, it represents 4-vector. Then taking into account, that in given case matrix \(|(I_ν)μ||\) of infinitesimal operator \(I_ν ≡ \bar{I}\) represents itself real number \(I = 1, \) we obtain for 4-vector \(θ_μ\) the following expression

\[ θ_μ = \frac{∂L}{∂(∂_μu_ν)}u_ν - \frac{∂L}{∂(∂_μu_ν^*)}u_ν^*, \]  \hspace{1cm} (23)

Components of 4-vector \(θ_μ\), which can be identified with additional 4-vector of charge-current density \(θ_μ ≡ f_μ^{[2]} =

i j^{[2]}_{\mu}, \text{ where } j^{[2]}_{\mu} \text{ is}

\[ j^{[2]}_{\mu} = i\left( \frac{\partial L}{\partial (\partial_{\mu} u)} u_i + \frac{\partial L}{\partial (\partial_{\mu} u')} u'_i \right), \tag{24} \]

satisfy to continuity equation

\[ \partial_{\mu} j^{[2]}_{\mu} = 0, \tag{25} \]

which represents itself the conservation law for 4-vector \( j^{[2]}_{\mu} \) in differential form. It distincts from known 4-vector of charge-current density \( j_{\mu} \) (see [2]), which is reasonable to redesignate to be \( j^{[1]}_{\mu} \), and which is

\[ j^{[1]}_{\mu} = -i\left( \frac{\partial L}{\partial (\partial_{\mu} u)} u_i - \frac{\partial L}{\partial (\partial_{\mu} u')} u'_i \right) \tag{26} \]

by the factor \( i \) and by sign of the first item. It means, that any complex field is characterized by total 4-vector \( j_{\mu} \), which is complex and can be represented in the form

\[ j_{\mu} = j^{[1]}_{\mu} + ij^{[2]}_{\mu}. \tag{27} \]

We see, that both real 4-vector-functions of the complex 4-current vector \( j_{\mu} \) are differ each other the only by sign of the first item.

Conserving quantity, corresponding to \( Q^{[2]} \), that is imaginary component of the charge, is equal to

\[ Q^{[2]} = iQ^{[2]} = -i \int \theta_4 d^3 x. \tag{28} \]

Consequently \( Q^{[2]} \) is determined by relationship

\[ Q^{[2]} = i \int \left( \frac{\partial L}{\partial (\partial_{\mu} u)} u_i + \frac{\partial L}{\partial (\partial_{\mu} u')} u'_i \right) d^3 x. \tag{29} \]

It is seen from relationship \( Q^{[2]} \), that obtained additional charge is really purely imaginary quantity. It follows from comparison with relationship for known conserved quantity for any complex field, for instance, for Dirac field. Let us remember, that real quantity - charge \( Q^{[1]} \), is the consequence of gauge symmetry, consisting in the invariance of Lagrange equations under the transformations

\[ u'(x) = exp(i \alpha) u(x) \tag{30} \]

and

\[ u'^+(x) = exp(-i \alpha) u^+(x). \tag{31} \]

In general case \( Q^{[1]} \), \( Q^{[2]} \), is

\[ Q = Q^{[1]} + iQ^{[2]} \tag{32} \]

with two real components \( Q^{[1]} \) and \( Q^{[2]} \). The statement is proven.

From the statement 3 we obtain the consequence, which seems to be essential and it is formulated in the form of the statement 4.

4. Conserving quantity - purely imaginary charge, which is invariant under total gauge transformations, corresponds to any real relativistic field (scalar, vector, spinor).

The proof is evident, if to take into account, that any real quantity, including relativistic field, is particular case of complex quantity.

In suggestion, that analogous statements are held true for quantized fields, we can conclude, that free EM-field quantum, that is photon, possesses along with the spin by the charge, which is purely imaginary in the case of real free EM-field. It becomes now to be physically understandable rather effective realization of EM-field interaction with the matter by means of given relativistic particles.

It becomes also to be understandable qualitatively the mechanism of appearance of real part of a charge when free real EM-field enter the matter. The velocity \( v \) of EM-field in the matter is less in comparison with the velocity \( c \) in vacuum. Consequently, hyperbolic rotation of coordinate system in, for example \((x_3, x_4)\)-plane of Minkowsky space and isomorphic to it rotation in \((Q_1, Q_2)\)-plane of complex charge space take place. It corresponds to appearance of real component of the charge, and it is consequence of additional hyperbolic symmetry of Maxwellian EM-field (see the next Section).

The same mechanism leads to appearance of imaginary part of EM-field vector-functions and currents. Naturally it is suggested, that life time of the photons, which are entered in the matter is rather long, that is rather strong electron-photon interaction takes place.

It seems to be clear, that Maxwell equations with all complex-valued vector and scalar variables give concrete realization of the connection between dual and gauge symmetries of EM-field.

It is remarkable, that, like to mechanics, a number of conservation laws, which can have EM-field, are optional in their simultaneous fulfillment. In particular, it is evident, that by automorphic transformation of the space of EM-field functions by relationship \( \theta_4 \), the conservation law for charge will always take place. At the same time the energy conservation law and the conservation of Poynting vector will be fulfilled, if given transformation is applied to EM-field potentials. The force characteristics, that is \( \vec{E}, \vec{H} \)-vector functions can be used to be basis for free EM-field description, since they will represent the full set in free EM-field case. However the energy conservation law and the conservation of Poynting vector, that is mathematical construction, to which enter \( \vec{E}, \vec{H} \)-vector functions, will not be fulfilled by transformation \( \theta_4 \) at arbitrary \( \beta \). Given situation is realized by the propagation of the EM-field in the matter with the velocity \( v \neq c \), that is with the velocity, which is not
equal to light velocity in vacuum. The charge remains to be Lorentz invariant quantity (see Sec.2), at the same time both the field characteristics, the energy and impulse (determined by Poynting vector) are not Lorentz invariant quantities. It is remarkable, that the conclusion on charge Lorentz invariance was formulated in [3] to be self-evident. Thus, we see, that the charge conservation law for EM-field is fulfilled even through the energy and impulse conservation laws do not take place. Therefore, the charge conservation law can be considered in given meaning to be more fundamental.

II. COMPOUND QUATERNION NATURE OF EM-FIELD WITH FOUR REAL COMPONENTS, HAVING DIFFERENT SPACE AND TIME PARITY

A. Generalized Maxwell Equations

Symmetry studies of electromagnetic (EM) field have a long history, which was started already in 19-th century from the work of Heaviside [4], where the existence of the symmetry of Maxwell equations under electrical and magnetic quantities was remarked for the first time. Mathematical formulation of given symmetry gave Larmor [5]. It is consisting in invariance of Maxwell equations for free EM-field under the transformations

$$\vec{E} \rightarrow \pm \hat{H}, \hat{H} \rightarrow \mp \vec{E},$$  \hspace{1cm} (34)

The transformations (34) are called duality transformations, or Larmor transformations. Larmor transformations (34) are particular case of the more general dual transformations, established by Rainich [6]. Dual transformations produce oneparametric abelian group $U_1$, which is subgroup of the group of chiral transformations of massless fields. Dual transformations correspond to irreducible representation of the group of chiral transformations of massless fields in particular case of quantum number $j = 1$ [3] and they are

$$\vec{E}' \rightarrow \vec{E} \cos \theta + \hat{H} \sin \theta$$

$$\hat{H}' \rightarrow \hat{H} \cos \theta - \vec{E} \sin \theta,$$ \hspace{1cm} (35)

where parameter $\theta$ is arbitrary continuous variable, $\theta \in [0, 2\pi]$. In fact the expression (35) is indication in implicit form on compound character of EM-field. Really at fixed $\theta$ the expression (35) will be mathematically correct, if vector-functions $\vec{E}$, $\hat{H}$ will have the same symmetry under improper rotations, that is concerning the parity $P$ under space inversion, both be polar or axial ones, or be both consisting of polar and axial components simultaneously. Analogous conclusion takes place regarding the parity $t$ under time reversal. The possibility to have the same symmetry, that is, the situation, when both the vector-functions $\vec{E}$, $\hat{H}$ are pure polar (axial) vector-functions, or both ones t-even (t-uneven) simultaneously contradicts to experiment. Consequently it remains the variant, that vector-functions $\vec{E}$, $\hat{H}$ in the expression (35) are compound and consists of the components with even and uneven parities under improper rotations. It is in agreement with overview on compound symmetry structure of EM-field vector-functions $\vec{E}(\vec{r},t)$, $\hat{H}(\vec{r},t)$, $\vec{D}(\vec{r},t)$, $\vec{B}(\vec{r},t)$, consisting of both the gradient and solenoidal parts, that is uneven and even parts under space inversion in [3], where compound symmetry structure of EM-field vector-functions is represented to be self-evident. It corresponds also to theoretical assumption in [7], where along with usual choice, that is, that electric field $\vec{E}$ is polar vector, magnetic field $\hat{H}$ is axial vector, the alternative choice is provided. The conclusion can be easily proved, if to represent relation (35) in matrix form

$$\begin{bmatrix} \vec{E}' \\ \hat{H}' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \vec{E} \\ \hat{H} \end{bmatrix}. \hspace{1cm} (36)$$

We see, that given matrix has the form, which allows set up in conformity to it the complex number according to biective mapping like to (1). Consequently, we have

$$\begin{bmatrix} \vec{E}' \\ \hat{H}' \end{bmatrix} = e^{-i\theta} \begin{bmatrix} \vec{E} \\ \hat{H} \end{bmatrix}, \hspace{1cm} (37)$$

that is

$$\vec{E}' = \vec{E} \cos \theta - i \vec{E} \sin \theta$$

$$\hat{H}' = \hat{H} \cos \theta - i \hat{H} \sin \theta. \hspace{1cm} (38)$$

It means, that to real plane, which is determined by the vectors $\vec{E}$ and $\hat{H}$ can be set in conformity the complex plane for the vectors $\vec{E}'$ and $\hat{H}'$.

It is evident now, that really both the vectors $\vec{E}'$ and $\hat{H}'$ are consisting of both, P-even and P-uneven components. So the first component of, for instance, $\vec{E}'$ will be P-even under reflection in the plane situated transversely to absciss-axis, the second component will be P-uneven.

Therefore, dual transformation symmetry of Maxwell equations, established by Rainich [6], indicates simultaneously on both complex nature of EM-field in general case, and that both electric and magnetic fields are consisting in general case of the components with various parity under improper rotations.

Let us find the invariants of dually transformed EM-field. It is easily to show, that the following relationship is taking place

$$\left[ \vec{E}^2 - \hat{H}^2 + 2i(\vec{E}\hat{H}) \right] e^{-2i\theta} = \text{inv}, \hspace{1cm} (39)$$

that is, we have at fixed parameter $\theta \neq 0$ two real EM-field invariants

$$(\vec{E}^2 - \hat{H}^2) \cos 2\theta + 2(\vec{E}\hat{H}) \sin 2\theta = \text{inv}$$

$$2(\vec{E}\hat{H}) \cos 2\theta - (\vec{E}^2 - \hat{H}^2) \sin 2\theta = \text{inv}. \hspace{1cm} (40)$$
It follows from relation (10) that, in particular, at $\theta = 0$ we have well known [8] EM-field invariants
\[
(E^2 - H^2) = \text{inv}
\]
\[
(EH) = \text{inv}.
\] (41)

It is interesting, that at $\theta = 45^\circ$ and at $\theta = 90^\circ$ the invariants of dually transformed EM-field are determined by the same relation (11) and by arbitrary $\theta$ we have two linearly independent combinations of given known invariants.

Subsequent extension of dual symmetry for the EM-field with sources leads also to known requirement of the existence of two type of other physical quantities - two type of charges and two type of intrinsic moments of the particles or the absorbing (dispersive) centers in condensed matter. They can be considered to be the components of complex charge, or dual charge (in another equivalent terminology) of so called dually charged particles, which were described theoretically (see [3]). We have to remark, that setting into EM-field theory of two type of charges and two type of intrinsic moments is actual, since recent discovery of dually charged quasiparticles [9] and the particles with pure imaginary electric intrinsic moments [10] in condensed matter, respectively, spin-Peierls $\pi$-solitons and Su-Schrieffer-Heeger $\sigma$-solitons in carbynoids, is clear experimental proof, that EM-field theory with complex charges and complex intrinsic moments has physical content. EM-field theory with complex charges and complex intrinsic moments ceases consequently to be the only formal model, which although is very suitable for many technical calculations, but was considered up to now to be mathematical abstraction, in which magnetic charges and magnetic currents are fictitious quantities. Similar conclusion concerns the conception of complex characteristics of EM-field in the matter.

Given conception agrees well with all practice of electric circuits’ calculations. Very fruitfull mathematical method for electric circuits’ calculations, which uses all complex electric characteristics, see for example [11], was also considered earlier the only to be formal, but convenient mathematical technique. Informality of given technique gets now natural explanation. It has to be taken into account however, that in the case $\theta = 0$ we have well known electrodynamics with odd P-parity of electric field and even P-parity of magnetic field. In given case all the EM-field characteristics are real and by using in calculations of the complex quantities, we have always add the corresponding complex conjugate quantities. At the same time in the case $\theta \neq 0$ it will be incorrect, since both the real observable quantities, which corresponds to any complex EM-field characteristics have to be retained.

Independent conclusion follows also from gauge invariance above considered (Sec.1). Really, the presence of complex charge means that 4-vector of current $j_\mu$ for any complex field is complex vector. In its turn, it means, that independently on starting origin of the charges and currents in the matter (they can be result of presence of Dirac field or another complex field) all the characteristics of EM-field in the matter have also to be complex-valued. Given conclusion follows immediately from Maxwell equations, since complex current $j_\mu$ enters explicitly Maxwell equations. It is also substantial, that Maxwell equations are invariant under the transformation of EM-field functions by relationship (10).

Let us designate the terms in (33) and (34) the system (35), (36), (37), (38), (39) can be rewritten
\[
\nabla \times \vec{E}(\vec{r}, t) = -\mu_0 \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} - \vec{j}_f(\vec{r}, t),
\] (43)
\[
\nabla \times \vec{H}(\vec{r}, t) = \epsilon_0 \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} + \vec{j}_e(\vec{r}, t),
\] (44)
\[
(\nabla \cdot \vec{E}(\vec{r}, t)) = \rho'_e(\vec{r}, t),
\] (45)
\[
(\nabla \cdot \vec{H}(\vec{r}, t)) = \rho'_g(\vec{r}, t),
\] (46)

where $\vec{j}_f(\vec{r}, t), \vec{j}_g(\vec{r}, t)$ are respectively electric and magnetic current densities, $\rho'_e(\vec{r}, t), \rho'_g(\vec{r}, t)$ are respectively electric and magnetic charge densities. Taking into account the relation (33) and (34) the system (35), (36), (37), (38), (39) can be rewritten
\[
\nabla \times (\vec{E}(\vec{r}, t) - i\vec{E}^2(\vec{r}, t)) = \\
-\mu_0 \left[ \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} - i\frac{\partial \vec{H}^2(\vec{r}, t)}{\partial t} \right] \\
- \vec{j}_f(\vec{r}, t) + i\vec{j}_g(\vec{r}, t),
\] (47)
\[
\nabla \times (\vec{H}(\vec{r}, t) - i\vec{H}^2(\vec{r}, t)) = \\
\epsilon_0 \left[ \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} - i\frac{\partial \vec{E}^2(\vec{r}, t)}{\partial t} \right] \\
+ \vec{j}_e(\vec{r}, t) - i\vec{j}_g(\vec{r}, t),
\] (48)
\[
(\nabla \cdot (\vec{E}(\vec{r}, t) - i\vec{E}^2(\vec{r}, t))) = \rho'_e(\vec{r}, t) - i\rho'_g(\vec{r}, t),
\] (49)
\[
(\nabla \cdot (\vec{H}(\vec{r}, t) - i\vec{H}^2(\vec{r}, t))) = \rho'_g(\vec{r}, t) - i\rho'_e(\vec{r}, t),
\] (50)

where $j_e(\vec{r}, t), j_e(\vec{r}, t), j_g(\vec{r}, t), j_g(\vec{r}, t)$ are correspondingly electric and magnetic current densities,
which by dual transformations are obeying to relation like to (38), they are designated like to (42), \( \rho^e_1(\vec{r}, t), \rho^e_2(\vec{r}, t), \rho^g_1(\vec{r}, t), \rho^g_2(\vec{r}, t) \) are correspondingly electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact in the system of equations (47), (48), (49), (50) are integrated Maxwell equations for two kinds of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering in equations, under space inversion. So, the component by parities of vector and scalar quantities, entering in equations, under space inversion. The components \( \vec{E}^1(\vec{r}, t), \vec{H}^1(\vec{r}, t), \vec{J}_e^1(\vec{r}, t) \) have uneven parity, \( \vec{E}^2(\vec{r}, t), \vec{H}^2(\vec{r}, t), \vec{J}_e^2(\vec{r}, t) \) have even parity, \( \rho^e_1(\vec{r}, t), \rho^g_2(\vec{r}, t) \) are scalars, \( \rho^e_2(\vec{r}, t), \rho^g_1(\vec{r}, t) \) are pseudoscalars. In the case, when \( J^g(\vec{r}, t) = 0, \rho^e_2(\vec{r}, t) = 0 \) we obtain the equations of usual singly charge electrodynamics for compound EM-field in mathematically correct form, which allows to separate the components of EM-field with various parities \( P \) under space inversion. It is remarkable, that the idea, that vector quantities, which characterize EM-field, are compound quantities and include both gradient and solenoidal parts, that is uneven and even parts under space inversion was put forward earlier in \( [3] \). At the same time in the equations of dual electrodynamics given idea was presented the only in implicit form. The representation in explicit form by equations (44), (45), (49), (50) seems to be actual, since field vector and scalar functions with various \( t \)- and \( P \)-parities are mathematically heterogeneous and, for instance, their simple linear combination, for instance, for \( P \)-uneven and \( P \)-even electric field vector-functions \( \vec{E}^1(\vec{r}, t), \vec{E}^2(\vec{r}, t) \)

\[
\alpha_1 \vec{E}^1(\vec{r}, t) + \alpha_2 \vec{E}^2(\vec{r}, t) = 0
\]

with coefficients \( \alpha_1, \alpha_2 \) from the field of real numbers, which is taking place in some theoretical and experimental works, is collage. Similar situation was discussed in \( [12] \) by analysis of Bloch vector symmetry under improper rotations. Mathematically the objects, which are like to (51) can exist. Actually to the set \( \{ \vec{E}^1(\vec{r}, t) \} \) and to the set \( \{ \vec{E}^2(\vec{r}, t) \} \) can be put in correspondence the affine space. However given affine space corresponds to direct sum of two usual vector spaces, consisting of different physically objects, that is, it represents in fact also collage. Really, given direct sum can be represented by direct sum of linear capsule

\[
\{ \alpha_k^1 \vec{E}^1(\vec{r}, t) | \alpha_k^1 \in R, k \in N \},
\]

representing itself three-dimensional vector space of the set \( \{ \vec{E}^1(\vec{r}, t) \} \) and linear capsule

\[
\{ \alpha_k^2 \vec{E}^2(\vec{r}, t) | \alpha_k^2 \in R, k \in N \},
\]

representing itself three-dimensional vector space of the set \( \{ \vec{E}^2(\vec{r}, t) \} \). It is substantial, that both the vector spaces cannot be considered to be subspaces of any three-dimensional or six-dimensional vector spaces, consisting of uniform objects. Moreover, it is evident, that the affine space, defined in that way cannot be metrizable, when considering it to be a single whole. It means in turn, that the set of objects, given by (51) are not vectors in usual algebraic meaning. Even Pythagorean theorem, for instance, cannot be used.

It can be shown, that the system, analogous to (47), (48), (49), (50) can be obtained for the second pair of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering in equations, under time reversal. Really it is easily to see, that Maxwell equations along with dual transformation symmetry, established by Rainich, given by relations (55) - (58), are symmetric relatively the dual transformations of another kind of all the vector and scalar quantities, characterizing EM-field, which, for instance, for electric and magnetic field strength vector-functions can be presented in the following matrix form

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} =
\begin{bmatrix}
\cosh \vartheta & i \sinh \vartheta \\
-i \sinh \vartheta & \cosh \vartheta
\end{bmatrix}
\begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix},
\]

(54)

where \( \vartheta \) is arbitrary continuous parameter, \( \vartheta \in [0, 2\pi] \). The relation (54) can be rewritten in the form

\[
\begin{bmatrix}
\vec{E}' \\
\vec{H}'
\end{bmatrix} =
\begin{bmatrix}
\cos i\vartheta & \sin i\vartheta \\
-\sin i\vartheta & \cos i\vartheta
\end{bmatrix}
\begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix}.
\]

(55)

In particular, if \( \vartheta \) is polar angle of coordinate system in the plane, determined by \( \vec{E} \) and \( \vec{H} \), the transformations (54) represent themselves hyperbolic rotations in \( (\vec{E}, \vec{H}) \)-plane. Let us call the transformations (54) by hyperbolic dual transformations. It represents the interest to consider the following particular case of hyperbolic dual transformations. We can define parameter \( \vartheta \) according to relation

\[
\tanh \vartheta = \frac{V}{c} = \beta,
\]

(56)

where \( V \) is velocity of the frame of reference, moving along \( x \)-axis in 3D subspace of \( I^4 \) Minkowski space. We can also to set up in conformity to the plane \( (x_2, x_3) \) in Minkowski space, the plane \( (\vec{E}, \vec{H}) \), in which \( \vec{E}, \vec{H} \) are orthogonal and are directed along absciss and ordinate axes correspondingly (or vice versa). Then we obtain

\[
|\vec{E}'| = \frac{|\vec{E}| + \beta|\vec{H}|}{\sqrt{1 - \beta^2}},
\]

\[
|\vec{H}'| = \frac{|\vec{H}| - \beta|\vec{E}|}{\sqrt{1 - \beta^2}},
\]

(57)

(or similar relations, in which \( \vec{E}, \vec{H} \) are interchanged by places). In vector form given transformations are

\[
\vec{E}' = \frac{\vec{E} + \frac{1}{\sqrt{1 - \beta^2}} \vec{H} \times \vec{V}}{\sqrt{1 - \beta^2}},
\]

\[
\vec{H}' = \frac{\vec{H} - \frac{1}{\sqrt{1 - \beta^2}} \vec{E} \times \vec{V}}{\sqrt{1 - \beta^2}},
\]

(58)
Therefore it is seen, that $\vec{E}$, $\vec{H}$ are transformed like to $x_0$ and $x_1$ coordinates (or vice versa) of the space $1R_4$. It follows from here, that both the vectors $\vec{E}^\eta$, $\vec{H}^\eta$ have $t$-even and $t$-uneven components in general case. We see also that Lorentz-invariance of Maxwell equations is particular case of hyperbolic dual symmetry. It means, that restriction to only Lorentz-invariance in consideration of Maxwell equations’ symmetry, which is usually used, constrains the concept on the EM-field itself and it is thereby constraining the possibilities for completeness of its practical usage. Taking into account (1) we obtain the relations, which are similar to (37), which can be rewritten in the similar to (38) form, that is, we have

$$\vec{E} = \vec{E}[1](r, t) + \vec{E}[2](r, t), \quad \vec{H} = \vec{H}[1](r, t) + \vec{H}[2](r, t)$$

and considering the vector-functions $(\vec{E}[1](r, t), \vec{E}[2](r, t))$ and $(\vec{H}[1](r, t), \vec{H}[2](r, t))$ to be definitional domain for the vector-functions $\vec{E}(r, t), \vec{H}(r, t)$ correspondingly, the Maxwell equations for the components of the field $(\vec{E}[1], \vec{H}[1])$ and $(\vec{E}[2], \vec{H}[2])$ have the same form and they are

$$\begin{align*}
\nabla \times (\vec{E}[3](r, t) - i \vec{E}[4](r, t)) &= -\frac{\partial \vec{H}[3](r, t)}{\partial t} - i \frac{\partial \vec{H}[4](r, t)}{\partial t} \\
&- \rho_0 \vec{E}[3](r, t) - \vec{j}^{[3]}(r, t), \quad \vec{j}^{[4]}(r, t) = 0,
\end{align*}$$

$$\begin{align*}
\nabla \times (\vec{H}[3](r, t) - i \vec{H}[4](r, t)) &= \epsilon_0 \left[ \frac{\partial \vec{E}[3](r, t)}{\partial t} - i \frac{\partial \vec{E}[4](r, t)}{\partial t} \right] \\
&+ \vec{j}^{[3]}(r, t) - i \vec{j}^{[4]}(r, t),
\end{align*}$$

$$(\nabla \cdot (\vec{E}[3](r, t) - i \vec{E}[4](r, t))) = \rho_e^{[3]}(r, t) - i \rho_e^{[4]}(r, t), \quad (\nabla \cdot (\vec{H}[3](r, t) - i \vec{H}[4](r, t))) = \rho_m^{[3]}(r, t) - i \rho_m^{[4]}(r, t),$$

where $\vec{j}^{[3]}(r, t), \vec{j}^{[4]}(r, t), \vec{j}^{[3]}(r, t), \vec{j}^{[4]}(r, t)$ are, correspondingly, electric and magnetic current densities, $\rho_e^{[3]}(r, t), \rho_e^{[4]}(r, t), \rho_m^{[3]}(r, t), \rho_m^{[4]}(r, t)$ are, correspondingly, electric and magnetic charge densities, which transformed and designated like to field strengths and currents. In fact the system of equations (61), (62), (63) represent itself correctly integrated Maxwell equations for two kinds of EM-fields (photon fields in quantum case), which differ by parities of vector and scalar quantities, entering equations, under time reversal. So, the components $\vec{E}^{[3]}(r, t), \vec{H}^{[4]}(r, t), \vec{j}^{[3]}(r, t)$ have uneven parity, $\vec{E}^{[4]}(r, t), \vec{H}^{[3]}(r, t), \vec{j}^{[4]}(r, t)$ have even parity, $\rho_e^{[3]}(r, t), \rho_m^{[4]}(r, t)$ are scalars, $\rho_e^{[4]}(r, t), \rho_m^{[3]}(r, t)$ are pseudoscalars. In the case, when $j^{[3]}(r, t) = 0, \rho^{[4]}(r, t) = 0$ we obtain the equations of usual singly charge electrodynamics for two components of EM-field with various parities under space inversion, at that either of the two consist also of two components of EM-field with various parity under time reversal.

It is easily to see, that invariants for EM-field, consisting of two hyperbolic dually symmetric parts, that is at $\vartheta \neq 0$ have the form, analogous to (39) and they can be obtained, if parameter $\theta$ to replace by $i\theta$. They are

$$\begin{align*}
\vec{E}^2 - \vec{H}^2 + 2i(\vec{E}\vec{H}) e^{2\vartheta} &= \text{inv}.
\end{align*}$$

Consequently, two real invariants at $\vartheta \neq 0$ have the form

$$\begin{align*}
(\vec{E}^2 - \vec{H}^2) e^{2\vartheta} &= I_1'' = \text{inv}, \\
2(\vec{E}\vec{H}) e^{2\vartheta} &= I_2'' = \text{inv}.
\end{align*}$$

It follows from relation (60), that in both the cases, that is at $\vartheta = 0$ and at fixed $\vartheta \neq 0$, we obtain in fact well known EM-field invariants, since factor $e^{2\vartheta}$ at fixed $\vartheta$ seems to be insufficient. At the same time at arbitrary $\vartheta$ the relation

$$\frac{I_1''}{I_2''} = \frac{I_1}{I_2} = W = \text{inv}$$

is taking place. It is seen, that the value of $W$ is independent on $\vartheta$. It means physically, that the absolute values of both the vector-functions $\vec{E}(r, t)$ and $\vec{H}(r, t)$ are changed synchronously by hyperbolic dual transformations.

So, the usage of complex number theory allows to represent correctly the electrodynamics for two photon fields, which differs by parities under space inversion or time reversal by the same single system of generalized Maxwell equations. At the same time we have two related sets, that is pairs of complex vector and scalar functions, which are ordered in their P- and t-parities. It corresponds to definition of quaternions. Really any quaternion number $x$ can be determined according to relation

$$x = (a_1 + ia_2) + (a_3 + ia_4)j,$$
elements of which are satisfying the conditions

\[(ij) = k, (ji) = -k, (ki) = j, (ik) = -j,\]

\[(ei) = i, (je) = j, (ek) = ke = k.\]

Let us designate the quantities

\[
\begin{align*}
(\vec{E}^{[1]}(\vec{r}, t) - i\vec{E}^{[2]}(\vec{r}, t)) + (\vec{E}^{[3]}(\vec{r}, t) - i\vec{E}^{[4]}(\vec{r}, t)) & = \vec{E}(\vec{r}, t) \\
(\vec{H}^{[1]}(\vec{r}, t) - i\vec{H}^{[2]}(\vec{r}, t)) + (\vec{H}^{[3]}(\vec{r}, t) - i\vec{H}^{[4]}(\vec{r}, t)) & = \vec{H}(\vec{r}, t) \\
(\vec{j}^{[1]}(\vec{r}, t) - i\vec{j}^{[2]}(\vec{r}, t)) + (\vec{j}^{[3]}(\vec{r}, t) - i\vec{j}^{[4]}(\vec{r}, t)) & = \vec{j}(\vec{r}, t) \\
(\vec{p}^{[1]}(\vec{r}, t) - i\vec{p}^{[2]}(\vec{r}, t)) + (\vec{p}^{[3]}(\vec{r}, t) - i\vec{p}^{[4]}(\vec{r}, t)) & = \vec{p}(\vec{r}, t)
\end{align*}
\]

are \(P\)-uneven, \(t\)-even,

\[
\begin{align*}
\vec{E}^{[2]}(\vec{r}, t), \vec{H}^{[1]}(\vec{r}, t), \vec{j}^{[2]}(\vec{r}, t), \vec{p}^{[1]}(\vec{r}, t), \rho^{[2]}_g(\vec{r}, t) & \\
\vec{j}^{[1]}(\vec{r}, t), \vec{p}^{[2]}(\vec{r}, t), \rho^{[3]}_g(\vec{r}, t) &
\end{align*}
\]

are \(P\)-uneven, \(t\)-uneven,

\[
\begin{align*}
\vec{E}^{[3]}(\vec{r}, t), \vec{H}^{[4]}(\vec{r}, t), \vec{j}^{[3]}(\vec{r}, t), \vec{p}^{[4]}(\vec{r}, t), \rho^{[3]}_g(\vec{r}, t) & \\
\vec{j}^{[4]}(\vec{r}, t), \vec{p}^{[3]}(\vec{r}, t), \rho^{[4]}_g(\vec{r}, t) &
\end{align*}
\]

are \(P\)-even, \(t\)-even,

\[
\begin{align*}
\vec{E}^{[4]}(\vec{r}, t), \vec{H}^{[3]}(\vec{r}, t), \vec{j}^{[4]}(\vec{r}, t), \vec{p}^{[3]}(\vec{r}, t), \rho^{[4]}_g(\vec{r}, t) & \\
\vec{j}^{[3]}(\vec{r}, t), \vec{p}^{[4]}(\vec{r}, t), \rho^{[3]}_g(\vec{r}, t) &
\end{align*}
\]

are \(P\)-even, \(t\)-uneven. According to definition of quaternions \(\vec{E}(\vec{r}, t), \vec{H}(\vec{r}, t), \vec{j}(\vec{r}, t), \vec{p}(\vec{r}, t), \rho_g(\vec{r}, t)\) are quaternions. It means, that EM-field has quaternion structure and dual and hyperbolic dual symmetry of Maxwell equations will take proper account, if all the vector and scalar quantities to represent in quaternion form. Consequently, we have

\[
\left[ \nabla \times (\vec{E}(\vec{r}, t)) \right] = -\mu_0 \left[ \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} \right] - \vec{J}_0(\vec{r}, t),
\]

\[
[\nabla \times (\vec{H}(\vec{r}, t))] = c_0 \left[ \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \right] + \vec{J}_c(\vec{r}, t),
\]

\[
(\nabla \cdot (\vec{E}(\vec{r}, t))) = \rho_c(\vec{r}, t),
\]

\[
(\nabla \cdot (\vec{H}(\vec{r}, t))) = \rho_g(\vec{r}, t)
\]

Therefore, symmetry of Maxwell equations under dual transformations of both the kinds allows along with generalization of Maxwell equations themselves to extend the field of application of Maxwell equations. It means also, that dual electrodynamics, developed by Tomilchick and co-authors, see for instance [3], obtains additional ground. Basic field equations in dual electrodynamics [3, 7], being to be written separately for two type of independent photon fields with various particles under space inversion or time reversal, will be isomorphic to Maxwell equations in complex form. It was in fact shown partly earlier in [3, 7], where complex charge was taken into consideration. At the same time all aspect of dual symmetry, leading to four-component quaternion form of Maxwell equations seem to be representing for the first time.

B. Cavity Dual Electrodynamics

Let us find the conserving quantities, which correspond to dual and hyperbolic dual symmetries of Maxwell equations. It seems to be interesting to realize given task on concrete practically essential example of cavity EM-field. At the same time to built the Lagrangian, which is adequate to given task it seems to be reasonable to solve the following concomitant task - to find dually symmetric solutions of Maxwell equations. It seems to be understandable, that the general solutions of differential equations can also possess by the same symmetry, which have starting differential equations, nevertheless dual symmetry of the solutions of Maxwell equations was earlier not found.

1. Classical Cavity EM-Field

Suppose EM-field in volume rectangular cavity without any matter inside it and made up of perfectly electrically conducting walls. Suppose also, that the field is linearly polarized and without restriction of commonness let us choose the one of two possible polarization of EM-field electrical component \(\vec{E}(\vec{r}, t)\) along \(x\)-direction. Then the vector-function \(E_\alpha(x, z)\vec{e}_x\) can be represented in well known form of Fourier sine series

\[
E^{[1]}(\vec{r}, t) = E_\alpha(x, z)\vec{e}_x = \sum_{\alpha=1}^{\infty} A_\alpha^E q_\alpha(t) \sin(k_\alpha z) \vec{e}_x,
\]

where \(q_\alpha(t)\) is amplitude of \(\alpha\)-th normal mode of the cavity, \(\alpha \in N, k_\alpha = \alpha \pi / L, A_\alpha^E = \sqrt{2\mu_0 \epsilon_0 / \pi} \),
\( \omega_c = \alpha \pi c/L, \) \( L \) is cavity length along z-axis, \( V \) is cavity volume, \( m_\alpha \) is parameter, which is introduced to obtain the analogy with mechanical harmonic oscillator. Let us remember, that the expansion in Fourier series instead of Fourier integral expansion is determined by known diskretness of \( \vec{k} \)-space, which is the result of finiteness of cavity volume. Particular sine case of Fourier series is consequence of boundary conditions

\[
\vec{n} \times \vec{E} \big|_S = 0, \quad (\vec{n} \vec{H}) \big|_S = 0,
\]

which are held true for the perfect cavity considered. Here \( \vec{n} \) is the normal to the surface \( S \) of the cavity. It is easily to show, that \( E_z(z,t) \) represents itself a standing wave along z-direction.

Let us analyse the solutions of Maxwell equations for EM-field in a cavity in comparison with known solutions from the literature to pay the attention to some mathematical details, which have however substantial physical nature. For given reasons, despite on analysis simplicity, conclusions, allowing to extend our insight to EM-field nature. For given reasons, despite on analysis simplicity, we will produce the consideration in detail.

Using the equation

\[
\epsilon_0 \frac{\partial \vec{E}(z,t)}{\partial t} = \left[ \nabla \times \vec{H}(z,t) \right],
\]

we obtain the expression for magnetic field

\[
\vec{H}(\vec{r},t) = \sum_{\alpha=1}^{\infty} A_\alpha E_\alpha k_\alpha \frac{dq_\alpha(t)}{dt} \cos(k_\alpha z) + f_\alpha(t) \vec{e}_y,
\]

where \( \{f_\alpha(t)\}, \alpha \in N, \) is the set of arbitrary functions of the time. It is evident, that the expression for \( \vec{H}(\vec{r},t) \) is satisfying to boundary conditions

\[
[\vec{n} \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}
\]

it is easily to find the class of field functions \( \{q_\alpha(t)\}. \) They will satisfy to differential equations

\[
\frac{d^2 q_\alpha(t)}{dt^2} + \frac{k_\alpha^2}{\mu_0 \epsilon_0} q_\alpha(t) = 0, \alpha \in N.
\]

Consequently, we have

\[
q_\alpha(t) = C_{1\alpha} e^{i\omega_\alpha t} + C_{2\alpha} e^{-i\omega_\alpha t}, \alpha \in N,
\]

where \( C_{1\alpha}, C_{2\alpha}, \alpha \in N \) are arbitrary constants. Thus, real-valued free Maxwell field equations result in well known in the theory of differential equations situation - the solutions are complex-valued functions. It means, that generally the field functions for free Maxwellian field in the cavity produce complex space. So we obtain additional independent argument, that the known conception, on the only real-quantity definiteness of EM-field, has to be corrected. On the other hand, the equation (84) has also the only real-valued general solution, which can be represented in the form

\[
q_\alpha(t) = B_\alpha \cos(\omega_\alpha t + \phi_\alpha),
\]

where \( B_\alpha, \phi_\alpha, \alpha \in N \) are arbitrary constants. It is substantial, that the functions in real-valued general solution have a definite t-parity.

Thus, we come independently on the previous consideration in Sec.I and Sec.II, Subsec.A to the conclusion, that classical Maxwellian EM-field can be both real-quantity defined and complex-quantity defined.

It is interesting, that there is the second physically substantial solution of Maxwell equations. Really, from general expression (82) for the field \( \vec{H}(\vec{r},t) \) it is easily to obtain differential equations for \( \{f_\alpha(t)\}, \alpha \in N, \)

\[
\frac{df_\alpha(t)}{dt} + A_\alpha E_\alpha \frac{\partial^2 q_\alpha(t)}{\partial t^2} \cos(k_\alpha z) - \frac{1}{\mu_0} A_\alpha k_\alpha q_\alpha(t) \cos(k_\alpha z) = 0.
\]

The formal solution of given equations in general case is

\[
f_\alpha(t) = A_\alpha E_\alpha \cos(k_\alpha z) \int_0^t q_\alpha(\tau) d\tau - \frac{d q_\alpha(t)}{dt} \frac{\epsilon_0}{k_\alpha}.
\]

Therefore, we have the second solution of Maxwell equations for \( \vec{H}(\vec{r},t) \) in the form

\[
\vec{H}^{[2]}(\vec{r},t) = -\left\{ \sum_{\alpha=1}^{\infty} A_\alpha H_\alpha'(q_\alpha(t)) \cos(k_\alpha z) \right\} \vec{e}_y,
\]

where \( A_\alpha H_\alpha' = \sqrt{2\omega_\alpha^2 m_\alpha / V \mu_0}. \) Similar consideration gives the second solution for \( \vec{E}(\vec{r},t) \)

\[
\vec{E}^{[2]}(\vec{r},t) = \left\{ \sum_{\alpha=1}^{\infty} A_\alpha E_\alpha''(q_\alpha(t)) \sin(k_\alpha z) \right\} \vec{e}_x.
\]

The functions \( q_\alpha'(t) \) and \( q_\alpha''(t) \) in relationships (89) and (90) are

\[
q_\alpha'(t) = \omega_\alpha \int_0^t q_\alpha(\tau) d\tau
\]

\[
q_\alpha''(t) = \omega_\alpha \int_0^t q_\alpha'(\tau') d\tau'
\]

correspondingly. Owing to the fact, that the solutions have simple form of harmonic trigonometrical functions, the second solution for electric field differs from the first solution the only by sign, that is substantial, and by inessential integration constants. Integration constants can be taken into account by means of redefinition of
factor \( m_\alpha \) in field amplitudes. It is also evident, that if vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \) are the solutions of Maxwell equations, then vector-functions \( \vec{T}\vec{E}(\vec{r}, t) \) and \( \vec{T}\vec{H}(\vec{r}, t) \), where \( \vec{T} \) is time inversion operator, are also the solutions of Maxwell equations. Moreover, if starting vector-function, to which operator \( \hat{T} \) is applied is \( t \)-even, then there is \( t \)-uneven solution, for instance for magnetic component in the form

\[
\frac{\hat{T}[t\vec{H}(\vec{r}, t)]}{t},
\]

where \( t \) is time. It can be shown in a similar way, that dually symmetric solutions, which are \( P \)-even and \( P \)-uneven are also existing.

Therefore, there are the solutions with various combinations of the signs for vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \), which are realized simultaneously, that is, their linear combination with coefficients from the field \( C \) of complex numbers will represent the solution of Cauchy problem for Maxwell equations in correspondence with known theorem, that the solution of Cauchy problem for any systems of homogeneous linear equations in partial derivatives exists and it is unique in the vicinity of any point of the initial surface (in the case, when the point selected is not characteristic point and the function, which determines given hypersurface is continuously differentiable). In other words, we obtain again the agreement with Maxwell equation symmetry consideration. Given property of EM-field seems to be essential, since it permits passing for the processes, which seemingly are forbidden by CPT-theorem. For example, let us consider the resonance system EM-field plus matter in the cavity, in particular, the so called dressed state of some quasiparticles’ system. Suppose, that wave function can be factorized, matter part is \( P \)- and \( t \)-even under space and time inversion transformations, while EM-field part is \( P \)-uneven. CPT-invariance will be preserved, since EM-field has simultaneously with \( t \)-even \( t \)-uneven component, determined by expression (92). Therefore \( t \)-parity of the function \( q_\alpha(t) \) can be various, and in the case, if we choose \( t \)-parity to be identical to the parity of the function \( q_\alpha(t) \), the solution will be different in the meaning, that the field vectors will have opposite \( t \)-parity in comparison with the first solution. It is evident, that boundary conditions are fulfilled for all the cases considered.

To build the Lagrangian we can choose the following sets of EM-field functions \( \{ u^s_\alpha(x) \} \), \( s = 1, 2, \alpha \in \mathbb{N} \),

\[
\{ u^1_\alpha(x) \} = \left\{ \sqrt{\alpha_0} A^E_\alpha \sin k_\alpha(x_3) [q_\alpha(x_4) \pm i q''_\alpha(x_4)] \right\},
\]

\[
\{ u^2_\alpha(x) \} = \left\{ \sqrt{\alpha_0} A^H_\alpha \cos k_\alpha(x_3) [-q_\alpha(x_4) \pm \frac{i}{\alpha} \frac{d\alpha(x_4)}{dx_4}] \right\}.
\]

The functions \( \{ u^s_\alpha(x) \} \), \( s = 1, 2, \alpha \in \mathbb{N} \) are built from the components of the expansion in Fourier series of the fields \( \vec{E}^{[1]}(\vec{r}, t), \vec{E}^{[2]}(\vec{r}, t) \) and \( \vec{H}^{[2]}(\vec{r}, t), \vec{H}^{[1]}(\vec{r}, t) \) correspondingly. At the same time the sets \( \{ u^s_\alpha(x) \} \), \( s = 1, 2, \alpha \in \mathbb{N} \) produce at fixed \( x \) two orthogonal countable bases, corresponding to \( s = 1, 2 \) in two Hilbert spaces, which are formed by vectors \( \vec{U}^{[s,\pm]}_\alpha(u^1_\alpha(x), u^2_\alpha(x), \ldots) \) for variable \( x \in \mathbb{R}^4 \). Really scalar product of two arbitrary vectors \( \vec{U}^{[s,\pm]}_\alpha(u^1_\alpha(x), u^2_\alpha(x), \ldots) \) and \( \vec{U}^{[s,\pm]}_\beta(v^1_\alpha(x), v^2_\alpha(x), \ldots) \) is equal to

\[
\langle \vec{U}^{[s,\pm]}_\alpha(x) | \vec{U}^{[s,\pm]}_\beta(x) \rangle = \int_0^L \int_0^L u^s_\alpha(x, z) v^s_\alpha(x, z) dz, \ s = 1, 2.
\]

Then vector distance is

\[
d(\vec{U}^{[s,\pm]}_\alpha(x), \vec{U}^{[s,\pm]}_\beta(x)) = ||\vec{U}^{[s,\pm]}_\alpha(x) - \vec{U}^{[s,\pm]}_\beta(x)||.
\]

So we obtain, that the vectors \( \{ \vec{U}^{[s,\pm]}_\alpha(x) \}, \ x \in \mathbb{R}^4 \) produce the space \( L^2 \) and taking into account the Riesz-Fisher theorem it means, that given vector space is complete, that in its turn means, that the spaces of vectors \( \{ \vec{U}^{[s,\pm]}_\alpha(x) \}, \ x \in \mathbb{R}^4, \ s = 1, 2, \) are Hilbert spaces. Consequently Lagrangian \( L(x) \) can be represented in the following form

\[
\begin{align*}
L(x) &= \sum_{s=1}^4 \sum_{\mu=1}^4 \sum_{\alpha=1}^\infty \frac{\partial u^s_\alpha(x)}{\partial x_\mu} \frac{\partial u^{s,\pm}_\alpha(x)}{\partial x_\mu} \\
&- \sum_{s=1}^4 \sum_{\mu=1}^4 \sum_{\alpha=1}^\infty K(x) u^{s,\pm}_\alpha(x) u^{s,\pm}_\alpha(x),
\end{align*}
\]

where \( K(x) \) is factor, depending on the set of variables \( x = \{ x_\mu \}, \mu = 1, 4 \).

Let us find the conserving quantity, corresponding to dual symmetry of Maxwell equations. Dual transformation, determined by relation (36) is the transformation the only in the space of field three-dimensional vector-functions \( \vec{E}, \vec{H} \), (let us designate it by \( (\vec{E}, \vec{H}) \)-space) and it does not touch upon the coordinates. It seems to be conveniet to define in given space the reference frame, then the transformation, given by (36) is the rotation of two component matrix vector-function

\[
||F|| = \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix}.
\]
Instead of two Hilbert space for two sets of vectors \( \{ \mathbf{U}^{(s,\pm)}(x) \} \), \( x \in R_4, s = 1, 2 \) we can also define one Hilbert space for row matrix vector function set

\[
\| \mathbf{U}(x) \| = \{ \mathbf{U}^{[1,\pm]}(x) \mathbf{U}^{[2,\pm]}(x) \}\]

(100)

with the set of components

\[
\{ \| U_\alpha(x) \| \} = \{ u^{1,\pm}_\alpha(x) u^{2,\pm}_\alpha(x) \},
\]

(101)

where \( \alpha \in N \). In general case instead parameter \( \theta \) we can define rotation angles \( \theta_{ik} \), \( i, k = 1, 2, 3 \) in 2D-planes of \((\vec{E}, \vec{H})\) functional space. It is evident, that \( \theta_{ik} \) are antisymmetric under the indices \( i, k \). According to Nöther theorem, the conserving quantity, corresponding to parameters \( \theta_{ik} \) in dual transformations \((100)\), that is at \( \theta_{ik} = \theta_{12} \) is determined by relations like to \((101)\) and \((102)\). So, we obtain

\[
S_{12}^\mu = - \left( \sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} \right) \| Y_\alpha \| + c.c.,
\]

(102)

where \( \mu = 1, 2 \) and it was taken into account, that \( \| X_\alpha \| \) in matrix relation \((102)\), which is like \((101)\), is equal to zero. The factor \( \frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} \) in \((102)\) is row matrix

\[
\frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} = \left[ \begin{array}{cc} \frac{\partial L}{\partial (\partial_{\alpha x_1} u^{1,\pm}_\alpha)} & \frac{\partial L}{\partial (\partial_{\alpha x_2} u^{1,\pm}_\alpha)} \\ \frac{\partial L}{\partial (\partial_{\alpha x_3} u^{2,\pm}_\alpha)} & \frac{\partial L}{\partial (\partial_{\alpha x_4} u^{2,\pm}_\alpha)} \end{array} \right],
\]

(103)

matrix \( \| Y_\alpha \| \) is product of matrices \( \| I_\alpha \| \) and \( \| U_\alpha(x) \| \), that is

\[
\| Y_\alpha \| = \| I_\alpha \| \left[ \begin{array}{cc} u^{1,\pm}_\alpha \\ u^{2,\pm}_\alpha \end{array} \right],
\]

(104)

where \( \| I_\alpha \| \) is the matrix, which corresponds to infinitesimal operator of dual or hyperbolic dual transformations of \( \alpha \)-th mode of cavity EM-field. It represents in general case the product of three matrices, corresponding to rotation along three mutually perpendicular axes in 3D functional space above defined. So \( \| I_\alpha \| = \| I_\alpha^1 \| \| I_\alpha^2 \| \| I_\alpha^3 \| \).

The transformations in the form, which is given by \((36)\) correspond to \( \theta_{23} = \theta, \theta_{12} = 0, \theta_{31} = 0 \), that is \( \| I_\alpha^2 \| = \| I_\alpha^3 \| = E \), where \( E \) is unit \([2 \times 2]\)-matrix. In the absence of dispersive medium in the cavity \( \| I_\alpha \| \) will be independent on \( \alpha \). Moreover, it is easy to see, that infinitesimal operator with matrix \( \| I_\alpha \| \) is the same for dual transformations, determined by \((35)\) and hyperbolic dual transformations, determined by \((43)\). Really \( \| I_\alpha \| \) in both the cases is

\[
\| I_\alpha \| = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right],
\]

(105)

for any \( \alpha \in N \).

Conserving quantity is

\[
S_{12}^\mu = - \frac{i}{c} \int \left( \sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} \| Y_\alpha \| \right) + c.c. \} d^3x
\]

(106)

The structure of \((106)\) unambiguously indicates, that it is the component of spin tensor \((2), (14)\), to which dual vector component can be set in the correspondence according to relation

\[
S_{12}^1 = \varepsilon_{ijk} S_{jk}^4 = - \frac{i}{c} \int \left( \sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} \| Y_\alpha \| \right) + c.c. \} d^3x,
\]

(107)

where \( \varepsilon_{ijk} \) is completely antisymmetric Levi-Civita 3-tensor.

Therefore we obtain, that the same physical conserving quantity corresponds to dual and hyperbolic dual symmetry of Maxwell equations. Taking into account the expressions for Lagrangian \((98)\) and for infinitesimal operator \((105)\), in the geometry choosed, when vector \( \vec{E} \) is directed along absciss axis, vector \( \vec{H} \) is directed along ordinate axis in \((\vec{E}, \vec{H})\) functional space, we have

\[
S_{12}^\mu = \sum_{\alpha=1}^{\infty} \frac{\partial u^{1,\pm}_\alpha}{\partial x_\mu} \frac{\partial u^{2,\pm}_\alpha}{\partial x_\mu} + c.c.,
\]

(108)

and

\[
S_3^\alpha = \varepsilon_{312} S_{12}^4 = - \frac{i}{c} \int \left( \sum_{\alpha=1}^{\infty} \frac{\partial L}{\partial (\partial_\mu \| U_\alpha \|)} \| Y_\alpha \| \right) + c.c. \} d^3x,
\]

(109)

It is projection of spin on the propagation direction. Therefore we have in given case right away physically significant quantity - spirality.

The relations \((102), (107), (108), (109)\) show, that spin of classical relativistic EM-field in the cavity and, correspondingly, spirality are additive quantities and they represent the sum of cavity spin and spirality modes. On the connection of the conserving quantity, which is invariant of dual symmetry with spin was indicated in \([3]\), where free EM-field was considered with traditional Lagrangian, which uses vector potentials to be field functions. The result obtained together with aforecited result in \([3]\) lift dilemma on the necessity of using of given quantity by consideration of classical EM-field. Really, the situation was to some extent paradoxical, and it can be displayed by the following conversation between two disputant physicists. "Spin exists" - has insisted the first, referring on the appearance of additional tensor component in total tensor of moment - intrinsic moment - to be consequence of Minkowsky space symmetry under Lorentz transformations, "Spin does not exists" - has insisted the second, referring on the metrized tensor of the moment, in which spin part is equal to zero \([2]\) in distinction from canonical tensor. In other words, both disputants were in one’s own way right. Dual symmetry leads to unambiguous conclusion "Spin exists" and has to be taken into consideration by the solution of tasks, concerning both classical and quantum electrodynamics. Moreover spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality
in the simplest case above considered) combine two subsystems of photon fields, that is the subsystem of two fields, which have definite $P$-parity (even and uneven) with the subsystem of two fields, which have definite $t$-parity (also even and uneven) into one system. In fact we obtain the proof for four component structure of EM-field to be a single whole, that is confirmation along with the possibility of the representation of EM-field in four component quaternion form, given by (142), (143), (144), the necessity of given representation. It extends the overview on the nature of EM-field itself. It seems to be remarkable, that given result on the special leading significance of spin is in agreement with result in [12], where was shown, that spin is quaternion vector of the state in Hilbert space, defined under ring of quaternions, of any quantum system (in the frame of the chain model considered) interacting with EM-field.

It is interesting, that the charge and currents, being to be the components of 4-vector, which are transformed by corresponding representation of Lorentz group, are invariants of hyperbolic dual transformations, that is, they are also Lorentz invariants, if to take into account, that Lorentz transformations are particular case of hyperbolic dual transformations. It is seen immediately from the expressions for 4-current and it means, that observers in various inertial frames will register the same value of the charge in correspondence with conclusion in [3]. It is connected with invariance of Lagrange equations and equations charge by multiplication of field functions on arbitrary complex number, established in Sec.1, since by hyperbolic dual transformations the multiplication of field functions on some complex number takes place.

2. Quantized Cavity EM-Field

The quantization of EM-field was proposed for the first time still at the earliest stage of quantum physics [12]. At the same time correspondence to each mode of radiation field the quantized harmonic oscillator, was proposed for the first time by Dirac [16] and it is widely used in QED including quantum optics [17], it is canonical quantization. EM-field potentials are used to be field functions by canonical quantization. At the same time to describe free EM-field it is sufficient to choose immediately the observable quantities - vector-functions $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ - to be field functions. We use further given idea by EM-field quantization. We can start like to canonical quantization, from classical Hamiltonian, which for the first partial classical solution of Maxwell equations is

$$\mathcal{H}^{[1]}(t) = \frac{1}{2} \int \int \int \left[ \epsilon_0 \vec{E}_z^2(z, t) + \mu_0 \vec{H}_y^2(z, t) \right] dxdydz$$

$$= \frac{1}{2} \sum_{a=1}^{\infty} \left[ m_\alpha \nu_\alpha q_\alpha^2(t) + \frac{p_\alpha^2(t)}{m_\alpha} \right] , p_\alpha = m_\alpha \frac{dq_\alpha(t)}{dt}. \quad (110)$$

So, taking into consideration the relationship for Hamiltonian $\mathcal{H}^{[1]}(t)$ we set in correspondence to canonical variables $q_\alpha(t), p_\alpha(t)$, determined by the first partial solution of Maxwell equations, the operators by usual way

$$[\hat{p}_\alpha(t), \hat{q}_\beta(t)] = i\hbar \delta_{\alpha\beta}$$

$$[\hat{q}_\alpha(t), \hat{q}_\beta(t)] = [\hat{p}_\alpha(t), \hat{p}_\beta(t)] = 0,$$ \quad (111)

where $\alpha, \beta \in N$. Introducing the operator functions of time $\hat{a}_\alpha(t)$ and $\hat{a}_\alpha^+(t)$

$$\hat{a}_\alpha(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha \hat{q}_\alpha(t) + i\hat{p}_\alpha(t) \right]$$

$$\hat{a}_\alpha^+(t) = \frac{1}{\sqrt{2\hbar m_\alpha \omega_\alpha}} \left[ m_\alpha \omega_\alpha \hat{q}_\alpha(t) - i\hat{p}_\alpha(t) \right], \quad (112)$$

we obtain the operator functions of canonical variables in the form

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2m_\alpha \omega_\alpha}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right]$$

$$\hat{p}_\alpha(t) = i \sqrt{\frac{\hbar m_\alpha \omega_\alpha}{2}} \left[ \hat{a}_\alpha^+(t) - \hat{a}_\alpha(t) \right]. \quad (113)$$

Then EM-field operator functions are obtained right away and they are

$$\hat{E}(\vec{r}, t) = \left( \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_\alpha}{V \epsilon_0}} \left[ \hat{a}_\alpha^+(t) + \hat{a}_\alpha(t) \right] \sin(k_\alpha z) \right) \vec{e}_x,$$ \quad (114)

$$\hat{H}(\vec{r}, t) = i \left( \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_\alpha}{V \mu_0}} \left[ \hat{a}_\alpha^+(t) - \hat{a}_\alpha(t) \right] \cos(k_\alpha z) \right) \vec{e}_y.$$ \quad (115)

Taking into account the relationships (114), (115) and Maxwell equations, it is easily to find an explicit form for the dependencies of operator functions $\hat{a}_\alpha(t)$ and $\hat{a}_\alpha^+(t)$ on the time. They are

$$\hat{a}_\alpha^+(t) = \hat{a}_\alpha^+(t = 0) e^{-i\omega_\alpha t},$$

$$\hat{a}_\alpha(t) = \hat{a}_\alpha(t = 0) e^{-i\omega_\alpha t}, \quad (116)$$

where $\hat{a}_\alpha^+(t = 0), \hat{a}_\alpha(t = 0)$ are constant, complex-valued in general case, operators.

It seems to be essential, that complex exponential dependencies in (116) cannot be replaced by the real-valued harmonic trigonometrical functions. Really, if to suggest, that

$$\hat{a}_\alpha(t) = \hat{a}_\alpha^+(t = 0) \cos \omega_\alpha t,$$ \quad (117)

then we obtain, that the following relation has to be taking place

$$[\hat{a}_\alpha^+(t = 0) - \hat{a}_\alpha(t = 0)]^{-1} [\hat{a}_\alpha^+(t) + \hat{a}_\alpha(t)] = \tan \omega_\alpha t.$$ \quad (118)
We see, that left-hand side in relation (118) does not depend on time, right-hand side is depending. The contradiction obtained establishes an assertion. Therefore, the quantized Maxwellian EM-field is complex-valued field in full correspondence with pure algebraic conclusion in Sec.1.

Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both complex-valued and real-valued functions. The fields \( \hat{E}^{[2]}(\vec{r}, t) \) and \( \hat{H}^{[2]}(\vec{r}, t) \) can be quantized in much the same way. The operators \( \hat{a}''_{\alpha}(t) \), \( \hat{a}''^{+}_{\alpha}(t) \) are introduced analogously to (112).

\[
\begin{align*}
\hat{a}''_{\alpha}(t) &= \frac{1}{\sqrt{2\hbar m_{\alpha}\omega_{\alpha}}} [m_{\alpha}\omega_{\alpha}\hat{q}''_{\alpha}(t) + i\hat{p}''_{\alpha}(t)] \\
\hat{a}''^{+}_{\alpha}(t) &= \frac{1}{\sqrt{2\hbar m_{\alpha}\omega_{\alpha}}} [m_{\alpha}\omega_{\alpha}\hat{q}''_{\alpha}(t) - i\hat{p}''_{\alpha}(t)] 
\end{align*}
\] (119)

For the field function operators we obtain

\[
\hat{E}^{[2]}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V\varepsilon_{0}}} [\hat{a}''^{+}_{\alpha}(t) + \hat{a}''_{\alpha}(t)] \sin(k_{\alpha}z) \hat{e}_{1},
\]

\[
\hat{H}^{[2]}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V\mu_{0}}} (-i) [\hat{a}''^{+}_{\alpha}(t) - \hat{a}''_{\alpha}(t)] \cos(k_{\alpha}z) \hat{e}_{2}.
\]

In accordance with definition of complex quantities we can built the following combination of solutions, satisfying Maxwell equations

\[
(\hat{E}^{[1]}(\vec{r}, t), \hat{E}^{[2]}(\vec{r}, t)) \rightarrow \hat{E}^{[1]}(\vec{r}, t) + i\hat{E}^{[2]}(\vec{r}, t) = \hat{F}(\vec{r}, t),
\]

(122)

\[
(\hat{H}^{[2]}(\vec{r}, t), \hat{H}^{[1]}(\vec{r}, t)) \rightarrow \hat{H}^{[2]}(\vec{r}, t) + i\hat{H}^{[1]}(\vec{r}, t) = \hat{H}(\vec{r}, t).
\]

(123)

Consequently, the electric and magnetic field operators for quantized EM-field, corresponding to general solution of Maxwell equations, are

\[
\hat{E}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V\varepsilon_{0}}} [\hat{a}''^{+}_{\alpha}(t) + \hat{a}''_{\alpha}(t)]
\]

\[
\quad + [\hat{a}''_{\alpha}(t) - \hat{a}''^{+}_{\alpha}(t)] \sin(k_{\alpha}z) \hat{e}_{x},
\]

and

\[
\hat{H}(\vec{r}, t) = \sum_{\alpha=1}^{\infty} \sqrt{\frac{\hbar \omega_{\alpha}}{V\mu_{0}}} [\hat{a}''^{+}_{\alpha}(t) - \hat{a}''_{\alpha}(t)]
\]

\[
\quad - [\hat{a}''_{\alpha}(t) + \hat{a}''^{+}_{\alpha}(t)] \cos(k_{\alpha}z) \hat{e}_{y},
\]

(124)

(125)

It is substantial, that both field operators \( \hat{E}(\vec{r}, t) \) and \( \hat{H}(\vec{r}, t) \) are Hermitian operators.

### 3. Cavity 4-currents

It represents the interest to calculate the 4-currents for given task. Let us place all the vector-functions in pairs in accordance with their parity. Then we have the following pairs

\[
(\hat{E}^{[1]}(\vec{r}, t), \hat{E}^{[2]}(\vec{r}, t)), (\hat{H}^{[2]}(\vec{r}, t), \hat{H}^{[1]}(\vec{r}, t))
\]

(126)

in which both the \( \hat{E} \)-vectors and \( \hat{H} \)-vectors have the same space parity (polar and axial correspondingly) and differ each other by t-parity, t-even and t-uneven in accordance with their numbers in pairs. It means, that they transform like to \( x_4 \) and \( x_1 \) coordinates in \( 1^R_4 \). In a similar manner can be set the vector-functions with opposite to the vector-functions in (126) space parity

\[
(\hat{E}^{[3]}(\vec{r}, t), \hat{E}^{[4]}(\vec{r}, t)), (\hat{H}^{[4]}(\vec{r}, t), \hat{H}^{[3]}(\vec{r}, t)).
\]

(127)

Then taking into account the definition of complex quantities to be pair of real defined quantities, taken in fixed order, we come in a natural way once again to concept of complex vector-functions, which describe Maxwellian EM-field equations. In other words, we have in fact the quantities

\[
\hat{E}^{[1]}(\vec{r}, t) + i\hat{E}^{[2]}(\vec{r}, t) = \hat{E}_{c,p}(\vec{r}, t), \\
\hat{H}^{[2]}(\vec{r}, t) + i\hat{H}^{[1]}(\vec{r}, t) = \hat{H}_{c,a}(\vec{r}, t),
\]

(128)

and

\[
\hat{E}^{[3]}(\vec{r}, t) + i\hat{E}^{[4]}(\vec{r}, t) = \hat{E}_{c,a}(\vec{r}, t), \\
\hat{H}^{[4]}(\vec{r}, t) + i\hat{H}^{[3]}(\vec{r}, t) = \hat{H}_{c,p}(\vec{r}, t),
\]

(129)

where complex plane put in correspondence to \( (y, z) \) real plane, subscripts a and p mean axial and polar respectively. It seems to be convenient to determine the space of EM-field vector-functions under the ring of quaternions with another basis in comparison with basis, given by (69). We will use now the quaternion basis \( \{e_i\}, i = 0, 3 \) with algebraic operations between elements, satisfying to relationships

\[
e_i e_j = \varepsilon_{ijk} e_k + \delta_{ij} e_0, e_0 e_i = e_i, e_0^2 = e_0, i, j, k = 1, 3,
\]

(130)

where \( \varepsilon_{ijk} \) is completely antisymmetric Levi-Chivita 3-tensor.

Let us define the vector biquaternion

\[
\hat{\Phi} = (\hat{E}^{[1]} + \hat{H}^{[2]}) + i(\hat{H}^{[1]} + \hat{E}^{[2]}),
\]

(131)

which can be represented to be the sum of the biquaternions

\[
\hat{\Phi} = \hat{F} + \hat{F},
\]

(132)

where \( \hat{F} = \hat{E}^{[1]} + i\hat{H}^{[1]}, \hat{F} = \hat{H}^{[2]} + i\hat{E}^{[2]} \). Then Maxwell equations for instance for two free photon fields with different t-parity are

\[
\nabla \hat{\Phi} = 0.
\]

(133)
The generalized Maxwell equations in quaternion form with quaternion basis, given by \[69\], can also be rewritten in fully quaternion form, if to use both the bases. It seems to be consequence of independence of basis definition for both the quaternion forms.

It is evident, that

\[ j_{\mu,\pm}(x) = j_{\mu,\pm}^{(1)}(x) + i j_{\mu,\pm}^{(2)}(x), \tag{134} \]

where subscript \(\pm\) corresponds to two possibilities for definition of complex vector-functions. Along with relationships \[128, 129\] they can be defined by the change of addition sign in \[128, 129\] to opposite. The quantity \(j_{\mu,\pm}^{(1)}(x)\) is well known quantity, and it is determined by

\[
j_{\mu,\pm}^{(1)}(x) = -\frac{ie}{\hbar c} \sum_{\alpha=1}^{2} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (u_{\mu}^{s,\pm}(x))} u_{\alpha}^{s,\pm}(x) \right]
+ \frac{ie}{\hbar c} \sum_{\alpha=1}^{2} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (u_{\mu}^{s,\pm}(x))} u_{\alpha}^{s,\pm}(x) \right], \tag{135} \]

where \(L(x)\) is Lagrange function and \(u_{\alpha}^{s,\pm}(x), s = 1, 2\) are

\[
u_{\alpha}^{1,\pm}(x) = \sqrt{\epsilon_0 A_{\alpha}^{E}} \sin k_{\alpha}(x)\left[ q_{\alpha}(x) \pm i q_{\alpha}'(x) \right] \\
u_{\alpha}^{2,\pm}(x) = \sqrt{\mu_0 A_{\alpha}^{H}} \cos k_{\alpha}(x)\left[ -q_{\alpha}'(x) + i \frac{d q_{\alpha}(x)}{dx} \right]. \tag{136} \]

The functions \(u_{\alpha}^{s,\pm}(x), s = 1, 2, \alpha \in N\) are built from the components of the expansion in Fourier series of the fields \(E_1^{[1]}(\vec{r},t), E_2^{[2]}(\vec{r},t)\) and \(H_1^{[1]}(\vec{r},t), H_1^{[2]}(\vec{r},t)\) correspondingly.

To determine the current density \(j_{\mu,\pm}^{(2)}(x)\) we have to take into consideration, that gauge symmetry group of EM-field is two-parametric group \(\Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathfrak{R}(\beta)\), where \(\mathfrak{R}(\beta)\) is abelian multiplicative group of real numbers (excluding zero). It leads also to existence for EM-field of complex 4-current densities including complex charge density component. Since the current density \(j_{\mu,\pm}^{(2)}(x)\) is

\[
j_{\mu,\pm}^{(2)}(x) = -\frac{ie}{\hbar c} \sum_{\alpha=1}^{2} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (u_{\mu}^{s,\pm}(x))} u_{\alpha}^{s,\pm}(x) \right]
- \frac{ie}{\hbar c} \sum_{\alpha=1}^{2} \sum_{s=1}^{2} \left[ \frac{\partial L(x)}{\partial (u_{\mu}^{s,\pm}(x))} u_{\alpha}^{s,\pm}(x) \right]. \tag{137} \]

It can be easily shown, that \(j_{3}^{1,\pm}(\vec{r},t)\) is always equal to zero for any set of twice continuously differentiable functions \(\{q_{\alpha}(t)\}, \alpha \in N\). The expression for arbitrary set of twice continuously differentiable functions \(\{q_{\alpha}(t)\}, \alpha \in N\), for \(j_{3}^{2,\pm}(\vec{r},t)\) is

\[
j_{3}^{2,\pm}(\vec{r},t) = -\frac{2ie}{\hbar c^2 V} \sum_{\alpha=1}^{N} m_{\alpha} \omega_{\alpha}^3 \sin 2k_{\alpha}z x \\
\times \left\{ i|q_{\alpha}(t)| \pm i \omega_{\alpha}^2 \int_{0}^{t} q_{\alpha}(t') dt' dt'' \right\}^2 \right] \tag{138} \]

The relationship \[(138)\] is true for both the variants in superposition

\[
\vec{H}^{[1]}(\vec{r},t) + i \vec{H}^{[2]}(\vec{r},t) = \vec{H}^{[1]}(\vec{r},t), i \neq j, i, j = 1, 2. \tag{139} \]

Taking into account relationship \[(68)\], that is the set \(\{q_{\alpha}(t)\}, \alpha \in N\), which satisfy the Maxwell equations we will have

\[
j_{4}^{1,\pm}(\vec{r},t) = -\frac{8ie}{\hbar c^2 V} \sum_{\alpha=1}^{N} m_{\alpha} \omega_{\alpha}^3 \sin 2k_{\alpha}z x \\
[C_{1\alpha} C_{2\alpha} e^{2\omega_{\alpha} t} + C_{2\alpha} e^{-2\omega_{\alpha} t}], \tag{140} \]

the expression for arbitrary set of twice continuously differentiable functions \(\{q_{\alpha}(t)\}, \alpha \in N\), for \(j_{4}^{1,\pm}(\vec{r},t)\) is

\[
j_{4}^{1,\pm}(\vec{r},t) = -\frac{2e}{\hbar c^2 V} \sum_{\alpha=1}^{N} m_{\alpha} \omega_{\alpha}^3 \sin 2k_{\alpha}z x \\
\times \left\{ \frac{dq_{\alpha}(t)}{dt} \mp \frac{dq_{\alpha}''(t)}{dt} \right\} \int_{0}^{t} q_{\alpha}(t') dt' dt'' \\
+ \cos^2 k_{\alpha}z \left[ \frac{1}{\omega_{\alpha}} \frac{d^2 q_{\alpha}'(t)}{dt^2} \pm i \omega_{\alpha} q_{\alpha}'(t) \right] \times \left[ \frac{1}{\omega_{\alpha}} \frac{dq_{\alpha}(t)}{dt} \mp i \omega_{\alpha} q_{\alpha}(t) \right] \times \left[ \frac{1}{\omega_{\alpha}} \frac{dq_{\alpha}''(t)}{dt} \mp i \omega_{\alpha} q_{\alpha}''(t) \right] \times \left[ \frac{1}{\omega_{\alpha}} \frac{dq_{\alpha}'''(t)}{dt} \mp i \omega_{\alpha} q_{\alpha}'''(t) \right]. \tag{141} \]

where \(q_{\alpha}''(t) = \omega_{\alpha}^2 \int_{0}^{t} q_{\alpha}(t') dt' dt''\). It is evident from relationship \[(141)\], that in the case of real-valued sets of twice continuously differentiable functions \(\{q_{\alpha}(t)\}, \alpha \in N\), \(j_{4}^{1,\pm}(\vec{r},t)\) is equal to zero. For complex-valued functions, determined by \[(68)\], we will have

\[
j_{4}^{1,\pm}(\vec{r},t) = \frac{8ie}{\hbar c^2 V} \sum_{\alpha=1}^{N} m_{\alpha} \omega_{\alpha}^3 \left( |C_{1\alpha}|^2 - |C_{2\alpha}|^2 \right). \tag{142} \]
It is seen from (142), that \( j^1_{\pm}(\vec{r}, t) \) in the case of Maxwellian EM-field is constant, which is equal to zero at \( |C_{1\alpha}| = |C_{2\alpha}| \), that is for all real-valued functions and for complex-valued functions \( \{q_\alpha(t)\}, \alpha \in \mathbb{N} \), which differ each other by arguments of constants \( C_{1\alpha} \) and \( C_{2\alpha} \).

\[
j^2_{\pm}(\vec{r}, t) = -\frac{2e}{\hbar c^2 V} \sum_{\alpha=1}^{\infty} m_\alpha \omega_\alpha^2 \sin^2 k_\alpha z \frac{d}{dt} (|q_\alpha(t)|^2) + \omega_\alpha^4 \left( \int_0^t \int q_\alpha(t') dt' dt'' \right) = \frac{d}{dt} \left[ \int_0^t q_\alpha(t') dt' \right] \frac{d}{dt} \left[ \int_0^t q_\alpha(t') dt' \right] \frac{1}{\omega_\alpha^2} \left( \frac{d}{dt} \left[ \int_0^t q_\alpha(t') dt' \right] \right)^2 \pm \omega_\alpha^2 \left( \frac{d}{dt} \left[ \int_0^t q_\alpha(t') dt' \right] \right)^2 + \omega_\alpha^2 \frac{d}{dt} \left( \int_0^t q_\alpha(t') dt' \right) \pm i \frac{d}{dt} \left( \int_0^t q_\alpha(t') dt' \right)
\]

(143)

For complex-valued functions, determined by (35), we obtain

\[
j^2_{\pm}(\vec{r}, t) = \frac{8ie}{\hbar c^2 V} \sum_{\alpha=1}^{\infty} m_\alpha \omega_\alpha^2 \cos 2k_\alpha z \times [C_{1\alpha} C^*_{2\alpha} e^{2i\omega_\alpha t} - C^*_{1\alpha} C_{2\alpha} e^{-2i\omega_\alpha t}].
\]

(144)

It can be shown, that continuity equation

\[
\frac{\partial j^1_{\pm}(x)}{\partial x_\mu} = 0
\]

(145)

is fulfilled for both general case and for Maxwellian EM-field functions considered.

III. CONCLUSIONS

It is shown on the basis of complex number theory, that any quantophysical quantity is complex quantity.

Additional gauge invariance of complex relativistic fields was found. It is based on invariance of generalized relativistic equations under the operations of additional gauge symmetry group - multiplicative group \( \mathbb{R} \) of all real numbers (without zero) and leads to appearance of purely imaginary component of charge. So, it was shown, that complex fields are characterized by complex charges. It gives key for correct generalization of field equations, in particular for electrodynamics. In application to EM-field it means that two-parametric group \( \Gamma(\alpha, \beta) = U_1(\alpha) \otimes \mathbb{R}(\beta) \) determines the gauge symmetry of EM-field and that free real EM-field is characterized by purely imaginary charge.

Additional hyperbolic dual symmetry of Maxwell equations is established, which includes Lorentz-invariance to be its particular case. The essence of additional hyperbolic dual symmetry of Maxwell equations is that, that Maxwell equations along with dual transformation symmetry, established by Rainich, given by (35) - (38), are symmetric relatively the dual transformations of another kind. Hyperbolic dual transformations for electric and magnetic field strength vector functions are

\[
\begin{bmatrix}
\tilde{E}^n \\
\tilde{H}^n
\end{bmatrix} = \begin{bmatrix}
\cosh \vartheta & i \sinh \vartheta \\
-i \sinh \vartheta & \cosh \vartheta
\end{bmatrix} \begin{bmatrix}
E \\
H
\end{bmatrix},
\]

(146)

where \( \vartheta \) is arbitrary continuous parameter, \( \vartheta \in [0, 2\pi] \).

Generalized Maxwell equations are obtained on the basis of both dual and hyperbolic dual symmetries of EM-field. It is shown, that in general case both scalar and vector quantities, entering equations, are quaternion quantities, four components of which have different parities under improper rotations.

Invariants for EM-field, consisting of dually symmetric parts, for both the cases of dual symmetry and hyperbolic dual symmetry are found. It is concluded, that Maxwell equations with all quaternion vector and scalar variables give concrete connection between dual and gauge symmetries of EM-field.

The example of free classical and quantized cavity EM-field is considered. It is shown, that the same physical conserving quantity corresponds to both dual and hyperbolic dual symmetry of Maxwell equations. It is spin in general case and spirality in the geometry choosed, when vector \( \vec{E} \) is directed along absciss axis, \( \vec{H} \) is directed along ordinate axis in \( (\vec{E}, \vec{H}) \) functional space. Spin takes on special leading significance among the physical characteristics of EM-field, since the only spin (spirality in the geometry considered) combine two subsystems of photon fields, which have definite \( P \)-parity (even and uneven) with the subsystem of two fields, which have definite \( t \)-parity (also even and uneven) into one system. It is considered to be the proof for four component structure of EM-field to be a single whole, that is, it is the confirmation along with the possibility of the representation of EM-field in four component quanton form, given by (13), (14), (17), (18), the necessity of given representation. It extends the overview on the nature of EM-field itself. It seems to be remarkable, that given result on the special leading significance of spin is in agreement with result in (12), where was shown, that spin is quaternion vector of the state in Hilbert space, defined under ring of quaternions, of any quantum system (in the frame of the chain model considered) interacting with EM-field.

New principle of EM-field quantization, which is based on choosing of immediately observable quantities - vector-functions \( \vec{E}(\vec{r}, t) \) and \( \vec{H}(\vec{r}, t) \) - to be field functions, is proposed. It is found, that quantized Maxwellian EM-field is the only complex-valued field. Consequently, there is difference between classical and quantized EM-fields, since classical EM-field can be determined by both
complex-valued and real-valued functions.

[1] Nöther E, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math-phys.Klasse, (1918) 235-257
[2] Bogush A A, Moroz L G, Introduction to the Theory of Classical Fields, M., Editorial URSS, 2004, 384
[3] Strazhev V.I, Tomilchick L.M, Electrodynamics with Magnetic Charge, Minsk, Nauka i Tekhnika, 1975, 336 pp
[4] Heaviside O, Phil.Trans.Roy.Soc.A, 183 (1893) 423-430
[5] Larmor J, Collected papers, London, 1928
[6] Rainich G Y, Trans.Am.Math.Soc., 27 (1925) 106
[7] Berezin A.V, Kurochkin Yu.A, Tolkachev Ye.A, Quaternions in Relativistic Physics, Minsk, Nauka i Tekhnika, 1989, 199 pp
[8] Landau L.D, Lifshitz E.M, Field Theory, M., Nauka, 504
[9] Yearchuck D, Yerchak Y, Kirilenko A, Popechts V, Doklady NANB, 52, N 1 (2008) 48-53
[10] Yearchuck D, Yerchak Y, Alexandrov A, Phys.Lett.A, 373 (2009) 489 - 495
[11] Andre Angot, Complements de Mathematiques, Paris, 1957, 778
[12] D.Yearchuck, Y.Yerchak, A.Dovlatova, Optics Communications, 283 (2010) 3448-3458
[13] Tolkachev E.A, Tomilchick L.M, Covariant Methods in Theoretical Physics, Minsk, 1981, pp 44-48
[14] Bogoliubov N N, Shirkov D V, Introduction in the theory of quantized fields, M., Nauka, 1973, 414 pp
[15] Born M, Jordan P, Zeitschrift fuer Physik, 34 (1925) 858-888
[16] Dirac P.AM, Proc.Roy.Soc., A, 114 (1927) 243-265
[17] Scully M O, Zubairy M S, Quantum Optics, Cambridge University Press, 1997, 650