Detecting fractional Josephson effect through $4\pi$ phase slip

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Abstract – Fractional Josephson effect is a unique character of Majorana Fermions in topological superconductor system. This effect is very difficult to detect experimentally because of the disturbance of quasiparticle poisoning and unwanted couplings in the superconductor. Here, we propose a scheme to probe fractional DC Josephson effect of semiconductor nanowire-based topological Josephson junction through $4\pi$ phase slip. By exploiting a topological RF SQUID system we find that the dominant contribution for Josephson coupling comes from the interaction of Majorana Fermions, resulting the resonant tunneling with $4\pi$ phase slip. Our calculations with experimentally reachable parameters show that the time scale for detecting the phase slip is two orders of magnitude shorter than the poisoning time of nonequilibrium quasiparticles. Additionally, with a reasonable nanowire length the $4\pi$ phase slip could overwhelm the topological trivial $2\pi$ phase slip. Our work is meaningful for exploring the effect of modest quantum fluctuations of the phase of the superconductor on the topological system, and provide a new method for quantum information processing.

Introduction. – Topological superconductor with p-wave pairing is a hot topic in condensed matter physics. The system can host at its boundaries one kind of exotic quasiparticles-Majorana Fermions (MFs), which are their own antiparticles. MFs has important applications in quantum information processing [1-4]. Two separate MFs could construct one physical qubit, named topological qubit. The non-locality makes topological qubit immune from local environment noise. Nowadays, intrinsic topological superconductor has yet to be found. Moreover, MFs are predicted to also exist in some complicate systems, e.g., topological insulator coupled to s-wave superconductor via proximity effect [5], or spin-orbit coupled semiconducting nanowire combined with superconductivity and magnetic field [6,7]. Recently, several groups have claimed that they had observed some important signatures of MFs in these systems [8-10]. However, the existence of MFs has not been confirmed due to the lack of a smoking-gun evidence.

A remarkable signature of MFs is fractional Josephson effect. It is well known that the supercurrent through a conventional Josephson junction is $2\pi$ periodic with the phase difference across the junction. However, this statement is not always true for topological Josephson junction, which is made with two weakly coupled topological superconductor instead of s-wave superconductor. Kitaev has predicted that the current-phase relation in topological Josephson junction should be $4\pi$ periodic [11]. This period doubling of the Josephson current is protected by fermion parity conservation. The fermion parity would not change unless a quasiparticle excitation occurs. Unfortunately, non-equilibrium quasiparticles were found in superconducting system at very low temperature, which is called quasiparticle poisoning [11, 12]. It can break the parity conservation of the system and restore the $2\pi$ period of the current in the characteristic time. Therefore, the experiment to probe the $4\pi$ periodicity should be accomplished within the characteristic time of quasiparticle poisoning. On the other side, the experimental duration time is limited by adiabatic condition and measurement speed.
Fast manipulation of the phase difference can excite transitions from the subgap Majorana bound states to the outgap continuum states due to the Landau-Zener transition. Therefore, it is challenging to experimentally detect the fractional Josephson effect. Recently, several theoretical proposals are brought forward to overcome the quasiparticle poisoning problem [13][15]. Although these proposals are nearly insensitive to quasiparticle poisoning, they all require that the junction works in the ballistic regime, where the nanowire is nearly transparent, i.e., the conductance $D \sim 1$. In this regime the nontopological Josephson junction can also produce the fractional Josephson effect due to the Landau-Zener transition [16][17]. Therefore, it is desirable to figure out a scheme working in the tunneling regime of the junction ($D \ll 1$). In addition, most of previous researches have paid attentions to AC Josephson effect where the junction is voltage or current biased. Actually, fractional DC Josephson effect, which does not bring dissipation, is more useful in the context of quantum information processing. For instance, it can be employed to couple topological qubits with conventional superconducting qubits.

Here we conceive a scheme for detecting fractional DC Josephson effect. Compared with its AC analog [18][20], the DC effect is more susceptible to parity-breaking excitations and other imperfections. Generally, three mechanisms, conventional Josephson coupling [21], quasiparticle poisoning, the coupling of MFs from one topological superconductor, result a conventional 2π phase slip which screens the 4π slip of topological Josephson energy. By elaborately designing the parameters of device and experiment, we can overcome these problems at the same time. Firstly, the conventional Josephson coupling could be neglected when the parameters of the superconducting circuit are proper, because the conventional Josephson energy $E_J$ relies on the parameters of the junction in a different manner with its topological analog $E_m$. When $E_J$ is much smaller than $E_m$, the 2π phase slips will be inhibited. Secondly, our scheme can be implemented in a time scale much shorter than the characteristic time of quasiparticle poisoning. At last, the circuit used in our scheme could be reasonably designed such that the interaction of MFs from one topological superconductor is much smaller than the topological Josephson coupling. In this case, the 4π phase slip can overwhelm the conventional 2π slip.

**System and Hamiltonian.** — The system we considered is a superconducting loop interrupted by a junction. The junction is made by putting a spin-orbit coupled semiconductor nanowire on two separate superconductors. The two pieces of the nanowire contacting with the superconductors underneath is superconducting due to proximity effect. Combining with a parallel magnetic field, the nanowire could be tuned into the topological phase. When the Zeeman splitting exceeds a critical value $B_c = \sqrt{\Delta^2 + \mu^2}$ ($\Delta$ and $\mu$ are the superconducting gap and the chemical potential, respectively), the two pieces of proximitized nanowire will transition to topological superconductors and two pairs of MFs emerge at their boundaries (see Fig. 1). Moreover, the two MFs at the junction couple with each other. The coupling Hamiltonian reads

$$H_m = i\gamma_1\gamma_2 E_m \cos \frac{\varphi}{2},$$

in which $\gamma_1, \gamma_2$ are Majorana operators, $\varphi$ is the phase difference across the junction, $E_m = \Delta\sqrt{D}$ is the amplitude of the topological Josephson coupling energy with $D$ the conductance of the quasi-one-dimensional nanowire. Besides, the conventional Josephson coupling of the junction may also exist, which is related to the quasi-continuum states above the superconducting gap. In the case of one-channel nanowire, the conventional Josephson coupling can be written as

$$H_J = -\Delta\sqrt{1 - D\sin^2 \frac{\varphi}{2}}.$$  

In the low conductance regime ($D \ll 1$), $H_J$ transforms to the celebrated tunneling Josephson coupling $H_J = -E_J \cos \varphi$ (up to a constant) with $E_J = \Delta D/4$. Therefore, it is straightforward to deduce the relation $E_J = E_m^2/4\Delta$. If $E_m$ is much smaller than the superconducting gap, we can get $E_J \ll E_m$. In this case, we can safely ignore $H_J$ term [22] and write the whole Hamiltonian as

$$H = E_c n^2 + E_L (\varphi - \varphi_c)^2 + H_m,$$

where $E_c = 2e^2/C$ is the charge energy of the junction, and $E_L = (\phi_0/2\pi)^2/2L$ is inductive energy of the circuit with $\phi_0$ being flux quantum. $\varphi_c = 2\pi \phi_c/\phi_0$, $\phi_c$ denotes the external flux threading the loop. The Hamiltonian is as same as that of a flux qubit except the Josephson coupling term. As well-known, a pair of MFs composes one Dirac fermion, and $H_m$ can be expressed as

$$H_m = E_m \cos \frac{\varphi}{2} (2f^\dagger f - 1),$$

in which we have defined $f = (\gamma_1 + i\gamma_2)/2$. The eigenvalue of $f^\dagger f$ (0 or 1) determines the parity of the Dirac fermion (even or odd). The topological Josephson coupling given by $H_m$ has two distinguishable characters. Firstly, the coupling is 4π periodic with phase difference. Resultantly, the charge tunneling the junction is in unit of single-electron instead of Cooper-pair. Very recently, an experiment [23] has examined this character in Coulomb blockade regime, in which $E_C \gg E_m$. In the opposite regime, that is $E_C \ll E_m$, the 4π phase slip dual with single electron tunneling can occur. Secondly, the coupling depends upon the fermion parity of the two MFs at the junction. This character makes the 4π phase slip sensitive to the fermion-parity breaking events, such as quasiparticle poisoning. In the following section, we will present our scheme for uncovering the unique 4π feature of MFs.
Scheme. – We now investigate how to observe the $4\pi$ phase slip with the system shown in the last section. Without losing generality, we assume that the parity of MFs is restricted in the even subspace. Later on, we will consider the effect of the unintended change of the parity on the phase slips. Under the circumstances, the potential energy of the whole Hamiltonian (Eq. (3)) is

$$U = E_L (\varphi - \varphi_e)^2 - E_m \cos \frac{\varphi}{2}.$$  \hspace{1cm} (5)

By tuning the parameter $\varphi_e$, we can control the configuration of the potential. If $\varphi_e = 0$, the potential has one global minimum at $\varphi = 0$ (see Fig.2A). If the flux is biased at $\varphi_e = 2\pi$, a symmetric double-well profile of the potential is formed, similar to the potential of a flux qubit biased at $\varphi_e \sim \pi$. However, the separation of the two minima of the double-well is $\sim 4\pi$ instead of $\sim 2\pi$ (see Fig.2B). The lowest two energy eigenstates in the double-well are symmetric and antisymmetric superpositions of left and right local states. The energy splitting of them is denoted by $\Delta E$. For probing the $4\pi$ phase slip, we initially set $\varphi_e = 0$. In low temperature limit, the system will be reset to the ground state in the well around $\varphi = 0$. Then, switch the bias to $\varphi_e = 2\pi$ quickly to make sure that the system localizes in the left well during this operation, and wait for a time $\Delta t \sim \frac{1}{\Delta E}$. In this period, the resonant tunneling of the phase difference between the double well can happen, and the state of the system is coherently oscillating between the two states of the double well. Finally, bias the circuit away from $\varphi_e = 2\pi$ and measure the total flux of the circuit. The resulting flux can either be $0$ or $2\phi_0$, corresponding to the left or right local state of the double well respectively. The possibility of finding $2\phi_0$ oscillates with $\Delta t$. In experiment, we can use the total flux of the loop with another RF SQUID [23]. The possibility of the system projecting to the $2\phi_0$ state can be obtained by repeating the above operations many times. Note that if the same operations are applied to a conventional or topological trivial RF SQUID, the final measured flux would definitely be $\phi_0$ independent of $\Delta t$, because of the $2\pi$ periodicity of their Josephson couplings [25]. Hence, the oscillating $4\pi$ phase slip is a distinctive signature of topological Josephson junction. However, in practice the superconducting circuit is subject to some unavoidable disturbance which might destroy the signature. Therefore, it is vital to investigate the robustness of our scheme.

Effect of quasiparticle poisoning. In Eq. (5), we have assumed that the parity of MFs is conserved in the whole process. Actually, the parity conservation can be broken by quasiparticle poisoning. Quasiparticles exist in various superconducting systems even at vary low temperature. One quasiparticle excitation event could alter the occupation of the in-gap states in a junction. For the topological Josephson junction, it would turn over the parity of MFs. In our case, we prepare the MFs at even parity state, thus an unwanted excitation will take it to odd state. If this happens when the circuit is biased at $\varphi = 2\pi$, the potential energy profile is changed. It is obvious that the circuit will eventually stay at the ground state of the well with minimum at $\varphi = 2\pi$. That is exactly the result in conventional RF SQUID in the same bias sequence. Thus, the $4\pi$ phase slip disappears. Therefore, anyone who is going to observe the $4\pi$ phenomenon must carry out the experiment in a period shorter than the quasiparticle poisoning time. Generally, the parity lifetime of the bound state in a proximitized semiconductor nanowire applied magnetic field exceeds $10\mu s$ [23]. The time needed to implement our scheme is on the order of $1/\Delta E$. Typically, we choose the parameters as follows: $E_m = 25\text{GHz} \times h$, $E_e = 3\text{GHz} \times h$, $E_L = 1\text{GHz} \times h$. With this parameter configuration, we have numerically calculated the splitting $\Delta E = 25\text{MHz}$. This value means that the phase slips happen in the time scale of $40\text{ns}$, which is at least two orders of magnitude shorter than the poisoning time. We stress that after each run of the experiment, the Fermion parity will be initialized to even subspace. Therefore, we can claim that the quasiparticles have little impact on our scheme.
A comment is in order. In our parameters set, the Josephson coupling energy is much larger than the inductive energy with ratio $E_m/E_L = 25$. Even though, the finiteness of the ratio would make the distance of the tunnelling minimum of the symmetric double well is not equal to 4π but rather smaller than it. In fact, the distance is about 3π with our parameters. From this view of point, the expression 4π phase slip is somewhat misleading. Similarly in a conventional RF SQUID the amplitude of the phase slip is not 2π either (< 2π). Actually, the names are stemming from the formation of the relate Josephson coupling. What is more, we can distinguish these two kinds of phase slips without any confusion.

Effect of finite length of topological superconductor

We know that the coupling of the two MFs of one topological superconductor is oscillating with the length of the superconductor [20,27]. The oscillation amplitude decreases exponentially with the length $L$, $\varepsilon = \varepsilon_0 e^{-L/\xi}$, where $\varepsilon_0$ is a prefactor, $\xi$ is superconducting coherence length. Generally, if the topological superconductor is much longer than its superconducting coherence length, this coupling is rather weak and can be neglected. That is why we have not put the interaction between $\gamma_1(\gamma_2)$ and $\gamma_3(\gamma_4)$ in Eq. (1). However, in practice, the length of a one-dimensional topological superconductor may be limited by the technique to make it or the size of the circuit. It is necessary to investigate the effect of the coupling between $\gamma_1(\gamma_2)$ and $\gamma_3(\gamma_4)$ on the 4π phase slips.

Let us first look at the Josephson coupling energy in absence of the interactions $\gamma_1\gamma_3$, $\gamma_2\gamma_4$, i.e., $H_m$ (Eq. (3)). When the phase difference takes values of $(2k + 1)\pi$ (k be integer), the even and odd parity states are degenerate. When the interactions present, the potential energy can be addressed as

$$U' = E_L(\phi - \phi_c)^2 - E_m \cos \frac{\phi}{2} \sigma_z + \varepsilon \sigma_x,$$  

where $\sigma_x,z$ are Pauli operators acting in the fermion parity space of $\gamma_1, \gamma_2$. $\varepsilon$ denotes the coupling strength of $\gamma_1\gamma_3$ ($\gamma_2\gamma_4$) which is much smaller than $E_m$. It is easy to see that the odd-even degeneracies at $\phi = (2k + 1)\pi$ are lift, and instead anticrossings arise, which leads to the mixing of the two parity states. When the circuit is biased at $\phi_c = 2\pi$ with the initial state be the ground state in the left well, there are two possible tunneling events. One is tunneling to the right well with same parity (named Tunneling 1), and the other is tunneling to the nearest well with opposite parity (Tunneling 2), as shown in Fig. 3. Tunneling 1 is the consequence of the topological Josephson coupling and signify the 4π phase slip. In contrast, Tunneling 2 denote the 2π phase slip which is always connected to the topological trivial Josephson junction. Therefore, if Tunneling 2 dominates the process, 4π phase slip is covered and we can not tell the topological phase from the topological trivial phase. To this end, one needs to clarify whether the Tunneling 2 is weak enough to be neglected under experimentally feasible condition.

Now we devote to estimate the tunneling rate of Tunneling 2. The coexistence of parity switching and quantum fluctuation of the phase difference make the task troublesome. We solve this problem in a quasiclassical manner. As Tunneling 2 will change the fermion parity, it is reasonable to believe that the tunneling rate should be related to the transition rate of the parity states when $\phi$ is considered as a classical quantity. While biasing the circuit at $\phi_c = 2\pi$, the system is initially located at left well with minimum of $\sim \pi/2$ (not 0 due to the finite of $E_m/E_L$) and parity is even. After Tunneling 2, the system localizes at $\phi = 2\pi$ and parity is odd. Therefore, the tunneling rate is limited by the transition rate of the fermion parity at $\phi = \pi/2$. For convenience, we assume they are approximately equal. The calculation of parity transition rate is a typical two-level-system problem. Starting with even parity, the population of odd parity state is oscillating with time between 0 and $P$, with $P = \varepsilon/\sqrt{\varepsilon^2 + (E_m \cos 2\pi/4)^2}$. According to Eq. (4) and the parameters in Ref. [25], when the nanowire is as long as $L = 2 \mu m$ which is reachable in experiment, the MFs coupling $\varepsilon$ is three orders of magnitude smaller than $E_m$. In this case, the maximum odd parity population $P \approx 0$, which means that even→odd transition rate is almost vanishing. One may argue that the initial state does not localize at $\phi = \pi/2$, but spreads on a range even including the anticrossing point $\phi = \pi$. In fact, the parity transition rate reach its maximum value of $\varepsilon$ at the anticrossing, which is the same order of magnitude as tunneling rate of Tunneling 1, i.e., $\Delta E$. However, the probability of the initial state be around the anticrossing is very small due to the large ratio $E_m/\varepsilon$, thereby Tunneling...
ing 2 would rarely occur in the period of Tunneling 1. In other words, $4\pi$ phase slip will not be covered by $2\pi$ phase slip.

**Discussion and Conclusion.**—We would like to discuss the feasibility of our scheme. The scheme is conceived based on the Hamiltonian of the system given by Eq. (3), in which we have neglected the conventional Josephson coupling of the topological junction. For justifying this approximation, we estimate the ratio $E_j/E_m$ with practical parameters. For the typical material NbN, its superconducting critical temperature is $\sim 10$ K, which equals eight times of the value of $E_m$ chosen in this paper. This condition in turn leads to $E_j = E_m/32$. Consequently, the conventional Josephson coupling has little effect on the $4\pi$ phase slips and can be ignored. In addition, the large ratio of $\Delta/E_m$ is helpful to prevent the subgap Majorana bound state being excited to the continuum states. The other issue is the viability of RF SQUID with a very small inductance energy. It is worth noting that a small value of the inductance energy and, thus, a large magnitude of $L$ is essential for the observation of the $4\pi$ phase slip, since the large ratio $E_m/E_L$ can make the distance of the minima of the double well of the superconducting phase far exceed $2\pi$. Taken $E_L = 1$ GHz, the inductance of the loop $L$ is up to 100 nH. In experiment, we can design a large area superconducting loop, or make use of a array of Josephson junctions playing the role of a superinductor, such as that in fluxonium qubit [21]. Indeed, the requirement of the large inductance could be loosed at the expense of slightly reducing the amplitude of the phase slip.

In conclusion, we have proposed a scheme for detecting fractional DC Josephson effect in topological RF SQUID system through $4\pi$ phase slip. To observe this phase slip, we take advantage of the resonant tunneling of the phase difference. Our calculations with reachable parameters show that the duration of the process of the scheme is much shorter than the quasiparticle poisoning time. More importantly, the $4\pi$ phase slip could overwhelm the topological trivial $2\pi$ phase slip with a practical nanowire length. Our scheme is experimentally feasible, and promising for exploring the interplay of topological superconductors and quantum computation.

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