EXAMPLES OF ENDPERIODIC MAPS

JOHN CANTWELL AND LAWRENCE CONLON

Abstract. We construct examples of end-periodic maps using Dehn twists. We sketch an alternative method to construct examples which gives better insight into the dynamics of the map.

1. Introduction

We construct examples of end-periodic maps \( h : L \rightarrow L \), with \( L \) a surface. We mainly use Dehn twists to construct the examples but we also sketch an alternative method of construction which gives better insight into the dynamics of the map. For all the examples, \( L \) orientable or not, \( h \) orientation preserving or not, there is an open three manifold \( M_0 \) which fibers over the circle with \( L \) as leaf and \( h \) the monodromy of the fibration. The manifold \( M_0 \) can be compactified yielding a (sutured) three manifold \( M \), by adding finitely many tangential boundary leaves, so that \( L \) is a leaf of a depth one foliation of \( M \).

This note is intended as a companion piece to [2], the various technical terms used here being defined in that paper.

Definition 1.1. A Dehn Twist homeomorphism in an annulus \( \{(t, \theta) \mid 0 \leq t \leq 1, 0 \leq \theta \leq 2\pi\} \), with oriented core curve \( C \) is given by \( d : I \times S^1 \rightarrow I \times S^1 \) with \( d(t, \theta) = (t, \theta + 2\pi t), \ 0 \leq t \leq 1, \ 0 \leq \theta \leq 2\pi \). For this to be well defined one stipulates an orientation on the annulus (a normal vector to the annulus). The curve \( d(0, \theta) = (0, \theta), \ 0 \leq \theta \leq 2\pi \), is then chosen to be the curve to the right of \( C \). With these conventions, a curve transverse to the annulus bends to the right, after the Dehn twist, as it enters the annulus from either side.

2. The Simple Example

The simple example consists of a surface \( L \) which is a planar strip with a sequence of open disks removed approaching each end. The translation \( t \) moves the juncture \( J_i \) to the juncture \( J_{i+1} \). The Dehn twist \( d \) is in the dotted circle in Figure 1 (top) and the endperiodic map is \( f(x) = d \circ t(x) \). One can construct the traintracks in Figure 1 (bottom). The laminations \( \Lambda_- \) and \( \Lambda_+ \) are obtained by blowing air in at the switches of the train tracks. The end \( \zeta \) of the semi-isolated leaf \( \lambda_+ \in \Lambda_+ \) and the end \( \eta \) of the semi-isolated leaf \( \lambda_- \in \Lambda_- \) are the only escaping ends. Thus, \( \lambda_- \) is the only semi-isolated leaf in \( \Lambda_- \), \( \lambda_+ \) is the only semi-isolated leaf in \( \Lambda_+ \), and there is a unique fixed point \( p \in \lambda_+ \cap \lambda_- \) giving neighborhoods \( [p, \zeta] \) and \( [p, \eta] \) of the escaping ends. The negative escaping set has one component, contains a neighborhood of the negative end, has topological frontier \( \Lambda_+ \), and border \( \lambda_+ \). Similarly, the positive escaping set has one component, contains a neighborhood of the positive end, has topological frontier \( \Lambda_- \), and border \( \lambda_- \). The complement of \( \Lambda_- \cup \Lambda_+ \) is the escaping set \( U \). That is, there are no principal regions. The two
(neighborhoods of) ends, $\zeta \cup \eta$, form a component of the border $\partial U$ of the first kind. The other component of $\partial U$ homeomorphic to $\mathbb{R}$ is of the second kind. The escaping set $U$ has infinitely many compact border components, each with two edges. Let $J_n$ be the junctures as in Figure 1. Let $B_n$ be the portion of $L$ between the junctures $J_n$ and $J_{n+1}$.

One sees that the triple $(\Gamma_+, \Gamma_-, f)$ is a Handel-Miller system associated to the endperiodic map $f$ by checking that Axioms I - VI of [2] are satisfied. Axioms I - IV are clearly satisfied. In Axiom VI, if $e_\pm$ are the negative and positive ends of the leaf $L$, one takes $J^0_{e_-}$ and $J_{e_+}$ to be the $J_0$ and $J_1$ of Figure 1 respectively. It is an easy exercise to verify Axiom V by verifying that $f$ preserves $\Lambda_{\pm}$.

![Figure 1. Simple Example](image1)

The positive and negative pre-Markov rectangles $R_\pm = M_1 \cup C_\pm \cup M_2$ with opposite edges $\alpha_\pm, \beta_\pm \subset \Lambda_-$ and $\gamma_\pm, \delta_\pm \subset \Lambda_+$ are shown in Figure 2. The upper rectangle is $R_-$ and the lower rectangle is $R_+$. Here $M_1$ and $M_2$, the left and right components of $R_- \cap R_+$, are the two Markov rectangles. The edges $\alpha_+, \beta_+$ are subsets of $\alpha_-$ which is the top edge of $R_-$ and the edges $\gamma_-, \delta_-$ are subsets of $\gamma_+$ which is the bottom edge of $R_+$. The rectangles $C_\pm$ are each equal to

![Figure 2. $R_- = M_1 \cup C_- \cup M_2$ (upper rectangle) and $R_+ = M_1 \cup C_+ \cup M_2$ (lower rectangle)](image2)
There is a unique homeomorphism $h$ which fixes $p$ (in Figure 2), preserves $\Lambda_{\pm}$, and sends the rectangle $R_-$ onto the rectangle $R_+$, satisfying $h(\alpha_-) = \alpha_+$, $h(\beta_-) = \beta_+$, $h(\gamma_-) = \gamma_+$, and $h(\delta_-) = \delta_+$.

In the traintracks of Figure 1, the rectangle $R_-$ shows up as the right arc from $p$ to $q$ while the rectangle $R_+$ shows up as the left arc from $p$ to $q$. Figure 3 shows what the image of $h(R_+)$ must be and in the traintracks of Figure 1, $h(C_+)$ appears as the loop based at $q$ going to the right. One can similarly see the loops $h^n(C_+)$ based at $p$, $n > 1$, in the traintracks of Figure 1 and imagine the rectangles $h^n(R_+)$, $n > 1$, in Figure 3. For example, in Figure 3, $h^2(C_+)$ is attached to $R_+$ along $h(\beta_+)$ and the rest of $h^2(R_+)$ is contained in $h(R_+)$. The rectangles $h^n(R_-)$ in Figure 3 and loops $h^n(C_-)$ in the traintracks of Figure 1 can be analogously visualized. From Figure 3 it is clear that there is a transverse invariant measure on $\Lambda_+$ with the transverse measure of $h^n(R_+ \cap \Lambda_+)$ equal to $1/2^n$, $n \in \mathbb{Z}$. Similar statements hold for $\Lambda_-$ and $R_-$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{\(h(R_+) = h(M_1) \cup h(C_+) \cup h(M_2)\) (shaded)}
\end{figure}

\textbf{Remark.} The surface $L$ and homeomorphism $h : L \to L$ is completely determined by the homeomorphism $h : R_- \to R_+$. This point of view can be used to define endperiodic maps. This way of defining endperiodic maps is less transparent than using Dehn twists but provides more insight into the dynamics of each map.

\textbf{Remark.} The simple example is an example of an irreducible endperiodic map $h$ of a surface $L$ with boundary. The map $h$ can be extended to the double of $L$ to give a reducible endperiodic map, the reducing curves being the original boundary curves of $L$. More generally, the examples we give with boundary below can be pasted together along their boundary curves to give other examples of reducible endperiodic maps.

\section{A Simple Orientation Reversing Example.}

Let $L$ be the planar strip with a sequence of open disks removed approaching each end and $d$ the Dehn twist in the dotted circle of Figure 1 (top). Let $t_f$ be the translation $t$ that moves the juncture $J_i$ to the juncture $J_{i+1}$ followed by a flip which interchanges the top and bottom edges of the planar strip and let the endperiodic map be $f(x) = d \circ t_f(x)$. The map $h$ has no fixed points but the points $p$ and $q$ are periodic points of period two. There is one ray $[p, e_+]$ of a leaf in $\Lambda_+$ that escapes to the positive end and one ray $[q, e_-]$ of a leaf in $\Lambda_-$ that escapes to the negative end. Thus, the laminations $\Lambda_{\pm}$ each have one semi-isolated leaf.

The positive and negative pre-Markov rectangles $R_{\pm}$ having opposite edges $\alpha_\pm, \beta_\pm \subset \Lambda_-$ and $\gamma_\pm, \delta_\pm \subset \Lambda_+$ are shown in Figure 3. The upper rectangle is...
Figure 4. Orientation Reversing version of the Simple Example

$R_-$ and the lower rectangle is $R_+$. The Markov rectangles are the two components of $R_- \cap R_+$. The edges $\alpha_+, \beta_+$ are subsets of $\alpha_-$ which is the top edge of $R_-$ and the edges $\gamma_-, \delta_-$ are subsets of $\gamma_+$ which is the bottom edge of $R_+$. There is a unique (orientation reversing) homeomorphism $h$ which takes $p$ to $q$ (in Figure 5), preserves $\Lambda_\pm$, and sends the rectangle $R_-$ onto the rectangle $R_+$ satisfying $h(\alpha_-) = \alpha_+$, $h(\beta_-) = \beta_+$, $h(\gamma_-) = \gamma_+$, and $h(\delta_-) = \delta_+$. This homeomorphism determines $L$ and $h : L \to L$.

Figure 5. $R_-$ (upper rectangle) and $R_+$ (lower rectangle)

2.2. A Simple Non-orientable Example. In this example, $L$ is the non-orientable surface given in Figure 6 (top), $t$ is translation to the right by one unit, $d$ is the Dehn twist in the dotted curve, and $h = d \circ t$ is the end-periodic map. The positive and negative laminations $\Lambda_\pm$ each consist of one leaf $\lambda_\pm$ with $p = \lambda_- \cap \lambda_+$ being just one point. The traintracks are drawn in Figure 6 (bottom). The negative and positive escaping sets $\mathcal{U}_\pm$ are each $L \smallsetminus \lambda_\pm$ and each of $\hat{\mathcal{U}}_\pm$ has border two copies of $\lambda_\pm$. The escaping set $\mathcal{U} = L \smallsetminus (\lambda_- \cup \lambda_+)$ and $\hat{\mathcal{U}}$ has four border components of the first kind. Two of these border components have homotopic lifts to the universal cover and are isotopic to the same reducing curve $\sigma_1$. The other two border curves of $\hat{\mathcal{U}}$ are isotopic to distinct reducing curves $\sigma_2$ and $\sigma_3$. The example reduces to two pieces $L_1$ and $L_2$. On $L_1$ the map is a (non-obvious) translation. The other piece $L_2$ is a neighborhood of $\lambda_- \cup \lambda_+$ and is a two legged chair (see Gabai [5, Example 5.1]). The map $h : L_2 \to L_2$ has fixed point $p$.

Remark. The non-orientable example is a primitive example in the sense of Definition 5.1.
2.3. A Simple Non-planar Example. In this example, $L$ is the non-planar surface with four boundary curves given in Figure 7 (top), $t$ is translation to the right by one unit, $d$ is the Dehn twist in the dotted curve, and $h = d \circ t$ is the end-periodic map. Under $h$, the straight (bottom) boundary curve and the curly boundary curve are both sent to themselves while the two wavy boundary curves are permuted. The positive and negative laminations $\Lambda_{\pm}$ each consist of one leaf $\lambda_{\pm}$ with $p = \lambda_- \cap \lambda_+$ being just one point. The leaves $\lambda_{\pm}$ are drawn in Figure 7 (bottom). The negative and positive escaping sets $U_{\pm}$ are each $L \smallsetminus \lambda_{\pm}$ and each of $\hat{U}_{\pm}$ has border two copies of $\lambda_{\pm}$.

The escaping set $U = L \smallsetminus (\lambda_- \cup \lambda_+)$ and $\hat{U}$ has four border components of the first kind. Two of these border components are peripheral and are isotopic to the two wavy boundary curves of $L$. The two other border components of $\hat{U}$ are isotopic...
to distinct reducing curves $\sigma_1$ and $\sigma_2$. The example reduces to two pieces $L_1$ and $L_2$. The surface $L_1$ is the portion of $L$ below the two reducing curves $\sigma_1$ and $\sigma_2$ that are isotopic to the two lower border leaves of $\hat{U}$. The surface $L_1$ is drawn in Figure 8. The map $h|L_1$ takes the juncture $J_n$ to $J_{n+1}$ for $n < 0$ and $n > 1$. On $L_1$, the map $h$ is a (non-obvious) translation. The other surface $L_2$ is a neighborhood of $\lambda_- \cup \lambda_+$ and thus has two positive and two negative trivial ends. This surface $L_2$ is a two legged chair (see Gabai [5, Example 5.1]). The map $h : L_2 \to L_2$ has fixed point $p$.

**Remark.** The surfaces $L_1$ and $L_2$ and the maps $h|L_1$ and $h|L_2$ are identical in both of the examples in Sections 2.2 and 2.3.

![Figure 8. $L_1$](image)

### 3. Fenley's Irreducible Example [3 Example 5]

This is an example of an irreducible, endperiodic map of a surface without boundary with one negative and one positive end. The map $f(x) = d_3 \circ d_2 \circ d_1 \circ t(x)$ is the composition of a translation and Dehn twists in the three circles labelled $d_1, d_2, d_3$, in Figure 9 (top). The resulting traintracks are given in Figure 9 (bottom). The laminations $\Lambda_-$ and $\Lambda_+$ are obtained by blowing air in at the switches of the train

![Figure 9. Fenley's Example [3 Example 5]](image)
tracks. There is one escaping end in $\Lambda_-$ and thus, there is one semi-isolated leaf $\lambda_+ \in \Lambda_+$. The negative escaping set has one component, contains a neighborhood of the negative end, has topological frontier $\Lambda_+$, and border $\lambda_+$. Similarly there is one escaping end in $\Lambda_+$ and one semi-isolated leaf $\lambda_- \in \Lambda_-$. The positive escaping set has one component, contains a neighborhood of the positive end, has topological frontier $\Lambda_-$, and border $\lambda_-$. The complement of $\Lambda_- \cup \Lambda_+$ is the escaping set $U$. That is, there are no principal regions. The non-rectangular components of $U$ consists of a family $\{C_n \mid n \in \mathbb{Z}\}$ of 12 sided regions where $C_n = f^n(C_0)$ for $n \in \mathbb{Z}$. The 12 sides of the border of $C_n$ are alternately in $\Lambda_-$ and $\Lambda_+$. The remainder of $U$ consists of infinitely many families of rectangles.

The positive and negative pre-Markov rectangles $R_\pm$ having opposite edges $\alpha_\pm, \beta_\pm \subset \Lambda_-$ and $\gamma_\pm, \delta_\pm \subset \Lambda_+$ are shown in Figure 10. The rectangle that opens to the left is $R_-$ and the rectangle that opens to the right is $R_+$. The Markov rectangles are the four components of $R_- \cap R_+$. The edges $\alpha_+, \beta_+$ are subsets of $\alpha_-$ which is the right edge of $R_-$ and the edges $\gamma_-, \delta_-$ are subsets of $\gamma_+$ which is the left edge of $R_+$. There is a unique orientation preserving homeomorphism $h$ sending the rectangle $R_-$ onto the rectangle $R_+$, satisfying $h(\alpha_-) = \alpha_+$, $h(\beta_-) = \beta_+$, $h(\gamma_-) = \gamma_+$, and $h(\delta_-) = \delta_+$, and preserving $\Lambda_\pm$ as in Figure 10. This homeomorphism determines $L$ and $h : L \to L$. As with the simple example in Section 2 from Figure 10 it is clear that there is a transverse invariant measure on $\Lambda_+$ with the transverse measure of $h^n(R_+ \cap \Lambda_+)$ equal $1/4^n$, $n \in \mathbb{Z}$. Similarly statements hold for $\Lambda_-$ and $R_-.$

3.1. A Reducible Example. The surface $L$ is a two ended ladder, the translation $t$ moves the juncture $J_i$ to the juncture $J_{i+1}$, and the two Dehn twists $d_1, d_2$ are as in Figure 9. The endperiodic map $f(x) = d_2 \circ d_1 \circ t(x)$. The resulting traintracks are given in Figure 11. The complement of $\Lambda_- \cup \Lambda_+$ is the escaping set $U$. That
is, there are no principal regions. There is one escaping leaf in $\Lambda_+$ and thus one semi-isolated leaf $\Lambda_-$ with fixed point $p$. There is one escaping leaf in $\Lambda_-$ and thus one semi-isolated leaf in $\Lambda_+$ with fixed point $q$. The one non-rectangular component of the escaping set $\mathcal{U}$ is homeomorphic to $(0, 1) \times \mathbb{R}$ and has two border components each of the second kind. There will be one reducing curve homotopic to both border components denoted by $\sigma$ in Figure 11.

The positive and negative pre-Markov rectangles $R_{\pm}$ with opposite edges

$$\alpha_{\pm}, \beta_{\pm} \subset \Lambda_- \quad \text{and} \quad \gamma_{\pm}, \delta_{\pm} \subset \Lambda_+$$

are shown in Figure 12. The rectangle that opens to the left is $R_-$ and the rectangle that opens to the right is $R_+$. The Markov rectangles are the three components of $R_- \cap R_+$. The edges $\alpha_+, \beta_+$ are subsets of $\alpha_-$ which is the right edge of $R_-$ and the edges $\gamma_-, \delta_-$ are subsets of $\gamma_+$ which is the left edge of $R_+$. There is a unique orientation preserving homeomorphism $h$ sending the rectangle $R_-$ onto the rectangle $R_+$, satisfying $h(\alpha_-) = \alpha_+$, $h(\beta_-) = \beta_+$, $h(\gamma_-) = \gamma_+$, and $h(\delta_-) = \delta_+$, and preserving $\Lambda_{\pm}$ as in Figure 12. This homeomorphism determines $L$ and $h : L \to L$. In the traintracks of Figure 11 the rectangle $R_-$ is represented by the edge $[p, r]$ which contains $q$ and the rectangle $R_+$ is represented by the edge $[q, r]$ that contains $p$.

4. Examples without Measure of Full Support

In contrast to the situation of homeomorphisms of compact surfaces, endperiodic homeomorphisms without measures of full support are common.
4.1. **Fenley’s Example without a Measure of Full Support** [4, Section 5]. Fenley gives a reducible example of an endperiodic map of a two ended non-planar surface where the laminations are not minimal. We give a reduced version of Fenley’s example. Let $L$ be a two ended surface with discs removed approaching both ends, $t$ a translation to the right by one unit, and $d_1, d_2$ Dehn twists in the dotted curves in Figure 13. Define $h = d_1^{-1} \circ d_2 \circ t$. Then the two laminations $\Lambda_{\pm}$ are not minimal. Details are given in Fenley [4].

![Figure 13. Dehn twists for Fenley’s Example with $\Lambda_{\pm}$ not Minimal](image)

4.2. **An Example with a Fixed Point on the Boundary.** Let $L$ be a two ended non-planar surface with a piece cut out from a neighborhood of the negative end as in Figure 14. Denote the boundary of $L$ by $\lambda_-$. If the endperiodic map is a “translation” $t$ that moves each handle to the right one unit, then the negative lamination will consist of $\lambda_-$, the positive lamination will consist of a half-infinite segment $\lambda_+$ with one end on $\lambda_-$, the other end approaching the positive end (see Figure 14), and $p = \lambda_- \cap \lambda_+$ the fixed point of $t$. If one composes $t$ with Dehn twists in the dotted curves in Figure 14 as in Fenley’s irreducible example (Section 3), the laminations are essentially the same as in Fenley’s irreducible example with the addition of the isolated half-leaf $\lambda_+$ to $\Lambda_+$ and the addition of the leaf $\lambda_-$ (which has both ends escaping and on which $\Lambda_- \setminus \{\lambda_-\}$ accumulates) to $\Lambda_-$. Thus neither of the laminations $\Lambda_{\pm}$ is minimal so there is no measure of full support.

![Figure 14. A Two Ended Non-planar Surface with Piece Cut Out](image)

4.3. **Double of the Previous Example.** If one doubles the surface $L$ of the previous example along its boundary one gets a surface with two positive ends and one negative end and an irreducible end-periodic map in which neither of the laminations $\Lambda_{\pm}$ is minimal, with the double of $\lambda_+$ an isolated leaf, and $\lambda_-$ having both ends escaping. The components of the negative and positive escaping sets $U_{\pm}$ are in one - one correspondence with the negative (resp. positive) ends.
Another Example. In this example the surface $L$ is an infinite strip with disks removed approaching both ends. The end-periodic map $h$ is the composition of a translation $t$ and Dehn twists in the two dotted circles as indicated in Figure 15. For more complicated examples one could do Dehn twists in any finite number of circles. Neither $\Lambda_{\pm}$ is minimal and there is no measure of full support. There will be one escaping leaf in each of $\Lambda_{\pm}$ and thus one semi-isolated leaf in each of $\Lambda_{\pm}$. The minimal positive lamination and the minimal negative lamination do not intersect.

5. Principal Regions and Primitive Examples

Definition 5.1. A primitive example of an endperiodic map is an endperiodic map in which $\Lambda_{\pm}$ each consist of finitely many isolated leaves.

If one takes a primitive example of an endperiodic map $t$ with principal regions and compose $t$ with some Dehn twists, the resulting endperiodic map will have basically the same principal regions as $t$. Many examples of endperiodic maps can be constructed by composing a primitive example of an endperiodic map with Dehn twists. The example in Section 5.5 is an example that can not be obtained by composing a primitive example with Dehn Twists.

5.1. An Example Whose Principal Regions have One Arm. Let $L$ be a surface with two non-planar ends and one disk removed. If $t$ is the “translation” that moves each handle forward one unit, then the negative lamination consists of one leaf $\lambda_-$ which is the boundary of the negative principal region enclosing the circle boundary of the deleted disk and the positive lamination consists of one leaf $\lambda_+$ which is the boundary of the positive principal region enclosing the circle boundary of the deleted disk as in Figure 16. If one composes $t$ with Dehn twists in the dotted curves in Figure 16 as in Fenley’s irreducible example (Section 3), the laminations are essentially the same as in Fenley’s irreducible example with the addition of the isolated $\Lambda_+$ (bounding the arm of the positive principal region which returns infinitely often to the core) to $\Lambda_+$ and the addition of the leaf $\lambda_-$ (bounding the escaping arm of the negative principal region $P_-$ and on which $\Lambda_- \setminus \{\lambda_-\}$ accumulates) to $\Lambda_-$. Neither of the laminations $\Lambda_{\pm}$ is minimal.

Remark. One can double along the circle boundary of $L$ and get a more complicated example.

5.2. An Irreducible Example with Two Negative Ends. Let $L$ be the surface in Figure 17 with two negative ends and one positive end. Let $t$ be the map that interchanges the two negative ends and takes the juncture $J_n$ to the juncture $J_{n+1}$ if
5.3. An Example whose Principal Regions have Two Arms. Let $L$ be the surface in Figure 17 but with a disk centered at the point $p$ removed and let $t$ be the map that interchanges the two negative ends and takes the juncture $J_n$ to the juncture $J_{n+1}$ if $n < 0$ or if $n > 0$. Then there are positive and negative principal regions with two arms. By composing $t$ with suitable Dehn twists in curves near the positive end, one can obtain an irreducible example such that the two arms of the negative principal region are escaping and the two arms of the positive principal region each return infinitely often to the core.

5.4. An Example whose Principal Regions have Three Arms. Let $L$ be the surface with one positive end and three negative ends and $t$ the endperiodic map that permutes the negative ends so that $t$ moves the handles along one unit in a neighborhood of the positive end and $t^3$ moves the handles along one unit in each neighborhood of the negative end. The endperiodic map $t$ is a primitive example and has one negative principal region with three arms and one positive principal region with three arms. The example is reducible. It reduces into four pieces. The main piece is a three legged chair (see Gabai [5, Example 5.1]). On the other three
pieces $t^3$ is a total translation. By composing $t$ with Dehn twists in suitable curves as in the previous example, one obtains an irreducible example with three negative ends such that the three arms of the negative principal region are escaping and the three arms of the positive principal region each return infinitely often to the core.

**Remark.** One can construct other, similar examples by,

1. Using more positive and/or negative ends and having the primitive example either permute or not permute the ends.
2. Deleting a disk from the nucleus of the principal regions and pasting in another example along the resulting circle boundary. For example, one could use the example in Section 5.1 or a compact surface with a boundary circle.

![Example Constructed without Using Dehn Twists](image.png)

**Figure 18.** Example Constructed without Using Dehn Twists

5.5. **An Example Constructed without Using Dehn Twists.** In Figure 18 the surface $L$ is obtained by pasting the segment $[x, y]$ on the left to the segment $[x, y]$ on the right. Let $T_a$ be the homeomorphism that moves the circle $a_n$ to the circle $a_{n+1}$, $-\infty < n < \infty$. Let $T_b$ be the homeomorphism that moves the circle
b_n to the circle b_{n+1}, -\infty < n < \infty. Define the endperiodic homeomorphism $h(x) = T_b \circ T_a(x)$. The positive lamination $\Lambda_+$ is given in boldface and the negative lamination $\Lambda_-$ is not in boldface. The endperiodic map $h(x)$ cannot be written as a composition of Dehn twists and a primitive example.

The positive lamination $\Lambda_+$ has three leaves, two of which are isolated. The third leaf is semi-isolated and approached on the non-isolated side by the other two leaves. There is a positive principal region $P_+$ with boundary these three leaves. Two arms of $P_+$ are escaping while the third arm which is bounded by the two isolated leaves approaches the semi-isolated leaf. Similar statements hold for the negative lamination $\Lambda_-$ and principal region $P_-$. In Figure 18, we have indicated the six vertices of $P_+ \cap P_-$ by dots. The two escaping arms of $P_+$ are the up and down boldface vertical line segments to the left. Similarly, the two escaping arms of $P_-$ are the up and down non-boldface vertical line segments to the right. The non-escaping arm of $P_+$ approaches the semi-isolated leaf in $\Lambda_+$ from the right. Similarly, the non-escaping arm of $P_-$ approaches the semi-isolated leaf in $\Lambda_-$ from the left.

6. Examples whose Ends are Repitions of Pairs of Pants

**Example 1.** This example is Example 1 of [1]. The orientation preserving “translation and flip” $t_f$ moves the juncture $J_i$ to the juncture $J_{i+1}$ and reverses the orientation of each juncture. The Dehn twists are given in Figure 19 and the endperiodic map $f(x) = d_2 \circ d_1 \circ t_f(x)$. The Dehn twist given in [1 Example 3] is not correct. The endperiodic map $f$ is the monodromy for a depth one foliation of the pretzel link $(2, 2, 2)$. Both $p$ and $q$ are periodic points of period 2 of $f(x)$.

**Example 2.** In this example we use the orientation preserving translation $t$ which moves the juncture $J_i$ to the juncture $J_{i+1}$. The endperiodic map $g(x) = d_2 \circ d_1 \circ t(x)$. Both $p$ and $q$ are fixed points of $g(x)$.

![Figure 19. Dehn Twists and Traintracks for Examples 1 and 2](image-url)
The laminations $\Lambda_{\pm}$ are the same in Examples 1 and 2. The complement of $\Lambda_{-} \cup \Lambda_{+}$ is the escaping set $U$, there are no principal regions, and the four non-rectangular components of $U$ are peripheral and unbounded. Two of the four non-rectangular components have border leaf of the first kind and two have border leaf of the second kind. In both examples, there are two semi-isolated leaves $\lambda_{1}^{-}, \lambda_{2}^{-} \subseteq \Lambda_{-}$, each with an end, $\eta_{1} \subseteq \lambda_{1}^{-}$ and $\eta_{2} \subseteq \lambda_{2}^{-}$ that does not return to the core. Therefore, there are only two semi-isolated leaves $\lambda_{1}^{-}, \lambda_{2}^{-} \subseteq \Lambda_{-}$.

The positive and negative pre-Markov rectangles $R_{\pm}$ has opposite edges $\alpha_{\pm}, \beta_{\pm} \subseteq \Lambda_{-}$ and $\gamma_{\pm}, \delta_{\pm} \subseteq \Lambda_{+}$ (Example 1) and are shown in Figure 20. The upper rectangle is $R_{-}$ and the lower rectangle is $R_{+}$. The Markov rectangles are the two components of $R_{-} \cap R_{+}$. The edges $\alpha_{+}$ (resp. $\beta_{+}$) are subsets of $\beta_{-}$ (resp. $\alpha_{-}$) and the edges $\gamma_{-}$ (resp. $\delta_{-}$) are subsets of $\delta_{+}$ (resp. $\gamma_{+}$). There is a unique homeomorphism $h$ which takes $p$ to $q$ (in Figure 19), preserves $\Lambda_{\pm}$, and sends the rectangle $R_{-}$ onto the rectangle $R_{+}$ satisfying $h(\alpha_{-}) = \alpha_{+}$, $h(\beta_{-}) = \beta_{+}$, $h(\gamma_{-}) = \gamma_{+}$, and $h(\delta_{-}) = \delta_{+}$. This homeomorphism determines $L$ and $h : L \to L$.

Figure 20 also gives the pre-Markov rectangles for Example 2. In that example, both the points $p$ and $q$ are fixed points. For Example 2, the labels on the edges $\alpha_{-}, \beta_{-}$ of $R_{-}$ must be interchanged so that $\alpha_{+} \subseteq \alpha_{-}$, $\beta_{+} \subseteq \beta_{-}$ and the labels on the edges $\gamma_{+}, \delta_{+}$ of $R_{+}$ must be interchanged so that $\gamma_{-} \subseteq \gamma_{+}$, and $\delta_{-} \subseteq \delta_{+}$. The unique homeomorphism $h : R_{-} \to R_{+}$, which fixes $p$ and $q$ and preserves $\Lambda_{\pm}$, completely determines $L$ and $h : L \to L$.

6.1. The General Situation. If a foliation has a pair of pants as compact leaf, an endperiodic end of a leaf $L$ at depth one can wind in on the pair of pants in complicated ways. In all these examples $L$ is orientable and the endperiodic map is orientation preserving.

![Figure 21. Junctures on a Pair of Pants](image-url)
The junctures $A, D$ on the pair of pants are given in Figure 21 and will have weight $m, n$ respectively. In the simple example (Section 2, $m = 1$ and $n = 0$. In Example 1, $m = 1$ and $n = 1$ and the pair of pants cut apart along the junctures is drawn in Figure 22 (left). In Example 2, $m = 1$ and $n = -1$ (compare with Figure 22 (right)). The symbolic picture in Figure 23 (left) can represent either the pair of pants cut apart along the junctures $A, D$ with weights $m = 1, n = 1$ as in Figure 22 (left) or the pair of pants cut apart along the junctures $A, D$ with weights $m = 1, n = -1$ as in Example 2. In the former case the surface represented symbolically has a twist.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig22.png}
\caption{Junctures $A, D$ with Weights 1, 1 (Left) and 2, $-3$ (Right)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{fig23.png}
\caption{Symbolic Picture of Surfaces in Figure 22}
\end{figure}

In Figure 22 (right), we draw the pair of pants cut apart along the junctures when the weights are $m = 2$ and $n = -3$ and in Figure 23 (right) we give symbolic pictures of this surface. Note that Figure 23 (right) represents the surface in Figure 22 (right). No twist is necessary. If we call the figure to the right in Figure 23 $X_0$ and let $X_i = f^i(X_0)$ and $U_i = \bigcup_{k=0}^{i} X_k$, $i \geq 0$, then Figure 24 represents the neighborhood $U_0$ of an end $e$ of the leaf $L$ of a foliation approaching the pair of pants with junctures $A, D$ weighted with $m = 2, n = -3$. Further, $f(U_i) = U_{i+1}$, $i \geq 0$, so the end $e$ is an attracting end [2, Definition 2.4] with the $X_i$, $i \geq 0$, fundamental domains for $e$ [2, Definition 2.6]. Unfortunately the $X_i$, $i \geq 1$, do not disconnect $U_0$. However if one takes $B_0 = X_0 \cup X_1 \cup X_2$, $B_n = f^{3n}(B_0)$, $J_0 = A_0 \cup A_1 \cup D_0 \cup D_1 \cup D_2$, and $J_n = f^{3n}(J_0)$, then the junctures $J_n$ and fundamental domains $B_n$ separate $U_0$, $n > 0$. In Figure 24, $B_0$ is represented with solid lines and $B_1$ represented by dashed lines and one can see that the junctures $J_n$ separate $U_0$, $n > 0$.

In [2] we want the junctures to separate a neighborhood of an end. The example in Figure 24 is why one must possibly choose a larger value of the integer $n_e$ [2, Lemma 2.5] to guarantee that the junctures separate.
Figure 24. Symbolic picture of End Approaching Pair of Pants with Junctures $A, D$ Weighted $m = 2, n = -3$

References

1. J. Cantwell and L. Conlon, Foliation cones, Geometry and Topology Monographs, Proceedings of the Kirbyfest, vol. 2, 1999, pp. 35–86.
2. , Handel-Miller theory and finite depth foliations, arXiv:1006.4525v1 [math.GT].
3. S. Fenley, Depth one foliations in hyperbolic 3-manifolds, Thesis, Princeton University, 1990.
4. , Endperiodic surface homeomorphisms and 3-manifolds, Math. Z. 224 (1997), 1–24.
5. D Gabai, Foliations and genera of links, Topology 23 (1984), 381–394.