On a "Type I Half–logistic Modified Weibull" Model: Some Extended Models and Applications

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**Abstract**

The cumulative distribution function (cdf) corresponding to the "four parameter extended type I half–logistic modified Weibull (TIHLMW) distribution" is [1]:

\[
M(t) = \frac{1 - e^{-\lambda(\alpha_1 t + \theta \beta_1)}}{1 + e^{-\lambda(\alpha_1 t + \theta \beta_1)}},
\]

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where $\lambda$, $\theta$, $\beta_1$ are positive shape parameters and $\alpha_1$ is a scale parameter. Also of interest to the specialists is the task of approximating the Heaviside function $h_{t_0}(t)$ where $t_0$ is the "median" by the new cumulative function in the Hausdorff sense. Following the results given in [2] we will generate the "new 6–parameters G–family of cdf – $Q(t)$ with baseline cdf – $M(t)$":

$$Q(t) = e^{-\frac{2\beta e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}}{1-e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}}} \left(2 - e^{-\frac{2\beta e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}}{1-e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}}}\right)^\alpha$$

We also study the "saturation" by this family and "confidential curves" $Q_1(t)$ and $Q_2(t)$ for which $Q_1(t) \leq Q(t) \leq Q_2(t)$.

Some numerical examples with real data from Biostatistics, Growth theory and Computer viruses propagation, using CAS MATHEMATICA illustrating our results are given.

It is shown that the study of the two characteristics - "confidential curves" and "super saturation" is a must when choosing the right model.

1 Introduction and Preliminaries

**Definition 1.** Consider the following cumulative distribution function (cdf) corresponding to the "Four parameter extended type I half-logistic modified Weibull (TIHLMW) distribution" [1]:

$$M(t) = \frac{1 - e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}}{1 + e^{-\lambda(\alpha_1 t + \theta t^\beta_1)}},$$

where $\lambda$, $\theta$, $\beta_1$ are positive shape parameters and $\alpha_1$ is a scale parameter.
Definition 2. The shifted Heaviside step function is defined by

\[ h_{t_0}(t) = \begin{cases} 
0, & \text{if } t < t_0, \\
[0, 1], & \text{if } t = t_0, \\
1, & \text{if } t > t_0 
\end{cases} \quad (2) \]

Definition 3. The Hausdorff distance (the H–distance) \( \rho(f, g) \) between two interval functions \( f, g \) on \( \Omega \subseteq \mathbb{R} \), is the distance between their completed graphs \( F(f) \) and \( F(g) \) considered as closed subsets of \( \Omega \times \mathbb{R} \). More precisely,

\[
\rho(f, g) = \max \{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \},
\]

wherein \( ||.|| \) is any norm in \( \mathbb{R}^2 \), e. g. the maximum norm \( ||(t, x)|| = \max\{|t|, |x|\} \); hence the distance between the points \( A = (t_A, x_A), B = (t_B, x_B) \) in \( \mathbb{R}^2 \) is \( ||A - B|| = \max(|t_A - t_B|, |x_A - x_B|) \).

Definition 4. In [2] Bantan, Jamal, Chesneau and Elgarhy introduced a new power family of distributions with c.d.f.

\[
Q(t) = e^{\alpha\beta(1 - \frac{1}{G(t)})} \left(2 - e^{\beta(1 - \frac{1}{G(t)})}\right)^\alpha \quad (3)
\]

where \( \alpha, \beta \in \mathbb{R}^+ \) and \( G(t) \) is a c.d.f. of a baseline continuous distribution. The following result shows some inequalities involving \( Q(t) \) (see, Proposition 1 [2]):

\[
e^{\alpha\beta(1 - \frac{1}{G(t)})} \left(2 - G(t)^\beta\right)^\alpha \leq Q(t) \leq 2^\alpha e^{\alpha\beta(1 - \frac{1}{G(t)})}. \quad (4)
\]

In this paper we study some properties of the cdf – \( M(t) \) and the family (3) with baseline cdf – \( G(t) = M(t) \).
2 Main Results.

When studying the intrinsic properties of the families $M(t)$ and $Q(t)$ (with baseline cdf – $M(t)$), it is also appropriate to study the ”saturation” to the horizontal asymptote.

2.1 The cdf $M(t)$.

In this Section we study the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_0}(t)$ by means of family (1).

Let $t_0$ is a positive root of the nonlinear equation $M(t_0) - \frac{1}{2} = 0$.

The one–sided Hausdorff distance $d$ satisfies the relation

$$M(t_0 + d) = \frac{1 - e^{-\lambda(a_1(t_0+d)+\theta(t_0+d)^{\beta_1})}}{1 + e^{-\lambda(a_1(t_0+d)+\theta(t_0+d)^{\beta_1})}} = 1 - d.$$  \(5\)

We illustrate the ”saturation” with the cdf (1) for various $\alpha_1, \beta_1, \lambda, \theta$ (see, Fig. 1)

2.1.1 Some Applications.

Example 1. Here we will present a new analysis of Conficker propagation in 2008 and we explore the Network Telescope project’s daily dataset [4], [5] collected on November 21, 2008.

We analyze the following data

```
data_Conficker :=

{{0.1, 10}, {1, 150}, {2, 300}, {3, 600}, {4, 2500}, {5, 5000},
{6, 7500}, {7, 13000}, {8, 19000}, {9, 25000}, {10, 31000},
{11, 37000}, {12, 44000}, {13, 52000}, {14, 58000}, {15, 66000},
{16, 74000}, {17, 81000}, {18, 86000}, {19, 89000}, {20, 92000},
{21, 92500}}
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Figure 1: a) $\alpha_1 = 2.2$, $\beta_1 = 6$, $\lambda = 0.2$, $\theta = 0.9$; $t_0 = 1.2104$; Hausdorff distance $d = 0.2133056$; b) $\alpha_1 = 1.2$, $\beta_1 = 12$, $\lambda = 0.1$, $\theta = 0.6$; $t_0 = 1.25856$; Hausdorff distance $d = 0.112747$; c) $\alpha_1 = 1.1$, $\beta_1 = 20$, $\lambda = 0.05$, $\theta = 0.4$; $t_0 = 1.21794$; Hausdorff distance $d = 0.0714225$. 
The $M^*(t) = \omega M(t)$ for $\omega = 92500; \alpha_1 = 0.05; \beta_1 = 2.69138; \lambda = 0.1$ and $\theta = 0.0123854$ is visualized on Fig. 2.

**Example 2.** We analyze the data given in [6].

The $M^*(t) = \omega M(t)$ for $\omega = 0.98; \alpha_1 = 0.01; \beta_1 = 2.55582; \lambda = 0.4136371$ and $\theta = 16,6566$ is visualized on Fig. 3.
2.2 Some properties of the family (3) with baseline cdf $M(t)$.

Formally, we will generate the "new" cdf $Q(t)$ with baseline cdf $M(t)$:

$$Q(t) = e^{-\frac{2\alpha\beta e^{-\lambda(\alpha_1 t + \theta \beta_1)}}{1-e^{-\lambda(\alpha_1 t + \theta \beta_1)}}} \left(2 - e^{-\frac{2\beta e^{-\lambda(\alpha_1 t + \theta \beta_1)}}{1-e^{-\lambda(\alpha_1 t + \theta \beta_1)}}}\right)$$  \hspace{1cm} (6)

**Example 3.** We examine the data for the growth of red abalone *Haliotis Rufescens* in Northern California [7].

For this data the fitted model $Q^*(t) = \omega Q(t)$ for $\omega = 194; \alpha_1 = 0.6; \beta_1 = 0.2; \lambda = 0.361; \theta = 2.3; \alpha = 0.0496998$ and $\beta = 57.7664$ is visualized on Fig. 4.

We study the Hausdorff approximation of the Heaviside step function $h_{t_0}(t)$ where $t_0$ is the "median" by family of type (6).

Following the ideas given in [2] we find the two–sided bounds:

$$Q_1(t) \leq Q(t) \leq Q_2(t)$$  \hspace{1cm} (7)

where
Figure 5: a) The two-sided bounds (7) for $\alpha_1 = 1.2; \beta_1 = 7; \lambda = 0.1; \theta = 0.6$ and $\alpha = 0.5; \beta = 1$; b) The model $Q(t)$ for $\alpha_1 = 1.2; \beta_1 = 7; \lambda = 0.1; \theta = 0.6$ and $\alpha = 0.5; \beta = 1$; H-distance $d = 0.153005$.

$$Q_1(t) = e^{-2\alpha_1 \beta_1 - \lambda(\alpha_1 t + \theta \beta_1)} \left( 2 - \frac{1 - e^{-\lambda(\alpha_1 t + \theta \beta_1)}}{1 + e^{-\lambda(\alpha_1 t + \theta \beta_1)}} \right)^{\alpha}$$

$$Q_2(t) = 2^{\alpha} e^{-2\alpha_1 \beta_1 - \lambda(\alpha_1 t + \theta \beta_1)} \left( 2 - \frac{1 - e^{-\lambda(\alpha_1 t + \theta \beta_1)}}{1 + e^{-\lambda(\alpha_1 t + \theta \beta_1)}} \right)^{\beta}.$$

The obtained two-sided estimations in particular case for $\alpha_1 = 1.2; \beta_1 = 7; \lambda = 0.1; \theta = 0.6$ and $\alpha = 0.5; \beta = 1$ are given in Fig. 5 a.

Let $t_0$ is the value for which $Q(t_0) = \frac{1}{2}$.  

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The Hausdorff distance $d$ between the function $h_{t_0}(t)$ and $Q(t)$ satisfies the relation

$$Q(t_0 + d) = 1 - d.$$  \hspace{1cm} (8)

For fixed $\alpha_1 = 1.2; \beta_1 = 7; \lambda = 0.1; \theta = 0.6$ and $\alpha = 0.5; \beta = 1$ from the nonlinear equation (8) we have $d = 0.153005$ (see, Fig. 5 b).

### 2.3 Concluding remarks.

The results obtained in this article can be successfully continued for generating of some new models based on known in literature ”G-families of cumulative distribution functions”.

The new model (6) has been applied widely in life testing experiments.

From Fig. 5 it can be seen that these estimations can be used as ”confidence bounds”, which are extremely useful for the specialists in the choice of model for cumulative data approximating in areas of Biostatistics, Population dynamics, Growth theory, Debugging and Test theory, Computer viruses propagation, Financial and Insurance mathematics.

Exploring both features - ”confidential curves” and ”super saturation” is a must when choosing the right model.

For other results, see [8]–[35].

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