Carleman estimate for the Navier–Stokes equations and an application to a lateral Cauchy problem

Mourad Bellassoued, Oleg Imanuvilov and Masahiro Yamamoto

1 University of Tunis El Manar, National Engineering School of Tunis, ENIT-LAMSIN, B.P. 37, 1002 Tunis, Tunisia
2 Department of Mathematics, Colorado State University 101 Weber Building, Fort Collins, CO 80523-1874, USA
3 Graduate School of Mathematical Sciences, The University of Tokyo 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

E-mail: mourad.bellassoued@fsb.rnu.tn, oleg@math.colostate.edu and myama@ms.u-tokyo.ac.jp

Received 4 July 2015, revised 2 November 2015
Accepted for publication 1 December 2015
Published 5 January 2016

Abstract

We consider the nonstationary linearized Navier–Stokes equations in a bounded domain and first we prove a Carleman estimate with a regular weight function. Second we apply the Carleman estimate to a lateral Cauchy problem for the Navier–Stokes equations and prove the Hölder stability in determining the velocity and pressure field in an interior domain. In the final section, we apply the results for the linearized Navier–Stokes equations to the fully nonlinear Navier–Stokes equations and establish a similar Hölder stability estimate within sufficiently smooth solutions, and prove the uniqueness of Leray–Hopf weak solutions by surface stresses on an arbitrarily chosen sub-boundary.

Keywords: Carleman estimate, Navier–Stokes equations, Cauchy problem

1. Introduction and the main Carleman estimate

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with smooth boundary $\partial \Omega$ (e.g., of $C^2$-class), and let $\nu = \nu(x)$ be the outward unit normal vector on $\partial \Omega$ at $x$. We set $Q := \Omega \times (0, T)$. 
We consider the fully nonlinear Navier–Stokes equations (1.1) and (1.2) and the linearized Navier–Stokes equations (1.2) and (1.3) for an incompressible viscous fluid:

\[
\begin{align*}
\partial_t v(x, t) - \kappa \Delta v(x, t) + (v \cdot \nabla)v + \nabla p &= F(x, t), \quad (x, t) \in Q, \\
\text{div} v(x, t) &= 0, \quad (x, t) \in Q,
\end{align*}
\]

(1.1) (1.2)

\[
\partial_t v(x, t) - \kappa \Delta v(x, t) + (A \cdot \nabla)v + (v \cdot \nabla)B + \nabla p &= F(x, t), \quad (x, t) \in Q.
\]

(1.3)

Here \( v = (v_1, \ldots, v_n)^T, n = 2, 3, 7 \) denotes the transpose of matrices, \( \kappa > 0 \) is a constant describing the viscosity, and for simplicity we assume that the density is one. Let \( \partial_j = \frac{\partial}{\partial x_j}, 1 \leq j \leq n, \Delta = \sum_{j=1}^n \partial_j^2, \nabla = (\partial_1, \ldots, \partial_n)^T, \nabla_{x,t} = (\nabla, \partial_t)^T, \partial_{ij} = \partial_i \partial_j = \partial_i^j, \partial_\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \) with \( \beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N} \cup \{0\})^n, |\beta| = \beta_1 + \cdots + \beta_n \),

\[
(w \cdot \nabla)v = \left( \sum_{j=1}^n w_j \partial_j v_1, \ldots, \sum_{j=1}^n w_j \partial_j v_n \right)^T
\]

for \( v = (v_1, \ldots, v_n)^T \) and \( w = (w_1, \ldots, w_n)^T \). Whenever we consider (1.3), we assume

\[
A \in W^{2,\infty}(Q), \quad \nabla B \in L^\infty(Q).
\]

(1.4)

In this paper, we establish a Carleman estimate for the linearized Navier–Stokes equations with a regular weight function and apply it to lateral Cauchy problems for the Navier–Stokes equations (1.1), (1.2) and (1.2), (1.3), and prove the Hölder stability in an arbitrarily given interior domain. For stating the main results, we introduce notations. Let \( I_n \) be the \( n \times n \) identity matrix and let the stress tensor \( \sigma(v, p) \) be defined by the \( n \times n \) matrix

\[
\sigma(v, p) = \kappa (\nabla v + (\nabla v)^T) - p I_n.
\]

Let a function \( d = d(x) \) satisfy

\[
d \in C^2(\overline{\Omega}), \quad |\nabla d(x)| > 0 \text{ on } \overline{\Omega}.
\]

(1.5)

We arbitrarily choose

\[
t_0 \in (0, T), \quad \beta > 0.
\]

(1.6)

We set

\[
\psi(x, t) = d(x) - \beta(t - t_0)^2, \quad \varphi(x, t) = e^{\lambda(t - t_0)}
\]

(1.7)

with a sufficiently fixed large constant \( \lambda > 0 \). We choose a non-empty relatively open sub-boundary \( \Gamma \subset \partial \Omega \) arbitrarily.

**Remark 1.** The choice of \( d \) satisfying (1.5) is very generous. One simple choice is \( d(x) = |x - x_0|^2 \) with fixed \( x_0 \not\in \mathbb{R}^n \setminus \overline{\Omega} \). However this choice does not yield the conditional stability for the unique continuation by data of \( (v, p) \) on \( \Gamma \times (0, T) \) with an arbitrarily fixed sub-boundary \( \Gamma \).

Let \( D \subset Q \) be a bounded domain with smooth boundary \( \partial D \) such that \( \partial D \cap (\partial \Omega \times (0, T)) = \Gamma \times (t_0, t_1) \) where \( t_0, t_1 \) are some constants satisfying \( 0 \leq t_0 < t_1 \leq T \).
For $k, \ell \in \mathbb{N} \cup \{0\}$ and $s > 0$, we set
\[ H^{k,\ell}(D) = \{ v \in L^2(D); \partial_{\beta}^j v \in L^2(D), |\beta| \leq k, \partial_j^\ell v \in L^2(D) \quad 0 \leq j \leq \ell \} \]
and
\[ \| (v, p) \|_{\mathcal{H}(\Omega)}^2 := \int_D \left\{ \frac{1}{s^2} \left( |\partial_{\beta} v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) \right. \]
\[ + |\nabla v|^2 + s^2 |v|^2 + \frac{1}{s} |\nabla p|^2 + s |p|^2 \left. \right\} e^{2sv} dx dt. \]

We are ready to state our Carleman estimate.

**Theorem 1.** There exist constants $s_0 > 0$ and $C > 0$, independent of $s$, such that
\[ \| (v, p) \|_{\mathcal{H}(\Omega)}^2 \leq C \int_D |F|^2 e^{2sv} dx dt + C \int_D \left( |h|^2 + |\nabla_x h|^2 \right) e^{2sv} dx dt \]
\[ + Ce^C s \left( \| v \|^2_{L^2(0,T;H^2(\Gamma))} + \| \partial_t v \|^2_{L^2(0,T;H^1(\Gamma))} + \| (v, p) \|^2_{L^2(0,T;H^1(\Gamma))} \right) \]
for all $s \geq s_0$ and $(v, p) \in H^{2,1}(D) \times H^{1,0}(D)$ satisfying (1.3),
\[ \text{div } v = h \quad \text{in } D \quad \text{with } h \in H^{1,1}(D), \]
and
\[ \left\{ \begin{array}{l}
 v(\cdot, 0) = v(\cdot, T) = 0 \quad \text{in } \Omega, \\
 |v| = |\nabla v| = |p| = 0 \quad \text{on } \partial D \setminus (\Gamma \times (0, T)).
\end{array} \right. \]

This is a Carleman estimate for the linearized Navier–Stokes equation (1.3) with boundary data on $\Gamma \subset \partial \Omega$.

Boulakia [2] proved a Carleman estimate with a weight function similar to ours for the homogeneous Stokes equations: $\partial_t v = \Delta v - \nabla p$ and $\text{div } v = 0$ with extra interior or boundary data. The Carleman estimate in [2] requires a stronger norm of boundary data than our Carleman estimate if it is applied to the case of the Stokes equations.

As for other Carleman estimates for the Navier–Stokes equations, we refer to Choulli *et al* [3], Fernández-Cara *et al* [6], where the authors use a weight function in the form
\[ \exp \left( \frac{2w(x)}{t(T-t)} \right) \]
with some regular function $w$. Thanks to that the weight function (1.10) vanishes at $t = 0, T$ with arbitrary $t$-derivatives, their Carleman estimates hold over the whole domain $Q$ for $v$ satisfying $v = 0$ on $\partial D$ but not necessarily $v(\cdot, 0) = v(\cdot, T) = 0$ in $\Omega$. The Carleman estimate with the weight in the form of (1.10) is convenient for proving the Lipschitz stability for an inverse source problem (e.g., [3]) and the exact null controllability [6]. On the other hand, since we have to assume the boundary values $v = 0$ on the whole boundary $\partial \Omega$, it is not suitable for proving the unique continuation. Moreover the weight (1.10) is never relevant for hyperbolic types of equations, in particular, for the mass conservation equation.
As for Carleman estimates for the Navier–Stokes equations, see also Fan et al [4] and Fan et al [5] with extra data in a neighborhood of the whole boundary, which is too much considering the parabolicity of the equations.

**Remark 2.** For a parabolic equation, Carleman estimates with the weights (1.7) and (1.10) of weight functions are already available. See e.g., Fursikov and Imanuvilov [7], Imanuvilov [9] as for (1.10), and theorem 3.1 in Yamamoto [27] as for (1.7). Our proof of theorem 1 is based on a parabolic Carleman estimate with the weight function (1.7). We note that the Carleman estimate with the weight (1.7) and any \( \sum_{j=1}^n a_{ij}(x, t) \frac{\partial}{\partial x_j} u - \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j} u - c(x, t) u = F \).

Here \( a_{ij} = a_{ji} \in C^1(\bar{Q}), \ b_j, c \in L^\infty(\bar{Q}), \ 1 \leq i, j \leq n \) and we assume \( \sum_{j=1}^n a_{ij}(x, t) \xi_j > 0 \) for any \( x \in \bar{Q} \) and \( \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \) such that \( |\xi| = 1 \). This difference between the Carleman estimates with (1.7) and (1.10) is the same also for the Navier–Stokes equations. The advantages and the disadvantages of the Carleman estimate with (1.10) are described as follows.

**Advantage.** We need not assume that \( u(\cdot, 0) = u(\cdot, t_0) = 0 \) in \( \Omega \). Thus it is more direct to prove the Lipschitz stability for an inverse source problem and the related exact controllability problem (e.g., [9, 27]). Moreover by the Carleman estimate with (1.10), we can prove the Lipschitz stability in determining \( u(\cdot, t_0) \) with arbitrary \( t_0 \in (0, T) \) by \( \nabla u \) on \( \Gamma \times (0, T) \) with arbitrarily chosen sub-boundary, provided that we know the boundary value on \( \partial \Omega \times (0, T) \).

**Disadvantages.**

1. Since we have to assume the complete boundary value on \( \partial \Omega \times (0, T) \), the Carleman estimate with (1.10) is not directly applied to the unique continuation where we do not know the boundary condition perfectly.
2. It does not hold for hyperbolic equations.

This paper is composed of four sections. In section 2, we prove the key Carleman estimate. Sections 3 and 4 are devoted to the lateral Cauchy problems for the linearized and the fully nonlinear Navier–Stokes equations, respectively.

**2. Proof of theorem 1**

**First step**

Let \( E \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial E \) and let \( E_0 = \{ x \in E; \text{dist} (x, \partial E) > \delta \} \) with small \( \delta > 0 \).

We prove

**Lemma 1.** Let \( p \in H^1(E) \) satisfy

\[
\Delta p = f_0 + \sum_{j=1}^n \partial_j f_j \quad \text{in } E
\]

and \( \text{supp } p \subset E_0 \). Let \( d_0 \in C^1(\mathbb{R}^n) \) satisfy \( d_0(x) \geq 0 \) for \( x \in E \) and \( |\nabla d_0(x)| > 0 \) for \( x \in \partial E \). We set \( \varphi_0(x) = e^{\lambda d_0(x)} \) with large constant \( \lambda > 0 \). Then there exist constants \( C > 0 \) and
\[ s_1 > 0 \text{ such that} \]
\[
\int_E \left( \frac{1}{s} |\nabla p|^2 + s |p|^2 \right) e^{2\gamma_0(x)} \, dx \leq C \int_E \left( \frac{1}{s^2} |f_0|^2 + \sum_{j=1}^n |f_j|^2 \right) e^{2\gamma_0(x)} \, dx
\]
for all \( s \geq s_1 \). The constants \( C \) and \( s_1 \) are independent of choices of \( p \).

**Proof.** Since \( \partial E \) is of \( C^3 \)-class, we choose a function \( \mu \in C^3(\mathbb{R}^n) \) such that \( 0 \leq \mu \leq 1 \), \( \mu > 0 \) in \( E \) and \( \mu = 1 \) in \( \mathbb{R}^n \setminus E \). We set \( \tilde{d}_0(x) = \mu(x)d_0(x) \) and \( \gamma_0(x) = e^{\gamma_0_0(x)} \) for \( x \in E \). Then \( \tilde{d}_0(x) = 0 \) for \( x \in \partial E \) and \( \tilde{d}_0 > 0 \), \( |\nabla \tilde{d}_0| = |\mu \nabla d_0 + d_0 \nabla \mu| = |\mu \nabla d_0| > 0 \) in \( E \). Hence the \( H^{-1} \)-Carleman estimate for an elliptic operator by Imanvillov and Puel \([10]\) yields

\[
\int_E \left( \frac{1}{s} |\nabla p|^2 + s |p|^2 \right) e^{2\gamma_0(x)} \, dx \leq C \int_E \left( \frac{1}{s^2} |f_0|^2 + \sum_{j=1}^n |f_j|^2 \right) e^{2\gamma_0(x)} \, dx
\]
for all \( s \geq s_1 \). Here we note that in theorem 1.2 in \([10]\), we set \( \omega = E \setminus E_\delta \) and use \( p \big|_{\partial E} = 0 \). Since \( p = 0 \) in \( E \setminus E_\delta \) and \( \tilde{d}_0 = d_0 \) in \( E_\delta \), we complete the proof of lemma 1. \( \square \)

**Lemma 2.** There exist constants \( s_0 > 0 \) and \( C > 0 \) such that

\[
\| (v, p) \|_{W^1_2(Q)} \leq C \int_Q |F|^2 e^{2\gamma_0} \, dx \, dt + C \int_Q \left( |h|^2 + |\nabla h|^2 \right) e^{2\gamma_0} \, dx \, dt
\]
for all \( s \geq s_0 \) and \( (v, p) \in H^{2,1}(Q) \times H^{1,0}(Q) \) satisfying (1.1),

\[
v(\cdot, 0) = v(\cdot, T) = 0 \quad \text{in} \quad \Omega,
\]

\[
|v| = |\nabla v| = |p| = 0 \quad \text{in} \quad \partial\Omega \times (0, T),
\]

and

\[
\text{div} v = h \quad \text{in} \quad Q
\]

with some \( h \in H^{1,1}(Q) \).

**Proof of lemma 2.** Thanks to the large parameter \( s > 0 \), in view of (1.4), it is sufficient to prove lemma 2 for \( B = 0 \) in (1.1). In fact, the Carleman estimate with \( B = 0 \) follows from the case of \( B = 0 \) by replacing \( F \) by \( F - (v \cdot \nabla)B \) and estimating \( |(F - (v \cdot \nabla)B)(x, t)| \leq |F(x, t)| + C|v(x, t)| \) for \( (x, t) \in Q \). Then, choosing \( s_0 > 0 \) large, we can absorb the term \( \int_Q |v|^2 e^{2\gamma_0} \, dx \, dt \) into the left-hand side of the Carleman estimate.

By the density argument, it is sufficient to prove the lemma for \( (v, p) \) such that supp \( v \) and supp \( p \) are compact in \( Q \). We consider

\[
\partial_t v = \kappa \Delta v - (A \cdot \nabla)v - \nabla p + F
\]
and

\[
\text{div} v = h \quad \text{in} \quad Q.
\]
Taking the divergence of (2.2) and using (2.3), we obtain
\[ \Delta p = - \sum_{j,k=1}^{n} \left\{ \partial_j \left( \partial_k A_j v_k \right) - \left( \partial_j \partial_k A_j \right) v_k \right\} + \text{div } F - \partial_t h - (A \cdot \nabla) h + \kappa \text{ div } (\nabla h) \quad \text{in } Q. \] (2.4)

Here we used
\[ \text{div } ((A \cdot \nabla) v) = \sum_{j,k=1}^{n} \partial_k \left( A_j \partial_j v_k \right) = \sum_{j=1}^{n} A_j \partial_j \left( \sum_{k=1}^{n} \partial_k v_k \right) + \sum_{j,k=1}^{n} \left( \partial_k A_j \right) \partial_j v_k \]
\[ = A \cdot \nabla (\text{div } v) + \sum_{j,k=1}^{n} \left\{ \partial_j \left( \partial_k A_j v_k \right) - \left( \partial_j \partial_k A_j \right) v_k \right\}. \] (2.5)

Moreover on the right-hand side of (2.4), the term \( \kappa \text{ div } (\nabla h) \) is not in \( L^2(Q) \) because we assume only \( h \in H^1(Q) \). Thus we cannot apply a usual Carleman estimate requiring \( \Delta p \in L^2(Q) \), and we need the \( -H^1 \)-Carleman estimate.

By a usual density argument, we can assume that \( \text{supp } p \subset Q \). By \( \text{supp } p \subset Q \), fixing \( t \in [0, T] \), we apply lemma 1 to (2.4) and obtain
\[ \int_{\Omega} \left( \frac{1}{s} |\nabla p(x, t)|^2 + s |p(x, t)|^2 \right) e^{2s \varphi(x, t)} \, dx \leq C \int_{\Omega} \left( |F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2 \right) e^{2s \varphi(x, t)} + C \int_{\Omega} |v(x, t)|^2 e^{2s \varphi(x, t)} \, dx \] (2.6)
for \( s \geq s_1 \) where \( s_1 > 0 \) is a sufficiently large constant.

Let \( s_0 := s_1 e^{\lambda \beta T^2} \). Here we note (1.6). Then, \( s \geq s_0 \) implies
\[ se^{-\lambda \beta (t-t_0)^2} \geq se^{-\lambda \beta T^2} \geq s_1 \]
for \( 0 \leq t \leq T \), so that for fixed \( t \in [0, T] \) by replacing \( s \) by \( se^{-\lambda \beta (t-t_0)^2} \), by (2.5) we can see
\[ \int_{\Omega} \left( \frac{1}{s} |\nabla p(x, t)|^2 + s |p(x, t)|^2 \right) \exp \left( 2 \left( se^{-\lambda \beta (t-t_0)^2} \right) \varphi(x, t_0) \right) \, dx \]
\[ \leq C \int_{\Omega} \left( |F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2 \right) \exp \left( 2 \left( se^{-\lambda \beta (t-t_0)^2} \right) \varphi(x, t_0) \right) \, dx \]
\[ + C \int_{\Omega} |v(x, t)|^2 \exp \left( 2 \left( se^{-\lambda \beta (t-t_0)^2} \right) \varphi(x, t_0) \right) \, dx, \]
that is
\[ \int_{\Omega} \left( \frac{1}{s} |\nabla p(x, t)|^2 + s |p(x, t)|^2 \right) e^{2s \varphi(x, t)} \, dx \]
\[ \leq C \int_{\Omega} \left( |F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2 \right) e^{2s \varphi(x, t)} + C \int_{\Omega} |v(x, t)|^2 e^{2s \varphi(x, t)} \, dx \]
for \( s \geq s_0 \) and \( 0 \leq t \leq T \). Integrating this inequality in \( t \) over \( (0, T) \), we have
\[ \int_{Q} \left( \frac{1}{s} |\nabla p|^2 + s |p|^2 \right) e^{2s \varphi} \, dx \, dt \]
\[ \leq C \int_{Q} \left( |F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2 \right) e^{2s \varphi} \, dx \, dt + C \int_{Q} |v|^2 e^{2s \varphi} \, dx \, dt \] (2.7)
for all \( s \geq s_0 \).
Next, regarding $F - \nabla p$ in (2.2) as non-homogeneous term, we apply a Carleman estimate for the parabolic operator $\partial_t v - \kappa \Delta v + (A \cdot \nabla) v$ (e.g., theorem 3.1 in Yamamoto [27]) to (2.2):

$$\frac{1}{s} \int_Q \left\{ \frac{1}{s} \left[ |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right] + s |\nabla v|^2 + s^3 |v|^2 \right\} e^{2s\tau} dx dt \leq C \int_Q \frac{1}{s} |\nabla p|^2 e^{2s\tau} dx dt + C \int_Q \frac{1}{s} |F|^2 e^{2s\tau} dx dt. \quad (2.8)$$

Substituting (2.7) into (2.8), we obtain

$$\int_Q \left\{ \frac{1}{s^2} |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right\} e^{2s\tau} dx dt \leq C \int_Q |F|^2 e^{2s\tau} dx dt + C \int_Q \left( |\partial h|^2 + |\nabla h|^2 + |h|^2 \right) e^{2s\tau} dx dt$$

$$+ C \int_Q |v|^2 e^{2s\tau} dx dt + \frac{C}{s} \int_Q |F|^2 e^{2s\tau} dx dt.$$

Choosing $s_0 > 0$ large, we can absorb the third term on the right-hand side into the left-hand side, again with (2.7), we complete the proof of lemma 2.

Second step

Without loss of generality, we can assume that $d > 0$ in $\Omega$ because we replace $d$ by $d + C_0$ with large constant $C_0 > 0$ if necessary.

In this step, we will prove

**Lemma 3.** There exist constants $s_0 > 0$ and $C > 0$ such that

$$\|v, p\|_{L^\infty(D)}^2 \leq C \int_D |F|^2 e^{2s\tau} dx dt + C \int_D \left( |h|^2 + |\nabla_c h|^2 \right) e^{2s\tau} dx dt$$

$$+ C e^{C_0 \left( \|v\|^2_{L^2(0,T;H^1(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^2(\Gamma))} + \|\partial_i v\|^2_{L^2(0,T;H^2(\Gamma))} + \|p\|^2_{L^2(0,T;H^2(\Gamma))} \right)}$$

for all $s \geq s_0$ and $(v, p) \in H^2(\Omega) \times H^1(\Omega)$ satisfying (1.3), (1.9) and

$$\text{div} \ v = h \quad \text{in} \ D. \quad (2.9)$$

**Proof of lemma 3.** We take the zero extensions of $v, p, A, F$ to $Q$ from $D$ and by the same letters we denote them:

$$v = \begin{cases} v & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases} \quad p = \begin{cases} p & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases}$$

By (1.9) we easily see that

$$\partial_t v = \begin{cases} \partial_t v & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases} \quad \partial_t p = \begin{cases} \partial_t p & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases}$$

$$\partial_i \partial_j v = \begin{cases} \partial_i \partial_j v & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases}$$

$$\partial_i \partial_j p = \begin{cases} \partial_i \partial_j p & \text{on } D, \\ 0 & \text{in } Q \setminus D, \end{cases}$$

$$\text{div} \ v = h \quad \text{in} \ D.$$
and
\[ \partial_p = \begin{cases} \partial_p, & \text{on } \partial \mathcal{D}, \\ 0, & \text{in } Q \setminus D \end{cases} \]
for \( 1 \leq i, j \leq n \). Moreover, since \( v = 0 \) on \( \partial D \setminus (\Gamma \times (0, T)) \) by (1.9), setting
\[ h = \begin{cases} h, & \text{on } \partial \mathcal{D}, \\ 0, & \text{in } Q \setminus D \end{cases}, \]
we see that \( h \in H^{1,1}(Q) \) and \( \text{div } v = h \) in \( Q \) (2.10) and
\[ \partial_t v = \kappa \Delta v + (A \cdot \nabla)v + \nabla p + F \quad \text{in } Q. \quad (2.11) \]

By the Sobolev extension theorem, there exist \( \tilde{p} \in L^2(0, T; H^1(\Omega)) \) and \( \tilde{v} \in H^{2,1}(Q) \) such that
\[ \begin{cases} \tilde{v} = v, \partial_t \tilde{v} = \partial_t v, \tilde{p} = p & \text{on } \partial \Omega \times (0, T), \\ \text{supp } \tilde{v}(x, \cdot ) \subset (0, T) & \text{for almost all } x \in \Omega \end{cases} \quad (2.12) \]
and
\[ \| \tilde{v} \|_{H^{2,1}(Q)} + \| \partial_t \tilde{v} \|_{L^2(0, T; H^1(\Omega))} + \| p \|_{L^2(0, T; H^1(\Omega))} \leq C \left( \| v \|_{L^2(0, T; H^2(\Gamma))} + \| v \|_{H^1(0, T; H^2(\Gamma))} + \| p \|_{L^2(0, T; H^2(\Gamma))} \right). \quad (2.13) \]
The last condition in (2.12) can be seen by \( v(\cdot, 0) = v(\cdot, T) = 0 \) in \( \Omega \) which follows from (1.9).

We set
\[ u = v - \tilde{v}, \quad q = p - \tilde{p} \quad \text{in } Q. \]
Then, in view of (2.10)–(2.12), we have
\[ |u| = |\nabla u| = |q| = 0 \quad \text{on } \partial \Omega \times (0, T) \quad (2.14) \]
and
\[ \partial_t u - \kappa \Delta u + \nabla q + (A \cdot \nabla)u = F - \left( \partial_t \tilde{v} - \kappa \Delta \tilde{v} + (A \cdot \nabla)\tilde{v} + \nabla \tilde{p} \right) = G \quad \text{in } Q, \quad (2.15) \]
\[ \text{div } u = h - \text{div } \tilde{v} \in H^{1,1}(Q). \quad (2.16) \]

We choose a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \) such that \( \Omega \supset \Omega \), \( \Gamma = \partial \Omega \cap \Omega \) and \( \partial \Omega \cap \overline{\Omega} = \partial \Omega \setminus \Gamma \). In other words, the domain \( \Omega \) is constructed by expanding \( \Omega \) only over \( \Gamma \) to the exterior such that the boundary \( \partial \Omega \) is smooth. We set
\[ \widehat{Q} = \Omega \times (0, T). \]

Let us recall that \( d \) satisfies (1.5). Since we can further choose \( \Omega \) such that \( \Omega \supset \Omega \) is included in a sufficiently small ball, we see that there exists an extension \( \tilde{d} \) in \( \Omega \) of \( d \) satisfying \( |\nabla \tilde{d}| > 0 \) in \( \Omega \).

We take the zero extensions of \( u, q, A, G \) and \( h - \text{div } \tilde{v} \) to \( \Omega \) and by the same letters we denote them. Therefore by (2.14)–(2.16), the zero extensions of \( u \) and \( h - \text{div } \tilde{v} \) satisfy
\[ \text{div } u = h - \text{div } \tilde{v} \in H^{1,1}(\widehat{Q}) \quad (2.17) \]
and
\[ \partial_t u - \kappa \Delta u + \nabla q + (A \cdot \nabla) u = G \quad \text{in} \ \tilde{Q}. \] (2.18)

By the zero extensions and (1.9), we obtain
\[ u(\cdot, 0) = u(\cdot, T) = 0 \quad \text{in} \ \Omega, \]
\[ |u| = |\nabla u| = |q| = 0 \quad \text{on} \ \partial \tilde{\Omega} \times (0, T). \] (2.19)

Therefore, by noting (2.19), we apply lemma 2 to (2.17) and (2.18), and we obtain
\[
\| (u, q) \|_{L^2_{\bar{\Omega}}}^2 \leq C \int_0^t |G|^2 e^{2\gamma \tau} dt
+ C \int_0^t \left( |h - \text{div} \, \tilde{\nu}|^2 + \left| \nabla_{\nu,\tilde{\nu}} (h - \text{div} \, \tilde{\nu}) \right|^2 \right) e^{2\gamma \tau} dt
\]
for \( s \geq s_0 \). Hence
\[
\| (v - \tilde{v}, p - \tilde{p}) \|_{L^2_{\bar{\Omega}}}^2 \leq C \int_0^t |F|^2 e^{2\gamma \tau} dt
+ C \int_0^t \left( |\partial_t \tilde{v} - \kappa \Delta \tilde{v} + (A \cdot \nabla) \tilde{v} + \nabla p|^2 e^{2\gamma \tau} dt
+ C \int_0^t \left( |h|^2 + |\nabla_{\nu,\tilde{\nu}} h|^2 + |\nabla \tilde{\nu}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{\nu}|^2 + |\nabla (\partial_i \tilde{\nu})|^2 \right) e^{2\gamma \tau} dt.
\]
Using \( |\partial_t \nu|^2 \leq 2 |\partial_t \tilde{v}|^2 + 2 |\partial_t (v - \tilde{v})|^2 \), etc on the left-hand side, we have
\[
\| (v, p) \|_{L^2_{\bar{\Omega}}}^2 \leq 2 \| (\tilde{v}, \tilde{p}) \|_{L^2_{\bar{\Omega}}}^2
+ 2C \int_0^t |F|^2 e^{2\gamma \tau} dt
+ 2C \int_0^t |\partial_t \tilde{v} - \kappa \Delta \tilde{v} + (A \cdot \nabla) \tilde{v} + \nabla p|^2 e^{2\gamma \tau} dt
+ 2C \int_0^t \left( |h|^2 + |\nabla_{\nu,\tilde{\nu}} h|^2 + |\nabla \tilde{\nu}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{\nu}|^2 + |\nabla (\partial_i \tilde{\nu})|^2 \right) e^{2\gamma \tau} dt
\leq C \int_0^t |F|^2 e^{2\gamma \tau} dt
+ C \left( \| \tilde{v} \|_{H^2(\Omega)}^2 + \| \tilde{p} \|_{H^1(\Omega)}^2 + \| \nabla \tilde{\nu} \|_{L^2(\Omega)}^2 \right)
+ C \int_0^t \left( |h|^2 + |\nabla_{\nu,\tilde{\nu}} h|^2 \right) e^{2\gamma \tau} dt
\]
for \( s \geq s_0 \). Since \( F \) and \( h \) are zero outside of \( D \), in view of (2.13), the proof of lemma 3 is completed.

Third step

For \( r > 0 \) and \( x_0 \in \mathbb{R}^n \), we set \( B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \). Then we prove

**Lemma 4.** Let \( v \in H^2(\Omega) \) and \( p \in H^1(\Omega) \).

(1) Case \( n = 3 \): For any \( x_0 \in \partial \Omega \), there exist \( r > 0 \) and a \( 10 \times 10 \) matrix \( A \in \mathcal{C}(B_r(x_0)) \) such that
where \( x(\theta_1, \theta_2) = (x_1(\theta_1, \theta_2), x_2(\theta_1, \theta_2), x_3(\theta_1, \theta_2)) \in \mathbb{R}^3 \), \( D_1 \subset \mathbb{R}^2 \) is a bounded domain and the functions \( x_1, x_2, x_3 \) with respect to \( \theta_1, \theta_2 \) are in \( C^2(D_1) \) and

\[
\det A(x(\theta_1, \theta_2)) = 0, \quad (\theta_1, \theta_2) \in \overline{D_1}
\]

and

\[
A(x(\theta_1, \theta_2)) = \begin{pmatrix}
    (\nabla_x v_1)(x(\theta_1, \theta_2)) \\
    (\nabla_x v_2)(x(\theta_1, \theta_2)) \\
    (\nabla_x v_3)(x(\theta_1, \theta_2)) \\
p(x(\theta_1, \theta_2))
\end{pmatrix} = \begin{pmatrix}
    \nabla_{\theta_1, \theta_2}(v_1(x(\theta_1, \theta_2))) \\
    \nabla_{\theta_1, \theta_2}(v_2(x(\theta_1, \theta_2))) \\
    \nabla_{\theta_1, \theta_2}(v_3(x(\theta_1, \theta_2))) \\
(\sigma(v, p)\nu)(x(\theta_1, \theta_2)) \\
\text{div } v(x(\theta_1, \theta_2))
\end{pmatrix}, \quad (\theta_1, \theta_2) \in D_1.
\]

(2) Case \( n = 2 \): For any \( x_0 \in \partial \Omega \), there exist \( r > 0 \) and a \( 5 \times 5 \) matrix \( A \in C^1(B_r(x_0)) \) such that

\[
\partial \Omega \cap B_r(x_0) = \{ x(\theta_1); \theta_1 \in I_1 \}
\]

where \( x(\theta_1) = (x_1(\theta_1), x_2(\theta_1)) \in \mathbb{R}^2 \), \( I_1 \subset \mathbb{R} \) is an open interval, and the functions \( x_1, x_2 \) are in \( C^2(I_1) \), and

\[
\det A(x(\theta_1)) = 0, \quad \theta_1 \in I_1
\]

and

\[
A(x(\theta_1)) = \begin{pmatrix}
    (\nabla_x v_1)(x(\theta_1)) \\
    (\nabla_x v_2)(x(\theta_1)) \\
p(x(\theta_1))
\end{pmatrix} = \begin{pmatrix}
    \frac{d}{d\theta_1}v_1(x(\theta_1)) \\
    \frac{d}{d\theta_1}v_2(x(\theta_1)) \\
p(x(\theta_1))
\end{pmatrix}, \quad \theta_1 \in I_1.
\]

Remark 3. The lemma guarantees that the boundary data \( (v, \partial_x v, p) \) and \( (\nu, \sigma(v, p)\nu) \) are equivalent (e.g., Ímanuvilov and Yamamoto [13]). As related papers on inverse boundary value problems for the Navier–Stokes equations in view of this equivalence, see Ímanuvilov and Yamamoto [12], Lai et al [17].

Proof of lemma 4. We prove only in the case of \( n = 3 \). The case of \( n = 2 \) is similar and simpler. It is sufficient to consider only on a sufficiently small sub-boundary \( \Gamma_0 \) of \( \partial \Omega \). Without loss of generality, we can assume that \( \Gamma_0 \) is represented by \( (x_1, x_2, \gamma(x_1, x_2)) \) where \( \gamma \in C^2(D_1), \theta_1 = x_1, \theta_2 = x_2, x_3 = \gamma(x_1, x_2) \) for \( (x_1, x_2) \in D_1 \). Moreover we assume that \( \Omega \) is located above \( x_3 = \gamma(x_1, x_2) \).

By the density argument, we can assume that \( v \in C^1(\overline{\Omega}) \) and \( p \in C(\overline{\Omega}) \).
We set $\gamma_1 := \partial_1 \gamma$ and $\gamma_2 := \partial_2 \gamma$. On $\Gamma_0$, we have

$$\nu(x) = \frac{1}{1 + \gamma_1^2 + \gamma_2^2} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$  \hfill (2.20)

By the definition, we have

$$\sigma(v, p) \nu = \kappa \begin{pmatrix} 2\partial_1 v_1 - \frac{p}{\kappa} & \partial_1 v_2 + \partial_2 v_1 \\ \partial_1 v_2 + \partial_2 v_1 & 2\partial_2 v_2 - \frac{p}{\kappa} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} - \begin{pmatrix} \partial_1 v_3 + \partial_3 v_1 \\ \partial_2 v_3 + \partial_3 v_2 \end{pmatrix} \begin{pmatrix} \nu_3 - \frac{p}{\kappa} \nu_2 \\ \nu_2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$  \hfill (2.21)

where

$$\begin{cases} q_1 \\ q_2 \\ q_3 \end{cases}.$$

We further set

$$q_4 := (\text{div} \nu)(x_1, x_2, \gamma(x_1, x_2)),$$  
\(g_k(x_1, x_2) := v_k(x_1, x_2, \gamma(x_1, x_2)), \quad k = 1, 2, 3.

Then

$$\partial_1 g_k = \partial_1 v_k + \gamma_1 (\partial_3 v_k)(x_1, x_2, \gamma(x_1, x_2)),$$  
$$\partial_2 g_k = \partial_2 v_k + \gamma_2 (\partial_3 v_k)(x_1, x_2, \gamma(x_1, x_2)),$$

that is

$$\begin{cases} \partial_1 v_k(x_1, x_2, \gamma(x_1, x_2)) = \partial_1 g_k - \gamma_1 (\partial_3 v_k)(x_1, x_2, \gamma(x_1, x_2)), \\ \partial_2 v_k(x_1, x_2, \gamma(x_1, x_2)) = \partial_2 g_k - \gamma_2 (\partial_3 v_k)(x_1, x_2, \gamma(x_1, x_2)), \quad k = 1, 2, 3, \end{cases}$$  \hfill (2.22)

and

$$\partial_3 v_k(x_1, x_2, \gamma(x_1, x_2)) = q_4 - (\partial_1 v_1 + \partial_2 v_2)(x_1, x_2, \gamma(x_1, x_2))$$  \hfill (2.23)

for $(x_1, x_2) \in D_1$. Setting

$$\begin{cases} h_1(x_1, x_2) = (\partial_3 v_1)(x_1, x_2), \\ h_2(x_1, x_2) = (\partial_3 v_2)(x_1, x_2), \end{cases}$$  \hfill (2.24)

by (2.22) and (2.23) we obtain

$$(\partial_3 v_3)(x_1, x_2, \gamma(x_1, x_2)) = q_4 - (\partial_1 v_1 + \partial_2 v_2)(x_1, x_2, \gamma(x_1, x_2))$$

$$= (q_4 - \partial_1 g_1 - \partial_2 g_2)(x_1, x_2) + (\gamma_1 \partial_3 v_1 + \gamma_2 \partial_3 v_2)(x_1, x_2, \gamma(x_1, x_2))$$

$$= g_0(x_1, x_2) + (\gamma_1 \partial_3 v_1 + \gamma_2 \partial_3 v_2)(x_1, x_2, \gamma(x_1, x_2))$$
\[ (g_0 + \gamma_1 h_1 + \gamma_2 h_2)(x_1, x_2) \]  

(2.25)

and so

\[
\begin{aligned}
(\partial v_1)(x_1, x_2, \gamma(x_1, x_2)) &= (\partial_1 g_1 - \gamma_1 h_1)(x_1, x_2, \gamma(x_1, x_2)), \\
(\partial v_2)(x_1, x_2, \gamma(x_1, x_2)) &= (\partial_2 g_1 - \gamma_2 h_1)(x_1, x_2, \gamma(x_1, x_2)), \\
(\partial v_3)(x_1, x_2, \gamma(x_1, x_2)) &= (\partial_1 g_2 - \gamma_1 h_2)(x_1, x_2, \gamma(x_1, x_2)), \\
(\partial v_4)(x_1, x_2, \gamma(x_1, x_2)) &= (\partial_2 g_2 - \gamma_2 h_2)(x_1, x_2, \gamma(x_1, x_2)), \\
(\partial v_5)(x_1, x_2, \gamma(x_1, x_2)) &= \partial_1 g_3 - \gamma_1 g_0 - \gamma_1^2 h_1 - \gamma_1^2 h_2, \\
(\partial v_6)(x_1, x_2, \gamma(x_1, x_2)) &= \partial_2 g_3 - \gamma_2 g_0 - \gamma_1^2 h_1 - \gamma_2^2 h_2, \\
& \quad (x_1, x_2) \in D_1.
\end{aligned}
\]

(2.26)

On the other hand, (2.21) yields

\[
\begin{aligned}
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_1 &= \left( 2\gamma_1 \partial_1 v_1 + \gamma_2 \partial_1 v_2 + \gamma_2 \partial_2 v_1 - \partial_1 v_3 - \partial_3 v_1 \right)(x_1, x_2, \gamma(x_1, x_2)) - \frac{\gamma_1}{\kappa} p, \\
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_2 &= \left( \gamma_1 \partial_1 v_2 + \gamma_2 \partial_2 v_2 + 2 \gamma_2 \partial_2 v_1 - \partial_2 v_3 - \partial_3 v_2 \right)(x_1, x_2, \gamma(x_1, x_2)) - \frac{\gamma_2}{\kappa} p
\end{aligned}
\]

and

\[
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_3 = \left( \gamma_1 \partial_1 v_3 + \gamma_1 \partial_3 v_1 + \gamma_2 \partial_2 v_3 + \gamma_2 \partial_3 v_2 - 2 \partial_3 v_3 \right)(x_1, x_2, \gamma(x_1, x_2)) + \frac{1}{\kappa} p,
\]

\[ (x_1, x_2) \in D_1. \]

Substitute (2.25) and (2.26), we have

\[
\begin{aligned}
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_1 &= -\left( 1 + \gamma_1^2 + \gamma_2^2 \right) h_1 - \frac{\gamma_1}{\kappa} p + G_1, \\
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_2 &= -\left( 1 + \gamma_1^2 + \gamma_2^2 \right) h_2 - \frac{\gamma_2}{\kappa} p + G_2, \\
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_3 &= -\gamma_1 \left( 1 + \gamma_1^2 + \gamma_2^2 \right) h_1 - \gamma_2 \left( 1 + \gamma_1^2 + \gamma_2^2 \right) h_2 + \frac{1}{\kappa} p + G_3.
\end{aligned}
\]

(2.27)

Here \( G_k, k = 1, 2, 3 \), are linear combinations of \( \partial g_k, q_1, q_2, q_3, q_4, j = 1, 2, k = 1, 2, 3 \), with coefficients given by \( \gamma \) and its first-order derivatives. We can uniquely solve (2.27) with respect to \( h_1, h_2, p \):

\[
\begin{pmatrix}
\frac{h_1(x_1, x_2)}{h_2(x_1, x_2)} \\
\frac{p(x_1, x_2, \gamma(x_1, x_2))}{h_2(x_1, x_2, \gamma(x_1, x_2))}
\end{pmatrix} = \tilde{A}(x_1, x_2) \begin{pmatrix}
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_1 - G_1 \\
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_2 - G_2 \\
\frac{1 + \gamma_1^2 + \gamma_2^2}{\kappa} q_3 - G_3
\end{pmatrix},
\]

\[ (x_1, x_2) \in D_1. \]
Here $\tilde{A} \in C^4(D_1)$ and $\det \tilde{A} = 0$ on $D_1$. The equations (2.25), (2.26) and (2.28) imply the existence of a $10 \times 10$ matrix $A \in C^4(D_1)$ satisfying the conditions in the lemma. Thus the proof of lemma 4 is completed.

Now, in terms of lemmata 3 and 4, we complete the proof of theorem 1 as follows. We consider only the case of $n = 3$. Without loss of generality, $\Gamma$ is given by 

$$\Gamma = \{ (x_1, x_2, x_3); x_1, x_2, x_3 \in D_1 \}$$

with $\gamma \in C^2(D_1)$.

We set $\nabla_{x, x, v} = (\partial_1 v_1, \partial_2 v_1, \partial_1 v_2, \partial_2 v_2, \partial_1 v_3, \partial_2 v_3)^T$. Then, by lemmata 2 and 3, we have

$$\begin{align*}
\frac{1}{1 + \gamma^2 + \gamma_2^2} \left( \begin{array}{c}
\nabla_{x, x, v}^2 \nabla_{x, x, v}^2 (x_1, x_2, \gamma (x_1, x_2)) \\
\sigma (v, p) \nu (x_1, x_2, \gamma (x_1, x_2))
\end{array} \right)
\end{align*}$$

with a $4 \times 6$ matrix $B_1 \in C^4(D_1)$. Therefore

$$\| \partial_3 v (\cdot, t) \|_{H^2 (\Gamma)} + \| p (\cdot, t) \|_{H^4 (\Gamma)}$$

by $B \in C^4(D_1)$. Consequently the interpolation inequality (e.g., theorem 7.7 (p 36) in Lions and Magenes [20]) yields

$$\| \partial_3 v (\cdot, t) \|_{L^2 (\Gamma)} \leq C \left( \begin{array}{c}
\nabla_{x, x, v}^2 \\
\sigma (v, p) \nu
\end{array} \right) (\cdot, t)$$

for $0 \leq t \leq T$. Hence

$$\| \partial_3 v \|_{L^2 (0, T; H^2 (\Gamma))} + \| p \|_{L^2 (0, T; H^4 (\Gamma))} \leq C \left( \begin{array}{c}
\nabla_{x, x, v}^2 \\
\sigma (v, p) \nu
\end{array} \right) (\cdot, t)$$

with this, lemma 3 completes the proof of theorem 1.

3. Conditional stability for the lateral Cauchy problem for the linearized Navier–Stokes equations

In this section, we discuss

Lateral Cauchy problem

We are given a sub-boundary $\Gamma$ of $\partial Q$ arbitrarily. Let $(v, p) \in H^{2,1}(Q) \times H^{1,0}(Q)$ satisfy (1.2) and (1.3). Determine $(v, p)$ in some sub-domain of $Q$ by $(\nu, \sigma (v, p) \nu)$ on $\Gamma \times (0, T)$ and $F$ in $Q$. 

13
In the case of the parabolic equation, there are very many works, and here we do not list up comprehensively and as restricted references, see Landis [18], Mizohata [21], Saut and Scheurer [22], Sogge [23] especially concerning the uniqueness. See also the monographs Beilina and Klibanov [1], Isakov [14], Klibanov and Timonov [15].

Combining a Carleman estimate and a cut-off function, we can prove

**Proposition 1.** Let \( \varphi(x, t) \) be given in theorem 1. We set
\[
Q(\varepsilon) = \{(x, t) \in \Omega \times (0, T); \varphi(x, t) > \varepsilon\}
\]
with \( \varepsilon > 0 \). Moreover we assume that
\[
\overline{Q(0)} \subset Q \cup (\Gamma \times [0, T])
\]
with sub-boundary \( \Gamma \subset \partial \Omega \). Then for any small \( \varepsilon > 0 \), there exist constants \( C > 0 \) and \( \theta \in (0, 1) \) such that
\[
\|v\|_{H^{\gamma}(Q(\varepsilon))} + \|p\|_{H^{\gamma}(Q(\varepsilon))} \leq C\left(\|v\|_{H^{\gamma}(Q)} + \|p\|_{L^2(Q)}\right)\theta + CG,
\]
where we set
\[
G := \left(\|F\|_{L^2(Q)}^2 + \|v\|_{L^2(0, T; H^2(\Gamma))}^2 + \|v\|_{H^1(0, T; L^2(\Gamma))}^2 + \|\sigma(v, p)v\|_{L^2(0, T; H^2(\Gamma))}^2\right)^{\frac{1}{2}}.
\]

As for the proof of proposition 1, see theorem 3.2.2 in section 3.2 of [14] for example.

Proposition 1 gives an estimate of the solution in \( Q(\varepsilon) \) by data on \( \Gamma \times (0, T) \), and \( Q(\varepsilon) \) and \( \Gamma \) are determined by an a priori given function \( d(x) \). Therefore this proposition does not give a suitable stability estimate to our lateral Cauchy problem as stated above, where we are requested to estimate the solution by data on as a small sub-boundary \( \Gamma \times (0, T) \) as possible.

In fact, in this section, we prove

**Theorem 2.** (Conditional stability)

Let \( \Gamma \subset \partial \Omega \) be an arbitrary non-empty sub-boundary of \( \partial \Omega \). For any \( \varepsilon > 0 \) and an arbitrary bounded domain \( \Omega_0 \) such that \( \overline{\Omega_0} \subset \Omega \cup \Gamma \), \( \partial \Omega_0 \cap \partial \Omega \) is a non-empty open subset of \( \partial \Omega \) and \( \partial \Omega_0 \cap \partial \Omega \) is a proper subset of \( \Gamma \); there exist constants \( C > 0 \) and \( \theta \in (0, 1) \) such that
\[
\|v\|_{H^{\gamma}(\Omega_0 \times (\varepsilon, T - \varepsilon))} + \|p\|_{H^{\gamma}(\Omega_0 \times (\varepsilon, T - \varepsilon))} \leq C\left(\|v\|_{H^{\gamma}(Q)} + \|p\|_{L^2(Q)}\right)^{\theta} + \left(\|F\|_{L^2(Q)} + \|v\|_{H^1(0, T; H^2(\Gamma))} + \|\sigma(v, p)v\|_{L^2(0, T; H^2(\Gamma))}\right)^{\theta}
\]
\[
+ C\left(\|F\|_{L^2(Q)} + \|v\|_{L^2(0, T; H^2(\Gamma))} + \|p\|_{H^1(0, T; L^2(\Gamma))} + \|\sigma(v, p)v\|_{L^2(0, T; L^2(\Gamma))}\right). \tag{3.1}
\]

In theorem 2, in order to estimate \( (v, p) \), we have to assume a priori bounds of \( \|v\|_{H^{1}(Q)} \) and \( \|p\|_{L^2(Q)} \). Thus estimate (3.1) is called a conditional stability estimate. We note that (3.1) is rewritten as
\[ \|v\|_{H^2((\Omega_0 \times (\varepsilon, T - \varepsilon)))} + \|p\|_{H^1((\Omega_0 \times (\varepsilon, T - \varepsilon)))} \]
\[ = O\left( \left( \|F\|_{L^2(\Omega)} + \|v\|_{L^2(0, T; H^1(\Gamma))} + \|\sigma(v, p)v\|_{L^2(0, T; H^1(\Gamma))} \right)^\theta \right) \]

as \( \|F\|_{L^2(\Omega)} + \|v\|_{L^2(0, T; H^1(\Gamma))} + \|\sigma(v, p)v\|_{L^2(0, T; H^1(\Gamma))} \to 0. \) Thus the estimate indicates stability of Hölder type.

For the homogeneous Stokes equations:
\[ \partial_t \nu - \Delta \nu + \nabla p = 0, \quad \text{div } v = 0 \quad \text{in } Q, \]

Boulakia [2] (proposition 2) proved the conditional stability in \( \Omega_0 \times (\varepsilon, T - \varepsilon) \) on the basis of a Carleman estimate in [2]. The norm of boundary data in [2] is stronger than our chosen norm.

The theorem does not directly give an estimate when \( \Omega_0 = \Omega \), but we can derive an estimate in \( \Omega \) by an argument similar to theorem 5.2 in Yamamoto [27] and we do not discuss details. Boulakia [2] (theorem 1) established a conditional stability estimate up to \( \partial \Omega \) by boundary or interior data. The argument is based on the interior estimate in \( \Omega_0 \times (\varepsilon, T - \varepsilon) \) and an argument similar to theorem 5.2 in [27].

Theorem 2 immediately implies the global uniqueness of the solution:

**Corollary 1.** Let \( \Gamma \subset \partial \Omega \) be an arbitrarily fixed sub-boundary. If \( (v, p) \in H^{2,1}(Q) \times H^{1,0}(Q) \) satisfies (1.2) and (1.3), and \( v = \sigma(v, p)v = 0 \) on \( \Gamma \times (0, T) \), then \( v = \sigma(v, p)v = 0 \) in \( \Omega \times (0, T) \).

**Proof of theorem 2.** Once a relevant Carleman estimate for the Navier–Stokes equations is proved, the proof is similar to theorem 5.1 in [27]. Thus, according to \( \Omega_0 \) and \( \Gamma \), we have to choose a suitable function \( d(x) \) defining the weight function \( \varphi \) (see (1.7)). For this, we show

**Lemma 5.** Let \( \omega \) be an arbitrarily fixed sub-domain of \( \Omega \) such that \( \overline{\omega} \subset \Omega \). Then there exists a function \( d(x) \in C^1(\Omega) \) such that
\[ d(x) > 0, \quad x \in \Omega, \quad d|_{\partial \Omega} = 0, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega} \setminus \overline{\omega}. \]

For the proof, see Fursikov and Imanuvilov [7], Imanuvilov [9], Imanuvilov et al [11].

We choose a bounded domain \( \Omega_1 \) with smooth boundary such that
\[ \Omega_1 \text{ is a proper subset of } \Omega_1, \quad \Gamma = \partial \Omega_1 \cap \Omega_1, \quad \partial \Omega_1 \setminus \Gamma \subset \partial \Omega_1, \]
and \( \Omega_1 \setminus \Gamma \) contains some non-empty open set. We note that \( \Omega_1 \) is constructed by taking a union of \( \Omega \) and a domain \( \Omega_1 \subset \mathbb{R}_n \setminus \overline{\Omega} \) such that \( \Omega_1 \cap \partial \Omega = \Gamma \). We choose \( \overline{\omega} \subset \Omega_1 \setminus \overline{\omega} \), and apply lemma 5 to obtain \( d \in C^2(\overline{\Omega}_1) \) satisfying
\[ d(x) > 0, \quad x \in \Omega_1, \quad d(x) = 0, \quad x \in \partial \Omega_1, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}_1. \]
Then, since \( \overline{\Omega}_0 \subset \Omega_1 \), we can take sufficiently large \( N > 1 \) such that
\[ \left\{ x \in \Omega_1; d(x) > \frac{4}{N} \|d\|_{C(\overline{\Omega}_1)} \right\} \cap \overline{\Omega} \supset \Omega_0. \]
Moreover we choose sufficiently large $\beta > 0$ such that
\[
\beta \epsilon^2 < ||d||_{C(\Sigma \cap \overline{\Omega})} < 2\beta \epsilon^2.
\] (3.5)

We arbitrarily fix $t_0 \in [\sqrt{2} \epsilon, T - \sqrt{2} \epsilon]$. We set $\varphi(x, t) = e^{\lambda(t, x)}$ with fixed large parameter $\lambda > 0$ and $\psi(x, t) = d(x) - \beta (t - t_0)^2$, $\mu_k = \exp \left( \lambda \left( \frac{k}{N} ||d||_{C(\overline{\Omega})} - \frac{\beta \epsilon^2}{N} \right) \right)$, $k = 1, 2, 3, 4,$ and $D = \{(x, t); x \in \overline{\Omega}, \varphi(x, t) > \mu_1 \}$.

Then we can verify that
\[
\Omega_0 \times \left( t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}} \right) \subset D \subset \Omega \times \left( t_0 - \sqrt{2} \epsilon, t_0 + \sqrt{2} \epsilon \right). \tag{3.6}
\]

In fact, let $(x, t) \in \Omega_0 \times \left( t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}} \right)$. Then, by (3.4) we have $x \in \overline{\Omega}$ and $d(x) > \frac{4}{N} ||d||_{C(\overline{\Omega})}$, so that
\[
d(x) - \beta (t - t_0)^2 > \frac{4}{N} ||d||_{C(\overline{\Omega})} - \frac{\beta \epsilon^2}{N},
\]
that is, $\varphi(x, t) > \mu_1$, which implies that $(x, t) \in D$ by the definition of $D$. Next let $(x, t) \in D$. Then $d(x) - \beta (t - t_0)^2 > \frac{1}{N} ||d||_{C(\overline{\Omega})} - \frac{\beta \epsilon^2}{N}$. Therefore
\[
||d||_{C(\overline{\Omega})} - \frac{1}{N} ||d||_{C(\overline{\Omega})} + \frac{\beta \epsilon^2}{N} > \beta (t - t_0)^2.
\]

Applying (3.5), we have $2 \left( 1 - \frac{1}{N} \right) \beta \epsilon^2 > \beta (t - t_0)^2$, which implies that $t_0 - \sqrt{2} \epsilon < t < t_0 + \sqrt{2} \epsilon$. The verification of (3.6) is completed.

Next we have
\[
\left\{ \begin{array}{l}
\partial D \subset \Sigma_1 \cup \Sigma_2, \\
\Sigma_1 \subset \Gamma \times (0, T), \\
\Sigma_2 = \{(x, t); x \in \Omega, \varphi(x, t) = \mu_1, t \in (0, T)\}. \tag{3.7}
\end{array} \right.
\]

In fact, let $(x, t) \in \partial D$. Then $x \in \overline{\Omega}$ and $\varphi(x, t) \geq \mu_1$. We separately consider the cases $x \in \overline{\Omega}$ and $x \in \partial \Omega$. First let $x \in \overline{\Omega}$. If $\varphi(x, t) > \mu_1$, then $(x, t)$ is an interior point of $D$, which is impossible. Therefore $\varphi(x, t) = \mu_1$, which implies $(x, t) \in \Sigma_2$. Next let $x \in \partial \Omega$. Let $x \in \partial \Omega \Gamma$. Then $x \in \partial \Omega \Gamma$ by the third condition in (3.2), and $d(x) = 0$ by the second condition in (3.3). On the other hand, $\varphi(x, t) \geq \mu_1$ yields that
\[
d(x) - \beta (t - t_0)^2 = -\beta (t - t_0)^2 \geq \frac{1}{N} ||d||_{C(\overline{\Omega})} - \frac{\beta \epsilon^2}{N},
\]
that is, $0 \leq \beta (t - t_0)^2 \leq \frac{1}{N} (-||d||_{C(\overline{\Omega})} + \beta \epsilon^2)$, which is impossible by (3.5). Therefore $x \in \Gamma$. By (3.6), we see that $0 < t < T$ and the verification of (3.7) is completed.

We apply theorem 1 in $D$. Henceforth $C > 0$ denotes generic constants independent of $s$ and choices of $v$, $p$. We need a cut-off function because we have no data on $\partial D \setminus (\Gamma \times (0, T))$. Let $\chi \in C^\infty(\mathbb{R}^{n+1})$ satisfying $0 \leq \chi \leq 1$ and
\[
\chi(x, t) = \begin{cases} 
1, & \varphi(x, t) > \mu_3, \\
0, & \varphi(x, t) < \mu_2.
\end{cases} \tag{3.8}
\]
We set \( y = \chi v \) and \( q = \chi p \). Then, by (1.1) and (1.2), we have
\[
\begin{align*}
\partial_t y - \kappa \Delta y + (A \cdot \nabla) y + (v \cdot \nabla) B + \nabla q
&= \chi F + v \partial_t \chi - 2\kappa \nabla \chi \cdot \nabla v - \kappa (\Delta \chi) v + (A \cdot \nabla \chi) v + p(\nabla \chi) 
\end{align*}
\]
and
\[
\begin{align*}
\text{div } y = \nabla \chi \cdot v 
\end{align*}
\]
in \( D \).

By (3.7) and (3.8), we see that \( |y| = |\nabla y| = |q| = 0 \) on \( \Sigma_2 \).

Hence theorem 1 yields
\[
\begin{align*}
\|y, q\|_{H^s(D)}^2 &\leq C \int_D |F|^2 e^{2\gamma t} \, dx \, dt \\
&+ C \int_D |v| \partial_t \chi - 2\kappa \nabla \chi \cdot \nabla v - \kappa (\Delta \chi) v + (A \cdot \nabla \chi) v + p(\nabla \chi)|^2 e^{2\gamma t} \, dx \, dt \\
&+ C \int_D (|\nabla \chi \cdot v|^2 + |\nabla \chi (\nabla \chi \cdot v)|^2) e^{2\gamma t} \, dx \, dt \\
&\quad + CE^s \left( \|\chi v\|_{L^2(0,T;H^2_0(\Gamma))}^2 + \|\chi v\|_{H^0(0,T;H^2_0(\Gamma))}^2 + \|\sigma(\chi, \chi p)v\|_{L^2(0,T;H^4_0(\Gamma))}^2 \right) 
\end{align*}
\]
(3.9)
for \( s \geq s_0 \). We can verify \( \|\chi v\|_{H^\gamma(\Gamma)} \leq C \|v\|_{H^\gamma(\Gamma)} \) with \( \gamma = 0, 1, 2 \), and for \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \), the interpolation inequality yields
\[
\begin{align*}
\|\chi v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 &\leq C \|v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 \quad \|\partial_t(\chi v)\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 \leq C \|\partial v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 
\end{align*}
\]
Therefore, since \( \sigma(\chi, \chi p)v = \chi \sigma(v, p)v + \kappa \left( (\partial_t \chi) v + (\partial_j \chi) v \right) \leq |\partial_t \chi| + |\partial_j \chi| \leq \nu \), we have
\[
\begin{align*}
\|\sigma(\chi, \chi p)v\|_{L^2(0,T;H^\gamma_0(\Gamma))} \leq \|\sigma(v, p)v\|_{L^2(0,T;H^\gamma_0(\Gamma))} + C \|v\|_{L^2(0,T;H^\gamma_0(\Gamma))} 
\end{align*}
\]
by \( \chi \in C^\infty(\mathbb{R}^{n+1}) \). Hence
\[
\begin{align*}
\|\chi v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 + \|\chi v\|_{H^0(0,T;H^\gamma_0(\Gamma))}^2 + \|\sigma(\chi, \chi p)v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 \\
\leq C \left( \|v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 + \|v\|_{H^0(0,T;H^\gamma_0(\Gamma))}^2 + \|\sigma(v, p)v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 \right) 
\end{align*}
\]
We recall that
\[
\begin{align*}
G^2 = \|F\|_{L^2(Q)}^2 + \|\chi v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 + \|\chi v\|_{H^0(0,T;H^\gamma_0(\Gamma))}^2 + \|\sigma(v, p)v\|_{L^2(0,T;H^\gamma_0(\Gamma))}^2 
\end{align*}
\]
The integrands of the second and the third terms on the right-hand side of (3.9) do not vanish only if \( \varphi(x, t) \leq \mu_1 \), because these coefficients include derivatives of \( \chi \) as factors and by (3.8) vanish if \( \varphi(x, t) > \mu_3 \). Therefore
\[
\begin{align*}
||\text{the second and the third terms on the right-hand side of (3.9)}|| \\
\leq C \left( \|v\|_{H^\gamma_0(\Gamma)}^2 + \|p\|_{L^2(\Gamma)}^2 \right) e^{2\gamma t}. 
\end{align*}
\]
Consequently (3.9) yields
\[
\|v, q\|\|_{\mathcal{X}(\Omega)}^2 \leq C \left( \|v\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) e^{2\mu_s} + C e^s G^2 \quad \forall \ s \geq s_0. \tag{3.10}
\]

By (3.4) and the definition of \(D\), we can directly verify that \((x, t) \in \Omega_0 \times \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right)\) implies \(\varphi(x, t) > \mu_q\). Therefore, noting (3.6) and (3.8), we see that
\[
\begin{aligned}
\|v, q\|\|_{\mathcal{X}(\Omega)}^2 & \geq \|v, p\|\|_{\mathcal{X}(\Omega_0 \times \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right))}^2 \\
& \geq e^{2\mu_s} \int_{t_0 - \frac{\varepsilon}{\sqrt{N}}}^{t_0 + \frac{\varepsilon}{\sqrt{N}}} \int_{\Omega_0} \left\{ \frac{1}{s} \left( |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + |\nabla v|^2 \\
& \quad + s^2 |v|^2 + \frac{1}{s^2} |\nabla p|^2 + s |p|^2 \right\} dx dt.
\end{aligned}
\]
Hence (3.10) yields
\[
\begin{aligned}
e^{2\mu_s} \int_{t_0 - \frac{\varepsilon}{\sqrt{N}}}^{t_0 + \frac{\varepsilon}{\sqrt{N}}} \int_{\Omega_0} \left\{ \frac{1}{s} \left( |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + |\nabla v|^2 \\
& \quad + s^2 |v|^2 + \frac{1}{s} |\nabla p|^2 + s |p|^2 \right\} dx dt \\
\leq C \left( \|v\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) e^{2\mu_s} + C e^s G^2.
\end{aligned}
\]

Therefore
\[
\begin{aligned}
\int_{t_0 - \frac{\varepsilon}{\sqrt{N}}}^{t_0 + \frac{\varepsilon}{\sqrt{N}}} \int_{\Omega_0} \left\{ \frac{1}{s} \left( |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 + |\nabla v|^2 + |v|^2 + |\nabla p|^2 + |p|^2 \right) \right\} dx dt \\
\leq C s^2 e^{-2s(\mu_q - \mu_s)} \left( \|v\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) + C e^s G^2 \quad \forall \ s \geq s_0.
\end{aligned}
\]

By \(\sup_{s \geq 0} e^{-st(\mu_q - \mu_s)} < \infty\), we estimate \(e^{-st(\mu_q - \mu_s)}\) by \(e^{-st(\mu_q - \mu_s)}\) on the right-hand side. Moreover, replacing \(C\) by \(Ce^s\), we have
\[
\begin{aligned}
\|v\|_{H^1(\Omega_0 \times \left(\mathcal{I}_T - \frac{\varepsilon}{\sqrt{N}}, T - \frac{\varepsilon}{\sqrt{N}}\right))}^2 + \|p\|_{L^2(\Omega_0 \times \left(\mathcal{I}_T - \frac{\varepsilon}{\sqrt{N}}, T - \frac{\varepsilon}{\sqrt{N}}\right))}^2 \\
\leq Ce^{-s(\mu_q - \mu_s)} \left( \|v\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) + C e^s G^2 \quad \forall \ s \geq s_0. \tag{3.11}
\end{aligned}
\]

for all \(s \geq 0\). Let \(m \in \mathbb{N}\) satisfy \(\sqrt{2} \varepsilon + \frac{m\varepsilon}{\sqrt{N}} \leq T - \sqrt{2} \varepsilon \leq \sqrt{2} \varepsilon + \frac{(m + 1)\varepsilon}{\sqrt{N}} \leq T\).

We here notice that the constant \(C\) in (3.11) is independent also of \(t_0\) provided that \(\sqrt{2} \varepsilon \leq t_0 \leq T - \sqrt{2} \varepsilon\). In (3.11), taking \(t_0 = \sqrt{2} \varepsilon + \frac{1}{\sqrt{N}} j = 0, 1, 2, \ldots, m\) and summing up \(j\), we have
\[
\begin{aligned}
\|v\|_{H^{1,1}(\Omega_0 \times \left(\mathcal{I}_T - \frac{\varepsilon}{\sqrt{N}}, T - \frac{\varepsilon}{\sqrt{N}}\right))}^2 + \|p\|_{L^2(\Omega_0 \times \left(\mathcal{I}_T - \frac{\varepsilon}{\sqrt{N}}, T - \frac{\varepsilon}{\sqrt{N}}\right))}^2 \\
\leq Ce^{-s(\mu_q - \mu_s)} \left( \|v\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) + C e^s G^2
\end{aligned}
\]
for all \( s \geq 0 \). Here we note that \( T - \sqrt{2} \varepsilon \leq \sqrt{2} \varepsilon + (m + 1) \frac{\varepsilon}{\sqrt{N}} \) implies \( T - \sqrt{2} \varepsilon - \frac{m \varepsilon}{\sqrt{N}} \leq \sqrt{2} \varepsilon + \frac{1}{\sqrt{N}} \varepsilon \). Replacing \( \left( \sqrt{2} + \frac{1}{\sqrt{N}} \right) \varepsilon \) by \( \varepsilon \), we have

\[
\|v\|_{H^2(\Omega_0 \times (\varepsilon, T - \varepsilon))} + \|p\|_{H^1(\Omega_0 \times (\varepsilon, T - \varepsilon))} \leq Ce^{-x(\mu - \mu_1)} (\|v\|_{L^2(Q)}^2 + \|p\|_{L^2(Q)}^2) + Ce^{C_0}G^2
\]

(3.12)

for all \( s \geq s_0 \).

First let \( G = 0 \). Then letting \( s \to \infty \) in (3.12), we see that \( |v| = |p| = 0 \) in \( \Omega_0 \times (\varepsilon, T - \varepsilon) \), so that the conclusion of theorem 2 holds true. Next let \( G \neq 0 \). First let \( G \geq \|v\|_{H^1(\Omega_0)} + \|p\|_{L^2(Q)} \). Then (3.12) implies \( \|v\|_{H^2(\Omega_0 \times (\varepsilon, T - \varepsilon))} + \|p\|_{H^1(\Omega_0 \times (\varepsilon, T - \varepsilon))} \leq Ce^{C_0}G \) for \( s \geq 0 \), which already proves the theorem. Second let \( G < \|v\|_{H^1(\Omega_0)} + \|p\|_{L^2(Q)} \). In order to make the right-hand side of (3.12) smaller, we choose \( s > 0 \) such that

\[
e^{-x(\mu - \mu_1)} (\|v\|_{L^2(Q)}^2 + \|p\|_{L^2(Q)}^2) = e^{C_0}G^2.
\]

By \( G \neq 0 \), we can choose

\[
s = \frac{1}{C + \mu_4 - \mu_3} \log \frac{\|v\|_{H^1(\Omega_0)}^2 + \|p\|_{L^2(Q)}^2}{G^2} > 0.
\]

Then (3.12) gives

\[
\|v\|_{H^2(\Omega_0 \times (\varepsilon, T - \varepsilon))} + \|p\|_{H^1(\Omega_0 \times (\varepsilon, T - \varepsilon))} \leq 2C \left( \|v\|_{H^1(\Omega_0)}^2 + \|p\|_{L^2(Q)}^2 \right)^{\frac{1}{2}} = 2C \left( \frac{1}{2} \right)^{\frac{1}{2}} G^{\frac{1}{2}}.
\]

The proof of theorem 2 is completed. \( \square \)

4. Lateral Cauchy problem for the fully nonlinear Navier–Stokes equations

Considering the Navier–Stokes system (1.1) and (1.2) and

\[
\partial_t \vec{v} - \kappa \Delta \vec{v} + \left( \vec{v} \cdot \nabla \right) \vec{v} + \nabla p = \vec{F}(x, t), \quad (x, t) \in Q
\]

(4.1)

and

\[
\text{div} \, \vec{v}(x, t) = 0, \quad (x, t) \in Q,
\]

(4.2)

we discuss

Lateral Cauchy problem for the fully nonlinear Navier–Stokes equations

Let \( \Gamma \subset \partial \Omega \) be a sub-boundary. Estimate \( \vec{v} - \bar{\vec{v}} \) and \( p - \bar{p} \) in a sub-domain of \( Q \) by \( (\vec{v} - \bar{\vec{v}}, \sigma(\nu, \vec{v}, p)\nu - \sigma(\bar{\vec{v}}, \bar{\nu})\nu) \) on \( \Gamma \times (0, T) \) and \( F - \bar{F} \) in \( Q \).

We can directly derive the conditional stability by theorem 2 if we assume enough regularity for \( v, \vec{v}, p, \bar{p} \).

**Theorem 3.** Let \( \Gamma \subset \partial \Omega \) be an arbitrary non-empty sub-boundary of \( \partial \Omega \). We assume that

\[
(v, p), (\vec{v}, \bar{p}) \in W^{2,\infty}(Q) \times H^{1,0}(Q)
\]

(4.3)

and

\[
\|v\|_{W^{2,\infty}(Q)} + \|\vec{v}\|_{W^{2,\infty}(Q)} + \|p\|_{L^2(Q)} + \|\bar{p}\|_{L^2(Q)} \leq M
\]

with some constant \( M > 0 \). For any \( \varepsilon > 0 \) and an arbitrary bounded domain \( \Omega_0 \) such that \( \Omega_0 \subset \Omega \setminus \Gamma \), \( \partial \Omega_0 \cap \partial \Omega \) is a non-empty open subset of \( \partial \Omega \) and \( \partial \Omega_0 \cap \partial \Omega \) is a proper subset of \( \Gamma \), there exist constants \( C > 0 \) and \( \theta \in (0, 1) \) such that
Theorem 3 asserts the conditional stability of Hölder type, provided that

\[ \|v - \bar{v}\|_{H^{1,0}(\partial \Omega, \partial \Omega - \varepsilon)} + \|p - \bar{p}\|_{H^{1,0}(\Omega, \Omega - \varepsilon)} \leq C M^{1-\theta} \left( \|F - \bar{F}\|_{L^2(Q)} + \|v - \bar{v}\|_{L^2(0, T; H^2(\Gamma))} \right) \]

\[ + \|v - \bar{v}\|_{H^1(0, T; H^2(\Gamma))} + \|\sigma(v, p)v - \sigma(\bar{v}, \bar{p})\sigma\|_{L^2(0, T; H^2(\Gamma))} \]

\[ + C \left( \|F - \bar{F}\|_{L^2(Q)} + \|v - \bar{v}\|_{L^2(0, T; H^2(\Gamma))} \right) \]

\[ + \|v - \bar{v}\|_{H^1(0, T; H^2(\Gamma))} + \|\sigma(v, p)v - \sigma(\bar{v}, \bar{p})\sigma\|_{L^2(0, T; H^2(\Gamma))} \]}

is a priori bounded.

**Proof.** Setting \( y = v - \bar{v}, q = p - \bar{p}, A = v, B = \bar{v} \), and subtracting (4.1) from (1.1), we have

\[ \partial_t y - \kappa \Delta y + (A \cdot \nabla) y + (y \cdot \nabla) B + \nabla q = F - \bar{F} \quad \text{in } Q \] (4.4)

and

\[ \text{div } y(x, t) = 0, \quad (x, t) \in Q. \] (4.5)

By the regularity assumption of \( v, \bar{v} \), we can apply theorem 2 to prove theorem 3. \( \square \)

Theorem 3 readily yields

**Corollary 2.** Let \( \Gamma \subset \partial \Omega \) be an arbitrary non-empty sub-boundary of \( \partial \Omega \). If \((v, p, (\bar{v}, \bar{p})) \in W^{2,\infty}(Q) \times H^{1,0}(Q)\) satisfy (1.1), (1.2) and (4.1), (4.2) respectively and

\[ v = \bar{v}, \quad \sigma(v, p)\nu = \sigma(\bar{v}, \bar{p})\nu \]

on \( \Gamma \times (0, T) \) and \( F = \bar{F} \) in \( Q \), then \( v = \bar{v} \) and \( p = \bar{p} \) in \( Q \).

Henceforth we assume

\[ F = \bar{F} = 0 \quad \text{in } Q \]

in (1.1) and (4.1). That is, we consider

\[ \begin{cases} \partial_t v - \kappa \Delta v + (v \cdot \nabla)v + \nabla p = 0 & \text{in } Q, \\ \text{div } v(x, t) = 0, & (x, t) \in Q, \\ v = 0 & \text{on } \partial \Omega \times (0, T), \\ v(x, 0) = a(x), & x \in \Omega. \end{cases} \] (4.6)

The regularity assumption (4.3) seems very strong for the Navier–Stokes equations. As for fundamental class of solutions to the Navier–Stokes equations, we can define the Leray–
Hopf weak solution (e.g., Ladyzhenskaya [16], Sohr [24], Temam [25, 26] as monographs and Giga [8] as a survey) as follows.

**Definition (weak solution)**

A vector-valued function \( v = v(x, t) \) is called a weak solution to (1.1) and (1.2) with \( v(\cdot, 0) = a \) if \( v = 0 \) on \( \partial \Omega \times (0, T) \),

\[
\int_0^T \left\{ \left( v, \partial_t \varphi \right)_{L^2(\Omega)} + (v, \kappa \Delta \varphi)_{L^2(\Omega)} + (v \cdot \nabla \varphi)_{L^2(\Omega)} \right\} \, dt + (a, \varphi(\cdot, 0))_{L^2(\Omega)} = 0
\]

and

\[
\int_0^T (v, \nabla \psi)_{L^2(\Omega)} \, dt = 0
\]

for all \( \varphi, \psi \in C^\infty(\Omega \times [0, T)) \) such that \( \text{div} \varphi = 0, \varphi \) and \( \psi \) vanish outside a compact set of \( \Omega \times [0, T) \).

Let

\[
X_2 = \{ u \in L^2(\Omega); \text{div} \ u = 0 \text{ in } \Omega, \quad u \cdot \nu = 0 \text{ on } \partial \Omega \}.
\]

Then it is known that for each \( a \in X_2 \), there exists a weak solution satisfying

\[
\|v(\cdot, t)\|_{L^2(\Omega)}^2 + 2\kappa \int_0^t \|\nabla v(\cdot, \xi)\|_{L^2(\Omega)}^2 \, d\xi \leq \|v(\cdot, t_0)\|_{L^2(\Omega)}^2
\]

for almost all \( t, t_0 \geq 0 \) with \( t > t_0 \) (e.g., theorem 3.6.2 (p 340) in Sohr [24]). This weak solution is called a Leray–Hopf weak solution to (4.6).

We note that it is a long-standing open problem whether a Leray–Hopf weak solution is unique in the case of three-dimensions \( n = 3 \).

Now we discuss a uniqueness result for Leray–Hopf weak solutions with extra boundary data on \( \Gamma' \times (0, T) \) in the case where the zero Dirichlet boundary condition is posed on \( \partial \Omega \times (0, T) \).

In addition to (4.6), we further consider

\[
\begin{align*}
\partial_t \tilde{v} - \kappa \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v} + \nabla \tilde{p} &= 0 \quad \text{in } Q, \\
\text{div} \tilde{v}(x, t) &= 0, \quad (x, t) \in Q, \\
\tilde{v} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\tilde{v}(x, 0) &= \tilde{a}(x), \quad x \in \Omega.
\end{align*}
\]

We assume that \( a, \tilde{a} \in X_2 \).

In (4.6) and (4.7), we can similarly discuss the case \( F = 0, v \big|_{\partial \Omega} = 0 \), etc., provided that 

\( F \) and \( v \big|_{\partial \Omega} \) satisfy some smoothness condition, but here we omit the details.

We would like to answer the following question. For the Leray–Hopf weak solutions \((v, p)\) and \((\tilde{v}, \tilde{p})\) to (4.6) and (4.7), can we conclude that \( v = \tilde{v} \) and \( p = \tilde{p} \) in \( Q := \Omega \times (0, T) \) from \( \sigma(v, p)\nu = \sigma(\tilde{v}, \tilde{p})\nu \) on \( \Gamma' \times (0, T) \)?

The following theorem gives the affirmative answer.

**Theorem 4.** Let \( n = 2, 3 \) and let \( \Gamma' \subset \partial \Omega \) be an arbitrarily chosen non-empty subboundary. We assume that \((v, p)\) and \((\tilde{v}, \tilde{p})\) are Leray–Hopf weak solutions to (4.6) and (4.7) respectively. If

\[
\sigma(v, p)\nu = \sigma(\tilde{v}, \tilde{p})\nu \quad \text{in } (C^\infty_0(\Gamma' \times (0, T)))',
\]

then \( v = \tilde{v} \) and \( p = \tilde{p} \) almost everywhere in \( Q \).
As lemma 6 shows, we really obtain $\sigma (v, p) \nu = \sigma (\tilde{v}, \tilde{p}) \nu$ almost everywhere on $\Gamma \times (0, T)$. The Leray–Hopf solution exists globally in time, but in the case of $n = 3$, the uniqueness of Leray–Hopf weak solution is not known. More precisely, the coincidence $\nu (\cdot, 0) = \tilde{v} (\cdot, 0)$ of initial values does not yield $v = \tilde{v}$ and $p = \tilde{p}$ in $Q$. This is a main reason why a strong solution which exists uniquely, should be discussed, but it is a very celebrated open problem whether or not the strong solution exists globally in time in general for $n = 3$.

In contrast to the initial-boundary value problem, the extra boundary information $(4.8)$ on $\Gamma \times (0, T)$ guarantees the uniqueness of Leray–Hopf weak solutions.

**Proof of theorem 4.** In the case of $n = 2$, since a Leray–Hopf weak solution is smooth for $t > 0$, in particular, $(v, p), (\tilde{v}, \tilde{p}) \in W^{2,\infty} (Q) \times H^{1,0} (Q)$ (e.g., [16, 25]), so that the proof is immediately seen by corollary 2.

Let $n = 3$. Then a Leray–Hopf weak solution may possess singularities but they are rarely distributed. More precisely,

**Lemma 6.** Let $(v, p)$ be a Leray–Hopf weak solution to $(4.6)$. Then there exists a closed set $E \subset (0, T)$ with the zero Lebesgue measure such that $(v, p)$ is smooth in $\overline{\Omega} \times ((0, T) \setminus E)$.

As for the proof, see e.g., section 5 of part I of Temam [26] where a sharper characterization of the measure of $E$ is established but for our proof it is sufficient that the Lebesgue measure of $E$ is zero.

By lemma 6, we choose a closed set $E_0 \subset (0, T)$ with the zero Lebesgue measure such that

$$(v, p), (\tilde{v}, \tilde{p}) \in C^2 \left( \overline{\Omega} \times \left( (0, T) \setminus E_0 \right) \right) \times C^1 \left( \overline{\Omega} \times \left( (0, T) \setminus E_0 \right) \right).$$

(4.9)

Let $t_0 \in (0, T) \setminus E_0$ be arbitrarily given. Since $E_0$ is closed, we can choose small $\varepsilon > 0$ such that $t_0 + \varepsilon \in I := (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\overline{I} \subset (0, T) \setminus E_0$. By (4.9) we see that

$$(v, p), (\tilde{v}, \tilde{p}) \in C^2 (\overline{\Omega} \times \overline{I}) \times C^1 (\overline{\Omega} \times \overline{I})$$

(4.10)

and

$$\sigma (v, p) \nu = \sigma (\tilde{v}, \tilde{p}) \nu \quad \text{on} \quad \Gamma \times \overline{I}.$$ (4.11)

Replacing $t$ by $t + (t_0 - \varepsilon)$, we can reduce all the systems for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ to the time interval $t \in (0, 2\varepsilon)$, and in terms of (4.10) and (4.11) we apply corollary 2. Hence $(v, p) = (\tilde{v}, \tilde{p})$ in $\overline{\Omega} \times (t_0 - \varepsilon, t_0 + \varepsilon)$.

Since $t_0 \in (0, T) \setminus E_0$ is arbitrary, we see that $(v, p) = (\tilde{v}, \tilde{p})$ in $\overline{\Omega} \times (0, T) \setminus E_0)$. Since the Lebesgue measure of $E_0$ is zero, we conclude that $(v, p) = (\tilde{v}, \tilde{p})$ almost everywhere in $\Omega \times (0, T)$. Thus the proof of theorem 4 is completed.

**Acknowledgments**

The authors thank the anonymous referees for their valuable comments. In particular, section 4 is added by a comment of the referee. As for section 4, the authors are grateful to Professor Yoshikazu Giga (The University of Tokyo) for useful discussions and information of the Navier–Stokes equations.
References

[1] Beilina L and Klibanov M V 2012 Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems (Berlin: Springer)
[2] Boulakia M Quantification of the unique continuation property for the nonstationary Stokes problem preprint
[3] Choulli M, Imanuvilov O Y, Puel J-P and Yamamoto M 2013 Inverse source problem for linearized Navier–Stokes equations with data in arbitrary sub-domain Appl. Anal. 92 2127–43
[4] Fan J, Di Cristo M, Jiang Y and Nakamura G 2010 Inverse viscosity problem for the Navier–Stokes equation J. Math. Anal. Appl. 365 750–7
[5] Fan J, Jiang Y and Nakamura G 2009 Inverse problems for the Boussinesq system Inverse Problems 25 085007
[6] Fernández-Cara E, Guerrero S, Imanuvilov O Y and Puel J-P 2004 Local exact controllability of the Navier–Stokes system J. Math. Pures Appl. 83 1501–42
[7] Fursikov A V and Imanuvilov O Y 1996 Controllability of Evolution Equations (Korea: Seoul National University)
[8] Giga Y 1983 Weak and strong solutions of the Navier–Stokes initial value problem Publ. RIMS Kyoto Univ. 19 887–910
[9] Imanuvilov O Y 1995 Controllability of parabolic equations Sb. Math. 186 879–900
[10] Imanuvilov O Y and Puel J-P 2003 Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems Int. Math. Res. Not. 16 883–913
[11] Imanuvilov O Y, Puel J-P and Yamamoto M 2009 Carleman estimates for parabolic equations with nonhomogeneous boundary conditions Chin. Ann. Math. 30B 333–78
[12] Imanuvilov O Y and Yamamoto M 2015 Global uniqueness in inverse boundary value problems for Navier–Stokes equations and Lamé system in two-dimensions Inverse Problems 31 035004
[13] Imanuvilov O Y and Yamamoto M 2015 Remark on boundary data for inverse boundary value problems for the Navier–Stokes equations (addendum to [12]) Inverse Problems 31 109401
[14] Isakov V 2006 Inverse Problems for Partial Differential Equations (Berlin: Springer)
[15] Klibanov M V and Timonov A A 2004 Carleman Estimates for Coefficient Inverse Problems and Numerical Applications (Utrecht: VSP)
[16] Ladyzhenskaya O A 1963 The Mathematical Theory of Viscous Incompressible Flow (New York: Gordon and Breach)
[17] Lai R Y, Uhlmann G and Wang J-N 2015 Inverse boundary value problem for the Stokes and the Navier–Stokes equations in the plane Arch. Ration. Mech. Anal. 215 811–29
[18] Landis E M 1962 Some questions in the qualitative theory of elliptic and parabolic equations AMS Transl. 2 20 173–238
[19] Leray J 1934 Sur le mouvement d’un liquide visqueux emplissant l’espace Acta Math. 63 193–248
[20] Lions J-L and Magenes E 1972 Non-Homogeneous Boundary Value Problems and Applications (Berlin: Springer)
[21] Mizohata S 1958 Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques Mem. Coll. Sci. Univ. Kyoto A 31 219–39
[22] Saut J-C and Scheurle J 1987 Unique continuation for some evolution equations J. Differ. Equ. 66 118–39
[23] Sogge C D 1990 A unique continuation theorem for second order parabolic differential operators Ark. Mat. 28 159–82
[24] Sohr H 2001 The Navier–Stokes Equations an Elementary Functional Analytic Approach (Birkhäuser, Basel)
[25] Temam R 1977 Navier–Stokes Equations (Amsterdam: North-Holland)
[26] Temam R 1995 Navier–Stokes Equations and Nonlinear Functional Analysis (Philadelphia, PA: SIAM)
[27] Yamamoto M 2009 Carleman estimates for parabolic equations and applications Inverse Problems 25 123013