INARIANT MEASURES OF STOCHASTIC DELAY LATTICE SYSTEMS

ZHANG CHEN
School of Mathematics, Shandong University
Jinan 250100, China

XILIANG LI
School of Mathematics and Information Science
Shandong Technology and Business University
Yantai, Shandong 264005, China

BIXIANG WANG*
Department of Mathematics
New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

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Abstract. This paper is concerned with the existence and uniqueness of invariant measures for infinite-dimensional stochastic delay lattice systems defined on the entire integer set. For Lipschitz drift and diffusion terms, we prove the existence of invariant measures of the systems by showing the tightness of a family of probability distributions of solutions in the space of continuous functions from a finite interval to an infinite-dimensional space, based on the idea of uniform tail-estimates, the technique of diadic division and the Arzela-Ascoli theorem. We also show the uniqueness of invariant measures when the Lipschitz coefficients of the nonlinear drift and diffusion terms are sufficiently small.

1. Introduction. This paper is concerned with invariant measures of the Ito stochastic delay lattice system defined on the integer set $\mathbb{Z}$:

$$
\begin{align*}
\frac{du_i(t)}{dt} & = (f_i(u_i(t-\rho)) + g_i)dt + \sum_{k=1}^{\infty} (h_{k,i} + \sigma_{k,i}(u_i(t)))dW_k(t), & t > 0,
\end{align*}
$$

with initial data

$$
u_i(s) = \xi_i(s), \quad s \in [-\rho, 0],
$$

where $u = (u_i)_{i \in \mathbb{Z}}$ is an unknown sequence, $\xi = (\xi_i)_{i \in \mathbb{Z}}$ is a given sequence, $\nu$, $\lambda$ and $\rho$ are positive constants, $g = (g_i)_{i \in \mathbb{Z}}$ and $h = (h_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}}$ are given deterministic sequences in $\ell^2$, $f_i, \sigma_{k,i} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions for all $i \in \mathbb{Z}$.

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* Corresponding author: Bixiang Wang.
and \( k \in \mathbb{N} \), and \((W_k)_{k \in \mathbb{N}}\) is a sequence of independent standard Wiener processes on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) which satisfies the usual condition.

Lattice systems can be used as model equations to describe propagation of nerve pulses, pattern formation and electric circuits, see, e.g., [7, 20, 21, 26, 38, 40, 41]. The solutions and their long-term dynamics of deterministic lattice systems have been studied in [1, 2, 3, 17, 24, 25, 26, 34, 67] and [4, 8, 16, 18, 19, 39, 52, 60], respectively. The random dynamics of stochastic lattice equations have been recently examined in [5, 6, 14, 15, 32, 33, 35, 61, 62] and the references therein. In this paper, we will investigate the dynamics of the stochastic delay lattice systems.

The delay equations arise from many practical systems where the current states of the systems depend on their past history, such as a memory process and an animal growth process in biology, see, e.g., [29, 47, 48, 55]. The solutions of delay equations have been extensively studied in the literature, see, e.g., [30, 31, 36, 56, 57] for deterministic delay equations, and [11, 12, 13, 27, 28, 37, 46, 49, 50, 51, 54, 58, 59, 64, 65, 66] for stochastic delay equations. In particular, invariant measures of stochastic delay equations have been investigated in [11, 27, 37, 54, 58, 59, 66] and the references therein.

Notice that in all these papers, the existence of invariant measures was only studied for finite-dimensional stochastic delay equations defined on \( \mathbb{R}^n \), where the compactness of closed bounded subsets of \( \mathbb{R}^n \) play a key role for proving the tightness of a family of distribution laws of solutions. As far as the authors are aware, there is no result reported in the literature regarding the existence of invariant measures for infinite-dimensional stochastic delay equations. The purpose of the present paper is to study this problem, and prove the existence of invariant measures for the infinite-dimensional stochastic delay lattice system (1)-(2) in \( C([-\rho, 0], \ell^2) \), which is the space of \( \ell^2 \)-valued continuous functions on \([-\rho, 0]\) with uniform norm. To that end, we must establish the tightness of probability distributions of the segments of solutions in \( C([-\rho, 0], \ell^2) \) which is the main obstacle of the paper. This type of difficulty is quite similar to that of proving the tightness of distributions of solutions to stochastic PDEs defined on unbounded domains where the standard Sobolev embeddings are not compact (see, e.g., [9, 10, 23, 42, 43, 44, 45, 53, 63]).

In this paper, we will develop a method to prove the tightness of a family of distributions of solutions in \( C([-\rho, 0], \ell^2) \) by combining the idea of uniform tail-estimates, the technique of diadic division and the Arzela-Ascoli theorem. More precisely, we will first show that under certain conditions the tails of the segments of the solutions are uniformly small for all time, and then use the technique of diadic division to prove the equicontinuity of solutions in \( C([-\rho, 0], \ell^2) \). Finally, we establish the tightness of probability distributions of the segments of solutions in \( C([-\rho, 0], \ell^2) \) (see Lemma 4.2), and hence the existence of invariant measures for Lipschitz drift and diffusion terms (see Theorem 4.3). It is worth mentioning that if there is no delay (i.e., \( \rho = 0 \)), the existence of invariant measures of stochastic lattice system in \( \ell^2 \) was recently studied in [61, 62] where the uniform tail-estimates are sufficient to derive the tightness of distributions of solutions in \( \ell^2 \). However, for the delayed stochastic system (1)-(2) with \( \rho > 0 \), the uniform tail-estimates alone are not enough to establish the tightness of the distributions of solutions in the space \( C([-\rho, 0], \ell^2) \). In this case, we must also employ the technique of diadic division as well as the Arzela-Ascoli theorem to derive the desired compactness of a family of distributions of solutions.
This paper is organized as follows. In the next section, we discuss the existence and uniqueness of solutions to the stochastic delay lattice system (1)-(2). Section 3 is devoted to the uniform estimates of solutions including those estimates on the tails of the solutions. In Section 4, we prove the existence of invariant measures in $C([-\rho,0],\ell^2)$ for (1)-(2) with Lipschitz drift and diffusion terms. In the last section, we show the uniqueness of invariant measures under further conditions.

From now on, we denote the inner product and the norm of $\ell^2$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The norm of the space $C([-\rho,0],\ell^2)$ will be written as $\| \cdot \|_{C_{\rho}}$.

2. Well-Posedness of stochastic lattice systems. In this section, we discuss the well-posedness of the stochastic delay lattice system (1)-(2) in $\ell^2$. To that end, we first reformulate system (1)-(2) as an abstract equation in $\ell^2$.

For the given sequences $g = (g_i)_{i \in \mathbb{Z}}$ and $h = (h_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}}$ in (1), we assume

$$
\|g\|^2 = \sum_{i \in \mathbb{Z}} |g_i|^2 < \infty \quad \text{and} \quad \|h\|^2 = \sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{Z}} |h_{k,i}|^2 < \infty. 
$$

For the nonlinear delay term in (1), we assume that $f_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$; that is, there exists a constant $L_0 > 0$ such that

$$
|f_i(s_1) - f_i(s_2)| \leq L_0 |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R} \text{ and } i \in \mathbb{Z}. 
$$

We further assume that $f_i : \mathbb{R} \to \mathbb{R}$ grows linearly; that is, for each $i \in \mathbb{Z}$, there exist positive numbers $\alpha_i$ and $\beta_0$ such that

$$
|f_i(s)| \leq \alpha_i + \beta_0 |s|, \quad \forall s \in \mathbb{R} \text{ and } i \in \mathbb{Z}, 
$$

where the sequence $(\alpha_i)_{i \in \mathbb{Z}}$ is in $\ell^2$.

Similarly, for the nonlinear diffusion in (1), we assume that $\sigma_{k,i} : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$; that is, for every $k \in \mathbb{N}$, there exists a constant $L_k > 0$ such that

$$
|\sigma_{k,i}(s_1) - \sigma_{k,i}(s_2)| \leq L_k |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R}, \text{ and } i \in \mathbb{Z}, \text{ and } k \in \mathbb{N},
$$

where the sequence $(L_k)_{k \in \mathbb{N}}$ belongs to $\ell^2$. We also assume that $\sigma_{k,i} : \mathbb{R} \to \mathbb{R}$ grows linearly; that is, for each $k \in \mathbb{N}$ and $i \in \mathbb{Z}$, there exist positive numbers $\delta_{k,i}$ and $\beta_k$ such that

$$
|\sigma_{k,i}(s)| \leq \delta_{k,i} + \beta_k |s|, \quad \forall s \in \mathbb{R}, \text{ and } i \in \mathbb{Z},
$$

where both sequences $(\delta_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}}$ and $(\beta_k)_{k \in \mathbb{N}}$ belong to $\ell^2$.

For convenience, we set

$$
\alpha = (\alpha_i)_{i \in \mathbb{Z}}, \quad L = (L_k)_{k \in \mathbb{N}}, \quad \beta = (\beta_k)_{k \in \mathbb{N}}, \quad \delta = (\delta_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}},
$$

and

$$
\|\alpha\|^2 = \sum_{i \in \mathbb{Z}} |\alpha_i|^2, \quad \|L\|^2 = \sum_{k \in \mathbb{N}} |L_k|^2, \quad \|\beta\|^2 = \sum_{k \in \mathbb{N}} |\beta_k|^2, \quad \|\delta\|^2 = \sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{Z}} |\delta_{k,i}|^2.
$$

In order to formulate the lattice system (1)-(2) as a differential equation in $\ell^2$, we write, for every $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$,

$$(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i, \quad i \in \mathbb{Z},$$

and

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad i \in \mathbb{Z}.$$  

Note that $A = BB^* = B^*B$ and

$$(B^*u, v) = (u, Bv), \quad \text{for all } u, v \in \ell^2.$$
where $(\cdot, \cdot)$ is the inner product of $\ell^2$.

For every $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, denote by $f(u) = (f_i(u_i))_{i \in \mathbb{Z}}$. By (4)-(5), one can verify that $f$ maps $\ell^2$ to $\ell^2$ and is globally Lipschitz continuous; more precisely, for all $u, v \in \ell^2$,

$$\|f(u)\|^2 \leq 2\|\alpha\|^2 + 2\beta^2\|u\|^2,$$

and

$$\|f(u) - f(v)\| \leq L_0\|u - v\|.$$  

Analogously, for each $k \in \mathbb{N}$ and each $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, denote by $\sigma_k(u) = (\sigma_{k,i}(u_i))_{i \in \mathbb{Z}}$. Then by (6)-(7), one can get

$$\sum_{k \in \mathbb{N}} \|\sigma_k(u)\|^2 \leq 2\|\delta\|^2 + 2\|\beta\|^2\|u\|^2,$$

and

$$\sum_{k \in \mathbb{N}} \|\sigma_k(u) - \sigma_k(v)\|^2 \leq \|L\|^2\|u - v\|^2,$$

which implies that $\sigma_k : \ell^2 \to \ell^2$ is globally Lipschitz continuous for every $k \in \mathbb{N}$.

Then problem (1)-(2) can be reformulated as a stochastic delay equation in $\ell^2$ for $t > 0$:

$$du(t) + \nu Au(t)dt + \lambda u(t)dt = (f(u(t - \rho)) + g)dt + \sum_{k=1}^{\infty} (h_k + \sigma_k(u(t)))dW_k(t),$$

with initial condition

$$u(s) = \xi(s), \quad s \in [-\rho, 0].$$

From now on, we denote the segment of $u$ by $u_t$ which is defined by

$$u_t(s) = u(t + s) \quad \forall s \in [-\rho, 0].$$

The solutions of system (12)-(13) will be understood in the following sense.

**Definition 2.1.** Suppose $\xi \in L^2(\Omega, C([\rho, 0], \ell^2))$ is $\mathcal{F}_0$-measurable. Then a continuous $\ell^2$-valued stochastic process $u(t)$ with $t \in [-\rho, \infty)$ is called a solution of system (12)-(13) if $(u_t)_{t \geq 0}$ is $\mathcal{F}_t$-adapted, $u_0 = \xi$, $u \in L^2(\Omega, C([-\rho, T], \ell^2))$ for all $T > -\rho$, and for each $t \geq 0$,

$$u(t) + \nu \int_0^t A u(s)ds + \lambda \int_0^t u(s)ds$$

$$= \xi(0) + \int_0^t (f(u(s - \rho)) + g)ds + \sum_{k=1}^{\infty} \int_0^t (h_k + \sigma_k(u(s)))dW_k(s)$$

in $\ell^2$ for almost all $\omega \in \Omega$.

Similar to stochastic delay equations in $\mathbb{R}^n$ (see, e.g., [49, p. 150, Theorem 2.2]), under conditions (3)-(7), one can prove that for any $\mathcal{F}_0$-measurable $\xi \in L^2(\Omega, C([-\rho, 0], \ell^2))$, system (12)-(13) has a solution $u \in L^2(\Omega, C([-\rho, T], \ell^2))$ for every $T > -\rho$ in the sense of Definition 2.1. Furthermore, this solution is unique in the sense that if $v$ is any other solution of (12)-(13), then

$$P \{\{u(t) = v(t) \quad \text{for all} \quad t \geq -\rho\} \} = 1.$$  

Actually, for any initial time $t_0 \geq 0$ and any $\mathcal{F}_{t_0}$-measurable $\xi \in L^2(\Omega, C([-\rho, 0], \ell^2))$, system (12)-(13) has a unique solution defined for $t \in [t_0 - \rho, \infty)$.

For later purpose, we now establish the Lipschitz continuity of solutions of problem (12)-(13) with respect to initial data in $L^2(\Omega, C([-\rho, 0], \ell^2))$. 


Lemma 2.2. Suppose (3)-(7) hold, and \( \xi_1, \xi_2 \in L^2(\Omega, C([-\rho, 0], C^2)) \). If \( u(t, \xi_1) \) and \( u(t, \xi_2) \) are the solutions of system (12)-(13) with initial data \( \xi_1 \) and \( \xi_2 \), respectively, then for any \( t \geq 0 \),

\[
E \left( \sup_{-\rho \leq s \leq t} \|u(s, \xi_1) - u(s, \xi_2)\|^2 \right) \leq (1 + M_0 e^{M_0 t}) E(\|\xi_1 - \xi_2\|_{C^2}^2),
\]

where \( M_0 \) is a positive constant independent of \( t, \xi_1 \) and \( \xi_2 \).

Proof. By (12)-(13), we get, for all \( t \geq 0 \),

\[
\begin{align*}
u \int_0^t A(u(s, \xi_1) - u(s, \xi_2))ds \\
+ \lambda \int_0^t (u(s, \xi_1) - u(s, \xi_2))ds \\
= \xi_1(0) - \xi_2(0) + \int_0^t (f(u(s, -\rho, \xi_1)) - f(u(s, -\rho, \xi_2)))ds \\
+ \sum_{k=1}^{\infty} (\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k.
\end{align*}
\]

Applying Ito’s formula to (14) we obtain, for all \( t \geq 0 \),

\[
\begin{align*}
\|u(t, \xi_1) - u(t, \xi_2)\|^2 + 2\nu \int_0^t \|B(u(s, \xi_1) - u(s, \xi_2))\|^2 ds \\
+ 2\lambda \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \\
= \|\xi_1(0) - \xi_2(0)\|^2 \\
+ 2 \int_0^t (u(s, \xi_1) - u(s, \xi_2), f(u(s, -\rho, \xi_1)) - f(u(s, -\rho, \xi_2))) ds \\
+ \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \\
+ 2 \sum_{k=1}^{\infty} \int_0^t (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k,
\end{align*}
\]

which implies that for \( t \geq 0 \),

\[
\begin{align*}
& E \left( \sup_{0 \leq r \leq t} \|u(r, \xi_1) - u(r, \xi_2)\|^2 \right) \\
\leq & E(\|\xi_1 - \xi_2\|_{C^2}^2) \\
+ & 2E \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\| \|f(u(s, -\rho, \xi_1)) - f(u(s, -\rho, \xi_2))\| ds \right) \\
+ & E \left( \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \right) \\
+ & 2E \left( \sup_{0 \leq r \leq t} \sum_{k=1}^{\infty} \left| \int_0^r (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k \right| \right). \tag{16}
\end{align*}
\]
Next, we estimate the right-hand side of (16). For the second term, by (9) we get

\[ 2\mathbb{E} \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\| \|f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))\| ds \right) \]
\[ \leq \mathbb{E} \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \right) + \mathbb{E} \left( \int_0^t \|f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))\|^2 ds \right) \]
\[ \leq \mathbb{E} \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \right) + L_0^2 \mathbb{E} \left( \int_0^t \|f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))\|^2 ds \right) \]
\[ \leq (1 + L_0^2) \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \right) + L_0^2 \mathbb{E} \left( \|u(s - \rho, \xi_1) - u(s - \rho, \xi_2)\|^2 \right). \] (17)

For the third term on the right-hand side of (16), by (11) we have

\[ \mathbb{E} \left( \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \right) \]
\[ \leq \|L\|^2 \mathbb{E} \left( \int_0^t \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \right) \]
\[ \leq \|L\|^2 \int_0^t \mathbb{E} \left( \sup_{0 \leq r \leq s} \|u(r, \xi_1) - u(r, \xi_2)\|^2 \right) ds. \] (18)

For the last term on the right-hand side of (16), by the BDG inequality we get

\[ 2\mathbb{E} \left( \sup_{0 \leq r \leq t} \left| \sum_{k=1}^{\infty} \int_0^r (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k \right| \right) \]
\[ \leq c_1 \mathbb{E} \left( \left( \int_0^t \sum_{k=1}^{\infty} \|(u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2)))\|^2 ds \right)^{\frac{1}{2}} \right) \]
\[ \leq c_1 \mathbb{E} \left( \left( \int_0^t \sum_{k=1}^{\infty} \|u(s, \xi_1) - u(s, \xi_2)\|^2 \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \right)^{\frac{1}{2}} \right) \]
\[ \leq c_1 \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u(s, \xi_1) - u(s, \xi_2)\| \left( \int_0^t \sum_{k=1}^{\infty} \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \right)^{\frac{1}{2}} \right) \]
\[ \leq \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u(s, \xi_1) - u(s, \xi_2)\|^2 \right) + \frac{1}{2} c_1^2 \mathbb{E} \left( \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \right). \]

which along with (18) implies that

\[ 2\mathbb{E} \left( \sup_{0 \leq r \leq t} \left| \sum_{k=1}^{\infty} \int_0^r (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k \right| \right) \]
\[
\begin{align*}
\leq & \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq r \leq t} \| u(r, \xi_1) - u(r, \xi_2) \|^2 \right) \\
& + \frac{1}{2} c_1^2 \| L \|^2 \int_0^t \mathbb{E} \left( \sup_{0 \leq r \leq s} \| u(r, \xi_1) - u(r, \xi_2) \|^2 \right) ds.
\end{align*}
\] (19)

It follows from (16)-(19) that for all \( t \geq 0 \),
\[
\begin{align*}
& \mathbb{E} \left( \sup_{0 \leq r \leq t} \| u(r, \xi_1) - u(r, \xi_2) \|^2 \right) \\
& \leq 2(1 + \rho L_0^2) \mathbb{E}(\| \xi_1 - \xi_2 \|_{C_p}^2) \\
& + 2 \left( 1 + L_0^2 + \| L \|^2 + \frac{1}{2} c_1^2 \| L \|^2 \right) \int_0^t \mathbb{E} \left( \sup_{0 \leq r \leq s} \| u(r, \xi_1) - u(r, \xi_2) \|^2 \right) ds.
\end{align*}
\] (20)

Applying the Gronwall inequality to (20), we obtain for all \( t \geq 0 \),
\[
\mathbb{E} \left( \sup_{0 \leq r \leq t} \| u(r, \xi_1) - u(r, \xi_2) \|^2 \right) \leq 2(1 + \rho L_0^2)e^{c_2 t} \mathbb{E}(\| \xi_1 - \xi_2 \|_{C_p}^2),
\]
where \( c_2 = 2 (1 + L_0^2 + \| L \|^2 + \frac{1}{2} c_1^2 \| L \|^2) \). This completes the proof. \( \Box \)

In the sequel, we assume that the constants \( \beta_0 \) and \( \beta \) in (8) and (10) are sufficiently small such that
\[
\lambda > 6 \beta_0 + 12 \| \beta \|^2.
\] (21)

By (21) one can verify that
\[
\lambda > \frac{4 \sqrt{2}}{3} \beta_0 + \frac{8}{3} \| \beta \|^2.
\] (22)

In addition, by (21) we get
\[
\lim_{\mu \to 0} \left( \mu + 24 \| \beta \|^2 + 2 \frac{4}{3} \beta_0 e^{\frac{1}{2} \mu \rho} - 2 \lambda \right) = 24 \| \beta \|^2 + 2 \frac{4}{3} \beta_0 - 2 \lambda < 0.
\] (23)

On the other hand, by (22) we obtain
\[
\lim_{\mu \to 0} \left( \mu + 2 \sqrt{2} \beta_0 e^{\frac{1}{2} \mu \rho} + 4 \| \beta \|^2 - \frac{3}{2} \lambda \right) = 2 \sqrt{2} \beta_0 + 4 \| \beta \|^2 - \frac{3}{2} \lambda < 0.
\] (24)

It follows from (23)-(24) that there exists a small number \( \mu > 0 \) such that
\[
\mu + 24 \| \beta \|^2 + 2 \frac{4}{3} \beta_0 e^{\frac{1}{2} \mu \rho} - 2 \lambda < 0 \quad \text{and} \quad \mu + 2 \sqrt{2} \beta_0 e^{\frac{1}{2} \mu \rho} + 4 \| \beta \|^2 - \frac{3}{2} \lambda < 0.
\] (25)

From now on, we fix such a \( \mu > 0 \) satisfying (25).

3. **Uniform estimates.** In this section, we derive uniform estimates of the solutions of problem (12)-(13) which are necessary for establishing the existence of invariant measures. In particular, we will show the tightness of a family of probability distributions of \( u_t \) in \( C([-\rho, 0], \ell^2) \).

We first discuss uniform estimates of solutions of problem (12)-(13) in \( L^2(\Omega, \ell^2) \) for all \( t \geq 0 \).

**Lemma 3.1.** Suppose (3)-(7) and (21) hold. If \( \xi \in L^2(\Omega, C([-\rho, 0], \ell^2)) \), then the solution \( u \) of (12)-(13) satisfies
\[
\sup_{t \geq -\rho} \mathbb{E}(\| u(t) \|^2) \leq M_1 \left( 1 + \mathbb{E}(\| \xi \|_{C_p}^2) \right)
\]
where \( M_1 \) is a positive constant independent of \( \xi \).
Proof. By (12) and Ito’s formula, we get for $t \geq 0$, 
\[
d(\|u(t)\|^2) + 2\nu\|Bu(t)\|^2 dt + 2\lambda\|u(t)\|^2 dt
= 2(u(t), f(u(t - \rho))) dt + 2(u(t), g) dt + \sum_{k=1}^{\infty} \|h_k + \sigma_k(u(t))\|^2 dt
+ 2 \sum_{k=1}^{\infty} (u(t), h_k + \sigma_k(u(t))) dW_k.
\] 
(26)

Let $\mu$ be the positive constant satisfying (25). Then we obtain from the above equality that for all $t \geq 0$, 
\[
e^{\mu t}\|u(t)\|^2 + 2\nu \int_{0}^{t} e^{\mu s} \|Bu(s)\|^2 ds
= \|\xi(0)\|^2 + (\mu - 2\lambda) \int_{0}^{t} e^{\mu s} \|u(s)\|^2 ds
+ 2 \int_{0}^{t} e^{\mu s}(u(s), f(u(s - \rho))) ds + 2 \int_{0}^{t} e^{\mu s}(u(s), g) ds
+ \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} \|h_k + \sigma_k(u(s))\|^2 ds + 2 \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s}(u(s), h_k + \sigma_k(u(s))) dW_k.
\]

Taking the expectation, we obtain for $t \geq 0$, 
\[
e^{\mu t}\mathbb{E}(\|u(t)\|^2) + 2\nu \int_{0}^{t} e^{\mu s}\mathbb{E}(\|Bu(s)\|^2) ds
= \mathbb{E}(\|\xi(0)\|^2) + (\mu - 2\lambda) \int_{0}^{t} e^{\mu s}\mathbb{E}(\|u(s)\|^2) ds
+ 2 \int_{0}^{t} e^{\mu s}\mathbb{E}(u(s), f(u(s - \rho))) ds + 2 \int_{0}^{t} e^{\mu s}\mathbb{E}(u(s), g) ds
+ \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} (\|h_k + \sigma_k(u(s))\|^2) ds.
\] 
(27)

We now deal with the right-hand side of (27). For the nonlinear drift term on the right-hand side of (27), by (8) we have 
\[
2 \int_{0}^{t} e^{\mu s}\mathbb{E}(u(s), f(u(s - \rho))) ds
\leq 2 \int_{0}^{t} e^{\mu s}\mathbb{E}(\|u(s)\| \|f(u(s - \rho))\|) ds
\leq \sqrt{2}\beta_0 e^{\frac{\mu}{2}} \int_{0}^{t} e^{\mu s}\mathbb{E}(\|u(s)\|^2) ds + \frac{1}{\sqrt{2}\beta_0 e^{\frac{\mu}{2}}} \int_{0}^{t} e^{\mu s}\mathbb{E}(\|f(u(s - \rho))\|^2) ds
\leq \sqrt{2}\beta_0 e^{\frac{\mu}{2}} \int_{0}^{t} e^{\mu s}\mathbb{E}(\|u(s)\|^2) ds + \frac{\sqrt{2}\|\alpha\|^2}{\beta_0 e^{\frac{\mu}{2}}} \int_{0}^{t} e^{\mu s} ds
+ \frac{\sqrt{2}\beta_0}{e^{\frac{\mu}{2}}} \int_{0}^{t} e^{\mu s}\mathbb{E}(\|u(s - \rho)\|^2) ds
\]
\[
\begin{align*}
&\leq \sqrt{2} \beta_0 e^{1/\beta} \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds + \frac{\sqrt{2} \|\alpha\|^2}{\beta_0 \mu e^{1/\beta}} e^{\mu t} \\
&\quad + \sqrt{2} \beta_0 e^{1/\beta} \int_{-\rho}^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds \\
&\leq \sqrt{2} \beta_0 e^{1/\beta} \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds + \frac{\sqrt{2} \|\alpha\|^2}{\beta_0 \mu e^{1/\beta}} e^{\mu t} \\
&\quad + \sqrt{2} \beta_0 e^{1/\beta} \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds + \sqrt{2} \beta_0 e^{1/\beta} \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds \\
&\leq 2 \sqrt{2} \beta_0 e^{1/\beta} \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds + \frac{\sqrt{2} \|\alpha\|^2}{\beta_0 \mu e^{1/\beta}} e^{\mu t} \\
&\quad + \sqrt{2} \beta_0 e^{1/\beta} \mathbb{P} (\xi_C^2). 
\end{align*}
\]

By Young’s inequality, we get
\[
2 \int_0^t e^{\mu s} \mathbb{E} (u(s), g) ds \leq \frac{1}{2} \lambda \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds + \frac{2}{\lambda \mu} \|g\|^2 e^{\mu t}.
\]

For the last term on the right-hand side of (27), by (10) we obtain
\[
\sum_{k=1}^{\infty} \int_0^t e^{\mu s} \mathbb{E} (\|h_k + \sigma_k (u(s))\|^2) ds \\
\leq \sum_{k=1}^{\infty} \int_0^t e^{\mu s} \mathbb{E} (2 \|h_k\|^2 + 2 \|\sigma_k (u(s))\|^2) ds \\
\leq 2 \sum_{k=1}^{\infty} \int_0^t e^{\mu s} \mathbb{E} (\|h_k\|^2) ds + 2 \sum_{k=1}^{\infty} \int_0^t e^{\mu s} \mathbb{E} (\|\sigma_k (u(s))\|^2) ds \\
\leq \frac{2}{\mu} \sum_{k=1}^{\infty} \|h_k\|^2 e^{\mu t} + 4 \int_0^t e^{\mu s} \mathbb{E} (\|\beta\|^2 + \|\beta\|^2 \|u(s)\|^2) ds \\
\leq \frac{2}{\mu} \left( \sum_{k=1}^{\infty} \|h_k\|^2 + 2 \|\beta\|^2 \right) e^{\mu t} + 4 \|\beta\|^2 \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds.
\]

It follows from (27)-(30) that, for \( t \geq 0 \),
\[
e^{\mu t} \mathbb{E} (\|u(t)\|^2) \leq (1 + \sqrt{2} \beta_0 e^{1/\beta}) \mathbb{E} (\|\xi\|^2) \\
+ \left( \mu + 2 \sqrt{2} \beta_0 e^{1/\beta} + 4 \|\beta\|^2 \right) \int_0^t e^{\mu s} \mathbb{E} (\|u(s)\|^2) ds \\
+ \left( \frac{\sqrt{2} \|\alpha\|^2}{\beta_0 e^{1/\beta}} + \frac{2}{\lambda} \|g\|^2 + 2 \sum_{k=1}^{\infty} \|h_k\|^2 + 4 \|\delta\|^2 \right) \frac{1}{\mu} e^{\mu t},
\]

which along with (25) shows that for all \( t \geq 0 \),
\[
\mathbb{E} (\|u(t)\|^2) \leq (1 + \sqrt{2} \beta_0 e^{1/\beta}) \mathbb{E} (\|\xi\|^2) e^{-\mu t} \\
+ \mu^{-1} \left( \frac{\sqrt{2} \|\alpha\|^2}{\beta_0 e^{1/\beta}} + \frac{2}{\lambda} \|g\|^2 + 2 \sum_{k=1}^{\infty} \|h_k\|^2 + 4 \|\delta\|^2 \right),
\]

Then the desired estimates follow from (31) and the fact that \( \sup_{-\rho \leq s \leq 0} \mathbb{E} (\|u(s)\|^2) \leq \mathbb{E} (\|\xi\|^2) \).
Lemma 3.2. Suppose (3)-(7) and (21) hold. If \( \xi \in L^4(\Omega, C([-\rho, 0], \ell^2)) \), then the solution \( u \) of (12)-(13) satisfies,

\[
\sup_{t \geq -\rho} \mathbb{E}(\|u(t)\|^4) \leq M_2 \left( 1 + \mathbb{E}(\|\xi\|_{L^4}^4) \right)
\]

where \( M_2 \) is a positive constant independent of \( \xi \).

Proof. Given \( n \in \mathbb{N} \), define a stopping time \( \tau_n \) by

\[
\tau_n = \inf\{ t \geq 0 : \|u(t)\| > n \},
\]

and \( \tau_n = +\infty \) if the set \( \{ t \geq 0 : \|u(t)\| > n \} = \emptyset \). Then by the continuity of solutions, we have

\[
\lim_{n \to \infty} \tau_n = +\infty.
\]

By (26) and Ito’s formula, we get for all \( t \geq 0 \),

\[
\begin{align*}
&d(\|u(t)\|^4) + 4\nu\|u(t)\|^2\|Bu(t)\|^2dt + 4\lambda\|u(t)\|^4dt \\
= &4\|u(t)\|^2(u(t), f(u(t) - \rho))dt + 4\|u(t)\|^2(u(t), g)dt \\
&+ 2\|u(t)\|^2 \sum_{k=1}^{\infty} \|h_k + \sigma_k(u(t))\|^2dt + 4\sum_{k=1}^{\infty} [(u(t), h_k + \sigma_k(u(t))]^2dt \\
&+ 4\sum_{k=1}^{\infty} \|u(t)\|^2(u(t), h_k + \sigma_k(u(t)))dW_k.
\end{align*}
\]

Let \( \mu \) be the positive number satisfying (25). Then by (32) we get

\[
\begin{align*}
&\mathbb{E}(\mu(t \wedge \tau_n)^4) + 4\nu\int_0^{t \wedge \tau_n} \mathbb{E}(u(s))\|Bu(s)\|^2 ds \\
= &\mathbb{E}(\xi(0)^4) + (\mu - 4\lambda)\int_0^{t \wedge \tau_n} \mathbb{E}(u(s))\|Bu(s)\|^2 ds + 4\int_0^{t \wedge \tau_n} \mathbb{E}(u(s), f(u(s) - \rho)) ds \\
&+ 4\int_0^{t \wedge \tau_n} \mathbb{E}(u(s), g) ds + 2\sum_{k=1}^{\infty} \int_0^{t \wedge \tau_n} \mathbb{E}(u(s))\|h_k + \sigma_k(u(s))\|^2 ds \\
&+ 4\sum_{k=1}^{\infty} \int_0^{t \wedge \tau_n} \mathbb{E}(u(s), h_k + \sigma_k(u(s)))^2 ds \\
&+ 4\sum_{k=1}^{\infty} \int_0^{t \wedge \tau_n} \mathbb{E}(u(s))^2(u(s), h_k + \sigma_k(u(s)))dW_k,
\end{align*}
\]

which implies that for all \( t \geq 0 \),

\[
\mathbb{E}(\mu(t \wedge \tau_n)^4) + 4\nu\mathbb{E}\left( \int_0^{t \wedge \tau_n} \mathbb{E}(u(s))^2\|Bu(s)\|^2 ds \right)
\]
For the last term on the right-hand side of (35) we have
\[ = \mathbb{E}(\|\xi(0)\|^4) + (\mu - 4\lambda)\mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) \]
\[ + 4\mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^2 (u(s), f(u(s - \rho))) ds \right) \]
\[ + 4\mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^2 (u(s), g) ds \right) \]
\[ + 2\mathbb{E} \left( \sum_{k=1}^{\infty} \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^2 \|h_k + \sigma_k(u(s))\|^2 ds \right) \]
\[ + 4\mathbb{E} \left( \sum_{k=1}^{\infty} \int_0^{t \wedge \tau_n} e^{\mu s} \|(u(s), h_k + \sigma_k(u(s)))\|^2 ds \right). \]

We now estimate the right-hand side of (33). Note that Young’s inequality implies
\[ a^3 b \leq \varepsilon a^4 + \varepsilon a^4 + \frac{27}{256\varepsilon^3} b^4, \quad \text{for all} \quad a \geq 0, \quad b \geq 0 \quad \text{and} \quad \varepsilon > 0. \] (34)

By (8) and (34), for the third term on the right-hand side of (33) we have
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) \]
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|f(u(s - \rho))\|^4 ds \right) \]
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} (8\|\alpha\|^4 + 8\beta_0^4\|u(s - \rho)\|^4) ds \right) \]
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) + 6^{\frac{3}{4}} \beta_0 e^{-\frac{3}{4} \mu \rho} \|\alpha\|^4 \mu^{-1} e^\mu \]
\[ + 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s - \rho)\|^4 ds \right). \] (35)

For the last term on the right-hand side of (35) we have
\[ 6^{\frac{3}{4}} \beta_0 e^{-\frac{3}{4} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s - \rho)\|^4 ds \right) \]
\[ = 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \mathbb{E} \left( \int_{-\rho}^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) \]
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{4} \mu \rho} \int_{-\rho}^{0} e^{\mu s} \left( \|\xi(s)\|^4 \right) ds + 6^{\frac{3}{4}} \beta_0 e^{2 \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) \]
\[ \leq 6^{\frac{3}{4}} \beta_0 e^{2 \mu \rho} \mathbb{E}(\|\xi\|^4_{C_{\tau_n}}) + 6^{\frac{3}{4}} \beta_0 e^{\frac{1}{2} \mu \rho} \mathbb{E} \left( \int_0^{t \wedge \tau_n} e^{\mu s} \|u(s)\|^4 ds \right) \]
which together with (35) shows that

\[
4E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 (u(s), f(u(s - \rho))) ds\right) \\
\leq 2 \times 34.5 \beta_0 e^{\frac{1}{2} \mu t} E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^4 ds\right) \\
+ 6 \beta_0^{-\frac{4}{3}} e^{-\frac{2}{3} \mu t} \|\alpha\|^4 \mu^{-1} e^{\mu t} + 6 \beta_0 e^{\frac{1}{2} \rho t} E\left(\|\xi\|^4\right). \tag{36}
\]

By (34), for the fourth term on the right-hand side of (33), we have

\[
4E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 (u(s), g) ds\right) \\
\leq 4E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^3 \|g\| ds\right) \\
\leq E\left(\int_0^{t\wedge\tau_n} e^{\mu s} (\lambda \|u(s)\|^4 + 27 \lambda^{-3} \|g\|^4) ds\right) \\
\leq \lambda E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^4 ds\right) + 27 \lambda^{-3} \mu^{-1} \|g\|^4 e^{\mu t}. \tag{37}
\]

For the last two terms on the right-hand side of (33), by (10) we get

\[
2E\left(\sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 \|h_k + \sigma_k(u(s))\|^2 ds\right) \\
+ 4E\left(\sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s} \|h_k + \sigma_k(u(s))\|^2 ds\right) \\
\leq 6E\left(\sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 \|h_k + \sigma_k(u(s))\|^2 ds\right) \\
\leq E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 \left(12 \sum_{k=1}^{\infty} \|h_k\|^2 + 24 \|\delta\|^2 + 24 \|\|u(s)\|^2\|\right) ds\right) \\
\leq E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^2 \left(12 \sum_{k=1}^{\infty} \|h_k\|^2 + 24 \|\delta\|^2\right) ds\right) \\
+ 24 \|\beta\|^2 E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^4 ds\right) \\
\leq E\left(\int_0^{t\wedge\tau_n} e^{\mu s} (\lambda \|u(s)\|^4 + \frac{1}{4} \lambda^{-1} (12 \sum_{k=1}^{\infty} \|h_k\|^2 + 24 \|\delta\|^2)^2) ds\right) \\
+ 24 \|\beta\|^2 E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^4 ds\right) \\
\leq (\lambda + 24 \|\beta\|^2) E\left(\int_0^{t\wedge\tau_n} e^{\mu s} \|u(s)\|^4 ds\right) \\
+ 36 \lambda^{-1} \mu^{-1} \sum_{k=1}^{\infty} \|h_k\|^2 + 2 \|\delta\|^2 e^{\mu t}. \tag{38}
\]
It follows from (33) and (36)-(38) that for all $t \geq 0$,
\[
\mathbb{E}\left(e^{\mu(t \wedge \tau_\nu)}\|u(t \wedge \tau_\nu)\|^4\right) \leq \left(1 + 6^2 \beta_0 \rho \frac{1}{4} \mu \rho\right) \mathbb{E}(\|\xi\|^4_{C_\nu}) + \left(\mu + 24\|\beta\|^2 + 2^2 3^2 \beta_0 e^{\frac{1}{4} \mu \rho} - 2\lambda\right) \mathbb{E}\left(\int_0^{t \wedge \tau_\nu} e^{\mu s} \|u(s)\|^4 ds\right)
\]
\[
+ \mu e^{\mu t} \left(6^2 \beta_0^{-3} e^{-\frac{3}{4} \mu \rho}\|\alpha\|^4 + 27\lambda^{-3}\|g\|^4 + 36\lambda^{-1}\sum_{k=1}^{\infty} \|h_k\|^2 + 2\|\delta\|^2\right)^2.
\]
By (25) we get from the above inequality that for all $t \geq 0$,
\[
\mathbb{E}\left(e^{\mu(t \wedge \tau_\nu)}\|u(t \wedge \tau_\nu)\|^4\right) \leq \left(1 + 6^2 \beta_0 \rho \frac{1}{4} \mu \rho\right) \mathbb{E}(\|\xi\|^4_{C_\nu})
\]
\[
+ \mu e^{\mu t} \left(6^2 \beta_0^{-3} e^{-\frac{3}{4} \mu \rho}\|\alpha\|^4 + 27\lambda^{-3}\|g\|^4 + 36\lambda^{-1}\sum_{k=1}^{\infty} \|h_k\|^2 + 2\|\delta\|^2\right)^2.
\]
Letting $n \to \infty$, we obtain from (39) that for all $t \geq 0$,
\[
e^{\mu t} \mathbb{E}\left(\|u(t)\|^4\right) \leq \left(1 + 6^2 \beta_0 \rho \frac{1}{4} \mu \rho\right) \mathbb{E}(\|\xi\|^4_{C_\nu})
\]
\[
+ \mu e^{\mu t} \left(6^2 \beta_0^{-3} e^{-\frac{3}{4} \mu \rho}\|\alpha\|^4 + 27\lambda^{-3}\|g\|^4 + 36\lambda^{-1}\sum_{k=1}^{\infty} \|h_k\|^2 + 2\|\delta\|^2\right)^2,
\]
and hence for all $t \geq 0$,
\[
\mathbb{E}\left(\|u(t)\|^4\right) \leq \left(1 + 6^2 \beta_0 \rho \frac{1}{4} \mu \rho\right) e^{-\mu t} \mathbb{E}(\|\xi\|^4_{C_\nu})
\]
\[
+ \mu^{-1} \left(6^2 \beta_0^{-3} e^{-\frac{3}{4} \mu \rho}\|\alpha\|^4 + 27\lambda^{-3}\|g\|^4 + 36\lambda^{-1}\sum_{k=1}^{\infty} \|h_k\|^2 + 2\|\delta\|^2\right)^2.
\]
Note that for all $t \in [-\rho, 0]$,\n\[
\mathbb{E}(\|u(t)\|^4) \leq \mathbb{E}(\|\xi\|^4_{C_\nu}),
\]
which along with (40) concludes the proof. \hfill \Box

Lemma 3.3. Suppose (3)-(7) and (21) hold. If $\xi \in L^4(\Omega, C([-\rho, 0], \ell^2))$, then the solution $u$ of (12)-(13) satisfies, for any $t > r \geq 0$,
\[
\mathbb{E}(\|u(t) - u(r)\|^4) \leq M_3(|t - r|^2 + |t - r|^4),
\]
where $M_3$ is a positive constant depending on $\xi$, but independent of $t$ and $r$.

Proof. It follows from (12) that for $t > r \geq 0$,
\[
u (u(t) - u(r)) = - \nu \int_r^t A u(s) ds - \lambda \int_r^t u(s) ds + \int_r^t (f(u(s - \rho)) + g) ds
\]
\[
+ \sum_{k=1}^{\infty} \int_r^t (h_k + \sigma_k(u(s))) dW_k(s)
\]
and hence
\[
\|u(t) - u(r)\| \leq (\lambda + 4\nu) \int_r^t \|u(s)\| ds + |t - r| \|g\| + \int_r^t \|f(u(s - \rho))\| ds
\]
\[
+ \sum_{k=1}^{\infty} \int_r^t \|h_k + \sigma_k(u(s))\| dW_k(s),
\]
(41)
By (41) we get for $t > r \geq 0$,
\[
\mathbb{E} \left( \|u(t) - u(r)\|^4 \right)
\leq 64(\lambda + 4\nu)^4 \mathbb{E} \left( \left( \int_r^t \|u(s)\| ds \right)^4 \right) \\
+ 64|t - r|^4 \|g\|^4 + 64\mathbb{E} \left( \left( \int_r^t \|f(u(s) - \rho)\| ds \right)^4 \right) \\
+ 64\mathbb{E} \left( \left\| \sum_{k=1}^{\infty} \int_r^t (h_k + \sigma_k(u(s))) dW_k(s) \right\|^4 \right).
\] (42)

Next, we deal with each term on the right-hand side of (42). For the first term, by Lemma 3.2 we obtain
\[
64(\lambda + 4\nu)^4 \mathbb{E} \left( \left( \int_r^t \|u(s)\| ds \right)^4 \right)
\leq 64(\lambda + 4\nu)^4 |t - r|^3 \int_r^t \mathbb{E}(\|u(s)\|^4) ds
\leq c_1 |t - r|^4,
\] (43)
where $c_1 > 0$ depends on $\xi$, but independent of $t$ and $r$.

For the third term on the right-hand side of (42), by (8) and Lemma 3.2 we obtain
\[
64\mathbb{E} \left( \left( \int_r^t \|f(u(s) - \rho)\| ds \right)^4 \right)
\leq 512|t - r|^3 \int_r^t \mathbb{E}(\|u(s)\|^4) ds
\leq 512|\alpha|^4 |t - r|^4 + 512\beta_0^4 |t - r|^3 \int_r^t \mathbb{E}(\|u(s)\|^4) ds
\leq 512|\alpha|^4 |t - r|^4 + 512\beta_0^4 |t - r|^3 \int_{t-\rho}^{t} \mathbb{E}(\|u(s)\|^4) ds
\leq c_2 |t - r|^4,
\] (44)
where $c_2 > 0$ is a constant depending on $\xi$.

For the last term on the right-hand side of (42), by (10), Lemma 3.2 and the BDG inequality we get
\[
64\mathbb{E} \left( \left\| \sum_{k=1}^{\infty} \int_r^t (h_k + \sigma_k(u(s))) dW_k(s) \right\|^4 \right)
\leq c_3 \mathbb{E} \left( \left( \int_r^t \sum_{k=1}^{\infty} \|h_k + \sigma_k(u(s))\|^2 ds \right)^2 \right)
\leq c_3 \mathbb{E} \left( \left( \int_r^t \sum_{k=1}^{\infty} (2\|h_k\|^2 + 2\|\sigma_k(u(s))\|^2) ds \right)^2 \right)
\]
Lemma 3.4. Suppose (3.4). Given \( n \) and \( \theta \).

Next, we derive uniform estimates on the tails of solutions to problem (12)-(13) which are crucial for establishing the tightness of a family of distributions of solutions.

**Lemma 3.4.** Suppose (3)-(7) and (21) hold. If \( \xi \in L^2(\Omega, C([-\rho, 0], \ell^2)) \), then the solution \( u \) of (12)-(13) satisfies

\[
\limsup_{n \to \infty} \sup_{t \geq -\rho} \sum_{|i| \geq n} \mathbb{E}(|u_i(t)|^2) = 0.
\]

**Proof.** Let \( \theta : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( 0 \leq \theta(s) \leq 1 \) for all \( s \in \mathbb{R} \) and

\[
\theta(s) = 0 \text{ for } |s| \leq 1; \quad \text{and } \theta(s) = 1 \text{ for } |s| \geq 2.
\]

Given \( n \in \mathbb{N} \), denote by \( \theta_n = \left( \frac{\theta}{n} \right)_{i \in \mathbb{Z}} \) and \( \theta_n u = \left( \theta\left( \frac{1}{n} \right) u_i \right)_{i \in \mathbb{Z}} \) for \( u = (u_i)_{i \in \mathbb{Z}} \). By (12) we get

\[
d(\theta_n u) + (\nu \theta_n Au + \lambda \theta_n u) dt = (\theta_n f(u(t - \rho)) + \theta_n g) dt + \sum_{k=1}^\infty (\theta_n h_k + \theta_n \sigma_k(u)) dW_k.
\]

By (47) and Ito’s formula we obtain

\[
\begin{aligned}
&d(\|\theta_n u(t)\|^2) + 2\nu(Bu(t), B^T(\theta_n u(t))) dt + 2\lambda\|\theta_n u(t)\|^2 dt \\
= &2(\theta_n u(t), \theta_n f(u(t - \rho))) dt + 2(\theta_n g, \theta_n u(t)) dt \\
&+ \sum_{k=1}^\infty \|\theta_n h_k + \theta_n \sigma_k(u(t))\|^2 dt + 2 \sum_{k=1}^\infty (\theta_n^2 u(t), h_k + \sigma_k(u(t))) dW_k.
\end{aligned}
\]

where \( c_4 > 0 \) is a constant depending on \( \xi \). It follows from (42)-(45) that for all \( t > r \geq 0 \),

\[
\mathbb{E}(\|u(t) - u(r)\|^4) \leq c_5(t - r)^2 + |t - r|^4,
\]

as desired. \( \square \)
Let $\mu$ be the positive constant satisfying (25). Then by (48) we obtain for all $t \geq 0$,
\[
e^{\mu t}\|\theta_n u(t)\|^2 = \|\theta_n \xi(0)\|^2 - 2\nu \int_0^t e^{\mu s}(Bu(s), B(\theta_n^2 u(s)))ds \\
+ (\mu - 2\lambda) \int_0^t e^{\mu s}\|\theta_n u(s)\|^2 ds \\
+ 2 \int_0^t e^{\mu s}(\theta_n u(s), \theta_n f(u(s - \rho)))ds \\
+ 2 \int_0^t e^{\mu s}(\theta_n u(s), \theta_n g)ds \\
+ \sum_{k=1}^{\infty} \int_0^t e^{\mu s}\|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 ds \\
+ 2 \sum_{k=1}^{\infty} \int_0^t e^{\mu s}(\theta_n^2 u(s), h_k + \sigma_k(u(s)))dW_k.
\]

Consequently, we get for all $t \geq 0$,
\[
e^{\mu t}E(\|\theta_n u(t)\|^2) = E(\|\theta_n \xi(0)\|^2) - 2\nu \int_0^t e^{\mu s}E(Bu(s), B(\theta_n^2 u(s)))ds \\
+ (\mu - 2\lambda) \int_0^t e^{\mu s}E(\|\theta_n u(s)\|^2) ds \\
+ 2 \int_0^t e^{\mu s}E(\theta_n u(s), \theta_n f(u(s - \rho)))ds \\
+ 2 \int_0^t e^{\mu s}E(\theta_n u(s), \theta_n g)ds \\
+ \sum_{k=1}^{\infty} \int_0^t e^{\mu s}E(\|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2) ds.
\]

Next, we estimate all terms on the right-hand side of (49). Note that $\xi \in L^2(\Omega, C([-\rho, 0], \ell^2))$, and hence $\xi(0) \in L^2(\Omega, \ell^2)$. From this we see that for every $\varepsilon > 0$, there exists $N_1 = N_1(\varepsilon, \xi) \geq 1$ such that for all $n \geq N_1$,
\[
\sum_{|i| \geq n} E(|\xi_i(0)|^2) \leq \varepsilon.
\]

By (50) we get for all $n \geq N_1$,
\[
E(\|\theta_n \xi(0)\|^2) = \sum_{i \in \mathbb{Z}} E\left(\left|\theta\left(\frac{i}{n}\right) \xi_i(0)\right|^2\right) \\
= \sum_{|i| \geq n} E\left(\left|\theta\left(\frac{i}{n}\right) \xi_i(0)\right|^2\right) \\
\leq \sum_{|i| \geq n} E(|\xi_i(0)|^2) \\
\leq \varepsilon.
\]
For the second term on the right-hand side of (49) we have

\[-2\nu \int_0^t e^{\mu s} E(Bu(s), B(\theta_n^2 u(s))) ds = -2\nu \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} (u_{i+1} - u_i) \left( \theta^2 \left( \frac{i+1}{n} \right) u_{i+1} - \theta^2 \left( \frac{i}{n} \right) u_i \right) \right) ds \]

\[= -2\nu \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \theta^2 \left( \frac{i+1}{n} \right) (u_{i+1} - u_i)^2 \right) ds \]

\[= -2\nu \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \theta^2 \left( \frac{i}{n} \right) (u_{i+1} - u_i) u_i \right) ds \]

\[\leq 2\nu \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \theta \left( \frac{i+1}{n} \right) + \theta \left( \frac{i}{n} \right) \right) \left| \theta \left( \frac{i+1}{n} \right) - \theta \left( \frac{i}{n} \right) \right| |u_{i+1} - u_i| |u_i| ds \]

\[\leq 74\nu \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \theta \left( \frac{i+1}{n} \right) - \theta \left( \frac{i}{n} \right) \right) |u_{i+1} - u_i| |u_i| ds \]

\[\leq \frac{c_1}{n} \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} |u_{i+1} - u_i| |u_i| \right) ds \]

\[\leq \frac{2c_1}{n} \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} (|u_{i+1}|^2 + |u_i|^2) \right) ds \]

\[= \frac{4c_1}{n} \int_0^t e^{\mu s} (|u(s)|^2) ds, \]

where $c_1 > 0$ depends only on $\theta$.

By (52) and Lemma 3.1 we get for all $t \geq 0$,

\[-2\nu \int_0^t e^{\mu s} E(Bu(s), B(\theta_n^2 u(s))) ds \leq \frac{4c_1 c_2}{n} \int_0^t e^{\mu s} ds \leq 4c_1 c_2 \mu^{-1} n^{-1} e^{\mu t}, \]  

where $c_2 > 0$ depends on $\xi$, but independent of $n$.

For the fourth term on the right-hand side of (49), we obtain

\[2 \int_0^t e^{\mu s} E(\theta_n u(s), \theta_n f(u(s - \rho))) ds \]

\[\leq 2 \int_0^t e^{\mu s} E(|\theta_n u(s)|||\theta_n f(u(s - \rho))||) ds \]

\[\leq \sqrt{2} \beta_0 e^{\frac{c_1}{2} \mu s} \int_0^t e^{\mu s} E(|\theta_n u(s)|^2) ds \]

\[+ \frac{1}{\sqrt{2} \beta_0 e^{\frac{c_1}{2} \mu s}} \int_0^t e^{\mu s} E(|\theta_n f(u(s - \rho))|^2) ds. \]
We now deal with the last term on the right-hand side of (54). By (5) we obtain

\[
\frac{1}{\sqrt{2}N_0 e^{2\mu N_0^2}} \int_0^t e^{\mu s} \mathbb{E} \left( \|\theta_n f(u(s))\|^2 \right) ds
= \frac{1}{\sqrt{2}N_0 e^{2\mu N_0^2}} \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \left( \frac{i}{N} \right)^2 f_i(u(s)) \right)^2 ds
\leq \frac{\sqrt{2}}{\beta_0 e^{2\mu N_0}} \left( \sum_{i \in \mathbb{Z}} \left( \frac{i}{N} \right)^2 \alpha_i^2 \right) \int_0^t e^{\mu s} ds
+ \frac{\sqrt{2}}{\beta_0 e^{2\mu N_0}} \int_0^t e^{\mu s} \left( \sum_{i \in \mathbb{Z}} \left( \frac{i}{N} \right)^2 u_i(s) \right)^2 ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \left( \sum_{i \geq n} \left( \frac{i}{N} \right)^2 \alpha_i^2 \right) + \frac{\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds.
\]

Since \( \alpha = (\alpha_i)_{i \in \mathbb{Z}} \in \ell^2 \), by (54)-(55) we find that there exists \( N_2 = N_2(\varepsilon) \geq 1 \) such that for all \( t \geq 0 \) and \( n \geq N_2 \),

\[
2 \int_0^t e^{\mu s} (\theta_n u(s), \theta_n f(u(s))) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds
\leq \frac{\sqrt{2}e^{2\mu t}}{\beta_0 e^{2\mu N_0^2}} \sum_{|i| \geq n} \alpha_i^2 + \frac{2\sqrt{2}}{\beta_0 e^{2\mu N_0^2}} \int_{-\rho}^t e^{\mu s} \left( \|\theta_n u(s)\|^2 \right) ds.
\]

Since \( \xi \in L^2(\Omega, C([\rho, 0], \ell^2)) \), we have \( \xi \in C([\rho, 0], L^2(\Omega, \ell^2)) \). Since \( [-\rho, 0] \) is compact, we see that the range of \( \xi \), \( \{\xi(s) \in L^2(\Omega, \ell^2) : s \in [-\rho, 0]\} \) is a compact subset of \( L^2(\Omega, \ell^2) \). Thus, given \( \varepsilon > 0 \), there exists \( s_1, \ldots, s_m \in [-\rho, 0] \) such that

\[
\{\xi(s) \in L^2(\Omega, \ell^2) : s \in [-\rho, 0]\} \subseteq \bigcup_{j=1}^m B(\xi(s_j), \frac{1}{2} \sqrt{\varepsilon}),
\]

where \( B(\xi(s_j), \frac{1}{2} \sqrt{\varepsilon}) \) is an open ball in \( L^2(\Omega, \ell^2) \) centered at \( \xi(s_j) \) with radius \( \frac{1}{2} \sqrt{\varepsilon} \). Since \( \xi(s_j) \in L^2(\Omega, \ell^2) \) for \( j = 1, \ldots, m \), there exists \( N_3 = N_3(\varepsilon, \xi) \geq \max\{N_1, N_2\} \) (independent of \( j = 1, \ldots, m \)) such that for all \( n \geq N_3 \) and \( j = 1, \ldots, m \),

\[
\sum_{|i| \geq n} \mathbb{E} (|\xi_i(s_j)|^2) \leq \frac{1}{4} \varepsilon.
\]
It follows from (57)-(58) that for all \( n \geq N_3 \) and \( s \in [-\rho, 0] \),
\[
\sum_{|i| \geq n} \mathbb{E}(|\xi_i(s)|^2) \leq \varepsilon,
\]
and hence for all \( n \geq N_3 \),
\[
\int_{-\rho}^{0} e^{\mu s} \mathbb{E} (|\theta_n \xi(s)|^2) \, ds \leq \int_{-\rho}^{0} e^{\mu s} \sum_{|i| \geq n} \mathbb{E}(|\xi_i(s)|^2) \, ds \leq \rho \varepsilon.
\]
By (56) and (60) we obtain, for all \( t \geq 0 \) and \( n \geq N_3 \),
\[
2 \int_{0}^{t} e^{\mu s} \mathbb{E}(\theta_n u(s), \theta_n f(u(s - \rho))) \, ds \leq \frac{\sqrt{2} e^{\mu t}}{\beta_0 e^{\frac{1}{2} \mu \rho}} \varepsilon + \sqrt{2} \beta_0 e^{\frac{1}{2} \mu \rho} \int_{0}^{t} e^{\mu s} \mathbb{E} (|\theta_n u(s)|^2) \, ds.
\]
On the other hand, by Young’s inequality, for the fifth term on the right-hand side of (49), we get
\[
2 \int_{0}^{t} e^{\mu s} \mathbb{E}(\theta_n u(s), \theta_n g) \, ds \leq \frac{1}{2} \lambda \int_{0}^{t} e^{\mu s} \mathbb{E} (|\theta_n u(s)|^2) \, ds + \frac{2}{\lambda \mu} e^{\mu t} \sum_{|i| \geq n} |g_i|^2.
\]
For the last term on the right-hand side of (49), by (7) we obtain
\[
\sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} \mathbb{E} (|\theta_n h_k + \theta_n \sigma_k(u(s))|^2) \, ds
\]
\[
\leq \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} (2 ||\theta_n h_k||^2 + 2 ||\theta_n \sigma_k(u(s))||^2) \, ds
\]
\[
\leq \frac{2}{\mu} e^{\mu t} \sum_{|i| \geq n} ||h_{k,i}||^2 + 2 \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} \mathbb{E} \left( \left| \theta \left( \frac{i}{n} \right) \sigma_{k,i}(u(s)) \right|^2 \right) \, ds
\]
\[
\leq \frac{2}{\mu} e^{\mu t} \sum_{|i| \geq n} \sum_{k=1}^{\infty} |h_{k,i}|^2 + 2 \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \int_{0}^{t} e^{\mu s} \mathbb{E} \left( \left| \theta \left( \frac{i}{n} \right) \sigma_{k,i}(u(s)) \right|^2 \right) \, ds
\]
\[
\leq \frac{2}{\mu} e^{\mu t} \sum_{|i| \geq n} \sum_{k=1}^{\infty} |h_{k,i}|^2 + 4 \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \delta_{k,i}^2 \int_{0}^{t} e^{\mu s} \, ds
\]
\[
+ 4 \left( \sum_{k=1}^{\infty} \beta_k^2 \right) \int_{0}^{t} e^{\mu s} \mathbb{E} \left( \sum_{i \in \mathbb{Z}} \theta \left( \frac{i}{n} \right) u_i(s) \right) \, ds
\]
\[
\leq \frac{2}{\mu} e^{\mu t} \sum_{|i| \geq n} \sum_{k=1}^{\infty} (|h_{k,i}|^2 + 2 \delta_{k,i}^2) + 4 ||\beta||^2 \int_{0}^{t} e^{\mu s} \mathbb{E} (||\theta_n u(s)||^2) \, ds.
\]
It follows from (49)-(50), (53) and (61)-(63) that, for \( t \geq 0 \) and \( n \geq N_3 \),
\[
\mathbb{E} (||\theta_n u(t)||^2) \leq \left( \mu + 2 \sqrt{2} \beta_0 e^{\frac{1}{2} \mu \rho} + 4 ||\beta||^2 \right) - \frac{3}{2} \lambda \int_{0}^{t} e^{\mu(s-t)} \mathbb{E} (||\theta_n u(s)||^2) \, ds
\]
Suppose Lemma 3.5. as desired. which implies that for all $C$
uniform estimates on the tails of solutions which are needed to establish the the

By (59) and (66) we obtain, for all $N$

By (25) and (64) we get, for all $g$

Since $g = (g_i)_{i \in \mathbb{Z}}$, $h = (h_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}}$ and $\delta = (\delta_{k,i})_{k \in \mathbb{N}, i \in \mathbb{Z}}$ belong to $\ell^2$, we infer from (65) that there exists $N_4 = N_4(\varepsilon, \xi) \geq N_3$ such that for all $t \geq 0$ and $n \geq N_4$,

which implies that for all $t \geq 0$ and $n \geq N_4$,

By (59) and (66) we obtain, for all $n \geq N_4$,

as desired.

We now improve the tail-estimates given by Lemma 3.4 to further derive the uniform estimates on the tails of solutions which are needed to establish the the tightness of probability distributions of the segments of solutions in the space $C([-\rho, 0], \ell^2)$.

**Lemma 3.5.** Suppose (3)-(7) and (21) hold. If $\xi \in L^2(\Omega, C([-\rho, 0], \ell^2))$, then the solution $u$ of (12)-(13) satisfies

$$
\limsup_{n \to \infty} \sup_{t \geq \rho} \left( \sup_{t-\rho \leq r \leq t} \sum_{|i| \geq n} |u_i(r)|^2 \right) = 0.
$$

**Proof.** Let $\theta : \mathbb{R} \to \mathbb{R}$ be the smooth function given by (46), $\theta_n = (\vartheta(\frac{1}{n}))_{i \in \mathbb{Z}}$ and $\theta_n u = (\vartheta(\frac{1}{n}) u_i)_{i \in \mathbb{Z}}$ for $u = (u_i)_{i \in \mathbb{Z}}$ as before. Then by (48), we get for all $t \geq \rho$ and
\( t - \rho \leq r \leq t, \)
\[ \|\theta_n u(r)\|^2 = \|\theta_n u(t - \rho)\|^2 \]
\[ - 2\nu \int_{t - \rho}^r (Bu(s), B(\theta_n^2 u(s))) ds \]
\[ - 2\lambda \int_{t - \rho}^r \|\theta_n u(s)\|^2 ds \]
\[ + 2 \int_{t - \rho}^r (\theta_n u(s), \theta_n f(u(s - \rho))) ds \]
\[ + 2 \int_{t - \rho}^r (\theta_n u(s), \theta_n g(s)) ds + \sum_{k=1}^{\infty} \int_{t - \rho}^r \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 ds \]
\[ + 2 \sum_{k=1}^{\infty} \int_{t - \rho}^r (\theta_n u(s), \theta_n h_k + \theta_n \sigma_k(u(s))) dW_k. \]
First taking the supremum with respect to \( r \) on \([t - \rho, t]\), and then taking the expectation of \((68)\), we get that for all \( t \geq \rho \),

\[
E \left( \sup_{t-\rho \leq r \leq t} \| \theta_n u(r) \|^2 \right)
\leq E(\| \theta_n u(t - \rho) \|^2) + \frac{c}{n} \int_{t-\rho}^t E(\|u(s)\|^2) ds
\]

\begin{align*}
&+ 2E \left( \int_{t-\rho}^t \| \theta_n u(s) \| \| \theta_n f(u(s - \rho)) \| ds \right) \\
&+ 2E \left( \int_{t-\rho}^t \| \theta_n u(s) \| \| \theta_n g \| ds \right) + \sum_{k=1}^{\infty} \int_{t-\rho}^t E \left( \| \theta_n h_k + \theta_n \sigma_k(u(s)) \|^2 \right) ds \\
&+ 2E \left( \sup_{t-\rho \leq r \leq t} \left| \sum_{k=1}^{\infty} \int_{t-\rho}^r (\theta_n u(s), \theta_n h_k + \theta_n \sigma_k(u(s))) dW_k \right| \right). 
\end{align*}

(69)

We now deal with each term on the right-hand side of \((69)\). First, by Lemma 3.4 we find that for any \( \varepsilon > 0 \), there exists \( N_1 = N_1(\varepsilon) \geq 1 \) such that for all \( n \geq N_1 \) and \( s \geq -\rho \),

\[
\sum_{|i| \geq n} E(|u_i(s)|^2) \leq \varepsilon,
\]

which implies that for all \( n \geq N_1 \) and \( s \geq -\rho \),

\[
E(\| \theta_n u(s) \|^2) = \sum_{|i| \geq n} E \left( \left| \theta \left( \frac{i}{n} \right) u_i(s) \right|^2 \right) \leq \sum_{|i| \geq n} E(|u_i(s)|^2) \leq \varepsilon. 
\]

(70)

By \((70)\), for the first term on the right-hand side of \((69)\) we have, for all \( n \geq N_1 \) and \( t \geq \rho \),

\[
E(\| \theta_n u(t - \rho) \|^2) \leq \varepsilon. 
\]

(71)

For the second term on the right-hand side of \((69)\), by Lemma 3.1 we see that there exists \( c_1 = c_1(\xi) > 0 \) such that for all \( t \geq \rho \),

\[
\frac{c}{n} \int_{t-\rho}^t E(\|u(s)\|^2) ds \leq \frac{cc_1\rho}{n}
\]

which shows that there exists \( N_2 = N_2(\varepsilon, \xi) \geq N_1 \) such that for all \( n \geq N_2 \) and \( t \geq \rho \),

\[
\frac{c}{n} \int_{t-\rho}^t E(\|u(s)\|^2) ds \leq \varepsilon. 
\]

(72)

For the third term on the right-hand side of \((69)\), by \((70)\) and \((5)\) we have

\[
2 \int_{t-\rho}^t E(\| \theta_n u(s) \| \| \theta_n f(u(s - \rho)) \| ) ds \\
\leq \int_{t-\rho}^t E(\| \theta_n u(s) \|^2) ds + \int_{t-\rho}^t E(\| \theta_n f(u(s - \rho)) \|^2) ds \\
\leq \rho \varepsilon + \int_{t-2\rho}^{t-\rho} E(\| \theta_n f(u(s)) \|^2) ds \\
= \rho \varepsilon + \int_{t-2\rho}^{t-\rho} E \left( \sum_{i \in \mathbb{Z}} \left| \theta \left( \frac{i}{n} \right) f_i(u_i(s)) \right|^2 \right) ds
\]
By (73) and the fact that \( \alpha = (\alpha_i)_{i \in \mathbb{Z}} \in \ell^2 \), we infer that there exists \( N_3 = N_3(\varepsilon, \xi) \geq N_2 \) such that for all \( t \geq \rho \) and \( n \geq N_3 \),

\[
2 \int_{t-\rho}^{t} \mathbb{E}(\|\theta_n u(s)\||\theta_n f(u(s-\rho))) ds \leq 2\rho \varepsilon + 2\beta_0^2 \rho \varepsilon. \tag{74}
\]

For the fourth term on the right-hand side of (69), by (70) we get

\[
2 \mathbb{E} \left( \int_{t-\rho}^{t} \|\theta_n u(s)\||\theta_n g ds \right) \leq \int_{t-\rho}^{t} \mathbb{E}(\|\theta_n u(s)\|^2) ds + \rho \sum_{|i| \geq n} |g_i|^2 \leq \rho \varepsilon + \rho \sum_{|i| \geq n} |g_i|^2. \tag{75}
\]

Since \( g = (g_i)_{i \in \mathbb{Z}} \in \ell^2 \), we find from (75) that there exists \( N_4 = N_4(\varepsilon, \xi) \geq N_3 \) such that for all \( n \geq N_4 \) and \( t \geq \rho \),

\[
2 \mathbb{E} \left( \int_{t-\rho}^{t} \|\theta_n u(s)\||\theta_n g ds \right) \leq 2 \rho \varepsilon. \tag{76}
\]

For the fifth term on the right-hand side of (69), by (70) and (7) we obtain

\[
\sum_{k=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E} \left( \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 \right) ds \\
\leq 2 \sum_{k=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E} \left( \|\theta_n h_k\|^2 + \|\theta_n \sigma_k(u(s))\|^2 \right) ds \\
\leq 2 \rho \sum_{|i| \geq n} \sum_{k=1}^{\infty} |h_{k,i}|^2 + 2 \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E} \left( \left| \theta \left( \frac{i}{n} \right) \sigma_{k,i}(u_i(s)) \right|^2 \right) ds \\
\leq 2 \rho \sum_{|i| \geq n} \sum_{k=1}^{\infty} |h_{k,i}|^2 + 4 \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E} \left( \theta^2 \left( \frac{i}{n} \right) (\delta_{k,i}^2 + \beta_k^2 |u_{k,i}(s)|^2) \right) ds \\
\leq 4 \rho \sum_{|i| \geq n} \sum_{k=1}^{\infty} (h_{k,i}^2 + \delta_{k,i}^2) + 4 \sum_{k=1}^{\infty} \beta_k \int_{t-\rho}^{t} \mathbb{E} \left( \|\theta_n u(s)\|^2 \right) ds \\
\leq 4 \rho \sum_{|i| \geq n} \sum_{k=1}^{\infty} (h_{k,i}^2 + \delta_{k,i}^2) + 4 \|\beta\|^2 \varepsilon,
\]

which implies that there exists \( N_5 = N_5(\varepsilon, \xi) \geq N_4 \) such that for all \( t \geq \rho \) and \( n \geq N_5 \),

\[
\sum_{k=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E} \left( \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 \right) ds \leq 4(\rho + \|\beta\|^2) \varepsilon. \tag{77}
\]
We now estimate the last term on the right-hand side of (49). By (77) and the BDG inequality we obtain

\[
2\mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \left| \sum_{k=1}^{\infty} \int_{t-\rho}^{r} (\theta_n u(s), \theta_n h_k + \theta_n \sigma_k(u(s))) dW_k \right| \right) 
\leq 2c\mathbb{E} \left( \left( \int_{t-\rho}^{t} \sum_{k=1}^{\infty} (\theta_n u(s), \theta_n h_k + \theta_n \sigma_k(u(s)))^2 ds \right)^{\frac{1}{2}} \right) 
\leq 2c\mathbb{E} \left( \left( \int_{t-\rho}^{t} \|\theta_n u(s)\|^2 \sum_{k=1}^{\infty} \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 ds \right)^{\frac{1}{2}} \right) 
\leq 2c\mathbb{E} \left( \sup_{t-\rho \leq s \leq t} \|\theta_n u(s)\| \left( \int_{t-\rho}^{t} \sum_{k=1}^{\infty} \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 ds \right)^{\frac{1}{2}} \right)
\leq \frac{1}{2} \mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \|\theta_n u(r)\|^2 \right) + 2c^2 \mathbb{E} \left( \left( \int_{t-\rho}^{t} \sum_{k=1}^{\infty} \|\theta_n h_k + \theta_n \sigma_k(u(s))\|^2 ds \right)^{\frac{1}{2}} \right) 
\leq \frac{1}{2} \mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \|\theta_n u(r)\|^2 \right) + 8c^2 (\rho + \|\beta\|^2) \varepsilon.
\]

By (69), (71), (72), (74), (76)-(78) we obtain that for all \( t \geq \rho \) and \( n \geq N_5 \),

\[
\mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \|\theta_n u(r)\|^2 \right) \leq c_1 \varepsilon,
\]
which implies that for all \( t \geq \rho \) and \( n \geq N_5 \),

\[
\mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \sum_{|i| \geq 2n} |u_i(r)|^2 \right) \leq \mathbb{E} \left( \sup_{t-\rho \leq r \leq t} \|\theta_n u(r)\|^2 \right) \leq c_1 \varepsilon.
\]

This concludes the proof. \( \square \)

4. Existence of invariant measures. In this section, we prove the existence of invariant measures of (12)-(13) on \( C([-\rho, 0], \ell^2) \). We first introduce the transition operators of the system and then establish the tightness of a family of probability distributions of solutions.

Recall that for any initial time \( t_0 \geq 0 \) and any \( \mathcal{F}_{t_0} \)-measurable \( \xi \in L^2(\Omega, C([-\rho, 0], \ell^2)) \), system (12)-(13) has a unique solution defined for \( t \in [t_0 - \rho, \infty) \). This solution is denoted by \( u(t, t_0, \xi) \) from now on. For convenience, given \( t \geq t_0 \) and \( \xi \in L^2(\Omega, C([-\rho, 0], \ell^2)) \), we use \( u_t(t_0, \xi) \) to denote the segment of the solution \( u(t, t_0, \xi) \) given by

\[
u_t(t_0, \xi)(s) = u(s + t, t_0, \xi) \quad \text{for all} \quad s \in [-\rho, 0].
\]

Then we have \( u_t(t_0, \xi) \in L^2(\Omega, C([-\rho, 0], \ell^2)) \) for all \( t \geq t_0 \).

If \( \varphi : C([-\rho, 0], \ell^2) \to \mathbb{R} \) is a bounded Borel function, then for \( 0 \leq r \leq t \) and \( \xi \in C([-\rho, 0], \ell^2) \), we set

\[
(p_r \varphi)(\xi) = \mathbb{E} (\varphi(u_t(r, \xi))).
\]

In particular, for \( \Gamma \in \mathcal{B} \{ C([-\rho, 0], \ell^2) \} \), \( 0 \leq r \leq t \) and \( \xi \in C([-\rho, 0], \ell^2) \), we set

\[
p(r, \xi; t, \Gamma) = (p_{r,t} 1_\Gamma)(\xi) = P \{ \omega \in \Omega : u_t(r, \xi) \in \Gamma \},
\]
where $1_\Gamma$ is the characteristic function of $\Gamma$. This show that $p(r, \xi; t, \cdot)$ is the probability distribution of $u(t, \xi)$ in $C([\rho, 0], \ell^2)$. For simplicity, the transition operator $p_{0,t}$ is written as $p_t$. Recall that a probability measure $\nu$ on $C([\rho, 0], \ell^2)$ is called an invariant measure of (12)-(13) if

$$\int_{C([\rho, 0], \ell^2)} (p_t \phi)(\xi) \, d\nu(\xi) = \int_{C([\rho, 0], \ell^2)} \phi(\xi) \, d\nu(\xi), \quad \forall t \geq 0.$$  

Next, we show the existence of invariant measures for problem (12)-(13), for which we need the following properties of the transition operators $\{p_{r,t}\}_{0 \leq r \leq t}$.

**Lemma 4.1.** Suppose (3)-(7) and (21) hold. Then we have:

(i) The family $\{p_{r,t}\}_{0 \leq r \leq t}$ is Feller; that is, if $\phi : C([\rho, 0], \ell^2) \to \mathbb{R}$ is bounded and continuous, then for any $0 \leq r \leq t$, the function $p_{r,t} \phi : C([\rho, 0], \ell^2) \to \mathbb{R}$ is also bounded and continuous.

(ii) The family $\{p_{r,t}\}_{0 \leq r \leq t}$ is homogeneous (in time); that is, for any $0 \leq r \leq t$, \[p(r, \xi; t, \cdot) = p(0, \xi, t-r, \cdot), \quad \forall \xi \in C([\rho, 0], \ell^2).\]

(iii) Given $r \geq 0$ and $\xi \in C([\rho, 0], \ell^2)$, the process $\{u_t(r, \xi)\}_{t \geq r}$ is a $C([\rho, 0], \ell^2)$-valued Markov process. Consequently, if $\phi : C([\rho, 0], \ell^2) \to \mathbb{R}$ is a bounded Borel function, then for any $0 \leq s \leq r \leq t$, $P$-a.s., \[p_{s,t}(\phi)(\xi) = (p_{s,r}(p_{r,t} \phi))(\xi), \quad \forall \xi \in C([\rho, 0], \ell^2),\]

and the Chapman-Kolmogorov equation is valid:

$$p(s, \xi; t, \Gamma) = \int_{C([\rho, 0], \ell^2)} p(s, \xi; r, dy) p(r, y; t, \Gamma)$$

for any $\xi \in C([\rho, 0], \ell^2)$ and $\Gamma \in \mathcal{B}(C([\rho, 0], \ell^2))$.

**Proof.** The Feller property (i) of $\{p_{r,t}\}_{0 \leq r \leq t}$ follows directly from Lemma 2.2, and (ii) and (iii) follow from the standard arguments as in [22, pp. 250-252]. The details are omitted.

We now establish the tightness of a family of probability distributions of segments of solutions to (12)-(13) which is needed for proving the existence of invariant measures.

**Lemma 4.2.** Suppose (3)-(7) and (21) hold. Then the distribution laws of the process $\{u_t(0,0)\}_{t \geq 0}$ is tight on $C([\rho, 0], \ell^2)$.

**Proof.** Note that in the present case, the initial condition $\xi = 0$ at initial time 0, and hence all uniform estimates of the solutions in the previous section are valid. For simplicity, we will write the solution $u(t, 0, 0)$ as $u(t)$ and the segment $u_t(0,0)$ as $u_t$ from now on.

It follows from Lemma 3.1 that there exists a constant $c_1 > 0$ such that $\mathbb{E}(\|u(t)\|^2) \leq c_1$ for all $t \geq -\rho$, which implies that

$$\mathbb{E}(\|u_t(0)\|^2) \leq c_1, \quad \forall t \geq 0. \tag{79}$$

By Chebyshev’s inequality, we get from (79) that for all $t \geq 0$,

$$P(\|u_t(0)\| \geq R) \leq \frac{1}{R^2} \mathbb{E}(\|u_t(0)\|^2) \leq \frac{c_1}{R^2} \to 0 \quad \text{as} \quad R \to 0,$$

and hence for every $\varepsilon > 0$, there exists $R_1 = R_1(\varepsilon) > 0$ such that

$$P(\|u_t(0)\| \geq R_1) \leq \frac{1}{3} \varepsilon, \quad \forall t \geq 0. \tag{80}$$
On the other hand, by Lemma 3.3 we know that for all $t \geq \rho$ and $r, s \in [-\rho, 0]$,
\begin{equation}
E \left( \|u(t + r) - u(t + s)\|^2 \right) \leq c_2(1 + |r - s|^2)|r - s|^2 \tag{81}
\end{equation}
for some number $c_2 > 0$. By (81) we get, for all $t \geq \rho$ and $r, s \in [-\rho, 0]$,
\begin{equation}
E \left( \|u_t(r) - u_t(s)\|^2 \right) \leq c_2(1 + \rho^2)|r - s|^2. \tag{82}
\end{equation}
Since the initial condition $\xi = 0$ in $C([-\rho, 0], \ell^2)$ in the present case, by (82) we infer that for all $t \geq 0$ and $r, s \in [-\rho, 0]$,
\begin{equation}
E \left( \|u_t(r) - u_t(s)\|^2 \right) \leq c_2(1 + \rho^2)|r - s|^2. \tag{83}
\end{equation}
By (83) and the usual technique of diadic division, one can verify that given $\epsilon > 0$, there exists $R_2 = R_2(\epsilon) > 0$ such that for all $t \geq 0$,
\begin{equation}
P \left( \sup_{-\rho \leq s < r \leq 0} \frac{\|u_t(r) - u_t(s)\|}{|r - s|^{\frac{1}{2}}} \leq R_2 \right) > 1 - \frac{1}{3}\epsilon. \tag{84}
\end{equation}
By Lemma 3.5, we infer that for each $\epsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $n_m = n_m(\epsilon, m) \geq 1$ such that for all $t \geq 0$,
\begin{equation}
E \left( \sup_{-\rho \leq s < r \leq 0} \sum_{|i| > n_m} |u_t(r)|^2 \right) < \frac{\epsilon}{2^{m+2}}, \tag{85}
\end{equation}
and hence for all $t \geq 0$ and $m \in \mathbb{N}$,
\begin{equation}
P \left( \sup_{-\rho \leq r \leq t, |i| > n_m} \sum_{|i| > n_m} |u_t(r)|^2 \geq \frac{1}{2^m} \right) \leq 2^m E \left( \sup_{-\rho \leq r \leq t, |i| > n_m} \sum_{|i| > n_m} |u_t(r)|^2 \right) < \frac{\epsilon}{2^{m+2}}. \tag{86}
\end{equation}
It follows from (85) that for all $t \geq 0$,
\begin{equation}
P \left( \bigcup_{m=1}^{\infty} \left\{ \sup_{-\rho \leq r \leq t, |i| > n_m} \sum_{|i| > n_m} |u_t(r)|^2 \geq \frac{1}{2^m} \right\} \right) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^{m+2}} \leq \frac{1}{4}\epsilon,
\end{equation}
which shows that for all $t \geq 0$,
\begin{equation}
P \left( \left\{ \sup_{-\rho \leq r \leq t, |i| > n_m} \sum_{|i| > n_m} |u_t(r)|^2 \leq \frac{1}{2^m} \text{ for all } m \in \mathbb{N} \right\} \right) > 1 - \frac{1}{3}\epsilon. \tag{87}
\end{equation}
Given $\epsilon > 0$, set
\begin{equation}
\mathcal{Y}_{1, \epsilon} = \left\{ v \in C([-\rho, 0], \ell^2) : \|v(0)\| \leq R_1(\epsilon) \right\}, \tag{88}
\end{equation}
\begin{equation}
\mathcal{Y}_{2, \epsilon} = \left\{ v \in C([-\rho, 0], \ell^2) : \sup_{-\rho \leq s < r \leq 0} \frac{\|v(r) - v(s)\|}{|r - s|^{\frac{1}{2}}} \leq R_2(\epsilon) \right\}, \tag{89}
\end{equation}
\begin{equation}
\mathcal{Y}_{3, \epsilon} = \left\{ v \in C([-\rho, 0], \ell^2) : \sup_{-\rho \leq r \leq t, |i| > n_m} \sum_{|i| > n_m} |v_t(r)|^2 \leq \frac{1}{2^m} \text{ for all } m \in \mathbb{N} \right\}, \tag{90}
\end{equation}
and
\begin{equation}
\mathcal{Y}_{\epsilon} = \mathcal{Y}_{1, \epsilon} \bigcap \mathcal{Y}_{2, \epsilon} \bigcap \mathcal{Y}_{3, \epsilon}. \tag{91}
\end{equation}
By (80), (84) and (86) we get, for all $t \geq 0$,
\begin{equation}
P \left( \{ u_t \in \mathcal{Y}_{\epsilon} \} \right) > 1 - \epsilon.
\end{equation}
It remains to show that \( \mathcal{Y}_\varepsilon \) is a precompact subset of \( C([-\rho, 0], \ell^2) \) for which, by the Arzela-Ascoli theorem, one needs to verify:

(i) For every \( r \in [-\rho, 0] \), the set \( \{ v(r) : v \in \mathcal{Y}_\varepsilon \} \) is precompact in \( \ell^2 \); and

(ii) \( \mathcal{Y}_\varepsilon \) is equicontinuous in \( C([-\rho, 0], \ell^2) \).

Note that the equicontinuity of \( \mathcal{Y}_\varepsilon \) follows from (88) and (90) immediately. Next, we show the precompactness of \( \{ v(r) : v \in \mathcal{Y}_\varepsilon \} \) in \( \ell^2 \) for every fixed \( r \in [-\rho, 0] \). By (87)-(88) we find that for every \( r \in [-\rho, 0] \),

\[
\|v(r)\| \leq \|v(r) - v(0)\| + \|v(0)\| \leq R_2(\varepsilon)|r|^{\frac{1}{2}} + R_1(\varepsilon) \leq \rho^{\frac{1}{2}} R_2(\varepsilon) + R_1(\varepsilon),
\]

which indicates that

\[
\text{the set } \{ v(r) : v \in \mathcal{Y}_\varepsilon \} \text{ is bounded in } \ell^2 \text{ for every fixed } r \in [-\rho, 0]. \quad (92)
\]

Given \( \kappa > 0 \), choose an integer \( m_0 = m_0(\kappa) \in \mathbb{N} \) such that \( 2^{m_0} > \frac{4}{\kappa^2} \). Then by (89) we obtain

\[
\sum_{|i| > m_0} |v_i(r)|^2 \leq \frac{1}{2^{m_0}} < \frac{\kappa^2}{8}, \quad \forall v \in \mathcal{Y}_\varepsilon. \quad (93)
\]

On the other hand, by (92) we see that the set \( \{ (v_i(r))_{i| \leq m_0} : v \in \mathcal{Y}_\varepsilon \} \) is bounded in the finite-dimensional space \( \mathbb{R}^{2m_0+1} \) and hence precompact. Consequently, \( \{ (v_i(r))_{i| \leq m_0} : v \in \mathcal{Y}_\varepsilon \} \) has a finite open cover of balls with radius \( \frac{1}{2}\kappa \), which along with (93) implies that the set \( \{ v(r) : v \in \mathcal{Y}_\varepsilon \} \) has a finite open cover of balls with radius \( \kappa \) in \( \ell^2 \). Since \( \kappa > 0 \) is arbitrary, we find that the set \( \{ v(r) : v \in \mathcal{Y}_\varepsilon \} \) is precompact in \( \ell^2 \). This completes the proof. \( \square \)

We are now in a position to prove the main result of this paper: existence of invariant measures for problem (12)-(13).

**Theorem 4.3.** Suppose (3)-(7) and (21) hold. Then problem (12)-(13) has an invariant measure on \( C([-\rho, 0], \ell^2) \).

**Proof.** To apply Krylov-Bogolyubov’s method, given \( n \in \mathbb{N} \), we set

\[
\mu_n = \frac{1}{n} \int_0^n p(0, 0; t, \cdot) dt. \quad (94)
\]

Then \( \{ \mu_n \}_{n=1}^\infty \) is tight on \( C([-\rho, 0], \ell^2) \) by Lemma 4.2. Consequently, there exists a probability measure \( \mu \) on \( C([-\rho, 0], \ell^2) \) such that, up to a subsequence,

\[
\mu_n \to \mu, \quad \text{as } n \to \infty. \quad (95)
\]

By (94)-(95) and the Chapman-Kolmogorov equation, we infer that for every \( t \geq 0 \) and every bounded and continuous function \( \phi : C([-\rho, 0], \ell^2) \to \mathbb{R} \),

\[
\int_{C([-\rho, 0], \ell^2)} \phi(y) d\mu(y) = \lim_{n \to \infty} \frac{1}{n} \int_0^n \left( \int_{C([-\rho, 0], \ell^2)} \phi(y) p(0, 0; s, dy) \right) ds
\]
variant measures. We first start with the uniform estimates in $L^1$ to each other at an exponential rate, which immediately imply the uniqueness of the nonlinear drift and diffusion terms are sufficiently small in the sense that

\[ \text{Uniqueness of invariant measures.} \]

5. **Uniqueness of invariant measures.** In this section, we discuss the uniqueness of invariant measures of system (12)-(13) under further conditions on the diffusion and drift terms. More precisely, we now assume that the Lipschitz coefficients of the invariant measures of system (12)-(13) under further conditions on the diffusion

We will show that under condition (96), any two solutions of (12)-(13) converge to each other at an exponential rate, which immediately imply the uniqueness of invariant measures. We first start with the uniform estimates in $L^2(\Omega, C([−\rho, 0], \ell^2))$.

**Lemma 5.1.** Suppose (3)-(7) and (96) hold, and $\xi_1, \xi_2 \in L^2(\Omega, C([−\rho, 0], \ell^2))$. If $u(t, \xi_1)$ and $u(t, \xi_2)$ are the solutions of system (12)-(13) with initial data $\xi_1$ and $\xi_2$, respectively, then for any $t \geq -\rho$,

\[
\mathbb{E} \left(\|u(t, \xi_1) - u(t, \xi_2)\|^2\right) \leq (1 + 4\lambda^{-1}L_0^2e^{\lambda t})\mathbb{E}(\|\xi_1 - \xi_2\|^2) e^{-\lambda t}.
\]

**Proof.** By (15) we find that for $t \geq 0$,

\[
e^{-\lambda t} \|u(t, \xi_1) - u(t, \xi_2)\|^2 + 2\int_0^t e^{\lambda s} \|B(u(s, \xi_1) - u(s, \xi_2))\|^2 ds \\
+ \lambda \int_0^t e^{\lambda s} \|u(s, \xi_1) - u(s, \xi_2)\|^2 ds \\
= \|\xi_1(0) - \xi_2(0)\|^2 + 2\int_0^t e^{\lambda s} (u(s, \xi_1) - u(s, \xi_2), f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))) ds \\
+ \sum_{k=1}^{\infty} \int_0^t e^{\lambda s} \|\sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))\|^2 ds \\
+ 2\sum_{k=1}^{\infty} \int_0^t e^{\lambda s} (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k. \tag{97}
\]

By (97) we get, for $t \geq 0$,

\[
\mathbb{E} \left(e^{\lambda t} \|u(t, \xi_1) - u(t, \xi_2)\|^2\right) \leq \mathbb{E}((\|\xi_1 - \xi_2\|^2) e^{-\lambda t}).
\]
For the third term on the right-hand side of (98), by (11) we have

\[ t \]

It follows from (98)-(100) that for all \( E \)

\[ \frac{\sum_{k=1}^{\infty} \int_{0}^{t} e^{\lambda s} \| \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2)) \|^2 ds}{E} \]

Next, we estimate the right-hand side of (98). For the second term, by (9) we obtain

\[ 2E \left( \int_{0}^{t} e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \| f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2)) \| ds \right) \]

\[ \leq \frac{1}{4} \lambda E \left( \int_{0}^{t} e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 ds \right) \]

\[ + \frac{4}{\lambda} E \left( \int_{0}^{t} e^{\lambda s} \| f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2)) \|^2 ds \right) \]

\[ \leq \frac{1}{4} \lambda E \left( \int_{0}^{t} e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 ds \right) \]

\[ + 4\lambda^{-1} L_0^2 \rho e^{\lambda \rho} E \left( \int_{-\rho}^{t} e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 ds \right) \]

\[ \leq \frac{1}{4} \lambda + 4\lambda^{-1} L_0^2 \rho e^{\lambda \rho} \ E \left( \int_{0}^{t} e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 ds \right) \]

\[ + 4\lambda^{-1} L_0^2 \rho e^{\lambda \rho} E \left( \| \xi_1 - \xi_2 \|^2 \right) \]

(99)

For the third term on the right-hand side of (98), by (11) we have

\[ E \left( \sum_{k=1}^{\infty} \int_{0}^{t} e^{\lambda s} \| \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2)) \|^2 ds \right) \]

\[ \leq \| L \|^2 \int_{0}^{t} E \left( e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 \right) ds \]

(100)

It follows from (98)-(100) that for all \( t \geq 0 \),

\[ E \left( e^{\lambda t} \| u(t, \xi_1) - u(t, \xi_2) \|^2 \right) \]

\[ \leq (1 + 4\lambda^{-1} L_0^2 \rho e^{\lambda \rho}) E(\| \xi_1 - \xi_2 \|^2) \]

\[ + \left( \frac{1}{4} \lambda + \| L \|^2 + 4\lambda^{-1} L_0^2 e^{\lambda \rho} \right) \int_{0}^{t} E \left( e^{\lambda s} \| u(s, \xi_1) - u(s, \xi_2) \|^2 \right) ds \]

(101)

Applying the Gronwall inequality to (101), we obtain for all \( t \geq 0 \),

\[ E \left( e^{\lambda t} \| u(t, \xi_1) - u(t, \xi_2) \|^2 \right) \leq (1 + 4\lambda^{-1} L_0^2 \rho e^{\lambda \rho}) E(\| \xi_1 - \xi_2 \|^2) e \left( \frac{1}{4} \lambda + \| L \|^2 + 4\lambda^{-1} L_0^2 e^{\lambda \rho} \right) t \]
and hence by (96) we obtain that for all \( t \geq 0, \)

\[
E \left( \| u(t, \xi_1) - u(t, \xi_2) \|_2^2 \right) \leq (1 + 4\lambda^{-1}L_0\rho e^{\lambda\rho})E(\| \xi_1 - \xi_2 \|_{C_{\rho}}^2)e^{-\frac{1}{2}\lambda t}.
\]  
\[ \text{(102)} \]

On the other hand, for \( t \in [-\rho, 0], \) we have

\[
E \left( \| u(t, \xi_1) - u(t, \xi_2) \|_2^2 \right) = E \left( \| \xi_1(t) - \xi_2(t) \|_2^2 \right) \leq E(\| \xi_1 - \xi_2 \|_{C_{\rho}}^2)e^{-\frac{1}{2}\lambda t},
\]

which together with (102) concludes the proof. \( \square \)

Next, we improve Lemma 5.1 to obtain the uniform estimates on the segments of solutions in \( C([-\rho, 0], \ell^2). \)

**Lemma 5.2.** Suppose (3)-(7) and (96) hold, and \( \xi_1, \xi_2 \in L^2(\Omega, C([-\rho, 0], \ell^2)). \) If \( u(t, \xi_1) \) and \( u(t, \xi_2) \) are the solutions of system (12)-(13) with initial data \( \xi_1 \) and \( \xi_2, \) respectively, then for any \( t \geq \rho, \)

\[
E \left( \sup_{t-\rho \leq r \leq t} \| u(r, \xi_1) - u(r, \xi_2) \|_2^2 \right) \leq M_4 E(\| \xi_1 - \xi_2 \|_{C_{\rho}}^2)e^{-\frac{1}{2}\lambda t},
\]

where \( M_4 \) is a positive number depending only on \( \lambda, \rho, L \) and \( L_0. \)

**Proof.** By (15) we obtain, for all \( t \geq \rho \) and \( r \geq t - \rho, \)

\[
\| u(r, \xi_1) - u(r, \xi_2) \|_2^2 + 2\nu \int_{t-\rho}^r \| B(u(s, \xi_1) - u(s, \xi_2)) \|_2^2 ds
\]

\[
+ \lambda \int_{t-\rho}^r \| u(s, \xi_1) - u(s, \xi_2) \|_2^2 ds
\]

\[
= \| u(t-\rho, \xi_1) - u(t-\rho, \xi_2) \|_2^2
\]

\[
+ 2 \int_{t-\rho}^r \| u(s, \xi_1) - u(s, \xi_2), f(u(s, \xi_1) - f(u(s, \xi_2)) \|_2^2 ds
\]

\[
+ \sum_{k=1}^{\infty} \int_{t-\rho}^r \| \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2)) \|_2^2 ds
\]

\[
+ 2 \sum_{k=1}^{\infty} \int_{t-\rho}^r (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k,
\]

and hence for all \( t \geq \rho, \)

\[
E \left( \sup_{t-\rho \leq r \leq t} \| u(r, \xi_1) - u(r, \xi_2) \|_2^2 \right)
\]

\[
\leq E \left( \| u(t-\rho, \xi_1) - u(t-\rho, \xi_2) \|_2^2 \right)
\]

\[
+ 2E \left( \int_{t-\rho}^t \| u(s, \xi_1) - u(s, \xi_2) \|_2^2 \| f(u(s, \xi_1)) - f(u(s, \xi_2)) \|_2^2 ds \right)
\]

\[
+ E \left( \sum_{k=1}^{\infty} \int_{t-\rho}^t \| \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2)) \|_2^2 ds \right)
\]

\[
+ 2E \left( \sup_{t-\rho \leq r \leq t} \left| \sum_{k=1}^{\infty} \int_{t-\rho}^r (u(s, \xi_1) - u(s, \xi_2), \sigma_k(u(s, \xi_1)) - \sigma_k(u(s, \xi_2))) dW_k \right| \right).
\]  
\[ \text{(103)} \]
For the first term on the right-hand side of (103), by Lemma 5.1 we see that for all $t \geq \rho$,

$$E\left(\|u(t - \rho, \xi_1) - u(t - \rho, \xi_2)\|^2\right) \leq c_1 E(\|\xi_1 - \xi_2\|_{C_\rho}^2)e^{-\frac{1}{2}\lambda(t-\rho)},$$

(104)

where $c_1 = (1 + 4\lambda^{-1}L_0^2pe^\lambda \rho$).

For the second term on the right-hand side of (103), by (9) and Lemma 5.1 we obtain

$$2E\left(\int_{t-\rho}^t \|u(s, \xi_1) - u(s, \xi_2)\||\|f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))\||ds\right)$$

$$\leq \int_{t-\rho}^t E\left(\|u(s, \xi_1) - u(s, \xi_2)\|^2\right) ds$$

$$+ \int_{t-\rho}^t E\left(\|f(u(s - \rho, \xi_1)) - f(u(s - \rho, \xi_2))\|^2\right) ds$$

$$\leq c_1 E(\|\xi_1 - \xi_2\|_{C_\rho}^2) \int_{t-\rho}^t e^{-\frac{1}{2}\lambda s} ds$$

$$+ L_0^2 \int_{t-\rho}^t E\left(\|u(s, \xi_1) - u(s, \xi_2)\|^2\right) ds$$

(105)

$$\leq 2c_1 \lambda^{-1} E(\|\xi_1 - \xi_2\|_{C_\rho}^2)e^{-\frac{1}{2}\lambda(t-\rho)} + L_0^2 \int_{t-2\rho}^{t-\rho} E\left(\|u(s, \xi_1) - u(s, \xi_2)\|^2\right) ds$$

$$\leq 2c_1 \lambda^{-1} E(\|\xi_1 - \xi_2\|_{C_\rho}^2)e^{-\frac{1}{2}\lambda(t-\rho)} + c_1 L_0^2 E(\|\xi_1 - \xi_2\|_{C_\rho}^2) \int_{t-2\rho}^{t-\rho} e^{-\frac{1}{2}\lambda s} ds$$

$$\leq 2c_1 \lambda^{-1} E(\|\xi_1 - \xi_2\|_{C_\rho}^2)e^{-\frac{1}{2}\lambda(t-\rho)} + 2c_1 \lambda^{-1} L_0^2 E(\|\xi_1 - \xi_2\|_{C_\rho}^2)e^{-\frac{1}{2}\lambda(t-2\rho)}$$

$$= 2c_1 \lambda^{-1} E(\|\xi_1 - \xi_2\|_{C_\rho}^2)(1 + L_0^2 e^\frac{1}{2}\lambda \rho)e^{-\frac{1}{2}\lambda(t-\rho)}.$$
invariant measure in Theorem 5.3. Suppose (13).

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by using the standard arguments (see, e.g., [66]).

Theorem 5.3. Suppose (3)-(7) and (96) hold. Then problem (12)-(13) has a unique invariant measure in $C([-\rho, 0], \mathbb{L}^2)$.

Proof. By Lemma 5.2 we see that for any $\xi_1, \xi_2 \in L^2(\Omega, C([-\rho, 0], \mathbb{L}^2))$, the segments of the solutions $u_t(\xi_1)$ and $u_t(\xi_2)$ of (12)-(13) satisfy, for all $t \geq \rho$,

$$E\left(\sup_{-\rho \leq s \leq t} \|u(s, \xi_1) - u(s, \xi_2)\|^2\right) \leq c_2 E\left(\|\xi_1 - \xi_2\|^2_{C^2}\right) e^{-\frac{1}{2} \lambda t},$$

as desired.

We are now ready to show the uniqueness of invariant measures of system (12)-(13).

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E-mail address: zchen@sdu.edu.cn
E-mail address: lixiliang@amss.ac.cn
E-mail address: bwang@nmt.edu