Vacuum solutions of Einstein’s equations in parabolic coordinates

Viaggiu Stefano

Dipartimento di Matematica, Università di Roma ‘Tor Vergata’, Via della Ricerca Scientifica, 1, I-00133 Roma, Italy

E-mail: viaggiu@mat.uniroma2.it or stefano.viaggiu@ax0rm1.roma1.infn.it

Received 14 December 2004, in final form 14 March 2005
Published 26 May 2005
Online at stacks.iop.org/CQG/22/2309

Abstract

We present a simple method to obtain vacuum solutions of Einstein’s equations in parabolic coordinates starting from ones with cylindrical symmetries. Furthermore, a generalization of the method to a more general situation is given together with a discussion of the possible relations between our method and the Belinsky–Zakharov soliton-generating solutions.

PACS numbers: 04.20.−q, 04.20.Jb

Introduction

Cylindrical solutions [1–5] have played an important role in the history of general relativity. In astrophysics, cylindrical solutions have been applied in the context of cosmic strings [6]. In the literature various techniques exist [7–11] for generating solutions of Einstein’s equations. In this paper we introduce a simple method for generating static and stationary ‘parabolic’ solutions starting from ones with cylindrical symmetries. Our starting point is the stationary axially symmetric line element in the form [12]

\[ ds^2 = f^{-1}[e^{2\gamma}((dx^1)^2 + (dx^2)^2) + \rho^2 d\phi^2] - f (dt - \omega d\phi)^2, \]

where \( x^1, x^2 \) are spatial coordinates, \( \phi \) is an angular coordinate, \( t \) is a time coordinate, \( \rho \) is the radius in a cylindrical coordinate system and \( f, \gamma, \omega \) are functions of \( x^1, x^2 \). The field equations [13] for the line element (1) can be written [14] in the form:

\[ \nabla^2 f - \frac{1}{f} \left( \frac{1}{f} \partial_a f^2 - \Phi_a \right) = 0, \quad \nabla^2 \Phi - \frac{2}{f} f_a \Phi_a = 0, \]

\[ \gamma_1 = -\frac{\Sigma \rho_1 + \Pi \rho_2}{4 \rho (\rho_1^2 + \rho_2^2)} + \frac{c}{\rho_1^2 + \rho_2^2}, \quad \gamma_2 = \frac{\Sigma \rho_2 - \Pi \rho_1}{4 \rho (\rho_1^2 + \rho_2^2)} + \frac{d}{\rho_1^2 + \rho_2^2}, \]

\[ c = \frac{2 \rho_1 \rho_2 + (\rho_1^2 - \rho_2^2) \rho_1}{(\rho_1^2 + \rho_2^2)} \rho_1, \quad d = \frac{2 \rho_1 \rho_2 (\rho_1^2 - \rho_2^2) \rho_2}{(\rho_1^2 + \rho_2^2)} \rho_2. \]
\[ \Sigma = \frac{\rho^2}{f^2} (f_2^2 - f_1^2) + f^2 (\omega_1^2 - \omega_2^2), \quad \Pi = -2\rho^2 \frac{f_1 f_2}{f^2} + 2 f^2 \omega_1 \omega_2, \]

\[ \omega_1 = -\frac{\rho}{f^2} \Phi_2, \quad \omega_2 = \frac{\rho}{f^2} \Phi_1, \]  

(2)

where a summation over \( \alpha \) is implicit with \( \alpha = 1, 2 \), i.e. \( x^1, x^2 \), and subindices denote partial derivatives. Further, the operator \( \nabla^2 \), with \( \nabla^2 = \partial^2_{\alpha\alpha} + \frac{\rho}{\rho} \partial_{\alpha} \), denotes the reduced tridimensional Laplacian (without \( \phi \)) up to a conformal factor: the bidimensional Laplacian is given by \( \Delta_1 = \partial^2_{\alpha\alpha} \). Consider now parabolic coordinates, which in terms of Cartesian ones are given by

\[ x = \lambda \mu \cos \phi, \quad y = \lambda \mu \sin \phi, \quad z = \frac{1}{2} (\lambda^2 - \mu^2). \]  

(3)

Because we are interested in axisymmetric solutions, we will use polar cylindrical coordinates \( \rho \) and \( z \) with

\[ \rho = \lambda \mu, \quad z = \frac{1}{2} (\lambda^2 - \mu^2). \]  

(4)

The inverse of transformations (4) is given by

\[ \lambda = \sqrt{z + \sqrt{\rho^2 + z^2}}, \quad \mu = \sqrt{\rho^2 + z^2} - z. \]  

(5)

Now, if we write the field equations in parabolic coordinates, i.e. \( x^1 = \lambda, x^2 = \mu \), then for the operator \( \nabla^2 \) in these coordinates we have

\[ \nabla^2 = \partial^2_{\alpha\alpha} + \frac{1}{\lambda} \partial_{\alpha} + \frac{1}{\mu} \partial_{\beta}. \]  

(6)

The same operator when expressed in cylindrical coordinates is

\[ \nabla^2 = \partial^2_{\rho\rho} + \partial^2_{z z} + \frac{1}{\rho} \partial_{\rho}. \]  

(7)

By comparing expressions (6) and (7) it is easy to see that if we take a solution of equations (2) with \( f = f(\rho), \Phi = \Phi(\rho) \), then also \( f = f(\lambda), \Phi = \Phi(\lambda) \) or \( f = f(\mu), \Phi = \Phi(\mu) \) are solutions. In this way, starting with a solution with cylindrical symmetries, we can obtain a parabolic one that is non-polynomial when expressed in cylindrical coordinates.

This paper is devoted to the discussion of the solutions so obtained together with an investigation of the limits of validity of the method.

In section 1 we apply the method using as the starting metric the Lewis [3] and the Papapetrou classes [12] of solutions and discuss possible physical interpretations of these solutions. In section 2 we show that, starting with a static spatially homogenous solution with a \( G_3 \) group of motion, a class of stationary solutions with a \( G_2 \) group of motion can be obtained which contains as a subclass the solution found in subsection 1.1. In section 3, we study the most general coordinate system permitted by our method. Finally, section 4 is devoted to a generalization of the method together with a study of the possible relations with the Belinsky–Zakharov (BZ) soliton-generating solutions.

1. Application of the method

1.1. Generating solutions from Lewis ones

For a first application we consider Lewis solutions [3] given by

\[ f = \frac{1}{(1 - B^2)} [P^2 \rho^\epsilon - B^2 Q^2 \rho^{2-\epsilon}], \quad \omega = \frac{B}{P Q} \frac{(Q^2 \rho^{2-\epsilon} - P^2 \rho^\epsilon)}{(P^2 \rho^\epsilon - B^2 Q^2 \rho^{2-\epsilon})}, \]

\[ e^{2\gamma} = \frac{\rho^{2-\epsilon}}{(1 - B^2)} [P^2 \rho^\epsilon - B^2 Q^2 \rho^{2-\epsilon}], \]

(8)

where \( B, P, Q \) and \( \epsilon \) are constants.
For our purpose, the function $\Phi$ for the Lewis solutions must be a function of $\rho$. From equations (2) we deduce

$$\Phi = \frac{(\epsilon - 1)2BPQz}{(1 - B^2)}. \quad (9)$$

Thus, a necessary condition to map cylindrical Lewis solutions into parabolic stationary ones is $\epsilon = 1$. Another possibility is that $B \to \infty$, but these solutions belong to the Levi-Civita static class that will be discussed later. Taking $\epsilon = 1$ in (8) we obtain $\omega = b$, where $b$ is a real constant. By integrating the field equations (2), we obtain the solution

$$f = a\lambda, \quad \omega = b, \quad e^{2\gamma} = \sqrt{\lambda(\lambda^2 + \mu^2)^3}. \quad (10)$$

where $a$ is a real positive constant. Also the function

$$f = a\mu \quad (11)$$

is a solution of system (2) and for $\gamma$ we obtain the same expression given in (10) with $\lambda \to \mu$. In this way we have used a subclass of Lewis solutions to obtain two solutions with parabolic-like symmetries.

Now we analyse the properties of (10). First of all, metric (10) is Petrov type I and has an Abelian $G_2$ group of motion with Killing vectors $\xi^1 = \partial_t, \xi^2 = \partial_\phi$. Besides, it has a coordinate singularity at $\lambda = 0$. In cylindrical coordinates it takes the form

$$ds^2 = \frac{a}{2^4} \left( z + \sqrt{\rho^2 + z^2} \right)^2 \left[ dp^2 + dz^2 \right] + \frac{\rho^2}{a} \frac{1}{\sqrt{z + \sqrt{\rho^2 + z^2}}} d\phi^2$$

$$- a \sqrt{z + \sqrt{\rho^2 + z^2}} (dt - b d\phi)^2. \quad (12)$$

Since at $\rho = 0$ (z-axis) $e^{2\gamma} = 1$ for $z > 0$ and $e^{2\gamma} \neq 1$ for $z \leq 0$, we conclude that (12) is regular on the axis only for $z > 0$.

This fact is confirmed by taking the relativistic invariants

$$R^{abde} R_{abde}, \quad R^{abde}e R_{abde,e}, \quad C^{abcd} C_{abcd} \cdots, \quad (13)$$

where $R$ denotes the Riemannian tensor, $C$ the Weyl one and ‘;' denotes the covariant derivative. The invariants are coordinate independent and when expressed in parabolic coordinates they are singular only for $\lambda = 0, \lambda^2 + \mu^2 = 0$, i.e. on the z-axis at $z \leq 0$. For example

$$R^{abde} R_{abde} = \frac{3a^2(4\lambda^2 + \mu^2)}{4\lambda^3(\lambda^2 + \mu^2)^3}$$

and

$$R^{abde}e R_{abde,e} = \frac{45a^3(3\mu^4 + 7\lambda^2 \mu^2 + 16\lambda^4)}{16(\lambda^2 + \mu^2)^2}. \quad (14)$$

To study the behaviour at spatial infinity the spherical coordinates are most appropriate. When (10) is expressed in such coordinates, it is easy to show that the line element is not regular at spatial infinity ($r \to \infty$) independently of the azimuthal angle $\theta$, and at $\theta = \pi$ ($\rho = 0, z \leq 0$) independently of $r$.

Now we study the physical interpretation of (10). As a first step note that it is possible to construct (see [15] and references therein) a ‘local’ formulation of general relativity and thus to define, in this ‘local’ frame, the analogue quantities of the Newtonian theory. In the stationary case we can define a ‘standard’ gravitational field $G$ in a reference frame $\Gamma$ adapted to the stationary spacetime (1) with a gravitational potential $U$ given by $f = e^{2U}$. Besides,
in \( \Gamma \) we can define a time parameter \( T \) analogue to the time defined in a Galilean reference frame, except for the fact that \( T \) is not defined globally but only on the geodesic of the particle. In terms of the proper time \( \tau \), with \( d\tau = \sqrt{-g_{\alpha\beta}} \, dx^\alpha \, dx^\beta \), we have

\[
d\tau = \sqrt{1 - v^2} \, dT
\]

where \( v^i = \frac{dv^i}{dT} \), \( i = 1-3 \), is the 3-velocity of a test particle with respect to \( T \).

Remember that, starting with the line element (1), a frame is adapted to the spacetime (1) if it is represented by coordinates \( x^\alpha = x^\alpha(x^i) \), \( i = 1-3 \). Further, if \( v^\alpha \) is the 4-velocity with respect to \( \tau \), then \( v^\alpha = \frac{\partial x^\alpha}{\partial \tau} \).

In particular, in this picture we can define the ‘relative’ energy \( H \) of a test particle as

\[
H = \frac{m_0}{\sqrt{1 - v^2}} \, \sqrt{f} = \frac{m_0}{\sqrt{1 - v^2}} \, a_\lambda.
\]

Thus, the surfaces with constant energy are rotational parabolic. Note that if \( \nu \simeq 0 \), then \( H \simeq 0 \). However, it is in principle possible to have a particle in the orbit with \( v^2 = 1 - a_\lambda \) which is therefore ultrarelativistic with \( H = m_0 \). Since surfaces with \( \lambda = \text{const} \) are equipotential, for the spacetime (10) orbits exist with energy \( H = m_0 \), i.e. the energy for a rest non-interacting particle.

Further, when \( \lambda a < 1 \), the potential \( U = \log \sqrt{T} < 0 \) and is thus attractive. When \( \lambda = \frac{1}{2} \), \( f = 1 \) and \( U = 0 \): in this case the particle has ‘local’ energy \( H = \frac{m_0}{\sqrt{1 - v^2}} \) of a free particle travelling in a Minkowskian spacetime with speed \( v \). Finally, for \( \lambda > 1 \) we have \( f > 1 \) and thus the potential \( U \) becomes repulsive.

Naturally, this does not mean that the source matter of (10) is a paraboloid, but this is an indication that parabolic symmetries have something to do with the solution (10). To enforce this reasoning, we consider the coordinate transformation [3] which changes solution (10) into the static form

\[
dx^2 = g^{2\nu} \left[ dx^\lambda + d\nu^2 \right] + \frac{\rho^2}{f} \, d\phi^2 - f \, dt^2
\]

with the same functions \( \nu \) and \( f \) given in (10). This is done by performing the transformation:

\[
dt = A \, dt' + B \, d\phi', \quad d\phi = C \, dt' + D \, d\phi',
\]

where \( A, B, C, D \) are functions of the non-ignorable coordinates \( \lambda, \mu \).

Imposing that (16) be equal to (10), we obtain the set of equations

\[
\begin{align*}
\frac{\rho^2}{f^2} \, CD &= AB + CD \omega^2 - AD \omega - BC \omega, \\
D^2 + \frac{f^2}{\rho^2} [2BD \omega - \omega^2 D^2 - B^2] &= 1, \\
-C^2 \frac{\rho^2}{f^2} + A^2 + \omega^2 C^2 - 2AC \omega &= 1.
\end{align*}
\]

System (18) has three independent equations for four variables and thus admits solutions. However, transformations (17) are purely local, i.e. non-integrable. This means that there exists a rotating reference frame such that the metric appears to be static, and the coordinates \( t', \phi' \) are admissible only on a geodetic. Further, note that system (18) depends on \( \frac{f}{\rho} \): for solution (10) this quantity is equal to \( \frac{\rho}{\nu} \). Therefore, the angular velocity of an observer in the frame \( \lambda, \mu, \phi', t' \) with respect to which the metric appears to be static, depends only on \( \mu \) (\( \omega \) is constant in spacetime (10)). Thus, the angular velocity of the source, as ‘seen’ from an observer with coordinates \( \lambda, \mu, \phi, t \), has a shape given by a paraboloid of rotation.
For example, with the ansatz $dt = A dt', d\phi = C dt' + D d\phi'$, with $Q = \frac{d\phi}{dt}$ and $Q' = \frac{d\phi'}{dt}$, thanks to (18) we obtain

$$Q = \frac{\alpha^2 b}{\alpha^2 b^2 - \mu^2} + Q' \frac{\mu^2}{-\alpha^2 b^2 + \mu^2}. \quad (19)$$

Expression (19) means that the source rotates with an angular velocity depending on $\mu$. In fact, for $Q' = 0$, it follows that $Q = \frac{\alpha^2 b}{\alpha^2 b^2 - \mu^2}$. Note that for $\mu \to \infty$, because in this limit $dt' \to dt$ and $d\phi' \to d\phi$, formula (19) reduces to $Q = Q'$. The main difficulty in analysing the nature of the source of (10) is that this metric does not admit asymptotical Minkowskian coordinates. Obviously, similar arguments follow for solution (11). This concludes our study of the physical interpretation of (10).

1.2. Generating solutions from Papapetrou ones

Another class of solutions that can be mapped into parabolic ones is given by Papapetrou [12]. Starting from metric (1) (Papapetrou gauge), these solutions are characterized by

$$f^2 + \Phi^2 = 1. \quad (20)$$

It is easy to see that the most general solution belonging to the Papapetrou class with $f = f(\rho), \Phi = \Phi(\rho)$ is

$$f = \frac{2\rho P}{1 + \rho^2 P}, \quad \Phi = \frac{\rho^2 P - 1}{\rho^2 + 1}, \quad (21)$$

where $P$ is a real constant.

Once equations (2) are solved for the metric functions $\omega$ and $\gamma$, we obtain

$$\omega = P \gamma + \beta, \quad e^{2\gamma} = \rho^{\frac{\gamma}{2}}. \quad (22)$$

Note that (21) has cylindrical symmetries only when $\phi = \text{constant}$, i.e. for planes passing through the $z$-axis. Actually, for our method to be applicable, we only need solutions with $f = f(\rho), \Phi = \Phi(\rho)$: from the field equations (2) it is easy to see that this implies that $\gamma = \gamma(\rho)$ and $\omega = \alpha z + \beta$, where $\alpha, \beta$ are real constants. Solution (21) is ‘mapped’ into

$$f = \frac{2\lambda P}{1 + \lambda^2 P}, \quad \Phi = \frac{\lambda^2 P - 1}{\lambda^2 + 1}. \quad (23)$$

By integrating the field equations (2) we get

$$\omega = \frac{1}{2} P \mu^2 + \beta, \quad e^{2\gamma} = \frac{\lambda^2 P}{(\lambda^2 + \mu^2)^{-1}}. \quad (24)$$

Also in this case we can obtain another solution by taking $\lambda \to \mu$. Generally, (23) admits a $G_2$ Abelian group of motion, i.e. $\xi^1 = \partial_\phi, \xi^2 = \partial_t$. Since condition (20) is again valid, the solution (23) belongs to the Papapetrou class.

With arguments similar to the ones used in subsection 1.1, it can be shown that metric (23) has a coordinate singularity at $\lambda = 0$, is regular on the $z$-axis only at $z > 0$ and is singular for $z \leq 0$, i.e. in the limit $\mu \to 0$ it follows that $e^{2\gamma} \to 1$. In fact, the invariants are singular for $\lambda = 0, \lambda^2 + \mu^2 = 0$. Solution (23) is Petrov type $O$ (flat) for $P = 0$, is Petrov type $D$ for $P = \pm 2$ and otherwise is Petrov type $I$. While at spatial infinity ($r \to \infty$) it is not regular. Besides, for the ‘relative’ energy $H$ of a test particle of inertial mass $m_0$ and velocity $v$ we get

$$H = \frac{m_0}{\sqrt{1 - v^2}} e^{U} = \frac{m_0}{\sqrt{1 - v^2}} \frac{\sqrt{2}}{\sqrt{\lambda^2 + \mu^2}}. \quad (25)$$
and thus, orbits with constant energy in a frame $\Gamma$ adapted to the stationary metric (1) are again rotational parabolic surfaces. Then, for $\lambda \simeq 0$, $H \simeq \frac{m_0}{\sqrt{1-v^2}} \sqrt{2}\lambda^{\frac{1}{2}}$ and for $\lambda \to \infty$, $H \simeq \frac{m_0}{\sqrt{1-v^2}} \lambda^{\frac{1}{2}}$, in both cases $H \to 0$, i.e. $U \to -\infty$. Moreover, for spacetime (23), $U \leq 0$ with equality only at $\lambda = 1$: therefore, at $\lambda = 1$, $H = \frac{m_0}{\sqrt{1-v^2}}$. Obviously, also in this case, this does not mean that the source matter of (23) is a paraboloid, because the shape of the configuration depends on the ‘match’ between the gravitational and the centrifugal force, which is not given a priori. In practice, the fact that orbits with parabolic symmetries are allowed does not guarantee that the source is a paraboloid but is a sign in this direction as well as the fact that the invariants of (24) are singular on the axis at $\lambda = 0$. Finally, since for (23) $\frac{\rho_2}{\rho} = F(\lambda, \mu)$, the arguments that lead to (19) are not valid, and consequently the shape of the angular velocity of the source, as ‘seen’ from an observer at rest in a general spacetime point with coordinates $\lambda, \mu, \phi, t$ (since the metric is not asymptotically flat, there does not exist a ‘privileged’ Minkowskian observer at spatial infinity), does not have parabolic symmetries.

2. Stationary solutions from static ones

To apply our method in the above section we have taken as the ‘starting’ metric a subclass of Lewis solutions with a $G_3$ group of motion and a $G_2$ subclass of Papapetrou solutions ($\alpha \neq 0$). All the generating solutions have a $G_2$ Abelian group of motion. In this section we show that it is possible to obtain a stationary $G_2$ solution starting with a static $G_3$ solution with Killing vectors $\xi^1 = \partial_t, \xi^2 = \partial_\phi, \xi^3 = \partial_z$. As an example we start with the most general cylindrically symmetrical static solution with an Abelian $G_3$ group of motion (found by Levi-Civita [1]) that can be obtained from Lewis solutions by setting $B = 0$ ($\omega = 0$). With the same technique used above we obtain

$$f = a\lambda^\epsilon, \quad e^{2\nu} = \frac{\lambda^{\frac{\epsilon}{2}}}{(\lambda^2 + \mu^2)^{2\nu-1}},$$

and the solution with $\lambda \to \mu$.

When expressed in terms of $\rho, z$, with the help of (5), all these parabolic solutions have complicated expressions. Note that, thanks to the analogy between the Laplacians (6) and (7) we can ‘map’ all static cylindrical solutions into parabolic ones. Solution (26) is Petrov type O for $\epsilon = 0, 2$ and otherwise is Petrov type I. Furthermore

$$f = a\lambda^\epsilon, \quad e^{2\nu} = \frac{\lambda^{\frac{\epsilon}{2}}}{(\lambda^2 + \mu^2)^{2\nu-1}}, \quad \omega = Q = \text{const}$$

is a stationary solution. This solution has similar features to (10), i.e. it has a physical coordinate independent singularity on the $z$-axis at $z \leq 0$ and is not regular at spatial infinity. Note that for $\epsilon = 1$ the class of solutions (27) reduces to (10). Further, for solution (27), $\nu' = \frac{\mu^2}{\mu^2 + \lambda^2} - 2\epsilon$ and thus the angular velocity of the source, in general, does not have parabolic symmetries. Now, the ‘local’ energy $H$ in the reference $\Gamma$ is

$$H = \frac{m_0}{\sqrt{1-v^2}} \sqrt{a\lambda^{\frac{\epsilon}{2}}},$$

and consequently $-\infty < U < +\infty$. The arguments used for (15) are still valid.

We now consider the interesting case of the static subclass of (26) with $\epsilon = 2$. This is a flat solution except on the $z$-axis at $z \leq 0$ and at spatial infinity where the metric is not regular. In parabolic coordinates we have

$$ds^2 = d\lambda^2 + d\mu^2 + \mu^2 d\phi^2 - \lambda^2 dt^2.$$
Performing the spatial coordinate transformation
\[ \tilde{x} = \mu \cos \phi, \quad \tilde{y} = \mu \sin \phi, \quad \tilde{z} = \lambda, \]  
(30)
metric (29) becomes
\[ ds^2 = d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 - \tilde{z}^2 \, dt^2. \]  
(31)
The solution admits a $G_4$ group of motion and is isomorphic to the flat static Das [16] solution, but with $\tilde{z} > 0$, and therefore covers the Das solution only for $z > 0$. In parabolic coordinates the four Killing vectors are
\[ \xi_1 = (0, \cos \phi, -\sin \phi, \mu, 0), \quad \xi_2 = (0, \sin \phi, \cos \phi, \mu, 0), \quad \xi_3 = (0, 0, 1, 0), \quad \xi_4 = (0, 0, 0, 1). \]
Also in this case we can generalize solution (31) to a stationary one obtaining
\[ ds^2 = d\lambda^2 + d\mu^2 + \mu^2 \, d\phi^2 - \lambda^2 (dt - Q \, d\phi)^2 \]  
(32)
where $Q$ is a constant.

3. Validity of the method

In this section we characterize the most general coordinate transformation for which our method works. First of all note that equations (2) have been written starting with the line element (1) (Papapetrou gauge). It is easy to see [14] that the Papapetrou gauge is preserved if and only if analytical coordinate transformations are considered, i.e. $x^1 + i x^2 = F(x^1 + i x^2)$. Obviously, the imposition of the Papapetrou gauge does not represent a loss of generality. Therefore, we can restrict our consideration to analytical coordinate transformations (note that if such coordinate transformations are not used, equations (2) assume a very complicated expression).

Since the starting point of our method is the analogy between Laplacians in cylindrical and parabolic coordinates, it is natural to ask if there exist some other coordinates $u, v$ such that the (reduced) Laplacian takes the form
\[ \nabla^2 = \partial_{uu} + \partial_{vv} + \frac{1}{u} \partial_u + \frac{1}{v} \partial_v. \]  
(33)
As a first step we consider separable analytical coordinate transformations:
\[ \rho = U(x^1) V(x^2), \quad z = z(x^1, x^2). \]  
(34)
Since the analyticity condition $\rho_{z^1} = z_{z^2}, \rho_{z^2} = -z_{z^1}$ must be imposed, we have $U_{z^i} V = z_{z^i}, U V_{z^i} = -z_{z^i}$: these lead to
\[ z = -V_{z^i} \int U \, dx^1 + h(x^2), \quad \frac{U_{z^1}}{U} = -\frac{V_{z^2}}{V}. \]  
(35)
System (35) only admits solutions given by
\[ U = \sinh x^1, \quad V = \sin x^2, \quad z = -\cosh x^1 \cos x^2, \]  
(36)
\[ U = \cosh x^1, \quad V = \cos x^2, \quad z = \sinh x^1 \sin x^2, \]  
(37)
\[ U = e^{x^1}, \quad V = \sin x^2, \quad z = -e^{x^1} \cos x^2, \]  
(38)
\[ U = x^1, \quad V = x^2, \quad z = \frac{1}{2}[(x^2)^2 - (x^1)^2]. \]  
(39)
Solutions (36) and (37) represent respectively spheroidal prolate and oblate coordinates, while (38) represents spherical coordinates and (39) parabolic coordinates. Coordinates (36)–(39) are the only analytical coordinates separable with respect to the operator $\nabla^2$. 
\( (\nabla^2 = \partial^2_{\mu\mu} + \partial^2_{\theta\theta} + \frac{\cosh \mu}{\sin \mu} \partial_{\mu} + \frac{\cos \theta}{\sin \theta} \partial_{\theta} ). \) Note that the parabolic coordinates are the only ones with respect to which the 'similarity' reasonings with the cylindrical polar coordinates are available. However, the 'similarity' reasoning with respect to the operator \( \nabla^2 \) can be done for another pair of coordinates. In fact, if we consider spheroidal prolate coordinates \( \mu, \theta \) with \( \rho = \sinh \mu \sin \theta, z = \cosh \mu \cos \theta, \) the operator \( \nabla^2 \) is

\[
\nabla^2 = \partial^2_{x^1} + \partial^2_{x^2} + \frac{\cosh \mu}{\sin \mu} \partial_{\mu} + \frac{\cos \theta}{\sin \theta} \partial_{\theta}.
\]

Now, if we take spherical coordinates compatible [14] with the Papapetrou gauge (1), i.e. \( \rho = e^\nu \sin \vartheta, z = e^\nu \cos \vartheta, \) we have

\[
\nabla^2 = \partial^2_{\nu\nu} + \partial^2_{\vartheta\vartheta} + \frac{\cos \vartheta}{\sin \vartheta} \partial_{\vartheta}.
\]

Thus, if we have spherical solutions with \( f = f(\vartheta), \Phi = \Phi(\vartheta), \) we can obtain prolate spheroidal ones with \( \vartheta \to \theta \). For example, we have the stationary solution in spherical coordinates belonging to the Papapetrou class given by

\[
f = \frac{\sin^2 \vartheta}{1 + \cos^2 \vartheta}, \quad \Phi = \frac{2 \cos \vartheta}{1 + \cos^2 \vartheta}.
\]

From solution (42) we can obtain another one with \( \vartheta \to \theta \). What happens if we consider coordinate transformations of the form \( \rho = G(x^1, x^2), z = H(x^1, x^2) \)? It is easy to show that the analyticity condition \( G_{x^1} = H_{x^2}, G_{x^2} = -H_{x^1} \) leads inevitably to the condition \( G(x^1, x^2) = U(x^1) V(x^2) \), that has been analysed above.

As a final consideration note that, since \( f = f(\lambda), \Phi = \Phi(\lambda), \) the term \( \frac{1}{\mu} \partial_{\mu} \) in the operator \( \nabla^2 \) as expressed by parabolic coordinates acts trivially on \( f(\lambda), \Phi(\lambda), \). This means that if we consider the coordinate transformation given by \( \rho = \lambda F(\mu), z = a^2 \lambda^2 - \frac{z^2}{2a}, a = \text{const}, \) the method is again applicable, with the inverse given by

\[
\lambda = \sqrt{\frac{1}{a}(\sqrt{\rho^2 + z^2} + z)}, \quad F = \sqrt{\frac{1}{a}(\sqrt{\rho^2 + z^2} - z)}.
\]

Condition (45) means that the Papapetrou gauge breaks down, but because of the independence of \( f \) and \( \Phi \) on \( \mu \) the field equations for these functions are again of the form given by the first two equations of (2) with \( \nabla^2 = \partial^2_{x^1} + \frac{1}{x^1} \partial_{x^1} \).

However, thanks to the second of equations (43), it is easy to see that the function \( F \) represents the possibility of performing a general coordinate transformation for \( \mu \) that does not lead to a different solution. No other possibilities are allowed.

### 4. Further improvements and final remarks

Our starting point has been the line element (1) written in the Papapetrou gauge. However, the most general form of the metric for a spacetime admitting a two-dimensional Abelian group of isometries with Killing vectors \( \partial_t, \partial_\phi \) is [16]

\[
dx^2 = \frac{1}{f(e^{2\nu}((dx^1)^2 + (dx^2)^2) + W^2 d\phi^2) - f(dt - \omega d\phi)^2}
\]

where \( W = W(x^1, x^2) \) with \( W > 0 \). It is easy to see [3] that the field equations imply

\[
\Delta W = W_{x^1 x^1} + W_{x^2 x^2} = 0.
\]

Condition (45) means that \( W \) can be chosen as a coordinate. Further, note that the determinant of the 2-metric \( g \) spanned by the Killing vectors \( \partial_t, \partial_\phi \) is

\[
(-\text{det}[g])^2 = W.
\]
The function $W$ characterizes a measure of the area of the orbits [17] of the isometry group and thus it has a geometrical significance. The case with $W = \lambda \equiv x^1$ has been analysed above. Therefore, we can start with a given $W = W(\rho, z) = \lambda$ satisfying condition (45) and with the other coordinate $U$ given by $W_\rho = U$, $W_\zeta = -U_\rho$ and then apply our method. The field equations for this new gauge are again given by (2) provided that $\rho$ is substituted with $x^1$. The similarity reasoning with the coordinates $W, U$ is possible only with the coordinates $\alpha, \beta$ given by

$$W = \alpha \beta, \quad U = \frac{1}{2}(\alpha^2 - \beta^2).$$

For example, we can take $W = \lambda$ with $\lambda = \alpha \beta, \mu = \frac{1}{2}(\alpha^2 - \beta^2)$ and use as a starting metric the solution (8) with $\epsilon = 1$ and with $\rho$ substituted with $\lambda$. Therefore, by changing $W(\rho, z)$, we can increase the class of coordinates with respect to which our method is applicable.

As a final consideration, we analyse the possible relations between the method presented in this paper and the well-known [10, 11] Belinsky–Zakharov soliton-generating technique. This method derives from the application of the ‘quantum inverse scattering method’ (QISM) to spacetimes admitting a two-dimensional Abelian group of isometries acting orthogonally transitively on two-dimensional spacelike or timelike orbits. For the spacelike case we have two spacelike Killing vectors and the Ernst equation becomes hyperbolic. In what follows we restrict our attention to the timelike case.

The starting point of the BZ method is the line element

$$ds^2 = F(\rho^2 + dz^2) + g_{AB} dx^A dx^B, \quad A, B = 3, 4.$$ 

(48)

For a given initial solution $(F_0, g_0)$ of the Einstein equations, by introducing a $2 \times 2$ complex matrix $\psi$, the field equations for $F_0, g_0$ can be reduced to a pair (Lax pair) of two first-order differential equations which are the integrability conditions for the Ernst equation with $(-\det ||g_0||)^{1/2} = \alpha, \Delta \alpha = 0$. By choosing $\alpha$ as one of the coordinates (the determinant of the 2-metric spanned by Killing vectors), as the other spatial coordinate $\beta$ we can take its conjugate: $\alpha_x^1 = \beta_{x^2}, \alpha_x^2 = -\beta_{x^2}$. The BZ method can generate new solutions $F, g$ by introducing a spectral parameter $\zeta$ together with the initial condition $\psi(\zeta = 0) = g_0$ (for more details see [10, 11, 17, 18]). Further, soliton solutions can be added depending on the pole singularities of the matrix $\psi(x^1, x^2, \zeta)$ in the complex plane of the spectral parameter $\zeta$. The poles of the $\psi$ matrix are a certain function $\mu_k = \mu(\rho, z)$ of the coordinates and the integer index $k$ denotes the number of solitons added to the initial solution. In BZ the pole trajectory, contrary to the original QISM, is not constant and is given by

$$\mu_k = \omega_k - z \pm \left[(\omega_k - z)^2 + \rho^2\right]^{1/2}$$

(49)

where the constant $\omega_k$ is the origin of the $z$ coordinates.

Finally, for the new solution $F, g$ we have again

$$(-\det ||g||)^{1/2} = \alpha.$$ 

(50)

There are formal analogies between the method presented in this paper and the BZ method. First of all, both of them need a starting metric and the group of isometries must be at least a $G_2$ Abelian group. Moreover, the area of the orbits on the isometries group (condition (50)) is preserved. Our method requires harmonic coordinates in order to achieve the conservation in the form of the line element and therefore of the field equations. For the BZ method the existence of such coordinates $\alpha, \beta$ is a consequence of the integrability condition of the Ernst equation (see [17]). Moreover, in such coordinates the field equation in BZ involving $\psi$ assumes the simplest form. However, these formal analogies are not sufficient to establish a direct link between the two methods. In fact, in spite of the similarity between the pole
trajectory formula and expression (5), we do not have at our disposal a simple way to relate a transformed solution in our method with the number of solitons that can be added in the BZ method.

However, it is interesting to note that in both cases, starting with a spatially homogenous metric with a $G_3$ group of motion, the resulting solution has a $G_2$ group of motion [18]. We conclude by saying that our method is also applicable for spacetimes admitting two spacelike Killing vectors. In this case the equation satisfied by $W$ is the wave equation instead of (45). Furthermore, for the coordinates $W, U$, the analytical condition is substituted with

$$W_x^1 = U_x^2, \quad W_x^2 = U_x^1,$$

implying that $W_x^1 x_1 - W_x^2 x_2 = 0$.

References

[1] Levi-Civita T 1917 Rend. Acc. Lincei 26 317
[2] Chazy J 1924 Bull. Soc. Math. France 52 17
[3] Lewis T 1932 Proc. R. Soc. A 136 176
[4] Linet B 1986 J. Math. Phys. 27 1817
[5] Santos N O 1993 Class. Quantum Grav. 10 2401
[6] Vilenkin A 1985 Phys. Rep. 121 263
[7] Geroch R 1972 J. Math. Phys. 13 233
[8] Xanthopoulos B C 1979 Proc. R. Soc. A 365 381
[9] Ehlers J 1962 Coll. Internationaux, CNRS 275
[10] Belinski V A and Zakharov V E 1978 Sov. Phys.—JETP 48 985
[11] Belinski V A and Zakharov V E 1979 Sov. Phys.—JETP 50 1
[12] Papapetrou V A 1953 Ann. Phys., Lpz. 6 12
[13] Ernst F J 1968 Phys. Rev. 167 5
[14] Bergamini R and Viaggiu S 2004 Class. Quantum Grav. 21 4567 (Preprint gr-qc/0305035)
[15] Ferrarese G 1994 Lezioni di Relatività Generale (Bologna: Pitagora Editrice)
[16] Kramer D, Stephani H, Herlt E and MacCallum M 1980 Exact Solutions of Einstein’s Field Equations
   (Cambridge: Cambridge University Press)
[17] Alekseev G A 2001 Physica D 152 97 (Preprint gr-qc/0001012)
[18] Jantzen R T 1980 Nuovo Cimento B 59 287