DEFORMATION QUANTIZATION VIA FELL BUNDLES

Beatriz Abadie*, and Ruy Exel**

May 1, 1997

ABSTRACT. A method for deforming C*-algebras is introduced, which applies to C*-algebras that can be described as the cross-sectional C*-algebra of a Fell bundle. Several well-known examples of non-commutative algebras, usually obtained by deforming commutative ones by various methods, are shown to fit our unified perspective of deformation via Fell bundles. Examples are the non-commutative spheres of Matsumoto, the non-commutative lens spaces of Matsumoto and Tomiyama, and the quantum Heisenberg manifolds of Rieffel. In a special case, in which the deformation arises as a result of an action of R^2, assumed to be periodic in the first d variables, we show that we get a strict deformation quantization.

1. Introduction. Deformations of C*-algebras, specially of the C*-algebra of continuous functions on the phase space of a classical physical system, have long been associated to the process of quantization and have been used to explain quantum mechanical phenomena such as the correspondence principle (see, for example, [16]).

One of the most popular processes for constructing these deformations is to describe the given C*-algebra by means of generators and relations, and, after introducing a deformation parameter into these relations, to consider the universal C*-algebra for the new relations. This process can be used, for example, for constructing the non-commutative torus [14], the soft torus [5], the quantum SU_2 groups [19], the non-commutative spheres [10], the non-commutative lens spaces [12], and the algebra of the q-canonical commutation relations [9].

However, C*-algebras arising from generators and relations are often intractable objects motivating one to search for alternate constructions. The goal of the present work is to show the usefulness of the techniques of Fell bundles (also known as C*-algebraic bundles [8]) in the study of deformations of C*-algebras.

The first step in our construction, as our title suggests, requires one to look for a Fell bundle structure for the given algebra B, over a locally compact topological group G. In the important special case of a discrete group G, by far the easiest to understand, this is roughly equivalent (see [7]) to finding a G-grading for B, that is, a decomposition of B as the closure of the direct sum \( \bigoplus_{t \in G} B_t \) of a family of closed linear subspaces \( B = \{ B_t \}_{t \in G} \), satisfying \( B_t B_s \subseteq B_{ts} \) and \( B_t^* = B_{t^{-1}} \) for all \( t \) and \( s \) in G. The collection \( B = \{ B_t \}_{t \in G} \), equipped with the operations of multiplication

\[ \cdot : B_t \times B_s \to B_{ts}, \]

and involution

\[ * : B_t \to B_{t^{-1}}, \]

inherited from \( B \), gives an important example of a Fell bundle. This Fell bundle contains, in many cases, the necessary information for the reconstruction of the algebra B. This is so, for example, when the group G is amenable. However, in some important cases where G does not posses this property, this is still true. See [7] and [13] for a thorough discussion of this problem.

The second step in constructing our deformation requires an action \( \theta \) of the group G on \( B \), called the deforming action, which is then used to deform the Fell bundle structure by means of introducing a new multiplication operation \( \times \) and a new involution \( * \), via the formulas

\[ a_t \times b_s := a_t \theta_t(b_s), \]

* Partially supported by CONICYT, Proyecto 2002 – Uruguay.
** Partially supported by CNPq – Brazil.
and
\[ a_t^s := \theta_t^{-1}(a_t^s), \]
for \( s, t \in G \), \( a_t \) in \( B_t \) and \( b_s \) in \( B_s \).

The norm and the linear structure of \( B \), on the other hand, are kept intact. The deformed algebra is then obtained by taking the cross-sectional C*-algebra of the deformed Fell bundle.

The invariance of the linear structure and the norm of the Fell bundle, under the deformation, is an important feature of our construction, because it allows us to embed part of the original algebra into its deformed version.

An important ingredient in the established theories of deformations of C*-algebras is the continuity of the collection of deformed algebras, as a function of the deformation parameter. This property is usually expressed by the fact that these form a continuous field of C*-algebras over the parameter space (see [15]). Our method of deformation is also shown to produce continuous fields of C*-algebras when one is given, not just one, but a family \( \{\theta^h\}_{h \in I} \) of actions of \( G \) on \( B \), as above, where \( I \) is an interval of real numbers. Since all of the applications presented here deal with discrete groups, we have opted to restrict our study of continuity for the discrete group case, although it seems plausible to expect that generalizations can be found for continuous groups.

The continuity results we obtain are, essentially, reworkings of Rieffel’s ideas in [15] for our more general situation of Fell bundles.

Our process of deformation is rather simple to follow, given that the necessary ingredients, the Fell bundle structure and the deforming action \( \theta \), are provided. One way to obtain these ingredients is via a deformation data for \( B \), that is, a triple \((G, \gamma, \theta)\), where \( G \) is a discrete abelian group and \( \gamma \) and \( \theta \) are commuting actions on \( B \), respectively of the Pontryagin dual \( \hat{G} \) of \( G \), and of \( G \) itself. Since \( \hat{G} \) is compact, the spectral decomposition of \( \gamma \) provides the Fell bundle structure, while \( \theta \) plays the role of the deforming action. This approach essentially consists of introducing a deformation parameter after taking a certain Fourier transform, a method that has already been used by other authors, including Rieffel (see, for example, the formula for the definition of \( s_h \) on page 541 of [16]). The advantage of emphasizing the Fell bundle structure is, perhaps, in making some formulas more transparent.

This construction, albeit rather elementary, provides some very interesting examples. We show, for instance, that the non-commutative spheres, the non-commutative lens spaces, and the quantum Heisenberg manifolds [16], can all be seen under the unified perspective of deformation of Fell bundles.

Even though our deformation is described for Fell bundles over general locally compact groups, all of our examples are restricted to the simpler case of discrete groups. This is partly due to the difficulty in identifying Fell bundle structures over non-discrete groups, but we feel that more general examples may be just as relevant.

We have already touched upon the need to search for alternate methods of deformations of C*-algebras. Perhaps one of the most successful approaches is the one due to Rieffel [17,18] where deformations arise from actions of \( \mathbb{R}^n \) on the given algebra. As indicated by Rieffel in page 84 of [18], when the action of \( \mathbb{R}^n \) factors through a compact group, one can describe the algebra as the cross-sectional algebra of the Fell bundle arising from the spectral subspaces of the given action. Rieffel’s deformation may then be seen as a deformation of the Fell bundle structure by means of the introduction of a 2-cocycle.

Roughly following this approach, our method may be viewed as a study of the situations in which one benefits from the simplifications arising from the general phenomena of “compactness”, or the dual notion of “discreteness”. However, our approach differs from Rieffel’s because our deformation is caused by a group action, as opposed to a 2-cocycle, and because we deform both the multiplication operation and the involution, while Rieffel’s deformation affects only the former.

One of the most important ingredients in the theory of deformation quantization is the computation of the first order term (the derivative at zero) of the deformed multiplication operation, as a function of the deformation parameter, often denoted by \( h \). We carry this out in the case where a C*-algebra \( B \) is deformed via a deformation data arising from an action \( \phi \) of \( \mathbb{R}^{2d} \) on \( B \). To be precise, let \( \phi \) be such an action. Our “compactness” assumption consists of supposing that \( \phi \) is periodic in its first \( d \) variables, hence inducing an action \( \gamma \) of the compact \( d \)-dimensional torus \( T^d \) on \( B \). Fixing a real number \( h \), we let \( \theta^h \) be the action of
We then show that the chosen to adopt. In what follows, we shall identify properties of these elements are necessary. The triple \((\mathbb{Z}^d, \gamma, \theta^h)\) is then a deformation data for \(B\), with which we construct the deformed algebra \(B^{(h)}\).

The derivation operators associated to the action \(\phi\) under the coordinate system \((x_1, \ldots, x_d, y_1, \ldots, y_d)\) for \(\mathbb{R}^{2d}\).

This is initially done for a very restrictive class of elements \(f\) and \(g\) in \(B\), namely the smooth elements belonging, each, to a single spectral subspace for \(\gamma\). The proof of this result is extremely simple and the formulas involved show, in a very transparent way, the roles of the various ingredients present in the context. In particular, the heavy machinery of oscillatory integrals of \([17]\) does not intervene, thanks, of course, to the simplification introduced by the periodicity assumption. Because of the simplicity of our formulas, our approach may be pedagogically relevant for the understanding of more sophisticated constructions.

The formula for the derivative of the deformed multiplication, above, is then extended to smooth elements \(f\) and \(g\), with a proof which permits one to see through the extent to which the differentiability properties of these elements are necessary.

This immediately implies that

\[
\lim_{h \to 0} \left\| \frac{f \times_h g - f g}{h} - \frac{1}{2\pi i} \sum_{j=1}^d \partial_{x_j}(f) \partial_{y_j}(g) \right\|_h = 0,
\]

where \(\times_h\) and \(\| \cdot \|_h\) refer to the deformed product and norm of \(B^{(h)}\), respectively, and \(\partial_{x_j}\) and \(\partial_{y_j}\) denote the derivation operators associated to the action \(\phi\) under the coordinate system \((x_1, \ldots, x_d, y_1, \ldots, y_d)\) for \(\mathbb{R}^{2d}\).

Combining this with the fact, shown below, that the \(B^{(h)}\), form a continuous field of \(C^*\)-algebras, we get a strict deformation quantization in the sense of Rieffel \([17, 18]\).

The authors would like to acknowledge the support of CONICYT (Uruguay) and FAPESP (Brazil) for funding numerous academic visits while this research was conducted.

2. The Deformation. Let \(G\) be a locally compact topological group and let \(\mathcal{B} = \{B_t\}_{t \in G}\) be a \(C^*\)-algebraic bundle over \(G\). The reader is referred to \([8]\) for a comprehensive treatment of the basic theory of \(C^*\)-algebraic bundles. These objects have recently been referred to as “Fell bundles”, a terminology we have chosen to adopt. In what follows, we shall identify \(\mathcal{B}\) with the total bundle space \(\bigcup_{t \in G} B_t\).

Let \(\mathcal{D} = \{D_t\}_{t \in G}\) be another Fell bundle over \(G\). A map \(\psi\) from \(\mathcal{B}\) to \(\mathcal{D}\) is called a homomorphism if

i) \(\psi\) is continuous,
ii) \(\psi(B_t) \subseteq D_t\), for all \(t\) in \(G\),
iii) \(\psi\) is linear on each \(B_t\),
iv) \(\psi(ab) = \psi(a)\psi(b)\), for all \(a, b\) in \(\mathcal{B}\), and
v) \(\psi(a^*) = \psi(a)^*\), for all \(a\) in \(\mathcal{B}\).

Let \(\psi\) be a homomorphism from \(\mathcal{B}\) to \(\mathcal{D}\). Observe that, since \(\psi\) restricts to a \(*\)-homomorphism between the \(C^*\)-algebras \(B_t\) and \(D_t\) (where \(e\) denotes the unit group element), then it is necessarily contractive there. Also, for each \(b_t\) in \(B_t\) we have

\[
\|\psi(b_t)\|^2 = \|\psi(b_t^* b_t)\| \leq \|b_t^* b_t\| = \|b_t\|^2,
\]

so that \(\psi\) is in fact norm-contractive everywhere.
If \( \psi \) is bijective, then \( \psi^{-1} \) is continuous as well [8,II.13.17] and hence it is also a homomorphism. In this case we say that \( \psi \) is an isomorphism. If, in addition, \( D = B \), then \( \psi \) is called an automorphism of \( B \). In particular, if \( \psi \) is an isomorphism, then it must be isometric. See also [8, VIII.3.3].

Given another locally compact topological group \( H \), by an action of \( H \) on \( B \), we shall mean an assignment \( \theta \) which, for each \( x \) in \( H \), gives an automorphism \( \theta_x \) of \( B \), satisfying \( \theta_x \theta_y = \theta_{xy} \), for \( x,y \in H \). We shall say that \( \theta \) is a continuous action if the map

\[
(x,b) \in H \times B \longmapsto \theta_x(b) \in B
\]

is jointly continuous.

Let us now suppose we are given a Fell bundle \( B = \{ B_t \}_{t \in G} \) over the locally compact group \( G \), as well as a continuous action \( \theta \) of the very same group \( G \) on \( B \). We wish to construct a new product on \( B \), denoted \( \times \), and a new involution, called \( {}^\circ \), providing a “deformed” bundle structure. In order to do so, define for \( a_t \) in \( B_t \) and \( b_s \) in \( B_s \),

\[ a_t \times b_s = a_t \theta_t(b_s), \]

and

\[ a_t^* = \theta_t^{-1}(a_t^*). \]

2.1. Proposition. If \( B \) keeps its linear, topological and norm structure, but is given the deformed operations \( \times \) and \( {}^\circ \), then it is a Fell bundle.

Proof. To check that the new multiplication operation is continuous, we shall use [8, VIII.2.4]. That is, given continuous sections \( \beta \) and \( \gamma \) of \( B \), we must show that the map

\[
(r,s) \in G \times G \longmapsto \beta(r) \times \gamma(s) \in B
\]

is continuous. Now, we have \( \beta(r) \times \gamma(s) = \beta(r)\theta_r(\gamma(s)) \), which is continuous by the continuity of \( \theta \) and of the original multiplication. A similar argument shows that the deformed involution is continuous.

Let us now verify the associativity of \( \times \). Given \( a_r \) in \( B_r \), \( b_s \) in \( B_s \), and \( c_t \) in \( B_t \) we have

\[
(a_r \times b_s) \times c_t = (a_r \theta_r(b_s)) \times c_t = a_r \theta_r(b_s)\theta_{rs}(c_t) = a_r \theta_r(b_s\theta_{rs}(c_t)) = a_r \times (b_s\theta_{rs}(c_t)) = a_r \times (b_s \times c_t).
\]

As for the anti-multiplcativity of the involution, let \( a_r \in B_r \) and \( b_s \in B_s \). Then

\[
(a_r \times b_s)^* = (a_r \theta_r(b_s))^* = \theta_{r^{-1}}(a_r \theta_r(b_s))^* = \theta_{r^{-1}}(\theta_r(b_s)a_r^*) = \theta_{r^{-1}}(b_s^*) \theta_{r^{-1}}(a_r^*) = \theta_{r^{-1}}(b_s^*) \times \theta_{r^{-1}}(a_r^*) = b_s^* \times a_r^*.
\]

The verification of the remaining axioms is routine and so is left as an exercise. \( \square \)

2.2. Definition. The bundle constructed above will be called the \( \theta \)-deformation of \( B \) and will be denoted \( B^\theta \).

Recall that a Fell bundle is said to be saturated [8,VIII.2.8] if \( B_{rs} = B_rB_s \) (closed linear span) for all \( r,s \). In the special case that \( G \) is equipped with a “length” function

\[ | \cdot | : G \to \mathbb{R}_+ \]

satisfying \( |e| = 0 \), and the triangular inequality \( |rs| \leq |r| + |s| \), then we say that \( B \) is semi-saturated (see [6,4.1, 4.8], [7,6.2]), if \( B_{rs} = B_rB_s \), whenever \( r,s \in G \) are such that \( |rs| = |r| + |s| \).

2.3. Proposition. If \( B \) is saturated (resp. semi-saturated) then so is \( B^\theta \).

Proof. It is enough to observe that \( B_r \times B_s = B_r\theta_r(B_s) = B_rB_s \). \( \square \)
3. Continuous fields arising from deformations. The purpose of this section is to show that the collection of deformed algebras, originated from a continuous family of group actions on a Fell bundle, gives rise to a continuous field of $C^*$-algebras.

We first establish some facts on Fell bundles over discrete groups that will enable us to extend the techniques in [15] to discuss upper semicontinuity. Let $\mathcal{B}$ and $\mathcal{D}$ be Fell bundles over a discrete group $G$, and let $\Phi : \mathcal{D} \to \mathcal{B}$ be a Fell bundle homomorphism. Since $\Phi$ is contractive, one can define $\Phi^1 : L^1(\mathcal{D}) \to L^1(\mathcal{B})$ by $[\Phi^1(f)](x) = \Phi[f(x)]$, for $f \in L^1(\mathcal{D})$, and $x \in G$. It is easily checked that $\Phi^1$ is a $*$-algebra homomorphism, so it gives rise to a $C^*$-algebra homomorphism $\Phi : C^*(\mathcal{D}) \to C^*(\mathcal{B})$.

A sequence of Fell bundle homomorphisms

$$0 \to \mathcal{E} \xrightarrow{i} \mathcal{D} \xrightarrow{\Pi} \mathcal{B} \to 0$$

is said to be exact if so are the sequences

$$0 \to \mathcal{E}_x \xrightarrow{i_x} \mathcal{D}_x \xrightarrow{\Pi|_x} \mathcal{B}_x \to 0$$

for all $x \in G$.

3.1. Lemma. Let $0 \to \mathcal{E} \xrightarrow{i} \mathcal{D} \xrightarrow{\Pi} \mathcal{B} \to 0$ be an exact sequence of Fell bundle homomorphisms over a discrete group $G$. Then $0 \to C^*(\mathcal{E}) \xrightarrow{i} C^*(\mathcal{D}) \xrightarrow{\Pi} C^*(\mathcal{B}) \to 0$ is also exact.

Proof. In view of [20, 2.29], and [8, VIII 5.11, 16.3], we only need to show that $0 \to L^1(\mathcal{E}) \xrightarrow{i} L^1(\mathcal{D}) \xrightarrow{\Pi^1} L^1(\mathcal{B}) \to 0$ is exact. It is apparent from the definition that $i^1$ is injective, and that $\text{Im}(i^1) = \ker(\Pi^1)$, so we need only show that $\Pi^1$ is onto. Fix $b_x \in \mathcal{B}_x$ and $\epsilon > 0$. Since $b_x \delta_x \in \text{Im} \Pi$, there exists $\bar{d} \in C^*(\mathcal{D})$ such that $\Pi(\bar{d}) = b_x \delta_x$, and

$$\|\bar{d}\|_{C^*(\mathcal{D})} \leq \|d + \ker \Pi\|_{C^*(\mathcal{D})/\ker \Pi} + \epsilon = \|b_x \delta_x\|_{C^*(\mathcal{B})} + \epsilon = \|b_x\|_{\mathcal{B}_x} + \epsilon.$$

Let $P^D_x$ (resp. $P^B_x$) denote the projection onto the $x^{th}$ spectral subspace of $\mathcal{D}$ (resp. $\mathcal{B}$). Then $P^B_x \Pi = \Pi P^D_x$, since the equality holds when restricted to $L^1(\mathcal{D})$. Now set $d = P^D_x(\bar{d})$. Then $d \in \mathcal{D}_x$, $\Pi(d) = \Pi P^D_x(\bar{d}) = P^B_x \Pi(\bar{d}) = b_x$, and $\|d\|_{\mathcal{D}_x} \leq \|\bar{d}\|_{C^*(\mathcal{D})/\ker \Pi} \leq \|b_x\|_{\mathcal{B}_x} + \epsilon$.

Now, if $\sum b_n \delta_{x_n} \in L^1(\mathcal{B})$, choose as above, for each positive integer $n$, $d_n \in \mathcal{D}_{x_n}$ so that $\Pi(d_n) = b_n$, and $\|d_n\|_{\mathcal{D}_{x_n}} \leq \|b_n\|_{\mathcal{B}_{x_n}} + n^{-2}$. Then $\Pi(\sum c_n \delta_{x_n}) = \sum b_n \delta_{x_n}$, so $\Pi^1$ is onto. \hfill $\square$

Back to the setting of the previous section, we consider a $C^*$-algebra $B$ that can be viewed as the cross-sectional $C^*$-algebra of a Fell bundle $\mathcal{B}$ over a discrete group $G$ whose $x^{th}$ fiber we denote by $B_x$. At this point we are ready to get a deformed version of $B$ by means of an action $\theta$ of $G$.

Notice that the algebra $B^\theta$ contains as a dense $*$-subalgebra the set $\bigoplus_{x \in G} B_x$ of compactly supported cross-sections. Although the $*$-algebra structure of $\bigoplus_{x \in G} B_x$ depends on $\theta$, its vector space structure does not.

Our purpose is to produce a continuous field of $C^*$-algebras, given a family $\{\theta^h\}$ of actions of $G$ on $\mathcal{B}$. The crucial point is to show that the map $h \mapsto \|\phi\|_h$ is continuous for any $\phi \in \bigoplus_{x \in G} B_x$, where $\|\phi\|_h$ denotes the norm of $\phi$ as an element of $C^*(B^\theta)$.

3.2. Notation. In the context above, let $I \subset \mathbb{R}$ be an open interval containing 0 and, for each $h \in I$, let $\theta^h$ be an action of $G$ on the Fell bundle $\mathcal{B}$ such that $\theta^0$ is the identity, and that the map $h \mapsto \theta^h_x(b)$ is continuous for any fixed $x \in G$, $b \in \mathcal{B}$. We denote the bundle $B^\theta$ by $B^h$, and by $\times_h$, $\otimes_h$ its product and involution, respectively. The norm in $C^*(B^h)$ is denoted by $\|\|_h$.

3.3. Proposition. The map $h \mapsto \|\phi\|_h$ is upper semicontinuous on $I$ for all $\phi \in \bigoplus_{x \in G} B_x$. 
Proof. The proof follows the lines of [15]. Let $\mathcal{D}$ be the Fell bundle over $G$ whose $x^{th}$ fiber is the Banach space $D_x = C_0(I, B_x)$, with multiplication and involution given by

$$(f_x \star f_y)(h) = f_x(h) \times_h f_y(h), \quad f_x^*(h) = (f_x(h))^\cap_h,$$

for $f_x \in D_x$, $f_y \in D_y$. For each $h \in I$ consider the Fell bundle homomorphism $\Pi^h : \mathcal{D} \to B$, given by $\Pi^h(f) = f(h)$. Since $\Pi^h$ is onto for any $h \in I$ we get, as in Lemma 3.1, the exact sequence

$$0 \to C^*(\mathcal{E}^h) \xrightarrow{\gamma_h} C^*(\mathcal{D}) \xrightarrow{\Pi^h} C^*(B^h) \to 0,$$

where $\mathcal{E}^h$ is the Fell bundle whose $x^{th}$ fiber is $E^h_x = \ker \Pi^h_x$, with the structure inherited from $\mathcal{D}$, and $\gamma_h$ denotes inclusion.

In order to apply [15.1.2], we next consider $C_0(I)$ as a $C^*$-subalgebra of the algebra of multipliers of $D_c$, in the obvious way, so we can view it ([8, VIII, 3.8]) as a central $C^*$-subalgebra of the multiplier algebra of $C^*(\mathcal{D})$.

Let $J_h = \{ f \in C_0(I) : f(h) = 0 \}$. It only remains to show that $C^*(\mathcal{E}^h) = C^*(\mathcal{D})_{J_h}$. For then, by [15.1.2], we will have that $\mathcal{D} \to \Pi^h(\phi)$ is upper semicontinuous for all $\phi \in C^*(\mathcal{D})$. This implies that $\mathcal{D} \to \Pi^h(\phi)$ is upper semicontinuous for any $\psi \in \bigoplus_{x \in G} B_x$. Now, it is apparent that $\phi_j \in L^1(\mathcal{E})$ for $j \in J_h$, and $\phi \in L^1(\mathcal{D})$, which shows that $C^*(\mathcal{D})_{J_h} \subset C^*(\mathcal{E}^h)$. On the other hand, if $\{ e_\lambda \}$ is a bounded approximate identity for $J_h$, then $\lim_\lambda \phi e_\lambda \to \phi$ for all $\phi \in C^*(\mathcal{E}^h)$; it suffices to show it for compactly supported maps $\phi$, since $\{ e_\lambda \}$ is assumed to be bounded. Notice that the statement holds for $\phi = f e_\lambda$, with $f \in E^h_x$, because $E^h_x = B_x \otimes J_h$. Now, if $\phi = f e_\lambda$ for some $f \in E^h_x$, we have

$$\|\phi e_\lambda - \phi\|^2 = \| (\phi e_\lambda - \phi)^*(\phi e_\lambda - \phi)\| \leq \|\phi\| \|\phi e_\lambda - \phi\|,$$

which goes to zero because $\phi^*\phi \in E^h_x$. This shows that $C^*(\mathcal{D})_{J_h} \subset C^*(\mathcal{E}^h)$. \qed

3.4. Proposition. If $G$ is also amenable, then the map $h \to \|\phi\|_h$ is lower semicontinuous on $I$ for all $\phi \in \bigoplus_{x \in G} B_x$.

Proof. Since $G$ is amenable, the left regular representation $\Lambda^h$ of $C^*(B^h)$ is faithful ([7,2.3 and 4.7]), so it suffices to show that $h \to \|\Lambda^h_x\|$ is lower semicontinuous for $\phi \in \bigoplus_{x \in G} B_x$.

As in [7], for $h \in I$ we denote by $L^2(B^h)$ the completion of $C_c(B^h)$ with its obvious right pre-Hilbert module structure over $B^h$, which yields the norm

$$\|\xi\|^2 = \| \sum_{x \in G} \phi(x) \times_h \xi(x) \|_{B^h} = \| \sum_x \theta_x^{-1} \phi(x)^* \xi(x) \|_{B^h},$$

for any $\xi \in \bigoplus_{x \in G} B_x$, the undecorated involution and multiplication denoting those in $B^0$.

The left regular representation $\Lambda^h$ of $\phi \in \bigoplus_{x \in G} B_x$ is the adjointable operator given by:

$$(\Lambda^h_\phi \xi)(y) = \sum_{x \in G} \phi(x) \times_h \xi(x^{-1} y) = \sum_x \phi(x) \theta_x^h [\xi(x^{-1} y)],$$

for $\xi \in \bigoplus_{x \in G} B_x \subset L^2(B^h)$. So we have

$$\|\Lambda^h_\phi \xi\|_{B^h} = \| \sum_{x,y} \theta_y \phi(x)^* \theta_x^h \phi(x)(\xi(x^{-1} y)) ) \|_{B^h}.$$
\{\frac{1}{m!}\xi_m\} also converges to \xi. So one can take \xi_0 \in \bigoplus_{x \in G} B_x, such that \|\xi_0\| = 1 and \|\xi - \xi_0\| < \frac{\epsilon}{2}\|\Lambda_\phi^{h_0}\|.

Then
\[ \|\Lambda_\phi^{h_0}\| - \frac{\epsilon}{2} < \|\Lambda_\phi^{h_0} \xi_0\| + \frac{\epsilon}{2}, \]
as required. It now follows that, for \bar{h} close enough to \bar{h}_0,
\[ \frac{\|\Lambda_\phi^{h} \xi_0\|}{\|\xi_0\|} > \|\Lambda_\phi^{h}\| - \epsilon, \text{ so } \|\Lambda_\phi^{h}\| > \|\Lambda_\phi^{h_0}\| - \epsilon. \]

\[\square\]

We summarize the previous results in the following theorem.

3.5. \textbf{Theorem.} Let \mathcal{B} be a Fell bundle over a discrete amenable group \(G\), and let \(B = C^*(\mathcal{B})\). If \(\{\bar{\theta} : h \in I\}\) and \(\mathcal{B}^h\) are as in 3.2, then \((C^*(\mathcal{B}^h), \Lambda)\) is a continuous field of \(C^*\)-algebras, such that \(C^*(\mathcal{B}^h) = B\), where \(\Lambda\) is the family of cross-sections obtained, as in [4,10.2.3], out of \(C_\nu(\mathcal{B}^h)\).

4. \textbf{Discrete abelian groups.} We would now like to describe a method for producing examples of the above situation. To reduce the technical difficulties to a minimum we will consider here exclusively the case of discrete abelian groups. Several interesting examples, however, will fit this context.

Fix, throughout this section, a discrete abelian group \(G\) and let \(\widehat{G}\) be its Pontryagin dual, so that \(\widehat{G}\) is a compact abelian group. We shall denote the duality between \(G\) and \(\widehat{G}\) by
\[(x,t) \in \widehat{G} \times G \mapsto \langle x,t \rangle \in S^1.\]

Let \(B\) be a \(C^*\)-algebra carrying a continuous action \(\gamma\) of \(\widehat{G}\). For each \(t\) in \(G\), the \(t\)-spectral subspace of \(B\) is defined by
\[B_t = \{b \in B : \gamma_x(b) = \langle x,t \rangle b, \text{ for all } x \in \widehat{G}\}.\]

It is an easy exercise to show that each \(B_t\) is a closed linear subspace of \(B\), that \(B_t B_s \subseteq B_{t,s}\), and that \(B_t^* = B_t^{-1}\). By imitating [6.2.5] one can show that \(B\) coincides with the closure of \(\bigoplus_{t \in G} B_t\) (we use the symbol \(\bigoplus\) to denote the algebraic direct sum, that is, the set of finiite sums) and that the formula
\[P_t(b) = \int_{\widehat{G}} \langle x,t \rangle^{-1} \gamma_x(b) \, dx, \quad \text{for } b \in B, t \in G,\]
defines a contractive projection \(P_t\), from \(B\) onto \(B_t\), where the integral is taken with respect to normalized Haar measure on \(\widehat{G}\). If \(\epsilon\) denotes the unit of \(G\), then \(P_\epsilon\) is in fact a positive conditional expectation onto \(B_\epsilon\).

The collection \(\mathcal{B} = \{B_t\}_{t \in G}\) therefore constitutes a Fell bundle over \(G\) and also makes \(B\) into a topologically \(G\)-graded algebra, as defined in [7,3.4]. If we take into account the fact that abelian groups are amenable, and use [7.4.7] in combination with [7.4.2], we then conclude that \(B\) is isomorphic to the (full) cross-sectional \(C^*\)-algebra of \(\mathcal{B}\) [8, VIII.17.2] as well as to its reduced cross-sectional \(C^*\)-algebra [7.2.3].

Now suppose that, in addition to the action \(\gamma\) above, we are given an action of \(G\) on \(B\) which commutes with \(\gamma\), in the sense that each \(\gamma_x\) commutes with each \(\theta_t\). It then follows that \(\theta_s(B_t) \subseteq B_t\) for each \(t, s \in G\), so that we can think of \(\theta\) as an action of \(G\) on the Fell bundle \(\mathcal{B}\).

This can in turn be fed to the construction described in the previous section, providing the \(\theta\)-deformed bundle \(\mathcal{B}_\theta^\gamma\). Taking a further step, one can form the cross-sectional algebra of this bundle.

4.1. \textbf{Definition.} Given commuting actions \(\gamma\) and \(\theta\), respectively of \(\widehat{G}\) and \(G\), on the \(C^*\)-algebra \(B\), the cross-sectional algebra of \(\mathcal{B}_\theta^\gamma\) will be called the \((\gamma, \theta)\)-deformation of \(B\) and will be denoted \(B_\theta^\gamma\).

It should be noted that, if \(\theta\) is the trivial action, then \(\mathcal{B}_\theta^\gamma\) is nothing but \(B\) itself and hence, by the comment made earlier, its cross-sectional \(C^*\)-algebra coincides with \(B\), and so \(B_\theta^\gamma = B\). Likewise, if \(\gamma\) is trivial then \(B_t = \{0\}\), for all \(t\), except for \(B_e\) which is the whole of \(B\) and, once more, one has that \(B_e^\gamma = B\). However, if neither group acts trivially, then the algebraic structure of \(B\) may suffer a significant transformation as it will become apparent after we discuss a few examples.

We like to think of this as if the pair \((\gamma, \theta)\) "causes" a deformation on \(B\). This motivates our next:
4.2. Definition. A deformation data for a $C^*$-algebra $B$ consists of a triple $(G, \gamma, \theta)$, where $G$ is a discrete abelian group, and $\gamma$ and $\theta$ are commuting actions, respectively of $\hat{G}$ and $G$, on $B$. The action $\gamma$ will be called the gauge action while $\theta$ will be referred to as the deforming action.

Unless otherwise noted, whenever we speak of the Fell bundle $B = \{B_t\}_{t \in G}$, in the presence of a deformation data $(G, \gamma, \theta)$ for a $C^*$-algebra $B$, we will be referring to the spectral decomposition of the gauge action, as above.

4.3. Remark. Observe that $B^\theta_\gamma$, being the cross-sectional $C^*$-algebra of $B^\theta$, contains the algebraic direct sum $\bigoplus_{t \in G} B_t$ as a dense *-sub-algebra. It should be noted that the set $\bigoplus_{t \in G} B_t$ itself, as well as its linear structure, depends exclusively on the gauge action. However, its involution and multiplication operations are strongly dependent of the deforming action. Also, since the process of deformation does not affect the norm structure of the Fell bundle, and since the fibers of that bundle embed isometrically into its cross-sectional algebra, we see that the norm of an element belonging to a single fiber remains unaffected by the deformation. However, there is not much we can say about the norm of other elements in $\bigoplus_{t \in G} B_t$. Summarizing, in case we are given several deformation data sharing the same gauge action, it will be convenient to think of the deformed algebras as completions of $\bigoplus_{t \in G} B_t$ under different norms and with different algebraic operations.

4.4. Proposition. Let $(G, \gamma, \theta)$ be a deformation data for a $C^*$-algebra $B$. Suppose $B$ carries a third continuous action $\alpha$, this time of a locally compact group $H$, which commutes both with $\gamma$ and $\theta$. Then there exists a continuous action $\bar{\alpha}$ of $H$ on $B^\gamma_\gamma$ which coincides with $\alpha$ on $\bigoplus_{t \in G} B_t$.

Proof. Since $\alpha$ commutes with the gauge action, each spectral subspace $B_t$ is invariant by $\alpha_h$, for each $h \in H$. So $\alpha_h$ can be thought of as an automorphism of the Fell bundle $B$. We claim it is also automorphic for the deformed structure. In fact, if $b_t \in B_t$ and $b_s \in B_s$ then

$$\alpha_h(b_t \times b_s) = \alpha_h(b_t \theta_t(b_s)) = \alpha_h(b_t) \theta_t(\alpha_h(b_s)) = \alpha_h(b_t) \times \alpha_h(b_s),$$

and

$$\alpha_h(b_t^\gamma) = \alpha_h(\theta_t^{-1}(b_t^\gamma)) = \theta_t^{-1}(\alpha_h(b_t)^*) = \alpha_h(b_t)^\gamma.$$

Thus $\alpha_h$ extends to an automorphism of $B^\gamma_\gamma$. The remaining verifications are left to the reader. \hfill $\Box$

Among the possible choices for the action $\alpha$ above one could take the gauge action itself, so one can speak of the “deformed gauge action”, that is $\bar{\gamma}$.

4.5. Proposition. For each $t$ in $G$, the $t$-spectral subspace for the deformed gauge action on $B^\theta_\gamma$ is precisely $B_t$.

Proof. Let us temporarily denote the $t$-spectral subspace for $\bar{\gamma}$ by $\bar{B}_t$. Since $\bar{\gamma}$ coincides with $\gamma$ on $\bigoplus_{t \in G} B_t$, it is clear that $\bar{\gamma}_x(b_t) = (x, t)b_t$ for each $b_t$ in $B_t$. So $B_t \subseteq \bar{B}_t$. Conversely, let $a \in \bar{B}_t$. Then, for each $\varepsilon > 0$, take a finite sum $\sum_{r \in G} b_r$ with $b_r \in B_r$, and such that $\|a - \sum_{r \in G} b_r\| < \varepsilon$. Considering the spectral projections

$$\bar{P}_t(b) = \int_{\hat{G}} (x, t)^{-1}\bar{\gamma}_x(b) \, dx, \quad \text{for} \ b \in B^\theta_t, \ t \in G,$$

we have $a = \bar{P}_t(a)$ while $\bar{P}_t(\sum_{r \in G} b_r) = b_t$. So $\|a - b_t\| = \|\bar{P}_t(a - \sum_{r \in G} b_r)\| < \varepsilon$. This says that $a$ is in the closure of $B_t$ within $B^\theta_t$. But since the norm on $B_t$ is not affected by the deformation, it remains a Banach space after the deformation is performed, and hence it is closed in $B^\theta_t$. Therefore $a \in B_t$. \hfill $\Box$

4.6. Theorem. Let $(G, \gamma, \theta)$ be a deformation data for a $C^*$-algebra $B$, and let $\alpha$ be an action of a group $H$ on $B$ which commutes both with $\gamma$ and $\theta$. Let $B^\theta$ be the fixed point sub-algebra of $B$ for $\alpha$, and let $\gamma^0$ and $\theta^0$ be the restrictions of $\gamma$ and $\theta$ to $B^\theta$, respectively. Then the deformed algebra $(B^\theta)^{\gamma^0}_{\theta^0}$ is isomorphic, in a natural way, to the fixed point sub-algebra of $B^\gamma_\gamma$ for $\bar{\alpha}$. 
5. The derivative of the deformed product. Let $\phi$ be a $C^*$-algebra carrying a strongly continuous action $\phi$ of $\mathbb{R}^{2d}$.

For each $j = 1, \ldots, 2d$, define the differential operator $\partial_{u_j}$ on $B$ by

$$\partial_{u_j}(f) = \frac{d}{d\lambda}\bigg|_{\lambda=0} \langle \phi_{(0, \ldots, \lambda, \ldots, 0)}(f) \rangle,$$

where the $\lambda$ in $(0, \ldots, \lambda, \ldots, 0)$ appears in the $j^{th}$ position. Of course $\partial_{u_j}(f)$ is only defined when $f$ is sufficiently smooth. In particular this is the case for the $\phi$-smooth elements, that is, those elements $f \in B$ such that

$$u \in \mathbb{R}^{2d} \mapsto \phi_u(f) \in B$$

is an infinitely differentiable Banach space valued function. It is well known that these elements form a dense subset of $B$ (see, e.g., [3.2.2.1]).

In what follows we shall adopt the coordinate system $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ on $\mathbb{R}^{2d}$ and hence we shall speak of the differential operators $\partial_{x_j}$ and $\partial_{y_j}$, for $j = 1, \ldots, d$.

In [17] (see also [18]) Rieffel showed how to construct a strict deformation quantization of $B$ "in the direction" of the Poisson bracket $\{\cdot, \cdot\}$ defined by

$$\{f, g\} = \sum_{j=1}^d \partial_{x_j}(f)\partial_{y_j}(g) - \partial_{y_j}(f)\partial_{x_j}(g),$$

in the important special case when $B$ is the algebra of continuous functions on a smooth manifold. Rieffel deals, in fact, with a more general situation, where the Poisson bracket involves the choice of a skew-symmetric matrix $J$.

Without attempting to develop a general theory, we would now like to describe a connection between Rieffel’s theory and ours. Our goal will be to compute the derivative of the deformed product on $B$, arising from a certain deformation data associated to $\phi$. The technical complications will be kept to a minimum by assuming that $\phi$ is periodic in the first $d$ variables.

Let $\gamma$ be the action of $\mathbb{R}^d$ given by the restriction of $\phi$ to its first $d$ variables, that is

$$\gamma(x_1, \ldots, x_d) = \phi(x_1, \ldots, x_d, 0, \ldots, 0), \quad \text{for } (x_1, \ldots, x_d) \in \mathbb{R}^d.$$ 

Our periodicity assumption, to make it precise, is that $\gamma$ is trivial on $\mathbb{Z}^d$ and hence it defines, by passage to the quotient $\mathbb{R}^d/\mathbb{Z}^d$, an action of the $d$ dimensional torus $T^d$ on $B$, which we will still denote by $\gamma$. 

On the other hand, consider the action \( \theta \) of \( \mathbb{R}^d \) on \( B \) defined by
\[
\theta_{(y_1,\ldots,y_d)} = \phi_{(0,\ldots,0,y_1,\ldots,y_d)}, \quad \text{for} \ (y_1,\ldots,y_d) \in \mathbb{R}^d.
\]

If \( h \) is a real number, we will let the action \( \theta^h \) of \( \mathbb{Z}^d \) on \( B \) be defined by
\[
\theta^h_{(n_1,\ldots,n_d)} = \theta_{(hn_1,\ldots,hn_d)}, \quad \text{for} \ (n_1,\ldots,n_d) \in \mathbb{Z}^d.
\]

Since both \( \gamma \) and \( \theta^h \) come from the action of the commutative group \( \mathbb{R}^{2d} \), it is clear that they commute with each other. Moreover, since the Pontryagin dual of the group \( \mathbb{Z}^d \) is precisely \( \mathbb{T}^d \), the triple \( (\mathbb{Z}^d, \gamma, \theta^h) \) is seen to be a deformation data for \( B \).

Let \( \mathcal{B} = \{ B_t \}_{t \in \mathbb{C}} \) be the Fell bundle arising from the spectral decomposition of \( \gamma \). We may then speak of the deformed bundle \( \mathcal{B} \circ \theta^h \) whose operations will be denoted by \( \times_h \) and \( \circ_h \). We also have the deformed algebra \( B^h \), which we will simply denote by \( B^{(h)} \).

**5.1. Proposition.** If \( f \) is \( \phi \)-smooth then \( P_t(f) \) is also \( \phi \)-smooth for all \( t \) in \( \mathbb{Z}^d \). In addition, for \( j = 1,\ldots,2d \), we have \( \partial_j(P_t(f)) = P_t(\partial_j(f)) \), and therefore each \( B_t \) is invariant under \( \partial_j \).

**Proof.** For \( u \in \mathbb{R}^{2d} \) we have
\[
\phi_u(P_t(f)) = \phi_u \left( \int_{\mathbb{T}^d} (x,t)^{-1}\gamma_x(f) \, dx \right) = \int_{\mathbb{T}^d} (x,t)^{-1}\gamma_x(\phi_u(f)) \, dx,
\]
which is therefore smooth as a function of \( u \). This shows that \( P_t(f) \) is \( \phi \)-smooth. We have
\[
\partial_j(P_t(f)) = \left. \frac{d}{d\lambda} (\phi_{(0,\ldots,\lambda,\ldots,0)}P_t(f)) \right|_{\lambda=0} = 0,
\]
\[
= \int_{\mathbb{T}^d} \frac{d}{d\lambda} (\phi_{(0,\ldots,\lambda,\ldots,0)} ) \left( (x,t)^{-1}\gamma_x(f) \right) \right|_{\lambda=0} \, dx = 0,
\]
\[
= \int_{\mathbb{T}^d} (x,t)^{-1}\gamma_x(\partial_j(f)) \, dx = P_t(\partial_j(f)). \quad \Box
\]

**5.2. Lemma.** Let \( t = (t_1,\ldots,t_d) \) and \( s = (s_1,\ldots,s_d) \) be in \( \mathbb{Z}^d \) and take \( f \in B_t \) and \( g \in B_s \). Suppose that \( g \) is smooth for \( \theta \). Then, for all real numbers \( h \)
\[
\left\| \frac{f \times_h g - fg}{h} - \frac{1}{2\pi i} \sum_{j=1}^d \partial_j(f)\partial_{y_j}(g) \right\| \leq |h| \| f \| \left\| \sum_{j,k=1}^d t_j t_k \partial_{y_j}(\partial_{y_k}(g)) \right\|.
\]

**Proof.** Initially we would like to stress that the term whose norm is referred to, in the left hand side above, lies in \( B_{t+s} \). This Banach space embeds isometrically into each \( B^{(h)} \), and hence its norm is unambiguously defined. We have
\[
f \times_h g - fg = f \theta^h_t(g) - fg.
\]

Now, consider the \( C^\infty \) map \( F : \mathbb{R} \to B \) given by
\[
F(h) := \theta^h_t(g) = \phi_{(0,\ldots,0,ht_1,\ldots,ht_d)}(g).
\]

Its first two derivatives are given by
\[
F'(h) = \phi_{(0,\ldots,0,ht_1,\ldots,ht_d)} \left( \sum_{j=1}^d t_j \partial_{y_j}(g) \right),
\]
\[
F''(h) = \phi_{(0,\ldots,0,ht_1,\ldots,ht_d)} \left( \sum_{j,k=1}^d t_j t_k \partial_{y_j}(\partial_{y_k}(g)) \right).
\]
and

\[ F'(h) = \phi_{(0,\ldots,0,h,t_1,\ldots,t_d)} \left( \sum_{j,k=1}^d t_j t_k \partial_{y_j}(\partial_{y_k}(g)) \right), \]

for all \( h \) in \( \mathbb{R} \). The first order Taylor expansion for \( F \) reads

\[ F(h) = F(0) + \frac{h}{\hbar} F'(0) + \int_0^h (h - \lambda) F''(\lambda) \, d\lambda, \]

from where we conclude that

\[ \left\| \frac{F(h) - F(0)}{h} - F'(0) \right\| \leq \|h\| \sup_{\lambda \in I} \|F''(\lambda)\|, \]

where the interval \( I \) is either \([0, h]\) or \([h, 0]\), depending on the sign of \( h \). Translating this back in terms of \( g \), we conclude that

\[ \left\| \frac{\theta^h_1(g) - g}{h} - \sum_{j=1}^d t_j \partial_{y_j}(g) \right\| \leq \|h\| \left\| \sum_{j,k=1}^d t_j t_k \partial_{y_j}(\partial_{y_k}(g)) \right\|. \]

Using the first equation obtained in the course of the present proof gives

\[ \left\| \frac{f \times_k g - fg}{h} - \sum_{j=1}^d t_j f \partial_{y_j}(g) \right\| \leq \|h\| \left\| f \right\| \left\| \sum_{j,k=1}^d t_j t_k \partial_{y_j}(\partial_{y_k}(g)) \right\|. \]

On the other hand, recall that \( f \) is in the \( t \)-spectral subspace of the gauge action. This means that, for \( x = (x_1,\ldots,x_d) \in \mathbb{R}^d \), we have that \( \gamma_x(f) = \langle x, t \rangle f \), or

\[ \gamma_x(f) = e^{2\pi ix_1 t_1} \cdots e^{2\pi ix_d t_d} f. \]

It follows that \( \partial_x(f) = 2\pi i t_j f \), and hence that \( t_j f = (2\pi i)^{-1} \partial_x(f) \), which, when plugged into the last inequality above, leads to the conclusion. \( \square \)

The purpose of this Lemma, as the reader may have anticipated, is to allow us to compute the derivative of \( f \times_k g \), with respect to \( h \), which is one of the most important ingredients in Rieffel’s theory of deformation quantization [17, 18]. However the expression \( f \times_k g \), strictly speaking, applies only for \( f \) and \( g \) belonging, each, to a single spectral subspace of the gauge action. The question we want to address is this:

**5.3. Question.** What is the biggest subset of \( B \) which can be mapped, in a natural way, into each deformed algebra \( B^{(h)} \)?

The remark made in 4.3 is relevant here, providing \( \bigoplus_{i \in \mathbb{Z}^d} B_i \) as a partial answer. Pushing this further, recall that the cross-sectional \( C^* \)-algebra of our Fell bundle \( B \) is defined to be the enveloping \( C^* \)-algebra of the \( L_1 \) cross-sectional algebra \( L_1(B) \). Therefore each \( B^{(h)} \) contains a copy of \( L_1(B) \) which, again by 4.3, does not depend on \( h \), as far as its normed linear space structure is concerned. A better answer to our question is thus \( L_1(B) \).

We do not claim, however, that this is the best possible answer. In fact, the word **natural** in 5.3 lacks a precise meaning, as it stands. The correct way to rephrase 5.3 could possibly be:

**5.4. Question.** For each \( h \), let \( \iota^h : L_1(B) \to B^{(h)} \) be the natural inclusion, viewed as a densely defined linear map on \( B \). Is \( \iota^h \) closable? That is, is the closure of its graph, the graph of a well defined linear map? If so, how to characterize the domain \( D^h \) of this map? Is there any relationship between the \( D^h \) for different \( h \)? What is the intersection of the \( D^h \) as \( h \) ranges in \( \mathbb{R} \)?
In defense of the $L_1$ cross-sectional algebra we must say that it includes the smooth elements for the gauge action: it is a well known fact that, for such an element $f$, one has that $f = \sum_{t \in \mathbb{Z}^d} P_t(f)$, where the series is absolutely convergent, since it satisfies Schwartz’s condition, namely that
\[
\sup_{t \in \mathbb{Z}^d} \| h(t)P_t(f) \| < \infty,
\]
for every complex polynomial $h$ in the $d$ variables $t_1, \ldots, t_d$.

5.5. Theorem. Let $f, g \in B$ be $\phi$-smooth elements. Then
\[
\lim_{\hbar \to 0} \left\| \frac{f \times_{\hbar} g - fg}{\hbar} - \frac{1}{2\pi i} \sum_{j=1}^d \partial_{x_j}(f)\partial_{y_j}(g) \right\|_{\hbar} = 0,
\]
where $\| \cdot \|_{\hbar}$ refers to the norm of the deformed algebra $B^{(\hbar)}$.

Proof. Initially we should observe that the terms appearing between the double bars, above, all have natural interpretations as elements of $B^{(\hbar)}$. This is because the smooth elements $f, g, fg$, and $\partial_{x_j}(f)\partial_{y_j}(g)$, may be seen as elements of $L_1(B)$, which, in turn, may be interpreted as a subset of $B^{(\hbar)}$, according to the comment above.

Write $f = \sum_{t \in \mathbb{Z}^d} P_t(f)$ and $g = \sum_{t \in \mathbb{Z}^d} P_t(g)$. For each $j = 1, \ldots, 2d$ we have that $\partial_{x_j}(f)$ is also smooth, hence it “Fourier series” converges:
\[
\partial_{x_j}(f) = \sum_{t \in \mathbb{Z}^d} P_t(\partial_{x_j}(f)) = \sum_{t \in \mathbb{Z}^d} \partial_{x_j}(P_t(f)),
\]
and similarly for $g$. So,
\[
\sum_{j=1}^d \partial_{x_j}(f)\partial_{y_j}(g) = \sum_{t,s \in \mathbb{Z}^d} \sum_{j=1}^d \partial_{x_j}(P_t(f))\partial_{y_j}(P_s(g)).
\]

Also
\[
\frac{f \times_{\hbar} g - fg}{\hbar} = \sum_{t,s \in \mathbb{Z}^d} P_t(f) \times_{\hbar} P_s(g) - P_t(f)P_s(g).
\]

Using 5.2, it follows that
\[
\left\| \frac{f \times_{\hbar} g - fg}{\hbar} - \frac{1}{2\pi i} \sum_{j=1}^d \partial_{x_j}(f)\partial_{y_j}(g) \right\|_{\hbar} \leq \sum_{t,s \in \mathbb{Z}^d} \left\| \frac{P_t(f) \times_{\hbar} P_s(g) - P_t(f)P_s(g)}{\hbar} - \frac{1}{2\pi i} \sum_{j=1}^d \partial_{x_j}(P_t(f))\partial_{y_j}(P_s(g)) \right\|_{\hbar} \leq \left| \hbar \right| \sum_{t,s \in \mathbb{Z}^d} \left\| P_t(f) \right\| \left\| \sum_{j,k=1}^d t_j t_k \partial_{y_k}(\partial_{y_j}(P_s(g))) \right\| \leq \left| \hbar \right| \sum_{j,k=1}^d \left( \sum_{t \in \mathbb{Z}^d} |t_j t_k| \left\| P_t(f) \right\| \left( \sum_{s \in \mathbb{Z}^d} \left\| P_s(\partial_{y_j}(\partial_{y_k}(g))) \right\| \right) \right).
\]

By our hypothesis, these infinite series converge, and hence the whole thing tends to zero as $\hbar \to 0$. \hfill \Box
5.6. Remark. If one is interested in determining the exact class of differentiability needed for the above result to hold, a quick look at the last displayed expression, in the proof above, gives the answer. That is, \( f \) should be supposed to be of class \( C^{2d+2} \) for \( \gamma \), and the second order differential of \( g \) with respect to \( \theta \) should be of class \( C^{2d} \) for \( \gamma \). These conditions imply the convergence of these infinite series, and hence the conclusion.

Our next result shows that the infinitesimal commutator, for the deformed product, is given by the Poisson bracket described at the beginning of this section. Its proof is an immediate consequence of 5.5.

5.7. Corollary. Let \( f, g \in B \) be smooth elements for \( \phi \). Then

\[
\lim_{h \to 0} \left\| \frac{f \times_h g - g \times_h f - [f, g]}{h} - \frac{1}{2\pi i} \{f, g\}_h \right\| = 0,
\]

where \([\cdot, \cdot]\) is the commutator for the original multiplication on \( B \), and \( \{\cdot, \cdot\} \) is the Poisson bracket defined near the beginning of this section.

Since the family \( \{\theta^h\}_{h \in \mathbb{R}} \) is obviously continuous in the sense of section 3, we get, by 3.5, a continuous field of \( C^* \)-algebras \( \{B^{(h)}\}_{h \in \mathbb{R}} \), and hence a strict deformation quantization in the sense of Rieffel [16, Definition 1.1], with the modification, required in the noncommutative situation, corresponding to the introduction of the term \([f, g]_0\) in the statement of 5.7. With this remark we have, for future reference, the following:

5.8. Corollary. The family \( \{B^{(h)}\}_{h \in \mathbb{R}} \) gives a strict deformation quantization for \( B \), in the direction of the Poisson bracket defined above.

6. Example: Non commutative 3-spheres. In [10] Matsumoto defined a family of \( C^* \)-algebras, denoted \( S_0^3 \), depending on a real parameter \( \vartheta \). This family is to be thought of as a deformation of the commutative \( C^* \)-algebra \( C(S^3) \) of all continuous complex valued functions on the 3-sphere \( S^3 \), because, when \( \vartheta = 0 \), \( S_0^3 \) is isomorphic to \( C(S^3) \).

The purpose of the present section is to show that \( S_0^3 \) can be constructed from a certain deformation data for the algebra \( C(S^3) \). Recall from [10] that \( S_0^3 \) may be defined as the universal \( C^* \)-algebra given by generators and relations as follows: for generators take symbols \( S \) and \( T \) and for relations consider

- M-1) \( S^*S = SS^*, \quad T^*T = TT^* \),
- M-2) \( \|S\| \leq 1, \quad \|T\| \leq 1 \),
- M-3) \( (1 - T^*T)(1 - S^*S) = 0, \) and
- M-4) \( TS = e^{2\pi i \vartheta ST} \).

An alternative description of \( S_0^3 \) is given by [10,8.1]. It says that \( S_0^3 \) is also the universal \( C^* \)-algebra on the generators \( B \) and \( C \) satisfying

- M-1') \( B^*B = BB^*, \quad C^*C = CC^* \),
- M-2') \( B^*B + C^*C = 1, \) and
- M-3') \( BC = e^{2\pi i \vartheta} BC \).

The relationship between these presentations is given by the formulas

\[
B = S(S^*S + T^*T)^{-\frac{1}{2}}, \quad C = T(S^*S + T^*T)^{-\frac{1}{2}}.
\]

Define an action \( \gamma \) of \( S^1 \) on \( S^3 \) by

\[
\gamma_\lambda(z, w) = (\lambda z, \lambda w),
\]

where \( z, w, \lambda \in \mathbb{C} \) satisfy \( |z|^2 + |w|^2 = 1 \) and \( |\lambda| = 1 \).
Also, fixing a real number \( \vartheta \), define an action \( \theta \) of \( \mathbb{Z} \) on \( S^3 \) by

\[
\theta_n(z, w) = (e^{2\pi i \vartheta}z, w), \quad \text{for } (z, w) \in S^3, \quad n \in \mathbb{Z}.
\]

These give actions of \( S^1 \) and \( \mathbb{Z} \) on \( C(S^3) \) by letting

\[
\gamma \lambda(f)\|_{(z, w)} = f(\lambda z, \lambda w), \quad \text{and} \quad \theta_n(f)\|_{(z, w)} = f(e^{2\pi i \vartheta}z, w)
\]

for \( f \in C(S^3) \), \( (z, w) \in S^3 \), \( \lambda \in S^1 \) and \( n \in \mathbb{Z} \). Noting that \( \gamma \) and \( \theta \) commute with each other, we see that we are facing a deformation data \((\mathbb{Z}, \gamma, \theta)\) for the algebra \( C(S^3) \).

### 6.1. Theorem

The deformed algebra \( C(S^3)^{\theta}_\gamma \) is isomorphic to Matsumoto’s algebra \( S^3_3 \).

**Proof.** Let \( Z, W \in C(S^3) \) be the functions defined by

\[
Z(z, w) = z, \quad \text{and} \quad W(z, w) = w,
\]

for \( (z, w) \in S^3 \). Since \( \gamma \lambda(Z) = \lambda Z \) and \( \gamma \lambda(W) = \lambda W \), we have that both \( Z \) and \( W \) belong to the first spectral subspace for \( \gamma \). Then, regarding the deformed product, we have

\[
Z \times W = Z\theta_1(W) = ZW \quad \text{and} \quad W \times Z = W\theta_1(Z) = e^{2\pi i \vartheta}WZ,
\]

so that

\[
W \times Z = e^{2\pi i \vartheta}Z \times W.
\]

This says that \( Z \) and \( W \) satisfy (M-3'). It is easy to check that they also satisfy (M-1') and (M-2') with respect to the deformed product an involution. So, by the universal property, there exists a \( C^* \)-algebra homomorphism

\[
\psi : S^3_\vartheta \to C(S^3)^{\theta}_\gamma,
\]

such that \( \psi(B) = Z \) and \( \psi(C) = W \), which we claim to be an isomorphism.

To show that \( \psi \) is surjective observe that, since \( Z \) and \( W \) belong to the image of \( \psi \), we just have to show that \( Z \) and \( W \) generate \( C(S^3)^{\theta}_\gamma \). In the special case of \( \vartheta = 0 \) this of course follows from the Stone–Weierstrass Theorem. Looking closer we can actually show that the \( n \)-spectral subspace for the action \( \gamma \) on \( C(S^3) \) is linearly spanned by the set

\[
\{Z^i Z^{*j} W^k W^{*l} : i, j, k, l \in \mathbb{N}, i - j + k - l = n\}.
\]

In fact, any \( f \in C(S^3) \) may be arbitrarily approximated by a linear combination of terms of the form \( Z^i Z^{*j} W^k W^{*l} \). Now, if \( f \) belongs to the \( n \)-spectral subspace, then \( f = P_n(f) \), where \( P_n \) is the corresponding spectral projection. On the other hand, if \( P_n \) is applied to the linear combination just mentioned, all terms will vanish except for those for which \( i - j + k - l = n \).

The somewhat curious fact that \( Z \) and \( W \) are also eigenvalues for \( \vartheta \) implies that

\[
Z^i \times Z^{*j} \times W^k \times W^{*l} = \mu Z^i Z^{*j} W^k W^{*l},
\]

for some complex number \( \mu \) of modulus one. Therefore one concludes that each spectral subspace for the deformed gauge action is contained in the sub-algebra of \( C(S^3)^{\theta}_\gamma \) generated by \( Z \) and \( W \). This shows that \( Z \) and \( W \) generate \( C(S^3)^{\theta}_\gamma \), and hence that \( \psi \) is surjective.

We next show that \( \psi \) is injective. Consider the circle action on \( S^3_3 \) specified, on the generators, by

\[
\alpha_\lambda(B) = \lambda B \quad \text{and} \quad \alpha_\lambda(C) = \lambda C,
\]

for some complex number \( \lambda \) of modulus one. Therefore one concludes that each spectral subspace for the deformed gauge action is contained in the sub-algebra of \( C(S^3)^{\theta}_\gamma \) generated by \( Z \) and \( W \). This shows that \( Z \) and \( W \) generate \( C(S^3)^{\theta}_\gamma \), and hence that \( \psi \) is surjective.

We next show that \( \psi \) is injective. Consider the circle action on \( S^3_3 \) specified, on the generators, by

\[
\alpha_\lambda(B) = \lambda B \quad \text{and} \quad \alpha_\lambda(C) = \lambda C,
\]
for \( \lambda \in S^1 \). The homomorphism \( \psi \), under scrutiny, is clearly equivariant for the action just defined on \( S^3_\theta \) and the deformed gauge action \( \bar{\gamma} \) on \( C(S^3_\theta) \). By using [6.2.9], it is now enough to verify that \( \psi \) is injective on the fixed point sub-algebra of \( S^3_\theta \) for \( \alpha \). Let us denote that sub-algebra by \( F \).

Recall that Matsumoto [11, Theorem 6] has shown that \( F \) is isomorphic to the commutative \( C^* \)-algebra of functions on the two-sphere \( S^2 \). More precisely, \( F \) turns out to be generated by the elements \( M \) and \( H \) of \( S^3 \) given by \( H = C^*C \) and \( M = CB^* \). It is easy to see that these operators satisfy the relations

\[
\begin{align*}
\text{i) } & H^* = H, \\
\text{ii) } & M^*M = MM^*, \\
\text{iii) } & MH = HM, \text{ and} \\
\text{iv) } & M^*M + H^2 = H.
\end{align*}
\]

Matsumoto has, in fact, shown that \( F \) is the universal \( C^* \)-algebra on generators \( H \) and \( M \) satisfying the above relations.

Now, the images of \( H \) and \( M \) under \( \psi \) are, of course,

\[
\psi(H) = W^0 \times W = W^*W
\]

and

\[
\psi(M) = W \times Z^0 = WZ^*,
\]

both of which lie in the fixed point sub-algebra, say \( B_0 \), for the deformed gauge action on \( C(S^3_\theta) \). The crucial point is that this algebra is impervious to the deformation, as one can easily see, so \( B_0 \) is just the algebra of continuous functions on the quotient space \( S^3/S^1 \), which is homeomorphic to \( S^2 \).

An explicit homeomorphism between \( S^3/S^1 \) and \( S^2 \) may be given by mapping the quotient class of \( (z, w) \in S^3 \) to the pair \( (h, m) \in \mathbb{R} \times \mathbb{C} \), defined by \( (h, m) = (w\bar{w}, w\bar{w}) \). It is elementary to check that \( (h, m) \) satisfies the equation

\[
|m|^2 + h^2 = h,
\]

which is precisely the equation defining the sphere of radius \( \frac{1}{2} \) centered at \( \left( \frac{1}{2}, 0 + i0 \right) \) in \( \mathbb{R} \times \mathbb{C} \). The map

\[
(z, w) \mapsto (w\bar{w}, w\bar{w})
\]

can now be shown to provide a homeomorphism from \( S^3/S^1 \) onto the above mentioned model for the 2-sphere.

Whenever a compact subset \( K \) of \( \mathbb{R} \times \mathbb{C} \) is defined via an equation (or even a system of equations), such as the sphere above, one can prove that \( C(K) \) is the universal \( C^* \)-algebra generated by symbols \( h \) and \( m \), subject to the conditions

\[
\begin{align*}
\text{i) } & h^* = h, \\
\text{ii) } & m^*m = mm^*, \\
\text{iii) } & mh = hm,
\end{align*}
\]

to which one should add the equations used to define \( K \). This implies that \( B_0 \) is the universal \( C^* \)-algebra generated by a pair of elements (namely \( h = W^*W \) and \( m = WZ^* \)) subject to the same relations as the ones defining \( F \), that is 6.2.

Therefore one sees that \( \psi \) is an isomorphism between \( F \) and \( B_0 \), hence injective. By [6.2.9], it follows that \( \psi \) is injective everywhere and thus it is an isomorphism. \( \square \)

In order to discuss the infinitesimal aspects of the deformation of \( S^3 \) recently described, let \( D_1 \) and \( D_2 \) denote the differential operators defined by

\[
D_1 f(z, w) = \frac{d}{d\lambda} (f(e^{2\pi i \lambda} z, w)) \bigg|_{\lambda=0},
\]

and

\[
D_2 f(z, w) = \frac{d}{d\lambda} (f(z, e^{2\pi i \lambda} w)) \bigg|_{\lambda=0},
\]

for \( (z, w) \in S^3 \) and \( f \in C^\infty(S^3) \).
6.3. Theorem. If \( f \) and \( g \) are in \( C^\infty(S^3) \) then
\[
\lim_{\vartheta \to 0} \left\| \frac{f \times_\vartheta g - g \times_\vartheta f}{\vartheta} - \frac{1}{2\pi i} (D_2(f)D_1(g) - D_1(f)D_2(g)) \right\|_\vartheta = 0,
\]
where \( \times_\vartheta \) and \( \| \|_\vartheta \) refer to the deformed multiplication and norm of \( S^3_\vartheta \). Therefore, the family \( \{S^3_\vartheta\}_{\vartheta \in \mathbb{R}} \) gives a strict deformation quantization for \( C(S^3) \), in the direction of the Poisson bracket \( D_2 \land D_1 \).

Proof. Let \( \phi \) be the action of \( \mathbb{R}^2 \) on \( S^3 \) defined by
\[
\phi(x,y)(z,w) = (\epsilon^{2\pi i(x+y)}z, \epsilon^{2\pi i x} w), \quad \text{for } (x,y) \in \mathbb{R}^2, \ (z,w) \in S^3.
\]

As in section 5 we may use \( \phi \) to obtain the deformation data \( (\mathbb{Z}, \gamma, \vartheta^h) \). However, one can easily see that this is precisely the deformation data used earlier in this section for \( h = \vartheta \). So, we may use 5.7 to treat the deformation \( S^3_\vartheta \). But, before that, let us remark that, since \( \phi \) is a smooth action of \( \mathbb{R}^2 \) on the compact manifold \( S^3 \), then any smooth function on \( S^3 \) will be \( \phi \)-smooth. Also, for \( f \) in \( C^\infty(S^3) \) we have, using the notation of section 5,
\[
\partial_\lambda f \bigg|_{(z,w)} = \frac{d}{d\lambda} \left( f(\epsilon^{2\pi i \lambda} z, \epsilon^{2\pi i \lambda} w) \right) \bigg|_{\lambda=0} = D_1 f(z,w) + D_2 f(z,w),
\]
and
\[
\partial_y f \bigg|_{(z,w)} = \frac{d}{d\lambda} \left( f(\epsilon^{2\pi i \lambda} z, w) \right) \bigg|_{\lambda=0} = D_1 f(z,w),
\]
that is, \( \partial_z = D_1 + D_2 \) while \( \partial_y = D_1 \). The Poisson bracket appearing in 5.7 then becomes
\[
\partial_\lambda (f) \partial_y (g) - \partial_y (f) \partial_\lambda (g) = (D_1(f) + D_2(f))D_1(g) - D_1(f)(D_1(g) + D_2(g)) =
\]
\[
= D_2(f)D_1(g) - D_1(f)D_2(g),
\]
concluding the proof. \( \square \)

7. Example: Non-commutative Lens spaces. Matsumoto and Tomiyama [12], building on [10], have introduced non-commutative versions of the classical lens spaces. This section is dedicated to proving that these can be described by using our method of deformation.

Recall that for nonzero co-prime integers \( p \) and \( q \), with \( p \neq 0 \), the lens space \( L(p,q) \) can be defined to be the quotient of the three-sphere \( S^3 \) by the action of the finite cyclic group \( \mathbb{Z}_p \) generated by the diffeomorphism
\[
\tau(z,w) = (\rho z, \rho^q w), \quad \text{for } (z,w) \in S^3,
\]
where \( \rho = \epsilon^{2\pi i/p} \).

Observe that, if one induces \( \tau \) to an automorphism of \( C(S^3) \) by the formula
\[
\tau(f) \bigg|_{(z,w)} = f(\rho z, \rho^q w), \quad \text{for } f \in C(S^3), \ (z,w) \in S^3,
\]
then we have \( \tau(Z) = \rho Z \) and \( \tau(W) = \rho^q W \), where \( Z \) and \( W \) are the coordinate functions on \( S^3 \) (defined in the proof of 6.1). Since \( Z \) and \( W \) generate \( C(S^3) \), these equations actually define \( \tau \). In addition one sees that the fixed point sub-algebra of \( C(S^3) \) for \( \tau \) coincides with the algebra of continuous functions on the quotient \( S^3/\mathbb{Z}_p = L(p,q) \).

Let \( \vartheta \) be a real number, fixed throughout. Consider the automorphism \( \sigma \) of \( S^3_\vartheta \) prescribed by \( \sigma(B) = \rho B \) and \( \sigma(C) = \rho^q C \), where \( B \) and \( C \) are as in the previous section. Among the many different characterizations of the non-commutative lens space \( L_\vartheta(p,q) \) presented in [12], one has:
7.1. **Definition.** The non-commutative lens space \( L_\theta(p, q) \) is defined to be the fixed point sub-algebra of \( S^3_\theta \) under the automorphism \( \sigma \).

Regarding the deformation data \((Z, \gamma, \theta)\) for the algebra \( C(S^3)\), defined shortly before 6.1, in terms of a given value for the parameter \( \theta \), observe that \( \gamma \) and \( \theta \) commute with \( \tau \) and hence both \( \gamma \) and \( \theta \) leave invariant the fixed point sub-algebra for \( \tau \), which we have seen to be a model for \( C(L(p, q)) \). By abuse of language we still denote by \( \gamma \) and \( \theta \) the corresponding restrictions of these to \( C(L(p, q)) \). So, this gives a deformation data for \( C(L(p, q)) \) and we may then form the deformed algebra \( C(L(p, q))_\gamma^\theta \).

7.2. **Theorem.** For each real \( \theta \) and co-prime integers \( p \) and \( q \), with \( p \neq 0 \), the \( C^*\)-algebras \( C(L(p, q))_\gamma^\theta \) and \( L_\theta(p, q) \) are isomorphic.

**Proof.** We shall derive this from 4.6. In fact, let the algebra \( B \), mentioned in the statement of 4.6 be \( C(S^3) \) with the deformation data \((Z, \gamma, \theta)\) referred to above. Then, as we have seen in 6.1, \( C(S^3)_\gamma^\theta \) is naturally isomorphic to \( S^3_\theta \). Still referring to the statement of 4.6, let \( H = Z_p \), which acts on \( B \) via \( \tau \). The natural extension of \( \tau \) to \( C(S^3)_\gamma^\theta \), provided by 4.4, coincides with \( \tau \) on the algebraic direct sum of the spectral subspaces for the deformed gauge action and hence satisfies

\[
\hat{\tau}(Z) = \rho Z \quad \text{and} \quad \hat{\tau}(W) = \rho^g W.
\]

But, since the isomorphism between \( C(S^3)_\gamma^\theta \) and \( S^3_\theta \) puts \( Z \) and \( W \) in correspondence with \( B \) and \( C \), respectively, we see that \( \hat{\tau} \) and \( \sigma \) correspond to each other under this isomorphism. In particular, the fixed point sub-algebra for \( \sigma \), a.k.a \( L_\theta(p, q) \), is isomorphic to the fixed point sub-algebra of \( C(S^3)_\gamma^\theta \) under \( \hat{\tau} \), which, by 4.6, is isomorphic to the deformed algebra \( C(L(p, q))_\gamma^\theta \).

Observe that \( \tau \) commutes with the action \( \phi \) referred to in the proof of 6.3. Therefore the operators \( D_1 \) and \( D_2 \) leave \( C(L(p, q)) \) invariant, when this is viewed as a subset of \( C(S^3) \).

7.3. **Theorem.** If \( f \) and \( g \) are in \( C^\infty(L(p, q)) \) then

\[
\lim_{\theta \to 0} \left\| f \times_\theta g - g \times_\theta f - \frac{1}{2\pi i} (D_2(f)D_1(g) - D_1(f)D_2(g)) \right\|_\theta = 0,
\]

where \( \times_\theta \) and \( \| \cdot \|_\theta \) refer to the deformed multiplication and norm of \( L_\theta(p, q) \). Therefore, the family \( \{L_\theta(p, q)\}_{\theta \in \mathbb{R}} \) gives a strict deformation quantization for \( C(L(p, q)) \), in the direction of the Poisson bracket \( D_2 \wedge D_1 \).

**Proof.** This is, in view of the comment above, a direct application of 6.3 for \( f \) and \( g \) in \( C(L(p, q)) \). \( \square \)

8. **Example: Non commutative Heisenberg manifolds.** For each positive integer \( c \), the Heisenberg manifold \( M^c \) consists of the quotient \( H/G^c \), where \( H \) is the Heisenberg group

\[
H = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\},
\]

viewed as a subgroup of \( SL_3(\mathbb{R}) \), and \( G^c \) is the discrete subgroup obtained when \( x, y \) and \( cz \) are required to be whole numbers.

To facilitate our notation we will use the coordinate system on \( H \) suggested by the association

\[
(x, y, z) \leftrightarrow \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}
\]
and hence we will identify $H$ with $\mathbb{R}^3$ without further warning. Under this coordinate system the multiplication in $H$ becomes

$$(x, y, z)(m, n, p) = (x + m, y + n, z + p + my).$$  \hfill{(8.1)}

So, $M^c$ can be described as the quotient of the Euclidean space $\mathbb{R}^3$ by the right action of $G^c$ given by 8.1.

In [16] Rieffel introduced a continuous field of $C^*$-algebras, denoted $D_{\mu,\nu}^c$, where $\mu$ and $\nu$ are real parameters, such that $D_{\mu,\nu}^c$ is isomorphic to $C(M^c)$ when $\mu = \nu = 0$.

Recall from [1] that $D_{\mu,\nu}^c$ can be defined to be the crossed product of $C(T^2)$, the algebra of continuous functions on the two-torus, by a certain Hilbert bimodule. There are, in fact, many descriptions for this construction. Perhaps the simpler such is provided by [2], where it is shown that

$$D_{\mu,\nu}^c \simeq C(T^2) \mathcal{A}_{X_{\alpha,\mu,\nu}}^c \mathbb{Z}.$$  

To describe this in detail, let $X^c$ be the set of continuous complex-valued functions on two real variables $x$ and $y$ satisfying

i) $f(x, y + 1) = f(x, y),$ and

ii) $f(x + 1, y) = e^{-2\pi i cy}f(x, y).$

Viewing the elements of $C(T^2)$ as periodic functions on two real variables, it is easy to check that, under pointwise multiplication, $C(T^2)X^c \subseteq X^c$. In this way, $X^c$ is given the structure of a $C(T^2)$-module. If we let, for $f$ and $g$ in $X^c$, $(f, g)_L = f\overline{g}$, then it is clear that $(f, g)_L$ is a periodic function on both variables, and hence belongs to $C(T^2)$. This gives $X^c$ the structure of a left Hilbert module over $C(T^2)$. Defining $(f, g)_R = \overline{f}g$, $X^c$ becomes a Hilbert bimodule.

Now, given real parameters $\mu$ and $\nu$, consider the automorphism $\alpha_{\mu,\nu}$ of $C(T^2)$ given by

$$\alpha_{\mu,\nu}(f)|_{(x, y)} = f(x + 2\mu, y + 2\nu).$$

The Hilbert bimodule $X_{\alpha,\mu,\nu}^c$, appearing above, is obtained by altering the right module operations of $X^c$ using $\alpha_{\mu,\nu}$ as in [2], that is: if $X$ is any Hilbert bimodule over a $C^*$-algebra $A$ and $\alpha$ is an automorphism of $A$, we let $X_\alpha$ denote the Hilbert bimodule over $A$ which coincides with $X$ as a left Hilbert module but which is equipped with the right module structure given by

$$x \cdot a = x\alpha(a), \quad x \in X, \ a \in A,$$

and right inner product $\langle \cdot, \cdot \rangle^{M^c}_R$ given by

$$\langle x, y \rangle^{M^c}_R = \alpha^{-1}(\langle x, y \rangle)_R, \quad x, y \in X.$$

So, $X_{\alpha,\mu,\nu}^c$ is a Hilbert bimodule over $C(T^2)$ and we may construct, as in [1], the crossed product $C(T^2)\mathcal{A}_{X_{\alpha,\mu,\nu}^c} \mathbb{Z}$ which, according to [2, Section 2], is isomorphic to $D_{\mu,\nu}^c$.

As in our earlier examples, we will show that $D_{\mu,\nu}^c$ can be described as a deformation of $C(M^c)$ relative to a certain deformation data.

Given $(x, y, z)$ in the Heisenberg group, we shall denote its class in $H/G^c$ by $[x, y, z]$.

Let $\mu$ and $\nu$ be fixed real numbers and consider the map $\phi$, from $\mathbb{R}^2$ into the Heisenberg group $H$, given by

$$\phi(a, b) := \exp \left( a \begin{bmatrix} 0 & 0 & 1/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 2\nu & 0 \\ 0 & 0 & 2\mu \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2b\nu & 2b^2\mu\nu + a/c \\ 0 & 1 & 2b\mu \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the two summands being exponentiated commute with each other, one sees that $\phi$ is a group homomorphism, yielding an action of $\mathbb{R}^2$ on $H$, by left multiplication. This action, which obviously commutes with the right action of $G^c$ on $H$, drops to the quotient, producing the following action of $\mathbb{R}^2$ on $M^c$:

$$\phi(a, b)[x, y, z] = [x + 2b\mu, y + 2b\nu, z + 2b\nu x + 2b^2\mu\nu + a/c],$$  \hfill{(8.2)}
for \((a, b) \in \mathbb{R}^2\) and \([x, y, z] \in M^c\).

If we now let \((\mathbf{Z}, \gamma, \theta^h)\) be the deformation data given by \(\phi\), as in section 5, the gauge action \(\gamma\), seen, as before, as an action of the circle group, will be

\[
\gamma_{e^{2\pi i t}}([x, y, z]) = [x, y, z + t/c],
\]

while the deforming actions \(\theta^h\) of \(\mathbf{Z}\) on \(M^c\) are given by iterating the diffeomorphism

\[
\theta^h([x, y, z]) = [x + 2h\mu, y + 2h\nu, z + 2h\nu x + 2h^2\mu\nu], \quad \text{for} \ [x, y, z] \in M^c.
\]

For the time being, let us assume that \(\hbar = 1\) or, what amounts to the same, that \(\mu\) and \(\nu\) are replaced, respectively, by \(h\mu\) and \(h\nu\). Correspondingly, let us denote the \(\theta^h\) above simply by \(\theta\).

For each integer \(k\) let us indicate by \(B_k\) the \(k\)-spectral subspaces for the gauge action \(\gamma\) on \(C(M^c)\). In particular, \(B_0\), which is the algebra of fixed points, coincides with the algebra of continuous functions on the quotient \(M^c/S^1\). It is a simple task to verify that the map

\[
[x, y, z] \in M^c \mapsto (e^{2\pi ix}, e^{2\pi iy}) \in \mathbb{T}^2
\]
drops to a homeomorphism from \(M^c/S^1\) to the 2-torus \(\mathbb{T}^2\). In other words, \(B_0\) is isomorphic to \(C(\mathbb{T}^2)\).

In general, for each \(k\) in \(\mathbf{Z}\), the \(k\)-spectral subspace \(B_k\) is given by the set of functions \(f : M^c \rightarrow \mathbb{C}\) satisfying \(\gamma_{\lambda}(f) = \lambda^k f\), for \(\lambda\) in \(S^1\) or, equivalently,

\[
f[x, y, z + t/c] = e^{2\pi i k t} f[x, y, z].
\]

This reflects the fact that \(\gamma\) is nothing but the dual action of \(C(M^c)\), when the latter is viewed as a Hilbert–bimodule crossed product \([1]\). Now, this implies that \(f[x, y, z] = e^{2\pi ikcz} f[x, y, 0]\). So, \(f\) is determined by its values on the elements \([x, y, 0]\). This suggests defining, for each such \(f\), the function \(g(x, y) := f[x, y, 0]\). Since \((x, y, 0)(0, 1, 0) = (x, y + 1, 0)\), we see that \(g\) is periodic in its second variable. Moreover, since \((x, y, 0)(1, 0, 0) = (x + 1, y, y)\), we have that

\[
g(x, y) = f[x, y, 0] = f[x + 1, y, y] = e^{2\pi icz} f[x + 1, y, 0] = e^{2\pi icz} g(x + 1, y).
\]

Summarizing, we have

i) \(g(x, y + 1) = g(x, y)\), and

ii) \(g(x + 1, y) = e^{-2\pi icz} g(x, y)\),

which the reader should compare with the equations defining the Hilbert bimodule \(X^c\), earlier in this section. Conversely, given any continuous function \(g : \mathbb{R}^2 \rightarrow \mathbb{C}\) satisfying (i) and (ii) above, one may define \(f[x, y, z] = e^{2\pi ikcz} g(x, y)\), and, after verifying that \(f\) is indeed well defined, show that \(f \in B_k\).

We next observe that the gauge action on \(M^c\) is semi-saturated, that is, \(C(M^c)\) is generated, as a \(C^*\)-algebra, by \(B_0\) and \(B_1\). This follows by the fact that \(C(M^c)\) is a Hilbert–bimodule crossed product \([1, \text{Theorem 3.1}]\) (see also \([6,4.1, 4.8]\) and \([7,6.2]\)).

It is not hard to show, using the Tietze extension Theorem, that the Fell bundle arising from any free action of \(\mathbb{T}^d\) on a locally compact space, such as the one we have, is actually saturated. However, we will not need this fact presently.

**8.3. Theorem.** For every integer \(c\), and real numbers \(\mu\) and \(\nu\), we have that \(C(M^c)^\gamma\) is isomorphic to \(D^\mu\nu\).

**Proof.** By 2.3 we have that \(B^\theta\) is also semi-saturated and hence, by \([1]\), in conjunction with \([7,4.2\text{ and }4.7]\), we conclude that \(C(M^c)^\gamma\), that is \(C^*(B^\theta)\), is given by the Hilbert bimodule crossed product \(B_0 \otimes_{B_1} \mathbf{Z}\). It is important to stress that the \(B_0\)–Hilbert bimodule structure of \(B_1\) we are referring to, is that coming from the operations of \(C^*(B^\theta)\), that is, the deformed bundle operations \(\times\) and \(\circ\) of \(B^\theta\). To make this more explicit,
let \( a \in B_0 \) and \( b, c \in B_1 \), which we may assume are given by \( a[x, y, z] = f(x, y) \), \( b[x, y, z] = e^{2\pi i cz} g(x, y) \), and \( c[x, y, z] = e^{2\pi i cz} h(x, y) \), where \( f \) is periodic and both \( g \) and \( h \) satisfy the conditions (i) and (ii) above for \( k = 1 \). The reader may then verify that

\[
\begin{align*}
    a \times b[x, y, z] &= e^{2\pi icz} f(x, y)g(x, y), \\
    b \times a[x, y, z] &= e^{2\pi icz} g(x, y)f(x + 2\mu, y + 2\nu), \\
    b^c \times c[x, y, z] &= g(x - 2\mu, y - 2\nu)h(x - 2\mu, y - 2\nu), \\
    b \times c^o[x, y, z] &= g(x, y)h(x, y).
\end{align*}
\]

These formulas tell us that the pair \((B_0, B_1)\) is identical to \((C(T^2), X^c_{\mu,\nu})\) as far as the Hilbert bimodule structure is concerned. Hence, as we already know that \( C(M^c)_{\gamma}^\theta = B_0 \rtimes B_1 \mathbb{Z} \) and that \( D^c_{\mu,\nu} = C(T^2) \rtimes X^c_{\mu,\nu} \mathbb{Z} \), it follows that \( C(M^c)^\theta_{\gamma} \simeq D^c_{\mu,\nu} \).

Let us now compute the differential operators \( \partial_x \) and \( \partial_y \), as in section 5, arising from the action \( \phi \) of \( \mathbb{R}^2 \) on \( M^c \). However, to avoid a notational conflict with the already established coordinate system \([x, y, z]\) for \( M^c \), we will denote them by \( \partial_a \) and \( \partial_b \), respectively. For a smooth function \( f \) on \( M^c \), we have

\[
\partial_a(f)[x, y, z] = \left. \frac{d}{da}(f[x, y, z + a/c]) \right|_{a=0} = c^{-1}\partial_a(f)[x, y, z],
\]

while

\[
\partial_b(f)[x, y, z] = \left. \frac{d}{db}(f[x + 2b\mu, y + 2b\nu, z + 2b^2\mu\nu + 2b\nu x]) \right|_{b=0} =
\]

\[
(2\mu\partial_1(f) + 2\nu\partial_2(f) + 2\nu x \partial_3(f))[x, y, z]
\]

where \( \partial_1 \), \( \partial_2 \) and \( \partial_3 \) correspond to the partial differentiation operators for the standard coordinates on \( \mathbb{R}^3 \).

The relevant Poisson bracket on \( M^c \) becomes

\[
\{\cdot, \cdot\} = \partial_x \wedge \partial_y = c^{-1}\partial_3 \wedge (2\mu\partial_1 + 2\nu\partial_2 + 2\nu x \partial_3) = 2c^{-1}\partial_3 \wedge (\mu \partial_1 + \nu \partial_2),
\]

which, up to a multiplicative factor, is precisely the Poisson bracket of interest in section 2 of \([16]\). We may therefore deduce from 5.7 and 8.3, one of the main results of \([16]\):

**8.4. Theorem.** The family \( \{D_{b\mu, b\nu}\}_{b \in \mathbb{R}} \) forms a strict deformation quantization in the direction of the Poisson bracket \( 2c^{-1}\partial_3 \wedge (\mu \partial_1 + \nu \partial_2) \).

**References**

[1] B. Abadie, S. Eilers and R. Exel, “Morita Equivalence for Crossed Products by Hilbert Bimodules”, preprint, University of Copenhagen, 1994, to appear in Trans. Amer. Math. Soc.

[2] B. Abadie and R. Exel, “Hilbert \( C^*\)-bimodules over commutative \( C^*\)-algebras and an isomorphism condition for quantum Heisenberg manifolds”, preprint, Universidade de São Paulo, 1996, to appear in Rev. Math. Phys.

[3] O. Bratteli, “Derivations, Dissipations and Group Actions on \( C^*\)-algebras”, Lecture Notes in Mathematics vol. 1229, Springer–Verlag, 1986.

[4] J. Dixmier, “\( C^*\)-Algebras”, North Holland, 1982.

[5] R. Exel, “The Soft Torus and Applications to Almost Commuting Matrices”, Pacific J. Math. 160 (1993), 207–217.

[6] R. Exel, “Circle Actions on \( C^*\)-Algebras, Partial Automorphisms and a Generalized Pimsner–Voiculescu Exact Sequence”, J. Funct. Analysis 122 (1994), 361–401.
[7] R. Exel, “Amenability for Fell Bundles”, preprint, Universidade de São Paulo, 1996.
[8] J. M. G. Fell and R. S. Doran, “Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles”, Pure and Applied Mathematics vol. 125 and 126, Academic Press, 1988.
[9] P. E. T. Jorgensen, L. M. Schmitt and R. F. Werner, “q-canonical commutation relations and stability of the Cuntz algebra”, Pacific J. Math. 165 (1994), 131–151.
[10] K. Matsumoto, “Noncommutative three-dimensional spheres”, Japan. J. Math. (N.S.) 17 (1991), 333–356.
[11] K. Matsumoto, “Noncommutative three-dimensional spheres. II. Noncommutative Hopf fibering”, Yokohama Math. J. 38 (1991), 103–111.
[12] K. Matsumoto and J. Tomiyama, “Noncommutative lens spaces”, J. Math. Soc. Japan 44 (1992), 13–41.
[13] C.-K. Ng, “Reduced Cross-sectional C*-algebras of C*-algebraic bundles and Coactions”, preprint, Oxford University, 1996.
[14] M. A. Rieffel, “C*-algebras associated with irrational rotations”, Pacific J. Math. 93 (1981), 415–429.
[15] M. A. Rieffel, “Continuous fields of C*-algebras coming from group cocycles and actions”, Math. Ann. 283 (1989), 631-643.
[16] M. A. Rieffel, “Deformation quantization of Heisenberg manifolds”, Commun. Math. Phys. 122 (1989), 531–562.
[17] M. A. Rieffel, “Deformation quantization for actions of Rd”, Mem. Amer. Math. Soc. 106 (1993), 93 pp.
[18] M. A. Rieffel, “Quantization and C*-algebras”, Contemp. Math. 167 (1994), 66–97.
[19] S. L. Woronowicz, “Twisted SU2 groups. An example of a non-commutative differential calculus”, Publ. RIMS, Kyoto Univ. 23 (1987), 117–181.
[20] G. Zeller-Meier, “Produits croisés d’une C*-algèbre par un group d’automorphismes”, J. Math. Pures Appl. 47 (1968), 101-239.

Centro de Matemáticas
Facultad de Ciencias
Eduardo Acevedo 1139
CP 11200, Montevideo – Uruguay
abadie@cmat.edu.uy

Departamento de Matemática
Universidade de São Paulo
Rua do Matão, 1010
05508-900 São Paulo – Brazil
exel@ime.usp.br

May 1997