**G₂-MANIFOLDS AND THE ADM FORMALISM**

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**Abstract.** We study a Hamiltonian function on the cotangent bundle of the space of Riemannian metrics on a 3-manifold \( M \) and prove the constrained Hamiltonian dynamics produces (Ricci-flat) \( G₂ \)-manifolds foliated by hypersurfaces diffeomorphic to \( M × SO(3) \).

1. Introduction

Let \( M \) be a closed oriented 3-manifold. Let \( \mathbf{M} \) denote the space of Riemannian metrics on \( M \), and \( (\gamma, \pi) \) an element of the cotangent bundle \( T^*\mathbf{M} \), where \( \gamma \in \mathbf{M} \) and \( \pi \) is its conjugate momentum, namely a \((2,0)\)-tensor field on \( M \) tensored with a volume form. The standard symplectic structure exists on \( T^*\mathbf{M} \). We shall study a Hamiltonian function

\[
\mathcal{H}(\gamma, \pi) = \int_M R(\gamma) \text{vol}(\gamma) + \int_M \det(\pi^i_j) \text{vol}(\gamma)^{-2},
\]

where \( R(\gamma) \) and \( \text{vol}(\gamma) \) denote the scalar curvature and the volume element of the metric \( \gamma \). Also, \( \pi^i_j = \gamma^{j\alpha} \pi^i_\alpha \), and \( \det(\pi^i_j) \) denotes the determinant of \( \pi \), which is contained in the 3-rd tensor powers of volume forms on \( M \). Here and through the paper we adopt Einstein’s convention. Our main result is the following (see Theorem 1 for details):

The constrained Hamiltonian dynamics of the Hamiltonian function \( \mathcal{H} \) produces \( G₂ \)-manifolds foliated by hypersurfaces diffeomorphic to \( M × SO(3) \).

Here the constraints are the conditions that \( \pi \) is positive-definite and \( \nabla_j \pi^{ij} = 0 \), where \( \nabla \) denotes the covariant differentiation of the Levi-Civita connection for each metric \( \gamma \). A \( G₂ \)-manifold is a Ricci-flat Riemannian 7-manifold with holonomy group contained in \( G₂ \). Note that as long as \( \det(\pi^i_j) \neq 0 \) and \( \nabla_j \pi^{ij} = 0 \) are satisfied, the dynamics produces Ricci-flat (semi-)Riemannian 7-manifolds. If \( \pi \) is indefinite, then the semi-Riemannian metrics have signature \((3, 4)\) and the holonomy groups contained in \( G₂^* \), where \( G₂^* \) denotes the non-compact counterpart of \( G₂ \) (see Remark 2).

The phase space \( T^*\mathbf{M} \) is that of the ADM formalism, a Hamiltonian formalism of general relativity discovered by Arnowitt, Deser and Misner in 1959 [ADM]. By choosing the lapse function \( N = 1 \) and the shift vector field \( N_i = 0 \) for \( i = 1, 2, 3 \), the constrained dynamics of the Hamiltonian function

\[
\mathcal{H}'(\gamma, \pi) = \int_M R(\gamma) \text{vol}(\gamma) + \int_M \left( \frac{1}{2} (\pi^i_i)^2 - \pi^{ij} \pi_{ij} \right) \text{vol}(\gamma)^{-1}
\]

produces 3+1 vacuum solutions of the Einstein equation foliated by spacelike hypersurfaces diffeomorphic to \( M \). Here the constraints are the conditions
that the sum of the integrands of the right-hand side in \( H' \) vanishes and 
\[ \nabla_j \pi^{ij} = 0. \]

These Hamiltonians have some similarities. Both integrands of them are the sums of the scalar curvature and the symmetric polynomials of the eigenvalues of the conjugate momenta. Moreover both dynamics of them preserve \( \nabla_j \pi^{ij} = 0 \) if it is satisfied at a point on the orbits.

The paper is a continuation of the author's previous result (see Theorem 1.2 of [C] or Proposition 3). The result was motivated by generalizing Bryant and Salamon's examples of complete \( G_2 \)-manifolds in Section 3 of [BS] by means of Hitchin’s flow approach to construction of \( G_2 \)-manifolds [H].

The paper is organized as follows. Section 2 reviews some notation and the main result (Theorem 1). Section 3 states the proof. Section 4 proves the scale-invariance of the dynamics of \( H \), a property of the volume growth under the dynamics, and observes examples obtained by scaling. These are essentially Bryant and Salamon’s examples in Section 3 of [BS]. In our Hamiltonian formulation, the examples constructed by hyperbolic 3-manifolds in [BS] smoothly extend to the indefinite Ricci-flat metrics with holonomy \( G_2' \) as the Hamiltonian dynamics of \( H \).

2. Notation and the main result

Let \( \otimes^p E, \wedge^p E \) and \( ^p E \) be the \( p \)-th tensor, symmetric and anti-symmetric power of a vector bundle \( E \) over a manifold \( N \), and \( \Omega^p(E) \) the \( E \)-valued \( p \)-forms on \( N \). Let \( M \) be a closed oriented 3-manifold and \( \mathbf{M} \) the space of Riemannian metrics
\[ \Omega^0(S^2 T^* M), \]
where \( S^2 T^* M \) is the positive-definite elements of \( S^2 T^* M \). For \( \gamma \in \mathbf{M} \), the tangent space \( T_{\gamma} \mathbf{M} \) is identified with \( \Omega^0(S^2 T^* M) \) and the cotangent space \( T^*_{\gamma} \mathbf{M} \) with \( \Omega^0_0(S^2 T M) \) via the pairing \( T^*_{\gamma} \mathbf{M} \otimes T_{\gamma} \mathbf{M} \rightarrow \mathbb{R} \) defined by
\[ \langle \pi, h \rangle = \int_M \pi^{ij} h_{ij} \]
for \( \pi \in \Omega^0_0(S^2 T^* M) \) and \( h \in \Omega^0(S^2 T^* M) \). Here \( \Omega^0_0(S^2 T M) = \Omega^0(S^2 T^* M) \otimes \wedge^3 T^* M \), where \( \wedge^3 T^* M \) denotes the positive elements of \( \wedge^3 M \). Thus we have
\[ (1) \quad T^* \mathbf{M} = \mathbf{M} \times \Omega^0_0(S^2 T^* M). \]

Then the standard symplectic form \( \Omega \) on \( T^* \mathbf{M} \) is described by
\[ \Omega_{(\gamma, \pi)}((h_1, \omega_1), (h_2, \omega_2)) = \int_M (h_1 \omega_2 - h_2 \omega_1) \]
for \( (h_1, \omega_1), (h_2, \omega_2) \in T_{(\gamma, \pi)} T^* \mathbf{M} \). Here \( h_1, h_2 \in \Omega^0(S^2 T^* M) \) and \( \omega_1, \omega_2 \in \Omega^0_0(S^2 T M) \).

Let us rewrite the phase space \( T^* \mathbf{M} \). Let \( P \) be a trivial principal \( SO(3) \)-bundle over \( M \) and \( \Omega^p(P; V) \) the \( V \)-valued \( p \)-forms on \( P \) for a vector space \( V \). Given a representation \( \rho : SO(3) \rightarrow GL(V) \), let \( \Omega^p(P; V)^F \) consist of \( \alpha \in \Omega^p(P; V) \) satisfying \( R^g \alpha = \rho(g^{-1}) \alpha \) and \( \iota(A^*) \alpha = 0 \) for \( g \in SO(3) \) and \( A \in \mathfrak{so}(3) \). Here \( \iota \) denotes the inner product and \( A^* \) the infinitesimal vector field of \( A \) on \( P \). Let us consider \( \mathbb{R}^3 \) with the matrix representation of \( SO(3) \).
A 1-form $e = (e^1, e^2, e^3) \in \Omega^1(P; \mathbb{R}^3)^F$ is called a solder 1-from if it satisfies the non-degenerate condition $e^1_u \wedge e^2_u \wedge e^3_u \neq 0$ for every point $u \in P$. Let $\mathcal{M}$ denote the solder 1-forms and $\mathcal{G}$ the gauge group of $P$. An element $\tau \in \mathcal{G}$ acts on $e \in \mathcal{M}$ by $\tau \cdot e = (\tau^{-1})^*e$. For $e \in \mathcal{M}$, we can associate a Riemannian metric such that $(s^*e^1, s^*e^2, s^*e^3)$ is a local orthonormal coframe for any section $s : U \to P$ on any domain $U$ of $M$. Then we have naturally an isomorphism

\begin{equation}
(2) \quad i : \mathcal{M}/\mathcal{G} \to M.
\end{equation}

Let $\mathcal{S}$ be the $3 \times 3$ real symmetric matrices on which $SO(3)$ acts by $g \cdot B = gBg^{-1}$ for $g \in SO(3)$ and $B \in \mathcal{S}$. For $e \in \mathcal{M}$, let $\bar{e} \in \mathcal{M}/\mathcal{G}$ be the element represented by $e$. The tangent space $T_{\bar{e}}(\mathcal{M}/\mathcal{G})$ is identified with $\Omega^0(P; \mathcal{S})^F$ via the following map

\begin{equation}
\bar{e}^i = T_{ij}e^j \mapsto T = (T_{ij})
\end{equation}

for $\bar{e} = (\bar{e}^1, \bar{e}^2, \bar{e}^3) \in T_{\bar{e}}\mathcal{M}$. The cotangent space $T^*_{\bar{e}}(\mathcal{M}/\mathcal{G})$ is identified with $\Omega^0_d(P; \mathcal{S})^F$ by the pairing $T^*_{\bar{e}}(\mathcal{M}/\mathcal{G}) \otimes T_{\bar{e}}(\mathcal{M}/\mathcal{G}) \to \mathbb{R}$ defined by

\begin{equation}
\langle U, T \rangle' = \int_M U_{ij}T_{ij}
\end{equation}

for $U \in \Omega^0_d(P; \mathcal{S})$ and $T \in \Omega^0(P; \mathcal{S})^F$. Here $\Omega^0_d(P; \mathcal{S})^F = \Omega^0(P; \mathcal{S})^F \otimes \Omega^0(\Lambda^3_d T^*M)$. Thus we have

\begin{equation}
T^*(\mathcal{M}/\mathcal{G}) = (\mathcal{M} \times \Omega^0_d(P; \mathcal{S})^F)/\mathcal{G},
\end{equation}

where $\tau \in \mathcal{G}$ acts by $\tau \cdot (e, U) = ((\tau^{-1})^*e, (\tau^{-1})^*U)$ on $(e, U) \in \mathcal{M} \times \Omega^0_d(P; \mathcal{S})^F$ and denote by $(e, U) \in T^*(\mathcal{M}/\mathcal{G})$ the element represented by $(e, U)$. Then the map (2) induces an isomorphism from (3) to (1).

Let $\mathcal{S}_+$ be the positive-definite elements of $\mathcal{S}$, and let

\begin{equation}
T^*(\mathcal{M}/\mathcal{G})^0 = (\mathcal{M} \times \Omega^0_d(P; \mathcal{S}_+)^F)/\mathcal{G}.
\end{equation}

This space is isomorphic to

\begin{equation}
T^*\mathcal{M}^0 = \mathcal{M} \times \Omega^0_d(\Lambda^3_d T\mathcal{M})
\end{equation}

via the isomorphism above. We write an element of $T^*(\mathcal{M}/\mathcal{G})$ as $(k_1e^i, k_2\mathcal{S}^i/\mu)$ by $k_1, k_2 \in \mathbb{R}_{>0}$, $e^i \in \mathcal{M}$, $\mathcal{S} \in \Omega^0(P; \mathcal{S})^F$ and $\mu \in \Omega^0(\Lambda^3_d T^*\mathcal{M})$. Fix a basis of $\mathfrak{so}(3)$ by $\{Y_1, Y_2, Y_3\}$ such that $[Y_1, Y_2] = Y_3$, $[Y_2, Y_3] = Y_1$ and $[Y_3, Y_1] = Y_2$. By the basis, we equivariantly identify $\Omega^p(P; \mathfrak{so}(3))$ with $\Omega^p(P; \mathbb{R}^3)$. For $e \in \mathcal{M}$, we have the Levi-Civita connection $a = a^iY_i \in \Omega^1(P; \mathfrak{so}(3))$ such that $de + [a \wedge e] = 0$, and denote by $\text{vol}(\mathcal{S})$ the volume element $e^1 \wedge e^2 \wedge e^3$. For $(2^{-\frac{1}{2}}, 2^\frac{1}{2}\text{vol}(\mathcal{S})) \in T^*(\mathcal{M}/\mathcal{G})^0$, we introduce a 2-form $\omega$ and a 3-form $\psi$ on $P$ by

\begin{equation}
(4) \quad \omega = S_{ij}a^i \wedge e^j,
\end{equation}

\begin{equation}
(5) \quad \psi = -(\text{det} \mathcal{S})e^1 \wedge e^2 \wedge e^3 + e^1 \wedge a^2 \wedge a^3 + e^2 \wedge a^3 \wedge a^1 + e^3 \wedge a^1 \wedge a^2.
\end{equation}

Here $\bar{B}$ denotes the cofactor matrix of a matrix $B$, and if $B$ is regular, $\bar{B} = \det B \cdot B^{-1}$. Then $(\omega, \psi)$ is well-defined and gives an $SU(3)$-structure on $P$ (see for example p. 6 and p. 9 of [CLSS]).
Let \( c(t) \) be a curve in \( T^*(\mathcal{M}/\mathcal{G})^0 \) defined on an interval \((t_1, t_2)\). For the curve \( c(t) \), we introduce a 3-from \( \phi \) on \( P \times (t_1, t_2) \) by
\[
\phi = \omega(t) \wedge dt + \psi(t),
\]
using (4) and (5) for each \( t \in (t_1, t_2) \). Then the 3-from \( \phi \) gives a \( G_2 \)-structure on \( P \times (t_1, t_2) \) (see for example p. 7 of [CLSS]). Generally, a \( G_2 \)-structure on a 7-manifold \( Y \) is called torsion-free if the Riemannian metric associated with the \( G_2 \)-structure has the holonomy group contained in \( G_2 \). Then the metric is Ricci-flat. See for example Section 1 of [BS] for basic properties of \( G_2 \)-structures.

Let us consider a Hamiltonian function
\[
H(\gamma, \pi) = \int_M R(\gamma)\text{vol}(\gamma) + \int_M \det ((\pi')_j)\text{vol}(\gamma)
\]
for \( (\gamma, \pi) \in T^*\mathcal{M} \), where \( \text{vol}(\gamma) \) is the volume element of \( \gamma, \pi = \pi' \text{vol}(\gamma) \) and \( (\pi')_j = (\pi')^{i_0} \gamma_{i_0j} \). Now we state our main result.

**Theorem 1.** Let \( c(t) \) be a curve in \( T^*\mathcal{M}^0 \) defined on an interval \((t_1, t_2)\). Assume that \( c(t) \) satisfies \( \nabla_j \pi^{ij} = 0 \) at some \( t_0 \in (t_1, t_2) \). Then the curve \( c(t) \) is an orbit of the Hamiltonian dynamics of (7) if and only if the \( G_2 \)-structure (6) is torsion-free.

Here we identify \( T^*\mathcal{M}^0 \) with \( T^*\mathcal{M}/\mathcal{G}^0 \) by the isomorphism induced by (2) and denote by \( \nabla \) the covariant differentiation of the Levi-Civita connection for each \( \gamma(t) \in \mathcal{M} \).

**Remark 2.** The Hamiltonian dynamics of (7) can be defined on \( T^*\mathcal{M} \). If \( \pi \) is indefinite and \( \det (\pi_j^i) \neq 0 \), then a \( G_2^* \)-structure is defined on \( P \times (t_1, t_2) \) by (6). Then Theorem 1 holds for \( G_2^* \)-structures. Generally, a torsion-free \( G_2^* \)-structure on a 7-manifold \( Y \) defines a Ricci-flat semi-Riemannian metric of signature (3, 4) on \( Y \). See p. 7 and Theorem 2.3 of [CLSS].

3. **Proof of Theorem 1**

Let \( \tilde{i} : T^*\mathcal{M}/\mathcal{G} \to T^*\mathcal{M} \) be the isomorphism induced by (2). Take a coordinate neighborhood \((U, x^1, x^2, x^3)\) of \( \mathcal{M} \) and a local section \( s : U \to P \). Given \( e \in \mathcal{M} \), let us define by \( s^*e^i = c_{ij}dx^j \) a \( 3 \times 3 \) matrix-valued function \( C = (c_{ij}) \) on \( U \). Then we have
\[
\tilde{i}((e, S\text{vol}(e))) = (\gamma, \pi' \text{vol}(\gamma)) = \left( \gamma'CC, \frac{1}{2}C^{-1}S'(C)^{-1}\text{vol}(e) \right)
\]
for \( (e, S\text{vol}(e)) \in T^*\mathcal{M}/\mathcal{G} \). Note \( \tilde{i}_*(T) = 2\text{CTC} \) for \( T \in T_e(\mathcal{M}/\mathcal{G}) \). Here we omits \( dx^idx^j \) and \( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \) of the symmetric tensors.

Let \( c = (e, S\text{vol}(e)) \). The Hamiltonian function (7) is
\[
H(e) = \int_M R(e)\text{vol}(e) + \frac{1}{8} \int_M \det S\text{vol}(e)
\]
on \( T^*(\mathcal{M}/\mathcal{G}) \) since \( \det ((\pi')_j^i) = \frac{1}{8} \det S \). Here \( R(e) \) is the scalar curvature of \( i(e) \in \mathcal{M} \), and the same symbol \( H \) is used. Also, the symplectic structure
where the vector field defined by \((12)\) and \((13)\) in Proposition 3 is

\[
\left(\frac{1}{8} \tilde{S}, (-2G + \frac{1}{4} \det S \cdot I) \text{vol}(e)\right)
\]

at \(c \in T^*(\mathcal{M}/\mathcal{G})\). By \((9)\) and \((10)\), we can thus see that the vector field above is the Hamiltonian vector field of \((8)\).
Moreover we have
\[
\frac{\partial}{\partial t} (S_{ij,j} e^1 \wedge e^2 \wedge e^3 \otimes Y_i)
\]
\[
= \frac{\partial}{\partial t} \left( d_H S \left( \frac{1}{2} [e \wedge e] \right) \right)
\]
\[
= d_H S ([\dot{e} \wedge e]) + \frac{1}{2} d_H \dot{S} ([e \wedge e]) + \frac{1}{2} [\dot{a}, S] ([e \wedge e]),
\]
and under the Hamiltonian flow of (12) and (13) in Proposition 3
\[
d_H S ([\dot{e} \wedge e]) = (S_{ia,\alpha} \tilde{S}_{\beta\beta} + S_{ia} \tilde{S}_{\alpha\beta,\beta} - (\det S)_{ia}) \text{vol}(e) \otimes Y_i,
\]
\[
\frac{1}{2} d_H \dot{S} ([e \wedge e]) = (-S_{ia} \tilde{S}_{\beta\beta,\alpha} - S_{ia,\alpha} \tilde{S}_{\alpha\beta} + 2(\det S)_{ia}) \text{vol}(e) \otimes Y_i,
\]
\[
\frac{1}{2} [\dot{a}, S] ([e \wedge e]) = (S_{ia\beta} \tilde{S}_{\beta\alpha,\alpha} - S_{a\beta} \tilde{S}_{\alpha\beta,\beta} - S_{ia} \tilde{S}_{\alpha\beta,\beta} + S_{ia} \tilde{S}_{\beta\beta,\alpha}) \text{vol}(e) \otimes Y_i.
\]
Here the dots denote the differentiation by time. By Lemma 4 and the
equalities above, we can see that (14) vanishes under the Hamiltonian flow.
Here we do not assume $S_{ij,\beta} = 0$. Hence the flow preserves $S_{ij,\beta} = 0$ if it is
satisfied at some $t_0 \in (t_1, t_2)$. This completes the proof of Theorem 1.

4. Properties and examples

Let us consider the flow defined by (12) and (13) in Proposition 3 on
$\mathcal{M} \times \Omega^0 (P; S)^F$, which contains indefinite or degenerate points in $S$. The flow has the following scale-invariance.

Proposition 5. Let $(e(t), S(t))$ be a solution of (12) and (13) defined on
an interval $(t_1, t_2)$. Then for any constant $\alpha > 0$, $(e'(t), S'(t'))$ such that $e'(t') = \alpha e(t')$ and $S'(t') = \alpha^{-\frac{2}{3}} S(\alpha^{-\frac{2}{3}} t')$ is also a solution defined on
$(\alpha^\frac{2}{3} t_1, \alpha^\frac{4}{3} t_2)$.

Proof. Take constants $\beta, \gamma > 0$ and set $e' = \alpha e$, $S' = \beta S$ and $t' = \gamma t$.
We have $G = \alpha^2 G'$, det $S = \beta^{-3} \text{det} S'$ and $\tilde{S} = \beta^{-2} \tilde{S}'$, where $G' \in \Omega^0 (P; S)^F$ denotes the components of the Einstein tensor of $e'$ with respect to $(e')^1, (e')^2, (e')^3$. Then we have
\[
\frac{\partial (e')^i}{\partial t'} = \frac{\partial t}{\partial t'} \frac{\partial (e')^i}{\partial t} = \beta^{-2} \gamma^{-1} \tilde{S}_{ij} (e')^j,
\]
\[
\frac{\partial S'}{\partial t'} = -\alpha^{-2} \beta \gamma^{-1} G' - \beta^{-2} \gamma^{-1} \text{tr} (\tilde{S}') S' + 2 \beta^{-2} \gamma^{-1} \text{det} S' \cdot I.
\]
Thus choosing $\beta = \alpha^{-\frac{2}{3}}, \gamma = \alpha^\frac{4}{3}$, we get the desired solution. \qed

The volume growth is related to the signature of $S(t)$ as follows.

Proposition 6. Let $(e(t), S(t))$ be a solution of (12) and (13). If $S(t_0)$
is semi-definite (resp. indefinite) at every point on $M$, Then the volume
defined by $e(t)$ satisfies
\[
\frac{\partial}{\partial t} \int_M \text{vol}(e(t)) \bigg|_{t = t_0} \geq 0 \quad \text{(resp.} \leq 0).\]

Proof. This is easily deduced by $S \tilde{S} = \text{det} S \cdot I$ and $\dot{e}^i = \tilde{S}_{ij} e^j$. \qed
Let us consider solutions of (12) and (13) in the form \((f(t)e_0, h(t)S_0)\), where \(f(t) \in \mathbb{R}_{>0}, h(t) \in \mathbb{R}\) and \((e_0, S_0) \in \mathcal{M} \times \Omega^0(P, S)^F\) such that \(\det S_0 \neq 0\) at every point. These are essentially Bryant and Salamon’s examples in Section 3 of [BS]. By simple computation, we can see \(S_0\) is constant diagonal, \(i(e_0)\) is constant curvature, and (12) and (13) are reduced to
\[
\dot{f} = s^2 fh^2, \quad \dot{h} = \frac{\sigma}{s} f^{-2} - s^2 h^3,
\]
where \(3s = \text{tr}(S_0)\) and \(\sigma\) denotes the sectional curvature of \(i(e_0)\). Moreover let \(x(t) = f(t)^2\) and \(y(t) = f(t)h(t)\). Then (15) is equivalent to the following.
\[
\dot{x} = 2s^2 y^2, \quad \dot{y} = \frac{\sigma}{s} x^{-\frac{3}{2}}, \quad x(t) > 0.
\]
This system has the global solution defined on the half-line for any initial value \((x_0, y_0)\) with \(x_0 > 0\). If \(\sigma > 0\), then the solutions become positive-definite \(G_2\)-structures more than sufficiently large time. On the other hand, if \(\sigma < 0\), then the solutions become indefinite \(G_2^*\)-structures more than sufficiently large time. If \(\sigma = 0\), then the solutions preserve the initial states.

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