Quasi-elementary $H$-Azumaya algebras arising from generalized (anti) Yetter-Drinfeld modules

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Abstract

Let $H$ be a Hopf algebra with bijective antipode, $\alpha, \beta \in Aut_{Hopf}(H)$ and $M$ a finite dimensional $(\alpha, \beta)$-Yetter-Drinfeld module. We prove that $\text{End}(M)$ endowed with certain structures becomes an $H$-Azumaya algebra, and the set of $H$-Azumaya algebras of this type is a subgroup of $BQ(k, H)$, the Brauer group of $H$.

Introduction

Let $H$ be a Hopf algebra with bijective antipode $S$ and $\alpha, \beta \in Aut_{Hopf}(H)$. An $(\alpha, \beta)$-Yetter-Drinfeld module, as introduced in [10], is a left $H$-module right $H$-comodule $M$ with the following compatibility condition:

$$(h \cdot m)(0) \otimes (h \cdot m)(1) = h_2 \cdot m(0) \otimes \beta(h_3)m(1)\alpha(S^{-1}(h_1)).$$

This concept is a generalization of three kinds of objects appearing in the literature. Namely, for $\alpha = \beta = id_H$, one obtains the usual Yetter-Drinfeld modules; for $\alpha = S^2$, $\beta = id_H$, one obtains the so-called anti-Yetter-Drinfeld modules, introduced in [3], [6], [7] as coefficients for the cyclic cohomology of Hopf algebras defined by Connes and Moscovici in [3], [4]; finally, an $(id_H, \beta)$-Yetter-Drinfeld module is a generalization of a certain object $H_\beta$ defined in [2]. The main result in [10] is that, if we denote by $\mathcal{YD}(H)$ the disjoint union of the categories $H\mathcal{YD}^H(\alpha, \beta)$ of $(\alpha, \beta)$-Yetter-Drinfeld modules, for all $\alpha, \beta \in Aut_{Hopf}(H)$, then $\mathcal{YD}(H)$ acquires the structure of a braided T-category (a concept introduced by Turaev in [12]) over a certain group $G$, a semidirect product between two copies of $Aut_{Hopf}(H)$. Moreover, the subcategory $\mathcal{YD}(H)_{fd}$ consisting of finite dimensional objects has left and right dualities.

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The Brauer group $BQ(k, H)$ of the Hopf algebra $H$ was introduced in \[\text{[1]},\] by taking equivalence classes of so-called $H$-Azumaya algebras in the braided category $\mathcal{H}_YD^H$ of Yetter-Drinfeld modules over $H$, and using the braided product inside this category to define the multiplication of the group. If $M \in \mathcal{H}_YD^H$ is a finite dimensional object, then $\text{End}(M)$ is an $H$-Azumaya algebra, representing the unit element in $BQ(k, H)$. Also, if $H$ is finite dimensional and $\beta \in Aut_{Hopf}(H)$, the object $H_\beta$ mentioned before is not an object in $\mathcal{H}_YD^H$ but nevertheless $\text{End}(H_\beta)$ with certain structures becomes an $H$-Azumaya algebra, and moreover the map $\beta \mapsto \text{End}(H_\beta)$ gives a group anti-homomorphism from $Aut_{Hopf}(H)$ to $BQ(k, H)$, see \[\text{[2]}.\]

The aim of this paper is to construct a new class of examples of $H$-Azumaya algebras, containing the two classes mentioned above as particular cases. Namely, we prove that if $\alpha, \beta \in Aut_{Hopf}(H)$ and $M \in \mathcal{H}_YD^H(\alpha, \beta)$ is finite dimensional, then $\text{End}(M)$ endowed with certain structures becomes an $H$-Azumaya algebra. The proof is rather technical and relies heavily on the fact that $\mathcal{YD}(H)_{fd}$ is a braided $T$-category with dualities. We also prove that, if we denote by $BA(k, H)$ the subset of $BQ(k, H)$ consisting of $H$-Azumaya algebras that can be represented as $\text{End}(M)$, with $M \in \mathcal{H}_YD^H(\alpha, \beta)$ finite dimensional, for some $\alpha, \beta \in Aut_{Hopf}(H)$, then $BA(k, H)$ is a subgroup of $BQ(k, H)$.

1 Preliminaries

We work over a ground field $k$. All algebras, linear spaces, etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Unless otherwise stated, $H$ will denote a Hopf algebra with bijective antipode $S$. We will use the version of Sweedler’s sigma notation: $\Delta(h) = h_1 \otimes h_2$. For unexplained concepts and notation about Hopf algebras we refer to \[\text{[8], [9], [11]}.\] By $\alpha, \beta, \gamma...$ we will usually denote Hopf algebra automorphisms of $H$. If $M$ is a vector space, a left $H$-module (respectively $H$-comodule) structure on $M$ will be usually denoted by $h \otimes m \mapsto h \cdot m$ (respectively $m \mapsto m_{(0)} \otimes m_{(1)}$).

We recall now some facts from \[\text{[11]}\] about $(\alpha, \beta)$-Yetter-Drinfeld modules.

**Definition 1.1** Let $\alpha, \beta \in Aut_{Hopf}(H)$. An $(\alpha, \beta)$-Yetter-Drinfeld module over $H$ is a vector space $M$, such that $M$ is a left $H$-module (with notation $h \otimes m \mapsto h \cdot m$) and a right $H$-comodule (with notation $M \rightarrow M \otimes H$, $m \mapsto m_{(0)} \otimes m_{(1)}$) with the following compatibility condition:

$$
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes \beta(h_3)m_{(1)}\alpha(S^{-1}(h_1)),
$$

(1.1)

for all $h \in H$ and $m \in M$. We denote by $\mathcal{H}_YD^H(\alpha, \beta)$ the category of $(\alpha, \beta)$-Yetter-Drinfeld modules, morphisms being the $H$-linear $H$-colinear maps.

**Remark 1.2** As for usual Yetter-Drinfeld modules, one can see that \[\text{[11]}\] is equivalent to

$$
(h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}\alpha(h_1).
$$

(1.2)

**Example 1.3** For $\alpha = \beta = id_H$, we have $\mathcal{H}_YD^H(id, id) = \mathcal{H}_YD^H$, the usual category of (left-right) Yetter-Drinfeld modules. For $\alpha = S^2$, $\beta = id_H$, the compatibility condition \[\text{[11]}\] becomes

$$
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3m_{(1)}S(h_1),
$$

(1.3)

hence $\mathcal{H}_YD^H(S^2, id)$ is the category of anti-Yetter-Drinfeld modules defined in \[\text{[3], [6], [7]}\].

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Example 1.4 For $β ∈ \text{Aut}_{\text{Hopf}}(H)$, define $H_β$ as in [2], that is $H_β = H$, with regular right $H$-comodule structure and left $H$-module structure given by $h · h' = β(h_2)h'S^{-1}(h_1)$, for all $h, h' ∈ H$. It was noted in [2] that $H_β$ satisfies a certain compatibility condition, which actually says that $H_β ∈ H\mathcal{YD}^H(id, β)$. More generally, if $α, β ∈ \text{Aut}_{\text{Hopf}}(H)$, define $H_{α, β}$ as follows: $H_{α, β} = H$, with regular right $H$-comodule structure and left $H$-module structure given by $h · h' = \beta(h_2)h'α(S^{-1}(h_1))$, for $h, h' ∈ H$. Then one can check that $H_{α, β} ∈ H\mathcal{YD}^H(α, β)$.

Example 1.5 Let $α, β ∈ \text{Aut}_{\text{Hopf}}(H)$ and assume that there exist an algebra map $f : H → k$ and a group-like element $g ∈ H$ such that

$$α(h) = g^{-1}f(h_1)β(h_2)f(S(h_3))g, \quad ∀ h ∈ H. \quad (1.4)$$

Then one can check that $k ∈ H\mathcal{YD}^H(α, β)$, with structures $h · 1 = f(h)$ and $1 → 1 ⊗ g$. More generally, if $V$ is any vector space, then $V ∈ H\mathcal{YD}^H(α, β)$, with structures $h · v = f(h)v$ and $v → v ⊗ g$, for all $h ∈ H$ and $v ∈ V$.

Definition 1.6 If $α, β ∈ \text{Aut}_{\text{Hopf}}(H)$ such that there exist $f, g$ as in Example 1.5, we say that $(f, g)$ is a pair in involution corresponding to $(α, β)$ (in analogy with the concept of modular pair in involution due to Connes and Moscovici) and the $(α, β)$-Yetter-Drinfeld modules $k$ and $V$ constructed in Example 1.5 are denoted by $f k^g$ and respectively $f V^g$.

For instance, if $α ∈ \text{Aut}_{\text{Hopf}}(H)$, then $(ε, 1)$ is a pair in involution corresponding to $(α, α)$.

2 Tensor products and duals

By [5], the tensor product of a Yetter-Drinfeld module with an anti-Yetter-Drinfeld module becomes an anti-Yetter-Drinfeld module. We generalize this result as follows:

Proposition 2.1 Let $α, β, γ ∈ \text{Aut}_{\text{Hopf}}(H)$ and $M, N$ two vector spaces which are left $H$-modules and right $H$-comodules.

(i) Endow $M ⊗ N$ with the left $H$-module structure $h · (m ⊗ n) = h_1 · m ⊗ h_2 · n$ and the right $H$-comodule structure $m ⊗ n → (m_{(0)} ⊗ n_{(0)}) ⊗ n_{(1)}m_{(1)}$ (we call these structures ”of type one”). If $M ∈ H\mathcal{YD}^H(α, β)$ and $N ∈ H\mathcal{YD}^H(β, γ)$, then $M ⊗ N ∈ H\mathcal{YD}^H(α, γ)$; in particular, if $M ∈ H\mathcal{YD}^H(S^2, id)$ and $N ∈ H\mathcal{YD}^H$, then $M ⊗ N ∈ H\mathcal{YD}^H(S^2, id)$, and if $M ∈ H\mathcal{YD}^H(id, β)$ and $N ∈ H\mathcal{YD}^H(β, id)$, then $M ⊗ N ∈ H\mathcal{YD}^H$.

(ii) Endow $M ⊗ N$ with the right $H$-module structure $h · (m ⊗ n) = h_2 · m ⊗ h_1 · n$ and the right $H$-comodule structure $m ⊗ n → (m_{(0)} ⊗ n_{(0)}) ⊗ m_{(1)}n_{(1)}$ (we call these structures ”of type two”). If $M ∈ H\mathcal{YD}^H(α, β)$ and $N ∈ H\mathcal{YD}^H(γ, α)$, then $M ⊗ N ∈ H\mathcal{YD}^H(γ, β)$; in particular, if $M ∈ H\mathcal{YD}^H$ and $N ∈ H\mathcal{YD}^H(S^2, id)$, then $M ⊗ N ∈ H\mathcal{YD}^H(S^2, id)$, and if $M ∈ H\mathcal{YD}^H(α, id)$ and $N ∈ H\mathcal{YD}^H(id, α)$, then $M ⊗ N ∈ H\mathcal{YD}^H$.

Proof. We include here a direct proof for (i) (while (ii) is similar and left to the reader), an indirect proof will appear below. We compute:

$$(h · (m ⊗ n))_{(0)} ⊗ (h · (m ⊗ n))_{(1)}$$

$$= (h_1 · m ⊗ h_2 · n)_{(0)} ⊗ (h_1 · m ⊗ h_2 · n)_{(1)}$$

$$= ((h_1 · m)_{(0)} ⊗ (h_2 · n)_{(0)}) ⊗ (h_2 · n)_{(1)}(h_1 · m)_{(1)}$$
\[
\begin{align*}
&= (h_{(1,2)} \cdot m_{(0)} \otimes h_{(2,2)} \cdot n_{(0)}) \otimes \gamma(h_{(2,3)}n_{(1)} \beta(S^{-1}(h_{(2,1)})) \beta(h_{(1,3)}) m_{(1)} \alpha(S^{-1}(h_{(1,1)}))) \\
&= (h_2 \cdot m_{(0)} \otimes h_5 \cdot n_{(0)}) \otimes \gamma(h_6) m_{(1)} \beta(S^{-1}(h_4) h_3) m_{(1)} \alpha(S^{-1}(h_1)) \\
&= (h_2 \cdot m_{(0)} \otimes h_3 \cdot n_{(0)}) \otimes \gamma(h_4) n_{(1)} m_{(1)} \alpha(S^{-1}(h_1)) \\
&= h_2 \cdot (m \otimes n)_{(0)} \otimes \gamma(h_3) (m \otimes n)_{(1)} \alpha(S^{-1}(h_1)),
\end{align*}
\]

that is \( M \otimes N \in H \mathcal{YD}^H(\alpha, \gamma) \).

In what follows, a tensor product with structures of type one will be denoted by \( \hat{\otimes} \), and one with structures of type two will be denoted by \( \hat{\otimes} \).

By 10, if \( M \in H \mathcal{YD}^H(\alpha, \beta) \) and \( N \in H \mathcal{YD}^H(\gamma, \delta) \), then \( M \otimes N \) becomes an object in \( H \mathcal{YD}^H(\alpha \gamma, \delta \gamma^{-1} \beta \gamma) \) with the following structures:

\[
h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n,
\]

\[
m \otimes n \mapsto (m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} := (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)} m_{(1)}.
\]

This tensor product defines, on the disjoint union \( \mathcal{YD}(H) \) of all categories \( H \mathcal{YD}^H(\alpha, \beta) \), a structure of a braided T-category (see 10), and will be denoted by \( \hat{\otimes} \) in what follows.

We want to see what is the relation between this tensor product and \( \hat{\otimes} \). We need a generalization of a result in 10, which states that, if \( \beta \in \text{Aut}_{H_{\text{opf}}}(H) \), then \( H \mathcal{YD}^H(\beta, \beta) \simeq H \mathcal{YD}^H \).}

**Proposition 2.2** If \( \alpha, \beta, \gamma \in \text{Aut}_{H_{\text{opf}}}(H) \), the categories \( H \mathcal{YD}^H(\alpha \beta, \gamma \beta) \) and \( H \mathcal{YD}^H(\alpha, \gamma) \) are isomorphic. A pair of inverse functors \( (F, G) \) is given as follows. If \( M \in H \mathcal{YD}^H(\alpha \beta, \gamma \beta) \), then \( F(M) \in H \mathcal{YD}^H(\alpha, \gamma) \), where \( F(M) = M \) as vector space, with structures \( h \mapsto m = \beta^{-1}(h) \cdot m \) and \( m \mapsto m_{<0>} \otimes m_{<1>} := m_{(0)} \otimes m_{(1)} \), for all \( h \in H \), \( m \in M \). If \( N \in H \mathcal{YD}^H(\alpha, \gamma) \), then \( G(N) \in H \mathcal{YD}^H(\alpha \beta, \gamma \beta) \), where \( G(N) = N \) as vector space, with structures \( h \mapsto n = \beta(h) \cdot n \) and \( n \mapsto n_{(0)} \otimes n_{(1)} := n_{(0)} \otimes n_{(1)} \), for all \( h \in H \), \( n \in N \). Both \( F \) and \( G \) act as identities on morphisms.

**Proof.** Everything follows by a direct computation.

**Corollary 2.3** We have isomorphisms of categories:

\[
H \mathcal{YD}^H(\alpha, \beta) \simeq H \mathcal{YD}^H(\alpha \beta^{-1}, \text{id}), \quad H \mathcal{YD}^H(\alpha, \beta) \simeq H \mathcal{YD}^H(\text{id}, \beta \alpha^{-1}),
\]

\[
H \mathcal{YD}^H(\alpha, \text{id}) \simeq H \mathcal{YD}^H(\text{id}, \alpha^{-1}), \quad H \mathcal{YD}^H(\text{id}, \beta) \simeq H \mathcal{YD}^H(\beta^{-1}, \text{id}),
\]

for all \( \alpha, \beta \in \text{Aut}_{H_{\text{opf}}}(H) \).

Let now \( M \in H \mathcal{YD}^H(\alpha, \beta) \) and \( N \in H \mathcal{YD}^H(\beta, \gamma) \). On the one hand, we can consider the tensor product \( M \hat{\otimes} N \), which is an object in \( H \mathcal{YD}^H(\alpha \beta, \gamma \beta) \). On the other hand, we have the tensor product \( M \hat{\otimes} N \), which is an object in \( H \mathcal{YD}^H(\alpha, \gamma) \). Using the above formulae, one can then check that we have:

**Proposition 2.4** \( M \hat{\otimes} N = F(M \hat{\otimes} N) \).

Let \( M \) be a finite dimensional vector space such that \( M \) is a left \( H \)-module and a right \( H \)-comodule. Denote by \( M^* \) the dual vector space \( M^* \), endowed with the following left \( H \)-module and right \( H \)-comodule structures:

\[
(h \cdot f)(m) = f(S(h) \cdot m),
\]
\[ f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)}) \otimes S^{-1}(m_{(1)}), \]

for all \( h \in H, \ f \in M^\circ, \ m \in M, \) and by \( ^{o}M \) the same vector space \( M^* \) endowed with the following left \( H \)-module and right \( H \)-comodule structures:

\[
(h \cdot f)(m) = f(S^{-1}(h) \cdot m), \\
f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)}) \otimes S(m_{(1)}),
\]

for all \( h \in H, \ f \in ^{o}M, \ m \in M \) (if \( M \) would be an object in \( _H\mathcal{YD}^H \), then \( M^\circ \) and \( ^{o}M \) would be the left and right duals of \( M \) in \( _H\mathcal{YD}^H \)).

**Proposition 2.5** If \( M \) is a finite dimensional object in \( _H\mathcal{YD}^H(\alpha, \beta) \), then \( M^\circ \) and \( ^{o}M \) are objects in \( _H\mathcal{YD}^H(\beta, \alpha) \).

**Proof.** Follows by direct computation (an alternative proof will appear below). \( \square \)

Recall now from [10] that, if \( M \) is a finite dimensional object in \( _H\mathcal{YD}^H(\alpha, \beta) \), then \( M \) has left and right duals \( M^* \) and respectively \( ^*M \) in the \( T \)-category \( \mathcal{YD}(H) \); in particular, \( M^* \) and \( ^*M \) are objects in \( _H\mathcal{YD}^H(\alpha^{-1}, \alpha \beta^{-1} \alpha^{-1}) \), defined as follows: as vector spaces they coincide both to the dual vector space of \( M \), with structures:

\[
(h \cdot f)(m) = f((\beta^{-1} \alpha^{-1} S(h)) \cdot m), \\
f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)}) \otimes S^{-1}(m_{(1)}),
\]

for \( M^* \), and

\[
(h \cdot f)(m) = f((\beta^{-1} \alpha^{-1} S^{-1}(h)) \cdot m), \\
f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)}) \otimes S(m_{(1)}),
\]

for \( ^*M \). We are interested to see how the objects \( M^\circ, \ M^* \) and respectively \( ^{o}M, \ ^*M \) are related. Consider the functor \( F \) as in Proposition 2.2, but this time between the categories \( _H\mathcal{YD}^H(\beta(\beta^{-1} \alpha^{-1}), \alpha(\beta^{-1} \alpha^{-1})) \) and \( _H\mathcal{YD}^H(\beta, \alpha) \). Then, using the expression for \( F \) and the above formulae, one can check that we have:

**Proposition 2.6** \( M^\circ = F(M^*) \) and \( ^{o}M = F(^*M) \).

**Lemma 2.7** Let \( M \in _H\mathcal{YD}^H(\alpha, \beta) \) and \( N \in _H\mathcal{YD}^H(\gamma, \delta) \) finite dimensional. Then the map

\[ \Psi : N^* \otimes M^* \rightarrow (M \otimes N)^*, \quad \Psi(n^* \otimes m^*)(m \otimes n) := m^*(m)n^*(n), \]

is an isomorphism in \( _H\mathcal{YD}^H(\gamma^{-1} \alpha^{-1}, \alpha \beta^{-1} \gamma^{-1} \delta^{-1} \gamma^{-1} \alpha^{-1}) \).

**Proof.** Straightforward computation. \( \square \)

### 3 Endomorphism algebras

Let \( A \) be an algebra in \( _H\mathcal{YD}^H \). We denote by \( A^{op} \) the (usual) opposite algebra, with multiplication \( a \bullet a' = a'a' \) for all \( a, a' \in A \), and by \( \overline{A} \) the \( H \)-opposite algebra (the opposite of \( A \) in the category \( _H\mathcal{YD}^H \)), which equals \( A \) as object in \( _H\mathcal{YD}^H \) but with multiplication \( a \ast a' = a'_{(0)}(a'_{(1)} \cdot a) \), for all \( a, a' \in A. \)
If $A, B$ are algebras in $\mathcal{HYD}^H$, then $A \otimes B$ becomes also an algebra in $\mathcal{HYD}^H$ with the following structures:

$$h \cdot (a \otimes b) = h_1 \cdot a \otimes h_2 \cdot b,$$

$$(a \otimes b) \mapsto (a(0) \otimes b(0)) \otimes b(1)a(1),$$

$$(a \otimes b)(a' \otimes b') = aa'(0) \otimes (a'(1) \cdot b)b'.$$

This algebra structure on $A \otimes B$ (which is just the braided tensor product of $A$ and $B$ in the braided category $\mathcal{HYD}^H$) is denoted by $A \# B$ and its elements are denoted by $a \# b$.

We introduce now endomorphism algebras associated to $(\alpha, \beta)$-Yetter-Drinfeld modules.

**Proposition 3.1** Let $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$ and $M \in \mathcal{HYD}^H(\alpha, \beta)$ finite dimensional. Then:

(i) $\text{End}(M)$ becomes an algebra in $\mathcal{HYD}^H$, with structures:

$$(h \cdot u)(m) = \alpha^{-1}(h_1) \cdot u(\alpha^{-1}(S(h_2)) \cdot m),$$

$$u(0)(m) \otimes u(1) = u(m(0))(0) \otimes S^{-1}(m(1))u(m(0))(1),$$

for all $h \in H$, $u \in \text{End}(M)$, $m \in M$.

(ii) $\text{End}(M)^{\text{op}}$ becomes an algebra in $\mathcal{HYD}^H$, with structures:

$$(h \cdot u)(m) = \beta^{-1}(h_2) \cdot u(\beta^{-1}(S^{-1}(h_1))) \cdot m),$$

$$u(0)(m) \otimes u(1) = u(m(0))(0) \otimes u(m(0))(1)S(m(1)),$$

for all $h \in H$, $u \in \text{End}(M)^{\text{op}}$, $m \in M$.

**Proof.** Everything follows by direct computation. Note that the structures of $\text{End}(M)$ can be obtained in two (equivalent) ways, namely either take $M \hat{\otimes} M^*$, which is in $\mathcal{HYD}^H$, and then transfer its structures to $\text{End}(M)$, or by taking first $M \hat{\otimes} M^{\circ}$, which is in $\mathcal{HYD}^H(\alpha, \alpha)$, transforming this into an object in $\mathcal{HYD}^H$ via the isomorphism $\mathcal{HYD}^H(\alpha, \alpha) \simeq \mathcal{HYD}^H$, and finally transferring the structures to $\text{End}(M)$. Similarly, the structures of $\text{End}(M)^{\text{op}}$ can be obtained either by transferring the structures from $^*M \hat{\otimes} M$, which is in $\mathcal{HYD}^H$, or by taking first $^\circ M \hat{\otimes} M$, which is in $\mathcal{HYD}^H(\beta, \beta)$, transforming this into an object in $\mathcal{HYD}^H$ via the isomorphism $\mathcal{HYD}^H(\beta, \beta) \simeq \mathcal{HYD}^H$ and finally transferring the structures to $\text{End}(M)^{\text{op}}$. 

**Remark 3.2** Assume that there exists a pair in involution $(f, g)$ corresponding to $(\alpha, \beta)$ and consider the $(\alpha, \beta)$-Yetter-Drinfeld module $f^k$ as in the Preliminaries. Then one can easily check that $\text{End}(f^k)$ coincides, as an algebra in $\mathcal{HYD}^H$, with $k$ with trivial Yetter-Drinfeld structures.

Let $\alpha, \beta, \gamma \in \text{Aut}_{\text{Hopf}}(H)$ and the functor $F$ as in Proposition 2.2. Then one can easily check that we have:

**Corollary 3.3** If $M$ is a finite dimensional object in $\mathcal{HYD}^H(\alpha, \beta, \gamma)$ and consider the object $F(M) \in \mathcal{HYD}^H(\alpha, \gamma)$, then $\text{End}(M) = \text{End}(F(M))$ and $\text{End}(M)^{\text{op}} = \text{End}(F(M))^{\text{op}}$ as algebras in $\mathcal{HYD}^H$.

**Corollary 3.4** If $M \in \mathcal{HYD}^H(\alpha, \beta)$ is finite dimensional, then

$$\text{End}(M^*) = \text{End}(M^\circ), \quad \text{End}(^* M) = \text{End}(^\circ M),$$

$$\text{End}(M^*)^{\text{op}} = \text{End}(M^\circ)^{\text{op}}, \quad \text{End}(^* M)^{\text{op}} = \text{End}(^\circ M)^{\text{op}},$$

as algebras in $\mathcal{HYD}^H$. 

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Corollary 3.5 Let $\alpha, \beta \in \text{Aut}_{H_{opf}}(H)$ and assume that $H$ is moreover finite dimensional. Then $\text{End}(H_{\alpha,\beta}) = \text{End}(H_{\beta\alpha^{-1}})$ as algebras in $H\mathcal{YD}^{H}$, where $H_{\alpha,\beta}$ and $H_{\beta\alpha^{-1}}$ are as in Example [1.4].

Proof. Follows from Corollary 3.3 using the fact that $H_{\alpha,\beta}$ and $H_{\beta\alpha^{-1}}$ correspond via the isomorphism of categories $H\mathcal{YD}^{H}(\alpha,\beta) \simeq H\mathcal{YD}^{H}(id,\beta\alpha^{-1})$.

Let $M$ be a finite dimensional vector space endowed with a left $H$-module and a right $H$-comodule structures. Consider on $\text{End}(M)$ the canonical left $H$-module and right $H$-comodule structures induced by the structures of $M$, that is

$$(h \cdot u)(m) = h_{1} \cdot u(S(h_{2}) \cdot m),$$

$$u_{(0)}(m) \otimes u_{(1)} = u(m_{(0)})(0) \otimes S^{-1}(m_{(1)}))u(m_{(0)})(1),$$

for all $h \in H$, $u \in \text{End}(M)$, $m \in M$. We recall the following concept from [2]: if $A$ is an algebra in $H\mathcal{YD}^{H}$, then $A$ is called quasi-elementary if there exists such an $M$ with the property that $\text{End}(M)$ with the above structures is an algebra in $H\mathcal{YD}^{H}$ which coincides with $A$ as an algebra in $H\mathcal{YD}^{H}$.

Proposition 3.6 Let $\alpha, \beta \in \text{Aut}_{H_{opf}}(H)$ and $M \in H\mathcal{YD}^{H}(\alpha,\beta)$ finite dimensional. Then $\text{End}(M)$ with structures as in Proposition 3.3 is a quasi-elementary algebra in $H\mathcal{YD}^{H}$.

Proof. This is obvious if $\alpha = \text{id}_{H}$, because of the formulae for the $H$-module and $H$-comodule structures of $\text{End}(M)$ given in Proposition 3.3 (i). For the general case, we consider the functor $F : H\mathcal{YD}^{H}(\alpha,\beta) \rightarrow H\mathcal{YD}^{H}(id,\beta\alpha^{-1})$ as in Proposition 2.2. We know from Corollary 3.3 that $\text{End}(M) = \text{End}(F(M))$ as algebras in $H\mathcal{YD}^{H}$, and since $\text{End}(F(M))$ is quasi-elementary it follows that so is $\text{End}(M)$. We emphasize that the $H$-module $H$-comodule object making $\text{End}(M)$ quasi-elementary is not $M$ itself, but $F(M)$.

Recall from [10] the group $G = \text{Aut}_{H_{opf}}(H) \times \text{Aut}_{H_{opf}}(H)$ with multiplication $(\alpha,\beta) \cdot (\gamma,\delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$. We have the obvious result:

Lemma 3.7 The map $G \rightarrow \text{Aut}_{H_{opf}}(H)$, $(\alpha,\beta) \mapsto \beta\alpha^{-1}$ is a group anti-homomorphism.

Proposition 3.8 If $H$ is finite dimensional, the map $(\alpha,\beta) \mapsto \text{End}(H_{\alpha,\beta})$ defines a group homomorphism from $G$ to the Brauer group $BQ(k,H)$ of $H$.

Proof. Using Corollary 3.5 the map $(\alpha,\beta) \mapsto \text{End}(H_{\alpha,\beta})$ is just the composition between the group anti-homomorphisms $G \rightarrow \text{Aut}_{H_{opf}}(H)$ from Lemma 3.7 and $\text{Aut}_{H_{opf}}(H) \rightarrow BQ(k,H)$, $\alpha \mapsto \text{End}(H_{\alpha})$ from [2].

Let $\beta \in \text{Aut}_{H_{opf}}(H)$ and $H_{\beta}$ as in Example 1.4 in [2] was defined another object, denoted by $H'_{\beta}$, as follows: it has the same left $H$-module structure as $H_{\beta}$, and right $H$-comodule structure given by $h \mapsto h_{1} \otimes \beta^{-1}(h_{2})$. It was proved then that $H'_{\beta}$ satisfies a certain compatibility condition, which actually says that $H'_{\beta} \in H\mathcal{YD}^{H}(\beta^{-1},id)$.

If instead of $H_{\beta}$ we take an arbitrary object $M \in H\mathcal{YD}^{H}(\alpha,\beta)$, with $\alpha,\beta \in \text{Aut}_{H_{opf}}(H)$, then the above result admits several possible generalizations; we choose the one that will serve our next purpose, which will be to identify the $H$-opposite of $\text{End}(M)$ (in case $M$ is finite dimensional), generalizing [2], Lemma 4.5 as well as [1], Proposition 4.2.
**Proposition 3.9** Let $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$ and $M \in \mathcal{HYDH}(\alpha, \beta)$. Define a new object $M'$ as follows: $M'$ coincides with $M$ as left $H$-modules, and has a right $H$-comodule structure given by

$$m \mapsto m_{<0>} \otimes m_{<1>} := m(0) \otimes \alpha_0^{-1}(m(1)),$$

where $m \mapsto m(0) \otimes m(1)$ is the comodule structure of $M$. Then $M' \in \mathcal{HYDH}(\alpha_0^{-1} \alpha, \alpha)$.

**Proof.** We compute:

$$(h \cdot m)_{<0>} \otimes (h \cdot m)_{<1>} = (h \cdot m)(0) \otimes \alpha_0^{-1}((h \cdot m)(1)) = h_2 \cdot m(0) \otimes \alpha_0^{-1}(\beta(h_3)m(1)\alpha(S^{-1}(h_1))) = h_2 \cdot m(0) \otimes \alpha(h_3)\alpha_0^{-1}(m(1))\alpha_0^{-1}(S^{-1}(h_1)) = h_2 \cdot m_{<0>} \otimes \alpha(h_3)m_{<1>}\alpha_0^{-1}(S^{-1}(h_1)), $$

that is $M' \in \mathcal{HYDH}(\alpha_0^{-1} \alpha, \alpha)$. $\square$

**Proposition 3.10** Let $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$ and $M \in \mathcal{HYDH}(\alpha, \beta)$ finite dimensional; consider also the object $M' \in \mathcal{HYDH}(\alpha_0^{-1} \alpha, \alpha)$ as above. Define the map

$$\tau : \text{End}(M) \to \text{End}(M')^{\text{op}}, \quad \tau(u)(m) = u(0)(\alpha_0^{-1}(u(1)) \cdot m),$$

for all $u \in \text{End}(M)$ and $m \in M'$, where $u \mapsto u(0) \otimes u(1)$ is the right $H$-comodule structure of $\text{End}(M)$. Then $\tau$ is an isomorphism of algebras in $\mathcal{HYDH}$.

**Proof.** We first prove that $\tau$ is an algebra map. We compute:

$$\tau(u \ast v)(m) = \tau(v(0)(v(1) \cdot u))(m) = (v(0)(v(1) \cdot u))(0)(\alpha_0^{-1}((v(0)(v(1) \cdot u))(1)) \cdot m) = v(0)(v(1) \cdot u)(0)(\alpha_0^{-1}((v(1) \cdot u)(1)v(0)(1)) \cdot m) = v(0)(v(1) \cdot u)(0)(\alpha_0^{-1}((v(1) \cdot u)(1)v(1)(1)) \cdot m) = v(0)(v(1) \cdot u)(0)(\alpha_0^{-1}(v(1)u(1)S^{-1}(v(1)u(1)) \cdot m) = v(0)(v(1) \cdot u)(0)(\alpha_0^{-1}(v(1)u(1)) \cdot m) = \tau(v)(u(0)(\alpha_0^{-1}(u(1)) \cdot m)) = \tau(v)(\tau(u)(m)) = (\tau(u) \ast \tau(v))(m), \quad q.e.d.$$

We prove now that $\tau$ is $H$-linear. We compute:

$$\tau(h \cdot u)(m) = (h \cdot u)(0)(\alpha_0^{-1}((h \cdot u)(1)) \cdot m) = (h_2 \cdot u(0))(\alpha_0^{-1}(h_3u(1)S^{-1}(h_1)) \cdot m) = \alpha_0^{-1}(h_2) \cdot u(0)(\alpha_0^{-1}(h_3)\alpha_0^{-1}(h_4u(1)S^{-1}(h_1)) \cdot m) = \alpha_0^{-1}(h_2) \cdot u(0)(\alpha_0^{-1}(u(1)) \cdot m) = \alpha_0^{-1}(h_2) \cdot u(0)(\alpha_0^{-1}(u(1)) \cdot m)$$


We prove now that $\tau$ is $H$-colinear. We have to prove that $\rho(\tau(u)) = \tau(u_{(0)}) \otimes u_{(1)}$, if we denote by $\rho$ the $H$-comodule structure of $\text{End}(M')^{op}$, that is, if we denote $\rho(v) = v^{(0)} \otimes v^{(1)}$, we have to prove that $\tau(u^{(0)}(m) \otimes \tau(u^{(1)}) = \tau(u_{(0)})(m) \otimes u_{(1)}$ for all $m \in M'$. Recall that the $H$-comodule structure of $M'$ is given by $m \mapsto m_{<0>} \otimes m_{<1>} = m_{(0)} \otimes \alpha \beta^{-1}(m_{(1)})$. First we compute:

$$\tau(u_{(0)})(m) \otimes u_{(1)} = u_{(0)}((\alpha^{-1}(u_{(1)}) \cdot m) \otimes u_{(1)})$$

for all $u \in \text{End}(M)$ and $m \in M$. Now we compute:

$$\tau(u^{(0)})(m) \otimes \tau(u^{(1)}) = \tau(u)(m_{<0>} \otimes \tau(u)(m_{<0>})_{<1>} \otimes \tau(u)(m_{<1>}), S(m_{<1>}))$$

$$= \tau(u)(m_{(0)})_{(0)} \otimes \alpha \beta^{-1}(\tau(u)(m_{(0)})_{(1)} S(m_{(1)}))$$

$$= (u_{(0)}((\alpha^{-1}(u_{(1)}) \cdot m_{(0)})_{(0)}$$

$$\otimes \alpha \beta^{-1}((u_{(0)}(\alpha^{-1}(u_{(1)}) \cdot m_{(0)})_{(1)} S(m_{(1)}))$$

$$\tag{3.1} m_{(0)}((\alpha^{-1}(u_{(1)}) \cdot m_{(0)})_{(1)} u_{(0)} S(m_{(1)})),$$

so the two terms are equal. The only thing left to prove is that $\tau$ is bijective; define the map

$$\tau^{-1} : \text{End}(M')^{op} \to \text{End}(M), \quad \tau^{-1}(v)(m) = v^{(0)}(\alpha^{-1}(S(v^{(1)}) \cdot m),$$

for all $v \in \text{End}(M')^{op}$ and $m \in M$. From the $H$-colinearity of $\tau$ it follows easily that $\tau^{-1} \tau = id$. We have not been able to prove directly that $\tau \tau^{-1} = id$; we need to prove first that $\tau^{-1}$ is also $H$-colinear, that is we have to prove that

$$\tau^{-1}(v)(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) \tau^{-1}(v)(m_{(0)})_{(1)} = \tau^{-1}(v^{(0)})(m) \otimes v^{(1)},$$

for all $v \in \text{End}(M')^{op}$ and $m \in M$. Note first that

$$v^{(0)}(m) \otimes v^{(1)} = v(m_{(0)})_{(0)} \otimes \alpha \beta^{-1}(v(m_{(0)})_{(1)} S(m_{(1)})),$$

which together with (3.1) imply

$$v(m_{(0)}) \otimes v(m_{(1)}) = v^{(0)}(m_{(0)}) \otimes \beta \alpha^{-1}(v^{(1)}) m_{(1)} \tag{3.2}.$$
\[ \tau^{-1}(v(m(0))_0) \otimes S^{-1}(m(1)) \tau^{-1}(v(m(0))_1) \]
\[ = (v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0)))_0 \otimes S^{-1}(m(1))(v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0)))_1 \]
\[ = v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0))_0 \otimes S^{-1}(m(1)) \beta \alpha^{-1}(v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0)))_1 \]
\[ = v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0))_0 \otimes S^{-1}(m(1)) \beta \alpha^{-1}(v(0)(\alpha^{-1}(S(v^{(1)}))_3 m(0))_1 S^{-1}(S(v^{(1)}))_1) \]
\[ = v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0))_0 \otimes S^{-1}(m(1)) \beta \alpha^{-1}(v(0)(\alpha^{-1}(S(v^{(1)}))_3 m(0))_1 (v^{(1)}))_4 \]
\[ = v(0)(\alpha^{-1}(S(v^{(1)})) \cdot m(0))_0 \otimes (v^{(1)}))_2 \]
\[ = \tau^{-1}(v(0))(m) \otimes v^{(1)}, \text{ q.e.d.} \]

Now from the $H$-colinearity of $\tau^{-1}$ it follows easily that $\tau \tau^{-1} = id$, hence $\tau$ is indeed an isomorphism with inverse $\tau^{-1}$.

It was proved in [1], Proposition 4.7 that, if $M$ is a finite dimensional Yetter-Drinfeld module, then $\text{End}(M)^{op}$ and $\text{End}(^\circ M)$ are isomorphic as algebras in $\mathcal{H}YD^H$. We generalize this result as follows:

**Proposition 3.11** Let $\alpha, \beta \in \text{Aut}_{\mathcal{H}op}(H)$ and $M \in \mathcal{H}YD^H(\alpha, \beta)$ finite dimensional. Then $\text{End}(M)^{op} \simeq \text{End}(^\circ M)(= \text{End}(^\circ M))$ as algebras in $\mathcal{H}YD^H$.

**Proof.** Define the map

\[ \iota : \text{End}(M)^{op} \rightarrow \text{End}(^\circ M), \quad \iota(u) = u^*, \]

which is obviously an algebra isomorphism. We prove now that it is $H$-linear. Let $u \in \text{End}(M)^{op}$, $h \in H$, $f \in ^\circ M$ and $m \in M$. Using the various formulae given before (and remembering that $^\circ M \in \mathcal{H}YD^H(\beta, \alpha)$) we compute:

\[ \iota(h \cdot u)(f)(m) = (f \circ (h \cdot u))(m) \]
\[ = f((h \cdot u)(m)) \]
\[ = f(\beta^{-1}(h_2) \cdot u(\beta^{-1}(S^{-1}(h_1)) \cdot m)), \]

\[ (h \cdot \iota(u))(f)(m) = (\beta^{-1}(h_1) \cdot \iota(u)(\beta^{-1}(S(h_2)) \cdot f))(m) \]
\[ = (\beta^{-1}(h_1) \cdot ((\beta^{-1}(S(h_2)) \cdot f) \circ u))(m) \]
\[ = (\beta^{-1}(S(h_2)) \cdot f) \circ u(\beta^{-1}(S^{-1}(h_1)) \cdot m) \]
\[ = (\beta^{-1}(S(h_2)) \cdot f)(u(\beta^{-1}(S^{-1}(h_1)) \cdot m)) \]
\[ = f(\beta^{-1}(h_2) \cdot u(\beta^{-1}(S^{-1}(h_1)) \cdot m)), \]

hence the two terms are equal. The $H$-colinearity of $\iota$ is easy to prove and left to the reader. □

We recall now some more facts from [10]. If $N \in \mathcal{H}YD^H(\gamma, \delta)$ and $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$, define the object $(\alpha, \beta)N = N$ as vector space, with structures

\[ h \rightarrow n = \gamma^{-1}\beta\gamma^{-1}(h) \cdot n, \]
\[ n \mapsto n_{<0>} \otimes n_{<1>} := n_{(0)} \otimes \alpha^{-1}(n_{(1)}). \]

Then \((\alpha, \beta)N \in H\mathcal{YD}^H(\alpha \gamma, \alpha^{-1}, \alpha^{-1} \beta^{-1} \beta \gamma^{-1}) = H\mathcal{YD}^H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1})\), where * is the multiplication in the group G recalled before. Let also \(M \in H\mathcal{YD}^H(\alpha, \beta)\) and denote by \(M N = (\alpha, \beta)N\); then the braiding in the T-category \(\mathcal{YD}(H)\) is given by the maps

\[ c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad c_{M,N}(m \otimes n) = n_{(0)} \otimes \beta^{-1}(n_{(1)}) \cdot m, \]

which are isomorphisms in \(H\mathcal{YD}^H((\alpha, \beta) * (\gamma, \delta))\). In particular, assume that \(\alpha = \beta = id_H\), so \(M \in H\mathcal{YD}^H\); then obviously \(M N = N\) as objects in \(H\mathcal{YD}^H(\gamma, \delta)\) and we have the isomorphism in \(H\mathcal{YD}^H(\gamma, \delta)\)

\[ c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad c_{M,N}(m \otimes n) = n_{(0)} \otimes n_{(1)} \cdot m, \]

with inverse \(c_{M,N}^{-1}(n \otimes m) = S(n_{(1)}) \cdot m \otimes n_{(0)}\).

It was proved in [1], Proposition 4.3 that, if \(M\) and \(N\) are finite dimensional Yetter-Drinfeld modules, then \(End(M) \# End(N) \simeq End(M \otimes N)\) as algebras in \(H\mathcal{YD}^H\). We generalize this result as follows:

**Proposition 3.12** If \(M \in H\mathcal{YD}^H(\alpha, \beta)\) and \(N \in H\mathcal{YD}^H(\gamma, \delta)\) both finite dimensional, then \(End(M) \# End(N) \simeq End(M \otimes N)\) as algebras in \(H\mathcal{YD}^H\).

**Proof.** Define the map \(\phi : End(M) \# End(N) \rightarrow End(M \otimes N)\) by the formula

\[ \phi(u \# v)(m \otimes n) = u(m_{(0)}) \otimes (m_{(1)} \cdot v)(n), \]

for all \(u \in End(M)\), \(v \in End(N)\), \(m \in M\), \(n \in N\), where - is the \(H\)-module structure of \(End(N)\) as in Proposition 3.1 (i). As in [1] one can prove that \(\phi\) is an algebra map. We prove now that \(\phi\) is \(H\)-linear. We compute:

\[
\begin{align*}
\phi(h \cdot (u \# v))(m \otimes n) &= \phi(h_{1} \cdot u \# h_{2} \cdot v)(m \otimes n) \\
&= (h_{1} \cdot u)(m_{(0)}) \otimes (m_{(1)} \cdot h_{2} \cdot v)(n) \\
&= \alpha^{-1}(h_{1}) \cdot u(\alpha^{-1}(S(h_{2})) \cdot m_{(0)}) \\
&\quad \otimes \gamma^{-1}(m_{(1)} \cdot h_{3}) \cdot v(\gamma^{-1}(S(m_{(1)} \cdot h_{4})) \cdot n) \\

(h \cdot \phi(u \# v))(m \otimes n)
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (\phi(u \# v)(\gamma^{-1}\alpha^{-1}(S(h_{2})) \cdot (m \otimes n))) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (\phi(u \# v)(\alpha^{-1}(S(h_{3})) \cdot m \otimes \gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \cdot n)) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (u((\alpha^{-1}(S(h_{3})) \cdot m)_{(0)}) \\
&\quad \otimes ((\alpha^{-1}(S(h_{3})) \cdot m)_{(1)} \cdot v)(\gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \cdot n)) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (u(\alpha^{-1}(S(h_{4})) \cdot m_{(0)}) \\
&\quad \otimes (\beta \alpha^{-1}(S(h_{3}))_{(1)} \cdot h_{5} \cdot v)(\gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \cdot n)) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (u(\alpha^{-1}(S(h_{4})) \cdot m_{(0)}) \\
&\quad \otimes (\beta \alpha^{-1}(S(h_{3}))_{(1)} \cdot h_{5} \cdot v)(\gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \cdot n)) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot u(\alpha^{-1}(S(h_{4})) \cdot m_{(0)}) \otimes \gamma^{-1}\beta \alpha^{-1}(S(h_{3})) \gamma^{-1}(m_{(1)} \cdot h_{5}) \gamma^{-1}(h_{(1,1)}) \\
&\quad \otimes v(\gamma^{-1}(S(h_{(1,2)})) \gamma^{-1}(S(m_{(1)})) \gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \cdot n)) \\
&= \gamma^{-1}\alpha^{-1}(h_{1}) \cdot (u(\alpha^{-1}(S(h_{3})) \cdot m_{(0)}) \otimes \gamma^{-1}\beta \alpha^{-1}(S(h_{2})) \gamma^{-1}(m_{(1)} \cdot h_{5}) \gamma^{-1}(h_{4}) \\
&\quad \otimes v(\gamma^{-1}(S(h_{5})) \gamma^{-1}(S(m_{(1)} \cdot h_{4})))) \\
\end{align*}
\]
\begin{align*}
\phi^{-1}(h_1) \cdot u(\phi^{-1}(S(h_4)) \cdot m_{(0)} \otimes \gamma^{-1}(h_2) \gamma^{-1}(h_3) \gamma^{-1}(m_{(1)} h_5) \\
\cdot v(\gamma^{-1}(S(m_{(1)} h_6)) \cdot n)
&= \alpha^{-1}(h_1) \cdot u(\alpha^{-1}(S(h_2)) \cdot m_{(0)} \otimes \gamma^{-1}(m_{(1)} h_3) \cdot v(\gamma^{-1}(S(m_{(1)} h_4)) \cdot n),
\end{align*}

and we see that the two terms are equal. We have to prove now that \( \phi \) is \( H \)-colinear, that is we have to prove that \( \phi(u \# v)_0 \otimes \phi(u \# v)_1 = \phi(u_0 \# v_0) \otimes v_1 u_1 \). We compute:

\[
\phi(u \# v)_0(m \otimes n) \otimes \phi(u \# v)_1
\]

and on the other hand:

\[
\phi(u_0 \# v_0)(m \otimes n) \otimes v_1 u_1
\]

hence the two terms are equal. The only thing left to prove is that \( \phi \) is bijective; we give a proof similar to the one in [1]. Namely, one can check that \( \phi \) coincides with the composition of the following isomorphisms:

\[
\text{End}(M) \otimes \text{End}(N) \cong M \otimes M^* \otimes N \otimes N^*
\]

where the first and the last are the canonical linear isomorphisms, the second is \( \text{id}_M \otimes c_{N \otimes N^*}^{-1} \) and the third is \( \text{id}_{M \otimes N} \otimes \Psi \), where \( \Psi \) is the isomorphism defined in Lemma [2,7].

Let us recall from [2] that, if \( A \) is an algebra in \( H \mathcal{YD}^H \) and \( \mu \in \text{Aut}_{H_{opf}}(H) \), we can define a new algebra in \( H \mathcal{YD}^H \), denoted by \( A(\mu) \), which equals \( A \) as an algebra, but with \( H \)-structures \( (A(\mu), \rightarrow, \rho') \) given by \( h \rightarrow a = \mu(h) \cdot a \) and \( \rho'(a) = a_{<0>} \otimes a_{<1>} := a_{(0)} \otimes \mu^{-1}(a_{(1)}) \), for all \( a \in A(\mu), h \in H \).
Proposition 3.13 Let $N \in _H\mathcal{YD}^H(\gamma, \delta)$ finite dimensional and $\alpha, \beta \in \text{Aut}_H^{\text{Hopf}}(H)$. Then $\text{End}^{(\alpha, \beta)}(N) = \text{End}(N)(\beta \alpha^{-1})$ as algebras in $_H\mathcal{YD}^H$.

Proof. We compute the structures of $\text{End}^{(\alpha, \beta)}(N)$. Let $h \in H$, $u \in \text{End}(N)$, $n \in N$; we have:

\[
(h \cdot u)(n) = \alpha \gamma^{-1} \gamma^{-1} \alpha^{-1}(h_1) \cdot u(\alpha \gamma^{-1} \gamma^{-1} \alpha^{-1}(S(h_2)) \cdot n)
\]

\[
= \gamma^{-1} \beta \gamma^{-1} \gamma^{-1} \alpha^{-1}(h_1) \cdot u(\gamma^{-1} \beta \alpha^{-1}(S(h_2)) \cdot n)
\]

\[
= \gamma^{-1} \beta \alpha^{-1}(h_1) \cdot u(\gamma^{-1} \beta \alpha^{-1}(S(h_2)) \cdot n),
\]

while the structures of $\text{End}(N)(\beta \alpha^{-1})$ are:

\[
(h \rightarrow u)(n) = (\beta \alpha^{-1}(h) \cdot u)(n)
\]

\[
= \gamma^{-1} \beta \alpha^{-1}(h_1) \cdot u(\gamma^{-1} \beta \alpha^{-1}(S(h_2)) \cdot n),
\]

\[
u_{<0>}(n) \otimes u_{<1>} = u_{(0)}(n) \otimes \alpha \beta^{-1}(u_{(1)})
\]

\[
= u_{(n(0))(0)} \otimes \alpha \beta^{-1}(S^{-1}(n_{(1)}))u_{(n(0))(1)}),
\]

and we are done. \qed

It was proved in \cite{1} that, if $M$ and $N$ are finite dimensional Yetter-Drinfeld modules, then $\text{End}(M) \# \text{End}(N) \simeq \text{End}(N) \# \text{End}(M)$ as algebras in $_H\mathcal{YD}^H$. By using Proposition 3.12 and the isomorphisms $c_{M,N}$ recalled above, we obtain the following generalization of this fact:

Proposition 3.14 Let $M \in _H\mathcal{YD}^H(\alpha, \beta)$ and $N \in _H\mathcal{YD}^H(\gamma, \delta)$, both finite dimensional. Then $\text{End}(M) \# \text{End}(N) \simeq \text{End}(M N) \# \text{End}(M)$ as algebras in $_H\mathcal{YD}^H$.

4 $H$-Azumaya algebras and a subgroup of the Brauer group

We begin by recalling several facts from \cite{1} about $H$-Azumaya algebras and the Brauer group of a Hopf algebra $H$.

Let $A$ be a finite dimensional algebra in $_H\mathcal{YD}^H$ and consider the maps

\[
F : A \# \overline{A} \rightarrow \text{End}(A), \quad F(a \# b)(c) = ac_{(0)}(c_{(1)} \cdot b),
\]

\[
G : \overline{A} \# A \rightarrow \text{End}(A)^{\text{op}}, \quad G(a \# b)(c) = a_{(0)}(a_{(1)} \cdot c)b,
\]

for all $a, b, c \in A$, which are algebra maps in $_H\mathcal{YD}^H$. In case $F$ and $G$ are bijective, $A$ is called $H$-Azumaya. If $M$ is a finite dimensional object in $_H\mathcal{YD}^H$, then $\text{End}(M)$ is an $H$-Azumaya algebra. If $A$ and $B$ are $H$-Azumaya, then so are $A \# B$ and $\overline{A}$. Two $H$-Azumaya algebras $A$ and $B$ are called Brauer equivalent (and denoted $A \sim B$) if there exist $M, N \in _H\mathcal{YD}^H$ finite dimensional such that $A \# \text{End}(M) \simeq B \# \text{End}(N)$ as algebras in $_H\mathcal{YD}^H$. The relation $\sim$ is an equivalence relation which respects the operation $\#$. The quotient set is a group with multiplication induced by $\#$ and inverse induced by $A \mapsto \overline{A}$. This group is denoted by $BQ(k, H)$ and called the Brauer group of $H$. The class of an $H$-Azumaya algebra $A$ in $BQ(k, H)$ is denoted by $[A]$.

We have now all the necessary ingredients to prove the main result of this paper:
Remark 4.3 Following \[2\], we denote \( \alpha \in Aut_{\text{Hopf}}(H) \) and \( M \in H YD^H(\alpha, \beta) \) a finite dimensional object. Then \( \text{End}(M) \), with structures as in Proposition 3.7, is an \( H\)-Azumaya algebra.

Proof. We prove that the map
\[
F : \text{End}(M) \# \hat{\text{End}}(M) \to \text{End}(\text{End}(M)),
\]
\[
F(a \# b)(c) = a c_{(0)}(c_{(1)} \cdot b), \quad \forall \ a, b, c \in \text{End}(M),
\]
is bijective (the proof that the other map \( G \) is bijective is similar and left to the reader). By Propositions 3.10, 3.11 and 3.12 we obtain that \( \text{End}(M) \# \hat{\text{End}}(M) \simeq \text{End}(M \otimes \text{End}(M)) \) as algebras in \( H YD^H \); in particular, it follows that \( \text{End}(M) \# \hat{\text{End}}(M) \) is a simple ring, and since \( F \) is an algebra map it follows that \( F \) is injective, and hence bijective, as \( \text{dim}_k(\text{End}(M) \# \hat{\text{End}}(M)) = \text{dim}_k(\text{End}(\text{End}(M))) \), finishing the proof. \( \square \)

Corollary 4.2 We denote by \( BA(k, H) \) the subset of \( BQ(k, H) \) consisting of \( H\)-Azumaya algebras that can be represented as \( \text{End}(M) \), with \( M \in H YD^H(\alpha, \beta) \) finite dimensional, for some \( \alpha, \beta \in Aut_{\text{Hopf}}(H) \). Then \( BA(k, H) \) is a subgroup of \( BQ(k, H) \). Moreover, if \( H \) is finite dimensional, the image of the group anti-homomorphism from \( \hat{\text{End}}(M) \) consisting of classes that are represented by quasi-elementary \( H\)-Azumaya algebras. It was noted in \( \hat{\text{Aut}}(\text{End}(M)) \) is a finite dimensional object in some \( H YD^H(\alpha, \beta) \), we do obtain a subgroup of \( BQ(k, H) \).

Proof. Follows immediately by using Propositions 3.10, 3.11 and 3.12. \( \square \)

Remark 4.3 Following \[2\], we denote \( BT(k, H) \) the subset of \( BQ(k, H) \) consisting of classes that are represented by \( \alpha \)-Yetter-Drinfeld modules, with \( H YD^H(\alpha, \beta) \) closed under multiplication, but it is not known whether it is a subgroup of \( BQ(k, H) \). By Proposition 3.6 \( BA(k, H) \subseteq BT(k, H) \). Thus, by considering only those quasi-elementary \( H\)-Azumaya algebras that are represented as \( \text{End}(M) \), with \( M \) a finite dimensional object in some \( H YD^H(\alpha, \beta) \), we do obtain a subgroup of \( BQ(k, H) \).

We recall from \[2\] that the construction \( A \mapsto A(\mu) \) recalled before defines a group action of \( Aut_{\text{Hopf}}(H) \) on \( BQ(k, H) \), by \( \mu([A]) = [A(\mu)] \) for \( \mu \in Aut_{\text{Hopf}}(H) \) and \( [A] \in BQ(k, H) \). As a consequence of Proposition 3.13 we obtain:

Corollary 4.4 The above action induces a group action of \( Aut_{\text{Hopf}}(H) \) on \( BA(k, H) \).

Proposition 4.5 Assume that there exists \((f, g)\) a pair in involution corresponding to \((\alpha, \beta)\) and let \( M \in H YD^H(\alpha, \beta) \) finite dimensional. Then \([\text{End}(M)] = 1\) in the Brauer group.

Proof. By \[10\], Theorem 5.1, \( M \) is isomorphic, as \((\alpha, \beta)\)-Yetter-Drinfeld modules, with \( f_{k^g} \otimes N \), where \( N \in H YD^H \). Thus \( \text{End}(M) \simeq \text{End}(f_{k^g} \otimes N) \simeq \text{End}(f_{k^g}) \# \text{End}(N) = k \# \text{End}(N) \), hence in the Brauer group we get \([\text{End}(M)] = [k][\text{End}(N)] = 1 \cdot 1 = 1\). \( \square \)

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