The dynamics of quantum phases in a spinor condensate

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We discuss the quantum phases and their diffusion dynamics in a spinor-1 atomic Bose-Einstein condensate. For ferromagnetic interactions, we obtain the exact ground state distribution of the phases associated with the total atom number \( N \), the total magnetization \( \mathcal{M} \), and the alignment (or hypercharge) \( \mathcal{Y} \) of the system. The mean field ground state is stable against fluctuations of atom numbers in each of the spin components, and the phases associated with the order parameter for each spin components diffuse while dynamically recover the two broken continuous symmetries \([U(1)\text{ and } SO(2)]\) when \( N \) and \( \mathcal{M} \) are conserved as in current experiments. We discuss the implications to the quantum dynamics due to an external (homogeneous) magnetic field. We also comment on the case of a spinor-1 condensate with anti-ferromagnetic interactions.

I. INTRODUCTION

Since the observation of Bose-Einstein condensation in trapped dilute alkali atoms \([1, 2, 3]\), the coherence properties of the condensate has become the focus of many theoretical studies \([1, 2, 4, 5, 6, 7, 8, 9]\). As was initially pointed out in Refs. \([4, 5]\), the number fluctuations associated with the assumption of a coherent state macroscopic condensate wavefunction leads to the “diffusion” (or spreading) of the initial phase. For a scalar, one component atomic condensate, this diffusion, the attempt to restore the \( U(1) \) symmetry of the underlying interacting system, can be formulated in terms of the dynamics of a zero mode, or the Goldstone mode \([4]\). More physically meaningful discussions in terms of the relative phases of two condensates were studied soon afterwards \([6, 7, 8, 9]\).

Starting with the remarkable direct observation of the first order coherence in an interference experiment \([7]\), direct relationship between the number fluctuation and quantum phase dynamics was first observed with a condensate in a periodic potential \([11]\), and more recently, in the remarkable Mott insulating state \([12, 13]\).

The emergence of spinor condensates \([14, 15]\) (of atoms with hyperfine quantum number \( F = 1 \)) has created new opportunities to understand quantum coherence and the associated phase dynamics within a three component condensate \([16, 17]\). In an earlier paper \([18]\), we have presented important physical insight into quantum phase diffusions of a spinor condensate. We have investigated the coupled zero mode dynamics as a consequence of the usual mean field treatment that calls for the breaking of two continuous symmetries: a \( U(1) \) gauge transformation \( e^{i\alpha} \) and \( SO(3) \) spin rotations \( U(\alpha, \beta, \tau) = e^{-iF_s \alpha} e^{-iF_l \beta} e^{-iF_s \tau} \) (in the absence of an external magnetic field) \([18]\).

This paper provides a detailed investigation of the quantum phase dynamics for a spinor-1 condensate. As before, we will focus on the case of ferromagnetic interactions, for which the validity of the single spatial mode approximation proves to be a convenient starting point for the emergence of transparent physical illustrations of the zero mode dynamics \([20]\). The fluctuations of atom numbers in each of the spin states will be shown to be connected directly, in this case, to the fluctuations of the pointing direction of the macroscopic spin (or magnetic dipole moment) of the condensate. This paper is organized as follows. In section II, after introducing the Hamiltonian for a spinor-1 condensate, we explicitly work out the Heisenberg equations for the atomic field operators. Following the standard Bogoliubov approach \([21]\), we obtain the coupled Gross-Pitaevskii equations for the mean field of the condensate and the coupled Bogoliubov-de Gennes equations for the quantum fluctuations. We then generalize the Hermitian operators for the condensate number and phase fluctuations (of the Goldstone mode) as introduced earlier \([4]\) to each of the spin component, and derive their dynamic equations under the rotating wave function approximation. In section III, we perform a detailed study of the stationary phase diffusion of a spinor-1 condensate within the single mode approximation valid exactly for ferromagnetic interactions. Section IV addresses the influence of an external magnetic field, and in section V we discuss the same phase dynamics for a condensate with anti-ferromagnetic interactions. We conclude in section VI.

II. FORMULATION

We consider a system of \( N \) spin-1 bosonic atoms interacting via only \( s \)-wave scattering \([10, 22, 23, 24]\). In the second quantized form, the Hamiltonian of our system becomes \([19, 22]\):

\[
H = \sum_{i} \int d\vec{r} \hat{\psi}^\dagger_i (\vec{r}) \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \hat{\psi}_i (\vec{r}) + \frac{\hbar^2}{2} \sum_{i,j} \int d\vec{r} \hat{\psi}_i^\dagger (\vec{r}) \hat{\psi}_j (\vec{r}) \hat{\psi}_j^\dagger (\vec{r}) \hat{\psi}_i (\vec{r}) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \int d\vec{r} \hat{\psi}_i^\dagger (\vec{r}) \hat{\psi}_j^\dagger (\vec{r}) \vec{F}_{ik} \cdot \vec{F}_{jl} \hat{\psi}_l (\vec{r}) \hat{\psi}_k (\vec{r}),
\]

where \( \hat{\psi}_j(\vec{r}) \) \((j = +, 0, -)\) denotes the annihilation operator for the \( j \)-th component of a spinor-1 field. The external trapping potential \( V_{\text{ext}}(\vec{r}) \) is assumed spin-independent as in a far off-resonant optical dipole force.
trap (FORT) so that atomic spin degrees of freedom are completely accessible. The pair interaction coefficients are \( c_0 = 4\pi \hbar^2 (a_0 + 2a_2)/3M \) and \( c_2 = 4\pi \hbar^2 (a_2 - a_0)/3M \), with \( a_0 \) (\( a_2 \)) the s-wave scattering length for two spin-1 atoms in the combined symmetric channel of total spin 0 (2), and \( M \) is the mass of atom. \( \hat{F} \) is spin 1 matrix representation. The interacting terms can be regrouped to show that they include respectively self-scattering, cross-scattering, and the spin-relaxation [24].

From the Hamiltonian (1), one can easily derive the Heisenberg equations for the field operators

\[
\dot{i}h\psi_+ = \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_+ + c_2 \psi_+^\dagger \psi_0^2 + c_0 (\psi_+^\dagger \psi_+ + \psi_0^\dagger \psi_0 + \psi_+^\dagger \psi_- - \psi_-^\dagger \psi_-) \psi_+ + c_2 (\psi_+^\dagger \psi_+ + \psi_0^\dagger \psi_0 - \psi_-^\dagger \psi_-) \psi_+ ,
\]

\[
\dot{i}h\psi_0 = \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_0 + 2c_2 \psi_0^\dagger \psi_+ \psi_-, + c_0 (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) \psi_0 + c_2 (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) \psi_0 ,
\]

\[
\dot{i}h\psi_- = \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_- + c_2 \psi_-^\dagger \psi_0^2 + c_0 (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) \psi_- + c_2 (\psi_-^\dagger \psi_- + \psi_0^\dagger \psi_0 - \psi_+^\dagger \psi_+) \psi_- .
\]

Following the Hartree mean-field theory we assume there are three ‘large’ condensate components \( \phi_0 \) around which we can study the small quantum fluctuations (off-condensate excitations) via the Bogoliubov approximation

\[
\psi_j(\vec{r}) = \sqrt{N} \phi_j(\vec{r}) + \delta \psi_j(\vec{r}) ,
\]

with \( N \) the total number of atoms. We note that such an approach can become questionable in a multi-component system where certain component becomes negligibly small and its quantum nature becomes important. For ferromagnetic interactions, however, our previous studies [24] show that the ground state wave functions can always be expressed as \( \phi_j \propto \phi \) and hence the Bogoliubov prescription can always be applied. By substituting Eq. (3) into (2), we find in the zeroth order, the coupled Gross-Pitaevskii equations (GPEs),

\[
\dot{i}h\phi_+ = [\mathcal{L} + c_0 (\rho + \rho + \rho) + c_2 (2\rho + \rho - \rho - \rho)] \phi_+ + c_2 N \phi_0^* \phi_0^2 ,
\]

\[
\dot{i}h\phi_0 = [\mathcal{L} + c_0 (\rho + \rho + \rho)] \phi_0 + 2c_2 N \phi_0^* \phi_0 \phi_-, \]

\[
\dot{i}h\phi_- = [\mathcal{L} + c_0 (\rho + \rho + \rho - \rho)] \phi_- + c_2 N \phi_0^* \phi_0^2 ,
\]

with \( \mathcal{L} = -\hbar^2 \nabla^2/2M + V_{\text{ext}}(\vec{r}) \), the \( j \)-th component condensate density \( \rho_j(\vec{r}) = N |\phi_j(\vec{r})|^2 \), and the total condensate density \( \rho(\vec{r}) = \rho_+(\vec{r}) + \rho_0(\vec{r}) + \rho_-(\vec{r}) \). \( \int d\vec{r} \rho_j(\vec{r}) = N_j \) is the number of condensed atoms in the \( j \)-th component.

In the first order, the quantum fluctuations obey the following set of Bogoliubov-de Gennes equations (BdGEs),

\[
\dot{i}h \frac{\partial \delta \psi_+}{\partial t} = [\mathcal{L} + c_0 (\rho + \rho + \rho) + c_2 (2\rho + \rho - \rho - \rho)] \delta \psi_+ + N(c_+ \phi_0^* \phi_+ + 2c_2 \phi_0 \phi_0^2) \delta \psi_0 ,
\]

\[
\dot{i}h \frac{\partial \delta \psi_0}{\partial t} = [\mathcal{L} + c_0 (\rho + \rho + \rho) + c_2 (2\rho + \rho - \rho - \rho)] \delta \psi_0 + N(c_+ \phi_0^* \phi_+ + 2c_2 \phi_0 \phi_0^2) \delta \psi_0 ,
\]

\[
\dot{i}h \frac{\partial \delta \psi_-}{\partial t} = [\mathcal{L} + c_0 (\rho + \rho + \rho) + c_2 (2\rho + \rho - \rho - \rho)] \delta \psi_- + N(c_+ \phi_0^* \phi_+ + 2c_2 \phi_0 \phi_0^2) \delta \psi_0 ,
\]

where we have defined \( c_\pm = c_0 \pm c_2 \).

To study the quantum phase dynamics, we introduce the number fluctuation operators [4]

\[
P_j = \int d\vec{r} \left[ \phi_j^* (\vec{r}) \delta \psi_j (\vec{r}) + \phi_j (\vec{r}) \delta \psi_j^* (\vec{r}) \right] .
\]

Using the GPEs (3) and the BdGEs (5) above, we find the evolution equations for these operators,

\[
\dot{i}h \frac{\partial P_j}{\partial t} = -\frac{\hbar}{2} \frac{\partial P_j}{\partial t} = \int d\vec{r} \left[ \delta \psi_j^* (\vec{r}) \delta \psi_j (\vec{r}) - \delta \psi_j (\vec{r}) \delta \psi_j^* (\vec{r}) \right] + 2c_2 \phi_0^* \phi_0 \phi_0^2 \delta \psi_0^* .
\]

Thus both \( P_{\text{tot}} = P_+ + P_0 + P_- \) and \( P_+ - P_- \) are constants of motion, which are in fact obvious since the Hamiltonian (1) conserves the total number of atoms and the total magnetization.

We define the conjugate phase operator \( Q_{\text{tot}} = Q_+ + Q_0 + Q_- \) according to

\[
Q_j = \int d\vec{r} \left[ \theta_j^* (\vec{r}) \delta \psi_j (\vec{r}) - \theta_j (\vec{r}) \delta \psi_j^* (\vec{r}) \right] .
\]

The canonical quantization condition \( [Q_{\text{tot}}, P_{\text{tot}}] = i\hbar \) is satisfied if the constraint \( J_+ + J_0 + J_- = 1 \) is enforced with \( J_j \equiv \int d\vec{r} \left[ \theta_j^* (\vec{r}) \phi_j (\vec{r}) + \phi_j^* (\vec{r}) \theta_j (\vec{r}) \right] \). From the defining equation for \( Q_{\text{tot}} \),

\[
\frac{dQ_{\text{tot}}}{dt} = N \dot{\bar{u}} ,
\]

the Goldstone mode inertial parameter \( \bar{u} \) can be determined. We obtain the dynamic equations for the phase.
functions as,

\[
\dot{W}_+ = [\mathcal{L} + c_0(\rho + \rho_+) + c_2(2\rho_+ + \rho_0 - \rho_-)] \theta_+ + N(c_+ \phi_0^2 + 2c_2 \phi^2 \phi_0) \theta_0 + c_- N \phi^2 \phi_-, \\
+ c_+ N \phi \phi_0 \theta_+^2 + c_+ N \phi \phi_0 \theta_-^2, \\
+ N(c_0 \phi_0^2 + 2c_2 \phi_0^2) \theta_0^2 - N \bar{\nu} \phi_0, \\
\dot{W}_- = [\mathcal{L} + c_0(\rho + \rho_-) + c_2(2\rho_- + \rho_0 - \rho_+)] \theta_-, \\
+ N(c_+ \phi_0^2 + 2c_2 \phi_0^2) \theta_0 - c_- N \phi^2 \phi_- + c_+ N \phi \phi_0 \theta_0, \\
+ N(c_0 \phi_0^2 + 2c_2 \phi_0^2) \theta_0^2 - N \bar{\nu} \phi_-, \\
\]

Combining Eqs. 4 and 10, the dynamic equations of $Q_j$ can be easily obtained to be

\[
\hat{Q}_+ = N \bar{\nu} P_+ + N \int d\vec{r} \left\{ - \left[ (c_+ \phi_0^* \phi_0 + 2c_2 \phi_0^* \phi_0) \theta_0^2 + (c_- \phi_0^* \phi_- + c_2 \phi_0^2) \theta_- + c_- \phi_0^* \phi_- \theta_- \right] \delta \psi_+ \\
+ \left( c_+ \phi_0^* \phi_+ \theta_+ + c_- \phi_0^* \phi_- \theta_+ + c_2 \phi_0^2 \phi_+ \theta_+ \right) \delta \psi_0 + \left( c_- \phi_0^* \phi_- \theta_+ + c_- \phi_0^* \phi_- \theta_- + c_2 \phi_0^2 \phi_- \theta_- \right) \delta \psi_-, \right\}
\]

\[
\hat{Q}_0 = N \bar{\nu} P_0 + N \int d\vec{r} \left\{ - \left[ (c_+ \phi_0^* \phi_0 + 2c_2 \phi_0^* \phi_0) \theta_0^2 + (c_- \phi_0^* \phi_- + c_2 \phi_0^2) \theta_- + c_- \phi_0^* \phi_- \theta_- \right] \delta \psi_0 \\
+ \left( c_+ \phi_0^* \phi_0 \theta_+ + c_- \phi_0^* \phi_- \theta_+ + c_2 \phi_0^2 \phi_0 \theta_+ \right) \delta \psi_+ + \left( c_- \phi_0^* \phi_0 \theta_- + c_- \phi_0^* \phi_- \theta_- + c_2 \phi_0^2 \phi_0 \theta_- \right) \delta \psi _, \right\}
\]

\[
\hat{Q}_- = N \bar{\nu} P_- + N \int d\vec{r} \left\{ - \left[ (c_+ \phi_0^* \phi_0 + 2c_2 \phi_0^* \phi_0) \theta_0^2 + (c_- \phi_0^* \phi_- + c_2 \phi_0^2) \theta_- + c_- \phi_0^* \phi_- \theta_- \right] \delta \psi_0 \\
+ \left( c_+ \phi_0^* \phi_0 \theta_+ + c_- \phi_0^* \phi_- \theta_+ + c_2 \phi_0^2 \phi_0 \theta_+ \right) \delta \psi_+ + \left( c_- \phi_0^* \phi_0 \theta_- + c_- \phi_0^* \phi_- \theta_- + c_2 \phi_0^2 \phi_0 \theta_- \right) \delta \psi _-, \right\}
\]

We note that $[Q_j, P_k] = i \hbar \delta_{jk} J_j$, $[\delta \psi_j(\vec{r}), P_k] = \delta_{jk} \phi_j(\vec{r})$, and $[\delta \psi_j(\vec{r}), Q_k] = -i \hbar \delta_{jk} \delta \psi_j(\vec{r})$. This prompts us to make the rotating wave approximation (RWA) \[3\] \[4\] by assuming that $\delta \psi_j(\vec{r}) = \sum_{a_1} a_{1j}(\vec{r}) P_j + \sum_{b_1} b_{1j}(\vec{r}) Q_j$, which leads to $a_1(\vec{r}) = \theta_j(\vec{r})/J_j$ and $b_1(\vec{r}) = \phi_j(\vec{r})/\bar{\nu} J_j$. Therefore, under the RWA, we obtain $\delta \psi_j(\vec{r}) = \theta_j(\vec{r}) P_j/J_j + \phi_j(\vec{r}) Q_j/\bar{\nu} J_j$. With this result, the complete dynamic equations for the number and phase fluctuation operators become

\[
\hat{P}_+ = -\frac{P_0}{2} = \hat{P}_- \\
\hat{Q}_+ = N \bar{\nu} P_+ - \frac{2 c_2 N}{\hbar} \left[ \left( \phi_+ \phi_- \phi_0^2 \right)^{'''} + 2 \left( \phi_0 \phi_+ + \phi_- \phi_0 \right) \phi_0^2 \frac{J_+}{J_0} \right] P_+ + \frac{4 N}{\hbar} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_+}{J_0}, \]

\[
\hat{Q}_0 = N \bar{\nu} P_0 - \frac{4 N}{\hbar} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_0}{J_+} \\
+ \frac{4 N}{\hbar} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_0}{J_+} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_+}{J_0}, \]

\[
\hat{Q}_- = N \bar{\nu} P_- - \frac{4 N}{\hbar} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_+}{J_0} \\
+ \frac{4 N}{\hbar} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_0}{J_+} \left[ \phi_0 \phi_0 + \phi_0 \phi_0 \right] \frac{J_+}{J_0}, \]
where we have introduced the following shorthand notation: \( \langle f | g \rangle = \int d^3 \mathbf{r} f^{*}(\mathbf{r}) g(\mathbf{r}) h(\mathbf{r}) \), \( (\cdot)' = \text{Re}(\cdot) \), \( (\cdot)'' = \text{Im}(\cdot) \), and \( I_{ml} = \int d^3 \mathbf{r} (\phi_{m} \theta_{n})' (\phi \theta)^{'} \). Equation (12) thus completely determines the zero mode condensate fluctuations. Before finding the corresponding compact zero mode Hamiltonian in quadratic forms of \( P_{j} \) and \( Q_{j} \), we will first attempt to simplify in the next section using general properties of \( \phi_{j}(\mathbf{r}) \) and \( \theta_{j}(\mathbf{r}) \) for a ferromagnetic spinor-1 condensate [24].

III. PHASE DIFFUSIONS AND THE ZERO MODES

In general, the ground state wave functions of a spinor condensate can be expressed as

\[
\phi_{j}(\mathbf{r}, t) = \phi_{j}(\mathbf{r}) e^{-i pt/\hbar + i \alpha_{j}},
\]

(13)

with a common chemical potential \( \mu \) for all spin components when the external magnetic field is negligible. Based on the minimization for the total energy Eq. (11) within the mean field theory, it was shown in Refs. [20, 24] that there exists an important relation among \( \alpha_{j} \), given by \( \alpha_{+} + \alpha_{-} - 2n_{0} = 0 \). We further proved in Ref. [24] that the ground state wave function for each of the spin component takes the same spatial shape for (ferromagnetic interactions), i.e.

\[
\phi_{j}(\mathbf{r}) = \sqrt{N_{j}} \phi(\mathbf{r}) e^{i m_{j}},
\]

(14)

with the real-valued mode function \( \phi(\mathbf{r}) \) (normalized to unity) governed by an equivalent scalar condensate GPE

\[
[\mathcal{L} + c_{+} N \phi^2(\mathbf{r})] \phi(\mathbf{r}) = \mu \phi(\mathbf{r}),
\]

(15)

of a scattering length \( a_{2} \) (note \( c_{+} \propto a_{2} \)). Atoms thus only collide in the symmetric total spin \( F = 2 \) channel in a ferromagnetic state in order to maintain their individual spins parallel. \( n_{j} = N_{j}/N \) is the ratio of the number of atoms in the \( i \)-th spin component to the total number of atoms. For any given magnetization \( M_{j} \), \( n_{j} \) is given explicitly by \( n_{+} = (1 + m)^{2}/4 \) and \( n_{0} = (1 - m^{2})/2 \) with \( m = M_{j}/N \) [21, 24].

Similarly, we look for phase functions of the form

\[
\theta_{j}(\mathbf{r}, t) = \theta_{j}(\mathbf{r}) e^{-i pt/\hbar + i \alpha_{j}}.
\]

(16)

Substituting Eq. (16) into Eq. (14), we find that \( \theta_{j}(\mathbf{r}, t) \) also evolve in time as \( e^{-i pt/\hbar} \). To focus on the steady state quantum fluctuation properties, we may therefore neglect the time-dependent part in \( \phi_{j}(\mathbf{r}, t) \) and \( \theta_{j}(\mathbf{r}, t) \) from now on. The steady state equations for phase functions now become

\[
[L - \mu + c_{0}(\rho + 2 \rho_{0}) + c_{2}(3 \rho_{0} + \rho_{+} - \rho_{-})] \xi_{+} + 2N(c_{+} \phi_{+}^{*} + c_{2} \phi_{-}) \phi_{0} \xi + N(2c_{-} \phi_{+}^{*} + c_{2} \phi_{-}) \xi = N \tilde{\nu} \phi_{+},
\]

\[
[L - \mu + c_{0}(\rho + 2 \rho_{0}) + c_{2}(\rho_{+} + 2N \phi_{+}^{*} - \rho_{-})] \xi_{0} = N \tilde{\nu} \phi_{+},
\]

\[
[L - \mu + c_{0}(\rho + 2 \rho_{0}) + c_{2}(3 \rho_{0} - \rho_{+} + \rho_{-})] \xi_{-} + 2N(c_{+} \phi_{+}^{*} + c_{2} \phi_{-}) \phi_{0} \xi + N(2c_{-} \phi_{+}^{*} + c_{2} \phi_{-}) \xi = N \bar{\nu} \phi_{-},
\]

(17)

where \( \xi_{j}(\mathbf{r}) = \theta_{j}(\mathbf{r}) e^{-i \alpha_{j}} \) are now real-valued functions. One can easily check that similar to the single mode approximation for \( \phi_{j}(\mathbf{r}) \), \( \xi_{j}(\mathbf{r}) \) can also be expressed as \( \xi_{j}(\mathbf{r}) = n_{j}^{1/2} \theta(\mathbf{r}) \) with \( \theta(\mathbf{r}) \) governed by the following equation

\[
[L - \mu + 3c_{+} N \phi^{2}(\mathbf{r})] \theta(\mathbf{r}) = \bar{\nu} \theta(\mathbf{r}),
\]

(18)

and \( \bar{\nu} \) is determined through the normalization constraint \( \int d^{3} \mathbf{r} \theta(\mathbf{r}) \theta^{*}(\mathbf{r}) = 1/2 \). Therefore, for ferromagnetic interactions, the phase functions can be generally expressed as

\[
\theta_{j}(\mathbf{r}) = \sqrt{n_{j}} \theta(\mathbf{r}) e^{im_{j}}.
\]

(19)

We note that Eq. (18) is in fact identical to the equation satisfied by the phase function of a scalar condensate [4], a point easily understood for ferromagnetic interactions when all spins align along the same direction. If we take this direction as the quantization axis, then the spinor condensate is essentially a scalar condensate since its total magnetization \( M \) is conserved. All overlap integrals in Eq. (12) are presented in Appendix A. They are all real quantities with the choice of phase parameters.

![FIG. 1: The N-dependence of \( \tilde{u} \) (in units of \( \hbar \omega_{\perp} \)) for a \( ^{87}\text{Rb} \) condensate in a cylindrically symmetric trap with \( \omega_{x} = \omega_{y} = \omega_{z} = (2 \pi)100 \) (Hz) and \( \omega_{z} = \lambda \omega_{\perp} \). We have used \( a_{0} = 101.8a_{B} \) and \( a_{2} = 100.4a_{B} \) (\( a_{B} \) Bohr radius) [23].](image)
It is then simply to verify that

\[ \varepsilon \]

\[ \varepsilon = \mathcal{O}_{\phi_0} \]

\[ \varepsilon = 2 \int d\theta \theta (\mathcal{L} - \mu + c_+ N \phi^2 (\vec{r})) \theta (\vec{r}). \]

(22)

We note that \( \varepsilon \equiv 0 \) for noninteracting atoms [\( \phi (\vec{r}) = \theta (\vec{r}) \)]. However, it’s nonzero in general and we can prove that \( \varepsilon > 0 \) since \( \theta (\vec{r}) \) is the solution of Eq. (18). By using Eq. (21), we then obtain

\[ \mathbf{p}^T \mathbf{A} \mathbf{p} = \frac{\varepsilon}{2} \left( \frac{p_{\perp}^2}{n_+} + \frac{p_{\perp}^2}{n_-} \right) + 2 \mathcal{O}_{\phi_0} \mathbf{p}^T \mathbf{A} \mathbf{p}, \]

where

\[ \mathbf{A}' = \begin{pmatrix} b_0 + \frac{4c_2}{1 + mb_0} & b_0 + \frac{2cm}{1 + m} & b_0 + \frac{2cm}{1 + m} \\ b_0 + \frac{2cm}{1 + m} & b_0 + \frac{2cm}{1 + m} & b_0 + \frac{2cm}{1 + m} \\ b_0 + \frac{2cm}{1 + m} & b_0 + \frac{2cm}{1 + m} & b_0 + \frac{2cm}{1 + m} \end{pmatrix} \]

It is then simply to verify that \( \mathbf{A}' \) is semi-positive definite, which guarantees that \( \mathbf{A} \) is positive definite. Thus the initial ground state, the mean field symmetry breaking state with a coherent condensate amplitude \( \sqrt{N} \phi (\vec{r}) \) is stable. The associated quantum fluctuations of the atom numbers can be studied by the linearization approximation Eq. (8).

We note that \( \mathbf{p}^T \mathbf{A} \mathbf{p} \)

\[ \mathbf{p}^T = \mathbf{p}'^T \mathbf{U} = \]

where

\[ \mathcal{U} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \]

(23)

which motivates the introduction of new number and phase fluctuation operators according to

\[ \mathbf{p}^T = \mathbf{p}'^T \mathbf{U}^T = \]

(24)

(25)

FIG. 2: The \( N \)-dependence of \( \varepsilon \) (in units of \( \hbar \omega \)) for the same parameters as in Fig. [19].

\[
\begin{align*}
(\rho_N, \rho_M, p_Y) \\
\text{and } q^T = (q_N, q_M, q_Y).
\end{align*}
\]

Not surprisingly, this transformation recovers the underline symmetry of a spinor-1 condensate as discovered recently in Ref. [28]. It turns out that \( \rho_N, \rho_M, \) and \( p_Y \) represent respectively the fluctuations of the total number of atoms, the magnetization, and the hypercharge. \( \mathbf{A} \) is not simultaneously diagonalized by the \( \mathbf{U} \), yet the zero mode Hamiltonian takes a much simpler form

\[
\frac{H_{\text{zero}}}{N} = ap_N^2 + bp_M^2 + cp_Y^2 \\
+ \alpha p_N p_Y + \beta p_M p_Y + \gamma p_N p_M \\
+ \eta q_Y^2.
\]

(24)

where all the coefficients are given in Appendix C. By replacing \( p_i \) with \( -i \partial \theta / \partial q_i \), we obtain an eigenvalue equation for the distribution of the various fluctuations in the condensate ground state

\[
\frac{E}{N \hbar^2} = \frac{\alpha^2 \varphi}{\partial q_N^2} - b \frac{\partial^2 \varphi}{\partial q_M^2} - c \frac{\partial^2 \varphi}{\partial q_Y^2} \\
- \alpha \frac{\partial^2 \varphi}{\partial q_N q_Y} - \beta \frac{\partial^2 \varphi}{\partial q_M q_Y} - \gamma \frac{\partial^2 \varphi}{\partial q_N q_M} \\
+ \frac{\eta}{\hbar^2} q_Y^2 \varphi.
\]

(25)

We look for eigenfunctions of the form

\[
\chi(qv) \text{e}^{i(k_N q_N + k_M q_M)}
\]

which upon substituting into (22) yields

\[
\frac{d^2 \chi}{dq_Y^2} + i k \frac{d \chi}{dq_Y} + \left( E' - \frac{\eta}{\hbar^2} q_Y^2 \right) \chi = 0,
\]

(26)

with

\[
\kappa = \alpha k_N + \beta k_M,
\]

\[
E' = \frac{E}{N \hbar^2} - (ak_N^2 + bk_M^2 + \gamma k_N k_M).
\]

(27)

(28)
The eigen-equation \[ \text{Eq. (27)} \] is essentially of the harmonic type, which can be easily solved and the resulting eigenvalues of Eq. \[ \text{Eq. (28)} \] are

\[
\frac{E_n}{N} = (2n + 1)\hbar (cn)^{1/2} + \Lambda(k_N, k_M),
\]

where \( n = 0, 1, 2, \ldots \) and

\[
\Lambda(k_N, k_M) = \hbar^2 (ak_N^2 + bk_M^2 + \gamma k_N k_M)
\]
\[
- \frac{\hbar^2}{4\epsilon}(ak_N + \beta k_M)^2.
\]

We note that \( \Lambda(k_N, k_M) \) must be greater than or equal to zero due to the positive definiteness of matrix \( \mathcal{A} \). Therefore for the ground state of Eq. \[ \text{Eq. (29)} \], we must have \( k_N = k_M = 0 \), the ground state \( n = 0 \) energy and eigenfunction are respectively

\[
E_0 = N\hbar (cn)^{1/2} = \frac{1}{2}\hbar \Omega,
\]

\[
\varphi_0(q_Y) = \pi^{-1/4}(\hbar \Delta Y_0)^{-1/2} \exp \left( -\frac{q_Y^2}{2\hbar^2 \Delta^2 Y_0} \right),
\]

with

\[
\Omega = \frac{2N}{\hbar} \left[ -c_2 \mathcal{O}_{\phi\phi}(\varepsilon - 4c_2 \mathcal{O}_{\phi\phi}) \right]^{1/2},
\]

and

\[
\Delta^2 Y_0 = \frac{1}{\hbar} \sqrt{\frac{c}{\eta}} = \frac{1}{3n_0^2} \left( \frac{\varepsilon - 4c_2 \mathcal{O}_{\phi\phi}}{-c_2 \mathcal{O}_{\phi\phi}} \right)^{1/2}.
\]

This ground state of \( H_{zero} \) can be easily understood. For a system starting with a fixed total number of atoms and magnetization, the dynamic conservation of the total number of atoms and total magnetization as dictated by the Hamiltonian \[ \text{Eq. (11)} \] requires \( \langle p_Y^2 \rangle = \langle q_Y^2 \rangle = 0 \), which leads to \( \langle q_Y^2 \rangle = \langle q_M^2 \rangle = \infty \), i.e., the ground state of \( H_{zero} \) has completely diffused phases for \( q_Y \) and \( q_M \). The inherent ferromagnetic interaction, nevertheless, prepares a correlated ground state such that both \( p_Y \) and \( q_Y \) takes a Gaussian form distribution with \( \langle p_Y^2 \rangle = 1/2\Delta^2 Y_0 \) and

\[
\langle q_Y^2 \rangle = \hbar^2 \Delta^2 Y_0 / 2,
\]

which is in fact the minimal uncertainty coherent state that is consistent with the symmetry breaking mean field assumptions of the ground state with three separate macroscopic wave functions for each of the spin components. However, it is experimentally difficult to produce a condensate having total fixed number and phase fluctuations as specified by such a state. We therefore turn to study the number and phase fluctuations of a general state. We start with the dynamic equations for \( p_j \) and \( q_j \),

\[
\dot{p}_N = 0,
\]

\[
\dot{p}_M = 0,
\]

\[
\dot{p}_Y = -2Nq_Y q_Y,
\]

\[
\dot{q}_N = N(2\alpha p_N + \gamma p_M + \alpha p_Y),
\]

\[
\dot{q}_M = N(\gamma p_N + 2\alpha p_M + \beta p_Y),
\]

\[
\dot{q}_Y = N(\alpha p_N + \beta p_M + 2\alpha p_Y).
\]

By noting that both \( p_N \equiv p_N(0) \) and \( p_M \equiv p_M(0) \) are constants of motion, we can express the solutions as

\[
x(t) = T(t)x(0),
\]

where \( x^T(t) = [p_N(t), p_M(t), p_Y(t), q_N(t), q_M(t), q_Y(t)] \), \( x^T(0) = [p_N(0), p_M(0), p_Y(0), q_N(0), q_M(0), q_Y(0)] \) is its initial value, and the evolution matrix \( T(t) \) takes following form.
The coefficients $\alpha'$, $\beta'$, $\zeta_a$, $\zeta_q$, and $\zeta_c$ are given in Appendix E. The explicit solutions of the time-dependent number and phase fluctuations are given in Appendix E. We note that the general solutions has a simple structure: in addition to oscillating terms of the form $\cos\Omega t$ and $\sin\Omega t$, the phase fluctuations of $q_N$ and $q_M$ also contains terms proportional to $Nt$. This indicates that our linearization approximation Eq. (3) is valid only for a finite duration. The time-dependent covariance can also be obtained to be

$$\langle x_i(t)x_j(t) \rangle = \sum_{kl} T_{ik}(t)T_{jl}(t)\langle x_k(0)x_l(0) \rangle. \quad (34)$$

Finally, let’s consider the diffusion of the direction of the macroscopic spin $\vec{f}(\vec{r}) = \sum_i \psi_i^\dagger(\vec{r})\vec{F}_{ij}\psi_j(\vec{r})$. As we have shown in Ref. [20], independent of the spatial coordinates, the spins of Bose condensed atoms are parallel for ferromagnetic interactions, i.e., essentially acts as a macroscopic magnetic dipole pointing along the same direction.

$$\vec{f}_0(\vec{r}) = N \sum_{ij} \phi_i^\dagger(\vec{r})\vec{F}_{ij}\phi_j(\vec{r}) = N\phi^2(\vec{r}) \begin{pmatrix} \sqrt{1 - m^2} \cos\Theta \\ -\sqrt{1 - m^2} \sin\Theta \end{pmatrix} \quad (35)$$

where $\Theta = \alpha_+ - \alpha_0 = \alpha_0 - \alpha_-$. Our initial choice of three separate macroscopic condensate wave functions (with given phases) for each of the spin components fixes the initial direction of the condensate spin direction. As the phase spreading dynamics attempts to restore the $U(1)$ symmetries of each of the phases, our initial choice of phases will become irrelevant. Under the linear approximation the fluctuation becomes

$$\delta\vec{f}(t) = \int d\vec{r} \left[ \vec{f}(\vec{r}) - \vec{f}_0(\vec{r}) \right], \quad (36)$$

or explicitly

$$\delta f_x = \frac{1}{\sqrt{2}} \int d\vec{r} (\phi_0^*\delta\psi_+ + \phi_+\delta\psi_0^\dagger + \phi_0\delta\psi_0^\dagger + \phi_\dagger\delta\psi_+ + h.c.),$$

$$\delta f_y = \frac{i}{\sqrt{2}} \int d\vec{r} (\phi_0^*\delta\psi_+ + \phi_+\delta\psi_0^\dagger + \phi_0\delta\psi_0^\dagger - \phi_\dagger\delta\psi_+ + h.c.),$$

$$\delta f_z = \int d\vec{r} (\phi_0^*\delta\psi_+ - \phi_\dagger\delta\psi_- + h.c.). \quad (37)$$

Although $\langle \delta\vec{f}(t) \rangle \equiv 0$, the dynamics of the zero mode causes the variance of $\delta\vec{f}(t)$ to be nonzero. It can be reexpressed in terms of the (time-dependent) number and phase fluctuation operators under the RWA as

$$\delta f_x(t) = \frac{\cos\Theta}{\sqrt{1 - m^2}} \left[ \sqrt{3}p_N(t) - \sqrt{2}mp_M(t) \right]$$

$$+ \frac{\sqrt{1 - m^2}}{2\hbar} \left[ \sqrt{3}q_N(t) + \sqrt{6}mq_N(t) \right],$$

$$\delta f_y(t) = \frac{-\sin\Theta}{\sqrt{1 - m^2}} \left[ \sqrt{3}p_M(t) - \sqrt{2}mp_M(t) \right]$$

$$+ \frac{\sqrt{1 - m^2}}{2\hbar} \left[ \sqrt{3}q_M(t) + \sqrt{6}mq_N(t) \right],$$

$$\delta f_z(t) = \sqrt{2}p_M(t), \quad (38)$$

where the time-dependent terms are

$$\sqrt{2}q_M(t) + \sqrt{6}mq_N(t) = \sqrt{2}q_M(0) + \sqrt{6}mq_N(0)$$

$$+ \frac{4\epsilon Nt}{1 - m^2} \left[ -\sqrt{3}mp_N(0) + \sqrt{2}p_M(0) \right]. \quad (39)$$

We see that $\delta f_z$ is a constant of motion due to the conservation of total magnetization. On the other hand, both $\delta f_x$ and $\delta f_y$ diffuse with time. To calculate the variance of $\delta\vec{f}(t)$, we simply take $\Theta = 0$, which yields

$$\delta f_x(t) = \frac{1}{\sqrt{1 - m^2}} \left[ \sqrt{3}p_N(0) - \sqrt{2}mp_M(0) \right],$$

$$\delta f_y(t) = \frac{1}{\sqrt{1 - m^2}} \left[ \sqrt{3}q_M(0) + \sqrt{6}mq_N(0) \right],$$

$$\delta f_z(t) = \sqrt{2}p_M(0), \quad (40)$$

i.e., with the direction of the initial macroscopic spin constrained in the $x$-$z$ plane. The results of the variances for condensate spin direction fluctuations are listed in Appendix E.

We can also project the fluctuation of the spin [Eq. (35)] onto the spherical coordinates $(\hat{r}, \theta, \phi)$ as illustrated in Fig. 5 and noting that

$$\hat{r} = \hat{x}\sin\theta \cos\phi + \hat{y}\sin\theta \sin\phi + \hat{z}\cos\theta,$$

$$\hat{\theta} = \hat{x}\cos\theta \cos\phi + \hat{y}\cos\theta \sin\phi - \hat{z}\sin\theta,$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi,$$

and

$$\sin\theta = \sqrt{1 - m^2}, \quad \cos\theta = m,$$

$$\sin\phi = -\sin\Theta, \quad \cos\phi = \cos\Theta,$$

$$\sin\Theta = \sqrt{1 - m^2}, \quad \cos\Theta = m,$$
We find
\[ \delta f_r(t) = \sqrt{3} p_N(0), \]
\[ \delta f_\phi(t) = \frac{1}{\sqrt{1 - m^2}} \left[ \sqrt{3} m p_N(0) - \sqrt{2} p_M(0) \right], \]
\[ \delta f_\phi(t) = \frac{1}{\sqrt{2} m} \left[ \sqrt{2} q_M(t) + \sqrt{6} m q_Y(t) \right]. \] (42)

We immediately see that both \( \delta f_r(t) \equiv \delta f_r(0) \) and \( \delta f_\phi(t) \equiv \delta f_\phi(0) \) are fixed due to the conservations of \( N \) and \( M \). Furthermore, with the use of Eq. (39), we find that \( \delta f_\phi(t) \) can be rewritten as
\[ \delta f_\phi(t) = \delta f_\phi(0) - \frac{2 \varepsilon N t}{\hbar} \delta f_\phi(0), \] (43)
where
\[ \delta f_\phi(0) = \frac{1}{\sqrt{2} m} \left[ \sqrt{2} q_M(0) + \sqrt{6} m q_Y(0) \right]. \]

We then see that \( \delta f_{f_\phi}(t) \) grows linearly with \( t \) and the diffusion rate is given by
\[ R_d = -\frac{2 \varepsilon N}{\hbar} \delta f_\phi(0), \] (44)
which is proportional to \( \delta f_\phi(0) \). A result easily understandable in terms of the single axis twisting of the isospin \( T^2 \) (of the Hamiltonian) of a spior-1 spinor condensate [28].

Given any initial condensate state, its variances in atom numbers for each individual spin component and their correlations and related phase fluctuations completely determine the subsequent fluctuations of \( \delta f_{\phi}(t) \). \( P_j(t) \) is in fact nothing but the atom number fluctuation of state \( |j\rangle \). In general we note that
\[ \int d\vec{r} \bar{\psi}_j^\dagger(\vec{r}) \psi_j(\vec{r}) = N + \sqrt{N} P_j + \int d\vec{r} \delta \bar{\psi}_j^\dagger(\vec{r}) \delta \psi_j(\vec{r}) \]
\[ = N + \sqrt{N} P_j + \frac{n_j}{2 J_j} P_j^2 + \frac{n_j}{2 \hbar^2 J_j} Q_j^2 - \frac{1}{2}, \] (45)
where \( s = \int d\vec{r} \bar{\theta}^2(\vec{r}) \) and we have used RWA in deriving the last line. One can easily find
\[ \delta N_j^2 = \langle N_j^2 \rangle - \langle N_j \rangle^2 \]
\[ = \left\langle \left[ \int d\vec{r} \bar{\psi}_j^\dagger(\vec{r}) \psi_j(\vec{r}) \right]^2 \right\rangle - \left\langle \int d\vec{r} \bar{\psi}_j^\dagger(\vec{r}) \psi_j(\vec{r}) \right\rangle^2 \]
\[ \approx N \langle P_j^2 \rangle = N \langle Y^2 \rangle, \quad (46) \]
to lowest order.

If we use \( \sigma_N^2, \sigma_M^2, \) and \( \sigma_Y^2 \) to denote the variances of the total atom number \( N \), the magnetization \( M \), and the hyper-charge \( Y \), we find that
\[ \left\langle [\delta f_r(t)]^2 \right\rangle = 3 \sigma_N^2, \]
\[ \left\langle [\delta f_\phi(t)]^2 \right\rangle = \frac{1}{1 - m^2} \left( 3 \sigma_N^2 + 2 \sigma_M^2 \right). \] (47)

To calculate the variance of \( \delta f_{\phi}(t) \), we assume a completely uncorrelated (between fluctuations of \( N, M, \) and \( Y \)) initial distribution in the form of a Gaussian wave packet
\[ \varphi(p_N, p_M, p_Y) = (2\pi)^{-3/4} (\sigma_N \sigma_M \sigma_Y)^{-1/2} \times \exp \left[ -\frac{1}{4} \left( \frac{p_N^2}{\sigma_N^2} + \frac{p_M^2}{\sigma_M^2} + \frac{p_Y^2}{\sigma_Y^2} \right) \right]. \] (48)

Since \( q_i = i \hbar \partial / \partial p_i \), we find for such a distribution
\[ \langle q_i(0) q_j(0) \rangle = \frac{\hbar^2}{4 \sigma_i^2} \delta_{ij}, \]
\[ \langle p_i(0) q_j(0) \rangle = \sigma_i^2 \delta_{ij}, \]
\[ \langle q_i(0) p_j(0) \rangle = 0, \quad \langle q_i(0) q_j(0) \rangle = 0, \quad \text{for} \ i \neq j, \] (49)
and \( i,j = N,M,Y \). Therefore
\[ \left\langle [\delta f_{\phi}(t)]^2 \right\rangle = \left\langle [\delta f_{\phi}(0)]^2 \right\rangle + \frac{4 \varepsilon^2 N^2 t^2}{\hbar^2} \left\langle [\delta f_{\phi}(0)]^2 \right\rangle \] (50)
with
\[ \left\langle [\delta f_{\phi}(0)]^2 \right\rangle = \frac{1 - m^2}{8} \left( \frac{1}{\sigma_M^2} + 3 m^2 \frac{1}{\sigma_Y^2} \right). \] (51)

If on the other hand, we take a distribution similar to Eq. (48) but with no correlations between the populations in the spin components \( j = +,0,-, \)
\[ \varphi(p_+, p_0, p_-) = (2\pi)^{-3/4} (\sigma_+ \sigma_0 \sigma_-)^{-1/2} \times \exp \left[ -\frac{1}{4} \left( \frac{p_+^2}{\sigma_+^2} + \frac{p_0^2}{\sigma_0^2} + \frac{p_-^2}{\sigma_-^2} \right) \right], \] (52)
with \( \sigma_j^2 \) denoting the corresponding atom number variance, we find
\[ \left\langle [\delta f_r(t)]^2 \right\rangle = \sigma_+^2 + \sigma_0^2 + \sigma_-^2, \]
\[ \left\langle [\delta f_\phi(t)]^2 \right\rangle = \frac{1}{1 - m^2} \left[ (1 - m)^2 \sigma_+^2 + m^2 \sigma_0^2 \right. + \left. (1 + m)^2 \sigma_-^2 \right], \] (53)
and Eq. (50) still holds but with
\[
\langle [\delta f \phi(0)]^2 \rangle = \frac{1 - m^2}{16} \left[ \left( \frac{1 + m^2}{\sigma^2} \right) + 4m^2 \frac{1}{\sigma^2} \right] + \left( \frac{1 - m^2}{\sigma^2} \right),
\]
(54)

In the Thomas-Fermi (TF) regime, we obtain most of the above results analytically. First we note that the ground wave function is
\[
\phi^2(\vec{r}) = \begin{cases} (c_+ N)^{-1} [\mu - V_{\text{ext}}(\vec{r})] & \text{if } \mu > V_{\text{ext}}(\vec{r}), \\ 0, & \text{otherwise.} \end{cases}
\]
(55)

where \( c_+ = 4\pi \hbar^2 a_2 / M \) and the chemical potential
\[
\mu = \frac{\hbar \omega_{\phi}}{2} \left( \frac{15 N a_2}{a_{ho}} \right)^{2/5}
\]
with \( \omega_{\phi} = (\omega_2 \omega_2 \omega_2)^{1/3} \) and \( a_{ho} = (\hbar / M \omega_{\phi})^{1/2} \). Similarly, ignoring the kinetic energy term in the Eq. (53), we obtain
\[
\theta(\vec{r}) = \begin{cases} [2V_0 \phi(\vec{r})]^{-1} & \text{if } \mu > V_{\text{ext}}(\vec{r}), \\ 0, & \text{otherwise.} \end{cases}
\]
(56)

and
\[
\hat{u} = \frac{c_+}{V_0}
\]
(57)

with
\[
V_0 = \int_{\mu - V_{\text{ext}}(\vec{r}) > 0} d\vec{r} \left[ \frac{4 \pi a_{ho}^3}{3} \left( \frac{15 a_2}{a_{ho}} \right)^{3/5} \right] N^{3/5}.
\]

All overlap integrals can be calculated analytically as
\[
O_{\phi\phi} = (4V_0)^{-1} = \frac{3}{16 \pi a_{ho}^3} \left( \frac{a_{ho}}{15 a_2} \right)^{3/5} N^{-3/5},
\]
\[
O_{\phi\theta} = \frac{32 \pi \mu \varphi / 2}{105 a_{ho}^4 N^2 \omega_{\phi ho}} \left( \frac{2}{M} \right)^{3/2}
= \frac{15^{2/5}}{14 \pi a_{ho}^3} \left( \frac{a_{ho}}{a_2} \right)^{3/5} N^{-3/5}.
\]

The parameter \( \varepsilon \) relates essentially to the kinetic energy operator \( E_k^{(\phi)} = -(\hbar^2 / 2M) \int d\vec{r} \varphi \nabla^2 \phi \) evaluated with respect to the phase function \( \theta(\vec{r}) \). In the TF approximation we find
\[
N \varepsilon = 2 E_k^{(\phi)}. \quad (58)
\]

Using Eqs. (55) and (55) we find
\[
(\mathcal{L} - \mu + 3c_+ N \phi^2) \theta = \frac{N \hat{u}}{\mu} (\mathcal{L} + c_+ N \phi^2) \phi,
\]
(59)

which simplifies to
\[
\left( \frac{-\hbar^2 \nabla^2}{2M} + 2c_+ N \phi^2 \right) \theta = \frac{N \hat{u}}{\mu} \left( \frac{-\hbar^2 \nabla^2}{2M} + \mu \right) \phi,
\]
(60)
in the TF limit. Multiplying this Eq. (60) from left by \( \theta \) or \( \phi \) respectively, and integrating over \( \vec{r} \), we obtain
\[
E_k^{(\phi)} + 2c_+ NO_{\phi\theta} = \frac{N \hat{u}}{\mu} \left( E_k^{(\phi)} + \mu \right),
\]
\[
E_k^{(\phi)} + 2c_+ N \int d\vec{r} \phi^2 \theta = \frac{N \hat{u}}{\mu} \left( E_k^{(\phi)} + \mu \right), \quad (61)
\]

where \( E_k^{(\phi)} = -(\hbar^2 / 2M) \int d\vec{r} \varphi \nabla^2 \phi \) is the kinetic energy per atom in the condensate, and \( E_k^{(\phi)} = -(\hbar^2 / 2M) \int d\vec{r} \varphi \nabla^2 \phi = -(\hbar^2 / 2M) \int d\vec{r} \varphi \nabla^2 \phi \). Eliminating \( E_k^{(\phi)} \) from above two equations, we then obtain
\[
E_k^{(\phi)} = \left( \frac{N \hat{u}}{\mu} \right)^2 E_k^{(\phi)} + \frac{N \hat{u}}{\mu} \left( \frac{N \hat{u}}{\mu} - 2c_+ N \omega_{\phi ho} \right) \int d\vec{r} \phi^2 \theta
\]
\[
= \left( \frac{N \hat{u}}{\mu} \right)^2 E_k^{(\phi)} + \frac{N \hat{u}}{\mu} \left( \frac{N \hat{u}}{\mu} - 2c_+ N \omega_{\phi ho} \right) \int d\vec{r} \phi^2 \theta
\]
\[
= \frac{4}{25} E_k^{(\phi)}. \quad (62)
\]

For a spherical trap, adopting the result of Ref. (20), we immediately find
\[
\varepsilon = \frac{4 \hbar^2}{5MN^2} \ln \left( \frac{R}{1.3a_{ho}} \right), \quad (63)
\]

with
\[
R = (15a_2 a_{ho}^4)^{1/5} N^{1/5},
\]

which seems to indicate that \( N \varepsilon \propto N^{-2/5} \), asymptotically goes to zero at large \( N \). By all measures, we find \( \varepsilon \) to be a very small quantity as compared to \( c_2 O_{\phi\theta} \). Taking \( \varepsilon \) as zero in the TF limit, we obtain
\[
n_0^2 \Delta Y_0 = \frac{1 \sqrt{7}}{3 \sqrt{10}},
\]
\[
\hbar \Omega = - \frac{\sqrt{3} \sqrt{14}}{15 \sqrt{10} \left( \frac{c_2}{\pi a_{ho}^2} \right)^{3/5} a_{ho}} \left( \frac{a_{ho}}{a_2} \right)^{3/5} N^{2/5}. \quad (64)
\]

To shed light on the above scaling results based on the TF approximation, we have performed extensive numerical calculations. As an example, we consider a spinor condensate of \(^{87}\)Rb atoms in a spherically symmetric trap with a harmonic frequency \( \omega = (2\pi)100 \) (Hz) and consider the regime of \( N \sim 10^4 \sim 2 \times 10^{-15} \). It turns out that our results indeed confirm the scaling relationship
\[
O_{\phi\phi}, O_{\phi\theta}, \hat{u}, \Omega / N \propto N^{-3/5}, \quad (65)
\]

and
\[
n_0^2 \Delta Y_0 \rightarrow \frac{1}{3 \sqrt{10}}, \quad (65)
\]
in the large \( N \) limit. We also find that
\[
E^{(\phi)}_k \propto N^{-0.3549},
\]
in rough agreement with the result of Ref. [20]. On the other hand, we find
\[
N \varepsilon \propto N^{0.1348}, \quad E^{(\theta)}_k \propto N^{0.1314},
\]
and
\[
N \varepsilon \approx 4 E^{(\theta)}_k,
\]
which do not agree with the results above as obtained with the TF approximation. A careful analysis as confirmed by our numerical results reveal that although
\[
\int d\bar{\mathbf{r}}(V_{\text{ext}} - \mu + c_+ N \phi^2) \phi^2
\]
is small, \( \int d\bar{\mathbf{r}}(V_{\text{ext}} - \mu + c_+ N \phi^2) \phi^2 \) is not, and approximately we find
\[
E^{(\theta)}_k \approx \frac{1}{2} \int d\bar{\mathbf{r}}(V_{\text{ext}} - \mu + c_+ N \phi^2) \phi^2.
\]
(66)

This leads to the opposite \( N \)-dependence for \( E^{(\theta)}_k \) and \( E^{(\phi)}_k \); while \( E^{(\phi)}_k \) increases with increasing values of \( N \), \( E^{(\theta)}_k \) decreases. Thus the identity Eq. (62) as obtained in the TF limit is invalid.

**IV. A NONZERO MAGNETIC FIELD**

Inside a nonzero homogeneous magnetic field (\( \bar{\mathbf{B}} \)), the atomic quantization axis becomes fixed, along the direction of \( \bar{\mathbf{B}} \), or more conveniently denoted as the \( z \)-axis. Mathematically, it turns out this is equivalent to the the requirement of the conservation of magnetization \( M = N_+ - N_- \). This reduces the SO(3) rotational symmetry to its subgroup SO(2), which is still a continuous symmetry. Thus the quantum dynamics of the phase remains formally the same as discussed before.

In the linear Zeeman regime, the original Hamiltonian \( H \) is augmented to
\[
H' = H - \hbar \omega_L \mathbf{F}_z.
\]
(67)
The Larmor precessing frequency is \( \omega_L = B \mu_B / \hbar \) with \( \mu_B \) the magnetic dipole moment for state \( |F = 1, M_F = 1 \rangle \). In this case, the only change is the phases of the spatial wave functions and the phase functions if of the following time-dependent form
\[
\alpha_j = \alpha_{j0} + j \omega_L t,
\]
for \( j = +, 0, \) and \( - \). \( \alpha_{+0} - \alpha_{-0} = \alpha_0 - \alpha_{-0} = 0 \) still holds true for a ferromagnetic spinor-1 condensate. A direct consequence of the magnetic field is thus a time-dependent relative phase
\[
\Theta(t) = \alpha_{+0} - \alpha_{-0} + \omega_L t = \alpha_0 - \alpha_{-0} + \omega_L t.
\]
(69)

When applied to the macroscopic condensate spin, we find the spin gains a precession around the \( z \)-axis.

Now the picture of the dynamics of the macroscopic condensate spin is clear: besides the precession induced by the external B-field, it also diffuses to restore the U(1) and SO(2) symmetries of the Hamiltonian.

The effect of quadratic Zeeman shift can also be simply addressed. Apart from an overall shift, it introduces a (positive) B-field dependent level shift \( \propto \hbar \delta B F^2_0 \) to states \(|\pm\rangle \) with respect to state \(|0\rangle \). This differential shift causes the precessing of the condensate spin to be twisted, i.e. the precessing of the \(|+\rangle \) (\(|-\rangle \)) component being slower (faster) by \( \delta B \).

**V. THE CASE OF ANTI-FERROMAGNETIC INTERACTIONS**

Our discussion of the phase diffusion in the previous section is based on the proof that the single spatial mode approximation is exact for ferromagnetic interactions. One might therefore conclude that a similar study for anti-ferromagnetic interactions would be difficult as, unlike for ferromagnetic interactions, it’s difficult to predict the ground state condensate structure. In reality, however, the quantum phase problem for anti-ferromagnetic interactions is much simpler, provided the mean field description for the ground state still applies.

First we note that when \( M \neq 0 \), the population in the 0-th component is zero in the ground state. Therefore, the zero mode Hamiltonian reduces to
\[
\frac{H_{\text{zero}}}{N} = -\frac{P_1^2}{J_1^2} \left( \frac{J_+}{2} \bar{u} + 2 c_- I_- \right) + \frac{4 c_+ P_1 P_2}{J_+ J_-} I_- + \frac{2}{J_+} I_+.
\]
(70)

which is exactly the same zero mode Hamiltonian for a binary condensate with no coupling between its two components. This has already been studied before [7]. We note that results obtained in Section II remain valid.

Now, let’s consider the \( M = 0 \) case. As was pointed out in Ref. [20], the single mode approximation again applies in this case, and we can assume
\[
\phi_j(\bar{r}) = \sqrt{n_j} \phi(\bar{r}) e^{i \alpha_j},
\]
where \( n_+ = n_- = (1 - n_0) / 2 \) and total energy of the system is independent of the value of \( n_0 \). A similar relation exists among the phases of the three components: \( \alpha_+ + \alpha_- - 2 \alpha_0 = \pi \). The spatial profile \( \phi(\bar{r}) \) satisfies following equation
\[
[\mathcal{L} + c_0 N \phi^2(\bar{r})] \phi(\bar{r}) = \mu \phi(\bar{r}),
\]
(72)
with a coefficient \( c_0 \) for the nonlinear term. The phase function then becomes
\[
\theta_j(\bar{r}) = \sqrt{n_j} \theta(\bar{r}) e^{i \alpha_j},
\]
(73)
where $\theta(\vec{r})$ is the solution of the following equation

$$[\mathcal{L} - \mu + 3c_0 N \phi^2(\vec{r})] \theta(\vec{r}) = N \ddot{\theta}(\vec{r}).$$ \hspace{2cm} (74)

Following the results of Sect. III, we define

$$\epsilon = \frac{2}{N} \int d\vec{r} \theta(\vec{r}) [\mathcal{L} - \mu + c_0 N \phi^2(\vec{r})] \theta(\vec{r}),$$ \hspace{2cm} (75)

and the zero mode Hamiltonian becomes

$$\frac{H_{\text{zero}}}{N} = \left[ \frac{\epsilon}{(1 - n_0) O_{\phi \theta}} + 2c_0 + \frac{2c_2}{1 - n_0} \right] p^2 O_{\phi \theta} + \left( \frac{\epsilon}{2n_0 O_{\phi \theta}} + 2c_0 \right) p^2 O_{\phi \theta} + \left[ \frac{\epsilon}{(1 - n_0) O_{\phi \theta}} + 2c_0 + \frac{2c_2}{1 - n_0} \right] p^2 O_{\phi \theta} + 4c_0 p_+^2 O_{\phi \theta} + 4c_0 p_-^2 O_{\phi \theta} + 4 \left( c_0 + \frac{2n_0}{1 - n_0} \right) p_+^2 O_{\phi \theta} + \frac{c_2 n_0 (1 - n_0) O_{\phi \theta}}{2\hbar^2} (q_+ - 2q_0')^2 \right] = p^T A p' + q^T B q', \hspace{2cm} (76)

the positive definiteness of matrices $A$ and $B$ can again be verified. By applying the same transformation \( \mathcal{T}(t) \), we obtain

$$\frac{H_{\text{zero}}}{N} = a p_N^2 + b p_M^2 + c p_Y^2 + \alpha p_N p_Y + \eta q_Y^2, \hspace{2cm} (77)$$

where all coefficients are now given in Appendix G. We see that fluctuations in $N$ are completely decoupled from that of $M$ and $Y$ in this case, a result that also happens for ferromagnetic interactions when $M = 0$. The ground state of (76) is similar to Eq. (20) except now that

$$\Omega = 2N \sqrt{c} = \frac{2N}{\hbar} \left[ c_2 O_{\phi \theta} (\epsilon + 4c_2 n_0^2 O_{\phi \theta}) \right]^{1/2},$$

$$\Delta_{Y_0}^2 = \frac{1}{3} \frac{c}{\eta} = \frac{1}{3} \left[ \epsilon + 4c_2 n_0^2 O_{\phi \theta} \right]^{1/2}.$$

The dynamic equations for $p_i$ and $q_i$ then become

$$\dot{p}_N = 0, \hspace{2cm} \dot{p}_M = 0, \hspace{2cm} \dot{p}_Y = -2N \eta q_Y, \hspace{2cm} \dot{q}_N = N (2a p_N + \alpha p_Y), \hspace{2cm} \dot{q}_M = 2N b p_M, \hspace{2cm} \dot{q}_Y = N (a p_N + 2c p_Y). \hspace{2cm} (79)$$

One can easily find the evolution matrix to be

$$\mathcal{T}(t) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-\alpha' (1 - \cos \Omega t) & 0 & \cos \Omega t & 0 & 0 & -\sin \Omega t / \Delta_{Y_0}^2 \sin \Omega t \\
\zeta_0 N t + \hbar \Delta_{Y_0}^2 \alpha' \sin \Omega t & 0 & 0 & 0 & 0 & 0 \\
\hbar \Delta_{Y_0}^2 \alpha' \sin \Omega t & 0 & 0 & 0 & 0 & 0 \\
\hbar \Delta_{Y_0}^2 \alpha' \sin \Omega t & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \hspace{2cm} (80)$$

where

$$\alpha' = \frac{\alpha}{2c} = \frac{(3n_0 - 1) \epsilon + 8c_2 n_0^2 O_{\phi \theta}}{\sqrt{2} (\epsilon + 4c_2 n_0^2 O_{\phi \theta})},$$

$$\zeta_0 = \frac{4ac - \alpha^2}{2c} = 12c_0 O_{\phi \theta} + \frac{3\epsilon (\epsilon + 4c_2 n_0^2 O_{\phi \theta})}{\epsilon + 4c_2 n_0^2 O_{\phi \theta}}.$$

As for $\delta f(t)$ we find

$$\delta f_x = \cos \Theta \sqrt{\frac{2n_0}{1 - n_0} p_M + \sin \Theta \frac{6n_0 (1 - n_0)}{h} q_Y},$$

$$\delta f_y = -\sin \Theta \sqrt{\frac{2n_0}{1 - n_0} p_M + \cos \Theta \frac{6n_0 (1 - n_0)}{h} q_Y},$$

$$\delta f_z = \sqrt{2} p_M. \hspace{2cm} (81)$$

VI. CONCLUSIONS

In conclusion, we have investigated in detail quantum phase diffusions of a spinor-1 condensate. When the elastic atomic interaction is of ferromagnetic type, the structure of the ground state is greatly simplified: the spatial mode functions of the different spin components are exactly the same and accordingly we also proved that the phase functions are exactly the same. This simplification allows us to construct the zero mode Hamiltonian which describes the number and phase fluctuations of a condensate while maintaining the conservations of the total number of atoms and the total magnetization. We have provided analytical results for the number and phase fluctuations due to both the quantum phase diffusion dynamics and the initial distribution of atom number and phase fluctuations. The structure of these fluctuations is rather simple: along with the oscillations due to $\sin \Omega t$
and $\cos\Omega t$ terms, quantum phase diffusion terms proportional to $Nt$ also exist. This suggests that the mean-field approach to spinor-1 condensates is only valid for a finite duration. Based on these results, we have studied the diffusion of the direction of a macroscopic condensate spin. Our investigation sheds important light on the studies of beyond mean field theory quantum correlations among Bose condensed atoms.

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APPENDIX A: OVERLAP INTEGRALS

In this appendix, we list the overlap integrals as involved in the dynamic Eq. [12]. Using the phase conventions as introduced in Eqs. [13] and [19], we find explicitly

\[ J_j = n_j, \]
\[ I_{jk} = n_j n_k \mathcal{O}_{\phi\theta}, \]
\[ (\phi_0 \theta_0 | \phi_\theta) = (\phi_0 \theta_0 | \phi_\theta) \]
\[ = (\phi_0^2 | \theta_\theta) \]
\[ = n_0^2 \mathcal{O}_{\phi\theta}/2 \]
\[ = 2n_0 n_0 \mathcal{O}_{\phi\theta}, \]
\[ (\phi_0^2 | \phi_\theta) = n_0^2 \mathcal{O}_{\phi\theta}/2. \]  

APPENDIX B: THE ZERO MODE HAMILTONIAN

The zero mode Hamiltonian is listed below.

\[ \frac{H_{zero}}{N} = -\frac{P^2}{J_+} \left[ \frac{J_+}{4} \tilde{u} + 2c - I_{++} + 2c_+ I_{++} + 2c_2 (\phi_0 \theta_0 | \phi_\theta) + c_2 (\phi_0^2 | \theta_\theta) \right] \]
\[ - \frac{P^2}{J_0} \left[ \frac{J_0}{4} \tilde{u} + c_+ I_{++} + c_+ I_{+0} + c_2 (\phi_\theta | \phi_0 \theta_0) + c_2 (\phi_\theta | \phi_0 \theta_0) \right] \]
\[ - \frac{P^2}{J_-} \left[ \frac{J_-}{4} \tilde{u} + 2c - I_{++} + 2c_+ I_{++} + 2c_2 (\phi_0 \theta_0 | \phi_\theta) + c_2 (\phi_0^2 | \theta_\theta) \right] \]
\[ + \frac{4P_+ P_0}{J_+ J_0} [c_+ I_{++} + c_2 (\phi_0 \theta_0 | \phi_\theta)] + \frac{4P_+ P_0}{J_+ J_0} [c_+ I_{++} + c_2 (\phi_0 \theta_0 | \phi_\theta)] \]
\[ + \frac{2P_+ P_0}{J_+ J_-} [2c - I_{++} + c_2 (\phi_0^2 | \theta_\theta)] - \frac{c_2}{\hbar^2} (\phi_0^2 | \phi_\theta) \left( \frac{Q_+}{J_+} - \frac{2Q_0}{J_0} + \frac{Q_-}{J_-} \right)^2. \]  

APPENDIX C: COEFFICIENTS IN THE ZERO MODE HAMILTONIAN FOR FERROMAGNETIC INTERACTIONS

The coefficients in the zero mode Hamiltonian [21] are

\[ a = \frac{1}{4n_0^2} \left[ \frac{\varepsilon}{3} (5 + 3m^2) \right. \]
\[ + 2\mathcal{O}_{\phi\theta} \left( 3c_0 (1 - m^2)^2 + \frac{8}{3} c_2 (1 - m^2)^2 \right) \left. \right], \]
\[ b = \frac{1}{2n_0^2} \left[ \varepsilon (1 + m^2) - 8c_2 \mathcal{O}_{\phi\theta} m^2 \right] \]  

\[ c = \frac{1}{3n_0^2} (\varepsilon - 4c_2 \mathcal{O}_{\phi\theta}), \]
\[ \alpha = \frac{\sqrt{2}}{6n_0^2} (1 + 3m^2)(\varepsilon - 4c_2 \mathcal{O}_{\phi\theta}), \]
\[ \beta = -\frac{2m}{\sqrt{3}n_0^2} (\varepsilon - 4c_2 \mathcal{O}_{\phi\theta}), \]
\[ \gamma = -\frac{2m}{n_0^2} \sqrt{\frac{2}{3}} \left[ \varepsilon - c_2 \mathcal{O}_{\phi\theta} (1 + 3m^2) \right], \]
\[ \eta = -3c_2 n_0^2 \mathcal{O}_{\phi\theta}/\hbar^2. \]
APPENDIX D: PARAMETERS IN EVOLUTION MATRIX

The list of parameters in the evolution matrix $T(t)$ are

$$
\alpha' = \frac{\alpha}{2c} = \frac{1 + 3m^2}{2\sqrt{2}},
$$

$$
\beta' = \frac{\beta}{2c} = -\sqrt{3}m,
$$

$$
\zeta_a = \frac{4ac - \alpha^2}{2c} = \frac{3(1 + m^2)\varepsilon}{1 - m^2},
$$

$$
\zeta_b = \frac{4bc - \beta^2}{2c} = \frac{4\varepsilon}{1 - m^2},
$$

$$
\zeta_c = \frac{2c\gamma - \alpha\beta}{2c} = -\frac{2\sqrt{6}m\varepsilon}{1 - m^2}.
$$

(D1)

---

APPENDIX E: EXPLICIT FORM OF NUMBER AND PHASE FLUCTUATION

Here we present the explicit form of the solutions for the number and phase fluctuations.
sate spin direction is given by

\[ \mathcal{M} = 0 \]

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\[ M = 0 \]

APPENDIX G: COEFFICIENTS IN THE ZERO MODE HAMILTONIAN FOR ANTI-FERROMAGNETIC INTERACTIONS

The coefficients of zero mode Hamiltonian (E1) for anti-ferromagnetic interactions (with \( \mathcal{M} = 0 \))

\[ a = \frac{(1 + 3n_0)\varepsilon}{6n_0(1 - n_0)} + \left[ 6c_0 - \frac{8c_2n_0}{3(1 - n_0)} \right] \mathcal{O}_{\phi\phi}, \]

\[ b = \frac{\varepsilon}{1 - n_0} + 4c_2\mathcal{O}_{\phi\phi}, \]

\[ c = \frac{\varepsilon}{3n_0(1 - n_0)} + \frac{4c_2n_0}{3(1 - n_0)} \mathcal{O}_{\phi\phi}, \]

\[ \alpha = \frac{2(3n_0 - 1)\varepsilon}{3\sqrt{2}n_0(1 - n_0)} + \frac{16c_2n_0}{3\sqrt{2}(1 - n_0)} \mathcal{O}_{\phi\phi}, \]

\[ \eta = \frac{3c_2n_0(1 - n_0)\mathcal{O}_{\phi\phi}/h^2}{\mathcal{O}_{\phi\phi}/h^2}. \]

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