Abstract

We study the problem of minimizing the number of critical simplices from the point of view of inapproximability and parameterized complexity. We first show inapproximability of Min-Morse Matching within a factor of $2^{\log(1-\epsilon)n}$. Our second result shows that Min-Morse Matching is $\text{W}[P]$-hard with respect to the standard parameter. Next, we show that Min-Morse Matching with standard parameterization has no FPT approximation algorithm for any approximation factor $\rho$. The above hardness results are applicable to complexes of dimension $\geq 2$.

On the positive side, we provide a factor $O(n \log n)$ approximation algorithm for Min-Morse Matching on 2-complexes, noting that no such algorithm is known for higher dimensional complexes. Finally, we devise discrete gradients with very few critical simplices for typical instances drawn from a fairly wide range of parameter values of the Costa–Farber model of random complexes.

1 Introduction

Classical Morse theory [56] is an analytical tool for studying topology of smooth manifolds. Forman’s discrete Morse theory is a combinatorial analogue of Morse theory that is applicable to simplicial complexes, and more generally regular cell complexes [29]. In Forman’s theory, discrete Morse functions play the role of smooth Morse functions, whereas discrete gradient vector fields are the analogues of gradient-like vector fields. The principal objects of study are, therefore, the so-called discrete gradient vector fields (or discrete gradients) on simplicial complexes. Discrete gradients are partial face-coface matchings that satisfy certain acyclicity conditions. Forman’s theory also has an elegant graph theoretic formulation [14], in which the acyclic matchings (or Morse matchings) in the Hasse diagram of a simplicial complex are in one-to-one correspondence with the discrete gradients on the simplicial complex. For this reason, we use the terms gradient vector fields and Morse matchings interchangeably.

Discrete Morse theory has become a popular tool in computational topology, image processing and visualization [6,8,13,36,43,58,62,68], and is actively studied in algebraic and topological combinatorics [37,44,45,55,66]. Over the period of last decade, it has emerged a powerful computational tool for several problems in topological data analysis [21,35,42]. Because of the wide array of applications there is a lot of practical interest in computing gradient vector fields on simplicial complexes with a (near-)optimal number of critical simplices [2,7,11,12,32,33,47]. The idea of using discrete Morse theory to speed up the computation of (co)homology [20,33,46], persistent homology [7,57], zigzag persistence [26,52], and multiparameter persistent homology [65] relies on the fact that discrete Morse theory can be employed to reduce the problem of computing the homology of an input simplicial complex to that of a much smaller chain complex.

The effectiveness of certain heuristics for Morse matching raises an important question: to what extent is it feasible to obtain near-optimal solutions for Morse matching in polynomial time? To this end, inapproximability results for Min-Morse Matching for simplicial complexes of dimension $d \geq 3$ were established in [9]. To this date, however, we are unaware of any hardness results for Min-Morse Matching on 2-complexes from the perspective of inapproximability or parameterized complexity (although the related Erasability problem was shown to be $\text{W}[P]$-hard by Burton et al. [12]). With this paper, we seek to close the knowledge gap. By establishing various hardness results, we demonstrate the limitations of polynomial time methods for computing near-optimal Morse matchings. On the other hand, by devising an approximation algorithm, we make it evident that

Parameterized inapproximability of Morse matching

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Min-Morse Matching on 2-complexes is not entirely inapproximable. We also observe that the typical Morse matching instances drawn from a wide range of parameter values of the Costa–Farber complexes are a lot easier in contrast to the discouraging worst case inapproximability bounds.

1.1 Related work

Joswig and Pfetsch [39] showed that finding an optimal gradient vector field is an NP-hard problem based on the relationship between erasability and Morse Matching observed by Lewiner [48, 49]. The erasability problem was first studied by Eğecioğlu and Gonzalez [24]. Joswig and Pfetsch also posed the question of approximability of optimal Morse matching as an open problem. On the positive side, Rathod et al. [60] devised the first approximation algorithms for Max-Morse Matching on simplicial complexes that provide constant factor approximation bounds for fixed dimension. Complementing these results, Bauer and Rathod [9] showed that for simplicial complexes of dimension \( d \geq 3 \) with \( n \) simplices, it is NP-hard to approximate Min-Morse Matching within a factor of \( O(n^{1-\epsilon}) \), for any \( \epsilon > 0 \). However, the question of approximability of Min-Morse Matching for 2-complexes is left unanswered in [9].

Next, Burton et al. [12] showed that the Erasability problem (that is, finding the number of 2-simplices that need to be removed to make a 2-complex erasable) is \( W[P] \)-complete. We note the \( W[P] \)-hardness of erasability can be inferred from our methods as well, and therefore our result can be seen as a strengthening of the hardness result from [12]. Moreover, our parameterized inapproximability results rely on the machinery developed by Eickmeyer et al. [25] and Marx [53].

Our reduction techniques have a flavor that is similar to the techniques used by Malgouryes and Francos [51] and Tancer [69] for proving NP-hardness of certain collapsibility problems. In particular, Tancer [69] also describes a procedure for “filling 1-cycles with disks” to make the complex contractible (and even collapsible for satisfiable inputs). Our technique of filling 1-cycles is however entirely different (and arguably simpler) than Tancer’s procedure. Our work is also related to [9] in that we use the so-called modified dunce hats for constructing the gadget used in the reduction. Recently, modified dunce hats were used to provide a simpler proof of NP-completeness of the shellability decision problem [64].

1.2 The Morse Matching Problems

The Max-Morse Matching problem (MaxMM) can be described as follows: Given a simplicial complex \( K \), compute a gradient vector field that maximizes the cardinality of matched (regular) simplices, over all possible gradient vectors fields on \( K \). Equivalently, the goal is to maximize the number of gradient pairs. For the complementary problem Min-Morse Matching (MinMM), the goal is to compute a gradient vector field that minimizes the number of unmatched (critical) simplices, over all possible gradient vector fields on \( K \). While the problem of finding the exact optimum are equivalent for MinMM and MaxMM, the approximation variants behave quite differently.

Additionally, we define another variant of the minimization problem for 2-dimensional complexes, namely Min-Reduced Morse Matching (MinrMM). For this problem, we seek to minimize the total number of critical simplices minus one. This variant is natural, since any discrete gradient necessarily has at least one critical 0-simplex. It corresponds to a variant definition of simplicial complexes commonly used in combinatorics, which also consider the empty set as a simplex of dimension −1.

1.3 Our contributions

| Approx. algorithm | Bauer, R. (2019) | This paper |
|-------------------|-----------------|------------|
| Min-Morse Matching (dim. = 2) | – | \( O(\frac{n}{\log n}) \) |

| Inapproximability | Bauer, R. (2019) | This paper |
|-------------------|-----------------|------------|
| Max-Morse Matching (dim. \( \geq 2 \)) | \( (1 - \frac{1}{2n}) + \epsilon \) | – |
| Min-Morse Matching (dim. \( \geq 3 \)) | \( O(n^{1-\epsilon}) \) | – |
| Min-Morse Matching (dim. \( = 2 \)) | – | \( 2^{\log^{(1-\epsilon)} n} \) |

Table 1: (In)approximability of Morse matching
This paper called its example, the vertex sets of the simplices in a geometric complex form an abstract simplicial complex, a topological space. A complex $K$ spans a simplex, a facet of $K$, of dimension $d$. We say that $K$ is a simplicial complex if it is a collection of simplices that satisfies the following conditions:

- any face of a simplex in $K$ also belongs to $K$, and
- the intersection of two simplices $\sigma_1, \sigma_2 \in K$ is either empty or a face of both $\sigma_1$ and $\sigma_2$.

For a complex $K$, we denote the set of $d$-simplices of $K$ by $K^{(d)}$. The $n$-skeleton of a simplicial complex $K$ is the simplicial complex $\bigcup_{m=0}^{n} K^{(m)}$. A simplex $\sigma$ is said to be a maximal face of a simplicial complex $K$ if it is not a strict subset of any other simplex $\tau \in K$. The underlying space of $K$ is the union of its simplices, denoted by $|K|$. The underlying space is implicitly used whenever we refer to $K$ as a topological space.

An abstract simplicial complex $\mathcal{S}$ is a collection of finite nonempty sets $A \in \mathcal{S}$ such that every nonempty subset of $A$ is also contained in $\mathcal{S}$. The sets in $\mathcal{S}$ are called its simplices. A subcomplex of $K$ is an abstract simplicial complex $L$ such that every face of $L$ belongs to $K$; denoted as $L \subseteq K$. For example, the vertex sets of the simplices in a geometric complex form an abstract simplicial complex, called its vertex scheme. Given an abstract simplicial complex $K$ with $n$ simplices, we can associate a

| $W[P]$-hardness of $\text{er}(K)$ (SP) | Burton et al. (2013) | This paper |
|--------------------------------|----------------|---------|
| $W[P]$-hardness of $\text{Min-Morse Matching}$ (SP) | -  | ✓ |
| FPT-inapproximability of $\text{Min-Morse Matching}$ (SP) | -  | ✓ |
| FPT-algorithm for $\text{Min-Morse Matching}$ (TW) | ✓  | - |

Table 2: Parameterized complexity of Morse matching. In the above table, SP denotes standard parameterization, whereas TW denotes treewidth parameterization.

In Section 5 we establish several hardness results for $\text{MinrMM}$, using a reduction from $\text{MinMCS}$. In particular, we show the following:

- $\text{MinMM}$ has no approximation within a factor of $2\log^{(1-\epsilon)} n$, for any $\epsilon > 0$, unless $\text{NP} \subseteq \text{QP}$ (throughout this paper, log denotes the logarithm by base 2),
- the standard parameterization of $\text{MinMM}$ is $W[P]$-hard, and
- $\text{MinMM}$ with standard parameterization has no FPT approximation algorithm for any approximation ratio function $\rho$, unless $\text{FPT} = W[P]$.

In Section 6 we first show that the $W[P]$-hardness result and FPT-inapproximability results easily carry over from $\text{MinMM}$ to $\text{MinMM}$. To the best of our knowledge, this constitutes the first FPT-inapproximability result in computational topology. Using the amplified complex construction introduced in [9], we observe that the inapproximability result also carries over from $\text{MinMM}$ to $\text{MinMM}$. In particular, we show that even for 2-complexes $\text{MinMM}$ cannot be approximated within a factor of $2\log^{(1-\epsilon)} n$, for any $\epsilon > 0$, unless $\text{NP} \subseteq \text{QP}$, where $n$ denotes the number of simplices in the complex.

Sections 7 and 8 are concerned with some positive results. First, in Section 7 we design an $O(n \log n)$-factor algorithm for $\text{MinMM}$ on 2-complexes. Then, in Section 8 we make the observation that Kahle’s techniques [10] for designing discrete gradients on random clique complexes generalize to Costa–Farber random complexes. Specifically, we show that for a wide range of parameter values, there exist discrete gradients for which the ratio of expected number of critical $r$-simplices to the expected number of $r$-simplices (for any fixed dimension $r$) tends to zero. Although these methods do not lead to approximation algorithms, they fall under the general paradigm of beyond worst-case analysis [63].

Note that we do not distinguish between abstract and geometric simplicial complexes since every abstract simplicial complex can be embedded in a Euclidean space of appropriate dimension. As a final remark, we believe that with this paper we tie all the loose ends regarding complexity questions in discrete Morse theory.

2 Topological preliminaries

2.1 Simplicial complexes

A $k$-simplex $\sigma = \text{conv} V$ is the convex hull of a set $V$ of $(k+1)$ affinely independent points in $\mathbb{R}^d$. We call $k$ the dimension of $\sigma$. We say that $\sigma$ is spanned by the points $V$. Any nonempty subset of $V$ also spans a simplex, a face of $\sigma$. A simplex $\sigma$ is said to be a coface of a simplex $\tau$ if and only if $\tau$ is face of $\sigma$. We say that $\sigma$ is a facet of $\tau$ and $\tau$ is a cofacet of $\sigma$ if $\sigma$ is a face of $\tau$ with $\dim \sigma = \dim \tau - 1$. A simplicial complex $K$ is a collection of simplices that satisfies the following conditions:

- any face of a simplex in $K$ also belongs to $K$, and
- the intersection of two simplices $\sigma_1, \sigma_2 \in K$ is either empty or a face of both $\sigma_1$ and $\sigma_2$.

For a complex $K$, we denote the set of $d$-simplices of $K$ by $K^{(d)}$. The $n$-skeleton of a simplicial complex $K$ is the simplicial complex $\bigcup_{m=0}^{n} K^{(m)}$. A simplex $\sigma$ is called a maximal face of a simplicial complex $K$ if it is not a strict subset of any other simplex $\tau \in K$. The underlying space of $K$ is the union of its simplices, denoted by $|K|$. The underlying space is implicitly used whenever we refer to $K$ as a topological space.

An abstract simplicial complex $\mathcal{S}$ is a collection of finite nonempty sets $A \in \mathcal{S}$ such that every nonempty subset of $A$ is also contained in $\mathcal{S}$. The sets in $\mathcal{S}$ are called its simplices. A subcomplex of $K$ is an abstract simplicial complex $L$ such that every face of $L$ belongs to $K$; denoted as $L \subseteq K$. For example, the vertex sets of the simplices in a geometric complex form an abstract simplicial complex, called its vertex scheme. Given an abstract simplicial complex $K$ with $n$ simplices, we can associate a
pointed simplicial complex to it by choosing an arbitrary vertex and regarding it as the distinguished basepoint of $K$. The $m$th wedge sum of $K$ is then the quotient space of a disjoint union of $m$ copies of $K$ with the distinguished basepoints of each of the copies of $K$ identified.

### 2.2 Discrete Morse theory and Erasability

We assume that the reader is familiar with simplicial complexes. Section 2.1 summarizes the key definitions. In this section, we provide a brief description of Forman’s discrete Morse theory on simplicial complexes. For a comprehensive expository introduction, we refer the reader to [30].

A real-valued function $f$ on a simplicial complex $K$ is called a discrete Morse function if

- $f$ is monotonic, i.e., $\sigma \subseteq \tau$ implies $f(\sigma) \leq f(\tau)$, and
- for all $t \in \text{im}(f)$, the preimage $f^{-1}(t)$ is either a singleton $\{\sigma\}$ (in which case $\sigma$ is a critical simplex) or a pair $\{\sigma, \tau\}$, where $\sigma$ is a facet of $\tau$ (in which case $(\sigma, \tau)$ form a gradient pair and $\sigma$ and $\tau$ are regular simplices).

Given a discrete Morse function $f$ defined on complex $K$, the discrete gradient vector field $V$ of $f$ is the collection of pairs of simplices $(\sigma, \tau)$, where $(\sigma, \tau)$ is in $V$ if and only if $\sigma$ is a facet of $\tau$ and $f(\sigma) = f(\tau)$.

Discrete gradient vector fields have a useful interpretation in terms of acyclic graphs obtained from matchings on Hasse diagrams, due to Chart [14]. Let $K$ be a simplicial complex, let $H_K$ be its Hasse diagram, and let $M$ be a matching in the underlying undirected graph $H_K$. Let $H_K(M)$ be the directed graph obtained from $H_K$ by reversing the direction of each edge of the matching $M$. Then $M$ is a Morse matching if and only if $H_K(M)$ is a directed acyclic graph. Every Morse matching $M$ on the Hasse diagram $H_K$ corresponds to a unique gradient vector field $V_M$ on complex $K$ and vice versa. For a Morse matching $M$, the unmatched vertices correspond to critical simplices of $V_M$, and the matched vertices correspond to the regular simplices of $V_M$.

A non-maximal face $\sigma \in K$ is said to be a free face if it is contained in a unique maximal simplex $\tau \in K$. If $d = \dim(\tau) = \dim(\sigma) + 1$, we say that $K' = K \setminus \{\sigma, \tau\}$ arises from $K$ by an elementary collapse, or an elementary $d$-collapse denoted by $K \prec_d K'$. Furthermore, we say that $K$ collapses to $L$, denoted by $K \prec L$, if there exists a sequence $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \prec_{d_i} K_{i+1}$ for all $i$. If $K$ collapses to a point, one says that $K$ is collapsible.

A simplicial collapse can be encoded by a discrete gradient.

**Theorem 2.1** (Forman [29], Theorem 3.3). Let $K$ be a simplicial complex with a vector field $V$, and let $L \subseteq K$ be a subcomplex. If $K \setminus L$ is a union of pairs in $V$, then $K \prec L$.

In this case, we say that the collapse $K \prec L$ is induced by the gradient $V$. As a consequence of this theorem, we obtain:

**Theorem 2.2** (Forman [29], Corollary 3.5). Let $K$ be a simplicial complex with a discrete gradient vector field $V$ and let $n_d$ denote the number of critical simplices of $V$ of dimension $d$. Then $K$ is homotopy equivalent to a CW complex with exactly $n_d$ cells of dimension $d$.

In particular, a discrete gradient vector field on $K$ with $n_d$ critical simplices of dimension $d$ gives rise to a chain complex having dimension $n_d$ in each degree $d$, whose homology is isomorphic to that of $K$. This condensed representation motivates the algorithmic search for (near-)optimal Morse matchings.

Following the terminology used in [9,21], we make the following definitions: A maximal face $\tau$ in a simplicial complex $K$ is called an internal simplex if it has no free face. If a 2-complex $K$ collapses to a 1-complex, we say that $K$ is erasable. Moreover, for a 2-complex $K$, the quantity $\text{er}(K)$ is the minimum number of internal 2-simplices that need to be removed so that the resulting complex collapses to a 1-complex. Equivalently, it is the minimum number of critical 2-simplices of any discrete gradient on $K$. Furthermore, we say that a subcomplex $L \subseteq K$ is an erasable subcomplex of $K$ (through the gradient $V$) if there exists another subcomplex $M \subseteq K$ with $K \setminus M$ (induced by the gradient $V$) such that the set of 2-dimensional simplices of these complexes satisfy the following relation: $L(2) \subseteq K(2) \setminus M(2)$. We call such a gradient $V$ an erasing gradient. Finally, we say that a simplex $\sigma$ in a complex $K$ is eventually free (through the gradient $V$) if there exists a subcomplex $L$ of $K$ such that $K \prec L$ (induced by $V$) and $\sigma$ is free in $L$. Equivalently, $K$ collapses further to a subcomplex not containing $\sigma$.

We recall the following results from [9].
Lemma 2.3 (Bauer, Rathod [9], Lemma 2.1). Let $K$ be a connected simplicial complex, let $p$ be a vertex of $K$, and let $\nabla_1$ be a discrete gradient on $K$ with $m_0 > 1$ critical simplices of dimension 0 and $m$ critical simplices in total. Then there exists a polynomial time algorithm to compute another gradient vector field $\bar{\nabla}$ on $K$ with $p$ as the only critical simplex of dimension 0 and $m - 2(m_0 - 1)$ critical simplices in total.

Lemma 2.4 (Bauer, Rathod [9], Lemma 2.3). If $K$ is an erasable complex, then any subcomplex $L \subset K$ is also erasable.

Lemma 2.5. Suppose that we are given a complex $K$ and a set $M$ of simplices in $K$ with the property that simplices in $M$ have no cofaces in $K$. Then $L = K \setminus M$ is a subcomplex of $K$ with the property that the gradient vector fields on $K$ with all simplices in $M$ critical are in one-to-one correspondence with gradient vector fields on $L$.

Proof. Given a gradient vector field on $L$, we extend it to a gradient vector field on $K$ by making all simplices in $M$ critical. Given a vector field $\nabla$ on $K$, the restriction $\nabla|_L$ is a gradient vector field on $L$.

Notation 1. For the remainder of the paper, we use $[m]$ to denote the set $\{1, 2, \ldots, m\}$ for any $m \in \mathbb{N}$, and $[i, j]$ to denote the set $\{i, i+1, \ldots, j\}$ for any $i, j \in \mathbb{N}$.

3 Algorithmic preliminaries

3.1 Approximation algorithms

An $\alpha$-approximation algorithm for an optimization problem is a polynomial-time algorithm that, for all instances of the problem, produces a solution whose objective value is within a factor $\alpha$ of the objective value of an optimal solution. The factor $\alpha$ is called the approximation ratio (or approximation factor) of the algorithm.

An approximation preserving reduction is a polynomial time procedure for transforming an optimization problem $A$ to an optimization problem $B$, such that an $\alpha$-approximation algorithm for $B$ implies an $f(\alpha)$-approximation algorithm for $A$, for some function $f$. Then, if $A$ is hard to approximate within factor $f(\alpha)$, the reduction implies that $B$ is hard to approximate within factor $\alpha$. A particularly well-studied class of approximation preserving reductions is given by the L-reductions, which provide an effective tool in proving hardness of approximability results [57][72].

Now, consider a minimization problem $A$ with a non-negative integer valued objective function $m_A$. Given an instance $x$ of $A$, the goal is to find a solution $y$ minimizing the objective function $m_A(x, y)$. Define $\text{OPT}_A(x)$ as the minimum value of the objective function on input $x$. An L-reduction (with parameters $\mu$ and $\nu$) from a minimization problem $A$ to another minimization problem $B$ is a pair of polynomial time computable functions $f$ and $g$, and fixed constants $\mu, \nu > 0$, satisfying the following conditions:

1. The function $f$ maps instances of $A$ to instances of $B$.
2. For any instance $x$ of $A$, we have 
   $$\text{OPT}_B(f(x)) \leq \mu \text{OPT}_A(x).$$
3. The function $g$ maps an instance $x$ of $A$ and a solution of the corresponding instance $f(x)$ of $B$ to a solution of $x$.
4. For any instance $x$ of $A$, and any solution $y$ of $f(x)$, we have 
   $$m_A(x, g(x, y)) - \text{OPT}_A(x) \leq \nu (m_B(f(x), y) - \text{OPT}_B(f(x))).$$

If $\mu = \nu = 1$, the reduction is strict.

We will use the following straightforward fact about L-reductions, which appears as Theorem 16.6 in a book by Williamson and Shnoys [72].

Theorem 3.1. If there is an L-reduction with parameters $\mu$ and $\nu$ from a minimization problem $A$ to another minimization problem $B$, and there is a $(1 + \delta)$-approximation algorithm for $B$, then there is a $(1 + \mu \nu \delta)$-approximation algorithm for $A$. 


3.2 Parameterized complexity

Parameterized complexity, as introduced by Downey and Fellows in [22], is a refinement of classical complexity theory. The theory revolves around the general idea of developing complexity bounds for instances of a problem not just based on their size, but also involving an additional parameter, which might be significantly smaller than the size. Specifically, we have the following definition.

Definition 3.1 (Parameter, parameterized problem [28]). Let $\Sigma$ be a finite alphabet.

1. A parameter of $\Sigma^*$, the set of strings over $\Sigma$, is a function $\rho : \Sigma^* \rightarrow \mathbb{N}$, attaching to every input $w \in \Sigma^*$ a natural number $\rho(w)$.
2. A Parameterized problem over $\Sigma$ is a pair $(P, \rho)$ consisting of a set $P \subseteq \Sigma^*$ and a (polynomial time computable) parametrization $\rho : \Sigma^* \rightarrow \mathbb{N}$.
3. A parameterized problem $(P, \rho)$ is said to be fixed-parameter tractable or FPT in the parameter $\rho$ if the question
   $$(x, p) \in \{(y, \rho(y)) \mid y \in P\}$$
   can be decided in running time $O(g(\rho) \cdot |x|^{O(1)})$, where $g : \mathbb{N} \rightarrow \mathbb{N}$ is an arbitrary computable function depending only on the parameter $p$.

FPT reductions provide a principal tool to establish hardness results in the parameterized complexity landscape.

Definition 3.2 (FPT reduction [28]). Given two parameterized problems $(P, k)$ and $(Q, k')$, we say that there is an FPT reduction from $(P, k)$ to $(Q, k')$, if there exists a functions $\varphi$ that transforms parameterized instances of $P$ to parameterized instances of $Q$ while satisfying the following properties:

1. $\varphi$ is computable by an FPT algorithm,
2. $\varphi(x)$ is a yes-instance of $(Q, k')$ if and only if $x$ is a yes-instance of $(P, k)$.
3. There exists a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $k'(\varphi(x)) \leq g(k(x))$.

The natural way of turning a minimization problem into a decision problem is to add a value $k$ to the input instance, and seek a solution with cost at most $k$. Taking this value $k$ appearing in the input as the parameter is called the standard parameterization of the minimization problem (sometimes also referred to as the natural parameterization). In general, the parameter can be any function of the input instance, for example, the treewidth of the input graph, or the maximum degree of the input graph.

Parameterized approximability is an extension of the notion of classical approximability. In formally, an FPT approximation algorithm is an algorithm whose running time is fixed parameter tractable for the parameter cost of the solution and whose approximation factor $\rho$ is a function of the parameter (and independent of the input size). For instance, every polynomial time approximation algorithm with constant approximation factor is automatically an FPT approximation algorithm, but an approximation algorithm with approximation factor $\Theta(\sqrt{n})$, where $n$ denotes the input size, is not an FPT approximation algorithm. Next, following [53], for standard parameterization of minimization problems, we provide definitions for FPT approximation algorithms and FPT cost approximation algorithms. Analogous definitions for maximization problems are also considered in [53].

Definition 3.3 (FPT approximation algorithm [53]). Let $P$ be an NP minimization problem, and let $\rho : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ be a computable function such that $k \mapsto k \cdot \rho(k)$ is nondecreasing. An FPT approximation algorithm for $P$ (over some alphabet $\Sigma$) with approximation ratio $\rho$ is an algorithm $A$ with the following properties:

1. For every input $(x, k)$ whose optimal solution has cost at most $k$, $A$ computes a solution for $x$ of cost at most $k \cdot \rho(k)$. For inputs $(x, k)$ without a solution of cost at most $k$, the output can be arbitrary.
2. The runtime of $A$ on input $(x, k)$ is $O(g(k) \cdot |x|^{O(1)})$ for some computable function $g$.

It is often convenient to work with a weaker notion of approximability where an algorithm is only required to compute the cost of an optimal solution rather than an actual optimal solution, and to work with decision rather than optimization problems. With that in mind, the notion of FPT cost approximability was introduced in [15].
Definition 3.4 (FPT cost approximation algorithm [53]). Let $P$ be an NP minimization problem (over the alphabet $\Sigma$), and $\rho : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ a computable function. For an instance $x$ of $P$, let $\min(x)$ denote its optimal value. Then, a decision algorithm $A$ is an FPT cost approximation algorithm for $P$ with approximation ratio $\rho$ if

1. For feasible instances $x$ of $P$ and parameterized instances $(x, k)$, $A$ satisfies:
   
   (a) If $k \geq \min(x) \cdot \rho(\min(x))$, then $A$ accepts $(x, k)$.
   
   (b) If $k < \min(x)$, then $A$ rejects $(x, k)$.

2. $A$ is an FPT algorithm. That is, there exists a computable function $f$ with the property that for an input $(x, k)$, the running time of $A$ is bounded by $f(k) \cdot |x|^{O(1)}$.

It can be readily checked that FPT-approximability implies FPT cost approximabibility with the same approximation factor. Please refer to Section 3.1 of [15] for more details.

Theorem 3.2 (Chen et al. [15]). Let $P$ be an NP minimization problem over the alphabet $\Sigma$, and let $\rho : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ be a computable function such that $k \cdot \rho(k)$ is nondecreasing and unbounded. Suppose that $P$ is FPT approximable with approximation ratio $\rho$. Then $P$ is FPT cost approximable with approximation ratio $\rho$.

An immediate consequence of the theorem above is that if $P$ is not FPT cost approximable with approximation ratio $\rho$ (under certain complexity theory assumptions), then $P$ is not FPT approximable with approximation ratio $\rho$ (under the same assumptions).

Gap problems and gap-preserving reductions were originally introduced in the context of proving the PCP theorem [5] – a cornerstone in the theory of approximation algorithms. These notions have natural analogues in the parameterized approximability setting. Below, we follow the definitions as provided by Eickmeyer et al. [25].

Definition 3.5 (gap instance of a parameterized problem [25]). Let $\delta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ be a function, $P$ a minimization problem, and $P'$ its standard parameterization. An instance $(x, k)$ is a $\delta$-gap instance of $P'$ if either $\min(x) \leq k$ or $\min(x) \geq k \cdot \delta(k)$.

Definition 3.6 (gap-preserving FPT reduction [25]). Let $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ be two computable functions, and let $P$ and $Q$ be two minimization problems. Let $P'$ and $Q'$ be the natural parameterizations of $P$ and $Q$, respectively. We say that a reduction $R$ from $P'$ to $Q'$ is a $(\alpha, \beta)$-gap-preserving FPT reduction if

1. $R$ is an FPT reduction from $P'$ to $Q'$,
2. for every $\alpha$-gap instance $(x, k)$ of $P'$, the instance $R(x, k)$ is a $\beta$-gap instance of $Q'$.

We use gap-preserving FPT reductions to establish FPT-inapproximability.

3.3 Circuits

First, we recall some elementary notions from Boolean circuits. In particular, by an and-node, we mean the digital logic gate that implements logical conjunction ($\land$); by an or-node, we mean the digital logic gate that implements logical disjunction ($\lor$), and by a not-node, we mean the digital logic gate that implements negation ($\neg$).

Definition 3.7 (Boolean circuit). A Boolean circuit $C$ is a directed acyclic graph, where each node is labeled in the following way:

1. every node with in-degree greater than 1 is either an and-node or an or-node,
2. each node of in-degree 1 is labeled as a negation node,
3. and each node of in-degree 0 is an input node.

Moreover, exactly one of the nodes with out-degree 0 is labeled as the output node.
Below, we recall some essential parameterized complexity results concerning circuits.

We use the terms gates and nodes interchangeably. We say that a gate has fan-in $k$ if its in-degree is at most $k$. We say that a gate is an ordinary gate if it is neither an input gate nor an output gate. We denote the nodes and edges in $C$ by $V(C)$ and $E(C)$ respectively. The size of a circuit $C$, denoted by $|C|$, is the total number of nodes and edges in $C$. That is, $|C| = |V(C)| + |E(C)|$. The Hamming weight of an assignment is the number of input gates receiving value 1. An assignment on the input nodes induces an assignment on all nodes. So given an assignment from the input nodes of circuit $C$ to $\{0, 1\}$, we say that the assignment satisfies $C$ if the value of the output node is 1 for that assignment. Let $I_f$ denote the set of input gates of $C$. Then, an assignment $A$ can be viewed as a binary vector of size $|I_f|$. In the Weighted Circuit Satisfiability (WCS) problem, we are given a circuit $C$ and an integer $k$, and the task is to decide if $C$ has a satisfying assignment of Hamming weight at most $k$. Accordingly, in the Min-Weighted Circuit Satisfiability (MinWCS) problem, we are given a circuit $C$, and the task is to find a satisfying assignment with minimum Hamming weight.

**Definition 3.8 (W[P]).** A parameterized problem $W$ belongs to the class $W[P]$ if it can be reduced to the standard parameterization of WCS.

A Boolean circuit is monotone if it does not contain any negation nodes. Let $C^+$ be the class of all monotone Boolean circuits. Then, Min-MONOTONE CIRCUIT SAT (MinMCS) is the restriction of the problem MinWCS to input circuits belonging to $C^+$.

The following result seems to be folklore and appears in the standard literature [23,27].

**Theorem 3.3 (Theorem 3.14 [28]).** The standard parameterization of MinMCS is $W[P]$-complete.

Furthermore, Eickmeyer et al. [25] showed that unless $W[P] = \text{FPT}$, MinMCS does not have an FPT approximation algorithm with polylogarithmic approximation factor $\rho$. The FPT-inapproximability result was subsequently improved by Marx [53] as follows.

**Theorem 3.4 (Marx [53]).** MinMCS is not FPT cost approximable, unless $\text{FPT} = W[P]$.

Combined with Theorem 3.2, the above theorem implies that MinMCS is not FPT-approximable for any function $\rho$, unless $\text{FPT} = W[P]$.

**Remark 1 (Fan-in 2 circuits).** We note that it is possible to transform a monotone circuit $C$ to another monotone circuit $C'$ such that both circuits are satisfied on the same inputs, and every gate of $C'$ has fan-in 2. This is achieved as follows: Each or-gate of in-degree $k$ in $C$ is replaced by a tree of or-gates with in-degree-2 in $C'$, and each and-gate of in-degree $k$ in $C$ is replaced by a tree of and-gates with in-degree-2 in $C'$. In each case, we transform a single gate having fan-in $k$ to a sub-circuit of $\Theta(k)$ gates having depth $\Theta(\log k)$ and fan-in 2. In fact, it is easy to check that $|C'|$ is a polynomial function of $|C|$, and $C'$ can be computed from $C$ in time polynomial in $C$. Since the number of input gates for $C$ and $C'$ is the same, for the rest of the paper we will assume without loss of generality that an input circuit instance has fan-in 2.

4 Reducing MinMCS to MinrMM

In this section, we describe how to construct a 2-complex $K(C)$ that corresponds to a monotone circuit $C(V, E)$. By Remark 1, we assume without loss of generality that $C$ has fan-in 2. For the rest of the paper, we denote the number of gates in $C$ by $n$. Also, throughout, we use the notation $j \in [a, b]$ to mean that $j$ takes integer values in the interval $[a, b]$.

Following the notation from Section 3.1, given a monotone circuit $C = (V, E)$ and the associated complex $K(C)$, let $\text{OPT}_{\text{MinMCS}}(C)$ denote the optimal value of the MinMCS problem on $C$, and let $\text{OPT}_{\text{MinMM}}(K(C))$ denote the optimal value of the MinrMM problem on $K(C)$. The value of the objective function $n_{\text{MinMM}}(K(C), \mathcal{I}(C, V))$ is the number of critical simplices in $V$ minus one; the value of the objective function $n_{\text{MinMM}}(C, \mathcal{I}(C, V))$ is the Hamming weight of the input assignment. In Section 4.2, we describe the map $K$ that transforms instances of MinMCS (monotone circuits $C$) to instances of MinMM (simplicial complexes $K(C)$), and the map $\mathcal{I}$ that transforms solutions of MinMM (discrete gradients $\mathcal{V}$ on $K(C)$) to solutions of MinMCS (satisfying input assignments $\mathcal{I}(C, V)$ of circuit $C$).
Figure 1: The figure (a) on the left depicts $D_{1, \ell}$ that is collapsible through one free face, namely $s_1 = \{3, 1\}$. The figure (b) on the right depicts $D_{2, \ell}$ that is collapsible through two free faces, namely $s_1 = \{3, 8\}$ and $s_2 = \{8, 1\}$. The edges $\{1, 2\}$ and $\{2, 3\}$ on the right and at the bottom of both subfigures are shown in light grey to indicate that they are identified to $\{1, 2\}$ and $\{2, 3\}$ on the left.

4.1 The building block for the gadget

We shall first describe a complex that serves as the principal building block for the gadget in our reduction. The building block is based on a modification of Zeeman’s dunce hat \[24\]. The dunce hat is a simplicial complex that is contractible (i.e. has the homotopy type of a point) but has no free faces and is therefore not collapsible. In contrast, we work with modified dunce hats \[31\] that are collapsible through either one or two free edges. The modified dunce hat has been previously used to show hardness of approximation of MAX-MORSE MATCHING \[9\] and W[P]-hardness of ERASABILITY EXPANSION HEIGHT \[10\], and is discussed extensively in these papers.

Figure 1 depicts two triangulations of modified dunce hats, which we denote by $D_{m, \ell}$. We use the subscript $m, \ell$ to designate the numbers of distinguished edges of $D_{m, \ell}$, which come in two types: the free edges of $D_{m, \ell}$ denoted by $s_i, 1 \leq i \leq m$, and the edges of type $t_j = \{y_j, z_j\}, 1 \leq j \leq \ell$. These distinguished edges are precisely the edges that are identified to edges from other building blocks. In Figure 1 we depict $D_{1, \ell}$ and $D_{2, \ell}$ with distinguished edges $s_i$, and $t_j$ highlighted.

Note that, in this paper, we only consider modified dunce hats with either one or two free edges. That is, for the purpose of this paper, $m \in \{1, 2\}$. Also, abusing terminology, we often refer to “modified dunce hats” as simply “dunce hats”.

Remark 2. It is easy to check that after executing a series of elementary 2-collapses, $D_{m, \ell}$ collapses to a complex induced by edges

$$\{\{1, 2\}, \{2, 3\}, \{2, 6\}, \{6, 5\}, \{6, 7\}, \{6, b_i\}, \{6, d_i\}, \{b_i, z_i\}, \{z_i, y_i\}, \{y_i, a_i\}\} \cup F$$

for $i \in [1, \ell]$, and,

- $F = \{\{v, 4\}\}$ if $m = 1$, where $v = 6$, if $\ell$ is even, and $v = (\ell - 1)/2 + 1$ if $\ell$ is odd,
- $F = \{s_1, \{4, 8\}\}$ if $m = 2$ and the collapse starts with a gradient pair involving $s_2$,
- $F = \{s_2, \{4, 8\}\}$ if $m = 2$ and the collapse starts with a gradient pair involving $s_1$.

The edges that are left behind after executing all the 2-collapses are highlighted using examples in Figure 1. (a) depicts the case where $\ell$ is odd and $m = 1$. (b) depicts the case where $\ell$ is even, $m = 2$ and the collapse starts with a gradient pair involving $s_1$.

4.2 Construction of the complex $K(C)$ and the map $\mathcal{I}(C, V)$

Given a circuit $C$, we first explain the construction of an intermediate complex $K'(C)$. We use the notation $D_{i, j}$ to refer to the copy of the dunce hat associated to gate $i$, having $m$ s-edges and $\ell$ t-edges. Sometimes, we suppress the subscript, and use the notation $D_{i, j}$ in place of $D_{m, \ell}$.
The complex $D_{1,3}$ collapses to the subcomplex induced by the highlighted edges. When $s_1$ cannot be made free (because of edge identifications from other dunce hats), then $s_1$ and $\Gamma_1$ are made critical. In any case, the remaining 2-collapses are executed as shown in Figure (a). The figure (b) on the right depicts $D_{2,2}$ that is collapsible through two free faces, namely $s_1 = \{3, 8\}$ and $s_2 = \{8, 1\}. There exists a collapsing sequence for $D_{2,2}$ starting from the gradient pair $(s_1, \Gamma_1)$ such that $D_{2,2}$ collapses to the subcomplex induced by the highlighted edges. A symmetric statement can be made for a collapse starting from $(s_2, \Gamma_2)$. 

As illustrated in Figure 3 to each input gate $G_i$ we associate a dunce hat $D^{(i,1)}$. To the output gate $G_o$, we associate $n$ copies of dunce hats $\{D^{(o,j)}\}_{j=1}^n$. Moreover, Figure 4 depicts how we associate to each ordinary gate $G_i$ $n$ blocks $\{1D^{(i,j)}, 2D^{(i,j)}, 3D^{(i,j)}\}_{j=1}^n$. The superscript to the left indexes dunce hats internal to the block. We call $2D^{(i,j)}$ the output component of block $j$ associated to $G_i$. Likewise, we call $1D^{(i,j)}$ and $2D^{(i,j)}$ the input components of block $j$ associated to $G_i$. If $G_p$ serves as one of the two inputs to $G_q$, we say that $G_p$ is a predecessor of $G_q$, and $G_q$ is the successor of $G_p$.

A simplex labeled $s$ in $D_{m,t}$ is correspondingly labeled as $s^{(p,j)}$ in $D^{(p,j)}_{m,t}$, (respectively as $k_s^{(p,j)}$ in $kD^{(p,j)}_{m,t}$). We call the unique $s$-edge of the dunce hat associated to an input gate $G_i$, namely $s_1^{(i,1)}$, its feedback edge. As depicted in Figure 4 for an ordinary gate $G_p$, for each $j \in [1, n]$, the $s_2$ edges of $1D^{(p,j)}$ and $2D^{(p,j)}$, namely $s_2^{(p,j)}$ and $s_2^{(p,j)}$, respectively, are called the feedback edges of the $j$-th block associated to $G_p$. For an ordinary gate $G_p$, for each $j \in [1, n]$, the $s_1$ edges of $1D^{(p,j)}$ and $2D^{(p,j)}$, namely $s_1^{(p,j)}$ and $s_1^{(p,j)}$, respectively, are called the input edges of the $j$-th block associated to $G_p$. For the output gate $G_o$, the $s_1$ and $s_2$ edges of $D^{(o,j)}$, namely $s_1^{(o,j)}$ and $s_2^{(o,j)}$, respectively, are called the input edges of the $j$-th copy associated to $G_o$.

To bring the notation of the edges closer to their function in the gadget, for the rest of the paper, we use the following alternative notation for $s$-edges. We denote the feedback edges $s_1^{(i,1)}$, $s_2^{(p,j)}$ and $s_2^{(p,j)}$ described above as $s_1^{(i,1)}$, $s_2^{(p,j)}$ and $s_2^{(p,j)}$, respectively. Also, we denote the input edges $s_1^{(p,j)}$, $s_1^{(p,j)}$, $s_1^{(o,j)}$ and $s_2^{(o,j)}$ described above by $s_1^{(p,j)}$, $s_2^{(p,j)}$, $s_1^{(o,j)}$ and $s_2^{(o,j)}$, respectively. Please see Figure 5 for an example.

We start with a disjoint union of dunce hats (or blocks) associated to each gate. Then, for an ordinary gate $G_p$ that is a predecessor of $G_q$, for all $j, k \in [1, n]$ two distinct $t$-edges from the $j$-th copy (output component of the block) associated to $G_q$ are identified to the two feedback edges of the $k$-th block associated to $G_p$. Also, for all $j, k \in [1, n]$ a $t$-edge from the output component of the $j$-th copy (output component of the block) associated to $G_q$ is identified to an input edge of the $j$-th copy (block) associated to $G_q$.

For an input gate $G_p$ that is a predecessor of $G_q$, for all $j \in [1, n]$, a $t$-edge from the $j$-th copy (output component of the block) associated to $G_q$ is identified to the feedback edge of the unique dunce hat associated to $G_p$. Also, for all $j \in [1, n]$, a $t$-edge from the dunce hat associated to $G_p$ is attached to the input edge of the $j$-th copy (block) associated to $G_q$. 

Figure 2: The figure (a) on the left depicts $D_{1,3}$ that is collapsible through a unique free face, namely $s_1 = \{3, 1\}. The complex $D_{1,3}$ collapses to the subcomplex induced by the highlighted edges. When $s_1$ cannot be made free (because of edge identifications from other dunce hats), then $s_1$ and $\Gamma_1$ are made critical. In any case, the remaining 2-collapses are executed as shown in figure (a). The figure (b) on the right depicts $D_{2,2}$ that is collapsible through two free faces, namely $s_1 = \{3, 8\}$ and $s_2 = \{8, 1\}. There exists a collapsing sequence for $D_{2,2}$ starting from the gradient pair $(s_1, \Gamma_1)$ such that $D_{2,2}$ collapses to the subcomplex induced by the highlighted edges. A symmetric statement can be made for a collapse starting from $(s_2, \Gamma_2)$.
Moreover, these identifications are done to ensure that: a feedback edge of a copy (block) associated to $G_p$ is free only if all the copies (output components of all the blocks) associated to all the successors of $G_p$ have been erased, and an input edge of a copy (block) associated to $G_q$ is free only if the unique copy (all the output components of all the block) associated to the predecessors of $G_p$ have been erased. Please refer to Figure 5 for an example illustrating the identifications.

It is important to note that the gluing is done so that the $s$-edges from two different copies (blocks) associated to the same gate are never identified as an outcome of gluing, nor do they intersect in a vertex. In particular if $G_p$ is a gate with $G_p = 2$ as inputs, where, for instance, if $G_p$ is an input gate and $G_p$ is an ordinary gate, then for every $k \in [1, n]$, $s_{i, k}^{(p,k)}$ is identified to a unique $t$-edge from the dunce hat associated to $G_{p, k}$, and $s_{2, k}^{(p,k)}$ is identified to $n$-edges each from a block associated to $G_{p, 2}$. These are the only identifications for edges $s_{i, k}^{(p,k)}$ and $s_{2, k}^{(p,k)}$. For every non-output gate $G_p$, let $\theta_p$ denote the number of successors of $G_p$. Then, for all $k \in [1, n]$, $s_{f_1}^{(p,k)}$ and $s_{f_2}^{(p,k)}$ each have $\theta_p$ identifications from $t$-edges coming from each of the blocks associated to each of the successors of $G_p$. These are the only identifications for $s_{f_1}^{(p,k)}$ and $s_{f_2}^{(p,k)}$. If $G_p$ is an input gate, then $s_{f_1}^{(p,k)}$ is identified to $\theta_p$ $t$-edges from blocks associated successors of $G_p$. Finally, the input $s$-edges of the $k$-th copy associated to the output gate $G_p$ is identified to either one or $n$-edges coming from dunce hats associated to predecessor gates, depending on whether the predecessor is an input gate or an ordinary gate. We refrain from providing indices for the identified $t$-edges as this would needlessly complicate the exposition.

For every non-input gate $G_i$, set $\phi_i = 1$ if the first input to $G_i$ is from an input gate, and set $\phi_i = 2n$ otherwise. Similarly, for every non-input gate $G_i$, set $\psi_i = 1$, if the second input to $G_i$ is from an input gate, and set $\psi_i = 2n$ otherwise.

Now we can readily check the following: In our construction, for a dunce hat $3D_{m, \ell}^{i,j}$ associated to an ordinary gate $G_i$, we have $m = 1$ or $m = 2$ (depending on whether it is an and-gate or an or-gate), and $\ell = \theta_i + \phi_i$. For dunce hats $2D_{m, \ell}^{i,j}$ and $1D_{m, \ell}^{i,j}$ associated to an ordinary gate $G_i$, we have $m = 2$, and $\ell = 1$. For a dunce hat $D_{m, \ell}^{i,j}$ associated to an output gate $G_i$, we have $m = 1$ or $m = 2$, and $\ell = \phi_i + \psi_i$. Finally, for the dunce hat $D_{m, \ell}^{i,j}$ associated to an input gate $G_i$, we have $m = 1$, and $\ell = \theta_i n$.

**Remark 3.** We reindex the dunce hats described above using the indexing set $\Xi$. That is, for every dunce hat in $K'(C)$ there exists a unique $\zeta \in [1, |\Xi|]$ such that $D_{m, \ell}^{i,j}$ identifies the dunce hat of interest. Sometimes in our exposition it is more convenient to refer to dunce hats with a single index as opposed to using two or three indices in the superscript.

Let $\zeta$ be the indexing variable, and $\Xi$ the indexing set as described in Remark 3 [3]. For a dunce hat $D_{m, \ell}^{i,j}$, we call the complex induced by the edges $\{(b^1, b^2), (b^2, b^3), (b^3, b^4), (b^4, b^5), (b^5, b^6), (b^6, b^7), (b^7, b^8), (b^8, b^9), (b^9, b^{10})\}$ (i.e., the pink edges of $D_{m, \ell}$ in Figure 3) the stem of $D_{m, \ell}^{i,j}$. Then, in complex $K'(C)$, let $H$ be the 1-dimensional subcomplex formed by the union of stems of $D_{m, \ell}^{i,j}$ for all $\zeta \in [1, |\Xi|]$.

**Remark 4 (Design choices for ordinary gates).** At this point we would like to remark that, in principle, one could construct a complex for circuits with arbitrary fan-ins wherein the or-gate and and-gate like behaviour can easily be implemented with a single (suitably sub-divided) dunce hat having two or more free edges. The problem with this approach is that it is much harder to control where the 1-cycles in the complex appear, and this makes the cycle filling procedure far more technical. This motivates our approach to first pass to circuits with fan-in two and then implement or-gates and and-gates with blocks of three instead of single dunce hats. As we shall see later, this leads to a straightforward instance-independent description of the 1-homology basis of $K'(C)$, which in turn simplifies cycle filling.

Given a gradient vector field $\nabla$ on $K$, we construct the map $\mathcal{I}(C, \nabla)$ as follows: For every input gate $G_i$ whose associated dunce hat has a critical 2-simplex in $\nabla$, we set $\mathcal{I}(C, \nabla)(G_i) = 1$. Please refer to Appendix A.2 for further details.
Figure 3: In all three figures, the distinguished edges are highlighted. The top figure shows an input gate and the dunce hat associated to it. We conceive the input gate as activated when the associated dunce hat has a critical 2-simplex in it. If the dunce hat doesn’t have critical 2-simplices, then $s_1$ must be paired to its coface for the dunce hat to be erased. The edge $s_1$ supports a feedback mechanism. In particular, if all the dunce hats associated to the output gate are erased without activating $G_i$, then we need an alternative means to erase $G_i$, which is provided by $s_1$. The figure in the middle (resp. bottom) shows an output or-gate (resp. an output and-gate) and the associated $n$ copies of dunce hats. In both cases, the $j$-th copy consists of a single dunce hat $D^{(i,j)}$, where $j \in [1,n]$. The idea behind the dunce hat associated to the or-gate is that if either $s_1$ or $s_2$ is free, then $D^{(i,j)}$ can be erased. The idea behind the dunce hat associated to the and-gate is that if $s_1$ is free, then $D^{(i,j)}$ can be erased. Finally, we have $n$ copies instead of a single copy per gate to ensure that optimum values of $\text{MinMCS}$ and $\text{MinrMM}$ are the same.
Figure 4: The figure on the top (resp. bottom) shows a non-output or-gate (resp. a non-output and-gate) and the associated $n$ blocks of dunce hats. In both cases $j$-th block consists of 3 dunce hats \( \{1D_{i,j}, 2D_{i,j}, 3D_{i,j}\} \), where $j \in [1,n]$. All distinguished edges are highlighted, and identical color coding indicates identifications. That is, red edges are glued to red edges and green to green. The arrows on the highlighted edges show the orientations of identifications. The idea behind the blocks associated to the or-gate is that if either the $s_1$ edge of \( 1D_{i,j} \) or the $s_1$ edge of \( 2D_{i,j} \) is free, then all three dunce hats in the $j$-th block can be erased. The idea behind the blocks associated to the and-gate is that if the $s_1$ edge of \( 1D_{i,j} \) and the $s_1$ edge of \( 2D_{i,j} \) are free, then all three dunce hats in the $j$-th block can be erased. For each block, the dark and the light blue $s_2$ edges of \( 1D_{i,j} \) and \( 2D_{i,j} \) respectively support a feedback mechanism. In particular, if the dunce hats associated to the output gate are erased then, we need an alternative means to erase all the dunce hats, since the satisfaction of the output gate is all we really care about. Finally, we have $n$ blocks instead of a single block per gate to ensure that optimum values of $\text{MinMCS}$ and $\text{MinrMM}$ are the same.
Figure 5: In this figure we depict the part of the complex associated to the (partial) circuit that implements \( z = (a \land b) \lor x \), where \( x \) is an input to the circuit. Identical color coding indicates identifications, and the arrows indicate orientations of identifications. Here we only show identifications for \( j \)-th block of \( G_q \) and \( k \)-th block of \( G_p \) for arbitrary \( j, k \in [1,n] \). Similar identifications occur across all respective associated blocks.
4.3 Construction of the complex $K(C)$

From the identifications described in Section 4.2, it is easy to check that $H = (V_H, E_H)$ is, in fact, a connected graph. Please refer to Lemma 4.4 for a simple proof. The procedure for constructing $K(C)$ is described in Algorithm 1.

**Algorithm 1** Procedure for constructing $K(C)$ from $K'(C)$

1. $K(C) \leftarrow K'(C)$.
2. $\triangleright$ Initially, $K(C)$ consists only of simplices from $K'(C)$.
3. $\triangleright$ Compute a cycle basis $B$ of $H$ with $Z_2$ coefficients as follows (Steps 4–13).
4. $i = 1$; $B = \emptyset$.
5. Let $\prec$ be an arbitrary total order on the edges of $H$.
6. **while** $E_H$ is non-empty **do**
   7. Successively remove every edge from $H$ that is incident on a vertex of degree 1.
   8. Choose a simple cycle in $H$ incident on the highest indexed edge w.r.t. $\prec$.
   9. Denote the cycle by $z_i$, and the highest indexed edge by $e_i^1$.
   10. $B = B \cup \{z_i\}$.
   11. Remove $e_i^1$ from $H$.
   12. $i = i + 1$.
   13. **end while**
14. Assemble the basis vectors of $B$ in a matrix $M$, where $\prec$ is used to index the rows of $M$, and the iterator variable $i$ from the while loop above is used to index the columns.
15. $\triangleright$ For every $z_i \in B$, let $n_i$ be the number of edges in $z_i$, and $\{e_j^1 \mid j \in [n_i]\}$ denote the edges in $z_i$.
16. **for** $i \leftarrow 1, |B| \ **do**$
    17. Add a new vertex $v_i$ to $K(C)$.
    18. **for** $j \leftarrow 1, n_i \ **do**$
        19. Add to $K(C)$ a 2-simplex $\sigma_j^i = e_j^1 * v_i$ for each edge $e_j^1$ of $z_i$.
        20. Add to $K(C)$ all of the faces of simplices $\sigma_j^i$.
    21. **end for**
22. **end for**
23. $\mathcal{D} \leftarrow K(C) \setminus K'(C)$.
24. RETURN $K(C), \mathcal{D}$.

**Remark 5.** By construction, the edges $e_i^1$, for $i \in [|B|]$, do not appear in the cycles $z_j$, for $j > i$. Hence, $M$ is upper-triangular.

Note that there exists a polynomial time subroutine to implement Step 8 of Algorithm 1. In particular, because the edges of $H$ that are not incident on any cycles of $H$ are removed in Step 7, every edge of $H$ is incident on some simple cycle contained in a minimum cycle basis (with unit weights on edges) of $H$.

**Remark 6.** In the construction described in Algorithm 1, the star of the vertex $v_i$ may be viewed as a “disk” that fills the cycle $z_i$. See Figure 6 for an illustration. Furthermore, it can be shown that

- the second homology groups $H_2(K'(C))$ and $H_2(K(C))$ are trivial,
- The cycles in $B$ form a basis for $H_1(K'(C))$,
- $K(C)$ is contractible.

However, our hardness results can be established without proving any of the statements in Remark 6. Having said that, it is important to bear in mind that the procedure of going from $K'(C)$ to $K(C)$ is, in fact, a 1-cycle filling procedure.

To establish hardness results, we introduce some additional notation. Given a monotone circuit $C = (\mathcal{V}, E)$ let $K(C)$ be its associated complex. Now let $\text{OPT}_{\text{MinMCS}}(C)$ denote the optimal value of the $\text{MinMCS}$ problem on $C$, and let $\text{OPT}_{\text{MinMM}}(K(C))$ denote the optimal value of the $\text{MinMM}$ problem on $K(C)$. The value of the objective function $m_{\text{MinMM}}(K(C), \mathcal{V})$ is the number of critical simplices in $\mathcal{V}$ minus one; the value of the objective function $m_{\text{MinMCS}}(C, I(C, \mathcal{V}))$ is the Hamming weight of the input assignment.
4.4 Reducing MinMCS to MinMM: Forward direction

Given a circuit $C$, suppose that we are given an input assignment $A$ that satisfies the circuit $C = (V(C), E(C))$. Let $\mathcal{I}$ be the set of gates that are satisfied by the assignment, and let $I(\mathcal{I})$ be the set of input gates that are assigned 1. Clearly, $I(\mathcal{I}) \subset \mathcal{I}$, and also the output gate $G_o \in \mathcal{I}$. Let $\mathcal{I} = V(C) \setminus \mathcal{I}$ denote the set of gates that are not satisfied by the input $A$. Clearly, the subgraph $G_{\mathcal{I}}$ of $C$ induced by the gates in $\mathcal{I}$ is a connected graph. Also, since $C$ is a directed acyclic graph, the induced subgraph $G_{\mathcal{I}}$ is also directed acyclic. Let $\prec_{\mathcal{I}}$ be some total order on $\mathcal{I}$ consistent with the partial order imposed by $G_{\mathcal{I}}$, and let $\prec_C$ be some total order on $V(C)$ consistent with the partial order imposed by $C$.

Next, given an assignment $A$ on $C$, we describe how to obtain a gradient vector field $V$ on $K(C)$. We denote the complex obtained after $i$-th step by $K_i(C)$.

**Step 1: Erase satisfied input gates**  First, for every input gate $G_i \in \mathcal{I}$, we make $G_i$ critical. By Lemma 2.5, this is akin to removing $\Gamma_1^{(i,1)}$ critical. Let $\mathcal{I} = V(C) \setminus \mathcal{I}$, denote the set of gates that are not satisfied by the assignment $A$. Clearly, the subgraph $G_{\mathcal{I}}$ of $C$ induced by the gates in $\mathcal{I}$ is a connected graph. Also, since $C$ is a directed acyclic graph, the induced subgraph $G_{\mathcal{I}}$ is also directed acyclic. Let $\prec_{\mathcal{I}}$ be some total order on $\mathcal{I}$ consistent with the partial order imposed by $G_{\mathcal{I}}$, and let $\prec_C$ be some total order on $V(C)$ consistent with the partial order imposed by $C$.

**Step 2: Forward collapsing**  Assume throughout Step 2 that the gates in $\mathcal{I}$ are indexed from 1 to $|\mathcal{I}|$ so that

for all $G_i, G_j \in \mathcal{I}$, $i < j \Leftrightarrow G_i \prec_{\mathcal{I}} G_j$.

**Lemma 4.1.** Let $G_p \in \mathcal{I} \setminus I(\mathcal{I})$. Suppose that all the gates in $I(\mathcal{I})$ have been erased, and for all gates $G_k \in \mathcal{I} \setminus I(\mathcal{I})$ with $k < p$ the associated dunce hats $3D^{(k,r)}_i$ for all $r \in [1,n]$ have been erased. Then, the dunce hats $3D^{(p,r)}_i$, for all $j \in [1,n]$ associated to $G_p$ can be erased.

**Proof.** Let $G_{p_1}$ and $G_{p_2}$ be inputs to $G_p$. Assume without loss of generality that $G_{p_1}, G_{p_2}$ are non-input gates. By our assumption on indexing, $p_1 < p_2 < p$. By construction, the only identifications to $s_i^{(p,1)} \in 1D^{(p,j)}_i$ are from $t$-edges that belong to $3D^{(p_1,r)}_i$ for all $r \in [1,n]$, and the only identifications to $s_i^{(p,2)} \in 2D^{(p,j)}_i$ are from $t$-edges that belong to $3D^{(p_2,r)}_i$ for all $r \in [1,n]$. We have two cases:

**Case 1.** Assume that $G_p$ is a satisfied or-gate. Then, either $G_{p_1} \in \mathcal{I}$ or $G_{p_2} \in \mathcal{I}$. Without loss of generality, we assume that $G_{p_1} \in \mathcal{I}$. Then, for all $j$, $s_i^{(p,j)}$ become free since, by assumption, the dunce hats $3D^{(p_1,r)}_i$ associated to $G_{p_1}$ have been erased. So using Lemma A.1 from Appendix A.1 for all $j$, $1D^{(p,j)}_i$ can be erased. For each $j$, the unique identification to $s_i^{(p,j)}$ is from a $t$-edge in $1D^{(p,j)}_i$. Hence, for all $j$, $3D^{(p,j)}_i$ becomes free, making it possible to erase $3D^{(p,j)}_i$ for all $j$.

**Case 2.** Assume that $G_p$ is a satisfied and-gate. Then, both $G_{p_1} \in \mathcal{I}$ and $G_{p_2} \in \mathcal{I}$. Thus, for all $j$, $s_i^{(p,j)}$ and $s_i^{(p,j)}$ become free since, by assumption, all the dunce hats $3D^{(p_1,r)}_i$ for all $p \in [1,n]$ associated to $G_{p_1}$ and all dunce hats $3D^{(p_2,r)}_i$ for all $q \in [1,n]$ associated to $G_{p_2}$ have been erased. So, using Lemma A.1 from Appendix A.1 for all $j \in [1,n]$, $1D^{(p,j)}_i$ and $2D^{(p,j)}_i$ can be erased. For all $j \in [1,n]$, the only two edges identified to $s_i^{(p,j)}$ belong to $1D^{(p,j)}_i$ and $2D^{(p,j)}_i$ respectively. Hence, for all $j \in [1,n]$, $s_i^{(p,j)}$ becomes free, making it possible to erase $3D^{(p,j)}_i$ for all $j \in [1,n]$. Thus, the dunce hats $3D^{(p,j)}_i$ for $j \in [1,n]$ associated to $G_p$ can be erased.

The argument is identical for the case when $G_{p_1}$ or $G_{p_2}$ is an input gate.

**Lemma 4.2.** All dunce hats associated to the output gate are erased.

**Proof.** Note that a satisfying assignment $A$ that satisfies the circuit, in particular, also satisfies the output gate. A simple inductive argument using Lemma 4.1 proves the lemma.

After applying Step 1, we apply Step 2, which comprises of executing the collapses described by Lemmas 4.1 and 4.2. This immediately gives us the following claim.

**Claim 4.1.** If there exists an assignment satisfying a circuit $C$ with Hamming weight $m$, then there exists a gradient vector field on $K(C)$ such that after making $m$ 2-cells critical, all the dunce hats $3D^{(p,j)}_i$ associated to the satisfied non-output gates $G_p$ and all dunce hats associated to the output gate can be erased.
The complex obtained after erasing executing Step 2 is denoted by $K^2(C)$. We have, $K^1(C) \setminus K^2(C)$.

Remark 7. Note that the forward collapses do not erase all the dunce hats associated to satisfied gates. For instance, for a satisfied or-gate $G_p$, if one of the input gates, $G_{p_1}$, is satisfied and the other, $G_{p_2}$, is not, then $1D^{(p;j)}$ and $3D^{(p;j)}$ will be erased, but $2D^{(p;j)}$ will not be erased. The dunce hats associated to the unsatisfied gates and the unerased dunce hats associated to the satisfied gates are erased in the next step while executing the backward collapses.

**Step 3: Backward collapsing** Assume throughout Step 3 that the gates in $V(C)$ are indexed from 1 to $n$ so that

for all $G_i, G_j \in V(C)$, $i < j \Leftrightarrow G_i \prec_C G_j$.

The idea behind backward collapsing is that the feedback edges become successively free when one starts the collapse from dunce hats associated to highest indexed gate and proceeds in descending order of index.

**Lemma 4.3.** If all the dunce hats associated to gates $G_k$, where $k > i$, have been erased, then the dunce hats associated to $G_i$ can be erased.

**Proof.** We have three cases to verify:

**Case 1.** First, assume that $G_i$ is an ordinary gate. The only identifications to edges $s_{f_1}^{(i,j)} \in 1D^{(i,j)}$ and $s_{f_2}^{(i,j)} \in 1D^{(i,j)}$ respectively are from the $t$-edges in dunce hats associated to successors of $G_i$.

By assumption, all dunce hats $3D^{(k;p)}$ associated to ordinary gates $G_k$, where $k > i$ have been erased, and all dunce hats $3D^{(k;p)}$ associated to the output gate $G_o$ have been erased. Hence, $s_{f_1}^{(i,j)}$ and $s_{f_2}^{(i,j)}$ are free, for every $j$. Therefore, for every $j$, dunce hats $1D^{(i,j)}$ and $2D^{(i,j)}$ can be erased.

**Case 2.** If $G_i$ is an unsatisfied gate, then for all $j$, the only identifications to $s$-edge(s) of $3D^{(i,j)}$ are from $t$-edges of $1D^{(i,j)}$ and $2D^{(i,j)}$. So the $s$-edge(s) of $3D^{(i,j)}$ become free for all $j$, allowing us to erace $3D^{(i,j)}$, for all $j$. Thus, all dunce hats associated to $G_i$ can be erased.

**Case 3.** Now assume that $G_i$ is an input gate. Then, the unique $s$-edge of the unique copy associated to $G_i$ is identified to $t$-edges of dunce hats associated to successors of $G_i$. Since, by assumption, all dunce hats associated to gates $G_k$, where $k > i$, have been erased, $s_{f_1}^{(i,1)}$ becomes free, allowing us to erace $D^{(i,1)}$.

Note that in the proof of this lemma, for ordinary satisfied gates only Case 1 may be relevant, whereas for ordinary unsatisfied gates both Case 1 and Case 2 apply.

**Claim 4.2.** If there exists an assignment satisfying a circuit $C$ with Hamming weight $m$, then there exists a gradient vector field on $K(C)$ with exactly $m$ critical 2-cells.

**Proof.** We prove the claim by induction. The base step of the induction is provided by Claim [4.1]. Then, we repeatedly apply the steps below until all gates in $K(C)$ are erased:

1. Choose the highest indexed gate whose associated dunce hats haven’t been erased.

2. Apply the collapses described in Lemma [4.3] to erase dunce hats associated to $G_k$.

The complex obtained after erasing all dunce hats in $K(C)$ is denoted by $K^3(C)$. We have, $K^1(C) \setminus K^2(C) \setminus K^3(C)$.

**Step 4: Deleting critical 1-simplices** Note that in complex $K^3(C)$, the $s$-edges $s_{f_1}^{(i,1)}$ from $D^{(i,1)}$, for all $G_i \in \mathcal{I}$ have no cofaces. Since they were already made critical in Step 1, by Lemma [2.5] we can delete $s_{f_1}^{(i,1)}$ from $K^3(C)$ for all $G_i \in \mathcal{I}$, and continue designing the gradient vector field on the subcomplex $K^4(C)$ obtained after the deletion.
Step 5: Removing dangling edges  Since the 2-collapses executed in Steps 1-3 are as described in Remark 2 and Figure 2, it is easy to check that for each $D_{m,\ell}^i \subset K(C)$, the edges that remain are of the form: \{1,2\}^{G_i}, \{2,3\}^{G_i}, \{2,6\}^{G_i}, \{5,6\}^{G_i}, \{7,6\}^{G_i}, \{b_k,6\}^{G_i}, \{e_k,6\}^{G_i}, \{d_k,6\}^{G_i}, \{b_k,z_k\}^{G_i}, \{z_k,y_k\}^{G_i}, \{y_k,a_k\}^{G_i}\} \cup F$ for $k \in [1,\ell]$, and,

- $F = \{\{v,4\}\}$ if $m = 1$, where $v = 6$, if $\ell$ is even, and $v = (\ell-1)/2+1$ if $\ell$ is odd,
- $F = \{s_1^i,\{4,8\}^{G_i}\}$ if $m = 2$ and $s_1^i$ is removed as part of a 2-collapse,
- $F = \{s_2^i,\{4,8\}^{G_i}\}$ if $m = 2$ and $s_1^i$ is removed as part of a 2-collapse.

We now execute the following 1-collapses (1-3 highlighted in green, and 4 highlighted in blue as illustrated in Figure 2).

1. Since $5^i, 7^i, 6^i, c_k^i, d_k^i$ are free for all $k \in [1,\ell]$, for all $D_{m,\ell}^i \subset K(C)$, we execute the following collapses for all $k \in [1,\ell]$, for all $D_{m,\ell}^i \subset K(C)$:

   $$(5^i,\{5,6\}^{G_i}), \ (7^i,\{7,6\}^{G_i}), \ (6^i,\{6,6\}^{G_i}), \ and \ (d_k^i,\{d_k,6\}^{G_i}).$$

2. Since the vertices $4^i$ are free for all $D_{m,\ell}^i \subset K(C)$, for all $D_{m,\ell}^i \subset K(C)$:

   - If $m = 1$, we execute the collapse $(4^i,\{4,v\}^{G_i})$, where $v = 6$, if $\ell$ is even, and $v = (\ell-1)/2+1$ if $\ell$ is odd,
   - If $m = 2$, we execute the collapse $(4^i,\{4,8\}^{G_i})$.

3. Now, $a_k^i$ become free for all $k \in [1,\ell]$, for all $D_{m,\ell}^i \subset K(C)$. So, we execute the collapses $(a_k^i,\{a_k,y_k\}^{G_i})$ for all $k \in [1,\ell]$, for all $D_{m,\ell}^i \subset K(C)$.

4. Now, $8^i$ become free for all $D_{2,\ell}^i \subset K(C)$. So, for all $D_{2,\ell}^i \subset K(C)$:

   - If $s_2^i$ was removed as part of a 2-collapse, we execute the collapse $(8^i, s_2^i)$,
   - else if $s_1^i$ was removed as part of a 2-collapse, we execute the collapse $(8^i, s_1^i)$.

Note that because of the identifications, there may exist several $D_{m,\ell}^i \subset K(C)$ with points $y_k^i \in D_{m,\ell}^i$ that are identical to $8^i$. So, the above collapses $(8^i, s_2^i)$, $r \in [1,2]$ may appear as $(y_k^i,\{y_k,z_k\}^{G_i})$ in other dunce hats $D_{m,\ell}^i \subset K(C)$.

The complex obtained after collapsing all the dangling edges is denoted by $K^3(C)$. So far, we have, $K^1(C) \subset K^2(C) \subset K^3(C)$ and $K^4(C) \subset K^5(C)$.

Step 6: Collapsing the cycle-filling disks The 1-complex $H$ formed by the union of stems of $D_{m,\ell}^i$, for all $i \in [1,|\Xi|]$ described in Section 4.3 is clearly a subcomplex of $K^5(C)$. Let $\mathcal{S} = K(C) \setminus K^5(C)$ be the set described in Algorithm 1 obtained while building $K(C)$ from $K^5(C)$. It is, in fact, easy to check that $K^5(C) = H \sqcup \mathcal{S}$. Next, we show that $H$ is a connected graph.

Lemma 4.4. $H$ is connected.

Proof. First note that for every $\xi \in [1,|\Xi|]$, the stem of $D_{m,\ell}^i \subset K(C)$ is connected. In particular, the stem of $D_{m,\ell}^i \subset K(C)$ connects $i^\ast$ and $3^\ast$ to $z_k^i$ all $k \in \ell$.

Suppose $G_i$ and $G_j$ are two gates in $C$ such that $G_i$ is the predecessor of $G_j$. Then, in every dunce hat associated to $G_i$, there exists a $2$-edge that is connected to an $s$-edge to every dunce hat associated to $G_j$. That is, for all $p,q \in [1,n]$ there exists a $\xi_{(i,p)}^{(j,q)}$ that is identified to either $1^{(j,q)}$ or $3^{(j,q)}$. Thus, the stems of $G_i$ are connected to the stems of $G_j$. Now, since $C$ itself is a connected directed acyclic graph, it follows that the complex $H$ which is the the union of stems of $D_{m,\ell}^i$, for all $\xi \in [1,|\Xi|]$ is also connected.

Now, as in Algorithm 1 let $M$ be the matrix whose columns represent a basis $B$ of the cycle space of $H$. The cycles $z_i$ of $B$ are represented by columns $M^i$. Let $n_i$ denote the number of edges in $z_i$. Let the vertices $v_j^i \in z_i, j \in [1,n_i]$ and the edges $e_j^i \in z_i, j \in [1,n_i]$ be indexed so that $e_j^i$ represents the lowest entry (that is the pivot) for column $M^i$, and $v_j^i$ and $v_{j+1}^i$ form the endpoints of $e_j^i$. Simplices $\sigma_i^j$ are indexed so that the vertices incident on $\sigma_i^j$ are $v_j^i$ and $v_{j+1}^i$ and $v_i$. Please refer to Figure 6 for
Figure 6: The above figure shows a triangulated disk that fills the cycle $z_i$. Here, $e^1_i$ is the pivot edge of $z_i$. The gradient field starts with a gradient pair that includes the pivot edge.

**Algorithm 2** Procedure for collapsing cycle-filling disks

1: for $i \leftarrow 1, |B|$ do
2:   Execute the collapse $(e^1_i, \sigma^1_i)$.
3:   for $j \leftarrow 2, n_i$ do
4:     Execute the collapse $(\{v_i, v^j_i\}, \sigma^j_i)$.
5:   end for
6:   Execute the collapse $(v_i, \{v_i, v^{1}_i\})$.
7: end for
8: Return $T$.

An example of a cycle $z_i$ with six edges. The procedure to collapse all the disks in $K^5(C) \subset K(C)$ corresponding to cycles $z_i \in B$ is described in Algorithm 2.

Note that in Algorithm 2 it is possible to execute the collapse $(e^1_i, \sigma^1_i)$ for each $i$ because the matrix $M$ of basis $B$ is upper-triangular. This guarantees that after collapsing all the disks corresponding to cycles $z_k, k \in [1, i-1]$, $e^1_i$ is free. Denote the complex obtained at the end of Algorithm 2 as $T$.

**Step 7: Collapsing the tree** Now observe that Algorithm 2 removes all simplices in $D$ from $K^5(C)$. So, in particular, $T \subset H \subset K^5(C)$. Moreover, the pivot edges $e^1_i$ from cycles $z_i$ are also removed as part of 2-collapses in Line 2 of Algorithm 2. In other words, $T = H \setminus \bigcup_{i=1}^{|B|} \{e^1_i\}$, where $B$ forms a basis for cycle space of $H$.

**Claim 4.3.** $T$ is a tree.

**Proof.** By Lemma 4.4, $H$ is connected. Removal of each edge $e^1_i$ from $H$, decreases the $\beta_1$ of $H$ by 1, whereas $\beta_0$ of $H$ is unaffected. Hence, $T$ is connected. Moreover, since we destroy all $|B|$ cycles of $H$, $T$ has no cycles, proving the claim.

Next, we greedily collapse the tree $T$ to a vertex $v_0 \in K(C)$, which can be done in time linear in the size of $T$. Finally, we make $v_0$ critical. Let $V$ be the collection of gradient pairs arising out of all the collapses from Steps 1-7. Also, note that $K^1(C)$ is obtained from $K(C)$ by deletion of $m$ critical 2-simplices. Then, $K^1(C) \subset K^3(C)$. Then, $K^3(C)$ is obtained from $K^3(C)$ by deleting $m$ critical 1-simplices. Then, $K^5(C)$ is obtained from $K^3(C)$ by executing some 1-collapses. Finally, $K^5(C) \subset T \subset v_0$. So, using Lemma 2.5, we conclude that given a circuit $C$ with a satisfying assignment $A$ of Hamming weight $m$, we can obtain a vector field $V$ on $K(C)$ with $m$ critical 2-simplices, $m$ critical 1-simplices and a single critical vertex. Now, for a circuit $C$ if the assignment.
A is, in fact, optimal, that is, assuming \( m = \text{OPT}_{\text{MinMCS}}(C) \), then \( \text{MinrMM} \) for complex \( K(C) \) has a solution of size \((2m + 1) - 1\) giving us the following proposition.

**Proposition 4.5.** \( \text{OPT}_{\text{MinrMM}}(K(C)) \leq 2 \cdot \text{OPT}_{\text{MinMCS}}(C) \).

We highlight the entire collapsing sequence in Figure 2(a) and (b). First we perform the 2-collapses as described in Steps 2-3. Then, the 1-collapses for highlighted edges (in green) are executed. This is followed by 1-collapses for highlighted edges (in blue), whenever these edges are available. These edges may not be available if they are involved in 2-collapses in other dunce hats, or if they are made critical. After executing the above collapses, for edges in green and blue, we first execute the 2-collapses to erase all the cycle-filling disks, which leaves behind a tree supported by the edges in pink. The tree is then collapsed to a point.

## 5 Hardness results for Min-Reduced Morse Matching

For maps \( K \) and \( I \) described in Section 4.2 we can establish the following relations.

**Proposition 5.1.** \( \text{OPT}_{\text{MinrMM}}(K(C)) \leq 2 \cdot \text{OPT}_{\text{MinMCS}}(C) \).

*Proof.* For proof, please refer to Proposition A.5 in Section A.2 \( \Box \)

**Proposition 5.2.** \( m_{\text{MinrMM}}(K(C), \hat{V}) \geq 2 \cdot m_{\text{MinMCS}}(C, I(C, \hat{V})) \)

*Proof.* For proof, please refer to Proposition A.5 in Appendix A.2 \( \Box \)

**Proposition 5.3.** \( \text{OPT}_{\text{MinrMM}}(K(C)) = 2 \cdot \text{OPT}_{\text{MinMCS}}(C) \).

*Proof.* For proof, please refer to Proposition A.7 in Appendix A.2 \( \Box \)

**Proposition 5.4.**

\[
m_{\text{MinMCS}}(C, I(C, \mathcal{V})) - \text{OPT}_{\text{MinMCS}}(C) \leq \frac{1}{2} m_{\text{MinrMM}}(K(C), \mathcal{V}) - \text{OPT}_{\text{MinrMM}}(K(C))
\]

*Proof.* Combining Propositions 5.1 and 5.2 proves the claim. \( \Box \)

We will use the following straightforward fact about L-reductions.

**Theorem 5.5 (Williamson, Shmoys [72]).** If there is an L-reduction with parameters \( \mu \) and \( \nu \) from a minimization problem \( A \) to a minimization problem \( B \), and there is a \((1 + \delta)\)-approximation algorithm for \( B \), then there is a \((1 + \mu \nu \delta)\)-approximation algorithm for \( A \).

Next, we shall use the following result by Alekhnovich et al. [1].

**Theorem 5.6 (Theorem 3, [1]).** Unless \( \text{NP} \subseteq \text{QP} \), there is no polynomial time algorithm which can approximate \( \text{MinMCS} \) within a factor of \( 2^{\log((1-\epsilon)^{-1})} n \), for any \( \epsilon > 0 \).

**Theorem 5.7.** \( \text{MinrMM} \) cannot be approximated within a factor of \( 2^{\log((1-\epsilon)^{-1})} n \), for any \( \epsilon > 0 \) unless \( \text{NP} \subseteq \text{QP} \).

*Proof.* From Proposition 5.2 and Proposition 5.3 we conclude that the reduction from MinMCS to MinrMM is a strict reduction with parameters \( \mu = 2 \) and \( \nu = \frac{1}{2} \). By Theorem 5.5, if there exists a \((1 + \delta)\)-approximation algorithm for MinrMM, then there exists a \((1 + \mu \nu \delta)\)-algorithm for MinMCS. Using Theorem 5.6 the claim follows. \( \Box \)

Denote the standard parameterizations for MinMCS and MinrMM by MinMCS’ and MinrMM’ respectively. Using the map \( K : C \rightarrow K(C) \) that transforms instances of MinMCS to instances of MinMm, we define a new map \( \hat{K} : (C, k) \rightarrow (K(C), k') \) that transforms instances of MinMCS’ to instances of MinrMM’, where we set \( k' = 2k \).

**Proposition 5.8.** The map \( \hat{K} \) from MinMCS’ to MinrMM’ is

1. an FPT reduction,
2. a \((\delta, \delta')\)-gap preserving reduction for every function \( \delta \), where \( \delta'(k) = \delta\left(\left\lfloor \frac{k}{2} \right\rfloor \right) \).
Proof. 1. First note that, using Proposition 5.8 for any value of parameter $k$, 
\[ \text{OPT}_{\text{MinMCS}}(C) \leq k \iff \text{OPT}_{\text{MinMM}}(K(C)) \leq 2k. \]
Then, the conclusion follows immediately from observing that complex $K(C)$ can be constructed in time polynomial in the size of $C$.

2. Suppose an instance $(C, k)$ is a $\delta$-gap instance of MinMCS'. That is, either $\text{OPT}_{\text{MinMCS}}(C) \leq k$ or $\text{OPT}_{\text{MinMCS}}(C) \geq k\delta(k)$. So, we have two cases to check:

Using Proposition 5.8
- $\text{OPT}_{\text{MinMCS}}(C) \leq k \Rightarrow \text{OPT}_{\text{MinMM}}(K(C)) \leq 2k = k'$.
- If $\text{OPT}_{\text{MinMCS}}(C) \geq k\delta(k) \Rightarrow \text{OPT}_{\text{MinMM}}(K(C)) \geq 2k\delta(k) = k'\delta(k') = k'\delta'(k')$. \hfill \qed

**Theorem 5.9.** 1. MinMM is $\text{W}[P]$-hard.

2. MinMM has no fixed-parameter tractable approximation algorithm with any approximation ratio function $\rho$, unless FPT = $\text{W}[P]$.

Proof. The first statement follows immediately from Proposition 5.8 and Theorem 3.3.

Eichmeyer et al. [25] provides a standard template to carry over FPT inapproximability results using gap preserving FPT reductions. Accordingly to prove the second statement, we closely follow the line of argument from [25, Corollary 12]. In this case, the strong FPT inapproximability result for MinMCS is carried over to MinMM. We first reduce MinMCS' to the approximation variant of MinMM'. Assume there exists an FPT cost approximation algorithm for MinMM with approximation ratio $\rho$, where $\rho$ is any computable function.

Given an input $(C, k)$ for MinMCS', we first use the construction described in the proof of Theorem 6. Using this construction, we obtain a circuit $C$ of size $|C| = f(k) \cdot |C|^{O(1)}$ for some computable function $f$ in FPT time (with parameter $k$), such that

- $(C', \alpha(k))$ is a $\delta$-gap instance for some $\alpha : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to \mathbb{R}_{>1}$,
- and $\rho(2\alpha(k)) \leq \delta(\alpha(k))$.

Note that satisfying the second condition becomes possible since we have no restriction on the function $\delta$.

Using the FPT gap-preserving reduction described in Proposition 5.8 from MinMCS to MinMM on the $\delta$-gap instance $(C, \alpha(k))$, we get a $\delta'$-gap instance $(K(C), 2\alpha(k))$ of MinMM' with $\delta'(2\alpha(k)) = \delta(\alpha(k))$. We run $A$ on $(K(C), \rho(2\alpha(k)) \cdot 2\alpha(k))$.

If $\text{OPT}_{\text{MinMM}}(K(C)) \leq 2\alpha(k)$, then
\[ \rho(2\alpha(k)) \cdot 2\alpha(k) \geq \rho(\text{OPT}_{\text{MinMM}}(K(C))) \cdot \text{OPT}_{\text{MinMM}}(K(C)) \]
and $A$ accepts. If, on the other hand, $\text{OPT}_{\text{MinMM}}(K(C)) \geq \delta'(2\alpha(k))2\alpha(k)$ then
\[ \rho(2\alpha(k)) \cdot 2\alpha(k) < \delta(\alpha(k)) \cdot 2\alpha(k) = \delta'(2\alpha(k)) \cdot 2\alpha(k) \leq \text{OPT}_{\text{MinMM}}(K(C)), \]
and $A$ rejects.

Hence, using such an algorithm $A$ we could devise an FPT cost approximable algorithm for MinMCS some computable function $\rho$, which in turn would imply $\text{W}[P] = \text{FPT}$ using Theorem 3.3. \hfill \qed

6 Hardness results for MIN-MORSE MATCHING

Denoting the standard parameterizations for MinMM by MinMM', we now consider the map $\tilde{K} : (K, p) \mapsto (K, p + 1)$ that transforms instances of MinMM' (simplicial complexes) to instances of MinMM' (identical simplicial complexes).

**Proposition 6.1.** The map $\tilde{K}$ from MinMM' to MinMM' is

1. an FPT reduction,
2. a $(\delta, \delta')$-gap preserving reduction for every function $\delta$, where $\delta'(p) = \frac{(p-1)\delta(p-1)+1}{p}$. 

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Proof. 1. By definition, \( \text{OPT}_{\text{MinMM}}(K) = \text{OPT}_{\text{MinMM}}(K) + 1 \). So, for any value of \( p \),

\[
\text{OPT}_{\text{MinMM}}(K) \leq p \Leftrightarrow \text{OPT}_{\text{MinMM}}(K) \leq p + 1.
\]

So, the conclusion follows immediately.

2. Suppose an instance \((K, p)\) is a \( \delta \)-gap instance of \( \text{MinMM} \). That is, either \( \text{OPT}_{\text{MinMM}}(K) \leq p \) or \( \text{OPT}_{\text{MinMM}}(K) \geq p\delta(p) \). So, we have two cases to check:

- If \( \text{OPT}_{\text{MinMM}}(K) \leq p \), then
  \[
  \text{OPT}_{\text{MinMM}}(K) \leq p + 1 = p'.
  \]

- If \( \text{OPT}_{\text{MinMM}}(K) \geq p\delta(p) \), then
  \[
  \text{OPT}_{\text{MinMM}}(K) \geq p\delta(p) + 1 = p'\delta(p').
  \]

Combining Theorem 5.9 and Proposition 6.1, we obtain the following result:

**Theorem 6.2.** \( \text{MinMM} \) is \( W[\mathbf{P}] \)-hard. Furthermore, it has no fixed-parameter tractable approximation algorithm within any approximation ratio function \( p \), unless \( \mathsf{FPT} = \mathsf{W}[\mathbf{P}] \).

**Definition 6.1** (Amplified complex). Given a pointed simplicial complex \( K \) with \( n \) simplices, the amplified complex \( \hat{K} \) is defined as the wedge sum of \( n \) copies of \( K \).

**Lemma 6.3.** For any 2-complex \( K \), \( \text{OPT}_{\text{MinMM}}(\hat{K}) = n \cdot \text{OPT}_{\text{MinMM}}(K) + 1 \).

**Proof.** It is easy to check that the optimal vector field on \( \hat{K} \) is obtained by repeating the optimal vector field on \( K \) on each of the \( n \) copies of \( K \) in \( \hat{K} \), while making the distinguished vertex of \( \hat{K} \) the unique critical vertex in \( \hat{K} \).

**Lemma 6.4.** Using a vector field \( \hat{V} \) on \( \hat{K} \) with \( m + 1 \) critical simplices, one can compute a vector field \( V \) on \( K \) with at most \( \lceil \frac{m}{n} \rceil + 1 \) critical simplices in polynomial time.

**Proof.** Using Lemma 2.3, we can assume without loss of generality that \( \hat{V} \) has the distinguished vertex as its unique critical simplex. Restricting \( \hat{V} \) to each of the \( n \) copies of \( K \), the claim follows.

**Proposition 6.5.** For a fixed \( \epsilon > 0 \), let \( p = f(n) \), where \( f(n) = o(n) \). Then, for any \( \delta \in (0, 1) \) and \( \sigma = f(n) - \delta \), if there exists a \( \sigma \)-factor approximation algorithm for \( \text{MinMM} \), then there exists a \( \sigma \)-factor approximation algorithm for \( \text{MinMM} \).

**Proof.** For a complex \( K \), the optimal value of \( \text{MinMM} \) on \( K \) is denoted by \( \text{OPT}_{\text{MinMM}}(K) \). Suppose that there exists a \( \sigma \)-factor approximation algorithm \( A \) for \( \text{MinMM} \). If we apply \( A \) on \( \hat{K} \), then using Lemma 6.3, we obtain a vector field with at most \( \sigma \cdot \text{OPT}_{\text{MinMM}}(K) + 1 \) critical simplices. Then, using Lemma 6.4, we can compute a vector field \( V \) on \( K \) with at most \( m(V) \) critical simplices, where

\[
m(V) \leq \left\lceil \frac{\sigma \cdot n \cdot \text{OPT}_{\text{MinMM}}(K) + \frac{\sigma - 1}{n}}{n} \right\rceil + 1
\]

\[
\leq \frac{\sigma \cdot n \cdot \text{OPT}_{\text{MinMM}}(K) + \frac{\sigma - 1}{n}}{n} + 1
\]

\[
\leq \sigma \cdot \text{OPT}_{\text{MinMM}}(K) + \frac{\sigma - 1}{n} + 2
\]

using \( [x + y] \leq [x] + [y] + 1 \)

\[
= \sigma \cdot \text{OPT}_{\text{MinMM}}(K) + \frac{\sigma - 1}{n}
\]

using \( \left\lfloor \frac{\sigma - 1}{n} \right\rfloor = 0 \) for large \( n \),

which gives us

\[
m(V) - 1 \leq \sigma \cdot \text{OPT}_{\text{MinMM}}(K) + 1
\]

\[
m(V) - 1 \leq \sigma \cdot \text{OPT}_{\text{MinMM}}(K) - \delta \cdot \text{OPT}_{\text{MinMM}}(K) + 1
\]

\[
\leq \sigma \cdot \text{OPT}_{\text{MinMM}}(K)
\]

assuming \( \text{OPT}_{\text{MinMM}}(K) > \frac{1}{\delta} \).
The above analysis shows that one can obtain a $p$-factor approximation algorithm for MinrMM assuming a $p$ factor approximation algorithm for MinMM. Note that $n^{\frac{2}{p}}$ is bounded by a polynomial in $n$ given the fact that $\frac{1}{p}$ is a constant. So, we can assume without loss of generality that $\text{OPT}_{\text{MinrMM}}(K) > \frac{1}{p}$ based on the observation by Joswig and Pfetsch [39] that if $\text{OPT}_{\text{MinrMM}}(K) \leq c$, for some constant $c$, then one can find the optimum in $O(n^c)$ time.

Combining Theorem 6.6 and proposition 6.5, we conclude that for a fixed $\epsilon > 0$, MinMM cannot be approximated within a factor of $2^{\log(1-\epsilon)n} - \delta$, for any $\delta > 0$, unless $\text{NP} \subseteq \text{QP}$. But, in order to get rid of the $\delta$-term in the inapproximability bound for MinMM, we can do slightly better by allowing $\epsilon$ to vary. To make this precise, suppose there exists an $\iota > 0$ such that MinMM can be approximated within a factor of $2^{\log(1-\epsilon)n}$, and let $\delta \in (0, 1)$. Then, using Proposition 6.5 this would give a $2^{\log(1-\epsilon)n} + \delta$ approximation algorithm for MinMM. However, one can always find an $\epsilon > 0$ such that $2^{\log(1-\epsilon)n} + \delta = O(2^{\log(1-\epsilon)n})$. Then, for sufficiently large $n$, $2^{\log(1-\epsilon)n} + \delta < 2^{\log(1-\epsilon)n}$.

Hence, the assumption of a $2^{\log(1-\epsilon)n}$-factor approximation algorithm for MinMM contradicts Theorem 5.7. We can thus make the following claim.

**Theorem 6.6.** For any $\epsilon > 0$, MinMM cannot be approximated within a factor of $2^{\log(1-\epsilon)n}$, unless $\text{NP} \subseteq \text{QP}$.

## 7 An approximation algorithm for MIN-MORSE MATCHING

In this section, we assume without loss of generality that the input complex $K$ is connected. The algorithm can be described as follows. Given a 2-complex $K$, let $n$ be the number of 2-simplices. Assume without loss of generality that $\log n$ is an integer that divides $n$. Partition the set of 2-simplices of $K$ arbitrarily into $\log n$ parts each of size $\frac{n}{\log n}$. Writing $S$ for the partition, we note that the power set $P(S)$ of the parts has $n$ elements. The 2-simplices that belong to a part $s \in S$ is denoted by $K_s^{(2)}$. Each element of $P(S)$ gives us a subset $\hat{S}$ of $S$. To each $\hat{S}$ we can associate a binary incidence vector $\hat{j}(\hat{S})$ of length $\log n$ in the natural way. Let $\hat{K}$ be a complex induced by the 2-simplices belonging to the parts that belong to some $\hat{S} \subseteq S$. In this case, we may also write $\hat{K}$ as $\hat{K} = K(\hat{j}(\hat{S}))$ to emphasize the data from which $\hat{K}$ can be constructed. Compute such a complex $\hat{K}$ for each subset $\hat{S}$, and let $\hat{S}_{\text{max}}$ be the subset of largest cardinality whose induced complex $\hat{K}_{\text{max}}$ is erasable. In particular, $\hat{K}_{\text{max}} \subseteq L$ where $L$ is a 1-complex. Make all the 2-simplices in $K \setminus \hat{K}_{\text{max}}$ critical. The gradient on $\hat{K}_{\text{max}}$ is comprised of the erasing gradient of $\hat{K}_{\text{max}}$, namely $\Psi$, combined with the optimal gradient for $L$, namely $\gamma^1$. In what follows, we will show that this simple algorithm provides a $O(\frac{n}{\log n})$-factor approximation for MIN-MORSE MATCHING on 2-complexes.

**Lemma 7.1.** Let $\hat{K}_{\text{max}} = K(\hat{j}(\hat{S}))$ for some $\hat{S}$. Let $w_j$ be the Hamming weight of $\hat{j}(\hat{S})$, and let $\gamma = \log n - w_j$. Then, every Morse matching on $K$ has at least $\gamma$ critical 2-simplices.

**Proof.** Suppose that there exists a gradient vector field $\gamma^2$ with $\mu$ critical 2-simplices where $\mu < \gamma$. Let $\Psi$ denote the critical 2-simplices of $\gamma^2$. Define $\hat{K}_{\text{ncw}}^{(2)}$ as follows:

$$\hat{K}_{\text{ncw}}^{(2)} = \bigcup_{s \in S, \Psi \cap K_s^{(2)} = \emptyset} K_s^{(2)}.$$

As before, let $\hat{K}_{\text{ncw}}$ be the complex induced by simplices in $\hat{K}_{\text{ncw}}^{(2)}$. Then, $\hat{K}_{\text{ncw}} \subseteq K \setminus \Psi$. However, $K \setminus \Psi$ is erasable via gradient $\gamma^2$. So, by Lemma 2.4, $\hat{K}_{\text{ncw}}$ is erasable. But this contradicts the maximality of $\hat{K}_{\text{max}}$, proving the claim. \hfill $\square$

We denote the critical $k$-simplices of $\gamma^2 \cup \gamma^1$ by $c_k$.

**Lemma 7.2.** The gradient vector field $\gamma^2 \cup \gamma^1$ over $K$ has at most $\beta_1 - \beta_2 + 1 + 2\gamma \cdot \left(\frac{n}{\log n}\right)$ critical simplices.

**Proof.** From Lemma 7.1, we have

$$c_2 = \gamma \cdot \left(\frac{n}{\log n}\right). \tag{1}$$

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By [39, Lemma 4.2], $K^1$ is connected, and one can compute a gradient vector field $V^1$ on $K^1$ with a single critical vertex in linear time using depth first search starting from an arbitrary vertex in $K^1$ (see, e.g., [60]). We have by [29, Theorem 1.7],

$$c_0 - c_1 + c_2 = \beta_0 - \beta_1 + \beta_2.$$ 

Since $\beta_0 = c_0 = 1$, we have,

$$c_2 - \beta_2 = c_1 - \beta_1.$$ 

Thus, combining Equation (1) and Equation (2), we have

$$c_1 = \beta_1 - \beta_2 + \gamma \cdot \left(\frac{n}{\log n}\right).$$

The claim follows.

**Theorem 7.3.** The exists a $O\left(\frac{n}{\log n}\right)$-factor approximation algorithm for MIN-MORSE MATCHING on 2-complexes.

**Proof.** To begin with, we know from Tancer [69, Proposition 5] that a 2-complex is erasable if and only if greedily collapsing triangles yields a 1-dimensional complex. That is, erasability of a complex can checked in polynomial time. Since we check the erasability of $O(n)$ complexes each of size $O(n)$, the algorithm terminates in polynomial time.

Now, let $V_{\text{min}}$ be an optimal gradient vector field. By Lemma 7.1, $V_{\text{min}}$ has at least $\gamma$ critical 2-simplices. By weak Morse inequalities [29, Theorem 1.7], the number of critical 1-simplices of $V_{\text{min}}$ is at least $\beta_1$, and the number of critical 0-simplices of $V_{\text{min}}$ is $\beta_0 = 1$. Thus, an optimal gradient vector field has at least $O(\gamma + \beta_1)$ critical simplices. Combining this observation with Lemma 7.2, it follows that the algorithm described in this section provides an $O\left(\frac{n}{\log n}\right)$-factor approximation for MIN-MORSE MATCHING on 2-complexes.

8 Morse matchings for Costa–Farber complexes

The strong hardness results established in Section 6 belie what is observed in computer experiments for both structured as well as random instances [33,60]. In particular, the structured instances generated by Lutz [38,50] and by the RedHom and CHomP groups [33] and the random instances that come from Linial–Meshulam model [54] and the Costa–Farber model (referred to as type-2 random complexes in [60]) turn out to be “easy” for Morse matching [60]. We use the terms “easy” and “hard” in an informal sense. Here, by easy instances, we mean those instances for which simple heuristics give near-optimal matchings, and by hard instances we mean instances for which known heuristics produce matchings that are far from optimal. Unfortunately, the approximation bounds in [60], and in Section 7 of this paper do not explain the superior performance of simple heuristics in obtaining near-optimal matchings. Below, we provide some justification for this phenomenon from the lens of random complexes. We describe two types of gradients on the Costa–Farber complexes, namely,

(a) the apparent pairs gradient and

(b) the random face gradient.

Finally, we summarize the behavior of these gradients on Linial–Meshulam complexes.

8.1 The apparent pairs gradient.

We start with the definition of the apparent pairs gradient [7], which originates from work by Kahle on random complexes [10] and is used as a powerful optimization tool in software for computing persistent homology of Rips filtrations, like Ripser [7] and Eirene [34].

Let $K$ be a $d$-dimensional simplicial complex, and let $V$ denote the set of vertices in $K$. Suppose that the vertices in $V$ are equipped with a total order $<$. For two simplices $\sigma, \tau \in K$, we write $\sigma \prec_K \tau$ if $\sigma$ comes before $\tau$ in the lexicographic ordering.
Following [7][40], we call a pair of simplices \((\sigma, \tau)\) of \(K\) an apparent pair of \(K\) (with respect to the lexicographic order on simplices) if both

- \(\sigma\) is the lexicographically highest facet of \(\tau\), and
- \(\tau\) is the lexicographically lowest cofacet of \(\sigma\).

As observed in [40], the collection of all the apparent pairs in \(X(n, p)\) forms a discrete gradient on \(X(n, p)\); see also [7] Lemma 3.5. We denote this gradient by \(\mathcal{V}_1\).

Kahle [40] introduced the lexicographic apparent pairs gradient to construct matchings with provably few critical simplices on Vietoris-Rips complexes built on a random set of points in space. An earlier variation of this construction [41] was used by the same author to study random clique complexes. The Morse numbers from this matching provides upper bounds on the Betti numbers of an even larger class of random combinatorial complexes, namely, the multiparameteric complexes of Costa and Farber [16][17][18][19]. Our analysis closely follows Kahle’s work on random clique complexes [40] Section 7.

The Costa–Farber complex \(X(n, p)\) on a vertex set \(V\) of size \(n\) and a probability vector \(p = \{p_1, \ldots, p_{n-1}\}\) can be described as follows. First, add all the vertices in \(V\) to the complex \(X(n, p)\). Next, include every possible edge independently with probability \(p_1\). So far, this is the same as the Erdős–Rényi graph \(G(n, p_1)\). That is, the 1-skeleton \(X_1 = G(n, p_1)\). Subsequently, for every 3-clique in \(G(n, p_1)\), include a 2-simplex independently with probability \(p_2\) to obtain the 2-skeleton \(X_2\). More generally, consider an \(r\)-simplex \(\sigma\) defined on an \(r + 1\)-element vertex set \(\mathcal{V}_r \subset V\). If all the simplices of the set \(\partial \sigma\) are present in \(X_{r-1}\), then include \(\sigma\) in \(X_r\) with probability \(p_r\). Do this for every \(r + 1\)-element subset of \(V\) to obtain the \(r\)-skeleton \(X_r\). Following this process for every \(r \in [n - 1]\) gives the complex \(X_{n-1} = X(n, p)\).

Let \(\mathcal{V}_r\) denote the total number of critical \(r\)-simplices. Assume without loss of generality that \(V = \{1, \ldots, n\}\). Since there are \(\binom{n}{r-1}\) possible choices for \(\sigma\) with \(v_0 = j\), and since \(v_0 \in [1, n-r]\), we obtain the following expression for \(\mathbb{E}(m_r)\):

\[
\mathbb{E}(m_r) \leq \sum_{j=1}^{n-r} \binom{n-j}{r} \prod_{i=1}^{r} p_i^{\binom{i}{r+1}} \left(1 - \prod_{\ell=1}^{r+1} p_\ell^{\binom{\ell}{r}}\right)^{j-1}.
\]

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\]

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- \(\sigma\) is the lexicographically highest facet of \(\tau\), and
- \(\tau\) is the lexicographically lowest cofacet of \(\sigma\).

As observed in [40], the collection of all the apparent pairs in \(X(n, p)\) forms a discrete gradient on \(X(n, p)\); see also [7] Lemma 3.5. We denote this gradient by \(\mathcal{V}_1\).

Kahle [40] introduced the lexicographic apparent pairs gradient to construct matchings with provably few critical simplices on Vietoris-Rips complexes built on a random set of points in space. An earlier variation of this construction [41] was used by the same author to study random clique complexes. The Morse numbers from this matching provides upper bounds on the Betti numbers of an even larger class of random combinatorial complexes, namely, the multiparameteric complexes of Costa and Farber [16][17][18][19]. Our analysis closely follows Kahle’s work on random clique complexes [40] Section 7.

The Costa–Farber complex \(X(n, p)\) on a vertex set \(V\) of size \(n\) and a probability vector \(p = \{p_1, \ldots, p_{n-1}\}\) can be described as follows. First, add all the vertices in \(V\) to the complex \(X(n, p)\). Next, include every possible edge independently with probability \(p_1\). So far, this is the same as the Erdős–Rényi graph \(G(n, p_1)\). That is, the 1-skeleton \(X_1 = G(n, p_1)\). Subsequently, for every 3-clique in \(G(n, p_1)\), include a 2-simplex independently with probability \(p_2\) to obtain the 2-skeleton \(X_2\). More generally, consider an \(r\)-simplex \(\sigma\) defined on an \(r + 1\)-element vertex set \(\mathcal{V}_r \subset V\). If all the simplices of the set \(\partial \sigma\) are present in \(X_{r-1}\), then include \(\sigma\) in \(X_r\) with probability \(p_r\). Do this for every \(r + 1\)-element subset of \(V\) to obtain the \(r\)-skeleton \(X_r\). Following this process for every \(r \in [n - 1]\) gives the complex \(X_{n-1} = X(n, p)\).

Let \(\mathcal{V}_r\) denote the total number of critical \(r\)-simplices. Assume without loss of generality that \(V = \{1, \ldots, n\}\). Since there are \(\binom{n-r}{j}\) possible choices for \(\sigma\) with \(v_0 = j\), and since \(v_0 \in [1, n-r]\), we obtain the following expression for \(\mathbb{E}(m_r)\):

\[
\mathbb{E}(m_r) \leq \sum_{j=1}^{n-r} \binom{n-j}{r} \prod_{i=1}^{r} p_i^{\binom{i}{r+1}} \left(1 - \prod_{\ell=1}^{r+1} p_\ell^{\binom{\ell}{r}}\right)^{j-1}.
\]
Let $c_r$ denote the total number of $r$-simplices in $X(n, p)$. Then, the expected number of $r$-simplices in $X(n, p)$ is given by

$$
\mathbb{E}(c_r) = \binom{n}{r+1} \prod_{i=1}^{r} p_i^{r+1-i}.
$$

Therefore,

$$
\frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} \leq \frac{\binom{n}{r} \prod_{i=1}^{r} p_i^{r+1-i}}{\binom{n}{r+1} \prod_{i=1}^{r+1} p_i^{r+1-i}} = \frac{(r+1)}{(n-r) \prod_{i=1}^{r+1} p_i^{r+1-i}}.
$$

When $r$ is a fixed constant,

$$
\frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = O\left(\frac{1}{n} \prod_{i=1}^{r+1} p_i^{-r+1}\right).
$$

Assuming the denominator $n \prod_{i=1}^{r+1} p_i^{-r+1} \to \infty$, we obtain $\frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = o(1)$.

**Remark 8.** While the asymptotics of the ratio $\frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)}$ is informative, it would still be interesting to also say something about the ratio $\mathbb{E}\left(\frac{m_r}{c_r}\right)$. In particular, our results do not say anything about the variance of $m_r$ vis-a-vis the variance of $c_r$. Towards this end, the authors of [70] show that for the apparent pairs gradient (referred to as the lexicographic gradient in [70]) on random clique complexes, the variance of the number of critical $r$-simplices is of the same order as the variance of the total number of $r$-simplices for random clique complexes, for fixed $r$. Furthermore, the authors of [70] prove a multivariate normal approximation theorem for the random vector $(m_2, m_3, \ldots, m_r)$ for a fixed $r$. We recommend the results of [70] to the interested reader.

### 8.2 The random face gradient.

Kahle [40, Section 7] describes an alternative method for designing gradients on random clique complexes with parameter $p_1$ for which the following holds true.

$$
\mathbb{E}(m_r) = \binom{n+2}{r+2} \binom{n}{r+2} p_1^r.
$$

We extend Kahle’s strategy to the scenario where some of the $(r-1)$-simplices may be matched to $(r-2)$-dimensional facets and may not be available to be matched to $r$-dimensional cofacets. We call such simplices inadmissible simplices. On the other hand, $(r-1)$-simplices that are not matched to their facets are called admissible simplices. An admissible simplex along with a cofacet forms an admissible pair, whereas an inadmissible simplex along with a cofacet forms an inadmissible pair. Randomly match every $r$-simplex to one of its admissible facets. This strategy doesn’t give you a discrete gradient on the nose as there will be $(r-1)$-simplices that are matched to more than one cofacets, and there might also be some cycles. These events are termed as bad events. It suffices to make one pair of simplices critical per bad event. Once the corresponding simplices associated to all bad events are made critical, one indeed obtains a discrete gradient $V_2$. Bounding the expected number of bad events $B_r$ therefore gives a bound on the expected number of critical simplices. It is then shown that the total number of bad events for dimension $r$ is given by

$$
\mathbb{E}(B_r) \leq \binom{r+2}{2} \binom{n}{r+2} p_1^{r+2} p_1^{r+1}.
$$

This is because each bad event contains at least one pair of $r$-simplices meeting in an admissible $(r-1)$-simplex. The total number of vertices involved are, therefore, $r+2$. So there are $\binom{r+2}{2}$ choices of $(r+2)$-vertex sets, and for every choice of an $(r+2)$-vertex set, there are at most $\binom{r+2}{2}$ admissible pairs. Finally, for an admissible pair to be a bad pair, all but one edge must be present among the $(r+2)$ vertices. The expected number of simplices of dimension $r$ is given by

$$
\mathbb{E}(c_r) = \binom{n}{r+1} p_1^{r+1}.
$$
Dividing the two, we obtain
\[ \frac{\mathbb{E}(B_r)}{\mathbb{E}(c_r)} \leq \frac{\binom{r+2}{2}\binom{n}{r+1}}{\binom{n}{r+1}} p_1^r. \] (5)

Note that if \( r \) is a fixed constant, and \( np_1^r \to 0 \), then
\[ \frac{\mathbb{E}(B_r)}{\mathbb{E}(c_r)} = o(1). \]

Kahle’s method [40] for constructing gradients in the sparse regime easily extends to the Costa–Farber model, and Equation (5) generalizes as follows:
\[ \frac{\mathbb{E}(B_r)}{\mathbb{E}(c_r)} \leq \frac{\binom{r+2}{2}\binom{n}{r+1}}{\binom{n}{r+1}} \prod_{j=1}^{r} p_j^{(j)}. \]

Let \( r \) be a fixed constant, and \( m_r \) denote the critical simplices of \( V_2 \). Then, we obtain
\[ \frac{\mathbb{E}(B_r)}{\mathbb{E}(c_r)} = \frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = O \left( n \prod_{j=1}^{r} p_j^{(j)} \right). \] (6)

When the parameters \( p \) in \( X(n, p) \) are such that
\[ n \prod_{j=1}^{r} p_j^{(j)} \to 0, \quad \text{we obtain} \quad \frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = o(1). \]

### 8.3 Linial–Meshulam complexes

From Equations (3) and (6), in both cases, we obtain very good discrete gradients for typical instances. In particular, we obtain the following theorem.

**Theorem 8.1.** Let \( r \) be a fixed dimension. Then, for the regimes of Costa–Farber complexes \( X(n, p) \) that satisfy
\[ n \prod_{\ell=1}^{r+1} p_\ell^{(\ell+1)} \to \infty \quad \text{or} \quad n \prod_{j=1}^{r} p_j^{(j)} \to 0 \]
there exist respective discrete gradients that satisfy \( \frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = o(1) \).

Specializing the above analysis to Linial–Meshulam complexes we obtain the following corollary.

**Corollary 1.** For the regimes of Linial-Meshulam complexes \( Y_d(n, p) \) that satisfy
\[ (np \to \infty \text{ and } r+1 = d) \quad \text{or} \quad (np \to 0 \text{ and } r = d) \]
there exist respective discrete gradients that satisfy \( \frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = o(1) \).

In other words, the above corollary says that if \( (p = o(d) \) and \( r+1 = d \), or if \( (p = \omega(d) \) and \( r = d \)) for Linial–Meshulam complexes \( Y_d(n, p) \), then there exist respective discrete gradients that satisfy \( \frac{\mathbb{E}(m_r)}{\mathbb{E}(c_r)} = o(1) \).

To further refine our analysis, we now define a gradient \( \mathcal{U} \) on the entire Linial–Meshulam complex \( Y_d(n, p) \) as follows:
- when \( np \to \infty \), let \( \mathcal{U} \) be the apparent pairs gradient on \( Y_d(n, p) \).
- when \( np \to 0 \), \( \mathcal{U} \) is comprised of
  - the apparent pairs gradient for matching \( k-1 \)-simplices to \( k \)-simplices for \( k \in [d-1] \)
  - the random face gradient for matching the admissible \( (d-1) \)-simplices to \( d \)-simplices.

Also, let \( \text{opt} \) be the optimal discrete gradient on \( Y_d(n, p) \). Then, the following holds true.

**Theorem 8.2.** For the regimes of Linial-Meshulam complexes \( Y_d(n, p) \) that satisfy
\[ np \to \infty \quad \text{or} \quad np \to 0 \]
the discrete gradient \( \mathcal{U} \) satisfies \( \frac{\mathbb{E}(\mathcal{U})}{\mathbb{E}(\text{opt})} \to 1 \) as \( n \to \infty \).
For proof, we refer the reader to Appendix B.

We would like to contrast the above observations with a known result from literature. We start with a definition. If a \( d \)-dimensional simplicial complex collapses to a \( d - 1 \)-dimensional complex, then we say that it is \( d \)-collapsible. The following result concerning the \( d \)-collapsibility threshold was established by Aronshtam and Linial [3,4]. See also [71, Theorem 23.3.17].

**Theorem 8.3** (Aronshtam, Linial [3,4]). There exists a dimension dependent constant \( c_d \) for Linial–Meshulam complexes \( \mathcal{Y}_d(n,p) \) such that

- If \( p \geq \frac{c}{n} \) where \( c > c_d \) then with high probability \( \mathcal{Y}_d(n,p) \) is not \( d \)-collapsible,
- and if \( p \leq \frac{c}{n} \) where \( c < c_d \) then \( \mathcal{Y}_d(n,p) \) is \( d \)-collapsible with probability bounded away from zero.

Therefore, from Theorems 8.2 and 8.3 we conclude that sufficiently away from the \( d \)-collapsibility threshold, we expect to have very good gradients. It is natural to ask what happens at the threshold? In relation to what is known for hard satisfiability instances [67,73], are complexes of dimension larger than 2 sampled at the collapsibility thresholds of \( \mathcal{Y}_d(n,p) \) and more generally \( \mathcal{X}(n,p) \) hard? The experiments in [60] do not address this question. Secondly, for 2-complexes is it possible to define a simple random model built out of gluing dunce hats geared specifically towards generating hard instances for MIN-MORSE MATCHING for a wide range of parameter values? We are optimistic about affirmative answers to both questions, but leave this topic for future investigation.

### 9 Conclusion and Discussion

In this paper, we establish several hardness results for MIN-MORSE MATCHING. In particular, we show that for complexes of all dimensions, MIN-MORSE MATCHING with standard parameterization is \( \text{W}[P] \)-hard and has no FPT approximation algorithm for any approximation factor. We also establish novel (in)approximability bounds for MIN-MORSE MATCHING on 2-complexes. While we believe that this paper provides a nearly complete picture of complexity of Morse matchings, we conclude the paper with two remarks.

**Strengthening of hardness results** We conjecture that for complexes of dimension \( d > 2 \), MIN-MORSE MATCHING does not admit an \( f(n) \)-approximation algorithm for any \( f = o(n) \). In particular, a result of this nature would show that while the problem is hard for complexes of all dimensions, it is, in fact, slightly harder for higher dimensional complexes when compared to 2-dimensional complexes, from an inapproximability standpoint.

**Hardness of other related combinatorial problems** In [10], the complexity of the following problem (ERASABILITY EXPANSION HEIGHT) was studied: Given a 2-dimensional simplicial complex \( K \) and a natural number \( p \), does there exist a sequence of expansions and collapses that take \( K \) to a 1-complex such that this sequence has at most \( p \) expansions? A more natural variant (EXPANSION HEIGHT) would be to study the complexity of determining sequences of expansions and collapses (with at most \( p \) expansions) that take \( K \) to a point. From what we understand, the only obstruction in [10] towards considering the complexity of determining whether \( K \) is simple homotopy equivalent to a point with bounded number of expansions is that the gadgets used in [10] have 1-cycles. We believe that an immediate application of the cycle filling method introduced in this paper would be towards establishing \( \text{W}[P] \)-completeness for EXPANSION HEIGHT.

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A Reducing MinMCS to MinrMM

A.1 Structural properties of the reduction

Note that Lemmas A.1 and A.2 appear as Lemmas 4.1 and 4.4 in [9], but with slightly different notation. For the sake of completeness, we restate the lemma with the notation introduced in this paper.

Lemma A.1. For a circuit \( C = (V(C), E(C)) \), let \( F \) be a discrete Morse function on \( K(C) \) with gradient \( \nabla \):

(i) If \( s^c \in D^c_{m,\ell} \) is eventually free in \( K(C) \), then \( D^c_{m,\ell} \) is erasable in \( K(C) \).

(ii) Suppose that \( D^c_{m,\ell} \) is erasable in \( K(C) \) through a gradient \( \nabla \),

- If \( m = 1 \), then \( (s^1_1, \Gamma^1_1) \) is a gradient pair in \( \nabla \), and for any simplex \( \sigma^c \in D^c_{m,\ell} \) such that \( \sigma^c \notin \{s^1_1, \Gamma^1_1\} \) we have \( F(s^1_1) > F(\sigma^c) \).

- If \( m = 2 \), then \( (s^2_1, \Gamma^2_1) \in \nabla \) or \( (s^2_2, \Gamma^2_2) \in \nabla \), and for any simplex \( \sigma^c \in D^c_{m,\ell} \) such that \( \sigma^c \notin \{s^2_1, \Gamma^2_1\} \) for \( i = 1, 2 \), then we have, \( \max(F(s^2_i), F(s^2_\ell)) > F(\sigma^c) \).

Proof. Suppose \( s^c_1 \) is eventually free in \( K(C) \). Then there exists a subcomplex \( L \) of \( K(C) \) such that \( K(C) \downarrow L \) and \( s^c_1 \) is free in \( L \). Note that, by construction of \( D^c_{m,\ell} \), this implies that \( D^c_{m,\ell} \) is a subcomplex of \( L \). Now using the gradient specified in Figure 2, all the 2-simplices of \( D^c_{m,\ell} \) can be collapsed, making \( D^c_{m,\ell} \) erasable in \( K(C) \). This proves the first statement of the lemma. The last two statements of the lemma immediately follow from observing that the \( s \)-edges are the only free edges in complex \( D^c_{m,\ell} \), the simplices \( \{\Gamma^1_1\} \) are the unique cofaces incident on edges \( \{s^c_1\} \) respectively, and \( D^c_{m,\ell} \) is erasable in \( K(C) \) through the gradient \( \nabla \) of \( F \).

Lemma A.2. For any input gate \( G_i \), the subcomplex \( D^{G_i} \setminus \{\Gamma^{G_i}_1\} \) is erasable in \( K(C) \).

Proof. Under the reindexing scheme described in Remark 3, let \( \zeta_1 \) be such that \( D^{G_i} = D^{\zeta_1} \). Consider the discrete gradient specified in Figure 2 (a) as a gradient \( \nabla^{G_i} \) on \( D^{G_i} \subseteq K(C) \). First note that \( D^{G_i} \setminus \{\Gamma^{G_i}_1\} \) is erasable in \( D^{G_i} \) through the gradient \( \nabla^{G_i} \setminus \{(s_1^{G_i}, \Gamma^{G_i}_1)\} \). Moreover, all 1-simplices of \( D^{G_i} \) that are paired in \( \nabla^{G_i} \) with a 2-simplex do not appear in \( D^{G_i} \) for any edge \( \zeta_1 \neq \zeta_2 \). It follows that \( D^{G_i} \setminus \{\Gamma^{G_i}_1\} \) is erasable in \( K(C) \).

A.2 Reducing MinMCS to MinrMM: Backward direction

We intend to establish an L-reduction from MinMCS to MinrMM. To this end, in Section 4.3 and Section 4.4, we described the map \( K : C \mapsto K(C) \) that transforms instances of MinMCS (monotone circuits) to instances of MinrMM (simplicial complexes). In this section, we seek to construct a map \( \mathcal{I} \) that transforms solutions of MinrMM (discrete gradients \( \nabla \) on \( K(C) \)) to solutions of MinMCS (satisfying input assignments \( \mathcal{I}(C, \nabla) \) of circuit \( C \)). Recall that \( m_{\text{MinMCS}}(K(C), \nabla) \) denotes the objective value of some solution \( \nabla \) on \( K(C) \) for MinrMM, whereas \( m_{\text{MinMCS}}(C, \mathcal{I}(C, \nabla)) \) denotes the objective value of a solution \( \mathcal{I}(C, \nabla) \) on \( C \) for MinMCS.

Suppose that we are given a circuit \( C = (V(C), E(C)) \) with \( n = |V(C)| \) number of nodes. Also, for a vector field \( \mathcal{V} \) on \( K(C) \), we denote the critical simplices of dimension 2, 1 and 0 by \( m_2(\mathcal{V}) \), \( m_1(\mathcal{V}) \) and \( m_0(\mathcal{V}) \) respectively. Then, by definition,

\[
m_{\text{MinMCS}}(K(C), \mathcal{V}) = m_2(\mathcal{V}) + m_1(\mathcal{V}) + m_0(\mathcal{V}) - 1.
\]
In Section 4.4 we designed a gradient vector field \( \mathcal{V} \) on \( K(C) \) with \( m_2(\mathcal{V}) = m, m_1(\mathcal{V}) = m \) and \( m_0(\mathcal{V}) = 1 \), for some \( m \leq n \). We have from [29] Theorem 1.7, 
\[
m_0(\tilde{\mathcal{V}}) - m_1(\tilde{\mathcal{V}}) + m_2(\tilde{\mathcal{V}}) = m_0(\mathcal{V}) - m_1(\mathcal{V}) + m_2(\mathcal{V}).
\]
which gives \( m_0(\tilde{\mathcal{V}}) - m_1(\tilde{\mathcal{V}}) + m_2(\tilde{\mathcal{V}}) = 1 \). Since \( m_0(\tilde{\mathcal{V}}) \geq 1 \), this gives, for any vector field \( \tilde{\mathcal{V}} \) on \( K(C) \), the following inequality 
\[
m_2(\tilde{\mathcal{V}}) \leq m_1(\tilde{\mathcal{V}}).
\]
In particular, from Equation (7) and Equation (8), we obtain 
\[
m_{\text{MinMM}}(K(C), \tilde{\mathcal{V}}) \geq 2m_2(\tilde{\mathcal{V}}).
\]
Now, if \( m_2(\tilde{\mathcal{V}}) \geq n \), we set \( \mathcal{I}(C, \tilde{\mathcal{V}}) \) to be the set of all input gates of \( C \). Clearly, this gives a satisfying assignment and using Equation (9) also satisfies 
\[
m_{\text{MinMM}}(K(C), \tilde{\mathcal{V}}) \geq 2 \cdot m_{\text{MinMCS}}(C, \mathcal{I}(C, \tilde{\mathcal{V}})).
\]
So, for the remainder of this section, we assume that \( m_2(\tilde{\mathcal{V}}) < n \). In particular, for any non-output gate \( G_i \), with \( n \) blocks, at most \( n - 1 \) of them may have critical 2-simplices.

**Definition A.1** (2-paired edges). Given a vector field \( \mathcal{V} \) on a 2-complex \( K \), we say that an edge \( e \in K \) is 2-paired in \( \mathcal{V} \) if it is paired to a 2-simplex in \( \mathcal{V} \).

**Definition A.2** (properly satisfied gates). Suppose that we are given a circuit \( C \), and a vector field \( \mathcal{V} \) on the associated complex \( K(C) \). Then,

1. an ordinary gate \( G_q \) is said to be properly satisfied if there exists a \( j \in [1, n] \) such that
   - for an or-gate \( G_q \) at least of the two edges \( s_1^{(q,j)}, s_2^{(q,j)} \) is 2-paired (in \( \tilde{\mathcal{V}} \)), or
   - for an and-gate \( G_q \) both the edges \( s_1^{(q,j)}, s_2^{(q,j)} \) are 2-paired (in \( \tilde{\mathcal{V}} \)),
   - in both cases, the \( j \)-th block has no critical 2-simplices;
2. an input gate \( G_i \) is said to be properly satisfied if the dunce hat associated to it contains at least one critical 2-simplex,
3. the output gate \( G_o \) is said to be properly satisfied if \( G_o \) is an or-gate and at least one of the two inputs gates of \( G_o \) is properly satisfied, or if \( G_o \) is an and-gate and both input gates of \( G_o \) are properly satisfied.

**Lemma A.3.** Suppose that \( G_k \) is a non-output gate that is properly satisfied. Then,

1. if \( G_k \) is an or-gate, then at least one of the two gates that serve as inputs to \( G_k \) is also properly satisfied.
2. if \( G_k \) is an and-gate, then both gates that serve as inputs to \( G_k \) are also properly satisfied.

**Proof.** Assume without loss of generality that \( G_k \) is an and-gate, and the two inputs that go into \( G_k \), namely \( G_\ell \) and \( G_j \), are both input gates. Since \( G_k \) is properly satisfied, there exists \( p \in [1, n] \) such that \( s_1^{(k,p)}, s_2^{(k,p)} \) are 2-paired and the \( p \)-th block of \( G_k \) has no critical 2-simplices. Now suppose that either \( G_\ell \) or \( G_j \) is not properly satisfied. For the sake of argument, suppose that \( G_\ell \) is not properly satisfied. That is, \( \mathcal{D}_{\ell}^{(k,1)} \) has no critical 2-simplices and \( s_1^{(f,1)} \) is 2-paired. Note that \( s_1^{(k,p)} \) is identified to a \( t \)-edge in \( \mathcal{D}_{\ell}^{(k,1)} \). Using Lemma A.1 we obtain \( \hat{F}(s_1^{(k,p)}) > \hat{F}(s_1^{(k,p)}) \). Combining the two inequalities we obtain
\[
\hat{F}(s_1^{(k,p)}) > \hat{F}(s_1^{(k,p)})
\]
Moreover, \( s_1^{(k,p)} \) is identified to a \( t \)-edge in \( \mathcal{D}_{\ell}^{(k,p)} \), and by assumption \( \mathcal{D}_{\ell}^{(k,p)} \) has no critical 2-simplices and, hence \( s_1^{(k,p)} \) is 2-paired. Therefore, once again, using Lemma A.1 we obtain
\[
\hat{F}(s_1^{(k,p)}) > \hat{F}(s_1^{(k,p)})
\]
Since Equation (10) and Equation (11) combine to give a contradiction, we conclude that \( G_\ell \) is properly satisfied. All combinations of \( G_k \) as an \{and-gate, or-gate\}, and \( G_\ell \) and \( G_j \) as \{or-gates, input gates, and-gates\} give similar contradictions, proving the claim.

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Lemma A.4. Given a Morse function $\hat{F}$ on $K(C)$ with vector field $\hat{V}$, the output gate $G_o$ is properly satisfied.

Proof. Assume without loss of generality that $G_o$ is an or-gate, and the two inputs to $G_o$, namely $G_\ell$ and $G_j$ are non-input and-gates. Let $k \in [1, n]$ be such that the $k$-th copy of $G_o$ has no critical 2-simplices. Such a $k$ exists because by assumption we have less than $n$ critical simplices. Now, suppose that neither $G_\ell$ nor $G_j$ is properly satisfied.

Since, $G_\ell$ is not properly satisfied there exists a $p \in [1, n]$ such that either $s_1^{(\ell,p)}$ or $s_2^{(\ell,p)}$ is not 2-paired and $p$-th block has no critical 2-simplices (because by assumption we have less than $n$ critical simplices). Assume without loss of generality that $s_1^{(\ell,p)}$ is not 2-paired. Then, $s_2^{(\ell,p)}$ is 2-paired. Using Lemma A.1, we obtain $\hat{F}(s_2^{(\ell,p)}) > \hat{F}(s_1^{(\ell,p)})$. Now, $s_2^{(1,k)}$ is identified to a $t$-edge in $\delta D^{(\ell,p)}$. So, using Lemma A.1, we obtain $\hat{F}(\hat{s}_1^{(\ell,p)}) > \hat{F}(s_1^{(o,k)}).$ Combining the two inequalities, we obtain,

$$\hat{F}(s_1^{(\ell,p)}) > \hat{F}(s_1^{(o,k)}).$$

Similarly, there exists a $q \in [1, n]$ such that either $s_1^{(j,q)}$ or $s_2^{(j,q)}$ is not 2-paired. Assume without loss of generality that $s_2^{(j,q)}$ is not 2-paired. Hence, we can show that

$$\hat{F}(s_2^{(j,q)}) > \hat{F}(s_2^{(o,k)}).$$

Combining Equation (12) and Equation (13), we obtain:

$$\max(\hat{F}(s_1^{(\ell,p)}), \hat{F}(s_2^{(j,q)})) > \max(\hat{F}(s_1^{(o,k)}), \hat{F}(s_2^{(o,k)})).$$

But, $s_1^{(\ell,p)}$ and $s_2^{(j,q)}$ are paired and $\ell,p$ are non-input and-gates. Let $G_{\ell,p}$ be such that the paired and-gate. Then, by Lemma A.3, both the inputs to $G_{\ell,p}$ are properly satisfied. As a consequence of our indexing we have $j \in [1, k - 1]$, and owing to the inductive hypothesis, $G_{j,q}$ is satisfied. But, since $G_{\ell,p}$ is an or-gate, this implies that $G_{\ell,p}$ is also satisfied. Suppose that $G_{i_k}$ be an and-gate. Then, by Lemma A.3, both the inputs to $G_{i_k}$, say $G_{i_k}, G_{i_p}$ are properly satisfied. As a
consequence of our indexing we have \( j, p \in [1, k - 1] \), and owing to the inductive hypothesis, \( G_{j^*}, G_{i^*} \) are satisfied. But, since \( G_{i^*} \) is an and-gate, this implies that \( G_{i_k} \) is also satisfied, completing the induction.

Finally, using Lemma [A.4] the output gate is properly satisfied, and by the argument above it is also satisfied.

An immediate consequence of Claim [A.1] is the following: 
\[
m_2(\tilde{V}) \geq m_{\text{MinMCS}}(C, I(C, \tilde{V}))
\]

**Proposition A.5.** \( m_{\text{MinMM}}(K(C), \tilde{V}) \geq 2 \cdot m_{\text{MinMCS}}(C, I(C, \tilde{V})) \)

**Proof.** This follows immediately by combining Equation [9] and Equation [16].

Now, if the gradient vector field \( \tilde{V} \) is, in fact optimal for \( K(C) \), then \( \tilde{V} \) has a single critical 0-simplex. That is, \( m_0(\tilde{V}) = 1 \) Recall that in Section 4.4, we designed a gradient vector field \( V \) on \( K(C) \) with \( m_2(V) = m, m_1(V) = m \) and \( m_0(V) = 1 \), for some \( m \leq n \). From [20] Theorem 1.7, we have  
\[
m_0(\tilde{V}) - m_1(\tilde{V}) + m_2(\tilde{V}) = m_0(V) - m_1(V) + m_2(V).
\]

which gives us
\[
-m_1(\tilde{V}) + m_2(\tilde{V}) = 0
\]

From Equation [17], we conclude that \( \text{OPT}_{\text{MinMM}}(K(C)) = 2m_2(\tilde{V}) \).

By Equation [16], we have
\[
m_2(\tilde{V}) \geq m_{\text{MinMCS}}(C, I(C, \tilde{V})).
\]

Since by definition
\[
m_{\text{MinMCS}}(C, I(C, \tilde{V})) \geq \text{OPT}_{\text{MinMCS}}(C),
\]

we have the following proposition

**Proposition A.6.** \( \text{OPT}_{\text{MinMM}}(K(C)) \geq 2\text{OPT}_{\text{MinMCS}}(C) \).

Combining Proposition [4.5] and Proposition [A.6] we obtain the following proposition.

**Proposition A.7.** \( \text{OPT}_{\text{MinMM}}(K(C)) = 2\text{OPT}_{\text{MinMCS}}(C) \).

## B Morse matchings for Linial–Meshulam complexes

For Linial–Meshulam complexes \( \Upsilon_d(n, p) \),

- when \( np \to \infty \), let \( \mathcal{U} \) be the apparent pairs gradient on \( \Upsilon_d(n, p) \).
- when \( np \to 0 \), \( \mathcal{U} \) is comprised of
  - the apparent pairs gradient for matching \( k - 1 \)-simplices to \( k \)-simplices for \( k \in [d - 1] \)
  - the random face gradient for matching the remaining \( (d - 1) \)-simplices to \( d \)-simplices.

The apparent pairs gradient, and the random face gradient are described in Section 8.

Let \( V \) be the vertex set of \( \Upsilon_d(n, p) \), and \( v' \) be the lexicographically lowest vertex of \( V \). For each \( r \in [0, d] \), let \( c_r \) denote the total number of \( r \)-dimensional simplices in \( \Upsilon_d(n, p) \). Let \( m_r \) denote the total number of critical \( r \)-simplices of \( \mathcal{U} \) and \( \overline{m}_r \) be the total number of regular simplices of \( \mathcal{U} \). Also, let \( n_r \) and \( \overline{n}_r \) denote the total number of critical \( r \)-simplices and regular \( r \)-simplices respectively of the optimal discrete gradient on \( \Upsilon_d(n, p) \).

**Lemma B.1.** All the \( k \)-simplices of \( \Upsilon_d(n, p) \) for \( k \in [0, d - 2] \) are matched by \( \mathcal{U} \). In particular, \( \overline{m}_k = m_k = c_k \) for \( k \in [d - 2] \), and \( \overline{m}_0 = m_0 = V - 1 \).

**Proof.** Let \( \sigma \) be a \( k \)-simplex, where \( k \in [d - 2] \). If \( v' \in \sigma \), then \( (\sigma \setminus \{v\}, \sigma) \in \mathcal{U} \), whereas if \( v' \notin \sigma \), then \( (\sigma, \sigma \cup \{v'\}) \in \mathcal{U} \). That is for \( k \in [0, d - 2] \), \( n_k = m_k = 0 \), and \( \overline{m}_k = \overline{m}_k = c_k \). Moreover, any vertex \( v \neq v' \) is matched to the edge \( \{v, v'\} \).
Note that
\[
\frac{\mathbb{E}(|\mathcal{U}|)}{\mathbb{E}(|\text{opt}|)} = \frac{\mathbb{E}(2|\mathcal{U}|)}{\mathbb{E}(2|\text{opt}|)} = \frac{\mathbb{E}(\sum_{k=0}^{d} m_k)}{\mathbb{E}(\sum_{k=0}^{d} m_k)}
\]

**Theorem.** For the regimes of Linial-Meshulam complexes \(Y_d(n,p)\) that satisfy
\(np \to \infty\) or \(np \to 0\)
the discrete gradient \(\mathcal{U}\) satisfies \(\mathbb{E}(|\mathcal{U}|)/\mathbb{E}(|\text{opt}|) \to 1\) as \(n \to \infty\).

**Proof.** We consider the following two cases:

**Case 1** \(np \to \infty\)

By definition,
\[
m_d \geq c_{d-1} - m_{d-1} - c_{d-2}
\]
Also since the complex is \(d\)-dimensional, we get
\[
m_d \leq c_{d-1}.
\]
Using Equations (18) and (19) and lemma B.1, we obtain
\[
\mathbb{E}(\sum_{k=0}^{d} m_k) \leq \mathbb{E}(\sum_{k=0}^{d} c_k + c_{d-1}) = \sum_{k=0}^{d} c_k + c_{d-1}.
\]
\[
\mathbb{E}(\sum_{k=0}^{d} m_k) \geq \mathbb{E}(\sum_{k=0}^{d-2} c_k + c_{d-1} - m_{d-1} + c_{d-1} - m_{d-1} - c_{d-2})
\]
\[
= \sum_{k=0}^{d-1} c_k + c_{d-1} - c_{d-2} - 2\mathbb{E}(m_{d-1}).
\]
Therefore,
\[
\frac{\mathbb{E}(|\mathcal{U}|)}{\mathbb{E}(|\text{opt}|)} = \frac{\mathbb{E}(\sum_{k=0}^{d} m_k)}{\mathbb{E}(\sum_{k=0}^{d} m_k)}
\]
\[
\geq \frac{\sum_{k=0}^{d-1} c_k + c_{d-1} - c_{d-2} - 2\mathbb{E}(m_{d-1})}{\sum_{k=0}^{d-1} c_k + c_{d-1}}
\]
\[
= 1 + \frac{-c_{d-2} - 2\mathbb{E}(m_{d-1})}{\sum_{k=0}^{d-1} c_k + c_{d-1}}.
\]
Using Corollary 1 and the fact that in \(Y_d(n,p), \frac{c_k}{c_h} \to 0\) for \(j < k\) and \(j, k \in [0, d-1]\), we conclude that
\[
\frac{\mathbb{E}(|\mathcal{U}|)}{\mathbb{E}(|\text{opt}|)} \to 1.
\]

**Case 2** \(np \to 0\)

Since a regular \((d-1)\)-simplex is paired to either a \(d\)-simplex or a \((d-2)\)-simplex, we obtain
\[
m_{d-1} \leq c_d + c_{d-2},
\]
Therefore, using Equations (20) and (21) and lemma B.1
\[
\frac{\mathbb{E}(|\mathcal{U}|)}{\mathbb{E}(|\text{opt}|)} = \frac{\mathbb{E}(\sum_{k=0}^{d} m_k)}{\mathbb{E}(\sum_{k=0}^{d} m_k)}
\]
\[
\geq \frac{\sum_{k=0}^{d-2} c_k + (\mathbb{E}(c_d) - \mathbb{E}(m_d)) + c_{d-2} - c_{d-3}) + (\mathbb{E}(c_d) - \mathbb{E}(m_d))}{\sum_{k=0}^{d-2} c_k + \mathbb{E}(c_d) + c_{d-2} + \mathbb{E}(c_d)}
\]
\[
= 1 + \frac{-c_{d-3} - 2\mathbb{E}(m_d)}{\sum_{k=0}^{d-2} c_k + c_{d-2} + 2\mathbb{E}(c_d)}.
\]
Using Corollary 1 and the fact that in $\mathcal{Y}_d(n, p)$, $\frac{c_j}{c_k} \to 0$ for $j < k$ and $j, k \in [0, d - 1]$, we conclude that

$$\frac{\mathbb{E}(|U|)}{\mathbb{E}(|opt|)} \to 1.$$  

$\square$