Integral in the sense of principal value as a distribution over parameters of integration

M. L. Nekrasov
Institute for High Energy Physics, 142284 Protvino, Russia

Abstract

An integral in the sense of principal value of a singular function or of product of singular functions can appear itself as a singular function in some range of values of integration parameters. In this case, if necessary subsequently to integrate with respect to parameters, the problem arises about interpretation of the initial integral as a distribution over the integration parameters. A solution to this problem is offered, which is initiated by actual applications in quantum field theory.

1 Introduction

In many applications of quantum field theory one has to deal with integrals determined in improper sense. A classical example is the integrals of Feynman diagrams where the propagators of particles are determined (in Minkowski space) with a specific rule of the bypass of mass-shell singularities. With the loop corrections are switched on also the ultraviolet divergences appear in such integrals. The consistent and mathematically verified solution to the problem of “elimination” of the ultraviolet divergences is based on the conception of the extending of linear continuous functionals—what the coefficient functions of $S$-matrix are—from a class of rapidly decreasing functions onto a class of arbitrary regular functions. As a matter of fact the solution is based on applying the distribution-theory methods. For the first time the idea of such an approach for solving the problem of ultraviolet divergences was put forward by N.N.Bogolyubov and afterwards was realized by him in collaboration with O.S.Parasiuk, and further by many other authors (see bibliography and the detailed account in monography).

A characteristic feature of the solution to the problem of ultraviolet divergences is the appearance of free finite parameters in the theory (in the renormalizable theories they are absorbed in favor of the renormalized constants). From the point of view of theory of distributions the mentioned property is entirely natural as it is connected with the operation of the extending of linear continuous functionals. Nevertheless, in many cases the solving by the same methods of other problems can be realized with the subsequent elimination of the ambiguities. Actually this occurs if the solution implies imposing of additional condition(s). Well-known example is the definition of a propagator in Minkowski space where the bypassing rule is fixed by the causality condition. In specific field-theory applications the elimination of
ambiguities can be carried out by the imposing of a self-consistency condition (see nontrivial example in [8]).

Among the problems whose solution is based on the extending of linear continuous functionals but is realized without the emergence of ambiguities, a particular place is occupied by the problem of asymptotic expansion in a parameter of an integral defining some quantity (for example, the amplitude or probability of a physical process). Really, if the expansion is carried out before the calculation of the integral, then the expansion can lead to the appearance of singular functions in the integrand, which are not integrable in the conventional sense. In this case for giving a sense to divergent integrals one can take advantage of the theory of the extending of linear continuous functionals (the theory of distributions). However, in contrast to the case of renormalizations, the ambiguities arising therewith must be completely eliminated—as the initial integral before performing the asymptotic expansion was well-determined (by our assumption) and its expansion contained no ambiguities.\footnote{The problem of asymptotic expansion under the symbol of integral arises when the integral cannot be calculated for technical reasons, but an expansion of the integral in a parameter is required.} A general recipe of the elimination of the ambiguities is explained in [9]. Nevertheless in some complicated cases that involve the calculation of the repeated integrals this recipe is insufficient. For example, the method of [9] does not allow one to determine the expansion of an integral in a parameter if as a result the integral itself becomes a singular function in other parameters which, in turn, are considered as variables of the subsequent integration. The situation becomes even more complicated if the expanded integrand includes a product of singular functions. Exactly this occurs in the case of pair (multiple) production and decay of unstable particles when the process is described basing on the expansion in the coupling constant of Breit-Wigner factors which stand in probability (not in amplitude) [10, 11, 12]. As was noted in [12] in this case already in the third order of the expansion the product of factors determined via the principal value (VP) emerges, and giving a sense to the product is a nontrivial problem.

The purpose of the present paper is to investigate the above mentioned situation. In narrower sense our efforts will be directed on adding a sense to expressions of the type of an integral in VP sense of a singular function or of product of singular functions if the integral itself is a singular function of integration parameters. The problem of elimination of the ambiguities arising at the determining of expressions of such type will be resolved by the imposing of a simple condition certainly satisfied in the actual applications standing beyond the present article. (The analysis and calculations directly in the framework of the mentioned applications see in [13].)

The structure of this paper is as follows. In the next section we find a solution to the problem of definition of integral in the case when the integrand contains a single singularity regularized by VP prescription but the integral itself is a singular function of integration parameters. Section 3 is devoted to the determining of an integral with the product of two VP in the integrand. In section 4 we demonstrate the effectiveness of the designed method on a specific nontrivial example of calculation of integral. In section 5 we obtain a generalization of the method to the case of product of three, four and greater numbers of VP. In Conclusion the basic outcomes of the article are specified.
2 Integral of a single VP-pole as a distribution over integration parameters

Let us consider an integral of VP with varying (upper) limit of integration,

\[ F_n(y) = \int_{-\infty}^{y} dx \ VP \frac{1}{x^n} u(x, y). \]  

(1)

Here \( u(x, y) \) is a weight function usually called a test function \[2, 3, 4, 5\]. It is supposed that \( u(x, y) \) is distinct from zero only in a bounded area, and is finite and differentiable the necessary number of times.

Through well-known formula for the principal value of a pole of degree \( n \),

\[ VP \frac{1}{x^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^n}{dx^n} \ln(|x|), \]  

(2)

supplied with the instruction to take an integral by the method of “integration by parts” \[3, 4, 5\], one can obtain

\[ F_n(y) = \frac{1}{(n-1)!} \left[ -\int_{-\infty}^{y} dx \ \ln(|x|) u^{(n)}(x, y) + \ln(|y|) u^{(n-1)}(y, y) - \sum_{k=1}^{n-1} (k-1)! \frac{1}{y^k} u^{(n-k-1)}(y, y) \right] \]  

(3)

Here \( u^{(k)}(x, y) \equiv \partial^k / \partial x^k u(x, y) \) and \( u^{(k)}(y, y) \equiv \partial^k / \partial x^k u(x, y)|_{x=y}. \)

Expression (3) defines function \( F_n(y) \) at any \( y \neq 0 \). At \( y = 0 \) the value of \( F_n(y) \) is not determined as at \( n = 1 \) expression (3) contains a logarithmic singularity, and at \( n > 1 \) a power one. Correspondingly, function \( F_n(y) \) at \( n = 1 \) is integrable in a neighborhood of \( y = 0 \), and at \( n \geq 2 \) is not such in the conventional sense. Nevertheless, the meaning of function \( F_n(y) \) can be extended to make it integrable in the sense of distributions. The necessity in this operation arises if \( F_n(y) \) should be further integrated with respect to \( y \).

As a recipe, a solution to the problem of extending the meaning of a function containing a pole consists in assigning a principal value to this pole and adding a functional concentrated at the point of singularity of the pole. Actually the mentioned functional must be a sum of Dirac delta function and of its derivatives with arbitrary coefficients, and the degree of the high-order derivative should be less by 1 than the degree of singularity of the pole \[4\] (the degree of the high-order derivative is fixed by the number of necessary subtractions in the test function after which the integral becomes well-determined in the conventional sense). In the case of formula (3) the solution consists in replacing each \( 1/y^k \) by \( VP(1/y^k) + \sum_{l=0}^{k-1} C_l \delta^{(l)}(y) \), where \( C_l \) are the coefficients describing the parametric ambiguity.

Now we show that the ambiguities in formula (3), arising at the extending of meaning of the poles, can be completely removed by the imposing of a condition of independence of the result of integration from the order of calculation of repeated integrals. In effect, the mentioned condition means equivalence of the result of the repeated integration to the result of the multiple integration. (Of course, this is not the only possible condition, but it naturally arises at solving enough wide class of problems.)
So, let us turn to the initial formula (1) and consider functional \( \tilde{F}_n[u] \) specified by repeated integration,

\[
\tilde{F}_n[u] = \int_{-\infty}^{\infty} dy \int_{-\infty}^{y} dx \, VP \frac{1}{x^n} u(x, y).
\]

Our aim is to determine this functional so that to make it be equal to a multiple integral,

\[
\int_{-\infty}^{\infty} dx \, VP \frac{1}{x^n} \theta(y - x) \, u(x, y),
\]

or to a functional determined in a similar way, but with opposite order of the repeated integration,

\[
F_n[u] = \int_{-\infty}^{\infty} dx \, VP \frac{1}{x^n} \int_{x}^{\infty} dy \, u(x, y).
\]

It should be emphasized that in contrast to (4) functional (6) is well-determined since the integral \( dy \) in (6) defines a function from the space of test functions of one variable if \( u(x, y) \) belongs to the space of test functions of two variables.

By making use of the definition (2) for principal value and carrying out obvious calculations, we obtain

\[
F_n[u] = -\frac{1}{(n-1)!} \int_{-\infty}^{\infty} dx \, \ln(|x|) \left\{ \int_{-\infty}^{\infty} dy \, \frac{d^n u(x, y)}{dx^n} - \sum_{k=0}^{n-1} \frac{d^k}{dx^k} \left[ \frac{\partial^{n-k-1}}{\partial x^{n-k-1}} u(x, y) \right]_{x=y} \right\}.
\]

In (7) all integrals are considered in a proper sense and converge. Therefore we can change the order of integration in the first term. In the second term we substitute \( y \) for \( x \). Then we get

\[
F_n[\phi] = -\frac{1}{(n-1)!} \int_{-\infty}^{\infty} dy \left\{ \int_{-\infty}^{y} dx \, \ln(|x|) \frac{d^n u(x, y)}{dx^n} - \ln(|y|) \sum_{k=0}^{n-1} \frac{d^k}{dy^k} \left[ \frac{\partial^{n-k-1}}{\partial x^{n-k-1}} u(x, y) \right]_{x=y} \right\}.
\]

With taking into consideration (2) the later formula may be rewritten as

\[
F_n[\phi] = -\frac{1}{(n-1)!} \int_{-\infty}^{y} dy \left\{ \int_{-\infty}^{\infty} dx \, \ln(|x|) \frac{d^n \phi(x, y)}{dx^n} - \ln(|y|) \sum_{k=0}^{n-1} \frac{d^k}{dy^k} \phi(x, y)_{|y=x} + \sum_{k=1}^{n-1} (k-1)! \, VP \frac{1}{y^k} \frac{\partial^{n-k}}{\partial x^{n-k}} \phi(x, y)_{|y=x} \right\}.
\]

\[\text{The introducing of formula (5) may be considered as a heuristic trick useful for the transition to formula (6). Nevertheless, a precise mathematical sense can be added to (6), as well. Really, since a singular distribution can be considered as an improper limit of a conventional function with respect to some parameter (by means of auxiliary regularization), the multiple integral in (5) can be considered as a conventional integral with the posterior transition to the limit. Formulas (5) and (6) are equivalent from this point of view. At the same time, functional (4) remains uncertain as long as the transition to the limit (removal of auxiliary regularization) before the calculation of integral \( dy \) leads to a meaningless result. In effect we determine functional (4) by the imposing of condition to remove auxiliary regularization after the calculation of all repeated integrals.}\]
It can easily be seen that the expression in the curly brackets in (9) coincides with that in the r.h.s. of formula (3), considered with substitution VP-poles for the ordinary poles and without the adding of δ-functions and its derivatives.

So, we have shown that the condition of independence of the result of the repeated integration of VP from the order of calculation of the repeated integrals (the requirement of equivalence between repeated and multiple integrations) implies the VP prescription for the poles arising at carrying out the first integration in the repeated integrals. In the case when both limits of integration are variables and when VP is appeared with the “shifted” argument, one can easily obtain the following general formula:

\[
\int_a^b dx \, VP \frac{1}{(x-y)^n} \ u(x, \ldots) = \\
\frac{1}{(n-1)!} \left\{ - \int_a^b dx \ \ln(|x-y|) \ u^{(n)}(x, \ldots) + \ln(|b-y|) \ u^{(n-1)}(b, \ldots) - \ln(|a-y|) \ u^{(n-1)}(a, \ldots) \\
- \sum_{k=1}^{n-1} (k-1)! \left( VP \frac{1}{(b-y)^k} \ u^{(n-k-1)}(b, \ldots) - VP \frac{1}{(a-y)^k} \ u^{(n-k-1)}(a, \ldots) \right) \right\}. \tag{10}
\]

Here \( u(x, \ldots) \equiv u(x, y, a, b) \) and the superscripts in brackets mean the partial derivatives of the corresponding order with respect to the first argument. In the case when an integration \( dy \) is supposed as the next one, it is convenient to represent formula (10) in the form

\[
\int_a^b dx \, VP \frac{1}{(x-y)^n} \ u(x, \ldots) = - \frac{1}{(n-1)!} \int_a^b dx \ \ln(|x-y|) \ u^{(n)}(x, \ldots) \\
+ \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{d^k}{dy^k} \ln(|b-y|) \ u^{(n-k-1)}(b, \ldots) - \frac{d^k}{dy^k} \ln(|a-y|) \ u^{(n-k-1)}(a, \ldots) \] . \tag{11}

Here the derivatives with respect to \( y \) are understood in the sense of distributions, i.e. they are to be moved by the rule of “integration by parts” at the next integration \( dy \).

Formulas (10) and (11) admit various writings in special cases. For example, if the dependence on \( y \) in the test function can be separated out into a factor \( \phi(y) \), then there is the following formula:

\[
\int_a^b dx \, VP \frac{1}{(x-y)^n} \ u(x, a, b) \ \phi(y) = - \frac{1}{(n-1)!} \ \phi(y) \frac{d^n}{dy^n} \int_a^b dx \ \ln(|x-y|) \ u(x, a, b) . \tag{12}
\]

Here the derivatives are again understood in the sense of distributions. The validity of (12) follows from the comparison of the result of applying (11) to the l.h.s. of (12) with the result appearing in the r.h.s. after the change of integration variable \( x \rightarrow x + y \) and carrying out then the direct calculation as in conventional integral.

The other important corollary of formula (10) appears in the case when the test function does not depend on variable \( x \) within the limits of integration \( a \ldots b \). In this case at \( n = 1 \) we get

\[
\int_a^b dx \ VP \frac{1}{x-y} \ u(y, a, b) = u(y, a, b) \left[ \ln(|b-y|) - \ln(|a-y|) \right] . \tag{13}
\]
At \( n \geq 2 \) we obtain

\[
\int_a^b \frac{1}{(x - y)^n} \, u(y, a, b) = - \frac{u(y, a, b)}{n-1} \left( VP \frac{1}{(b - y)^{n-1}} - VP \frac{1}{(a - y)^{n-1}} \right). \tag{14}
\]

Let us remember once again that in formulas (10)-(14) the quantities \( a, b, y \) (or, at least, some of them) are considered as variables of a subsequent integration but not as parameters. In the case when the mentioned quantities are considered as parameters, the symbol \( VP \) in the r.h.s of the above formulas should be omitted and the formulas themselves have a sense only at uncoincident arguments occurring in the logarithms and/or poles.

### 3 Product of two \( VP \)

Now let us consider an integral with a more complicated structure containing a product of two \( VP \)-poles:

\[
\int_a^b \frac{1}{(x - z_1)^{n_1}} \frac{1}{(x - z_2)^{n_2}} \, u(x, \ldots). \tag{15}
\]

Unfortunately, the formulas of the previous Section do not allow one to add a sense to expression (15) since at \( z_1 = z_2 \) the integrand containing a product of singular functions is undetermined. Nevertheless, the method of the previous Section, basically, can be applied in this case, as well. Really, we can at first determine integral (15) on the assumption that \( z_1 \) and \( z_2 \) are the parameters of integration, not equal one another. As a result we obtain a singular function of \( z_1 \) and \( z_2 \). Then we extend the meaning of this function in the sense of distributions. For removal of ambiguities originating therewith we proceed to a repeated integral of triple multiplicity containing integration with respect to \( z_1 \) and \( z_2 \), and determine this integral by the imposing of condition of independence of the result from the order of integration. (It should be used the property that the integral is well determined if at first the integration is carried out with respect to \( z_1 \) and \( z_2 \), and only then with respect to \( x \).)

However, the above mentioned method in the case of two poles is found too cumbersome and hardly probable is justified, especially if there is a greater number of \( VP \) in the integrand. So let us use of a trick based on an independent determination of the product of two \( VP \) poles. (We emphasize, once again, that the necessity of reference to integral (15), instead of at once to the triple repeated integral with another order of integration, is caused by the fact that the indicated order of integration provides a practical solvability of a problem in some applications.)

At first we consider the case \( n_1 = n_2 = 1 \), and at \( x \neq z_1, x \neq z_2, z_1 \neq z_2 \) examine the following formula with the ordinary poles realized in the sense of conventional functions:

\[
\frac{1}{x - z_1} \frac{1}{x - z_2} = \frac{1}{z_1 - z_2} \left( \frac{1}{x - z_1} - \frac{1}{x - z_2} \right). \tag{16}
\]

The expressions in the both sides of formula (16) may be also considered as functionals determined on space of test functions vanishing at the coincidence of any pair of arguments. On such space of test functions there is still an equality (15) between both functionals.
Now let us state a problem about the extending of these functionals onto the all space of test functions. The solution we realize in two steps. At first we determine each pole by assigning \( VP \) and adding the \( \delta \)-function with arbitrary factor. Then we determine the product of \( VP \)-poles. For solving the latter problem we use the remarkable property of formula (16) which consists in the fact that its l.h.s. may be determined as a product of two \( VP \) if at first the integration is meant with respect to \( z_1 \) and/or \( z_2 \) and only then with respect to \( x \). At the same time, the r.h.s. of (16) with \( VP \) attached to the poles is well determined if, conversely, the integration with respect to \( x \) at first is supposed. Let us call the order of integration by regular if it coincides with the above mentioned one, and by irregular otherwise. Then we see that the r.h.s. of formula (16) can be used for determining the l.h.s. in the case of irregular order of integration, and the l.h.s. can be used for determining the r.h.s. in the corresponding case. In both cases the equating of one side of the formula to another side should be treated as the extending of a linear continuous functional. As was directed above, this operation is not unambiguous and can make sense only up to a functional concentrated at the point of singularity (uncertainty) of the initial unextended functional. In our case this means the necessity of adding the product of two \( \delta \)-functions with arbitrary coefficient to one of the sides of the resulting relation.

Below we present the result obtained with the taking into consideration of the symmetry at reading the formula from left to right:

\[
VP \frac{1}{x - z_1} VP \frac{1}{x - z_2} \overset{def}{=} VP \frac{1}{z_1 - z_2} \left[ VP \frac{1}{x - z_1} - VP \frac{1}{x - z_2} \right] + VP \frac{C_1}{z_1 - z_2} \left[ \delta(x - z_1) - \delta(x - z_2) \right] + C_2 \delta(x - z_1)\delta(x - z_2).
\]

Here the following notes are in order. First, we have not shown the contribution of \( \delta \)-function that has to be added to \( VP (z_1 - z_2)^{-1} \) in the r.h.s. since this contribution is zero in view of nulling the expression in square brackets at \( z_1 = z_2 \). Second, we have not written the contributions of \( \delta \)-functions added to single poles in the l.h.s. since by virtue of the symmetry the structure of the relevant contributions coincides with the structure of the second term in the r.h.s. of the formula. (Therefore, the mentioned contributions are absorbed by the second term in the r.h.s.) Third, we consider the product of \( \delta \)-functions in the r.h.s. of (17) to be determined so that the result of its integration is independent from the order of calculation of integrals. The latter requirement implies that \( \delta(x - z_1)\delta(x - z_2) = \delta(z_1 - z_2)\delta(x - z_2) = \delta(z_1 - z_2)\delta(x - z_1) \). Under this condition formula (17) can be read in the opposite direction, from right to left. In the latter case all terms with \( \delta \)-functions are to be transferred to the l.h.s. of the relation.

Coefficient \( C_1 \) in (17) can be determined through condition of the recursive invariance of the formula. Namely, let us demand the invariance of the formula at the removal of square

\(^3\)A general theory of extending of linear continuous functionals (regularization of singular functions) is discussed with enclosing of numerous illustrations in [4] and [5]. In the context of the problem under consideration it is appropriate to carry out an analogy with the theory of renormalizations in the field theory: the replacing of the ordinary poles by \( VP \)-poles and the adding to them of \( \delta \)-functions can be compared with the elimination of UV divergences accompanied by emergence of the ambiguities in subdiagrams; the determining of the product of two \( VP \) and the adding of the product of \( \delta \)-functions corresponds to the elimination of UV divergences and emergence of the ambiguities of the overall type in the diagram (see Section 29 in monography [1]).
brackets in the r.h.s. and determine the arising products of $VP$ by means of the formula (17) itself. Then we get

$$C_1 = 0.$$  \hfill (18)

Unfortunately, in doing so the coefficient $C_2$ remains uncertain, i.e. it cannot be fixed by the reasons of symmetry. So, the determining of $C_2$ is only possible by the imposing of an additional condition. As such condition we consider the requirement of independence of the result of integration from the order of calculation of the repeated integrals of the r.h.s. of (17). In the case of irregular order of integration this requirement will fix $C_2$ by equating the result of integration to that obtained at the regular order of the calculation of integrals. The easiest way to actualize this condition is to carry out calculations at some special choice of the test function. The elementary choice is the unit function in the integral with finite limits.

So, let us consider the following repeated integral of triple multiplicity:

$$I_{x,z_1,z_2} = \int_0^1 dx \int_0^1 dz_1 \int_0^1 dz_2 \frac{VP}{x - z_1} \frac{VP}{x - z_2}. \hfill (19)$$

Its direct calculation with the use of (13) leads to result

$$I_{x,z_1,z_2} = \frac{1}{3} \pi^2.$$ \hfill (20)

On the other hand, again by direct calculating we can get

$$I_{z_1,z_2,x} = \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 dx \frac{VP}{z_1 - z_2} \left[ VP \frac{1}{x - z_1} - VP \frac{1}{x - z_2} \right] = -\frac{2}{3} \pi^2. \hfill (21)$$

From (20), (21) and (17) we conclude

$$C_2 = \pi^2.$$ \hfill (22)

Result (22) can be obtained also on the basis of some skilful manipulation with Sokhotsky formula. Really, let us consider a product of two simple poles with the “causal” bypass of singularity (see below, formula (23)). It should be noted that this product is well determined on the space of test functions under consideration [5]. If integration $dz_1$ and $dz_2$ at first is implied (with any test function), then the Sokhotsky formula can be applied to each multiplier. Moreover, in the resulting expression the brackets can be removed, as well. Eventually we obtain

$$\frac{1}{x - z_1 + i0} \frac{1}{x - z_2 + i0} \leq VP \frac{1}{x - z_1} \frac{1}{x - z_2}$$

$$-1 \pi \left[ VP \frac{1}{x - z_1} \delta(x - z_2) + VP \frac{1}{x - z_2} \delta(x - z_1) \right] - \pi^2 \delta(x - z_1) \delta(x - z_2). \hfill (23)$$

Here the point placed above the equality symbol recalls that the equality has a sense only at the particular order of calculating the repeated integrals.
On the other hand, provided that the integration $\mathrm{d}x$ at first is carried out, the expression in the l.h.s. of (23) can be transformed to the form (see the first note below formula (17))

$$\text{VP} \frac{1}{z_1 - z_2} \left[ \frac{1}{x - z_1 + \mathrm{i}0} - \frac{1}{x - z_2 + \mathrm{i}0} \right].$$

(24)

Again by applying Sokhotsky formula and removing the brackets we get

$$\frac{1}{x - z_1 + \mathrm{i}0} - \frac{1}{x - z_2 + \mathrm{i}0} = \text{VP} \frac{1}{z_1 - z_2} \left[ \text{VP} \frac{1}{x - z_1} - \text{VP} \frac{1}{x - z_2} \right] - \pi \text{VP} \frac{1}{z_1 - z_2} \left[ \delta(x - z_1) - \delta(x - z_2) \right].$$

(25)

The point under the equality symbol in (25) recalls that here another (particular) order of calculation of the repeated integral is supposed.

Now by virtue of independence from the order of calculation of the repeated integrals of the product of “causal” factors, we equate the r.h.s. in (23) and (25). Then again we obtain (22).

So, ultimately formula (17) takes the form of

$$\text{VP} \frac{1}{x - z_1} \text{VP} \frac{1}{x - z_2} = \text{VP} \frac{1}{z_1 - z_2} \left[ \text{VP} \frac{1}{x - z_1} - \text{VP} \frac{1}{x - z_2} \right] + \pi^2 \delta(x - z_1)\delta(x - z_2).$$

(26)

The meaning of this formula is as follows: its r.h.s. defines the l.h.s. in the case of irregular order of calculation of integrals, at first $\mathrm{d}x$ and only then $\mathrm{d}z_1$ and/or $\mathrm{d}z_2$. This definition provides the equality of the result of integration of the r.h.s. to that obtained at the regular order of integration of the l.h.s., at first $\mathrm{d}z_1$ and/or $\mathrm{d}z_2$ and only then $\mathrm{d}x$. (It should be emphasized, once again, that the imposing of another condition may change the value of $C_2$.)

A generalization of formula (26) to the case of the product of poles of an arbitrary degree can be easily generated through the use of relation

$$\text{VP} \frac{1}{(x - z_1)^{n_1}} \text{VP} \frac{1}{(x - z_2)^{n_2}} = \frac{1}{(n_1 - 1)! (n_2 - 1)!} \frac{\mathrm{d}^{n_1 - 1}}{\mathrm{d}z_1^{n_1 - 1}} \frac{\mathrm{d}^{n_2 - 1}}{\mathrm{d}z_2^{n_2 - 1}} \left[ \text{VP} \frac{1}{x - z_1} \text{VP} \frac{1}{x - z_2} \right].$$

(27)

By substituting (26) into (27) we obtain after simple calculation

$$\text{VP} \frac{1}{(x - z_1)^{n_1}} \text{VP} \frac{1}{(x - z_2)^{n_2}} = \frac{\pi^2}{(n_1 - 1)! (n_2 - 1)!} \delta^{(n_1 - 1)}(x - z_1)\delta^{(n_2 - 1)}(x - z_2)
+ \sum_{k=0}^{n_1-1} \binom{n_2+k-1}{k} (-)^k \text{VP} \frac{1}{(z_1 - z_2)^{n_2+k}} \text{VP} \frac{1}{(x - z_1)^{n_1-k}}
+ \sum_{k=0}^{n_2-1} \binom{n_1+k-1}{k} (-)^k \text{VP} \frac{1}{(z_2 - z_1)^{n_1+k}} \text{VP} \frac{1}{(x - z_2)^{n_2-k}}.$$

(28)

Returning to integral (15) introduced in the beginning of this Section, we see that it can be determined by means of formula (28) and then can be calculated with the aid of the formulas of the previous Section.
In conclusion of the present Section we note that formula (26) in effect is not completely new. In particular, in [14] a similar formula was derived, presented in the form of a relation between repeated integrals of double multiplicity, and in [15] an equivalent up to notation formula was presented (but without derivation and comments). However the derivation in [14], which is based on calculation of the conditional limits with respect to parameter of the conventional integrals, substantially differs from our derivation which is based on the extending of linear continuous functionals. The basic advantage of our derivation is the universality and flexibility of the mathematical tools in operation. This reflects, in particular, in a possibility of automatic generalization of the results to the case of any multiplicity of the integrals, and also in the extreme transparency and brevity of the proposed solution.

4 Nontrivial example of calculation of integrals

Let us consider an example of the use of formula (26) close to that which appears in some actual applications. Namely, let us consider an integral over a simplex

\[ I(z) = \int_0^\infty \int_0^\infty dx \, dy \, \theta(2 + z - x - y) \, VP \frac{1}{x - 1} \, VP \frac{1}{y - 1}. \]  \hspace{1cm} (29)

At once we note that at \( z > 0 \) the point of singularity of both poles \( \{ x = 1, \, y = 1 \} \) certainly falls on the integration area. At \( z < 0 \) only one of the poles can be singular. The case \( z = 0 \) in some sense is transitional. Simultaneously this case is specific-singular because at \( z = 0 \) the ambiguity of \( \theta \)-function is joined to the singularity in integrand. (So, at \( z = 0 \) integral (29) requires of additional determining. A possible way is noted in the footnote on page [4].)

Integrals of the type of (29) arise at calculating the probabilities of the processes of pair production and decay of unstable particles in the approach of a modified perturbation theory (MPT) based on expansion in the coupling constant of the Breit-Wigner factors standing in the probability (not in amplitude). Both \( VP \)-poles in (29) in this connection correspond to the particular contribution emerging from the product of two Breit-Wigner factors in the next-next-to-leading order of the expansion of the cross-section. The points \( x = 1 \) and \( y = 1 \) correspond to the positions of the mass-shells of resonances. Quantity \( z \) stands for the energy of exclusive process counted off from the threshold of the pair production. (See [12] for the establishing of correspondence, and [13] for computation of an actual processes.)

Let us represent (26) in the form of repeated integral and take advantage of formula (13). Then we obtain

\[ I(z) = \int_0^{2+z} dx \, VP \frac{1}{x - 1} \int_0^{2+z-x} dy \, VP \frac{1}{y - 1} = \int_0^{2+z} dx \, VP \frac{1}{x - 1} \ln |1 + z - x|. \]  \hspace{1cm} (30)

Unfortunately, at \( z = 0 \) the last integral in (30) is not determined. Nevertheless at \( z \neq 0 \) its calculation can be carried out in a direct way by separating the range of integration onto the sub-ranges. (Note that in some complicated cases this is not always possible to make.) Omitting tiresome calculations, we write down the result at \( z > 0 \):

\[ I(z) = 2 \text{dilog}(1 + z^{-1}) + \ln^2(z) - \frac{\pi^2}{6}, \]  \hspace{1cm} (31)
\[
dilog(z) \equiv \int_1^z dt \, \frac{\ln(t)}{1 - t}.
\] (32)

The more perfect method of calculation of \( I(z) \) is based on the change of variables \( x + y = \xi, \ x - y = 2\eta \) (therewith the symmetry of the going through of the integration area is achieved) and on the usage of formula (26):

\[
I(z) = 2 + z \int_0^{\xi/2} d\xi \int_{-\xi/2}^{\xi/2} d\eta \left\{ VP \frac{1}{\eta + \xi/2 - 1} - VP \frac{1}{\eta - \xi/2 + 1} - \pi^2 \delta(\xi - 2)\delta(\eta) \right\}
\]

\[
= 2 + z \int_0^{\xi/2} d\xi \left\{ \frac{\ln|\xi - 1|}{\xi - 2} - \pi^2 \theta(z) \right\}.
\] (33)

The last integral in (33) can be calculated at any \( z \). In particular, at \( z > -1 \) we get

\[
I(z) = -2 \log(1 + z) + \frac{\pi^2}{2} - \pi^2 \theta(z).
\] (34)

At \( z > 0 \) the expressions in the r.h.s. of (31) and (34) are equal each other by virtue of the relation

\[
2 \log\left(1 + z^{-1}\right) + 2 \log(1 + z) + \ln^2(z) + \frac{\pi^2}{3} = 0.
\] (35)

The validity of (35) can easily be verified by the differentiation of the l.h.s. with taking into consideration (32) and by calculating separately its particular value, for example, at \( z = 1 \).

Thus, at \( z > 0 \) the both above mentioned calculations are equivalent. Nevertheless, the second method of calculation of integral (29) allows one to solve the problem of the going through the range of the threshold of pair production of unstable particles, a stumbling-block from the point of view of calculations of [12]. (In actual applications a singularity arises at \( z = 0 \), which needs in a regularization.) Besides, outside the threshold (at \( z \neq 0 \)) the calculation by the second method is considerably simpler, which is very important from the point of view of the practical solvability of a problem.

5 Product of several VP

The results of Section 3 may be generalized by induction to the case of any number of VP-poles. So, the product of three poles of degree 1 can be determined via multiplying both sides of formula (26) by one more pole. As a result in the r.h.s. we obtain the product of no more than two VP-poles of \( x \). By virtue of (26) this product is well determined. Furthermore, by sequential applying (26) the result can be reduced to the form of a sum of single poles of \( x \). Written in the completely symmetric form, the result is

\[
\prod_{n=1}^{3} VP \frac{1}{x - z_n} = \sum_{n=1}^{3} VP \frac{1}{x - z_n} \left[ \prod_{k \neq n}^{3} VP \frac{1}{z_n - z_k} - \frac{\pi^2}{3} \prod_{k \neq n}^{3} \delta(z_n - z_k) + \pi^2 \prod_{k \neq n}^{3} \delta(x - z_k) \right].
\] (36)
Formula for the product of four poles can be obtained by multiplying both parts of (36) by one more pole, or by multiplying formula (26) by itself. Both methods after reducing to the form of symmetrized sum of single poles of $x$ lead to result:

$$\prod_{n=1}^{4} VP \frac{1}{x-z_n} = \sum_{n=1}^{4} VP \frac{1}{x-z_n} \left[ \prod_{k \neq n} VP \frac{1}{z_n-z_k} + \frac{\pi^2}{3} \sum_{l \neq n} VP \frac{1}{z_l-z_n} \prod_{k \neq n, k \neq l} \delta(z_n-z_k) \right]$$

$$+ \frac{\pi^2}{4} \sum_{P(z_1,\ldots,z_4)} VP \frac{\delta(x-z_1)\delta(x-z_2)}{(z_1-z_3)(z_2-z_4)} - \pi^4 \prod_{n=1}^{4} \delta(x-z_n).$$

(37)

Here indices $k$ and $l$ run values 1,2,3,4 except for the values indicated under the symbols of sum or product. The summation in the second term in the r.h.s. is carried out over the all permutations of $\{z_1, z_2, z_3, z_4\}$.

The process of multiplying by a simple VP-pole can be continued. In doing so at each step of the induction in the r.h.s. we obtain the product of only two VP-poles of $x$, which in view of (26) is a well-determined quantity. After the necessary number of steps we can obtain a formula that determine the product of any number of simple VP-poles. Then by the differentiating with respect to parameters, similarly as in (27), we can derive the result for the product of any number of VP-poles of any degree. In effect this procedure is trivial. In view of awkwardness we do not write down the ultimate result.

6 Conclusion

So, if an integral in the sense of principal value of a singular function or of product of singular functions appears by singular function of the parameters of integration, then the integral can be determined in the sense of distributions. The ambiguities arising therewith are in control and can be eliminated by the imposing of the condition of independence of the result of integration from the order of calculation of the repeated integrals. (The mentioned condition, of course, is not unique but naturally arises at solving the wide class of problems.) In the case of a single pole with VP-prescription in the integrand, the result is described by formulas (10) and (11). The essence of these formulas consists in assigning VP-prescription, again, for the poles arising at the conventional calculating of integral. The case of product of two VP-poles in the integrand is nontrivial, but it can be reduced to the case with a single VP-pole via the reduction formulas (26) and (28). The case with many VP-poles is easily considered by induction.

The results of the present article are extremely important for a systematic description of the processes of pair (multiple) production and decays of unstable particles in the high orders of perturbation theory. Furthermore, in view of the obvious universality they can be applied in other applications, as well, containing the repeated integration of singular functions determined in the sense of principal value.

The author is grateful to V.A.Petrov for valuable notes, and also to A.I.Alekseev for the indication to Ref. [15], and A.Bassetto for the indication to Ref. [14].
References

[1] N.N.Bogolyubov, D.V.Shirkov. Introduction to the theory of quantized fields. 3rd ed. New York: Wiley, 1980.

[2] S.L.Sobolev. Mat. sbornik I, 43 (1936) 39.

[3] L.Schwartz. Theorie des Distribution. I, II, Paris, 1950-51.

[4] I.M.Gelfand and G.E.Shilov. Generalized functions. V.1. Properties and operations. New York: Academic Press, 1964; V.2. Spaces of fundamental and generalized functions. New York: Academic Press, 1967; V.S.Vladimirov. Generalized functions in mathematical physics. Moskow: Nauka, 1976.

[5] N.N.Bogolyubov, A.A.Logunov, I.T.Todorov. Introduction to axiomatic quantum field theory. Benjamin. 1975; N.N.Bogolyubov, A.A.Logunov, A.I.Oksak, I.T.Todorov. General principles of quantum field theory. Dordrecht: Kluwer. 1990.

[6] N.N.Bogolyubov. Doklady USSR Acad. Sci. 82 (1952) 217.

[7] N.N.Bogolyubov, O.S.Parasiuk. Izv. USSR Acad. Sci. Ser. Mat. 20 (1956) 585; N.N.Bogolyubov, O.S.Parasiuk. Acta Math. 97 (1957) 227.

[8] M.L.Nekrasov, V.E.Rochev. Theor. Math. Phys. 74 (1988) 108; M.L.Nekrasov, V.E.Rochev. Dynamical chiral symmetry breaking by QCD infrared singularities. Preprint IHEP 86-186. Serpukhov. 1986.

[9] F.V.Tkachov. Int.J.Mod.Phys. A8 (1993) 2047; F.V.Tkachov. Phys.Lett. B412 (1997) 350.

[10] M.L.Nekrasov. Eur.Phys.J. C19 (2001) 441.

[11] M.L.Nekrasov. Gauge-invariant description of W-pair production in NLO approximation. in: Proc. of XV International Workshop QFTHEP’2000, ed. by M.N.Dubinin et al., SINP MSF, Moscow, 2000, p.218 [hep-ph/0102284]

[12] M.L.Nekrasov. Phys.Lett. B545 (2002) 119.

[13] M.L.Nekrasov, in preparation.

[14] N.I.Muskhelishvili. Singular Integral Equations, Noordhoff LTD, Groningen, 1953.

[15] A.Bassetto, R.Soldati. Nucl.Phys. B276 (1986) 517.