COVARIANCE MATRICES OF LENGTH POWER FUNCTIONALS OF RANDOM GEOMETRIC GRAPHS – AN ASYMPTOTIC ANALYSIS

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ABSTRACT. Asymptotic properties of a vector of length power functionals of random geometric graphs are investigated. More precisely, its asymptotic covariance matrix is studied as the intensity of the underlying homogeneous Poisson point process increases. This includes a systematic discussion of matrix properties like rank, definiteness, determinant, eigenspaces or decompositions of interest. For the formulation of the results a case distinction is necessary. Indeed, in the three possible regimes the respective covariance matrix is of quite different nature which leads to different statements. Finally, stochastic consequences for random geometric graphs are derived.

1. INTRODUCTION

Studying data seen as point clouds is of great interest in data analysis, machine learning, and related fields. One very interesting recent approach is topological data analysis where geometric as well as topological information is investigated for applications like the use of persistent homology or visualization methods (see, e.g., [4], [5], [7], [8], [24]). In particular, in the aforementioned areas, simplicial complexes and their properties are of significance like Vietoris-Rips as well as Čech complexes (see [7]). Their common 1-skeleton is a geometric graph and is one of the main objects in the following studies.

More precisely, given a point cloud $X \subseteq \mathbb{R}^d$ and a given value $\delta$, the geometric graph $\mathcal{G}(X, \delta)$ is defined as follows. Vertices are the points from $X$ and there exists an edge between two such points if and only if the distance between them is smaller or equal to $\delta$.

A central idea of stochastic geometry to study generic behaviours of objects of interest is the use of point processes to construct random ones. In particular, this yields random simplicial complexes and geometric graphs (see, e.g., [1], [2], [3], [12], [14], [18]), where the vertices originate from a Poisson point process $\eta$ (see [15]). Such graphs have been studied from various points of views (see, e.g., [19]). Note that here the original approach is due to Gilbert for modelling networks [10].

In this paper we consider a sequence of homogeneous Poisson point processes $\eta_t$ with intensity $t$. It is always assumed that the processes are supported in a convex compact set $W \subseteq \mathbb{R}^d$. Given the processes $\eta_t$ and a sequence of distances $\delta_t$, we study asymptotic behaviours of the induced random graphs $\mathcal{G}(\eta_t, \delta_t)$ as $\delta_t \to 0$ for $t \to \infty$.

2020 Mathematics Subject Classification. Primary: 15B52, 60D05; Secondary: 05C80, 60G55.

Key words and phrases. covariance matrix, length power functional, Poisson point process.
Recall, that in the random geometric graph $G(\eta_t, \delta_t)$ there exists an edge between vertices $x, y \in \eta_t$ with $x \neq y$ if and only if $\|x - y\| \leq \delta_t$. In the past, Poisson functionals have been investigated at various places (see, e.g., [16], [17] or [22]). For example, research work about a length power functional $L_{\tau}^{(\tau)}$ of a random geometric graph has been carried out. In particular, Reitzner–Schulte–Thäle [19] studied certain algebraic properties of the asymptotic covariance matrix $\Sigma_n$ of a vector $(\tilde{L}_{\tau}^{(\tau_1)}, \ldots, \tilde{L}_{\tau}^{(\tau_n)})$ of normalized and centered length power functionals. Taking especially this work as a motivation the main goals of this paper are:

(i) Expanding the analysis of $\Sigma_n$ as extensive as possible concerning algebraic aspects like rank, definiteness, determinant, eigenspaces and selected decompositions.

(ii) Using the gained algebraic results in (i) for a stochastic analysis on the entry wise length power functionals in the considered vector.

In dependence to the asymptotic behavior of the term $t\delta^d_t$, which occurs as the expected degree of a vertex in the random geometric graph, the following regimes arise:

- **subcritical (sparse) regime** for $t\delta^d_t \xrightarrow{t \to \infty} 0$,
- **critical (thermodynamic) regime** for $t\delta^d_t \xrightarrow{t \to \infty} c \in \mathbb{R}_{>0}$,
- **supercritical (dense) regime** for $t\delta^d_t \xrightarrow{t \to \infty} \infty$.

The structure of the covariance matrix $\Sigma_n$ differs significantly in each regime. This is introduced in Section 2. Remarkably, the matrix in the critical regime can be decomposed into the corresponding matrices of the other two regimes. For the convenience of the reader we summarize in Section 3 all main results of the manuscript. The asymptotic analysis needs tools related to symmetric positive semidefinite matrices, which are discussed in Section 4. The core part of our investigations is started right afterwards. Due to the different matrix structures in the regimes, there are separate examinations in Sections 5–7. Every question of interest is answered in the supercritical regime. Beside substantial progress, in the other regimes there remain also open problems. Cross-regime observations related to decompositions are discussed in Section 8. In Section 9 we apply our results to the original stochastic situation. It is not too difficult to show (see Theorem 9.1) that in the supercritical regime

$$\tilde{L}_{\tau}^{(\tau)} - \frac{d}{\tau + d} \tilde{L}_{\tau}^{(0)} \xrightarrow{\mathbb{P}} 0 \text{ as } t \to \infty.$$ 

Thus, asymptotically the number of edges $\tilde{L}_{\tau}^{(0)}$ determines all other length power functionals. More involved are the investigations in the other regimes. In Theorem 9.2 we prove that

$$\tilde{L}_{\tau}^{(\tau)} - \frac{d}{\tau + d} \tilde{L}_{\tau}^{(0)} \xrightarrow{\mathbb{P}} \frac{1}{\max\{\sqrt{c}, 1\}} Z \text{ as } t \to \infty,$$

where $Z$ is a normal distributed random variable independent of $\tilde{L}_{\tau}^{(0)}$. Hence, the length power functionals asymptotically decompose into the number of edges and some independent noise. Finally, in Section 10 open questions and possible strategies to solve them are discussed.
2. Preliminaries

A Poisson point process is always given as \( \eta_t : \Omega \to \mathbb{N}(W) \), \( \omega \to \sum_{i=1}^{N(\omega)} \delta_{X_i(\omega)} \), where \( W \subseteq \mathbb{R}^d \) is a convex compact set, \( X_i : \Omega \to W \) for \( i \in [N(\omega)] \) as well as \( N : \Omega \to \mathbb{N} \) are random variables, and \( \mathbb{N}(W) \) is the set of finite counting measures on \( W \). Here we also write \( [n] = \{1, \ldots, n\} \) for \( n \in \mathbb{N} \). Throughout the paper we use:

**Notation 2.1.**

(i) The notation \( \eta_t(\omega) \) means the counting measure \( \sum_{i=1}^{N(\omega)} \delta_{X_i(\omega)} \) and simultaneously also the point set \( \{X_1(\omega), \ldots, X_{N(\omega)}(\omega)\} \) understood as the support of the measure.

(ii) Realizations are simply written as \( \eta_t = \eta_t(\omega) \), \( x_i = X_i(\omega) \), and \( N = N(\omega) \).

(iii) Given a Borel set \( A \in \mathcal{B}(W) \) the abbreviation \( \eta_t(A) = \eta_t(\omega)(A) = \sum_{i=1}^{N} \delta_{x_i}(A) \) is induced by (ii) and is the number of random points in \( A \). If we identify \( \eta_t \) with its support, then \( \eta_t \cap A = \eta_t(\omega) \cap A = \{x_1, \ldots, x_n\} \cap A \) stands for the points lying in \( A \).

(iv) \( \eta^m_{t, \neq}(\omega) = \{(x_1, \ldots, x_m) \mid x_i = X_{j_i}(\omega) \text{ pairwise distinct for } j_i \in [N(\omega)]\} \) is the set of all \( m \)-tuples of pairwise distinct points in \( \eta_t \). We also write \( \eta^m_{t, \neq}(\omega) \) for \( \eta^m_{t, \neq}(\omega) \).

The discussion in the following is based on *Poisson functionals* in \( W \). Such a functional with an underlying Poisson point process \( \eta : \Omega \to \mathbb{N}(W) \) is a random variable

\[ F : \Omega \to \mathbb{R} \text{ with a decomposition } F = f \circ \eta \text{ P-a.s.} \]

for a measurable function \( f : \mathbb{N}(W) \to \mathbb{R} \). For further details see, e.g., [21]. In particular, applications in the context of our work are discussed in [2], [11], [16], and [19]. As above we also shorten \( F(\omega) = f(\eta(\omega)) \) by \( F = f(\eta) \). The central Poisson functional of this paper is the so-called *length power functional*

\[ L^{(\tau)}_t = L^{(\tau)}(\mathcal{S}(\eta_t, \delta_t)) = \frac{1}{2} \sum_{(x_1, x_2) \in \eta^2_{t, \neq}} \mathbb{1}(\|x_1 - x_2 \| \leq \delta_t) \|x_1 - x_2 \|^\tau \text{ for } \tau \in \mathbb{R}, \]

where here and in the following \( \mathbb{1}(\cdot) \) is an indicator function. The functional sums up the \( \tau \)-th powers of the edge lengths in the random geometric graph \( \mathcal{S}(\eta_t, \delta_t) \). For \( \tau = 1 \) this corresponds to the total edge length and for \( \tau = 0 \) to the number of edges in the random geometric graph. Note that this number and generalizations of it were already investigated in [11] from a stochastic point of view. Our main goal is an extensive examination of the asymptotic covariance matrix of a functional vector

\[ (L^{(\tau_1)}_t, \ldots, L^{(\tau_n)}_t) \]

for powers \(-d/2 < \tau_1 < \ldots < \tau_n\) of *centered and normalized length power functionals*

\[ L^{(\tau)}_t := \frac{\left(L^{(\tau)}_t - \mathbb{E}(L^{(\tau)}_t)\right)}{\max\{t^{\delta_t d/2}, t^{3/2} \delta_t^{d/2}\}}. \]

This matrix was explicitly determined in [19, Thm. 3.3] by Reitzner–Schulte–Thäle:
Theorem 2.2 ([19]). Let \(-d/2 < \tau_1 < \ldots < \tau_n\). The random vector \((\tilde{L}_t(\tau_1), \ldots, \tilde{L}_t(\tau_n))\) has the asymptotic covariance matrix

\[
\Sigma_n = \begin{cases} 
\Sigma_{nb} & \text{for } \lim_{t \to \infty} r_t \delta_t^d = 0, \\
\Sigma_{nb} + c \Sigma_{np} & \text{for } \lim_{t \to \infty} r_t \delta_t^d = c \in (0, 1], \\
\frac{1}{c} \Sigma_{nb} + \Sigma_{np} & \text{for } \lim_{t \to \infty} r_t \delta_t^d \in (1, \infty), \\
\Sigma_{np} & \text{for } \lim_{t \to \infty} r_t \delta_t^d = \infty
\end{cases}
\]

with \(\Sigma_{nb} := V(W) \left( \frac{d \cdot \kappa_d}{\kappa_{d+1}} \right)\), \(\Sigma_{np} := V(W) \left( \frac{d^2 \cdot \kappa_d^2}{(\tau_i + d)(\tau_j + d)} \right)\) \(\in \mathbb{R}^{n \times n}\) where \(V(W)\) and \(\kappa_d\) are the volumes of \(W\) and of the \(d\)-dimensional unit ball (w.r.t. the Lebesgue measure).

For example, for \(n = 1\) one obtains \(\Sigma_n = (\sigma_{11})\) for a certain constant \(\sigma_{11} > 0\) described above. All questions studied in this paper are trivial in this case. Therefore we assume \(n \geq 2\) in the remaining discussion. \(\Sigma_n\) is symmetric, positive semidefinite and hence diagonalizable. We denote its real eigenvalues sorted and in increasing order by \(\lambda_1 \leq \ldots \leq \lambda_n\).

3. Main results

The following tables present an overview of our algebraic results for the asymptotic covariance matrix \(\Sigma_n\) for \(n \geq 2\), which are described in each regime. Open cases are also mentioned. In Sections 5–7 first properties are discussed regime-dependent and include rank, determinant, definiteness, and eigenspaces. A summary is:

|                  | supercritical regime | subcritical regime | critical regime |
|------------------|----------------------|-------------------|-----------------|
| rank             | rank(\(\Sigma_n\)) = 1 (5.3(i)) | maximal rank (6.2(ii)) | maximal rank (7.1(ii)) |
| invertibility   | singular (5.3) | regular (6.2), inverse in 6.3 | regular (7.1), inverse in 7.2 |
| Jordan normal form | diagonalizable | diagonalizable | diagonalizable |
| determinant      | det(\(\Sigma_n\)) = 0 (5.3(ii)) | det(\(\Sigma_n\)) > 0 (6.2(ii)), explicit formula in (9) | det(\(\Sigma_n\)) > 0 (7.1(ii)), explicit formula in 7.3 |
| definiteness     | positive semidefinite not positive definite (5.3(iii)) | positive definite (6.2(ii)) | positive definite (7.1(iii)) |
| characteristic polynomial | explicit formula in (5.3 (iv)) | explicit formula in (6.6) | formula in (7.5) |
| eigenvalues      | explicit formula in (5.2) | bounds in (6.7) and (7.7), explicit formula open | open |
| eigenspaces      | explicit formula in (5.2) | open | open |

Table 1. first results in all regimes
The supercritical regime is completely solved in the sense that we obtain for every aspect either explicit formulas or qualitative statements. In the other two regimes some questions are still open. We would like to point out that from a qualitative point of view the results in the subcritical and critical regime coincide, since the critical case is just a rank-1-disturbance of the subcritical one. In Sections 5–7 we study also matrix decompositions and prove:

| Decomposition         | Supercritical Regime | Subcritical Regime | Critical Regime                  |
|-----------------------|----------------------|--------------------|----------------------------------|
| LU decomposition      | explicit formula in (5.5(i)) | explicit formula in (6.9) | explicit formula for the vector of natural increasing powers in (7.11) |
| Cholesky decomposition| explicit formula in (5.5(ii)) | explicit formula in (6.10) | explicit formula for the vector of natural increasing powers in (7.13) |
| Square root           | explicit formula in (5.5(iii)) | open               | open                             |

Table 2. Decompositions in all regimes

Observe that LU and Cholesky decompositions are derived in the supercritical as well as in the subcritical regime. In the critical regime they are investigated only in the special case of interest, where $\tau_1 = 0, \ldots, \tau_n = n - 1$ and $d = 2$. Significant connections between the matrices and their decompositions are additionally discussed in Section 8. We conclude the paper in Sections 9 and 10 with applications and open problems as mentioned before.

4. Positive semidefinite symmetric matrices

Here we recapitulate facts related to a matrix $\Sigma_n$ of length power functionals (see Theorem 2.2), which are used all over the manuscript. Recall that $\Sigma_n$ has the following structure:

$$
\begin{pmatrix}
\lim_{t \to \infty} \mathbb{V}(\tilde{L}_1^{(\tau_1)}) & \lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_1)}, \tilde{L}_1^{(\tau_2)}) & \cdots & \lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_1)}, \tilde{L}_1^{(\tau_n)}) \\
\lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_2)}, \tilde{L}_1^{(\tau_1)}) & \lim_{t \to \infty} \mathbb{V}(\tilde{L}_1^{(\tau_2)}) & \cdots & \lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_2)}, \tilde{L}_1^{(\tau_n)}) \\
\vdots & \vdots & \ddots & \vdots \\
\lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_n)}, \tilde{L}_1^{(\tau_1)}) & \lim_{t \to \infty} \mathbb{Cov}(\tilde{L}_1^{(\tau_n)}, \tilde{L}_1^{(\tau_2)}) & \cdots & \lim_{t \to \infty} \mathbb{V}(\tilde{L}_1^{(\tau_n)})
\end{pmatrix} \in \mathbb{R}^{n \times n}.
$$

A sum $C = A + B$ of symmetric matrices $A, B$ is again symmetric and matrix properties of interest of $C$ can be obtained by the ones of $A, B$ as follows:

**Lemma 4.1.** Let $A, B \in \mathbb{R}^{n \times n}$ be two symmetric matrices and $C = A + B \in \mathbb{R}^{n \times n}$ with corresponding eigenvalues $\lambda_1^A \leq \ldots \leq \lambda_n^A, \lambda_1^B \leq \ldots \leq \lambda_n^B$ and $\lambda_1^C \leq \ldots \leq \lambda_n^C$. Then:

(i) $\lambda_1^A + \lambda_1^B \leq \lambda_j^C \leq \lambda_1^A + \lambda_j^B$ for $j \in [n]$.
(ii) If $A$ is positive definite and $B$ is positive semidefinite, then $C$ is positive definite.
(iii) If $A$ and $B$ are positive semidefinite, then $\text{rank}(C) \geq \text{max}\{\text{rank}(A), \text{rank}(B)\}$.

**Proof.** (i) follows from the theorem of Courant–Fischer. This yields then (ii) and (iii). □
Bounds for eigenvalues of positive definite symmetric matrices can, e.g., be found in [25, (2.3)] (upper bounds) and [25, (2.54)] (lower bounds). More precisely, they are:

**Lemma 4.2.** Let \( A \in \mathbb{R}^{n \times n} \) be a positive definite symmetric matrix with eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_n \), \( m = \text{tr}(A)/n \), and \( s^2 = \text{tr}(A^2)/n - m^2 \). Then

\[
\det(A) \left( m + \frac{s}{\sqrt{n-1}} \right)^{n-1} \leq \lambda_1 \leq \ldots \leq \lambda_n \leq m + s\sqrt{n-1}.
\]

Matrix decompositions of \( \Sigma_n \) are studied in Sections 5–7. Recall that an LU decomposition of a matrix \( A \in \mathbb{R}^{n \times n} \) is an equation \( PA = LU \), where \( L = (l_{ij}) \in \mathbb{R}^{n \times n} \) is a normalized lower triangular matrix and \( U = (u_{ij}) \in \mathbb{R}^{n \times n} \) is an upper triangular matrix, which doesn’t need to be normalized, i.e. also entries \( u_{ii} = 0 \) are allowed. Moreover, \( P \) is a matrix obtained from the unit matrix by a certain number of row switches. For the next theorem see, e.g., [20].

**Theorem 4.3.** Let \( A \in \mathbb{R}^{n \times n} \) be a matrix. There exists an algorithm which terminates and returns the matrices \( P, L \) and \( U \) for a correct LU decomposition \( PA = LU \). If \( A \) is regular and \( P = I_n \), this decomposition is unique and \( u_{ii} \neq 0 \) for \( i \in [n] \).

In this manuscript \( I_n \) is the \( n \times n \) unit matrix. Besides LU decompositions we also investigate a Cholesky decomposition \( A = GG^\dagger \) of a positive semidefinite symmetric matrix \( A \in \mathbb{R}^{n \times n} \) into a product of a lower triangular matrix \( G = (g_{ij}) \in \mathbb{R}^{n \times n} \) and its transpose. See, e.g., [20] or [23, Theorem 4.2.6.] for a proof of:

**Algorithm 4.4.**

**Input:** Matrix \( A \in \mathbb{R}^{n \times n} \) positive semidefinite and symmetric.

**Output:** Matrix \( G = (g_{ij}) \in \mathbb{R}^{n \times n} \) s.t. \( A = GG^\dagger \).

**Do:**

Compute \( G = (g_{ij}) \in \mathbb{R}^{n \times n} \) column by column as

\[
g_{ij} = \begin{cases} 
0 & \text{for } i < j, \\
\sqrt{a_{ii} - \sum_{k=1}^{i-1} g_{ik}^2} & \text{for } i = j, \\
\frac{1}{g_{jj}} (a_{ij} - \sum_{k=1}^{j-1} g_{ik} g_{jk}) & \text{for } i > j, \text{ and } g_{jj} \neq 0, \\
0 & \text{for } i > j \text{ and } g_{jj} = 0.
\end{cases}
\]

If \( A \) is positive definite, then \( G \) is uniquely determined and \( g_{ii} > 0 \) for all \( i \in [n] \).

An LU decomposition \( A = LU \) and a Cholesky decomposition \( A = GG^\dagger \) of \( A \in \mathbb{R}^{n \times n} \) are both decompositions into a lower and an upper triangular matrix. If \( A \) is symmetric and positive definite, then the unique decompositions can be merged into another. Indeed:

(i) The matrix \( G \) from the Cholesky-decomposition of \( A \) is given by \( G = LD^\dagger \), where \( D^\dagger = (\sqrt{d_{ij}}) \in \mathbb{R}^{n \times n} \) is a diagonal matrix with \( d_{ii} = u_{ii} \).

(ii) In the LU decomposition of \( A \) one has \( L = G\tilde{D} \) and \( U = \tilde{D}^{-1}G^\dagger \), where \( \tilde{D} = (\tilde{d}_{ij}) \in \mathbb{R}^{n \times n} \) is a diagonal matrix with \( \tilde{d}_{ii} = 1/g_{ii} \).
5. Supercritical Regime

Recall Theorem 2.2 and the notation used in (3). In this section $\Sigma_n$ is always considered in the supercritical regime (i.e. $t \delta_l^d \to \infty$) for $n \geq 2$. First examinations regarding eigenvalues and eigenspaces of $\Sigma_n$ are:

**Example 5.1.** We consider the asymptotic covariance matrix $\Sigma_2$ for any dimension $d$ with $\tau_2 > \tau_1 > -d/2$ and volume $V(W)$. A calculation of the characteristic polynomial yields

$$\chi(\Sigma_2) = (\lambda - \lambda_1)(\lambda - \lambda_2) \in \mathbb{R}[\lambda]$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \frac{V(W)d^2 \chi_d^2((\tau_1 + d)^2 + (\tau_2 + d)^2)}{(\tau_1 + d)^2(\tau_2 + d)^2}$. The eigenspaces are

$$\text{eig}(\Sigma_2, \lambda_1) = \{r(\frac{\tau_1 + d}{\tau_2 + d}, -1)^t \mid r \in \mathbb{R}\} \text{ and } \text{eig}(\Sigma_2, \lambda_2) = \{r(\frac{\tau_2 + d}{\tau_1 + d}, 1)^t \mid r \in \mathbb{R}\}.$$  

This examples can be generalized to arbitrary $n \geq 2$. We introduce the abbreviation:

$$a_i = \tau_i + d \text{ for } i \in [n] \text{ and } b = \sum_{k=1}^{n} \prod_{l \in [n] \setminus \{k\}} a_l^2.$$  

According to (3) we have in the supercritical regime for the symmetric and positive semi-definite matrix $\Sigma_n$ that

$$\Sigma_n = V(W)d^2 \chi_d^2 \begin{pmatrix}
\frac{1}{a_1^2} & \frac{1}{a_1 a_2} & \cdots & \frac{1}{a_1 a_n} \\
\frac{1}{a_2 a_1} & \frac{1}{a_2^2} & \cdots & \frac{1}{a_2 a_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{a_n a_1} & \frac{1}{a_n a_2} & \cdots & \frac{1}{a_n^2}
\end{pmatrix}.$$

**Theorem 5.2.** Let $\Sigma_n$ be as in (3) in the supercritical regime for $n \geq 2$. Then its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = V(W)d^2 \chi_d^2(\Sigma_n^{n-1} 1/a_i^2)$. Let

$$v_1 = (\frac{a_1}{a_2}, -1, 0, \ldots, 0)^t, \ldots, v_{n-1} = (\frac{a_1}{a_n}, 0, \ldots, 0, -1)^t, \text{ and } v_n = (a_n/a_1, a_n/a_2, \ldots, 1)^t.$$  

Then the eigenspaces of $\Sigma_n$ are $\text{eig}(\Sigma_n, \lambda_1) = \langle v_1, \ldots, v_{n-1} \rangle$ and $\text{eig}(\Sigma_n, \lambda_2) = \langle v_n \rangle$.

**Proof.** For $\lambda_2$ one sees that $v_n$ is an eigenvector, since

$$v_n = \begin{pmatrix}
\frac{1}{a_1^2} & \frac{1}{a_1 a_2} & \cdots & \frac{1}{a_1 a_n} \\
\frac{1}{a_2 a_1} & \frac{1}{a_2^2} & \cdots & \frac{1}{a_2 a_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{a_n a_1} & \frac{1}{a_n a_2} & \cdots & \frac{1}{a_n^2}
\end{pmatrix} v_n = \begin{pmatrix}
\frac{a_n}{a_1} + \frac{a_n}{a_1 a_2} + \cdots + \frac{1}{a_1 a_n} \\
\frac{a_n}{a_2} + \frac{a_n}{a_2 a_1} + \cdots + \frac{1}{a_2 a_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{a_n} + \frac{1}{a_n a_1} + \cdots + \frac{1}{a_n a_n}
\end{pmatrix} v_n = \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right) v_n.$$  

A similar computation shows that the $(n-1)$-many (obviously) linearly independent vectors $v_1, \ldots, v_{n-1}$ are eigenvectors for $\lambda_1$. This concludes the proof. \hfill \Box
Corollary 5.3. With the the assumptions of Theorem 5.2 it follows: (i) \( \text{rank}(\Sigma_n) = 1 \); (ii) \( \det(\Sigma_n) = 0 \); (iii) \( \Sigma_n \) is positive semidefinite but, not positive definite; (iv) The characteristic polynomial of \( \Sigma_n \) is \( \chi(\Sigma_n) = (\lambda - \lambda_1)^{n-1}(\lambda - \lambda_2) \in \mathbb{R}[\lambda] \).

The preceding outcome of the rank and definiteness was already found in [19, Proposition 3.4]. The results related to eigenvalues and eigenspaces of \( \Sigma_n \) in Theorem 5.2 lead to the next fact.

Corollary 5.4. With the the assumptions of Theorem 5.2 set \( D = \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2) \in \mathbb{R}^{n \times n} \) and \( S \in \mathbb{R}^{n \times n} \) with column vectors \( v_1, \ldots, v_n \). Then \( \Sigma_n \) is similar to \( D \) and \( D = S^{-1} \Sigma_n S \), where \( S^{-1} = (\tilde{s}_{ij}) \in \mathbb{R}^{n \times n} \) with entries

\[
\tilde{s}_{ij} = \begin{cases} 
\frac{\Pi_{k=1}^n a_k^2}{a_{ij}a_n b} & \text{for } i = n, \\
-\frac{\sum_{k \in [n] \setminus \{i\}} a_{k}^2 \Pi_{l \in [n] \setminus \{i,j\}} a_{l}^2}{b} & \text{for } j = i + 1, \\
\frac{\Pi_{k=1}^n a_k^2}{a_{i+1}a_j b} & \text{else}.
\end{cases}
\]

Proof. Let \( \tilde{S} \) be the matrix with entries \( \tilde{s}_{ij} \). There are four distinct cases for entries of \( SS \).

The first one arises by the multiplication of the first row of \( \tilde{S} \) with the first column of \( \tilde{S} \):

\[
\left( \frac{a_1}{a_k+1} \right) \sum_{k \in [n-1] \setminus \{j-1\}} \frac{a_{k}}{a_{k+1}} \cdot \tilde{s}_{k1} + \frac{a_n}{a_1} \cdot \tilde{s}_{n1} = \left( \frac{a_1}{a_k+1} \right) \sum_{k \in [n] \setminus \{j\}} \frac{a_{k}}{a_{k+1}} \cdot \tilde{s}_{k1} + \frac{a_n}{a_1} \cdot \tilde{s}_{n1} = 1.
\]

The multiplication of the first row of \( S \) with the \( j \)-th column of \( \tilde{S} \) for \( j \neq 1 \) is

\[
\sum_{k \in [n-1] \setminus \{j-1\}} \frac{a_1}{a_k+1} \cdot \tilde{s}_{k1} + \frac{a_n}{a_1} \cdot \tilde{s}_{n1} = \sum_{k \in [n] \setminus \{j\}} \frac{a_1 a_j \sum_{k \in [n] \setminus \{j,k+1\}} a_k^2}{b} - \frac{a_1 a_j \sum_{k \in [n] \setminus \{j\}} a_k^2}{b} + \frac{a_1 a_j \prod_{k \in [n] \setminus \{1,j\}} a_k^2}{b} = 0.
\]

and this is easily seen to be 0. The multiplication of the \( i \)-th row of \( S \) with the \( j \)-th column of \( \tilde{S} \) for \( i \neq 1 \) and \( j \neq i \) gives

\[
-\tilde{s}_{i-1,j} + \frac{a_n}{a_i} \cdot \tilde{s}_{nj} = -\frac{\sum_{k \in [n] \setminus \{i\}} a_k^2 \prod_{l \in [n] \setminus \{i,k\}} a_l^2}{a_ia_j b} + \frac{a_n}{a_i} \cdot \frac{\prod_{k=1}^n a_k^2}{a_ia_n b} = 0.
\]

Finally, the multiplication of the \( i \)-th row of \( S \) with the \( j \)-th column of \( \tilde{S} \) for \( i = j \neq 1 \) yields

\[
-\tilde{s}_{i-1,j} + \frac{a_n}{a_i} \cdot \tilde{s}_{nj} = \frac{\sum_{k \in [n] \setminus \{i\}} a_k^2 \prod_{l \in [n] \setminus \{i,k\}} a_l^2}{b} + \frac{a_n}{a_i} \cdot \frac{\prod_{k=1}^n a_k^2}{a_ia_n b} = 1.
\]

All in all thereby arises the unit matrix. Hence, \( \tilde{S} = S^{-1} \). \( \square \)
In the preceded result we found the Schur decomposition
\[ \Sigma_n = SDS^{-1} \]
of \( \Sigma_n \). See [13] for more details. Other interesting decompositions are:

**Theorem 5.5.** Let \( \Sigma_n \) be as in (3) in the supercritical regime for \( n \geq 2 \).

(i) An LU decomposition \( \Sigma_n = LU \) of \( \Sigma_n \) is given by \( L = (l_{ij}) \in \mathbb{R}^{n \times n} \) and \( U = (u_{ij}) \in \mathbb{R}^{n \times n} \) with entries

\[ l_{ij} = \begin{cases} \frac{a_i}{a_j} & \text{for } i = j, \\ 0 & \text{for } i > 1 \text{ and } j = 1, \\ & \text{else} \end{cases} \quad \text{and } u_{ij} = \begin{cases} \frac{\sqrt{V(W)(d_{\mathcal{Z}_d})^2}}{a_j} & \text{for } i = 1, \\ 0 & \text{else}. \end{cases} \]

(ii) A Cholesky decomposition \( \Sigma_n = GG' \) of \( \Sigma_n \) is given by \( G = (g_{ij}) \in \mathbb{R}^{n \times n} \) with entries

\[ g_{ij} = \begin{cases} \frac{\sqrt{V(W)(d_{\mathcal{Z}_d})^2}}{a_i} & \text{for } j = 1, \\ 0 & \text{else}. \end{cases} \]

(iii) A matrix root \( B = (b_{ij}) \in \mathbb{R}^{n \times n} \) of \( \Sigma_n \) (i.e., \( B^2 = \Sigma_n \)) has entries

\[ b_{ij} = \frac{\sqrt{V(W)(d_{\mathcal{Z}_d})^2}}{a_i a_j \sqrt{\sum_{k=1}^{n} \prod_{i} a_k^2}}. \]

**Proof.**

(i) Computing the entries \( \sigma_{ij} \) of the given product \( LU \) leads to

\[ \sigma_{1j} = 1 \cdot \frac{V(W)(d_{\mathcal{Z}_d})^2}{a_1 a_j} = \frac{V(W)(d_{\mathcal{Z}_d})^2}{a_1 a_j} \text{ for } j \in [n] \] and

\[ \sigma_{ij} = \frac{a_i}{a_j} \cdot \frac{V(W)(d_{\mathcal{Z}_d})^2}{a_1 a_j} = \frac{V(W)(d_{\mathcal{Z}_d})^2}{a_1 a_j} \text{ for } i > 2 \text{ and } j \in [n], \]

which are exactly the entries of \( \Sigma_n \).

(ii) We determine the entries of \( G \) column by column by using formula (4). Without a computation one obtains \( g_{ij} = 0 \) for \( j > i \). Additionally, we get the first column of \( G \) by

\[ g_{11} = \sqrt{\sigma_{11}} = \sqrt{V(W)(d_{\mathcal{Z}_d})^2/a_1} \quad \text{and} \quad g_{1i} = \sigma_{1i}/\sqrt{V(W)(d_{\mathcal{Z}_d})^2/a_1} = \frac{\sqrt{V(W)(d_{\mathcal{Z}_d})^2}}{a_i} \text{ for } i \geq 2. \]

For \( i \geq 2 \), one has \( g_{ii} = \sqrt{(d_{\mathcal{Z}_d})^2 V(W)/a_i^2} - g_{i1}^2 = 0 \), so the remaining columns are 0.

(iii) Let \( S, D \) and \( S^{-1} \) be the matrices from Corollary 5.4. Let \( B = SD^{1/2}S^{-1} \). Then \( BB = SDS^{-1} = \Sigma_n \). The product \( SD^{1/2} = R = (r_{ij}) \in \mathbb{R}^{n \times n} \) has entries

\[ r_{ij} = \begin{cases} \frac{\sqrt{V(W)(d_{\mathcal{Z}_d})^2}}{a_i} \sqrt{\sum_{k} \prod_{i} a_k^2} & \text{for } j = n, \\ 0 & \text{else}. \end{cases} \]

The claimed entries for \( B \) are determined from the product \( RS^{-1} \). \( \square \)
The next example illustrates the decompositions found above for a vector of length $n = 3$:

**Example 5.6.** Consider $\Sigma_3$ for $d = 2, V(W) = 1, \tau_1 = 0, \tau_2 = 1$ and $\tau_3 = 2$:

$$
\Sigma_3 = 4\pi^2 \begin{pmatrix}
0.25 & 0.5 & 0.5 \\
0.5 & 0.67 & 0.68 \\
0.5 & 0.68 & 0.8
\end{pmatrix}.
$$

According to Theorem 5.5(i) an LU decomposition of $\Sigma_3$ is given by

$$
LU = 4\pi^2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1.25 & 1 & 1/2 \\
1 & 0.5 & 1/2 \\
0 & 0 & 1
\end{pmatrix}.
$$

Theorem 5.5(ii) leads to a Cholesky decomposition of $\Sigma_3$:

$$
GG^t = 4\pi^2 \begin{pmatrix}
1/2 & 0 & 0 \\
0 & 1/3 & 0 \\
0 & 0 & 1/4
\end{pmatrix} \begin{pmatrix}
1 & 1/3 & 1/4 \\
1/3 & 0 & 0 \\
1/4 & 0 & 0
\end{pmatrix}.
$$

Theorem 5.5(iii) yields $\Sigma_3$ as the following product of matrix roots:

$$
BB = 4\pi^2 \begin{pmatrix}
6/\sqrt{244} & 4/\sqrt{244} & 3/\sqrt{244} \\
4/\sqrt{244} & 8/3\sqrt{244} & 2/\sqrt{244} \\
3/\sqrt{244} & 2/\sqrt{244} & 3/2\sqrt{244}
\end{pmatrix} \begin{pmatrix}
6/\sqrt{244} & 4/\sqrt{244} & 3/\sqrt{244} \\
4/\sqrt{244} & 8/3\sqrt{244} & 2/\sqrt{244} \\
3/\sqrt{244} & 2/\sqrt{244} & 3/2\sqrt{244}
\end{pmatrix}.
$$

6. **Subcritical Regime**

In this section $\Sigma_n$ is always considered in the subcritical regime (i.e. $t \delta_d \to 0$) for $n \geq 2$.

Recall that then this matrix has the following structure:

$$
\Sigma_n = V(W) \left( \frac{d \cdot \kappa_d}{2(\tau_i + \tau_j + d)} \right) \in \mathbb{R}^{n \times n}.
$$

By re-scaling and substituting

$$
\tau_i := \tau_i + d/2 \text{ for } i \in [n]
$$

one obtains the following *Cauchy matrix* (see, e.g., [13]):

$$
\Omega_n = (\omega_{ij}) := \frac{2}{d \cdot \kappa_d V(W)} \Sigma_n = \frac{1}{(\tau_i + \tau_j + d)} \frac{1}{(x_i - (-x_j))} \in \mathbb{R}^{n \times n}.
$$

It follows from [13, 0.9.12.1] that:

**Lemma 6.1.** Let $\Omega_n$ be as in (8). Then

$$
\det(\Omega_n) = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2}{\prod_{1 \leq i, j \leq n} (x_i + x_j)} > 0.
$$
Corollary 6.2. Let \( \Sigma_n \) be as in (3) in the subcritical regime for \( n \geq 2 \). Then:

\[
\det(\Sigma_n) = \left( \frac{V(W)dx_d}{2} \right)^n \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{1 \leq i, j \leq n} (x_i + x_j) > 0.
\]

In particular, (i) \( \text{rank}(\Sigma_n) = n \); (ii) \( \Sigma_n \) is positive definite.

The aforementioned conclusion about the rank and the definiteness was given before in [19, Proposition 3.4]. The inverse for an arbitrary Cauchy matrix is explicitly known, e.g., by [9, Corollary 3.1] since there are special cases of Cauchy-Vandermonde matrices. For \( \Sigma_n \) this implies:

**Proposition 6.3.** Let \( \Sigma_n \) be as in (3) in the subcritical regime for \( n \geq 2 \). The inverse \( \hat{\Sigma}_n = (\hat{\sigma}_{ij}) \in \mathbb{R}^{n \times n} \) of \( \Sigma_n \) has entries

\[
\hat{\sigma}_{ij} = \begin{cases} 
\frac{-8x_i x_j (x_i + x_j) \prod_{e \in [n](i,j), k \in [i,j]} (x_k + x_j)}{V(W)dx_d(2x_i - x_j)^2 \prod_{e \in [n](i,j), k \in [i,j]} (x_k - x_i)} & \text{for } i \neq j, \\
\frac{4x_i \prod_{e \in [n](i)} (x_k + x_i)^2}{V(W)dx_d \prod_{e \in [n](i)} (x_k - x_i)^2} & \text{for } i = j.
\end{cases}
\]

Next we consider eigenvalues and eigenspaces of \( \Sigma_n \). For this it is useful to observe that for an eigenvalue \( \lambda \in \mathbb{R} \) and a corresponding eigenvector \( v \in \mathbb{R}^n \) of \( \Omega_n \) one has

\[
\Sigma_n v = (V(W)dx_d/2)\Omega_n v = (V(W)dx_d/2)\lambda v.
\]

**Example 6.4.** Consider \( \Omega_2 \) as in (8). Its characteristic polynomial \( \chi(\Omega_2) \in \mathbb{R}[\lambda] \) is

\[
\chi(\Omega_2) = \det \begin{pmatrix} \frac{1}{x_1 + x_2} - \lambda & \frac{1}{x_1 + x_2} \\ \frac{1}{x_2 + x_2} - \lambda \end{pmatrix} = \lambda^2 - \lambda \left( \frac{1}{2x_1} + \frac{1}{2x_2} \right) + \frac{(x_1 - x_2)^2}{2x_1 2x_2 (x_1 + x_2)^2}.
\]

Thus, according to (10) all eigenvalues of \( \Sigma_2 \) are

\[
\lambda_{1,2} = \frac{V(W)dx_d}{8x_1 x_2 (x_1 + x_2)} \left( (x_1 + x_2)^2 \pm \sqrt{x_1^4 + 14x_1^2 x_2^2 + x_2^4} \right).
\]

Experiments and further examples show that it is difficult to determine explicitly the eigenspaces of \( \Sigma_n \) for \( n \geq 3 \). Instead, we prove in the following a formula for the characteristic polynomial and afterwards bounds for the eigenvalues of \( \Sigma_n \). Similar as above for eigenvalues and eigenvectors, we observe the following relationship between \( \chi(\Omega_n) \) and \( \chi(\Sigma_n) \). Let \( \Omega_n \) be as in (8) with given characteristic polynomial

\[
\chi(\Omega_n) = a_n \lambda^n + \ldots + a_1 \lambda + a_0 \in \mathbb{R}[\lambda].
\]

Then \( \chi(\Sigma_n) = \det(\Sigma_n - \lambda I_n) = \left( \frac{V(W)dx_d}{2} \right)^n \det(\Omega_n - \frac{2}{V(W)dx_d} \lambda I_n) \) equals to

\[
a_n \lambda^n + a_{n-1} \lambda^{n-1} \left( \frac{V(W)dx_d}{2} \right) + \ldots + a_1 \lambda \left( \frac{V(W)dx_d}{2} \right)^{n-1} + a_0 \left( \frac{V(W)dx_d}{2} \right)^n.
\]
The following lemma is a variation of a well-known fact for the characteristic polynomial of an arbitrary quadratic matrix, which is useful for the investigated matrices in this paper. For example, it is an immediate consequence of [13, Theorem 1.2.16.].

**Lemma 6.5.** Let \( \Omega_{R,S} = \{ z_{r,s} \} \in \mathbb{R}^{n \times n} \) with \( R = (r_1, \ldots, r_n), S = (s_1, \ldots, s_n) \in \mathbb{N}^n \) and characteristic polynomial \( \chi(\Omega_{R,S}) = (-1)^n \lambda^n + a_{R,S}^{n-1} \lambda^{n-1} + \cdots + a_{R,S}^1 \lambda + a_{R,S}^0 \in \mathbb{R}[\lambda] \). Then the coefficients \( a_{R,S}^{k,n} \) for \( k \in [n-1] \) can be computed via the formula

\[
a_{R,S}^{k,n} = (-1)^k \sum_{1 \leq i_1 < \cdots < i_{n-k} \leq n} a^{0,n-k}_{(r_1, \ldots, r_{n-k}), (s_1, \ldots, s_{n-k})}.
\]

Note that \( a^{0,n-k}_{(r_1, \ldots, r_{n-k}), (s_1, \ldots, s_{n-k})} \) is the determinant of the matrix \((z_{r_1 s_i})_{u,v \in [n-1]}\). Using the previous lemma and the formula for the determinant in Lemma 6.1 we can calculate \( \chi(\Omega_n) = a^{(n)}_n \lambda^n + \cdots + a^{(0)}_n \) of \( \Omega_n \) given as in (8), since \( \Omega_n = \Omega_{[n],[n]} \) with \( z_{r,s} = \frac{1}{x_r + x_s} \) and \( a^{(k)}_n = a_{(1, \ldots, n), (1, \ldots, n)}^{k,n} \). Taking additionally (11) into account, we obtain:

**Corollary 6.6.** Let \( \Sigma_n \) be as in (3) in the subcritical regime for \( n \geq 2 \) with \( \chi(\Sigma_n) = b^{(n)}_n \lambda^n + b^{(n-1)}_n \lambda^{n-1} + \cdots + b^{(1)}_n \lambda + b^{(0)}_n \in \mathbb{R}[\lambda] \). The coefficients of \( \chi(\Sigma_n) \) are

\[
b^{(k)}_n = (-1)^k \left( \frac{V(W) d \lambda d \sigma_d}{2} \right)^{n-k} \sum_{1 \leq i_1 < \cdots < i_{n-k} \leq n} \prod_{l,k \in \{1, \ldots, n-k\}} \left( 1 \leq i \leq l \leq n \right) \prod_{l,k \in \{1, \ldots, n-k\}} (x_l - x_k)^2, k \in [n-1].
\]

Determining the eigenvalues of \( \Sigma_n \) explicitly is an open problem. However, we have:

**Theorem 6.7.** Let \( \Sigma_n \) be as in (3) in the subcritical regime for \( n \geq 2 \) with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). We have \( S_n \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \tilde{S}_n \) with

\[
\tilde{S}_n = \sqrt{(n-1) \left( \sum_{i=1}^{n} \left( \frac{1}{x_i} \right)^2 \right) - \left( \sum_{i=1}^{n} \frac{1}{x_i} \right)^2}.
\]

\[
S_n = \frac{V(W) d \lambda d \sigma_d}{2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2,
\]

\[
2 \prod_{1 \leq i, j \leq n} (x_i + x_j) \left( \sum_{i=1}^{n-1} \frac{1}{x_i} + \sqrt{\sum_{i=1}^{n} \frac{1}{x_i} - \left( \sum_{i=1}^{n} \frac{1}{x_i} \right)^2} \right)^{n-1}.
\]

**Proof.** Apply Lemma 4.2 to the matrix \( \Omega_n \) according to (8). We have \( m = \sum_{i=1}^{n} (2nx_i)^{-1} \). The matrix \( \Omega_n^2 \) has entries \( \sum_{k=1}^{n} (x_k + x_i)(x_k + x_j)^{-1} \) at places \((i, j)\) for \( i, j \in [n] \). Hence, \( \text{tr}(\Omega_n^2) = \sum_{k=1}^{n} \sum_{i=1}^{n} (x_k + x_i)^{-2} \) and, thus,

\[
s^2 = \frac{\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{x_k + x_i}^2}{n} - \left( \frac{\sum_{i=1}^{n} \frac{1}{x_i}}{n} \right)^2 = \frac{\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{n}{x_k + x_i}^2}{n^2} - \left( \frac{\sum_{i=1}^{n} \frac{1}{x_i}}{n} \right)^2.
\]
Using the results for det(\(\Omega_n\)) (according to Lemma 6.1), \(n\) and \(s\) in Lemma 4.2 and applying (10) by adding the multiplicative constant \(V(W)d\sigma_0/2\) leads to the claimed bounds. \(\square\)

The given bounds in Theorem 6.7 can be tight and appear as the smallest and largest eigenvalue of \(\Sigma\). For \(n = 2\) this occurs for any \(d\), \(V(W)\) and \(\tau_1 < \tau_2\), which follows from the description of \(\Omega_2\) in (8) and the known eigenvalues of \(\Sigma_2\) from Example 6.4. We have

\[
S_2 = \frac{V(W)d\sigma_0}{2} \left( \frac{x_2 + x_1}{4x_1x_2} + \sqrt{\frac{14x_1^2x_2^2 + x_1^4 + x_1^4}{16x_1^2x_2^2(x_1 + x_2)^2}} \right) = \lambda_2
\]

and for \(S_2\) one gets

\[
V(W)d\sigma_0 \cdot \left( \frac{(x_1 - x_2)^2}{4x_1x_2(x_1 + x_2)^2} + \sqrt{\frac{14x_1^2x_2^2 + x_1^4 + x_1^4}{16x_1^2x_2^2(x_1 + x_2)^2}} \right) = V(W)d\sigma_0 \cdot \left( \frac{(x_1 - x_2)^2}{4x_1x_2(x_1 + x_2)^2} + \sqrt{\frac{14x_1^2x_2^2 + x_1^4 + x_1^4}{16x_1^2x_2^2(x_1 + x_2)^2}} \right) = \lambda_1.
\]

It would be quite interesting to investigate in a further analysis, if \(S_n = \lambda_1\) and \(S_n = \lambda_n\) is true for \(n \geq 3\) or whether the bounds are not tight in general.

**Conjecture 6.8.** Let \(\Sigma_n\) be as in (3) in the subcritical regime for \(n \geq 2\) with eigenvalues \(\lambda_1 \leq \ldots \leq \lambda_n\). Then \(\lambda_i \neq \lambda_j\) for all \(i \neq j \in [n]\).

Due to the structure of \(\Sigma_n\) we determine its LU decomposition as:

**Theorem 6.9.** Let \(\Sigma_n = (\sigma_{ij})\) be as in (3) in the subcritical regime for \(n \geq 2\). The LU decomposition \(\Sigma_n = LU\) is given by \(L = (l_{ij})\) and \(U = (u_{ij})\) with entries

\[
l_{ij} = \begin{cases} 0 & \text{for } i < j, \\ \frac{2x_j\prod_{k=1}^{i-1}(x_j-x_k)(x_i-x_k)}{\prod_{k=1}^{i-1}(x_j-x_k)\prod_{k=1}^{i-1}(x_i-x_k)} & \text{for } i \geq j, \end{cases},
\]

\[
u_{ij} = \begin{cases} 0 & \text{for } i > j, \\ \frac{V(W)d\sigma_0\prod_{k=1}^{i-1}(x_i-x_k)(x_j-x_k)}{2\prod_{k=1}^{i-1}(x_i-x_k)\prod_{k=1}^{i-1}(x_j-x_k)} & \text{for } i \leq j. \end{cases}
\]

**Proof.** We first show the following formula for any \(q \in \mathbb{N}, q \geq 1\) and \(x_i, x_j, x_1, \ldots, x_q \in \mathbb{R}\):

\[
\prod_{m=1}^{q}(x_i - x_m)(x_j - x_m) = \prod_{m=1}^{q}(x_i + x_m)(x_j + x_m)
\]

\[\quad \quad \quad - \sum_{m=1}^{q} 2x_m(x_i + x_j) \prod_{k=m+1}^{q}(x_i + x_k)(x_j + x_k) \prod_{k=1}^{m-1}(x_i - x_k)(x_j - x_k).
\]

(13)
We perform an induction on $q \geq 1$, where the base case $q = 1$ is trivial. Next one sees

$$\prod_{m=1}^{q} (x_i - x_m)(x_j - x_m) = (x_i - x_q)(x_j - x_q) \prod_{m=1}^{q-1} (x_i - x_m)(x_j - x_m)$$

$$= (x_i + x_q)(x_j + x_q) \prod_{m=1}^{q-1} (x_i - x_m)(x_j - x_m) - (2x_i x_q + 2x_j x_q) \prod_{m=1}^{q-1} (x_i - x_m)(x_j - x_m)$$

$i.h. - 2x_q (x_i + x_j) \prod_{m=1}^{q-1} (x_i - x_m)(x_j - x_m) + (x_i + x_q)(x_j + x_q) \left( \prod_{m=1}^{q-1} (x_i + x_m)(x_j + x_m) \right.$

$\left. - \sum_{m=1}^{q-1} 2x_m (x_i + x_j) \prod_{k=m+1}^{q-1} (x_i + x_k)(x_j + x_k) \prod_{k=1}^{m-1} (x_i - x_k)(x_j - x_k) \right)$

$= - 2x_q (x_i + x_j) \prod_{m=1}^{q-1} (x_i - x_m)(x_j - x_m) + \prod_{m=1}^{q} (x_i + x_m)(x_j + x_m)$

$- \sum_{m=1}^{q-1} 2x_m (x_i + x_j) \prod_{k=m+1}^{q} (x_i + x_k)(x_j + x_k) \prod_{k=1}^{m-1} (x_i - x_k)(x_j - x_k)$

$= \prod_{m=1}^{q} (x_i + x_m)(x_j + x_m) - \sum_{m=1}^{q} 2x_m (x_i + x_j) \prod_{k=m+1}^{q} (x_i + x_k)(x_j + x_k) \prod_{k=1}^{m-1} (x_i - x_k)(x_j - x_k).$

We compute the entries $\sigma_{ij} = \sum_{m=1}^{n} l_{im} u_{mj}$ of the product $LU$. Let $q = \min\{i, j\}$. It is $l_{im} u_{mj} = 0$ for $m \in \{q + 1, \ldots, n\}$. Thus,

$$\sigma_{ij} = \sum_{m=1}^{q} l_{im} u_{mj} = \sum_{m=1}^{q} 2x_m \prod_{k=1}^{m-1} (x_m + x_k)(x_i - x_k) \prod_{k=1}^{m} (x_i + x_k) \frac{V(W) d x_d \prod_{k=1}^{m-1} (x_m - x_k)(x_j - x_k)}{2 \prod_{k=1}^{m-1} (x_m + x_k) \prod_{k=1}^{m} (x_j + x_k)}$$

$$= \frac{V(W) d x_d}{2} \sum_{m=1}^{q} 2x_m \prod_{k=1}^{m-1} (x_i - x_k) \prod_{k=1}^{m} (x_j + x_k) \prod_{k=1}^{m} (x_i + x_k) \prod_{k=1}^{q} (x_i + x_k)(x_j + x_k)$$

$$= \frac{V(W) d x_d}{2} \sum_{m=1}^{q} 2x_m \prod_{k=1}^{m-1} (x_i - x_k)(x_j - x_k) \prod_{k=1}^{q} (x_i + x_k)(x_j + x_k)$$

$$= \frac{V(W) d x_d}{2} \frac{\prod_{m=1}^{q} (x_i + x_m)(x_j + x_m) - \prod_{m=1}^{q} (x_i - x_m)(x_j - x_m)}{(x_i + x_j) \prod_{k=1}^{q} (x_i + x_k)(x_j + x_k)}$$

$$= \frac{V(W) d x_d}{2} \frac{\prod_{m=1}^{q} (x_i + x_m)(x_j + x_m)}{(x_i + x_j) \prod_{k=1}^{q} (x_i + x_k)(x_j + x_k)} = \frac{V(W) d x_d}{2(x_i + x_j)}.$$  

These are the entries of $\Sigma_n$ given as in (7). \qed
Corollary 6.10. Let \( \Sigma_n = (\sigma_{ij}) \) be as in (3) in the subcritical regime for \( n \geq 2 \). Then the Cholesky decomposition \( \Sigma_n = GG^t \) is given by \( G = (g_{ij}) \) with entries
\[
g_{ij} = \begin{cases} 0 & \text{for } i < j, \\ \sqrt{\frac{V(W)d \xi_d}{2}} \frac{\sqrt{2\tau_i \prod_{k=1}^{i-1} (x_i-x_k)}}{\prod_{k=1}^{i} (x_i+x_k)} & \text{for } i \geq j. \end{cases}
\]

Proof. There exists a unique Cholesky decomposition of \( \Sigma_n \) due to Algorithm 4.4. According to the remark after this algorithm, it can be derived from the LU decomposition of \( \Sigma_n \). For this we define a diagonal matrix \( D = (d_{ij}) \) using entries from \( U \) by
\[
d_{ij} = \begin{cases} 0 & \text{for } j \neq i, \\ \frac{V(W)d \xi_d \prod_{k=1}^{i-1} (x_j-x_k)^2}{4x_j \prod_{k=1}^{i-1} (x_j+x_k)^2} & \text{for } j = i. \end{cases}
\]
Then \( G \) equals \( G = LD^\frac{1}{2} \) where \( L \) is given as in Theorem 6.9. A straightforward calculation concludes the proof.

Example 6.11. Consider \( \Sigma_2 \) for \( \tau_1 = 0 \) and \( \tau_2 \in \mathbb{R} \). Then
\[
\Sigma_2 = GG^t = \sqrt{\frac{V(W)d \xi_d}{2}} \begin{pmatrix} \frac{\sqrt{d}}{\sqrt{2x}} & 0 \\ \frac{\sqrt{2x}}{a_2} & \frac{\bar{a}}{\sqrt{2x}a_2} \end{pmatrix} \sqrt{\frac{V(W)d \xi_d}{2}} \begin{pmatrix} \frac{\sqrt{d}}{\sqrt{2x}} & 0 \\ \frac{\sqrt{2x}}{a_2} & \frac{\bar{a}}{\sqrt{2x}a_2} \end{pmatrix}.
\]

7. Critical Regime

Concluding the discussions of the last two sections we consider \( \Sigma_n \) in the critical regime (i.e. \( t \delta^d_n \to c \in (0, \infty) \)) for \( n \geq 2 \). Recall that
\[
\Sigma_n = \begin{cases} V(W)d \xi_d \frac{a_{a_i} + 2cd \xi_d(x_i+x_j)}{2(x_i+x_j)a_i a_j} & \text{for } c \in (0, 1], \\ V(W)d \xi_d \frac{a_{a_i} + 2cd \xi_d(x_i+x_j)}{2(x_i+x_j)a_i a_j} & \text{for } c \in (1, \infty). \end{cases}
\]

If matrices from various regimes appear, they are labeled as \( \Sigma_n^{sb} \), \( \Sigma_n^{sp} \), and \( \Sigma_n^{ct} \) for the subcritical, supercritical, and critical regime. The abbreviations \( x_i = \tau_i + d/2 \) and \( a_i = \tau_i + d \) for \( i \in [n] \) used before are taken up again. The matrices \( \Sigma_n^{sb} \) by (6) and \( \Sigma_n^{sp} \) by (7) using the same data \( x_i \) as well as \( a_i \) are called corresponding matrices of \( \Sigma_n^{ct} \). Recall that by Theorem 2.2 the matrix \( \Sigma_n = \Sigma_n^{ct} \) is always a sum of \( \Sigma_n^{sp} \) and \( \Sigma_n^{sb} \) using certain coefficients.

The results in the subcritical and supercritical regime yield:

Corollary 7.1. Let \( \Sigma_n \) be as in (3) in the critical regime for \( n \geq 2 \). Then (i) \( \text{rank}(\Sigma_n) = n \); (ii) \( \det(\Sigma_n) > 0 \); (iii) \( \Sigma_n \) is positive definite.

Proof. Theorem 2.2 and Corollaries 4.1(iii), 5.3(i), 6.2(ii) yield for \( c \in (0, 1] \) that:
\[
\text{rank}(\Sigma_n^{ct}) \geq \max\{\text{rank}(c \Sigma_n^{sp}), \text{rank}(\Sigma_n^{sb})\} = \max\{1, n\} = n.
\]
For \( c \in (1, \infty) \) a similar argument proves the claim.

In [19, Proposition 3.4] the authors proved already the fact above. Additionally, we are able to give formulas for the inverse matrix and determinant of $\Sigma_n^{cr}$ using the fact that it is a rank-1-correction of $\Sigma_n^{sh}$ as well as the already known results about the adjoint and the inverse matrix in the subcritical regime.

Indeed, the inverse matrix of $\Sigma_n^{cr}$ can be computed with the Woodbury matrix identity (see, e.g., [13, 0.7.4.2]). It implies the following theorem by observing that $c\Sigma_n^{sp} = w^t$ where $v = \sqrt{cV(W)d\varphi_d(1/a_1, \ldots, 1/a_n)^t}$, $\Sigma_n^{sp} = ww^t$ where $w = \sqrt{V(W)d\varphi_d(1/a_1, \ldots, 1/a_n)^t}$,

$$(\Sigma_n^{cr})^{-1} = (v \cdot 1 + v^{sh})^{-1}$$

for $c \in (0, 1]$, and $$(\Sigma_n^{cr})^{-1} = (w \cdot 1 + 1/c\Sigma_n^{sh})^{-1}$$

for $c \in (1, \infty)$.

**Theorem 7.2.** Let $\Sigma_n^{cr}$ be as in (3) in the critical regime for $n \geq 2$. For $c \in (0, 1]$ we have

$$(\Sigma_n^{cr})^{-1} = (\Sigma_n^{sh})^{-1} - (\Sigma_n^{sh})^{-1}v(1 + v^{sh}(\Sigma_n^{sh})^{-1})^{-1}v^{sh}(\Sigma_n^{sh})^{-1}$$

and for $c \in (1, \infty)$ one has

$$(\Sigma_n^{cr})^{-1} = (1/c\Sigma_n^{sh})^{-1} - (1/c\Sigma_n^{sh})^{-1}w(1 + w^{sh}(1/c\Sigma_n^{sh})^{-1}w)^{-1}w^{sh}(1/c\Sigma_n^{sh})^{-1}.$$  

Note that the expressions $(1 + v^{sh}(\Sigma_n^{sh})^{-1})^{-1}$ and $(1 + w^{sh}(1/c\Sigma_n^{sh})^{-1}w)^{-1}$ exist since the entries of $(\Sigma_n^{sh})^{-1}$ multiplied with $1/(a_ia_j)$ are $\not= -1$ according to Proposition 6.3.

**Theorem 7.3.** Let $\Sigma_n$ be as in (3) in the critical regime with $n \geq 2$. For $c \in (0, 1]$ we have

$$\det(\Sigma_n) = D \left( \frac{V(W)d\varphi_d}{2} \right)^n$$

and for $c \in (1, \infty)$ there is (within the brackets) an additional $c$ in the denominator. Here

$$D = \left(1 + \sum_{j=1}^n \frac{c d\varphi_d}{a_j} \left(\frac{4x^j_k \prod_{k \in [n] \setminus \{j\}} (x_k + x_j)^2}{a_j \prod_{k \in [n] \setminus \{j\}} (x_k - x_j)^2}\right) + \sum_{i \in [n] \setminus \{j\}} \frac{8x^i_j(x_i + x_j) \prod_{i \in [n] \setminus \{i,j\}, k \in [i,j]} (x_k + x_i)}{a_i (x_j - x_i)^2} \prod_{i \in [n] \setminus \{i,j\}, k \in [i,j]} (x_k - x_i) \right) \prod_{1 \leq i < j \leq n} (x_i - x_j)^2,$$

**Proof.** With the notations for $v$ and $w$ introduced before Theorem 7.2 it follows from the matrix determinant lemma (also called Cauchy’s formula, see [13, 0.8.5.11.]) using the inverse of the already known concentration matrix $\Sigma_n^{sh}$ for $c \in (0, 1]$ that

$$\det(\Sigma_n^{cr}) = \det(\Sigma_n^{sh} + c\Sigma_n^{sp}) = (1 + v^{sh}(\Sigma_n^{sh})^{-1}) \det(\Sigma_n^{sh})$$

and for $c \in (1, \infty)$ that

$$\det(\Sigma_n^{cr}) = \det(1/c\Sigma_n^{sh} + \Sigma_n^{sp}) = (1 + w^{sh}(1/c\Sigma_n^{sh})^{-1})w(1/c)^n \det(\Sigma_n^{sh}).$$

Next we insert the determinant in the subcritical regime from Corollary 6.2 and compute $v^{sh}(\Sigma_n^{sh})^{-1}v$ in the case $c \in (0, 1]$ and $w^{sh}(1/c\Sigma_n^{sh})^{-1}w$ in the case $c \in (1, \infty)$. In the first case
Proposition 6.3 implies that the entry in place $j$ of the $1 \times n$-matrix $v' (\Sigma_n^{sb})^{-1}$ is
\[
\left( \frac{\sqrt{c \cdot V(W)} d x_d 4 x_j \prod_{k \in [n] \setminus \{j\}} (x_k + x_j)^2}{a_j V(W) d x_d \prod_{k \in [n] \setminus \{j\}} (x_k - x_j)^2} \right) + \sum_{i \in [n] \setminus \{j\}} \left( -\frac{\sqrt{c \cdot V(W)} d x_d 8 x_i x_j (x_i + x_j) \prod_{l \in [n] \setminus \{i,j\}, k \in \{i,j\}} (x_k + x_l)}{a_i V(W) d x_d (x_j - x_i)^2 \prod_{l \in [n] \setminus \{i,j\}, k \in \{i,j\}} (x_k - x_l)} \right).
\]
A multiplication of this row vector with $v$ from the right yields
\[
\sum_{j=1}^n cd x_d \left( \frac{4 x_j \prod_{k \in [n] \setminus \{j\}} (x_k + x_j)^2}{a_j \prod_{k \in [n] \setminus \{j\}} (x_k - x_j)^2} + \sum_{i \in [n] \setminus \{j\}} \frac{8 x_i x_j (x_i + x_j) \prod_{l \in [n] \setminus \{i,j\}, k \in \{i,j\}} (x_k + x_l)}{a_i (x_j - x_i)^2 \prod_{l \in [n] \setminus \{i,j\}, k \in \{i,j\}} (x_k - x_l)} \right).
\]
For $c \in (1, \infty)$ an analogue computation concludes the proof.

As a next goal we are discussing questions related to eigenvalues of $\Sigma_n$ in the critical regime.

**Example 7.4.** Consider for $n = 2$ and $c \in (0, 1]$ the covariance matrix
\[
\Sigma_2 = V(W) \begin{pmatrix}
d x_d (a_1^2 + 2 x d x_d 2 x_1) & d x_d (a_1 a_2 + 2 x d x_d (x_1 + x_2)) \\
d x_d (a_2 a_2 + 2 x d x_d (x_1 + x_2)) & 2 (x_1 + x_2) a_1 a_2 \\
\end{pmatrix}.
\]
The case where $d = 2$, $V(W) = 1$ and (non-negative increasing powers) $\tau_1 = 0$, $\tau_2 = 1$ is of special interest. In particular, we consider $c = 1$. Then a calculation of the characteristic polynomial with eigenvalues $\lambda_1, \lambda_2$ of $\Sigma_2$ leads to
\[
\chi(\Sigma_2) = \lambda^2 - \lambda \cdot \text{tr}(\Sigma_2) + \det(\Sigma_2) = \lambda^2 - \lambda \left( \frac{13 \pi^2}{9} + \frac{3 \pi}{4} \right) + \frac{\pi^2}{36} + \frac{\pi^2}{72} \in \mathbb{R}[\lambda].
\]
We obtain $\lambda_{1,2} = \frac{\pi}{12} \left( 52 \pi + 27 \pm \sqrt{52^2 \pi^2 + 2664 \pi + 657} \right)$.

The characteristic polynomial can be determined for the critical regime as in the other possible regimes. Lemma 6.5 and Theorem 7.3 yield the following:

**Corollary 7.5.** Let $\Sigma_n = (\sigma_{ij})$ be as in (3) in the critical regime for $n \geq 2$ with characteristic polynomial $\chi(\Sigma_n) = (-1)^n \lambda^n + a_n^{(n-1)} \lambda^{n-1} + \ldots + a_n^{(1)} \lambda + a_n^{(0)} \in \mathbb{R}[\lambda]$. For $c \in (0, 1]$ we have
\[
a_n^{(k)} = (-1)^k D_k \left( \frac{V(W) d x_d}{2} \right)^{n-k} \text{ for } k \in [n-1].
\]
For $c \in (1, \infty)$ there is an additional $c$ in the denominator of $V(W) d x_d / 2$. Here
\[
D_k = \sum_{1 \leq i_1 < \ldots < i_{n-k} \leq n} \left( 1 + \sum_{j \in \{i_1, \ldots, i_{n-k}\}} \frac{c d x_d}{a_j} \frac{4 x_j \prod_{k \in \{i_1, \ldots, i_{n-k}\} \setminus \{j\}} (x_k + x_j)^2}{a_j \prod_{k \in \{i_1, \ldots, i_{n-k}\} \setminus \{j\}} (x_k - x_j)^2} \right).
\]
+ \sum_{i \in \{1, \ldots, n\} \setminus \{j\}} \frac{8x_i x_j (x_i + x_j) \prod_{l \in \{i_1, \ldots, i_{n-1}\} \setminus \{j\}, k \in \{i, j\}} (x_k + x_l)}{a_i (x_i - x_j)^2 \prod_{l \in \{i_1, \ldots, i_{n-1}\} \setminus \{j\}, k \in \{i, j\}} (x_k - x_l)}

\prod_{i, j \in \{i_1, \ldots, i_{n-1}\}} (x_i - x_j)^2

\prod_{i, j \in \{i_1, \ldots, i_{n-1}\}} (x_i + x_j)

\text{for } k \in [n-1].

As in the subcritical regime we can not determine eigenvalues explicitly. Upper and lower bounds for them are given below. See also, e.g., [6] for related results using other methods.

**Theorem 7.6.** Let \( \Sigma_n^{cr} \) be as in (3) in the critical regime for \( n \geq 2 \) with eigenvalues \( \lambda_1^{cr} \leq \ldots \leq \lambda_n^{cr} \) and let \( \tilde{S}_n \) and \( \bar{S}_n \) be as in Theorem 6.7. We have

\[ U_n \leq \lambda_1^{cr} \leq \ldots \leq \lambda_n^{cr} \leq \bar{U}_n \]

(i) \( U_n = \tilde{S}_n + \sum_{i=1}^n \frac{c V(W) d^2 \kappa_n^2}{a_i^2} \) for \( c \in (0, 1) \),

(ii) \( U_n = \frac{\bar{S}_n}{c} \) and \( \bar{U}_n = \frac{\bar{S}_n}{c} + \sum_{i=1}^n \frac{V(W) d^2 \kappa_n^2}{a_i^2} \) for \( c \in (1, \infty) \).

**Proof.** Consider \( c \in (0, 1] \). Let \( \lambda_1^t \leq \lambda_2^t \leq \ldots \leq \lambda_n^t \) for \( t \in \{sb, sp\} \) be all eigenvalues of the corresponding matrices \( \Sigma_n^{sb} \) and \( \Sigma_n^{sp} \) of the matrix \( \Sigma_n^{cr} \). Due to Theorem 5.2 the eigenvalues of \( \Sigma_n^{sp} \) are \( \lambda_1^{sp} = \lambda_2^{sp} = \ldots = \lambda_{n-1}^{sp} = 0 \) and \( \lambda_n^{sp} = V(W) d^2 \kappa_n^2 (\sum_{i=1}^n 1/a_i^2) \). Theorem 6.7 provides an upper bound \( \tilde{S}_n \) and a lower bound \( \bar{S}_n \) for the eigenvalues \( \lambda_n^{sp} \leq \ldots \leq \lambda_n^{sb} \) of \( \Sigma_n^{sb} \). Using Lemma 4.1(i), one can state bounds for the eigenvalues of \( \Sigma_n^{cr} \). Note that the multiplicative constant \( c \) occurring in the sum decomposition of \( \Sigma_n^{cr} \) must be taken into account. Thus,

\[ \tilde{S}_n + c \cdot 0 \leq \lambda_1^{sb} + c \lambda_1^{sp} \leq \lambda_1^{cr} \leq \lambda_1^{cr} \leq \lambda_n^{cr} \leq \lambda_n^{sb} + c \lambda_n^{sp} \leq \bar{S}_n + c \sum_{i=1}^n \frac{V(W) d^2 \kappa_n^2}{a_i^2}. \]

The proof of the case \( c \in (1, \infty) \) is done in an analogue way.

**Remark 7.7.** It is of interest to improve the bounds from the previous theorem. A possible approach for this is to use eigenvalues \( \lambda_i^{sb} \) of \( \Sigma_n^{sb} \) themselves. Then Lemma 4.1(i) yields:

(i) If \( c \in (0, 1] \), then \( \lambda_i^{sb} \leq \lambda_i^{cr} \leq \lambda_i^{sb} + c V(W) d^2 \kappa_n^2 (\sum_{i=1}^n 1/a_i^2) \) for \( i \in [n] \),

(ii) If \( c \in (1, \infty) \), then \( \frac{1}{c} \lambda_i^{sb} \leq \lambda_i^{cr} \leq \frac{1}{c} \lambda_i^{sb} + V(W) d^2 \kappa_n^2 (\sum_{i=1}^n 1/a_i^2) \) for \( i \in [n] \).

Note that this remark yields at the moment an improvement only for \( n = 2 \), since the eigenvalues \( \lambda_n^{sb} \) for \( n > 2 \) have not been explicitly determined yet.

**Example 7.8.** Consider \( \Sigma_2^{cr} \) from Example 7.4 with \( c = 1, d = 2, V(W) = 1 \) and (non-negative increasing powers) \( \tau_1 = 0 \) and \( \tau_2 = 1 \). The bounds from Theorem 7.6 lead to

\[ U_2 = \frac{\pi}{24} (9 - \sqrt{73}) \leq \lambda_1^{cr} \leq \lambda_2^{cr} \leq \frac{\pi}{24} (9 + \sqrt{73}) + \frac{13 \pi^2}{9} = \bar{U}_2. \]
Observe that Remark 7.7 is applicable. By Example 7.4 the eigenvalues of \( \Sigma_2^{sb} \) are \( \lambda_1^{sb} = \frac{\pi}{24}(9 - \sqrt{73}) \) and \( \lambda_2^{sb} = \frac{\pi}{24}(9 + \sqrt{73}) \). According to Theorem 5.2 all eigenvalues of \( \Sigma_2^{sp} \) are given as \( \lambda_1^{sp} = 0 \) and \( \lambda_2^{sp} = 13\pi^2/9 \). Hence,

\[
\frac{\pi}{24}(9 - \sqrt{73}) \leq \lambda_1^{ct} \leq \frac{\pi}{24}(9 - \sqrt{73}) + \frac{13\pi^2}{9}
\]

\[
\frac{\pi}{24}(9 + \sqrt{73}) \leq \lambda_2^{ct} \leq \frac{\pi}{24}(9 + \sqrt{73}) + \frac{13\pi^2}{9}.
\]

Note that according to these bounds the eigenvalues are potentially lying in an interval of length \( \lambda_2^{sp} = \frac{13\pi^2}{9} < 2 \cdot 10^2 \). However, we know that

\[
\lambda_{i,2}^{ct} = \frac{\pi}{72} \left( 52\pi + 27 \mp \sqrt{52^2\pi^2 + 2664\pi + 657} \right)
\]

and these numbers are close to the end points of that interval, since they have only a distance \( \leq 10^{-3} \) to \( \bar{U}_2 \) or \( U_2 \). This examples motivates to study improvements for \( \bar{U}_n \) and \( U_n \) to obtain better bounds.

We state the following conjecture regarding eigenvalues of the considered matrices.

**Conjecture 7.9.** Let \( \Sigma_n \) be as in (3) in the critical regime for \( n \geq 2 \) with eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_\ell_n \). Then \( \lambda_i \neq \lambda_j \) for all \( i \neq j \in [n] \).

Finally, we investigate decompositions of \( \Sigma_n \) in the critical regime. They become very complicated in this regime. That’s why we focus here on an interesting special case:

**Definition 7.10.** Let \( \tau_i = i - 1 \) for \( i = 1, \ldots, n \) and \( d = 2 \). The vector of length power functionals \( (\tilde{L}_1^{(\tau_1)}, \ldots, \tilde{L}_2^{(\tau_2)}) \) according to (2) is called vector of natural increasing powers.

For the vector of natural increasing powers the LU decomposition is:

**Proposition 7.11.** Let \( \Sigma_n \) be as in (3) in the critical regime with \( n \geq 2 \) with respect to the vector of natural increasing powers. The LU decomposition \( \Sigma_n = LU \) with \( L = (l_{ij}) \) and \( U = (u_{ij}) \) is given by

\[
l_{ij} = \begin{cases} 
0 & \text{for } i < j, \\
\frac{\pi(i+1)\pi(j+1)}{\prod_{k=1}^{i-1}(i-k)\prod_{k=1}^{j-1}(j-k)} & \text{for } i > j,
\end{cases}
\]

and

\[
u_{ij} = \begin{cases} 
\frac{\pi(1+2c\pi)}{\pi(1+2\pi)} & \text{for } i = 1 \text{ and } c \in (0, 1], \\
\frac{\pi(1+2c\pi)}{c(j+1)} & \text{for } i = 1 \text{ and } c \in (1, \infty), \\
0 & \text{for } i > j,
\end{cases}
\]

\[
\begin{cases} 
\frac{\pi\prod_{k=1}^{i-1}(i-k)(j-k)}{(i+j)\prod_{k=1}^{i-1}(i-k)\prod_{k=1}^{j-1}(j-k)} & \text{for } 1 < i \leq j \text{ and } c \in (0, 1], \\
\frac{\pi\prod_{k=1}^{i-1}(i-k)(j-k)}{c(i+j)\prod_{k=1}^{i-1}(i-k)\prod_{k=1}^{j-1}(j-k)} & \text{for } 1 < i \leq j \text{ and } c \in (1, \infty).
\end{cases}
\]
Proof. We compute the entries $\sigma_{ij} = \sum_{m=1}^{n} l_{im} u_{mj}$ of the given matrix $LU$ for $c \in (0, 1]$. For this let $q = \min\{i, j\}$. One sees $l_{im} u_{mj} = 0$ for $m \in \{q + 1, \ldots, n\}$ and $l_{i1} u_{1j} = 2\pi(1 + 2c\pi)/(i + 1)(j + 1)$. Thus,

$$\sigma_{ij} = \frac{2\pi(1 + 2c\pi)}{(i + 1)(j + 1)} + \sum_{m=2}^{q} \frac{2\pi m \prod_{k=1}^{m-1} (i - k)(j - k)}{\prod_{k=1}^{m} (i + k)(j + k)} \prod_{k=1}^{m} (m - k)(j - k)$$

$$= \frac{2\pi(1 + 2c\pi)}{(i + 1)(j + 1)} + \sum_{m=2}^{q} \frac{2\pi m \prod_{k=1}^{m-1} (i - k)(j - k)}{\prod_{k=1}^{m} (i + k)(j + k)}$$

$$= \frac{2\pi(1 + 2c\pi)}{(i + 1)(j + 1)} + \frac{\pi}{(i + j)} - \frac{2\pi}{(i + 1)(j + 1)} = \frac{\pi(4c\pi(i + j) + (i + 1)(j + 1))}{(i + j)(i + 1)(j + 1)}.$$

These are the entries of $\Sigma_{cr}$ as in (14) for the vector of natural increasing powers. The case for $c \in (1, \infty)$ is treated analogously.

The former results implies:

**Corollary 7.12.** Let $\Sigma_n$ be as in (3) in the critical regime with $n \geq 2$ with respect to the vector of natural increasing powers. For $c \in (0, 1]$ the determinant of $\Sigma_n$ is given by

$$\det(\Sigma_n) = \frac{\pi(1 + 2c\pi)}{2} \prod_{i=2}^{n} \frac{\pi \prod_{k=1}^{i-1} (i - k)^2}{2i \prod_{k=1}^{i-1} (i + k)^2}.$$ 

For $c \in (1, \infty)$ there is an additional factor $c^n$ in the denominator of $\pi(1 + 2c\pi)/2$.

The LU decomposition of $\Sigma_n$ with respect to the vector of natural increasing powers yields immediately the Cholesky decomposition of $\Sigma_n$:

**Corollary 7.13.** Let $\Sigma_n$ be as in (3) in the critical regime for $n \geq 2$ with respect to the vector of natural increasing powers. The Cholesky decomposition $\Sigma_n = GG^t$ is given by $G = (g_{ij})$, such that entries for $c \in (0, 1]$ are given by

$$g_{ij} = \begin{cases} 0 & \text{for } j > i, \\
\frac{2}{i + 1} \sqrt{\frac{\pi(1 + 2c\pi)}{2}} & \text{for } j = 1 \text{ and } i \geq 1, \\
 \prod_{k=1}^{j-1} (i - k) \prod_{k=1}^{j-1} (j - k) \sqrt{\frac{\pi \prod_{k=1}^{i-1} (j - k)^2}{2i \prod_{k=1}^{i-1} (j + k)^2}} & \text{for } 1 < j \leq i. 
\end{cases}$$

For $c \in (1, \infty)$ there is an additional $c$ in every denominator of all occurring roots.

Proof. The unique Cholesky decomposition of $\Sigma_n$ exists due to Algorithm 4.4 and it can be derived from the LU decomposition $\Sigma_n = LU$ determined in Proposition 7.11.
For this, define the diagonal matrix \( D = (d_{ij}) \) with diagonal entries from \( U \) given by

\[
d_{ij} = \begin{cases} 
0 & \text{for } j \neq i, \\
\frac{\pi(1+2\pi^2)}{(j+1)} & \text{for } j = i = 1 \text{ and } c \in (0, 1], \\
\frac{\pi(1+2\pi^2)}{c(j+1)} & \text{for } j = i = 1 \text{ and } c \in (1, \infty), \\
\frac{\pi \prod_{k=1}^{j-1}(j-k)^2}{2j \prod_{k=1}^{j-1}(j-k)^2} & \text{for } j > i \text{ and } c \in (0, 1], \\
\frac{\pi \prod_{k=1}^{j-1}(j-k)^2}{2c_j \prod_{k=1}^{j-1}(j-k)^2} & \text{for } j > i \text{ and } c \in (1, \infty).
\end{cases}
\]

This yields \( G = LD^\frac{1}{2} \). This concludes the proof. \( \square \)

Finally, we compute an example of a Cholesky decomposition of \( \Sigma_2 \) for arbitrary powers, which will be used in Section 9.

**Example 7.14.** Consider \( \Sigma_2 \) in the critical regime. For \( c \in (0, 1] \) we see

\[
\Sigma_2 = \frac{V(W) \sqrt{\tau_1}}{2} \begin{pmatrix} 
\frac{4d x_d cx_1 + a_1^2}{2x_1 a_1^2} & \frac{2d x_d c(x_1 + x_2) + a_1 a_2}{(x_1 + x_2) a_1 a_2} & \frac{4d x_d c x_2 + a_2^2}{2x_2 a_2^2} \\
\frac{2d x_d c(x_1 + x_2) + a_1 a_2}{(x_1 + x_2) a_1 a_2} & \frac{4d x_d c(x_1 + x_2) + a_1 a_2}{(x_1 + x_2) a_1 a_2} & \frac{(2d x_d c(x_1 + x_2) + a_1 a_2) \sqrt{2x_1}}{2a_2(x_1 + x_2) \sqrt{4d x_d c x_1 + a_1^2}} \\
\frac{4d x_d c x_2 + a_2^2}{2x_2 a_2^2} & \frac{(2d x_d c(x_1 + x_2) + a_1 a_2) \sqrt{2x_1}}{2a_2(x_1 + x_2) \sqrt{4d x_d c x_1 + a_1^2}} & 0
\end{pmatrix}.
\]

and therefore

\[
\Sigma_2 = GG^t
\]

with the following matrix \( G \):

\[
\sqrt{\frac{V(W) \sqrt{\tau_1}}{2}} \begin{pmatrix} 
\sqrt{\frac{4d x_d c x_1 + a_1^2}{2x_1 a_1^2}} & 0 & 0 \\
\frac{2d x_d c(x_1 + x_2) + a_1 a_2}{(x_1 + x_2) a_1 a_2} & \sqrt{\frac{2x_c + 1}{d}} & 0 \\
\frac{2d x_d c(x_1 + x_2) + a_1 a_2}{(x_1 + x_2) a_1 a_2} & 0 & \frac{x^2}{2a_2^2}
\end{pmatrix}.
\]

For \( \tau_1 = 0 \) and \( \tau_2 \in \mathbb{R} \) we get for instance

\[
G = \sqrt{\frac{V(W) \sqrt{\tau_1}}{2}} \begin{pmatrix} 
\sqrt{\frac{2x_c + 1}{d}} & 0 & 0 \\
\frac{x^2}{2a_2^2}
\end{pmatrix}.
\]

For \( c \in (1, \infty) \) there is an additional \( \sqrt{c} \) in each denominator of the entries of the matrix \( G \).

### 8. Across-regimes investigations

In this Section some across regimes results regarding decompositions are presented. As before, we use labels cr, sb or sp as well as notations \( l_{ij}, u_{ij} \) and \( g_{ij} \) for the entries of corresponding matrices in decompositions of interest.
Theorem 8.1. Let $\Sigma_n^{cr}$ be as in (3) in the critical regime for $n \geq 2$ with respect to the vector of natural increasing powers. Consider the LU decompositions $\Sigma_n^{sb} = L^{sb}U^{sb}$ and $\Sigma_n^{sp} = L^{sp}U^{sp}$. Then the entries of the matrices $L^{cr}$, $U^{cr}$ of the LU decomposition of $\Sigma_n^{cr}$ are given as:

(i) For $c \in (0, 1]$ we have $l_{ij}^{cr} = l_{ij}^{sb}$ for $i, j \in [n]$ and $u_{ij}^{cr} = u_{ij}^{sb} + cu_{ij}^{sp}$.

(ii) For $c \in (1, \infty)$ we have $l_{ij}^{cr} = l_{ij}^{sb}$ for $i, j \in [n]$ and $u_{ij}^{cr} = \frac{1}{c}u_{ij}^{sb} + u_{ij}^{sp}$.

Proof. Let $c \in (0, 1]$. Theorem 2.2 yields

$$(L^{sp})^{-1}\Sigma_n^{cr} = (L^{sp})^{-1}\Sigma_n^{sb} + (L^{sp})^{-1}c\Sigma_n^{sp}. $$

Since the first column of the matrices $L^{sp}$ and $L^{sb}$ coincide, it follows

$$(L^{sp})^{-1}\Sigma_n^{cr} = (U^{(1)})^{sb} + cU^{sp}. $$

Here on the right hand, the multiplication with $(L^{sp})^{-1}$ corresponds to the use of the first (Frobenius) matrix in an LU algorithm (see Theorem 4.3). For the first summand $(U^{(1)})^{sb}$ denotes the resulting matrix by multiplying with $\Sigma_n^{sb}$. For the second summand we have $U^{sp} = (L^{sp})^{-1}\Sigma_n^{sp}$.

Indeed, according to Theorem 5.5(i) the matrix $U^{sp}$ has only zeros below the first row, so transformations in the algorithm using left multiplications of further (Frobenius) matrices (which are zero in the places $(i, j)$ with $i > j = 1$) do not change anything.

We apply the remaining $n - 2$ steps of the LU algorithm determining the LU decomposition of $\Sigma_n^{sb}$ to both sides of the Equation (15), i.e. multiplications from the left with suitable Frobenius matrices to transform $\Sigma_n^{sb}$ into an upper triangle matrix. Multiplying all these lower triangular matrices yields exactly $(L^{sb})^{-1}$. Hence,

$$(L^{sb})^{-1}\Sigma_n^{cr} = (L^{sb})^{-1}\Sigma_n^{sb} + cU^{sp} = U^{sb} + cU^{sp}. $$

The right side of the latter equation is an upper triangular matrix and the left side is a lower triangular matrix multiplied with $\Sigma_n^{cr}$. This leads to the LU decomposition of $\Sigma_n^{cr}$:

$$\Sigma_n^{cr} = L^{sb}(U^{sb} + cU^{sp}).$$

For $c \in (1, \infty)$ one uses the same transformations and sees $\Sigma_n^{cr} = L^{sb}(\frac{1}{c}U^{sb} + U^{sp})$. □

Remark 8.2. Since $cu_{1j}^{sp} = 2c\pi\frac{\pi}{j+1} = 2c\pi u_{1j}^{sb}$ and $u_{1j}^{sp} = 2\pi^2\frac{u_{1j}^{sb}}{j+1} = 2c\pi^2\frac{u_{1j}^{sb}}{c}$, all entries of $L^{cr}$ and $U^{cr}$ can be computed using only the entries of $L^{sb}$ and $U^{sb}$. Thus, for $i, j \in [n]$ we have

$l_{ij}^{cr} = l_{ij}^{sb}$ and $u_{ij}^{cr} = mu_{ij}^{sb}$ with $m = \begin{cases} 1 + \frac{2c\pi}{c} & \text{for } c \in (0, 1], \\ \frac{1 + 2c\pi}{c} & \text{for } c \in (1, \infty). \end{cases}$

It is easy to see that Theorem 8.1 can not be generalized to an arbitrary vector of length power functionals. For example, the choices $\tau_1 = 1$, $\tau_2 = 3$, $\tau_3 = 5$, $d = 2$, and $c = 1$ leads to a counterexample. For Cholesky decompositions we get:
Theorem 8.3. Let $\Sigma_n^{cr}$ be as in (3) in the critical regime for $n \geq 2$ to the vector of natural increasing powers. Then for Cholesky factors $G^{cr}$ of $\Sigma_n^{cr}$ we have $g_{ij}^{cr} = g_{ij}^{sb}$ for $j > 1$ and $g_{i1}^{cr} = \nu g_{i1}^{sb}$ with

$$v = \begin{cases} \sqrt{1 + cu_{i1}^{sb}/u_{i1}^{cr}} & \text{for } c \in (0, 1], \\ \sqrt{1/c + u_{i1}^{sb}/u_{i1}^{cr}} & \text{for } c \in (1, \infty). \end{cases}$$

Proof. A comparison of Corollaries 6.10 and 7.13 shows that $g_{ij}^{cr}$ and $g_{ij}^{sb}$ coincide in all entries, except for those in the first column. The factor by which $g_{i1}^{cr}$ and $g_{i1}^{sb}$ differ, arises by the use of a diagonal matrix similar to the one used in the proof of Corollary 7.13. Indeed, the matrices $L^{cr}$ and $L^{sb}$ are identical. The same is true for the matrices $U^{cr}$ and $U^{sb}$, except for their first rows. Here we have $u_{ij}^{cr} = u_{ij}^{sb} + cu_{ij}^{sp}$ for $c \in (0, 1]$ and $u_{ij}^{cr} = \frac{1}{c}u_{ij}^{sb} + u_{ij}^{sp}$ for $c \in (1, \infty)$ according to Theorem 8.1. Set

$$m = \frac{u_{i1}^{cr}}{u_{i1}^{sb}} = \begin{cases} 1 + \frac{cu_{i1}^{sp}}{u_{i1}^{sb}} & \text{for } c \in (0, 1], \\ \frac{1}{c} + \frac{u_{i1}^{sp}}{u_{i1}^{sb}} & \text{for } c \in (1, \infty). \end{cases}$$

Observe that the entries $d_{11}^{cr}$ of $D^{cr}$ and $d_{11}^{sb}$ of $D^{sb}$ from the proof of Corollary 7.13 are related as $d_{11}^{cr} = md_{11}^{sb}$. Hence,

$$g_{i1}^{sb} = l_{i1}^{sb}\sqrt{d_{11}^{sb}} \quad \text{and} \quad g_{i1}^{cr} = l_{i1}^{cr}\sqrt{d_{11}^{cr}} = l_{i1}^{sb}\sqrt{md_{11}^{sb}}.$$  

Thus, setting $v = \sqrt{m}$ one obtains $g_{i1}^{cr} = \nu g_{i1}^{sb}$. \qed

Regarding the Cholesky decomposition there arise observations similar to Remark 8.2:

Remark 8.4. It is possible to give a formula for $G^{cr}$ with respect to the vector of natural increasing powers without using $U^{sp}$ (but still $U^{sb}$) in contrast to the one determined in Theorem 8.3. For this observe at first that comparing the entries $u_{i1}^{sb}$ and $u_{i1}^{sp}$ one gets $u_{i1}^{sp} = 2\pi u_{i1}^{sb}$. Using the notation $m$ and $v$ as in the proof of Theorem 8.3 this leads to

$$m = \begin{cases} 1 + 2c\pi & \text{for } c \in (0, 1], \\ \frac{1}{c} + 2\pi & \text{for } c \in (1, \infty) \end{cases}, \quad \text{and} \quad v = \begin{cases} \sqrt{1 + 2c\pi}, & \text{for } c \in (0, 1], \\ \sqrt{\frac{1 + 2c\pi}{c}}, & \text{for } c \in (1, \infty). \end{cases}$$

Additionally note that exactly this constant $m$ occurred already in Remark 8.2 concerning the LU decomposition of $\Sigma_n^{cr}$.
9. Stochastic applications

In this section we consider some of the main results of this manuscript from a stochastic point of view. According to [19, Theorem 5.2.] \((\tilde{L}^{(\tau_1)}, \ldots, \tilde{L}^{(\tau_n)})\) is asymptotic normal distributed. Recall that \(a_i = \tau_i + d\) and abbreviate \(a = \tau + d\) as well as \(x = \tau + d/2\). With this we can state the following results:

**Theorem 9.1.** Assume that \(t \delta_t^d \to \infty\) and \(\tau_1, \tau_2 > -d/2\). Then

\[
D_t^{(\tau_1, \tau_2)} = a_1 \tilde{L}_t^{(\tau_1)} - a_2 \tilde{L}_t^{(\tau_2)} \overset{P}{\to} 0 \text{ as } t \to \infty.
\]

**Proof.** From the Schur decomposition of \(\Sigma^b_2\) in Corollary 5.4 we gain the rotation matrix

\[
R = \begin{pmatrix}
a_1 & a_2 \\
\sqrt{a_1^2 + a_2^2} & \sqrt{a_1^2 + a_2^2}
\end{pmatrix},
\]

such that \(R^{-1}(\tilde{L}_t^{(\tau_1)}, \tilde{L}_t^{(\tau_2)})' \to N\) with \(N \sim \mathcal{N}(0, \Sigma)\), where

\[
\Sigma = \begin{pmatrix}
0 & 0 \\
0 & V(W)(d\xi_d)^2(\sum_{i=1}^n 1/a_i)
\end{pmatrix}.
\]

The claim follows from the entries of the first row of \(R\).

\[\square\]

In particular, setting \(\tau_1 = 0\), we see that the number of edges \(L_t^{(0)}\) asymptotically determines \(L_t^{(\tau)}\) for all \(\tau\). More precisely, we have

\[(16) \quad \tilde{L}_t^{(\tau)} = d \tilde{L}_t^{(0)} / a + D_t^{(\tau)} \text{ with some } D_t^{(\tau)} \overset{P}{\to} 0.\]

If \(Y \sim \mathcal{N}(0, \Sigma)\) with \(\Sigma\) a positive definite \(2 \times 2\)-matrix (i.e. in the critical and subcritical regime), then for the Cholesky factor \(G\) with \(\Sigma = GG'\) the equation \(G^{-1}\Sigma(G^{-1})' = I_2\) holds. So

\[G^{-1}Y = (Z_1, Z_2),\]

where the \(Z_i\) are independent \(\mathcal{N}(0, 1)\)-random variables. If \(Y = \lim_{t \to \infty}(\tilde{L}_t^{(0)}, \tilde{L}_t^{(\tau)})\), then

\[G^{-1}Y = \lim_{t \to \infty}(b_0 \tilde{L}_t^{(0)}, c_0 \tilde{L}_t^{(0)} + c \tilde{L}_t^{(\tau)}).\]

This leads in particular to:

**Theorem 9.2.** Assume that \(t \delta_t^d \to c \in [0, \infty), \tau > -d/2\), and that \(Z \sim \mathcal{N}(0, 1)\) is independent of \(\tilde{L}_t^{(0)}\). Then

\[(17) \quad \tilde{L}_t^{(\tau)} = \frac{d}{a} \tilde{L}_t^{(0)} + \frac{\tau \sqrt{V(W)}d\xi_d}{a \sqrt{4x \max\{c, 1\}}} Z + D_t^{(\tau)} \text{ with some } D_t^{(\tau)} \overset{P}{\to} 0,\]

as \(t \to \infty.\)
Proof. In the subcritical regime and using Example 6.11 we see for the Cholesky factor $G$ of $\Sigma_2$, that

$$G^{-1} = \sqrt{\frac{2}{V(W)d_{\infty,d}}} \begin{pmatrix} \sqrt{a} & 0 \\ \frac{d\sqrt{2}}{t} & \frac{a\sqrt{2}}{t} \end{pmatrix}.$$ 

Thus, $G^{-1}Y = \sqrt{\frac{2}{V(W)d_{\infty,d}}} \lim_{t \to \infty} (\sqrt{dL_t^{(0)}} - \frac{d\sqrt{2}}{t} L_t^{(0)} + \frac{a\sqrt{2}}{t} \tilde{L}_t^{(\tau)})'$, which implies the claim immediately. In the critical regime we observe from Example 7.14 for $c \in (0,1] that$

$$G^{-1} = \sqrt{\frac{2}{V(W)d_{\infty,d}}} \begin{pmatrix} d & 0 \\ \frac{2}{a\sqrt{\tau^2}} & \frac{2}{\sqrt{\tau^2}} \end{pmatrix}.$$ 

Hence, $G^{-1}Y = \sqrt{\frac{2}{V(W)d_{\infty,d}}} \lim_{t \to \infty} (\sqrt{dL_t^{(0)}} - \frac{d}{a\sqrt{\tau^2}} L_t^{(0)} + \sqrt{\frac{2}{\tau^2}} \tilde{L}_t^{(\tau)})'$. For $c \in (1,\infty)$ according to Example 7.14 a similar computation leads to the same results with an additional $c$ in the nominator of all entries of $2/V(W)d_{\infty,d}$. This shows the claim in the critical regime for both cases for $c$. □

It would be of high interest to determine the order of $D_t^{(\tau)}$ in Theorems 9.1 and 9.2 as $t \to \infty$, or even its asymptotic distribution. This seems to be out of reach at the moment.

10. Outlook

Future research activities start by studying open problems from Tables 1 and 2 for a complete analysis of $\Sigma_n$ with respect to all algebraic aspects considered before. For example, though we known in the critical and subcritical regime the characteristic polynomial, in general, we have no formula for the eigenvalues. A detailed analysis of them would be of interest. For this, a possible strategy is to use matrix decompositions and related methods as discussed in this manuscript. Of course such decompositions are also interesting in their own right and we would like to know more of them like Singular value or QR decompositions. Another example for a research goal is related to the bounds for eigenvalues of $\Sigma_n$ described in Theorem 6.7 and Theorem 7.6. There are not tight and should be improved.

Stochastic research objectives are related to generalizations of random geometric graphs. As already observed before, they are a special case of random geometric (simplicial) complexes. Moreover, other Poisson functionals must be considered. In particular, volume power functionals (see [2]) and their covariance matrices will be studied in the future as well as $k$-simplex counting functionals.

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