Polyanalytic reproducing Kernels on the quantized annulus

Nizar Demni

1 Institut de Mathématiques de Marseille (I2M, UMR 7373), Aix-Marseille Université, CNRS, Marseille, France

2 Department of Mathematics, Faculty of Sciences and Technics (M’Ghila), Sultan Moulay Slimane, PO. Box 523, Béni Mellal, Morocco

E-mail: mouayn@usms.ma

Received 20 August 2020, revised 16 November 2020
Accepted for publication 19 November 2020
Published 10 December 2020

Abstract
While dealing with the constant-strength magnetic Laplacian on the annulus, we complete Peetre’s work. In particular, the eigenspaces associated with its discrete spectrum turn out to be polyanalytic spaces with respect to the invariant Cauchy–Riemann operator, and we write down explicit formulas for their reproducing kernels. When the magnetic field strength is an integer, we compute the limits of the obtained kernels when the outer radius of the annulus tends to infinity and express them by means of the fourth Jacobi theta function and of its logarithmic derivatives. Under the same quantization condition, we also derive their transformation rule under the action of the automorphism group of the annulus.

Keywords: Quantized annulus, magnetic laplacian, reproducing kernels, polyanalytic spaces, invariant Cauchy–Riemann operator, fourth Jacobi theta function

1. Introduction

A Riemann surface \( \mathcal{M} \) is called hyperbolic if its holomorphic universal covering space is the unit disk \( \mathbb{D} = \{ \zeta \in \mathbb{C}, |\zeta| < 1 \} \). If additionally its fundamental group \( \pi_1 (\mathcal{M}) \) is commutative then \( \mathcal{M} \) is isomorphic to either \( \mathbb{D} \), or to the punctured unit disk \( \mathbb{D} \setminus \{0\} \) or to the annulus
\[
\Omega_{1,R} := \{ z \in \mathbb{C}, 1 < |z| < R \}.
\]

Such surfaces are referred to as exceptional hyperbolic Riemann surfaces [12] and are involved in physics as phase spaces for the classical mechanics of systems with Hamiltonian functions. A special interest is given to the annulus \( \Omega_{1,R} \) and stems from the key role it plays in the...
quantization of the Hall effect. Indeed, because of its complete universality, the quantization must be insensitive to continuous deformations of the sample geometry. Using this freedom degree and due to its additional symmetry, the annulus geometry was proposed by Laughlin [20] as a substitute of the standard ‘Hall bar’ one.

Recently, the annular domain appeared in the study of the confinement of exciton–polariton condensate in relation to the Meissner effect [9]. In this respect, topological spin Meissner states can be observed at arbitrary high magnetic fields. For more details on this phenomenon, we refer the reader to (p 154) where a magnetic flux quantization is also discussed. The annulus appeared as well in relation to the problem of a Josephson junction in a superconducting loop subject to a uniform magnetic field (see [3] and references therein). In a nutshell, the occurrence of the annular geometry has increased in both theoretical and experimental physics.

Geometrically, $\Omega_{1,R}$ can be covered by a horizontal strip using the exponential map and the density of its corresponding Poincaré metric reads (see e.g. [29, 30]):

$$\omega_R(z) := \left( \frac{\log R}{\pi} \right) |z| \sin \left( \frac{\pi \log |z|}{\log R} \right).$$

(1.1)

We can therefore consider for any $B > 1/2$ the associated weighted $L^2$-space $\mathcal{H}_B(\Omega_{1,R})$ of functions $\phi : \Omega_{1,R} \to \mathbb{C}$ with finite squared norm:

$$\int_{\Omega_{1,R}} |\phi(z)|^2 (\omega_R(z))^2 B^{-2} \, d\mu(z) < \infty,$$

d$\mu(z)$ being the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. The holomorphic subspace $\mathcal{A}(\Omega_{1,R})$ of $\mathcal{H}_B(\Omega_{1,R})$ was considered by Peetre in [30] where the correspondence principle ([5]) in the semi-classical limit $B \to +\infty$ was proved (with $B$ playing the role of the inverse Planck constant $\hbar$). This principle was connected with the holomorphic Berezin transform whose integral kernel is expressible in terms of the reproducing kernel of $\mathcal{A}(\Omega_{1,R})$. The latter was obtained in [31] (see section 7 there) and may be written in a more compact form using Euler’s reflection formula for the Gamma function ([2]) as:

$$K^{(R,B)}_{\Omega}(z, w) := \frac{(2\pi)^{2B-3}}{\Gamma(2B-1) R^{2B}(\log R)^{2B-1}} \sum_{j \in \mathbb{Z}} \Gamma \left( B + i \frac{\log R}{\pi} (j + B) \right) \left( \frac{zw}{R} \right)^j$$

(1.2)

where $i$ is the pure imaginary unit. The subspace $\mathcal{A}(\Omega_{1,R})$ fits also the null space

$$\mathcal{E}_0(\Omega_{1,R}) := \{ \phi \in \mathcal{H}_B(\Omega_{1,R}) : \Delta_B \phi = 0 \}$$

(1.3)

of the so-called invariant Laplacian operator with weight $B$ t ([30]):

$$\Delta_B := - (\omega_R(z))^2 \partial^2 - 2B \omega_R(z) (\partial \omega_R(z)) \partial.$$  

(1.4)

This is a densely-defined operator on $\mathcal{H}_B(\Omega_{1,R})$ whose discrete spectrum consists of the following finite set of eigenvalues:

$$\lambda_{B,m} := - m (2B - m - 1), \quad m = 0, 1, \ldots, \lfloor B - (1/2) \rfloor,$$

each being of infinite multiplicity ($\lfloor x \rfloor$ stands for the greatest integer not exceeding $x$).

In this paper, we are concerned with higher Landau-levels eigenspaces:

$$\mathcal{E}_m(\Omega_{1,R}) := \{ \phi \in \mathcal{H}_B(\Omega_{1,R}) : \Delta_B \phi = \lambda_{B,m} \phi \}, \quad m = 0, 1, \ldots, \lfloor B - (1/2) \rfloor.$$  

(1.5)
For a fixed Landau level \( \lambda, E_m(\Omega_{1,R}) \) turns out to be the \( m \)th true polyanalytic space \( \Omega_{1,R} \) with respect to the invariant Cauchy–Riemann operator \((\omega_R(z))^2\partial\bar{\partial}\). For this eigenspace, we shall extend Peetre’s formula (1.2) and write down its polyanalytic reproducing kernel \( K^{(R,B)}_m(z,w) \) as well as the limit of the latter as \( R \to \infty \). Actually, the non-orthonormal basis elements were expressed in [30] through Routh–Romanovski polynomials [36, 37] and we shall compute below their \( L^2 \)-norms. Recall from [30] that these eigenfunctions were determined after carrying the eigenvalue problem of \( \Delta_B \) into the one of the Schrödinger operator associated with the hyperbolic Scarf potential [1]. In this respect, it is worth noting that the boundedness of this potential implies the existence of a continuous spectrum for \( \Delta_B \) corresponding to scattering states.

Back to bound states, it is known that Routh–Romanovski polynomials may be represented through Jacobi polynomials with imaginary arguments and parameters. Making use of this relation and of Bateman’s product formula, we shall derive another expression of \( K^{(R,B)}_m(z,w) \) as a finite and alternating-sign sum of elliptic-like series. In particular, one retrieves the known fact that \( K^{(R,1)}_0(z,w) \) is closely connected to Weierstrass elliptic function \( \wp \) associated with the rectangular lattice (16). More generally, we shall prove that under the quantization condition \( B \in \mathbb{N} \setminus \{0\} \), the polyanalytic reproducing kernel \( K^{(R,B)}_m(z,w) \) may be expressed through higher derivatives of the fourth Jacobi’s theta function \( \theta_4 \) associated with the rectangular lattice. Under the same condition, we shall also derive the transformation rule of \( K^{(R,B)}_m(z,w) \) under the action of the inversion with respect to the circle centered at the origin and of radius \( R \). Since this kernel is readily seen to be invariant under rotations, this transformation rule exhausts its quasi-invariance under the automorphism group of the annulus. Of course, one cannot expect a strong analogy with the Poincaré disc case since the automorphism group of the latter model is much more larger than the one of the annulus.

The paper is organized as follows. In section 2, we recall some geometrical facts about the invariant Laplacian \( \Delta_B \) as well as some of its needed spectral properties. In section 3, we write down the orthonormal basis of the eigenspaces (1.5) associated with the discrete spectrum of \( \Delta_B \), discuss their polyanalyticity property and derive explicit expressions for the corresponding reproducing kernels. Section 4 is devoted to the relation of these kernels to the fourth Jacobi theta functions and to their invariance properties under the quantization condition. Section 5 contains concluding remarks with a particular emphasis on probabilistic aspects of reproducing kernels of polyanalytic Hilbert spaces. Proofs of our results are detailed in three appendices.

2. \( L^2 \) spectral theory of \( \Delta_B \)

The Poincaré metric of the annulus \( \Omega_{1,R} \) is written in local coordinates as:

\[
ds = \frac{|dz|}{\omega_R(z)},
\]

where \( \omega_R(z) \) is defined in (1.2). It allows to define the (1,0)-connection:

\[
\varpi = \partial_z + B\partial_{\bar{z}}(\log \omega_R(z))
\]

to which is associated the Bochner Laplacian:

\[
H_B = -(\omega_R(z))^2(\partial_z + B\partial_{\bar{z}}(\log \omega_R(z)))(\partial_{\bar{z}} - B\partial_z(\log \omega_R(z)))
\]

\[
= -(\omega_R(z))^2(\partial_{\bar{z}} - B\partial_z(\log \omega_R(z)))(\partial_z + B\partial_{\bar{z}}(\log \omega_R(z)))
\]

\[
+ 2B(\omega_R(z))^2\partial_{\bar{z}}(\log \omega_R(z)).
\]
This is a densely-defined and essentially self-adjoint operator on the weighted $L^2$-space ([39])

$$\mathcal{S}_0 (\Omega_{1,R}) = L^2 (\Omega_{1,R}, (\omega_R (z))^{-2} \, d\mu (z)).$$

Moreover, by analogy with [39], p 124, the ground state transformation

$$Q_B : \mathcal{S}_0 (\Omega_{1,R}) \to \mathcal{S}_B (\Omega_{1,R})$$

developed by:

$$Q_B [f] (z) = (\omega_R (z))^{-B} f (z), \quad z \in \Omega_{1,R},$$

is a unitary map and intertwines the operators $H_B$ and $\Delta_B$:

$$(Q_B)^{-1} \circ H_B \circ Q_B = \Delta_B.$$
Since this potential is bounded, then $\mathcal{L}_B$ admits a continuous spectrum whose eigenfunctions (scattering states) may be found in [17].

**Remark 2.1.** The Routh–Romanovski polynomials were discovered by Routh [37] and rediscovered by Romanovski [36] within the context of probability distributions. They are also named Romanovski of type IV or ‘finite Romanovski’ due to their finite-orthogonality.

**Remark 2.2.** The spectral theory of $\Delta_B$ bears some similarities with that of the Schrödinger operator with uniform magnetic field in the Poincaré upper half-plane $\mathbb{H}^2 = \{(x, y), x \in \mathbb{R}, y > 0\}$. The latter is actually given in suitable units by $(-M_B + B^2)/2$, where

$$M_B := y^2 \left( \partial_x^2 + \partial_y^2 \right) - 2iBy\partial_x$$

is the $B$-weight mass Laplacian [25, 26]. Moreover, up to the variable change $s = -\log y$, and when acting on functions of the form $g(s, x) := \exp\left(-i\gamma x - \frac{1}{8}s\right) \psi(s), \gamma \in \mathbb{R}$, this operator is mapped to the one-dimensional Schrödinger operator with Morse potential ([28]):

$$-\frac{1}{2}\partial_s^2 g + \left( \frac{1}{2} y^2 \exp(-2s) + \gamma B \exp(-s) \right) g + \left( \frac{1}{2} B^2 + \frac{1}{8} \right) g.$$

For more details on this connection, we refer the reader to [21, 27]. The structure of the spectrum is also similar: provided that $B > 1/2$, a finite number of eigenvalues usually known as hyperbolic Landau levels arises. Physically, this phenomenon means that the magnetic field has to be strong enough to capture the particle in a closed orbit, giving rise to *bound states* in which the particle cannot leave the system without additional energy ([7]).

### 3. Reproducing kernel of $\mathcal{E}_m(\Omega_{1,B})$ and its limit

Let $B > 1/2$ and fix $m = 0, 1, \ldots, [B - 1/2]$. Then an orthogonal basis of the Hilbert space $\mathcal{E}_m(\Omega_{1,B})$ is given by ([30]):

$$\phi_j(z) = z^j R^{(m+1,B,1-B)}_m \left( \cot \left( \frac{\pi \log(|z|)}{\log R} \right) \right), \quad z \in \Omega_{1,B}, \ j \in \mathbb{Z}. \quad (3.1)$$

The squared norm of $\phi_j$ in $\mathcal{H}_B(\Omega_{1,B})$ admits the following expression (see appendix A for the proof):

$$||\phi_j||^2 = 2^{3-2(B-m)} R^B \frac{(\log R)^{2B-1} m!\Gamma(2B-m)}{\pi^{2B-3} (2(B-m)-1)! \Gamma(B-m + \frac{1}{2} i\alpha(j,B))} |R|^{\frac{R}{2}}. \quad (3.2)$$

Furthermore, these basis elements satisfy the polyanalyticity property with respect to the invariant Cauchy–Riemann operator (see [32], pp 241–243):

$$D^m_z := (\omega(z))^2 \partial_z.$$  

In particular, for $m = 0$, the null space $\mathcal{E}_0(\Omega_{1,B})$ coincides with the space $\mathcal{A}(\Omega_{1,B})$ of analytic functions in $\Omega_{1,B}$ belonging to $\mathcal{H}_B(\Omega_{1,B})$. However, one readily checks by direct computations that unless $m = 0$, it may happen that:

$$(\partial^m_z \mathcal{R}^{(m+1,B,1-B)}_m \left( \cot \left( \frac{\pi \ln(|z|)}{\ln(R)} \right) \right)) \neq 0.$$
More generally, denote
\[ \mathcal{F}^{(m)}(\Omega_{1,R}) := \{ \phi \in \mathcal{S}_B(\Omega_{1,R}), (D^2 \phi)^{m+1} \phi = 0 \} \]
the polyanalytic space of order \( m \). Then, the eigenspace in (1.5) may be decomposed as:
\[ \mathcal{E}_m(\Omega_{1,R}) = \mathcal{F}^{(m+1)}(\Omega_{1,R}) \oplus \mathcal{F}^{(m)}(\Omega_{1,R}), \]
where \( \oplus \) stands for the orthogonal difference of two sets. This fact is a direct consequence of the factorization property proved in [32] and valid for arbitrary Riemann surfaces. Accordingly, \( \mathcal{E}_m(\Omega_{1,R}) \) will be referred to as the \( m \)th true-polyanalytic space on the annulus \( \Omega_{1,R} \) with respect to \( D^2 \). In appendix B, we shall prove the following formula for the reproducing kernel of this eigenspace.

**Theorem 3.1.** Let \( B > 1/2, m = 0, 1, \ldots, [B - (1/2)] \). Then, the reproducing kernel of the \( m \)th eigenspace \( \mathcal{E}_m(\Omega_{1,R}) \) reads
\[
K_m^{(B)}(z, w) = \frac{(2\pi)^{2B - 3}(2B - 2m - 1)}{R^2(\log R)^{2B - 1}} V(2B - m) \\
\times \sum_{j=0}^{m} \sum_{k=0}^{m-j} \frac{(1 - 2B + m)_{k+j}}{(m - k - l)!} \frac{V_{k+l} V_l}{k! l!} \sigma_{l,j}^{(B)}(z, w)
\]
where
\[
\sigma_{l,j}^{(B)}(z, w) = \sum_{j \in \mathbb{Z}} \Gamma(B - k + i\alpha(j, B)/2) \Gamma(B - l - i\alpha(j, B)/2) \left( \frac{z}{R} \right)^j
\](3.3)
and
\[
V = \frac{1}{4} \left( 1 + i \cot \left( \frac{\pi \log(|z|)}{\log(R)} \right) \right) \left( 1 + i \cot \left( \frac{\pi \log(|w|)}{\log(R)} \right) \right)
\](3.4)
for every \( z, w \in \Omega_{1,R} \).

### 3.1. Limiting polyanalytic reproducing kernels as \( R \to \infty \)

When \( m = 0 \), the finite sum above contains only one term \( l = k = 0 \) and we recover the (analytic) reproducing kernel \( K_0^{(B)}(z, w) \) in (1.2). This is in agreement with the computations done in ([31], p 263), if we identify the parameter \( \alpha \) there with \( 2B - 2 \) here and if we use Euler’s reflection formula:
\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]
In particular, if \( B \geq 1 \) is an integer then the index change \( j + B \to j \) together with formula (8) from [11], p 4:
\[
\Gamma\left( B + i\frac{\log R}{\pi} \right) \Gamma\left( B - i\frac{\log R}{\pi} \right) = \frac{2 \log(R)\Gamma(B)^2}{R^{2j} - 1} \prod_{q=1}^{B-1} \left( 1 + \frac{(j \log R)^2}{\pi^2 q^2} \right)
\]

\]
yield
\[
K_0^{(R,B)}(z, w) = \frac{(2\pi)^{2B-2} [\Gamma(B)]^2}{\pi \Gamma(2B - 1)} \frac{\sum_{j=1}^{R^2} \, j \, R_{B-1}^{2-q^2}}{1 \prod_{q=1}^{B-1} (\pi w)^j}.
\]

Here, the term \( j = 0 \) is understood as the limit:
\[
\lim_{j \to 0} \frac{j}{R^2 - 1} \prod_{q=1}^{B-1} \left( 1 + \frac{(j \log R)^2}{\pi^2 q^2} \right) (\pi w)^j = \frac{1}{2 \log(R)},
\]

and an empty product equals one. As a result, the following limit holds:
\[
\lim_{R \to \infty} K_0^{(R,B)}(z, w) = \frac{2^{2B-2}}{\pi \Gamma(2B - 1)} \sum_{j \geq 1} \left( \frac{2B-1}{(\pi w)^j} \right),
\]

which is the (analytic) reproducing kernel of the complementary of the closed unit disc (or after the variable change \( \pi w \to 1/(\pi w) \) the one of the punctured open unit disc).

More generally, we can determine the limiting expression of \( K_n^{(R,B)} \) as \( R \to \infty \) for higher Landau level \( 1 \leq m \leq [B - (1/2)] \). However, we found it simpler to rely on the very definition of \( K_n^{(R,B)} \) given in (5.8) below, rather than the expression proved in theorem 3.1. Indeed, the latter is the sum of alternating-sign terms since \( (1 - 2B - 2m) \in -\mathbb{N} \), each one of them blows-up as \( R \to \infty \). Back to the limit as \( R \to \infty \), we perform the index change \( j + B \to j \) as above and appeal again to formula (8) from [11], p 4 to derive:
\[
\left| \Gamma \left( B - m + i \frac{\log R}{\pi} \right) \right|^2 = \frac{2 \log(R)(\Gamma(B - m))^2}{R^{2j - 1}} \prod_{q=1}^{B - m - 1} \left( 1 + \frac{(j \log R)^2}{\pi^2 q^2} \right).
\]

Now,
\[
\lim_{R \to \infty} \frac{\Gamma(B - m))^2}{(\log(R))^{2B - 2m - 2}} \prod_{q=1}^{B - m - 1} \left( 1 + \frac{(j \log R)^2}{\pi^2 q^2} \right) = \left( \frac{j}{\pi} \right)^{2B - 2m - 2},
\]

while the expansion (5.3) below of Routh–Romanovski polynomials yields the following limit:
\[
\lim_{R \to \infty} \frac{1}{(\log(R))^m} \frac{\pi^m}{\Gamma(m \log(R)/\pi)} \frac{\pi^m \log(|z|)}}{\log(R)} \left( \cot \left( \frac{\pi \log(|z|)}{\log(R)} \right) \right)
\]
\[
= \frac{(-2i)^m m!}{\pi^m} \sum_{l=0}^{m} (m - 2B + 1)l (i)^{m-l} \frac{l!(m - l)!}{(2 \log(|z|))^l}
\]
\[
= \frac{(2j)^m}{\pi^m} F_0 \left( -m, m - 2B + 1; \frac{i}{2j \log(|z|)} \right)
\]
\[
= \frac{(2j)^m}{\pi^m} F_0 \left( \frac{i}{2j \log(|z|)} \right)
\]

\( m \geq m \).
for any \( j \neq 0 \). Similarly,

\[
\lim_{R \to \infty} \frac{1}{(\log(R))^m} R^{j} \left( \frac{i}{2j \log(|w|)} \right)
\]

Since

\[
\lim_{R \to \infty} \frac{j}{R^{2j} - 1} = 0, \quad j \geq 0,
\]

where we recall that the term \( j = 0 \) is understood as \((1/2 \log(R))\), and

\[
\lim_{R \to \infty} \frac{j}{R^{2j} - 1} = -j, \quad j \leq -1,
\]

then we end up with

**Corollary 3.1.** The polyanalytic reproducing kernel of \( \Omega_{1,\infty} \) associated to the \( m \)th Landau level is given by:

\[
\lim_{R \to \infty} K^{(R, B)}(z, w) = \frac{4^{B-m-1}(2B-2m-1)}{\pi m! \Gamma(2B-m)} \sum_{j \geq 1} \frac{J_m(B)^2 j \log(|z|)}{2j \log(|w|)}.
\]

**Remark 3.1.** \( J_m(B) \) is a generalized Bessel polynomial and originates in the study of the wave equation in spherical coordinates ([18], see also [13]). As for Routh–Romanovski polynomials, the family formed by Bessel polynomials is finitely orthogonal with respect to the inverse Gamma distribution.

### 4. Some properties of \( K^{R,B}(z, w) \)

In this section, we shall assume that \( B \geq 1 \) is integer and investigate two properties of \( K^{(R, B)}_m \). Thinking of \( B \) as the flux of the (closed two-form) magnetic field, this integrality assumption is indeed a quantization condition.

#### 4.1. Poly-analytic Bergman kernel and fourth Jacobi theta function

If \( B = 1 \) then (3.5) reduces to:

\[
K^{(R, 1)}_0(z, w) = \frac{1}{\pi \bar{z} \bar{w}} \sum_{j \in \mathbb{Z}} \frac{j}{1 - 1/R^{2j}} \left( \frac{\bar{z} \bar{w}}{R^2} \right)^j
\]

Since \( z/R, y/R \) belong to the annulus \( \Omega_{1/R, 1} \), then we retrieve the known formula of the analytic reproducing kernel in \( \Omega_{1/R, 1} \) endowed with its Lebesgue measure (see [6], p 9, (29)).
is known in this case that $K_0^{(R,1)}(z, w)$ may be expressed through the Weierstrass elliptic function $\wp$ ([6], p 10). Below, we shall extend this connection with elliptic functions and prove under the quantization condition $B \in \mathbb{N} \setminus \{0\}$ that $K_m^{(R,B)}(z, w)$ is a linear combination of higher logarithmic derivatives of Jacobi’s fourth theta function.

To this end, let

$$\theta_4(z, \tau) := 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{i k^2 \tau} \cos(2kz), \quad \Im(\tau) > 0.$$ 

be the fourth theta function de Jacobi ([11], p 355). Then, we shall prove in appendix C the following result:

**Proposition 4.1.** For any $m = 0, 1, \ldots, \lfloor B - 1/2 \rfloor$ the polyanalytic reproducing kernel $K_m^{(R,B)}(z, w)$ may be written as a finite sum of higher logarithmic derivatives of the fourth Jacobi’s theta function $\theta_4(z, \tau)$ associated with the rectangular lattice parameter $\tau = i \log(R)/\pi$ and evaluated at the point $i(2j + 1)(2, 1):$

$$
\sigma_{0,0}^{(R,2)}(z, w) = 1 + \frac{\log(R)}{i} \partial_0(\log(\theta_4)) \left( \frac{i}{2} \log \left( \frac{z}{R} \right) \right) + \frac{(\log(R))^3}{i \pi^2} \partial_0^3(\log(\theta_4)) \left( \frac{i}{2} \log \left( \frac{z}{R} \right) \right),
$$

$$
\sigma_{0,1}^{(R,2)}(z, w) = 1 + \frac{(\log(R))^3}{\pi} \partial_0^3(\log(\theta_4)) \left( \frac{i}{2} \log \left( \frac{z}{R} \right) \right) + \frac{(\log(R))^4}{\pi^5} \partial_0^4(\log(\theta_4)) \left( \frac{i}{2} \log \left( \frac{z}{R} \right) \right).
$$

Here is an illustration of the statement of this proposition for $(B, m) = (2, 1)$:

$$
\sigma_{0,0}^{(R,2)}(z, w) = \sigma_{0,1}^{(R,2)}(w, z).
$$

4.2. Invariance under the automorphism group

The automorphism group of the annulus $\Omega_{1,R}$ the direct product of the rotation group and of the cyclic group generated by the inversion $I_R$. It is much more smaller than the Mobius group and does not act transitively on $\Omega_{1,R}$. While the kernel $K_m^{(R,B)}(z, w)$ is obviously seen to be rotation-invariant (see (5.8) in appendix B below), its transformation rule under inversion is not clear unless $B \geq 1$ is an integer. Indeed, the index change $j \mapsto j - B$ in (5.8) shows in this case that

$$
K_m^{(R,B)}(z, w) = \left( \frac{z}{R} \right)^{2B} K_m^{(R,B)}(z, w),
$$

or equivalently

$$
K_m^{(R,B)}(z, w) = \left[ (I_R)'(z)(I_R)'(w) \right]^B K_m^{(R,B)}(I_R(z), I_R(w)).
$$
Written in this form, this transformation rule reminds the one satisfied by the reproducing kernel of the hyperbolic disc under the action of the Möbius group (see e.g. [14]).

For general values of $B > 1/2$, it is readily seen from the expression (5.9) below that the inversion $I_R$ has the effect to transform $\alpha(j, B)$ to $\alpha(j, -B)$ which is meaningless since $B$ is positive.

5. Concluding remarks

In this paper, we derived the reproducing kernel corresponding to eigenspaces of the magnetic Laplacian on the annulus. We also expressed them by means of the fourth Jacobi theta function when the magnetic flux is an integer and investigated their transformation rule under the automorphism group of the annulus. At the probabilistic side, reproducing kernels of Hilbert spaces provide very interesting examples of determinantal point processes (DPP). For instance, for $l = 1, 2, \ldots$, the kernel $(1 - \zeta \zeta')^{l+1}$, $\zeta, \zeta' \in \mathbb{D}$, governs the determinantal correlation functions of the zeros of entire series whose coefficients are independent $l \times l$ Ginibre matrices [19, 33]. This example was generalized in [8] using the magnetic Laplacian in the hyperbolic disc, however no connection to random matrices was found yet in general. The flat counterpart of this DPP is related to the Fock space and was introduced and studied in [38], extending the celebrated Ginibre point process. Remarkably, the reproducing kernel of the hyperbolic DPP studied in [8] is polyanalytic with respect to both the Wirtinger operator $\partial_\zeta$ ([14]) and its weighted counterpart (that is the invariant Cauchy–Riemann operator, [32]). Though this property is obvious in the flat setting since both operators coincide, it remains intriguing for the hyperbolic disc geometry since powers of both operators are clearly different (see [10] for a comparison of these operators). By direct calculations, one checks that the poly-analyticity property with respect to $\partial_\zeta$ fails for the basis elements of $E_m(\Omega_{1,R})$, $m \neq 0$ while it still holds true for the invariant Cauchy–Riemann operator. Finally, we would like to point out the recent preprint [16] where the authors relate zeroes of Laurent series with Gaussian coefficients to a hyper-determinantal point process governed by the Szegö kernel of the annulus (see also [15] for other connections of DPP to elliptic functions).

Appendix A

Proof. of (3.2) Recall the squared $L^2$-norm with respect to the weight $(\omega_R(z))^{2B-2}$:

$$\|\phi_j\|^2 = \int_{\Omega_{1,R}} \bar{\phi}_j(z) \phi_j(z)(\omega_R(z))^{2B-2} \, d\mu(z)$$

where $\phi_j$ is given by (3.1). Using elementary variable changes, this integral is expressed as the $L^2$-norm of the $m$th Routh–Romanovskii polynomial with respect to the orthogonality weight (2.3) as

$$\|\phi_j\|^2 = 2 \frac{(\log R)^{2B-1}}{\pi^{2B-2}} \int_{-\infty}^{+\infty} R_m^{(\alpha,\beta),1-B}(\xi) R^{\beta}(\xi) \, d\xi.$$  

(5.1)

Now, the Routh–Romanovskii polynomials admits the Rodrigues representation ([29], p 259):

$$R_m^{(\alpha,\beta)}(\xi) = \frac{1}{\omega^{(\alpha,\beta)}(\xi)} \frac{d^m}{d\xi^m} \left( \omega^{(\alpha,\beta)}(\xi) \left(1 + \xi^2 \right)^{\frac{\beta}{2}} \right),$$

with

$$\omega^{(\alpha,\beta)}(\xi) := e^{-\alpha \cot^{-1}\xi} \left(1 + \xi^2 \right)^{\beta-1}.$$
Substitute the polynomial \( R_m^{(\alpha(j,B),1-B)}(\xi) \) in (5.1) by its Rodrigues representation and performing \( m \) integration by parts, one gets:

\[
\|\phi_m\|^2 = (-1)^m m! a_m^{(\alpha(j,B),1-B)} \frac{(\log R)^{2B-1}}{\pi^{2B-2}} \int_{-\infty}^{+\infty} \theta_j^R(\xi) \left(1 + \xi^2\right)^m d\xi
\]

(5.2)

where \( a_m^{(\alpha(j,B),1-B)} \) is the leading coefficient of \( R_m^{(\alpha(j,B),1-B)}(\xi) \), which depends only on \( B \) and \( m \). Indeed, the Routh–Romanovski and the Jacobi polynomials with complex-conjugate imaginary parameters are interrelated via ([23]):

\[
R_m^{(\alpha(j,B),1-B)}(x) = (-2i)^m m! P_m^{(-B+i\alpha(j,B)/2,-B-i\alpha(j,B)/2)}(ix)
\]

\[
= (-2i)^m m! \sum_{l=0}^{m} \frac{(m-2B+1)(-B+i\alpha(j,B)/2+l+1)_{m-l}}{l!(m-l)!} (ix-1)^l 2^l
\]

whence we readily get the following expression for the leading coefficient:

\[
a_m^{(\alpha(j,B),1-B)} = (-1)^m \frac{\Gamma(2B-m)}{\Gamma(2B-2m)}.
\]

(5.4)

Now, the variable change \( \xi = \cot \theta \) yields:

\[
\int_{-\infty}^{+\infty} (1 + \xi^2)^m \theta_j^R(\xi) d\xi = \int_0^\pi e^{\alpha(j,B)\theta} (\sin \theta)^{2(B-m)-2} d\theta
\]

\[
= \frac{\pi e^{\alpha(j,B)/2} \Gamma(2(B-m) - 1)}{2^{2m-2} \Gamma(B-m+i\alpha_j/2) \Gamma(B - m - i\alpha_j/2)}
\]

(5.5)

where we applied the Cauchy Beta integral ([34], p 445, [22]):

\[
\int_0^\pi e^{-p \sin \theta} d\theta = \frac{2^{-\nu} \pi e^{-\pi p/2}}{(\nu + 1)B \left(\frac{\nu + ip}{2}, \frac{\nu - ip}{2} + 1\right)} \text{,} \quad \nu > -1,
\]

with \( p = -\alpha(j,B) \) and \( \nu = 2(B-m) - 2 \). Combining (5.1), (5.2) and (5.5), we end up with:

\[
\|\phi_m\|^2 = \frac{(\log R)^{2B-1}}{\pi^{2B-3}} 2^{3-2(B-m)} (-1)^m m! a_m^{(\alpha(j,B),1-B)} \times \frac{e^{\alpha(j,B)/2} \Gamma(2(B-m) - 1)}{\Gamma(B-m+i\alpha(j,B)/2)^2}.
\]

Keeping in mind (2.2) and (5.4), the expression (3.2) of the squared norm follows.

\(\square\)
Appendix B

Proof of theorem 3.1. According to (3.2), an orthonormal basis for $\mathcal{E}_m(\Omega_{1,R})$ given by:

$$\Phi_j(z) := \left( \frac{(\log R)^{2B-1} m! \Gamma(2B-m)}{2(B-m)-1} \right)^{-1/2} \frac{2^{3-2(B-m)} R^{j+B}}{\Gamma(B-m + i \left( \frac{\log R}{\pi} \right) (j+B))^2} \left( \frac{\pi \log |z|}{\log R} \right) \Bigg|_{\zeta_j \in \Omega_{1,R}, \quad j \in \mathbb{Z}}. \tag{5.6}$$

Then, by the general theory of reproducing kernels ([41], p 119):

$$K^{(R,B)}_m(z,w) = \sum_{j \in \mathbb{Z}} \Phi_j(z) \Phi_j(w). \tag{5.7}$$

Inserting (5.6) into (5.7), we get:

$$K^{(R,B)}_m(z,w) = \left\{ \begin{array}{ll}
\frac{2^{3-2(B-m)} \Gamma(2B-m)}{m! \Gamma(2B-m)} & (z,w) \\
\frac{2^{3-2(B-m)} R^{j+B}}{\Gamma(B-m + i \left( \frac{\log R}{\pi} \right) (j+B))^2} \left( \frac{\pi \log |z|}{\log R} \right) \Bigg|_{\zeta_j \in \Omega_{1,R}, \quad j \in \mathbb{Z}}, & (z,w) \\
\end{array} \right. \tag{5.8}$$

Using again the relation ([23]):

$$\mathcal{R}_m^{(a,b)}(x) = (-2i)^m m! \Gamma_m^{(b-1+ia/2,b-1-ia/2)}(ix), \quad m \geq 0,$$

then (5.8) takes the form:

$$K^{(R,B)}_m(z,w) = \left\{ \begin{array}{ll}
\frac{2^{3-2(B-m)} m! \Gamma(2B-m)}{R^{6m} (\log R)^{2B-1} \Gamma(2B-m)} & (z,w) \\
\frac{2^{3-2(B-m)} R^{j+B}}{\Gamma(B-m + i \left( \frac{\log R}{\pi} \right) (j+B))^2} \left( \frac{\pi \log |z|}{\log R} \right) \Bigg|_{\zeta_j \in \Omega_{1,R}, \quad j \in \mathbb{Z}}, & (z,w) \\
\end{array} \right. \tag{5.9}$$

For sake of simplicity, we introduce the following notations:

$$\mu_j := \frac{i}{2} \alpha(j,B) = \frac{i}{\pi} (j+B) \log(R), \quad t = \frac{\zeta_m}{R}.$$
and

\[ X = \cot \left( \frac{\pi \log |z|}{\log R} \right), \quad Y = \cot \left( \frac{\pi \log |w|}{\log R} \right). \]

Consequently, (5.9) reads:

\[
K_m^{(R,B)}(z, w) = \gamma_m^{(R,B)} \sum_{j \in \mathbb{Z}} j! \Gamma(B - m + \mu j)^2 P_m^{(-B - \mu j, -B + \mu j)}(iX) \\
\times P_m^{(-B + \mu j, -B - \mu j)}(-iY),
\]

where

\[
\gamma_m^{(R,B)} : = \frac{(2\pi)^{2B-3} m! (2B - 2m - 1)}{R^B (\log R)^{2B-1} \Gamma(2B - m)}.
\]

Equivalently, the symmetry relation ([2], p 305)

\[
P_m^{(-B + \mu j, -B - \mu j)}(-iY) = (-1)^m P_m^{(-B - \mu j, -B + \mu j)}(iY)
\]

entails

\[
K_m^{(R,B)}(z, w) = (-1)^m \gamma_m^{(R,B)} \sum_{j \in \mathbb{Z}} j! \Gamma(B - m + \mu j)^2 P_m^{(-B + \mu j, -B - \mu j)}(iX) \\
\times P_m^{(-B - \mu j, -B + \mu j)}(iY).
\]

Now, recall Bateman’s formula ([4], p 392):

\[
P_m^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(y) = \sum_{k=0}^{m} (-1)^m \frac{(\alpha + \beta + m + 1)}{m! (m - k)!} \binom{m}{k} \binom{\alpha + \beta}{k} \\
\times \frac{\Gamma(\alpha + m + 1) \Gamma(\beta + m + 1)}{\Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} P_{m-k}^{(\alpha, \beta)} \left( \frac{1 + xy}{x + y} \right)
\]

as well as the expression of the Jacobi polynomial ([42], p 67)

\[
P_k^{(\alpha, \beta)}(x) = \sum_{l=0}^{k} \binom{k + \alpha}{k - l} \binom{k + \beta}{l} \left( \frac{x - 1}{2} \right)^l \left( \frac{x + 1}{2} \right)^{k-l}
\]

to write

\[
P_m^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(y) = (-1)^m \frac{\Gamma(\alpha + m + 1) \Gamma(\beta + m + 1)}{n!} \\
\times \sum_{k=0}^{m} (-1)^m \frac{(\alpha + \beta + m + 1)_k}{4^k (m - k)!} \\
\times \sum_{l=0}^{k} \frac{(1 + xy - x - y)^l (1 + xy + x + y)^{k-l}}{l! (k-l)! \Gamma(\alpha + l + 1) \Gamma(\beta + k - l + 1)}. \quad (5.12)
\]

(5.13)
Next, writing
\[
\Gamma(-\alpha - n) = (-1)^n \frac{\Gamma(-\alpha)\Gamma(1+\alpha)}{\Gamma(\alpha + n + 1)}, \quad n \in \mathbb{Z}_+, \quad \alpha + n \notin \mathbb{Z}_-.
\] (5.14)
and similarly for \(\beta\), Bateman’s formula takes the following form:
\[
\Gamma(-\alpha - m)\Gamma(-\beta - m) P_m^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(y)
\]
\[
= \frac{(-1)^m m!}{\Gamma(m+\alpha+\beta+1)} \sum_{l=0}^{m} \frac{(\alpha + \beta + m + 1)_l}{4^l (m-k)!} \sum_{k=0}^{m-l} \sum_{l=0}^{m-l} \sum_{k=0}^{m-l} (1 + xy - x - y)^l (1 + xy + x + y)^{l-k} \Gamma(-\alpha - l) \Gamma(-\beta - l - k).
\]
Equivalently, changing the summation order and performing the index change \(k \mapsto k + l\), we obtain:
\[
\Gamma(-\alpha - m)\Gamma(-\beta - m) P_m^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(y)
\]
\[
= \frac{(-1)^m m!}{\Gamma(m+\alpha+\beta+1)} \sum_{l=0}^{m} \frac{(\alpha + \beta + m + 1)_l}{4^l (m-k)!} \sum_{k=0}^{m-l} \sum_{l=0}^{m-l} \sum_{k=0}^{m-l} \frac{(1 - x)(1 - y)^l (1 + x + y)^k}{l! k!} \Gamma(-\alpha - l) \Gamma(-\beta - l).
\]
Specializing this form of Bateman’s formula to \(\alpha = -B + \mu_j\beta = -B - \mu_j\), \(x = iX\), and \(y = iY\), we get:
\[
|\Gamma(B - m + \mu_j)|^2 P_m^{(-B+B+\mu_j)}(iX) P_m^{(-B+B+\mu_j)}(iY)
\]
\[
= \frac{(-1)^m m!}{\Gamma(m+\alpha+\beta+1)} \sum_{l=0}^{m} \frac{(1 - 2B + m)_l}{(m-k-l)!} \sum_{k=0}^{m-l} \sum_{l=0}^{m-l} \sum_{k=0}^{m-l} \frac{(1 - x)(1 - y)^l (1 + x + y)^k}{l! k!} \Gamma(B - l - \mu_j) \Gamma(B - k + \mu_j).
\]
Keeping in mind (5.10), theorem 3.1 is proved. \(\square\)

Appendix C

Proof of proposition 4.1. Recall \(\alpha(j, B) = 2(j + B)\log(R) / \pi\) and consider the series (3.3):
\[
\sigma_{k,l}^{(R,B)}(z, w) = \sum_{j \in \mathbb{Z}} \Gamma \left( B - k + i \frac{\alpha(j, B)}{2} \right) \Gamma \left( B - l - i \frac{\alpha(j, B)}{2} \right) \left( \frac{z}{R} \right)^j,
\]
where without loss of generality, we assume \(0 \leq k \leq l \leq m\) since
\[
\sigma_{k,l}^{(R,B)}(z, w) = \sigma_{l,k}^{(R,B)}(w, z).
\]
Now, perform the index change \(j \mapsto j - B\) to write it as
\[
\sigma_{kk}^{(R,B)}(z,u) = \left( \frac{R}{z^{\pi}} \right)^B \sum_{j \in \mathbb{Z}} \Gamma \left( B - k + \frac{i j \log(R)}{\pi} \right) \times \Gamma \left( B - l - \frac{i j \log(R)}{\pi} \right) \left( \frac{z^{\pi}}{R} \right)^j \\
= \left( \frac{R}{z^{\pi}} \right)^B \sum_{j \in \mathbb{Z}} \prod_{s=1}^{l-k} \left( B - (k + s) + \frac{i j \log(R)}{\pi} \right) \times \Gamma \left( B - l + \frac{i j \log(R)}{\pi} \right) \Gamma \left( B - l - \frac{i j \log(R)}{\pi} \right) \left( \frac{z^{\pi}}{R} \right)^j,
\]
where an empty product equals one. Next, we shall appeal to formula (8) from [11], p 4:
\[
\Gamma \left( B - l + \frac{i j \log R}{\pi} \right) \Gamma \left( B - l - \frac{i j \log R}{\pi} \right) = \frac{2 \log(R)[\Gamma(B - l)]^2}{R^{2j} - 1} \prod_{q=1}^{B-l-1} \left( 1 + \left( \frac{j \log R}{\pi q^2} \right) \right)
\]
to get further
\[
\sigma_{kk}^{(R,B)}(z,u) = 2 \log(R)[\Gamma(B - l)]^2 \left( \frac{R}{z^{\pi}} \right)^B \sum_{j \in \mathbb{Z}} \prod_{s=1}^{l-k} \left( B - (k + s) + \frac{i j \log(R)}{\pi} \right) \times \prod_{q=1}^{B-l-1} \left( 1 + \left( \frac{j \log R}{\pi q^2} \right) \right) \left( \frac{z^{\pi}}{R} \right)^j.
\]
Now, note that
\[
\prod_{q=1}^{B-l-1} \left( 1 + \left( \frac{j \log R}{\pi q^2} \right) \right) \frac{j R^l}{R^{2j} - 1}
\]
is invariant under the flip \( j \mapsto -j \), while
\[
\prod_{s=1}^{l-k} \left( B - (k + s) + \frac{i j \log(R)}{\pi} \right)
\]
is a complex polynomial in \( j \). The odd part of the latter leads to series of the form:
\[
\sum_{j \geqslant 1} j^{B-l-1} \prod_{q=1}^{\infty} \left( 1 + \left( \frac{j \log R}{\pi q^2} \right) \right) \frac{j R^l}{R^{2j} - 1} \left[ \left( \frac{z^{\pi}}{R} \right)^j - \left( \frac{R}{z^{\pi}} \right)^j \right], \quad s \geqslant 1 \quad (5.16)
\]
and its even part to
\[
\sum_{j \geqslant 1} j^{B-l-1} \prod_{q=1}^{\infty} \left( 1 + \left( \frac{j \log R}{\pi q^2} \right) \right) \frac{j R^l}{R^{2j} - 1} \left[ \left( \frac{z^{\pi}}{R} \right)^j + \left( \frac{R}{z^{\pi}} \right)^j \right], \quad s \geqslant 0. \quad (5.17)
\]
But
\[ 4i \sum_{j \geq 1} \frac{R^j}{j^2} \sinh \left( j \log \left( \frac{z \sqrt{R}}{R} \right) \right) \]
is the logarithmic derivative of the theta function \( \theta_4(\cdot, i \log(R)/\pi) \) evaluated at \( (i/2) \ln(z \sqrt{R}/R) \) ([11], p 358), and in turn
\[ 4i \sum_{j \geq 1} \frac{j R^j}{j^2} \cosh \left( j \log \left( \frac{z \sqrt{R}}{R} \right) \right) \]
is its second logarithmic derivative. Since
\[ j^2 \prod_{q=1}^{B-j-1} \left( 1 + \frac{(j \log R)^2}{\pi^2 q^2} \right), \quad s \geq 0, \]
are even polynomials in \( j \), then the series (5.16) and (5.17) are higher logarithmic derivatives of \( \theta_4(\cdot, i \log(R)/\pi) \) as well evaluated at \( (i/2) \ln(z \sqrt{R}/R) \). Keeping in mind the expression of the poly-analytic Bergman kernel proved in theorem 3.1, proposition 4.1 is proved. □

ORCID iDs
Zouhair Mouayn https://orcid.org/0000-0003-3510-1971

References

[1] Alvarez-Castillo D E and Kirchbach M 2007 Exact spectrum and wave functions of the hyperbolic scarf potential in terms of finite Romanovski polynomials Rev. Mex. Fís. E. 53 143–54
[2] Andrews G E, Askey R and Roy R 1999 Special Functions (Cambridge: Cambridge University Press)
[3] Badía-Majos A 2018 Josephson critical currents in annular superconductors with Pearl vortices Phys. Rev. B 98 184518
[4] Bateman H 1932 Partial Differential Equations of Mathematical Physics (Cambridge: Cambridge University Press)
[5] Berezin F A 1975 General concept of quantization Commun. Math. Phys. 40 153–74
[6] Bergmann S 1970 The Kernel Function and Conformal Mapping (Mathematical Surveys and Monographs vol 5) (Providence, RI: American Mathematical Society)
[7] Comtet A 1987 On the Landau levels on the hyperbolic plane Ann. Phys. 173 185–209
[8] Demni N and Lazag P 2019 The hyperbolic-type point process J. Math. Soc. Japan 71 1137–52
[9] Gulevich R D, Skryabin V D, Alodjants A P and Ivan Shelykh A 2016 Topological spin Meissner effect in spinor exciton–polariton condensate: constant amplitude solutions, half-vortices, and symmetry breaking Phys. Rev. B 94 115407
[10] Englisch M and Zhang G 2017 Toeplitz operators on higher Cauchy–Riemann spaces Doc. Math. 22 1081–116
[11] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 Higher Transcendental Functions vol 2 (New York: McGraw-Hill) p xvii+396
[12] Farkas H M and Kra I 1992 Riemann Surfaces (Graduate Texts in Mathematics vol 71) 2nd edn (Berlin: Springer) p xvi+363
[13] Grosswald E 1978 Bessel Polynomials (Lecture Notes in Mathematics vol 698) (Berlin: Springer)
[14] Hachadi H and Youssifi E H 2019 The polyanalytic reproducing kernels Complex Anal. Oper. Theor. 13 3457–78
[15] Katori M 2017 Elliptic determinantal processes and elliptic Dyson models SIGMA Symmetry Integrability Geom. Methods Appl. 13 36
[16] Katori M and Shirai T Zeros of the i.i.d. Gaussian Laurent series on an annulus: weighted Szegö kernels and permanental-determinantal point processes (arXiv:2008.04177)

[17] Khare A and Sukhatme U P 1988 Scattering amplitudes for supersymmetric shape-invariant potentials by operator methods J. Phys. A: Math. Gen. 21 501–8

[18] Krall H L and Frink O 1949 A new class of orthogonal polynomials: the Bessel polynomials Trans. Am. Math. Soc. 65 100

[19] Krishnapur M 2009 From random matrices to random analytic functions Ann. Probab. 37 314–46

[20] Laughlin R B 1981 Quanitzed Hall conductivity in two dimensions Phys. Rev. B 23 5632–3

[21] Linetsky V 2006 Pricing equity derivatives subject to bankruptcy Math. Finance 16 255–82

[22] Masjed-Jamei M, Marcellán F and Huertas E J 2014 A finite class of orthogonal functions generated by Routh–Romanovski polynomials Complex Var. Elliptic Equ. 59 162–71

[23] Martínez-Finkelshtein A, Silva Ribeiro L L, Ranga A S and Tyaglov M 2019 Complementary Romanovski–Routh polynomials: from orthogonal polynomials on the unit circle to Coulomb wave functions Proc. Am. Math. Soc. 147 2625–40

[24] Mouayn Z 2005 Coherent states attached to Landau levels on the Poincaré disc J. Phys. A: Math. Gen. 38 9309–16

[25] Mouayn Z 2003 Characterization of hyperbolic Landau states by coherent state transforms J. Phys. A: Math. Gen. 36 9071–6

[26] Mouayn Z 2008 An integral transform connecting spaces of hyperbolic Landau states with a class of weighted Bergman spaces Complex Var. Elliptic Equ. 53 1083–92

[27] Morse P M 1929 Diatomic molecules according to the wave mechanics. II. Vibrational levels Phys. Rev. 34 57

[28] Peetre J 1988 Hankel forms on multiply-connected plane domains. Part one: the case of connectivity two Complex Var. Elliptic Equ. 10 123–39

[29] Peetre J 1993 Correspondence principle for the quantized annulus, Romanovski polynomials, and Morse potential J. Funct. Anal. 117 377–400

[30] Peetre J and Zhang G 1991 Projective structures on an annulus and Hankel forms Glasgow Math. J. 33 247–66

[31] Peetre J and Zhang G 1993 Harmonic analysis on the quantized Riemann sphere Int. J. Math. Math. Sci. 16 225–43

[32] Peres Y and Virág B 2005 Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process Acta Math. 194 1–35

[33] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 Integrals and Series: More Special Functions vol 1 (London: Gordon and Breach)

[34] Raposi A, Weber H J, Alvarez-Castillo D and Kirchbach M 2007 Romanovski polynomials in selected physics problems Central Eur. J. Phys. 5 253–84

[35] Romanovski V 1929 Sur quelques classes nouvelles de polynomes orthogonaux C. R. Acad. Sci. Paris 188 1023–25

[36] Routh E J 1884 On some properties of certain solutions of a differential equation of second order Proc. London Math. Soc. 16 245–61

[37] Shirai T 2015 Ginibre-type point processes and their asymptotic behavior J. Math. Soc. Japan 67 763–87

[38] Shigekawa I 1987 Eigenvalue problems of Schrödinger operator with magnetic field on compact Riemannian manifold J. Funct. Anal. 75 92–127

[39] Shubin M 2001 Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds J. Funct. Anal. 186 92–116

[40] Saitoh S and Sawano Y 2016 Theory of Reproducing Kernels and Applications (Berlin: Springer)

[41] Szegö G 1939 Orthogonal Polynomials vol 23 (Providence, RI: American Mathematical Society)

[42] Kita T 2015 Statistical Mechanics of Superconductivity (Graduate Texts in Physics) (Berlin: Springer)