On Non-Abelian Holonomies

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Abstract. We provide a method and the results for the calculation of the holonomy of a Yang-Mills connection in an arbitrary triangular path, in an expansion (developed here to fifth order) in powers of the corresponding segments. The results might have applications in generalizing to Yang-Mills fields previous calculations of the corrections to particle dynamics induced by loop quantum gravity, as well as in the field of random lattices.

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1. Introduction

Constraints in Lorentz covariance violations have been experimentally studied since a long time ago [1] by obtaining observational bounds upon the violating parameters. Recent experiments have shown an impressive increase in their sensitivities thus producing even more stringent bounds [2].

In order to correlate such experimental results, Kostelecky and collaborators have proposed a phenomenological extension of the standard model, which incorporates the most general Lorentz-violating interactions compatible with power counting renormalizability together with the particle structure of the standard model [3]. An impressive number of applications to very different processes have been already considered, as can be seen in Ref. [4], for example.

Different models accounting for minute Lorentz violations have recently arisen in the context of quantum gravity induced corrections to the propagation and interactions of particles [5, 6, 7, 8, 9]. This amounts to realize the generic violating parameters appearing in the standard model extension in terms of specific quantities involving the Planck length \( \ell_P \) together with additional physically relevant objects. Moreover, the high precision obtained in the determination of the experimental bounds has brought quantum gravity induced effects to the level of observable phenomena [10, 11, 12, 13].

On the other hand, in the early 80’s, the growing importance of computer simulations of gauge theories required a short distance cutoff of geometrical origin such as a lattice. However, regular lattices break essential symmetries of continuum theories such as translational and rotational invariance. Motivated by the need of maintaining these symmetries, the field theory on a random lattice was suggested [14]. Further on, the connection of random lattices with quantum gravity and strings was studied and low dimensional systems on random lattices were solved using matrix model techniques [15].

In this work we concentrate on some aspects arising in the process of generalizing the loop quantum gravity inspired model described in [6, 7] to Yang-Mills fields, in order to obtain the non-Abelian generalization of the corrections previously found for the dynamics of photons. Namely, corrections to standard matter dynamics are obtained by means of calculating non-Abelian holonomies, either of gravitational or Yang-Mills type, around triangular paths. To this end we have to revise and extend the procedure of Ref. [16] that was applied to the case of rectangular cells. It turns out that the method we present in this paper contains as a particular case the result of Ref. [16], though it is applicable to arbitrary cells made up of triangles. The basic building block in our analysis is the holonomy along a straight line segment, which characteristic property of been path-ordered is consistently maintained to all orders in our expansion.

The problem we deal with here is closely related to the non Abelian Stokes theorem which has been repeatedly discussed in the literature [16, 17].

This paper is organized as follows: in Section 2 we state the problem to be dealt with and introduce some notation. Section 3 summarizes the results for the Abelian case which we intend to generalize here. The non-Abelian case is subsequently discussed in Section
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4 which contains our main results. Using the procedure of Ref. [16] the corresponding
calculation is performed in Section 5, which allows us to show some discrepancies that
arise between the two methods. Finally we close with a summary and discussion in Section
6.

The computations involved in deriving $h_{\alphaIJ}^{(5)}$ were done using FORM [18].

2. Statement of the problem

The proposed method to obtain the quantum gravity induced corrections to the
Yang-Mills Lagrangian requires the calculation of the object

$$T_\rho = tr \left( G_\rho h_{\alphaIJ} \right),$$

where $G_\rho$ are the generators of the corresponding Lie algebra and $h_{\alphaIJ}$ is the holonomy
of the Yang-Mills connection $A_a = A^\rho_a G_\rho$ in the triangle $\alphaIJ$, with vertex $v$, defined by
the vectors $\vec{s}_I$ and $\vec{s}_J$, arising from the vertex $v$ in the way described below.

Our main task will be to construct an expansion of $T_\rho$ in powers of the segments
$s^a_I, s^b_J$.

To be more precise, we have

$$h_{\alphaIJ} = P \exp \left( \oint_{\alphaIJ} A_a(\vec{x}(s)) \frac{dx^a}{ds} ds \right),$$

where $P$ is a path-ordered product to be specified later. As shown in Fig.1, the closed
path $\alphaIJ$, parameterized by $\vec{x}(s)$, is defined in the following way: we start from the vertex
$v$ following a straight line with the direction and length of $\vec{s}_I$, then follow another straight
line in the direction and length of $\vec{s}_J - \vec{s}_I$, and finally return to $v$ following $-\vec{s}_J$. From
the definition of the holonomy, we have the transformation property

$$h_{\alphaIJ} \rightarrow U(v) h_{\alphaIJ} U(v)^{-1},$$

under a gauge transformation of the connection, where $U(v)$ is a group element evaluated
at the vertex $v$. In other words, $h_{\alphaIJ}$ transforms covariantly under the group.

3. The Abelian case

The corresponding calculation was performed in Ref. [6] and here we summarize the results
in order to have the correct expressions to which the non-Abelian result must reduce when
taking the commuting limit. In this case Eq. (1) reduces to

$$T = \exp(\Phi_{IJ}) - 1,$$

where $\Phi_{IJ}$ is the magnetic flux through the area of the triangle, given by

$$\Phi^B_{IJ} = \oint_{\alphaIJ} dt \, \vec{s}^a(t) A_a(t)$$

$$= \int_{\vec{v} + \vec{s}_I} A_a dx^a + \int_{\vec{v} + \vec{s}_J} A_a dx^a + \int_{\vec{v} + \vec{s}_J} A_a dx^a,$$
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where the connection $A_a(\vec{x}(s))$ is now a commuting object.

The basic building block in (5) is

$$\int_{\vec{v}} A_a(\vec{x}) \, dx^a = \int_0^1 A_a(\vec{v}_1 + t (\vec{v}_2 - \vec{v}_1)) (\vec{v}_2 - \vec{v}_1)^a \, dt$$

$$= \int_0^1 A_a(\vec{v}_1 + t \vec{\Delta}) \, \Delta^a \, dt$$

$$= \left(1 + \frac{1}{2!} \Delta^b \partial_b + \frac{1}{3!} (\Delta^b \partial_b)^2 + \ldots\right) \Delta^a A_a(v),$$

(6)

with $\Delta^a = (\vec{v}_2 - \vec{v}_1)^a$. The infinite series in parenthesis is

$$F(x) = 1 + \frac{1}{2!} x + \frac{1}{3!} x^2 + \frac{1}{4!} x^3 + \ldots = \frac{e^x - 1}{x},$$

(7)

yielding

$$\int_{\vec{v}} A_a(\vec{x}) \, dx^a = F(\Delta^a \partial_a) (\Delta^a A_a(\vec{v}_1)).$$

(8)

In the following we employ the notation $\Delta^a V^a = \vec{\Delta} \cdot \vec{V}$. Using the above result in the three integrals appearing in (5) and after some algebra, we obtain

$$\Phi^B(F_{IJ}) = F_1(\vec{s}_I \cdot \nabla, \vec{s}_J \cdot \nabla) \, s^a_s^b \epsilon_{abc} B_c(v),$$

(9)

where the gradient acts upon the coordinates of $\vec{v}$. The function $F_1$ is

$$F_1(x, y) = \frac{y(e^x - 1) - x(e^y - 1)}{x y (y - x)} = -\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \frac{x^n - y^n}{x - y}.$$

(10)

Let us emphasize that $F_1(x, y)$ is just a power series in the variables $x$ and $y$. Expanding in powers of the segments $s^a_s$ we obtain

$$\Phi^B(F_{IJ}) = \left(1 + \frac{1}{3}(s^c_I + s^c_J) \partial_c + \frac{1}{12} (s^e_I s^d_I + s^e_J s^d_J + s^e_J s^d_J) \partial_e \partial_d + \ldots\right) \times$$

$$\times \frac{1}{2} s^a_s^b \epsilon_{abc} B_c(v).$$

(11)

Notice that the combination

$$\frac{1}{2} s^a_s^b \epsilon_{abc} = \mathcal{A} n_c,$$

(12)

is just the oriented area of the triangle with vertex $v$ and sides $s^a_s$, $s^a_J$, joining at this vertex, having value $\mathcal{A}$ and unit normal vector $n_c$.

To conclude we have to calculate

$$\left(e^{\Phi^B(F_{IJ})} - 1\right) = \sum_{n=2}^{\infty} \frac{1}{n!} (\Phi^B(F_{IJ}))^n = \sum_{n=2}^{\infty} M_{nIJ},$$

(13)

where the subindex $n$ labels the corresponding power in the vectors $s^a$. The results are

$$M_{2IJ} : = s^a_s^b \frac{1}{2!} F_{ab},$$

(14)

$$M_{3IJ} : = s^a_s^b \frac{1}{3!} (s_I + s_J) F_{ab},$$

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In a similar way to the Abelian case we separate the calculation of the holonomy up to fifth order. We are using the notation where \( s_I^a \) is produced by the replacement

\[
A_a \to A_a = A_a^\rho G_{\rho}, \quad \partial_a \to D_a = \partial_a - [A_a, ], \quad F_{ab} \to F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b].
\]

We expect that the non-Abelian generalization of the quantities (14), (15), (16), (17) is produced by the replacement

\[
A_a \to A_a = A_a^\rho G_{\rho}, \quad \partial_a \to D_a = \partial_a - [A_a, ],
\]

\[F_{ab} \to F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b].\]

Nevertheless, at this level there are potential ordering ambiguities which will be resolved in the next sections.

4. The non-Abelian case

In a similar way to the Abelian case we separate the calculation of the holonomy \( h_{\alpha_{IJ}} \) in three basic pieces through the straight lines along the sides of the triangle \( \alpha_{IJ} \). We have

\[h_{\alpha_{IJ}} = P(e^{L_1})P(e^{L_2})P(e^{L_3}) \equiv U_3 U_2 U_1,\]

where

\[L_1 = \int_0^1 dt A_a(\vec{v} + t \vec{s_I}) s_I^a, \quad L_2 = \int_0^1 dt A_a(\vec{v} + \vec{s_I} + t (\vec{s_J} - \vec{s_I})) (s_J^a - s_I^a), \quad L_3 = \int_0^1 dt A_a(\vec{v} + \vec{s_J} - t \vec{s_J}) (-s_J^a).\]

Here we have parameterized each segment with \( 0 \leq t \leq 1 \).

4.1. The basic building block

Let us consider in detail the contribution

\[U_1 = P(e^{L_1}), \quad L_1 = \int_0^1 dt A_a(\vec{v} + t \vec{s_I}) s_I^a,\]

with \( \vec{s_I} = \{s_I^a\} \).

Using the definition

\[U_1 = 1 + \int_0^1 dt A_a(\vec{v} + t \vec{s_I}) s_I^a + \int_0^1 dt \int_0^t dt' A_a(\vec{v} + t \vec{s_I}) A_b(\vec{v} + t' \vec{s_I}) s_J^b s_I^a + \ldots,\]

for the path ordering, we arrive at the following expression

\[U_1 = 1 + I_1(x, A_a(v)) s_I^a + I_2(x, \bar{x}, A_a(v)) A_b(v) s_J^b s_I^a + \ldots + I_3(x, \bar{x}, \bar{x}) A_a(v) A_b(v) A_c(v) s_I^a s_J^b s_L^c + \ldots\]
Here we are adopting the conventions
\[ x = s_i^c \partial_c, \quad \bar{x} = s_i^c \bar{\partial}_c, \quad \bar{\bar{x}} = s_i^c \bar{\bar{\partial}}_c, \]
\[ I_1 = F(x), \quad I_2 = \frac{F(x + \bar{x}) - F(x)}{\bar{x}}, \]
\[ I_3 = \frac{1}{\bar{x} + \bar{\bar{x}}} \left[ \frac{1}{\bar{x}} (F(x + \bar{x} + \bar{\bar{x}}) - F(x)) - \frac{1}{\bar{\bar{x}}} (F(x + \bar{x} + \bar{\bar{x}}) - F(x)) \right]. \]

with \( F(x) \) given by Eq. (7). The notation in Eq. (26) is that each operator \( x, \bar{x}, \bar{\bar{x}} \) acts only in the corresponding field \( A, \bar{A}, \bar{\bar{A}} \) respectively. We write
\[ U_1 = \sum_N U_1^{(N)}, \]
where the superindex \( N \) indicates the powers of \( s_i^a \) contained in each term. A detailed calculation produces
\[ U_1^{(1)} = s_i^a A_a, \quad U_1^{(2)} = \frac{1}{2} (xs_i^a A_a + s_i^a s_i^b A_a A_b), \]
\[ U_1^{(3)} = \frac{1}{3!} (x^2 s_i^a A_a + (\bar{x} + 2x) s_i^a s_i^b A_b A \bar{A} + s_i^a s_i^b s_i^c A_a A_b A_c), \]
\[ U_1^{(4)} = \frac{1}{4!} \left[ x^3 s_i^a A_a + (3x^2 + 3x \bar{x} + \bar{x}^2) s_i^a s_i^b A_a A \bar{A} + (3x + 2\bar{x} + \bar{\bar{x}}) s_i^a s_i^b s_i^c A_a A_b A_c \right. \]
\[ \left. + s_i^a s_i^b s_i^c s_i^d A_a A_b A_c A_d \right]. \]

Specializing to the case \( \bar{s}_i = (a, 0, 0) \) and to third order in \( a \) we obtain
\[ U_1 = 1 + a A_1 + \frac{1}{2} a^2 (\partial_1 A_1 + A_1^2) + \frac{1}{3!} a^3 (\partial_1^2 A_1 + 3 \partial_1 A_1 + 2 (\partial_1 A_1) A_1 + 3 A_1^3). \]

Next we compare our result (34) with the calculation according to the method of Ref. [16]. Using the equation (3.15a) of such reference for \( L_1 \) we obtain
\[ e^{L_1} = 1 + a A_1 + \frac{1}{2} a^2 (\partial_1 A_1 + A_1^2) + \frac{1}{3!} a^3 \left( \partial_1^2 A_1 + \frac{3}{2} \left( A_1 \partial_1 A_1 + (\partial_1 A_1) A_1 \right) + A_1^3 \right). \]

which does not agree with the expression (34). This basic discrepancy propagates when combining more segments and ultimately it is the source of the differences between our results and those obtained via the application of the methods in Ref. [16].

It is interesting to remark that in the Abelian limit both (34) and (35) reduce to
\[ U_1 = 1 + a A_1 + \frac{1}{2} a^2 (\partial_1 A_1 + A_1^2) + \frac{1}{3!} a^3 (\partial_1^2 A_1 + 3 A_1 \partial_1 A_1 + A_1^3), \]
which is obtained from a direct calculation using the expression (8).

4.2. The holonomy \( h_{\alpha_1} \)

Now we put the remaining pieces together in order to calculate \( h_{\alpha_1} = U_3 U_2 U_1 \). Using the notation
\[ y = s_j^a \partial_a \]
and starting from the basic structure (26) we obtain, mutatis mutandis,
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\[ U_2^{(1)} = (s_j^a - s_i^a)A_a, \quad (38) \]
\[ U_2^{(2)} = \frac{1}{2}[(x + y)(s_j^a - s_i^a)A_a + (s_j^a - s_i^a)(s_j^b - s_i^b)A_aA_b], \quad (39) \]
\[ U_2^{(3)} = \frac{1}{3!}[(x^2 + y^2 + xy)(s_j^a - s_i^a)A_a + (x + y + 2\bar{x})(s_j^a - s_i^a)(s_j^b - s_i^b)A_a\tilde{A}_b \]
\[ + (s_j^a - s_i^a)(s_j^c - s_i^c)(s_j^d - s_i^d)A_aA_bA_c], \quad (40) \]
\[ U_2^{(4)} = \frac{1}{4!}[(x^3 + y^3 + 3x^2y + 3x^2)(s_j^a - s_i^a)A_a + (x\bar{y} + 3x\bar{x} + x^2 + 2xy + 2x\bar{y} + 3\bar{x}^2 \]
\[ + 3y^2 + \bar{y}^2 + 3y\bar{y} + 5\bar{x}\bar{y})(s_j^a - s_i^a)(s_j^b - s_i^b)A_a\tilde{A}_b + (x + 2\bar{x} + 3\bar{y} + 2\bar{y} + \bar{x}) \]
\[ \times (s_j^a - s_i^a)(s_j^c - s_i^c)(s_j^d - s_i^d)A_aA_b\tilde{A}_c + (s_j^a - s_i^a)(s_j^b - s_i^b)(s_j^c - s_i^c)(s_j^d - s_i^d) \]
\[ (s_j^e - s_i^e)(s_j^f - s_i^f)A_aA_bA_cA_d], \quad (41) \]

for \( U_2 \), together with
\[ U_3^{(1)} = -s_j^aA_a, \quad (42) \]
\[ U_3^{(2)} = \frac{1}{2}(-ys_j^aA_a + s_j^a s_j^bA_aA_b), \quad (43) \]
\[ U_3^{(3)} = \frac{1}{3!}(-y^2 s_j^aA_a + (2y + 3\bar{x})s_j^a s_j^bA_a\tilde{A}_b - s_j^a s_j^b s_j^cA_aA_bA_c), \quad (44) \]
\[ U_3^{(4)} = \frac{1}{4!}[-y^3 s_j^aA_a + (3y^2 + 3\bar{y} + \bar{y}) s_j^a s_j^bA_a\tilde{A}_b - (3\bar{y} + 2\bar{y} + y)s_j^a s_j^b s_j^cA_a\tilde{A}_b\tilde{A}_c \]
\[ + s_j^a s_j^b s_j^d s_j^eA_aA_bA_cA_d], \quad (45) \]

for \( U_3 \). Let us emphasize that in all the expressions above for \( U_1, U_2 \) and \( U_3 \), the connection is evaluated at the vertex \( v \). The bars only serve to indicate the position in which the corresponding derivative acts.

Next we write the contributions to the holonomy in powers of the segments. According to Eqs. (20) and (30) we obtain
\[ h_{\alpha_{ij}}^{(2)} = \frac{1}{2} s_i^a s_j^b A_{ij} F_{ab}, \quad (46) \]
\[ h_{\alpha_{ij}}^{(3)} = \frac{1}{3!} (s_i^a + s_j^a) s_i^a s_j^b D_c A_{ab}, \quad (47) \]
\[ h_{\alpha_{ij}}^{(4)} = \frac{1}{4!} (s_i^a s_i^d + s_i^c s_j^d + s_j^c s_j^d) s_i^a s_j^b D_c D_d F_{ab} + \frac{1}{8} s_i^a s_j^b s_i^c s_j^d D_c D_d F_{ab}, \quad (48) \]
\[ h_{\alpha_{ij}}^{(5)} = \frac{1}{5!} (s_i^a s_i^d s_i^e + s_i^c s_j^d s_j^e + 2(s_i^a s_i^d s_i^e + s_i^c s_j^d s_j^e - s_i^a s_j^d s_i^e - s_j^c s_j^d s_i^e) s_i^a s_j^b D_c D_d D_e F_{ab} \]
\[ + \frac{1}{4!} s_i^a s_j^b s_i^c s_j^d (F_{ab} (s_i^c + s_j^c) D_e F_{de} + (s_i^c + s_j^c) (D_c F_{de}) F_{ab}). \quad (49) \]

Eq. (48) resolves the ordering ambiguity which apparently arose in covariantizing the first term in the RHS of Eq. (16). Nevertheless, as we subsequently show there is really no such ambiguity at this order. Let us consider the combination
\[ s_i^a s_j^b s_i^c s_j^d (D_c D_d - D_d D_c) F_{ab} = s_i^a s_j^b s_i^c s_j^d [D_c, D_d] F_{ab} \]
\[ = -s_i^a s_j^b s_i^c s_j^d [F_{cd}, F_{ab}] = [F, F] = 0 \quad (50) \]
where we have used the notation $F = s_i^a s_j^b F_{ab}$ together with the property
\[ [D_c, D_d] G = - [F_{cd}, G] \]  
valid for any object $G$ in the adjoint representation (Jacobi identity).

The results (46), (47) and (48), which we have obtained by direct calculation, constitute in fact the unique gauge covariant generalization of the corresponding Abelian expressions (14), (15) and (16). This provides a strong support to our method of calculation.

5. The holonomy according to the method of Ref. [16]

In this section we calculate the holonomy $h_{a_1j}$ using the method of Ref. [16] adapted for our case of three segments: $L_1, L_2, L_3$. From Eqs. (3.1), (3.3), (3.4), (3.12) and (3.14), with $\lambda = 1$, of that reference it follows that

\[ h_{a_1j} = e^{L_3} e^{L_2} e^{L_1} = e^{\Sigma + \sum_{n=2} F_n} = e^H. \]  
(52)

The basic building blocks are

\[ L_1 = s_i^a F(x) A_a(v), \quad L_2 = (s_i^a - s_i^b) F(y - x) e^x A_a(v), \quad L_3 = -s_j^a F(y) A_a(v). \]  
(53)

where the vertex $v$ generalizes de point $(x_1, x_2)$ in the notation of [16]. The calculational method indicated in Eqs. (52) and (53) is clearly not equivalent to the correct prescription (20): $h_{a_1j} = U_3 U_2 U_1$, with the $U$’s given by Eq. (26) together with the corresponding extensions that take into account the change of the starting point in the corresponding path.

Let us write here those expressions arising from the method in Ref. [16] that we will use in our calculation

\[ \Sigma = L_1 + L_2 + L_3, \]  
(54)
\[ 2!F_2 = [(L_3 + L_2), L_1] + [L_3, L_2], \]  
(55)
\[ 3!F_3 = - [\Sigma, F_2] + [(L_3 + L_2)^2 L_1] + [L_3 L_2], \]  
(56)
\[ 4!F_4 = - 3! [\Sigma, F_3] - [\Sigma^2 F_2] + [(L_3 + L_2)^3 L_1] + [L_3^3 L_2]. \]  
(57)

with the notation $[A^2B] = [A, [A, B]]$ and so on.

Expanding each segment in powers of the vectors $s^a$, which number is denoted by the superindex, leads to (up to third order)

\[ L_1^{(1)} = s_i^c A_c, \quad L_1^{(2)} = \frac{1}{2!} s_i^a s_j^b \partial_b A_a, \]  
(58)
\[ L_1^{(3)} = \frac{1}{3!} s_i^a s_j^b s_k^c \partial_b \partial_c A_a, \]  
(59)
\[ L_2^{(1)} = (s_j^a - s_i^a) A_a, \quad L_2^{(2)} = \frac{1}{2!} (s_k^b + s_j^b) (s_j^a - s_i^a) \partial_b A_a, \]  
(60)
\[ L_2^{(3)} = \frac{1}{3!} [(s_j^b s_k^c + s_j^c s_k^b) (s_i^a - s_j^a) \partial_b \partial_c A_a, \]  
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\[ L_3^{(1)} = -s^c_j A_c, \quad L_3^{(2)} = -\frac{1}{2!} s^a_j s^b_j \partial_b A_a. \]  

\[ L_3^{(3)} = -\frac{1}{3!} s^c_j s^b_j s^a_j \partial_b \partial_c A_a. \]  

Next we write the contributions to \( H \) in powers of the segments. Using Eq. (52), the first order contribution \( H^{(1)} \) vanishes and the remaining contributions are

\[ H^{(2)} = \frac{1}{2} s^a_i s^b_j F_{ab}, \]  

\[ H^{(3)} = \frac{1}{3!} (s^c_i + s^c_j) s^a_i s^b_j D_c F_{ab} + \Sigma^{(3)}, \]  

with

\[ \Sigma^{(3)} = \frac{1}{12} (-s^b_j s^a_i s^c_i + s^b_i s^a_j s^c_j + s^b_j s^a_i s^c_j + s^b_j s^a_j s^c_i - s^b_j s^a_i s^c_j - s^b_i s^a_j s^c_i) [\partial_b A_a, A_c]. \]  

The reader can verify that the term \( \Sigma^{(3)} \) is not covariant under the gauge group. This would not be the case if one had used the path-ordering prescription (that is to say Eq. (20) instead of Eq. (52)) in the construction of the holonomies. Indeed, gauge covariance should hold to each order in the expansion in powers of the segments.

6. Summary and discussion

We have calculated the holonomy \( h_{\alpha_{IJ}} \) of the Yang-Mills connection \( A_a \) in the triangle \( \alpha_{IJ} \) with vertex \( v \) and sides \( s^a_i, s^b_j \) joining at that vertex, as shown in Fig. (1). Our results, to fifth order in the segments, are given in Eqs. (46) (47) (48) and (49) of section 4. The direct calculation shows that, to fourth order, the results are directly given by the replacement \( \partial_a \rightarrow D_a, F_{ab} \rightarrow F_{ab} \) in the corresponding formulas for the Abelian case. This is so because, as explained at the end of section 4, the potential order ambiguity in the fourth order term is absent. From the fifth order on such ordering ambiguities arise, so that is not possible to guess the correct answer from the Abelian case. Clearly then, one has to perform the full calculation in order to obtain the correct result.

In section 5 the same calculation was performed using the method of Ref. [16]. The results for the triangle coincide to second order and start to differ from the third order on. This difference can be traced back to that arising in the calculation of the holonomy for a straight-line segment according to Eqs. (34) and (35), which the reader can easily verify. Contrary to what is expected, the calculation according to the method of Ref. [16] produces non-covariant contributions starting at third order.

The specific results presented in Ref. [16] correspond to the calculation of the holonomy for a rectangle of sides \( a \) and \( b \), respectively. Using the method described in section 4 we have verified that, up to fourth order, the result given in Eq. (3.19) of that reference is correct.

Nevertheless, since the calculational method of Ref. [16] does not properly take into account the path ordering, it is not possible to guarantee the multiplicative composition law of holonomies. In fact, one might think of obtaining the holonomy for the rectangle...
by composing the results of properly chosen triangles. Even though, as commented above, one should not expect to obtain the correct result, we have explored this possibility. To this end let us consider a quadrilateral $ABCD$ with vertex $v$ at the point $A$, as shown in Fig. (2). The sides $AB$ and $CD$ are parallel, but $DA$ and $BC$ are not. We are interested in calculating the holonomy for the path $ABCDA$ as indicated in Fig. (2). This can be done by composing, via matrix multiplication, the holonomies corresponding to the triangles $ABC$ (spanned by the vectors $\vec{s}_I, \vec{s}_J$) and $ACD$ (spanned by the vectors $\vec{\bar{s}}_I, \vec{\bar{s}}_J$), each of which is calculated according to expressions (64) and (65), together with their corresponding $\Sigma$’s. In other words,

$$h_{ABCDA} = h_{ACDA} h_{ABCA}. \quad (67)$$

We introduce the further notation in the plane of the quadrilateral

$$\vec{AB} = \vec{s}_I = (a, 0), \quad \vec{AC} = \vec{s}_J = (a', b), \quad \vec{AC} = \vec{\bar{s}}_I = (a', b), \quad \vec{AD} = \vec{\bar{s}}_J = (a'', b). \quad (68)$$

We only pay attention to the non-covariant contributions. To third order they just add up and we obtain

$$\Sigma_{ABCA}^{(3)} + \Sigma_{ACDA}^{(3)} = \frac{(a'' - a' + a)}{12} \{3a'(a'' - a)[\partial_1 A_1, A_1]\}$$
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\[ + b(a'' + a' - a) \left[ [\partial_1 A_1, A_2] + [\partial_1 A_2, A_1] + [\partial_2 A_1, A_1] \right] \]
\[ + b^2 \left[ [\partial_2 A_1, A_2] + [\partial_2 A_2, A_1] + [\partial_1 A_2, A_2] \right]. \]  \tag{70}\]

We see that the above expression is not zero in general. Nevertheless, in the symmetrical case of a parallelepiped, characterized by the condition
\[ a'' - a' + a = 0, \]  \tag{71}\]
the non-covariant piece \( (70) \) vanishes. Of course, the above condition includes that of a rectangle which is \( a = a' \) and \( a'' = 0 \).

Moreover, to fourth order in the expansion, it is possible to show that following the calculational method of Ref. [16], the composition law \( (67) \) yields gauge covariance violations even for the case of the parallelepiped.

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