Zitterbewegung of Klein-Gordon particles and its simulation by classical systems

Tomasz M. Rusin and Włodzick Zawadzki

1 Orange Customer Service sp. z o. o., ul. Twarda 18, 00-105 Warsaw, Poland
2 Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-688 Warsaw, Poland

(Dated: May 1, 2014)

The Klein-Gordon equation is used to calculate the Zitterbewegung (ZB, trembling motion) of spin-zero particles in absence of fields and in the presence of an external magnetic field. Both Hamiltonian and wave formalisms are employed to describe ZB and their results are compared. It is demonstrated that, if one uses wave packets to represent particles, the ZB motion has a decaying behavior. It is also shown that the trembling motion is caused by an interference of two sub-packets composed of positive and negative energy states which propagate with different velocities. In the presence of a magnetic field the quantization of energy spectrum results in many interband frequencies contributing to ZB oscillations and the motion follows a collapse-revival pattern. In the limit of non-relativistic velocities the interband ZB components vanish and the motion is reduced to cyclotron oscillations. The exact dynamics of a charged Klein-Gordon particle in the presence of a magnetic field is described on an operator level. The trembling motion of a KG particle in absence of fields is simulated using a classical model proposed by Morse and Feshbach – it is shown that a variance of a Gaussian wave packet exhibits ZB oscillations.

PACS numbers: 03.65.Pm, 11.40.-q, 03.65.-w

I. INTRODUCTION

The phenomenon of Zitterbewegung (ZB, trembling motion) goes back to Schrodinger who proposed it in 1930 for free relativistic electrons in a vacuum [1]. Schrodinger observed that, due to non-commutativity of the velocity operators with the Dirac Hamiltonian, relativistic electrons experience a trembling motion in absence of external fields. It was later recognized that ZB is due to an interference of electron states with positive and negative electron energies. A very high frequency of ZB in a vacuum, corresponding to \( \hbar \omega_Z = 2mc^2 \), and its very small amplitude on the order of the Compton wavelength \( \lambda_c = \hbar / m_c \approx 3.86 \times 10^{-3} \) Å made it impossible to observe this effect in its original form with the currently available experimental methods. However, in a recent work Gerritsma et al. [2] simulated the 1+1 Dirac equation and the resulting Zitterbewegung with the use of trapped ions excited by laser beams. The important advantage of this method is that one can simulate also the basic parameters of the Dirac equation and tailor their desired values. The result of Gerritsma et al. allows one to expect that observable effects for relativistic particles in a vacuum can be convincingly reproduced with more "user friendly" parameters. In general, there has been recently a revival of interest in the relativistic-type equations related to "the rise of graphene" [3], topological insulators and similar systems in narrow-gap semiconductors [4].

The purpose of our paper is to describe the phenomenon of Zitterbewegung for charged Klein-Gordon (KG) spin-zero particles in absence of fields and in the presence of a magnetic field [2, 8]. The Zitterbewegung of KG particles in absence of fields was described before, see [5, 10]. However, in our treatment we introduce a number of additional elements. First, we describe the particles by wave packets and show that this feature leads to a transient character of the resulting ZB motion. Second, we use both the Hamiltonian and wave forms of Klein-Gordon equation (KGE) and show the equivalence of the two approaches. Third, we point out that ZB is a result of interference between positive and negative energy sub-packets propagating with different velocities. Fourth, we simulate classically the ZB motion using a simple mechanical system proposed by Morse and Feshbach [11]. Still, our main objective is to consider in detail the dynamics of a charged KG particle in the presence of an external uniform magnetic field and describe the phenomenon of ZB in this situation. To the best of our knowledge this problem has not been treated before.

The one-particle Klein-Gordon equation for spin-zero particles leads to some well known difficulties [10, 12]. The KG equation involves second time derivative, the probability density is not positively definite, there are problems with the position operator or vanishing square of the velocity operator. For this reason in the present work we calculate ZB of \textit{average current} which has well defined meaning in the theory of KG equation. For charged particles the average current is proportional to average particle velocity, so in our work we calculate one of these two quantities. In previous treatments of ZB for Dirac equation, simulation by trapped ions or solid-state systems, the authors usually calculated ZB of the position operator.

In our considerations we encounter another interesting anomaly of KG equation, namely, that particle velocities can exceed the speed of light for sufficiently large momenta. In other words it appears that, in contrast to
the Dirac equation for electrons, KGE does not possess an automatic “safety brake” for velocities to keep them below $c$. To our knowledge this feature has not been remarked before, so we mention it throughout our work.

Our paper is organized as follows. In Section II we calculate ZB of a wave packet using the Hamiltonian formalism, in Section III we obtain similar results with the use of KG waves and discuss explicitly physical background for the transient behavior of ZB motion. Section IV contains a description of ZB for a charged KG particle in a magnetic field. In Section V we simulate classically the ZB phenomenon using a system proposed by Morse and Feshbach. In Section VI we discuss our results, the paper is concluded by a summary. Appendix A contains a derivation of particle dynamics in the presence of a magnetic field, Appendix B discusses the problem of high particle velocities, in Appendices C and D we give some mathematical details.

II. ZITTERBEWEGUNG IN VACUUM

We begin by considering a Klein-Gordon particle in absence of external fields. The Klein-Gordon equation in the Hamiltonian form is [13]

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi. \quad (1)$$

Here the Hamiltonian is

$$\hat{H} = \frac{\tau_3 + i\tau_2}{2m} \hat{p}^2 + \tau_3 mc^2, \quad (2)$$

where $m$ is particle mass, $\hat{p}$ is particle momentum and $\tau_j \ (j = 1, 2, 3)$ are the Pauli matrices $\sigma_j$, respectively. The wave function $\Psi$ is a two-component vector

$$\Psi = \left( \begin{array}{c} \varphi \\ \chi \end{array} \right). \quad (3)$$

In the Hamiltonian form one can introduce the Heisenberg picture [13]. The $z$-th component of the time-dependent velocity operator is

$$\hat{v}_z(t) = e^{i\hat{H}t/\hbar}\hat{v}_z(0)e^{-i\hat{H}t/\hbar}, \quad (4)$$

where $\hat{v}_z(0) = \partial \hat{H}/\partial \hat{p}_z$. In this representation $\hat{v}_z(t)$ is a $2 \times 2$ matrix operator. Expanding $e^{i\hat{H}t/\hbar} = 1 + \hat{H}t + (1/2!)\hat{H}^2 + \ldots$ and noting that $\hat{H}^2 = E^2$, where the energy is $E = \pm cp_0$ with

$$p_0 = \pm \sqrt{m^2c^2 + p^2}, \quad (5)$$

we obtain

$$e^{i\hat{H}t/\hbar} = \cos(Et/\hbar) + \frac{\hbar \hat{H}}{E} \sin(Et/\hbar). \quad (6)$$

The velocity operator in Eq. (4) is a product of three matrices. Its $(1,1)$ component is

$$\langle \hat{v}_z(t) \rangle_{11}(t) = \frac{\hat{p}_z}{m} + \frac{\hat{p}_z^2 \hat{p}_z}{2m p_0^2} \left[ \cos(2Et/\hbar) - 1 \right]. \quad (7)$$

The remaining elements of $\hat{v}_z(t)$ are calculated similarly. The $\hat{v}_x$ and $\hat{v}_y$ components of the velocity operator are obtained from $\hat{v}_z(t)$ by the replacement $\hat{p}_z \rightarrow \hat{p}_x, \hat{p}_y$, respectively. In the non-relativistic limit $p \ll mc$ we obtain in Eq. (7) the classical motion $\langle \hat{v}_z \rangle_{11}(t) \approx \hat{p}_z/m$. In absence of external fields $p_i$ are good quantum numbers. We introduce $p = \hbar k$ and $q = \lambda z$, where the effective Compton wavelength is $\lambda_c = \hbar / mc$. Also, we introduce a useful frequency $\omega_0 = (mc^2)/\hbar$. Both $\lambda_c$ and $\omega_0$ refer to particles of mass $m$. In the above notation Eq. (7) becomes

$$\langle \hat{v}_z \rangle_{11}(t) = cq_z + \frac{\hbar}{2m} \frac{q^2 q_z}{1 + q^2} \left[ \cos(2\omega_0 t \sqrt{1 + q^2}) - 1 \right]. \quad (8)$$

The first term in Eq. (8) corresponds to the classical motion of a particle while the second term describes rapid oscillations of the velocity. The velocity oscillates from $v_{max} = cq_z$ to $v_{min} = cq_z/(1 + q^2)$. Since the maximum velocity of the particle is $c$, there must be $|q| \leq 1$. We notice that, in principle, Eq. (8) admits velocities above the speed of light. We discuss this issue in more detail in Appendix B. The frequency of oscillations varies from $\omega = 2\omega_0$ for low $q$ to $\omega = 2\sqrt{2}\omega_0$ for $|q| = 1$. The velocity oscillations taking place in absence of external fields are called Zitterbewegung.

Integrating $\langle \hat{v}_z \rangle_{11}(t)$ in Eq. (8) over time we have

$$\langle \hat{v}_z \rangle = \langle \hat{v}_z(0) + cq_z t - \frac{\hbar}{2m} \frac{q^2 q_z}{1 + q^2} t + \frac{\lambda_c}{4} \frac{q^2 q_z}{(1 + q^2)^{3/2}} \sin(2\omega_0 t \sqrt{1 + q^2}) \rangle. \quad (9)$$

The amplitude of ZB oscillations of the position operator is on the order of $\lambda_c$. The operator $\hat{z}_1(t)$ is obtained in the formal way, physical limitations to the position operator will be discussed below.

In order to obtain physical observables one needs to average the operator quantities over the wave packet. The average velocity $\langle \hat{v}_z(t) \rangle$ of the wave packet $|W\rangle$ is

$$\langle \hat{v}_z(t) \rangle = \langle W | \tau_3 \hat{v}_z(t) | W \rangle = \sum_{pp'} \langle W | p \rangle \langle p | \tau_3 \hat{v}_z(t) | p' \rangle \langle p' | W \rangle. \quad (10)$$

For KGE in the Hamiltonian form the matrix elements of operators include an additional $\tau_3$ factor [13]. We take the wave packet in the form of a two-component vector $|r\rangle = (1, 0)^T |w\rangle$ with one non-vanishing component. Here $|r\rangle |w\rangle \equiv w(r)$ is a Gaussian function with a nonzero momentum $\hbar k_0$

$$w(r) = \frac{1}{(d\sqrt{\pi})^{3/2}} \exp[-r^2/(2d^2)] + i\hbar k_0 r. \quad (11)$$

There is $w(k) = \int e^{-ikr/\hbar} w(r) d^3r$ and we have

$$\langle k | w \rangle = (2d\sqrt{\pi})^{3/2} \exp[-d^2 (k - k_0)^2/2]. \quad (12)$$
where the positive and negative energy states. First we intro-
duce the unity operator

\[ \hat{1} = \sum_{\mathbf{k}s} |\mathbf{k}s\rangle \langle \mathbf{k}s| \tau_3 s, \]

(14)

where \( s = \pm 1, \) and

\[ \langle r | \mathbf{k}s \rangle = \frac{e^{ikr}}{2\sqrt{mc_p}} \left( \frac{mc + sp_0}{mc - sp_0} \right) \]

(15)

are the two eigenstates of \( \hat{H} \) corresponding to the pos-
tive and negative energies \( E_s = sp_0. \) These states are

normalized according to \( \langle \mathbf{k}s | \tau_3 | \mathbf{k}'s' \rangle / (2\pi)^{3/2} = \delta_{\mathbf{k}k'} \delta_{ss'}. \)

Then

\[ |W\rangle = \sum_{\mathbf{k}s} s |\mathbf{k}s\rangle \langle \mathbf{k}s| \tau_3 |W\rangle = \sum_{\mathbf{k}s} s |\mathbf{k}s\rangle W_{\mathbf{k}s}, \]

(16)

where \( W_{\mathbf{k}s} = \langle \mathbf{k}s | \tau_3 |W\rangle. \) The sub-packet of pos-
tive energy states is \( |W+\rangle = \sum_{\mathbf{k}+} |\mathbf{k}+\rangle W_{\mathbf{k}+}, \) while the

sub-packet of negative energy states is \( |W-\rangle = \sum_{\mathbf{k}-} |\mathbf{k}-\rangle W_{\mathbf{k}-}. \)

Using Eqs. (15) and (16) we find

\[ W_{\mathbf{k}s} = (2d\sqrt{\pi})^{3/2} \frac{(mc + sp_0)}{2\sqrt{mc_p}} e^{-d^2(k-k_0)^2}/2. \]

(17)

The average packet velocity is

\[ \langle \hat{v}_z(t) \rangle = \sum_{\mathbf{k}k's'} s s' W_{\mathbf{k}s} W_{\mathbf{k}'s'} \langle \mathbf{k}s | \tau_3 \hat{v}_z(t) | \mathbf{k}'s' \rangle \]

\[ = \sum_{\mathbf{k}k's'} s s' W_{\mathbf{k}s} W_{\mathbf{k}'s'} e^{i(\omega_s - \omega_{s'})t} \langle \mathbf{k}s | \tau_3 \partial_{\mathbf{p}_z} | \mathbf{k}'s' \rangle. \]

(18)

We defined \( \omega_s = s\omega_0 \sqrt{1 + (k_0/c_p)^2} \) and used the equality

\[ \langle \mathbf{k}s | \tau_3 e^{i\hat{H}t/\hbar} = \langle \mathbf{k}s | e^{i\hat{H}t/\hbar} \tau_3 = e^{i\omega_s t} |\mathbf{k}s\rangle \tau_3, \]

(19)

which follows from the properties: \( \hat{H} = \tau_3 \hat{H} \tau_3 \)

and \( |\mathbf{k}s\rangle |\hat{H}\rangle = (\hat{H} |\mathbf{k}s\rangle) = E_s |\mathbf{k}s\rangle. \) Another proof of the

identity [19] is given in Appendix C. There is also

\[ \langle \mathbf{k}s | \tau_3 \frac{\partial \hat{H}}{\partial \mathbf{p}_z} | \mathbf{k}'s' \rangle = (2\pi)^3 \frac{\delta_{\mathbf{k}k'}}{p_0} \]

(20)

which does not depend on \( s \) and \( s'. \) Combining Eqs. (18) - (20) we obtain

\[ \langle \hat{v}_z(t) \rangle = \frac{2d^3\pi^{3/2}}{(2\pi)^3 m} \int \frac{p_z}{p_0} e^{-d^2(k-k_0)^2} d^3k \]

\[ \times \sum_{s,s'} ss' (mc + sp_0)(mc + s'p_0)e^{i(\omega_s - \omega_{s'})t}. \]

(21)

The average velocity in Eq. (21) is a sum of four terms.

The term with \( s = s' = +1 \) describes the motion of pos-
tive energy sub-packet, while the term with \( s = s' = -1 \) corre-
sponds to the negative energy sub-packet

\[ \langle \hat{v}_z \rangle = \frac{d^3}{4m\pi^{3/2}} \int \left( \frac{me}{p_0} \right)^2 p_z e^{-d^2(k-k_0)^2} d^3k. \]

(22)
Thus the two sub-packets move with different velocities. Their relative velocity is
\[
\langle \dot{v}_x \rangle^{rel} = \frac{cd^3}{\pi^{3/2}} \int \frac{P_z e^{-d^2(k-k_0)^2}}{P_0^2} d^3k.
\] (23)

Two terms in Eq. (21) with \( s \neq s' \), corresponding to an interference of the two packets, give rise to an oscillatory term
\[
\langle \dot{v}_z(t) \rangle^{osc} = \frac{d^3}{4m_\pi 3/2} \int \left( 1 - \frac{m_\pi c^2}{P_0^2} \right) P_z \times \cos(2\omega_k t) e^{-d^2(k-k_0)^2} d^3k,
\] (24)

where \( \omega_k = \sqrt{1+(k\lambda_c)^2} \). According to the Riemann-Lesbegues theorem this term has a transient character \[14]. Performing integrations in Eqs. (22) and (23), we obtain again Eq. (19). Thus we showed that the ZB oscillations arise from the interference of positive and negative energy states. After a certain time the two sub-packets are sufficiently far away from each other and the overlap between them vanishes, which results in the disappearance of ZB oscillations. This explains the behavior of velocity shown in Fig. 1.

To evaluate the decay of ZB oscillations, we estimate the time after which the two sub-packets will be separated from each other by the distance \( 2d \). Assuming that \( k_0\lambda_c \approx 1 \), the relative velocity between the two sub-packets is \( \langle \dot{v}_z \rangle^{rel} \approx c(k_0\lambda_c) \). The time interval after which the distance between the sub-packets exceeds \( 2d \) is
\[
t_d \approx \frac{2d}{ck_0\lambda_c}.
\] (25)

It is seen in Fig. 1 that the ZB oscillations nearly disappear after \( t_d \). For example there is \( t_d = 5\tau_c \) for \( d = 2\lambda_c \). Since the ZB frequency is \( 2\omega_0 = 2mc^2/h \), a number of non-vanishing oscillations is approximately
\[
N_{osc} \approx \frac{2\omega_0 t_d}{2\tau_c} = \frac{2}{\lambda_c} \left( \frac{1}{k_0\lambda_c} \right).
\] (26)

The above estimation correctly evaluates the number of ZB oscillations seen in Fig. 1. The optimal conditions for an appearance of ZB are: wide packets and small values of \( |k_0| \). On the other hand, for too small values of \( |k_0| \) one of the two sub-packets disappears, see Eq. (22), which reduces amplitude of ZB oscillations.

### III. WAVE FORM OF KGE

Now we intend to demonstrate a relation between the ZB oscillations of the average packet velocity calculated above with the use of the Hamiltonian form of KGE and an average current obtained from the wave form of KGE. In absence of external fields the Klein-Gordon equation has the wave equation form
\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(x) - \nabla^2 \phi(x) + \frac{m^2c^2}{\hbar^2} \phi(x) = 0,
\] (27)

where \( x = (ct, \mathbf{r}) \) is the position four-vector \[10]. The solution of this equation is
\[
\phi(x) = \frac{1}{(2\pi)^3} \int \sqrt{\frac{mc}{p_0}} \left( a(k) e^{-ik \cdot x} + b^*(k) e^{ik \cdot x} \right) d^3k,
\] (28)

where \( k = (\omega_k/c, \mathbf{k}) \), \( \omega_k = \omega_0 \sqrt{1+(k\lambda_c)^2} \), and \( a(k), b^*(k) \) are complex coefficients. Function \( \phi \) is normalized to
\[
\int d^3k \left[ \phi^* \frac{\partial \phi}{\partial t} - \left( \frac{\partial \phi^*}{\partial t} \right) \phi \right] = Q,
\] (29)

where \( Q = \pm 1 \) for charged particles and \( Q = 0 \) for neutral particles. In the following we select \( Q = +1 \), which leads to
\[
\int d^3k \left( a^*(k)a(k) - b^*(k)b(k) \right) = 1.
\] (30)

To determine the coefficients \( a(k) \) and \( b^*(k) \) we need two boundary conditions for \( \phi \) and \( \partial \phi/\partial t \) at \( x = (0, \mathbf{r}) \). Having specified \( a(k) \) and \( b^*(k) \) one can calculate the current density \( j(x) \)
\[
j(x) = \frac{\hbar}{2i\pi} \left[ \phi^* (\nabla \phi) - (\nabla \phi^*) \phi \right],
\] (31)

and the average current \( \langle j(t) \rangle = \int j(x) d^3r \).

Our aim is to find a correspondence between the average packet velocity calculated in Eq. (13) and the average current \( \langle j(t) \rangle \) given in Eq. (51). To this end we select the coefficients \( a(k) \) and \( b^*(k) \) in such a way that the function \( \phi \) in the wave form of KGE corresponds to the wave packet \( (w(r), 0)^T \) in the Hamiltonian form of KGE. Relations between \( \phi, \partial \phi/\partial t \) and the two-component wave function \( \Psi = (\varphi, \chi)^T \) in the Hamiltonian form of KGE are
\[
\phi = \varphi + \chi,
\] (32)
\[
i\partial \phi/\partial t = mc^2(\varphi - \chi)/h.
\] (33)

Since \( (\varphi, \chi)^T = (w(r), 0)^T \) we find the coefficients \( a(k) \) and \( b^*(k) \) from Eqs. (32) - (33) by setting \( \varphi(t = 0, \mathbf{r}) = w(r) \) and \( \chi = 0 \). From Eq. (32) we have
\[
\int \sqrt{\frac{mc}{p_0}} \left[ a(k)e^{ik \cdot r} + b^*(k)e^{-ik \cdot r} \right] d^3k
\] (34)

while from Eq. (33) we have
\[
\int \sqrt{\frac{mc}{p_0}} \left[ a(k)e^{ik \cdot r} - b^*(k)e^{-ik \cdot r} \right] p_0 d^3k
\] (35)

In terms including \( b^*(k) \) we replace \( k \rightarrow -k \), solve equations (34) and (35) for \( a(k) \) and \( b^*(k) \), and obtain
\[
\phi(r, t) = \frac{(2d/2\pi)^{3/2}}{2mc} \int d^3k e^{-d^2(k-k_0)^2/2 + ik \cdot r} \times \left[ \left( 1 + \frac{mc}{p_0} \right) e^{-i\omega_0 t} + \left( 1 - \frac{mc}{p_0} \right) e^{i\omega_0 t} \right].
\] (36)
The above function $\phi$ includes both positive and negative energy amplitudes. For $p \to 0$ there is $1 + mc/p_0 \approx 2$ and $1 - (mc/p_0) \approx p^2/(2mc^2)$. Thus the second term in Eq. (36) is much smaller than the first. In this limit the packet consists of the positive energy states alone.

In Fig. 2 we plot the time evolution of the wave packet $\phi$ in one dimension. The packet propagates according to a one-dimensional version of Eq. (36). The initial packet is assumed in a Gaussian form

$$\phi(x, 0) = \frac{d}{\sqrt{\pi}} e^{-x^2/(2d^2)} + ik_0 x. \quad (37)$$

Its absolute value is indicated in Fig. 2 by the thick line. Each thin line describes $|\phi(x, t)|$ in successive time intervals $2t_c = 2\hbar/(mc^2)$. It is seen that the packet splits into two sub-packets moving with different velocities. The sub-packet at the right corresponds to positive energies while the sub-packet at the left corresponds to negative energies. The difference in the amplitudes of sub-packets results from different contributions of the positive and negative energy states in the initial packet at $t = 0$, see Eqs. (34) - (35). The Zitterbewegung occurs only when the sub-packets overlap. Each of the sub-packets slowly spreads in time, but the spreading time is much larger than the overlapping time, so the ZB vanishes much faster than the spreading of sub-packets.

Now we continue the calculation of average current given in Eq. (31) using function $\phi$ of Eq. (35). This function has the form of an integral over $k$. To calculate the spatial derivative $\nabla \phi$ we change the order of integration and differentiation, which can be done for any function decaying exponentially for $k \to \infty$. Using the identity: $1 + (mc/p_0)^2 = 2 - (p/p_0)^2$, we obtain for the first term of the average current

$$\frac{\hbar}{2im} \int \phi_0 \frac{\partial \phi}{\partial z} d^3r = \frac{d^3\hbar}{8im\pi^{3/2}} \int d^3ke^{-d^2(k-k_0)^2}(ik_z) \times$$

$$= \frac{d^3}{2\pi^{3/2}} \int d^3ke^{-d^2(k-k_0)z} \left\{ \frac{p_z^2}{m^2} + \frac{p_z^2}{2mp_0}[\cos(2\omega t) - 1] \right\}. \quad (38)$$

Calculation of the second term in the current: $\hbar/(2im) \int (\partial \phi^*/\partial z) \phi d^3r$, gives the same result but with an opposite sign, so that both terms in Eq. (31) add together. Comparing Eq. (38) with Eqs. (31) and (13) we conclude that the current density $\langle j_z(t) \rangle$ averaged over the packet $\phi(x)$ in Eq. (28) equals to the average velocity $\langle v_z(t) \rangle$ of the packet in the Hamiltonian form of KGE multiplied by the particle charge. This way we establish an equivalence of Zitterbewegung in the Hamiltonian and wave equation formalisms.

The above equivalence is valid for the average values only. In the Hamiltonian form of KGE one can define the time dependent velocity operator $\hat{v}(t) = e^{iHt/\hbar} \hat{v}(0) e^{-iHt/\hbar}$, which can be expressed in a closed form without specifying of the wave packet, see Eq. (7). But an analogous current operator in the wave form of KGE can be defined as a current density $\hat{j}(x)$, which strongly depends on the form of function $\phi$.

Even more significant differences between the Hamiltonian and wave descriptions of ZB appear in the analysis of the position operator $\hat{r}(t)$. In the Hamiltonian form of KGE the position operator written in the Heisenberg picture is $\hat{r}(t) = e^{iHt/\hbar} \hat{r}(0) e^{-iHt/\hbar}$ and, for the field-free KGE, it can be calculated in a compact form, see Eq. (9) and Ref. [8]. On the other hand, there is no well defined position operator $\hat{\mathbf{r}}$ for the wave form of KGE since this operator is not hermitian, see Ref. [10]. However, one can calculate an average position operator for the wave form of KGE by integrating the average current over time

$$\langle \mathbf{r}(t) \rangle = \langle \mathbf{r}(0) \rangle + \frac{1}{Q} \int \langle j(t) \rangle dt, \quad (39)$$

where the charge $Q \neq 0$. This example indicates that the equivalence between the Zitterbewegung for the Hamiltonian and wave equation formalisms holds for the average values only.

IV. ZITZERBEWEGUNG IN A MAGNETIC FIELD

In the presence of a magnetic field the KG Hamiltonian for a charged particle reads (10)

$$\hat{H} = \frac{\tau_3 + i\tau_2}{2m}(\mathbf{p} - q\mathbf{A})^2 + \tau_3 mc^2, \quad (40)$$
where \( q \) is the particle charge and \( \mathbf{A} \) is the vector potential of a magnetic field. We assume the magnetic field \( \mathbf{B} \) to be parallel to the \( z \) axis and describe it by the asymmetric gauge \( \mathbf{A} = B(-y, 0, 0) \). Eigenstates of the Hamiltonian are of the form

\[
\Psi(r) = e^{i(k_xx + i\tau_2)z} \Phi(y),
\]

and the resulting eigenenergy equation is \( H \Psi = E \Psi \) with

\[
H = (\tau_3 + i\tau_2) \frac{1}{2m} \left[ (\hbar k_y + qBy)^2 + \hbar^2 k_y^2 + \hbar^2 k_x^2 \right].
\]

We introduce the magnetic radius \( L = \sqrt{\hbar/|q|B} \) and define \( \xi = k_x L + \eta_0 y/L \), where \( \eta_0 = \pm 1 \) is the sign of \( q \). Then there is \( \eta_0 y = \xi L - k_x L^2 \) and \( \partial/\partial y = (1/L)\partial/\partial \xi \). The eigenenergies are \( E_n = sE_{n,k_z} \), where

\[
E_{n,k_z} = \sqrt{m^2 c^4 + 2mc^2 \hbar \omega_n(n + 1/2) + (\hbar c k_z)^2}.
\]

The corresponding eigenstates \( |n\rangle \) are characterized by four quantum numbers: \(|n\rangle = |n, k_x, k_z, s\rangle\), where \( n \) labels the Landau levels, \( k_x \) and \( k_z \) are wave vector components and \( s = \pm 1 \) label positive and negative energy branches. The wave functions are

\[
\Psi_n(r) \equiv \langle r|n\rangle = \frac{e^{i(k_xx + i\tau_2)z}}{4\pi} \phi_n(\xi) \left( \frac{\mu^{+}_{n,k_x,s}}{\mu_{n,k_x,s}} \right),
\]

where \( \phi_n(\xi) \) are the harmonic oscillator functions

\[
\phi_n(\xi) = \frac{1}{\sqrt{L^2 C_n}} H_n(\xi) e^{-\xi^2/2c^2},
\]

in which \( H_n(\xi) \) are the Hermite polynomials and \( C_n = \sqrt{2^n n! \sqrt{\pi}} \). We defined \( \mu^{+}_{n,k_x,s} = \nu_{n,k_x} \pm s/\nu_{n,k_x} \), where

\[
\nu_{n,k_x} = \sqrt{m^2 c^2 / E_{n,k_z}}.
\]

We want to calculate an average packet velocity in a magnetic field. We can, as before, introduce the Heisenberg picture for the time-dependent velocity operator. Then the \( j \)-th component of the average velocity is, see Eq. (18),

\[
\langle \hat{v}_j(t) \rangle = \langle W| \tau_3 e^{i\hat{H}_t/\hbar} \hat{v}_j e^{-i\hat{H}_t/\hbar} |W \rangle,
\]

where \( \hat{v}_j = \partial \hat{H}/\partial \hat{p}_j \). For the Hamiltonian (40) in the asymmetric gauge we find

\[
\hat{v}_x = (\tau_3 + i\tau_2) \left( \frac{\hat{p}_x - qBy}{m} \right),
\]

\[
\hat{v}_y = (\tau_3 + i\tau_2) \frac{\hat{p}_y}{m},
\]

\[
\hat{v}_z = (\tau_3 + i\tau_2) \frac{\hat{p}_z}{m}.
\]

The unity operator is now

\[
\hat{1} = \sum_n |n\rangle \langle n| s_n \tau_3,
\]

where the states \( \langle r|n \rangle \) are given in Eq. (44) and \( s_n = \pm 1 \) are the quantum numbers associated with the states \(|n\rangle \). The proof of the above identity is given in Appendix C. Using the unity operator we expand the packet \(|W \rangle\) in term of the eigenstates of \( \hat{H} \) [see Eq. (10)]

\[
|W \rangle = \sum_n s_n |n\rangle \langle n| \tau_3 |W \rangle \equiv \sum_n s_n |n\rangle W_n,
\]

where \( W_n = \langle n| \tau_3 |W \rangle \). Inserting \(|W \rangle\) into Eq. (46) one obtains [see Eq. (13)]

\[
\langle \hat{v}_j(t) \rangle = \sum_{nm} s_n s_m W_n^{*} W_m \langle n\tau_3 e^{i\hat{H}_t/\hbar} \hat{v}_j e^{-i\hat{H}_t/\hbar} |m \rangle.
\]

There is \( e^{-i\hat{H}_t/\hbar} |n\rangle \) = \( e^{-i\omega_n t} |n\rangle \), where \( \omega_n = s_n E_{n,k_z}/\hbar \). Proceeding the same way as in Section III we have

\[
\langle n\tau_3 e^{i\hat{H}_t/\hbar} |n\rangle = e^{i\omega_n t} |n\rangle\tau_3,
\]

which finally gives

\[
\langle \hat{v}_j(t) \rangle = \sum_{nm} s_n s_m W_n^{*} W_m e^{i(\omega_n - \omega_m)t} \langle n\tau_3 \hat{v}_j |m \rangle.
\]

The matrix elements of velocity operators calculated between the states \(|n\rangle, |m \rangle\) are

\[
\langle n| \tau_3 \hat{v}_y |m \rangle = \frac{c}{i\sqrt{2L}} \nu_{n,k_x} \nu_{m,k_z} \delta_{k_x,k_z} \delta_{k_x',k_z'} \times
\]

\[
(\sqrt{n + 1} \delta_{m,n+1} - \sqrt{n} \delta_{m,n-1}),
\]

\[
\langle n| \tau_3 \hat{v}_x |m \rangle = \frac{c}{\sqrt{2L}} \nu_{n,k_z} \nu_{m,k_z} \delta_{k_x,k_z} \delta_{k_x',k_z'} \times
\]

\[
(\sqrt{n + 1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}),
\]

\[
\langle n| \tau_3 \hat{v}_z |m \rangle = \frac{p_m}{m} \nu_{n,k_x} \nu_{m,k_z} \delta_{k_x,k_z} \delta_{k_x',k_z'} \delta_{m,n,n}.
\]

The matrix elements of \( \hat{v}_y \) and \( \hat{v}_x \) are nonzero for the states with \( n = m \pm 1 \) and arbitrary indexes \( s_n \) and \( s_m \). The matrix elements of \( \hat{v}_z \) are nonzero for \( m = n \) and arbitrary indexes \( s_n \) and \( s_m \). To simplify the further analysis we assume the initial wave packet \(|W \rangle\) to be in a separable form

\[
|W \rangle = W_{xy}(x,y)W_z(z).
\]

Then there is

\[
W_n = \langle n| \tau_3 |W \rangle = \mu_{n,k_z}^+ g_z(k_z) F_n(k_z),
\]

where

\[
F_n(k_z) = \frac{1}{\sqrt{2L C_n}} \int_{-\infty}^{\infty} g_{xy}(k_x, y) e^{-\frac{1}{2} \xi^2 H_n(\xi) dy},
\]

in which

\[
g_{xy}(k_x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w_{xy}(x, y) e^{ik_x x} dx,
\]
The velocity of the packet oscillates with many frequencies \(\omega_{n+1} \pm \omega_n\) (or \(2\omega_n\) for \(\hat{v}_z\)), but in practice the spectrum is limited to a few frequencies related to the largest coefficients \(U_{n+1,n}\) and \(U_{n,n}\). The frequencies \(\omega_{n+1} - \omega_n\) correspond to the intraband transitions and they can be interpreted as the cyclotron resonances. These frequencies do not appear in \(\hat{v}_z\) velocity. On the other hand, the frequencies \(\omega_{n+1} + \omega_n\) and \(2\omega_n\) (for \(\hat{v}_z\)) correspond to interband transitions and they can be interpreted as the Zitterbewegung components of the motion in analogy to the situation at zero field. The motion in the \(x-y\) directions requires that \(k_{0z} \neq 0\) because for \(k_{0z} = 0\) all the coefficients \(U_{n+1,n}\) and \(U_{n,n}\) vanish. For the motion in the \(z\) direction one needs only that \(k_{0z} \neq 0\), because the coefficients \(U_{n,n}\) are nonzero for any \(k_{0z}\) vector \(\mathbf{\lambda}\).

Considering the non-relativistic limit in Eqs. (63) - (66), there is \(\hbar \omega_c \ll mc^2\) and \(\hbar k_z \ll mc\), so that \(\omega_{n+1} - \omega_n \approx \hbar \omega_c\) and \(\omega_{n+1} + \omega_n \approx 2mc^2/\hbar\). In this limit there is \(\nu_{n+1} \approx \nu_n \approx 1\), and the ZB part of velocity is nearly zero. In this case we may decouple in Eqs. (63) - (66) the summation over \(n\) and integration over \(k_z\). This gives

\[
\sum_{n=0}^{\infty} \sqrt{n + 1} U_{n+1,n} = -\frac{k_{0z} L}{\sqrt{2}}
\]

Integrating over \(k_z\) one gets

\[
\langle \hat{v}_y(t) \rangle \approx \frac{\hbar k_{0z}}{m} \sin(\omega_c t),
\]

\[
\langle \hat{v}_x(t) \rangle \approx \frac{\hbar k_{0z}}{m} \cos(\omega_c t),
\]

\[
\langle \hat{v}_z(t) \rangle \approx \frac{\hbar k_{0z}}{m}.
\]

Thus in the non-relativistic limit the particle moves on a circular orbit with the cyclotron frequency in the \(x-y\) plane and a constant velocity in the \(z\) direction. Let us introduce a measure of intensity of a magnetic field by its relation to an effective Schrödinger field \(\hbar B_s/m = mc^2\) or, equivalently, by \(L_s = \hbar/mc\). There is \(B_s = 4.41 \times 10^3 (m_e/m_e)^2\) T, where \(m_e\) is the electron mass. Below we perform calculations for pions \(\pi^+\) having the mass \(m \simeq 273.1\ m_e\), so the effective Schrödinger field is \(B_s = 3.29 \times 10^{14}\) T.

In Fig. 3 we plot the average packet velocity for three values of magnetic field. The ellipsoidal packet is selected with a nonzero initial momentum \(k_{0z}\). We assume that the five parameters: \(d_x\), \(d_y\), \(d_z\), \(L\) and \(k_{0z}\) have similar orders of the magnitude which are the optimal conditions for the appearance of Zitterbewegung phenomenon. In Fig. 3 we selected parameters: \(d_x = 0.91(B_s/B)\lambda_c\), \(d_y = 0.82(B_s/B)\lambda_c\), \(d_z = 0.68(B_s/B)\lambda_c\), \(k_{0z} = 0.7(B_s/B)\lambda_c^{-1}\) and \(k_{0z} = 0\), where \(B_s\) is the effective Schrödinger field. For \(B = 4.5B_s\) we set \(k_{0z} = \lambda_c^{-1}\). For low fields \((B = 0.0045B_s)\) the packet moves on a circular orbit, see Eqs. (69) and (70). For such fields the ZB components of the motion are negligible. For higher fields the packet motion includes both the intraband and interband (ZB) components so that several frequencies give significant contributions to the motion. In all cases

\[
g_z(k_z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w_z(z) e^{ik_z z} dz.
\]

For \(\langle \hat{v}_y(t) \rangle\) we obtain

\[
\langle \hat{v}_y(t) \rangle = c \frac{\lambda_c}{2\sqrt{2L}} \sum_{n,m=0}^{\infty} \int_{-\infty}^{\infty} dk_z \langle g_z(k_z)^2 \rangle \left( \sqrt{n + 1} U_{n+1,n} + U_{n,n+1} \right) \\{ (1 + \nu_m^2 + \nu_n^2) \cos(\omega_{n+1} t - \omega_n t) + (\nu_m^2 + 1) \sin(\omega_{n+1} t - \omega_n t) \} dk_z.
\]

For \(\langle \hat{v}_x(t) \rangle\) we have

\[
\langle \hat{v}_x(t) \rangle = c \frac{\lambda_c}{2\sqrt{2L}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_z \langle g_z(k_z)^2 \rangle \left( \sqrt{n + 1} U_{n+1,n} + U_{n,n+1} \right) \\{ (1 + \nu_{n+1}^2 + \nu_n^2) \cos(\omega_{n+1} t - \omega_n t) + (\nu_{n+1}^2 + 1) \sin(\omega_{n+1} t + \omega_n t) \} dk_z.
\]

\[
\langle \hat{v}_z(t) \rangle = c \frac{\lambda_c}{2} \sum_{n=0}^{\infty} U_{n,n} \int_{-\infty}^{\infty} dk_z \langle g_z(k_z)^2 \rangle \left( (1 + \nu_m^2) + (1 - \nu_m^2) \cos(2\omega_n t) \right) dk_z.
\]
V. SIMULATION OF ZB

The phenomenon of Zitterbewegung for relativistic particles in a vacuum has an unfavorable high frequency corresponding to the energy gap between the positive and negative energy branches: $\hbar\omega_0 \approx 2mc^2$, and a very small amplitude on the order of the effective Compton wavelength $\Delta r \approx \hbar/(mc)$, see Eq. (9). Thus, similarly to the case of relativistic electrons, one can not hope at present to observe directly the ZB in a vacuum. However, it was recently demonstrated by Gerritsma et al. that one can simulate the ZB of electrons in a vacuum using trapped ions interacting with laser beams \cite{2}. In this experiment the authors simulated the linear momentum $\hat{p}_i$ appearing in the Dirac equation with the use of Jaynes-Cumminngs interaction between the electrons on trapped ion levels and the electromagnetic radiation. The decisive advantage of such a simulation is that one can tailor the frequency and amplitude of ZB making them considerably more favorable than the values for a vacuum. Clearly, it would be of interest to simulate the ZB of a Klein-Gordon particle using similar methods. The prob-
The Klein-Gordon equation appears in several classical systems, usually as a modification of the wave equation $\Box \phi = 0$. Under some conditions KGE is used to describe sound waves in ducts [18, 19], electromagnetic waves in the ionosphere [20, 21], transverse modes of wave guides [22] and oceanic waves [23]. Below we examine in more detail a model proposed by Morse and Feshbach in which one can simulate KGE with the use of a piano string and a thin rubber sheet. We demonstrate similarities and differences between ZB in the relativistic KGE and its classical analogues.

Let us consider flexible one dimensional string in the $x$ direction, see Fig. 6. We assume that the string is uniform with a linear density $\rho$. A uniform tension $T$ is applied to each element $dx$ of the string. We neglect all other forces acting on the string (e.g. gravity) and the stiffness of the string. Let $y(x,t)$ be a displacement of the element $dx$ of the string from its equilibrium position at an instant $t$. We assume that $y(x,t)$ is small compared to the length of the string and to the distances to each end of the string. The restoring force acting on each element $dx$ of the string is $F_T = T \frac{\partial^2 y}{\partial x^2} dx$ and displacement $y(x,t)$ of the released string changes according to the wave equation [11]

$$\frac{1}{u^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \nu^2 y,$$

where $\nu^2 = K/T$. Equation (73) has the form of wave KGE with the light speed replaced by $\nu$ and the mass term $m^2 c^2 / h^2$ replaced by $\nu^2$. Comparing Eq. (73) with Eq. (27) we find the following correspondence between parameters of the two systems

$$\frac{T}{\rho} \leftrightarrow c^2,$$

$$\frac{K}{T} \leftrightarrow \frac{m^2 c^2}{h^2} = \lambda_c^{-2}$$

Thus one can simulate values of $c$ and $\lambda_c$ by changing material parameters $\rho$, $K$ and $T$.

However, there exist also limitations of such a simulation and they affect a possibility of observation of ZB motion in classical analogues of KGE. The first difference between the relativistic KGE and its classical counterpart is that the wave function $\phi$ in the relativistic KGE is not an observable. On the other hand, all classical analogues of $\phi$ (such as a displacement of the string, the pressure of sound or the oceanic waves, the intensity of electromagnetic field etc.) are observable quantities. The second difference is that the relativistic function $\phi$ is a function of complex variable, while its classical counterpart is a function of real variable. A direct consequence of these limitations for observation of ZB in classical systems is that, for any real function $\xi(r,t)$ being the solution of KGE, the current density associated with this function is always zero: $j \propto \left[ \xi^* \nabla \xi - (\nabla \xi)^* \xi \right] = 0$. Therefore we are not able to simulate directly the current or velocity oscillations calculated in the previous sections.

To overcome this problem let us consider the motion of a neutral particle described by a real field $\xi$. For simplicity we assume a one-dimensional KGE that can be simulated by a flexible string attached to an elastic substrate described above. In our calculations we use the relativistic form of KGE but the final results will be presented for parameters corresponding to the flexible string.
model. We assume the initial wave packet to be a real Gaussian function without an initial momentum

$$w_0(x) = \frac{1}{(2\sqrt{\pi})^{1/2}} \exp[-x^2/(2d^2)]. \quad (76)$$

Its Fourier transform is

$$w_0(k) = (2d\sqrt{\pi})^{1/2} \exp[-dk^2/2]. \quad (77)$$

A real solution $\xi(x,t)$ of KGE is

$$\xi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_0(k) \cos(kx - \omega_k t) dk, \quad (78)$$

where $\omega_k = \omega_0 \sqrt{1 + (k\lambda_c)^2}$. The average current for the real wave packet of Eq. (77) is zero and no ZB occurs. Thus we turn to other physical operators which do not commute with the KG Hamiltonian [1]. Namely, we calculate a variance of the position operator for the above real function $\xi(x,t)$

$$V = \langle \xi | \dot{x}^2 | \xi \rangle - \langle \phi | \dot{x} | \phi \rangle^2 = \langle \xi | \dot{x}^2 | \xi \rangle, \quad (79)$$

since $\langle \phi | \dot{x} | \phi \rangle = 0$. Assuming $\xi(x,t)$ in the form (78) we have

$$V = \iiint \infty w_0(k)w_0(k') \cos(kx - \omega_k t) \times$$

$$\cos(k'x - \omega_{k'}t)x^2 dx dk dk'$$

$$= \iiint \infty \left[ B_k B_{k'} e^{ikx}e^{ik'x} + B_k B_{k'} e^{-ikx}e^{ik'x} \right] x^2 dx dk dk'. \quad (80)$$

where $B_k = w_0(k)e^{-i\omega_k t}/(4\pi)$. Consider the first of the four terms given above. Because $w_0(k)$ and $B_k$ decay exponentially for $k \to \pm \infty$, one can change the order of integration over $x$, $k$ and $k'$ and replace $x^2 \to (\partial/\partial k)(\partial/\partial k')$. Then we integrate by parts over $k$ and $k'$ and obtain

$$= \iiint \infty \left[ \frac{\partial B_k}{\partial k} \frac{\partial B_{k'}}{\partial k'} e^{ikx}e^{ik'x} \right] x^2 dx dk dk'$$

$$= 2\pi \int_{-\infty}^{\infty} \left| \frac{\partial B_k}{\partial k} \frac{\partial B_{k'}}{\partial k'} \right|_{k'=k} dk. \quad (81)$$

The other three terms in Eq. (80) are calculated similarly. After some manipulations we find

$$V = V_1^c + V_1^{osc} + V_2^c + V_2^{osc} + V_3,$$  \quad (82)

where

$$V_1^c = \frac{d^3}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-d^2k^2} (kd)^2 dk,$$

$$V_1^{osc} = \frac{d^3}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-d^2k^2} (kd)\cos(2\omega_k t) dk,$$

$$V_2^c = \frac{d(ct)^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-d^2k^2} (k\lambda_c)^2 \cos(2\omega_k t) dk,$$

$$V_2^{osc} = \frac{d(ct)^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-d^2k^2} (k\lambda_c)^2 \cos(2\omega_k t) dk,$$

$$V_3 = \frac{d^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-d^2k^2} \sin(2\omega_k t) dk.$$  \quad (87)

The term $V_3$ is odd in $k$ and it vanishes upon the integration. For $t = 0$ the variance in Eq. (82) is equal to the variance $V_0 = d^2/2$ of the initial packet $w_0(x)$. The variance given in Eq. (82) consists of oscillating and non-oscillating terms. For large times the non-oscillating terms grow in time as $d^2/2 + C t^2$, where $C$ is a constant depending on $d$. The quadratic dependence of the variance on time is similar to that of a Gaussian wave packet in the non-relativistic quantum mechanics. The oscillations in Eqs. (84) and (85) have the same interband frequency $2\omega_k$ as the velocity oscillation given in Eq. (83). Therefore the oscillations of variance of the position operator can be interpreted as a signature of Zitterbewegung in classical systems. The term $V_2^{osc}$ has a decaying character and it vanishes after a few oscillations. The $V_2^{osc}$ term gives persistent oscillations because of the presence of $t^2$ factor in front of the integral. To estimate the time dependence of these oscillations we consider the limit of large packet widths $d \gg \lambda_c$. In this case the Gaussian function in Eq. (80) restricts the integration to small values of $k$. Then we may disregard $(k\lambda_c)^2$ term in the denominator of integrand and expand $\omega_k$ under the cosine function. This gives approximately

$$V_2^{osc} \simeq \frac{d(ct)^2}{4} \int_{-\infty}^{\infty} e^{-d^2k^2} (k\lambda_c)^2 \cos[\omega_0 t(2 + k^2\lambda_c^2)] dk$$

$$\simeq \frac{d(ct)^2}{4} \sum_{\eta = \pm 1} \frac{e^{2i\eta\omega_0 t}}{(d^2 + i\eta\omega_0 t)^{3/2}}. \quad (88)$$

For large time we may approximate in Eq. (85)

$$V_2^{osc} \simeq -C_d t^{1/2} \cos(2\omega_0 t), \quad (89)$$

where $C_d$ is a constant depending on $d$. Thus the oscillations of variance are persistent, their amplitude increases with time as $t^{1/2}$ and their frequency is $2\omega_0$. Since non-oscillating terms $V_2^c$ increase as $t^2$, the total variance of the packet has a quadratic time dependence with superimposed oscillations. This behavior is illustrated in Fig. 7. In our classical considerations we do not face the problem of negative variances that can occur for some quantum systems, see Refs. [24, 25]. For $t < 5t_c$ the oscillations have an irregular character because of the contribution of $V_2^{osc}$ term.
Estimating the characteristic frequency $2\omega_0$ for the flexible string attached to elastic substrate we have

$$\omega_0^2 = \frac{m^2c^4}{\hbar^2} = e^2 \times \frac{1}{\lambda_c^2} \leftrightarrow \left( \frac{T}{\rho} \right) \left( \frac{K}{T} \right) = \frac{K}{\rho}, \quad (90)$$

so that the analogue of the relativistic frequency $\omega_0$ does not depend on the applied tension. Taking a piano copper string of the bulk density $\rho_{3D} = 8940 \text{ kg/m}^3$ and having cross section of radius $r = 1 \text{ mm}$ one gets a linear density $\rho = \pi r^2 \rho_{3D} = 2.81 \times 10^{-2} \text{ kg/m}$. We identify the rubber elastic constant $K$ with the Young modulus $K = 0.05 \times 10^{10} \text{ N/m}^2$. Then the analogue of ZB frequency given in Eq. (90) is $2\omega_0^* = 8.44 \times 10^4 \text{ s}^{-1}$, i.e. the corresponding frequency is $f_0 = 13.43 \text{ kHz}$, which can be heard by the human ear. The characteristic time of ZB oscillations is $t_c^* = 1/\omega_0^* = 2.37 \times 10^{-5} \text{ s}$. Assuming the tension of the string $T = 1000 \text{ N}$ we find from Eq. (74) that the simulated Compton wavelength is $\lambda_c^* = 4.47 \text{ mm}$. The initial wave packet should have widths $d$ on the order of a few $\lambda_c^*$, i.e. of a few centimeters, and it will move with the velocity $u = 188.7 \text{ m/s}$, see Eq. (71). Thus, it is really possible to simulate and observe the Zitterbewegung phenomenon in this system. Finally, we observe that in classical simulations all the involved quantities are well defined observables. Since classical KGE does not reproduce but only simulates the quantum KGE, we are allowed to consider quantities which are not well defined in the quantum world.

**FIG. 7:** Calculated classical variance of position of wave packet propagating according to KGE. Dashed line: non-oscillating part of variance; solid line: total variance. For string oscillations analyzed in the text there is $\omega_0^* = 4.47 \text{ mm}$ and $t_c^* = 2.37 \times 10^{-5} \text{ s}$.

VI. DISCUSSION

Our main results for ZB of KG particles in absence of fields are shown in Fig. 1 and in the presence of a magnetic field in Figs. 3 - 5. It is not our purpose here to consider difficulties of the one-particle Klein-Gordon equation but we keep them in mind. In particular, we do not consider particle trajectories as they are believed to be not well defined, see [12]. On the other hand, we describe average particle velocities and currents both in the Hamiltonian and wave formalisms. The results can be compared to those for relativistic electrons in a vacuum described by the Dirac equation as well as for electrons in solids.

Similarly to the Dirac electrons, the ZB phenomenon of KG particles is due to the interference of positive and negative energy states. In the non-relativistic limit one of the two components progressively vanishes and the ZB contribution to the motion disappears. This can be clearly seen in Figs. 3 and 5 as well as in Fig. 3 of Ref. [17] for the Dirac electrons. If particles are described by wave packets the ZB motion decays in time, see our Fig. 1 for KE particles and Fig. 2 of Ref. [26] for the Dirac electrons. This is a general consequence of the Riemann-Lesbegues theorem, as indicated by Lock [14], calculated by the present authors [27] and experimentally confirmed by Gerritsma et al. [2]. In all cases the basic frequency of ZB oscillations is given by the energy difference between the positive and negative energy branches: $\hbar\omega_Z \simeq 2mc^2$ with the corresponding particle mass. The main difference with the Dirac electrons is the spin. For KG particles the interband ZB frequencies in a magnetic field do not include the spin energies, one does not deal with the Fermi sea for the negative energy branches, etc. The KG Hamiltonian is quadratic in momenta which does not allow a direct simulation with the use of Jaynes-Cummings interaction.

As to the Zitterbewegung of electrons in narrow-gap semiconductors and in particular in zero-gap monolayer graphene, one should emphasize that, although it is also described using a two-band model of band-structure [3], its physical nature is completely different from ZB of particles in a vacuum. The ZB in semiconductors or in graphene results from the electron motion in a periodic potential [28]. In zero-gap situation in graphene the ZB frequency is given by the difference of energies between positive and negative energy branches, etc. The KG Hamiltonian is quadratic in momenta which strongly resembles the KG particle in a vacuum is presented by electrons in carbon nanotubes: one can neglect the electron spin dealing with an energy gap controlled by the tube’s diameter [24]. The resulting ZB frequency and amplitude have values easily accessible experimentally. On the other hand, it is at present not clear how to follow dynamics of a single electron in a solid. As to KG particles in a vacuum, one is bound to recourse to simulations since the ZB frequency and amplitude as well as
field intensities necessary to see ZB effects in the presence of a magnetic field, exceed the present experimental possibilities.

We present a classical simulation of ZB by using a mechanical system and calculate the oscillating variance of position of the wave packet. The variance of position operator for the Dirac Hamiltonian was calculated by Barut and Malin [31] who found it to be the reminiscence of ZB of electrons in a vacuum. The present authors analyzed in Ref. [27] the variance of position operator in bilayer graphene and found its oscillating character with the frequency equal to that of ZB.

One should finally remark that the attempts are constantly made in the literature to overcome the above mentioned difficulties in the interpretation of position operator in KG equation. In particular, Mostafazadeh [31] proposed a redefinition of the scalar product of solutions to KGE which allows one to obtain positively defined probability distribution of position. Semenov et al. [27] proposed to limit the allowed solutions of KG equation to those having positive-definite probability distributions. They showed that the physical solutions of KGE fulfill this criterion. If the above attempts are accepted one could analyze ZB of the position operator for KG particles, see Eq. (9).

VII. SUMMARY

We considered the trembling motion (Zitterbewegung) of relativistic spin-zero particles in absence of fields and in the presence of a magnetic field using the Klein-Gordon equation. We aimed to describe physical observables (currents and velocities) calculating quantities averaged with the use of Gaussian wave packets. Surprisingly, the calculated particle velocities can exceed the velocity of light for sufficiently large momenta indicating that KGE does not posses an automatic restriction of relativity. We showed that the trembling motion has a decaying character resulting from an interference of positive and negative energy sub-packets moving with different velocities. In the presence of a magnetic field there exist many interband frequencies that contribute to Zitterbewegung. On the other hand, in the limit of non-relativistic energies the interband ZB components vanish while the intraband components reduce to the cyclotron motion with a single frequency. The trembling motion was simulated using the classical system obeying the Klein-Gordon equation – a stretched string attached to a rubber sheet. The calculated variance of position of the string shaped initially as a Gaussian packet exhibits oscillations corresponding to Zitterbewegung with the correct frequency.

Appendix A

In this Appendix we calculate an exact time dependence of current operators for a KG particle in a magnetic field. We define the creation and annihilation operators

\[ \hat{a} = (\xi + \partial / \partial \xi) / \sqrt{2}, \]

\[ \hat{a}^+ = (\xi - \partial / \partial \xi) / \sqrt{2}, \]

and rewrite Eq. (40) in the form

\[ \dot{H} = T \left[ \hbar \omega_c (\hat{a}^+ \hat{a} + 1 / 2) + \hbar^2 k^2 / 2m \right] + \tau_3 mc^2, \]

where \( \omega_c = qB/m \) is the cyclotron frequency and \( T = (\tau_3 + i\tau_2) \). The current density is

\[ j = \frac{\hbar}{2im} \left[ \psi^\dagger \tau_3 T \nabla \psi - (\nabla \psi^\dagger) \tau_3 T \psi \right] \]

\[ - \frac{e}{mc} A \psi^\dagger \tau_3 T \psi, \]

and the average current is \( \langle j \rangle = \int j \, d^3r \). We introduce the current operator \( \hat{J} \) in such way that for \( j \) given in Eq. (A3) there is

\[ \langle \psi | \tau_3 \hat{J} | \psi \rangle = \int j \, d^3r. \]

Note the presence of \( \tau_3 \) in the matrix element. In the asymmetric gauge one has

\[ \langle j_x \rangle = - \frac{i\hbar}{2m} \int \left( \psi^\dagger \tau_3 T \frac{\partial \psi}{\partial x} - \frac{\partial \psi^\dagger}{\partial x} \tau_3 T \psi \right) \, d^3r \]

\[ - \frac{qB}{mc} \int \left( \psi^\dagger \tau_3 T \psi \right) \, d^3r, \]

\[ \langle j_y \rangle = - \frac{i\hbar}{2m} \int \left( \psi^\dagger \tau_3 T \frac{\partial \psi}{\partial y} - \frac{\partial \psi^\dagger}{\partial y} \tau_3 T \psi \right) \, d^3r. \]

Below we assume the function \( \psi \) to be Gaussian-like. In that case we may simplify the above expressions for the average current by integrating by parts the terms including derivatives of \( \psi^\dagger \)

\[ \langle j_x \rangle = - \frac{i\hbar}{m} \int \left( \psi^\dagger \tau_3 T \frac{\partial \psi}{\partial x} \right) \, d^3r - \frac{qB}{mc} \int \left( \psi^\dagger \tau_3 T \psi \right) \, d^3r, \]

\[ \langle j_y \rangle = - \frac{i\hbar}{m} \int \left( \psi^\dagger \tau_3 T \frac{\partial \psi}{\partial y} \right) \, d^3r, \]

so the components of the current operator are

\[ \hat{J}_x = - \frac{i\hbar}{m} \frac{T}{\partial x} \frac{\partial}{\partial x} - \frac{qB}{mc} \hat{y}, \]

\[ \hat{J}_y = - \frac{i\hbar}{m} \frac{T}{\partial y} \frac{\partial}{\partial y}. \]

In the Heisenberg picture the time-dependent current operator is

\[ \hat{J}(t) = e^{i\hat{H}t / \hbar} \hat{J}(0) e^{-i\hat{H}t / \hbar}, \]

where \( \hat{H} \) is given in Eq. (A2). Our task is to calculate the time evolution of the current operator \( \hat{J}_x(t) \) and \( \hat{J}_y(t) \).
By averaging these operators over the state $\psi$, as shown in Eq. (A11), one obtains the time-dependent current corresponding to $\psi$.

It is convenient to rewrite current operators in Eqs. (A9) and (A10) in the form

$$\hat{J}_x = -\frac{i\hbar}{m}\partial_x + \frac{qB}{\sqrt{2mc}}(\hat{J} + \hat{J}^+),$$  \hspace{1cm} (A12)
$$\hat{J}_y = -\frac{i\hbar}{\sqrt{2m}}(\hat{J} - \hat{J}^+),$$  \hspace{1cm} (A13)

where we introduce three auxiliary operators:

$$\hat{P} = (\tau_3 + i\tau_2)\frac{\partial}{\partial x} \equiv \tau \frac{\partial}{\partial x},$$  \hspace{1cm} (A14)
$$\hat{J} = (\tau_3 + i\tau_2)\hat{a} \equiv \tau \hat{a},$$  \hspace{1cm} (A15)
$$\hat{J}^+ = (\tau_3 + i\tau_2)\hat{a}^+ \equiv \tau \hat{a}^+. $$  \hspace{1cm} (A16)

We calculate the time dependence of $\hat{J}$, $\hat{J}^+$ and $\hat{P}$ in a way similar to that described in Ref. [11]. Consider first the operator $\hat{P}$. From the equation of motion $\hat{P}_t = (i/\hbar)[\hat{H}, \hat{P}]$ one has

$$\hat{P}_t = \frac{imc^2}{\hbar}[\tau_3, \hat{P}] = 2i\hbar\tau_1 \frac{\partial}{\partial x}, \hspace{1cm} (A17)$$

where we used $\hat{T}^2 = 0$. Since $[\hat{H}, \hat{P}_t] = 0$, there is $[\hat{H}, \hat{P}_t] = 2\hat{H}\hat{P}_t - [\hat{H}, \hat{P}_t] = 2\hat{H}\hat{P}_t$, and one obtains

$$\hat{P}_{tt} = \frac{2i}{\hbar}\hat{H}\hat{P}_t. $$  \hspace{1cm} (A18)

We solve this equation for $\hat{P}_t$ and then integrate the solution over time

$$\hat{P}(t) = \frac{\hbar}{2i\hbar} e^{2i\hat{H}/\hbar} \hat{P}_t(0) + \hat{C},$$ \hspace{1cm} (A19)

where $\hat{C}$ is a constant of integration. Applying the initial conditions: $\hat{P}(0) = \hat{T}(\partial/\partial x)$, $\hat{P}_t(0) = 2i\hbar\tau_1(\partial/\partial x)$, and using the identity $H^{-1} = \hat{T}/E^2$ we have

$$\hat{P}(t) = \hat{T} \frac{\partial}{\partial x} + \frac{\hbar\omega_0}{E^2} [e^{2i\hat{H}/\hbar} - 1] \tau_1 \frac{\partial}{\partial x}. $$  \hspace{1cm} (A20)

It is seen that $\hat{P}(t)$ in Eq. (A20) satisfies the initial conditions for $\hat{P}(0)$ and $\hat{P}_t(0)$. The form of $\hat{P}(t)$ given above resembles results obtained for the position operator in the field-free case by Fuda and Furlani [9].

Now we turn to the operators $\hat{J}$ and $\hat{J}^+$. From Eqs. (A15) and (A16) one has

$$\hat{J}_t = 2i\omega_0\tau_1 \hat{a}, $$  \hspace{1cm} (A21)
$$\hat{J}^+_t = 2i\omega_0\tau_1 \hat{a}^+, $$  \hspace{1cm} (A22)

where $\omega_0 = mc^2/\hbar$. Using $[\hat{a}, \hat{a}^+] = 1$ one obtains

$$\{\hat{H}, \hat{J}_t\} = -2i\omega_0\hbar\omega_c \hat{J},$$  \hspace{1cm} (A23)
$$\{\hat{H}, \hat{J}^+_t\} = +2i\omega_0\hbar\omega_c \hat{J}^+. $$  \hspace{1cm} (A24)

Upon applying the identities

$$[\hat{H}, \hat{J}_t] = +2\hbar \hat{J}_t - \{\hat{H}, \hat{J}_t\}, $$  \hspace{1cm} (A25)
$$[\hat{H}, \hat{J}^+_t] = -2\hbar \hat{J}^+_t + \{\hat{H}, \hat{J}^+_t\}, $$  \hspace{1cm} (A26)

we get

$$\hat{J}_{tt} = (2i/\hbar)\hat{H}\hat{J}_t - 2\omega_0\omega_c \hat{J}, $$  \hspace{1cm} (A27)
$$\hat{J}^+_{tt} = -(2i/\hbar)\hat{H}\hat{J}^+_t - 2\omega_0\omega_c \hat{J}^+. $$  \hspace{1cm} (A28)

In Eqs. (A27) and (A28) we eliminate terms with the first derivatives using the substitutions $\hat{J} = \exp(+i\hat{H}t/\hbar)\hat{B}$ and $\hat{J}^+ = R^+ \exp(-i\hat{H}t/\hbar)$, respectively. This gives

$$\hat{B}_{tt} = -\hbar^2 \hat{R}_t \hat{B} + 2\omega_0 \hbar \omega_c \hat{B}, $$  \hspace{1cm} (A29)
$$\hat{B}^+_{tt} = -\hbar^2 \hat{R}^+_t \hat{B}^+ + 2\omega_0 \hbar \omega_c \hat{B}^+, $$  \hspace{1cm} (A30)

where $\hat{B} = \hat{\Omega} = \hat{H}/\hbar$. In the above equations the operator

$$\hat{M}^2 = \hat{\Omega}^2 + 2\omega_0 \omega_c $$  \hspace{1cm} (A31)

stands on the left-hand side of $\hat{B}$, but on the right-hand side of $\hat{B}^+$. Solutions to Eqs. (A29) and (A30) are

$$\hat{B} = e^{-i\hat{M}t}\hat{C}_1 + e^{i\hat{M}t}\hat{C}_2, $$  \hspace{1cm} (A32)
$$\hat{B}^+ = \hat{C}_1^+ e^{-i\hat{M}t} + \hat{C}_2^+ e^{i\hat{M}t}, $$  \hspace{1cm} (A33)

where $\hat{M} = +\sqrt{\hat{M}^2}$ is the positive root of $\hat{M}^2$. Both $\hat{C}_1$ and $\hat{C}_2$ and their complex conjugates are time-independent operators.

Using the initial conditions: $\hat{B}(0) = \hat{J}(0) = \hat{T}\hat{a}$ and $\hat{B}_t(0) = \hat{J}_t(0) = 2i\hbar\omega_0\tau_1\hat{a}$ [see Eq. (A21)] and similar expressions for $\hat{B}^+(0)$ and $\hat{B}^+_t(0)$, we find that $\hat{J}(t) = \hat{J}_1(t) + \hat{J}_2(t)$, where

$$\hat{J}_1(t) = \frac{1}{2} e^{i\hat{M}t} e^{-i\hat{M}t} \left[ \hat{J}(0) + \hat{M}^{-1} \hat{J}(0) \hat{\Omega} \right], $$  \hspace{1cm} (A34)
$$\hat{J}_2(t) = \frac{1}{2} e^{i\hat{M}t} e^{i\hat{M}t} \left[ \hat{J}(0) - \hat{M}^{-1} \hat{J}(0) \hat{\Omega} \right]. $$  \hspace{1cm} (A35)

Similarly, one can express $\hat{J}^+(t) = \hat{J}^+_1(t) + \hat{J}^+_2(t)$, where

$$\hat{J}^+_1(t) = \frac{1}{2} \left[ \hat{J}^+(0) + \hat{\Omega} \hat{J}^+(0) \hat{M}^{-1} \right] e^{i\hat{M}t} e^{-i\hat{M}t}, $$  \hspace{1cm} (A36)
$$\hat{J}^+_2(t) = \frac{1}{2} \left[ \hat{J}^+(0) - \hat{\Omega} \hat{J}^+(0) \hat{M}^{-1} \right] e^{i\hat{M}t} e^{-i\hat{M}t}. $$  \hspace{1cm} (A37)

The results are given in terms of the operators $\hat{\Omega}$ and $\hat{M}$. To finalize the description, one needs to specify the physical sense of functions appearing in Eqs. (A33) - (A37).

For a reasonable function $f(D)$ of an operator $D$, its eigenenergies $\lambda_\alpha$ and its eigenstates $|d\rangle$, there exists the following relationship: $f(D)|d\rangle = f(\lambda_\alpha)|d\rangle$, provided that $f(\lambda_\alpha)$ exists. To find the meanings of the operators $M^{-1}$ and $e^{\pm i\lambda_\alpha M T}$ we express them as functions of the operator $\hat{M}^2 = \hat{H}^2/\hbar^2 + 2\omega_0 \omega_c \hat{C}$, see Eq. (A31). From the definition of $\hat{M}^2$ it follows that its eigenstates are
equal to the eigenstates \(|n\rangle\) of \(\hat{H}\). The eigenvalues \(\lambda_n^2\) of the operator \(\hat{M}^2\) are 
\[
\lambda_n^2 = E_{n+1,k}\kappa \quad \text{and we obtain}
\]
\[
\hat{M}^\pm_{n}\langle n\rangle = (\hat{M}^2)^{1/2}\langle n\rangle = \eta E_{n+1,k}\kappa |n\rangle, \quad (A38)
\]
\[
e^{\pm i\hat{M}t}|n\rangle = e^{\pm i(\hat{M}^2)^{1/2}t}|n\rangle = e^{\pm i\eta E_{n+1,k}\kappa}|n\rangle, (A39)
\]
where \(\eta = +1\) or \(\eta = -1\). As seen from Eqs. (A34) - (A37), the sums \(\hat{J}_1(t) + \hat{J}_2(t)\) and \(\hat{J}_1^+(t) + \hat{J}_2^+(t)\) do not depend on the sign of \(\eta\), so we select \(\eta = +1\).

Finally we show that the matrix elements of the operator \(\hat{J}(t) = \hat{J}_1(t) + \hat{J}_2(t)\) are equal to the matrix elements of the current operator \(\hat{J}(t) = e^{\alpha it}\hat{J}(0)e^{-\alpha it}\) in the Heisenberg picture. The operator \(\hat{J}\) is proportional to the annihilation operator \(\hat{a}\) whose non-vanishing matrix elements are \(\langle n' | \hat{a}| n \rangle = \sqrt{n + 1} \delta_{n', n+1}, \) so we select two eigenstates of KG Hamiltonian \(|n\rangle = \{|n, s\rangle\}\) and \(|n'\rangle = |n + 1, z\rangle\), see Eq. (11). Here we omitted quantum numbers \(k_s\) and \(k_z\). For \(\hat{J}(t)\) one has
\[
\langle n|\hat{J}_3\hat{J}_H(t)|n\rangle = e^{i\omega_n t} e^{-i\omega_{n+1} t} \hat{J}(0)_{nm}, \quad (A40)
\]
where we define \(\hat{J}(0)_{nm} = \langle n|\hat{J}_3\hat{J}(0)|n'\rangle\). To calculate the matrix elements of \(\hat{J}_1(t)\) we use Eqs. (A38) - (A39) and obtain
\[
\langle n|\hat{M}^{-1}\hat{J}(0)\hat{J}_1|n\rangle = \frac{\hbar}{E_{n+1}} \hat{J}(0)_{nn} \frac{zE_{n+1}}{\hbar} = z\hat{J}(0)_{nn}, \quad (A41)
\]
which finally gives
\[
\langle n|\hat{J}_1(t)|n\rangle = \frac{1 + z}{2} \hat{J}(0)_{nn} e^{i\omega_n t} e^{-i\omega_{n+1} t}, \quad (A42)
\]
\[
\langle n|\hat{J}_2(t)|n\rangle = \frac{1 - z}{2} \hat{J}(0)_{nn} e^{i\omega_n t} e^{+i\omega_{n+1} t}. \quad (A43)
\]

The matrix elements of \(\hat{J}_1(t)\) are nonzero for \(z = +1\) only, while the matrix elements of \(\hat{J}_2(t)\) are nonzero for \(z = -1\) only. Comparing Eqs. (A42) and (A43) with Eq. (A40) we see that for each of four combinations of \(s = \pm 1\) and \(z = \pm 1\) the matrix elements of \(\hat{J}_H(t)\) are equal to the matrix elements of \(\hat{J}(t) = \hat{J}_1(t) + \hat{J}_2(t)\), which is what we wanted to show. Calculations for \(\hat{J}^+(t)\) are similar to those for \(\hat{J}(t)\). The compact equations (A34) - (A37) are our final results for the time dependence of \(\hat{J}_1(t)\) and \(\hat{J}^+(t)\) operators. These equations are exact and they are quite fundamental for relativistic spin-0 particles in a magnetic field. If we calculate average currents of Eqs. (A12) and (A13) with the use of expressions (A12) - (A13) and the wave packet (11), one obtains results corresponding to the velocities given in Section IV.

Appendix B

In this Appendix we analyze in more detail the relation of the particle velocity to the speed of light. We consider (1, 1) component of the velocity operator for a KG particle given in Eq. (7). For the wave packet \(|\hat{v}_{11}\rangle|\rho\rangle = w(r) (1, 0)^T\) with one nonzero component the average velocity is given by the average of \(\langle \hat{v}_{11}\rangle\) over the function \(w(r)\). The unexpected feature of operator \(\langle \hat{v}_{11}\rangle\) is that for large \(p\) this velocity can exceed the speed of light c.

There are two possible ways to overcome this problem. We can additionally assume that \(|p| \leq mc\), which ensures that the velocity \(\langle \hat{v}_{11}\rangle\) does not exceed \(c\). This condition is equivalent to \(|q| \leq 1\) in the text, see Eq. (7). Alternatively, one can take the initial wave packet \(w(r)\) which does not contain components with \(|p| > mc\). Then the Gaussian packet in Eq. (12) must be replaced by a non-Gaussian packet \(w'(r)\) of the form
\[
\langle k|w'\rangle = (2d\sqrt{\pi})^{3/2} \exp[-d^2 (k - k_0)^2/2], \quad (B1)
\]
where \(\Theta(\xi)\) is the step function.

For the Dirac Hamiltonian \(\hat{H}_D = c \sum_j \hat{a}_j \hat{p}_j + mc^2 \hat{\beta}\), the situation is different. Expanding \(e^{i\hat{H}_D t/\hbar}\) in a power series one obtains an expression analogous to \(e^{i\hat{H}t/\hbar}\) given in Eq. (6). After some algebra we find
\[
\langle \hat{v}_{11}\rangle|\rho\rangle = \frac{mc^2 p_z}{mc^2 c^2 + p^2} \left[ 1 - \cos(2Et/\hbar) \right]. \quad (B2)
\]
In contrast to the KG case, the velocity operator given in Eq. (12) has correct relativistic behavior for all values of \(p\). In Eq. (B2) the expression in square brackets oscillates between zero and two. The factor \(1/\sqrt{c^2 + p^2}\) tends to zero for both large and small values of \(p\). Its maximum is at \(p_{\text{max}} = (0, 0, mc)\) for which one obtains
\[
\langle \hat{v}_{11}\rangle|\rho\rangle = \frac{c}{2} \left[ 1 - \cos(2Et/\hbar) \right]. \quad (B3)
\]
The above velocity never exceeds the speed of light. Therefore, when calculating the average velocity of the wave packet for the Dirac Hamiltonian, there is no need for an artificial truncation of the high momentum components of the wave packet, as proposed in Eq. (5) for a KG particle.

Appendix C

We prove here some identities appearing in the previous sections. We begin with the identity in Eq. (50). Closing Eq. (50) with the use of states \(|r\rangle\) and \(|r'\rangle\), employing Eq. (14) and writing explicitly the summations and integrations over the quantum numbers we obtain
\[
\delta_{r,r'} = \sum_n \langle r|n\rangle \langle n|r'\rangle s_n t_{r} = \frac{1}{16\pi^2} \sum_{n=0}^{\infty} \phi_n(\xi) \phi_n(\xi') \times \int_{-\infty}^{\infty} e^{i k_s (x - x')} dk_s \int_{-\infty}^{\infty} e^{i k_s (z - z')} dk_z \times \sum_{s=\pm 1} \mu_s \mu_s' s_{r} t_{s}, \quad (C1)
\]
where \( \mu^\pm = \mu^\pm_{n,k_x,s} \). In the above equation the summation over \( n \) gives \( \delta_{E,E'} \), the product of the two integrals is \( 4\pi^2\delta_{x,x'}\delta_{z,z'} \), so the product of the three terms equals \( 4\pi^2\delta_{\nu,\nu'} \). Taking the explicit form of \( \mu^\pm = \nu \pm s/\nu \) where \( \nu = \sqrt{mc^2/E_{n,k_x}} \), we obtain for the last line of Eq. (C1)

\[
\sum_{s=\pm 1} \left( s(\nu + s/\nu)^2 s(\nu^2 - 1/\nu^2) s(\nu - s/\nu)^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 4.
\]

(C2)

Collecting all numerical factors we see that the right hand side of Eq. (C1) equals \( \delta_{\nu,\nu'} \).

Next we prove the identity used in the derivation of Eqs. (19) and (53). Let \( \hat{O} \) be any operator for which \( \hat{O} = \tau_3 \hat{O}^\dagger \tau_3 \), where dagger signifies the Hermitian conjugate. We want to show that

\[
\tau_3 e^\hat{O} = e^{\hat{O}^\dagger} \tau_3.
\]

To this end we expand the exponents and get

\[
\tau_3 \left[ 1 + \frac{\hat{O}}{1!} + \frac{\hat{O}^2}{2!} \ldots \right] = \left[ 1 + \frac{\hat{O}^\dagger}{1!} + \frac{\hat{O}^{2\dagger}}{2!} \ldots \right] \tau_3.
\]

(C4)

Since \( \hat{O} = \tau_3 \hat{O}^\dagger \tau_3 \), there is \( \hat{O}^\dagger = \tau_3 \hat{O}^\dagger \tau_3 \). Then for \( n \geq 0 \) there is \( \hat{O}^n = \tau_3 \hat{O}^n \tau_3 \) and we obtain for the RHS of Eq. (C4)

\[
\left[ 1 + \frac{\hat{O}^\dagger}{1!} + \frac{\hat{O}^{2\dagger}}{2!} \ldots \right] \tau_3 = \left[ 1 + \frac{\hat{O}}{1!} + \frac{\hat{O}^2}{2!} \tau_3 \ldots \right] \tau_3
\]

\[
= \tau_3 \left[ 1 + \frac{\hat{O}}{1!} + \frac{\hat{O}^2}{2!} \ldots \right] = \tau_3 e^\hat{O},
\]

which is the desired result.

**Appendix D**

In this appendix we quote for completeness all formulas necessary for a calculation of coefficients \( U_{m,n} \) in Eqs. (63) - (69). Here we assume the initial wave vector in the form \( k_0 = (k_{0x}, 0, k_{0z}) \). Using the definitions of \( g_{xy}(k_x, y) \), \( F_n(k_x) \) and \( U_{m,n} \), we obtain (see Ref. [17])

\[
g_{xy}(k_x, y) = \sum_{x'} \frac{d_x}{\pi d_y} e^{-\frac{x'^2}{2}} e^{-\frac{y^2}{2}} e^{-\frac{(k_x - k_{0x})^2}{2}},
\]

(D1)

and

\[
F_n(k_x) = \frac{A_n \sqrt{Ld_y}}{\sqrt{2\pi d_x} C_n} e^{-\frac{y^2}{2}} e^{-\frac{(k_x - k_{0x})^2}{2}} \frac{H_n(-k_x c)}{n/2},
\]

(D2)

where \( D = L^2/\sqrt{L^2 + d_y^2} \), \( c = L^3/\sqrt{L^4 - d_y^2} \), and

\[
A_n = \frac{\sqrt{2\pi d_y}}{\sqrt{L^2 + d_y^2}} \frac{L^2 - d_y^2}{L^2 + d_y^2} \sum_{m,n} \frac{Q_m}{\sqrt{C_m C_n d}}
\]

\[
\times (1 - (cQ)^2)^{(m+n-2l)/2} H_m + n - 2l \left( \frac{-cQY}{\sqrt{1 - (cQ)^2}} \right),
\]

(D4)

in which \( Q = 1/\sqrt{d_x^2 + D^2} \), \( W = d_y DQ \), \( k_{0x} \), and \( Y = d_z k_{0z} Q \). For the special case of \( d_y = L \), the formula for \( U_{m,n} \) is much simpler:

\[
U_{m,n} = \frac{2\sqrt{\pi} (-i)^{m+n} d_x}{C_m C_n L} \frac{L}{2P} \frac{m+n+1}{P} \times \exp \left( \frac{-d_x^2 k_{0x} L^2}{2P^2} \right) H_{m+n} \left( \frac{-i d_z k_{0z}}{P} \right),
\]

(D5)

where \( P = \sqrt{d_x^2 + \frac{1}{4}L^2} \). In the above expressions the coefficients \( U_{m,n} \) are real numbers and they are symmetric in \( m, n \) indices. For further discussion of of \( U_{m,n} \) see Refs. [17, 32].

[1] E. Schrödinger, Sitzungsber., Preuss. Akad. Wiss. Phys. Math. Kl. 24, 418 (1930). Schrödinger’s derivation is reproduced in A. O. Barut and A. J. Bracken, Phys. Rev. D 23, 2454 (1981).

[2] R. Gerritsma, G. Kirchmair, F. Zahringer, E. Solano, R. Blatt, and C. F. Roos, Nature 463, 68 (2010).

[3] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, Science 306, 666 (2004).

[4] W. Zawadzki and T. M. Rusin, J. Phys. Cond. Matt. 23, 143201 (2011).

[5] O. Klein, Z. Physik 37, 895 (1926).

[6] W. Gordon, Z. Physik 40, 117 (1926).

[7] V. Fock, Z. Physik 38, 242 (1926) and 39, 226 (1926).

[8] M. G. Fuda and E. Furlani, Am. J. Phys. 50, 545 (1982).

[9] W. Greiner Relativistic Quantum Mechanics (Springer, Berlin, 1994).

[10] A. Wachter Relativistic Quantum Mechanics (Springer, Berlin, 2010).

[11] P. M. Morse and H. Feshbach Methods of Theoretical Physics (McGraw-Hill, New York, 1953).

[12] S. S. Schweber An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Co., Evanston, 1961).

[13] H. Feshbach and F. Villars, Rev. Mod. Phys. 30, 24 (1958).

[14] J. A. Lock, Am. J. Phys. 47, 797 (1979).
[15] V. Radovanovic *Problem Book in Quantum Field Theory* (Springer, Berlin, 2008).

[16] N. S. Witte, R. L. Dawe, and K. C. Hines, J. Math. Phys. 28, 1864 (1987).

[17] T. M. Rusin and W Zawadzki, Phys. Rev. D 82, 125031 (2010).

[18] B. J. Forbes, E. R. Pike and D. B. Sharp, J. Acoust. Soc. Am. 114, 1291 (2003).

[19] B. J. Forbes and E. R. Pike, Phys. Rev. Lett. 93, 054301 (2004).

[20] J. D. Jackson *Classical Electrodynamics* (Wiley, New York, 1999).

[21] S. V. Tsynkov, SIAM J. Imaging Sciences 2, 140 (2009).

[22] W. Geyi *Foundations of Applied Electrodynamics* (Wiley, New York, 2010).

[23] D. Wurmser, G. J. Orris and R. Duskena, J. Acoust. Soc. Am. 101, 1309 (1997).

[24] B. I. Lev, A. A. Semenov, C. V. Usenko, and J. R. Klauder, Phys. Rev. A 66, 022115 (2002).

[25] A. A. Semenov, C. V. Usenko and B. I. Lev, Phys. Lett. A 372, 4180 (2008).

[26] M. Merkl, F. E. Zimmer, G. Juzeliunas, and P. Ohberg, EPL 83, 54002 (2008).

[27] T. M. Rusin and W. Zawadzki, Phys. Rev. B 76, 195439 (2007).

[28] W. Zawadzki and T. M. Rusin, Phys. Lett. A 374, 3533 (2010).

[29] W. Zawadzki, Phys. Rev. B 74, 205439 (2006).

[30] A. O. Barut and S. Malin, Rev. Mod. Phys. 40, 632 (1968).

[31] A. Mostafazadeh, Ann. Phys. (N.Y.) 309, 1 (2004).

[32] T. M. Rusin and W. Zawadzki, Phys. Rev. B 78, 125419 (2008).