On the connectedness of spectral sets and irreducibility of spectral cones in Euclidean Jordan algebras

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Abstract

Let $V$ be a Euclidean Jordan algebra of rank $n$. A set $E$ in $V$ is said to be a spectral set if there exists a permutation invariant set $Q$ in $\mathbb{R}^n$ such that $E = \lambda^{-1}(Q)$, where $\lambda : V \to \mathbb{R}^n$ is the eigenvalue map that takes $x \in V$ to $\lambda(x)$ (the vector of eigenvalues of $x$ written in the decreasing order). If the above $Q$ is also a convex cone, we say that $E$ is a spectral cone. This paper deals with connectedness and arcwise connectedness properties of spectral sets. By relying on the result that in a simple Euclidean Jordan algebra, every eigenvalue orbit $[x] := \{y : \lambda(y) = \lambda(x)\}$ is arcwise connected, we show that if a permutation invariant set $Q$ is connected (arcwise connected), then $\lambda^{-1}(Q)$ is connected (respectively, arcwise connected). A related result is that in a simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.

Key Words: Euclidean Jordan algebra, spectral set, connectedness, irreducible cone

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1 Introduction

Let $V$ be a Euclidean Jordan algebra of rank $n$. The eigenvalue map $\lambda : V \to \mathbb{R}^n$ takes $x \in V$ to $\lambda(x)$, the vector of eigenvalues of $x$ written in the decreasing order. A set $E$ in $V$ is said to be a spectral set if there exists a permutation invariant set $Q$ in $\mathbb{R}^n$ such that $E = \lambda^{-1}(Q)$. If the above $Q$ is also a convex cone, we say that $E$ is a spectral cone. A function $F : V \to \mathbb{R}$ is said to be a spectral function if it is of the form $F = f \circ \lambda$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a permutation invariant function.

The above concepts are generalizations of similar concepts that have been studied in the settings of Euclidean $n$-space $\mathbb{R}^n$ and $S^n(\mathbb{H}^n)$, the space of all $n \times n$ real (respectively, complex) Hermitian matrices, see for example, [2], [3], [4], [5], [12], [13], [17], [18], [19], and the references therein. In the case of $\mathbb{R}^n$, spectral sets/cones/functions are related to permutation invariance, and in $S^n(\mathbb{H}^n)$ they are precisely those that are invariant under linear transformations of the form $X \to UXU^*$, where $U \in \mathbb{R}^{n \times n}$ is an orthogonal (respectively, unitary) matrix. In the general setting of Euclidean Jordan algebras, they have been studied in several works, see [1], [9], [14], [15], [16], [20], [23], and [24].

Focusing on topological/convexity/linearity properties of spectral sets/cones, in two recent papers Jeong and Gowda [15], [16] show that the multivalued map $\lambda^{-1}$ from $\mathbb{R}^n$ to $V$ behaves like a linear isomorphism on certain types of permutation invariant sets. Specifically, the following results are shown (where part of the first result is due to Baes [1]):

**Proposition 1.1** Let $Q$, $Q_1$, and $Q_2$ be permutation invariant sets in $\mathbb{R}^n$ with $Q_1$ and $Q_2$ convex. Let $\alpha \in \mathbb{R}$. Then

(a) $\lambda^{-1}(Q)$ is open/closed/compact/convex/cone in $V$ if and only if $Q$ is so in $\mathbb{R}^n$. Moreover, $\lambda^{-1}(Q^\#) = [\lambda^{-1}(Q)]^\#$, where $\#$ denotes any operation of taking closure, interior, boundary, or convex/conic hull.

(b) $\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$ and $\lambda^{-1}(\alpha Q_1) = \alpha \lambda^{-1}(Q_1)$.

(c) When $Q$ is a convex cone, $\lambda^{-1}(Q)$ is pointed if and only if $Q$ is pointed.

Motivated by these results, we ask if connectedness and arcwise (=pathwise) connected properties of permutation invariant $Q$ carry over to $\lambda^{-1}(Q)$. In this paper, we answer these affirmatively by relying on the result that in a simple Euclidean Jordan algebra, for any element $x$, the eigenvalue orbit $[x] := \{y : \lambda(y) = \lambda(x)\}$ is arcwise connected. This result will also be used to show that in any simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.
2 Preliminaries

We let $\mathcal{R}^n$ denote the Euclidean $n$-space (where the vectors are regarded as column vectors or row vectors depending on the context). In $\mathcal{R}^n$, we denote the standard coordinate vectors by $v_1, v_2, \ldots, v_n$, where $v_i$ is the vector with one in the $i$th slot and zeros elsewhere. We use the notation $\Sigma_n$ to denote the set of all $n \times n$ permutation matrices. For any set $S$ in $\mathcal{R}^n$, we let $\Sigma_n(S) := \{ \sigma(s) : \sigma \in \Sigma_n, s \in S \}$. For any vector $v \in \mathcal{R}^n$ and a set $Q \subseteq \mathcal{R}^n$, $v^\downarrow$ denotes the decreasing rearrangement of $v$ (i.e., the entries of $v^\downarrow$ satisfy $q_1^\downarrow \geq q_2^\downarrow \geq \cdots \geq q_n^\downarrow$) and $Q^\downarrow := \{ q^\downarrow : q \in Q \}$.

A non-empty set $Q$ in $\mathcal{R}^n$ is said to be permutation invariant if $\sigma(Q) = Q$ for all $\sigma \in \Sigma_n$.

We assume that the reader is familiar with standard topological notions/results dealing with (ar-cwise) connectedness, components, etc. Recall that a set $S$ (say, in a topological space) is connected if it is not the union of two nonempty separated sets (where separation takes the form $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, with the ‘overline’ indicating the closure) [21]. The set $S$ is arcwise (= pathwise) connected if any two points of $S$ can be joined by a continuous arc (= continuous image of an interval in $\mathcal{R}$) that lies inside $S$. Connected (arcwise connected) components of a set $S$ are maximal connected (respectively, arcwise connected) sets in $S$.

For basic things related to Euclidean Jordan algebras, we refer to [8] or [16] for a summary. Throughout this paper, $\mathcal{V}$ denotes a Euclidean Jordan algebra of rank $n$. Recall that a Euclidean Jordan algebra $\mathcal{V}$ is simple if it is not a direct product of nonzero Euclidean Jordan algebras (or equivalently, if it does not contain any non-trivial ideal). It is known (see [8], Prop. III.4.4) that any nonzero Euclidean Jordan algebra is, in a unique way, a direct product/sum of simple Euclidean Jordan algebras. Moreover, there are (in the isomorphic sense) only five simple algebras, two of which are: $\mathcal{S}^n$, the algebra of $n \times n$ real symmetric matrices, and $\mathcal{H}^n$, the algebra of $n \times n$ complex Hermitian matrices. The other three are: $n \times n$ quaternion Hermitian matrices, $3 \times 3$ octonion Hermitian matrices, and the Jordan spin algebra. In $\mathcal{V}$, each element $x$ has a spectral decomposition: $x = q_1e_1 + q_2e_2 + \cdots + q_ne_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$ and the real numbers $q_1, q_2, \ldots, q_n$ are (called) the eigenvalues of $x$. Then $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ denotes the vector of eigenvalues of $x$ written in the decreasing order. We note that

$$\lambda : \mathcal{V} \to \mathcal{R}^n$$

is continuous ([1], Corollary 24) and $\lambda(\mathcal{V}) = (\mathcal{R}^n)^{\downarrow}$.

For any $x \in \mathcal{V}$, we let

$$[x] := \{ y \in \mathcal{V} : \lambda(y) = \lambda(x) \}$$

denote the ‘eigenvalue orbit’ of $x$. (The notation $[x]_\mathcal{V}$ will be used when more than one algebra is involved.) Suppose $\mathcal{V}$ is a Cartesian product (or a direct sum) $\mathcal{V} = \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \cdots \times \mathcal{V}^{(N)}$, where each $\mathcal{V}^{(i)}$ is simple. It is easy to see that in $\mathcal{V}$, an element $c$ is a primitive idempotent if and only if it is of the form $(0, 0, \ldots, 0, c_i, 0, \ldots, 0)$, where $c_i$ is a primitive idempotent in $\mathcal{V}^{(i)}$ for some $i$. Applying this to elements of a Jordan frame in $\mathcal{V}$, we see that the eigenvalues of any
Given a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$, we assume that its listing/enumeration is fixed and define, for any $q = (q_1, q_2, \ldots, q_n) \in \mathcal{R}^n$,

$$q \ast \mathcal{E} := q_1e_1 + q_2e_2 + \cdots + q_ne_n.$$ 

We note that

$$\lambda(q \ast \mathcal{E}) = q^T$$

and when $\mathcal{E}$ is fixed, $\Theta : q \mapsto q \ast \mathcal{E}$ is a continuous map from $\mathcal{R}^n$ to $\mathcal{V}$.

Recall that a nonempty set in $\mathcal{V}$ is a spectral set if it is of the form $\lambda^{-1}(Q)$ for some permutation invariant set $Q$ in $\mathcal{R}^n$. We will freely use the results in the following (easily verifiable) proposition.

**Proposition 2.1** Let $P$ and $Q$ be nonempty subsets of $\mathcal{R}^n$ with $Q$ permutation invariant. Then,

- $Q = \Sigma_n(Q^\perp)$.
- $x \in \lambda^{-1}(Q)$ if and only if $x = q \ast \mathcal{E}$ for some Jordan frame $\mathcal{E}$ and $q \in Q$.
- $\lambda(\lambda^{-1}(P)) = P \cap P^\perp$ and $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\perp)$. In particular, for any $\Omega \subseteq (\mathcal{R}^n)^\perp$, $\lambda(\lambda^{-1}(\Omega)) = \Omega$ and $\lambda^{-1}(Q) = \lambda^{-1}(Q^\perp)$.
- Spectral sets can be generated by taking $\lambda$–inverse images of subsets of $(\mathcal{R}^n)^\perp$. In fact, any set of the form $\lambda^{-1}(P)$ is a spectral set.
- The correspondence $\Omega \mapsto \lambda^{-1}(\Omega)$ is one-to-one and onto between nonempty subsets of $\mathcal{R}^n$ and spectral sets in $\mathcal{V}$.
- For a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$, the set $\Theta(Q) = \{q \ast \mathcal{E} : q \in Q\}$ is independent of the listing of elements in $\mathcal{E}$.
- $\lambda^{-1}(Q)$ is the union of sets of the form $\Theta(Q)$ as $\mathcal{E}$ varies over all Jordan frames.

Recall that an automorphism $\phi$ of $\mathcal{V}$ is an invertible linear transformation on $\mathcal{V}$ that satisfies the condition $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in \mathcal{V}$. The set of all such transformations is denoted by $\text{Aut}(\mathcal{V})$ and we let $G$ denote the connected component of the identity transformation (henceforth called the component of identity) in $\text{Aut}(\mathcal{V})$. For example, $\text{Aut}(S^n)$ consists of transformations of the form $\phi(X) := UUX^T$ ($X \in S^n$) where $U \in \mathcal{R}^{n \times n}$ is an orthogonal matrix. When $U$ has deter-
minant one, such a \( \phi \) belongs to the corresponding \( G \). Also, \( \text{Aut}(\mathcal{H}^n) \) consists of transformations of the form \( \phi(X) := UXU^* \) \( (X \in \mathcal{H}^n) \), where \( U \) is a unitary matrix. In this case, \( G = \text{Aut}(\mathcal{H}^n) \) ([10], page 15).

We need the following result connecting eigenvalue orbits and spectral sets.

**Proposition 2.2** (i) The following inclusions hold:

\[
\{ \phi(x) : \phi \in G \} \subseteq \{ \phi(x) : \phi \in \text{Aut}(\mathcal{V}) \} \subseteq [x] \quad (x \in \mathcal{V}).
\]

These become equalities when \( \mathcal{V} \) is simple.

(ii) A set \( E \) in \( \mathcal{V} \) is a spectral set if and only if for all \( x \in E \), \( [x] \subseteq E \).

(iii) If \( E \) is a spectral set in \( \mathcal{V} \), then for all \( \phi \in \text{Aut}(\mathcal{V}) \), \( \phi(E) = E \). Converse holds when \( \mathcal{V} \) is simple or \( \mathbb{R}^n \).

**Proof.** (i) As automorphisms preserve Jordan frames and eigenvalues, the stated inclusions follow. Now suppose \( \mathcal{V} \) is simple and let \( y \in [x] \) so that \( \lambda(x) = \lambda(y) \). We write the spectral decompositions \( x = \sum_i^n \lambda_i(x)e_i \) and \( y = \sum_i^n \lambda_i(y)f_i \), where \( \{e_1, e_2, \ldots, e_n\} \) and \( \{f_1, f_2, \ldots, f_n\} \) are Jordan frames in \( \mathcal{V} \). Since \( \mathcal{V} \) is simple, by Corollary IV.2.7 in [8], there is an automorphism in \( G \) that takes one Jordan frame to the other. Hence, we may write \( y = \phi(x) \) for some \( \phi \in G \). Thus, by (i), \( [x] = \{ \phi(x) : \phi \in G \} \).

(ii) This is given in Theorem 1, [16].

(iii) This is given in Theorem 2, [16]. \( \square \)

In view of Item (iii) in the above theorem, we see that in \( S^n \), a set \( E \) is spectral if and only if

\[ X \in E \Rightarrow UXU^T \in E \quad \text{for all orthogonal matrices } U \in \mathbb{R}^{n \times n}. \]

## 3 Main connectedness results

To motivate our discussion and results, we start with two examples.

**Example (1)** Let \( n > 1 \) and consider the non-simple algebra \( \mathcal{V} = \mathbb{R}^n \). Then, \( \text{Aut}(\mathbb{R}^n) = \Sigma_n \) and \( G \) consists of just the identity matrix. For any \( q \in \mathbb{R}^n \), \( \lambda(q) = q^k \). Also, for any permutation invariant set \( Q \) in \( \mathbb{R}^n \), \( \lambda^{-1}(Q) = Q \). Thus, in this setting, if a permutation invariant set is connected (arcwise connected), then its \( \lambda \)-inverse image is connected (respectively, arcwise connected). Can we improve this by assuming only the connectedness (arcwise connectedness) of \( Q^k \)? To answer this, let \( Q := \{c_1, c_2, \ldots, c_n\} \) denote the set of standard coordinate vectors in \( \mathbb{R}^n \). Then \( Q \) is permutation invariant and \( Q^k = \{c_1\} \). We see that while \( Q^k \) is connected (arcwise connected), \( \lambda^{-1}(Q) \) is not connected.
Example (2) Let \( n > 1 \) and consider the simple algebra \( \mathcal{V} = \mathcal{S}^n \). As noted previously, \( \text{Aut}(\mathcal{S}^n) \) consists of transformations of the form \( \phi(X) := UXU^T \ (X \in \mathcal{S}^n) \) where \( U \in \mathbb{R}^{n \times n} \) is an orthogonal matrix. Now consider the set \( Q := \{c_1, c_2, \ldots, c_n\} \) of Example 1. As noted earlier, \( Q \) is a permutation invariant set with \( Q^1 \) connected. With \( D \) denoting the diagonal matrix with \( c_1 \) on its diagonal, we have

\[
\lambda^{-1}(Q) = \lambda^{-1}(Q^1) = \left\{ UDU^T : U \in \mathbb{R}^{n \times n} \text{ is orthogonal} \right\} = \left\{ uu^T : u \in \mathbb{R}^n, ||u|| = 1 \right\}.
\]

(The vector \( u \) that appears above is the first column of \( U \).) As \( n > 1 \), the unit sphere in \( \mathbb{R}^n \) is arcwise connected. Hence, \( \lambda^{-1}(Q) \) is arcwise connected. Thus, in contrast to Example 1, a weaker hypothesis involving connectedness (arcwise connectedness) of \( Q^1 \) suffices to get the connectedness (arcwise connectedness) of \( \lambda^{-1}(Q) \). We show below that in the setting of a simple algebra, such a statement holds for any permutation invariant set.

The following is a basic (possibly known) result about the connectedness of eigenvalue orbits. Recall that \( G \) is the connected component of identity in \( \text{Aut}(\mathcal{V}) \).

**Theorem 3.1** The following statements hold:

(a) \( G \) is arcwise connected.

(b) When \( \mathcal{V} \) is simple, for any \( x \in \mathcal{V} \), \([x]\) is arcwise connected.

(c) For any \( x \in \mathcal{V} \), \([x]_r\) is arcwise connected.

**Proof.** (a) As \( \text{Aut}(\mathcal{V}) \) is a (matrix) Lie group (see [S], page 36), its connected component of identity, namely \( G \), is arcwise connected, see [G], Theorem 1.9.1.

(b) Now suppose \( \mathcal{V} \) is simple and consider any \( x \in \mathcal{V} \). Then, by Item (i) in the previous proposition, \([x] = \{\phi(x) : \phi \in G\}\). As \( G \) is arcwise connected and the map \( \phi \mapsto \phi(x) \) is continuous, \([x]\) is also arcwise connected.

(c) If \( \mathcal{V} \) is simple, \([x]_r = [x]\) for any \( x \). Then, the result follows from (b). Suppose \( \mathcal{V} \) is non-simple, and let \( \mathcal{V} \) be a product of simple algebras: \( \mathcal{V} = \mathcal{V}(1) \times \mathcal{V}(2) \times \cdots \times \mathcal{V}(N) \), where each \( \mathcal{V}(i) \) is simple. Let \( x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \in \mathcal{V} \). Then, by (b), each \([x^{(i)}]_{\mathcal{V}(i)}\) is arcwise connected. Hence, the set

\[
[x]_r = [x^{(1)}]_{\mathcal{V}(1)} \times [x^{(2)}]_{\mathcal{V}(2)} \times \cdots \times [x^{(N)}]_{\mathcal{V}(N)},
\]

being a product of arcwise connected sets, is arcwise connected. \( \Box \)

To illustrate Items (b) and (c) in the above result, we let \( x \) be a primitive idempotent in a simple algebra \( \mathcal{V} \). Then, \([x]\), which is the set of all primitive idempotents, is arcwise connected. (This result is known, see [S], page 71.) This may fail if \( \mathcal{V} \) is not simple: let \( \mathcal{V} = \mathbb{R}^n \ (n > 1) \) and \( x = c_1 \).
Then, \([x] = \{c_1, c_2, \ldots, c_n\}\) is not connected. However, \([x]_r = \{c_1\}\) (a singleton set) is arcwise connected.

We now state a necessary condition for a \(\lambda\)-inverse image to be connected (arcwise connected).

**Proposition 3.2** Let \(P \subseteq \mathbb{R}^n\). If \(\lambda^{-1}(P)\) is connected (arcwise connected) in \(\mathcal{V}\), then \(P \cap P^\perp\) is connected (arcwise connected) in \(\mathbb{R}^n\). In particular, if \(Q\) is permutation invariant and \(\lambda^{-1}(Q)\) is connected (arcwise connected) in \(\mathcal{V}\), then \(Q^\perp\) is connected (arcwise connected) in \(\mathbb{R}^n\).

**Proof.** The first statement follows from the continuity of the eigenvalue map \(\lambda\) and the equality \(P \cap P^\perp = \lambda(\lambda^{-1}(P))\). The second one follows from the equality \(Q \cap Q^\perp = Q^\perp\). \(\square\)

The next two results deal with sufficient conditions.

**Theorem 3.3** Suppose \(Q \subseteq \mathbb{R}^n\) is permutation invariant and one of the following conditions holds.

(a) \(\mathcal{V}\) is simple and \(Q^\perp\) is connected (arcwise connected).

(b) \(Q\) is connected (arcwise connected).

Then \(\lambda^{-1}(Q)\) is connected (respectively, arcwise connected) in \(\mathcal{V}\).

**Proof.** Suppose condition (a) holds so that \(\mathcal{V}\) is simple and \(Q^\perp\) is connected (arcwise connected). We fix a Jordan frame \(\mathcal{E} = \{e_1, e_2, \ldots, e_n\}\) and consider the continuous map \(\Theta : \mathbb{R}^n \to \mathcal{V}\) defined by \(\Theta(q) := q \ast \mathcal{E}\), see Section 2. Then, the image \(\Theta(\mathcal{E})\), which is a subset of \(\lambda^{-1}(Q^\perp)\), is connected (respectively, arcwise connected). Now let \(x \in \lambda^{-1}(Q^\perp)\) be arbitrary. Then, \(\lambda(x) \in Q^\perp\) and there is a Jordan frame \(\{f_1, f_2, \ldots, f_n\}\) such that \(x = \lambda_1(x)f_1 + \lambda_2(x)f_2 + \cdots + \lambda_n(x)f_n\). As \(\mathcal{V}\) is simple, from Theorem 3.1 \([x]\) is arcwise connected. Also,

\[
\lambda_1(x)e_1 + \lambda_2(x)e_2 + \cdots + \lambda_n(x)e_n \in \Theta(Q^\perp) \cap [x].
\]

So the sets \(\Theta(Q^\perp)\) and \([x]\) are connected (respectively, arcwise connected) and their intersection is nonempty. It follows that their union is also connected (respectively, arcwise connected). As \(x\) is arbitrary in \(\lambda^{-1}(Q^\perp)\), the connected component (respectively, arcwise connected component) of \(\lambda^{-1}(Q^\perp)\) that contains \(\Theta(Q^\perp)\) must be \(\lambda^{-1}(Q^\perp)\) itself. This proves that \(\lambda^{-1}(Q^\perp)\) is connected (respectively, arcwise connected) under the condition (a). The stated assertion follows since \(\lambda^{-1}(Q) = \lambda^{-1}(Q^\perp)\).

Now suppose condition (b) holds. If \(\mathcal{V}\) is simple, we can use the previous argument as \(Q^\perp\) (which is the image of \(Q\) under the continuous map \(q \mapsto q^\perp\)) is connected (arcwise connected). So, assume that \(\mathcal{V}\) is non-simple and write it as an orthogonal direct sum \(\mathcal{V} = \mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)} \oplus \cdots \oplus \mathcal{V}^{(N)}\), where each \(\mathcal{V}^{(i)}\) is simple. In \(\mathcal{V}\), we fix a Jordan frame \(\mathcal{E} := \{e_1, e_2, \ldots, e_n\}\) and consider the continuous map \(\Theta : \mathbb{R}^n \to \mathcal{V}\) defined by \(\Theta(q) = q \ast \mathcal{E}\). As \(Q\) is connected (arcwise connected) and \(\Theta\)
is continuous, $\Theta(Q)$ is connected (arcwise connected). Also, as $Q$ is permutation invariant, any rearrangement/listing of elements of $E$ will not alter the set $\Theta(Q)$. Hence, we may assume without loss of generality that $E = \bigcup_1^N E(i)$, where each $E(i)$ is a Jordan frame in $V(i)$. Writing $q \in R^n$ in the block form $q = (q(1), q(2), \ldots, q(N))$, we see that $q \ast E = q(1) \ast E(1) + q(2) \ast E(2) + \ldots + q(N) \ast E(N)$. Now, let $x \in \lambda^{-1}(Q)$. Then, there exist a Jordan frame $F$ in $V$ and a $p \in Q$ such that $x = p \ast F$. Similar to the above, we may write $x = p \ast F = p(1) \ast F(1) + p(2) \ast F(2) + \ldots + p(N) \ast F(N)$, where $F(i)$ is a Jordan frame in $V(i)$. Let $y := p \ast E = p(1) \ast E(1) + p(2) \ast E(2) + \ldots + p(N) \ast E(N)$. As $\lambda(p(i) \ast E(i)) = \lambda(p(i) \ast F(i))$ for all $i$, it follows that $y \in [x]_r$. Hence, $y \in \Theta(Q) \cap [x]_r$. As $\Theta(Q)$ is connected (arcwise connected) and $[x]_r$ is arcwise connected (by Theorem 3.1), it follows that $\Theta(Q) \cup [x]_r$ is connected (respectively, arcwise connected). Since $x$ is arbitrary in $\lambda^{-1}(Q)$, the connected (arcwise connected) component of $\lambda^{-1}(Q)$ must be $\lambda^{-1}(Q)$. This proves that $\lambda^{-1}(Q)$ is connected (respectively, arcwise connected).

Corollary 3.4 When $V$ is simple, the following statements hold:

(i) If $\Omega$ is any subset of $(R^n)^{\perp}$ that is connected (arcwise connected), then $\lambda^{-1}(\Omega)$ is connected (respectively, arcwise connected). 

(ii) If $P$ is any subset of $R^n$ such that $P \cap P^{\perp}$ is connected (arcwise connected), then $\lambda^{-1}(P)$ is connected (respectively, arcwise connected).

Proof. (i) Let $\Omega$ be connected (arcwise connected) subset of $(R^n)^{\perp}$. Then $Q := \Sigma_n(\Omega)$ is permutation invariant and $Q^{\perp} = \Omega$. Hence condition (a) in the above theorem applies. Since $\lambda^{-1}(\Omega) = \lambda^{-1}(Q)$, we see that $\lambda^{-1}(\Omega)$ is connected (arcwise connected).

(ii) Suppose $P$ is any subset of $R^n$ such that $P \cap P^{\perp}$ is connected (arcwise connected). As $P \cap P^{\perp}$ is a subset of $(R^n)^{\perp}$ and $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^{\perp})$, the stated result follows from (i) applied to $\Omega := P \cap P^{\perp}$.

Remarks (1) Suppose $V$ is simple. In view of Proposition 3.2, for any set $P$ in $R^n$, connectedness (arcwise connectedness) of $P \cap P^{\perp}$ is necessary and sufficient for that of $\lambda^{-1}(P)$. Thus, a spectral set $E$ in a simple algebra $V$ is connected (arcwise connected) if and only if $\lambda(E)$ is connected (arcwise connected) in $R^n$.

(2) The conclusions in the above corollary may fail if $V$ is not simple. For example in $R^3$, $c_1 = (1,0,0)$ is on the boundary of $(R^3)^{\perp}$. Letting $V = R^3$ and $\Omega = \{c_1\}$, we see that $\lambda^{-1}(\Omega) = \{c_1, c_2, c_3\}$ (set of standard coordinate vectors in $R^3$) is not connected. In the same setting, it is easy to construct an arcwise connected set $P$ such that $P \cap P^{\perp} = \{c_1\}$. (For example, $P$ could be a circle through $c_1$ such that $P \backslash \{c_1\}$ is outside of $(R^3)^{\perp}$.) For such a $P$, $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^{\perp}) = \{c_1, c_2, c_3\}$ is not connected.
Motivated by the above two results, one may ask if $\lambda$-inverse image of a simply connected permutation invariant set $Q$ in $\mathcal{R}^n$ is simply connected in $\mathcal{V}$. While the answer to this is not clear, we note that in $\mathcal{S}^2$, the set of all primitive idempotents is given by \((\mathbb{S}, \text{page 71})\)

$$\left\{ \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}, 0 \leq \theta \leq \pi \right\}.$$ 

This is homeomorphic to a circle, hence not simply connected. However, it is of the form $\lambda^{-1}(Q^\downarrow)$, where $Q$ is the set of standard coordinate vectors in $\mathcal{R}^2$. While this $Q$ is not simply connected, $Q^\downarrow$, consisting of only one element, is simply connected. So the counterpart of (a) in Theorem 3.3 may not hold for simple connectedness.

We end this section by mentioning a classical result of Ky Fan. Suppose $Q$ is a permutation invariant set that satisfies one of the conditions of Theorem 3.3. Assume further that $Q$ is compact. Then, $\lambda^{-1}(Q)$ is connected and (by Proposition 1.1) compact in $\mathcal{V}$. Hence, for any $c \in \mathcal{V}$, the image of $\lambda^{-1}(Q)$ under the continuous function $x \mapsto \langle c, x \rangle$ is connected and compact in $\mathcal{R}$. So, there exists real numbers $\delta$ and $\Delta$ such that

$$\left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = [\delta, \Delta]. \quad (1)$$

(With some additional work, it is possible to describe the forms of $\delta$ and $\Delta$.) To see a special case, consider two matrices $C$ and $A$ in $\mathcal{H}^n$ and let $Q = \Sigma_n(\{\lambda(A)\})$. Clearly, $Q$ is permutation invariant and compact (actually, finite). Since $\mathcal{H}^n$ is simple and $Q^\downarrow$ (a singleton set) is connected, by Theorem 3.3 (or by Theorem 3.1), $\lambda^{-1}(Q) = [A] = \{UAU^* : U \in \mathcal{C}^{n \times n} \text{ is unitary}\}$ is connected in $\mathcal{H}^n$. As $\langle X, Y \rangle = \text{tr}(XY)$ in $\mathcal{H}^n$, (1) reads:

$$\left\{ \text{tr}(CUA^*) : U \in \mathcal{C}^{n \times n} \text{ is unitary} \right\} = [\delta, \Delta].$$

With $\Delta := \langle \lambda(C), \lambda(A) \rangle$ and $\delta := \langle \lambda^+(C), \lambda(A) \rangle$, where $\lambda^+(C)$ denotes the increasing rearrangement of $\lambda(C)$, this statement is due to Fan [7] (see also [25], Corollary 1.6).

### 4 Components of a spectral set

We now describe connected (arcwise connected) components of a spectral set in a simple algebra.

**Theorem 4.1** Let $\mathcal{V}$ be simple and $E := \lambda^{-1}(Q)$ be a spectral set in $\mathcal{V}$, where $Q$ is permutation invariant in $\mathcal{R}^n$. Then, every connected (arcwise connected) component of $E$ is a spectral set. Moreover, $C \rightarrow \lambda(C)$ is a one-to-one correspondence between connected (arcwise connected) components of $E$ and those of $Q^\downarrow$.

**Proof.** We consider the case of connected components. The case of arcwise connected components
is similar. Let $C$ be a connected component of $E$. As $\mathcal{V}$ is simple, for any $x \in C$, $[x]$ is (arcwise) connected (by Theorem 3.1) and $[x] \cap C$ contains $x$. Hence, $[x] \subseteq C$. By Proposition 2.2(ii), $C$ is a spectral set; we may now write $C$ as $C = \lambda^{-1}(P)$, where $P$ is a permutation invariant set in $\mathcal{R}^n$. As $C$ is connected, its image $\lambda(C) = P^\perp$ is connected. We claim that this set is a connected component of $Q^\perp$. To simplify the notation, let $\Omega := \lambda^{-1}(P)$. Then,

$$C \subseteq E \Rightarrow P^\perp = \lambda(C) \subseteq \lambda(E) = Q^\perp \Rightarrow \Omega \subseteq Q^\perp.$$ 

So, $\Omega$ is a connected subset of $Q^\perp$. Let $\Omega^*$ be the connected component of $Q^\perp$ that contains $\Omega$, so

$$\Omega \subseteq \Omega^* \subseteq Q^\perp.$$ 

Then,

$$C = \lambda^{-1}(\Omega) \subseteq \lambda^{-1}(\Omega^*) \subseteq \lambda^{-1}(Q^\perp) = \lambda^{-1}(Q) = E.$$ 

Since $\mathcal{V}$ is simple, by Item (i) in Corollary 3.11, $\lambda^{-1}(\Omega^*)$ is connected. By our assumption that $C$ is a connected component of $E$ we get $C = \lambda^{-1}(\Omega^*)$, leading to $\Omega = \lambda(C) = \lambda((\lambda^{-1}(\Omega^*))) = \Omega^*$, the last equality is due to Proposition 2.1. Hence, $\Omega$ is a connected component of $Q^\perp$.

Now we show that every connected component of $Q^\perp$ arises this way. Let $\Omega$ be a connected component of $Q^\perp$. By Item (i) in Corollary 3.11, $C := \lambda^{-1}(\Omega)$ is connected in $E$. Suppose $C^*$ is the connected component of $E$ that contains $C$. We show that $C = C^*$. Now,

$$C \subseteq C^* \subseteq E \Rightarrow \lambda(C) \subseteq \lambda(C^*) \subseteq \lambda(E).$$

We have $\lambda(C) = \lambda(\lambda^{-1}(\Omega)) = \Omega$, where the second equality is due to Proposition 2.1. It follows that $\Omega \subseteq \lambda(C^*) \subseteq Q^\perp$. As $\Omega$ is a connected component and $\lambda(C^*)$ is connected, we must have $\Omega = \lambda(C^*)$, that is, $\Omega = \lambda(C) = \lambda(C^*)$. So, $C = \lambda^{-1}(\Omega) = \lambda^{-1}(\lambda(C^*)) \supseteq C^*$. Since $C \subseteq C^*$, we conclude that $C = C^*$ and that $C$ is a connected component of $E$. Finally, the concluding statement in the theorem about one-to-one and onto correspondence is easily verified.

Remarks (4) The conclusions in the above result may fail if $\mathcal{V}$ is not simple, see Example 1.

5 Irreducibility of spectral cones

We now provide another application of Theorem 3.11. Recall that a (nonempty) set $K$ is a convex cone if it is convex and $tx \in K$ for all $x \in K$ and $t \geq 0$ in $\mathcal{R}$. If, in addition, $K \cap -K = \{0\}$, then $K$ is said to be a pointed convex cone. We say that a convex cone $K$ is reducible if it can be written as a sum $K = K_1 + K_2$ where $K_1$ and $K_2$ are nonzero convex cones with $\text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. A convex cone that is not reducible is said to be irreducible. (These concepts hold in $\mathcal{R}^n$ as well.)

We now address the irreducibility of spectral cones. In $\mathcal{R}^n$, spectral cones (which are just permutation invariant convex cones) can be reducible or irreducible. In fact, $\mathcal{R}_+^n$ ($n > 1$) is reducible, while
\( \mathbb{R}_+ \mathbf{1} \) is irreducible, where \( \mathbf{1} \) represents the vector of ones in \( \mathbb{R}^n \). In a simple Euclidean Jordan algebra, the corresponding symmetric cone (= the cone of squares which is a closed convex self-dual homogeneous cone) is irreducible (see [8], Prop. III.4.5), even though it is of the form \( \lambda^{-1}(\mathbb{R}_+^n) \) with \( \mathbb{R}_+^n \) reducible for \( n > 1 \). Below, we show that any pointed spectral cone in a simple Euclidean Jordan algebra is irreducible. First, we recall a well-known result ([11], slightly modified to suit our setting): If \( K \) is a nonzero pointed reducible convex cone in \( \mathcal{V} \), then there exists a unique set of nonzero irreducible convex cones \( K_i, i = 1, 2, \ldots, r \), such that

\[
K = K_1 + K_2 + \cdots + K_r
\]

with \( \text{span}(M) \cap \text{span}(N) = \{0\} \), whenever \( M \) denotes the sum of some \( K_i \)s and \( N \) denotes the sum of the rest of the \( K_i \)s. Moreover, the above representation is unique up to permutation of indices.

In this setting, we say that \( K \) is a direct sum of \( K_i \)s.

**Theorem 5.1** Suppose \( \mathcal{V} \) is simple. Then, every pointed spectral cone in \( \mathcal{V} \) is irreducible.

**Proof.** If possible, suppose \( K \) is a nonzero pointed spectral cone in \( \mathcal{V} \) which is a direct sum of irreducible convex cones:

\[
K = K_1 + K_2 + \cdots + K_r
\]

with \( r > 1 \). We claim that each \( K_i \) is a pointed spectral cone. The pointedness of \( K_i \) is clear, as \( K \) is pointed. We show that \( K_1 \) is a spectral cone. Since \( \mathcal{V} \) is simple, because of Items (i) and (ii) in Proposition 2.2, it is enough to show that for all \( x \in K_1 \), \([x] = \{\phi(x) : \phi \in G\} \subseteq K_1 \). Let \( \phi \in G \). Since \( K \) is a spectral cone, \( \phi(K) = K \) (by Item (iii) in Proposition 2.2). This implies that \( K = \phi(K_1) + \phi(K_2) + \cdots + \phi(K_r) \) is another direct sum representation in terms of irreducible convex cones. By the uniqueness of factors in the decomposition, \( \phi(K_1) \) must be equal to some \( K_j \). So,

\[
\phi(K_1) \subseteq \bigcup_{j=1}^r K_j, \forall \phi \in G.
\]

Now, let \( 0 \neq x \in K_1 \). Then, all the elements in \([x]\) are nonzero and

\[
[x] = \{\phi(x) : x \in G\} \subseteq A \cup B,
\]

where \( A := K_1 \setminus \{0\} \) and \( B := \cup_j K_j \setminus \{0\} \). Since \( K_1 \cap \overline{B} \subseteq \text{span}(K_1) \cap \text{span}(B) = \{0\} \), we see that \( A \cap \overline{B} = \emptyset \). (Here, ‘overline’ denotes the closure.) Similarly, \( B \cap \overline{A} = \emptyset \). So the sets \( A \) and \( B \) are separated [21]. Now, by Theorem 3.3 (or by Theorem 5.1), \([x]\) is connected. As \([x] \subseteq A \cup B \) and \( 0 \neq x \in A \), we must have \([x] \subseteq A \). We conclude that \([x] \subseteq K_1 \). This inclusion also holds for \( x = 0 \) so

\[
[x] \subseteq K_1, \forall x \in K_1.
\]
Thus, $K_1$ is a spectral cone by Item (ii) in Proposition 2.2. A similar argument works for all other cones $K_i$. Now, all the spectral cones $K$, $K_1$, $K_2$, $\ldots$, $K_r$ are pointed spectral cones. Hence, by Theorem 7.3 in [15], $e$ (the unit element of $\mathcal{V}$) or $-e$ belongs to all of them. Since $K$ is pointed, either $e$ or $-e$, but not both, can belong to all. Moreover, since the cones $K_i$ have zero as the only common element, we conclude that $r = 1$. This violates our assumption that $r > 1$. Hence, $K$ is irreducible.

The above result gives an alternate proof of the fact that in any simple algebra, the symmetric cone is irreducible. Also, every pointed convex cone $K$ in $\mathcal{S}^n$ satisfying the condition

$$X \in K \Rightarrow U X U^T \in K$$

for all orthogonal matrices $U \in \mathbb{R}^{n \times n}$ is irreducible. We provide two more examples.

**Example (3)** Let $n \geq 3$ and $m \in \{1, 2, \ldots, n-1\}$. For each $q \in \mathbb{R}^n$, let $s_m(q)$ be the sum of the smallest $m$ entries of $q$. Then, the ‘rearrangement cone’

$$Q^*_m = \{q \in \mathbb{R}^n : s_m(q) \geq 0\},$$

is a permutation invariant pointed closed convex cone in $\mathbb{R}^n$ [14]. Hence, by Item (c) in Proposition 1.1,

$$\lambda^{-1}(Q^*_m) = \{x \in \mathcal{V} : \lambda_n(x) + \lambda_{n-1}(x) + \cdots + \lambda_{n-m+1}(x) \geq 0\}$$

is a pointed convex cone. (Note that when $m = 1$, this cone is the symmetric cone of $\mathcal{V}$.) By the above theorem, this cone, in a simple algebra, is irreducible.

**Example (4)** Let $n \geq 3$ and consider the following permutation invariant cone (see Example 2 in [15])

$$Q := \left\{q \in \mathbb{R}^n : tr(q) \geq \sqrt{\frac{n}{2}} \|q\|_2\right\}$$

where $tr(q)$ denotes the sum of all entries of $q$ and $\|q\|_2$ denotes the 2-norm of $q$. This cone is a proper cone (that is, it is a closed convex pointed cone with nonempty interior). By the above theorem, when $\mathcal{V}$ is a simple algebra of rank $n$, the proper spectral cone

$$K := \lambda^{-1}(Q) = \left\{x \in \mathcal{V} : tr(x) \geq \sqrt{\frac{n}{2}} \|x\|_2\right\}$$

is irreducible, where $tr(x)$ denotes the sum of all eigenvalues of $x$ and $\|x\|_2 := \|\lambda(x)\|_2$. Using the linearity of the trace and the strict convexity of $\| \cdot \|_2$, it is easy to show that every boundary vector in $K$ is an extreme vector. As $n$ (the rank of $\mathcal{V}$) is at least 3, such a property is false for the symmetric cone of $\mathcal{V}$ (take a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and consider $e_1 + e_2$ which is on the boundary of the symmetric cone of $\mathcal{V}$, but not an extreme vector); so, $K$ and the symmetric cone.
of $\mathcal{V}$ are not isomorphic. This shows that in every simple algebra of rank $n \geq 3$, there is a proper (irreducible) spectral cone that is not isomorphic to the corresponding symmetric cone.

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