A note on 2-dimensional Finsler manifold

Morteza Faghfouri · Rahim Hosseinoghli

Abstract We solve the following problem for $n = 2$: Is any $n$-dimensional Finsler manifold $(M, F)$ with a function $f$ which is nonconstant and smooth on $M$ satisfying
\[ \frac{\partial g_{ij}}{\partial y^k} \frac{\partial f}{\partial x^i} = 0, \]
a Riemannian manifold? The problem for $n > 2$ remains open.

Mathematics Subject Classification (2010) 53C60, 53C25.

Keywords Berwaldian metric, Finsler manifold, partial differential equation, doubly warped product metric.

1 Introduction

The notion of doubly warped product manifolds has an important role in Riemannian geometry and its applications. For example, Beem-Powell in [2] studied this product for Lorentzian manifolds. Then Allison in [1] considered global hyperbolicity of doubly warped products and null pseudo convexity of Lorentzian doubly warped products and recent years in [3], [5] and [6] extended some properties of warped product, submanifolds and geometric inequality in warped product manifolds for doubly warped product submanifolds into arbitrary Riemannian manifolds. In 2001, Kozma- Peter-Varga in [7] defined their warped product for Finsler metrics and concluded that completeness of a doubly warped product can be related to completeness of its components.

In [8, Theorem 3] E. Peyghan and A. Tayebi proved that: Let $M_1 f_1 \times_{f_1} M_2$ be a $DWP$-Finsler manifold and $f_1$ is constant on $M_1$ ($f_2$ is constant on $M_2$).

M. Faghfouri
Faculty of Mathematics, University of Tabriz, Tabriz, Iran.
E-mail: faghfouri@tabrizu.ac.ir

R. Hosseinoghli
Faculty of Mathematics, University of Tabriz, Tabriz, Iran.
E-mail: r.hosseinoghli90@ms.tabrizu.ac.ir
Then $M_1 \times_{f_1} M_2$ is Berwaldian if and only if $M_1$ is Riemannain, $M_2$ is Berwaldian and

$$C_k^{ij} \frac{\partial f_1}{\partial x^i} = 0.$$  

($\ast$)

($M_2$ is Riemannian, $M_1$ is Berwaldian and $C_k^{ij} \frac{\partial f_2}{\partial x^i} = 0$).

Now we can ask this question: Is there any nonconstant smooth function on Finsler manifold $M$, which satisfies ($\ast$)?

In this paper by using partial differential equation properties, we show that, if $M$ is a 2-dimensional Finsler manifold and the equality ($\ast$) holds for a nonconstant function $f$, then $M$ is a Riemannian manifold.

2 Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM := \bigcup_{x \in M} T_x M$ the tangent bundle of $M$, and by $TM^0 = TM - \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \rightarrow [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$;
(iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,

where

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} \| F^2(y + su + tv) \|_{y, t=0}. $$

Let $(M, F)$ be a Finsler manifold. The second and third order derivatives of $\frac{1}{2} F^2 := \frac{1}{2} F^2(x, y)$ at $y \in T_x M^0$ are the symmetric forms $g_y$ and $C_y$ on $T_x M$, which called the fundamental tensor and Cartan torsion, respectively. In other notation,

$$C_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$$

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \{ g_{y+tw}(u, v) \} \big|_{t=0}, \quad u, v, w \in T_x M$$

the family $C := \{C_y\}_{y \in TM_0}$ is called the cartan torsion, it is well known that $C = 0$ if and only if $F$ is Riemannian. let $b_i$ be a local frame for $TM$, and $g_{ij} := g_y(b_i, b_j)$, $C_{ijk} := C_y(b_i, b_j, b_k)$. then $g_{ij}(x, y) = \frac{\partial^2 F^2(x, y)}{2 \partial y^i \partial y^j}$ and

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^2 F^2}{\partial y^i \partial y^j \partial y^k}. $$

For a Finsler manifold $(M, F)$, a global vector field $G$ is induced by $F$ on $TM^0$, which in a standard coordinate $(x^i, y^i)$ for $TM^0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2C^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i = \frac{1}{4} y^i \{ [F^2]_{x^i y^j} y^k - [F^2]_{y^i} \}, \quad y \in T_x M.$$
The $G$ is called the spray associated to $(M, F)$. A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2} T^i_{jk}(x) y^j y^k$ is quadratic in $y \in T_x M$ for any $x \in M$. For a tangent vector $y \in T_x M^0$, define $B_y : T_x M \times T_x M \times T_x M \to T_x M$, $E_y : T_x M \times T_x M \to \mathbb{R}$ and $D_y : T_x M \times T_x M \to T_x M$ by

$$B_y(u, v, w) := B^i_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^l}, \quad E_y(u, v) := E^i_{jk}(y) u^i v^j v^k$$

and

$$D_y(u, v, w) := D^i_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^l}$$

where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E^i_{jk} = \frac{1}{2} B^m_{jkm},$$

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n + 1} \left( E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + \frac{\partial E^i_{jkl}}{\partial y^j} y^j \right).$$

$B$, $E$ and $D$ are called the Berwald curvature, mean Berwald curvature and Douglas curvature, respectively. Then $F$ is called a Berwald metric, weakly Berwald metric and a Douglas metric if $B = 0$, $E = 0$ and $D = 0$, respectively [4]. The notion of warped product manifold was introduced in [3] where it served to give new examples of Riemannian manifolds. On the other hand, Finsler geometry is just Riemannian geometry without the quadratic restriction. Thus it is natural to extend the construction of warped product manifolds for Finsler geometry [7].

Let $(M_1, F_1)$ and $(M_2, F_2)$ be two Finsler manifolds and $f_i : M_i \to \mathbb{R}^+$, $i = 1, 2$ are smooth functions. Let $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ be the natural projection maps. The product manifold $M_1 \times M_2$ endowed with the metric $F : TM_1^0 \times TM_2^0 \to \mathbb{R}$ given by

$$F(y, v) = \sqrt{f_2^2(\pi_2(y)) F_2^2(y) + f_1^2(\pi_1(y)) F_1^2(v)}$$

is considered, where $TM_1^0 = TM_1 - \{0\}$ and $TM_2^0 = TM_2 - \{0\}$. The metric defined above is a Finsler metric. The product manifold $M_1 \times M_2$ with the metric $F(y, v)$ for $(y, v) \in TM_1^0 \times TM_2^0$ defined above will be called the doubly warped product (DWP) of the manifolds $M_1$ and $M_2$ and $f_i$, $i = 1, 2$ will be called the warping function. We denote this warped by $M_1 f_2 \times f_1 M_2$. If $f_2 = 1$, then we have a warped product manifold. If $f_i$, $i = 1, 2$ is not constant, then we have a proper DWP-manifold.

Let $(M_1, F_1)$ and $(M_2, F_2)$ be two Finsler manifolds. Then the functions

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F_1^2(x, y)}{\partial y^i \partial y^j}, \quad g_{\alpha \beta}(u, v) = \frac{1}{2} \frac{\partial^2 F_2^2(u, v)}{\partial u^\alpha \partial v^\beta},$$

define a Finsler tensor field of type $(0, 2)$ on $TM_1^0$ and $TM_2^0$, respectively. Now let $M_1 \times_{f_1} M_2$ be a warped Finsler manifold and let $x \in M$ and $y \in T_x M$,
where \( x = (x, u), y = (y, v), M = M_1 \times M_2 \) and \( T_x M = T_x M_1 \oplus T_u M_2 \). Then we conclude that

\[
g_{ab}(x, u, y, v) = \begin{pmatrix} \frac{1}{2} \frac{\partial^2 F^2(x, u, y, v)}{\partial y^a \partial y^b} \end{pmatrix} = \begin{pmatrix} g_{ij} & 0 \\ 0 & f_1^2 g_{\alpha \beta} \end{pmatrix}
\]

where \( y^a = (y^i, v^\alpha), y^b = (y^j, v^\beta) \) and \( g_{ij} = g_{ij}, g_{ab} = f_1^2 g_{\alpha \beta}, g_{ij} = g_{ij} = 0 \) and

\[
i, j, \ldots \in \{1, 2, \ldots, n_1\}, \alpha, \beta, \ldots \in \{1, 2, \ldots, n_2\}, a, b, \ldots \in \{1, 2, \ldots, n_1 + 1, \ldots, n_1 + n_2\},
\]

where

\[
\dim(M_1) = n_1, \quad \dim(M_2) = n_2, \quad \dim(M_1 \times M_2) = n_1 + n_2.
\]

So the spray coefficients of warped product are given by

\[
G^i(x, u, y, v) = G^i(x, y) - \frac{1}{4} g^{ib} \frac{\partial f_1^2}{\partial x^b} F_2,
\]

\[
G^\alpha(x, u, y, v) = G^\alpha(u, v) + \frac{1}{4} f_1^2 g_{\alpha \beta} \frac{\partial \partial f_1^2}{\partial x^\beta} y^\beta.
\]

The Berwald curvature of \((M_1 \times f M_2)\) is as follows:

\[
B^i_{ijl} = B^k_{ijl} - \frac{1}{4} g^{kh} \frac{\partial f_1^2}{\partial x^h} \frac{\partial f_1^2}{\partial x^k} F_2^2,
\]

\[
B^\alpha_{i\beta\lambda} = B^\alpha_{i\beta\lambda} - \frac{1}{4} g_{h\alpha} g_{\beta \lambda} \frac{\partial f_1^2}{\partial x^h} F_2^2,
\]

\[
B^\alpha_{i\beta\lambda} = -\frac{1}{4} f_1^2 g_{h\alpha} g_{\beta \lambda} C_{\alpha \beta \lambda},
\]

\[
B^\alpha_{ij\lambda} = -\frac{1}{4} f_1^2 g_{h\alpha} g_{\beta \lambda} C_{\alpha \beta \lambda},
\]

\[
B^\alpha_{ijk} = 0.
\]

**Theorem 21 (\[8\])** Let \( M_1 f_2 \times f_1 M_2 \) be a DW P-Finsler manifold and \( f_1 \) is constant on \( M_1 \) (\( f_2 \) is constant on \( M_2 \)). Then \( M_1 f_2 \times f_1 M_2 \) is Berwaldian if and only if \( M_1 \) is Riemannian, \( M_2 \) is Berwaldian and \( C^i_{jk} \frac{\partial f_1}{\partial x^i} = 0 \) (\( M_2 \) is Riemannian, \( M_1 \) is Berwaldian and \( C^i_{jk} \frac{\partial f_2}{\partial x^i} = 0 \)).

**Corollary 22 (\[8\])** Let \((M_1 \times f_1 M_2, F)\) be a proper WP-Finsler manifold. Then \((M_1 \times f_1 M_2, F)\) is Berwaldian if and only if \( M_2 \) is Riemannian, \( M_1 \) is Berwaldian and

\[
C^i_{jk} \frac{\partial f_1}{\partial x^i} = -2 \frac{\partial g^{ij}}{\partial y^k} \frac{\partial f_1}{\partial x^i} = 0.
\]
Theorem 23 ([9]) Let \((M_1 f_2, M_2, F)\) be a proper DWP-Finsler manifold. Then \((M_1 f_2, M_2, F)\) is weakly Berwald if and only if \(M_2\) and \(M_1\) are weakly Berwalds and
\[
C^i_j \frac{\partial f_1}{\partial x^i} = C^i_{\gamma \nu} \frac{\partial f_2}{\partial u^\nu} = 0.
\]

Corollary 24 ([9]) Let \((M_1 \times_f M_2, F)\) be a proper WP-Finsler manifold. Then \((M_1 \times_f M_2, F)\) is Douglas if and only if \(M_2\) is Riemannian, \(M_1\) is Berwaldian and
\[
C^i_j \frac{\partial f_1}{\partial x^i} = 0.
\]

Theorem 25 A DWP-Finsler manifold \((M_1 \times_f M_2, F)\) with isotropic mean Berwald curvature is a weakly Berwald manifold provided that
\[
C^i_j \frac{\partial f_1}{\partial x^i} = 0 \text{ or } C^i_{\gamma \nu} \frac{\partial f_2}{\partial u^\nu} = 0.
\]

3 Main result

Theorem 31 If \((M, F)\) is a 2-dimensional Finsler manifold and \(f\) is nonconstant smooth function on \(M\) satisfying
\[
\frac{\partial g^{ij}}{\partial y^k} \frac{\partial f}{\partial x^i} = 0, \tag{1}
\]
then \(M\) is a Riemannian manifold.

Proof Let \(f\) be a nonconstant smooth function on \(M\) which satisfies (1). Then we have
\[
g^{ij} \frac{\partial f}{\partial x^j} = c^i(x), \tag{2}
\]
where \(c^i(x)\) is a smooth function on \(M\). The above Equ. (2) implies that
\[
\frac{\partial f}{\partial x^k} = \frac{1}{2} c^i(x) \frac{\partial^2 F^2}{\partial y^i \partial y^j}.
\]
By integrating (3) with respect to \(y^j\) we obtain
\[
2 \frac{\partial f}{\partial x^j} y^j = c^i(x) \frac{\partial F^2}{\partial y^i}.
\]
By Choosing \(u = F^2, A_j = \frac{\partial f}{\partial x^j}\) in the equation (4), we have the following PDE equation
\[
c^i_j \frac{\partial u}{\partial y^j} + c^2 \frac{\partial u}{\partial y^2} = 2A_1 y^1 + 2A_2 y^2. \tag{5}
\]
The general solution of PDE equation (5) are given by

\[ F^2 = u = A_1 \frac{c_1}{c^1} (y^1)^2 + A_2 \frac{c_2}{c^2} (y^2)^2 + \varphi (c^2 y^1 - c^1 y^2) \]  

(6)

where \( \varphi \) is an arbitrary smooth one variable function \[10\]. Since \( u \) is homogeneous of degree 2 so \( \varphi \) is homogeneous of degree 2, too. That \( \varphi \) is one variable implies that \( \varphi(t) = at^2 \), where \( a \) is a real constant, so we have

\[ F^2 = A_1 \frac{c_1}{c^1} (y^1)^2 + A_2 \frac{c_2}{c^2} (y^2)^2 + a(c^2 y^1 - c^1 y^2)^2. \]  

(7)

The Cartan torsion

\[ C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = 0 \]

thus \( M \) is Riemannian.

**Corollary 32** Let \((M_1, F_1)\) and \((M_2, F_2)\) be Finsler manifolds with \( \text{dim} \ M_1 = 2 \), \( \text{dim} \ M_2 = n_2 \) and \( f_i : M_i \rightarrow \mathbb{R}, i = 1, 2 \) are positive smooth functions.

1. A proper \((2 + n_2)\)-dimensional WP-Finsler manifold \( M_1 \times f_1 M_2 \) is a Berwald manifold, if and only if it is a Riemannian manifold.
2. A proper \((2 + 2)\)-dimensional DW-P-Finsler manifold \( M_1 f_2 \times f_1 M_2 \) is a weakly Berwald manifold, if and only if it is Riemannian manifold (\( \text{dim} \ M_2 = 2 \)).
3. A proper \((2 + n_2)\)-dimensional WP-Finsler manifold \( M_1 \times f_1 M_2 \) is a Douglas manifold, if and only if it is a Riemannian manifold.

The authors have not succeeded in finding a counterexample to the following problem. They conjecture that it might be true but, unfortunately, they have been unable to provide a proof for it.

**Problem.**

Is any \( n \)-dimensional \((n > 2)\) Finsler manifold \((M, F)\) with function \( f \) which is nonconstant and smooth on \( M \) satisfying

\[ \frac{\partial^2 g^{ij}}{\partial y^i} \frac{\partial f}{\partial x^j} = 0, \]

a Riemannian manifold?

**References**

1. D. Allison, Pseudoconvexity in Lorentzian doubly warped products, Geom. Dedicata, 1991, 39: 223-227, doi: 10.1007/BF00182295.
2. J. K. Beem and T. G. Powell, Geodesic completeness and maximality in Lorentzian warped products, Tensor (N.S.), 1982, 39: 31-36.
3. R. L. Bishop and B. O’Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., 1969, 145: 1-49.
4. S.-S. Chern and Z. Shen, Riemann-Finsler geometry, vol. 6 of Nankai Tracts in Mathematics, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
5. M. Faghfouri and N. Ghaffarzadeh, On doubly warped product submanifolds of generalized $(\kappa, \mu)$-space forms, Afrika Matematika, 2014, doi:10.1007/s13370-014-0299-y.
6. M. Faghfouri and A. Majidi, On doubly warped product immersions, Journal of Geometry, 2014, doi: 10.1007/s00022-014-0245-z.
7. L. Kozma, R. Peter, and C. Varga, Warped product of Finsler manifolds, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 2001, 44: 157-170.
8. E. Peyghan and A. Tayebi, On doubly warped product Finsler manifolds, Nonlinear Anal. Real World Appl., 2012, 13:1703-1720, doi: 10.1016/j.nonrwa.2011.12.002.
9. E. Peyghan, A. Tayebi, and B. Najafi, Doubly warped product Finsler manifolds with some non-Riemannian curvature properties, Ann. Polon. Math., 2012, 105: 293-311, doi: 10.4064/ap105-3-6.
10. I. N. Sneddon, Elements of partial differential equations, Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1957 original.
11. A. Tayebi and B. Najafi, On isotropic Berwald metrics, Ann. Polon. Math., 2012, 103: 109-121, doi: 10.4064/ap103-2-1.
12. A. Tayebi and H. Sadeghi, On Cartan torsion of Finsler metrics, Publ. Math. Debrecen, 2013, 82: 461-471, doi: 10.5486/PMD.2013.5379.
13. B. Unal, Doubly warped products, Differential Geom. Appl., 2001, 15: 253-263, doi: 10.1016/S0926-2245(01)00051-1.